Magnetohydrodynamic hybrid nanofluid flow with the effect of Darcy–Forchheimer theory and slip conditions over an exponential stretchable sheet

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Abstract
The characteristics of water-based hybrid nanofluid flow when passing over an exponentially stretchable sheet with velocity and thermal slip factors are presented. This article provides a concept for a hybridized fluid comprising copper and cobalt iron oxide nanoparticles (NPs) dispersed into a base fluid (water). In addition, physical observations of the heat absorption behavior, the Darcy effect, the thermal radiation, and viscous dissipation are also taken into account. Because of their strong thermophysical properties, copper and cobalt iron oxide NPs are used in a wide range of applications in the engineering and medical fields. To study the dynamics of these NPs, a system of partial differential equations (PDEs) has been generated that forms a highly nonlinear coupled model. The PDE system is converted into nondimensional ordinary differential equations (ODEs) with the aid of similarity replacements. The semi-analytical homotopy analysis method (HAM) is applied to the set of dimensionless ODEs obtained to find the solution. Two engineering parameters, the Nusselt number and the skin friction, are plotted versus various parameters of the hybridized fluid using bar charts. It was observed that the no-slip condition, the suction parameter, and the Darcy–Forchheimer medium enhanced the thermal profile of the hybridized fluid.

Keywords
Hybrid nanofluid, stretching sheet, chemical reaction, homotopy analysis method solution, magnetohydrodynamics, heat source

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Introduction
To reduce both costs and energy use significantly, intensification of thermal transfer is essential. Currently, advancements in science and technology are encouraging orders for compact devices with extraordinary features, the highest possible performance, precise operating characteristics, and long lifetimes. As a result, engineers and scientists have worked extensively on heat transmission analysis to upgrade the existing industrial work. A new class of nanotube suspensions created by Choi demonstrated the enormous improvement in the thermal transport characteristics of these...
solid suspensions, which were called nanofluids, when compared with ordinary fluids. Nanofluid technology has proven to be a helpful tool in the development of oils and lubricants with higher performance. Nanofluids play vital roles in various industrial applications, including fabrication of paints and coatings, paper printing, power generation, cancer therapy, ceramics manufacture, drug delivery, and food production. Punith et al.\textsuperscript{5} investigated nanofluid (ferromagnetic) flow with regard to the impact of a magnetic dipole over a stretching sheet and calculated that the thermal gradient was improved by augmentation of the Brownian motion and thermophoresis, with the Stefan blowing condition causing the nanofluid to display high heat transfer. Bhatti and Michaelides\textsuperscript{6} analyzed nanofluid flow over a Riga plate in terms of the Arrhenius activation energy and found that the fluid’s concentration profile was improved by augmentation of the activation energy; in this regard, the Brownian motion increased the temperature profile. Ullah\textsuperscript{7} found that a reduction in the Nusselt number occurred as a function of the Brownian motion and thermophoresis parameter with a decreasing temperature gradient; however, a reduction in the Sherwood number also occurred as a function of the Brownian motion and the thermophoresis parameter under a similarly decreasing temperature gradient. Alsallami et al.\textsuperscript{8} clearly explained the thermal transfer gradient and nanofluid flow behavior above a porous plate in terms of the impact of slip conditions; they claimed that addition of nanoscale particles with high thermal conductivity and a large volume fraction represents the best way to increase heat flux in nanofluids, but their simulation results contradicted the results reported previously by other researchers.\textsuperscript{9–14}

With regard to the development of these technologies, a new class of fluids for heat transfer has been introduced in the literature and has led to technological updates worldwide; these fluids are known as hybrid nanofluids. A stretchable sheet with the fluid motion characteristics described above in fluid mechanics terms is highly significant for both biomedical sciences and industrial processes. The ability to obtain a similar solution for fluid flow passing over a stretching sheet was first studied by Crane.\textsuperscript{33} However, when compared with the stretching scenario itself, the flow passing over such a stretchy sheet received little attention. Alharbi et al.\textsuperscript{34} stated that the flow over a shrinking sheet is a backward flow. With regard to this statement, Rashid et al.\textsuperscript{35} explored the research on viscous fluid flow over a nonlinearly stretching sheet and found that the flow features revealed many interesting behavioral aspects that confirmed further study of the effects of nonlinear stretching on the flow characteristics is warranted. Allhowaity et al.\textsuperscript{36} investigated the homotopy perturbation method and axisymmetric flow over a stretching sheet, and comparison of their results with the exact solutions showed that the homotopy perturbation investigated a computational model for hybridized fluid dynamics under convective conditions on a rotating sheet. Their results showed that as the stretching ratio increased, the velocity profiles of the CuO and TiO\textsubscript{2} hybrid base fluids decreased, while their velocity curves increased. Rashid et al.\textsuperscript{18} provided insight into the flow behavior of hybrid nanofluids with a physical perception of mixed convection over a vertical surface fixed by a porous medium, revealing that there are dual solutions for opposing flows. Zhao et al.\textsuperscript{19} studied the improvement in the thermal transmission of a hybridized fluid containing Ag + CuO + H\textsubscript{2}O NPs; their research showed that the chemical reaction, thermal gradient, and heat generation characteristics of the hybrid nanofluid were higher than those of nanofluids based on a single nanoparticle type, even in the presence of radiation. Tlili et al.\textsuperscript{20} proposed a model of hybrid nanofluid (methanol) 3D flow based on magnetohydrodynamics (MHD) over a surface with uneven thickness and slip factors. Their results determined that the thermal transfer rate of the hybridized fluid was significantly greater than that of the corresponding nanofluid. The Lorentz force effect was also smaller on the hybridized fluid when compared with the effect on the nanofluid. Usman et al.\textsuperscript{21} proposed a model of hybrid nanofluid MHD thin film flow over an unsteady spinning disk that considered the impact of couple stress, and the percentage improvement caused by heat transfer was calculated and shown graphically. Their investigation established that the hybridized fluid has a higher heat transmission capability when compared with that of conventional nanofluids. Ullah\textsuperscript{22} also investigated nanofluid and hybridized fluid flow with entropy generation in thermal structures. Many more articles are available in the literature that specify that the hybrid nanofluid represents an important alternative solution to traditional thermal systems.\textsuperscript{23–32}

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method gives outstanding results. The analysis of Bilal et al.\textsuperscript{37} of temperature field inflow over a stretching sheet with uniform heat flux showed that the temperature at a point declines with an upsurge in the Prandtl number. Zhao et al.\textsuperscript{38} explained the effects of slip on MHD viscous flow over a stretching sheet and obtained an exact solution to show that there is only one physical solution for any combination of the slip, the magnetic parameters, and the mass transfer parameters. Chu et al.\textsuperscript{39} explored the concept of a thermal gradient rate in viscoelastic fluid MHD flow over a stretchy sheet under the combined impact of variable thermal conductivity and a nonuniform heat source. Their results showed that in the boundary layer region, the combined effects of the radiation, the variable thermal conductivity, and the nonuniform heat source had a noteworthy effect on the control of the thermal gradient rate. The flow of fluid over a stretching sheet with respect to the effect of a magnetic field was studied using a power law model by Zhao et al.\textsuperscript{40} and they concluded that in this case, a thinner boundary layer occurred because of the effect of the magnetic field, thus causing increased wall friction.

As a result of the wide range of its potential applications, hybridized fluid motion through permeable porous surfaces has now become the center of attention for many researchers. Possible application areas of flow through permeable porous surfaces include the environmental field, the solar sciences field, nuclear engineering, and materials science. To understand this type of porous surface flow, Darcy’s law is erroneously but widely used. To use the permeability to calculate the effect of inertia, Forchheimer\textsuperscript{41} devised a second-order polynomial for use in the momentum equation. Muskat and Wyckoff\textsuperscript{42} undertook a study to identify the Forchheimer component. Darcy–Forchheimer concepts are used in many studies of the regularity of hybrid nanofluids, and several researchers have studied porous medium flow. A number of these studies are cited here. Ullah et al.\textsuperscript{43} investigated the Darcy–Forchheimer hybrid nanofluid flow through a porous medium with variable features. Ullah et al.\textsuperscript{44} analyzed the flow of a hybridized fluid under the influence of a Marangoni convective pattern through a Darcy–Forchheimer medium. Sun et al.\textsuperscript{45} performed a study of a Darcy–Forchheimer hybrid nanofluid with MoS\textsubscript{2} and SiO\textsubscript{2} nanoparticles with extension of the entropy generation. Bilal et al.\textsuperscript{46} proposed a model of heat and mass transfer through a Darcy–Forchheimer hybrid nanofluid flow over a stretching curved surface. Li et al.\textsuperscript{47} investigated a dusty hybrid nanofluid with two phases via extension of the viscous dissipation over a cylinder. Nazeer et al.\textsuperscript{48} proposed a model of the hybridized fluid dynamics due to solar radiation in a Darcy–Forchheimer permeable medium over a flat plate.

Inspired by the approaches described above, this work presents the results of development of a model of a sophisticated form of hybridized fluid that contains nanoparticles of Cu (copper) and CoFe\textsubscript{2}O\textsubscript{4} (cobalt ferrite) flowing over an exponentially stretching sheet. The core of the proposed model is the analysis of the effects of different factors on the flow and heat transport behavior of the hybrid nanofluid. In addition, the effects of various factors on the dynamics of the hybridized fluid regimes are sketched both graphically and statistically.

Novelty is added to the recent work by extending the published work of Ahmed and Akbar\textsuperscript{49} by addressing the following points:

- The recent work is based on a hybridized fluid composed of nanoparticles of Cu (copper) and CoFe\textsubscript{2}O\textsubscript{4} (cobalt ferrite) in a base fluid of H\textsubscript{2}O (water).
- The influence of the magnetic field is added to the recent work.
- The effects of thermal expansion, heat absorption, the Darcy–Forchheimer medium, and the velocity and thermal slip conditions of the fluid are added to the recent work.
- The focus of the recent work has been the thermal enhancement applications of this hybridized fluid of Cu and CoFe\textsubscript{2}O\textsubscript{4} nanoparticles in H\textsubscript{2}O.
- The role of the present work is to fill the gaps in the literature by attempting to solve the transformed system of equations (ODEs) with the aid of the homotopy analysis method (HAM) method\textsuperscript{50,51} using Mathematica software (Wolfram Research).

Mathematical Formulation

The electrically conducting and incompressible steady-state dynamics of a 2D Darcy–Forchheimer flow of H\textsubscript{2}O-based hybridized fluid past an exponentially stretching sheet are considered. The effects of partial slip and the thermal jump conditions on the dynamics are investigated. The nanoscale particles of copper (Cu) and cobalt ferrite (CoFe\textsubscript{2}O\textsubscript{4}) are immersed in the base fluid of water (H\textsubscript{2}O). To develop a mathematical model, a Cartesian coordinate system is proposed in which the stretchable sheet surface is aligned with the x-axis and the y-axis represents the vertical direction, with the flow field being constrained to the area where y>0. The geometry of the problem is depicted in Figure 1. In
establishing the governing equations for the flow of the hybridized fluid, the following assumptions are made:

- Because of the ambient velocity \( \varepsilon_1(x) \), the hybrid nanofluid flow permeates over the stretchy surface.
- We ignore a weak induced magnetic field generated by application of a magnetic field to the flow at an angle \( \theta \).
- The nanoparticles (Cu and CoFe\(_2\)O\(_4\)) immersed in the base fluid (H\(_2\)O) are supposed to be in the state of thermal equilibrium, and therefore no-slip phenomena occur between them.
- Combining the concepts of optically thick radiation, heat absorption, and viscous dissipation enhances the heat transfer rate.
- \( T_h(x) \) represents the temperature of the hybridized fluid at the stretchable sheet surface, while \( T_e \) represents the ambient temperature of the hybrid nanofluid.
- Because no externally applied electric field is present, the polarization effect can be overlooked.

The constitutive momentum flow equation and the energy equation for the hybrid nanofluid are then determined as follows using the assumptions above:\(^5\)\(^2\)

\[
\frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu_{\text{hnf}} \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_{\text{hnf}}}{\rho_{\text{hnf}}} B_0^2 u \sin^2(\theta) - \frac{v_{\text{hnf}}}{K^*} u, \tag{2}
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{\text{hnf}} \frac{\partial^2 T}{\partial y^2} - \frac{1}{(\rho C_P)_{\text{hnf}}} \frac{\partial q_r}{\partial y} + \frac{\mu_{\text{hnf}}}{(\rho C_P)_{\text{hnf}}} \left( \frac{\partial u}{\partial y} \right)^2 - \frac{H_0}{(\rho C_P)_{\text{hnf}}} (T - T_e) \tag{3}
\]

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left( \frac{\alpha_{\text{hnf}}}{\rho C_P} + \frac{16T_0^2 \sigma^*}{3(\rho C_P)_{\text{hnf}} k^*} \right) \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{\text{hnf}}}{(\rho C_P)_{\text{hnf}}} \left( \frac{\partial u}{\partial y} \right)^2 - \frac{H_0}{(\rho C_P)_{\text{hnf}}} (T - T_e) + \frac{\sigma_{\text{hnf}}}{(\rho C_P)_{\text{hnf}}} B_0^2 u^2 \sin^2(\theta). \tag{6}
\]

Figure 1. Hybrid nanofluid flow over an extending sheet.
The initial and boundary conditions are given by:

\[ u = e_1(x) + e v_{hnf} \frac{\partial \theta}{\partial y}, \quad \nu = e_2(x), \quad T = T_0(x) + \delta T \frac{\partial T}{\partial y} \]  

at \( y = 0 \).

\[ u \to 0, \quad \nu \to 0, \quad T \to T_\infty \] as \( y \to \infty \).

(7)

The experimental values of the parameters for water, Cu, and CoFe2O4 are reported in Table 1. In Table 2, the subscript \( f \) indicates properties of the base fluid (water) and \( hnf \) indicates properties of the hybrid nanofluid (H2O + Cu + CoFe2O4). In addition, \( \phi_{Cu} \) and \( \phi_{CoFe2O4} \) signify the volume fractions of the \( Cu \) and CoFe2O4 NPs, respectively. \( \rho_{Cu} \) and \( \rho_{CoFe2O4} \) signify the densities of the \( Cu \) and CoFe2O4 NPs, respectively, \( \sigma_{Cu} \) denotes the electrical conductivity of \( Cu \), and \( \sigma_{CoFe2O4} \) is the electrical conductivity of CoFe2O4. \( \rho_{(C_p)_{Cu}} \) and \( \rho_{(C_p)_{CoFe2O4}} \) indicate the heat capacitances, whereas \( k_{Cu} \) and \( k_{CoFe2O4} \) represent the thermal conductivities.

The transformation variables are:

\[ u = \frac{\partial \phi}{\partial y}, \quad v = \frac{v_{hnf} \sqrt{2Re}}{l} \left( \frac{\eta F'(\eta)}{F(\eta)} + n(\eta, x) \right), \quad T(x, y) = T_\infty - (T_e - T_0) e^{\alpha/2} \frac{\Theta(\eta)}{l} \]  

(8)

By incorporating equation (8) into equations (2) and (3), we obtain:

\[ \frac{k_{hnf}}{k_f} \theta_{hnf} + \frac{\theta}{r} \Theta'(\eta) + Pr \left( \frac{\nu_{hnf}}{2} \alpha F''(\eta) + \frac{2\nu_{hnf}}{\alpha} M F''(\eta) \right) \]  

(9)

Terms for the local Nusselt number \( Nu_t \) and the skin-friction coefficient \( SF_s \) are calculated in both dimensional and dimensionless forms. The dimensional forms of \( Nu_t \) and \( SF_s \) are given by:

\[ Nu_t = \frac{x r_w}{k_f(T_\infty - T_\infty)} \]  

(10)

\[ SF_s = \frac{F_w}{e T \beta f} \]  

(11)
Table 2. Thermal properties of the hybrid nanofluid \((\phi_1 = \phi_{Cu}, \phi_2 = \phi_{CoFe_2O_4})^{53-56}\)

| Properties                        | Models                                                                 |
|-----------------------------------|------------------------------------------------------------------------|
| Viscosity \(\mu_{nf}/\mu_f\)     | \(\frac{1}{(1 - \phi_{Cu} - \phi_{CoFe_2O_4})}\)                      |
| Density \(\rho_{nf}/\rho_f\)     | \(\phi_{Cu}(\rho_{Cu}) + \phi_{CoFe_2O_4}(\rho_{CoFe_2O_4}) + (1 - \phi_{Cu} - \phi_{CoFe_2O_4})\) |
| Thermal Capacity \(\mu_{nf}/\rho_f\) | \(\phi_{Cu}(\mu_{Cu}/\rho_{Cu}) + \phi_{CoFe_2O_4}(\mu_{CoFe_2O_4}/\rho_{CoFe_2O_4}) + (1 - \phi_{Cu} - \phi_{CoFe_2O_4})\) |
| Thermal Expansion \(\mu_{nf}/\rho_f\) | \(\phi_{Cu}(\mu_{Cu}/\rho_{Cu}) + \phi_{CoFe_2O_4}(\mu_{CoFe_2O_4}/\rho_{CoFe_2O_4}) + (1 - \phi_{Cu} - \phi_{CoFe_2O_4})\) |
| Thermal Conductivity \(k_{nf}/k_f\) | \(\frac{\phi_{Cu}k_{Cu} + \phi_{CoFe_2O_4}k_{CoFe_2O_4}}{\phi_{Cu} + \phi_{CoFe_2O_4}} + 2k_{nf} + 2(\phi_{Cu}k_{Cu} + \phi_{CoFe_2O_4}k_{CoFe_2O_4})\) |
| Electrical Conductivity \(\sigma_{nf}/\sigma_f\) | \(\frac{\phi_{Cu}\sigma_{Cu} + \phi_{CoFe_2O_4}\sigma_{CoFe_2O_4}}{\phi_{Cu} + \phi_{CoFe_2O_4}} + \frac{2\sigma_{nf} - k_{Cu}\phi_{Cu} + k_{CoFe_2O_4}\phi_{CoFe_2O_4}}{2(k_{Cu} + k_{CoFe_2O_4})}\) |

where the wall heat flux and the shear stress are represented by \(\tau_w = -k_{nf}(\frac{d\Theta}{dy})_{y=0} + (q_r)_{y=0}\) and \(F_w = \mu_{nf}(\frac{d\nu}{dy})_{y=0}\) respectively. \(Nu_s\) and \(SF_s\) also have dimensionless forms as shown below:

\[
\begin{align*}
Nu_s &= \left(\frac{\mu_{nf}}{\mu_f} + T_b\right)\frac{1}{\sqrt{2Re}}E^m(0), \\
SF_s &= \frac{\mu_{nf}}{\mu_f} \sqrt{\frac{Re}{2}}F^m(0).
\end{align*}
\]

HAM Solution

The HAM has been used to determine the solution for the proposed model.\(^{57}\) For this purpose, the linear operators are given as follows:

\[
\begin{align*}
F_0(\eta) &= 1 - \exp(-\eta), \quad \Theta_0(\eta) = \exp(-\eta), \\
\mathcal{L}(F) &= F'' - F' \quad \text{and} \quad \mathcal{L}(\Theta) = \Theta'' - \Theta.
\end{align*}
\]

The extended forms of \(\mathcal{L}\) and \(\mathcal{L}_\Theta\) are given as:

\[
\mathcal{L}(c_1 + c_2 \exp(-\eta) + c_3 \exp(-2\eta)) = 0, \\
\mathcal{L}_\Theta(c_4 \exp(-\eta) + c_5 \exp(-2\eta)) = 0.
\]

By using Taylor’s expansion, we obtain:

\[
F(\eta; \rho) = F_0(\eta) + \sum_{x=1}^{\infty} F_x(\eta)\rho^x
\]

\[
\Theta(\eta; \rho) = \Theta_0(\eta) + \sum_{x=1}^{\infty} \Theta_x(\eta)\rho^x,
\]

Next, we obtain:

\[
F_x(\eta) = \frac{1}{\rho!} \left. \frac{dF(\eta; \rho)}{d\eta} \right|_{\rho = 0}, \quad \Theta_x(\eta) = \frac{1}{\rho!} \left. \frac{d\Theta(\eta; \rho)}{d\eta} \right|_{\rho = 0}.
\]

The equations above can also be written as:

\[
\mathcal{L}(F_x(\eta)) = \mathcal{L}_e F_{x-1}(\eta), \\
\mathcal{L}_\Theta(\Theta_x(\eta)) = \mathcal{L}_e \Theta_{x-1}(\eta).
\]

where the local Reynolds number is given by \(Re_e = c_1(x)x/\nu_f\).

Results and Discussion

Analysis of Results

Fluid flow phenomena over a stretching surface under the influence of various physical effects, including heat sources, a porous surface, and application of an inclined magnetic field, have been studied. The flow scenario has been modeled in the form of nonlinear partial differential equations, which have been handled analytically using the HAM. The results are presented graphically in the figures, while the relevant numerical values are tabulated in the tables.
Discussion of Results

This section evaluates the physical mechanism behind each figure in the velocity and temperature profiles. The following observations have been made.

Velocity and Energy Profiles. Figure 2(a) and (b) shows the relationships of the nondimensional magnetic parameter $M$ with the velocity field $F_0(\eta)$ and the temperature field $Y_h(\eta)$, respectively, for both slip ($L = 0.5$) and no-slip ($L = 0$) conditions. Note that intensifying the value of $M$ causes $F_0(\eta)$ to decrease, while the value of $\Theta(\eta)$ is enhanced. Physically, a gradual improvement in $M$ creates a resistance force to the flow of the hybrid nanofluid called the Lorentz force, through which the frictional drag force is incremented.

Figure 3 is drawn to represent the effect of the suction parameter $S$ on the velocity profile $F_0(\eta)$ and the temperature profile $Y_h(\eta)$, respectively, for the cases of the slip ($L = 0.5$) and no-slip ($L = 0$) conditions. When $S > 0$, the values of $F_0(\eta)$ and $\Theta(\eta)$ are reduced, but the corresponding profiles in Figure 3(c) and (d) show the opposite behavior when the injection parameter ($S < 0$) is less than

![Figure 2](image-url)

**Figure 2.** (a) Velocity field $F(\eta)$ and (b) temperature field $\Theta(\eta)$ vs. the magnetic parameter $M$.

![Figure 3](image-url)

**Figure 3.** (a), (c) Velocity field $F'(\eta)$ and (b), (d) temperature field $\Theta'(\eta)$ vs. suction ($S$)/injection ($-S$) parameters.
zero. Physically, the momentum boundary layer is fixed to the stretchy sheet surface in the scenario when suction occurs, and this breaks the flow momentum and reduces the profiles of both $F_0^h$ and $Y_0^h$ for the hybrid nanofluid. Injection, in contrast, improves the fluid flow by causing a lateral mass flux to occur across the stretchy sheet, which adds momentum to the fluid flow. As a result, the hybrid nanofluid’s $F_0^h$ and $Y_0^h$ profiles improve in this case.

Figure 4(a) and (b) illustrate the impact of the thermal radiation $T_b$ and heat absorption parameter $H_a$, respectively, on the temperature profile $\Theta(\eta)$ of the hybrid nanofluid. Injection, in contrast, improves the fluid flow by causing a lateral mass flux to occur across the stretchy sheet, which adds momentum to the fluid flow. As a result, the hybrid nanofluid’s $F_0^h$ and $Y_0^h$ profiles improve in this case.

Figure 5(a) and (b) illustrate the relationships of the angle $\theta$ to the velocity profile $F_0^h(\eta)$ and the temperature profile $\Theta(\eta)$ of the hybrid nanofluid in the cases of slip ($L = 0.5$) and no-slip ($L = 0$) conditions. Increasing the value of $\theta$ causes the $F_0^h(\eta)$ to vanish while enhancing the $\Theta(\eta)$. Physically, the magnetic field strength increases with increasing $\theta$, which generates a greater resistive force to $F_0^h(\eta)$, causing the fractional force to increase and $\Theta(\eta)$ to improve.

Figure 6(a) and (b) capture the relationships of the temperature profile $\Theta(\eta)$ to the Eckert number $E_c$ and the thermal slip parameter $d$, respectively, in the cases of slip ($L = 0.5$) and no-slip ($L = 0$) conditions for the hybrid nanofluid. The plot in Figure 6(a) shows that enhancement of $E_c$ reduces the $\Theta(\eta)$. Physically, $E_c$ represents the enthalpy and kinetic energy relation, and it therefore reduces the viscous fluid stresses by translating kinetic energy into internal energy. An increment in $E_c$ increases the internal energy, and in this way, the fluid’s $\Theta(\eta)$ is enhanced. For increases in $d$, the heat flows from the stretchable sheet toward the hybrid nanofluid and $\Theta(\eta)$ is improved.
Figure 6. Temperature profile $\Theta(\eta)$ vs. (a) Eckert number $Ec$ and (b) slip parameter $d$.

Figure 7. (a) Velocity and (b) temperature profile $\Theta(\eta)$ vs. Forchheimer parameter $Fr$.

Figure 8. (a) Velocity and (b) temperature profile $\Theta(\eta)$ vs. local porosity parameter $\xi$.

Figure 7(a) shows the relationship between the velocity profile $F'(\eta)$ and the Forchheimer parameter $Fr$ in the cases of slip ($L = 0.5$) and no-slip ($L = 0$) conditions for the hybrid nanofluid. A larger value of $Fr$ causes the $F'(\eta)$ to decline. Physically, an increment in $Fr$ produces an inertial drag force, which acts as a barrier to the $F'(\eta)$, and thus the velocity profile vanishes. Figure 7(b) shows the variation in the temperature profile $\Theta(\eta)$ of the hybrid nanofluid when the Forchheimer parameter $Fr$ is varied. Increments in $Fr$ produce a strong $\Theta(\eta)$ and the thermal layer thickness increases for the hybrid nanofluid.

Figure 8(a) shows the relationship between the velocity profile $F'(\eta)$ and the local porosity parameter $\xi$ in
the cases of slip \((L = 0.5)\) and no-slip \((L = 0)\) conditions for the hybridized fluid. \(F'(\eta)\) declines when \(\lambda\) is increased because \(\xi\) is inversely related to the Darcian drag force; therefore, the hybrid nanofluid flow permeability improves, causing \(F'(\eta)\) to vanish. Figure 8(b) show the relationship of \(\xi\) versus \(\Theta(\eta)\) for the hybrid nanofluid in the cases of slip and no-slip conditions. In the presence of porous media, the resistance to hybrid nanofluid flow is enhanced, which induces higher temperatures and increased thermal layer thickness.

Figure 9(a) illustrates the effect of the Grashof number \(Gr\) on the velocity profile \(F'(\eta)\) in the cases of slip \((L = 0.5)\) and no-slip \((L = 0)\) conditions for the hybrid nanofluid, which shows that \(F'(\eta)\) decreases as \(Gr\) increases. Figure 9(b) shows that an increase in \(Gr\) causes the \(\Theta(\eta)\) to decline. This occurs because augmentation of \(Gr\) causes a convection current enhancement, thus improving \(\Theta(\eta)\) while also reducing the hot surface area of the sheet.

Figures 1 to 9 show that the temperature profile \(\Theta(\eta)\) has a greater enhancement effect under the no-slip
Bar Chart 2. Nusselt number $Nu_x$ vs. (a) Eckert number $Ec$, (b) slip parameter $d$, (c) thermal radiation $Tb$, (d) angle $\theta$, (e) suction parameter $S$, (f) injection parameter $-S$, (g) magnetic parameter $M$, and (h) absorption parameter $Ha$.
(L = 0.5) condition as compared with that under the slip condition (L = 0), while in the case of the velocity profile \( F'(\eta) \), the opposite phenomena can be observed. Physically, in the slip condition case, there is a fraction produced between the sheet surface and the fluid that enhances the temperature profile and reduces the velocity profile.

**Skin Friction and Nusselt Number.** Bar Chart 1(a) to (d) show the effect of increasing the value of the skin friction \( SF_x \) versus the suction parameter \( S \), the injection parameter \( -S \), the magnetic parameter \( M \), and the angle \( \theta \) in the cases of the slip and no-slip conditions. Similarly, Bar Chart 2(a) to (h) illustrate the relationships of the Nusselt number \( Nu_x \) to the Eckert number \( Ec \), the slip parameter \( d \), the thermal radiation \( T_R \), the angle \( \theta \), the suction parameter \( S \), the injection parameter \( -S \), the magnetic parameter \( M \), and the absorption parameter \( Ha \). Note that increased values of \( T_R, S, \) and \( Ha \) enhance \( Nu_x \), while increased values of \( Ec, d, \theta, -S, \) and \( M \) reduce \( Nu_x \).

**Conclusion**

The analysis presented here reports the dynamic characteristics of a water-based hybridized fluid composed of copper (Cu) and cobalt ferrite (CoFe\(_2\)O\(_4\)) NPs when passing over an exponentially stretchable sheet. The previously published work is extended by using the concepts of thermal and velocity slip factors, the heat absorption coefficient, the magnetic field effect, Darcy–Forchheimer parameters, and a hybridized nanomaterial. The core findings from these observations are presented as follows:

- The fluid motion \( F'(\eta) \) of a hybrid nanofluid is slowed substantially by elevating the magnetic field angle and strength, but this elevation has a reverse effect on the energy profile \( \Theta(\eta) \).
- Raising the injection factor causes the fluid dynamics profile \( F'(\eta) \) and energy profile \( \Theta(\eta) \) to be enhanced, while the reverse effect is produced by raising the suction parameter and the Darcy–Forchheimer parameters.
- The thermal profile \( \Theta(\eta) \) of the hybrid nanofluid can be increased by increasing the thermal slip factor \( d \).
- The contributions of the heat absorption \( Ha \) and the thermal radiation \( T_R \) boost the heat transmission rate.
- The dispersion of the copper and cobalt ferrite NPs in the base fluid magnifies the velocity and the energy transmission rate remarkably because of the fluid’s remarkable thermophysical properties.
- The transformation of mechanical energy into the thermal energy of the nanoparticles occurs when the Eckert number \( Ec \) increases. As a result, the temperature field \( \Theta(\eta) \) is enhanced by increasing \( Ec \).
- In the case of the slip condition (L = 0.5), the thermal profile curve is higher than in the case of the no slip (L = 0) condition, because a fraction is created between the sheet surface and the fluid as a result of the slip condition that helps to enhance the thermal profile \( \Theta(\eta) \).

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\( \alpha \)  
\( (\rho c) \)  
\( k \)  
\( T_\infty \)  

**Thermal diffusivity** \( (m^2 s^{-1}) \)  
**Heat capacity** \( (J kg^{-1} k^{-1}) \)  
**Thermal conductivity** \( (wm^{-1} k^{-1}) \)  
**Ambient temperature** \( (k) \)

**Nondimensional Quantities**

\( F \)  
\( \Theta \)  
\( Ha \)  
\( M \)  
\( \chi \)  
\( Tb \)  
\( Pr \)  
\( \eta \)  
\( Re \)  

**Velocity profile**  
**Temperature profile**  
**Heat absorption**  
**Magnetic parameter**  
**Dimensionless coordinates**  
**Radiation parameter**  
**Prandtl number**  
**Similarity variable**  
**Reynolds number**

\( Ec \)  
\( L \)  
\( d \)  
\( S \)  
\( Fr \)  
\( \xi \)  
\( Gr \)

**Eckert number**  
**Velocity slip**  
**Thermal slip**  
**Suction/injection parameter**  
**Forchheimer parameter**  
**Porosity parameter**  
**Grashof number**

**Abbreviations**

\( f \)  
\( hnf \)  
\( HAM \)  
\( MHD \)

**Base fluid**  
**Hybrid nanofluid**  
**Homotopy analysis method**  
**Magnetohydrodynamics**

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