THE COLORED JONES POLYNOMIALS
AND
THE SIMPLICIAL VOLUME OF A KNOT

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Abstract. We show that the set of colored Jones polynomials and the set of generalized Alexander polynomials defined by Akutsu, Deguchi and Ohtsuki intersect non-trivially. Moreover it is shown that the intersection is (at least includes) the set of Kashaev’s quantum dilogarithm invariants for links. Therefore Kashaev’s conjecture can be restated as follows: The colored Jones polynomials determine the hyperbolic volume for a hyperbolic knot. Modifying this, we propose a stronger conjecture: The colored Jones polynomials determine the simplicial volume for any knot. If our conjecture is true, then we can prove that a knot is trivial if and only if all of its Vassiliev invariants are trivial.

In [13], R.M. Kashaev defined a family of complex valued link invariants indexed by integers \( N \geq 2 \) using the quantum dilogarithm. Later he calculated the asymptotic behavior of his invariant and observed that for the three simplest hyperbolic knots it grows as \( \exp(\text{Vol}(K)N/2\pi) \) when \( N \) goes to the infinity, where \( \text{Vol}(K) \) is the hyperbolic volume of the complement of a knot \( K \) [14]. This amazing result and his conjecture that the same also holds for any hyperbolic knot have been almost ignored by mathematicians since his definition of the invariant is too complicated (though it uses only elementary tools).

The aim of this paper is to reveal his mysterious definition and to show that his invariant is nothing but a specialization of the colored Jones polynomial. The colored Jones polynomial is defined for colored links (each component is decorated with an irreducible representation of the Lie algebra \( sl(2,\mathbb{C}) \)). The original Jones polynomial corresponds to the case that all the colors are identical to the 2-dimensional fundamental representation. We show that Kashaev’s invariant with parameter \( N \) coincides with the colored Jones polynomial in a certain normalization with every color the \( N \)-dimensional representation, evaluated at the primitive \( N \)-th root of unity. (We have to normalize the colored Jones polynomial so that the value for the trivial knot is one, for otherwise it always vanishes).

On the other hand there are other colored polynomial invariants, the generalized multivariable Alexander polynomial defined by Y. Akutsu, T. Deguchi and T. Ohtsuki [1]. They used the same Lie algebra \( sl(2,\mathbb{C}) \) but a different hierarchy of representations. Their invariants are parameterized by \( c+1 \) parameters; an integer

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and complex numbers \( p_i \) \( (i = 1, 2, \ldots, c) \) decorating the components, where \( c \) is the number of components of the link. In the case where \( N = 2 \), their invariant coincides with the multivariable Alexander polynomial and their definition is the same as the second authors’ \([21]\). Using the Akutsu–Deguchi–Ohtsuki invariants we have another coincidence. We will show that if all the colors are \((N − 1)/2\) then the generalized Alexander polynomial is the same as Kashaev’s invariant since it coincides with the specialization of the colored Jones polynomial as stated above. Therefore the set of colored Jones polynomials and the set of generalized Alexander polynomials of Akutsu–Deguchi–Ohtsuki intersect at Kashaev’s invariants.

The paper is organized as follows. In the first section we recall the definition of the link invariant defined by Yang–Baxter operators. In \( \S 2 \) we show that the Akutsu–Deguchi–Ohtsuki invariant coincides with the colored Jones polynomial when the colors are all \((N − 1)/2\) by showing that their representation becomes the usual representation corresponding to the irreducible \( N \)-dimensional representation of \( \mathfrak{sl}(2, \mathbb{C}) \). In \( \S 3 \) we show that if we transform the \( \hat{R} \)-matrix used in the colored Jones polynomial by a Vandermonde matrix then it has a form very similar to Kashaev’s \( \hat{R} \)-matrix. In fact it is proved in \( \S 4 \) that these two \( \hat{R} \)-matrix differ only by a constant. We also confirm the well-definedness of Kashaev’s invariant by using this fact.

In the final section we propose our ”dröm i Djursholm”. We use M. Gromov’s simplicial volume for a knot to generalize Kashaev’s conjecture. Observing that the simplicial volume is additive and unchanged by mutation, we conjecture that Kashaev’s invariants (= specializations of the colored Jones polynomial = specializations of the Akutsu–Deguchi–Ohtsuki invariant) determine the simplicial volume. If our dream comes true, then we can show that a knot is trivial if and only if all of its Vassiliev invariants are trivial.

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1. Preliminaries

In this section we recall the definitions of Yang–Baxter operators and associated link invariants. If an invertible linear map \( R : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N \) satisfies the following Yang–Baxter equation, it is called a Yang–Baxter operator.

\[
(R \otimes \text{id})(\text{id} \otimes R)(R \otimes \text{id}) = (\text{id} \otimes R)(R \otimes \text{id})(\text{id} \otimes R),
\]

where \( \text{id} : \mathbb{C}^N \to \mathbb{C}^N \) is the identity. If there exists a homomorphism \( \mu : \mathbb{C}^N \to \mathbb{C}^N \) and scalars \( \alpha, \beta \) satisfying the following two equations, the quadruple \( S = (R, \mu, \alpha, \beta) \) is called an enhanced Yang-Baxter operator \([20]\).

\[
(\mu \otimes \mu)R = R(\mu \otimes \mu),
\]
for a basis \{n\} of \(\mathbb{C}^k\) and associate the homomorphism \(b_R(B) : (\mathbb{C}^N)^\otimes n \to (\mathbb{C}^N)^\otimes n\) by replacing \(\sigma_i^{\pm 1}\) (the usual \(i\)-th generator of the braid group) in \(\xi\) with

\[
\frac{id \otimes \cdots \otimes id \otimes R^{\pm 1} \otimes id \otimes \cdots \otimes id}{i-1 \cdots N-i-1}.
\]

Then taking the operator trace \(n\) times we define

\[
T_S(\xi) = \alpha^{-w(\xi)} \beta^{-n} \text{Sp}_1 \left( \text{Sp}_2 \left( \cdots \left( \text{Sp}_n \left( b_R(\xi) \mu^\otimes n \right) \right) \right) \right),
\]

where \(w(\xi)\) is the sum of the exponents. Then \(T_S(\xi)\) defines a link invariant and denoted by \(T_S(L)\).

To define the (generalized) Alexander polynomial from an enhanced Yang–Baxter operator we have to be more careful, since \(T_S\) always vanishes in this case. If the following homomorphism

\[
T_{S,1}(\xi) = \alpha^{-w(\xi)} \beta^{-n} \text{Sp}_2 \left( \text{Sp}_3 \left( \cdots \left( \text{Sp}_n \left( b_S(\xi) \mu^\otimes (n-1) \right) \right) \right) \right) \in \text{End}(\mathbb{C}^N)
\]

is a scalar multiple and

\[
\text{Sp}_1(\mu) T_{S,1}(\xi) = T_S(\xi)
\]

for any \(\xi\), then the scalar defined by \(T_{S,1}(\xi)\) becomes a link invariant (even if \(\text{Sp}_1(\mu) = 0\)) and is denoted by \(T_{S,1}(L)\). Note that this invariant can be regarded as an invariant for \((1, 1)\)-tangles, where a \((1, 1)\)-tangle is a link minus an open interval.

2. **The Intersection of the Generalized Alexander Polynomials and the Colored Jones Polynomials**

In [1] Akutsu, Deguchi and Ohtsuki defined a generalization of the multivariable Alexander polynomial for colored links. First we will briefly describe their construction only for the case where all the colors are the same according to [1].

Fix an integer \(N \geq 2\) and a complex number \(p\). Put \(s = \exp(\pi \sqrt{-1}/N)\) and \([k] = (s^k - s^{-k})/(s - s^{-1})\) for a complex number \(k\). Note that \([N] = 0\) and \([N - k] = [k]\).
Let $U_q(\mathfrak{sl}(2, \mathbb{C}))$ be the quantum group generated by $X, Y, K$ with the following relations.

\[ KX = sXK, \quad KY = s^{-1}YK, \quad XY - YX = \frac{K^2 - K^{-2}}{s - s^{-1}}. \]

Let $F(p)$ be the $N$-dimensional vector space over $\mathbb{C}$ with basis \{ $f_0, f_1, \ldots, f_{N-1}$ \}. We give an action of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ on $F(p)$ by

\[ X(f_i) = \sqrt{[2p - i + 1][i]}f_{i-1}, \]
\[ Y(f_i) = \sqrt{[2p - i][i + 1]}f_{i+1}, \]
\[ K(f_i) = sp^{-i}f_i. \]

Using Drinfeld’s universal $R$-matrix given in [7], we can define a set of enhanced Yang–Baxter operators $S_A(p)$ with complex parameter $p$. Then Akutsu–Deguchi–Ohtsuki’s generalized Alexander polynomial is defined to be $T_{S_A(p), 1}$ by using the notation in the previous section. We denote it by $\Phi_N(L, p)$ for a link $L$. Note that if $N = 2$ the invariant $\Phi_2(L, p)$ is the same as the multivariable Alexander polynomial [21].

Next we review the colored Jones polynomial at the $N$-th root of unity. There is another $N$-dimensional irreducible representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$, corresponding to the usual $N$-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. Let $E$ be the $N$-dimensional complex vector space with basis \{ $e_0, e_1, \ldots, e_{N-1}$ \} and we define the action of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ by

\[ X(e_i) = [i + 1]e_{i+1}, \]
\[ Y(e_i) = [i]e_{i-1}, \]
\[ K(e_i) = sp^{-(N-1)/2}e_i. \]

(See for example [17, (2.8)].) By using Drinfeld’s universal $R$-matrix again we have another enhanced Yang–Baxter operator $S_J$. Then the invariant $T_{S_J, 1}$ coincides with the colored Jones polynomial of a link each of whose component decorated by the $N$-dimensional irreducible representation evaluated at $t = s^2 = \exp(2\pi\sqrt{-1}/N)$. Note that before evaluating at $s^2$, we have to normalize the colored Jones polynomial so that its value of the trivial knot is one for otherwise the invariant would be identically zero. This is well-defined since the colored Jones polynomial defines a well-defined $(1, 1)$-tangle invariant ([17, (3.9) Lemma]). We will denote $T_{S_J, 1}$ by $J_N$.

Now we put $p = (N - 1)/2$ in the Akutsu–Deguchi–Ohtsuki invariant. Then since $[N - k] = [k]$, we have

\[ X(f_i) = [i]f_{i-1}, \]
\[ Y(f_i) = [i + 1]f_{i+1}, \]
\[ K(f_i) = s^{(N-1)/2-1}f_i \]

and so the two representation $F((N - 1)/2)$ and $E$ are quite similar. In fact if we exchange $X$ and $Y$, and replace $K$ with $K^{-1}$ then these two coincide. (This automorphism is known as the Cartan automorphism [13, p. 123, Lemma VI.1.2].) Therefore they determine the same Yang–Baxter operator and the same link invariant, that is, we have the following theorem.
**Theorem 2.1.** The Akutsu–Deguchi–Ohtsuki invariant with all the colors \( p = (N - 1)/2 \) coincides with the colored Jones polynomial corresponding to the \( N \)-dimensional irreducible representation evaluated at \( \exp(2\pi \sqrt{-1}/N) \). More precisely, we have \( \Phi_N(L, (N - 1)/2) = J_N(L) \) for every link \( L \).

**Remark 2.2.** After finishing this work we were informed by Deguchi that it has already observed \[5\] that the \( R \)-matrices given by \( F((N - 1)/2) \) and \( E \) coincide.

3. \( \check{R} \)-matrix for the colored Jones polynomial at the \( N \)th root of unity

Let \( R_J \) be the \( \check{R} \)-matrix shown in \[17\], Corollary 2.32], which is the \( N^2 \times N^2 \) matrix with \( (i, j) \)th entry

\[
(R_J)_{ij} = \min(N-1-i,j) \sum_{n=0}^{\min(N-1-i,j)} \delta_{i,n} \delta_{k,n} \frac{(s-s^{-1})^n [i+n]! \ [N-1+n-j]!}{[i]! \ [N-1-j]!} \\
\times s^{2(i-N-1/2)(j-N-1/2)-n(i-j)-n(n+1)/2},
\]

where \([k]! = [k][k-1] \cdots [2][1]\). Note that our matrix \( R_J \) corresponds to \( \check{R} \) in \[17\], Definition 2.35]. This matrix is used to define an enhanced Yang–Baxter operator and the link invariant \( J_N \) described in the previous section.

The aim of this section is to transform it to a matrix similar to Kashaev’s \( \check{R} \)-matrix. Let \( W \) and \( D \) be the \( N \times N \) matrices with \( (i,j) \)th entry \( W_{ij} = s^{2ij} \) and \( D_{ij} = \delta_{i,j}s^{(N-1)j} \) respectively, where \( \delta_{i,j} \) is Kronecker’s delta. We will calculate the product \( \check{R}_J = (W \otimes W)(id \otimes D)R_J(id \otimes D^{-1})(W^{-1} \otimes W^{-1}) \) with \( id \) the \( N \times N \) identity matrix and show the following proposition.

**Proposition 3.1.**

\[
(\check{R}_J)^{cd}_{ab} = \begin{cases} 
\rho(a, b, c, d)(-1)^{a+b+c+1}[d-c-1]! [N-1+c-a]! & \text{if } d \geq b > a \geq c, \\
\rho(a, b, c, d)(-1)^{a+c}[b-d-1]! [N-1+c-a]! & \text{if } b > a \geq c \geq d, \\
\rho(a, b, c, d)(-1)^{b+d}[N-1+b-d]! [c-a-1]! & \text{if } c \geq d \geq b > a, \\
\rho(a, b, c, d)(-1)^{c+d}[N-1+b-d]! [a-b]! & \text{if } a \geq c \geq d \geq b, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \rho(a, b, c, d) = s^{-N^2/2+1/2+c+d-2b+(a-d)(c-b)}[N-1]! (s-s^{-1})^{2(N-1)}/N^2 \).

**Proof.** Since \( (W \otimes W)^{cf}_{ab} = s^{2ac+2bf} \), \( (id \otimes D)^{kl}_{ef} = \delta_{e,k} \delta_{f,l} s^{(N-1)l} \), \( (id \otimes D^{-1})^{gh}_{ij} = \delta_{g,i} \delta_{h,j} s^{-(N-1)j} \), \( (W^{-1} \otimes W^{-1})^{cd}_{gh} = s^{-2cg-2dh}/N^2 \), we have

\[
N^2 (\check{R}_J)^{cd}_{ab}
\]
where

\[
\begin{align*}
&= \sum_{i,j=0}^{N-1} \sum_{n=0}^{N-1-i,j} \frac{\delta_{e,k,ij} \delta_{g,i} \delta_{h,j-n} s^{2ae+2bf-2cg-2dh+(N-1)(i-j)}}{N-1-i,j} \\
&\times \left( \frac{(s-s^{-1})^n}{[n]!} \right) \left( \frac{[N-1+n-j]!}{[N-1-j]!} \right)
\end{align*}
\]

with

\[
\frac{x}{y} = \frac{[x]!}{[y]![x-y]!}.
\]

Since the summation \( \sum_{i,j=0}^{N-1} \sum_{n=0}^{N-1-i,j} \) is the same as \( \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-i,j} \sum_{n=0}^{N-1-n} \), we have

\[
N^2 \left( \hat{R}_{ij} \right)_{cd}^{ab} = s^{(N-1)^2/2} \sum_{n=0}^{N-1-n} \sum_{j=0}^{N-1-n} s^{(2b-2a+N)n-n^2/2-3n/2+(2a-2d+n+2)j} (s-s^{-1})^n
\]

\[
\times \left( \frac{[N-1+n-j]!}{[N-1-j]!} \right) S(n, 2(b+c+j)-n)
\]

with \( S(\alpha, \beta) = \sum_{i=0}^{N-1-n} s^i \left( \frac{\alpha+i}{i} \right) \).

Replacing \( j \) with \( k \), the summation turns out be \( \sum_{k=0}^{N-1} \sum_{n=0}^{N-1-k} \) and we have

\[
N^2 \left( \hat{R}_{ij} \right)_{cd}^{ab} = s^{(N-1)^2/2} \sum_{k=0}^{N-1-k} s^{2(a-d+1)k}[N-1-k]X(k),
\]

where

\[
X(k) = \sum_{n=0}^{N-1-k} (-1)^n s^{2(b-d)n+k+n(n+1)/2} \left( \frac{(s-s^{-1})^n}{[N-1-k-n]!} \right) S(n, 2(b+c+k)+n).
\]

Note that from Lemma A.1, we have

(3.1) \( S(n, 2(b+c+k)+n) = \)

\[
(1-s^{2(b-c+k-1)}) (1-s^{2(b-c+k-2)}) \cdots (1-s^{2(b-c+k+n-N+1)}).
\]

As easily seen from Lemma A.1, \( S(\alpha, \beta) \) and \( T(\alpha, \beta) \) vanishes if \((\beta - \alpha - 2) \geq 0 \geq (\beta + \alpha - 2N + 2) \) and if \((\beta + \alpha - 1) \geq 0 \geq (\beta - \alpha + 1) \) respectively. We will use this fact repeatedly from now on and divide the proof into some cases according to the order of \( a, b, c, d \).

First we divide the proof into two cases: \( b > c \) and \( c \geq b \).

**Case 1** \( b > c \). In this case \( b-c+k-1 \geq 0 \) and so from (3.1), \( S(n, 2(b-c+k)+n) = 0 \) if \( n \leq N-1-k-b+c \). If \( n > N-1-k-b+c \), we see that

\[
S(n, 2(b-c+k)+n) =
\]
\[ (-1)^{N-1-n} s^{(N-1-n)(2(b-c+k)+n-N)/2} (s-s^{-1})^{N-1-n} \frac{[b-c+k-1]!}{[b-c+k+n-N]!}. \]

Therefore

\[ X(k) = (-1)^{N+1} (s-s^{-1})^{N-1-s(b-c+k)(N-1)-N(N-1)/2} \frac{[b-c+k-1]!}{[b-c-1]!} \]

\[ \times \sum_{n=N-1-k-b+c}^{N-1-k} (-1)^n s^{(b+c-2d)n} \left[ \begin{array}{c} b-c-1 \\ N-1-k-n \end{array} \right] \]

(putting \( i = N-1-k-n \))

\[ = (-1)^k (s-s^{-1})^{N-1-s(2b-2d+k)(N-1)-(b+c-2d)k-N(N-1)/2} \frac{[b-c+k-1]!}{[b-c-1]!} \]

\[ \times \sum_{i=0}^{b-c-1} (-1)^i s^{(2d-b-c)i} \left[ \begin{array}{c} b-c-1 \\ i \end{array} \right] \]

\[ = (s-s^{-1})^{N-1-s(2d-2b)-(b+c-2d+1)k-N(N-1)/2} \frac{[b-c+k-1]!}{[b-c-1]!} \]

\[ \times T(b-c-1, 2d - b - c), \]

with \( T(\alpha, \beta) = \sum_{i=0}^{\alpha} (-1)^i s^\beta \binom{n}{i} \). From Lemma \[ A.1 \] we have

\[ X(k) = (s-s^{-1})^{N-1-s^2(2d-c-1)(1-s^2(b-c-2) \cdots (1-s^2(2d-b+1)))} \]

We divide the case into two subcases; \( d \geq b \) and \( b > d \).

**Subcase 1.1** \((d \geq b)\). Since

\[ (1-s^2(2d-c-1))(1-s^2(2d-c-2)) \cdots (1-s^2(2d-b+1)) \]

\[ = (-1)^{b-c-1} s^{(2d-b-c)(b-c-1)/2} (s-s^{-1})^{b-c-1} \frac{[d-c-1]!}{[d-b]!}, \]

we have

\[ X(k) = (-1)^{b+c+1} (s-s^{-1})^{N+b-c-2} s^{2(d-b)-N(N-1)/2+(2d-b-c)(b-c-1)/2} \]

\[ \times \frac{[d-c-1]!}{[d-b]!} s^{(2d-b-c-1)k} \frac{[N-1-k]![b-c+k-1]!}{[b-c-1]!} \]

and so

\[ N^2 (\hat{R})_a^d \]

\[ = s^{(N^2+1)/2} (-1)^{b+c} (s-s^{-1})^{N+b-c-2} s^{2(d-b)-N(N-1)/2+(2d-b-c)(b-c-1)/2} \]

\[ \times \frac{[d-c-1]!}{[d-b]!} \sum_{k=0}^{N-1} s^{(2a-b-c+1)k} \frac{[N-1-k]![b-c+k-1]!}{[b-c-1]!} \]

\[ = s^{(N^2+1)/2} (-1)^{b+c} (s-s^{-1})^{N+b-c-2} s^{2(d-b)-N(N-1)/2+(2d-b-c)(b-c-1)/2} \]

\[ \times \frac{[d-c-1]!}[d-b!] S(b-c-1, 2a-b-c+1) \]
\[= s^{(N^2 + 1)/2}(1)^{b+c}(s - s^{-1})^N b c - 2 s^{2(d-b) - N}(N-1)/2 + (2d-b-c)(b-c-1)/2 \]
\[\times \frac{[d - c - 1]!![N - 1]!!}{[d - b]!!} (1 - s^{2(a-b)})(1 - s^{2(a-b-1)}) \ldots (1 - s^{2(a-c-N+1)}),\]

where the second equality follows from \([N - 1 - k]! = [N - 1]!/|k|!\).

Noting that \(a - c - N + 1 \leq 0\), we see that
\[(1 - s^{2(a-b)})(1 - s^{2(a-b-1)}) \ldots (1 - s^{2(a-c-N+1)})\]
\[= \begin{cases} 0 & \text{if } a \geq b, \\ (s - s^{-1})^N b c + b - c - N + 1)(N + c - b - c - N + 1) \ldots (N + c - b - c - N + 1) & \text{if } b > a. \end{cases}\]

Therefore we have
\[\left(\tilde{R}_j\right)_{ab}^{cd} = \rho(a, b, c, d)(-1)^{a+b+1} \frac{[d - c - 1]!![N - 1 + c - a]!}{[d - b]!![b - a - 1]!}\]

if \(b > a\) and zero otherwise. Thus the proof is complete for the case \(d \geq b > c\).

(Note that \([N - 1 + c - a] = 0\) if \(c > a\).)

**Subcase 1.2 \((b > d)\).** In this case we have
\[(1 - s^{2(d-c-1)})(1 - s^{2(d-c-2)}) \ldots (1 - s^{2(d-b+1)})\]
\[= s^{2(d-b-c)(b-c-1)/2}(s - s^{-1})^N b c - 1 \frac{[b - d - 1]!}{[c - d]!}\]

if \(c \geq d\) and 0 otherwise. Therefore \(\left(\tilde{R}_j\right)_{ab}^{cd} = 0\) if \(d > c\). If \(c \geq d\), we have

\[N^2 \left(\tilde{R}_j\right)_{ab}^{cd}\]
\[= s^{(N-1)^2/2}(s - s^{-1})^N b c - 2 s^{2(d-b) - N}(N-1)/2 + (2d-b-c)(b-c-1)/2 \]
\[\times \frac{[N - 1]!![b - d - 1]!}{[c - d]!} S(b - c - 1; 2a - b - c + 1)\]
\[= s^{-(N-1)^2/2}(s - s^{-1})^N b c - 2 s^{2(d-b) + (2d-b-c)(b-c-1)/2} \]
\[\times \frac{[N - 1]!![b - d - 1]!}{[c - d]!} (1 - s^{2(a-b)})(1 - s^{2(a-b-1)}) \ldots (1 - s^{2(a-c-N+1)})\]

This vanishes if \(a \geq b\) and equals \(N^2 \rho(a, b, c, d)(-1)^{a+c+b-d-1}[N - 1 + c - a]!/\frac{[c - d]!![b - a - 1]!}{[c - d]!![b - a - 1]!}\) otherwise, completing the proof for the case \(b > d\) and \(b > c\) since \([N - 1 + c - a] = 0\) if \(c > a\).

**Case 2 \((c \geq b)\).** First note from \([I, 3]\), \(X(k) = 0\) if \(k \geq c - b + 1\). If \(k < c - b + 1\), we have

\[S(n, 2(b - c + k) + n)\]
\[= s^{(2(b-c+k)+n-N)(N-1-n)/2}(s - s^{-1})^N - n \frac{[N - 1 - n + c - b - k]!}{[c - b - k]!}\]

and so
\[X(k) = (s - s^{-1})^N - s^{(2(b-c+k)-N)(N-1)/2} \frac{[c - b]!}{[c - b - k]!}\]
\[ \times \sum_{n=0}^{N-1-k} s^{(b+c-2d)n} \left[ \frac{N - 1 - k - n + c - b}{c - b} \right] \]

(putting \( i = N - 1 - k - n \))

\[ = (s - s^{-1})^{N-1} s^{(2(b-c+k)-N)(N-1)/2+(b+c-2d)(N-1-k)} \frac{[c-b]!}{[c - b - k]!} \]

\[ \times S(c - b, 2d - b - c) \]

\[ = (s - s^{-1})^{N-1} s^{(2(b-c+k)-N)(N-1)/2+(b+c-2d)(N-1-k)} \frac{[c-b]!}{[c - b - k]!} \]

\[ \times (1 - s^{2(d-c-1)})(1 - s^{2(d-c-2)}) \cdots (1 - s^{2(2b-N+1)}). \]

Since \( d - b - N + 1 \leq 0 \), \( X(k) \) vanishes and so does \( \left( \tilde{R}_J \right)_{cd}^{ab} \) if \( d > c \). Therefore we assume that \( c \geq d \).

In this case since

\[ (1 - s^{2(d-c-1)})(1 - s^{2(d-c-2)}) \cdots (1 - s^{2(2b-N+1)}) \]

\[ = s^{(2d-b-c-N)(N-1+b-c)/2}(s - s^{-1})^{N-1+b-c}[N - 1 + b - d]! \frac{[c-b]!}{[c - d]!}, \]

we have

\[ X(k) = (s - s^{-1})^{2N-2+b-c} \]

\[ \times s^{(2b-2c-N)(N-1)/2+(b+c-2d)(N-1)+(2d-b-c-N)(N-1+b-c)/2} \]

\[ \times \frac{[N - 1 + b - d]! [c - b]!}{[c - d]!} \]

\[ \times s^{(N-1+2d-b-c)k} \frac{1}{[c - b - k]!}. \]

Therefore

\[ N^2 \left( \tilde{R}_J \right)_{cd}^{ab} = s^{(N-1)^2/2}(s - s^{-1})^{2N-2+b-c} \]

\[ \times s^{(2b-2c-N)(N-1)/2+(b+c-2d)(N-1)+(2d-b-c-N)(N-1+b-c)/2} \]

\[ \times \frac{[N - 1]! [N - 1 + b - d]!}{[c - d]!} T(c - b, 2a - b + c + 1) \]

\[ = s^{(N-1)^2/2}(s - s^{-1})^{2N-2+b-c} \]

\[ \times s^{(2b-2c-N)(N-1)/2+(b+c-2d)(N-1)+(2d-b-c-N)(N-1+b-c)/2} \]

\[ \times (1 - s^{2(a-b)})(1 - s^{2(a-b-1)}) \cdots (1 - s^{2(a-c+1)}). \]

There are two subcases; \( b > a \) and \( a \geq b \).

**Subcase 2.1 (b > a).** Since

\[ (1 - s^{2(a-b)})(1 - s^{2(a-b-1)}) \cdots (1 - s^{2(a-c+1)}) = \]

\[ s^{(2a-b-c+1)(c-b)}(s - s^{-1})^{c-b} \frac{[c - a - 1]!}{[b - a - 1]!}, \]
we have
\[(\tilde{R}_J)_{a b}^{c d} = \rho(a, b, c, d) (-1)^{b+d} \frac{(N - 1 + b - d)! [c - a - 1]!}{[c - d]! [b - a - 1]!} \]
and the proof for the case where \(c \geq b\) and \(b > a\) is complete. (Note that \((\tilde{R}_J)_{a b}^{c d}\) vanishes unless \(c \geq d \geq b\).)

**Subcase 2.2 \((a \geq b)\)**. In this case we only have to consider the case where \(a \geq c\) for otherwise \((\tilde{R}_J)_{a b}^{c d} = 0\). Now
\[1 - s^{2(a-b)}(1 - s^{2(a-b-1)}) \cdots (1 - s^{2(a-c+1)}) = (-1)^{b+c} s^{(2a-b-c+1)(c-b)}(s-s^{-1})^{c-b} \frac{[a-b]!}{[a-c]!} \]
and so we have
\[(\tilde{R}_J)_{a b}^{c d} = \rho(a, b, c, d) (-1)^{c+d} \frac{(N - 1 + b - d)! [a - b]!}{[c - d]! [a - c]!}. \]
The proof for the case where \(c \geq b\) and \(a \geq b\) is now complete. (Note again that \((\tilde{R}_J)_{a b}^{c d} = 0\) unless \(c \geq d \geq b\).)

## 4. Kashaev’s R-matrix and his invariant

In this section we will calculate Kashaev’s \(\tilde{R}\)-matrix given in [13] and prove that it coincides with the matrix \(\tilde{R}_J\) up to a constant given in the previous section.

We prepare notations following [13]. Fix an integer \(N \geq 2\). Put \((x)_n = \prod_{i=1}^{n} (1 - x^i)\) for \(n \geq 0\). Define \(\theta : \mathbb{Z} \to \{0, 1\}\) by
\[\theta(n) = \begin{cases} 1 & \text{if } N > n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \]

For an integer \(x\), we denote by \(\text{res}(x) \in \{0, 1, 2, \ldots, N-1\}\) the residue modulo \(N\).

Now Kashaev’s \(\tilde{R}\)-matrix \(R_K\) is given by
\[(R_K)_{a b}^{c d} = Nq^{1+c-b+(a-d)(c-b)} \frac{\theta(\text{res}(b-a-1) + \text{res}(c-d))\theta(\text{res}(a-c) + \text{res}(d-b))}{(q^{\text{res}(b-a-1)}(q^{-1})^{\text{res}(a-c)}(q^{\text{res}(c-d)}(q^{-1})^{\text{res}(d-b)})} \]
with \(q = s^2\). Note that we are using \(P \circ R\) with \(R\) defined in [13, 2.12] rather than \(R\) itself where \(P\) is the homomorphism from \(C^N \otimes C^N\) to \(C^N \otimes C^N\) sending \(x \otimes y\) to \(y \otimes x\).

We will show the following proposition.
Proposition 4.1.

\[
(R_K)_{cd}^{ab} = \begin{cases} 
\lambda(a, b, c, d)(-1)^{a+b+1}\frac{[d-c-1]![N-1+c-a]!}{[d-b]![b-a-1]!} & \text{if } d \geq b > a \geq c, \\
\lambda(a, b, c, d)(-1)^{a+c}\frac{[b-d-1]![N-1+c-a]!}{[c-d]![b-a-1]!} & \text{if } b > a \geq c \geq d, \\
\lambda(a, b, c, d)(-1)^{b+d}\frac{[N-1+b-d]![c-a-1]!}{[c-d]![b-a-1]!} & \text{if } c \geq d > b > a, \\
\lambda(a, b, c, d)(-1)^{c+d}\frac{[N-1+b-d]![a-b]!}{[c-d]![a-c]!} & \text{if } a \geq c \geq d \geq b, \\
0 & \text{otherwise},
\end{cases}
\]

where \(\lambda(a, b, c, d) = s^{-N^2/2+N/2+c+d-2(a-d)(c-b)(s-s^{-1})^{1-N}/([N-1]!)^2}\).

Proof. Since \(N-1 \geq b-a-1 \geq -N,\) \(N-1 \geq c-d \geq -N+1,\) \(N-1 \geq a-c \geq -N+1\) and \(N-1 \geq d-b \geq -N + 1,\) we see that \((R_K)_{cd}^{ab}\) vanishes except for the following four cases, which have already appeared in Proposition 3.1: (i) \(d \geq b > a \geq c,\) (ii) \(b > a \geq c \geq d,\) (iii) \(c \geq d \geq b \geq a\) and (iv) \(a \geq c \geq d \geq b.\)

We will only prove the first case because the other cases are similar. Noting that

\[
(q)_n = (-1)^n s^{n(n+1)/2}(s-s^{-1})^n[n]!, \\
(q^{-1})_n = s^{-n(n+1)/2}(s-s^{-1})^n[n]!,
\]

we have

\[
(R_K)_{cd}^{ab} = (-1)^{a+b+c+d}N s^{2+2c-2b+2(a-d)(c-b)+a-d+2N/2+1/2}(s-s^{-1})^{-N+1} \\
\times \frac{1}{[d-b]![b-a-1]![N+c-d]![a-c]!}
\]

since res\((b-a-1) = b-a-1,\) res\((a-c) = a-c,\) res\((c-d) = N+c-d\) and res\((d-b) = d-b.\) Now since \([N-n] = [n],\) we see that

\[
\frac{1}{[d-b]![b-a-1]![N+c-d]![a-c]!} = \frac{1}{([N-1])^2} \frac{[d-c-1]![N+c-a-1]!}{[b-a-1]![d-b]!}.
\]

Therefore

\[
(R_K)_{ab}^{cd} = \lambda(a, b, c, d)(-1)^{a+b+1}\frac{[d-c-1]![N-1+c-a]!}{[d-b]![b-a-1]!}
\]
as required. \(\square\)

Therefore \(R_K\) and \(R_J\) are equal up to a constant depending only on \(N.\) More precisely we have

Proposition 4.2. Let \(R_K\) and \(R_J\) be the \(\hat{R}\)-matrices defined by as above. Then we have

\[
R_K = s^{-(N+1)(N-3)/2}(W \otimes W)(id \otimes D)R_J(id \otimes D^{-1})(W^{-1} \otimes W^{-1})
\]
for any \(N \geq 2.\)
Proof. From Propositions 3.1 and 4.1, we only have to check that
\[
\rho(a, b, c, d)/\lambda(a, b, c, d) = s^{(N+1)(N-3)/2} \left( \frac{(s - s^{-1})^{N-1}[N-1]}{N} \right)^3
\]
but this coincides with \( s^{(N+1)(N-3)/2} \) as shown below.

We have
\[
(s - s^{-1})^{N-1}[N-1]! = \prod_{k=1}^{N-1} (2\sqrt{-1} \sin(k\pi/N))
\]
\[
= \sqrt{-1}^{N-1} \prod_{k=1}^{N-1} (2\sin(k\pi/N)).
\]
On the other hand from [3, I.392-1, p. 33], we have
\[
\sin(Nx) = 2^{N-1} \prod_{k=0}^{N-1} \sin(x + k\pi/N).
\]
Divided by \( \sin x \) and taking the limit \( x \to 0 \), we have
\[
N = \prod_{k=1}^{N-1} (2\sin(k\pi/N)).
\]
Therefore we have
\[
(-1)^N s^{(N-3)/2} \left( \frac{(s - s^{-1})^{N-1}[N-1]}{N} \right)^3 = (-1)^N s^{(N-3)/2} \sqrt{-1}^{(N-1)}
\]
\[
= s^{N^2+(N-3)/2+3(N-1)/2} = s^{(N+1)(N-3)/2},
\]
completing the proof.

We will show that the matrix \( R_K \) also satisfies the Yang–Baxter equation. To do that we prepare a lemma.

Lemma 4.3. The matrices \( D \) and \( D^{-1} \) can go through \( R_J \) in pair, that is, the following equality holds.
\[
(id \otimes D)R_J(id \otimes D^{-1}) = (D^{-1} \otimes id)R_J(D \otimes id).
\]

Proof. It is sufficient to show that \((D \otimes D)R_J = R_J(D \otimes D)\). Since \( D_j^i = \delta_{i,j}s^{(N-1)j} \),
\[
((D \otimes D)R_J)^{ij}_{kl} = \sum_{a,b} \delta_{a,k}\delta_{b,l}s^{(N-1)k}s^{(N-1)l}(R_J)_{ab}^{ij}
\]
\[
= s^{(N-1)(k+l)}(R_J)_{kl}^{ij}
\]
and
\[
(R_J(D \otimes D))^{ij}_{kl} = \sum_{a,b} \delta_{a,i}\delta_{b,j}s^{(N-1)i}s^{(N-1)j}(R_J)_{ab}^{ij}
\]
\[
= s^{(N-1)(i+j)}(R_J)_{kl}^{ij}.
\]
But these two coincide since \((R_J)^{ij}_{kl}\) vanishes unless \( k + l = i + j \) (the charge conservation law), completing the proof.

Using Lemma 4.3 we can give another proof of the following proposition.
**Proposition 4.4 (Kashaev).** Kashaev’s $\tilde{R}$-matrix $R_K$ satisfies the Yang–Baxter equation, that is,

$$(R_K \otimes \text{id})(\text{id} \otimes R_K)(R_K \otimes \text{id}) = (\text{id} \otimes R_K)(R_K \otimes \text{id})(\text{id} \otimes R_K).$$

**Proof.** From Proposition 4.2, we have

$$(R_K \otimes \text{id})(\text{id} \otimes R_K)(R_K \otimes \text{id})
= (W \otimes W \otimes \text{id})(\text{id} \otimes D \otimes \text{id})(R_J \otimes \text{id})(\text{id} \otimes D^{-1} \otimes \text{id})(W^{-1} \otimes W^{-1} \otimes \text{id})
\times (\text{id} \otimes W \otimes W)(\text{id} \otimes D \otimes \text{id})(R_J \otimes \text{id})(\text{id} \otimes D^{-1} \otimes \text{id})(W^{-1} \otimes W^{-1} \otimes \text{id})
\times (W \otimes W \otimes \text{id})(\text{id} \otimes D \otimes \text{id})(R_J \otimes \text{id})(\text{id} \otimes D^{-1} \otimes \text{id})(W^{-1} \otimes W^{-1} \otimes \text{id})
= (W \otimes W \otimes W)(\text{id} \otimes D \otimes D)(R_J \times \text{id})
\times (\text{id} \otimes D^{-1} \otimes D)(\text{id} \otimes R_J)(\text{id} \otimes R_J)
\times (R_J \otimes \text{id})(\text{id} \otimes D^{-1} \otimes D^{-1})(W^{-1} \otimes W^{-1} \otimes W^{-1})
= (W \otimes W \otimes W)(\text{id} \otimes D \otimes D^2)
\times (R_J \otimes \text{id})(\text{id} \otimes R_J)(R_J \otimes \text{id})
\times (\text{id} \otimes D^{-1} \otimes D^{-2})(W^{-1} \otimes W^{-1} \otimes W^{-1}).$$

Similar calculation shows

$$(\text{id} \otimes R_K)(R_K \otimes \text{id})(\text{id} \otimes R_K)
= (W \otimes W \otimes W)(\text{id} \otimes D \otimes D^2)
\times (\text{id} \otimes R_J)(\text{id} \otimes R_J)(R_J \otimes \text{id})
\times (\text{id} \otimes D^{-1} \otimes D^{-2})(W^{-1} \otimes W^{-1} \otimes W^{-1}).$$

From the Yang–Baxter equation for $R_J$ these two coincide, completing the proof. \qed

To show that $R_J$ and $R_K$ define the same link invariant, we will construct enhanced Yang–Baxter operators precisely by using them.

Let $\mu_J$ be the $N \times N$-matrix with $(i, j)$-entry $(\mu_J)^i_j = \delta_{i, j}s^{2i-N+1}$. Then the quadruple $S_J = (R_J, \mu_J, s^{N^2-1}, 1)$ is a Yang–Baxter operator and the following lemma holds.

**Lemma 4.5.**

$$(\mu_J \otimes \mu_J)R_J = R_J(\mu_J \otimes \mu_J),$$

$$\sum_{j=0}^{N-1} ((R_J)^{\pm 1}(\text{id} \otimes \mu_J))^i_j = (s^{N^2-1})^{\pm 1}\text{id}.$$ 

Next we will give a Yang–Baxter operator using $R_K$. Let $\mu_K$ be the $N \times N$-matrix with $(i, j)$-entry $(\mu_K)^i_j = -s\delta_{i, j+1}$. Then we have
Lemma 4.6.

\[ W D \mu J D^{-1} W^{-1} = \mu_K. \]

Proof. Since \( W^a = s^{2a} \), \( D^b = \delta_{a,b}s^{(N-1)b} \), \( (\mu_J)^c = \delta_{b,c}s^{2c-N+1} \), \( (D^{-1})^d = \delta_{c,d}s^{-(N-1)d} \) and \( (W^{-1})^i = \delta_{d,i}s^{-2d/N} \), we have

\[
(W D \mu J D^{-1} W^{-1})^j_1 \nu = \frac{1}{N} (-s) \sum_{a=0}^{N-1} s^{2(j-i+1)a} = -s \delta_{i,j+1},
\]

completing the proof.

Combining Lemmas 4.5 and 4.6, we show that \( S_K = (R_K, \mu_K, -s, 1) \) is also an enhanced Yang–Baxter operator.

Lemma 4.7.

\[ (\mu_K \otimes \mu_K)R_K = R_K(\mu_K \otimes \mu_K), \]

\[ \sum_{j=0}^{N-1} ((R_K)^{\pm 1}(id \otimes \mu_K))^{ij}_{kj} = (-s)^{\pm 1} id. \]

Proof. Noting that \( \mu_J \) and \( D \) commutes since they are diagonal, the first equality follows immediately from that in Lemma 4.5.

The second equality follows from

\[
(R_K)^{\pm 1}(id \otimes \mu_K) = s^{\mp(N+1)(N-3)/2}(W \otimes W)(id \otimes D)R_J(id \otimes \mu_J)(id \otimes D^{-1})(W^{-1} \otimes W^{-1}),
\]

completing the proof.

Note that the lemma can also be proved by using [13] (2.8) and (2.17).

Now we see that \( S_J \) and \( S_K \) define the same link invariant by using the following lemma.

Lemma 4.8. Let \( \xi \) be an \( n \)-braid. Then

\[ b_{R_K}(\xi) = (W^{\otimes n})(D^{k_1} \otimes D^{k_2} \otimes \cdots \otimes D^{k_n}) b_{R_J}(\xi) \times (D^{-k_1} \otimes D^{-k_2} \otimes \cdots \otimes D^{-k_n}) \left( (W^{-1})^{\otimes n} \right) \]

for some non-negative integers \( k_1, k_2, \ldots, k_n \).

Proof. In fact we can show that \( (k_1, k_2, \ldots, k_n) \) is of the form

\[ (0, 1, 2, \ldots, d_1, 0, 1, 2, \ldots, d_2, \ldots, 0, 1, 2, \ldots, d_h) \]

by using Lemma 4.3 repeatedly to ‘push’ \( D \) and \( D^{-1} \) from left to right (See the proof of Proposition 4.4). Details are omitted.

Since we know that \( J_N = T_{S_J, 1} \) is well-defined as described in [13], from the previous lemma \( T_{S_{K, 1}}(L) \) is also a link invariant, which we denote by \( \langle L \rangle_N \). Note that it is implicitly stated in [13] that the invariant can be regarded as an invariant for \((1, 1)\)-tangles. Note also that though the invariant was defined only up to a multiple of \( s \) in [13], we can now define it without ambiguity.
Since
\[ b_{R_k}(\xi)(id \otimes \mu_K^{\otimes(n-1)}) = (W^{\otimes n}) \left( D^{k_1} \otimes D^{k_2} \otimes \cdots \otimes D^{k_n} \right) b_{R,J}(\xi) \]
\times \left( id \otimes \mu_J^{\otimes(n-1)} \right) \left( D^{-k_1} \otimes D^{-k_2} \otimes \cdots \otimes D^{-k_n} \right) \left( (W^{-1})^{\otimes n} \right) \]
from Lemma 4.8 we conclude that \( S_J \) and \( S_K \) define the same link invariant.

**Theorem 4.9.** For any link \( L \) and any integer \( N \geq 2 \), \( \langle L \rangle_N \) and \( J_N(L) \) coincide.

5. **Relation between the simplicial volume and the colored Jones polynomials.**

Let \( K \) be one of the three simplest hyperbolic knots \( 4_1, 5_2 \) and \( 6_1 \). Kashaev found in [14] that the hyperbolic volume of \( S^3 \setminus K \), denoted by \( \text{Vol}(K) \), coincides numerically with the growth rate of the absolute value of \( \langle K \rangle_N \) with respect to \( N \).

More precisely,
\[ \text{Vol}(K) = 2\pi \lim_{N \to \infty} \frac{\log |\langle K \rangle_N|}{N}. \]

We would like to modify his conjecture taking Gromov’s simplicial volume (or Gromov norm) [10] into account. Let us consider the torus decomposition of the complement of a knot \( K \) [11, 12]. Then the simplicial volume of \( K \), denoted by \( \|K\| \) is equal to the sum of the hyperbolic volumes of hyperbolic pieces of the decomposition divided by \( v_3 \), the volume of the ideal regular tetrahedron in \( \mathbb{H}^3 \), the three-dimensional hyperbolic space. Recall that it is additive under the connect sum [25]
\[ \|K_1 \sharp K_2\| = \|K_1\| + \|K_2\|, \]
and that it does not alter by mutation [24, Theorem 1.5].

Noting that \( J_N \) is multiplicative under the connect sum, that is,
\[ J_N(K_1 \sharp K_2) = J_N(K_1) J_N(K_2) \]
and that it does not alter by mutation [24, Corollary 6.2.5], we propose the following conjecture.

**Conjecture 5.1 (Volume conjecture).** For any knot \( K \),
\[ \|K\| = \frac{2\pi}{v_3} \lim_{N \to \infty} \frac{\log |J_N(K)|}{N}. \]

**Remark 5.2.** First note that if Kashaev’s conjecture is true then our conjecture holds for hyperbolic knots and their connect sums. It is also true for torus knots since Kashaev and O. Tirkkonen [private communication] showed that the right hand side of (5.1) vanishes in this case by using H. Morton’s formula [19] (see also [22]).

**Remark 5.3.** Note however that the volume conjecture does not hold for links since \( J_N \) of the split union of two links vanishes.

As a consequence of the volume conjecture, we anticipate the following simplest case of V. Vassiliev’s conjecture [27, 6.1 Stabilization conjecture] (see also [16, Chapter 1, Part V (L), Conjecture]).
Conjecture 5.4 (V. Vassiliev). Assume that every Vassiliev (finite-type) invariant of a knot is identical to that of the trivial knot. Then it is unknotted.

We show that the volume conjecture implies Conjecture 5.4 by using the following two lemmas.

Lemma 5.5 ([8, Corollary 4.2]). If $\|K\| = 0$ then $K$ is obtained from the trivial knot by applying a finite number (possibly zero) of the following two operations:
1) making a connect sum,
2) making a cable.

Lemma 5.6. If $\|K\| = 0$, then the Alexander polynomial $\Delta(K)$ of $K$ is trivial if and only if $K$ is the trivial knot.

Proof. This lemma comes from Lemma 5.5 and the following three facts [3, §2] (see also [4, 8.23 Proposition]).
i) the Alexander polynomial of a non-trivial torus knot is not trivial.
i) the Alexander polynomial is multiplicative under the connect sum. Therefore if $\Delta(K_1)$ and $\Delta(K_2)$ are non-trivial, then $\Delta(K_1 \# K_2)$ is also non-trivial.
iii) if $K'$ is a knot obtained from $K$ by a cabling operation, then $\Delta(K')$ is $\Delta(K)f(t)$ with some Laurent polynomial $f(t)$. Hence if $\Delta(K)$ is non-trivial, so is $\Delta(K')$.

Proof that the volume conjecture implies Conjecture 5.4. First note that every coefficient of both colored Jones polynomial and Alexander polynomial as a power series in $h = \log t$ is a Vassiliev invariant. So a knot $K$ with every Vassiliev invariant trivial has the trivial colored Jones polynomial for any color and the trivial Alexander polynomial. In particular $J_N(K) = 1$ for any $N$. Therefore assuming the volume conjecture, $\|K\|$ vanishes. From Lemma 5.6, $K$ should be trivial, completing the proof.

Remark 5.7. It was pointed out by Vaintrob and Bar-Natan that using the Melvin–Morton–Rozansky conjecture [18, 23] proved by Bar-Natan and S. Garoufalidis [2], we can also show that a knot is trivial if and only if all of its colored Jones polynomials are trivial since the Melvin–Morton–Rozansky conjecture says that the Alexander polynomial can be determined by the colored Jones polynomials.

A. Appendix

In this appendix we prove some technical formulas used in the paper. Put

$$S(\alpha, \beta) = \sum_{i=0}^{N-1} s^{\beta i} \left[ \frac{\alpha + i}{i} \right],$$

$$T(\alpha, \beta) = \sum_{i=0}^{\alpha} (-1)^i s^{\beta i} \left[ \frac{\alpha}{i} \right].$$

Note that the summation in $S(\alpha, \beta)$ is essentially from 0 to $N - 1 - \alpha$. Then we have
Lemma A.1. The following formulas hold.

\[ S(\alpha, \beta) = \prod_{j=1}^{N-\alpha-1} (1 - s^{\beta - \alpha - 2j}) = (1 - s^{\beta - \alpha - 2})(1 - s^{\beta - \alpha - 4}) \cdots (1 - s^{\beta + \alpha - 2N+2}), \]

\[ T(\alpha, \beta) = \prod_{j=1}^{\alpha} (1 - s^{\beta + \alpha + 1 - 2j}) = (1 - s^{\beta + \alpha - 1})(1 - s^{\beta + \alpha - 3}) \cdots (1 - s^{\beta + \alpha + 1}). \]

Proof. We only prove the equality for \( S(\alpha, \beta) \) since the other case is similar. We use the following quantized Pascal relation.

\[ \binom{\alpha + i}{i} = s^{-\alpha} \left[ \alpha + i - 1 \atop i - 1 \right] + s^i \left[ \alpha + i - 1 \atop i \right]. \]

Then since

\[ S(\alpha, \beta) = s^{-\alpha} \sum_{i=0}^{N-1} s^{\beta i} \left[ \alpha + i - 1 \atop i - 1 \right] + \sum_{i=0}^{N-1} s^{(\beta+1)i} \left[ \alpha + i - 1 \atop i \right] \]

(putting \( k = i - 1 \) in the first term)

\[ = s^{\beta - \alpha} \sum_{k=-1}^{N-2} s^k \left[ \alpha + k \atop k \right] + S(\alpha - 1, \beta + 1) \]

\[ = s^{\beta - \alpha} S(\alpha, \beta) + S(\alpha - 1, \beta + 1), \]

we have the following recursive formula.

\[ S(\alpha - 1, \beta + 1) = (1 - s^{\beta - \alpha}) S(\alpha, \beta). \]

Now the required formula follows since \( S(N - 1, \gamma) = 1 \) for any integer \( \gamma \). \( \square \)

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