Next-to-next-to-leading-order QCD corrections to $e^+e^- \rightarrow J/\psi + \eta_c$ at $B$ factories

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Within the nonrelativistic QCD (NRQCD) factorization framework, we compute the long-awaited $\mathcal{O}(\alpha_s^2)$ correction for the exclusive double charmonium production process at $B$ factories, i.e., $e^+e^- \rightarrow J/\psi + \eta_c$ at $\sqrt{s} = 10.58$ GeV. For the first time, we confirm that NRQCD factorization does hold at next-to-next-to-leading-order (NNLO) for exclusive double charmonium production. It is found that including the NNLO QCD correction considerably reduces the renormalization scale dependence, and also implies the reasonable perturbative convergence behavior for this process. Our state-of-the-art prediction is consistent with the BABAR measurement within errors.

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1. Introduction. Back in the beginning of this century, one particularly pressing dilemma of Standard Model is the severe discrepancy between the BELLE measurement [1] and the subsequent predictions [2, 3] for the exclusive double-charmonium production $e^+e^- \rightarrow J/\psi + \eta_c$ at $\sqrt{s} = 10.58$ GeV. This disquieting discrepancy has triggered a flurry of theoretical explorations in the following years. Although some explanations invoke certain exotic scenarios [5, 6], the consensus is that this puzzle is rooted in our inadequate knowledge about quarkonium production mechanism. The mainstream investigations from the first principles of QCD are based on the light-cone factorization [7, 10] and NRQCD factorization [11, 14]. Unfortunately, apart from poorly known light-cone distribution amplitudes of charmonia, some unsurmountable difficulty in the former approach, e.g., the endpoint singularity, renders a next-to-leading order (NLO) perturbative calculation to such a helicity-flipped exclusive process impossible [13]. In contrast, for the hard exclusive process $e^+e^- \rightarrow J/\psi + \eta_c$, the NRQCD approach [10] provides a more predictive framework that is amenable to systematically incorporating the higher-order perturbative and relativistic corrections.

One key progress in alleviating the tension is brought by the NLO perturbative calculation for $e^+e^- \rightarrow J/\psi + \eta_c$ in NRQCD approach, where a significant positive $\mathcal{O}(\alpha_s)$ correction is found [11, 12]. The relative $\mathcal{O}(v^2)$ correction to $e^+e^- \rightarrow J/\psi + \eta_c$ has also been addressed [2, 13, 14], where $v$ denotes the typical velocity of the $c$ quark inside a charmonium. Notwithstanding large uncertainty inherent to various NRQCD matrix elements, it was suggested that [13, 14], by including both $\mathcal{O}(\alpha_s)$ and (a partial re-summation of) relativistic corrections, one may largely resolve the discrepancy. Later the joint perturbative and relativistic order-$\alpha_s v^2$ correction was also investigated, which was found to modestly enhance the existing NRQCD predictions [17, 18].

The recently commissioning Belle II experiment will certainly conduct more precise measurement for this double quarkonium production channel. Therefore, it is desirable to have more accurate theoretical prediction available. Given the substantial $\mathcal{O}(\alpha_s)$ correction to the cross section, one cannot resist speculating whether the magnitude of the NNLO perturbative correction is abnormally large or not. Would the $\mathcal{O}(\alpha_s^2)$ correction for $e^+e^- \rightarrow J/\psi + \eta_c$ be as significant as the recently available NNLO perturbative corrections for $\gamma^*\gamma \rightarrow \eta_c$ [19] and $\eta_c \rightarrow$ light hadrons [20]? Undoubtedly, the NNLO correction for a $1 \rightarrow 4$ process involving massive quarks represents a cutting-edge challenge in the area of multi-loop calculation. To sense the daunting difficulty, we quote the authoritative review of quarkonium physics in 2011 [21]: “the calculation of . . . is perhaps beyond the current state of the art”. Notwithstanding enormous technical obstacles, in this paper we will report our endeavour in accomplishing this NNLO calculation.

2. NRQCD factorization for cross section. It is convenient to define the time-like electromagnetic (EM) form factor $F(s)$ through

$$
\langle J/\psi(P_1, \lambda) + \eta_c(P_2)|J_{EM}^\mu|0\rangle = i F(s) \epsilon^{\mu\nu\rho\sigma} P_1^\nu P_2^\rho \epsilon^*_\sigma(\lambda),
$$

where $J_{EM}^\mu$ is the quark EM current, and $s = (P_1 + P_2)^2$. The tensor structure specified in [11] is uniquely con-
strained by Lorentz and parity invariance. The outgoing $J/\psi$ must be transversely polarized, i.e., $\lambda = \pm 1$.

For hard exclusive reaction involving quarkonium, NRQCD factorization also holds at amplitude level. Specifically speaking, the EM form factor in $|f|^2$ can be expressed as

\[ F(s) = \sqrt{4M_{J/\psi}M_{\eta_c}(J/\psi|\psi^1\sigma \cdot \epsilon\chi|0)\langle\eta_c|\psi^1\chi|0\rangle} \times [f + g_{J/\psi}(v^2)_{J/\psi} + g_{\eta_c}(v^2)_{\eta_c} + \cdots], \tag{2} \]

where the perturbatively calculable effects are encoded in the short-distance coefficients (SDCs) $f$ and $g_{H} (H = J/\psi, \eta_c)$, and the long-distance effects encapsulated in the nonperturbative vacuum-to-charmonium matrix elements, which are often modeled by the phenomenological charmonium wave functions at the origin. Note by default the charmonium states in the NRQCD matrix elements, which are often modeled by the phenomenological charmonium wave functions at the origin. The prefactor in $f$ compensates the fact that these states are relativistically normalized in $\langle \eta_c^2 \rangle_{J/\psi}$ and $\langle \eta_c^2 \rangle_{\eta_c}$ are defined as the dimensionless ratios of two NRQCD matrix elements for $J/\psi$ and $\eta_c$.

Substituting (2) into (1), it is straightforward to deduce the cross section, which can be further divided into the $O(v^0)$ and $O(v^2)$ pieces:

\[ \sigma[e^+e^- \to J/\psi + \eta_c] = \frac{4\pi\alpha_e^2}{3} \left( \frac{|P|}{\sqrt{s}} \right)^3 |F(s)|^2 \]

\[ = \sigma_0 + \sigma_2 + O(\alpha_e v^4), \tag{3} \]

where $|P|$ signifies the magnitude of the three-momentum carried by the $J/\psi$ in the center-of-mass frame, and

\[ \sigma_0 = 8\pi\alpha_e^2 m^2(1 - 4r)^{3/2} \langle O_{J/\psi} \rangle_{J/\psi} \cdot \langle O_{\eta_c} \rangle_{\eta_c} |f|^2, \tag{4a} \]

\[ \sigma_2 = 4\pi\alpha_e^2 m^2(1 - 4r)^{3/2} \langle O_{J/\psi} \rangle_{J/\psi} \cdot \langle O_{\eta_c} \rangle_{\eta_c} \times \sum_{H = J/\psi, \eta_c} \left( \frac{1 - 10r}{1 - 4r} |f|^2 + 4 \text{Re}(fg^*_H) \right) (v^2)_H, \tag{4b} \]

where a dimensionless ratio

\[ r = 4m^2/s \]

is introduced for brevity. To condense the notation, we have also introduced the following symbols: $\langle O_{J/\psi} \rangle_{J/\psi} = |\langle J/\psi|\psi^1\sigma \cdot \epsilon\chi|0\rangle|^2$, and $\langle O_{\eta_c} \rangle_{\eta_c} = |\langle \eta_c|\psi^1\chi|0\rangle|^2$. In deriving (4a), we have employed the Gremm-Kapustin relation $M_H^2 \approx 4m^2(1 + (v^2)_H)$ to eliminate the explicit occurrence of the charmonium masses.

Thanks to the weaker strong coupling constant $\alpha_s$, at the scale of the charm quark Compton wavelength or shorter, the SDCs $f$ and $g_H$ are subject to perturbative expansion in $\alpha_s$:

\[ f = f^{(0)} + \frac{\alpha_s}{\pi} f^{(1)} + \frac{\alpha_s^2}{\pi^2} f^{(2)} + \cdots, \tag{5a} \]

\[ g_{H} = g_{H}^{(0)} + \frac{\alpha_s}{\pi} g_{H}^{(1)} + \cdots. \tag{5b} \]

Substituting (5) back to (4), we can organize $\sigma_0$ and $\sigma_2$ in perturbation series in $\alpha_s$. For example, using

\[ |f|^2 = |f^{(0)}|^2 + \frac{\alpha_s}{\pi} 2\text{Re} \left( f^{(0)} f^{(1)*} \right) \]

\[ + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 2\text{Re} \left( f^{(0)} f^{(2)*} \right) + |f^{(1)}|^2 \right], \tag{6} \]

we can decompose $\sigma_0 = \sigma_0^{(0)} + \sigma_0^{(\alpha_s)} + \sigma_0^{(\alpha_s^2)} + \cdots$.

The tree-level SDCs through $O(v^2)$ have been available long ago. Here we list their values:

\[ f^{(0)} = \frac{32\pi C_F c_{\alpha_s}}{N_c m^2}, \tag{7a} \]

\[ g_{J/\psi}^{(0)} = \frac{3 - 10r}{6} f^{(0)}, \quad g_{\eta_c}^{(0)} = \frac{2 - 5r}{3} f^{(0)}, \tag{7b} \]

where $c_{\alpha_s} = 2/3$ is the electric charge of the charm quark, $N_c = 3$ is the number of colors and $C_F = (N_c^2 - 1)/(2N_c)$.

The NLO perturbative corrections to those SDCs, for both $f$ and $g_{H}$, have also been known for a while. Since their analytic expressions are rather lengthy, here we just list the asymptotic expression for $f^{(1)}$ as $s \gg m^2$:

\[ f^{(1)} \approx f^{(0)} \left\{ \beta_0 \left[ -\frac{1}{4} \ln \frac{s}{4\mu_R^2} + \frac{5}{12} \right] + \frac{13}{24} \ln^2 r \right. \]

\[ + \frac{4}{5} \ln 2 \ln r - \frac{41}{24} \ln r - \frac{53}{24} \ln^2 2 + \frac{65}{8} \ln 2 - \frac{1}{36} \pi^2 \]

\[ \left. - \frac{19}{4} \right] + i \pi \left( \frac{1}{4} \beta_0 + \frac{13}{12} \ln r + \frac{5}{4} \ln 2 - \frac{41}{24} \right), \tag{8} \]

where $\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f$ is the one-loop coefficient of the QCD $\beta$ function, and $n_f = 4$ denotes the number of active quark flavors. $\mu_R$ denotes the renormalization scale, with the natural choice around $\sqrt{s}/2$. The double logarithm $\alpha_s \ln^2 r$ is believed to account for large positive NLO correction, which was first discovered in $[23]$, and carefully analyzed in $[24]$. The asymptotic expressions for $g_{J/\psi}^{(1)}$ and $g_{\eta_c}^{(1)}$ can be found in $[17]$.

**FIG. 1:** Representative diagrams for $\gamma^* \to c\bar{c}(3S_1^{(1)}) + c\bar{c}(1S_0^{(1)})$ through NNLO in $\alpha_s$.

3. Outline of calculation and main result. To compute $f^{(2)}$, we take the shortcut by directly calculating the quark-level amplitude for $\gamma^* \to c\bar{c}(3S_1^{(1)}, P_1) + c\bar{c}(1S_0^{(1)}, P_2)$. To LO accuracy in $v$, we neglect the relative momentum in each $c\bar{c}$ pair prior to carrying out the
loop integration, which amounts to directly extracting the NRQCD SDCs from the hard region. We work in $d = 4 - 2\epsilon$ spacetime dimensions to regularize both UV and IR divergences. About 2000 NNLO Feynman diagrams, as well as the corresponding amplitudes are generated by Qgraf/FeynArts. Some representative diagrams through NNLO are sampled in Fig. 1. Since the center-of-mass energy at $B$ factory exceeds twice bottom quark mass, we explicitly include the bottom loops in both regular and light-by-light diagrams. It is legitimate to drop those “light-by-light” diagrams in which $J_{\text{EM}}$ directly couples with the light quark, since $\sum_{q=u,d,s} e_q = 0$. The covariant projector technique is utilized to project each $\bar{c}c$ pair onto the intended quantum number. We then employ the packages FeynCalc/FormLink to conduct the trace over Dirac and SU(N) color matrices. After the integration-by-parts (IBP) reduction with the aid of Apart and FIRE, we end up with about 700 master integrals (MIs). Originally, we first tried to employ the sector decomposition (SD) method to evaluate these MIs numerically and find it extremely challenging to obtain reliable results within tolerable amount of time. Fortunately, a powerful new algorithm dubbed Auxiliary Mass Flow (AMF) has recently been developed by Liu and Ma. This algorithm is based on numerical differential equation method, which is tailored to tackle multi-scale multi-loop MIs with high numerical precision in a short time. We utilize the recently released package AMFlow to compute all the MIs.

To eliminate UV divergences, we employ the field-strength and mass renormalization, with two-loop expressions of $Z_2$ and $Z_m$ taken from and renormalize the strong coupling constant in the MS scheme to two-loop order. However, the renormalized NNLO QCD amplitude is found to still contain a single IR pole, yet with the coefficient exactly equal to the sum of the anomalous dimensions for the NRQCD bilinear operators carrying the quantum number of $J/\psi$ and $\eta_c$. This pattern is exactly what we anticipate for NRQCD factorization for double quarkonium production at NNLO. This IR pole can be factored into the corresponding NRQCD matrix elements under the MS scheme, which are actually scale-dependent quantities whose evolution is given by the following equation:

$$\frac{d}{d\ln \mu_A^2} \ln \frac{\langle \sigma \cdot \mathbf{c} \rangle}{\sigma} = - \left( \frac{\alpha_s}{\pi} \right)^2 \gamma_{J/\psi} + O \left( \alpha_s^3, \epsilon^2 \right),$$

$$\frac{d}{d\ln \mu_A^2} \ln \frac{\langle \eta_c \rangle}{\sigma} = - \left( \frac{\alpha_s}{\pi} \right)^2 \gamma_{\eta_c} + O \left( \alpha_s^3, \epsilon^2 \right),$$

with

$$\gamma_{J/\psi} = - \frac{\pi^2}{12} C_F \left( 2 C_F + 3 C_A \right),$$

$$\gamma_{\eta_c} = - \frac{\pi^2}{4} C_F \left( 2 C_F + C_A \right),$$

where $\mu_A$ is referred to as the NRQCD factorization scale, whose value lies somewhere between $m\bar{v}$ and $m$. Finally, the UV, IR-finite $O(\alpha_s^2)$ SDC reads

$$f^{(2)} = f^{(0)} \left\{ \frac{\beta_0^2}{16} \ln \frac{s}{4\mu_R^2} \left( \frac{\beta_1}{16} + \frac{1}{2} \beta_0 \hat{f}^{(1)}(1) \right) \ln \frac{s}{4\mu_R^2} 
\right. 
\left. + (\gamma_{J/\psi} + \gamma_{\eta_c}) \ln \frac{m^2}{\mu^2} + F(r) \right\},$$

(11)

Moreover, $\hat{f}^{(1)} = f^{(1)} / f^{(0)} |_{\mu = \sqrt{s}/2}$. $\beta_1 = \frac{34}{3} C_A - \frac{20}{3} C_A T_F n_f - 4 C_F T_F n_f$ is the two-loop coefficient of the QCD $\beta$ function. The occurrence of $\ln \mu_R$ is dictated by the renormalization group invariance.

The non-logarithmic term is embedded in the function $F(r)$ in (11). It is of our primary interest to ascertain this term as precise as possible, in order to pin down the impact of the NNLO perturbative correction.

4. Phenomenology. The production rate initially measured by BELLE is $\sigma(e^+ e^- \to J/\psi + \eta_c) \times B_{\geq 4} = 33.7 \pm 9$ fb [1], later shifted to $\sigma(J/\psi + \eta_c) \times B_{>2} = 25.6 \pm 2.8 \pm 3.4$ fb [40], where $B_{>2}$ denotes the branching fraction for the $\eta_c$ into $n$ charged tracks. An independent measurement by BABAR in 2005 yields $\sigma(J/\psi + \eta_c) \times B_{>2} = 17.6 \pm 2.8 \pm 1.5$ fb [11].

In the numerical analysis, We concentrate on the $B$ factories center-of-mass energy $\sqrt{s} = 10.58$ GeV, and take charm quark pole mass $m = 1.5$ GeV and bottom mass $m_b = 4.7$ GeV. The QED coupling constant $\alpha(\sqrt{s}) = 1/130.9$ [14], and the QCD running coupling constant is evaluated to two-loop accuracy with the aid of the package RunDec. The NRQCD matrix elements are taken from [14]: $\langle J/\psi | \lambda \rangle = 0.440$ GeV$^3$ and $\langle \eta_c | \lambda \rangle = 0.437$ GeV$^3$. For simplicity, we omit the relativistic corrections in phenomenological analysis.

In Fig. 2 we plot the dependence of the predicted cross section on $\mu_R$ and $\mu_A$, including numerous individual contribution from different perturbative order. We observe flatter $\mu_R$ dependence of the NNLO cross section.

FIG. 2: The cross section, predicted with various level of precision, as function of $\mu_R$. We take $m = 1.5$ GeV. The brown bands represent the uncertainty due to varying $\mu_A$ from $1$ GeV to $m$, where the lower bound corresponds to $\mu_A = 1$ GeV and upper bound $\mu_A = m$. 
In Table II we enumerate the individual contribution to the cross section at various levels of perturbative accuracy in NRQCD factorization.

From Table I we observe that the NNLO correction to this double charmonium production process is sizable, but not yet as substantial as the NLO correction (Note this is not the case in [19, 20]). It is reassuring that the perturbative expansion exhibits a convergent signature. Depending on the choice of $\mu_R$, the NNLO correction may range from 33% to 51%. Nevertheless, the total NNLO cross section possesses a milder dependence on the renormalization scale than the NLO prediction. We note that a recent study [14] based on principle of maximum conformality claims a favored setting of $\mu_R$ to be around 2 – 3 GeV, which can bring NLO NRQCD prediction consistent with the B factory measurements.

In Table I we also include the dependence of the double charmonium production cross section on charm quark mass. Table II indicates that the NLO and NNLO NRQCD predictions are quite sensitive to the charm pole mass, and smaller mass yields a prediction much closer to the data.

To sense the profile of $F(r)$, we list the values of this function at some benchmark energy points in Table I. The terms labeled with subscript “lbl” denote the contributions from the “light-by-light” diagrams, as illustrated by the representative diagram in Fig. I. The numerical difficulty to obtain accurate predictions increases enormously as $\sqrt{s}$ increases. Our results can be applied to predict the exclusive $J/\psi + \eta_c$ production at future very high energy $e^+e^-$ colliders such as Z factory, CEPC/FCC-ee, ILC, in which the exclusive double charmonium production rates would be too small to be observed. From the data of Table II we attempt to fit the coefficient of the anticipated endpoint logarithm $\alpha_s^2 \ln^2 r$. Pitifully, perhaps because the maximum value of $\sqrt{s}$ (500 GeV) is still not asymptotically high, we fail to determine this coefficient in an unambiguous manner.

5. Summary and outlook. More than one decade after the NLO correction became first available [11], we eventually accomplish the long-awaited calculation of the $O(\alpha_s^2)$ correction to $e^+e^- \rightarrow J/\psi + \eta_c$ at B factories. We verify that NRQCD factorization does hold at NNLO in $\alpha_s$ for exclusive double S-wave charmonium production. Including the NNLO QCD correction reduces the dependence on the renormalization scale, and exhibits reasonable perturbative convergence behavior. Our state-of-the-art prediction is compatible with the BABAR measurement, but still somewhat smaller than the BELLE measurement. The future remeasurement of this process at BELLE II experiment will be crucial to clarify the situation. The future work along this direction includes precisely deducing the endpoint logarithm $\propto \alpha_s^2 \ln^2 r$ in $F(r)$, and strive to resum these types of endpoint logarithms to all orders.

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TABLE I: Individual contributions to the predicted $\sigma[e^+e^- \rightarrow J/\psi+\eta_c]$ (in units of fb) at $\sqrt{s} = 10.58$ GeV. We take $\mu_R = \sqrt{s}/2$, and $\mu_\Lambda = 1$ GeV. The first error is obtained by varying $m_c$ from 1.3 to 1.7 GeV, and the second error is deduced by varying $\mu_R$ from $2m_c$ to $\sqrt{s}$.

| $m$(GeV) | $\mu_R$ | LO | NLO | NNLO |
|----------|---------|-----|-----|------|
| 1.5      | $\sqrt{s}/2$ | 5.05 $^{+0.92+2.31}_{-0.99-1.49}$ | 10.54 $^{+2.86+3.92}_{-2.60-2.66}$ | 15.00 $^{+5.03+4.29}_{-4.14-3.14}$ |

TABLE II: Values of $F(r)$ at some different center-of-mass energies with $m = 1.5$ GeV.

| $\sqrt{s}$(GeV) | $F(r)$ |
|-----------------|---------|
| 10.58           | $\left(25.300 - 19.883i\right) - \left(0.18765 + 0.01218i\right)$ |
| 91.2            | $\left(589.13 - 308.17i\right) - \left(0.012676 + 0.010692i\right)$ |
| 240             | $\left(1178.68 - 556.97i\right) - \left(0.00092905 + 0.0013069i\right)$ |
| 350             | $\left(1490.9 - 678.7i\right) - \left(0.0006519 - 0.0017494i\right)$ |
| 500             | $\left(1835.9 - 807.6i\right) - \left(0.0105894 - 0.0044144i\right)$ |

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