The structure of two-valued strategy-proof social choice functions with indifference

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February 14, 2020

Abstract

We give a structure theorem for all coalitionally strategy-proof social choice functions whose range is a subset of cardinality two of a given larger set of alternatives. We provide this in the case where the voters/agents are allowed to express indifference and the domain consists of profiles of preferences over a society of arbitrary cardinality. The theorem, that takes the form of a representation formula, can be used to construct all functions under consideration.

JEL Code: D71
AMS Subject Classification: 91B14
Keywords: social choice functions, strategy-proofness, coalitions, weak orderings.

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1 Introduction

The purpose of this paper is to present a representation formula for all strategy-proof social choice functions

\[ \phi : \mathcal{P} \rightarrow A, \]

under the following assumptions:

- \( A \) is a set of alternatives having cardinality at least two;
- \( V \) is an arbitrary set of voters (i.e. the society \( V \) is not necessarily finite);
- \( \mathcal{P} \) is the set of all profiles \( P = (P_v)_{v \in V} \) (universal domain hypothesis), consisting of weak orders \( P_v \) on \( A \) - one for every member \( v \) of the society;
- the range of \( \phi \) is of cardinality two.

We adopt the term two-valued social choice function for a \( \phi \) with range of cardinality two. If \( A \) is itself of cardinality two and \( \phi \) is onto, we shall use the term binary social choice function. We consider social choice functions that are coalitionally strategy-proof, i.e. no group of agents has incentives to form a coalition that can manipulate the social choice for their own advantage with false reporting (see Definition 2.3).

Representation formulas are more specific than characterization results since they de facto characterize, but also furnish an operational way to produce the desired functions. Therefore we are addressing a rather natural question and, more, we are considering this question even in the simplest case of functions whose range is of cardinality two. Despite all this, until now not too many representation formulas can be found in the literature for tackling the case wherein voters may express indifference between alternatives.

When we assume that every voter can only express strict orderings, namely when we limit the domain of \( \phi \) by considering only strict profiles\(^2\), we have the following representation results (Theorem 1.1 (i) and Theorem 1.1 (ii)).

**Theorem 1.1** The strategy-proof social choice functions defined on the set of all strict profiles, are all, and only, the following functions \( P \mapsto \phi_{\mathcal{F}}(P) \).

(i) Functions with range \( \{a, b\} \subseteq A \) of cardinality two:

\[ \phi_{\mathcal{F}}(P) = \begin{cases} 
    a, & \text{if } \{v \in V : a \succ b, \text{ according to } P_v\} \in \mathcal{F} \\
    b, & \text{if } \{v \in V : a \succ b, \text{ according to } P_v\} \notin \mathcal{F}
\end{cases} \]

\(^1\)Rational preference relations, or just preference relations, according to the books by Mas-Colell et al. [10] and by Aliprantis and Border [1] respectively. So, throughout the sequel, a profile \( P = (P_v)_{v \in V} \), consists of complete, transitive relations \( P_v \) on a set \( A \) of alternatives.

\(^2\)If every \( P_v \) is also antisymmetric, i.e. it is a strict order (one would say strict preference according to [10] or, reflexive linear order according to [1]), then we say that the profile P is strict.
where \( \mathcal{F} \) is a nonempty collection of coalitions of voters which is closed under super-
sets.

(ii) Functions with finite range \( A^* \subseteq A \) of cardinality three or more:

\[
\phi_{\mathcal{F}}(P) = \text{the unique alternative } a \in A^* \text{ such that } D(a, P) \in \mathcal{F},
\]

where \( \mathcal{F} \) is an ultrafilter of coalitions, and, for \( x \in A^* \), we denote by \( D(x, P) \) the set
\[\{v \in V : v \text{ has } x \text{ as his top choice among the alternatives in } A^*, \text{ according to } P_v\}\].

The formulation of (i) and (ii) can be found in Rao et al. [14, Proposition 4.1, and Theorem
3.5, respectively]. In the case of a finite society \( V \) and with \( A = \{a, b\} \), (i) is due to Larsson
and Svensson [9, Theorem 2]\(^3\), whereas (ii) combines results from Ishikava and Nakamura
[7], Mihara [11], Pazner and Wesley [12].

The scenario in which voters are allowed to express indifference among alternatives, is
much more complex. This is the setting of Barberà et al. [4], where a strategy-proof
characterization is offered in [4, Theorem 1] for two-valued social choice functions. To the
best of our knowledge, only recently Lahiri and Pramanik [8, Theorem 1] considered the
problem of obtaining a representation formula. They gave a rather difficult formula for
representing binary social choice functions in the framework of a finite society. For an
arbitrary society (i.e. finite or infinite) a simpler representation theorem for binary social
functions has been obtained by Basile et al. [5, Theorem 4.2]. However, we do not know
representation formulas when indifference is permitted for cases such as functions with
range of cardinality at least three and even two-valued functions. In other words, we do
not know results that generalize in a straightforward manner Theorem 1.1 to the case of
weak orderings. In the following we aim to give for the first time a formula for two-valued
strategy-proof social choice functions.

We will be considering the case of the range of cardinality three or more in a subsequent
paper.

Facing two-valued functions rather than merely binary ones, entails new serious difficulties
that must be tackled. This is so because the strategy-proofness is not sufficient to guarantee
that the irrelevant alternatives, i.e. those not belonging to the range, play no role. In
general, a strategy-proof social choice function defined on all weak profiles may not be
independent of irrelevant alternatives. Take for example, \( A = \{a, b, c\} \) and let one profile
\( P \) have every voter being indifferent between \( a \) and \( b \) but prefers them over \( c \). Let a second
profile \( Q \) have every voter being indifferent between \( a \) and \( b \) but prefers \( c \) over \( a \) and \( b \). Then
\( \phi(P) \) need not be equal to \( \phi(Q) \) and such \( \phi \) with range \( \{a, b\} \) can be easily constructed
(see Example 2.4, infra).

\(^3\)The Authors describe the rule \( \phi_{\mathcal{F}} \) as voting by committees (cfr. Peleg [13], Barberà et al. [2]).
In the binary setting a central role is played by coalitions of indifferent agents. Here, due to the presence of further alternatives besides those in the range, we needed to give the analogous role to some profiles of preferences that we named partial indifference profiles. A consistent extension to the present setting of the characterization of strategy-proofness by means of the concept of compatibility with dominance introduced in [5], was hence necessary. Since our setting coincides with that of Barberà et al. [4], we also compare their characterization result with ours.

In [9], [8] and [5], the structure of the representation theorems relies on the concept of nonempty superset closed families of coalitions (named committees in the Social Choice literature). Here a slight modification is necessary to treat the complexity of all possible situations: it is necessary to consider the possibility of empty families and also that the sets involved in the families are empty. Given that, the structure of two-valued strategy-proof social choice function we have discovered in Definition 4.1, Theorems 4.4 and 5.2 looks formally exactly like that of the special case we first achieved in [5, Theorem 4.2]. However we wish to emphasize that it is by no means a straight extension of the previous representation, being the step from binary to two-valued functions quite considerable. Finally, we mention that, as already known from [5], our representation is not unique. Moreover we also point out that the possibility of fixing a priori, freely, the parameter $x$ of the representation\(^4\) has to be considered jointly with the fact that the other parameter - what we have called the double collection $(\mathcal{I}, \mathcal{F})$, Definition 3.4 - will coherently change.

The paper is organized as follows. In Section 2 we introduce the dominance relation between profiles and the concept of compatibility of a social choice function with the dominance. The characterization of strategy-proofness by means of compatibility is in Theorem 2.10 and the Section ends with a comparison of compatibility with the notions of essentially ab-based and ab-monotonic due to Barberà et al [4]. Section 3 presents some notions which are preparatory to the definition of index of a profile, and the index itself. This is a tool for the definition in Section 4 of some special functions, that we call of type $\psi$. Theorem 4.4 shows that such functions are strategy-proof, whereas to the fact that there are no other two-valued strategy-proof social choice functions is devoted Section 5. The general representation result is Theorem 5.2.

### 2 Strategy-proofness

Let $\mathcal{W}(A)$ be the set of weak orderings over the set $A$ of alternatives. Functions $\pi$ from subsets of $V$ to $\mathcal{W}(A)$ are named partial profiles of preferences. A partial profile is therefore $\pi = (\pi_v)_{v \in I}$ where $I$ is a subset of $V$ and $\pi_v \in \mathcal{W}(A)$ for every $v \in I$. In case $I = V$, we speak of a (total) profile of preferences, omitting “total”. For the class of all (total) profiles we use the notation $\mathcal{P}$. For the case in which the domain is the empty set, we

\(^4\)One can choose ex ante $x$ as the value of the scf on any profile of unanimous indifference between the alternatives in the range.
reserve the name of empty profile. For a profile \( P = (P_v)_{v \in V} \) we shall also use the notation \( P = [P_I, P_{I^c}] \) if \( I \) is a subset of \( V \) and \( P_I, P_{I^c} \) are the obvious restrictions \( P_I = (P_v)_{v \in I} \), \( P_{I^c} = (P_v)_{v \notin V} \) of \( P \). Extending this notation to arbitrary partitions of \( V \) or to partial profiles is straightforward.

With reference to an ordering \( W \) on the set \( A \) of alternatives, as usual, the notation \( x \succ_W y \) stands for \( (x, y) \in W \), the notation \( x \succ_W y \) stands for \( [(x, y) \in W \text{ and } (y, x) \notin W] \), and the notation \( x \sim_W y \) stands for \( [(x, y) \in W \text{ and } (y, x) \in W] \).

**Definition 2.1** Let \( a \) and \( b \) be two distinct alternatives. A **partial \( \{a, b\}\)-indifference profile** is a partial profile \( \pi \) such that for every voter \( v \in \text{dom}(\pi) \) (the domain of \( \pi \)) one has \( a \sim_{\pi_v} b \).

Trivially, the empty profile is a partial \( \{a, b\}\)-indifference profile. The class of all partial \( \{a, b\}\)-indifference profiles will be denoted by \( \mathcal{P}(a \sim b) \). It contains profiles of unanimous indifference between \( a \) and \( b \) also, namely profiles \( P = (P_v)_{v \in V} \) such that \( a \sim_{P_v} b, \forall v \in V \).

Notice that if \( A \) consists of two alternatives \( a, b \) only, then \( \mathcal{P}(a \sim b) \) can be identified, via the mapping \( \pi \mapsto \text{dom}(\pi) \), with the power set of \( V \).

Throughout the sequel, \( \phi : \mathcal{P} \to A \) stands for a social choice function (scf, for short). Dealing with societies of any cardinality, the notion of strategy-proofness we adopt is based on coalitions of individuals rather than on a single individual. As usual, coalition is synonymous to nonempty subset of \( V \).

**Definition 2.2** Let \( \phi \) be scf. We say that a coalition \( D \) can manipulate a profile \( P \) under \( \phi \) if there is another profile \( Q \) such that

- every voter \( v \) in \( D^c \) has the same preference order in both \( P \) and \( Q \), i.e. \( P_v = Q_v \);
- every voter \( v \) in \( D \) prefers \( \phi(Q) \) to \( \phi(P) \) according to \( P_v \), i.e. \( \phi(Q) \succ_{P_v} \phi(P) \).

When the two conditions above are verified we say that the coalition \( D \) manipulates the profile \( P \) by presenting (or reporting) the profile \( Q \).

A desirable property of a scf is that no group of agents has incentives to form a coalition that, with false reporting, can manipulate the social outcome for their own advantage. This is at the basis of the following definition.

**Definition 2.3** We say that a scf \( \phi \) is **coalitionally strategy-proof** (CSP, for short) if no coalition of voters can manipulate any profile under \( \phi \).
If in the previous definitions we replace arbitrary coalitions with singletons, the notion we identify is that of individual strategy-proofness. It is obvious that every CSP social choice function is individually strategy-proof. On the other hand it is well known that the two concepts of strategy-proofness, individual and coalitional, coincide for the case of $V$ being finite, whereas they do not coincide if $V$ is infinite.

Example 2.4 The purpose of this example is to clarify that strategy-proofness does not imply independence of irrelevant alternatives. Let $V, A$ be consisting of two agents $\{v_1, v_2\}$ and three alternatives $\{a, b, c\}$. A strategy-proof scf $\phi$ with range $\{a, b\}$ can be defined as follows. If for the profile $P$ either one of the following holds true, then we define $\phi(P) = b$. In all the other cases we define $\phi(P) = a$.

```
Profile P
v1  v2
a ~ b ~ c  a ~ b ~ c  φ → a
```

This is so even if the function $\phi$ is strategy-proof and the profiles $P$ and $Q$ are identical on $\{a, b\}$. The fact that $\phi$ cannot be manipulated can be easily shown. Suppose, on the contrary, that a voter $v$ can manipulate a profile $P$ reporting a profile $Q$, we have: $\phi(Q) \succ P_v \phi(P)$ and $P' = Q'$. Without loss of generality we can suppose that $\phi(Q) = b$.

Hence we have $b = P_v \phi(Q) \succ P_v \phi(P) = a$. The voter $v$ cannot be $v_1$ otherwise $\phi(P)$ is $b$ by definition of $\phi$. Hence we have that $v = v_2$ and therefore that $b = P_{v_2} \phi(Q) \succ P_{v_2} \phi(P) = a$ and $P_{v_1} = Q_{v_1}$. By the definition of $\phi$, the fact that $\phi(Q) = b$, $P_{v_1} = Q_{v_1}$, and $\phi(P) = a$, we have that necessarily $a \sim b \succ c$ but this together with $b \succ a$ would give that $\phi$ on $P$ takes value $b$, a contradiction.

Remark 2.5 It is immediate that a scf $\phi$ which is CSP, enjoys the following property:

$\{y \in \text{Range}(\phi) : y \succ P_v \phi(P), \forall v \in V\} = \emptyset$, for every profile $P \in \text{dom}(\phi)$
which is the well known weak Pareto optimality.

Since our purpose is to find a representation formula for CSP social choice functions that have range of cardinality two, we adopt the following approach: we fix a set \(\{a,b\} \subseteq A\) consisting of two distinct alternatives and produce a formula for CSP social choice functions with range \(\{a,b\}\). With this in mind, we introduce the following notations. Let \(a\) and \(b\) be distinct alternatives and \(P\) be a profile, then we set:

- the indifference set of voters: \(I(P) = \{v \in V : a \sim_{P_v} b\}\),
- the set of voters that prefer \(a\) to \(b\): \(D(a,P) = \{v \in V : a \succ_{P_v} b\}\),
- the set of voters that prefer \(b\) to \(a\): \(D(b,P) = \{v \in V : b \succ_{P_v} a\}\).

Also useful is the following notion of \(\{a,b\}\)-equivalence set.

**Definition 2.6** Let \(P, Q\) be two profiles. The \(\{a,b\}\)-equivalence set of \(P\) and \(Q\) consists of the voters \(v \in V\) for which either one of the following three statements hold true:

- \(P_v = Q_v\);
- restricted to \(\{a,b\}\), the preferences \(P_v\) and \(Q_v\) are identical and coincide with \(a \succ b\);
- restricted to \(\{a,b\}\), the preferences \(P_v\) and \(Q_v\) are identical and coincide with \(b \succ a\);

and it is denoted by \(E(P,Q)\).

**Comment 2.7 (about notation)** For the four sets we have introduced, a more appropriate notation should involve the set \(\{a,b\}\) also: \(I_{\{a,b\}}(P)\), \(D_{\{a,b\}}(a,P)\), \(D_{\{a,b\}}(b,P)\), \(E_{\{a,b\}}(P,Q)\). The notational abuse is for the sake of simplicity. Throughout the paper the reference to the two alternatives \(a\) and \(b\) will be unambiguous.

We now introduce the following dominance relation between profiles.

**Definition 2.8** Let \(\{a,b\}\) be two distinct alternatives in \(A\). Given two profiles \(P\) and \(Q\), we say that \(P\) dominates \(Q\) and write \(P \sube_{\{a,b\}} Q\) if either \(V = E(P,Q) \cup [I(P) \cap D(a,Q)]\) or \(V = E(P,Q) \cup [I(P) \cap D(b,Q)]\).

**Comment 2.7 (continuation)** It is for the already invoked sake of simplicity that we do not explicitly speak of \(\{a,b\}\)-dominance and, moreover, we write \(P \sube Q\) instead of \(P \sube_{\{a,b\}} Q\).
When $V = E(P, Q) \cup [I(P) \cap D(a, Q)]$, we adopt the notation $P \{a, b\}_{a} \sqsupset Q$ that, however, we simplify as $P \sqsupset a Q$. We shall say that $P$ $a$-dominates $Q$.

Analogously for $V = E(P, Q) \cup [I(P) \cap D(b, Q)]$ we write $P \{a, b\}_{b} \sqsupset Q$ that we simplify as $P \sqsupset b Q$. We shall say that $P$ $b$-dominates $Q$.

Clearly the unions in the definition of dominance are disjoint unions. Notice that the following are obviously equivalent:

- The simultaneous validity of the two dominances, $P \sqsupset a Q$ and $P \sqsupset b Q$,
- The simultaneous validity of the two dominances, $P \sqsupset a Q$ and $Q \sqsupset a P$,
- The simultaneous validity of the two dominances, $P \sqsupset b Q$ and $Q \sqsupset b P$,
- $V = E(P, Q)$

So in general the dominance $\sqsupset a$ (and $\sqsupset b$) is not antisymmetric ($P \sqsupset a Q$ and $Q \sqsupset a P \not\Rightarrow P = Q$) although it is reflexive and transitive (not complete).

**Definition 2.9** Let $\phi$ be a two-valued scf, and let $\{a, b\}$ be its range. We say that $\phi$ is compatible with the dominance relation $\sqsupset$ when

$$P \begin{array}{c}\phi(P)\end{array} \sqsupset Q \implies \phi(Q) = \phi(P).$$

We shall now characterize coalitional strategy-proofness in terms of dominance.

**Theorem 2.10** A scf $\phi$ with range $\{a, b\}$ is CSP if and only if it is compatible with $\sqsupset$.

**Proof:** Suppose $\phi$ is compatible with $\sqsupset$ and assume that it is not CSP. This means that profiles $P, Q$ exist and a coalition $D$ such that $P$ and $Q$ are identical on $D^c$ and for $v \in D$ we have $\phi(Q) \succ P_v \phi(P)$.

Let us fix, without loss of generality, that $\phi(P) = a$ and $\phi(Q) = b$. Define the following partition of $D$:

$$D_1 = D(b, Q) \cap D, \quad D_2 = D(a, Q) \cap D, \quad D_3 = I(Q) \cap D.$$

and a new profile $R = [R_{D^c}, R_{D_1}, R_{D_2}, R_{D_3}]$, identical to $P$ (and $Q$) on $D^c$, to $P$ on $D_1$, to $Q$ on $D_3$, and arbitrary on $D_2$ as long as it guarantees $a \sim R_v b$ for $v \in D_2$. The figure below describes the situation.
It is straightforward to observe that
\[ E(R, Q) = D_c \cup D_1 \cup D_3; \quad E(R, P) = D_c \cup D_1. \]

This entails
\[ R \sqsupseteq_a Q \text{ and } R \sqsupseteq_b P. \]

Because of the compatibility assumption, \( \phi(R) \) must be \( a \) and \( b \) at the same time, this produces a contradiction. Thus, if \( \phi \) is compatible with \( \sqsupseteq \) then it is CSP.

Now we prove the converse. Let us suppose that \( \phi \) is CSP. We have to prove that
\[ P \mathrel{\phi(P)} \sqsupseteq Q \Rightarrow \phi(Q) = \phi(P). \]

Since \( \phi(P) \in \{a, b\} \), we consider the case \( \phi(P) = a \). The other case can be treated similarly. Let us consider the following partition \( \{V_1, V_2, V_3, V_4\} \) of \( V \):
\[
\begin{align*}
V_1 &= \{v \in V : P_v = Q_v\}; \\
V_2 &= \{v \in V : P_v \neq Q_v; P_v|_{\{a, b\}} = Q_v|_{\{a, b\}} = a \succ b\}; \\
V_3 &= \{v \in V : P_v \neq Q_v; P_v|_{\{a, b\}} = Q_v|_{\{a, b\}} = b \succ a\}; \\
V_4 &= V \setminus E(P, Q),
\end{align*}
\]

since clearly \( \{V_1, V_2, V_3\} \) is a partition of \( E(P, Q) \). The figure below illustrate the situation we have:
\[
\begin{align*}
P &= [P_{v_1}, P_{v_2}, P_{v_3}, P_{v_4}]; \\
Q &= [P_{v_1}, Q_{V_2}, Q_{V_3}, Q_{V_4}]; \\
P_{v_2} &\neq Q_{v_2}; \\
P_{v_3} &\neq Q_{v_3}; \\
[\forall v \in V_2 \Rightarrow a \succ b \& a \succ b]; \\
[\forall v \in V_3 \Rightarrow b \succ a \& b \succ a]; \\
[\forall v \in V_4 \Rightarrow a \sim b \& a \sim b].
\end{align*}
\]
We assume $V_1$ is a proper subset of $V$, otherwise the assertion we need to prove, i.e. $\phi(Q) = a$, would be trivial since $P$ and $Q$ would coincide. One of the sets $\{V_2, V_3, V_4\}$ is nonempty. Without loss of generality we assume $V_2$ is nonempty, and, in every one of the four possible cases determined by the nonemptyness of $V_3, V_4$, we shall prove the assertion.

Case $V_3 = V_4 = \emptyset$.
We have $P = [P_{V_1}, P_{V_2}]$ and $Q = [P_{V_1}, Q_{V_2}]$, therefore if $\phi(Q) = b$, the coalition $V_2$ manipulates $Q$ presenting $P$.

Case $V_3 = \emptyset; V_4 \neq \emptyset$.
We have $P = [P_{V_1}, P_{V_2}, P_{V_4}]$ and $Q = [P_{V_1}, Q_{V_2}, Q_{V_4}]$. Define the profile $S = [P_{V_1}, Q_{V_2}, P_{V_4}]$ over which the value of $\phi$ must be $a$ otherwise $V_2$ presents $P$ and manipulates $S$. At this point also $\phi(Q) = a$ otherwise the coalition $V_4$ can manipulate $Q$ presenting $S$.

Case $V_3 \neq \emptyset; V_4 = \emptyset$.
We have $P = [P_{V_1}, P_{V_2}, P_{V_3}]$ and $Q = [P_{V_1}, Q_{V_2}, Q_{V_3}]$. Define the profile $S = [P_{V_1}, Q_{V_2}, P_{V_3}]$ over which the value of $\phi$ must be $a$ otherwise $V_2$ presents $P$ and manipulates $S$. At this point also $\phi(Q) = a$ otherwise the coalition $V_3$ can manipulate $S$ presenting $Q$.

Case $V_3 \neq \emptyset; V_4 \neq \emptyset$.
In this last case we see that $\phi([P_{V_1}, Q_{V_2}, P_{V_3}, P_{V_4}]) = a$, otherwise the coalition $V_2$ manipulates $[P_{V_1}, Q_{V_2}, P_{V_3}, P_{V_4}]$ presenting $P$. Hence, necessarily $\phi([P_{V_1}, Q_{V_2}, P_{V_3}, Q_{V_4}]) = a$, otherwise $V_4$ manipulates $[P_{V_1}, Q_{V_2}, P_{V_3}, Q_{V_4}]$ presenting $[P_{V_1}, Q_{V_2}, P_{V_3}, P_{V_4}]$. Finally we get $\phi(Q) = a$, otherwise if $\phi(Q) = b$, $V_3$ manipulates $[P_{V_1}, Q_{V_2}, P_{V_3}, Q_{V_4}]$ presenting $Q$. \hfill \Box

**Corollary 2.11** Let $\phi$ be a CSP scf with range $\{a, b\}$. Then

$$V = E(P, Q) \implies \phi(P) = \phi(Q).$$

**Remark 2.12** In Barberà et al. [4], where the society $V$ is finite, the notion of coalitional strategy-proofness is called weak group strategy-proofness and it is characterized as follows.

[4, Theorem 1]: if the range of $\phi$ is $\{a, b\}$, then $\phi$ is CSP if and only if it is **essentially ab-based** and **essentially ab-monotonic**.

The purpose of this Remark is to show directly that a function $\phi$ with range $\{a, b\}$ has both the above properties if and only if it is compatible with the $\{a, b\}$-dominance relation.

We remind that according to [4],

- $\phi$ is essentially ab-based, if the following implication holds true:

$$P, Q \in \mathcal{P}; \quad v \in I(P) \Rightarrow P_v = Q_v; \quad D(a, P) = D(a, Q); \quad D(b, P) = D(b, Q) \quad \Rightarrow \quad \phi(P) = \phi(Q);$$

- $\phi$ is essentially ab-monotonic if the following implications (1) and (2) hold true:
Now we observe that the simultaneous validity of essential ab-based and ab-monotonic conditions can be just restated as the validity of both implication (1) and (2) above without requiring that at least one of the inclusions involved is strict.

**Proposition 2.13** \(\phi\) is essentially ab-based and essentially ab-monotonic if and only if following implications \((1')\) and \((2')\) hold true:

\[
\begin{align*}
(1') \quad P, Q \in \mathcal{P}; & \quad v \in I(P) \cap I(Q) \Rightarrow P_v = Q_v; \quad \phi(P) = a; \\
& \quad D(a, Q) \supseteq D(a, P); D(b, P) \supseteq D(b, Q)
\end{align*}
\]
\[
\Rightarrow \phi(Q) = a.
\]

\[
\begin{align*}
(2') \quad P, Q \in \mathcal{P}; & \quad v \in I(P) \cap I(Q) \Rightarrow P_v = Q_v; \quad \phi(P) = b; \\
& \quad D(b, Q) \supseteq D(b, P); D(a, P) \supseteq D(a, Q)
\end{align*}
\]
\[
\Rightarrow \phi(Q) = b.
\]

On the basis of the previous Proposition, we can proceed to the direct comparison between compatibility with dominance and essential ab-based and ab-monotonic conditions.

**Proposition 2.14** Let \(\phi\) be a scf with range \(\{a, b\}\). Then, \(\phi\) is essentially ab-based and essentially ab-monotonic if and only if it is compatible with \(\{a, b\}\).

**Proof:** Let us assume the compatibility. In order to prove that \(\phi\) is essentially ab-based and ab-monotonic, we show, without loss of generality, the implication \((1')\):

\[
\begin{align*}
(1') \quad P, Q \in \mathcal{P}; & \quad v \in I(P) \cap I(Q) \Rightarrow P_v = Q_v; \quad \phi(P) = a; \\
& \quad D(a, Q) \supseteq D(a, P); D(b, P) \supseteq D(b, Q)
\end{align*}
\]
\[
\Rightarrow \phi(Q) = a.
\]

Once we notice that we cannot find voters \(v\) in \(I(P)\) for which \(b \succ Q_v a\), whereas for agents \(v \in I(P)\) if we have that \(a \sim Q_v b\), then \(P_v = Q_v\), we can represent the left hand side of the implication we have to prove as in the figure below.

| Profile P | \(D(a, P)\) | \(I(P)\) | \(D(b, P)\) | \(\phi\) | \(a\) |
|-----------|--------------|----------|--------------|--------|-----|
| Profile Q | \(a \succ b\) | \(P = Q\) | \(a \succ b\) | \(a \succ b\) | \(a \sim b\) |
|           |              |          |              |        |     |
|           | \(V_1\) | \(V_2\) | \(V_3\) | \(V_4\) | \(V_5\) |
| Profile R | \(a \succ b\) | \(R = P\) | \(a \sim b\) | \(b \succ a\) | \(R = Q\) |

\(^6\)V need not to be finite.

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Let us introduce a new profile $R$. With reference to the partition $\{D(a, P), V_1, V_2, V_3, V_4, V_5\}$ of the society $V$, we take any profile that ensures: on $D(a, P)$ that $a \succ b$ on $I(P) = V_1 \cup V_2$ is identical to $P$, on $V_3$ ensures that $a \sim R_b$, on $V_4$ ensures that $b \succ a$, is $Q$ on the set $V_5$.

It is straightforward to check that $V = E(R, P) \cup [I(R) \cap D(b, P)] = E(R, Q) \cup [I(R) \cap D(a, Q)]$.

what means that the profile $P$ is $b$-dominated by $R$ and the profile $Q$ is $a$-dominated by $R$.

By the compatibility assumption we first have that $\phi(R) = a$ and then that $\phi(Q) = a$ as desired.

For the converse, we take into account that we can assume the validity of the implications (1) and (2) above, without requiring the strictness of the inclusions involved. Let us suppose now that $P \sqsubseteq Q$. We have to obtain that $\phi(Q) = \phi(P)$. Without loss of generality consider $\phi(P) = a$. We observe that we must have $D(a, Q) \supseteq D(a, P)$. Indeed, if this is not the case we find, $v$ with $a \succ P_v b$ and either $a \sim Q_v b$ or $b \succ Q_v a$. Since $V = E(P, Q) \cup [I(P) \cap D(a, Q)]$, and $v \notin E(P, Q)$ we get a contradiction. \(\square\)

### 3 The two classes $\mathcal{P}_a(\pi, \mathcal{F})$ and $\mathcal{P}_b(\pi, \mathcal{F})$ of profiles and the notion of index of a profile

In [5, Definition 2.5], we defined superset closed families of coalitions of voters (also known as committees in the Social Choice literature) as nonempty subsets $\mathcal{F}$ of $2^V \setminus \{\emptyset\}$ that are closed under supersets, i.e. $E \supseteq F \in \mathcal{F} \Rightarrow E \in \mathcal{F}$. It will be useful to have a slight extension of this concept for our case of two-valued rather than binary functions.

**Definition 3.1** $\mathcal{F}$ is a superset closed family on a set $V$ - and we write $\mathcal{F} \in SSCF(V)$ - if: $\mathcal{F} \subseteq 2^V$, and $E \supseteq F \in \mathcal{F} \Rightarrow E \in \mathcal{F}$.

We shall write $\mathcal{F} \in SSCF$ to mean that for some subset $D$ of $V$, $\mathcal{F} \in SSCF(D)$.

Notice that if $\mathcal{F}$ is empty, then it is a superset closed family.

According to the fact that an $\mathcal{F} \in SSCF(V)$ which is nonempty either contains $\emptyset$ or not, we obviously have: in the first case $\mathcal{F}$ is necessarily the power set $2^V$; in the second case it is a superset closed family of coalitions, the largest one being $2^V \setminus \{\emptyset\}$.

If $\mathcal{F} \in SSCF(V)$, setting, as in [5], $E \in \mathcal{F}^\circ \iff E^c \notin \mathcal{F}$ we define the dual of $\mathcal{F}$.

Trivially: $\mathcal{F}^\circ \in SSCF(V)$; $\mathcal{F}^{\circ\circ} = \mathcal{F}$; the empty set ($\mathcal{F} = \emptyset$) and the power set ($\mathcal{G} = 2^V$) are dual to each other; the dual of a superset closed family of coalitions is also a superset closed family of coalitions.
In other words to the dual pairs \((\mathcal{F}, \mathcal{F}^\circ)\) of superset closed family of coalitions considered, solely, in [5] we have just added the pairs \((\emptyset, 2^V)\), \((2^V, \emptyset)\).

Fix two distinct alternatives \(a, b \in A\). We shall now define two useful classes of profiles, denoted by \(\mathcal{P}_a(\pi, \mathcal{F})\) and \(\mathcal{P}_b(\pi, \mathcal{F})\), for \(\pi \in \mathcal{P}(a \sim b)\) and \(\mathcal{F} \in \text{SSCF}(V \setminus \text{dom}(\pi))\).

**Definition 3.2** Let \(\pi\) a partial \((a, b)\)-indifference profile whose domain is the set \(I\). Let \(\mathcal{F}\) a superset closed family on \(I^c\).

We say that a profile \(P \in \mathcal{P}\) belongs to the class \(\mathcal{P}_a(\pi, \mathcal{F})\) if the following two conditions hold:
\[
\begin{align*}
\{ & D(a, P) \cap I^c \in \mathcal{F}; \\
& v \in I \Rightarrow \text{ either } P_v = \pi_v, \text{ or the restriction of } P_v \text{ to } \{a, b\} \text{ is } a \succ b.
\}
\]

Analogously, we say that \(P \in \mathcal{P}\) belongs to \(\mathcal{P}_b(\pi, \mathcal{F})\) if:
\[
\begin{align*}
\{ & D(b, P) \cap I^c \in \mathcal{F}^c; \\
& v \in I \Rightarrow \text{ either } P_v = \pi_v, \text{ or the restriction of } P_v \text{ to } \{a, b\} \text{ is } b \succ a.
\}
\]

**Remark 3.3** Note that:
1. \(\mathcal{P}_a(\pi, \emptyset)\) is empty; \(P \in \mathcal{P}_b(\pi, \emptyset)\) iff \(v \in I \Rightarrow \text{ either } P_v = \pi_v, \text{ or } b \succ a;\)
2. \(P \in \mathcal{P}_a(\pi, 2^I)\) iff \(v \in I \Rightarrow \text{ either } P_v = \pi_v, \text{ or } a \succ b; \quad \mathcal{P}_b(\pi, 2^I)\) is empty.

The relations above cover the case that \(\pi\) is a (total) profile of unanimous indifference between \(a\) and \(b\). Indeed in this case \(I^c\) is empty and the possible choices for \(\mathcal{F}\) on \(I^c\) are only two (each other dual): \(\mathcal{F}\) is empty or the power set of \(I^c\) (\(\mathcal{F} = \{\emptyset\}\)).

3. If \(\pi\) is the empty profile, and hence \(\mathcal{F} \in \text{SSCF}(V)\), we have: \(P \in \mathcal{P}_a(\emptyset, \mathcal{F})\) iff \(D(a, P) \in \mathcal{F}; \quad P \in \mathcal{P}_b(\emptyset, \mathcal{F})\) iff \(D(b, P) \in \mathcal{F}^\circ\).

4. The classes \(\mathcal{P}_a(\pi, \mathcal{F})\) and \(\mathcal{P}_b(\pi, \mathcal{F})\) are disjoint. Indeed, suppose \(P \in \mathcal{P}_a(\pi, \mathcal{F}) \cap \mathcal{P}_b(\pi, \mathcal{F})\). This gives the validity of the conditions: \(D(a, P) \cap I^c \in \mathcal{F}; \quad D(b, P) \cap I^c \in \mathcal{F}^\circ\). In particular: \(D(b, P) \cap I^c \in \mathcal{F}^\circ\) is the same as \(I^c \setminus [D(b, P) \cap I^c \in \mathcal{F}^\circ] \notin \mathcal{F}\). But the latter set is \(I^c \setminus D(b, P)\) and includes \(D(a, P) \cap I^c \in \mathcal{F}\), a contradiction.

5. In case \(A\) contains only the two alternatives \(a\) and \(b\), for the conditions considered in the Definition 3.2 above to define the classes \(\mathcal{P}_a(\pi, \mathcal{F})\) and \(\mathcal{P}_b(\pi, \mathcal{F})\), the second lines reduce simply to \(D(b, P) \cap I = \emptyset\) and to \(D(a, P) \cap I = \emptyset\) respectively. \(\Box\)

For an ordinal \(\beta \geq 1\) (finite or not), let \(\Lambda\) denote the well ordered set \(\{\lambda : 0 \leq \lambda < \beta\}\). For every \(\lambda \in \Lambda\), let \(\pi^\lambda\) be a partial \((a, b)\)-indifference profile. We give a map \(\Pi\) from a well ordered set to \(\mathcal{P}(a \sim b)\), i.e. to \(\Pi = (\pi^\lambda)_{\lambda \in \Lambda} = \{\pi^\lambda : 0 \leq \lambda < \beta\}\), the name of well ordered collection of partial \((a, b)\)-indifference profiles. Similarly, a map \(\mathcal{F}\) from a well ordered set to \(\text{SSCF}\) will be called well ordered collection of superset closed families.
Definition 3.4 Let $a$ and $b$ two distinct alternatives. We call double collection with respect to $\{a,b\}$ a pair $\langle \Pi, F \rangle$ consisting of a well ordered collection $\Pi$ of partial $\{a,b\}$-indifference profiles, and a well ordered collection $F$ of superset closed families such that:

(i) $\Pi$ and $F$ have a common well ordered domain $\Lambda$, i.e. $\Pi = (\pi^\lambda)_{\lambda \in \Lambda}$ and $F = (F_\lambda)_{\lambda \in \Lambda}$

(ii) $F_\lambda \in SSCF(I_\lambda)$ for every $\lambda \in \Lambda$, where $I_\lambda = \text{dom} (\pi^\lambda)$.

We are now ready to introduce the notion of index of a profile. This notion is more general than the corresponding notion considered in [5] for the case of $A = \{a,b\}$.

Definition 3.5 Given the two distinct alternatives $a$ and $b$, let $\langle (\pi^\lambda)_{\lambda \in \Lambda}, (F_\lambda)_{\lambda \in \Lambda} \rangle$ be a double collection with respect to $\{a,b\}$.
We say that a profile $P \in \mathcal{P}$ has $\{a,b\}$-index $\lambda_{\{a,b\}} \in \Lambda$ if the set

$$\{ \lambda \in \Lambda : P \in \mathcal{P}_a(\pi^\lambda, F_\lambda) \cup \mathcal{P}_b(\pi^\lambda, F_\lambda) \}$$

is nonempty and $\lambda_{\{a,b\}}$ is its first element.
If the above set is empty, we say that the index of the profile is $\infty$.

The symbol $\lambda_{\{a,b\}}(P)$ for the index of $P$ will be shortened to $\lambda(P)$ since there is no ambiguity. Notice that if, as in [5], there are only two alternatives $a$ and $b$, since in this case partial indifference profiles can be identified with their domain, the concept of double collection here, coincide with that in [5]. Moreover, by 5. in Remark 3.3, Definition 3.5 above reduces to [5, Definition 3.1].

4 Building strategy-proof social choice functions

In this section we introduce a formula (formula (⋆) below) for the construction of CSP scfs whose ranges are of cardinality two. This formula makes use of the index $\lambda$ defined before. We shall see in the next section that every CSP scf with a range of cardinality at most two can be constructed in this way. In other words the formula we are speaking about will give a representation (next Theorem 5.2) for all two-valued CSP social choice functions.

Definition 4.1 A scf will be said to be of $\psi$-type if for a suitable choice of:

- $\{a,b\} \subseteq A$,
- $x \in \{a,b\}$, and
- a double collection $\langle \Pi, F \rangle$ with respect to $\{a,b\}$,
for every profile \( P \) the corresponding value is determined as follows:

\[
(\star) \quad P \in \mathcal{P} \mapsto \begin{cases} 
  a, & \text{if } P \in \mathcal{P}_a(\pi_0(P), \mathcal{F}_0(P)) \\
  b, & \text{if } P \in \mathcal{P}_b(\pi_0(P), \mathcal{F}_0(P)) \\
  x, & \text{if } \lambda(P) = \infty
\end{cases}
\]

A scf of \( \psi \)-type will be denoted by \( \psi \), omitting, for our sake of notational simplicity, the reference to \( \{a, b\} \), \( x \) and \( (\Pi, \mathcal{F}) \).

The function \( \phi \) of Example 2.4.

Let us consider the function \( \phi \) of Example 2.4. We recognize it as defined by means of \((\star)\) if we take:

- the partial \( \{a, b\}\)-indifference profiles \( \pi^0, \pi^1 \) defined both on \( \{v_1\} \) respectively as follows: \( a \sim b \succ c; \ c \succ a \sim b \);
- the corresponding families \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) on \( \{v_2\} \) respectively as follows: \( \mathcal{F}_0 = \{\{v_2\}\} = \mathcal{F}_0^c; \mathcal{F}_1 = \text{empty}, \mathcal{F}_1^c = \{\emptyset, \{v_2\}\} \).

Then we see that

a profile \( P \) belongs to \( \mathcal{P}_a(\pi^0, \mathcal{F}_0) \) iff \( a \succ P_{v_2} \) and either one of the following is true:

\( P_{v_1} = \pi^0_{v_1} \) or \( a \succ \overset{P_{v_2}}{P_{v_1}} \);

a profile \( P \) belongs to \( \mathcal{P}_b(\pi^0, \mathcal{F}_0) \) iff \( b \succ P_{v_2} \) and either one of the following is true:

\( P_{v_1} = \pi^0_{v_1} \) or \( b \succ \overset{P_{v_2}}{P_{v_1}} \);

\( \mathcal{P}_a(\pi^1, \mathcal{F}_1) \) is empty whereas a profile \( P \) belongs to \( \mathcal{P}_b(\pi^1, \mathcal{F}_1) \) iff either one of the following is true:

\( P_{v_1} = \pi^1_{v_1} \) or \( b \succ \overset{P_{v_2}}{P_{v_1}} \).

Finally, the function \( \phi \) is the \( \psi \)-type scf associated to the double collection \( (\pi^0, \pi^1), (\mathcal{F}_0, \mathcal{F}_1) \) and to \( x = a \).

Proposition 4.2 Suppose \( \psi \) is associated to \( x \in \{a, b\} \), and \( (\Pi, \mathcal{F}) \). Suppose the double collection consists just of one element \( (\pi^0, \mathcal{F}_0) \) (say \( \beta \) is 1). Then, the range of \( \psi \):

1. contains \( a \) and \( x \), if \( \mathcal{F}_0 \) is the power set of \( I_0^c \) (and the dual family \( \mathcal{F}_0^c \) is empty)
2. contains \( b \) and \( x \), if \( \mathcal{F}_0 \) is empty (and the dual family \( \mathcal{F}_0^c \) is the power set of \( I_0^c \))
3. is \( \{a, b\} \), if both \( \mathcal{F}_0 \) and \( \mathcal{F}_0^c \) are nonempty.

In cases (1) and (2) the scf \( \psi \) may be constant.
The assumption that \( P(v) \) voter

With this purpose in mind, we prove the following two claims under the assumption that

\[
[Q \in \mathcal{P}_a(\pi, \mathcal{F}) \iff I^c \in \mathcal{F} \iff \mathcal{F} \text{ nonempty}] \quad \text{and} \quad Q \notin \mathcal{P}_b(\pi, \mathcal{F}).
\]

Similarly, if \( R \) ensures that \( D(b, R) = V \), then

\[
R \notin \mathcal{P}_a(\pi, \mathcal{F}) \quad \text{and} \quad [R \in \mathcal{P}_b(\pi, \mathcal{F}) \iff I^c \in \mathcal{F}^c \iff \mathcal{F}^c \text{ nonempty}].
\]

If the family \( \mathcal{F} \) is the power set of \( I^c \), we saw (Remark 3.3) that necessarily the class \( \mathcal{P}_b(\pi, \mathcal{F}) \) is empty\(^7\), and this means that we cannot guarantee a priori the value \( b \) for \( \psi \). If the family \( \mathcal{F}^c \) is the power set of \( I^c \), we saw that necessarily the class \( \mathcal{P}_a(\pi, \mathcal{F}) \) is empty\(^8\), and this means that we cannot guarantee a priori the value \( a \) for \( \psi \). Combining all the above remarks, the statements (1), (2), and (3) follow.

\[ \square \]

**Corollary 4.3** Suppose \( \psi \) is associated to \( x \in \{a, b\} \), and \( \langle \Pi, \mathcal{F} \rangle \). Suppose that both \( \mathcal{F}_0 \) and \( \mathcal{F}_0^c \) are nonempty, then, the range of \( \psi \) is \( \{a, b\} \).

Notice that in [5] it is always the case that for the first index 0 of \( \Lambda \) both \( \mathcal{F}_0 \) and \( \mathcal{F}_0^c \) are nonempty. The \( \psi \)-type functions introduced here specialize to those defined in [5].

We now have the main result of this section.

**Theorem 4.4** Every \( \psi \)-type function is coalitionally strategy-proof.

**Proof:** Suppose \( \psi \) is associated to \( x \in \{a, b\} \), and the double collection \( \langle \Pi, \mathcal{F} \rangle \) with respect to \( \{a, b\} \). We assume the range is \( \{a, b\} \). By appealing to Theorem 2.10 we have to show that \( P \triangleright \_ Q \Rightarrow \psi(Q) = \psi(P) \).

With this purpose in mind, we prove the following two claims under the assumption that \( P \triangleright \_ Q \),

\[
(\text{claim 1}) \quad [\lambda(P) < \infty, \psi(P) = b] \Rightarrow [Q \in \mathcal{P}_b(\pi^{\lambda(P)}, \mathcal{F}_{\lambda(P)})].
\]

\[
(\text{claim 2}) \quad [\lambda(Q) < \infty, \psi(Q) = a] \Rightarrow [P \in \mathcal{P}_a(\pi^{\lambda(Q)}, \mathcal{F}_{\lambda(Q)})].
\]

The assumption that \( P \triangleright \_ Q \), i.e. \( V = \mathcal{E}(P, Q) \cup [I(P) \cap D(b, Q)] \), means that for every voter \( v \) one of the following four circumstances holds true:

(i) \( P_v = Q_v \);

\(^7\)Also: \( P \in \mathcal{P}_a(\pi, \mathcal{F}) \) is the same as \( [v \in I \Rightarrow \text{either } P_v = \pi_v \text{ or } a \succ b] \); notice that \( \phi(\pi) = \phi(Q) = a \) and \( \phi(P) = x \) if the profile \( P \in \mathcal{P}(a \sim b) \) and \( P_t \neq \pi \).

\(^8\)Also: \( P \in \mathcal{P}_b(\pi, \mathcal{F}) \) is the same as \( [v \in I \Rightarrow \text{either } P_v = \pi_v \text{ or } b \succ a] \).

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(ii) the restrictions to \( \{a, b\} \) of the preferences \( P_v \) and \( Q_v \) are identical and coincide with \( a \succ b \);
(iii) the restrictions to \( \{a, b\} \) of the preferences \( P_v \) and \( Q_v \) are identical and coincide with \( b \succ a \);
(iv) \( a \sim_{P_v} b \) and \( b \succ_{Q_v} a \).

**Proof of claim 1:** By definition of \( \psi \), we have that \( P \in \mathcal{P}_b(\pi_{\lambda(P)}, \mathcal{F}_\lambda(P)) \). In its turn this means:

\[
\begin{align*}
\{ D(b, P) \cap I^c_{\lambda(P)} \in \mathcal{F}_\lambda(P) \} \\
[*] v \in I_{\lambda(P)} \Rightarrow \text{either } P_v = \pi_{\lambda(P)} \text{, or the restriction of } P_v \text{ to } \{a, b\} \text{ is } b \succ a.
\end{align*}
\]

Since for voters in \( D(b, P) \) we can only have either (i) or (iii), we recognize that \( D(b, P) \subseteq D(b, Q) \), and by the superset closedness we have

\[
D(b, Q) \cap I^c_{\lambda(P)} \in \mathcal{F}_\lambda(P).
\]

We also have that

\[
[**] v \in I_{\lambda(P)} \Rightarrow \text{either } Q_v = \pi_{\lambda(P)} \text{, or the restriction of } Q_v \text{ to } \{a, b\} \text{ is } b \succ a,
\]

and these two facts give the assertion of claim 1.

To check \([**]\), notice that if \( v \in I_{\lambda(P)} \) and \( P_v \neq Q_v \), because of \([*]\) either (iii) or (iv) is true and consequently the restriction of \( Q_v \) to \( \{a, b\} \) is \( b \succ a \). \( \Box \)

**Proof of claim 2:** Quite similarly to what before, by definition of \( \psi \), we have that \( Q \in \mathcal{P}_a(\pi_{\lambda(Q)}, \mathcal{F}_\lambda(Q)) \). In its turn this means:

\[
\begin{align*}
\{ D(a, Q) \cap I^c_{\lambda(Q)} \in \mathcal{F}_\lambda(Q) \} \\
[*'] v \in I_{\lambda(Q)} \Rightarrow \text{either } Q_v = \pi_{\lambda(Q)} \text{, or the restriction of } Q_v \text{ to } \{a, b\} \text{ is } a \succ b.
\end{align*}
\]

Since for voters in \( D(a, Q) \) we can only have either (i) or (ii), we recognize that \( D(a, Q) \subseteq D(a, P) \), and by the superset closedness we have

\[
D(a, P) \cap I^c_{\lambda(Q)} \in \mathcal{F}_\lambda(Q).
\]

We also have that

\[
[**'] v \in I_{\lambda(Q)} \Rightarrow \text{either } P_v = \pi_{\lambda(Q)} \text{, or the restriction of } P_v \text{ to } \{a, b\} \text{ is } a \succ b,
\]

and these two facts give the assertion of claim 2.

For \([**']\), notice that if \( v \in I_{\lambda(Q)} \) and \( P_v \neq Q_v \), because of \([*'\] we have (ii) and consequently the restriction of \( P_v \) to \( \{a, b\} \) is \( a \succ b \). \( \Box \)
We can now move to the proof of

\[ P \models_{\psi(P)} Q \Rightarrow \psi(Q) = \psi(P), \]

in the case that \( \psi(P) = b \). We are not losing any generality in this since the case that \( \psi(P) = a \) can be treated similarly by means of an obvious modification of claims 1 and 2. Suppose the index \( \lambda(P) \) is finite. Then we have by claim 1 that \( \lambda(Q) \leq \lambda(P) \). Consequently if \( \psi(Q) = a \), by claim 2 we get \( \lambda(Q) = \lambda(P) \) and also \( P \in \mathcal{P}_a(\pi^{\lambda(P)}, \mathcal{F}_{\lambda(P)}) \) which contradicts the assumption that \( \psi(P) = b \).

Suppose now that \( \lambda(P) = \infty \). This means, by definition of \( \psi \), that \( \psi(P) = b = x \). It follows that if \( \psi(Q) = a \), it must be true that \( \lambda(Q) < \infty \), then again claim 2 applies and \( \lambda(P) \) should be finite. \( \square \)

## 5 The structure theorem

In this section we shall see that every CSP scf \( \phi \) defined on all weak profiles, whose range is \( \{a, b\} \), can be represented by means of the formula (★) in Definition 4.1. In order to achieve this result we shall first identify particular classes of profiles based on top choice only (choice between \( a \) and \( b \)), that necessarily have to go to prescribed values. This is essentially the content of the following lemma.

Let \( \{\phi = x\} \) denote the set of all profiles \( P \) attaining value \( x \) under \( \phi \), i.e. \( \phi(P) = x \).

**Proposition 5.1** Let \( \phi : \mathcal{P} \to A \) be CSP. Assume that the range of \( \phi \) is \( \{a, b\} \). Then there exists a unique superset closed family \( \mathcal{F} \) of coalitions such that, if \( \pi \) is the empty profile, one has: \( \mathcal{P}_a(\pi, \mathcal{F}) \subseteq \{\phi = a\} \), and \( \mathcal{P}_b(\pi, \mathcal{F}) \subseteq \{\phi = b\} \).

**PROOF:** Take the family \( \mathcal{F} \) accordingly to Theorem 1.1 (i) applied to the restriction of \( \phi \) to the subset of \( \mathcal{P} \) of all strict profiles. We show the implication

\[ (+) \quad D(a, P) \in \mathcal{F} \Rightarrow \phi(P) = a. \]

Similarly one can obtain the other implication

\[ (+++) \quad D(b, P) \in \mathcal{F} \Rightarrow \phi(P) = b. \]

Let \( P \) be an arbitrary profile. We can assume that the coalition \( D(a, P) \) is a proper subset of \( V \). 9. Let us take two strict orderings \( S_1 \) and \( S_2 \) on \( A \) that guarantee \( a \succ_{S_1} b \) and \( b \succ_{S_2} a \), and define a strict profile \( \widehat{P} \) in the following way.

---
9Otherwise we just use the Pareto optimality (see Remark 2.5).
\[ \hat{P}_v = \begin{cases} S_1, & \text{for } v \in D(a,P) \\ S_2, & \text{for } v \notin D(a,P) \end{cases} \]

Since \( D(a,P) = D(a,\hat{P}) \) and the implication we need to prove is true for strict profiles (as Theorem 1.1 (i) says), we have that \( \phi(\hat{P}) = a \). We also have \( \phi([P_{D(a,P)},\hat{P}_{D(a,P)^c}]) = a \), otherwise the coalition \( D(a,P) \) manipulates the profile \([P_{D(a,P)},\hat{P}_{D(a,P)^c}]\) by presenting \( \hat{P} \).

We can conclude \( \phi(P) = a \) as desired since otherwise the coalition \( D(a,P)^c \) manipulates the profile \([P_{D(a,P)},\hat{P}_{D(a,P)^c}]\) by presenting \( P \).

According to point 3. of Remark 3.3, we can rewrite the above implications (+) and (++) respectively as \( P_a(\pi,F) \subseteq \{ \phi = a \} \) and \( P_b(\pi,F) \subseteq \{ \phi = b \} \). For the uniqueness of \( F \), the same argument used in [5, Proposition 4.1 ] applies. \( \square \)

We are now ready to show that all CSP scf are necessarily of \( \psi \)-type. Let us recall that:

**Theorem 5.2** Let \( \phi : \mathcal{P} \to A \) be a scf which is coalitionally strategy-proof and has range \( \{a,b\} \subseteq A \) of cardinality two. Let \( \pi \) be a profile of unanimous indifference between the alternatives \( a \) and \( b \), and set \( x = \phi(\pi) \).

Then, we can find a double collection \( \langle \Pi, F \rangle = \langle (\pi^\lambda)_{0 \leq \lambda < \beta}, (F_\lambda)_{0 \leq \lambda < \beta} \rangle \) with respect to \( \{a,b\} \) such that the function \( \phi \) coincides with the \( \psi \)-type function associated with \( \langle \Pi, F \rangle \) and \( x \).

Moreover, the dual families \( F_0 \) and \( F^0 \) are both nonempty, \( ^{10} \) and the ordinal \( \beta \) is finite if \( V \) and \( A \) are both finite.

The proof of this theorem will be achieved by transfinite induction. Before going into the details of the proof, it is convenient to adopt some notation.

\( (N_1) \) All profiles will be enumerated as \( P^0, P^1, \ldots, P^n, \ldots : \eta < \delta \) where \( \delta \) is some ordinal. We shall denote this enumeration by \( \mathcal{E} \). In other words we have \( \mathcal{P} = \{ P^n : \eta < \delta \} \) for a suitable ordinal \( \delta \). Note that \( \delta \) is finite if \( V \) and \( A \) are finite. To fix ideas we can assume that \( P^0 \) is the profile \( \pi \).

\( (N_2) \) If we have a partial \( \{a,b\} \)-indifference profile \( \pi^\lambda \), and \( F_\lambda \in SCF(V \setminus \text{dom}(\pi^\lambda)) \), for the set of profiles \( \mathcal{P}_a(\pi^\lambda,F_\lambda) \cup \mathcal{P}_b(\pi^\lambda,F_\lambda) \) we shall use the notation \( \mathcal{P}_\lambda \), for the sake of simplicity.

\( (N_3) \) For every ordinal \( \alpha \), the set \( \Delta_\alpha \) is defined as

\[ \{ \eta < \delta : P^n \in \mathcal{P} \setminus \bigcup_{0 \leq \lambda < \alpha} \mathcal{P}_\lambda \text{ and } \phi(P^n) \neq x \} =: \Delta_\alpha. \]

\( ^{10} \)Namely, they consist of coalitions.

Naturally, compare Theorem 5.2 with the combination of Corollary 4.3 and Theorem 4.4.
When $\Delta_\alpha \neq \emptyset$, $Q^\alpha$ stands for the profile whose index in the enumeration $\mathcal{E}$ is the first element of $\Delta_\alpha$. In other words $Q^\alpha = P^{\eta_\alpha}$, where $\eta_\alpha := \min \Delta_\alpha$.

Notice that if $\alpha < \beta$, then $\Delta_\alpha \supseteq \Delta_\beta$, hence $\eta_\alpha \leq \eta_\beta$.

The following is the key technical tool in order to obtain our representation theorem stated in Theorem 5.2.

**Lemma 5.3** Let $\phi : \mathcal{P} \rightarrow A$ be a scf which is coalitionally strategy-proof and has range $\{a, b\} \subseteq A$ of cardinality two. Let $\pi$ be a profile of unanimous indifference between the alternatives $a$ and $b$, and set $x = \phi(\pi)$. Assume that

(i) $\pi^0$ is the empty profile, $\mathcal{F}_0$ is a superset closed family of coalitions on $V$ with the following property:

$$\{\phi = a\} \supseteq \mathcal{P}_a(\pi^0, \mathcal{F}_0) \quad \text{and} \quad \{\phi = b\} \supseteq \mathcal{P}_b(\pi^0, \mathcal{F}_0)$$

(ii) $\beta$ is an ordinal,

(iii) $0 < \alpha < \beta \Rightarrow \Delta_\alpha \neq \emptyset$, and $\alpha \leq \eta_\alpha$,

(iv) for every $0 < \alpha < \beta$, correspondingly to the profile $Q^\alpha$, there exists a family $\mathcal{F}_\alpha \in \text{SSCF}(V \setminus \mathcal{I}(Q^\alpha))$ such that, setting $I_\alpha := I(Q^\alpha)$, $\pi^\alpha_v := Q^\alpha_v$ (for all $v \in I_\alpha$), we have:

$$Q^\alpha \in \mathcal{P}_\alpha \setminus \bigcup_{0 < \alpha < \beta} \mathcal{P}_\alpha,$$

$$\{\phi = a\} \supseteq \bigcup_{0 < \alpha < \beta} \mathcal{P}_a(\pi^\alpha, \mathcal{F}_\alpha), \quad \text{and} \quad \{\phi = b\} \supseteq \bigcup_{0 < \alpha < \beta} \mathcal{P}_b(\pi^\alpha, \mathcal{F}_\alpha).$$

Under the above assumptions we have that either one of the following holds true:

1. $\Delta_\beta = \emptyset$, and the function $\phi$ coincides with the $\psi$-type function associated with $\langle \Pi, \mathcal{F} \rangle$ and $x$, where the double collection $\langle \Pi, \mathcal{F} \rangle$ consists of the $\pi^\alpha$, and the $\mathcal{F}_\alpha$ for all $0 \leq \alpha < \beta$;

2. $\Delta_\beta \neq \emptyset$, and for $Q^\beta$ there exists a family $\mathcal{F}_\beta \in \text{SSCF}(V \setminus \mathcal{I}(Q^\beta))$ such that, setting $I_\beta := I(Q^\beta)$, $\pi^\beta_v := Q^\beta_v$ (for all $v \in I_\beta$), we have:

$$Q^\beta \in \mathcal{P}_\beta \setminus \bigcup_{0 \leq \lambda < \beta} \mathcal{P}_\lambda,$$

$$Q^\beta \notin \{Q^\alpha : 0 < \alpha < \beta\}, \quad \pi \notin \mathcal{P}_\beta, \quad \beta \leq \eta_\beta,$$

$$\{\phi = a\} \supseteq \mathcal{P}_a(\pi^\beta, \mathcal{F}_\beta), \quad \text{and} \quad \{\phi = b\} \supseteq \mathcal{P}_b(\pi^\beta, \mathcal{F}_\beta).$$

\[11\text{Hence, by the definition (N4) of the profile } Q^\alpha, \text{ we have } Q^\alpha \in \mathcal{P}_\alpha \setminus \left( \bigcup_{0 \leq \lambda < \alpha} \mathcal{P}_\lambda \right).\]
PROOF: We can distinguish two cases that will give rise to either 1. or 2.
Suppose that over \( \mathcal{P} \setminus (\bigcup_{0 \leq \alpha < \beta} \mathcal{P}_\alpha) \) the function \( \phi \) takes one value only. This value is necessarily \( x \). Then what stated in 1. is obviously true, given the assumptions.

Hence, let us assume that over \( \mathcal{P} \setminus (\bigcup_{0 \leq \alpha < \beta} \mathcal{P}_\alpha) \) the function \( \phi \) has range \( \{a, b\} \). According to (\( N_3 \)) this is the same as \( \Delta_\beta \neq \emptyset \).

By definition (\( N_4 \)), \( Q^\beta \in \mathcal{P} \setminus (\bigcup_{0 \leq \alpha < \beta} \mathcal{P}_\alpha) \) and \( \phi(Q^\beta) \neq \phi(\pi) \). To fix ideas, we can assume, without loss of generality, that \( \phi(Q^\beta) = a \).

Set \( I_\beta := I(Q^\beta) \) and notice that this is a nonempty set. Also notice that \( Q^\alpha / \notin \{Q^\alpha : 0 < \alpha < \beta \} \) since every \( Q^\alpha \in \mathcal{P}_\alpha \). Since we know that \( \eta_\alpha \leq \eta_\beta \), we entail that \( \eta_\alpha < \eta_\beta \), otherwise we get \( Q^\alpha = Q^\beta \) which is false. Moreover \( \beta \leq \eta_\beta \). Indeed, if not, we have \( \beta > \eta_\beta \), then, setting \( \eta_\beta = \alpha \), by assumption, from \( \alpha < \beta \), we have \( \alpha \leq \eta_\alpha \), hence \( \eta_\beta \leq \eta_\alpha \), whereas we have just seen that the converse is true.

The restriction of \( Q^\beta \) to \( I_\beta \) is a partial \( \{a, b\} \)-indifference profile that we denote by \( \pi^\beta \), so \( \pi^\beta = (Q^\beta)_v \in I_\beta \).

We now associate to \( \pi^\beta \) a family \( F_\beta \) on \( I^c_\beta \) according to the following procedure. In case \( V = I_\beta \) we set \( F_\beta = \{\emptyset\} \) (hence the dual family \( F^\circ_\beta \) is empty). In case \( I_\beta \) is properly contained in \( V \), we distinguish two cases according to the cardinality of the range of the restriction

\[
\phi_{I^c_\beta}(P_{I^c_\beta}) := \phi(\pi^\beta, P_{I^c_\beta})
\]

of \( \phi \) to the coalition \( I^c_\beta \).

Such a restriction alway takes value \( a \) since \( \phi_{I^c_\beta}(Q^\beta_{I^c_\beta}) = \phi(\pi^\beta, Q^\beta_{I^c_\beta}) = \phi(Q^\beta) \). It may be possible that \( b \) is never attained by means of the restriction. In this case we define \( F_\beta \) as the power set of \( I^c_\beta \) (hence the dual family \( F^\circ_\beta \) is empty).

Notice that (see Remark 3.3) in both cases considered so far, we have

\[
P \in \mathcal{P}_a(\pi^\beta, F_\beta) \text{ iff for every voter } v \in I_\beta \text{ either } P_v = \pi^\beta_v = Q^\beta_v \text{ or } a \succ b,
\]

(though \( \mathcal{P}_b(\pi^\beta, F_\beta) \) is empty) from which we get

\[
(\diamond) \quad \mathcal{P}_a(\pi^\beta, F_\beta) \subseteq \{\phi = a\}.
\]

Indeed, for every \( P \in \mathcal{P}_a(\pi^\beta, F_\beta) \) we have

| Profile P | Q^\beta | a \succ b |
|-----------|--------|--------|
| I^c_\beta | I^c_\beta | I^c_\beta |

\( ^{12} \) For a profile \( P \) the condition \( I(P) = \emptyset \) gives \( P \in \mathcal{P}_0 \). This is a consequence of (i) and what observed in point 3. of Remark 3.3.

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where \( I'_{\beta} = \{ v \in I_{\beta} : P_v = Q_{\beta}^b \} \) and \( I''_{\beta} = \{ v \in I_{\beta} : v \notin I'_{\beta} \& a \succ b \} \), and by strategy-proofness \( \phi(P) = a \) otherwise the coalition \( I''_{\beta} \) (only the case this set is nonempty is significant) manipulates \( P \) by presenting \( Q_{I''_{\beta}} \).

In case the function \( \phi_{I''_{\beta}} \) takes both values \( a \) and \( b \) we can apply Proposition 5.1 to this function determining a superset closed family \( F_{\beta} \) of subcoalitions of \( I''_{\beta} \) such that

\[
D(a, P_{I''_{\beta}}) \in F_{\beta} \Rightarrow \phi_{I''_{\beta}}(P_{I''_{\beta}}) = a,
\]

\[
D(b, P_{I''_{\beta}}) \in F_{\beta} \Rightarrow \phi_{I''_{\beta}}(P_{I''_{\beta}}) = b.
\]

From Definition 3.2 we have again as before the inclusion (\( \phi \)) to which we can add the parallel inclusion

\[
(\phi') \quad P_b(\pi^\beta, F_{\beta}) \subseteq \{ \phi = b \}.
\]

Indeed, for every \( P \in P_b(\pi^\beta, F_{\beta}) \) we have

\[
\begin{array}{c|c|c|c}
Profile & P & Q^a & b \succ a \\
\hline
I_{\beta} & I_{\beta} & I_{\beta} &
\end{array}
\]

where the only change is that \( I''_{\beta} = \{ v \in I_{\beta} : v \notin I'_{\beta} \& b \succ a \} \), and by strategy-proofness \( \phi(P) = b \) otherwise the coalition \( I''_{\beta} \) (only the case this set is nonempty is significant) manipulates \( P \) by presenting \( Q_{I''_{\beta}} \).

Having, at this stage, defined \( \pi^\beta \) and \( F_{\beta} \), by construction we have\(^{13} \) \( Q^\beta \in P_{\beta} = P_a(\pi^\beta, F_{\beta}) \cup P_b(\pi^\beta, F_{\beta}) \) and we have seen that \( P_a(\pi^\beta, F_{\beta}) \subseteq \{ \phi = a \} \), and \( P_b(\pi^\beta, F_{\beta}) \subseteq \{ \phi = b \} \).

To see that \( \pi \notin P_{\beta} \), we notice what follows.

In case that \( F_{\beta} \) is the power set of \( I_{\beta} \): then, because of formula (\( \phi \)) we have \( \pi \notin P_a(\pi^\beta, F_{\beta}) \) and, on the other hand, \( P_b(\pi^\beta, F_{\beta}) \) is empty.

In case that \( F_{\beta} \) comes from the application of Proposition 5.1 and, therefore, is a superset closed family of subcoalitions of \( I_{\beta} \): observe that \( D(a, \pi) \cap I_{\beta} = D(b, \pi) \cap I_{\beta} = \emptyset \), so \( \pi \notin P_a(\pi^\beta, F_{\beta}) \) and \( \pi \notin P_b(\pi^\beta, F_{\beta}) \).

We can now move to the proof of Theorem 5.2

**Proof:**

Setting \( \lambda = 0 \), \( I_0 = \emptyset \), and \( \pi^0 = \text{the empty profile} \), we define the family \( F_0 \) by appealing to Proposition 5.1.

Notice that none of the profiles unanimously indifferent between \( a \) and \( b \) belongs to \( P_0 \).

We can now distinguish two cases according to the fact that:

\(^{13}\)See Definition 3.2. Precisely, due to the assumption that \( \phi(Q^\beta) = a \), the profile \( Q^\beta \in P_a(\pi^\beta, F_{\beta}) \).
• Case 1: over $\mathcal{P} \setminus \mathcal{P}_0$ the function $\phi$ takes one value only.

• Case 2: over $\mathcal{P} \setminus \mathcal{P}_0$ the function $\phi$ has range $\{a, b\}$.

Suppose we are in case 1. Then, the value attained by $\phi$ is necessarily $x$. Hence, the theorem is proved since $\phi$ coincides with the $\psi$-type function associated with the double collection $\langle \pi^0, \mathcal{F}_0 \rangle$ and $x$ (i.e. $\beta$ is one).

On the contrary, suppose that over $\mathcal{P} \setminus \mathcal{P}_0$ the function $\phi$ has range $\{a, b\}$. In this case if we set $\beta = 2$, all the assumptions of Lemma 5.3 are satisfied. Indeed, note that the present case exactly says that $\Delta_1 \neq \emptyset$, and what is needed to show is that for the profile $Q^1$ there is a family $\mathcal{F}_1 \in \text{SSCF}(I^1)$ such that setting $\pi^1 = Q^1_{I^1}$, we have $Q^1 \in \mathcal{P}_1$, $\pi \notin \mathcal{P}_1$, $\{\phi = a\} \supseteq \mathcal{P}_a(\pi^1, \mathcal{F}_1)$, and $\{\phi = b\} \supseteq \mathcal{P}_b(\pi^1, \mathcal{F}_1)$.

The argument is pretty much the same as in the proof of Lemma 5.3, and we summarize it in the following.

The set $I_1 := I(Q^1)$ is nonempty. To the partial $\{a, b\}$-indifference profile $\pi^1 = (Q^1_v)_{v \in I_1}$ we associate a family $\mathcal{F}_1$ on $I^c_1$ according to the following procedure. In case $V = I_1$ we set $\mathcal{F}_1 = \{\emptyset\}$ (hence the dual family $\mathcal{F}_1^c$ is empty). In case $I_1$ is properly contained in $V$, we distinguish two cases according to the cardinality of the range of the restriction

$$\phi_{I^c_1}(P_{I^c_1}) := \phi([\pi^1, P_{I^c_1}])$$

of $\phi$ to the coalition $I^c_1$.

Such a restriction always takes value $\phi(Q^1)$ since $\phi_{I^c_1}(Q^1_{I^c_1}) = \phi([\pi^1, Q^1_{I^c_1}]) = \phi(Q^1)$. It is possible that $x$ is never attained by $\phi_{I^c_1}$. In this case we define $\mathcal{F}_1$ as the power set of $I^c_1$ (hence the dual family $\mathcal{F}_1^c$ is empty).

In case the function $\phi_{I^c_1}$ takes both values $a$ and $b$ we can apply Proposition 5.1 to this function determining a superset closed family $\mathcal{F}_1$ of subcoalitions of $I^c_1$ such that $D(a, P_{I^c_1}) \in \mathcal{F}_1 \Rightarrow \phi_{I^c_1}(P_{I^c_1}) = a$, and $D(b, P_{I^c_1}) \in \mathcal{F}_1 \Rightarrow \phi_{I^c_1}(P_{I^c_1}) = b$.

Having, at this stage, defined $\mathcal{F}_1$, we can verify that

$$(\diamond) \quad \mathcal{P}_a(\pi^1, \mathcal{F}_1) \subseteq \{\phi = a\},$$

$$(\diamond') \quad \mathcal{P}_b(\pi^1, \mathcal{F}_1) \subseteq \{\phi = b\},$$

$Q^1 \in \mathcal{P}_1 = \mathcal{P}_a(\pi^1, \mathcal{F}_1) \cup \mathcal{P}_b(\pi^1, \mathcal{F}_1)$, and $\pi \notin \mathcal{P}_1$.

Once the assumptions of Lemma 5.3 have been proved, by this lemma, either we get that $\phi$ is the $\psi$-type function associated to the double collection $\langle (\pi^0, \pi^1), (\mathcal{F}_0, \mathcal{F}_1) \rangle$ and $x$, or it will be possible to repeat the application of Lemma 5.3 (with $\beta = 3$) since we shall have $\Delta_2 \neq \emptyset$ and the family $\mathcal{F}_2$, the profile $\pi^2 = Q^2_{I_2}$ with:

\footnote{Trivially 1 ≤ $\eta_1$.}
$Q^2 \notin \{Q^1\}, \ 2 \leq \eta_2, Q^2 \in \mathcal{P}_2 \setminus (\mathcal{P}_0 \cup \mathcal{P}_1), \ \pi \notin \mathcal{P}_2, \ \mathcal{P}_a(\pi^2, \mathcal{F}_2) \subseteq \{\phi = a\}, \ \mathcal{P}_b(\pi^2, \mathcal{F}_2) \subseteq \{\phi = b\}.$

Then, either we get that $\phi$ is the $\psi$-type function associated to the double collection $\langle (\pi^0, \pi^1, \pi^2), (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) \rangle$ and $x$, or we again can apply Lemma 5.3 ($\beta = 4$) ... We can proceed by transfinite induction. The induction procedure will stop at an ordinal $\gamma \leq \delta$ when the function $\phi$ takes only one value on the set $\{\eta < \delta : P^n \in \mathcal{P} \setminus (\bigcup_{0 \leq \lambda < \gamma} \mathcal{P}_\lambda)\}$.

If $A$ and $V$ are finite the process clearly stops after finitely many steps. \qed
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