The Exact Tachyon Beta-Function for the Wess-Zumino-Witten Model

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We derive an exact expression for the tachyon $\beta$-function for the Wess-Zumino-Witten model. We check our result up to three loops by calculating the three-loop tachyon $\beta$-function for a general non-linear $\sigma$-model with torsion, and then specialising to the case of the WZW model.
1. Introduction

The Wess-Zumino-Witten (WZW) model[1] is a particularly interesting example of a conformal field theory. Indeed it is currently believed that all rational conformal field theories may be derived from the WZW model by the Goddard-Kent-Olive (GKO) construction[2] (or equivalently by gauging[3]). The WZW model on a Lie group manifold \( G \) is parametrised by the level (which is constrained to be an integer for a compact \( G \)). The properties which characterise the conformal field theory—the central charge and the conformal dimensions of the primary fields—have exact non-perturbative expressions in terms of the level and the Casimirs for the Lie algebra of \( G \).[4]

In general, we may consider perturbing the WZW model by adding a potential term to the action. This potential term is usually taken to be a primary field of the WZW model; however we would like to consider the case where this restriction is not applied. Our motivation for this comes from non-abelian Toda field theories[5], which have recently been receiving some attention[6][7]. The action for these models consists precisely of a WZW model coupled to a potential term. In a recent paper[8] we discussed the conformal properties of the non-abelian Toda field theory at the quantum level. The conformal behaviour of the potential term is described by the tachyon \( \beta \)-function (in terminology derived from string theory) and therefore we needed an exact expression for this \( \beta \)-function. Our aim in this paper is to derive this. In the particular case where the potential is simply \( \text{tr}(g) \), where \( g \in G \), the tachyon \( \beta \)-function is simply related to the conformal dimension of \( \text{tr}(g) \), and we shall exploit this to deduce the general form of the tachyon \( \beta \)-function. We shall then check this result by performing an explicit perturbative calculation up to three-loop order. We first obtain the result for a general non-linear \( \sigma \)-model, and then specialise to the case of the WZW model. In the WZW case, our perturbative calculation is similar to a three-loop calculation of the conformal dimension of \( \text{tr}(g) \) carried out some time ago[9]. However, we use a different prescription[10][11] for continuing the two-dimensional alternating symbol to \( d \) dimensions in the context of dimensional regularisation. This prescription has the conceptual advantage of avoiding the need for introducing extra evanescent terms into the action, and also preserves explicit \( O(d) \) covariance. It now appears to have been accepted as the standard prescription for calculations of this type.
2. Exact result for tachyon $\beta$-function

In this Section we derive our principal result, namely the exact tachyon $\beta$-function for the WZW model. We first write down the action for the Wess-Zumino-Witten (WZW) model defined on a group manifold $M_G$:

$$kS_{WZW}(g) = -\frac{k}{8\pi} \int_S d^2 x \text{tr}(g^{-1} \partial_\mu g^{-1} \partial^\mu g) + \frac{ik}{12\pi} \int_B d^3 x \epsilon^{\mu\nu\rho} \text{tr}(g^{-1} \partial_\mu g^{-1} \partial_\nu g^{-1} \partial_\rho g) \quad (2.1)$$

where $g$ is a group element in the defining representation of $G$, whose generators satisfy

$$[T_a, T_b] = i f_{abc} T_c. \quad (2.2)$$

$B$ is a 3-dimensional ball whose surface is the two-dimensional worldsheet $S$. We assume that the group generators satisfy $\text{tr}(T_a T_b) = \delta_{ab}$. We are using here the conventions of Ref. [12]. The level $x$ is defined in terms of $k$ by $x = \frac{2k}{\psi^2}$, where $\psi$ is the highest root in $G$. The central charge is given by

$$c = \frac{k \dim G}{k + \frac{1}{2} c^G}, \quad (2.3)$$

and $g$ is a primary field with conformal dimensions given by [4]

$$h = \bar{h} = \frac{c^R}{k + \frac{1}{2} c^G}, \quad (2.4)$$

where $c^G$ and $c^R$ are the eigenvalues of the quadratic Casimir in the adjoint and defining representations respectively, so that

$$f_{acd} \bar{f}_{bcd} = c^G \delta_{ab}, \quad T_a T_a = c^R 1. \quad (2.5)$$

The WZW model is a particular example of a two-dimensional non-linear $\sigma$-model, whose action is given in general by

$$S(\phi) = \frac{\lambda}{8\pi} \int d^2 x \sqrt{\gamma} \{ \gamma^{\mu\nu} G_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j + \epsilon^{\mu\nu} B_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j + \frac{1}{\lambda} D(\phi) R^{(2)} + V(\phi) \} \quad (2.6)$$

where $\gamma_{\mu\nu}$ is the two-dimensional metric, $\gamma$ is its determinant and $\epsilon^{\mu\nu}$ is the two-dimensional alternating symbol. $\{\phi^i\}$ represent co-ordinates on some target manifold with metric $G_{ij}$ and antisymmetric tensor field $B_{ij}$ defined on it, $D$ is the dilaton field coupling to the two-dimensional scalar curvature $R^{(2)}$, and $V$ is the tachyon field. (The terminology derives from string theory; our conventions are equivalent to taking $\alpha' = \frac{2}{\lambda}$ in Ref. [11], where
\( \alpha' \) is the string coupling.) The conformal invariance conditions for the \( \sigma \)-model Eq. (2.6) may be written[11]

\[
\begin{align*}
B^G_{ij} &\equiv \beta^G_{ij} + \frac{2}{\lambda} \nabla_i \partial_j D + 2 \partial_i (W_j) = 0 \quad (2.7a) \\
B^B_{ij} &\equiv \beta^B_{ij} + \frac{2}{\lambda} H^k_{ij} \partial_k D + 2 H^k_{ij} W_k = 0 \quad (2.7b) \\
B^V &\equiv \beta^V - 2V + \frac{1}{\lambda} \partial^k D \partial_k V + W^k \partial_k V = 0 \quad (2.7c)
\end{align*}
\]

where \( \beta^G_{ij}, \beta^B_{ij} \) and \( \beta^V \) are the standard renormalisation group \( \beta \)-functions for \( G_{ij}, B_{ij} \) and \( V \), and \( H_{ijk} \) is the torsion, defined by \( H_{ijk} = 3 \nabla [i B_{jk}] \). \( W_i \) is a vector field which can be determined perturbatively within a given renormalisation scheme. We shall be using a renormalisation scheme in which \( W_i \) vanishes for the WZW model.

When \( B^G_{ij} \) and \( B^B_{ij} \) both vanish, the quantity \( B^D \) given by

\[
B^D \equiv \beta^D + \frac{1}{\lambda} \partial^k D \partial_k D + W^k \partial_k D, \quad (2.8)
\]

where \( \beta^D \) is the dilaton \( \beta \)-function, becomes constant [12][11] and is then related to the central charge \( c \) for the conformal field theory by

\[
c = 3 B^D. \quad (2.9)
\]

The tachyon \( \beta \)-function is of the form

\[
\beta^V = \Omega V \quad (2.10)
\]

where \( \Omega \) is a differential operator, in general of arbitrary order. We may write \( \Omega \) in the form

\[
\Omega = \sum X^{(n)k_1...k_n} \nabla_{k_1} \ldots \nabla_{k_n} \quad (2.11)
\]

where \( X^{(n)k_1...k_n} \) is an \( n \)th rank tensor constructed from the Riemann tensor, the torsion and their derivatives contracted together.

In the case of the WZW model Eq. (2.1), the target manifold is the group \( M_G \). There is no tachyon field \( V \), and the metric \( G_{ij} \) and \( B_{ij} \) may be read off by comparing Eqs. (2.1) and (2.6). We have[14]

\[
G_{ij} = e_{ai} e_{aj}, \quad \lambda = k, \quad (2.12)
\]

where the vielbein \( e_{ai} \) is defined by

\[
i g^{-1} \partial_i g = e_{ia} T_a \quad (2.13)
\]

\( \lambda = k \)
and satisfies

\[ e_a^i e_{bi} = \delta_{ab}. \]  

(2.14)

In fact \(B_{ij}\) can only be defined locally, but we have a globally defined expression for \(H_{ijk}\) \[14\],

\[ H_{ijk} = \frac{1}{2} f_{abc} e_{ai} e_{bj} e_{ck}. \]  

(2.15)

We also have

\[ \nabla_i e_{aj} = f_{abc} e_{bi} e_{cj}. \]  

(2.16)

from which it follows that the Riemann tensor is given by \[14\]

\[ R_{ijkl} = \frac{1}{4} f_{abc} f_{cde} e_{ai} e_{bj} e_{ck} e_{dl}. \]  

(2.17)

and also that

\[ \nabla_l H_{ijk} = 0. \]  

(2.18)

We will denote quantities evaluated for the particular case of the WZW model by subscripts \(WZW\). \(\Omega_{WZW}\) has an expansion similar to Eq. (2.11), but in terms of \(X^{(n)k_1...k_n}_{WZW}\) obtained from \(X^{(n)k_1...k_n}\) by substituting the expressions in Eqs. (2.15) and (2.17) for the Riemann tensor and torsion. \(X^{(n)k_1...k_n}_{WZW}\) is constructed purely from the structure constants and the vielbeins.

If \(V\) is a primary field, it will be an eigenfunction of \(\Omega_{WZW}\) whose eigenvalue is the conformal dimension of \(V\). In particular, for \(V = \text{tr}(g)\), we must have

\[ \Omega_{WZW} \text{tr}(g) = \frac{c_R}{k + \frac{1}{2} c_G} \text{tr}(g). \]  

(2.19)

This information is sufficient to determine \(\Omega_{WZW}\). We can assume without loss of generality that \(X^{(n)k_1...k_n}_{WZW}\) is symmetric in \(k_1...k_n\) (indeed, \(X^{(n)k_1...k_n}_{WZW}\) is naturally given in this form by the explicit calculation–see later). So we have, using Eqs. (2.13) and (2.16),

\[ X^{(n)k_1...k_n}_{WZW} \nabla_{k_1} \cdots \nabla_{k_n} \text{tr}(g) = i^n X^{(n)k_1...k_n}_{WZW} \text{tr}(g T_{a_1} \cdots T_{a_n}) e_{a_1 k_1} \cdots e_{a_n k_n}, \]  

(2.20)

since the structure constant terms in Eq. (2.16) vanish by symmetry here. Eq. (2.19) can be expanded in powers of \(\frac{1}{k}\) (or equivalently in powers of \(c_G\)), and this expansion must correspond to the perturbation expansion for \(\Omega\). Hence we see that
$X^{(n)k_1...k_n}_{WZWT \alpha_1 ... T \alpha_n e_{a_1 k_1} ... e_{a_n k_n}}$ must reduce to a function of $c^R$ and $c^G$ linear in $c^R$. The only way in which this can happen is if

$$X^{(n)k_1...k_n}_{WZW} = 0, \quad i \neq 2 \quad (2.21)$$

$$X^{(2)k_1 k_2}_{WZW} T_{\alpha_1} T_{\alpha_2} e_{a_1 k_1} e_{a_2 k_2} = - \frac{c^R}{k + \frac{1}{2}c^G}. \quad (2.22)$$

We then must have

$$X^{(2)k_1 k_2}_{WZW} e_{a_1 k_1} e_{a_2 k_2} = - \frac{c^R}{k + \frac{1}{2}c^G} \delta_{a_1 a_2}, \quad (2.23)$$

and hence, from Eq. (2.12),

$$X^{(2)k_1 k_2}_{WZW} = - \frac{c^R}{k + \frac{1}{2}c^G} g^{k_1 k_2}. \quad (2.24)$$

So we finally have the desired result

$$\Omega_{WZW} = - \frac{c^R}{k + \frac{1}{2}c^G} \nabla^2, \quad (2.25)$$

and hence

$$\beta^V_{WZW} = - \frac{c^R}{k + \frac{1}{2}c^G} \nabla^2 V. \quad (2.26)$$

3. Three-loop perturbative calculation

In this Section we go some way towards corroborating our exact result for the tachyon $\beta$-function for the WZW model, derived in the previous Section, by performing a perturbative calculation up to three-loop order. We start by doing the computation for the general non-linear $\sigma$-model of Eq. (2.6), before specialising to the WZW model. It is most convenient to perform these computations using dimensional regularisation; however, the crucial issue which then arises for a $\sigma$-model of the form Eq. (2.6), one with a term involving the antisymmetric tensor $B_{ij}$, is how to extend the two-dimensional alternating symbol, $\epsilon^{\mu\nu}$, away from two dimensions. In two dimensions one has the relation

$$\epsilon^{\mu\nu} \epsilon^{\rho\sigma} = \gamma^{\mu\sigma} \gamma^{\nu\rho} - \gamma^{\mu\rho} \gamma^{\nu\sigma}. \quad (3.1)$$

However, if one tries to apply this relation for $d \neq 2$, one encounters inconsistencies. One solution is to regard $\epsilon^{\mu\nu}$ as strictly two-dimensional even within dimensional regularisation\[9\]. The drawback of this approach is that the tangent-space group is reduced from
$O(d)$ to $O(d-2) \times O(2)$, and as a consequence one is obliged to introduce additional, “evanescent” couplings which were not present in the original two-dimensional action. Physical results independent of these evanescent couplings can be obtained, but only at the expense of additional calculation. This is the method used in Ref. [9] to compute the anomalous dimension of $\text{tr}(g)$ up to three loops.

An alternative approach, which has been fairly widely used[10][11], is to abandon Eq. (3.1) away from two dimensions, but to assume the existence of a tensor $\epsilon^{\mu\nu}$ in general $d$ dimensions with the property

$$\epsilon^{\mu\rho}\epsilon_{\rho\nu} = -\gamma^{\mu\nu}. \quad (3.2)$$

We should stress that the $\gamma^{\mu\nu}$ which appears on the RHS of Eq. (3.2) is the $d$-dimensional metric. Evanescent couplings are required in this case also for the rigorous discussion of renormalisability[11]; however, the important difference as compared to the previous prescription is that these evanescent couplings completely decouple from physical results. Moreover it turns out that a relation of the form Eq. (3.2) is sufficient for perturbative calculations.

The most convenient means for discussing the quantisation of the non-linear $\sigma$-model is the use of the background field method[15]. The field $\phi^i$ is expanded around a classical background configuration as

$$\phi^i = \phi^i_0 + \pi^i. \quad (3.3)$$

However, the field $\pi^i$ is not very convenient for use as the quantum field variable, since it does not transform as a vector. It is customary to write $\pi^i$ in terms of the field $\xi^i$, the tangent vector to the geodesic linking $\phi^i_0$ to $\phi^i_0 + \pi^i$, and to use $\xi^i$ as the quantum field[14][16]. This guarantees a manifestly covariant perturbation expansion written in terms of tensor quantities such as the Riemann tensor, the torsion and their covariant derivatives. The expansion of the action for the non-linear $\sigma$-model in Eq. (2.6) in terms of $\xi$ takes the following form (setting the dilaton term, which will not concern us, to zero)[15][17][16]:

$$S(\phi + \pi) = S(\phi) + \sum_{i=1}^{N} S^{(i)}(\phi, \xi), \quad (3.4)$$

where

$$S^{(1)} = \int d^2x \left( g_{ij} \partial_\mu \phi^i \partial_\nu \phi^j D^\mu \xi^j + \epsilon^{\mu\nu} H_{ijk} \partial_\mu \phi^i \partial_\nu \phi^j \xi^k + \nabla_1 V \xi^i \right), \quad (3.5a)$$

$$S^{(2)} = \frac{1}{2} \int d^2x \left( g_{ij} D^\mu \xi^i D^\nu \xi^j + (\gamma^{\mu\nu} R_{iklj} + \epsilon^{\mu\nu} \nabla_1 H_{ijk}) \partial_\mu \phi^i \partial_\nu \phi^j \xi^k \xi^l \right).$$
\[ S^{(3)} = \int d^2x \left( \frac{1}{3} \epsilon^{\mu\nu} H_{ijk} D_\mu \phi^i D^\mu \phi^j \xi^k + \nabla_i \nabla_j V \xi^i \xi^j \right), \] 
\[ S^{(4)} = \int d^2x \left( \frac{1}{6} \epsilon^{\mu\nu} H_{ijk} D_\mu \phi^i D^\mu \phi^j \xi^k + \frac{1}{6} \nabla_i \nabla_j \nabla_k V \xi^i \xi^j \xi^k \right) + \ldots, \] 
\[ \Omega^{(1)} = -\frac{1}{\lambda} \nabla^2, \] 
\[ \Omega^{(2)} = \frac{2}{\lambda^2} H^{kmn} H^l_{mn} \nabla_k \nabla_l \]
This leads to the following contribution to $\Omega$ at 3 loops:

$$
\Omega^{(3)} = \frac{1}{\lambda^3} \left( -\frac{3}{2} R^{kmpn} R^l_{\ mnp} - 7 R^{kpqm} H^{nl}_{\ l p m} H_{nq} 
- 5 H^{k m n} H^{l p q} H_{m p r} H_{n q} - 7 H^{p r s} H^{q t r s} H_{p m}^k H_{q m}^l + \frac{3}{2} \nabla^p H^{k m n} \nabla_p H_{l m n} 
- \frac{5}{6} \nabla^k H_{m n p} \nabla^l H^{m n p} \nabla_k \nabla_l + \frac{4}{\lambda^3} \nabla^k H^{l n p} H^m_{\ n p} \nabla_{(k} \nabla_{l} \nabla_{m)} \right)
$$

(3.9)

Specialising to the case of the WZW model, we readily find, using Eqs. (2.15), (2.17) and (2.18),

$$
\Omega^{WZW(1)} = - \frac{1}{k} \nabla^2 
$$

(3.10a)

$$
\Omega^{WZW(2)} = \frac{1}{2k^2} e^G \nabla^2 
$$

(3.10b)

$$
\Omega^{WZW(3)} = - \frac{1}{4k^3} e^{2G} \nabla^2 
$$

(3.10c)

which is consistent with the expansion of Eq. (2.25) up to $O(\frac{1}{k^3})$.

4. Conclusions

Our central result is the exact expression for the tachyon $\beta$-function in the WZW model, given by Eq. (2.26). This quantity played a crucial role in a recent paper deriving the exact conformally invariant action for the non-abelian Toda field theory. We checked this result up to 3rd order in perturbation theory by calculating the tachyon $\beta$-function at this order for a general non-linear $\sigma$-model, and then specialising to the case of the WZW model. The result for the general case seems to us to be of interest since it is the first full, direct computation of a renormalisation group quantity at three loops for the general non-linear $\sigma$-model with torsion. (The three-loop contribution to the dilaton $\beta$-function was, however, calculated indirectly in Refs. [20], [21], [11].)

A more stringent check on the validity of Eq. (2.25) would be provided by a four-loop calculation of $\beta^V$. For instance, Eq. (2.21) is trivially satisfied at three loops since the term involving three derivatives in $\Omega^{(3)}$ in Eq. (3.3) manifestly vanishes upon specialisation to the WZW model. On the other hand, at four loops there are possible four-derivative terms in the general result for $\Omega^{(4)}$ (see Ref. [22] for a calculation of $\Omega^{(4)}$ in the torsion free case), and these do not obviously vanish immediately upon specialisation to the WZW case. A four-loop calculation would be formidably difficult, however.

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**Figure Captions**

Fig. 1. One-loop diagram contributing to tachyon $\beta$-function (with tachyon insertion denoted by cross).

Fig. 2. Two-loop diagram contributing to tachyon $\beta$-function.

Fig. 3. Three-loop diagrams contributing to tachyon $\beta$-function.
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