A refined primal-dual analysis of the implicit bias

Ziwei Ji  Matus Telgarsky
{ziweiji2,mjt}@illinois.edu
University of Illinois, Urbana-Champaign

Abstract

Recent work shows that gradient descent on linearly separable data is implicitly biased towards the
maximum margin solution. However, no convergence rate which is tight in both \( n \) (the dataset size) and
\( t \) (the training time) is given. This work proves that the normalized gradient descent iterates converge to
the maximum margin solution at a rate of \( O \left( \frac{\ln(n)}{\ln(t)} \right) \), which is tight in both \( n \) and \( t \). The proof is via
a dual convergence result: gradient descent induces a multiplicative weights update on the (normalized)
SVM dual objective, whose convergence rate leads to the tight implicit bias rate.

1 Introduction

Recent work has shown that in deep learning, the solution found by gradient descent not only gives a low
training error, but also has low complexity and thus generalizes well (Zhang et al., 2016; Bartlett et al.,
2017). This motivates the study of the implicit bias of gradient descent; i.e., among all the solutions with
low training error, which solution is found by gradient descent.

When the data is linearly separable, Soudry et al. (2017) characterize the implicit bias using the maximum
margin solution: the gradient descent iterates \( w_t \) grow unboundedly, and \( w_t / \|w_t\| \) converges to the maximum
margin solution at rate \( O \left( 1 / \ln(t) \right) \) almost surely. However, since the denominator only grows at a rate of
\( \ln(t) \), a fine-grained bound on the numerator is needed. Ideally, the numerator should be \( O \left( \ln(n) \right) \), where
\( n \) is the size of dataset, since otherwise the convergence could essentially be exponential in the dataset size.
Ji and Telgarsky (2018b) give a \( O \left( \sqrt{\frac{\ln(n)}{\ln(t)}} \right) \) rate for the same convergence, but they do not get the
optimal dependence on \( t \).

In this paper, we prove that \( w_t / \|w_t\| \) indeed converges to the maximum margin solution at a rate of
\( O \left( \ln(n) / \ln(t) \right) \) almost surely. The proof is via a dual convergence analysis: the primal iterate \( w_t \) induces a
dual iterate \( q_t \), and moreover primal gradient descent induces dual multiplicative weights update (mirror
descent with KL divergence). The dual convergence rate then allows us to get an implicit bias rate which is
tight in both \( n \) and \( t \).

Theorem 1.1. Suppose the data is linearly separable, and let \( \bar{u} \) denote the maximum margin solution. Let
\( R \) denote the (training) risk, and use constant step size \( \eta \leq \min \{ 1, 1/R(w_0) \} \). Then almost surely,
\[
\frac{w_t}{\|w_t\|} - \bar{u} \leq O \left( \frac{\ln(n)}{\ln(t)} \right).
\]

Moreover, this rate is tight in both \( n \) and \( t \).

The paper is organized as follows. The introduction continues with related work and notation. Section 2
gives a generic dual convergence result. Section 3 gives the tight rate for the implicit bias. Finally, Section 4
discusses some open problems.

1.1 Related work

Mirror descent is a classical technique in convex optimization (Bubeck, 2015). The multiplicative weights
method, a special case of mirror descent, is also well-studied (Arora et al., 2012). (Note that although a
uniform initialization is often required, it is not required in this paper.) Bach (2015) show the equivalence
between mirror descent and generalized conditional gradient method, via convex duality. A strongly convex regularization is considered there, which is not considered in this paper.

There is extensive recent work on the implicit bias of gradient descent. As discussed above, Soudry et al. (2017) and Ji and Telgarsky (2018b) show that gradient descent on linearly separable data is implicitly biased towards the maximum margin solution. The $O(1/\ln(t))$ rate given in (Soudry et al., 2017) is obtained under the same “almost sure” condition as in this paper. Gunasekar et al. (2018b) study the implicit bias of generic optimization algorithms such as steepest descent. Nacson et al. (2018) show that a normalized gradient descent can give a faster margin maximization rate. (This normalized gradient descent is equivalent to letting $\eta_t = 1$ in our notation, and also leads to a faster dual convergence; see the discussion at the end of Section 2.) Gunasekar et al. (2018b) and Ji and Telgarsky (2018a) show that gradient descent on linear networks also implicitly finds the maximum margin solution.

Margin maximization and implicit bias were heavily studied in the context of boosting methods (Schapire et al., 1997; Schapire and Freund, 2012; Shalev-Shwartz and Singer, 2008). Boosting methods are themselves a form of coordinate descent, one whose convergence is difficult to analyze (Schapire, 2010); interestingly, the original proof of AdaBoost’s empirical risk convergence also used an analysis in the dual (Collins et al., 2002), though without any rate. This same dual analysis, and also work by Kivinen and Warmuth (1999), point out that AdaBoost, in the dual, performs iterative Bregman projection. In fact, using notation introduced below, coordinate descent induces dual multiplicative weights update on the objective $\|A^\top q\|_2^2 / 2$.

1.2 Notation

We use $\| \cdot \|$ to denote the $\ell_2$ norm.

The dataset is denoted by $\{x_i, y_i\}_{i=1}^n$, where $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, +1\}$. We assume $\|x_i\| \leq 1$ and the data is linearly separable. Let $\gamma := \max_{\|u\| \leq 1} \min_{1 \leq i \leq n} y_i \langle u, x_i \rangle > 0$ denote the maximum margin, and $\hat{u} := \arg\max_{\|u\| \leq 1} \min_{1 \leq i \leq n} y_i \langle u, x_i \rangle$ denote the maximum margin solution.

We consider the unbounded, unregularized empirical risk minimization problem with the exponential loss:

$$\min_{w \in \mathbb{R}^d} \mathcal{R}(w) := \frac{1}{n} \sum_{i=1}^n \exp \left( -y_i \langle w, x_i \rangle \right) = \frac{1}{n} \sum_{i=1}^n \exp \left( -\langle w, z_i \rangle \right),$$

where $z_i := y_i x_i$ is used for simplicity.

Given $w \in \mathbb{R}^d$, we consider the following dual variable $q \in \Delta_n$: for $1 \leq i \leq n$,

$$q_i = \frac{\exp \left( -\langle w, z_i \rangle \right)}{\sum_{i=1}^n \exp \left( -\langle w, z_i \rangle \right)} = \frac{\exp \left( -\langle w, z_i \rangle \right)}{n \mathcal{R}(w)}.$$

Furthermore, we consider the dual objective $f : \Delta_n \to \mathbb{R}$ defined as below:

$$f(q) := \frac{1}{2} \sum_{i=1}^n q_i z_i^\top z_i = \frac{1}{2} \left\| A^\top q \right\|^2,$$

where $A \in \mathbb{R}^{n \times d}$ has $z_i^\top$ as its $i$-th row. $f(q)$ is closely related to the SVM dual objective $(\|A^\top \alpha\|^2 / 2 - 1^\top \alpha)$ over $\mathbb{R}_+^n$. It has minimum $f(\hat{q}) = \gamma^2 / 2$, where $\hat{q}$ is the normalized SVM dual optimal solution.

2 Dual convergence

We analyze gradient descent on the (primal) risk $\mathcal{R}(w)$. Gradient descent starts with some initialization $w_0$, and sets $w_{t+1} := w_t - \eta_t \nabla \mathcal{R}(w_t)$ for $t \geq 0$. For each gradient descent iterate $w_t$, the corresponding dual variable is denoted by $q_t$. By definition,

$$w_{t+1} = w_t - \eta_t \nabla \mathcal{R}(w_t) = w_t + \hat{\eta}_t A^\top q_t,$$

where $\hat{\eta}_t := \eta_t \mathcal{R}(w_t)$ is a convenient way to write the rescaling which induces $q_t \in \Delta_n$. 

2
One key observation is that primal gradient descent induces a multiplicative weights update (mirror descent with KL divergence) on the dual objective $f(q)$ (Bubeck 2015, Section 4.3): by definition $q_{t,i} \propto \exp(-\langle w_t, z_i \rangle)$ and $q_{t+1,i} \propto \exp(-\langle w_{t+1}, z_i \rangle)$, and

$$
\exp(-\langle w_{t+1}, z_i \rangle) = \exp(-\langle w_t, z_i \rangle) \exp(-\langle \eta_t A^\top q_t, z_i \rangle)
$$

$$
= \exp(-\langle w_t, z_i \rangle) \exp(-\eta_t (AA^\top)_{i,i})
$$

$$
= \exp(-\langle w_t, z_i \rangle) \exp(-\eta_t \nabla f(q_t)_{i,i})
$$

This motivates us to analyze dual convergence using mirror descent techniques, which gives the following result.

**Theorem 2.1.** For any $t \geq 1$, any $q \in \Delta_n$, if $\hat{\eta}_j = \eta_j R(w_j) \leq 1$ for all $j < t$, then

$$
f(q_t) - f(q) \leq \frac{D_{KL}(q_0, q_t) - D_{KL}(q, q_t)}{\sum_{j < t} \hat{\eta}_j} \leq \frac{\ln(n)}{\sum_{j < t} \hat{\eta}_j}.
$$

In particular, if $\eta_j = \eta \leq 1/R(w_0)$ for all $j < t$ (thus $\hat{\eta}_j = \eta_j R(w_j) \leq R(w_j)/R(w_0)$), then

$$
f(q_t) - \min_{q \in \Delta_n} f(q) = f(q_t) - \frac{1}{2} \gamma^2 \leq \ln(n) \ln(1 + \frac{\eta R(w_0) \gamma^2}{2}).
$$

The first step is to show that $f(q)$ is 1-smooth w.r.t. the $\ell_1$ norm (cf. Lemma 2.2). The smoothness of $f$ and the strong convexity of negative entropy (Pinsker’s inequality) lead to a per-step bound which is then telescoped. Smoothness also ensures a decreasing dual objective (i.e., $f(q_{t+1}) \leq f(q_t)$), which is not necessarily true for general mirror descent (cf. Lemma 2.3). Finally, the proof is finished by a lower bound on the sum of $\hat{\eta}_t$ (cf. Lemma 2.3). Below are the proof details.

The first lemma establishes the smoothness of $f(q)$.

**Lemma 2.2.** The dual objective $f(q)$ is 1-smooth w.r.t. the $\ell_1$ norm.

**Proof.** For any $p, q \in \Delta_n$, using the Cauchy-Schwarz inequality and $\|z_i\| \leq 1$,

$$
\|\nabla f(p) - \nabla f(q)\|_\infty = \left\| AA^\top (p - q) \right\|_\infty = \max_{1 \leq i \leq n} \left| \langle A^\top (p - q), z_i \rangle \right|
$$

$$
\leq \max_{1 \leq i \leq n} \left\| A^\top (p - q) \right\| \|z_i\|
$$

$$
\leq \left\| A^\top (p - q) \right\|.
$$

Furthermore by the triangle inequality and $\|z_i\| \leq 1$,

$$
\left\| A^\top (p - q) \right\| \leq \sum_{i=1}^n |p_i - q_i| \|z_i\| \leq \sum_{i=1}^n |p_i - q_i| = \|p - q\|_1.
$$

Therefore by definition, $f(q)$ is 1-strongly smooth w.r.t. the $\ell_1$ norm. □

The next lemma gives a per-step bound, and also shows the monotonicity of $f(q_t)$.

**Lemma 2.3.** For any $t \geq 0$, any $q \in \Delta_n$, if $\hat{\eta}_t = \eta_t R(w_t) \leq 1$, then

$$
\hat{\eta}_t (f(q_{t+1}) - f(q_t)) \leq D_{KL}(q_t, q) - D_{KL}(q, q_{t+1}),
$$

and $f(q_{t+1}) \leq f(q_t)$. 

3
\textbf{Proof.} An alternative way to write down the mirror descent update is
\[ q_{t+1} = \arg \min_{q \in \Delta_\gamma} \left( f(q) + \langle AA^\top q_t, q - q_t \rangle + \frac{1}{\eta_t} D_{\text{KL}}(q, q_t) \right), \tag{2.4} \]
with a standard mirror descent guarantee
\[ \hat{\eta}_t (f(q_t) - f(q)) \leq \hat{\eta}_t \langle AA^\top q_t, q_t - q_{t+1} \rangle + D_{\text{KL}}(q_t, q_t) - D_{\text{KL}}(q_t, q_{t+1}) - D_{\text{KL}}(q_t, q_t). \tag{2.5} \]

On the other hand, Lemma 2.7 ensures that \( f(q) \) is 1-smooth w.r.t. the \( \ell_1 \) norm, while \( D_{\text{KL}}(p, q) \geq \|p - q\|_1^2/2 \) by Pinsker’s inequality. As a result,
\[ f(q_{t+1}) \leq f(q_t) + \langle AA^\top q_t, q_t - q_{t+1} \rangle + \frac{1}{2}\|q_{t+1} - q_t\|_1^2 \quad \text{(smoothness of } f) \]
\[ \leq f(q_t) + \langle AA^\top q_t, q_t - q_t \rangle + D_{\text{KL}}(q_t, q_t) \quad \text{(Pinsker’s inequality)} \]
\[ \leq f(q_t) + \langle AA^\top q_t, q_t - q_t \rangle + \frac{1}{\hat{\eta}_t} D_{\text{KL}}(q_t, q_t) \quad \text{ (} \hat{\eta}_t \leq 1 \text{)} \]
\[ \leq f(q_t). \quad \text{(by eq. (2.4))} \]

In other words, \( f(q_t) \) is non-increasing. Moreover, by rearranging terms,
\[ \hat{\eta}_t \langle AA^\top q_t, q_t - q_{t+1} \rangle - D_{\text{KL}}(q_t, q_{t+1}) \leq \hat{\eta}_t (f(q_t) - f(q_{t+1})). \tag{2.6} \]

Combining eqs. (2.5) and (2.6), we get
\[ \hat{\eta}_t (f(q_t) - f(q)) \leq \hat{\eta}_t (f(q_t) - f(q_{t+1})) + D_{\text{KL}}(q_t, q_t) - D_{\text{KL}}(q_t, q_{t+1}). \]

Rearranging terms finishes the proof. \hfill \Box

Telescoping the per-step bound given in Lemma 2.3 gives the first result in Theorem 2.1. The other result in Theorem 2.1 for constant \( \eta_t \) requires a lower bound on the sum of \( \hat{\eta}_t \). To this end, we first introduce the following result from prior work. It is also used later to get a refined implicit bias rate.

\textbf{Lemma 2.7 (Ji and Telgarsky 2018b Lemma 3.5).} For any \( t \geq 0 \), if \( \hat{\eta}_t = \eta_t R(w_t) \leq 1 \), then
\[ R(w_{t+1}) \leq R(w_t) - \eta_t \left( 1 - \frac{\eta_t R(w_t)}{2} \right) \|\nabla R(w_t)\|^2. \]

With Lemma 2.7 in hand, we give a lower bound on the sum of \( \hat{\eta}_t \).

\textbf{Lemma 2.8.} For any \( t \geq 1 \), if \( \eta_j = \eta \leq 1/R(w_0) \) for all \( j < t \), then
\[ \sum_{j < t} \hat{\eta}_j \geq \ln \left( 1 + \frac{\eta R(w_0) \gamma^2}{2} \right). \]

To prove Lemma 2.8, a key observation is that \( \ln R \) is also convex, since it is the composition of \( \ln \text{-sum-exp} \) (itself convex) and a linear mapping. Therefore the convexity of \( \ln R \) gives
\[ \ln R(w_{j+1}) - \ln R(w_j) \geq \langle \nabla \ln R(w_j), w_{j+1} - w_j \rangle = -\hat{\eta}_j \|\nabla \ln R(w_j)\|^2 = -\hat{\eta}_j \|A^\top q_j\|^2. \]

The triangle inequality gives that \( \|A^\top q_j\| \leq 1 \), which implies \( \hat{\eta}_j \geq \ln R(w_j) - \ln R(w_{j+1}) \), and
\[ \sum_{j < t} \hat{\eta}_j \geq \ln R(w_0) - \ln R(w_t). \]

As a result, we only further need an upper bound of \( R(w_t) \), which is obtained using Lemma 2.7 and a gradient lower bound. The full proof of Lemma 2.8 is given in the appendix.

Combining Lemmas 2.3 and 2.8 gives the second result in Theorem 2.1. A formal proof is given in the appendix. Another remark is that if instead we let \( \hat{\eta}_j \) be a constant, then we can get a \( O(1/t) \) dual convergence rate. However, our main focus is on convergence induced by primal gradient descent.

Theorem 2.1 gives an intuitive explanation why the maximum margin solution is implicitly favored by primal gradient descent. Since \( f(q_t) \) minimizes the dual SVM objective, and \( \sum_{j < t} \hat{\eta}_j \) is unbounded, the primal gradient descent iterate \( w_t \) will indeed converge in direction to the maximum margin solution. In fact, as shown in the next section, Theorem 2.1 can give a tight rate for the convergence of \( w_t/\|w_t\| \) to \( \bar{u} \).
3 A refined rate for finding the implicit bias

As discussed in the introduction, prior work does not establish a rate which is tight in both \( n \) and \( t \). We give such a tight rate in this section, using the dual convergence result Theorem 2.1.

We first introduce some additional notation. Recall that \( \tilde{u} \) is the maximum margin direction; given any vector \( a \in \mathbb{R}^d \), let \( \Pi_{\perp}[a] := a - \langle a, \tilde{u} \rangle \tilde{u} \) denote its component orthogonal to \( \tilde{u} \). Given a gradient descent iterate \( w_t \), let \( v_t := \Pi_{\perp}[w_t] \). Given a data point \( z_i \), let \( z_{i,\perp} := \Pi_{\perp}[z_i] \).

Let \( S := \{ z_i : \langle \tilde{u}, z_i \rangle = \gamma \} \) denote the set of support vectors, and let

\[
R_{\gamma}(w) := \frac{1}{n} \sum_{z_i \in S} \exp\left(-\langle w, z_i \rangle\right)
\]

denote the risk induced by support vectors, and

\[
R_{\succ\gamma}(w) := \frac{1}{n} \sum_{z_i \not\in S} \exp\left(-\langle w, z_i \rangle\right)
\]

denote the risk induced by non-support vectors. In addition, let \( S_{\perp} := \{ z_{i,\perp} : z_i \in S \} \), and

\[
R_{\perp}(w) := \frac{1}{n} \sum_{z_i \in S} \exp\left(-\langle w, z_{i,\perp} \rangle\right) = \frac{1}{n} \sum_{z_i \in S_{\perp}} \exp\left(-\langle w, z \rangle\right)
\]

denote the risk induced by components of support vectors orthogonal to \( \tilde{u} \). By definition, \( R_{\perp}(w) = R_{\succ\gamma}(w) \exp\left(\gamma \langle w, \tilde{u} \rangle\right) \).

Lastly, let \( \gamma' := \min_{z_i \not\in S} \langle \tilde{u}, z_i \rangle - \gamma \) denote the margin between support vectors and non-support vectors. If there are no non-support vectors, let \( \gamma' = \infty \).

Below is our main result.

**Theorem 3.1.** If the data points \( z_i \) are sampled from a density w.r.t. the Lebesgue measure, then almost surely \( R_{\perp} \) has a unique minimizer \( \bar{v} \) over span\( (S_{\perp}) \). If additionally \( \eta_j = \eta \leq \min\{1, 1/R(w_0)\} \) for all \( j < t \), then

\[
\|v_t - \bar{v}\| \leq \max\{\|v_0 - \bar{v}\|, 2\} + \frac{\ln(n)}{\gamma'} + R(w_0) + 1.
\]

The almost sure existence and uniqueness of \( \bar{v} \) is proved in the appendix. Quantities related to \( \bar{v} \) are also analyzed in prior work (e.g., \( \bar{w} \) in (Soudry et al., 2017)). Below we prove the second part of Theorem 3.1.

The key potential is \( \|v_t - \bar{v}\|^2 \). The change in this potential comes from three parts: (i) a part due to support vectors, which does not increase this potential; (ii) a part due to non-support vectors, which is controlled by the dual convergence result Lemma 2.7; (iii) a squared gradient term, which is controlled by the risk bound Lemma 2.7.

**Proof of second part of Theorem 3.1.** For technical reasons, we consider a range of steps during which \( \|v_j - \bar{v}\| \geq 1 \). If \( \|v_t - \bar{v}\| \leq 1 \), then the proof is done. Otherwise let \( t_{-1} \) denote the last step before \( t \) such that \( \|v_{t_{-1}} - \bar{v}\| \leq 1 \); if such a step does not exist, let \( t_{-1} = -1 \). Furthermore, let \( t_0 = t_{-1} + 1 \). Since it always holds that \( \|\eta \nabla R(w_j)\| \leq 1 \), we have \( \|v_{t_0} - \bar{v}\| \leq \max\{\|v_0 - \bar{v}\|, 2\} \).

Note that

\[
\begin{align*}
\|v_{j+1} - \bar{v}\|^2 &= \|v_j - \bar{v} - \eta \Pi_{\perp}[\nabla R(w_j)]\|^2 \\
&\leq \|v_j - \bar{v} - \eta \nabla R(w_j)\|^2 \\
&= \|v_j - \bar{v}\|^2 - 2\eta \langle \nabla R(w_j), v_j - \bar{v} \rangle + \eta^2 \|\nabla R(w_j)\|^2.
\end{align*}
\]  (3.2)
Furthermore, the inner product term in eq. (3.2) can be decomposed into two parts, for support vectors and non-support vectors respectively:

\[- \langle \nabla R(w_j), v_j - \bar{v} \rangle = \left\langle \frac{1}{n} \sum_{z_i \in S} \exp(-\langle w_j, z_i \rangle) z_i, v_j - \bar{v} \right\rangle + \left\langle \frac{1}{n} \sum_{z_i \not\in S} \exp(-\langle w_j, z_i \rangle) z_i, v_j - \bar{v} \right\rangle. \tag{3.3} \]

The support vector part in eq. (3.3) is non-positive, due to convexity of $R_\perp$:

\[\left\langle \frac{1}{n} \sum_{z_i \in S} \exp(-\langle w_j, z_i \rangle) z_i, v_j - \bar{v} \right\rangle = \left\langle \frac{1}{n} \sum_{z_i \in S} \exp(-\langle w_j, z_i \rangle) z_i, v_j - \bar{v} \right\rangle = \exp(-\gamma \langle w_j, \bar{v} \rangle) \left\langle \frac{1}{n} \sum_{z_i \in S} \exp(-\langle w_j, z_i \rangle) z_i, v_j - \bar{v} \right\rangle \]

\[\leq \exp(-\gamma \langle w_j, \bar{v} \rangle) \langle -\nabla R_\perp(v_j), v_j - \bar{v} \rangle \]

\[\leq \exp(-\gamma \langle w_j, \bar{v} \rangle) \langle R_\perp(\bar{v}) - R(v_j) \rangle \leq 0. \]

The part for non-support vectors in eq. (3.3) is bounded using Cauchy-Schwarz:

\[\left\langle \frac{1}{n} \sum_{z_i \not\in S} \exp(-\langle w_j, z_i \rangle) z_i, v_j - \bar{v} \right\rangle \leq \frac{1}{n} \sum_{z_i \not\in S} \exp(-\langle w_j, z_i \rangle) \| z_i \| \| v_j - \bar{v} \| \]

\[\leq R_{>\gamma}(w_j) \| v_j - \bar{v} \|. \tag{3.4} \]

For $t_0 \leq j < t$, combining eqs. (3.2) and (3.4), and invoking $\| v_j - \bar{v} \| \geq 1$,

\[\| v_{j+1} - \bar{v} \|^2 \leq \| v_j - \bar{v} \|^2 + 2\eta R_{>\gamma}(w_j) \| v_j - \bar{v} \| + \eta^2 \| \nabla R(w_j) \|^2 \]

\[\leq \| v_j - \bar{v} \|^2 + 2\eta R_{>\gamma}(w_j) \| v_j - \bar{v} \| + \eta^2 \| \nabla R(w_j) \|^2 \| v_j - \bar{v} \| \]

\[\leq \left( \| v_j - \bar{v} \| + \eta R_{>\gamma}(w_j) + \frac{\eta^2}{2} \| \nabla R(w_j) \|^2 \right)^2, \]

and thus

\[\| v_{j+1} - \bar{v} \| \leq \| v_j - \bar{v} \| + \eta R_{>\gamma}(w_j) + \frac{\eta^2}{2} \| \nabla R(w_j) \|^2. \tag{3.5} \]

The sum of $R_{>\gamma}$ is bounded using Lemma 2.3. First we have

\[\frac{1}{2} \left\| A^\top q_j \right\|^2 \geq \frac{1}{2} \left\langle A^\top q_j, \bar{v} \right\rangle^2 = \frac{1}{2} (A \bar{u}, q_j)^2 \geq \frac{1}{2} \left( \gamma + \gamma \frac{R_{>\gamma}(w_j)}{R(w_j)} \right)^2 \geq \frac{1}{2} \gamma^2 + \gamma \gamma' \frac{R_{>\gamma}(w_j)}{R(w_j)}. \]

As a result, let $\bar{q}$ denote an optimal dual solution of $f$, then Lemma 2.4 gives

\[D_{KL}(\bar{q}, q_j) - D_{KL}(\bar{q}, q_{j+1}) \geq \hat{v}_f \left( f(q_{j+1}) - \frac{1}{2} \gamma^2 \right) \]

\[\geq \eta R_{>\gamma}(w_{j+1}) \gamma \gamma' \frac{R_{>\gamma}(w_{j+1})}{R(w_{j+1})} = \eta \gamma \gamma' R_{>\gamma}(w_{j+1}). \]

Telescoping gives

\[\sum_{j=0}^{\infty} \eta R_{>\gamma}(w_j) = \eta R_{>\gamma}(w_0) + \sum_{j=1}^{\infty} \eta R_{>\gamma}(w_j) \leq 1 + \frac{D_{KL}(\bar{q}, q_0)}{\gamma \gamma'} \leq 1 + \frac{\ln(n)}{\gamma \gamma'}. \tag{3.6} \]
The squared gradient term in eq. (3.5) is bounded using Lemma 2.7. It implies
\[ R(w_j) - R(w_{j+1}) \geq \eta \left( 1 - \frac{\eta R(w_j)}{2} \right) \| \nabla R(w_j) \|^2 \geq \frac{\eta^2}{2} \| \nabla R(w_j) \|^2. \]
Since \( \eta \leq 1 \), we have
\[ \sum_{j=0}^{\infty} \frac{\eta^2}{2} \| \nabla R(w_j) \|^2 \leq \sum_{j=0}^{\infty} \frac{\eta}{2} \| \nabla R(w_j) \|^2 \leq \sum_{j=0}^{\infty} (R(w_j) - R(w_{j+1})) \leq R(w_0). \] (3.7)
Combining eqs. (3.5) to (3.7) gives
\[ \| v_t - \bar{v} \| \leq \| v_0 - \bar{v} \| + \frac{\ln(n)}{\gamma' \gamma'} + R(w_0) + 1, \]
which finishes the proof. □
Soudry et al. (2017) also show that \( \| v_t - \bar{v} \| \) is bounded almost surely. However, they do not show that \( \| v_t - \bar{v} \| \) is \( O(\ln(n)) \). As noted in the introduction, without such a \( O(\ln(n)) \) bound, the convergence could essentially be exponential in the dataset size. Below we further show that this bound is tight: \( \| v_t - \bar{v} \| \) could be \( \Omega(\ln(n)) \) for certain datasets.

**Theorem 3.8.** Consider the dataset in \( \mathbb{R}^2 \) where \( z_1 = (0.1, 0) \) and \( z_2, \ldots, z_n \) are all \((0.2, 0.2)\). Then \( \bar{v} = (0, 0) \), and starting from \( w_0 = (0, 0) \), for large enough \( t \), we have
\[ \| v_t - \bar{v} \| = \| v_t \| \geq \ln(n) - \ln(2). \]

The proof of Theorem 3.8 is given in the appendix.

With Theorems 3.1 and 3.8 we can prove Theorem 1.1.

**Proof of Theorem 1.1.** Theorem 3.1 implies that \( \| v_t \| \leq \| \bar{v} \| + \| v_0 - \bar{v} \| + O(\ln(n)) \). It is also proved in prior work (Soudry et al., 2017; Ji and Telgarsky, 2018) that \( \| w_t \| = \Theta(\ln(t)) \). This gives
\[ \frac{1}{\| w_t \|} \leq \frac{1}{\| w_t - \bar{w} \|} = \frac{\| w_t - \bar{w} \|}{\| w_t \|} \leq \frac{2\| v_t \|}{\| w_t \|} \leq O\left(\frac{\ln(n)}{\ln(t)}\right). \]

The tightness of the above rate is due to Theorem 3.8 and the fact that \( \| w_t \| = \Theta(\ln(t)) \). □

4 Open problems

An interesting open problem is if any of the above results can be extended to neural networks. Although such an exact characterization of the implicit bias might be too strong to hope for, the above results may still give some inspiration on the generalization properties of gradient descent applied to neural networks, which may further help us design better algorithms.

Another open problem is to get a tight rate in both \( n \) and \( t \) for all datasets (rather than merely almost all). Applying these styles of proof to general data require a complicated iterative decomposition process (Soudry et al., 2017). Hopefully their techniques can be combined with the techniques presented here to give an exact rate.

Lastly, the analysis here is tied to the exponential loss, whereas the similar logistic and cross entropy losses are more common in practice. Is there a way to adapt the present analysis to those cases?
References

Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing*, 8(1):121–164, 2012.

Francis Bach. Duality between subgradient and conditional gradient methods. *SIAM Journal on Optimization*, 25(1):115–129, 2015.

Peter L Bartlett, Dylan J Foster, and Matus J Telgarsky. Spectrally-normalized margin bounds for neural networks. In *Advances in Neural Information Processing Systems*, pages 6240–6249, 2017.

Sébastien Bubeck. Convex optimization: Algorithms and complexity. *Foundations and Trends in Machine Learning*, 2015.

Michael Collins, Robert E. Schapire, and Yoram Singer. Logistic regression, AdaBoost and Bregman distances. *Machine Learning*, 48(1-3):253–285, 2002.

Suriya Gunasekar, Jason Lee, Daniel Soudry, and Nathan Srebro. Characterizing implicit bias in terms of optimization geometry. *arXiv preprint arXiv:1802.08246*, 2018a.

Suriya Gunasekar, Jason D Lee, Daniel Soudry, and Nati Srebro. Implicit bias of gradient descent on linear convolutional networks. In *Advances in Neural Information Processing Systems*, pages 9461–9471, 2018b.

Ziwei Ji and Matus Telgarsky. Gradient descent aligns the layers of deep linear networks. *arXiv preprint arXiv:1810.02032*, 2018a.

Ziwei Ji and Matus Telgarsky. Risk and parameter convergence of logistic regression. *arXiv preprint arXiv:1803.07300v2*, 2018b.

Jyrki Kivinen and Manfred K. Warmuth. Boosting as entropy projection. In *COLT*, pages 134–144, 1999.

Mor Shpigel Nacson, Jason Lee, Suriya Gunasekar, Nathan Srebro, and Daniel Soudry. Convergence of gradient descent on separable data. *arXiv preprint arXiv:1803.01905*, 2018.

Robert E. Schapire. The convergence rate of AdaBoost. In *COLT*, 2010.

Robert E. Schapire and Yoav Freund. *Boosting: Foundations and Algorithms*. MIT Press, 2012.

Robert E. Schapire, Yoav Freund, Peter Bartlett, and Wee Sun Lee. Boosting the margin: A new explanation for the effectiveness of voting methods. In *ICML*, pages 322–330, 1997.

Shai Shalev-Shwartz and Yoram Singer. On the equivalence of weak learnability and linear separability: New relaxations and efficient boosting algorithms. In *COLT*, pages 311–322, 2008.

Daniel Soudry, Elad Hoffer, Mor Shpigel Nacson, Suriya Gunasekar, and Nathan Srebro. The implicit bias of gradient descent on separable data. *arXiv preprint arXiv:1710.10345*, 2017.

Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals. Understanding deep learning requires rethinking generalization. *arXiv preprint arXiv:1611.03530*, 2016.
A Omitted proofs from Section 2

Below is the proof of the lower bound on the sum of \( \hat{\eta}_j \).

**Proof of Lemma 2.8** To prove Lemma 2.8 we first need a risk upper bound. Recall that Lemma 2.7 ensures that for any \( j < t \), if \( \hat{\eta}_j = \eta_j \mathcal{R}(w_j) \leq 1 \), then

\[
\mathcal{R}(w_{j+1}) \leq \mathcal{R}(w_j) - \eta_j \left( 1 - \frac{\eta_j \mathcal{R}(w_j)}{2} \right) \| \nabla \mathcal{R}(w_j) \|^2. \tag{A.1}
\]

As a result, if we let \( \eta_j = \eta \leq 1/\mathcal{R}(w_0) \), then \( \mathcal{R}(w_j) \) never increases, and the requirement \( \hat{\eta}_j = \eta_j \mathcal{R}(w_j) \leq 1 \) of eq. (A.1) always holds.

Dividing both sides of eq. (A.1) and rearranging terms gives

\[
\frac{1}{\mathcal{R}(w_{j+1})} \geq \frac{1}{\mathcal{R}(w_j)} + \eta \left( 1 - \frac{\eta \mathcal{R}(w_j)}{2} \right) \frac{\| \nabla \mathcal{R}(w_j) \|^2}{\mathcal{R}(w_j) \mathcal{R}(w_{j+1})},
\]

which implies

\[
\frac{1}{\mathcal{R}(w_{j+1})} \geq \frac{1}{\mathcal{R}(w_j)} + \eta \left( 1 - \frac{\eta \mathcal{R}(w_j)}{2} \right) \mathcal{R}(w_{j+1}) \geq \frac{1}{\mathcal{R}(w_j)} + \eta \left( 1 - \frac{\eta \mathcal{R}(w_j)}{2} \right) \gamma^2. \tag{A.2}
\]

Since \( \mathcal{R}(w_j) \leq 1 \), eq. (A.2) implies

\[
\frac{1}{\mathcal{R}(w_{j+1})} \geq \frac{1}{\mathcal{R}(w_j)} + \eta \left( 1 - \frac{\eta \mathcal{R}(w_j)}{2} \right) \mathcal{R}(w_{j+1}) \geq \frac{1}{\mathcal{R}(w_j)} + \frac{\eta}{2} \gamma^2,
\]

and thus

\[
\mathcal{R}(w_t) \leq 1/ \left( \frac{1}{\mathcal{R}(w_0)} + \frac{\eta \gamma^2}{2} t \right). \tag{A.3}
\]

Now consider the result in Lemma 2.8. Notice that \( \ln \mathcal{R} \) is also convex, since it is the composition of \( \ln \text{sum-exp} \) and a linear mapping. Therefore the convexity of \( \ln \mathcal{R} \) gives

\[
\ln \mathcal{R}(w_{j+1}) - \ln \mathcal{R}(w_j) \geq \langle \nabla \ln \mathcal{R}(w_j), w_{j+1} - w_j \rangle = -\hat{\eta}_j \| \nabla \ln \mathcal{R}(w_j) \|^2 = -\hat{\eta}_j \| A^\top q_j \|^2.
\]

The triangle inequality ensures \( \| A^\top q_j \| \leq \sum_{i=1}^n \| q_{j,i} \| \| z_i \| \leq 1 \), which implies \( \ln \mathcal{R}(w_{j+1}) - \ln \mathcal{R}(w_j) \geq -\hat{\eta}_j \), and thus

\[
\sum_{j<t} \hat{\eta}_j \geq \ln \mathcal{R}(w_0) - \ln \mathcal{R}(w_t). \tag{A.4}
\]

Combining eqs. (A.3) and (A.4) gives

\[
\sum_{j<t} \hat{\eta}_j \geq \ln \mathcal{R}(w_0) + \ln \left( \frac{1}{\mathcal{R}(w_0)} + \frac{\eta \gamma^2}{2} t \right) = \ln \left( 1 + \frac{\eta \mathcal{R}(w_0) \gamma^2}{2} t \right).
\]

Below is the proof of the main dual convergence result.
Proof of Theorem 2.1. Lemma 2.3 gives that for any \( j < t \),
\[
\hat{\eta}_j (f(q_{j+1}) - f(q)) \leq D_{KL}(q, q_j) - D_{KL}(q, q_{j+1}).
\]
Take the sum of the above inequality from 0 to \( t - 1 \), we get
\[
\sum_{j<t} \hat{\eta}_j (f(q_{j+1}) - f(q)) \leq D_{KL}(q, q_0) - D_{KL}(q, q_t).
\]
Lemma 2.3 also ensures that \( f(q_{j+1}) \leq f(q_j) \) for all \( j < t \), thus
\[
\left( \sum_{j<t} \hat{\eta}_j \right) (f(q_t) - f(q)) \leq D_{KL}(q, q_0) - D_{KL}(q, q_t),
\]
which proves the first part of Theorem 2.1. The second result of Theorem 2.1 follows immediately after invoking Lemma 2.8. \( \square \)

B Omitted proofs from Section 3

Below is the proof of the first part of Theorem 3.1, that almost surely \( \mathcal{R}_\perp \) has a unique minimizer \( \bar{v} \).

Proof of first part of Theorem 3.1. Theorem 3.1 of (Ji and Telgarsky, 2018b) ensures that \( S_\perp \) can be decomposed into two subsets \( B \) and \( C \), with the following properties:

- The risk induced by \( B \)
  \[
  \mathcal{R}_B(w) := \frac{1}{n} \sum_{z \in B} \exp \left( -\langle w, z \rangle \right)
  \]
  is strongly convex over span(\( B \)).

- If \( C \) is nonempty, then there exists a vector \( \tilde{u} \), such that \( \langle z, \tilde{u} \rangle = 0 \) for all \( z \in B \), and \( \langle z, \tilde{u} \rangle \geq \tilde{\gamma} > 0 \) for all \( z \in C \).

On the other hand, Lemma 12 of (Soudry et al., 2017) proves that, almost surely there are at most \( d \) support vectors, and furthermore the \( i \)-th support vector \( z_i \) has a positive dual variable \( q_i \), such that
\[
\sum_{z_i \in S} q_i z_i = \gamma \bar{u}.
\]
As a result,
\[
\sum_{z_i \perp \in \mathcal{S} \perp} q_i z_i, \perp = \sum_{z_i, \perp \in \mathcal{S} \perp} q_i z_i, \perp = 0.
\]
Note that
\[
0 = \left\langle \sum_{z_i, \perp \in \mathcal{S} \perp} q_i z_i, \perp, \tilde{u} \right\rangle = \sum_{z_i, \perp \in \mathcal{C}} q_i \langle z_i, \perp, \tilde{u} \rangle \geq \tilde{\gamma} \sum_{z_i, \perp \in \mathcal{C}} q_i,
\]
which implies \( C \) is empty.

Therefore \( R_{\mathcal{S} \perp} = \mathcal{R}_\perp \) is strongly convex over span(\( S_\perp \)). The existence and uniqueness of the minimizer \( \bar{v} \) follows from strong convexity. \( \square \)

Below is the proof of the lower bound on \( \| v_t - \bar{v} \| \).

Proof of Theorem 3.8. By construction, the only support vector is \( z_1 = (0, 1, 0) \), and \( z_{1, \perp} = (0, 0) \). Therefore \( \text{span}(S_\perp) = \text{span} \left( \{ (0, 0) \} \right) = \{ (0, 0) \} \), and \( \bar{v} = (0, 0) \). Moreover,
\[
\mathcal{R}_{\gamma}(w) = \frac{1}{n} \exp (-0.1 w_1), \quad \text{and} \quad \mathcal{R}_{>\gamma}(w) = \frac{n-1}{n} \exp (-0.2(w_1 + w_2)),
\]
and
and for any $t \geq 0$,

$$\nabla R(w_t)_1 = 0.1R_{\gamma}(w_t) + 0.2R_{> \gamma}(w_t), \quad \text{and} \quad -\nabla R(w_t)_2 = 0.2R_{> \gamma}(w_t). \quad (B.1)$$

Recall that $w_0 = 0$, and thus eq. (B.1) implies that $w_{t,1} \geq 0$ and $R_{\gamma}(w_t) \leq 1/n$ for all $t$. As a result, as long as $R(w_t) \geq 2/n$, it holds that $R_{> \gamma}(w_t) \geq R_{\gamma}(w_t)$ and $|\nabla R(w_t)_2| \geq |\nabla R(w_t)_1|/2$.

Let $\tau$ denote the first step when the risk is less than $2/n$:

$$\tau = \min \{t : R(w_t) < 2/n\}.$$

Since $|\nabla R(w_t)_2| \geq |\nabla R(w_t)_1|/2$ for all $t < \tau$, we have

$$w_{\tau,2} \geq w_{\tau,1}/2.$$

On the other hand, since $\|z_i\| \leq 1/3$, it holds that $R(w_\tau) \geq \exp(-\|w_\tau\|/3)$, which implies that

$$\|w_\tau\| \geq 3 \ln(n/2).$$

As a result,

$$w_{\tau,2} \geq \ln(n/2).$$