REDUCED INVARIANT SETS

GERALD W. SCHWARZ

In honor of Dick Palais

Abstract. Let $K$ be a compact Lie group and $W$ a finite-dimensional real $K$-module. Let $X$ be a $K$-stable real algebraic subset of $W$. Let $\mathcal{I}(X)$ denote the ideal of $X$ in $\mathbb{R}[W]$ and let $\mathcal{I}_K(X)$ be the ideal generated by $\mathcal{I}(X)^K$. We find necessary conditions and sufficient conditions for $\mathcal{I}(X) = \mathcal{I}_K(X)$ and for $\sqrt{\mathcal{I}_K(X)} = \mathcal{I}(X)$. We consider analogous questions for actions of complex reductive groups.

1. Introduction

Let $K$ be a compact Lie group, let $W$ be a finite-dimensional real $K$-module and let $X \subset W$ be $K$-invariant and real algebraic (the zero set of real polynomial functions on $W$). Let $\mathcal{I}(X)$ denote the ideal of $X$ in $\mathbb{R}[W]$. Let $\mathbb{R}[W]^K$ denote the $K$-invariants in $\mathbb{R}[W]$ and let $\mathcal{I}_K(X)$ be the ideal generated by $\mathcal{I}(X)^K := \mathcal{I}(X) \cap \mathbb{R}[W]^K$. We say that $X$ is $K$-reduced if $\mathcal{I}_K(X) = \mathcal{I}(X)$ and almost $K$-reduced if $\sqrt{\mathcal{I}_K(X)} = \mathcal{I}(X)$. Let $Kw$ be an orbit in $W$. Then the slice representation at $w$ is the action of the isotropy group $K_w$ on $N_w$, where $N_w$ is a $K_w$-complement to $T_w(Kw)$ in $W \cong T_w(W)$. An orbit $Kw \subset W$ is principal (resp. almost principal) if the image of $K_w$ in $\text{GL}(N_w)$ is trivial (resp. finite). We denote the principal (resp. almost principal) points of $W$ by $W_{\text{pr}}$ (resp. $W_{\text{apr}}$) and we set $X_{\text{pr}} := W_{\text{pr}} \cap X$ and $X_{\text{apr}} := W_{\text{apr}} \cap X$. The strata of $W$ are the collections of points $S \subset W$ whose isotropy groups are conjugate. There are finitely many strata. If $\mathbb{R}[W]$ is a free $R[W]^K$-module, then we say that $W$ is cofree. In the following, when we talk about one set being dense in another, we are referring to the Zariski topology.

Here are our main results:

Theorem 1.1. If $X$ is $K$-reduced (resp. almost $K$-reduced), then $X_{\text{pr}}$ (resp. $X_{\text{apr}}$) is dense in $X$, and conversely if $W$ is cofree.

Theorem 1.2. Let $w \in W$. Then the orbit $Kw$ is $K$-reduced (resp. almost $K$-reduced) if and only if $Kw$ is principal (resp. almost principal).

To prove the results above and to obtain further results we need to complexify. Let $V = W \otimes_{\mathbb{R}} \mathbb{C}$ and $G = K_{\mathbb{C}}$ be the complexifications of $W$ and $K$. We have the quotient morphism $\pi: V \to V//G$ where $\pi$ is surjective, $V//G$ is an affine variety and $\pi^*\mathbb{C}[V//G] = \mathbb{C}[V]^G$. We have the Luna strata of the quotient $V//G$ whose inverse images in $V$ are the strata of $V$. The strata of $V$ are in 1-1 correspondence with those of $W$ [Sch80, §5]. Let $Y = X_{\mathbb{C}}$ be the complexification of $X$ (the Zariski closure of $X$ in $V$). We say that $Y$ is $G$-saturated if $Y = \pi^{-1}(\pi(Y))$ and that $Y$ is $G$-reduced if the ideal $\mathcal{I}(Y)$ of $Y$ is generated by $\mathcal{I}(Y)^G$. We can define $Y_{\text{apr}}$ and $Y_{\text{pr}}$ as above (see §3). If $f_1, \ldots, f_k$ are functions on a complex variety, let $\mathcal{I}(f_1, \ldots, f_k)$ denote the ideal they generate.

Theorem 1.3. (1) $X$ is almost $K$-reduced if and only if $Y$ is $G$-saturated.
(2) $X$ is $K$-reduced if and only if $Y$ is $G$-reduced.
(3) $X_{\text{apr}}$ (resp. $X_{\text{pr}}$) is dense in $X$ if and only if $Y_{\text{apr}}$ (resp. $Y_{\text{pr}}$) is dense in $Y$.
Theorem 1.4. Assume that $Y \parallel G \subset V \parallel G$ is the zero set of $f_1, \ldots, f_k$.

1. Suppose that $Y_{apr}$ is dense in $Y$ and that for any stratum $S$ of $V$ which intersects $Y \setminus Y_{apr}$ the codimension of $S$ in $V$ is at least $k + 1$. Then $Y$ is $G$-saturated.

2. Suppose that $Y_{pr}$ is dense in $Y$ and that $Y$ is $G$-saturated. In addition, suppose that $\mathcal{I}(\pi(Y)) = \mathcal{I}(f_1, \ldots, f_k)$ where $Y$ has codimension $k$ in $V$. Then $Y$ is $G$-reduced.

Corollary 1.5. If (1) above holds, then $X$ is almost $K$-reduced. If (2) holds, then $X$ is $K$-reduced.

In sections 2–4 we consider when a general $G$-invariant $Y \subset V$ is $G$-saturated or $G$-reduced and we establish Theorem 1.4. In section 5 we treat the real case by complexifying. At the end of section 5 we establish Theorems 1.1, 1.2 and 1.3.

D. Ž. Djoković posed the question of identifying the $X$ which are $K$-reduced. Our results give a partial answer. We thank M. Raïss for transmitting the question to us. We thank the referee for a careful reading of the manuscript, helpful suggestions and Lemma 3.3.

2. The complex case

Let $G$ be a complex reductive group and $Y$ an affine algebraic set with an algebraic $G$-action. Dual to the inclusion $\mathbb{C}[Y]^G \subset \mathbb{C}[Y]$ we have the quotient morphism $\pi_Y : Y \to Y \parallel G$. Let $V$ be a finite-dimensional $G$-module and let $Y$ be a $G$-stable algebraic subset of $V$ (the zero set of an ideal of $\mathbb{C}[V]$). We shall denote $\pi_Y$ simply by $\pi$. Then $\pi_Y = \pi|_Y$ and $\pi(Y) \simeq Y \parallel G$ is an algebraic subset of $V \parallel G$. We say that $Y$ is $G$-saturated if $Y = \pi^{-1}(\pi(Y))$. Let $\mathcal{I}(Y)$ denote the ideal of $Y$ in $\mathbb{C}[V]$ and let $\mathcal{I}_G(Y)$ denote the ideal generated by $\mathcal{I}(Y)^G$. We say that $Y$ is $G$-reduced if $\mathcal{I}(Y) = \mathcal{I}_G(Y)$. The null cone $\mathcal{N}(V)$ of $V$ is the fiber $\pi^{-1}(\pi(0))$. Then $\mathcal{N}(V)$ is (scheme theoretically) defined by the ideal $\mathcal{I}_G(\{0\})$ so that the scheme $\mathcal{N}(V)$ is reduced if and only if the set $\mathcal{N}(V)$ is $G$-reduced, in which case we say that $V$ is coreduced. See [KS11] for more on coreduced representations.

The points of $V \parallel G$ are in one-to-one correspondence with the closed $G$-orbits in $V$. The Luna strata of $V \parallel G$ are the sets of closed orbits whose isotropy groups are all $G$-conjugate. There are finitely many strata in $V \parallel G$, and we consider their inverse images in $V$ to be the strata of $V$. Let $v \in V$ such that $Gv$ is closed. Then the isotropy group $G_v$ is reductive, and there is a $G_v$-stable complement $N_v$ to $T_v(Gv)$ in $V \simeq T_v(V)$. We call the action of $G_v$ on $N_v$ the slice representation at $v$.

We start with some examples.

Example 2.1. Let $(V, G) = (k\mathbb{C}^n, \text{SL}_n)$, $k \geq n$. The invariants are generated by the determinants $\det_{i_1, \ldots, i_n}$ where the indices $1 \leq i_1 < \cdots < i_n \leq k$ tell us which $n$ copies of $\mathbb{C}^n$ to take. Then $V$ is coreduced since $\mathcal{N}(V)$ is the determinantal variety of $(k \times n)$-matrices of rank at most $n - 1$. See also [KS11]. All orbits outside the null cone are closed with trivial isotropy group, hence are principal.

Example 2.2. Let $G \subset \text{GL}(V)$ be finite and nontrivial. Then $\mathcal{N}(V)$ is the origin which is $G$-saturated but not $G$-reduced.

Part (2) of the proposition below follows from Serre’s criterion for reducedness [Mat80, Ch. 7]. Part (1) also follows, using the Jacobian criterion for smoothness.

Proposition 2.3. Let $Y \subset V$ be a $G$-saturated algebraic set.

1. If $Y$ is $G$-reduced, then for every irreducible component $Y_k$ of $Y$ there is a point of $Y_k$ where rank $f = \text{codim } Y_k$. Here $f = (f_1, \ldots, f_d) : V \to \mathbb{C}^d$ and the $f_i$ generate $\mathcal{I}_G(Y)$.

2. If $\mathcal{I}_G(Y) = \mathcal{I}(f_1, \ldots, f_d)$ where the $f_i \in \mathbb{C}[V]^G$ and $Y$ has codimension $d$, then $Y$ is $G$-reduced if and only if the rank condition of (1) is satisfied.
Example 2.4. Let $G = \SO_3(\mathbb{C})$ acting as usual on $V = 2\mathbb{C}^3$. Then the invariants are generated by inner products $f_{ij}, 1 \leq i \leq j \leq 2$. Each copy of $\mathbb{C}^3$ has a weight basis $\{v_2, v_0, v_{-2}\}$ relative to the action of the maximal torus $T = \mathbb{C}^*$ where $v_j$ has weight $j$. The null cone $Y := \mathcal{N}(V)$ is the $G$-orbit of all the vectors $v = (\alpha v_2, \beta v_2)$ for $\alpha, \beta \in \mathbb{C}$. But one easily calculates that the rank of $(f_{11}, f_{22}, f_{12}) : V \to \mathbb{C}^3$ at $v$ is at most 2 while $Y$ has codimension 3. Thus the null cone is not $G$-reduced.

3. The case where $Y_{pr}$ or $Y_{apr}$ is dense in $Y$

Throughout this section we assume that $V$ is a stable representation of $G$, i.e., there is a nonempty open subset of closed orbits. This is always the case when $(V,G) = (W_{\mathbb{C}}, K_{\mathbb{C}})$ is a complexification ([Lun72] or [Sch80, Cor. 5.9]). Let $Gv$ be a closed orbit. We say that $Gv$ is principal if the slice representation $(N_{v}, G_{v})$ is a trivial representation and that $Gv$ is almost principal if $G_{v} \to \GL(N_{v})$ has finite image. We denote the principal (resp. almost principal) points of $V$ by $V_{pr}$ (resp. $V_{apr}$). If $Y \subset V$ is $G$-stable, we set $Y_{pr} = Y \cap V_{pr}$ and $Y_{apr} = Y \cap V_{apr}$. Both $Y_{apr}$ and $Y_{pr}$ are open in $Y$. In general, the fiber of $\pi$ through a closed orbit $Gv \subset V$ is $G \times^{G_{v}} \mathcal{N}(N_{v})$ (the $G$-fiber bundle with fiber $\mathcal{N}(N_{v})$ associated to the $G_{v}$-principal bundle $G \to G/G_{v}$). Thus the fiber is set-theoretically the orbit if and only $\mathcal{N}(N_{v})$ is a point. This happens if and only if the image $G_{v} \to \GL(N_{v})$ is finite, i.e., $v \in V_{apr}$. Hence $Y_{apr}$ is always $G$-saturated. Similarly, the fiber is scheme-theoretically the orbit if and only if $\mathcal{N}(N_{v})$ is schematically a point which is equivalent to $G_{v}$ acting trivially on $N_{v}$, i.e., we have $v \in V_{pr}$. Hence $Y_{pr}$ is always $G$-reduced. To sum up we have

Proposition 3.1. Let $Gv$ be a closed orbit and let $Y \subset V$ be a $G$-stable algebraic set.

1. If $Y = Y_{apr}$, then $Y$ is $G$-saturated. In particular, $Gv$ is $G$-saturated if and only if it is almost principal.
2. If $Y = Y_{pr}$, then $Y$ is $G$-reduced. In particular, $Gv$ is $G$-reduced if and only if it is principal.
3. The fiber $\pi^{-1}(\pi(v))$ is $G$-reduced if and only if the slice representation $(N_{v}, G_{v})$ is coreduced.

Corollary 3.2. If the isotropy groups of $G$ acting on $Y$ are all finite, then $Y$ is $G$-saturated and if $G$ acts freely on $Y$, then $Y$ is $G$-reduced.

Of course, it is possible that $Y$ is $G$-saturated (resp. $G$-reduced) even if $Y_{apr}$ (resp. $Y_{pr}$) is empty. But in the case of a complexification $Y = X_{\mathbb{C}}$ it is necessary for $G$-saturation (resp. $G$-reducedness) that $Y_{apr}$ (resp. $Y_{pr}$) is dense in $Y$ (see §5). We consider the case that $Y_{apr}$ or $Y_{pr}$ is not dense in $Y$ in the next section.

Unfortunately, we do not have the analogues of Proposition 3.1(1) and (2) for $X$. See Example 5.3 below.

Recall that $V$ is cofree if $\mathbb{C}[V]$ is a free $\mathbb{C}[V]^{G}$-module. Equivalently, $\pi : V \to V/\!/G$ is flat, or $\mathbb{C}[V]^{G}$ is a regular ring and the codimension of $\mathcal{N}(V)$ is $\dim \mathbb{C}[V]^{G}$ [Sch80, Proposition 17.29].

We owe the following lemma to the referee.

Lemma 3.3. Let $V$ be a cofree $G$-module and let $U \subset V/\!/G$ be locally closed.

1. We have $\pi^{-1}(\overline{U}) = \pi^{-1}(\overline{U})$.
2. If $\pi^{-1}(U)$ is reduced, then so is $\pi^{-1}(\overline{U})$.

Proof. For (1) set $Z := \pi^{-1}(\overline{U})$. Then $\pi(Z)$ is closed [Kra84, II.3.2], hence $\pi(Z) = \overline{U}$. Since $\pi$ is flat, so is $\pi^{-1}(\overline{U}) \to \overline{U}$. Set $S := \pi^{-1}(\overline{U}) \setminus Z$. Then $S$ is open, hence $\pi(S)$ is open in $\overline{U}$ (by flatness). By construction, $\pi(S)$ does not meet $U$, hence we must have $S = \emptyset$, establishing (1).
For (2) we can assume that $U = \overline{U} = \{u \in \overline{U} \mid f(u) \neq 0\}$ for $f \in C[U]$. Set $Z := \pi^{-1}(\overline{U})$, the schematic inverse image of $\overline{U}$. Since $C[U] \rightarrow C[\overline{U}] = C[U]_{f}$ is injective and $C[Z]$ is flat over $C[U]$, it follows that $C[Z] \rightarrow C[Z]_{f} = C[\pi^{-1}(U)]$ is also injective. Since the latter ring is reduced, so is $C[Z]$ and we have established (2). \hfill \square

**Corollary 3.4.** Suppose that $(V,G)$ is cofree and that $Y \subset V$ is a $G$-stable algebraic set such that $Y_{\text{apr}}$ is dense in $Y$. Then $Y$ is $G$-saturated.

**Example 3.5.** Let $(V,G) = (4C^{2}, SL_{2})$ and let $Y = 2C^{2} \times \{0\}$. Then $Y_{\text{pr}} = Y_{\text{apr}}$ is dense in $Y$ (it is the set of linearly independent vectors in $Y$) but $Y$ is not $G$-saturated since it does not contain the null cone. The $G$-module $V$ is not cofree, so we don’t contradict Corollary 3.4. Note that this example is the complexification of the case where $X = C^{2} \times \{0\} \subset W := C^{2} \oplus C^{2}$ and $K = SU(2, \mathbb{C})$. Thus $X_{\text{pr}} = X_{\text{apr}}$ is dense in $X$ but $X$ is not almost $K$-reduced. (We use Theorem 1.3.) This shows that cofreeness is also necessary in Theorem 1.1.

**Theorem 3.6.** Suppose that $Y \subset V$ is $G$-stable such that

1. $Y_{\text{apr}}$ is dense in $Y$.
2. $Y/\![G \subset V/\!G$ is the zero set of $f_{1}, \ldots, f_{k}$ where the minimal codimension of a non almost principal stratum of $V$ which intersects $Y$ is at least $k + 1$.

Then $Y$ is $G$-saturated.

**Proof.** Let $\tilde{Y}$ denote $\pi^{-1}(\pi(Y))$. Then each irreducible component of $\tilde{Y}$ has codimension less than or equal to $k$. Let $S$ be a non almost principal stratum of $V$ which intersects $Y$. Then $S \cap \tilde{Y}$ is nowhere dense in $\tilde{Y}$. Thus $Y_{\text{apr}}$ is dense in $\tilde{Y}$. Now $\tilde{Y}_{\text{apr}}$ and $Y_{\text{apr}}$ have the same image in $Y/\!G$. Hence $Y_{\text{apr}} = \tilde{Y}_{\text{apr}}$ and $Y = \tilde{Y}$ is saturated. \hfill \square

**Example 3.7.** Let $(V,G) = (kC^{2}, SL_{2})$, $k \geq 2$. The codimension of the null cone is $k - 1$ and the subset $Y$ where the first copy of $C^{2}$ is zero is not saturated, but corresponds to the subset of $V/\!G$ where the determinant invariants $\det_{12}, \ldots, \det_{1k}$ vanish (see Example 2.1). Thus the codimension condition in Theorem 3.6(2) is sharp.

Here is an example that is a complexification.

**Example 3.8.** Let $(V,G) = (2C^{2}, SO_{2}(\mathbb{C}))$ and let $Y = C^{2} \times \{0\} \cup \{0\} \times C^{2}$. Then $Y_{\text{apr}}$ is dense in $Y$ since any point not in $N(V)$ is on a principal orbit and $N(V)$ is nowhere dense in $Y$. However, $Y$ is not $G$-saturated since it does not contain $N(V)$. Note that $\mathcal{I}(Y/\!G)$ is generated by $\text{det}$ (the determinant), $f_{12}$ and $f_{11}f_{22}$ where the $f_{ij}$ are the inner product invariants. Since $\det^{2} = f_{11}f_{22} - f_{12}^{2}$, $\mathcal{I}(Y/\!G)$ is the radical of the ideal generated by $f_{12}$ and $f_{11}f_{22}$. The null cone has codimension 2. Again this shows that the codimension condition in Theorem 3.6 is sharp.

We now have the following corollary of Lemma 3.3

**Corollary 3.9.** Suppose that $(V,G)$ is cofree and that $Y \subset V$ is $G$-stable such that $Y_{\text{pr}}$ is dense in $Y$. Then $Y$ is $G$-reduced.

**Remark 3.10.** For $Y$ to be $G$-reduced, it is not sufficient that every slice representation of $V$ is coreduced. (This is the same as saying that every fiber of $\pi: V \rightarrow V/\!G$ is reduced.) Just consider Example 3.5 again. Here $Y_{\text{pr}}$ is dense in $Y$ but $Y$ is not $G$-saturated, let alone $G$-reduced.

**Theorem 3.11.** Let $V$ be a $G$-module and let $Y \subset V$ be $G$-saturated such that $Y_{\text{pr}}$ is dense in $Y$. Suppose that $\pi(Y) \subset V/\!G$ is the zero set of $f_{1}, \ldots, f_{k}$ where the codimension of $Y$ is $k$. Then $Y$ is $G$-reduced.
Proof. The rank of the differential of \( f = (f_1, \ldots, f_d) : V \to \mathbb{C}^d \) is maximal at a point of each irreducible component of \( Y \) since \( Y \) is reduced at all points of \( Y_{pr} \). Thus we can apply Serre’s criterion (Proposition 2.3).

Example 3.12. Let \((V, G) = (4\mathbb{C}^2, \text{SL}_2(\mathbb{C}))\) and let \( Y \) be the zero set of two of the determinant invariants \( \det_{ij} \). Then \( Y_{pr} \) is dense in \( Y \) since the only non-principal stratum is \( \mathcal{N}(V) \) which has codimension 3 while \( Y \) has codimension 2. By Theorem 3.11, \( Y \) is \( G \)-reduced.

4. The case where \( Y_{pr} \) or \( Y_{apr} \) is not dense in \( Y \)

We can say something in the case that \( Y_{apr} \) or \( Y_{pr} \) is not dense in \( Y \). We are certainly in this case if \( V \) is not stable, since then \( V_{pr} \) and \( V_{apr} \) are empty. Let \( v \in Y \) such that \( Gv \) is closed. Let \((N_v, G_v)\) be the slice representation and \( S \) the corresponding stratum of \( V \). We say that \((N_v, G_v)\) is a generic slice representation for \( Y \) if \( S \cap Y \) is dense in an irreducible component of \( Y \). We also say that \( S \) is generic for \( Y \).

Proposition 4.1. Let \((N_v, G_v)\) be a generic slice representation of \( Y \) corresponding to the stratum \( S \) of \( V \). If \( Y \) is \( G \)-saturated, then \( Y \cap S = \pi^{-1}(\pi(Y \cap S)) \). If \( Y \) is \( G \)-reduced, then \((N_v, G_v)\) is coreduced.

Proof. If \( Y \) is \( G \)-saturated, then we obviously must have that \( Y \cap S = \pi^{-1}(\pi(Y \cap S)) \). Let \( Z \) denote \( \pi(S) \). Then \( \pi^{-1}(Z) \to Z \) is a fiber bundle with fiber \( G \times^{G_v} \mathcal{N}(N_v) \). If \( Y \) is \( G \)-reduced, then the bundle is reduced, hence \((N_v, G_v)\) is coreduced.

Let \( S \) be a stratum of \( V \). We say that \( Y \) is \( S \)-saturated if \( Y \cap S = \pi^{-1}(\pi(Y \cap S)) \). We say that \( Y \) is \( S \)-reduced if \( Y \) is \( S \)-saturated and the slice representation \((N_v, G_v)\) associated to \( S \) is coreduced. Corresponding to Corollaries 3.4 and 3.9 and Theorems 3.6 and 3.11 we have the following result whose proof we leave to the reader.

Theorem 4.2. Let \( Y \subset V \) be a \( G \)-stable algebraic set.

1. If \( V \) is cofree and \( Y \) is \( S \)-saturated for every stratum \( S \) which is generic for \( Y \), then \( Y \) is \( G \)-saturated.
2. If \( V \) is cofree and \( Y \) is \( S \)-reduced for every stratum \( S \) which is generic for \( Y \), then \( Y \) is \( G \)-reduced.
3. Suppose that \( Y \) is \( S \)-saturated for every every generic stratum \( S \) of \( Y \). Further assume that the minimal codimension of the strata of \( V \) which intersect \( Y \) but are not generic for \( Y \) is greater than \( k \) and that \( Y/G \) is the zero set of \( f_1, \ldots, f_k \). Then \( Y \) is \( G \)-saturated.
4. Suppose that \( Y \) is \( G \)-saturated and that the ideal of \( \pi(Y) \subset V/G \) is generated by \( f_1, \ldots, f_k \) where the codimension of \( Y \) in \( V \) is \( k \). Also assume that \( Y \) is \( S \)-reduced for every generic stratum \( S \) of \( Y \). Then \( Y \) is \( G \)-reduced.

5. The real case

Let \( W \) be a real \( K \)-module where \( K \) is compact. Let \( X \subset W \) be real algebraic and \( K \)-stable. Now \( K \) is naturally a real algebraic group and the action on \( W \) is real algebraic. Moreover, every orbit of \( K \) in \( W \) is a real algebraic set [Sch01]. Let \( Y := X_{\mathbb{C}} \) denote the complexification of \( X \) inside \( V := W \otimes_{\mathbb{R}} \mathbb{C} \) and let \( G \) denote the complexification \( K_{\mathbb{C}} \) of \( K \). Then \( G \) is reductive and \( V \) is a stable \( G \)-module ([Lun72] or [Sch80, Cor. 5.9]). We say that a slice representation \((N_w, K_w)\) is a generic slice representation for \( X \) if \( w \in X \) and the corresponding stratum contains a nonempty open subset of \( X \). Equivalently, the complexification of \((N_w, K_w)\) is generic for \( Y \).

Proposition 5.1. (1) \( X \) is almost \( K \)-reduced if and only if \( Y \) is \( G \)-saturated.
(2) \( X \) is \( K \)-reduced if and only if \( Y \) is \( G \)-reduced.
(3) The set $X_{apr}$ (resp. $X_{pr}$) is dense in $X$ if and only if the set $Y_{apr}$ (resp. $Y_{pr}$) is dense in $Y$.

(4) $X$ is almost $K$-reduced implies that $X_{apr}$ is dense in $X$.

(5) $X$ is $K$-reduced implies that $X_{pr}$ is dense in $X$.

Proof. The ideal of $Y$ is $I(X) \otimes_{\mathbb{R}} \mathbb{C} \subset \mathbb{C}[W] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[V]$ and $I_K(X) \otimes_{\mathbb{R}} \mathbb{C} = I_G(Y)$. Thus $I(Y) = I_G(Y)$ if and only if $I(X) = I_K(X)$, and $I(Y) = \sqrt{I_G(Y)}$ if and only if $I(X) = \sqrt{I_K(X)}$. Hence we have (1) and (2). For (3), note that $X_{apr}$ is open in $X$ and that $Y_{apr}$ is open in $Y$. If a stratum $S$ of $W$ is dense in an irreducible component of $X$, then the corresponding stratum $S_C$ of $V$ is dense in an irreducible component of $Y$. Thus if $X_{apr}$ is not dense in $X$, then $Y_{apr}$ is not dense in $Y$. Clearly, if $X_{apr}$ is dense in $X$, $Y_{apr}$ is dense in $Y$. The argument for $X_{pr}$ and $Y_{pr}$ is similar, hence we have (3). Now suppose that $X$ is almost $K$-reduced. Then for $S$ a generic stratum of $X$ and $x \in S \cap X$, the complexification $G x \simeq G/G_x$ of $K x$ is Zariski dense in the fiber $G \times^{G_x} \mathcal{N}(W_x \otimes_{\mathbb{R}} \mathbb{C})$ where $G_x = (K_x)_{\mathbb{C}}$. Thus $\mathcal{N}(W_x \otimes_{\mathbb{R}} \mathbb{C})$ is a point, i.e., the stratum consists of almost principal orbits. Hence we have (4), and (5) is proved similarly. □

Corollary 5.2. Let $X = K w$ be an orbit. Then $X$ is almost $K$-reduced if and only if $K w$ is almost principal and $X$ is $K$-reduced if and only if $K w$ is principal.

Unfortunately, it is not true that $X = X_{pr}$ (or $X = X_{apr}$) implies the same equality for $Y$.

Example 5.3. Let $K = \text{SU}_2(\mathbb{C})$ and $W = 2 \mathbb{C}^2 \oplus \mathbb{R}$ where $K$ acts as usual on the copies of $\mathbb{C}^2$ and trivially on $\mathbb{R}$. We consider $W$ to be $2 \mathbb{H} \oplus \mathbb{R}$ where $\mathbb{H}$ denotes the quaternions. Then $K \simeq S^3$, the unit quaternions, and the action on $2 \mathbb{H}$ is given by $k(p, q) = (kp, kq)$, $p, q \in \mathbb{H}$, $k \in S^3$. Let $p \mapsto \bar{p}$ denote the usual conjugation of quaternions. The invariants of $K$ acting on $2 \mathbb{H}$ are generated by $(p, q) \mapsto \bar{(pq)}$ where the first two invariants lie in $\mathbb{R}$ and the last in $\mathbb{H}$. Let $\alpha$ and $\beta$ denote the first two invariants and let $\gamma$ be the real part of $\bar{q} p$. Let $\delta$, $\epsilon$ and $\zeta$ be the invariants which are the $i$, $j$ and $k$ components of $\bar{q} p$, respectively. Then there are certainly points in $2 \mathbb{H}$ where $\delta$, $\epsilon$ and $\zeta$ vanish and where $\alpha = \beta = \gamma$ is any positive real number. Let $x$ be a coordinate on the copy of $W$ in $R$ and let $X$ be the subset of $W$ defined by $\delta = \epsilon = \zeta = 0, \alpha = \beta = \gamma$ and $(\alpha - 1)^2 + x^2 = 1/2$. Then $\alpha$ never vanishes on $X$ which implies that the isotropy group at the corresponding point of $W$ is trivial, so we have that $X = X_{apr}$. The quotient $X/K$ is a smooth curve, hence $X$ is smooth of dimension 4. The complexification $Y$ of $X$ also has dimension four and contains some of the points $(s, t, \pm \sqrt{-3/4})$ where $(s, t)$ lies in the null cone of $2 \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq 4 \mathbb{C}^2$ for the action of $K_{\mathbb{C}} \simeq \text{SL}_2(\mathbb{C})$. But this null cone has dimension 5. Hence $Y$ is not $G$-saturated, let alone $G$-reduced, and $Y \neq Y_{apr}$. Moreover, $X$ is neither $K$-reduced nor almost $K$-reduced.

Now we recover the theorems of the introduction. Theorem 1.2 is just Corollary 5.2. Theorem 1.3 is a consequence of Proposition 5.1 and Theorem 1.4 follows from Theorems 3.6 and 3.11.

Proof of Theorem 1.1. Suppose that $X$ is $K$-reduced. Then Proposition 5.1 shows that $X_{pr}$ is dense in $X$. Conversely, if $(W, K)$ is cofree (equivalently, $(V, G)$ is cofree) and $X_{pr}$ is dense in $X$, then $Y_{pr}$ is dense in $Y$ by Proposition 5.1 and $Y$ is $G$-reduced by Corollary 3.9. Hence $X$ is $K$-reduced. The proof in the almost $K$-reduced case is similar. □

References

[Kra84] Hanspeter Kraft, Geometrische Methoden in der Invariantentheorie, Aspects of Mathematics, D1, Friedr. Vieweg & Sohn, Braunschweig, 1984.

[KS11] Hanspeter Kraft and Gerald W. Schwarz, Reduced null cones, to appear.

[Lun72] Domingo Luna, Sur les orbites fermées des groupes algébriques réductifs, Invent. Math. 16 (1972), 1–5.

[Mat80] Hideyuki Matsumura, Commutative algebra, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.
[Sch80] Gerald W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 51, 37–135.

[Sch01] , *Algebraic quotients of compact group actions*, J. Algebra 244 (2001), no. 2, 365–378.

Department of Mathematics, Brandeis University, Waltham, MA 02454-9110

E-mail address: schwarz@brandeis.edu