On $H_3(1)$ Hankel determinant for certain subclass of analytic functions

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Abstract: The objective of this paper is to obtain an upper bound to Hankel determinant of third order for any function $f$, when it belongs to certain subclass of analytic functions, defined on the open unit disc in the complex plane.

Key words: Analytic function, upper bound, third Hankel determinant, positive real function.

1. INTRODUCTION

Let $A$ denotes the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.1)

in the open unit disc $E = \{ z : |z| < 1 \}$. Let $S$ be the subclass of $A$ consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as Coefficient conjecture, which states that for a univalent function its $n^{th}$ Taylor’s coefficient is bounded by $n$ (see [4]). The bounds for the coefficients of these functions give information about their geometric properties. For example, the $n^{th}$ coefficient gives information about the area where as the second coefficient of functions in the family $S$ yields the growth and distortion properties of the function. A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. The Hankel determinant of $f$ for...
$q \geq 1$ and $n \geq 1$ was defined by Pommerenke [20], which has been investigated by many authors, as follows.

$$H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}. \quad (1.2)$$

It is worth of citing some of them. Ehrenborg [7] studied the Hankel determinant of exponential polynomials. Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $S$ with bounded boundary rotation. The Hankel transform of an integer sequence and some of its properties were discussed by Layman (see [13]). It is observed that $H_2(1)$, the Fekete-Szegö functional is the classical problem settled by Fekete-Szegö [8] is to find for each $\lambda \in [0, 1]$, the maximum value of the coefficient functional, defined by $\phi_\lambda(f) := |a_3 - \lambda a_2^2|$ over the class $S$ and was proved by using Loewner method. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where $t$ is real, for the inverse function of $f$ defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when $f^{-1} \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order $\alpha$ ($0 < \alpha \leq 1$). In recent years, the research on Hankel determinants has focused on the estimation of $|H_2(2)|$, where

$$H_2(2) = \begin{vmatrix}
a_2 & a_3 & a_4 \\
a_3 & a_4 & a_5 \\
\end{vmatrix} = a_2a_4 - a_3^2,$$

known as the second Hankel determinant obtained for $q = 2$ and $n = 2$ in (1.2). Many authors obtained an upper bound to the functional $|a_2a_4 - a_3^2|$ for various subclasses of univalent and multivalent analytic functions. It is worth citing a few of them. The exact (sharp) estimates of $|H_2(2)|$ for the subclasses of $S$ namely, bounded turning, starlike and convex functions denoted by $R$, $S^*$ and $K$ respectively in the open unit disc $E$, that is, functions satisfying the conditions $\text{Re} f'(z) > 0$, $\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$ and $\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0$ were proved by Janteng et al. [11] [10] and determined the bounds as $4/9$, $1$ and $1/8$ respectively. For the class $S^*(\psi)$ of Ma-Minda starlike functions, the exact bound of the second Hankel determinant was obtained by Lee et al. [15]. Choosing $q = 2$ and $n = p + 1$ in (1.2), we obtain the second Hankel determinant for the $p$-valent function (see [24]), as follows.

$$H_2(p+1) = \begin{vmatrix}
a_{p+1} & a_{p+2} & a_{p+3} \\
a_{p+2} & a_{p+3} & a_{p+4} \\
\end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2,
The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted to the third order Hankel determinant denoted by $H_3(1)$, obtained for $q = 3$ and $n = 1$ in (1.2), also called as Hankel determinant of third kind, namely

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \ (a_1 = 1).$$

Expanding the determinant, we have

$$H_3(1) = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2), \quad (1.3)$$
equivalently

$$H_3(1) = H_2(3) + a_2J_2 + a_3H_2(2),$$

where $J_2 = (a_3a_4 - a_2a_5)$ and $H_2(3) = (a_3a_5 - a_4^2)$.

Babalola [2] is the first one, who tried to estimate an upper bound for $|H_3(1)|$ for the classes $R$, $S^*$ and $K$. As a result of this paper, Raza and Malik [22] obtained an upper bound to the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [23] derived an upper bound to the third kind Hankel determinant for a subclass of analytic functions. Bansal et al. [3] improved the upper bound for $|H_3(1)|$ for some of the classes estimated by Babalola [2] to some extent. Recently, Zaprawa [25] improved all the results obtained by Babalola [2]. Further, Orhan and Zaprawa [19] obtained an upper bound to the third kind Hankel determinant for the classes $S^*$ and $K$ functions of order alpha. Very recently, Kowalczyk et al. [12] calculated sharp upper bound to $|H_3(1)|$ for the class of convex functions $K$ and showed as $|H_3(1)| \leq \frac{4}{135}$, which is far better than the bound obtained by Zaprawa [25]. Lecko et al. [14] determined sharp bound to the third order Hankel determinant for starlike functions of order $1/2$. Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [5]), in this paper, we are making an attempt to obtain an upper bound to the functional $|H_3(1)|$ for the function $f$ belonging to the class, defined as follows.

**Definition 1.1.** A function $f(z) \in A$ is said to be in the class $Q(\alpha, \beta, \gamma)$ with $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$, if it satisfies the condition that

$$\text{Re} \left\{ \frac{f(z)}{z} + \beta f'(z) \right\} \geq \gamma, \quad z \in E. \quad (1.4)$$
This class was considered and studied by Zhi- Gang Wang et al. [26].
In obtaining our results, we require a few sharp estimates in the form of lemmas valid for functions with positive real part.
Let $\mathcal{P}$ denotes the class of functions consisting of $g$, such that

$$g(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (1.5)$$

which are analytic in $E$ and $\text{Re} g(z) > 0$ for $z \in E$. Here $g$ is called the Carathéodory function [6].

**Lemma 1.2.** ([9]) If $g \in \mathcal{P}$, then the sharp estimate $|c_k - \mu c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, with $n > k$ and $\mu \in [0, 1]$.

**Lemma 1.3.** ([17]) If $g \in \mathcal{P}$, then the sharp estimate $|c_k - c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N}$, with $n > k$.

**Lemma 1.4.** ([21]) If $g \in \mathcal{P}$ then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $g(z) = \frac{1+z}{1-z}$, $z \in E$.

In order to obtain our result, we refer to the classical method devised by Libera and Zlotkiewicz [16], used by several authors.

2. Main result

**Theorem 2.1.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, $(\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1)$ then

$$|H_3(1)| \leq 4t_1^2 \left[ \frac{k_1 \alpha^6 + k_2 \alpha^5 + k_3 \alpha^4 \beta + k_4 \alpha^3 \beta^2 + k_5 \alpha^2 \beta^3 + k_6 \alpha \beta^4 + k_7 \beta^5}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right],$$

where $k_1 = 2$, $k_2 = 2(18\beta + 1)$, $k_3 = 2(132\beta + 15)$, $k_4 = 2(511\beta + 87)$, $k_5 = (2179\beta + 490)$, $k_6 = 12(203\beta + 56)$, $k_7 = 12(93\beta + 30)$ and $t_1 = (\alpha + \beta - \gamma)$.

**Proof.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$. By virtue of Definition 1.1, there exists an analytic function $g \in \mathcal{P}$ in the open unit disc $E$ with $g(0) = 1$ and $\text{Re}\{g(z)\} > 0$ such that

$$\frac{1}{\alpha + \beta - \gamma} \left\{ \frac{f(z)}{z} + \beta f'(z) - \gamma \right\} = g(z) \quad (2.1)$$
Using the series representation for $f$ and $g$ in (2.1), upon simplification, we obtain
\[
\sum_{n=2}^{\infty} (\alpha + n\beta) a_n z^{n-2} = (\alpha + \beta - \gamma) \sum_{n=1}^{\infty} c_n z^{n-1}. \quad (2.2)
\]
The coefficient of $z^{t-2}$, where $t$ is an integer with $t \geq 2$ in (2.2) is given by
\[
a_t = \frac{(\alpha + \beta - \gamma) c_{t-1}}{(\alpha + t\beta)}, \text{ with } t \geq 2. \quad (2.3)
\]
Substituting the values of $a_2$, $a_3$, $a_4$ and $a_5$ from (2.3) in the functional given in (1.3), it simplifies to
\[
H_3(1) = (\alpha + \beta - \gamma)^2 \left[ \frac{c_2 c_4}{(\alpha + 3\beta)(\alpha + 5\beta)} - \frac{(\alpha + \beta - \gamma)c_3}{(\alpha + 3\beta)^3} - \frac{c_3^2}{(\alpha + 4\beta)^2} \right.
\]
\[\left. - \frac{(\alpha + \beta - \gamma)c_3^2 c_4}{(\alpha + 2\beta)^2(\alpha + 5\beta)} + \frac{2(\alpha + \beta - \gamma)c_1 c_2 c_3}{(\alpha + 2\beta)(\alpha + 3\beta)(\alpha + 4\beta)} \right]. \quad (2.4)
\]
On grouping the terms in the expression (2.4), in order to apply the lemmas, we have
\[
H_3(1) = t_1^2 \left[ \frac{c_4(c_2 - t_1 c_2^2)}{(\alpha + 2\beta)^2(\alpha + 5\beta)} - \frac{c_3}{(\alpha + 4\beta)^2} \left\{ c_3 - \frac{t_1(\alpha + 4\beta)c_1 c_2}{(\alpha + 2\beta)(\alpha + 3\beta)} \right\} \right.
\]
\[\left. + \frac{c_2(c_4 - t_1 c_2^2)}{(\alpha + 3\beta)^3} - \frac{c_2}{(\alpha + 3\beta)(\alpha + 4\beta)^2} \left\{ c_4 - \frac{t_1(\alpha + 4\beta)c_1 c_3}{(\alpha + 2\beta)(\alpha + 4\beta)} \right\} \right.
\]
\[\left. + \frac{(d_1 \alpha^6 + d_2 \alpha^5 + d_3 \alpha^4 \beta + d_4 \alpha^3 \beta^2 + d_5 \alpha^2 \beta^3 + d_6 \alpha \beta^4 + d_7 \beta^5)c_2 c_4}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right], \quad (2.5)
\]
with $d_1 = 1$, $d_2 = 18(\beta - 1)$, $d_3 = 133\beta - 19$, $d_4 = 4(129\beta - 35)$, $d_5 = 2(554\beta - 249)$, $d_6 = 8(156\beta - 107)$, $d_7 = 4(144\beta - 143)$ and $t_1 = (\alpha + \beta - \gamma)$. On applying the triangle inequality in (2.5), we have
\[
\left| H_3(1) \right| \leq t_1^2 \left[ \frac{|c_4||c_2 - t_1 c_2^2|}{(\alpha + 2\beta)^2(\alpha + 5\beta)} + \frac{|c_3|}{(\alpha + 4\beta)^2} \left| c_3 - \frac{t_1(\alpha + 4\beta)c_1 c_2}{(\alpha + 2\beta)(\alpha + 3\beta)} \right| \right.
\]
\[\left. + \frac{|c_2||c_4 - t_1 c_2^2|}{(\alpha + 3\beta)^3} + \frac{|c_2|}{(\alpha + 3\beta)(\alpha + 4\beta)^2} \left| c_4 - \frac{t_1(\alpha + 4\beta)c_1 c_3}{(\alpha + 2\beta)(\alpha + 4\beta)} \right| \right.
\]
\[\left. + \frac{|d_1 \alpha^6 + d_2 \alpha^5 + d_3 \alpha^4 \beta + d_4 \alpha^3 \beta^2 + d_5 \alpha^2 \beta^3 + d_6 \alpha \beta^4 + d_7 \beta^5||c_2||c_4|}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right]. \quad (2.6)
\]
Upon using the lemmas given in (1.2), (1.3) and (1.4) in the inequality (2.6), it simplifies to
\[ |H_3(1)| \leq 4t_1 \left[ \frac{k_1 \alpha^6 + k_2 \alpha^5 + k_3 \alpha^4 \beta + k_4 \alpha^3 \beta^2 + k_5 \alpha^2 \beta^3 + k_6 \alpha \beta^4 + k_7 \beta^5}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right], \] (2.7)

with \( k_1 = 2, k_2 = 2(18 \beta + 1), k_3 = 2(132 \beta + 15), k_4 = 2(511 \beta + 87), k_5 = (2179 \beta + 490), k_6 = 12(203 \beta + 56), k_7 = 12(93 \beta + 30) \) and \( t_1 = (\alpha + \beta - \gamma) \). This completes the proof of the theorem.

Remark 2.2. For the values \( \alpha = 1 - \sigma, \beta = \sigma, \gamma = 0 \), so that \( (\alpha + \beta - \gamma) = 1 \) in (2.7), we obtain
\[ |H_3(1)| \leq 4 \left[ \frac{63 \sigma^6 + 312 \sigma^5 + 411 \sigma^4 + 414 \sigma^3 + 188 \sigma^2 + 44 \sigma + 4}{(1 + \sigma)^2(1 + 2\sigma)^3(1 + 3\sigma)^2(1 + 4\sigma)} \right]. \] (2.8)

Remark 2.3. Choosing \( \sigma = 1 \) in the expression (2.8), it coincides with the result obtained by Zaprawa [25].

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