Arbitrarily varying and compound classical-quantum channels and a note on quantum zero-error capacities

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Abstract. We consider compound as well as arbitrarily varying classical-quantum channel models. For classical-quantum compound channels, we give an elementary proof of the direct part of the coding theorem. A weak converse under average error criterion to this statement is also established. We use this result together with the robustification and elimination technique developed by Ahlswede in order to give an alternative proof of the direct part of the coding theorem for a finite classical-quantum arbitrarily varying channels with the criterion of success being average error probability. Moreover we provide a proof of the strong converse to the random coding capacity in this setting. The notion of symmetrizability for the maximal error probability is defined and it is shown to be both necessary and sufficient for the capacity for message transmission with maximal error probability criterion to equal zero.

Finally, it is shown that the connection between zero-error capacity and certain arbitrarily varying channels is, just like in the case of quantum channels, only partially valid for classical-quantum channels.

1 Introduction

Channel uncertainty is omnipresent and mostly unavoidable in real-world applications and one of the major technological challenges is the design of communication protocols that are robust against it. The incarnation of that challenge on the theoretical side delivers a plethora of interesting structural and methodological problems for Information Theory. Despite these facts it happened only recently that this range of problems received the necessary attention in Quantum Information Theory and especially in Quantum Shannon Theory [7], [15], [9], [11], [6]. In this paper we revisit two basic models for communication under channel uncertainty, the compound and arbitrarily varying channels with classical input and quantum output and give essentially self-contained derivations of
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coding theorems for them. These results were originally obtained in [7] and [9].
The contributions of the paper and the difference to existing work are the following. First, in [9] a capacity result with strong converse for compound channels with a classical input and quantum output (compound cq-channel for short) under the maximum error criterion has been derived. However, the achievability proof given there lacks transparency and does not show that good codes with the uniformly bounded exponentially decreasing maximal error exist. Indeed, in [9] it is merely shown that good codes exist with uniformly super-polynomially decreasing maximal error probability. Here we prove that sharper result for the average error criterion and, at the same time, give a significantly simpler proof of the achievability part of the coding theorem based on a universal hypothesis testing result which is a generalization of the technique developed by Hayashi and Ogawa in [25]. The passage to the maximal error criterion can be carried out via a standard argument which can be found in [9].

It is interesting to compare this result with related work of Hayashi [21] and Datta and Hsieh [17]. The works [21] and [17] aim at showing the existence of codes depending on the input distribution and a prescribed rate only and achieving an exponential but channel dependent decay of error probability for all cq-channels whose Holevo information is strictly larger than that prescribed rate. The good codes in our approach depend on the input distribution and the set of cq-channels generating the compound cq-channel. Additionally we obtain a uniform exponential bound on error probabilities, a property that seems highly desirable in case that the channel is unknown.

Moreover, we prove the weak converse to the coding theorem under average error criterion by a reduction to the strong converse for the maximal error via a lemma of Ahlswede and Wolfowitz from [2].

Second, once we have the achievability result for compound cq-channels we can obtain the corresponding results for arbitrarily varying cq-channels (AVcqC) in a straight-forward fashion via Ahlswede’s powerful elimination [4] and robustification [5] techniques. This way, we obtain an alternative approach to the coding theorem for AVcqCs which was originally proven by Ahlswede and Blinovsky in [7].

Finally, we show that a naive quantum analog of Ahlswede’s beautiful relation [3] between Shannon’s zero-error capacity [27] and the capacity of arbitrarily varying channels subject to maximal error criterion does hold neither for AVcqCs when employing the maximal error criterion nor for the strong subspace transmission over arbitrarily varying quantum channels. The latter communication scenario is widely acknowledged as a fully quantum counterpart to message transmission subject to the maximal error criterion.

2 Notation and Conventions

All Hilbert spaces are assumed to have finite dimension and are over the field \( \mathbb{C} \). The set of linear operators from \( \mathcal{H} \) to \( \mathcal{H} \) is denoted \( \mathcal{B}(\mathcal{H}) \). The adjoint of \( b \in \mathcal{B}(\mathcal{H}) \) is marked by a star and written \( b^* \). The notation \( \langle \cdot, \cdot \rangle_{HS} \) is reserved
for the Hilbert-Schmidt inner product on $\mathcal{B}(\mathcal{H})$.

$S(\mathcal{H})$ is the set of states, i.e. positive semi-definite operators with trace 1 acting on the Hilbert space $\mathcal{H}$. Pure states are given by projections onto one-dimensional subspaces. A vector $x \in \mathcal{H}$ of unit length spanning such a subspace will therefore be referred to as a state vector, the corresponding state will be written $|x\rangle\langle x|$. For a finite set $X$ the notation $\mathcal{P}(X)$ is reserved for the set of probability distributions on $X$, and $|X|$ denotes its cardinality. For any $l \in \mathbb{N}$, we define $X^l := \{(x_1, \ldots, x_l) : x_i \in S \forall i \in \{1, \ldots, l\}\}$, we also write $x^l$ for the elements of $X^l$. For any natural number $N$, we define $[N]$ to be the shortcut for the set $\{1, \ldots, N\}$

The set of classical-quantum channels (cq-channels) mapping a finite alphabet $X$ to a Hilbert space $\mathcal{H}$ is denoted $CQ(X, \mathcal{H})$. Since $CQ(X, \mathcal{H})$ is the set of functions $W : X \rightarrow S(\mathcal{H})$. It is naturally equipped with the norm $\| \cdot \|_{cq}$ (which is inherited from the usual one-norm $\| \cdot \|_1$ on operators) and is defined by

$$\|W\|_{cq} := \max_{x \in X} \|W(x)\|_1 \quad (W \in CQ(X, \mathcal{H})).$$

It is common, to embed the set $\mathcal{P}(X)$ of probability distributions into $\mathcal{B}(\mathbb{C}^{|X|})$, i.e. to fix an orthonormal basis $\{e_x\}_{x \in X}$ in $\mathbb{C}^{|X|}$ and assign to every $p \in \mathcal{P}(X)$ an element of $\mathcal{B}(\mathbb{C}^{|X|})$ which is diagonal in this basis. For a channel $W \in CQ(X, \mathcal{H})$ and a given input probability distribution $p \in \mathcal{P}(X)$ one defines the corresponding state on $\mathbb{C}^{|X|} \otimes \mathcal{H}$ by

$$\rho := \sum_{x \in X} p(x)|e_x\rangle\langle e_x| \otimes W(x). \quad (1)$$

The set of measurements with $N \in \mathbb{N}$ different outcomes is written $\mathcal{M}_N(\mathcal{H}) := \{ (D_1, \ldots, D_N) : \sum_{i=1}^N D_i \leq 1_\mathcal{H} \text{ and } D_i \geq 0 \forall i \in [N]\}$. To every $(D_1, \ldots, D_N) \in \mathcal{M}_N(\mathcal{H})$ there corresponds a unique operator defined by $D_0 := 1_\mathcal{H} - \sum_{i=1}^N D_i$.

The von Neumann entropy of a state $\rho \in S(\mathcal{H})$ is given by

$$S(\rho) := -\text{tr}(\rho \log \rho), \quad (2)$$

where $\log(\cdot)$ denotes the base two logarithm which is used throughout the paper (accordingly, $\exp(\cdot)$ is reserved for the base two exponential). For two states $\rho, \sigma \in S(\mathcal{H})$, the quantum relative entropy is defined by

$$D(\rho||\sigma) := \begin{cases} \text{tr}(\rho \log \rho - \rho \log \sigma) & \text{if } \ker \sigma \subseteq \ker \rho \\ +\infty & \text{else}. \end{cases} \quad (3)$$

The Holevo information is for a given channel $W \in CQ(X, \mathcal{H})$ and input probability distribution $p \in \mathcal{P}(X)$ defined by

$$\chi(p, W) := S(\mathbb{W}) - \sum_{x \in X} p(x)S(W(x)) = \sum_{x \in X} p(x)D(W(x)||\mathbb{W}), \quad (4)$$
where $\bar{W}$ is defined by $\bar{W} := \sum_{x \in X} p(x) W(x)$. This quantity is concave w.r.t. the input probability distribution and convex w.r.t. the channel. Its concavity property follows directly from the concavity of the von Neumann entropy, its convexity in the channel is by joint convexity of the quantum relative entropy. For an arbitrary set $\mathcal{W} \subset \mathcal{CQ}(X, \mathcal{H})$ we denote its convex hull by $\text{conv}(\mathcal{W})$ (for the definition of the convex hull, [28] is a useful reference). In fact, for a set $\mathcal{W} := \{W_s\}_{s \in S}$

$$\text{conv}(\mathcal{W}) = \left\{ W_q \in \mathcal{CQ}(X, \mathcal{H}) : W_q = \sum_{s \in S} q(s) W_s, \; q \in P(S), |\text{supp}(q)| < \infty \right\}$$

(5)

because of Carathéodory's Theorem.

3 Definitions

3.1 The compound classical-quantum channel

Let $\mathcal{W} \subset \mathcal{CQ}(X, \mathcal{H})$. The memoryless compound cq-channel associated with $\mathcal{W}$ is given by the family $\{W_{\otimes l}\}_{l \in \mathbb{N}, W \in \mathcal{W}}$. With slight abuse of notation it will be denoted $W$ or, if necessary, 'the compound cq-channel $W$' for short. In the remainder, using arbitrary index sets $T$, we will often write $\mathcal{W} = \{W_t\}_{t \in T}$ to enhance readability. Before we continue, let us put a brief remark in order to explain why this subsection contains no definition of random codes (while subsection 3.2 does):

Remark 1. We abstain from defining random codes for compound cq-channels, the reason for this being that they do offer no increase in capacity. For the reader interested in the topic, we briefly outline one way of arriving at this conclusion. First, the capacity of compound channels, seen as a function from the power set of the set of channels with given input and output systems to the reals, is continuous (this can fact can be proven by an argument very similar to the one given for compound quantum channels in Sect. 8 of [11] together with continuity of the single channel classical capacity, cf. [23]). This allows for an arbitrarily good (speaking in terms of their capacity) approximation of infinite compound cq-channels by finite ones, so that we can restrict our discussion to finite compound cq-channels.

Second, given such a finite compound cq-channel $\{W_t\}_{t \in T}$ and a sequence of random codes which achieve a given rate $r$ with asymptotically vanishing average error, we may simply use it for the memoryless cq-channel $\bar{W} := \frac{1}{|T|} \sum_{t \in T} W_t$. Since the average error is a convex function of the channel, this implies the existence of a sequence of deterministic codes at the same asymptotic rate with vanishing average error for $\bar{W}$.

Using affinity of the average error criterion once more, we see that the very same sequence of deterministic codes also has vanishing average error for the cq-compound channel $\{W_t\}_{t \in T}$, only with a slightly slower convergence. As in
the definition of $W$, the assumption that $|T| < \infty$ holds is crucial at this point of the argument. This shows that random codes cannot have higher asymptotic rates than deterministic ones, if one insists on asymptotically vanishing average error.

For the maximal error criterion, it is enough to note that both the random and the deterministic capacity for transmission of messages over a compound cq-channel using that criterion are upper bounded by the respective capacities for the average error criterion.

**Definition 1.** An $(l, M_l)$-code for message transmission over a compound cq-channel $W \subset CQ(X, H)$ is a family $(x^l_m, D^l_m)_{m=1}^{M_l}$, where $x^l_1, \ldots, x^l_{M_l} \in X^l$ and $(D^l_1, \ldots, D^l_{M_l}) \in \mathcal{M}_{M_l}(H^\otimes l)$.

**Definition 2.** For $\lambda \in [0, 1)$, a non-negative number $R$ is called a $\lambda$-achievable rate for transmission of messages over the compound cq-channel $W = \{W_t\}_{t \in T}$ using the average error criterion if there is a sequence $\{(u^l_m, D^l_m)_{m=1}^{M_l}\}_{l \in \mathbb{N}}$ of $(l, M_l)$-codes with

$$
\lim_{l \to \infty} \frac{1}{l} \log M_l \geq R \quad \text{and} \quad \lim_{l \to \infty} \sup_{t \in T} \frac{1}{M_l} \sum_{m=1}^{M_l} \text{tr}(W_t^\otimes l(u^l_m)(1_{H^\otimes l} - D^l_m)) \leq \lambda.
$$

**Definition 3.** For $\lambda \in [0, 1)$, a non-negative number $R$ is called a $\lambda$-achievable rate for transmission of messages over the compound cq-channel $W = \{W_t\}_{t \in T}$ using the maximal error criterion if there is a sequence $\{(u^l_m, D^l_m)_{m=1}^{M_l}\}_{l \in \mathbb{N}}$ of $(l, M_l)$-codes with

$$
\lim_{l \to \infty} \frac{1}{l} \log M_l \geq R \quad \text{and} \quad \lim_{l \to \infty} \sup_{t \in T} \max_{m \in [M_l]} \text{tr}(W_t^\otimes l(u^l_m)(1_{H^\otimes l} - D^l_m)) \leq \lambda.
$$

**Definition 4.** For $\lambda \in [0, 1)$, the $\lambda$-capacity for message transmission using the average error criterion of a compound cq-channel $W$ is given by

$$
\overline{C}_C(W, \lambda) := \sup \left\{ R : \begin{array}{l} R \text{ is a } \lambda\text{-achievable rate for} \\ \text{transmission of messages over } W \\ \text{using the average error probability criterion} \end{array} \right\}. \tag{6}
$$

The number $\overline{C}_C(W, 0)$ is called the weak capacity for message transmission using the average error criterion of $W$ and abbreviated $\overline{C}_C(W)$.

**Definition 5.** For $\lambda \in [0, 1)$, the $\lambda$-capacity for message transmission using the maximal error criterion of a compound cq-channel $W$ is given by

$$
C_C(W, \lambda) := \sup \left\{ R : \begin{array}{l} R \text{ is a } \lambda\text{-achievable rate for transmission} \\ \text{of messages over } W \\ \text{using the maximal error probability criterion} \end{array} \right\}. \tag{7}
$$
The number $C_C(W, 0)$ is called the weak capacity for message transmission using the maximal error criterion of $W$ and abbreviated $C_C(W)$.

3.2 The arbitrarily varying classical-quantum channel

Let $A \subset CQ(X, H)$. In the remainder we will write $A = \{A_s\}_{s \in S}$, where $S$ denotes an index set, in order to enhance readability. We also set

$$A_s' := \otimes_{i=1}^l A_{s_i}.$$  

The arbitrarily varying classical-quantum channel associated with $A$ is given by the family $\{A_{s,l}\}_{s,l \in \mathbb{N}}$. Again, with slight abuse of notation it will be denoted $A$ or, if necessary, ‘the AVcqC A’ for short.

In this work, we will always consider the set $S$ to be finite. Generalizations of our results to the case of arbitrary sets can be done by standard techniques (see [6]). We will now define random codes and the random capacity emerging from them. In order to do so, we have to clarify a few things.

A code for an AVcqC $A$ will, for some choice of $l, N \in \mathbb{N}$, be given by a probability measure $\mu_l$ on the set $((X_l)^N \times M_N(H^{\otimes l}), \Sigma_l)$, where $\Sigma_l$ is a suitably chosen sigma-algebra. It has to be taken care that a function $f$ defined by

$$((x_l, D_l), (x_l', D_l')) \mapsto \min_{s,l \in \mathbb{N}} \frac{1}{N} \sum_{i=1}^N \text{tr} \{W_{s,l}(x_l^i) D_l^i\}$$

is measurable w.r.t. $\Sigma_l$. Also, in order to define deterministic codes later, $\Sigma_l$ has to contain all the singleton sets. In the remainder, we shall assume that such a choice is always made.

An explicit example of such a sigma-algebra is given by the Borel sigma-algebra defined using the topology induced by the metric

$$(x, D) \mapsto 1 - \delta(x, x') + \|D - D'\|_2$$

and equal to zero else, and for sake of simplicity, we set $l = N = 1$. Finally, we note that the function $f$ mentioned above is continuous w.r.t. to that metric.

In the following definitions, let $\lambda \in [0, 1)$.

Definition 6. An $(l, M_l)$-random code for message transmission over $A = \{A_s\}_{s \in S}$ is a probability measure $\mu_l$ on $((X_l)^{M_l} \times M_N(H^{\otimes l}), \Sigma_l)$. In order to shorten our notation, we write elements of $((X_l)^{M_l} \times M_N(H^{\otimes l}))$ in the form $(x_l^1, D_l^1)_{i=1}^{M_l}$.

Definition 7. An $(l, M_l)$-deterministic code for message transmission over $A = \{A_s\}_{s \in S}$ is given by a random code for message transmission over $A$ with $\mu_l$ assigning probability one to a singleton set.

Definition 8. A non-negative number $R$ is called $\lambda$-achievable for transmission of messages over the AVcqC $A = \{A_s\}_{s \in S}$ with random codes using the average error criterion if there is a sequence $(\mu_l)_{l \in \mathbb{N}}$ of $(l, M_l)$-random codes such that the following two lines are true:

$$\lim_{l \to \infty} \frac{1}{l} \log M_l \geq R$$
\[
\limsup_{l \to \infty} \max_{s' \in S'} \int \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr} \left( A_{x_i'} (x_i') (1_{H^{S'}} - D_l'_{i}) \right) d\mu_l((u'_i, D_l'_{i})_{i=1}^{M_l}) \leq \lambda. \quad (10)
\]

**Definition 9.** A non-negative number \(R\) is called \(\lambda\)-achievable for transmission of messages over the AVcqC \(\mathcal{A} = \{A_s\}_{s \in S}\) with deterministic codes using the average error criterion if it is \(\lambda\)-achievable with random codes by a sequence \((\mu_l)_{l \in \mathbb{N}}\) which are deterministic codes.

**Definition 10.** The \(\lambda\)-capacity for message transmission using random codes and the average error criterion of an AVcqC \(\mathcal{A}\) is given by

\[
C_{\mathcal{A}, r}(\mathcal{A}, \lambda) := \sup \left\{ R : R \text{ is a } \lambda\text{-achievable rate for transmission of messages over } \mathcal{A} \text{ with random codes using the average error probability criterion} \right\}. \quad (11)
\]

The number \(C_{\mathcal{A}, r}(\mathcal{A}, 0)\) is called the weak capacity for message transmission using random codes and the average error criterion of \(\mathcal{A}\) and abbreviated \(C_{\mathcal{A}, r}(\mathcal{A})\).

**Definition 11.** The \(\lambda\)-capacity for message transmission using deterministic codes and the average error criterion of an AVcqC \(\mathcal{A}\) is given by

\[
C_{\mathcal{A}, d}(\mathcal{A}, \lambda) := \sup \left\{ R : R \text{ is a } \lambda\text{-achievable rate for transmission of messages over } \mathcal{A} \text{ with deterministic codes using the average error probability criterion} \right\}. \quad (12)
\]

The number \(C_{\mathcal{A}, d}(\mathcal{A}, 0)\) is called the weak capacity for message transmission using deterministic codes and the average error criterion of \(\mathcal{A}\) and abbreviated \(C_{\mathcal{A}, d}(\mathcal{A})\).

**Definition 12.** A non-negative number \(R\) is called \(\lambda\)-achievable for transmission of messages over the AVcqC \(\mathcal{A} = \{A_s\}_{s \in S}\) with deterministic codes using the maximal error probability criterion if there is a sequence of \((l, M_l)\)-random codes with each \(\mu_l\) being a deterministic code such that the following two lines are true:

\[
\liminf_{l \to \infty} \frac{1}{l} \log M_l \geq R \quad (13)
\]

\[
\limsup_{l \to \infty} \max_{l' \in S'} \max_{i=1}^{M_l} \int \text{tr} \left( A_{x_i'} (x_i') (1_{H_{l'}^{S'}} - D_{l'}_{i}) \right) d\mu_l((u'_i, D_{l'}_{i})_{i=1}^{M_{l'}}) \leq \lambda. \quad (14)
\]

**Definition 13.** The \(\lambda\)-capacity for message transmission using deterministic codes and the maximal error probability criterion of an AVcqC \(\mathcal{A}\) is given by

\[
C_{\mathcal{A}, d}(\mathcal{A}, \lambda) := \sup \left\{ R : R \text{ is a } \lambda\text{-achievable rate for transmission of messages over } \mathcal{A} \text{ with deterministic codes using the maximal error probability criterion} \right\}. \quad (15)
\]

The number \(C_{\mathcal{A}, d}(\mathcal{A}, 0)\) is called the weak capacity for message transmission using deterministic codes and the maximal error criterion of \(\mathcal{A}\) and abbreviated \(C_{\mathcal{A}, d}(\mathcal{A})\).
The following definition will turn out to be useful to decide whether a given AVcqC has nonzero capacity for transmission of messages using average error criterion and deterministic codes.

**Definition 14.** Let \( \mathcal{A} = \{ A_s \}_{s \in S} \subseteq CQ(X, \mathcal{H}) \) be an AVcqC. If, for every \( x, x' \in X \), we have

\[
\text{conv}(\{ A_s(x) \}_{s \in S}) \cap \text{conv}(\{ A_s(x') \}_{s \in S}) \neq \emptyset,
\]

then \( \mathcal{A} \) is called \( m \)-symmetrizable.

### 3.3 Zero-error capacity

**Definition 15.** An \((l, M_l)\) zero-error code for a stationary memoryless cq-channel defined by \( V \in CQ(X, \mathcal{H}) \) is given by a family \( (x^i_l, D^i_l)_{l=1}^{M_l} \), where \( x^i_1, \ldots, x^i_{M_l} \in X_l \) and \( (D^i_1, \ldots, D^i_{M_l}) \in \mathcal{M}_{M_l}(\mathcal{H}^\otimes l) \) satisfy \( \text{tr}(V^{\otimes l}(x^i_l)D^i_l) = 1 \) for every \( i \in [M_l] \).

**Definition 16.** The zero-error capacity for message transmission over the cq-channel \( V \in CQ(X, \mathcal{H}) \) is given by

\[
C_0(V) := \lim_{l \to \infty} \frac{1}{l} \log \max \{ M_l : \exists (l, M_l) \text{ zero-error code for } V \}. \tag{17}
\]

### 4 Main Results

We now enlist the main results contained in this work. We will not state the results obtained in Subsection 6.3. These evolve around the relation between zero-error capacities and arbitrarily varying channels. They include both message transmission and entanglement transmission. Rather than stating a positive result, in this section we argue that certain straightforward quantum analogues of results that are valid in the classical theory do not hold. As always, this is a delicate task that involves much more than just embedding a commutative subalgebra into a non-commutative one. We therefore encourage the reader to consider this last subsection as something that should be read separately and in one piece.

Our first result is the following.

**Theorem 1 (cq Compound Coding Theorem).** For every compound cq-channel \( W \in CQ(X, \mathcal{H}) \) it holds

\[
\mathcal{V}_C(W) = \max_{p \in \mathcal{P}(X)} \inf_{W \in W} \chi(p, W). \tag{18}
\]

In subsection 6.1 an analogue of the Ahlswede dichotomy from [4] for arbitrarily varying classical-quantum channels will be derived. This statement has originally been obtained by Ahlswede and Blinovsky in [7]. The precise mathematical formulation reads as follows.
Theorem 2 (Ahlswede-Dichotomy for AVcqCs). Let \( A = \{ A_s \}_{s \in S} \subset CQ(X, H) \) be an AVcqC. Then

1) \( \overline{C}_{A,r}(A) = \overline{C}_C(\text{conv}(A)) \) \hspace{1cm} (19)

2) If \( \overline{C}_{A,d}(A) > 0 \), then \( \overline{C}_{A,d}(A) = \overline{C}_{A,r}(A) \). \hspace{1cm} (20)

Also, this section contains the following statement, which asserts, that every sequence of random codes with error strictly smaller than 1 for all but finitely many blocklengths will not achieve rates higher than the rightmost term in (19).

Theorem 3 (Strong converse). Let \( A := \{ A_s \}_{s \in S} \) be an AVcqC. For every \( \lambda \in [0, 1) \)

\[ \overline{C}_{A,r}(A, \lambda) \leq \overline{C}_C(\text{conv}(A)) \] \hspace{1cm} (21)

holds.

Remark 2. The result can be gained for arbitrary (infinite) AVcqCs with only trivial modifications of the proof given below.

In the next subsection 5.2, we show that the capacity for message transmission over an AVcqC using deterministic codes and the maximal error probability criterion is zero if and only if the AVcqC is \( m - \text{symmetrizable} \).

This is an analog of [22, Theorem 1]. It can be formulated as follows.

Theorem 4. Let \( A = \{ A_s \}_{s \in S} \subset CQ(X, H) \) be an AVcqC. Then \( C_{A,d}(A) \) is equal to zero if and only if \( A \) is \( m \)-symmetrizable.

5 Compound cq-channels

In this section, we consider compound cq-channels and give a rigorous proof for the achievability part of the coding theorem under the average error criterion together with a weak converse. The channel coding problem for compound cq-channels was treated, restricted to achievability, by Datta and Hsieh [17] for a certain class of compound channels, and Hayashi [21]. In our proof, we exploit the close relationship between channel coding and hypothesis testing which was utilized by Hayashi and Nagaoka [20] before. With focus set on the maximal error criterion, the compound cq channel coding theorem was proven in [9] already where also a strong converse theorem was proven for this setting.

For orientation of the reader, we sketch the contents of this section. In Lemma 1 we reduce the problem of finding good channel codes for a finite compound channel to the problem of finding good hypothesis tests for certain quantum states generated by this channel. The existence of hypothesis tests with a performance sufficient for our purposes is shown in Lemma 5. In order to establish the coding theorem for arbitrary compound channels, we recall some approximation results in Lemma 6. With these preparations, we are able to prove the direct part of the
coding theorem. Additionally, we give a proof of the weak converse (for which we utilize the strong converse result for the maximal error criterion given in \[19\]) in Theorem 1. A strong converse for coding under the average error criterion does not hold in general for compound cq-channels (for further information, see Remark 4).

We consider a compound channel \( \mathcal{W} := \{ W_t \}_{t \in T} \subset \text{CQ}(\mathbf{X}, \mathcal{H}) \) where \( T \) is a finite index set. We fix an orthonormal basis \( \{ e_x \}_{x \in \mathbf{X}} \) in \( \mathbb{C}^{|\mathbf{X}|} \). For \( \mathcal{W} \) and a given input probability distribution \( p \in \mathcal{P}(\mathbf{X}) \) we define for every \( t \in T \) states

\[
\rho_t := \sum_{x \in \mathbf{X}} p(x) |e_x\rangle \langle e_x| \otimes W_t(x), \quad \text{and} \quad \sigma_t := p \otimes \sigma_t,
\]

for the probability distribution \( p := \sum_{x \in \mathbf{X}} p(x) |e_x\rangle \langle e_x| \), and \( \sigma_t := \sum_{x \in \mathbf{X}} p(x) W_t(x) \).

With some abuse of notation, we use the letter \( p \) for the probability distribution as well as for the according quantum state defined above. Moreover, we define for every \( l \in \mathbb{N} \) states

\[
\rho_l := \frac{1}{|T|} \sum_{t \in T} v_l \rho_t^\otimes l v_l^*,
\]

\[
\tau_l := \frac{1}{|T|} \sum_{t \in T} v_l \sigma_t^\otimes l v_l^* = p^\otimes l \otimes \frac{1}{|T|} \sum_{t \in T} \sigma_t^\otimes l
\]

where \( v_l : (\mathbb{C}^{|\mathbf{X}|} \otimes \mathcal{H})^\otimes l \to (\mathbb{C}^{|\mathbf{X}|})^\otimes l \otimes \mathcal{H}^\otimes l \) is the isomorphism permuting the tensor factors. The next lemma is a variant of a result by Hayashi and Nagaoka in \[20\], which states that good hypothesis tests imply good message transmission codes for the average error criterion. Here it is formulated and proven for the states \( \rho_l \) and \( \tau_l \).

**Lemma 1.** Let \( \mathcal{W} := \{ W_t \}_{t \in T} \subset \text{CQ}(\mathbf{X}, \mathcal{H}) \) be a compound cq-channel with \( |T| < \infty \), \( p \in \mathcal{P}(\mathbf{X}) \), and \( l \in \mathbb{N} \). Let further \( \rho_l, \tau_l \) be the states associated to \( \mathcal{W}, p \) as defined in \[24\] and \[25\]. If for \( \lambda \in [0, 1] \), and \( a > 0 \) exists a projection \( q_l \in \mathcal{B}((\mathbb{C}^{|\mathbf{X}|})^\otimes l \otimes \mathcal{H}^\otimes l) \) which fulfills the conditions

1. \( \text{tr}(q_l \rho_l) \geq 1 - \lambda \)
2. \( \text{tr}(q_l \tau_l) \leq 2^{-la} \),

then for any \( \gamma \) with \( a \geq \gamma > 0 \) and \( M_l := \lceil 2^l(a-\gamma) \rceil \) there is an \((l, M_l)\)-code \( (x_m^l, D_m^l)_{m \in [M_l]} \) with

\[
\max_{t \in T} \frac{1}{M_l} \sum_{m=1}^{M_l} \text{tr}(W_t^\otimes l(x_m^l) (1_{\mathcal{H}^\otimes l} - D_m^l)) \leq |T|(2\lambda + 4 \cdot 2^{-l\gamma})
\]

The following operator inequality is a crucial ingredient in the proof of the lemma above, it was given in a more general form by Hayashi and Nagaoka in \[20\].
Lemma 2. Let \( a, b \in \mathcal{B}(\mathcal{H}) \) be operators on \( \mathcal{H} \) with \( 0 \leq a \leq 1 \) and \( b \geq 0 \). Then
\[
\mathbb{1}_H - (a + b)^{-\frac{1}{2}} a (a + b)^{-\frac{1}{2}} \leq 2(\mathbb{1}_H - a) + 4b,
\] (27)
where \((\cdot)^{-1}\) denotes the generalized inverse.

Proof. See Lemma 2 in [20]. \(\square\)

Proof (of Lemma 1). Let \( l \in \mathbb{N} \), \( q_l \) a projection such that the assumptions of the lemma are fulfilled, and \( \gamma \) a number with \( 0 < \gamma \leq a \). According to the assumptions, \( q_l \) takes the form
\[
q_l = \sum_{x^l \in \mathbf{X}^l} |e_{x^l}\rangle\langle e_{x^l}| \otimes q_{x^l},
\] (28)
where \( q_{x^l} \in \mathcal{B}(\mathcal{H}^\otimes l) \) is a projection for every \( x^l \in \mathbf{X}^l \). Set \( M_l := \lfloor 2^{(a - \gamma)} \rfloor \), and let \( U_1, ..., U_{M_l} \) be i.i.d. random variables with values in \( \mathbf{X}^l \), each distributed according to the \( l \)-fold product \( p^\otimes l \) of the given distribution \( p \). We define a random operator
\[
D_m := \left( \sum_{n=1}^{M_l} q_{U_n} \right)^{-\frac{1}{2}} q_{U_m} \left( \sum_{n=1}^{M_l} q_{U_n} \right)^{-\frac{1}{2}}
\] (29)
for every \( m \in [M_l] \) (we omit the superscript \( l \) here), where again generalized inverses are taken. The particular form of the decoding operators \( D_1, ..., D_{M_l} \) in eq. (29) guarantees, that
\[
\sum_{m=1}^{M_l} D_m \leq \mathbb{1}_{\mathcal{H}^\otimes l}
\]
holds for every outcome of \( U_1, ..., U_{M_l} \), and therefore \( (U_m, D_m)_{m \in [M_l]} \) is a random code of size \( M_l \). The remaining task is to bound the expectation value of the average error of this random code. We introduce an abbreviation for the average of the channels in \( \mathcal{W} \) by
\[
\overline{\mathcal{W}}(\cdot) := \frac{1}{T} \sum_{t=1}^{T} \mathcal{W}_t^\otimes l(\cdot).
\]
The error probability of the random code is bounded as follows. By virtue of Lemma [2]
\[
\mathbb{E} \left[ \text{tr} \left( \overline{\mathcal{W}}(U_m)(\mathbb{1}_{\mathcal{H}^\otimes l} - D_m) \right) \right] \leq 2 \mathbb{E} \left[ \text{tr} \left( \overline{\mathcal{W}}(U_m)(\mathbb{1}_{\mathcal{H}^\otimes l} - q_{U_m}) \right) \right] + 4 \cdot \sum_{m \in [M_l]} \mathbb{E} \left[ \text{tr} \left( \overline{\mathcal{W}}(U_m)q_{U_m} \right) \right]
\] (30)
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holds. The calculation of the expectation values on the r.h.s. of the above equation is straightforward, we obtain for every $m \in [M]$

$$\mathbb{E}[\text{tr}(W(U_m)(\mathbb{I}_N - q_{U_m}))] = \text{tr}(\rho(U_m - q)),$$

and, for $n \neq m$,

$$\mathbb{E}\left[\text{tr}\left(W(U_m)q_{U_n}\right)\right] = \text{tr}(\rho).$$

Together with the assumptions of the lemma, eqns. (31) and (32) imply

$$\mathbb{E}\left[\text{tr}\left(1|T|\sum_{t \in T} W_t(U_m)(\mathbb{I}_N - D_m)\right)\right] \leq 2\lambda + 4 \cdot M \cdot 2^{-la} \leq 2\lambda + 4 \cdot 2^{-la}\gamma$$

Because this error measure is an affine function of the channel we conclude, that there exists a cq-code $(x_l^m, D_m)_{m=1}^M$ for $W$ with average error bounded by

$$\frac{1}{M} \sum_{m=1}^M \text{tr}(W_t(U_m)(\mathbb{I}_N - D_m)) \leq |T|(2\lambda + 4 \cdot 2^{-la})$$

for every $t \in T$, which is what we aimed to prove.

The next two lemmata contain facts which are important for later considerations. The first lemma presents a bound on the cardinality of the spectrum of operators on a tensor product space which are invariant under permutations of the tensor factors. The group $S_l$ of permutations on $[l]$ is, on $H \otimes l$, represented by defining (with slight abuse of notation) for each $\sigma \in S_l$

$$\sigma \in B(H \otimes l) v_1 \otimes \ldots \otimes v_l := v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(l)}.$$ 

Let us write $Y = \sum_{i,j} Y_{i,m,j,n}$, where $Y_{i,m,j,n} : H_{i,j} \mapsto H_{i,m}$. Then according to Schur’s lemma, $Y_{i,m,j,n} = 0, (i \neq j)$ and $Y_{i,m,i,n} = c_{i,m,n} Q_{i,m,n}$ for all valid

$$|\text{spec}(Y)| \leq (l + 1)^d\gamma.$$ 

Proof. It is clear that, under the action of $S_l$, $H \otimes l$ decomposes into a finite direct sum $H \otimes l = \bigoplus_{m=1}^M \bigoplus_{j=1}^{m} H_{i,j}$, where the $H_{i,j}$ are irreducible subspaces of $S_l$, $m_j \in \mathbb{N}$ their multiplicity and $M \in \mathbb{N}$. Moreover, $H_{i,j} \simeq H_{i,k}$ f.a. $i \in [M]$, $j, k \in [m_i]$ and to every such choice of indices there exists a linear operator $Q_{i,j,k} : H_{i,k} \mapsto H_{i,j}$ such that $\sigma Q_{i,j,k} = Q_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$ f.a. $\sigma \in S_l$. Let us write $Y = \sum_{i,j} Y_{i,m,j,n}$, where $Y_{i,m,j,n} : H_{j,n} \mapsto H_{i,m}$. Then according to Schur’s lemma, $Y_{i,m,j,n} = 0, (i \neq j)$ and $Y_{i,m,i,n} = c_{i,m,n} Q_{i,m,n}$ for all valid
choices of indices and unique complex numbers $c_{i,m,n} \in \mathbb{C}$.

Thus, defining the self-adjoint operators $Y_i := \sum_{m,n=1}^{m_i} c_{i,m,n}Q_{i,m,n}$, we see that

$$Y = \sum_{i=1}^{M} Y_i$$  \hspace{1cm} (36)

holds. Obviously, $Y_{i,m,i,m} = \mathbb{1}_{H_{i,m}}$. Thus, with an appropriate choice of bases in every single one of the $H_{i,m}$ and defining the matrices $C_i$ by $(C_i)_{mn} := c_{i,m,n}$, we can write a matrix representation $\tilde{Y}_i$ of $Y_i$ as

$$\|B\| H_{i,m} \cdot \mathbb{1}_m.$$

Clearly then, each of the $Y_i$ can have no more than $m_i$ different eigenvalues. Since $\text{supp}(Y_i) \perp \text{supp}(Y_j)$ ($i \neq j$), we get

$$|\text{spec}(Y)| \leq \sum_{i=1}^{M} m_i.$$  \hspace{1cm} (37)

Now, taking a look at [13], equation (1.22), we see that $m_i \leq (l+1)^{d^2/2}$ holds. The number $M$ is the number of different Young tableaux occurring in the representation of $S_\delta$ on $H^\otimes l$, and obeys the bound $M \leq N_T([d]_l^l)$, where $N_T([d]_l^l)$ is the number of different types on $[d]_l^l$, that itself obeys $N_T([d]_l^l) \leq (l + 1)^d$ (Lemma 2.2 in [14]). For $d \geq 2$ we thus have

$$|\text{spec}(Y)| \leq \sum_{i=1}^{M} m_i \leq (l + 1)^{d^2/2}(l + 1)^d \leq (l + 1)^d.$$  \hspace{1cm} (38)

Lemma 5 provides the result which will, together with Lemma 1, imply the existence of optimal codes for $W$. We give a proof which is based on an idea of Ogawa and Hayashi which originally appeared in [25]. An important ingredient of their proof is the operator inequality stated in the following lemma.

**Lemma 4** ([19]). Let $\chi$ be a state on on a Hilbert space $K$, and $\mathcal{M} := \{P_k\}_{k=1}^{K} \subset \mathcal{B}(K)$ be a collection of projections on $K$ with $\sum_{k=1}^{K} P_k = \mathbb{1}_{K}$. Then the operator inequality

$$\chi \leq K \cdot \sum_{k=1}^{K} P_k \chi P_k$$  \hspace{1cm} (39)

holds.

**Lemma 5.** For every $\delta > 0$, finite compound cq-channel $W := \{W_t\}_{t \in T} \subset CQ(X, \mathcal{H})$ and $p \in \mathcal{P}(X)$ there exists a constant $\tilde{c}$, such that for every sufficiently large $l \in \mathbb{N}$ there exists a projection $q_{l,\delta} \in \mathcal{B}((\mathbb{C}^{|X|})^{\otimes 1} \otimes H^{\otimes 1})$ which fulfills

1. $\text{tr}(q_{l,\delta} \rho) \geq 1 - |T| \cdot 2^{-|T|}$, and
2. $\text{tr}(q_{l,\delta} \sigma_l) \leq 2^{-l(a-\delta)}$
where $\rho_l, \tau_l$ are the states belonging to $W, p$ according to [24] and [25], and $a$ is defined by $a := \min_{t \in [T]} D(\rho_t || p \otimes \sigma_t)$.

Proof. Let $\delta > 0$ be fixed, for $l \in \mathbb{N}$, we have $\text{ran}(\rho_l) \subseteq \text{ran}(\tau_l) := \mathcal{H}_l$, which allows us to restrict ourselves to $\mathcal{H}_l$, where $\tau_l$ is invertible. For every $\varepsilon \in (0, 1)$, we define a regularized version $\rho_{l, \varepsilon}$ to $\rho_l$ by

$$\rho_{l, \varepsilon} := (1 - \varepsilon)\rho_l + \varepsilon \tau_l.$$  

(40)

These operators are invertible on $\mathcal{H}_l$ and approximate $\rho_l$, i.e.

$$\|\rho_{l, \varepsilon} - \rho_l\|_1 \leq 2\varepsilon.$$  

(41)

holds for every $\varepsilon > 0$. We also define an operator

$$\overline{\rho}_{l, \varepsilon} := \sum_{\lambda \in \text{spec}(\tau_l) \setminus \{0\}} E_{\lambda} \rho_{l, \varepsilon} E_{\lambda},$$  

(42)

which is the pinching of $\rho_{l, \varepsilon}$ to the eigenspaces of $\tau_l$ (here $E_{\lambda}$ is the projection which projects onto the eigenspace belonging to the eigenvalue $\lambda$ for every $\lambda \in \text{spec}(\tau_l)$). This definition guarantees

$$\tau_l \overline{\rho}_{l, \varepsilon} = \overline{\rho}_{l, \varepsilon} \tau_l.$$  

(43)

With $a$ as assumed in the lemma, we define the operator

$$T_{\varepsilon} := \overline{\rho}_{l, \varepsilon} - 2l(a - \delta)\tau_l$$  

(44)

with spectral decomposition

$$T_{\varepsilon} = \sum_{\mu \in \text{spec}(T_{\varepsilon})} \mu P_{\mu}.$$  

(45)

The projection $q_{l, \delta}$ onto the nonnegative part of $T_{\varepsilon}$, defined by

$$q_{l, \delta} := \sum_{\mu \in \text{spec}(T_{\varepsilon}) : \mu \geq 0} P_{\mu},$$  

(46)

will now be shown to suffice the bounds stated in the lemma. Clearly, $q_{l, \delta} T_{\varepsilon} q_{l, \delta}$ is a positive semidefinite operator, therefore, with Lemma 14 the inequality

$$q_{l, \delta} \tau_l q_{l, \delta} \leq 2^{-l(a - \delta)} q_{l, \delta} \overline{\rho}_{l, \varepsilon} q_{l, \delta}.$$  

(47)

is valid. Taking traces in (47) yields

$$\text{tr}(q_{l, \delta} \tau_l) \leq 2^{-l(a - \delta)} \text{tr}(q_{l, \delta} \overline{\rho}_{l, \varepsilon})$$  

(48)

$$\leq 2^{-l(a - \delta)}$$  

(49)

which shows, that $q_{l, \delta}$ fulfills the second bound in the lemma. We shall now prove, that $q_{l, \delta}$ for $l$ large enough actually also suffices the first one. To this end
we derive an upper bound on $\text{tr}( (1 - q_{l, \delta}) \rho_{l, \varepsilon} )$ for any given $\varepsilon > 0$, which implies (together with (11)) a bound on $\text{tr}( (1 - q_{l, \delta}) \rho_{l, \varepsilon} )$. In fact it is sufficient to find an upper bound on $\text{tr}( (1 - q_{l, \delta}) \bar{\rho}_{l, \varepsilon} )$, which can be seen as follows. Because $\bar{\rho}_{l, \varepsilon}$ and $\tau_{l}$ commute by construction (see eq. (41)), $T_{\varepsilon}$ and $\tau_{l}$ commute as well. This in turn implies that $q_{l, \delta}$ commutes with the operators $E_{1}, \ldots, E_{\text{spec}(\tau_{l})}$ in the spectral decomposition of $\tau_{l}$ which eventually ensures us, that

$$\text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \bar{\rho}_{l, \varepsilon} ) = \text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \rho_{l, \varepsilon} )$$

holds. For an arbitrary but fixed number $s \in [0, 1]$ we have

$$\text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \bar{\rho}_{l, \varepsilon} ) = \text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \rho_{l, \varepsilon} )$$

$$\leq 2^{-ts(a-\delta)} \text{tr}( T_{\varepsilon}^{t} (1 - H_{\otimes l} - q_{l, \delta}) )$$

$$\leq 2^{-ts(a-\delta)} \text{tr}( T_{\varepsilon}^{t} (1 - H_{\otimes l} - q_{l, \delta}) )$$

The inequality in (52) is justified by the following argument. Since $\rho_{l, \varepsilon}$ and $\tau_{l}$ commute, they are both diagonal in the same orthonormal basis $\{ g_{i} \}_{i=1}^{d}$, i.e. they have spectral decompositions of the form

$$\rho_{l, \varepsilon} = \sum_{i=1}^{d} \chi_{i} |g_{i}\rangle\langle g_{i}|,$$

$$\tau_{l} = \sum_{i=1}^{d} \theta_{i} |g_{i}\rangle\langle g_{i}|.$$  

(54)

Because $q_{l, \delta}$ projects onto the eigenspaces corresponding to nonnegative eigenvalues of $T_{\varepsilon}$, we have

$$1_{H_{\otimes l}} - q_{l, \delta} = \sum_{i \in N} |g_{i}\rangle\langle g_{i}|,$$

(55)

where the set $N$ is defined by $N := \{ i \in [d] : \chi_{i} - 2^{l(a-\delta)} \theta_{i} < 0 \}$. It follows

$$\chi_{i}^{s} \leq 2^{ls(a-\delta)} \theta_{i}^{s}$$

(56)

for all $i \in N$ and $s \in [0, 1]$. This in turn implies, via (43) and (54),

$$\text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \bar{\rho}_{l, \varepsilon} ) \leq 2^{la(a-\delta)} \text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \rho_{l, \varepsilon} )$$

(57)

which shows (52). Combining eqns. (50) and (55) we obtain

$$\text{tr}( (1 - H_{\otimes l} - q_{l, \delta}) \rho_{l, \varepsilon} ) \leq 2^{la(a-\delta)} \text{tr}( T_{\varepsilon}^{t} (1 - H_{\otimes l} - q_{l, \delta}) )$$

$$= 2^{la(a-\delta)} \text{tr}( T_{\varepsilon}^{t} (1 - H_{\otimes l} - q_{l, \delta}) )$$

$$= 2^{la(a-\delta)} \text{tr}( T_{\varepsilon}^{t} (1 - H_{\otimes l} - q_{l, \delta}) )$$

(58)

Here we used the fact, that $\bar{\rho}_{l, \varepsilon}$ and $\tau_{l}$ commute in the first equality. Eq. (58) is justified, because the eigenprojections of $\tau_{l}$ which appear in the definition of $\bar{\rho}_{l, \varepsilon}$
are absorbed by $\tau_{l,\varepsilon}^{\frac{s}{2}}$. We can further upper bound the above expressions in the following way. Note, that

$$\rho_{l,\varepsilon} \leq |\text{spec}(\tau_{l})| \rho_{l,\varepsilon}.$$  \hspace{1cm} (59)

holds by Lemma 4. Because $-(\cdot)^{-s}$ is an operator monotone function for every $s \in [0,1]$ (see e.g. [8]), (59) implies

$$\tau_{l,\varepsilon}^{s} \leq |\text{spec}(\tau_{l})|^{s} \rho_{l,\varepsilon}.$$  

Using the above relation, one obtains

$$\text{tr}(\rho_{l,\varepsilon} \tau_{l}^{s} \rho_{l,\varepsilon}^{s} \tau_{l}^{s}) \leq |\text{spec}(\tau_{l})|^{s} \text{tr}(\rho_{l,\varepsilon} \tau_{l}^{s} \rho_{l,\varepsilon}^{s} \tau_{l}^{s}).$$

By combination with (58) this leads to

$$\text{tr}((\mathbb{1}_{H^\otimes l} - \rho_{l,\varepsilon}) \rho_{l,\varepsilon}) \leq |\text{spec}(\tau_{l})|^{s} 2^{l(s-(a-\delta))} \text{tr}(\rho_{l,\varepsilon} \tau_{l}^{s} \rho_{l,\varepsilon}^{s} \tau_{l}^{s})$$

where $d := \text{dim } H$. In (60), we used the definition

$$\psi_{l,\varepsilon}(s) := -\log \text{tr}(\rho_{l,\varepsilon} \tau_{l}^{s} \rho_{l,\varepsilon}^{s} \tau_{l}^{s}).$$

By the mean value theorem it suffices to show that $f'_{l,\varepsilon}(0) < 0$ for small enough $\varepsilon > 0$. For the derivative, we have

$$f'_{l,\varepsilon}(0) = a - \delta - \frac{1}{l} D(\rho_{l,\varepsilon} \parallel \tau_{l}).$$

The relative entropy term in (60) can be lower bounded as follows. It holds

$$D(\rho_{l,\varepsilon} \parallel \tau_{l}) = -S(\rho_{l,\varepsilon}) - \text{tr}(\rho_{l,\varepsilon} \log \tau_{l})$$

$$= -S(\rho_{l,\varepsilon}) + lS(p) + S \left( \frac{1}{|T|} \sum_{t \in T} \sigma_{l}^{\otimes l} \right)$$

$$\geq -S(\rho_{l,\varepsilon}) + lS(p) + \frac{1}{|T|} \sum_{t \in T} lS(\sigma_{l}).$$  \hspace{1cm} (68)
Notice that the equality in (67) indeed holds, because the marginals on $(\mathbb{C}^{|X|})^\otimes l$ and $\mathcal{H}^\otimes l$ of $\rho_1$ and $\tau_1$ are equal and therefore equal to the marginals of $\rho_{1,\epsilon}$ by definition for each $\epsilon \in (0,1)$. The inequality in (68) is valid due to concavity of the von Neumann entropy. Because (69) holds, 

$$S(\rho_{1,\epsilon}) \leq S(\rho_1) + 2\epsilon \log \frac{\dim(H_l)}{2\epsilon}$$

is valid for $\epsilon < \frac{1}{2\epsilon}$, since for two states $\rho, \sigma \in S(\mathcal{H})$ with $\|\rho - \sigma\|_1 \leq \epsilon \leq \frac{1}{\epsilon}$, Fannes’ inequality [18],

$$|S(\rho) - S(\sigma)| \leq \epsilon \log \frac{\dim \mathcal{H}}{\epsilon},$$

is valid. Together with (69), (68) implies

$$D(\rho_{1,\epsilon}||\tau_1) \geq -S(\rho_1) - 2\epsilon \log \frac{d}{2\epsilon} + lS(p) + \frac{1}{|T|} \sum_{t \in T} lS(\sigma_t) \geq -\frac{1}{|T|} \sum_{t \in T} lS(\rho_t) - \log |T| - 2\epsilon \log \frac{d}{2\epsilon} + lS(p) + \frac{1}{|T|} \sum_{t \in T} lS(\sigma_t) = \frac{l}{|T|} \sum_{t \in T} D(\rho_{t}||p \otimes \sigma_t) - \log |T| - 2\epsilon \log \frac{d}{2\epsilon}. \quad (71)$$

The inequality in (71) results from the fact, that the von Neumann entropy is an almost convex function, i.e.

$$S(\rho) \leq \sum_{i=1}^{N} p_i S(\rho_i) + \log(N) \quad (73)$$

for any mixture $\rho = \sum_{i=1}^{N} p_i \rho_i$ of states. Inserting (72) in (66) gives

$$f_{l,\epsilon}(0) \leq \min_{t \in T} D(\rho_{t}||p \otimes \sigma_t) - \delta - \frac{1}{|T|} \sum_{t \in T} D(\rho_{t}||p \otimes \sigma_t) + 2\epsilon \log \frac{d}{2\epsilon} + \frac{l}{l} \log |T| \leq -\frac{\delta}{2} + \frac{l}{l} \log |T|, \quad (74)$$

provided that $0 < \epsilon < \epsilon_0(\delta)$ where $\epsilon_0$ is small enough to ensure $2\epsilon \log \frac{d}{2\epsilon} < \frac{\delta}{2}$. The mean value theorem shows that for $s \in (0,1]$

$$f_{l,\epsilon}(s) = f_{l,\epsilon}(0) + f_{l,\epsilon}'(s') \cdot s$$

holds for some $s' \in (0,s)$. Since $f_{l,\epsilon}(0) = 0$, (74) shows that we can guarantee

$$f_{l,\epsilon}(s) < \left( -\frac{\delta}{2} + \frac{l}{l} \log |T| \right) s \quad (75)$$
for small enough \( s \). By (62) and (75) we obtain for \( \varepsilon < \varepsilon_0(\delta) \) and \( l \) large enough to make \( \psi_l(s) < \delta s \) valid,

\[
\text{tr}
\left(
(1 - q_{l,\delta} \rho_{l,\varepsilon})
\right) 
\leq
\exp\left\{ -l \left( \frac{\delta}{4} s - w(l) \right) \right\}
\leq
\exp\left\{ -l \delta s \right\}
\leq
\left| T \right| \cdot \exp\left\{ -l \delta s \right\}.
\]

Using (41), we have (with \( \varepsilon < \varepsilon_0 \))

\[
\text{tr}
\left(
(1 - q_{l,\delta} \rho_{l,\varepsilon})
\right) 
\leq
\| \rho_{l,\varepsilon} - \rho_l \|_1 
+ \text{tr}
\left(
(1 - q_{l,\delta} \rho_{l,\varepsilon})
\right)
\leq
2 \varepsilon + \left| T \right| \exp\left\{ -l \delta s \right\}.
\]

We can in fact, choose the parameter \( \varepsilon \) dependent on \( l \) in a way that \( (\varepsilon_l)_{l=1}^\infty \) decreases exponentially in \( l \), which proves the second claim of the lemma. \( \square \)

In order to prove the direct part of the coding theorem for general sets of channels we have to approximate arbitrary sets of channels by finite ones. For \( \alpha > 0 \), an \( \alpha \)-net in \( \mathcal{CQ}(X, \mathcal{H}) \) is a finite set \( N_\alpha := \{ W_i \}_{i=1}^{N_\alpha} \subset \mathcal{CQ}(X, \mathcal{H}) \) with the property, that for every channel \( W \in \mathcal{CQ}(X, \mathcal{H}) \) there exists an index \( i \in [N_\alpha] \) such that

\[
\| W - W_i \|_{cq} < \alpha
\]

holds. For a given set \( W \subset \mathcal{CQ}(X, \mathcal{H}) \) an \( \alpha \)-net \( N_\alpha \) in \( \mathcal{CQ}(X, \mathcal{H}) \) generates an approximating set \( \tilde{W}_\alpha \) defined by

\[
\tilde{W}_\alpha := \{ W_i \in N_\alpha : B_{cq}(\alpha, W_i) \cap W \neq \emptyset \}.
\]

The next lemma states that we find good approximations of arbitrary compound cq-channels among such sets as defined above. The proof can be given by minor variations of the corresponding results in [9], [10], and we omit it here.

**Lemma 6.** Let \( W := \{ W_{t} \}_{t \in T} \subset \mathcal{CQ}(X, \mathcal{H}) \) and \( \alpha \in (0, \frac{1}{e}) \). There exists a set \( T_\alpha \subseteq T \) which fulfills the following conditions

1. \( |T_\alpha| < \left( \frac{1}{\alpha} \right)^{2|X|^d} \),
2. given any \( l \in \mathbb{N} \), to every \( t \in T \) one finds an index \( t' \in T_\alpha \) such that

\[
\| W_{t}^{\otimes l}(x) - W_{t'}^{\otimes l}(x) \|_1 < 2 \alpha.
\]

holds for every \( x \in X^l \). Moreover,
3. for every $p \in \mathfrak{P}(X)$,
\[
\left| \min_{p' \in \mathcal{T}_n} \chi(p, W_{t'}) - \inf_{t \in T} \chi(p, W_t) \right| \leq 2\alpha \log \frac{d}{2\alpha} \tag{81}
\]
holds.

The following lemma is from [2] and will be used to establish the weak converse in Theorem 1. It states that codes which have small average error probability for a finite compound cq-channel contain subcodes with good maximal error probability of not substantially smaller size.

**Lemma 7 (cf. [2], Lemma 1).** Let $W : \{W_t\}_{t \in T} \subset CQ(X, \mathcal{H})$ be a compound channel with $|T| < \infty$ and $l \in \mathbb{N}$. If $(u_l, D_l)_{l=1}^{M_l}$ is an $(l, M_l)$-code with
\[
\max_{t \in T} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(W_t^{\otimes l}(u_l^i)(\mathbb{1}_\mathcal{H} - D_l^i)) \leq \overline{\chi} \tag{82}
\]
Then there exists for every $\epsilon > 0$ a subcode $(u_{l, \epsilon}, D_{l, \epsilon})_{l=1}^{M_{l, \epsilon}}$ of size $M_{l, \epsilon} = \left\lfloor \frac{\epsilon}{1-\epsilon} \right\rfloor M_l$ with
\[
\max_{t \in T} \max_{j \in [M_{l, \epsilon}]} \text{tr}(W_t^{\otimes l}(u_{l, \epsilon}^j)(\mathbb{1}_\mathcal{H} - D_{l, \epsilon}^j)) \leq |T| (\overline{\chi} + \epsilon) \tag{83}
\]
Finally, we have gathered all the prerequisites to prove Theorem 1.

**Proof (of Theorem 1).** The direct part (i.e. the assertion that the r.h.s. lower-bounds the l.h.s. in (18)) is proven by combining Lemma 1 with Lemma 5. Let $p = \arg \max_{p' \in \mathfrak{P}(X)} \inf_{t \in T} \chi(p', W_t)$. We show that for any $\delta > 0$,
\[
\inf_{t \in T} \chi(p, W_t) - \delta \tag{84}
\]
is an achievable rate. We can restrict ourselves to the case, where $\inf_{t \in T} \chi(p, W_t) > \delta > 0$ holds, because otherwise the above statement is trivially fulfilled. The above mentioned lemmata consider finite sets of channels, therefore we choose an approximating set $\mathcal{W}_\alpha$ (of cardinality $T_\alpha$) according to Lemma 6 for every $l \in \mathbb{N}$, where we leave the sequence $\alpha_1, \alpha_2, ...$ initially unspecified. For every $l \in \mathbb{N}$ and $t' \in T_\alpha$, let $\rho_{t'}, \sigma_{t'}$ be defined according to eq. (22) and (23), and further define states
\[
\rho_l := \frac{1}{|T_\alpha|} \sum_{t' \in T_\alpha} v_t^l \rho_{t'}^{\otimes l} v_t^{\ast} \tag{85}
\]
and
\[
\tau_l := \rho^{\otimes l} \otimes \frac{1}{|T_\alpha|} \sum_{t' \in T_\alpha} \sigma_{t'}^{\otimes l} \tag{86}
\]
For a given number \( \eta \) with \( 0 < \eta < a_l \), Lemma 5 guarantees (for large enough \( l \)), with a suitable constant \( \tilde{c} > 0 \), the existence of a projection \( q_{l,\eta} \in B((C|X|)^{\otimes l} \otimes \mathcal{H}^{\otimes l}) \) with
\[
\text{tr}(q_{l,\eta}\rho_l) \geq 1 - |T_{\alpha_l}| \cdot 2^{-\tilde{c}l}
\] (87)
and
\[
\text{tr}(q_{l,\eta}\tau_l) \leq 2^{-(a_l - \eta)}
\] (88)
where we defined \( a_l := \min_{t' \in T_{\alpha_l}} D(p_{t'}||p \otimes \sigma_{t'}) \). This by virtue of Lemma 1 implies for every \( \gamma > 0 \) such that \( \eta + \gamma < a_l \) the existence of a cq-code \( (x_{l,m}, D_{l,m})_{m \in [M]} \) of size
\[
M_l = \lfloor 2^{(a_l - \eta - \gamma)} \rfloor
\] (89)
and average error bounded by
\[
\max_{t' \in T_{\alpha_l}} \frac{1}{M_l} \sum_{m=1}^{M_l} \text{tr}(W_{t'}^{\otimes l}(u_{m}^{l})(\mathbb{1}_{\mathcal{H}^{\otimes l}} - D_{m}^{l})) \leq 2|T_{\alpha_l}|2^{-\tilde{c}l} + 4|T_{\alpha_l}|2^{-l\gamma}.
\] (90)
Notice, that for other positive numbers \( \gamma, \delta \), trivial codes have \( M_l = 1 \geq \lfloor 2^{(a_l - \eta - \gamma)} \rfloor \). Using (89) we obtain,
\[
\frac{1}{l} \log M_l \geq \min_{t' \in T_{\alpha_l}} \chi(p, W_{t'}) - \eta - \gamma \geq \inf_{t \in T} \chi(p, W_{t}) - \eta - \gamma - 4\alpha_l \log \frac{d}{2\alpha_l},
\] (91)
where the second inequality follows from Lemma 6. For the average error, it holds,
\[
\sup_{t' \in T_{\alpha_l}} \frac{1}{M_l} \sum_{m \in [M_l]} \text{tr} \left( W_{t'}^{\otimes l}(u_{m}^{l})(\mathbb{1}_{\mathcal{H}^{\otimes l}} - D_{m}^{l}) \right) \leq \max_{t' \in T_{\alpha_l}} \frac{1}{M_l} \sum_{m \in [M_l]} \text{tr} \left( W_{t'}^{\otimes l}(u_{m}^{l})(\mathbb{1}_{\mathcal{H}^{\otimes l}} - D_{m}^{l}) \right) + 2\alpha_l
\] (93)
\[
\leq 2|T_{\alpha_l}|2^{-\tilde{c}l} + 4|T_{\alpha_l}|2^{-l\gamma} + 2\alpha_l.
\] (94)
The first of the above inequalities follows from Lemma 6, the second one is by (90). Because we chose the approximating sets according to Lemma 6, we have,
\[
|T_{\alpha_l}| \leq \left( \frac{6}{\alpha_l} \right)^{2|X|d^2}
\] (96)
holds. In fact, if we specify \( \alpha_l \) to be \( \alpha_l := 2^{-\tilde{c}l} \) for every \( l \in \mathbb{N} \), where \( \tilde{c} \) is a constant with \( 0 < \tilde{c} < \min \left\{ \frac{\tilde{c}}{4|X|d^2}, \frac{\eta}{2|X|d^2} \right\} \), the r.h.s of (96) decreases exponentially for \( l \to \infty \). If we additionally choose \( \eta \) and \( \gamma \), small enough to validate
\[ \delta > \eta + \gamma + 2\alpha l \log \frac{\delta}{2\alpha l}, \text{ for sufficiently large } l, \text{ the rate defined in (84) is shown to be achievable by (92) and (91). Since } \delta \text{ was arbitrary, the direct statement follows.} \]

It remains to prove the converse statement. For the proof, we will construct a good code for transmission under the maximal error criterion and invoke the strong converse result given in [9] (see Remark 3). We show, that for any \( \delta > 0 \),

\[ C_C(W) < \max_{p \in \Psi(X)} \inf_{t \in T} \chi(p, W_t) + \delta. \]  

(97)

Let \( \delta > 0 \) and assume that for some fixed \( l \in \mathbb{N} \), \( C_l := (u_m^l, D_m^l)_{m=1}^{M_l} \) is an \((l, M_l)\)-code with

\[
\sup_{t \in T} \frac{1}{M_l} \sum_{m=1}^{M_l} \text{tr}(W_t \otimes (u_m^l)(1_{\mathcal{H}^\otimes l} - D_m^l)) \leq \lambda_l. \tag{98}
\]

We always can find a finite subset \( \hat{T} \subset T \) such that

\[
\left| \max_{p \in \Psi(X)} \inf_{t \in \hat{T}} \chi(p, W_t) - \max_{p \in \Psi(X)} \min_{t \in \hat{T}} \chi(p, W_t) \right| \leq \frac{\delta}{2}. \tag{99}
\]

(99)

holds (e.g. a set \( T_\alpha \) as in Lemma 6 for suitable \( \alpha \)). We set \( \epsilon := \frac{1}{2|\hat{T}|} \). By virtue of Lemma 7 we find a subcode \( (u_j^l, D_j^l)_{j=1}^{M_{l,\epsilon}} \subseteq C_l \) of \( C_l \) which has size

\[
M_{l,\epsilon} := \left\lfloor \frac{\epsilon}{1 - \epsilon} M_l \right\rfloor \tag{100}
\]

and maximal error bounded by

\[
\max_{t \in \hat{T}} \max_{j \in M_{l,\epsilon}} \text{tr} \left( W_t \otimes (u_j^l)(1_{\mathcal{H}^\otimes l} - D_j^l) \right) \leq \lambda_{|\hat{T}|} + \frac{1}{2}. \tag{101}
\]

(101)

If \( l \) is sufficiently large, the r.h.s. is strictly smaller than one. Therefore, by the strong converse theorem for coding under the maximal error criterion (see [9], Theorem 5.13), we have (with some constant \( K > 0 \))

\[
\frac{1}{l} \log M_{l,\epsilon} \leq \max_{p \in \Psi(X)} \min_{t \in \hat{T}} \chi(p, W_t) + K \frac{1}{\sqrt{l}} \tag{103}
\]

\[
\leq \max_{p \in \Psi(X)} \inf_{t \in \hat{T}} \chi(p, W_t) + \frac{\delta}{2} + K \frac{1}{\sqrt{l}}. \tag{104}
\]

The second line above follows from (99). On the other hand, by (100), we have

\[
\log M_l \leq \log M_{l,\epsilon} + \log \left( \frac{\epsilon}{2(1 - \epsilon)} \right). \tag{105}
\]
Dividing both sides of (105) by $l$ and combining the result with (104) shows that for sufficiently large $l$

$$\frac{1}{l} \log M_l \leq \max_{p \in \Psi(X)} \inf_{t \in T} \chi(p, W_t) + \frac{\delta}{2} + K \frac{1}{\sqrt{l}} + \frac{1}{l} \log \left( \frac{\varepsilon}{2(1 - \varepsilon)} \right)$$

(106)

$$\leq \max_{p \in \Psi(X)} \inf_{t \in T} \chi(p, W_t) + \delta$$

(107)

holds, which shows (97). Since $\delta$ was an arbitrary positive number, we are done. 

$\square$

Remark 3. While the achievability part for cq-compound channels regarding the maximal error criterion given in [9] required technical effort, the strong converse proof was rather uncomplicated. It was given there by a combination of Wolfowitz’ proof technique for the strong converse in case of classical compound channels and a lemma from [29].

Remark 4. We remark here, that a general strong converse does not hold for the capacity of compound cq-channels if the average error is considered as criterion for reliability of the message transmission. This can be seen by a counterexample given by Ahlswede in [1] (Example 1) regarding classical compound channels. However, we will see in the proof of Theorem 3 that in certain situations (especially, where $W$ is a convex set) a strong converse proof can be established.

As a corollary to the achievability part of Theorem 1 above, we immediately obtain a direct coding theorem for the capacity of a finite cq-compound channel under the maximal error criterion.

Corollary 1. For a finite compound cq-channel $W := \{W_t\}_{t \in T} \subset CQ(X, H)$ we have

$$C_{CC}(W) \geq \max_{p \in \Psi(X)} \min_{t \in T} \chi(p, W_t)$$

(108)

Proof. For an arbitrary number $\delta > 0$, we show, that

$$\max_{p \in \Psi(X)} \min_{t \in T} \chi(p, W_t) - \delta$$

(109)

is an achievable rate. Let $\{C_l\}_{l \in \mathbb{N}}$, $C_l := (u_{m,l}, D_{m,l})_{m=1}^{M_l} \forall l \in \mathbb{N}$, be a sequence of $(l, M_l)$-codes with

$$\liminf_{l \to \infty} \frac{1}{l} \log M_l \geq \max_{p \in \Psi(X)} \min_{t \in T} \chi(p, W_t) - \frac{1}{\delta}$$

(110)

and

$$\max_{t \in T} \frac{1}{M_l} \sum_{m=1}^{M_l} \text{tr} \left( W_t^{\otimes l} (u_{m,l}^{l}) (1_{\mathcal{H}} \otimes W_t - D_{m,l}) \right) \leq \lambda_l$$

(111)
for every \( l \in \mathbb{N} \), where \( \lim_{l \to \infty} \lambda_l = 0 \). Such codes exist by virtue of Theorem 1. Because of Lemma 7, we find for each \( l \in \mathbb{N} \) a subcode \( \tilde{C}_l := (u_{l,m_i}^l, D_{l,m_i}^l)_{i \in [\tilde{M}_l]} \subseteq C_l \) of size \( \tilde{M}_l := \left\lfloor \frac{1}{1 - \epsilon_l} M_l \right\rfloor \) and maximal error

\[
\max_{t \in T} \max_{i \in [\tilde{M}_l]} \text{tr} \left( W_i^{\otimes l}(u_{l,m_i}^l)(\mathbb{1}_{H^{\otimes l}} - D_{l,m_i}^l) \right) \leq (\lambda_l + \epsilon_l)|T|. \tag{112}
\]

with the sequence \( (\epsilon_l)_{l=1}^{\infty} \) defined by \( \epsilon_l := 2^{-l \delta} \) f.a. \( l \in \mathbb{N} \), it is clear that we find a sequence of \((l, \tilde{M}_l)\)-subcodes \( \{\tilde{C}_l\}_{l \in \mathbb{N}} \), where \( \tilde{C}_l := (u_{l,m_i}^l, D_{l,m_i}^l)_{i=1}^{\tilde{M}_l} \) f.a. \( l \in \mathbb{N} \), which fulfills

\[
\lim_{l \to \infty} \max_{t \in T} \max_{i \in [\tilde{M}_l]} \text{tr} \left( W_i^{\otimes l}(u_{l,m_i}^l)(\mathbb{1}_{H^{\otimes l}} - D_{l,m_i}^l) \right) = 0 \tag{113}
\]

and

\[
\liminf_{l \to \infty} \frac{1}{l} \log \tilde{M}_l = \liminf_{l \to \infty} \frac{1}{l} \log M_l \geq \max_{p \in \mathcal{P}(X)} \min_{t \in T} \chi(p, W_t) - \delta. \tag{114}
\]

Remark 5. The above corollary, although proven here for finite sets, can be extended to arbitrary compound sets by approximation arguments, as carried out in [9]. Moreover, an inspection of the proofs in this section shows that the speed of convergence of the errors remains exponential.

6 AVCQC

6.1 The Ahlswede-Dichotomy for AVcqCs

In this section, we prove Theorem 2 and Theorem 3. The proof of Theorem 2 is carried out via robustification of codes for a suitably chosen compound cq-channel. More specifically, to a given AVcqC \( A \) we take a sequence of codes for the compound channel \( W := \text{conv}(A) \) that operates close to the capacity of \( W \). Thanks to Theorem 1, we know that there exist codes for \( W \) that, additionally, have an exponentially fast decrease of average error probability. The robustification technique then produces a sequence of random codes for \( A \) that have a discrete, but super-exponentially large support and, again, an exponentially fast decrease of average error probability.

An intermediate result here is the (tight) lower bound on \( C_{A,r}(A) \).

A variant of the elimination technique of [3] is proven that is adapted to AVcqCs and reduces the amount of randomness from super-exponential to polynomial, while slowing down the speed of convergence of the average error probability from exponential to polynomial at the same time.

Then, under the assumption that \( C_{A,d}(A) > 0 \) holds, the sender can send the required amount of subexponentially many messages in order to establish sufficiently much common randomness. After that, sender and receiver simply use
the random code for $A$.

We now start out on our predescribed way. The following Theorem 5 and Lemma 8 will be put to good use, but are far from being new so we simply state them without proof.

Let, for each $l \in \mathbb{N}$, $\text{Perm}_l$ denote the set of permutations acting on $\{1, \ldots, l\}$. Let us further suppose that we are given a finite set $S$. We use the natural action of $\text{Perm}_l$ on $S^l$ given by $\sigma : S^l \rightarrow S^l$, $\sigma(s)_i := s_{\sigma(i)}$.

Let $T(l, S)$ denote the set of types on $S$ induced by the elements of $S^l$, i.e. the set of empirical distributions on $S$ generated by sequences in $S^l$. Then Ahlswede’s robustification can be stated as follows.

**Theorem 5 (Robustification technique, cf. Theorem 6 in [5]).**

Let $S$ be a set with $|S| < \infty$ and $l \in \mathbb{N}$. If a function $f : S^l \rightarrow [0, 1]$ satisfies

$$\sum_{s \in S} f(s^l)q(s^l)^{q(s^l)} \geq 1 - \gamma$$

for all $q \in T(l, S)$ and some $\gamma \in [0, 1]$, then

$$\frac{1}{l!} \sum_{\sigma \in \text{Perm}_l} f(\sigma(s^l)) \geq 1 - (l + 1)^{|S|} \cdot \gamma \quad \forall s^l \in S^l.$$  

The original theorem can, together with its proof, be found in [5]. A proof of Theorem 5 can be found in [6]. The following Lemma is borrowed from [4].

**Lemma 8.** Let $K \in \mathbb{N}$ and real numbers $a_1, \ldots, a_K, b_1, \ldots, b_K \in [0, 1]$ be given. Assume that

$$\frac{1}{K} \sum_{i=1}^{K} a_i \geq 1 - \epsilon \quad \text{and} \quad \frac{1}{K} \sum_{i=1}^{K} b_i \geq 1 - \epsilon,$$

hold. Then

$$\frac{1}{K} \sum_{i=1}^{K} a_i b_i \geq 1 - 2\epsilon.$$  

We now come to the promised application of the robustification technique to AVcqCs.

**Lemma 9.** Let $A = \{A_s\}_{s \in S}$ be an AVcqC. For every $\eta > 0$ there is a sequence of $(l, M_l)$-codes for the compound channel $W := \text{conv}(A)$ and an $l_0 \in \mathbb{N}$ such that the following two statements are true.

$$\liminf_{l \rightarrow \infty} \frac{1}{l} \log M_l \geq \overline{C}(W) - \eta$$

$$\min_{s \in S^l} \frac{1}{l!} \sum_{\sigma \in \text{Perm}_l} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s(\sigma^{-1}(x^l_i))\sigma^{-1}(D^l_i)) \geq 1 - (l + 1)^{|S|} \cdot 2^{-lc} \quad \forall l \geq l_0$$

with a positive number $c = c(|\mathcal{X}|, \dim \mathcal{H}, \mathcal{A}, \eta)$. 

Remark 6. The above result can be gained for arbitrary, non-finite sets $S$ as well. A central idea then is the approximation of $\text{conv}(A)$ from the outside by a convex polytope. Since $CQ(X, H)$ is not a polytope itself (except for trivial cases), an additional step consists of applying a depolarizing channel $N_p$ and approximate $N_p(\text{conv}(A))$, a set which does not touch the boundary of $CQ(X, H)$, instead of $\text{conv}(A)$.

This step can then be absorbed into the measurement operators, i.e. one uses operators $N_p(D_i^l)$ instead of the original $D_i^l$ ($i = 1, \ldots, M_l$).

A thorough application of this idea can be found in [6], where the robustification technique gets applied in the case of entanglement transmission over arbitrarily varying quantum channels.

Proof. According to Lemma 5, there is a sequence of $(l, M_l)$ codes for the compound channel $\text{conv}(A) = \{W_q : W_q = \sum_{s \in S} q(s) A_s, q \in \mathcal{P}(S)\}$ fulfilling

$$\liminf_{l \to \infty} \frac{1}{l} \log M_l \geq C_C(\text{conv}(A)) - \eta$$  \hspace{1cm} (121)$$

and

$$\exists l_0 \in \mathbb{N} : \inf_{W \in \text{conv}(A)} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(W \otimes^l (x_i^l) D_i) \geq 1 - 2^{-lc} \quad \forall l \geq l_0. \hspace{1cm} (122)$$

The idea is to apply Theorem 5. Let us, for the moment, fix an $N \ni l \geq l_0$ and define a function $f_l : S^l \to [0, 1]$ by

$$f_l(s^l) := \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s \sigma^{-1}(x_i^l)) D_i).$$  \hspace{1cm} (123)$$

Then for every $q \in \mathcal{P}(S)$ we have

$$\sum_{s^l \in S^l} f_l(s^l) \prod_{i=1}^{l} q(s_i) = \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(W_q \otimes^l (x_i^l) D_i) \geq 1 - 2^{-lc}. \hspace{1cm} (124)$$

It follows from Theorem 5 that

$$1 - (l + 1)^{|S|} \cdot 2^{-lc} \leq \frac{1}{|\Pi|} \sum_{\sigma \in \text{Perm}_l} f_l(\sigma(s^l))$$  \hspace{1cm} (125)$$

$$= \frac{1}{|\Pi|} \sum_{\sigma \in \text{Perm}_l} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s \sigma^{-1}(x_i^l)) \sigma^{-1}(D_i)) \quad \forall s^l \in S^l \hspace{1cm} (126)$$

holds, where

$$\sigma(B_1 \otimes \ldots \otimes B_l) := B_{\sigma^{-1}(1)} \otimes \ldots \otimes B_{\sigma^{-1}(l)} \quad \forall B_1, \ldots, B_l \in \mathcal{B}(H). \hspace{1cm} (127)$$

defines, by linear extension, the usual representation of $\text{Perm}_l$ on $\mathcal{B}(H)^\otimes l$ and the action of $\text{Perm}_l$ on $X^l$ is analogous to that on $S^l$. \hfill \Box
It is easily seen from the above Lemma 9 and Theorem 1, that the following theorem holds.

**Theorem 6.** For every AVcqC \( \mathcal{A} \),

\[
C_{A,r}(\mathcal{A}) \geq C_{C}(\text{conv}(\mathcal{A})) = \max_{p \in \Psi(X)} \inf_{A \in \text{conv}(\mathcal{A})} \chi(p, A).
\]

In the following we give a proof of the remaining inequality in (19). In fact, we prove the stronger statement Theorem 3:

**Proof (of Theorem 3).** We define \( W := \text{conv}(\mathcal{A}) \). Since \( |S| \) is finite, this set is compact. The function \( \chi(\cdot, \cdot) \) is a concave-convex function (see eq. (4)), therefore by the Minimax Theorem,

\[
\max_{p \in \Psi(X)} \min_{W \in \mathcal{W}} \chi(p, W) = \min_{W \in \mathcal{W}} \max_{p \in \Psi(X)} \chi(p, W)
\]

holds. Both sides of the equality are well defined, because we are dealing with a compact set. Let an arbitrary \( W_q \in \mathcal{W} \) be given by the formula

\[
W_q = \sum_{s \in S} q(s) A_s,
\]

where \( q \in \Psi(X) \). Set, for every \( l \in \mathbb{N} \), \( q^{\otimes l}(s^l) := \prod_{i=1}^l q(s_i) \). Let \( \lambda \in [0, 1) \), \( \delta > 0 \) and \((\mu_l)_{l \in \mathbb{N}}\) be a sequence of \((l, M_l)\)-random codes such that both

\[
\liminf_{l \to \infty} \frac{1}{l} \log M_l = C_{A,r}(\mathcal{A}, \lambda) - \delta
\]

and

\[
\liminf_{l \to \infty} \min_{s^l \in S^l} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s(u^i) D^i_l) \, d\mu_l((u^i, D^i_l)_{i=1}^{M_l}) \geq 1 - \lambda.
\]

For every \( l \in \mathbb{N} \) it holds that

\[
\int \sum_{i=1}^{M_l} \text{tr}(W_q^{\otimes l}(u^i) D^i_l) \, d\mu_l((u^i, D^i_l)_{i=1}^{M_l}) = \sum_{s^l \in S^l} q^{\otimes l}(s^l) \int \sum_{i=1}^{M_l} \text{tr}(A_s(u^i) D^i_l) \, d\mu_l((u^i, D^i_l)_{i=1}^{M_l}) \geq \min_{s^l \in S^l} \int \sum_{i=1}^{M_l} \text{tr}(A_s(u^i) D^i_l) \, d\mu_l((u^i, D^i_l)_{i=1}^{M_l}),
\]

which shows, that

\[
\liminf_{l \to \infty} \int \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(W_q^{\otimes l}(u^i) D^i_l) \, d\mu_l((u^i, D^i_l)_{i=1}^{M_l}) \geq 1 - \lambda
\]
Arbitrarily varying & compound classical-quantum channel s... holds. It follows the existence of a sequence \((u'_l, D'_l)_{l\in\mathbb{N}}\) of \((l, M_l)\)-codes for the discrete memoryless cq-channel \(W_q\) satisfying
\[
\liminf_{l \to \infty} \frac{1}{l} \log M_l = C_{A,d}(A, \lambda) - \delta \quad \text{and} \quad (137)
\]
\[
\liminf_{l \to \infty} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(W_q^\otimes l(u'_l D'_l)) \geq 1 - \lambda. \quad (138)
\]
By virtue of the strong converse theorem for single cq-DMCs given in [29] (also to be found and independently obtained in [26]), for any \(\lambda \in [0, 1), \delta > 0\) it follows
\[
\overline{C}_{A,r}(A, \lambda) - \delta = \liminf_{l \to \infty} \frac{1}{l} \log M_l \quad (139)
\]
\[
\leq \max_{p \in \mathcal{P}(X)} (p, W_q) \quad (140)
\]
and, since \(W_q \in \mathcal{W}\) was arbitrary,
\[
\overline{C}_{A,r}(A, \lambda) - \delta \leq \min_{W \in \mathcal{W}} \max_{p \in \mathcal{P}(X)} \chi(p, W). \quad (141)
\]
The equality in (142) holds by (129). Since \(\delta\) was an arbitrary positive number, we are done. \(\square\)

The following lemma contains the essence of the derandomization procedure.

**Lemma 10 (Random Code Reduction).** Let \(A = \{A_s\}_{s \in S}\) be an AVcqC, \(l \in \mathbb{N}\), \(\mu_l\) an \((l, M_l)\)-random code for \(A\) and \(1 > \varepsilon_l \geq 0\) with
\[
e(\mu_l, A) := \inf_{s' \in \mathcal{S}^l} \int \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s(x'_i D'_i))d\mu_l((x'_i, D'_i)_{i=1}^{M_l}) \geq 1 - \varepsilon_l. \quad (143)
\]
Let \(n, m \in \mathbb{R}\). Then if \(4\varepsilon_l \leq l^{-m} \text{ and } 2 \log |S| < l^{n-m-1}\) there exist \(l^n\) \((l, M_l)\)-deterministic codes \((x_{1,j}^l, \ldots, x_{M_l,j}^l, D_{1,j}^l, \ldots, D_{M_l,j}^l)\) \((1 \leq j \leq l^n)\) for \(A\) such that
\[
\frac{1}{l^n} \sum_{j=1}^{l^n} \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s(x'_{i,j} D'_i)) \geq 1 - l^{-m} \quad \forall s' \in \mathcal{S}^l. \quad (144)
\]

**Proof.** Set \(\varepsilon := 2l^{-m}\). By the assumptions of the lemma we have
\[
e(\mu_l, A) := \min_{s' \in \mathcal{S}^l} \int \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_s(x'_i D'_i))d\mu_l((x'_i, D'_i)_{i=1}^{M_l}) \geq 1 - \varepsilon_l. \quad (145)
\]
For a fixed \(K \in \mathbb{N}\), consider \(K\) independent random variables \(A_i\) with values in \((\mathcal{X}^l)^{M_i}) \times \mathcal{M}_{M_l}(H^{\otimes l})\) which are distributed according to \(\mu_l\).
Define, for each $s^l \in S^l$, the function $p_{s^l} : (\mathcal{X}^l)^{M_l} \times \mathcal{M}_{M_l}(\mathcal{H}^{\otimes l}) \to [0, 1]$, $(x_1^l, \ldots, x_{M_l}^l, D_1^l, \ldots, D_{M_l}^l) \mapsto \frac{1}{M_l} \sum_{i=1}^{M_l} \text{tr}(A_{s^l}(x_i^l)D_i^l)$.

We get, by application of Markov’s inequality, for every $s^l \in S^l$:

$$P(1 - \frac{1}{K} \sum_{j=1}^{K} p_{s^l}(A_j) \geq \frac{\varepsilon}{2}) = P(2^{K - \sum_{j=1}^{K} p_{s^l}(A_j)} \geq 2^{K/2}) \leq 2^{-K/2} E(2^{K - \sum_{j=1}^{K} p_{s^l}(A_j)})$$ (146)

The $A_i$ are independent and it holds $2^t \leq 1 + t$ for every $t \in [0, 1]$ as well as

$$\log(1 + \varepsilon) \leq 2\varepsilon$$

and so we get

$$P(1 - \frac{1}{K} \sum_{j=1}^{K} p_{s^l}(A_j) \geq \frac{\varepsilon}{2}) \leq 2^{-K/2} E(2^{K - \sum_{j=1}^{K} p_{s^l}(A_j)}) \leq 2^{-K/2} E(2^{K - \sum_{j=1}^{K} p_{s^l}(A_j)})$$ (147)

Therefore,

$$P(\frac{1}{K} \sum_{j=1}^{K} p_{s^l}(A_j) \geq 1 - \frac{\varepsilon}{2}) \geq 1 - |S|^2 2^{-K/4}.$$ (155)

By assumption, $2\log |S| \leq l^m$ and thus the above probability is larger than zero, so there exists a realization $A_1, \ldots, A_l$ such that

$$\frac{1}{l^m} \sum_{i=1}^{l^m} \frac{1}{M_l} \text{tr}(W_{s^l}(x_i^l)D_i^l) \geq 1 - \frac{1}{l^m}.$$ (156)

Now we pass to the proof of Theorem 2: If $C_{A,r}(A) = 0$ or $C_{A,d}(A) = 0$ there is nothing to prove. So, let $C_{A,r}(A) > 0$ and $C_{A,d}(A) > 0$. Then we know that, to every $l \in \mathbb{N}$, there exists a deterministic code for $A$ that, for sake of simplicity, is denoted by $(x_1^l, \ldots, x_{l^2}^l, D_1^l, \ldots, D_{l^2}^l)$, such that

$$\min_{s^l \in S^l} \frac{1}{l} \sum_{i=1}^{l^2} \text{tr}(A_{s^l}(x_i^l)D_i^l) \geq 1 - \varepsilon_l$$ (157)

and $\varepsilon_l \searrow 0$. Also, by Lemma 3 to every $\varepsilon > 0$ there is a sequence $(\mu_m)_{m \in \mathbb{N}}$ of random codes for transmission of messages over $A$ using the average error

$$\text{tr}(A_{s^l}(x_i^l)D_i^l) \geq 1 - \varepsilon_l$$ (157)
probability criterion and an \( m_0 \in \mathbb{N} \) such that
\[
\liminf_{m \to \infty} \frac{1}{m} \log M_m \geq C_{A,r}(A) - \varepsilon
\] (158)

\[
\int \frac{1}{M_m} \sum_{j=1}^{M_m} \text{tr}(A_{s^m}((x_j^m)^{\frac{1}{m}})) d\mu_m((x_1^m, \ldots, x_{M_m}^m, D_1^l, \ldots, D_{M_m}^l)) \geq 1 - 2^{-mc}
\] (159)

for all \( m \geq m_0 \) with a suitably chosen (and possibly very small) \( c > 0 \). This enables us to define the following sequence of codes: Out of the random code, by application of Lemma 10 and for a suitably chosen \( m_1 \geq m_0 \) such that the preliminaries of Lemma 10 are fulfilled, we get for every \( m \geq m_1 \) a discrete random code supported only on the set \( \{ (y_{ij}^m)_j, (E_{ij})_j \}_{j=1}^{M_m} \) such that
\[
\liminf_{m \to \infty} \frac{1}{m} \log M_m \geq C_{A,r}(A) - \varepsilon
\] (160)

\[
\frac{1}{m^2} \sum_{j=1}^{m^2} \frac{1}{M_m} \sum_{i=1}^{M_m} \text{tr}(A_{s^m}(y_{ij}^m)) E_{ij} \geq 1 - \frac{1}{m} \quad \forall m \geq m_1.
\] (161)

Now all we have to do is combine the two codes: For \( l, m \in \mathbb{N} \), define an \((l^2 M_m, 1)^2\)-deterministic code with the doubly-indexed message set \( \{ (x^l_j, y_{ij}^m) \}_{j=1}^{M_m} \) by the following sequence:
\[
\left( \left( x^l_j, y_{ij}^m \right), D_i \otimes E_{ij} \right)_{j=1}^{M_m}.
\] (162)

For the average success probability, by Lemma 8 it then holds
\[
\min_{(s^l, s^m) \in \mathbb{N}^{l+m}} \frac{1}{l^2 M_m} \sum_{i=1}^{l^2 M_m} \sum_{j=1}^{M_m} \text{tr}(A_{s^l, s^m})(\left( x^l_j, y_{ij}^m \right)) D_i \otimes E_{ij} \geq 1 - 2 \max \{ \varepsilon_l, \frac{1}{m} \}.
\] (163)

Now let there be sequences \((l_t)_{t \in \mathbb{N}}\) and \((m_t)_{t \in \mathbb{N}}\) such that \( l_t = \alpha(t) \) and \( l_t + m_t = t \) f.a. \( t \in \mathbb{N} \). Define a sequence of \((l_t, m_t)\)-deterministic codes \((\hat{x}_1^l, \ldots, \hat{x}_{l^2 M_m}^l, \hat{D}_1^l, \ldots, \hat{D}_{l^2 M_m}^l) \) for \( A \) by applying, for each \( t \in \mathbb{N} \), the above described procedure with \( m = m_t \) and \( l = l_t \). Then
\[
\liminf_{t \to \infty} \frac{1}{l_t} \log l_t^2 M_{m_t} \geq R \quad \text{and} \quad (164)
\]

\[
\liminf_{l \to \infty} \frac{1}{l^2 M_m} \sum_{k=1}^{l^2 M_m} \text{tr}(A_{s^l}(x_k^l)) \hat{D}_k = 1.
\] (165)

### 6.2 M-Symmetrizability

In this section, we prove Theorem 4.
Proof. We adapt the strategy of [22], that has already been successfully used in [6]. Assume \( \mathcal{A} \) is \( m \)-symmetrizable. Let \( l \in \mathbb{N} \). Take any \( a^l, b^l \in X^l \). Then there exist corresponding probability distributions \( p(a_1), \ldots, p(a_l), p(b_1), \ldots, p(b_l) \in \mathcal{P}(S) \) such that the probability distributions \( p(a^l), p(b^l) \in \mathcal{P}(S^l) \) defined by \( p(s^l|a^l) := \prod_{i=1}^{l} p(s_i|a_i) \), \( p(s^l|b^l) := \prod_{i=1}^{l} p(s_i|b_i) \) satisfy
\[
\sum_{s^l \in S^l} p(s^l|a^l)A_{s^l}(a^l) = \sum_{s^l \in S^l} p(s^l|b^l)A_{s^l}(b^l) \tag{166}
\]
and thereby lead, for every two measurement operators \( D_a, D_b \geq 0 \) satisfying \( D_a + D_b \leq 1_{H^l} \), to the following inequality:
\[
\sum_{s^l \in S^l} p(s^l|a^l)\text{tr}(A_{s^l}(a^l)D_a) = \sum_{s^l \in S^l} p(s^l|b^l)\text{tr}(A_{s^l}(b^l)D_a) \leq \sum_{s^l \in S^l} p(s^l|b^l)\text{tr}(A_{s^l}(b^l)(1_{H^l} - D_b)) = 1 - \sum_{s^l \in S^l} p(s^l|b^l)\text{tr}(A_{s^l}(b^l)D_b). \tag{167}
\]

Let a sequence of \((l, M_l)\) codes for message transmission over \( \mathcal{A} \) using the maximal error probability criterion satisfying \( M_l \geq 2 \) and \( \min_{i \in [M_l]} \min_{s^l \in S^l} \text{tr}(A_{s^l}(x^l_i)D_i^l) = 1 - \varepsilon_l \) be given, where \( \varepsilon_l \searrow 0 \). Then from the above inequality we get
\[
1 - \varepsilon_l \leq 1 - (1 - \varepsilon_l) \quad \Leftrightarrow \quad \varepsilon_l \geq 1/2. \tag{170}
\]
Therefore, \( C_{\mathcal{A},d}(\mathcal{A}) = 0 \) has to hold.

Now, assume that \( \mathcal{A} \) is not \( m \)-symmetrizable. Then there are \( x, y \in X \) such that
\[
\text{conv}(\{A_{x}(x)\}_{s \in S}) \cap \text{conv}(\{A_{y}(y)\}_{s \in S}) = \emptyset. \tag{171}
\]
The rest of the proof is identical to that in [6] with \( \hat{l} \) set to one. \( \square \)

### 6.3 Relation to the zero-error capacity

A remarkable feature of classical arbitrarily varying channels is their connection to the zero-error capacity of (classical) d.m.c.s, which was established by Ahlswede in [3, Theorem 3].

We shall first give a reformulation of Ahlswede’s original result and then consider two straightforward generalizations of it result, one for cq-channels, the other for quantum channels. In both cases it is shown, that no such straightforward generalization is possible.
Ahlswee’s original result. Ahlswee’s result can be formulated using the following notation. For two finite sets \( \mathbf{A}, \mathbf{B} \), \( C(\mathbf{A}, \mathbf{B}) \) stands for the set of channels from \( \mathbf{A} \) to \( \mathbf{B} \), i.e. each element of \( W \in C(\mathbf{A}, \mathbf{B}) \) defines a set of output probability distributions \( \{ W(a) \}_{a \in \mathbf{A}} \). With slight abuse of notation, for each \( D \subseteq \mathbf{B} \) and \( a \in \mathbf{A} \), \( W(D) := \sum_{b \in D} W(b|a) \). The (finite) set of extremal points of the (convex) set \( C(\mathbf{A}, \mathbf{B}) \) will be written \( E(\mathbf{A}, \mathbf{B}) \).

For two channels \( W_1, W_2 \in C(\mathbf{A}, \mathbf{B}) \), their product \( W_1 \otimes W_2 \in C(\mathbf{A}^2, \mathbf{B}^2) \) is defined through \( W_1 \otimes W_2(b_1^2|a_2^2) := W_1(b_1|a_1)W_2(b_2|a_2) \). An arbitrarily varying channel (AVC) is, in this setting, defined through a set \( \mathcal{W} = \{ W_s \}_{s \in \mathbf{S}} \subset C(\mathbf{A}, \mathbf{B}) \) (we assume \( \mathbf{S} \) and, hence, \( |\mathbf{S}| \), to be finite). The different realizations of the channel are written

\[
W_{s^l} := W_{s^l_1} \otimes \ldots \otimes W_{s^l_s}, \quad (s^l \in \mathbf{S}^l) \tag{172}
\]

and, formally, the AVC \( \mathcal{W} \) consists of the set \( \{ W_{s^l} \}_{s^l \in \mathbf{S}^l, l \in \mathbb{N}} \).

An \((l, M_l)\)-code for the AVC \( \mathcal{W} \) is given by a set \( \{ \mathcal{N}_l^i \}_{i=1}^{M_i} \subset \mathbf{A}^l \) called the ‘code-words’ and a set \( \{ D^i_l \}_{i=1}^{M_i} \) of subsets of \( \mathbf{B}^l \) called the ‘decoding sets’, that satisfies

\[
D^i_l \cap D^j_l = \emptyset, \quad i \neq j.
\]

A nonnegative number \( R \in \mathbb{R} \) is called an achievable maximal-error rate for the AVC \( \mathcal{W} \), if there exists a sequence of \((l, M_l)\) codes for \( \mathcal{W} \) such that both

\[
\liminf_{l \to \infty} \frac{1}{l} \log M_l \geq R \quad \text{and} \quad \limsup_{l \to \infty} \min_{s^l \in \mathcal{W}} \min_{1 \leq i \leq M_l} W_{s^l}(D^i_l|s^l) = 1. \tag{173}
\]

The (deterministic) maximal error capacity \( C_{\max}(\mathcal{W}) \) of the AVC \( \mathcal{W} \) is, as usually, defined as the supremum over all achievable maximal-error rates for \( \mathcal{W} \).

Much stronger requirements concerning the quality of codes can be made. An \((l, M_l)\)-code is said to have zero error for the AVC \( \mathcal{W} \), if for all \( 1 \leq i \leq M_l \) and \( s^l \in \mathbf{S}^l \) the equality \( W_{s^l}(D^i_l|s^l) = 1 \) holds.

The zero error capacity \( C_0(\mathcal{W}) \) of the AVC \( \mathcal{W} \) is defined as

\[
C_0(\mathcal{W}) := \lim_{l \to \infty} \max \left\{ \frac{1}{l} \log M_l : \exists (l, M_l) \text{-code with zero error for } \mathcal{W} \right\}. \tag{174}
\]

The above definitions carry over to single channels \( W \in C(\mathbf{A}, \mathbf{B}) \) by identifying \( W \) with the set \( \{ W \} \).

In short form, the connection \[3\] Theorem 3] between the capacity of certain arbitrarily varying channels and the zero-error capacity of stationary memoryless channels can now be reformulated as follows:

**Theorem 7.** Let \( W \in C(\mathbf{A}, \mathbf{B}) \) have a decomposition \( W = \sum_{s \in \mathbf{S}} q(s) W_s \), where \( \{ W_s \}_{s \in \mathbf{S}} \subset E(\mathbf{A}, \mathbf{B}) \) and \( q(s) > 0 \ \forall s \in \mathbf{S} \). Then for the AVC \( \mathcal{W} := \{ W_s \}_{s \in \mathbf{S}} \):

\[
C_0(\mathcal{W}) = C_{\max}(\mathcal{W}). \tag{175}
\]

Conversely, for every AVC \( \mathcal{W} = \{ W_s \}_{s \in \mathbf{S}} \subset E(\mathbf{A}, \mathbf{B}) \) and every \( q \in \mathbb{P}(\mathbf{S}) \) with \( q(s) > 0 \ \forall s \in \mathbf{S} \), equation (172) holds for the channel \( W := \sum_{s \in \mathbf{S}} q(s) W_s \).
Remark 7. Let us note at this point, that the original formulation of the theorem did not make reference to extremal points of the set of channels, but rather used the equivalent notion "channels of $0 - 1$-type".

Remark 8. By choosing $W \in E(A, B)$, one gets the equality $C_0(W) = C_{\text{max}}(W)$. The quantity $C_{\text{max}}(W)$ being well-known and easily computable, it may seem that Theorem 7 solves Shannon's zero-error problem. This is not the case, as one can verify by looking at the famous pentagon channel that was introduced in [27, Figure 2]. The pentagon channel is far from being extremal. That its zero-error capacity is positive [27] is due to the fact that it is not a member of the relative interior $riE(A, B)$.

Recently, in [6], this connection was investigated with a focus on entanglement and strong subspace transmission over arbitrarily varying quantum channels. The complete problem was left open, although partial results were obtained.

A no-go result for cq-channels. We will show below that, even for message transmission over AVcqCs, there is (in general) no equality between the capacity $C_0(W)$ of a channel $W \in CQ(X, H)$ and any AVcqC $A = \{A_s\}_{s \in S}$ constructed by choosing the set $\{A_s\}_{s \in S}$ to be a subset of the set of extremal points of $CQ(X, H)$ such that

$$W = \sum_{s \in S} \lambda(s)A_s$$

holds for a $\lambda \in \mathcal{P}(S)$. Observe that the requirement that each $A_s$ ($s \in S$) be extremal in $CQ(X, H)$ is a natural analog of the decomposition into channels of $0 - 1$-type that is used in the second part of [3].

A first hint why the above statement is true can be gained by looking at the method of proof used in [3], especially equation (22) there. The fact that the decoding sets of a code for an arbitrarily varying channel as described in [3] have to be mutually disjoint, together with the perfect distinguishability of different non-equal outputs of the special channels that are used in the second part of this paper, is at the heart of the argumentation.

The following lemma shows why, in our case, it is impossible to make a step that is comparable to that from [3, equation (21)] to [3, equation (22)].

Lemma 11. Let $A = \{A_s\}_{s \in S}$ be an AVcqC with $C_{A,d}(A) > 0$ and $0 < R < C_{A,d}(A)$. To every sequence of $(l, M_l)$ codes satisfying

$$\liminf_{l \to \infty} \frac{1}{l} \log M_l \geq R$$

and

$$\lim_{l \to \infty} \min_{s \in [M_l]} \min_{x_i \in S} \text{tr}(A_s(x_i) D_l^{s}) = 1$$

there is another sequence of $(l, M_l)$ codes with modified decoding operators $\tilde{D}_l^{s}$ such that

1) $$\liminf_{l \to \infty} \frac{1}{l} \log M_l \geq R$$

2) $$\lim_{l \to \infty} \min_{s \in [M_l]} \min_{x_i \in S} \text{tr}(A_s(x_i) \tilde{D}_l^{s}) = 1$$

3) $\forall i \in [M_l], l \in \mathbb{N}, \text{tr}(A_s(x_i) \tilde{D}_l^{s}) < 1$
Proof. Just use, for some $c > 0$, the transformation $\tilde{D}_l := (1 - 2^{-c})D_l + 2^{-c} \frac{1}{M} (1_{\mathcal{H}^\otimes l} - D_0^l)$. \hfill \Box

After this preliminary statement, we give an explicit example that shows where the construction in equation (176) must fail.

Lemma 12. Let $X = \{1, 2\}$ and $\mathcal{H} = \mathbb{C}^2$. Let $\{e_1, e_2\}$ be the standard basis of $\mathcal{H}$ and $\psi_+ := \sqrt{1/2}(e_1 + e_2)$. Define $W \in \text{CQ}(X, \mathcal{H})$ by $W(1) = |e_1\rangle \langle e_1|$ and $W(2) = |\psi_+\rangle \langle \psi_+|$. Then the following hold.

1. $W$ is extremal in $\text{CQ}(X, \mathcal{H})$
2. For every set $\{A_s\}_{s \in S} \subset \text{CQ}(X, \mathcal{H})$ and every $\lambda \in \Psi(S)$ such that (176) holds, $\{A_s\}_{s \in S} = \{W\}$.
3. $C_0(W) = 0$, but $C_{A,d}(\{W\}) > 0$.

Proof. 1) Let, for an $x \in (0, 1)$ and $W_1, W_2 \in \text{CQ}(X, \mathcal{H})$,

\[ W = xW_1 + (1 - x)W_2. \tag{180} \]

Then, clearly,

\[ |e_1\rangle \langle e_1| = xW_1(1) + (1 - x)W_2(1) \implies W_1(1) = W_2(1) = W(1) \tag{181} \]

and

\[ |\psi_+\rangle \langle \psi_+| = xW_1(2) + (1 - x)W_2(2) \implies W_1(2) = W_2(2) = W(2), \tag{182} \]

so $W = W_1 = W_2$.
2) is equivalent to 1).
3) It holds $\text{tr}\{W(i)W(j)\} > 1/2$ ($i, j \in X$). Let $l \in \mathbb{N}$. Assume there are two codewords $a^l, b^l \in X^l$ and corresponding decoding operations $C, D \geq 0$, $C + D \leq 1_{\mathbb{C}^2}$, such that

\[ \text{tr}\{W^\otimes l(a^l)C\} = \text{tr}\{W^\otimes l(b^l)D\} = 1 \implies \text{tr}\{W^\otimes l(a^l)D\} = \text{tr}\{W^\otimes l(b^l)C\} = 0. \tag{183} \]

Then we may add a third operator $E := 1_{\mathbb{C}^2}^\otimes l - C - D$ and it holds that

\[ \text{tr}\{W^\otimes l(a^l)E\} = \text{tr}\{W^\otimes l(b^l)E\} = 0. \tag{184} \]

From equations (184) and (183) we deduce the following:

\[ \sqrt{E}W^\otimes l(a^l)\sqrt{E} = \sqrt{E}W^\otimes l(b^l)\sqrt{E} = \sqrt{D}W^\otimes l(a^l)\sqrt{D} = \sqrt{C}W^\otimes l(b^l)\sqrt{C} = 0. \tag{185} \]

With these preparations at hand, we are led to the following chain of inequalities:
Definition 17. An (l, k)–strong subspace transmission code for \( \mathcal{J} \) is a pair \((\mathcal{P}, \mathcal{R})\) of maps \( \mathcal{P} : \mathcal{F}_{l} \rightarrow \mathcal{K} \otimes \mathcal{F}_{1}^{\otimes l} \) such that \( \mathcal{R} \) is an achievable strong subspace transmission rate for the AVQC \( \mathcal{J} = \{ \mathcal{N}_s \}_{s \in \mathcal{S}} \) if there is a sequence of \( (l, k_l) \)–strong subspace transmission codes such that

1. \( \lim_{l \rightarrow \infty} \inf_{s \in \mathcal{S}} \frac{1}{l} \log k_l \geq R \) and
2. \( \lim_{l \rightarrow \infty} \inf_{s \in \mathcal{S}} \min_{\psi \in \mathcal{S}(\mathcal{F}_{1})} \langle \psi, \mathcal{R}^{l} \circ \mathcal{P}^{l} | \psi \rangle | \psi \rangle \mathcal{N}_s \mathcal{P}^{l} | \psi \rangle = 1. \)

The random strong subspace transmission capacity \( \mathcal{A}_{s,\text{random}}(\mathcal{J}) \) of \( \mathcal{J} \) is defined by

\[
\mathcal{A}_{s,\text{det}}(\mathcal{J}) := \sup \left\{ R : R \text{ is an achievable strong subspace transmission rate for } \mathcal{J} \right\}.
\]
Self-evidently, we will also need a notion of zero-error capacity:

**Definition 18.** An $(l,k)$ zero-error quantum code (QC for short) $(\mathcal{F},\mathcal{P},\mathcal{R})$ for $\mathcal{N} \in \mathcal{C}(\mathcal{H},\mathcal{K})$ consists of a Hilbert space $\mathcal{F}$, $\mathcal{P} \in \mathcal{C}(\mathcal{F},\mathcal{H}^\otimes 2)$, $\mathcal{R} \in \mathcal{C}(\mathcal{K}^\otimes 2,\mathcal{F})$ with $\dim \mathcal{F} = k$ such that

$$
\min_{x \in \mathcal{F} : ||x|| = 1} \langle x, \mathcal{R} \circ \mathcal{N}^\otimes l \circ \mathcal{P}(|x\rangle\langle x|)x \rangle = 1.
$$

(193)

The zero-error quantum capacity $Q_0(\mathcal{N})$ of $\mathcal{N} \in \mathcal{C}(\mathcal{H},\mathcal{K})$ is now defined by

$$
Q_0(\mathcal{N}) := \lim_{l \to \infty} \frac{1}{l} \log \max \{ \dim \mathcal{F} : \exists (l,k) \text{ zero-error QC for } \mathcal{N} \}.
$$

(194)

**Conjecture 1.** Let $\mathcal{N} \in \mathcal{C}(\mathcal{H},\mathcal{K})$ have a decomposition $\mathcal{N} = \sum_{s \in \mathcal{S}} q(s)\mathcal{N}_s$, where each $\mathcal{N}_s$ is extremal in $\mathcal{C}(\mathcal{H},\mathcal{K})$ and $q(s) > 0 \forall s \in \mathcal{S}$. Then for the AVQC $\mathcal{J} := \{\mathcal{N}_s\}_{s \in \mathcal{S}}$,

$$
Q_0(\mathcal{N}) = \mathcal{A}_{s,\text{det}}(\mathcal{J}).
$$

(195)

Conversely, for every AVQC $\mathcal{J} = \{\mathcal{N}_s\}_{s \in \mathcal{S}}$ with $\mathcal{N}_s$ being extremal for every $s \in \mathcal{S}$ and every $q \in \mathcal{P}(\mathcal{S})$ with $q(s) > 0 \forall s \in \mathcal{S}$, equation (195) holds for the channel $\mathcal{N} := \sum_{s \in \mathcal{S}} q(s)\mathcal{N}_s$.

**Remark 9.** One could formulate weaker conjectures than the one above. A crucial property of extremal classical channels that was used in the proof of Theorem 7 was that $W_{x'}(|x|)$ is a dirac-measure for every codeword $x'$, if only $\{W_{x'}\}_{x \in \mathcal{S}} \subset \mathcal{E}(A,B)$.

This property gets lost for the extremal points of $\mathcal{C}(\mathcal{H},\mathcal{K})$ (see the channels that are used in the proof of Theorem 8), but could be regained by restriction to channels consisting of only one single Kraus operator.

This conjecture leads us to the following theorem:

**Theorem 8.** Conjecture 7 is wrong.

**Remark 10.** As indicated in Remark 9 there could still be interesting connections between (for example) the deterministic strong subspace transmission capacity of AVQC’s and the zero-error entanglement transmission of stationary memoryless quantum channels.

**Proof.** Let $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$. Let $\{e_0,e_1\}$ be the standard basis of $\mathbb{C}^2$. Consider, for a fixed but arbitrary $x \in [0,1]$ the channel $\mathcal{N}_x \in \mathcal{C}(\mathcal{H},\mathcal{K})$ defined by Kraus operators $A_1 := \sqrt{1-x^2}|e_0\rangle\langle e_1|$ and $A_2 := |e_0\rangle\langle e_0| + x|e_1\rangle\langle e_1|$. As was shown in [30], this channel is extremal in $\mathcal{C}(\mathcal{H},\mathcal{K})$. It is also readily seen from the definition of Kraus operators, that it approximates the identity channel $id_{\mathbb{C}^2} \in \mathcal{C}(\mathcal{H},\mathcal{K})$:

$$
\lim_{x \to 1} ||\mathcal{N}_x - id_{\mathbb{C}^2}||_1 = 0.
$$

(196)

Now, on the one hand, $\mathcal{N}_x$ being extremal implies span($\{A_i^*A_j\}_{i,j=1}^{2}$) = $\mathcal{M}(\mathbb{C}^2)$ for all $x \in [0,1]$ (where $\mathcal{M}(\mathbb{C}^2)$ denotes the set of complex $2 \times 2$ matrices) by
[12] Theorem 5. This carries over to the channels \( \mathcal{N}_x^{\otimes l} \) for every \( l \in \mathbb{N} \): Let the Kraus operators of \( \mathcal{N}_x^{\otimes l} \) be denoted \( \{ A_{i,l} \}, i, j \in \{ 1, 2 \}^l \), then

\[
\text{span}(\{ A_{i,l}^* A_{j,l} \}, i, j \in \{ 1, 2 \}^l) = \{ M : M \text{ is complex } 2^l \times 2^l - \text{matrix} \}. \tag{197}
\]

On the other hand, it was observed e.g. in [16], that for two pure states \( |\phi\rangle\langle \phi|, |\psi\rangle\langle \psi| \in S((\mathbb{C}^2)^{\otimes l}) \), the subspace spanned by them can be transmitted with zero error if and only if

\[
|\psi\rangle\langle \phi| \perp \text{span}(\{ A_{i,l}^* A_{j,l} \}, i, j \in \{ 1, 2 \}^l).
\tag{198}
\]

This is in obvious contradiction to equation (197), therefore \( Q_0(\mathcal{N}_x) = 0 \ \forall x \in [0, 1) \). On the other hand, from equation (196) and continuity of \( A_{s,\det}() \) in the specifying channel set (15), though indeed only the continuity results of [23] that were also crucial in the development of corresponding statements in [6] are really needed here) we see that there is an \( X \in [0, 1) \) such that for all \( x \geq X \) we have \( A_{s,\det}(\mathcal{N}_x) > 0 \). Letting \( x = X \) we obtain \( Q_0(\mathcal{N}_X) = 0 \) and \( A_{s,\det}(\mathcal{N}_X) > 0 \), so \( Q_0(\mathcal{N}_X) \neq A_{s,\det}(\mathcal{N}_X) \) in contradiction to the statement of the conjecture.

\[ \square \]

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