On the inter-critical inhomogeneous generalized Hartree equation

Abstract It is the purpose of this work to study the Choquard equation

\[ i\dot{u} - (-\Delta)^s u = \pm |x|^{\gamma} (I_\alpha \ast |\cdot|^{\gamma}|u|^p)|u|^{p-2}u \]

in the space \( \dot{H}^s \cap \dot{H}^{s_c} \), where \( 0 < s_c < s \) corresponds to the scale invariant homogeneous Sobolev norm. Here, one considers two separate cases. The first one is the classical case \( s = 1 \) and the second one is the fractional regime \( 0 < s < 1 \) with radial data. One tries to develop a local theory using a new adapted sharp Gagliardo–Nirenberg estimate. Moreover, one investigates the concentration of non-global solutions in \( L^\infty T^* (\dot{H}^{s_c}) \). One needs to deal with the lack of a mass conservation, since the data are not supposed to be in \( L^2 \). This note gives a complementary to the previous works about the same problem in the energy space \( H^1 \).

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1 Introduction

This paper is concerned with the Cauchy problem for a Choquard equation

\[
\begin{align*}
\begin{cases}
  i\dot{u} + \Delta u & = \epsilon |x|^{\gamma} (I_\alpha \ast |\cdot|^{\gamma}|u|^p)|u|^{p-2}u; \\
  u|_{t=0} & = u_0,
\end{cases}
\end{align*}
\]

where \( u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \), for some integer \( N \geq 1, \epsilon = \pm 1 \) refers to the attractive/repulsive regime. The above problem is said inhomogeneous because of the singular quantity \( |\cdot|^{\gamma} \), where \( \gamma < 0 \). The Riesz potential is defined on \( \mathbb{R}^N \) by

\[ I_\alpha := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2} - 2\alpha}} |\cdot|^{N-\alpha}, \quad 0 < \alpha < N. \]
Here and hereafter, one assumes \([1]\) the following conditions
\[
\min\{\alpha, -\gamma, -\gamma + N, \gamma + N, 2 + \alpha + 2\gamma\} > 0. \tag{1.2}
\]
Moreover, one considers the fractional inhomogeneous generalized Hartree problem
\[
\begin{cases}
i\dot{u} - (-\Delta)^s u = \epsilon |x|^{\gamma} (I_\alpha * |\cdot|^q |u|^q)|u|^{q-2}u; \\
u|_{t=0} = u_0,
\end{cases} \tag{1.3}
\]
where \(N/2 < s < 1\) and the fractional Laplacian operator stands for
\[
(-\Delta)^s := \mathcal{F}^{-1}(|\cdot|^{2s}\mathcal{F}).
\]
The different parameters of the above problem will be assumed to satisfy \([15]\),
\[
\min\{-\gamma, \alpha, N - \alpha, N + \gamma - s, 2s + 2\gamma + \alpha, N + \alpha + 2\gamma - 2s\} > 0. \tag{1.4}
\]
The particular case \(s = 1, q = 2\) and \(\gamma = 0\) in (1.3) models an approximation of the Hartree–Fock theory for a plasma with one component \([13]\). If \(s = 1\) and \(\gamma = 0\), then (1.3) is called Choquard equation and is used in quantum theory \([11]\). The problem (1.3), with \(s = 1\) and \(\gamma \neq 0\) arises in the physics of laser beams and the physics of multiple-particle systems, \([5, 14]\). The case \(0 < s < 1\) is called fractional Schrödinger equation and extends the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths \([9, 10]\).

The Eq. (1.1) enjoys the scaling invariance
\[
u_\lambda = \lambda^\frac{2+2\gamma+\alpha}{2(p-1)} u(\lambda^2 \cdot, \lambda \cdot), \quad \lambda > 0.
\]
Thanks to the identity
\[
\|\nu_\lambda(t)\|_{\dot{H}_\mu} = \lambda^{\mu-s_c} \|u(\lambda^2 t)\|_{\dot{H}_\mu},
\]
the unique critical exponent giving an invariant Sobolev norm under the previous scaling is
\[
s_c := \frac{N}{2} - \frac{2 + 2\gamma + \alpha}{2(p-1)}.
\]
Two particular cases are widely investigated in the literature. The first one \(s_c = 0\) is related to the mass conservation and gives the \(L^2\)-critical exponent \(p_* := 1 + \frac{2+2\gamma+\alpha}{N}\). The second one \(s_c = 1\) gives the \(\dot{H}^1\)-critical exponent
\[
p^* := \begin{cases}
1 + \frac{2+2\gamma+\alpha}{N-2} & \text{if } N \geq 3; \\
\infty & \text{if } N = 2.
\end{cases}
\]
This case is also related to the energy conservation law
\[
E[u_0] = E[u(t)] := \int_{\mathbb{R}^N} \left(|\nabla u(t)|^2 - \frac{1}{p} |x|^{\gamma} (I_\alpha * |\cdot|^q |u(t)|^p)|u(t, x)|^p \right) dx
\]
\[
:= \int_{\mathbb{R}^N} |\nabla u(t, x)|^2 dx - \frac{1}{p} \mathcal{P}[u].
\]
Denote also the continuous Sobolev injection \(\dot{H}^1(\mathbb{R}^N) \hookrightarrow L^{p_c}(\mathbb{R}^N)\), where
\[
p_c := \frac{2N}{N - 2s_c} = \frac{2N(p-1)}{2 + 2\gamma + \alpha}.
\]
The critical Sobolev exponent in the fractional case reads
\[
s_c' := \frac{N}{2} - \frac{2s + 2\gamma + \alpha}{2(q-1)}.
\]
Thus, the mass-critical and energy-critical exponents are

\[ q_* := 1 + \frac{\alpha + 2s + 2\gamma}{N}, \quad q^* := 1 + \frac{2s + 2\gamma + \alpha}{N - 2s}. \]

Denote also the continuous Sobolev injection \( \dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{q_c}(\mathbb{R}^N) \), where

\[ q_c := \frac{2N}{N - 2s} = \frac{2N(q - 1)}{2s + 2\gamma + \alpha}. \]

The conserved energy reads

\[
E_s[u_0] := E_s[u(t)] := \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} u(t)|^2 \frac{1}{q} |x|^\gamma (I_\alpha * |\cdot|^\gamma |u(t)|^q) |u(t,x)|^q \right) dx \nonumber
\]

\[
:= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(t,x)|^2 dx - \frac{1}{q} Q[u].
\]

To the authors knowledge, there is few papers dealing with the inhomogeneous generalized Hartree problem. Indeed, a sharp threshold of global well-posedness and scattering vs finite time blow-up of energy solutions was obtained by the first author in [1,16]. The spherically symmetric assumption for the energy scattering was removed in [18]. The instability of standing waves was treated in [7]. The local theory was developed in [15] for the fractional inhomogeneous Choquard problem with radial data.

The Hartree problems (1.1) and (1.3) are locally well posed in the respective energy spaces \( H^1 \) and \( H^s_{rd} \), see [1, Theorem 5.2] and [15, Theorem 2.4]. Here, one studies these problems respectively in \( \dot{H}^1 \cap \dot{H}^s \) and \( \dot{H}^s_{rd} \cap \dot{H}^s \). The spherically symmetric assumption in the fractional case is due to the loss of regularity in the non-radial fractional Strichartz estimate [8]. It is the aim of this paper to develop a local theory by removing the mass conservation since the data is no longer in \( L^2 \). There exist mainly three technical difficulties to handle. The first one is related to the presence of a singular inhomogeneous term \( |\cdot|^\gamma \), the second one is the non-local source term and the last one is the presence of a non-local fractional Laplace operator. In the case of a Schrödinger problem with an inhomogeneous local source term, the local theory in \( \dot{H}^1 \cap H^s \) was developed recently in [2]. It seems that [2, Proposition 4.2] needs the restriction \( \sigma < \frac{2 - b}{N - (2 - b)} \). This gives some supplementary conditions in the given results.

The rest of this paper is organized as follows. The next section contains the main results and some useful estimates needed in the sequel. Section 3 contains the proofs of the results concerning the inhomogeneous Choquard problem (1.1). The last section contains the proofs of the results concerning the fractional inhomogeneous Choquard problem (1.3).

Denote for simplicity the Lebesgue space \( L^r := L^r(\mathbb{R}^N) \) with the usual norm \( \| \cdot \|^r := \| \cdot \|_{L^r} \) and \( \| \cdot \| := \| \cdot \|_2 \). Take \( H^r := H^r(\mathbb{R}^N) \) be the usual inhomogeneous Sobolev space endowed with the complete norm

\[
\| \cdot \|_{H^r} := \left( \| \cdot \|^2 + \| (-\Delta)^{\frac{s}{2}} \cdot \|^2 \right)^{\frac{1}{2}}.
\]

If \( X \) is an abstract space \( C_T(X) := C([0, T], X) \) stands for the set of continuous functions valued in \( X \) and \( X_{rd} \) is the set of radial elements in \( X \), moreover for an eventual solution to (1.1), \( T^* > 0 \) denotes its lifespan. Finally, \( x^\pm \) are two real numbers near to \( x \) satisfying \( x^+ > x \) and \( x^- < x \).

**2 Background material**

Let us start with the contribution of this note.

2.1 Main results

This subsection contains two parts.
2.1.1 Results about the Choquard problem (1.1)

First, one gives a sharp Gagliardo–Nirenberg-type inequality.

**Theorem 2.1** Let \( N \geq 2, \gamma, \alpha \) satisfying (1.2) such that \( 1 + \gamma > 0 \) and \( p > 1 \). Then,

1. There exists a positive constant \( C(N, p, \gamma, \alpha) \), such that for any \( u \in \dot{H}^1 \cap L^{pc} \),
   \[
   P[u] \leq C(N, p, \gamma, \alpha) \|u\|^{2(p-1)}_{L^{pc}} \|\nabla u\|^2.
   \]  
   (2.5)

Moreover, if \( 1 + \frac{\alpha + 2\gamma}{N} < p < 1 + \frac{2+2\gamma+\alpha}{N-2-2\gamma} \), then

2. The best constant in the above estimate is attained in some \( \psi \in \dot{H}_{rd}^1 \) satisfying \( C_{opt} = P[\psi] \) and
   \[
   -\Delta \psi - (p-1)|\psi|^{p_{pc}-2}\psi + \frac{p}{C_{opt}} |x|^\gamma (I_\alpha \ast |\cdot|^\gamma) |\psi|^{p-2}\psi = 0;
   \]  
   (2.6)

3. Furthermore, there is \( 0 \neq \phi \in \dot{H}^1 \cap L^{pc} \) such that
   \[
   0 = -\Delta \phi + |\phi|^{p_{pc}-2}\phi - |x|^\gamma (I_\alpha \ast |\cdot|^\gamma) |\phi|^{p-2}\phi;
   \]  
   (2.7)
   \[
   C_{opt} = p \|\phi\|^{-2(p-1)}_{L^{pc}}. \]  
   (2.8)

**Remark 2.2** The restriction \( p < 1 + \frac{2+2\gamma+\alpha}{N-2-2\gamma} \) is needed in the proof of a compactness embedding, which gives the convergence of a minimizing sequence. See Lemma 3.1. This restriction does not appear [1] in the estimate on \( H^1 \), available for \( 1 + \frac{\alpha + 2\gamma}{N} < p < p^* \),

\[
P[u] \lesssim \|\nabla u\|^{Np-N\gamma-2\gamma} \|u\|^{2p-(Np-N\gamma-2\gamma)}_{L^{pc}}.
\]

Second, one considers the evolution regime. The Schrödinger problem (1.1) is locally well posed in \( \dot{H}^1 \cap \dot{H}^{\infty} \).

**Theorem 2.3** Let \( N \geq 2, \alpha, \gamma \) satisfy (1.2), \( 2 \leq p < p^* \) such that \((N-2)(p-1) - \alpha - 2\gamma > 0\). If \( N = 2 \), one assumes that \( 2 + 2\gamma + \alpha < 1 \) and \( p < \frac{1}{2(2+2\gamma+\alpha)} \), or \( 2 + 2\gamma + \alpha > 1 \). Then, if \( u_0 \in \dot{H}^1 \cap \dot{H}^{\infty} \), there exist \( T^* > 0 \) and a unique maximal solution to (1.1),

\[
u \in C([0, T^*], \dot{H}^1 \cap \dot{H}^{\infty}) \cap L^q([0, T^*], W^{1,r} \cap \dot{W}^{\infty,r}) \cap L^q([0, T^*], L^{1,r}),
\]

where \((q, r) \in \Gamma \) and \((q_1, r_1) \in \Gamma_{\infty} \). Moreover,

1. \( T^* = \infty \) in the defocusing case;
2. \( T^* = \infty \) if \( 1 + \gamma > 0, p < 1 + \frac{2+2\gamma+\alpha}{N-2-2\gamma} \) and \( \|\nu\|_{L^{\infty}_{T^*}(\dot{H}^{\infty})} < \infty \) satisfies \( \|\nu\|_{L^{pc}_{T^*}(L^{pc})} < \|\phi\|_{pc} \), where \( \phi \) is a ground state of (2.7);
3. The energy is conserved.

**Remarks 2.4** Note that

1. The assumption \( p \geq 2 \) avoids a singularity in the source term;
2. There is an extra restriction in the bi-dimensional case;
3. In the second point, the blow-up is due to the gradient concentration in \( L^2 \).

Finally, one is concerned with the finite time blow-up of the focusing solutions. Thanks to the Sobolev injection \( \dot{H}^1 \hookrightarrow \dot{H}^1 \cap \dot{H}^{\infty} \) and the variance identity, there is non-global solutions to (1.1) with data in \( \dot{H}^1 \cap \dot{H}^{\infty} \). In the following, one gives a concentration result of finite time blow-up solutions in \( L^{\infty}_{T^*}(\dot{H}^{\infty}) \).

**Theorem 2.5** Take the same conditions in Theorem 2.3. Let a maximal solution to (1.1),

\[
u \in C_{T^*}(\dot{H}^1 \cap \dot{H}^{\infty}).
\]

Assume that \( T^* < \infty \) and \( \|\nu\|_{L^{\infty}_{T^*}(\dot{H}^{\infty})} < \infty \). If

\[
\lim_{t \to T^*} \mu(t) \|\nabla u(t)\|^{-\frac{1}{p-\gamma}}_{L^{pc}} = \infty
\]
thus,

$$\liminf_{T^*} \int_{|x|<\rho(t)} |u(t,x)|^{p_0} \, dx \geq \|\phi\|_{p_*}^{p_*},$$

where $\phi$ is a ground state of (2.7).

**Remark 2.6** Using the scaling

$$u^0(t,x) := \|\nabla u(t_0)\|^{-\frac{s}{\gamma}} (-\Delta)^{\frac{\gamma}{2}} u\left(t_0 + t\|\nabla u(t_0)\|^{-\frac{\gamma}{2}}, \|\nabla u(t_0)\|^{-\frac{1}{\gamma}} x\right),$$

one has $\|\nabla u^0(0)\| = 1$ and $\|(-\Delta)^{\frac{\gamma}{2}} u^0(0)\| = \|(-\Delta)^{\frac{\gamma}{2}} u(t_0)\|$. Thus, $t_0 + t\|\nabla u(t_0)\|^{-\frac{\gamma}{2}} < T^*$. This implies that

$$\|\nabla u(t_0)\| \gtrsim \frac{1}{(T^* - t_0)^{\frac{1}{\gamma}}}.$$  

### 2.1.2 Results about the fractional Choquard problem (1.3)

First, one gives a sharp Gagliardo–Nirenberg-type inequality.

**Theorem 2.7** Let $N \geq 2$, $\gamma, \alpha$ satisfying (1.4) such that $s + \gamma > 0$ and $q > 1$. Then,

1. There exists a positive constant $C(N, q, \gamma, \alpha, s)$, such that for any $v \in \dot{H}^s \cap L^{q^*}$,

$$Q[v] \leq C(N, q, \gamma, \alpha, s) \|v\|_{q^*}^{2(q-1)} \|(-\Delta)^{\frac{\gamma}{2}} v\|^2. \quad (2.9)$$

Moreover, if $1 + \frac{\alpha+2\gamma}{N} < q < 1 + \frac{2s+2\gamma+\alpha}{N-2s-2\gamma}$, then

2. The best constant in the above estimate is attained in some $\psi \in \dot{H}^s_{rd}$ such that $C_{opt,s} = Q[\psi]$ and

$$(-\Delta)^{\frac{\gamma}{2}} \psi - (q-1)|\psi|^{q-2} \psi + \frac{q}{C_{opt,s}} |x|^{\gamma} (I_{a} \ast |\cdot|^\gamma) |\psi|^{q-2} \psi = 0; \quad (2.10)$$

3. Furthermore, there is $0 \neq \phi \in \dot{H}^s \cap L^{q^*}$ such that

$$0 = (-\Delta)^{\frac{\gamma}{2}} \phi + |\phi|^{q^*-2} \phi - |x|^{\gamma} (I_{a} \ast |\cdot|^\gamma) |\phi|^{q^*-2} \phi; \quad (2.11)$$

$$C_{opt,s} = q \|\phi\|_{q^*}^{-2(q-1)}. \quad (2.12)$$

**Remark 2.8** The restriction $q < 1 + \frac{2s+2\gamma+\alpha}{N-2s-2\gamma}$ is needed in the proof of a compactness embedding, which gives the convergence of a minimizing sequence. See Lemma 4.1. This restriction does not appear [15] in the estimate on $\dot{H}^s$, available for $1 + \frac{\alpha+2\gamma}{N} < q < q^*$

$$Q[u] \lesssim \|(-\Delta)^{\frac{\gamma}{2}} u\|^{\frac{Nq-N-a-2\gamma-2\gamma}{s}} \|u\|^{2p-\frac{s-2p-N-a-2\gamma}{s}}.$$  

Second, one considers the evolution regime. The Schrödinger problem (1.3) is locally well posed in $\dot{H}^s_{rd} \cap \dot{H}^s_{k^*}$.

**Theorem 2.9** Let $N \geq 2$, $\frac{N}{2N-1} < s < 1$, $\alpha, \gamma$ satisfying (1.4), $1 + \frac{\alpha+2\gamma}{N-2s} < q < q^*$ and $u_0 \in \dot{H}^s_{rd} \cap \dot{H}^s_{k^*}$. Assume that if $N = 2$, $\left(2s + 2\gamma + \alpha > 1 \text{ or } 2s + 2\gamma + \alpha < 1 \text{ and } q < 1 + \frac{1}{1-(2s+2\gamma+\alpha)}\right)$. Then, there exists $T^* > 0$ and a unique maximal solution to (1.3),

$$u \in C([0, T^*), \dot{H}^s_{rd} \cap \dot{H}^s_{k^*}) \cap L^q([0, T^*), \dot{W}^{s,r} \cap \dot{W}^{k,r}) \cap L^{q^*}([0, T^*), L^{r^*}),$$

where $(q, r) \in \Gamma$ and $(q_1, r_1) \in \Gamma^{k_s}$. Moreover,

1. $T^* = \infty$ in the defocusing case;
2. $T^* = \infty$ if $q < 1 + \frac{2r + 2y + q}{N - 2y} - 2s$, $\|v\|_{L_{x,y}^\infty(\dot{H}^s_x)} < \infty$ and $\|v\|_{L_x^\infty(L_y^\infty)} < \|\phi\|_{qc}$, where $\phi$ is a ground state of (2.11);
3. the energy is conserved.

Remarks 2.10 Note that
1. The assumption $q \geq 2$ avoids a singularity in the source term;
2. There is an extra restriction in the bi-dimensional case;
3. In the second point the blow-up is due to the fractional $s$-order gradient concentration in $L^2$;
4. The radial assumption avoids a loss of regularity in Strichartz estimates [8].

Finally, one is concerned with the finite time blow-up of the focusing solutions. Thanks to the Sobolev injection $H^s_{rd} \hookrightarrow \dot{H}^s \cap \dot{H}^{\frac{N}{s} - \frac{3}{2}}$ and the localized variance identity [15], there is non-global solutions to (1.3) with data in $H^s_{rd} \cap \dot{H}^{\frac{N}{s} - \frac{3}{2}}$. In the following, one gives a concentration result about finite time blow-up solutions in $L_{x,y}^\infty(\dot{H}^s_x)$.

Theorem 2.11 Take the same conditions in Theorem 2.9. Let a maximal solution to (1.3),

$$v \in C_{T^*}(\dot{H}^s_{rd} \cap \dot{H}^{\frac{N}{s} - \frac{3}{2}}).$$

Assume that $T^* < \infty$ and $\|u\|_{L_x^\infty(L_y^\infty)} < \infty$. If

$$\lim_{t \to T^*} \mu(t) \|(-\Delta)^\frac{s}{2} v(t)\|_{L_y^\infty} = \infty,$$

thus,

$$\liminf_{T^*} \int_{|x| < \mu(t)} |v(t, x)|^{q_i} \, dx \geq \|\phi\|_{qc}^{q_i},$$

where $\phi$ is a ground state of (2.11).

Remark 2.12 Using the scaling

$$v^0(t, x) := \|(-\Delta)^\frac{s}{2} v(t_0)\|^{-\frac{1}{s-\frac{N}{2}}} \|(-\Delta)^\frac{s}{2} v(t_0)\|^{-\frac{2}{s-\frac{N}{2}}} \|(-\Delta)^\frac{s}{2} v(t_0)\|^{-\frac{1}{s-\frac{N}{2}}} x,$$

one has $\|(-\Delta)^\frac{s}{2} v^0(0)\| = 1$ and $\|(-\Delta)^\frac{s}{2} v^0(t)\| = \|(-\Delta)^\frac{s}{2} v(t)\|$. Thus, $t_0 + t \|(-\Delta)^\frac{s}{2} v(t_0)\|^{-\frac{2}{s-\frac{N}{2}}} < T^*$. This implies that

$$\|(-\Delta)^\frac{s}{2} v(t_0)\| \geq \frac{1}{(T^* - t_0)^{\frac{1}{s-\frac{N}{2}}} \|(-\Delta)^\frac{s}{2} v(t_0)\|^{-\frac{2}{s-\frac{N}{2}}} \|(-\Delta)^\frac{s}{2} v(t_0)\|^{-\frac{1}{s-\frac{N}{2}}} x.$$
Lemma 2.14 Let $1 < p \leq q < \infty$, $N \geq 1$, $0 < s < N$ and $\beta \geq 0$ satisfy the conditions

$$\beta < \frac{N}{q} \quad \text{and} \quad s = \frac{N}{p} - \frac{N}{q} + \beta.$$ 

Then, for any $u \in W^{s,p}$, one has

$$\|x|^{-\beta}u\|_q \leq C(\beta, p, q, N, s)\|(-\Delta)^{\frac{s}{2}}u\|_p.$$ 

Let us write a fractional chain rule [3].

Lemma 2.15 Let $s \in (0, 1]$ and $1 < p_i, q_i < \infty$ satisfying $\frac{1}{p_i} = \frac{1}{p_i} + \frac{1}{q_i}$. Then,

1. $\|(-\Delta)^{\frac{s}{2}}G(u)\|_p \lesssim \|G'(u)\|_{p_1} \|(-\Delta)^{\frac{s}{2}}u\|_{q_1}$, if $G \in C^1(C)$;
2. $\|(-\Delta)^{\frac{s}{2}}(uv)\|_p \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{p_1} \|v\|_{q_1} + \|(-\Delta)^{\frac{s}{2}}v\|_{p_2} \|u\|_{q_2}$.

Now, one gives an essential estimate in the Schrödinger context.

Definition 2.16 A couple of real numbers $(q, r)$ is said to be $\mu$ admissible (admissible if $\mu = 0$) if

$$N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2}{q} + \mu,$$

where

$$\begin{cases} \frac{2N}{N-2\mu} < r \leq \left(\frac{2N}{N-2}\right)^{-}, & \text{if } N \geq 3; \\ \frac{2}{1-\mu} < r \leq \left(\frac{2}{1-\mu}\right)^{+}, & \text{if } N = 2; \\ \frac{2}{1-2\mu} < r \leq \infty, & \text{if } N = 1. \end{cases}$$

Here, $(a^+) = \frac{a \vee a}{a \vee a}$. Finally, one says that $(q, r)$ is said to be $-\mu$ admissible if

$$N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2}{q} - \mu,$$

where

$$\begin{cases} \left(\frac{2N}{N-2\mu}\right)^{+} < r \leq \left(\frac{2N}{N-2}\right)^{-}, & \text{if } N \geq 3; \\ \left(\frac{2}{1-\mu}\right)^{-} < r \leq \left(\frac{2}{1-\mu}\right)^{+}, & \text{if } N = 2; \\ \left(\frac{2}{1-2\mu}\right)^{+} < r \leq \infty, & \text{if } N = 1. \end{cases}$$

For simplicity, one denotes $\Gamma^\mu$ the set of $\mu$ admissible pairs. Let also

$$\| \cdot \|_{S^\mu(I)} := \sup_{(q,r) \in \Gamma^\mu} \| \cdot \|_{L^q(I,L^r)}, \quad \| \cdot \|_{(S^\mu)'(I)} := \inf_{(q,r) \in \Gamma^\mu} \| \cdot \|_{L^q(I,L^r)}.$$ 

Take the particular case $\Gamma := \Gamma^0$ and $S(I) := S^0(I)$.

A standard tool to control solutions of (1.1) is the Strichartz estimate [4,6].

Proposition 2.17 Let $N \geq 1$ and $\gamma \in \mathbb{R}$. Then, there exists $C > 0$ such that

1. $\|e^{\Delta t}u\|_{S^\mu} \leq C\|\nabla|\mu|u\|;$
2. $\|\int_0^t e^{i(-\tau)^{\Delta}} f(\tau) \, d\tau\|_{S^\mu} \leq C\|f\|_{(S^\mu)'\gamma}$.

Definition 2.18 A couple of real numbers $(q, r)$ such that $q, r \geq 2$ is said to be radial admissible if

$$\frac{4N+2}{2N-1} \leq q \leq \infty, \quad \frac{2}{q} + \frac{2N-1}{r} \leq N - \frac{1}{2},$$

or

$$2 \leq q \leq \frac{4N+2}{2N-1}, \quad \frac{2}{q} + \frac{2N-1}{r} < N - \frac{1}{2}.$$
Recall some Strichartz estimates [8] for the fractional Schrödinger problem.

**Proposition 2.19** Let \( N \geq 2, \mu \in \mathbb{R}, \frac{N}{2N-1} < s < 1 \) and \( u_0 \in H^\mu_{\text{rad}} \). Then

\[
\|u\|_{L^p_t(L^r_x)} \leq \|u_0\|_{\dot{H}^\mu} + \|i\dot{u} - (-\Delta)^s u\|_{L^q_t(L^\tilde{r}_x)},
\]

if \((q, r)\) and \((\tilde{q}, \tilde{r})\) are radial-admissible pairs such that \((\tilde{q}, \tilde{r}, N) \neq (2, \infty, 2)\) or \((q, r, N) \neq (2, \infty, 2)\) and satisfy the condition

\[
\frac{2s}{q} + \mu = N\left(\frac{1}{2} - \frac{1}{r}\right), \quad \frac{2s}{\tilde{q}} - \mu = N\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right).
\]

**Remarks 2.20** Note that

1. If we take \( \mu = 0 \) in the previous inequality, we obtain the classical Strichartz estimate;
2. One denotes the sets \( \Gamma^\mu_s := \{(q, r), \text{ radial-admissible}, (q, r, N) \neq (2, \infty, 2) \text{ and } \frac{2s}{q} + \mu = N\left(\frac{1}{2} - \frac{1}{r}\right)\} \) and \( \Gamma_s := \Gamma^0_s \), and the norms

\[
\|\cdot\|_{S^\mu_s(I)} := \sup_{(q, r) \in \Gamma^\mu_s} \|\cdot\|_{L^q(I, L^r)}, \quad \|\cdot\|_{(S^\mu_s)^*(I)} := \inf_{(q, r) \in \Gamma^\mu_s} \|\cdot\|_{L^{q'}(I, L^{r'})}.
\]

### 3 The Choquard equation

In this section, one proves the results about the generalized Hartree problem (1.1). In this section, one defines the real numbers

\[
B := Np - N - \alpha - 2\gamma, \quad A := 2p - B,
\]

and the source term

\[
\mathcal{N}[u] := \mathcal{N}(\rho, \gamma, \alpha)[u] := |x|^\gamma (I_\alpha * |\cdot|^\gamma |u|^p)|u|^{p-2}u.
\]

#### 3.1 Gagliardo–Nirenberg inequality

The goal of this sub-section is to prove Theorem 2.1. Let us start with a compact injection.

**Lemma 3.1** Let \( N \geq 1, \gamma, \alpha \) satisfying (1.2) and \( 1 + \frac{\alpha + 2\gamma}{N} < p < 1 + \frac{2 + 2\gamma + \alpha}{N - 2 - 2\gamma} \). Then, the following injection

\[
\dot{H}^1 \cap L^{p_c} (\mathbb{R}^N) \hookrightarrow \hookrightarrow L^{\frac{2Np}{N + \alpha}} (|x|^{\frac{2N\gamma}{N + \alpha}} \text{ dx}). \quad (3.13)
\]

**Proof** Take a sequence of function such that \( \sup_n \left( \|u_n\|_{p_c} + \|\nabla u_n\| \right) < \infty \), converging weakly to zero. One proves that \( \|u_n\|_{L^{\frac{2Np}{N + \alpha}} (|x|^{\frac{2N\gamma}{N + \alpha}}, \text{ dx})} \to 0 \). Since \( p_c < \frac{2N}{N - 2} \), an interpolation gives, one has

\[
\sup_n \|u_n\|_q < \infty, \quad \forall q \in (p_c, \frac{2N}{N - 2}).
\]

Take \( R > e^{\frac{q + N}{2N\gamma}} > 0 \) and write using \( p_c < \frac{2Np}{N + \alpha} < \frac{2N}{N - 2} \),

\[
\int_{|x| > R} |u_n|^{\frac{2Np}{N + \alpha}} |x|^{\frac{2N\gamma}{N + \alpha}} \text{ dx} \leq R^{\frac{2N\gamma}{N + \alpha}} \int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N + \alpha}} \text{ dx} \leq c\epsilon.
\]

On the other hand, with Poincaré inequality and compact Sobolev injections, one has

\[
\|u_n\|_{L^q(|x| \leq R)} \to 0, \quad \forall q \in (2, \frac{2N}{N - 2}).
\]
Here $\frac{2N}{N-2} = \infty$ for $N = 2$. By Hölder estimate for $1 < a < \frac{a+N}{2p}$, one writes
\[
\int_{|x| < R} |u_n|^{\frac{2Np}{N-2}} |x|^{\frac{2N\gamma}{N-2}} \, dx \leq \| |x|^{\frac{2Np}{N-2}} \|_{L^a(|x| < R)} \| u_n \|^{\frac{2Np}{N-2}}_{L^{a'}(|x| < R)} \leq C(N, \gamma, R) \| u_n \|^{\frac{2Np}{N-2}}_{L^{a'}(|x| < R)}.
\]
Since $1 + \frac{a+2\gamma}{N} < p < 1 + \frac{2+2\gamma+a}{N-2-2\gamma}$, one can choose $a = \frac{a+N}{2p} - \epsilon$ such that $2 < \frac{Np}{N+a} < \frac{2N}{N-2}$. Then, for large $n$,
\[
\int_{|x| < R} |u_n|^{\frac{2Np}{N-2}} |x|^{\frac{2N\gamma}{N-2}} \, dx < \epsilon.
\]
The proof is ended. \(\square\)

3.2 Proof of the interpolation inequality (2.5)

First, using Lemmas 2.13 and 2.14, one has
\[
\int_{\mathbb{R}^N} |x|^{\gamma} (I^{\alpha} |\cdot| |\cdot| u|^p) |u|^p \, dx \lesssim \| |x|^{\gamma} u |u|^{p-1} \|^{\frac{2Np}{N-2}}_{L^p} \| u \|^{2(p-1)}_{L^p} \lesssim \| \nabla u \|^{2} \| u \|^{2(p-1)}_{L^p}.
\]
The estimate (2.5) is proved.

3.3 Proof of the Eq. (2.6)

Denote
\[
\beta := \frac{1}{C_{\text{opt}}} = \inf_{H^1 \cap L^p} \frac{\| \nabla u \|^{2} \| u \|^{2(p-1)}_{L^p}}{\mathcal{P}[u]}.
\]
Using (2.5), there exists a sequence $(v_n)$ in $\dot{H}^1 \cap L^p$ such that
\[
\beta = \lim_n J(v_n) := \lim_n \frac{\| \nabla v_n \|^{2} \| v_n \|^{2(p-1)}_{L^p}}{\mathcal{P}[v_n]}.
\]
Denoting for $a, b \in \mathbb{R}$, the scaling $u^{a,b} := au(b \cdot)$, one computes
\[
\| \nabla u^{a,b} \|^2 = a^2 b^{2-N} \| \nabla u \|^2; \\
\| u^{a,b} \|_{L^p} = ab^{-\frac{N}{p}} \| u \|_{L^p}; \\
\mathcal{P}[u^{a,b}] = a^{2p} b^{-N-a-2\gamma} \mathcal{P}[u].
\]
It follows that
\[
J(u^{a,b}) = J(u).
\]
Now, we choose
\[
\mu_n := \left( \frac{\| v_n \|_{L^p}}{\| \nabla v_n \|} \right)^{\frac{1}{1-\gamma}} \quad \text{and} \quad \lambda_n := \frac{\| v_n \|_{L^p}}{\| \nabla v_n \|^{\frac{N-2}{N-2+2\gamma}}}. \]
Thus, \( \psi_n := \nu_{n}^{\lambda_n,\mu_n} \) satisfies
\[
\| \psi_n \|_{P^c} = \| \nabla \psi_n \| = 1 \quad \text{and} \quad \beta = \lim_{n} J(\psi_n).
\]

Then, \( \psi_n \rightharpoonup \psi \) in \( \dot{H}^1 \cap L^{P^c} \) and using Sobolev injection (3.13) via Lemma 2.13, we get for a sub-sequence denoted also \( (\psi_n) \),
\[
P[\psi_n] \rightarrow P[\psi].
\]
This implies that, when \( n \) goes to infinity
\[
J(\psi_n) = \frac{1}{P[\psi_n]} \rightarrow \frac{1}{P[\psi]}.
\]
Using lower semi continuity of the \( \dot{H}^1 \cap L^{P^c} \) norm, we get
\[
\| \psi \|_{P^c} \leq 1 \quad \text{and} \quad \| \nabla \psi \| \leq 1.
\]
Then, \( J(\psi) < \beta \) if \( \| \psi \|_{P^c} \| \nabla \psi \| < 1 \), which implies that
\[
\| \psi \|_{P^c} = 1 \quad \text{and} \quad \| \nabla \psi \| = 1.
\]
It follows that
\[
\psi_n \rightarrow \psi \quad \text{in} \quad \dot{H}^1 \cap L^{P^c}
\]
and
\[
\beta = J(\psi) = \frac{1}{P[\psi]}.
\]
The minimizer satisfies the Euler equation
\[
\partial_{\varepsilon} J(\psi + \varepsilon \eta)_{\varepsilon=0} = 0, \quad \forall \eta \in C_{0}^{\infty}(\mathbb{R}^N).
\]
Hence, \( \psi \) satisfies
\[
-\Delta \psi + (p-1)\psi^{p-2}\psi - \beta p (I_{a} \ast |x|^p |\phi|^p) \phi = 0.
\]
This completes the proof.

### 3.4 Proof of the Eq. (2.8)

Thanks to the previous subsection, we know that \( C_{opt} = \frac{1}{b} = P[\psi] \), where \( \psi \) is given in (2.6). Take, for \( a, b \in \mathbb{R} \), the scaling \( \psi = \phi^{a,b} := a \phi(b \cdotp) \). Then, the previous equation gives
\[
-\frac{1}{p-1} a^{2-p^c} b^2 \Delta \phi + |\phi|^{p^c-2} \phi - \frac{\beta p}{p-1} a^{2-p^c} b^{-a-2} (I_{a} \ast |x|^p |\phi|^p) \phi \phi^{p-2} = 0.
\]
Choosing
\[
a := \left( \beta p \left( \frac{1}{p-1} \right)^{\frac{2+2p^c+a}{2}} \right)^{\frac{2N}{2N+p^c(1-N-2\gamma)}};
\]
\[
b := \left( (p-1)a^{p^c-2} \right)^{\frac{1}{2}}
\]
\[
= \left( p-1 \right)^{\frac{1}{2}} \left( \beta p \left( \frac{1}{p-1} \right)^{\frac{2+2p^c+a}{2}} \right)^{\frac{N(1-p^c)}{2N+p^c(1-N-2\gamma)}}.
\]
it follows that
\[ \Delta \phi + |\phi|^{p-2}\phi - (I_\alpha * \cdot |p|^p)|x|^p|\phi|^{p-2}\phi = 0. \]

Now, since
\[ \| \psi \|_{p_c} = 1 = ab^{-\frac{N}{p_c}} \| \phi \|_{p_c}, \]
we get
\[ \beta = \frac{1}{p} \| \phi \|_{p_c}^{2(p-1)}. \]
The proof is closed.

3.5 Well-posedness

In this sub-section, one proves Theorem 2.3. Let us give some estimates of the source term.

**Lemma 3.2** Let \( N \geq 2, \gamma, \alpha \) satisfy (1.2) and \( p_2 < p < p^* \) such that \( p \geq 2 \) and \((N-2)(p-1)-\alpha-2\gamma > 0\). Then, there exist \( c, \theta > 0 \) and \( 0 < \theta_2 < 2(p-1) \) such that for any real interval \(|I| \leq 1\), one has

1. \[ \| \nabla \nabla u \|_{S'(I)} \leq c|I|^{\theta} \| \nabla u \|^{2p-1}_{S(I)}; \]
2. \[ \| (-\Delta)^{\frac{\gamma}{2}} \nabla u \|_{S'(I)} \leq c|I|^{\theta} \| (-\Delta)^{\frac{\gamma}{2}} u \|^{2p-1}_{S(I)} \| \nabla u \|_{S(I)}, \]
3. \[ \| \nabla u \|_{S^{-\alpha}(I)} \leq c|I|^{\theta} \| \nabla u \|^{p_2}_{L^{\infty}(I)} \| u \|^{2p-1-\theta_2}_{S^{\gamma}(I)} \]

**Proof** 1. Let us write with triangular inequality
\[ |\nabla \nabla u| \lesssim |x|^\gamma (I_\alpha * |\gamma|^p |u|^p) |u|^{p-2} |\nabla u| + |x|^\gamma (I_\alpha * |\gamma|^p |u|^p) |u|^{p-1} \]
\[ + |x|^\gamma (I_\alpha * |\gamma|^p |u|^p) |u|^{p-1} + |x|^\gamma (I_\alpha * |\gamma|^p |u|^p) |u|^{p-1} \]
\[ := (I) + (II) + (III) + (IV). \]
The first and last terms are controlled similarly. Also the second and third one. Let us decompose the first term as follows
\[ (I) = |x|^\gamma (I_\alpha * \chi_{|x|<1} |\cdot|^p |u|^p) |u|^{p-2} |\nabla u| + |x|^\gamma (I_\alpha * \chi_{|x|\geq1} |\cdot|^p |u|^p) |u|^{p-2} |\nabla u| \]
\[ := (I_1) + (I_2). \]
By Lemma 2.13 via Sobolev injections, one has
\[ \| (I_2) \|_{L^{p'}(|x|>1)} \lesssim \| |x|^p \|^{2}_{L^{\infty}(|x|>1)} \| \nabla u \|_r \| u \|^{2p-1}_{S_{N-\gamma}} \]
\[ \lesssim \| |x|^p \|^{2}_{L^{\infty}(|x|>1)} \| \nabla u \|^{2p-1}_{r}. \]
Here
\[ \frac{1}{p'} + \frac{\alpha}{N} = \frac{2}{a} + \frac{1}{r} + \frac{2(p-1)(N-r)}{Nr}. \]
The integrability condition reads
\[ -2\gamma > \frac{2N}{a} = \alpha + N - \frac{2np}{r} + 2(p-1). \]

Let the admissible pair
\[ (q, r) := \left( \frac{4p}{(N-2)(p-1)-\alpha-2\gamma}^+, \frac{2Np}{\alpha + N + 2\gamma + 2(p-1)^+} \right) \in \Gamma. \]
Then,

$$
\|(I_2)\|_{L^p(L^{r'}(|x|>1))} \lesssim T^\theta \|\nabla u\|^{2p-1}_{L^2(L^r)}
$$

where one takes $\theta := 1 - \frac{2p}{q} > 0$. For the second term, one computes with Strichartz estimates and Lemma 2.13 via Sobolev injections

$$
\|(I_1)\|_{L^{r'}(|x|>1)} \lesssim \|x|^{\gamma-p-1}\|L^{\alpha}(|x|>1)\|\|x|^{\gamma-1}\|L^{\alpha}(|x|<1)\|\|\nabla u\|_{r_1} \|u\|^{2p-1}_{L^{\alpha}(L^r)}.
$$

Here

$$
\frac{1}{r_1} + \frac{\alpha}{N} = \frac{1}{c} + \frac{1}{d} + \frac{2(\gamma - 1)(N - r_1)}{N r_1}.
$$

Take $\frac{N}{\gamma} = -\gamma - 2\epsilon$ and $\frac{N}{\alpha} = -\gamma + \epsilon$, for some $0 < \epsilon < < 1$. Letting $\frac{a}{\gamma} := \frac{1}{c} + \frac{1}{d}$, one gets

$$
1 + \frac{\alpha}{N} = \frac{2}{a} + \frac{2(\gamma - 1)(N - r_1)}{N r_1} > -\gamma.
$$

This is the same condition (3.14). Thus, one keeps the same admissible pair $(q, r) = (q_1, r_1)$. Thus,

$$
\|(I_1)\|_{L^q(L^{r'}(|x|>1))} \lesssim T^\theta \|\nabla u\|^{2p-1}_{L^2(L^r)}
$$

The notations of the real numbers $a, c, q, r, \theta$ may change from term to another. Let us decompose the second term as follows

$$(III) = \|x|^{\gamma-1}(I_a \ast \chi_{|x|<1}) \cdot |x|^{\gamma}|u|^p|u|^{p-1}| + |x|^{\gamma-1}(I_a \ast \chi_{|x| \geq 1}) \cdot |x|^{\gamma}|u|^p|u|^{p-1}|$$

$$
:= (III_1) + (III_2).
$$

By Strichartz and Hardy–Littlewood–Sobolev estimates via Sobolev injections

$$
\|(III_2)\|_{L^{r'}(|x|>1)} \lesssim \|x|^{\gamma-p-1}\|L^a(|x|>1)\|\|x|^{\gamma-1}\|L^a(|x|>1)\|\|\nabla u\|_{r'}^{2p-1}\|u\|^{2p-1}_{L^2(L^r)}.
$$

Here

$$
\frac{1}{r'} + \frac{\alpha}{N} = \frac{1}{a} + \frac{1}{c} + \frac{(2p - 1)(N - r)}{N r}.
$$

Thus,

$$
1 + \frac{\alpha}{N} = \frac{1}{a} + \frac{1}{c} + \frac{(2p - 1)(N - r)}{N r}.
$$

The integrability condition reads

$$
-2\gamma + 1 > N\left(\frac{1}{a} + \frac{1}{c}\right) = \alpha + N - \frac{2Np}{r} + 2p - 1.
$$

This is the same (3.14). Thus taking previous admissible pairs, one gets

$$
\|(III_2)\|_{L^q(L^{r'}(|x|>1))} \lesssim T^\theta \|\nabla u\|^{2p-1}_{L^2(L^r)}.
$$
For the second term, one computes with Strichartz and Hardy–Littlewood–Sobolev estimates via Sobolev injections
\[ \| (II_1) \|_{L^p_t(L^{p-1}_x)} \lesssim \| x \|^{p-1}_{L^p(\{|x|>1\})} \| \nabla x \|^p_{L^p(\{|x|<1\})} \| u \|^{2p-1}_{L^1(\{|x|<1\})} \]
\[ \lesssim \| x \|^{p-1}_{L^p(\{|x|>1\})} \| \nabla x \|^p_{L^p(\{|x|<1\})} \| \nabla u \|^{2p-1}_{L^1(\{|x|<1\})}. \]

Here,
\[ \frac{1}{r_1} + \frac{\alpha}{N} = \frac{1}{c} + \frac{1}{d} + \frac{(2p-1)(N-r_1)}{N r_1}. \]

Take \( \frac{N}{c} = -\gamma - 2\epsilon \) and \( \frac{N}{d} = -\gamma + \epsilon \), for some \( 0 < \epsilon < 1 \). Letting \( \frac{2}{a} := \frac{1}{c} + \frac{1}{d} \), one gets
\[ -2\gamma > \frac{2N}{a} = \alpha + N - \frac{2pN}{r_1} + 2p - 1. \]

This is the same condition (3.14). Thus,
\[ \| (II_1) \|_{L^{p'}_t(L^{p-1}_x)} \lesssim T^\theta \| \nabla u \|^{2p-1}_{L^{p}_t(L^r)} \]

The estimates on the unit ball follow similarly. This finishes the proof of the first point.

2. Using Strichartz estimates and Lemma 2.14, one has with previous notations
\[ \| (-\Delta)^{\frac{\gamma}{2}} N[u] \|_{L^{2N(\frac{p-1}{p})}((|x|<1))} \lesssim \| \nabla N[u] \|_{L^{2N(p-1)}((|x|<1))} \]
\[ \lesssim \| (I) \|_{L^{2N(p-1)}((|x|<1))} + \| (II) \|_{L^{2N(p-1)}((|x|<1))} \]
\[ + \| (III) \|_{L^{2N(p-1)}((|x|<1))} + \| (IV) \|_{L^{2N(p-1)}((|x|<1))}. \]

Compute, using Lemma 2.13 and Sobolev injections
\[ \| (I) \|_{L^{2N(p-1)}((|x|<1))} \lesssim \| x \|^{2p-1}_p \| (I_0 \ast \chi_{|x|<1} \cdot |N|^{p-1}) u \|^{2p-2} \| u \| \]
\[ \lesssim \| x \|^{2p-1}_p \| u \|^{2p-1}_r \| u \|^{2p-1}_{\frac{N}{2N-1}}. \]

Here,
\[ -2\gamma < \frac{2N}{a} = 2 + \alpha + \frac{2 + 2\gamma + \alpha}{2} \frac{N(2p-1)}{r} + 2p - 1, \]

\( r := \left( \frac{2N(2p-1)(p-1)}{2 + 2\gamma + \alpha + 2N(p-1)^2} \right)^+ \) and \( q := \left( \frac{4(2p-1)(p-1)}{N(p-1) - (2 + 2\gamma + \alpha)} \right)^- \).

With a direct computation, one has \( 2 < r < \frac{2N}{N-2} \). So, for \( N \geq 4 \), one has \( s_r r < q \). In the case \( N = 2 \), this condition reads
\[ 2 + 2\gamma + \alpha > (1 - (2 + 2\gamma + \alpha))x, \quad x = p - 1. \]

This is satisfied if \( 2 + 2\gamma + \alpha > 1 \) or \( 2 + 2\gamma + \alpha < 1 \) and \( p < 1 + \frac{1}{1-(2+2\gamma+\alpha)} \). For \( N = 3 \), the condition \( N > r \) is equivalent to
\[ 2x^2 - 2x + 2 + \alpha + 2\gamma > 0, \quad x := p - 1. \]

This is satisfied if \( p \geq 2 \). In conclusion, the condition \( p < p^* \) gives \( \frac{2}{a} - \frac{2p-1}{q} > 0 \) and
\[ \| (I) \|_{L^{2N(p-1)}((|x|<1))} \lesssim \| \nabla u \| \| (-\Delta)^{\frac{\gamma}{2}} u \|^{2p-1}_r \| L^{2N(\frac{p-1}{p})}(\omega_{2p-1}) \lesssim T^\theta \| \nabla u \|^{2p-1}_{L^{2N(p-1)}(\omega_{2p-1})} \]
\[ \| (-\Delta)^{\frac{\gamma}{2}} u \|^{2p-1}_{L^{2N(p-1)}(\omega_{2p-1})}. \]
Let us control $\| (I_2) \|_{L^2(I, L^{2N/(p-1)}(\{x|<1\}))}$. Compute, using Lemma 2.13,

\[
\| (I_2) \|_{L^2(I, L^{2N/(p-1)}(\{x|<1\}))} \lesssim \| |x|^p (I_{\alpha} \ast |x| \chi_{\{x|>1\}}) |u|^p |\nabla u| \|_{L^2(I, L^{2N/(p-1)}(\{x|<1\}))} \lesssim \| |x|^p \|_{L^2(\{x|<1\})} \| |x|^p \|_{L^2(\{x|>1\})} \| \nabla u \| \| u \|_{L^{2(p-1)}(\{x|<1\})}^{2(p-1)/N}.
\]

Here, taking for some $0 < \epsilon << 1$, $\frac{N}{c} = -\gamma + 2\epsilon$ and $\frac{N}{d} = -\gamma - \epsilon$, one gets

\[-2b < \frac{N}{c} + \frac{N}{d} = 2 + \alpha + \frac{2 + 2\gamma + \alpha}{2(p-1)} - \frac{N(2p-1)}{r} + 2(p-1)s_c, \quad N > rs_c.
\]

Taking account of (3.15), one keeps the above admissible couple and

\[
\| (I_2) \|_{L^2(I, L^{2N/(p-1)}(\{x|<1\}))} \lesssim T^{\frac{1}{2} - \frac{2p-1}{q}} \| \nabla u \|_{L^q(I,L^r)} \| (-\Delta)^{\frac{s}{2}} u \|_{L^2(I,L^r)}^{2(p-1)}.
\]

Let us estimate the following term as above

\[
\| (I_1) \|_{L^4(\{x|<1\})} \lesssim \| |x|^{-1} (I_{\alpha} \ast |x| \chi_{\{x|<1\}}) |u|^p |\nabla u| \|_{L^4(\{x|<1\})} \lesssim \| |x|^{-1} \|_{L^4(\{x|<1\})} \| |x|^p \|_{L^4(\{x|<1\})} \| \nabla u \| \| u \|_{L^{2(p-1)}(\{x|<1\})}^{2(p-1)/N}.
\]

Here,

\[-2\gamma + 1 < \frac{N}{a} + \frac{N}{b} = 2 + \alpha + \frac{2 + 2\gamma + \alpha}{2(p-1)} - \frac{N(2p-1)}{r} + 2(p-1)s_c, \quad N > rs_c.
\]

This is the same condition as (3.15). Thus, similarly

\[
\| (I_1) \|_{L^4(\{x|<1\})} \lesssim T^{\frac{1}{2} - \frac{2p-1}{q}} \| \nabla u \|_{L^q(I,L^r)} \| (-\Delta)^{\frac{s}{2}} u \|_{L^2(I,L^r)}^{2(p-1)}.
\]

Let us estimate the following quantity.

\[
\| (I_1) \|_{L^4(\{x|>1\})} \lesssim \| |x|^{-1} (I_{\alpha} \ast |x| \chi_{\{x|>1\}}) |u|^p |\nabla u| \|_{L^4(\{x|>1\})} \lesssim \| |x|^{-1} \|_{L^4(\{x|>1\})} \| |x|^p \|_{L^4(\{x|>1\})} \| \nabla u \| \| u \|_{L^{2(p-1)}(\{x|>1\})}^{2(p-1)/N}.
\]

Here, taking $\frac{N}{a} = -\gamma + 1 - \epsilon$ and $\frac{N}{b} = -\gamma + 2\epsilon$, for some $0 < \epsilon << 1$, one gets the condition

\[-2\gamma + 1 < \frac{N}{a} + \frac{N}{b} = 2 + \alpha + \frac{2 + 2\gamma + \alpha}{2(p-1)} - \frac{N(2p-1)}{r} + 2(p-1)s_c, \quad N > r.
\]

This is the same condition as (3.15). Thus, similarly

\[
\| (I_1) \|_{L^4(\{x|>1\})} \lesssim T^{\frac{1}{2} - \frac{2p-1}{q}} \| \nabla u \|_{L^q(I,L^r)} \| (-\Delta)^{\frac{s}{2}} u \|_{L^2(I,L^r)}^{2(p-1)}.
\]

The estimation of the other terms follow similarly.
3. Taking account of Hardy–Littlewood–Sobolev and Sobolev inequalities, one has for \( (q, r) \in \Gamma^{A} \) and \( (q, r) \in \Gamma^{A} \),

\[
\| \mathcal{N}[u] \|_{L_{x}^{r}(L_{y}^{q}(|x|<1))} \leq c \| |x|^{2} \|_{L_{y}^{2}(|x|<1)} \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| u(t) \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{2p-1-\theta} \| L_{y}^{\gamma}(0,T) \\
+ c \| |x|^{\gamma} \|_{L^{q}(|x|>1)} \| |x|^{\gamma} \|_{L^{r}(|x|<1)} \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| u(t) \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{2p-1-\theta} \| L_{y}^{\gamma}(0,T) \\
\leq c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| L_{y}^{\gamma}(0,T) \\
+ c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| L_{y}^{\gamma}(0,T) \\
\leq c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| L_{y}^{\gamma}(0,T) \\
\end{equation}

Here, one takes \( \theta = \theta', \frac{2}{a} = \frac{1}{b} + \frac{1}{c} \) and \( \frac{1}{q} - \frac{2p-1-\theta}{q} > 0 \) with

\[
-2\gamma < \frac{2N}{a} = \alpha + N \frac{(N-2)}{2} = \frac{N(2p-\theta)}{r}. \tag{3.16}
\]

The first inequality is equivalent to \( q > \frac{2p-\theta}{1-s_{c}} \). Let us take \( 0 < \theta << 1 \) and

\[
q := \left( \frac{2p-\theta}{1-s_{c}} \right)^{+}, \quad r := \left( \frac{2N(2p-\theta)}{(N-2s_{c})(2p-\theta)-4(1-s_{c})} \right)^{-}.
\]

A direct computation gives (3.16). Thus,

\[
\| \mathcal{N}[u] \|_{L_{x}^{r}(L_{y}^{q}(|x|<1))} \leq c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| L_{y}^{\gamma}(0,T).
\]

Let us do similar estimations on the complementary of the unit ball. Take the same notations and write

\[
\| \mathcal{N}[u] \|_{L_{x}^{r}(L_{y}^{q}(|x|>1))} \leq c \| |x|^{2} \|_{L_{y}^{2}(|x|>1)} \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|>1))}^{\theta} \| u(t) \|_{L_{y}^{r}(L_{x}^{q}(|x|>1))}^{2p-1-\theta} \| L_{y}^{\gamma}(0,T) \\
+ c \| |x|^{\gamma} \|_{L^{q}(|x|>1)} \| |x|^{\gamma} \|_{L^{r}(|x|<1)} \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{\theta} \| u(t) \|_{L_{y}^{r}(L_{x}^{q}(|x|<1))}^{2p-1-\theta} \| L_{y}^{\gamma}(0,T) \\
\leq c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|>1))}^{\theta} \| L_{y}^{\gamma}(0,T) \\
+ c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|>1))}^{\theta} \| L_{y}^{\gamma}(0,T) \\
\leq c T^{\gamma} \left| \frac{2p-1-\theta}{q} \right| \| u \|_{L_{y}^{r}(L_{x}^{q}(|x|>1))}^{\theta} \| L_{y}^{\gamma}(0,T)
\]

Here, one takes \( \theta = \theta', \frac{2}{a} = \frac{1}{b} + \frac{1}{c} \) and \( \frac{1}{q} - \frac{2p-1-\theta}{q} > 0 \) with

\[
-2\gamma > \frac{2N}{a} = \alpha + N \frac{(N-2)}{2} = \frac{N(2p-\theta)}{r}.
\]

The following choice satisfies the above conditions

\[
(q, r) = (\infty, \frac{2N}{N-2s_{c}}).
\]

This closes the proof.
3.6 Local existence

Now, one proves Theorem 2.3. The proof follows using Strichartz estimates in Proposition 2.17, the integral formula and a classical Picard fixed point method. Take the function

\[ f(u) := e^{i\Delta}u_0 + \int_0^\iota e^{i(s-\Delta)N[u]}\,ds. \]

Let, for \( T, R > 0 \) the centered ball \( B_T(R) \) with radius \( R \) of the space

\[ X_T := \left( \bigcap_{(q, r) \in \Gamma} L^q_T\left(\dot{W}^{1, r} \cap \dot{W}^{s, r}\right) \right) \cap \left( \bigcap_{(q_1, r_1) \in \Gamma} L^q_T\left(\dot{L}^{r_1}_T\right) \right), \]

equipped with the complete distance

\[ d(u, v) := \sup_{(q, r) \in \Gamma} \|u - v\|_{L^q_T(L^r)}. \]

Using the triangular inequality, let us write for \( w := u - v \),

\[ |\mathcal{N}[u] - \mathcal{N}[v]| \lesssim |x|^r \left[ (I_{a_1} | \cdot | \|u\|^{p-2} |w|) + (I_{a_2} | \cdot | \|u\|^{p-1} + |w| |u|^{p-1}) |w||u|^{p-1} \right]. \]

Thanks to Lemma 3.2 via Strichartz estimate, one has

\[ d(f(u), f(v)) \lesssim \|\mathcal{N}[u] - \mathcal{N}[v]\|_{S^{-\alpha}(\mathbb{R})} \lesssim T^{\alpha_2} \|\nabla u\|_{L^\infty_T(L^2)} \|u\|_{S^\alpha(I)} \|w\|_{S^{-\alpha}(I)} \lesssim cT^{\alpha_2} R^{2(p-1)} d(u, v). \]

Moreover, taking account of Stritarz estimates via Lemma 3.2, one writes

\[ \sup_{(q, r) \in \Gamma} \|\nabla f(u)\|_{L^q_T(L^r)} + \sup_{(q, r) \in \Gamma} \|(-\Delta)^{\alpha_2} f(u)\|_{L^q_T(L^r)} + \sup_{(q, r) \in \Gamma} \|f(u)\|_{L^q_T(L^r)} \leq c\|u_0\|_{\dot{H}^{1} \cap \dot{H}^{s_\varepsilon}} + c(T^\theta + T^\theta_2) R^{2p-1}. \]

Choose \( R := 2c\|u_0\|_{\dot{H}^{1} \cap \dot{H}^{s_\varepsilon}} \) and \( T > 0 \) such that \( c(T^\theta + T^\theta_2) < \frac{1}{2R^{2p-1}} \), one gets a contraction of \( B_T(R) \). The proof follows with a Picard argument.

3.7 Global existence

Assume that \( \|u\|_{L^\infty_t(L^{s_\varepsilon}(\dot{H}^{s_\varepsilon}))} < \infty \) and \( \|u\|_{L^\infty_t(L^p_p)} < \|\phi\|_{p_c} \) and \( T^* < \infty \). Then, Theorem 2.1 gives

\[ E[u] = \|\nabla u\|^2 - \frac{\epsilon}{p} \int_{\mathbb{R}^N} \mathcal{N}[u] \,dx \geq \|\nabla u\|^2 - \frac{C_{opt}}{p} \|u\|_{p_c}^{2(p-1)} \|\nabla u\|^2 \geq \|\nabla u\|^2 \left( 1 - \left( \frac{\|u\|_{p_c}}{\|\phi\|_{p_c}} \right)^{2(p-1)} \right). \]

Thus, \( \sup_{t \in (0, T^*)} \|\nabla u(t)\| < \infty \). This contradiction finishes the proof.
3.8 Non-global solutions

In this sub-section, one proves Theorem 2.5. Take a sequence of positive real numbers \( t_n \to T^* \) and the sequences

\[
\beta_n := \| \nabla u(t_n) \|^{- \frac{1}{2} s_c}, \quad v_n := \beta_n^{\frac{2(1-s_c)}{2(p-1)}} u(t_n, \beta_n). \]

A computation gives

\[
\begin{align*}
\| (-\Delta)^{\frac{1}{2}} v_n \| &= \| (-\Delta)^{\frac{1}{2}} u(t_n) \|; \\
\| \nabla v_n \| &= 1; \\
E(v_n) &= \beta_n^{2(1-s_c)} E(u_0).
\end{align*}
\]

Thus,

\[
\sup_n \| v_n \|_{ \dot{H}^{s_c} \cap \dot{H}^1} < \infty, \quad E(v_n) \to 0.
\]

Take a weak limit of \( v_n \) in \( \dot{H}^{s_c} \cap \dot{H}^1 \) denoted by \( v \). With the weak limit lower semi-continuity and \( \lambda(t_n) \gg \beta_n \), one has for any \( R > 0 \),

\[
\int_{|x| < R} |v|^p_c \, dx \leq \liminf_n \int_{|x| < R} |v_n|^p_c \, dx = \liminf_n \int_{|x| < R \beta_n} |u(t_n)|^p_c \, dx \\
\leq \liminf_n \int_{|x| < \lambda(t_n)} |u(t_n)|^p_c \, dx.
\]

Now, with the lower semi-continuity and the compact embedding (3.13), one has

\[
0 = \liminf_n E(v_n) \geq \frac{1}{2} \| \nabla v \|^2 (1 - \left[ \frac{\| v \|_{p_c}}{\| \phi \|_{p_c}} \right]^{2(p-1)}).
\]

So,

\[
\liminf_n \int_{|x| < \lambda(t_n)} |u(t_n)|^p_c \, dx \geq \| \phi \|_{p_c}^p.
\]

The proof is complete.

4 The fractional Choquard equation

In this section, one proves the results about the Choquard problem with a non-local Laplacian operator. In this section, one defines the real numbers

\[
s_B := Nq - N - \alpha - 2\gamma, \quad A_s := 2q - B_s.
\]

Moreover, one decomposes the source term as follows

\[
\mathcal{N}[u] := \mathcal{N}_q[u] \\
:= \mathcal{N}_1[u] + \mathcal{N}_2[u] \\
:= |x|^\gamma (I_\alpha * \chi_{|.|<1}) \cdot |x|^\gamma |u|^q |u|^{q-2} u + |x|^\gamma (I_\alpha * \chi_{|.|>1}) \cdot |y|^\gamma |u|^q |u|^{q-2} u.
\]
4.1 Gagliardo–Nirenberg inequality

The goal of this sub-section is to prove Theorem 2.7. Let us start with a compact injection.

**Lemma 4.1** Let $N \geq 2, \gamma, \alpha$ satisfying (1.4) and $1 + \frac{\alpha + 2\gamma}{N} \leq q < 1 + \frac{2\alpha + \gamma + \alpha}{N - 2\gamma}$. Then, the following injection is compact

$$\dot{H}^{s} \cap L^{q_{c}}(\mathbb{R}^{N}) \hookrightarrow \hookrightarrow L^{\frac{2Nq_{c}}{N - \alpha}}(|x|^{\frac{2N}{N + \alpha}}).$$

(4.17)

**Proof** Take a sequence of functions such that $\sup_{n} \left( \|u_{n}\|_{q_{c}} + \|(-\Delta)^{\frac{s}{2}}u_{n}\| \right) < \infty$, converging weakly to zero. One proves that $\|u_{n}\|_{L^{\frac{2Nq_{c}}{N + \alpha}}(|x|^{\frac{2N}{N + \alpha}})} \to 0$. Since $q < q^{*}$, one has $q_{c} < \frac{2N}{N - 2s}$ and an interpolation argument gives

$$\sup_{n} \|u_{n}\|_{q} < \infty, \quad \forall \, q \in (q_{c}, \frac{2N}{N - 2s}).$$

Take $R > \epsilon^{\frac{\alpha + N}{N - \alpha}} > 0$ and write using $q_{c} < \frac{2Nq}{N + \alpha} < \frac{2N}{N - 2s}$,

$$\int_{|x| > R} |u_{n}|_{\frac{2Nq}{N + \alpha}} \frac{2N}{N + \alpha} \, dx \leq R^{\frac{2Nq}{N + \alpha}} \int_{\mathbb{R}^{N}} |u_{n}|_{\frac{2Nq}{N + \alpha}} \, dx \leq c \epsilon.$$  

On the other hand, with Poincaré inequality and compact Sobolev injections, one has

$$\|u_{n}\|_{L^{q}(|x| < R)} \to 0, \quad \forall \, q \in (2, \frac{2N}{N - 2s}).$$

By Hölder estimate for $1 < a < \frac{\alpha + N}{2\gamma}$, one writes

$$\int_{|x| < R} |u_{n}|_{\frac{2Nq}{N + \alpha}} \frac{2Nq}{N + \alpha} \, dx \leq \|u_{n}\|_{L^{a}(|x| > R)} \|u_{n}\|_{\frac{2Nq}{N + \alpha}} \frac{2Nq}{N + \alpha} \leq C(N, \gamma, R) \|u_{n}\|_{L^{\frac{2Nq^{*}}{N - 2\gamma}}(|x| > R)}.$$  

Since $1 + \frac{\alpha + 2\gamma}{N} < q < q^{*}$, one can choose $a = \frac{\alpha + N}{2\gamma} - \epsilon$ such that $2 < \frac{2Nq^{*}}{N + \alpha} < \frac{2N}{N - 2s}$. Then, for large $n$,

$$\int_{|x| < R} |u_{n}|_{\frac{2Nq}{N + \alpha}} \frac{2Nq}{N + \alpha} \, dx < \epsilon.$$  

The proof is ended. \qed

4.2 Proof of the interpolation inequality (2.9)

First, using Lemmas 2.13 and 2.14, one has

$$\int_{\mathbb{R}^{N}} |x|^{\gamma} (I_{\alpha} \ast | \cdot |^{\gamma} |u|^{q}) |u|^{q} \, dx \lesssim \| |x|^{\gamma} u|u|^{q-1}\|_{\frac{2N}{N + \alpha}}^{2N} \lesssim \| |x|^{\gamma} u\|_{\frac{2Nq^{*}}{N - 2\gamma}}^{2N} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{q_{c}}^{2(q-1)} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{q_{c}}^{2(q-1)}.$$  

The estimate (2.9) is proved.
4.3 Proof of the Eq. (2.10)

Denote
\[ \beta_s := \frac{1}{C_{opt,s}} := \inf_{H^s \cap L^\infty} \|(-\Delta)^{\frac{s}{2}} u\|^2 \frac{\|u\|^{2(q-1)}}{Q[u]} . \]

Using (2.5), there exists a sequence \((v_n)\) in \(\dot{H}^s \cap L^\infty\) such that
\[ \beta_s = \lim_{n \to \infty} J_s(v_n) := \lim_{n \to \infty} \frac{\|(-\Delta)^{\frac{s}{2}} v_n\|^2 \|v_n\|^{2(1+q)}}{Q[v_n]} . \]

Denoting for \(a, b \in \mathbb{R}\), the scaling \(u^{a,b} := au(b)\), one computes
\[ \|(-\Delta)^{\frac{s}{2}} u^{a,b}\|^2 = a^2 b^{2s-N} \|(-\Delta)^{\frac{s}{2}} u\|^2 ; \]
\[ \|u^{a,b}\|_{q_c} = ab^{-\frac{N}{q}} \|u\|_{q_c} ; \]
\[ Q[u^{a,b}] = a^2 b^{-N-\alpha-2\gamma} Q[u] . \]

It follows that
\[ J_s(u^{a,b}) = J_s(u) . \]

Now, we choose
\[ \mu_n := \left( \frac{\|v_n\|_{q_c}}{\|(-\Delta)^{\frac{s}{2}} v_n\|} \right)^{\frac{1}{2-s}} \text{ and } \lambda_n := \frac{\|v_n\|_{q_c}^{\frac{N-2s}{2s-2\gamma}}}{{\|(-\Delta)^{\frac{s}{2}} v_n\|^2}^{\frac{2s-2\gamma}{2s-2\gamma+\mu_n}}} . \]

Thus, \(\psi_n := u_n^{\mu_n,\lambda_n}\) satisfies
\[ \|\psi_n\|_{q_c} = \|(-\Delta)^{\frac{s}{2}} \psi_n\| = 1 \text{ and } \beta_s = \lim_{n \to \infty} J_s(\psi_n) . \]

Then, \(\psi_n \rightharpoonup \psi\) in \(\dot{H}^s \cap L^\infty\) and using Sobolev injection (4.17), we get for a sub-sequence denoted also \((\psi_n)\),
\[ Q[\psi_n] \to Q[\psi] . \]

This implies that, when \(n\) goes to infinity
\[ J(\psi_n) = \frac{1}{P[\psi_n]} \to \frac{1}{P[\psi]} . \]

Using lower semicontinuity of the \(\dot{H}^s \cap L^\infty\) norm, we get
\[ \|\psi\|_{q_c} \leq 1 \text{ and } \|(-\Delta)^{\frac{s}{2}} \psi\| \leq 1 . \]

Then, \(J_s(\psi) < \beta\) if \(\|\psi\|_{q_c} \|(-\Delta)^{\frac{s}{2}} \psi\| < 1\), which implies that
\[ \|\psi\|_{q_c} = 1 \text{ and } \|(-\Delta)^{\frac{s}{2}} \psi\| = 1 . \]

It follows that
\[ \psi_n \to \psi \text{ in } \dot{H}^s \cap L^\infty \]
and
\[ \beta_s = J_s(\psi) = \frac{1}{Q[\psi]} . \]

The minimizer satisfies the Euler equation
\[ \partial_{\varepsilon} J_s(\psi + \varepsilon \eta)|_{\varepsilon=0} = 0, \quad \forall \eta \in C^\infty_0(\mathbb{R}^N) . \]

Hence, \(\psi\) satisfies
\[ (-\Delta)^s \psi + (q-1)|\psi|^{q-2}\psi - \beta_s q(I_q * |\cdot|^q |\psi|^q)|x| |\psi|^{q-2} \psi = 0 . \]

This completes the proof.
4.4 Proof of the Eq. (2.12)

Thanks to the previous sub-section, we know that \( C_{opt,s} = \frac{1}{|b|} = Q[\psi] \), where \( \psi \) is given in (2.10). Take, for \( a, b \in \mathbb{R} \), the scaling \( \psi = \phi^{a,b} := a\phi(b \cdot) \). Then, the previous elliptic equation gives

\[
\frac{1}{q - 1} a^{2q - 2} b^{2s} (\Delta)^s \phi + |\phi|^{q^* - 2} \phi - \frac{\beta_s q}{q - 1} a^{2q - 2} b^{-2s} (I_a * | \cdot |^q |\psi|^q |x|^\gamma |\phi|^{q^* - 2} \phi = 0.
\]

Choosing

\[
a := \left( \beta_s q \left( \frac{1}{q - 1} \right) \frac{2s + 2\gamma + \alpha}{2s} \right)^\left( \frac{2s}{2s + 2\gamma + \alpha} \right)^N N \phi = 2 \left( -N \right) \phi b \left( \phi b \right)
\]

\[
b := \left( (q - 1) a^{q^* - 2} \right)^\frac{1}{q^* s}
\]

\[
= (q - 1)^\frac{1}{q^* s} \left( \beta_s q \left( \frac{1}{q - 1} \right) \frac{2s + 2\gamma + \alpha}{2s} \right)^\left( \frac{2s}{2s + 2\gamma + \alpha} \right)^N N \phi = 2 \left( -N \right) \phi b \left( \phi b \right)
\]

it follows that

\[-(\Delta)^s \phi + |\phi|^{q^* - 2} \phi - (I_a * | \cdot |^q |\psi|^q |x|^\gamma |\phi|^{q^* - 2} \phi = 0.
\]

Now, since

\[\|\psi\|_{q^*} = 1 = ab \frac{N}{\phi} \|\phi\|_{q^*},\]

one gets

\[\beta_s = \frac{1}{q} \|\phi\|_{q^*}^{2(q - 1)}.\]

The proof is closed.

4.5 Proof of Theorem 2.9

In this sub-section, one proves Theorem 2.9. The next result regroups some local estimates about the source term.

**Lemma 4.2** Let \( N \geq 2, s \in \left( 0, \frac{N}{2(N - 1)} \right) \), \( \gamma, \alpha \) satisfy \((I.2), 1 + \frac{\alpha + \gamma}{N - 2} < q < q^* \) such that \( q \geq 2 \). Then, there exist \( c, \theta > 0 \) and \( 0 < \theta_2 < 2(q - 1) \) such that for any real interval \( |I| \leq 1 \), one has

1. \( \|(-\Delta)^\frac{s}{2} N[u]\|_{S^r(I)} \leq c |I|^\theta \|(-\Delta)^\frac{s}{2} u\|_{S^r(I)}^{2q - 1}; \)
2. \( \|(-\Delta)^\frac{s}{2} N[u]\|_{S^r(I)} \leq c |I|^\theta \|(-\Delta)^\frac{s}{2} u\|_{S^r(I)}^{2q - 1} \)
3. \( \|N^2[u]\|_{S^{r_2}(I)} \leq c |I|^\theta \|(-\Delta)^\frac{s}{2} u\|_{S^{r_2}(I)}^{2q - 1} \)

**Proof** 1. Thanks to Hardy–Littlewood–Sobolev and Hölder estimates via Sobolev injection, one has

\[\|x|\| (I_a * | \cdot |^q |u|^q |x|^\gamma |u|^{q^* - 2} \|_{L^q(x|>1)} \]

\[\|x|\| \|x|\| (\Delta)^\frac{s}{2} u \|_{r_1} \|u\|_{N^r(I)}^{2q - 1} \]

\[\|x|\| \|x|\| (\Delta)^\frac{s}{2} u \|_{r_1}^{2q - 1} \]

\[\|(-\Delta)^\frac{s}{2} u \|_{r_1}^{2q - 1}. \]
Here

\[
1 + \frac{\alpha}{N} = \frac{2}{a} + \frac{2}{r_1} + \frac{2(q - 1)(N - sr_1)}{Nr_1}.
\]

The integrability condition reads

\[
-2\gamma > \frac{2N}{a} = \alpha + N + 2s(q - 1) - \frac{2Nq}{r_1}.
\]  \hspace{1cm} (4.18)

Let the admissible pair

\[(q_1, r_1) := \left(\left[\frac{4sq}{(N - 2s)(q - 1) - \alpha - 2\gamma}\right] + \left[\frac{2Nq}{\alpha + N + 2\gamma + 2s(q - 1)}\right]\right) \in \Gamma.
\]

This gives \(\frac{N}{a} < -\gamma\). Moreover, \(q < q^*\) implies that \(\theta := 1 - \frac{2q}{q_1} > 0\) and

\[
\|\|x\|^{\theta} (I_a * \chi_{|\cdot| > 1} \cdot |\cdot| u^q) |u|^{q-2} (-\Delta)^{\frac{s}{2}} u\|_{L^q_t(L^r_x(|x| > 1))} \lesssim T^\theta \|(-\Delta)^{\frac{s}{2}} u\|_{L^q_t(L^r_x)}^{2q-1}.
\]

Now, one computes with Strichartz and Hölder estimates via Sobolev injections

\[
\|\|x\|^{\theta} (I_a * \chi_{|\cdot| > 1} \cdot |\cdot| u^q) |u|^{q-1}\|_{L^2(|x| > 1)} \lesssim \|\|x\|^{\theta} \|L^r(|x| > 1)\|L^q(|x| > 1)\|\|x\|^{\theta} \|L^r(|x| > 1)\|(-\Delta)^{\frac{s}{2}} u\|_{L^q}^{2q-1}.
\]

Here

\[
\frac{1}{r} \frac{\alpha}{N} = \frac{1}{d} + \frac{1}{c} + \frac{2(q - 1)(N - sr_2)}{Nr_2}.
\]

Thus,

\[
1 + \frac{\alpha}{N} = \frac{1}{d} + \frac{1}{c} + \frac{1}{r_2} + \frac{2(q - 1)(N - sr_2)}{Nr_2}.
\]

The integrability condition reads

\[
-2\gamma + s > N\left(\frac{1}{d} + \frac{1}{c}\right) = \alpha + N + s(2q - 1) - \frac{2Nq}{r_2}.
\]

This is the same condition (4.18). So, following similarly, one gets

\[
\|\|x\|^{\theta^*} (I_a * \chi_{|\cdot| > 1} \cdot |\cdot| u^q) |u|^{q-1}\|_{L^q_t(L^r_x(|x| > 1))} \lesssim T^\theta \|(-\Delta)^{\frac{s}{2}} u\|_{L^q_t(L^r_x)}^{2q-1}.
\]

Now, let us estimate the term

\[
\|\|x\|^{\theta} (I_a * \chi_{|\cdot| > 1} \cdot |\cdot| u^q) |u|^{q-2} (-\Delta)^{\frac{s}{2}} u\|_{L^r(|x| < 1)} \lesssim \|\|x\|^{\theta} \|L^r(|x| > 1)\|L^q(|x| < 1)\|\|x\|^{\theta} \|L^r(|x| > 1)\|(-\Delta)^{\frac{s}{2}} u\|_{L^r}^{2q-1} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{L^r}^{2q-1}.
\]

Here taking \(0 < \epsilon \ll 1\), \(\frac{N}{a_2} = -\gamma + \epsilon\) and \(\frac{N}{a_1} = -\gamma - 2\epsilon\), such that \(\frac{2N}{a} := \frac{N}{a_1} + \frac{N}{a_2} < -2\gamma\) and

\[
1 + \frac{\alpha}{N} = \frac{2}{a} + \frac{2}{r_1} + \frac{2(q - 1)(N - sr_1)}{Nr_1}.
\]
Thus, with previous computation, one has
\[ \|x\|^q (I_a * \chi_{|\cdot|>1} \cdot |\cdot|^q |u|^q) |u|^{q-2} (-\Delta)^{\frac{r}{2}} u \|_{L^{\infty}_t L^{q'}_{x'}([|x|<1])} \lesssim T^q \|(-\Delta)^{\frac{r}{2}} u\|_{L^2_t(L^{2}_{x'}([|x|<1]))}^{2q-1}. \]

With the same reasoning, one gets
\[ \|x\|^q (I_a * \chi_{|\cdot|>1} \cdot |\cdot|^q |u|^q) |u|^{q-1} \|_{L^{\infty}_t L^{q'}_{x'}([|x|<1])} \lesssim T^q \|(-\Delta)^{\frac{r}{2}} u\|_{L^2_t(L^{2}_{x'}([|x|<1]))}^{2q-1}. \]

The proof of the first follows with the fractional chain rule in Lemma 2.15.

2. Using Lemma 2.14, Strichartz and Hölder estimates, one has
\[ \|(-\Delta)^{\frac{r}{2}} N[u]\|_{L^{2N}_t(L^{2N}_{x'})([|x|<1])} \lesssim \|(-\Delta)^{\frac{r}{2}} N[u]\|_{L^{2N(q-1)}_t(L^{2N(q-1)}_{x'})([|x|<1])} \lesssim \|(-\Delta)^{\frac{r}{2}} N_1[u]\|_{L^{2N(q-1)}_t(L^{2N(q-1)}_{x'})([|x|<1])} + \|(-\Delta)^{\frac{r}{2}} N_2[u]\|_{L^{2N(q-1)}_t(L^{2N(q-1)}_{x'})([|x|<1])}. \]

Compute, using Lemma 2.13,
\[ \|x\|^q (I_a * \chi_{|\cdot|<1} \cdot |\cdot|^q |u|^q) |u|^{q-2} (-\Delta)^{\frac{r}{2}} u \|_{L^{2N(q-1)}_t(L^{2N(q-1)}_{x'})([|x|<1])} \lesssim \|x\|^q \|(-\Delta)^{\frac{r}{2}} u\|_{L^2_t(L^{2N}_{x'})([|x|<1])}^{2(q-1)}. \]

Here,
\[ -2\gamma < \frac{2N}{a} = 2s + \alpha + \frac{2s + 2\gamma + \alpha}{2(q-1)} - \frac{N(q-1)}{r} + 2(q-1)s_1^r, \quad N > s_1^r. \quad (4.19) \]

Take the admissible pair \((q_1, r_1) \in \Gamma_x\) such that
\[ r_1 := \left( \frac{2N(2q-1)(q-1)}{2s + 2\gamma + \alpha + 2N(q-1)^2} \right)^+, \quad q_1 := \left( \frac{4s(2q-1)(q-1)}{N(q-1) - (2s + 2\gamma + \alpha)} \right)^-. \]

With a direct computation, one has \(2 < r_1 < \frac{2N}{N-2s}\). So, for \(N \geq 4\), one has \(r_1 < \frac{2N}{N-2s} \leq N\). In the case \(N = 2\), the condition \(N > s_1^r r_1\) reads
\[ 2s + 2\gamma + \alpha > (1 - (2s + 2\gamma + \alpha))x, \quad x := q - 1. \]
This is equivalent to \(2s + 2\gamma + \alpha > 1\) or \(2s + 2\gamma + \alpha < 1\) and \(q < 1 + \frac{1}{1 - (2s + 2\gamma + \alpha)}\). In the case \(N = 3\), the condition \(N > r_1\) reads
\[ 2x^2 - 2x + 2s + \alpha + 2\gamma > 0. \]
This is satisfied if \(q \geq 2\). Thus, by Sobolev injections
\[ \|x\|^q (I_a * \chi_{|\cdot|<1} \cdot |\cdot|^q |u|^q) |u|^{q-2} (-\Delta)^{\frac{r}{2}} u \|_{L^2_t(L^{2N(q-1)}_x([|x|<1])})} \lesssim \|(-\Delta)^{\frac{r}{2}} u\|_{L^2_t(L^{2N(q-1)}_x([|x|<1]))}^{2(q-1)}. \]

The condition \(q < q^*\) gives \(\frac{1}{2} - \frac{2q-1}{q_1} > 0\). Now, let us control the term
\[ \|x\|^q (I_a * \chi_{|\cdot|<1} \cdot |\cdot|^q |u|^q) |u|^{q-2} (-\Delta)^{\frac{r}{2}} u \|_{L^2_t(L^{2N(q-1)}_x([|x|<1])})} \lesssim \|x\|^q \|L^q([|x|<1])\| \|x\|^q \|\|L^{q'}([|x|<1])\| \|(-\Delta)^{\frac{r}{2}} u\|_{L^2_t(L^{2N(q-1)}_x([|x|<1]))}^{2(q-1)}. \]
Thus, one keeps the above admissible couple and

$$-2\gamma < N\left(\frac{1}{c} + \frac{1}{d}\right) = 2s + \alpha + \frac{2 + 2\gamma + \alpha}{2(q - 1)} - \frac{N(2q - 1)}{r} + 2(q - 1)s_c, \quad N > r_{s_c}'.$$

This is the same condition (4.19). Thus, similarly

$$\|x\|^\gamma(I_{a} * \chi_{|x|>1} \cdot |\gamma| |u|^q)|u|^{q-2}(-\Delta)^\frac{q}{2} u\|_{L^2(I,L_{s_c}^{2N/(q-1)+2\gamma} \gamma (|x|<1))}
\lesssim T^{\frac{1}{2} - \frac{2\gamma - 1}{q}} \|(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})} ||(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})}.$$

Let us estimate the following term

$$\|x\|^\gamma-s(I_{a} * \chi_{|x|<1} \cdot |\gamma| |u|^q)|u|^{q-1}\|_{L^2(I,L_{s_c}^{2N/(q-1)+2\gamma} \gamma (|x|<1))}
\lesssim \|x\|^\gamma-s \|L^s(|x|<1)\| \|x\|^\gamma \|L^s(|x|<1)\| \|x\|^\gamma \|L^s(|x|<1)\| \|(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})} ||(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})}.$$

Here,

$$-2\gamma + s < N\left(\frac{1}{a} + \frac{1}{b}\right) = 3s + \alpha + \frac{2s + 2\gamma + \alpha}{2(q - 1)} - \frac{N(2q - 1)}{r_1} + 2(q - 1)s_c', \quad N > r_{s_c}'.$$

This is the same condition (4.19). Thus, similarly

$$\|x\|^\gamma-s(I_{a} * \chi_{|x|<1} \cdot |\gamma| |u|^q)|u|^{q-1}\|_{L^2(I,L_{s_c}^{2N/(q-1)+2\gamma} \gamma (|x|<1))}
\lesssim T^{\frac{1}{2} - \frac{2\gamma - 1}{q}} \|(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})} ||(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})}.$$

Let us estimate the following quantity.

$$\|x\|^\gamma-s(I_{a} * \chi_{|x|>1} \cdot |\gamma| |u|^q)|u|^{q-1}\|_{L^2(I,L_{s_c}^{2N/(q-1)+2\gamma} \gamma (|x|>1))}
\lesssim \|x\|^\gamma-s \|L^s(|x|>1)\| \|x\|^\gamma \|L^s(|x|<1)\| \|x\|^\gamma \|L^s(|x|<1)\| \|(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})} ||(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})}.$$

Here, taking $\frac{N}{q} = -\gamma + s - \epsilon$ and $\frac{N}{q} = -\gamma + 2\epsilon$, for some $0 < \epsilon << 1$, one gets the condition

$$-2\gamma + s < N\left(\frac{1}{a} + \frac{1}{b}\right) = 3s + \alpha + \frac{2 + 2\gamma + \alpha}{2(q - 1)} - \frac{N(2q - 1)}{r_1} + 2(q - 1)s_c', \quad N > r_{s_c}'.$$

This is the same condition (4.19). Thus, similarly

$$\|x\|^\gamma-s(I_{a} * \chi_{|x|>1} \cdot |\gamma| |u|^q)|u|^{q-1}\|_{L^2(I,L_{s_c}^{2N/(q-1)+2\gamma} \gamma (|x|>1))}
\lesssim T^{\frac{1}{2} - \frac{2\gamma - 1}{q}} \|(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})} ||(-\Delta)^\frac{q}{2} u\|_{L^{2N/(q-1)}(I,L_{s_c}^{q_1})}.$$

The estimation of the other terms follow similarly. This point follows with the fractional chain rule in Lemma 2.15 and Strichartz estimate via the above calculus.
3. Taking account of H"older and Sobolev inequalities, one has for $(\tilde{q}_1, r_1) \in \Gamma_{s-c}^{-\tilde{q}_1}$ and $(q_1, r_1) \in \Gamma_{s-c}^{r_1}$,

$$
\| N[u] \|_{L_{r_1}^{q_1'}(L^{r_1'}(|x| < 1))} \leq c \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \| u(t) \|_{L_{r_1}^{q_1}(0, T)}^{2q_1 - \theta} \\
+ c \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u(t) \|_{L_{r_1}^{q_1}(0, T)}^{2q_1 - \theta} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \\
\leq c T^{q_1} \frac{1}{2q_1 - \theta} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \\
+ c T^{q_1} \frac{1}{2q_1 - \theta} \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \\
\leq c T^{q_1} \frac{1}{2q_1 - \theta} \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta}.
$$

Here, one takes $\theta = \theta'$, $\frac{2}{a} = b + \frac{1}{c}$ and $\frac{1}{q_1} = \frac{2q_1 - \theta}{q_1} > 0$ with

$$
-2\gamma < \frac{2N}{a} = \alpha + N - \frac{\theta(N - 2s)}{2} = \frac{N(2q - \theta)}{r_1}.
$$

The first inequality is equivalent to $q_1 > \frac{s(2q - \theta)}{s - s_c}$. Let us take $0 < \theta << 1$ and

$$
q_1 := \left( \frac{s(2q - \theta)}{s - s_c} \right)^{+}, \quad r_1 := \left( \frac{2N(2q - \theta)}{(N - 2s)(2q - \theta) - 4(s - s_c)} \right)^{+}.
$$

A direct computation gives (4.20) because $q < q^*$. Thus,

$$
\| N[u] \|_{L_{r_1}^{q_1'}(L^{r_1'}(|x| < 1))} \leq c T^{q_1} \frac{1}{2q_1 - \theta} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(L^{q_1}(0, T)) \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta}.
$$

Let us do similar estimations on the complementary of the unit ball. Take the same notations and write

$$
\| N[u] \|_{L_{r_1}^{q_1'}(L^{r_1'}(|x| > 1))} \leq c \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \| u(t) \|_{L_{r_1}^{q_1}(0, T)}^{2q_1 - \theta} \\
+ c \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u(t) \|_{L_{r_1}^{q_1}(0, T)}^{2q_1 - \theta} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \\
\leq c T^{q_1} \frac{1}{2q_1 - \theta} \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \\
+ c T^{q_1} \frac{1}{2q_1 - \theta} \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta} \\
\leq c T^{q_1} \frac{1}{2q_1 - \theta} \| |x|^{\gamma} \|_{L^{s}(|x| > 1)} \| |x|^{\gamma} \|_{L^{s}(|x| < 1)} \| (-\Delta)^{\gamma} u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))} \| u \|_{L_{r_1}^{q_1}(L^{q_1}(0, T))}^{2q_1 - \theta}.
$$

Here, one takes $\theta = \theta'$, $\frac{2}{a} = b + \frac{1}{c}$ and $\frac{1}{q_1} = \frac{2q_1 - \theta}{q_1} > 0$ with

$$
-2\gamma < \frac{2N}{a} = \alpha + N - \frac{\theta(N - 2s)}{2} = \frac{N(2q - \theta)}{r_1}.
$$

The following choice satisfies the above conditions

$$(q_1, r_1) = (\infty, \frac{2N}{N - 2s_c}).$$

This closes the proof. \hfill \Box
4.6 Local existence

Now, one proves Theorem 2.9. The proof follows using Strichartz estimates in Proposition 2.19, the integral formula and a classical Picard fixed point method. Take the function

\[ f(u) := e^{-i(-\Delta)t}u_0 + \int_0^t e^{-i(-\Delta)(t-\tau)}N[u] \, d\tau. \]

Let, for \( T, R > 0 \) the centered ball \( B_T(R) \) with radius \( R \) of the space

\[ X_T := \left( \bigcap_{(q,r)\in\Gamma} L_T^q \left( \hat{W}^{s,r} \cap \hat{W}^{s,c} \right) \right) \cap \left( \bigcap_{(q_1,r_1)\in\Gamma_s} L_T^{q_1} \left( L^{r_1} \right) \right), \]

endowed with the complete distance

\[ d(u, v) := \sup_{(q,r)\in\Gamma_s^c} \| u - v \|_{L_T^q(L^r)}. \]

Using the triangular inequality, let us write for \( w := u - v \),

\[ |N[u] - N[v]| \lesssim |x|^\gamma (I_\alpha \ast \cdot |\cdot |^q |u|^q |u|^{q-2} |w| + (I_\alpha \ast \cdot |\cdot | |u|^{q-1} + |v|^{q-1}) |w|) |v|^{q-1}. \]

Thanks to Lemma 4.2 via Strichartz estimate, one has

\[ d(f(u), f(v)) \lesssim \| N[u] - N[v] \|_{(S^n_{-\gamma},\gamma)'(0,T)} \]
\[ \lesssim T^{\theta_2} \| (-\Delta)^{\frac{\gamma}{2}} u \|_{L_T^q(L^2)} \| u \|_{S^n_{-\gamma}(0,T)}^{2(q-1) - \theta_2} \| w \|_{S^n_{-\gamma}(0,T)} \]
\[ \leq c T^{\theta_2} R^{2(q-1)} d(u, v). \]

Moreover, taking account of Stritarz estimates via Lemma 4.2, one writes

\[ \sup_{(q_1,r_1)\in\Gamma_s^c} \| (-\Delta)^{\frac{\gamma}{2}} f(u) \|_{L_T^{q_1}(L^{r_1})} \]
\[ + \sup_{(q_1,r_1)\in\Gamma_s^c} \| (-\Delta)^{\frac{\gamma}{2}} f(u) \|_{L_T^{q_1}(L^{r_1})} \]
\[ \leq c \| u \|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}} + c (T^{\theta_1} + T^{\theta_2}) R^{2q-1}. \]

Choose \( R := 2c \| u_0 \|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}} \) and \( T > 0 \) such that \( c (T^{\theta_1} + T^{\theta_2}) < \frac{1}{2R^{2q-1}} \), one gets a contraction of \( B_T(R) \). The proof follows with a Picard argument.

4.7 Global existence

Assume that \( \| u \|_{L_T^\infty(\dot{H}^{s_1})} < \infty \) and \( \| u \|_{L_T^\infty(\dot{H}^{s_2})} < \| \phi \|_{\dot{H}^s} \) and \( T^* < \infty \). Then, Theorem 2.7 gives

\[ E[u] = \| (-\Delta)^{\frac{\gamma}{2}} u \|^2 - \frac{\epsilon}{q} \int_{\mathbb{R}^N} N[u] \, dx \]
\[ \geq \| (-\Delta)^{\frac{\gamma}{2}} u \|^2 - C_{\text{opt},s} \| u \|_{\dot{H}^{s_1}}^{2(q-1)} \| (-\Delta)^{\frac{\gamma}{2}} u \|^2 \]
\[ \geq \| (-\Delta)^{\frac{\gamma}{2}} u \|^2 \left( 1 - \frac{\| u \|_{\dot{H}^{s_1}}}{\| \phi \|_{\dot{H}^s}}^{2(q-1)} \right). \]

Thus, \( \sup_{t \in [0,T^*)} \| (-\Delta)^{\frac{\gamma}{2}} u(t) \| < \infty \). This contradiction finishes the proof.
4.8 Finite time blow-up

In this sub-section, one proves Theorem 2.11. Take a sequence of positive real numbers $t_n \to T^*$ and the sequences

$$\beta_n := \|(-\Delta)^{\frac{s}{2}} u(t_n)\|^{-\frac{1}{s-\frac{q}{2}}}$$

$$v_n := \beta_n^{\frac{2s+2s+\gamma}{2(s-1)}} u(t_n, \beta_n).$$

A computation gives

$$\|(-\Delta)^{\frac{s}{2}} v_n\| = \|(-\Delta)^{\frac{s}{2}} u(t_n)\|;$$

$$\|(-\Delta)^{\frac{s}{2}} v_n\| = 1;$$

$$E(v_n) = \beta_n^{2(s-\frac{q}{2})} E(u_0).$$

Thus,

$$\sup_n \|v_n\|_{H^s \cap \dot{H}^s} < \infty, \ E(v_n) \to 0.$$ 

Take a weak limit of $v_n$ in $H^s \cap \dot{H}^s$ denoted by $v$. With the weak limit lower semi-continuity and $\lambda(t_n) \gg \beta_n$, one has for any $R > 0$,

$$\int_{|x| < R} |v|^q \, dx \leq \liminf_n \int_{|x| < R} |v_n|^q \, dx$$

$$= \liminf_n \int_{|x| < R\beta_n} |u(t_n)|^q \, dx$$

$$\leq \liminf_n \int_{|x| < \lambda(t_n)} |u(t_n)|^q \, dx.$$ 

Now, with the lower semi-continuity and the compact embedding (4.17), one has

$$0 = \liminf_n E(v_n) \geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|^2 \left(1 - \left[\frac{\|v\|_{q_c}}{\|\beta\|_{q_c}}\right]^{2(q-1)}\right).$$

So,

$$\liminf_n \int_{|x| < \lambda(t_n)} |u(t_n)|^q \, dx \geq \|\beta\|^q_{q_c}.$$ 

The proof is complete.

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