INTRODUCTION

Calculating the optical forces acting on mesoscopic particles typically relies on defining a balance equation between the momentum of the electromagnetic field and the momentum of the mesoscopic particle. This can be done using various definition of the electromagnetic stress tensor such as Chu, Maxwell-Lorentz, Minkowski, Abraham, Einstein-Laub. All these approaches deliver the same total momentum transfer to rigid particles. Furthermore, in all these cases the momentum is proportional to the incident intensity. The differences between each stress tensor definition lies in their precise microscopic interpretation. In the following, we consider only Maxwell-Lorentz stress tensor, however all the discussions are applicable to any of the other definitions.

To transform classical fields into quantum compatible wave-function fields, we introduce the quantum conversion (QC). The QC is based on two parts, a mathematical transformation, including units, of the classical fields to wave-function equivalent fields. The QC ensures the correct units in the physical context. The transforma-
sion ensures the correct momentum transfer to rigid particles. Further, in all these cases the momentum is proportional to the incident intensity. The differences between each stress tensor definition lies in their precise microscopic interpretation. In the following, we consider only Maxwell-Lorentz stress tensor, however all the discussions are applicable to any of the other definitions.

The electromagnetic momentum transferred to scattering particles is proportional to the intensity of the incident fields, however, the momentum of single photons (√hv) does not naturally appear in these classical expressions. Here, we discuss an alternative to Maxwell’s stress tensor that renders the classical electromagnetic field momentum compatible to the quantum mechanical one. This is achieved through the introduction of the quantum conversion which allows the transformation, including units, of the classical fields to wave-function equivalent fields.

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\[
\int_{-\infty}^{\infty} f_0(t) g_0(t) dt = \int_{-\infty}^{\infty} \sqrt{\hbar \nu} \hat{f}^*(\nu) \hat{g}(\nu) d\nu = \int_{-\infty}^{\infty} f^*(t) \left( i \hbar \frac{d}{dt} \right) g(t) dt
\]

which represents the main purpose of the QC. In effect, the QC introduces a first order time domain derivation whose role is to make the integral intensity measure proportional to the frequency of the field. As we will see in the following, this will introduce a proportionality between the energy/momentum measures and the frequency/wave-vector of the fields considered. The proportionality coefficient is Planck’s constant and the QC converts using a different coefficients each spectral component of the field to encode the energy information onto the frequency of the field. This is possible as each frequency component is orthogonal to each other.

For completeness, we can define the inverse QC as:

\[
f_0(t) = (Q^{-1} \circ f)(t) = i \hbar \frac{d}{dt} (Q \circ f)(t)
\]

and exemplify the effect of the QC on a harmonic function:

\[
Q(\exp(i\omega \tau))(t) = \text{sgn}(\omega) i \exp(i\omega t)/\sqrt{\hbar \omega}
\]

where \(\text{sgn}(\omega)\) corresponds to the sign function.

In the following, we consider the QC of the electromagnetic fields solutions of Maxwell’s equations (Gaussian units) in free space.

\[
\nabla \cdot E_0 = 0,
\n\nabla \cdot B_0 = 0,
\nc \nabla \times E_0 = -\partial_t B_0,
\nc \nabla \times B_0 = \partial_t E_0,
\]

where \(E_0\) and \(B_0\) are the electric and magnetic vector fields and where \(c\) is the speed of light. We define the QC of the electromagnetic fields as \(E = Q \circ E_0\) and \(B = Q \circ B_0\) where \(E_0\) and \(B_0\) are considered to be zero.
mean analytic signal functions i.e. contain only strictly positive frequencies.

Taking property (3) into account we can define the converted field energy density as

$$\rho_\text{E} = \frac{1}{8\pi} (E^*(i\hbar \partial_t)E + H^*(i\hbar \partial_t)H)$$

which will is globally identical to the standard definition of energy density i.e. the total time integrated energy in a region is the same for both definitions. The time dependence of the fields has change however using the normalised fields gives exactly the same total intensity/energy of the pulse. What is more this approach maintains relative intensities which means that through relative measures we are not able to distinguish physically between the classic and the quantum version. The origin of property is the orthogonality of the different spectral components of the pulse i.e. the total energy content of two monochromatic waves is equal to the sum of their individual energies.

The flow of energy is described by

$$J_\text{E} = \frac{c}{8\pi} (E^* \times (i\hbar \partial_t)H + (i\hbar \partial_t)E \times H^*)$$

Using this definition of the flow allows us to determine the energy transferred to a volume $V$ surrounded by a surface $S$ as

$$\frac{d\mathcal{E}}{dt} = \int_S J_\text{E} \cdot n \, ds$$

where $n$ is the surface normal and $\mathcal{E} = \int_V \rho_\text{E} \, dv$ the integrated energy density in the volume $v$.

Using the quantum converted fields we can redefine the linear momentum density and its current/flux. To do this, we notice the parallelism between the definition of energy and momentum. They only differ in the generating differential operator. What is more, taking into account the dispersion relationship between the momentum flux integral (10) and Maxwell’s stress tensor, which in this context determines the variation of photon flux due to a scattering body.

This integral delivers identical average forces as Maxwell’s stress tensor. It shows the equivalence between quantum converted fields in conjunction with the operator approach and classical electromagnetic forces. In both cases, the observable macroscopic mechanical force is the same.

We further remark that the density defined by the identity operator $\rho = \langle E^*E + H^*H \rangle/(8\pi)$ corresponds to the photon density. This explains physically the equivalence between the momentum flux integral (10) and Maxwell’s stress tensor, which in this context determines the variation of photon flux due to a scattering body.

To complete the equivalence picture between the classical fields and the quantum fields we further need to take into account the optical eigenmodes defined either over the whole space [2, 3] or over a finite region of interest [4]. Indeed, using this approach one can draw a direct equivalence between quantum mechanics and classical electromagnetism where the optical eigenmodes play the role of photon wavefunctions [5].

Further, the quantum conversion can easily be modified to deal with non analytic functions and negative frequencies when considering relative energy transfers. In physical terms, the quantum conversion corresponds to the introduction of frequency dependent units or measurement metric. This conversion generalises the concept of position dependent metric to the Hilbert space. Indeed, each orthogonal basis vector/function can have its own units just like each point in space can have its own distance measurement. Finally, we note that the inverse quantum conversion can be applied to transform quantum wave functions into their classical form opening up to the possibility of a new interpretation of quantum mechanics.

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