HOMOLOGICAL SUBSETS OF Spec

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Abstract. We investigate homological subsets of the prime spectrum of a ring, defined by the help of the Ext-family \(\{\text{Ext}^g_R(\cdot, R)\}\). We extend Grothendieck’s calculation of \(\dim(\text{Ext}^g_R(M, R))\). We compute support of \(\text{Ext}^g_R(M, R)\) in many cases. Also, we answer a low-dimensional case of a problem posed by Vasconcelos on the finiteness of associated prime ideals of \(\{\text{Ext}^g_R(M, R)\}\). An application is given.

1. Introduction

Throughout the paper \(R\) is a commutative noetherian local ring of dimension \(d\), and \(M\) a finitely generated module of grade \(g\) and dimension \(r\), otherwise specializes. This paper deals with the invariants attached to the Ext-modules. Associated prime ideals of \(\text{Hom}_R(\cdot, \sim)\) computed several years ago. As far as we know, the first computation of \(\text{Ass}(\text{Ext}^{\infty}_R(F, G))\) appeared in the (LC) by Grothendieck. Here, (LC) referred to local cohomology [12] (also, see [13]).

Problem 1.1. (Vasconcelos) Is \(\text{E-Ass}_R(M)\) finite?

Denote the homological support by \(\bigcup \text{Supp}(\text{Ext}^i_R(\cdot, R))\). In Section 2 we show that the homological support have strange properties compared to the classical support. Despite of this, we show over finitely generated modules, homological support is the classical support. This drops \(p \cdot \dim(M) < \infty\) from a result of Peskine and Szpiro, see [17]. As an application, we extend an implicit result of Grothendieck [12, 6.4.4] by avoiding scheme-theory and (LC) (see Corollary 2.9):

Observation 1.2. If \(\dim(\text{Ext}^{d-i}_R(M, R)) \leq i\) for all \(i\) and \(g = d - r\), then \(\dim(\text{Ext}^{d-r}_R(M, R)) = r\).

In the light of (LC) and in Proposition 2.12 we observe:

Observation 1.3. The formula \(\dim(\text{Ext}^g_R(M, R)) = \dim M\) holds for all \(M\) if and only if \(R\) is Cohen-Macaulay.

In general, \(M\) is not supported in the support of \(\text{Ext}^g_R(M, R)\), even over regular rings, see Example 2.10 (i). However, we give situations for which \(M\) is supported in \(\text{Supp}(\text{Ext}^g_R(M, R))\), see Corollary 2.7, 2.8, and Example 2.10. If we focus on modules of finite projective dimension over formally equidimensional rings, the game is changed: the dimension formula holds for all of such modules, see also [7, Proposition 2.2] by Beder.

Suppose \(R\) is a homomorphic image of a Gorenstein ring \(S\). Recall from [23] that the homological associated prime ideals of \(M\) are defined by the set \(\text{h-Ass}_R(M) := \bigcup \text{Ass}_R(\text{Ext}^i_R(M, S))\). Trivially, this is a finite set and coincides with the former \(\text{E-Ass}_R(M)\) in the Gorenstein case. Following (LC), we observe in Section 3 that \(\text{Ass}(M) \subseteq \text{h-Ass}(M)\). Set \(M_{(i_1, \ldots, i_p)} := \text{Ext}^{d-i_p}_S(\ldots(\text{Ext}^{d-i_1}_S(M, S), \ldots, S), S)\).

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*All the results that we need from [12] are also in [13]. Concerning this we only cite to the english lecture note.
This comes from Vasconcelos’ investigation of the notions of homological degree and well-hidden associated prime ideals. We continue Section 3 to understand $\bigcup_{|i_1,\ldots,i_p\rangle} \text{Ass}(M_{(i_1,\ldots,i_p)}p)$. We determine it in the diagonal case with $p > 2$: $\bigcup_{p=0}^{\infty} \text{Ass}(M_{(i_1,\ldots,i_p)p}) = \bigcup_{p=0}^{\infty} \text{Ass}(M_{(i_1,\ldots,i_p)p})$, see Corollary 3.12. Also, we determine it in the Cohen-Macaulay case (see Corollary 3.10).

Section 4 investigates different notions of homological annihilators. We connect them to the classical annihilator. The first one is the Bridger’s ideal $\gamma(-) := \bigcap_{i>0} \text{rad}(\text{Ann}_R \text{Ext}^1_R(-, R))$. The second one is the Auslander-Buchsbaum invariant factor $\alpha(-)$ [3]. The third one is $h\text{-Ann}(-) := \prod_{i=0}^{d} \text{Ann}_R \text{Ext}^i_R(-, R)$. By using homological support, we remark that $\gamma(-)$ is support sensitive in the category of finitely generated modules of positive grade. We derive a similar result for $\alpha(-)$. Motivated from Auslander’s comments on the functor Ext, we collect several remarks on the annihilators of Ext-modules.

Section 5 deals with Problem 1.1. We first reduce it to the class of cyclic modules. We show a little more, please see Lemma 5.5. Then, we show

Observation 1.4. i) Problem 1.1 is true over 3-dimensional excellent normal local domains.

ii) Problem 1.1 is true over two dimensional reduced excellent local rings.

The final section motivated from a result of Macaulay at 1904. First, we recall a more general version of this by Serre, please see Theorem 6.1 and 6.2. These results presented in the Ext-from by Griffith and Evans: Let $R$ be a regular local ring containing a field and $I$ be a height two prime ideal such that $\text{Ext}^i_R(R/I, R)$ is cyclic. Then $I$ is two generated.

Observation 1.5. Let $(R, m)$ be a Cohen-Macaulay local ring and $I$ be a Cohen-Macaulay ideal of height two and of finite projective dimension. Then $\mu(\text{Ext}^i_R(R/I, R)) = \mu(I) - 1$.

### 2. Homological Interpretation of Support

By $\text{p.dim}_R(-(\text{resp. id}_R(-)))$ we mean projective (resp. injective) dimension. The notation $E\text{-Supp}(-)$ stands for $\bigcup_{i \geq 0} \text{Supp}(\text{Ext}^i_R(-, R))$. This may be empty for modules with quite large support:

Example 2.1. Let $R$ be a complete local integral domain of positive dimension. Let $F$ be the fraction field of $R$. It is shown by Auslander [11] Page 166] that $\text{Ext}^i_R(F, R) = 0$ for all $i$. So, $E\text{-Supp}(F) = \emptyset \neq \text{Supp}(F) = \text{Spec}(R)$.

Example 2.2. The complete-local assumption is important. Note that $\text{Ext}^2_Z(Q, \mathbb{Z})$ is related to the adèlle groups from number theory. By accepting continuum hypothesis, one has $\text{Ext}^2_Z(Q, \mathbb{Z}) = R$ as a vector space over $Q$. So, $E\text{-Supp}_Z(Q) = \text{Supp}_Z(Q)$.

Interestingly, may be homological support is quite large against to the classical support:

Example 2.3. Adopt the notation of Example 2.1. It is shown in [11] Page 166] that $R \simeq \text{Ext}^1_R(F/R, R)$. So, $E\text{-Supp}(F/R) = \text{Spec}(R) \supsetneq \text{Spec}(R) \setminus \{(0)\} = \text{Supp}(F/R)$.

Lemma 2.4. Let $L$ and $N$ be finitely generated and nonzero. Then $\text{Ext}^i_R(L, N) \neq 0$ for some $i \leq \text{dim} N$.

Proof. For each ideal $I$ recall that $\text{ht}_N(I)$ defines by $\inf\{\text{dim}(N_p) : p \in \text{Supp}(N) \cap V(I)\}$, where $V(I)$ is the set of all prime ideals containing $I$. Since $R$ is local, we have $L \otimes N \neq 0$. Consequently, $\text{Supp}(L) \cap \text{Supp}(N) \neq \emptyset$. Repell this as $\text{Supp}(N) \cap V(\text{Ann} L) \neq \emptyset$. Deduce from this that $\text{ht}_N(\text{Ann} L) < \infty$. It sufficient to recall that $\inf\{i : \text{Ext}^i_R(L, N) \neq 0\} = \text{grade}(\text{Ann} L, N) \leq \text{ht}_N(\text{Ann} L) < \infty$. 


Corollary 2.5. Keep the above notation in mind. Then $\text{Supp}(L \otimes N) \subset \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}^i_R(L, N))$.

Proof. Let $p \in \text{Supp}(L \otimes N)$. Then $L_p$ and $N_p$ are nonzero. In view of Lemma 2.4, there is an $i \leq \dim N_p \leq \dim N$ such that $\text{Ext}^i_{R_p}(L_p, N_p) \neq 0$. Note that $\text{Ext}^i_R(L, N)_p \simeq \text{Ext}^i_{R_p}(L_p, N_p)$, because of the finiteness of $L$ and $N$. So, $p \in \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}^i_R(L, N))$.

Proposition 2.6. Let $M$ and $N$ be finitely generated and nonzero. Then

$$\text{Supp}(M \otimes N) = \bigcup_{i=0}^{\infty} \text{Supp}(\text{Ext}^i_R(M, N)) = \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}^i_R(M, N)).$$

Proof. We bring the following trivial facts

1) $\text{Supp}(M \otimes N) = \text{Supp}(M) \cap \text{Supp}(N)$,
2) $\text{Supp}(\text{Ext}^i_R(M, N)) \subset \text{Supp}(M) \cap \text{Supp}(N)$, and
3) $\bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}^i_R(M, N)) \subset \bigcup_i \text{Supp}(\text{Ext}^i_R(M, N))$.

We look at

$$\text{Supp}(M \otimes N) \supset \text{Supp}(M) \cap \text{Supp}(N) \supset \bigcup_i \text{Supp}(\text{Ext}^i_R(M, N)) \supset \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}^i_R(M, N)) \supset \text{Supp}(M \otimes N).$$

The proof is now complete.

$M$ is called quasi-perfect if $\inf\{i : \text{Ext}^i_R(M, R) \neq 0\} = \sup\{i : \text{Ext}^i_R(M, R) \neq 0\}$.

Corollary 2.7. Let $M$ be quasi-perfect of grade $g$. Then $\text{Supp}(M) = \text{Supp}(\text{Ext}^g_R(M, R))$.

Corollary 2.8. Let $(R, m)$ be a $d$-dimensional Cohen-Macaulay local ring, Gorenstein on its punctured spectrum, and $M$ be an one-dimensional $R$-module. Then $\text{Ext}^{d-1}_R(M, R)$ has same support as of $M$. In particular, $\dim(\text{Ext}^{d-1}_R(M, R)) = 1$.

Proof. We have $\text{id}(R_p) = \text{ht}(p)$ for all non-maximal prime ideal $p$. This yields that $\text{Supp}(\text{Ext}^i_R(M, R)) \subset \{m\}$. We are going to apply the grade conjecture. This is the place that we use the Cohen-Macaulay assumption, because the conjecture verified over such a ring without any stress on the finiteness of projective dimension, see [17]. Thus,

$$\text{grade}(M) = \dim R - \dim M = d - 1,$$

e.g. $\text{Ext}^{d-1}_R(M, R) \neq 0$. In particular, $m$ belongs to its support. In view of Proposition 2.6, we have

$$\text{Supp}(M) = \text{Supp}(\text{Ext}^{d-1}_R(M, R)) \cup \text{Supp}(\text{Ext}^d_R(M, R)) = \text{Supp}(\text{Ext}^{d-1}_R(M, R)) \cup \{m\} = \text{Supp}(\text{Ext}^{d-1}_R(M, R)).$$

This is what we want to prove.

Corollary 2.9. Let $(R, m)$ be a $d$-dimensional local ring and $M$ be $r$-dimensional. If $\dim(\text{Ext}^{d-i}_R(M, R)) \leq i$ for all $i$ and that $g = d - r$, then $\dim(\text{Ext}^g_R(M, R)) = r$. 

The first condition holds when \( p \cdot \dim(M) < \infty \) or \( R \) is Gorenstein. The second condition holds either \( R \) is Cohen-Macaulay or \( p \cdot \dim(M) < \infty \) and \( R \) is formally equidimensional.

**Proof.** Set \( b_i := \text{Ann}(\text{Ext}_R^i(M, R)) \). In view of Proposition 2.6, \( \text{Supp}(M) = V(b_2) \cup \cdots \cup V(b_\ell) \). Let \( p_0 \subset \cdots \subset p_r = m \) be a maximal and strict chain of prime ideals in the \( \text{Supp}(M) \). We claim that \( p_0 \in V(b_\ell) \).

Suppose on the contradiction that \( p_0 \notin V(b_\ell) \). Hence \( p_0 \in V(b_\ell) \) for some \( \ell > g \), i.e., \( p_0 \supset b_\ell \). Clearly, \( p_j \in V(b_j) \) for all \( j \). By definition of \( \dim \), we have \( \dim(R/b_\ell) \geq r = d - g \). Due to our assumption, \( V(b_\ell) \) is of dimension at most \( d - \ell \). Combining these, we observe \( d - g > d - \ell \geq \dim(R/b_\ell) \geq d - g \), which is a contradiction. Thus, \( p_0 \in V(b_\ell) \). Therefore, \( r = \dim(M) \geq \dim(\text{Ext}_R^i(M, R)) \geq r \). So, \( \dim(\text{Ext}_R^i(M, R)) = r \).

\( \square \)

**Example 2.10.** i) Let \( R := \mathbb{Q}[x, y, z] \) and let \( I := (xy, xz) \). Then \( \text{Supp}(R/I) \neq \text{Supp}(\text{Ext}_R^i(R/I, R)) \).

ii) Let \( R \) be Gorenstein and \( L := \text{Ext}_R^i(\text{Ext}_R^i(M, R), R) \) for each \( i \). Then \( \text{Supp}(\text{Ext}_R^i(L, R)) = \text{Supp}(L) \).

iii) Let \( R \) be Gorenstein and \( M \) a Cohen-Macaulay module of grade \( g \). Then \( \text{Supp}(\text{Ext}_R^i(M, R)) = \text{Supp}(M) \).

**Proof.** i) Set \( M := R/I \). We look at the minimal free resolution of \( M \):

\[
0 \longrightarrow R \longrightarrow \mathllap{-z} \begin{bmatrix} y \\ xz \end{bmatrix} \longrightarrow R^2 \longrightarrow \mathllap{xy} \begin{bmatrix} z \\ xz \end{bmatrix} \longrightarrow R \longrightarrow 0.
\]

Apply \( \text{Hom}_R(-, R) \) to it, we get to the following complex

\[
R \mathllap{-z} \begin{bmatrix} y \\ xz \end{bmatrix} \longrightarrow R^2 \longrightarrow \mathllap{xy} \begin{bmatrix} z \\ xz \end{bmatrix} \longrightarrow R \longrightarrow 0.
\]

Note that \( \text{Ass}(\text{Hom}_R(M, R)) = \text{Supp}(M) \cap \text{Ass}(R) = \emptyset \). Let us compute the \( \text{Ext}_R^i(M, R) \). As \( y, z \) is a regular sequence in \( R \), the Koszul complex on \( \{y, z\} \) presents the free resolution of \( (y, z) \). Thus, the kernel of \( (-z, y) : R^2 \longrightarrow R \) is generated by \( (y, z) \) and so \( \text{Ext}_R^1(M, R) \simeq (y, z)R/(xy, xz)R \). This yields that \( \text{Ann}(\text{Ext}_R^1(M, R)) = (x) \). Therefore, \( g := \text{grade}(M) = 1 \) and \( (y, z) \in \text{Supp}(M) \setminus V(x) \). We deduce from this that \( \text{Supp}(\text{Ext}_R^i(M, R)) = V(x) \subset \text{Supp}(M) \).

ii) This follows by [3, 7.60], where Bridger has shown the following amusing result:

\[
L = \text{Ext}_R^i(\text{Ext}_R^i(M, R), R) \simeq \text{Ext}_R^i(\text{Ext}_R^i(\text{Ext}_R^i(M, R), R), R) = \text{Ext}_R^i(\text{Ext}_R^i(L, R), R).
\]

iii) This follows from the Ext-duality [6, 3.3.10]: \( M \simeq \text{Ext}_R^i(\text{Ext}_R^i(M, R), R) \).

\( \square \)

**Discussion 2.11.** (Grothendieck [12, 6.4.4]) Let \( (R, m) \) be a \( d \)-dimensional Cohen-Macaulay local ring with a canonical module and \( M \) a finitely generated module. Then \( \dim(\text{Ext}_R^{d-\dim M}(M, \omega_R)) = \dim M \).

**Proposition 2.12.** Let \( (R, m) \) be a local ring and \( M \) a finitely generated module of grade \( g \). Then \( \dim(\text{Ext}_R^{g}(M, R)) = \dim M \) for all \( M \) if and only if \( R \) is Cohen-Macaulay.

**Proof.** Suppose first that \( R \) is not Cohen-Macaulay. We show the dimension formula does not hold for the residue field. Note that \( g := \text{grade}(R/m) = \text{depth}(R) \). So \( \dim(\text{Ext}_R^{g}(R/m, R)) = 0 \neq d - g \).

Suppose now that \( R \) is Cohen-Macaulay. In order to prove the formula without loss of generality, we assume that \( R \) is complete. Then \( R \) has a canonical module [6, Theorem 3.3.6]. By Discussion 2.11 \( \dim(\text{Ext}_R^{d-\dim M}(M, \omega_R)) = \dim M \). In view of the grade conjecture over Cohen-Macaulay rings and without any stress about finiteness of projective dimension we have \( g = d - \dim M \). Let \( \{x_1, \ldots, x_g\} \) be a maximal \( R \)-sequence in \( \text{Ann}(M) \). Put \( \overline{R} := R/\mathfrak{m}R \) and note that \( M \) can view as an \( \overline{R} \)-module. As
\(\omega_R\) is maximal Cohen-Macaulay, \(x\) is a \(\omega_R\)-sequence in \(\text{Ann}(M)\). Due to the Rees lemma \[16\] Page 140], there is the isomorphism

\[
\text{Ext}_R^d(M, \omega_R) \cong \text{Ext}_R^d(M, \omega_R/\omega_R) \cong \text{Ext}_R^0(M, \omega_R).
\]
The last isomorphism is in \[6\] Theorem 3.3.5], e.g., \(\omega_R \cong \omega_R/\omega_R\). In a similar vein there is an isomorphism \(\text{Ext}_R^d(M, R) \cong \text{Ext}_R^0(M, R)\).

Claim: One has \(\dim(\text{Hom}_R^\omega(M, R)) = \dim(\text{Hom}_R^\omega(M, \omega_R))\). To this end we show they have the same associated prime ideals. As \(R\) and \(\omega_R\) are Cohen-Macaulay over \(R\), their associated prime ideals are the minimal primes of their support. On the other hand \(R\) and \(\omega_R\) have a same set as the support, because \((\omega_R)_p = \omega_{R_p}\), see \[6\] Theorem 3.3.5]. Thus, \(\text{Ass}_R(R) = \text{Ass}_R(\omega_R)\). In view of \[6\] Ex. 1.2.27],

\[
\text{Ass}_R(\text{Hom}_R^\omega(M, R)) = \text{Supp}_R(M) \cap \text{Ass}_R(R) = \text{Supp}_R(M) \cap \text{Ass}_R(\omega_R) = \text{Ass}_R(\text{Hom}_R^\omega(M, \omega_R)).
\]

Combining these we get

\[
\dim M = \dim(\text{Ext}_R^\omega(M, \omega_R)) = \dim(\text{Hom}_R^\omega(M, \omega_R)) = \dim(\text{Hom}_R^\omega(M, R)) = \dim(\text{Ext}_R^0(M, R)).
\]

\[\square\]

3. HOMOLOGICAL ASSOCIATED PRIMES

**Example 3.1.** Let \(R\) be either \(Q[x, y, z]\) or \(Q[[x, y, z]]\) and let \(M := R/(xy, xz)\). Then

\[
\bigcup \text{Min}(\text{Ext}_R^i(M, R)) = \text{Ass}_R(M) = \{(x), (y), (z)\}.
\]

**Proof.** In view of Example 2.10 we have:

\[
\text{Ext}_R^i(M, R) \cong \begin{cases} 
\frac{(-y, z)}{(xy, xz)} & \text{if } i = 1 \\
\frac{g}{(y, z)} & \text{if } i = 2 \\
0 & \text{otherwise}
\end{cases}
\]

Thus, \(\bigcup \text{Min}(\text{Ext}_R^i(M, R)) = \text{Ass}(M)\).

\[\square\]

**Discussion 3.2.** (Grothendieck) Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \(d\) with a canonical module. Let \(M\) be a finitely generated module of dimension \(n\) and depth \(t\). The following holds.

i) \(\text{Ext}_R^i(M, \omega_R) = 0\) for \(i \notin [d - n, d - t]\). Also, \(\text{Ext}_R^i(M, \omega_R) \neq 0\) for \(i = d - n\) and \(i = d - t\).

ii) \(\dim(\text{Ext}_R^i(M, \omega_R)) \leq d - i\).

The proof of Discussion 3.2(ii) implies the following more general fact:

**Proposition 3.3.** Let \(M\) and \(N\) be finitely generated modules such that either \(p, \dim(M) < \infty\) or \(\text{id}(N) < \infty\) over any commutative noetherian ring. Let \(i \leq d\). Then \(\dim(\text{Ext}_R^i(M, N)) \leq d - i\).

**Proof.** Without loss of generality we assume that \(d := \dim R\) is finite. Suppose \(p\) is of coheight \(> d - i\). Suppose first that \(\text{id}(N) < \infty\). Due to a theorem of Bass \[6\] Theorem 3.1.17], we observe that

\[
\text{id}(N_p) \leq \text{depth}(R_p) \leq \dim R_p = \text{ht}(p) \leq d - \dim R/p < d - (d - i) = i.
\]

Thus \(\text{Ext}_R^i_p(M_p, N_p) = 0\). Suppose now that \(p, \dim(M) < \infty\). By Auslander-Buchsbaum formula and in a similar way as above, \(p, \dim(M_p) < i\). Consequently, \(\text{Ext}_R^i_p(M_p, N_p) = 0\). We show in each cases that \(\text{Supp}((\text{Ext}_R^i(M, N))\) is of dimension less or equal than \(d - i\). This proves the desired fact.

\[\square\]
Example 3.4. i) The finitely generated assumption is really needed. In view of Example 2.2, Ext^2_R(Q, Z) is a nonzero rational vector space. So, dim(Ext^2_R(Q, Z)) = 1 > 0 = d - 1.

ii) The finiteness of homological dimensions is important. Look at R := Q[[X, Y]]/(X^2). It is easy to see Ext^i_R(R/xR, R/xR) ∼ R/xR for all i. So, dim(Ext^i_R(R/xR, R/xR)) = 1 > 0 = d - 1.

In the above example Ext^i_R(R/(x), R) = 0 for all i > 0. One may search the validity of dim(Ext^i_R(−, R)) ≤ d − i. In general, this is not the case as the next example says.

Example 3.5. Let R := Q[X, Y, Z]/(X^2, XY, XZ). This is a 2-dimensional ring and min(R) = {(x)}. Due to the formula rad(y, z) = (x, y, z), one can show that {y, z} is a system of parameters. Since {y, z} is not a regular sequence, R is not Cohen-Macaulay. Set M := R/xR. Note that neither p. dim(M) < ∞ nor id(R) < ∞. We are going to show dim(Ext^2_R(M, R)) = 2 > 0 = d - 2. We restate the 3 relations x^2 = xy = xz = 0 of {x, y, z} by

\[
\begin{pmatrix}
  x & 0 & 0 \\
  0 & x & 0 \\
  0 & 0 & x
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

We restate the Koszul relations of {x, y, z} by

\[
\begin{pmatrix}
  -y & -z & 0 \\
  x & 0 & -z \\
  0 & x & y
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

Then we put all of these relations of {x, y, z} to the following package

\[A := \begin{pmatrix}
  x & 0 & 0 & -y & -z & 0 \\
  0 & x & 0 & x & 0 & -z \\
  0 & 0 & x & 0 & x & y
\end{pmatrix}.\]

Look at the free resolution of M:

\[\ldots \rightarrow R^6 \xrightarrow{A} R^3 \xrightarrow{[x \ y \ z]} R \xrightarrow{x} R \xrightarrow{\cdot} M \xrightarrow{\cdot} 0.\]

Delete M from the right and apply Hom_R(−, R) we get to the following complex

\[0 \rightarrow R \xrightarrow{x} R \xrightarrow{[x \ y \ z]} R^3 \xrightarrow{A'} R^6.\]

Let \(a, b, c\) be such that \([a, b, c]A = 0\). It is solution of the following system of six equations

\[
\begin{align*}
  ax &= bx = cx = 0 \\
  -ay + bx &= 0 \\
  -az + cx &= 0 \\
  -bz + cy &= 0
\end{align*}
\]

It is easy to see that \(\{(x, 0, 0), (0, x, 0), (0, 0, x), (0, y, z)\}\) are the solutions. Hence

\[
\text{Ext}^2_R(M, R) = \frac{\ker(A')}{\text{im}([x \ y \ z])} \cong \frac{\langle (x, 0, 0), (0, x, 0), (0, 0, x), (0, y, z) \rangle}{(x, y, z) R}.
\]

Thus,

\[
(x) \subseteq \text{Ann}(\text{Ext}^2_R(R/(x), R)) \subseteq \text{Ann}(\langle (x, 0, 0), (0, x, 0), (0, 0, x), (0, y, z) \rangle/(x, y, z) R) = (x).
\]

We observe that dim(Ext^2_R(M, R)) = dim R/(x) = 2.
Proposition 3.6. Let \((R, m)\) be a local ring which is a homomorphic image of a Gorenstein ring \((S, n)\) of dimension \(d\). Let \(M\) be a finitely generated \(R\)-module of dimension \(n\). Then \(\text{Ass}_R(M) \subseteq \bigcup_{i=d-n}^n \text{Min}_R(\text{Ext}_S^i(M, S))\). The equality holds if \(M\) is Cohen-Macaulay.

Proof. We use an idea taking from Grothendieck [12] Proposition 6.6. By \((-)\) we mean the Matlis dual functor. We view \(M\) as an \(S\)-module via the map \(S \to R\). In the light of the local duality theorem, we have \(H_m^i(M)^\vee \cong \text{Ext}_S^{d-i}(M, S)\). Also, \(H_m^i(M) \cong H_n^i(M)\) as an \(R\)-module. Since \(H_m^i(M)^\vee \cong H_m^i(M)\) we observe that

\[
a_i(M) := \text{Ann}_R(H_m^i(M)) = \text{Ann}_R(H_n^i(M)) = \text{Ann}_R(\text{Ext}_S^{d-i}(M, S)).
\]

In view of Discussion 3.2(ii), \(\dim \text{Ext}_S^{d-i}(M, S) \leq i\). Let \(p \in \text{Ass}_R(M)\) of dimension \(i\) and let \(q\) be its image in \(S\). Then \(\dim(S_q) = d - i\). By Discussion 3.2(i)

\[
p \in \text{Ass}(M) \iff \text{depth}(M_p) = 0
\]

\[
\iff \text{depth}(M_q) = 0
\]

\[
\iff \text{Ext}_S^{d-i}(M_q, S_q) \neq 0
\]

\[
\iff \text{Ext}_S^{d-i}(M, S)_p \neq 0
\]

\[
\iff p \in V(a_i(M)).
\]

We remark that \((\ast)\) is “if and only if” when \(M\) is Cohen-Macaulay. As minimal elements of support are the associated primes, we get the claim by recalling again that \(\dim(\text{Ext}_S^{d-i}(M, S)) \leq i\).

Example 3.7. i) The finitely generated assumption is important: Let \(R\) be a complete regular local ring of positive dimension with the fraction field \(F\). Clearly, \(\text{Ass}(F) \neq \emptyset\). Therefore, \(\text{Ass}(F) \not\subseteq h\text{-Ass}(F) = \emptyset\).

ii) The Cohen-Macaulayness is important: We revisit Example 3.1. We take \(R = S\). Note that \(\emptyset \neq \text{Ass}(I) \subseteq \text{Ass}(R) = \{0\}\), i.e., \(\text{Ass}(I) = \{0\}\). In view of Example 2.10 we observe that

\[
(y, z) \in \left(\bigcup_{i=d-n}^n \text{Min}_R(\text{Ext}_S^i(I, S))\right) \setminus \text{Ass}_R(I).
\]

Corollary 3.8. Let \(M\) be finitely generated. Then \(\text{Ass}_M(M) \subseteq h\text{-Ass}_M(M)\).

Recall from [24] the hidden associated prime ideals of \(M\) is \(\text{Hidd}_R(M) := h\text{-Ass}_R(M) \setminus \text{Ass}_R(M)\). Now, we recover a result of Foxby [10] Proposition 3.4b] via two different arguments:

Corollary 3.9. Let \(R\) be a Gorenstein local ring and \(M\) a Cohen-Macaulay module. Then \(\text{Hidd}(M) = \emptyset\).

First proof. As \(M\) is Cohen-Macaulay, \(\text{Ext}_R^i(M, R)\) is either zero or Cohen-Macaulay. This may be respell by \(\text{Min}(\text{Ext}_R^i(M, R)) = \text{Ass}_R(\text{Ext}_R^i(M, R))\). Proposition 3.6 yields the claim.

Second proof. Apply Corollary 2.9.

We drop the later \(p\) from \((i_1, \ldots, i_p)\) if there is no danger of confusion.

Corollary 3.10. Let \((R, m)\) be a Gorenstein and \(M\) Cohen-Macaulay. Then \(\bigcup_p \bigcup_{(i_1, \ldots, i_p)} \text{Ass}(M_{(i_1, \ldots, i_p)}) = \text{Ass}(M)\).

Example 3.11. We revisit Example 2.10. Let \(S := \mathbb{Q}[x, y, z]\) and let \(M := S/(xy, xz)\). Let \(p \geq 1\). Then

\[
M_{(i_1, \ldots, i_{p-1}, 2)} := \begin{cases} 
  S/(y, z) & \text{if } i_1 = \ldots = i_{p-1} = 2 \\
  0 & \text{otherwise}
\end{cases}
\]
Proof. In view of Example 2.10, \(\text{Ext}_R^3(M, S) \simeq \frac{S}{(y, z)}\). Since \(y, z\) is a regular sequence on \(S\), from its Koszul complex we obtain that \(\text{Ext}_R^3(\text{Ext}_R^3(M, S), S) \simeq \frac{S}{(y, z)}\delta_{j,i}\), where \(\delta\) is the Kronecker delta. From this we get the claim. \(\square\)

Theorem 4.3. Let \(R\) be a Gorenstein local ring, \(M\) a finitely generated module and \(i\) be any integer. Then for \(n > 2\) we have \(\bigcup_{p=0}^n \text{Ass}(M(i_{i,...,i}p)) = \bigcup_{p=0}^n \text{Ass}(M(i_{i,...,i}p))\).

Proof. We revisit Bridger’s amusing formula \(M_{i_{i,...,i}} \simeq M_{i_{i,...,i}}\). Apply \(\text{Ext}_R^d(-, R)\) to this, \(M_{i_{i,...,i}} \simeq M_{i_{i,...,i}}\). Similarly, we detect \(M_{i_{i,...,i}}\) from \(M_{i_{i,...,i}}\) for some \(q < p\). The induction completes the proof. \(\square\)

4. HOMOLOGICAL ANNihilators

In the Introduction we attached three types of ideals to a module, namely \(\gamma(-), h-\text{Ann}(-)\) and \(a(-)\). Let us present the definition of \(a(M) := \bigcap_{\alpha(\text{Ann}(\Lambda'(M))) \neq 0} \text{Ann}(-, M)\). Also, \(\beta(M) := \{x \in R : M_x \text{ is}\ R_x\text{-projective}\}\) introduced by Auslander and Buchsbaum. First, we compute things via an example.

Example 4.1. Let \(R := \mathbb{Q}[x, y, z]\) and set \(M := R/(xy, xz)\). One has

\[\gamma(M) = \text{Ann}(M) = h-\text{Ann}(M) = \beta(M) = a(M).\]

Also, \(\gamma(xy, xz) \neq \text{Ann}(xy, xz)\).

Proof. In the case of modules of finite projective dimension, the claims \(\gamma(M) = \beta(M) = a(M)\) is in \([5]\). In view of Example 2.10, \(\gamma(M) = \text{Ann}(M) = h-\text{Ann}(M)\). Again, Example 2.10 implies that \(\gamma(xy, xz) \neq \text{Ann}(xy, xz)\). \(\square\)

Proposition 4.2. Let \(M\) be a finitely generated module and of positive grade. Then \(\gamma(M) = \text{rad}(\text{Ann}(M))\).

Proof. The grade condition says that \(\text{Ext}_R^d(M, R) = 0(\dagger)\). Set \(c_i(M) = \text{rad}(\text{Ann}(\text{Ext}_R^d(M, R)))\). In view of Proposition 2.6, we conclude that

\[\text{V}(\text{Ann}(M)) = \text{Supp}(M) \implies (\dagger)\]

\[= \bigcup_{i=1}^d \text{Supp}(\text{Ext}_R^d(M, R)) = \bigcup_{i=1}^d \text{V}(c_i(M)) = \text{V}(\bigcap_{i=1}^d c_i(M)).\]

Since \(\text{Ann}(M) \subset \bigcap_{i=1}^d c_i(M)\), we get from the displayed item that \(\bigcap_{i=1}^d c_i(M) = \text{rad}(\text{Ann}(M))\). Again we look at \(\text{rad}((\text{Ann}(M))) \subset c_i(M)\). Immediately, we deduce that

\[\text{rad}(\text{Ann}(M)) \subset \bigcap_{i=1}^d c_i(M) \subset \bigcap_{i=1}^d c_i(M) = \text{rad}(\text{Ann}(M)).\]

Consequently, \(\gamma(M) = \bigcap_{i=1}^\infty c_i(M) = \text{rad}(\text{Ann}(M)).\) \(\square\)

Example 4.3. The finitely generated assumption is important: Adopt the notation of Example 2.1. Note that \(\text{Hom}_R(F, R) = 0\). Thus \(F\) is of positive grade and \(\gamma(F) = R\). It remains to note that \(\text{Ann}(F) = 0\).

Corollary 4.4. Let \(M\) and \(N\) be modules of positive grade with the same support. Then \(\gamma(M) = \gamma(N)\).

Corollary 4.5. Let \(M\) be a finitely generated module of finite projective dimension. Then \(\gamma(M) = \text{rad}(\text{Ann}(M))\) if and only if \(M\) is of positive grade.
Proof. One direction is in Proposition 3.2 without any assumption on the projective dimension. To see
the other side implication, we may assume that \( \gamma(M) \neq \text{rad}(\text{Ann}(M)) \). Since \( \text{rad}(\text{Ann}(M)) \subsetneq \gamma(M) \),
there is an \( M \)-regular element in \( \gamma(M) \). In view of [4] Theorem 7.57, \( M \) is a 1-syzygy module. It turns
out that \( M \) is subset of a free module. So, \( \text{Hom}_R(M, R) \neq 0 \), i.e., \( \text{grade}(M) \) is zero. \( \square \)

**Corollary 4.6.** Let \( M \) and \( N \) be modules of positive grade and of finite projective dimension with the same support.
Then \( \beta(M) = \beta(N) \).

**Proof.** In view of [4] Proposition 4], \( \beta(M) = \gamma(M) \). So, the claim follows from the support sensitivity of \( \gamma(\cdot) \). \( \square \)

**Discussion 4.7.** i) By Proposition 2.6 \( h\text{-Ann}(M) \) and \( \text{Ann}(M) \) have a same radical.

ii) Over Gorenstein local rings \( h\text{-Ann}(M) \subset \text{Ann}(M) \). To see this its enough to apply local duality along with [18, Page 350]. We should remark that, over polynomial rings, this is in [14 Page 215, Remark], where it referred to an unpublished work of Eisenbud and Evans.

iii) Suppose \( (R, m) \) is Gorenstein and local. Then \( \text{Ann}(M)^{Gdim(M) - \ell + 1} \subset h\text{-Ann}(M) \subset \text{Ann}(M) \).

**Corollary 4.8.** Suppose \( R \) is Gorenstein and \( M \) is quasi-perfect. Then \( h\text{-Ann}(M) = \text{Ann}(M) \).

**Example 4.9.** The quasi-perfect assumption is important: Suppose \( (R, m) \) is a polynomial ring over an
infinite field and of dimension \( d > 3 \). Let \( 2 \leq i_1 < \ldots < i_{\ell} \leq d - 2 \). Evans and Griffith [9] Theorem A] constructed a prime ideal \( p \) which is not maximal such that \( \text{Ext}^{d-i+1}_R(R/p, R) \simeq R/m \) for \( i \in \{i_1, \ldots, i_{\ell} \} \) and zero exactly when \( i \notin \{i_1, \ldots, i_{\ell}, d - 1 \} \). The undetermined \( \text{Ext} \) is \( \text{Ext}^{2}_R(R/p, R) \).

Put \( a_2 := \text{Ann}(\text{Ext}^1_R(R/p, R)) \). By Observation 1.2 \( \dim(R/a_2) = d - 2 \). As \( p \) is a \((d - 2)\)-dimensional prime ideal and \( p \subseteq a_2 \), we have \( a_2 = p \). Thus, \( h\text{-Ann}(R/p) = m^t p \neq p = \text{Ann}(R/p) \).

**Remark 4.10.** If \( M \) is finitely generated over an integral domain, then \( 0 \neq \gamma(M) \). Indeed, by generic of freeness, there is \( a \in R \) such that \( M_a \) is free as an \( R_a \)-module. Let \( x \) be a suitable power of \( a \). Due to [5], \( x \text{Ext}^1_R(M, R) = 0 \) for all \( i > 0 \).

In particular, \( \gamma(M) \) and \( \beta(M) \) are nonzero.

**Example 4.11.** With the same notation as of Example 2.1, \( \text{Ext}^1_R(F/R, R) \simeq R \). Thus, \( \gamma(F/R) = 0 \).

Let \( J \) be the jacobian ideal of a complete local ring \( R \). Wang gives an \( \epsilon \in \mathbb{N} \) such that \( \text{Ext}^1_R(M, R\text{-mod}) = 0 \) for all \( i > \dim R \), where \( R\text{-mod} \) is the category of finitely generated modules. It may be \( J = 0 \). For example, \( R := \mathbb{Q}[X]/(X^2) \).

**Lemma 4.12.** Let \( M \) be a finitely generated module over any commutative ring and \( L \) be any module. Then \( \text{Ext}^1_R(M, L) \) has no nonzero projective submodule.

**Proof.** We use the induction. The case \( i = 1 \) is in [1] Theorem 4.1]. We make use of the standard shifting isomorphisms. Let \( E(L) \) be the injective envelope of \( L \). For the higher ext, its enough to look at \( 0 \rightarrow L \rightarrow E(L) \rightarrow E(L)/L \rightarrow 0 \) and apply the induction hypothesis throughout natural identification \( \text{Ext}^{i-1}_R(M, E(L)/L) \simeq \text{Ext}^1_R(M, L) \) for all \( i > 1 \). \( \square \)

**Corollary 4.13.** Let \( M \) be finitely generated and \( L \) be any module. Then each element of \( \text{Ext}^i_R(M, L) \) has a nontrivial annihilation for all \( i > 0 \).

**Proof.** Suppose there is \( x \in \text{Ext}^i_R(M, L) \) which is a faithful element. Then \( R = \text{Ext}^i_R(M, L) \simeq xR \hookrightarrow \text{Ext}^i_R(M, L) \).

By Lemma 4.12 \( \text{Ext}^i_R(M, L) \) has no nonzero projective submodule. This yields the contradiction. \( \square \)
Corollary 4.14. Let $R$ be reduced and $M$ finitely generated. Then $\text{Ass}(R) \cap \text{E-Ass}(M) = \text{Ass}(R) \cap \text{Supp}(M)$.

Proof. Suppose there is $p \in \text{Ass}(R) \cap \text{E-Ass}(M)$. Then $p = (0 : r)$ for some $r \in R$ and $R/p$ embedded into $\text{Ext}^i_R(M, R) = \text{Ext}^i_R(\Omega^{i-1}(M), R)$. Here $\Omega^{i-1}(M)$ is the $(i-1)$-th syzygy module of $M$. Suppose $i \neq 0$. By [1] Corollary 4.2, trace of $R/p$ is nilpotent. As $R$ is reduced, $\text{tr}(R/p) = 0$. But $0 \neq r \in \text{tr}(R/p)$. This contradiction says that $i = 0$. Since $\text{Ass}(\text{Ext}^i_R(M, R)) = \text{Ass}(R) \cap \text{Supp}(M)$, we get the claim. □

5. COMMENTS ON E-Ass$(−)$

Lemma 5.1. Let $R$ be any commutative ring and $I \lhd R$. Then $\text{Ext}^i_R(I, −) \simeq \text{Ext}^{i+1}_R(R/I, −)$ for all $i > 0$. In particular, $I \subseteq \gamma(I)$.

Proof. Look at the induced long exact sequence of Ext-modules induced from the short exact sequence $0 \to I \to R \to R/I \to 0$.

We start by 3 reductions of Problem 1.1. The first reduction is a weak inductive method.

Proposition 5.2. Let $R$ be a $d$-dimensional local ring such that Problem 1.1 is true over $d − 1$-dimensional rings. Then $|\text{E-Ass}_R(I)| < \infty$ for all ideal $I$ of positive grade.

Proof. Recall from Lemma 5.1 that $I \text{Ext}^0_R(I, R) = 0$. Let $y \in I$ be a regular element. Look at the exact sequence

$$0 \to I \to I \to I/yI \to 0.$$  

This induces the exact sequence

$$0 \to \text{Ext}^i_R(I, R) \to \text{Ext}^{i+1}_R(I/yI, R) \to \text{Ext}^{i+1}_R(I, R) \to 0.$$  

Thus $|\text{E-Ass}_R(I)| < \infty$ provided $|\text{E-Ass}_R(I/yI)| \leq \infty$. By [16] Page 140, there is the isomorphism

$$\text{Ext}^{i+1}_R(I/yI, R) \simeq \text{Ext}^0_{\hat{R}}(I/yI, \hat{R})$$  

we observe that E-Ass$_R(I)$ is finite if E-Ass$_{\hat{R}}(I/yI)$ is finite. □

Remark 5.3. i) Problem 1.1 reduces to complete case.

ii) Problem 1.1 reduces to maximal Cohen-Macaulay modules, when $R$ is Cohen-Macaulay.

Proof. i) Let $p$ be a prime ideal and let $q \in \text{Ass}(\hat{R}/p\hat{R})$. Indeed, if $x \in R \setminus p$, then $R/p \to \hat{R}/p\hat{R}$ is injective and so $\hat{R}/p\hat{R} \to \hat{R}/p\hat{R}$ is injective. Thus, $x \notin \bigcup_{Q \in \text{Ass}(\hat{R}/p\hat{R})} Q$. Therefore, $q \cap R = p$. This show that if $p_1 \neq p_2$ then $\text{Ass}(\hat{R}/p_1\hat{R}) \neq \text{Ass}(\hat{R}/p_2\hat{R})$ (+). Now suppose that $\bigcup_{Q \in \text{Ass}(\hat{R}/p\hat{R})} \text{Ass}(\hat{R}/p\hat{R})$ is finite. In view of [16] Theorem 23.2,

$$\text{Ass}(\hat{R}/(M, R) \otimes_R \hat{R}) = \bigcup_{p \in \text{Ass}(\text{Ext}^i_R(M, R))} \text{Ass}(\hat{R}/p\hat{R}).$$

Combine this along with (+) we deduce that $\bigcup_{Q \in \text{Ass}(\hat{R}/p\hat{R})} \text{Ass}(\hat{R}/p\hat{R})$ is finite.

ii) Assume that dim $R = d$ and $M$ be nonzero. We assume that $p, \text{dim}(M) = \infty$. We look at the following exact sequence:

$$0 \to \Omega_d \to F_{d−1} \to \cdots \to F_0 \to M \to 0$$

where $F_i$ is finitely generated free for all $i = 0, \ldots, d$. Then $\Omega_d \neq 0$. In view of [5] Exercise 2.1.26 $\Omega_d$ is maximal Cohen-Macaulay. Also, $\text{Ext}^{i+d}_R(M, R) \simeq \text{Ext}^i_R(\Omega_d, R)$. It turns out that finiteness of $\text{E-Ass}_R(\Omega_n)$ implies the finiteness of $\text{E-Ass}_{\hat{R}}(M)$. Without loss of the generality, we may assume that $M$ is maximal Cohen-Macaulay. □
We say a has max-height at most $t$ if for all $p \in \text{Ass}(R/\mathfrak{a})$, one has $ht(p) \leq t$.

**Theorem 5.4.** ([2] Theorem B.b]) Let $R$ be an integrally closed Cohen-Macaulay ring and $N$ a finitely generated $R$-module. There is an ideal $b$ of max-height at most 2 such that $\text{Ext}^i_R(b, -) \simeq \text{Ext}^{i+2}_R(N, -)$ $\forall i \geq 2$.

Let us reduce things to the cyclic modules.

**Lemma 5.5.** Let $R$ be an integrally closed Cohen-Macaulay local ring. The following are equivalent:

i) $|\text{E-Ass}(M)| < \infty$ for all finitely generated $M$.

ii) $|\text{E-Ass}(b)| < \infty$ for all ideal $b$ of max-height at most two.

iii) $|\text{E-Ass}(R/b)| < \infty$ for all ideal $b$ of max-height at most two.

**Proof.** Lemma 5.1 says that $\text{Ext}^i_R(b, -) \simeq \text{Ext}^{i+1}_R(R/b, -)$ for all $i > 0$. By Auslander’s remark and for all $i \geq 2$, we have $\text{Ext}^{i+1}_R(R/b, -) \simeq \text{Ext}^{i+2}_R(M, -)$. We note that the initial terms $\text{Ext}^0_R(M, -)$ are not effective, because the associated prime ideals of a finitely generated module is a finite set. This yields the claim.

**Fact 5.6.** i) The set $|\text{E-Ass}(M)| < \infty$, if $p \cdot \dim(M) < \infty$. The same claim is true if $\text{Gdim}(M) < \infty$.

ii) The problem 1.1 is true if $R$ is Gorenstein on its punctured spectrum.

**Proof.** i) The first claim is clear. Recall that if $\text{Gdim}(M) < \infty$, then $\text{Gdim}(M) = \sup \{i : \text{Ext}^i_R(M, R) \neq 0\}$.

So, $\text{E-Ass}(M)$ is finite.

ii) Let $i \geq \dim R$, which is finite as we may assume that $R$ is local. Let $p \neq m$ be a prime ideal. Since $R_p$ is Gorenstein, $\text{id}(R_p) = ht(p) < \infty$. Thus, $\text{Ext}^i_R(M_p, R_p) = 0$. Consequently, $\text{Supp}(\text{Ext}^i_R(M, R)) \subset \{m\}$.

Then $\text{E-Ass}(M) \subset \bigcup_{i=0}^{\dim R-1} \text{Ass}(\text{Ext}^i_R(M, R)) \cup \{m\}$. The later is a finite set.

**Corollary 5.7.** Problem 1.1 is true over two dimensional normal local domains.

**Proof.** This follows by Fact 5.6(ii), because $R_p$ is regular for all prime ideal $p$ of height one.

**Proposition 5.8.** i) Problem 1.1 is true over 3-dimensional excellent normal local domains.

ii) Problem 1.1 is true over two dimensional reduced excellent local rings.

**Proof.** Let us bring a general phenomena. The Gorenstein locus $\text{Gor}(X)$ of $X := \text{Spec}(R)$ is the set of all primes $p$ such that $R_p$ is Gorenstein. Since $R$ is excellent and in view of [11], there is an ideal $I$ such that $X \setminus \text{Gor}(X) = V(I)$.

i) Normal rings are $R(1)$. As $R$ is normal, $ht(I) > 1$. Due to $\dim R = 3$ we get that $\dim R/I \leq 1$. Thus, for all $i \geq 3$ we have $\text{Supp}(\text{Ext}^i_R(M, R)) \subset V(I)$. Therefore, $\text{E-Ass}(M) \subset \bigcup_{i=0}^{2} \text{Ass}(\text{Ext}^i_R(M, R)) \cup V(I)$.

The later is a finite set.

ii) Reduced rings are $R(0)$. This implies that $ht(I) > 0$. The reminder of the proof go ahead as of part i).

**Lemma 5.9.** Let $R$ be a 2-dimensional local ring. Suppose $\text{E-Ass}(I)$ is finite for all unmixed ideals of codimension zero. Then $\text{E-Ass}(I)$ is finite for all $I$.

**Proof.** We may assume that $ht(I) = 0$. Suppose $I$ is not unmixed and the claim is hold for unmixed part.

we write $I = I_1 \cap I_2$, where $I_1$ is the unmixed part and $I_2$ is the mixed part. As $ht(I_2) > 0$, then by the assumption $|\text{E-Ass}(R/I_2)| < \infty$. By the same reasoning, the claim is true for the ideal $I_1 + I_2$. We look
at the exact sequence \( 0 \to R/I \to R/I_1 \oplus R/I_2 \to R/(I_1 + I_2) \to 0 \). This induces the following long exact sequence of Ext-modules

\[
\cdots \to \text{Ext}^{i-1}_R(R/(I_1 + I_2), R) \xrightarrow{\partial_i} \text{Ext}^i_R(R/I, R) \xrightarrow{\sigma_i} \text{Ext}^i_R(R/I_1, R) \oplus \text{Ext}^i_R(R/I_2, R) \to \cdots
\]

Although "\( |\text{Ass}| < \infty \)" is not well-behaved with respect to the quotient, "\( |\text{Supp}| < \infty \)" behaves well with respect to the quotient. Set \( E_i := \frac{\text{Ext}^{i-1}_R(R/(I_1 + I_2), R)}{\ker(\partial_i)} \). In particular, \( \bigcup_i \text{Supp}(E_i) \) is finite. Set \( D_i := \text{im}(\sigma_i) \). Then \( \bigcup_i \text{Ass}(D_i) \) is finite. Now look at the exact sequence \( 0 \to E_i \to \text{Ext}^i_R(R/I, R) \to D_i \to 0 \). This implies that \( \text{Ass}(\text{Ext}^i_R(R/I, R)) \subseteq \text{Ass}(E_i) \cup \text{Ass}(D_i) \), as claimed. Hence, without loss of generality we assume that \( I \) is unmixed and is of zero height. \( \square \)

**Example 5.10.** Look at the Fermat’s curve \( F_p[[X,Y,Z]]/(X^p + Y^p + Z^p) \). Then \( \text{E-Ass}(I) \) is finite for all \( I \).

**Proof.** In view of the above lemma, we assume \( I \) is unmixed and of codimension zero. Note that \( R := F_p[[X,Y,Z]]/(X^p + Y^p + Z^p) \). Set \( \xi := (x+y+z)^i \) and \( \zeta := (x+y+z)^p−i \). Without loss of the generality we assume that \( I = \xi R \). Its free resolution is given by:

\[
\cdots \xrightarrow{\xi} R \xrightarrow{\xi} R \xrightarrow{\xi} R \xrightarrow{\xi} I.
\]

We deduce from this that \( |\text{E-Ass}(I)| < \infty \). \( \square \)

**Example 5.11.** \( R := F[[X,Y,Z]]/(X^2,XYZ) \). Then \( \text{E-Ass}(I) \) is finite for all ideal \( I \).

**Proof.** In view of the above lemma, we may assume that \( I = (x) \). Set \( a := yz \). Note that \( R_a \simeq F[[y,z,y^{-1},z^{-1}]] \) which is regular. It turns out that \( \text{Ext}^i_R(I, R) \) is annihilated by some uniform power of \( a \) for all \( i \geq 2 \), see Fact 5.6. Thus \( \text{Supp}(\text{Ext}^i_R(I, R)) \subset V(I + yz) \) for all \( i \geq 2 \). Note that \( V(I + yz) \) is finite. So, \( \text{E-Ass}_R(I) \subseteq \bigcup_{i=0}^1 \text{Ass}(\text{Ext}^i_R(I, R)) \cup V(I + yz) \), which is a finite set. \( \square \)

**Remark 5.12.** Let \( R \) be a normal closed Cohen-Macaulay local domain of dimension four. Suppose \(|\text{E-Ass}(R/b)| < \infty \) for almost complete-intersection ideal \( b \) of height 1. The following are equivalent:

i) \(|\text{E-Ass}(R/b)| < \infty \) for all ideal \( b \) of height exactly two.

ii) \(|\text{E-Ass}(M)| < \infty \) for all finitely generated \( M \).

iii) \(|\text{E-Ass}(R/b)| < \infty \) for all unmixed ideal \( b \) of height exactly two.

**Proof.** In view of Corollary 5.7 we may assume that \( \dim R > 2 \).

i) \( \implies \) ii): Keep the above lemma in mind and let \( b \) be an ideal of max-height at most two. If \( \text{ht}(b) = 2 \) we are done by the assumption. Suppose \( \text{ht}(b) = 1 \). We claim that things reduce to the unmixed case. To this end, we write \( b = b_1 \cap b_2 \), where \( b_1 \) is the unmixed part and \( b_2 \) is the mixed part. We assume that the claim is true for \( b_1 \). If \( \text{ht}(b_2) = 2 \), then by the assumption \(|\text{E-Ass}(R/b_2)| < \infty \). Thus, \( \text{ht}(b_2) > 2 \). Consequently, \( \dim R/b_2 \leq 1 \). Here, we used our low-dimensional assumption. This yields that \( \text{E-Ass}(R/b_2) \subset V(b_2) \) which is a finite set. By the same reasoning, the claim is true for the ideal \( b_1 + b_2 \), because \( \text{ht}(b_1 + b_2) > 1 \). We look at the exact sequence \( 0 \to R/b \to R/b_1 \oplus R/b_2 \to R/(b_1 + b_2) \to 0 \). This induces the following long exact sequence of Ext-modules

\[
\cdots \to \text{Ext}^{i-1}_R(R/(b_1 + b_2), R) \xrightarrow{\partial_i} \text{Ext}^i_R(R/b, R) \xrightarrow{\sigma_i} \text{Ext}^i_R(R/b_1, R) \oplus \text{Ext}^i_R(R/b_2, R) \to \cdots
\]

The reasoning given by Lemma 5.9 allow us to assume that \( b \) is unmixed and is of height one. In view of Fact 5.6 we may assume that \( b \) is not principal. Let \( b \) such that it generates \( b \) at the minimal primes of \( b \). Look at the irredundant decomposition of \( (b) = b \cap c \). Let \( c \) be in \( c \) but not in minimal primes of \( b \).
If $\text{ht}(b, c) = 2$, then $(b, c)$ is complete-intersection and the claim follows by Fact 5.5. So, $(b, c)$ is almost complete intersection.

Claim A. One has $b = (b : c)$. Indeed, let $x \in b$. Then $xc \in b \cap c = (b)$. For the other side inclusion, let $x \in (b : c)$. Then $xc \in (b) = \bigcap q_i \cap \bigcap \tilde{q}_i$ where $q_i$ (resp. $\tilde{q}_i$) are primary components of $b$ (resp. $c$).

Since $b$ is unmixed, $\{\text{rad}(q_i)\}_i = \min(b)$. Due to the choose of $c$, $c \notin \text{rad}(q_i)$ for all $i$. Since $xc \in q_i$ and $c \notin \text{rad}(q_i)$, we deduce from the definition of primary ideals that $x \in q_i$ for all $i$, i.e., $x \in (b)$. So, $(b : c) \subseteq b$. This proves the claim.

Claim A) induces the following exact sequence

$$0 \longrightarrow \frac{R}{b} \xrightarrow{c} \frac{R}{(b)} \longrightarrow \frac{R}{(b, c)} \longrightarrow 0.$$  

Keep in mind that $p. \dim(R/bR) = 1$. The induced long exact sequence tells us $\text{Ext}^i_R\left(\frac{R}{b}, -\right) \cong \text{Ext}^{i+1}_R\left(\frac{R}{(b, c)}, -\right)$ for all $i > 1$. This completes the proof of the implication $i) \implies ii)$.  

$ii) \implies iii)$: This is trivial.

$iii) \implies i)$: Suppose $\text{ht}(b) = 2$. We need to reduce things to the unmixed part. The proof is a repetition of the above argument. We left it to the reader. \hfill \square

We left to the reader to find more reductions to almost complete-intersection ideals.

6. Ext and the number of generators

Denote the minimal number of generators of a module by $\mu(-)$.

**Theorem 6.1.** (Macaulay 1904, Vasconcelos 1967, and Smith 2013) Let $F$ be a field and $I$ be an ideal in $S := F[X, Y]$ such that $S/I$ is a Poincaré duality algebra. Then $\mu(I) = 2$.

Also, see [24 Proposition 2.4]. The Poincaré duality algebra is Gorenstein.

**Theorem 6.2.** (Serre 1960) Let $S$ be a regular local ring and $I$ be a height two ideal such that $R := S/I$ is Gorenstein. Then $\mu(I) = 2$.

The Gorenstein condition implies that $\omega_R = \text{Ext}_R^2(R, S)$ is cyclic.

**Lemma 6.3.** Let $(R, m)$ be local and $M$ be such that $p := p. \dim(M) < \infty$. Then $\mu(\text{Ext}_R^p(M, R)) = \beta_p(M)$.

**Proof.** Recall that $\beta_p(-)$ is the Betti number. Look at the minimal free resolution of $M$

$$0 \longrightarrow R^{\beta_0(M)} \xrightarrow{X} R^{\beta_1(M)} \longrightarrow \ldots \longrightarrow R^{\mu(M)} \longrightarrow M \longrightarrow 0.$$  

Delete $M$ from the right and apply $\text{Hom}_R(-, -)$ we arrive to the following complex

$$0 \longrightarrow R^{\mu(M)} \longrightarrow \ldots \longrightarrow R^{\beta_{p-1}(M)} \xrightarrow{X^t} R^{\beta_p(M)} \longrightarrow 0.$$  

Here, $(-)^t$ denotes the transpose. Set $x_i := (x_{i1}, \ldots, x_{ip}(M))$. Thus $\text{Ext}_R^p(M, R) = \frac{R^{\beta_p(M)}}{(x_{i1}, \ldots, x_{ip})}$. This shows that $\mu(\text{Ext}_R^p(M, S)) \leq \mu(R^{\beta_p(M)})$. One has $(x_{i1}, \ldots, x_{ip})R^{\beta_p(M)} \subseteq mR^{\beta_p(M)}$. Consequently, $\frac{R^{\beta_p(M)}}{(x_{i1}, \ldots, x_{ip})} = mR^{\beta_p(M)}/(x_{i1}, \ldots, x_{ip})$. By [16 Page 35], $\mu(E) = \dim_R/E/mE$ for an $R$-module $E$. Therefore,

$$\mu(\text{Ext}_R^p(M, R)) = \dim_R/m \left( \frac{R^{\beta_p(M)}}{(x_{i1}, \ldots, x_{ip})} \right) = \dim_R/m \left( \frac{mR^{\beta_p(M)}}{R^{\beta_p(M)}} \right) = \mu(R^{\beta_p(M)}) = \beta_p(M).$$
Observation 6.4. Let \((S, \mathfrak{m})\) be a Cohen-Macaulay local ring and \(I\) a Cohen-Macaulay ideal of height two and of finite projective dimension. Then \(\mu(\text{Ext}_S^2(S/I, S)) = \mu(I) - 1\).

Proof. Set \(R := S/I\). By Auslander-Buchsbaum formula,
\[
p. \dim_S(R) = \text{depth} S - \text{depth}_S R = \text{depth} S - \text{depth}_R R = \dim S - \dim S/I = \text{ht}(I) = 2.
\]
Set \(\mu := \mu(I)\). By Hilbert-Burch, there is a matrix \(X\) with entries from \(m\) such that the minimal free resolution of \(S/I\) is
\[
0 \longrightarrow S^{\mu-1} \xrightarrow{X} S^\mu \longrightarrow S \longrightarrow S/I \longrightarrow 0.
\]
Thus \(\beta_2(S/I) = \mu - 1\). In view of Lemma 6.3, we get the claim.

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