THE LOCAL STRUCTURE OF GENERALIZED CONTACT BUNDLES

JONAS SCHNITZER AND LUCA VITAGLIANO

Abstract. Generalized contact bundles are odd dimensional analogues of generalized complex manifolds. They have been introduced recently and very little is known about them. In this paper we study their local structure. Specifically, we prove a local splitting theorem similar to those appearing in Poisson geometry. In particular, in a neighborhood of a regular point, a generalized contact bundle is either the product of a contact and a complex manifold or the product of a symplectic manifold and a manifold equipped with an integrable complex structure on the gauge algebroid of the trivial line bundle.

Contents

Introduction 1
1. Preliminaries 3
   1.1. The gauge algebroid 3
   1.2. Jacobi bundles and their characteristic foliations 5
2. Generalized contact and Dirac-Jacobi geometry 6
   2.1. The omni-Lie algebroid and its symmetries 6
   2.2. Generalized contact and Dirac-Jacobi bundles 9
3. The transversal to a leaf 20
   3.1. Contact leaves and their transversals 20
   3.2. Lcs leaves and their transversals 24
4. Splitting theorems 26
   4.1. Splitting around a contact point 28
   4.2. Splitting around an lcs point 30
5. The regular case 32
   5.1. Local normal form around a regular contact point 32
   5.2. Local normal form around a regular lcs point 32
Appendix A. Complex structures on the gauge algebroid 34
   A.1. Local normal form 34
   A.2. Dolbeault-Atiyah cohomology 35
References 37

Introduction

Generalized complex manifolds have been introduced by Hitchin in [17] and further investigated by Gualtieri in [16], and the literature about them is now rather wide. Generalized
complex manifolds are necessarily even dimensional and they encompass symplectic and complex manifolds as extreme cases. A natural question is what is the odd dimensional analogue of a generalized complex manifold. Several answers to this question appeared already in the literature but the works on generalized geometry in odd dimensions are still sporadic \cite{18, 30, 26, 36, 27, 2}. Recently, A. Wade and the second author proposed a partially new definition of an odd dimensional analogue of a generalized complex manifold, called a \textit{generalized contact bundle} \cite{33}. Generalized contact bundles are a slight generalization of Iglesias-Wade integrable generalized almost contact structures \cite{18} to the realm of (generically non-trivial) line bundles, and encompass not necessarily coorientable contact manifolds as an extreme case. At the other extreme they encompass line bundles equipped with an integrable complex structure on their gauge algebroid. In turn, such line bundles are intrinsic models for so called \textit{normal almost contact manifolds} \cite{4}. In our opinion, generalized contact bundles have an advantage over previous proposals of a generalized contact geometry: they have a firm conceptual basis in the so called \textit{homogenization scheme} \cite{35}, which is, in essence, a dictionary from contact and related geometries to symplectic and related geometries. In principle, applying the dictionary is straightforward: it is enough to replace functions on a manifold $M$ with sections of a line bundle $L \to M$, vector fields over $M$ with derivations of $L$, etc. In practice, applying the dictionary can be actually challenging, and may lead to interesting new features \cite{15, 31, 32, 20, 21, 6, 29, 28, 34, 35}.

In \cite{33} the authors define generalized contact bundles, and study their structure equations, showing, in particular, that every generalized contact bundle is a Jacobi bundle \cite{19, 22, 25}. This puts odd dimensional generalized geometry in the framework of Jacobi geometry. In this paper we begin a systematic study of generalized contact bundles by studying their local structure. Our main results are two splitting theorems. In this introduction we provide for them rough statements to be better explained and made precise in the bulk of the paper.

**Theorem (A).** Let $M$ be a manifold equipped with a generalized contact bundle, and let $x_0 \in M$ be a point in an odd dimensional characteristic leaf of $M$. Then, locally around $x_0$, $M$ is isomorphic, up to a B-field transformation, to the product of a contact manifold and a homogeneous generalized complex manifold whose homogeneous Poisson structure vanishes at a point.

**Theorem (B).** Let $M$ be a manifold equipped with a generalized contact bundle, and let $x_0 \in M$ be a point in an even dimensional characteristic leaf of $M$. Then, locally around $x_0$, $M$ is isomorphic, up to a B-field transformation, to the product of a symplectic manifold and a manifold with a generalized contact bundle whose Jacobi structure vanishes at a point.

We also explicitly discuss the local structure of a generalized contact bundle in a neighborhood of a regular point, proving the following two local normal form theorems.

**Theorem (C).** Let $M$ be a $(2n + 2d + 1)$-dimensional manifold equipped with a generalized contact bundle, and let $x_0 \in M$ be a point in a $(2d + 1)$-dimensional characteristic leaf of $M$. If $x_0$ is a regular point, then, locally around $x_0$, $M$ is isomorphic, up to a B-field transformation, to the product of the standard $(2d + 1)$-dimensional contact manifold $(\mathbb{R}^{2d+1}, \theta_{\text{can}})$ and the standard complex space $\mathbb{C}^n$.

**Theorem (D).** Let $M$ be a $(2n + 2d + 1)$-dimensional manifold equipped with a generalized contact bundle, and let $x_0 \in M$ be a point in a $2d$-dimensional characteristic leaf of $M$. If $x_0$
is a regular point, then, locally around $x_0$, $M$ is isomorphic, up to a $B$-field transformation, to the product of the standard $2d$-dimensional symplectic space $(\mathbb{R}^{2d}, \Omega_{can})$ and the cylinder $\mathbb{R} \times \mathbb{C}^n$ equipped with the canonical complex structure on the gauge algebroid of the trivial line bundle $(\mathbb{R} \times \mathbb{C}^n) \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{C}^n$.

The proof of Theorem (D) requires proving that certain Dolbeault-like cohomologies associated with a complex structure on the gauge algebroid of a line bundle are locally trivial.

The paper is organized as follows. In Section 1 we collect the necessary preliminaries on gauge algebroids and Jacobi structures. In Section 2 we recall the notions of generalized contact bundle \[33\] and complex Dirac-Jacobi structure \[32\]. In this section we also discuss in details symmetries of the omni-Lie algebroid, which plays for generalized contact and Dirac-Jacobi bundles a similar role as the generalized tangent bundle plays for generalized complex and Dirac manifolds. Finally we discuss homogeneous generalized complex structures, and a suitable notion of product of Dirac-Jacobi bundles, which appears to be unavoidable in a precise formulation of our splitting theorems. In Section 3 we describe in details the structures induced on the characteristic leaves of a generalized contact structure, and on their transversals. Section 4 contains our main results: the splitting theorems around a point in a contact and around a point in a locally conformal symplectic leaf. In the last Section 5 we prove, as corollaries, local normal form theorems around a regular point. Finally, in Appendix A we discuss a very special class of generalized contact structures: complex structures on the gauge algebroid of a line bundle. We prove a local normal form theorem analogous to the Newlander-Nirenberg theorem, and the local vanishing of an associated Dolbeault-like cohomology. Both are consequences of their standard even dimensional counterparts.

We assume the reader is familiar with the fundamentals of Lie algebroids, Dirac manifolds and generalized complex structures.

1. Preliminaries

1.1. The gauge algebroid. A derivation of a vector bundle $E \rightarrow M$ is an $\mathbb{R}$-linear operator $\Delta : \Gamma(E) \rightarrow \Gamma(E)$ satisfying the following Leibniz rule

$$\Delta(f \varepsilon) = X(f)\varepsilon + f \Delta \varepsilon, \quad f \in C^\infty(M), \quad \varepsilon \in \Gamma(E),$$

for a, necessarily unique, vector field $X \in \mathfrak{x}(M)$, called the symbol of $\Delta$ and denoted by $\sigma(\Delta)$. Derivations are sections of a Lie algebroid $DE \rightarrow M$, called the gauge algebroid of $E$, whose anchor is the symbol, and whose bracket is the commutator of derivations \[23\]. The fiber $D_xE$ of $DE$ over a point $x \in M$ consists of $\mathbb{R}$-linear maps $\Delta : \Gamma(E) \rightarrow E_x$ satisfying the Leibniz rule $\Delta(f \varepsilon) = v(f)\varepsilon_x + f(x)\Delta \varepsilon$ for some tangent vector $v \in T_xM$: the symbol of $\Delta$.

Correspondence $E \mapsto DE$ is functorial, in the following sense. Let $F \rightarrow N$ and $E \rightarrow M$ be two vector bundles, and let $\Phi : F \rightarrow E$ be a regular vector bundle map, i.e. a bundle map, covering a smooth map $\phi : N \rightarrow M$, which is an isomorphism on fibers. Then $\Phi$ induces a (generically non-regular) vector bundle map $D\Phi : DF \rightarrow DE$ via formula

$$D\Phi(\Delta)\varepsilon = (\Phi \circ \Delta)(\Phi^* \varepsilon)$$

for all $\Delta \in DF$, and $\varepsilon \in \Gamma(E)$. Here $\Phi^* \varepsilon$ is the pull-back of $\varepsilon$ along $\Phi$, i.e. it is the section of $F$ given by $(\Phi^* \varepsilon)_y = \Phi_{|\phi(y)}^{-1}(\varepsilon_{\phi(y)})$, $y \in N$. The vector bundle map $D\Phi$ will be sometimes
denoted by $\Phi_*$ if there is no risk of confusion. Correspondence $\Phi \mapsto D\Phi$ preserves identity and compositions.

Derivations of a vector bundle $E$ can be seen as linear vector fields on $E$, i.e. vector fields generating a flow by vector bundle automorphisms. Namely, for every derivation $\Delta$ of $E$, there exists a unique flow $\{\Phi_t\}$ by vector bundle automorphisms $\Phi_t : E \to E$ such that

$$\Delta \varepsilon = \frac{d}{dt}|_{t=0} \Phi_t^* \varepsilon$$

for all $\varepsilon \in \Gamma(E)$.

The gauge algebroid acts tautologically on the vector bundle $E$. Accordingly, there is a de Rham complex of $DE$ with coefficients in $E$, denoted $(\Omega^*_E, d_D)$. Cohomains in $(\Omega^*_E, d_D)$ will be referred to as Atiyah forms. They are vector bundle maps $\wedge^* DE \to E$. Differential $d_D$ is given by the usual formula. Atiyah forms can be pulled-back along regular vector bundle maps. Namely, let $F \to N$ and $E \to M$ be vector bundle, and let $\Phi : F \to E$ be a regular vector bundle map covering a smooth map $\phi : M \to N$. For $\omega \in \Omega^*_F$, we define $\Phi^* \omega \in \Omega^*_E$ via

$$(\Phi^* \omega)_y(\Delta_1, \ldots, \Delta_k) = \Phi|_{F_y}^{-1} \circ \omega_{\phi(y)}(\Phi_* \Delta_1, \ldots, \Phi_* \Delta_k)$$

for all $y \in N$, and $\Delta_1, \ldots, \Delta_k \in D_y F$. One can also take the Lie derivative $L_\Delta := [\iota_\Delta, d_D]$ of Atiyah forms along a derivation $\Delta$, and all these operators satisfy the usual Cartan calculus identities. Additionally, there is a distinguished derivation, namely the identical one: $\mathbb{1} : \Gamma(E) \to \Gamma(E)$, $\varepsilon \to \varepsilon$, and contraction $\iota_\mathbb{1}$ of Atiyah forms with $\mathbb{1}$ is a contracting homotopy for $(\Omega^*_E, d_D)$. In particular, $(\Omega^*_E, d_D)$ is acyclic.

In the case of a line bundle $L \to M$, every first order differential operator $\Gamma(L) \to \Gamma(L)$ is a derivation. Consequently, there are vector bundle isomorphisms $DL \simeq \text{Hom}(J^1 L, L)$, and $J^1 L \simeq \text{Hom}(DL, L)$, a non-degenerate pairing $\langle - , - \rangle : J^1 L \otimes DL \to L$, where $J^1 L \to M$ is the first jet bundle of $L$. In this case, the identical derivation $\mathbb{1}$ spans the kernel of the symbol and there is a short exact sequence:

$$0 \longrightarrow \mathbb{R}_M \longrightarrow DL \overset{\sigma}{\longrightarrow} TM \longrightarrow 0, \quad (1.1)$$

where $\mathbb{R}_M = M \times \mathbb{R}$ is the trivial line bundle over $M$. Dually, there is a short exact sequence

$$0 \longleftarrow L \longleftarrow J^1 L \longleftarrow T^* M \otimes L \longleftarrow 0. \quad (1.2)$$

The embedding $T^* M \otimes L \hookrightarrow J^1 L$, extends to an embedding

$$\Omega^*(M, L) \hookrightarrow \Omega^*_L,$$

of $L$-valued forms on $M$ into Atiyah forms on $L$, consisting in composing with the symbol $\sigma : DL \to TM$. We will often interpret $\Omega^*(M, L)$ as a subspace in $\Omega^*_L$ without further comments. Notice that $\Omega^1_L = \Gamma(J^1 L)$, and the first differential $d_D : \Gamma(L) \to \Omega^1_L$ agrees with the first jet prolongation $j_L^1 : \Gamma(L) \to \Gamma(J^1 L)$.

Remark 1.1.1 (Atiyah forms on the trivial line bundle). When $L = \mathbb{R}_M$ is the trivial line bundle, then sections of $L$ are just functions on $M$, both sequences (1.1) and (1.2) splits canonically via the standard flat connection in $\mathbb{R}_M$, and we have

$$D \mathbb{R}_M = TM \oplus \mathbb{R}_M,$$

$$J^1 \mathbb{R}_M = T^* M \oplus \mathbb{R}_M.$$
In this case, a generic derivation is of the form \( X + f \) where \( X \) is a vector field and \( f \) is a function. Similarly a generic section of \( J^1\mathbb{R}_M \) is of the form \( \eta + g \cdot j^11 \), where \( \eta \) is a 1-form, \( g \) is a function, and \( j^11 \) is the first jet prolongation of the constant function \( 1 \in C^\infty(M) \). In the following, we will denote \( j^1 = j^11 \). Then we have

\[
j^1 f = df + f \cdot j^1, \quad f \in C^\infty(M).
\]

More generally, any Atiyah form \( \omega \in \Omega^\bullet_{\mathbb{R}_M} \) can be uniquely written as

\[
\omega = \omega_0 + \omega_1 \wedge j^1, \quad (1.3)
\]

with \( \omega_0, \omega_1 \in \Omega^\bullet(M) \), and we used the symbol to give Atiyah forms the structure of a graded \( \Omega^\bullet(M) \)-module. Correspondence \( \omega \mapsto (\omega_0, \omega_1) \) establishes an isomorphism of graded \( \Omega^\bullet(M) \)-modules:

\[
\Omega^\bullet_{\mathbb{R}_M} \cong \Omega^\bullet(M) \oplus \Omega^{\bullet-1}(M).
\]

In terms of decomposition (1.3) the natural operations on Atiyah forms read as follows:

\[
\begin{align*}
d\omega &= d\omega_0 + (d\omega_1 + (-)^{\omega_1} \omega_0) \wedge j \\
\iota_X f \omega &= \iota_X \omega_0 + (-)^{\omega_1} f \omega_1 + \iota_X \omega_1 \wedge j \\
\mathcal{L}_X f \omega &= \mathcal{L}_X \omega_0 + f \omega_0 + \omega_1 \wedge df + (\mathcal{L}_X \omega_1 + f \omega_1) \wedge j.
\end{align*}
\]

for all \( X + f \in \Gamma(D\mathbb{R}_M) \).

1.2. Jacobi bundles and their characteristic foliations. Jacobi manifolds were introduced by Kirillov [19], and independently, Lichnerowicz [22], as generalizations of Poisson manifolds. Here we adopt to Jacobi manifolds the slightly more intrinsic approach via Jacobi bundles [25] (see also [28]). Jacobi bundles encompass (not necessarily coorientable) contact manifolds as non-degenerate instances.

Let \( L \to M \) be a line bundle. A Jacobi structure on \( L \) is a Lie bracket

\[
\{ - , - \} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)
\]

which is also a bi-differential operator or, equivalently, a bi-derivation. The bracket \( \{ - , - \} \) is also called the Jacobi bracket. A Jacobi bundle is a line bundle equipped with a Jacobi structure. A Jacobi bracket \( \{ - , - \} \) can be regarded as a 2-form

\[
J : \wedge^2 J^1 L \to L
\]

satisfying an additional integrability condition, and in the following we will often take this point of view.

**Example 1.2.1.** Every contact manifold is canonically equipped with a Jacobi bundle containing a full information on the contact structure. Indeed, let \( (M, H) \) be a contact manifold, i.e. \( H \subset TM \) is a maximally non-integrable hyperplane distribution, and consider the normal line bundle \( L := TM/H \). The distribution \( H \) can be equivalently encoded in a line bundle valued 1-form \( \theta \in \Omega^1(M, L) \): the canonical projection \( \theta : TM \to L \). In its turn \( \theta \) can be seen as an Atiyah 1-form on \( L \). One can prove that \( \omega := d\theta \in \Omega^2_{\mathbb{R}_L} \) is a non-degenerate (and closed) Atiyah 2-form (see, e.g., [32]). Here, the non-degeneracy means that the induced vector bundle map

\[
\omega_\flat : DL \to J^1 L
\]
is invertible. Its inverse $\omega^{-1}$ is the sharp map $J^2 : J^1 L \to DL$ of a (unique, non-degenerate) Jacobi structure $J := \omega^{-1} : \wedge^2 J^1 L \to L$. Conversely, every non-degenerate Jacobi structure on a line bundle $L \to M$, determines a contact structure $H \subset TM$ on $M$, with $TM/H = L$. For some more details, see the discussion at the beginning of Subsection 3.1.

**Example 1.2.2.** Every locally conformal symplectic (lcs) manifold is canonically equipped with a Jacobi bundle containing a full information on the lcs structure. We adopt a slightly more intrinsic approach to lcs manifolds. Namely, in this paper, a lcs structure on a line bundle $L \to M$ is a pair $(\Omega, \nabla)$, where $\nabla$ is a flat connection in $L$, and $\Omega$ is an $L$-valued 2-form on $M$, which is 1) non-degenerate and 2) closed with respect to the connection differential $d^\nabla : \Omega^1(M, L) \to \Omega^2(M, L)$. When $L = \mathbb{R} M$ is the trivial line bundle we recover the usual definition. So let $L \to M$ be a line bundle equipped with an lcs symplectic structure $(\nabla, \Omega)$. The bracket
\[ \{-, -\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L), \quad (\lambda, \mu) \mapsto \Omega^{-1}(d^\nabla \lambda, d^\nabla \mu) \]
is a Jacobi bracket. Interpret it as a 2-form $J : \wedge^2 J^1 L \to L$. It is easy to see that the rank of $J$ is $\dim M$. Conversely, every Jacobi structure $J$ on a line bundle $L \to M$ such that rank $J = \dim M$ determines an lcs structure on $L$. For some more details, see the discussion at the beginning of Subsection 3.2.

Similarly as a Poisson manifold, a manifold $M$ equipped with a Jacobi bundle $(L, J)$ possesses a canonical (generically singular) foliation, called the characteristic foliation and defined as follows. Consider the sharp map associated to $J$, $\sigma_J : J^1 L \to TM$, whose image is an involutive distribution on $M$. The integral foliation $\mathcal{F}$ of $\im \sigma_J$ is the characteristic foliation of the Jacobi bundle $(L, J)$, and its leaves are characteristic leaves. Odd dimensional leaves of $\mathcal{F}$ are naturally contact manifolds, while even dimensional leaves are lcs manifolds. For more details about properties of characteristic leaves in Jacobi geometry see, e.g., [28] (see also Section 3).

When $L = \mathbb{R} M \to M$ is the trivial line bundle, a Jacobi bracket $\{-, -\}$ on $L$ is equivalent to a Jacobi pair, i.e. a pair $(\Lambda, E)$, consisting of a bivector $\Lambda \in \mathfrak{X}^2(M)$ and a vector field $E \in \mathfrak{X}(M)$ such that
\[ [\Lambda, \Lambda]^{SN} = 2E \wedge \Lambda, \quad \text{and} \quad [E, \Lambda]^{SN} = 0, \]
where $[-, -]^{SN}$ is the Schouten-Nijenhuis bracket of multivectors. The equivalence is provided by the following formula:
\[ \{f, g\} = \Lambda(f, g) + E(f)g - fE(g), \quad f, g \in C^\infty(M). \]

**Example 1.2.3.** On $\mathbb{R}^{2d+1}$, with coordinates $(x^1, \ldots, x^d, p_1, \ldots, p_d, u)$, there is a canonical Jacobi pair $(\Lambda_{\text{can}}, E_{\text{can}})$ given by
\[ \Lambda_{\text{can}} := \frac{\partial}{\partial p_1} \wedge \left( \frac{\partial}{\partial x^1} + p_1 \frac{\partial}{\partial u} \right), \quad \text{and} \quad E_{\text{can}} = \frac{\partial}{\partial u}. \]
We denote by $J_{\text{can}}$ the Jacobi structure corresponding to the Jacobi pair $(\Lambda_{\text{can}}, E_{\text{can}})$.

## 2. Generalized contact and Dirac-Jacobi geometry

### 2.1. The omni-Lie algebroid and its symmetries.

The natural arena for generalized geometry in odd dimensions is the omni-Lie algebroid $\mathbb{D}L$ of a line bundle $L \to M$ [9]. Recall
that $\mathbb{D}L = DL \oplus J^1L$, where $DL \to M$ is the gauge algebroid. The omni-Lie algebroid possesses the following structures:

\> a natural projection

$$\text{pr}_D : \mathbb{D}L \to DL;$$

(2.1)

\> a non-degenerate, symmetric, split signature $L$-valued 2-form

$$\langle\langle -, - \rangle\rangle : \mathbb{D}L \otimes \mathbb{D}L \to L$$

given by:

$$\langle\langle (\Delta, \psi), (\square, \chi) \rangle\rangle := \langle \chi, \Delta \rangle + \langle \psi, \square \rangle;$$

\> a (non-skew symmetric, Dorfman-like) bracket

$$[\lbrack [-, -] \rbrack] : \Gamma(\mathbb{D}L) \times \Gamma(\mathbb{D}L) \to \Gamma(\mathbb{D}L)$$

given by:

$$\lbrack [\lbrack (\Delta, \psi), (\square, \chi) \rbrack] \rbrack := \lbrack [\lbrack \Delta, \square \rbrack, L_\Delta \chi - \iota_\square d_D \psi \rbrack \rbrack$$

for all $\Delta, \square \in DL$, and all $\varphi, \psi \in \Gamma(J^1L)$. These structures satisfy certain identities that we do not report here (for more details see, e.g., [32]). Most of them are just the obvious analogues of those holding for the standard Courant algebroid: the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$. Accordingly, the rest of this subsection is just an adaptation from similar features of $\mathbb{T}M$.

We now describe symmetries of the omni-Lie algebroid $\mathbb{D}L$. First of all, for a vector bundle $E \to M$, we denote by $\text{Aut}(E)$ the group of its automorphisms.

**Definition 2.1.1.** A Courant-Jacobi automorphism of $\mathbb{D}L$ is a pair $(\Phi, \Phi)$ consisting of

1. an automorphism $\Phi$ of the vector bundle $\mathbb{D}L$, and
2. an automorphism $\Phi$ of $L$,

such that $\Phi$ and $\Phi$ cover the same diffeomorphism $\phi : M \to M$, and, additionally,

$$D\Phi \circ p_D = p_D \circ \Phi$$

$$\Phi^\ast \langle\langle \alpha, \beta \rangle\rangle = \langle\langle \Phi^\ast \alpha, \Phi^\ast \beta \rangle\rangle$$

$$\Phi^\ast [\alpha, \beta] = [\Phi^\ast \alpha, \Phi^\ast \beta]$$

for all $\alpha, \beta \in \Gamma(\mathbb{D}L)$. The group of Courant-Jacobi automorphisms is denoted by $\text{Aut}_{\text{CJ}}(\mathbb{D}L)$.

**Example 2.1.2.** Let $B$ be a closed Atiyah 2-form, i.e. $B \in \Omega^2_L$ and $d_D B = 0$ (in particular, $B$ is exact). Denote by $e^B : \mathbb{D}L \to \mathbb{D}L$ the vector bundle automorphism defined by

$$e^B(\Delta, \psi) := (\Delta, \psi + \iota_\Delta \Omega), \quad (\Delta, \psi) \in \mathbb{D}L.$$  

Using decomposition $\mathbb{D}L = DL \oplus J^1L$ we can write $e^B$ in matrix form:

$$e^B = \begin{pmatrix} \text{id} & 0 \\ B_\psi & \text{id} \end{pmatrix}. \quad (2.3)$$

An easy computation shows that $(e^B, \text{id})$ is a Courant-Jacobi automorphism. We will refer to it as a $B$-field transformation, adopting the same terminology as for Courant automorphisms of the generalized tangent bundle. Clearly $e^0 = \text{id}$, $e^{B_1} \circ e^{B_2} = e^{B_1 + B_2}$, and $(e^B)^{-1} = e^{-B}$, for all closed Atiyah 2-forms $B, B_1, B_2$, showing that $B$-field transformations form an abelian subgroup of $\text{Aut}_{\text{CJ}}(\mathbb{D}L)$ isomorphic to $Z^2_L$: the group of 2-cocycles in $(\Omega^2_L, d_L)$.  

$\diamondsuit$
Example 2.1.3. Let $\Phi : L \to L$ be a vector bundle automorphism covering a diffeomorphism $\phi : M \to M$. Define a vector bundle automorphism $\mathbb{D}\Phi : \mathbb{D}L \to \mathbb{D}L$ via

$$\mathbb{D}\Phi(\Delta, \psi) := (\Phi_*(\Delta), (\Phi^{-1})^*\psi), \quad (\Delta, \psi) \in \mathbb{D}L.$$ 

It is easy to see that $(\mathbb{D}\Phi, \Phi)$ is a Courant-Jacobi automorphism. Additionally $\mathbb{D}\text{id} = \text{id}$, $\mathbb{D}\Phi_1 \circ \mathbb{D}\Phi_2 = \mathbb{D}(\Phi_1 \circ \Phi_2)$, and $(\mathbb{D}\Phi)^{-1} = \mathbb{D}(\Phi^{-1})$, for all $\Phi, \Phi_1, \Phi_2 \in \text{Aut}(L)$, showing that Courant-Jacobi automorphisms of the form $\mathbb{D}\Phi$ form a subgroup of $\text{Aut}_{CJ}(\mathbb{D}L)$ isomorphic to $\text{Aut}(L)$. Finally, let $B \in Z^2_L$ and $\Phi \in \text{Aut}(L)$. Then

$$e^B \circ \mathbb{D}\Phi = \mathbb{D}\Phi \circ e^{\Phi^*B}.$$ 

In particular we see that $B$-field transformations and automorphisms of $L$ generate a subgroup in $\text{Aut}_{CJ}(\mathbb{D}L)$ isomorphic to $Z^2_L \rtimes \text{Aut}(L)$ where $\text{Aut}(L)$ acts on $Z^2_L$ (from the right) via pull-backs.

Actually, exactly as for the generalized tangent bundle, $B$-field transformations and automorphisms of $L$ generate the full group of Courant-Jacobi automorphisms, according to the following proposition which we report here for completeness.

Proposition 2.1.4. Let $L \to M$ be a line bundle. Then

$$\text{Aut}_{CJ}(\mathbb{D}L) \simeq Z^2_L \rtimes \text{Aut}(L).$$

Proof. We identify $Z^2_L \rtimes \text{Aut}(L)$ with a subgroup of $\text{Aut}_{CJ}(\mathbb{D}L)$ via the embedding

$$Z^2_L \rtimes \text{Aut}(L) \hookrightarrow \text{Aut}_{CJ}(\mathbb{D}L), \quad (B, \Phi) \mapsto (e^B \circ \mathbb{D}\Phi, \Phi).$$

Our task is showing that the latter is onto. So let $(\Phi, \Phi) \in \text{Aut}_{CJ}(\mathbb{D}L)$. Define a new automorphism $(\Theta, \Theta)$ via

$$(\Theta, \Theta) = (\mathbb{D}\Phi^{-1}, \Phi^{-1}) \circ (\Phi, \Phi) = (\mathbb{D}\Phi^{-1} \circ \Phi, \text{id}).$$

Now consider $\mathbb{D}\Phi^{-1} \circ \Phi$. For $(\Delta, \psi) \in \mathbb{D}L$ we put

$$(\mathbb{D}\Phi^{-1} \circ \Phi)(\Delta, \psi) = (\Psi_D(\Delta, \psi), \Psi_J(\Delta, \psi)), $$

and

$$B(\Delta, \Box) := \langle \Psi_J(\Delta, 0), \Box \rangle.$$ 

Compatibility with $\langle -,- \rangle$ implies that $B$ is skew-symmetric, so it is an Atiyah 2-form. Compatibility with the bracket then implies that $B$ is closed. It is now easy to see that $\Phi = e^B \circ \mathbb{D}\Phi$ and this concludes the proof. \hfill $\Box$

We now pass to infinitesimal symmetries of $\mathbb{D}L$. First of all, for a vector bundle $E \to M$, denote by $\text{aut}(E)$ the Lie algebra of its infinitesimal automorphisms. As already remarked, $\text{aut}(E)$ is canonically isomorphic to the Lie algebra $\Gamma(\mathbb{D}E)$ of derivations $E$.

Definition 2.1.5. An infinitesimal Courant-Jacobi automorphism of $\mathbb{D}L$ is a pair $(\Delta, \Delta)$ consisting of

1. a derivation $\Delta$ of $\mathbb{D}L$, and
2. a derivation $\Delta$ of $L$,
such that $\Delta$ and $\Delta$ have the same symbol, and, additionally
\[
[\Delta, p_D\alpha] = p_D(\Delta\alpha)
\]
\[
\Delta\langle\alpha, \beta\rangle = \langle\Delta\alpha, \beta\rangle + \langle\alpha, \Delta\beta\rangle
\]
\[
\Delta[\alpha, \beta] = [\Delta\alpha, \beta] + [\alpha, \Delta\beta]
\]
for all $\alpha, \beta \in \Gamma(\mathbb{D}L)$. Equivalently $(\Delta, \Delta)$ generates a flow by Courant-Jacobi automorphisms of $\mathbb{D}L$. The Lie algebra of infinitesimal Courant-Jacobi automorphisms is denoted by $\text{aut}_{CJ}(\mathbb{D}L)$.

**Example 2.1.6.** Let $B$ be a closed Atiyah 2-form, i.e. $B \in Z^2_L$. Denote by $\mathcal{B}$ the endomorphism of $\mathbb{D}L$ given by
\[
\mathcal{B} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.
\]
(2.4)
Then $(\mathcal{B}, 0)$ is an infinitesimal Courant-Jacobi automorphism, exponentiating to the $B$-field transformation corresponding to $B$.

**Example 2.1.7.** Let $\square$ be a derivation of $L$. Define a derivation $\mathcal{L}_\square$ of $\mathbb{D}L$ via
\[
\mathcal{L}_\square(\Delta, \psi) := ([\square, \Delta], \mathcal{L}_\square\psi), \quad (\Delta, \psi) \in \Gamma(\mathbb{D}L).
\]
It is easy to see that $(\mathcal{L}_\square, \square)$ is an infinitesimal Courant-Jacobi automorphism. Infinitesimal automorphisms of the form $(\mathcal{L}_\square, \square)$ together with those of the form $(\mathcal{B}, 0)$ from previous example generate a Lie subalgebra in $\text{aut}_{CJ}(\mathbb{D}L)$ isomorphic to $\text{aut}(L) \ltimes Z^2_L$ in the obvious way. Here $\text{aut}(L)$ acts on $Z^2_L$ via Lie derivatives.

**Proposition 2.1.8.** Let $L \to M$ be a line bundle. Then
\[
\text{aut}_{CJ}(\mathbb{D}L) \simeq \text{aut}(L) \ltimes Z^2_L
\]

**Proof.** The proof is similar to that of Proposition 2.1.4 and it is left to the reader.

**Remark 2.1.9.** Infinitesimal Courant-Jacobi automorphisms in $\text{aut}(L) \ltimes Z^2_L$ generate Courant-Jacobi automorphisms in $Z^2_L \rtimes \text{Aut}(L)$. Namely, let $(B, \square) \in \text{aut}(L) \ltimes Z^2_L$, and let $(\mathcal{L}_\square + \mathcal{B}, \square)$ be the corresponding infinitesimal Courant-Jacobi automorphism. If $\square$ generates the flow $\{\Phi_t\}$ by vector bundle automorphism of $L$, then $(\mathcal{L}_\square + \mathcal{B}, \square)$ generates the flow
\[
\{(e^{C_t} \circ \mathcal{D}\Phi_t, \Phi_t)\}
\]
by Courant-Jacobi automorphisms corresponding to $\{(C_t, \Phi_t)\} \subset Z^2_L \rtimes \text{Aut}(L)$ where
\[
C_t := -\int_0^t (\Phi_s^*B)\,d\epsilon.
\]

2.2. Generalized contact and Dirac-Jacobi bundles.
2.2.1. Generalized contact bundles. A generalized contact bundle is a line bundle $L \to M$ equipped with a generalized contact structure, i.e. and endomorphism $\mathbb{K} : DL \to DL$ of the omni-Lie algebroid such that

- $\mathbb{K}$ is almost complex, i.e. $\mathbb{K}^2 = -\text{id}$,
- $\mathbb{K}$ is skew-symmetric, i.e. $\langle \mathbb{K}\alpha, \beta \rangle + \langle \alpha, \mathbb{K}\beta \rangle = 0$ for all $\alpha, \beta \in DL$, and
- $\mathbb{K}$ is integrable, i.e. $[\mathbb{K}\alpha, \mathbb{K}\beta] - [\alpha, \beta] - \mathbb{K}[\alpha, \mathbb{K}\beta] = 0$ for all $\alpha, \beta \in \Gamma(DL)$.

Let $(L \to M, \mathbb{K})$ be a generalized contact bundle. Then $M$ is odd-dimensional. Actually, generalized contact bundles are odd-dimensional analogues of generalized complex manifolds and they encompass contact manifolds and complex structures on the gauge algebroid of $L$ as extreme cases. To see this, use direct sum decomposition $DL = DL \oplus J^1L$ to present $\mathbb{K}$ in the form

$$
\mathbb{K} = \begin{pmatrix}
\varphi & J^1 \\
\omega & -\varphi^\dagger
\end{pmatrix}.
$$

Then

- $\varphi : DL \to DL$ is a vector bundle endomorphism,
- $\varphi^\dagger : J^1L \to J^1L$ is its adjoint, i.e. $\langle \varphi^\dagger \psi, \Delta \rangle = \langle \psi, \varphi, \Delta \rangle$, $(\Delta, \psi) \in DL$,
- $J : \wedge^2 J^1L \to L$ is a 2-form with sharp map $J^2 : J^1L \to DL$, and
- $\omega : \wedge^2 DL \to L$ is an Atiyah 2-form with flat map $\omega : DL \to J^1L$.

Additionally, $\varphi, J, \omega$ satisfy some identities. In particular, $J$ is a Jacobi bracket, so that $(L, J)$ is a Jacobi bundle. When $\varphi = 0$, then $\omega^{-1} = -J^2$ and $J$ is the Jacobi bracket of a (unique) contact structure $H \subset M$ such that $TM/H = L$ (see Example A). When $J = \omega = 0$, then $\varphi$ is an integrable complex structure on the gauge algebroid $DL$ (see Appendix A).

Remark 2.2.1. Let $\mathbb{K}$ be a generalized contact structure on $L$, and let $(\Phi, \Psi)$ be a Courant-Jacobi automorphism of $DL$. Then $\Phi \circ \mathbb{K} \circ \Phi^{-1}$ is a generalized contact structure as well. In particular, for $(\Phi, \Psi) = (e^B, \text{id})$, the $B$-field transformation corresponding to a closed Atiyah 2-form $B$, we obtain that $e^B \circ \mathbb{K} \circ e^{-B}$ is a generalized contact structure. The latter will be denoted by $\mathbb{K}B$.

2.2.2. Dirac-Jacobi bundles. Similarly as for generalized complex structures, generalized contact structures can be seen as (particularly nice) complex Dirac-Jacobi structures, i.e. complex Dirac structures in the omni-Lie algebroid. Recall that a Dirac-Jacobi structure on $L$ is a vector subbundle $\mathcal{L} \subset DL$ such that

- $\mathcal{L}$ is maximally isotropic wrt $\langle -,- \rangle$;
- $\mathcal{L}$ is involutive, i.e. $[\Gamma(\mathcal{L}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L})$.

Example 2.2.2.

- Let $L \to M$ be a line bundle and let $J : \wedge^2 J^1L \to L$ be a bi-differential operator on $\Gamma(L)$. Then $\text{graph } J := \{(J^2 \psi, \psi) : \psi \in J^1L\} \subset DL$ is a maximally isotropic subbundle, and it is a Dirac-Jacobi structure iff $J$ is a Jacobi structure.
- Let $L \to M$ be a line bundle and let $\omega \in \Omega^2_D$ be an Atiyah 2-form on $L$. Then $\text{graph } \omega := \{(\Delta, t\Delta \omega) : \Delta \in DL\} \subset DL$ is a maximally isotropic subbundle, and it is a Dirac-Jacobi structure iff $d_D \omega = 0$.

Remark 2.2.2.
Now, let \( \mathcal{K} \) be a generalized contact structure on \( L \). Consider the complexified omni-Lie algebroid \( DL \otimes \mathbb{C} \), and let \( \mathcal{K}_K \subset DL \otimes \mathbb{C} \) be the +i-eigenbundle of \( \mathcal{K} \). Then \( \mathcal{K}_K \) is a (complex) Dirac-Jacobi structure such that \( \mathcal{K}_K \cap \overline{\mathcal{K}}_K = 0 \), in particular \( DL \otimes \mathbb{C} = \mathcal{K}_K \oplus \overline{\mathcal{K}}_K \). Additionally, \( \mathcal{K}_K \) contains the full information about \( \mathcal{K} \). Finally, all Dirac-Jacobi structures \( \mathcal{L} \subset DL \otimes \mathbb{C} \) such that \( \mathcal{L} \cap \overline{\mathcal{L}} = 0 \) arise in this way, and we will call them complex Dirac-Jacobi structures of generalized contact type.

Remark 2.2.3. Let \( \mathcal{L} \) be a Dirac-Jacobi structure on \( L \), and let \( (\Phi, \Phi) \) be a Courant-Jacobi automorphism of \( DL \). Then \( \Phi(\mathcal{L}) \) is a Dirac-Jacobi structure as well. In particular, \( e^B(\mathcal{L}) \) is a Dirac-Jacobi structure, denoted by \( \mathcal{L}^B \), for every closed Atiyah 2-form \( B \). If \( \mathcal{L} = \mathcal{K}_K \) is the +i-eigenbundle of a generalized contact structure \( \mathcal{K} \), then \( e^B(\mathcal{L}) = \mathcal{L}_K^B = \mathcal{K}_K^B \). We stress that, in general, \( \mathcal{K}^B = e^B \circ \mathcal{K} \circ e^{−B} \) is not a honest generalized contact structure, unless \( B \) is a real Atiyah form (see, e.g., Remark 1.1.1).

Lemma 2.2.4. Let \( \mathcal{K} \) be a generalized contact structure as in (2.5), and let \( \mathcal{L} = \mathcal{K}_K \subset DL \otimes \mathbb{C} \) be its +i-eigenbundle. Then

\[
pDL \cap pDL \overline{\mathcal{L}} = \text{im } J^2 \otimes \mathbb{C}.
\]

Proof. First prove that \( \text{im } J^2 \otimes \mathbb{C} \subset pDL \cap pDL \overline{\mathcal{L}} \). It is enough to show that \( \text{im } J^2 \subset pDL \cap pDL \overline{\mathcal{L}} \).

So let \( \psi \in J^2L \). As \( DL \otimes \mathbb{C} = \mathcal{L} \oplus \overline{\mathcal{L}} \), then there exist unique \( (\Delta, \chi), (\Delta', \chi') \in \mathcal{L} \) such that

\[
(0, \psi) = \frac{1}{2} \left( (\Delta, \chi) + (\Delta', \chi') \right).
\]

As \( \psi \) is real, we also have

\[
(0, \psi) = (0, \psi^\dagger) = \frac{1}{2} \left( (\Delta, \chi) + (\Delta', \chi') \right).
\]

So \( \Delta = \Delta' \), and \( \chi = \chi' \), and

\[
(0, \psi) = \frac{1}{2} \left( (\Delta, \chi) + (\Delta, \chi) \right),
\]

in particular \( \text{Re } \Delta = 0 \). Now, apply \( \mathcal{K} \) to both sides of (2.6) to find

\[
(J^2 \psi, -\varphi^1 \psi) = \mathcal{K}(0, \psi) = \frac{1}{2} \mathcal{K} \left( (\Delta, \chi) + (\Delta, \chi) \right) = \frac{1}{2} \left( (\Delta, \chi) - (\Delta, \chi) \right),
\]

so that \( J^2 \psi = -\text{Im } \Delta = i\Delta = i\overline{\Delta} \). As \( \Delta \in pDL \mathcal{L} \), we conclude that \( J^2 \psi \in pDL \mathcal{L} \cap pDL \overline{\mathcal{L}} \).

Now show that \( pDL \mathcal{L} \cap pDL \overline{\mathcal{L}} \subset \text{im } J^2 \otimes \mathbb{C} \). So let \( \Delta \in pDL \mathcal{L} \cap pDL \overline{\mathcal{L}} \). Then there exist \( \chi, \chi' \in J^1L \otimes \mathbb{C} \) such that \( (\Delta, \chi), (\Delta, \chi') \in \mathcal{L} \). Put \( \psi = \frac{1}{2}(\chi - \chi') \). Then

\[
(J^2 \psi, -\varphi^1 \psi) = \mathcal{K}(0, \psi) = \frac{1}{2} \mathcal{K} \left( (\Delta, \chi) - (\Delta, \chi') \right) = \frac{1}{2} \left( (\Delta, \chi) + (\Delta, \chi') \right).
\]

In particular \( i\Delta = J^2 \psi \). This concludes the proof.

2.2.3. The 2-form of a complex Dirac-Jacobi structure. Let \( L \to M \) be a line bundle and let \( \mathcal{L} \subset DL \otimes \mathbb{C} \) be a complex Dirac-Jacobi structure on \( L \). There is a canonical skew-symmetric, \( L \otimes \mathbb{C} \)-valued bilinear map \( \varpi \) defined pointwise on the smooth, but not necessarily regular, subbundle \( pDL \mathcal{L} \) as follows:

\[
\varpi : \wedge^2 pDL \mathcal{L} \to L \otimes \mathbb{C}, \quad (\Delta, \nabla) \mapsto (\psi, \nabla),
\]

(2.7)
here \( \psi \in J^1L \otimes \mathbb{C} \) is any 1-jet such that \( (\Delta, \psi) \in \mathcal{L} \). It immediately follows from the definition of \( \varpi \), that
\[
\mathcal{L} = \{ (\Delta, \psi) \in DL \otimes \mathbb{C} : \langle \psi, \nabla \rangle = \varpi(\Delta, \nabla) \text{ for all } \nabla \in p_DL \}.
\]
(2.8)

Similarly as in generalized complex geometry \([1]\), when \( \mathcal{L} \) is of generalized contact type, we can relate \( \varpi \) to the corresponding generalized contact structure \( \mathcal{K} \). First consider the complex conjugate form
\[
\varpi : \wedge^2 p_DL \to L \otimes \mathbb{C}, \quad (\Delta, \nabla) \mapsto \overline{\varpi(\Delta, \nabla)}.
\]
The real and imaginary parts of \( \varpi \):
\[
\text{Re} \varpi := \frac{1}{2}(\varpi + \overline{\varpi}) \quad \text{and} \quad \text{Im} \varpi := \frac{1}{2i}(\varpi - \overline{\varpi})
\]
are only defined on the intersection \( p_DL \cap p_DL = \text{im} J^\# \), and we have the following

**Lemma 2.2.5.** Let \( \mathcal{K} \) be a generalized contact structure as in (2.8), let \( \mathcal{L} = \mathcal{L}_K \) be its +i-eigenbundle, and let \( \varpi \) be the canonical 2-form on \( p_DL \). Then
\[
-J(\psi, \psi') = \text{Im} \varpi(J^2\psi, J^2\psi')
\]
(2.9)

for all \( \psi, \psi' \in J^1L \otimes \mathbb{C} \).

**Proof.** It is enough to prove (2.9) for \( \psi, \psi' \in J^1L \). First of all notice that the rhs of (2.9) makes sense in view of Lemma 2.2.4. Now, let \( \psi, \psi' \in J^1L \), put \( \nabla = J^\# \psi' \), and take \( (\Delta, \chi) \in \mathcal{L} \) such that
\[
(0, \psi) = \frac{1}{2}((\Delta, \chi) + (\overline{\Delta}, \overline{\chi}))
\]
as in the proof of Lemma 2.2.4. In particular Re \( \Delta = 0 \), \( J^2\psi = i\Delta \), and Re \( \chi = \psi \). Finally, let \( \nabla \in \text{im} J^\# \). Then
\[
J(\psi, \psi') = -\langle \psi, \nabla \rangle = -(\text{Re} \chi, \nabla) = -\text{Re}(\chi, \nabla) = -\text{Re} \varpi(\Delta, \nabla) = -\text{Im} \varpi(i\Delta, \nabla) = -\text{Im} \varpi(\nabla, \nabla).
\]

\( \square \)

**Remark 2.2.6.** As recalled in Example 2.2.2, every Jacobi structure \( J \) on a line bundle \( L \to M \) determines a real Dirac-Jacobi structure: \( \mathcal{L}_J := \text{graph} J \subset DL \oplus J^1L = DL \), and we have \( p_DL \mathcal{L}_J = \text{im} J^\# \). In turn, for every real Dirac-Jacobi structure \( \mathcal{L} \) on a \( L \to M \), there is a canonical \( L \)-valued 2-form \( \wedge^2 p_DL \mathcal{L} \to L \), defined by the same formula (2.7) as \( \varpi \). Formula (2.9) then states that the imaginary part of \( \varpi \) agrees with the (complexification of the) 2-form \( \omega_J : \wedge^2 \text{im} J^2 \to L \) induced by the Jacobi structure \( J \) underlying the generalized contact structure \( \mathcal{K} \):
\[
\text{Im} \varpi = \omega_J.
\]

Notice that \( \omega_J \), hence \( \text{Im} \varpi \), is (pointwise) non-degenerate. \( \diamond \)
2.2.4. Backward images of (complex) Dirac-Jacobi structures. Let \((L \to M, \mathbb{K})\) be a generalized contact bundle, and let \(\mathcal{L} = \mathcal{L}_\mathbb{K} \subset \mathbb{D}L \otimes \mathbb{C}\) be its \(+i\)-eigenbundle. Like in generalized complex geometry, not all submanifolds of \(M\) inherits from \(\mathbb{K}\) a generalized contact bundle structure. However, all submanifolds of \(M\) inherits from \(\mathcal{L}\) a complex Dirac-Jacobi structure (up to regularity issues), via the backward image construction which we now recall for later use. We describe backward images for real Dirac-Jacobi structures. The following considerations extend straightforwardly to complex Dirac-Jacobi structures. So let \(L \to M\) be a line bundle equipped with a Dirac-Jacobi structure \(\mathcal{L} \subset \mathbb{D}L\). Consider another line bundle \(L_N \to N\) together with a regular vector bundle map \(\Phi : L_N \to L\) covering a smooth map \(\phi : N \to M\). Define a subbundle \(\Phi^!\mathcal{L} \subset \mathbb{D}L_N\) via
\[
\Phi^!\mathcal{L} := \{ (\Delta, \Phi^*\psi) \in \mathbb{D}L_N : (\Phi^*\Delta, \psi) \in \mathcal{L} \}.
\]
The bundle \(\Phi^!\mathcal{L}\) is always maximal isotropic, but needs not to be smooth. Nonetheless, when it is smooth, it is also regular, and, even more, it is a Dirac-Jacobi structure on \(L_N\), called the backward image of \(\mathcal{L}\) along \(\Phi\). There is a simple sufficient condition for smoothness, sometimes referred to as the clean intersection condition \([7, 32]\). For the purposes of this paper, we only need to know that:
- if \(\phi\) is a submersion, the clean intersection condition is automatically satisfied, hence \(\Phi^!\mathcal{L}\) is a Dirac-Jacobi structure;
- if \(\phi\) is the immersion of a(n immersed) submanifold \(S \hookrightarrow M\), it is easy to see that the clean intersection condition boils down to
\[
\text{rank}(p_D\mathcal{L}|_S + DL|_S) = \text{constant}. \tag{2.10}
\]
We refer to \([7, 32]\) for more details.

2.2.5. Products of Dirac-Jacobi structures. Let \((M_1, \mathcal{L}_1)\) and \((M_2, \mathcal{L}_2)\) be manifolds equipped with (standard) Dirac structures, i.e. maximally isotropic, and involutive subbundles \(\mathcal{L}_i\) of the generalized tangent bundles \(\mathbb{T}M_i = TM_i \oplus T^*M_i\). Then \(\mathcal{L}_1 \times \mathcal{L}_2 \subset \mathbb{T}M_1 \times \mathbb{T}M_2 = \mathbb{T}(M_1 \times M_2)\) is a Dirac structure on the product \(M_1 \times M_2\), called the product of \(\mathcal{L}_1\) and \(\mathcal{L}_2\). It is not immediately obvious how to extend this simple construction to line bundles and Dirac-Jacobi structures. In this section we propose such an extension. The splitting theorems of Section \([4]\) will be formulated in terms of the product of Dirac-Jacobi structures as defined here.

Begin with two Dirac-Jacobi structures \(\mathcal{L}_1, \mathcal{L}_2 \subset \mathbb{D}L\) on the same line bundle \(L \to M\). Let \(\mathcal{L}_1 \ast \mathcal{L}_2 \subset \mathbb{D}L\) be the (not necessarily regular) subbundle defined by
\[
\mathcal{L}_1 \ast \mathcal{L}_2 := \{ (\Delta, \psi_1 + \psi_2) : (\Delta, \psi_i) \in \mathcal{L}_i, i = 1, 2 \}.
\]

**Lemma 2.2.7.** If \(\mathcal{L}_1 \ast \mathcal{L}_2 \subset \mathbb{D}L\) is a smooth subbundle, then it is a Dirac-Jacobi structure.

**Proof.** The proof is an adaptation of the analogous proof for Dirac structures \([24]\). We report it here for completeness. It is clear that \(\mathcal{L}_1 \ast \mathcal{L}_2 \subset \mathbb{D}L\) is an isotropic subbundle. Now projection \(p_D : \mathbb{D}L \to DL\) restricts to a surjection \(p_D : \mathcal{L}_1 \ast \mathcal{L}_2 \to p_D\mathcal{L}_1 \cap p_D\mathcal{L}_2\) whose kernel is
\[
K := (\mathcal{L}_1 \cap J^1L) + (\mathcal{L}_2 \cap J^1L).
\]
But for any Dirac-Jacobi structure \(\mathcal{L}\), we have \(\mathcal{L} \cap J^1L = \text{Ann}(p_D(\mathcal{L}))\), where, for a subbundle \(V \subset DL\), we denote \(\text{Ann}(V) := \{ \psi \in J^1L : \langle \psi, \Delta \rangle = 0, \text{ for all } \Delta \in V \}\). Hence
\[
K = \text{Ann}(p_D\mathcal{L}_1) + \text{Ann}(p_D\mathcal{L}_2) = \text{Ann}(p_D\mathcal{L}_1 \cap p_D\mathcal{L}_2).
\]
Lemma 2.2.9. The following conditions are equivalent:

\[ \text{M} \]

Remark is the open and dense subset we are looking for. This concludes the proof.

For the involutivity, recall that a (regular) maximal isotropic subbundle \( \mathcal{L} \subset DL \) is involutive iff

\[ \Upsilon(\alpha, \beta, \gamma) := \langle [\alpha, \beta], \gamma \rangle = 0 \]

for all \( \alpha, \beta, \gamma \in \Gamma(\mathcal{L}) \). As \( \Upsilon \) is a (skew-symmetric) 3-linear form: \( \Upsilon : \wedge^3 \mathcal{L} \rightarrow L \), in order to check whether \( \mathcal{L} \) is involutive or not, it is enough to check whether \( \Upsilon \) vanishes pointwise or not on an open and dense subset of \( M \). Now, when \( \mathcal{L} = \mathcal{L}_1 \ast \mathcal{L}_2 \), it is straightforward to see that \( \Upsilon \) vanishes on sections of the special form \( (\Delta, \psi_1 + \psi_2) \), where \( (\Delta, \psi_i) \in \Gamma(\mathcal{L}_i) \), \( i = 1, 2 \). Hence, it is enough to show that there exists an open and dense subset \( U \subset M \) such that, for any point \( a \in \mathcal{L}_1 \ast \mathcal{L}_2 \) over some point \( x \in U \), \( a \) is the value at \( x \) of a local section \( \alpha \in \Gamma(\mathcal{L}_1 \ast \mathcal{L}_2) \) of the special form. It is easy to see that

\[ U := \{ x \in M : \text{both } pD\mathcal{L}_1 \text{ and } pD\mathcal{L}_2 \text{ have constant rank around } x \} \]

is the open and dense subset we are looking for. This concludes the proof. \( \Box \)

Remark 2.2.8. If \( \mathcal{L}_2 = DL \subset DL \), then \( \mathcal{L}_1 \ast \mathcal{L}_2 = \mathcal{L}_1 \) for any \( \mathcal{L}_1 \).

We are now ready to define a notion of product of Dirac-Jacobi structures. So let \( L_i \rightarrow M_i \) be line bundles equipped with Dirac-Jacobi structures \( \mathcal{L}_i \subset DL_i \), \( i = 1, 2 \). We assume we have an additional datum, namely a line bundle \( L \rightarrow M_1 \times M_2 \) over the product, together with regular vector bundle maps \( P_i : L \rightarrow L_i \) covering the canonical projections \( p_i : M_1 \times M_2 \rightarrow M_i \), \( i = 1, 2 \): \( \text{(2.11)} \)

In this situation we can consider back-ward images, \( P_1^* \mathcal{L}_1, P_2^* \mathcal{L}_2 \subset DL \), and they are regular because the \( p_i \) are submersions, \( i = 1, 2 \). Finally consider

\[ P_1^* \mathcal{L}_1 \ast P_2^* \mathcal{L}_2. \]

If it is regular, it is a Dirac-Jacobi structure on \( L \).

Now notice that, in view of diagram \( \text{(2.11)} \), \( L \) comes with (partial) connections \( D_i \), along \( \ker p_i, \ i = 1, 2 \), and we can define a genuine connection \( D^\times \) in \( L \), by putting

\[ D^\times_{X_1 + X_2} \lambda = (D_1)_{X_1} \lambda + (D_2)_{X_2} \lambda, \quad X_i \in \ker p_i, \quad i = 1, 2. \]

Lemma 2.2.9. The following conditions are equivalent:

(1) Connection \( D^\times \) is flat.

(2) Around every point of \( M_1 \times M_2 \) there is a nowhere vanishing local section \( \lambda \in \Gamma(L) \) such that \( \lambda = P_1^* \lambda_1 = P_2^* \lambda_2 \) for some local sections \( \lambda_i \in \Gamma(L_i), \ i = 1, 2 \).
(3) For every \((\bar{x}_1, \bar{x}_2) \in M_1 \times M_2\), there are local trivializations \(L_i \cong \mathbb{R}_{M_i}\), around \(\bar{x}_i\), \(i = 1, 2\), and \(L \cong \mathbb{R}_{M_1 \times M_2}\), around \((\bar{x}_1, \bar{x}_2)\), such that the \(P_i : L \to L_i\) identify with projections \(\mathbb{R}_{M_1 \times M_2} \to \mathbb{R}_{M_i}\), \((x_1, x_2; r) \mapsto (x_i; r)\), where \((x_1, x_2) \in M_1 \times M_2\), and \(r \in \mathbb{R}\).

Proof.\
(1) \(\implies\) (2). Choose as \(\lambda\) a nowhere vanishing flat section wrt \(D^x\).\
(2) \(\implies\) (3). Choose the (local) trivializations \(L \cong \mathbb{R}_{M_1 \times M_2}\), and \(L_i \cong \mathbb{R}_{M_i}\), that identify \(\lambda\), and \(\lambda_i\), with the constant functions 1, \(i = 1, 2\).\
(3) \(\implies\) (1). Obvious.

When one, hence all three, of the conditions in Lemma 2.2.9 hold, we say that the product is flat. If, additionally, \(P_1^0 \mathfrak{L}_1 \ast P_2^0 \mathfrak{L}_2\) is regular, we call it the (flat) product of \(\mathfrak{L}_1\) and \(\mathfrak{L}_2\) (wrt \(P_1, P_2\)) and denote it by \(\mathfrak{L}_1 \times^! \mathfrak{L}_2\).

We will provide examples of products of Dirac-Jacobi structures later on. For now we only remark that, if we apply an analogous construction to a pair of Dirac structures, we get exactly their standard product.

Remark 2.2.10. The above discussion applies to complex Dirac-Jacobi structures without modifications.

Remark 2.2.11. Let \(\mathfrak{L}_i\) be Dirac Jacobi structures on the line bundles \(L_i \to M_i\), \(i = 1, 2\), and let \(\mathfrak{L}_1 \times^! \mathfrak{L}_2\) be a flat product of them wrt to projections \(P_1, P_2\) as in diagram (2.11). Finally, let \(B\) be a closed Atiyah 2-form on \(L_1\). It is easy to see that
\[(\mathfrak{L}_1 \times^! \mathfrak{L}_2)^P_1 B = \mathfrak{L}_1^B \times^! \mathfrak{L}_2.\]

2.2.6. Homogeneous generalized complex structures. As already mentioned, unlike Poisson manifolds, manifolds \(M\) with a Jacobi bundle \((L \to M, J)\) possess two kinds of characteristic leaves. Odd dimensional ones inherit from \(J\) a canonical contact structure, and we call them contact leaves. Even dimensional leaves inherit from \(J\) an lcs structure, and we call them lcs leaves. Let \(\mathcal{O}\) be a leaf and \(x_0 \in \mathcal{O}\). By a transversal to \(\mathcal{O}\) at \(x_0\), we mean a submanifold \(N\) such that \(x_0 \in N\), and \(T_{x_0} M = T_{x_0} N \oplus T_{x_0} \mathcal{O}\). It turns out that transversals to lcs leaves, with the restricted line bundle, possess a canonical Jacobi structure around \(x_0\). Additionally, this Jacobi structure vanishes at \(x_0\). On the other hand, transversals to contact leaves, possess a canonical homogeneous Poisson structure (up to the choice of a nowhere vanishing section of \(L\) around \(x_0\)). The homogeneous Poisson structure vanishes at \(x_0\). Recall that a homogeneous Poisson structure on a manifold \(M\) is a pair \((\pi, Z)\) where \(\pi\) is a Poisson bi-vector, and \(Z\) is a vector field, called the homogeneity vector field, such that \(\mathcal{L}_Z \pi = -\pi\).

Example 2.2.12. On \(\mathbb{R}^{2d}\), with coordinates \((x^1, \ldots, x^d, p_1, \ldots, p_d)\), there is a canonical homogeneous Poisson structure \((\pi_{\text{can}}, Z_{\text{can}})\) given by
\[
\pi_{\text{can}} = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i}, \quad \text{and} \quad Z_{\text{can}} = p_i \frac{\partial}{\partial p_i}.
\]
The theory of Jacobi structures is strongly related to that of homogeneous Poisson structures, as the example of transversals to contact leaves shows (see also [11]). In a similar way generalized contact geometry is strongly related to the theory of homogeneous generalized complex structures which we define now. Let $M$ be a manifold.

**Definition 2.2.13.** A **homogeneous generalized complex structure** on $M$ is a pair $(\mathcal{J}, Z)$, where

\[
\mathcal{J} = \begin{pmatrix} A & \pi^\sharp \\ \sigma & -A^* \end{pmatrix} \in \text{End}(\mathbb{T}M)
\]

is a generalized complex structure, and $Z = (Z, \zeta)$ is a section of the generalized tangent bundle $\mathbb{T}M$ such that

\[
\begin{align*}
\mathcal{L}_Z A &= \pi^\sharp \circ (d\zeta)_3, \\
\mathcal{L}_Z \pi &= -\pi \text{ (in particular $(\pi, Z)$ is a homogeneous Poisson structure),} \\
\mathcal{L}_Z \sigma &= \sigma - \iota_A d\zeta,
\end{align*}
\]

where $\iota_A d\zeta$ is the 2-form defined by

\[
(\iota_A d\zeta)(X, Y) = d\zeta(AX, Y) + d\zeta(X, AY),
\]

for all $X, Y \in \mathfrak{X}(M)$.

The main motivation for this definition is that the transversal to a contact leaf in the base of a generalized contact bundle is a homogeneous generalized complex manifold, as we will show in Section 3.1.

Definition 2.2.13 can be rephrased in terms of the complex Dirac structure associated to $\mathcal{J}$, i.e. the $+i$-eigenbundle $\mathcal{L}_3$ of $\mathcal{J}$ in the complexified generalized tangent bundle $\mathbb{T}M \otimes \mathbb{C}$. Namely, we have the following

**Proposition 2.2.14.** Let

\[
\mathcal{J} = \begin{pmatrix} A & \pi^\sharp \\ \sigma & -A^* \end{pmatrix}
\]

be a generalized complex structure on $M$, and let $Z = (Z, \zeta)$ be a section of the generalized tangent bundle $\mathbb{T}M$. Then the following conditions are equivalent.

1. $(\mathcal{J}, Z)$ is a homogeneous generalized complex structure;
2. $([Z, X], \mathcal{L}_Z \eta - \eta + \iota_X d\zeta) \in \Gamma(\mathcal{L}_3)$ for all $(X, \eta) \in \Gamma(\mathcal{L}_3)$;
3. $(\mathcal{L}_Z \eta + \iota_X d\zeta) \in \Gamma(\mathcal{L}_3)$ for all $(X, \eta) \in \Gamma(\mathcal{L}_3)$.

**Proof.** It is clear that (2) and (3) are equivalent. It remains to prove that (1) $\iff$ (3). Assume first that $(\mathcal{J}, Z)$ is a homogeneous generalized complex structure, let $\alpha = (X, \eta) \in \Gamma(\mathcal{L}_3)$ and compute

\[
\mathcal{J} \begin{pmatrix} [Z, X] + X \\ \mathcal{L}_Z \eta + \iota_X d\zeta \end{pmatrix} = \begin{pmatrix} A & \pi^\sharp \\ \sigma & -A^* \end{pmatrix} \begin{pmatrix} [Z, X] + X \\ \mathcal{L}_Z \eta + \iota_X d\zeta \end{pmatrix} = \begin{pmatrix} A[Z, X] + AX + \pi^\sharp \mathcal{L}_Z \eta + \pi^\sharp \iota_X d\zeta \\ \sigma [Z, X] + \sigma X - A^* \mathcal{L}_Z \eta - A^* \iota_X d\zeta \end{pmatrix}.
\]

(2.12)
The first entry is

\[ A[Z,X] + AX + \pi^z L_Z \eta + \pi^\ell_X d\zeta \]

\[ = [Z,AX] - (L_Z A)X + AX + [Z,\pi^z \eta] - (L_Z \pi)^z \eta + (\pi^z \circ (d\zeta)_b)X \]

\[ = [Z,AX + \pi^z \eta] + AX + \pi^z \eta \]

\[ = \iota([Z,X] + X), \]

where we used that \( AX + \pi^z \eta \) is the first entry of \( \mathcal{J} \alpha \). Similarly, the second entry in (2.12) is

\[ \sigma_0[Z,X] + \sigma_0 X - A^* L_Z \eta - A^* \iota_X d\zeta \]

\[ = L_Z (\sigma_0 X) - (L_Z \sigma)_0 X + \sigma_0 X - L_Z (A^* \eta) + (L_Z A)^* \eta - (A^* \circ (d\zeta)_b)X \]

\[ = L_Z (\sigma_0 X - A^* \eta) + (d\zeta)_b (AX + \pi^z \eta) \]

\[ = \iota(L_Z \eta + \iota_X d\zeta), \]

showing that \( ([Z,X] + X, L_Z \eta + \iota_X d\zeta) \) is a +i-eigensection of \( \mathcal{J} \).

Conversely, let \( ([Z,X] + X, L_Z \eta + \iota_X d\zeta) \) be a +i-eigensection of \( \mathcal{J} \) for all +i-eigensections \( \alpha = (X, \eta) \). One can show that \( (\mathcal{J}, \mathcal{Z}) \) is a homogeneous generalized complex structure with a similar computation as above (but in the reverse order). \( \square \)

Every homogeneous generalized complex structure \( (\mathcal{J}, \mathcal{Z}) \) determines a complex Dirac-Jacobi structure on the trivial line bundle \( \mathbb{R}_M := M \times \mathbb{R} \to M \) according to the following

**Proposition 2.2.15.** Let \( (\mathcal{J}, \mathcal{Z}) \) be a homogeneous generalized complex structure on \( M \), with \( \mathcal{Z} = (Z, \zeta) \). In \( \mathbb{D} \mathbb{R}_M \otimes \mathbb{C} = \mathbb{D} \mathbb{R}_M \otimes \mathbb{C} \) consider the subbundle \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \) spanned over \( \mathbb{C} \) as follows:

\[ \mathcal{L}(\mathcal{J}, \mathcal{Z}) := \langle [1 - Z, \zeta + \zeta(Z) \cdot i], (X, \eta + (\eta(Z) - \zeta(X)) \cdot i) \rangle \quad \text{for} \quad (X, \eta) \in \mathcal{L}_{\mathcal{J}} \quad (2.13) \]

(\text{where we use the same notations as in Remark 1.1.1). Then \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \) is a (complex) Dirac-Jacobi structure.}

**Proof.** A direct computation with the generators shows that \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \) is isotropic. As its rank is \( \dim M + 1 \), it is also maximal isotropic. For the involutivity, we will show that the trilinear form

\[ \Upsilon : \wedge^3 \mathcal{L}(\mathcal{J}, \mathcal{Z}) \to \mathbb{R}_M, \quad (\alpha, \beta, \gamma) \mapsto \langle \alpha, [\beta, \gamma] \rangle \]

vanishes on generators. To do this, first denote by \( \langle -,- \rangle_{\wedge^2 \mathcal{L}} \) and \( [-,-]_{\wedge^2 \mathcal{L}} \) the bilinear form and the Dorfman bracket in the generalized tangent bundle, and notice that, for all

\[ (X_i + f_i, \eta_i + g_i \cdot i) \in \Gamma(\mathbb{D} \mathbb{R}_M \otimes \mathbb{C}), \]

with \( X_i \in \mathfrak{X}(M), \eta_i \in \Omega^1(M), \text{ and } f_i, g_i \in C^\infty(M) \), \( i = 1, 2 \), we have:

\[ \langle (X_1 + f_1, \eta_1 + g_1 \cdot i), (X_2 + f_2, \eta_2 + g_2 \cdot i) \rangle = \langle (X_1, \eta_1), (X_2, \eta_2) \rangle_{\wedge^2 \mathcal{L}} + f_1 g_2 + f_2 g_1, \quad (2.14) \]

and

\[ [(X_1 + f_1, \eta_1 + g_1 \cdot i), (X_2 + f_2, \eta_2 + g_2 \cdot i)] = [(X_1, \eta_1), (X_2, \eta_2)]_{\wedge^2 \mathcal{L}} + (X_1 (f_1) - X_2 (f_1), g_2 df_1 + g_1 df_2 + (X_1 (g_2) - X_2 (g_2) + g_2 f_1 + \eta_1 (X_2)) \cdot i) \quad (2.15) \]

Now, let \( \alpha = (1 - Z, \zeta + \zeta(Z) \cdot i) \) and for \( (X_i, \eta_i) \in \Gamma(\mathcal{L}_g) \), let

\[ \beta_i = (X_i, \eta_i + (\eta_i (Z) - \zeta(X_i)) \cdot i) \]
be the corresponding generator of \( \Gamma(\mathcal{L}(\mathcal{J}, \mathcal{Z})) \), \( i = 1, 2, 3 \). A straightforward computation exploiting \((2.13)\) and \((2.15)\) shows that
\[
\mathcal{Y}(\beta_1, \beta_2, \alpha) = \{(X, \eta_1), ([X, X_2], \mathcal{L}_Z \eta_2 - \eta_2 + i_{X_2} \mathcal{d} \zeta)\}_{TM}
\]
and the rhs vanishes in view of Proposition \((2.2.14)\). Finally, again from \((2.14)\) and \((2.15)\) we get
\[
\mathcal{Y}(\beta_1, \beta_2, \beta_3) = \{(X, \eta_1), ([X, X_2], (X_3, \beta_3))\}_{TM} = 0
\]
and this concludes the proof. \( \Box \)

Complex Dirac-Jacobi structures on \( \mathbb{R}_M \) of the form \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \) for some homogeneous generalized complex structure \( (\mathcal{J}, \mathcal{Z}) \) can be characterized as follows. First of all, denote by \( p_\mathbb{R} : \mathbb{D} \mathbb{R}_M = TM \oplus \mathbb{R}_M \to \mathbb{R}_M \) the natural projection.

**Proposition 2.2.16.** A complex Dirac-Jacobi structure \( \mathcal{L} \subset \mathbb{D} \mathbb{R}_M \otimes \mathbb{C} \) is of the form \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \) for some homogeneous generalized complex structure \( (\mathcal{J}, \mathcal{Z}) \) iff it satisfies the following conditions

\[
\begin{align*}
(1) \quad & \text{rank}_\mathbb{C}(\mathcal{L} \cap \mathcal{F}) = 1, \\
(2) \quad & p_\mathbb{D} \mathcal{L} + p_\mathbb{D} \mathcal{F} = \mathbb{D} \mathbb{R}_M \otimes \mathbb{C}, \\
(3) \quad & p_\mathbb{R} \circ p_\mathbb{D} : \mathcal{L} \cap \mathcal{F} \to \mathbb{M} \times \mathbb{C} \text{ is surjective (hence an isomorphism)}.
\end{align*}
\]

**Proof.** Begin with a homogeneous generalized complex structure \( (\mathcal{J}, \mathcal{Z}) \), \( \mathcal{Z} = (Z, \zeta) \), and the associated complex Dirac-Jacobi structure \( \mathcal{L} = \mathcal{L}(\mathcal{J}, \mathcal{Z}) \) as in \((2.13)\). It is easy to see that \( \mathcal{L} \cap \mathcal{F} \) is spanned by \((1 - Z, \zeta + \zeta(Z) \cdot j)\), in particular \( \mathcal{L} \) satisfies property \((1)\) in the statement. For property \((2)\) notice that \( p_\mathbb{D} \mathcal{L} + p_\mathbb{D} \mathcal{F} \) is spanned by \( 1 - Z \) and
\[
p_T \mathcal{L}_\mathcal{J} + p_T \mathcal{F}_\mathcal{J} = p_T(\mathcal{L}_\mathcal{J} + \mathcal{F}_\mathcal{J}) = TM \otimes \mathbb{C},
\]
where we denoted by \( p_T : TM \to TM \) the projection. So \( \mathcal{L} \) satisfies also \((2)\). Property \((3)\) now follows from the fact that
\[
p_\mathbb{R}(1 - Z) = 1 \neq 0.
\]
This concludes the “only if” part of the proof.

For the “if” part, let \( \mathcal{L} \subset \mathbb{D} \mathbb{R}_M \otimes \mathbb{C} \) be a complex Dirac-Jacobi structure satisfying properties \((1)-(3)\) in the statement. It follows from \((1)\) and \((3)\) that there exists a unique, necessarily real, section \( \alpha \) of \( \mathcal{L} \cap \mathcal{F} \) such that \( (p_\mathbb{R} \circ p_\mathbb{D}) \alpha = 1 \). In particular, \( \alpha \) is of the form \((1 - Z, \zeta + g \cdot j)\), for a real vector field \( Z \), a real 1-form \( \zeta \), and a real function \( g \). From isotropy, \( g = \zeta(Z) \), so
\[
\alpha = (1 - Z, \zeta + \zeta(Z) \cdot j), \quad Z \in \mathfrak{X}(M), \quad \zeta \in \Omega^1(M).
\]
We put \( \mathcal{Z} := (Z, \zeta) \). Next we want to construct a generalized complex structure \( \mathcal{J} : TM \to TM \). To do this, we first define
\[
\mathcal{L}_\mathcal{T} := \{(X, \eta) \in TM \otimes \mathbb{C} : (X, \eta + (\eta(Z) - \zeta(X)) \cdot j) \in \mathcal{L}\}.
\]
We claim that \( \mathcal{L}_\mathcal{T} \) is a complex Dirac structure such that \( \mathcal{L}_\mathcal{T} \cap \overline{\mathcal{L}_\mathcal{T}} = 0 \). From \((2.14)\), \( \mathcal{L}_\mathcal{T} \) is (pointwise) maximal isotropic. So it is a regular vector subbundle provided only it is the image of a vector bundle map. Our next aim is constructing such a map. First of all, consider the endomorphism
\[
F : \mathbb{D} \mathbb{R}_M \otimes \mathbb{C} \to \mathbb{D} \mathbb{R}_M \otimes \mathbb{C}, \quad (X + f, \eta + g \cdot j) \mapsto (X + fZ + f, \eta - f\zeta + g \cdot j),
\]
and the natural projection
\[ p_T : \mathbb{D}R_M \otimes \mathbb{C} \to \mathbb{T}M \otimes \mathbb{C}, \quad (X + f, \eta + g \cdot j) \mapsto (X, \eta). \]
We want to show that
\[ \mathcal{L}_T = (p_T \circ F) \mathcal{L}. \]
As \( F \) fixes elements of the form \((X, \eta + g \cdot j)\), it is clear that \( \mathcal{L}_T \subset (p_T \circ F) \mathcal{L} \). In order to check the reverse inclusion, begin with \( \beta = (X + f, \eta + g \cdot j) \in \mathcal{L} \). It follows from isotropy that \( g = \eta(Z) - \zeta(X) - f \zeta(Z) \). Now compute
\[ (\tilde{X}, \tilde{\eta}) := (p_T \circ F) \beta = (X + f Z, \eta - f \zeta). \]
But \((\tilde{X}, \tilde{\eta}) \in \mathcal{L}_T\), indeed
\[
\begin{align*}
(\tilde{X}, \tilde{\eta}) &= (\tilde{X} + (\tilde{\eta}(Z) - \zeta(\tilde{X})) \cdot j) \\
&= (X + f Z, \eta - f Z + (\eta(Z) - \zeta(X) - 2 f \zeta(Z)) \cdot j) \\
&= \beta - f \alpha
\end{align*}
\]
which belongs to \( \mathcal{L} \). So \( \mathcal{L}_T \) is a regular maximal isotropic subbundle of \( \mathbb{T}M \otimes \mathbb{C} \). Involutivity follows from (2.15) and the involutivity of \( \mathcal{L} \). Next we check \( \mathcal{L}_T \cap \mathcal{L}_T = 0 \). So let \((X, \eta) \in \mathcal{L}_T \cap \mathcal{L}_T \). This means that
\[ (X, \eta + (\eta(Z) - \zeta(X)) \cdot j) \in \mathcal{L} \cap \overline{\mathcal{L}}. \]
As \( p_R X = 0 \) this can only be if \((X, \eta) = 0 \). We conclude that \( \mathcal{L}_T \) is the \( +i \)-eigenbundle of a generalized complex structure \( \mathcal{J} \) on \( M \). Using (2.15) again, and Proposition 2.2.14 it is easy to see that \((\mathcal{J}, \mathcal{Z})\) is a homogeneous generalized complex structure in a similar way as in the proof of Proposition 2.2.15. Finally, it is obvious that \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \subset \mathcal{L} \). As they are both maximal isotropic, they actually coincide. This concludes the proof. \( \square \)

Notice that conditions (1) and (2) in Proposition 2.2.16 make sense for every complex Dirac-Jacobi structure. So we give the following

**Definition 2.2.17.** A complex Dirac-Jacobi structure \( \mathcal{L} \subset \mathbb{D}L \otimes \mathbb{C} \) on a line bundle \( L \to M \) is of homogeneous generalized complex type if

1. \( \text{rank}_D(\mathcal{L} \cap \overline{\mathcal{L}}) = 1 \),
2. \( p_D \mathcal{L} + p_D \overline{\mathcal{L}} = DL \otimes \mathbb{C} \).

The above definition is motivated by the following

**Proposition 2.2.18.** Let \( \mathcal{L} \subset \mathbb{D}L \otimes \mathbb{C} \) be a complex Dirac-Jacobi structure of homogeneous generalized complex type on a line bundle \( L \to M \). Then, locally, around every point of \( M \), there exists a trivialization \( L \simeq \mathbb{R}M \) identifying \( \mathcal{L} \) with the complex Dirac-Jacobi structure \( \mathcal{L}(\mathcal{J}, \mathcal{Z}) \subset \mathbb{D}R_M \otimes \mathbb{C} \) induced by a homogeneous generalized complex structure \((\mathcal{J}, \mathcal{Z})\).

**Proof.** Let \( \mathcal{L} \) be as in the statement, and let \( x_0 \in M \). Choose a nowhere vanishing local section \( \alpha = (\Delta, \psi) \in \Gamma(\mathcal{L} \cap \overline{\mathcal{L}}) \) around \( x_0 \). We can choose \((\Delta, \psi)\) to be real. Then we have \( \Delta \neq 0 \). Indeed, if \( \Delta_x = 0 \) for some \( x \), then
\[
0 \neq \psi_x \in \mathcal{L} \cap J^1 L \subset \text{Ann}(p_D \mathcal{L}) \cap \text{Ann}(p_D \overline{\mathcal{L}}) = \text{Ann}(p_D \mathcal{L} + p_D \overline{\mathcal{L}}) = \text{Ann}(DL \otimes \mathbb{C}) = 0,
\]
a contradiction. So \( \Delta \) is a non-vanishing (local) derivation. It is easy to see that, for a non-vanishing derivation of \( L \), locally, around every point, there always exists a trivialization
3. The Transversal to a Leaf

Let \((L \to M, \mathbb{K})\) be a generalized contact bundle with
\[
\mathbb{K} = \begin{pmatrix} \varphi & J^t \\ \omega & -\varphi^t \end{pmatrix},
\]
and let \(\mathcal{L}\) be its +i-eigenbundle. In this section, as a preparation for the splitting theorems, we study special classes of submanifolds of \(M\). Specifically, characteristic leaves of the underlying Jacobi structure \(J\) and their transversals. As we already outlined, in this paper, by a transversal to a leaf \(\mathcal{O}\) at a point \(x_0 \in \mathcal{O}\), we will always understand a minimal dimension transversal, i.e., a submanifold \(N\) through \(x_0\) such that \(T_{x_0}M = T_{x_0}N \oplus T_{x_0}\mathcal{O}\). We begin with contact leaves.

3.1. Contact leaves and their transversals. Recall that an odd dimensional characteristic leaf \(\mathcal{O}\) of \(J\) possesses a canonical contact structure \(H \subset T\mathcal{O}\). This can be seen as follows. First of all, \(J\) restricts to a Jacobi structure \(J_\mathcal{O}\) on the restricted line bundle \(L_\mathcal{O} := L_O \to \mathcal{O}\). Now \(\sigma J^t_\mathcal{O} : J^1 L_\mathcal{O} \to T\mathcal{O}\) is surjective (by definition of characteristic leaf), and it follows from \(\dim \mathcal{O} = \text{odd}\) that \(J^t_\mathcal{O} : J^1 L_\mathcal{O} \to DL_\mathcal{O}\) is surjective, hence an isomorphism. Let \(\omega_\mathcal{O} = J^{-1}_\mathcal{O} \in \Omega^2_{L_\mathcal{O}}\) be the Atiyah 2-form inverting \(J_\mathcal{O}\), i.e., \((\omega_\mathcal{O})_0 := (J^t_\mathcal{O})^{-1}\). Notice that \(DL_\mathcal{O} = (\text{im} J^t)|_\mathcal{O}\) and \(\omega_\mathcal{O}\) agrees with the pointwise restriction to \(\mathcal{O}\) of the 2-form \(\omega : \wedge^2 \text{im} J^t \to L\) from Remark 2.2.6. Now, the integrability condition for \(J_\mathcal{O}\) is equivalent to \(d_B \omega_\mathcal{O} = 0\), and \(\iota_\mathcal{O} \omega_\mathcal{O} \in \Omega^1_{L_\mathcal{O}}\) is necessarily of the form \(\iota_\mathcal{O} \omega_\mathcal{O} = \theta_\mathcal{O} \circ \sigma\) for a unique \(L_\mathcal{O}\)-valued 1-form \(\theta_\mathcal{O} : T\mathcal{O} \to L_\mathcal{O}\). The kernel \(H\) of \(\theta_\mathcal{O}\) is a contact structure containing the full information on \(J_\mathcal{O}\). This contact structure can be equivalently encoded in a generalized contact structure
\[
\mathbb{K}_\mathcal{O} = \begin{pmatrix} 0 & J^t_\mathcal{O} \\ -\left(J^{-1}_\mathcal{O}\right)_0 & 0 \end{pmatrix}
\]
on \(L_\mathcal{O}\). Let \(\mathcal{L}_\mathcal{O} \subset DL_\mathcal{O} \otimes \mathbb{C}\) be the +i-eigenbundle of \(\mathbb{K}_\mathcal{O}\). We have
\[
\mathcal{L}_\mathcal{O} := \left\{ (J^t_\mathcal{O}(\psi), iv\psi) \in DL_\mathcal{O} \otimes \mathbb{C} : \psi \in J^1 L_\mathcal{O} \otimes \mathbb{C} \right\}.
\]
In the following, for an (immersed) submanifold \(S \hookrightarrow M\), we simply denote by \(L_S\) the restricted line bundle \(L|_S\), and by \(I_S : L|_S \hookrightarrow L\) the natural (injective) immersion. It is a regular vector bundle map covering the injective immersion \(i_S : S \hookrightarrow M\).

Proposition 3.1.1. Let \(\mathcal{O}\) be an odd dimensional leaf of \(J\). The backward image of the complex Dirac-Jacobi structure \(\mathcal{L}\) along the immersion \(I_\mathcal{O} : L_\mathcal{O} \hookrightarrow L\) is a Dirac-Jacobi structure of generalized contact type on \(L_\mathcal{O}\), denoted \(I^t_\mathcal{O} \mathcal{L}\). Additionally, it is a B-field transformation of \(\mathcal{L}_\mathcal{O}\):
\[
I^t_\mathcal{O} \mathcal{L} = \mathcal{L}_\mathcal{O}^B
\]
for some \(B \in \mathbb{Z}_{L_\mathcal{O}}^2\).
Proof. We divide the proof in several steps. First we prove that $I^1_O \mathfrak{L} \subset D L_O \otimes \mathbb{C}$ is a regular subbundle, checking the clean intersection condition (2.10):

$$\text{rank}_O (p_D \mathfrak{L}|_O + D L_O \otimes \mathbb{C}) = \text{constant}.$$ 

To do this notice that

$$D L_O \otimes \mathbb{C} = \text{im} J^1|_O \otimes \mathbb{C} = p_D \mathfrak{L}|_O \cap p_D \overline{\mathfrak{L}}|_O \subset p_D \mathfrak{L}|_O.$$

Hence

$$p_D \mathfrak{L}|_O + D L_O \otimes \mathbb{C} = p_D \mathfrak{L}|_O$$

which is constant rank because

1. $p_D \mathfrak{L}|_O + p_D \overline{\mathfrak{L}}|_O = (D L)|_O \otimes \mathbb{C}$ is constant rank,
2. $p_D \mathfrak{L}|_O \cap p_D \overline{\mathfrak{L}}|_O = D L_O \otimes \mathbb{C}$ is constant rank, and
3. $p_D \mathfrak{L}|_O$ and $p_D \overline{\mathfrak{L}}|_O$ have the same rank.

This proves the first part of the statement.

Next we show that $I^1_O \mathfrak{L}$ is a Dirac-Jacobi structure of generalized contact type. For this, it is enough to check that

$$I^1_O \mathfrak{L} \cap I^1_O \overline{\mathfrak{L}} = I^1_O \mathfrak{L} \cap I^1_O \mathfrak{L} = 0.$$

So let $(\Delta, \psi) \in (I^1_O \mathfrak{L} \cap I^1_O \overline{\mathfrak{L}})_x$ for some $x \in O$. This means that, there exist $\chi', \chi'' \in J^1 x L \otimes \mathbb{C}$ such that $\psi = I^1_O \chi' = I^1_O \chi''$, and, additionally, $(\Delta, \chi') \in \Sigma_x$ and $(\Delta, \chi'') \in \overline{\Sigma}_x$. Now, define $\chi \in J^1 x L \otimes \mathbb{C}$ by putting

$$(\chi, \nabla) = \begin{cases} 
\langle \chi', \nabla \rangle & \text{if } \nabla \in p_D \Sigma_x \\
\langle \chi'', \nabla \rangle & \text{if } \nabla \in p_D \overline{\Sigma}_x
\end{cases}.$$

As both $\chi'$ and $\chi''$ agree with $\psi$ on $(p_D \mathfrak{L} \cap p_D \overline{\mathfrak{L}})_x = (\text{im} J^1 x L \otimes \mathbb{C})_x = D x L_O \otimes \mathbb{C}$, then $\chi$ is well-defined. It immediately follows from (2.8) that $(\Delta, \chi) \in (\Sigma \cap \overline{\Sigma})_x = 0$. So $(\Delta, \chi) = 0$, hence $(\Delta, \psi) = 0$. We conclude that $I^1_O \mathfrak{L}$ is a Dirac-Jacobi structure of generalized contact type. In particular, there is an underlying Jacobi structure $J$ on $L_O$.

As a third step, we prove that the Jacobi structure underlying $I^1_O \mathfrak{L}$ is precisely $J O$: the restriction to $O$ of the Jacobi structure $J$. In other words, $J = J O$. First of all,

$$p_D I^1_O \mathfrak{L} = p_D \mathfrak{L} \cap (D L_O \otimes \mathbb{C}) = D L_O \otimes \mathbb{C}.$$

In particular the $L_O$-valued 2-form $\varpi O$ induced by $I^1_O \mathfrak{L}$ on $p_D I^1_O \mathfrak{L}$ is a genuine (complex) Atiyah 2-form on $L_O$. Notice that $\varpi O$ actually agrees with $\varpi$ on $D L_O$. Indeed let $\Delta, \nabla \in D L_O$. There is $\psi \in J^1 L$ such that $(\Delta, \psi) \in I^1_O \mathfrak{L}$. Compute

$$\varpi O(\Delta, \nabla) = \langle I^1_O \psi, \nabla \rangle = \langle \psi, \nabla \rangle = \varpi(\Delta, \nabla).$$

Now, let $\psi \in J^1 L O$, and let $\Psi \in J^1 L$ be such that $I^1_O \Psi = \psi$. We want to compare $J^1_O \Psi$ and $J^1 \psi$. To do this pick $\nabla \in D L_O$ and compute

$$\text{Im } \varpi O(J^1 \psi, \nabla) = \langle \psi, \nabla \rangle = \langle I^1_O \Psi, \nabla \rangle = \langle \Psi, \nabla \rangle$$

$$= \text{Im } \varpi(J^1 \psi, \nabla) = \text{Im } \varpi(J^1 O \psi, \nabla) = \text{Im } \varpi O(J^1 O \psi, \nabla).$$

But $\text{Im } \varpi O$ is non-degenerate (Remark 2.2.6) so $J^1 O \psi = J^1 \psi$. In particular $\text{Im } \varpi O = J^1 O^{-1}$. 

Finally, we prove that $I^i_O \mathcal{L}$ is a $B$-field transformation of $\mathcal{L}_O$. From $p_D I^i_O \mathcal{L} = DL_O \otimes \mathbb{C}$, we have

$$I^i_O \mathcal{L} = \{(\Delta, \iota_\Delta \varpi_O) : \Delta \in DL_O\} = \text{graph}(\varpi_O) \subset DL_O \otimes \mathbb{C}. \tag{3.2}$$

Let $B = \text{Re} \varpi_O \in \Omega^2_{DL_O}$. From (3.2), and the involutivity of $I^i_O \mathcal{L}$ we have $d_D \varpi_O = 0$, hence $d_D B = 0$. Finally compute

$$(I^i_O \mathcal{L})^{-B} = \text{graph}(\varpi_O - \text{Re} \varpi_O) = \text{graph}(i \text{Im} \varpi_O) = \mathcal{L}_O,$$

where we used (3.1) and the fact that $\text{Im} \varpi_O = J^{-1}_O$. \hfill \Box

We now pass to transversals. A transversal to a characteristic leaf of a generalized complex manifold inherits a generalized complex structure, at least around the intersection point with the leaf. The precise analogue cannot be true for contact leaves of a generalized contact structure, simply because, in this case, transversals are even dimensional.

**Proposition 3.1.2.** Let $N$ be a transversal at $x_0 \in \mathcal{O}$ to an odd dimensional leaf $\mathcal{O}$ of $J$. Around $x_0$, the backward image of the complex Dirac-Jacobi structure $\mathcal{L}$ along the embedding $I_N : L_N \hookrightarrow L$ is a complex Dirac-Jacobi structure $I^i_N \mathcal{L}$ of homogeneous generalized complex type such that

$$\left(I^i_N \mathcal{L} \cap \overline{I^i_N \mathcal{L}}\right)_{x_0}$$

is spanned by a vector of the form $(1 \times x_0, \psi)$. In particular, any local trivialization $L_N \simeq \mathbb{R}_N$ around $x_0$, identifies $I^i_N \mathcal{L}$ with the complex Dirac-Jacobi structure corresponding to a homogeneous generalized complex structure.

**Proof.** First of all we prove that $I^i_N \mathcal{L} \subset DL_N$ is a regular subbundle, hence a Dirac-Jacobi structure on $L_N$, checking the clean intersection condition (2.10):

$$\text{rank}_\mathbb{C}(p_D \mathcal{L}|_N + DL_N \otimes \mathbb{C}) = \text{constant}.$$ 

We have

$$p_D \mathcal{L}|_N \supset p_D \mathcal{L}|_N \cap p_D \overline{\mathcal{L}}|_N = \text{im} J^i|_N \otimes \mathbb{C}.$$

At the point $x_0$ we have $(\text{im} J^i|_{x_0}) \otimes \mathbb{C} = D_{x_0} L_O$, so

$$p_D \mathcal{L}|_{x_0} + D_{x_0} L_N \otimes \mathbb{C} \supset D_{x_0} L_O + D_{x_0} L_N = D_{x_0} L.$$ 

But $p_D \mathcal{L}|_N + DL_N \otimes \mathbb{C} \subset (DL)|_N$ is a smooth, possibly non-regular subbundle, hence its rank can only increase around $x_0$, and we conclude that

$$p_D \mathcal{L}|_N + DL_N \otimes \mathbb{C} = (DL)|_N \otimes \mathbb{C} \tag{3.3}$$

in a whole neighborhood of $x_0$. In particular, the lhs has constant rank.

Next we show that $\text{rank}_\mathbb{C}(I^i_N \mathcal{L} \cap \overline{I^i_N \mathcal{L}}) = 1$ around $x_0$. Denote by $\nu N = TM|_N /TN$ and $\nu^* N = \text{Ann}(TN) \subset T^* M|_N$ the normal and the conormal bundle to $N$, respectively. It is useful to consider the following skew-symmetric bilinear map

$$\mu : \wedge^2 (\nu^* N \otimes L_N) \to L_N, \quad (\eta, \theta) \mapsto (J^i \eta, \theta),$$

and the associated vector bundle map $\mu^\sharp : \nu^* N \otimes L_N \to \nu N$ implicitly defined by

$$(\mu^\sharp \eta, \theta) = \mu(\eta, \theta), \quad \eta, \theta \in \nu^* N \otimes L_N.$$
In other words $\mu^2$ is the composition

$$\nu^*N \otimes L_N \xrightarrow{J^1} (DL)|_N \xrightarrow{\sigma} TM|_N \to \nu N,$$

where the last arrow is the natural projection (with kernel $TN$). We want to show that $\mu$ has maximal rank around $x_0$: $\text{rank} \mu = \text{rank}(\nu N) - 1 = \dim \mathcal{O} - 1 = \text{even}$. To do this it is enough to show that $\text{rank}_{x_0} \mu = \dim \mathcal{O} - 1$, in other words $\dim(\ker \mu^2_{x_0}) = 1$. So compute

$$\ker \mu^2_{x_0} = \left\{ \eta \in \nu^*x_0 N \otimes L_{x_0} : J^1_{x_0} \eta \in DL_N \right\}.$$

But $\nu^*x_0 N \otimes L_{x_0} = T_{x_0} \mathcal{O} \otimes L_{x_0}$, and $(\text{im} J^1_{x_0}) = D_{x_0} L_{x_0}, 3$ so we find

$$\ker \mu^2_{x_0} = \left\{ \eta \in T_{x_0} \mathcal{O} \otimes L_{x_0} : J^1_{x_0} \eta = r \cdot 1_{x_0}, \text{for some } r \in \mathbb{R} \right\}$$

(3.4)

Now, we go back to $I^*_N \mathcal{L}$ and consider the real (a-priori not necessarily regular) subbundle

$$R := \left\{ \text{Re } \alpha : \alpha \in I^*_N \mathcal{L} \cap I^*_N \overline{\mathcal{L}} \right\} \subset I^*_N \mathcal{L} \cap I^*_N \overline{\mathcal{L}}.$$

clearly $I^*_N \mathcal{L} \cap I^*_N \overline{\mathcal{L}}$ is (canonically isomorphic to) the complexification of $R$. We want to show that there is a (pointwise) exact sequence:

$$0 \to R \xrightarrow{\kappa} \nu^*N \otimes L_N \xrightarrow{\mu^2} \nu N,$$

(3.5)

proving that, around $x_0$, $\text{rank}_R R = \text{rank}_\mathbb{C}(I^*_N \mathcal{L} \cap I^*_N \overline{\mathcal{L}}) = 1$ as claimed. Define $\kappa : R \to \nu^*N \otimes L_N$ as follows. Take $\alpha = (\Delta, \psi) \in R$. This means that $(\Delta, \psi) \in DL_N$ is such that there exists $\chi \in J^1L \otimes \mathbb{C}$ with $(\Delta, \chi) \in \mathcal{L}|_N$ (hence $(\Delta, \overline{\chi}) \in \overline{\mathcal{L}}|_N$ and $I^*_N \chi = \psi$. Actually, $\chi$ is unique. Indeed, let $\chi' \in J^1L \otimes \mathbb{C}$ be such that $(\Delta, \chi') \in \mathcal{L}|_N$ and $I^*_N \chi' = \psi$. Then, on one side

$$\chi - \chi' \in \left( (J^1L)|_N \otimes \mathbb{C} \right) \cap \mathcal{L}|_N = \text{Ann}(p_D \mathcal{L}|_N).$$

On the other side, $I^*_N (\chi - \chi') = 0$, i.e.

$$\chi - \chi' \in \text{Ann}(DL_N \otimes \mathbb{C}),$$

so

$$\chi - \chi' \in \text{Ann}(p_D \mathcal{L}|_N + DL_N \otimes \mathbb{C}) = 0$$

where we used $[3.3]$ (which holds true around $x_0$). So if we work around $x_0$, $\chi = \chi'$, we put

$$\kappa(\Delta, \psi) := \text{Im } \chi,$$

and, from $\psi = I^*_N \chi = I^*_N \overline{\chi}$, it belongs to $\nu^*N \otimes L_N = \text{Ann}(DL_N) \subset (J^1L)|_N$.

Before proving that sequence $[3.5]$ is exact, the following remark is useful. Let $(\Delta, \psi) \in R$ and let $\chi$ be as above. Then

$$\Delta = J^1(\text{im } \chi).$$

(3.6)

Indeed, from $\mathbb{K}(\Delta, \chi) = i(\Delta, \chi)$, we find $i\Delta = \varphi \Delta + J^1 \chi$ (take just the first component). Similarly, from $\mathbb{K}(\Delta, \overline{\chi}) = -i(\Delta, \overline{\chi})$, we find $i\Delta = -\varphi \Delta - J^1 \overline{\chi}$. So $\Delta = J^1(\chi - \overline{\chi})/2i = J^1(\text{im } \chi)$.

Now, we prove that $[3.5]$ is exact. First of all $\kappa$ in injective. Indeed, if $\kappa(\Delta, \psi) = \text{im } \chi = 0$, then $\chi = \overline{\chi}$ and $(\Delta, \chi) \in \mathcal{L} \cap \overline{\mathcal{L}} = 0$, so $\chi = 0$, and, from $[3.6]$, $(\Delta, \psi) = (0, I^*_N \chi) = 0$. It
remains to show that $\ker \mu^\sharp = \im \kappa$. So let $(\Delta, \psi) \in R$, let $\chi$ be as above, and let $\eta \in \nu^*N \otimes L_N$. Compute
\[
\langle \mu^\sharp(\im \chi), \eta \rangle = \langle J^\sharp(\im \chi), \eta \rangle = (\Delta, \eta) = 0,
\]
where we used (3.6) again, and the fact that $\Delta \in DL_N$. So $\ker \mu^\sharp \subset \im \kappa$. Finally, let $\eta \in \nu^*N \otimes L_N$ be such that $\mu^\sharp \eta = 0$. This means that $\Delta := J^\sharp \eta \in DL_N$. Put
\[
\alpha := (\Delta, -I^\Delta_N(\varphi^\dagger \eta)).
\]
We claim that $\alpha \in R$, and $\eta = \kappa(\alpha)$. To see this notice that
\[
(\Delta, i\eta - \varphi^\dagger \eta) = i (\id - iK) (0, \eta) \in \mathfrak{L}
\]
hence $(\Delta, I^\Delta_N(i\eta - \varphi^\dagger \eta)) = \alpha \in R$. Additionally
\[
\kappa(\alpha) = \im (i\eta - \varphi^\dagger \eta) = \eta.
\]
We conclude that $\ker \mu^\sharp = \im \kappa$, and $\rank_{\mathbb{C}}(I^\Delta_N \mathfrak{L} \cap I^\Delta_N \mathfrak{T}) = 1$ as claimed.

To prove that $I^\Delta_N \mathfrak{T}$ is a complex Dirac-Jacobi structure of homogeneous generalized complex type, it remains to show that $p_D I^\Delta_N \mathfrak{L} + p_D I^\Delta_N \mathfrak{T} = DL_N$. To do this we compute
\[
\Ann \left( p_D I^\Delta_N \mathfrak{L} + p_D I^\Delta_N \mathfrak{T} \right) = \Ann \left( p_D I^\Delta_N \mathfrak{L} \right) \cap \Ann \left( p_D I^\Delta_N \mathfrak{T} \right) = (J^1L_N \otimes \mathbb{C}) \cap I^\Delta_N \mathfrak{L} \cap I^\Delta_N \mathfrak{T}.
\]
But the above discussion, together with formula (3.14), reveals that, at the point $x_0$, $R$, hence $I^\Delta_N \mathfrak{L} \cap I^\Delta_N \mathfrak{T}$, is spanned by an element of the form $(1_{x_0}, \zeta)$. In particular,
\[
(J^1L_N \otimes \mathbb{C}) \cap I^\Delta_N \mathfrak{L} \cap I^\Delta_N \mathfrak{T} = 0
\]
at the point $x_0$, hence in a whole neighborhood of $x_0$. This concludes the proof. \hfill \square

3.2. Lcs leaves and their transversals. We now pass to lcs leaves. As already mentioned, an even dimensional characteristic leaf of $J$ possesses a canonical lcs structure. To see this one can argue as follows. As before, $J$ restricts to a Jacobi structure $J_{\mathcal{O}}$ on the restricted line bundle $L_{\mathcal{O}} \to \mathcal{O}$. Now $\sigma J^\sharp_\mathcal{O} : J^1L_{\mathcal{O}} \to T\mathcal{O}$ is surjective again, and, as $\dim \mathcal{O} = \text{even}$, then $J^\sharp_\mathcal{O} : J^1L_{\mathcal{O}} \to DL_{\mathcal{O}}$ takes values in a $\dim \mathcal{O}$-dimensional subbundle $C_{\mathcal{O}} \subset DL_{\mathcal{O}}$ transversal to $\mathbb{R}_{\mathcal{O}} \subset DL_{\mathcal{O}}$. In other words, $C_{\mathcal{O}}$ is the image of a linear connection $\nabla : T\mathcal{O} \to DL_{\mathcal{O}}$. Additionally, $J_{\mathcal{O}}$ induces a non-degenerate $L_{\mathcal{O}}$-valued 2-form on $C_{\mathcal{O}}$, hence on $T\mathcal{O}$. So we get a non-degenerate $\Omega_{\mathcal{O}} \in \mathcal{O}^2(\mathcal{O}, L_{\mathcal{O}})$. Notice that $C_{\mathcal{O}} = \im J^\sharp_{\mathcal{O}}$, and $\Omega_{\mathcal{O}}$ (viewed as a 2-form on $C_{\mathcal{O}}$) agrees with the pointwise restriction to $\mathcal{O}$ of the 2-form $\omega_J : \lambda^2 \im \varphi^\dagger \im J^\sharp \to L$ from Remark 2.2.6. Now the integrability condition for $J_{\mathcal{O}}$ is equivalent to $\nabla$ being flat and $d^\nabla \Omega_{\mathcal{O}} = 0$. So $(\Omega_{\mathcal{O}}, \nabla)$ is an lcs structure on $L_{\mathcal{O}}$ and it contains the full information on $J_{\mathcal{O}}$. This lcs structure $(\Omega_{\mathcal{O}}, \nabla)$ can be equivalently encoded in a complex Dirac-Jacobi structure $\mathfrak{L}_{\mathcal{O}}$ given by the same formula (3.1) as before.

Remark 3.2.1. Let $\mathcal{O}$ be an even dimensional leaf of $J$. Then the subbundle $\mathfrak{L}_{\mathcal{O}} \subset DL_{\mathcal{O}} \otimes \mathbb{C}$ given by formula (3.1) is a complex Dirac-Jacobi structure such that
\[
p_D \mathfrak{L}_{\mathcal{O}} = p_D \mathfrak{T}_{\mathcal{O}} = C_{\mathcal{O}} \otimes \mathbb{C} \quad \text{and} \quad \mathfrak{L}_{\mathcal{O}} \cap \mathfrak{T}_{\mathcal{O}} = \Ann(C_{\mathcal{O}} \otimes \mathbb{C}).\]
Proposition 3.2.2. Let $\mathcal{O}$ be an even dimensional leaf of $J$. The backward image of the complex Dirac-Jacobi structure $\mathfrak{L}$ along the immersion $I_{\mathcal{O}} : L_{\mathcal{O}} \hookrightarrow L$ is a Dirac-Jacobi structure on $L_{\mathcal{O}}$, denoted $I_{\mathcal{O}}^* \mathfrak{L}$, such that

1. $\text{rank}_C (I_{\mathcal{O}}^* \mathfrak{L} \cap I_{\mathcal{O}}^* \mathfrak{L}) = 1$, 
2. $1 \not\in p_D I_{\mathcal{O}}^* \mathfrak{L} + p_D I_{\mathcal{O}}^* \mathfrak{L}$, and 
3. $(p_D I_{\mathcal{O}}^* \mathfrak{L} + p_D I_{\mathcal{O}}^* \mathfrak{L}) \cap (1) = DL_{\mathcal{O}} \otimes \mathbb{C}$.

Additionally, it is locally a B-field transformation of $\mathfrak{L}_{\mathcal{O}}$:

$$I_{\mathcal{O}}^* \mathfrak{L} \cong \mathfrak{L}_{\mathcal{O}}^B$$

for some $B \in Z^2_{L_{\mathcal{O}}}$. 

Proof. First of all, we prove that $I_{\mathcal{O}}^* \mathfrak{L} \subset DL_{\mathcal{O}} \otimes \mathbb{C}$ is a regular subbundle. As usual, we check the clean intersection condition:

$$\text{rank}_C (p_D \mathfrak{L}_{\mathcal{O}}| + DL_{\mathcal{O}} \otimes \mathbb{C}) = \text{constant}.$$ 

So notice that, in this case

$$DL_{\mathcal{O}} \otimes \mathbb{C} = (1) \oplus C_{\mathcal{O}} = (1) \oplus \text{im} J^1|_{\mathcal{O}} \otimes \mathbb{C} \subset (1) + p_D \mathfrak{L}|_{\mathcal{O}}.$$ 

But $1 \not\in p_D \mathfrak{L}|_{\mathcal{O}}$, otherwise, from $1 = \mathfrak{L}$, and $\text{im} J^1 \otimes \mathbb{C} = p_D \mathfrak{L} \cap p_D \mathfrak{L}$ we would get $1 \in C_{\mathcal{O}}$ which is not the case. We conclude that

$$p_D \mathfrak{L}|_{\mathcal{O}} + DL_{\mathcal{O}} \otimes \mathbb{C} = (1) + p_D \mathfrak{L}|_{\mathcal{O}}$$

which is constant rank in the same way as for contact leaves (see the proof of Proposition 3.1.1). So $I_{\mathcal{O}}^* \mathfrak{L}$ is a Dirac-Jacobi structure on $L_{\mathcal{O}}$.

Next we show that

$$p_D I_{\mathcal{O}}^* \mathfrak{L} = p_D I_{\mathcal{O}}^* \mathfrak{L} = C_{\mathcal{O}} \otimes \mathbb{C}. \quad (3.7)$$

We will get, in particular, properties (2) and (3) in the statement. From $p_D I_{\mathcal{O}}^* \mathfrak{L} = DL_{\mathcal{O}} \cap p_D \mathfrak{L}$, and $p_D \mathfrak{L} \cap p_D \mathfrak{L} = \text{im} J^1 \otimes \mathbb{C}$ we get $C_{\mathcal{O}} \otimes \mathbb{C} \subset p_D I_{\mathcal{O}}^* \mathfrak{L} \cap p_D I_{\mathcal{O}}^* \mathfrak{L}$. Now let $(\Delta, \psi) \in I_{\mathcal{O}}^* \mathfrak{L}$, so that $\Delta \in p_D I_{\mathcal{O}}^* \mathfrak{L}$. In particular, $\Delta \in DL_{\mathcal{O}} \otimes \mathbb{C}$, meaning that $\Delta = \Delta_0 + z \cdot 1$ for some $\Delta_0 \in C_{\mathcal{O}} \otimes \mathbb{C}$, and some $z \in \mathbb{C}$. It follows that $\Delta - \Delta_0 \in p_D \mathfrak{L}|_{\mathcal{O}}$. As $1 \not\in p_D \mathfrak{L}|_{\mathcal{O}}$, we have $z = 0$, and $\Delta = \Delta_0 \in C_{\mathcal{O}} \otimes \mathbb{C}$. So $p_D I_{\mathcal{O}}^* \mathfrak{L} \subset C_{\mathcal{O}} \otimes \mathbb{C}$, and, similarly, $p_D I_{\mathcal{O}}^* \mathfrak{L} \subset C_{\mathcal{O}} \otimes \mathbb{C}$.

Now we show that

$$I_{\mathcal{O}}^* \mathfrak{L} \cap I_{\mathcal{O}}^* \mathfrak{L} = \text{Ann}(C_{\mathcal{O}} \otimes \mathbb{C}).$$

We will get, in particular, property (1) in the statement. So let $(\Delta, \psi) \in (I_{\mathcal{O}}^* \mathfrak{L} \cap I_{\mathcal{O}}^* \mathfrak{L})_x$ for some $x \in \mathcal{O}$. This means that there exist $\chi', \chi''$ as in the proof of Proposition 3.1.1, and we can even construct $\chi$ exactly as there. As $\mathfrak{L}$ is of generalized contact type, actually $(\Delta, \chi) = 0$, i.e. $\Delta = 0$, and $\chi \in \text{Ann}(C_{\mathcal{O}} \otimes \mathbb{C})$. This shows that $I_{\mathcal{O}}^* \mathfrak{L} \cap I_{\mathcal{O}}^* \mathfrak{L} \subset \text{Ann}(C_{\mathcal{O}} \otimes \mathbb{C}) \subset J^1 L_{\mathcal{O}} \otimes \mathbb{C}$.

The reverse inclusion $\text{Ann}(C_{\mathcal{O}} \otimes \mathbb{C}) \subset I_{\mathcal{O}}^* \mathfrak{L} \cap I_{\mathcal{O}}^* \mathfrak{L}$ immediately follows from (3.7).

It remains to show that, locally, $I_{\mathcal{O}}^* \mathfrak{L}$ is a B-field transformation of

$$\mathfrak{L}_{\mathcal{O}} := \{ (J^1_{\mathcal{O}}(\psi), i_{\psi}) \in DL_{\mathcal{O}} \otimes \mathbb{C} : \psi \in J^1 L_{\mathcal{O}} \otimes \mathbb{C} \}, \quad (3.8)$$

where $J_{\mathcal{O}}$ is the restriction to $L_{\mathcal{O}}$ of the Jacobi structure $J$. To do this, denote by $\varpi_{\mathcal{O}}$ the $L_{\mathcal{O}}$-valued 2-form induced by $I_{\mathcal{O}}^* \mathfrak{L}$ on $p_D I_{\mathcal{O}}^* \mathfrak{L} = C_{\mathcal{O}} \otimes \mathbb{C}$. We can extend $\varpi_{\mathcal{O}}$ to a genuine Atiyah 2-form on $L_{\mathcal{O}}$, by putting $\iota_1 \varpi_{\mathcal{O}} = 0$. Similarly as in the case of a contact leaf, $\varpi_{\mathcal{O}}$
actually agrees with $\varpi$ on $C_O$. It follows that the imaginary part $\text{Im} \varpi_O$ agrees with the lcs form $\Omega_O$. From (2.8) we get

$$I_O^L = \{(\Delta, \iota_\Delta \varpi_O + A) : \Delta \in C_O \otimes \mathbb{C} \text{ and } A \in \text{Ann}C_O\}.$$  

(3.9)

Using $C_O \simeq T\mathcal{O}$, we can also think of $\varpi_O$ as an $L_O \otimes \mathbb{C}$-valued 2-form on $\mathcal{O}$. Then, if we denote by $\nabla : T\mathcal{O} \to DL_O$ the flat connection in $L_O$ whose image is $C_O$, from (3.9) and involutivity, we get $d^{\nabla} \varpi_O = 0$. In particular, $d^{\nabla} \Omega_O = 0$, and, locally, $\text{Re} \varpi_O = d^{\nabla} \eta$, for some $\eta \in \Omega^1(\mathcal{O}, L_O) \subset \Gamma(J^1L_O)$. Put $B := dD\eta$. An easy computation shows that

$$B = \text{Re} \varpi_O + \mathcal{C} \wedge \eta$$

where $\mathcal{C} : DL_O \to \mathbb{R}_{\mathcal{O}}$ is the unique 1-form with kernel $C_O$, and such that $\langle \mathcal{C}, 1 \rangle = 1$. Hence

$$(I_O^L)^{-B} = \{(\Delta, \iota_\Delta \text{Im} \varpi_O + A) : \Delta \in C_O \otimes \mathbb{C} \text{ and } A \in \text{Ann}C_O\} = \mathcal{L}_O.$$

For the very last step we used (3.8), and the fact that $\text{Im} \varpi_O = \Omega_O$ (together with the relationship between $\Omega_O$ and $J_O$ discussed at the beginning of this subsection).

Proposition 3.2.3. Let $N$ be a transversal at $x_0 \in \mathcal{O}$ to an even dimensional leaf $\mathcal{O}$ of $J$. Around $x_0$, the backward image of the complex Dirac-Jacobi structure $\mathcal{L}$ along the embedding $I_N : L_N \hookrightarrow L$ is a complex Dirac-Jacobi structure $I_N^L$ of generalized contact type.

Proof. One can prove that $I_N^L \subset \mathbb{D}L_N$ is a regular subbundle in a very similar way as for the transversal to a contact leaf (proof of Proposition 3.1.2) and we leave it to the reader to take care of the obvious adaptations. Now, we show that

$$\left( I_N^L \cap I_N^L \right)_{x_0} = 0.$$

It will follow that $I_N^L \cap I_N^L = 0$ in a whole neighborhood of $x_0$. So let $(\Delta, \psi) \in (I_N^L \cap I_N^L)_{x_0}$. Then $\Delta \in pD\mathcal{L}_{x_0} \cap pD\overline{\mathcal{L}}_{x_0} \cap D_{x_0}L_N \otimes \mathbb{C} = ((C_O)_{x_0} \cap D_{x_0}L_N) \otimes \mathbb{C} = 0,$

and we find $\chi', \chi'' \in J_{x_0}^1L \otimes \mathbb{C}$, such that $(0, \chi') \in \mathcal{L}_{x_0}$ (i.e. $\chi' \in \text{Ann}(pD\mathcal{L}_{x_0})$), $(0, \chi'') \in \overline{\mathcal{L}}_{x_0}$ (i.e. $\chi'' \in \text{Ann}(pD\overline{\mathcal{L}}_{x_0})$), and, additionally, $\psi = I_N^L \chi' = I_N^L \chi''$. Hence

$$\chi' - \chi'' \in \text{Ann}(pD\mathcal{L}_{x_0} \cap pD\overline{\mathcal{L}}_{x_0}) \cap \text{Ann}(D_{x_0}L_N \otimes \mathbb{C}) = \text{Ann}((C_O)_{x_0} + D_{x_0}L_N) \otimes \mathbb{C} = 0.$$

It follows that $(0, \chi') = (0, \chi'') \in \mathcal{L}_{x_0} \cap \overline{\mathcal{L}}_{x_0} = 0$, so that $\psi = 0$ as well. This concludes the proof.

4. Splitting theorems

In this section we prove a local splitting theorem for generalized contact bundles analogous to Weinstein splitting theorem for Poisson structures [37], and similar splitting theorems in Poisson-related geometries: Jacobi geometry [11], Dirac geometry [5] (see also [13]), Lie algebroid geometry [12][14][38], generalized complex geometry [1] (see also [3] for an important refinement of Abouzaid-Boyarchenko result). As expected, our splitting theorem is similar to that for generalized complex manifold on one side, and to that for Jacobi bundles on the other side. In particular, we actually prove two splitting theorems: one about the local structure around a point in a contact leaf and one about the local structure around a point in a lcs
leaf. Our proof is different in spirit from that of Abouzaid and Boyarchenko, and it is rather inspired by the recent work of Bursztyn, Lima and Meinrenken [8], who provided a unified approach to splitting theorems in Poisson (and related) geometries.

We begin recalling the splitting theorems of Dazord, Lichnerowicz and Marle for Jacobi bundles [11].

**Theorem 4.0.1.** Let \((L \to M, J)\) be a Jacobi bundle, and let \(N\) be a sufficiently small transversal at \(x_0 \in O\) to a \((2d + 1)\)-dimensional characteristic leaf \(O\) of \(J\). Then, there are

\(\triangleright\) a homogenous Poisson structure \((\pi_N, Z_N)\) on \(N\),

\(\triangleright\) an open neighborhood \(V\) of 0 in \(\mathbb{R}^{2d+1}\), and

\(\triangleright\) a line bundle isomorphism \(\Phi : L \to \mathbb{R}_{N\times V}\), covering a diffeomorphism \(\phi : M \to N\times V\),

locally defined around \(x_0\), such that

1. \(\phi\) identifies \(N\) with \(N \times \{0\}\), and (a neighborhood of \(x_0\) in) \(O\) with \(\{x_0\} \times V\),

2. \(\Phi\) identifies \(J\) with the Jacobi structure \(J^\times\) corresponding to the Jacobi pair \((\Lambda^\times, E^\times)\) given by

\[
\Lambda^\times = \Lambda_{can} + \pi_N - E_{can} \wedge Z_N, \quad \text{and} \quad E^\times = E_{can},
\]

where \((\Lambda_{can}, E_{can})\) is the Jacobi pair from Example 1.2.3.

**Remark 4.0.2.** Formula (4.1) has a nice interpretation in terms of Dirac-Jacobi structures. Namely, let \(L_{can} = \text{graph} J_{can} \subset D\mathbb{R}V\) be the Dirac-Jacobi structure induced by \(J_{can}\) on the trivial line bundle, and let \(L_N \subset D\mathbb{R}N\) be the Dirac-Jacobi structure spanned as follows:

\[
L_N = \left\{ (1 - Z_N, 0), (\pi_N^2 \eta, \eta + \eta(Z_N) \cdot j) : \eta \in T^*N \right\}.
\]

Additionally, let \(L^\times = \text{graph} J^\times \subset D\mathbb{R}N\times V\). Then \(L^\times\) is the flat product of \(L_N\) and \(L_{can}\) wrt to the standard projections \(\mathbb{R}_{N\times V} \to \mathbb{R}_N\), and \(\mathbb{R}_{N\times V} \to \mathbb{R}_V\):

\[
L^\times = L_N \times^1 L_{can}.
\]

**Theorem 4.0.3.** Let \((L \to M, J)\) be a Jacobi bundle, and let \(N\) be a sufficiently small transversal at \(x_0 \in O\) to a \(2d\)-dimensional characteristic leaf \(O\) of \(J\). Then, there are

\(\triangleright\) a Jacobi pair \((\Lambda_N, E_N)\) on \(N\),

\(\triangleright\) an open neighborhood \(V\) of 0 in \(\mathbb{R}^{2d}\), and

\(\triangleright\) a line bundle isomorphism \(\Phi : L \to \mathbb{R}_{N\times V}\), covering a diffeomorphism \(\phi : M \to N\times V\),

locally defined around \(x_0\), such that

1. \(\phi\) identifies \(N\) with \(N \times \{0\}\), and (a neighborhood of \(x_0\) in) \(O\) with \(\{x_0\} \times V\),

2. \(\Phi\) identifies \(J\) with the Jacobi structure \(J^\times\) corresponding to the Jacobi pair \((\Lambda^\times, E^\times)\) given by

\[
\Lambda^\times = \Lambda_N + \pi_{can} - E_{can} \wedge Z_{can}, \quad \text{and} \quad E^\times = E_N,
\]

where \((\pi_{can}, Z_{can})\) is the homogeneous Poisson structure from Example 2.2.12.
Remark 4.0.4. Again, formula (1.2) has an interpretation in terms of Dirac-Jacobi structures. Namely, let \( \mathcal{D}(\pi_{can}, Z_{can}) \subset \mathbb{D}R_V \) be the Dirac-Jacobi structure spanned as follows:

\[
\mathcal{D}(\pi_{can}, Z_{can}) = \left\{ (1 - Z_{can}, 0), (\pi_{can}^\sharp \eta, \eta + \eta(Z_{can}) \cdot i) : \eta \in T^*V \right\}
\]

and let \( \mathcal{L}_N = \text{graph } J_N \subset \mathbb{D}R_N \) be the Dirac-Jacobi structure induced by \( J_N \). Finally, let \( \mathcal{L}^\times = \text{graph } J^\times \subset \mathbb{D}R_{N \times V} \). Then \( \mathcal{L}^\times \) is the flat product of \( \mathcal{D}(\pi_{can}, Z_{can}) \) and \( \mathcal{L}_N \) wrt to the standard projections \( \mathbb{R}_{N \times V} \to \mathbb{R}_N \), and \( \mathbb{R}_{N \times V} \to \mathbb{R}_V \):

\[
\mathcal{L}^\times = \mathcal{D}(\pi_{can}, Z_{can}) \times_1 \mathcal{L}_N.
\]

\[\diamondsuit\]

4.1. Splitting around a contact point. We are finally ready to prove our main results. We begin with a remark.

Remark 4.1.1. The Jacobi structure \( J_{can} \) from Example 1.2.3 is non degenerate. Hence it corresponds to a contact structure \( H_{can} \). Namely, let \( \omega_{can} = J_{can}^{-1} \) be the Atiyah 2-form inverting \( J_{can} \). Then

\[
\omega_{can} = dx^i \wedge dp_i - (du - p_i dx^i) \wedge i,
\]

and \( \iota_\omega \omega_{can} \) agrees with

\[
\theta_{can} = du - p_i dx^i,
\]

the canonical contact 1-form on \( \mathbb{R}^{2d+1} \), and \( H_{can} = \ker \theta_{can} \) is the canonical contact structure on \( \mathbb{R}^{2d+1} \). The latter can be equivalently encoded in a generalized contact structure

\[
\begin{pmatrix}
0 & J_{can}^2 \\
-(\omega_{can}), & 0
\end{pmatrix}
\]

(4.3)

whose \(+i\)-eigenbundle is

\[
\mathcal{L}_{can}^{odd} = \left\{ (J_{can}^i(\psi), i\psi) : \psi \in J^1\mathbb{R}_{\mathbb{R}^{2d+1} \otimes \mathbb{C}} \right\}.
\]

Clearly, we also have

\[
\mathcal{L}_{can}^{odd} = \left\{ (\Delta, i \cdot i \Delta \omega_{can}) : \Delta \in \mathbb{D}R_{\mathbb{R}^{2d+1} \otimes \mathbb{C}} \right\} = (\mathbb{D}R_{\mathbb{R}^{2d+1} \otimes \mathbb{C}})^{i\omega_{can}},
\]

(4.4)

i.e. \( \mathcal{L}_{can}^{odd} \) can be seen as the complex \( B \)-field transformation of the complex Dirac-Jacobi structure \( \mathbb{D}R_{\mathbb{R}^{2d+1} \otimes \mathbb{C}} \subset \mathbb{D}R_{\mathbb{R}^{2d+1} \otimes \mathbb{C}} \) by means of the closed complex Atiyah 2-form \( i\omega_{can} \). This simple remark will be useful below. Actually, similar considerations hold for any non-degenerate Jacobi structure. Notice, however, that (4.4) does not mean that there is a Courant-Jacobi automorphism intertwining (4.3) with some other generalized contact structure. Yet in other words, \( \mathbb{D}R_{\mathbb{R}^{2d+1} \otimes \mathbb{C}} \) is not a complex Dirac-Jacobi structure of generalized contact type, and the obvious reason is that only real \( B \)-field transformations are Courant-Jacobi automorphisms, while \( i\omega_{can} \) is a purely imaginary Atiyah 2-form. \[\diamondsuit\]

Theorem 4.1.2. Let \( (L \to M, \mathbb{K}) \) be a generalized contact bundle, let \( \mathcal{D} \subset \mathcal{D}L \otimes \mathbb{C} \) be the \(+i\)-eigenbundle of \( \mathbb{K} \), and let \( N \) be a sufficiently small transversal at \( x_0 \in \mathcal{O} \) to a \((2d + 1)\)-dimensional characteristic leaf \( \mathcal{O} \). Then, there are

- an open neighborhood \( U \) of \( 0 \) in \( \mathbb{R}^{2d+1} \),
- a line bundle isomorphism \( \Phi : L \to \mathbb{R}_{N \times U} \), covering a diffeomorphism \( \phi : M \to N \times U \), locally defined around \( x_0 \), and
a closed Atiyah 2-form $B$ on $\mathbb{R}_{N \times U}$ such that

1. $\phi$ identifies $N$ with $N \times \{0\}$, and (a neighborhood of $x_0$ in) $\mathcal{O}$ with $\{x_0\} \times U$,
2. the Courant-Jacobi automorphism $e^B \circ \mathbb{D} \Phi$ identifies $\mathfrak{L}$ with

$$\mathfrak{L}_N \times \mathfrak{L}_{\text{can}}^\text{odd},$$

(4.5)

the flat product of $\mathfrak{L}_N$ and $\mathfrak{L}_{\text{can}}^\text{odd}$ wrt the standard projections $P_N : \mathbb{R}_{N \times U} \to \mathbb{R}_N$, and $P_U : \mathbb{R}_{N \times U} \to \mathbb{R}_U$. Here $\mathfrak{L}_N = I_N^* \mathfrak{L}$ is the complex Dirac-Jacobi structure of homogeneous generalized complex type induced by $\mathfrak{L}$ on $N$ (see Proposition 3.1.2), and $\mathfrak{L}_{\text{can}}^\text{odd}$ is the complex Dirac-Jacobi structure of generalized contact type from Remark 4.7.

Proof. The present proof and, similarly, the proof of Theorem 4.2.2 below, are inspired by a general technique recently proposed by Bursztyn, Lima and Meinrenken to prove splitting theorems in Poisson and related geometries. Without loss of generality, we can assume that $M = N \times V$, $L = \mathbb{R}_{N \times V}$ is the trivial line bundle, and the Jacobi structure underlying $\mathbb{E}$ is $J^x$, where $V$, $N$ and $J^x$ are as in Theorem 4.0.1. Now let $\psi \in \Gamma(J^1 \mathbb{R}_{N \times V})$ be given by

$$\psi = x' dp_i - p_i dx^i + (x' p_i - u) \cdot i.$$

Put $\mathcal{E} := J^1 \psi$. Then

$$\mathcal{E} = x^i \frac{\partial}{\partial x^i} + u \frac{\partial}{\partial u} + p_i \frac{\partial}{\partial p_i}$$

(4.6)

is the Euler vector field on $V$. More precisely, it is the covariant derivative along the Euler vector field wrt the canonical flat connections in $\mathbb{R}_{N \times V}$. By changing $\psi$ into $f \psi$ with $f \in C^\infty(V \times N)$ a suitable bump function equal to 1 around $N$, we can arrange that $\mathcal{E}$ is complete, while $\mathbb{E}$ still holds around $N$. Denote by $\{\Phi_t\}$ the flow of $\mathcal{E}$ on $\mathbb{R}_{N \times U}$, and let $\{\phi_t\}$ be its projection to $N \times V$. Then, for all $t \leq 0$ we have

$$\Phi_t(x, v ; r) = (x, e^t \cdot v ; r), \quad (x, v ; r) \in N \times V \times \mathbb{R},$$

at least when $v$ is small enough. Put

$$U := \left\{ v \in V : \lim_{t \to -\infty} \phi_t(x, v) \in N \times \{0\} \text{ for all } x \in N \right\}.$$

Then $U \subset V$ is an open subset and $\mathcal{E}$ remains complete when restricted to $U$. Additionally, the family of maps

$$K_s := \Phi_{\log(s)} : \mathbb{R}_{N \times U} \to \mathbb{R}_{N \times U}$$

extends smoothly to $s = 0$, and $K_0 = I_N \circ P_N$, where $P_N : \mathbb{R}_{N \times U} \to \mathbb{R}_N$ is the canonical projection and $I_N : \mathbb{R}_N \to \mathbb{R}_{N \times U}$ is the embedding at $u = 0 \in U$.

Now consider the $+i$-eigensection $\alpha$ of $K$ given by

$$i(0, \psi) + K(0, \psi) = (\mathcal{E}, \chi) \in \Gamma(\mathfrak{L}),$$

(4.7)

where we put $\chi := i\psi - \varphi^\dagger \psi$. Consider also the infinitesimal Courant-Jacobi automorphism

$$(\mathcal{L}_\mathcal{E} - dD\chi, \mathcal{E}) = ([[\mathcal{E}, \chi], -], \mathcal{E}).$$

(4.8)

From (4.7), (4.8), and involutivity, the flow of $(\mathcal{L}_\mathcal{E} - dD\chi, \mathcal{E})$ preserves $\mathfrak{L}$. From Remark 2.1.9 this flow is

$$\{ (e^{C_t} \circ \mathbb{D} \Phi_t, \Phi_t) \}, \quad \text{where } C_t = \int_0^t \Phi^*_t (d\mathcal{L}_\chi) d\epsilon.$$
In particular

\[ \mathcal{L} = (\mathbb{D} \Phi_{-\log(s)}) \mathcal{L}^{C_{-\log(s)}} = (K_0 \mathcal{L})^{C_{-\log(s)}}, \]  

(4.9)

for all \( s > 0 \). Put \( B_s := C_{-\log(s)} \) and compute

\[ B_s = \int_0^{-\log(s)} \Phi^*(dD\chi)de = \int_1^1 \tau^{-1} K^*_\tau(dD\chi)d\tau = i \int_1^1 \tau^{-1} K^*_\tau(dD\psi)d\tau - \int_1^1 \tau^{-1} K^*_\tau(dD\varphi^\dagger\psi)d\tau. \]

In a possibly smaller neighborhood of \( N \times \{0\} \) we have

\[ K^*_\tau(dD\psi) = K^*_\tau(2dx^i \wedge dp_i + (2p_idx^i - du) \wedge j) = 2\tau^2 dx^i \wedge dp_i + (2\tau^2 p_idx^i - \tau du) \wedge j, \]

for all \( \tau \in [0,1] \). Hence, for all \( s \in (0,1] \),

\[ \int_s^1 \tau^{-1} K^*_\tau(dD\psi)d\tau = (1 - s) \((1 + s)dx^i \wedge dp_i - (du - (1 + s)p_idx^i) \wedge j) , \]

which extends to \( s = 0 \). We conclude that, in a possibly smaller neighborhood of \( N \times \{0\} \), \( B_0 \) is well-defined, and, more precisely,

\[ B_0 = B + i\omega_{can} \]

where \( B \) is a certain real closed Atiyah 2-form. Finally, from (4.9), by continuity, we get, in a neighborhood of \( N \times \{0\} \)

\[ \mathcal{L} = (K_0^{-1} \mathcal{L})^{B_0} \]

\[ = (P_N^{-1} f_N, \mathcal{L})^{B + i\omega_{can}} \]

\[ = (P_N \mathcal{L} \star D \mathbb{R} N \times U \otimes \mathbb{C})^{B + i\omega_{can}} \quad \text{(Remark 2.2.8)} \]

\[ = (P_N \mathcal{L} \star P_U(D \mathbb{R} U \otimes \mathbb{C}))^{B + i\omega_{can}} \]

\[ = (\mathcal{L} \times^! D \mathbb{R} U \otimes \mathbb{C})^{B + i\omega_{can}} \]

\[ = (\mathcal{L} \times^! (D \mathbb{R} U \otimes \mathbb{C})^{i\omega_{can}})^B \quad \text{(Remark 2.2.11)} \]

\[ = (\mathcal{L} \times^! \mathcal{L}_{odd})^B \quad \text{(Equation (4.4))}. \]

\[ \square \]

4.2. Splitting around an lcs point.

**Remark 4.2.1.** Consider the homogeneous Poisson structure \((\pi_{can}, Z_{can})\) from Example 2.2.12.

The Poisson structure \(\pi_{can}\) is non-degenerate and its inverse is \(\Omega_{can} = dx^i \wedge dp_i\), the canonical symplectic structure on \(\mathbb{R}^{2d}\). In its turn \(\Omega_{can} = -d\Theta_{can}\), where

\[ \Theta_{can} = p_i dx^i \]

is the Liouville 1-form. The pair \((\Omega_{can}, Z_{can})\) is a homogeneous symplectic structure in the sense that \(L_{Z_{can}, \Omega_{can}} = \Omega_{can}\), and we can encode it in a complex Dirac-Jacobi structure of homogeneous generalized complex type

\[ \mathcal{L}^{eu}_{can} := \left\{ (1 - Z_{can}, 0), (\pi^\sharp_{can} \eta, i \cdot \eta) : \eta \in T^* \mathbb{R}^{2d} \otimes \mathbb{C} \right\}. \]
Now, consider the exact Atiyah 2-form $\xi_{\text{can}} = -d\Theta_{\text{can}}$. It is easy to see that

$$\mathcal{L}_{\text{can}}^\text{ev} = \{(\Delta, i \cdot \iota_\Delta \xi_{\text{can}}) : \Delta \in D\mathbb{R}^{2d} \otimes \mathbb{C}\} = (D\mathbb{R}^{2d} \otimes \mathbb{C})^\iota_{\text{can}},$$

i.e. $\mathcal{L}_{\text{can}}^\text{ev}$ is the complex $B$-field transformation of the complex Dirac-Jacobi structure $D\mathbb{R}^{2d} \otimes \mathbb{C} \subset D\mathbb{R}^{2d} \otimes \mathbb{C}$ by means of the complex closed Atiyah 2-form $i\xi_{\text{can}}$. Similar considerations hold for any homogeneous Poisson structure $(\pi, Z)$ such that $\pi$ is non-degenerate. We leave the simple details to the reader.

\textbf{Theorem 4.2.2.} Let $(L \to M, \mathcal{K})$ be a generalized contact bundle, let $\mathcal{L} \subset D\mathbb{C} \otimes \mathbb{C}$ be the $+i$-eigenbundle of $\mathcal{K}$, and let $N$ be a sufficiently small transversal at $x_0 \in \mathcal{O}$ to a $2d$-dimensional characteristic leaf $\mathcal{O}$. Then, there are

\begin{itemize}
  \item an open neighborhood $U$ of $0$ in $\mathbb{R}^{2d}$,
  \item a line bundle isomorphism $\Phi : L \to N \times U$, covering a diffeomorphism $\phi : M \to N \times U$, locally defined around $x_0$, and
  \item a closed Atiyah 2-form $B$ on $\mathcal{N} \times U$
\end{itemize}

such that

\begin{enumerate}
  \item $\phi$ identifies $N$ with $N \times \{0\}$, and $\mathcal{O}$ with $\{x_0\} \times U$,
  \item the Courant-Jacobi automorphism $e^B \circ \mathbb{D}\Phi$ identifies $\mathcal{L}$ with $\mathcal{L}_N \times \mathcal{L}_{\text{can}}^\text{ev}$,
\end{enumerate}

the flat product of $\mathcal{L}_N$ and $\mathcal{L}_{\text{can}}^\text{ev}$ wrt the standard projections $P_N : \mathbb{R}^{N \times U} \to \mathbb{R}_N$, and $P_U : \mathbb{R}^{N \times U} \to \mathbb{R}_u$. Here $\mathcal{L}_N = \mathcal{L}^1 N$ is the complex Dirac-Jacobi bundle structure of generalized contact type induced by $\mathcal{L}$ on $N$ (see Proposition 3.2.3), and $\mathcal{L}_{\text{can}}^\text{ev}$ is the complex Dirac-Jacobi structure of homogeneous generalized complex type from Remark 4.2.1.

\textbf{Proof.} Without loss of generality, we assume that $M = N \times V$, $L = \mathbb{R}^{N \times V}$ is the trivial line bundle, and the Jacobi structure underlying $\mathcal{K}$ is $J^\times$, where $V$, $N$ and $J^\times$ are as in Theorem 4.0.3. Let $\psi \in \Gamma(J^1 \mathbb{R}^{N \times V})$ be given by

$$\psi = x^i dp_i - p_i dx^i + (x^i p_i) \cdot j.$$ 

So that

$$\mathcal{E} := J^1 \psi = x^i \frac{\partial}{\partial x^i} + p_i \frac{\partial}{\partial p_i}$$

is the Euler vector field on $V$. We define $U$, $K_0$, $B_0$ exactly as in the proof of theorem 4.1.2. A direct computation then shows that $B_0$ is well defined around $N \times \{0\}$ and it is given by

$$B_0 = B + i\xi_{\text{can}}$$

for some real closed Atiyah 2-form $B$. Exactly as in the proof of Theorem 4.1.2, we now get

$$\mathcal{L} = (K_0^1 \mathcal{L})^{B_0} = (\mathcal{L}_N \times \mathcal{L}_{\text{can}}^\text{ev})^B.$$

\qed
5. The regular case

Let \((L \to M, \mathbb{K})\) be a generalized contact bundle, and let \(J\) be the Jacobi structure underlying \(\mathbb{K}\). A point \(x_0 \in M\) is a regular point for \(\mathbb{K}\) if the characteristic leaves of \(J\) has constant dimension around \(x_0\). Similarly as in the generalized complex case [16], when \(x_0 \in M\) is a regular point, Splitting Theorems 4.1.2 and 4.2.2 simplify and we get honest local normal form theorems around \(x_0\).

5.1. Local normal form around a regular contact point.

Remark 5.1.1. Denote by \(A_{can}\) the standard complex structure on \(\mathbb{C}^n\). It can be encoded in a generalized complex structure

\[
\begin{pmatrix}
A_{can} & 0 \\
0 & -A_{can}^* 
\end{pmatrix}
\]

whose \(+i\)-eigenbundle is \(T^{1,0}\mathbb{C}^n \oplus (T^{0,1}\mathbb{C}^n)^*\). The generalized complex structure (5.1) is homogeneous wrt to the zero section \((0,0) \in \Gamma(\mathbb{T}\mathbb{C}^n)\), and we get the following complex Dirac-Jacobi structure of homogeneous generalized complex type on \(\mathbb{C}^n\):

\[
\mathfrak{L}_{\mathbb{C}^n} := \langle (1,0), (X, \eta) : X \in T^{1,0}\mathbb{C}^n, \text{ and } \eta \in (T^{0,1}\mathbb{C}^n)^* \rangle.
\]

\[\diamond\]

Theorem 5.1.2. Let \((L \to M, \mathbb{K})\) be a generalized contact bundle with \(\dim M = 2(n+d)+1\). Let \(\mathfrak{L} \subset \mathbb{D}L \otimes \mathbb{C}\) be the \(+i\)-eigenbundle of \(\mathbb{K}\), and let \(x_0 \in M\) be a regular point in a \((2d+1)\)-dimensional characteristic leaf. Then, locally, around \(x_0\), \(\mathfrak{L}\) is isomorphic to the flat product

\[
\mathfrak{L}_{\mathbb{C}^n} \times L_{\text{can}}
\]

wrt the standard projections \(\mathbb{R}\mathbb{C}^n \times \mathbb{R}^{2d+1} \to \mathbb{R}\mathbb{C}^n\), and \(\mathbb{R}\mathbb{C}^n \times \mathbb{R}^{2d+1} \to \mathbb{R}\mathbb{R}^{2d+1}\), up to a \(B\)-field transformation.

Proof. Let \(N\) be a sufficiently small transversal to the characteristic leaf through \(x_0\). From Theorem 4.1.2 it is enough to show that the induced Dirac-Jacobi structure \(\mathfrak{L}_{\gamma} = i_{\gamma} \mathfrak{L}\) on \(N\) is isomorphic to \(\mathfrak{L}_{\mathbb{C}^n}\) around \(x_0\) up to a \(B\)-field transformation. From the proof of Proposition 3.1.2 and the fact that \(x_0\) is a regular point, it easily follows that \(\mathfrak{L}_{\gamma} \cap \mathfrak{L}_{\gamma}^N\) is (everywhere, not only at \(x_0\)) spanned by a section of the form \((1, \zeta)\), with \(\zeta \in T^*N \otimes L_N\), and, from the proof of Proposition 2.2.16, \(\mathfrak{L}_{\gamma}^N\) is isomorphic to a Dirac-Jacobi structure of generalized complex type of the form \(\mathfrak{L}_{(\gamma, Z)}^N\) (see (2.13)) with \(Z = 0\). In particular,

\[
\begin{align*}
(1) & \quad \pi = 0, \\
(2) & \quad A \text{ is a complex structure on } N, \text{ and} \\
(3) & \quad \sigma = i_A d\zeta
\end{align*}
\]

(see Definition 2.2.13). A direct computation exploiting (1) and (3) shows that, after the \(B\)-field transformation \((e^{d\zeta}, \text{id})\), we achieve \(\zeta = \sigma = 0\). Finally, with a diffeomorphism, we achieve \(A = A_{can}\), showing that \(\mathfrak{L}_{\gamma}^N\) is isomorphic to \(\mathfrak{L}_{\mathbb{C}^n}\) up to a \(B\)-field transformation. \(\square\)

5.2. Local normal form around a regular lcs point.
Remark 5.2.1. Consider the standard complex structure $\varphi_{can}$ on the gauge algebroid of the trivial line bundle over the cylinder $\mathbb{R} \times \mathbb{C}^n$ from Example A.0.1 in the Appendix. It can be encoded in a generalized contact structure

$$
\begin{pmatrix}
\varphi_{can} & 0 \\
0 & -\varphi_{can}^\dagger
\end{pmatrix}
$$

(5.2)

whose $+i$-eigenbundle is

$$
\mathcal{L}_{\mathbb{R} \times \mathbb{C}^n} = D^{1,0}\mathbb{R} \times \mathbb{C}^n \oplus (D^{0,1}\mathbb{R} \times \mathbb{C}^n)^\ast
$$

(see Appendix A for more details).

Theorem 5.2.2. Let $(L \to M, \mathbb{K})$ be a generalized contact bundle with $\dim M = 2(n + d) + 1$. Let $\mathcal{L} \subset \mathcal{D}L \otimes \mathbb{C}$ be the $+i$-eigenbundle of $\mathbb{K}$, and let $x_0 \in M$ be a regular point in a $2d$-dimensional characteristic leaf. Then, locally, around $x_0$, $\mathcal{L}$ is isomorphic to the flat product

$$
\mathcal{L}_{\mathbb{R} \times \mathbb{C}^n} \times \mathbb{C} \mathcal{L}_{ev}
$$

wrt the standard projections $\mathbb{R}(\mathbb{R} \times \mathbb{C}^n) \times \mathbb{R}^{2d} \to \mathbb{R} \times \mathbb{C}^n$, and $\mathbb{R}(\mathbb{R} \times \mathbb{C}^n) \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, up to a $B$-field transformation.

Proof. Let $N$ be a sufficiently small transversal to the characteristic leaf through $x_0$. From Theorem [2,2] it is enough to show that the induced Dirac-Jacobi structure $\mathcal{L}_N = L_N^1 \mathcal{L}$ on $N$ is isomorphic to $\mathcal{L}_{\mathbb{R} \times \mathbb{C}^n}$ around $x_0$ up to a $B$-field transformation. As $x_0$ is a regular point, the characteristic foliation $\mathcal{F}$ is a regular lcs foliation around $x_0$. In particular $p_D L \cap p_D \mathcal{L} = \text{im} \nabla_F$ where $\nabla_F$ is a flat leaf-wise connection along $\mathcal{F}$ in $L$. Hence $p_D L \cap p_D \mathcal{L} = p_D L \cap p_D \mathcal{L} \cap D L_N = \text{im} \nabla_F \cap D L_N = 0$. This means that $\mathcal{L}_N$ is the $+i$-eigenbundle of a generalized contact structure $\mathbb{K}_N$ of the form

$$
\mathbb{K}_N = \begin{pmatrix}
\varphi_N & 0 \\
(\omega_N)_b & -\varphi_N^\dagger
\end{pmatrix}.
$$

(5.3)

In particular $\varphi_N$ is an integrable complex structure on the Atiyah algebroid $D L_N$. In the following we refer to the Appendix for notation and the main properties of such a complex structure. First of all, notice that from [5,3] we have $p_D L_N = D^{(1,0)}L_N$ (recall from the Appendix that $D^{(1,0)}L_N$ denotes the $+i$-eigenbundle of $\varphi_N$). Define a complex Atiyah 2-form $\gamma \in \Omega^{(2,0)}_L \otimes \mathbb{C}$ by putting

$$
\gamma(\Delta, \nabla) = \langle \psi, \nabla \rangle, \quad \Delta, \nabla \in D^{(1,0)}L_N,
$$

where $\psi \in J^1L \otimes \mathbb{C}$ is any element such that $\langle \Delta, \psi \rangle \in \mathcal{L}_N$. A straightforward computation using the involutivity of $\mathcal{L}_N$ shows that $\partial_D \gamma = 0$, and, from Remark A.2.2 locally around $x_0$, there is $\rho \in \Omega^{(1,0)}_L$ such that $\gamma = \partial_D \rho$. It is easy to see that

$$
B := -2\text{Re} \left( \gamma + \overline{\partial_D \rho} \right)
$$

is a (real) closed Atiyah 2-form. We claim that

$$
\mathcal{L}_N^B = D^{(1,0)}L_N \oplus \text{Hom}(D^{(0,1)}L_N, L_N).
$$

(5.4)

This follows, after a simple computation, from the remark that

$$
\mathcal{L}_N = \text{graph} \gamma \oplus \text{Hom}(D^{(0,1)}L_N, L_N)$$
and the fact that \((\gamma + B)\) takes values in \(\text{Hom}(D^{(0,1)}L_N, L_N)\).

Notice that (5.4) means that \(\mathfrak{B}_N^{\#}\) is the +i-eigenbundle of the generalized contact structure
\[
\begin{pmatrix}
\varphi_N & 0 \\
0 & -\varphi_N^\dagger
\end{pmatrix}.
\]

Finally, in view of Theorem A.1.1, with a line bundle isomorphism we can achieve \(\varphi_N = \varphi_{\text{can}}\), and this concludes the proof. □

Appendix A. Complex structures on the gauge algebroid

Let \(L \to M\) be a line bundle. In this appendix we study the local properties of a generalized contact structure of complex type, i.e. a generalized contact structure \(\mathcal{K}\) on \(L\), of the form

\[
\mathcal{K} = \begin{pmatrix}
\varphi & 0 \\
0 & -\varphi^\dagger
\end{pmatrix}.
\]  (A.1)

In this case \(\varphi : DL \to DL\) is a(n integrable) complex structure on the gauge algebroid \(DL\), i.e.

\(\varphi\) is almost complex, i.e. \(\varphi^2 = -\text{id}\),
\(\varphi\) is integrable, i.e. its Lie algebroid Nijenhuis torsion \(N_\varphi\) vanishes.

Here \(N_\varphi : \Lambda^2 DL \to DL\) is the skew-symmetric bilinear map defined by

\[
N_\varphi(\Delta, \nabla) = [\varphi\Delta, \varphi\nabla] - [\Delta, \nabla] - \varphi([\varphi\Delta, \nabla] + [\Delta, \varphi\nabla]), \quad \Delta, \nabla \in \Gamma(DL).
\]

Conversely, given a complex structure on \(DL\), (A.1) defines a generalized contact structure.

Example A.0.1. Consider the cylinder \(\mathbb{R} \times \mathbb{C}^n\) over the standard complex space \(\mathbb{C}^n\). Let \(u\) be the standard real coordinate on the first factor, and let \(z_i = x_i^1 + iy_i^1, i = 1, \ldots, n\), be the standard complex coordinates on the second factor. There is a canonical integrable complex structure \(\varphi_{\text{can}}\) on the gauge algebroid of the trivial line bundle \(\mathbb{R} \times \mathbb{C}^n\) defined by

\[
\varphi_{\text{can}} \mathbb{1} = \frac{\partial}{\partial u}, \quad \text{and} \quad \varphi_{\text{can}} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i^1}.
\]

A.1. Local normal form.

Theorem A.1.1. Let \(L \to M\) be a line bundle equipped with a complex structure \(\varphi : DL \to DL\) on the gauge algebroid. Then, locally, around every point of \(M\), there are

\(\triangleright\) coordinates \(u, x^1, \ldots, x^n, y^1, \ldots, y^n\) on \(M\), and
\(\triangleright\) a flat connection \(\nabla\) in \(L\), such that

\[
\varphi = \nabla_{\partial/\partial u}, \quad \varphi\nabla_{\partial/\partial x^i} = \nabla_{\partial/\partial y_i^1}.
\]  (A.2)

In other words, locally, around every point of \(M\), there is a line bundle isomorphism \(L \to \mathbb{R} \times \mathbb{C}^n\) identifying \(\varphi\) with \(\varphi_{\text{can}}\) from Example A.0.1.

The proof will essentially follow from the Newlander-Nirenberg theorem after applying the homogenization trick [35] which we now recall. First of all, consider the principal \(\mathbb{R}^\times\)-bundle \(\widetilde{M} = L^* \setminus 0 \to M\), and denote by \(\mathcal{E}\) the restriction to \(\widetilde{M}\) of the Euler vector field. A section \(\lambda\) of \(L\) corresponds to a linear function on \(L^*\), and, by restriction, to a homogeneous function \(\lambda\) on \(\widetilde{M}\), where by “homogeneous” we mean that \(\mathcal{E}(\lambda) = \lambda\). Every homogeneous function on
arises in this way. Secondly, let \( \Delta \) be a derivation of \( L \). Then there exists a unique vector field \( \tilde{\Delta} \) on \( \tilde{M} \) such that
\[
\tilde{\Delta}(\tilde{\lambda}) = \tilde{\Delta} \lambda, \quad \text{for all } \lambda \in \Gamma(L).
\]
Vector field \( \tilde{\Delta} \) commutes with \( \mathcal{E} \) and every vector field commuting with \( \mathcal{E} \) arises in this way. In particular \( \tilde{\Delta} \) is projectable onto \( M \) and its projection is \( \sigma(\Delta) \). Notice that \( \tilde{\Delta} = E \). Thirdly, let \( \varphi : DL \to DL \) be a vector bundle endomorphism. Then there exists a unique \((1,1)\)-tensor \( \tilde{\varphi} : T\tilde{M} \to T\tilde{M} \) such that
\[
\tilde{\varphi}\tilde{\Delta} = \tilde{\varphi}\Delta, \quad \text{for all } \Delta \in \Gamma(DL).
\]
(A.3)

The Lie derivative \( L_E \tilde{\varphi} \) vanishes, and every \((1,1)\)-tensor \( A \) on \( \tilde{M} \) such that \( L_E A = 0 \) arises in this way. Additionally, \( \varphi \) is an integrable complex structure iff \( \tilde{\varphi} \) is a complex structure on \( \tilde{M} \).

**Proof of Theorem A.1.1.** Now, let \( \varphi : DL \to DL \) be an integrable complex structure. Consider \( \tilde{\varphi} \). It is a complex structure on \( \tilde{M} \). As \( \mathcal{E} \) is nowhere vanishing, it can be locally completed to a holonomic complex frame, i.e. locally, around every point of \( \tilde{M} \), there are coordinates \( T, U, X^1, \ldots, X^n, Y^1, \ldots, Y^n \) such that
\[
E = \frac{\partial}{\partial T}, \quad \tilde{\varphi}E = \frac{\partial}{\partial U}, \quad \text{and } \tilde{\varphi} \frac{\partial}{\partial X^i} = \frac{\partial}{\partial Y^i}.
\]
As all coordinate vector fields commute with \( \mathcal{E} \), they all come from (commuting) derivations of \( L \). In particular
\[
\begin{align*}
&\quad \triangleright U, X^1, \ldots, X^n, Y^1, \ldots, Y^n, \text{ are pull-backs via projection } \tilde{M} \to M \text{ of uniquely defined coordinates } u, x^1, \ldots, x^n, y^1, \ldots, y^n \text{ on } M, \text{ and} \\
&\quad \triangleright \text{there exists a unique flat connection } \nabla \text{ in } L \text{ such that } \\
&\quad \quad \quad \frac{\partial}{\partial U} = \nabla_{\partial/\partial u}, \quad \ldots, \quad \frac{\partial}{\partial X^i} = \nabla_{\partial/\partial x^i}, \quad \ldots, \quad \frac{\partial}{\partial Y^i} = \nabla_{\partial/\partial y^i}, \quad \ldots
\end{align*}
\]
From (A.3), coordinates \( u, x^1, \ldots, x^n, y^1, \ldots, y^n \) on \( M \) and flat connection \( \nabla \) possess all the required properties. \( \square \)

**A.2. Dolbeault-Atiyah cohomology.** Let \( L \to M \) be a line bundle, and let \( \varphi : DL \to DL \) be an integrable complex structure on the gauge algebroid of \( L \). Similarly as in the case of a complex manifold, there is a cohomology theory attached to \( \varphi \). Namely, consider the complexification \( DL \otimes \mathbb{C} \) of the gauge algebroid and denote by \( D^{(1,0)}L \) and \( D^{(0,1)}L \) the \( +i \) and the \( -i \)-eigenbundles of \( \varphi \) respectively, so that
\[
DL \otimes \mathbb{C} = D^{(1,0)}L \oplus D^{(0,1)}L,
\]
and complex Atiyah forms \( \Omega^*_L \otimes \mathbb{C} \) splits as
\[
\Omega^*_L \otimes \mathbb{C} = \bigoplus_{r,s} \Omega^{(r,s)}_L,
\]
where we denoted by \( \Omega^{(r,s)}_L \) the sections of (complex) vector bundle
\[
\wedge^r(D^{(1,0)}L)^* \otimes \wedge^s(D^{(0,1)}L)^* \otimes L.
\]
The de Rham differential $d_D$ splits, in the obvious way, as $d_D = \partial_D + \overline{\partial}_D$, where 
\[ \partial_D : \Omega^k_L \to \Omega^{k+1}_L, \quad \text{and} \quad \overline{\partial}_D : \Omega^k_L \to \Omega^{k+1}_L, \]
and the integrability of $\varphi$ is equivalent to 
\[ \partial_D^2 = \overline{\partial}_D^2 = \partial_D \overline{\partial}_D + \overline{\partial}_D \partial_D = 0. \]
We call cohomology of $\overline{\partial}_D$ the Dolbeault-Atiyah cohomology.

**Theorem A.2.1.** The Dolbeault-Atiyah cohomology vanishes locally.

**Proof.** In view of Theorem A.1.1 it is enough to work in the case when $M = \mathbb{R} \times \mathbb{C}^n$. Let $u$ be the standard (real) coordinate on the first factor and let $z^i = x^i + iy^i, i = 1, \ldots, n,$ be the standard complex coordinates on the second factor. We can also assume that $L = \mathbb{R}_M$ is the trivial line bundle and (A.2) holds with $\nabla$ being the canonical flat connection on $\mathbb{R}_M$. In this case $D^{1,0}_L$ is spanned by complex derivations
\[ \Box := \frac{1}{2} (\mathbb{1} - i \nabla_{\partial/\partial u}), \quad \text{and} \quad \nabla_i = \frac{1}{2} (\nabla_{\partial/\partial x^i} - i \nabla_{\partial/\partial y^i}), \quad i = 1, \ldots, n. \quad (A.4) \]
It is easy to see that every complex Atiyah form $\omega$ on $\mathbb{R}_M$ can be uniquely written as 
\[ \omega = \omega_0 + \omega_1 \wedge \ell \]
where $\omega_0, \omega_1$ are standard complex forms on $M$ and 
\[ \ell = j + i \cdot du. \]
A long but straightforward computation exploiting (A.4), shows that 
\[ \overline{\partial}_D \omega = \overline{\partial}_\omega_0 + \left( \overline{\partial}_\omega_1 + (-)^{\omega_0} (\omega_0 + L_Y \omega_0) \right) \wedge \ell \]
where 
\[ Y := \sigma(\Box) = \frac{i}{2} \frac{\partial}{\partial u}, \]
and $\overline{\partial}$ is the standard Dolbeault differential on $\mathbb{C}^n$ (acting on forms on $\mathbb{R} \times \mathbb{C}^n$ in the obvious way). So $\omega$ is $\overline{\partial}_D$-closed iff 
\[ \overline{\partial}_D \omega_0 = \overline{\partial}_\omega_1 + (-)^{\omega_0} (\omega_0 + L_Y \omega_0) = 0. \]
In this case, use the vanishing of standard Dolbeault cohomology (with a real parameter $u$), to choose a form $\rho_0$ such that $\overline{\partial}_\rho_0 = \omega_0$. As the Lie derivative along $Y$ commutes with $\overline{\partial}$ we find 
\[ \overline{\partial} \left( \omega_1 - (-)^{\rho_0} (\rho_0 + L_Y \rho_0) \right) = 0, \]
and we can choose $\rho_1$ such that $\overline{\partial}_\rho_1 = \omega_1 - (-)^{\rho_0} (\rho_0 + L_Y \rho_0)$. It is now easy to see that 
\[ \overline{\partial}_D (\rho_0 + \rho_1 \wedge \ell) = \omega. \]
This concludes the proof. \qed

**Remark A.2.2.** It immediately follows from Theorem A.2.1 that the cohomology of $\partial_D$ does also vanish locally. \hfill \diamond
References

[1] M. Abouzaid, and M. Boyarchenko, Local structure of generalized complex manifolds, *J. Sympl. Geom.* 4 (2006), 43–62; e-print: arXiv:math/0412084.

[2] M. Aldi, and D. Grandini, Generalized contact geometry and T-duality, *J. Geom. Phys.* 92 (2015), 78–93; e-print: arXiv:1312.7471.

[3] M. Bailey, Local classification of generalized complex structures, *J. Differ. Geom.* 95 (2013), 1–37; e-print: arXiv:1201.4887.

[4] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Birkhäuser, Boston, 2002.

[5] C. Blohmann, Removable presymplectic singularities and the local splitting of Dirac structures, *IMRN* (2016), to appear in; e-print: arXiv:1410.5298.

[6] A. J. Bruce, and A. G. Tortorella, Kirillov structures up to homotopy, *Diff. Geom. Appl.* 48 (2016), 72–86; e-print: arXiv:1507.00454.

[7] H. Bursztyn, A brief introduction to Dirac manifolds, in: Geometric and Topological Methods for Quantum Field Theory, Proceedings of the 2009 Villa de Leyva Summer School (A. Cardona, I. Contreras, and A.F. Reyes-Lega eds.), Cambridge Univ. Press, Cambridge, 2013, pp.4–38; e-print: arXiv:1112.5037.

[8] H. Bursztyn, H. Lima, and E. Meinrenken, Splitting theorems for Poisson and related structures, *J. Reine angew. Math.* to appear in; e-print: arXiv:1605.05386.

[9] Z. Chen, and Z. Liu, Omni-Lie algebroids, *J. Geom. Phys.* 60 (2010), 799–808; e-print: arXiv:0710.1923.

[10] Z. Chen, Z. Liu, Y. Sheng, Dirac structures of omni-Lie algebroids, *Int. J. Math.* 22 (2011), 1163–1185; e-print: arXiv:0802.3819.

[11] P. Dazord, A. Lichnerowicz, and C.-M. Marle, Structures locale des variétés de Jacobi, *J. Math. Pures Appl.* 70 (1991), 101–152.

[12] J.-P. Dufour, Normal forms for Lie algebroids, in: Lie algebroids and related topics in differential geometry (Warsaw, 2000), *Banach Center Publ.* 54, Polish Acad. Sci. Inst. Math., Warsaw, 2001, pp. 35–41.

[13] J.-P. Dufour, A. Wade, On the local structure of Dirac manifolds, *Compos. Math.* 144 (2008), 774–786; e-print: arXiv:math/0405257.

[14] R. Fernandes, Lie algebroids, holonomy and characteristic classes, *Adv. Math.* 170 (2002), 119–179; e-print: arXiv:math/0007132.

[15] J. Grabowski, Graded contact manifolds and contact Courant algebroids, *J. Geom. Phys.* 68 (2013), 27–58; e-print: arXiv:1102.0759.

[16] M. Gualtieri, Generalized complex geometry, *Ann. Math.* 174 (2011), 75–123; e-print: arXiv:math/0703298.

[17] N. Hitchin, Generalized Calabi-Yau manifolds, *Quart. J. Math.* 54 (2003), 281–308; e-print: arXiv:math/0209099.

[18] D. Iglesias-Ponte, and A. Wade, Contact manifolds and generalized complex structures, *J. Geom. Phys.* 53 (2005), 249–258; e-print: arXiv:math/0404519.

[19] A. A. Kirillov, Local Lie algebras, *Russian Math. Surveys* 31 (1976), 57–76.

[20] H. V. Lê, Y.-G. Oh, A. G. Tortorella, and L. Vitagliano, Deformations of coisotropic submanifolds in abstract Jacobi manifolds, *J. Sympl. Geom.*, in press; e-print: arXiv:1410.8446.

[21] H. V. Lê, A. G. Tortorella, and L. Vitagliano, Jacobi bundles and the BFV complex, *J. Geom. Phys.* 121 (2017), 347–377; e-print: arXiv:1601.04540.

[22] A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées, *J. Math. Pures Appl.* 57 (1978), 453–488.

[23] K. C. H. Mackenzie, General theory of Lie groupoids and Lie algebroids, Cambridge Univ. Press, Cambridge, 2005.

[24] I. Mărcut, Mini-course on Dirac Geometry, 10th International Young Researcher Workshop on Geometry, Mechanics and Control, Paris, January 2016, lecture notes; available at http://www.math.ru.nl/~imarcut/index_files/Dirac.pdf.

[25] C. M. Marle, On Jacobi manifolds and Jacobi bundles, in: Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), *Math. Sci. Res. Inst. Publ.* 20, Springer, New York, 1991, pp. 227–246.

[26] Y. S. Poon, and A. Wade, Generalized contact structures, *J. London Math. Soc.* 2011 333–352; e-print: arXiv:0912.5314.
[27] K. Sekiya, Generalized almost contact structures and generalized Sasakian structures, *Osaka J. Math.* 52 (2015) 43–59; e-print: arXiv:1212.6064.

[28] A. G. Tortorella, Deformations of coisotropic submanifolds in Jacobi manifolds, *Ph.D. thesis*, University of Florence, 2017; e-print: arXiv:1705.08962.

[29] A. G. Tortorella, Rigidity of integral coisotropic submanifolds of contact manifolds, *Lett. Math. Phys.* (2017), in press; e-print: arXiv:1605.00411.

[30] I. Vaisman, Generalized CRF-structures, *Geom. Dedicata* 133 (2008) 129–154; e-print: arXiv:0705.3934.

[31] L. Vitagliano, $L_\infty$-algebras from multicontact geometry, *Diff. Geom. Appl.* 39 (2015), 147–165; e-print: arXiv:1311.2751.

[32] L. Vitagliano, Dirac-Jacobi bundles, *J. Sympl. Geom.*, in press; e-print: arXiv:1502.05420.

[33] L. Vitagliano, and A. Wade, Generalized contact bundles, *C. R. Math.* 354 (2016), 313–317; e-print: arXiv:1507.03973.

[34] L. Vitagliano, and A. Wade, Holomorphic Jacobi manifolds; e-print: arXiv:1609.07737.

[35] L. Vitagliano, and A. Wade, Holomorphic Jacobi manifolds and complex contact groupoids; e-print: arXiv:1710.03300.

[36] A. Wade, Local structure of generalized contact manifolds, *Diff. Geom. Appl.* 30 (2012), 124–135.

[37] A. Weinstein, The local structure of Poisson manifolds, *J. Diff. Geom.* 18 (1983), 523–557.

[38] A. Weinstein, Almost invariant submanifolds for compact group actions, *J. Eur. Math. Soc.* 2 (2000), 53–86; e-print: arXiv:math/9908133.

**DipMat, Università degli Studi di Salerno, via Giovanni Paolo II n° 123, 84084 Fisciano (SA) Italy. E-mail address: jschnitzer@unisa.it**

**DipMat, Università degli Studi di Salerno, via Giovanni Paolo II n° 123, 84084 Fisciano (SA) Italy. E-mail address: lvitagliano@unisa.it**