MODULAR FUNCTORS ARE DETERMINED BY THEIR GENUS ZERO DATA.

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ABSTRACT. We prove in this paper that the genus zero data of a modular functor determines the modular functor. We do this by establishing that the $S$-matrix in genus one with one point labeled arbitrarily can be expressed in terms of the genus zero information and we give an explicit formula. We do not assume the modular functor in question has duality or is unitary, in order to establish this.

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1. INTRODUCTION

A modular functor is a functor $Z$ from the category of smooth surfaces with certain extra structure, namely the category of labeled marked surfaces, to the category of finite dimensional vector spaces over the complex numbers. In section 2 we shall give the precise axioms for a modular functor following K. Walker’s topological reformulation 25 of G. Segal’s axioms for a conformal field theory 19.

The objects of the category of labeled marked surfaces are pairs consisting of a marked surface and a labeling. A marked surface here refers to a quadruple of structures: A smooth closed oriented surface, a finite set of "marked" points, a tangent direction at each of the marked points together with a Lagrangian subspace of the first homology of the surface. Labeled means that each marked point of the surface is labeled by an element from a certain finite label-set $\Lambda$ which is specific to the modular functor $Z$. This label set is further required to have an involution $^\dagger$ and a preferred $0 \in \Lambda$, such that $0^\dagger = 0$.

By the factorization property of a modular functor, we can express the vector space associated to any label marked surface as a direct sum of tensor products of
vector spaces associated to 2-spheres with three marked points labeled appropriately. This is done by choosing a pair of pants decomposition of the surface.

Let \( S^2 = \mathbb{C} \cap \{ \infty \} \) and \( v_t \) be the direction along the positive real axis at \( t \in \mathbb{R} \cup \{ \infty \} \subset S^2 \). We let \( Y = (S^2; 0, 1, \infty; v_0, v_1, v_\infty) \). For \( \lambda, \mu, v \in \Lambda \), we define
\[
Z_{\lambda, \mu, v} = Z(Y, \lambda, \mu, v).
\]

As will be explained in section \( \textsection 3 \) this gives us an isomorphism
\[
F \left[ \begin{array}{cc} h & \zeta \\ \lambda & k \end{array} \right] : \bigoplus_{\nu \in \Lambda} Z_{\nu, \mu, \lambda} \otimes Z_{\nu^+, \kappa, \xi} \to \bigoplus_{\nu \in \Lambda} Z_{\nu, \lambda, \kappa} \otimes Z_{\nu^+, \xi, \mu}.
\]

The second kind of flip is the change from the pair of pants decomposition of the torus with one marked point given by \( \alpha \) to the one given by \( \beta \) in figure \( \textsection \).

For each label \( \lambda \in \Lambda \), we get an isomorphism
\[
S(\lambda) : \bigoplus_{\mu \in \Lambda} Z_{\lambda, \mu, \mu^+} \to \bigoplus_{\nu \in \Lambda} Z_{\lambda, \nu, \nu^+}.
\]

For \( \lambda = 0 \), we simply write \( S = S(0) \). From the axioms of a modular functor it follows that \( \dim Z_{0, \mu, \mu^+} = 1 \) for all \( \mu \in \Lambda \). Further, we will see in section \( \textsection 3 \) that the
axioms of a modular functor determines a unique non-zero vector in $Z_{0,\mu,\mu^\dagger}$, and so we get a matrix $S = (S_{\mu,\nu})_{\mu,\nu \in \Lambda}$.

A flip relating two pair of pants decompositions gives an isomorphism between the corresponding two direct sums of tensor products of the vector spaces $Z_{\lambda,\mu,\nu}$ for appropriately varying $\lambda, \mu, \nu \in \Lambda$.

For the first flip, the corresponding isomorphism is clearly determined by the functor applied to genus zero surfaces with less than or equal to four marked points. On the other hand the second flip involves a genus one surface with one marked point. However, we will prove in this paper that $S(\lambda)$ for all $\lambda \in \Lambda$ is determined by the restriction of the modular functor to genus zero surfaces. We call this restriction to genus zero surfaces of a modular functor the genus zero data of the modular functor.

**Theorem 1.1.** The genus zero data of any modular functor determines the modular functor.

The key ingredient in the proof of this theorem is what we call the curve operators. Given an oriented simple closed curve $\gamma$ on a labeled marked surface and a label $\lambda_\gamma \in \Lambda$, we construct an endomorphism of the vector space the modular functor associates to the labeled marked surface. Loosely speaking they are obtained by creating two points labeled by $\lambda_\gamma$ and $\lambda_\gamma^\dagger$ near each other along $\gamma$. Move one of them around $\gamma$ and then annihilate them again. For the precise definition see sections [4]. The next step is to express the automorphism induced by a Dehn-twist in a simple closed curve as a linear combination of curve operators for the same curve. In fact this linear combination is seen to be universal by factoring along the boundary of a tubular neighbourhood of the curve. Moreover, it is then clear the linear combination is also determined by genus zero data. Now, since curve operators are determined completely by genus zero data, as we argue in section [6] we see that the Dehn-twist in any simple closed curve is determined by genus zero data. Using the standard presentation of the mapping class group of surface of genus one with one marked point, we conclude that the matrices $S(\lambda)$ are also determined by the genus zero data.

In fact, we will establish the following explicit formula for $S(\lambda)$.

Pick a basis $\xi_j(\lambda, \mu, \nu), j = 1, \ldots, \dim Z_{\lambda,\mu,\nu}$ for $Z_{\lambda,\mu,\nu}$. For $\nu = \mu^\dagger$ and $\lambda = 0$, we will assume that $\xi_1(0, \mu, \mu^\dagger)$ is this preferred vector in $Z_{0,\mu,\mu^\dagger}$, as discussed above.
We then define

\[ F_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right] (\zeta_i(\nu, \mu, \lambda) \otimes \zeta_j(v^+, \kappa, \xi)) \]

\[ = \sum_{k, l} F_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right]_{ij}^{kl} \zeta_k(\nu, \lambda, \kappa) \otimes \zeta_l(v^+, \xi, \mu) \]

and

\[ S(\lambda)(\zeta_i(\lambda, \mu, \mu^+)) = \sum_{\nu, j} S(\lambda)_{\nu, j}^{\mu, i} \zeta_j(\lambda, v, v^+). \]

Also we need the following three self diffeomorphisms of \( \Upsilon \). The diffeomorphism \( R : \Upsilon \rightarrow \Upsilon \) is \( R(z) = (z - 1)/z \). It induces a linear isomorphism

\[ R : Z_{\lambda, \mu, \nu} \rightarrow Z_{\mu, \nu, \lambda} \]

with the matrix presentation

\[ R(\zeta_i(\lambda, \mu, \nu)) = \sum \zeta_j(\mu, \nu, \lambda). \]

The diffeomorphism \( B : \Upsilon \rightarrow \Upsilon \) is the composition of \( z \mapsto z/(z - 1) \) with a negative half-twist at 0 and \( \infty \) and a positive half-twist at 1. Again it induces a linear isomorphism

\[ B : Z_{\lambda, \mu, \nu} \rightarrow Z_{\lambda, \nu, \mu} \]

with the matrix presentation

\[ B(\zeta_i(\lambda, \mu, \nu)) = \sum \zeta_j(\lambda, \nu, \mu). \]

Finally, the Dehn twist in a circle centered in 0 and of radius bigger than 1 induces a further endomorphism of \( \Upsilon \). In particular it induces an isomorphism of \( Z_{\lambda, \mu, \nu} \) to itself. But since this space is one dimensional, this isomorphism is simply given by multiplication by a non-zero complex number \( d_\mu \).

We introduce the so called twisted \( F \)-isomorphism

\[ \tilde{F}_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right] : \bigoplus_{\nu \in \Lambda} Z_{\nu, \mu, \lambda} \otimes Z_{v^+, \kappa} \rightarrow \bigoplus_{\nu \in \Lambda} Z_{\kappa, \lambda, \nu} \otimes Z_{v^+, \nu, \kappa} \]

defined by

\[ \tilde{F}_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right] = (BR^2 \otimes \text{Id})F_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right] (\text{Id} \otimes BR). \]

In the matrix presentation we get

\[ \tilde{F}_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right]_{ki}^{jm} = \sum_{p, r, s, w} R_{ip} B_{pr} R_{ru}^2 w B_{wj} F_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \mu \\ \lambda \\ \kappa \end{array} \right]_{kr}^{sm}. \]

**Theorem 1.2.** For any modular functor we have the following formula

\[ S(\lambda)^{\nu, j}_{\mu, i} = \sum_{\kappa, m, k} d_{\kappa}^{-1} d_{\mu} S_{\kappa, \mu, \nu} \tilde{F}_{\nu, \tilde{\nu}} \left[ \begin{array}{c} \kappa \\ v^+ \\ \lambda \end{array} \right]_{ki}^{jm} R_{mk}. \]
The formula alone does not completely prove that $S(\lambda)$ is determined by genus zero data, since it involves $S_{\kappa^1, \beta}$. However, we will in fact first argue that $S(\lambda)$ for all $\lambda \in \Lambda$ is determined by genus zero data, in particular so is $S_{\kappa^1, \beta} = S(0)_{\kappa^1, 1}$.

This paper is organized as follows. We present the axioms of a modular functor in section 2. In section 3 we recall the notion of basic data as defined by Kevin Walker. The curve operators are introduced in section 4. In section 5 we establish that the Dehn twist in any curve can be expressed as a linear combination of the curve operators associated to the curve. This rests on Proposition 5.2, which is proved in section 7. In section 6 it argued that the Dehn twist in the curve operators associated to the curve. This rests on Propo sition 5.2, which is proved in section 7. A couple of formulae involving $F$ and $S$ are derived in section 7 followed by a proof of Proposition 5.2. In section 8 we derive the formula for the $S(\lambda)$-matrix.

The result of this paper is used in the final paper in the series of three papers [2], [3] and [4]. In the first two papers we construct modular functors from Conformal Field Theory and in the third we identifying the resulting modular functors which underlies the Reshetikhin-Turaev TQFT [17], [18] and [21] via the Skein theory realizations of Blanchet, Habegger, Masbaum and Vogel [8], [9] and [7].

2. THE AXIOMS FOR A MODULAR FUNCTOR

We shall in this section give the axioms for a modular functor. These are due to G. Segal and appeared first in [19]. We present them here in a topological form, which is due to K. Walker [25]. See also [12]. We note that similar, but different, axioms for a modular functor are given in [21] and in [6]. It is however not clear if these definitions of a modular functor is equivalent to ours.

Let us start by fixing a bit of notation. By a closed surface we mean a smooth real two dimensional manifold. For a closed oriented surface $\Sigma$ of genus $g$ we have the non-degenerate skew-symmetric intersection pairing

$$(\cdot, \cdot) : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \rightarrow \mathbb{Z}.$$  

Suppose $\Sigma$ is connected. In this case a Lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Z})$ is by definition a subspace, which is maximally isotropic with respect to the intersection pairing. A $\mathbb{Z}$-basis $(\vec{a}, \vec{b}) = (a_1, \ldots, a_g, \beta_1, \ldots, \beta_g)$ for $H_1(\Sigma, \mathbb{Z})$ is called a symplectic basis if

$$(a_i, \beta_j) = \delta_{ij}, \quad (a_i, a_j) = (\beta_i, \beta_j) = 0,$$

for all $i, j = 1, \ldots, g$.

If $\Sigma$ is not connected, then $H_1(\Sigma, \mathbb{Z}) = \bigoplus_i H_1(\Sigma_i, \mathbb{Z})$, where $\Sigma_i$ are the connected components of $\Sigma$. By definition a Lagrangian subspace is in this paper a subspace of the form $L = \bigoplus_i L_i$, where $L_i \subset H_1(\Sigma_i, \mathbb{Z})$ is Lagrangian. Likewise a symplectic basis for $H_1(\Sigma, \mathbb{Z})$ is a $\mathbb{Z}$-basis of the form $((\vec{a}^i, \vec{b}^i))$, where $(\vec{a}^i, \vec{b}^i)$ is a symplectic basis for $H_1(\Sigma_i, \mathbb{Z})$.

For any real vector space $V$, we define $PV = (V - \{0\})/\mathbb{R^+}$.

**Definition 2.1.** A pointed surface $(\Sigma, P)$ is an oriented closed surface $\Sigma$ with a finite set $P \subset \Sigma$ of points. A pointed surface is called stable if the Euler characteristic of each component of the complement of the points $P$ is negative. A pointed surface is called saturated if each component of $\Sigma$ contains at least one point from $P$. 
Definition 2.2. A morphism of pointed surfaces \( f : (\Sigma_1, P_1) \to (\Sigma_2, P_2) \) is an isotopy class of orientation preserving diffeomorphisms which maps \( P_1 \) to \( P_2 \). Here the isotopy is required not to change the induced map of the first order Jet at \( P_1 \) to the first order Jet at \( P_2 \).

Definition 2.3. A marked surface \( \Sigma = (\Sigma, P, V, L) \) is an oriented closed smooth surface \( \Sigma \) with a finite subset \( P \subset \Sigma \) of points with projective tangent vectors \( V \in \sqcup_{p \in P} PT_p \Sigma \) and a Lagrangian subspace \( L \subset H_1(\Sigma, \mathbb{Z}) \).

The notions of stable and saturated marked surfaces are defined just like for pointed surfaces.

In the case of genus zero we omit the Lagragian subspace from the discussion, since it in this case can only be zero subspace.

Definition 2.4. A morphism \( f : \Sigma_1 \to \Sigma_2 \) of marked surfaces \( \Sigma_i = (\Sigma_i, P_i, V_i, L_i) \) is an isotopy class of orientation preserving diffeomorphisms \( f : \Sigma_1 \to \Sigma_2 \) that maps \( (P_1, V_1) \) to \( (P_2, V_2) \) together with an integer \( s \). Hence we write \( f = (f, s) \).

Remark 2.1. Any marked surface has an underlying pointed surface, but a morphism of marked surfaces does not quite induce a morphism of pointed surfaces, since we only require that the isotopies preserve the induced maps on the projective tangent spaces.

Remark 2.2. If in the notation above, we only specify \( f : \Sigma_1 \to \Sigma_2 \), then it is assumed that the integer \( s = 0 \).

Let \( \sigma \) be Wall’s signature cocycle for triples of Lagrangian subspaces of \( H_1(\Sigma, \mathbb{R}) \) (see \[27\]).

Definition 2.5. Let \( f_1 = (f_1, s_1) : \Sigma_1 \to \Sigma_2 \) and \( f_2 = (f_2, s_2) : \Sigma_2 \to \Sigma_3 \) be morphisms of marked surfaces \( \Sigma_i = (\Sigma_i, P_i, V_i, L_i) \) then the composition of \( f_1 \) and \( f_2 \) is \( f_2 f_1 = (f_2 f_1, s_2 + s_1 - \sigma((f_2 f_1)_*, L_1, f_2, L_2, L_3)) \).

With the objects being marked surfaces and the morphism and their composition being defined as in the above definition, we have constructed the category of marked surfaces.

The mapping class group \( \Gamma(\Sigma) \) of a marked surface \( \Sigma = (\Sigma, L) \) is the group of automorphisms of \( \Sigma \). One can prove that \( \Gamma(\Sigma) \) is a central extension of the mapping class group \( \Gamma(\Sigma) \) of the surface \( \Sigma \) defined by the 2-cocycle \( c : \Gamma(\Sigma) \to \mathbb{Z} \), \( c(f_1, f_2) = \sigma((f_1 f_2)_*, L, f_1, L, L) \). One can also prove that this cocycle is equivalent to the cocycle obtained by considering two-framings on mapping cylinders (see \[5\] and \[11\]).

Notice also that for any morphism \( (f, s) : \Sigma_1 \to \Sigma_2 \), one can factor
\[
(f, s) = ((\text{Id}, s') : \Sigma_2 \to \Sigma_2) \circ (f, s - s') = (f, s - s') \circ ((\text{Id}, s') : \Sigma_1 \to \Sigma_1).
\]

In particular \( (\text{Id}, s) : \Sigma \to \Sigma \) is \( (\text{Id}, 1)^s \).

Definition 2.6. The operation of disjoint union of marked surfaces is
\[
(\Sigma_1, P_1, V_1, L_1) \sqcup (\Sigma_2, P_2, V_2, L_2) = (\Sigma_1 \sqcup \Sigma_2, P_1 \sqcup P_2, V_1 \sqcup V_2, L_1 \sqcup L_2).
\]

Morphisms on disjoint unions are accordingly \( (f_1, s_1) \sqcup (f_2, s_2) = (f_1 \sqcup f_2, s_1 + s_2) \).

We see that disjoint union is an operation on the category of marked surfaces.
We then have natural continuous maps $q : \Sigma_1 \to \Sigma_2$ we let the orientation reversed morphism be given by $-f = (f, -s) : -\Sigma_1 \to -\Sigma_2$.

We also see that orientation reversal is an operation on the category of marked surfaces. Let us now consider glueing of marked surfaces.

Let $(\Sigma, \{p_-, p_+\} \sqcup P, \{v_-, v_+\} \sqcup V, L)$ be a marked surface, where we have selected an ordered pair of marked points with projective tangent vectors $((p_-, v_-), (p_+, v_+))$, at which we will perform the glueing.

Let $c : P(T_p, \Sigma) \to P(T_p, \Sigma)$ be an orientation reversing projective linear isomorphism such that $c(v_-) = v_+$. Such a $c$ is called a glueing map for $\Sigma$. Let $\Sigma$ be the oriented surface with boundary obtained from $\Sigma$ by blowing up $p_-$ and $p_+$, i.e.,

$$\Sigma = \{p_-, \{p_+, \Sigma\}\} \cup P(T_p, \Sigma) \cup P(T_p, \Sigma),$$

with the natural smooth structure induced from $\Sigma$. Let now $\Sigma_c$ be the closed oriented surface obtained from $\Sigma$ by using $c$ to glue the boundary components of $\Sigma$. We call $\Sigma_c$ the glueing of $\Sigma$ at the ordered pair $((p_-, v_-), (p_+, v_+))$ with respect to $c$.

Let now $\Sigma'$ be the topological space obtained from $\Sigma$ by identifying $p_-$ and $p_+$. We then have natural continuous maps $\gamma : \Sigma_c \to \Sigma'$ and $n : \Sigma \to \Sigma'$. On the first homology group $n$ induces an injection and $\gamma$ a surjection, so we can define a Lagrangian subspace $L_c \subset H_1(\Sigma_c, \mathbb{Z})$ by $L_c = q_c^{-1}(n_c(L))$. We note that the image of $P(T_p, \Sigma)$ (with the orientation induced from $\Sigma$) induces naturally an element in $H_1(\Sigma_c, \mathbb{Z})$ and as such it is contained in $L_c$.

**Remark 2.3.** If we have two glueing maps $c_i : P(T_p, \Sigma) \to P(T_p, \Sigma)$, $i = 1, 2$, we note that there is a diffeomorphism $f$ of $\Sigma$ inducing the identity on $((p_-, v_-) \sqcup (p_+, v_+))$ which is isotopic to the identity among such maps, such that $(df_{p_+})^{-1}c_2df_{p_-} = c_1$. In particular $f$ induces a diffeomorphism $f : \Sigma_{c_1} \to \Sigma_{c_2}$ compatible with $f : \Sigma \to \Sigma$, which maps $L_{c_1}$ to $L_{c_2}$. Any two such diffeomorphisms of $\Sigma$ induces isotopic diffeomorphisms from $\Sigma_1$ to $\Sigma_2$.

**Definition 2.7.** Let $\Sigma$ be a marked surface. We denote by $-\Sigma$ the marked surface obtained from $\Sigma$ by the operation of reversal of the orientation. For a morphism $f = (f, s) : \Sigma_1 \to \Sigma_2$ we let the orientation reversed morphism be given by $-f = (f, -s) : -\Sigma_1 \to -\Sigma_2$.

We observe that glueing also extends to morphisms of marked surfaces which preserves the ordered pair $((p_-, v_-), (p_+, v_+))$, by using glueing maps which are compatible with the morphism in question.

**Remark 2.4.** Let $\Sigma = (\Sigma, P, V, L)$ be marked surface. Assume that $\gamma$ is an oriented simple closed curve on $\Sigma - P$, such that $[\gamma] \in L$. Assume further we have a point $p$ on $\gamma$. We can then cut $\Sigma$ along $\gamma$ and obtain a surface with two boundary components, which are naturally identified with $\gamma$. By identifying each of the two boundary component to a point, say $\{p', p''\}$, we get a new closed surface $\Sigma'$ with a set of marked points $\tilde{P} = \emptyset \cup \{p', p''\}$ and tangent vectors $\tilde{V} = V \cup \{v', v''\}$. Here $v'$ and $v''$ are induced by $p \in \gamma$. Let $\Sigma'$ be...
obtained from $\Sigma$ by identifying $p'$ with $p''$. We have the quotient map $q : \Sigma \rightarrow \Sigma'$ and the identification map $n : \tilde{\Sigma} \rightarrow \Sigma'$. We specify a Lagragian subspace in $\tilde{\Sigma} \subset H_1(\tilde{\Sigma}, \mathbb{Z})$ by $\tilde{L} = n_*^{-1} q_* (L)$.

We say that $\tilde{\Sigma} = (\tilde{\Sigma}, \tilde{P}, \tilde{V}, \tilde{L})$ is obtained from $\Sigma$ by factoring $\Sigma$ along $(\gamma, p)$. The operation of factoring is an inverse to glueing.

We can now give the axioms for a 2 dimensional modular functor.

**Definition 2.9.** A label set $\Lambda$ is a finite set furnished with an involution $\lambda \mapsto \lambda^\dagger$ and a trivial element $0$ such that $0^\dagger = 0$.

**Definition 2.10.** Let $\Lambda$ be a label set. The category of $\Lambda$-labeled marked surfaces consists of marked surfaces with an element of $\Lambda$ assigned to each of the marked points and morphisms of labeled marked surfaces are required to preserve the labelings. An assignment of elements of $\Lambda$ to the marked points of $\Sigma$ is called a labeling of $\Sigma$ and we denote the labeled marked surface by $(\Sigma, \lambda)$, where $\lambda$ is the labeling.

We define a labeled pointed surface similarly.

**Remark 2.5.** The operation of disjoint union clearly extends to labeled marked surfaces. When we extend the operation of orientation reversal to labeled marked surfaces, we also apply the involution $(\cdot)^\dagger$ to all the labels.

**Definition 2.11.** A modular functor based on the label set $\Lambda$ is a functor $V$ from the category of labeled marked surfaces to the category of finite dimensional complex vector spaces satisfying the axioms MF1 to MF5 below.

**MF1. Disjoint union axiom:** The operation of disjoint union of labeled marked surfaces is taken to the operation of tensor product, i.e. for any pair of labeled marked surfaces there is an isomorphism

$$V((\Sigma_1, \lambda_1) \sqcup (\Sigma_2, \lambda_2)) \cong V(\Sigma_1, \lambda_1) \otimes V(\Sigma_2, \lambda_2).$$

The identification is associative.

**MF2. Glueing axiom:** Let $\Sigma$ and $\Sigma_c$ be marked surfaces such that $\Sigma_c$ is obtained from $\Sigma$ by glueing at an ordered pair of points and projective tangent vectors with respect to a glueing map $c$. Then there is an isomorphism

$$V(\Sigma_c, \lambda) \cong \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \mu^\dagger, \lambda),$$

which is associative, compatible with glueing of morphisms, disjoint unions and it is independent of the choice of the glueing map in the obvious way (see remark 2.3).

**MF3. Empty surface axiom:** Let $\emptyset$ denote the empty labeled marked surface. Then

$$\dim V(\emptyset) = 1.$$

**MF4. Once punctured sphere axiom:** Let $\Sigma = (S^2, \{p\}, \{v\}, 0)$ be a marked sphere with one marked point. Then

$$\dim V(\Sigma, \lambda) = \begin{cases} 
1, & \lambda = 0 \\
0, & \lambda \neq 0.
\end{cases}$$
**MF5. Twice punctured sphere axiom:** Let $\Sigma = (S^2, \{p_1, p_2\}, \{v_1, v_2\}, \{0\})$ be a marked sphere with two marked points. Then

$$\dim V(\Sigma, (\lambda, \mu)) = \begin{cases} 1, & \lambda = \mu^\dagger \\ 0, & \lambda \neq \mu^\dagger. \end{cases}$$

In addition to the above axioms one may has extra properties, namely

**MF-D. Orientation reversal axiom:** The operation of orientation reversal of labeled marked surfaces is taken to the operation of taking the dual vector space, i.e for any labeled marked surface $(\Sigma, \lambda)$ there is a pairing

$$\langle \cdot, \cdot \rangle : V(\Sigma, \lambda) \otimes V(\Sigma, \lambda^\dagger) \to C,$$

compatible with disjoint unions, glueings and orientation reversals (in the sense that the induced isomorphisms $V(\Sigma, \lambda) \cong V(\Sigma, \lambda^\dagger)^*$ and $V(\Sigma, \lambda^\dagger) \cong V(\Sigma, \lambda)^*$ are adjoints).

and

**MF-U. Unitarity axiom**

Every vector space $V(\Sigma, \lambda)$ is furnished with a hermitian inner product

$$\langle \cdot, \cdot \rangle : V(\Sigma, \lambda) \otimes \overline{V(\Sigma, \lambda)} \to C$$

so that morphisms induces unitary transformation. The hermitian structure must be compatible with disjoint union and glueing. If we have the orientation reversal property, then compatibility with the unitary structure means that we have a commutative diagram

$$\begin{array}{ccc}
V(\Sigma, \lambda) & \xrightarrow{\cong} & V(\Sigma, \lambda^\dagger)^* \\
\downarrow\cong & & \downarrow\cong \\
\overline{V(\Sigma, \lambda)^*} & \xrightarrow{\cong} & \overline{V(\Sigma, \lambda^\dagger)},
\end{array}$$

where the vertical identifications come from the hermitian structure and the horizontal ones from the duality.

### 3. Basic data

Following Walker [25], we will review the notion of basic data.

Fix throughout the rest of this paper a modular functor $Z$ with label set $(\Lambda, 0, \dagger)$. We do not assume that this modular functor $Z$ has a duality structure as in axiom MF-D or a unitary structure as in axiom MF-U.

We let $\Delta = (S^2; \infty; v_\infty), \Xi = (S^2; 0, \infty; v_0, v_\infty)$ and recall that $\Upsilon = (S^2; 0, 1, \infty; v_0, v_1, v_\infty)$. For $\lambda, \mu, v \in \Lambda$ we define

$$Z_0 = Z(\Delta, 0), \quad Z_{\lambda, \lambda^\dagger} = Z(\Xi, \lambda, \lambda^\dagger)$$

and recall that

$$Z_{\lambda, \mu, v} = Z(\Upsilon, \lambda, \mu, v).$$

Here $\dim Z_0 = \dim Z_{\lambda, \lambda^\dagger} = 1$ and we define

$$N_{\lambda, \mu}^v = \dim Z_{\lambda, \mu, v}^\dagger.$$
The morphisms of the marked surfaces $\Delta$, $\Xi$ and $Y$ acts on the vector spaces $Z_{\alpha}$, $Z_{\alpha,1}^+$ and $Z_{\alpha,\mu,\nu}$.

Recall the self morphisms $R$ and $B$ of $Y$. They, together with the Dehn-twist around 0 and the morphisms $(\text{Id},1)$, generate the mapping class groupoids of $Y$ with all its possible labelings. By factorization, these also determine the action of the mapping class groupoid of both $\Delta$ and $\Xi$ with all possible labelings. We shall make use of the following notation:

$$B_{12} = RBR^{-1}, \quad B_{13} = R^{-1}BR \quad \text{and} \quad B_{23} = B$$

Further we denote by $T_1, T_2$ and $T_3$ the Dehn-twists around 0 and the morphisms $(\text{Id},1)$, generate the mapping class groupoids of $Y$ with all its possible labelings. By factorization, these also determine the action of the mapping class groupoid of both $\Delta$ and $\Xi$ with all possible labelings. We shall make use of the following notation:

$$B_{12} = RBR^{-1}, \quad B_{13} = R^{-1}BR \quad \text{and} \quad B_{23} = B$$

Factoring $\Psi$ along $(\gamma_1,3/2)$ we obtain the marked surface $\Psi_1$ say and likewise factoring $\Psi$ along $(\gamma_2,5/2)$ we obtain the marked surface $\Psi_2$ say. There is a unique diffeomorphism from $Y \coprod Y$ to $\Psi_1$ which takes the real axes to parts of the real axes on each component, infinity to the respective quotients of $\gamma_i$ and maps the first copy of $Y$ to the interior of $\gamma_i$ in $\mathbb{C}$.

We thus get two isomorphisms

$$\Phi_1(\mu,\lambda) : \bigoplus_{v \in \Lambda} Z(Y; v, \mu, \lambda) \otimes Z(Y; v^+, \kappa, \xi) \to Z(\Psi, \lambda, \mu, \xi, \kappa)$$

and

$$\Phi_2(\mu, \lambda) : \bigoplus_{\nu \in \Lambda} Z(Y; \nu, \lambda, \kappa) \otimes Z(Y; \nu^+, \xi, \mu) \to Z(\Psi, \lambda, \mu, \xi, \kappa).$$

We then define

$$F \left[ \begin{array}{ccc} \mu & \xi \\ \nu & \kappa \end{array} \right] : \bigoplus_{v \in \Lambda} Z_{v,\mu,\lambda} \otimes Z_{v^+,\kappa,\xi} \to \bigoplus_{\nu \in \Lambda} Z_{\nu,\lambda,\kappa} \otimes Z_{\nu^+,\xi,\mu}$$

by

$$F = \Phi_2^{-1} \circ \Phi_1.$$

Let $\Theta$ be an oriented genus one surface. Let $p$ be a point on $\Theta$ and $v_p$ be a tangent vector at $p$. Choose two oriented simple closed curves $(\alpha, \beta)$ on $\Theta - \{p\}$ as indicated in figure 3.

Let $L_\alpha = \text{Span}\{[\alpha]\}$ and $L_\beta = \text{Span}\{[\beta]\}$ be the Lagrangian subspaces generated in the first homology group. Let $\Theta^\alpha = (\Theta, p, v_p, L_\alpha)$ and $\Theta^\beta = (\Theta, p, v_p, L_\beta)$

Let $\Theta_\alpha$ and $\Theta_\beta$ be marked surfaces, which results from factorizing $\Theta^\alpha$ along $\alpha$, respectively $\Theta^\beta$ along $\beta$. By factorization we get isomorphisms

$$\Phi_\alpha : Z(\Theta^\alpha, \lambda) \to \bigoplus_{\mu} Z(\Theta_\alpha, \lambda, \mu, \mu^+)$$

and

$$\Phi_\beta : Z(\Theta^\beta, \lambda) \to \bigoplus_{\nu} Z(\Theta_\beta, \lambda, \nu, \nu^+).$$

Pick diffeomorphisms

$$f_\alpha : Y \to \Theta_\alpha$$

and

$$f_\beta : Y \to \Theta_\beta$$

which maps the real axis onto $\beta$, respectively $\alpha$. 

Then we can define
\[
S(\lambda) : \bigoplus_{\mu} Z_{\lambda,\mu,\mu^t} \rightarrow \bigoplus_{\nu} Z_{\lambda,\nu,\nu^t}
\]
by
\[
S(\lambda) = Z(f_\beta)^{-1} \Phi_\beta Z(Id) \Phi^{-1}_{\alpha} Z(f_\alpha),
\]
where \( Id : (\Theta^{\alpha}, \lambda) \rightarrow (\Theta^{\beta}, \lambda) \).

Following Walker ((3.6) in [25]), we define basic data as follows.

**Definition 3.1** (Walker). *Basic data for the modular functor \( Z \) consist of the following data:

**A:** The vector spaces \( Z_0, Z_{\lambda,\lambda^t} \) and \( Z_{\lambda,\mu,\nu} \) together with the induced actions of the groupoids of morphism of marked surface acting on them.

**B:** The linear isomorphism
\[
F \left[ \begin{array}{ccc}
\mu & \xi \\
\Lambda & \kappa
\end{array} \right] : \bigoplus_{\nu \in \Lambda} Z_{\nu,\mu,\lambda} \otimes Z_{\nu^t,\kappa,\xi} \rightarrow \bigoplus_{\nu \in \Lambda} Z_{\nu,\lambda,\kappa} \otimes Z_{\nu^t,\xi,\mu}.
\]

**C:** The linear isomorphism
\[
S(\lambda) : \bigoplus_{\mu} Z_{\lambda,\mu,\mu^t} \rightarrow \bigoplus_{\nu} Z_{\lambda,\nu,\nu^t}.
\]

**Lemma 3.1.** *The basic data determines the modular functor \( Z \) uniquely.*

This lemma is proved in section 5 of [25]. Below we outline the main construction behind that.

Of course A and B are by definition part of the genus zero data of \( Z \). Further A and B clearly determines the genus zero part of \( Z \). By definition \( S(\lambda) \) requires genus one as well. But the main result of this paper is that C is in fact determined by A and B. Hence

**Theorem 3.1.** *The basic data under A and B determines the modular functor \( Z \).*

This theorem follows from Theorem 1.1 which is proved below.

Let us now fix a vector \( \zeta_0 \in Z_0 \) and \( \zeta(\lambda) \in Z_{\lambda,\lambda^t} \). By the glueing axiom we get natural isomorphisms
\[
Z_0 \cong Z_{0,0} \otimes Z_0
\]
and
\[
Z_{\lambda,\lambda^t} \cong Z_{\lambda,\lambda^t} \otimes Z_{\lambda,\lambda^t}.
\]
Under these isomorphisms, we require that
\[
\zeta_0 = \zeta(0) \otimes \zeta_0
\]
and
\[
\zeta(\lambda) = \zeta(\lambda) \otimes \zeta(\lambda).
\]
This condition uniquely fixes \( \zeta(\lambda) \) for all \( \lambda \in \Lambda \).

We shall also need to do a computation in a basis of \( Z_{\lambda,\mu,\nu} \), so we fix a basis \( \zeta_j(\lambda, \mu, \nu), j = 1, \ldots, N^t_{\lambda,\mu} \) for each of these vector spaces. In case \( N^t_{\lambda,\mu} = 1 \), we use the notation \( \zeta(\lambda, \mu, \nu) = \zeta_1(\lambda, \mu, \nu) \). We will require that under the isomorphism
\[
Z_{\lambda,\lambda^t} \cong Z_{\lambda,\lambda^t,0} \otimes Z_0
\]
we have that

(3) \[ \zeta(\lambda) = \xi_1(\lambda, \lambda^\dagger, 0) \otimes \xi_0. \]

We write \( S = S(0) \) and with respect to the prefereed basis \( \zeta(\lambda) \in Z_{\lambda, \lambda^\dagger}, \lambda \in \Lambda \),
we have the matrix presentation \( S = (S_{\lambda, \mu})_{\lambda, \mu \in \Lambda} \).

Let us now recall the reconstruction of a modular functor from its basic data.
On a marked surface \( \Sigma \), one considers pairs \((C, \Pi)\), where

- \( C \) is a finite collections of disjoint simple closed curves, each equipped with
  a base point, such the result of factoring \( \Sigma \) along \( C \) results in a disjoint
  union of marked surfaces \( \Sigma_i, i \in I \) for some finite set \( I \). Further we assume
  that the Lagrangian subspace of \( \Sigma \) is generated by the curves in \( C \).
- \( \Pi \) is a disjoint union of morphisms of marked surfaces from an ordered
  disjoint union of \( \Delta \)'s, \( \Xi \)'s and \( \Upsilon \)'s to the \( \Sigma_i, i \in I \), covering each \( \Sigma_i \) exactly
  once.

Such a pair \((C, \Pi)\) is called an overmarking of \( \Sigma \) in [25]. If \( C \) is such that \(|C|\) is
minimal, then \((C, \Pi)\) is called a marking in [25] following [13]. We shall call these
pairs decompositions in this paper so as not to confuse them with the structure of a
marked surface as introduced in the previous section.

It follows from the results of [13], that any two pairs of decompositions of
a given marked surface are related by a finite sequence of one of the following
changes of decompositions from say \((C', \Pi')\) to \((C'', \Pi'')\):

\( \mathcal{M} \): The collection of curves are the same \( C' = C'' \), and there are automor-
phisms of the \( \Delta \)'s, \( \Xi \)'s and \( \Upsilon \)'s, which relates \( \Pi' \) to \( \Pi'' \), namely \((\Pi'')^{-1} \Pi' \).

\( \mathcal{A} \): Insertion or removal of a component of \( C' \) to obtain \( C'' \), which results in a
 corresponding insertion or removal of a copy of \( \Xi \).

\( \mathcal{D} \): Insertion or removal of a component of \( C' \) to obtain \( C'' \), which results in the
 replacement of one \( \Xi \) by one new \( \Delta \) and one new \( \Upsilon \) or the converse.

\( \mathcal{F} \): There are \( \gamma' \in C' \) and \( \gamma'' \in C'' \) with the property that \( C' - \{\gamma'\} = C'' - \{\gamma''\} \) and the factorization along \( C' - \{\gamma'\} \) contains a component which
via \( \Pi' \) and \( \Pi'' \) is identified with \( \Psi \), such that \( \gamma_1 \) goes to \( \gamma' \) and \( \gamma_2 \) goes to \( \gamma'' \).

\( \mathcal{S} \): There are \( \gamma' \in C' \) and \( \gamma'' \in C'' \) with the property that \( C' - \{\gamma'\} = C'' - \{\gamma''\} \) and the factorization along \( C' - \{\gamma'\} \) contains a component which
via \( \Pi' \) and \( \Pi'' \) is identified with \((\Theta, p, v_p)\), such that \( a \) goes to \( \gamma' \) and \( b \) goes to \( \gamma'' \).

Let \((C, \Pi)\) be a decomposition of a labeled marked surface \((\Sigma, \lambda)\). Let \( \Sigma_C \) be
the marked surface, one obtains from factoring \( \Sigma \) along \( C \). Let \( \Lambda_c = \Lambda^{\times c} \), where
\( c = |C| \). The factorization axiom gives us an isomorphism

\[ Z(\Sigma, \lambda) \cong \bigoplus_{\mu \in \Lambda_c} Z(\Sigma_C, \lambda, \mu) \]

For each \( \mu \in \Lambda_c \) we let \( Z(\lambda, \mu) \) be the corresponding tensor product of the vector
spaces \( Z_{\lambda'} \)'s, \( Z_{\lambda', (\lambda')^\dagger} \)'s and \( Z_{\lambda', \mu, \nu} \)'s. Using \( \Pi \) we then get an isomorphism

\[ Z(\Pi) : Z(\lambda, \mu) \rightarrow Z(\Sigma_C, \lambda, \mu) \]

which induces an isomorphism

\[ Z(C, \Pi) : \bigoplus_{\mu \in \Lambda_c} Z(\lambda, \mu) \rightarrow Z(\Sigma, \lambda). \]
Suppose now \((C', \Pi')\) and \((C'', \Pi'')\) are two decompositions of the same pointed surface \((\Sigma, P, V)\), which are related by one of the changes \(\mathcal{M}\) to \(S\). Let \(L'\) and \(L''\) be the Lagrangian subspaces generated by respectively the \(C\) surface \((\Sigma', P, V)\) and \(M\) surface \((\Sigma, P, V, L')\). The isomorphism is determined by the basic data as follows:

\[
\Sigma \text{ and } \Sigma \text{ related by a sequence of changes given by } \mathcal{F} \text{ we have that } L' = L'', \text{ which is not the case for } S. \text{ Let } \Sigma' = (\Sigma, P, V, L') \text{ and } \Sigma'' = (\Sigma, P, V, L''). \text{ We then get an induced isomorphism }
\]

\[
Z((C', \Pi'), (C'', \Pi'')) : \bigoplus_{\mu \in \Lambda, \nu} Z(\lambda, \mu) \rightarrow \bigoplus_{\mu \in \Lambda, \nu} Z(\lambda, \mu)
\]
given by

\[
Z((C', \Pi'), (C'', \Pi'')) = Z(C'', \Pi'')^{-1} Z(\text{Id} : (\Sigma', \lambda) \rightarrow (\Sigma', \lambda)) Z(C', \Pi')
\]

This isomorphism is determined by the basic data as follows:

For change of type:

- \(\mathcal{M}\): The linear map is given by a direct sum of tensor products of the linear maps induced by the morphism \((\Pi'')^{-1} \circ \Pi'\) between the appropriate \(\Sigma\)'s, \(\Delta\)'s and \(Y\)'s.
- \(\mathcal{A}\): The linear map is induced by insertion or "removal" of the vector \(\xi(\lambda) \in Z_{\Lambda, \Lambda^+}\) for the appropriate \(\Xi\).
- \(\mathcal{D}\): The linear map is induced by the identity tensor the isomorphism \(\mathcal{J}\) (or its inverse) inserted at the appropriate place.
- \(\mathcal{F}\): The linear map is induced by the identity tensor the isomorphism \(\mathcal{I}\) inserted at the appropriate place.
- \(\mathcal{S}\): The linear map is induced by the identity tensor the isomorphism \(\mathcal{L}\) inserted at the appropriate place.

Let \((C_1, \Pi_1)\) and \((C_2, \Pi_2)\) be any two decompositions of the same labeled pointed surface \((\Sigma, P, V, \lambda)\). Let \(L_i\) be the induced Lagrangian subspaces on \(\Sigma\) and \(\Sigma_i = (\Sigma, P, V, L_i)\) the corresponding marked surfaces. Since \((C_1, \Pi_1)\) and \((C_2, \Pi_2)\) are related by a sequence of changes \(\mathcal{M}\) to \(S\), we get that the isomorphism

\[
Z((C_1, \Pi_1), (C_2, \Pi_2)) : \bigoplus_{\mu_1 \in \Lambda, \nu_1} Z(\lambda, \mu_1) \rightarrow \bigoplus_{\mu_2 \in \Lambda, \nu_2} Z(\lambda, \mu_2)
\]
given by

\[
Z((C_1, \Pi_1), (C_2, \Pi_2)) = Z(C_2, \Pi_2)^{-1} Z(\text{Id} : (\Sigma_1, \lambda) \rightarrow (\Sigma_2, \lambda)) Z(C_1, \Pi_1)
\]
is also determined by the basic data.

Suppose now that \(f : (\Sigma_1, \lambda_1) \rightarrow (\Sigma_2, \lambda_2)\) is a morphism of marked surfaces and \((C, \Pi)\) is a decomposition of \(\Sigma_1\). Then \((f(C), f \circ \Pi)\) is a decomposition of \(\Sigma_2\) and we get a commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{\mu \in \Lambda} Z(\lambda_1, \mu) & \xrightarrow{Z(C, \Pi)} & Z(\Sigma_1, \lambda_1) \\
\text{Id} \downarrow & & \downarrow Z(f) \\
\bigoplus_{\mu \in \Lambda} Z(\lambda_2, \mu) & \xrightarrow{Z(f(C), f \circ \Pi)} & Z(\Sigma_2, \lambda_2)
\end{array}
\]

Suppose now \((\gamma, p)\) is an oriented simple closed curve with a preferred point \(p\) on a marked surface \(\Sigma\). Let \(\Sigma\) be obtained from \(\Sigma\) by factoring along \((\gamma, p)\). Suppose that \((C, \Pi)\) is a decomposition of \(\Sigma\), such that \(\gamma \in C\). Then \((C, \Pi)\) also
induces a decomposition of $\Sigma$, say $(\tilde{C}, \tilde{\Pi})$ and we get the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\mu \in \Lambda} Z(\lambda, \mu) & \xrightarrow{Z(\tilde{C}, \tilde{\Pi})} & Z(\Sigma, \lambda) \\
\text{Id} & \Downarrow \cong & \\
\bigoplus_{\mu' \in \Lambda} \bigoplus_{\mu'' \in \Lambda_{-1}} Z(\lambda, \mu', \mu'') & \xrightarrow{Z(\tilde{C}, \tilde{\Pi})} & \bigoplus_{\mu' \in \Lambda} Z(\tilde{\Sigma}, \lambda, \mu', (\mu')^+) 
\end{array}$$

Likewise we trivially get a similar diagram for the case of the disjoint union isomorphism.

From this it follows that the basic data determines the modular functor because the action of any morphism, the factorization and the disjoint union isomorphisms are all determined by the basic data.

By considering various finite sequences of decompositions of certain marked surfaces, we generate nontrivial relations on the basic data. Then such universal relations are given on page 55 in [25]. We will use some of them in section 7.

4. CURVE OPERATORS

Let $\Sigma = (\Sigma, P, V, L)$ be a general marked surface. Let $\lambda$ be a labeling of $\Sigma$.

Let $\gamma$ be an oriented simple closed curve on $\Sigma - P$ and $\lambda, \gamma \in \Lambda$ a fixed label. We now define an operator

$$Z(\gamma, \lambda, \gamma) : Z(\Sigma, \lambda) \to Z(\Sigma, \lambda)$$

canonically associated to the pair $(\gamma, \lambda, \gamma)$.

Choose an embedding $i : D \to \Sigma - P$ of the unit disc $D$ into $\Sigma - P$, such that $i([-1, 1]) = \gamma \cap i(D)$ as indicated in figure 4. Let $(\gamma_i, p_i) = (i(\partial D), i(1))$. Let $P' = \{i(-\frac{1}{2}), i(\frac{1}{2})\}$ and $V'$ be directions along $\gamma$ in the positive direction at $P'$. Let $\tilde{P} = P' \cup P$, $\tilde{V} = V \cup V'$ and $\Sigma = (\Sigma, P', \tilde{V}, L)$.

![Figure 4. The curve $\gamma$ and the disk $i(D)$ on $\Sigma$.](image)

The factorization of $\Sigma$ along $\gamma_i$ has two connected components which we denote $\tilde{\Sigma}'$ and $\tilde{\Sigma}''$. Here $\tilde{\Sigma}'$ is obtained from $i(D)$ by identifying $\gamma_i$ to a point $p'$ with $p_i \in \gamma_i$ inducing a tangent direction $v'$ at $p'$. Likewise $\tilde{\Sigma}''$ is the quotient of $\Sigma - i(D - \partial D)$, where we identify $\gamma_i$ to a point $p''$ again with $p_i \in \gamma_i$ inducing a tangent direction $v''$ at $p''$.

Let $\tilde{P}' = P' \cup \{p'\}$ and $\tilde{V}' = V' \cup \{v'\}$ set $\tilde{\Sigma}' = (\tilde{\Sigma}', \tilde{P}', \tilde{V}')$. Let $\tilde{P}'' = P \cup \{p''\}$ and $\tilde{V}'' = V \cup \{v''\}$ set $\tilde{\Sigma}'' = (\tilde{\Sigma}'', \tilde{P}'', \tilde{V}'')$. 


The glueing and disjoint union axiom gives isomorphisms
\[
Z(\Sigma, \lambda, \lambda_\gamma^+, \lambda_\gamma) \cong \bigoplus_{\mu \in \Lambda} Z(\Sigma', \lambda_\gamma^+, \lambda_\gamma, \mu) \otimes Z(\Sigma'', \mu^+, \lambda)
\]
and
\[
Z(\tilde{\Sigma}, \lambda, \lambda_\gamma, \lambda_\gamma^+) \cong \bigoplus_{\mu \in \Lambda} Z(\tilde{\Sigma}', \lambda_\gamma, \lambda_\gamma^+, \mu) \otimes Z(\tilde{\Sigma}'', \mu^+, \lambda).
\]
The embedding \(i\) induces an isomorphism of marked curves \(i : Y \to \tilde{\Sigma}'\), which therefore gives isomorphisms
\[
Z(i) : Z(Y, \lambda_\gamma^+, \lambda_\gamma, 0) \to Z(\tilde{\Sigma}', \lambda_\gamma^+, \lambda_\gamma, 0)
\]
and
\[
Z(i)^\dagger : Z(\tilde{\Sigma}', \lambda_\gamma, \lambda_\gamma^+, 0)^* \to Z(Y, \lambda_\gamma, \lambda_\gamma^+, 0)^*.
\]
Also by the glueing and disjoint union axiom combined with axiom MF4, we get an isomorphism
\[
Z(\tilde{\Sigma}'', 0, \lambda) \cong Z(\Sigma, \lambda)
\]
which is unique up to scale.

The vector \(\tilde{\xi}_1(\lambda_\gamma^+, \lambda_\gamma, 0) \in Z(Y, \lambda_\gamma^+, \lambda_\gamma, 0)\) together with the isomorphisms \(Z(i)\) gives us an inclusion
\[
I_i(\lambda_\gamma) : Z(\Sigma, \lambda) \to Z(\tilde{\Sigma}, \lambda, \lambda_\gamma^+, \lambda_\gamma).
\]
A vector vector \(\alpha_\gamma \in Z(Y, \lambda_\gamma, \lambda_\gamma^+, 0)^*\) together with the isomorphism \(Z(i)^\dagger\) gives a projection
\[
P_i(\lambda_\gamma) : Z(\tilde{\Sigma}, \lambda, \lambda_\gamma, \lambda_\gamma^+) \to Z(\Sigma, \lambda).
\]
We shall normalize the forms \(\alpha_\mu \in Z(Y, \mu, \mu^+, 0)^*\), \(\mu \in \Lambda\) as follows:

We require that
\[
\alpha_\mu(Z(B)(\tilde{\xi}_1(\mu^+, \mu, 0))) = \frac{S_{\mu, \mu}}{S_{0,0}}.
\]

We shall now consider the following diffeomorphism \(\Phi\) of \(\tilde{\Sigma}\). Fix a tubular neighbourhood of \(\gamma\) inside \(\Sigma - P\). The diffeomorphism \(\Phi\) will be the identity outside this tubular neighbourhood. Inside the tubular neighbourhood it rotates and stretches \(i([-\frac{1}{2}, \frac{1}{2}])\) onto \(\gamma - i((-\frac{1}{2}, \frac{1}{2}))\) and \(\gamma - i((-\frac{1}{2}, \frac{1}{2}))\) onto \(i([-\frac{1}{2}, \frac{1}{2}])\). We see that
\[
Z(\Phi) : Z(\tilde{\Sigma}, \lambda, \lambda_\gamma^+, \lambda_\gamma) \to Z(\Sigma, \lambda, \lambda_\gamma, \lambda_\gamma^+).
\]

**Definition 4.1.** The curve operator associate to \((\gamma, \lambda_\gamma)\) is by definition
\[
Z(\gamma, \lambda_\gamma) = P_i(\lambda_\gamma) \circ Z(\Phi) \circ I_i(\lambda_\gamma).
\]

We observe that \(Z(\gamma, \lambda_\gamma)\) does not depend on the choice of \(i\). In fact \(Z(\gamma, \lambda_\gamma)\) only depends on the free homotopy class of \(\gamma\).

We clearly have the following lemma.

**Lemma 4.1.** Suppose \(f : \Sigma_1 \to \Sigma_2\) is a morphism of marked surface and \(\gamma_i\) are closed oriented curves on \(\Sigma_i - P_i\), \(i = 1, 2\), such that \(f(\gamma_1) = \gamma_2\). Then
\[
Z(f)^{-1}Z(\gamma_2, \lambda)Z(f) = Z(\gamma_1, \lambda)
\]
for all \(\lambda \in \Lambda\).
If $\gamma$ is contractible on the marked surface $\Sigma$, then

$$Z(\gamma, \lambda_\gamma) = \frac{S_{0,\lambda_\gamma}}{S_{0,0}} \text{Id}_{Z(\Sigma, \lambda)}.$$  

5. The relation between curve operators and Dehn Twists

In this section we give a formula for the Dehn-Twist operator in terms of the curve operators associated to any oriented simple closed curve.

Let $(\Sigma, \lambda)$ be a labeled marked surface. Let $\gamma$ be an oriented simple closed curve on $\Sigma - P$. Let $\varphi_\gamma : (\Sigma, \lambda) \to (\Sigma, \lambda)$ be the Dehn twist in the curve $\gamma$. By construction $\varphi_\gamma$ is the identity outside some tubular neighbourhood of $\gamma$. Similarly, $Z(\gamma, \lambda_\gamma)$ is also a local construction within a tubular neighbourhood of $\gamma$. Pick a point $p$ on $\gamma$ and let $\tilde{\Sigma}$ be obtained from $\Sigma$, by factoring $\Sigma$ along $(\gamma, p)$. Then we get an isomorphism

$$Z(\Sigma, \lambda) \cong \bigoplus_\mu Z(\tilde{\Sigma}, \lambda, \mu, \mu^\dagger).$$

Both of the operators $Z(\varphi_\gamma)$ and $Z(\gamma, \lambda_\gamma)$ are diagonal with respect to this direct sum decomposition and acts by multiples of the identity on each of the summands. This follows immediately from factoring along the boundary of a tubular neighborhood of $\gamma$. One also sees this way that these multiples by which these operators acts by are independent of both $\gamma$ and $(\Sigma, \lambda)$.

Proposition 5.1. There exist uniquely determined constants $c_{\lambda_\gamma} \in \mathbb{C}$ such that for any simple closed oriented curve $\gamma$ on any labeled marked surface $(\Sigma, \lambda)$ we have that

$$Z(\varphi_\gamma) = \sum_{\lambda_\gamma \in \Lambda} c_{\lambda_\gamma} Z(\gamma, \lambda_\gamma).$$

Recall that $\Xi = (S^2; 0, \infty; v_0, v_\infty)$ and let $\gamma$ be the unit circle oriented in the positive direction.

We define $C_{\lambda, \mu} \in \mathbb{C}$ to be the scalar by which $Z(\gamma, \lambda)$ acts on $Z(\Xi, \mu, \mu^\dagger) = Z_{\mu, \mu^\dagger}$, for $\lambda, \mu \in \Lambda$. Further we recall that $d_\mu \in \mathbb{C}$ is the scalar by which $Z(\varphi_\gamma)$ acts on $Z(\Xi, \mu, \mu^\dagger) = Z_{\mu, \mu^\dagger}$.

Proposition 5.2. The matrix $C_{\lambda, \mu}$ is invertible.

We will prove this Proposition in section 7.

Proof of Proposition 5.1. The constants $c_{\lambda_\gamma}$, which we seek has to satisfy

$$d_\mu = \sum_{\lambda} c_{\lambda} C_{\lambda, \mu}.$$

By Proposition 5.2 the matrix $C$ is invertible, so there is a unique solution to this set of equations.

We remark that since $C$ is invertible, the $c_{\lambda}$ are determined by the genus zero data. Formula 8 below gives an explicit formula for the $c_{\lambda}$’s.
6. **The Reduction from a Once Punctured Genus One Surfaces to Genus Zero**

Recall the simple closed curves $\alpha$ and $\beta$ on $\Theta - \{p\}$ from figure 3. Let $S : \Theta^a \to \Theta^a$ be the morphism of marked surfaces, which satisfies that

$$S(\alpha) = \beta, \quad S(\beta) = \alpha^{-1},$$

where we here interpret $\alpha$ and $\beta$ as generators of the fundamental group of $\Theta - \{p\}$ base at their intersection point.

**Theorem 6.1.** The morphism

$$Z(S) = Z(\Theta^a, \lambda) \to Z(\Theta^a, \lambda)$$

is determined by the genus zero part of $Z$ for all $\lambda \in \Lambda$.

**Proof.** We recall that the mapping class group $\Gamma$ of $\Theta - \{p\}$ is

$$\Gamma \cong \{S, T | (ST)^3 = S^2\},$$

where $S$ is as specified above and $T$ is the Dehn-twist in $\alpha$.

We see that

$$S = T^{-1}ST^{-1}S^{-1}T^{-1}.$$ 

Hence if we let $T'$ be the Dehn-twist in $\beta$, then

$$S = T^{-1}(T')^{-1}T^{-1}.$$ 

But by Proposition we have that there are constants $\tilde{c}_\lambda$ such that

$$Z(T')^{-1} = \sum_{\lambda \beta \in \Lambda} \tilde{c}_{\lambda \beta} Z(\beta, \lambda \beta).$$

So

$$Z(S) = \sum_{\lambda \beta \in \Lambda} \tilde{c}_{\lambda \beta} Z(T^{-1})Z(\beta, \lambda \beta)Z(T^{-1}).$$

Let $f : A \to \Theta - \{p\}$ be an embedding of an annulus $A$ into $\Theta - \{p\}$ as shown in figure 5.

![Figure 5](image-url). A once punctured surface of genus 1 with an annulus $A$ around the $\alpha$ curve.
Let \( \alpha_1 \) and \( \alpha_2 \) be the two boundary curves of \( j(A) \), with base points say \( q_1 \in \alpha_1 \) and \( q_2 \in \alpha_2 \). Let \( \Theta_1 \) respectively \( \Theta_2 \) be obtained from \( \Theta \) by factoring along \( \alpha_1 \) respectively along \( \alpha_2 \). We denote the resulting base point and tangent direction by \((P_1, V_1) = (\{q_1', q''_1\}, \{v_{q'_1}, v_{q''_1}\})\) and \((P_2, V_2) = (\{q'_2, q''_2\}, \{v_{q'_2}, v_{q''_2}\})\) on respectively \( \Theta_1 \) and \( \Theta_2 \). Let \( \Sigma \) be obtained from \( \Theta_1 \), by factoring along \( \alpha_2 \) or equivalently from \( \Theta_2 \) by factoring along \( \alpha_1 \). Pick a point \( r \in j(A - \partial A) \cap \gamma \) and the tangent direction \( v_r \) at \( r \) along \( \beta \) in the positive direction.

Now choose an embedding \( i : D \rightarrow \Theta - (j(A) \cup \{p\}) \) and a diffeomorphism \( \Phi \) with the properties required in the construction of \( Z(\beta, \lambda_\beta) \). In fact we will choose a \( \Phi \) which is a composite of two diffeomorphisms \( \Phi_1 \) and \( \Phi_2 \) as follows. Let \( p'_{-\beta} = i(-\frac{1}{2}) \) and \( p'_{+\beta} = i(\frac{1}{2}) \) and \( v_{p'_{-\beta}} \), \( v_{p'_{+\beta}} \) respectively \( v_{p'_{\beta}} \) be the induced tangent directions along \( \beta \).

Let \( P = \{p, p'_{-\beta}, p'_{+\beta}\} \) and \( V = \{v_p, v_{p'_{-\beta}}, v_{p'_{+\beta}}\} \). Let \( P' = \{p, p'_{+\beta}, r\} \) and \( V' = \{v_p, v_{p'_{+\beta}}, v_r\} \). Further set
\[
\tilde{P}_i = P \cup P_i, \tilde{V}_i = V \cup V_i
\]
for \( i = 1, 2 \).

We pick
\[
\Phi_1 : (\Theta_2, \tilde{P}_2, \tilde{V}_2) \rightarrow (\Theta_2, \tilde{P}'_2, \tilde{V}'_2)
\]
such that \( \Phi_1 \) is the identity outside a tubular neighbourhood of the piece of \( \beta \) from \( p'_{-\beta} \) to \( r \) and it maps \( \Phi_1(p'_{+\beta}) = p'_{+\beta} \) and \( \Phi_1(p'_{\beta}) = r \). The map
\[
\Phi_2 : (\Theta_1, \tilde{P}'_1, \tilde{V}'_1) \rightarrow (\Theta_1, \tilde{P}_1, \tilde{V}_1)
\]
is chosen such that it is the identity outside a neighbourhood of the piece of \( \beta \) from \( r \) to \( p'_{+\beta} \) and it maps \( \Phi_2(r) = p'_{+\beta} \). By re-glueing \( \Theta_1 \) and \( \Theta_2 \) to obtain \( \Theta \) we see that \( \Phi_1 \) and \( \Phi_2 \) induces diffeomorphisms \( \Phi_1' \) and \( \Phi_2' \) of \( \Theta \), which preserves \( \alpha_2 \) respectively \( \alpha_1 \). We let
\[
\Phi = \Phi_2' \circ \Phi_1' : (\Theta, P, V) \rightarrow (\Theta, P, V).
\]

Thus we have the linear maps for all choices of \( \mu, \mu' \in \Lambda \)
\[
Z(\Phi_1)_\mu : Z(\Theta_2, \tilde{P}_2, \tilde{V}_2, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}, \mu, \mu^\dagger) \rightarrow Z(\Theta_2, \tilde{P}'_2, \tilde{V}'_2, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}, \mu, \mu^\dagger)
\]
and
\[
Z(\Phi_2)_\mu' : Z(\Theta_1, \tilde{P}'_1, \tilde{V}'_1, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}, \mu', (\mu')^\dagger) \rightarrow Z(\Theta_1, \tilde{P}_1, \tilde{V}_1, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}, \mu', (\mu')^\dagger).
\]

We further have the following two commutative diagrams
\[
\begin{array}{ccc}
Z(\Theta, P, V, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}) & \xrightarrow{Z(\Phi_1)} & Z(\Theta, P', V', \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}) \\
\oplus_{\mu \in \Lambda} Z(\Theta_2, \tilde{P}_2, \tilde{V}_2, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}, \mu, \mu^\dagger) & \xrightarrow{\oplus_{\mu \in \Lambda} Z(\Phi_1)_\mu} & \oplus_{\mu \in \Lambda} Z(\Theta_2, \tilde{P}'_2, \tilde{V}'_2, \lambda, \lambda^\dagger_{\gamma_r}, \lambda_{\gamma_r}, \mu, \mu^\dagger)
\end{array}
\]
Since both of these diffeomorphisms are the identity in a neighborhood of $m$, we get the formula

$$Z(\Theta, P', V', \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t) \xrightarrow{Z(\Phi_{\gamma}')} Z(\Theta, P, V, \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t)$$

$$\bigoplus_{\mu' \in \Lambda} Z(\Theta, \tilde{P}_1', \tilde{V}_1', \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t, (\mu')^t) \xrightarrow{\bigoplus_{\mu' \in \Lambda} Z(\Phi_{\gamma}')} \bigoplus_{\mu' \in \Lambda} Z(\Theta, \tilde{P}_1, \tilde{V}_1, \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t, \mu, (\mu')^t)$$

where the vertical arrows are the factorization isomorphisms.

By the commutativity of factorization we also have the following commutative diagram of isomorphisms:

$$Z(\Theta, P', V', \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t) \xrightarrow{\bigoplus_{\mu' \in \Lambda} Z(\Phi_{\gamma}')} \bigoplus_{\mu \in \Lambda} Z(\Theta_1, \tilde{P}_1', \tilde{V}_1', \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t, \mu, (\mu')^t)$$

$$\bigoplus_{\mu' \in \Lambda} Z(\Theta_2, \tilde{P}_2', \tilde{V}_2', \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t, (\mu')^t) \xrightarrow{\bigoplus_{\mu, \mu' \in \Lambda} Z((\Sigma), \tilde{P}, \tilde{V}, \lambda, \lambda_{\gamma}, \lambda_{\gamma}^t, \mu, \mu', (\mu')^t)}$$

Now consider the curve operator $Z(\beta, \lambda_{\beta})$. Since factorization along non-intersecting curves commute we see that both $P_1(\lambda_{\beta})$ and $I_1(\lambda_{\beta})$ is determined by genus zero morphology by factoring along say $\alpha_1$, which commutes with the factorization along $\gamma_1$.

By representing $T$ as the Dehn Twist in $\alpha_1$ respectively $\alpha_2$, we see get diffeomorphisms

$$T_1 : (\Theta_1, \tilde{P}_1', \tilde{V}_1') \to (\Theta_2, \tilde{P}_2', \tilde{V}_2')$$

and

$$T_2 : (\Theta_2, \tilde{P}_2', \tilde{V}_2') \to (\Theta_1, \tilde{P}_1', \tilde{V}_1')$$

Since both of these diffeomorphisms are the identity in a neighbourhood of $i(D)$, we get the formula

$$Z(S) = \sum_{\lambda_{\beta} \in \Lambda} \tilde{e}_{\lambda_{\beta}} P_1(\lambda_{\beta}) Z(T_2^{-1}(\Phi_{\beta}')) Z(\Phi_{\gamma}^t T_1^{-1}) I_1(\lambda_{\beta})$$

Tracing through the previous three commutative diagrams, we see that this is also the case for $Z(\Phi_{\gamma}^t T_1^{-1})$ and $Z(T_2^{-1}(\Phi_{\beta}'))$ are determined by genus zero data.

We observe the same argument can be used to show the following: The action of a Dehn twist along any curve on a marked surface equipped with a fixed decomposition is determined by the isomorphisms $M$ to $F$.

**Proof of Theorem 1.1.** We consider the morphism

$$S : (\Theta^a, \lambda) \to (\Theta^b, \lambda)$$

and observe that it is compatible with the factorizations in $\alpha$ and $\beta$ and via $f_{\alpha}$ and $f_{\beta}$ is compatible with the identity morphism $\text{Id} : Y \to Y$. We have the composition identity

$$S : (\Theta^a, \lambda) \to (\Theta^b, \lambda)$$

$$= (S : (\Theta^a, \lambda) \to (\Theta^b, \lambda)) \circ (\text{Id} : (\Theta^b, \lambda) \to (\Theta^a, \lambda))$$.  

From this we conclude that

$$Z(f^{-1}_a)Z(S : (\Theta^a, \lambda) \rightarrow (\Theta^a, \lambda))Z(f_a) = S(\lambda)^{-1}.$$  

Since $$Z(S : (\Theta^a, \lambda) \rightarrow (\Theta^a, \lambda))$$ is determined by genus zero data, we see that so is $$S(\lambda)$$ for all $$\lambda \in \Lambda$$.

\[\square\]

7. Consequences of the Universal Relations on the Basic Data.

In this section we will prove the formula in Proposition 5.2. It follows immediately from Proposition 7.2 below. However, first we need to derive a couple of consequences from the pentagon relation for $$F$$ and relations between $$F$$ and $$S$$.

Consider the isomorphism (1). A special case is if one of the labels $$\lambda, \mu, \xi, \kappa$$ equals 0 \(\in \Lambda\). In this case all of the terms in the direct sum of the domain and codomain of (1) are zero, except for one of them.

Lemma 7.1. We have the formulae

- $$\lambda = 0$$: $$F(\zeta(\mu^\dagger, 0, \mu) \otimes v) = R^2(v) \otimes \zeta(\xi^\dagger, \xi, 0)$$
- $$\mu = 0$$: $$F(\zeta(\lambda^\dagger, \lambda, 0) \otimes v) = \zeta(\kappa^\dagger, 0, \kappa) \otimes R(v)$$
- $$\xi = 0$$: $$F(v \otimes \zeta(\kappa^\dagger, 0, \kappa)) = R(v) \otimes \zeta(\lambda^\dagger, 0, \lambda)$$
- $$\kappa = 0$$: $$F(v \otimes \zeta(\xi^\dagger, 0, \xi)) = \zeta(\mu^\dagger, \mu, 0) \otimes R^2(v)$$

Proof. The case $$\xi = 0$$ is precisely relation 5. on page 55 of [25]. They are all obtained by considering the two ways of decomposing $$\mathcal{Y}$$ along a system of two curves, related by an $$F$$ change and such that the factorization of both gives two copies of $$\mathcal{Y}$$ and one copy of $$\Delta$$, as illustrated in figure 27 on page 53 of [25].

\[\square\]

For each $$\lambda \in \Lambda$$ we define

$$E_\lambda = F_{0,0} \left[ \begin{array}{c} \lambda \\ \lambda^\dagger \\ \lambda \end{array} \right].$$

Lemma 7.2. We have the following formula

$$S_{0,0}E_\lambda = S_{0,\lambda^\dagger}.$$  

for all $$\lambda \in \Lambda$$. 

Figure 6. A relation between $S$ and $F$.

Proof. We consider relation 3. on page 55 of [25], which is obtained from considering six decompositions of a genus one surface with two marked points as depicted in figure 6. See also figure 18 on page 43 and figure 26 on page 52 of [25].

Starting at the domain of the morphism $S$ on the right of figure 6 and going counter clockwise around to the codomain of the same morphism, we get that

\[
\begin{align*}
\zeta(0, \lambda^\dagger, \lambda) &\otimes \zeta(0, 0, 0) \\
\zeta(\lambda^\dagger, 0, 0, 0) &\otimes \zeta(\lambda, \lambda^\dagger, 0) \\
\zeta(0, 0, 0, 0) &\otimes \zeta(\lambda, \lambda^\dagger, 0)
\end{align*}
\]

\[
\begin{align*}
\xrightarrow{F} &
\xrightarrow{B_2^{-1} \otimes B_2^{-1}} \\
\xrightarrow{B_3^{-1} \otimes B_3^{-1}} \\
\xrightarrow{\text{Id} \otimes T_1^{-1} T_2} \\
\xrightarrow{\text{Id} \otimes T_1} \\
\xrightarrow{R^{-1} \otimes R} \\
\xrightarrow{R^{-1} \otimes R} \\
\xrightarrow{E}
\end{align*}
\]
Where as the morphism $S$ of course gives
\[ \zeta(0, \lambda^+, \lambda) \otimes \zeta(0, 0, 0) \xrightarrow{\text{Id} \otimes S} S_{0, \lambda^+} \zeta(0, \lambda^+, \lambda) \otimes \zeta(0, \lambda^+, \lambda) \]
From which we conclude the formula
\[ S_{0,0} E_\lambda = S_{0,\lambda^+}. \]

Corollary 7.1. We have that $S_{0,0} \neq 0$ for any modular functor.

Lemma 7.3. For all $\lambda, \mu, \nu \in \Lambda$, we have the relation
\[ E_{\lambda^+} \sum_{i,j,r} F_{0,\mu} \left[ \begin{array}{c} \lambda \\ \lambda^+ \\ \nu \\ \nu^+ \end{array} \right]_{11}^{ij} R_{\mu \nu} F_{\nu,0} \left[ \begin{array}{c} \lambda \\ \mu^+ \\ \mu \\ \mu^+ \end{array} \right]_{rl}^{11} R_{ls}^2 = \delta_{sl}. \]

Proof. The pentagon relation as depicted in figure 7 gives a relation between five applications of the $F$-isomorphism between five decompositions of a genus zero surface with five marked points. This is relation 1. on page 55 of [25] which is also illustrated in figure 24 on page 50 of [25].
Starting in the upper left hand corner and going counter clockwise around we get that

\[
\zeta(0, \lambda, \lambda^+) \otimes \zeta(0, v, v^+) \otimes \zeta_l(v^+, \mu, \lambda^+) \overset{F \otimes \text{Id}}{\longrightarrow}
\]

\[
\sum_{k,l} F_{0,k} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} \zeta_l(\kappa, \lambda^+, v^+) \otimes \zeta_l(\kappa^+, v, \lambda) \otimes \zeta_l(v^+, \mu, \lambda^+) \overset{\text{Id} \otimes R \otimes \text{Id}}{\longrightarrow}
\]

\[
\sum_{k,l} F_{0,k} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} \zeta_l(\kappa, \lambda^+, v^+) \otimes \zeta_l(v, \lambda, \lambda^+) \otimes \zeta_l(v^+, \mu, \lambda^+) \overset{\text{Id} \otimes F}{\longrightarrow}
\]

\[
\sum_{k,l} F_{0,k} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} \zeta_l(\kappa, \lambda^+, v^+) \otimes \zeta_l(\mu, \lambda^+, v^+) \otimes \zeta_l(0, \lambda^+, \lambda) \overset{\text{Id} \otimes R \otimes \text{Id}}{\longrightarrow}
\]

\[
\sum_{k,l} F_{0,k} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} \zeta_l(\mu, \lambda^+, v^+) \otimes \zeta_l(\mu, 0, \lambda) \otimes \zeta_l(0, \lambda^+, \lambda) \overset{p^{(13)}(\text{Id} \otimes R \otimes \text{Id})}{\longrightarrow}
\]

\[
\sum_{k,l} F_{0,k} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} R_{is} \zeta_l(\lambda, \lambda^+, v^+) \otimes \zeta_l(\lambda, 0, \lambda^+) \otimes \zeta_l(0, \lambda^+, \lambda) \overset{\text{Id} \otimes R \otimes \text{Id}}{\longrightarrow}
\]

\[
\sum_{k,l} F_{0,k} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} R_{is} \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(\lambda^+, v^+, \mu) \overset{\text{Id} \otimes F}{\longrightarrow}
\]

\[
E_{\lambda^+} \sum_{i,j,r} F_{0,r} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} R_{i\beta} \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(\lambda^+, v^+, \mu) \overset{\text{Id} \otimes R \otimes \text{Id}}{\longrightarrow}
\]

\[
E_{\lambda^+} \sum_{i,j,r} F_{0,r} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} R_{i\beta} \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(\lambda^+, 0, \lambda^+) \otimes \zeta_l(\lambda^+, v^+, \mu) \overset{\text{Id} \otimes F}{\longrightarrow}
\]

\[
E_{\lambda^+} \sum_{i,j,r} F_{0,r} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} R_{i\beta} \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(\lambda^+, 0, \lambda^+) \otimes \zeta_l(\lambda^+, v^+, \mu) \overset{\text{Id} \otimes R \otimes \text{Id}}{\longrightarrow}
\]

\[
E_{\lambda^+} \sum_{i,j,r} F_{0,r} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} R_{i\beta} \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(0, \lambda^+, \lambda) \otimes \zeta_l(\lambda^+, v^+, \mu) \overset{\text{Id} \otimes R \otimes \text{Id}}{\longrightarrow}
\]

This proves the stated formula.

\[\square\]

From formula (5) we conclude that \(E_{\lambda} \neq 0\) and therefore also that \(S_{0,\lambda} \neq 0\) for all \(\lambda \in \Lambda\). Furthermore we also get the relation

\[\sum_{i,j,r} F_{0,r} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} \overset{\text{Id} \otimes \text{Id} \otimes \text{Id}}{\longrightarrow} R_{\text{Id} \otimes \text{Id} \otimes \text{Id}} = E_{\lambda^+} \delta_{\lambda^+}.
\]

By summing over \(l\) in formula (5) we get

\[\sum_{i,j,r} F_{0,r} \left[ \frac{\lambda}{\lambda^+} \frac{v}{v^+} \right]^{ij} R_{jr} F_{r,0} \left[ \frac{\lambda}{\kappa^+} \frac{\lambda^+}{\mu} \right]^{11} \overset{\text{Id} \otimes \text{Id} \otimes \text{Id}}{\longrightarrow} R_{\text{Id} \otimes \text{Id} \otimes \text{Id}} = E_{\lambda^+} \delta_{\lambda^+}.
\]

As above, \(\Theta\) is an oriented genus one surface and \(\alpha\) and \(\beta\) are simple closed curves as indicated in figure 3. Let \(\Theta^\alpha\) and \(\Theta^\beta\) be marked surfaces, which results
from factoring \((\Sigma, L_\alpha)\) along \(\alpha\), respectively \((\Sigma, L_\beta)\) along \(\beta\). By factorization we get isomorphisms

\[
\Phi'_\alpha : Z(\Theta, L_\alpha) \to \bigoplus_\lambda Z(\Theta'_{\alpha}, \lambda, \lambda^t)
\]

and

\[
\Phi'_\beta : Z(\Theta, L_\beta) \to \bigoplus_\mu Z(\Theta'_{\beta}, \mu, \mu^t).
\]

Pick diffeomorphisms

\[
f'_\alpha : \Xi \to \Theta'_{\alpha}
\]

and

\[
f'_\beta : \Xi \to \Theta'_{\beta}
\]

which maps the real axis onto \(\beta\), respectively \(\alpha\).

Then we get a basis \(\xi_\alpha^\lambda\) for \(Z(\Sigma, L_\alpha)\) and \(\xi_\beta^\mu\) for \(Z(\Sigma, L_\beta)\) by

\[
\xi_\alpha^\lambda = (\Phi'_\alpha)^{-1} Z(f'_\alpha)(\xi_\lambda)
\]

and

\[
\xi_\beta^\mu = (\Phi'_\beta)^{-1} Z(f'_\beta)(\xi_\mu).
\]

We have of course that

\[
\zeta_\beta^\mu = \sum_\mu S_{\lambda,\mu} \xi_\alpha^\lambda.
\]

**Proposition 7.1.** We have the following formula

\[
Z(\beta, \lambda) \xi_\mu^\alpha = \sum_\nu N_{\mu,\nu}^\lambda \xi_\nu^\alpha.
\]

**Proof.** We compute the action of \(Z(\beta, \lambda)\) on \(\xi_\mu^\alpha\) by computing the compositions indicated in figure 8.

\[
\zeta_\mu^\alpha \mapsto \zeta(0, \lambda, \lambda^t) \otimes \zeta(0, \mu, \mu^t) \mapsto_{F}^E
\]

\[
\sum_{v,i,j} F_{0,v} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \begin{array}{c} \lambda \\ \mu \end{array} \right]_{ij} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \begin{array}{c} \lambda \\ \mu \end{array} \right]_{ij} \zeta_j(v, \lambda, \lambda^t) \otimes \zeta_i(v, \mu, \mu^t, \lambda) \xrightarrow{R^2 \otimes R}^{R}
\]

\[
\sum_{v,i,j} F_{0,v} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \begin{array}{c} \lambda \\ \mu \end{array} \right]_{ij} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \begin{array}{c} \lambda \\ \mu \end{array} \right]_{ij} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \begin{array}{c} \lambda \\ \mu \end{array} \right]_{ij}^{R^2 \otimes R} \zeta_j(v, \lambda, \lambda^t) \otimes \zeta_i(v, \mu, \mu^t) \mapsto_{F}^E
\]

\[
\sum_{v,i,j} F_{0,v} \left[ \begin{array}{c} \mu \\ \lambda \end{array} \begin{array}{c} \mu \\ \lambda \end{array} \right]_{ij} \left[ \begin{array}{c} \mu \\ \lambda \end{array} \begin{array}{c} \mu \\ \lambda \end{array} \right]_{ij} \left[ \begin{array}{c} \mu \\ \lambda \end{array} \begin{array}{c} \mu \\ \lambda \end{array} \right]_{ij}^{R^2 \otimes R} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \begin{array}{c} \lambda \\ \mu \end{array} \right]_{ij}^{11} \zeta(0, v, \lambda, \lambda^t, \lambda) =
\]

\[
\sum_{v} E_{\lambda^t}^{-1} N_{\mu,\nu}^\lambda \zeta(0, v, \lambda^t, \lambda) \otimes \zeta(0, \lambda, \lambda^t, \lambda) \mapsto_{F}^E \sum_{v} E_{\lambda^t}^{-1} N_{\mu,\nu}^\lambda \xi_\nu^\alpha
\]

Hence

\[
Z(\beta, \lambda)(\xi_\mu^\alpha) = \sum_{\nu} N_{\mu,\nu}^\lambda \xi_\nu^\alpha
\]

since

\[
\alpha_{\lambda}(Z(B)\zeta(\lambda, \lambda^t, 0)) = E_{\lambda^t}
\]
Proposition 7.2. We have the following formula

$$C_{\lambda, \mu} = S_{\mu, \lambda} / S_{\mu, 0}. $$

Proof. From the formula in Proposition 7.1 we deduce that

$$S_{\mu, 0} C_{\lambda, \mu} = \sum_{\nu} N_{\nu, \mu}^\lambda S_{\mu, \nu}. $$

Letting $q = 0$, we get that

$$S_{\mu, 0} C_{\lambda, \mu} = S_{\mu, \lambda},$$

which proves the stated formula, since we also conclude that $S_{\mu, 0} \neq 0$ from this.

Corollary 7.2. We have the following formula for the coefficients $c_{\kappa}$:

$$c_{\kappa} = d_{\kappa, 0} S_{\kappa, 0}. $$

Proof. One easily checks this formula by substitution.
8. THE FORMULA FOR $S(\lambda)$.

We begin by establishing a formula for the curve operator in terms of the $F$, $R$, $B$ and the $d_\mu$'s.

Lemma 8.1. For all $\kappa, \lambda \in \Lambda$ and all $j = 1, \ldots N_{\kappa, v}^v$ we have that

$$Z(\beta, \kappa)(\xi_{ij}(\lambda, \mu, \mu^t)) = \sum_{v, j, k, m, s, w, p} d^{-1}_{v, k} R_{ip} B_{pr} F_{v, t, \mu} \left[ \begin{array}{c} \kappa \\ \mu^t \\ \lambda \end{array} \right] R_{mk} R_{sw} B_{w, j} (\kappa, v, v^t).$$

Proof. The curve operator $Z(\beta, \lambda)$ acts as follows:
\( \zeta_i(\lambda, \mu, \mu^\dagger) \mapsto \)

\[
\zeta(\lambda^\dagger, \lambda, 0) \otimes \zeta_i(\lambda, \mu, \mu^\dagger) \otimes \zeta(0, \kappa, \kappa^\dagger) \quad (\text{Id} \otimes R \otimes \text{Id}) p_{231}(F \otimes \text{Id})
\]

\[
\sum_p R_{ip} \zeta(0, \kappa, \kappa^\dagger) \otimes \zeta(0, \mu, \mu^\dagger) \otimes \zeta_p(\mu, \mu^\dagger, \lambda) \xrightarrow{F \otimes \text{Id}}
\]

\[
\sum_{\tau, p, I, m} R_{ip} F_{0, \tau} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ \kappa^\dagger & \mu \end{array} \right]_{11}^{lm} \zeta(\tau, \kappa^\dagger, \mu) \otimes \zeta_m(\tau^\dagger, \mu^\dagger, \kappa) \otimes \zeta_p(\mu, \mu^\dagger, \lambda) \xrightarrow{\text{Id} \otimes R \otimes B_{23}}
\]

\[
\sum_{\tau, p, I, m, r} R_{ip} (B_{23})_{pr} R_{mk} F_{0, \tau} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ \kappa^\dagger & \mu \end{array} \right]_{11}^{lm} \zeta(\tau, \kappa^\dagger, \mu) \otimes \zeta_k(\mu^\dagger, \kappa, \tau^\dagger) \otimes \zeta_r(\mu, \lambda, \mu^\dagger) \xrightarrow{\text{Id} \otimes F}
\]

\[
\sum_{\tau, \nu, l, m, p, k, r, s, t} R_{ip} (B_{23})_{pr} R_{mk} F_{0, \tau} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ \kappa^\dagger & \mu \end{array} \right]_{11}^{lm} F_{\mu^\dagger, v} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ \tau^\dagger & \lambda \end{array} \right]_{st}^{kr} R_{iu}^2 R_{iv}
\]

\[
\zeta_s(\nu, \tau^\dagger, \lambda) \otimes \zeta_0(\mu^\dagger, \kappa, \nu^\dagger) \xrightarrow{\text{Id} \otimes F}
\]

\[
\sum_{v, l, m, p, k, r, s, t, u, v} R_{ip} (B_{23})_{pr} R_{mk} F_{0, v} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ \kappa^\dagger & \mu \end{array} \right]_{11}^{lm} F_{\mu^\dagger, v} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ v^\dagger & \lambda \end{array} \right]_{st}^{kr} R_{iu}^2 R_{iv}
\]

\[
F_{\mu^\dagger, 0} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ v^\dagger & \nu \end{array} \right]_{sv}^{11} \zeta_s(\nu, v^\dagger, \lambda) \otimes \zeta(0, v^\dagger, \nu) \otimes \zeta(0, \kappa^\dagger, \kappa)
\]

We now apply formula (6) to this expression and we get that

\[
\zeta_i(\lambda, \mu, \mu^\dagger) \mapsto \sum_{v, l, m, p, k, r, s} R_{ip} (B_{23})_{pr} F_{\mu^\dagger, v} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ v^\dagger & \lambda \end{array} \right]_{kr}^{sm} R_{mk}
\]

\[
\zeta_s(\nu, v^\dagger, \lambda) \otimes \zeta(0, v^\dagger, \nu) \otimes \zeta(0, \kappa^\dagger, \kappa) \mapsto \sum_{v, l, m, p, k, r, s} R_{ip} (B_{23})_{pr} F_{\mu^\dagger, v} \left[ \begin{array}{c} \kappa & \mu^\dagger \\ v^\dagger & \lambda \end{array} \right]_{kr}^{sm} R_{mk}
\]

\[
(B_{12})_{sv} R_{aw}^2 d_{\mu^\dagger}^{-1} \zeta(\lambda, v, v^\dagger)
\]

from which the formula follows.

\[
\square
\]

**Proof of Theorem** By the calculations in section and Proposition we get that

\[
S(\lambda)_{\mu, i} = d_v \sum_{\kappa} c_{\kappa} Z(\beta, \kappa)_{\mu, i} d_{\mu}.
\]

Combining this with formula (8) and the formula for \( Z(\beta, \kappa) \) from Proposition we obtain the formula stated in Theorem.
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