Applications of Orbit Equivalence to Actions of Discrete Amenable Groups

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Abstract

Since the work of Ornstein and Weiss in 1987 (Entropy and isomorphism theorems for actions of amenable groups, J. Analyse Math., 48 (1987)) it has been understood that the natural category for classical ergodic theory would be probability measure preserving actions of discrete amenable groups. A conclusion of this work is that all such actions on nonatomic Lebesgue probability spaces were orbit equivalent. From this foundation two broad developments have been built. First, a full generalization of the various equivalence theories, including Ornstein’s isomorphism theorem itself, exists. Fixing the amenable group $G$ and an action of it, one can define a metric-like notion on the full-group of the action, called a size. A size breaks the orbit equivalence class of a single action into subsets, those reachable by a Cauchy sequence (in the size) of full group perturbations. These subsets are the equivalence classes associated with the size. Each size possesses a distinguished “most random” set of classes, the “Bernoulli” classes of the relation. An Ornstein-type theorem can be obtained. Many naturally occurring equivalence relations can be described in this way. Perhaps most interesting, entropy itself can be so described. Second, one can use the characterization of discrete amenable actions as those which are orbit equivalent to an action of $Z$ to lift theorems from actions of $Z$ to those of arbitrary amenable groups. The most interesting of these are first, that actions of completely positive entropy (called $K$-systems for $Z$ actions) are mixing of all orders (proven jointly with B. Weiss) and that such actions have countable Haar spectrum (proven by Golodets and Dooley). As all ergodic actions are orbit equivalent, only ergodicity is preserved by orbit equivalences in general, but by considering orbit equivalences restricted to be measurable with respect to a sub-σ algebra, many properties relative to that algebra are preserved. This provides the tool for this method to succeed.

2000 Mathematics Subject Classification: 28D15, 37A35.
Keywords and Phrases: Amenable group, Orbit equivalence, Entropy.

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1. Definitions and examples of sizes

Our goal in this section is to describe a metric-like notion on the full group of a measure preserving action of an amenable group and show how this leads to various restricted orbit equivalence theories. This work can be found in complete detail in Restricted Orbit Equivalence for Actions of Discrete Amenable Groups by D.J. Rudolph and J. Kammeyer, Cambridge Tracts in Mathematics #146.

Let \((X, \mathcal{F}, \mu)\) be a fixed nonatomic Lebesgue probability space. Let \(G\) be an infinite discrete amenable group. Let \(O \subseteq X \times X\) be an ergodic, measure preserving, hyperfinite equivalence relation. For our purposes, this simply means that \(O = \{(x, T_g(x))\}_{g \in G}\) where \(T : G \times X \to G\) (written of course \(T_g(x)\)) is some ergodic and free, measure preserving action of \(G\) on \(X\).

**Definition 1.1** Let \(G\) be an infinite countable discrete amenable group. A \(G\)-arrangement \(\alpha\) is any map from \(O\) to \(G\) that satisfies:

(i) \(\alpha\) is 1-1 and onto, in that for a.e. \(x \in X\), for all \(g \in G\), there is a unique \(x' \in X\) with \(\alpha(x, x') = g\). We write \(x' = T^\alpha_g(x)\);

(ii) \(\alpha\) is measurable and measure preserving, i.e. for all \(A \in \mathcal{F}, g \in G\), both \(T^\alpha_g(A) \in \mathcal{F}\) and \(\mu(T^\alpha_g(A)) = \mu(A)\); and

(iii) \(\alpha\) satisfies the cocycle equation \(\alpha(x_2, x_3)\alpha(x_1, x_2) = \alpha(x_1, x_3)\).

As \(G\) will not vary for our considerations we will abbreviate this as an arrangement. Let \(A\) denote the set of all such arrangements.

**Lemma 1.2** \(\alpha\) is a \(G\)-arrangement if and only if there is a measure preserving ergodic free action of \(G, T\), whose orbit relation is \(O\) such that \(\alpha(x, T_g(x)) = g\) for all \((x, T_g(x)) \in O\).

Thus the vocabulary of \(G\)-arrangements on \(O\) is precisely equivalent to the vocabulary of \(G\)-actions whose orbits are \(O\). For a \(G\)-arrangement \(\alpha\), we write \(T^\alpha\) for the corresponding action. For a \(G\)-action \(T\), we write \(\alpha_T\) for the corresponding \(G\)-arrangement.

**Definition 1.3** The full group of \(O\) is the group (under composition) \(\Gamma\) of all measure preserving invertible maps \(\phi : X \to X\) such that for \(\mu\)-a.e. \(x \in X\), \((x, \phi(x)) \in O\).

**Definition 1.4** A \(G\)-rearrangement of \(O\) is a pair \((\alpha, \phi)\), where \(\alpha\) is a \(G\)-arrangement of \(O\) and \(\phi \in \Gamma\). As \(G\) is fixed for our purposes we will abbreviate this as a rearrangement. Let \(Q\) denote the set of all such rearrangements.

Intuitively, a rearrangement is simply a change (i.e. rearrangement) of an orbit from the arrangement \(\alpha\) to the arrangement \(\alpha\phi\), where \(\alpha\phi(x, x') = \alpha(\phi(x), \phi(x'))\). One can formalize such a rearrangement in three different ways. Set \(B\) to be the set of bijections of \(G\) and \(B\) the subgroup of \(G\) fixing the identity. Both are topologized via the product topology on \(G^G\). Notice there is a homomorphism \(H : B \to G\) given by \(H(q)(g) = q(id)^{-1}q(g)\). The kernel of \(H\) consist of the left translation maps.
To a rearrangement we can associate a family of functions $q_{x}^\alpha \phi \in B$ where

$$q_{x}^\alpha \phi(g) = \alpha(x, \phi(T_{g}^\alpha(x))).$$

Now suppose $\alpha$ and $\beta$ are two arrangements of the orbits $O$. Regard the first as an initial arrangement and the second as a terminal arrangement. We can associate to this pair and any point $x$ a bijection from $G$ fixing the identity that describes how the arrangement of the orbit has changed:

$$h_{x}^{\alpha,\beta}(g) = \beta(x, T_{g}^\alpha(x)).$$

Notice here that $\hat{H}(q_{x}^\alpha \phi) = h_{x}^{\alpha,\alpha \phi}$. Write $h_{x}^{\alpha,\beta}$:

$$X \rightarrow G.$$

The third way to view a rearrangement pair has a symbolic dynamic flavor. For each orbit $O(x) = \{x'; (x, x') \in O\}$, a rearrangement $(\alpha, \phi)$ also gives rise to a natural map $G \rightarrow G$ (not a bijection though), given by

$$f_{x}^{\alpha,\phi}(g) = \alpha(T_{g}^\alpha(x), \phi(T_{g}^\alpha(x))).$$

Visually, regarding $O(x)$ laid out by $\alpha$ as a copy of $G$, $\phi$ translates the point at position $g$ to position $f_{x}^{\alpha,\phi}(g) \cdot g$.

There is a natural link between the three functions $h_{x}^{\alpha,\beta}$, $q_{x}^\alpha \phi$ and $f_{x}^{\alpha,\phi}$ as follows. For any map $f : G \rightarrow G$ we define

$$Q(f)(g) = f(g) \cdot g$$

and

$$H(f)(g) = f(g) \cdot f(id)^{-1}.$$ 

It is an easy calculation that

$$H(f_{x}^{\alpha,\phi}) = h_{x}^{\alpha,\alpha \phi} \text{ and } Q(f_{x}^{\alpha,\phi}) = q_{x}^\alpha \phi.$$ 

Let $\{F_{i}\}$ be a fixed Følner sequence for $G$. We will describe a number of concepts in terms of the $F_{i}$.

We now consider three pseudometrics on the set of rearrangements. These all arise from natural topologies on functions $G \rightarrow G$. As $G$ is countable the only reasonable topology is the discrete one, using the discrete 0,1 valued metric. This topologizes $G^{G}$ as a metrizable space with the product topology. This is the weakest topology for which the evaluations $g : f \rightarrow f(g)$ are continuous functions. Notice that $H$ is a continuous map from $G^{G}$ to itself and the map $h \rightarrow h^{-1}$ on $G$ is continuous.

Define a metric $d$ on $G$ as follows. List the elements of $G$ as $\{g_{1} = \text{id}, g_{2}, \ldots\}$ and let $d_{0}$ be the 0,1 valued metric on $G$. Set

$$d(h_{1}, h_{2}) = \sum_{i} [d_{0}(h_{1}(g_{i}), h_{2}(g_{i}))+ d_{0}(h_{1}^{-1}(g_{i}), h_{2}^{-1}(g_{i})))2^{-(i+1)}].$$

Notice that if $h_{1}, h_{2}, h_{1}^{-1},$ and $h_{2}^{-1}$ agree on $g_{1}, \ldots, g_{i}$ then $d(h_{1}, h_{2}) \leq 2^{-i}$. On the other hand if $d(h_{1}, h_{2}) < 2^{-i}$ then $h_{1}, h_{2}$ and their inverses agree on this list of $i$ terms.
Lemma 1.5 The metric $d$ on $G$ gives the restricted product topology and makes $G$ a complete metric space.

We can use this to define a complete $L^1$ metric on arrangements:

$$\|\alpha, \beta\|_1 = \int d(h^{\alpha, \beta}, \text{id}) \, d\mu.$$ 

As $d(h_1, h_2) = d(h_2^{-1} h_1, \text{id})$ and $(h_x^{\alpha, \beta})^{-1} = h_x^{\beta, \alpha}$ we see that this is a metric.

We can also define a metric similar to $d$ on $G^G$ itself making it a complete metric space by just taking half of the terms in $d$:

$$d_1(f_1, f_2) = \sum_i d_0(f_1(g_i), f_2(g_i)) 2^{-i}.$$ 

This also leads to an $L^1$ metric on $G^G$-valued functions on a measure space:

$$\|f_1, f_2\|_1 = \int d_1(f_1, f_2) \, d\mu.$$ 

These two $L^1$ distances now give us two families of $L^1$ distances on the full-group, one a metric the other a pseudometric, associated with an arrangement $\alpha$:

$$\|\phi_1, \phi_2\|_w^\alpha = \int d(h^{\alpha, \phi_1}, h^{\alpha, \phi_2}) = \|\alpha \phi_1 , \alpha \phi_2\|_1$$

and

$$\|\phi_1, \phi_2\|_s^\alpha = \int d_1(f^{\alpha, \phi_1}, f^{\alpha, \phi_2}) \, d\mu = \|f^{\alpha, \phi_1}, f^{\alpha, \phi_2}\|_1.$$ 

The weak $L^1$ distance, $\|\cdot\|_w^\alpha$, is only a pseudometric but the strong $L^1$ distance, $\|\cdot\|_s^\alpha$, is a metric.

To describe the weak*-distance between two arrangements let $G^* = G \cup \{\ast\}$ be the one point compactification of $G$. Now $(G^*)^G$ is a compact metric space and hence the Borel probability measures on $(G^*)^G$, which we write as $M_1(G^*)$, are a compact and convex space in the weak* topology. Let $D(\mu_1, \mu_2)$ be an explicit metric giving this topology.

We define the distribution pseudometric between two rearrangements by

$$\|(\alpha, \phi), (\beta, \psi)\|_* = D((f^{\alpha, \phi})^*(\mu), (f^{\beta, \psi})^*(\nu)).$$

We can combine the two $L^1$-metrics on arrangements and the full group to define a product metric on rearrangements in the form

$$\|(\alpha_1, \phi_1), (\alpha_2, \phi_2)\|_1 = \|\alpha_1, \alpha_2\|_1 + \mu(\{x : \phi_1(x) \neq \phi_2(x)\}).$$ 

We end this Section by relating this complete $L^1$-metric on rearrangements to the distribution pseudometric.

We now define the notion of a size $m$ on rearrangements $(\alpha, \phi)$ as a family of pseudometrics $m_\alpha$ on the full-group satisfying some simple relations to the metrics and pseudometrics we just defined.
A size is a function 
\[ m : Q \to \mathbb{R}^+ \]
such that, if we write
\[ m_\alpha(\phi_1, \phi_2) = m(\alpha \phi_1, \phi_1^{-1} \phi_2), \]
then \( m \) satisfies the following three axioms.

**Axiom 1** For each \( \alpha \in A \), \( m_\alpha \) is a pseudometric on \( \Gamma \).

**Axiom 2** For each \( \alpha \in A \), the identity map 
\[ (\Gamma, m_\alpha) \to (\Gamma, \|\cdot\|_\alpha) \]
is uniformly continuous. In particular this means that if 
\[ m_\alpha(\phi_1, \phi_2) = 0 \]
then the two arrangements \( \alpha \phi_1 \) and \( \alpha \phi_2 \) are identical.

**Axiom 3** \( m \) is upper semi-continuous with respect to the distribution metric. That 
is to say, for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon, \alpha, \phi) \), such that if 
\[ \|(\alpha, \phi), (\beta, \psi)\|_* < \delta \]
then 
\[ m(\beta, \psi) < m(\alpha, \phi) + \varepsilon. \]

This last condition implies that if the two measures \((f^{\alpha,\phi})^*(\mu)\) and \((f^{\beta,\psi})^*(\nu)\) are the same, then \( m(\alpha, \phi) = m(\beta, \psi) \). Hence the value \( m \) is well defined on those 
measures on \( G^G \) which arise as such an image, and we can write

\[ m(\alpha, \phi) = m((f^{\alpha,\phi})^*(\mu)). \]

We can now define \( m \)-equivalence of two arrangements.

**Definition 1.6** We say \( \alpha \) and \( \beta \) are \( m \)-equivalent arrangements if there exist \( \phi_i \) which are \( m_\alpha \)-Cauchy, \( \phi_i^{-1} \) are \( m_\beta \)-Cauchy and \( \alpha \phi_i \) converges in probability to \( \beta \).

One can now define \( m \)-equivalence of actions on distinct spaces as meaning 
there are conjugate versions of the actions as arrangements on the same orbit space 
where the arrangements are \( m \)-equivalent in the sense of the definition.

We now give a list of examples to indicate the range of equivalence relations 
that can be brought under this perspective.

Many examples of sizes have the common feature of being integrals of some 
pointwise calculation of the distortion of a single orbit. To make this precise we first 
review some material about bijections of \( G \). Remember that \( B \) is the space of all 
bijections of the group \( G \) with the product topology, \( G \) is the space of bijections fixing \( \text{id} \) and we metrized both with a complete metric \( d \). The group \( G \) can be regarded as a subgroup of \( B \) acting by left multiplication, \( (g(g') = gg') \). The map \( H : B \to G \) 
given by \( H(q) = qq(id)^{-1} \) is a contraction in \( d \). Also \( G \) acting by right multiplication 
conjugates \( B \) to itself giving an action of \( G \) on \( B \). \((T_g(q)(g') = q(g')g^{-1}) \) We view 
this action by representing an element \( q \in B \) by a map \( f : G \to G, f(g) = q(g)g^{-1}. \)
Those maps $f \in G^G$ that arise from bijections are a $G_\delta$ and hence a Polish space we call $F$. The map $q \rightarrow f$ is obviously a homeomorphism from $\mathcal{B}$ to $F$. For $f \in F$ let $Q(f)$ be the associated bijection and for $q \in \mathcal{B}$ let $F(q)$ be the associated name in $G^G$. The action of $G$ on $\mathcal{B}$ in its representation as $F$ is the shift action $\sigma_g(f)(g') = f(g'g)$. Any rearrangement pair $(\alpha, \phi)$ then gives rise to an ergodic shift invariant measure on this Polish subset of $G^G$ and any ergodic shift invariant measure is an ergodic action of $G$ with a canonical rearrangement pair. The probability measures on a Polish space are weak* Polish and hence the invariant and ergodic measures on this Polish space are weak* Polish.

We will now define a general class of sizes that arise as integrals of valuations made on the bijections $q_{\alpha,\phi}^x$.

**Definition 1.7** A Borel $D : \mathcal{B} \rightarrow \mathbb{R}^+$ is called a size kernel if it satisfies:

1. $D(q) \geq 0$.
2. $D(id) = 0$.
3. $D(q(id)^{-1}q(id)) = D(q)$.
4. $D(q_1(id)q_2q_1^{-1}(id)q_1) \leq D(q_1) + D(q_2)$.
5. For every $\varepsilon > 0$ there is a $\delta > 0$ so that if $D(q) < \delta$ then $d(id, H(q)) < \varepsilon$.
6. The function $\mu \rightarrow \int D(q(x)) \, d\mu(x)$ is weak* continuous on the space of shift invariant measures $\mu$ on the Polish space $F$.

**Note.** An element of $G$ is regarded as an element of $\mathcal{B}$ acts by left multiplication.

For a size kernel $D$ we define

$$m^D(\alpha, \phi) = \int D(q_{\alpha,\phi}^x) \, d\mu(x).$$

We call such an $m^D$ an integral size.

**Example 1 (Conjugacy and Orbit Equivalence)**

These first two examples are the extremes of what is possible. For one the equivalence class will be simply the full group orbit and for the other it will be the entire set of arrangements. Both of the pseudometrics $d(q, id)$ and $d(H(q), id)$ are easily seen to be size kernels and so both

$$m^1(\alpha, \phi) = \|(\alpha, \phi), (\alpha, id)\|_s^\alpha$$

and

$$m^0(\alpha, \phi) = \|(\alpha, \phi), (\alpha, id)\|_w^\alpha$$

are sizes.

As $d$ makes $\mathcal{B}$ complete, relative to $m^1$ sequence phi is $m^1$ Cauchy iff $\phi_i \rightarrow \phi$ in probability. Thus $\phi \sim^1 \beta$ iff $\beta = \alpha\phi$ i.e. they differ by an element of the full group and the equivalence class of $\alpha$ is exactly its full group orbit. As for $m^0$, for any $\alpha$ and $\beta$ one can construct a sequence of $\phi_i$ with $\alpha\phi_i \rightarrow \beta$ in $L^1$ with the sequence $\phi_i$ an $m_\alpha$ Cauchy sequence. Thus all arrangements are $m^0$ equivalent.

**Example 2 (Kakutani Equivalence)**
For this example let $G = \mathbb{Z}^n$ and $B_N = [-N, N]^n$ be the standard Følner sequence of boxes centered at 0. We begin with a metric on $\mathbb{Z}^n$ given by

$$\tau(\vec{u}, \vec{v}) = \min\left(\frac{\|\vec{u}\|}{\|\vec{u}\|} - \frac{\|\vec{v}\|}{\|\vec{v}\|} + |\ln(\|\vec{v}\|) - \ln(\|\vec{u}\|)|, 1\right)$$

(assuming $\vec{0}/\|\vec{0}\| = \vec{0}$). What is important about $\tau$ are the following two properties:

1. $\tau$ is a metric on $\mathbb{Z}^n$ bounded by 1 and
2. $\vec{u}$ and $\vec{v}$ are $\tau$ close iff the norm of their difference is small in proportion to both of their norms.

For $h \in G$ set $B_N(h) = \{\vec{v} \in B_N | h(\vec{v}) \in B_N\}$ (those elements of $B_N$ mapped into $B_N$ by $h$). Now set

$$k(h) = \sup_N \left(\frac{1}{\#B_N} \sum_{\vec{v} \in B_N(h)} \tau(\vec{v}, h(\vec{v})) + \#\{\vec{v} \in B_N | h(\vec{v}) \notin B_N\}\right).$$

Now set $K(q) = k(H(q))$.

**Lemma 1.8** The function $K$ is a size kernel.

For $d = 1$ standard arguments imply that this size yields even Kakutani equivalence. For $d > 1$ it is leads to an analogous equivalence relation among Katok cross-sections of $\mathbb{R}^d$-actions.

Our last example moves beyond size kernels.

**Example 3 (Entropy as a Size)**

We discuss this example only for actions of $\mathbb{Z}$ although the ideas extend to general countable amenable groups.

The size at its base will simply be the entropy of the rearrangement itself. We make this precise as follows. The function $g(\alpha, \phi)(x) = \alpha(x, \phi(x))$ takes on countably many values and hence can be regarded as a countable partition $g(\alpha, \phi)$ of $X$. Set $\Gamma_0^\alpha$ to be those $\phi$ for which $g(\alpha, \phi)$ is finite. It is not difficult to see that $\Gamma_0^\alpha$ is a subgroup and moreover $\Gamma_0^{\alpha \psi} = \psi^{-1} \Gamma_0^\alpha \psi$ as $g(\alpha \psi, \psi^{-1} \phi \psi)(\psi^{-1}(x)) = g(\alpha, \phi)(x)$. It can be shown that the $\Gamma_0^\alpha$ are all $m_1^\alpha$ dense in $\Gamma$. For $\phi \in \Gamma_0^\alpha$ one can use the entropy of the process $h(T^\alpha, g(\alpha, \phi))$ to start the definition of a size defining

$$e(\alpha, \phi) = \inf_{\phi' \in \Gamma_0^\alpha} h(T^\alpha, g(\alpha, \phi')) + \mu\{x | \phi(x) \neq \phi'(x)\}.$$ 

Now set the size to be

$$m^e(\alpha, \phi) = e(\alpha, \phi) + m^0(\alpha, \phi).$$

**Proposition 1.9** Two $\mathbb{Z}$-actions are $m^e$ equivalent iff they have the same entropy.
2. Transference via orbit equivalence

A second natural type of restriction can be placed on an orbit equivalence. Here the interest is in two arrangements $\alpha$ and $\beta$ of perhaps distinct groups. Suppose $\mathcal{A}$ is an invariant sub $\sigma$-algebra for the action $T^\alpha$ of $G_1$ and $\beta$ is a $G_2$ arrangement of the same orbit space. We say the orbit equivalence from $\alpha$ to $\beta$ is $\mathcal{A}$-measurable if the function $h^{\alpha,\beta}(x,g_1) = \beta(x,T_g^\alpha(x))$ describing or the orbit is rearranged, is $\mathcal{A}$ measurable for all choices of $g_1$. Up to conjugacy we can regard all ergodic actions of infinite discrete and amenable groups as residing on the same orbit space, so beyond ergodicity no dynamical property will be preserved by orbit equivalence. On the other hand, many dynamical properties have versions “relative to” an invariant sub $\sigma$-algebra and many such properties are indeed invariant under orbit equivalences that are measurable with respect to that sub $\sigma$-algebra.

This method first arose in work with B. Weiss showing that actions of discrete amenable groups that have completely positive entropy (cpe), commonly called $K$-systems, are mixing of all orders. This transference method has been applied to a variety of questions. Here is an outline of the argument to this first result to exhibit the format. The complete argument can be found in Entropy and mixing for amenable group actions by D.J. Rudolph and B. Weiss, Annals of Mathematics, 151, (2000)m pp. 1119-1150.

**Lemma 2.1** If $T$ is an action of a discrete amenable group $G$ and $T \times B$, its direct product with a Bernoulli action $B$ of $G$, is relatively cpe with respect to the Bernoulli second coordinate, then $T$ must be cpe.

**Lemma 2.2** If $\hat{T}$ and $\hat{S}$ are ergodic actions on the same orbits of two discrete amenable groups $G_1$ and $G_2$ and the orbit equivalence between them is $\mathcal{A}$ measurable where $\mathcal{A}$ is a $\hat{T}$ invariant sub $\sigma$-algebra, then for any partition $P$, the conditional entropies $h(T,P|\mathcal{A})$ and $h(S,P|\mathcal{A})$ are equal.

Let $S_i$ be a list of finite subsets of either $G_i$. We say the $S_i$ spread if any particular $\gamma \neq \text{id}$ belongs to at most finitely many of the sets $S_iS_i^{-1}$. If the sets $S_i(x)$ are random sequences of finite sets depending on $x$, we can again say they are spread if for a.e. $x$ they form a spread sequence. A classical characterization of the $K$-systems, which we state in a relative form says

**Theorem 2.3** $T$, a $\mathbb{Z}$-action, is relatively cpe with respect to a sub $\sigma$-algebra $\mathcal{A}$ iff for all partitions $P$ and all $\mathcal{A}$ measurable and spread random sequences of sets $S_i(x)$,

$$\frac{1}{\#S_i} \left[ h( \bigvee_{g \in S_i} T_g(P)|\mathcal{A}) - \sum_{g \in S_i} h(T_g(P)|\mathcal{A}) \right] \to 0$$

in $L^1$.

That is to say, the translates of $P$ become conditionally ever more independent the more spread they become. Refer to this property as $\mathcal{A}$-relative uniform mixing if it holds for all $P$. 
Lemma 2.4 If $T$ and $S$ are $A$-measurably orbit equivalent actions of perhaps distinct groups and $T$ is $A$-relatively uniformly mixing. Then $S$ is also.

We now describe the orbit transference proof that cpe actions of discrete amenable groups are always mixing of all orders. Suppose $T$ is a cpe action of the group $G$. Take $T \times B$ where $B$ is a Bernoulli action of $G$. This direct product will be ergodic and in fact relatively cpe with respect to the Bernoulli coordinate. Now $B$ is orbit equivalent to a $\mathbb{Z}$ action and this orbit equivalence lifts to an orbit equivalence of $T \times B$ to some ergodic $\mathbb{Z}$ action $S$. The orbit equivalence will be $A$ measurable where $A$ is this Bernoulli coordinate algebra. Now $S$ will still be $A$ relatively cpe and hence $A$ relatively uniformly mixing. But now this tells us $T \times B$ is also $A$ relatively uniformly mixing. Restricting this to partitions that are measurable with respect to the first coordinate tells us $T$ itself is uniformly mixing (without any conditioning) and hence mixing of all orders.

A second and quite significant application of this method, due to Dooley and Golodets, is to show that cpe actions have countable Haar spectrum. In as yet unwritten work, again with B. Weiss, one can show that weakly mixing isometric extensions of Bernoulli actions must be Bernoulli.

This remains an area of very active work. We end on an open question. Consider the known result for $\mathbb{Z}$ actions, that a weakly mixing and isometric extension of a base action that is mixing must itself be mixing. Is this result true for general amenable group actions? To apply the transference method one needs a relativized version of the result for $\mathbb{Z}$ actions. That is to say, one needs to know that a relatively weakly mixing relatively isometric extension of a relatively mixing action is still relatively mixing. What seems an obstacle here is simply the definition of relative mixing over a sub $\sigma$-algebra $A$. Certainly it means that for any sets $A$ and $B$ that

$$\left| E(I_A I_B \circ T^j | A) - E(I_A | A) E(I_B \circ T^j | A) \right| \to 0.$$

The question is, in what sense should it tend to zero. Pointwise convergence behaves well for orbit equivalence but the above relativized question seems answerable only for mean convergence.