Cohomology of the variational bicomplex on the infinite order jet space

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Abstract. We obtain the cohomology of the variational bicomplex on the infinite order jet space of a smooth fiber bundle in the class of exterior forms of finite jet order. This provides a solution of the global inverse problem of the calculus of variations of finite order on fiber bundles.

2000 Mathematics subject classification. Primary 58A20, 58E30; Secondary 55N30.

Key words and phrases. Jet manifolds, infinite order jet space, differential graded algebra, variational complex, cohomology of sheaves.

1. Introduction

Let \( Y \to X \) be a smooth fiber bundle. We obtain cohomology of the variational bicomplex on the infinite order jet space \( J^\infty Y \) of \( Y \to X \) in the class of exterior forms of finite jet order. This is cohomology of the vertical differential \( d_V \), the horizontal (or total) differential \( d_H \) and the variational operator \( \delta \).

The two differential calculus of exterior forms \( \mathcal{O}^*_\infty \) and \( \mathcal{Q}^*_\infty \) are usually considered on \( J^\infty Y \). The \( \mathcal{O}^*_\infty \) is the direct limit of graded differential algebras of exterior forms on finite order jet manifolds. Its cohomology, except de Rham cohomology and a particular result of [18] on \( \delta \)-cohomology, remains unknown. At the same time, \( \mathcal{O}^*_\infty \) is most interesting for applications because it consists of exterior forms on finite order jet manifolds. The \( \mathcal{Q}^*_\infty \) is the structure algebra of the sheaf of germs of exterior forms on finite order jet manifolds. There is the \( \mathbb{R} \)-algebra monomorphism \( \mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty \). The \( d_H \)- and \( \delta \)-cohomology of \( \mathcal{Q}^*_\infty \) has been investigated in [1] [10]. Due to Lemma 3 below, we simplify this investigation and complete it by the study of \( d_V \)-cohomology of \( \mathcal{Q}^*_\infty \). We prove that the graded differential
algebra $\mathcal{O}_\infty^*$ has the same $d_H$- and $\delta$-cohomology as $\mathcal{Q}_\infty^*$ (see Theorem 13 below). This provides a solution of the global inverse problem of the calculus of variations in the class of exterior forms of finite jet order.

Note that the local exactness of the calculus of variations has been proved in the class of exterior forms of finite order by use of homotopy operators which do not minimize the order of Lagrangians (see, e.g., [13, 17]). The infinite variational complex of such exterior forms on $J^\infty Y$ has been studied by many authors (see, e.g., [3, 4, 13, 15, 17]). However, these forms on $J^\infty Y$ fail to constitute a sheaf. Therefore, the cohomology obstruction to the exactness of the calculus of variations has been obtained in the class of exterior forms of locally finite order which make up the above mentioned algebra $\mathcal{Q}_\infty^*$ [2, 16]. A solution of the global inverse problem in the calculus of variations in the class of exterior forms of a fixed jet order has been suggested in [1] by a computation of cohomology of the fixed order variational sequence (see [10, 19] for another variant of such a variational sequence). The key point of this computation lies in the local exactness of the finite order variational sequence which however requires rather sophisticated ad hoc technique in order to be reproduced (see also [11]). We show that the obstruction to the exactness of the finite order calculus of variations is the same as for exterior forms of locally finite order, without minimizing an order of Lagrangians. The main point for applications is that this obstruction is given by closed forms on the fiber bundle $Y$, and is of first order.

The article is organized as follows. In Section 2, the differential calculus $\mathcal{O}_\infty^*$ and $\mathcal{Q}_\infty^*$ on $J^\infty Y$ are introduced in an algebraic way. In Section 3, the variational bicomplex on $J^\infty Y$ is set. Section 4 is devoted to cohomology of the differential calculus $\mathcal{Q}_\infty^*$ on $J^\infty Y$. In Section 5, the isomorphism of $d_H$- and $\delta$-cohomology of $\mathcal{Q}_\infty^*$ to that of $\mathcal{O}_\infty^*$ is proved. In Sections 6, a solution of the global inverse problem in the calculus of variations in different classes of exterior forms is provided.

2. THE DIFFERENTIAL CALCULUS ON $J^\infty Y$

Smooth manifolds throughout are assumed to be real, finite-dimensional, Hausdorff, paracompact, and connected. Put further $\dim X = n \geq 1$. We follow the standard terminology of jet formalism [7, 12, 14].

Recall that the infinite order jet space $J^\infty Y$ of a smooth fiber bundle $Y \to X$ is defined
as a projective limit \((J^\infty Y, \pi^\infty_r)\) of the inverse system

\[
X \xleftarrow{\pi} Y \xleftarrow{\pi_0} \ldots \xleftarrow{\pi^{r-1}} J^{r-1} Y \xleftarrow{\pi^{r-1}} J^r Y \xleftarrow{\pi} \ldots
\]

of \(J^r Y\) of finite order jet manifolds \(J^r Y\) of \(Y \to X\), where \(\pi^{r-1}\) are affine bundles. Bearing in mind Borel’s theorem, one can say that \(J^\infty Y\) consists of the equivalence classes of sections of \(Y \to X\) identified by their Taylor series at points of \(X\). Endowed with the projective limit topology, \(J^\infty Y\) is a paracompact Fréchet manifold \([16]\). A bundle coordinate atlas \(\{U_Y, (x^\lambda, y^i)\}\) of \(Y \to X\) yields the manifold coordinate atlas

\[
\{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\}, \quad 0 \leq |\Lambda|,
\]

of \(J^\infty Y\), together with the transition functions

\[
y^{i}_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x^\lambda} d_\lambda y^i_\Lambda,
\]

where \(\Lambda = (\lambda_k \ldots \lambda_1)\), \(\lambda + \Lambda = (\lambda\lambda_k \ldots \lambda_1)\) are multi-indices and \(d_\lambda\) denotes the total derivative

\[
d_\lambda = \partial_\lambda + \sum_{|\Lambda|\geq 0} y^i_{\lambda+\Lambda} \partial^A.
\]

With the inverse system \([1]\), one has the direct system

\[
\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*_0 \xrightarrow{\pi^*_0} \mathcal{O}^*_1 \xrightarrow{\pi^*_1} \ldots \xrightarrow{\pi^*_r} \mathcal{O}^*_r \to \cdots
\]

of graded differential \(\mathbb{R}\)-algebras \(\mathcal{O}^*_r\) of exterior forms on finite order jet manifolds \(J^r Y\), where \(\pi^*_r\) are pull-back monomorphisms. The direct limit of this direct system is the above mentioned graded differential \(\mathbb{R}\)-algebra \((\mathcal{O}^*_\infty, \pi^\infty_r)\) of exterior forms on finite order jet manifolds modulo the pull-back identification. The \(\mathcal{O}^*_\infty\) is a differential calculus over the \(\mathbb{R}\)-ring \(\mathcal{O}^0\) of continuous real functions on \(J^\infty Y\) which are the pull-back of smooth real functions on finite order jet manifolds by surjections \(\pi^\infty_r\). Passing to the direct limit of the de Rham complexes of exterior forms on finite order jet manifolds, de Rham cohomology of the graded differential algebra \(\mathcal{O}^*_\infty\) has been found, and coincides with de Rham cohomology of the fiber bundle \(Y\) \([4, 3]\). However, this is not a way of studying other cohomology of the algebra \(\mathcal{O}^*_\infty\).
To solve this problem, let us enlarge $O^0_\infty$ to the $\mathbb{R}$-ring $Q^0_\infty$ of continuous real functions on $J^\infty Y$ such that, given $f \in Q^0_\infty$ and any point $q \in J^\infty Y$, there exists a neighborhood of $q$ where $f$ coincides with the pull-back of a smooth function on some finite order jet manifold. The reason lies in the fact that the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $Q^0_\infty$. Therefore, sheaves of $Q^0_\infty$-modules on $J^\infty Y$ are fine and, consequently, acyclic. Then, the abstract de Rham theorem on cohomology of a sheaf resolution can be called into play.

Remark 1. Throughout, we follow the terminology of [16] where by a sheaf $S$ over a topological space $Z$ is meant a sheaf bundle $S \to Z$. Accordingly, $\Gamma(S)$ denotes the canonical presheaf of sections of the sheaf $S$, and $\Gamma(Z, S)$ is the group of global sections of $S$. All sheaves below are ringed spaces, but we omit this terminology if there is no danger of confusion.

Let us define a differential calculus over the ring $Q^0_\infty$. Let $O^*_r$ be a sheaf of germs of exterior forms on the $r$-order jet manifold $J^r Y$ and $\Gamma(O^*_r)$ its canonical presheaf. There is the direct system of canonical presheaves 

$$
\Gamma(O^*_r) \xrightarrow{\pi^*_r} \Gamma(O^*_0) \xrightarrow{\pi^*_1} \Gamma(O^*_1) \xrightarrow{\pi^*_2} \cdots \xrightarrow{\pi^*_r} \Gamma(O^*_r) \xrightarrow{\pi^*_1} \cdots,
$$

where $\pi^*_r$ are pull-back monomorphisms with respect to open surjections $\pi^*_r$. Its direct limit $O^*_\infty$ is a presheaf of graded differential $\mathbb{R}$-algebras on $J^\infty Y$. Let $Q^*_\infty$ be a sheaf constructed from $O^*_\infty$ and $\Gamma(Q^*_\infty)$ its canonical presheaf. There is the $\mathbb{R}$-algebra monomorphism of presheaves $Q^*_\infty \to \Gamma(Q^*_\infty)$. The structure algebra $Q^*_\infty = \Gamma(J^\infty Y, Q^*_\infty)$ of the sheaf $Q^*_\infty$ is a desired differential calculus over the $\mathbb{R}$-ring $Q^0_\infty$.

For short, we agree to call elements of $Q^*_\infty$ the exterior forms on $J^\infty Y$. Restricted to a coordinate chart $(\pi^*_0)^{-1}(U_Y)$ of $J^\infty Y$, they can be written in a coordinate form, where horizontal forms $\{dx^i\}$ and contact 1-forms $\{\theta^i = dy^i_\Lambda - y^i_\lambda + \Lambda dx^\lambda\}$ constitute the set of generators of the algebra $Q^*_\infty$. There is the canonical splitting 

$$
Q^*_\infty = \bigoplus_{k,s} Q_{k,s}^\infty, \quad 0 \leq k, \quad 0 \leq s \leq n,
$$

of $Q^*_\infty$ into $Q^0_\infty$-modules $Q_{k,s}^\infty$ of $k$-contact and $s$-horizontal forms, together with the corresponding projections 

$$
h_k : Q^*_\infty \to Q^k_\infty, \quad 0 \leq k, \quad h^s : Q^*_\infty \to Q^s_\infty, \quad 0 \leq s \leq n.
$$
Accordingly, the exterior differential on $Q^*_\infty$ is decomposed into the sum $d = d_H + d_V$ of horizontal and vertical differentials such that

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H(\phi) = dx^\Lambda \wedge d\Lambda(\phi), \quad \phi \in Q^*_\infty,$$

$$d_V \circ h_s = h_s \circ d \circ h_s, \quad d_V(\phi) = \theta_i^\Lambda \wedge \partial_i \Lambda \phi.$$

3. **The variational bicomplex**

Being nilpotent, the differentials $d_V$ and $d_H$ provide the natural bicomplex $\{\Omega^{k,m}_\infty\}$ of the sheaf $\Omega^*_\infty$ on $J^\infty Y$. To complete it to the variational bicomplex, one defines the projection $\mathbb{R}$-module endomorphism

$$\tau = \sum_{k>0} \frac{1}{k} h_k \circ h^n,$$

$$\tau(\phi) = (-1)^{|\Lambda|} \theta^i \wedge [d\Lambda(\partial_i \Lambda \phi)], \quad 0 \leq |\Lambda|, \quad \phi \in \Gamma(\Omega^{0,n}_\infty),$$

of $\Omega^*_\infty$ such that

$$\tau \circ d_H = 0, \quad \tau \circ d \circ \tau - \tau \circ d = 0.$$

Introduced on elements of the presheaf $\Omega^*_\infty$ (see, e.g., [3, 7, 17]), this endomorphism is induced on the sheaf $\Omega^*_\infty$ and its structure algebra $Q^*_\infty$. Put

$$E_k = \tau(Q^{k,n}_\infty), \quad E_k = \tau(Q^{k,n}_\infty), \quad k > 0.$$

Since $\tau$ is a projection operator, we have isomorphisms

$$\Gamma(E_k) = \tau(\Gamma(Q^{k,n}_\infty)), \quad E_k = \Gamma(J^\infty Y, E_k).$$

The variational operator on $\Omega^*_\infty$ is defined as the morphism $\delta = \tau \circ d$. It is nilpotent, and obeys the relation

$$\delta \circ \tau - \tau \circ d = 0.$$

Let $\mathbb{R}$ and $\Omega^*_X$ denote the constant sheaf on $J^\infty Y$ and the sheaf of exterior forms on $X$, respectively. The operators $d_V, d_H, \tau$ and $\delta$ give the following variational bicomplex
of sheaves of differential forms on $J^\infty Y$:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \Omega_{\infty}^{k,0} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{k,1} & \overset{d_{V}}{\rightarrow} & \cdots & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{k,m} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{k,n} & \overset{\tau}{\rightarrow} & E_{k} & \rightarrow & 0 \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & \Omega_{\infty}^{1,0} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{1,1} & \overset{d_{V}}{\rightarrow} & \cdots & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{1,m} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{1,n} & \overset{\tau}{\rightarrow} & E_{1} & \rightarrow & 0 \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & \Omega_{\infty}^{0} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{0,1} & \overset{d_{V}}{\rightarrow} & \cdots & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{0,m} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{0,n} & \overset{\tau}{\rightarrow} & E_{\infty} & \rightarrow & \cdots \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & \mathbb{R} & \overset{\pi^\infty}{\rightarrow} & \Omega_{\infty}^{0} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{0,1} & \overset{d_{V}}{\rightarrow} & \cdots & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{0,m} & \overset{d_{V}}{\rightarrow} & \Omega_{\infty}^{0,n} & \overset{\tau}{\rightarrow} & E_{\infty} & \rightarrow & \cdots \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & \mathbb{R} & \overset{d_{V}}{\rightarrow} & \mathcal{D}_{X} & \overset{d}{\rightarrow} & \cdots & \overset{d}{\rightarrow} & \cdots & \overset{d}{\rightarrow} & 0 \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots \\
\end{array}
$$

The second row and the last column of this bicomplex form the variational complex

$$
0 \rightarrow \mathbb{R} \rightarrow \Omega_{\infty}^{0} \overset{d_{V}}{\rightarrow} \Omega_{\infty}^{0,1} \overset{d_{V}}{\rightarrow} \cdots \overset{d_{V}}{\rightarrow} \Omega_{\infty}^{0,m} \overset{d_{V}}{\rightarrow} \Omega_{\infty}^{0,n} \overset{\tau}{\rightarrow} E_{\infty} \rightarrow \cdots .
$$

The corresponding variational bicomplexes $\{Q_{\infty}^{\ast}, E_{k}\}$ and $\{O_{\infty}^{\ast}, E_{k}\}$ of the differential calculus $Q_{\infty}^{\ast}$ and $O_{\infty}^{\ast}$ take place.

There are the well-known statements summarized usually as the algebraic Poincaré lemma (see, e.g., [13, 17]).

**Lemma 2.** If $Y$ is a contractible fiber bundle $\mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n}$, the variational bicomplex $\{O_{\infty}^{\ast}, E_{k}\}$ of the graded differential algebra $O_{\infty}^{\ast}$ is exact.

It follows that the variational bicomplex of sheaves (4) is exact for any smooth fiber bundle $Y \rightarrow X$. Moreover, all sheaves $\Omega^{k,m}$ in this bicomplex are fine, and so are the sheaves $E_{k}$ in accordance with the following lemma.

**Lemma 3.** Sheaves $E_{k}$, $k > 0$, are fine.

**Proof.** Though $\mathbb{R}$-modules $E_{k>1}$ fail to be $Q_{\infty}^{0}$-modules [17], one can use the fact that the sheaves $E_{k>0}$ are projections $\tau(\Omega^{k,n}_{\infty})$ of sheaves of $Q_{\infty}^{0}$-modules. Let $U = \{U_{i}\}_{i \in I}$ be a locally finite open covering of $J^{\infty}Y$ and $\{f_{i} \in Q_{\infty}^{0}\}$ the associated partition of unity. For any open subset $U \subset J^{\infty}Y$ and any section $\varphi$ of the sheaf $\Omega^{k,n}_{\infty}$ over $U$, let us put
(φ) = f_i φ. Then, \{h_i\} provide a family of endomorphisms of the sheaf \( \Omega_{k,n}^\infty \), required for \( \Omega_{k,n}^\infty \) to be fine. Endomorphisms \( h_i \) of \( \Omega_{k,n}^\infty \) also yield the \( \mathbb{R} \)-module endomorphisms

\[ \overline{h}_i = \tau \circ h_i : \mathcal{E}_k \xrightarrow{\text{in}} \Omega_{k,n}^\infty \xrightarrow{h_i} \Omega_{k,n}^\infty \xrightarrow{\tau} \mathcal{E}_k \]

of the sheaves \( \mathcal{E}_k \). They possess the properties required for \( \mathcal{E}_k \) to be a fine sheaf. Indeed, for each \( i \in I \), there is a closed set \( \text{supp} f_i \subset U_i \) such that \( \overline{h}_i \) is zero outside this set, while the sum \( \sum_{i \in I} \overline{h}_i \) is the identity morphism.

This Lemma simplify essentially our cohomology computation of the variational bi-complex in comparison with that in [1, 16]. Since all sheaves except \( \mathbb{R} \) and \( \pi_*^\infty \Omega_X^\infty \) in the bicomplex (4) are fine, the abstract de Rham theorem ([3], Theorem 2.12.1) can be applied to columns and rows of this bicomplex in a straightforward way. We will quote the following variant of this theorem.

**Theorem 4.** Let

\[ 0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \quad p > 1, \]

be an exact sequence of sheaves on a paracompact topological space \( Z \), where the sheaves \( S_p \) and \( S_{p+1} \) are not necessarily acyclic, and let

\[ 0 \to \Gamma(Z, S) \xrightarrow{h_*} \Gamma(Z, S_0) \xrightarrow{h^0_*} \Gamma(Z, S_1) \xrightarrow{h^1_*} \cdots \xrightarrow{h^{p-1}_*} \Gamma(Z, S_p) \xrightarrow{h^p_*} \Gamma(Z, S_{p+1}) \]

be the corresponding cochain complex of structure groups of these sheaves. The \( q \)-cohomology groups of the cochain complex (7) for \( 0 \leq q \leq p \) are isomorphic to the cohomology groups \( H^q(Z, S) \) of \( Z \) with coefficients in the sheaf \( S \).

The \( d_H \)- and \( \delta \)-cohomology of the differential calculus \( \mathcal{Q}_\infty^* \) on \( J_\infty^\infty Y \) has been found in [10]. Its \( \delta \)-cohomology at terms \( \mathcal{Q}_\infty^{0,n} \) and \( E_1 \) has been also obtained in [1] (several statements without proof were announced in [2]). We recover this cohomology in a short way due to Lemma 3, and complete it by the \( d_V \)-cohomology of \( \mathcal{Q}_\infty^* \) corresponding to columns of the bicomplex (4).

**4. Cohomology of \( \mathcal{Q}_\infty^* \)**

We start from the following facts.
Lemma 5. There is an isomorphism

\[ H^*(J^\infty Y, \mathbb{R}) = H^*(Y, \mathbb{R}) = H^*(Y) \]

between cohomology \( H^*(J^\infty Y, \mathbb{R}) \) of \( J^\infty Y \) with coefficients in the constant sheaf \( \mathbb{R} \), that \( H^*(Y, \mathbb{R}) \) of \( Y \), and de Rham cohomology \( H^*(Y) \) of \( Y \).

Proof. A fiber bundle \( Y \) is a strong deformation retract of \( J^\infty Y \). Then, the first isomorphism in (8) follows from the Vietoris–Begle theorem ([4], Theorem 11.4; [9], Corollary 2.7.7), while the second one is a consequence of the well-known de Rham theorem. \( \square \)

Lemma 6. There is an isomorphism

\[ H^*(J^\infty Y, \pi^\infty* \mathcal{O}_X^m) = H^*(Y, \pi^* \mathcal{O}_X^m) \]

between cohomology \( H^*(J^\infty Y, \pi^\infty* \mathcal{O}_X^m) \) of \( J^\infty Y \) with coefficients in the pull-back sheaf \( \pi^\infty* \mathcal{O}_X^m \) and that of \( Y \) with coefficients in the sheaf \( \pi^* \mathcal{O}_X^m \).

Proof. The isomorphism (9) also follows from the facts that \( Y \) is a strong deformation retract of \( J^\infty Y \) and that \( \pi^\infty* \mathcal{O}_X^m \) is the pull-back onto \( J^\infty Y \) of the sheaf \( \pi^* \mathcal{O}_X^m \) on \( Y \) ([9], Corollary 2.7.7).

Remark 7. Lemma 5 and Lemma 6 are corollary of Lemma 2 as follows. Let us consider the open surjection \( \pi^\infty_0: J^\infty Y \to Y \) and the direct images \( \pi^\infty_0\mathbb{R} \) and \( \pi^\infty_0(\pi^\infty* \mathcal{O}_X^m) \) of sheaves \( \mathbb{R} \) and \( \pi^\infty* \mathcal{O}_X^m \) on \( J^\infty Y \). They are isomorphic to the sheaves \( \mathbb{R} \) and \( \pi^* \mathcal{O}_X^m \) on \( Y \), respectively. Lemma 5 shows that, every point \( y \in Y \) has a base of open neighbourhoods \( \{U_y\} \) whose inverse images \( (\pi^\infty_0)^{-1}(U_y) \) are acyclic for the sheaves \( \mathbb{R} \) and \( \pi^\infty* \mathcal{O}_X^m \). Then, a weak version of the Leray theorem states the cohomology isomorphisms (8) and (9) ([8]). Moreover, since other sheaves in the bicomplex (4) are acyclic on \( (\pi^\infty_0)^{-1}(U_y) \) and the bicomplexes of their sections over \( (\pi^\infty_0)^{-1}(U_y) \) are exact, we have the exact direct image \( \{\pi^\infty_0* \mathcal{O}^*, \pi^\infty_0* \mathcal{E}_k\} \) on \( Y \) of the bicomplex (4), whose rows and columns are resolutions of sheaves on \( Y \). Due to the \( \mathbb{R} \)-algebra isomorphism \( \mathcal{Q}^*_\infty = \Gamma(Y, \pi^\infty* \mathcal{O}_X^*) \), one can study cohomology of the graded differential \( \mathbb{R} \)-algebra \( \mathcal{Q}^*_\infty \) by use of this variational bicomplex on \( Y \). In particular, it follows that cohomology of \( \mathcal{Q}^*_\infty \) of degree \( q > \text{dim} Y \) vanishes.

Turn to de Rham cohomology of the algebra \( \mathcal{Q}^*_\infty \). Let us consider the de Rham complex of sheaves

\[ 0 \to \mathbb{R} \to \Omega^0_\infty \xrightarrow{d} \Omega^1_\infty \xrightarrow{d} \cdots \]
on $J^\infty Y$ and the de Rham complex of their structure algebras

$$0 \to \mathbb{R} \to Q^0_\infty \xrightarrow{d} Q^1_\infty \xrightarrow{d} \cdots.$$  

**Proposition 8.** There is an isomorphism

$$H^*(Q^*_\infty) = H^*(Y)$$

of de Rham cohomology $H^*(Q^*_\infty)$ of the graded differential algebra $Q^*_\infty$ to that $H^*(Y)$ of the fiber bundle $Y$.

**Proof.** The proof is obvious. The complex (10) is exact due to the Poincaré lemma, and is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$ since $\Omega^*_\infty$ are sheaves of $Q^0_\infty$-modules. Then, Theorem 4 and Lemma 5 complete the proof. $\square$

It follows that every closed form $\phi \in Q^*_\infty$ splits into the sum

$$\phi = \varphi + d\xi, \quad \xi \in Q^*_\infty,$$

where $\varphi$ is a closed form on the fiber bundle $Y$. This splitting plays an important role in the sequel. Since the graded differential algebras $O^*_\infty$ and $Q^*_\infty$ have the same de Rham cohomology, we further agree to call

$$H^*(J^\infty Y) \overset{\text{def}}{=} H^*(Q^*_\infty) = H^*(O^*_\infty)$$

the de Rham cohomology of $J^\infty Y$.

Let us consider the vertical exact sequence of sheaves

$$0 \to \Omega^m_X \xrightarrow{\pi^*_X} \Omega^0,m_\infty \xrightarrow{d_V} \cdots \xrightarrow{d_V} \Omega^k,m_\infty \xrightarrow{d_V} \cdots, \quad 0 \leq m \leq n,$$

in the variational bicomplex (4) and the complex of their structure algebras

$$0 \to \mathcal{O}^m(X) \xrightarrow{\pi^*_X} Q^0,m_\infty \xrightarrow{d_V} \cdots \xrightarrow{d_V} Q^k,m_\infty \xrightarrow{d_V} \cdots.$$

**Proposition 9.** There is an isomorphism

$$H^*(m, d_V) = H^*(Y, \pi^*\mathcal{O}^m_X)$$
of the cohomology groups $H^*(m, d_V)$ of the complex (13) to the cohomology groups $H^*(Y, \pi^*\mathfrak{D}_X^m)$ of $Y$ with coefficients in the pull-back sheaf $\pi^*\mathfrak{D}_X^m$ on $Y$.

**Proof.** The exact sequence (14) is a resolution of the pull-back sheaf $\pi^\infty*\mathfrak{D}_X^m$ on $J^\infty Y$. Then, by virtue of Theorem 3, we have a cohomology isomorphism

$$H^*(m, d_V) = H^*(J^\infty Y, \pi^\infty*\mathfrak{D}_X^m).$$

Lemma 6 completes the proof. \qed

**Corollary 10.** Cohomology groups $H^{>\dim Y}(m, d_V)$ vanish.

The cohomology groups $H^*(m, d_V)$ have a $C^\infty(X)$-module structure. For instance, let

$$Y \cong X \times V \to X$$

be a trivial fiber bundle with a typical fiber $V$. There is an obvious isomorphism of $\mathbb{R}$-modules

$$H^*(m, d_V) = \mathfrak{D}_X^m \otimes H^*(V).$$

Turn now to the rows of the variational bicomplex (4). We have the exact sequence of sheaves

$$0 \to \mathfrak{Q}_{\infty}^{k,0} \xrightarrow{d_H} \mathfrak{Q}_{\infty}^{k,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}_{\infty}^{k,n} \xrightarrow{\tau} \mathfrak{E}_k \to 0, \quad k > 0.$$

Since the sheaves $\mathfrak{Q}_{\infty}^{k,0}$ and $\mathfrak{E}_k$ are fine, this is a resolution of the fine sheaf $\mathfrak{Q}_{\infty}^{k,0}$. It states immediately the following.

**Proposition 11.** The cohomology groups $H^*(k, d_H)$ of the complex

$$0 \to \mathfrak{Q}_{\infty}^{k,0} \xrightarrow{d_H} \mathfrak{Q}_{\infty}^{k,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}_{\infty}^{k,n} \xrightarrow{\tau} E_k \to 0, \quad k > 0,$$

are trivial.

This result at terms $\mathfrak{Q}_{\infty}^{k, \leq n}$ recovers that of [13]. The exactness of the complex (17) at the term $\mathfrak{Q}_{\infty}^{k,n}$ means that, if

$$\tau(\phi) = 0, \quad \phi \in \mathfrak{Q}_{\infty}^{k,n},$$

10
then
\[ \phi = d_H \xi, \quad \xi \in Q_{\infty}^{k,n-1}. \]

Since \( \tau \) is a projection operator, there is the \( \mathbb{R} \)-module decomposition
\[ Q_{\infty}^{k,n} = E_k \oplus d_H(\mathbb{Q}_{\infty}^{k,n-1}). \]

**Remark 12.** One can derive Proposition 11 from Theorem 4, without appealing to that sheaves \( E_k \) are acyclic.

Let us consider the exact sequence of sheaves
\[ 0 \rightarrow \mathbb{R} \rightarrow Q_{\infty}^0 \xrightarrow{d_H} Q_{\infty}^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q_{\infty}^{0,n} \]
where all sheaves except \( \mathbb{R} \) are fine. Then, from Theorem 4 and Lemma 5, we state the following.

**Proposition 13.** Cohomology groups \( H^r(d_H), r < n, \) of the complex
\[ 0 \rightarrow \mathbb{R} \rightarrow Q_{\infty}^0 \xrightarrow{d_H} Q_{\infty}^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q_{\infty}^{0,n} \]
are isomorphic to de Rham cohomology groups \( H^r(Y) \) of \( Y \).

This result recovers that of [16], but let us say something more.

**Proposition 14.** Any \( d_H \)-closed form \( \sigma \in Q_{\infty}^{*,<n} \) is represented by the sum
\[ \sigma = h_0 \varphi + d_H \xi, \quad \xi \in Q_{\infty}^*, \]
where \( \varphi \in \mathcal{O}_{\infty}^{*,0} \) is a closed form on the fiber bundle \( Y \).

**Proof.** Due to the relation
\[ h_0 d = d_H h_0, \]
the horizontal projection \( h_0 \) provides a homomorphism of the de Rham complex (11) to the complex
\[ 0 \rightarrow \mathbb{R} \rightarrow Q_{\infty}^0 \xrightarrow{d_H} Q_{\infty}^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q_{\infty}^{0,n} \xrightarrow{d_H} 0. \]
Accordingly, there is a homomorphism
\[ h_0^*: H^r(J^\infty Y) \rightarrow H^r(d_H), \quad 0 \leq r \leq n, \]
of cohomology groups of these complexes. Proposition 8 and Proposition 13 show that, for \( r < n \), the homomorphism (23) is an isomorphism (see the relation (29) below for the case \( r = n \)). It follows that a horizontal form \( \psi \in Q^{0, < n} \) is \( d_H \)-closed (resp. \( d_H \)-exact) if and only if \( \psi = h_0 \phi \) where \( \phi \) is a closed (resp. exact) form. The decomposition (12) and Proposition 11 complete the proof.

Proposition 15. If \( \phi \in Q^{0, < n} \) is a \( d_H \)-closed form, then \( d_V \phi = d \phi \) is necessarily \( d_H \)-exact.

Proof. Being nilpotent, the vertical differential \( d_V \) defines a homomorphism of the complex (22) to the complex

\[
0 \to Q^{1,0}_\infty \xrightarrow{d_H} Q^{1,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q^{1,n}_\infty \xrightarrow{d_H} 0
\]

and, accordingly, a homomorphism of cohomology groups \( H^*(d_H) \to H^*(1,d_H) \) of these complexes. Since \( H^{<n}(1,d_H) = 0 \), the result follows.

Let us prolong the complex (19) to the variational complex

\[
0 \to \mathbb{R} \to Q^0_\infty \xrightarrow{d_H} Q^{1,0}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q^{0,n}_\infty \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \to \cdots
\]

(24)

of the graded differential algebra \( Q^*_\infty \). In accordance with Lemma 3, the variational complex (5) is a resolution of the constant sheaf \( \mathbb{R} \) on \( J^\infty Y \). Then, Theorem 4 and Proposition 8 give immediately the following.

Proposition 16. There is an isomorphism

\[
H^*_\text{var} = H^*(Y)
\]

between cohomology \( H^*_\text{var} \) of the variational complex (24) and de Rham cohomology of the fiber bundle \( Y \).

The isomorphism (23) recovers the result of [16] and that of [1] at terms \( Q^{0,n}_\infty \), \( E_1 \), but let us say something more. The relation (3) for \( \tau \) and the relation (21) for \( h_0 \) define a homomorphisms of the de Rham complex (11) of the algebra \( Q^*_\infty \) to the variational complex (24). The corresponding homomorphism of their cohomology groups is an isomorphism. Then, in accordance with the splitting (12), we come to the following assertion which complete Proposition 14.
Proposition 17. Any \( \delta \)-closed form \( \sigma \in Q^{k,n}_\infty \), \( k \geq 0 \), is represented by the sum

\[
\sigma = h_0 \varphi + d_H h_0 \xi, \quad k = 0, \tag{26}
\]

\[
\sigma = \tau(\varphi) + \delta(\xi), \quad k > 0, \tag{27}
\]

where \( \varphi \) is a closed \((n + k)\)-form on \( Y \) and \( \xi \in Q^*_\infty \).

5. Cohomology of \( O^*_\infty \)

Thus, we have the whole cohomology of the graded differential algebra \( Q^*_\infty \). The following theorem provide us with \( d_H \)- and \( \delta \)-cohomology of the graded differential algebra \( O^*_\infty \).

Theorem 18. Graded differential algebra \( O^*_\infty \) has the same \( d_H \)- and \( \delta \)-cohomology as \( Q^*_\infty \).

Proof. Let the common symbol \( D \) stand for the coboundary operators \( d_H \) and \( \delta \) of the variational bicomplex. Bearing in mind the decompositions (20), (26) and (27), it suffices to show that, if an element \( \phi \in O^*_\infty \) is \( D \)-exact with respect to the algebra \( Q^*_\infty \) (i.e., \( \phi = D \varphi, \varphi \in Q^*_\infty \)), then it is \( D \)-exact in the algebra \( O^*_\infty \) (i.e., \( \phi = D \varphi', \varphi' \in O^*_\infty \)).

Lemma 2 states that, if \( Y \) is a contractible fiber bundle and a \( D \)-exact form \( \phi \) on \( J^\infty Y \) is of finite jet order \([\phi] \) (i.e., \( \phi \in O^*_\infty \)), there exists an exterior form \( \varphi \in O^*_\infty \) on \( J^\infty Y \) such that \( \phi = D \varphi \). Moreover, a glance at the homotopy operators for \( d_V, d_H \) and \( \delta \) shows that the jet order \([\varphi] \) of \( \varphi \) is bounded for all exterior forms \( \phi \) of fixed jet order. Let us call this fact the finite exactness of the operator \( D \). Given an arbitrary fiber bundle \( Y \), the finite exactness takes place on \( J^\infty Y \) over any open subset \( U \) of \( Y \) which is homeomorphic to a convex open subset of \( \mathbb{R}^{\dim Y} \). Now, we show the following.

(i) Suppose that the finite exactness of the operator \( D \) takes place on \( J^\infty Y \) over open subsets \( U, V \) of \( Y \) and their non-empty overlap \( U \cap V \). Then, it is also true on \( J^\infty Y \mid_{U \cup V} \).

(ii) Given a family \( \{U_\alpha\} \) of disjoint open subsets of \( Y \), let us suppose that the finite exactness takes place on \( J^\infty Y \mid_{U_\alpha} \) over every subset \( U_\alpha \) from this family. Then, it is true on \( J^\infty Y \) over the union \( \bigcup_\alpha U_\alpha \) of these subsets.

If the assertions (i) and (ii) hold, the finite exactness of \( D \) on \( J^\infty Y \) takes place since one can construct the corresponding covering of the manifold \( Y \) (Lemma 9.5).
Proof of (i). Let $\phi = D\varphi \in \mathcal{O}_Y^*$ be a $D$-exact form on $J^\infty Y$. By assumption, it can be brought into the form $D\varphi_U$ on $(\pi_0^\infty)^{-1}(U)$ and $D\varphi_V$ on $(\pi_0^\infty)^{-1}(V)$, where $\varphi_U$ and $\varphi_V$ are exterior forms of finite jet order. Due to the decompositions (20), (26) and (27), one can choose the forms $\varphi_U$, $\varphi_V$ such that $\varphi - \varphi_U$ on $(\pi_0^\infty)^{-1}(U)$ and $\varphi - \varphi_V$ on $(\pi_0^\infty)^{-1}(V)$ are $D$-exact forms. Let us consider the difference $\varphi_U - \varphi_V$ on $(\pi_0^\infty)^{-1}(U \cap V)$. It is a $D$-exact form of finite jet order which, by assumption, can be written as $\varphi_U - \varphi_V = D\sigma$ where an exterior form $\sigma$ is also of finite jet order. Lemma 19 below shows that $\sigma = \sigma_U + \sigma_V$ where $\sigma_U$ and $\sigma_V$ are exterior forms of finite jet order on $(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively. Then, putting

$$\varphi'_U = \varphi_U - D\sigma_U, \quad \varphi'_V = \varphi_V + D\sigma_V,$$

we have the form $\phi$ equal to $D\varphi'_U$ on $(\pi_0^\infty)^{-1}(U)$ and $D\varphi'_V$ on $(\pi_0^\infty)^{-1}(V)$, respectively. Since the difference $\varphi'_U - \varphi'_V$ on $(\pi_0^\infty)^{-1}(U \cap V)$ vanishes, we obtain $\phi = D\varphi'$ on $(\pi_0^\infty)^{-1}(U \cup V)$ where

$$\varphi' \overset{\text{def}}{=} \begin{cases} \varphi'|_U = \varphi'_U, \\ \varphi'|_V = \varphi'_V \end{cases}$$

is of finite jet order.

Proof of (ii). Let $\phi \in \mathcal{O}_Y^*$ be a $D$-exact form on $J^\infty Y$. The finite exactness on $(\pi_0^\infty)^{-1}(\cup U_\alpha)$ holds since $\phi = D\varphi_\alpha$ on every $(\pi_0^\infty)^{-1}(U_\alpha)$ and, as was mentioned above, the jet order $[\varphi_\alpha]$ is bounded on the set of exterior forms $D\varphi_\alpha$ of fixed jet order $[\phi]$.

Lemma 19. Let $U$ and $V$ be open subsets of a fiber bundle $Y$ and $\sigma \in \mathcal{O}_Y^*$ an exterior form of finite jet order on the non-empty overlap $(\pi_0^\infty)^{-1}(U \cap V) \subset J^\infty Y$. Then, $\sigma$ splits into a sum $\sigma_U + \sigma_V$ of exterior forms $\sigma_U$ and $\sigma_V$ of finite jet order on $(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively.

Proof. By taking a smooth partition of unity on $U \cup V$ subordinate to the cover $\{U, V\}$ and passing to the function with support in $V$, one gets a smooth real function $f$ on $U \cup V$ which is 0 on a neighborhood of $U - V$ and 1 on a neighborhood of $V - U$ in $U \cup V$. Let $(\pi_0^\infty)^* f$ be the pull-back of $f$ onto $(\pi_0^\infty)^{-1}(U \cup V)$. The exterior form $((\pi_0^\infty)^* f)\sigma$ is zero on a neighborhood of $(\pi_0^\infty)^{-1}(U)$ and, therefore, can be extended by 0 to $(\pi_0^\infty)^{-1}(U)$. Let us denote it $\sigma_U$. Accordingly, the exterior form $(1 - (\pi_0^\infty)^* f)\sigma$ has an extension $\sigma_V$ by 0.
to \((\pi_0^\infty)^{-1}(V)\). Then, \(\sigma = \sigma_U + \sigma_V\) is a desired decomposition because \(\sigma_U\) and \(\sigma_V\) are of finite jet order which does not exceed that of \(\sigma\). \(\square\)

It is readily observed that Theorem 18 can be applied to de Rham cohomology of \(O_{\infty}^*\) whose isomorphism (13) to that of \(Q_{\infty}^*\) has been stated.

6. The global inverse problem

The variational complex (24) provides the algebraic approach to the calculus of variations on fiber bundles in the class of exterior forms of locally finite jet order [3, 7, 17]. For instance, the variational operator \(\delta\) acting on \(Q_{\infty}^{0,n}\) is the Euler–Lagrange map, while \(\delta\) acting on \(E_1\) is the Helmholtz–Sonin map. Let \(L \in Q_{\infty}^{0,n}\) be a horizontal density on \(J_{\infty}Y\). One can think of \(L\) as being a Lagrangian of local finite order. Then, the decomposition (18) leads to the first variational formula

\[
dL = \tau(dL) + (\text{Id} - \tau)(dL) = \delta(L) + dH(\phi), \quad \phi \in Q_{\infty}^{1,n-1},
\]

where \(\delta(L)\) is the Euler–Lagrange form associated with the Lagrangian \(L\).

Let us relate the cohomology isomorphism (25) to the global inverse problem of the calculus of variations. As a particular repetition of Proposition 17, we come to its following solution in the class of exterior forms of locally finite jet order.

**Theorem 20.** A Lagrangian \(L \in Q_{\infty}^{0,n}\) is variationally trivial, i.e., \(\delta(L) = 0\) if and only if

\[
L = h_0\varphi + dHh_0\xi, \quad \xi \in Q_{\infty}^*,
\]

where \(\varphi\) is a closed \(n\)-form on \(Y\) (see the expression (26)).

**Theorem 21.** An Euler–Lagrange-type operator \(E \in E_1\) satisfies the Helmholtz condition \(\delta(E) = 0\) if and only if

\[
E = \delta(L) + \tau(\phi), \quad L \in Q_{\infty}^{0,n},
\]

where \(\phi\) is a closed \((n + 1)\)-form on \(Y\) (see the expression (27)).

Theorem 21 contains the similar result of [1, 16].
Remark 22. As a consequence of Theorem 20, one obtains that the cohomology group
$H^n(d_H)$ of the complex (22) obeys the relation

\[(29) \quad H^n(d_H)/H^n(Y) = \delta(\mathcal{Q}_\infty^{0,n}),\]

where $\delta(\mathcal{Q}_\infty^{0,n})$ is the $\mathbb{R}$-module of Euler–Lagrange forms on $J^\infty Y$.

Theorem 18 provides the similar solution of the global inverse problem in the class of
exterior forms of finite jet order. The theses of Theorem 20 and Theorem 21 remain true
if all exterior forms belong to $\mathcal{O}_\infty^*$. Theorem 21 contains the result of [18].

As was mentioned above, a solution of the global inverse problem in the calculus of
variations in the class of exterior forms of a fixed jet order has been suggested in [1] by
a computation of cohomology of the fixed order variational sequence. The first thesis of
[1] agrees with Theorem 20 for finite order Lagrangians, but says that the jet order of the
form $\xi$ in the expression (28) is $k - 1$ if $L$ is a $k$-order variationally trivial Lagrangian.
The second one states that a $2k$-order Euler–Lagrange operator can be always associated
with a $k$-order Lagrangian. However, because of the sophisticated technique, these results
were not widely recognized.

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