Discrimination between $\Lambda$CDM, quintessence, and modified gravity models using wide area surveys
Houri Ziaeepour
Max Planck Institut für Extraterrestrische Physik (MPE), Giessenbachstraße, Postfach 1312, 85741 Garching, Germany.
houriziaeepour@gmail.com

Abstract
In the past decade or so observations of supernovae, Large Scale Structures (LSS), and the Cosmic Microwave Background (CMB) have confirmed the presence of what is called dark energy - the cause of accelerating expansion of the Universe. They have also measured its density as well as the value of other cosmological parameters according to the concordance $\Lambda$CDM model with few percent uncertainties. Next generation of surveys should allow to constrain this model with better precisions, or distinguish between a $\Lambda$CDM and alternative models such as modified gravity and (interacting)-quintessence models. In this work we parametrize both homogeneous and anisotropic components of matter density in the context of interacting dark energy models with the goal of discriminating between $f(R)$ modified gravity and its generalizations, and interacting dark energy models, for which we also propose a phenomenological description of energy-momentum conservation equations inspired by particle physics. It is based on the fact that the simplest interactions between particles/fields are elastic scattering and decay. The parametrization of growth rate proposed here is nonetheless general and can be used to constrain other interactions. As an example of applications, we present an order of magnitude estimation of the accuracy of the measurement of these parameters using Euclid and Planck surveys data, and leave a better estimation to a dedicated work.

1 Introduction
Nowadays it is a well established fact that according to the Einstein theory of gravity $\sim 73\%$ of the mass and energy in the Universe is in a strange form with unusual properties inconsistent with any type of matter known to us. It is generally called dark energy (see e.g. [1][2][3][4] for recent reviews). In the last two decades or so numerous models have been suggested to explain this mysterious and dominant constituent of the Universe. The majority of these theories can be classified in one the following three categories: 1) Models based on a scalar field e.g. quintessence [5][6][7][8] and its variants such as K-essence [9][10][11][12][13] in which the kinetic term in the Lagrangian has a non-standard form, and varying neutrino mass models [14][15] in which the accelerating expansion of the Universe is generated by the variation of neutrinos mass due to their interaction with a light scalar field; 2) Modified gravity models in which dark energy is explained as the deviation of gravitational interaction from Einstein theory of gravity. Examples of such models include scalar-tensor [16][17][18] and $f(R)$ gravity [19][20][21], Chameleon [22][23], and DGP [24]; 3) A cosmological constant - introduced by Einstein himself [25] and interpreted by Lemaître as the energy density of vacuum [26]. It is phenomenologically the simplest of three categories, and is still the best fit to all observational data [27][28][29]. However, naive estimations of vacuum energy are $\sim 42$ to 123 orders of magnitude larger than the observed dark energy [27]. For this reason, alternative explanations have been explored even before the observation of the accelerating expansion of the Universe [5][6]. The main task of cosmologists today is discriminating between these models, in particular distinguishing the first two categories mentioned above from a cosmological constant.

A notable difference between a cosmological constant and some of alternative models is the presence of a weak interaction between matter and dark energy. Pure quintessence models, in which there is no interaction between the scalar field and matter are somehow pathological because all known fundamental particles, including neutrinos which have very small couplings, interact non-gravitationally with some other particles. Even axiomatic weakly coupled particles such as axions [28][29] are expected to interact with gauge bosons such as gluons. Fields in candidate extensions of the Standard Model are
related to each others by symmetries, thus either by gauge interaction or by mass mixing. On the other hand, if a field such as quintessence interacts only with gravity, then naturally it should be considered to belong to the gravity sector. An example of such fields is dilaton which was first introduced in the context of Kaluza-Klein model for the unification of gravity and electromagnetic forces \[30, 31\], and is also associated with conformal gravity models, see e.g. \[32\] and references therein. But gravity is a universal force and interacts with all other particles. Thus, in contradiction with the assumption above, the quintessence field must have an interaction with other particles. In fact dilaton does have non-minimal interaction with other species, see for instance \[33, 34\]. This makes the task of finding a candidate for a non-interacting quintessence field very difficult. A more problematic issue with pure quintessence models is the fact that they do not solve the coincidence problem of dark energy, i.e. why it becomes dominant only at late times and after galaxy formation. Interacting quintessence models in which the quintessence field has a weak interaction with some matter species, in particular with dark matter, can solve or at least soften the huge fine-tuning of dark energy density with respect to matter in the early Universe \[35, 36\].

In modified gravity models the deviation from the Einstein theory of gravity can be, either written explicitly, or presented by introducing new fields, usually scalars, in the matter sector. The first presentation is called Jourdan frame and the second Einstein frame. Because in the latter case the model looks very similar to an interacting quintessence model\[1\], it is necessary to find a proper definition that discriminates between what is called modified gravity and what is called interacting quintessence. In modified gravity, the scalar field is usually related to a dilaton field, thus it has a geometrical origin and arises from a broken conformal symmetry \[16\]. For this reason, the scalar field always interacts with the trace of energy-momentum tensor of matter \[67\]. The situation is not so straightforward for interaction between matter and scalar field in interacting quintessence models for which various types of interactions are considered in the literature, see for instance \[70\]. In many of these models in analogy with modified gravity, in particular \(f(R)\) models, the interaction term is considered to be proportional to the trace of the energy-momentum tensor of matter\[2\].

In this work we try to determine the interaction between dark matter and dark energy in a collisional description of interactions inspired by particle physics. Using the Boltzmann equation with a collisional term and some results from studies of the microphysics of dark energy condensate \[71\], we show that the interaction can be described only approximately by spacetime dependent functions, and in general one needs the distribution in the phase space \(f(x,p)\) where \(p\) is the 4-momentum, see Sec. \[8\] for more details. However, at present and for foreseeable future, we cannot observe the phase space distribution of dark energy. Considering this fact, we use thermodynamical description of average energy-momentum and velocity to obtain approximate covariant expressions for interactions between matter and a scalar field as dark energy. This leads to a modification of energy-momentum conservation equation which explicitly deviates from modified gravity. Their difference can be used as a mean for classifying models and discriminating these two categories in the data.

On the observational side, one has to find the best way of parametrizing cosmological evolution equations such that they admit discrimination between at least the three major categories of models discussed above. In preference they should not depend on the details which are neither well understood nor can be targeted with the precision of present and near future surveys. Observations show that dark energy has negligible clustering (see e.g. \[72, 73, 74, 75\] for latest results). Therefore, its dominant contribution is in the homogeneous component of the Einstein and conservation equations. It also affects the evolution of anisotropies mainly through their dependence on the background cosmology. For this reason, irrespective of the way we measure the equation of state of dark energy - from observations of supernovae that are only sensitive to background cosmology, or from observations

\[1\]Note that when we talk about interacting quintessence models we mean models in which the scalar field interact with some other constituents of the Universe. All quintessence models have a self-interaction which is not explicitly considered in the formulation presented in this work.

\[2\]For the reasons described in detail in Sec. \[8\] when we talk about the interaction term, we mean the modification of energy-momentum conservation equation due to an interaction.
of matter perturbations by measuring lensing or galaxy distribution - we must extract parameters of background cosmology to determine the contribution of dark energy. Consequently, it is crucial to understand how different models affect this component through a proper parametrization that facilitates the discrimination between various models. This is another goal of the present work. Although there are few popular parametrizations \cite{76, 77, 78, 79} in the literature, specially for testing modification of the Einstein theory of gravity at large scales, as we will show in this work, they are not suitable for discriminating between modified gravity and (interacting)-quintessence models. We should remind that for ΛCDM model the growth rate $f$ is roughly scale independent. Therefore, observation of the violation of this property would be a clear signature of inconsistency with standard cosmology. But the measurement of $f$ and the expansion rate $H$ by themselves are not enough for discriminating between modified gravity, quintessence, and interacting quintessence, and a parametrization that does not depend on the details of these models is necessary to highlight their differences. Evidently, one can simply fit the data with models and compare their goodness of fit. But, this does not take into account the degeneracies and similarities. Therefore, a smart parametrization and better data analysing methods are necessary. Moreover, the fact that most popular modified gravity models can be formulated as an scalar field theory means that their differences from (interacting)-quintessence must be understood and the parametrization must be performed in a way that it highlights these differences and help discrimination.

In this work we propose a new set of parameters to describe, in a model independent way, the effect of an interacting dark energy on the evolution of the expansion rate of the Universe and another set of parameters for the growth rate. These quantities are the most sensitive measurables for discriminating between dark energy models. Consequently, the ultimate goal of various measurement methods is to constrain cosmological and dark energy models by measuring one or both these quantities. For instance, galaxy distribution and lensing surveys determine the power spectrum of fluctuations for one or multiple redshift bins. Future large surface and sensitive spectroscopic surveys such as Euclid allow to determine the matter power spectrum for a statistically significant number of redshift bands, and thereby extract the growth rate, see e.g. \cite{80} for the methodology applied to The WiggleZ Dark Energy Survey. The BAO measurements determine the expansion rate and angular diameter distance at one or multiple redshift bins, and can thereby estimate variation of these quantities. Supernovae data measure the expansion rate directly, and the variation of $H(z)$ can be extracted. Therefore, parametrizations that we will discuss here are relevant for all measurement methods in cosmology.

In Sec. \ref{sec:parametric} we present a new parametrization for Friedman equation in the context of a general interacting dark energy model. In Ref. \cite{37} we defined a quantity $B(z) \propto \rho_{de}/P_{de}$ and proposed it for the measurement of the equation of state of dark energy defined as $w \equiv P_{de}/\rho_{de}$ where $P_{de}$ and $\rho_{de}$ are pressure and energy density of dark energy respectively. It is specially suitable for measuring the deviation from a cosmological constant, see Appendix A for the definition of $B(z)$ and a review of its properties. In addition, we argued that in what concerns the sign of $\gamma(z)$ (see equation \ref{eq:gamma} below for its definition), this quantity has distinct geometrical properties which make it less sensitive to uncertainties of other quantities such as $H_0$ or $\Omega_m$, respectively the present value of Hubble constant and the density fraction of matter. The sign of $\gamma(z)$ is the discriminator between what is called phantom models which have $w < -1$, and normal scalar fields (quintessence) models and a cosmological constant for which $w \geq -1$.

Using this parametrization and properties of $B(z)$, we show that in presence of an interaction between dark energy and other components, one obtains different estimation for $\gamma_{eff}^{de}$ (see next section for its definition) from $H(z)$ and from $B(z)$ when data is analyzed with the null hypothesis of a ΛCDM model as dark energy. In this way one can predict the sensitivity of surveys to interacting dark energy models in a model-independent manner. Then, we discuss the properties of parameters for each category of models, their differences, and how this information can be used to discriminate between various dark energy models. In Sec. \ref{sec:phenom} we describe phenomenological interactions for interacting...
quintessence models and compare it with modified gravity. This leads to an approximate description for non-gravitational interactions between dark matter and dark energy.

In Sec. 4 we present evolution equations of over-density and velocity fields in each category of models for the interactions obtained in Sec. 3. Then we describe how one can discriminate between interacting quintessence and modified gravity models by using matter power spectrum and its evolution, i.e. the growth rate of anisotropies. Because the growth rate plays a special role in the discrimination between various dark energy models, in Sec. 5 we parametrize its evolution, and as an example of application, we obtain an order of magnitude estimate for the discrimination ability of the Euclid mission \[43\] that measures both parameters of the homogeneous component (the background cosmology) and the evolution of growth rate of matter anisotropies. In addition, we compare our parametrization with other parametrizations that can be found in the literature which are usually designed to test the Einstein theory of gravity. Conclusions and outlines are summarized in Sec. 6. Properties of the functions \( B(z) \) (and \( A(z) \)) are reviewed in Appendix A. Fisher matrix for dark energy without parametrization of its equation of state \( w(z) \) is described in Appendix B. A summary of covariant formulation of a classical scalar fields as a perfect fluid is given in Appendix C. In Appendix D we calculate an approximate analytical solution for the growth rate of matter anisotropies.

Here we must emphasize that the predictions for future missions obtained in this work are only representative and order of magnitude estimations of what is expect from future surveys. They should be considered as a QD (Quick and Dirty), hand-shaking predictions. Their purpose is only to show that it is possible to measure the new parameters with reasonable uncertainties. A proper prediction for future observation projects needs detailed consideration of instrumental response, simulation of data analysing procedure, and understanding of the sources of systematic and statistical errors. Fulfilling these requests necessitates a dedicated investigation which is out of the scope of the present work that targets theoretical issues related to the discrimination between various dark energy models. In fact, a number of authors have performed predictions for uncertainties of various measured quantities by future missions, see for instance \[68, 81, 82, 83\]. They usually consider models that can be classified as modified gravity according to the classification criteria discussed in Sec. 3. Nonetheless, some of their parameters can be related to quantities defined in this work, thus their predictions can be used to obtain a rough estimation for the expected uncertainties for the new parameters.

Throughout this work we use Einstein frame for modified gravity models unless it is explicitly specified. In this way, a unified description can be made for all interacting dark energy models based on a scalar field formulation.

## 2 Friedman equation in interacting dark matter-dark energy models

Apriori the measurement of the equation of state of dark energy is simple. It is enough to measure the expansion rate of the Universe \( H(z) \equiv \dot{a}(z)/a \), or a quantity related to it such as the luminosity distance \( D(z) \) at different redshifts. Then, by modeling known constituents of the Universe as non-interacting perfect fluids, one can fit the data and measure the effective equation of state \( w(z) \), defined as \( P_{eff}(z)/\rho_{eff}(z) \). The suffix “eff” is used to remind that pressures and densities obtained in this way can be effective quantities rather than physical pressure and density of constituents, because we have neglected any interaction between components. Therefore, from now on effective quantities mean quantities determined from data by considering a null hypothesis.

In practice however the life is not so simple. The density of a perfect fluid changes with redshift as \((1 + z)^{3\gamma} \) (\( \gamma \) is defined in \[22\]). Therefore, at low redshifts when \( z \rightarrow 0 \), the total density is not very sensitive to the value of \( \gamma \) or equivalently \( w(z) \) and their variation with \( z \), see Appendix A for more details. This statement is independent of the type of data or proxy used for determining \( H(z) \) or \( D(z) \). On the other hand, at high redshifts where \( H(z) \) is more sensitive to the equation of state, dark energy is subdominant. Moreover, it is more difficult to measure \( H(z) \) and \( D(z) \) at higher
redshifts and measurement uncertainties can make the estimation of \( w(z) \) and its evolution unusable for discrimination between models.

If constituents of the Universe do not interact with each other, Friedman equation which determines the evolution of expansion function \( a(t) \) can be written as:

\[
\frac{H^2}{H_0^2} = \frac{\rho(z)}{\rho_0} = \Omega_m(1+z)^3 + \Omega_b(1+z)^{3w} + \Omega_K(1+z)^2 + \Omega_{de}(1+z)^{3\gamma(z)}, \quad \rho_c(z) \equiv \frac{3H^2}{8\pi G} \tag{1}
\]

\[
\gamma(z) = \frac{1}{\ln(1+z)} \int_0^z dz' \frac{1+w(z')}{1+z'}, \quad P_{de}(z) \equiv w(z)\rho_{de}
\tag{2}
\]

In this class of models matter and radiation densities evolve only due to the expansion. This is a good approximation for all redshifts \( z < z_{\text{cmb}} \sim 1100 \).

In interacting dark energy models matter and radiation terms in the right hand side of the Friedman equation (1) can contain an additional redshift-dependent factor:

\[
\frac{H^2}{H_0^2} = \frac{\rho_c(z)}{\rho_{c0}} = \sum_i \Omega_i F_i(z)(1+z)^{3\gamma_i} \quad i = m, b, h, k, \text{ and } de
\tag{3}
\]

Without lack of generality we assume that \( F_{de} = 1 \) and all redshift dependent terms are included in \( \gamma(z) \). In quintessence models the coefficient of the curvature term also is constant because it is assumed to be related to geometry/gravity and independent of the behaviour of other components. At present observations are consistent with only gravitational interaction between various components in (3), thus additional interactions must be very weak. By definition and without lack of generality we consider \( F_i(z = 0) = 1 \). Observations also show that \( \Omega_k \approx 0 \), therefore throughout this work we assume \( \Omega_k = 0 \) unless it is explicitly mentioned. Note that in the case of modified gravity models, a parametrization similar to (3) can be defined both in Einstein and Jordan frames.

A simple example for which an approximate expression for \( F_i(z) \) coefficients can be found is a model with a cosmological constant as dark energy and a slowly decaying dark matter. The decay remnants are assumed to be visible relativistic particles [39]. In this case:

\[
\frac{H^2}{H_0^2} \approx \Omega_m(1+z)^3 \exp\left(\frac{t_0 - t}{\tau}\right) + \Omega_b(1+z)^3 + \Omega_h(1+z)^4 + \Omega_m(1+z)^4 \left(1 - \exp\left(\frac{t_0 - t}{\tau}\right)\right) + \Omega_{\Lambda}
\tag{4}
\]

\[
F_m(t) \approx \exp\left(\frac{t_0 - t}{\tau}\right) + (1+z)\left(1 - \exp\left(\frac{t_0 - t}{\tau}\right)\right), \quad \tau \gg \tau_0, \quad F_b = F_h = 1 \quad \gamma(z) = 0
\tag{5}
\]

where \( \tau \) is the lifetime of dark matter and \( \tau_0 \) is the age of the Universe. It is demonstrated that in this example, if the decay/interaction of dark matter is not considered in the data analysis, a \( w_{\text{eff}} < -1 \) can be obtained for dark energy, see [40, 42] for more details about the set up and the proof.

Note that in [40], we have included the contribution of relativistic remnants in \( F_m \). However, as this component has a redshift dependence similar to hot matter, it also makes sense to consider it as hot matter and add it to hot component. It is even possible to add this term to dark energy contribution, as long as it is small and induces only a slight deviation from a cosmological constant. In this case, one can show that the effective dark energy will have \( w_{\text{eff}} < -1 \) [40]. The reason for such freedom is that we do not measure or take into account the decay remnants. This example clearly shows that parametrization [3] is not unique when all the components and their interactions are not know. Therefore, one has to be very careful about degeneracies when data are analyzed and interpreted. In particular, prior assumptions such as stability of matter and radiation components can affect measurements and conclusions. This example also show that for ruling out \( \Lambda \)CDM model, it is enough to prove that at least one of \( F_i \neq 1 \), or \( \gamma_{de} \neq 1 \).

\(^4\)This statement is true if baryon pressure is negligible. Future surveys can be sensitive to small baryon pressure. In this case it must be taken into account before any conclusion about \( \Lambda \)CDM model is made.
Extension of this example to quintessence models without coupling to matter is straightforward and one simply needs to consider $\gamma(z) \neq 0$. A more interesting extension is to assume that the quintessence scalar field is one of the remnants of the decay of dark matter, which during cosmological time condensates and makes a classical quintessence field. In this case, it has been shown [36, 85, 71] that coefficients $F_m$, $F_h$, and equation of state of dark energy $w(z)$ (or equivalently $\gamma(z)$) are not independent. However, their relations are too sophisticated and cannot be described in an analytical form and numerical techniques should be employed [36].

According to (5):

$$F_m(z) > F_m(z = 0)$$

and because $\tau \gg \tau_0$, $F_i(z)$ coefficients are close to 1 at all redshifts. In general, for an interaction which transfers energy from dark matter to other components, the inequality (6) is applied because at high redshifts one expects a larger contribution of dark matter in the total density than in a non-interacting model. Inversely, if energy is transferred from other components, for instance from dark energy, to dark matter:

$$F_m(z) < F_m(z = 0)$$

An example of such models is scaling dark energy [55, 56] in which at early times dark energy has a much larger contribution in the total energy density, but it gradually decays to dark matter and only recently its equation of state approaches $w \sim -1$. Another example is the class of models called early dark energy. Although the original model [57, 58, 59] is a pure quintessence/k-essence, there are variants of this model in which, there is an interaction in the dark sector [60] or between dark energy and visible sector [61].

In models with elastic interaction between two sectors, no energy is transferred between them, and $F_m(z) = F_h(z) = 1$. Nonetheless, the phase space of matter and dark energy in these models can change and thereby $w_{de}$ can depend on $z$.

For $f(R)$ modified gravity models homogeneous Einstein equations and energy conservation equation in Jordan frame are [67, 68],

$$\left(1 + f_R\right)H^2 + \frac{1}{6} f - \frac{a''}{a^3} f_R + \frac{f''}{a} f_R H = \frac{8 \pi G}{3} \sum_i \rho_i$$

$$\frac{a''}{a^3} = - \frac{4 \pi G}{3} \sum_i (\rho_i + 3P_i) + \left(1 + f_R\right)H^2 + \frac{\dot{f}}{6} - \frac{H f_R'}{a} - \frac{f_R''}{2a^2}$$

$$\dot{\rho}_i + 3H \rho_i = - \frac{\dot{f}_R}{2(1 + f_R)} (\rho_i - 3P_i), \quad i = m, h, k$$

where $a' = \dot{a}$. Dot and prime mean derivation with respect to comoving and conformal time, respectively. Subscript $R$ means derivation with respect to scalar curvature $R \equiv R_{\mu\nu} g^{\mu\nu}$. We remind that at linear order the effect of matter perturbations on $R$ is zero, thus $R$ only depends on $z$ and the effect of $f(R) \neq 0$ on the evolution of perturbations manifests itself by changing the background cosmology.

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1. When equations apply to both dark matter and baryons, we indicate them collectively with subscript $m$.

2. Here we have written Einstein and conservation equations in Jordan frame because they lead to expressions for $F_i$ coefficients which are explicitly very different from quintessence case.
After solving density conservation equation (10), Friedmann equation (8) can be written as the following:

\[
\rho_i(z) = \rho_i(z=0)(1+z)^{3\gamma_i}\left(\frac{1+f_R(z=0)}{1+f_R(z)}\right)^{-\frac{1-3w_i}{2}}
\]  

(11)

\[
\frac{H^2}{H_0^2} = \frac{\rho_c(z)}{\rho_{c0}} = \sum_i \Omega_i F_i(z)(1+z)^{3\gamma_i}
\]  

(12)

\[
F_i(z) = \left(\frac{1+f_R(R(z=0))}{1+f_R(R(z))}\right)^{-\frac{1-3w_i}{2}}, \quad w_m = 0, \quad w_h = \frac{1}{3}, \quad w_k = -\frac{1}{3}
\]  

(14)

Equation (13) is the energy density of effective dark energy in f(R) gravity models. Similar to quintessence models we can assume F_{de} = 1. The only explicit difference between (12) and the same equation for an interacting quintessence model is the presence of a nontrivial coefficient for the curvature term if \(\Omega_k \neq 0\). Nonetheless, the evolution of coefficients \(F_i(z)\) with redshift is different from their counterparts in interacting quintessence models, in particular from models in which energy is transferred to dark energy at low redshifts, see equation (6). In fact, the function f(R) is not completely arbitrary and must satisfy a number of constraints. Notably, \(f(R)|_{R=\infty} \rightarrow 0\) to make the model consistent with Einstein theory of gravity in mild or strong gravity fields, and \(f_R > 0\) due to stability constraint \([44]\). Under these conditions:

\[
F_i(z) > F_i(z=0)
\]  

(15)

Comparing (15), (6), and (7) one can immediately conclude that the measurement of \(F_m(z)\) and its evolution with redshift can discriminate between dark energy models in which energy is transferred from dark energy to dark matter such as scaling models, and f_R modified gravity models. But it cannot discriminate modified gravity from models in which energy is transferred from dark matter to dark energy such as the model discussed in \([36, 85, 71]\). To discriminate the latter and other models of this category from f_R modified gravity, the coefficient of relativistic (hot) component \(F_h(z)\) and its evolution must be measured. Evidently, such measurements are very difficult. For instance, one has to measure very precisely the temperature of CMB at high redshifts or \(H(z)\) at a large number of redshift bins and fit the data with \(F_h \neq 1\). In Einstein frame the evolution of matter density is the same as in equation (11) \([67]\), but evolution equation of hot matter is similar to ΛCDM. In what concerns the discrimination from interacting quintessence what is discussed from Jourdan is applicable.

2.1 Model-independent discrimination of interacting dark energy models

In this section we show that if ΛCDM or a simple quintessence are considered as null hypothesis, measurements of effective dark energy density and effective equation of state from \(H(z)\) and the function \(A(z)\) defined in Appendix A separately, give different values for these quantities if dark energy interacts with matter. Similarity of \(F_m(z)\), specially if the curvature of the Universe is zero, means that we cannot distinguish between interacting quintessence and modified gravity models in a model-independent manner - except for the cases explained above. For this reason in this section we only study the discrimination between interacting dark energy models parametrized as in equation (12) and a cosmological constant and/or non-interacting quintessence.

For analyzing cosmological data, ΛCDM with a stable and non-interacting dark matter is usually used as null hypothesis. Nonetheless, the methodology explained below is not sensitive to redshift dependence of \(\gamma_{de}\), and we can consider the more general case of non-interacting quintessence as the null hypothesis. The expansion of the Universe for such cosmologies is ruled by equation (11). Therefore, we rearrange terms in equation (12) such that it looks similar to equation (11). Then, we
determine effective quantities which are measured by fitting a ΛCDM or a non-interacting quintessence
model to data:

\[
\frac{H^2}{H_0^2} = \sum_i \Omega_i (1 + z)^{3\gamma_i} + \sum_i \Omega_i (\mathcal{F}_i(z) - 1)(1 + z)^{3\gamma_i} + \Omega_{de}(1 + z)^{3\gamma_{de}(z)}
\]  \hspace{1cm} (16)

In null hypothesis model only \(\gamma_{de}\) is redshift dependent and \(\gamma_i, \ i = m, \ h, \ k\) are constant. By comparing (16) with (11) the effective contribution of dark energy is expressed as:

\[
\Omega_{eff}(1 + z)^{3\gamma_{eff}(z)} = \sum_i \Omega_i (\mathcal{F}_i(z) - 1)(1 + z)^{3\gamma_i} + \Omega_{de}(1 + z)^{3\gamma_{de}(z)}
\]  \hspace{1cm} (17)

In both interacting quintessence and modified gravity models coefficients \(\mathcal{F}_i\)'s are defined such that \(\mathcal{F}_i(z = 0) = 1\), therefore at \(z = 0\) the first term in (17) is null and we can separate \(\Omega_{eff}\) and \(\gamma_{de}(z)\):

\[
\Omega_{eff}(H) = \Omega_{de}, \quad \gamma_{eff}(H) = \gamma_{de}(z = 0)
\]  \hspace{1cm} (18)

\[
\gamma_{eff}(z) = \log\left(\frac{\sum_i \Omega_i \mathcal{F}_i(z) - 1)(1 + z)^{3\gamma_i} + \Omega_{de}(1 + z)^{3\gamma_{de}(z)}}{3 \log(1+z)}\right)
\]  \hspace{1cm} (19)

where superscript \((H)\) means measured from Hubble constant \(H\).

Suppose we can also measure \(A(z)\) defined in (109), and use it to determine the effective density and equation of state of dark energy. For an interacting dark energy model parametrized according to (16) quantities \(B(z)\) and \(A(z)\) are:

\[
B(z) \equiv \frac{1}{3(1+z)^2 \rho_0} \frac{d\rho}{dz} = \sum_{i=m,h,k} \Omega_i \left(\gamma_i \mathcal{F}_i(z) + (1 + z) \frac{d\mathcal{F}_i}{dz}\right)(1 + z)^{3(\gamma_i-1)} + \Omega_{de}(w(z) + 1)(1 + z)^{3(\gamma_{de}(z)-1)}
\]  \hspace{1cm} (20)

\[
A(z) \equiv B(z) - \sum_{i=m,h,k} \Omega_i \gamma_i(1 + z)^{3(\gamma_i-1)} = \sum_{i=m,h,k} \Omega_i \left(\gamma_i (\mathcal{F}_i(z) - 1) + (1 + z) \frac{d\mathcal{F}_i}{dz}\right)(1 + z)^{3(\gamma_i-1)} + \Omega_{de}(w(z) + 1)(1 + z)^{3(\gamma_{de}(z)-1)}
\]  \hspace{1cm} (21)

Using (109) in Appendix A as the definition of \(A(z)\), we find the following expression for its parameters:

\[
\Omega_{eff}(A)(w_{eff}(z) + 1)(1 + z)^{3\gamma_{eff}(z)} = \sum_i \Omega_i \left(\gamma_i (\mathcal{F}_i(z) - 1) + (1 + z) \frac{d\mathcal{F}_i}{dz}\right)(1 + z)^{3\gamma_i} + \Omega_{de}(w(z) + 1)(1 + z)^{3\gamma_{de}(z)} = (1 + z)A(z)
\]  \hspace{1cm} (22)

where superscript \((A)\) means measured from \(A(z)\). Equations (17) and (22) are fundamentally different. In particular:

\[
\Omega_{eff}(A) = \sum_i \Omega_i \frac{\frac{d\mathcal{F}_i(z=0)}{dz} + \Omega_{de}(w(z = 0) + 1)}{w_{eff}(A)(z = 0) + 1}
\]  \hspace{1cm} (23)

which in contrast to \(\Omega_{eff}(H)\), in general is not equal to \(\Omega_{de}\). Equality arises only when \(\mathcal{F}_i\) do not vary with redshift i.e. \(\mathcal{F}_i = 1\) at all redshifts. This condition is satisfied by the null hypothesis ΛCDM and by non-interacting quintessence models. Therefore, assuming that \(\Omega_m\) and \(\Omega_k\) are known (e.g. from CMB), simultaneous measurements of \(H(z)\) and \(A(z)\) at even one \(z > 0\) is apriori enough for testing the presence of an interaction between dark matter and dark energy independent of the underlying model. Evidently, in practice the measurements must be performed at many redshift bins to improve statistics and to compensate for measurement errors.
Apriori one can use other quantities such as angular diameter distance $D_A$ or luminosity distance $D_L$ which are easier to measure rather than $A(z)$. However, both these quantities are functional of $H(z)$ through integration of $1/H^{1/2}(z)$. Thus, in general they do not have an analytical expression. Besides, their derivatives depend on $F_i$’s only, in contrast to (22) that depends on both $F_i$’s and their derivatives. Therefore, $\Omega_{\text{eff}}$ and $\gamma_{\text{eff}}$ obtained from $dD_A/dz$ or $dD_L/dz$ will be equal to ones determined from $H(z)$ irrespective of the underlying cosmology. This shows that the function $A(z)$ (or equivalently $B(z)$) introduced in (37) has special properties and is well suited for discriminating between dark energy models. It can be measured from supernovae data, see (37) for the methodology. As for LSS data, one needs to determine both $H(z)$ and its evolution $dH(z)/dz$ to be able to calculate $A(z)$, for instance from the BAO and the power spectrum of matter fluctuations (51). This is not an easy task. As an example consider supernovae observations that measure the luminosity distance $D_L$ to a supernova from its standardized apparent magnitude. The angular luminosity distance $D_A$ is related to the luminosity distance, see (114). To determine $dD_A/dz$ apriori one can use the measured $D_A$, and determine its derivative (slope). However, due to scattering and discreteness of data, such a measurement will have large uncertainties. The same problem arises for $dH(z)/dz$ or $A(z)$ because they depend on derivatives of $D_L$, see equations (110) and (113). Nonetheless, there are various methods such as binning of data, using a fit in place of discrete data, etc. that allow to improve the estimation. Near future large area surveys such as Euclid (43), BigBOSS (22), LSST (23) will be able to determine these quantities with relatively good precision, see also Sec. 5 for measurement methodology. In particular, large surface spectroscopic and lensing surveys such as Euclid are able to determine the variation of total density with redshift $d\rho/dz \propto B(z)$ with good precision. In Appendix B we obtain the Fisher matrix for dark energy parameters without considering a specific parametrization for the equation of state $w(z)$.

2.2 Discrimination precision

Measurements of cosmological parameters show that $w_{de}^{obs} \sim -1$ irrespective of which proxy or measurement method - supernovae, CMB, or LSS has been used. This means that $|F_i(z) - 1| \approx 0$ and $dF_i(z)/dz \approx 0$. Moreover, addition of $F_i(z)$ to the model increases the number of parameters. Giving the fact that we have essentially two observables: $H(z)$ and one of $D_A(z)$, $D_L(z)$ or $B(z)$, greater number of parameters means also greater degeneracy, thus more uncertainty for discrimination between $\Lambda$CDM, a non-interacting quintessence, and interacting dark energy models.

One way of measuring the presence of interaction without having to fit data to the large number of parameters in equations (10) and (21), is to measure how different $\Omega^{(H)}_{\text{eff}}$, $\Omega^{(A)}_{\text{eff}}$, $\gamma^{(H)}_{\text{eff}}$, and $\gamma^{(A)}_{\text{eff}}(z)$ are, because as we discussed in the previous section, when $F_i \neq 1$ these effective quantities are not the same. To this end, a natural criteria is:

$$
\Theta(z) \equiv \frac{\Omega^{(A)}_{\text{eff}}(w^{(A)}_{\text{eff}}(z) + 1)(1 + z)^{3\gamma_{\text{eff}}^{(A)}(z)} - \Omega^{(H)}_{\text{eff}}(w^{(H)}_{\text{eff}}(z) + 1)(1 + z)^{3\gamma_{\text{eff}}^{(H)}(z)}}{\Omega^{(H)}_{\text{eff}}(w^{(H)}_{\text{eff}}(z) + 1)(1 + z)^{3\gamma_{\text{eff}}^{(H)}(z)}}
$$

(24)

This quantity can be explained explicitly as a function of $\Omega_i$, $F_i$, $\gamma_i$, and is zero when $F_i = 1$, $dF_i/dz = 0$. Note that we have chosen expression (22) for comparison rather than (17) because it is not possible to determine $\Omega^{(A)}_{\text{eff}}$ in a model independent manner, see equation (23). By contrast $\Omega^{(H)}_{\text{eff}} = \Omega_{de}$, thus $\gamma_{\text{eff}}^{(H)}$ and thereby $w^{(H)}_{\text{eff}}$ can be determined without any reference to $F_i$ coefficients. In (37) we suggested to use the sign and evolution of $A(z)$ to discriminate between dark energy with $\gamma(z) \neq 0$ and a cosmological constant. Here $\Theta(z)$ plays a similar role for discriminating between interacting or non-interacting dark energy.

Assuming that $\Omega_m$ and $\Omega_k$ are determined independently and with very good precision, for instance from CMB anisotropies with marginalization over $\gamma_{de}$, $\Theta$ can be determined from the measurement of $H(z)$ and $B(z)$. The latter can be measured from whole sky or wide area spectroscopic surveys data
such as Euclid, or multi-band photometric surveys such as DES. Evidently determination of $B(z)$ that depends on $dH/dz$ is very difficult. However, it is easy to see that there is no other quantity which can be measured more easily and discriminates between ΛCDM and dynamical dark energy models with a better precision. For instance, the BAO method determines $H(z)$ and $D_A(z)$ directly. But, $D_A(z)$ depends on $w(z)$ or equivalently $\gamma(z)$ through an integral, see equation (114). Therefore, it is less sensitive to the variation of $\gamma(z)$ with redshift. This is analogue to binning a data. Evidently, a binned data is less noisy and has a smaller uncertainty. But, if the goal is to measure the variation of data, the binning can completely smear out small variations. Therefore, irrespective of methods and measured proxies, we are limited by inherent properties of the physical system. In this respect, the precision with which $\Theta(z)$ can be measured gives the ultimate sensitivity of an observation/data set to deviation from ΛCDM.

3 Interactions

In the previous section we used Friedman equation for parametrizing interaction between matter and dark energy. Evolution of their densities is ruled by energy-momentum conservation. But, in presence of non-gravitational interactions between constituents the energy-momentum tensor of each component $T_{\mu\nu}^i$ is not separately conserved, and conservation equation can be only written for the total energy-momentum tensor $T_{\mu\nu}^{\text{free}}$ defined as:

$$T_{\mu\nu} = \sum_i T_{\mu\nu}^i + T_{\mu\nu}^{\text{int}}$$

where $T_{\mu\nu}^i$ is the energy-momentum tensor of component $i$ in absence of interaction with other components, i.e. $T_{\mu\nu}^{i\text{free}, \mu\nu} = 0$, and $T_{\mu\nu}^{\text{int}}$ is the energy-momentum tensor of interaction and $T_{\mu\nu}^{\text{int}, \mu\nu} = 0$. In the literature on interacting dark energy models (see e.g. [35]) when only two constituents - matter and dark energy - are considered, the energy-momentum conservation equations are usually written as:

$$T_{\mu\nu}^{\text{free}, \mu\nu} = Q^\mu, \quad T_{\mu\nu}^{\phi, \mu\nu} = -Q^\mu$$

for an interaction current $Q^\mu$. Comparing (26) and with (27), it is clear that tensors in the left hand side of equations in (27) do not correspond to free energy-momentum tensors, and along with $Q^\mu$ they are obtained somehow arbitrarily by division of (26). In fact, equations in (27) are inspired by perturbation theory in which for each perturbative order, the right hand sides of these equations are estimated by using quantities from one perturbative order lower. Thus, they constitute an iterative set of equations from zero order (free) model in which $Q^\mu = 0$, up to higher orders. This approach is not suitable for dark energy where we ignore, not only interactions but also the free model. Therefore, a more general expression should be used:

$$T_{\mu\nu}^{m, \mu\nu} = -Q_m^\mu, \quad T_{\mu\nu}^{\phi, \mu\nu} = -Q_\phi^\mu, \quad T_{\mu\nu}^{\text{int}, \mu\nu} = Q_m^\mu + Q_\phi^\mu$$

In these equations matter and dark energy tensors $T_{\mu\nu}^{m}$ and $T_{\mu\nu}^{\phi}$ have the same expression as in the absence of interaction, but with respect to fields which are not free. These expressions can be justified by considering the Lagrangian of the model. In Einstein frame the Lagrangian for a weakly interacting system can be divided to free and interaction parts:

$$\mathcal{L} = \sum_i \mathcal{L}_i + \mathcal{L}_{\text{int}}$$

7Non-gravitational interactions between cosmological constituents must be weak. Therefore, separation of energy-momentum tensor to free and interaction component is allowed.
Considering only local interactions, in the dynamics equations for the fields partial derivative of $\mathcal{L}_{\text{int}}$ with respect to each field determines the interaction term. Dynamic equations can be related to energy-momentum conservation equations \cite{27, 30}. Therefore, interaction currents $Q^\mu_m$ and $Q^\mu_\varphi$ are generated by partial derivatives of $\mathcal{L}_{\text{int}}$ with respect to the corresponding field.

In the previous section we explained that the scalar field in scalar-tensor modified gravity models is related to a dilaton. Consequently, the interaction term is proportional to the trace of matter, see equation \cite{10} for an explicit example of $f(R)$ models. In this case there is no interaction between scalar field and relativistic particles, and it can be shown that $Q^\mu_m = -Q^\mu_\varphi \cite{27}$, i.e. $T^\mu_{\text{int} \nu} = 0$ and conservation equations in \cite{27} can be used. Interaction current $Q^\mu$ for these models can be written as:

$$Q^\mu = \mathcal{C}(\varphi)T_m \partial^\mu \varphi$$  \hspace{1cm} (30)

where $T_m = g_{\mu\nu} T_m^{\mu\nu}$. In $f(R)$ models the coupling $\mathcal{C}$ is a constant. Here we consider $\varphi$-dependence to cover more general cases. Some authors have also considered $Q^\mu \propto T_m u_m^\mu$ for interacting quintessence models \cite{70}. In fact, the interaction current of interacting dark energy models in literature is usually considered to be $Q^0 \propto \rho_m = T_m$ for cold dark matter i.e. similar to what is obtained for $f(R)$ modified gravity models \cite{77}. However, giving the fact that these models share some important properties with modified gravity models, such as the absence of interaction between relativistic matter and scalar field, we classify them in the modified gravity category. In fact, interactions in interacting quintessence models can be more diverse than this simple case. In the rest of this section we describe how they can be formulated without considering their details.

In the context of quantum field theory, the Lagrangian $\mathcal{L}$ can be easily written for various types of fields and their interactions, see e.g. \cite{71}. But these formulations are usually complicated, and are necessary if the microphysics of dark energy models is studied. There are various ways to write $\mathcal{L}$ and/or $T^{\mu\nu}$ with respect to macroscopic quantities which are apriori measurable from cosmological observations. For instance, one can use a fluid description for components. The Lagrangian of a fluid is defined as \cite{54}:

$$\mathcal{L}_f = \frac{1}{2}(\rho + \rho g)u^\mu u^\nu + \frac{1}{4}(\rho - \rho g) g^{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Pi^{\mu\nu}$$ \hspace{1cm} (31)

$$\rho \equiv K + V, \quad P \equiv K - V$$ \hspace{1cm} (32)

where $K$ and $V$ are respectively kinetic and potential energy, and $\Pi^{\mu\nu}$ is the traceless shear tensor. Note that if we impose the traceless condition on the Lagrangian, the last term in the right hand side of equation \cite{31} becomes zero. Therefore, this term must be considered as a Lagrange multiplier, and traceless condition is imposed after determination of $T^{\mu\nu}_f$ \cite{54}. It is easy to check that the Lagrangian $\mathcal{L}_f$ leads to the familiar expression for the energy-momentum tensor of a fluid:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \left[ \partial\sqrt{-g} \mathcal{L} \right]_{\partial g_{\mu\nu}} - \partial_{\mu} \left( \frac{\partial \sqrt{-g} \mathcal{L}}{\partial \partial_{\nu} g_{\mu\nu}} \right)$$, \hspace{1cm} $T^{\mu\nu}_f = (\rho + P) u^\mu u^\nu - g^{\mu\nu} \rho + \Pi^{\mu\nu}$ \hspace{1cm} (33)

Transformation of a Lagrangian written with respect to fields to a fluid description is easy, and one can determine the energy-momentum of interaction $T^{\mu\nu}_{\text{int}}$ and the current $Q^\mu$ defined in \cite{28} directly and without any ambiguity, see Appendix C. However, their descriptions as a function of density and pressure of the fluid depend on the self-interaction potential $V(\varphi)$. For instance, a Higgs-like interaction between a scalar and a fermion $\propto \varphi \psi \bar{\psi}$ is described as $\propto (\rho_{\psi} - P_{\psi})(\rho_{\varphi} - P_{\varphi})^{1/2}$ if $V(\varphi) \propto \varphi^2$, and as $\propto (\rho_{\psi} - P_{\psi})(\rho_{\varphi} - P_{\varphi})^{1/4}$ if $V(\varphi) \propto \varphi^4$. Therefore, when the objective is a general parametrization of interactions without considering details of the underlying model, this type of description is not very suitable.

A more serious problem of fluid description of interaction Lagrangian is the fact that conservation equations in \cite{28} are equivalent to field equations and can be obtained from them \cite{36}. Therefore, they do not contain quantum processes such as decay and scattering. It is well known that the Boltzmann equation plays the role of intermediate between quantum and classical description of interacting systems \cite{86, 87, 88, 89, 90}. In this case, components are defined by their phase space distribution $f(p, x)$.
where $p$ and $x$ are respectively momentum and spacetime coordinates. Interactions are included as collision terms in the right hand side of the Boltzmann equation \[91\] \[92\] \[93\], from which one can obtain energy-momentum and number conservation equations directly:

$$p^{\mu} \partial_{\mu} f_i(p, x) - \Gamma^{\mu}_{\nu\rho} p^{\nu} \frac{\partial f_i}{\partial p^{\rho}} \equiv L[f_i] = C_i(p, x) \quad (34)$$

$$n^\mu_{;i} = \int d\tilde{p} \: C_i(p, x), \quad d\tilde{p} = \frac{g}{(2\pi)^3} d^4 p (E^2 - \tilde{p}^2 - m_i^2) \quad (35)$$

$$T^\mu_{\nu; i} = \int d\tilde{p} \: p^\mu C_i(p, x) \quad (36)$$

where $g$ is the number of internal degrees of freedom (e.g. spin) of species $i$. Conservation equations \[35\] and \[36\] are obtained by using the following property of the Boltzmann operator $L$ defined in \[34\], see e.g. \[92\]:

$$\left[ \int d\tilde{p} \: p^{p_1} p^{p_2} \ldots p^{p_n} f(p, x) \right]_{,i} = \left[ \int d\tilde{p} \: p^{p_1} p^{p_2} \ldots p^{p_n} L[f(p, x)] \right] \quad (37)$$

Collisional terms can be written by using cross-sections of interactions which can be determined separately from the quantum formulation of the model \[91\] \[99\]. In the context of interacting dark energy models, the simplest examples of collisional terms are elastic scattering between dark matter and dark energy, and dark energy and radiation. For these interactions the collisional terms are:

$$C_m(p, x) = -\Gamma_m m_m f_m(p, x) - f_m(p, x) \int d\tilde{p} \: f_\varphi(p_\varphi, x) \: A_k(p, p_\varphi) \: \sigma_m(p, p_\varphi) + \int d\tilde{p} \: d\tilde{p} \: f_m(p_m, x) \: f_\varphi(p_\varphi, x) \: A_k(p_m, p_\varphi) \frac{d\sigma_m(p_m, p_\varphi, p)}{d\tilde{p}} \quad (38)$$

$$C_\varphi(p, x) = \Gamma_m m_m \int d\tilde{p} \: f_m(p_m, x) \frac{d\mathcal{M}(p_m, p)}{d\tilde{p}} - f_\varphi(p, x) \int d\tilde{p} \: f_m(p_m, x) \: A_k(p, p_m) \sigma_\varphi(p_m, p) + \int d\tilde{p} \: d\tilde{p} \: f_m(p_m, x) \: f_\varphi(p_\varphi, x) \: A_k(p_m, p_\varphi) \frac{d\sigma_\varphi(p_m, p_\varphi, p)}{d\tilde{p}} \quad (39)$$

$$A_k(p_1, p_2) = [(p_1 p_2)^2 - m_1^2 m_2^2]^{1/2} \quad (40)$$

where $\Gamma_m$ is the total decay width of dark matter, $\mathcal{M}(p_m, p)$ is the multiplicity of $\varphi$ with momentum $p$ in the decay remnants of dark matter particles with momentum $p_m$, and $\sigma_m(p_m, p_\varphi)$ is the total cross-section of interaction between dark matter and dark energy with momentum $p_m$ and $p_\varphi$, respectively.\[8\]

The disadvantage of this approach is that it needs phase space distribution of components which is not always available, specially for dark energy. Moreover, the absence of an explicit description for the Lagrangian means that the total energy-momentum tensor needed for determining Einstein equations and metric evolution, can be obtained only by solving equation \[30\] for all components. These equations are differentio-integral and usually don’t have analytical solution. Thus, in practice interacting models can be studied only numerically, otherwise one needs to consider some approximation. For instance, dark energy interaction with matter must be very weak. Thus, $|T_{int}^{\mu\nu}| \ll |\sum_i T_i^{\mu\nu}|$. Therefore, we can neglect its contribution in the total energy-momentum tensor and Einstein equations \[10\].

\[8\]In models where energy is transferred from dark energy to dark matter, the interaction must be nonlinear and very sophisticated such that a very light quintessence field be able to produce massive dark matter particles. At present no fundamental description for such models is available.

\[8\]Note that although dark energy is a condensate i.e. its particles have the same energy, presumably zero momentum, a general condensate state can contain very large number of particles in different energy levels, see \[91\] for more details.

\[10\]In some dark energy models such as early dark energy it is assumed that the density of dark energy at high redshifts is much larger, and only at low redshifts it is reduced. Although at redshifts relevant for dark energy surveys cosmology must be very close to $\Lambda$CDM, one must be aware that in many models of this type, the approximation of weak interaction can be applied only at low redshifts. It is also expected that these models leave a detectable signature on the CMB spectrum \[57\] \[58\].
As for the integration of collision term in equations (35) and (36), under some physically motivated assumptions they can be simplified and integrated. For instance, when dark matter is assumed to be a scalar, the expression for the scattering cross-section is very simple, see e.g. [36]. It is simply proportional to the coupling constant and delta functions for energy-momentum conservation. It is expected that the mass of quintessence field be very small, especially much smaller than the mass of dark matter. The momentums of both components are also expected to be small. In this case their distribution at large momentums is strongly suppressed, and the cross-section around the peak of distribution can be considered to be approximately constant. Under these simplifications, it is easy to see that scattering term in the right hand side of (36) is proportional to integrals of the form:

\[
\int \, d\vec{p}_1 \, d\vec{p}_2 \, P^\mu_1 \, f_1(p_1, x) \, f_2(p_2, x) = n^\mu_1 \int \, d\vec{p}_2 \, f_2(p_2, x) \approx \frac{n^\mu_1 \, u^\mu_2}{m_2} \int \, d\vec{p}_2 \, P_2^\mu \, f_2(p_2, x) = \frac{n^\mu_1 \, u^\mu_2 \, n^\mu_2}{m_2}
\]

where \(n^\mu_i\) and \(u^\mu_i\) are number density and velocity of species \(i\), respectively. Approximate expression for \(n^\mu\) in (43) is valid when the distribution in momentum space is concentrated around a peak. Using similar approximations the decay terms in the right hand side of (36) can be also described as a function of velocity and number vectors. Finally, after grouping all the constant or approximately constant factors together, energy-momentum conservation equations for dark matter and dark energy can be written as:

\[
T_{\mu\nu}^{m} \cdot n_m \approx -L_m n^\mu_m + A_{m} n^\mu_m u^\nu_\varphi n^\nu_\varphi \equiv Q^\mu_m
\]

\[
T_{\mu\nu}^{\varphi} \cdot n_\varphi \approx L_\varphi n^\mu_\varphi + A_{\varphi} n^\mu_\varphi u^\nu_m n^\nu_m \equiv Q^\mu_\varphi
\]

where constants \(L_i\) and \(A_{is}\) are decay width and scattering amplitude for species \(i\). In the rest of this work we use these equations as an approximation for energy-momentum conservation equations irrespective of dark matter type (spin) and details of interaction between two dark components. They affect constants \(L_i\) and \(A_{is}\) which are used as parameters. One can also add dark matter self-annihilation term to (43). But, it is easy to show that self-annihilation is proportional to \(|n_m|^2\). Thus, it is significant only in dense regions i.e. at small spatial scales such as the central region of dark matter halos, which are in nonlinear regime and are not studied in the present work. Here we only consider homogeneous and linear perturbations. Therefore, the effect of annihilation is negligible. We remind that equation (43) is not restricted to cold dark matter and can be also used for relativistic matter, for instance neutrinos in early universe, or a hot component at low redshifts.

Although in the rest of this work we consider the interaction terms described in this section, for what concerns the study of differences between modified gravity and interacting quintessence models, the formulation of anisotropies and discrimination methods explained in the next two sections can be applied to other choices of interactions. It is enough to find an interaction current similar to what we have found for decay and scattering above and add them to the right hand side of equations (43) and (44).

4 Cosmology and evolution of anisotropies

In this section we first determine \(F_i\) coefficients defined in Sec. 2 for both modified gravity and quintessence models according to interaction currents and energy-momentum conservation equations obtained in the previous section. Then, we consider the effect of interactions on the evolution of anisotropies, and describe how interaction parameters can be extracted from data.
4.1 Interaction coefficients in the Friedman equation

4.1.1 Modified gravity

Using the energy momentum conservation equation (28) and the interaction current for modified gravity models, the scalar field equation and the evolution equation of the homogeneous matter density can be determined as the followings [67]:

\[ \ddot{\varphi}'' + 2H \ddot{\varphi} + a^2 \varphi' + a^2 C(\varphi) \sum_i (\dot{\rho}_i - 3\dot{P}_i), \quad \mathcal{H} = \frac{a'}{a} \]  

(45)

\[ \ddot{P}_i + 3H(\dot{P}_i + \dot{\rho}_i) = \mathcal{C}(\varphi) \varphi' (\dot{\rho}_i - 3\dot{P}_i) \quad i = m, b, h \]  

(46)

where barred quantities are homogeneous components, \( \varphi \) in subscript means derivative with respect to \( \varphi \). Note that here we have generalized the original calculation in [67] by considering a \( \varphi \)-dependent \( \mathcal{C}(\varphi) \) coefficient in the right hand side of these equations to cover a larger class of modified gravity models, see e.g. [33]. For \( f(R) \) models \( C = \sqrt{4\pi G/3} [67] \). Equations (45) and (46) are coupled and an analytical solution cannot be found without considering an explicitly \( V(\varphi) \). Therefore, to solve the equation for \( \ddot{\rho} \), which is in fact the only directly observable quantity, we simply consider the right hand side of the equation as a time-dependent source. The solution of equation (46) can be written as:

\[ \ddot{\rho}_i(z) = \ddot{\rho}_i(z_0)(1+z)^{3(1+w_i)}e^{(1-3w_i)F(\varphi)}, \quad F(\varphi) \equiv \int \mathcal{C}(\varphi) d\varphi, \quad i = m, b, h \]  

(47)

where \( w_i \equiv \dot{P}_i/\dot{\rho}_i \) for all species except dark energy are assumed to be constant and are given in equation (14). Comparing this solution with (47) all constant coefficients including \( (1 + f_R(z_0))^{-1/3} \) are included in \( \ddot{\rho}_i(z_0) \). Apriori one can test the presence of a \( f(R) \) modified gravity by measuring simultaneously \( F_m(z), F_h(z), \) and equation of state of dark energy from equation (13). In fact in this equation if we neglect the last term that depends on the time derivative, the effective dark energy density becomes:

\[ \rho_{de} \approx \frac{3}{8\pi G} \frac{f_R}{1 + f_R} \left( -\frac{d\ln f(R)}{6} - \frac{R}{6} \right) \]  

(51)

To be consistent with observations \( f(R) \) cannot be a fast varying function of \( R \). Therefore, the dominant term in (51) is the term proportional to \( R \) which makes the relation between \( \rho_{de}, R \), and \( f(R) \) very simple. Other \( F_i \)'s and evolution of corresponding densities have also known expressions, notably \( F_h(z) = 1 \). Therefore, apriori simultaneous fitting of these quantities can test \( f(R) \) modified gravity models. More generally, in modified gravity models dark energy term in the Friedman equation is an effective contribution generated from non-conventional interaction between matter and gravity. Therefore, it is more correlated to matter than in \( \Lambda \)CDM or (interacting)-quintessence models. In the former apriori there is no correlation in the dark sector, and in the latter case the interaction can be very small and is only necessary for reducing fine-tunings and making the model more natural. Similar correlation tests can be performed for other modified gravity models too. Evidently, giving the small deviation of dark energy from a cosmological constant, the measurements and calculation of correlations are not trivial tasks. Furthermore, the discrimination must be cross-checked by using anisotropies for distinguishing between dark energy models, explained in Sec. 4.2.

14
4.1.2 Interacting quintessence

In the same way, we can determine $F_i$ coefficients for (interacting)-quintessence using equation [13]. We replace $n^\mu$ with approximation [12] and include $1/m$ factors in the $L$ and $A_S$ coefficients. After these simplifications, the evolution equation for the density of interacting quintessence models becomes:

$$\dot{\rho}_i + 3H(\dot{\rho}_i + \dot{P}_i) = -L_i a \dot{\rho}_i + A_{si} a \ddot{\rho}_i \dot{\varphi}$$  \hspace{1cm} (52)

where $i$ indicates any cold matter or relativistic species that interact with quintessence field [1]. A clear difference between interaction term in (52) and (46) is that the former does not explicitly depend on the scalar field, and therefore we do not need to know and solve a field equation similar to (45). The solution of this equation and corresponding $F_i$'s are:

$$\rho_i(z) = \rho_i(z_0)(1 + z)^{3(1 + w_i)} \exp\left(L_i(\tau(z) - \tau(z_0)) + A_{si} \int dz \frac{\dot{\varphi}^2(z)}{(1 + z)H(z)}\right)$$  \hspace{1cm} (53)

$$F_i(z) = \exp\left(-L_i(\tau(z) - \tau(z_0)) + A_{si} \int dz \frac{\dot{\varphi}^2(z)}{(1 + z)H(z)}\right) \approx 1 + L_i(\tau(z_0) - \tau(z)) + A_{si} \int_{z_0}^z dz \frac{\dot{\varphi}^2(z)}{(1 + z)H(z)}$$  \hspace{1cm} (54)

where $\tau(z)$ is the age of the Universe at redshift $z$. Note that even in absence of expansion, the density of dark matter at high redshifts is higher if $L_i > 0$.

Along with consistency relation explained above for modified gravity models, explicit dependence of (54) on measurable quantities $\dot{\rho}_i(z)$ and $H(z)$ apriori allows to discriminate between interacting quintessence and modified gravity models. Note that the prior knowledge about the evolution of these quantities are mandatory for distinguishing the underlying model and without such information one cannot single out any of these models.

4.2 Matter perturbations in interacting dark energy cosmologies

Although dark energy influences the evolution of perturbations mainly through quantities related to the homogeneous component - background cosmology - the study of anisotropies can be a powerful mean both for measuring the equation of state and for discriminating between candidate models. Standard candles, such as supernovae type Ia, allow direct measurements of distances, and thereby cosmological parameters. However, they are rare events, can deviate from being standard due to absorption or late detection [62], sub-types, and dependence of their light curve on other properties such as metallicity, mass, and magnetic field of progenitors [63]. Determination of dark energy properties from evolution of perturbations provides additional information and a mean for cross-check of the two methods.

Matter perturbations in presence of an interacting dark energy [64] and in $f(R)$ modified gravity models [19, 20, 67, 68] have been calculated by various authors, thus here we do not repeat them and simply use their results. Our main objective is to find and discuss features that can be used for discrimination between dark energy models.

Considering only scalar perturbations, we define the first-order metric in conformal gauge as the following:

$$ds^2 = a^2(\eta)[(1 + 2\psi(x))d\eta^2 - (1 - 2\phi(x))\delta_{ij}dx^i dx^j]$$  \hspace{1cm} (55)

\[1\] If species $i$ has interaction with another component, for instance is scattered by another species, we can add a second scattering term to (53). The best example is the scattering of photons or neutrinos by baryons. Here for the sake of simplicity we neglect such interactions which are not the main concern of this work. However, in a full formulation of the problem they should be considered, specially if they can mimic an interaction with dark energy.

\[12\] For $f(R)$ modified gravity in which $\mathcal{C}$ is constant $\beta'$ in (55) can be replaced by an expression depending on density and pressure, and there is no need for solving field equation of the scalar field either.
As we mentioned in the Introduction, for modified gravity models we write evolution equations in Einstein frame. Thus, here only their interaction terms distinguish them from quintessence models.

We use fluid description for both matter and dark energy. After linearizing energy-momentum conservation equations and taking their Fourier transform with respect to spatial coordinates, evolution equations for density and velocity perturbations of matter component $i$ and dark energy can be written as:

$$
\delta \rho_i'(i) + 3H\delta \rho_i(i)(1 + C^2_{s(i)}) + (1 + w(i))\bar{\rho}(i)(3\phi' - i k_j v_j^i) = \delta Q_{i0}(56)
$$

$$
((1 + w(i))\bar{\rho}(i)v_i v_j)' + 4H(1 + w(i))\bar{\rho}(i)v_i v_j - i k_j C^2_{s(i)}\delta \rho_i - i k_l \Pi_{ij}^l - i k_j (1 + w(i))\bar{\rho}(i)\psi = \delta Q_{ij}(57)
$$

$$
\delta \rho_{\varphi}' + 3H\delta \rho_{\varphi}(1 + C^2_{s\varphi}) - (1 + w_{\varphi})\bar{\rho}_{\varphi}(3\phi' - i k_j v_j^\varphi) = \delta Q_{\varphi0}(58)
$$

$$
((1 + w_{\varphi})\bar{\rho}_{\varphi}v_{\varphi})' + 4H(1 + w_{\varphi})\bar{\rho}_{\varphi}v_{\varphi} - i k_j C^2_{s\varphi}\delta \rho_{\varphi} - i k_l \Pi_{\varphi}^l - i k_j (1 + w_{\varphi})\bar{\rho}_{\varphi}\psi = \delta Q_{\varphi j}(59)
$$

where $C^2_{s(i)} \equiv \delta P_i/\delta \rho_i(i)$ is the speed of sound for species $i$ \[13\], $v_i(i)$ is its velocity, and $\Pi_{ij}^l$ is its anisotropic shear. The perturbation of interaction current for modified gravity and quintessence models derived from [30], [43] and [44] are as the followings:

**Modified gravity:**

$$
\delta Q_{(i)0} = \rho_i(1 - 3w_i)C_{\varphi}(\bar{\varphi})\bar{\varphi}'\delta \varphi + C(\bar{\varphi})\left((1 - 3w_i)\delta \varphi' + (1 - 3C^2_{s(i)})\bar{\varphi}'\delta i\right)
$$

$$
\delta Q_{(i)j} = i k_j C(\bar{\varphi})\bar{\rho}(i)(1 - 3w_i)\delta \varphi = -\delta Q_{\varphi j}, \quad i = m, b, h
$$

**Interacting quintessence:**

$$
\delta Q_{(i)0} = -aL_{(i)}(\delta \rho_i + \bar{\rho}(i)\psi) + aA_{s(i)}\left[\bar{\rho}_\varphi\delta \rho_i + \bar{\rho}_i(\delta \rho_{\varphi} + \bar{\rho}_\varphi\psi)\right]
$$

$$
\delta Q_{(i)j} = a v_{(i)j}(-L_{(i)}\bar{\rho}_i + A_{s(i)}\bar{\rho}_i\bar{\rho}_\varphi)
$$

$$
\delta Q_{\varphi0} = aL_{\varphi}(\delta \rho_i + \psi \bar{\rho}_i) + aA_{s\varphi}\left[\delta \rho_{\varphi}\bar{\rho}_i + \bar{\rho}_i\delta \rho_{\varphi} - \bar{\rho}_i(\delta \rho_{\varphi} + \bar{\rho}_\varphi\psi)\right]
$$

$$
\delta Q_{\varphi j} = a v_{(i)j}(L_{\varphi}\bar{\rho}_i + A_{s\varphi}\bar{\rho}_i\bar{\rho}_\varphi), \quad i = \text{All matter interacting with } \varphi
$$

In [60], $C_{\varphi}$ is the derivative of $C(\varphi)$ with respect to $\varphi$. To obtain these equations we have used the following definition and properties:

$$
\rho_{\varphi} = u(\varphi)^\mu u(\varphi)^\nu T_{\varphi}^{\mu\nu} = \frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi + V(\varphi)
$$

$$
P_{\varphi} = \frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi - V(\varphi)
$$

$$
u(\varphi)^\mu = \frac{\partial^{\mu}\varphi}{\partial_{\nu}\varphi\partial^{\nu}\varphi} = \partial^{\mu}\varphi\left(\rho_{\varphi} + P_{\varphi}\right)^2
$$

$$
\frac{\delta \rho_{\varphi} + \delta P_{\varphi}}{\rho_{\varphi} + P_{\varphi}} = 2\left[\frac{a \partial^0(\delta \varphi)}{\left(\rho_{\varphi} + P_{\varphi}\right)^2} + \psi\right]
$$

Evidently, these equations are valid for both modified gravity and interacting quintessence. They are also highly coupled, thus it is impossible or very difficult to find an analytical solution for them.

To complete evolution equations for modified gravity, we also need the evolution of $\delta \varphi$. This can be

\[13\]To prevent confusion between spacetime indices and indices indicating the species, when there is a risk of confusion we put the latter inside brackets.
obtained by expanding the field $\varphi = \bar{\varphi} + \delta \varphi$ and using the covariant field equation, see e.g. [71]:

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \varphi) + V_{\varphi}(\varphi) = C(\varphi) T_{m}$$

(70)

$$\delta \varphi'' + 2 \mathcal{H} \delta \varphi' + \psi'(\bar{\varphi'} + 2 \mathcal{H} \bar{\varphi}) + \psi(\bar{\varphi''} + 2 \mathcal{H} \bar{\varphi'} + 2 \frac{a''}{a} \bar{\varphi}) - (k^2 - \frac{a''}{a}) \delta \varphi = C(\varphi) \delta T_{m} + C_{\varphi} \mathcal{T} \delta \varphi$$

(71)

As we mentioned in previous sections, solving these equations is not the main aim of present work. Our goal is to single out differences of these models that can be used for discriminating them from other models. For instance, in modified gravity models the perturbation of interaction current does not depend on the metric perturbations $\psi$ and $\phi$. By contrast, in interacting dark energy the current perturbation depends on the metric perturbation and it is easy to that:

$$\frac{\text{term} \propto \psi}{\text{term} \propto \delta} = 1$$

(72)

Because according to observations $\delta \varphi$, $\varphi'$, and $\delta \varphi'$ are very small, in both models the terms proportional to $\delta_i \equiv \delta \rho_i / \rho_i$ are dominant. In this case, it is easy to see that for modified gravity $\delta Q_{(i)0} \propto C(\varphi)$ and for interacting quintessence $\delta Q_{(i)0} \propto (-L_{\varphi} + A_{(i)} \bar{\rho}_{\varphi})$. Although apriori these quantities evolve differently, both of them are expected to vary very slowly. Thus, it is not possible to distinguish them, specially in a model independent way. Other properties such as (72) cannot be used directly either. Nonetheless, they influence the growth rate $\propto \delta_{i} / \delta$, density power spectrum, and density-velocity correlations, etc. In the next section we discuss how these measurable quantities can be related to interaction current, and thereby allow to discriminate between modified gravity and quintessence models.

Perturbation equations [62] to [65] depend on metric perturbations $\psi$ and $\phi$, and their time derivatives. These quantities can be determined from Einstein equations for perturbations (see e.g. [95]):

$$k^2 \delta \phi + 3 \mathcal{H} (\dot{\phi} + \mathcal{H} \psi) = 4 \pi G a^2 \sum_{i} \delta \rho_{i}$$

(73)

$$k^2 (\dot{\phi'} + \mathcal{H} \psi) = -4 \pi G a^2 \sum_{i} k_{j} \psi_{(i)} (\dot{\rho}_{(i)} + \ddot{P}_{(i)})$$

(74)

$$\phi'' + \mathcal{H} (\psi + 2 \phi') + \left( \frac{2 a''}{a} - \frac{a'^2}{a^2} \right) \psi + \frac{k^2}{3} = -4 \pi G a^2 \sum_{i} \delta P_{i}$$

(75)

$$k^2 (\phi - \psi) = -12 \pi G a^2 \sum_{i} (k_{l} k_{l} - \frac{1}{3} \delta_{i}^{l} \Pi_{(i)l}^{l})$$

(76)

Note that in these equations the interaction energy is neglected. The reason is that we need $T_{\text{int}}^{\mu \nu}$, which in the phenomenological description of interactions is not known. Nonetheless, its omission in equations [73] to [76] should not induce large errors because present observations show that any non-gravitational interaction between various constituents of the Universe - if any - must be very small, and therefore this approximation is justified. Metric perturbations $\psi$ and $\phi$ cannot be directly observed, except through lensing. Otherwise, they can be extracted from these equations when density-density and density-velocity correlations, and induced anisotropic shear $\Pi_{(i)l}^{l}$ are determined from LSS data.

Although phenomenological interaction currents [43] and [44] are inspired from well understood scattering of particles, one cannot rule out other types of interaction. Even for these cases apriori one should be able to write equations similar to [62] - [65], and [72]. The fact that the latter relations are independent of the strength of the coupling between dark energy and matter proves that finding a different proportionality between $\psi$ and $\delta$ terms would be a clear signature of an unusual quintessence model, e.g. one with a non-minimal interaction with gravity. Evidently, such measurements are not easy. Nonetheless, with the huge amount of data expected from near future surveys and their better precision, more accurate measurements of parameters should be possible, and precision analysis necessary for detailed examination of dark energy models should be achievable.
5 Estimation of forecast precision for surveys

In this section we first describe how in practice the background cosmology parameters defined in Sec. 2 are calculated. Their uncertainties determine how well a survey can discriminate between modified gravity and (interacting)-quintessence models, independent of the data type or observation method. Then, we calculate and parametrize the evolution equation of the growth rate of matter anisotropies and discuss its measurement uncertainty. As an example we make an order of magnitude estimate for the expected uncertainty of these quantities for the Euclid mission. As we mentioned in the Introduction, a proper forecast needs detailed study of observational effects and uncertainties which is out of the scope of present work.

5.1 Discriminating between a cosmological constant and other models

As we discussed in Sec. 2, discrimination ability of surveys between a cosmological constant and a redshift dependent dark energy can be evaluated by using the function \( \Theta(z) \) defined in (12). To calculate the quantity \( \Theta \) and its uncertainty, we need to know uncertainties of the estimation of effective background cosmological parameters. The function \( \Theta \) depends on \( \Omega_{\text{eff}}^{(H)} \), \( w_{\text{eff}}^{(H)}(z) \), \( \Omega_{\text{eff}}^{(A)} \), and \( w_{\text{eff}}^{(A)}(z) \), effective dark energy fractional density and equation of state dark energy determined, by fitting \( H(z) \) and \( A(z) \), respectively. By measuring \( H(z) \), from either supernovae or BAO data, one can determine \( w_{\text{eff}}^{(H)}(z) \) and \( \Omega_{\text{eff}}^{(H)} \) relatively easily. On the other hand, measurements of \( w_{\text{eff}}^{(A)}(z) \) and \( \Omega_{\text{eff}}^{(A)} \) are less straightforward, because one has to determine \( dH/dz \), or equivalently \( dD_A/dz \) and \( d^2D_A/dz^2 \) (see Appendix A for relation between these quantities). For this reason, the uncertainty of \( \Theta \) is dominated by uncertainties of \( w_{\text{eff}}^{(A)}(z) \) and \( \Omega_{\text{eff}}^{(A)} \). Finally, coefficients \( F_i \)'s that present the evolution of equation of state of various constituents, are determined by fitting the deviation of \( H(z) \) from the null hypothesis of a \( \Lambda \)CDM cosmology. However, as we argued in Sec. 2, there are strong degeneracies between \( F_i \)'s and \( \gamma(z) \) which can be resolved only with using other types of data, in particular matter anisotropies, see Sec. 5.2 for more details.

As an example, we estimate the uncertainty of \( \Theta \) for the Euclid mission. For the parametrization \( w_{\text{eff}}(z) = w_p + w_a z/(1+z) \), according to the Euclid-Red Book, the standard deviation for these coefficients are expected to be \( \sigma_{w_p} \sim 0.015 \) and \( \sigma_{w_a} \sim 0.15 \) for Euclid data alone, and \( \sigma_{w_p} \sim 0.007 \) and \( \sigma_{w_a} \sim 0.035 \) for Euclid+Planck data. No forecast for the expected uncertainty of \( dH/dz \) is yet available. For this reason, we simply use error propagation rules to determine a rough estimation for \( \sigma_{dH/dz} \) from available forecasts. We approximate \( dH/dz \) with its definition as a difference ratio: \( dH/dz \approx \Delta H/\Delta z \), then we use the general uncertainty propagation rule to a function of \( n \) variables \( f(x_1, \ldots, x_n) \):

\[
\sigma_f^2 = \sum_{i,j=1,\ldots,n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} C_{ij}
\]

where \( C_{ij} \) is the covariance matrix for random variables \( x_1, \ldots, x_n \). Assuming \( \sigma_H/H \sim 1\% \), negligible error for \( z \), and \( F_i \sim 1 \), the dominant source of error in \( dH/dz \) from \( w(z) \). Because the coefficients of derivatives with respect to these parameters in (77) is roughly of the order of one, we estimate \( \sigma_{dH/dz}/(dH/dz) \sim 10 - 15\% \). Functions \( A(z) \) and \( B(z) \) are related to \( dH/dz \), see (108), and when the uncertainties of \( H \), density fractions \( \Omega_i \)'s and redshift \( z \) are much smaller, \( \sigma_{w_{eff}^{(A)}}/w_{eff}^{(A)}(z) \sim \sigma_{\Omega_{eff}^{(A)}}/\Omega_{eff}^{(A)} \sim \sigma_B \sim \sigma_{dH/dz} \sim 10\% \) around optimal redshift of \( z \sim 0.5 \). Measurement precisions of \( F_i \)'s also are of the order of precision of \( dH/dz \), i.e. \( \sigma_{F_i}/F_i \sim \sigma_{dH/dz}/(dH/dz) \sim 10-15\% \).

Evidently, uncertainties obtained here are very rough estimations. The aim of these exercises is just to show what level of error we expect from near future surveys. A proper prediction needs detailed simulation of measurements and data analysing methods, instrumental effects, and systematic and statistical errors. They need a dedicated study that we leave to a future work.

Finally we want to make a remark about the redshift dependence of \( w(z) \), which in the literature
is usually parametrized \[51\]. In Appendix [A] we show that for the same value of \(w\) at two different redshifts, different parametrizations lead to very different evolution for \(A(z)\). Inversely, if we estimate \(w(z)\) from the measurement of \(A(z)\), parametrization of \(w(z)\) can lead to very different evolution for this function, despite employment of the same data for \(A(z)\). Therefore, we must estimate \(w\) at each redshift without parametrizing it. As for the estimation of uncertainties, for instance from the Fisher matrix, they can be determined from the set of \(\{w(z), \gamma(z), z\}\) at every redshift bin rather than from a parametrization, see Appendix [B].

### 5.2 Discrimination between modified gravity and interacting quintessence models

If we observe a non-zero \(\Theta\), then we must use the power spectrum and growth rate of perturbations to investigate the nature and origin of deviation from a cosmological constant. The comparison between evolution equation of modified gravity and interacting quintessence models in Sec. [122] showed that their interaction currents are very different, and thereby the evolution of matter anisotropies and dark energy density in these models are not the same. In fact, if we could decompose the interaction current to terms proportional to scalar metric perturbations and matter density fluctuations, it were possible to distinguish between these models. However, in practice measured quantities are matter power spectrum and its growth rate \(f(z, k)\) defined as:

\[
f(z, k) = \frac{d \ln D}{d \ln a} = \frac{\delta_m}{H \delta_m}, \quad D = \frac{\delta_m(z, k)}{\delta_m(z = 0, k)}
\]

(78)

The function \(f(z, k)\) is usually extracted from the power spectrum using a model \([96, 97, 98]\), for instance a power-law for the primordial spectrum including its modification by Kaiser effect \([99, 100, 101, 102]\) and redshift distortion due to the velocity dispersion \([103]\).

To obtain the evolution equation of \(f(z, k)\), we replace potentials \(\psi\) and \(\phi\) by expressions depending only on \(\delta_m = \delta \rho_m/\bar{\rho}_m\) and \(\theta_m = i k_j v_j (m)\). Assuming a negligible anisotropic shear at \(z \lesssim \mathcal{O}(1)\) which concerns galaxy surveys, scalar metric perturbations - gravitational potentials - \(\psi\) and \(\phi\) can be determined from Einstein equations \([14, 45, 58, 104]\):

\[
\phi = \psi = \frac{4\pi G \bar{\rho}_m}{k^2} \left( \delta_m + 3(1 + w_m) \frac{\mathcal{H} \theta_m}{k^2} \right) + \Delta \psi
\]

(79)

\[
\Delta \psi = \frac{4\pi G}{k^2} \left( \delta \rho_\phi - 3 \mathcal{H} \delta \phi (\bar{\rho}_\phi + \bar{P}_\phi)^{\frac{2}{3}} \right)
\]

(80)

\[
\phi' = -\frac{4\pi G \bar{\rho}_m}{k^2} \left( \delta_m + (3 + \frac{k^2}{\mathcal{H}^2})(1 + w_m) \frac{\mathcal{H} \theta_m}{k^2} \right) + \Delta \phi'
\]

(81)

\[
\Delta \phi' = -\mathcal{H} \Delta \psi + 4\pi G a^2 \delta \phi (\bar{\rho}_\phi + \bar{P}_\phi)^{\frac{1}{3}}
\]

(82)

Note that in \([79\) and \([81\] we have separated terms which vanish for \(\Lambda\)CDM model and written them as \(\Delta \psi\) and \(\Delta \phi'\). As observations show that dark energy behaves very similar to a cosmological constant - at least for \(z \lesssim \mathcal{O}(1)\), both these quantities are expected to be very small. It is why we write them as a variation of \(\psi\) and \(\phi'\). For future use it is also better to redefine them as followings:

\[
\epsilon_0 \equiv \frac{\delta \rho_\phi}{\bar{\rho}_m}, \quad \epsilon_1 \equiv \frac{\mathcal{H} (\bar{\rho}_\phi + \bar{P}_\phi)^{\frac{1}{3}} \delta \phi}{\bar{\rho}_m}
\]

(83)

\[
\Delta \psi = \frac{4\pi G \bar{\rho}_m}{k^2} (\epsilon_0 - 3 \epsilon_1)
\]

(84)

\[
\Delta \phi' = -\frac{4\pi G \bar{\rho}_m \mathcal{H}}{k^2} \left( \epsilon_0 - (3 + \frac{k^2}{\mathcal{H}^2}) \epsilon_1 \right)
\]

(85)

\[\text{In this section for the sake of simplicity of notation we consider that } F_i\text{'s factors for species are included in } w_i\text{'s, i.e. } (1 + z)^{3\gamma} F_i \text{ is redefined as } (1 + z)^{3\gamma_i(z)} \text{ and } w_i \text{ is obtained from } \text{[2]} \text{ using this redefined } \gamma_i. \text{ Therefore, for interacting dark energy models } w_m \text{ is nonzero and in general depends on redshift.}\]
After replacing \( \phi' \) and \( \psi \) in (56) and (57) with (51) and (10) respectively, evolution equation of matter and velocity perturbations can be written as:

\[
\begin{align*}
\delta'_{m} + \frac{\dot{\rho}_{m}}{\rho_{m}} + 3H\left( (1 + C_{sm}^{2})\delta_{m} + (1 + w_{m}) \frac{3\Omega_{m}H^{2}}{2k^{2}} \left[ \delta_{m} + (3 + \frac{k^{2}}{H^{2}})(1 + w_{m}) \frac{\theta_{m}}{k^{2}} \right] \right) + \left( \epsilon_{0} - 3 + \frac{k^{2}}{H^{2}} \right) + (1 + w_{m}) \theta_{m} = \delta Q_{m0} \\
\theta'_{m} + \frac{\dot{w}_{m}}{1 + w_{m}} \delta_{m} + \frac{\dot{\theta}_{m}}{\rho_{m}} + 4H\theta_{m} - \frac{C_{sm}k^{2}}{1 + w_{m}} \delta_{m} - 3\Omega_{m}H^{2}\left( \delta_{m} + 3(1 + w_{m}) \frac{\theta_{m}}{k^{2}} + \epsilon_{0} - 3\epsilon_{1} \right) \\
= ik \delta Q_{(m)}^{i} \tag{86}
\end{align*}
\]

where \( \delta Q_{m0} \) and \( ik \delta Q_{(m)}^{i} \) are interaction currents and \( \Omega_{m} \equiv 8\pi Ga^{2}\bar{\rho}_{m}/3H^{2} \). Moreover, in present and near future wide area surveys such as DES and Euclid the value of \( H/ck \ll 1 \). For instance, for Euclid \( H/ck \ll 0.01 \). Therefore, we can neglect terms proportional to \( H/k \). Under these approximations, evolution equations of density and velocity become:

**Modified gravity:**

\[
\begin{align*}
\delta'_{m} + 3H(C_{sm}^{2} - w_{m})\delta_{m} + (1 + w_{m})\theta_{m} = & \frac{3\Omega_{m}(1 - 3w_{m})C_{\phi}(\bar{\phi})}{8\pi G} a\dot{H}_{1} + C(\bar{\phi}) \left( \frac{3\Omega_{m}(1 - 3w_{m})}{8\pi G} \right) \frac{1}{2} \left( \frac{C_{sm}^{2}}{\Omega_{m}(1 + w_{m})} \right) aH_{0} \tag{88}
\\
\theta'_{m} + \frac{H\theta_{m}}{1 + w_{m}} - \frac{C_{sm}k^{2}}{2} \delta_{m} = & \frac{3\Omega_{m}H^{2}(\delta_{m} + \epsilon_{0} - 3\epsilon_{1})}{2} - \frac{\sqrt{3k^{2}(1 - 3w_{m})\Omega_{m}}}{(8\pi G(1 + w_{m})\Omega_{m})} \frac{1}{2} \left( \frac{C(\bar{\phi})}{\Omega_{m}} \right) \epsilon_{1} \tag{89}
\end{align*}
\]

**Interacting quintessence:**

\[
\begin{align*}
\delta'_{m} + 3H(C_{sm}^{2} - w_{m})\delta_{m} + (1 + w_{m})\theta_{m} = & aA_{sm}\epsilon_{0} \tag{90}
\\
\theta'_{m} + \frac{H\theta_{m}}{1 + w_{m}} - \frac{C_{sm}k^{2}}{2} \delta_{m} = & -\frac{w_{m}}{1 + w_{m}} \left( -L_{m} + A_{sm}\bar{\rho}_{\phi} \right) a\theta_{m} \tag{91}
\end{align*}
\]

Now that we have the evolution equations for \( \delta_{m} \) and \( \theta_{m} \), we can determine the evolution of growth rate. The procedure for calculating \( df(z, k)/dz \) is straightforward. We replace \( \theta_{m} \) in (89) and (91) with its value obtained from (88) and (91), respectively for modified gravity and interacting quintessence models. Then, we replace \( \bar{\delta}'_{m} \) with its value from equation (78). The final equation has the following general form:

\[
\begin{align*}
f'\mathcal{H} + f(\mathcal{H}' + \mathcal{H}^{2}) + f^{2}\mathcal{H}^{2} + 3(C_{sm}^{2} - w_{m})(\mathcal{H}' + f\mathcal{H}') + 3(C_{sm}^{2} - w_{m})\mathcal{H}^{2} + \frac{3}{2} \Omega_{m}(1 + w_{m})^{2}\mathcal{H}^{2} + k^{2}C_{sm}^{2}E_{0}\mathcal{H} + E_{1}k^{2} + E_{2}\mathcal{H} + E_{3}\mathcal{H}^{2} + E_{4} = 0 \tag{92}
\end{align*}
\]

Coefficients \( E_{0}, E_{1}, E_{2}, E_{3}, E_{4} \) depend on \( z \) and \( k \), and have the following values for the two models discussed here:

\[\text{Note that the speed of light } c = 1 \text{ is assumed in metric } [54], \text{ and therefore it does not explicitly appear in our calculations.}\]
Modified gravity:

\[
E_0 \equiv C(\dot{\varphi}) a(\ddot{\rho}_\varphi + \ddot{P}_\varphi) \frac{1}{3} \left( 3(C^2_{s\varphi} - w_m)(1 + 3w_m) - 1 + 3w_m \right) \tag{93}
\]

\[
E_1 \equiv C(\dot{\varphi}) \frac{(1 + w_m)(1 - 3w_m)\ddot{\rho}_m}{Ha(\ddot{\rho}_\varphi + \ddot{P}_\varphi)} \tag{94}
\]

\[
E_2 \equiv 3(1 + w_m)(C^2_{s\varphi} - w_m)C(\dot{\varphi}) a(\ddot{\rho}_\varphi + \ddot{P}_\varphi) \frac{1}{3} - C(\dot{\varphi})(1 - 3w_m)\frac{a\ddot{\rho}_m}{H\delta_m} - \frac{1}{2}(1 - 3w_m)(1 + C^2_{s\varphi})\ddot{\rho}_m \tag{95}
\]

\[
E_3 \equiv \frac{3\Omega_m(1 + w_m)^2}{2} \left( \frac{\epsilon_0}{\delta_m} - \frac{3\epsilon_1}{\delta_m} \right) \tag{96}
\]

\[
E_4 \equiv \frac{1}{\delta_m} \left( C(\dot{\varphi})(1 - 3w_m)\frac{a\ddot{\rho}_m}{H} + C(\dot{\varphi}) a(\ddot{\rho}_\varphi + \ddot{P}_\varphi) \frac{1}{3} - C(\dot{\varphi})(1 - 3w_m)\frac{a\ddot{\rho}_m}{H\delta_m} - \frac{1}{2}(1 - 3w_m)(1 + C^2_{s\varphi})\ddot{\rho}_m \right)' + \left[ C(\dot{\varphi})(1 - 3w_m)\frac{a\ddot{\rho}_m}{H\delta_m} + C(\dot{\varphi}) a(\ddot{\rho}_\varphi + \ddot{P}_\varphi) \frac{1}{3} - C(\dot{\varphi})(1 - 3w_m)\frac{a\ddot{\rho}_m}{H\delta_m} - \frac{1}{2}(1 - 3w_m)(1 + C^2_{s\varphi})\ddot{\rho}_m \right] \tag{97}
\]

Interacting quintessence:

\[
E_0 \equiv w_m a(-L_m + A_{s\varphi}\ddot{\rho}_\varphi) \tag{98}
\]

\[
E_1 \equiv 0 \tag{99}
\]

\[
E_2 \equiv 3w_m a(C^2_{s\varphi} - w_m)(-L_m + A_{s\varphi}\ddot{\rho}_\varphi) + A_{s\varphi} a\ddot{\rho}_m(1 + 3w_m) \frac{\epsilon_0}{\delta_m} \tag{100}
\]

\[
E_3 \equiv \frac{3\Omega_m(1 + w_m)^2}{2} \left( \frac{\epsilon_0}{\delta_m} - \frac{3\epsilon_1}{\delta_m} \right) \tag{101}
\]

\[
E_4 \equiv -A_{s\varphi} a\ddot{\rho}_m \left( \frac{\epsilon_0}{\delta_m} + \frac{2a\epsilon_0}{\delta_m}(-L_m + A_{s\varphi}\ddot{\rho}_\varphi) \right) \tag{102}
\]

In the calculation of (92)-(102) we have neglected terms proportional to \( H/ck \).

For \( \Lambda \)CDM model \( E_i = 0, \quad i = 0, 1, \ldots, 4 \). For a non-interacting quintessence model all \( E_i \) coefficients are zero except \( E_3 \). A notable difference between modified gravity and interacting quintessence models is the coefficient \( E_1 \) which is strictly zero for interacting dark energy models and nonzero for modified gravity that leaves an additional scale dependent signature on the evolution of matter anisotropies.

The other explicitly scale dependent term is common for all models and is expected to be very small because it is proportional to the square of sound speed which is very small for cold matter. In addition, in contrast to the rest of \( E_i \) coefficients, \( E_1 \) and \( E_3 \) are dimensionless. Evidently, the contribution of \( E_1 k^2 \) term with respect to other terms in equation (92) increases for larger \( k \), i.e. at short distances. But, the effect of nonlinearities, i.e. mode coupling also increases at large \( k \), see e.g. [104]. They can imitate interactions and lead to misinterpretation of data. For this reason, it is suggested that observation of galaxy clusters is a good discriminator between dark energy models [105],[106], because clusters are still close to linear regime, but have relatively large \( k \).

Discriminating power of a survey can be estimated by the precision of \( E_1 \) and \( E_3 \) measurements. However, one expects some degeneracies when equation (92) is fitted to determine \( E_i \)'s. Moreover, in galaxy surveys, \( f \) and \( f' \) (or more exactly \( df/dz \)) are determined from the measurement of power spectrum from galaxy distribution, and \( H \) and \( H' \) from the BAO effect on the spectrum. Thus, these measurements are not completely independent. An independent measurement of \( H \) and \( H' \), e.g. using supernovae will help to reduce degeneracies and error propagation from measured quantities to the
estimation of \( E_i \)’s. The relation between \( \mathcal{H}' \) and \( B(z) \) defined in \( \text{(110)} \) shows the logical connection of parametrization of homogeneous component - background cosmology - and evolution of fluctuations, specially in what concerns discrimination between dark energy models. In fact, anisotropies depend on the equation of state of matter, which in the context of interacting dark energy models, is modified by its interaction with dark energy. Thus, their independent measurements optimize their employment in distinguishing between various models.

Although apriori \( df/dz \) can be determined directly from data by differentiating \( f \), usually due to shot noise the errors would be very large unless we extensively rebin the data. However, rebinning smears the redshift-dependence, which is the most important information for discriminating between models. Another approach is to solve equation \( \text{(92)} \) analytically. It does not have an analytical solution for the general case, but as we show in Appendix \( \text{D} \) when \( w_m \) and \( C_{2m}^2 \) are approximated by constant values, and cosmology is matter, radiation or cosmological constant dominated, i.e. up to desired precision only one component determines its evolution, an approximate solution can be found. At present epoch where matter and dark energy have comparable contributions, coefficients in \( \text{(92)} \) even for \( \Lambda \text{CDM} \) vary with redshift. Nonetheless, their variation arrives very quickly to saturation. Therefore, the true solution is not very different from the approximate analytical one under the explained conditions, and it is possible to determine perturbations around the analytical solution by linearizing equation \( \text{(92)} \), see Appendix \( \text{D} \) for more details.

A rough estimation of the uncertainties of \( E_i \)’s measured by Euclid can be performed in the same manner as what is presented in Sec. 5.1 for \( \Theta \) and \( \mathcal{F}_i \)’s. It is expected that growth rate \( f \) can be reconstructed from Euclid+Planck data with an uncertainty \( \sigma_f/f \lesssim 3\% \). Considering equation \( \text{(92)} \) and estimation of uncertainty of \( \mathcal{H}' \) obtained in Sec. 5.1 the uncertainty of \( \sigma_{f'/f'} \) must be \( \sim 10\% – 15\% \). This limits our ability to distinguish between a \( \Lambda \text{CDM} \) model where \( E_i = w_m = 0 \), and quintessence or interacting dark energy models where these quantities are not zero. Considering the linear equation obtained in Appendix \( \text{D} \) from expansion of \( f \) around its solution for \( \Lambda \text{CDM} \), the total uncertainty of deviation from this model is roughly the same as what is obtained for \( f' \), i.e. \( \sim 10\% – 15\% \). But, the uncertainty in the estimation of each \( E_i \) is expected to be larger because of the degeneracy of these parameters. Evidently, determination of \( f \) and \( f' \) at multiple redshifts should help somehow reduce degeneracies and improve discrimination between models. More precise estimations as well as the estimation of the effect of nonlinearities and the optimal choice of scale range need detail simulation of surveys. We leave these tasks for future works.

5.3 Interpretation and comparison with other parametrizations

It would be useful to have a better insight on the physical meaning of the parameters defined in the previous section, and to compare them with what is used in the literature for parametrizing dark energy models.

We begin with \( \epsilon_0 \) and \( \epsilon_1 \) defined in \( \text{(83)} \). Their definitions show that the former depends only on dark energy density anisotropy and the latter only on the peculiar velocity of dark energy field, i.e. on its kinematics, see \( \text{(124)} \). They follow each other closely and approach zero when the field approaches its minimum value. However, their exponent close to the minimum depends on the interaction. Therefore, their measurements give us information about the potential and interactions of the scalar field. Moreover, the difference in the dependence of evolution equation of anisotropies and growth factor to \( \epsilon_0 \) and \( \epsilon_1 \) shows that only by separation of kinematics and dynamics of dark energy - scalar field - it would be possible to distinguish between modified gravity and other scalar field models.

The deviation of gravity potentials \( \phi \) and \( \psi \) from their value in \( \Lambda \text{CDM} \) \( \Delta \psi \) is the quantity which can be measured directly from lensing data \( \text{(107)} \). For this reason various authors have used \( \Delta \psi \) to parametrize the deviation from \( \Lambda \text{CDM} \). \( \text{(76, 77, 78, 79)} \). However, equations \( \text{(70)} \) and \( \text{(80)} \) show that although \( \Delta \psi \neq 0 \) is by definition a signature of deviation from \( \Lambda \text{CDM} \), in contrast to claims in the literature, it is not necessarily the signature of a modified gravity model because quintessence models,
both interacting and non-interacting, also induce $\Delta \psi \neq 0$. This is also another manifestation of the difference between kinematics and dynamical effects of interacting dark energy models described above.

Because we have used Einstein frame for both quintessence and modified gravity models, in absence of an anisotropic shear $\phi = \psi$ even in non-$\Lambda$CDM models. In linear approximation gravitational lensing effect depends on the total potential $\Phi \equiv \phi + \psi$ (see e.g. [107] for a review). Therefore, in Einstein frame

$$\Phi = 2\phi = 2\psi = \Phi_{\Lambda CDM} + 2\Delta \psi,$$

$$\Phi_{\Lambda CDM} \equiv \frac{4\pi G \bar{\rho}_m}{k^2} \left( \delta_m + 3(1 + w_m) \frac{H \theta_m}{k^2} \right) \equiv \frac{4\pi G \bar{\rho}_m}{k^2 \Delta_m}$$  \hspace{1cm} (103)

In the notation of [76] $\Phi = 2\Sigma \Phi_{\Lambda CDM}$, thus:

$$\Sigma = 1 + \frac{\Delta \psi}{\Phi_{\Lambda CDM}} = \frac{\epsilon_0 - 3\epsilon_1}{k^2 \Delta_m}$$ \hspace{1cm} (104)

The other quantity which affects the evolution of lensing and directly depends on cosmology is the growth factor of matter anisotropies which determines the evolution of $\Delta_m$ defined in (103). This quantity can be obtained from integration of growth rate $f$ defined in [78] and is usually parametrized as $\Omega_m^\gamma$. For $\Lambda$CDM $\gamma \approx 0.55$ [108]. In this respect there is no difference between our formulation and what is used in the literature. Evidently, this simple parametrization cannot distinguish between various dark energy models. By contrast, the more sophisticated decomposition proposed in Sec. 5.2 is able to distinguish between quintessence and modified gravity. Note that in Jordan frame there are two other parameters: $\eta \equiv (\psi - \phi)/\phi$ and $Q = \phi/\phi_{\Lambda CDM}$. The parameter $\Sigma = Q(1 + \eta/2)$, thus it is not independent. In Einstein frame $\eta = 1$ unless there is an anisotropic shear. At first sight it seems that there is less information in Einstein frame about modified gravity than in Jordan frame. However, one should notice that in Einstein frame the fundamental parameters are $\epsilon_0$ and $\epsilon_1$ and other quantities such as $\Delta \psi$ and $f$ can be explained as a function of these parameters. Therefore, the amount of information in Einstein and Jordan frame about modified gravity - if it is what we call dark energy - is the same. The advantage of formulation in Einstein frame and definition of $\epsilon_0$ and $\epsilon_1$ is that they can be used for both major categories of models. Moreover, they have explicit physical interpretations that can be easily related to the underlying model of dark energy.

More recently based on an original work by C. Skordis [109], two groups [110, 111] have suggested new parametrizations which are basically only for discriminating modified gravity models from $\Lambda$CDM. Both groups use the following approximate description for the Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + U_{\mu\nu}$$ \hspace{1cm} (105)

The tensor $U_{\mu\nu}$ is called \textit{Energy-momentum tensor of dark energy} [109], and originally its definition has been for formulation of all modifications of the Einstein theory of gravity. In [110] this tensor is expanded with respect to potentials $\psi$ and $\phi$, and coefficients of this expansion are used for parametrizing the underlying modified gravity model.

Note that equation (105) is at all scales an approximation because the right hand side is explicitly proportional to the Newton coupling constant. Considering $f(R)$ models which are the simplest modification of the Einstein theory of gravity, in contrast to (105), the coupling to matter is modified in both frames, see equations (8)-(10) for Jordan frame, and the formulation of $f(R)$ model in Einstein frame in [67]. In fact, in Einstein frame the modification is explicit in the energy-momentum conservation equation. This means that if a deviation from $\Lambda$CDM is observed, it would be very difficult to verify the consistency of the model at short distances because the deviation of coupling from Newton constant $G$ is put by hand to zero. Moreover, this formulation and parametrization by definition does not help to detect interaction between dark energy and matter, because it depends only on the total variation of metric potentials. In addition, in this formulation $U_{\mu\nu}$ is assumed to be a conserved component, which as we discussed in Sec. 3 is not consistent because in contrast e.g.
to perturbative quantum field theories, we never measure the free component. Furthermore, equation (105) has exactly the same form for quintessence models, thus in this framework it is not possible to discriminate between this class and modified gravity models without knowing the underlying model in detail.

The formulation in [111] uses a Lagrangian formalism with quadratic and higher order deviations from the Einstein theory of gravity. The energy-momentum tensor of dark energy $U_{\mu\nu}$ is obtained be using variational methods from this Lagrangian. It is a function of $g_{\mu\nu}$ or the set $\{g_{\mu\nu}, \varphi, \partial_\mu \varphi\}$ when the dark-(energy) sector includes also a scalar field. Then, they use 3+1 spacetime decomposition, thus all coefficients of the above expansion depend only on time, and apply variational methods to determine perturbations $\delta U_{\mu\nu}$ around an arbitrary background. Their formulation is technically and theoretically interesting, specially for studying various modified gravity models, but there is neither a model independent parametrization for dark energy nor for observables.

6 Outline

We have parametrized the interaction between dark energy and matter for modified gravity and interacting quintessence models as modifications of the evolution of matter and radiation background and perturbations densities, and the equation of state of dark energy. We have showed that when the interaction is ignored in the data analysis, the effective value of parameters are not the same if we calculate them from Friedman equation or from a function proportional to the derivative with respect to redshift of total mean energy density of the Universe. We have also defined a single quantity that evaluates the strength of the interaction. Its observational uncertainty can be used to estimate the discriminating power of a cosmological survey.

We have obtained a phenomenological description for the interaction current in the context of interacting quintessence models motivated by particle physics. Based on these results, we have suggested to distinguish between modified gravity and (interacting)-quintessence dark energy models of non-gravitational origin by the way they modify energy-momentum conservation equation. If the interaction current is proportional to the trace of the energy-momentum tensor of matter, we classify the model as modified gravity, otherwise, as (interacting)-quintessence and its variants, such as K-essence, quintom, cosmon, etc.

We have determined the modification of evolution equation of density and velocity perturbations in the context of modified gravity and interacting quintessence models discussed above, and used them to obtain a parametrized description of evolution equation of the growth factor that can be used for both these models as well as a simple $\Lambda$CDM model, which has been considered as the null hypothesis in our discussions. The difference between the value of these parameters can distinguish between aforementioned models. We have also obtained order of magnitude estimations for uncertainties on these quantities measured with the Euclid mission. A better forecast for these uncertainties needs simulations of the survey and the data analysis that we have left to future works.

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A Properties of $A(z)$

One of the principle aims of LSS surveys is the measurement of Hubble constant $H(z)$, angular diameter distance $D_A$, and luminosity distance $D_L$, mainly by measuring Baryon Acoustic Oscillations (BAO) which play the role of a reference distance scale \[46\]. The maximum effect of BAO on the power spectrum is at redshift $\sim 0.3 \ [46]$. However, as we mentioned in the Introduction a direct determination of $\gamma(z)$ from Hubble constant, $D_A$, or $D_L$ when $z \to 0$ is not possible. In fact, using equation (1) and the definition of angular diameter distance, it is easy to see that:

$$\ln \left( \frac{d}{dz}((1+z)D_A) \right)^{-1} - \Omega_m(1+z)^3 - \Omega_h(1+z)^4 - \Omega_K(1+z)^2 = \ln \Omega_{de} + 3\gamma(z) \log(1+z)$$ \hspace{1cm} (106)

At small redshifts the last term on the r.h.s. of (106) which contains $\gamma(z)$ approaches zero, and the effect of the latter becomes negligibly small. Now, consider the following quantities:

$$H^2(z) = \frac{8\pi G}{3} \rho(z)$$ \hspace{1cm} (107)

$$B(z) = \frac{1}{3(1+z)^2 \rho_0} \frac{d\rho}{dz} = \frac{2H(z)}{3H_0^2(1+z)^2} \frac{dH}{dz} = \frac{2H(z)}{3(1+z)H_0} \left( \frac{(1+z)dH}{dz} + H \right)$$ \hspace{1cm} (108)

$$A(z) = B(z) - \Omega_m - \frac{4}{3} \Omega_h(1+z) - \frac{2\Omega_K}{3(1+z)}$$

$$= \Omega_{de} \left( \gamma + (1+z) \ln(1+z) \frac{d\gamma}{dz} \right)(1+z)^{3(\gamma-1)} = \Omega_{de}(w(z) + 1)(1+z)^{3(\gamma-1)}$$ \hspace{1cm} (109)

where $H(z) = \dot{a}/a$ is the expansion rate of the Universe and $\rho(z)$ is the total density at redshift $z$. It is clear that $A(z)$ is proportional to the deviation of dark energy from a cosmological constant at any redshift including $z = 0$. In addition, its sign determines whether dark energy has normal or phantom-like equation of state at a given redshift. It can be shown \[37\] that when $dw/dz \ll 3w(z)(w(z) + 1)/(1+z)$, the sign of $dA/dz$ is opposite to the sign of $w(z) + 1$. This condition is satisfied at low redshifts - see examples of models in Fig. 1. It means that $A(z)$ is a concave or convex function of redshift, respectively for positive or negative $w(z) + 1$. Observations show that the
contribution of $\Omega_k$ and $\Omega_h$ at low redshifts is much smaller than the uncertainty of $\Omega_m$. The function $dA/dz$ does not depend on $\Omega_m$. Thus, the uncertainty on the value of $\Omega_m$ can shift the value of $A(z)$ but it does not change its slope and its shape i.e. its concavity or convexity will be preserved.

The function $B(z)$ can be easily related to directly measurable quantities:

$$B(z) \equiv \frac{1}{3(1+z)^2} \frac{d\rho}{dz} = \frac{2}{3(1+z)^2} \left[ \frac{dD_A}{d\ln a} - \frac{D_A}{(1+z)^2} \right]$$

$$D_l = (1+z) H_0 \int_0^z \frac{dz}{H(z)}$$

or equivalently with respect to normalized angular distance:

$$B(z) = -\left( \frac{2}{3(1+z)^2} \left[ D_A + (1+z) \frac{dD_A}{dz} \right]^2 \right)$$

$$D_A = \frac{H_0}{1+z} \int_0^z \frac{dz}{H(z)} = \frac{D_l}{(1+z)^2}$$

Note that these equations are written for a flat universe, but can be easily extended to the cases where $\Omega_k \neq 0$.

### B About Fisher matrix for equation of state of dark energy

Fisher matrix evaluates the sensitivity - information content - of a measured quantity to variables and parameters that define the underlying model [47]. Under special conditions, e.g. Gaussianity of distributions, Fisher matrix can be related to the covariance matrix of measurements. In LSS surveys the main measured quantity is the power spectrum of matter density anisotropies. Application of Fisher matrix to CMB [48] and galaxy surveys [96, 97, 98] is well studied and widely used. In what concerns the measurement of dark energy density, its variation, and its equation of state from galaxy surveys, one has to extract $H(z)$ and $D_A(z)$ either from BAO [112, 113, 114] or by fitting the complete power spectrum [51]. Fisher matrix for 2-dimensional power spectrum is determined by Seo & Eisenstein [96, 97, 98] with $H(z)$ and $D_A(z)$ as parameters. A transformation from these quantities to coefficients of a parametrized equation of state, for instance $w(z) = w_0 + w_a z/(1+z)$ allow to determine the covariant matrix for the measurement of $w_0$ and $w_a$ [51].

Although apriori the value of these quantities can be determined at any redshift, in practice the limited volume and deepness of surveys allow to determine the power spectrum at the average redshift of the survey or for some bins of redshift in the case of large deep surveys. In the latter case, the estimation of $w(z)$ as a function of redshift depends strongly on its parametrization. Fig. 1 shows the plot of $A(z)$ for examples in which $w(z) = w_0 + w_a z/(1+z)$ allow to determine the covariant matrix for the measurement of $w_0$ and $w_a$ [51].

Simpson and Peacock [115] use $\{w_0, w_a, \Omega_\Lambda, \Omega_k, \Omega_m h^2, \Omega_h h^2, n_s, A_s, \beta, \gamma', \sigma_p\}$ as independent parameters for estimating cosmological parameters from the measurement of the galaxy power spectrum. Here $w_a \equiv -dw/d\ln a$, $\beta(z) \equiv f(z)/b(z)$ where $f(z)$ is the growth rate of scalar fluctuations and $b(z)$ is the linear bias, and $\gamma'$ is the parameter that define an approximate parametrization for $f(z) \approx \Omega_m^{\gamma'}(z)$ for $\Lambda$CDM [108]. It can be also shown that in what concerns the determination of the Fisher matrix
Figure 1: $A(z)$ as a function of redshift. To see how well $A(z)$ can distinguish between various models and how systematic and statistical errors as well as parametrization affect the reconstructed model, we consider 3 parametrizations as written on the plot above. Note that parametrizations for the plot in the center and on the right are equivalent up to a redefinition of coefficients $w_0$ and $w_1$. We first consider a given value for $w(z)$ at $z = 0$ and $z = 3$, determine corresponding coefficients $w_{0i}$ and $w_{1i}$ where index $i$ is for initial. Then to simulate systematic errors we plot the following models: $w_0 = -1 + |w_{0i} + 1|$, $w_1 = w_{1i}$ (dotted line), $w_0 = -1 - |w_{0i} + 1|$, $w_1 = -w_{1i}$ (dot-dash), $w_0 = -1 + |w_{0i} + 1|$, $w_1 = -w_{1i}$ (dashed) and $w_0 = w_{0i}$ and $w_1 = w_{1i}$ (full line). Colored vertical bars present statistical errors. The uncertainty of $a z$ is $1 \sigma_{A(z=0)} = 0.01$ (top row) and $1 \sigma_{A(z=0)} = 0.05$ (bottom row) at $z = 0$ and evolves with redshift as $\sigma_A(z) = \sigma_A(z = 0)(1 + z)^2$. It seems to be possible to distinguish between normal and phantom dark energy models easily, if uncertainties are limited to few percents. Evidently, achieving such a precision is challenging even for space missions such as Euclid.
for dark energy, \(w(z)\) and \(dw/dz\) alone lead to a singularity \(^{16}\).

In place of parametrizing \(w(z)\), we suggest to use \(w(z)\), \(\gamma(z)\) and \(z\) to determine the Fisher matrix for dark energy parameters. It can be easily shown that Fisher matrix becomes singular if the first two quantities are considered \(^{51}\), because \(w(z)\) and \(\gamma(z)\) are not independent - if one knows \(w(z)\), then \(\gamma(z)\) can be determined from \(^{2}\). This problem does not arise when \(w\) is parametrized because expansion parameters are explicitly independent. The relationship of \(w(z)\) and \(\gamma(z)\) is very similar to the relation between \(H(z)\) and \(D_A(z)\). Fisher matrix for \(\{H(z), D_A(z), z\}\) set of parameters is calculated in \(^{51}\). Using this formulation, a parameter transformation gives the Fisher matrix for \(\{w(z), \gamma(z), z\}\). Relation between Fisher matrices with 2 sets of parameters for dark energy parameters. It can be easily shown that Fisher matrix becomes singular if the first \(w\) and \(\gamma\) parameters can be replaced by \(A(z) = \Omega_{de}(w(z)+1)(1+z)^{3(\gamma-1)}\). In fact, it is preferable to replace \(\gamma(z)\) with \(A(z)\), because at low redshifts the \(\gamma(z)\)-dependent term has very small effect on the evolution \(H(z)\) and \(D_A\). By contrast, the deviation of \(A(z)\) from its value in \(\Lambda\)CDM model is maximized for \(z \to 0\), see Fig. \(^{1}\).

### C Fluid description of a scalar field

Energy momentum tensor of a scalar field is:

\[
T_{\varphi}^{\mu\nu} = -\frac{1}{2}g^{\mu\sigma}g^{\rho\sigma}\partial_\rho \varphi \partial_\sigma \varphi + g^{\mu\nu}V(\varphi) + \partial^\mu \varphi \partial^\nu \varphi
\]

Using definition \(^{33}\) of a perfect fluid, the density and pressure are defined as:

\[
\rho_{\varphi} \equiv u_\mu u_\nu T_{\varphi}^{\mu\nu} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + V(\varphi), \quad P_{\varphi} \equiv \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi)
\]

\(u^\mu\) is the velocity vector and \(u^\mu u_\mu = 1\). It is easy to verify that with above definitions for \(\rho_{\varphi}\) and \(P_{\varphi}\):

\[
u_\mu = \frac{\partial u^\varphi}{(\rho_{\varphi} + P_{\varphi})^{\frac{1}{2}}}
\]

\(^{16}\)For the sake of simplicity in the discussion of Fisher matrix here, we neglect other cosmological parameters, i.e we assume that dark energy parameters can be factorized from other quantities. In practice, one has to consider a single matrix Fisher matrix containing all parameters. Thus there would be one single covariant matrix that includes correlation of all uncertainties.
D Solution of evolution equation of growth rate

For $\Lambda$CDM cosmology, $E_i = 0$, $i = 0, \ldots, 4$. We also consider $w_m = C^2_{sm} = 0$. In this case after dividing equation (132) by $\mathcal{H}^2$, the evolution equation of growth rate becomes:

$$\frac{f'}{\mathcal{H}} + f \left( \frac{\mathcal{H}'}{\mathcal{H}^2} + 1 \right) + f^2 + \frac{3}{2} \Omega_m = 0$$

(125)

After changing the variable from $\eta$ to $\ln a$, this equation changes to:

$$\frac{df}{dx} + \left( \frac{x''}{x'} + 1 \right) f + f^2 + \frac{3}{2} \Omega_m = 0, \quad x = \ln \frac{a(\eta)}{a_0(\eta)}$$

(126)

By integrating the Friedman equation for flat $\Lambda$CDM one obtains:

$$\mathcal{H} = \frac{d \ln \left( \frac{a}{a_0} \right)}{d \eta} = x' = \frac{\mathcal{H}_0 a}{a_0} \sqrt{\Omega_m(a_0) \left( \frac{a}{a_0} \right)^3 + \Omega_\Lambda}$$

(127)

$$\mathcal{E} \equiv \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{x''}{x'^2} = \frac{\Omega_\Lambda - \Omega_m(a_0) \frac{a_0^3}{a^3}}{\Omega_\Lambda + \Omega_m(a_0) \frac{a_0^3}{a^3}} = \frac{\Omega_\Lambda - \Omega_m(a_0) e^{-3x}}{\Omega_\Lambda + \Omega_m(a_0) e^{-3x}}$$

(128)

For $z = 0$, $\mathcal{E} = -1$ and for $z \rightarrow \infty$, $\mathcal{E} = \Omega_\Lambda(a) - \Omega_m(a)$. To be able to solve (129) analytically we must assume $\mathcal{E}$ is a constant. This is a good approximation if we are interested only on a small range of redshifts. Under this assumption, the solution of (126) can be obtained by integration:

$$f_{\Lambda CDM}(z) = \frac{-(\mathcal{E} + 1 - \frac{\alpha_1}{2}) + (\mathcal{E} + 1 + \frac{\alpha_1}{2})(1 + z)^{\alpha_1}}{1 - (1 + z)^{\alpha_1}}, \quad \alpha_1 = \sqrt{(\mathcal{E} + 1)^2 - 6\Omega_m}$$

(129)

For $-\sqrt{6\Omega_m} - 1 < \mathcal{E} < \sqrt{6\Omega_m} - 1$, $\alpha_1$ is imaginary and according to this approximation solution $f(z)$ has an oscillating component. A simple attempt to make (129) more precise is to take into account that $\mathcal{E}$ depends on redshift.

To obtain an approximate solution for interacting dark energy models parametrized by coefficients $E_i = 0$, $i = 0, \ldots, 4$ in (122), under the assumption that these corrections are small, we can linearize this equation around $f_{\Lambda CDM}$. Note that in general it is expected that in interacting dark energy models $w_m$ and $C^2_{sm}$ are not zero. Therefore, we add also their contribution to the linearized model:

$$f = f_{\Lambda CDM} + \Delta f$$

$$\Delta f' + \left[ \frac{\mathcal{H}'}{\mathcal{H}} + E_0 + \mathcal{H} \left( 1 + 3(C^2_{sm} - w_m) + 2 f_{\Lambda CDM} \right) \right] \Delta f + 3(C^2_{sm} - w_m) \frac{\mathcal{H}'}{\mathcal{H}} + 3\mathcal{H} \left( C^2_{sm} - w_m + \frac{\Omega_m}{2} w_m(2 + w_m) \right) + (C^2_{sm} + E_1) \frac{k^2}{\mathcal{H}} + E_2 + E_3 \mathcal{H} + E_4 \frac{1}{\mathcal{H}} = 0$$

(131)

Solution of this linearized equation is straightforward and can be formally written as the following:

$$\Delta f(z) = \frac{\mathcal{H}}{(1 + z)(1 + 3(C^2_{sm} - w_m))} \exp \left[ \int \frac{dz}{1 + z} \left( \frac{E_0}{\mathcal{H}} + 2 f_{\Lambda CDM} \right) \right] \times \left\{ 1 + \int dz \frac{(1 + z)(1 + 3(C^2_{sm} - w_m))}{(1 + z)^2} \exp \left[ - \int \frac{dz}{1 + z} \left( \frac{E_0}{\mathcal{H}} + 2 f_{\Lambda CDM} \right) \right] \right\} \left[ 3(C^2_{sm} - w_m) \frac{\mathcal{H}'}{\mathcal{H}} + 3\mathcal{H} \left( C^2_{sm} - w_m + \frac{\Omega_m}{2} w_m(2 + w_m) \right) + \frac{k^2}{\mathcal{H}} (C^2_{sm} + E_1) + E_2 + E_3 \mathcal{H} + E_4 \frac{1}{\mathcal{H}} \right]$$

(132)

Determination of integrals in (132) needs details of redshift dependence of coefficients $E_i$’s which is model dependent. Nonetheless, they depend on the scalar field which must vary very slowly with redshift. Therefore, at zero order, they can be considered as constant. Although even with this simplification it is difficult to determine (132) analytically, a numerical determination allow to write it as an expansion with respect to $E_i$ coefficient. This expansion would be suitable for compression with data and determination of $E_i$. 

34