Theory of the crossover from lasing to steady state superradiance

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Lasing and steady state superradiance are two phenomena that may appear at first glance to be distinct. In a laser, phase information is maintained by a macroscopic intracavity light field, and the robustness of this phase is what leads to the coherence of the output light. In contrast, the coherence of steady-state superradiant systems derives from the macroscopic collective dipole of a many-atom ensemble. In this paper, we develop a quantum theory that connects smoothly between these two extreme limits. We show that lasing and steady-state superradiance should be thought of as the two extreme limits of a continuous crossover. The properties of systems that lie in the superradiance, lasing, and crossover parameter regions are compared. We find that for a given output intensity a narrower linewidth can be obtained by operating closer to the superradiance side of the crossover. We also find that the collective phase is robust against cavity frequency fluctuations in the superradiant regime and against atomic level fluctuations in the lasing regime.

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I. INTRODUCTION

Since its first demonstration in 1960 [1], the laser has had a profound impact on fundamental science research and has found widespread applications in society in general. Although many different types of lasers exist, with their characteristic parameters (such as power, linewidth, pulse duration, and physical size) spanning many orders of magnitude, all lasers share a common conceptual foundation. A laser is a cavity quantum electrodynamics (QED) system consisting of a gain medium inside an optical cavity [2]. We will oftentimes refer to the gain medium as “atoms” for brevity. Lasers typically operate in the good cavity regime of cavity QED where the linewidth of the cavity is much narrower than the bandwidth of the gain medium. The atoms generate a coherent electromagnetic field in the cavity by means of stimulated emission [3]. Stimulated emission is a quantum mechanical interference effect in which the presence of a large number of photons in a particular mode of a light field increases the probability that an atom will emit into that mode. In a laser, the macroscopic phase information that is associated with the coherence of the generated radiation is encoded in the light field.

Around the same time as the laser was first demonstrated, the effect of superradiance was predicted [4], and soon thereafter experimentally demonstrated [5]. Superradiance is a quantum mechanical interference effect in which correlations between atoms lead to collective emission. Superradiance has most commonly been considered as a transient phenomenon. Atoms in an ensemble are prepared in the excited state. Spontaneous emission is then enhanced via the growth of atom-atom correlations. However, it has been known for some time that superradiance can also occur in steady state [6-9] by placing the atomic ensemble inside a cavity. In contrast to lasers, superradiance in steady-state occurs in a cavity with a much broader linewidth than the atomic linewidth. This regime is referred to as the bad-cavity limit of cavity QED [10-14]. The radiation produced in steady state superradiance is also coherent. However, in contrast to a laser, the coherence is encoded in the atomic medium. Progress has recently been made towards the experimental realization of steady-state superradiant systems [15-17].

An important application of lasers is as a stable local oscillator for optical atomic clocks and precision spectroscopy [18]. These lasers rely on stabilization against reference cavities. The most advanced such lasers reach linewidths below 0.1 Hz corresponding to quality factors of $Q > 10^{15}$ [19]. The principal limiting factor in the way of further improvement of these local oscillators is thermal vibrations of the dielectric coatings on the cavity mirrors [20]. To overcome this technical challenge, researchers have proposed an alternate approach using an active system based on steady state superradiance on a clock transition to create an even more stable light source [6,12]. However, this proposal has challenges of its own. First of all, in spite of the enhancement that occurs due to superradiance, the produced intensity is orders of magnitude lower than for a conventional laser. Second of all, perturbations of atomic transition frequencies can potentially lead to phase and frequency perturbations in the generated field.

In this paper we develop a unified theory of lasers and steady state superradiance. We show that lasers and steady state superradiance are the extreme limits of a continuous crossover. The theory allows us to directly compare and contrast lasers, steady state superradiant

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systems, and systems in the crossover region using common language. Our analysis further clarifies the qualitative and quantitative differences between lasing and steady state superradiance. From the perspective of applications, the unified theory enables us to determine the optimal system for ultra-stable local oscillators and precision measurement applications.

We analyze the model using different levels of approximation: an exact method using Monte-Carlo trajectories and SU(4) operators, a semi-classical method based on c-number Langevin equations, a quantum phase diffusion model, and a mean-field model. The different approaches provide insight into different aspects of the problem. Highly simplified models like the mean field equations and phase diffusion yield a qualitative understanding of the general characteristics of systems throughout the crossover. By comparison between the approximations we can differentiate between truly critical physical effects and less important details. We find that fluctuations and correlations play an important role in the noise properties of the system (e.g. the linewidth of the generated light), but can be modelled semi-classically. Comparison with the exact SU(4) method for small numbers of atoms shows that c-number Langevin equations provide an accurate description of the system. Due to their much smaller computational complexity, we are then able to use the c-number Langevin equations to quantitatively study much larger systems relevant for experiment.

The rest of this paper is organized as follows. In Section II we summarize the physical model upon which our analysis is based. In Section III we discuss several approximation methods. We compare the approximations with one another to determine their accuracy and to evaluate their ability to capture the various physical signatures. In Section IV we define a crossover parameter which characterizes the relative importance of stimulated emission to collective atomic effects in a cavity QED system. In Section V we discuss our results on the crossover.

II. MODEL

As noted in the introduction, the fundamental ingredients of lasers and superradiance systems are an electromagnetic field and atoms serving as a gain medium. A minimal model consists of a single mode cavity field and an ensemble of N-two level atoms. The atoms couple to the cavity field via the dipole interaction. Energy is supplied to the system by means of incoherent repumping mechanisms. In practice this necessitates auxiliary atomic levels that rapidly decay and can be adiabatically eliminated. The resulting effect is incoherent transfer of population from the ground to excited state. The incoherent repumping, along with the atomic spontaneous emission, cavity decay, and other relaxation processes, make the system fundamentally an open quantum system that requires a quantum master equation treatment.

Mathematically, our model is described by the quantum master equation derived in the Born and Markov approximations for the reduced density matrix of the system \( \rho \),

\[
\frac{d}{dt} \rho = \frac{1}{i\hbar} [\hat{H}, \rho] + \hat{L} [\rho],
\]

where,

\[
\hat{H} = \frac{\hbar \omega_c}{2} \sum_{j=1}^{N} \hat{\sigma_j^+} \hat{\sigma_j^-} + \hbar \Omega \sum_{j=1}^{N} (\hat{a}^\dagger \hat{\sigma_j^-} + \hat{\sigma_j^+} \hat{a}) .
\]

The Hamiltonian \( \hat{H} \) describes the coherent evolution of the coupled atom cavity system, where \( \omega_c \) is the atomic transition frequency and \( \omega_c \) is the frequency of the cavity mode. The Pauli spin matrices for the \( j \)-th atom are \( \hat{\sigma}_j^x, \hat{\sigma}_j^y \) and \( \hat{\sigma}_j^z \), and \( \hat{a} \) is the annihilation operator of the cavity mode. The atom-cavity coupling strength is \( \Omega \). In general, the atom-cavity coupling depends on the location of the atom in the cavity field. To simplify the discussion, we ignore the spatial dependence since it results in quantitative changes but does not alter the basic physical properties. In principle, a constant \( \Omega \) could be realized experimentally by confining the atoms to locations of equal amplitude of the cavity mode by means of a superimposed optical lattice.

The Liouvillian superoperator \( \hat{L} [\rho] \) describes the various non-Hermitian processes,

\[
\hat{L} [\rho] = -\frac{\kappa}{2} (\hat{a}^\dagger \hat{\rho} \hat{a} \hat{\rho} - 2 \hat{\rho} \hat{a}^\dagger \hat{a} \hat{\rho}) \\
- \frac{\gamma}{2} \sum_{j=1}^{N} (\hat{\sigma}_j^+ \hat{\rho} \hat{\sigma}_j^- + \hat{\rho} \hat{\sigma}_j^- \hat{\sigma}_j^+ - 2 \hat{\sigma}_j^\dagger \hat{\sigma}_j \hat{\rho}) \\
- \frac{w}{2} \sum_{j=1}^{N} (\hat{\sigma}_j^x \hat{\sigma}_j^y \hat{\rho} + \hat{\rho} \hat{\sigma}_j^y \hat{\sigma}_j^x - 2 \hat{\sigma}_j^x \hat{\sigma}_j^y \hat{\rho}) ,
\]

where \( \kappa \) is the decay rate of the cavity, \( \gamma \) is the free-space spontaneous emission rate of the atoms, \( w \) is the repumping rate, and \( 1/T_2 \) is the rate of inhomogeneous dephasing.

III. SOLUTION METHODS

Obtaining a direct numerical solution to Eq. (1) is impossible for experimentally relevant numbers of particles because the dimension of the Hilbert space of the system scales as \( 2^N \). In this Section we introduce several solution methods to overcome the exponential scaling of the size of the Hilbert space. They can be grouped into three categories: Exact methods (SU(4) method with Monte-Carlo simulation), semi-classical methods (c-number Langevin equations, phase diffusion), and mean-field treatments.
The exact solution methods solve the quantum mechanical problem directly without further approximations, but are limited in applicability to small numbers of atoms. The SU(4) method provides an exact numerical solution of Eq. (1) by exploiting an underlying permutation symmetry to drastically reduce the Hilbert space dimension [21,22]. Details of the method have been described previously in [22]. Here we extend the approach to solve the quantum master equation in the SU(4) representation using the quantum jump method [23,24]. We give details of the SU(4) quantum jump method in Appendix A.

Semi-classical methods aim to capture the physics of the system correctly for large atom number. They include a classical representation of fluctuations and correlations. Comparison with direct solution methods for small atom number allows us to verify the validity and accuracy of the semi-classical approaches.

The mean-field methods neglect fluctuations to arrive at equations for averaged quantities. These equations are sufficiently simple that it is straightforward to obtain closed form solutions that provide valuable qualitative insights into the system behavior.

A. Quantum Langevin Equations

For the derivation of the semi-classical equations corresponding to Eq. (1), it is convenient to work in the Heisenberg picture. The resulting equations are the quantum Langevin equations

\[
\begin{align*}
\frac{d}{dt} \hat{a} &= \frac{1}{2} (\kappa + 2i\omega_c) \hat{a} - \frac{i N \Omega}{2} \hat{S}^- + \hat{F}_a, \\
\frac{d}{dt} \hat{S}^- &= \frac{1}{2} (\Gamma + 2i \omega_a) \hat{S}^- + \frac{i \Omega}{2} \hat{a} \hat{S}^z + \hat{F}^-, \\
\frac{d}{dt} \hat{S}^z &= - (w + \gamma) (\hat{S}^z - d_0) + i \Omega (\hat{a}^\dagger \hat{S}^z - \hat{a} \hat{S}^z) + \hat{F}^z,
\end{align*}
\]

where \( \delta = \omega_a - \omega_c \) is the atom-cavity detuning, \( \Gamma \equiv w + \gamma + 2/T_2 \) is the generalized single-atom decoherence, and \( d_0 = (w - \gamma)/(w + \gamma) \) characterizes the atomic inversion that would be obtained for a single-atom in the absence of the cavity. We have defined the collective operators,

\[
\hat{S}^\pm = \frac{1}{N} \sum_{k=1}^{N} \hat{\sigma}_k^\pm, \\
\hat{S}^z = \frac{1}{N} \sum_{k=1}^{N} \hat{\sigma}_k^z.
\]

The quantum noise operators \( \hat{F}_\mu \) have zero mean and second-order correlations given by

\[
\langle \hat{F}_\mu(t) \hat{F}_\nu(t') \rangle = 2 D^\mu\nu \delta(t - t').
\]

The diffusion matrix elements \( D^\mu\nu \) are obtained using the Einstein relations [2].

\[
\begin{align*}
2D^{aa^\dagger} &= \kappa, \\
2D^{+ -} &= \frac{1}{N} \left( w + \frac{1}{T_2} \left( 1 + \langle \hat{S}^z \rangle \right) \right), \\
2D^-+ &= \frac{1}{N} \left( \gamma + \frac{1}{T_2} \left( 1 - \langle \hat{S}^z \rangle \right) \right), \\
2D^{zz} &= - \frac{2w}{N} \langle \hat{S}^z \rangle, \\
2D^{z+} &= \frac{2\gamma}{N} \langle \hat{S}^+ \rangle, \\
2D^{z-} &= \frac{2\gamma}{N} \langle \hat{S}^- \rangle, \\
2D^{zz} &= \frac{2\gamma}{N} \left( 1 + \langle \hat{S}^z \rangle \right) + \frac{2w}{N} \left( 1 - \langle \hat{S}^z \rangle \right).
\end{align*}
\]

B. \( \mathcal{C} \)-number Langevin Equations

Quantum Langevin equations are operator valued stochastic differential equations. As such they are difficult to numerically simulate. To obtain numerically tractable equations we construct a semi-classical theory by replacing the operators in the quantum Langevin equations by complex numbers,

\[
\begin{align*}
\frac{d}{dt} a &= - \frac{1}{2} (\kappa + 2i \omega_c) a - \frac{i N \Omega}{2} S^- + F^a, \\
\frac{d}{dt} S^- &= - \frac{1}{2} (\Gamma + 2i \omega_a) S^- + \frac{i \Omega}{2} a S^z + F^-, \\
\frac{d}{dt} S^z &= - (w + \gamma) (S^z - d_0) + i \Omega (a^* S^- - a S^+) + F^z,
\end{align*}
\]

where there are no hats over the variables to signify that they are \( \mathcal{C} \)-numbers and not operators. The noise terms \( F^a, F^-, \) and \( F^z \) should be interpreted according to the rules of Ito calculus. It is easier to construct the semi-classical equations by introducing real variables according to

\[
\begin{align*}
q &= \frac{1}{2} (a^* + a), \\
p &= \frac{1}{2} (a^* - a), \\
S^z &= \frac{1}{2} (S^+ + S^-), \\
S^y &= \frac{1}{2i} (S^+ - S^-).
\end{align*}
\]
The equations of motion in terms of these variables are
\[
\frac{d}{dt}q = -\kappa q - 2\omega_c p - N \Omega S^y + F^y, \tag{22}
\]
\[
\frac{d}{dt}p = -\kappa p + 2\omega_c q + N \Omega S^x + F^x, \tag{23}
\]
\[
\frac{d}{dt}S^x = -\Gamma S^x - 2\omega_c S^y + \Omega p S^z + F^z, \tag{24}
\]
\[
\frac{d}{dt}S^y = -\Gamma S^y + 2\omega_c S^x - \Omega q S^z + F^y, \tag{25}
\]
\[
\frac{d}{dt}S^z = -(w + \gamma)(S^z - d_0) + 2\Omega (q S^y - p S^x) + F^z. \tag{26}
\]

The noise terms have zero mean and delta-correlations given by
\[
\langle F^{\mu}(t) F^{\nu}(t') \rangle = 2 \mathcal{D}^{\mu\nu}\delta(t - t'). \tag{27}
\]

The correspondence between the semi-classical and quantum mechanical Langevin equations is established by requiring that they produce identical equations for first and second moments of the system operators. Comparison of the second moments allows us to find the classical diffusion matrix elements \(\mathcal{D}^{\mu\nu}\). In order to make this procedure well defined we have to choose a specific ordering of the quantum mechanical operators. We choose to make the correspondence using symmetric ordering defined by the symmetric expectation value
\[
\langle \hat{A}^{\mu}\hat{A}^{\nu} \rangle_s = \frac{1}{2} \left( \langle \hat{A}^{\mu}\hat{A}^{\nu} \rangle + \langle \hat{A}^{\nu}\hat{A}^{\mu} \rangle \right), \tag{28}
\]
where \(\hat{A}^{\mu}\) and \(\hat{A}^{\nu}\) are system operators. We point out that in this formulation, the classical Langevin equations are equivalent to a Fokker-Planck equation for the Wigner quasi-probability distribution. The resulting diffusion matrix elements are
\[
2 \mathcal{D}^{yy} = 2 \mathcal{D}^{pp} = \frac{\kappa}{4},  \\
2 \mathcal{D}^{xx} = 2 \mathcal{D}^{yy} = \frac{\Gamma}{4N},  \\
2 \mathcal{D}^{xz} = 2 \mathcal{D}^{yz} = -\frac{w + \gamma}{N} \langle S^z \rangle,  \\
2 \mathcal{D}^{yz} = 2 \mathcal{D}^{xy} = -\frac{w + \gamma}{N} \langle S^y \rangle,  \\
2 \mathcal{D}^{zz} = \frac{2}{N} ((w + \gamma) + (-w + \gamma) \langle S^z \rangle). \tag{29}
\]

We solve the stochastic differential equations, Eqs. (22)-(26) by means of an explicit second order weak scheme [26]. We find empirically that the symmetrically ordered diffusion matrix is positive definite when the system is above the first threshold (defined in Sec. 31C). Below this threshold, the symmetrically ordered diffusion matrix is not positive definite, and divergent trajectories can occur. We numerically evolve an ensemble of trajectories simultaneously and we compute the expectation values appearing in Eq. (29) as ensemble averages. This allows us to use the additive form of the explicit second order weak scheme, which is simpler to implement than the general form. Typically, an ensemble of 1000 trajectories is sufficient to achieve convergence to within a few percent.

A specialization of the c-number Langevin approach that includes fluctuations in the phase (but not amplitude) of the photon field has been presented by Haken [14], and provides a closed-form solution to obtain the spectral linewidth of the output field. We will refer to this as the phase diffusion method, and the details are given in Appendix 13.

C. Mean-Field Treatment

The mean field equations capture many of the most important features of the physical system because the noise terms scale in general as \(\sqrt{N}\) while the expectation values scale as \(N\). In the limit of large numbers of atoms the noise terms are therefore typically less important for certain quantities.

By taking expectation values of the semi-classical Eqs. (22–26) we obtain mean-field equations written in the reference frame rotating at frequency \(\omega\)
\[
\frac{d}{dt}a_0 = -\frac{1}{2} (\kappa + 2i(\omega_c - \omega))a_0 - iN \Omega S^-_0, \tag{30}
\]
\[
\frac{d}{dt}S^-_0 = -\frac{1}{2} (\Gamma + 2i(\omega_a - \omega)) S^-_0 + iN \frac{\alpha_0 S^-_0}{2}, \tag{31}
\]
\[
\frac{d}{dt}S^z_0 = -(w + \gamma) (S^z_0 - d_0) + 2iN (a_0 S^-_0 - a_0 S^+_0), \tag{32}
\]

where the 0 subscript denotes the mean value, e.g. \(\langle \hat{a} \rangle = a_0\). Noise terms do not appear since they have zero average.

A closed-form solution of Eqs. (30–32) can be obtained in steady-state by setting the left hand sides to zero. We find
\[
S^-_0 = \frac{(\kappa + 2i(\omega_c - \omega))(\Gamma + 2i(\omega_a - \omega))}{N \Omega^2}, \tag{33}
\]
for the steady state inversion. The oscillation frequency of the atom-cavity coupled system \(w\) can be determined using the condition that \(S^-_0\) must be real, giving
\[
\omega = \frac{\kappa \omega_a + \Gamma \omega_c}{\kappa + \Gamma}. \tag{34}
\]

Simple expressions for atomic inversion and intracavity photon number can be obtained in the limit of \(\delta = \omega_a - \omega_c \ll \Gamma, \kappa\). We find
\[
S^z_0 \approx \frac{1}{C},  \\
|a_0|^2 \approx \frac{N(w + \gamma)}{2\kappa} \left( d_0 - \frac{1}{C} \right). \tag{35}
\]
where \( \mathcal{C} \equiv \frac{N \Omega^2}{\kappa} \) is the generalized many-atom cooperativity parameter. We refer to this as a generalized parameter since the cooperativity is typically defined in terms of the single-atom linewidth \( \gamma \), but here the effective linewidth \( \Gamma \) includes the dephasing \( 1/T_2 \) and incoherent repumping \( w \) as well.

The zeros of the intra cavity photon number Eq. (35) determine where the system reaches threshold. The first threshold is obtained at

\[
w_1 = \gamma ,
\]

which corresponds to the condition that energy must be supplied to the system at a rate sufficient to maintain population inversion of the atoms. A coherent macroscopic field in the cavity emerges and is accompanied by the formulation of a collective atomic dipole. A second threshold occurs at a higher rate of pumping,

\[
w_2 = \frac{N \Omega^2}{\kappa} ,
\]

where we have assumed the collective decay rate \( \mathcal{C} \Gamma \) is much larger than the single atom rates \( \gamma \) and \( 1/T_2 \). The second threshold corresponds to the situation where the repumping is so strong that \( S_\gamma \) is close to unity, and the noise due to the strong repumping prevents the formation of both a macroscopic photon field in the cavity and a macroscopic dipole in the atomic ensemble.

The photon number in the cavity reaches its maximum at an approximate repumping strength of

\[
w = w_{\text{opt}} = \frac{N \Omega^2}{2\kappa} - \gamma - \frac{1}{T_2} .
\]

Again assuming the collective decay rate \( \mathcal{C} \Gamma \) is much larger than the single atom rates \( \gamma \) and \( 1/T_2 \), we find a simple expression for the maximum photon number,

\[
\langle |a_0|^2 \rangle_{\text{opt}} = \frac{N^2 \Omega^2}{8\kappa^2} .
\]

**IV. CHARACTERIZATION OF THE Crossover**

The crossover from superradiance to lasing is characterized by a transition from coherence encoded in an atomic ensemble to coherence encoded in the light field. The key parameter in identifying the regime is the ratio of the photon number to atom number. With this motivation, we introduce a crossover parameter as

\[
\xi \equiv \frac{\langle |a_0|^2 \rangle_{\text{opt}}}{N} ,
\]

that is, the dimensionless ratio of the maximum intracavity photon number to the number of atoms. The parameter \( \xi \) quantifies the relative importance of stimulated emission to collective atomic spontaneous emission. If \( \xi \ll 1 \), the system is in the bad cavity or superradiant regime. If \( \xi \gg 1 \) the system is in the good cavity or laser regime. In the crossover or intermediate region, \( \xi \approx 1 \), the system possesses features of both.

The mean field equations allow us to rewrite this expression in an alternate way that illuminates the role of the system parameters in determining the crossover regime. Using Eq. (39), we can rewrite Eq. (40) as

\[
\xi = \frac{N \Omega^2}{8\kappa^2} .
\]

The interpretation of this is that the crossover is also characterized by the ratio of the collective coupling between the many atom ensemble and the photon mode, \( \sqrt{N} \Omega \), to the linewidth of the cavity, \( \kappa \).

**V. RESULTS**

In this section, we present results throughout the crossover from lasing to superradiance for the field intensity and linewidth, and for the atomic inversion and correlations. We begin by comparing different solution methods for small atom numbers. This comparison shows that the \( c \)-number theory gives an accurate description of first and second moments of the system operators. With the validity of the semiclassical method established we then apply it to experimentally relevant systems with large atom number.

**A. Comparison of Different Solution Methods**

In order to determine the validity of the approximate solution methods, we begin by comparing to the exact SU(4) Monte-Carlo simulation for \( N = 40 \) atoms. This is small enough to still be tractable by exact SU(4) Monte-Carlo simulations and at the same time it is large enough to expect the approximate solution methods to be reasonably accurate.

Fig. 1 shows several observables obtained using the mean field Langevin method, the phase diffusion method, the \( c \)-number Langevin method, and exact SU(4) Monte-Carlo simulations for three different values of the crossover parameter: \( \xi = 0.2 \), \( \xi = 1 \), and \( \xi = 5 \). These values of \( \xi \) place the system in the superradiance, crossover, and lasing parameter regions, respectively.

Figs. 1(a) and (c) show that the mean field equations are accurate near the peak of the intracavity photon number, \( w = w_{\text{opt}} \), but they are less accurate outside that region. Fig. 1(d) shows that the phase diffusion model for the linewidth also agrees with the exact solution in the region around \( w = w_{\text{opt}} \), but disagrees outside that region, where the phase diffusion approximation breaks down. Although they do not quantitatively agree with the exact SU(4) method, the analytic solutions obtained
by the mean field and phase diffusion models capture the correct qualitative behavior of the system.

Fig. 1 shows excellent agreement between c-number Langevin and the exact SU(4) theory in all parameter regions for all of the considered observables. Therefore, the c-number Langevin theory can be relied upon for larger atom numbers inaccessible to the exact numerical solution.

B. Many-Atom Characteristics of the Crossover

Now that the accuracy of the semi-classical c-number Langevin method has been established by comparison with the exact SU(4) theory, we study the semiclassical approach in more experimentally realistic systems with \( N = 10^4 \). The results of these simulations are shown in Fig. 2. We also include the mean-field Langevin theory, and the phase diffusion method for the linewidth (see Appendix B). We consider both the case of vanishing inhomogeneous broadening, \( 1/T_2 = 0 \), as well as \( 1/T_2 = w_{\text{opt}}/5 \).

As seen in Fig. 2 (a), the inversion \( \langle \hat{a}^\dagger \hat{a} \rangle \), the correlation between atoms \( \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \), the intracavity photon number \( \langle \hat{a}^\dagger \hat{a} \rangle \), and the intensity correlation function \( G^{(2)}(0) \) all show universal behavior in the superradiance, crossover, and lasing regimes after appropriate scaling. The calculations for different values of \( T_2 \) show that throughout the crossover the system is insensitive to atomic dephasing provided that the repumping rate is larger than the dephasing rate.

The linewidth \( \Delta \nu \) does not show universal behavior in the superradiance, crossover, and lasing regimes. As seen in Fig. 2 (b), in the superradiance region, when \( 1/T_2 = 0 \), \( \Delta \nu \) is constant in the region of \( w < w_{\text{opt}} \). In contrast, the linewidth in the lasing regime, shown in Fig. 2 (d), linearly decreases as \( w \) increases towards \( w_{\text{opt}} \). This is the typical Schawlow-Townes behavior of the laser. In the crossover region, shown in Fig. 2 (c), we see that for \( w < w_{\text{opt}} \), \( \Delta \nu \) is constant, and as \( w \) approaches \( w_{\text{opt}} \), \( \Delta \nu \) starts to linearly decrease in a similar manner to its behavior in the lasing regime. This is a consequence of the property that a system in the crossover region displays characteristics of both superradiance and lasing.

When \( 1/T_2 \) is increased to \( 1/T_2 = w_{\text{opt}}/5 \), Fig. 2 (b) shows that \( \Delta \nu \) increases for \( w < w_{\text{opt}} \), but is not significantly affected by \( 1/T_2 \) as \( w \) approaches \( w_{\text{opt}} \). In the crossover region, seen in Fig. 2 (c), the behavior is similar. In the lasing regime, as shown in Fig. 2 (d), a qualitatively different result is observed. The linewidth decreases in the region slightly below \( w = w_{\text{opt}} \) for \( 1/T_2 = w_{\text{opt}}/5 \) when compared to the \( 1/T_2 = 0 \) case. This reduction has also been observed for smaller atom...
numbers using the exact SU(4) method, so this interesting and counterintuitive result is not a consequence of the failure of the semiclassical approximation.

Fig. 2 shows the potential advantages of operating in the crossover regime, rather than in the regime of a conventional laser. Most conventional lasers are limited by available repump power, and cannot operate at the repump rate that would achieve the greatest output power and smallest spectral linewidth. It is therefore interesting to compare crossover and lasing systems operating at the same absolute repump rate. As seen in Fig. 2 (a), for the same pump rate \( w \), a system in the crossover region can operate with \( w = w_{opt} \), whereas for a lasing system, that repump rate would imply \( w \ll w_{opt} \). For this same absolute repump rate, the crossover system may obtain a linewidth that is orders of magnitude smaller than the linewidth of the system in the lasing parameter regime. Fig. 2 (b) shows that this improvement in linewidth can be achieved without paying the penalty of a greatly reduced output intensity. At this \( w \), the output intensities of the two systems are comparable.

### C. Robustness Against Frequency Shifts

The sensitivity to frequency shifts is another figure of merit of an ultrastable light source, especially in the context of precision measurements. The linewidth of the emitted light as discussed so far assumes a perfectly stable cavity frequency and atomic transition frequency. However, in the real world these frequencies can vary. For example, thermal fluctuations of the cavity mirrors or of the dielectric coatings on the mirror surfaces can cause fluctuations of the cavity resonance frequency. Fluctuating electromagnetic fields, either through stray fields or through the variation of the black-body radiation that can arise due to temperature variations, can cause atomic level shifts. In this Section, we explore the robustness of...
the ultrastable light sources to these imperfections.

We illustrate in Fig. 4(a) the sensitivity of the line-center of the spectrum of the output light with respect to both the cavity resonance frequency and the atomic resonance frequency as a function of the crossover parameter. The sensitivity is characterized by the derivatives of the line-center frequency with respect to $\omega_c$ and $\omega_a$. On the superradiance side of the crossover, $\xi \ll 1$, we observe that the system is sensitive to fluctuations of the atomic resonance frequency but robust against fluctuations of the cavity resonance. The situation is reversed on the lasering side. As a consequence, by continuously varying the crossover parameter, one has control of the relative importance of stimulated emission and collective superradiance. We showed that this encapsulates the relative importance of stimulated emission and collective superradiance. We developed a semiclassical method based on the emitted light, intensity correlation functions, and the sensitivity to perturbations of the cavity and atomic resonance frequencies. We find that when the repump rate is constrained, a system in the crossover regime may operate with a much smaller intrinsic linewidth and be less sensitive to cavity pulling than a comparable system operating as a conventional laser.

VI. CONCLUSION

In this paper we have theoretically studied the continuous crossover from steady state superradiance to lasing. We have defined a dimensionless crossover parameter that characterizes the regime as the ratio of the maximum intracavity photon number to the atom number. We showed that this encapsulates the relative importance of stimulated emission and collective superradiance. We developed a semiclassical method based on $c$-number Langevin equations and verified the accuracy of this method by comparison with exact numerical solutions.

We have systematically investigated a range of important observables: the output intensity, the linewidth of the emitted light, intensity correlation functions, and the sensitivity to perturbations of the cavity and atomic resonance frequencies. We find that when the repump rate is constrained, a system in the crossover regime may operate with a much smaller intrinsic linewidth and be less sensitive to cavity pulling than a comparable system operating as a conventional laser.

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Appendix A: SU(4) Simulation of the Quantum Master Equation

In this appendix we provide a short summary of the method we are using for the direct numerical simulation of the open quantum. Although the general aspects of the formalism are detailed in reference [22], here we extend the SU(4) method to allow us to simulate systems with a moderate number of photons and atoms. We do this by unraveling the quantum master equation into quantum trajectories—a standard method in quantum optics. The unusual feature here is that the unraveling is performed in Louville space rather than in Hilbert space, because it is in Louville space that the SU(4) method operates.

The key insight is to exploit the invariance of the master equation [1] under particle exchange. This permutation symmetry allows us to write the equations of motion in terms of generators of the SU(4) group. The Hamiltonian becomes

$$\frac{1}{i\hbar} [H, \hat{\rho}] = -2i\omega_c\Sigma_3\hat{\rho} - i\omega_a[\hat{a}^\dagger \hat{a}, \hat{\rho}]$$

$$-i\Omega [a(M_++N_+)\hat{\rho} + a^\dagger(M_-+N_-)\hat{\rho}]$$

$$+i\Omega [(U_+ + V_+)\hat{\rho}a^\dagger + (U_- + V_-)\hat{a}\hat{\rho}], \quad (A1)$$

and the dissipation terms become

$$\frac{1}{2} \sum_{j=1}^{N} (2\sigma^z_j \hat{\rho} \sigma^z_j - \sigma^+ \sigma^- \hat{\rho} - \hat{\rho} \sigma^+ \sigma^-) = -\frac{N}{2} \Delta Q_3 + Q_\mp , \quad (A2)$$

for the population changing terms, and

$$\sum_{j=1}^{N} (\sigma^z_j \hat{\rho} \sigma^z_j - \hat{\rho}) = 4M_3 - 2Q_3 - 2\Sigma_3 - N , \quad (A3)$$

for the dephasing term. In these equations, $Q_\pm, M_\pm, N_\pm, U_\pm, V_\pm, Q_3, M_3,$ and $\Sigma_3$ are superoperators [22].

We expand the density matrix in terms of the fully symmetrical multiplet $P_{q,q_3,\sigma_3} [22]$ of the SU(4) group,

$$\hat{\rho} = \sum_{q,q_3,\sigma_3, m,n} C_{q,q_3,\sigma_3}^{m,n} P_{q,q_3,\sigma_3} |m\rangle \langle n| , \quad (A4)$$
that number of excited atoms. In Ref. [22] we have shown during the evolution of a single trajectory, \( N \) phasing quantum jumps leave \( N \) by one. When a spontaneous emission or a cavity-decay event occurs, the atomic state for a particular fully symmetrical Hamiltonian [23–25]. The random jumps are chosen with probabilities such that the correct density operator is deduced. Therefore, the atomic state for a particular fully symmetrical Hamiltonian is \( \langle q_3 + \sigma_3 + N/2 \rangle \). And thus the corresponding photon state is \( \langle m \rangle \langle n \rangle = \langle N_q - (q_3 + \sigma_3 + N/2) \rangle \langle N_q - (q_3 + \sigma_3 + N/2) \rangle \).

The quantum Monte Carlo method decomposes the density operator evolution into a set of quantum trajectories where, between applications of random jumps into random channels, the system evolves under an effective Hamiltonian. Random jumps are chosen with probabilities such that the correct density operator evolution is obtained when an average is taken over trajectories. To construct a single trajectory, we first need to identify the jump operators. In our problem, there are four decay channels: repumping, spontaneous emission, dephasing, and cavity decay. The corresponding jump operators \( \mathcal{J}_i \) are

\[
\mathcal{J}_1 \dot{\hat{\rho}} = w \sum_{j=1}^{N} (\sigma_j^+ \rho_j^-) = w Q_+ \dot{\hat{\rho}}, \\
\mathcal{J}_2 \dot{\hat{\rho}} = \gamma \sum_{j=1}^{N} (\sigma_j^- \rho_j^+) = \gamma Q_- \dot{\hat{\rho}}, \\
\mathcal{J}_3 \dot{\hat{\rho}} = \frac{1}{2T_2} \sum_{j=1}^{N} (\sigma_j^\sigma_j^+ + \sigma_j^- \rho_j^-) = \frac{1}{2T_2} (4M_3 - 2Q_3 - 2\Sigma_3) \dot{\hat{\rho}}, \\
\mathcal{J}_4 \dot{\hat{\rho}} = \kappa \alpha \rho \hat{a}^\dagger.
\]

When a repumping quantum jump occurs, \( N_q \) increases by one. When a spontaneous emission or a cavity-decay quantum jump happens, \( N_q \) decreases by one. The dephasing quantum jumps leave \( N_q \) unchanged. Therefore, during the evolution of a single trajectory, \( N_q \) is uniquely determined at every time step by keeping track of the numbers of jumps of the different types. With knowledge of \( N_q \), the photon number does not need to be treated as an independent variable but is determined from the number of excited atoms. In Ref. [22] we have shown that

\[
\mathcal{J}_s \dot{\hat{p}}_{q,q_3,\sigma_3} = (q_3 + \sigma_3) \mathcal{P}_{q,q_3,\sigma_3}, \\
\mathcal{P}_{q,q_3,\sigma_3} \dot{\hat{J}}_z = (q_3 - \sigma_3) \mathcal{P}_{q,q_3,\sigma_3},
\]

where \( \dot{\hat{J}}_z = \sum_{j=1}^{N} \sigma_j^z / 2 \) is the collective spin operator. Therefore, the atomic state for a particular fully symmetrical atomic basis state in terms of the number of excited atoms is \( |q_3 + \sigma_3 + N/2 \rangle \). And thus the corresponding photon state is

\[
| m \rangle \langle n | = | N_q - (q_3 + \sigma_3 + N/2) \rangle \langle N_q - (q_3 + \sigma_3 + N/2) |. 
\]

The annihilation operator \( \hat{a} \) is decomposed according to

\[
\hat{a} = (a_0 + \hat{\rho}) e^{i\hat{\phi}}.
\]
Above threshold, amplitude fluctuations are small so that $\hat{p}$ can be neglected. We then obtain for the two time correlation function of the field amplitude

$$\langle \hat{a}^\dagger(t)\hat{a}(0) \rangle = a_0^2 \exp\left( \frac{i}{2} \int_0^t d\tau \Im[\hat{F}] \right).$$

\begin{equation}
(\text{B5})
\end{equation}

After substituting Eq. \text{[B4]} into Eq. \text{[B1]}, we take the imaginary part to first order in products of operators, and find,

$$\hat{\phi} = -\frac{1}{2} (\kappa + \Gamma) \hat{\phi} + \frac{1}{a_0} \Im[\hat{F}],$$

\begin{equation}
(\text{B6})
\end{equation}

where a factor of $e^{-i\theta}$ has been absorbed into $\hat{F}$. Equation \text{[B6]} is then integrated, assuming that $(\kappa + \Gamma)$ is large, to arrive at

$$\hat{\phi}(t) - \hat{\phi}(0) = \frac{2}{a_0(\kappa + \Gamma) \gamma} \int_0^t dt' \Im \left[ \frac{\Gamma}{2} \hat{F}^a - i \frac{\Omega}{2} \hat{F}^b \right].$$

\begin{equation}
(\text{B7})
\end{equation}

Since $\hat{F}^a$ and $\hat{F}^b$ are Gaussian, we can use

$$\langle e^{i(\hat{\phi}(t) - \hat{\phi}(0))} \rangle = e^{-\frac{1}{2} \langle (\hat{\phi}(t) - \hat{\phi}(0))^2 \rangle}.$$

\begin{equation}
(\text{B8})
\end{equation}

Therefore, we use Eq. \text{[B7]}, along with Eqs. \text{[17]} to find

$$\langle (\hat{\phi}(t) - \hat{\phi}(0))^2 \rangle = \frac{(C + 1) \Gamma}{2(Cd_0 - 1)(w + \gamma)} \frac{\Omega^2 \kappa}{(\kappa + \Gamma)^2} t,$$

\begin{equation}
(\text{B9})
\end{equation}

so that the linewidth $\Delta \nu$ given by

$$\Delta \nu = \frac{(C + 1) \Gamma}{2(Cd_0 - 1)(w + \gamma)} \frac{\Omega^2 \kappa}{(\kappa + \Gamma)^2}.$$

\begin{equation}
(\text{B10})
\end{equation}