MUMFORD-THADDEUS PRINCIPLE ON THE MODULI SPACE OF VECTOR BUNDLES ON AN ALGEBRAIC SURFACE

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§0. Introduction.

The purpose of this paper is to study what we call the “Mumford-Thaddeus principle” which states that Geometric Invariant Theory (henceforth, “GIT”) quotients undergo specific transformations (birational and similar to Mori’s flip under some mild conditions) when the polarization (i.e. the linearized ample line bundle) is varied (cf.[MFK94], [Thaddeus93,94], and [Dolgachev-Hu93]). The case we consider is the moduli space of vector bundles on an algebraic surface.

Let $X$ be a nonsingular projective surface over an algebraically closed field $k$ of characteristic zero (this assumption about characteristic is used only in analyzing the behavior of semistability with respect to Galois covers in §5). To construct the moduli space, [Gieseker77] first introduces the notion of $H$-Gieseker-semistability for torsion-free coherent sheaves; a refinement of Mumford-Takemoto $H$-slope-semistability. This depends on the choice of an ample line bundle $H$ (actually, only the numerical class of $H$) on the surface $X$. Then he considers an appropriate GIT problem and proves that the projective quotient $M(r, c_1, c_2, H)$ coarsely represents the Seshadri equivalence classes of torsion-free coherent sheaves of fixed rank $r$, first and second Chern classes $c_1$ and $c_2$, which are $H$-Gieseker-semistable. Once the existence of such a moduli space is established, a basic and natural problem is to study the change of $M(r, c_1, c_2, H)$ as $H$ varies.

The aim of this paper is to prove the

Main Theorem. As the polarization $H$ varies through the ample cone of $X$, the moduli space $M(r, c_1, c_2, H)$ goes through a locally finite sequence of Thaddeus-type flips (and contraction morphisms and their inverses) in the category of projective moduli spaces $M((r, c_1, c_2) \otimes \mathcal{L}, A)$ of $\mathcal{L}$-twisted $A$-Gieseker-semistable sheaves for
rational line bundles $L \in \text{Pic}(X) \otimes \mathbb{Q}$ and ample line bundles $A$ (see below for the definition of $L$-twisted semistability and the definition of a Thaddeus-type flip).

In this paper a Thaddeus-type flip is defined to be a diagram

$$
\begin{array}{ccc}
Q_H^{ss} / / G & \to & Q_H^{ss} / / G \\
\psi & \searrow & \psi^+ \\
& Q_A^{ss} / / G \\
\end{array}
$$

where $Q_H^{ss} / / G$ (resp. $Q_H^{ss} / / G, Q_A^{ss} / / G$) is the GIT quotient of the locus of semistable points $Q_H^{ss}$ (resp. $Q_H^{ss}, Q_A^{ss}$) with respect to the linearization $H$ (resp. $H', A$) of the action of a reductive group $G$ on a projective scheme $Q$, and the morphisms $\psi$ and $\psi^+$ are induced from the inclusion $Q_H^{ss} \subset Q_A^{ss} \supset Q_H^{ss}$.

We remark that we do not require the morphisms to be birational or small in the definition, though [Dolgachev-Hu93] and [Thaddeus94] claim these and other nice properties for the morphisms under some mild conditions.

It should be mentioned that there exists another compactification of the moduli space of vector bundles, the Uhlenbeck compactification, whose projective variety structure is described in [J.Li93]. Its variation with respect to the polarization has been considered, at least in rank 2 case, in [Hu-W.P.Li94]. The introduction in the present paper of the notion of $L$-twisted $A$-Gieseker-semistable sheaves is the novelty required to study Gieseker’s space.

We now outline the structure of this paper: In §1 we prove some basic finiteness results. [Gieseker77] shows that if we fix the polarization $H$, then the set $S(r, c_1, c_2, H)$ of all coherent torsion-free sheaves with given numerical data and which are $H$-Gieseker-semistable is bounded, i.e., parametrized by a scheme of finite type over $k$. Global finiteness, by contrast, does not hold in general, i.e., the union of the sets $S(r, c_1, c_2, H)$ when $H$ varies over all ample numerical classes may not be bounded (see Remark 1.7 (1) for an example). The best one can hope for is local finiteness, and we prove this for slope semistability (which implies the result for Gieseker-semistability): Let $\Delta$ be a convex polyhedral cone in the ample cone of $X$ and $\mu(r, c_1, c_2, H)$ denote the set of all $H$-slope-semistable torsion-free sheaves of given numerical data. Then the union of the sets $\mu(r, c_1, c_2, H)$ when $H$ varies over $\Delta - \{0\}$ is bounded.

We also have the following finiteness result: The set of all subsheaves $F \subset E$ where $E \in \mu(r, c_1, c_2, H)$ for some $H \in \Delta - \{0\}$ with torsion-free quotient $E/F$ satisfying

$$
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H = 0
$$

is bounded.

Hence the set of all hyperplanes

$$
L_F := \{ z \in N^1(X)_{\mathbb{Q}}; \left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot z = 0 \}
$$

is finite, where $F$ runs over all such subsheaves as above with

$$
\frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)}
$$
being not numerically trivial. The $L_F$’s thus give a stratification $\Delta - \{0\} = \bigsqcup \Delta_s$ into cells which characterizes the change of the set $\mu(r,c_1,c_2,H)$.

In §2, we study the GIT problem. Ideally, one would like to use the boundedness result above to embed all the sheaves in $\cup_{H \in \Delta - \{0\}} S(r,c_1,c_2,H)$ into one big space independent of $H$ (e.g. a Grothendieck Quot scheme), observe the correspondence between $H$-Gieseker-semistable sheaves and GIT semistable points for an appropriate linearization, and then apply the Mumford-Thaddeus principle. Though this turns out to be too naive a picture and indeed false if one applies the known construction directly, it will nevertheless be our guiding principle throughout.

Gieseker’s original construction using the covariant method is not suitable for our purpose since the relevant spaces are highly dependent of the polarization, whereas Simpson’s direct use of the Grothendieck Quot scheme [Simpson92] provides us with a more adequate setting. Briefly, the method for a FIXED polarization is the following: First, one embeds all the sheaves $E \in S(r,c_1,c_2,H)$ into the Quot scheme parametrizing surjections $O_X^{\otimes l} \to E \otimes H^a \to 0$ for large $a$. Second, one proves the correspondence between $H$-Gieseker-semistable sheaves and the GIT semistable points with respect to the action of $SL(l)$ and the linearization induced from the embedding of the Quot scheme into a Grassmannian with its Plücker coordinates. The latter embedding comes from taking a further high multiple of $H$ and a surjection $H^0(O_X^{\otimes l} \otimes H^m) \to H^0(E \otimes H^a \otimes H^m) \to 0$. The GIT quotient then recovers $M(r,c_1,c_2,H)$.

Our Key GIT Lemma (Lemma 2.4), which is a modification of Simpson’s method, goes along the same line – one simply uses DIFFERENT ample bundles $A$ and $H$ for the first and second embeddings – but this turns out to be quite subtle and provides us with some unexpected consequences. First, by the boundedness result it is a simple matter to embed all the sheaves in $\cup_{H \in \Delta - \{0\}} S(r,c_1,c_2,H)$ into a Quot scheme by taking a high multiple of $A \in \Delta - \{0\}$ and a surjection $O_X^{\otimes l} \to E \otimes A^a \to 0$, though it will be observed that this only reflects the properties local around $A$. If one then considers the linearization obtained by embedding the Quot scheme into a Grassmannian via a surjection $H^0(O_X^{\otimes l} \otimes H^m) \to H^0(E \otimes A^a \otimes H^m) \to 0$, the question of what sheaves the semistable points correspond to becomes quite delicate. The answer is the following: the set of semistable points with respect to the second embedding by $H$ corresponds not to $S(r,c_1,c_2,H)$ but to a subset $S(r,c_1,c_2,A)_H$ of the $A$-Gieseker-semistable sheaves consisting of those $E$ for which any subsheaf $F \subset E$ having the same averaged Euler characteristic with respect to $A$

$$p(F,A,n) = \frac{\chi(F \otimes A^n)}{rk(F)} = \frac{\chi(E \otimes A^n)}{rk(E)} = p(E,A,n)$$

also satisfies

$$\frac{c_1(F)}{rk(F)} \cdot H \geq \frac{c_1(E)}{rk(E)} \cdot H.$$  

(Note that the direction of the inequality is opposite from the one in the definition of $H$-Gieseker-semistability.) We denote the GIT quotient $M(r,c_1,c_2,A)_H$. This correspondence, while seemingly technical and somewhat irregular, is the important ingredient in the proof of the Main Theorem.

§3 begins the analysis of “twists”, which not only gives us a second stratification of $\Delta - \{0\}$, built on top of the previous one, characterizing the changes in the
set $S(r,c_1,c_2,H)$ as $H$ varies, but also tells us how the (birational) map should be factorized into a sequence of Thaddeus-type flips as in the Main Theorem. A torsion-free coherent sheaf $E$ is said to be $L$-twisted $A$-Gieseker-semistable for $L \in \text{Pic}(X) \otimes \mathbb{Q}$ if and only if for all $F \subset E$

$$\frac{\chi(F \otimes L \otimes A^n)}{rk(F)} \leq \frac{\chi(E \otimes L \otimes A^n)}{rk(E)}$$

for $n >> 0$, where we compute the Euler characteristics formally using the Riemann-Roch formula. We denote the set of such $E$ with given numerical data by $S((r,c_1,c_2) \otimes L,A)$, and in §4 and §5 we shall construct a projective moduli space $M((r,c_1,c_2) \otimes L,A)$ coarsely representing the Seshadri equivalence classes of this set with respect to the $L$-twisted $A$-Gieseker-stability.

Now consider the previous stratification $\Delta - \{0\} = \coprod_s \Delta_s$, and denote by $V(\Delta_s)$ the $\mathbb{Q}$-span of $\Delta_s$ in $N^1(X)_{\mathbb{Q}}$. Let $\Delta_s$ and $\Delta_s'$ be two $d$-dimensional cells such that $V(\Delta_s) = V(\Delta_s')$ and that they are separated by a $d - 1$-dimensional cell $W$. Let $A$ be an ample line bundle on $W$. Then our analysis states that there exists a stratification

$$V(\Delta_s) = \coprod_i L_i \coprod_j M_j$$

consisting of a finite number of hyperplanes $L_i$ parallel to $V(W)$ and connected components $M_j$ of $V(\Delta_s) - \coprod_i L_i$ which determines the change of the set $S((r,c_1,c_2) \otimes L,A)$ as $L$ varies in $V(\Delta_s)$. Moreover, it follows that

$$S(r,c_1,c_2,A) = S(r,c_1,c_2,H) \quad \text{for } A \in W \text{ and } H \in \Delta_s$$

$$\iff$$

$$\Delta_s \text{ is contained in one of the strata } M_j$$

and this gives a criterion, by induction on the dimension $d$ of the cells, to determine the second stratification of $\Delta - \{0\}$ characterizing the changes in the set $S(r,c_1,c_2,H)$ as $H$ varies in $\Delta - \{0\}$.

Our aim then is to show that for a sequence of rational ample line bundles $L_i \in L_i$ and $M_j \in M_j$ there exists a sequence of Thaddeus-type flips:

$$M((r,c_1,c_2) \otimes M_i,A) \quad \quad M((r,c_1,c_2) \otimes M_{i+1},A)$$

$$\swarrow \quad \quad \searrow$$

$$M((r,c_1,c_2) \otimes L_i,A)$$

for $i = 0,1,\cdots,l$ arising from the Key GIT Lemma. This, together with the observation

$$M(r,c_1,c_2,H) = M((r,c_1,c_2) \otimes M_0,A)$$

$$M(r,c_1,c_2,H') = M((r,c_1,c_2) \otimes M_{i+1},A)$$

$$M(r,c_1,c_2,A) = M((r,c_1,c_2) \otimes M_i,A) \text{ or } M((r,c_1,c_2) \otimes L_i,A)$$

for some $i$,

where $H \in \Delta_s$ and $H \in \Delta_s'$ (in fact, we may take $M_0 = H^n$ and $M_{i+1} = H'^{n'}$ for $n, n'$ sufficiently large) solves the problem of factorizing the (birational) map between $M(r,c_1,c_2,H)$ and $M(r,c_1,c_2,H')$, and thus proves the Main Theorem.
§4 is devoted to the Integral Case of the construction of the flip. When $L_i$ may be chosen to be an integral line bundle, then $M((r, c_1, c_2) \otimes L_i, A)$ is nothing but the classical moduli space $M(r, c_1, c_2, c_{2c_i}, A)$ where
\[
c_{1c_i} = c_1 + c_1(L_i) \\
c_{2c_i} = c_2 + (r - 1)c_1 + \frac{r(r - 1)}{2} c_1(L_i)^2.
\]
Moreover, we shall see that
\[
M((r, c_1, c_2) \otimes M_i, A) = M(r, c_1, c_2, c_{2c_i}, A)^{\prime} \\
M((r, c_1, c_2) \otimes M_{i+1}, A) = M(r, c_1, c_2, c_{2c_i}, A)^{\prime}.
\]
Therefore, the flip arises immediately from the construction in §2 and the Key GIT Lemma.

§5 deals with the more delicate case where $L_i$ is only a RATIONAL line bundle. Our strategy is to use Kawamata’s technique of finding some nice Galois cover $\phi: Y \rightarrow X$ so that the set of $L_i$-twisted $A$-Gieseker-semistable sheaves on $X$ correspond to the subset of $Y^\prime$-twisted $\phi^* A$-Gieseker-semistable sheaves on $Y$, where $Y^\prime$ is now an INTEGRAL line bundle on $Y$. (Note that $Y^\prime$ is not exactly the pull back $\phi^* L_i$ but $Y^\prime = \phi^* L_i + \frac{1}{2} R$ where $R$ is the ramification divisor of $\phi$.) Then by looking at the appropriate locus of the Quot scheme associated to $Y$ and taking the action of the Galois group into consideration, we reduce the construction of the flip to the Mumford-Thaddeus principle and the Key GIT Lemma applied to the sheaves on $Y$ instead of $X$.

Finally starting from the $d$-cells of maximal dimension and descending inductively on the dimension $d$, i.e., going from $\Delta$ to $W$ and repeating, and moreover letting $\Delta$ vary in $\text{Amp}(X)_Q$, we can factorize all the transformations among the moduli spaces $M(r, c_1, c_2, H)$ with varying polarizations $H \in \text{Amp}(X)_Q$ into sequences of Thaddeus-type flips. Our proof shows more generally that all the transformations among the moduli spaces $M((r, c_1, c_2) \otimes L, H)$ with varying polarizations $H$ and rational twists $L$ can be factorized into sequences of Thaddeus-type flips.

We remark that instead of fixing the first chern class $c_1 \in N^1(X)_Q$ we may fix the determinant of coherent torsion-free sheaves to be $I \in \text{Pic}(X)$. Then we obtain $M((r, I, c_2, H)$ (resp. $M((r, I, c_2) \otimes L, A)$) as a closed subscheme of $M(r, c_1, c_2, H)$ (resp. $M((r, I, c_2) \otimes L, A)$) where $c_1(I) = c_1$. The results above hold for these moduli spaces with a fixed determinant without any change (cf.[Gieseker-Li94]).

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§1. Some Basic Finiteness Results.

Let $X$ be a nonsingular projective surface over an algebraically closed field $k$ of characteristic zero, $N^1(X)_Q$ the Neron-Severi group of $X$ tensored by $Q$, $\text{Amp}(X)_Q$ the convex cone in $N^1(X)_Q$ generated by ample divisors.
Proof of Theorem 1.3.

is to say, all \(E\) of finite type over \(S\).

Fix a numerical class \(c\) of \(F\) over \(S\) fiber of \(H\).

A set \(\mathcal{E}\) is \(H\)-Gieseker-semistable (resp. \(H\)-Gieseker-stable) iff for all coherent subsheaves \(E\) of \(F\) (resp. \(F \subset E\))

\[
\frac{c_1(F)}{rk(F)} \cdot H \leq \frac{c_1(E)}{rk(E)} \cdot H \quad (\text{resp. } <).
\]

Let \(H\) be a polarisation. A coherent torsion-free sheaf \(E\) of rank \(rk(E)\) is \(H\)-slope-semistable (resp. \(H\)-slope-stable) iff for all subsheaves \(F\) of \(E\) (resp. \(F \subset E\))

\[
p(F, H, n) := \frac{\chi(F \otimes H^n)}{rk(F)} = \frac{1}{2} H^2 n^2 + \left( \frac{c_1(F)}{rk(F)} \cdot H - \frac{1}{2} K_X \cdot H \right) n
\]

\[
\leq p(E, H, n) = \frac{\chi(E \otimes H^n)}{rk(E)} = \frac{1}{2} H^2 n^2 + \left( \frac{c_1(E)}{rk(E)} \cdot H - \frac{1}{2} K_X \cdot H \right) n
\]

\[
+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)} + \chi(O_X)
\]

\[
(\text{resp. } p(F, H, n) < p(E, H, n))
\]

for \(n \gg 0\).

We denote by \(\mu(r, c_1, c_2, H)\) the set of all coherent torsion-free sheaves of a fixed rank \(r\), first and second chern classes \(c_1\) and \(c_2\), which are \(H\)-slope-semistable, and use the notation \(S(r, c_1, c_2, H)\) for \(H\)-Gieseker-semistable ones.

Definition 1.2. A set \(T\) of coherent sheaves on \(X\) is bounded iff there is a scheme \(S\) of finite type over \(k\) and a coherent sheaf \(F\) over \(S \times X\) flat over \(S\) so that for all \(E \in T\) there exists a closed point \(s \in S\) such that the coherent sheaf \(F_s\) on the fiber of \(S \times X\) over \(s\) is isomorphic to \(E\).

Theorem 1.3. Let \(\Delta\) be a convex cone generated by a finite number of ample classes \(H_1, H_2, \cdots, H_l \in Amp(X)_Q - \{0\}\), i.e.,

\[
\Delta = \{t_1 H_1 + t_2 H_2 + \cdots + t_l H_l \in Amp(X)_Q; t_1, t_2, \cdots, t_l \in \mathbb{Q}, t_1 \geq 0, t_2 \geq 0, \cdots, t_l \geq 0\}
\]

Fix a numerical class \(c_1 \in N^1(X)_Q\), an integer \(c_2 \in \mathbb{Z}\) and another integer \(r \in \mathbb{N}\).

Then the set of all coherent torsion free sheaves \(E\) of \(rk(E) = r\), \(c_1(E) = c_1\) and \(c_2(E) = c_2\) s.t. \(E\) is \(H\)-slope-semistable for some \(H \in \Delta - \{0\}\), is bounded. That is to say, \(\cup_{H \in \Delta - \{0\}} \mu(r, c_1, c_2, H)\) is bounded.

Proof of Theorem 1.3.

We prove by induction on the rank \(r\).

When \(r = 1\), a coherent torsion-free sheaf \(E\) of \(rk(E) = 1\), \(c_1(E) = c_1\) and \(c_2(E) = c_2\), is isomorphic to \(\mathcal{L} \otimes \mathcal{I}_Z\) where \(\mathcal{I}_Z\) is the ideal sheaf defining an artinian scheme \(Z\) of length \(c_2\) and \(\mathcal{L}\) is a line bundle with \(c_1(\mathcal{L}) = c_1\). Therefore,
$S(1, c_1, c_2, H)$ is parametrized by Pic$^{c_1}(X) \times \text{Hilb}^{c_2}(X)$, where Pic$^{c_1}(X)$ is a connected component of the Picard scheme of $X$ whose corresponding line bundles have the same numerical class $c_1$, and Hilb$^{c_2}(X)$ is the Hilbert scheme parametrizing artinian schemes on $X$ of length $c_2$. Thus the family of these sheaves is bounded. (Note that in the rank 1 case, the $H$-slope-semistability is automatic.)

Next assuming the boundedness for the case of rank $\leq r - 1$, we prove the boundedness for rank $= r$, by considering the following two separate subsets.

Case A: The subset consisting of those $E$ which are strictly $H$-slope-semistable for some $H \in \Delta - \{0\}$, i.e., there exists an exact sequence

$$0 \to F \to E \to G \to 0$$

where $F$ is a nonzero subsheaf of $E$ with $r(F) < rk(E) = r$, $G$ is torsion free with the condition that

$$\frac{c_1(F)}{rk(F)} \cdot H = \frac{c_1(E)}{r} \cdot H = \frac{c_1(G)}{rk(G)} \cdot H.$$ 

(If for the original choice of $F$ the cokernel $G$ is not torsion-free, then we take the saturation of $F$ in $E$ to make the cokernel torsion-free without changing the first chern class of $F$ or $G$.)

For all such $E$ in this subset, we have

$$c_1(F) \equiv \frac{rk(F)}{r} c_1(E) + \mathcal{L},$$

$$c_1(G) \equiv \frac{rk(G)}{r} c_1(E) - \mathcal{L},$$

where $\mathcal{L} \in \text{Pic}(X) \otimes \frac{1}{r} \mathbb{Z}$ with $\mathcal{L} \cdot H = 0$, and $\equiv$ denotes the numerical equivalence. Thus

$$(*) \quad c_2(E) = c_1(F)c_1(G) + c_2(F) + c_2(G)$$

$$= -\mathcal{L}^2 + \frac{rk(G) - rk(F)}{r} c_1(E) \cdot \mathcal{L} + \frac{rk(F)rk(G)}{r^2} c_1(E)^2$$

$$+ c_2(F) + c_2(G).$$

We remark here that the Bogomolov inequality holds not only for vector bundles but also for torsion-free sheaves.

**Lemma 1.4.** Let $E$ be a coherent torsion-free sheaf of rank $r$, $H$-slope-semistable for some $H \in \text{Amp}(X)_\mathbb{Q} - \{0\}$. Then

$$c_2(E) \geq \frac{r - 1}{2r} c_1(E)^2.$$

**Proof of Lemma 1.4.**

Let

$$0 \to E \to E^{**} \to Q \to 0$$
be the inclusion of $E$ into its double dual $E^{**}$ with the cokernel $Q$ having a support on a finite number of points. Since $E^{**}$ is also $H$-slope-semistable, by applying the Bogomolov inequality to the vector bundle $E^{**}$ we conclude

$$c_2(E) = c_2(E^{**}) + \text{length of } Q \geq c_2(E^{**}) \geq \frac{r - 1}{2r} c_1(E^{**})^2 = \frac{r - 1}{2r} c_1(E)^2.$$ 

Since both $F$ and $G$ are $H$-slope-semistable, by Lemma 1.4 we have

$$c_2(F) \geq \frac{rk(F) - 1}{2rk(F)} c_1(F)^2 = \frac{rk(F) - 1}{2rk(F)} \left( L^2 + \frac{rk(F)}{r} c_1(E) \cdot L + \left( \frac{rk(F)}{r} \right)^2 c_1(E)^2 \right),$$

$$c_2(G) \geq \frac{rk(G) - 1}{2rk(G)} c_1(G)^2 = \frac{rk(G) - 1}{2rk(G)} \left( L^2 - \frac{rk(G)}{r} c_1(E) \cdot L + \left( \frac{rk(G)}{r} \right)^2 c_1(E)^2 \right).$$

Plugging these into the formula for $c_2(E)$, we obtain the inequality

$$c_2 = c_2(E) \geq (-1 + \frac{rk(F) - 1}{2rk(F)} + \frac{rk(G) - 1}{2rk(G)}) L^2 + 0 \cdot c_1(E) \cdot L + \frac{(rk(F) + rk(G))(rk(F) + rk(G) - 1)}{2r^2} c_1(E)^2 \geq (1 - h)(-L^2) + l,$$

where $h$ and $l$ are constants depending only on $r$ and $c_1$

$$1 > h = \min\{rk(F) + rk(G) = r\} \left\{ \frac{rk(F) - 1}{2rk(F)} + \frac{rk(G) - 1}{2rk(G)} \right\},$$

$$l = \min\{rk(F) + rk(G) = r\} \left\{ \frac{(rk(F) + rk(G))(rk(F) + rk(G) - 1)}{2r^2} c_1(E)^2 \right\}.$$

Therefore, we have

$$0 \leq -L^2 \leq \frac{c_2 - l}{1 - h}.$$ 

**Lemma 1.5.** Fix $N \in \mathbb{N}$. The number of lattice points

$$\sharp\{x \in Pic(X) \otimes \frac{1}{r} \mathbb{Z}; -x^2 \leq N, x \cdot H = 0 \text{ for some } H \in \Delta - \{0\}\}$$

is finite.

**Proof of Lemma 1.5.**

Observe that locally with respect to $H \in \Delta - \{0\}$ we can find an orthonormal basis of $N^1(X)_\mathbb{Q}$ consisting of $e^H_1 = \frac{H}{\sqrt{H \cdot H}}, e^H_2, \cdots, e^H_{\dim N^1(X)_\mathbb{Q}}$, which vary continuously
with respect to $H$ as seen, e.g., by the method of Gram-Schmidt. This implies that the set
\[
\{(H, x) \in \Delta \times N^1(X)_\mathbb{Q}; -x^2 \leq N, x \cdot H = 0, \\
H = t_1 H_1 + t_2 H_2 + \cdots + t_l H_l, t_1 + t_2 + \cdots + t_l = 1\}
\]
is compact by the Hodge Index Theorem. Thus its image in $N^1(X)_\mathbb{Q}$ is also compact, and the number of the lattice points contained in it is finite.

By applying this lemma we conclude that the set of the possible numerical classes of $L$ is finite, which implies that both the set of the possible numerical classes of $c_1(F)$ and that of the possible numerical classes of $c_1(G)$ are finite.

Moreover, the finiteness of the numerical classes of $L$ implies that both $c_2(F)$ and $c_2(G)$ are bounded from below, which follows from the Bogomolov inequalities as above again. This with the equality (\*) in turn implies that both $c_2(F)$ and $c_2(G)$ have only a finite number of possibilities.

Now by inductive hypothesis on the rank and noting both $F$ and $G$ are $H$-slope-semistable, we conclude that both $F$ and $G$ form bounded families and so does $E \in \text{Ext}^1(F, G)$.

We state the conclusion of this case in the following form.

**Proposition 1.6.** *The set of all torsion-free coherent sheaves $F$ on $X$ such that there exists an exact sequence\[
0 \to F \to E \to G \to 0,
\]
where $E \in \mu(r, c_1, c_2, H)$ for some $H \in \Delta - \{0\}$ ($r > \text{rk}(F)$), and where the quotient $G$ is a coherent torsion-free sheaf, satisfying\[
\frac{c_1(F)}{\text{rk}(F)} \cdot H = \frac{c_1(E)}{r} \cdot H,
\]
is bounded, and so is the set of all such $G$. In particular, the sets of the pairs of the first and second chern classes of such sheaves \[
\{(c_1(F), c_2(F))\} \text{ and } \{(c_1(G), c_2(G))\}
\]
consist of a finite number of elements.*

We now deal with the second subset.

Case B: The complement of the previous subset, i.e., the subset consisting of such $E$ that for ANY $H \in \Delta - \{0\}$ is either $H$-slope-STABLE or $H$-slope-UNSTABLE (equivalently, NOT $H$-slope-semistable).

The condition of being $H$-slope-stable and that of being $H$-slope-unstable are both open with respect to $H \in \Delta - \{0\}$. Since $E$ is $H$-slope-semistable for some $H \in \Delta - \{0\}$, we conclude that $E$ must be $H$-slope-stable for ALL $H \in \Delta - \{0\}$. In particular, $E$ is, say, $H_1$-slope-stable. Thus $E$ is $H_1$-Gieseker-semistable. Therefore, [Gieseker77,Corollary1.3] implies that the family of such sheaves $E$ is bounded.

This completes the proof of Theorem 1.3.
Remark 1.7.

(1) The “local-boundedness” as in Theorem 1.2 is the best we can hope for IN GENERAL (though for some restricted classes of surfaces the global-boundedness does hold as we will see in (2)). In fact, we can construct a counter-example as follows to the statement that the set of all coherent torsion free sheaves $E$ of fixed rank $rk(E) = r$, fixed first and second Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$, $H$-slope-semistable for some $H \in \text{Amp}(X)_\mathbb{Q} - \{0\}$, is bounded. (Actually our example presents an unbounded family of coherent torsion free sheaves $E$ of fixed rank $r$, fixed first and second Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$, $H$-Gieseker-semistable for some $H \in \text{Amp}(X)_\mathbb{Q} - \{0\}$. Thus even if we restrict ourselves to the Gieseker-semistable sheaves the global boundedness does not hold.)

Take $X = E \times E$, where $E$ is a generic elliptic curve (with no nontrivial automorphism fixing the origin other than the involution) so that $\dim_{\mathbb{Q}} N^1(X)_\mathbb{Q} = 3$. Then

$$\text{Pic}(X) = \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}D \text{ (in } N^1(X)_\mathbb{Q})$$

where $E_1 = E \times \{p\}, E_2 = \{q\} \times E$ and $D$ is the diagonal. Their intersection pairings are

$$E_1^2 = E_2^2 = D^2 = 0$$
$$E_1 \cdot E_2 = E_1 \cdot D = E_2 \cdot D = 1$$

and thus we have the intersection matrix

$$\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.$$ 

By diagonalizing the matrix, we have the corresponding sublattice of rank 3 of $\text{Pic}(X)$ generated by

$$U = E_1 + E_2 + D$$
$$V = E_1 - D$$
$$W = E_2 - D.$$ 

Their intersection pairings are

$$U^2 = 6, V^2 = W^2 = -2$$
$$U \cdot V = V \cdot W = W \cdot U = 0.$$ 

If we introduce the coordinate system $(u, v, w) = uU + vV + wW$ in $N^1(X)_\mathbb{Q}$, then the ample cone $\text{Amp}(X)_\mathbb{Q} - \{0\}$ is defined as the connected component of

$$\{(u, v, w) \in N^1(X)_\mathbb{Q}; 6u^2 - 2v^2 - 2w^2 > 0\}$$

containing $(1, 0, 0)$.

Note that if we take a line bundle $\mathcal{L}$ whose numerical class is given by

$$\mathcal{L} = uU + vV + wW \text{ in } N^1(X)_\mathbb{Q},$$
then $c_2(E)$ of the rank 2 vector bundle

$$E = \mathcal{L} \oplus \mathcal{L}^{-1}$$

is given by

$$c_2(E) = -\mathcal{L}^2 = -(6u^2 - 2v^2 - 2w^2).$$

Now we focus our attention to the plane \{ $w = 0$ \}.

We recall an elementary lemma from number theory.

**Lemma 1.8.** Let $\omega \in \mathbb{R}_{>0}$ be an irrational number. Then there exist infinitely many rational numbers

$$\frac{q}{p} \text{ where } p, q \in \mathbb{N}, \text{gcd}(p, q) = 1$$

such that

$$0 < \frac{q}{p} - \omega < \frac{1}{p^2}.$$ 

We apply this lemma to $\omega = \sqrt{3}$. Thus we have infinitely many rational numbers

$$\frac{q}{p} \text{ where } p, q \in \mathbb{N}, \text{gcd}(p, q) = 1$$

such that

$$0 < \frac{q}{p} - \sqrt{3} < \frac{1}{p^2}.$$ 

By excluding finitely many rational numbers we may assume

$$q < 2p.$$ 

Thus for a constant $c \geq 8$, we have

$$0 < q - \sqrt{3}p < \frac{1}{p} = \frac{8}{2(2p + 2p)} < \frac{c}{2(q + \sqrt{3}p)},$$

which implies

$$0 < 2(q + \sqrt{3}p)(q - \sqrt{3}p) = -(6p^2 - 2q^2) < c.$$
Therefore, there exists \( c_2 \in \mathbb{N} \) with
\[
0 < c_2 < c
\]
such that there are infinitely many \((p, q) \in \mathbb{N}^2\) with
\[
-(6p^2 - 2q^2) = c_2.
\]
Finally take line bundles \( L_{(p, q)} \) s.t.
\[
L_{(p, q)} = pU + qV \text{ in } N^1(X)_Q.
\]
For the pairs \((p, q)\) with \( q \gg 0\), the point \((p, q, 0) \in N^1(X)_Q\) is very close to the boundary \( \sqrt{3}u = v \) of the ample cone (intersected with the plane \( \{w = 0\} \)). Therefore, it follows that there exists an \( (\mathbb{Q})\)-ample divisor \( H_{(p, q)} \) s.t.
\[
L_{(p, q)} \cdot H_{(p, q)} = 0.
\]
(Actually \( H_{(p, q)} \) gets very close inside of the ample cone to the line \( \sqrt{3}u = v \) as \( q \) becomes large.)

The rank 2 vector bundles
\[
E_{(p, q)} = L_{(p, q)} \oplus L_{(p, q)}^{-1}
\]
have the property
\[
c_1(E_{(p, q)}) = 0, c_2(E_{(p, q)}) = c_2
\]
and that \( E_{(p, q)} \) is \( H_{(p, q)} \)-slope-semistable for \( H_{(p, q)} \in Amp(X)_Q - \{0\} \). Actually \( E_{(p, q)} \) is \( H_{(p, q)} \)-Gieseker-semistable since for any \( n \) we have
\[
\frac{1}{2} \chi(E_{(p, q)} \otimes H_{(p, q)}^n) = \chi(L_{(p, q)} \otimes H_{(p, q)}^n) = \chi(L_{(p, q)}^{-1} \otimes H_{(p, q)}^n)
\]
\[
= \frac{1}{2} n^2 H^2_{(p, q)} + \frac{1}{2} c_2.
\]

It is easy to see for any fixed ample line bundle \( A \) on \( X \), the intersection number \( A \cdot L_{(p, q)} \) is unbounded as \( q \) becomes larger, thus the sheaf \( E_{(p, q)} \) which contains \( L_{(p, q)} \) as a subsheaf is unbounded.

We remark also that what is essential in the example above is the irrationality of the slope \( \sqrt{3} \) and not the circular shape of the cone.

(2) For some restricted classes of surfaces, the global finiteness does hold. We claim the global finiteness for the class of surfaces whose nef cones are rational and polyhedral (e.g. Del Pezzo surfaces). To see this, we prove a variant of Lemma 1.5.

**Lemma 1.5’.** Let \( X \) be a nonsingular projective surface whose nef cone is rational and polyhedral, i.e., there exists a finite number of nef line bundles \( M_1, M_2, \ldots, M_m \in \text{Pic}(X) \) such that
\[
Amp(X)_\mathbb{R} = \mathbb{R}_{\geq 0}M_1 + \mathbb{R}_{\geq 0}M_2 + \cdots + \mathbb{R}_{\geq 0}M_m.
\]
Fix $N \in \mathbb{N}$ and $r \in \mathbb{N}$. Then the number of lattice points
\[ \sharp \left\{ x \in \text{Pic}(X) \otimes \mathbb{Q} : -x^2 \leq N, x \cdot H = 0 \text{ for some } H \in \text{Amp}(X)_{\mathbb{Q}} - \{0\} \right\} \]
is finite.

**Proof of Lemma 1.5’.**

To prove Lemma 1.5’ we cannot use the argument of the proof of Lemma 1.5 directly. But essentially the only thing that may give rise to the infinite number of lattice points is the existence of some “irrational” edge on the boundary of the ample cone as in the previous counterexample, which is excluded by the assumption. Since after the application of Hodge index Theorem the proof is just an analysis of lattice points bounded by some quadratic hypersurface (and since we will not use Lemma 1.5’ or Theorem 1.3’ in the arguments of later chapters), we leave the details of the proof to the reader.

Once we have the finiteness of the lattice points above, the rest of the argument goes through without change. Thus we have

**Theorem 1.3’:** Let $X$ be a nonsingular projective surface whose nef cone is rational and polyhedral (e.g., a Del Pezzo surface, a relatively minimal ruled surface over a smooth curve (cf. [CKM88]) or a special kind of $K3$ surface (cf. [Kovács94])). Fix a numerical class $c_1 \in N_1(X)_{\mathbb{Q}}$, an integer $c_2 \in \mathbb{Z}$ and another integer $r \in \mathbb{N}$. Then the set of all coherent torsion-free sheaves $E$ of $rk(E) = r$, $c_1(E) = c_1$ and $c_2(E) = c_2$ s.t. $E$ is $H$-slope-semistable for some $H \in \text{Amp}(X)_{\mathbb{Q}} - \{0\}$, is bounded.

We also have the boundedness for Gieseker-semistable sheaves.

**Corollary 1.9.** Under the same assumption as in Theorem 1.3 (resp. Theorem 1.3’), the set of all coherent torsion free sheaves $E$ of $rk(E) = r$, $c_1(E) = c_1$ and $c_2(E) = c_2$, $H$-Gieseker-semistable for some $H \in \Delta - \{0\}$ (resp. $H \in \text{Amp}(X)_{\mathbb{Q}} - \{0\}$), is bounded. That is to say, $\cup_{H \in \Delta - \{0\}} S(r, c_1, c_2, H)$ (resp. $\cup_{H \in \text{Amp}(X)_{\mathbb{Q}} - \{0\}} S(r, c_1, c_2, H)$) is bounded.

**Proof of Corollary 1.9.**

If $E$ is $H$-Gieseker-semistable, then $E$ is $H$-slope-semistable. That is to say, the set of $H$-Gieseker-semistable sheaves is a subset of $H$-slope-semistable sheaves. Now the assertion of the corollary is immediate from that of Theorem 1.3 (resp. Theorem 1.3’).

We will use the boundedness results of this section in §3 to determine the stratification of $\Delta - \{0\}$ which describes the change of the set $\mu(r, c_1, c_2, H)$ as $H$ varies in $\Delta - \{0\}$ and that of the set $S(r, c_1, c_2, H)$. 

§2. Key GIT Lemma after Simpson

In §2, we focus our attention on the key GIT lemma, modifying the ideas of [Simpson92], which gives us the main machinery needed to reduce the problem of factorizing the (birational) transformations among various moduli spaces to the Mumford-Thaddeus principle. We use the same notation as in §1. We shall present the entire argument; while the reader may find this a bit repetitive of [Simpson92], we believe the necessary modifications too subtle and too many to just leave to reference.
Step 1: Embedding into a Grothendieck Quot scheme.

Let $\cup_{H \in \Delta - \{0\}} S(r, c_1, c_2, H)$ be the set of all coherent torsion-free sheaves of rank $r$, fixed first and second Chern classes $c_1$ and $c_2$. $H$-Gieseker-semistable for SOME $H \in \Delta - \{0\}$. Then the result of §1 tells us that the set $\cup_{H \in \Delta - \{0\}} S(r, c_1, c_2, H)$ is bounded, i.e., there exists a scheme $S$ of finite type over $k$ and coherent sheaf $F$ on $S \times X$ flat over $S$ such that for any $E \in \cup_{H \in \Delta - \{0\}} S(r, c_1, c_2, H)$ there is $s \in S$ with $E \cong F_s$. Let $\pi_1 : S \times X \to S$ and $\pi_2 : S \times X \to X$ be the first and second projections respectively.

Let $A$ be an ample divisor on $X$ with $A \in \Delta - \{0\}$.

Then there exists $a_0 \in \mathbb{N}$ such that for all $a \geq a_0$

$$\pi_1^* \pi_2^* (A^a) \to F \otimes \pi_2^* A^a$$

is surjective, and

$$R^i \pi_2^* (A^a) = 0 \text{ for } i > 0.$$

Furthermore, for all $s \in S$ we have that

$$R^i \pi_1^* (F \otimes \pi_2^* A^a) \otimes k(s) \to H^i (X, F \otimes A^a)$$

is an isomorphism for $i \geq 0$ and that $\pi_1^* (F \otimes \pi_2^* A^a)$ is locally free of rank $l = \chi_{r,c_1,c_2}(A^a)$ (cf. [Hartshorne, Cohomology and Base Change, Theorem 12.11]), where we introduce the notation

$$\chi_{r,c_1,c_2}(A^a) = \frac{r}{2} A^2 a^2 + (c_1 \cdot A - \frac{r}{2} K_X \cdot A) a$$

$$+ \frac{1}{2} (c_1^2 - 2c_2 - c_1 \cdot K_X) + r \chi(\mathcal{O}_X).$$

We may assume $S$ is a finite disjoint union of affine schemes $S_\alpha$ such that

$$\pi_1^* (F \otimes \pi_2^* A^a)|_{S_\alpha} \cong \mathcal{O}^\text{fl}_{S_\alpha}$$

and thus we have a surjection

$$\mathcal{O}^\text{fl}_{S \times X} \cong \pi_1^* \pi_2^* (F \otimes \pi_2^* A^a) \to F \otimes \pi_2^* A^a.$$

Therefore, we obtain a morphism

$$\phi : S \to Quot(\mathcal{O}^\text{fl}_{X}/\chi_{r,c_1',c_2'})$$

such that for all $s \in S$ we have

$$(F \otimes \pi_2^* A^a)_s \cong (Univ)_{\phi(s)},$$

where $Quot(\mathcal{O}^\text{fl}_{X}/\chi_{r,c_1',c_2'})$ (Later we will use the abbreviation $Quot$) is the Grothendieck’s Quot scheme parametrizing the quotients of $\mathcal{O}^\text{fl}_{X}$ whose Hilbert polynomial with respect to an ample line bundle $H$ is given by $\chi_{r,c_1',c_2'}(H^m)$, and

$$c_1' = c_1 + rac_1(A)$$

$$c_2' = c_2 + (r - 1)ac_1 \cdot c_1(A) + \frac{r(r - 1)}{2} a^2 c_1(A)^2.$$
Step 2: Inducing the linearized polarizations onto the Quot scheme.

We take an ample line bundle $H$ on $X$.

Let $\pi_1 : Quot \times X \to Quot$ and $\pi_2 : Quot \times X \to X$ be the first and second projections. We have the universal quotient sheaf $Univ$ over $Quot \times X$

$$0 \to \ker \to \mathcal{O}^\oplus_{Quot \times X} \to Univ \to 0.$$ 

There exists $M_H \in \mathbb{N}$ such that for all $m \geq M_H$ we have

$$R^i \pi_1_*(Univ \otimes \pi_2^*H^m) = 0$$ $$R^i \pi_1_*(\mathcal{O}^\oplus_{Quot \times X} \otimes \pi_2^*H^m) = 0$$ $$R^i \pi_1_*(\ker \otimes \pi_2^*H^m) = 0$$

for $i > 0$.

and for all closed points $q \in Quot$

$$R^i \pi_1_*(Univ \otimes \pi_2^*H^m) \otimes k(q) \to H^i(X, (Univ)_q \otimes H^m)$$ $$R^i \pi_1_*(\mathcal{O}^\oplus_{Quot \times X} \otimes \pi_2^*H^m) \otimes k(q) \to H^i(X, \mathcal{O}_X^\oplus \otimes H^m)$$ $$R^i \pi_1_*(\ker \otimes \pi_2^*H^m) \otimes k(q) \to H^i(X, (\ker)_q \otimes H^m)$$

are all isomorphisms for $i \geq 0$.

Furthermore,

$$\pi_1_*(Univ \otimes \pi_2^*H^m)$$

is locally free of rank $R_m = \chi_{r,c_1,c_2}(H^m),$

$$\pi_1_*(\mathcal{O}^\oplus_{Quot \times X} \otimes \pi_2^*H^m) \cong \mathcal{O}^\oplus_{Quot} \otimes_k H^0(X, H^m)$$

is locally free and so is

$$\pi_1_*(\ker \otimes \pi_2^*H^m).$$

Therefore we have the surjection

$$\mathcal{O}^\oplus_{Quot} \otimes_k H^0(X, H^m) \cong \pi_1_*(\mathcal{O}^\oplus_{Quot \times X} \otimes \pi_2^*H^m) \to \pi_1_*(Univ \otimes \pi_2^*H^m) \to 0$$

with $\pi_1_*(Univ \otimes \pi_2^*H^m)$ being locally free, which induces a morphism from the Quot scheme into the Grassmannian of the $R_m$-dimensional quotients of $k^\oplus \otimes_k H^0(X, H^m)$

$$\phi_m : Quot \to \text{Grass}(k^\oplus \otimes_k H^0(X, H^m), R_m).$$

By taking $M_H$ sufficiently large, we may assume that $\phi_m$ is an embedding.

The algebraic group $SL(l)$ naturally acts both on $Quot$ and $\text{Grass}(k^\oplus \otimes_k H^0(X, H^m), R_m)$, and $\phi_m$ is $SL(l)$-linear ($SL(l)$-equivariant). There is a natural $SL(l)$-linear embedding of $\text{Grass}(k^\oplus \otimes_k H^0(X, H^m), R_m)$ by the Plücker embedding, which gives an $SL(l)$-linearization on the ample line bundle on $Quot$

$$\text{det}\{\pi_1_*(Univ \otimes \pi_2^*H^m)\}.$$ 

Thus Mumford’s notion of semistable (stable) points w.r.t. this linearized ample bundle is defined on $Quot$.

The Hilbert-Mumford numerical criterion for semistability (stability) for the Grassmannian $\text{Grass}(V \otimes W, R)$ of the $R$-dimensional quotients of the vector space $V \otimes W$ under the action of $SL(V)$ with the linearization induced by the Plücker embedding is given by the following lemma.
Lemma 2.1. A point \( p : V \otimes W \rightarrow U \rightarrow 0 \) in \( \text{Grass}(V \otimes W, R) \) is semistable (resp. stable) for the action of \( \text{SL}(V) \) and the linearization induced by the Plücker embedding if and only if for all nonzero proper subspaces \( 0 \neq L \subset V \) we have 
\[ p(L \otimes W) \neq 0 \quad \text{and} \quad \frac{\dim L}{\dim p(L \otimes W)} \leq \frac{\dim V}{\dim U} \quad (\text{resp.} <) \]

See [Simpson92, Proposition 1.14] and for the proof [MF82, Proposition 4.3].

[Simpson92, Lemma 1.15] also gives a criterion for a point \( q \in \text{Quot} \) to be semistable.

Lemma 2.2. Given the situation as in Step 1 and Step 2 above. Then there exists \( M_H \in \mathbb{N} \) such that for all \( m \geq M_H \) the following holds: Suppose \( q : O_X^\oplus \rightarrow E \otimes A^a \rightarrow 0 \) is a point in \( \text{Quot} \) such that 
\[ k^\oplus \cong H^0(O_X^\oplus) \rightarrow H^0(E \otimes A^a) \]
is an isomorphism, and that 
\[ \frac{h^0(F_L \otimes A^a)}{\chi(F_L \otimes A^a \otimes H^m)} \leq \frac{h^0(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)} = \frac{1}{\chi_{r,c_1,c_2}(H^m)} \quad (\text{resp.} <) \]
for all subsheaves \( F_L \subset E \) where \( F_L \otimes A^a \) is generated by a nonzero (resp. nonzero proper) linear subspace \( L \) of \( H^0(E \otimes A^a) \). Then \( q \) is semistable (resp. stable) w.r.t. the action of \( \text{SL}(V) \) and the linearization induced from the Plücker embedding of \( \text{Grass}(k^\oplus \otimes_k H^0(X,H^m), R_m) \).

Proof of Lemma 2.2.

For all nonzero linear subspaces \( 0 \neq L \subset V = k^\oplus = H^0(O_X^\oplus) \cong H^0(E \otimes A^a) \), let \( F_L \) be the subsheaf of \( E \) such that \( F_L \otimes A^a \) is generated by \( L \). Note 
\[ \dim L \leq h^0(F_L \otimes A^a). \]

\( \text{Quot} \) is embedded into \( \text{Grass}(k^\oplus \otimes_k H^0(X,H^m), R_m) = \text{Grass}(V \otimes W, R) \) (if we take \( M_H \) large enough), which in turn is embedded into a projective space by the Plücker embedding.

Remark that once \( a \) is fixed the family of subsheaves \( F_L \otimes A^a \) of \( E \otimes A^a = (\text{Univ})_q \) generated by a linear subspace \( L \) as \( q \) varies among all points in \( \text{Quot} \) and \( L \) varies among all subspaces of \( H^0(O_X^\oplus) \) is bounded; similarly, the family of such kernels \( K_L \) of the exact sequences 
\[ 0 \rightarrow K_L \rightarrow L \otimes_k O_X \rightarrow F_L \otimes A^a \rightarrow 0 \]
is bounded.

Now take \( M_H \in \mathbb{N} \) so that for all \( m \geq M_H \) we have 
\[ h^0(F_L \otimes A^a) \leq h^0(F_L \otimes A^a \otimes H^m) = \chi(F_L \otimes A^a \otimes H^m) \]
and 
\[ h^1(K_L \otimes H^m) = 0 \]
for all such $F_L$ and $K_L$. Thus from the exact sequence

$$0 \to K_L \otimes H^m \to L \otimes_k H^m \to F_L \otimes A^a \otimes H^m \to 0$$

we have

$$L \otimes_k H^0(X, H^m) \to H^0(F_L \otimes A^a \otimes H^m) \to H^1(K_L \otimes H^m) = 0,$$

which implies

$$\dim p(L \otimes W) = h^0(F_L \otimes A^a \otimes H^m) = \chi(F_L \otimes A^a \otimes H^m).$$

Therefore, we conclude

$$\frac{\dim L}{\dim p(L \otimes W)} \leq \frac{h^0(F_L \otimes A^a)}{\dim p(L \otimes W)} = \frac{h^0(F_L \otimes A^a)}{\chi(F_L \otimes A^a \otimes H^m)} \leq \frac{h^0(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)} \leq \frac{\dim V}{\dim U}.$$ (resp. If $L$ is a proper subspace, then either $F_L = E$ and the first inequality is strict $<$, or $F_L$ is a proper subsheaf of $E$ and the second inequality is strict $<$ by assumption.)

Thus by Lemma 2.1 $q$ is semistable (resp. stable).

The following lemma, a variant of [Simpson92,Lemma1.16], which characterizes a semistable point $q : \mathcal{O}_X^{\oplus l} \to E \otimes A^a \to 0$ plays a crucial role with the previous lemma in establishing the key GIT lemma of this section.

**Lemma 2.3.** For a fixed $a \in \mathbb{N}(\geq a_0)$ and an ample line bundle $H$ on $X$, there exists $M_H \in \mathbb{N}$ s.t. for all $m \geq M_H$, the following holds: If a point $q : \mathcal{O}_X^{\oplus l} \to E \otimes A^a \to 0$ is semistable with respect to the action of $SL(l)$ and the linearization induced from the Plücker embedding of Grass($k^{\oplus l} \otimes_k H^0(X, H^m), R_m$), then the natural homomorphism

$$k^{\oplus l} = H^0(\mathcal{O}_X^{\oplus l}) \to H^0(E \otimes A^a)$$

is injective, and for any nonzero quotient

$$E \otimes A^a \to G \otimes A^a \to 0$$

with $rk(G) > 0$, we have

$$\frac{h^0(G \otimes A^a)}{rk(G)} \geq \frac{\chi(E \otimes A^a)}{rk(E)} = \frac{l}{r}.$$
Moreover, suppose that \( G \) over \( T \times X \) is a bounded family of coherent sheaves on \( X \) (independent of \( a \)). Then there exists \( a_2 \in \mathbb{N} \geq a_0 \) such that for all \( a \geq a_2 \) and an ample line bundle \( H \) on \( X \), there exists \( M_H \in \mathbb{N} \) such that for all \( m \geq M_H \) the following holds: If a point 
\[
q: \mathcal{O}_X^{\oplus l} \to E \otimes A^a \to 0
\]
is semistable with respect to the action of \( SL(l) \) and the linearization induced from the Plücker embedding of Grass\((k^\oplus l) \otimes_k H^0(X,H^m), R_m\) and if the natural homomorphism 
\[
k^\oplus l = H^0(\mathcal{O}_X^{\oplus l}) \to H^0(E \otimes A^a)
\]
is an isomorphism, then for any nonzero quotient 
\[
E \otimes A^a \to G \otimes A^a \to 0
\]
with \( G \cong G_t \) for some \( t \in T \), we have
\[
\frac{h^0(G \otimes A^a)}{\chi(G \otimes A^a \otimes H^m)} \geq \frac{\chi(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)}.
\]

**Proof of Lemma 2.3.**

Suppose a point \( q: \mathcal{O}_X^{\oplus l} \to E \otimes A^a \to 0 \) is semistable. Then Lemma 2.1 tells us that for any nonzero linear subspace \( 0 \neq L \subset V = H^0(\mathcal{O}_X^{\oplus l}) \) \( p \) being the map given by  
\[
p : L \otimes W(= H^0(H^m)) \to U(= H^0(E \otimes A^a \otimes H^m)) \]
\[
\downarrow \quad \nearrow \quad \downarrow \quad \nearrow
\]
\[
H^0(E \otimes A^a) \otimes H^0(H^m)
\]
we have \( p(L \otimes W) \neq 0 \). Thus the homomorphism \( L \to H^0(E \otimes A^a) \) is nonzero. Since \( L \) is an arbitrary nonzero linear subspace, we have the desired injectivity of 
\[
k^\oplus l = H^0(\mathcal{O}_X^{\oplus l}) \to H^0(E \otimes A^a).
\]

Suppose 
\[
E \otimes A^a \to G \otimes A^a \to 0
\]
is a nonzero quotient with \( rk(G) > 0 \) and 
\[
\frac{h^0(G \otimes A^a)}{rk(G)} < \frac{\chi(E \otimes A^a)}{rk(E)} = \frac{l}{r}.
\]
Let \( L(\subset V) \) be the kernel of the homomorphism  
\[
V = H^0(\mathcal{O}_X^{\oplus l})(\to H^0(E \otimes A^a)) \to H^0(G \otimes A^a),
\]
which is nonzero since we have the strict inequality above and \( rk(G) \leq rk(E) \) and hence 
\[
h^0(G \otimes A^a) < l = h^0(\mathcal{O}_X^{\oplus l}).
\]
Let $F_L$ be the subsheaf of $E$ such that $F_L \otimes A^a$ is generated by (the image of) $L$ in $H^0(E \otimes A^a)$.

Note that from the exact sequence

$$0 \to K_G \otimes A^a \to E \otimes A^a \to G \otimes A^a \to 0$$

and the inclusion

$$F_L \otimes A^a \hookrightarrow K_G \otimes A^a,$$

it follows

$$(*) \quad rk(F_L) + rk(G) \leq rk(E).$$

Moreover, we have

$$(** \quad \dim L \geq l - h^0(G \otimes A^a).)$$

Therefore, the strict inequality of the assumption together with $(*)$ and $(**)$ implies

$$\dim L \frac{rk(F_L)}{rk(E)} \geq l.$$  \hfill (♥)

Thus we obtain

$$\frac{\dim L}{\chi(F_L \otimes A^a \otimes H^m)} > \frac{l}{\chi(E \otimes A^a \otimes H^m)} \hfill (\heartsuit)$$

for $m >> 0$.

Note that once $a$ is fixed, such $F_L$ (as well as $F_L \otimes A^a$) ranges over a bounded family and thus we have a finite number of possibilities for $\chi(F_L \otimes A^a \otimes H^m)$ as polynomials in $m$.

Take $M_H \in \mathbb{N}$ such that for all $m \geq M_H$ we have

$$h^0(F_L \otimes A^a \otimes H^m) = \chi(F_L \otimes A^a \otimes H^m)$$

$$h^0(E \otimes A^a \otimes H^m) = \chi(E \otimes A^a \otimes H^m),$$

$$h^1(K_L \otimes H^m) = 0$$

where

$$0 \to K_L \to L \otimes_k \mathcal{O}_X \to F_L \otimes A^a \to 0,$$

and that $(\heartsuit)$ holds if we have such $G$.

Then we have

$$\dim p(L \otimes W) = h^0(F_L \otimes A^a \otimes H^m)$$

$$= \chi(F_L \otimes A^a \otimes H^m),$$

which implies

$$\frac{\dim L}{\dim p(L \otimes W)} \geq \frac{l}{h^0(E \otimes A^a \otimes H^m)}.$$  \hfill (♥)

This contradicts the criterion for semistability of $q$ given in Lemma 2.1.

Now we proceed to the proof of the second half of the lemma.
Note first that since the family of such $G$ is bounded by assumption, so is the family of kernels $K_G$ of the sequence

$$0 \to K_G \to E \to G \to 0.$$ 

Therefore, there exists $a_2 \in \mathbb{N}(\geq a_0)$ such that for all $a \geq a_2$ and exact sequences

$$0 \to K_G \otimes A^a \to E \otimes A^a \to G \otimes A^a \to 0,$$

we have the exactness of

$$0 \to H^0(K_G \otimes A^a) \to H^0(E \otimes A^a) \to H^0(G \otimes A^a) \to 0$$

by the virtue of

$$H^1(K_G \otimes A^a) = 0,$$

and that $K_G \otimes A^a$ is generated by the global sections $H^0(K_G \otimes A^a)$.

Since we assume $k^{\oplus l} \cong H^0(K_G \otimes A^a)$ is an isomorphism, we have

$$L = H^0(K_G \otimes A^a)$$

and $F_L = K_G$ in the notation of the first half of our proof.

Therefore, we have

$$(*)' \chi(F_L \otimes A^a \otimes H^m) + \chi(G \otimes A^a \otimes H^m) = \chi(E \otimes A^a \otimes H^m)$$

and

$$(**') \dim L = l - h^0(G \otimes A^a).$$

Take $M_H \in \mathbb{N}$ so that for all $m \geq M_H$ we have

$$\chi(F_L \otimes A^a \otimes H^m) = h^0(F_L \otimes A^a \otimes H^m) > 0$$

$$\chi(G \otimes A^a \otimes H^m) = h^0(G \otimes A^a \otimes H^m) > 0$$

$$\chi(E \otimes A^a \otimes H^m) = h^0(E \otimes A^a \otimes H^m) > 0,$$

and that

$$h^1(K_L \otimes H^m) = 0$$

where

$$0 \to K_L \to L \otimes_k \mathcal{O}_X \to F_L \otimes A^a \to 0.$$

Suppose

$$\frac{h^0(G \otimes A^a)}{\chi(G \otimes A^a \otimes H^m)} < \frac{\chi(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)} = \frac{l}{\chi(E \otimes A^a \otimes H^m)}.$$ 

This together with $(*)'$ and $(**')$ implies

$$(\heartsuit) \quad \frac{\dim L}{\chi(F_L \otimes A^a \otimes H^m)} > \frac{\chi(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)}.$$
Now
\[ h^1(K_L \otimes H^m) = 0 \]
implies
\[ \dim p(L \otimes W) = h^0(F_L \otimes A^a \otimes H^m) \]
\[ = \chi(F_L \otimes A^a \otimes H^m). \]
Thus (\heartsuit) leads to
\[ \frac{\dim L}{\dim p(L \otimes W)} > \frac{l}{h^0(E \otimes A^a \otimes H^m)}, \]
which again contradicts the criterion of semistability of \( q \) given in Lemma 2.1.

Ideally and quite naively one would expect after having embedded all the sheaves in
\[ \cup_{H \in \Delta-\{0\}} S(r, c_1, c_2, H) \]
into the Quot scheme Quot by taking a sufficiently high multiple of an ample line bundle \( A \in \Delta - \{0\} \), the semistable points with respect to the \( SL(l) \)-action and the linearization induced from the Plücker embedding twisting by a high multiple of an ample line bundle \( H \in \text{Amp}(X)_0 - \{0\} \) would correspond to the \( H \)-Gieseker-semistable sheaves. In fact in the classical setting with the fixed polarization \( A = H \) this is indeed the case. But in our setting with changing polarizations, the situation becomes more subtle. Steps 1 and 2 are not independent but are actually quite intertwined, and the naive expectation as above turns out to be false. But the study of the change of the sets of the semistable points as we change the second polarization \( H \) gives the vital information to analyze the transformation among the various moduli spaces.

**Key GIT Lemma 2.4.** Let \( A \) be a **VERY** ample line bundle with \( A \in \Delta - \{0\} \). Then there exists \( a_A \in \mathbb{N} \) such that for all \( a \geq a_A \) the following holds:

(i) All the sheaves in \( \cup_{H \in \Delta-\{0\}} S(r, c_1, c_2, H) \) can be embedded into a Quot scheme by taking a multiple \( A^a \) of \( A \), i.e., for all \( E \in \cup_{H \in \Delta-\{0\}} S(r, c_1, c_2, H) \) there exists a point
\[ q \in \text{Quot}(\mathcal{O}_X^{\oplus l}/\chi_{r,c_1,c_2'}) \]
s.t.
\[ E \otimes A^a \cong (\text{Univ})_q \]
where
\[ l = \chi_{r,c_1,c_2}(A^{a}) \]
\[ c_1' = c_1 + rac_1(A) \]
\[ c_2' = c_2 + (r - 1)c_1 \cdot ac_1(A) + \frac{r(r - 1)}{2} \cdot a^2 c_1(A)^2. \]

(ii) We denote by \( Q \) the closure in Quot\((\mathcal{O}_X^{\oplus l}/\chi_{r,c_1,c_2'}) \) of the set of points \( q \) such that \((\text{Univ})_q\) is torsion free with \( c_1((\text{Univ})_q) = c_1' \) and \( c_2((\text{Univ})_q) = c_2' \). Note that \( Q \) is \( SL(l) \)-invariant.
For any ample line bundle \( H \in \text{Amp}(X)_{\mathbb{Q}} - \{0\} \), there exists \( M_H \in \mathbb{N} \) such that for all \( m \geq M_H \) the following equivalence holds:

A point \( q \in Q \) is semistable with respect to the action of \( \text{SL}(l) \) and the linearization induced from the Plücker embedding of \( \text{Grass}(k^{\oplus l} \otimes H^0(X, H^m), R_m) \)

if and only if

\((\text{Univ})_q = E \otimes A^a\) is a coherent torsion free sheaf of rank \( \text{rk}(E) = r, c_1(E) = c_1, c_2(E) = c_2 \) such that \( E \) is \( A \)-Gieseker-semistable and that for all subsheaves \( F \subset E \) having the same averaged Euler characteristics \( p(F, A, a) = p(E, A, a) \) (as polynomials in \( a \)) we have

\[
\frac{c_1(F)}{\text{rk}(F)} \cdot H \geq \frac{c_1(E)}{\text{rk}(E)} \cdot H,
\]

i.e.,

\( E \in S(r, c_1, c_2, A)_H \),

and furthermore the natural homomorphism

\( k^{\oplus l} = H^0(\mathcal{O}_X^{\oplus l}) \to H^0((\text{Univ})_q) \)

is an isomorphism.

**Proof of Claim 2.5.**

First note that by taking the saturation of \( F \) in \( E \) we may assume that the quotient \( E/F \) is also torsion free.
Take an $A$-slope-Harder-Narasimhan filtration of $F$

$$0 = F_0 \subset F_1 \subset \cdots \subset F_e = F.$$  

Set $Q_i = F_i/F_{i-1}$. Then we have

$$h^0(F \otimes A^a) \leq \Sigma_i h^0(Q_i \otimes A^a),$$

and by the definition of an $A$-slope-Harder-Narasimhan filtration we have for all $i = 1, 2, \ldots, e$

$$\mu_A(Q_i) = \frac{c_1(Q_i)}{rk(Q_i)} \cdot A \leq \mu_A(Q_1) \leq \frac{c_1(E)}{rk(E)} \cdot A,$$

noting the second inequality comes from the fact that $E$ being $A$-Gieseker-semistable implies $E$ being $A$-slope-semistable. We also have

$$\Sigma_i rk(Q_i) = rk(F).$$

Here we quote [Simpson92, Corollary1.7] without proof.

**Lemma 2.6.** There exists $b \in \mathbb{N}$ depending only on $r, X$ and $A$ such that for all $A$-slope-semistable coherent torsion free sheaves $F$ of $rk(F) \leq r$ on $X$

$$h^0(F) \leq \frac{rk(F)}{2} \deg_A X \left\{ \frac{\mu_A(F)}{\deg_A X} + b \right\}^2.$$

(Note also that if $\mu_A(F) < 0$ then $h^0(F) = 0$.)

Going back to the proof of Claim 2.5, the lemma above implies

$$h^0(Q_i \otimes A^a) \leq \frac{rk(Q_i)}{2} \deg_A X \left\{ \frac{\mu_A(Q_i \otimes A^a)}{\deg_A X} + b \right\}^2 = \frac{rk(Q_i)}{2} \deg_A X \left\{ \frac{\mu_A(Q_i)}{\deg_A X} + a + b \right\}^2.$$

Thus we have

$$h^0(F \otimes A^a) \leq \Sigma_i h^0(Q_i \otimes A^a) \leq \frac{rk(F)}{2} \deg_A X \left\{ \frac{\mu_A(E)}{\deg_A X} + a + b \right\}^2 + \frac{1}{2} \deg_A X \left\{ \frac{\mu_A(Q_e)}{\deg_A X} + a + b \right\}^2.$$

We remark that for all $\alpha \in \mathbb{Q}$ there exists $\beta \in \mathbb{Q}$ and $a_\beta \in \mathbb{N}$ s.t. if

$$\frac{\mu_A(Q_e)}{\deg_A X} \leq \frac{\mu_A(E)}{\deg_A X} - \beta$$

then for $a \geq a_\beta$

$$\frac{rk(F)}{2} \deg_A X \left\{ \frac{\mu_A(E)}{\deg_A X} + a + b \right\}^2 + \frac{1}{2} \deg_A X \left\{ \frac{\mu_A(Q_e)}{\deg_A X} + a + b \right\}^2 \leq \frac{rk(F)}{2} \deg_A X \left\{ a - \alpha \right\}^2.$$
For this we only have to take $\beta$ so that
\[
\frac{rk(F) - 1}{2} \deg_A X \{2(\frac{\mu_A(E)}{\deg_A X} + b)\} + \frac{1}{2} \deg_A X \{2(\frac{\mu_A(Q_e)}{\deg_A X} - \beta)\}
\leq \frac{rk(F)}{2} \deg_A X \{2(-\alpha)\}
\]
and then the existence of such $\alpha_\beta$ is clear.

On the other hand, we can choose $\alpha$ and $a_\alpha \in \mathbb{N}$ so that for all $a \geq a_\alpha$ we have
\[
\frac{1}{2} \deg_A X (a - \alpha)^2 < \frac{1}{rk(E)} \chi(E \otimes A^a).
\]
Let $\beta$ be the number as above. Then for $F \subset E$ with
\[
\frac{\mu_A(Q_e)}{\deg_A X} \leq \frac{\mu_A(E)}{\deg_A X} - \beta
\]
we have for $a \geq a_\alpha, a_\beta$
\[
\frac{h^0(F \otimes A^a)}{rk(F)} < \frac{\chi(E \otimes A^a)}{rk(E)}.
\]
Moreover, since $S(r,c_1,c_2,A)$ is bounded, the set of coherent sheaves $F$ such that $F \subset E$ for some $E \in S(r,c_1,c_2,A)$ with $E/F$ being torsion free and that
\[
\frac{\mu_A(Q_e)}{\deg_A X} > \frac{\mu_A(E)}{\deg_A X} - \beta
\]
is bounded (cf.
Lemma 2.8). Therefore, we may choose $a_\gamma$ so that for all $a \geq a_\gamma$ we have
\[
h^0(F \otimes A^a) = \chi(F \otimes A^a).
\]
Furthermore, the set of the Hilbert polynomials for all such $F$ is finite, so we may choose $a_\delta$ so that for all $a \geq a_\delta$ we have
\[
\frac{\chi(F \otimes A^a)}{rk(F)} \leq \frac{\chi(E \otimes A^a)}{rk(E)}
\]
and that the equality holds iff
\[
\frac{\chi(F \otimes A^a \otimes A^m)}{rk(F)} = \frac{\chi(E \otimes A^a \otimes A^m)}{rk(E)} \quad \text{for all } m \in \mathbb{Z},
\]
i.e., iff
\[
p(F, A, a) = p(E, A, a)
\]
as polynomials in $a$.
Take $a_1 = \max\{a_0, a_\alpha, a_\beta, a_\gamma, a_\delta\}$.
This completes the proof of Claim 2.5.

Now we go to the proof of the “if” part of (ii) in the Key GIT Lemma 2.4.

First we choose $a_1$ as in Claim 2.5 and take $a \geq a_1$. 

We take an ample line bundle $H \in \text{Amp}(X)_\mathbb{Q}$.

Suppose that a point 

$q : \mathcal{O}_X^\oplus \to (\text{Univ})_q \to 0$

has the property that $(\text{Univ})_q = E \otimes A^a$ is a coherent torsion free sheaf of rank $rk(E) = r$, $c_1(E) = c_1$, $c_2(E) = c_2$ such that $E$ is $A$-Gieseker-semistable and that for all subsheaves $F \subset E$ having the same averaged Euler characteristics $p(F, A, a) = p(E, A, a)$ (as polynomials in $a$) we have

$$\frac{c_1(F)}{rk(F)} \cdot H \geq \frac{c_1(E)}{rk(E)} \cdot H,$$

i.e.,

$$E \in S(r, c_1, c_2, A),$$

and furthermore that

$$k^\oplus = H^0(\mathcal{O}_X^\oplus) \to H^0((\text{Univ})_q)$$

is an isomorphism.

By virtue of Lemma 2.2 we only have to show there exists $M_H \in \mathbb{N}$ (bigger than the old $M_H$ in Lemma 2.2) such that for all $m \geq M_H$ we have

$$\frac{h^0(F \otimes A^a)}{\chi(F \otimes A^a \otimes H^m)} \leq \frac{h^0(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)}$$

for all nonzero coherent subsheaves $F \subset E$ where $F \otimes A^a$ is generated by some linear subspace of $H^0(E \otimes A^a)$.

Once $a$ is fixed, the set of all such $F$ that $F \subset E$ for some $E \in S(r, c_1, c_2, A)$ and that $F \otimes A^a$ is generated by some linear subspace of $H^0(E \otimes A^a)$ is bounded. Thus the set of the polynomials (in terms of $m$) $\chi(F \otimes A^a \otimes H^m)$ is finite. These polynomials all have the leading term

$$\frac{rk(F)}{2} \text{deg}_H X \cdot m^2,$$

so we may choose $M_H$ big enough so that for $F$ with

$$\frac{h^0(F \otimes A^a)}{rk(F)} \leq \frac{h^0(E \otimes A^a)}{rk(E)}$$

we get the desired inequality

$$\frac{h^0(F \otimes A^a)}{\chi(F \otimes A^a \otimes H^m)} \leq \frac{h^0(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)}.$$

By the previous Claim 2.5, for those $F$ with the equality

$$\frac{h^0(F \otimes A^a)}{rk(F)} = \frac{h^0(E \otimes A^a)}{rk(E)}$$
we have 
\[ \frac{\chi(F \otimes A^a \otimes A^m)}{rk(F)} = \frac{\chi(E \otimes A^a \otimes A^m)}{rk(E)} \]
for all \( m \in \mathbb{Z} \),
i.e.,
\[ p(F, A, a) = p(E, A, a) \]
as polynomials in \( a \). By assumption, for those \( F \) we have
\[ c_1(F) \cdot H \geq c_1(E) \cdot H, \]
and thus by rechoosing \( M_H \in \mathbb{N} \) big enough if necessary we have for all \( m \geq M_H \)
\[ \frac{\chi(F \otimes A^a \otimes H^m)}{rk(F)} \]
\[ = \frac{1}{2} H^2 m^2 + \left( \frac{c_1(F)}{rk(F)} \cdot H - \frac{1}{2} K_X \cdot H \right) m + p(F, A, a) \]
\[ \geq \frac{1}{2} H^2 m^2 + \left( \frac{c_1(E)}{rk(E)} \cdot H - \frac{1}{2} K_X \cdot H \right) m + p(E, A, a) \]
\[ = \frac{\chi(E \otimes A^a \otimes H^m)}{rk(E)} \]
\[ = \frac{h^0(E \otimes A^a \otimes H^m)}{rk(E)} > 0. \]
Thus again for those \( F \) we have
\[ \frac{h^0(F \otimes A^a)}{\chi(F \otimes A^a \otimes H^m)} \leq \frac{h^0(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)}. \]
Therefore, the point \( q \in Q \) is semistable with respect to the action of \( SL(l) \) and the linearization induced from the Plücker embedding of \( Grass(k^{\otimes l} \otimes H^0(X, H^m), R_m) \).
This concludes the proof of the “if” part.
Now we turn to the proof of the second half “only if” part of (ii).
Suppose that
\[ q : O_X^{\otimes l} \to (Univ)_q = E \otimes A^a \to 0 \]
is a point of \( Q \) which is semistable with respect to the action of \( SL(l) \) and the linearization induced from the Plücker embedding of \( Grass(k^{\otimes l} \otimes H^0(X, H^m), R_m) \) for \( m \geq M_H \) where \( M_H \) is the integer given in Lemma 2.3.
First we prove that \( E \) is torsion free.
Let \( T \subset E \) be the torsion of \( E \).
We quote the following lemma of [Simpson92, Lemma 1.17].

**Lemma 2.7.** Let
\[ q : O_X^{\otimes l} \to (Univ)_q = E \otimes A^a \to 0 \]
be a point of \( Q \) and \( T \subset E \) be the torsion part of \( E \). Then there exists a coherent TORSION FREE sheaf \( E' \) of rank \( rk(E') = rk(E) \) such that the Hilbert polynomials (in terms of \( a \)) are the same
\[ \chi(E' \otimes A^a) = \chi(E \otimes A^a) \]
and that we have an inclusion

\[ 0 \rightarrow E/T \rightarrow E'. \]

Take

\[ 0 \rightarrow E/T \rightarrow E' \]

as in Lemma 2.7.

Then for any quotient

\[ E' \xrightarrow{\psi} G \rightarrow 0 \text{ with } rk(G) > 0 \]

we have

\[
\frac{h^0(G \otimes A^a)}{rk(G)} \geq \frac{h^0(\psi(E/T) \otimes A^a)}{rk(\psi(E/T))} \\
\geq \frac{\chi(E \otimes A^a)}{rk(E)} \quad \text{(by Lemma 2.3)} \\
= \frac{\chi(E' \otimes A^a)}{rk(E')}.
\]

Let \( G \) be the \( A \)-slope-semistable quotient of \( E' \) with the smallest

\[
\mu_A(G) = \frac{c_1(G)}{rk(G)} \cdot A,
\]

in other words the last step quotient of an \( A \)-slope-Harder-Narasimhan filtration of \( E' \).

We apply Lemma 2.6 to conclude that there is \( b \in \mathbb{N} \) depending only on \( r, X \) and \( A \) such that

\[
\frac{\chi(E' \otimes A^a)}{rk(E')} \leq \frac{h^0(G \otimes A^a)}{rk(G)} \\
\leq \frac{1}{2} \deg_A X \left( \frac{\mu_A(G)}{\deg_A X} + a + b \right)^2.
\]

Since the polynomial in \( a \)

\[
\chi(E' \otimes A^a) = \chi_{r,c_1,c_2}(A^a)
\]

is fixed, there is a constant \( c \in \mathbb{Q} \) and \( a_c \in \mathbb{N}(a_c \geq c) \) such that for all \( a \geq a_c \) we have

\[
\frac{\chi(E' \otimes A^a)}{rk(E')} \geq \frac{1}{2} \deg_A X (a - c)^2.
\]

Thus

\[
(a - c)^2 \leq \left( \frac{\mu_A(G)}{\deg_A X} + a + b \right)^2.
\]
Since
\[ a - c \geq 0, \quad \frac{\mu_A(G \otimes A^a)}{\deg_A X} = \frac{\mu_A(G)}{\deg_A X} + a \geq 0 \]
\[ \left( \frac{h^0(G \otimes A^a)}{rk(G)} \right) \geq \chi_{r,c_1,c_2}(A^a) > 0 \]
and since \( b \in \mathbb{N} \), we have
\[ a - c \leq \frac{\mu_A(G)}{\deg_A X} + a + b, \]
which gives us
\[ \mu_A(G) \geq \deg_A X (-b - c). \]
Remark that \( b \) and \( c \) are constants taken independent of \( a \) or \( m \).

**Lemma 2.8.** Let \( L \in \mathbb{Q} \) be a fixed constant, \( P(a) \) a fixed polynomial in \( a \) and \( A \) an ample line bundle on \( X \). Then the set of all coherent torsion free sheaves \( E \) such that
\[ \chi(E \otimes A^a) = P(a) \]
and that
\[ \mu_A(G) \geq L \]
where \( G \) is the last step quotient of an \( A \)-slope-Harder-Narasimhan filtration of \( E \), is bounded.

**Proof of Lemma 2.8.**

This is [Simpson92,Theorem1.1] proved using a lemma in [Maruyama81]. Here we give a simple alternative proof in the case where \( X \) is a surface, using Bogomolov inequality.

Take the Harder-Narasimhan filtration of \( E \) with respect to \( A \)-Gieseker-stability
\[ 0 = F_0 \subset F_1 \subset \cdots \subset F_e = E. \]
(Note that the Harder-Narasimhan filtration of \( E \) with respect to \( A \)-slope-stability is obtained from the one above with respect to \( A \)-Gieseker-stability by clustering together the factors with the same averaged slope with respect to \( A \).)

Set
\[ G_i = F_i / F_{i-1}. \]
Write
\[ c_1(G_i) \equiv \frac{c_1(G_i) \cdot A}{A^2} A + \mathcal{L}_i \]
with
\[ \mathcal{L} \in \frac{1}{A^2} \text{Pic}(X) \text{ and } \mathcal{L}_i \cdot A = 0. \]
Then since
\[ \mu_A(G_1) \geq \mu_A(G_2) \geq \cdots \mu_A(G_e) = \mu_A(G) \geq L \]
is bounded from below and
\[ \text{rk}(G_1)\mu_A(G_1) + \text{rk}(G_2)\mu_A(G_2) + \cdots + \text{rk}(G_n)\mu_A(G_n) = c_1(E) \cdot A \]
is constant determined by the fixed polynomial \( \chi(E \otimes A^a) = P(a) \), there are only
finite number of possibilities for \( c_1(G_i) \cdot A \).

Now
\[ \chi(G_i \otimes A^a) = \frac{1}{2} r_i A^2 a^2 + \frac{1}{2} (2c_1(G_i) \cdot A - r_i K_X \cdot A) a \]
\[ + \frac{1}{2} \{ c_1(G_i)^2 - 2c_2(G_i) - K_X \cdot c_1(G_i) \} + r_i \chi(O_X) \]
and
\[ \chi(E \otimes A^a) = \Sigma \chi(G_i \otimes A^a). \]

By looking at the constant term, we have
the constant term of \( P(a) - r\chi(O_X) \)
\[ = \Sigma \frac{1}{2} \{ c_1(G_i)^2 - 2c_2(G_i) - K_X \cdot c_1(G_i) \} \]
\[ \leq \Sigma \frac{1}{2} \{ c_1(G_i)^2 - 2r_i - \frac{1}{2} c_1(G_i)^2 - K_X \cdot c_1(G_i) \} \text{ by Bogomolov inequality} \]
\[ = \Sigma \frac{1}{2} \{ c_1(G_i)^2 - K_X \cdot k_i + \frac{1}{r_i} \frac{(c_1(G_i) \cdot A)^2}{A^2} - \frac{(c_1(G_i) \cdot A)(K_X \cdot A)}{A^2} \}. \]

Since we have finitely many possibilities for \( c_1(G_i) \cdot A = r_i \mu_A(G_i) \) and \( r_i, K_X \cdot A \)
and \( A^2 \) are constants, and since \( \mathcal{L}_i^2 \) is quadratic in terms of \( \mathcal{L}_i \in A^\perp \) with negative
definite form, while \( K_X \cdot \mathcal{L}_i \) is linear, we conclude
\[ \mathcal{L}_i \in A^\perp \cap \frac{1}{A^2} \text{Pic}(X) \subset N^1(X) \mathbb{Q} \]
has only a finite number of possibilities. This implies \( C_1(G_i) \) has only a finite
number of possibilities, and so does \( c_2(G_i) \) again by Bogomolov inequality. Thus
by [Gieseker77, Corollary1.3] the family of \( G_i \) is bounded, and finally so is the family
of \( E \).

Applying this lemma, we conclude that the set of such sheaves \( E' \) remain in a
bounded family which is independent of \( a \) or \( m \). Thus we may retake \( a_1 \) big enough
so that for all \( a \geq a_1 \) we have
\[ h^0(E' \otimes A^a) = \chi(E' \otimes A^a) \]
and \( E' \otimes A^a \) is generated by global sections.

Now taking the quotient sheaf in Lemma 2.3 to be \( G = E/T \), we get
\[ h^0(E/T \otimes A^a) \geq \chi_{r,c_1,c_2}(A^a). \]

Since
\[ \chi(E' \otimes A^a) = h^0(E' \otimes A^a) \geq h^0(E/T \otimes A^a) = \chi_{r,c_1,c_2}(A^a), \]
we conclude that the natural inclusion
\[ H^0(E/T \otimes A^a) \hookrightarrow H^0(E' \otimes A^a) \]
is an isomorphism. But since \( E/T \hookrightarrow E' \) and since \( H^0(E/T \otimes A^a) = H^0(E' \otimes A^a) \)
generates \( E' \), we conclude \( E/T = E' \). Finally since the Hilbert polynomials of \( E \) and \( E' \) are the same, we conclude \( T = 0 \), i.e., \( E \) is torsion free. Furthermore, the equality
\[ h^0(E \otimes A^a) = h^0(E/T \otimes A^a) = h^0(E' \otimes A^a) = \chi_{r,c_1,c_2}(A^a) \]
together with the injectivity of Lemma 2.3
\[ k^{\oplus l} \hookrightarrow H^0(E \otimes A^a) \]
implies
\[ k^{\oplus l} \rightarrow H^0(E \otimes A^a) \]
is an isomorphism.

Next we prove that \( E \) is \( A \)-Gieseker-semistable. Suppose not. Then there exists a nonzero torsion free quotient
\[ E \rightarrow G \rightarrow 0 \]
with \( \text{rk}(G) > 0 \) such that
\[ \frac{\chi(G \otimes A^a)}{\text{rk}(G)} < \frac{\chi(E \otimes A^a)}{\text{rk}(E)} \quad \text{for } a >> 0. \]

Lemma 2.8 implies that the set of all such \( G \) remains in a bounded family independent of \( a \) or \( m \). Therefore, we may choose \( a_A \in \mathbb{N} \) (\( a_A \geq a_1, a_2 \) where \( a_2 \) is the one in the second half of Lemma 2.3) such that for all \( a \geq a_A \) we have
\[ \chi(G \otimes A^a) = h^0(G \otimes A^a). \]

Note also that the boundedness of all such \( G \) implies that the set of Hilbert polynomials \( \chi(G \otimes A^a \otimes H^m) \) as polynomials in \( m \) is finite, and that their leading terms are all
\[ \frac{\text{rk}(G)}{2} \deg H X m^2. \]

Thus we may choose \( M_H \) (taken bigger than the one in the second half of Lemma 2.3) such that for all \( m \geq M_H \) we have
\[ \frac{h^0(G \otimes A^a)}{\chi(G \otimes A^a \otimes H^m)} < \frac{\chi(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)}, \]
contradicting the conclusion of the second half of Lemma 2.3.
Finally let $F \subset E$ be a subsheaf of $E$ having the same averaged Euler characteristics
\[ p(F, A, a) = p(E, A, a) \text{ (as polynomials in $a$).} \]

Suppose
\[ \frac{c_1(F)}{rk(F)} \cdot H < \frac{c_1(E)}{rk(E)} \cdot H. \]

Let $G$ be the quotient
\[ 0 \to F \to E \to G \to 0. \]

Note that the conditions imply that $G$ is also a coherent torsion free sheaf which is $A$-Gieseker-semistable with
\[ p(G, A, a) = p(E, A, a) \text{ (as polynomials in $a$),} \]

and that
\[ \frac{c_1(G)}{rk(G)} \cdot H > \frac{c_1(E)}{rk(E)} \cdot H. \]

Then for $m \in \mathbb{N}$ we have
\[
\begin{align*}
p(G \otimes A^a, H, m) &= \frac{1}{2} H^2 m^2 + \left( \frac{c_1(G)}{rk(G)} \cdot H - \frac{1}{2} K_X \cdot H \right) m + p(G, A, a) \\
&> \frac{1}{2} H^2 m^2 + \left( \frac{c_1(E)}{rk(E)} \cdot H - \frac{1}{2} K_X \cdot H \right) m + p(E, A, a) \\
&= p(E \otimes A^a, H, m).
\end{align*}
\]

Since the set of all such $G$ is bounded independent of $a$, we may choose $a_A$ so that for all $a \geq a_A$
\[
\frac{h^0(G \otimes A^a)}{rk(G)} = p(G, A, a) = p(E, A, a) = \frac{h^0(E \otimes A^a)}{rk(E)} > 0.
\]

Again since the set of all such $G$ is bounded, we may assume that for $m \geq M_H$ we have
\[
0 < \frac{h^0(G \otimes A^a \otimes H^m)}{rk(G)} = p(G \otimes A^a, H, m) > p(E \otimes A^a, H, m).
\]

These imply
\[
\frac{h^0(G \otimes A^a)}{\chi(G \otimes A^a \otimes H^m)} = \frac{p(G, A, a)}{p(G \otimes A^a, H, m)} \leq \frac{\chi(E \otimes A^a)}{\chi(E \otimes A^a \otimes H^m)},
\]

contradicting again the conclusion of the second half of Lemma 2.3.

This completes the proof of the Key GIT Lemma.

§3. Rationally Twisted Gieseker-Semistability and Stratifications

In this section, we analyse the change of the set of the $A$-Gieseker-semistable sheaves (the polarization $A$ being fixed) when tensored by a line bundle $\mathcal{L} \in \text{Pic}(X)$.
The analysis naturally leads us to the notion of Gieseker-semistability twisted by a rational line bundle $L \in \text{Pic} \otimes \mathbb{Q}$. We will give a stratification of the space of twists, which describes the change of the set of $L$-twisted $A$-Gieseker-semistable sheaves as $L$ varies, based upon the stratification of $\Delta - \{0\} = \bigsqcup_s \Delta_s$. These two stratifications not only completely describe the change of $S(r, c_1, c_2, H)$ as $H$ varies, but also illustrate how the transformation is factorized into a sequence of flips.

Consider the set of all coherent torsion free sheaves $F$ on $X$ as in Proposition 1.6 such that there exists an exact sequence

$$0 \to F \to E \to G \to 0$$

where $E \in \mu(r, c_1, c_2, H)$ for some $H \in \Delta - \{0\}$, and where the quotient $G$ is a coherent torsion free sheaf, satisfying

$$\frac{c_1(F)}{rk(F)} \cdot H = \frac{c_1(E)}{rk(E)} \cdot H$$

but

$$0 \neq \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \in N^1(X)_{\mathbb{Q}}.$$

Take the set of all hyperplanes

$$L_F := \{ z \in N^1(X)_{\mathbb{Q}}; z \cdot (\frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)}) = 0 \}$$

for all such sheaves $F$. Proposition 1.6 implies that $\{L_F\}$ is a finite set.

**Definition 3.1.** A $d$-cell of $\Delta - \{0\}$ associated to $\{L_F\}$ ($d = 1, 2, \ldots, \rho_\Delta = \dim_{\mathbb{Q}} V(\Delta)$, where $V(\Delta)$ is the vector subspace of $N^1(X)_{\mathbb{Q}}$ generated by $\Delta$.) is a connected component of

$$\Delta \cap L_{F_1} \cap \cdots \cap L_{F_{\Delta-d}} \setminus (\cup_{F \neq F_i, i=1,\ldots,\rho_\Delta-d} L_F).$$

Now we have a stratification of $\Delta - \{0\}$

$$\Delta - \{0\} = \bigsqcup_s \Delta_s$$

into the $d$-cells where $d = 1, \cdots, \rho_\Delta$.

Before going into further discussion, we introduce the notion of rationally twisted Gieseker-semistability, which just slightly generalizes the classical notion of Gieseker-semistability but provides us with the right category to factorize the transformations among the various moduli spaces.

**Definition 3.2.** Let $H$ be an ample line bundle. A coherent torsion free sheaf $E$ is said to be $L$-twisted $H$-Gieseker-semistable (resp. $L$-twisted $H$-Gieseker-stable) for $L \in \text{Pic}(X) \otimes \mathbb{Q}$ iff for all $F \subset E$ (resp. for all $F \subsetneq E$)

$$\frac{\chi(F \otimes L \otimes H^n)}{rk(F)} \leq \frac{\chi(E \otimes L \otimes H^n)}{rk(E)} \text{ for } n \gg 0$$

(resp. <)
where we compute the Euler characteristics formally using the Riemann-Roch formula, e.g.,

\[
\frac{\chi(E \otimes L \otimes H^n)}{\text{rk}(E)} = \frac{1}{2} H^2 n^2 + \left\{ \frac{c_1(E)}{\text{rk}(E)} \cdot H + L \cdot H - \frac{1}{2} K_X \cdot H \right\} n + \frac{1}{2} L^2 + \frac{c_1(E)}{\text{rk}(E)} \cdot L - \frac{1}{2} K_X \cdot L + \frac{1}{2} c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X \cdot L + \frac{1}{2} c_1(L)^2 - c_1(L) \cdot K_X \cdot L + \chi(O_X).
\]

We denote by \( S((r, c_1, c_2) \otimes L, H) \) the set of all coherent torsion free sheaves \( E \) of \( \text{rk}(E) = r, c_1(E) = c_1, c_2(E) = c_2 \), that are \( L \)-twisted \( H \)-Gieseker-semistable.

**Remark 3.3.**

(i) As is clear from the definition, \( L \)-twistedness only depends on the class of \( L \) in \( N^1(X) \).

(ii) When \( L \) is an INTEGRAL line bundle \( L \in \text{Pic}(X) \), then \( S((r, c_1, c_2) \otimes L, H) \) can be identified with \( S((r, c_1, c_2, H) \) where

\[
c_{1L} = c_1 + r c_1(L)
\]

\[
c_{2L} = c_2 + (r - 1)c_1 \cdot c_1(L) + \frac{r(r - 1)}{2} c_1(L)^2
\]

under the one to one correspondence

\[
E \in S((r, c_1, c_2) \otimes L, H) \leftrightarrow E \otimes L \in S((r, c_1, c_2, H).
\]

(iii) The Harder-Narasimhan filtration with respect to rationally twisted Gieseker- (semi)stability can be taken just as in the classical case and the notion of \( L \)-twisted Seshadri equivalence is also well defined.

The stratification \( \Delta - \{0\} = \bigsqcup \Delta_s \) gives a basis to study the change of \( S(r, c_1, c_2, H) \) when \( H \) varies and the change of \( S((r, c_1, c_2) \otimes L, H) \) when \( L \) changes, as is indicated by the next lemma, whose proof is immediate from the construction.

**Lemma 3.4.**

(i) For \( H, H' \in \Delta_s \), we have

\[
\mu(r, c_1, c_2, H) = \mu(r, c_1, c_2, H')
\]

\[
S(r, c_1, c_2, H) = S(r, c_1, c_2, H')
\]

Moreover, the Seshadri equivalence classes of \( S(r, c_1, c_2, H) \) with respect to \( H \)-Gieseker-stability are the same as the Seshadri equivalence classes of \( S((r, c_1, c_2, H') \) with respect to \( H' \)-Gieseker-stability.

(ii) Fix \( H \in \Delta_s \). Then for any \( L \in V(\Delta_s) \), we have

\[
S(r, c_1, c_2, H) = S((r, c_1, c_2) \otimes L, H).
\]
Moreover, the Seshadri equivalence classes of $S(r, c_1, c_2, H)$ with respect to $H$-Gieseker-stability are the same as the Seshadri equivalence classes of $S((r, c_1, c_2) \otimes L, H)$ with respect to $L$-twisted $H$-Gieseker-stability.

Let $\Delta_s$ and $\Delta_{s'}$ be two $d$-cells such that $V(\Delta_s) = V(\Delta_{s'})$ and that $\Delta_s$ and $\Delta_{s'}$ are separated by a $d-1$-cell $W$, i.e., $W$ is the unique $d-1$-cell contained in $\Delta_s \cap \Delta_{s'}$. We take an ample line bundle $A \in W$.

**Proposition 3.5.** There exists a stratification

$$V(\Delta_s) = \coprod L_i \coprod M_j$$

consisting of a finite number of hyperplanes $L_i$ parallel to $V(W)$ and connected components $M_j$ of $V(\Delta_s) - (\coprod L_i)$, which determines the change of $S((r, c_1, c_2) \otimes L, A)$ as $L$ varies in $V(\Delta_s)$, i.e., for $L, L' \in V(\Delta_s)$,

$$S((r, c_1, c_2) \otimes L, A) = S((r, c_1, c_2) \otimes L', A)$$

if and only if $L$ and $L'$ belong to the same stratum $L_i$ or $M_j$. This stratification is independent of the choice of $A \in W$.

Moreover,

$$S(r, c_1, c_2, A) = S(r, c_1, c_2, H)$$

if and only if $\Delta_s$ is contained in one of the strata $M_j$. By induction on the dimension $d$ of the $d$-cells, these rules completely determine the change of the set $S(r, c_1, c_2, H)$ as $H$ varies in $\Delta - \{0\}$.

**Proof of Proposition 3.5.**

The condition for a coherent torsion free sheaf

$$E \in \mu(r, c_1, c_2, A)$$

to be $L$-twisted $A$-Gieseker-semistable is that

$$\chi(F \otimes L \otimes A^n) \leq \chi(E \otimes L \otimes A^n)$$

for all nonzero subsheaves $F \subset E$. By taking the saturation we only have to check the condition for those subsheaves $F$ with the torsion free quotient $E/F$.

If there is a subsheaf $F \subset E$ with

$$\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \} \cdot A > 0,$$

then the condition (♦) is never satisfied for any $L \in V(\Delta_s)$ for this $F$ and $E$ is not $L$-twisted $A$-Gieseker-semistable for any $L \in V(\Delta_s)$.

For a subsheaf $F \subset E$ with

$$\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \} \cdot A < 0,$$

the condition (♦) is always satisfied for any $L \in V(\Delta_s)$.
Therefore, the only subsheaves $F \subset E$ for which the validity of the condition $(\diamondsuit)$ depends on $L$ are the ones with
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A = 0.
\]
For such $F$, the condition $(\diamondsuit)$ is equivalent to
\[
\left( \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right) \cdot L
+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)}
- \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} 
\leq 0.
\]
If $\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\}$ is numerically equivalent to 0, then again the condition $(\diamondsuit)$ is independent of $L \in V(\Delta_s)$.

Take the set of hyperplanes
\[
M_F = \left\{ z \in V(\Delta_s) ; \left( \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right) \cdot z
+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)}
- \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} \right\} = 0
\]
for all subsheaves $F \subset E$ for some $E \in \mu(r, c_1, c_2, A)$ with $E/F$ torsion free,
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A = 0
\]
and $\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\}$ being not numerically equivalent to 0. Proposition 1.6 says that $\{M_F\}$ is a finite set. Note that the $M_F$ are all parallel to $V(W)$, since $A \in W$ where $W$ is one of the $d-1$-cells of the stratification of $\Delta - \{0\}$.

Fix $E \in \mu(r, c_1, c_2, A)$. Then the previous argument shows that the locus of $L$ in $V(\Delta_s)$ where $E$ is $L$-twisted $A$-Gieseker-semistable is either empty, one of the hyperplanes $M_F$, a closed subspace sandwiched between two of the hyperplanes $M_F$, a closed half subspace whose boundary is one of the hyperplanes $M_F$ or the entire space $V(\Delta_s)$.

Let $L$ be one of the boundary hyperplanes and $L \in L$. Since $L$ is on the boundary, there exists a nonzero proper subsheaf $F \subset E$ with torsion free $E/F$ such that
\[
\frac{\chi(F \otimes L \otimes A^n)}{rk(F)} = \frac{\chi(E \otimes L \otimes A^n)}{rk(E)} = \frac{\chi(E/F \otimes L \otimes A^n)}{rk(E/F)} \text{ for all } n,
\]
that both $F$ and $E/F$ are $L$-twisted $A$-Gieseker-semistable and that
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot \mathcal{M} \leq 0
\]
if and only if
\[ \left\{ \frac{c_1(E/F)}{rk(E/F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot M \geq 0, \]
with
\[ \left\{ \frac{c_1(E)}{rk(E)} - \frac{c_1(F)}{rk(F)} \right\} \text{(and thus also} \left\{ \frac{c_1(E/F)}{rk(E/F)} - \frac{c_1(E)}{rk(E)} \right\} \]
not being numerically trivial on \( V(\Delta) \). This implies that the sheaf
\[ F \oplus E/F \]
is \( M \)-twisted \( A \)-Gieseker-semistable if and only if \( M \in L \).

This completes the argument for the proof that there is a stratification as desired
\[ V(\Delta_s) = \coprod L_i \coprod M_j \]
consisting of a finite number of hyperplanes \( L_i \) (a part of the \( M_F \)) parallel to \( V(W) \) and connected components \( M_j \) of \( V(\Delta_s) - (\coprod L_i) \) which determines the change of the set \( S((r, c_1, c_2) \otimes L, A) \) as \( L \) varies in \( V(\Delta_s) \).

The last part of the proposition follows immediately from the fact that for any \( H \in \Delta_s \)
\[ S(r, c_1, c_2, H) = S((r, c_1, c_2) \otimes H^n, A) \text{ for } n >> 0, \]
which will be shown in the next lemma.

This completes the proof of Proposition 3.5.

We remark that since for any \( L \in V(\Delta_s) \) we have
\[ S((r, c_1, c_2) \otimes L, A) = S((r, c_1, c_2) \otimes L \otimes A^n, A) \]
and since \( L \otimes A^n \) is ample for \( n >> 0 \), each stratum of the stratification above has an ample representative, and that it is actually determined by its intersection with the ample cone
\[ V(\Delta_s) \cap \text{Amp}(X)_\mathbb{Q} = (\coprod L_i \coprod M_j) \cap \text{Amp}(X)_\mathbb{Q}. \]

**Lemma 3.6.** Take an ample line bundle \( H \in \Delta_s \) (resp. \( H' \in \Delta_{s'} \)). Then
\[ S((r, c_1, c_2) \otimes H^n, A) = S(r, c_1, c_2, H) \text{ for } n >> 0 \]
(resp. \( S((r, c_1, c_2) \otimes H'^{n'}, A) = S(r, c_1, c_2, H') \text{ for } n' >> 0 \)).

Moreover, the Seshadri equivalence classes of \( S((r, c_1, c_2) \otimes H^n, A) \) (resp. \( S((r, c_1, c_2) \otimes H'^{n'}, A) \)) with respect to \( H^n \)-twisted (resp. \( H'^{n'} \)-twisted) \( A \)-Gieseker-stability are the same as the Seshadri equivalence classes of \( S(r, c_1, c_2, H) \) (resp. \( S(r, c_1, c_2, H') \)) with respect to \( H \)-Gieseker-stability (resp. \( H' \)-Gieseker-stability).

**Proof of Lemma 3.6.**

Let \( P = \{(c_1(F), c_2(F))\} \) be the set of the pairs of numerical classes for such sheaves \( F \subset E \) with \( E/F \) torsion free for some \( E \in \mu(r, c_1, c_2, A) \) that
\[ \left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A = 0. \]
$P$ is a finite set by Proposition 1.6, and thus we can choose $n > 0$ so that for all $(c_1(F), c_2(F)) \in P$, if
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H < 0
\]
then
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H^n + \frac{1}{2} c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X - \frac{1}{2} c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X < 0,
\]
and if
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H > 0
\]
then
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H^n + \frac{1}{2} c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X - \frac{1}{2} c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X > 0.
\]
Take $E \in S(r, c_1, c_2, H)$. Then for all subsheaves $F \subset E$ ($E/F$ torsion free) we have either
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H < 0
\]
or
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H = 0
\]
together with the condition
\[
\frac{1}{2} c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X - \frac{1}{2} c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X \leq 0.
\]
In the first case, we have
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A \leq 0.
\]
If the strict inequality $<$ holds, then the condition $(♦)$ is immediate. If the equality $=$ holds, then the choice of $n$ guarantees $(♦)$. 
In the second case, we have
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A = 0.
\]
Since
\[
\frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)} - \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} \leq 0,
\]
again the condition (\(\diamondsuit\)) is satisfied.

Therefore, we conclude \(E \in S((r, c_1, c_2) \otimes H^n, A)\).

Now suppose \(E \in S((r, c_1, c_2) \otimes H^n, A)\). Note that \(E \in \mu(r, c_1, c_2, A)\). If \(E / S(r, c_1, c_2, H)\), then there exists a subsheaf \(F \subset E\) such that either
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H > 0
\]
or
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H = 0
\]

Together with the condition
\[
\frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)} - \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} > 0.
\]

In the first case, if
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A > 0,
\]
then \(E \not\in S((r, c_1, c_2) \otimes H^n, A)\), a contradiction! If
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A = 0,
\]
then the choice of \(n\) implies
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H^n + \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)} - \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} > 0,
\]
and thus \(E \not\in S((r, c_1, c_2) \otimes H^n, A)\), again a contradiction!
In the second case, we have

$$\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot A = 0,$$

and

$$\frac{1}{2} \left( c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X \right)$$

implies

$$\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H^n$$

$$+ \frac{1}{2} \left( c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X \right)$$

$$- \frac{1}{2} \left( c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X \right) > 0.$$

Thus $E \notin S((r, c_1, c_2) \otimes H^n, A)$, a contradiction!

Therefore, we conclude $E \in S(r, c_1, c_2, H)$.

The statement about the Seshadri equivalence classes follows similarly.

Let $M_0$ and $M_{l+1}$ be the two strata containing $H^n$ (for $n \gg 0$) and $H^{n'}$ (for $n' \gg 0$) respectively. Then Lemma 3.6 indicates that at least set-theoretically these two strata correspond to the moduli spaces $M(r, c_1, c_2, H)$, $M(r, c_1, c_2, H')$. One would expect that going from $M_0$ in the stratification $V(\Delta_e) = \bigsqcup L_i \bigsqcup M_j$ to $M_{l+1}$ corresponds to a sequence of moduli spaces of rationally twisted $A$-Gieseker-semistable sheaves. It should be noted that some of the strata may not contain any integral points corresponding to $\text{Pic}(X)$ and that this is the reason we are naturally led to introduce the notion of rationally twisted Gieseker-semistability.

§4 and §5 will be devoted to showing not only set-theoretically but scheme-theoretically that this is indeed the case. We construct the moduli space of rationally twisted Gieseker-semistable sheaves. Moreover we show that there is a sequence of flips among them, each of which is governed by the Mumford-Thaddeus principle of GIT: Let $M_0, L_0, L_1, M_1, \ldots, L_l, M_{l+1}$ be a sequence of strata starting with $M_0$ containing $H^n$ (for $n \gg 0$) and ending with $M_{l+1}$ containing $H^{n'}$ (for $n' \gg 0$) such that

$$M_i \cap M_{i+1} = L_i$$

for $i = 0, 1, \ldots, l$. (Note that these strata actually exhaust all the strata in $V(\Delta_e) = \bigsqcup L_i \bigsqcup M_j$.) Choose representatives $L_i, M_i \in \text{Pic}(X)_Q$ with $L_i \in L_i, M_i \in M_i$.

Our aim is to show that there is a sequence of Thaddeus-type flips

$$M((r, c_1, c_2) \otimes M_i, A) \quad M((r, c_1, c_2) \otimes M_{i+1}, A)$$

$$\searrow \quad \searrow$$

$$\quad M((r, c_1, c_2) \otimes L_i, A)$$
for $i = 1, 2, \cdots, l$, each of which is a transformation constructed by the Key GIT Lemma of §2 and thus governed by the Mumford-Thaddeus principle.

§4. Construction of Flip: Integral Case

We use the notation in §3.

In this section, we construct the flip

$$M((r, c_1, c_2) \otimes M_i, A) \to M((r, c_1, c_2) \otimes M_{i+1}, A)$$

when a representative $L_i \in L_i$ can be taken to be an INTEGRAL line bundle, i.e., $L \in \text{Pic}(X)$, using the Key GIT Lemma of §2. (We will deal with the more delicate case of the construction of the flip when $L_i \in L_i$ is only a RATIONAL line bundle in the next section.)

Theorem 4.1.

(i) The one to one correspondence

$$E \in S((r, c_1, c_2) \otimes L_i, A) \leftrightarrow E \otimes L_i \in S((r, c_1 L_i, c_2 L_i, A)$$

gives the one to one correspondence between

$$S((r, c_1, c_2) \otimes M_i, A) \text{ and } S((r, c_1 L_i, c_2 L_i, A)_H$$

and that between

$$S((r, c_1, c_2) \otimes M_{i+1}, A) \text{ and } S((r, c_1 L_i, c_2 L_i, A)_H.$$

(ii) $M((r, c_1, c_2) \otimes M_i, A)_H$ (resp. $M((r, c_1, c_2) \otimes M_i, A)_H$) constructed in §2 gives the moduli space $M((r, c_1, c_2) \otimes M_i, A)$ (resp. $M((r, c_1, c_2) \otimes M_{i+1}, A)$) which coarsely represents the Seshadri equivalence classes of $S((r, c_1, c_2) \otimes M_i, A)$ (resp. $S((r, c_1, c_2) \otimes M_{i+1}, A)$) with respect to $M_i$-twisted (resp. $L_i$-twisted, $M_{i+1}$-twisted) $A$-Gieseker-stability.

(iii) The diagram of morphisms

$$M((r, c_1, c_2) \otimes M_i, A) \to M((r, c_1, c_2) \otimes M_{i+1}, A)$$

which arises from the Key GIT Lemma gives the desired flip

$$M((r, c_1, c_2) \otimes M_i, A) \to M((r, c_1, c_2) \otimes M_{i+1}, A)$$

Proof of Theorem 4.1.
(i) Suppose $E \in S((r,c_1,c_2) \otimes \mathcal{M}_i, A)$. Since $\mathcal{L}_i$ is on the boundary $L_i$ of the stratum $M_i$ containing $\mathcal{M}_i$, $S((r,c_1,c_2) \otimes \mathcal{M}_i, A) \subset S((r,c_1,c_2) \otimes \mathcal{L}_i, A)$.

Therefore, $E \otimes \mathcal{L}_i$ is $A$-Gieseker-semistable. If $F \otimes \mathcal{L}_i$ is a subsheaf of $E \otimes \mathcal{L}_i$ with $p(F \otimes \mathcal{L}_i, A, a) = \frac{\chi(F \otimes \mathcal{L}_i \otimes A^n)}{rk(F)} = \frac{\chi(E \otimes \mathcal{L}_i \otimes A^n)}{rk(E)} = p(E \otimes \mathcal{L}_i, A, a)$

and

$$\frac{c_1(F)}{rk(F)} \cdot H' < \frac{c_1(E)}{rk(E)} \cdot H',$$

then

$$\frac{c_1(F)}{rk(F)} \cdot A = \frac{c_1(E)}{rk(E)} \cdot A$$

$$\frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)} = \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)}$$

and

$$\frac{c_1(F)}{rk(F)} \cdot \mathcal{M}_i > \frac{c_1(E)}{rk(E)} \cdot \mathcal{M}_i.$$

This implies

$$\frac{\chi(F \otimes \mathcal{M}_i \otimes A^n)}{rk(F)} > \frac{\chi(E \otimes \mathcal{M}_i \otimes A^n)}{rk(E)}$$

for $n \gg 0$, contradicting the $\mathcal{M}_i$-twisted $A$-Gieseker-semistability of $E$.

Thus $E \otimes \mathcal{L}_i \in S(r, c_1 \mathcal{L}_i, c_2 \mathcal{L}_i, A)_{H'}$.

On the other hand, suppose $E \otimes \mathcal{L}_i \in S(r, c_1 \mathcal{L}_i, c_2 \mathcal{L}_i, A)_{H'}$. Take a subsheaf $F \otimes \mathcal{L}_i \subset E \otimes \mathcal{L}_i$. Since $E \otimes \mathcal{L}_i$ is $A$-Gieseker-semistable, we have

$$\frac{\chi(F \otimes \mathcal{L}_i \otimes A^n)}{rk(F)} \leq \frac{\chi(E \otimes \mathcal{L}_i \otimes A^n)}{rk(E)}$$

for $n \gg 0$.

Therefore,

$$\frac{c_1(F)}{rk(F)} \cdot A \leq \frac{c_1(E)}{rk(E)} \cdot A$$

and if the equality holds, then

$$\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot \mathcal{L}_i$$

$$+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)}$$

$$- \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} \leq 0.$$

If

$$\frac{c_1(F)}{rk(F)} \cdot A < \frac{c_1(E)}{rk(E)} \cdot A,$$
then clearly
\[
\frac{\chi(F \otimes \mathcal{M}_i \otimes A^n)}{rk(F)} < \frac{\chi(E \otimes \mathcal{M}_i \otimes A^n)}{rk(E)} \quad \text{for } n >> 0.
\]

If
\[
\frac{c_1(F)}{rk(F)} \cdot A = \frac{c_1(E)}{rk(E)} \cdot A
\]
and
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot \mathcal{L}_i
+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)}
- \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} < 0,
\]
then
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot \mathcal{M}_i
+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)}
- \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} < 0
\]
and thus
\[
\frac{\chi(F \otimes \mathcal{M}_i \otimes A^n)}{rk(F)} < \frac{\chi(E \otimes \mathcal{M}_i \otimes A^n)}{rk(E)} \quad \text{for } n >> 0.
\]

If
\[
\frac{c_1(F)}{rk(F)} \cdot A = \frac{c_1(E)}{rk(E)} \cdot A
\]
and
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot \mathcal{L}_i
+ \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)}
- \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} = 0,
\]
i.e., if
\[
p(F \otimes \mathcal{L}_i, A, a) = p(E \otimes \mathcal{L}_i, A, a),
\]
then we have
\[
\left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H' \geq 0.
\]
If
\[ \left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H' > 0, \]
then
\[ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \cdot M_i + \frac{1}{2} \frac{c_1(F)^2 - 2c_2(F) - c_1(F) \cdot K_X}{rk(F)} \]
\[ - \frac{1}{2} \frac{c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{rk(E)} < 0 \]
and hence
\[ \frac{\chi(F \otimes M_i \otimes A^n)}{rk(F)} < \frac{\chi(E \otimes M_i \otimes A^n)}{rk(E)} \text{ for } n >> 0. \]
If
\[ \left\{ \frac{c_1(F)}{rk(F)} - \frac{c_1(E)}{rk(E)} \right\} \cdot H' = 0, \]
then
\[ \frac{\chi(F \otimes M_i \otimes A^a)}{rk(F)} = \frac{\chi(E \otimes M_i \otimes A^a)}{rk(E)} \text{ for all } a. \]
Thus \( E \in S((r, c_1, c_2) \otimes M_i, A). \)

A similar argument works for the other cases inside of the (resp.). This proves (i).

First note that the existence of a good categorical quotient \( M(r, c_1L_i, c_2L_i, A)_{H'} \) follows from the standard GIT [MFK94]. Since the locus \( Q \) is a closed sub-scheme of the projective Quot scheme \( Quot(O_X^{\oplus l}/\chi_{r,c_1L_i,c_2L_i}) \), the GIT quotient \( M(r, c_1L_i, c_2L_i, A)_{H'} \) is also a projective scheme. Also note that set-theoretically we can identify \( S((r, c_1, c_2) \otimes M_i, A) \) with \( S(r, c_1L_i, c_2L_i, A)_{H'} \) by (i). Now we verify that for \( M_i \)-twisted \( A \)-Gieseker-semistable sheaves \( E \) and \( E' \), the closures of the orbits corresponding to \( E \otimes L_i \otimes A^a \) and \( E' \otimes L_i \otimes A^a \) in \( Q^{ss} \) intersect if and only if \( gr(E) \cong gr(E') \), where \( Q^{ss} \) is the locus of semistable points in \( Q \) (See §2.) with respect to the linearization induced from the Plücker embedding by taking a high multiple of \( H' \) and where \( gr(E) \) is the direct sum of the quotients of the Harder-Narasimhan filtration of \( E \) with respect to \( M_i \)-twisted \( A \)-Gieseker-stability. Given an extension
\[ 0 \to F \to E \to G \to 0 \]
we can find a family of extensions \( E_t \) of \( G \) by \( F \), parametrized by \( t \in A^1_k \), such that for each \( t \neq 0 \) the extension is the given one, and for \( t = 0 \) the extension is trivial. Now applying this repeatedly to the Harder-Narasimhan filtration we see that the orbit corresponding to \( gr(E) \otimes L_i \otimes A^a \) is in the closure of the orbit corresponding to \( E \otimes L_i \otimes A^a \). Also note that if \( E \) is \( M_i \)-twisted \( A \)-Gieseker-semistable, then so is \( gr(E) \) and thus the orbit corresponding to \( gr(E) \otimes L_i \otimes A^a \) is in \( Q^{ss} \). So if \( gr(E) = gr(E') \) then the closures of the orbits of \( E \otimes L_i \otimes A^a \) and \( E' \otimes L_i \otimes A^a \) intersect.

Conversely we see that the orbit corresponding to \( gr(E) \otimes L_i \otimes A^a \) is closed as follows. Suppose \( R \) is a discrete valuation ring over \( k \) with the field of fractions
Proposition 5.1. Let \( \mathcal{L}_i = \frac{q}{m} \Lambda_i \) where \( \Lambda_i \) is a very ample line bundle on \( X \). Take a positive integer \( m \in \mathbb{N} \) such that \( q \) divides \( m \) and that \( mA - \Lambda_i \) is very ample. There exists a Galois cover \( \phi : Y \to X \) from a nonsingular projective surface \( S \) s.t.

(i) the Galois group \( G \cong \mathbb{Z}/m \mathbb{Z} \) and thus \( \phi \) is a Kummer extension,

(ii) \( \mathcal{L}_i^Y = \phi^* \mathcal{L}_i + \frac{1}{2} R \) is represented by an integral line bundle which is naturally a \( G \)-sheaf in the sense of Mumford (See [Mumford70, P.69]), where \( R \) is the ramification divisor \( R_Y = \phi^* K_X + R \), and

(iii) a coherent sheaf on \( X \) is torsion free and \( \mathcal{L}_i^Y \)-twisted \( A \)-Gieseker-semistable if and only if \( \phi^* E \) is torsion free and \( \mathcal{L}_i^Y \)-twisted \( \phi^* A \)-Gieseker-semistable.

Proof of Proposition 5.1.

First we construct the Galois cover following [KMM87, §1-1].
Take smooth irreducible members

\[ \Gamma_1, \Gamma_2 \in |\Lambda_i| \]

and

\[ H_1^{(1)}, H_2^{(1)} \in |mA - \Gamma_1| \]
\[ H_1^{(2)}, H_2^{(2)} \in |mA - \Gamma_2| \]

such that

\[ \Gamma_1 \cup \Gamma_2 \cup \cup_{k,i} H_k^{(i)} \]

has only simple normal crossings. Now let \( X = \cup U_\alpha \) be an affine open cover of \( X \) with the transition functions of \( A \)

\[ \{a_\alpha; a_{\alpha\beta} \in H^0(U_\alpha \cap U_\beta, \mathcal{O}_X^*)\} \]

and local sections

\[ \{\varphi_{k\alpha}^{(i)}; \varphi_{k\alpha}^{(i)} \in H^0(U_\alpha, \mathcal{O}_X)\} \]

such that

\[ H_k^{(i)} + \Gamma_1|U_\alpha = \text{div}(\varphi_{k\alpha}^{(i)}) \text{ on } U_\alpha \]

and that

\[ \varphi_{k\alpha}^{(i)} = a_{\alpha\beta}^m \cdot \varphi_{k\beta}^{(i)}. \]

We take the normalization of \( X \) in \( \text{Rat}(X)[\cup_{k,i}(\varphi_{k\alpha}^{(i)})\bar{\mathbb{F}}] \) (for some \( \alpha \)) as \( Y \). (Note that \( \text{Rat}(X)[\cup_{k,i}(\varphi_{k\alpha}^{(i)})\bar{\mathbb{F}}] = \text{Rat}(X)[\cup_{k,i}(\varphi_{k\beta}^{(i)})\bar{\mathbb{F}}] \) for any \( \alpha, \beta \).)

[KMM87,Theorem 1-1-1 & Lemma 1-1-2] and the construction imply that \( \phi : Y \rightarrow X \) is a Galois cover from a nonsingular surface \( Y \) with the Galois group \( G \cong (\mathbb{Z}/m)^4 \) such that

\[ \phi^*(\Gamma_i) = m\{\phi^*(\Gamma_i)_{\text{red}}\} \]
\[ \phi^*(H_k^{(i)}) = m\{\phi^*(H_k^{(i)})_{\text{red}}\} \]

and that

\[ K_Y = \phi^*K_X + R \]

where

\[ R = (m - 1)\{\phi^*(\Gamma_1)_{\text{red}} + \phi^*(H_1^{(1)})_{\text{red}} + \phi^*(H_2^{(1)})_{\text{red}} \]
\[ + \phi^*(\Gamma_2)_{\text{red}} + \phi^*(H_1^{(2)})_{\text{red}} + \phi^*(H_2^{(2)})_{\text{red}}\} \]
\[ = \frac{m - 1}{m}(2\phi^*(2mA - \Gamma_1)). \]

Therefore,

\[ L_i^Y = \phi^*L_i + \frac{1}{2}R \]
\[ = \frac{p}{q}m\{(\phi^*\Gamma_1)_{\text{red}}\} + (m - 1)\phi^*(2mA) + (m - 1)\{(\phi^*\Gamma_1)_{\text{red}}\} \]
is integral since \( q \) divides \( m \). Moreover, since \((\phi^* \Gamma_1)_{red}\) is \( G \)-invariant, the line bundle \( \mathcal{L}_Y^i \) can be given naturally the structure of a \( G \)-sheaf in the sense of Mumford by embedding it in the constant sheaf \( \text{Rat}(X) \). This proves (i) and (ii).

In order to prove (iii), first consider the exact sequence for a coherent torsion free sheaf \( E \) on \( X \)

\[
0 \to E \to E^{**} \to \text{Coker} \to 0,
\]

which gives rise to another exact sequence on \( Y \)

\[
0 \to \phi^* E \to \phi^* E^{**} \to \phi^* \text{Coker} \to 0,
\]

since \( \phi \) is faithfully flat. But since \( E^{**} \) is locally free and thus so is \( \phi^* E^{**} \), we conclude \( \phi^* E \) is torsion free. The converse is also immediate.

Now the computation of the difference

\[
\frac{\chi(\phi^* F \otimes \mathcal{L}_Y^i \otimes \phi^* A^n)}{\text{rk}(\phi^* F)} - \frac{\chi(\phi^* E \otimes \mathcal{L}_Y^i \otimes \phi^* A^n)}{\text{rk}(\phi^* E)} = \text{deg}_{\phi} \left( \frac{\chi(F \otimes \mathcal{L}_i \otimes A^n)}{\text{rk}(F)} - \frac{\chi(E \otimes \mathcal{L}_i \otimes A^n)}{\text{rk}(E)} \right)
\]

shows that if \( \phi^* E \) is \( \mathcal{L}_Y^i \)-twisted \( \phi^* \text{A-Gieseker-semistable} \), then \( E \) is \( \mathcal{L}_Y^i \)-twisted \( \text{A-Gieseker-semistable} \). To verify the converse we only have to show that if \( F_Y \subset \phi^* E \) is the first piece of the Harder-Narasimhan filtration of \( \phi^* E \) with respect to the \( \mathcal{L}_Y^i \)-twisted \( \phi^* \text{A-Gieseker-semistability} \), then \( F_Y = \phi^* F \) for some subsheaf \( F \subset E \).

First remark that \( \phi^* E \) is a \( G \)-sheaf and that the maximality of \( F_Y \) implies \( F_Y \) is \( G \)-invariant and thus \( F_Y \) is a \( G \)-subsheaf of \( \phi^* E \). We will actually prove \( F_Y = \phi^* (\phi_*(F_Y)^G) \).

We claim

\[
\phi^* (\phi_*(F_Y)^G) \subset F_Y \subset \phi^* (\{\phi_*(F_Y)^G\}^s)
\]

where \( \phi_*(F_Y)^G \subset \{\phi_*(F_Y)^G\}^s \subset E \) is the saturation of \( \phi_*(F_Y)^G \) in \( E \), and we will prove this by a local analysis of the Galois cover \( \phi : Y \to X \).

Take \( x \in X \). Set \( R = \mathcal{O}_{X,x} \).

We deal with the case where

\[
\begin{align*}
x & \in \Gamma_1, H_1^{(2)} \\
x & \notin \Gamma_2, H_2^{(1)}, H_1^{(1)}, H_2^{(2)}.
\end{align*}
\]

(Other cases can be treated similarly.)
From the construction of the Galois cover, for a set of local equations
\[
\varphi_1^{(1)} \text{ of } \Gamma_1 + H_1^{(1)} \\
\varphi_2^{(1)} \text{ of } \Gamma_1 + H_2^{(1)} \\
\varphi_1^{(2)} \text{ of } \Gamma_2 + H_1^{(2)} \\
\varphi_2^{(2)} \text{ of } \Gamma_2 + H_2^{(2)},
\]
we have
\[
(\phi_*\mathcal{O}_Y)_x = \oplus_{i,j,k,l} R \cdot (\varphi_1^{(1)})^{\frac{i}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{j}{\varphi_2^{(2)}}} (\varphi_1^{(1)})^{\frac{k}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{l}{\varphi_2^{(2)}}},
\]
This decomposition corresponds to the decomposition of \((\phi_*\mathcal{O}_Y)_x\) into the eigenspaces under the action of \(G\). (cf. [KMM87,Theorem 1-1-1 and Lemma 1-1-2].)

Now let \(E_x = M\). Then
\[
\{\phi_*(\phi^*E)\}_x = \oplus_{i,j,k,l} M \otimes R \cdot (\varphi_1^{(1)})^{\frac{i}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{j}{\varphi_2^{(2)}}} (\varphi_1^{(1)})^{\frac{k}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{l}{\varphi_2^{(2)}}}.
\]
Let
\[
\{\phi_*(\phi^*E)\}_x \supset \phi_*(F_Y)_x = \oplus_{i,j,k,l} F_{i,j,k,l} \otimes R \cdot (\varphi_1^{(1)})^{\frac{i}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{j}{\varphi_2^{(2)}}} (\varphi_1^{(1)})^{\frac{k}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{l}{\varphi_2^{(2)}}}.
\]
Note that
\[
\{\phi_*(F_Y)^G\}_x = F_{0,0,0,0}.
\]
Since \(\phi_*(F_Y)_x\) is a \((\phi_*\mathcal{O}_Y)_x\)-module, we have
\[
F_{0,0,0,0} \subset F_{i,j,k,l} \\
(\varphi_1^{(1)})^{\frac{i}{\varphi_2^{(1)}}} (\varphi_1^{(2)})^{\frac{j}{\varphi_2^{(2)}}} F_{i,j,k,l} \subset F_{0,0,0,0}.
\]
This implies \(F_Y/\phi^*(\phi_*(F_Y)^G)\) is torsion. As
\[
\phi_*(F_Y)^G \subset \{\phi_*(F_Y)^G\}^s
\]
and
\[
\phi^*E/\phi^*(\{\phi_*(F_Y)^G\}^s) \cong \phi^*(E/\{\phi_*(F_Y)^G\}^s)
\]
is torsion free, we obtain the claim
\[
F_Y \subset \phi^*(\{\phi_*(F_Y)^G\}^s).
\]
Therefore, we have
\[
\text{rk}(\phi_*(F_Y)^G) = \text{rk}(\phi^*(\phi_*(F_Y)^G)) = \text{rk}(F_Y) \\
= \text{rk}(\phi^*(\{\phi_*(F_Y)^G\}^s)) = r([\phi_*(F_Y)^G]),
\]
and in particular
\[ \chi(F_Y \otimes L_Y^1 \otimes \phi^* A^n) < \frac{\phi^*([\phi_*(F_Y)^G]^n \otimes L_Y^1 \otimes \phi^* A^n)}{rk(F_Y)} \text{ for } n >> 0. \]

By the maximality of \( F_Y \), we obtain
\[ F_Y = \phi^*([\phi_*(F_Y)^G]^s), \]
and thus
\[ \phi^*([\phi_*(F_Y)^G] = F_Y = \phi^*([\phi_*(F_Y)^G]^s). \]

This completes the proof of (iii) and Proposition 5.1.

Now we take the Quot scheme
\[ Quot_Y = Quot(\mathcal{O}_Y^{\oplus Y} / \chi_{(r, \phi^* c_1, \phi^* c_2)} \otimes L_Y \otimes \phi^* A^n) \]
which parametrizes the quotient of \( \mathcal{O}_Y^{\oplus Y} \) whose Hilbert polynomial is the same as
\[ \phi^* E \otimes L_Y^i \otimes \phi^* A^n \text{ for } E \in S((r, c_1, c_2) \otimes L_i, A) \text{ with } l_Y = \chi(\phi^* E \otimes L_Y^i \otimes \phi^* A^n). \]

Note that we take \( n \) appropriately large according to the construction in §2 applied to this situation.

There is a natural action of \( G \) on \( Quot_Y \), namely if
\[ q : \mathcal{O}_Y^{\oplus Y} \to E_Y \to 0 \]
\[ (t_1, t_2, \cdots, t_{l_Y}) \to (s_1, s_2, \cdots, s_{l_Y}) \]
is a point sending the sections \( (t_1, t_2, \cdots, t_{l_Y}) \) corresponding to the direct summands to the sections \( (s_1, s_2, \cdots, s_{l_Y}) \) of \( E_Y \), then for \( g \in G \)
\[ gg : \mathcal{O}_Y^{\oplus Y} \to g^* E_Y \to 0 \]
\[ (t'_1, t'_2, \cdots, t'_{l_Y}) \to (g^* s_1, g^* s_2, \cdots, g^* s_{l_Y}) \]
is the point sending the sections \( (t'_1, t'_2, \cdots, t'_{l_Y}) \) corresponding to the direct summands to the sections \( (g^* s_1, g^* s_2, \cdots, g^* s_{l_Y}) \) of \( g^* E_Y \). Note that we identify the element \( g \in G \) with the automorphism \( q : Y \to Y \) and that according to this definition of the action, we have the reversed order identity \((gh)q = h(qg)\) a priori, but having \( G \) being abelian we have the usual identity for the action \((gh)q = g(hq)\).

There is also a natural action of \( GL(l_Y) \) on \( Quot_Y \). If
\[ q : \mathcal{O}_Y^{\oplus Y} \to E_Y \to 0 \]
\[ (t_1, t_2, \cdots, t_{l_Y}) \to (s_1, s_2, \cdots, s_{l_Y}) \]
is a point sending the sections \( (t_1, t_2, \cdots, t_{l_Y}) \) corresponding to the direct summands to the sections \( (s_1, s_2, \cdots, s_{l_Y}) \) of \( E_Y \), then for \( M \in GL(l_Y) \)
\[ Mq : \mathcal{O}_Y^{\oplus Y} (\to \mathcal{O}_Y^{\oplus Y}) \to E_Y \to 0 \]
\[ (t'_1, t'_2, \cdots, t'_{l_Y}) \to (s_1, s_2, \cdots, s_{l_Y})M \]
is the point sending the sections \((t'_1, t'_2, \cdots, t'_{t_Y})\) corresponding to the direct summands to the sections \((s_1, s_2, \cdots, s_{t_Y})M\).

Suppose

\[ E \in S((r, c_1, c_2) \otimes L, A). \]

Then by Proposition 5.1 (iii)

\[ \phi^* E \in S((r, \phi^*c_1, \phi^*c_2) \otimes L^Y, \phi^* A). \]

Moreover, Proposition 5.1 (ii) implies

\[ \phi^* E \otimes L_i^Y \otimes \phi^* A^a = \phi^* (E \otimes A^a) \otimes L_i^Y \]

has a natural \(G\)-sheaf structure, and thus \(G\) acts on

\[ H^0(\phi^* E \otimes L_i^Y \otimes \phi^* A^a). \]

We have the decomposition into eigenspaces

\[ H^0(\phi^* E \otimes L_i^Y \otimes \phi^* A^a) = V = \oplus V_{i,j,k,l} \]

under the action of \(G\), and taking the basis from the eigenspaces we obtain the matrix representation \(g \leftrightarrow L_g\), where \(L_g\) is block diagonal with blocks of size \(\dim V_{i,j,k,l}\), corresponding to the simultaneous diagonalization of the matrix representation of \(G\), i.e., for the basis

\[ s_1, s_2, \cdots, s_{t_Y} \in H^0(\phi^* E \otimes L_i^Y \otimes \phi^* A^a) \]

chosen from the eigenspaces \(V_{i,j,k,l}\) we have

\[ (gs_1, gs_2, \cdots, gs_{t_Y}) = (s_1, s_2, \cdots, s_{t_Y})L_g. \]

Note that the above action of \(g\) can be expressed as \(\phi_g \circ g^* \) where

\[ \phi^* E \otimes L_i^Y \otimes \phi^* A^a \xrightarrow{\phi_g} g^* (\phi^* E \otimes L_i^Y \otimes \phi^* A^a) \]

is the isomorphism satisfying the cocycle condition in the definition of the \(G\)-sheaf structure of \(\phi^* E \otimes L_i^Y \otimes \phi^* A^a\) (cf. [Mumford70]).

Remark that the dimension of \(V_{i,j,k,l}\) is independent of \(E \in S((r, c_1, c_2) \otimes L, A)\) and if we fix the order of the \(V_{i,j,k,l}\) then \(L_g\) is also independent of \(E \in S((r, c_1, c_2) \otimes L, A)\) (at least on a connected component of the Quot scheme, and we construct the moduli space connected component by component).

Now take a point

\[ q : O_Y^{t_Y} \rightarrow \phi^* E \otimes L_i^Y \otimes \phi^* A^a \rightarrow 0 \]

\[(t_1, t_2, \cdots, t_{t_Y}) \rightarrow (s_1, s_2, \cdots, s_{t_Y}) \]

where the sections \((s_1, s_2, \cdots, s_{t_Y})\) are chosen as above, then the isomorphism \(\phi_g\) makes the following diagram commutative

\[ ggq : O_Y^{t_Y} \rightarrow g^* (\phi^* E \otimes L_i^Y \otimes \phi^* A^a) \rightarrow 0 \]

\[(t'_1, t'_2, \cdots, t'_{t_Y}) \rightarrow (s_1, s_2, \cdots, s_{t_Y}) \]

\[ L_gq : O_Y^{t_Y} \rightarrow O_Y^{t_Y} \rightarrow \phi^* E \otimes L_i^Y \otimes \phi^* A^a \rightarrow 0 \]

\[(t'_1, t'_2, \cdots, t'_{t_Y}) \rightarrow (s_1, s_2, \cdots, s_{t_Y})M \]

and thus

\[ ggq = L_gq \in \text{Quot}_Y. \]
Lemma 5.2. Define
\[ D = \{ q \in Quot_Y; gg = L_gq \text{ for } \forall g \in G \} \subset Quot_Y. \]

(i) \( D \) is naturally a closed subscheme of \( Quot_Y \).

(ii) For
\[ q : O_Y^{\oplus Y} \rightarrow E_Y \otimes L_i^Y \otimes \phi^* A^a \rightarrow 0 \in D \]
by definition there exists an isomorphism \( \phi_g \) which makes the following diagram commutative
\[ gg : O_Y^{\oplus Y} \rightarrow g^*(E_Y \otimes L_i^Y \otimes \phi^* A^a) \rightarrow 0 \]
\[ L_gq : O_Y^Y (\rightarrow O_Y^Y) \rightarrow E_Y \otimes L_i^Y \otimes \phi^* A^a \rightarrow 0. \]

Then \( \{ \phi_g; g \in G \} \) gives the structure of \( G \)-sheaf on \( E_Y \otimes L_i^Y \otimes \phi^* A^a \) and thus on \( E_Y \). Actually \( \phi_g \) is the restriction of the isomorphism \( \Phi_g \) defined over the universal quotient sheaf \( (Univ)_D \) over \( D \) and \( \{ \Phi_g; g \in G \} \) gives the structure of \( G \)-sheaf on \( (Univ)_D \).

Proof of Lemma 5.2.

(i) \( D = \cap (g \times L_g)^{-1} \Delta \) is naturally a closed subscheme of \( Quot_Y \), where \( g \times L_g : Quot_Y \rightarrow Quot_Y \times Quot_Y \) is the product morphism and \( \Delta \subset Quot_Y \) is the diagonal.

(ii) Let
\[ q : O_Y^{\oplus Y} \rightarrow E_Y \otimes L_i^Y \otimes \phi^* A^a \rightarrow 0 \in D \]
\[ (t_1, t_2, \cdots, t_{\ell_Y}) \rightarrow (s_1, s_2, \cdots s_{\ell_Y}) \]
be a point in \( D \) sending the sections \( (t_1, t_2, \cdots, t_{\ell_Y}) \) corresponding to the direct summands to the sections \( (s_1, s_2, \cdots, s_{\ell_Y}) \) of \( E_Y \otimes L_i^Y \otimes \phi^* A^a \). Then
\[ \phi_{gh}((gh)^*s_1, (gh)^*s_2, \cdots, (gh)^*s_{\ell_Y}) \]
\[ = (s_1, s_2, \cdots, s_{\ell_Y})L_{gh} \]
\[ = (s_1, s_2, \cdots, s_{\ell_Y})L_hL_g \]
\[ = \{ \phi_h(h^*s_1, h^*s_2, \cdots, h^*s_{\ell_Y}) \}L_g \]
\[ = \phi_g \circ g^*(\phi_h(h^*s_1, h^*s_2, \cdots, h^*s_{\ell_Y})}. \]

Since \( (s_1, s_2, \cdots, s_{\ell_Y}) \) generate \( E_Y \otimes L_i^Y \otimes \phi^* A^a \), this shows that \( \{ \phi_g; g \in G \} \) satisfies the cocycle condition and thus gives \( E_Y \otimes L_i^Y \otimes \phi^* A^a \) the structure of a \( G \)-sheaf. The assertion on \( (Univ)_D \) over \( D \) can be checked similarly.

For \( q : O_Y^{\oplus Y} \rightarrow (Univ)_q \rightarrow 0 \in D \), \( (Univ)_q \) is a \( G \)-sheaf by Lemma 5.2 (ii) and hence \( (Univ)_q \otimes L_i^{Y-1} \) is also given a natural \( G \)-sheaf structure. There is a natural map
\[ \phi^* \{ \phi_*((Univ)_q \otimes L_i^{Y-1})^G \} \rightarrow (Univ)_q \otimes L_i^{Y-1}. \]
We define the locus in $D$

$$\mathcal{P}^o = \{ q \in D : \phi^* \{ \phi_* ((Univ)^q \otimes L_i^{-1})^G \} \to (Univ)^q \otimes L_i^{-1} $$

is an isomorphism},

which is easily seen to be open in $D$, and $\mathcal{P}$ to be its (scheme theoretic) closure in $D$.

Now let $Q_Y$ be the closure in $Quot_Y$ of points corresponding to torsion free sheaves with given chern classes as in §2. We define

$$VD = \mathcal{P} \cap Q_Y \subset D,$$

which is $SL(\oplus GL(l_{i,j,k,l}))$-invariant, where $SL(\oplus GL(l_{i,j,k,l}))$ is the subgroup of $SL(l_Y)$ formed by the elements which are block diagonal, with blocks in $GL(l_{i,j,k,l})$. Here,

$$l_{i,j,k,l} = \dim_k V_{i,j,k,l}.$$ 

Theorem 5.3. (i)

$$VD_{\phi^*H''}^{ss} = Q_{Y\phi^*H''}^{ss} \cap VD$$

(resp. $VD_{\phi^*A}^{ss} = Q_{Y\phi^*A}^{ss} \cap VD$

$$VD_{\phi^*H'}^{ss} = Q_{Y\phi^*H'}^{ss} \cap VD$$

where $Q_{Y\phi^*H''}^{ss}$ (resp. $Q_{Y\phi^*A}^{ss}$, $Q_{Y\phi^*H'}^{ss}$) is the locus of the semistable points of $Q_Y$ with respect to the action of $SL(l_{i,j,k,l})$ and the linearization induced from the Plücker embedding of $Grass(k^{\otimes l_Y} \otimes \phi^*H^m, R_m)$ (resp. $Grass(k^{\otimes l_Y} \otimes \phi^*A^m, R_m)$, $Grass(k^{\otimes l_Y} \otimes \phi^*H^m, R_m)$) as in §2 (applied to $Y$ instead of $X$) and $VD_{\phi^*H''}^{ss}$ (resp. $VD_{\phi^*A}^{ss}$, $VD_{\phi^*H'}^{ss}$) is the locus of the semistable points with respect to the action of $SL(\oplus GL(l_{i,j,k,l}))$ and the same linearization ($SL(\oplus GL(l_{i,j,k,l})) \subset SL(l_Y)$).

(ii) The diagram of morphisms

$$VD_{\phi^*H'}^{ss} \cap SL \quad \downarrow \quad \cap \quad \cap \quad \cap$$

$$VD_{\phi^*A}^{ss} \cap SL$$

where $SL = SL(\oplus GL(l_{i,j,k,l}))$, gives the desired flip

$$M((r, c_1, c_2) \otimes M_i, A) \quad \psi_i \quad \cap \quad \psi_i^+ \quad \cap \quad \psi_i^+$$

$$M((r, c_1, c_2) \otimes L_i, A).$$
Proof of Theorem 5.3.

(i) First note that
\[ VD_{\phi^* H}^{ss} \supset QY_{\phi^* H}^{ss} \cap VD \]
follows from the definition of semistable points, since \( SL(l) \)-invariant sections are automatically \( SL(\oplus \GL(l_{i,j,k,l})) \)-invariant. We show the opposite inclusion as follows.

Let
\[ V = \oplus V_{i,j,k,l} \]
be the decomposition of the vector space \( V \) into the direct summands \( V_{i,j,k,l} \) of \( \dim V_{i,j,k,l} = l_{i,j,k,l} \),
\[ v_{i,j,k,l} : V \to V_{i,j,k,l} \]
the projection onto the \( \{ i,j,k,l \} \)-th factor.

The Hilbert-Mumford numerical criterion for semistability for the Grassmannian \( \text{Grass}(V \otimes W, R) \) of the \( R \)-dimensional quotients of the vector space \( V \otimes W \) under the action of \( SL(\oplus \GL(l_{i,j,k,l})) \) with the linearization induced by the Plücker embedding now reads:

**Lemma 2.1'.** If a point \( p : V \otimes W \to U \to 0 \) in \( \text{Grass}(V \otimes W, R) \) is semistable for the action of \( SL(\oplus \GL(l_{i,j,k,l})) \) and the linearization induced by the Plücker embedding, then for all nonzero subspaces \( L \subset V \) which preserves the decomposition, i.e., \( L = \oplus v_i(L) \) we have \( p(L \otimes W) \neq 0 \) and
\[
\frac{\dim L}{\dim p(L \otimes W)} \leq \frac{\dim V}{\dim U}.
\]

Using this Lemma 2.1’ we can prove the following slightly modified version of Lemma 2.3 in a similar way presented in §2.

**Lemma 2.3'.** For a fixed \( a \in \mathbb{N} (\geq a_0) \) and an ample line bundle \( \phi^* H \) on \( Y \), there exists \( M_{\phi^* H} \in \mathbb{N} \) s.t. for all \( m \geq M_{\phi^* H} \), the following holds: If a point
\[ q : O_Y^{\oplus l_Y} \to E_Y \otimes L_Y^\phi \otimes \phi^* A^a \in VD \]
is semistable with respect to the action of \( SL(\oplus \GL(l_{i,j,k,l})) \) and the linearization induced from the Plücker embedding of \( \text{Grass}(k^{\oplus l_Y} \otimes_k H^0(Y, \phi^* H^m, R_m)) \), then the natural homomorphism
\[ k^{\oplus l_Y} = H^0(O_Y^{\oplus l_Y}) \to H^0(E_Y \otimes L_Y^\phi \otimes \phi^* A^a) \]
is injective, and for any nonzero \( G \)-sheaf quotient
\[ E_Y \otimes L_Y^\phi \otimes \phi^* A^a \to G_Y \otimes L_Y^\phi \otimes \phi^* A^a \to 0 \]
(the surjection is compatible with the \( G \)-sheaf structure of both \( E_Y \) and \( G_Y \)) with \( rk(G_Y) \neq 0 \), we have
\[
\frac{h^0(G_Y \otimes L_Y^\phi \otimes \phi^* A^a)}{rk(G_Y)} \geq \frac{\chi(E_Y \otimes L_Y^\phi \otimes \phi^* A^a)}{rk(E_Y)}.
\]
Moreover, suppose that $\mathcal{G}$ over $T \times Y$ is a bounded family of coherent sheaves on $Y$ (independent of $a$). Then there exists $a_2 \in \mathbb{N}(\geq a_0)$ such that for all $a \geq a_2$ and an ample line bundle $\phi^* H$ on $Y$ the following holds: If a point

$$ q : \mathcal{O}_{\mathcal{Y}}^{\oplus l_Y} \to E_Y \otimes L_i^Y \otimes \phi^* A^a \to 0 \in \mathcal{V}\mathcal{D} $$

is semistable with respect to the action of $SL(\oplus GL(l_{i,j,k,l}))$ and the linearization induced from the Plücker embedding of Grass$(k^{\oplus l_Y} \otimes_k H^0(Y, \phi^* H^m), R_m)$ and if the natural homomorphism

$$ k^{\oplus l_Y} = H^0(\mathcal{O}_{\mathcal{Y}}^{\oplus l_Y}) \to H^0(E_Y \otimes L_i^Y \otimes \phi^* A^a) $$

is an isomorphism, then for any nonzero $G$-sheaf quotient

$$ E_Y \otimes L_i^Y \otimes \phi^* A^a \to G_Y \otimes L_i^Y \otimes \phi^* A^a \to 0 $$

(the surjection is compatible with the $G$-sheaf structure of both $E_Y$ and $G_Y$) with $G_Y \cong \mathcal{G}_t$ for some $t \in T$, we have

$$ \frac{h^0(G_Y \otimes L_i^Y \otimes \phi^* A^a)}{\chi(G_Y \otimes L_i^Y \otimes \phi^* A^a \otimes (\phi^* H^m)^*)} \geq \frac{\chi(E_Y \otimes L_i^Y \otimes \phi^* A^a)}{\chi(E_Y \otimes L_i^Y \otimes \phi^* A^a \otimes (\phi^* H^m)^*)}. $$

Now suppose

$$ q : \mathcal{O}_{\mathcal{Y}}^{\oplus l_Y} \to E_Y \otimes L_i^Y \otimes \phi^* A^a \in \mathcal{V}\mathcal{D}_{\phi^* H_{ss}}, $$

i.e., $q \in \mathcal{V}\mathcal{D}$ is semistable with respect to the action of $SL(\oplus GL(l_{i,j,k,l}))$ and the same linearization induced by a multiple of $\phi^* H$ as for the action of $SL(l_Y)$. If we assume that $E_Y$ has a torsion $T_Y$, then noting that $T_Y$ is $G$-invariant and thus $G$-subsheaf of $E_Y$ and using Lemma 2.3’ we derive a contradiction in the same way as in the proof of “only if” part of (ii) in Key GIT Lemma of §2. If we assume that $E_Y$ is not $L_i^Y$-twisted $\phi^* A$-Gieseker-semistable, then noting that the maximal destabilizing subsheaf $F_Y$ is $G$-invariant and thus a $G$-subsheaf and that the quotient $E_Y / G_Y$ is a $G$-sheaf with the surjection $E_Y \to E_Y / F_Y \to 0$ compatible with the $G$-sheaf structure, again we derive a contradiction in the same way as in the proof of “only if” part of (ii) in Key GIT Lemma of §2.

Remark that we can also prove in a similar way to the proof of (i) in Theorem 4.1 (by retaking $\mathcal{L}_i, \mathcal{M}_i, A, H, H'$ we may assume they are all on a line in $V(\Delta_s)$) that

$$ E_Y \in S((r, \phi^* c_1, \phi^* c_2) \otimes L_i^Y, \phi^* A)_{\phi^* H} $$

if and only if

$$ E_Y \in S((r, \phi^* c_1, \phi^* c_2) \otimes \mathcal{M}_i^Y, \phi^* A). $$

Now suppose

$$ E_Y \notin S((r, \phi^* c_1, \phi^* c_2) \otimes L_i^Y, \phi^* A)_{\phi^* H}. $$

Then

$$ E_Y \notin S((r, \phi^* c_1, \phi^* c_2) \otimes \mathcal{M}_i^Y, \phi^* A). $$
Take \( F_Y \) to be the maximal destabilizing subsheaf. Then \( F_Y \) is \( G \)-invariant and thus a \( G \)-subsheaf. (We may also assume that \( F_Y \) has the same averaged Euler characteristics
\[
\frac{\chi(F_Y \otimes \mathcal{L}^Y_i \otimes \phi^* A^\alpha)}{\text{rk}(F_Y)} = \frac{\chi(E_Y \otimes \mathcal{L}^Y_i \otimes \phi^* A^\alpha)}{\text{rk}(E_Y)}
\]
(as polynomials in \( a \)). The rest of the argument goes without change as in the last part of the proof of “only if” part of (ii) in Key GIT Lemma of §2 to derive a contradiction. Therefore, we conclude
\[ E_Y \in S((r, \phi^* c_1, \phi^* c_2) \otimes \mathcal{L}_i^Y, \phi^* A)_{\phi^* H}, \]
and thus
\[ \mathcal{V}\mathcal{D}_{\phi^* H^{ss}} = Q_{Y \phi^* H^{ss}} \cap \mathcal{V}D. \]
The other cases in (resp.) can be proved similarly.

(ii) First we prove the following lemma.

**Lemma 5.4.** For
\[ q : \mathcal{O}_Y^{\oplus r} \to (\text{Univ})_q \to 0 \in \mathcal{V}\mathcal{D}_{\phi^* A^{ss}}, \]
(and thus automatically for
\[ q : \mathcal{O}_Y^{\oplus r} \to (\text{Univ})_q \to 0 \in \mathcal{V}\mathcal{D}_{\phi^* H^{ss}} \]
and
\[ q : \mathcal{O}_Y^{\oplus r} \to (\text{Univ})_q \to 0 \in \mathcal{V}\mathcal{D}_{\phi^* H^{ss}} \])
the natural morphism
\[ \phi^* \{ \phi_*((\text{Univ})_q \otimes \mathcal{L}_i^{Y-1})^G \} \to (\text{Univ})_q \otimes \mathcal{L}_i^{Y-1} \]
is an isomorphism.

**Proof of Lemma 5.4.**

Since by definition,
\[ \mathcal{V}D = \mathcal{P} \cap Q_Y \subset \mathcal{D} \]
where \( \mathcal{P} \) is the closure of the locus
\[ \mathcal{P}^\circ = \{ q \in \mathcal{D} : \phi^* \{ \phi_*((\text{Univ})_q \otimes \mathcal{L}_i^{Y-1})^G \} \to (\text{Univ})_q \otimes \mathcal{L}_i^{Y-1} \text{ is an isomorphism} \}, \]
we only have to show that in \( \mathcal{V}\mathcal{D}_{\phi^* A^{ss}} \) the condition
\[ \phi^* \{ \phi_*((\text{Univ})_q \otimes \mathcal{L}_i^{Y-1})^G \} \to (\text{Univ})_q \otimes \mathcal{L}_i^{Y-1} \]
being an isomorphism is a closed condition in parameter \( q \).
Note first that for $q \in \mathcal{V}D_{\phi^* A^s}$ since $(\text{Univ})_q$ is torsion free, the natural morphism
\[
\phi^* \{(\text{Univ})_q \otimes L_i^{Y-1}\}^G \to (\text{Univ})_q \otimes L_i^{Y-1}
\]
is injective. Suppose that
\[
\phi^* \{(\text{Univ})_s \otimes L_i^{Y-1}\}^G \to (\text{Univ})_s \otimes L_i^{Y-1}
\]
is an isomorphism for all $s \in S$ but possibly at $s_0 \in S$, where $S$ is any one parameter subspace in $\mathcal{V}D_{\phi^* A^s}$.

We have the decomposition
\[
(\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes p_2^s(L_i^{Y-1})) = \oplus_{i,j,k,l} \mathcal{E}_{i,j,k,l}
\]
into the eigenspaces under the action of $G$ and
\[
(\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes p_2^s(L_i^{Y-1}))^G = \mathcal{E}_{0,0,0,0}.
\]
Since $(\text{Id}_S \times \phi)_*((\text{Univ})_S)$ is flat over $S$, $(\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes p_2^s(L_i^{Y-1}))$ is also flat over $S$. This in turn implies $(\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes p_2^s(L_i^{Y-1}))^G = \mathcal{E}_{0,0,0,0}$ is flat over $S$. Therefore, we conclude that $(\text{Id}_S \times \phi)^*((\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes p_2^sL_i^{Y-1}))^G$ is flat over $S$. Now since
\[
(\text{Univ})_{s_0} \otimes L_i^{Y-1}/\phi^*\{(\text{Univ})_{s_0} \otimes L_i^{Y-1}\}^G
\]
is torsion (cf. Proposition 5.1 (iii)) and since
\[
\chi((\text{Univ})_{s_0} \otimes L_i^{Y-1} \otimes L_i^{Y} \otimes \phi^* A^a) = \chi((\text{Univ})_s \otimes L_i^{Y-1} \otimes L_i^{Y} \otimes \phi^* A^a)
\]
\[
= \chi((\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes L_i^{Y-1})^G \otimes L_i^{Y} \otimes \phi^* A^a)
\]
\[
= \chi((\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes L_i^{Y-1})^G \otimes L_i^{Y} \otimes \phi^* A^a)
\]
\[
= \chi((\text{Id}_S \times \phi)_*((\text{Univ})_S \otimes L_i^{Y-1})^G \otimes L_i^{Y} \otimes \phi^* A^a)
\]
we conclude
\[
\phi^* \{(\text{Univ})_{s_0} \otimes L_i^{Y-1}\}^G = (\text{Univ})_{s_0} \otimes L_i^{Y-1}.
\]

Lemma 5.4 implies that for
\[
q : \mathcal{O}^{\otimes Y}_X \to E_Y \otimes L_i^{Y} \otimes \phi^* A^a \in \mathcal{V}D_{\phi^* A^s},
\]
we have
\[
E_Y = \phi^*(\phi_*(E_Y)^G),
\]
and that moreover
\[
(\text{Id} \times \phi)^*\{(\text{Id} \times \phi)_*((\text{Univ}) \otimes p_2^s(L_i^{Y} \otimes \phi^* A^a)^{-1})_{\mathcal{V}D_{\phi^* A^s} \times X}\}^G
\]
\[
\to ((\text{Univ}) \otimes p_2^s(L_i^{Y} \otimes \phi^* A^a)^{-1})_{\mathcal{V}D_{\phi^* A^s} \times X}^G
\]
is an isomorphism.

In order to prove (ii)

$$M((r, c_1, c_2) \otimes M_i, A) \cong \mathcal{V}D_{\phi_{\cdot}}H^{ss}/\mathbb{S}L(\oplus GL(l_{i,j,k,l})), $$

we first verify that for

$$q : \mathcal{O}_{\mathcal{T}}^\otimes_{\mathcal{Y}} \to E_Y \otimes L_Y \otimes \phi^* A^a \in \mathcal{V}D_{\phi_{\cdot}}H^{ss}$$

and

$$q' : \mathcal{O}_{\mathcal{T}}^\otimes_{\mathcal{Y}} \to E_Y' \otimes L_Y' \otimes \phi^* A^a \in \mathcal{V}D_{\phi_{\cdot}}H^{ss},$$

the closures of the orbits of $q$ and $q'$ in $\mathcal{V}D_{\phi_{\cdot}}H^{ss}$ intersect if and only if $gr(\phi_{\cdot}(E_Y)^G) = gr(\phi_{\cdot}(E_Y')^G)$ where $gr(\phi_{\cdot}(E_Y)^G)$ (resp. $gr(\phi_{\cdot}(E_Y')^G)$) is the direct sum of the quotients of the Harder-Narasimhan filtration of $\phi_{\cdot}(E_Y)^G$ (resp. $\phi_{\cdot}(E_Y')^G$) with respect to the $M_i$-twisted $A$-Gieseker-stability. The proof goes without any change from that for (ii) in Theorem 4.1. Secondly if we have a family $\mathcal{E}$ of $M_i$-twisted $A$-Gieseker-semistable sheaves on $S \times X$, then by taking $(Id_S \times \phi)^* (\mathcal{E} \otimes p_1^* (L_{i,l} \otimes \phi^* A^a))$ over $S \times Y$ and locally choosing a frame for $p_1^* ((Id_S \times \phi)^* (\mathcal{E} \otimes p_2^* (L_{i,l} \otimes \phi^* A^a)))$ which corresponds to taking the eigenspace decomposition, we have a map

$$S \to \mathcal{V}D_{\phi_{\cdot}}H^{ss}$$

unique up to the action of $SL(\oplus GL(l_{i,j,k,l}))$. Therefore, it gives a map

$$S \to \mathcal{V}D_{\phi_{\cdot}}H^{ss}/\mathbb{S}L(\oplus GL(l_{i,j,k,l})).$$

By uniqueness the map is defined globally on $S$. $\mathcal{V}D_{\phi_{\cdot}}H^{ss}/\mathbb{S}L(\oplus GL(l_{i,j,k,l}))$ is universal for this property since it is a categorical quotient, and this proves

$$M((r, c_1, c_2) \otimes M_i, A) \cong \mathcal{V}D_{\phi_{\cdot}}H^{ss}/\mathbb{S}L(\oplus GL(l_{i,j,k,l})).$$

The other cases in (resp.) can be proved similarly.

(iii) is now immediate from (i) (ii) above. We remark that the morphism $\psi_i$ (resp. $\psi_i^+$) is induced from the universal property of the moduli spaces $M((r, c_1, c_2) \otimes M_i, A)$ (resp. $M((r, c_1, c_2) \otimes M_{i+1}, A)$) and $M((r, c_1, c_2) \otimes L_i, A)$, and that the point in $M((r, c_1, c_2) \otimes M_i, A)$ (resp. $M((r, c_1, c_2) \otimes M_{i+1}, A)$) corresponding to the Seshadri equivalence class of $E$ with respect to the $M_i$-twisted (resp. $M_{i+1}$-twisted) $A$-Gieseker-stability is mapped under $\psi_i$ (resp. $\psi_i^+$) to the point corresponding to the Seshadri equivalence class of $E$ with respect to the $L_i$-twisted $A$-Gieseker-stability.

This completes the proof of Theorem 5.3.

Combining Theorem 4.1 and Theorem 5.3 we have

**Theorem 5.4.** Let $M_0, L_0, L_1, M_1, \ldots, L_i, M_{i+1}$ be a sequence of strata starting with $M_0$ containing $H^n$ (for $n >> 0$) and ending with $M_{i+1}$ containing $H^{n'}$ (for $n' >> 0$) such that

$$\overline{M_i \cap M_{i+1}} = L_i$$
for \( i = 0, 1, \cdots, l \). (Note that these strata actually exhaust all the strata in \( V(\Delta_s) = \prod L_i \prod M_j \).) Then

\[
M(r,c_1,c_2,H) = M((r,c_1,c_2) \otimes M_0,A) \\
M(r,c_1,c_2,H') = M((r,c_1,c_2) \otimes M_{i+1},A) \\
M(r,c_1,c_2,A) = M((r,c_1,c_2) \otimes M_i,A) \text{ or } M((r,c_1,c_2) \otimes L_i,A) \text{ for some } i,
\]

and there is a sequence of Thaddeus-type flips

\[
\begin{array}{c}
M((r,c_1,c_2) \otimes M_i,A) \\
\downarrow \psi_i \\
M((r,c_1,c_2) \otimes L_i,A)
\end{array}
\quad \quad \quad
\begin{array}{c}
M((r,c_1,c_2) \otimes M_{i+1},A) \\
\downarrow \psi_i^+ \\
M((r,c_1,c_2) \otimes L_i,A)
\end{array}
\]

for \( i = 1, 2, \cdots, l \), each of which is a transformation constructed by the Key GIT Lemma of \( \S 2 \) and thus governed by the Mumford-Thaddeus principle.

**Remark 5.5.**

The structure of each flip

\[
\begin{array}{c}
M((r,c_1,c_2) \otimes M_i,A) \\
\downarrow \psi_i \\
M((r,c_1,c_2) \otimes L_i,A)
\end{array}
\quad \quad \quad
\begin{array}{c}
M((r,c_1,c_2) \otimes M_{i+1},A) \\
\downarrow \psi_i^+ \\
M((r,c_1,c_2) \otimes L_i,A)
\end{array}
\]

can be fairly explicitly described. For example, if \( r = 2 \), then over a point

\[
p \in M((r,c_1,c_2) \otimes L_i,A)
\]

where \( \psi_i \) is not isomorphic, \( p \) corresponds to the Seshadri equivalence class of the form

\[
E = L \oplus L'
\]

with \( L \) and \( L' \) being rank one torsion free sheaves s.t. \( E \) is \( L_i \)-twisted \( A \)-Gieseker-semistable, but \( L \) destabilizes \( E \) with respect to the \( M_{i+1} \)-twisted \( A \)-Gieseker-semistability and \( L' \) destabilizes \( E \) with respect to \( M_i \)-twisted \( A \)-Gieseker-semistability.

We have the morphisms which are set-theoretically bijective

\[
P(\text{Ext}^1(L,L')) \to \psi_i^{-1}(p) \\
P(\text{Ext}^1(L',L)) \to \psi_i^{+1}(p),
\]

i.e., the flip corresponds to flipping the factors of the extension class (at least set-theoretically). More detailed and scheme-theoretic analysis of the flip will be published elsewhere.

Finally starting from the \( d \)-cells of maximal dimension and descending inductively on the dimension \( d \), i.e., going from \( \Delta_s \) to \( W \) and repeating, and moreover letting \( \Delta \) vary in \( \text{Amp}(X)_0 \), we have the main theorem.
**Theorem 5.6 = Main Theorem.** The moduli space $M(r, c_1, c_2, H)$ goes through a sequence of flips (and contraction morphisms & their inverses) in the category of moduli spaces $M((r, c_1, c_2) \otimes L, A)$ of $L$-twisted $A$-Gieseker-semistable sheaves for rational line bundles $L \in \operatorname{Pic}(X) \otimes \mathbb{Q}$ and $A \in \operatorname{Amp}(X) \otimes \mathbb{Q}$, all of which are governed by the Mumford-Thaddeus principle of GIT.

Though we restricted ourselves to the subvector space $V(\Delta_s) \subset N^1(X) \otimes \mathbb{Q}$ in Proposition 3.5 and Theorem 5.4, it can be shown in the same manner that there exists a stratification of $N^1(X) \otimes \mathbb{Q}$ which describes the change of $L$-twisted $A$-Gieseker-semistable sheaves when $L$ varies in $N^1(X) \otimes \mathbb{Q}$, and the corresponding moduli spaces are connected by sequences of flips governed by Mumford-Thaddeus principle.

**Theorem 5.7.** The moduli spaces $M((r, c_1, c_2) \otimes L, A)$ of $L$-twisted $A$-Gieseker-semistable sheaves exist for rational line bundles $L \in \operatorname{Pic}(X) \otimes \mathbb{Q}$ and polarizations $A \in \operatorname{Amp}(X) \otimes \mathbb{Q}$ and are connected by sequences of flips (and contraction morphisms & their inverses), all of which are governed by Mumford-Thaddeus principle of GIT.

Note added in proof: While revising the paper after circulating the first version in July 1994, we learned in October 1994 that [Ellingsrud-Göttsche94] obtained similar results in the rank 2 case on particular (although from the point of view of Donaldson theory the most interesting) types of surfaces, but with a much finer analysis of flips. Though our results have no restrictions on surfaces or rank, the price had to be paid in getting only a general description. The aforementioned paper uses the notion of parabolic semistability, and the construction of moduli spaces depends on [Maruyama-Yokogawa93][Yokogawa93]. Their semistability coincides with our notion of rationally-twisted Gieseker semistability as we will explain below. We also direct the reader’s attention to the recent papers [Friedman-Qin94] and [Yoshioka94].

We use the notation in §3, 4 and 5. Let $H \in \Delta_s$ be an ample line bundle. Then a sufficiently high multiple $H^n$ is in the stratum $M_0$ whereas $H^{-n}$ is in the stratum $M_{l+1}$. We take an effective Cartier divisor $D \in H^n$. Parabolic semistability with parameter $a \in \mathbb{Q}$ is measured by the polynomial

$$ Par_a(E, n) = (1 - a) \frac{\chi(E \otimes O_X \otimes A^n)}{rk(E)} + a \frac{\chi(E \otimes O_X(D) \otimes A^n)}{rk(E)} $$

whereas our $L$-twisted $A$-Gieseker-semistability for $L = (1 - a)H^{-n} + aH^n \in \operatorname{Pic}(X) \otimes \mathbb{Q}$ is measured by the polynomial

$$ Twist_L(E, n) = \frac{\chi(E \otimes L \otimes A^n)}{rk(E)}.$$

Now it is straightforward to see that for a subsheaf $F \subset E$

$$ Par_a(E, n) - Par_a(F, n) = Twist_L(E, n) - Twist_L(F, n) $$

and thus both semistabilities coincide. In any case, we observe the change of semistability according to the stratification

$$ V(\Delta_s) = \coprod L_i \coprod M_j $$
starting from the point $H^n$ to the point $H^{-n}$ along the line joining them.

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