Submillimeter constraints for non-Newtonian gravity from spectroscopy

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Abstract

In this work, we consider the Yukawa-type and power-type non-Newtonian corrections, which induce amplification of gravitational interaction on submillimeter scales, and analytically calculate deviations produced by the atomic gravitational field on the energy levels of hydrogen-like ions. Analyzing ionic transitions between Rydberg states, we derive prospective constraints for non-Newtonian corrections. It is shown that the results also provide stronger constraints, due to the high accuracy for Rydberg transition measures into optical spectrum frequency range, than the current empirical bounds following from Casimir force measurements.

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I. INTRODUCTION

Several theoretical scenarios in different contexts have predicted possible deviations from Newtonian gravity at short-distance scales \[1\]. Their initial formulation was invoked for the fifth force theory, known for predicting a new long-range force that would influence the interactions between the particles up to a certain distance \[2\]. In this framework, the detection of corrections to Newton’s inverse square law of gravity, for instance, would imply the existence of a fifth force in addition to known interactions predicted by the Standard Model \[1–3\].

In its turn, direct tests of the non-Newtonian gravity have attracted attention in recent years due to the predictive accuracy of these theories, compared to the current level of experimental uncertainty, and also motivated by the possibility of find traces of new physics \[4–6\]. Recent experiments using torsion-balance in a regime of submillimeter scales have attempted to examine since the validity of some physical principles, such as the Weak Equivalence Principle, e.g., see \[7\], even the existence of a dark-energy length scale as the regulator of a new fundamental length scale for gravity \[6\]. In this case, the results impose limits on corrections to Newton’s gravity law, showing the absence of deviations up to a submillimeter scale \[8, 9\].

From measuring the Casimir force, constraints to predicted deviations to Newtonian gravity have also been extracted \[9–11\]. In this context, the non-Newtonian interaction is assumed to present Yukawa-type, and power-type corrections arising from the exchange of particles, such as light scalar particles \[10, 11\].

In another way, there is a possibility that the non-Newtonian theory of gravity can be described as a theory of gravitation modified by the existence of extra dimensions \[9\]. Therefore, the Newtonian gravitational potential must present corrections on short-range scales due to compact extra dimensions, whose compactification radius \(R\) is of the sub-micrometer order. However, the four-dimensional behavior of gravitational potential is recovered if the distances regime \(r\) is greater than the radius \(R\) \[12, 13\]. In this theoretical framework, the usual Newtonian gravitational potential has a Yukawa-type correction, for instance, that presents an explicit dependence with the space-like dimensionality and the compactification scale \[14\]. Hence, if experimental deviations are measures, the fifth force could be evidence for empirical traces from extra dimensions.
In the scenario of extra-dimensional theories proposed by unification schemes, one can highlight the known braneworld models, which have an elegant geometric description of the ordinary universe inspired by string theories. According to this theoretical model, our 4-dimensional observable universe is a submanifold, in which the standard model fields are confined, embedding in a higher-dimensional space with δ large extra dimensions \( R \gg l_{\text{Planck}} \) \[12, 13\]. In this context, the spectroscopy has proved as an independent way to test deviations from the inverse-square law \[15–23\]. For instance, in ref. \[15\], there have been obtained stronger constraints to compactification radius and higher-dimensional Planck mass from analyzing the 1s – 2s electronic transition in the hydrogen by considering the thick braneworld scenario. Furthermore, in recent work \[16\], it was shown that extra dimensions with a thickness \( \sigma \lesssim 10^{-19} m \) could provide the excess energy found in the recent measurements of muonic hydrogen 2S – 2P transitions, and so solve the known proton radius puzzle \[24, 25\]. Thus, the proton radius puzzle would signal new physics coming from large extra dimensions, which would imply a violation of Newton’s gravity inverse-square law \[16\]. Moreover, recently the spectroscopy data analysis of the hydrogen-like atoms in Rydberg states has been applied to study the influence of a new Yukawa-type interaction on energy levels, and the atomic spectra deviations have been obtained \[17\].

This work aims to discuss prospective constraints on deviations from the non-Newtonian gravitational law of power-type and Yukawa-type from spectroscopic data and compare them. This paper is structured as follows: In the next section, we present the power-type correction to Newton’s gravity law and explicitly calculate the gravitational contribution for the ionic energy levels. Then, considering states with high angular momentum, we study specific transitions in the neon electronic and muonic atoms aiming to obtain constraints on the parameters of this model by assuming that the empirical predicted uncertainty to transitions localizing in the optical frequency spectrum limits the found deviations. In the third section, we consider the non-Newtonian potential with a Yukawa-type correction and calculate gravitational energy contribution to transitions between Rydberg states for the electronic and muonic hydrogen-like ions. From this analysis, we have presented constraints to the theoretical model and discuss the results. The bounds have been obtained requiring that the gravitational energy contribution does not exceed the promised empirical uncertainty, considering the leading term of atomic Hamiltonian. In its turn, in the fourth section, we have compared the deviations on the atomic energy spectra generated by both proposed
non-Newtonian parameterizations. Finally, in the last section, we present the conclusions.

II. POWER-LAW NON-NEWTONIAN POTENTIAL PARAMETRIZATION

Inspired by the proposed Arkani-Dimopoulos-Dvali (ADD) model of large extra dimensions, one can write the corrections to Newtonian gravitational potential due to higher dimensionality of spacetime according to the following equation [12, 13],

\[ V(r) = \begin{cases} \frac{-G_{4+\delta}M}{r^{4+\delta}}, & \text{if } r \ll \lambda \\ \frac{-G_N M}{r}, & \text{if } r \gg \lambda \end{cases} \]

(1)

where \( \lambda \) is the compactification scale related to the extra dimension radius. In this case, we have assumed that the spacetime has \( \delta \) extra dimensions.

The \((4 + \delta)\)-dimensional gravitational constant, \( G_{4+\delta} \), can be easily rewritten in terms of the usual 4-dimensional Newtonian constant, \( G_N \), by assuming that at regime \( r \gg \lambda \), we should recover the four-dimensional behavior for gravitational potential. Thus, we find

\[ G_{4+\delta} = \lambda^\delta G_N. \]

(2)

Conveniently, let us express the eq. (1) through a function that interpolates and describes the Newton’s gravitational potential behaviors into two regimes \((r \ll \lambda \text{ and } r \gg \lambda)\). Thus, the power-type parametrization which generalize the gravitational potential can be obtained directly [9]:

\[ V(r) = -\frac{G_N M}{r} \left( 1 + \left( \frac{\lambda}{r} \right)^\delta \right), \]

(3)

where \( \delta = 0, 1, 2, \ldots \), \( \lambda \) is an interaction constant associated with the extra-dimensional radius and, consequently, has length dimension. From experimental data will constrain both parameters. Let us mention that several extensions of General Relativity also predict this parametrization and even \( f(R) \) theories [26].

A. Shifted non-Newtonian gravitational atomic energy

At this point, we are interested in analyzing transitions between Rydberg states with high angular momentum. Therefore, the effects of the nuclear structure are negligible, and
the uncertainty originated from the proton radius is small, so we do not take into account the fact that the proton has a finite structure \[20\]. Hence, in the first approach, the atomic energy levels will be perturbed by gravitational effects, which get amplified in certain regimes \((r \ll \lambda)\) as a result of the gravitational potential correction \([\text{3}]\). We begin our analysis by considering the perturbation as being described in terms of the gravitational Hamiltonian,

\[ H_g = m_i V(r), \quad (4) \]

where \(m_i\) is the lepton mass.

In order to obtain the constraints provided by this model \([\text{3}]\), we will find the contribution of gravitational energy to the energy levels by applying perturbation formalism. In this case, it is important to stress that the eq. \((4)\) corresponds to a small contribution to the total atomic Hamiltonian. Under this assumption, one can analytically determine the general expression for the gravitational energy contribution on energy levels of hydrogen-like ion with \(Z\) atomic number and then express it as

\[
\mathcal{E}_{g,i}(n, l) \equiv \langle H_g \rangle_{n,l} = -\frac{G_N m_i M}{a_{0,i}} \frac{Z}{n^2} \left(1 + _3F_2(n-l, 1 - \delta, -2l - \delta; -2l, n - l - \delta + 1; 1)\right)
\]

\[
\times \frac{(-2)^\delta \Gamma(2l + 1) \Gamma(n - l)}{n \Gamma(\delta) \Gamma(2l + \delta + 1) \Gamma(n - l - \delta + 1)} \left(\frac{2Z\lambda}{a_{0,i}n}\right)^\delta, \quad (5)
\]

where the average \(\langle \rangle_{n,l}\) is calculated with respect to well-known unperturbed wave functions for the hydrogen atom, \(M \cong Am_p\) is the nuclear mass, \(a_{0,i}\) is the Bohr radius, \(_pF_q(a; b; c)\) is the generalized hypergeometric function, and the \(i\)-index labels the lepton orbiting the hydrogen-like ion, i.e., \(i = e (\mu)\) for an electronic (muonic) ion.

Let us now consider a hydrogen-like ion with \(Z\) atomic number found an \(n\)-state with \(l = n - 1\) angular momentum. In this case, the gravitational non-Newtonian potential energy, \(E_{g,i}(n) = \mathcal{E}_{g,i}(n, n - 1)\), can be reduced to:

\[
E_{g,i}(n) = -\frac{G_N m_i M}{a_{0,i}} \frac{Z}{n^2} \left(1 + \frac{\Gamma(2n - \delta)}{\Gamma(2n)} \left(\frac{2Z\lambda}{a_{0,i}n}\right)^\delta\right), \quad (6)
\]

where \(\Gamma(n) = (n - 1)!\) is the Gamma function. The analytic expression \((6)\) provides us with the shift on energy levels of the Rydberg atom in the \(n\)-state \((l = n - 1)\) due to the gravitational potential energy contribution. We should note that the expected classical
result to 4-dimensional gravitational energy due to the atomic nucleus is recovered when \( \lambda = 0 \). Through the eq. (1), one can see that gravitational energy is amplified at scale \( \lambda > a_{0,i}n/2Z \), if the gravitational potential has power-type correction.

B. Constraints on power-law-type deviation

A promising spectroscopic measurement technique involving optical frequency combs proposes a method to measure frequency transitions with remarkably high empirical precision. In this sense, the transition measurements between Rydberg’s states, localizing in the optical frequency range (100THz - 1000THz), will be performed with a promised relative uncertainty of the order of \( 10^{-19} \) [27, 28]. Thus, the limitation in the extraction of the Rydberg constant, due to high uncertainties originated by the atomic structure, and which is evidenced in the proton radius problem [24, 25], is circumvented [29]. In this way, a previous paper [20], showed that measurements of transitions between Rydberg states located in the optical frequency spectrum of muonic ions would present measurable gravitational effects in braneworld scenarios and, therefore, has demonstrated to be an alternative way to search for extra dimensions empirical traces.

Following this idea, let us find the prospective constraints by considering that transitions occur between adjacent states \((n, l = n - 1)\) and \((n - 1, l = n - 2)\), and, in the case of Rydberg states, will be limited by the general expression (7). Thus, the gravitational contribution to energy levels should be less than the promised empirical uncertainty for the leading term contribution of the atomic Hamiltonian

\[
\Delta E_{g,i} \equiv |E_{g,i}(n) - E_{g,i}(n-1)| \leq \delta E_{\text{exp}}^n, \tag{7}
\]

where \( \delta E_{\text{exp}}^n \) is the empirical uncertainty promised for transition and can be written as

\[
\delta E_{\text{exp}}^n = 10^{-19}hcR_\infty Z^2 \left( \frac{1}{n^2} - \frac{1}{(n-1)^2} \right), \tag{8}
\]

and \( R_\infty \) is the Rydberg constant, \( h \) is the Planck constant, and \( c \) is the speed of light. The eq. (7) will provide us the theoretical constraints on the \( \lambda \)-parameter for certain \( \delta \)-values fixed.

Here, one can present the constraints on the gravitational contribution (7) by analyzing the transition \( n = 15 \) to \( n = 14 \), for instance, in the electronic and muonic neon ion \((^{20}\text{Ne}^{+9})\) -
see, e.g., \[30\], assuming that such contribution is of the order of, or lower than the promised uncertainty of the leading term of this transition involving Rydberg states. These selected states ensure that frequency transitions lie about the optical band \[27, 28\]. For the sake of simplicity, we must still mention that the muonic hydrogen-like ions are atoms whose lepton orbiting the nucleus is the muon, e.g., see \[24, 25\].

\[ \begin{align*}
\text{FIG. 1: Constraints on parameters of the power-law gravitational potential. The symbols } & \text{“○” (”□”) indicate the fixed value to } \delta = 2, 3, 4, 5, 6 \text{ and } 7 \text{ in muonic (electronic) Neon ion. These curves (LHC and Casimir), were obtained from Fig. 11 of Ref. } [9] \text{ using plot-digitizer software } [31] . \\
\text{In fig. } & \text{ one can observe that for } \delta > 3, \text{ the spectroscopic constraints of the electronic ion (} \Delta E_{g,e} \text{ curve) are more restrictive than the bounds obtained through the Casimir effect. In its turn, the constraints found for the muonic ion transition spectroscopic data lead to stronger bounds than those obtained from the Casimir physics, for all values of } \delta \text{ considered. }
\end{align*} \]

Before proceeding, it is important to stress that although the constraints found by spectroscopic analysis for this particular non-Newtonian potential are weaker than those found by the LHC, this study presents another independent way of testing deviations to Newton’s gravity law. For more details on obtaining the LHC and Casimir curves, see Ref. \[9\].

III. BOUNDS FROM YUKAWA-TYPE POTENTIAL

Let us analyze another parametrization that describes deviations to Newton’s gravity law which has been motivated, at first, by theories predicting a new long-range interaction
of Yukawa-type. These deviations from the predictions to gravitational potential have also been proposed in some scenarios as extra-dimensional theories \cite{12–14}, and even in Casimir Physics \cite{9–11}, for instance. The Yukawa parametrization presents a generalized form to the gravitational potential generated by a mass M which can be written as:

\[ V_{Y_u}(r) = -G_N \frac{M}{r} \left(1 + \alpha e^{-r/\lambda}\right) \equiv V_N(r) + \delta V_{Y_u}(r), \tag{9} \]

where \( G_N \) is the 4-dimensional Newtonian gravitational constant, \( V_N(r) \) is the usual Newtonian gravitational potential, and \( \delta V_{Y_u}(r) \) is the Yukawa correction to usual potential. According to extra-dimensional scenarios, the \( \lambda \)-parameter will describe the compactification scale of the higher-dimensional space, which is assumed to be on the sub-mm scale to keep consistency with the currently empirical data obtained. At the same time, \( \alpha \) is a coupling constant related to hidden spacetime dimensions \cite{14}.

Such as we proceeded in the previous subsection \textbf{II B} one will apply the perturbation method to calculate the gravitational contribution to transition frequencies by considering atoms in Rydberg’s states. Therefore, one can consider the potential \textbf{(9)} as a perturbation to atomic energy levels and rewrite it in the Hamiltonian form \( \tilde{H}_g = m_i V_{Y_u}(r) \). In this case, we are interested in calculating the non-Newtonian gravitational contribution for a given energy level \( n \) of a hydrogen-like atom in Rydberg’s state \((n \text{ and } l = n - 1)\), which can be evaluated analytically and thus yields the expression,

\[ \tilde{E}_{g,i}(n) = -\frac{G_N m_i M Z}{a_{0,i}} \frac{n^2}{n^2} \left[1 + \alpha \left(\frac{2\lambda Z/n a_{0,i}}{1 + 2\lambda Z/n a_{0,i}}\right)^{2n}\right]. \tag{10} \]

We emphasize that for the \( \alpha \to 0 \), one can recover the expected four-dimensional behavior for the gravitational potential energy. In this case, the \( \alpha \) parameter turns off the non-Newtonian effects generated by the Yukawa correction.

Finally, we will find the bounds on free parameters \( \alpha \) and \( \lambda \) from the study of transitions involving adjacent states by an analogous procedure to that of the previous subsection \textbf{II B}. Thus, admitting that the gravitational contribution to energy levels on transitions in hydrogen-like Rydberg atoms must be less than promised empirical uncertainty, one obtains the constraints on parameters from Yukawa corrected potential. This condition ensures that no traces of new physics have been detected so far. For this study, we analyze the transition from \( n = 14 \) to \( n = 15 \) for the electronic and muonic ion of Neon \((^{20}\text{Ne}^{+9})\), and we find the
bounds for the parameters $\lambda$ and $\alpha$, see fig. 2.

The gravitational contribution to transition energy has been provided by the equation

$$\Delta \tilde{E}_{g,i} \equiv \left| \tilde{E}_{g,i}(n) - \tilde{E}_{g,i}(n - 1) \right|.$$  

It is important to highlight that the non-Newtonian gravitational energy will increase the energy gap between adjacent states with the principal quantum numbers $n$ and $n - 1$. In fig. 2 we have compared the obtained bounds from this study with a wide variety of empirical constraints originated from different sources such as neutron scattering, spectroscopy of exotic atoms, collider, see, e.g., [8, 9, 32, 33]. From Ref. [9], the collider data have been extracted using plot-digitizer software [31]. The additional empirical bounds can be found in Ref. [32]. As is seen, the constraints obtained for $\lambda < 10^{-8}$ m are stronger for both cases of Rydberg state transitions (electronic and muonic) than the bounds obtained through the Casimir effect analysis. The shaded areas represent the excluded region for the existence of non-Newtonian effects.

At last, let us emphasize that, although we have obtained stronger bounds as they are prospective constraints, one may hope that these results will be weakened since the empirical data, together with their respective uncertainty ($\delta E_{\text{exp}}$), are provided. By taking into account, the theoretical uncertainty ($\delta E_{\text{th}}$) obtained by study all terms of atomic Hamiltonian, one also expects a weakening of the obtained constraints. In this case, the new constraints can be found by using the combined uncertainty, defined by $\delta E = \sqrt{\delta E_{\text{th}}^2 + \delta E_{\text{exp}}^2}$, to limit non-Newtonian deviations on atomic energy levels.
IV. NON-NEWTONIAN PARAMETRIZATIONS CONSTRAINTS COMPARISON

As discussed in the preceding sections, the inverse square law corrections are proposed in many scenarios considering different parametrizations. In principle, the different proposed deviations to Newton’s gravitational law would produce unlike effects on the atomic energy levels. In turn, it is possible, for instance, to compare the gravitational energy contribution with the leading term contribution to the energy levels of Rydberg atoms \( E_n = -cR_\infty Z^2/n^2 \). In this case, the expected deviation for atomic energy levels due to the non-Newtonian contribution is estimated, as discussed, e.g., in [17], where a new Yukawa-type interaction has been considered.

In another way, we aim to compare the effects produced by Yukawa-type potential and power-law potential on the energy levels of muonic and electronic atoms. In this case, we study the relationship between the effects produced in the two proposed scenarios, \( E_{g,i}(n)/E_{g,i}(n) \), considering hydrogen-like ions in Rydberg states (\( l = n - 1 \)). As shown in fig. 3, the deviations found for the two studied parametrizations have contributed distinguishably to atomic energy levels. Therefore, if such deviations are measured, we could, in principle, determine the parametrization that best fits the empirical data and, hence, predicts the possible origin of the inferred correction for the atomic levels.

FIG. 3: Estimation of the ratio between non-Newtonian effects due to power-law-type \( (E_{g,i}) \) and Yukawa-type \( (\tilde{E}_{g,i}) \) parameterizations on the atomic energy levels. The “□” symbol identifies the electronic atom while the “▽” symbol labels the muonic atom.
To obtain fig. 3 we have considered an hydrogen-like ion with $Z = 10$, $\lambda = 10^{-8}$m, and $\alpha = 10^{20}$. The corrections originated from the Yukawa parametrization showed to be stronger than those obtained by power-law-type correction, except in the case of muonic atoms in states with $n < 10$ e $\delta > 4$. We also infer that for high values of $n$ ($n \gg 50$), the correction produced is approximately $\delta$-independent, as seen in the electronic atom. We notice that the effects on muonic atoms are amplified compared to electronic atoms, mainly due to the smaller Bohr radius of the muonic system ($a_{0,\mu}$). Thus, spectroscopy of muonic atoms has been shown as an exciting approach in searching for new physics.

V. CONCLUDING REMARKS

Summarily we have discussed the deviations of Newton’s inverse-square law proposed from several scenarios. The fifth force theory, for instance, predicts Newton’s gravitational law deviations revealed by a new long-range interaction on the submillimeters scale. Furthermore, in Casimir physics, the claimed deviations to Newtonian gravity have been studied and would be manifest in short-distance scaling experiments due to the exchange of light or massless particles via an interaction of the Yukawa-type or power-law-type. On the other hand, recent higher-dimensional theories postulate in their description the existence of modifications of the usual $(3+1)$-dimensional Newtonian potential. In known braneworld models, deviations from inverse-square law originate from the existence of hidden spatial dimensions. According to such models, the Newtonian gravitational potential would take the form of Yukawa potential at a short-range scale.

In this work, we have explored the possibility of making a theoretical prediction on deviations of Newtonian gravity. Thus, one have presented constraints by considering the Yukawa and power-law parametrizations as corrections to Newton’s gravitational law. We have calculated deviations to energy levels of electronic and muonic ions in Rydberg states due to gravitational potential energy contribution corrected by the Yukawa and power-law-type parametrizations. Then, we presented and discussed the bounds obtained for the free parameters of both models ($\delta, \alpha$, and $\lambda$), comparing them with constraints from different origins.

Although the bounds found in this analysis are stronger than results obtained from the Casimir force measurements, it is important to stress that we found prospective constraints
in this study. For the bounds obtained through the study of power-law potential, we find stronger constraints for muonic ion than the results of the electronic ion and data from the Casimir physics. As shown, the Yukawa-type potential presents stronger bounds to $\lambda < 10^{-8}$m, excluding any deviations to Newton’s inverse-square law to $\alpha > 10^{20}$. For this analysis, we required that the gravitational energy contribution corrected by the Yukawa-type and power-type parametrizations does not exceed the theoretical uncertainty of the leading term of atomic Hamiltonian, ensuring relative promised uncertainty to be of order $10^{-19}$. In turn, the uncertainty has been predicted by spectroscopy techniques of high-precision optical frequency combs.

Finally, we compare the deviations on Rydberg ion energy levels generated from the two parametrizations studied. In this case, considering a hydrogen-like ion with $Z = 10$, we found that the Yukawa-type gravitational energy contribution is stronger than the correction obtained for the power-law-type potential for $n > 10$ and $\delta \leq 4$ (fixing $\lambda = 10^{-8}$m, and $\alpha = 10^{20}$). The detecting deviations in ionic energy levels would indicate a fifth force of nature. Since the discussed corrections are described for different regimes and proposed in unlike scenarios, from the measurement deviations, it would be possible to infer, in principle, the origin of deviations.

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