A determination of the QED coupling at the $Z$ pole

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Abstract

We critically examine the evaluation of the hadronic contribution to the running of the QED coupling, $\alpha$, from $Q^2 = 0$ to $Q^2 = M_Z^2$. Using data for $e^+e^- \rightarrow$ hadrons we find that $\alpha(M_Z^2)^{-1} = 128.99 \pm 0.06$, as compared to the existing value of $128.87 \pm 0.12$. 

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The improvement in the measurements of electroweak quantities allows high precision tests of the Standard Model in which the measured $Z$ boson mass is related to other observables (see, for example, the recent reviews in Refs. [1, 3]). Surprisingly, out of the three accurately measured quantities ($\alpha$, $G_F$, $M_Z$) which determine the Standard Model, the largest uncertainty comes from the running of $\alpha$ from $Q^2 = 0$, where it is precisely known, up to the $Z$ pole, which is the scale relevant for the electroweak precision tests. Indeed, other electroweak quantities are being measured with an accuracy comparable to that associated with $\alpha(M_Z^2)$. The source of the ambiguity in the value of $\alpha(M_Z^2)$ is the hadronic contribution to the photon vacuum polarization $\Pi(s)$. This contribution is determined by the dispersive sum of all possible hadronic states produced in $e^+e^-$ annihilation into hadrons via an intermediate photon

$$\text{Re} \, \Pi_h(s) = \frac{\alpha s}{3\pi} P \int_{4m_e^2}^{\infty} \frac{R^{\gamma\gamma}(s')}{s'(s' - s)} ds' , \quad (1)$$

with $\alpha^{-1} = 137.036$ and

$$R^{\gamma\gamma} = \frac{\sigma(e^+e^- \rightarrow \gamma \rightarrow \text{hadrons})}{\sigma_{\mu\mu}}. \quad (2)$$

Here $\sigma_{\mu\mu} = 4\pi\alpha(s)^2 / 3s$ is the lowest order point-like $e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-$ cross section. The $\mu\mu$ cross section is expressed in terms of the running coupling $\alpha(s)$ in order to eliminate any QED effects from the hadronic contribution to the current-current two-point function $\Pi_h/\alpha$.

The currently accepted determination of the running of $\alpha$ is based on the analysis of Burkhardt et al. [4]. They used data for $R^{\gamma\gamma}$, supplemented by narrow resonance contributions, to obtain a hadronic contribution to the running of $\alpha$ of

$$\Delta\alpha_h(M_Z^2) = -\text{Re} \, \Pi_h(M_Z^2) = 0.0288 \pm 0.0009. \quad (3)$$

When the leptonic contribution,

$$\Delta\alpha_\ell(M_Z^2) = \frac{\alpha}{3\pi} \sum_\ell \left[ \ln \frac{M_Z^2}{m_\ell^2} - \frac{5}{3} + \mathcal{O}\left(\frac{m_\ell^2}{M_Z^2}\right) \right] = 0.03142, \quad (4)$$

is added, their result gives

$$-\text{Re} \, \Pi(M_Z^2) \equiv \Delta\alpha = \Delta\alpha_h(M_Z^2) + \Delta\alpha_\ell(M_Z^2) = 0.0602 \pm 0.0009, \quad (5)$$

which translates to

$$\alpha(M_Z^2)^{-1} = (1 - \Delta\alpha)\alpha^{-1} = 128.78 \pm 0.12. \quad (6)$$
The analysis[4] was subsequently updated by Jegerlehner [5] to give

\[
\Delta \alpha_h(M_Z^2) = 0.0282 \pm 0.0009, \tag{7}
\]

\[
\alpha(M_Z^2)^{-1} = 128.87 \pm 0.12. \tag{8}
\]

Here we are using the effective QED coupling, which is denoted by \( \bar{\alpha} \) in the Review of Particle Properties [1, 2].

For electroweak precision tests it is important to see if the determination of \( \text{Re} \Pi_h(M_Z^2) \) (and hence of \( \alpha(M_Z^2) \)) can be improved and, in particular, the error reduced. In the following we compare our analysis with the original work of Ref. [4] since it lists the various contributions to \( \Delta \alpha_h \) in detail. Indeed the first column of Table 1, which is taken directly from Ref. [4], shows the contributions to \( \text{Re} \Pi_h \) from different \( W \equiv \sqrt{s} \) regions of the dispersion integral, together with the associated errors. We note that the largest error arises from the region \( 2.3 < W < 9 \text{ GeV} \), which contains the \( c\bar{c} \) resonance region together with the ‘continuum’ regions below both the \( c\bar{c} \) and the \( b\bar{b} \) threshold. The evaluation of the contributions from this region, used in Ref. [4]†, relied on the original MARK I [7] data for \( R(W) \). In fact the major uncertainty in the determination of \( \text{Re} \Pi_h \) is associated with the normalization errors of the measurements of \( \sigma(e^+e^- \to \text{hadrons}) \). Thus the 10% error associated with the \( 2.3 < W < 9 \text{ GeV} \) contribution of Ref. [4] reflects the 10% normalization uncertainty of the MARK I data.

In the continuum regions well above the \( q\bar{q} \) thresholds we are now in a position to use perturbative QCD to predict \( R^{\gamma\gamma} \) extremely accurately. First our knowledge of the QCD coupling has considerably improved since the previous calculation [4] of \( \text{Re} \Pi_h \). Even if we take a conservative view of the average of all of the measurements of \( \alpha_s \) [2], we can conclude \( \alpha_s(M_Z^2) = 0.118 \pm 0.007 \). Moreover \( R^{\gamma\gamma} \) is known up to, and including, the \( \mathcal{O}(\alpha_s^3) \) contributions, and the running of \( \alpha_s \) is known to 3 loops. Finally we now have experimental evidence of the value of the top quark mass \( m_t \). Thus, for example, if we evaluate \( R^{\gamma\gamma} \) at values of \( W = 3, 9 \) and 150 GeV just below the \( c\bar{c}, b\bar{b} \) and \( t\bar{t} \) thresholds respectively, we find

\[\dagger\]The updated analysis [\#] uses data from the Crystal Ball collaboration [\#].
\[ R^{\gamma\gamma} = 2.17 \pm 0.03 \quad \text{at } W = 3 \text{ GeV}, \quad (9) \]
\[ = 3.58 \pm 0.04 \quad \text{at } W = 9 \text{ GeV}, \quad (10) \]
\[ = 3.80 \pm 0.01 \quad \text{at } W = 150 \text{ GeV}. \quad (11) \]

We allow for the change in the number of flavors at each \( q\bar{q} \) threshold both in \( \alpha_s(s) \) and in \( R^{\gamma\gamma}(s) \). The errors include the ±0.007 uncertainty in the input value of \( \alpha_s(M_Z^2) \), an uncertainty from yet unknown \( \mathcal{O}(\alpha_s^4) \) terms (taken to be equal in size to the \( \alpha_s^3 \) contribution to \( R^{\gamma\gamma} \)), and uncertainties from threshold effects. The main threshold uncertainties arise in the \( b\bar{b} \) channel because we combine data on the resonance contributions with the QCD formula. We estimate these effects by comparing a naive \( \beta(3-\beta^2)/2 \) threshold behaviour with the full \( \mathcal{O}(\alpha_s) \) QCD formula \( [8] \) and find that \( b\bar{b} \) threshold uncertainties contribute very little to the error on \( \alpha(M_Z^2) \).

Given that \( R^{\gamma\gamma} \) is known so precisely in the continuum regions, we may use it to improve the normalization of the experimental measurements of \( \sigma(e^+e^- \rightarrow \text{hadrons}) \). Such a program was carried out for 21 experiments by Marshall \([9]\) in a detailed study performed in 1988. He noted that many experiments which partially overlap the MARK I region have smaller systematic errors than the MARK I data. As a result of a global QCD fit to the world data for \( R \) he concluded that the experimental normalization of 1.00 ± 0.10 of the MARK I data should be adjusted to 0.850 ± 0.019 to bring them into line with the world data (which was by far the biggest adjustment of data that he obtained). In their paper \([7]\) MARK I quote a normalization error of ±20% at \( W = 2.6 \) GeV decreasing smoothly to ±10% for \( W > 6 \) GeV. Marshall’s adjustment brought the MARK I data into excellent agreement with QCD expectations in the continuum regions below the \( b\bar{b} \) and \( c\bar{c} \) thresholds. Clearly such an adjustment will have a dramatic effect on the value, and the accuracy, of \( \alpha(M_Z^2) \). Incidentally at the time of Marshall’s analysis the coefficient of the \( (\alpha_s/\pi)^3 \) term in \( R^{\gamma\gamma} \) was erroneously large and of the wrong sign (+65 instead of −12.8 for five flavors). As a consequence the renormalization factors found by Marshall should have been even smaller.

To evaluate \([4]\) we assume that \( R^{\gamma\gamma} \) is given by perturbative QCD in the continuum regions \( 3 < W < 3.9 \) GeV and \( 6.5 < W < \infty \), apart from the \( \psi \) and \( \Upsilon \) resonance contributions.
Typical errors on $R^{\gamma\gamma}$ in these regions are shown in Eqs. (9)–(11). Following Marshall’s procedure, we also use the continuum values of $R^{\gamma\gamma}$ to normalize the various data sets. For the MARK I data we find that an overall renormalization of $0.83 \pm 0.02$ is required. In fact fitting to the MARK I data in the $W < 3.85$ GeV continuum region gives essentially the same renormalization factor as fitting to their $W > 6.5$ GeV continuum data, see Fig. (a). One option would be to use the renormalized MARK I data to evaluate the contribution to (9) from the interval $3.9 < W < 6.5$ GeV, between the two continuum regions. To be precise we could use the curve of Fig. (a) in which the portion between $4.5 < W < 6.5$ GeV is shown dotted. To check this ‘MARK I’ curve, we compare with the more recent and precise Crystal Ball measurements. First we renormalize the Crystal Ball (‘90) measurements using their $W > 6.5$ GeV data. A factor $1.06 \pm 0.02$ is found. Indeed with this renormalization all the Crystal Ball (‘90) data are well described by $R(QCD)$, see Figs. (a,b). From Fig. (a) we see that the more precise Crystal Ball (‘90) data lie significantly above the renormalized MARK I data for $W \approx 5$ GeV. Evidence in favor of the higher Crystal Ball values and, in particular, for taking the continuous (rather than the dotted) curve for $R$, comes from two first-generation experiments, DASP and PLUTO, see Figs. (c,d) respectively. The data are shown after they have been renormalized to $R(QCD)$ at $W \approx 3.6$ GeV. On the other hand for $W < 4.5$ GeV the data of Figs. (b,c,d) show that the line drawn through the MARK I data is a reasonable representation of $R$ in the $c\bar{c}$ resonance region. We note that both the Crystal Ball (‘86) and the DASP data are able to resolve the $\psi(4.04)$ and $\psi(4.16)$ states and, especially, that our curve is a good average of $R$ for this resonance region. The contributions to $\Re \Pi_h$, obtained from integrating $R$ over the continuous curve in Fig. (a), are shown in Tables (1) and (2), together with the contributions of the families of $\psi$ and $\Upsilon$ resonances. The resonance contributions are determined from

$$\Pi_{\text{res}} = -\frac{3\Gamma_{ee}}{M} \frac{\alpha}{\alpha(M^2)},$$

where $M$ and $\Gamma_{ee}$ are the mass and leptonic width of the resonance, respectively, and where we account for the running of the effective QED coupling at the resonance scale. Equation (12) follows from integration over a narrow Breit-Wigner resonance form.
The errors associated with the continuum contributions are estimated as for Eqs. (9)–(11). For the intervening interval, \(3.9 < W < 6.5\) GeV, we estimate the error by repeating the calculation using the dotted line in Fig. 1(a). The contribution reduces from \(2.90 \times 10^{-3}\) to \(2.71 \times 10^{-3}\), and we regard this change as representative of the uncertainty of this interval.

We now turn to the region below \(W = 3\) GeV. Here there are many experiments measuring \(e^+e^-\) annihilation to specific hadronic channels. We evaluate the contribution to Re \(\Pi_h\) from this region in four separate parts; see Table 2. First we calculate the contribution from \(e^+e^- \rightarrow \pi^+\pi^-\) by integrating over \(R\) obtained from detailed measurements [13] of the pion form factor \(F_\pi(s)\) via

\[
R(s) = \frac{1}{4} \left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} |F_\pi(s)|^2 .
\]

To be specific we integrate over the curve shown in Fig. 2 and assign to this contribution a \(\pm 4\%\) uncertainty arising primarily from the experimental normalization of \(|F_\pi(s)|^2\). This region is dominated by the \(\rho\) resonance (with the resonance shape mutilated by \(\rho - \omega\) interference). For example the intervals \(1 < W < 1.4\) and \(1.4 < W < 2\) GeV only give contributions to \(-\text{Re} \Pi_h\) of \(0.16 \times 10^{-3}\) and \(0.02 \times 10^{-3}\) respectively. Secondly, we include the contribution due to the \(\omega(782)\) resonance using (12), where the error reflects the uncertainty in the observed leptonic width [2]. Thirdly, we calculate the \(e^+e^- \rightarrow K\bar{K}\) contribution to Re \(\Pi_h\) using the parametric form for the kaon form factor determined by Bisello et al. [14] from a fit to \(e^+e^- \rightarrow K^+K^-\) data [14, 15]. By far the dominant contribution here comes from the \(\phi\) resonance, though there are small contributions from the \(\rho, \omega \rightarrow K\bar{K}\) resonance tails and an even smaller contribution from the 1.5–1.6 GeV resonance region; see the dotted curve in Fig. 3. The error reflects the uncertainty in the \(\phi\) leptonic width [2]. To a good approximation the total contribution is twice the \(K^+K^-\) contribution. On the \(\phi\) resonance this allows for \(K^0\bar{K}^0\) and \(3\pi\) contributions, while above the resonance it represents the total \(K\bar{K}\) contribution since, away from threshold, the effect of the \(K^+K^- - K^0\bar{K}^0\) mass difference is suppressed. Finally we have the contributions to Re \(\Pi_h\) from the region above \(W = 0.9\) GeV due to multi (\(\geq 3\)) pion production. For the region \(0.9 < W < 1.45\) GeV we use the sum of the data for the exclusive channels [16]. The dominant contribution comes from \(\pi^+\pi^-\pi^0\pi^0\) and \(\pi^+\pi^-\pi^+\pi^-\) production. Above \(W = 1.45\) GeV we use a line through the \(R\) data of Refs. [17], joining on to the QCD value at \(W = 3\) GeV, as shown in Fig. 3.
It is instructive to compare our results with those of Ref. [4]. From Table 1 we see that the main improvement in accuracy comes from the $2.3 < W < 9$ GeV and $12 < W < \infty$ regions, due mainly to our use of $R^{\gamma\gamma}$(QCD). The difference in size of the $2.3 < W < 9$ GeV contribution can be attributed to our use of renormalized MARK I and Crystal Ball '90 data. To make an exact comparison with Ref. [4] we should let $m_t \to \infty$, rather than taking the value $m_t = 174$ GeV that we have used to include the $e^+e^- \to t\bar{t}$ contribution. The resulting effect is that $-\text{Re } \Pi_h(M_Z^2)$ would increase very slightly to $27.39 \times 10^{-3}$.

Although we have numerically integrated over \( \rho \) and \( \phi \) resonant shapes, and carefully considered individual contributions to $e^+e^-$ production processes, we find that there is a comparatively modest reduction in the error of the contribution to $\text{Re } \Pi_h$ from the low energy, $W < 3$ GeV, region. From the first five rows of Table 1 we find that the contribution for $W < 2.3$ GeV is $-(6.06 \pm 0.25) \times 10^{-3}$ as compared to $-(6.34 \pm 0.43) \times 10^{-3}$ of the previous calculation [4]. In fact we see from Table 2 that the error $\pm 0.33 \times 10^{-3}$ on the $W < 3$ GeV contribution limits the present accuracy of $\alpha(M_Z^2)$.

In conclusion we find that

$$\alpha(M_Z^2)^{-1} = 128.99 \pm 0.06 \quad (14)$$

The large reduction in the error, in comparison to that found in Refs. [4, 5], arises because we use the precise perturbative QCD prediction for $R^{\gamma\gamma}$ in the regions well above $q\bar{q}$ thresholds. As a consequence we find (i) that the contribution from the high $W \equiv \sqrt{s}$ region is well determined, and (ii) that we are able to reliably normalize the data in the $c\bar{c}$ resonance region which significantly reduces the value and the uncertainty of $\alpha(M_Z^2)$. Although perturbative QCD stabilizes the contribution from the $W \gtrsim 3$ GeV region, the contribution from lower values of $W$ relies directly on the available data for $e^+e^-$ annihilation into hadrons. In fact the dominant uncertainty arises from the $1 < W < 3$ GeV region and, in view of the importance of an accurate value of $\alpha(M_Z^2)$ for precision tests of the Standard Model, it is crucial to improve the accuracy of the low energy measurements of $\sigma(e^+e^- \to \text{hadrons})$.

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Note added: After this work was completed we received a paper entitled “Re-evaluation of the Hadronic Contribution to \( \alpha(M^2_Z) \)” by M. L. Swartz, SLAC-PUB-6710, November 1994, in which he obtained \( \Delta \alpha_h(M^2_Z) = (26.66 \pm 0.75) \times 10^{-3} \), and hence \( \alpha(M^2_Z)^{-1} = 129.08 \pm 0.10 \). We note that the error estimates would be expected to differ, since the two (completely independent) calculations differ in their reliance on \( R(\text{QCD}) \).

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Table 1: Contributions to $-1000 \text{Re} \Pi_h(M_Z^2)$.

| $W$ range (GeV) | Burkhardt et al. [4] | This work |
|-----------------|-----------------------|-----------|
| $2m_\pi$–1.0    | —                     | 3.21 ± 0.14 |
| $\rho$          | 3.484 ± 0.171         | —         |
| $\omega$        | 0.347 ± 0.021         | 0.31 ± 0.01 |
| 1.0–2.3         | 1.981 ± 0.391         | 1.95 ± 0.21‡ |
| $\phi$          | 0.528 ± 0.024         | 0.59 ± 0.02† |
| 2.3–9.0         | 7.208 ± 0.721         | 6.56 ± 0.22‡ |
| $\psi$’s        | 1.084 ± 0.057         | 0.92 ± 0.05 |
| 9.0–12.0        | 1.686 ± 0.169         | 1.67 ± 0.06* |
| $\Upsilon$’s    | 0.118 ± 0.005         | 0.12 ± 0.01 |
| 12.0–∞          | 12.368 ± 0.371        | 11.99 ± 0.09* |
| Total           | 28.8 ± 0.9††          | 27.32 ± 0.42 |
| $\alpha(M_Z^2)^{-1}$ | 128.78 ± 0.12††   | 128.99 ± 0.06 |

†Includes the full $e^+e^- \rightarrow K\bar{K}$ contribution.

*These errors have to be added linearly.

‡Part of these errors have to added linearly and a part in quadrature.

††An updated analysis [5] gives $-1000 \text{Re} \Pi_h(M_Z^2) = 28.2 ± 0.9$ and $\alpha(M_Z^2)^{-1} = 128.87 ± 0.12$. 
Table 2: Summary of the evaluation of Re Πₜ(M₂)

| W range (GeV)       | Information used                                      | −1000 Re Πₜ(M₂) |
|---------------------|-------------------------------------------------------|-----------------|
| 2mₚ–3.0             |                                                       |                 |
| (i) π⁺π⁻ (W < 2)    | Data for |Fₚ|²                     | 3.39 ± 0.14     |
| (ii) ω → 3π         | Narrow resonance formula                             | 0.31 ± 0.01     |
| (iii) K¯K inc. φ    | Parametrization of e⁺e⁻ → K¯K data                    | 0.59 ± 0.02†    |
| (iv) e⁺e⁻ → hadrons  | R(data), e⁺e⁻ → 4π etc.                               | 2.72 ± 0.30     |
|                     | excluding ππ, K¯K                                    |                 |
| 3.0–3.9             | R(QCD)                                                | 0.88 ± 0.01     |
|                     | +J/ψ, ψ', ψ(3.77)                                    | 0.92 ± 0.05     |
| 3.9–6.5             | Renormalized R(data)                                 | 2.90 ± 0.19     |
| 6.5–∞               | R(QCD)                                                | 15.49 ± 0.15    |
|                     | +Υ(nS), n = 1, . . . , 6                              | 0.12 ± 0.01     |
| Total               |                                                       | 27.32 ± 0.42    |

†Includes the φ → 3π contribution.

The errors are added in quadrature, except those for R(QCD).
Figure Captions

Figure 1: The (a) MARK I [7], (a,b) Crystal Ball ’90 [6], (c) DASP [11], and (d) PLUTO [12] data for $R$, respectively renormalized by 0.83, 1.06, 0.94, and 0.96, – factors which are found by fitting to $R^{\gamma \gamma}_{(QCD)}$ in the continuum regions $W > 6.5$ GeV and $W < 3.9$ GeV. The Crystal Ball ’86 data [10] in (b) are not renormalized. The continuous curve is the same in each plot and is the representation of the data used to calculate $\alpha(M_2^2)$. The dotted curve is a representation of the MARK I data for $4.5 < W < 6.5$ GeV which is used to estimate the uncertainty. The $J/\psi$, $\psi'$ and $\psi(3.77)$ contributions are included using (12).

Figure 2: Data on the pion form factor [13]. The curve is the representation of the data which is used to evaluate the $\pi\pi$ contribution.

Figure 3: The dashed and dotted curves are a representation of the data for the contribution to $R$ from $e^+e^- \rightarrow \pi^+\pi^-$ and $K\bar{K}$ respectively. The continuous curve represents the data for multi ($\geq 3$) pion production. For $W < 1.45$ GeV it represents the sum of the multipion exclusive channels. Above $W = 1.45$ GeV the multipion data are taken from Refs. [7, 17]. Also shown are three MARK I measurements [7], renormalized by a factor of 0.83. We numerically integrate over the $\rho$ and $\phi$ resonance forms, but include the $\omega(782)$ contribution via (12).
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