Subdivision Invariant Models in Lattice Gauge Theory

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**Abstract**

A class of lattice gauge theories is presented which exhibits novel topological properties. The construction is in terms of compact Wilson variables defined on a simplicial complex which models a four dimensional manifold with boundary. The case of $Z_2$ and $Z_3$ gauge groups is considered in detail, and we prove that at certain discrete values of the coupling parameter, the partition function in these models remains invariant under subdivision of the underlying simplicial complex. A variety of extensions is also presented.

ITFA-93-03 / YCTP-P4-93
February 1993

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1 Introduction

In this paper, we undertake a study of certain lattice gauge theories which have special properties with respect to subdivision of the underlying lattice. The motivation for such a search has its roots in topological field theory (see \[1\] for a review), where quantum field theories have been constructed whose observables are topological or smooth invariants of the underlying spacetime manifold. In the pure Chern-Simons theory \[2, 3\], one has, in particular, a partition function which is a topological invariant of a framing of the spacetime 3-manifold. Our interest originates primarily from the desire to see these types of structures emerge from a traditional lattice approach.

Calculations in lattice gauge theory, and statistical mechanics generally, are concerned with the behaviour of systems in a continuum limit, where the underlying lattice is subdivided into smaller and smaller units. At any given stage of subdivision, one has only a crude approximation to the continuum theory. Topological field theories are, on the other hand, quite different. The topology of any manifold can be captured in terms of a lattice (simplicial complex), and further subdivisions of that lattice in no way enhance ones topological picture of the space. It is of interest to construct lattice models which also reflect this property; models in which the observables are invariant under lattice subdivision. There is then no need to be concerned with a continuum limit, as the model would already compute - exactly - the relevant quantities. In other words, one would already be at the continuum limit.

Here, we construct models which have the property of subdivision invariance at certain discrete values of the coupling parameter. While our motivation for these particular examples stems primarily from the Chern-Simons theory, we will not establish any firm link with that theory here. Our approach is entirely self-contained and we will have no need to refer to results in any continuum model, or to invoke general folklore in quantum field theory. Although nothing in our construction forces us to consider discrete gauge groups, our analysis here will focus on these simpler examples. We find that the partition function of our model, which is defined on a simplicial complex which models a 4-manifold with boundary, is invariant under the type 4 Alexander subdivision \[4\]. This is essentially a local property which we can prove by looking at the Boltzmann weight on a single 4-simplex. This

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special property is restricted to discrete values of the coupling parameter. We also consider other types of lattice subdivision, and show that the partition function of our theory on a disk is invariant under all of the Alexander moves.

We begin in the next section with an overview of lattice gauge theory in terms of compact Wilson variables, and provide some background on simplicial complexes. Our model is then defined and we move on to consider specific cases in succeeding sections. A simple two dimensional version is considered first, and then we treat the $Z_3$ and $Z_2$ gauge groups in four dimensions. While all our detailed calculations are for discrete groups, we discuss some obvious extensions both to continuous groups, and to higher dimensional analogs of the models presented here. We close then with some concluding remarks.

2 General Properties

We first recall the essential definitions needed in a Wilson formulation of lattice gauge theory on a simplicial complex. For a complete account of the latter, see [5].

Let $[v_0, \ldots, v_n]$ denote the oriented n-simplex spanned by the geometrically independent set of points $\{v_i\}$, called its vertices. One can picture these simplices as points, line segments, triangles and tetrahedrons for $n$ equal to zero through three. A simplex which is spanned by any subset of the vertices is called a face of the original simplex.

A simplicial complex $K$ is a collection of simplices which are glued together under two restrictions. Any face of a simplex in $K$ is required to be a simplex in $K$, and the intersection of any two simplices in $K$ must be a face of each of them.

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The basic fields which enter a formulation of lattice gauge theory are group valued maps on the 1-simplices (denoted $[a, b]$) with the rule that $U_{ba} = U_{ab}^{-1}$. In order to define a theory, one requires that the group be compact with an invariant measure, and one takes the action on the “link” variables - the gauge transformations - to be given by:

$$U_{ab} \rightarrow g_a U_{ab} g_b^{-1}.$$  \hspace{1cm} (1)
Here, $g_a$ is a group element associated with the vertex $a$.

We take the action of our theory to be a gauge invariant function $S$ of the above link variables. The partition function is then

$$Z = \prod_\alpha \int dU_\alpha \exp[\beta S(U)] ,$$

(2)

where the index $\alpha$ indicates the set of independent 1-simplices. In the case of a discrete gauge group, the group integration (whose volume we normalize to unity) is a discrete sum,

$$\int dU \rightarrow \frac{1}{|G|} \sum_U ,$$

(3)

where $|G|$ denotes the order of the group. One can further define correlation functions of the link variables,

$$< U_{\gamma_1} \cdots U_{\gamma_p} > = \prod_\alpha \int dU_\alpha U_{\gamma_1} \cdots U_{\gamma_p} \exp[\beta S(U)] .$$

(4)

In the above, a group trace is understood, whenever necessary. The key point is that, in general, all of these quantities depend not only on the coupling parameter $\beta$, but also on the simplicial complex $K$.

The above was an abstract description of lattice gauge theory; however, the motivation for it arises as follows. Normally in a gauge theory the basic dynamical variable is a connection on some principal $G$ bundle over spacetime. One can construct the Wilson link variables

$$W_{C_{ab}} = P \exp[\int_{C_{ab}} A] ,$$

(5)

where $C_{ab}$ is some path in spacetime between the endpoints $a$ and $b$, and $A$ is a connection which takes values in the Lie algebra of the group $G$. As usual, the symbol $P$ denotes the path ordering. It is straightforward to establish that the the link variables are solutions to the differential equation:

$$D W_{C_{ab}} = 0 ,$$

(6)

where $D$ is the covariant derivative.
Let $U_{abc} = U_{ab} U_{bc} U_{ca}$ be the holonomy, based at the first vertex $a$, around the triangle determined by $a$, $b$ and $c$, and traversed in the order from left to right. In terms of the Wilson link variables, this is represented by choosing a closed contour $C$ in (5), and taking a group trace if necessary. A property of holonomy is that in the limit when the loop becomes infinitesimally small, $U_C$ approaches

$$\exp[\int F] \ ,$$

where the integral is over a surface which has $C$ as its boundary.

The usual Wilson action is given by

$$S = \frac{1}{2} \sum_{U} (U - 1) + (U^{-1} - 1) \ ,$$

where the sum is over all the basic holonomies on the lattice. Notice that each term in this sum depends on a single holonomy. It is a standard exercise [6] to show that the continuum limit of the above action is the Yang-Mills theory:

$$S \to \frac{1}{2} \int F^2 \ .$$

In the present discussion, we are motivated to consider a discrete version of the Chern form, $F \wedge F$. The first observation is to note that one can produce $F$ by combinations such as $U - U^{-1}$ and $U - 1$, in the continuum limit. We will base our action on a combination of two independent holonomies, which are tied together at a point, and have no edges in common. Let us first, however, digress to review a product which will serve to form the analog of the wedge product of differential forms.

Denote by $P$, one of the $(r + s + 1)!$ permutations of the set of vertices $\{v_0, \cdots, v_{r+s}\}$, which span some $(r + s)$-simplex, and by $Pv_i$ the value of that permutation on $v_i$. Let $c^r$ and $c^s$ be group valued maps on $r$- and $s$-simplices, respectively. The $\star$-product $c^r \star c^s$ yields a group valued map defined on $(r + s)$-simplices, and is defined by:
\[
\langle c^r \star c^s, [v_0, \ldots, v_{r+s}] \rangle = \frac{1}{(r + s + 1)!} \sum_P (-1)^{|P|} \left< c^r, [Pv_0, \ldots, Pv_r] \right> \cdot \left< c^s, [Pv_r, \ldots, Pv_{r+s}] \right>,
\]

when the order \( v_0 \ldots v_{r+s} \) is in the equivalence class of the orientation of the simplex \([v_0, \ldots, v_{r+s}]\) (this determines the overall sign of the product), and where the sum is over all permutations of the vertices. The notation \( \langle \cdot, \cdot \rangle \) is used to indicate the evaluation of the map on the accompanying simplex, and the product on the right hand side of (10) refers to multiplication in the relevant group or ring. For a more complete definition of the \( \star \)-product, we refer to [7]; we simply note here that it is a variation of the standard cup product which achieves graded commutativity at the expense of associativity (the usual cup product is graded commutative only on cohomology classes).

The actual number of independent terms in the above sum is given by the number of ways one can partition the set of vertices into two parts which contain one vertex in common, and an easy counting yields

\[
\frac{(r + s + 1)!}{r! s!}.
\]

Now we are equipped to return to our action. We take this to be a sum over holonomy pairs

\[
S = \sum (U - U^{-1}) \star (U - U^{-1});
\]

a trace is understood when required, and the sum here is over all elementary 4-simplices in the simplicial complex. The \( \star \)-product ensures that the two factors are independent holonomies with one point in common. Of course, one could equally well consider an action of the form

\[
S = \sum (U - 1) \star (U - 1),
\]

or several other possibilities, each of which yield the same continuum limit. Note, however, that the quantity \((U - U^{-1})_{abc}\) has the additional feature that it is antisymmetric in its last two indices; this follows simply from \(U_{acb} = U_{abc}^{-1}\).

As we have seen, the holonomy \(U\) is a group valued map on 2-simplices. The actions defined above, in terms of two independent holonomies, are therefore naturally defined on a 4-simplex.
We should emphasize that achieving a continuum limit akin to the form $F \wedge F$ is purely motivational, and we shall not draw on any properties of continuum theories here. It is not clear, a priori, whether any of the familiar continuum properties will translate onto a finite lattice. We note that an action similar to the above was studied for the case of a torus, in a different context [8]. Our aim is simply to take the above lattice definition, and prove that it has certain topological features.

When we evaluate the star product on a given 4-simplex, the antisymmetrization produces generically $5!$ terms. With the action (12), the symmetries present reduce that to 15 distinct terms. We will take as the Boltzmann weight of this theory, evaluated on the simplex $[v_0, v_1, v_2, v_3, v_4]$, the normalization given by:

$$ W[0, 1, 2, 3, 4] = B[0, 1, 2, 3, 4] B[0, 1, 3, 4, 2] B[0, 1, 4, 2, 3] $$

$$ + B[1, 0, 2, 4, 3] B[1, 0, 3, 2, 4] B[1, 0, 4, 3, 2] $$

$$ + B[2, 0, 1, 3, 4] B[2, 0, 3, 4, 1] B[2, 0, 4, 1, 3] $$

$$ + B[3, 0, 1, 4, 2] B[3, 0, 2, 1, 4] B[3, 0, 4, 2, 1] $$

$$ + B[4, 0, 1, 2, 3] B[4, 0, 2, 3, 1] B[4, 0, 3, 1, 2] \quad , \quad (14) $$

where,

$$ B[0, 1, 2, 3, 4] = \exp[\beta (U - U^{-1})_{v_0 v_1 v_2} (U - U^{-1})_{v_3 v_4}] \quad . \quad (15) $$

Our analysis deals with the general case of complex coupling $\beta$. We will, however, still use the phrase “Boltzmann weight” in this more general context.

Using the Boltzmann weight defined above, we can compute the partition function for the theory defined on a simplicial complex $K$. The central issue of interest here is to examine how this function behaves upon subdivision of the complex. For ease of illustration, let us consider a single 4-simplex $[v_0, v_1, v_2, v_3, v_4]$. A convenient basis of subdivision operations, known as Alexander moves [4], are available, and these allow a direct analysis of this question. The Alexander moves can be described in turn by:

Type 1 Alexander subdivision:

$$ [v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] \quad , \quad (16) $$
Type 2 Alexander subdivision:

\[ [v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] + [v_0, v_1, x, v_3, v_4] \quad \text{(17)} \]

Type 3 Alexander subdivision:

\[ [v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] + [v_0, v_1, x, v_3, v_4] + [v_0, v_1, v_2, x, v_4] \quad \text{(18)} \]

Type 4 Alexander subdivision:

\[ [v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] + [v_0, v_1, x, v_3, v_4] + [v_0, v_1, v_2, x, v_4] + [v_0, v_1, v_2, v_3, x] \quad \text{(19)} \]

One can picture the move of type 1 as the introduction of an additional vertex \( x \), which is placed at the center of the 1-simplex \([v_0, v_1]\), and is then joined to all the remaining vertices of the 4-simplex. Moves 2 to 4 involve a similar construction, where \( x \) is placed at the center of the simplices \([v_0, v_1, v_2]\), \([v_0, v_1, v_2, v_3]\), and finally \([v_0, v_1, v_2, v_3, v_4]\). There is, in addition, a type 0 move which is effected by replacing a vertex of the simplicial complex by a new vertex. This can be considered as a degenerate case, and need not concern us in the following.

According to Alexander [4], two simplicial complexes are said to be equivalent if and only if it is possible to transform one into the other by a sequence of these moves. Hence, any function of \( K \) which is invariant under these moves yields a combinatorial invariant of the simplicial complex.

### 3 A Toy Model in Two Dimensions

Let us consider a simpler two dimensional version of the type of models we want to explore. Here, it is natural to consider an action which depends on a single holonomy, and for the gauge group \( Z_3 = \{1, \exp[2\pi i/3], \exp[4\pi i/3]\} \), we will take the Boltzmann weight evaluated on the 2-simplex \([v_0, v_1, v_2]\) to be given by:

\[ W[v_0, v_1, v_2] = \exp[\beta (U - U^{-1})_{v_0v_1v_2}] \quad \text{(20)} \]
where $U_{v_0v_1v_2}$ is the holonomy combination $U_{v_0v_1} U_{v_1v_2} U_{v_2v_0}$.

It is now a simple matter to explore the subdivision properties of this theory. First, consider the Boltzmann weight under the type 2 Alexander move where

$$[v_0, v_1, v_2] \rightarrow [v_3, v_1, v_2] + [v_0, v_3, v_2] + [v_0, v_1, v_3].$$  \hfill (21)

The picture of this subdivision operation is simply that of adding a new $v_3$ vertex to the center of the original, and then connecting that to the other vertices by the addition of three new links. This present theory enjoys the special property that:

$$W_{v_0, v_1, v_2} = \frac{1}{27} \sum_{U_{v_0v_3} U_{v_1v_3} U_{v_2v_3}} W_{v_3, v_1, v_2} W_{v_0, v_3, v_2} W_{v_0, v_1, v_3},$$ \hfill (22)

when the coupling takes values such that $s^3 = 1$, where $s = e^{i \beta \sqrt{3}}$ is a convenient scale parameter. This property is not obvious, but can be checked, by hand, in a straightforward way. Notice that at the special points, the Boltzmann weight itself is $Z_3$-valued for every link configuration, and this will be the pattern in all our examples.

The situation under type 1 Alexander subdivision is slightly trickier. Here, a given 2-simplex $[v_0, v_1, v_2]$ is broken into two pieces:

$$[v_3, v_1, v_2] + [v_0, v_3, v_2].$$ \hfill (23)

The picture is that of splitting the $[v_0, v_1]$ edge by the introduction of the $v_3$ vertex at its center, and then connecting that to $v_2$ with a new link. One can ask how the Boltzmann weight or partition function behaves under this move. Again, one can verify by hand that

$$W_{v_0, v_1, v_2} = \frac{1}{3} \sum_{U_{v_2v_3}} W_{v_3, v_1, v_2} W_{v_0, v_3, v_2},$$ \hfill (24)

when $s^3 = 1$ and $U_{v_0v_1} = U_{v_0v_3} U_{v_3v_1}$. We were fortunate in this case to have a simple relation between the boundaries of the original simplex, and that of the simplices after type 1 subdivision; the extra constraint on the product of two of the new link variables just reflects that relationship.
Of course, the action (20) vanishes for the group $Z_2$, and so, in line with the analysis of the previous section, one may wish to examine the $Z_2$ model with Boltzmann weight:

$$W[v_0, v_1, v_2] = \exp[\beta (U - 1)v_0v_1v_2] .$$

(25)

The immediate observation, however, is that the above is simply the standard Wilson action for this group. Nevertheless, if one examines this model with a complex coupling $\beta$, then a simple analysis reveals that the Boltzmann weight is invariant under the Alexander moves of type 1 and 2. This takes place when the coupling satisfies $s^2 = 1$, where $s = e^{-2\beta}$, and $U_{v_0v_1} = U_{v_0v_3}U_{v_3v_1}$.

It is well known that two-dimensional Yang-Mills theory is invariant under area-preserving subdivisions, although such a result is obtained for $\beta$ real and positive, and using a heat kernel form of the action [9, 10]. This simple example indicates that it might be fruitful to re-examine the usual Wilson action and search for special subdivision properties at complex couplings.

One other point that is especially transparent in this model concerns our choice to include the $-1$ term in the defining action; one might ask whether that really has any significance. Suppose we had defined the Boltzmann weight

$$W'[v_0, v_1, v_2] = \exp[\beta U_{v_0v_1v_2}] .$$

(26)

Then a simple scaling of our previous result yields the relation,

$$W'[v_0, v_1, v_2] = \frac{1}{8} s \sum_{U_{v_0v_3}, U_{v_1v_3}, U_{v_2v_3}} W'[v_3, v_1, v_2] W'[v_0, v_3, v_2] W'[v_0, v_1, v_3] ,$$

(27)

which we can interpret as subdivision invariance up to a scaling factor. This lends some support to our original geometrical motivation which suggested the combination $(U - 1)$. If one were to search more generally for other models with interesting subdivision properties, it would be important not to discard models which had this additional scaling behavior.
4 The $Z_3$ Model

We will consider in this section the type of model we outlined in the section on General Properties; i.e., a gauge theory whose action is based on two independent holonomies which are tied together at a point and have no edges in common. Pictorially, one might refer to this as a “bowtie” configuration. The setting here is on a simplicial complex which models a four dimensional manifold with boundary (which can also be empty) and we will focus on the group $Z_3$, which we represent multiplicatively as the cube roots of unity, $1$, $\exp[2\pi i/3]$ and $\exp[-2\pi i/3]$. We were motivated in our choice of action by the familiar Chern form, and we saw earlier that we had some freedom in writing a discrete analog. Our calculations in this section will be based on the action (12), and we will take the Boltzmann weight for a given ordering of vertices to be given by:

$$B[0,1,2,3,4] = \exp[\beta(U - U^{-1})_{v_0v_1v_2} (U - U^{-1})_{v_0v_3v_4}] ,$$

and we will insert that into the expression (14) to get a quantity $W[0,1,2,3,4]$ which takes into account all the different permutations of the ★-product. It will be useful in the following analysis to introduce a scale parameter which we take in this model to be the quantity $s = \exp[-3\beta]$.

The behaviour of the theory under the type 4 Alexander move parallels that of the simple two dimensional model under the type 2 move. Let us investigate the Boltzmann weight of this model in the same way. Take the 4-simplex $[v_0, v_1, v_2, v_3, v_4]$ and its corresponding type 4 subdivision which is the sum of five 4-simplices:

$$[v_5, v_1, v_2, v_3, v_4] + [v_0, v_5, v_2, v_3, v_4] + [v_0, v_1, v_5, v_3, v_4] + [v_0, v_1, v_2, v_5, v_4] + [v_0, v_1, v_2, v_3, v_5] .$$

The property of this theory is that

$$W[v_0, v_1, v_2, v_3, v_4] = \frac{1}{3^5} \sum_{v_5} W[v_5, v_1, v_2, v_3, v_4] W[v_0, v_5, v_2, v_3, v_4] W[0, v_0, v_1, v_2, v_3, v_4] W[v_0, v_1, v_2, v_5, v_4] W[0, v_0, v_1, v_2, v_3, v_5] ,$$

when $s^3 = 1$ and where the sum is over the 5 links which join to the new $v_5$ vertex. We do not know an elementary way of seeing this at the present time.
However, it is simple enough to write a computer program to check this kind of relation and verify it for all choices of “boundary data”, and this we have done using Mathematica \([1]\). The relation is quite simple owing to the fact that both the boundary of the original simplex and its type 4 subdivision are identical. This property guarantees that when we evaluate the partition function of the theory on any simplicial complex, that the number one gets is invariant under all type 4 subdivisions. We emphasize again that this special property only holds for the values of the coupling \(\beta\) such that \(s^3 = 1\).

While we do not yet have such a complete understanding of the subdivision properties of this model under the other Alexander moves at the level of Boltzmann weights, we can nevertheless offer some computational evidence why it is interesting. Here, we will compute exactly the partition function of the theory on the 4-disk which provides an example with a boundary topologically equivalent to \(S^3\). One can model the disk as a simplicial complex with a single 4-simplex, or through more complex subdivisions.

The following results for the partition function (2) were calculated using Mathematica. One aspect of lattice gauge theory that is important to take advantage of in these computer studies is the freedom to gauge fix some link components. The issue of gauge fixing in the Wilson formulation is particularly simple and elegant and does not introduce any murky questions which could undermine the rigour of our analysis. The construction of the partition function in terms of group integrations and a gauge invariant action allows one to fix arbitrarily the links on a maximal tree \([6]\). Roughly speaking, this is any collection of links which does not include a closed path and which cannot be extended by the addition of other links. From a practical perspective, this significantly reduces the number of group integrations (which are just finite sums in this case) that we must perform. Let us now list, in turn, the results of our calculation for the 4-disk.

For the representation of the disk in terms of a single 4-simplex, we find the partition function:

\[
Z = \frac{1}{3^6} (221 + 120(s + s^{-1}) + 60(s^2 + s^{-2}) + 54(s^5 + s^{-5}) + 20(s^6 + s^{-6}))
\]

(31)
Under a type 1 Alexander subdivision of that simplex, we find:

\[
Z = \frac{1}{3^9} (4215 + 2256(s + s^{-1}) + 1596(s^2 + s^{-2}) + 720(s^3 + s^{-3})
+ 660(s^4 + s^{-4}) + 1068(s^5 + s^{-5}) + 664(s^6 + s^{-6}) + 456(s^7 + s^{-7})
+ 48(s^8 + s^{-8}) + 162(s^{10} + s^{-10}) + 72(s^{11} + s^{-11}) + 32(s^{12} + s^{-12})) .
\]

Under a type 2 Alexander subdivision, the computation yields:

\[
Z = \frac{1}{3^{10}} (9641 + 5544(s + s^{-1}) + 4482(s^2 + s^{-2}) + 2610(s^3 + s^{-3})
+ 2178(s^4 + s^{-4}) + 3222(s^5 + s^{-5}) + 2286(s^6 + s^{-6}) + 1764(s^7 + s^{-7})
+ 738(s^8 + s^{-8}) + 522(s^9 + s^{-9}) + 666(s^{10} + s^{-10}) + 270(s^{11} + s^{-11})
+ 294(s^{12} + s^{-12}) + 54(s^{13} + s^{-13}) + 30(s^{15} + s^{-15}) + 18(s^{16} + s^{-16})
+ 18(s^{17} + s^{-17}) + 8(s^{18} + s^{-18})) .
\]

For the case of the disk represented by four 4-simplices which are the type 3 subdivision of the original simplex, we have:

\[
Z = \frac{1}{3^{10}} (10293 + 4680(s + s^{-1}) + 3756(s^2 + s^{-2}) + 2064(s^3 + s^{-3})
+ 2508(s^4 + s^{-4}) + 2544(s^5 + s^{-5}) + 1840(s^6 + s^{-6}) + 1992(s^7 + s^{-7})
+ 1638(s^8 + s^{-8}) + 1104(s^9 + s^{-9}) + 1080(s^{10} + s^{-10}) + 600(s^{11} + s^{-11})
+ 320(s^{12} + s^{-12}) + 72(s^{13} + s^{-13}) + 60(s^{14} + s^{-14}) + 16(s^{15} + s^{-15})
+ 24(s^{16} + s^{-16}) + 72(s^{18} + s^{-18}) + 8(s^{21} + s^{-21})) ,
\]

and finally for the partition function on the simplicial complex resulting from the fourth Alexander move, we have:

\[
Z = \frac{1}{3^{10}} (11841 + 5460(s + s^{-1}) + 2640(s^2 + s^{-2}) + 780(s^3 + s^{-3})
+ 2250(s^4 + s^{-4}) + 2034(s^5 + s^{-5}) + 600(s^6 + s^{-6}) + 1560(s^7 + s^{-7})
+ 2520(s^8 + s^{-8}) + 2970(s^9 + s^{-9}) + 1560(s^{10} + s^{-10}) + 600(s^{11} + s^{-11})
+ 180(s^{12} + s^{-12}) + 90(s^{13} + s^{-13}) + 60(s^{15} + s^{-15}) + 60(s^{16} + s^{-16})
+ 120(s^{17} + s^{-17}) + 60(s^{18} + s^{-18}) + 60(s^{19} + s^{-19})) .
\]
Although these results may appear at first glance to have neither rhyme nor reason, we expect that they will have special properties at the points $s^3 = 1$. Indeed, all five of the functions in (31) - (33) reduce to the simple form,

$$Z(s) = \frac{1}{3^4} (29 + 26 (s + s^{-1})),$$

(36)

at those particular values of $s$. When $s = 1$, or equivalently $\beta = 0$, we have a trivial subdivision invariant point, and $Z = 1$, but at the other two cube roots of unity, we find the value $Z = 1/27$.

As a second example, let us consider the 4-dimensional sphere $S^4$. We model $S^4$ as the boundary of a 5-simplex:

$$\partial [v_0, v_1, v_2, v_3, v_4, v_5] = [v_1, v_2, v_3, v_4, v_5] + [v_0, v_2, v_3, v_4, v_5] + [v_0, v_1, v_3, v_4, v_5] + [v_0, v_1, v_2, v_4, v_5] + [v_0, v_1, v_2, v_3, v_5] + [v_0, v_1, v_2, v_4, v_3] + [v_0, v_1, v_2, v_5, v_4],$$

(37)

where we have chosen to write it in a way such that each 4-simplex is positively oriented according to the order given by its vertices. The precise definition of the boundary operator $\partial$ can be found in [5]. The result of this computation is

$$Z = \frac{1}{3^{10}} (33309 + 12300(s^9 + s^{-9}) + 570(s^{18} + s^{-18})).$$

(38)

When we restrict $s$ to be a cube root of unity, we find that $Z = 1$.

### 5 A Novel $Z_2$ Model

We already considered a two dimensional $Z_2$ based model in an earlier section; here we would like to extend that to four dimensions. Of course, actions which depend on the combination $U - U^{-1}$ necessarily lead to a trivial theory for this group, but there is another problem if we use the $U - 1$ combination in concert with the $\star$-operator. For abelian groups generally, the holonomy $U_{abc}$ is invariant under cyclic permutations of the indices, so that the base point of the holonomy does not enter. In the case of $Z_2$, we also have that $U = U^{-1}$
for all group elements, so the holonomy combination is in fact symmetric in all indices. We tacitly avoided this in two dimensions and simply defined how to evaluate the Boltzmann weight on a given 2-simplex. It is interesting that one can do something similar in four dimensions as well, and we will take as a matter of definition the expression (14) for \( W[0, 1, 2, 3, 4] \) together with a new quantity,

\[
B[0, 1, 2, 3, 4] = \exp[\beta (U - 1) v_0 v_1 v_2 (U - 1) v_0 v_3 v_4] .
\]

This has the effect of selecting essentially half the terms which would appear in the \( \ast \)-product, and “discarding” those which would have entered with opposite sign. The bottom line as to whether this is a fruitful line of thought is whether we can achieve similar success in terms of subdivision properties. As in all the models, we will find a certain scale combination convenient, and we take \( s = \exp[4 \beta] \) in this section.

The first order of business is to analyze the subdivision properties under the type 4 Alexander move. It turns out that this model also enjoys the simple relation (30) for its Boltzmann weight at the points \( s^2 = 1 \). Although the number of link variables is much smaller than in the \( Z_3 \) example, the number is nevertheless quite large, and we also employed a computer program to verify this claim.

It is also straightforward to analyze the other subdivision properties of this model in the explicit calculation of the partition function on a 4-disk. We find, in the same way, that for a single 4-simplex this theory yields the partition function,

\[
Z = \frac{1}{2^6} (11 + 15 s^2 + 27 s^5 + 10 s^6 + s^{15}) .
\]

Under the first Alexander move, where there are two 4-simplices, we find,

\[
Z = \frac{1}{2^9} (31 + 54 s^2 + 33 s^4 + 48 s^5 + 12 s^6 + 96 s^7 + 24 s^8 + 105 s^{10} + 72 s^{11} + 22 s^{12} + 6 s^{20} + 8 s^{21} + s^{30}) .
\]

The type 2 Alexander move applied to the original simplex leads to,

\[
Z = \frac{1}{2^{10}} (29 + 45 s^2 + 54 s^4 + 45 s^5 + 27 s^6 + 108 s^7 + 18 s^8 + 72 s^9 + 54 s^{10} + 18 s^{11} + 153 s^{12} + 36 s^{13} + 27 s^{14} + 102 s^{15} + 135 s^{16} + 45 s^{17} + 13 s^{18} + 9 s^{25} + 18 s^{26} + 12 s^{27} + 3 s^{36} + s^{45}) .
\]
while for the type 3 move we find,

$$Z = \frac{1}{2^{10}} (19 + 12s^2 + 48s^4 + 24s^5 + 16s^6 + 48s^7 + 15s^8 + 72s^9 + 42s^{10}$$
$$+ 84s^{12} + 48s^{13} + 96s^{14} + 24s^{15} + 30s^{16} + 96s^{17} + 28s^{18} + 75s^{20} + 104s^{21}$$
$$+ 84s^{22} + 4s^{24} + 4s^{30} + 24s^{31} + 12s^{32} + 8s^{33} + 6s^{42} + s^{60}) \ . \ (43)$$

Lastly, the type 4 Alexander move, which we already know is an invariance of the partition function at $s^2 = 1$ from our more general analysis, leads to,

$$Z = \frac{1}{2^{10}} (16 + 30s^4 + 15s^5 + 30s^6 + 60s^9 + 45s^{10} + 70s^{12} + 90s^{14} + 60s^{15}$$
$$+ 70s^{18} + 120s^{19} + 30s^{20} + 90s^{22} + 27s^{25} + 75s^{26} + 130s^{27} + 20s^{36}$$
$$+ 30s^{37} + 5s^{39} + 10s^{48} + s^{75}) \ . \ (44)$$

All of these results become much more transparent when we restrict them to the points where $s^2 = 1$. It is remarkable that all of the above polynomials reduce to the simple formula,

$$Z(s) = \frac{1}{2^4} (9 + 7s) \ . \ (45)$$

The two roots of unity, $+1$ and $-1$ yield the values 1 and $1/8$ respectively for the partition function of the disk.

Again, we can compute the partition function on $S^4$, and in this instance we find:

$$Z = \frac{1}{2^{10}} (16 + 60s^6 + 45s^{10} + 15s^{12} + 180s^{14} + 20s^{18} + 180s^{20} + 180s^{24}$$
$$+ 45s^{28} + 27s^{30} + 180s^{32} + 60s^{42} + 15s^{54} + s^{90}) \ . \ (46)$$

Observe that the polynomial for $S^4$ contains even powers of $s$ only. When $s^2 = 1$, the partition function therefore assumes the value $Z = 1$.

6 Generalizations

Here, we discuss some generalizations and extensions of the models considered above. The framework we have outlined is obviously quite general, and
one can immediately consider corresponding theories based on the traditional continuous gauge groups. The only constraint, as we have noted, is that the group should have an invariant measure and finite volume, or new ideas are required. On a finite lattice, the partition function is a well defined number, and the issue is whether these other gauge groups allow for special subdivision invariant points. The behaviour of these theories in the continuum limit for generic couplings is a far more difficult question, and we will not address that here. From a purely topological perspective, one might adopt the point of view that behaviour away from the special points is irrelevant.

As our final example, let us consider a model in six dimensions. Clearly, one can define a model in any even dimension, either with discrete or continuous gauge groups. The construction is quite straightforward, and simply involves taking the star product of three independent holonomies, viz.,

\[ S = \sum (U - U^{-1}) \star (U - U^{-1}) \star (U - U^{-1}). \]  

(47)

Such an action will be evaluated on a 6-simplex, and the sum is over all the elementary 6-simplices in the simplicial complex. Again, it would be of interest to examine the subdivision properties of the associated partition functions.

7 Concluding Remarks

As we have seen, the interesting subdivision properties of the models presented here are based upon a few key ingredients. In particular, one is guided to employ the star product operator in seeking a lattice transcription of the Chern-form. As a result, one finds (in four dimensions) an action which depends on two holonomies. Furthermore, a crucial ingredient is to consider these models for a general complex coupling parameter; indeed, the interesting subdivision properties are present when the associated scale parameter is a root of unity. It should be emphasized that the Boltzmann weight itself is a group valued object at these special points, and this plays an important role in the analysis.

The novel features uncovered in these models should warrant further investigation. Of most importance, perhaps, is a detailed examination of the
properties of the Boltzmann weight under the remaining Alexander moves. The challenge to come to a similar level of understanding for more complicated groups is also well defined. One would also like to see if a relation exists with the models presented in [12, 13], and to explore the subdivision properties of other correlation functions.

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