Definitions of solutions to the IBVP
for multiD scalar balance laws

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Abstract
We consider four definitions of solution to the initial–boundary value problem for a scalar balance laws in several space dimensions. These definitions are generalised to the same most general framework and then compared. The first aim of this paper is to detail differences and analogies among them. We focus then on the ways the boundary conditions are fulfilled according to each definition, providing also connections among these various modes. The main result is the proof of the equivalence among the presented definitions of solution.

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1 Introduction
This paper is concerned with the relations among different definitions of solution to the Initial–Boundary Value Problem (IBVP) for a general scalar balance law in several space dimensions:

\[
\begin{aligned}
\partial_t u(t,x) + \nabla \cdot f(t,x,u(t,x)) &= F(t,x,u(t,x)) & (t,x) \in \mathbb{R}_+ \times \Omega \\
u(0,x) &= u_0(x) & x \in \Omega \\
u(t,\xi) &= u_b(t,\xi) & (t,\xi) \in \mathbb{R}_+ \times \partial \Omega.
\end{aligned}
\]

Above and hereinafter, \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \), with smooth boundary \( \partial \Omega \), and \( \mathbb{R}_+ = [0, +\infty] \). The way the boundary condition is satisfied is going to be precised further on and constitutes a key issue addressed in this paper.

The pioneering work by Bardos, le Roux and Nédélec [1] introduces a definition of solution to (1.1) following the spirit of the one given by Kružkov in [7] in the case without boundary. The idea of the authors is to include in a unique integral inequality both Kružkov definition and the boundary condition. However, the BLN–definition considers only functions admitting a trace at the boundary, for instance \( \text{BV} \) functions. In [1], the authors explain the way the boundary condition has to be understood and introduce a key inequality on the boundary,
which we call the BLN condition, relating the boundary datum to the trace of the solution, see (5.6).

It is also possible to consider a definition of solution to (1.1) analogous to the BLN–one, though involving classical (regular) entropy–entropy flux pairs, see [2, Definition 2.5] and Definition 5.2. In this way there is a sort of symmetry between, on one side, the BLN–definition and this one for IBVPs as (1.1) and, on the other side, Kružkov definition and the definition of weak entropy solution for initial value problems on all \( \mathbb{R}^N \). On all \( \mathbb{R}^N \), solutions to the initial value problem for a general scalar balance law are usually found in \( L^\infty \), therefore a question naturally arises: is it possible to find a concept of solution to the IBVP (1.1) in this function space? The key difficulty is that, in general, a function in \( L^\infty \) does not necessarily admit a trace at the boundary. A first proposal to overcome this issue is given by Otto in his PhD thesis (a summary is presented in [10], while more details and proofs can be found in [8] Chapter 2). In the case of autonomous scalar conservation laws, Otto replaces the Kružkov entropy–entropy flux pairs as exploited in [1] with the so-called boundary entropy–entropy flux pairs and bases on them his definition of solution. In this paper we provide a generalisation of this definition to deal with non autonomous fluxes and arbitrary source terms, see Definition 3.3.

Still looking for solutions to (1.1) in \( L^\infty \), in the case of scalar conservation laws with divergence free flux, Vovelle introduces in [15] a definition of solution using the so-called Kružkov semi-entropy–entropy flux pairs. This definition is then extended by Martin in [9] to deal with general scalar balance laws. The resulting MV–definition, exploiting Kružkov semi entropies, resembles the BLN–definition, although not requiring the existence of the trace of the solution at the boundary.

To sum up, there are mainly four definitions of solution to (1.1), which can be classified as follows: involving the trace at the boundary (BLN–solutions and classical entropy–solutions, i.e. Definition 5.2) or not (Otto-type solutions, i.e. Definition 3.3 and MV–solutions); dealing with regular entropies (entropy–solutions and Otto–type ones) or with Lipschitz ones (BLN and MV–solutions). Definition 3.3 and MV–definition share the interesting feature of being stable under \( L^1 \)-convergence, see Remark 3.6 and also [8, Chapter 2, Remark 7.33].

In this paper, we prove first the equivalence of Definition 3.3 and MV–definition, and then focus on the way the boundary conditions are fulfilled according to those definitions.

The main result of this paper is the proof of the equivalence among all the presented definitions of solution to (1.1). Of course, this can be done only when the existence of the trace of the solution at the boundary is assumed. Further information on the existence of the trace at the boundary can be found in [11, 12, 14], see Section 5 for more details. As an intermediate step, we also prove the equivalence among the way the boundary conditions are understood according to the various definitions.

The paper is organised as follows. Section 2 collects the notation used throughout the paper. Sections 3 and 4 are devoted to the Otto-type definition of solution and Martin–Vovelle–one: the first section contains the definitions themselves and the theorem stating the equivalence between them, while results on the way the boundary conditions are fulfilled constitute the latter one. Section 5 deals with the definitions of solution with traces and provides also the main equivalence result. In Section 6 we give the definition of strong solution to (1.1) and a related results. Section 7 provides further details on the one dimensional case,
while Section 8 summarises the existence results that can be found in the literature. We collect the detailed proofs of our results in Section 9.

2 Notation

The space dimension $N$, with $N \geq 1$, is fixed throughout. We set $\mathbb{R}_+ = [0, +\infty]$. We denote by $\nu(\xi)$ the exterior normal to $\xi \in \partial \Omega$. For $w, k \in \mathbb{R}$ set

$$I[w, k] = \{ z \in \mathbb{R} : (w - z)(z - k) \geq 0 \} = \{ \vartheta w + (1 - \vartheta)k : \vartheta \in [0, 1] \}. \quad (2.1)$$

In other words, $I[w, k]$ denotes the closed interval with end points $w$ and $k$.

For the divergence of a vector field, possibly composed with another function, we use the following notation:

$$\nabla \cdot f(t, x, u(t, x)) = \text{div} f(t, x, u(t, x)) + \partial_u f(t, x, u(t, x)) \cdot \nabla u(t, x).$$

We use below the following standard assumptions:

- **(IC)** $u_0 \in L^\infty(\Omega; \mathbb{R})$;
- **(BC)** $u_b \in L^\infty([0, T] \times \partial \Omega; \mathbb{R})$;
- **(f)** $f \in C^2([0, T] \times \overline{\Omega} \times \mathbb{R}; \mathbb{R}^N)$, $\partial_u f \in L^\infty_{\text{loc}}([0, T] \times \Omega \times \mathbb{R}; \mathbb{R}^N)$ and $\nabla \cdot \partial_u f \in L^\infty([0, T] \times \Omega \times \mathbb{R}; \mathbb{R}^{N \times N})$;
- **(F)** $F \in C^2([0, T] \times \overline{\Omega} \times \mathbb{R}; \mathbb{R})$ and $\partial_u F \in L^\infty([0, T] \times \Omega \times \mathbb{R}; \mathbb{R})$.

Following [9, 15], we set

$$\text{sgn}^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases} \quad \text{sgn}^-(s) = \begin{cases} 0 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0, \end{cases} \quad s^+ = \max\{s, 0\}, \quad s^- = \max\{-s, 0\}. \quad (2.2)$$

We often use below the equalities $(-s)^- = s^+$ and $(-s)^+ = s^-$. Introduce moreover the following notation: if $g : \mathbb{R}^2 \to \mathbb{R}$, for all $z, w \in \mathbb{R}$, set

$$\partial_1 g(z, w) = \lim_{h \to 0} \frac{g(z + h, w) - g(z, w)}{h}, \quad \partial_2 g(z, w) = \lim_{h \to 0} \frac{g(z, w + h) - g(z, w)}{h},$$

and similarly for functions of more arguments.

3 $L^1$-Stable Definitions

Before introducing the first definition of solution to (1.1), we need to recall the notion of (classical) entropy–entropy flux pair, see [8] Chapter 2, Definition 3.22.

**Definition 3.1.** The pair $(\eta, q) \in C^2(\mathbb{R}; \mathbb{R}) \times C^2([0, T] \times \overline{\Omega} \times \mathbb{R}; \mathbb{R}^N)$ is called an entropy–entropy flux pair with respect to $f$ if

i) $\eta$ is convex, i.e. $\eta''(z) \geq 0$ for all $z \in \mathbb{R}$;
ii) for all $t \in [0,T]$, for all $x \in \Omega$, for all $x \in \mathbb{R}$, $\partial_t q(t,x,z) = \eta(z) \partial_3 f(t,x,z)$.

The notion of boundary entropy–entropy flux pair is first introduced by Otto in [10], see also [8], for autonomous scalar conservation laws on bounded domains, and then extended to a more general case in [9] [15]. We recall it here for completeness.

**Definition 3.2.** The pair $(H,Q) \in C^2(\mathbb{R}^2;\mathbb{R}) \times C^2([0,T] \times \overline{\Omega} \times \mathbb{R};\mathbb{R}^N)$ is called a boundary entropy–entropy flux pair with respect to $f$ if

i) for all $w \in \mathbb{R}$ the function $z \mapsto H(z, w)$ is convex;

ii) for all $t \in [0,T]$, $x \in \overline{\Omega}$ and $z, w \in \mathbb{R}$, $\partial_3 Q(t,x,z,w) = \partial_1 H(z,w) \partial_3 f(t,x,z)$;

iii) for all $t \in [0,T]$, $x \in \overline{\Omega}$ and $w \in \mathbb{R}$, $H(w, w) = 0$, $Q(t,x,w,w) = 0$ and $\partial_1 H(w, w) = 0$.

Note that if $H$ is as above, then $H \geq 0$.

We now extend the definition given by Otto (see [10] Proposition 2 and also [8] Theorem 7.31)) to account for non autonomous fluxes and arbitrary source terms. The concept of boundary entropy–entropy flux pairs introduced above characterises the definition.

**Definition 3.3.** A regular entropy solution (RE–solution) to the initial–boundary value problem \([\Box]\) on the interval $[0,T]$ is a map $u \in L^\infty([0,T] \times \Omega;\mathbb{R})$ such that for any boundary entropy–entropy flux pair $(H,Q)$, for any $k \in \mathbb{R}$ and for any test function $\varphi \in C^1_c(\mathbb{R} \to \mathbb{R}^N;\mathbb{R}_+)$

\[
\int_0^T \int_{\Omega} \left[ H\left(u(t,x),k\right) \partial_t \varphi(t,x) + Q\left(t,x,u(t,x),k\right) \cdot \nabla \varphi(t,x) \right] \, dx \, dt \\
+ \int_0^T \int_{\Omega} \partial_1 H\left(u(t,x),k\right) \left[F\left(t,x,u(t,x)\right) - \text{div} f\left(t,x,u(t,x)\right)\right] \varphi(t,x) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \text{div} Q\left(t,x,u(t,x),k\right) \varphi(t,x) \, dx \, dt + \int_{\Omega} H\left(u_0(x),k\right) \varphi(0,x) \, dx \\
+ \|\partial_u f\|_{L^\infty([0,T] \times \partial \Omega;\mathbb{R}^N)} \int_0^T \int_{\partial \Omega} H\left(u_b(t,\xi),k\right) \varphi(t,\xi) \, d\xi \, dt \geq 0,
\]

where $\mathcal{U}$ is the interval $\mathcal{U} = [-U,U]$, with $U = \|u\|_{L^\infty([0,T] \times \Omega;\mathbb{R})}$.

A comment on the constant appearing in the last line of the integral inequality above is at the end of Section 7.

**Remark 3.4.** Observe that an equivalent definition of solution can be obtained considering test functions $\varphi \in C^1_c(\mathbb{R} \times \mathbb{R}^N;\mathbb{R}_+)$ and the following integral inequality:

\[
\int_0^T \int_{\Omega} \left[ H\left(u(t,x),k\right) \partial_t \varphi(t,x) + Q\left(t,x,u(t,x),k\right) \cdot \nabla \varphi(t,x) \right] \, dx \, dt \\
+ \int_0^T \int_{\Omega} \partial_1 H\left(u(t,x),k\right) \left[F\left(t,x,u(t,x)\right) - \text{div} f\left(t,x,u(t,x)\right)\right] \varphi(t,x) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \text{div} Q\left(t,x,u(t,x),k\right) \varphi(t,x) \, dx \, dt \\
+ \int_{\Omega} H\left(u_0(x),k\right) \varphi(0,x) \, dx - \int_{\Omega} H\left(u(T,x),k\right) \varphi(T,x) \, dx \\
+ \|\partial_u f\|_{L^\infty([0,T] \times \partial \Omega;\mathbb{R}^N)} \int_0^T \int_{\partial \Omega} H\left(u_b(t,\xi),k\right) \varphi(t,\xi) \, d\xi \, dt \geq 0.
\]
A similar definition of solution is given by Vovelle in [15, Definition 1], see also [9, Definition 1], using the so-called Kružkov semi-entropy–entropy flux pairs, which are Lipschitz continuous functions, thus less regular than the boundary entropies considered in the definition of RE-solution.

**Definition 3.5.** A semi-entropy solution (MV-solution) to the initial–boundary value problem (1.1) on the interval [0, T] is a map $u \in L^\infty([0, T] \times \Omega; \mathbb{R})$ such that for any $k \in \mathbb{R}$ and for any test function $\varphi \in C^1_\infty([-\infty, T] \times \mathbb{R}^N; \mathbb{R}_+)$

\[
\int_0^T \int_\Omega (u(t, x) - k)^\pm \partial_t \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{sgn}^\pm(u(t, x) - k) \left(f(t, x, u(t, x)) - f(t, x, k)\right) \cdot \nabla \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{sgn}^\pm(u(t, x) - k) \left[F(t, x, u(t, x)) - \text{div} f(t, x, k)\right] \varphi(t, x) \, dx \, dt \\
+ \int_\Omega (u_0(x) - k)^\pm \varphi(0, x) \, dx \\
+ \|\partial_u f\|_{L^\infty([0, T] \times \Omega)} \int_0^T \int_{\partial\Omega} (u_b(t, \xi) - k)^\pm \varphi(t, \xi) \, d\xi \, dt \geq 0,
\]

where $U$ is the interval $U = [-U, U]$, with $U = \|u\|_{L^\infty([0, T] \times \Omega; \mathbb{R})}$.

Before entering into the details of the link between these two definitions of solution to (1.1), we emphasise a feature they share.

**Remark 3.6.** Both Definitions 3.3 of RE-solution and 3.5 of MV-solution are stable under $L^1$-convergence. This remarkable feature is underlined in [8, Chapter 2, Remark 7.33] for the particular definition given by Otto, but it is immediate to see that it extends to both Definitions 3.3 and 3.5. More precisely, let $u^n_i$ and $u^n_b$ be sequences of initial and boundary data converging in $L^1$ to $u_0$ and $u_b$ respectively. Let $u^n$ be a solution to (1.1), according to either of the two definitions, with initial datum $u^n_0$ and boundary datum $u^n_b$. Assume that $u^n$ converges to $u$ in $L^1$. Then, this limit function $u$ is a solution to (1.1), according to the same definition, with initial datum $u_0$ and boundary datum $u_b$.

Our first aim is to establish a connection between Definition 3.3 of RE-solution and Definition 3.5 of MV-solution. An intermediate step is constituted by the following Lemma, which gives a link between the boundary entropy–entropy flux pairs exploited in Definition 3.3 and the Kružkov semi-entropy–entropy flux pairs used in Definition 3.5.

**Lemma 3.7 ([15, Lemma 1] and [9, Lemma 3]).** Let $\eta \in C^2(\mathbb{R}; \mathbb{R})$ be a convex function such that there exists $w \in [A, B]$ with $\eta(w) = 0$ and $\eta'(w) = 0$. Then $\eta$ can be uniformly approximated on $[A, B]$ by applications of the kind

$$s \mapsto \sum_{i=1}^p \alpha_i (s - \kappa_i)^- + \sum_{j=1}^q \beta_j (s - \tilde{\kappa}_j)^+,$$

where $\alpha_i \geq 0$, $\beta_j \geq 0$, $\kappa_i$, $\tilde{\kappa}_j \in [A, B]$.

Conversely, there exists a sequence of boundary entropy–entropy flux pairs which converges to the Kružkov semi-entropy–entropy flux pairs.
Thanks to Lemma 3.7, the equivalence between RE–solution and MV–solution follows immediately. For the detailed proof we refer to Section 9.

**Theorem 3.8.** Let $u \in L^\infty([0, T[ \times \Omega; \mathbb{R})$. Then $u$ is a RE–solution to (1.1), in the sense of Definition 3.3, if and only if $u$ is a MV–solution to (1.1), in the sense of Definition 3.5.

### 4 Behaviour at the Boundary

We now focus our attention on the way the boundary conditions are fulfilled according to the definitions of solution to (1.1) introduced in Section 3. All the proofs are deferred to Section 9.

The following Lemma is a generalisation to problem (1.1) of [8, Lemma 7.34]. It states the way the boundary conditions are satisfied in the case of a RE–solution to (1.1).

**Lemma 4.1.** Let $u \in L^\infty([0, T[ \times \Omega; \mathbb{R})$ be a RE–solution to (1.1), according to Definition 3.3. Then, for all boundary entropy–entropy flux pairs $(H, Q)$ and for all $\beta \in L^1([0, T[ \times \partial \Omega; \mathbb{R}_+)$

$$\text{ess lim}_{\rho \to 0^+} \int_0^T \int_{\partial \Omega} Q(t, \xi, u(t, \xi - \rho \nu(\xi)) , u_b(t, \xi)) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt \geq 0, \quad (4.1)$$

$\nu(\xi)$ being the exterior normal to $\xi \in \partial \Omega$.

**Remark 4.2.** Due to the equivalence between RE–solution and MV–solution proved in Theorem 3.8, the boundary conditions are satisfied in the sense of (4.1) also in the case of a MV–solution to (1.1). See [9, Lemma 4], and also [15, Remark 3], for a different proof of (4.1), starting from Kružkov semi-entropy–entropy flux pairs.

An alternative formulation of the boundary conditions is also possible, both in the case of RE–solution and MV–solution, see [8, Lemma 7.12] and [9, Lemma 16].

**Lemma 4.3.** Let $u \in L^\infty([0, T[ \times \Omega; \mathbb{R})$ be a RE–solution (or MV–solution) to (1.1) in the sense of Definition 3.3 (Definition 3.5). Define the function $F \in C^0([0, T[ \times \overline{\Omega} \times \mathbb{R}^3; \mathbb{R}^N)$:

$$F(t, x, z, w, k) = \begin{cases} f(t, x, w) - f(t, x, z) & \text{for } z \leq w \leq k, \\ 0 & \text{for } w \leq z \leq k, \\ f(t, x, z) - f(t, x, k) & \text{for } w \leq k \leq z, \\ f(t, x, k) - f(t, x, z) & \text{for } z \leq k \leq w, \\ f(t, x, z) - f(t, x, w) & \text{for } k \leq z \leq w, \\ f(t, x, w) - f(t, x, k) & \text{for } k \leq w \leq z. \end{cases} \quad (4.2)$$

Then, for all $\beta \in L^1([0, T[ \times \partial \Omega; \mathbb{R}_+)$ and for all $k \in \mathbb{R}$

$$\text{ess lim}_{\rho \to 0^+} \int_0^T \int_{\partial \Omega} F(t, \xi, u(t, \xi - \rho \nu(\xi)) , u_b(t, \xi), k) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt \geq 0. \quad (4.3)$$

**Remark 4.4.** Observe that the function $F$ defined in (4.2) can be written also as follows

$$F(t, x, z, w, k) = \frac{1}{2} \left[ \text{sgn}(z - w) \left( f(t, x, z) - f(t, x, w) \right) \right]$$
\[- \operatorname{sgn}(k - w) \left( f(t, x, k) - f(t, x, w) \right) \\
+ \operatorname{sgn}(z - k) \left( f(t, x, z) - f(t, x, k) \right),
\]
and also
\[
F(t, x, z, w, k) = \operatorname{sgn}^+ \left( z - \max\{w, k\} \right) \left( f(t, x, z) - f(t, x, \max\{w, k\}) \right) \\
+ \operatorname{sgn}^- \left( z - \min\{w, k\} \right) \left( f(t, x, z) - f(t, x, \min\{w, k\}) \right).
\]

5 Solutions with Traces

So far we have considered two definitions of solution to (1.1), sought in \(L^\infty\), and proved their equivalence. The RE–definition involves regular entropies, while the MV–definition deals with Lipschitz continuous ones. In this Section we present two additional definitions of solution to (1.1), in which the trace of the solution at the boundary appears explicitly. The idea is to draw a parallel with RE and MV–solutions: indeed, the two definitions we are going to introduce are characterised by regular and Lipschitz continuous entropies respectively, and we prove that they are equivalent.

Since the existence of the trace of the solution is required, more regularity is needed on the solution with respect to Definitions 3.3 of RE–solutions and 3.5 of MV–solutions. To this aim, introduce the following space:

Definition 5.1. A function \(u\) belongs to the space \(TR^\infty([0, T] \times \Omega; \mathbb{R})\) if there exists a function \(\operatorname{tr}u \in L^\infty([0, T] \times \partial \Omega; \mathbb{R})\) such that
\[
\lim_{r \to 0^+} \int_0^T \int_{\partial \Omega} \left| u(t, \xi - r \nu(\xi)) - \operatorname{tr}u(t, \xi) \right| d\xi dt = 0.
\]

We remark the following. Bardos, le Roux and Nédélec in [1] consider solutions in \(BV([0, T] \times \Omega; \mathbb{R}) \subset TR^\infty([0, T] \times \Omega; \mathbb{R})\): indeed, \(BV\) functions admit a trace at the boundary reached by \(L^1\) convergence, see [1, Lemma 1], [5, Paragraph 5.3], [6, Chapter 2] and [2, Appendix]. Consider now the case of \(L^\infty\) solutions, which are functions \(u\) satisfying in the sense of distribution on \([0, T] \times \Omega\) the inequality
\[
\partial_t \eta(u) + \operatorname{div} q(t, x, u) \leq 0,
\]
for any \((\eta, q)\) (classical) entropy–entropy flux pair (with respect to \(f\), see Definition 3.1). Panov proves in [12, Theorem 1.4 and Remark 6.3] the existence of the trace at the boundary for \(L^\infty\) solutions to (1.1), under the following non-degeneracy condition on the flux: the function \(f\) is continuous and such that for a.e. \((t, \xi) \in \mathbb{R}_+ \times \partial \Omega\) and all \((s, y) \in \mathbb{R}_+ \times \mathbb{R}^N \setminus \{(0, 0)\}\) the function \(u \to s u + y f(t, \xi, u)\) is not constant on non-degenerate intervals, i.e. \(f\) satisfies the following genuine non linearity condition
\[
\mathcal{L} \left( \left\{ u | s + y \partial_u f(t, \xi, u) = 0 \right\} \right) = 0 \quad \text{for every} \quad (s, y) \in \mathbb{R}_+ \times \mathbb{R}^N \setminus \{(0, 0)\},
\]
where \(\mathcal{L}\) is the Lebesgue measure and \((t, \xi) \in \mathbb{R}_+ \times \partial \Omega\). As pointed out also in [14], the above assumption allows to avoid flux functions whose restriction to an open subset is linear.

The following definition uses the (classical) entropy–entropy flux pairs, see Definition 3.1. It extends the particular case of scalar conservation laws with autonomous fluxes considered in [3, Chapter 6, Definition 6.9.1] and in [13, Chapter 15], see also [2, Definition 2.5].
Definition 5.2. An entropy solution (E–solution) to the initial–boundary value problem (1.1) on the interval [0, T] is a map \( u \in (L^\infty \cap \mathcal{T}^\infty)([0, T] \times \Omega; \mathbb{R}) \) such that for any entropy–entropy flux pair \((\eta, q)\) and for any test function \( \varphi \in C^1_c([t_0, t_1]) \) on the interval \([0, T] \times \mathbb{R}^N; \mathbb{R}_+\)

\[
\int_0^T \int_\Omega \left[ \eta \left( u(t, x) \right) \partial_t \varphi(t, x) + q \left( t, x, u(t, x) \right) \cdot \nabla \varphi(t, x) \right] \, dx \, dt \\
+ \int_0^T \int_\Omega \eta' \left( u(t, x) \right) \left[ F \left( t, x, u(t, x) \right) - \text{div} \ f \left( t, x, k \right) \right] \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{div} \left( q \left( t, x, u(t, x) \right) \right) \varphi(t, x) \, dx \, dt \\
= \eta \left( u(0, x) \right) \varphi(0, x) \, dx - \int_0^T \int_{\partial \Omega} q \left( t, \xi, u_0(t, \xi) \right) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt \\
+ \int_0^T \int_{\partial \Omega} \eta' \left( u_0(t, \xi) \right) \left( f \left( t, \xi, u_0(t, \xi) \right) - f \left( t, \xi, \text{tr} \ u(t, \xi) \right) \right) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt \geq 0.
\] (5.2)

We now recall the definition of solution to (1.1) due to Bardos, le Roux and Nédélec [1, p. 1028], which exploits the classical Kružkov entropy–entropy flux pairs.

Definition 5.3. A Kružkov entropy solution (BLN–solution) to the initial–boundary value problem (1.1) on the interval [0, T] is a map \( u \in (L^\infty \cap \mathcal{T}^\infty)([0, T] \times \Omega; \mathbb{R}) \) such that for any \( k \in \mathbb{R} \) and for any test function \( \varphi \in C^1_c([t_0, t_1]) \) on the interval \([0, T] \times \mathbb{R}^N; \mathbb{R}_+\)

\[
\int_0^T \int_\Omega \left| u(t, x) - k \right| \partial_t \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left( f \left( t, x, u(t, x) \right) - f \left( t, x, k \right) \right) \cdot \nabla \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left[ F \left( t, x, u(t, x) \right) - \text{div} \ f \left( t, x, k \right) \right] \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_{\partial \Omega} \left| u_0(x) - k \right| \varphi(0, x) \, dx \\
- \int_0^T \int_{\partial \Omega} \text{sgn} \left( u_0(t, \xi) - k \right) \left( f \left( t, \xi, \text{tr} \ u(t, \xi) \right) - f \left( t, \xi, k \right) \right) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt \geq 0.
\] (5.3)

E–solutions, as in Definition 5.2 and BLN–solutions, as in Definition 5.3 are actually equivalent, see [24, Proposition 2.6]. The proof of the equivalence between these two Definitions of solution is based on an analogous of Lemma 3.7 and is briefly sketched in Section 9.

Theorem 5.4. The map \( u \in (L^\infty \cap \mathcal{T}^\infty)([0, T] \times \Omega; \mathbb{R}) \) is an E–solution to (1.1), in the sense of Definition 5.2, if and only if \( u \) is a BLN–solution to (1.1), in the sense of Definition 5.3.

Before studying the relation among all the considered definitions, we provide the analogous to Lemma 4.1 and Lemma 4.3 explaining the way E–solutions and BLN–solutions to (1.1) fulfil the boundary conditions. Concerning E–solutions, the following Lemma holds.

Lemma 5.5. Let \( u \in (L^\infty \cap \mathcal{T}^\infty)([0, T] \times \Omega; \mathbb{R}) \) be an E–solution to (1.1) in the sense of Definition 5.2. Then, for all (classical) entropy–entropy flux pairs \((\eta, q)\) and for a.e. \((t, \xi) \in [0, T] \times \partial \Omega\),

\[
\left[ \eta \left( u(t, \xi) \right) \varphi(t, \xi) \left( u(t, \xi) \right) - \eta' \left( u(t, \xi) \right) \left( f \left( t, \xi, \text{tr} \ u(t, \xi) \right) - f \left( t, \xi, u_0(t, \xi) \right) \right) \cdot \nu(\xi) \right] \geq 0.
\] (5.4)
The proof is deferred to Section 9. Observe that condition (5.4) is the generalisation to the multidimensional case of the boundary entropy inequality due to Dubois and LeFloch [4, Theorem 1.1].

In the following Lemma we recall the well-known BLN condition, linking the boundary datum and the trace of the solution. The proof is in Section 9.

Lemma 5.6. Let \( u \in (L^\infty \cap TR^\infty)([0, T] \times \Omega; \mathbb{R}) \) be a BLN–solution to (1.1) in the sense of Definition 5.3. Then, for all \( k \in \mathbb{R} \) and for a.e. \( (t, \xi) \in ]0, T[ \times \partial \Omega \),

\[
\left( \text{sgn} \left( \text{tr} u(t, \xi) - k \right) - \text{sgn} \left( u_b(t, \xi) - k \right) \right) \left( f(t, \xi, \text{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \geq 0.
\] (5.5)

Moreover, condition (5.4) is equivalent to the following: for all \( k \in \mathcal{I}[\text{tr} u(t, \xi), u_b(t, \xi)] \) and a.e. \( (t, \xi) \in ]0, T[ \times \partial \Omega \)

\[
\text{sgn} \left( \text{tr} u(t, \xi) - u_b(t, \xi) \right) \left( f(t, \xi, \text{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \geq 0.
\] (5.6)

The following Proposition constitutes the basis for the proof of the equivalence of all the definitions of solution to (1.1) presented so far. It is a generalisation of [8, Lemma 7.24] to problem (1.1): it takes into account non autonomous fluxes and arbitrary source terms. In particular, this Proposition provides a connection among the ways the boundary conditions are understood according to the various definitions of solution introduced so far. However, we need to require the existence of the trace of the solution at the boundary. For further details about the trace, see the references at the beginning of Section 5. The proof is deferred to Section 9.

Proposition 5.7. Let \( u_b \in L^\infty([0, T] \times \partial \Omega; \mathbb{R}) \) and \( u \in (L^\infty \cap TR^\infty)([0, T] \times \Omega; \mathbb{R}) \). Then the following statements are equivalent:

1. (4.1) holds, for any boundary entropy–entropy flux pair \((H, Q)\) and for any \( \beta \in L^1([0, T[ \times \partial \Omega; \mathbb{R}_+); \)
2. (4.3) holds, for any \( \beta \in L^1([0, T[ \times \partial \Omega; \mathbb{R}_+) \) and for all \( k \in \mathbb{R} \);
3. for a.e. \( (t, \xi) \in ]0, T[ \times \partial \Omega \) and for all \( k \in \mathbb{R} \) it holds

\[
\mathcal{F}(t, \xi, \text{tr} u(t, \xi), u_b(t, \xi), k) \cdot \nu(\xi) \geq 0,
\] (5.7)

with \( \mathcal{F} \) as in (4.2);
4. (5.3) holds for a.e. \( (t, \xi) \in ]0, T[ \times \partial \Omega \) and for all \( k \in \mathcal{I}[\text{tr} u(t, \xi), u_b(t, \xi)] \);
5. (5.4) holds for a.e. \( (t, \xi) \in ]0, T[ \times \partial \Omega \) and for any entropy–entropy flux pair \((\eta, q)\);
6. for a.e. \( (t, \xi) \in ]0, T[ \times \partial \Omega \) and all entropy–entropy flux pair \((\eta, q)\), such that \( \eta'(u_b(t, \xi)) = 0 \) and \( q(t, \xi, u_b(t, \xi)) = 0 \), it holds

\[
q(t, \xi, \text{tr} u(t, \xi)) \cdot \nu(\xi) \geq 0.
\] (5.8)

We can now state our main result: given that \( u \) admits a trace in the sense of (5.1), we prove that the Definitions of solution presented in this Section, that is E–solution and BLN–solution, are equivalent to the Definitions of solution introduced in Section 5, i.e. RE–solution and MV–solution.
Theorem 5.8. Let \( u \in (L^\infty \cap TR^\infty)([0,T] \times \Omega; \mathbb{R}) \). Then \( u \) is a RE–solution to (1.1) according to Definition 3.5, or equivalently a MV–solution to (1.1) in the sense of Definition 5.2 if and only if \( u \) is an E–solution to (1.1) according to Definition 5.2, or equivalently a BLN–solution to (1.1) in the sense of Definition 5.3.

The proof is deferred to Section 9. Remark that, according to the results by Panov [12] recalled at the beginning of this section, \( L^\infty \) solutions admit a trace at the boundary in the case of non-degenerate fluxes, thus in those cases there is no need to consider the intersection with the space \( TR^\infty([0,T] \times \Omega; \mathbb{R}) \).

6 Strong Solutions

For completeness, we recall below the definition of strong (smooth) solution to (1.1).

Definition 6.1. A strong solution to the initial–boundary value problem (1.1) on the interval \([0,T]\) is a map \( u \in C^1([0,T] \times \Omega; \mathbb{R}) \cap C^0([0,T] \times \overline{\Omega}; \mathbb{R}) \) which satisfies pointwise the equation and the initial condition, and it is such that, for all \((t,\xi) \in [0,T] \times \partial \Omega\) and for all \( k \in I[u(t,\xi),u_b(t,\xi)] \),

\[
\text{sgn} \left( u(t,\xi) - u_b(t,\xi) \right) \left( f \left( t,\xi, u(t,\xi) \right) - f \left( t,\xi, k \right) \right) \cdot \nu(\xi) \geq 0.
\] (6.1)

Note that condition (6.1) reduces to (5.6): the difference is that strong solutions are defined up to the boundary, and therefore the notion of trace is not needed. For further details on the boundary conditions for smooth solution, including an heuristic derivation, see [8, Chapter 2, Section 6].

The following result holds.

Proposition 6.2. Let \( u \in C^1([0,T] \times \Omega; \mathbb{R}) \cap C^0([0,T] \times \overline{\Omega}; \mathbb{R}) \) be a strong solution to (1.1) in the sense of Definition 6.1. Then \( u \) is also a RE–solution to (1.1), in the sense of Definition 3.5.

Obviously, due to the equivalence among the definitions of solution proven in Theorem 5.8, every strong solution is also a MV–solution, an E–solution and a BLN–solution.

The proof follows the line of the second part of the proof of Theorem 5.8 and it is hence omitted. The main difference is that the solution itself at the boundary is considered, instead of its trace.

7 The 1 Dimensional Case

In this section we focus on the case \( N = 1 \), i.e. \( \Omega \) is the interval \([a,b]\), with \( a, b \in \mathbb{R} \). The boundary datum is assigned at the end points of the interval: for \( t \in \mathbb{R}_+ \)

\[
u(t,a) = u_b(t,a), \quad u(t,b) = u_b(t,b).
\]

We write explicitly how the last line of the integral inequality in the definition of solution reads in the case of the RE–definition and of the E–definition, the other two cases being completely analogous. Observe that the exterior normal to \( \partial \Omega \) in \( a \) is \(-1\), while in \( b \) is \(+1\).
• RE–definition:

\[ + \| \partial_u f \|_{L^\infty([0,T] \times \Omega \times \mathcal{U}; \mathbb{R})} \int_0^T \left[ H(u_b(t,a),k) \varphi(t,a) + H(u_b(t,b),k) \varphi(t,b) \right] \, dt. \]

• E–definition:

\[ + \int_0^T q(t,a,u_b(t,a)) \varphi(t,a) \, dt - \int_0^T q(t,b,u_b(t,b)) \varphi(t,b) \, dt \]
\[ - \int_0^T \eta'(u_b(t,a)) \left( f(t,a,u_b(t,a)) - f(t,a,\text{tr} u(t,a)) \right) \varphi(t,a) \, dt \]
\[ + \int_0^T \eta'(u_b(t,b)) \left( f(t,b,u_b(t,b)) - f(t,b,\text{tr} u(t,b)) \right) \varphi(t,b) \, dt. \]

What is immediately evident is the presence of the minus sign in the last case, while the first contains only sums. The minus sign is due to the scalar product with the exterior normal to \( \partial \Omega \), which occurs in the E–definition, and in the BLN–definition as well. It can be seen that there is no need for a minus sign neither in the RE–definition nor in the MV–definition.

We can exploit the one dimensional setting to analyse a feature of the RE–definition and the MV–definition. Indeed, the integral inequalities of these two definitions involve the constant \( \| \partial_u f \|_{L^\infty([0,T] \times \Omega \times \mathcal{U}; \mathbb{R})} \), where \( \mathcal{U} = [-U,U] \), with \( U = \| u \|_{L^\infty([0,T] \times \Omega; \mathbb{R})} \). This is nothing but the Lipschitz constant of \( f \) with respect to \( u \), therefore one may wonder whether it is possible to consider a different constant, either smaller or larger, and to still get a solution.

One can see, through the following one dimensional example, that the above constant is indeed the smallest possible. Consider the following problem

\[
\begin{cases}
\partial_t u(t,x) + \partial_x f(u(t,x)) = 0 & (t,x) \in \mathbb{R}_+ \times [a,b[ \\
u(0,x) = 1 & x \in ]a,b[ \\
u(t,a) = u_b(t,a) = 1 & t \in \mathbb{R}_+ \\
u(t,b) = u_b(t,b) = -1 & t \in \mathbb{R}_+ 
\end{cases}
\]

The solution is constant and equal to 1. Fix \( T > 0 \). Observe that \( \| u \|_{L^\infty([0,T] \times [a,b]; \mathbb{R})} = 1 \). Consider the integral inequality (3.2) of the MV–definition, and use the positive constant \( c \) instead of \( \| \partial_u f \|_{L^\infty([0,T] \times [a,b] \times \mathcal{U}; \mathbb{R})} = 1 \). Since we know that \( u = 1 \) is the solution, simple computations show that \( c \) should be grater or equal to 1.

Obviously, given a solution \( u \), choosing a greater value of the Lipschitz constant still ensures that \( u \) is a solution: indeed the constant is multiplied by a non negative term, both in the RE and in the MV–definition.

8 Notes on the Existence of Solutions

We recap the results present in the literature concerning the existence of solutions to problem (1.1), specifying in each case the considered definition of solution.
As far as it concerns the case of an autonomous scalar conservation laws, i.e. \( f = f(u) \) and \( F = 0 \), Otto proves the existence and uniqueness of a RE–solution to (1.1), see [10] and also [8, Chapter 2] for more detailed proofs.

In [9], Martin proves the existence and uniqueness of a MV–solution to the general problem (1.1), though imposing on the flux and the source terms a condition which leads to a (simple) maximum principle, see [9, Assumption 1.(iv)].

Bardos, le Roux and Nédélec prove in [1] existence and uniqueness of a BLN–solution to (1.1) in the case of homogeneous boundary conditions, i.e. \( u_b = 0 \). The result is then generalised in [2] to allow for (regular) non zero boundary data. The case of an autonomous scalar conservation laws with BLN–definition is considered also by Serre in [13, Chapter 15]. Dafermos in [3, Chapter 6] focuses on the same autonomous problem, although under homogeneous boundary conditions, and exploits the RE–definition of solution.

9 Technical Details

Proof of Theorem 3.8.

A RE–solution is a MV–solution. It is enough to consider the following sequences of boundary entropy–entropy flux pairs

\[
H_n(z, k) = \left( (z - k)^\pm + \frac{1}{n^2} \right)^{1/2} - \frac{1}{n}
\]

\[
Q_n(t, x, z, k) = \int_k^z \partial_1 H_n(w, k) \partial_u f(t, x, w) \, dw.
\]

Indeed, as \( n \) goes to \(+\infty\),

\[
H_n(z, k) \to (z - k)^\pm
\]

\[
Q_n(t, x, z, k) \to \text{sgn}(z - k)^\pm \left( f(t, x, z) - f(t, x, k) \right),
\]

so that, in the limit, (3.1) becomes

\[
\int_0^T \int_\Omega (u(t, x) - k)^\pm \partial_t \varphi(t, x) \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \text{sgn} (u(t, x) - k)^\pm \left( f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \text{sgn} (u(t, x) - k)^\pm \left( F(t, x, u(t, x)) - \text{div} f(t, x, u(t, x)) \right) \varphi(t, x) \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \text{div} f(t, x, u(t, x)) - \text{div} f(t, x, k) \, dx \, dt
\]

\[
+ \int_\Omega (u_0(x) - k)^\pm \varphi(0, x) \, dx
\]

\[
+ \|\partial_u f\|_{L^\infty([0,T] \times \Omega; \mathbb{R}^N)} \int_0^T \int_{\partial \Omega} (u_b(t, \xi) - k)^\pm \varphi(t, \xi) \, d\xi \, dt \geq 0,
\]

where \( \mathcal{U} = [-U, U] \), with \( U = \|u\|_{L^\infty([0,T] \times \Omega; \mathbb{R})} \). Combining the second and the third lines in the inequality above yields exactly (3.2).
A MV–solution is a RE–solution. Since \( u \in L^\infty([0,T] \times \Omega; \mathbb{R}) \), there exist \( A, B \in \mathbb{R} \), with \( A < B \), such that \( A \leq u(t,x) \leq B \) for a.e. \((t,x) \in [0,T] \times \Omega \), and hence \( u \in L^\infty([0,T] \times \Omega; [A,B]) \). We can then apply Lemma 3.7. Each boundary entropy–entropy flux pair is uniformly approximated by a linear combination with positive coefficients of Kružkov semi-entropy–entropy flux pairs. Thus the inequality in \( \text{(3.12)} \) is preserved and \( \text{(3.11)} \) holds.

Proof of Lemma 4.1. The proof extends [3] Lemma 7.34] to consider non autonomous fluxes and general source terms.

Let \((H,Q)\) be a boundary entropy–entropy flux pair, \(k \in \mathbb{R}\). The analogous of [3] Lemma 7.34, hypothesis (7.35)] is the following:

\[
\begin{align*}
\int_0^T \int_\Omega [H(u(t,x),k) \partial_t \varphi(t,x) + Q(t,x,u(t,x),k) \cdot \nabla \varphi(t,x)] \, dx \, dt \\
+ \int_0^T \int_\Omega \partial_t H(u(t,x),k) \left[ F(t,x,u(t,x)) - \text{div} f(t,x,u(t,x)) \right] \varphi(t,x) \, dx \, dt \\
+ \int_\Omega \int_\Gamma \text{div} Q(t,x,u(t,x),k) \varphi(t,x) \, dx \, dt \\
+ \| \partial_t f \|_{L^\infty([0,T] \times \Omega; \mathbb{R}^N)} \int_0^T \int_{\partial\Omega} H(u_b(t,\xi),k) \varphi(t,\xi) \, d\xi \, dt \geq 0,
\end{align*}
\]

which follows directly from the Definition 3.3 of RE–solution when considering a test function \( \varphi \in C_0^\infty([0,T] \times \Omega; \mathbb{R}^+) \). Similarly, the analogous of [3] Lemma 7.34, hypothesis (7.36)] is the following:

\[
\begin{align*}
\int_0^T \int_\Omega [H(u(t,x),k) \partial_t \varphi(t,x) + Q(t,x,u(t,x),k) \cdot \nabla \varphi(t,x)] \, dx \, dt \\
+ \int_0^T \int_\Omega \partial_t H(u(t,x),k) \left[ F(t,x,u(t,x)) - \text{div} f(t,x,u(t,x)) \right] \varphi(t,x) \, dx \, dt \\
+ \int_\Omega \int_\Gamma \text{div} Q(t,x,u(t,x),k) \varphi(t,x) \, dx \, dt \\
+ \| \partial_t f \|_{L^\infty([0,T] \times \Gamma; \mathbb{R}^N)} \int_0^T \int_{\partial\Omega} H(u_b(t,\xi),k) \varphi(t,\xi) \, d\xi \, dt \geq 0,
\end{align*}
\]

where \( \mathcal{U} = [-U,U] \) with \( U = \|u\|_{L^\infty([0,T] \times \Omega; \mathbb{R})} \), which follows directly from the Definition 3.3 of RE–solution when considering a test function \( \varphi \in C_0^\infty([0,T] \times \Omega; \mathbb{R}^+) \).

We proceed as in the proof of [3] Lemma 7.34]. For the sake of simplicity, we restrict ourselves to the case of a half–space, i.e.

\[
\begin{align*}
\Omega &= \left\{ x = (x',s) \in \mathbb{R}^{N-1} \times \mathbb{R} : y < 0 \right\}, \\
\nu &= (0,\ldots,0,1) \in \mathbb{R}^N, \\
\Gamma &= \{ 0,T \times \mathbb{R}^{N-1} \}, \quad r = (t,x') \in \Gamma, \\
Q_T &= \{ p = (r,s) : r \in \Gamma, s < 0 \}.
\end{align*}
\]

The general case can then be obtained by a covering argument, i.e. by considering that the boundary \( \partial\Omega \) can be locally replaced by the border of a half-space.

For any boundary entropy–entropy flux pair \((H,Q)\), denote, for \((t,x) \in [0,T] \times \Omega, w \in \mathbb{Q}\),

\[
\eta(z) = H(z,w), \quad q(t,x,z) = Q(t,x,z, w).
\]
Choosing \( \varphi(t, x) = \varphi(r, s) = \beta(r) \alpha(s) \), with \( \alpha \in C^1_c([-\infty, 0[; \mathbb{R}_+), \) in (9.1) yields

\[
- \int_{-\infty}^0 \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr \, \alpha'(s) \, ds 
\leq C \int_{-\infty}^0 \alpha(s) \, ds,
\]

where

\[
C = \|\eta(u)\|_{L^\infty(Q_T; \mathbb{R})} \int_\Gamma |\partial_1 \beta(r)| \, dr + \|q(\cdot, \cdot, u)\|_{L^\infty(Q_T; \mathbb{R}^N)} \int_\Gamma |\nabla_x \beta(r)| \, dr
\]

\[
+ \left[ \|\eta'(u)\|_{L^\infty(Q_T; \mathbb{R})} \|F - \text{div} f(\cdot, \cdot, u)\|_{L^\infty(Q_T; \mathbb{R})} + \|\text{div} q(\cdot, \cdot, u)\|_{L^\infty(Q_T; \mathbb{R})} \right] \int_\Gamma |\beta(r)| \, dr.
\]

Thanks to integration by parts on the left hand side of (9.3) and to the fact that \( \alpha \geq 0 \), we obtain that the function

\[
s \mapsto \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr - C \, s
\]

is non increasing on \( ]-\infty, 0[ \). Moreover, we have

\[
\text{ess lim inf}_{s \to 0^-} \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr \geq - \text{ess sup}_{Q_T} \left| q(\cdot, \cdot, u) \right| \int_\Gamma \beta(r) \, dr.
\]

Monotonicity (9.1) and boundedness from below (9.5) imply that the following quantity is finite

\[
\text{ess lim}_{s \to 0^-} \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr.
\]

From (9.2), for all \( \alpha \in C^1_c(\mathbb{R}; \mathbb{R}_+) \) we get

\[
- \int_{-\infty}^0 \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr \, \alpha'(s) \, ds
\]

\[
\leq C \int_{-\infty}^0 \alpha(s) \, ds + \|\partial_uf\|_{L^\infty([0, T] \times \Omega \times U; \mathbb{R}^N)} \int_\Gamma \eta(u_b(r)) \beta(r) \, dr \, \alpha(0).
\]

Choose \( \alpha_n(s) = (n + 1) \chi_{[-1/n, 0[} \) mollify it properly and insert it in (9.6): in the limit \( n \to \infty \) we obtain

\[
\text{ess lim}_{s \to 0^-} \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr \geq -\|\partial_uf\|_{L^\infty([0, T] \times \Omega \times U; \mathbb{R}^N)} \int_\Gamma \eta(u_b(r)) \beta(r) \, dr.
\]

Let \( J \subseteq C^1_c(\Gamma; \mathbb{R}_+) \) be a countable set of functions such that for all \( \beta \in L^1(\Gamma; \mathbb{R}_+) \) there is a sequence \( (\beta_n) \) in \( J \) such that \( \lim_n \beta_n = \beta \) in \( L^1(\Gamma; \mathbb{R}_+) \). Therefore,

\[
\lim_n \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta_n(r) \, dr = \int_\Gamma q(r, s, u(r, s)) \cdot \mathbf{v} \beta(r) \, dr
\]

uniformly in \( s \in ]-\infty, 0[ \) and

\[
\lim_n \int_\Gamma \eta(u_b(r)) \beta(r) \, dr = \int_\Gamma \eta(u_b(r)) \beta(r) \, dr.
\]
Due to (9.6), there exists a set $E_w$ of measure zero such that for all $\beta \in J$ there exists
\[ \lim_{s \to 0^-} \int_\Gamma q(r, s, u(r, s)) \cdot \nu \beta(r) \, dr \]
and moreover
\[ \lim_{s \to 0^-} \int_\Gamma q(r, s, u(r, s)) \cdot \nu \beta(r) \, dr \geq -\|\partial_u f\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^N)} \int_\Gamma \eta(u_b(r)) \beta(r) \, dr. \]

Note that the set $E_w$ depends on $w$ because $\eta$ and $q$ depend on $w$. The above result can be extended to functions $\beta \in L^1(\Gamma; \mathbb{R}_+)$, so that for all $w \in \mathbb{Q}$ the quantity
\[ \lim_{s \to 0^-} \int_\Gamma Q(r, s, u(r, s), w) \cdot \nu \beta(r) \, dr \]
e
exists and
\[ \lim_{s \to 0^-} \int_\Gamma Q(r, s, u(r, s), w) \cdot \nu \beta(r) \, dr \geq -\|\partial_u f\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^N)} \int_\Gamma H(u_b(r), w) \beta(r) \, dr. \hspace{1cm} (9.8) \]

Let $v \in L^\infty(\Gamma; \mathbb{R})$ and $\beta \in L^1(\Gamma; \mathbb{R}_+)$ be given, and let $(v_n)$ be a sequence of simple functions with values in $\mathbb{Q}$ which converges to $v$ almost everywhere in $\Gamma$. Obviously, (9.8) holds for all $w = v_n$. Moreover,
\[ \lim_{n} \int_\Gamma Q(r, s, u(r, s), v_n(r)) \cdot \nu \beta(r) \, dr = \int_\Gamma Q(r, s, u(r, s), v(r)) \cdot \nu \beta(r) \, dr \]
nuniformly in $s \in \] - \infty, 0[$ and
\[ \lim_{n} \int_\Gamma H(u_b(r), v_n(r)) \beta(r) \, dr = \int_\Gamma H(u_b(r), v(r)) \beta(r) \, dr. \]

Hence, the following inequality holds
\[ \lim_{s \to 0^-} \int_\Gamma Q(r, s, u(r, s), v(r)) \cdot \nu \beta(r) \, dr \geq -\|\partial_u f\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^N)} \int_\Gamma H(u_b(r), v(r)) \beta(r) \, dr. \]

Choosing $v = u_b$ and recalling the properties of the boundary entropy $H$ (see Definition 3.2) conclude the proof. \(\square\)

**Proof of Lemma 4.3.** The proof follows immediately from Lemma 4.1. Indeed, for $k \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$, using the notation introduced in (2.1), define the maps
\[ \Delta^k(u, w) = \min_{z \in I^I[u,k]} |u - z| \]
\[ H^k_n(u, w) = \left( \left( \Delta^k(u, w) \right)^2 + \frac{1}{n^2} \right)^{1/2} - \frac{1}{n} \]
\[ Q^k_n(t, x, u, w) = \int_{w}^u \partial_1 H^k_n(z, w) \partial_u f(t, x, z) \, dz. \]

It can be easily proved that, for all $k \in \mathbb{R}$, the sequence of boundary entropy–entropy flux pairs $(H^k_n(u, w), Q^k_n(t, x, u, w))$ converges uniformly to $(\Delta^k(u, w), F(t, x, u, w, k))$ as $n$ goes to $+\infty$. Applying (4.1), with $Q$ replaced by $Q^k_n$, yields the thesis in the limit $n \to +\infty$, for all $k \in \mathbb{R}$ and $\beta \in L^1([0,T]\times\partial\Omega; \mathbb{R}_+)$. \(\square\)
Proof of Theorem 5.4.

An E–solution is a BLN–solution. It is sufficient to consider the following sequence of (classical) entropies: for $k \in \mathbb{R}$

$$\eta_n(z) = \sqrt{(z-k)^2 + \frac{1}{n}},$$

the corresponding entropy fluxes $q_n$ being defined as in point 2. of Definition 3.1. A standard limiting procedure allows to obtain, in the limit $n \to +\infty$,

$$\eta_n(z) \to |z-k|$$

$$q_n(t,x,z) \to \text{sgn}(z-k) \left( f(t,x,z) - f(t,x,k) \right),$$

so that, in the limit $n \to +\infty$, (5.2) becomes

$$\int_0^T \int_\Omega \left[ \left| u(t,x) - k \right| \partial_t \varphi(t,x) + \text{sgn} \left( u(t,x) - k \right) \left( f(t,x,u(t,x)) - f(t,x,k) \right) \cdot \nabla \varphi(t,x) \right] \, dx \, dt$$

$$+ \int_0^T \int_\Omega \text{sgn} \left( u(t,x) - k \right) \left[ F(t,x,u(t,x)) - \text{div} f(t,x,k) \right] \varphi(t,x) \, dx \, dt$$

$$+ \int_0^T \int_\Omega \text{sgn} \left( u(t,x) - k \right) \text{div} \left( f(t,x,u(t,x)) - f(t,x,k) \right) \varphi(t,x) \, dx \, dt$$

$$+ \int_0^T \int_\Omega |u_0(x) - k| \varphi(0,x) \, dx$$

$$- \int_0^T \int_{\partial \Omega} \text{sgn} \left( u_b(t,\xi) - k \right) \left( f(t,\xi,u_b(t,\xi)) - f(t,\xi,k) \right) \cdot \nu(\xi) \varphi(t,\xi) \, d\xi \, dt$$

$$+ \int_0^T \int_{\partial \Omega} \text{sgn} \left( u_b(t,\xi) - k \right) \left( f(t,\xi,u_b(t,\xi)) - f(t,\xi,\text{tr } u(t,\xi)) \right) \cdot \nu(\xi) \varphi(t,\xi) \, d\xi \, dt \geq 0.$$  

Combining in the inequality above the second line with the third and the fifth one with the sixth yields exactly (5.3).

A BLN–solution is an E–solution. Assume that $\|u\|_{L^\infty([0,T] \times \Omega;\mathbb{R})} \leq U$. It is immediate to see that $u$ satisfies (5.2) with $\eta(u) = \alpha|u-k| + \beta$, for any $\alpha > 0$ and $k, \beta \in \mathbb{R}$. Moreover, if $u$ satisfies (5.2) for two distinct locally Lipschitz continuous pairs $(\eta_1,q_1)$ and $(\eta_2,q_2)$, then the same inequality (5.2) holds for $u$ with $(\eta_1 + \eta_2,q_1 + q_2)$. It can be proved by induction that $u$ satisfies (5.2) for any pair $(\eta,q)$ with $\eta$ piecewise linear and continuous on $[-U,U]$. Furthermore, if $u$ satisfies (5.2) for the continuous pairs $(\eta_n,q_n)$ and the $\eta_n$ converge uniformly to $\eta$ on $[-U,U]$, then $u$ fulfills (5.2) also for the pair $(\eta,q)$, where $q$ is defined as in point 2. of Definition 3.1. To conclude, since any convex entropy $\eta$ is the uniform limit on $[-U,U]$ of piecewise linear and continuous functions, we obtain the proof.

Proof of Lemma 5.5. The proof follows the lines of that of [2] Proposition 2.3. Indeed, let $\Phi \in C^1_b([0,T] \times \Omega;\mathbb{R})$ and $\psi_h \in C^1_b([0,1])$, with $\psi_h(\xi) = 1$ for all $\xi \in \partial \Omega$, $\psi_h(x) = 0$ for all $x \in \Omega$ with $B(x,h) \subseteq \Omega$, and $\|\nabla \psi_h\|_{L^\infty(\Omega;\mathbb{R})} \leq 2/h$. Write (5.2) with $\varphi(t,x) = \Phi(t,x) \psi_h(x)$ and take the limit as $h \to 0$. For any entropy–entropy flux pair $(\eta,q)$, thanks to the Dominated Convergence Theorem and to [2] Lemma A.4 and Lemma A.6, we get

$$\int_0^T \int_\Omega q(t,\xi,\text{tr } u(t,\xi)) \cdot \nu(\xi) \Phi(t,\xi) \, d\xi \, dt - \int_0^T \int_\Omega q(t,\xi,u_b(t,\xi)) \cdot \nu(\xi) \Phi(t,\xi) \, d\xi \, dt$$

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The above formula and (5.6) imply (5.4).

Therefore, there is a set $E$.

Proof of Proposition 5.7. The proof is completed. □

Hence, for any entropy–entropy flux pair $(\eta, q)$, (5.4) holds almost everywhere on $[0, T] \times \partial \Omega$.

Proof of Lemma 5.6. Proving that (5.5) holds is done in the same way as in the proof of [2] Proposition 2.3.

It is immediate to prove that (5.5) reduces to (5.6) when $k \in I[\text{tr} u(t, \xi), u_b(t, \xi)]$. On the other hand, assume that (5.6) holds and consider the various possibilities.

- If $k \leq \min \{\text{tr} u(t, \xi), u_b(t, \xi)\}$ or $k \geq \max \{\text{tr} u(t, \xi), u_b(t, \xi)\}$: the quantity

$$\text{sgn} (\text{tr} u(t, \xi) - k) - \text{sgn} (u_b(t, \xi) - k)$$

is equal to 0, so (5.5) clearly holds.

- If $\text{tr} u(t, \xi) \leq k \leq u_b(t, \xi)$: (5.6) reads $-(f(t, \xi, \text{tr} u(t, \xi)) - f(t, \xi, k)) \cdot \nu(\xi) \geq 0$, while $\text{sgn} (\text{tr} u(t, \xi) - k) = -1$ and $\text{sgn} (u_b(t, \xi) - k) = +1$, so that (5.5) clearly holds.

- If $u_b(t, \xi) \leq k \leq \text{tr} u(t, \xi)$: (5.6) reads $(f(t, \xi, \text{tr} u(t, \xi)) - f(t, \xi, k)) \cdot \nu(\xi) \geq 0$, while $\text{sgn} (\text{tr} u(t, \xi) - k) = +1$ and $\text{sgn} (u_b(t, \xi) - k) = -1$, so that (5.5) clearly holds.

The proof is completed. □

Proof of Proposition 5.7

1 \Rightarrow 2. It is proved in Lemma 4.3.

2 \Rightarrow 3. From (4.3) it follows that, for any $\beta \in L^1([0, T] \times \partial \Omega; \mathbb{R}^+)$ and for any $k \in \mathbb{R}$,

$$\int_0^T \int_{\partial \Omega} \mathcal{F} (t, \xi, \text{tr} u(t, \xi), u_b(t, \xi), k) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt = \text{ess lim}_{\rho \to 0^+} \int_0^T \int_{\partial \Omega} \mathcal{F} (t, \xi, u(t, \xi - \rho \nu(\xi)), u_b(t, \xi), k) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt.$$ 

Therefore, there is a set $E \subseteq [0, T] \times \partial \Omega$ of zero measure such that, for all $k \in \mathbb{R}$ and for all $(t, \xi) \in ([0, T] \times \partial \Omega) \setminus E$

$$\mathcal{F} (t, \xi, \text{tr} u(t, \xi), u_b(t, \xi), k) \cdot \nu(\xi) \geq 0.$$

3 \Rightarrow 4. It follows immediately from the definition (4.2) of $\mathcal{F}$.

4 \Rightarrow 5. For any entropy–entropy flux pair $(\eta, q)$ and for any $(t, \xi) \in [0, T] \times \partial \Omega$, it holds

$$q(t, \xi, z) = q(t, \xi, w) + \int_w^z \eta'(\lambda) \partial_u f(t, \xi, \lambda) \, d\lambda$$

$$= q(t, \xi, w) + \eta'(w) (f(t, \xi, z) - f(t, \xi, w)) + \int_w^z \eta''(\lambda) (f(t, \xi, z) - f(t, \xi, \lambda)) \, d\lambda.$$

The above formula and (5.6) imply (5.4).
\[5 \Rightarrow 6\] It is sufficient to apply (5.4) to any entropy–entropy flux pair \((\eta, q)\) with
\[
\eta'(u_b(t, \xi)) = 0 \quad \text{and} \quad q(t, \xi, u_b(t, \xi)) = 0.
\]
\[6 \Rightarrow 1\] For any boundary entropy–entropy flux pair \((H, Q)\) and \(\beta \in L^1([0, T] \times \partial \Omega; \mathbb{R}_+)\) it holds
\[
\int_0^T \int_{\partial \Omega} Q\left(t, \xi, tr u(t, \xi), u_b(t, \xi)\right) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt
\]
\[
= \text{ess lim}_{\rho \to 0+} \int_0^T \int_{\partial \Omega} Q\left(t, \xi, u(t, \xi - \rho \nu(\xi)), u_b(t, \xi)\right) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt.
\]
The right hand side above is clearly positive, due to Definition 3.2 and (5.8), proving (4.1).

**Proof of Theorem 5.8** Thanks to Theorem 3.8 and Theorem 5.4, we know that the following relations hold

\[ \text{RE–solutions} \iff \text{MV–solutions} \quad \text{and} \quad \text{E–solutions} \iff \text{BLN–solutions}. \]

Therefore, we now prove that a MV–solution is a BLN–solution and that an E–solution is a RE–solution.

**A MV–solution is a BLN–solution.** Let \(k \in \mathbb{R}\) and \(\varphi \in C^{1}_c([\xi] - \infty, T[\times \mathbb{R}^N; \mathbb{R}_+].\) Adding (3.2) with '+' and (3.2) with '−' yields the following inequality:
\[
\int_0^T \int_\Omega |u(t, x) - k| \partial_t \varphi(t, x) \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left( f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left[ F(t, x, u(t, x)) - \text{div} f(t, x, k) \varphi(t, x) \right] \, dx \, dt
\]
\[
+ \int_\Omega |u_0(x) - k| \varphi(0, x) \, dx \geq 0.
\]

Fix \(h > 0\) and consider as a test function \(\Phi_h(t, x) = \varphi(t, x) (1 - \psi_h(x))\), with \(\varphi \in C^{1}_c([-\infty, T[\times \mathbb{R}^N; \mathbb{R}_+]\) and \(\psi_h \in C^{1}_c(\Omega; [0, 1])\), with \(\psi_h(\xi) = 1\) for all \(\xi \in \partial \Omega\), \(\psi_h(x) = 0\) for all \(x \in \Omega\) with \(B(x, h) \subseteq \Omega\), and \(\|\nabla \psi_h\|_{L^\infty(\Omega, \mathbb{R}^N)} \leq 2/h\). Note that \(\lim_{h \to 0} (1 - \psi_h(x)) = \chi_\Omega(x)\) and \(\Phi_h \in C^{1}_c([-\infty, T[\times \mathbb{R}^N; \mathbb{R}_+]\). Using \(\Phi_h\) into (9.9) yields
\[
\int_0^T \int_\Omega |u(t, x) - k| (1 - \psi_h(x)) \partial_t \varphi(t, x) \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left( f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) (1 - \psi_h(x)) \, dx \, dt
\]
\[
- \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left( f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \varphi(t, x) \nabla \psi_h(x) \, dx \, dt
\]
pair with respect to $f$ as follows:

\[
\text{boundary entropy–entropy flux pair} \ (\phi, \text{div } \nu, \text{div } \omega, \text{div } \eta, \nu, \omega, \eta) \quad \text{by Definition 3.2 of boundary entropy–entropy flux pair, (9.5) and the positivity of the test function (9.6).}
\]

Let now $h$ tend to 0. Thanks to [2, Lemma A.4 and Lemma A.6] we obtain

\[
\begin{align*}
\int_0^T \int_\Omega |u(t, x) - k| \partial_t \phi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{sgn}(u(t, x) - k) \left[ F (t, x, u(t, x)) - \text{div } f (t, x, k) \right] \phi(t, x) (1 - \psi_h(x)) \, dx \, dt \\
+ \int_\Omega |u_o(x) - k| \phi(0, x) (1 - \psi_h(x)) \, dx \geq 0.
\end{align*}
\]

Consider in particular the third line above:

\[
- \int_0^T \int_{\partial \Omega} \text{sgn}(\text{tr } u(t, \xi) - k) \left( f (t, \xi, \text{tr } u(t, \xi)) - f (t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt.
\]

Since $u$ is a MV–solution to (1.1), we can apply Lemma 4.1 or equivalently 4.3. Moreover, $u \in (L^\infty \cap \text{T}V_\infty)([0, T] \times \Omega; R)$ and thus Proposition 5.7 and Lemma 5.6 hold. Therefore, thanks to (5.5) and the positivity of the test function $\varphi$, we get

\[
[\text{(9.11)}] \leq - \int_0^T \int_{\partial \Omega} \text{sgn}(u_o(t, \xi) - k) \left( f (t, \xi, \text{tr } u(t, \xi)) - f (t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt,
\]

which inserted into (9.10) yields (5.3), concluding the proof.

**An E–solution is a RE–solution.** Let $\varphi \in C^1([-\infty, T[ \times \Omega; \mathbb{R}^+)$ and $k \in \mathbb{R}$. For any boundary entropy–entropy flux pair $(H, Q)$, set for any $t \in [0, T]$, $x \in \Omega$ and $z \in \mathbb{R}$

\[
\eta(z) = H(z, k), \quad q(t, x, z) = Q(t, x, z, k).
\]

By Definition 3.2 of boundary entropy–entropy flux pair, $(\eta, q)$ is an entropy–entropy flux pair with respect to $f$. Notice moreover that $\eta(k) = 0$. Since $u$ is an E–solution to (1.1), it satisfies (5.2) with the above choice of the test function, which, thanks to (9.12), now reads as follows:

\[
\begin{align*}
\int_0^T \int_\Omega \left[ H (u(t, k)) \partial_t \varphi(t, x) + Q (t, x, u(t, x), k) \cdot \nabla \varphi(t, x) \right] \, dx \, dt \\
+ \int_0^T \int_\Omega \partial_t H (u(t, x), k) \left[ F (t, x, u(t, x)) - \text{div } f (t, x, k) \right] \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_\Omega \text{div } Q (t, x, u(t, x), k) \varphi(t, x) \, dx \, dt \\
+ \int_\Omega H (u_o(x), k) \varphi(0, x) \, dx \geq 0.
\end{align*}
\]

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Apply now Lemma 5.3 and Proposition 5.7. In particular, (4.1) holds for any boundary entropy–entropy flux pair \((H, Q)\) and for any \(\beta \in \mathbf{L}^1 ([0, T] \times \partial \Omega; \mathbb{R}_+)\). We now follow the lines of the second part of the proof of [8]. Theorem 7.31 in order to prove that \(u\) satisfies (5.1). The idea is to show that every \(u\) which is a solution inside the domain \(\Omega\), that is (9.13) holds, and which satisfies the boundary condition in a suitable way, is indeed a RE–solution.

Define the following maps: for \(z, w \in \mathbb{R}\)

\[
\tilde{H}(z, w) = \begin{cases} 
\eta(z) - \eta(w) & \text{if } z \leq w \leq k, \\
0 & \text{if } w \leq z \leq k, \\
\eta(z) & \text{if } w \leq k \leq z, \\
\eta(z) & \text{if } z \leq k \leq w, \\
0 & \text{if } k \leq z \leq w, \\
\eta(z) - \eta(w) & \text{if } k \leq w \leq z,
\end{cases}
\]

and, for \(t \in [0, T], x \in \overline{\Omega}\),

\[
\tilde{Q}(t, x, z, w) = \begin{cases} 
q(t, x, z) - q(t, x, w) & \text{if } z \leq w \leq k, \\
0 & \text{if } w \leq z \leq k, \\
q(t, x, z) & \text{if } w \leq k \leq z, \\
q(t, x, z) & \text{if } z \leq k \leq w, \\
0 & \text{if } k \leq z \leq w, \\
q(t, x, z) - q(t, x, w) & \text{if } k \leq w \leq z.
\end{cases}
\]

It is easy to see that \((\tilde{H}, \tilde{Q}) \in \mathbf{C}^0(\mathbb{R}^2; \mathbb{R}) \times \mathbf{C}^0([0, T] \times \overline{\Omega} \times \mathbb{R}^2; \mathbb{R}_+^N)\). Define, for \(n \in \mathbb{N} \setminus \{0\}\),

\[
H_n(z, w) = \begin{cases} 
\eta(z) - \eta\left(w - \frac{1}{n}\right) & \text{if } z \leq w - \frac{1}{n}, \\
0 & \text{if } w - \frac{1}{n} \leq z \leq k + \frac{1}{n}, \\
\eta(z) - \eta\left(k + \frac{1}{n}\right) & \text{if } k + \frac{1}{n} \leq z, \\
\eta(z) - \eta\left(k - \frac{1}{n}\right) & \text{if } z \leq k - \frac{1}{n}, \\
0 & \text{if } k - \frac{1}{n} \leq z \leq w + \frac{1}{n}, \\
\eta(z) - \eta\left(w + \frac{1}{n}\right) & \text{if } w + \frac{1}{n} \leq z.
\end{cases}
\]

Then, \((\tilde{H}, \tilde{Q})\) can be locally uniformly approximated by \((H_n, \tilde{Q}_n)\), defined as follows

\[
\tilde{H}_n(z, w) = \int_{\mathbb{R}} H_n(\lambda, w) \rho_{1/n}(z - \lambda) \, d\lambda, \\
\tilde{Q}_n(t, x, z, w) = \int_{w}^{z} \partial_t \tilde{H}_n(\lambda, w) \partial_x f(t, x, \lambda) \, d\lambda,
\]

where \(\rho_{1/n}\) is a smooth mollifier. The pair \((H_n, \tilde{Q}_n)\) is clearly a boundary entropy–entropy flux pair. Since (4.1) holds, we have, for all \(\beta \in \mathbf{L}^1([0, T] \times \partial \Omega; \mathbb{R}_+)\),

\[
\text{ess lim}_{\rho \to 0^+} \int_{0}^{T} \int_{\partial \Omega} \tilde{Q}_n \left( t, \xi, u \left(t, \xi - \rho \nu(\xi)\right), u_b(t, \xi) \right) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt \geq 0,
\]

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which becomes, as \( n \to +\infty \),

\[
\int_0^T \int_{\partial \Omega} \tilde{Q} (t, \xi, \text{tr} u(t, \xi), u_b(t, \xi)) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt \geq 0,
\]

(9.14)

where we use the hypothesis that \( u \) admits a trace at the boundary. Going through all the cases in the definition of \( \tilde{Q} \) and exploiting the properties of \( \eta \) yield

\[
\left| \tilde{Q}(t, z, w) - q(t, x, z) \right| \leq \|\partial_u f\|_{L^\infty([0, T] \times \Omega \times \mathbb{R}^N)} \eta(w),
\]

where \( \mathcal{U} \) is the interval \( \mathcal{U} = [-U, U] \) with \( U = \|u\|_{L^\infty([0, T] \times \Omega \times \mathbb{R})} \). Therefore, by (9.14), for all \( \beta \in C^1([0, T] \times \partial \Omega; \mathbb{R}_+) \)

\[
\begin{align*}
\int_0^T \int_{\partial \Omega} q \left( t, \xi, \text{tr} u(t, \xi) \right) \cdot \nu(\xi) \beta(t, \xi) \, d\xi \, dt \\
\geq - \|\partial_u f\|_{L^\infty([0, T] \times \Omega \times \mathbb{R}^N)} \int_0^T \int_{\partial \Omega} \eta (u_b(t, \xi)) \beta(t, \xi) \, d\xi \, dt.
\end{align*}
\]

(9.15)

Fix \( h > 0 \). Consider (9.13) with the test function \( \Phi_h(t, x) = \varphi(t, x) (1 - \psi_h(x)) \), with \( \psi_h \) as in the first part of the proof of this Theorem, so that we obtain

\[
\begin{align*}
\int_0^T \int_{\partial \Omega} H \left( u(t, x), k \right) \left( 1 - \psi_h(x) \right) \partial_t \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_{\Omega} Q \left( t, x, u(t, x), k \right) \cdot \nabla \varphi(t, x) \left( 1 - \psi_h(x) \right) \, dx \, dt \\
- \int_0^T \int_{\Omega} Q \left( t, x, u(t, x), k \right) \cdot \varphi(t, x) \nabla \psi_h(x) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \partial_t H \left( u(t, x), k \right) \left[ F \left( t, x, u(t, x) \right) - \text{div} f \left( t, x, u(t, x) \right) \right] \varphi(t, x) \left( 1 - \psi_h(x) \right) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \text{div} Q \left( t, x, u(t, x), k \right) \varphi(t, x) \left( 1 - \psi_h(x) \right) \, dx \, dt \\
+ \int_0^T \int_{\partial \Omega} H \left( u_0(x), k \right) \varphi(0, x) \left( 1 - \psi_h(x) \right) \, dx \geq 0.
\end{align*}
\]

Let now \( h \) tend to 0: by [2, Lemma A.4 and Lemma A.6] we get

\[
\begin{align*}
\int_0^T \int_{\partial \Omega} H \left( u(t, x), k \right) \partial_t \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_{\Omega} Q \left( t, x, u(t, x), k \right) \cdot \nabla \varphi(t, x) \, dx \, dt \\
- \int_0^T \int_{\Omega} Q \left( t, \xi, \text{tr} u(t, \xi), k \right) \cdot \varphi(t, \xi) \, d\xi \, dt \\
+ \int_0^T \int_{\Omega} \partial_t H \left( u(t, x), k \right) \left[ F \left( t, x, u(t, x) \right) - \text{div} f \left( t, x, u(t, x) \right) \right] \varphi(t, x) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \text{div} Q \left( t, x, u(t, x), k \right) \varphi(t, x) \, dx \, dt
\end{align*}
\]
\[ + \int_{\Omega} H \left( u_0(x), k \right) \varphi(0, x) \, dx \geq 0. \]

Thanks to the definition of \( q(t, x, z) = Q(t, x, z, k) \) and to (9.15) we obtain (3.1), concluding the proof. \( \square \)

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**References**

[1] C. Bardos, A. Y. le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.

[2] R. M. Colombo and E. Rossi. Rigorous estimates on balance laws in bounded domains. *Acta Math. Sci. Ser. B Engl. Ed.*, 35(4):906–944, 2015.

[3] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[4] F. Dubois and P. LeFloch. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J. Differential Equations*, 71(1):93–122, 1988.

[5] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[6] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[7] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[8] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[9] S. Martin. First order quasilinear equations with boundary conditions in the \( L^\infty \) framework. *J. Differential Equations*, 236(2):375–406, 2007.

[10] F. Otto. Initial-boundary value problem for a scalar conservation law. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(8):729–734, 1996.

[11] E. Y. Panov. Existence of strong traces for generalized solutions of multidimensional scalar conservation laws. *J. Hyperbolic Differ. Equ.*, 2(4):885–908, 2005.

[12] E. Y. Panov. Existence of strong traces for quasi-solutions of multidimensional conservation laws. *J. Hyperbolic Differ. Equ.*, 4(4):729–770, 2007.

[13] D. Serre. *Systems of conservation laws. 2*. Cambridge University Press, Cambridge, 2000. Geometric structures, oscillations, and initial-boundary value problems, Translated from the 1996 French original by I. N. Sneddon.

[14] A. Vasseur. Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Ration. Mech. Anal.*, 160(3):181–193, 2001.

[15] J. Vovelle. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numer. Math.*, 90(3):563–596, 2002.