Breakdown of Tan’s relation in lossy one-dimensional Bose gases

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In quantum gases with contact repulsion, the distribution of momenta of the atoms typically decays as $\sim 1/|p|^4$ at large momentum $p$. At thermal equilibrium, Tan’s relation connects the amplitude of that $1/|p|^4$ tail to the adiabatic derivative of the energy with respect to the gas’ coupling constant or scattering length. Here it is shown that the relation breaks down in the one-dimensional Bose gas with contact repulsion subject to atom losses. Immediately after a loss event, the wavefunction has a ghost singularity whose effect on the momentum distribution is unaccounted for by the usual term in Tan’s relation. Because of the existence of infinitely many conserved quantities, even after relaxation, this effect remains. It is reflected in the rapidity distribution, which itself acquires a $1/|p|^4$ tail. In the momentum distribution, that tail adds to the usual term. This phenomenon is discussed for arbitrary interaction strengths, and it is supported by exact calculations in the two asymptotic regimes of infinite and weak repulsion.

Introduction. In a quantum gas, contact interactions can impart large momenta to the particles: the singularity of the many-body wavefunction when two particles are at the same position is reflected in the tails of their momentum distribution $w(p)$, which decay as $w(p) \sim 1/|p|^4$. It contrasts with the gaussian decay that would be expected from the Boltzmann distribution in an ideal gas. The $1/|p|^4$ tails were first noticed for hard-core one-dimensional (1D) Bosons by Minguzzi et al \cite{2}, then studied in 1D gases of arbitrary interaction strength by Olshanii and Dunjko \cite{3}, and by Tan in three-dimensional (3D) fermionic gases \cite{1}. \cite{4}. For a general analysis in two and three dimensions for bosons, fermions and mixtures, see Refs. \cite{6} \cite{7}. Remarkably, the amplitude of the tail, $C := \lim_{p \to \infty} |p|^4 w(p)$, is a thermodynamic quantity \cite{2} \cite{4}. Tan’s ‘adiabatic sweep theorem’ \cite{4}, or simply ‘Tan’s relation’, connects the amplitude $C$ to the adiabatic derivative \cite{8} of the energy with respect to the gas’ contact interaction parameter. For Bose gases, Tan’s relation reads \cite{1}

$$C = c_c, \quad \text{with} \quad c_c := \frac{m^2}{(2\pi \hbar)^d} 2g^2 \partial (E/V) / \partial g.$$  \tag{1}

Here $m$ is the particles’ mass, $E$ is the energy of the gas, $V$ is its volume, and $g$ is the interaction coupling constant \cite{7}. The momentum distribution is normalized as $\int d^d p w(p) = N/V$, where $N$ is the total atom number and $d$ is the dimension of the system. $c_c$, defined by the second equality of Eq. (1), is called the ‘contact density’.

Tails in the momentum distribution have been observed experimentally in 3D fermionic gases and Tan’s relation has been verified \cite{10} \cite{11}. It has also been verified, using spectroscopy, in 3D Bose gases \cite{12}. On the theory side, Tan’s relation and its extensions have been thoroughly investigated \cite{6} \cite{7} \cite{13} \cite{16}. Recent works have focused on the 1D Bose gas \cite{17} \cite{19}, exploiting the relation between the contact density and the zero-distance two-body correlation function (Eq. (3) below).

Tan’s relation (1) is based on the assumption that the tails of the momentum distribution are entirely due to the contact two-body interaction. In this Letter, we point out that this assumption is not always valid. It holds true at thermal equilibrium, and thus for chaotic systems since they relax towards thermal states. However, it may break down in the 1D Bose gas with contact repulsion. There, thermalization is prevented by the existence of an extensive number of conservation laws \cite{20} \cite{21}, so the gas typically lies in a non-thermal state (see e.g. reviews in Ref. \cite{22}). In that case, we show that the momentum distribution can acquire extra contributions that do not come from the two-body interaction, yet also decay as $\sim 1/|p|^4$. Then Tan’s relation (1) breaks down, and one has $C > c_c$.

Bose gases with contact repulsion in 1D are realized in cold atoms experiments, when the transverse confinement is sufficiently strong so that transverse degrees of freedom are frozen \cite{23} \cite{24}. In this Letter we show that, under atom losses, the gas evolves towards a non-thermal state with a momentum distribution violating the equality (1). Thus, an important implication of our findings is that Tan’s relation will most probably be violated experimentally in 1D Bose gases, since cold atom gases, even when they are very well isolated from their environment, always suffer from losses \cite{25} \cite{27}.

The essence of the breakdown of Tan’s relation is as follows. Immediately after a loss event, the wavefunction has a singularity at the position of the lost atoms, in addition to the singularities when two of the remaining particles meet. In the momentum distribution, this additional singularity is reflected as a $1/|p|^4$ term which adds to the usual contact term. If the gas were chaotic, then it would relax to a new thermal state. The effect on the momentum distribution would therefore be observable only at short time after the loss; after relaxation Tan’s relation would be recovered. However, the 1D Bose gas is not chaotic and the effect remains present at any time. In this Letter we elaborate on that scenario, and
support it by an analysis performed in two asymptotic regimes of the gas. In the hard-core regime, we use recent exact results on losses [28]. In the quasicondensate regime, we use the techniques developed in Refs. [28] [30]. In both cases, we show that the amplitude of the tail of the momentum distribution becomes substantially larger than the value predicted by Tan’s relation.

The contact in the 1D Bose gas. We consider bosons with contact repulsion in a periodic system of size $L$. The Hamiltonian is (with $|\Psi(z), \Psi^*(z')| = \delta(z - z')$)

$$H = \int_0^L dz \, \Psi^+(z) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{g}{2} \Psi^+(z)\Psi(z) \right) \Psi(z).$$

We start by recalling the effects of the contact interaction on the tails of the momentum distribution, following Ref. [2]. Because of the contact interaction, the many-body wavefunction $\psi(z_1, \ldots, z_N) = \langle 0 | \Psi(z_1) \cdots \Psi(z_N) | \psi \rangle$ has a cusp singularity whenever two positions coincide [31]: $\partial_{z_i} \psi_{z_1 \rightarrow z_j} - \partial_{z_j} \psi_{z_1 \rightarrow z_j} = (mg/\hbar^2) \psi(z_1, \ldots, z_i = z_j, \ldots)$. When one takes the Fourier transform, those cusps become $1/p^4$ tails, which give a $\sim 1/p^4$ contribution to the momentum distribution after taking the squared modulus of the wavefunction. When this calculation is done carefully (as in Ref. [2]), it shows that the contact interaction contributes to the tail of the momentum distribution $w(p)$ as $C_c/p^4$ with

$$C_c = m^2/(2\pi\hbar)g^2n^2g^2(0).$$

Here $n = N/L$ is the atom density and $g(j)(0) = \langle \Psi(z)^{+j} \Psi(z)^j \rangle/n^j$, where $j \in \mathbb{N}$, is the normalized zero-distance $j$-body correlation function, independent of $z$ in a translation invariant system. Eq. (3) is an alternative definition of the contact density in 1D, equivalent to the one in Eq. (1) for stationary states, i.e. for diagonal density matrices. Indeed, if $|\psi\rangle$ is an eigenstate, a straightforward application of the Hellmann-Feynman theorem leads to $n^2g^2(0) = 2 \langle \psi | \partial H/\partial |\psi\rangle^2/L = 2\partial (E/L)/\partial g$.

Actually, Eq. (3) is more general than Eq. (1) since it also applies to non-stationary states, as long as the cusp singularity condition is fulfilled.

For a gas at thermal equilibrium, other equivalent expressions for the contact density are $C_c = m^2/(2\pi\hbar)^2g^2(\partial(F/V)/\partial g)_{T,r}$ and $C_c = m^2/(2\pi\hbar)^2g^2(\partial(\Omega/V)/\partial g)_{T,\mu}$ where $F$ and $\Omega$ are the free energy and the grand canonical potential respectively, and $T$ and $\mu$ are the temperature and chemical potential of the gas. Such expressions were used for instance in Refs. [17] [33]. However, as discussed in the introduction, the 1D Bose gas is in general not at thermal equilibrium. We now explain why this leads to larger amplitudes of the tails of the momentum distribution.

The rapidity distribution, its tails, and tails of the momentum distribution. Because of the extensive number of its conserved quantities, the 1D Bose gas typically relaxes to a Generalized Gibbs Ensemble (see e.g. the volume [22]) which is parametrized by its rapidity distribution [32] [34]. The rapidities are conserved by the Hamiltonian dynamics: they characterize the eigenstates of the Hamiltonian [2], which take the form of Bethe states [35] [36]. The rapidities are the asymptotic momenta of the atoms if one lets the gas expand freely in 1D [37] [41]. They are conveniently thought of as the momenta of quasiparticles with infinite lifetime [42] [43], dubbed ‘Bethe quasiparticles’ in this Letter. After relaxation to a Generalized Gibbs Ensemble, expectation values of local observables are functionals of the rapidity distribution $\rho(q)$ [32] [34]. In the following, we normalize the rapidity distribution as $\int dq \rho(q) = N/L$.

We stress that the rapidity distribution is not equal to the momentum distribution of the atoms. This is well illustrated by the ground state of the system: its rapidity distribution $\rho(k)$ vanishes outside a finite interval [35] [36], while its momentum distribution $w(p)$ presents the aforementioned $1/p^4$ tails that extend to infinity [2].

Nevertheless, the momentum distribution may reflect features of the rapidity distribution, and vice-versa. In particular, this is the case when the rapidity distribution $\rho(q)$ has a non-vanishing component for rapidities $q$ much larger than its typical width $Q$. The Bethe quasiparticles with very large rapidities $q \gg Q$ must correspond to atoms with large momenta $p \simeq q$. Thus we expect that such rapidities will contribute to the momentum distribution around the same wavevector. The reverse is also expected: if the tails of the momentum distribution are larger than the ones coming from the contact interaction, then we expect them to be caused by large-rapidity Bethe-quasiparticles, so they will also be present in the rapidity distribution.

Let us imagine that the rapidity distribution of the gas has tails decaying as $1/q^4$ (we will argue below that atom losses naturally produce such tails), and let $C_r := \lim_{q \to \infty} q^4 \rho(q)$ be their amplitude. Because the large-$q$ Bethe-quasiparticles correspond to an additional population of atoms with large momenta $p \simeq q$, we expect the amplitude of the tails in the momentum distribution to be

$$C := \lim_{p \to \infty} p^4w(p) = C_c + C_r.$$

It holds both for non-stationary states [44] and for systems which have relaxed. This is our key conjecture, and the first main formula of this Letter. At thermal equilibrium, it matches Tan’s relation (1) because the rapidity distribution decays exponentially at large $q$ so $C_r$ vanishes. On the other hand, a non-vanishing $C_r$ results in the breakdown of Eq. (1).

We do not know how to prove formula (4) for generic states, since evaluating the momentum distribution in the Generalized Gibbs Ensemble parametrized by $\rho(k)$ is a very hard task in general (see e.g. Refs. [20] [45]). However we can check that formula (4) holds true in three different asymptotic regimes of the 1D Bose gas: the ideal Bose gas, hard-core, and quasi-condensate regimes.
First, in the ideal Bose gas regime —i.e. when the typical energy per atom is much larger than both $mg^2/h^2$ and $gn$—, the atoms behave as non-interacting bosons. The rapidities are the bosons’ momenta, while $C_r$ vanishes. Thus, Eq. (4) is trivially satisfied. Second, in the hard-core regime —where $g \to \infty$—, we exploit results of Ref. [46] to compute $w(p)$ from $\rho(q)$ [47]. Eq. (4) is then verified numerically, see Fig. 1. Third, in the quasicondensate regime of weak interactions —i.e. when the typical kinetic energy per atom is much smaller than $gn$, and $\gamma := mg/(h^2n) \ll 1$—, one can use Bogoliubov theory [48]. In this effective description the Hamiltonian reduces to a collection of independent harmonic modes, labeled by their momentum $q$, and the integrals of motion are the occupation $\alpha_q$ of each mode. The link between Bogoliubov excitations and Bethe quasiparticles is not obvious. However, this issue has been discussed by Lieb [49] (see also Ref. [50]), who identifies, for states close to the ground state, the large-$q$ Bogoliubov excitations to Bethe quasiparticles with rapidities $k \approx q$. Therefore the $C_r/q^4$ tail in the rapidity distribution translates to Bogoliubov mode occupations $\alpha_q \approx 2\pi h C_r/q^4$ for large $q$ [51]. For a set of occupations $\{\alpha_q\}$, the momentum distribution $w(p)$ can be calculated analytically using the results of Ref. [48]. One finds [47] that Eq. (4) is again verified.

From now on, we assume that formula (4) holds. Our next task is to show that $C_r > 0$ because of losses.

\[
(C_c + C_r)/C
\]

\[
C_c/q^4
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\[
C_r/g
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\[
Gt
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\[
(C_c + C_r)/C
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C_r/g
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Gt
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(C_c + C_r)/C
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C_c/q^4
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C_r/g
\]

\[
Gt
\]

FIG. 1. Results for one-body losses in the hard-core limit. The initial state is at thermal equilibrium at temperature $T = 1.02n_0$ and chemical potential $\mu = 5T$, where $n_0$ is the initial atom density. Inset: evolution of $C_c$, $C_r$, and $C_c$ and $C_r$ are computed directly from Eq. (4), while $C$ is extracted numerically by a fitting procedure [47]. Main figure: the ratio $C_c/C$ is compared to $e^{-Gt}$ (red dashed line), its value predicted from Eq. (11). We also check, comparing $(C_c + C_r)/C$ to 1, that formula (4) holds true.

Losses and $1/q^4$ tails of the rapidity distribution. We consider the general case of local $K$-body losses, where $K = 1, 2, 3, \ldots$ is the number of atoms lost in each loss event. Depending on the experiment, losses are typically dominated by $K = 1$, $K = 2$ [52] [53] or $K = 3$ processes [20] [27], but it is convenient to keep $K$ arbitrary. The atom density then decays as $dn/dt = -KG\rho^{(K)}(0)n^K$, where $G$ —in units of $length^{K-1} time^{-1}$— is a constant characterizing the loss rate. Following Ref. [28] (see also Refs. [54] [55]), we assume that the loss rate $Gn^K$ is much smaller than the relaxation time, so that the gas relaxes to a Generalized Gibbs Ensemble after each loss event. This allows to represent the evolution of the gas under losses by its time-dependent rapidity distribution [28].

Let us assume that, at $t = 0$ the gas’ rapidity distribution has no $1/q^4$ tails, i.e. $C_r(t = 0) = 0$. For instance, the gas could be in a thermal state. We want to show that at $t = 0$, $dC_r/dt > 0$, implying that the rapidity distribution will develop non vanishing $1/q^4$ tails.

To do this, we elaborate on the microscopic mechanism presented in the introduction. Consider the many-body wavefunction $\psi_{t=t_1}(z_1, \ldots, z_N)$ just before a loss event occurring at time $t_1$ and position $z_1$. Right after the loss, the wavefunction of the remaining $N - K$ atoms is $\psi_{t=t_1}(z_1, \ldots, z_{N-K}) = L^{K/2}\psi_{t=t_1}(z_1, \ldots, z_{N-K}, z_{N-K+1} = z_1, \ldots, z_N = z_2)$. As a reminiscence of its cusp singularities before the loss, the wavefunction $\psi_{t=t_1}$ still has a cusp at $z_1 = z_j$ ($j = 1, \ldots, N - K$). Following the calculation of Ref. [23], we find that it results in a contribution $C^{(1 loss)}/p^4$ to the momentum distribution, with the amplitude

\[
C^{(1 loss)} = \frac{\hbar^3}{2\pi} L^{K-1}(N - K) \int dz_2 \ldots dz_{N-K} \left| \partial_{z_1} \psi_{t=t_1}^{(1)} - \partial_{z_1} \psi_{t=t_1}^{(1)} \right|^2,
\]

where the variables $z_{N-K+1}, \ldots, z_N$ in the integrand are taken equal to $z_1$. The boundary condition imposed by the contact interaction gives $\partial_{z_1} \psi_{t=t_1}^{(1)} = \partial_{z_1} \psi_{t=t_1}^{(1)} = Kmg/h^2 \psi(z_1 = z_1, z_2, \ldots, z_{N-K+1} = z_1, \ldots, z_N = z_2)$. Then, using the expression of $g(K+1)(0)$ in first quantization, we get

\[
C^{(1 loss)} = \frac{m^2}{2\pi h} \frac{nK^2}{L} g^2 g(K+1)(0).
\]

Here we have used the fact that, as $N \to \infty$, $N - K \approx N$ and $N, \ldots, (N - K) \approx N^{K+1}$.

Next, we rely on formula (4) and argue that the contribution (6) of one loss event to the momentum distribution translates into the same contribution to the rapidity distribution. Indeed, the contribution (6) is not taken into account in the contact density $C_c$ at time $t = t_1^-$, therefore according to formula (4) it must appear in the tail of the rapidity distribution:

\[
C_r(t=t_1^-) - C_r(t=t_1^-) = C^{(1 loss)}.
\]

Like $\rho(k)$, $C_r$ is conserved by the Hamiltonian dynamics, so this increase of $C_r$ remains after relaxation to a Generalized Gibbs Ensemble. Finally, we multiply this result by $LG^{(K)}(0)dt$, the number of loss events occurring in the system during a short time interval $dt$. This leads to the initial growth rate

\[
\frac{dC_r}{dt}(t = 0) = \frac{m^2}{2\pi h} Gn^{K+1} K^2 g^2 g(K)(0)g(K+1)(0).
\]
This equation is the second main formula of this Letter. It shows that \( \frac{dC_t}{dt}|_{t=0} > 0 \), such that \( C_t \) becomes non zero. Together with Eq. (4), it implies that the momentum distribution develops tails that are larger than what is expected from Tan’s relation.

We stress that Eq. (3) gives only the initial growth rate of the tail of the rapidity distribution. At later times, its evolution will also involve additional damping effects. Indeed, under atom losses the gas ultimately evolves to the vacuum, therefore the whole rapidity distribution — including its tails— will go to zero at very long times. The calculation of the damping of \( C_t \) at longer times is not obvious. Below we obtain further results in the hard-core and quasicondensate regimes.

**Exact results in the hard-core regime.** In the hard-core regime \((g \to \infty)\), only one-body losses are relevant, since \( g^{(K)}(0) = 0 \) for \( K > 1 \). Thus, in this paragraph we fix \( K = 1 \). We exploit the following result of Ref. 28: given an initial rapidity distribution \( \rho_0(q) \) at time \( t = 0 \), the distribution at time \( t \) is

\[
\rho(q) = \text{Re} \left[ \frac{i e^{-Gt}}{1 - 2(1 - e^{-Gt})} \int \frac{d\alpha}{(q - \lambda)/\hbar + 2m_0(1 - e^{-Gt})} \right],
\]

where \( n_0 = \int \rho_0(q) dq \) is the initial density. The asymptotic expansion of \( \rho(q) \) at large \( q \) is of the form \( \rho(q) = C_t q^2 + o(1/q^4) \), with a time-dependent amplitude

\[
C_t(t) = 4\hbar m/\pi |n_0 e_0 - j_0^2/(2m)| e^{-Gt} (1 - e^{-Gt}).
\]

Here \( j_0 = \int q \rho_0(q) dq \) and \( e_0 = \int q^2/(2m) \rho_0(q) dq \) are the initial momentum and energy density respectively 58]. The initial growth rate is \( dC_t/dt|_{t=0} = 4\hbar m/\pi G |n_0 e_0 - j_0^2/(2m)| \), which is consistent with our general formula (2) in the \( g \to \infty \) limit. Indeed, it follows from the identity \( \lim_{g \to \infty} g^2 q^{(2)}(0) = 8\hbar^2/(m^2 \pi) |ne - j^2/(2m)| \). 47

The same identity can be used to calculate the contact density (1) as \( C_c = 4\hbar m/\pi \left| i e - j^2/(2m) \right| \). Furthermore, using Eq. (9), it is possible to show that the particle, momentum, and energy densities evolve as \( n(t) = n_0 e^{-Gt} \), \( j(t) = j_0 e^{-Gt} \), \( e(t) = e_0 e^{-Gt} \) respectively 47. Thus Eq. (10) can be written as

\[
C_t(t)/C_c(t) = \exp(Gt) - 1.
\]

The ratio between the amplitude \( C_t \) and the contact density \( C_c \) grows exponentially as time increases. This is our third main result: not only does the term \( C_t/p^4 \) contribute to the momentum distribution, it also becomes dominant compared to the contact term. It is also illustrated in Fig. 1.

**Results for the quasicondensate.** In the quasicondensate regime, correlations between atoms are weak and \( g^{(1)}(0) \approx 1 \) for all \( j \). A convenient representation of that regime is obtained phase-density representation 48: \( n_0 \) writes the atomic field \( \Psi \) as \( \sqrt{n + \delta n} e^{i\theta} \) where \( \theta \) and \( \delta n \) are the phase and density fluctuation fields, which fulfill \( \delta n(z, \theta(z')) = i\theta(z - z') \), and \( \delta n, \delta \theta/\partial z \ll n \). The Bogoliubov approximation then leads to a collection of independent harmonic modes 47, 48. One can decompose \( \delta n \) and \( \theta \) in Fourier modes \( \delta n_q = \sqrt{1/L} \int dz e^{-iqz/k} \delta n(z) \) and \( \theta_q = \sqrt{1/L} \int dz e^{iqz/hk} \theta(z) \), where \( \varepsilon_q = \sqrt{q^2/2m + 2gn} \) and \( f_q = \varepsilon_q/(\sqrt{2m}) \). The population of the mode \( q \) reads \( \alpha_q = (\langle H_q \rangle - \varepsilon_q^2/2\varepsilon_q) \) where \( \varepsilon_q/2 \) is the zero-point energy.

Using equipartition of energy—which is verified when we are assuming slow losses—, one finds that the time evolution of \( n \) and \( f_q \) do not contribute to \( d\alpha_q/dt \), which reduces to 47

\[
\frac{d\alpha_q}{dt} = \frac{f_q d(\delta n_q \delta n_{-q})}{4n} + \frac{n d(\theta_q \theta_{-q})}{f_q dt}.
\]

This expression would vanish under pure Hamiltonian dynamics. But density fluctuations and phase fields are affected by the losses, as analyzed in Refs. 29, 30, 47, 58 (see also 47: their evolution is given by \( d(\delta n(z) \delta n(z'))/dt = K^2 G_n K^{-1} [\delta n(z - z') - 2\delta n(z) \delta n(z')] \)).

Using the Fourier transforms, plugging them into Eq. (12), and using again equipartition of energy, one gets \( d\alpha_q/dt = K^2 Gn K^{-1} (\alpha_q - 1/2 + 1/(f_q + f_{-q})) \) 59. Finally, we rely once more on the identification of Bogoliubov excitations with Bethe quasiparticles valid at large \( q \) 49, 50, such that \( \rho(q) = \alpha_q/(2\pi\hbar) \). Using the large \( q \) expansion of \( f_q \), we find that the amplitude of the \( 1/q^4 \) tails of \( \rho(q) \) evolves according to

\[
\frac{dC_r}{dt} = K^2 G_n K^{-1} \left( -C_r + \frac{m^2}{2\pi \hbar} g^2 n^2 \right).
\]

This is the fourth main result of this Letter. Notice that Eq. (13) reproduces the short-time behavior (1) for \( C_t(t = 0) = 0 \) as expected (recall that \( g^{(1)}(0) = 1 \)).

Eq. (13) allows to compute \( C_t(t) \) at all times 47. Using the fact that \( C_c = m^2/(2\pi\hbar) g^2 n^2 \) in the quasicondensate regime (see Eq. (9)), we find the long time behavior

\[
C_c(t) = C_c(0) e^{-Gt} / (K(t - 1)).
\]

For \( K = 1 \), one finds the same behavior as in the hard-core regime. For \( K \geq 3 \), the ratio takes an asymptotic value. For instance, the ratio \( C_c/C_t \) goes to 3 for three-body losses, so at large time the tail of the momentum distribution \( C/p^4 \) is four times larger than its value predicted by Tan’s relation (1).

**Conclusion.** An experimental test of the predictions of this paper is within reach with current cold atoms setup. There exist different ways of measuring the momentum distribution of 1D gases 59, 62. Because of the small amplitude of the tails, such a measurement requires
a high dynamical range, which can be achieved for instance using metastable atoms [63]. Usually, gases in experiments are non-uniform. Within a local density approximation, results presented here are however easily generalized [37].

On the theoretical side, our results open several research lines. First, for quantitative comparison with experiment, one should compute the evolution of the rapidity tails in intermediate regimes of the 1D gas. For this, one can in principle rely on the method presented in Ref. [28], although an improvement of the numerical efficiency of that method would be required. Second, it would be interesting to investigate the effects of losses in higher dimension. The singularity of the wavefunction at the position of the lost atoms is also expected to have an effect that remains to be elucidated. Finally, it would be interesting to study loss processes that are not purely local or not purely Markovian. How would this impact the development of the momentum tails?

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**Appendix A: Computing the momentum distribution \( w(p) \) from the rapidity distribution \( \rho(q) \) in the hard-core limit**

In the main text (Fig. 1) we refer to a numerical procedure which allows us to evaluate the momentum distribution \( w(p) \) of hard-core bosons as a functional of their rapidity distribution \( \rho(q) \). Here we explain how we implement that procedure. In this section we set \( \hbar = m = 1 \).

We exploit formulas (14)-(15) of Ref. [46], which gives the one-body density matrix as follows:

\[
\langle \Psi^\dagger(x) \Psi(y) \rangle = \sum_{i,j=0}^{\infty} \varphi_i(x) \sqrt{n_i} Q_{ij}(x,y) \sqrt{n_j} \varphi_j^*(y),
\]

where the \( \varphi_i(x) \) (\( i = 0, \ldots, \infty \)) are the single-particle eigenfunctions of the Schrödinger operator for an infinite system in an external potential, \( -\hbar^2/(2m) \partial_x^2 + V(x) \), and \( n_i \in [0,1] \) is the occupation of each orbital. In Ref. [46], it is assumed that the \( n_i \) are the occupations of a Gibbs ensemble at a given temperature and chemical potential. But Eq. (A1) is more general, and it holds true for any occupations, corresponding to a Generalized Gibbs Ensemble. The semi-infinite matrix \( Q(x,y) \) is defined as \( Q(x,y) = (P^{-1})^T \det P \), with

\[
P_{ij}(x,y) = \delta_{ij} - 2 \text{sign}(y-x) - \sqrt{n_i n_j} \int_x^{y} \phi_i(z) \phi_j^*(z) dz.
\]

We stress that this formula is based on the mapping from hard-core bosons to free fermions, and that it works for an infinite system. In principle, it does not apply to a finite system with periodic boundary conditions. The reason is that hard-core bosons with periodic boundary conditions map to periodic/anti-periodic boundary conditions for the fermions, depending on the whether the total number of fermions is odd/even respectively. Since formula (A1) works for arbitrary occupation numbers, the parity of the number of fermions is not fixed (unless all \( n_i \) are equal to 0 or 1). However, the one-body density matrix typically decays quickly with the distance \( |x-y| \). Moreover, we are mostly interested in its short-distance behavior, because this is what fixes the large-\( p \) tail of the momentum distribution. Therefore, we can work with \( x, y \in [-L/2, L/2] \) with periodic boundary conditions for the fermions as long as \( L \) is large enough. Thus, we can use plane waves \( \varphi_i(x) = e^{i q_i x} / \sqrt{L} \) with \( q_j \in 2\pi\mathbb{Z} / L \), such that

\[
\langle \Psi^\dagger(x) \Psi(0) \rangle = \frac{2\pi}{L} \sum_{q_i, q_j} e^{i q_i x} \sqrt{\rho(q_i) \rho(q_j)} Q_{ij}(x,0).
\]

Here we have used the fact that the occupation of each fermionic mode is given by the rapidity den-
FIG. 2. Top: rapidity distribution in the hard-core limit, given by Eq. (9) in the main text. The initial rapidity distribution \( \rho_0(q) \) is the thermal distribution at temperature \( T = 1.02n_0^3 \) and chemical potential \( \mu = 5T \), after some fraction of the atoms have been lost (\( n_0 \) is the initial density of atoms). The other curves are the rapidity distributions after some fraction of the atoms have been lost. The inset shows a zoom on the tails of the curves. The rapidity distributions corresponding to Eq. (9) in the main text have been obtained using this method. The Hellmann-Feynman theorem, together with thermodynamic Bethe Ansatz calculations (see e.g. Ref. [64], or the supplementary methods of Ref. [63]), lead to the following formula for \( g^{(2)}(0) \), or equivalently for the density of interaction energy \( e_1 := g\partial(E/L)/\partial g \):

\[
e_1 = \frac{1}{2} n^2 g g^{(2)}(0) = \int \left[ q/m - v_{\text{eff}}(q) \right] q \rho(q) dq.
\]

Appendix B: Calculation of the product \( g^2 g^{(2)}(0) \) in the \( g \to \infty \) limit

In the main text, we use the relation

\[
\lim_{g \to \infty} n^2 g^2 g^{(2)}(0) = 8\hbar^2/m [ne - j^2/(2m)],
\]

(BO.1)

where \( n = \int \rho(q) dq \) is the particle density, \( j = \int q \rho(q) dq \) is the momentum density, and \( e = \int q^2/(2m) \rho(q) dq \) is the energy density in a state of arbitrary rapidity density \( \rho(q) \). This identity can be derived as follows. We first consider finite \( g \). The Hellmann-Feynman theorem, together with thermodynamic Bethe Ansatz calculations (see e.g. Ref. [64], or the supplementary methods of Ref. [63]), lead to the following formula for \( g^{(2)}(0) \), or equivalently for the density of interaction energy \( e_1 := g\partial(E/L)/\partial g \):

\[
e_1 = \frac{1}{2} n^2 g g^{(2)}(0) = \int \left[ q/m - v_{\text{eff}}(q) \right] q \rho(q) dq.
\]

Here \( v_{\text{eff}}(q) \) is the ‘effective velocity’ defined by the thermodynamic Bethe Ansatz formula

\[
v_{\text{eff}}(q) = \frac{1}{m} \frac{\text{d}r(q)}{\text{d}q},
\]

(BO.3)
where id(q) = q, 1(q) = 1, and the ‘dressing’ of a function f(q) is defined as

\[ f^{dr}(q) = f(q) + \int \varphi(q - q') f^{th}(q') \frac{1}{1(dq')} \rho(q') dq'. \]  

(B4)

Here \( \varphi(q) = 2mg/(mg/h^2 + q^2) \) is the Lieb-Liniger kernel [35 49]. Expanding at first order in 1/g, one finds

\[ 1^{dr}(q) = 1 + 2h^2 n/(mg) + O(1/g^2) \]

and id(q) = q + 2h^2/\( mg \) + O(1/g^2), so

\[ v^{eff}(q) = \frac{q}{m} - \frac{2h^2}{m^2 g} (qn - j) + O(1/g^2). \]  

(B5)

Inserting this into Eq. (B2), one gets the relation (B1).

Appendix C: Evolution of the atom density, momentum density and energy density under one-body losses in the hard-core limit

In the main text we argue that, using the evolution equation for the rapidity distribution under one-body losses in the hard-core limit (9), it can be shown that the atom density, momentum density and energy density evolve with time as \( n(t) = e^{-GT} n_0 \), \( j(t) = e^{-GT} j_0 \), \( e(t) = e^{-GT} e_0 \) respectively.

This can be derived as follows. First, one uses the rapidity distribution (9) to define a generating function for the conserved charges (following Ref. [28]),

\[ Q(z) := \frac{i}{\pi} \int \frac{\rho(q) dq}{z - q}, \]

(C1)

for \( z \in \mathbb{C}, \text{Im} z > 0 \). \( Q(z) \) is analytic for \( \text{Im} z > 0 \). Moreover, for \( q = 1 \), we have

\[ \lim_{z \to 1} \text{Re}[Q(z)] = \rho(q). \]  

(C2)

Thus, we see that Eq. (9) in the main text is equivalent to

\[ Q(z) = \frac{i e^{-GT} \int \frac{\rho_0(\lambda) d\lambda}{(z - \lambda)/(h + 2i n_0(1 - e^{-GT}))}}{1 - i2(1 - e^{-GT}) \int \frac{\rho_0(\lambda) d\lambda}{(z - \lambda)/(h + 2i n_0(1 - e^{-GT}))}}, \]

(C3)

for \( \text{Im} z > 0 \).

The atom density \( n = \int \rho(q) dq \), the momentum density \( j = \int q \rho(q) dq \) and the energy density \( e = \int q^2 \rho(q) dq/(2m) \) appear in the asymptotic expansion of Eq. (C1) at large \( z \):

\[ Q(z) \approx \frac{i}{\pi} \left( \frac{n}{z + j/2} + \frac{2me}{z^3} + \ldots \right). \]  

(C4)

Expanding Eq. (C3) to order \( O(1/z^3) \), one finds

\[ Q(z) \approx \frac{i}{\pi} \left( \frac{e^{-GT} n_0}{z} + \frac{e^{-GT} j_0}{z^2} + \frac{2me^{-GT} e_0}{z^3} + \ldots \right), \]

(C5)

which gives the time-dependence claimed above for the three densities.

Appendix D: Bogoliubov theory in the quasicondensate regime (after Mora and Castin)

We follow the conventions of Mora and Castin [48]. Inserting a phase-amplitude representation of the annihilation operator, \( \Psi(z) = \sqrt{n} + \sqrt{n} e^{i\theta} \), one finds that \( \delta n(z), \theta(z') = i\delta(z - z') \), in the Hamiltonian (2), one finds to second order:

\[ H - \mu N \simeq \int \left[ \frac{\hbar^2}{8m n} (\partial_z \delta n)^2 + \frac{\mu}{2} \delta n^2 + \frac{\hbar^2 n}{2m} (\partial_z \theta)^2 \right] dz. \]

This quadratic Hamiltonian allows to grasp quantum fluctuations around the classical profile which solves the Gross-Pitaevskii equation, \( n = N/L = \mu / g \) where \( \mu \) is the chemical potential. One can define a boson annihilation field \( B(z) = \frac{i}{2\sqrt{n}} \delta n(z) + i\sqrt{n} \theta(z) \) such that \( [B(z), B^\dagger(z')] = \delta(z - z') \), and its Fourier modes \( B_q = \int e^{-i\pi z/\hbar} B(z) dz/\sqrt{L} \) with \( q \in (2\pi n/L) \mathbb{Z} \). Then the quadratic Hamiltonian becomes, up to constant terms,

\[ H - \mu N \simeq \frac{1}{2} \sum_q \left( \frac{B_q}{B^\dagger_q} \right)^2 \left( \frac{\mu}{2m} + \frac{\mu^2}{2m} + \mu \right) \left( B_q \right)^2, \]

where we have used \( \mu = gn \). Finally, the Hamiltonian \( H_q \) is diagonalized by a Bogoliubov transformation

\[ \left( \begin{array}{c} B_q \\ B^\dagger_q \end{array} \right) = \left( \begin{array}{cc} \tilde{u}_q & \tilde{v}_q^* \\ \tilde{v}_q & \tilde{u}_q^* \end{array} \right) \left( \begin{array}{c} b_q \\ b_q^\dagger \end{array} \right), \]

with \( |\tilde{u}_q|^2 - |\tilde{v}_q|^2 = 1 \). Here a convenient choice is \( \tilde{u}_q = \tilde{v}_q^* = \cosh(\theta_q/2) \) and \( \tilde{v}_q = \tilde{u}_q^* = -\sinh(\theta_q/2) \), which gives

\[ H - \mu N \simeq \sum_q \varepsilon_q b_q b_q^\dagger + \text{const.}, \]

with a dispersion relation \( \varepsilon_q = \sqrt{\frac{\mu}{2m}} \left( \frac{q^2}{2m} + \mu \right) \).

1. Population of Bogoliubov modes and momentum distribution

Let us consider a state where the population of each Bogoliubov mode is \( \alpha_q = \langle b_q^\dagger b_q \rangle \). The one-particle density matrix is (see Ref. [18], formula (184)):

\[ g^{(1)}(z) = \exp \left[ -\frac{1}{n} \int \frac{dq}{2\pi \hbar} \left( |\tilde{u}_q|^2 + |\tilde{v}_q|^2 \right) \alpha_q + |\tilde{v}_q|^2 \right] / (1 - \cos(qz/\hbar)). \]

Following Lieb [39], we identify quasiparticle excitations with large rapidities with the large-q Bogoliubov modes. Then we are interested in the case when \( n_q \) decays as \( 2\pi \hbar C_q / q^4 \) at large \( q \), where \( C_q \) is the same constant as in the main text. We note that

\[ \langle (\tilde{u}_q^2 + \tilde{v}_q^2) \alpha_q + \tilde{v}_q^2 \rangle \simeq 2\pi \hbar C_q L / q^4, \]

(D2)
which follows from the fact that $\bar{v}_q^2 = v_q^2 - 1 = m^2 \mu^2 / q^4 + O(1/q^6)$, and $m^2 \mu^2 = 2\pi \hbar C_c$ (valid in the quasicondensate regime). In general, $1/k^4$ tails result in a discontinuity of the third derivative of the Fourier transform, according to $\partial^3_z \left( \int \frac{dk}{2\pi} e^{ikx} \right)_{|z| \to 0} = 1$. Thus, the discontinuity of the argument of the exponential in (D1) is

$$\partial^3_z \left( \frac{1}{n} \int \frac{dk}{2\pi \hbar} \left[ \bar{v}_q^2 + v_q^2 \right] \alpha_q + (1 - \cos(qz/\hbar)) \right)_{|z| \to 0+} = 2 \beta_\gamma + \beta_\gamma^2 \rho_0.$$

Consequently, $g^{(1)}(z)$ also possesses a discontinuity in its third derivative,

$$\partial^3_z g^{(1)}_{|z| \to 0+} - \partial^3_z g^{(1)}_{|z| \to 0-} = \frac{2 \pi}{\hbar^2} (C_t + C_c) \rho_0.$$

Taking the Fourier transform, one finds that the momentum distribution has a tail with coefficient $C_t + C_c$, as claimed in the main text:

$$w(p) = \frac{n}{2\pi} \int_0^\infty e^{ipz/\hbar} g^{(1)}(z) dz \sim (C_t + C_c)/p^4.$$  \hfill (D4)

2. The effect of losses on Bogoliubov modes

The effect of losses in the quasicondensate regime has been investigated in Refs. [29, 30, 57, 58]. For the convenience of the reader, we recall the results that are useful for this Letter.

In terms of the Fourier modes of the phase and density fluctuation fields, $\theta_q = (1/\sqrt{L}) \int dz \theta(z)e^{-iqz/\hbar}$ and $\delta n_q = (1/\sqrt{L}) \int dz \delta n(z)e^{-iqz/\hbar}$, the population $\alpha_q$ of the Bogoliubov mode $q$ reads

$$\alpha_q = \frac{f_q}{4n} \langle \delta n_{-q} \delta n_q \rangle + \frac{n}{f_q} \langle \theta_q \theta_{-q} \rangle - \frac{1}{2}.$$

where $f_q = \sqrt{(q^2/(2m) + \gamma q^4)/(2m)}$.

Under losses, the density $n$ and the coefficient $f_q$ become time-dependent, as well as the phase and density fluctuations $\langle \delta n_{-q} \delta n_q \rangle$ and $\langle \theta_q \theta_{-q} \rangle$. One finds

$$\frac{d\alpha_q}{dt} = \frac{f_q}{4n} \frac{d\langle \delta n_{-q} \delta n_q \rangle}{dt} + \frac{n}{f_q} \frac{d\langle \theta_q \theta_{-q} \rangle}{dt} + \frac{1}{f_q/n} \frac{d(f_q/n)}{dt} \left[ \frac{f_q}{4n} \langle \delta n_{-q} \delta n_q \rangle - \frac{n}{f_q} \langle \theta_q \theta_{-q} \rangle \right].$$

We are assuming slow losses. Then, to compute $d\alpha_q/dt$, which is a slowly varying quantity, one can average over a time $2\pi/\varepsilon_q$. This time-average ensures equipartition of energy between the two conjugate variables $\delta n_q$ and $\theta_{-q}$.

Consequently, the second line in the equation vanishes, and we have

$$\frac{d\alpha_q}{dt} = \frac{f_q}{4n} \frac{d\langle \delta n_{-q} \delta n_q \rangle}{dt} + \frac{n}{f_q} \frac{d\langle \theta_q \theta_{-q} \rangle}{dt},$$

which is the equation used in the main text. Note that the fact that $d\alpha_q/dt$ is not affected by the slow time evolution of $n$ and $f_q$ (i.e. the vanishing of the second line of Eq. (D6)) can also be interpreted as the result of adiabatic following of the eigenstates of $H_q = \varepsilon_q (\hbar^2 q^4 + 1/2)$. We now recall the effect of losses on density and phase fluctuations, analyzed in Refs. [29].

a. Effect of losses on density fluctuations

The goal of this section is to derive the formula for the evolution of the density fluctuations,

$$\frac{d\langle \delta n(z) \delta n(z') \rangle}{dt} = K^2 G n K^2 \langle \delta n(z) \delta n(z') \rangle$$  \hfill (D8)

which is used in the main text.

To do this, we consider a cell of length $\ell$, much smaller than the typical length scale of variation of the phase $\theta$, but large enough so that it contains a number of atoms $N \gg 1$. We note $N = n \ell$ the atom number corresponding to the mean atomic density $n$ in the gas. We are interested in the effect of losses during a time interval $\Delta t$ satisfying $N^{-K} \ll \gamma \Delta t \ll N^{1-K}$ where $\gamma := G/\ell^K$ is the loss rate in the cell. This ensures that the number of lost atoms is much larger than one, but much smaller than $N$.

We consider an initial state with an atom number distribution $P_0(N)$. Here fluctuations can be either of statistical or of quantum nature. Let $P_0(M)$ the probability to have $M$ loss events until time $\Delta t$. One has

$$P_0(M) = \sum_N P_0(N) P(M|N),$$

where $P(M|N)$ is the probability to have $M$ loss events conditioned to an initial number of atoms $N$. Under the assumption $\gamma \Delta t \ll N^{1-K}$, this is well approximated by a Poisson distribution [58]

$$P(M|N) = \frac{1}{M!} e^{-\gamma \Delta t N^K} (\gamma \Delta t N^K)^M.$$  \hfill (D10)

Furthermore, for $\gamma \Delta t \gg N^{-K}$, the Poissonian becomes a Gaussian,

$$P(M|N) \approx \frac{e^{-(M-N^K \gamma \Delta t)^2}}{\sqrt{2\pi \sigma}}.$$  \hfill (D11)

The variance can be approximated by its value for $N = \bar{N}$, which is

$$\sigma = \sqrt{\gamma \Delta t \bar{N}^K}.$$  \hfill (D12)
The probability to have \( N \) atoms in the cell at time \( \Delta t \) is then

\[
P(N) = \sum_M P_0(N + KM)P(M|N + KM)
\]

\[
\simeq \int dMP_0(N + KM)P(M|N + KM), \tag{D13}
\]

where we have used the fact that both \( N \) and \( M \) are typically large to replace the sum by an integral.

We are now ready to compute the atom number fluctuations at time \( \Delta t \). For this we introduce \( \tilde{N}(\Delta t) = N - \bar{N} \) at time 0, and \( \delta N(\Delta t) = \tilde{N}(\Delta t) - \bar{N}(\Delta t) \) at time \( \Delta t \), where \( \bar{N}(\Delta t) = \bar{N} - K\gamma\Delta t \bar{N}^K \) is the atom number corresponding to the gas mean density after \( \Delta t \). Using (D13), one gets

\[
\langle \delta N(\Delta t)^2 \rangle = \int d\tilde{N} P_0(\tilde{N}) \int dM(\tilde{N} - KM - \bar{N}(\Delta t))^2P(M|\tilde{N}).
\]

Then the Gaussian approximation of \( P(M|\tilde{N}) \) (Eq. (D11)) gives

\[
\langle \delta N(\Delta t)^2 \rangle = K^2\gamma\Delta t \bar{N}^K + \int d\tilde{N} P_0(\tilde{N})(\tilde{N} - K\gamma\Delta t \bar{N}^K - \bar{N} + K\gamma\Delta t \bar{N}^K)^2.
\]

Using the fact that the atom number fluctuations around \( \tilde{N} \) are small, one can expand to lowest order in \( \delta N(0) = \tilde{N} - \bar{N} \). Then the expression inside the parenthesis becomes \((1 - K^2\gamma\Delta t \bar{N}^{K-1})\delta N(0)\); the square of that expression is \((1 - 2K^2\gamma\Delta t \bar{N}^{K-1})\delta N(0)^2\) at first order in \( \gamma\Delta t \bar{N}^{K-1} \). Thus we obtain

\[
\langle \delta N(\Delta t)^2 \rangle = K^2\gamma\Delta t \bar{N}^K + (1 - 2K^2\gamma\Delta t \bar{N}^{K-1}) \langle \delta N^2 \rangle. \tag{D16}
\]

This lead to the differential form

\[
\frac{d\langle \delta N^2 \rangle}{dt} = K^2Gn^K\ell - 2K^2Gn^{K-1}\langle \delta N^2 \rangle, \tag{D17}
\]

where we have used \( \gamma\bar{N}^{K-1} = Gn^{K-1} \).

Let us now consider two different cells located around \( z_\alpha \) and \( z_\beta \). For given atom numbers \( N_\alpha \) and \( N_\beta \) in the cell located in \( z_\alpha \) and \( z_\beta \), respectively, the fluctuations of the number of loss events in both cells are not correlated. Then similar calculations as above give

\[
\frac{d\langle \delta N_\alpha \delta N_\beta \rangle}{dt} = -2K^2\gamma\Delta t \bar{N}^{K-1}\langle \delta N_\alpha \delta N_\beta \rangle. \tag{D18}
\]

Eq. (D17) and (D18) imply that the evolution of the fluctuations of the density field \( \delta n(z) \simeq \delta N/\ell \) (for a cell around at position \( z \)) is given by Eq. (D8) as claimed.

### 3. Evolution of the momentum distribution

Although losses do not depend on the phase variable, losses do have an impact on the phase fluctuations \( \langle \theta(z)^2 \rangle \). This is due to the broadening of the phase as one gains knowledge on the atom number \( N \), its conjugate variable. This ensures the preservation of quantum uncertainty relations. Losses increase our knowledge of \( N \) because if one records the losses, then one gains knowledge on \( N \). This effect can be exploited in a feedback scheme to cool down the Bogoliubov modes \([58]\). The quantitative evaluation of this effect is done in Ref. [29], and the result reads:

\[
\frac{d\langle \theta(z)\theta(z') \rangle}{dt} = \frac{1}{4}K^2Gn^{K-2}\delta(z - z'). \tag{D19}
\]

This is the equation used in the main text. We point out that Eq. (D19), as well as Eq. (D8), can also be derived from stochastic equations, see Ref. [29].

![FIG. 3. Momentum distribution of a quasicondensate submitted to one-body losses of rate \( G \). The initial state is a thermal state at a linear density \( n_0 = 10/\sqrt{mgn_0}/h \) and at a temperature \( T = gn_0 \). Its momentum distribution is shown as the blue solid line. The dashed blue line is \( C_{c,0}/p^4 \), where \( C_{c,0} = (mn_0g^2)/(2\pi) \) is the initial contact density \( (g^2(0)) \simeq 1 \) in the quasicondensate regime. The red solid line is the momentum distribution after a time \( t = 1/G \). The dashed red line is \( C(t)/p^4 \), where \( C(t) = e^{Gt}(mn_0g^2)/(2\pi) \).

We performed numerical calculations for one-body losses \( (K = 1) \), starting from a thermal state with linear density \( n_0 \) and temperature \( T \). For this purpose one needs to compute the time evolution of the Bogoliubov mode’s populations \( \{\alpha_q\} \). As written in the main text, taking the Fourier transform of Eq. (D8) and Eq. (D19) and injecting into Eq. (D7) we find

\[
d\alpha_q/dt = K^2Gn^{K-1}(-\alpha_q - 1/2 + 1/4(f_q + f_q^{-1})). \tag{D20}
\]
This equation, together with the equation $n = n_0 e^{-Gt}$, allows to compute $\alpha(t)$. We then compute the first order correlation function using Eq. (D1). We finally take its Fourier transform to extract the momentum distribution $w(p)$. Fig. 3 shows resulting momentum distributions, in log-log scale, at time $t = 0$ and at time $t = 1/\Gamma$. We see that, for those parameters, the $1/p^4$ behavior appears for momenta larger than $\approx 3\sqrt{m\Gamma n_0}$. The amplitude of the tails is in agreement with the analytic prediction $C(t) = e^{Gt}(mn(t)g)^2/(2\pi)$.

4. Solution of the differential equation (13) for losses in the quasicondensate regime

We use the dimensionless variable $\tau = Kn_0^{-1}Gt$, where $n_0$ is the atom density at $t = 0$. In the quasicondensate regime, we have $g^{(K)}(0) = 1$, so the atom density $n(\tau)$ evolves according to

$$d(n/n_0)/d\tau = -(n/n_0)^K.$$  \hspace{1cm} (D21)

The differential equation (13) in the main text is

$$dC_t/d\tau = -KC_t - KCC_0(n/n_0)K+1,$$  \hspace{1cm} (D22)

with $C_{c,0} = m^2g^2n_0^2/(2\pi\hbar)$. Using Eq. (D21) one can easily check that the solutions of that differential equation are (for $K \neq 2$)

$$C_t(\tau) = \frac{KC_{c,0}}{K-2}(n/n_0)^2 + A(n/n_0)^K,$$  \hspace{1cm} (D23)

for any constant $A$. The constant $A$ is then fixed in terms of the initial condition $C_t(t = 0) = 0$ (this is the initial condition assumed in the main text). This gives (for $K \neq 2$):

$$C_t(\tau) = \frac{KC_{c,0}}{K-2}(n/n_0)^2 \left[1 - (n/n_0)^K\right].$$  \hspace{1cm} (D24)

If $K = 2$, then we have instead

$$(K = 2)\quad C_t(\tau) = -2C_{c,0}(n/n_0)^2 \log(n/n_0).$$  \hspace{1cm} (D25)

Recall that $C_c(\tau) = m^2g^2n(\tau)^2/(2\pi\hbar)$. Then we get

$$C_t(\tau)/C_c(\tau) = \begin{cases} \frac{K}{(K-2)} \left[1 - (n/n_0)^K\right] & \text{if } K \neq 2, \\ -2 \log(n/n_0) & \text{if } K = 2. \end{cases}$$  \hspace{1cm} (D26)

Finally, we note that the solution of Eq. (D21) is

$$n(\tau)/n_0 = \begin{cases} [1 + (K-1)\tau]^{1/(1-K)} & \text{if } K > 1, \\ e^{-\tau} & \text{if } K = 1. \end{cases}$$  \hspace{1cm} (D27)

Eqs. (D26) and (D27) give the large $\tau$ behavior reported in Eq. (14) in the main text.

Appendix E: Generalization to non-uniform gases

In most experimental situations, gases are confined into a slowly-varying longitudinal potential, often of quadratic form. The confinement is however usually weak enough to ensure the validity of the Generalized Hydrodynamics approach \cite{42,60} (which corresponds, in the case of stationary states, to the well known Local Density Approximation). The rapidity distribution then becomes a two dimensional function $\rho(q, z)$, where, for a given $z$, $\rho(q, z)$ is the local rapidity distribution. The coefficient $C_r = \lim_{r \to \infty} q^4 \rho(q)$ becomes $z$-dependent and we note it $C_r(z)$. Moreover we introduce the extensive quantity $W(p) = \int dz w(p, z)$, where $w(p, z)$ is the local momentum distribution, and $C = \lim_{p \to \infty} p^4 W(p)$. $W(p)$ is normalized to $\int dp W(p) = N$ where $N$ is the total atom number. Eq. (4) of the main text then becomes

$$C = \int dz (C_c(z) + C_r(z))$$  \hspace{1cm} (E1)

where $C_c(z) = m^2g^2n(z)^2g^{(2)}(0, z)/(2\pi\hbar)$ is the local contact density. Here $g^{(2)}(0, z) = \langle \hat{\psi}^+(z)\hat{\psi}^+(z)\hat{\psi}(z)\hat{\psi}(z) \rangle$ is the zero-distance two-body correlation function, computed at position $z$. For a given $z$, $C_c(z)$ is a functional of $\rho(p, z)$, see Eq. (B2). Thus $C$ can be computed once the function $\rho(p, z)$ is known.

As losses occur, $\rho(p, z)$ is locally modified by losses. The system is then, in general, brought to a non-stationary solution of the Generalized Hydrodynamics equations and one should compute the time-evolution of $\rho(p, z)$ using Eq. (16) of Ref. \cite{28}.

\hspace{1cm}