Quantum and Classical Data Transmission Through Completely Depolarising Channels in a Superposition of Cyclic Orders

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Completely depolarising channels are often regarded as the prototype of physical processes that are useless for communication: any message that passes through them along a well-defined trajectory is completely erased. When two such channels are used in a quantum superposition of two alternative orders, they become able to transmit some amount of classical information, but still no quantum information can pass through them. Here we show that the ability to place $N$ completely depolarising channels in a superposition of $N$ alternative causal orders enables a high-fidelity, heralded transmission of quantum information with error vanishing as $1/N$. This phenomenon highlights a fundamental difference with the $N = 2$ case, where completely depolarising channels are unable to transmit quantum data, even when placed in a superposition of causal orders. The ability to place quantum channels in a superposition of orders also leads to an increase of the classical communication capacity with $N$, which we rigorously prove by deriving an exact single-letter expression. Our results highlight the more complex patterns of correlations arising from multiple causal orders, which is similar to the more complex patterns of entanglement arising in multipartite quantum systems.

Introduction. Shannon’s information theory was initially developed under the assumption that the information carriers were classical systems [1]. At the fundamental level, however, physical systems obey the laws of quantum mechanics, which enable radically new communication protocols [2, 3] and give rise to a variety of new communication capacities [4].

Traditionally, the extension of Shannon’s theory to the quantum domain assumed that the configuration of the communication devices was fixed. In principle, however, quantum theory is compatible with scenarios where the communication devices are arranged in a coherent superposition of alternative configurations. For example, the available devices could act in different orders, and the choice of order could be controlled by the state of a quantum system, using a primitive known as the quantum SWITCH [5, 6]. Similarly, the devices could be used as alternatives to one another, and the choice of which device is used for communication could be controlled by the state of a quantum system, giving rise to a superposition of alternative quantum evolutions [7, 12].

The ability to superpose different configurations of communication devices can be exploited to achieve advantages over the standard model of quantum Shannon theory, where the configuration of the channels is fixed. Advantages of the superposition of orders have been shown in Refs. [13–19], while advantages of the superposition of channels have been shown in Refs [9–11]. At a conceptual level, these advantages can be rigorously formalised in a resource-theoretic framework, where the resources are communication devices, and the allowed operations on them include placement operations, which determine the arrangement of the communication devices in space and time [20]. Different advantages can then be understood as the result of different ways to enlarge the set of placement operations allowed by standard quantum Shannon theory. At a more practical level, new communication protocols with superpositions of configurations have been experimentally realised [21, 25]. The information-theoretic advantages of the superposition of causal orders have also inspired a new line of investigation in quantum thermodynamics [20].

One of the most striking advantages of the superposition of configurations is the ability to communicate through channels that completely block information when used in a definite configuration. The prototype
of such channel is the completely depolarising channel, which outputs white noise independently of its input. Strikingly, it was shown that two completely depolarising channels can be used for transmitting classical information when arranged in a superposition of two alternative orders \cite{13}. On the other hand, this phenomenon is limited to the transmission of classical bits: in this Letter we will show that, when two completely depolarising channels are combined in the quantum SWITCH, the resulting channel cannot be used to transmit quantum data.

While the communication advantages of the quantum SWITCH of two channels are well known, much less is known about the advantages of the quantum SWITCH of multiple channels. Recent works \cite{16,17} considered the amount of classical bits transmitted through \( N \) completely depolarising channels, showing an increase of the Holevo information \cite{27}. However, the Holevo information is only a lower bound to the actual capacity \cite{28,29}, and an increase in the Holevo information does not necessarily imply an increase in the capacity. Moreover, the increase in the capacity, while technically interesting, would only be a quantitative improvement in a task that can already be accomplished with \( N = 2 \) channels. A natural question is whether there exists some communication task that cannot be achieved at all by superposing the order of two channels but instead becomes possible when multiple channels are used.

Here, we answer the question in the affirmative, providing a concrete example of a communication task that can only achieved when \( N > 2 \) causal orders are superposed. We consider \( N \) completely depolarising channels combined in a superposition of \( N \) causal orders related to each other by cyclic permutations. We show that a high-fidelity heralded transmission of quantum bits can be achieved with error vanishing as \( 1/N \). Our finding is in stark contrast to the impossibility of quantum data transmission through \( N = 2 \) completely depolarising channels, and highlights a genuinely new feature arising from \( N > 2 \) channels in alternative causal orders. The high-fidelity heralded transmission of quantum data is also potentially relevant for the task of entanglement distribution in quantum networks \cite{30} and for the task of private classical communication \cite{31,32}.

In addition to establishing the possibility of heralded quantum communication, we analytically determine the classical communication capacity of \( N \) completely depolarising channels in a superposition of \( N \) causal orders, and we demonstrate that the capacity increases monotonically with \( N \). To this purpose, we establish a connection between the quantum SWITCH of completely depolarising channels and the universal quantum NOT gate \cite{33,34}. We then use this connection to prove a single-letter formula for the classical capacity. Our result demonstrates that increasing the number of “useless” channels leads to an increase in the number of bits that can be reliably transmitted. To the best of our knowledge, this is the first rigorous demonstration of a task where the benefit of the superposition of causal orders grows monotonically with the number of configurations that are superposed.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Communication through \( N \) channels in a superposition of \( N \) cyclic orders. A sender, located at node 1 of a quantum communication network, sends messages to a receiver, located at node \( N + 1 \), through a sequence of intermediate nodes, labelled as 2, \ldots, \( N \). The intermediate nodes are connected by \( N \) quantum channels \( C^{(1)}, \ldots, C^{(N)} \), which have been placed in one of \( N \) configurations related by cyclic permutations, as shown in the graphic. The choice of configuration is controlled by a quantum system in a coherent superposition.}
\end{figure}

Communication devices in a quantum superposition of alternative orders. A communication device transmitting a quantum system is described by a quantum channel, that is, a completely positive trace preserving linear map \( C \) transforming linear operators on the system’s Hilbert space \( \mathcal{H} \). Any such map can be written in the Kraus representation \( C(\rho) = \sum_i C_i \rho C_i^\dagger \), where the Kraus operators \( \{ C_i \} \) satisfy \( \sum_i C_i^\dagger C_i = I \).

Here, we consider the application of \( N \) channels in a coherent superposition of different alternative orders. The superposition of orders is constructed using the quantum SWITCH \cite{5,6}, a higher-order operation that transforms two quantum channels into a new quantum channel, in which the two input channels are executed in one of two alternative orders, depending on the state of control qubit, called the order qubit. Here we adopt the original definition of the quantum SWITCH \cite{5}, where the two channels act in two subsequent time steps, possibly allowing for intermediate operations. Mathematically, the quantum SWITCH transforms two input quantum channels \( C^{(1)} \) and \( C^{(2)} \) into the output channel

\[ S[C^{(1)}, C^{(2)}](\cdot) = \sum_{j_1,j_2} W_{j_1,j_2} \cdot W_{j_1,j_2}^\dagger, \]

whose Kraus operators \( W_{ij} \) are defined as

\[ W_{ij} := |0\rangle\langle 0| \otimes C^{(1)}_{j_1} \otimes C^{(2)}_{j_2} + |1\rangle\langle 1| \otimes C^{(2)}_{j_2} \otimes C^{(1)}_{j_1}, \]

where the three systems in the tensor product on the right-hand side are the order qubit, the input system in the first time step, and the input system in the second time step. Here, \( \{ C^{(1)}_{j_1} \} \) and \( \{ C^{(2)}_{j_2} \} \) are Kraus operators for channels \( C^{(1)} \) and \( C^{(2)} \), respectively. Note that,
while the individual Kraus operators \(W_{ij}\) depend on the choice of Kraus representation for \(C^{(1)}\) and \(C^{(2)}\), the overall quantum channel \(S[C^{(1)}, C^{(2)}]\) depends only on the channels \(C^{(1)}\) and \(C^{(2)}\) themselves, making the quantum SWITCH a well-defined operation on quantum channels \([6,37]\).

It is worth stressing that, while the order of the two processes \(C^{(1)}\) and \(C^{(2)}\) inside the quantum SWITCH is indefinite, the channel \(S[C^{(1)}, C^{(2)}]\) produced by the quantum SWITCH has a well-defined causal structure: the input of the first time slot is provided first, followed by the output of the first time slot, the input of the second time slot, and, finally, the output of the second time slot. Accordingly, a communication protocol using the channel \(S[C^{(1)}, C^{(2)}]\) will have a well-defined causal structure: first, the sender inputs a state in the first time slot, then the first time slot is connected to the second with some intermediate operation, and finally the receiver collects the output of the second time slot.

When \(N > 2\) channels are available, the quantum SWITCH operation \([1]\) can be applied to each pair of channels, thus generating all possible permutations of their orders \([35]\). In a resource theory of communication, the quantum SWITCH can be viewed as an operation performed by a communication provider, who places the available communication devices between the sender and receiver \([20]\). Here, we consider a placement of the \(N\) devices in a network with \(N - 1\) intermediate nodes, as illustrated in Fig. 1. Again, note that the causal structure of the process generated by the quantum SWITCH is well-defined, even though the \(N\) channels inside the quantum SWITCH act in an indefinite order. As a consequence, the causal structure of the communication protocol in the network of Fig. 1 is well-defined: first, the sender inputs the state in the first node, then the first intermediate party receives the output at the second node and sends it to the third node, and so on until, finally, the receiver receives the output at the last node.

We will assume that the order qubits are inaccessible to the sender and are initialised by the communication provider in a fixed state before the beginning of the communication protocol. Also, we will take the intermediate nodes in Fig. 1 to contain identity operations, so that the effective channel available to the sender and receiver becomes

\[
C_{\text{eff}}(\rho) = \sum_{\pi, \pi' \in S} \omega_{\pi, \pi'} |\pi\rangle \langle \pi'| \otimes C_{\pi\pi'}(\rho), \tag{3}
\]

where \(S\) is a set of permutations, \(\omega\) is the state of the order qubits (with matrix elements \(\omega_{\pi, \pi'}\) and support in a subspace spanned by an orthonormal basis \(|\pi\rangle\pi' \in S\) labelled by permutations in \(S\), and

\[
C_{\pi\pi'}(\rho) := \sum_{j_1, \ldots, j_N} C^{\pi(1) \ldots \pi(N)}_{j_{\pi(1)} \ldots j_{\pi(N)}} \rho C^{\pi'(1) \ldots \pi'(N)}_{j'_{\pi'(1)} \ldots j'_{\pi'(N)}} \tag{4}
\]

with the notation \(C^{i_1 \ldots i_N}_{j_1 \ldots j_N} := C^{(i_1)}_{j_1} \ldots C^{(i_N)}_{j_N}\), where \(\{C^{(i)}_{j_i}\}\) are Kraus operators for channel \(C^{(i)}\).

Heralded quantum communication through completely depolarising channels. When the configuration of the channels is fixed, the completely depolarising channel \(D(\cdot) := I/d \text{Tr}[\cdot]\) is the prototype of a useless channel: since its output is independent of the input, this channel does not permit the transmission of any data, be it classical or quantum.

Now, suppose that \(N\) completely depolarising channels are combined by the quantum SWITCH, generating the effective channel \(C_{\text{eff}}\) in Equation (3). In the following we will take \(S\) to be the set of cyclic permutations \(\pi\), mapping the index \(a\) into the index \(\pi(a) = (a + k) \mod N\) for some given \(k\) \(\in\{0, \ldots, N - 1\}\), and we will set \(\omega = |e_0\rangle \langle e_0|\), with \(|e_0\rangle = \sum_\pi |\pi\rangle/\sqrt{N}\).

A convenient Kraus representation of the completely depolarising channel is a uniform mixture of an orthogonal unitary basis \(\{U_i\}_{i=1}^d\), namely \(D(\rho) = \sum_{i=1}^d U_i \rho U_i^{\dagger} / d^2\), where \(d\) is the dimension of the system and \(\text{Tr}[U_i U_j] = d \delta_{i,j}\). Using this representation, we derive the relations

\[
C_{\pi\pi}(\rho) = \frac{I}{d} \quad \text{and} \quad C_{\pi\pi'}(\rho) = \frac{\rho}{d^2} \quad \forall \pi \neq \pi', \tag{5}
\]

(see Appendix A). Inserting these relations into Eq. (3) yields the expression

\[
C_{\text{eff}}(\rho) = \frac{I}{N} \otimes \frac{I}{d} + \sum_{\pi \neq \pi'} |\pi\rangle \langle \pi'| \otimes \frac{\rho}{N d^2} \tag{6}
\]

the second equality following from the relations \(N|e_0\rangle \langle e_0| = \sum_\pi |\pi\rangle \langle \pi|\) and \(I = \sum_\pi |\pi\rangle \langle \pi|\). Rearranging the terms in Eq. (6), we rewrite the effective channel as

\[
C_{\text{eff}}(\rho) = (1 - p) \rho_0 \otimes \mathcal{E}_0(\rho) + p \rho_1 \otimes \mathcal{E}_1(\rho), \tag{7}
\]

where \(\rho_0 := |e_0\rangle \langle e_0|\) and \(\rho_1 := (I - |e_0\rangle \langle e_0|)/(N - 1)\) are orthogonal states of the control system, \(\mathcal{E}_0\) and \(\mathcal{E}_1\) are the quantum channels defined by

\[
\mathcal{E}_0(\rho) := \frac{N - 1}{N - 1 + d^2} \rho + \frac{d^2}{N - 1 + d^2} \frac{I}{d}, \tag{8}
\]

and

\[
\mathcal{E}_1(\rho) := \frac{d^2}{d^2 - 1} \rho - \frac{1}{d^2 - 1} I, \tag{9}
\]

respectively, and \(p := (N - 1)(d^2 - 1)/(N d^2)\). Two alternative ways to generate the channel \(C_{\text{eff}}\) from depolarising channels in a superposition cyclic orders are discussed in Appendix B.

Equation (7) shows that the effective channel \(C_{\text{eff}}\) is a mixture of two channels \(\mathcal{E}_0\) and \(\mathcal{E}_1\), flagged by two orthogonal states of the order qubits. By measuring the order qubits, it is then possible to herald the occurrence of the channels \(\mathcal{E}_0\) and \(\mathcal{E}_1\).
The channel $E_1$ is independent of $N$. For $d = 2$, it is the universal NOT channel introduced by Bužek, Hillery, and Werner [33] and experimentally realised in a series of works [34–36]. The universal NOT gate is known to be an entanglement-breaking channel [37], or equivalently, a “measure-and-reprepare” channel, which can be realised by measuring the input and preparing an output state depending on the measurement outcome [40]. Since entanglement-breaking channels have zero quantum capacity [41], no quantum information can be transmitted through the channel $E_1$. For $d > 2$, the channel $E_1$ is a generalisation of the universal NOT, and can be characterised as the channel that minimises the fidelity between a generic input state $|\psi\rangle$ and the corresponding output state $E_1(|\psi\rangle\langle\psi|)$ (see Appendix C). In Appendix C we show that $E_1$ is entanglement-breaking and therefore unable to transmit any quantum data.

The channel $E_0$, instead, is a depolarising channel, with probability of depolarisation equal to $d^2/(N + d^2 - 1)$. Remarkably, this probability vanishes as $d^2/N$ in the large $N$ limit, enabling a perfect transmission of quantum data. It is also remarkable that the probability of high-fidelity transmission does not vanish in the large $N$ limit: such a probability remains larger than $1/d^2$ for every value of $N$. For qubits, this means that the state of the target system has a probability at least 25% of reaching the receiver with an error smaller than $4/N$.

The heralded, high-fidelity transmission of quantum information could be exploited for the distribution of entanglement in quantum networks [39], which in turn serves as a primitive for distributed quantum computation [42]. Our results could also be useful for cryptographic purposes, such as private classical communication [31, 32], or the generation of secret keys via the BB84 protocol [3] or E91 protocols [3]. A discussion of these applications is provided in Appendix D.

For finite $N$, it is possible to show that channel $E_0$ has a non-zero quantum capacity for all values of $N$ larger than a given finite value $N_0 > 2$. For example, for $d = 2$ and $N > 13$ it is possible to show that the probability of depolarisation is less than 1/4, which guarantees that the depolarising channel $E_0$ has a non-zero quantum capacity [41]. In turn, the non-zero quantum capacity of channel $E_0$ ensures that the overall channel $\mathcal{C}_\text{eff}$ has a non-zero quantum capacity assisted by two-way classical communication [43], as shown in Appendix E. In Appendix E we also show that quantum data transmission with the assistance of two-way classical communication is possible through the quantum SWITCH of $N$ cyclic permutations if and only if $N \geq d + 2$.

The possibility of quantum information transmission is a fundamental difference between the bipartite and the multipartite quantum SWITCH in a way that is somewhat reminiscent of the difference between bipartite and multipartite entanglement. For $N = 2$, we prove that no superposition of causal orders permits the transmission of quantum bits through completely depolarising channels, under the natural assumption that the sender does not use the control system to establish entanglement with the receiver (see Appendix C for the details).

All the results presented so far concerned the superposition of completely depolarising channels. A natural question is whether any of our conclusions would change if we were to consider partially depolarising channels. In particular, one could ask whether the quantum SWITCH could enable the transmission of quantum information using $N = 2$ partially depolarising channels that individually have zero quantum capacity. In Appendix H we answer the question in the negative, showing that the quantum capacity of each depolarising channel is a bottleneck for the amount of quantum information one can send through the quantum SWITCH. An interesting open question is whether the use of partially depolarising channels could reduce the number of channels needed to achieve quantum data transmission starting from channels with zero capacity. More broadly, the study of quantum communication with partially depolarising channels in an indefinite causal order remains as an interesting problem for future research.

Enhanced transmission of classical information. We now quantify the amount of classical bits transmittable through $N$ depolarising channels in a superposition of $N$ alternative orders. By the Holevo-Schumacher-Westmoreland theorem [28, 29], the classical capacity of a generic noisy channel $\mathcal{N}$ is given by $C(\mathcal{N}) = \lim_{n \to \infty} \chi(N^\otimes n)/n$, where $\chi$ is the Holevo information [27], defined as $\chi(N) = \max_{(\rho_x)} \sum_{x} p_X S(\rho_x) - \sum_{x} p_x S(\rho_x)$, $\rho_x$ being an arbitrary ensemble of quantum states, and $S(\rho) = -\text{Tr}[\rho \log \rho]$ being the von Neumann entropy. In Appendix L we prove that the Holevo information of the effective channel $\mathcal{C}_\text{eff}$ is additive, and therefore the classical capacity has the single-letter formula $C(\mathcal{C}_\text{eff}) = \chi(\mathcal{C}_\text{eff})$, for which we provide an exact expression.

The classical capacity is plotted in Fig. 2, for different values of $N$ and $d$. The capacity increases monotonically with $N$, rigorously demonstrating the benefit of increasing the number of alternative orders. In Appendix L we provide an asymptotic expression for the capacity in the

![FIG. 2. Classical capacity of the effective channel $C_{\text{eff}}$, plotted with respect to $N$ for message systems of dimension $d = 2, 3, 4$ and 5.](image-url)
large $N$ limit, showing that it decreases with $d$, tending to zero for $d \to \infty$. For $N = 2$, the decrease with $d$ was observed for the Holevo information $[13]$, although it was not known whether the actual channel capacity was also decreasing.

Conclusions. We demonstrated a communication advantage of the superposition of multiple causal orders, by showing a communication task that cannot be achieved by superposing two orders, but becomes possible when the number of orders is sufficiently large. Specifically, we demonstrated that the placement of $N$ completely depolarising channels in a superposition of $N$ cyclic orders enables a high-fidelity heralded transmission of quantum information, with error vanishing as $1/N$. For finite $N$, we found that a non-zero quantum capacity assisted by two-way classical communication can be achieved with $N$ qubit depolarising channels whenever $N \geq 4$.

The possibility of quantum data transmission through completely depolarising channels highlights a fundamental difference with the $N = 2$ scenario, where no quantum information can pass through completely depolarising channels. A recent experiment $[14]$ on the superposition of $N = 4$ channels suggests that an experimental demonstration of non-zero quantum capacity assisted by two-way classical communication could be achieved in the near future. Most importantly, our results motivate an investigation of the operational features of the different types of quantum superpositions arising when multiple causal orders are superposed.

It is intriguing to imagine that the distinction between the superposition of two and multiple causal orders could mirror the distinction between bipartite and multipartite entanglement, whose study has led to the discovery of a wealth of new quantum information protocols. In this respect, our result indicates that the superposition of multiple causal orders is a genuinely new resource that is not reducible to the superposition of $N = 2$ causal orders, just as genuine multipartite entanglement is not reducible to bipartite entanglement. We hope that our work will stimulate future explorations of the analogy between superpositions of causal orders and multipartite entanglement, thereby leading to a deeper understanding of the interplay between causality and quantum physics.

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Note added. After the completion of this work, we became aware of Ref. $[45]$, which independently derived the Holevo information of $N$ completely depolarising channels in a superposition of $N$ cyclic permutations.

[1] C. E. Shannon, The Bell System Technical Journal 27, 379 (1948).
[2] C. H. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing (India, 1984) p. 175.
[3] A. K. Ekert, Physical Review Letters 67, 661 (1991).
[4] M. M. Wilde, Quantum information theory (Cambridge University Press, 2013).
[5] G. Chiribella, G. D’Ariano, P. Perinotti, and B. Valiron, arXiv preprint arXiv:0912.0195 (2009).
[6] G. Chiribella, G. M. D’Ariano, P. Perinotti, and B. Valiron, Physical Review A 88, 022318 (2013).
[7] Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, Physical Review Letters 64, 2965 (1990).
[8] D. K. Oi, Physical Review Letters 91, 067902 (2003).
[9] N. Gisin, N. Linden, S. Massar, and S. Popescu, Physical Review A 72, 012328 (2005).
[10] A. A. Abbott, J. Wechs, D. Horsman, M. Mhalla, and C. Branciard, Quantum 4, 333 (2020).
[11] G. Chiribella and H. Kristjánsson, Proceedings of the Royal Society A 475, 20180003 (2019).
[12] Q. Dong, S. Nakayama, A. Soeda, and M. Murao, arXiv preprint arXiv:1911.01645 (2019).
[13] D. Ebler, S. Salek, and G. Chiribella, Physical Review Letters 120, 120502 (2018).
[14] S. Salek, D. Ebler, and G. Chiribella, arXiv preprint arXiv:1809.06655 (2018).
[15] G. Chiribella, M. Banik, S. S. Bhattacharya, T. Guha, M. Alimuddin, A. Roy, S. Saha, S. Agrawal, and G. Kar, New Journal of Physics 23, 033039 (2021).
[16] L. M. Procopio, F. Delgado, M. Enríquez, N. Belabas, and J. A. Levenson, Entropy 21, 1012 (2019).
[17] L. M. Procopio, F. Delgado, M. Enríquez, N. Belabas, and J. A. Levenson, Physical Review A 101, 012346 (2020).
[18] N. Loizeau and A. Grinbaum, Physical Review A 101, 012340 (2020).
[19] S. S. Bhattacharya, A. G. Maity, T. Guha, G. Chiribella, and M. Banik, PRX Quantum 2, 020350 (2021).
[20] H. Kristjánsson, G. Chiribella, S. Salek, D. Ebler, and M. Wilson, New Journal of Physics 22, 073014 (2020).
[21] L.-P. Lamoureux, E. Brainis, N. Cerf, P. Emplit, M. Haelterman, and S. Massar, Physical Review Letters 94, 230501 (2005).
Appendix A: Off-diagonal terms of the cyclic switch

We consider the terms for which $\pi \neq \pi'$ in

$$C_\text{eff}(\rho) = \sum_{\pi, \pi' \in S} \omega_{\pi, \pi'} \vert \pi \rangle \langle \pi' \vert \otimes C_{\pi \pi'}(\rho), \quad (A1)$$
where given the Kraus decomposition \( \{ \frac{1}{d} U_{j_{(i)}} \pi(i) \}_{i=1}^d \) of each completely depolarising channel \( D^{(i)} \)

\[
C_{\pi\pi'} = \sum_{j_1, \ldots, j_N} \frac{U_{j_{(1)}}^{\pi(1)} \cdots U_{j_{(N)}}^{\pi(N)} A_{j_{(1)}}^{\pi'(1)} \cdots A_{j_{(N)}}^{\pi'(N)}}{d^{2N}-1}. \quad (A2)
\]

We show that any \( C_{\pi\pi'} \), with \( \pi \neq \pi' \) cyclic permutations, evaluates to the same expression

\[
C_{\pi\pi'} = \frac{\rho}{d^2}. \quad (A3)
\]

The derivation relies on the property of the Kraus decomposition of a completely depolarising channel used in [13] to derive the output of the 2 party switch of two completely depolarising channels. Specifically, they form an orthonormal unitary basis on the \( d^2 \) dimensional space of linear operators where the orthonormality is with respect to the Hilbert Schmidt product. Since \( \pi \) and \( \pi' \) are cyclic permutations, \( \pi' \) itself is a cyclic permutation relative to \( \pi \). Each \( C_{\pi\pi'} \) can be evaluated using only the knowledge that the relative permutation between \( \pi \) and \( \pi' \) is a non-trivial cyclic permutation. Adopting an ordered product notation

\[
\prod_{a=\pi(1)}^{\pi(N)} U_{j_a}^a \equiv U_{j_{\pi(1)}}^{\pi(1)} \cdots U_{j_{\pi(N)}}^{\pi(N)},
\]

the output can be written as

\[
C_{\pi\pi'} = \frac{1}{d^{2N}} \sum_{\{j_k \in N\}} d^2 \left( \prod_{a=\pi(1)}^{\pi(N)} U_{j_a}^a \right) \rho \left( \prod_{b=\pi'(1)}^{\pi'(N)} U_{j_b}^{b\dagger} \right). \quad (A5)
\]

Using the cyclic property of \( \pi' \) on the right hand side we can rewrite this as

\[
C_{\pi\pi'} = \frac{1}{d^{2N}} \sum_{\{j_k \in N\}} d^2 \left( \prod_{a=\pi(1)}^{\pi(N-1)} U_{j_a}^a \right) \rho \left( \prod_{b=\pi'(1)}^{\pi'(N)} U_{j_b}^{b\dagger} \right) \left( \prod_{c=\pi(N-1)}^{\pi'(1)} U_{j_c}^{c\dagger} \right) \quad (A6)
\]

Next, summing over \( j_{\pi(N)} \) through the use of the identity

\[
\frac{1}{d^2} \sum_{j_{\pi(N)}=1}^{d^2} U_{j_{\pi(N)}}^{\pi(N)} \rho U_{j_{\pi(N)}}^{\pi(N)} = \text{Tr}[\rho] \frac{I}{d} \quad (A7)
\]

gives

\[
C_{\pi\pi'} = \frac{d}{d^{2N}} \sum_{\{j_k \in N, k \neq \pi(N)\}} d^2 \left( \prod_{a=\pi(1)}^{\pi(N-1)} U_{j_a}^a \right) \text{Tr} \left[ \rho \prod_{b=\pi'(1)}^{\pi'(N)} U_{j_b}^{b\dagger} \right] \left( \prod_{c=\pi(N-1)}^{\pi'(1)} U_{j_c}^{c\dagger} \right) \quad (A8)
\]

Then summing over \( j_{\pi'(N)} \) through the use of the identity

\[
\sum_{j_{\pi'(N)}=1}^{d^2} U_{j_{\pi'(N)}}^{\pi'(N)} \text{Tr} \left[ \rho U_{j_{\pi'(N)}}^{\pi'(N)} \right] = \rho d \quad (A9)
\]

gives

\[
C_{\pi\pi'} = \frac{1}{d^{2(N-1)}} \sum_{\{j_k \in N, k \neq \pi(N), \pi'(N)\}} d^2 \left( \prod_{a=\pi(1)}^{\pi'(N)-1} U_{j_a}^a \right) \left( \prod_{b=\pi'(N)-1}^{\pi'(1)} U_{j_b}^{b\dagger} \right) \rho \quad (A10)
\]

\[
\sum_{\{j_k \in N, k \neq \pi(N), \pi'(N)\}} \rho = \frac{\rho}{d^2}.
\]
The above relations can also be derived by diagrammatic means, as done in Ref. [46], where the case of arbitrary non-cyclic permutations is also analysed. Notably, cyclic permutations appear to be those for which the off-diagonal terms in $C_{\pi\pi'}$ have the highest weight.

Appendix B: Two realisations of the effective channel $C_{\text{eff}}$

Here we discuss two alternative scenarios that give rise to the superposition of $N$ depolarising channels corresponding to channel $C_{\text{eff}}$ in Eq. (7) of the main text. For simplicity of presentation, we will focus on the realisation of the depolarising channels as random mixtures of unitary processes, although all the arguments below can be extended to arbitrary realisations.

FIG. 3. Realisation of the effective channel $C_{\text{eff}}$ through coherent control of paths. A quantum system traverses $N$ regions at $N$ distinct moments of time, following a path controlled by an $N$-dimensional control system (bottom wire). The control system is initialised in the uniform superposition state $|e_0\rangle = \sum_{k=0}^{N-1} |k\rangle / \sqrt{N}$, and the control of the paths is implemented by unitary channels $\text{ctrl} - S^{k+i}$, corresponding to unitary operators $\text{ctrl} - S^{k+i} := \sum_{k=0}^{N-1} S^{k+i} \otimes |k\rangle \langle k|$, where $S$ is the unitary operator that shifts cyclically the $N$ inputs by one position. In the $k$-th region, the system undergoes a unitary process $U^{(k)}$, chosen at random according to a suitable probability distribution. When the system does not pass through the $k$-th region, the input to that region is the vacuum state $|\text{vac}\rangle$. Overall, the process in the region is described by a unitary channel $\widetilde{U}^{(k)}$, acting on the direct sum of a one-particle subspace and of the vacuum, and coinciding with $U^{(k)}$ on the states of the one-particle subspace [11]. The path of the system through the $N$ regions is controlled by a quantum system (bottom wire in the figure), which permutes the system with the vacuum state of $N-1$ modes. The overall evolution resulting from this scheme coincides with the effective channel $C_{\text{eff}}$ in Eq. (7) of the main text. This way of reproducing the output of the quantum SWITCH was introduced in [11] for $N = 2$ and is generalised here to $N > 2$.

It is important to note that this realisation of the channel $C_{\text{eff}}$ requires the unitary process $\widetilde{U}^{(k)}$ to be the same at all time steps. This situation can be engineered in a photonic table-top scenario, where the unitary processes are implemented by optical devices, such as waveplates, whose behaviour is stable over the timescale of the experiment. To reproduce the completely depolarising channel, the choice of waveplates is randomised, or simply taken to be unknown to the sender and receiver. On the other hand, depolarising noise in real world applications often arises from uncontrolled fluctuations, which may occur on a short time scale. In this setting, the unitary processes taking place in each region may vary during the transmission of the system from the sender to the receiver, especially when $N$ is large.

A more radical way to generate the channel $C_{\text{eff}}$ is to use a quantum circuit with loops, as illustrated in Figure 4. Mathematically, the loops in the circuit correspond to postselected teleportation protocols, where the probability of success of one of the outcomes is artificially scaled up to 1 [5, 47]. Physically, circuits with loops could arise in scenarios...
FIG. 4. Realisation of the effective channel $C_{\text{eff}}$ through a circuit with loops. A quantum system traverses $N$ regions via a quantum circuit including loops. The path of the system is determined by an $N$-dimensional quantum system (bottom wire), initialised in the superposition state $|e_0\rangle = \sum_{k=0}^{N-1} |k\rangle / \sqrt{N}$. The control of the paths is implemented by unitary channels $\text{ctrl}_k - S_k$ and $\text{ctrl}_{k+1}$ corresponding to unitary operators $\text{ctrl}_k - S_k := \sum_{k=0}^{N-1} S_k \otimes |k\rangle \langle k|$, where $S$ is the unitary operator that shifts cyclically $N$ quantum systems by one position. In the $k$-th region, the system undergoes a unitary process $U(k)$, chosen at random according to a suitable probability distribution. The overall evolution from the input to the output is given by the channel $C_{\text{eff}}$ in Eq. (7) of the main text.

Involving closed timelike curves, or scenarios where postselection is taken to be fundamental [47–50]. Alternatively, they could arise probabilistically from ordinary quantum circuits using quantum teleportation. In this realisation, however, the probability of successfully producing channel $C_{\text{eff}}$ decreases exponentially with $N$: for $N$ qubit channels, the success probability is $1/4^{N-1}$, due to the $N-1$ loops in the circuit.

The realisation in Figure 4 was presented in [5] for $N = 2$ and is generalised here to $N > 2$. An important feature is that it does not require correlations between processes happening at different times. Ideally, the processes $U(k)$ could even be taken to be instantaneous, and still the scheme in Figure 4 would provide a realisation of the channel $C_{\text{eff}}$. This is in stark contrast with the realisation in Figure 3 which requires each unitary gate $\tilde{U}(k)$ to be the same at $N$ different moments of time.

The scheme in Figure 4 also admits a more straightforward extension from unitary to non-unitary processes: explicitly, it allows one to realise the quantum SWITCH of $N$ arbitrary channels by simply inserting such channels in the green slots of the circuit. In this respect, this scheme reflects more closely the definition of the quantum SWITCH as a higher order operation that takes in input $N$ quantum channels, and generates a new quantum channels by connecting them in a superposition of orders [5, 6].

Appendix C: Generalisation of the universal $\text{NOT}$ gate to $d > 2$

The channel $E_1$ defined in Equation (9) of the main text is a generalisation of the universal $\text{NOT}$ gate to dimension $d > 2$. It can be equivalently expressed as

$$E_1(\rho) = \int d\psi \psi_\perp \text{Tr}[P_\psi \rho], \quad \psi_\perp := \frac{I - |\psi\rangle\langle \psi|}{d-1}, \quad P_\psi := d |\psi\rangle\langle \psi|,$$

(C1)

where $d\psi$ is the unitarily invariant probability distribution over the set of pure states. Operationally, the channel $E_1$ can be realised by measuring the input system with measurement operators $\{P_\psi\}$, and preparing the output state $\psi_\perp$ conditionally on the measurement outcome $\psi$. This implies that $E_1$ is an entanglement-breaking channel [39], and therefore cannot transmit any quantum information [41], even with the assistance of two-way classical communication.

Note that the channel $E_1$ is covariant with respect to the action of the group $\text{SU}(d)$, that is, it satisfies the condition

$$U \circ E_1 = E_1 \circ U \quad \forall U : \rho \mapsto U \rho U^\dagger, U \in \text{SU}(d).$$

(C2)
The covariance of $E_1$ follows immediately from Eq. (C1): for every operator $\rho \in L(\mathbb{C}^d)$, one has
\[
(E_1 \circ U)(\rho) = \int d\psi \frac{I - |\psi\rangle\langle\psi|}{d - 1} \operatorname{Tr}[d(|\psi\rangle\langle\psi|) U \rho U^\dagger] 
= \int d\psi U \left( \frac{I - U^\dagger|\psi\rangle\langle\psi|U}{d - 1} \right) U^\dagger \operatorname{Tr}[d(U^\dagger|\psi\rangle\langle\psi|U) \rho] 
= \int d\psi U \left( \frac{I - |\psi^\prime\rangle\langle\psi^\prime|}{d - 1} \right) U^\dagger \operatorname{Tr}[d(|\psi^\prime\rangle\langle\psi^\prime|) \rho] 
= (U \circ E_1)(\rho),
\] (C3)
where the first equality follows from from Eq. (C1), the second equality follows from the cyclic property of the trace, the third equation follows from setting $|\psi^\prime\rangle := U^\dagger|\psi\rangle$ and by using the unitary invariance of $d\psi$, and the fourth equation follows again from Eq. (C1).

The channel $E_1$ can also be characterised as the channel for which the output state is maximally orthogonal to the input state. More precisely, $E_1$ minimises the fidelity between a generic input state $|\psi\rangle$ and the corresponding output state $E_1(|\psi\rangle\langle\psi|)$, namely
\[
F(C) := \int d\psi \langle \phi | C(|\psi\rangle\langle\psi|) |\psi\rangle.
\] (C4)
Without loss of generality, the channel $C$ that minimises the fidelity $F(C)$ can be chosen to be covariant with respect to the action of the group $SU(d)$.

Note that covariant channels form a convex set. Since the fidelity (C4) is a linear function of $C$, the minimum is attained on an extreme point of the convex set of covariant channels. The extreme points have been classified in Ref. [51] in terms of the Choi representation, which associates channel $C$ to the Choi operator
\[
C := \sum_{i,j} C(|i\rangle\langle j|) \otimes |i\rangle\langle j|.
\] (C5)
Specifically, Theorem 1 of Ref. [51] shows that the extreme covariant channels have Choi operators of the form
\[
C = d \frac{P}{\operatorname{Tr}[P]},
\] (C6)
where $P$ is a projector on an irreducible subspace of the representation $\{U \otimes U\}_{U \in SU(d)}$. This representation has two irreducible subspaces: the first is one-dimensional and consists of vectors proportional to the maximally entangled state $|\Phi^+\rangle := \sum_i |i\rangle|i\rangle/\sqrt{d}$, while the second is the orthogonal complement of $|\Phi^+\rangle$.

The corresponding Choi operators are $C_1 = d |\Phi^+\rangle\langle\Phi^+| |\Phi^+\rangle\langle\Phi^+|$ and $C_2 = d/(d^2-1) (I \otimes I - |\Phi^+\rangle\langle\Phi^+|)$. Direct inspection shows that they are the Choi operators of the identity channel and of channel $E_1$, respectively. The identity channel is the maximiser of the fidelity (C4), while channel $E_1$ is the minimiser.

Appendix D: Application to private communication and key distribution

In the main text we have shown that the quantum SWITCH of $N$ completely depolarising channels (with sufficiently large $N$) enables a heralded transmission of quantum information, with probability of success larger than $1/d^2$ and with fidelity approaching $1$ as $1/N$. The ability to open up a reliable channel for the transmission of quantum data is potentially useful for cryptographic applications, including (but not limited to) private classical communication and quantum key distribution. In the following we focus on these two applications, clarifying the underlying assumptions for the application of our results, and outlining possible ways to go beyond these assumptions.

Private classical communication. The most direct cryptographic application of our results is the transmission of private classical messages [31,32]. In this task, a sender and a receiver use a given quantum channel to communicate classical messages with the guarantee that no eavesdropper can decode the message by accessing the environment of the channel. The rate at which classical communication can be privately transmitted is called the private capacity, and the quantum capacity of a channel is known to be a lower bound to its private capacity. Hence, a quantum channel with non-zero quantum capacity can transmit private classical messages at a non-zero rate. This fact can be used to argue a non-zero private capacity in the context of our communication protocol, in the scenario where two-way classical communication between sender and receiver is allowed [13].
The argument can be outlined as follows. The insertion of $N$ completely depolarising channels into the quantum SWITCH gives rise to the effective channel $C_{\text{eff}}$ in Eq. (7) of the main text. Then, a lower bound on the private capacity of $C_{\text{eff}}$ assisted by two-way classical communication follows from two observations:

1. There exists a two-way communication protocol that uses the channel $C_{\text{eff}}$ to achieve quantum data transmission at a rate that is at least the average of the two-way assisted quantum capacities of channels $E_0$ and $E_1$, with the probabilities given in the main text. The details of the protocol are presented in Section E of this supplemental material.

2. Channel $E_0$ has non-zero quantum capacity assisted by two way communication if and only if $N \geq d + 2$. The details of the protocol are presented in Section F of this supplemental material.

The above results show that the effective channel $C_{\text{eff}}$ has a non-zero quantum capacity assisted by two-way classical communication whenever $N \geq d + 2$. Since the (two-way assisted) quantum capacity is a lower bound to the (two-way assisted) private capacity, this argument shows that the quantum SWITCH of $N$ completely depolarising channels guarantees private classical communication whenever $N \geq d + 2$.

Quantum key distribution. In the main text we have shown that the quantum SWITCH of $N$ completely depolarising channels permits a perfect heralded transmission of single qubit states in the $N \to \infty$ limit. In turn, the heralded transmission of single qubits could be used to implement the BB84 [2] or E91 protocols [3] for the generation of secret keys. An obvious question is whether this heuristic applies also in the finite $N$ scenario. In principle, the fact that the transmission error vanishes as $1/N$ suggests an affirmative answer, as one can always find a sufficiently large finite $N$ such that the noise level in the heralded channel $E_0$ is below any desired error threshold. For example, if one uses the standard error threshold of 11% for the BB84 protocol, then every value $N > 15$ provides a sufficiently clean communication channel. Of course, one may want to consider more stringent thresholds that arise when the eavesdropper is allowed to perform more general types of attacks (possibly including attacks where the eavesdropper itself is allowed to perform operations in an indefinite order). Nevertheless, as long as the threshold is finite, there would still exist a finite value of $N$ such that the quantum SWITCH provides a heralded channel above the threshold.

The above analysis relies on the assumption that the transmission line between the sender and receiver is described by a specific quantum channel, namely the channel $C_{\text{eff}}$ in Eq. (7) of the main text. To guarantee this condition, the sender and receiver can adopt a channel certification protocol [52–54], which would allow them to decide whether or not the transmission line between them is sufficiently close to the channel $C_{\text{eff}}$. Recall that the effective channel $C_{\text{eff}}$ is a mixture of a partially depolarising channel and of a universal NOT channel, flagged by orthogonal states of the control system. In the communication protocol, only the partially depolarising channel channel is used. Hence, the communicating parties only need to certify a single-qubit, partially depolarising channel. This can be done e.g. by applying the channel to one side of a maximally entangled state, and by certifying the resulting state with a tomographic method. Particularly suited for this purpose is the method by Renner and Christandl [55], which provides reliable error bars even in the presence of an adversary. Another approach, which does not require a full tomography, is to observe that the partially depolarising channel is covariant under the $SU(2)$ group. To check whether an unknown channel is close to the desired depolarising channel, one can apply random unitary gates at the input and output, thus converting the unknown channel into an $SU(2)$-covariant channel, and then use the fact that every $SU(2)$-covariant channel is a random mixture of the identity channel and of the universal NOT (see e.g. [51]). At this point, the certification of the partially depolarising channel can be achieved by estimating its fidelity with the identity channel, or equivalently, by estimating the fidelity of its Choi state with the maximally entangled state [56].

Operationally, the condition that the transmission line is described by a specific channel (in our case, $C_{\text{eff}}$) places a constraint on the possible attacks that an eavesdropper may perform: the eavesdropper can have full access to the environment of the channel and to the classical communication between the sender and the receiver, but cannot modify the overall channel connecting them. In the context of key distribution, however, it is desirable to relax this assumption. Several relaxations are possible, and in the following we briefly outline some of them for the benefit of researchers interested in further exploring this line of research.

A first relaxation is to allow the eavesdropper to alter the quantum channels placed into the quantum SWITCH, while still assuming that the quantum SWITCH correctly describes the way these channels are combined. For example, one may consider a spacetime situation where the quantum SWITCH describes the causal structure connecting the laboratories of $N$ eavesdroppers, as in the “closed laboratory” model of Oreshkov, Costa, and Brukner [57]. In this framework, the eavesdroppers are free to perform any operation in their respective laboratories, but do not interfere with the overall causal structure. Mathematically, this situation can be modelled with the circuit with loops in Figure 4 assuming that the green boxes are chosen freely by $N$ eavesdroppers, while the blue portion of the circuit is trusted.

While the assumption that the eavesdroppers do not interfere with the superposition of causal orders may be justified in the above spacetime scenario, it is hard to justify it ordinary realisations based on coherent control over multiple paths, such as the realisation in Figure 5 or other realisations employed in table-top implementations of
quantum communication with indefinite causal order \[22\] \[25\]. In the scenario of Figure 3, an eavesdropper acting in one of the \(N\) regions can break the superposition of orders by performing measurements that determine whether or not the region is traversed by a particle at a given time. Note that, of course, the collapse of coherence will alter the overall channel between the communicating parties, who can in principle detect the alteration, and abort the protocol if necessary. An interesting question is whether there exists a test that is weaker than a full channel certification and has the property that, if passed, it guarantees security with an arbitrary adversarial choice of channels. In the scheme of Figure 3, addressing this question requires replacing the unitary channels \(\tilde{U}^{(k)}\) with arbitrary quantum channels, possibly changing over time, exhibiting correlations between one time step and the others, and/or correlations between one region and another.

Note that the model discussed in the previous paragraph still assumes a trusted mechanism controlling the path of the system through the \(N\) regions (referring to Figure 3, the assumption is that the blue parts of the circuit are trusted). Further relaxations of this assumption include giving the eavesdropper access to the control system, e.g. replacing the state \(|e_0\rangle\) with an arbitrary initial state, and/or performing adversarial operations on the control system at some of the intermediate steps before the transmission is concluded. Finally, one could give the eavesdropper complete control both over the channels and over the way they are combined. For the communicating parties, the transmission line would then become a black box, on which no assumption is made except for the dimensionality of its input and output systems. This situation is an instance of semi-device-independent cryptography, which aims at proving security under mild assumptions on the devices used in the protocol. Key distribution protocols with bounds on the dimension have been previously developed \[58\] \[59\] and proved to be secure against individual attacks, possibly involving the use of quantum memories. An interesting open problem in this research direction is the extension of the security proofs to more general attacks.

### Appendix E: Quantum capacity assisted by two-way classical communication

Here we show that, for \(N > d + 1\), the effective channel \(C_{\text{eff}}\) has a non-zero quantum capacity assisted by two-way classical communication between sender and receiver, as in the model of Ref. \[43\].

Since both channels \(E_0\) and \(E_1\) are covariant with respect to the defining representation of SU(d), they can be implemented by quantum teleportation, using the Choi state as the resource state for the teleportation protocol \[43\] \[44\].

Hence, the sender and the receiver can use the following communication protocol:

1. the sender prepares \(n\) pairs of \(d\)-dimensional systems, with each pair in the maximally entangled state \(|\Phi^+\rangle = \sum_i |i\rangle \otimes |i\rangle/\sqrt{d}\),
2. for each pair, the sender sends the first system of the pair to the receiver through the effective channel \(C_{\text{eff}}\) (all together, this means that the channel \(C_{\text{eff}}\) is used \(n\) times),
3. the receiver measures all the control systems, identifying two subsets: one subset containing of \(k\) systems acted upon by channel \(E_0\), and another subset containing \((n-k)\) systems acted upon by channel \(E_1\),
4. the receiver communicates to sender which systems have been acted upon channel \(E_0\), and which ones have been acted upon channel \(E_1\) (at this point, the sender and receiver share \(k\) copies of the Choi state of \(E_0\), and \(n-k\) copies of the Choi state of \(E_1\), according to some suitable permutation known both to the sender and to the receiver),
5. the sender encodes information using the optimal encoding for two-way assisted quantum communication with channel \(E_0^\otimes k \otimes E_1^{\otimes (n-k)}\) (with suitable permutation of the systems, depending on the location of the systems acted upon by channels \(E_0\) and \(E_1\), respectively),
6. the sender and the receiver use the Choi states as a resource for quantum teleportation, and thereby achieving the transmission the encoded state through the channel \(C_0^\otimes k \otimes E_1^{\otimes (n-k)}\) (with suitable permutation of the systems),
7. the sender and receiver use two-way classical communication to achieve optimal two-way assisted quantum communication with channel \(C_0^\otimes k \otimes E_1^{\otimes (n-k)}\) (with suitable permutation of the systems).

For large \(n\), the above protocol achieves a rate of at least \((1-p)Q_{\text{oe}}(E_0) + pQ_{\text{oe}}(E_1)\), where \(Q_{\text{oe}}(C)\) denotes the two-way assisted quantum capacity of a generic channel \(C\). In the particular case under consideration, we have \(Q(E_1) = 0\), so the contribution to the quantum capacity comes only from channel \(E_0\). Hence, the two-way assisted capacity of channel \(C_{\text{eff}}\) is non-zero whenever the two-way assisted capacity of channel \(E_0\) is non-zero. In the next section of this Supplemental Material, we will show that \(Q_{\text{oe}}(E_0)\) is non-zero whenever \(N\) is strictly larger than \(d + 1\).
Appendix F: $Q_{\omega}(E_0) > 0$ if and only if $N > d + 1$

The quantum channel $E_0$ in Eq. (8) of the main text is a partially depolarising channel, of the form $E_0 = \lambda_{N,d} I + (1 - \lambda_{N,d}) D$ with $\lambda_{N,d} := (N - 1)/(N - 1 + d^2)$. Its Choi operator is

$$E_0 = \frac{1}{d} \sum_{m,n} E_0(|m\rangle\langle n|) \otimes |m\rangle\langle n|$$

$$= \lambda_{N,d} |\Phi^+\rangle\langle \Phi^+| + (1 - \lambda_{N,d}) \frac{I \otimes I}{d^2}, \quad (F1)$$

where $|\Phi^+\rangle := \sum_m |m\rangle \otimes |m\rangle/\sqrt{d}$ is the canonical maximally entangled state.

States of the form (F1) are known as isotropic, and their entanglement properties have been studied in Refs. [61, 62]. In particular, these works have shown that isotropic states of the form $\rho_\lambda = \lambda |\Phi^+\rangle\langle \Phi^+| + (1 - \lambda) \frac{I \otimes I}{d^2}$ are entangled if and only if $\lambda > 1/(d + 1)$. Moreover, they showed that isotropic states are entangled if and only if they are distillable, that is, if and only if one can extract perfect Bell states from them at a non-zero rate and with with asymptotically negligible error.

If the Choi state $E_0$ is separable, then the quantum channel $E_0$ is entanglement-breaking [39], and therefore it has zero quantum capacity, even with the assistance of two-way classical communication. If the Choi state $E_0$ is entangled, then it is distillable, and the distillation rate provides a lower bound to the two-way quantum capacity. Hence, the condition for the two-way assisted quantum capacity to be positive is $\lambda_{N,d} > 1/(d + 1)$, or equivalently, $N > d + 1$, as it follows from the equality $\lambda_{N,d} = (N - 1)/(N - 1 + d^2)$.

Appendix G: No quantum information transmission with $N = 2$ completely depolarising channels

Here we show that the transmission of quantum information is impossible when $N = 2$ completely depolarising channels are combined in a superposition of orders. We prove this result in three scenarios, listed in order of increasing generality:

1. the two quantum channels are arranged in a superposition of two different orders, and arbitrary operations on the target system are allowed at the intermediate node between them, as in Figure 1 of the main text, with $N = 2$,

2. the two quantum channels are arranged in a superposition of two different orders, and controlled operations are allowed between them,

3. the two quantum systems are arranged in a superposition of two different orders, the sender can perform operations that do not establish entanglement between her lab and the control system, and the intermediate party can perform arbitrary controlled operations.

The impossibility of transmitting quantum information in these three cases is demonstrated in the following subsections.

1. Intermediate party without access to the control

Suppose that two completely depolarising channels are placed in a superposition of two alternative orders, and that the intermediate party acting between them performs an operation described by a quantum channel $R$.

Mathematically, the two depolarising channels are first combined by the (original) quantum SWITCH, thus obtaining the channel

$$S[C^{(1)},C^{(2)}](\cdot) = \sum_{j_1,j_2} W_{j_1,j_2} \cdot W_{j_1,j_2}^\dagger, \quad (G1)$$

with Kraus operators $W_{ij}$

$$W_{j_1,j_2} := |0\rangle\langle 0| \otimes C_{j_1}^{(1)} \otimes C_{j_2}^{(2)} + |1\rangle\langle 1| \otimes C_{j_2}^{(2)} \otimes C_{j_1}^{(1)}. \quad (G2)$$
Then, the quantum channel $\mathcal{R}$ is inserted between the first and second time-slot, resulting into a new channel with Kraus operators

$$W'_{j_1 j_2 k} := |0\rangle\langle 0| \otimes C_{j_1}^{(1)} R_k C_{j_2}^{(2)} + |1\rangle\langle 1| \otimes C_{j_2}^{(2)} R_k C_{j_1}^{(1)},$$

where $\{R_k\}$ are the Kraus operators of channel $\mathcal{R}$.

When the control is initialised in the state $\omega$, the resulting channel $C_{\text{eff}, \omega, \mathcal{R}}$ can be computed using Equations (A7) and (A9), and is given by

$$C_{\text{eff}, \omega, \mathcal{R}}(\rho) := \omega_{00} |0\rangle\langle 0| \otimes \frac{I}{d} + \omega_{11} |1\rangle\langle 1| \otimes \frac{I}{d} + \left(\omega_{01} |0\rangle\langle 1| + \omega_{10} |1\rangle\langle 0|\right) \otimes \frac{\mathcal{R}^\dagger(\rho)}{d^2},$$

where $\mathcal{R}^\dagger$ is the adjoint of $\mathcal{R}$, defined as $\mathcal{R}^\dagger(\rho) := \sum_k R_k^\dagger \rho R_k$ where $\mathcal{R}(\rho) = \sum_k R_k \rho R_k^\dagger$ is an arbitrary Kraus decomposition of $\mathcal{R}$ (the definition of $\mathcal{R}^\dagger$ is independent of the choice of Kraus decomposition).

Since the channel $\mathcal{R}$ satisfies the normalisation condition $\sum_k R_k^\dagger R_k = I$, the above relation can be rewritten as

$$C_{\text{eff}, \omega, \mathcal{R}} = (\mathcal{I}_C \otimes \mathcal{R}^\dagger) \circ C_{\text{eff}, \omega},$$

where $\mathcal{I}_C$ is the identity on the control system, and $C_{\text{eff}, \omega} := C_{\text{eff}, \omega, \mathcal{I}}$.

Now, the action of the channel $C_{\text{eff}, \omega}$ can be expressed as

$$C_{\text{eff}, \omega}(\rho) = J_{\text{eff}}(\omega \otimes \rho)$$

where $J$ is a quantum channel acting jointly on the control and the target, given by

$$J_{\text{eff}}(\omega \otimes \rho) := p_+ \omega \otimes \mathcal{E}_+(\rho) + p_- Z \omega Z \otimes \mathcal{E}_-(\rho),$$

with $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, $p_\pm := (d^2 \pm 1)/(2d^2)$, and

$$\mathcal{E}_\pm(\rho) := \frac{1}{d^2 \pm 1} \left( d I \pm \rho \right).$$

Note that the channels $\mathcal{E}_+$ and $\mathcal{E}_-$ coincide with the channels $\mathcal{E}_0$ and $\mathcal{E}_1$ in the main text, in the special case $N = 2$. As it turns out, both channels $\mathcal{E}_+$ and $\mathcal{E}_-$ are entanglement-breaking [39], and therefore cannot be used to transmit quantum information. For $\mathcal{E}_+$, the proof is provided in Appendix $\mathcal{E}_+$, while for $\mathcal{E}_-$ the proof is provided in Appendix $\mathcal{E}_-$. Hence, also the channel $J_{\text{eff}}$ in Equation (G7) is entanglement-breaking, and so are the channels $C_{\text{eff}, \omega}$ and $C_{\text{eff}, \omega, \mathcal{R}}$ in Equation (G5).

### 2. Intermediate party performing controlled operations

Suppose that two completely depolarising channels are placed in a superposition of two alternative orders, and that the intermediate party acting between them can perform controlled operations, of the form

$$\text{ctrl} - \mathcal{R}(\omega \otimes \rho) := \sum_k \left( |0\rangle\langle 0| \otimes R_k + |1\rangle\langle 1| \otimes R_k^\dagger \right) (\omega \otimes \rho) \left( |0\rangle\langle 0| \otimes R_k + |1\rangle\langle 1| \otimes R_k^\dagger \right)^\dagger,$$

with $\sum_k R_k^\dagger R_k = \sum_k R_k R_k^\dagger = I$.

When the control is initialised in the state $\omega$, the effective channel is

$$C_{\text{eff}, \omega, \text{ctrl} - \mathcal{R}}(\rho) = \omega_{00} |0\rangle\langle 0| \otimes \frac{I}{d} + \omega_{11} |1\rangle\langle 1| \otimes \frac{I}{d} + \left(\omega_{01} |0\rangle\langle 1| + \omega_{10} |1\rangle\langle 0|\right) \otimes \left( \frac{\sum_k R_k^\dagger \rho R_k}{d^2} \right) + \omega_{10} |1\rangle\langle 0| \otimes \left( \frac{\sum_k R_k^\dagger \rho R_k}{d^2} \right),$$

$$= \text{ctrl} - \mathcal{R}^\dagger \circ C_{\text{eff}, \omega}(\rho).$$

In the previous subsection we showed that $C_{\text{eff}, \omega}$ is entanglement-breaking. Hence, $C_{\text{eff}, \omega, \text{ctrl} - \mathcal{R}}$ is entanglement breaking as well.
3. Sender performing operations that do not create entanglement between her lab and the control

We now consider the scenario where two completely depolarising channels are placed in a superposition of two alternative orders, and all the parties involved in the communication protocols (sender, receiver, and intermediate party acting between the two depolarising channels) can perform joint operations on the control system, subject to the constraints that (i) the sender cannot establish entanglement between her laboratory and the control system, and (ii) the intermediate party performs a controlled operation.

Let us denote by $A$, $\text{ctrl} - \mathcal{R}$, and $B$ the channels performed by sender, intermediate party, and receiver, respectively. The condition that the sender does not use the control to establish entanglement is captured by the following definition:

**Definition 1** A channel $A$ from an input system $C \otimes S$ to an output system $C' \otimes S'$ does not transfer entanglement from $S$ to $C'$ if, for every initial state $\omega$ of system $C$, and every entanglement-breaking channel $\mathcal{E}$ acting on system $S'$, the channel $(I_C \otimes \mathcal{E}) \circ A(\omega \otimes I_S)$ is entanglement-breaking.

Intuitively, the definition means that, once the entanglement with system $S'$ is destroyed, no entanglement remains in the output.

The effective channel resulting from the superposition of orders and from the parties’ operations is

$$C_{\text{eff}, \omega, A, R, B}(\rho) = B \circ \text{ctrl} - \mathcal{R}^\dagger \circ J_{\text{eff}} \circ A \circ (I_C \otimes \omega),$$

where $J_{\text{eff}}$ is the channel defined in Equation (G7). Note that one has

$$J_{\text{eff}} \circ A \circ (I_C \otimes \omega) = p_+ (I_C \otimes \mathcal{E}_+) \circ A \circ (I_C \otimes \omega) + p_- (I_C \otimes \mathcal{E}_-) \circ A \circ (I_C \otimes \omega),$$

where $\mathcal{Z}$ is the unitary channel defined by $\mathcal{Z}(\rho) := Z \rho Z$. Now, recall that both $\mathcal{E}_+$ and $\mathcal{E}_-$ are entanglement-breaking. Using the fact that $A$ does not transfer entanglement to the control, we obtain that the two terms in the r.h.s. of Eq. (G12) are entanglement-breaking channels. Hence, the whole channel $J_{\text{eff}} \circ A \circ (I_C \otimes \omega)$ is entanglement-breaking, and so is the effective channel $C_{\text{eff}, \omega, A, R, B}$. In conclusion, no transmission of quantum information is possible unless the sender transfers entanglement to the control.

**Appendix H: Partially depolarising channels in the $N = 2$ scenario**

We have shown that the combining $N = 2$ completely depolarising channels in the quantum SWITCH does not enable a transmission of quantum data. An interesting question is whether the quantum SWITCH could still permit quantum data transmission using $N = 2$ partially depolarising channels that, individually, have zero quantum capacity. Here we answer the question in the negative, showing that the quantum capacity of each partially depolarising channel is a bottleneck for quantum capacity achievable through the quantum SWITCH (even in a heralded setting). In particular, this result implies that no information can be sent through the quantum SWITCH unless the original depolarising channels already had a positive capacity.

Consider the partially depolarising channel $D_\lambda := \lambda I + (1 - \lambda) D$, where $\lambda$ is the probability that no depolarisation takes place, and $D$ is the completely depolarising channel. Suppose that two identical channels $C^{(1)} = C^{(2)} = D_p$ are placed in the quantum SWITCH, with the control qubit initially in the state $|+\rangle$. Then, the resulting channel is

$$S(C^{(1)}, C^{(2)}) = \lambda^2 |+\rangle \langle +| \otimes I + 2\lambda (1 - \lambda) |+\rangle \langle +| \otimes D + (1 - \lambda)^2 (p_0 |+\rangle \langle +| \otimes \mathcal{E}_0 + p_1 | -\rangle \langle -| \otimes \mathcal{E}_1),$$

where $\mathcal{E}_0 = \frac{1}{d^2 + 1} I + \frac{d^2}{d^2 + 1} D$, and $\mathcal{E}_1 = \frac{1}{d^2 - 1} I - \frac{1}{d^2 - 1} I$ are the quantum channels defined in Eqs. (8) and (9) of the main text, while $p_0 = (d^2 + 1)/(2d^2)$ and $p_1 = (d^2 - 1)/(2d^2)$ are the corresponding probabilities (cf. Eq. (7) in the main text).

The channel $S(C^{(1)}, C^{(2)})$ can also be written as

$$S(C^{(1)}, C^{(2)}) = [1 - (1 - \lambda)^2 p_1] |+\rangle \langle +| \otimes \mathcal{E}_0' + (1 - \lambda)^2 p_1 | -\rangle \langle -| \otimes \mathcal{E}_1,$$

$$\mathcal{E}_0' := \frac{p^2 I + 2\lambda (1 - \lambda) D + (1 - \lambda)^2 p_0 \mathcal{E}_0}{1 - (1 - \lambda)^2 p_1}. \quad \text{(H2)}$$

As in the fully depolarising case, it is possible to separate the channel $\mathcal{E}_0'$ and $\mathcal{E}_1$ by measuring the order qubit on the basis $\{|+\rangle, | -\rangle\}$. When such a measurement is performed, the outcome $+$ heralds the occurrence of the channel $\mathcal{E}_0'$, while the outcome $-$ heralds the occurrence of the channel $\mathcal{E}_1$. 


Now, recall that the channel $\mathcal{E}_1$ is entanglement breaking, and therefore cannot transmit any quantum information. Hence, we will focus our attention on the channel $\mathcal{E}_0'$, heralded by the outcome $. The heralded channel $\mathcal{E}_0'$ is a partially depolarising channel, of the form $\mathcal{E}_0' = \lambda' I + (1 - \lambda') \mathcal{D}$, with

$$\lambda' = \frac{\chi^2 + (1-\lambda)^2}{1 - \frac{(1-\lambda)^2 (d^2-1)}{2d^2}}.$$  \hfill (H3)

We now show that the heralded channel $\mathcal{E}_0'$ has less quantum capacity than a single depolarising channel $\mathcal{D}_\lambda$. In other words, the quantum capacity of the each individual depolarising channel remains a bottleneck in the $N = 2$ case. To establish this result, we compare the new probability $\lambda'$ in Eq. (H3) with the probability $\lambda$ appearing in the original depolarising channels $\mathcal{D}_\lambda$. By solving the inequality $\lambda' \geq \lambda$, we obtain the solutions $\lambda = 1$ and $\lambda \leq \frac{\sqrt{d^2 + 8 - d^2} - 2}{2(d^2 - 1)}$.

A first observation is that the quantum SWITCH does not break the $\mathcal{D}_\lambda$ bottleneck in the low noise regime. For $\lambda \approx 1$, one has $\lambda' < \lambda$, meaning that, in fact, the heralded channel $\mathcal{E}_0'$ is more noisy than the original depolarising channel $\mathcal{D}_\lambda$.

Let us now look into the high noise regime $\lambda \leq \frac{\sqrt{d^2 + 8 - d^2} - 2}{2(d^2 - 1)} =: \lambda_{\text{max}}$. Recall that a depolarising channel $\mathcal{D}_\lambda$ is entanglement breaking whenever $\lambda \leq 1/(d + 1)$ (see e.g. Supplementary Note [F]). As it turns out, one has $\lambda_{\text{max}} < 1/(d + 1)$, meaning that the quantum SWITCH reduces the amount of noise only when the initial channel $\mathcal{D}_\lambda$ is entanglement breaking. In this regime, however, also the heralded channel $\mathcal{E}_0'$ is also entanglement breaking. Summarising, the quantum SWITCH of $N = 2$ depolarising channels reduces the amount of noise for $\lambda \leq \lambda_{\text{max}}$, but this noise reduction is not sufficient to enable the transmission of quantum data: if the depolarising channel $\mathcal{D}_\lambda$ has zero quantum capacity, then the heralded channel $\mathcal{E}_0'$ is entanglement breaking and therefore cannot transmit any quantum information. In general, the quantum capacity of the channel $\mathcal{D}_\lambda$ remains as a bottleneck for the quantum capacity of the heralded channel $\mathcal{E}_0'$.

In passing, we note that, while the heralded channel $\mathcal{E}_0'$ cannot offer any advantage over a single depolarising channel $\mathcal{D}_\lambda$, it still offers an advantage over the channel $\mathcal{D}^2_\lambda$, arising from the use of the two depolarising channels $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ in a fixed order without any intermediate operation between them. Since $\mathcal{D}^2_\lambda$ is a depolarising channel of the form $\mathcal{D}^2_\lambda = \lambda^2 I + (1 - \lambda^2) \mathcal{D}$, we can observe that the heralded channel $\mathcal{E}_0'$ is generally less noisy than $\mathcal{D}^2_\lambda$. Indeed, the condition $\lambda' \geq \lambda^2$ is satisfied for every $p \in [0, 1]$. This condition implies that channel $\mathcal{E}_0'$ achieves quantum data transmission for larger values of $\lambda$ compared to channel $\mathcal{D}^2_\lambda$. Note that channel $\mathcal{D}^2_\lambda$ can be obtained from the switched channel $S(\mathcal{C}^{(1)}, \mathcal{C}^{(2)})$ by decohering the order qubit in the basis $\{|0\rangle, |1\rangle\}$. In this respect, the advantage of the channel $\mathcal{E}_0'$ compared to the channel $\mathcal{D}^2_\lambda$ can be interpreted as a benefit of quantum coherence in the order qubit.

### Appendix I: Classical capacity of the effective channel

Here we determine the classical capacity of the effective channel $\mathcal{C}_{\text{eff}}$ defined in Equation (3) of the main text. According to the Holevo-Schumacher-Westmoreland theorem [28, 29], the classical capacity of a generic quantum channel $\mathcal{C}$ is

$$C(\mathcal{C}) = \lim_{n \to \infty} \inf \frac{\chi(\mathcal{C}^{\otimes n})}{n},$$  \hfill (I1)

where $\chi(\mathcal{D})$ is the Holevo information of a generic quantum channel $\mathcal{D}$, and is defined as $\chi(\mathcal{D}) := \sup_{\{p_x, \rho_x\}} S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$, the maximum being over all ensembles $\{p_x, \rho_x\}$ where $\rho_x$ is a quantum state and $p_x$ is a probability.

Now, the effective channel has the form $\mathcal{C}_{\text{eff}} = p \rho_0 \otimes \mathcal{E}_0 + (1 - p) \rho_1 \otimes \mathcal{E}_1$, where $\rho_0$ and $\rho_1$ are orthogonal states of the control system, and $\mathcal{E}_0$ and $\mathcal{E}_1$ are two channels acting only on the target system. Hence, the $n$-fold product $\mathcal{C}_{\text{eff}}^{\otimes n}$ has the form

$$\mathcal{C}_{\text{eff}}^{\otimes n} = \sum_i p_i \rho_i^{(n)} \otimes \mathcal{E}_i^{(n)},$$  \hfill (I2)

where $\{\rho_i^{(n)}\}$ are orthogonal states of the control system, and $\mathcal{E}_i^{(n)}$ is a channel acting on $n$ copies of the target system (specifically, each state $\rho_i^{(n)}$ is the tensor product of $k$ copies of the state $\rho_0$ and $(n - k)$ copies of the state $\rho_1$, while each channel $\mathcal{E}_i^{(n)}$ is the tensor product of $k$ copies of channel $\mathcal{E}_0$ and $(n - k)$ copies of the channel $\mathcal{E}_1$, for some $k \in \{0, \ldots, n\}$).
By convexity of the Holevo information, one has the inequality
\[
\chi(C_{\text{eff}}^\otimes n) \leq \sum_i p_i^{(n)} \chi \left( \rho_i^{(n)} \otimes \mathcal{E}_i^{(n)} \right) = \sum_i p_i^{(n)} \chi \left( \mathcal{E}_i^{(n)} \right),
\]
the second equality being due to the fact that the state \( \rho_i^{(n)} \) is independent of the input of the channel. Note that, at this point, the state of the control system has disappeared from our upper bound. In fact, the only role of the control system is to guarantee the achievability of the upper bound: since the states \( \rho_i^{(n)} \) are orthogonal, it turns out that the above inequality is actually an equality.

At this point, we observe that each channel \( \mathcal{E}_i^{(n)} \) is covariant with respect to the defining representation of the group \( \text{SU}(d)^\otimes n \), namely
\[
\mathcal{E}_i^{(n)} \circ (U_1 \otimes \cdots \otimes U_n) = (U_1 \otimes \cdots \otimes U_n) \circ \mathcal{E}_i^{(n)},
\]
where, for every \( i \in \{1, \ldots, n\} \), \( U_i \) is a unitary channel defined by \( U_i \lambda = U_i \cdot U_i^\dagger \), and \( U_i \) is an arbitrary element of \( \text{SU}(d) \), chosen independently for every value of \( i \). Covariance of the channel \( \mathcal{E}_i^{(n)} \) with respect to the representation \( \{U_1 \otimes \cdots \otimes U_n\} \) is derived from the followings observations

- each channel \( \mathcal{E}_i^{(n)} \) has the product form \( \mathcal{E}_i^{(n)} = C_1 \otimes C_2 \otimes \cdots \otimes C_n \), where each channel in the product on the right-hand-side is either the channel \( \mathcal{E}_0 \) or the channel \( \mathcal{E}_1 \);
- The channels \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) are both covariant with respect to the fundamental representation of \( \text{SU}(d) \), that is \( \mathcal{U} \circ \mathcal{E}_i = \mathcal{E}_i \circ \mathcal{U} \) for every \( i \in \{0,1\} \), and for every \( \mathcal{U} : \rho \mapsto U \rho U^\dagger, U \in \text{SU}(d) \). Indeed, \( \mathcal{E}_0 \) was proven to be covariant in Eq. \( (\text{I}3) \), and \( \mathcal{E}_1 \) is a depolarizing channel, whose covariance is immediate from the definition.
- Since each channel in the product \( C_1 \otimes C_2 \otimes \cdots \otimes C_n \) is covariant, the product is also covariant: for arbitrary and independently chosen unitary operators \( U_1, U_2, \ldots, U_n \), one has \( (U_1 \otimes \cdots \otimes U_n) \circ (C_1 \otimes C_2 \otimes \cdots \otimes C_n) = (U_1 \circ C_1) \otimes (U_2 \circ C_2) \otimes \cdots \otimes (U_n \circ C_n) = (C_1 \circ U_1) \otimes (C_2 \circ U_2) \otimes \cdots \otimes (C_n \circ U_n) = (C_1 \otimes C_2 \otimes \cdots \otimes C_n) \circ (U_1 \otimes \cdots \otimes U_n) \).

Note that the representation \( \{U_1 \otimes \cdots \otimes U_n\} \) is irreducible. For a generic channel \( C \), if \( C \) is covariant with respect to an irreducible representation, then the Holevo information has the form
\[
\chi(C) = \log d - S_{\text{min}}(C),
\]
where \( S_{\text{min}}(C) := \min_{\rho} S(C(\rho)) \) is the minimum output entropy \( [63] \). Hence, the channels \( \mathcal{E}_i^{(n)} \) satisfy the condition
\[
\chi \left( \mathcal{E}_i^{(n)} \right) = n \log d - S_{\text{min}} \left( \mathcal{E}_i^{(n)} \right),
\]
\[
\text{(I6)}
\]

Now, it only remains to determine the minimum output entropy of the channels \( C^{(n)} \). Each channel \( \mathcal{E}_i^{(n)} \) is the product of \( n \) channels, each of which is one of the two channels \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) in the main text.

Channel \( \mathcal{E}_0 \) is a depolarizing channel, and the minimum output entropy \( S_{\text{min}}(C_0^{\otimes n}) \) has been evaluated by King \( [64] \), who showed the additivity property
\[
S_{\text{min}}(C_0^{\otimes n}) = n S_{\text{min}}(C_0),
\]
\[
\text{(I7)}
\]
and evaluated the minimum output entropy
\[
S_{\text{min}}(\mathcal{E}_0) = \left( \lambda_{N,d} + \frac{1 - \lambda_{N,d}}{d} \right) \log \left( \lambda_{N,d} + \frac{1 - \lambda_{N,d}}{d} \right) - \left( \frac{d - 1}{d} \right) \left( 1 - \lambda_{N,d} \right) \log \left( \frac{1 - \lambda_{N,d}}{d} \right),
\]
\[
\text{(I8)}
\]
where \( \lambda_{N,d} \) is the probability of the identity channel in the decomposition \( \mathcal{E}_0(\rho) = \lambda_{N,d} \rho + (1 - \lambda_{N,d}) I/d \).

Channel \( \mathcal{E}_1 \), the universal NOT gate, transforms the input state \( \rho \) into the output state \( \mathcal{E}_1(\rho) = \rho^{2}/(d^2 - 1) I/d - \rho/(d^2 - 1) \). Decomposing the input state as \( \rho = \sum_i q_i |\psi_i\rangle \langle \psi_i| \) for some probabilities \( \{q_i\} \), and using the concavity of the von Neumann entropy, one obtains
\[
S(\mathcal{E}_1(\rho)) \geq \sum_i q_i S(\mathcal{E}_1(\langle \psi_i | \psi_i \rangle)) = S(\mathcal{E}_1(\langle \psi | \psi \rangle)) \quad \forall \langle \psi | \psi \rangle \in \mathcal{H}, \|\psi\| = 1.
\]
\[
\text{(I9)}
\]
Hence, the minimum output entropy of $\mathcal{E}_1$ is given by

$$S_{\min}(\mathcal{E}_1) = S(\mathcal{E}_1(\langle \psi | \psi \rangle)) \quad \forall \langle \psi | \in H, \| | \psi \| = 1$$

$$= -\frac{1}{d+1} \log \left( \frac{1}{d+1} \right) - \frac{d}{d+1} \log \left( \frac{d}{d^2-1} \right) .$$  \quad (I10)

Now, recall that $\mathcal{E}_1$ is entanglement-breaking. Shor [65] showed the additivity property

$$S_{\min}(\mathcal{A} \otimes \mathcal{B}) = S_{\min}(\mathcal{A}) + S_{\min}(\mathcal{B}) ,$$  \quad (I11)

for every pair of channels $(\mathcal{A}, \mathcal{B})$ such that at least one of the channels is entanglement breaking. Hence, every channel $\mathcal{E}_i^{(n)}$ of the form $\mathcal{E}_i^{(n)} = \mathcal{E}_0^k \otimes \mathcal{E}_1^{(n-k)}$, $k < n$, up to permutations of the Hilbert spaces, will satisfy the condition

$$S_{\min} \left( \mathcal{E}_i^{(n)} \right) = S_{\min} \left( \mathcal{E}_0^k \right) + S_{\min} \left( \mathcal{E}_1^{(n-k)} \right) = k S_{\min}(\mathcal{E}_0) + (n - k) S_{\min}(\mathcal{E}_1) .$$  \quad (I12)

Hence, we obtained the bound

$$\chi(\mathcal{C}_{\text{eff}}) \leq \sum_{k=0}^{n} (1-p)^k p^{n-k} \left[ n \log d - k S_{\min}(\mathcal{E}_0) - (n - k) S_{\min}(\mathcal{E}_1) \right]$$

$$= n \left[ \log d - (1-p) S_{\min}(\mathcal{E}_0) - p S_{\min}(\mathcal{E}_1) \right] ,$$  \quad (I13)

and therefore

$$C(\mathcal{C}_{\text{eff}}) \leq \log d - (1-p) S_{\min}(\mathcal{E}_0) - p S_{\min}(\mathcal{E}_1) .$$  \quad (I14)

The above bound is achievable by the ensemble of orthogonal states $\{|x\rangle \langle x|\}_{x=1}^d$ with uniform probabilities $p_x = 1/d$. Using the definition of $\mathcal{C}_{\text{eff}}$ in the main text, we have

$$S(\mathcal{C}_{\text{eff}}(|x\rangle \langle x|)) = (1-p) S_{\min}(\mathcal{E}_0) + p S_{\min}(\mathcal{E}_1) + H(p) + p \log(N-1) \quad \forall x \in \{1, \ldots, d\} ,$$  \quad (I15)

with $H(p) := -p \log p - (1-p) \log(1-p)$. Moreover, we have

$$\mathcal{C}_{\text{eff}} \left( \sum_x p_x |x\rangle \langle x| \right) = \frac{I}{d} \otimes \left( (1-p) \rho_0 + p \rho_1 \right) ,$$  \quad (I16)

from which we obtain

$$S \left( \mathcal{C}_{\text{eff}} \left( \sum_x p_x |x\rangle \langle x| \right) \right) = \log d + H(p) + p \log(N-1) .$$  \quad (I17)

Hence, we have

$$\chi(\mathcal{C}_{\text{eff}}) \geq S \left( \mathcal{C}_{\text{eff}} \left( \sum_x p_x |x\rangle \langle x| \right) \right) - \sum_x p_x S(\mathcal{C}_{\text{eff}}(|x\rangle \langle x|))$$

$$= \log d - (1-p) S_{\min}(\mathcal{E}_0) - p S_{\min}(\mathcal{E}_1) .$$  \quad (I18)

Since one has $\chi(\mathcal{C}) \leq C(\mathcal{C})$ for every channel $\mathcal{C}$, the bounds (I14) and (I18) imply the equality

$$C(\mathcal{C}_{\text{eff}}) = \log d - (1-p) S_{\min}(\mathcal{E}_0) - p S_{\min}(\mathcal{E}_1)$$

$$= \log d + \frac{N + d^2 - 1}{Nd^2} \left[ \frac{N - 1 + d}{N - 1 + d^2} \log \left( \frac{N - 1 + d}{N - 1 + d^2} \right) + \frac{d(d-1)}{N - 1 + d^2} \log \left( \frac{d}{N - 1 + d^2} \right) \right]$$

$$+ \frac{(N-1)(d^2-1)}{Nd^2} \left[ \frac{1}{d+1} \log \left( \frac{1}{d+1} \right) + \frac{d}{d+1} \log \left( \frac{d}{d^2-1} \right) \right]$$

$$= \log d + \frac{N - 1 + d}{Nd^2} \log \left( \frac{N - 1 + d}{N - 1 + d^2} \right) + \frac{d(d-1)}{Nd^2} \log \left( \frac{d}{N - 1 + d^2} \right)$$

$$+ \frac{(N-1)(d^2-1)}{Nd^2} \left[ \frac{1}{d+1} \log \left( \frac{1}{d+1} \right) + \frac{d}{d+1} \log \left( \frac{d}{d^2-1} \right) \right] .$$  \quad (I19)
For every fixed \( N \), the capacity vanishes as \( O(1/d^2) \) in the large \( d \) limit. In the large \( N \) limit, the channel \( \mathcal{E}_0 \) becomes noiseless, and the capacity has the asymptotic expression

\[
C(\mathcal{C}_{\text{eff}}) = \frac{\log(d+1)}{d^2} - \left( 1 - \frac{1}{d} \right) \log \left( 1 - \frac{1}{d^2} \right) - \frac{1}{d} \log \left( 1 + \frac{1}{d} \right) - O \left( \left( 1 - \frac{1}{d} \right) \frac{\log N}{N} \right). \tag{120}
\]

This expression is decreasing with \( d \), and also converges to 0 for \( d \to \infty \).