Actions of compact groups on coherent sheaves

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Let $K$ be a compact Lie group with complexification $K^\mathbb{C}$ and $X$ a reduced Stein space endowed with a continuous action of $K$ by holomorphic transformations. In this set up there is a complex space $X//K$ and a surjective $K$-invariant holomorphic map $\pi: X \to X//K$ such that the structure sheaf of $X//K$ is the sheaf of invariants $(\pi_*\mathcal{O}_X)^K$. Furthermore, the $K$-action on $X$ can be complexified in the following sense: $X$ can be realized as an open $K$-stable subset of a Stein space $X^c$ which is endowed with a holomorphic $K^\mathbb{C}$-action. Moreover, the inclusion $X \subset X^c$ is universal, i.e., every $K$-equivariant holomorphic map $\phi$ from $X$ into a holomorphic $K^\mathbb{C}$-space $Z$ extends uniquely and $K^\mathbb{C}$-equivariantly to a holomorphic map $\phi^c: X^c \to Z$ (see [H]).

In the present paper we extend the above results to Stein spaces $X$ which are not necessarily reduced. More importantly, we show that every continuous coherent $K$-sheaf over $X$ can be extended to a coherent $K^\mathbb{C}$-sheaf over $X^c$. This is also proved in the context of non-reduced spaces.

The constructions of quotients and complexifications are compatible with the reduced structures. Thus we have $\text{red}(X)//K = \text{red}(X//K)$ and $\text{red}(X)^c = \text{red}(X^c)$, where red associates to a complex space its underlying reduced space. Moreover, the results are also valid for $K$-spaces $X$ with $\text{red}(X)$ admitting a semistable quotient (see Section 1). In this context we show that the sheaf $(\pi_*\mathcal{S})^K$ of invariant sections of a coherent $K$-sheaf $\mathcal{S}$ over $X$ is a coherent analytic sheaf over the quotient space $X//K$ (see Section 1). Two direct applications of our results are

a) For every closed complex $K$-subspace $A$ of $X$ the image $\pi(A)$ is a closed subspace of $X//K$ such that $A//K \cong \pi(A)$. Moreover, $A$ extends to a closed $K^\mathbb{C}$-subspace $A^c$ of $X^c$.

b) Every holomorphic $K$-vector bundle over $X$ is given by the restriction of a holomorphic $K^\mathbb{C}$-vector bundle over the complexification $X^\mathbb{C}$.

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1. Formulation of the results

In this paper a complex space $X$ is a not necessarily reduced complex space with countable topology. By $\text{red}(X)$ we denote the underlying reduced space, i.e., set-theoretically $\text{red}(X) = X$ and $\mathcal{O}_{\text{red}(X)} = \mathcal{O}_X/\mathcal{N}_X$ on the level of sheaves. Here $\mathcal{N}_X$ denotes the nilradical of the structure sheaf $\mathcal{O}_X$ of $X$.

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Let $G$ be a real Lie group and assume that $G$ acts continuously on $X$ by holomorphic transformations. Then $G$ acts also on the structure sheaf $\mathcal{O} := \mathcal{O}_X$ of $X$: A given $g \in G$ induces for every $x \in X$ an isomorphism $\mathcal{O}_x \to \mathcal{O}_{g \cdot x}$, $f \mapsto g \cdot f$, of stalks. We call $X$ a complex $G$-space if for all open sets $N \subset G$ and $U_1, U_2 \subset X$ with $U_2 \subset \bigcap_{g \in N} g \cdot U_1$ the map

\begin{equation}
(1) \quad \Phi: N \times \mathcal{O}(U_1) \to \mathcal{O}(U_2), \quad (g, f) \mapsto (g \cdot f)|_{U_2}
\end{equation}

is continuous with respect to the canonical Fréchet topology on $\mathcal{O}(U_j)$, $j = 1, 2$ (see [G-R], Chapter V, §6). If $G$ is a complex Lie group and all above maps $\Phi$ are holomorphic, then we say that $X$ is a holomorphic $G$-space.

Now let $\mathcal{S}$ be a $G$-sheaf on $X$, i.e., $G$ acts on $\mathcal{S}$ such that the projection $\mathcal{S} \to X$, $s_x \mapsto x$ is $G$-equivariant. Reformulating (1) for a coherent sheaf $\mathcal{S}$, we obtain the notion of a continuous (resp. holomorphic) coherent $G$-sheaf on $X$.

A holomorphic map $\phi: X \to X'$ of two complex $G$-spaces $X$ and $X'$ is called $G$-equivariant, if it is $G$-equivariant as a map of sets and the comorphism $\phi^0$ is compatible with the actions of $G$ on the structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_{X'}$. The map $\phi$ is called $G$-invariant if it is equivariant with respect to the trivial action of $G$ on $X'$.

Remark 1. If $X$ is reduced, then it is a complex $G$-space. If moreover $G$ is a complex Lie group and the action $G \times X \to X$ is holomorphic, then $X$ is a holomorphic $G$-space (see [K]). A holomorphic map of reduced $G$-spaces is equivariant if and only if it is equivariant as a map of sets.

Now let $K$ be a compact real Lie group and assume that $X$ is a complex $K$-space. A $K$-invariant holomorphic map $\pi: X \to X/K$ onto a complex space $X/K$ is said to be a semistable quotient of the $K$-space $X$, if

i) the structure sheaf $\mathcal{O}_{X/K}$ of $X/K$ is the sheaf of invariants $(\pi_* \mathcal{O}_X)^K$, i.e., for every open set $Q \subset X/K$ we have $\mathcal{O}_{X/K}(Q) = \mathcal{O}_X(\pi^{-1}(Q))^K$ and

ii) $\pi$ is a Stein map, i.e., for every Stein open set $Q \subset X/K$ the inverse image $\pi^{-1}(Q)$ is also Stein.

The above definition generalizes the notion introduced in [H-M-P] to the non-reduced case. Examples of semistable quotients occur in Geometric Invariant Theory. Our first main result is the following

**Quotient Theorem.** A semistable quotient $\pi: X \to X/K$ for $X$ exists if and only if it exists for $\text{red}(X)$. If this is the case, then $\text{red}(\pi): \text{red}(X) \to \text{red}(X/K)$ is a semistable quotient for $\text{red}(X)$.

By [H], Section 6.5, every reduced Stein $K$-space has a semistable quotient and the associated quotient space is again Stein. This, together with the above result, implies the following

**Corollary 1.** If $X$ is a Stein $K$-space, then the semistable quotient $\pi: X \to X/K$ exists. Moreover, $X/K$ is a Stein space.
Since the reduction of a semistable quotient is a semistable quotient of the associated reduced $K$-space, [H], 2.3, yields that semistable quotients are universal with respect to $K$-invariant holomorphic maps. In particular, the quotient space $X//K$ is unique up to isomorphism.

If $X$ admits a semistable quotient $\pi: X \to X//K$, then we denote for a $K$-sheaf $S$ on $X$ by $(\pi_*S)^K$ the sheaf of invariants on $X//K$, i.e., $(\pi_*S)^K(Q) := S(\pi^{-1}(Q))^K$. The following fact is a generalization of a result of Roberts (see [R]) for holomorphic $K^C$-sheaves.

**Coherence Theorem.** Assume that $X$ has a semistable quotient $\pi: X \to X//K$ and let $S$ be a continuous coherent $K$-sheaf on $X$. Then the sheaf $(\pi_*S)^K$ on $X//K$ is coherent.

Let $K^C$ denote the complexification of $K$. A *complexification* $X^c$ of a complex $K$-space $X$ is a holomorphic $K^C$-space $X^c$ which contains $X$ as a $K$-stable open subset such that every $K$-equivariant holomorphic map $\phi$ from $X$ into a holomorphic $K^C$-space $Z$ extends uniquely to a $K^C$-equivariant holomorphic map $\phi^c: X^c \to Z$. Note that a complexification $X^c$ is unique up to isomorphism and that $X^c = K^C \cdot X$.

**Complexification Theorem.** If $X$ has a semistable quotient $\pi: X \to X//K$, then the complexification $X^c$ exists. Moreover, the extension $\pi^c: X^c \to X//K$ is a semistable quotient for the $K^C$-space $X^c$ and $\text{red}(X^c) = \text{red}(X)^c$ holds.

This result generalizes the Theorem in Section 6.6 of [H], where the existence of complexifications is proved for reduced Stein $K$-spaces.

Now, let $S$ be a continuous coherent $K$-sheaf on $X$ and assume that $X$ has a semistable quotient $\pi: X \to X//K$. A holomorphic $K^C$-sheaf $S^c$ on $X^c$ with $S^c|X = S$ is called a $K^C$-extension of $S$.

**Extension Theorem.** Assume that $X$ has a semistable quotient and let $S$ be a continuous coherent $K$-sheaf $S$ on $X$. Then, up to $K^C$-equivariant isomorphism, there is a unique $K^C$-extension of $S$.

For a $K$-stable closed subspace $A$ of $X$ which is defined by a $K$-invariant sheaf $\mathcal{I}$ of ideals, the Coherence and Extension Theorem imply:

**Corollary 2.** The sheaf $(\pi_*\mathcal{I})^K$ of ideals endows $\pi(A)$ with the structure of closed subspace of $X//K$. The restriction of $\pi$ to $A$ is a semistable quotient for $A$. Moreover, $A$ has a complexification $A^c$ by a closed $K^C$-subspace of $X^c$ and $\pi(A) = \pi^c(A^c)$.

If $S$ is a locally free continuous $K$-sheaf over $X$ with $K^C$-extension $S^c$, then the complement $E$ of the set of points in $X^c$ at which $S^c$ is not locally free is a proper analytic $K^C$-stable subset of $X^c$. Since $X^c = K^C \cdot X$, this implies that $E$ is empty. In other words we have the following

**Corollary 3.** The extension of a locally free $K$-sheaf over $X$ is a locally free $K^C$-sheaf over $X^c$. In particular, every holomorphic $K$-vector bundle over $X$ extends to a holomorphic $K^C$-vector bundle over $X^c$. 

□
2. Equivariant Resolution

Let $K$ be a compact real Lie group and let $X$ be a complex $K$-space. In this section we assume that the associated reduced $K$-space $\text{red}(X)$ has a semistable quotient $\pi_r: \text{red}(X) \to \text{red}(X)//K$. Let $S$ be a continuous coherent $K$-sheaf on $X$. Note that we have a continuous representation of $K$ on the Fréchet space $S(X)$.

A section $s \in S(X)$ of $S$ is said to be $K$-finite if the vector subspace of $S(X)$ generated by $K \cdot s$ is of finite dimension. We denote the vector subspace of $K$-finite elements of $S(X)$ by $S(X)_{\text{fin}}$. By a theorem of Harish-Chandra (see [HC], Lemma 5), $S(X)_{\text{fin}}$ is dense in $S(X)$.

**Equivariant Resolution Lemma.** Every $y \in \text{red}(X)//K$ has an open neighborhood $Q$ such that over $U := \pi_r^{-1}(Q)$ there is an exact sequence $\mathcal{V} \to S|U \to 0$ of $K$-sheaves. Here $\mathcal{V}$ is a finite dimensional representation space of $K$ and $\mathcal{V} := \mathcal{O} \otimes V$ is endowed with the action defined by $k \cdot (f \otimes v) := k \cdot f \otimes k \cdot v$.

*Proof.* Since $\pi_r$ is a Stein map, we may assume that $X$ is a Stein space. By Proposition 3.2 and Corollary 2 in Section 2.3 of [H], there is a unique $K$-invariant analytic set $A$ of minimal dimension contained in $\pi_r^{-1}(y)$. We fix a point $x \in A$. Then, since $X$ is a Stein space, there are sections $s_1, \ldots, s_r \in S(X)$ which generate $S_x$ as an $\mathcal{O}_x$-module.

The $K$-finite elements of $S(X)$ are dense. Thus we may assume that the $s_i$ generate a finite dimensional $K$-submodule $V$ of $S(X)$. Hence there is an equivariant homomorphism $\alpha: \mathcal{V} \to S$ of $K$-sheaves which is defined by $f \otimes s \to fs$ where $s$ denotes the section of $S$ defined by $s \in V \subset S(X)$.

Since $\alpha$ is a homomorphism of coherent $K$-sheaves, $B := \text{Supp}(S/\alpha(\mathcal{V}))$ is a closed $K$-invariant analytic subset of $X$. By definition, $\alpha$ is surjective over $X \setminus B$. Furthermore, we have $\pi_r(A) \cap \pi_r(B) = \emptyset$. Hence $Q := \text{red}(X)//K \setminus \pi_r(B)$ is open, contains $y$ and $\alpha$ is surjective over $U := \pi_r^{-1}(Q)$. □

As a consequence of the Equivariant Resolution Lemma we give here a proof of the Coherence Theorem in the reduced Stein case (see also [R]). We use two well known Lemmas. The proof of the first one can be found for example in [R] or [Sch].

**Lemma 1.** Let $V$ and $W$ be finite dimensional complex $K^C$-modules. Then there exist $K$-equivariant polynomials $p_i: W \to V$, $i = 1, \ldots, r$ which generate $(\pi_{W*}\mathcal{V})^K$ as an $\mathcal{O}_W//K^C$-module. □

**Lemma 2.** For every exact sequence $S_1 \to S_2 \to S_3$ of continuous coherent $K$-sheaves on $X$ the induced sequence $(\pi_{r*}S_1)^K \to (\pi_{r*}S_2)^K \to (\pi_{r*}S_3)^K$ on $\text{red}(X)//K$ is also exact.

*Proof.* Integration over $K$ defines a projection operator $S(X) \to S(X)^K$. Since exactness is a local property and $\pi_r: \text{red}(X) \to \text{red}(X)//K$ is a Stein map, the assertion follows. □

**Proposition 1.** Let $X$ be a reduced Stein $K$-space and denote by $\pi: X \to X//K$ the semistable quotient. Then $(\pi_*S)^K$ is a coherent sheaf on $X//K$. 

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Proof. First consider the case \( \mathcal{S} = \mathcal{V} := \mathcal{O}_X \otimes V \) with some \( K \)-module \( V \). By [H], 6.6, we may assume that \( X \) is a holomorphic \( K^\mathbb{C} \)-space. Moreover, by [H], 5.4 and 6.2, we may assume that \( X \) is a closed \( K \)-subspace of an open Stein \( \pi_W \)-saturated \( K \)-subspace of some finite dimensional \( K \)-module \( W \). So, for the case \( \mathcal{S} = \mathcal{V} \), Lemma 1 provides \( p_1, \ldots, p_r \in \mathcal{S}(X)^K \) that generate \((\pi_\ast \mathcal{S})^K\), i.e., the associated sequence

\[
\mathcal{O}_X^r \big/ K \xrightarrow{\alpha} (\pi_\ast \mathcal{S})^K \to 0.
\]

is exact. The kernel of \( \alpha \) is \((\pi_\ast \mathcal{R})^K\), where \( \mathcal{R} \subset \mathcal{O}_X^r \) denotes the \( K \)-sheaf of relations of the generators \( p_i \). Applying Lemma 2 to an equivariant resolution of \( \mathcal{R} \) yields that also \( \text{Ker}(\alpha) \) is finitely generated. Consequently \((\pi_\ast \mathcal{S})^K\) is coherent. For general \( \mathcal{S} \), apply Lemma 2 to an equivariant resolution \( \mathcal{V}' \to \mathcal{V} \to \mathcal{S} \to 0. \)

\[\square\]

3. \( K^\mathbb{C} \)-Extensions

In this section we apply the Equivariant Resolution Lemma to investigate extensions of coherent \( K \)-sheaves. Assume that \( X \) is a complex \( K \)-space such that there is a semistable quotient \( \pi_\ast \text{red}(X) \to \text{red}(X) \big/ K \) and a complexification \( X^c \) with the following properties:

i) \( \text{red}(X^c) \) is the complexification of \( \text{red}(X) \) and \( \pi_\ast ^\mathbb{C} \text{:red}(X^c) \to \text{red}(X) \big/ K \) is a semistable quotient.

ii) For every open subspace of the form \( U := \pi_\ast ^{-1}(Q) \) with \( Q \subset \text{red}(X) \big/ K \) open, the open subspace \( U^c := K^\mathbb{C} \cdot U = (\pi_\ast ^\mathbb{C})^{-1}(Q) \) of \( X^c \) is a complexification of \( U \).

Note that, by [H], Sections 3.3 and 6.6, these assumptions are valid if \( X \) is a reduced Stein \( K \)-space.

Identity Principle. Let \( \mathcal{T} \) be a holomorphic coherent \( K^\mathbb{C} \)-sheaf over \( X^c \). Then the restriction map \( R : \mathcal{T}(X^c)^\text{fin} \to \mathcal{T}(X)^\text{fin} \) is bijective.

Proof. First we show that \( R \) is injective. So, let \( s \in \mathcal{T}(X^c) \) be \( K \)-finite such that \( s\vert X = 0. \) Since \( u\vert X = 0 \) for all \( u \in V := \text{Lin}_C(K \cdot s) = \text{Lin}_C(K^\mathbb{C} \cdot s) \), we have \( (g \cdot s)\vert X = 0 \) for all \( g \in K^\mathbb{C} \). Hence \( s\vert g^{-1} \cdot X = 0 \) for every \( g \in K^\mathbb{C} \). Since \( X^c = K^\mathbb{C} \cdot X \), it follows that \( s = 0 \).

In order to show that \( R \) is surjective, we first assume that \( X \) is Stein and that there is an equivariant resolution \( \mathcal{V} \xrightarrow{\alpha} \mathcal{T} \to 0 \) with \( V \) a finite-dimensional \( K \)-module and \( V := \mathcal{O}_{X^c} \otimes V \). Every \( K \)-finite \( F \in \mathcal{V}(X) \) defines a finite-dimensional \( K \)-module \( W := \text{Lin}_C(K \cdot f) \).

Evaluating the elements of \( W \) determines a \( K \)-equivariant holomorphic map \( \Phi : X \to \text{Hom}(W, V) \). Since \( \Phi \) extends to a \( K^\mathbb{C} \)-equivariant holomorphic map \( \Phi^\mathbb{C} : X^c \to V \), it follows that \( F \) extends to \( X^c \) as a holomorphic map. This yields surjectivity of the restriction \( \mathcal{V}(X^c)^\text{fin} \to \mathcal{V}(X)^\text{fin} \).

Surjectivity of \( R \) is obtained as follows. Let \( s \in \mathcal{T}(X)^\text{fin} \). Since \( X \) was assumed to be Stein, we find an \( s_1 \in \mathcal{V}(X) \) with \( \alpha(s_1) = s \). Note that \( \alpha \) maps \( \text{Lin}_C(K \cdot s_1) \) onto the finite-dimensional vector space \( \text{Lin}_C(K \cdot s) \). Hence we may assume that \( s_1 \) is \( K \)-finite. As seen above, \( s \) has an extension \( s_1^f \in \mathcal{V}(X^c)^\text{fin} \). Then \( \alpha(s_1^f) \in \mathcal{T}(X^c)^\text{fin} \) is an extension of \( s \).
Now, in the general case, let $s \in T(X)_{\text{fin}}$. The Equivariant Resolution Lemma and the above consideration yield a cover of $\text{red}(X//K)$ by open sets $Q_i$ such that we can extend $s$ over each $U_i = \pi_r^{-1}(Q_i)$ to $s_i^c \in T(U_i^c)_{\text{fin}}$. By injectivity of $R$, we obtain that any two such extensions $s_i^c$ and $s_j^c$ coincide over $U_i^c \cap U_j^c$. Thus the $s_i^c$ patch together to an extension $s^c \in T(X^c)$ of $s$. As before we can achieve that $s^c$ is $K$-finite. \hfill \Box

Let $S^1$ and $S^2$ be holomorphic coherent $K^C$-sheaves on $X^c$. Then $\mathfrak{H}om(S^1, S^2)$ is a holomorphic $K^C$-sheaf. The action of $g \in K^C$ on $F \in \mathfrak{H}om(S^1, S^2)_x$ is given by $$(g \cdot F)(s) := g \cdot (F(g^{-1} \cdot s)).$$
where $s \in S^1_{g \cdot x}$. The $K^C$-invariant global sections of $\mathfrak{H}om(S^1, S^2)$ are precisely the $K^C$-equivariant homomorphisms from $S^1$ to $S^2$. So the Identity Principle yields:

**Homomorphism Lemma.** Every homomorphism $\alpha : S^1|X \to S^2|X$ of $K$-sheaves extends uniquely to a homomorphism $\alpha^c : S^1 \to S^2$ of $K^C$-sheaves. \hfill \Box

**Local Extension Lemma.** Let $S$ be a continuous coherent $K$-sheaf on $X$. Then every $y \in \text{red}(X)//K$ has an open neighborhood $Q$ such that the restriction $S|_U$ of $S$ to $U := \pi_r^{-1}(Q)$ has a $K^C$-extension $S^c|_U$.

*Proof.* By the Equivariant Resolution Lemma we find an open Stein neighborhood $Q \subset \text{red}(X)//K$ of $y$ such that over $U$ there exists an equivariant resolution $\mathcal{V}_1 \xrightarrow{\alpha} \mathcal{V}_2 \to S|_U \to 0$, where the $\mathcal{V}_i$ are finite-dimensional representation spaces of $K$ and $\mathcal{V}_i$ are the associated $K$-sheaves on $X$.

Set $\mathcal{V}_i^c := \mathcal{O}_{X^c} \otimes \mathcal{V}_i$ and endow each $\mathcal{V}_i^c$ with the diagonal $K^C$-action. By the Homomorphism Lemma, there is a unique $K^C$-equivariant extension $\alpha^c : \mathcal{V}_1^c \to \mathcal{V}_2^c$ of $\alpha$. Identify $S|_U$ with $\mathcal{V}_2/\text{Im}(\alpha)$ and set $S^c := \mathcal{V}_2^c/\text{Im}(\alpha^c)$. \hfill \Box

4. Invariant subspaces of reduced spaces

Let $K$ be a compact real Lie group and $X$ a complex $K$-space. Assume that the associated reduced $K$-space $\text{red}(X)$ has a semistable quotient $\pi_r : \text{red}(X) \to (\text{red}(X)//K)$. A technical ingredient for the proofs of our results is the following

**Local Embedding Lemma.** Every point $y \in (\text{red}(X)//K$ has an open Stein neighborhood $Q \subset (\text{red}(X)//K$ such that the open subspace $U := \pi_r^{-1}(Q)$ of $X$ can be realized as a closed $K$-subspace of a reduced Stein $K$-space.

*Proof.* We may assume that $X$ is a Stein space. As in the reduced case (see [H], 6.2), we can find a $K$-equivariant holomorphic map $\phi$ from $X$ into a complex finite dimensional $K$-module $V$ such that $\phi$ is an immersion along the fiber $\pi_r^{-1}(y)$ and $\text{red}(\phi)$ embeds some $\pi_r$-saturated open neighborhood $\text{red}(U)$ of $x$ properly into an open $K$-stable subset $Z'$ of $V$. 


By Siu’s Theorem (see [S]), $Z'$ contains a Stein open neighborhood $Z''$ of $\phi(U)$. Set $Z := \bigcap_{k \in K} k \cdot Z''$. Then $Z$ is an open $K$-stable Stein neighborhood of $\phi(U)$. Now, the set $A \subset U$ consisting of all points $a \in U$ for which $\phi$ is not an immersion is a $K$-stable analytic subset of $\text{red}(U)$ which does not intersect $\pi^{-1}(y)$. Hence we may shrink $U$ and $Z$ such that they are still Stein and $\phi|U:U \to Z$ is a closed embedding. \hfill \Box

In the sequel, let $Z$ be a reduced Stein $K$-space and assume that $X$ is a $K$-subspace of $Z$. Then $X$ is defined by a continuous coherent $K$-sheaf $I$ of ideals in $\mathcal{O}_Z$. Let $\kappa: Z \to Z//K$ denote the semistable quotient.

**Proposition 2.** The ideal $(\kappa_* I)^K$ defines a closed subspace $X//K$ of $Z//K$ and the restriction $\pi := \kappa|X: X \to X//K$ is a semistable quotient for $X$. Moreover, $\text{red}(\pi): \text{red}(X) \to \text{red}(X//K)$ is a semistable quotient for $\text{red}(X)$.

**Proof.** By Proposition 1, $(\kappa_* I)^K$ is coherent and hence $X//K$ is a complex subspace of $Z//K$. The fact that $\pi$ and $\text{red}(\pi)$ are semistable quotients now follows from applying Lemma 2 to the following two exact sequences of $K$-sheaves:

$$0 \to I \to \mathcal{O}_Z \to \mathcal{O}_X \to 0, \quad 0 \to \sqrt{I} \to \mathcal{O}_Z \to \mathcal{O}_{\text{red} X} \to 0.$$ \hfill \Box

Now assume that the ideal $I$ has an extension to $Z^c$ by a holomorphic $K^C$-ideal $I^c$ of $\mathcal{O}_{Z^c}$. Note that for the $K^C$-subspace $X^c$ of $Z$ defined by $I^c$ we have $X^c = K^C \cdot X$. In particular, [H], Section 3.3, implies that $\text{red}(X^c)$ is the complexification of $\text{red}(X)$. This statement holds also for the corresponding non-reduced spaces:

**Proposition 3.** Let $Q \subset X//K$ be open and $U := \pi^{-1}(Q)$. Then the open subspace $U^c := K^C \cdot U$ of $X^c$ is the complexification of $U$.

**Proof.** We have to show that $U^c$ is universal with respect to $K$-equivariant holomorphic maps $\phi: U \to Y$ into holomorphic $K^C$-spaces $Y$.

As a set $U$ is of the form $U = U^c \cap X$. In particular, $U$ is orbit-convex (see [H], 3.2) and hence $\text{red}(U^c)$ is the complexification of $\text{red}(U)$. So the map $\phi$ extends to a $K^C$-equivariant continuous map $\phi^c: U^c \to Y$.

Let $\phi^0: \mathcal{O}_Y \to \phi_* \mathcal{O}_U$ denote the comorphism of $\phi$. Then we define the comorphism $(\phi^c)^0: \mathcal{O}_Y \to \phi^c_* \mathcal{O}_{U^c}$ as follows: For $y \in Y$ and $x \in U$ with $\phi^c(x) = y$ choose a $g_0 \in K^C$ with $g_0 \cdot x \in U$ and set

$$(\phi^c)^0(f) := g_0^{-1} \cdot \phi^0(g_0 \cdot f)$$

for every germ $f \in \mathcal{O}_Y$ at $y$. We have to show that this definition does not depend on the choice of $g_0$. Set

$$N(x) := \{g \in K^C; g \cdot x \in U\}.$$
Recall that $U$ is an orbit convex subset of $U^c$ and therefore $N(x)$ is connected mod $K$ (see [H], 1.5, 3.2 and 6.6). Now, let $h := g_0 \cdot f$ and consider the set

$$M := \{ g \in N(x); (gg_0^{-1})^{-1} \cdot \phi^0(gg_0^{-1} \cdot h) = \phi^0(h) \}.$$ 

Then $Kg_0 \subset M$ holds. Moreover, $M$ is closed in $N(x)$ because the $K^C$-sheaves $\mathcal{O}_{X^c}$ and $\mathcal{O}_Y$ are continuous. We claim that $M$ is also open in $N(x)$. Using the holomorphy of the $K^C$-sheaves $\mathcal{O}_{X^c}$ and $\mathcal{O}_Y$ this is seen as follows: If $g_1 \in M$ and $g_2 := g_1g_0^{-1}$, then

$$(2) \quad \phi^0(h) = (gg_2)^{-1} \cdot \phi^0(gg_2 \cdot h)$$

holds for all $g \in K$. After representing $h$ by some section $h$ defined in a neighborhood of $\phi(g_0 \cdot x)$ and restricting (2) to a sufficiently small neighborhood of $g_0 \cdot x$, the right hand side of (2) depends holomorphically on $g$.

Now the Identity Theorem yields that (2) is satisfied for $g$ in some neighborhood of $e \in K^C$. This implies that $M$ is open in $N(x)$. Consequently $M = N(x)$ holds, i.e., $(\phi^c)^0$ is well defined.

5. Proof of the Theorems

Proof of the Quotient Theorem. Assume first that a semistable quotient $\pi: X \to X//K$ exists. Consider the exact sequence $0 \to N_X \to \mathcal{O}_X \to \mathcal{O}_X/N_X \to 0$ of $K$-sheaves. By Lemma 2, the associated sequence of sheaves of invariants on $X//K$ is also exact. This implies that red($\pi$): red($X$) $\to$ red($X//K$) is a semistable quotient for red($X$).

Now assume that there is a semistable quotient $\pi_r$ : red($X$) $\to$ red($X//K$) on the level of topological spaces set $X//K :=$ red($X$)//$K$ and endow $X//K$ with the structure sheaf $\mathcal{O}_{X//K} := \pi_* (\mathcal{O}_X)^K$. Then it follows from Proposition 2 and the Local Embedding Lemma that $X//K$ is a complex space. Moreover, by construction, $\pi := \pi_r$, interpreted as a morphism of the ringed spaces $X$ and $X//K$, is the quotient map.

Proof of the Coherence Theorem. By the Local Embedding Lemma and Proposition 2, we may assume that $X$ is a $K$-stable closed subspace of a reduced Stein $K$-space $Z$. Noting that the trivial extension of $\mathcal{S}$ to $Z$ is a continuous coherent $K$-sheaf, we obtain the assertion from Proposition 1.

Proof of the Complexification Theorem. Let $\pi : X \to X//K$ be the semistable quotient for $X$. Choose a cover of $X//K$ by Stein open sets $Q_i$ as in the Local Embedding Lemma. According to the Local Extension Lemma we may assume that every $U_i := \pi^{-1}(Q_i)$ satisfies the assumptions of Proposition 3.

Let $U^c_i$ be the complexification of $U_i := \pi^{-1}(Q_i)$. Note that every complexification $(U_i \cap U_j)^c$ of $U_i \cap U_j$ is contained as an open subspace in both, $U^c_i$ and $U^c_j$. Consequently the $U^c_i$ can be glued together over $X//K$ to a complexification $X^c$ of $X$. 

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By construction, \( \text{red}(X^c) \) is the complexification of \( \text{red}(X) \). So we obtain \( \text{red}(\pi^c) = (\text{red} \pi)^c \) for the \( K \)-invariant holomorphic \( \pi^c: X^c \to X//K \) extension of \( \pi \). In particular, \( \pi^c \) is a Stein map and hence a semistable quotient. □

Note that, by construction, the complexification \( X^c \) satisfies the technical assumptions i) and ii), made in Section 3.

Proof of the Extension Theorem. By the Homomorphism Lemma, we only have to prove the existence of a \( K^C \)-extension. According to the Local Extension Lemma we can cover \( X//K \) by open sets \( Q_i \) such that on each \( U_i := \pi^{-1}(Q_i) \) the restriction \( S_i := S|U_i \) has an \( K^C \)-extension \( S^c_i \) to \( U^c_i = (\pi^c)^{-1}(Q_i) \).

The Homomorphism Lemma yields glueing homomorphisms \( S^c_i|U^c_i \cap U^c_j \to S^c_j|U^c_i \cap U^c_j \) that extend the identity map \( S_i|U_i \cap U_j \to S_j|U_i \cap U_j \). Hence the \( S^c_i \) can be glued together to a \( K^C \)-sheaf \( S^c \) on \( X^c \). It is straightforward to check that \( S^c \) is a \( K^C \)-extension of \( S \). □

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