Abstract. We consider a minimal realization of a rational matrix functions $R(\lambda)$ of the form

$$R(\lambda) = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1} B =: P(\lambda) + C(\lambda E - A)^{-1} B,$$

where $A_i \in \mathbb{C}^{n \times n}$, and $A, E, C, B$ are constant matrices of appropriate dimensions. We perturb the polynomial part and either $C$ or $B$ from the realization part. We derive explicit computable expressions of backward errors of approximate eigenvalue of $R(\lambda)$. We also determine minimal perturbations for which approximate eigenvalue are exact eigenvalue of the perturbed matrix rational functions.

Key words. Rational matrix, realization, matrix polynomial, eigenvalue, eigenvector, Fiedler pencil, linearization, backward error.

AMS subject classifications. 65F15, 15A18, 15B57, 15A22

1. Introduction. In this paper we study about the perturbation analysis of rational eigenvalue problem. Given a rational matrix in realization form $R(\lambda)$ the main purpose of this paper is to investigate backward error analysis of $R(\lambda)$. Particularly, we consider a minimal realization of $R(\lambda)$ of the form

$$R(\lambda) = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1} B =: P(\lambda) + C(\lambda E - A)^{-1} B,$$  

(1.1)

where $A_i \in \mathbb{C}^{n \times n}$, and $A, E, C, B$ are constant matrices of appropriate dimensions and perturb the polynomial part and either the matrix $C$ or $B$ from the realization part. Then we derive explicit computable expressions of backward errors of approximate eigenvalue of $R(\lambda)$.

Rational eigenvalue problems arises in many applications such as calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid-solid structures and in control theory, see [16, 17, 7, 12] and the references therein. For example, the rational matrix eigenvalue problem [12]

$$R(\lambda)x := -Ax + \lambda Bx + \sum_{j=1}^{k} \frac{\rho_j}{\lambda - \mu_j} C_j x = 0$$

where the matrices $A, B$ and $C_j$ are symmetric and positive (semi-) definite, and they are typically large and sparse arises vibrations of a tube bundle immersed in an inviscid compressible fluid are governed under some simplifying assumptions by an elliptic eigenvalue problem with non-local boundary conditions.

A similar problem [11]

$$R(\lambda)x = -Kx + \lambda Mx + \lambda^2 \sum_{j=1}^{k} \frac{1}{\omega_j - \lambda} C_j x = 0,$$

arises when a generalized linear eigenproblem is condensed exactly.

Considering a realization of $R(\lambda)$ given in [14], it is shown in [14] that the eigenvalues and the eigenvectors of $R(\lambda)$ can be computed by solving the generalized
We perturb the polynomial part and some of the rational part. The n we derive the
approximate eigenvalue and eigentriple of

\[ C \]

this paper is to undertake a detailed backward perturbation analys is of approximate
tional eigenvalue problem(see [14] and the references therein). T he main purpose of
significance due to the fact that linearization is a standard approac h to solving a ra-
tion of an optimum linearization of an rational eigenvalue problem. This assumes
of rational eigenvalue problems. Further, it also plays an important r ole in the selec-
ion play an important role in the accuracy assessment of co mputed solutions
computed solution is an exact solution of the perturbed problem. Bac kward perturba-
tively.

results on matrix polynomial. In section 3 we define the backward error
\eta R of an approximate eigenvalue and eigentriple of R for the rational eigenvalue problem R(λ).
We perturb the polynomial part and some of the rational part. Then we derive the
eigenvalue problem for the pencil

\[ C_1(\lambda) := \lambda \begin{bmatrix} A_m & I_n \\ & \ddots & \vdots \\ & & I_n \\ -E \end{bmatrix} + \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & -I_n & 0 \\ B & A \end{bmatrix} C \]  

(1.2)

where the void entries represent zero entries. The pencil C(λ) referred to as a companion linearization of R(λ) in [14].

For computing zeros (eigenvalues) and poles of rational matrix, linearizations of
rational matrix have been introduced recently in [16] via matrix-fraction descriptions
(MFD) of rational matrix. Let \( G(\lambda) = N(\lambda)D(\lambda)^{-1} \) be a right coprime MFD of G(λ),
where N(λ) and D(λ) are matrix polynomials with D(λ) being regular. Then the zero
structure of G(λ) is the same as the eigenstructure of N(λ) and the pole structure
of G(λ) is the same as the eigenstructure of D(λ), see [10]. Also G(λ) can be uniquely
written as G(λ) = P(λ) + Q(λ), where P(λ) is a matrix polynomial and Q(λ) is
strictly proper [10]. We define \( \text{deg}(G) := \text{deg}(P) \), the degree of the polynomial part
of G(λ).

**Definition 1.1 (Linearization, [1]).** Let G(λ) be an \( n \times n \) rational matrix function
(regular or singular) and let \( G(\lambda) = N(\lambda)D(\lambda)^{-1} \) be a right coprime MFD of G(λ).
Set \( r := \text{deg}(\text{det}(D(\lambda))) \), \( p := \max(n, r) \) and \( m := \text{deg}(G(\lambda)) \). If \( m \geq 1 \) then an
\( (mn + r) \times (mn + r) \) matrix pencil \( \mathbb{L}(\lambda) \) of the form

\[
\mathbb{L}(\lambda) := \begin{bmatrix} X - \lambda Y & C \\ B & A - \lambda E \end{bmatrix}
\]

(1.3)
is said to be a linearization of G(λ) provided that there are \( (mn + r) \times (mn + r) \) uni-
matrix pencils \( U(\lambda) \) and \( V(\lambda) \), \( A \times p \) unimodular matrix polynomials
Z(λ) and W(λ) such that \( U(\lambda) \text{diag}(I_{s-(mn+r)}, \mathbb{L}(\lambda)) V(\lambda) = \text{diag}(I_{s-n}, N(\lambda)) \) and
Z(λ)\text{diag}(I_{p-r}, A - \lambda E)W(λ) = \text{diag}(I_{p-n}, D(\lambda)) \) for \( \lambda \in \mathbb{C} \), where \( A - \lambda E \) is an
\( r \times r \) pencil with \( E \) being nonsingular and \( s := \max(mn + r, 2n) \).

Thus the zeros and poles of G(λ) are the eigenvalues of \( \mathbb{L}(\lambda) \) and \( A - \lambda E \), respec-
tively.

Backward perturbation analysis determines the smallest perturbation for which a
computed solution is an exact solution of the perturbed problem. Backward perturba-
tion analysis play an important role in the accuracy assessment of computed solutions
of rational eigenvalue problems. Further, it also plays an important role in the selec-
tion of an optimum linearization of an rational eigenvalue problem. This assumes
significance due to the fact that linearization is a standard approach to solving a ra-
tional eigenvalue problem(see [14] and the references therein). The main purpose of
this paper is to undertake a detailed backward perturbation analysis of approximate
eigenvalues of rational matrix functions.

In the present work we addressed the backward error of the REP. We perturbed
only the polynomial part and either C or B matrix from realization part given in
[11].

The paper is organized as follows: Section 2 contains some basic definitions and
results on matrix polynomial. In section 3 we define the backward error \( \eta R \) of an
approximate eigenvalue and eigentriple of R for the rational eigenvalue problem R(λ).
We perturb the polynomial part and some of the rational part. Then we derive the
explicit computable expressions of backward error. We also find out the minimal perturbations for which approximate eigenelements are exact eigenelements of the perturbed matrix rational functions. Finally, in section 4 we derive the backward error of companion linearization of rational eigenvalue problem and analyze the comparison.

\textbf{Notation:} We consider the Hölder $p$-norm on $\mathbb{C}^n$ defined by $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$, and $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$. We denote the set of $n \times n$ matrices with complex entries by $\mathbb{C}^{n \times n}$. We consider the spectral norm $\|\cdot\|_2$ and the Frobenius norm $\|\cdot\|_F$ on $\mathbb{C}^{n \times n}$ given by

$$\|A\|_2 := \max_{\|x\|=1} \|Ax\|_2 \quad \text{and} \quad \|A\|_F := (\text{Trace} A^* A)^{1/2}.$$

We denote the largest nonzero singular value of $A \in \mathbb{C}^{m \times n}$ by $\sigma_{\max}$ and smallest nonzero singular value of $A \in \mathbb{C}^{m \times n}$ by $\sigma_{\min}$ and pseudoinverse of $A \in \mathbb{C}^{m \times n}$ by $A^+$.

Consider $R(\lambda) + \Delta R(\lambda) = \sum_{j=0}^m (A_j + \Delta A_j)\lambda^j + W(C + \Delta C, A + \Delta A, E + \Delta E, B + \Delta B)$. Denote $\lambda_m = (1, \lambda, \ldots, \lambda^{m-1}, \lambda^m)$.

\textbf{2. Preliminaries.} Consider the polynomial eigenvalue problem (PEP) $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$, where

$$P(\lambda) = \sum_{i=0}^m \lambda^i A_i, \quad A_i \in \mathbb{C}^{n \times n}, \ A_m \neq 0$$

is a matrix polynomial of degree $m$. Here $x$ and $y$ are right and left eigenvectors corresponding to eigenvalue $\lambda$. The pair $(\lambda, x)$ is referred to as an eigenpair and the triple $(\lambda, x, y)$ is referred to as an eigentriple. We will assume throughout that $P$ is regular, that is $P(\lambda) \neq 0$.

\textbf{Definition 2.1.} Let $\mathbb{L}_m(\mathbb{C}^{n \times n})$ be the vector space of $n \times n$ matrix polynomials of degree at most $m$. Let $\|\cdot\|$ be a norm on $\mathbb{L}_m(\mathbb{C}^{n \times n})$. For $1 \leq p \leq \infty$, define $\|\cdot\|_p : \mathbb{L}_m(\mathbb{C}^{n \times n}) \to \mathbb{R}$ by

$$\|P\| := \|\|A_0\|, \ldots, \|A_m\||_p, \quad 1 \leq p \leq \infty, \quad (2.1)$$

where $P(z) = \sum_{i=0}^m z^i A_i$ and $\|\cdot\|_p$ is the Hölder’s $p$-norm. Then $\|\cdot\|$ is a norm on $\mathbb{L}_m(\mathbb{C}^{n \times n})$ and $\langle \cdot, \cdot \rangle_m : \mathbb{L}_m(\mathbb{C}^{n \times n}) \times \mathbb{L}_m(\mathbb{C}^{n \times n}) \to \mathbb{C}$ given by

$$\langle P_1, P_2 \rangle_m := \sum_{i=0}^m \langle A_i, B_i \rangle$$

is an innerproduct on $\mathbb{L}_m(\mathbb{C}^{n \times n})$, where $P_1(\lambda) = \sum_{j=0}^m \lambda^j A_i$ and $P_2(\lambda) = \sum_{j=0}^m \lambda^j B_i$. Then the dual norm $\|\cdot\|_*$ of $\|\cdot\|$ is given by

$$\|Y\|_* = \max \{ |\langle X, Y \rangle_m| : \|X\|_1 = 1 \}.$$

Given a norm $\|\cdot\|$ on $\mathbb{L}_m(\mathbb{C}^{n \times n})$, we define the normwise backward error of an approximate eigenpair $(x, \lambda)$ of $P(\lambda)$, where $\lambda$ is finite, is defined by

$$\eta_\lambda(x, \lambda) := \min \{ \|\Delta P\| : \Delta P \in \mathbb{L}_m(\mathbb{C}^{n \times n}), \ \det(P(\lambda) + \Delta P(\lambda)) = 0 \}$$

where $\Delta P(\lambda) = \sum_{i=0}^m \lambda^i \Delta A_i$. Then explicitly $\eta_\lambda(x, \lambda)$ is given by

$$\eta_\lambda(x, \lambda) = \min_{\|x\|_1 = 1} \left\{ \frac{\|P(\lambda)x\|}{\|\lambda, \ldots, \lambda^m\|_q} : x \in \mathbb{C}^n \right\} \leq \|P\|_p, \quad (2.2)$$
where $1/p + 1/q = 1$. Particularly, for 2-norm and frobenious norm the backward error is same and is given by

$$\eta_2(\lambda, P) = \eta_F(\lambda, P) = \frac{\sigma_{\text{min}}(P(\lambda))}{\|\{1, \lambda, \ldots, \lambda^n\}\|_q}.$$ 

Also, the explicit formula of the backward error is obtained in [8] is given by

$$\eta(x, \lambda) = \frac{\|P(\lambda)x\|_2}{(\sum_{i=0}^m |\lambda^i||A_i||_2\|x\|_2)}.$$  

(2.3)

3. Backward Errors for Rational Eigenvalue Problem. Consider the rational eigenvalue problem (REP) $R(\lambda)x = 0$ and $y^*R(\lambda) = 0$, where

$$R(\lambda) = \sum_{j=0}^m A_j\lambda^j - C(A - \lambda E)^{-1}B := P(\lambda) + W(\lambda),$$  

(3.1)

where $W(\lambda) = C(A - \lambda E)^{-1}B$, the size of $A$ and $E$ is $r \times r$, the size of $C$ is $n \times r$ and size of $B$ is $r \times n$. Here $x$ and $y$ are right and left eigenvectors corresponding to the eigenvalue $\lambda$. The pair $(\lambda, x)$ is referred to as an eigenpair and the triple $(\lambda, x, y)$ is referred to as an eigentriple. The standard way of solving this problem is to convert $R$ into a linear polynomial see [14, 11]

$$C_1(\lambda)z = 0,$$  

(3.2)

where $C_1(\lambda) = \lambda X + Y$, $Y, X \in \mathbb{R}^{nm+r}$ given in [12] and

$$z = \begin{bmatrix} \lambda^{m-1}x \\ \lambda^{m-2}x \\ \vdots \\ x \\ y \end{bmatrix}$$

and $y = -(A - \lambda E)^{-1}Bx$, with the same spectrum as $R$ and solve the eigenproblem for $C_1(\lambda)z = (\lambda X + Y)z = 0$, where $z$ is the right eigenvector of $C_1(\lambda)$, and $C_1$ is the companion linearization of $R$ see [11].

We now develop a general framework for perturbation analysis for rational eigenvalue problem (REP). We use the following notations throughout this chapter as follows:

Let $P(\lambda) = \sum_{i=0}^m A_i\lambda^i$ is a matrix polynomial of degree $\leq m$, $A_i \in \mathbb{C}^{n \times n}, i = 0 : m$, $R(\lambda) = P(\lambda) + C(A - \lambda E)^{-1}B$, where $C \in \mathbb{C}^{n \times r}, A, E \in \mathbb{C}^{r \times r}, B \in \mathbb{C}^{r \times n}$. Now perturb the coefficient of the matrix $R(\lambda)$ by $\Delta A_i, i = 0 : m, \Delta C, \Delta A, \Delta E, \Delta B$. Now question is how one can develop a general framework for perturbation analysis of rational eigenvalue problem (REP) ? Now consider the space of rational matrix functions of degree $n$, $\mathcal{X} = \langle (\mathbb{C}^{n \times n})^{m+1}, \mathbb{C}^{n \times m}, \mathbb{C}^{m \times n}, \mathbb{C}^{m \times m}, \mathbb{C}^{m \times n} \rangle$. Let $R \in \mathcal{X}$,

$$R(\lambda) = P(\lambda) + C(A - \lambda E)^{-1}B,$$

where $P(\lambda) = \sum_{i=0}^m A_i\lambda^i, A_i \in \mathbb{C}^{n \times n}, C \in \mathbb{C}^{n \times r}, A, E \in \mathbb{C}^{r \times r}, B \in \mathbb{C}^{r \times n}$. Now $\mathcal{X}$ is vector space under the componentwise addition and scalar multiplication. We assume throughout that the matrix rational function $R(\lambda)$ is regular, that is, det $R(\lambda) \neq 0,$
for some $\lambda \in \mathbb{C}$. The roots of $q_i(\lambda)$ are the poles of $R(\lambda)$. $R(\lambda)$ is not defined on these poles. Needless to mention that for perturbation analysis it is necessary to choose a norm so as to measure the magnitude of perturbations. Thus when $X$ is equipped with a norm the resulting normed linear space provides a general framework for perturbation analysis of matrix rational functions in $X$. So let $\|\cdot\|$ be a norm on $X$. Then the normed linear space $(X, \|\cdot\|)$ provides a natural framework for spectral perturbation analysis of matrix rational functions in $X$. We now employ this framework and analyze perturbation theory of matrix rational functions. So one may ask: Is there a natural norm on $X$ that facilitates systematic analysis of perturbation analysis of matrix rational functions? We will see that there are plenty of norms on the space of matrix rational functions $X$. Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n \times n}$. For $1 \leq p \leq \infty$, we define $\|\cdot\| : X \to \mathbb{R}$ by

$$
\|R\|_p := \|\|A_0\|, \cdots, \|A_m\|, \|C\|, \|A\|, \|E\|, \|B\|\|_p
$$

is a norm, where $R(z) := \sum_{i=0}^{m} A_i z^i + C(A - zE)^{-1} B$ and $\|\cdot\|_p$ is the Hölder’s $p$-norm on $\mathbb{C}^{n+5}$. We denote the space $X$ when equipped with the norm $\|\cdot\|_p$ by $(X_p, \|\cdot\|)$. Now consider $\mathbb{C}^{n \times n}$ and define $(\cdot, \cdot) : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ by $(X, Y) := \text{Trace}(Y^* X)$. Then $(\cdot, \cdot)$ defines an inner product on $\mathbb{C}^{n \times n}$ and $\|X\|_F := \sqrt{(X, X)}$ is the Frobenius norm on $\mathbb{C}^{n \times n}$. For a fixed $Y \in \mathbb{C}^{n \times n}$, the map $X \to (X, Y)$ is a linear functional on $\mathbb{C}^{n \times n}$ then (by the Riesz representation theorem) there exits a unique $Z \in \mathbb{C}^{n \times n}$ such that $F(X) = (X, Z)$. Now let $\|\cdot\|$ be a norm on $\mathbb{C}^{n \times n}$. Then $\|\cdot\|_* : \mathbb{C}^{n \times n} \to \mathbb{R}$ given by

$$
\|Y\|_* = \sup\{(X, Y) : X \in \mathbb{C}^{n \times n}, \|X\| = 1\}
$$

defines a norm and is referred to as the dual norm of $\|\cdot\|$. This shows that, For $R_1, R_2 \in (X_2, \|\cdot\|_f)$,

$$
\langle R_1, R_2 \rangle_X := \sum_{i=0}^{m} \langle A_i, A_i' \rangle + \langle C_1, C_2 \rangle + \langle A_1, A_2 \rangle + \langle E_1, E_2 \rangle + \langle B_1, B_2 \rangle,
$$

where $R_1(z) := \sum_{i=0}^{m} A_i z^i + C_1(A_1 - zE_1)^{-1} B_1$ and $R_2(z) := \sum_{i=0}^{m} A_i' z^i + C_2(A_2 - zE_2)^{-1} B_2$ defines an inner product on $(X_2, \|\cdot\|_f)$. Thus $(X_2, \|\cdot\|_f)$ is a Hilbert space and $\|R\|_2 = \sqrt{(R, R)_X}$. Let $\|\cdot\|$ be a norm on $X$ and $\|\cdot\|_*$ the dual norm

$$
\|Y\|_* = \sup\{(X, Y) : \|X\| = 1\}.
$$

In particular for $p$-norm the dual is given by

$$
\|R\|_* := \|(\|A_0\|, \cdots, \|A_m\|, \|C\|, \|A\|, \|E\|, \|B\|)\|_q.
$$

Now, we derive the backward error of approximate eigenelements of rational matrix function where we perturbed the polynomial part and the matrix either $C$ or $B$ matrix from realization part given in (1.1). Corresponding to this we define the backward error as is follows:

**Definition 3.1.** The normwise backward error of an approximate eigenelement $\lambda$ of $R(\lambda)$, where $\lambda$ is finite, is defined by

$$
\eta_p(\lambda, R) := \min\{\|\Delta P \cdot \Delta B \cdot \Delta C\| : \Delta R \in X, \det(R(\lambda) + \Delta R(\lambda)) = 0\}
$$

$$
\eta_p(\lambda, R) := \min\{\|\Delta P \cdot \Delta B \cdot \Delta C\|^T \cdot \Delta R \in X, \det(R(\lambda) + \Delta R(\lambda)) = 0\}
$$
and the matrix

Now consider the matrix rational function with perturbing only the polynomial part

\[ R = \begin{bmatrix} R^{-1}(\lambda) \\ \lambda R^{-1}(\lambda) \\ W_1(\lambda) R^{-1}(\lambda) \end{bmatrix}, \quad W_1(\lambda) = (A - \lambda E)^{-1}B, \quad \Delta = \begin{bmatrix} \Delta A_0 & \Delta A_1 & \Delta C \end{bmatrix} \quad \text{and} \quad v = R(\lambda)x. \]

Then the following statements are equivalent.

(i) \( \det(R(\lambda) + \Delta R(\lambda)) = 0 \)

(ii) \( \Delta T(\lambda)v = -v. \)

**Proof.** Let \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta C(A - \lambda E)^{-1}B. \) Set \( W_1(\lambda) = (A - \lambda E)^{-1}B. \) Then \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta C W_1(\lambda). \) Now \( \det(R(\lambda) + \Delta R(\lambda)) = 0. \) That means there exists a nonzero vector \( x \) such that \( \|x\| = 1 \) and \( R(\lambda)x + \Delta R(\lambda)x = 0. \) So we have \( (I + \Delta R(\lambda)R^{-1}(\lambda))R(\lambda)x = 0. \) Put \( v = R(\lambda)x, \) implies that \( x = R^{-1}(\lambda)v. \) Then \( \Delta R(\lambda)R^{-1}(\lambda)v = -v. \) Hence the result follows. \( \square \)

**Corollary 3.3.** Let \( R \) satisfies all the given conditions of Lemma 3.2. Then

\[ \eta(\lambda, v, R) := \min \{ \|\Delta\| : \Delta T(\lambda)v = -v \}. \]

**Theorem 3.4.** Consider the subordinate matrix norm \( \|\cdot\| \) on \( \mathbb{C}^{n \times n}. \) Let \( R(\lambda) = A_0 + \lambda A_1 + C(A - \lambda E)^{-1}B \) and \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta C(A - \lambda E)^{-1}B. \) Set

\[ \Delta = \begin{bmatrix} \Delta A_0 & \Delta A_1 & \Delta C \end{bmatrix}. \quad \text{Chose} \ x = R(\lambda)^{-1}v. \quad \text{Set} \ T(\lambda) = \begin{bmatrix} R^{-1}(\lambda) \\ \lambda R^{-1}(\lambda) \\ W_1(\lambda) R^{-1}(\lambda) \end{bmatrix} \quad \text{and} \quad W_1(\lambda) = (A - \lambda E)^{-1}B. \]

Then we have

\[ \eta(\lambda, R) = \min_{\|v\| = 1} \left\{ \frac{v(T(\lambda)v)^*}{\|T(\lambda)v\|^2} \right\}. \]

Then we have

\[ \eta(\lambda, R) = \min_{\|v\| = 1} \left\{ \frac{1}{\|T(\lambda)v\|^2} \right\}. \]

**Proof.** Let \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta C(A - \lambda E)^{-1}B. \) Set \( W_1(\lambda) = (A - \lambda E)^{-1}B. \) Then \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta C W_1(\lambda). \) Now \( \det(R(\lambda) + \Delta R(\lambda)) = 0. \) That means there exists a nonzero vector \( x \) such that \( \|x\| = 1 \) and \( R(\lambda)x + \Delta R(\lambda)x = 0. \) So we have \( (I + \Delta R(\lambda)R^{-1}(\lambda))R(\lambda)x = 0. \) Put \( v = R(\lambda)x, \) implies that \( x = R^{-1}(\lambda)v. \) Then \( \Delta R(\lambda)R^{-1}(\lambda)v = -v. \Rightarrow \left[ \begin{array}{ccc} \Delta A_0 & \Delta A_1 & \Delta C \end{array} \right] T(\lambda)v = -v. \) Define

\[ \Delta A_0 := \frac{-v[R^{-1}(\lambda)v]^*}{\sigma_{\max}(T(\lambda))}, \quad \Delta A_1 := \frac{-\lambda v[R^{-1}(\lambda)v]^*}{\sigma_{\max}(T(\lambda))}, \quad \text{and} \quad \Delta C := \frac{-v[W(\lambda)R^{-1}(\lambda)v]^*}{\sigma_{\max}(T(\lambda))}. \]

Now consider the matrix rational function with perturbing only the polynomial part and the matrix \( C \) only. Consider \( \Delta R(\lambda) := \Delta A_0 + \lambda \Delta A_1 + \Delta C(A - \lambda E)^{-1}B. \) Then by the construction, we get \( R(\lambda)v + \Delta R(\lambda)v = 0 \) and \( \|\Delta\| = \frac{1}{\|T(\lambda)v\|}. \) \( \square \)

**Theorem 3.5.** Let \( R(\lambda) = A_0 + \lambda A_1 + C(A - \lambda E)^{-1}B \) be regular. Chose \( x = R(\lambda)^{-1}v. \) Set \( \Delta = \begin{bmatrix} \Delta A_0 & \Delta A_1 & \Delta C \end{bmatrix}, \quad T(\lambda) = \begin{bmatrix} R^{-1}(\lambda) \\ \lambda R^{-1}(\lambda) \\ W_1(\lambda) R^{-1}(\lambda) \end{bmatrix} \quad \text{and} \quad W_1(\lambda) = (A - \lambda E)^{-1}B. \)
\( (A - \lambda E)^{-1}B \). Then for Frobenious norm \( \| . \|_F \) and 2-norm on \( \mathbb{C}^{n \times n} \) we have

\[
\eta(\lambda, R) = \frac{1}{\sigma_{\text{max}}(T(\lambda))},
\]

**Proof.** Let \( v \) be unit right singular vector of \( T(\lambda) \). Let \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta C(A - \lambda E)^{-1}B \). Set \( W_1(\lambda) = (A - \lambda E)^{-1}B \). Then \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \Delta CW_1(\lambda) \). Now \( \det(R(\lambda) + \Delta R(\lambda)) = 0 \). That means there exists a nonzero vector \( x \) such that \( \|x\| = 1 \) and \( R(\lambda)x + \Delta R(\lambda)x = 0 \). So we have \( [I + \Delta R(\lambda)R^{-1}(\lambda)]R(\lambda)x = 0 \). Put \( v = R(\lambda)x \), implies that \( x = R^{-1}(\lambda)v \). Then \( \Delta R(\lambda)R^{-1}(\lambda)v = -v \). Define

\[
\Delta A_0 := \frac{-v[R^{-1}(\lambda)v]^*}{\sigma_{\text{max}}(T(\lambda))}, \quad \Delta A_1 := \frac{-\bar{v}[R^{-1}(\lambda)v]^*}{\sigma_{\text{max}}(T(\lambda))}, \quad \text{and} \quad \Delta C := \frac{-v[W(\lambda)R^{-1}(\lambda)v]^*}{\sigma_{\text{max}}(T(\lambda))}.
\]

Now consider the matrix rational function with perturbing only the polynomial part and the matrix \( L \) only. Consider \( \Delta R(\lambda) := \Delta A_0 + \lambda \Delta A_1 + \Delta C(A - \lambda E)^{-1}B \). Then by the construction, we get \( R(\lambda)x + \Delta R(\lambda)x = 0 \). Since each \( \Delta A_0, \Delta A_1 \) and \( \Delta C \) is a rank 1 matrix, the spectral and the Frobenius norms of \( \Delta A_0, \Delta A_1 \) and \( \Delta C \) are same. Consequently, \( \| \Delta R \|_F \) is same for the spectral and the Frobenius norms on \( \mathbb{C}^{n \times n} \). Now, we get \( \| \Delta R \|_2 = \frac{1}{\sigma_{\text{max}}(T(\lambda))} \). Hence the result follows. \( \square \)

**Theorem 3.6.** Let \( R(\lambda) = \sum_{i=0}^{m} \lambda^i A_i + C(A - \lambda E)^{-1}B \) be regular and \( \Delta R(\lambda) = \sum_{i=0}^{m} \lambda^i \Delta A_i + \Delta C(A - \lambda E)^{-1}B \). Chose \( x = R(\lambda)^{-1}v \). Set \( T(\lambda) = \left[ \begin{array}{c} A_m^T \otimes R^{-1}(\lambda) \\ W_1(\lambda)R^{-1}(\lambda) \end{array} \right] \), \( W_1(\lambda) = (A - \lambda E)^{-1}B \), and \( \Delta = [ \Delta A_0 \cdots \Delta A_m \Delta C ] \). Then for Frobenious norm \( \| . \|_F \) and 2-norm on \( \mathbb{C}^{n \times n} \) we have

\[
\eta_2(\lambda, R) = \eta_F(\lambda, R) = \frac{1}{\sigma_{\text{max}}(T(\lambda))}
\]

**Proof.** Define

\[
\Delta A_i = \frac{-\text{sign}(\lambda^i)|\lambda|^i v[R^{-1}(\lambda)v]^*}{\sigma_{\text{max}}(T(\lambda))}, \quad i = 0 : m, \quad \Delta C := \frac{-v[W(\lambda)R^{-1}(\lambda)v]^*}{\sigma_{\text{max}}(T(\lambda))}
\]

Then by the construction, we get \( [R(\lambda) + \Delta R(\lambda)]x = 0 \).

**Lemma 3.7.** Let \( R(\lambda) = A_0 + \lambda A_1 + C(A - \lambda E)^{-1}B \), \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + C(A - \lambda E)^{-1}\Delta B \), and \( \lambda \in \mathbb{C} \). Assume that \( R(\lambda) \) is nonsingular. Set \( S(\lambda) = \left[ \begin{array}{c} R^{-1}(\lambda) \lambda R^{-1}(\lambda) & -R^{-1}(\lambda)W(\lambda) \\ \Delta A_0 & \Delta A_1 \end{array} \right] \). Then \( \tilde{W}(\lambda) = C(A - \lambda E)^{-1} \) and \( \Delta = \left[ \begin{array}{c} \Delta A_0 \\ \Delta A_1 \\ \Delta B \end{array} \right] \).

Then the following statements are equivalent.

(i) \( \det(R(\lambda) + \Delta R(\lambda)) = 0 \)

(ii) \( S(\lambda)\Delta x = -x \).

**Proof.** Let \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + C(A - \lambda E)^{-1}\Delta B \). Now \( \det(R(\lambda) + \Delta R(\lambda)) = 0 \). That means there exists a nonzero vector \( x \) such that \( \|x\| = 1 \) and \( R(\lambda)x + \Delta R(\lambda)x \neq 0 \). So we have \( [I + R^{-1}(\lambda)\Delta R(\lambda)]x = 0 \). Then \( R^{-1}(\lambda)\Delta R(\lambda)x = -x \). Hence the result follows. \( \square \)

**Corollary 3.8.** Let \( R \) satisfies all the given conditions of Lemma 3.7. Then

\[
\eta(\lambda, x, R) = \min\{\|\Delta\| : S(\lambda)\Delta x = -x\}
\]
Theorem 3.9. Consider the subordinate matrix norm \( \| \cdot \| \) on \( \mathbb{C}^{n \times n} \). Let \( R(\lambda) = A_0 + \lambda A_1 + C(A - \lambda E)^{-1}B \) and \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + C(A - \lambda E)^{-1}B \). Set

\[
\Delta = \begin{bmatrix} \Delta A_0 \\ \Delta A_1 \\ \Delta B \end{bmatrix}
\]

and \( S(\lambda) = \begin{bmatrix} R^{-1}(\lambda) & \lambda R^{-1}(\lambda) & R^{-1}(\lambda)\hat{W}(\lambda) \end{bmatrix} \). If \( S(\lambda)S(\lambda)^+x = x \). Then we have

\[
\eta(\lambda, R) = \min_{\|x\|=1} \left\{ \|S(\lambda)^+xx^*\| \right\}.
\]

Proof. Let \( \hat{W}(\lambda) = C(A - \lambda E)^{-1} \) and \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \hat{W}(\lambda)\Delta B \). Now \( \det(R(\lambda) + \Delta R(\lambda)) = 0 \). Then there exists \( x \neq 0 \) with \( \|x\| = 1 \) such that \( [R(\lambda) + \Delta R(\lambda)]x = 0 \). Then \( [I + R^{-1}(\lambda)\Delta R(\lambda)]x = 0 \). Define \( Z(\lambda) = [S(\lambda)S(\lambda)^+]^{-1} \) and

\[
\Delta A_0 = R^{-1}(\lambda)^*Z(\lambda)x^* \quad \Delta A_1 = \lambda R^{-1}(\lambda)^*Z(\lambda)x^* \quad \Delta B = (R^{-1}(\lambda)\hat{W}(\lambda))^*Z(\lambda)x^*.
\]

Then by the construction, we get \( [I + R^{-1}(\lambda)\Delta R(\lambda)]x = 0 \) and \( \|\Delta\| = \|S(\lambda)^+xx^*\|^{-1} \).

Theorem 3.10. Consider the space \( (\mathbb{X}, \| \cdot \|) \) corresponding to subordinate matrix norm \( \| \cdot \| \) on \( \mathbb{C}^{n \times n} \). Let \( R(\lambda) = A_1 + \lambda A_2 + C(A - \lambda E)^{-1}B \). Perturbing only the pencil part and the matrix \( B \). Set

\[
S(\lambda) = \begin{bmatrix} R^{-1}(\lambda) & \lambda R^{-1}(\lambda) & R^{-1}(\lambda)\hat{W}(\lambda) \end{bmatrix} \). If \( S(\lambda)S(\lambda)^+v = v \). Then we have

\[
\eta_p(\lambda, R) = \min_{\|x\|=1} \left\{ 1 \right\}
\]

where \( 1/p + 1/q = 1 \).

Proof. Let \( \hat{W}(\lambda) = C(A - \lambda E)^{-1} \) and \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \hat{W}(\lambda)\Delta B \). Now \( \det(R(\lambda) + \Delta R(\lambda)) = 0 \). Then there exists \( x \neq 0 \) with \( \|x\| = 1 \) such that \( [R(\lambda) + \Delta R(\lambda)]x = 0 \). Then \( [I + R^{-1}(\lambda)\Delta R(\lambda)]x = 0 \). Define \( Z(\lambda) = [S(\lambda)S(\lambda)^+]^{-1} \) and

\[
\Delta A_0 = R^{-1}(\lambda)^*Z(\lambda)x^* \quad \Delta A_1 = \lambda R^{-1}(\lambda)^*Z(\lambda)x^* \quad \Delta B = (R^{-1}(\lambda)\hat{W}(\lambda))^*Z(\lambda)x^*.
\]

Then by the construction, we get \( [I + R^{-1}(\lambda)\Delta R(\lambda)]x = 0 \) and \( \|\Delta\| = \|S(\lambda)^+xx^*\|^{-1} \).

Theorem 3.11. Let \( R(\lambda) = A_0 + \lambda A_1 + C(A - \lambda E)^{-1}B \) be regular, \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + C(A - \lambda E)^{-1}B \) and \( \lambda \in \mathbb{C} \). Set \( \Delta = \begin{bmatrix} \Delta A_0 \\ \Delta A_1 \\ \Delta B \end{bmatrix} \),

\[
S(\lambda) = \begin{bmatrix} R^{-1}(\lambda) & \lambda R^{-1}(\lambda) & R^{-1}(\lambda)\hat{W}(\lambda) \end{bmatrix} \) and \( \hat{W}(\lambda) = C(A - \lambda E)^{-1} \). If \( S(\lambda)S(\lambda)^+v = v \). Then for Frobenious norm \( \| \cdot \|_F \) and 2-norm on \( \mathbb{C}^{n \times n} \) we have

\[
\eta_2(\lambda, R) = \eta_F(\lambda, R) = \frac{1}{\sigma_{\min}(S(\lambda))}.
\]

Proof. Let \( v \) be the right singular vector of \( S(\lambda) \). Let \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \hat{W}(\lambda)\Delta B \). Set \( \hat{W}(\lambda) = C(A - \lambda E)^{-1} \). Then \( \Delta R(\lambda) = \Delta A_0 + \lambda \Delta A_1 + \hat{W}(\lambda)\Delta B \). Define \( Z(\lambda) = [S(\lambda)S(\lambda)^+]^{-1} \) and

\[
\Delta A_0 = -R^{-1}(\lambda)^*Z(\lambda)v^* \quad \Delta A_1 = -\lambda R^{-1}(\lambda)^*Z(\lambda)v^* \quad \Delta B = -(R^{-1}(\lambda)\hat{W}(\lambda))^*Z(\lambda)v^*.
\]

Then by the construction, we get \( [I + R^{-1}(\lambda)\Delta R(\lambda)]v = 0 \). Since each \( \Delta A_0, \Delta A_1 \) and \( \Delta B \) is a rank 1 matrix, the spectral and the Frobenius norms of \( \Delta A_0, \Delta A_1 \) and \( \Delta B \) are same. Consequently, \( \|\Delta R\|_2 \) is same for the spectral and the Frobenius norms on \( \mathbb{C}^{n \times n} \) and \( \|\Delta\|_2 = \frac{1}{\sigma_{\min}(S(\lambda))} \). Hence the result follows.

Theorem 3.12. Let \( R(\lambda) = \sum_{i=0}^{m} \lambda^i A_i + C(A - \lambda E)^{-1}B \) be regular and \( \Delta R(\lambda) = \sum_{i=0}^{m} \lambda^i \Delta A_i + C(A - \lambda E)^{-1}B \). Set \( \Delta = \begin{bmatrix} \Delta A_0 \\ \vdots \\ \Delta A_m \\ \Delta B \end{bmatrix} \), \( S(\lambda) = \begin{bmatrix} A_m \otimes R^{-1}(\lambda) & R^{-1}(\lambda)\hat{W}(\lambda) \end{bmatrix} \).
and $\tilde{W}(\lambda) = C(A - \lambda E)^{-1}$. If $S(\lambda)S(\lambda)^+v = v$. Then for Frobenious norm $\|\cdot\|_F$ and matrix 2-norm on $\mathbb{C}^{n \times n}$ we have

$$\eta_2(\lambda, R) = \eta_F(\lambda, R) = \frac{1}{\sigma_{\min}(S(\lambda))}.$$ 

**Theorem 3.13.** Let $R$ be regular. Perturbing only polynomial part and keep the rational part as it is. Then for matrix $\mathbf{2}$-norm we have

$$\eta(\lambda, R) \leq \frac{\sigma_{\min}(R(\lambda))}{\| (1, \lambda, \ldots, \lambda^m) \|_2}.$$ 

4. **Companion Linearization.** Let $C_1(\lambda) = \lambda \chi + \mathcal{V}$ be the first companion linearization of $R(\lambda)$. Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n \times n}$. For $1 \leq p \leq \infty$, $\|C\| = \|((\|\mathcal{V}\|, \|\chi\|))_p$. Therefore by Hölder’s inequality we have

$$\|C_1(\lambda)\| \leq \|C\| \| (1, \lambda) \|_q.$$ 

**Proposition 4.1.** Let $\|\cdot\|$ be the subordinate matrix norm on $\mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$. Then

$$\eta_p(\lambda, C_1) = \min_{\|z\|=1} \left\{ \frac{\|C_1(\lambda) z\|}{\| (1, \lambda) \|_q} : z \in \mathbb{C}^{n^{-m+1}} \right\} = \left( \| (1, \lambda) \|_q \|C_1(\lambda)^{-1}\|_1^1 \right) \leq C_1\|q,$$

where $1/p + 1/q = 1$. For matrix 2-norm and Frobenius norm we get

$$\eta_2(\lambda, C_1) = \frac{\sigma_{\min}(C_1(\lambda))}{\| (1, \lambda) \|_2}.$$ 

**Theorem 4.2.** Let $\eta_2(\lambda, R)$ is given in Theorem 4.13. Then

$$\frac{\eta_2(\lambda, C_1)}{\eta_2(\lambda, R)} \geq \frac{\sqrt{m} \sigma_{\min}(C_1(\lambda))}{\sigma_{\min}(R(\lambda))}.$$ 

*Proof. Let $\Lambda_m = (1, \lambda, \ldots, \lambda^m)$. Then

$$\frac{\eta_2(\lambda, C_1)}{\eta_2(\lambda, R)} \geq \frac{\sigma_{\min}(C_1(\lambda))}{\sigma_{\min}(R(\lambda))} \| (1, \lambda) \|_2 \geq \frac{\sigma_{\min}(C_1(\lambda))}{\sigma_{\min}(R(\lambda))} \| (1, \lambda) \|_2.$$ 

We know that $\frac{1}{\sqrt{2}} \leq \frac{\| \Lambda_m \|}{\| (1, \lambda) \|_2} \leq 1$. Now

$$\frac{\| \Lambda_m \|}{\| (1, \lambda) \|_2} = \frac{\| \Lambda_m \|}{\| \Lambda_{m-1} \|_2} \frac{1}{\| (1, \lambda) \|_2} \geq \frac{1}{\sqrt{2}} \| \Lambda_{m-1} \|_2 = \frac{1}{\sqrt{2}} \left( \sum_{i=0}^{m-1} \lambda^i \right)^{1/2}.$$ 

If $|\lambda| \geq 1$ then $\frac{\| \Lambda_m \|}{\| (1, \lambda) \|_2} \geq \frac{1}{\sqrt{2}} \sqrt{m}$. So

$$\frac{\eta_2(\lambda, C_1)}{\eta_2(\lambda, R)} \geq \frac{\sqrt{m} \sigma_{\min}(C_1(\lambda))}{\sqrt{2} \sigma_{\min}(R(\lambda))}.$$ 

$\square$
**Conclusion** We defined the backward error $\eta_R$ of an approximate eigenvalue and eigentriple of $R$ for the rational eigenvalue problem $R(\lambda)$ given in (1.1). Then we derived the explicit computable expressions of backward error of approximate eigenvalue of $R(\lambda)$. We also found out the minimal perturbations for which approximate eigenelements are exact eigenelements of the perturbed matrix rational functions. Finally, we derived the backward error of companion linearization of rational eigenvalue problem and analyzed the comparison with the original one.

**REFERENCES**

[1] R. Alam and N. Behera, *Linearizations for Rational Matrix Functions and Rosenbrock System Polynomials*, SIAM J. Matrix Analysis Appl., 37(2016), pp.354-380.

[2] R. Alam and N. Behera, *Recovery of eigenvectors of rational matrix functions from Fiedler-like linearizations*, Linear Algebra Appl., 510(2016), pp.373-394.

[3] R. Alam and N. Behera, *Generalized Fiedler pencils for Rational Matrix functions*, SIAM J. MATRIX ANAL. APPL., 39(2018), pp. 587-610.

[4] A. Amparan, F. M. Dopico, S. Marcaida, and I. Zaballa, *Strong Linearizations of Rational Matrices*, SIAM J. MATRILINEAR ANAL. APPL., 39(2018), pp. 1670-1700.

[5] E. N. Antoniou and S. Vologiannidis, *A new family of companion forms of polynomial matrices*, Electron. J. Linear Algebra, 11(2004), pp.78-87.

[6] N. Behera, *Fiedler linearizations for LTI state-space systems and for rational eigenvalue problems*, PhD Thesis, IIT Guwahati, 2014.

[7] Betcke, T., Higham, N. J., Mehrmann, V., Schröder, C., and Tisseur, F., *NLEVP: A collection of nonlinear eigenvalue problems*, ACM Trans. Math. Softw. 39 (2) (2013).

[8] Nicholas J. Higham, Ren-Cang Li, and Françoise Tisseur, *Backward error of polynomial eigenproblems solved by linearization*, SIAM J. Matrix Anal. Appl., 29(4) 2007, pp. 1218–1241.

[9] F. De Terán, F. M. Dopico, and D. S. Mackey, *Fiedler companion linearizations and the recovery of minimal indices*, SIAM J. Matrix Anal. Appl., 31(2010), pp. 2181-2204.

[10] T. Kailath, *Linear systems*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1980.

[11] V. Mehrmann and H. Voss, *Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods*, GAMM Mitt. Ges. Angew. Math. Mech., 27(2004), pp.121-152.

[12] Conca, C. and Planchard, J. and Vanninathan, M., *Existence and location of eigenvalues for fluid-solid structures*, Comput. Methods Appl. Mech. Engrg., 77(3), 1989, pp.253–291.

[13] H. H. Rosenbrock, *State-space and multivariable theory*, John Wiley & Sons, Inc., New York, 1970.

[14] Y. Su and Z. Bai, *Solving rational eigenvalue problems via linearization*, SIAM J. Matrix Anal. Appl., 32(2011), pp.201-216.

[15] A. I. G. Vardoulakis, *Linear multivariable control*, John Wiley & Sons Ltd., 1991.

[16] H. Voss, *A rational spectral problem in fluid-solid vibration*, Electron. Trans. Numer. Anal., 16(2003), pp.93-105.

[17] H. Voss, *Iterative projection methods for computing relevant energy states of a quantum dot*, J. Comput. Phys., 217(2006), pp.824-833.