Heuristic for solving cyclic bandwidth sum problem by following the structure of the graph

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Abstract

The cyclic bandwidth sum problem (CBSP) consists of finding a labeling of the vertices of an undirected and unweighted graph with a fixed number of vertices with distinct integers such that the sum of (cyclic) difference of labels of adjacent vertices is minimized. Although theoretical results exist that give optimal value of cyclic bandwidth sum (CBS) for standard graphs, there are neither results in the general case, nor explicit methods to reach this optimal result. In addition to this lack of theoretical knowledge, only a few methods have been proposed to approximately solve the CBSP. In this paper, we introduce a new algorithm to find an approximate solution for CBSP and an associated labeling. The idea is to find a labeling which follows the structure of the graph. The heuristic is a two-step algorithm: the first step consists of traveling the graph to find a set of paths which follow the structure of the graph, using a similarity criterion based on the Jaccard index to jump from one vertex to the next one. The second step consists of merging all the previously obtained paths, based on a greedy approach that extends a partial solution by inserting a new path at the position that minimizes the cyclic bandwidth sum. The effectiveness of the proposed heuristic, both in terms of performance and time execution, is shown through experiments on graphs whose optimal value of CBS is known as well as on real-world graphs.

1 Introduction

Graph labeling consists of assigning labels to vertices or edges of a graph. There exists a large variety of labeling problems that are related to distinctive applications in several fields, such as graph layout design [13], or information retrieval [3, 30]. We focus in this article on the labeling of vertices (or nodes) of a simple connected and undirected graph, with the objective to find a graph labeling which reflects the topology of the graph. Such a labeling shall minimize the distances between labels of adjacent vertices. Finding a labeling which follows the structure of the graph could be a challenge with high stake for many applications. Among them, we can point out those related to distributed inference over networks [22], diffusion [5] or visualization of networks [1]. We propose in the following to travel the graph by solving the cyclic bandwidth sum problem, as described below.
1.1 General framework of graph labeling

Let \( G = (V,E) \) be a simple connected and undirected graph with \( V \) the set of vertices, and \( E \) the set of edges. The number of vertices is noted \( n = \#V \). Chung \cite{Chung1984} proposed a framework which encompasses many graph labeling problems. It is based on a mapping between \( V \) and the set of vertices of a host graph \( H = (N,E_H) \) with \( N = \{0,\ldots,n-1\} \). Graph labeling problems are then defined as finding the best mapping \( \pi \) from \( V \) to \( N \), subject to conditions, often based on a minimization or maximization of distances between labels taken between pairs of adjacent vertices of \( G \).

This distance, noted \( d_H \), is defined as the length of the shortest path between the corresponding vertices in the host graph \( H \). Two conditions of interests are often used:

1. the maximum distance \( d_H \) between the labels of two adjacent vertices of \( G \) is minimized, i.e. finding a labeling \( \hat{\pi} \) such that:
   \[
   \hat{\pi} = \arg\min_{\pi} \max_{\{u,v\} \in E} d_H(\pi[u],\pi[v])
   \]

2. the sum of distances \( d_H \) between all pairs of adjacent vertices of \( G \) is minimized, i.e. finding \( \hat{\pi} \) such that:
   \[
   \hat{\pi} = \arg\min_{\pi} \sum_{\{u,v\} \in E} d_H(\pi[u],\pi[v])
   \]

The resulting graph labeling problems have been extensively studied in the case where the host graph is a path graph \( P \) where \( E_P = \{\{i,i+1\} \mid i = 0 \ldots n-2\} \). The length of the shortest path between two nodes \( u \) and \( v \) in this graph is given by

\[
d_P(\pi[u],\pi[v]) = |\pi[u] - \pi[v]|
\]

The minimization of this distance results in labeling vertices such that high jumps are avoided. These problems are called bandwidth problem (condition 1) and bandwidth sum problem (condition 2).

Lin \cite{Lin1992} and \cite{Lin1991} introduced the problems where the host graph is a cycle \( C \) with \( n \) vertices, where \( E_C = \{\{i,i+1\} \mid i = 0 \ldots n-2\} \cup \{n-1,0\} \). In this case, the distance between two vertices \( u,v \in V \) is given by

\[
d_C(\pi[u],\pi[v]) = \min(|\pi[u] - \pi[v]|,n-|\pi[u]-\pi[v]|)
\]

Such a distance enables the labeling to take into account more complex structures such as cyclic structures. The resulting problems are called cyclic bandwidth problem (condition 1) and cyclic bandwidth sum problem (condition 2). We focus in this paper on the cyclic bandwidth sum problem (CBS) defined as

\[
\min_{\pi} CBS(G) = \min_{\pi} \sum_{\{u,v\} \in E} d_C(\pi[u],\pi[v])
\]

Examples of optimal labeling solving Eq. \( 5 \) are shown in Fig. \[ for some standard graphs. We can see that the labeling closely follows the structure.

Lin \cite{Lin1992} showed that the cyclic bandwidth problem, and by extension the cyclic bandwidth sum problem, are two NP-hard problems.
1.2 Related work

Many works have been done on the study of labeling graph problems: the bandwidth problem and bandwidth sum problem have been extensively studied: Chung [8] provided a comprehensive study of this specific problem, describing especially the work in Cuthill [10], which introduced an algorithm to solve the bandwidth problem. Similarly, some studies have also been realized on other graph labeling problems, such that cyclic bandwidth problem [23][26], antibandwidth problem [4] or cyclic antibandwidth problem [24], both in terms of theoretical results and algorithms.

Conversely, only few results are available in the literature for solving the cyclic bandwidth sum problem. Two articles focus on the mathematical aspects of this problem: Jianxiu [21] introduced cyclic bandwidth sum problem and proposed theoretical optimal values for some standard graphs, such as wheel or k-regular graphs, as well as lower and upper bounds for all graphs with known number of nodes and edges. Later on, Chen [6] studied theoretical optimal value of the CBS for special cases of graphs, as for complete bipartite graphs. If these theoretical results do not help to get the optimal labeling of a graph, they are nonetheless useful to check the correct behavior of a heuristic build to solve CBSP. The lack of knowledge is particularly obvious with regard to the algorithms to solve CBSP. To the best of our knowledge, only one heuristic has been proposed to solve the cyclic bandwidth sum problem, published in Satsangi et al. [28] and described more fully in Satsangi [27]. The heuristic is based on a general variable neighborhood search (GVNS). The idea of GVNS is to change the neighborhood to descent to local minima of CBS and to escape from the valleys which contain them in order to browse a large part of the solution space. Two phases are defined: A shaking phase in which the neighborhood is changed and which consists in applying several shaking operations where the vertices are shifted, reversed, flipped or swapped without taking into account the proximity of vertices. This operation enables the algorithm to escape from valleys and to browse the solution space. A local search is then performed to descent in a valley to a local minima and is realized by switching consecutive vertices or swapping vertices considering edges which have the highest contribution to the CBS.

Preliminary versions of the algorithm MCBS have been proposed in [17, 19, 18, 20]: they outline in particular the motivation for designing a heuristic solving CBSP, and how the relabeling is useful for studying the evolution of the structure of a temporal network over time (see Section 6.2 for more details). These proceedings are nevertheless really early work, giving only the main ideas of the algorithm. Hence, neither a detailed description is given in these articles, nor any experiments are made to test the validity and performance of the proposed heuristic in his ability to solve the cyclic bandwidth sum problem.

1.3 Contribution

The paper is organized as follows. Section 2 sketches the principles of the proposed method. Detailed algorithms are presented in Section 3 while a worst-case complexity study is given in Section 4. The performances of the algorithm are investigated in Section 5 through the analysis of experiments on graphs whose optimal value of CBS
is known, a comparison between our results and the results obtained using GVNS, a statistical analysis on graphs whose optimal is unknown and finally, the comparison between the heuristic MCBS with four variants in order to justify choices of design, qualitative study to visually study the performance of the heuristic to follow the structure of the graph. Sections 6 and 7 conclude this paper and discuss extensions of the proposed method to handle weighted graphs as well as an example of application of this heuristic.

2 Heuristic to minimize the Cyclic Bandwidth Sum of a graph

![Graphs Examples](image)

Figure 1: Examples of standard graphs with optimal labeling minimizing the CBS. The labeling visually browses each of the graphs according to its structure as much as possible.

The structure of the graph is a predominant element in the labeling. Especially, the presence of regular structures or denser parts, like community structures, has to be considered to constrain the vertex labels. The aim of the algorithm is to browse the graph following its structure. For instance, in the simple case of a cycle (see Fig. 1a), the correct behavior of the algorithm should be as follows: Starting from one random vertex, to jump to one of its two neighbors, then to the next one, and so forth, following the cycle vertex by vertex until reaching the first vertex and stops. In the less trivial case where the graph is organized by several cliques (see Fig. 1f), the algorithm should browse all the vertices inside a clique before jumping to another one. More generally, the algorithm has to adapt its search to the structure of the graph, whatever the structure is.

A solution to achieve this goal is to perform a random walk on the graph that successively numbers the vertices when they are reached. However, this approach has
several disadvantages. First, a random walk can reach a vertex several times and thus has to be controlled to avoid going to vertices already numbered. Second, the choice of the next vertex depends only on the neighborhood of the current vertex, and not on a more extended neighborhood. Third, if we prevent the walk to go to an already numbered vertex, the walk can stop before visiting all the vertices.

The heuristic we propose below fills in the gaps of a random walk and consists of a two-step algorithm. The first step performs local searches in order to find a collection of independent paths with respect to the local structure of the graph, while the second step determines the best way to arrange the paths such that the CBS is minimized. Details on these two steps are given below.

2.1 **Step 1: Guiding the search towards locally similar vertices**

The first step consists in finding a collection of paths in the graph, that is to say some sequences of vertices consecutively connected. The algorithm performs a depth-first search in which the next vertex is chosen based on its similarity to the current vertex. This similarity depends on the intersection of the two vertex neighborhoods.

The search is executed as follows. Starting from a vertex, the algorithm jumps to one of its unvisited neighbors, and so until there is no more accessible vertices. Then, the algorithm starts a new path from a vertex which has not been yet inserted in a path, and then continues to build paths until all the vertices are in a path. At the end of this step, a collection of paths is obtained that partitions the graph vertex set.

2.1.1 **Initialization**

Any vertices not yet inserted in a path can be used as starting node. However, to favor the computation of longer paths, vertices that are at the periphery of the graph are preferred. The incentive behind this choice lies on the fact that the path should start to one of the extremity of the graph. For example, let us consider a simple path graph: Starting from a vertex in the middle of the path will generate two paths, although it is obvious that the graph can be traveled using a single path. There are several measures to determine the centrality of a vertex, that can also be used to find vertices that are outer of the graph. We chose the simplest one by namely using the degree of the vertices: the unvisited vertex with the smaller degree is selected to start the path. Other centrality measures could be used, as for instance the closeness or the betweenness centrality measures. However, based on our experiments, the results are not significantly improved facing the higher computational complexity of such methods (see results in Section 5.4).

2.1.2 **Selection of the next vertex**

The depth-first search is performed so that the next vertex is chosen according to the similarity of its neighborhood to the one of the current vertex. The more it is similar, the more likely it has to be picked up. The neighborhood similarity of two vertices is evaluated based on the Jaccard index:
where $u$ and $v$ are the two considered vertices, $\text{adj}[u]$ and $\text{adj}[v]$ are their respective adjacent vertex sets. $J(u,v)$ is equal to the number of common neighbors including $u$ and $v$, divided by the total number of neighbors. Therefore, when the two vertices have the same set of neighbors, this measure equals to 1. It may happen that two neighbors of the current node $u$ have the same similarity index with $u$. In this case, the algorithm selects the vertex with the smaller degree, according to the same motivation as for the initialization of the path. If the two vertices have the same degree, then the vertex is selected according to the initial labeling, the chosen node being the first node encountered by the algorithm.

It is worth noticing that vertices of degree 1 in the neighborhood cannot be chosen as following nodes because it will end up the path. Instead, the vertex of degree 1 are immediately inserted after their unique neighbor to guarantee that the nodes are as close in the labeling as they are in the graph.

One refinement is considered in the computation of the similarity measure that highly improves the efficiency of our procedure. It consists in restricting the set of considered neighbors to the ones not yet included in a path. The intuition to understand this choice is to consider a vertex neighborhood whose vertices are all labeled but one. While considering the whole set of neighbors, the unlabeled vertex is “hidden” by the other vertices and then forgotten. The impact of this refinement is evaluated in Section 5.4.

2.1.3 End of the search

The search for a path ends when all the neighbors of the current node have been inserted in a path. The algorithm starts a new path using the remaining vertices, until all the nodes belong to a path.

2.2 Step 2: Greedy merge of paths

The second step aims at aggregating these paths in a unique labeling in such a way that the CBS is minimized, the position of the node in the labeling giving its label. We perform a greedy search that takes the locally optimal choice while merging a new path in the partial labeling under construction: The algorithm tries to insert the path and the reverse path at each possible index in the current labeling and retains the insertion that minimizes the CBS. The paths are selected in turns according to their length, the largest one being selected first. This choice is done to maximize the number of combinations: the shortest path have more possibilities of insertion into a long path since a path cannot be broken when it is inserted. A variant where the sort is reverse is studied in Section 5.4.

The computation of the CBS, as given in Eq. (5), is demanding as it requires that every edge of the graph be traveled. In step 2, the CBS is computed twice (forward
and backward) for each possible index of the current labeling. However, most of the CBS value remained unchanged when changing the index where the path is inserted. Therefore, we propose in the following a way to incrementally update the CBS value without considering all the edges of the graph but only the ones that are impacted by changing the insertion position. Section 5.4 shows some experiments that illustrate the gain of this approach over the direct computation of the CBS.

### 2.2.1 Incremental computing of the CBS: principle

Let us consider the insertion of a path $P$ into the current labeling $O$ as described in Fig. 2. Line 1 represents the current cycle made of a sequence of vertices $O_1$ followed by the vertex $k$ at position $i$ and ended by the sequence of vertices $O_2$. $P$ (line 2) is the sequence of vertices that is currently inserted at the position $i$ (line 3), i.e. just before vertex $k$. Thus, the current ordering begins by the path $O_1$, is followed by $P$, then comes the vertex $k$ and the path $O_2$. Line 4 gives the current cycle when $P$ is inserted at the position $i+1$.

| 1 | labeling $O$ | $\cdots - - - O_1 - - - | - - - k - | - - - O_2 - - - |$ |
| 2 | path $P$ | $\cdots - - - P - - - |$ |
| 3 | insertion of $P$ at index $i$ | $\cdots - - - O_1 - - - | - - - P - - - | - - - k - | - - - O_2 - - - |$ |
| 4 | insertion of $P$ at index $i+1$ | $\cdots - - - O_1 - - - | - - - k - | - - - P - - - | - - - O_2 - - - |$ |

Figure 2: Schema of the insertion of path $P$ in the current labeling $O$

Let $\#P = p$ and $k$ be the vertex that moves in top of $P$ after incrementing the index. $CBS^{(i)}$ is the value of the cyclic bandwidth sum when $P$ is inserted at index $i$ and $\pi[\cdot]$ refers to the label of vertex $\cdot$ in this configuration. Eq. 7 gives a decomposition of the CBS expression according to the identified groups of vertices:

$$CBS^{(i)} = CBS^{(i)}(O_1, O_1) + CBS^{(i)}(O_2, O_2) + CBS^{(i)}(O_1, O_2) + CBS^{(i)}(P, P) + CBS^{(i)}(k, O_1) + CBS^{(i)}(k, O_2) + CBS^{(i)}(k, P) + CBS^{(i)}(P, O_1) + CBS^{(i)}(P, O_2)$$

with $CBS^{(i)}(X, Y) = \sum_{u \in X, v \in Y, [u, v] \in E} d_C(\pi_i[u], \pi_i[v])$.

To express $CBS^{(i+1)}$ in terms of $CBS^{(i)}$, we can observe that the increase of the index leads to the following changes in the labeling:

$$\forall u \in P, \pi_{i+1}[u] = \pi_i[u] + 1$$

$$\forall u \in O_1, \pi_{i+1}[u] = \pi_i[u]$$

$$\forall u \in O_2, \pi_{i+1}[u] = \pi_i[u]$$

It induces the identification of an invariant and variant part in the CBS computation as detailed below.
2.2.2 Invariant terms

Considering that vertices in $O_1$ and $O_2$ have the same labels in $\pi_i$ and $\pi_{i+1}$, we have

$$CBS^{(i+1)}(O_1, O_1) = CBS^{(i)}(O_1, O_1)$$  \hspace{1cm} (12)
$$CBS^{(i+1)}(O_2, O_2) = CBS^{(i)}(O_2, O_2)$$  \hspace{1cm} (13)
$$CBS^{(i+1)}(O_1, O_2) = CBS^{(i)}(O_1, O_2)$$  \hspace{1cm} (14)

Similarly, as the labels of vertices in $P$ are all incremented by 1, we have:

$$CBS^{(i+1)}(P, P) = CBS^{(i)}(P, P)$$  \hspace{1cm} (15)

2.2.3 Variant terms

Following the definition of $d_e$ given in Eq. (4), for each edge we have to determine which term between $|\pi[u] - \pi[v]|$ and $n - |\pi[u] - \pi[v]|$ is the minimum, both at index $i$ and $i + 1$. This leads to the following theorems. Only Theorem 1 is proven, since the proofs of Theorems 2, 3, 4 and 5 follow the same reasoning.

Edges between $k$ and the vertices of $O_1$:

**Theorem 2.1.** Let $u \in O_1$ and $\Delta = \pi_i[k] - \pi_i[u]$. We have:

1. If $\Delta \leq \frac{n}{2}$ then $CBS^{(i+1)}(k, u) = CBS^{(i)}(k, u) - p$.
2. If $\Delta \geq \frac{n}{2} + p$ then $CBS^{(i+1)}(k, u) = CBS^{(i)}(k, u) + p$.
3. If $\frac{n}{2} < \Delta < \frac{n}{2} + p$ then $CBS^{(i+1)}(k, u) = CBS^{(i)}(k, u) + 2\Delta - (n + p)$

**Proof.** For all $u \in O_1$, we have $\pi_{i+1}[u] = \pi_i[i] < \pi_{i+1}[k] < \pi_i[k]$ and thus $0 < \pi_{i+1}[k] - \pi_{i+1}[u] < \Delta$. This allows us to remove the absolute value in Eq. (4). The first term of the minimum function in Eq. (4) is used for $CBS^{(i)}(u, k)$ if

$$\pi_i[k] - \pi_i[u] \leq n - (\pi_i[k] - \pi_i[u])$$  \hspace{1cm} (16)
$$2(\pi_i[k] - \pi_i[u]) \leq n$$
$$\pi_i[k] - \pi_i[u] \leq \frac{n}{2}$$
$$\Delta \leq \frac{n}{2}$$

and for $CBS^{(i+1)}(u, k)$ if

$$\pi_i[k] - p - \pi_i[u] \leq n - (\pi_i[k] - p - \pi_i[u])$$  \hspace{1cm} (17)
$$2(\pi_i[k] - p - \pi_i[u]) \leq n$$
$$\pi_i[k] - p - \pi_i[u] \leq \frac{n}{2}$$
$$\Delta \leq \frac{n}{2} + p$$

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Symmetrically the second term of the minimum in Eq. (4) is used for CBS\(^{(i)}(u,k)\) if \(\Delta \geq \frac{n}{2}\) and for CBS\(^{(i+1)}(u,k)\) if \(\Delta \geq \frac{n}{2} + p\).

Then, using Eq. (8) and Eq. (10), there are 3 possible cases:

1. If \(\Delta \leq \frac{n}{2}\)

   \[
   CBS^{(i+1)}(k,u) - CBS^{(i)}(k,u) = (\pi_{i+1}[k] - \pi_{i+1}[u]) - (\pi_i[k] - \pi_i[u])
   = (\pi_i[k] - p - \pi_i[u]) - (\pi_i[k] - \pi_i[u])
   = -p
   \]  

2. If \(\Delta \geq \frac{n}{2} + p\)

   \[
   CBS^{(i+1)}(k,u) - CBS^{(i)}(k,u) = (n - (\pi_{i+1}[k] - \pi_{i+1}[u])) - (n - (\pi_i[k] - \pi_i[u])
   = -(\pi_i[k] - p - \pi_i[u]) + (\pi_i[k] - \pi_i[u])
   = p
   \]

3. \(\frac{n}{2} < \Delta < \frac{n}{2} + p\)

   \[
   CBS^{(i+1)}(k,u) - CBS^{(i)}(k,u) = (\pi_{i+1}[k] - \pi_{i+1}[u]) - (n - (\pi_i[k] - \pi_i[u])
   = (\pi_i[k] - p - \pi_i[u]) - n + (\pi_i[k] - \pi_i[u])
   = 2\Delta - (n + p)
   \]

\[\square\]

Edges between \(k\) and the vertices of \(O_2\):

**Theorem 2.2.** Let \(u \in O_2\) and \(\Delta = \pi_i[u] - \pi_i[k]\). We have:

1. If \(\Delta \leq \frac{n}{2} - p\) then \(CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) + p\).
2. If \(\Delta \geq \frac{n}{2}\) then \(CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - p\).
3. If \(\frac{n}{2} - p < \Delta < \frac{n}{2}\) then \(CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - 2\Delta + (n - p)\)

Edges between \(k\) and the vertices of \(P\):

**Theorem 2.3.** Let \(u \in P\) and \(\Delta = \pi_i[k] - \pi_i[u]\). We have:

1. If \((p + 1) - \frac{n}{2} \leq \Delta \leq \frac{n}{2}\) then \(CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - 2\Delta + (p + 1)\).
2. If \(\Delta > \frac{n}{2}\) then \(CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) - n + (p + 1)\).
3. If \(\Delta < (p + 1) - \frac{n}{2}\) then \(CBS^{(i+1)}(k,u) = CBS^{(i)}(k,u) + n - (p + 1)\)
Edges between $P$ and the vertices of $O_1$:

**Theorem 2.4.** Let $u \in P$, $v \in O_1$ and $\Delta = \pi_i[u] - \pi_i[v]$. We have:

1. If $\Delta \leq \frac{n}{2} - 1$ then $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) + 1$.
2. If $\Delta \geq \frac{n}{2}$ then $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) - 1$.
3. If $\frac{n}{2} - 1 < \Delta < \frac{n}{2}$ then $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v)$

Edges between $P$ and the vertices of $O_2$:

**Theorem 2.5.** Let $u \in P$, $v \in O_2$ and $\Delta = \pi_i[v] - \pi_i[u]$. We have:

1. If $\Delta \leq \frac{n}{2}$ then $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) - 1$.
2. If $\Delta \geq \frac{n}{2} + 1$ then $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v) + 1$.
3. If $\frac{n}{2} < \Delta < \frac{n}{2} + 1$ then $CBS^{(i+1)}(u,v) = CBS^{(i)}(u,v)$

Therefore, the CBS value can be updated just in considering the edges that involve $k$ or a vertex of $P$, thus greatly reducing the computation of the CBS.

### 2.3 Comments

#### 2.3.1 Influence of the initialization

Given an initial labeling, the algorithm is completely deterministic and several executions will lead to the same solution. The algorithm can nevertheless return different solutions for a same graph by changing the initial labeling. Three steps of the heuristic produce a stochastic behavior and all of them originate from the same statement: When a sort is realized, whatever the criterion of sorting, if several elements have the same value, then the first encountered by the algorithm is selected before the other ones. This happens when (1) the nodes are sorting according to the degree to select the first node of a path, (2) two neighbors of a given vertex have the same similarity value, (3) when the path insertion at several positions leads to the same CBS value, and (4) when several paths have the same length.

#### 2.3.2 Locally search against global search

A drawback of the heuristic is that it performs local search over the graph, the algorithm is prevented to jump to a node which is not a neighbor of the previous one. The labeling is hence really close to the structure of the graph. Nevertheless, some cases would lead to an optimal solution with a loss of regularity in the labeling of the graph, which cannot be achieved with the heuristic.
Algorithm 1 Minimization_Cyclic_Bandwidth_Sum

Require: $G = (V, E)$
Ensure: $\pi$ a one-to-one and onto mapping of $V$ to $\{0 \ldots n - 1\}$.

1: for all $u \in V$ do
2:   color[$u$] $\leftarrow$ white
3:   degree[$u$] $\leftarrow$ Degree$(G, u)$
4: end for
5: $S \leftarrow V$
6: Paths $\leftarrow$ List()
7: while $S \neq \emptyset$ do
8:   $u_0 \leftarrow \arg\min_{u \in S} \text{degree}[u]$
9:   $S \leftarrow S \setminus \{u_0\}$
10: if color[$u_0$] = white then
11:   $P \leftarrow \text{Find_best_path}(u_0, \text{color})$
12:   List-Insert(Paths, $P$)
13: end if
14: end while
15: Order $\leftarrow \text{arg\, max}_{P \in \text{Paths}} \text{length}(P)$
16: List-Remove(Paths, Order)
17: while Paths $\neq \emptyset$ do
18:   $P_0 \leftarrow \text{arg\, max}_{P \in \text{Paths}} \text{length}(P)$
19:   Order $\leftarrow \text{Merge_paths}(\text{Order}, P_0)$
20: end while
21: $i \leftarrow 0$
22: while Order.next $\neq \text{nil}$ do
23:   $\pi[\text{Order.key}] \leftarrow i$
24:   $i \leftarrow i + 1$
25: end while
26: Order $\leftarrow \text{Order.next}$
27: end while
28: return $\pi$

3 Detailed algorithm

Algorithm 1 presents the heuristic algorithm that labels the graph vertices to minimize Eq. (5). Let $G = (V, E)$ be a graph supposed to be connected with $n$ vertices. The algorithm outputs a one-to-one mapping $\pi$ from $V$ to $\{0, \ldots, n - 1\}$. While building the graph labeling, the algorithm tags the graph vertices in the following way: when a vertex has not been considered so far, its color is set to white and it becomes gray when it is inserted into a path. From lines 1 to 4, Algorithm 1 initializes the vertex color to white and computes the degree (i.e. the numbers of neighbors) of each node using the function Degree. From line 5 to 14, the first step of the heuristic is done. Every vertex is considered in the increasing order of degree: the less connected it is, the earliest it is considered. The sort is realized in line 8, where one of the vertices with
the minimal value of degree is retained. If this vertex has not been considered so far (line 10), a path is started from this node is searched using \texttt{Find\_best\_path} described in Algorithm 2 (line 11) and stored (line 12) in \texttt{Paths}, a list containing all the found paths. The second step of the heuristic is performed in lines 15 to 21: all the paths are merged to form the final labeling. The longest path is first considered and inserted in an empty list called \texttt{Order} (lines 15) and then removed from the list of paths \texttt{Paths}. All the remaining paths are then considered one-by-one from the longest one to the shortest one and inserted in the list \texttt{Order} using the function \texttt{Merge\_paths} described in Algorithm 3 (lines 15 and 21). Each path is inserted in the list \texttt{Order}. Finally, in lines 22 to 28, the labeling \( \pi \) is constructed by assigning the labels to the vertices following their order of appearance in the list \texttt{Order}.

\begin{algorithm}
\caption{\texttt{Find\_best\_path}(u, color)}
\begin{algorithmic}[1]
\Ensure \( P \) a path that starts from \( u \) and whose vertices are even closer than they are connected.
\State \texttt{End\_Of\_Path} \leftarrow \text{False}
\State \( P \leftarrow \text{nil} \)
\While {\texttt{End\_Of\_Path} = \text{False}}
\State \texttt{List\_Insert}(P, u)
\State \texttt{color}[u] \leftarrow \text{gray}
\State \( H \leftarrow \emptyset \)
\ForAll {\( v \in \text{adj}[u] \)}
\If {\texttt{color}[v] = \text{white}}
\If {\texttt{degree}[v] = 1}
\State \texttt{List\_Insert}(P, v)
\State \texttt{color}[v] \leftarrow \text{gray}
\Else
\State \( H \leftarrow H \cup \{v\} \)
\EndIf
\EndIf
\EndFor
\If {\( H \neq \emptyset \)}
\State \( J \leftarrow \{v \in H \mid MIJ(u, v) = \max_{w \in H} MIJ(u, w)\} \)
\State \( u \leftarrow \arg \min_{w \in J} \text{degree}[w] \)
\Else
\State \texttt{End\_Of\_Path} \leftarrow \text{True}
\EndIf
\EndWhile
\State \Return \( P \)
\end{algorithmic}
\end{algorithm}

Algorithm 2 computes a path whose first node \( u \) is given in input. Lines 1 and 2 initialize the Boolean variable \texttt{End\_Of\_Path} which indicates if the path is finished, and \( P \) as an empty list describing the path. While the path can be extended, the current node \( u \) is added at the end of the path \( P \) (line 4) and its color is set to \texttt{gray} (line 5). A list \( H \) is initialized in line 6 and will store all the vertices that can potentially extend
the path. From line 7 to line 16, the vertices in \text{adj}[u] of neighbors of \( u \) whose color is white, are either immediately inserted into the path, if they are only connected to \( u \) (lines 9 to 11), or inserted into the list \( H \). From line 17 to 22, this list is iterated to select the vertices with the highest similarity index with \( u \) using the function \( \text{MIJ} \) described in Section 2, and among these vertices (line 18), to select the one with the smaller degree (line 20). If the list \( H \) is empty, i.e. the node \( u \) has no neighbor not yet inserted in a path, then the variable \text{End}_\text{Of}_\text{Path} is set to \text{True} and the function returns \( P \).

Algorithm 3 \text{Merge}_\text{paths}(\text{Order}, \text{P})

\textbf{Ensure:} Best\text{Order}, an arrangement of the vertices of \text{Order} and \text{P} that minimizes CBS.
\begin{itemize}
    \item line 1: Forward\text{Best\_Position}, Forward\text{Best\_CBS} \leftarrow \text{Incremental\_CBS}(\text{Order}, \text{P})
    \item line 2: Backward\text{Best\_Position}, Backward\text{Best\_CBS} \leftarrow \text{Incremental\_CBS}(\text{Order}, \text{Reverse}(\text{P}))
    \item line 3: if Forward\text{Best\_CBS} \leq Backward\text{Best\_CBS} then
    \item line 4: return \text{Insert\_list}(\text{Order}, \text{P}, \text{Forward\_Best\_Position})
    \item line 5: else
    \item line 6: return \text{Insert\_list}(\text{Order}, \text{Reverse}(\text{P}), \text{Backward\_Best\_Position})
\end{itemize}

Algorithm 3 looks for the best position to insert \( \text{P} \) into \text{Order}. The algorithm consists in inserting \( \text{P} \), both forward and backward, at every position in the order list and to retain the one that minimizes the CBS (see Eq. (5)) evaluated over the edges whose adjacent vertices belong to \text{Order}. The CBS is computed incrementally for all the possible indices as described in Section 2.2.1. The function \text{Incremental\_CBS}, not described here, returns the best position to insert \( \text{P} \) in \text{Order} such that the CBS, restricted to the nodes in \text{Order} and \( \text{P} \), is minimized, as well as the value of the CBS. This function is run for the path forward (line 1) and the path backward (line 2). According to the value of the CBS in both cases, the suitable merger is returned (lines 3 to 7).

4 Worst case complexity of the algorithm

We now examine the worst-case time complexity of Algorithm 1 when applied on a connected graph \( G(V, E) \) with \( \#V = n \) and \( \#E = m \). The loop on lines 1-4 takes time \( O(n) \). The set \( S \) initialized line 5 can be implemented as a min-priority queue with a binary min-heap. The time to build the binary min-heap is \( O(n) \) (line 5). Lines 8 and 9 can be done using the \text{EXTRACT-MIN} function that takes time \( O(\log n) \). Similarly, the set \( \text{Paths} \) can be implemented as a max-priority queue with a binary max-heap and line 12 takes in the worst case a time proportional to the logarithm of the number of paths, that is in the worst case \( O(\log n) \). Therefore, the loop lines 7 to 14 takes time \( O(n\log n) \) without considering the time required by \text{Find\_best\_path}.

Using aggregate analysis, the while loop of the function \text{Find\_best\_path} is executed at most once for each vertex of \( V \), since the vertex \( u \) is always white at the
beginning of the loop and the instruction line 5 of the function consists in painting the vertex in gray. The function List-Insert is in constant time. The set $H$ of white vertices that are adjacent to $u$ is implemented as a max-priority queue using a binary heap data structure that makes possible to run MAX_HEAP_INSERT, that inserts a new element into $H$ (line 13) while maintaining the heap property of $H$ in $O(\log(#H))$, that is in the worst case in $O(\log(#\text{adj}[u]))$. Thus, the loop on lines 7-16 in Algorithm 2 is executed #adj[u] times and at each iteration (1) the modified Jaccard index computation takes time $\Theta(\min(#\text{adj}[u],#\text{adj}[v]))$ and (2) MAX_HEAP_INSERT takes time $O(\log(#\text{adj}[u]))$. Therefore, loop lines 7-16 is in $O(#\text{adj}[u]^2)$. Lines 18 and 19 can be done using EXTRACT_MAX in time $O(\log(#\text{adj}[u]))$ and the total complexity of lines 3-22 is in $O(#\text{adj}[u]^2)$. Consequently, the total time spend by Find_best_path is $O(\sum_{u \in V} #\text{adj}[u]^2)$. As K. Das established in [11] that

$$\sum_{u \in V} #\text{adj}[u]^2 \leq m \left( \frac{2m}{n-1} + n - 2 \right)$$

we can conclude that the total cost of Algorithm 2 is bounded by $O(mn)$. We can also use an aggregate analysis to evaluate the time taken by the Merge_paths function (see Algorithm 3). The function is called for each path computed by the calls of Find_best_path and each vertex belongs to one and only one of these paths. Lines 1 and 2 takes time $O(n)$, when almost all the vertices have already been merged in Order. Incremental_CBS runs through (1) all the edges between the vertices of the current path $P$ and the ones of Order and (2) between the vertex of Order at position position and the other vertices of Order $\cup P$:

(1) takes $O(mn)$ since all the edges of the graph are examined when aggregating the analysis over the all paths: the adjacency list of each vertex is examined once. Furthermore, for each of these $m$ edges, the $n$ positions of Order are evaluated.

(2) is also in $O(mn)$ since aggregating the adjacency lists of the vertices of Order leads to the $m$ edges of the graph that are evaluated for each of the at most $n$ paths.

The other instructions of the loop are executed in constant time. Therefore, the total time spend in Merge_paths function is $O(mn)$.

Finally, coming back to Algorithm 1 we can observe that this algorithm has a time complexity in $O(mn)$ since the loop lines 1 to 6 takes $O(n)$, the loop on lines 7-14 is in $O(n \log n + mn) = O(mn)$, lines 15-16 takes $O(\log(n))$, the loop on lines 17-22 takes also $O(mn)$ and the last loop on lines 23-28 is in $O(n)$.

5 Computational experiments

This section describes the computational experiments that we carried out to assess the performance of the heuristic Minimization_Cyclic_Bandwidth_Sum (MCBS) proposed in the previous sections. The aim of this part is to test the ability of the algorithm to obtain a good approximate solution for the CBSP, in a reasonable amount of time. This is achieved by devising four types of experiments:
1. **Performance on graphs with known optimal value of CBS**: the efficiency of the heuristic is studied by comparing the value of CBS obtained after relabeling with the optimal value which is known for some regular graphs. A study of the computational time for these graphs is also realized to highlight the behavior of the heuristic on extreme cases.

2. **Comparison between GNVS and MCBS algorithms**: We compare the MCBS algorithm with the GNVS heuristic proposed in [28]. This study is realized on a restricted number of graphs when the comparison is feasible.

3. **Statistical analysis of the MCBS heuristic**: a more extensive analysis of the behavior of the algorithm is performed on a representative collection of graphs, where the minimal value of CBS is reported as well as its distribution in respect of the number of repetitions.

4. **Comparison between the MCBS algorithms and some of its variants**: We compare the heuristic with some variants so as to highlight some choices in the design of the algorithm, as described in Section 2.

The code for the MCBS algorithm has been implemented in Python 2.7 using the module Networkx [16] for the graph representation and tools from the graph theory. The code for the GVNS has been provided by the authors and has been run using Matlab© R2012b. All the experiments were conducted on a 2.60 GHz Intel© Core™ i7 with 8 GB of RAM.

5.1 **Performance on graphs with known optimal value of CBS**

[21] and [6] proposed theoretical studies of the optimal values for the CBS when the graph has strong constraints on its structure. The following list gives the graph whose optimal value of CBS is known. For each graph, a brief description is given as well as the mathematical expression of the optimal value of CBS. $n$ indicates the number of nodes in the graph. A graphic representation of these graphs is given in Fig. 1. More details about these graphs can be found in [25].

**Path**  A path $P_n$ in a simple graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence.

$$CBS_{opt}(P_n) = n - 1$$

**Cycle**  A cycle $C_n$ is a path graph whose start node and the end node are the same.

$$CBS_{opt}(C_n) = n$$

**Wheel**  A wheel $W_n$ consists in a cycle of length $n - 1$ in which each node is linked to a central node called *hub*.

$$CBS_{opt}(W_n) = n + \lfloor \frac{1}{4}n^2 \rfloor$$
Complete bipartite

A complete bipartite graph $K_{m_1, m_2}$ is a simple bipartite graph with $n = m_1 + m_2$ such that two vertices are linked if and only if they are in different sets.

$$\text{CBS}_{opt}(K_{m_1, m_2}) = \begin{cases} 
\frac{m_1 m_2^2 + m_2^3}{4} & \text{if } m_1 \text{ and } m_2 \text{ are even} \\
\frac{m_1^2 m_2 + m_1^2 m_2 + m_1}{4} & \text{if } m_1 \text{ is even and } m_2 \text{ is odd} \\
\frac{m_1^2 m_2^2 + m_2^3 m_2 + m_1 m_2}{4} & \text{if } m_1 \text{ and } m_2 \text{ are odd} \\
\frac{m_1^2 m_2 + m_2^3 m_2 + m_2}{4} & \text{if } m_1 \text{ is odd and } m_2 \text{ is even}
\end{cases}$$

Performances

Two procedures have been used for the experiments on the graphs described above: For the path, the cycle and the wheel, the number of nodes $n$ has been tested for all values between 10 to 500. For the complete bipartite graph, three different values of the ratio $\frac{m_1}{m_2}$ have been tested: $1, \frac{1}{3}$ and $\frac{1}{7}$. All values of $n$ between 10 and 500 leading to whole values for $m_1$ and $m_2$ have been retained. For both procedure, 20 repetitions have been made for each $n$, and an initial labeling is randomly drawn before each repetition.

For all the instances, the MCBS algorithm has been able to achieve the optimal cyclic bandwidth sum given by the theoretical results.

| $n$ | Path | Cycle | Wheel | CBP 1 | CBP $\frac{1}{3}$ | CBP $\frac{1}{7}$ |
|-----|------|-------|-------|-------|-------------------|-------------------|
| 64  | 0.00 | 0.00  | 0.00  | 0.01  | 0.49              | 0.35              |
| 128 | 0.00 | 0.00  | 0.02  | 0.22  | 6.21              | 4.34              |
| 192 | 0.00 | 0.00  | 0.05  | 1.19  | 27.35             | 19.22             |
| 256 | 0.01 | 0.01  | 0.14  | 3.74  | 80.52             | 56.35             |
| 320 | 0.01 | 0.01  | 0.24  | 9.23  | 189.03            | 131.16            |
| 384 | 0.01 | 0.01  | 0.38  | 19.52 | 386.64            | 261.15            |
| 448 | 0.01 | 0.01  | 0.60  | 36.85 | 685.64            | 475.96            |

Table 1: Averaged execution time in seconds of the algorithm for different values of $n$ for the standard graphs

Tab.1 shows the averaged execution time in seconds of the algorithm for different values of $n$ for all the studied graphs. The computational cost of the algorithm hugely increases when the graph is the complete bipartite graph, which can be explained by the peculiar structure of these graphs. Indeed, the algorithm will first of all compute a first path containing all the nodes of the smaller subset and the same number of nodes in the other one. All the remaining nodes will be considered as a path of length 1 (since they are isolated when the smaller subset is removed), and the algorithm will spend a huge amount of time to merge one by one all these nodes with the first path. This very greedy step makes explode the computational cost when $n$ increases for these graphs.
5.2 Comparison between GNVS and MCBS algorithms

A comparison between the MCBS algorithm, proposed in this paper, and the GVNS heuristic, introduced in [28], is realized in the following. Two collections of graph are used to compare the two methods: the first one consists of the Cartesian products of nodes, leading to highly regular graphs, and the second one is called the Harwell-Boeing collection, a set of matrices based on real-world applications. In order to fairly compare the two methods, both in terms of efficiency and time execution, we adopted the following common procedure for the two heuristics:

- the number of repetitions is set to 20.
- before each repetition, the labeling is randomly drawn.

It is worth to note that the two heuristics have not been implemented using the same technology: while the code for the GNVS procedure, provided by the authors, has been implemented using Matlab ©, our procedure has been implemented using Python. Nonetheless, the same computational environment has been used and enables us to compare the order of magnitude of the computational cost of the two heuristics.

5.2.1 Cartesian products

The Cartesian product of two graphs $G = (V_G, E_G)$, with $\#V_G = n$, and $H = (V_H, E_H)$, with $\#V_H = m$, is denoted $G \times H$, and is the graph with vertex set $V_G \times V_H = \{(u, v) | u \in V_G, v \in V_H\}$. The vertices $(u_G, u_H)$ and $(v_G, v_H)$ are adjacent if and only if $u_G = v_G$ and $(u_H, v_H) \in V_H$ or $u_H = v_H$ and $(u_G, v_G) \in V_G$.

[21] gives tight upper bound of the optimal value of CBS for the Cartesian products of graph using the optimal values of CBS and BS (bandwidth sum) of the graph $G$ and $H$ are known:

$$CBS_{opt}(G \times H) \leq \min\{nBS_{opt}(G) + m^2CBS_{opt}(H), mBS_{opt}(H) + n^2CBS_{opt}(G)\}$$

Note however that the bound gives no idea about the best labeling as for graphs whose optimal value is known.

We focus on our experiments on the Cartesian products of paths and cycles. The optimal value of CBS for the two graphs are given above. The optimal value of BS for the path and the cycle are easily obtained: $BS_{opt}(P_n) = n - 1$ and $BS_{opt}(C_n) = 2(n - 1)$. The number of vertices for the paths and the cycles varies from 5 to 9.

Tab. 2 gives the results of the two heuristics for the Cartesian products of paths and cycles. For each heuristic, the columns $minCBS$ and $medianCBS$ give respectively the minimal value and the median of the CBS over the 20 repetitions while the column $avgT$ returns the execution time of each relabeling averaged over the 20 repetitions. The column $ub$ gives the theoretical upper bound.

For all the Cartesian products, except for the smaller ones, the MCBS heuristic is better than GVNS. The performance of MCBS is nevertheless still acceptable since the obtained CBS are lower than the theoretical upper bound, are really close to the
Table 2: Comparison between MCBS and GNVS on Cartesian products of cycles and paths. Each line corresponds to the graph $G \times H$, where $P_n$ indicates a path graph with $n$ nodes and $C_n$ a cycle graph with $n$ nodes. For each heuristic, the columns \textit{minCBS} and \textit{medianCBS} give respectively the minimal value and the median of the CBS over 20 repetitions while the column \textit{avgT} returns the execution time of each relabeling averaged over 20 repetitions. The column \textit{ub} gives the theoretical upper bound. The value in bold refers to the minimal value of CBS obtained between the two procedures.
obtained value using GVNS. However, despite the good performance of MCBS compared to GVNS, the CBS obtained for Cartesian products of paths and cycles is always greater than the theoretical upper bound. It can be explained by the high regularity of these graphs, leading to optimal solutions which do not follow exactly the structure of the graph.

The comparison of the averaged execution time of the two heuristics gives a clear advantage to the MCBS algorithm, one instance being almost instantaneous.

### 5.2.2 Harwell-Boeing collection

Experiments on real-world graphs have been made using graphs from the Harwell-Boeing Sparse Matrix Collection Graphs [14] which consist of a set of standard matrices arising from various problems in engineering and scientific fields. Graphs are derived from these matrices as follows: Let $M_{ij}$ be the element of the $i$th row and the $j$th column of a sparse matrix $M$ of size $n \times n$, the resulting graph has $n$ vertices such that there is an edge between vertices $i$ and $j$ if and only if $M_{ij} \neq 0$ and $i \neq j$. We selected 10 matrices from this collection, leading to a collection of 10 graphs, all of them are unweighted and undirected ones.

| Graph | Gn | m | minCBS | medianCBS | avgT | minCBS | medianCBS | avgT |
|-------|----|---|--------|-----------|------|--------|-----------|------|
| can62 | 62 | 78 | 243    | 263       | 0.02 | 557    | 697       | 62.81|
| can73 | 73 | 152| 1000   | 1067      | 0.09 | 1723   | 1966      | 131.33|
| can61 | 61 | 248| 1364   | 1611      | 0.02 | 2304   | 2577      | 213.39|
| can24 | 24 | 68 | 216    | 259       | 0.01 | 215    | 255       | 19.28 |
| can144| 144| 576| 2250   | 2256      | 0.01 | 15040  | 15937     | 1278.03|
| ash85 | 85 | 219| 1152   | 1358      | 0.12 | 2276   | 2994      | 194.47|
| dwt72 | 72 | 75 | 204    | 225       | 0.04 | 566    | 687       | 51.88 |
| bcsprw01|39|46|106    |117        |0.01 |211    |246        |28.24 |
| bcsprw02|49|59|159    |173        |0.01 |338    |422        |42.96 |
| bcsprw03|118|179|820    |974        |0.22 |2950   |3382       |275.90|

Table 3: Comparison between MCBS and GNVS on graphs of the Harwell-Boeing collection. The name of graphs are given in the first column. $n$ indicates the number of nodes and $m$ the number of edges. For each heuristic, the columns $minCBS$ and $medianCBS$ give respectively the minimal value and the median of the CBS over 20 repetitions while the column $avgT$ returns the execution time of each relabeling averaged over 20 repetitions. The value in bold refers to the minimal value of CBS obtained between the two procedures.

Tab. 3 gives the results of the two heuristics for the Cartesian products of paths and cycles. As previously, for each heuristic, the columns $minCBS$ and $medianCBS$ give respectively the minimal value and the median of the CBS over the 20 repetitions while the column $avgT$ returns the execution time of each relabeling averaged over the 20 repetitions.
As for Cartesian products, the results clearly show the superiority of MCBS over GVNS, both in terms of performance and execution time.

### 5.3 Statistical analysis of the MCBS heuristic

The two previous experiments show the ability of the MCBS algorithm to achieve a good solution which minimizes the CBS. Nevertheless, as described in Section 2.3.1, the initial labeling has an influence on the performance of the heuristic. This point is dealt with by repeating the algorithm a number of times noted \( k \), and by selecting the best value of CBS over these \( k \) repetitions.

In the following, the correct way to choose the number of repetitions \( k \) is studied, and by extension the impact of the initial labeling on the value of CBS. It consists of performing 100 repetitions of the following process: repeat the algorithm \( k \) times with an initial labeling randomly drawn before each repetition, and select the best value of CBS achieved. A distribution of the minimal value of CBS obtained when repeating the algorithm \( k \) times is therefore analyzed. This technique is applied on three types of graph: Cartesian products (see Section 5.2.1), a small subset from the Harwell-Boeing collection (see Section 5.2.2) and random graphs generated using an Erdős Rényi model [25].

After computation, the median \( q_{50} \) and the 90th percentile \( q_{90} \) of the distribution are returned. These results are expressed as relative distance with respect to the minimal value of CBS \( \text{minCBS} \) obtained over all the instances, set as reference for the studied graph. This relative distance \( dq \) is defined as follows for the value \( q_{50} \):

\[
dq_{50} = \frac{q_{50} - \text{minCBS}}{\text{minCBS}}
\]

and correspondingly for \( q_{90} \). The smaller \( dq \) is, the better the algorithm is.

#### 5.3.1 Cartesian products

Tab. 4 gives the results of the statistical analysis for the Cartesian products of cycles and paths. We can make the following comments. First the relabeling of Cartesian products of a path with a cycle leads, whatever the number of vertices, to the same value of CBS. This is due to the peculiar topology of these graphs: the vertices of the Cartesian product of a path lie on a cylinder, and then the algorithm has always the same behavior: that is starting from one vertex of the first cycle, travel around the cycle and then jump to the next cycle and so on. Second, the maximal relative distance \( dq_{90} \) is obtained for the Cartesian products of a cycle with 7 vertices and a cycle with 8 vertices and is equal to 0.29. It means that the fluctuation of 90 % of the minimal values of CBS over \( k \) repetitions is at most 30 % greater than the minimal value of CBS obtained over numerous repetitions, even when \( k \) is low. Finally, the increase of the number of repetitions \( k \) leads unsurprisingly to an increase in the performance of the algorithm. Nevertheless, as described previously, even when \( k \) is low the deviation compared to the best results obtained is limited.
Table 4: Results of the statistical analysis for the Cartesian products of cycles and paths. Each line corresponds to the graph $G \times H$, where $P_n$ indicates a path graph with $n$ nodes and $C_n$ a cycle graph with $n$ nodes. The column $dq_{50}$ (respectively $dq_{90}$) represents the relative distance between the median (respectively the 90th percentile) of the distribution of the minimal value of CBS obtained over $k$ repetitions, and the minimal value over all the repetitions.
5.3.2 Harwell-Boeing collection

Tab. 5 gives the results of the statistical analysis for a reduced subset of the Harwell-Boeing collection. We choose the “bcspwr” graphs so as to have a collection of increasing number of nodes and number of edges (up to 443 nodes). The first three columns give the name of the graph as well as its number of nodes and number of edges. An important result is that the higher the number of nodes is, the higher the deviation is. This can be explained by high amount of possibilities of labeling. A solution is hence to adapt the number of repetitions to the number of vertices in the graph. Nonetheless even with a low number of repetitions, the deviation are still lower than a third of the reference value.

5.3.3 Random graphs

A random graph is a graph where the edges between the vertices is set randomly. There exists many models to build random graphs [25]. We selected the simplest one called Erdős-Rényi model: for each pair of vertices, an edge between the two vertices has a probability $p$ to appear, with $p$ is an input of the model. The obtained graph does not have a well-structured topology. The performance of MCBS is studied on these random graphs, using the same procedure as previously. Three values for $p$ have been chosen, from the less dense graphs ($p = 0.3$) to the denser ($p = 0.7$).

Tab. 6 gives the results of the statistical analysis for the random graphs. As for previous graphs, the results show that the deviation from the minimal value obtained is limited even when the number $k$ of repetitions is low.

5.4 Comparison between the MCBS algorithms and some of its variants

We study in this section four variants of the algorithm MCBS in order to justify the validity of the design of the heuristic. Each variant answers to a choice of design:

| Graph | $k = 10$ | $k = 20$ | $k = 50$ |
|-------|---------|---------|---------|
|       | $q_{50}$ | $q_{90}$ | $q_{50}$ | $q_{90}$ | $q_{50}$ | $q_{90}$ |
| bcsfwr01 | 0.04 0.06 | 0.02 0.05 | 0.04 0.04 |
| bcsfwr02 | 0.02 0.05 | 0.01 0.03 | 0.00 0.01 |
| bcsfwr03 | 0.15 0.19 | 0.10 0.17 | 0.11 0.13 |
| bcsfwr04 | 0.13 0.24 | 0.13 0.18 | 0.08 0.08 |
| bcsfwr05 | 0.18 0.24 | 0.15 0.20 | 0.13 0.17 |

Table 5: Results of the statistical analysis for a reduced subset of the Harwell-Boeing collection. The first three columns give the name of the graph as well as its number of nodes and number of edges. The column $d_{q_{50}}$ (respectively $d_{q_{90}}$) represents the relative distance between the median (respectively the 90th percentile) of the distribution of the minimal value of CBS obtained over $k$ repetitions, and the minimal value over all the repetitions.
Table 6: Results of the statistical analysis for three graphs obtained using the Erdős-Rényi model with \( p \) equals to 0.3, 0.5 and 0.7. The column \( dq_{50} \) (respectively \( dq_{90} \)) represents the relative distance between the median (respectively the 90th percentile) of the distribution of the minimal value of CBS obtained over \( k \) repetitions, and the minimal value over all the repetitions.

|       | \( k = 10 \) | \( k = 20 \) | \( k = 50 \) |
|-------|-------------|-------------|-------------|
| \( p \) | \( dq_{50} \) | \( dq_{90} \) | \( dq_{50} \) | \( dq_{90} \) | \( dq_{50} \) | \( dq_{90} \) |
| 0.3   | 0.04 0.11   | 0.03 0.04   | 0.03 0.05   |
| 0.5   | 0.06 0.09   | 0.04 0.06   | 0.02 0.03   |
| 0.7   | 0.10 0.16   | 0.06 0.08   | 0.05 0.06   |

**Variant 1: Selection of the first node** As described in Section 2.1.1 there are several methods to select the first node of a path in the first step of the algorithm. We compare here the original version where the unvisited vertex with the smaller degree is selected, with a variant where the degree is replaced by the betweenness centrality \( [25] \), a measure giving for each node, its importance in the graph.

**Variant 2: Sort of the paths** The step 2 of the algorithm 2.2 consists of merging all the paths obtained in step 1. We compare in the following the original version where the paths are considered from the longest ones to the shortest ones with a version where the sort is reverse.

**Variant 3: Computation of the similarity index** In step 1, the next node in the depth-first search is chosen according to the similarity of its neighborhood to the one of the current vertex (see 2.1.2). We proposed a refinement where the neighborhood is restricted to the unvisited nodes. A variant without this refinement is tested.

**Variant 4: Calculation of CBS** In the step 2 of the algorithm 2.2 two paths are merged by inserting one of the path in the other one. The position in which the path is inserted is chosen such that the CBS is minimized. It is hence necessary to compute the CBS for all possible positions. We proposed an incremental version of this computation, compared with a variant of MCBS where the CBS is directly computed for all positions.

All the variants are compared to the original algorithm MCBS on a reduced but representative collection of graphs:

- \( (cbp) \) Complete bipartite graph with \( m_1 = 20 \) and \( m_2 = 60 \) (cbp)
- \( (cp) \) Cartesian products \( P_{20} \times P_{15} \)
- \( (can) \) The graph can144 with 144 nodes and 576 edges, from the Harwell-Boeing collection
- \( (er) \) A random graph of type Erdős-Rényi with 100 vertices and a density of edges of 0.4
| G   | MCBS CBS | MCBS avgT | Variant 1 CBS | Variant 1 avgT | Variant 2 CBS | Variant 2 avgT | Variant 3 CBS | Variant 3 avgT | Variant 4 CBS | Variant 4 avgT |
|-----|----------|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| cbp | 24000    | 1.10      | 24000          | 1.20           | 26680          | 0.45           | 24000          | 1.04           | 24000          | 20.98          |
| cart| 9139     | 0.02      | 12049          | 0.38           | 8989           | 0.02           | 5455           | 2.86           | 9421           | 0.02           |
| can | 2250     | 0.02      | 2253           | 0.12           | 2250           | 0.02           | 11110          | 0.01           | 2250           | 0.02           |
| er  | 48960    | 0.28      | 50300          | 0.57           | 50088          | 0.21           | 49114          | 0.21           | 48960          | 1.61           |

Table 7: Comparison between MCBS and four variants. The columns CBS and avgT indicate respectively the minimal value of CBS and the average execution time over 50 repetitions. The results clearly show that all variants leads to degraded performances, either in terms of CBS or in terms of computational cost.

Tab. 4 gives the results of experiments: for the original version of MCBS and each variant, the columns CBS and avgT indicate respectively the minimal value of CBS and the average execution time over 50 repetitions. Comparing Variant 1 and MCBS shows that the performance are really close, except that Variant 1 is much more slow on cart and can. This is due to the higher computational complexity of the betweenness centrality, which requires the calculation of all shortest paths, compared to the degree. Graphs cbp and er are not affected since the diameter of those graphs are really small, and hence the computation of the shortest paths between all pairs of vertices is faster than graphs where the diameter is larger. Variant 2 is also not relevant to consider since it fails to achieve the optimal solution for the complete bipartite graphs when the theoretical optimal value of CBS is known. Variant 3 leads to more ambiguous results, as the obtained CBS is clearly worse compared to the original MCBS for the graph “can” while the performance is better for the Cartesian product. Nonetheless this result is achieved with a high computational cost. Finally, Variant 4 leads to similar results in terms of CBS, but with a much higher execution time, especially for the complete bipartite graph which require many mergers of paths due to its peculiar structure.

6 Extension and application

6.1 The case of weighted graphs

We focus our study on unweighted graphs, which is the most common case to consider graph labeling problem. It is relevant nevertheless to consider these problems for more general graphs. Up to our knowledge, there is no theoretical study about the cyclic bandwidth sum problem when the graph is considered as weighted. It is nonetheless possible to define this problem by taking into account the weight of each edge in the sum of difference of labels. If we note $w_{uv}$ the weight between adjacent vertices $u$ and $v$, we have

$$\min_{\pi} f(\pi) \quad \text{with} \quad f(\pi) = \sum_{\{u,v\}\in E} w_{uv}d_H(\pi(u), \pi(v))$$

$$\text{(21)}$$

The algorithm we propose to deal with weighted graph is very similar to the one in the unweighted case. Two minor modifications have to be taken into account: the
computation of the Jaccard index and the incremental computation of CBS in step 2. The first problem can be addressed by defining a weighted Jaccard index between two vertices $u$ and $v$ as the following:

$$J_w(u, v) = \frac{N(u, v)}{D(u, v)}$$  \hspace{1cm} (22)$$

where $N(u, v)$ represents the weight of neighbors shared by the two vertices and is defined by

$$N(u, v) = 2w_{u,v} + \sum_{x \in V \atop x \sim u \atop x \sim v} \min(w_{ux}, w_{vx})$$  \hspace{1cm} (23)$$

and $D(u, v)$ represents the total weight of neighborhood of $u$ and $v$, and is defined by

$$D(u, v) = 2w_{u,v} + \sum_{x \in V \atop x \sim u \atop x \sim v} \frac{w_{ux} + w_{vx}}{2} + \sum_{x \in V \atop x \sim u \atop x \sim v} w_{ux} + \sum_{x \in V \atop x \sim u \atop x \sim v} w_{vx}$$  \hspace{1cm} (24)$$

We can note that if all weights are set to 1 i.e. the graph is unweighted, we have $N(u, v) = 2 + \sharp\{\text{Common neighbors of } u \text{ and } v\}$ and $D = 2 + \sharp\{\text{All neighbors of } u \text{ and } v\}$ which corresponds to the Jaccard index defined previously.

Adaptation of the incremental CBS is also easy, as it only needs to multiply each term we add and remove by the weight of the considered edge.

![Figure 3: Weighted grid 5×15: the edges in columns have a weight of 10 while the edges in lines have a weight of 1. The numbering is coded as a color: two vertices with close colors are close in the labeling. The algorithm clearly colors vertex by following the column of the grid, i.e. the edges with a high weights.](image)

(a) Random labeling  (b) Weighted relabeling  (c) Unweighted relabeling

Figure 3: Weighted grid $5 \times 15$: the edges in columns have a weight of 10 while the edges in lines have a weight of 1. The numbering is coded as a color: two vertices with close colors are close in the labeling. The algorithm clearly colors vertex by following the column of the grid, i.e. the edges with a high weights.

There is no theoretical study about the weighted cyclic bandwidth sum problem to test the validity of the heuristic in the weighted case. Besides it could be quite tricky to
characterize the structure of a weighted graph. We propose here a simple approach to highlight the good behavior of the algorithm, by building a graph in the following way: starting from a grid with 15 lines and 5 columns, the links between vertices in the same column are set to a weight equal to 10 while the links between vertices in the same line are set to 1. Hence, the underlying structure is a grid whose columns have a higher importance (and hence should be favored by the algorithm when it will browse the vertices) than lines. Fig. 3 shows the results of the procedure on this graph. The numbering is coded as a color: two vertices with close colors are close in the labeling. The algorithm clearly colors vertex by following the column of the grid, i.e. the edges with a high weights. follows the columns.

6.2 Application in transforming graphs in time series

Graphs are objects which are well-fitted to represent networks, whether physical, biological or social. The study of these objects enables us to describe the underlying networks and hence describe, explain and model the studied system. If there exists many techniques to find properties on static graphs, for instance to find communities of vertices or to define diffusion processes, the study of temporal networks i.e. networks whose vertices and edges evolve over time, is a relatively new field of study. We proposed a signal processing approach to study temporal networks, based on a method introduced by [29]. Their method intends to transform a graph into a collection of signals using classical multidimensional scaling [2]. In [17, 18, 19], we extended this method to exhibit specific frequency patterns in temporal networks by applying the method on each snapshot of a dynamic graph and linking these frequency patterns with graph properties. It enables us to visualize the evolution of the frequency patterns over time and hence to monitor how the structure of the graph evolves at each time step.
A major issue of the transformation from graphs to signals concerns the indexation of signals, which is based on the labeling of vertices. If two neighboring vertices in the labeling are not adjacent in the graph, then their values in signals are different and the signals are blurred: It leads to abrupt variations of the signals over vertices and then it is more difficult to find relevant frequency patterns. To avoid these brutal variations in the signal, the labeling must then take into account the proximity between vertices and so on the structure of the graph. Fig. 4 shows an example of the consequence of a poor labeling to the resulting signals: transformation of a cycle graph leads to harmonic oscillations if the labeling follows the cycle, but to high-frequency signals if the labeling is random. This example highlights the fact that the labeling which solves CBSP is the labeling which is adapted to index the signals: the structure of the graph i.e. the relation of distance between vertices of the graph, is well-preserved in the CBSP labeling.

7 Conclusion

Finding a labeling which follows the structure of the graph is not an easy task, considering the high diversity of possible structures. We proposed in the previous sections a heuristic to solve this problem by considering a classical graph labeling problem called cyclic bandwidth sum problem. Based on a depth-first search controlled by a modified Jaccard index, this heuristic is able to achieve the optimal cyclic bandwidth sum given by theoretical studies for standard graphs, as well as to find good approximation for real networks, in a reasonable execution time. The labeling try at best to fit the topology of the graph without imposing any assumption on the structure. Many extensions of this algorithm can be considered, for weighted and directed graphs, with the same idea of taking into account the global structure of the network. Studies of the cyclic bandwidth sum problem are nonetheless complicated by the lack of theoretical knowledge on such objects.

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