Discretization of Virasoro Algebra

Ryuji KEMMOKU† and Satoru SAITO‡§

Department of Physics, Tokyo Metropolitan University
Minami-Osawa, Hachioji, Tokyo 192-03, Japan

Abstract

A $q$-discretization of Virasoro algebra is studied which reduces to the ordinary Virasoro algebra in the limit of $q \to 1$. This is derived starting from the Moyal bracket algebra, hence is a kind of quantum deformation different from the quantum groups. Representation of this new algebra by using $q$-parametrized free fields is also given.

†E-mail:kemmoku@phys.metro-u.ac.jp
‡E-mail:saito@phys.metro-u.ac.jp
§This work is supported in part by the Grant-in-Aid for general Scientific Research from the Ministry of Education, Science and Culture, Japan (No.02640234)
1 Introduction

In recent years, one of the main themes of the study of nonlinear integrable systems is to make clear the physical and mathematical meaning of quantum groups [1]. Physically, quantum groups have been regarded in some sense as hidden symmetries of the integrable systems (strings, conformal fields, statistical models, etc.). Many people discussed deformation of such models as extension or generalization of these models. Mathematically, the representation theory of quantum groups has been investigated based on the results of Lie algebra. Especially, theories based on $U_q(\hat{sl}_2)$, the quantum enveloping affine Lie algebra, has been developed vigorously through statistical models, and conformal field theories [2]. From the progress of them, we have understood that the representation theory of $q$-deformed universal enveloping algebras had also a deep connection with many other fields of mathematics which were thought to be disconnected before.

In the study of integrable systems Virasoro algebra plays an essential part. There have been some attempts to construct $q$-deformed version of Virasoro algebra [3, 4, 5]. None of the efforts were quite successful and the problem is left open. This situation should be contrasted with one of the Kac-Moody case, in which the $q$-deformed version has been studied in detail.

Some years ago, we proposed an algebra which is supposed to be a candidate of the $q$-deformation of Virasoro algebra \[3, 5\]:

\[ [L_{mn}, L_{m'n'}] = C_{m,n}^{m',n'} L_{m+m',n+n'} + C_{m',n'}^{-m,n} L_{m+m',n-n'} \] (1)

\[ C_{m',n'}^{m,n} = \frac{[nm'-nm]}{[n][n']} [n+n'] \]

where \([x] = (q^x - q^{-x})/(q - q^{-1})\). This is an infinite dimensional Lie algebra which reduces to the ordinary Virasoro in the $q \to 1$ limit;

\[ [L_m, L_{m'}] = (m' - m)L_{m+m'} \] (2)

irrespective to the values of $n$ and $n'$. If we restrict the generators to $L_{1,1}$, $L_{-1,1}$ and $L_{0,2}$, they form the quantum group $U_q(sl_2)$. In this sense, (1) is a Lie algebra which includes $U_q(sl_2)$ as a “subalgebra”. The central extension of (1) was discusced in ref [6]. Some years ago Chaichian and Prešnajdar showed that the central extended version of the algebra (1) can be represented.
by $q$-deformed fermionic fields starting from the quantum Heisenberg algebra \cite{7}. But strictly speaking, it is not a so-called quantum group because it also contains trivial Hopf structure. Then, what is this algebra? This is our problem discussed in this article.

We like to clarify the meaning of this algebra from the viewpoint of integrable deformation of Lie algebras. Such deformation takes place in our case through discretization of differential operators in the representation of Virasoro algebra. The new algebras are reduced to the algebra of Chaichian et al. and ours under some conditions. It also enables us to understand various steps which connect the Moyal algebra and ours. It will be shown that the discretized Virasoro algebra can be represented in terms of free fields. The symmetry of the algebra will be shown to correspond to the statistics of the fields. We will also discuss the central extension of our algebra.

\section{The $q$-discretization of Virasoro algebra}

To begin with let us notice the differential representation of $L_{mn}$ in (1), i.e.

$$L_{mn} = z^m q^n (z \partial_z + \Delta_m) - q^{-n} (z \partial_z + \Delta_m) \equiv z^m \left[ z \partial_z + \Delta_m \right]_n$$

(3)

$$\Delta_m = \frac{m}{2} + \delta \quad \forall \delta,$$

where $[x]_n \equiv [x]/[n]$. Note that $L_{1,1}$, $L_{-1,1}$ and $L_{0,2}$ are $U_q(sl_2)$ generators and form the quantum group. We see that the main part of the operator is

$$\frac{q^n (z \partial_z + \Delta) - q^{-n} (z \partial_z + \Delta)}{q^n - q^{-n}},$$

(4)

which is nothing but a $q$-difference analog of the differential operator $z \partial_z + \Delta_m$. So the algebra generated by $L_{mn}$ is a discrete version of the corresponding Lie algebra. We will discuss about algebras of this type and which reduce to the Virasoro algebra as $q$ goes to 1.

Firstly we pay attention to the fact that $L_{mn}$ consists of a shift operator $l_{mn}^\alpha = z^m q^n z \partial_z + \alpha$, which constitutes an algebra among themselves:

$$[ l_{mn}^\alpha, l_{m'n'}^{\alpha'} ] = (q^{m'n'} - q^{mn'}) l_{m+m',n+n'}^{\alpha+\alpha'}.$$  

(5)
We can link up this generator with other object, called the Moyal bracket algebra. For a function such as

\[ f(x, y) = \lambda \sum_{m,n} f_{mn}(q) e^{mx} e^{-ny}, \]  

we define \( K_f \) by

\[ K_f \equiv \frac{1}{2} f(x + 2\lambda \partial_y , y - 2\lambda \partial_x). \]

Then we find \( K_f (\forall f) \) satisfies

\[ [K_f, K_g] = K(\sinh 2\lambda \{f,g\}), \]

\[ \sinh 2\lambda \{f, g\} \equiv \sinh \left[ 2\lambda \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \right) \right] f(x_f, y_f) g(x_g, y_g) \bigg|_{x_f=x_g=x} \bigg|_{y_f=y_g=y}. \]

If we put \( x = \ln z, y = \ln w \) and \( \lambda = \ln q/2 \) in (7), \( K_f \) becomes

\[ \frac{\lambda}{2} \sum_{mn} f_{mn} \frac{l^\alpha_m l^{-\alpha}_n}{z^m w^n}. \]

This fact shows that \( l^\alpha_{mn} \) is considered as a block of the generators \( K_f \). Eq.(8) is nothing but the Moyal bracket algebra. Therefore the algebra (5) constitutes a part of the Moyal algebra.

The Moyal bracket was first introduced as a quantum mechanical analog or an extension of the Poisson bracket for the purpose of denoting statistical distributions on quantum phase space. In this sense, our prescription of replacing the differential operators in the Virasoro algebra by the difference operators provides an alternative “quantization” scheme of the \( x \)-plane.

Keeping this in mind we try to make certain combinations of \( l^\alpha_{mn} \) which form a closed algebra and reduce to the Virasoro algebra in the \( q \to 1 \) limit. We notice that

\[ \frac{l^\alpha_{mn} - l^{-\alpha}_m - n}{q - q^{-1}} = z^m [nz\partial_z + \alpha] \]

do not close an algebra in general. But the following particular combinations

\[ \frac{l^m_{j+r} - l^m_{j-r}}{q - q^{-1}} \pm \frac{l^m_{j-r} - l^m_{-j+r}}{q - q^{-1}} \]

are not closed.
do for any fixed \( r \); namely, if we rewrite this combination in more convenient way as

\[
\mathcal{L}^{(j,r;\pm)}_m = z^m \left[ \frac{j \left( z \partial_z + \frac{m}{2} \right)}{[j]_\mp} \right] r \partial_z \pm \equiv z^m \left[ z \partial_z + \frac{m}{2} \right]_{j;\mp} \left[ z \partial_z \right]_{r;\pm},
\]

(13)

where \([ \ ]_\pm \) are defined as follows;

\[
[x]_+ \equiv \frac{q^x + q^{-x}}{2}, \quad [x]_- \equiv [x],
\]

then we have

\[
\left[ \mathcal{L}^{(j,r;\pm)}_m, \mathcal{L}^{(j',r';\pm)}_{m'} \right] = C^{(m, j+r)}_{m', j'+r} + C^{(m, j-r)}_{m', j'-r} + C^{(m, j'+r)}_{m', j-r} + C^{(m, j-r)}_{m', j'+r} + C^{(m, j-r)}_{m', j'+r} + C^{(m, j'+r)}_{m', j-r}.
\]

(14)

for each value of \( r \). The double signs on both sides correspond each other. The structure constant \( C \) is defined as

\[
C^{(m, j+r)}_{m', j'+r} \equiv \frac{\left[ (j+r)m'-(j'+r)m \right]}{2 [j]_\mp [j']_\mp [r]_\pm}.
\]

(15)

These algebras contain \( L_{mn} \) algebra of (1) and the algebra of Chaichian et al. Actually, the generators of the former can be identified with \( \mathcal{L}^{(j,0;\pm)}_m \), and the latter with \( \mathcal{L}^{(j,1;\pm)}_m \), respectively.

In the limit of \( q \to 1 \), \( \mathcal{L}^{(j,r;\pm)}_m \) reduce such as

\[
\mathcal{L}^{(j,r;\pm)}_m \to z^m \left( z \partial_z + \frac{m}{2} \right) \equiv L^{(\pm)}_m, \quad \mathcal{L}^{(j,r;\pm)}_m \to z^{m+1} \partial_z \equiv L^{(\mp)}_m,
\]

(16)

respectively. These \( L^{(\pm)}_m \) satisfy (2), that is they are Virasoro generators. Thus we get \( q \)-discretized Virasoro algebras, which we call \( D \)-Virasoro algebras (or \( D \)-Virasoro for short).

More generally, if we sum over \( j \) and \( r \) under the proper weight, i.e. we define

\[
K^{(f;\pm)}_m = z^m \sum_{j,r} \varepsilon^{(\pm)}_{j,r} F^{(\pm)}_{m,j} \left[ z \partial_z + \frac{m}{2} \right]_{j;\mp} \left[ z \partial_z \right]_{r;\pm},
\]

(17)

these operators also form a closed algebra:

\[
\left[ K^{(f;\pm)}_m, K^{(f';\pm)}_{m'} \right] = z^{m+m'} \sum_{j,r} \varepsilon^{(\pm)}_{j,r} H^{(\pm)}_{m+m',j} \left[ z \partial_z + \frac{m+m'}{2} \right]_{j;\mp} \left[ z \partial_z \right]_{r;\pm} = K^{(h;\pm)}_{m+m'}
\]

(18)
where

\[
H_{m+m'}^{(\pm)} \ j = \sum_{j+j'+s=J} \varepsilon_s^{(\pm)} F_{m \ j}^{(\pm)} G_{m' \ j'}^{(\pm)} C_{(m' \ j'+s)^\pm}^{(m \ j+s)^\pm} + \sum_{j+j'-s=J} \varepsilon_s^{(\pm)} F_{m \ j}^{(\pm)} G_{m' \ j'}^{(\pm)} C_{(m' \ j'-s)^\pm}^{(m \ j-s)^\pm} + \sum_{j-j'+s=J} \varepsilon_s^{(\pm)} F_{m \ j}^{(\pm)} G_{m' \ j'}^{(\pm)} C_{(m' \ j'-s)^\pm}^{(m \ j+s)^\pm} + \sum_{j-j'-s=J} \varepsilon_s^{(\pm)} F_{m \ j}^{(\pm)} G_{m' \ j'}^{(\pm)} C_{(m' \ j'-s)^\pm}^{(m \ j-s)^\pm}.
\]

The last line of (19) is derived by setting \( \varepsilon_s = \varepsilon_{-s} \) and \( F_j = F_{-j} \) \((G_j = G_{-j})\).

The Moyal deformation has also been discussed as a deformation of \( w_\infty \) algebra within the context of the area preserving maps on a torus \([10, 11]\). This is natural from our point of view, because the \( w_\infty \) algebra contains Virasoro as a subalgebra \((N = 2)\) \([12]\). Hence \( D\text{-Virasoro} \) might have some connections with deformations of \( w_\infty \) algebra.

### 3 The free field representation of \( D\text{-Virasoro} \)

In this chapter, we consider the free field representation of \( D\text{-Virasoro} \) in the bilinear form of free fields. The difference between ordinary Virasoro and \( D\text{-Virasoro} \) is that the latter has new indices \( j, r \) and \( \pm \). Though \( j \) is different for each generator, \( r \) and \( \pm \) are common for all. This means that there are different types of \( D\text{-Virasoro} \) algebras specified by \( r \) and \( \pm \) which are independent each other. Hence it is natural to label the fields by them.

We first define \( \hat{L}_m \) by

\[
\hat{L}_m^{(j,r)} = \frac{1}{2} \sum_k A_k^{(j)} m-k \ (q) \ \alpha_k^{(r)} \ \alpha^\dagger_k^{(r)}, \tag{20}
\]

Next, we assume that the free fields \( \alpha_k^{(r)} \) are bosonic, i.e. they satisfy

\[
[\alpha_k^{(r)}, \alpha_{k'}^{(r)}] \equiv D_k^{(r)} (q) \ \delta_{k+k',0}, \tag{21}
\]

where \( D_k^{(r)} (q) \) is a function of \( q \) which is to be determined later. Accordingly, \( A_k^{(j)} \) and \( D_k^{(r)} \) must satisfy the following symmetry conditions;

\[
A_k^{(j)} m-k = A_k^{(j)} m-k \ , \ D_k^{(r)} = -D_k^{(r)}. \tag{22}
\]
so that \( \hat{L}_m^{(r)} \) is defined properly. We now ask if this \( \hat{L}_m \) surely the bosonic free field representation of the D–Virasoro. To answer this let us calculate the commutator of \( \hat{L}_m \):

\[
\left[ \hat{L}_m^{(j,r)}, \hat{L}_m^{(j',r)} \right] = -\frac{1}{2} \sum_k \left( A_k^{(j)} A_k^{(j')} D^{(r)}_m - A_k^{(j')} A_k^{(j)} D^{(r)}_m \right) \alpha_k^{(r)} \alpha_k^{(r)} ,
\]

(23)

On the other hand, if we substitute \( \hat{L}_m \) to \( L_m \) and calculate the r.h.s. of (14), we find

\[
\frac{1}{2} \sum_k \left( C_{m' j' r}^m \pm A_k^{(j+j') r} \right) \pm C_{m' j' r}^m \pm A_k^{(j+j') r} \right) \alpha_k^{(r)} \alpha_k^{(r)} .
\]

(24)

Comparing (23) and (24), we find a simple solution for \( A \) and \( D \), if we take the index “−”, i.e. the structure constant \( C_{-} \) in (24):

\[
A_k^{(j)} = -\left[ \frac{2k - m}{2} \right]_{j+} , \quad D_k^{(r)} = [k]_{r-} .
\]

(25)

We cannot find such solution in “+” case. This means that \( \hat{L}_m^{(j,r;-)} \) can be represented by the use of bosonic operators only.

Then corresponding to \( \hat{L}_m^{(j,r;+)} \), it is natural to assume that they are represented by the fermionic free fields. Actually, if we define the anticommutation relation of them by

\[
[\alpha_k^{(r)}, \alpha_{k'}^{(r)}]_+ \equiv D_k^{(r)}(q) \delta_{k+k',0} ,
\]

(26)

where \([,]_+\) denotes the anticommutator, then we can show that \( \hat{L}_m^{(j,r)} \) forms an algebra isomorphic to (14) with “+”, for the following \( A \) and \( D \):

\[
A_k^{(j)} = -\left[ \frac{2k - m}{2} \right]_{j-} , \quad D_k^{(r)} = [k]_{r+} .
\]

(27)

From these results, we see that the index ± in (13) comes from the difference of the statistics of the free field representation of them. If we take the limit of \( q \to 1 \), \( \hat{L} \) reproduces the original Virasoro generators represented by boson and fermion for − and +, respectively.
4 Central extension

Though we have neglected the center of $D$–Virasoro in our discussion so far, it is straightforward to incorporate it to our theory. Let us investigate the bosonic case. For later convenience, we change the notation and denote the bosonic field by use of the Hermite conjugate operators

$$\alpha_k^{(r)} = -iB_k^{(r)}(q) \ a_k^{(r)} \ , \ \alpha_{-k}^{(r)} = i\bar{B}_k^{(r)}(q) \ a_k^{(r)}\ ,$$

where $B_k(\bar{B}_k)$ is a function of $q$ and tends to $\sqrt{k}$ in the $q \to 1$ limit. The commutation relation (21) becomes

$$[a_k^{(r)}, a_{k'}^{(r)}\dagger] = \frac{\tilde{D}_k^{(r)}}{B_k^{(r)}\bar{B}_k^{(r)}} \ \delta_{kk'}.$$  

(29)

For the zero mode, we assume for the operators to satisfy the relation;

$$[x, p] = i \frac{q^r - q^{-r}}{2r \ln q} \equiv iD_0(q).$$

(30)

(Hereafter, for simplicity, we will not write the index $r$ explicitly.) In this notation, after the normal-ordering, $\hat{L}$ is represented by

$$\hat{L}_m = iA_m^{(j)} \ D_0^{-1} \ B_m \ p \ a_m + \frac{1}{2} \sum_{k=1}^{m-1} A_{k \ m-k}^{(j)} \ B_k \ a_m \ a_k

- \sum_{k=m+1}^{\infty} A_{k \ m-k}^{(j)} \ B_{k-m} \ a_{k-m}^\dagger \ a_k$$

$$\hat{L}_{-m} = -iA_m^{(j)} \ D_0^{-1} \ B_m \ p \ a_m^\dagger + \frac{1}{2} \sum_{k=1}^{m-1} A_{k \ m-k}^{(j)} \ B_k \ a_{m-k} \ a_k^\dagger

- \sum_{k=m+1}^{\infty} A_{k \ m-k}^{(j)} \ B_k \ B_{k-m} \ a_k \ a_{k-m}^\dagger$$

$$\hat{L}_0 = -\frac{D_0^{-2}}{2} [j]_+ p^2 + \sum_{k=1}^{\infty} A_{k \ -k}^{(j)} \ B_k \ B_k \ a_k^\dagger \ a_k$$

(31)

(32)

(33)

for $m > 0$. The center can be obtained from the commutator $[\hat{L}_m, \hat{L}_{-m}]$ ($m \neq 0, \pm 1$) by taking the normal-ordering of $\hat{L}_0$;

$$\frac{1}{2} \sum_{k=1}^{m-1} A_{k \ m-k}^{(j)} A_{k \ m-k}^{(j')} \ D_k \ D_{m-k} \ .$$

(34)
When we substitute $A$ and $D$ of (25), it becomes

$$\frac{1}{2} \sum_{k=1}^{m-1} \left[ \frac{2k-m}{2} \right]_{j,+} \left[ \frac{2k-m}{2} \right]_{j',+} [k]_{r,+} [m-k]_{r,-}. \quad (35)$$

It is easy to check that this central extension satisfies Jacobi identity and the center (35) reduces to $m(m^2 - 1)/12$ in $q \rightarrow 1$. For the fermionic case, we can also get it in a similar manner.

5 Discussions

In this paper we have constructed the $q$-discretized Virasoro algebra, in short, $D$–Virasoro. Though we use the term “$q$-discretization”, its full meaning has not been clarified. At the present stage, we can only say that the $q$-discretization means the discretization of Riemann surfaces on which fields are defined.

On the other hand, our scheme is certainly “quantization” at the algebraic level because it is originated from the Moyal bracket algebra which was quantum mechanical analog of the classical Poisson algebra. From this point of view, the motivation of the $q$-discretization is similar to the one of $q$-deformation [13]. But $D$–Virasoro is not a nontrivial quantum group because it contains the trivial Hopf algebra even at $q \neq 1$. (In ref. [4, 7] it is shown that their algebras have the non-trivial Hopf structure. Then we expect $D$–Virasoro also has such one.) As the $q$-deformation of Virasoro has not been discovered yet, we need make clear the relation between these two different schemes of “quantizing” algebras. $D$–Virasoro may give some informations for the study of it. The break through seems to be in the affine Kac-Moody algebra, too. The research of $U_q(\hat{sl}_2)$ has been much progressed as mentioned above [1, 4]. If we get its $q$-discretized form, we must be able to show the connection between the $q$-discretization and the $q$-deformation.

Following the introduction of $D$–Virasoro, we represented it by the use of the free fields. It is nothing but a $q$-discretized Sugawara construction. There it was shown that the bosonic and fermionic representations could be understood in a unified way. From this fact we are led to the concept of supersymmetric algebras. If we consider the supersymmetric Riemann surfaces, two different types of $D$–Virasoro might be represented by a superalgebra [4]. Moreover, use of such representation enables us to discuss deformation of integrable field theories. At present it is uncertain whether such deformation preserves integrability. But there are some indications...
which seem to support the integrability of the $q$-discretized models. For example, in string theory, a braiding of a kind of deformed strings is shown to yield a Yang-Baxter equation \[5, 14\]. This problem is also connected to the representation theory of $D$–Virasoro. It is expected that the investigation of the representation theory reveals the physical meanings of $q$-discretization.

We would like to thank Chikara Itoh and Katsumi Shigura for useful calculation.
References

[1] See, for example, Infinite Analysis, eds. A.Tsuchiya, T.Eguchi and M.Jimbo (World Scientific, 1992) ; Differential Geometric Methods in Theoretical Physics, eds. S.Catto and A.Rocha (World Scientific, 1992)

[2] V.G.Drinfel’d, Sov.Math.Dokl. 36 (1988) 212 ;
V.Chari and A.Pressley, Commun.Math.Phys. 142 (1991) 261 ;
I.B.Frenkel and N.Yu.Reshetikhin, Commun.Math.Phys. 146 (1992) 1 ;
B.Davies, O.Foda, M.Jimbo, T.Miwa and A.Nakayashiki, Commun.Math.Phys. 151 (1993) 89

[3] T.L.Curtright and C.K.Zachos, Phys.Lett. B243 (1990) 237 ;
A.P.Polychronakos, Phys.Lett. B256 (1991) 35 ;
N.Aizawa and H.Sato, Phys.Lett. B256 (1991) 185 ;
Ch.Devchand and M.V.Saveliev, Phys.Lett. B258 (1991) 364 ;
M.Chaichian, A.P.Isaev, J.Lukierski, Z.Popowicz and P.Prešnajdor, Phys.Lett. B262 (1991) 32

[4] H.Hiro-oka, O.Matsui, T.Naito and S.Saito, TMUP-HEL-9004 (1990), unpublished

[5] S.Saito, Integrability of Strings, in: Nonlinear Fields: Classical, Random, Semiclassical, eds. P.Garbaczewski and Z.Popowicz (World Scientific,1991) p.286 ; q-Virasoro and q-Strings, in: Quarks, Symmetries and Strings, eds. M.Kaku, A.Jevicki and K.Kikkawa (World Scientific, 1991) p.231

[6] H.Sato, Hiroshima Univ. preprint HUPD-9201 (1992)

[7] M.Chaichian and P.Prešnajdor, Phys.Lett. B277 (1992) 109

[8] D.B.Fairlie and C.K.Zachos, Phys.Lett. B224 (1989) 101 ; The authors would like to thank A.Polychronakos who pointed out the connection between (3) and the Moyal algebra.

[9] J.E.Moyal, Proc.Cambridge.Phil.Soc. 45 (1949) 99
[10] C.N.Pope, L.J.Romans and X.Shen, Phys.Lett. B236 (1990) 173 ; Nucl.Phys. B339 (1990) 191

[11] D.B.Fairlie and J.Nuyts, Commun.Math.Phys. 134 (1990) 413

[12] I.Bakas, Commun.Math.Phys. 134 (1990) 487

[13] V.G.Drinfel’d, Quantum Groups, in: ICM proceedings (Berkley, 1986) p.798

[14] H.Câteau and S.Saito, Phys.Rev.Lett. 65 (1990) 2487