SIMPLY CONNECTED MANIFOLDS OF DIMENSION $4k$ WITH TWO SYMPLECTIC DEFORMATION EQUIVALENCE CLASSES

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Abstract. We present smooth simply connected closed $4k$-dimensional manifolds $N := N_k$, for each $k \in \{2, 3, \cdots \}$, with distinct symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$. To distinguish $[[\omega_i]]$'s, we used the symplectic $Z$ invariant in [4] which depends only on the symplectic deformation equivalence class. We have computed that $Z(N, [[\omega_1]]) = \infty$ and $Z(N, [[\omega_2]]) < 0$.

1. Introduction

An almost-Kähler metric on a smooth manifold $M^{2n}$ of real dimension $2n$ is a Riemannian metric $g$ compatible with a symplectic structure $\omega$, i.e. $\omega(X, Y) = g(X, JY)$ for an almost complex structure $J$, where $X, Y$ are tangent vectors at a point of the manifold. Two symplectic forms $\omega_0$ and $\omega_1$ on $M$ are called deformation equivalent, if there exists a diffeomorphism $\psi$ of $M$ such that $\psi^* \omega_1$ and $\omega_0$ can be joined by a smooth homotopy of symplectic forms, [5]. For a symplectic form $\omega$, its deformation equivalence class shall be denoted by $[[\omega]]$. We denote by $\Omega_{[[\omega]]}$ the set of all almost Kähler metrics compatible with a symplectic form in $[[\omega]]$. Examples of smooth manifolds with more than one symplectic deformation class have been an interesting subject to study; refer to [6], [7] or [8].

For a smooth closed manifold $M$ of dimension $2n \geq 4$ which admits a symplectic structure $\omega$, we have defined a symplectic invariant $Z$ in [4]:

$$Z(M, [[\omega]]) = \sup_{g \in \Omega_{[[\omega]]}} \frac{\int_M s_g d\text{vol}_g}{(\text{Vol}_g)^{\frac{n-1}{n}}}$$

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where $d\text{vol}_g$, $s_g$, $\text{Vol}_g$ are the volume form, the scalar curvature and the volume of $g$ respectively.

In [4], we presented a six dimensional non-simply connected closed manifold which admits two symplectic deformation classes $[[\omega_i]]$, $i = 1, 2$, such that their $Z$ values have distinct signs. Then in [3], we showed an eight dimensional simply connected closed manifold with the same property.

The main result in this article is to present a simply connected manifold of dimension $4k$, for each $k \in \{2, 3, \cdots\}$, with the above property.

2. Examples in Dimension $4k$

Here we shall prove the following:

**Theorem 2.1.** For each integer $k \geq 2$, there exists a smooth closed simply connected $4k$-dimensional manifold $N$ with symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$ such that $Z(N, [[\omega_1]]) = \infty$ and $Z(N, [[\omega_2]]) < 0$.

The manifold $N$ is (diffeomorphic to) the product of $k$ copies of a complex surface of general type with ample canonical line bundle which is homeomorphic to $R^8$, the blow up of the complex projective plane $\mathbb{CP}^2$ at 8 points in general position. This general type complex surface may be obtained as a small deformation of Barlow’s explicit complex surfaces [1]. When $k = 2$, the manifold $N$ in the theorem can be the one studied by Catanese and LeBrun [2].

To prove this theorem, we need the following;

**Proposition 1.** Let $W$ be a complex surface of general type with ample canonical line bundle, homeomorphic to $R^8$. Consider a Kähler Einstein metric of negative scalar curvature on $W$ with Kähler form $\omega_W$ on $W$. Set $N := W \times \cdots \times W$, the $k$-fold product of $W$.

Then $Z(N, [[\omega_W + \cdots + \omega_W]]) = -4\sqrt{2}k$, and it is attained by a Kähler Einstein metric.

**Proof.** The argument here follows the scheme in [4, Section 3] and is similar to that in [3]. We recall one known fact about $W$ from [7, Section 4]; there is a homeomorphism of $W$ onto $R^8$ which preserves the Chern class $c_1$. And there is a diffeomorphism of $N$ onto $R^8(k) := R^8 \times \cdots \times R^8$, the $k$-fold product of $R^8$ [2, Section 4].

Note that $R^8$ is well known to admit a Kähler Einstein metric of positive scalar curvature obtained by Calabi-Yau solution.
Then, the first Chern class of $W$ can be written as $c_1(W) = 3E_0 - \sum_{i=1}^{8} E_i \in H^2(W, \mathbb{R}) \cong \mathbb{R}^9$, where $E_i$, $i = 0, \ldots, 8$, is the Poincare dual of a homology class $\tilde{E}_i$, $i = 0, \ldots, 8$ so that $\tilde{E}_i \cdot E_j = \epsilon_i \delta_{ij}$, where $\epsilon_0 = 1$ and $\epsilon_i = -1$ for $i \geq 1$. So, in this basis the intersection form becomes

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$ 

We have the orientation of $W$ induced by the complex structure and the fundamental class $[W] \in H_4(W, \mathbb{Z}) \cong \mathbb{Z}$. As $\omega_W$ is the Kähler form of a Kähler Einstein metric $g_W$ of negative scalar curvature, we may get $[\omega_W] = -3E_0 + \sum_{i=1}^{8} E_i$ by scaling if necessary.

By Künneth theorem $H^2(N, \mathbb{R}) \cong \bigoplus_{j=1}^{k} \pi_j^*H^2(W) \cong \mathbb{R}^9 \oplus \cdots \oplus \mathbb{R}^9$, where $\pi_j$ is the projection of $N$ onto the $j$-th factor. Then,

$$c_1(N) = \sum_{j=1}^{k} \pi_j^*c_1(W) = \sum_{j=1}^{k} \pi_j^*(3E_0 - \sum_{i=1}^{8} E_i).$$

Consider any smooth path of symplectic forms $\omega_t$, $0 \leq t \leq \delta$, on $N$ such that $\omega_0 = \omega_W + \cdots + \omega_W$. We may write

$$[\omega_t] = \sum_{j=1}^{k} \sum_{i=0}^{8} n_j^i(t) \pi_j^*E_i \in H^2(N, \mathbb{R})$$

for some continuous functions $n_j^i(t)$ in $t$, $i = 0, \cdots, 8$. As $\{\omega_t\}$ is connected, their first Chern class $c_1(\omega_t) = c_1(N)$ does not depend on $t$. Using the intersection form we do a combinatorial computation;

$$(2.1) \quad [\omega_t]^{2k}([N]) = \left( \sum_{j=1}^{k} \sum_{i=0}^{8} n_j^i(t) \pi_j^*E_i \right)^{2k}([W \times \cdots \times W])$$

$$= C_k^{2k}C_{2k-2}^2 \cdots C_2^2 \prod_{j=1}^{k} \{n_0^j(t)^2 - \sum_{i=1}^{8} n_i^j(t)^2 \} > 0,$$

where $C_k^n = \frac{n!}{(n-k)!k!}$.

Set $[\omega^j(t)] = \sum_{i=0}^{8} n_i^j(t) E_i \in H^2(W, \mathbb{R})$, so that $[\omega_t] = \sum_{j=1}^{k} \pi_j^*[\omega^j(t)]$. We put $A_j := A_j(t) = [\omega^j(t)]^2[W] = n_0^j(t)^2 - \sum_{i=1}^{8} n_i^j(t)^2$. As $A_j(0) = [\omega_W]^2[W] > 0$ and
\[
\prod_{j=1}^{k} A_j(t) > 0 \text{ from (2.1), we have } A_j(t) > 0. \text{ Then } n_0(t)^2 > \sum_{i=1}^{k} n_i(t)^2 \text{ and as } n_0(0) = -3 < 0, \text{ so } n_0(t) < 0.
\]

We also put \( B_j := B_j(t) = (c_1(W) \cdot [\omega^j(t)])[W] = 3n_0(t) + \sum_{i=1}^{k} n_i(t). \)

Since \( n_0(t)^2 > \sum_{i=1}^{k} n_i(t)^2 \) and \( |\sum_{i=1}^{k} n_i(t)| \leq \sqrt{3} \sum_{i=1}^{k} n_i(t)^2 \), we get

\[
(2.2) \quad 3n_0(t) + \sum_{i=1}^{k} n_i(t) \leq 3n_0(t) + 2\sqrt{2} \sqrt{\sum_{i=1}^{k} n_i(t)^2} < 3n_0(t) + 2\sqrt{2} \sqrt{n_0(t)^2} = (3 - 2\sqrt{2})n_0(t) < 0.
\]

As \( c_1(\omega_t) = \pi_1^t c_1(W) + \cdots + \pi_k^t c_1(W) \), by combinatorial computation we obtain;

\[
(2.3) \quad c_1(\omega_t) \cdot [\omega^j_{t}]^{2k-1}([N]) = \sum_{j=1}^{k} (2k - 1)C_2^j C_2^{2k-2} \cdots C_2^2 (A_1 A_2 \cdots A_k) \cdot B_j \cdot A_j.
\]

Putting \( A = A_1 \cdots A_k \) and \( C = C_2^1 C_2^{2k-2} \cdots C_2^2 \), from (2.1) and (2.3) we have;

\[
\frac{c_1(\omega_t) \cdot [\omega^j_{t}]^{2k-1}}{[\omega^j_{t}]^{2k-1} = \frac{\sum_{j=1}^{k} (CA) \cdot B_j \cdot A_j}{k} = \frac{(CA)^\frac{1}{k}}{k} \sum_{j=1}^{k} B_j \cdot A_j.
\]

From the AM-GM (Arithmetic Mean - Geometric Mean) inequality; \( \sqrt[\sum_{i=1}^{k} x_i} \geq \frac{x_1 + x_2 + \cdots + x_n}{n} \), setting \( x_j = -B_j \cdot A_j > 0 \), we get

\[
(2.4) \quad \sum_{j=1}^{k} B_j \cdot A_j \leq -k \left( \frac{(-1)^k B_1 \cdots B_k}{A_k^k} \right) \cdot \frac{1}{A_k^x}.
\]

So,

\[
\frac{c_1(\omega_t) \cdot [\omega^j_{t}]^{2k-1}}{[\omega^j_{t}]^{2k-1}} \leq -C_{\frac{1}{k}} \left( \frac{(-1)^k B_1 \cdots B_k}{A_1 \cdots A_k} \right)^\frac{1}{A_k^x}.
\]

From (2.2),

\[
(2.5) \quad \frac{B_j^2}{A_j} \geq \frac{3n_0(t) + 2\sqrt{2} \sum_{i=1}^{k} n_i(t)^2}{n_0(t)^2 - \sum_{i=1}^{k} n_i(t)^2} \frac{(3 - 2\sqrt{2} \sqrt{y})^2}{1 - y},
\]

where \( y = \sum_{i=1}^{k} n_i(t)^2 y \). By calculus, \( \frac{(3 - 2\sqrt{2} \sqrt{y})^2}{1 - y} \geq 1 \) for \( y \in [0, 1] \) with equality at \( y = \frac{8}{9} \). So, we get \( \frac{B_j^2}{A_j} \geq 1 \) and \( \frac{-B_j}{\sqrt{A_j}} \geq 1 \).

From this we have

\[
(2.6) \quad \frac{c_1(\omega_t) \cdot [\omega^j_{t}]^{2k-1}}{[\omega^j_{t}]^{2k-1}} \leq -C_{\frac{1}{k}}.
\]
There is a basic inequality for any symplectic structure $\omega$ on a closed manifold $M$ of dimension $2n$ [4];

\begin{equation}
Z(M, [[\omega]]) \leq \sup_{\omega \in [[\omega]]} \frac{4\pi c_1(\omega) \cdot \frac{[\omega]^{n-1}}{(n-1)!}}{(\frac{[\omega]^n}{n!})^{\frac{n-1}{n}}}. \tag{2.7}
\end{equation}

As the expression $\frac{4\pi c_1(\omega) \cdot \frac{[\omega]^{n-1}}{(n-1)!}}{(\frac{[\omega]^n}{n!})^{\frac{n-1}{n}}}$ is invariant under a change $\omega \mapsto \phi^* (\omega)$ by any diffeomorphism $\phi$, so from (2.6) and the definition of $Z$, we get

\[ Z(N, [[\omega_W + \cdots + \omega_W]]) \leq -4\pi \frac{(2k)!^2}{(2k-1)!} C^{\frac{1}{2k}} = -4\sqrt{2}\pi k. \]

We consider the Kähler form $\omega_W + \cdots + \omega_W$ of the product Kähler Einstein metric $g_W + \cdots + g_W$ of negative scalar curvature on $N = W \times \cdots \times W$. One can readily check that this symplectic form satisfies the equality of both (2.6) and (2.7). So, we conclude $Z(N, [[\omega_W + \cdots + \omega_W]]) = -4\sqrt{2}\pi k$. \hfill \Box

**Proof of Theorem 2.1.** Consider the positive Kähler Einstein metric on $R_8$ and let $\omega_1$ be the Kähler form of the product positive Kähler Einstein metric on $R_8 \times \cdots \times R_8$, which is diffeomorphic to $N$. We have $Z(N, [[\omega_1]]) = \infty$ (scaling by different constants on each factor gives $\infty$). And let $\omega_2$ be $\omega_W + \cdots + \omega_W$. Then $Z(N, [[\omega_2]]) < 0$ from Proposition 1. From the fact that these values are different, we conclude that $[[\omega_1]]$ and $[[\omega_2]]$ are distinct symplectic deformation equivalence classes. This proves Theorem 2.1. \hfill \Box

In this article I demonstrated examples in $4k$ dimension. But by refining the argument of [4], one may try to get, for each $k \geq 1$, examples of closed symplectic $(4k + 2)$-dimensional manifolds admitting two symplectic deformation equivalence classes with distinct signs of $Z(\ , [[\cdot]])$ invariants.

So far we only used the Catanese-LeBrun manifold as building blocks. But one may use other 4-dimensional closed simply connected symplectic manifolds of smaller Euler characteristic.

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