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Global compactness for a class of quasi-linear elliptic problems

by

C. Mercuri, M. Squassina
GLOBAL COMPACTNESS FOR A CLASS OF QUASI-LINEAR ELLIPTIC PROBLEMS

CARLO MERCURI AND MARCO SQUASSINA

ABSTRACT. We prove a global compactness result for Palais-Smale sequences associated with a class of quasi-linear elliptic equations on exterior domains.

I. INTRODUCTION AND MAIN RESULT

Let $\Omega$ be a smooth domain of $\mathbb{R}^N$ with a bounded complement and $N > p > m > 1$. The main goal of this paper is to obtain a global compactness result for the Palais-Smale sequences of the energy functional associated with the following quasi-linear elliptic equation

$$-\operatorname{div}(L(x,Du)) - \operatorname{div}(M(x,Du)) + M_s(u,Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in } \Omega,$$

where $u \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$, meant as the completion of the space $D(\Omega)$ of smooth functions with compact support, with respect to the norm $||u||_{W^{1,p}(\Omega) \cap D^{1,m}(\Omega)} = ||u||_p + ||u||_m$, having set $||u||_p := ||u||_{W^{1,p}(\Omega)}$ and $||u||_m := ||Du||_{L^m(\Omega)}$. We assume that $V$ is a continuous function on $\Omega$,

$$\lim_{|x| \to \infty} V(x) = V_\infty \quad \text{and} \quad \inf_{x \in \Omega} V(x) = V_0 > 0.$$

As known, lack of compactness may occur due to the lack of compact embeddings for Sobolev spaces on $\Omega$ and since the limiting equation on $\mathbb{R}^N$

$$-\operatorname{div}(L(x,Du)) - \operatorname{div}(M(x,Du)) + M_s(u,Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in } \mathbb{R}^N,$$

with $u \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$, is invariant by translations. A particular case of (1.1) is

$$-\Delta_p u - \operatorname{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m} a'(u)|Du|^m + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega,$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$, for a suitable function $a \in C^1(\mathbb{R};\mathbb{R}^+)$, or the even simpler case where $a$ is constant, namely

$$-\Delta_p u - \Delta_m u + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega.$$

Since the pioneering work of Benci and Cerami [2] dealing with the case $L(\xi) = |\xi|^2/2$ and $M(s,\xi) \equiv 0$, many papers have been written on this subject, see for instance the bibliography of [12]. Quite recently, in [12], the case $L(\xi) = |\xi|^p/p$ and $M(s,\xi) \equiv 0$ was investigated. The main point in the present contribution is the fact that we allow, under suitable assumptions, a quasi-linear term $M(u,Du)$ depending on the unknown $u$ itself. The typical tools exploited in [2, 12], in addition to the point-wise convergence of the gradients, are some decomposition (splitting) results both for the energy functional and for the equation, along a given bounded Palais-Smale sequence $(u_n)$. To this regard, the explicit dependence on $u$ in the term $M(u,Du)$ requires a rather careful analysis. In particular, we can handle it for

$$\nu|\xi|^m \leq M(s,\xi) \leq C|\xi|^m, \quad p-1 < m < p-1 + p/N.$$
The restriction on $m$, together with the sign condition (1.9) provides, thanks to the presence of $L$, the needed a priori regularity on the weak limit of $(u_n)$, see Theorems 3.2 and 3.4.

Besides the aforementioned motivations, which are of mathematical interest, it is worth pointing out that in recent years, some works have been devoted to quasi-linear operators with double homogeneity, which arise from several problems of Mathematical Physics. For instance, the reaction diffusion problem $u_t = -\text{div} (\mathbb{D}(u)Du) + \ell(x,u)$, where $\mathbb{D}(u) = d_p|Du|^{p-2} + d_m|Du|^{m-2}$, $d_p > 0$ and $d_m > 0$, admitting a rather wide range of applications in biophysics [10], plasma physics [16] and in the study of chemical reactions [1]. In this framework, $u$ typically describes a concentration and $\text{div} (\mathbb{D}(u)Du)$ corresponds to the diffusion with a coefficient $\mathbb{D}(u)$, whereas $\ell(x,u)$ plays the rôle of reaction and relates to source and loss processes. We refer the interested reader to [5] and to the reference therein. Furthermore, a model for elementary particles proposed by Derrick [9] yields to the study of standing wave solutions $\psi(x,t) = u(x)e^{i\omega t}$ of the following nonlinear Schrödinger equation

$$i\psi_t + \Delta_2 \psi - b(x)\psi + \Delta_p \psi - V(x)|\psi|^{p-2}\psi + |\psi|^\sigma \psi = 0 \quad \text{in } \mathbb{R}^N,$$

for which we refer the reader e.g. to [3].

In order to state the first main result, assume $N > p > m \geq 2$ and

$$p - 1 \leq m < p - 1 + p/N, \quad p < \sigma < p^*,$$

and consider the $C^2$ functions $L : \mathbb{R}^N \to \mathbb{R}$ and $M : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that both the functions $\xi \mapsto L(\xi)$ and $\xi \mapsto M(s,\xi)$ are strictly convex and

$$\nu|\xi|^p \leq |L(\xi)| \leq C|\xi|^p, \quad |L_\xi(\xi)| \leq C|\xi|^{p-1}, \quad |L_{\xi\xi}(\xi)| \leq C|\xi|^{p-2},$$

for all $\xi \in \mathbb{R}^N$. Furthermore, we assume

$$\nu|\xi|^m \leq |M(s,\xi)| \leq C|\xi|^m, \quad |M_s(s,\xi)| \leq C|\xi|^m, \quad |M_\xi(s,\xi)| \leq C|\xi|^{m-1},$$

$$|M_{ss}(s,\xi)| \leq C|\xi|^m, \quad |M_{s\xi}(s,\xi)| \leq C|\xi|^{m-1}, \quad |M_{\xi\xi}(s,\xi)| \leq C|\xi|^{m-2},$$

for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and that the sign condition (cf. [14])

$$M_s(s,\xi)s \geq 0,$$

holds for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Also, $G : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function with $G'(s) := g(s)$ and

$$|G'(s)| \leq C|s|^{\sigma-1}, \quad |G''(s)| \leq C|s|^{\sigma-2},$$

for all $s \in \mathbb{R}$. We define

$$j(s,\xi) := L(\xi) + M(s,\xi) - G(s),$$

and on $W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ with $\|u\|_{W^{1,p}(\Omega) \cap D^{1,m}(\Omega)} = \|u\|_p + \|u\|_m$ the functional

$$\phi(u) := \int_\Omega j(u, Du) + \int_\Omega V(x)|u|^p/p.$$

Finally, on $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)} = \|u\|_p + \|u\|_m$ we define

$$\phi_\infty(u) := \int_{\mathbb{R}^N} j(u, Du) + \int_{\mathbb{R}^N} V_\infty |u|^p/p.$$

See Section 2 for some properties of the functionals $\phi$ and $\phi_\infty$. The first main global compactness type result is the following
Theorem 1.1. Assume that (1.5)-(1.11) hold and let \((u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) be a bounded sequence such that

\[
\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*.
\]

Then, up to a subsequence, there exists a weak solution \(v_0 \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) of

\[-\text{div}(L_\xi(Du)) - \text{div}(M_\xi(u, Du)) + M_\xi(u, Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in } \Omega,
\]

a finite sequence \(\{v_1, \ldots, v_k\} \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)\) of weak solutions of

\[-\text{div}(L_\xi(Du)) - \text{div}(M_\xi(u, Du)) + M_\xi(u, Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in } \mathbb{R}^N
\]

and \(k\) sequences \(\{y^i_n\} \subset \mathbb{R}^N\) satisfying

\[
|y^i_n| \to \infty, \quad |y^i_n - y^j_n| \to \infty, \quad i \neq j, \quad \text{as } n \to \infty,
\]

\[
\|u_n - v_0\| \rightarrow \sum_{i=1}^k v_i((\cdot - y^i_n))\|W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \to 0, \quad \text{as } n \to \infty,
\]

\[
\|u_n\|_p \rightarrow \sum_{i=1}^k \|v_i\|_p, \quad \|u_n\|_m \rightarrow \sum_{i=1}^k \|v_i\|_m, \quad \text{as } n \to \infty,
\]

as well as

\[
\phi(v_0) + \sum_{i=1}^k \phi_\infty(v_i) = c.
\]

Let us now come to a statement for the cases \(1 < m \leq 2\) or \(1 < p \leq 2\). Let us define

\[
\mathfrak{L}(\xi, h) := \frac{|L_\xi(\xi + h) - L_\xi(\xi)|}{|h|^{p-1}}, \quad \text{if } 1 < p < 2,
\]

\[
\mathfrak{G}(s, t) := \frac{|G'(s + t) - G'(s)|}{|t|^{\sigma-1}}, \quad \text{if } 1 < \sigma < 2,
\]

\[
\mathfrak{M}(s, \xi, h) := \frac{|M_\xi(s, \xi + h) - M_\xi(s, \xi)|}{|h|^{m-1}}, \quad \text{if } 1 < m < 2.
\]

If either \(p < 2, \sigma < 2\) or \(m < 2\), we shall weaken the twice differentiability assumptions, by requiring \(L_\xi \in C^1(\mathbb{R}^N \setminus \{0\}), G' \in C^1(\mathbb{R} \setminus \{0\}), M_\xi \in C^1(\mathbb{R} \times (\mathbb{R}^N \setminus \{0\}))\), \(M_\xi \in C^0(\mathbb{R} \times \mathbb{R}^N)\) and \(M_{ss} \in C^0(\mathbb{R} \times \mathbb{R}^N)\). Moreover we assume the same growth conditions for \(L_\xi, M_\xi, G''\) by the following hypotheses:

\[
\sup_{h \neq 0, \xi \in \mathbb{R}^N} \mathfrak{L}(\xi, h) < \infty, \quad \text{(1.12)}
\]

\[
\sup_{t \neq 0, s \in \mathbb{R}} \mathfrak{G}(s, t) < \infty, \quad \text{(1.13)}
\]

\[
\sup_{h \neq 0, (s, \xi) \in \mathbb{R} \times \mathbb{R}^N} \mathfrak{M}(s, \xi, h) < \infty. \quad \text{(1.14)}
\]

Conditions (1.12)-(1.13), in some more concrete situations, follow immediately by homogeneity of \(L_\xi\) and \(G'\) (see, for instance, [12, Lemma 3.1]). Similarly, (1.14) is satisfied for instance when \(M\) is of the form \(M(s, \xi) = a(s)\mu(\xi)\), being \(a : \mathbb{R} \to \mathbb{R}^+\) a bounded function and \(\mu : \mathbb{R}^N \to \mathbb{R}^+\) a \(C^1\) strictly convex function such that \(\mu_\xi\) is homogeneous of degree \(m - 1\).

Theorem 1.2. Under the additional assumptions (1.12)-(1.14) in the sub-quadratic cases, the assertion of Theorem 1.1 holds true.

As a consequence of the above results we have the following compactness criterion.
Corollary 1.3. Assume (2.1) below for some $\delta > 0$ and $\mu > p$. Under the hypotheses of Theorem 1.1 or 1.2, if $c < c^*$, then $(u_n)$ is relatively compact in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ where

$$c^* := \min \left\{ \frac{\delta}{\mu}, \frac{\mu - p}{\mu p}V_{\infty} \right\} \left[ \min\{\nu, V_{\infty}\} \frac{\|u\|_{\nu}^{\sigma}}{C_g} \right]^{\frac{p}{\sigma - p}},$$

and $S_{p,\sigma}$ and $C_g$ are constants such that $S_{p,\sigma}\|u\|^\sigma \geq \|u\|_{\nu}^{\sigma}$ and $|g(s)| \leq C_g|s|^{\sigma - 1}$.

Remark 1.4. It would be interesting to get a global compactness result in the case $L = 0$ and $p = m$, namely for the model case

$$(1.15) \quad - \text{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m} a'(u)|Du|^m + V(x)|u|^{m-2}u = |u|^{\sigma - 2}u \quad \text{in } \Omega.$$ 

Notice that, even assuming $a'$ bounded, $a'(u)|Du|^m$ is merely in $L^1(\Omega)$ for $W_0^{1,m}(\Omega)$ distributional solutions. In general, in this setting, the splitting properties of the equation are hard to formulate in a reasonable fashion.

Remark 1.5. The restriction of between $m$ and $p$ in assumption (1.5) is no longer needed in the case where $M$ is independent of the first variable $s$, namely $M_s \equiv 0$.

Remark 1.6. We prove the above theorems under the a-priori boundedness assumption of $(u_n)$. This occurs in a quite large class of problems, as Proposition 2.2 shows.

Remark 1.7. With no additional effort, we could cover the case where an additional term $W(x)|u|^{m-2}u$ appears in (1.1) and the functional framework turns into $W_0^{1,p}(\Omega) \cap W_0^{1,m}(\Omega)$.

In the spirit of [11], we also get the following

Corollary 1.8. Let $N > p \geq m > 1$ and assume that $\xi \mapsto L(\xi)$ is $p$-homogeneous, $\xi \mapsto M(\xi)$ is $m$-homogeneous, $L(\xi) \geq |\xi|^p$, $M(\xi) \geq |\xi|^m$ (we put $\nu = 1$) and set

$$(1.16) \quad S_\Omega := \inf_{\|u\|_{L^\sigma(\Omega)} = 1} \int_\Omega L(Du) + M(Du) + V(x)|u|^p,$$

$$S_\mathbb{R}^N := \inf_{\|u\|_{L^\sigma(\mathbb{R}^N)} = 1} \int_{\mathbb{R}^N} |Du|^p + |u|^p,$$

with $V(x) \to 1$ as $|x| \to \infty$. Assume furthermore that

$$(1.17) \quad S_\Omega < \left( \frac{\sigma - p}{m} \right)^{\frac{p}{\sigma - p}} S_\mathbb{R}^N.$$ 

Then (1.16) admits a minimizer.

Remark 1.9. We point out that, some conditions guaranteeing the nonexistence of nontrivial solutions in the star-shaped case $\Omega = \mathbb{R}^N$ can be provided. For the sake of simplicity, assume that $L$ is $p$-homogeneous and that $\xi \mapsto M(s, \xi)$ is $m$-homogeneous. Then, in view of [13, Theorem 3], that holds for $C^1$ solutions by virtue of the results of [8], we have that (1.1) admits no nontrivial $C^1$ solution well behaved at infinity, namely satisfying condition (19) of [13], provided that there exists a number $a \in \mathbb{R}^+$ such that a.e. in $\mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$(N - p(a + 1))L(\xi) + (N - m(a + 1))M(s, \xi) + (asg(s) - NG(s))$$

$$+ \frac{(N - ap)V(x) + x \cdot DV(x)}{p} |s|^p - aM_s(s, \xi)s \geq 0,$$

holding, for instance, if there exists $0 \leq a \leq \frac{N - p}{p}$ such that

$$asg(s) - NG(s) \geq 0, \quad (N - ap)V(x) + x \cdot DV(x) \geq 0, \quad M_s(s, \xi)s \leq 0,$$
for a.e. $x \in \mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Also, in the more particular case where $g(s) = |s|^{\sigma-2}s$ and $V(x) = V_\infty > 0$, then the above conditions simply rephrase into
\[ \sigma \geq p^*, \quad M_s(s, \xi)s \leq 0, \]
for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. In fact, in (1.9), we consider the opposite assumption on $M_s$.

2. Some preliminary facts

It is a standard fact that, under condition (1.6) and (1.10), the functionals
\[ u \mapsto \int_\Omega L(Du), \quad u \mapsto \int_\Omega V(x)|u|^p, \quad u \mapsto \int_\Omega G(u) \]
are $C^1$ on $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Analogously, although $M$ depends explicitly on $s$, the functional
\[ \mathbb{M} : W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \to \mathbb{R}, \quad \mathbb{M}(u) = \int_\Omega M(u, Du), \]
admits, thanks to condition (1.5), directional derivatives along any $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ and
\[ \mathbb{M}'(u)(v) = \int_\Omega M_s(u, Du) \cdot Dv + \int_\Omega M_s(u, Du)v, \]
as it can be easily verified observing that $p \leq \frac{p}{p-m} \leq p^*$ and that, for $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, by Young’s inequality, for some constant $C$ it holds
\[ |M_s(u, Du) \cdot Dv| \leq \mu|Du|^m + C|Dv|^m \in L^1(\Omega), \]
\[ |M_s(u, Du)v| \leq \mu|Du|^p + C|v|^{p+m} \in L^1(\Omega). \]
Furthermore, if $u_k \to u$ in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ as $k \to \infty$ then $\mathbb{M}'(u_k) \to \mathbb{M}'(u)$ in the dual space $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, as $k \to \infty$. Indeed, for $\|v\|_{W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)} \leq 1$, we have
\[ |\mathbb{M}'(u_k)(v) - \mathbb{M}'(u)(v)| \]
\[ \leq \int_\Omega |M_s(u_k, D\xi) - M_s(u, D\xi)| |Dv| + \int_\Omega |M_s(u_k, D\xi) - M_s(u, D\xi)| |v| \]
\[ \leq \|M_s(u_k, D\xi) - M_s(u, D\xi)\|_{L^{p/m}} \|Dv\|_{L^m} + \|M_s(u_k, D\xi) - M_s(u, D\xi)\|_{L^p/m} \|v\|_{L^{p/(p-m)}} \]
\[ \leq \|M_s(u_k, D\xi) - M_s(u, D\xi)\|_{L^{p/m}} + \|M_s(u_k, D\xi) - M_s(u, D\xi)\|_{L^p/m}. \]
This yields the desired convergence, using (1.7) and the Dominated Convergence Theorem. Notice that the same argument carried out before applies either to integrals defined on $\Omega$ or on $\mathbb{R}^N$. Hence the following proposition is proved.

**Proposition 2.1.** In the hypotheses of Theorems 1.1 and 1.2, the functionals $\phi$ and $\phi_\infty$ are $C^1$.

In addition to the assumptions on $L, M$ and $g, G$ set in the introduction, assume now that there exist positive numbers $\delta > 0$ and $\mu > p$ such that
\[ (2.1) \quad \mu M(s, \xi) - M_s(s, \xi)s - M_s(s, \xi) \cdot \xi \geq \delta|\xi|^m, \quad \mu L(\xi) - L_\xi(\xi) \cdot \xi \geq \delta|\xi|^p, \quad sg(s) - \mu G(s) \geq 0, \]
for any $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$. This hypothesis is rather well established in the framework of quasi-linear problems (cf. [14]) and it allows an arbitrary Palais-Smale sequence $(u_n)$ to be bounded in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, as shown in the following
Proposition 2.2. Let \( j \) be as in (1.11) and assume that (1.5) holds. Let \( (u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) be a sequence such that
\[
\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*.
\]
Then, if condition (2.1) holds, \( (u_n) \) is bounded in \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \).

Proof. Let \( (w_n) \subset (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \) with \( w_n \to 0 \) as \( n \to \infty \) be such that \( \phi'(u_n)(v) = \langle w_n, v \rangle \), for every \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \). Whence, by choosing \( v = u_n \), it follows
\[
\int_\Omega j_\varepsilon(u_n, Du_n) \cdot Du_n + \int_\Omega j_s(u_n, Du_n)u_n + \int_\Omega V(x)|u_n|^p = \langle w_n, u_n \rangle.
\]
Combining this equation with \( \mu \phi(u_n) = \mu c + o(1) \) as \( n \to \infty \), namely
\[
\int_\Omega \mu j(u_n, Du_n) + \frac{\mu}{p} \int_\Omega V(x)|u_n|^p = \mu c + o(1),
\]
recalling the definition of \( j \), and using condition (2.1), yields
\[
\frac{\mu - p}{p} \int_\Omega V(x)|u_n|^p + \delta \int_\Omega |Du_n|^p + \delta \int_\Omega |Du_n|^m \leq \mu c + \|w_n\||u_n||W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) + o(1),
\]
as \( n \to \infty \), implying, due to \( V \geq V_0 \) that
\[
\|u_n\|^p_{W^{1,p}(\Omega)} + \|u_n\|^m_{D^{1,m}(\Omega)} \leq C + C\|u_n\|_{W^{1,p}(\Omega)} + C\|u_n\|_{D^{1,m}(\Omega)} + o(1),
\]
as \( n \to \infty \). The assertion then follows immediately. \( \square \)

From now on we shall always assume to handle \textit{bounded} Palais-Smale sequences, keeping in mind that condition (2.1) can guarantee the boundedness of such sequences.

Proposition 2.3. Let \( j \) be as in (1.11) and assume that \( 1 < m < p < N \) and \( p < \sigma < p^* \). Let \( (u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) bounded be such that
\[
\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*.
\]
Then, up to a subsequence, \( (u_n) \) converges weakly to some \( u \) in \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \), \( u_n(x) \to u(x) \) and \( Du_n(x) \to Du(x) \) for a.e. \( x \in \Omega \).

Proof. It is sufficient to justify that \( Du_n(x) \to Du(x) \) for a.e. \( x \in \Omega \). Given an arbitrary bounded subdomain \( \omega \subset \subset \Omega \) of \( \Omega \), from the fact that \( \phi'(u_n) \to 0 \) in \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \), we can write
\[
\int_\omega a(u_n, Du_n) \cdot Du = \langle w_n, v \rangle + \langle f_n, v \rangle + \int_\omega v \, d\mu_n, \quad \text{for all } v \in D(\omega),
\]
where \( (w_n) \subset (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \) is vanishing, and hence in particular \( w_n \in W^{-1,p'}(\omega) \), with \( w_n \to 0 \) in \( W^{-1,p'}(\omega) \) as \( n \to \infty \) and we have set
\[
a_n(x, s, \xi) := L_\xi(\xi) + M_\xi(s, \xi), \quad \text{for all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,
\]
\[
f_n := -V(x)|u_n|^{p-2}u_n + g(u_n) \in W^{-1,p'}(\omega), \quad n \in \mathbb{N},
\]
\[
\mu_n := -M_\xi(u_n, Du_n) \in L^1(\omega), \quad n \in \mathbb{N}.
\]
Due to the strict convexity assumptions on the maps \( \xi \mapsto L(\xi) \) and \( \xi \mapsto M(s, \xi) \) and the growth conditions on \( L_\xi, M_\xi, M_s \) and \( g \), all the assumptions of [6, Theorem 1] are fulfilled. Precisely,
\[
|a_n(x, s, \xi)| \leq |L_\xi(\xi)| + |M_\xi(s, \xi)| \leq C|\xi|^{p-1} + C|\xi|^{m-1} \leq C + C|\xi|^{p-1},
\]
for a.e. \( x \in \omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\), and
\[
\begin{align*}
  f_n &\to f, \quad f := -V(x)|u|^{p-2}u + g(u), \quad \text{strongly in } W^{-1,p'}(\omega), \\
  \mu_n &\rightharpoonup \mu, \quad \text{weakly* in } \mathcal{M}(\omega), \quad \text{since } \sup_{n \in \mathbb{N}} \|M_s(u_n, Du_n)\|_{L^1(\omega)} < +\infty.
\end{align*}
\]

Then, it follows that \( Du_n(x) \to Du(x) \) for a.e. \( x \in \omega \). Finally, a simple Cantor diagonal argument allows to recover the convergence over the whole domain \( \Omega \).

Next we prove a regularity result for the solutions of equation (1.1).

**Proposition 2.4.** Let \( j \) be as in (1.11) and assume (1.5) and (1.9). Let \( u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \) be a solution of (1.1). Then
\[
u \in \bigcap_{q \geq p} L^q(\Omega), \quad u \in L^\infty(\Omega) \quad \text{and} \quad \lim_{|x| \to \infty} u(x) = 0.
\]

**Proof.** Let \( k, i \in \mathbb{N} \). Then, setting \( v_{k,i}(x) := (u_k(x))^i \) with \( u_k(x) := \min\{u^+(x), k\} \), it follows that \( v_{k,i} \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \) can be used as a test function in (1.1), yielding
\[
\begin{align*}
  \int_{\Omega} L_{\xi}(Du) \cdot Dv_{k,i} + \int_{\Omega} M_s(u, Du) \cdot Dv_{k,i} &+ \int_{\Omega} M_s(u, Du) v_{k,i} + \int_{\Omega} V(x)|u|^{p-2}u v_{k,i} = \int_{\Omega} g(u) v_{k,i}.
\end{align*}
\]
Taking into account that \( Dv_{k,i} \) is equal to \( i u^{i-1} Du\chi_{\{0 < u < k\}} \), by convexity and positivity of the map \( \xi \mapsto M(s, \xi) \) we deduce that \( M_s(u, Du) \cdot Dv_{k,i} \geq 0 \). Moreover, by the sign condition (1.9) it follows \( M_s(u, Du) v_{k,i} \geq 0 \) a.e. in \( \Omega \). Then, we reach
\[
\begin{align*}
  \int_{\Omega} i(u_k)^{i-1}L_{\xi}(Du_k) \cdot Du_k + \int_{\Omega} V(x)|u|^{p-2}u(u_k(x))^i \leq \int_{\Omega} g(u)(u_k(x))^i,
\end{align*}
\]
yielding in turn, by (1.10), that for all \( k, i \geq 1 \)
\[
\begin{align}
  \nu \int_{\Omega}(u_k)^{i-1}|Du_k|^p &\leq C \int_{\Omega}(u^+(x))^{\sigma-1+i}.
\end{align}
\]
If \( \tilde{u}_k := \min\{u^-(x), k\} \), a similar inequality
\[
\begin{align}
\nu \int_{\Omega}(\tilde{u}_k)^{i-1}|Du_k|^p &\leq C \int_{\Omega}(u^-(x))^{\sigma-1+i},
\end{align}
\]
can be obtained by using \( \tilde{v}_{k,i} := -(\tilde{u}_k)^i \) as a test function in (1.1), observing that by (1.9),
\[
M_s(u, Du) \tilde{v}_{k,i} = -M_s(u, Du)\chi_{\{-k < u < 0\}}(-u)^i \geq 0,
\]
\[
M_s(u, Du) \cdot Dv_{k,i} = i(-u)^{i-1}\chi_{\{-k < u < 0\}}M_s(u, Du) \cdot Du \geq 0.
\]
Once (2.2)-(2.3) are reached, the assertion follows exactly as in [15, Lemma 2, (a) and (b)]. \( \square \)

We now recall the following version of [7, Lemma 4.2] which turns out to be a rather useful tool in order to establish convergences in our setting. Roughly speaking, one needs some kind of sub-criticality in the growth conditions.

**Lemma 2.5.** Let \( \Omega \subset \mathbb{R}^N \) and \( h : \Omega \times \mathbb{R} \times \mathbb{R}^N \) be a Carathéodory function, \( p, m > 1, \mu \geq 1, \ p \leq \sigma \leq p^* \) and assume that, for every \( \varepsilon > 0 \) there exist \( a_\varepsilon \in L^p(\Omega) \) such that
\[
|h(x, s, \xi)| \leq a_\varepsilon(x) + \varepsilon |s|^{\sigma/\mu} + \varepsilon |\xi|^{p/\mu} + \varepsilon |\xi|^m/\mu,
\]
a.e. in \( \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\). Assume that \( u_n \to u \) a.e. in \( \Omega \), \( Du_n \to Du \) a.e. in \( \Omega \) and \( (u_n) \) is bounded in \( W_0^{1,p}(\Omega) \), \( (u_n) \) is bounded in \( D_0^{1,m}(\Omega) \).
Then $h(x, u_n, Du_n)$ converges to $h(x, u, Du)$ in $L^\mu(\Omega)$.

**Proof.** The proof follows as in [7, Lemma 4.2] and we shall sketch it here for self-containedness. By Fatou’s Lemma, it immediately holds that $u \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$. Furthermore, there exists a positive constant $C$ such that

$$|h(x, s, \xi) - h(x, s_2, \xi_2)|^\mu \leq C (a_\infty(x))^{\mu} + C\varepsilon^{\mu}|s_1|^{\sigma} + C\varepsilon^{\mu}|s_2|^{\sigma} + C\varepsilon^{\mu}|\xi_1|^{m} + C\varepsilon^{\mu}|\xi_2|^{m} + C\varepsilon^{\mu}|\xi_1|^{p} + C\varepsilon^{\mu}|\xi_2|^{p},$$

a.e. in $\Omega$ and for all $(s_1, \xi_1) \in \mathbb{R} \times \mathbb{R}^N$ and $(s_2, \xi_2) \in \mathbb{R} \times \mathbb{R}^N$. Then, taking into account the boundedness of $(Du_n)$ in $L^p(\Omega) \cap L^m(\Omega)$ and of $(u_n)$ in $L^\sigma(\Omega)$ by interpolation being $p \leq \sigma \leq p^*$, the assertion follows by applying Fatou’s Lemma to the sequence of functions $\psi_n : \Omega \to [0, +\infty)$

$$\psi_n(x) := -|h(x, u_n, Du_n) - h(x, u, Du)|^\mu + C(a_\infty(x))^{\mu} + C\varepsilon^{\mu}|u_n|^{\sigma} + C\varepsilon^{\mu}|u|^{\sigma} + C\varepsilon^{\mu}|Du_n|^{m} + C\varepsilon^{\mu}|Du|^{m} + C\varepsilon^{\mu}|Du_n|^{p} + C\varepsilon^{\mu}|Du|^{p},$$

and, finally, exploiting the arbitrariness of $\varepsilon$. \qed

3. **Proof of the result**

3.1. **Energy splitting.** The next result allows to perform an energy splitting for the functional

$$J(u) = \int_\Omega j(u, Du), \quad u \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega),$$

along a bounded Palais-Smale sequence $(u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$. The result is in the spirit of the classical Brezis-Lieb Lemma [4].

**Lemma 3.1.** Let the integrand $j$ be as in (1.11) and

$$p-1 \leq m < p - 1 + p/N, \quad p \leq \sigma \leq p^*.$$ 

Let $(u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ with $u_n \rightharpoonup u$, $u_n \rightarrow u$ a.e. in $\Omega$ and $Du_n \rightarrow Du$ a.e. in $\Omega$. Then

$$\lim_{n \rightarrow \infty} \int_\Omega j(u_n - u, Du_n - Du) - j(u_n, Du_n) + j(u, Du) = 0. \quad (3.1)$$

**Proof.** We shall apply Lemma 2.5 to the function

$$h(x, s, \xi) := j(s - u(x), \xi - Du(x)) - j(s, \xi),$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Given $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, consider the $C^1$ map $\varphi : [0, 1] \rightarrow \mathbb{R}$ defined by setting

$$\varphi(t) := j(s - tu(x), \xi - tDu(x)), \quad \text{for all } t \in [0, 1].$$

Then, for some $\tau \in [0, 1]$ depending upon $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, it holds

$$h(x, s, \xi) = \varphi(1) - \varphi(0) = \varphi'(\tau)$$

$$= -j\zeta(s - \tau u(x), \xi - \tau Du(x))u(x) - j\zeta(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x)$$

$$= -L_{\xi}(\xi - \tau Du(x)) \cdot Du(x)$$

$$- M_{s}(s - \tau u(x), \xi - \tau Du(x))u(x)$$

$$- M_{\xi}(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x) + G^\tau(s - \tau u(x))u(x).$$
Hence, for a.e. \( x \in \Omega \) and all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \), it follows that
\[
|h(x, s, \xi)| \leq |L(x)(\xi - \tau Du(x))| + |Du(x)| + |M_s(s - \tau u(x), \xi - \tau Du(x))| + |G'(s - \tau u(x))||u(x)|
\]
\[
+ |M_s(s - \tau u(x), \xi - \tau Du(x))| + |G'(s - \tau u(x))||u(x)|
\]
\[
\leq C(|\xi|^{p-1} + |Du(x)|^{p-1})|Du(x)| + C(|\xi|^m + |Du(x)|^m)|u(x)|
\]
\[
+ C(|\xi|^m - |Du(x)|^m)|Du(x)| + C(|\xi|^m + |u(x)|^m)|u(x)|
\]
\[
\leq \varepsilon|\xi|^p + C_\varepsilon|Du(x)|^p + \varepsilon|\xi|^p + C_\varepsilon|Du(x)|^p + C_\varepsilon|u(x)|^p
\]
\[
+ \varepsilon|\xi|^m + C_\varepsilon|Du(x)|^m + \varepsilon|s|^p + C_\varepsilon|u(x)|^\sigma
\]
\[
= a_\varepsilon(x) + \varepsilon|s|^p + \varepsilon|\xi|^m,
\]
where \( a_\varepsilon : \Omega \to \mathbb{R} \) is defined a.e. by
\[
a_\varepsilon(x) := C_\varepsilon|Du(x)|^p + C_\varepsilon|Du(x)|^m + C_\varepsilon|u(x)|^p
\]
\[
+ \varepsilon|\xi|^m + C_\varepsilon|u(x)|^\sigma.
\]
Notice that, as \( p - 1 \leq m < p - 1 + p/N \) it holds \( p \leq p/(p - m) \leq p^* \), yielding \( u \in L^{p/(p-m)}(\Omega) \) and in turn, \( a_\varepsilon \in L^1(\Omega) \). The assertion follows directly by Lemma 2.5 with \( \mu = 1 \).

We have the following splitting result

**Theorem 3.2.** Let the integrand \( j \) be as in (1.11) and
\[
p - 1 \leq m \leq p - 1 + p/N, \quad p < \sigma < p^*.
\]
Assume that \( (u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \) is a bounded Palais-Smale sequence for \( \phi \) at the level \( c \in \mathbb{R} \) weakly convergent to some \( u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \). Then
\[
\lim_{n \to \infty} \left( \int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \right) = c - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p},
\]
namely
\[
\lim_{n \to \infty} \phi_\infty(u_n - u) = c - \phi(u),
\]
being \( u_n \) and \( u \) regarded as elements of \( W_0^{1,p}(\mathbb{R}^N) \cap D_0^{1,m}(\mathbb{R}^N) \) after extension to zero out of \( \Omega \).

**Proof.** In light of Proposition 2.3, up to a subsequence, \( (u_n) \) converges weakly to some function \( u \) in \( W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \), \( u_n(x) \to u(x) \) and \( Du_n(x) \to Du(x) \) for a.e. \( x \in \Omega \). Also, recalling that by assumption \( V(x) \to V_\infty \) as \( |x| \to \infty \), we have [4, 17]
\[
(3.2) \quad \lim_{n \to \infty} \int_\Omega V(x)|u_n - u|^p - V_\infty|u_n - u|^p = 0,
\]
\[
(3.3) \quad \lim_{n \to \infty} \int_\Omega V(x)|u_n - u|^p - V(x)|u_n|^p + V(x)|u|^p = 0.
\]
Therefore, by virtue of Lemma 3.1, we conclude that
\[
\lim_{n \to \infty} \phi_\infty(u_n - u) = \lim_{n \to \infty} \left( \int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \right)
\]
\[
= \lim_{n \to \infty} \left( \int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V(x) \frac{|u_n - u|^p}{p} \right)
\]
\[
= \lim_{n \to \infty} \left( \int_\Omega j(u_n, Du_n) + \int_\Omega V(x) \frac{|u_n|^p}{p} \right) - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p}
\]
\[
= \lim_{n \to \infty} \phi(u_n) - \phi(u) = c - \phi(u),
\]
concluding the proof. \( \square \)
Remark 3.3. In order to shed some light on the restriction (1.5) of $m$, it is readily seen that it is a sufficient condition for the following local compactness property to hold. Assume that $\omega$ is a smooth domain of $\mathbb{R}^n$ with finite measure. Then, if $(u_n)$ is a bounded sequence in $W^{1,p}_0(\omega)$, there exists a subsequence $(u_{n_k})$ such that

$$\mathcal{Y}(x, u_{n_k}, Du_{n_k})$$

converges strongly to some $\mathcal{Y}_0$ in $W^{-1,p'}(\omega)$,

where $\mathcal{Y}(x, s, \xi) = g(s) - M_s(s, \xi) - V(x)|s|^{p-2}s$. In fact, taking into account the growth condition on $g$ and $M_s$, this can be proved observing that for every $\varepsilon > 0$, there exists $C_\varepsilon$ such that

$$|\mathcal{Y}(x, s, \xi)| \leq C_\varepsilon + \varepsilon |s|^{-\frac{N(p-1)+\varepsilon}{\varepsilon p}} + \varepsilon |\xi|^{p-1+p/N},$$

for a.e. $x \in \omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

3.2. Equation splitting I (super-quadratic case). We shall assume that $m, p \geq 2$ and that conditions (1.7)-(1.8) hold. The following Theorem 3.4 and the forthcoming Theorem 3.5 (see next subsection) are in the spirit of the Brezis-Lieb Lemma [4], in a dual framework. For the particular case

$$M(s, \xi) = 0 \quad \text{and} \quad L(\xi) = \frac{|\xi|^p}{p},$$

we refer the reader to [12].

Theorem 3.4. Assume that (1.5)-(1.11) hold and that

$$p - 1 \leq m < p - 1 + N/p, \quad p < \sigma < p^*.$$  

Assume that $(u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ is such that $u_n \rightharpoonup u, u_n \rightarrow u$ a.e. in $\Omega$, $Du_n \rightarrow Du$ a.e. in $\Omega$ and there is $(w_n)$ in the dual space $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$ such that $w_n \rightarrow 0$ as $n \rightarrow \infty$ and, for all $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$,

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_nv = \langle w_n, v \rangle. \quad (3.4)$$

Then $\phi'(u) = 0$. Moreover, there exists a sequence $(\xi_n)$ that goes to zero in $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$, such that

$$\langle \xi_n, v \rangle := \int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv$$

$$- \int_\Omega j_s(u_n, Du_n)v - \int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u, Du)v + \int_\Omega j_\xi(u, Du) \cdot Dv,$$

for all $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$.

Furthermore, there exists a sequence $(\zeta_n)$ in $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$ such that

$$\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v = \langle \zeta_n, v \rangle$$

for all $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ and $\zeta_n \rightarrow 0$ as $n \rightarrow \infty$, namely $\phi'_\infty(u_n - u) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Fixed some $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$, let us define for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$f_v(x, s, \xi) := j_s(s - u(x), \xi - Du(x))v(x)$$

$$+ j_\xi(s - u(x), \xi - Du(x)) \cdot Dv(x) - j_s(s, \xi)v(x) - j_\xi(s, \xi) \cdot Dv(x).$$

In order to prove 3.5 we are going to show that

$$\lim_{n \rightarrow \infty} \sup_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \left| \int_\Omega f_v(x, u_n, Du_n) - f_v(x, u, Du) \right| = 0. \quad (3.6)$$
As it can be easily checked, there holds
\[-f_{v}(x, s, \xi) = \int_{0}^{1} j_{ss}(s - \tau u(x), \xi - \tau Du(x))u(x)v(x)d\tau
+ \int_{0}^{1} j_{s\xi}(s - \tau u(x), \xi - \tau Du(x)) \cdot [Du(x)v(x) + Dv(x)u(x)]d\tau
+ \int_{0}^{1} [j_{\xi\xi}(s - \tau u(x), \xi - \tau Du(x)) Du(x)] \cdot Dv(x)d\tau.\]

Hence, by plugging the particular form of \(j\) in the above equation yields
\[-f_{v}(x, s, \xi) = a(x, s, \xi)v(x) + b(x, s)v(x) + c_{1}(x, s, \xi) \cdot Dv(x) + c_{2}(x, s, \xi) \cdot Dv(x) + d(x, \xi) \cdot Dv(x)\]
where
\[a(x, s, \xi) := \int_{0}^{1} [M_{ss}(s - \tau u(x), \xi - \tau Du(x))u(x) + M_{s\xi}(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x)]d\tau,\]
\[b(x, s) := -\int_{0}^{1} G''(s - \tau u(x))u(x)d\tau,\]
\[c_{1}(x, s, \xi) := \int_{0}^{1} M_{s\xi}(s - \tau u(x), \xi - \tau Du(x))u(x)d\tau,\]
\[c_{2}(x, s, \xi) := \int_{0}^{1} M_{\xi\xi}(s - \tau u(x), \xi - \tau Du(x)) Du(x)d\tau,\]
\[d(x, \xi) := \int_{0}^{1} L_{\xi\xi}(\xi - \tau Du(x)) Du(x)d\tau.\]

We claim that, as \(n \to \infty\), it holds
\[a(\cdot, u_{n}, Du_{n}) \to a(\cdot, u, Du) \quad \text{in } L^{(p^*)'}(\Omega),\]
\[b(\cdot, u_{n}) \to b(\cdot, u) \quad \text{in } L^{p'}(\Omega),\]
\[c_{1}(\cdot, u_{n}, Du_{n}) \to c_{1}(\cdot, u, Du) \quad \text{in } L^{p'}(\Omega),\]
\[c_{2}(\cdot, u_{n}, Du_{n}) \to c_{2}(\cdot, u, Du) \quad \text{in } L^{m'}(\Omega),\]
\[d(\cdot, Du_{n}) \to d(\cdot, Du) \quad \text{in } L^{p'}(\Omega).\]

Then, using Hölder’s inequality and the embeddings of \(W^{1,p}_{0}(\Omega) \cap D_{0}^{1,m}(\Omega)\) into \(L^{p}(\Omega)\) and \(L^{p*}(\Omega)\) we obtain
\[
\sup_{\|v\|_{W^{1,p}_{0}(\Omega) \cap D_{0}^{1,m}(\Omega)} \leq 1} \left| \int_{\Omega} f_{v}(x, u_{n}, Du_{n}) - f_{v}(x, u, Du) \right| \\
\leq C\|a(\cdot, u_{n}, Du_{n}) - a(\cdot, u, Du)\|_{L^{(p^*)'}(\Omega)} \\
+ C\|b(\cdot, u_{n}) - b(\cdot, u)\|_{L^{p'}(\Omega)} \\
+ C\|c_{1}(\cdot, u_{n}, Du_{n}) - c_{1}(\cdot, u, Du)\|_{L^{p'}(\Omega)} \\
+ C\|c_{2}(\cdot, u_{n}, Du_{n}) - c_{2}(\cdot, u, Du)\|_{L^{m'}(\Omega)} \\
+ C\|d(\cdot, Du_{n}) - d(\cdot, Du)\|_{L^{p'}(\Omega)},
\]
yielding the desired conclusion (3.6). It remains to prove the convergences we claimed above. For each term, we shall exploit Lemma 2.5. Since \(m < p - 1 + p/N\), we can set
\[\alpha := \frac{m}{p^* - 1}, \quad \beta := \frac{pN}{pN - N + p - mN}.\]
it follows $\beta > 0$ and $m < m + \alpha < p$. Young’s inequality yields in turn
\[
y^{(m+\alpha)/(p')'} \leq Cy^{m/(p')'} + Cy^{p/(p')'}, \quad \text{for all } y \geq 0.
\]
Since $\beta/(p')' > 1$ and $(m+\alpha)/(p')' > 1$, by the growths of $M_{ss}$ and $M_{\xi}$, we have
\[
|a(x, s, \xi)| \leq C(|\xi|^m + |Du(x)|^m)|u(x)| + C(|\xi|^{m-1} + |Du(x)|^{m-1})|Du(x)|
\leq \varepsilon|\xi|^{p/(p')'} + C_{\varepsilon}|u(x)|^{p/(p')'} + C_{\varepsilon}|Du(x)|^{p/(p')'} + \varepsilon|\xi|^{(m+\alpha)/(p')'} + C_{\varepsilon}|Du(x)|^{(m+\alpha)/(p')'}.
\]
Furthermore,
\[
|b(x, s)| \leq C(|s|^{\sigma-2} + |u(x)|^{\sigma-2})u(x)| \leq \varepsilon|s|^{\sigma/\sigma'} + C_{\varepsilon}|u|^{\sigma/\sigma'},
\]
\[
|c_1(x, s, \xi)| \leq C(|\xi|^{m-1} + |Du(x)|^{m-1})|u(x)|
\leq \varepsilon|\xi|^{p/(p')} + C_{\varepsilon}|u(x)|^{p/(p')} + C_{\varepsilon}|Du(x)|^{p/p'},
\]
\[
|c_2(x, s, \xi)| \leq C(|\xi|^{m-2} + |Du(x)|^{m-2})|Du(x)|
\leq \varepsilon|\xi|^{m/m'} + C_{\varepsilon}|Du(x)|^{m/m'},
\]
\[
|d(x, \xi)| \leq C(|\xi|^{p-2} + |Du(x)|^{p-2})|Du(x)| \leq \varepsilon|\xi|^{p/p'} + C_{\varepsilon}|Du(x)|^{p/p'}.
\]
From the point-wise convergence of the gradients and the growth estimates of $j_\xi, j_\sigma$ and $g$ that $u$ is a week solutions to the problem, namely for all $v \in W^1_p(\Omega) \cap D_0^{1+m}(\Omega)$
\[
\int_\Omega L_\xi(Du) \cdot Dv + \int_\Omega M_\xi(u, Du) \cdot Dv + \int_\Omega M_s(u, Du)v + \int_\Omega V(x)|u|^{p-2}uv = \int_\Omega g(u)v.
\]
To get this, recall that $v \in L^{p/(p-m)'}(\Omega)$ and the sequence $(M_s(u_n, Du_n))$ is bounded in $L^{p/m'}(\Omega)$ and hence it converges weakly to $M_s(u, Du)$ in $L^{p/m'}(\Omega)$. Thanks to Proposition 2.4 (recall that $\beta \geq p$ if and only if $m \geq p - 2 + p/N$ and this is the case since $m \geq p - 1$), we have $L^{\beta}(\Omega)$. Hence, $u \in L^{\sigma}(\Omega) \cap L^{p/m}(\Omega) \cap L^{\beta}(\Omega)$,

being $p \leq p/(p-m) < p^*$ and $p < \sigma < p^*$. By the previous inequalities the claim follows by Lemma 2.5 with the choice $\mu = (p^*)', \sigma', p', m'$ and $p'$ respectively. Let us now recall a dual version of properties (3.2)-(3.3) (cf. [17]), namely there exist two sequences $(\mu_n)$ and $(\nu_n)$ in $(W_0^{1,p}(\Omega) \cap D_0^{1+m}(\Omega))^*$ which converge to zero as $n \to \infty$ and such that
\[
\int_\Omega V(x)|u|^{p-2}(u - u_n)v = \int_\Omega V(x)|u|^{p-2}(u - u_n)v + \langle \mu_n, v \rangle,
\]
\[
\int_\Omega V(x)|u|^{p-2}(u - u_n)v = \int_\Omega V(x)|u|^{p-2}u_n v - \int_\Omega V(x)|u|^{p-2}uv + \langle \mu_n, v \rangle,
\]
for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1+m}(\Omega)$. Whence, by collecting (3.4), (3.5), (3.6), (3.7), we get
\[
\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_\sigma(u_n - u, Du_n - Du)v + \int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v
\]
\[
= \int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_\sigma(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v
\]
\[
- \int_\Omega j_\xi(u, Du) \cdot Dv - \int_\Omega j_\sigma(u, Du)v - \int_\Omega V(x)|u|^{p-2}uv + \langle \xi_n + \mu_n + \nu_n, v \rangle = \langle \zeta_n, v \rangle,
\]
where $\langle \zeta_n, v \rangle := \langle w_n + \xi_n + \mu_n + \nu_n, v \rangle$ and $\zeta_n \to 0$ as $n \to \infty$. This concludes the proof. \qed
3.3. Equation splitting II (sub-quadratic case). We assume that (1.12)-(1.14) hold.

**Theorem 3.5.** Assume (1.9), let the integrand j be as in (1.11) and $p \leq 2$ or $m \leq 2$ or $\sigma \leq 2$,

$$p - 1 \leq m < p - 1 + p/N, \quad p < \sigma < p^*.$$  

Assume that $(u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ is such that $u_n \to u$, $u_n \to u$ a.e. in $\Omega$, $Du_n \to Du$ a.e. in $\Omega$ and there exists $(w_n)$ in $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$ such that $w_n \to 0$ as $n \to \infty$ and, for every $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$,

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_\xi(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_nv = \langle w_n, v \rangle.$$

Then $\phi'(u) = 0$. Moreover, there exists a sequence $(\hat{\xi}_n)$ that goes to zero in $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$, such that

$$\langle \hat{\xi}_n, v \rangle := \int_\Omega j_\xi(u_n - u, Du_n - Du)v + \int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv$$

$$- \int_\Omega j_\xi(u_n, Du_n)v - \int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_\xi(u, Du)v + \int_\Omega j_\xi(u, Du) \cdot Dv,$$  

for all $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$.

Furthermore, there exists a sequence $(\hat{\xi}_n)$ in $W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ with

$$\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_\xi(u_n - u, Du_n - Du)v + \int_\Omega V(x)|u_n - u|^{p-2}(u_n - u)v = \langle \hat{\xi}_n, v \rangle$$

for all $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ and $\hat{\xi}_n \to 0$ as $n \to \infty$, namely $\phi'_{\infty}(u_n - u) \to 0$ as $n \to \infty$.

**Proof.** Keeping in mind the argument in proof of Theorem 3.4, here we shall be more sketchy. For every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ we plug $L, M, G$ into the equation

$$f_v(x, s, \xi) = j_s(s - u(x), \xi - Du(x))v(x)$$

$$+ j_\xi(s - u(x), \xi - Du(x)) \cdot Dv(x) - j_\xi(s, \xi)v(x) - j_\xi(s, \xi) \cdot Dv(x),$$

thus obtaining

$$f_v(x, s, \xi) = (M_s(s - u(x), \xi - Du(x)) - M_s(s, \xi))v(x) - (G'(s - u(x)) - G'(s))v(x)$$

$$+ (M_\xi(s - u(x), \xi - Du(x)) - M_\xi(s, \xi)) \cdot Dv(x) + (L_\xi(\xi - Du(x)) - L_\xi(\xi)) \cdot Dv(x)$$

$$= a'v(x) + b'v(x) + c' \cdot Dv(x) + d' \cdot Dv(x).$$

We write the term $M_\xi(s - u(x), \xi - Du(x)) - M_\xi(s, \xi)$ in a more suitable form, namely

$$c' = M_\xi(s - u(x), \xi - Du(x)) - M_\xi(s, \xi)$$

$$= \underbrace{M_\xi(s - u(x), \xi - Du(x)) - M_\xi(s, \xi - Du(x))}_{c'_1(x, s, \xi)} + \underbrace{M_\xi(s, \xi - Du(x)) - M_\xi(s, \xi)}_{c'_2(x, s, \xi)},$$

so that

$$f_v(x, s, \xi) = a'(x, s, \xi)v(x) + b'(x, s)v(x) + (c'_1(x, s, \xi) + c'_2(x, s, \xi)) \cdot Dv(x) + d'(x, \xi) \cdot Dv(x).$$

The term $a'$ admits the same growth condition of $a$, cf. the proof of Theorem 3.4. Also, since

$$c'_1(x, s, \xi) = - \int_0^1 M_{\xi_u}(s - \tau u(x), \xi - Du(x))u(x) d\tau,$$

as for the term $c_1$ in the proof of Theorem 3.4 we obtain

$$|c'_1(x, s, \xi)| \leq \varepsilon |\xi|^p/p' + C_\varepsilon |u(x)|^{p/(p-m)p'} + C_\varepsilon |Du(x)|^{p/p'}.$$
On the other hand, directly from assumptions (1.12)- (1.14) we get
\[ |b'(x, s)| \leq C|u(x)|^{\sigma'/\alpha}, \quad |e'_2(x, s, \xi)| \leq C|Du(x)|^{m'/m}, \quad |d'(x, \xi)| \leq C|Du(x)|^{p'/p'} \]
The conclusion follows then by the same argument carried out in Theorem 3.4.

In the spirit of [17, Lemma 8.3], we have the following

**Lemma 3.6.** Under the hypotheses of Theorem 1.1 or 1.2, let \((y_n) \subset \mathbb{R}^N\) with \(|y_n| \to \infty\),
\[
\begin{align*}
&u_n(\cdot + y_n) \rightharpoonup u \quad \text{in} \ W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N), \\
u_n(\cdot + y_n) \rightharpoonup v \quad \text{a.e. in} \ \mathbb{R}^N, \\
&Du_n(\cdot + y_n) \rightharpoonup Du \quad \text{a.e. in} \ \mathbb{R}^N, \\
&\phi_\infty(u_n) \to c, \\
&\phi_\infty'(u_n) \to 0 \quad \text{in} \ (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*,
\end{align*}
\]
Then \(\phi_\infty'(u) = 0\) and, setting \(v_n := u_n - u(\cdot - y_n)\), we have
\[
\begin{align*}
\phi_\infty(v_n) &\to c - \phi_\infty(u) \quad (3.9) \\
\phi_\infty'(v_n) &\to 0 \quad \text{in} \ (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*, \quad (3.10)
\end{align*}
\]
and \(\|v_n\|_p^p = \|u_n\|_p^p - \|u\|_p^p + o(1)\) and \(\|v_n\|_m^m = \|u_n\|_m^m - \|u\|_m^m + o(1)\) as \(n \to \infty\).

**Proof.** The energy splitting (3.9) follows by Theorem 3.2 applied with \(\Omega = \mathbb{R}^N\) and the sequence \((u_n)\) replaced by \((u_n(\cdot + y_n))\). Take now \(\varphi \in D(\Omega)\) with \(\|\varphi\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \leq 1\) and define \(\varphi_n := \varphi(\cdot + y_n)\). Then \(\varphi_n \in D(\Omega_n)\), where \(\Omega_n = \Omega - \{y_n\} \subset \Omega\) for \(n\) large. For any \(n \in \mathbb{N}\), we get
\[
\langle \phi_\infty'(v_n), \varphi \rangle = \langle \phi_\infty'(u_n(\cdot + y_n) - u), \varphi \rangle.
\]
By the splitting argument in the proof of Theorem 3.4, it follows that
\[
\langle \phi_\infty'(u_n(\cdot + y_n) - u), \varphi \rangle = \langle \phi_\infty'(u_n(\cdot + y_n)), \varphi \rangle - \langle \phi_\infty'(u), \varphi \rangle + \langle \zeta_n, \varphi \rangle,
\]
where \(\zeta_n \to 0\) in the dual of \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\). If we prove that \(u\) is critical for \(\phi_\infty\), then the right-hand side reads as \(\langle \phi_\infty'(u_n), \varphi \rangle + \langle \zeta_n, \varphi \rangle\), and also the second limit (3.10) follows. To prove that \(\phi_\infty'(u) = 0\) we observe that, for all \(\varphi \in D(\mathbb{R}^N)\),
\[
\langle \phi_\infty'(u_n(\cdot + y_n)), \varphi \rangle \to \langle \phi_\infty'(u), \varphi \rangle, \quad \left(\langle \phi_\infty'(u_n(\cdot + y_n)), \varphi \rangle \right) \leq \|\phi_\infty'(u_n)\|_p \|\varphi\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \to 0.
\]
Indeed, defining \(\tilde{\varphi}_n := \varphi(\cdot - y_n)\), since \(|y_n| \to \infty\) as \(n \to \infty\), we have \(\text{supp} \tilde{\varphi}_n \subset \Omega\) for \(n\) large enough and \(\|\tilde{\varphi}_n\|_{W^{1,p}_0(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)} = \|\tilde{\varphi}\|_{W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)}\). The last assertion follows by using Brezis-Lieb Lemma [4].

We can finally come to the proof of the main results.

**4. Proof of Theorems 1.1 and 1.2 completed**

We follow the scheme of the proof given in [17, p. 121]. Let \((u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) be a bounded Palais-Smale sequence for \(\phi\) at the level \(c \in \mathbb{R}\). Hence, there exists a sequence \((w_n)\) in the dual of \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) such that \(w_n \to 0\) and \(\phi(u_n) \to c\) as \(n \to \infty\) and, for all \(v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\), we have
\[
\int_\Omega L_\xi(Du_n) \cdot Dv + \int_\Omega M_\xi(u_n, Du_n) \cdot Dv + \int_\Omega M_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_nv = \int_\Omega g(u_n)v + \langle w_n, v \rangle.
\]
Since \( (u_n) \) is bounded in \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \), up to a subsequence, it converges weakly to some function \( v_0 \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) and, by virtue of Proposition 2.3, \((u_n)\) and \((Du_n)\) converge to \( v_0 \) and \( Dv_0 \) a.e. in \( \Omega \), respectively. In turn (see also the proof of Theorem 3.4) it follows

\[
\int_\Omega L(\xi(Du_0)) \cdot Du + \int_\Omega M(\xi(v_0, Du_0)) \cdot Du + \int_\Omega M_s(v_0, Du_0)v + \int_\Omega V(x)|v_0|^{p-2}v_0v = \int_\Omega g(v_0)v,
\]

for any \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \). By combining Theorem 3.2 and Theorem 3.4, setting \( u_1^n := u_n - v_0 \) and thinking the functions on \( \mathbb{R}^N \) after extension to zero out of \( \Omega \), get

\[
(4.1) \quad \phi_{\infty}(u_1^n) \rightarrow c - \phi(v_0), \quad n \rightarrow \infty,
\]

\[
(4.2) \quad \int_{\mathbb{R}^N} L(\xi(Du_1^n)) \cdot Du + \int_{\mathbb{R}^N} M(\xi(u_1^n, Du_1^n)) \cdot Du + \int_{\mathbb{R}^N} M_s(u_1^n, Du_1^n)v + \int_{\mathbb{R}^N} V_\infty|u_1^n|^{p-2}u_1^n v = \int_{\mathbb{R}^N} g(u_1^n)v + \langle w^n, v \rangle.
\]

where \( (w^n) \) is a sequence in the dual of \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) with \( w_1^n \rightarrow 0 \) as \( n \rightarrow \infty \). In turn, it follows that \((u_1^n)\) is Palais-Smale sequence for \( \phi_{\infty} \) at the energy level \( c - \phi(v_0) \). In addition,

\[
\|u_1^n\|_p^p = \|u_n\|_p^p - \|v_0\|_p^p + o(1), \quad \|u_1^n\|_m^m = \|u_n\|_m^m - \|v_0\|_m^m + o(1), \quad as \ n \rightarrow \infty,
\]

by the Brezis-Lieb Lemma \([4]\). Let us now define

\[
\varpi := \lim\sup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_1^n|^p.
\]

If it is the case that \( \varpi = 0 \), then, according to \([11, \text{Lemma I.1}]\), \((u_1^n)\) converges to zero in \( L^r(\mathbb{R}^N) \) for every \( r \in (p, p^*) \). Then, one obtains that

\[
\lim_{n \rightarrow \infty} \int_{\Omega} g(u_1^n)u_1^n = 0, \quad \int_{\Omega} M_s(u_1^n, Du_1^n)u_1^n \geq 0,
\]

where the inequality follows by the sign condition (1.9). In turn, testing equation (4.2) with \( v = u_1^n \), by the coercivity and convexity of \( \xi \mapsto L(\xi), M(s, \xi) \), we have

\[
\limsup_{n \rightarrow \infty} \left[ \nu \int_{\mathbb{R}^N} |Du_1^n|^p + \nu \int_{\mathbb{R}^N} |Du_1^n|^m + V_\infty \int_{\mathbb{R}^N} |u_1^n|^p \right]
\leq \limsup_{n \rightarrow \infty} \left[ \int_{\mathbb{R}^N} L(\xi(Du_1^n)) \cdot Du_1^n + \int_{\mathbb{R}^N} M(\xi(u_1^n, Du_1^n)) \cdot Du_1^n + \int_{\mathbb{R}^N} V_\infty|u_1^n|^p \right] \leq 0,
\]

yielding that \((u_1^n)\) strongly converges to zero in \( W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \), concluding the proof in this case. If, on the contrary, it holds \( \varpi > 0 \), then, there exists an unbounded sequence \((y_1^n) \subset \mathbb{R}^N \) with \( \int_{B(y_1^n,1)} |u_1^n|^p > \varpi / 2 \). Whence, let us consider \( v_1^n := u_1^n(\cdot + y_1^n) \), which, up to a subsequence, converges weakly and pointwise to some \( v_1 \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \), which is nontrivial, due to the inequality \( \int_{B(0,1)} |v_1|^p \geq \varpi / 2 \). Notice that, of course,

\[
\lim_{n \rightarrow \infty} \phi_{\infty}(v_1^n) = \lim_{n \rightarrow \infty} \phi_{\infty}(u_1^n) = c - \phi(v_0).
\]
Moreover, since $|y_n| \to \infty$ and $\Omega$ is an exterior domain, for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we have $\varphi(-y_n^1) \in \mathcal{D}(\Omega)$ for $n \in \mathbb{N}$ large enough. Whence, in light of equation (4.2), for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we get

$$
\int_{\mathbb{R}^N} L_\xi(Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_\xi(v_n^1, Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_s(v_n^1, Dv_n^1)\varphi
+ \int_{\mathbb{R}^N} V_\infty |v_n^1|^{p-2}(v_n^1)\varphi - \int_{\mathbb{R}^N} g(v_n^1)\varphi = \int_{\mathbb{R}^N} L_\xi(Du_n^1) \cdot D\varphi(-y_n^1)
+ \int_{\mathbb{R}^N} M_\xi(u_n^1, Du_n^1) \cdot D\varphi(-y_n^1) + \int_{\mathbb{R}^N} M_s(u_n^1, Du_n^1)\varphi(-y_n^1) + \int_{\mathbb{R}^N} V_\infty |u_n^1|^{p-2}(u_n^1)\varphi(-y_n^1)
- \int_{\mathbb{R}^N} g(u_n^1)\varphi(-y_n^1) = \langle w_n^1, \varphi(-y_n^1) \rangle.
$$

Defining the form $\langle \tilde{w}_n^1, \varphi \rangle := \langle w_n^1, \varphi(-y_n^1) \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we conclude that

$$
\int_{\mathbb{R}^N} L_\xi(Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_\xi(v_n^1, Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_s(v_n^1, Dv_n^1)\varphi
+ \int_{\mathbb{R}^N} V_\infty |v_n^1|^{p-2}(v_n^1)\varphi - \int_{\mathbb{R}^N} g(v_n^1)\varphi = \langle \tilde{w}_n^1, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).
$$

Since $(\tilde{w}_n^1)$ converges to zero in the dual of $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$, it follows by Proposition 2.3 (with $V = V_\infty$ and $\Omega = \mathbb{R}^N$) that the gradients $Dv_n^1$ converge point-wise to $Dv_1$, namely

$$
Dv_n^1(x) \to Dv_1(x), \quad \text{a.e. in } \mathbb{R}^N.
$$

Setting $u_n^2 := u_n^1 - v_1(-y_n^1)$, in light of (4.1)-(4.2) and (4.3), we can apply Lemma 3.6 to the sequence $(v_n^1)$, getting

$$
\lim_{n \to \infty} \phi_\infty(u_n^2) = c - \phi(v_0) - \phi_\infty(v_1),
$$

as well as $\phi_\infty(v_1) = 0$ and, furthermore, for every $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$, we have

$$
\int_{\mathbb{R}^N} L_\xi(Du_n^2) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(u_n^2, Du_n^2) \cdot Dv + \int_{\mathbb{R}^N} M_s(u_n^2, Du_n^2)v
+ \int_{\mathbb{R}^N} V_\infty |u_n^2|^{p-2}u_n^2v - \int_{\mathbb{R}^N} g(u_n^2)v = \langle v_n^2, v \rangle,
$$

where $(\zeta_n^2)$ goes to zero in the dual of $W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$. In turn, $(u_n^2) \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ is a Palais-Smale sequence for $\phi_\infty$ at the energy level $c - \phi(v_0) - \phi(v_1)$. Arguing on $(u_n^2)$ as it was done for $(u_n^1)$, either $u_n^2$ goes to zero strongly in $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ or we can generate a new $(u_n^3)$. By iterating the above procedure, one obtains diverging sequences $(y_i^k)$, $i = 1, \ldots, k - 1$, solutions $v_i$ on $\mathbb{R}^N$ to the limiting problem, $i = 1, \ldots, k - 1$ and a sequence

$$
u_n^k = u_n - v_0 - v_1(-y_n^1) - v_2(-y_n^2) - \cdots - v_{k-1}(-y_n^{k-1}),
$$

such that (recall again Lemma 3.6) as $n \to \infty$

$$
\|u_n^k\|_p^p = \|u_n\|_p^p - \|v_0\|_p^p - \|v_1\|_p^p - \cdots - \|v_{k-1}\|_p^p + o(1),
\|u_n^k\|_m^m = \|u_n\|_m^m - \|v_0\|_m^m - \|v_1\|_m^m - \cdots - \|v_{k-1}\|_m^m + o(1),
$$

as well as $\phi'_\infty(u_n^k) \to 0$ in $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$ and

$$
\phi_\infty(u_n^k) \to c - \phi(v_0) - \sum_{j=1}^{k-1} \phi_\infty(v_j).
$$
Notice that the iteration is forced to end up after a finite number \( k \geq 1 \) of steps. Indeed, for every nontrivial critical point \( v \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \) of \( \phi_\infty \) we have,

\[
\int_{\mathbb{R}^N} L_\xi(Dv) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(Dv) \cdot Dv + \int_{\mathbb{R}^N} M_\mu(v, Dv) v + \int_{\mathbb{R}^N} V_\infty |v|^p = \int_{\mathbb{R}^N} g(v) v,
\]
yielding by the sign condition, the coercivity-convexity conditions and the growth of \( g \),

\[
\min \{ \nu, V_\infty \} |v|^p + \| Dv \|^m_{L^m(\mathbb{R}^N)} \leq C_g \| v \|^\sigma_{L^\sigma(\mathbb{R}^N)} \leq C_g S_{p,\sigma} \| v \|^\sigma_p,
\]
so that, due to \( \sigma > p \), it holds

\[
\| v \|^p_p \geq \left[ \frac{\min \{ \nu, V_\infty \}}{C_g S_{p,\sigma}} \right] \frac{\sigma}{p} =: \Gamma_\infty > 0,
\]
thus yielding from (4.4)

\[
\| u_n^k \|^p_p \leq \| u_n \|^p_p - \| v_0 \|^p_p - (k - 1) \Gamma_\infty + o(1).
\]
By boundedness of \( (u_n) \), \( k \) has to be finite. Hence \( u_n^k \to 0 \) strongly in \( W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \) at some finite index \( k \in \mathbb{N} \). This concludes the proof. \( \square \)

5. Proof of Corollary 1.3

As a byproduct of the proof of the Theorems 1.1 and 1.2, since the \( p \) norm is bounded away from zero on the set of nontrivial critical points of \( \phi_\infty \), cf. (4.5), we can estimate \( \phi_\infty \) from below on that set. In order to do so, we use condition (2.1). For any nontrivial critical point of the functional \( \phi_\infty \), we have (see the proof of Proposition 2.2)

\[
\mu \phi_\infty(v) \geq \delta \int_{\Omega} |Dv|^p + \frac{\mu - 1}{p} V_\infty \int_{\mathbb{R}^N} |v|^p \geq \min \left\{ \delta, \frac{\mu - 1}{p} V_\infty \right\} \| v \|^p_p.
\]
An analogous argument applies to \( \phi \), yielding for any nontrivial critical point

\[
\mu \phi(u) \geq \delta \int_{\Omega} |Du|^p + \frac{\mu - 1}{p} V_0 \int_{\Omega} |u|^p \geq \min \left\{ \delta, \frac{\mu - 1}{p} V_0 \right\} \| u \|^p_p.
\]
Now notice that, recalling (4.6) and a similar variant for the norm of the critical points of \( \phi \) in place of \( \phi_\infty \), setting also

\[
e_\infty := \min \left\{ \frac{\delta}{\mu}, \frac{\mu - 1}{\mu p} V_\infty \right\} \Gamma_\infty, \quad e_0 := \min \left\{ \frac{\delta}{\mu}, \frac{\mu - 1}{\mu p} V_0 \right\} \Gamma_0, \quad \Gamma_0 := \left[ \frac{\min \{ \nu, V_0 \}}{C_g S_{p,\sigma}} \right] \frac{\sigma}{p} > 0,
\]
from Theorems 1.1 or 1.2 we have \( c \geq \ell e_0 + k e_\infty \) for some \( \ell \in \{ 0, 1 \} \) and non-negative integer \( k \). Condition \( c < c^* := e_\infty \) implies necessarily \( k < 1 \), namely \( k = 0 \). This provides the desired compactness result, using Theorems 1.1 or 1.2. \( \square \)

6. Proof of Corollary 1.8

Defining the functionals \( J, M : W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \to \mathbb{R} \) by

\[
J(u) := \frac{1}{p} \int_{\Omega} L(Du) + \frac{1}{m} \int_{\Omega} M(Du) + \frac{1}{p} \int_{\Omega} V(x) |u|^p, \quad Q(u) := \frac{S_{\Omega}}{\sigma} \int_{\Omega} |u|^\sigma,
\]
and given a minimization sequence \( (u_n) \) for problem (1.16), by Ekeland’s variational principle, without loss of generality we can replace it by a new minimization sequence, still denoted by \( (u_n) \) for which there exists a sequence \( (\lambda_n) \subset \mathbb{R} \) such that for all \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \)

\[
J'(u_n)(v) - \lambda_n Q'(u_n)(v) = \langle w_n, v \rangle, \quad \text{with } w_n \to 0 \text{ in the dual of } W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega).
\]
Taking into account the homogeneity of $L$ and $M$, choosing $v = u_n$ this means
\[
\int_{\Omega} L(Du_n) + \int_{\Omega} M(Du_n) + \int_{\Omega} V(x)|u_n|^p - S_{\Omega} \lambda_n \int_{\Omega} |u_n|^\sigma = \langle w_n, u_n \rangle.
\]
Since $\|u_n\|_{L^\sigma(\Omega)} = 1$ for all $n$ and $\int_{\Omega} L(Du_n) + M(Du_n) \to S_{\Omega}$ as $n \to \infty$, this means that $(u_n)$ is a Palais-Smale sequence for the functional $I(u) := J(u) - Q(u)$ at an energy level
\[
(6.1) \quad c \leq \frac{\sigma - m}{\sigma m} S_{\Omega},
\]
since it holds (recall that $p \geq m$), as $n \to \infty$,
\[
I(u_n) = \frac{1}{p} \int_{\Omega} L(Du_n) + \frac{1}{m} \int_{\Omega} M(Du_n) + \frac{1}{p} \int_{\Omega} V(x)|u_n|^p - \frac{S_{\Omega}}{\sigma} \leq \frac{1}{m} \int_{\Omega} L(Du_n) + \frac{1}{m} \int_{\Omega} M(Du_n) + \frac{1}{m} \int_{\Omega} V(x)|u_n|^p - \frac{S_{\Omega}}{\sigma} = \left( \frac{1}{m} - \frac{1}{\sigma} \right) S_{\Omega} + o(1).
\]
From Corollary 1.3 (applied with $L(Du)$ replaced by $L(Du)/p$, $M(u, Du)$ replaced by $M(Du)/m$ and $G \equiv 0$), the compactness of $(u_n)$ holds provided that (in the notations of Corollary 1.3)
\[
c < \min \left\{ \frac{\delta}{\mu}, \frac{\mu - p}{\mu} V_{\infty} \right\} \left[ \min \{ \nu, V_{\infty} \} \right]^{1 - \frac{p}{\sigma}}.
\]
In our case, we can take $\mu = \sigma$, $\delta = \frac{\sigma - p}{\mu}$, $C_0 = S_{\Omega}$, $V_{\infty} = 1$, $\nu = 1$, $S_{p, \sigma} = S_{R^N}^{\sigma - p}$, yielding
\[
c < \frac{\sigma - p}{\sigma p} S_{R^N}^{\sigma - p} / S_{\Omega}^{\sigma - p}.
\]
Hence, finally, by combining this conclusion with (6.1) the compactness (and in turn the solvability of the minimization problem) holds if (1.17) holds, concluding the proof.

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