A Global Geometric View of Splaying

Parinya Chalermsook, Mayank Goswami, László Kozma, Kurt Mehlhorn, and Thatchaphol Saranurak

Abstract. Splay trees (Sleator and Tarjan [8]) satisfy the so-called access lemma. Many of the nice properties of splay trees follow from it. What makes self-adjusting binary search trees (BSTs) satisfy the access lemma? We propose a global view of BST algorithms, which allows us to identify general structural properties of algorithms that satisfy the access lemma. As an application of our techniques, we present:

(i) Unified and simpler proofs that capture all known BST algorithms satisfying the access lemma including splay trees and their generalization to the class of local algorithms (Subramanian [9], Georgakopoulos and McClurkin [6]), as well as Greedy BST, introduced by Demaine et al. [4] and shown to satisfy the access lemma by Fox [5].

(ii) A new family of algorithms based on “strict” depth-halving, shown to satisfy the access lemma.

(iii) An extremely short proof for the $O(\log n \log \log n)$ amortized access cost for the path-balance heuristic (proposed by Sleator), matching the best known bound (Balasubramanian and Raman [2]) to a lower-order factor.

Key to our results is a geometric view of how BST algorithms rearrange the search path, which reveals the underlying features that make the amortized cost logarithmic w.r.t. the sum-of-logs (SOL) potential. We obtain further structural results from this insight: for instance, we prove that locality is necessary for satisfying the access lemma, as long as the SOL potential is used. We believe that our work makes a step towards a full characterization of efficient self-adjusting BST algorithms, in particular, it shows the limitations of the SOL potential as a tool for analysis.

1 Introduction

The binary search tree (BST) is a fundamental data structure for the dictionary problem. Self-adjusting BST algorithms re-arrange the tree in response to data accesses, and are thus able to adapt to the distribution of queries. Splay trees, introduced in 1983 by Sleator and Tarjan [8], transform the tree after each access through a sequence of local operations on the search path. Splay trees have a number of attractive properties, including logarithmic amortized cost, static optimality, static finger, and working set properties. These four properties are corollaries of the access lemma, a statement that bounds the amortized cost of a single restructuring operation.

*Work done while at Saarland University.
Despite their efficiency both in theory and practice, splay trees are considered rather mysterious [4] or at least “intriguing” [3] by many authors. The rearrangement of the search path proceeds through a sequence of double-rotations (“zig-zigs” and “zig-zags”). The access lemma relies on the sum of logarithms (SOL) of subtree sizes as a potential function and the proof involves a sum that “magically” telescopes through the local operations. Subramanian [9] and Georgakopoulos and McClurkin [6] extend splay trees to a class of local algorithms, and identify sufficient conditions for such algorithms to satisfy the access lemma (roughly speaking, local algorithms, at any moment of their execution, rearrange only constantly many elements on the search path). But, again, the proof that these algorithms satisfy the access lemma relies on the mysterious sum that simply telescopes through local operations. It is not clear how and whether it is possible to provide a global view of splay trees, as well as the more general local algorithms. The question of analyzing the BST algorithms globally has been raised by both authors.

In order to analyze the amortized cost of BST algorithms, one usually employs the potential function method, which is provably universal for the purpose of amortized analysis [7, Thm 3.4]. The literature of the access lemma has so far focused on the SOL function or essentially identical variants. Given its past success, it is intriguing to investigate its power and limitation in a more general context, for instance, whether (i) the function is strong enough for the purpose of proving the access lemma for any self-adjusting BST algorithm, or (ii) its power is limited to what we know already.

Our contributions and techniques: In this paper we adopt a global view of the BST model. We consider the entire class of minimally self-adjusting BST algorithms: algorithms that rearrange only the search path during each access. Such an algorithm can be seen as a mapping from the search path (called “before-path”) to the resulting tree (called “after-tree”). Observe that all subtrees that are disjoint from the before-path can be reattached to the after-tree in a unique way governed by the ordering of the elements, so we do not need any information about the elements outside of the search path. Indeed, the fact that they do not need any additional bookkeeping besides the pointers of the tree, is one of the attractive features of minimally self-adjusting algorithms. This global model is natural, as it is well known that any BST rearrangement can be performed using a linear number of rotations, and thus the cost of rearrangement is absorbed in the access cost.

Our first contribution is the identification of general global conditions on the transformation from “before-path” to “after-tree”. We show that for any BST algorithm that respects these conditions, the access lemma holds, together with all its corollaries; in fact, our conditions allow a fine-grained analysis: we can bound the amortized cost as a function of the decomposition-complexity of the after-tree.

**Theorem 1 (Informal).** Let $A$ be a BST algorithm that touches only the search path and brings the accessed element $s$ to the root. Then $A$ satisfies the access lemma if (i) the number of newly created leaves is $\Omega(|P| - z)$ where $P$ is the search path and $z$ is the number of “side alternations” [4] in $P$ and (ii) for any element $t > s$ (resp. $t < s$),  

$^{4}$The number of times the search path switches between elements larger than the search key and elements smaller than the search key.
the search path of \( t \) in the after-tree turns right (resp. left) at most a constant number of times.

This theorem is sufficient to derive the access lemma for all previous BST algorithms that satisfy the access lemma.

**Corollary 2 (Informal).** The following BST algorithms satisfy the access lemma: (i) Splay tree, as well as its generalizations to local algorithms (ii) Greedy BST, and (iii) new heuristics based on “strict” depth-halving.

The first and second results unify and simplify previously known BST access lemma results. Our third result partially addresses an open question raised by several authors [2, 6, 9] about whether some form of depth reduction is sufficient to guarantee the access lemma. To prove these results, we apply a new geometric view of how BST algorithms rearrange the search path, which is particularly helpful for the purpose of amortized analysis by the SOL potential function.

The geometric approach extends the range of applicability of our results. Combined with results of Demaine et al. [4], we can prove the access lemma for geometric algorithms whose counter-part in tree-view is not minimally self-adjusting. Indeed, this is the case for online Greedy BST. At the same time, our view is rather different from the geometric model of Demaine et al., in that it allows a reconstruction of the tree in its current state, but it does not capture the entire access history.

As a by-product, we illustrate a global geometric view of splay trees which unfolds the result after a sequence of zig-zig and zig-zag operations. We find this new description intuitive and of independent interest. We also observe that splay trees are “minimal” in the following sense: they perform the minimum possible work to satisfy our sufficient conditions, and nothing more.

Our second contribution is rather conceptual: We make new observations about the power and limitations of the SOL potential function. In particular, we provide an explanation of why the access lemma has only been shown for local algorithms.

**Theorem 3 (Informal).** If a minimally self-adjusting BST algorithm \( A \) satisfies the access lemma by the sum-of-logs potential, then algorithm \( A \) can be made local.

The term local hides a subtle point: existing generalizations of splay trees are local both in the sense that they act on constant size portions of the search path, and in the sense that the local transformations have no information about the relative positions of other elements on the search path. The family of algorithms captured by our result is more general; our algorithms act locally but think globally: they can use advice about which local transformation to perform, and can also be non-deterministic.

Finally, locality has a nice geometric representation. We show that a BST algorithm is local if and only if its geometric counterpart admits a decomposition into a constant number of “monotone point sets”.

## 2 Geometry of BST Algorithms

A main component of our paper is a geometric illustration of how BST algorithms rearrange the search path.
Any binary search tree $T$ on $[n]$ can be described by a height diagram (or height function) $h_T : [n] \to \mathbb{N}$ as follows: Let $H$ be the height of $T$. For each vertex $v \in T$, we define $h_T(v) = H - d(v)$ where $d(v)$ is the distance from root to vertex $v$. Conversely, any height function $h : [n] \to \mathbb{N}$ with “tree structure” can also be transformed into a BST on $[n]$: more formally, we say that a height diagram $h$ has tree structure if for any interval $[a, b] \subseteq [1, n]$, there is a unique maximum in $[a, b]$. See Figure 1 for an example height diagram.

Given a height diagram $h : [n] \to \mathbb{N}$, let $H = \max_{a \in [n]} h(a)$. The stair of $h$ at $a \in [n]$, denoted by $\text{stair}_h(a)$, is defined to contain all elements in $[n]$ that are not “blocked” when viewed from $(a, H + 1)$, i.e. formally, $b \in \text{stair}_h(a)$ if and only if the rectangular region formed by $(a, H + 1)$ and $(b, h(b))$ does not contain any point $(b', h(b'))$ for $b' \in [n]$. The reader familiar with the geometric view of Demaine et al. might recognize here the relation with the concept of unsatisfied rectangles. See Figure 1 for the geometric view of stairs.

**Proposition 4.** Let $T$ be a tree and $h$ its height diagram. Then, for any element $a \in [n]$, $\text{stair}_h(a)$ contains exactly the elements on the search path of $a$ in $T$.

Now we define the notion of neighborhood. This definition is essentially the same as the one used by Fox. Let $h$ be a height diagram. The neighborhood $N_h(a)$ is the maximal open interval $(x, y)$ such that $a \in (x, y)$ and there is no element $b \in (x, y)$ where $h(b) \geq h(a)$; we remark that the neighborhood is thought of as an interval of reals. It is very instructive to see this in the geometric view (Figure 1).

**Proposition 5.** Let $h$ be a height diagram for tree $T$. Then for any element $a \in [n]$, $N_h(a) \cap [n]$ contains exactly the elements in the subtree of $T$ rooted at $a$.

**Self-adjusting BST algorithms:** A self-adjusting BST algorithm $A$ can be described by a collection of path rearrangement rules $\{\tau_k\}$ that map a (search) path of length $k$ into a binary search tree with $k$ nodes. We focus on the family of BST algorithms that rearrange only the search path and do not touch other nodes.

Let $T$ be a binary search tree on $[n]$. Let $w : [n] \to \mathbb{R}_{>0}$ and for any set $S$, $w(S) = \sum_{a \in S \cap [n]} w(a)$. Sleator and Tarjan defined the potential function $\Phi_T = \sum_{a \in [n]} \log w(T_a)$ where $T_a$ is the subtree of $T$ rooted at $a$. We say that an algorithm $A$ satisfies the access lemma (via the SOL potential function) if for all $T'$ that can be obtained as a rearrangement done by algorithm $A$ after some element $s$ is accessed, we have

$$\Phi_T - \Phi_{T'} + O(1 + \log \frac{W}{w(s)}) \geq \Omega(|P|)$$

where $P$ is the search path when accessing $s$ in $T$ and $W = w(T)$.
Geometric BST algorithms: Now we turn the concepts of self-adjusting BST algorithms and access lemma into geometric ones. A geometric BST algorithm is a collection of rules that describe how the heights of the elements on a stair can be adjusted. More formally, we represent each stair on elements by a height diagram $h_{\text{stair}} : \{-l, -l + 1, \ldots, 0, \ldots, r\} \rightarrow \mathbb{N}$. We think of 0 as the accessed element, and negative and positive numbers as elements on the left and right stairs respectively.

A geometric BST algorithm is a collection of mappings $A = \{\tau_{l,r}\}$ such that $\tau_{l,r}$ maps the height diagram $h_{\text{stair}}$ on $\{-l, \ldots, 0, \ldots, r\}$ to an adjusted height diagram $h'_{\text{stair}} = \tau_{l,r}(h_{\text{stair}})$ on the same set $\{-l, \ldots, 0, \ldots, r\}$.

These rules are applied as follows. Let $h : [n] \rightarrow \mathbb{N}$ be a height function (i.e. a tree). If an element $s \in [n]$ is accessed, then the algorithm $A$ adjusts the heights of elements in $\text{stair}_h(s) = \{a_{-l}, \ldots, a_0 = s, \ldots, a_r\}$ by outputing $h' : [n] \rightarrow \mathbb{N}$ with $h'(a) = h(a)$ for $a \notin \text{stair}_h(s)$ and $h'(a_j) = H + h'_{\text{stair}}(j)$ where $H = \max_{a \in [n]} h(a)$ and $h_{\text{stair}}(j) = h(a_j)$, for all $j \in \{-l, \ldots, 0, \ldots, r\}$. For technical reasons, we only allow height adjustments that lift elements in the stair higher than the non-stair elements.

We say that a geometric BST algorithm is natural if it guarantees that $h'_{\text{stair}}$ has tree structure.

**Proposition 6.** There is a correspondence between self-adjusting BST algorithms and natural geometric BST algorithms.

We observe that the correspondence can be made one-to-one, with the choice of a suitable canonical height function.

**Proposition 7 ([4]).** For any (possibly not natural) geometric BST algorithm $A$, there is a BST algorithm whose amortized cost is at most $O(1)$ times the cost of $A$.

These correspondences imply that, in order to analyze the asymptotic cost of a BST algorithm, one only needs to analyze the cost for its geometric counterpart.

**Geometric Access Lemma:** We define the geometric variant of the Sleator-Tarjan potential as $\Phi_h = \sum_{a \in [n]} \log \text{w}(N_h(a))$. Let $A$ be a geometric BST algorithm. Let $h : [n] \rightarrow \mathbb{N}$ be a height diagram and let $h'$ be the output of algorithm $A$ when accessing element $s \in [n]$. Algorithm $A$ satisfies the access lemma (via the SOL potential function) if it always holds that

$$\Phi_h - \Phi_{h'} + O(1 + \log \frac{W}{\text{w}(s)}) \geq \Omega(|\text{stair}_h(s)|).$$

### 3 Tool: Disjoint and Monotone Sets

In this section, we introduce our main technical tool. It allows us to partition the change in potential into independent parts that are easy to analyze individually. Let $A$ be a geometric BST algorithm. Let $h$ be a height diagram and $h'$ be a new height diagram obtained from $A$ after accessing some element $s \in [n]$. The main task is to lower bound the term $\Phi_h - \Phi_{h'}$. We define a partial potential on $X \subseteq [n]$ by $\Phi_h(X) = \sum_{a \in X} \log \text{w}(N_h(a))$. 

5
Lemma 8. Let \( a \notin \text{stair}_h(s) \). Then \( w(N_h(a)) = w(N_{h'}(a)) \). Therefore, \( \Phi_h - \Phi_{h'} = \Phi_h(S) - \Phi_{h'}(S) \), where \( S = \text{stair}_h(s) \).

It is thus sufficient to concentrate on the elements in the stair of the accessed element. The following proposition is obvious.

Proposition 9. Let \( X = \bigcup_{i=1}^{k} X_i \) where the sets \( X_i \) are pairwise disjoint. Then \( \Phi_h - \Phi_{h'} = \sum_{i=1}^{k} (\Phi_h(X_i) - \Phi_{h'}(X_i)) \).

A subset \( X' \subseteq X \) is neighborhood-disjoint if \( N_{h'}(a) \cap N_{h'}(a') = \emptyset \) for all \( a \neq a' \in X' \). We bound the change of partial potential for neighborhood-disjoint sets.

Lemma 10. Let \( X' \) be a neighborhood-disjoint subset of \( X \). Let \( x_0 \in X' \) be the element with minimum height \( h(x_0) \) in \( X' \), and let \( I \subseteq X \) be any interval (i.e. set of the form \( \{i', \ldots, r'\} \) that contains the after-neighborhoods of all elements in \( X' \). Then

\[
|X'| \leq 2 + 8 \cdot \log \frac{w(I)}{w(N_h(x_0))} + \Phi_h(X') - \Phi_{h'}(X').
\]

Proof: We consider the negative and nonnegative elements separately, i.e. \( X' = X'_{<0} \cup X'_{\geq 0} \).

We show \( 1 + \Phi_h(X'_{<0}) - \Phi_{h'}(X'_{<0}) + 4 \log \frac{w(I)}{w(N_h(x_0))} \geq |X'_{\geq 0}| \), and the same holds for \( X'_{\geq 0} \). We only give the proof for \( X'_{\geq 0} \).

Denote \( X'_{\geq 0} \) by \( Y = \{a_0, a_1, \ldots, a_q\} \) where \( 0 \leq a_0 < \ldots < a_q \). Note that \( x_0 \) is a descendant of \( a_0 \) before the access and hence \( N_h(x_0) \subseteq N_h(a_0) \). Let \( N_h(a_0) = (c, d) \). Then \( (0, a_0) \subseteq (c, d) \) and \( d \leq a_1 \). For \( j \geq 0 \), define \( \sigma_j \) as the largest index \( \ell \) such that \( w((c, a_j]) \leq 2^j w(N_h(a_j)) \). Then \( \sigma_0 = 0 \) since weights are positive and \( (c, d) \) is a proper subset of \( (c, a_1] \). It is possible that \( \sigma_j = \sigma_{j+1} \). The set \( \{\sigma_0, \ldots\} \) contains at most \( \lceil \log \frac{w(I)}{w(N_h(a_0))} \rceil \) distinct elements. It contains \( 0 \) and \( q \).

Now we count the number of \( i \) with \( \sigma_j \leq i < \sigma_{j+1} \). We call such an element \( a_i \) heavy if \( w(N_h(a_i)) > 2^{j-1} w(N_h(a_0)) \). There can be at most 3 heavy elements as otherwise \( w((c, a_{i+1}]) > \sum_{\sigma_k \leq \sigma_i < \sigma_{i+1}} w(N_h(a_k)) > 4 \cdot 2^{j-1} w(N_h(a_0)) \), a contradiction.

Next we count the number of light (= non-heavy) elements. For each such light element \( a_i \), we have \( w(N_h(a_i)) \leq 2^{j-1} w(N_h(a_0)) \). We also have \( w(N_h(a_{i+1})) \geq w((c, a_{i+1}]) > w((c, a_{i+1})] \) and thus \( w(N_h(a_{i+1})) > 2^j w(N_h(a_0)) \) by the definition of \( \sigma_j \). Thus \( r_i = w(N_h(a_{i+1})) / w(N_h(a_i)) \geq 2 \) whenever \( a_i \) is a light element. Moreover, for any \( i = 0, \ldots, q - 1 \) (not necessarily light), we have \( r_i \geq 1 \). Thus,

\[
2 \text{number of light elements} \leq \prod_{0 \leq i \leq q - 1} r_i = \left( \prod_{0 \leq i \leq q} \frac{w(N_h(a_i))}{w(N_h(a_{i+1}))} \right) \cdot \frac{w(N_h(a_q))}{w(N_h(a_0))}.
\]

So the number of light elements is at most \( \Phi_h(Y) - \Phi_{h'}(Y) + \log \frac{w(I)}{w(N_h(a_0))} \).

Putting the bounds together, we obtain, writing \( L \) for \( \log \frac{w(I)}{w(N_h(a_0))} \):

\[
|Y| \leq 1 + 3(\lceil L \rceil - 1) + \Phi_h(Y) - \Phi_{h'}(Y) + L \leq 1 + 4L + \Phi_h(Y) - \Phi_{h'}(Y).
\]

\[\square\]
A subset $X' \subseteq X$ is monotone if for all $a, a' \in X'$, we have that $h(a) \leq h(a')$ implies $h'(a) \leq h'(a')$, i.e. the ordering of heights is not reversed. The proof of the following lemma is deferred to the appendix.

**Lemma 11.** Let $X'$ be a monotone subset of $X$ with either $x > 0$ for all $x \in X'$ or $x < 0$ for all $x \in X'$. If $h'(0) \geq h'(x)$ for all elements $x \in X'$ then

$$\Phi_h(X') - \Phi_{h'}(X') + \log \frac{W}{w(s)} \geq 0.$$ 

**Theorem 12.** Suppose that, for every access for an element $s$, a geometric BST algorithm $A$ maps the height of stair $h$ to the new height $h'$ such that, we can partition $S = \text{stair}_h(s) = (\bigcup_{i \leq k} D_i) \cup (\bigcup_{i \leq \ell} M_i)$ where the $D_i$'s are neighborhood-disjoint sets, and the $M_i$'s are monotone sets. Then,

$$\Phi_h(S) - \Phi_{h'}(S) + 2k + (8k + \ell) \log \frac{W}{w(s)} \geq \sum_{i \leq k} |D_i|.$$ 

The proof of Theorem 12 follows immediately from Lemma 10 and 11 with $I = [n]$. We next give some easy applications. From now on, when the height diagrams $h$ and $h'$ are clear from context, we simply write $\Phi_h$ and $\Phi_{h'}$ as $\Phi$ and $\Phi'$ respectively. Figure 2 illustrates the applications.

**Greedy BST:** Greedy BST becomes very simple in the language of height diagrams. It maps any input stair $h$ to a constant function $h'$. Let $S = \{-l, \ldots, 0, \ldots, r\}$ denote the elements in the stair.

**Theorem 13.** $\Phi'(S) - \Phi(S) \leq O(1 + \log \frac{W}{w(s)}) - |S|$. Thus, Greedy BST satisfies the access lemma.

**Proof:** Notice that element $i \in S$ has neighborhood $N_{h'}(i) = (i - 1, i + 1)$. We decompose $S = S_{\text{odd}} \cup S_{\text{even}}$ where $S_{\text{odd}}$ and $S_{\text{even}}$ are the odd and even elements in $S$. Both sets are neighborhood-disjoint. Application of Theorem 12 yields the claim. □

**Path-Balance:** The path-balance algorithm maps any path to a balanced BST. Let $S = \{-l, \ldots, 0, \ldots, r\}$ be the input stair and let $c = \lceil \log_2 (1 + |S|) \rceil$. The height function $h$ is mapped to $h'$, where $h'$ is the height function of any tree with root 0 of height $c$ on $S$.

**Lemma 14.** $|S| \leq \Phi(S) - \Phi'(S) + O((1 + \log |S|)(1 + \log(W/w(s))))$.

**Proof:** We decompose $S = \bigcup_{k \leq c} S_k$ where $S_k$ contains all elements $a$ with $h'(a) = k$. Since $S_k$ is neighborhood-disjoint for every $k$, an application of Theorem 12 completes the proof. □

**Theorem 15.** Path-Balance has amortized cost at most $O(\log n \log \log n)$. 

---

Fig. 2: Decomposition of after-tree for Greedy BST (left) and Path-Balance (right). Elements with the same color are in the same disjoint set.
Proof: We choose the uniform weight function: \( w(a) = 1 \) for all \( a \). Let \( c_i \) be the cost of the \( i \)-th access, \( 1 \leq i \leq m \), and let \( C = \sum_{1 \leq i \leq m} c_i \) be the total cost of the accesses. Note that \( \prod_{i} c_i \leq (C/m)^m \). The potential of a tree with \( n \) items is at most \( n \log n \). Thus \( C \leq n \log n + \sum_{1 \leq i \leq m} O((1 + \log c_i)(1 + \log n)) = O((n + m) \log n) + O(m \log n) \cdot \log(C/m) \) by Lemma 14. Assume \( C = K(n + m) \log n \) for some \( K \). Then \( K = O(1) + O(1) \cdot \log(K \log n) \) and hence \( K = O(\log \log n) \). \( \square \)

4 Another Tool: Zigzag Sets

In this section, it will be convenient for us to consider the stair elements ordered increasingly by their heights. That is, let \( s \) be an accessed element, and \( S = \text{stair}_h(s) \). We write \( S^{'}, s = \{a_1, \ldots, a_{|S|-1}\} \) where \( h(a_i) < h(a_{i+1}) \) for all \( i < |S| - 1 \). The notion of zigzag set of \( S \) is defined as follows: For each \( i \), define the set \( Z_i = \{a_i, a_{i+1}\} \) if either \( a_i < s < a_{i+1} \) or \( a_i > s > a_{i+1} \), or \( Z_i = \emptyset \) otherwise. The zigzag set \( Z_S \) is defined as \( Z_S = \bigsqcup_i Z_i \). In words, the number of non-empty sets \( Z_i \) is exactly the number of “side alternations” among the stair elements, and the cardinality of \( Z_S \) is the number of elements involved in such alternations.

Recall that \( h \) and \( h' \) are the height diagrams before and after adjustment respectively.

Again, we use the simplified notation \( \Phi = \Phi_h \) and \( \Phi' = \Phi_{h'} \).

Rotate to Root: Since we deal with BST algorithms that bring the accessed element to the root, we first analyze the rotate-to-root algorithm (Allen, Munro 11), that brings the accessed element \( s \) to the root and leaves all other heights unchanged, i.e., \( h'(s) > \max_{a \in S \setminus s} h'(a) \) and \( h'(a) = h(a) \) for \( a \neq s \).

Lemma 16. \( \Phi(S) - \Phi'(S) + O(1 + \log \frac{w}{w(N(s))}) \geq |Z| \).

Proof: Because \( s \) is made the root and no other height is changed, we have that, if \( a < s \), \( N'(a) = N(a) \cap (-\infty, s) \), else \( N'(a) = N(a) \cap (s, \infty) \). The proof of the following claim is deferred to the appendix.

Claim. \( \Phi(Z_i) - \Phi'(Z_i) + \log \frac{w(N(a_{i+1}))}{w(N(a_i))} \geq 2 \).

Let \( Z_{\text{even}} (Z_{\text{odd}}) \) be the union of the \( Z_i \) with even (odd) indices. One of the two sets has cardinality at least \( |Z|/2 \). Assume that it is the former; the other case is symmetric. We sum the statement of the claim over all \( i \) in \( Z_{\text{even}} \) and obtain

\[
\sum_{i \in Z_{\text{even}}} \left( \Phi(Z_i) - \Phi'(Z_i) + \log \frac{w(N(a_{i+1}))}{w(N(a_i))} \right) \geq 2 |Z_{\text{even}}| \geq |Z|.
\]

Notice that the elements in \( S \setminus Z_{\text{even}} \) form two monotone sets and hence \( \Phi(S \setminus Z_{\text{even}}) - \Phi'(S \setminus Z_{\text{even}}) + 2 \log(W/w(s)) \geq 0 \). This completes the proof. \( \square \)

The following theorem combines all three tools we have introduced: disjoint, monotone, and zigzag sets.
Theorem 17. Suppose that, for every access for element \( s \), a geometric BST algorithm \( \mathcal{A} \) maps the height of stair \( h \) to the new height \( h' \) such that (i) \( h'(s) = \max_{a \in S \setminus s} h'(a) \), and (ii) we can partition \( S \setminus s = (\bigcup_{i \leq k} D_i) \cup (\bigcup_{\ell \leq \ell} M_i) \) where \( D_i \)'s are neighborhood-disjoint sets, and \( M_i \)'s are monotone sets. Then,

\[
\Phi(S) - \Phi'(S) + O((k + \ell)(1 + \log \frac{W}{w(s)})) \geq \sum_{i \leq k} |D_i| + |Z_S|.
\]

Proof: Let \( h'' \) be the intermediate height function after executing rotate-to-root, i.e. we look at the transformation performed by algorithm \( \mathcal{A} \) from \( h \) to \( h' \) as a two-step transformation from \( h \) to \( h'' \) and then to \( h' \). Denote the potential of \( h'' \) by \( \Phi'' \). We analyze \( \Phi(S) - \Phi''(S) \) and \( \Phi''(S) - \Phi'(S) \).

By Lemma 16, \( \Phi(S) - \Phi''(S) = O(1 + \log \frac{W}{w(N(s))}) \geq |Z_S| \). By Theorem 12, \( \Phi''(S) - \Phi'(S) = O((k + \ell)(1 + \log \frac{W}{w(N(s))})) \geq \sum_{i \leq k} |D_i| \). Summing the two inequalities completes the proof. \( \square \)

**Splay:** Splay extends rotate-to-root: It brings the accessed element \( s \) to the root and it swaps the heights in each pair \( \{a_{2i+1}, a_{2i+2}\} \) that are on the same side of \( s \). More precisely, \( h'(s) = \max_{a \in S \setminus s} h'(a) + 1 \), and for each pair \( \{a_{2i+1}, a_{2i+2}\} \) where \( s < \min\{a_{2i+1}, a_{2i+2}\} \) or \( s > \max\{a_{2i+1}, a_{2i+2}\} \), we set \( h'(a_{2i+2}) = h(a_{2i+1}) + 1 \), \( h'(a_{2i+1}) = h(a_{2i+2}) \) (we refer to these pairs as swapped pairs); otherwise, do nothing. See Figure 3 for an illustration.

**Proposition 18.** The above description of splay is equivalent to the Sleator-Tarjan description.

**Theorem 19.** \( \Phi(S) - \Phi'(S) + O(1 + \log \frac{W}{w(N(s))}) \geq \Omega(|S|) \). Thus, splay satisfies access lemma.

Proof: Let \( P_{\text{swap}} \) and \( P_{\text{not}} \) denote the sets of pairs \( \{a_{2i+1}, a_{2i+2}\} \) whose height are swapped and not swapped by splay respectively. Note that \( |P_{\text{swap}}| + |P_{\text{not}}| \geq |S|/2 - 1 \). Observe that \( P_{\text{not}} \) is a subset of the zigzag set \( Z_S \) of \( S \). Next, \( D = \{a_{2i+2} \mid \{a_{2i+1}, a_{2i+2}\} \in P_{\text{swap}} \} \) is a neighborhood-disjoint set, because the swap always guarantees that \( h'(a_{2i+2}) < \min\{h'(a_l), h'(a_r)\} \) where \( a_l \) and \( a_r \) are the two adjacent elements of \( a_{2i+2} \) in \( S \). Also, \( S \setminus (D + s) \) forms two monotone sets, one on each side of \( s \).

Invoking Theorem 17 with \( k = 1 \) and \( \ell = 2 \) completes the proof. \( \square \)

5 Sufficient Condition for Minimally Self-Adjusting BST

In this section, we translate Theorem 17 from geometric view to tree-view, when the theorem guarantees the access lemma, i.e. for \( k, \ell = O(1) \). For simplicity, we restrict ourselves to algorithms that move the accessed element to the root.
Suppose that the before-path \( P \) is mapped to the after-tree \( T \) rooted at 0; again 0 is the element accessed. Let \( h \) and \( h' \) be the height functions of \( P \) and \( T \), respectively.

For any element \( x \) in \( T \), suppose that \((0 = s, s_1, \ldots, s_k = x)\) is the search path of \( x \) in \( T \). If \( s_i < s_{i+1} \), then we say that there is a right turn at \( s_i \); otherwise there is a left turn at \( s_i \). Let \( T_\prec \) (\( T_\succ \)) contain the elements of \( T \) that are smaller (larger) than \( s \). First we prove a structural lemma that will be useful in making a connection between the tree-view and the geometric view. Roughly speaking, the monotone sets can be interpreted as turns “away from the root” in the tree-view. Using the following lemma (proof in Appendix), it is easy to derive our sufficient conditions.

**Lemma 20.** Let \( S \subseteq \{-l, \ldots, 0, \ldots, r\} \) be a subset of stair, let \( S_\succ = S \cap \{0, \ldots, r\} \) and \( S_\prec = S \cap \{-l, \ldots, -1\} \). Then \( S_\succ \) (resp. \( S_\prec \)) can be decomposed into \( \ell \) increasing (resp. decreasing) monotone sets if and only if the search path of every element \( x \in T_\succ \) (\( T_\prec \)) contains at most \( \ell \) right (resp. left) turns.

**Theorem 21.** Suppose that, for every access for \( s \), a BST algorithm \( A \) rearranges a search path that contains \( z \) side alternations, into a tree \( T \) such that (i) \( s \) is the root of \( T \), (ii) the number of leaves of \( T \) is \( \Omega(|T| - z) \), (iii) for every element \( x \) larger (smaller) than \( s \), the search path of \( x \) in \( T \) contains at most \( O(1) \) right (left) turns. Then \( A \) satisfies the access lemma.

**Proof:** Let \( B \) be the set of leaves of \( T \) and let \( b = |B| \). By assumption (ii), there is a positive constant \( c \) such that \( b \geq (|T| - z)/c \). Then \( |T| \leq cb + z \). We decompose \( S \setminus s \) into \( B \) and \( \ell \) monotone sets. By assumption (iii), \( \ell = O(1) \). An application of Theorem 17 with \( k = 1 \) and \( \ell = O(1) \) completes the proof. \( \Box \)

### 6 Limitations of the SOL Potential: Local Algorithms

In this section we define a class of minimally self-adjusting BST algorithms that we call **local**. We show that an algorithm is local exactly if all after-trees it creates can be decomposed into constantly many monotone sets. Our definition of local algorithm is inspired by similar definitions by Subramanian [9] and Georgakopoulos and McClurkin [6]. Our locality criterion subsumes both previous definitions, apart from a technical condition not needed in these works: we require the transformation to bring the accessed element to the root. We require this (rather natural) condition in order to simplify the proofs. We mention that it can be removed at considerable expense in technicalities. Apart from this point, our definition of locality is more general: while existing local algorithms are oblivious to the global structure of the after-tree, our definition of local algorithm allows external global advice, as well as non-determinism.

Consider the before-path \( P \) and the after-tree \( T \). A **decomposition** of the transformation \( P \rightarrow T \) is a sequence of BSTs \( (P = Q_0 \xrightarrow{P_0} Q_1 \xrightarrow{P_1} \cdots \xrightarrow{P_{k-1}} Q_k = T) \), such that for all \( i \), the tree \( Q_{i+1} \) can be obtained from the tree \( Q_i \), by rearranging a path \( P_i \) contained in \( Q_i \) into a tree \( T_j \), and linking all the attached subtrees in the unique way given by the element ordering. Clearly, every transformation has such a decomposition, since a sequence of rotations fulfills the requirement. The decomposition is **local** with window-size \( w \), if it satisfies the following conditions:
We call a minimally self-adjusting algorithm \( A \) local, if all the before-path \( \rightarrow \) after-tree transformations performed by \( A \) have a local decomposition with constant-size window. Let \( T \) be a BST, and let \( T_\succ \), and \( T_\prec \) be the right (resp. left) subtree of the root of \( T \). We say that \( T \) can be decomposed into \( w \) monotone sets, if there exist \( w_L \) and \( w_R \), such that \( w_R + w_L < w \), and \( T_\succ \) can be decomposed into \( w_L \) decreasing sets, and \( T_\prec \) can be decomposed into \( w_R \) increasing sets.

The following theorem shows that local algorithms are exactly those that respect the monotone condition of Theorem 21 (proof in Appendix).

**Theorem 22.** Let \( A \) be a minimally self-adjusting algorithm. (i) If \( A \) is local with window size \( w \), then all the after-trees created by \( A \) are decomposable into \( 2w \) monotone sets. (ii) If all the after-trees created by \( A \) are decomposable into \( w \) monotone sets, then \( A \) is local with window-size \( w \).

**Necessity of \( O(1) \) monotone sets:** We show that the access lemma with the SOL potential function does not hold if the after-trees cannot be decomposed into constantly many monotone sets.

**Theorem 23.** Consider a transformation from before-path \( P \) to after-tree \( T \) by algorithm \( A \). If there is no height function \( h \) of \( T \) for which the elements in \( T \setminus \{ \) can be decomposed into constantly many monotone sets, then \( A \) does not satisfy the access lemma with the SOL potential.

**Proof:** Let \( a_{-1}, \ldots, a_0 = s, \ldots, a_r \) be the elements in \( T \) where \( a_i < a_{i+1} \). Let \( x > s \) be an element for which the search path of \( x \) in \( T \) contains a maximum number of right turns. By Lemma 20 we may assume that the number of right turns is \( k = \omega(1) \). Let \( a_{i_1}, \ldots, a_{i_k} \) be the elements where the right turns occur. Observe that all these nodes are descendants of \( x \) in the before-path.

We now define a weight assignment to the elements of \( T \) and the pendent trees for which the access lemma does not hold with the SOL potential. We assign weight zero to all pendent trees, weight one to all proper descendants of \( x \) in \( P \) and weight \( K \) to all ancestors of \( x \) in \( P \). Here \( K \) is a big number. The total weight \( W \) then lies between \( K \) and \( |T| K \).

We next bound the potential change. Let \( r(a_i) = w(N'(a_i))/w(N(a_i)) \) be the ratio of the weight of the neighborhood of \( a_i \) in the after-tree and in the before-path. For any element \( a_{i_j} \) at which a right turn occurs, we have \( w(N(a_{i_j})) \leq |T| \) and \( w(N'(a_{i_j})) \geq K \). So \( r(a_{i_j}) \geq K/|T| \). Consider now any other \( a_i \). If it is an ancestor of \( x \) in the before-path, then \( w(N(a_i)) \leq W \) and \( w(N'(a_i)) \geq K \). If it is a descendant of \( x \), then \( w(N(a_i)) \leq |T| \) and \( w(N'(a_i)) \geq 1 \). Thus \( r(a_i) \geq 1/|T| \) for every \( a_i \). We conclude

\[
\Phi'(T) - \Phi(T) \geq k \cdot \log \frac{K}{|T|} - |T| \log |T|.
\]
If \( A \) satisfies the access lemma with the SOL potential function, then we must have
\[
\Phi'(T) - \Phi(T) = O(\log \frac{W}{w(s)} - |T|) = O(\log(K|T|)).
\]
However, if \( K \) is large enough and \( k = \omega(1) \), then
\[
k \cdot \lg \frac{K}{|T|} - |T| \lg |T| \gg O(\log(K|T|)).
\]
Because of the equivalence between monotone sets and local algorithms, we have

**Theorem 24.** If a minimally self-adjusting BST algorithm \( A \) satisfies the access lemma with the SOL potential, then \( A \) can be made local.

### 7 New Heuristics: Depth reduction

In this section we look at new heuristics for path-rearrangement that are provably efficient (as before, we restrict ourselves to heuristics that move the accessed element to the root). As already observed by Sleator and Tarjan in the original splay paper [8], the property that makes splaying efficient is depth-halving, i.e. the fact that every element on the access path reduces its distance to the root by a factor of approximately two. It is tempting to try abstracting away the desirable property of depth-reduction from the concrete local operations employed in splay trees. In other words, we ask whether a suitable global depth-reduction property is sufficient to guarantee the access lemma. This question has been raised in various forms by several authors [2, 6, 9]. Based on Theorem 21 we give both positive and negative results in this direction.

Let \( x \) and \( y \) be two arbitrary nodes on the search path. If \( y \) is an ancestor of \( x \) in the search path, but not in the after-tree, then we say that \( x \) has lost the ancestor \( y \), and \( y \) has lost the descendant \( x \). Similarly we define gaining an ancestor or a descendant. We stress that only nodes on the search path (resp. the after-tree) are counted as descendants, and not the nodes of the pendant trees.

Let \( d(x) \) denote the number of ancestors of \( x \) in the search path. We give a sufficient condition for a good heuristic, stated below. The proof is deferred to the appendix.

**Theorem 25.** Let \( A \) be a minimally self-adjusting BST algorithm that satisfies the following conditions: (i) Every node \( x \) on the search path loses at least \( (\frac{1}{2} + \epsilon) \cdot d(x) - c \) ancestors, for fixed constants \( \epsilon > 0, c > 0 \), and (ii) every node on the search path, except the accessed element, gains at most \( d \) new descendants, for a fixed constant \( d > 0 \). Then \( A \) satisfies the access lemma.

One may ask how tight are the conditions of Theorem 25. If we relax the constant in the first condition from \( (\frac{1}{2} + \epsilon) \) to \( \frac{1}{2} \), the conditions of Theorem 21 are no longer implied. Figure 4 in the appendix shows a rearrangement in which every node loses a \( \frac{1}{2} \)-fraction of its ancestors, gains at most two ancestors or descendants, yet both the number of side alternations and the number of leaves created are \( O(\sqrt{|T|}) \), where \( T \) is the after-tree. If we further relax the ratio to \( (\frac{1}{2} - \epsilon) \), we can construct an example where the number of alternations and the number of leaves created are only \( O(\log |T|/\epsilon) \).

Allowing more gained descendants and limiting instead the number of gained ancestors is also beyond the strength of Theorem 21. In the example of Figure 4 in the appendix every node loses an \( (1 - o(1)) \)-fraction of ancestors, yet the number of leaves created is only \( O(\sqrt{|T|}) \) (there are no alternations in the before-path).
Finally, we observe that depth-reduction alone is likely not sufficient: one can restructure the access path in such a way that every node reduces its depth by a constant factor, yet the resulting after-tree has an anti-monotone path of linear size. Figure 5 in the appendix shows such an example for depth-halving. Based on Theorem 23, this means that if such a restructuring were to satisfy the access lemma in its full generality, the SOL potential would not be able to show it.

Open Questions: We showed that a minimally self-adjusting BST algorithm can satisfy the access lemma under the SOL potential only if it is local. This corresponds to condition (iii) of Theorem 21. We ask whether condition (ii) of Theorem 21 (i.e., on the number of leaves and side alternations) is also necessary. At one extreme, creating only constantly many leaves can be very inefficient (as in the rotate-to-root heuristic). At the other end, can the access lemma still hold with a sublinear number of zig-zags and leaves?

More generally, one may ask whether locality is a necessary feature of all efficient BST algorithms. We have shown that some natural heuristics (e.g., path-balance or depth reduction) do not share this property. A full understanding of such “truly nonlocal” heuristics seems to require further insight.

Acknowledgement: The authors thank Raimund Seidel for suggesting the study of depth-reducing heuristics and for useful insights about BSTs and splay trees.

References

1. Brian Allen and J. Ian Munro. Self-organizing binary search trees. J. ACM, 25(4):526–535, 1978.
2. R. Balasubramanian and Venkatesh Raman. Path balance heuristic for self-adjusting binary search trees. In Proceedings of FSTTCS, pages 338–348, 1995.
3. Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. Introduction to Algorithms (3. ed.). MIT Press, 2009.
4. Erik D. Demaine, Dion Harmon, John Iacono, Daniel M. Kane, and Mihai Patrascu. The geometry of binary search trees. In SODA 2009, pages 496–505, 2009.
5. Kyle Fox. Upper bounds for maximally greedy binary search trees. In WADS 2011, pages 411–422, 2011.
6. George F. Georgakopoulos and David J. McClurkin. Generalized template splay: A basic theory and calculus. Comput. J., 47(1):10–19, 2004.
7. Kurt Mehlhorn and Peter Sanders. Algorithms and Data Structures: The Basic Toolbox. Springer, 2008.
8. Daniel Dominic Sleator and Robert Endre Tarjan. Self-adjusting binary search trees. J. ACM, 32(3):652–686, 1985.
9. Ashok Subramanian. An explanation of splaying. J. Algorithms, 20(3):512–525, 1996.
A Proof Omitted from Section 3

A.1 Proof of Lemma 11

By definition, \( \Phi_h(X') - \Phi_{h'}(X') = \log \prod_{x \in X'} \frac{w(N_h(a))}{w(N_{h'}(a))} \). We order the elements in \( X' = \{a_1, \ldots, a_q\} \) such that \( 0 < a_1 < a_2 < \ldots < a_q \). Then \( h'(a_i) \leq h'(a_{i+1}) \) by monotonicity and \( \Phi_h(X') - \Phi_{h'}(X') = \log \frac{w(N_h(a_0))}{w(N_{h'}(a_0))} + \sum_{i=1}^{q-1} \log \frac{w(N_h(a_{i+1}))}{w(N_{h'}(a_i))} \).

Now, notice that \( N_{h'}(a_i) \subset (0, a_{i+1}) \) since \( h'(0) \geq h'(a_i) \) and \( h'(a_i) \leq h'(a_{i+1}) \), while \( N_h(a_{i+1}) \) contains the interval \( (0, a_{i+1}) \). Therefore, we have that \( N_{h'}(a_i) \subset N_h(a_{i+1}) \). This implies that the second sum is nonnegative. Thus \( \Phi_h(X') - \Phi_{h'}(X') \geq \log \frac{w(N_h(a_0))}{w(N_{h'}(a_0))} \geq \log \frac{w(s)}{W} \).

B Proof Omitted from Section 4

B.1 Proof of the Claim in Lemma 16

We give the proof for the case where \( a_i > 0 \) and \( a_{i+1} < 0 \); the other case is symmetric. Let \( a' \) be the element preceding \( a_{i+1} \) in the stair of \( s \) and let \( a'' \) be the element following \( a_i \) in the stair. If these elements do not exist, they are \( -\infty \) and \( +\infty \), respectively. Let \( W_1 = w((a', 0)) \), \( W_2 = w((0, a'')) \), and \( W' = w((a_{i+1}, 0)) \). The weights of the before-neighborhoods of \( a_i \) and \( a_{i+1} \) are \( w(N(a_i)) = W' + w(s) + W_2 \) and \( w(N(a_{i+1})) = W_1 + w(s) + W_2 \). The after-neighborhoods of these elements have weights \( w(N'(a_i)) = W_2 \) and \( w(N'(a_{i+1})) = W_1 \).

Thus \( \Phi(Z_i) - \Phi'(Z_i) + \log \frac{W_1 + w(s) + W_2}{W'' + w(s) + W'} \geq \log (W_1 + w(s) + W_2) - \log W_1 + \log (W_2 + w(s) + W') - \log W_2 + \log \frac{W_1 + w(s) + W_2}{W'' + w(s) + W'} \geq 2 \log (W_1 + W_2) - \log W_1 - \log W_2 \geq 2, \) since \( (W_1 + W_2)^2 \geq 4W_1W_2 \) for all positive numbers \( W_1 \) and \( W_2 \).

C Proof Omitted from Section 5

C.1 Proof of Lemma 20

The proofs for the positive and negative sets are symmetric, so we only treat the positive set.

\((\Rightarrow)\) Suppose that there is some element \( x > 0 \), whose search path in the after-tree \( T \) contains \( \ell' > \ell \) right turns. Let \( R = \{s_1, \ldots, s_{\ell'}\} \) be the nodes where the right turns occur in the search path from root to \( x \), and consider \( s_i \) and \( s_j \) in \( R \) with \( i < j \). Then \( h'(s_i) > h'(s_j) \) since \( s_i \) is an ancestor of \( s_j \) in \( T \) and \( h(s_i) < h(s_j) \) since \( 0 < s_i < s_j \) and \( s_i \) and \( s_j \) belong to the right side of the stair of \( s \). Thus \( s_i \) and \( s_j \) cannot belong to the same monotone set.

\((\Leftarrow)\) Suppose that the search path of every element \( x \in T_\succ \) contains at most \( r \) right turns. We decompose \( T_\succ = T_1 \cup \ldots \cup T_r \), where \( T_i \) is the set of elements whose search path contains exactly \( i \) right turns and show how to modify the height function \( h' \) into an equivalent height function \( h \) for which each \( T_i \) is a monotone set. For any \( x \) in \( T_\succ \), let \( RL_u^{k_1}RL_{u_1}^{k_2} \ldots RL_{u_r}^{k_r} \) encode the search path for \( x \); here \( R \) and \( L \) indicate the
right and left turn respectively; for instance, if the RL²RR encodes the search path that follows one right turn, then two lefts and two rights respectively. We define $h$ by

$$h(x) = H - d(x)$$

where $d(x) = \sum_{0 \leq j \leq r} (k_j^{(x)} + 1)n^{r-j}$

We prove that each set $T_i$ is monotone w.r.t. this height function. Consider two elements $x, y \in T_i$ and assume that $x < y$. If $y$ is an ancestor of $x$, then clearly $d(x) > d(y)$ because the encoding of $x$ can only start with the encoding of $y$, followed by consecutive $L$’s. Otherwise, if $y$ is not an ancestor of $x$, let $z$ be their least common ancestor, so $x$ and $y$ are in the left and right subtrees of $z$ respectively. Then, there is a $j$ such that $k_j^{(x)} = k_j^{(y)}$ for $j < j$ and $k_j^{(x)} > k_j^{(y)}$. Thus $d(x) > d(y)$.

D  Proofs Omitted from Section [6]

D.1  Proof of Theorem [22]

Let $s$ denote the accessed element in the before-path $P$ (i.e. the root of $T$).

(i) Suppose for contradiction that the after-tree $T$ is not decomposable into $2w$ monotone sets. As a corollary of Lemma [20] $T$ contains a sequence of elements $x_1, x_2, \ldots, x_{w+1}$ such that either (a) $s < x_1 < \cdots < x_{w+1}$, or (b) $x_{w+1} < x_w < \cdots < x_1 < s$ holds, and $x_{i+1}$ is a descendant of $x_i$ for all $i$. Assume that case (a) holds; the other case is symmetric.

Let $i'$ be the first index for which $x_{w+1} \in P_{i'}$. From the (window-size) condition we know that $P_{i'}$ contains at most $w$ elements, and thus there exists some index $j < w + 1$ such that $x_j \notin P_{i'}$. As $x_j$ is a descendant of $x_{w+1}$ in the before-path, it was on some path $P_{i''}$ for $i'' < i'$, and due to the (no-revisit) condition it will not be on another path in the future. Thus, it is impossible that $x_j$ becomes an ancestor of $x_{w+1}$, so no local algorithm can create $T$ from $P$.

(ii) We give an explicit local algorithm $A$ that creates the tree $T$ from path $P$. As in the proof of Lemma [20] we decompose $T_\geq = R_1 \cup \ldots \cup R_{w_l}$, and $T_\leq = L_1 \cup \ldots \cup L_{w_l}$, where $R_i$ (resp. $L_i$) is the set of elements whose search path contains exactly $i$ right (resp. left) turns. Let $L_0 = R_0 = \{s\}$. Let $P = (x_1, x_2, \ldots, x_k = s)$ be the search path for $s$, i.e., $x_1$ is the root of the current tree and $x_{j+1}$ is a child of $x_j$. For any $j$, let $t_j(R_i)$ be the element in $R_i \cap \{x_j, \ldots, x_k\}$ with minimal index; $t_j(L_i)$ is defined analogously.

For any node $x$ of $T$, let the first right ancestor $FRA(x)$ be the first ancestor of $x$ in $T$ that is larger than $x$ (if any) and let the first left ancestor $FLA(x)$ be the first ancestor of $x$ smaller than $x$ (if any).

Lemma 26. Fix $j$, let $X = \{x_j, \ldots, x_k\}$, consider any $i \geq 1$, and let $x = t_j(R_i)$.

(i) If $x$ is a right child in $T$ then its parent belongs to $X \cap R_{i-1}$.
(ii) If $x$ is a left child in $T$, then $FLA(x)$ is equal to $t_j(X_{i-1})$ and $FRA(x) \notin X$.
(iii) If $x$ is a right child and $FRA(x) \in X$ then all nodes in the subtree of $T$ rooted at $x$ belong to $X$.
(iv) If $FRA(x) \in X$ then $FRA(t_j(R_e)) \in X$ for all $e \geq i$.  

15
Proof:
(i) The parent of $x$ lies between $s$ and $x$ and hence belongs to $X$. By definition of the $R_i$’s, it also belongs to $R_{i-1}$.

(ii) $\text{parent}(x) \in R_i$ and hence, by definition of $t_j(R_i)$, $\text{parent}(x) \notin X$. $\text{FLA}(x) < x$ and hence $\text{FLA}(x) \in X \cap R_{i-1}$. The element in $R_{i-1}$ after $\text{FLA}(x)$ is larger than $\text{parent}(x)$ and hence does not belong to $X$. The second claim holds since $\text{FRA}(x) \in R_i$ if $x$ is a left child.

(iii) The elements between $s$ and $\text{FRA}(x)$ (inclusive) belong to $X$.

(iv) Since $z = \text{FRA}(x) \in X$, $x$ is a right child and $z$ belongs to $R_i$ for some $\ell < i$. Since $x = t_j(R_i)$, the right subtree of $z$ contains no element in $X \cap R_i$. Consider any $\ell > i$. Then $t_j(R_{\ell})$ must lie in the left subtree of $z$ and hence $\text{FRA}(t_j(R_{\ell})) \not\subseteq z$. Thus $\text{FRA}(t_j(R_{\ell})) \in X$.

We are now ready for the algorithm. We traverse the search path $P$ to $s$ backwards towards the root. Let $P = (x_1, x_2, \ldots, x_k = s)$. Assume that we have reached node $x_j$. Let $X = \{x_1, \ldots, x_k\}$. We maintain an active set $A$ of nodes. It consists of all $t_j(R_i)$ such that $\text{FRA}(t_j(R_i)) \not\subseteq X$ and all $t_j(L_i)$ such that $\text{FLA}(t_j(L_i)) \not\subseteq X$. When $j = k$, $A = \{s\}$. Consider any $y \in A$ and assume $\text{parent}(y) \in X$. Then $y$ must be a right child by (ii) and $\text{FRA}(y) \not\subseteq X$. Since $\text{FRA}(y)$ is also $\text{FRA}(\text{parent}(y))$, the parent is also active.

By part (iv) of the preceding Lemma, there are indices $\ell$ and $r$ such that exactly the nodes $t_j(L_{-\ell})$ to $t_j(R_r)$ are active. When $j = k$, only $t_j(R_0) = s$ is active. We maintain the active nodes in a path $P’$. By the preceding paragraph, the nodes in $X \setminus A$ form subtrees of $T$. We attach them to $P’$ at the appropriate places and we also attach $P’$ to the initial segment $x_1$ to $x_{j-1}$ of $P$.

What are the actions required when we move from $x_j$ to $x_{j-1}$? Assume $x_{j-1} > s$ and let $X’ = \{x_{j-1}, \ldots, x_k\}$. Also assume that $x_{j-1}$ belongs to $R_i$ and hence $x_{j-1} = t_{j-1}(R_i)$. For all $\ell \neq i$, $t_j(R_\ell) = t_{j-1}(R_i)$. Notice that $x_{j-1}$ is larger than all elements in $X$ and hence $\text{FRA}(x_{j-1}) \not\subseteq X’$. Thus $x_{j-1}$ becomes an active element and the $t_j(R_{\ell})$ for $\ell < i$ are active and will stay active. All $t_j(R_\ell), \ell > j$, with $\text{FRA}(t_j(R_\ell)) = x_{j-1}$ will become inactive and part of the subtree of $T$ formed by the inactive nodes between $t_{j-1}(R_{i-1})$ and $x_{j-1}$. We change the path $P’$ accordingly.

Remark: The algorithm in the proof of Theorem [22] relies on advice about the global structure of the before-path to after-tree transformation, in particular, it needs information about the nearest left- or right- ancestor of a node in the after-tree $T$. This fact makes Theorem [22] more generally applicable. We observe that a limited amount of information about the already-processed structure of the before-path can be encoded in the shape of the path $P’$ that contains the active set $A$ (the choice of the path shape is rather arbitrary, as long as the largest or the smallest element is at its root).

D.2 Discussion of Known Local Algorithms

This section further illustrates the generality of Theorem [21]. For any element $x$ in $T$, the neighbors of $x$ are the predecessor of $x$ and the successor of $x$. 

16
Subramanian local algorithm [9]: This type of algorithm is such that 1) there is a constant $D$ such that the leaf of $P_{i+D}$ is not a leaf of $T_i$, 2) if the depth of the leaf $l_i$ of $P_i$ is $d_i$, then the depth of $l_i$ and neighbor of $l_i$ in $T_i$ is less than $d_i$.

Georgakopoulos and McClurkin local algorithm [6]: This type of algorithm is such that 1) the leaf of $P_{i+1}$ cannot be a leaf of $T_i$, 2) if there are $k$ transformations yielding $T_1, \ldots, T_k$, then there are $\Omega(k)$ many $T_i$’s which are not paths.

Theorem 27. Any Subramanian local algorithm is a Georgakopoulos and McClurkin local algorithm.

Proof: The first condition of Subramanian’s implies the first condition of Georgakopoulos and McClurkin’s by “composing” $D$ transformations together. From now on we can assume that, for every $i$, the leaf of $P_{i+1}$ cannot be a leaf of $T_i$ even for Subramanian’s algorithm.

For the second condition, suppose that, for $i \in \{i_0, i_0 + 1\}$, the depth of the leaf $l_i$ of $P_i$ is $d_i$ and the depth of $l_i$ and neighbors of $l_i$ in $T_i$ is less than $d_i$, but $T_i$ is a path.

We claim that composing the $i_0$-th and $i_0 + 1$-th transformations give us a non-path tree. Let $l'_{i_0}$ be the leaf of $T_{i_0}$. Let $\text{pred}$ and $\text{succ}$ be the predecessor and the successor of $l_{i_0+1}$ in $P_{i_0+1}$. As $T_{i_0}$ is a path, $\text{pred} < l'_{i_0}$ if $\text{pred}$ exists, and $l'_{i_0} < \text{succ}$ if $\text{succ}$ exists.

There must exist another element $x \neq l_{i_0+1}, \text{pred}, \text{succ}$ in $P_{i_0+1}$. Otherwise, $P_{i_0+1}$ is of size either 2 or 3. Then there is no transformation such that $T_{i_0+1}$ is a path and satisfies Subramanian’s condition.

Since $x$ exists, we know that either $x < \text{pred}$ or $\text{succ} < x$. Assume w.l.o.g. that $x < \text{pred}$. There must, moreover, exist $x$ such that $x < \text{pred}$ and $x$ is below $\text{pred}$ in $T_{i_0+1}$. Otherwise, $\text{pred}$ or $l_{i_0+1}$ would have depth $d_{i_0+1}$ violating Subramanian’s condition.

Now $\text{pred}$ is higher than both $x$ and $l'_{i_0}$, where $x < \text{pred} < l'_{i_0}$. Therefore, there is a branching in the “composed” transformation. So composing the $i_0$-th and $i_0 + 1$-th transformations give us a non-path tree. \qed

Theorem 28. A Georgakopoulos and McClurkin local algorithm that brings the accessed element to the root satisfies the conditions of Theorem 27. 
Hence it satisfies the access lemma.

Proof: By Theorem 22 we just need to show that the after-tree $T$ has $\Omega(k - z)$ leaves, when $P$ contains $z$ side alternations (zig-zag) and there are $k$ transformations. To do this, we claim that all non-path $T_i$’s, except $O(z)$ many, contribute a leaf to $T$.

For each non-path $T_i$, suppose that there are two leaves $l_1$ and $l_2$ in $T_i$ which are on the same side. That is, both are less or more than the accessed element $s$. Then $T_i$ would contribute one branching to $T$, because the leaf of $P_{i+1}$ cannot be $l_1$ or $l_2$ and so there will be another element between $l_1$ and $l_2$ placed higher than both of them, which is a branching. A branching in $T$ contributes a leaf in $T$.

Now if $T_i$ is not a path but there are no two leaves on the same side: this means that there is exactly one leaf on left and right side of $s$. However, there can be at most $w \cdot z = O(z)$ many of this kind of $T_i$’s. This is because for each side alternation of $P$, the algorithm can bring up at most $w$ elements from another side. \qed
E  Proof Omitted from Section 7

E.1  Proof of Theorem 25

We show that \( A \) satisfies the three conditions of Theorem 21. Condition (i) is satisfied by definition.

Let \( s \) be the accessed element, and let \( L_1 \) be its left child in the after-tree. Let \( (L_1, \ldots, L_t) \) denote the longest sequence of nodes such that for all \( i < t \), \( L_{i+1} \) is the right child of \( L_i \) in the after-tree, and let \( T_i \) denote the left subtree of \( L_i \) for all \( i \leq t \). Observe that the nodes in \( T_i \) are ancestors of \( L_i \) in the before-path, therefore, \( L_i \) has gained them as descendants. Thus, from condition (ii), we have that \(|T_i| \leq d\) for all \( i \).

Since there are at most \( d \) nodes in each subtree, the largest number of left-turns in the left subtree of \( s \) is \( d \). A symmetric statement holds for the right subtree of \( s \). This proves condition (iii) of Theorem 21.

Next, we show that a linear number of leaves are created, verifying condition (ii) of Theorem 21.

We claim that there exists a left-ancestor of \( s \) in the before-path that loses \( \epsilon d(s)/2 - (c + 1) \) left-ancestors, or a right-ancestor of \( s \) that loses this number of right-ancestors.

Suppose that there exists such a left-ancestor \( L \) of \( s \) (the argument on the right is entirely symmetric). Observe that the left-ancestors that \( L \) has not lost form a right-path, with subtrees hanging to the left; the lost left-ancestors of \( L \) are contained in these subtrees. From the earlier argument, each of these subtrees is of size at most \( d \). Since the subtrees contain in total at least \( \epsilon d(s)/2 - (c + 1) \) elements, there are at least \((\epsilon d(s)/2 - (c + 1))/d = \Omega(d(s))\) many of them, thus creating \( \Omega(d(s)) \) new leaves.

It remains to prove the claim that some ancestor of \( s \) loses many ancestors “on the same side”. Let \( L \) and \( R \) be the nearest left- (respectively right-) ancestor of \( s \) on the before-path. W.l.o.g. assume that \( L \) is the parent of \( s \) in the search path. For any node \( y \), let \( d_l(y), d_r(y) \) denote the number of left- respectively right-ancestors of a node \( y \) in the search path. We consider two cases:

- If \( d_l(s) > d_r(s) \), then \( d_r(L) \leq d(s)/2 \). Since \( L \) loses \((\frac{1}{2} + \epsilon) \cdot d(L) - c \geq (\frac{1}{2} + \epsilon) \cdot d(s) - (c + 1) \) ancestors, it must lose at least \( \epsilon d(s) - (c + 1) \) left-ancestors.

- Suppose now that \( d_l(s) \leq d_r(s) \). Then \( d_l(R) < d_r(R) \) and hence \( d_l(R) \leq d(R)/2 \). At the same time \( d(R) \geq d_l(R) \). Therefore \( d_l(R) \geq \epsilon \cdot (d(s) - 2)/2 - c \geq \epsilon \cdot d(s)/2 - (c + 1) \) right-ancestors.
Let $z, \ell$ be the number of side alternations in the before-path and the number of leaves in the after-tree respectively. Let $n = |T|$.

(left) A rearrangement in which every node loses half of its ancestors and gains only one new descendant. However, $z, \ell = O(\sqrt{n})$.

(right) A rearrangement in which every node loses a $(1 - o(1))$-fraction of its ancestors and gains only one new ancestor. However, $z = 0, \ell = O(\sqrt{n})$.

Fig. 5: A rearrangement in which every node approximately halves its depth. However, there is an element $x < y$ whose search path contains $\Omega(n)$ left turns. By Theorem 22 and Theorem 23, this rearrangement cannot satisfy access lemma with the SOL potential.