How simple is simple pendulum?

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Abstract

The simple pendulum is one of the first experiments that students of higher physics do. There are certain precautions which the students are asked to take while performing the experiment. In this note we will try to explain as to why these precautions are taken up and what will happen if we relax them.
The simple pendulum is one of the first experiments that physics students do in higher secondary. It's a fairly simple experiment, which is then repeated in form of "finding g by bar/ katar's pendulum" in graduation. Yet a feature noticed is the lack of appreciation by the students that "the initial angular displacement of the pendulum must be small", which is disturbing. At the end of the session during viva examination of the students reveal the ignorance of the students about this precaution. Answers ranging from "to minimize air resistance" to "make sure the pendulum won't wobble" are given. In this article we will approach the problem of convincing students as to why this precaution is necessary. Also we will try to give some insight of what could be the possible results if we don't take this precaution.

In the first section we are be going to discuss the simple pendulum which the student know (where the initial displacement is small). Here in this section we will set up the relevant equation first and then solve this equation analytically according to the given initial conditions. In second section we will set up the equation of the motion of simple pendulum, when we relax the condition that the initial displacement from mean position is small. In the third section we will try to solve the equation which we had set up in second section. In last section we will derive the conclusions of the whole exercise.

1 Pendulum with small initial displacement

Before proceeding to set up the differential equation of a pendulum with a large initial displacement from the mean position, first in this section we see how the usual simple harmonic equation is a result of the imposed condition that the initial displacement from the mean position is small. Consider the Figure (1), which shows the bob of the pendulum suspended by a string of length 'L'. The angular displacement is Θ and the weight’s \(mg\) resolved component pointing towards the means position is \(mgsin\Theta\). This is the force, trying to bring back the
bob to its mean position, \( i.e. \)

\[ F = - mgsin\Theta \]  \hspace{1cm} (1)

This force is a restoring force (this is indicated by the negative sign in eq.(1)), since it is trying to bring back the body to its initial (mean) position, since more you move away from the mean position the larger is the force. That is, the force is directly proportional to the displacement of the pendulum from its mean position. Here the displacement is the length of chord, marked AB. Since initially, we assume the angular displacement to be very small, the displacement (length of the chord) and the length of the arc AB can be assumed equal. Also, as a result of our assumption the mgsin\( \Theta \) component which acts tangentially at point 'B', can be assumed to be along the chord. Now the length of chord is difficult to compute, however, the length of the arc is obtained from the relation

\[ \Theta = \frac{\text{arc}}{\text{radius}} \]  \hspace{1cm} (2)
Since, the radius of the circle under consideration here is the length of the thread suspending the bob (with the assumption that the radius of the bob is insignificant as compared to the length of the thread, i.e. the bob can be considered to be a point mass), the arc length is given as

\[ \text{arc } AB = L \Theta \]  

(3)

hence, with our assumption that the angular displacement is very small, the length of the chord (s) is equal to arc AB, i.e.

\[ s = L \Theta \]  

(4)

The equation of motion eq.(1), now can be written as

\[ m \frac{d^2 s}{dt^2} = - mgsin\Theta \]  

(5)

substituting eqn(4) with the knowledge that the length of the string is constant we have

\[ \frac{d^2 \Theta}{dt^2} = - \frac{g}{L} \sin\Theta \]  

(6)

In the above equation considering the case of small initial angular displacement i.e. \( \Theta \) is small, then \( \sin\Theta \sim \Theta \) and substituting \( \omega^2 = \frac{g}{L} \), we have

\[ \frac{d^2 \Theta}{dt^2} = - \omega^2 \Theta \]  

(7)

This is the familiar simple harmonic motion (SHM) equation, whose general solution is given by

\[ \Theta(t) = A \sin(\omega t) + B \cos(\omega t) \]  

(8)

where A and B are constants (there would be in general two constants because we are trying to solve a second order differential eqn.(7)). We can get the values of the constants by choosing suitable initial conditions (in that case we call the solution to be a particular solution). We will discuss this again in last section.

The time period of oscillation can be obtained from the relationship (\( \omega = \frac{2\pi}{T} \))

\[ \omega = \sqrt{\frac{g}{L}} \]  

(9)
leading to

$$T = 2\pi \sqrt{\frac{T}{g}}$$  \hspace{1cm} (10)

2 Pendulum with large initial displacement

To set up the differential equation for a pendulum which is given a large displacement, again we have to find the length of the chord AB. In general, the length of the chord, or the displacement is given as

$$s = 2L \sin \frac{\Theta}{2}$$  \hspace{1cm} (11)

considering this as the displacement and using the EOM (eq.(1))

$$m \frac{d^2 s}{dt^2} = mL \cos \frac{\Theta}{2} \frac{d^2 \Theta}{dt^2} - \frac{mL}{2} \sin \frac{\Theta}{2} \left( \frac{d\Theta}{dt} \right)^2$$

$$= -mg \sin \Theta$$  \hspace{1cm} (12)

so the EOM for large initial displacement is (simplifying eq.(12))

$$\cos \frac{\Theta}{2} \frac{d^2 \Theta}{dt^2} - \frac{1}{2} \sin \frac{\Theta}{2} \left( \frac{d\Theta}{dt} \right)^2 = -\omega^2 \sin \Theta$$  \hspace{1cm} (13)

Now our aim is to estimate the error which would be introduced if the displacement is large. For this we have to solve the above eqn.(13). The term $\left( \frac{d\Theta}{dt} \right)^2$ complicates the simple second order differential equation we have dealt with. Such equations are called non-linear differential equations and usually are difficult to solve analytically. Indeed most non-linear equations can only be solved numerically. We will try to solve this equation numerically in next section and will try to compare the two solutions.

3 Numerical results

As we mentioned in previous section we can’t solve the eqn.(13) analytically. So we will try to solve the equation numerically. Now if we want to solve eqn. numerically we can’t get what is
called general solution. To get numerical solution we require the exact numerical values of all the constants and initial conditions.

The initial conditions (conditions at $t = 0$) would be imposed on $\Theta$ and $d\Theta/dt$. These initial conditions would help us in finding out the particular solution from the general solution eq.(8). To solve eq.(13) numerically we also require the value of $\omega$.

The initial conditions which we are choosing are

$$\Theta(t = 0) = a \quad ; \quad \frac{d\Theta(t = 0)}{dt} = 0$$

(14)

where $a$ is a constant whose value we will take as input. So to completely solve the equation numerically we have to specify the values of $a$ and $\omega$.

Fig(2) and Fig(3) is a plot exhibiting the evolution of $\Theta$, the angular displacement with time and $\omega$ respectively, based on the results of our numerical solutions of equation (7) and (13). For comparing the possible error that creeps in due to the injudicious choice, i.e. use eqn(7).
Figure 3: Plot of $\Theta$ vs $\omega$, other parameters are time = 20 seconds, $a = 1.4$ (in radians)

for large displacement, the numerical solution of both equations were done for the same initial displacement $a$, with the assumption that the ratio of $g$ to $L$ is 16 ($i.e.$ $\omega = 4$). Remember, small initial displacement condition is when you can substitute $\Theta$ instead of $2\sin(\Theta/2)$. It is evident from fig(5), where the straight line shows linear displacement as a function of angular displacement for small oscillation condition (eqn (4)) while actual displacement follows eqn(11). In figure(5) we have tried to show the region (of the angular displacement) in which a motion of pendulum can be considered to be that of SHM. Keeping this in mind, to emphasis our point to show errors that are bound to occur, we have done our calculations with $a = 1.4$ rad. The variation in the angular displacement with time, $\Theta(t)$ in fig(2), for the two cases are very different, especially with increasing time. This indicates disparity between the two cases. It would hence, be interesting to note that initially there is not much difference in position at a given instant between the two cases, as can be seen. However, as time progresses disparity increases. This disparity is found to depend on the size of the initial displacement, as shown in Fig(4). Fig(4) is the angular displacement of pendulum after 40 sec of the sustained oscillations.
for increasing initial displacement. The choice is just to show disparity between the two cases soon after oscillations have commenced. Figure (4) shows how much the disparity between the two cases increases with increasing initial angular displacement, in turn the deviation from “small displacement condition”.

4 Conclusion

Now let’s summarize the conclusions of the whole exercise. We know that only small initial displacement can exhibit SHM motion, while large initial displacement will execute Non SHM. A simple pendulum experiment is usually done to establish the value of the acceleration due to gravity. To appreciate the error that will occur if SHM equations are used while actually the initial displacement warrants the use of the non-linear equations, we calculate the average time period with large displacements using the above data. For SHM case the time period would be a constant for any value of initial displacement. For large initial displacement as
we can see from Figure(2) the motion of pendulum won’t be simple harmonic. Typically a student find out time period by taking the time taken for a certain no. of oscillations, then dividing this time by no. of oscillations. For Figure(3) we have numerically found the time period by calculating time taken for 23 oscillations. In last graph (Figure(4)) we have shown the plot of the variation of acceleration due to gravity with the initial displacement based on the time periods thus calculated from fig(3) and substituting in eqn(10). As we can see for SHM motion this would be a straight line. But once one consider large initial displacement this would no longer be the case, and the variation the student can expect in his exercise could be fairly large.

In summary we have tried to show the importance of the condition of small initial displacement in the whole exercise of performing a experiment of calculation of acceleration due to gravity by simple pendulum.
Figure 6: Plot of Time period with the initial displacement (in radians), other parameter is $\omega (= 3.13)$

Figure 7: Plot of acceleration due to gravity with initial displacement other parameters are same as that of Figure(6)