Analytical Computation of Cosmological Correlators from Symmetries and Singularities

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by

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Cosmological correlators are very important objects in cosmology as they offer us a huge amount of information of the early universe. They reside on the future boundary of a de Sitter space and can be calculated by two methods. First method involves our old Lagrangian picture where we evaluate the bulk time integrals to find the correlator at the future boundary. The other method which is our main topic of interest is to find the correlators by imposing constraints from symmetries and singularities on the future boundary. Particularly, we obtain the differential equation satisfied by the correlator by imposing de Sitter isometries which act on the future boundary just like the conformal symmetry. Then we analytically solve this differential equation by imposing boundary conditions coming from the correct normalization of the physical singularities and absence of the unphysical ones. In the process we see emergence of the effects of time dependent background like particle production and also obtain their observable signatures in the correlators. Finally we compare our analytical solution with the numerical ones.
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Chapter 1

Introduction

The early stages of hot big bang cosmology seems to be the highest energy observable natural process. No particle accelerator on earth is anywhere near to the energy range involved in the birth of our universe, nor there is any promise for such high energy accelerators in the near future. Hence, we can’t simulate the birth and subsequent evolution of our universe in any lab.

What we can do instead is to look for patterns or spatial correlations in the late-time cosmological structures like the large scale structure (LSS) of galaxies or the anisotropies of cosmic microwave background (CMB), which can possibly tell us about the physics in the early times. Going from the physics of the early universe to the observations we make today, or in other words, to build a consistent history of the universe that could explain the correlations is a challenging task.

In inflationary cosmology [1, 2, 3], we can trace back all the spatial correlations to the origin of the hot big-bang or the end of inflation (see figure 1.1). This end of inflation forms a boundary of an approximate de Sitter(dS) spacetime. The question is, as posed earlier, how can we reproduce these spatial correlations which are given by cosmological observations on the future boundary. One way to do this is to follow the time evolution of the correlations in the bulk spacetime. Another way is to determine these inflationary correlators by imposing constraints through symmetries and singularities at the future boundary without any reference to time-evolution. In this treatise, though we will briefly discuss the first one, our primary focus will be on the latter one. One motivation behind this is that the notion of time itself breaks down in the initial big-bang singularity and thus a perfect time evolution picture may be erroneous.

Our discussion on cosmological correlator will mainly revolve around the four-point function of the conformally coupled scalars (scalar fields for which the trace of the energy tensor is zero) in de Sitter space, mediated by the tree-level exchange of massive scalars. The theoretically motivated reason being that these four-point functions act as building blocks for constructing inflationary correlators corresponding to exchange of particles of general mass and spin [4, 5].

We are interested in the inflationary correlators because it can give us information about the physics of the early universe. Perturbations are created quantum-mechanically on sub-horizon scales when different parts of the universe were still in causal contact. During inflation, the comoving Hubble radius , \((aH)^{-1}\), shrinks
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Figure 1.1: Illustration of the evolution of the scalar fluctuations and how information of the early universe gets encoded in the cosmological structures.

while the comoving scales \( k^{-1} \) remains constant (see figure 1.1). As a result, the perturbations exit the horizon. With the end of inflation, the hot big-bang era commences when the comoving Hubble radius starts monotonically increasing. Thus, the perturbations again enter the horizon and in between horizon exit and re-entry, it gets frozen in the sense that it doesn’t evolve. On re-entry, these fluctuations unfold into anisotropies in CMB and perturbations in the LSS [3, 6]. Thus, from these cosmological structures we can derive observables which are sensitive to phenomena in the inflationary era.

The central goal of this thesis is to evaluate the boundary four-point function of conformally coupled scalars with massive scalar exchange, by using symmetries of the de Sitter space and the singularities of the correlators without any reference to time-evolution following the paper [4]. The main text is divided into four chapters. In chapter 2, the immensely studied topic of scattering amplitudes is presented in a form which is more in par with our ultimate goal. In chapter 3, a very brief discussion of quantum fields in de Sitter field is given to show the differences with Minkowski space. In chapter 4, the time evolution picture is presented and some explicit calculations are done to evaluate the correlators. This will give us hindsight and intuition of what actually is happening in the boundary perspective (next chapter). Finally in chapter 5, the correlation functions are calculated by using symmetries and singularities. In the concluding chapter (6), we compare the de -Sitter solutions with our flat space scattering amplitude. We also explicitly write down what observable signatures we can get from these correlators and some relevant future prospects.

A number of appendices contain relevant technical details and derivations. In Appendix A, we derive the behaviour of the quantum fields at the future boundary of a de Sitter space from quasinormal modes in static patch. In Appendix B, we explicitly calculate the scalar mode functions in a de Sitter background. In Appendix C, we obtain the OPE limit of the four-point correlator and show how it factorizes into a product of three-point functions. In Appendix D, the singularity from the excited initial states is analyzed. In Appendix E, a brief discussion of how, in general, particles are produced in a time-dependent background is presented.
Notation and conventions

We follow the exact notation and conventions used by the paper [4]. Our metric signature is \((-, +, +, +)\). Throughout the paper, we used natural units, \(\hbar = c \equiv 1\). We denote massive scalar fields by \(\sigma\) and use \(\varphi\) for conformally coupled scalars. We use Greek letters for spacetime indices, \(\mu = 0, 1, 2, 3\), and Latin letters for spatial indices, \(i = 1, 2, 3\). Three-dimensional vectors will be denoted in boldface \((\mathbf{k})\) or with Latin superscripts \((k^i)\). The magnitude of vectors is defined as \(k = |\mathbf{k}|\). We will use the subscript ‘flat’ for Mandelstam variables in flat space to distinguish them from the exchange momenta \(s \equiv |k_1 + k_2|\) and \(t \equiv |k_2 + k_3|\).
Chapter 2
Scattering amplitudes in flat space

Since its inception the lagrangian formalism of QFT have been very successful in giving precise physical predictions. Having explicit correspondence with the classical approach and algorithmized by Feynman techniques, this new tool became all time favourite for particle physicists. However, we have to keep in mind that these lagrangians are not found in nature, instead they are thought of by physicists to model the machinery of our universe. It turns out that the convenience of describing nature by these man-made structures has its shortcomings. The lagrangian description is plagued by redundancies. As such, the lagrangian has no fundamental meaning. The fundamentality of a QFT theory can be attributed to invariant structures like the S-matrix (scattering amplitude). Now, the S-matrix is invariant under arbitrary field redefinitions of the form \[ \phi \rightarrow f(\phi) \]

Thus by applying any field redefinitions we can map one lagrangian to infinitely many other lagrangians describing essentially the same physics. To get rid of these redundancies, symmetries like gauge and diffeomorphism invariance are incorporated which end up making the calculations and physical interpretations more complicated.

In the recent decades, physicists have been focussing on the symmetries and singularities of a theory, which are the rudimentary structures. There is a huge progress in the understanding of these symmetries and singularities and how they dictate the analytic structure of our fundamental object, the S-matrix[8, 9, 10]. In the following, a brief discussion of these constraints and their implications are presented.

- **Lorentz invariance**: Being a fundamental object describing a theory, we expect the content of S-matrix to be same in any translated frame, rotated frame or any other inertial frame having a constant relative velocity (boosts). All these set of transformations can be compactly represented by Lorentz transformations. Thus our scattering amplitude must be Lorentz invariant. This implies that it can only be functions of some Lorentz invariant quantities which we take to be the Mandelstam variables \( s, t \) and \( u \).

- **Locality**: We further assume that our theory is local which means that physics at a spacetime point is affected only by the immediate neighbourhood and not
from any phenomena happening at large distances from that point. Locality is somewhat related to causality which states that information cannot travel through spacetime faster than light. For the small period of measurements we perform in our labs, causality thus ensures that our experiment has no bearings on happenings from very far away places which is nothing but the assumption of locality. Locality restricts the scattering amplitude to have only simple poles in Mandelstam variables up to tree-level\cite{11}. This pole singularity signals the exchange of an intermediate state over a macroscopic physical distance in spacetime.

- **Unitarity:** Unitarity is just the conservation of probability. In terms of scattering theory it states that the total probability for an arbitrary final state to arise from some initial states and vice-versa is unity. Unitarity implies that when the simple poles are approached, one or more particles go on-shell and the residue at these poles factorizes into two lower order amplitudes.

Now, armed with these powerful tools, let’s analyze the analytic structure of the four point scattering amplitude ($A_4$). The bootstrapping technology which sophisticatedly deal with this is spinor-helicity formalism\cite{9, 10}. We will not go to the details of that formalism, however, we will use it as a motivation to do the job. Let’s consider the four-point scattering amplitude of scalars of mass $m$ and dimension $d$. All other particles are taken to be significantly heavier than $m$. In all our discussions from here on, we will restrict ourselves up to the tree-level. For lower energies, the theory is described by the contact interactions which are characterized by the simplest singularity structure possible and thus corresponds to polynomial in Mandelstam variables. At higher energies, we have contributions from exchange of massive particles. For example, the existence of a particle of mass $M$ and spin $S$ will lead to poles at $s, t, u \rightarrow M^2$. From locality these poles are simple poles and unitarity causes the amplitude to factorize into three particle amplitudes $A_3^\lambda$ involving two external particles and the massive internal particle with helicity $\lambda$. Near s-channel, the amplitude can be written as

$$A_4(s, t, u) \xrightarrow{s \rightarrow M^2} \frac{1}{s - M^2} \sum_\lambda A_3^\lambda(p_1, p_2, p_I) A_3^{-\lambda}(p_1, p_2, -p_I) \quad (2.1)$$

Now, we just have to calculate the three-particle amplitudes which can also be done using our physical constraints, as done in these papers\cite{12, 13}. But we will now switch to our lagrangian description and calculate the amplitudes from Feynman techniques, simply because it is much more convenient and straightforward in this case.

For non-vanishing three-particle amplitude, the exchanged spin-$S$ field ($R_{\mu_1..\mu_S}$) must couple to a conserved current of the external particles. For complex scalars their is only one conserved current given by $\left(\left(\partial_\mu \phi^\dagger\right) \phi - \phi^\dagger \partial_\mu \phi\right)$. Thus the coupling lagrangian becomes\cite{14}

$$L_{\text{coupling}} = g \left[\left(\partial^{\mu_1} \phi^\dagger\right) \phi - \phi^\dagger \partial^{\mu_1} \phi\right] \cdots \left[\left(\partial^{\mu_S} \phi^\dagger\right) \phi - \phi^\dagger \partial^{\mu_S} \phi\right] R_{\mu_1..\mu_S} \quad (2.2)$$

Writing the fields in operator form with creation, annihilation operators and polarization tensors and acting with the in and out states, we get the amplitude to be
\[ A_4^\lambda(p_1, p_2, p_1) = g (p_1 - p_2)^{\mu_1} \cdots (p_1 - p_2)^{\mu_s} \epsilon_{\mu_1\cdots\mu_S}^\lambda (p_I) \] (2.3)

Thus, the helicity sum of eqn.(2.1) can be written as

\[ \sum_{\lambda} k^{\mu_1} \cdots k^{\mu_s} q^{\nu_1} \cdots q^{\nu_s} \epsilon_{\mu_1\cdots\mu_S}^\lambda \epsilon_{\nu_1\cdots\nu_S}^{-\lambda} \] (2.4)

where \( k \equiv p_1 - p_2 \) and \( q \equiv p_3 - p_4 \) and \( \epsilon_{\mu_1\cdots\mu_S}^\lambda \) is the polarization tensor of the spin-S particle. The polarization tensor can be written as a product of polarization vectors of spin-1 particles\[15\]-

\[ \epsilon_{\mu_1\cdots\mu_S}^\lambda = \prod_{\lambda_1, \ldots, \lambda_S = -1}^1 \delta_{\lambda_1 + \cdots + \lambda_S, \lambda} \times \sqrt{\frac{2^S (S + \lambda)! (S - \lambda)!}{(2S)! \prod_{i=1}^S (1 + \lambda_i)! (1 - \lambda_i)!}} \prod_{i=1}^S \epsilon_{\mu_i}^{\lambda_i} \] (2.5)

Some of the needed properties of the polarization tensor is highlighted below:

\[ p^\nu \epsilon_{\nu S}^{\mu_1 \cdots \mu_S} = 0 \quad \text{(transversality)} \] (2.6)

\[ \sum_{\nu} \epsilon_{\nu S}^{\mu_1 \cdots \mu_S} = 0 \quad \text{(tracelessness)} \] (2.7)

From transversality condition, we see that for a particle at rest \((p^\nu = (M, 0, 0, 0))\), the time component of the polarization tensor vanishes. To exploit this property, we evaluate the sum in the rest frame of the exchanged particle. Hence, the sum becomes

\[ P_{d,S}(k, q) \equiv \sum_{\lambda} k^{i_1} \cdots k^{i_S} q^{j_1} \cdots q^{j_S} \epsilon_{i_1 \cdots i_S}^{\lambda} \epsilon_{j_1 \cdots j_S}^{-\lambda} \equiv |k|^S |q|^S P_{d,S}(\cos \theta) \] (2.8)

where \( \cos \theta \equiv k \cdot q / |k| |q| \) and \( i \)'s and \( j \)'s are spatial indices. Now, we use the tracelessness property to get

\[ \sum_{i=1}^d \nabla_i^2 P_{d,S}(k, q) = \sum_{m, n} \sum_{\lambda} k^{i_m} \cdots k^{i_m+1} k^{i_{m+1}} \cdots k^{i_{n+1}} k^{i_{n+1}} \cdots k^{i_S} q^{j_1} \cdots q^{j_S} \epsilon_{i_1 \cdots i_S}^{-\lambda} \epsilon_{j_1 \cdots j_S}^{\lambda} \times \left( \sum_{i} \epsilon_{i_1 \cdots i_{m-1} i_{m+1} \cdots i_{n-1} i_{n+1} \cdots i_S}^{\lambda} \right) = 0 \]

The solution to this \( d \)-dimensional Laplace equation is

\[ \frac{1}{|k - q|^{d-2}} = \frac{1}{(|k|^2 - 2 |k| |q| \cos \theta + |q|^2)^{d/2-1}} = \frac{1}{|q|^{d-2}} \sum_{n=0}^{\infty} \left( \frac{|k|}{|q|} \right)^n C_n^{(d/2-1)}(\cos \theta) \] (2.9)

where \( C_n^{(d/2-1)}(\cos \theta) \) is the \( d \)-dimensional Gegenbauer polynomial. Matching the coefficient of \( |k|^S \), we see that the polynomial \( P_{d,S} \) is essentially a Gegenbauer polynomial. Thus our four-point function becomes

\[ A_4 = g^2 \frac{|k|^S |q|^S}{S - M^2} P_{d,S}(\cos \theta) \] (2.10)
The final step is to connect the above quantities to the Lorentz invariance ones. For that we make the observation that the rest frame of the massive particle can be taken as the centre of mass (CM) frame of the system. The configuration is shown as above in figure 2.1.

In CM frame we have

\[ s \equiv -(p_1 + p_2)^2_{\text{CM-frame}} = (E_1 + E_2)^2 = (E_3 + E_4)^2 \]  

(2.11)

Using \( E_1 = E_2 \) (due to \( m_1 = m_2 = m \)), we get

\[ s = 4E_1^2 = 4p_1^2 + 4m^2 \Rightarrow |p_1| = \frac{1}{2} \sqrt{s - 4m^2} \]

Similarly, for the other momenta we have

\[ |p_2| = |p_3| = |p_4| = \frac{1}{2} \sqrt{s - 4m^2} \]

\[ \therefore |p_1 - p_2| = |p_3 - p_4| = \sqrt{s - 4m^2} = |k| = |q| \]  

(2.12)

Also, from the exchange Mandelstam variable \( t \), we get

\[ t = -(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2E_1E_3 + 2 |p_1| |p_3| \cos \theta \]

Putting all the relevant values, we get the expression for \( \cos \theta \) as

\[ \cos \theta = 1 + \frac{2t}{s - 4m^2} \]  

(2.13)

and thus get the final expression for amplitude for \( s \)-channel. Similar evaluation can be done for \( t \) and \( u \)-channel. Along with the contact terms, we get the most general form of the analytic structure of the four-particle amplitude (upto tree-level)-

\[ A_4(s, t, u) = g^2 \left( \frac{4m^2 - M^2}{s - M^2} \right)^s P_{d, s} \left( 1 + \frac{2t}{M^2 - 4m^2} \right) + \text{t- and u-channels+contact} \]

Thus, the symmetries and singularities can be used to fix the structure of the amplitude near physical poles in flat space (for a more formal treatment see [13]). Before going to the details of how a similar logic could be implemented in determining the correlators in de Sitter space; we briefly review the characteristics of quantum fields in de Sitter space in the next chapter.
Chapter 3

Quantum fields in de Sitter space

In flat slicing, the metric of the four-dimensional dS spacetime can be written as

\[ ds^2 = -d\eta^2 + \frac{dx^2}{(H\eta)^2} \] (3.1)

where \( H \) is the Hubble scale and \( \eta = \int \frac{dt}{a} \) is the conformal time (\( t \) is the proper or cosmic time). To avoid clutter, we will take \( H = 1 \) for now. We will retain \( H \) while doing explicit calculations in Chapter 3. As is evident from the above expression, the metric is manifestly invariant under spatial translations and rotations. The less obvious isometries are the dilation and special conformal transformation (SCTs), whose associated Killing vectors are

\[ D \equiv -\eta \partial_\eta - x^i \partial_{x^i} \] (3.2)

\[ K^i \equiv 2x^i \eta \partial_\eta + 2x^i x^j \partial_{x^j} + \left( \eta^2 - |x|^2 \right) \partial_{x^i} \] (3.3)

In most of our study, we will be interested in the late time limit of the correlators where \( \eta \to 0 \) keeping \( x \) fixed. At this future boundary, a massive scalar field \( \phi(\eta, x) \) can be written as a sum of exponentials of proper time (\( e^{-\Delta t} \) or \( \eta^\Delta \))

\[ \lim_{\eta \to 0} \phi(\eta, x) = O^+(x)\eta^{\Delta^+} + O^-(x)\eta^{\Delta^-} \] (3.4)

where the scaling dimensions are

\[ \Delta^\pm = \frac{3}{2} \pm i\mu \quad , \quad \mu \equiv \sqrt{m^2 - \frac{9}{4}} \] (3.5)

This is related to the quasi normal mode behaviour in the static patch (more on this in Appendix A).

The action of the generators (3.2) and (3.3) on the boundary operators \( O^{\pm}(x) \) become

\[ D \equiv -\Delta^\pm - x^i \partial_{x^i} \] (3.6)

\[ K^i \equiv 2x^i \Delta^\pm + 2x^i x^j \partial_{x^j} - |x|^2 \partial_{x^i} \] (3.7)

which are the generators of the three dimensional conformal group. Thus we see that de Sitter isometries act on the late-time expectation conformal values in the same way as the...
Chapter 3: Quantum fields in de Sitter space

To exploit the translation invariance, we will work in Fourier space where this translational invariance translates to momentum conservation. Thus, any correlator in Fourier space can be written as

\[ \langle O_1 \cdots O_N \rangle = (2\pi)^3 \delta^3 \left( \sum_{i=1}^{n} k_i \right) \langle O_1 \cdots O_N \rangle' \]  

where we have defined \( O_n \equiv O(k_n) \) and the term \( \langle \cdots \rangle' \) is written by stripping off the three-dimensional delta function.

There is another advantage of working in Fourier space. Observations in cosmology is made difficult by the fact that we observe only one universe (cosmic variance), hence we don’t have statistical data for different outcomes. This problem can be partially solved by taking averages on different spatial patches of the sky. When we are computing correlators in Fourier space we are doing this averaging already. The Fourier transforms of the generators (3.6) and (3.7) are

\[ D \equiv - (\Delta^\pm - 3) + k^i \partial_{k^i} \]  

\[ K^i \equiv 2 (\Delta^\pm - 3) \partial_{k^i} - 2k^j \partial_{k^j} \partial_{k^i} + k^i \partial_{k^j} \partial_{k^j} \]  

which are the generators acting on \( O^\pm(k) \), Fourier transform of \( O(x) \).

**Fourier transform:** The appearance of 3 in the above expressions is a little bit subtle. Let us denote the Fourier transform of a function \( f(x) \) by \( \mathcal{F}[f(x)] \). We know that the Fourier transforms of \( x \) and \( \partial_x \) are given by

\[ \mathcal{F}[xf(x)] = -i \partial_x \mathcal{F}[f(x)] \quad , \quad \mathcal{F}[\partial_x f(x)] = -ik \mathcal{F}[f(x)] \]  

Using the above equations we get

\[ \mathcal{F}[x^i \partial_x^i f(x)] = -i \partial_{k^i} \mathcal{F}[\partial_x f(x)] = -\partial_{k^i} k^i \mathcal{F}[f(x)] = (-3 - k^i \partial_{k^i}) \mathcal{F}[f(x)] \]  

We will mostly be interested in the correlation functions of conformally coupled scalars, for which \( m = \sqrt{2}H \) or \( \Delta^- = 2 \). The reason being that for these scalars many of the analytical calculations are simple and clean. Also, they act as an building block from which many other cases can be derived as we will touch upon at the end of this excerpt (see Chapter 6). For conformally coupled scalars, \( \mu \) is imaginary and thus the first part of the solution (3.4) dominates for \( \eta \to 0 \). But we will work with \( O^- \) (and drop the superscript) as it is easy to handle due to some convenient cancellations for \( \Delta^- = 2 \). The correlation functions of \( O^+ \) can be simply obtained from those of \( O^- \) by simple momentum-dependent rescaling-

\[ \langle O_1^+ \cdots O_N^+ \rangle = \frac{\langle O_1^- \cdots O_N^- \rangle}{(k_1 k_2 \cdots k_N)^{2\Delta^-}} \]  

Quantum correlators must be invariant under the action of the isometries:

\[ \delta \langle O_1 \cdots O_N \rangle = \sum_{n=1}^{N} \langle O_1 \cdots \delta O_n \cdots O_n \rangle = 0 \]  

13
where

\[ \delta O_n \propto G O_n \]  

(3.15)

where \( G \) is the generator of the transformation. If we denote the generators corresponding to the operators \( O_n \) as \( D_n \) and \( K_n \), then we can write the invariance as

\[ D \langle O_1 \cdots O_N \rangle = \sum_n D_n \langle O_1 \cdots O_N \rangle = \sum_{n=1}^N \left( - (\Delta_n - 3) + k_n^i \partial_{k_n^i} \right) \langle O_1 \cdots O_N \rangle = 0 \]  

(3.16)

\[ K^i \langle O_1 \cdots O_N \rangle = \sum_n K^i_n \langle O_1 \cdots O_N \rangle \]

\[ = \sum_{n=1}^N \left( 2 (\Delta_n - 3) \partial_{k_n^i} - 2 k_n^j \partial_{k_n^j} \partial_{k_n^i} + k_n^i \partial_{k_n^j} \partial_{k_n^j} \partial_{k_n^i} \right) \langle O_1 \cdots O_N \rangle \]

(3.17)

= 0

These are essentially the conformal Ward identities. As we will see in the later chapters, these constraint equations highly restrain the structure of the correlation functions, to the point that two and three-point correlators are completely determined from them.


Chapter 4

Cosmological correlators from bulk perspective

The correlators calculated from a cosmological perspective are, for an obvious reason, called cosmological correlators. Calculating cosmological correlators is different from the corresponding analysis (evaluating the S-matrix) in our familiar QFT, routinely done by the particle physicists. These differences are highlighted below:

- In particle physics we are interested in calculating the S-matrix, which is the transition amplitude for a state $|in\rangle$ from a distant past to evolve into a state $|out\rangle$ in the far future.

$$\langle out | S | in \rangle = \langle out (+\infty) | in (-\infty) \rangle$$

While, in cosmology, we want to evaluate the expectation values of the product of operators at a fixed time, between two in-states.

$$\langle Q \rangle = \langle in | Q(\tau) | in \rangle$$

- Boundary conditions are imposed at very early and late times in S-matrix theory when the scattering particles are very far from the interaction region. The asymptotic states are taken to be the vacuum states of the free Hamiltonian $H_0$. On the contrary, boundary conditions are only imposed at very early times for the cosmological correlators, when the wavelength is deep inside the horizon and the interaction picture fields should have the same form as in Minkowski spacetime. Thus, the in-state is taken to be the vacuum state of the interacting theory in the far past. This will lead to the definition of the Bunch-Davis vacuum which we will see later.

Figure 4.1 schematically show the features of both the formalism. The task of calculating cosmological correlators is best done by something called the in-in formalism, also known as the Schwinger-Keldysh formalism[16, 17, 18] (for more recent reviews see [19, 20, 21]). We briefly discuss the salient features of this formalism below and then go on to explicitly calculate some correlation functions using this formalism.
(a) in-out transition amplitude

(b) in-in expectation values

Figure 4.1: Schematics to illustrate the salient features of the two formalism. Particle physicists primarily compute the in-out amplitude while cosmologists work in the in-in formalism.

4.1 The in-in formalism

The in-in formalism described here is followed from Weinberg’s presentation [17]. Let us denote the Hamiltonian of the system by

$$H[\phi(x,t),\pi(x,t)] \equiv \int d^3x \mathcal{H}[\phi(x,t),\pi(x,t)]$$ (4.1)

where, as we can see, the Hamiltonian is a functional of the fields $\phi$ and their conjugate momenta $\pi$. In Heisenberg picture, the evolution of the fields is generated by $H$ as

$$\dot{\phi}(x,t) = i[H,\phi] \quad , \quad \dot{\pi}(x,t) = i[H,\pi]$$ (4.2)

As we will be working on a cosmological setting, the background metric in general will be time-dependent. Thus we assume the existence of a time-dependent c-number solution\(^1\), $\bar{\phi}$ and $\bar{\pi}$, satisfying the classical equations of motion

$$\dot{\bar{\phi}}(x,t) = \frac{\partial \mathcal{H}}{\partial \bar{\pi}} \quad , \quad \dot{\bar{\pi}}(x,t) = -\frac{\partial \mathcal{H}}{\partial \bar{\phi}}$$ (4.3)

We now expand around these solution

$$\phi(x,t) \equiv \bar{\phi}(x,t) + \delta\phi(x,t) \quad , \quad \pi(x,t) \equiv \bar{\pi}(x,t) + \delta\pi(x,t)$$ (4.4)

Correspondingly, we expand the Hamiltonian

$$H[\phi,\pi] = H[\bar{\phi},\bar{\pi}] + \int d^3x \frac{\partial \mathcal{H}}{\partial \phi} \delta\phi + \int d^3x \frac{\partial \mathcal{H}}{\partial \pi} \delta\pi + \tilde{H}[\delta\phi,\delta\pi; t]$$ (4.5)

where we accumulated all the quadratic and higher order terms in perturbations under the $\tilde{H}$ term. We will call this the fluctuation Hamiltonian. We see that the way we constructed $\tilde{H}$ give it an explicit time-dependence.

Putting (4.4), (4.5) in (4.2) and using the classical equation of motions (4.3), we get the evolution equation of the perturbations as

$$\delta\dot{\phi}(x,t) = i[\tilde{H},\delta\phi] \quad , \quad \delta\dot{\pi}(x,t) = i[\tilde{H},\delta\pi]$$ (4.6)

\(^1\)In cosmology, $\bar{\phi}$ would describe the Robertson–Walker metric and the expectation values of various scalar fields.
Thus, we see that the perturbations are evolved by $\hat{H}$. The time evolution is made complicated by the interactions inside $\hat{H}$, which lead to non-linear equation of motion. Thus, we introduce the interaction picture in which we split the fluctuation Hamiltonian into the quadratic part $H_0$ (kinetic term) and the interaction Hamiltonian $H_{\text{int}}$ (containing all the higher order terms starting from the cubic term).

$$\hat{H}[\delta\phi(t), \delta\pi(t); t] = H_0[\delta\phi(t), \delta\pi(t); t] + H_{\text{int}}[\delta\phi(t), \delta\pi(t); t]$$

*Note: In our familiar QFT, such a split would generally involved $H_0$ being time-independent with $H_{\text{int}}$ carrying the small time dependence. But in the in-in formalism both of them have explicit time dependence which is why this formalism is favourable for our cosmological calculations which generally have time-dependent background.*

In this interaction picture, the leading time-dependence of the interaction picture fields $\phi^I$ is determined by the quadratic Hamiltonian $H_0$ (which gives linear equation of motion)

$$\dot{\phi}^I(x, t) = i[H_0^I, \phi^I] \quad, \quad \dot{\pi}^I(x, t) = i[H_0^I, \pi^I] \quad (4.7)$$

where $H_0^I \equiv H_0[\phi^I(t), \pi^I(t); t]$ and the initial conditions

$$\phi^I(t_0) = \delta\phi(t_0) \quad, \quad \pi^I(t_0) = \delta\pi(t_0) \quad (4.8)$$

The solution to the above equation can be written in Fourier space (in quantized form) as

$$\phi_k^I(t) = \phi_k(t)\hat{a}_k + \phi_k^*(t)\hat{a}_k^\dagger \quad (4.9)$$

where the mode functions (we will see some explicit examples in Appendix B), $\phi_k(t)$, are the solutions of the free-field equation and the operators $\hat{a}_k$ define the free-field vacuum. Corrections to the evolution of the operators can then be treated perturbatively in $H_{\text{int}}$. Using the evolution operators from (4.6) and (4.7) we can express operators in the Heisenberg picture ($Q(t)$) in terms of operators in the interaction picture ($Q^I(t)$)

$$Q(t) = F^{-1}(t, t_i)Q^I(t)F(t, t_i)$$

where

$$F(t, t_i) \equiv Te^{-i\int_{t_i}^{t} H_{\text{int}}(v) dv} \quad (4.10)$$

where $H_{\text{int}}^I(t) \equiv H_{\text{int}}[\phi^I(t), \pi^I(t); t]$ and $T$ denotes time ordered product. The operator $F$ can be thought of as the evolution operator for quantum states in the interaction picture.

$$|\Omega(t)\rangle = F(t, t_i) |\Omega(t_i)\rangle$$

where $|\Omega(t_i)\rangle \equiv |\Omega\rangle$ is the vacuum of the interacting theory. To relate this to the vacuum of the free field theory $|0\rangle$, we will use a little trick. Inserting a complete set of energy eigenstates ($|\Omega\rangle$, $|n\rangle$) of the full theory, where $|n\rangle$ are the excited states, we have

$$|0\rangle = |\Omega\rangle \langle\Omega|0\rangle + \sum_n |n\rangle \langle\Omega|n\rangle$$

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and correspondingly
\[ e^{-i\hat{H}(t-t_i)}|0\rangle = e^{-i\hat{H}(t-t_i)}|\Omega\rangle \langle \Omega| 0 \rangle + \sum_n e^{-iE_n(t-t_i)}|n\rangle \langle n| \Omega| 0 \rangle. \]

Adding a small imaginary part to the initial time and taking the early time to negative infinity, \( t_i \to -\infty (1 - i\epsilon) \equiv -\infty^- \), will dampen the excited states, \( e^{-iE_n(t-t_i)} \to 0 \). We are then left with only the interacting vacuum. Thus, this \( i\epsilon \) prescription has effectively turned off the interaction at early times and project the interacting vacuum onto the free vacuum. Finally we get our working formula:

\[ \langle Q(t) \rangle = \langle 0 | \hat{T} e^{i \int_{-\infty}^t dt' H_{\text{int}}^I(t')} Q^I(t) \hat{T} e^{-i \int_{-\infty}^t dt' H_{\text{int}}^I(t')} |0 \rangle. \] (4.11)

Physically, the expression is simple to interpret and we can do the interpretation in two ways. First we can think of the time ordered exponential to evolve the vacuum from the initial time to the current time of evaluation \( t \). The anti-time ordered product do the same for the hermitian conjugate of the vacuum. Finally we sandwich the operator between this two vacuum states (which are at time \( t \) now) giving the expectation value of the operator at time \( t \). Another way to think of this is that we evolve the operator itself back to the initial time where we sandwich the operator with the initial vacuum to get the expectation value.

Using (4.11), we can calculate correlation functions upto the required order in perturbations of \( H_{\text{int}} \). Let’s first write down the expansion of the (anti-) time ordered exponential as\(^2\)

\[
T e^{-i \int_{-\infty}^t dt' H_{\text{int}}^I(t')} = 1 - i \int_{-\infty}^t dt' H_{\text{int}}^I(t') + (-i)^2 \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' H_{\text{int}}^I(t') H_{\text{int}}^I(t'') + \ldots
\]

\[
\bar{T} e^{i \int_{-\infty}^t dt' H_{\text{int}}^I(t')} = 1 + i \int_{-\infty}^t dt' H_{\text{int}}^I(t') + (i)^2 \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' H_{\text{int}}^I(t') H_{\text{int}}^I(t'') + \ldots
\]

The expectation values of the correlation functions can then be written, upto second order, as

\[
\langle Q(t) \rangle = \langle Q^I(t) \rangle - i \int_{-\infty}^t dt' \langle [Q^I(t), H_{\text{int}}^I(t')] \rangle + \frac{(+i)(-i)}{2} \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle (H_{\text{int}}^I(t')Q^I(t)H_{\text{int}}^I(t'') + H^I(t'')Q^I(t)H_{\text{int}}^I(t')) \rangle + (i)^2 \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle (Q^I(t)H_{\text{int}}^I(t'')H_{\text{int}}^I(t') + H_{\text{int}}^I(t'')H_{\text{int}}^I(t')Q^I(t)) \rangle
\] (4.12)

It is evident that the leading terms for the exchange interactions are the last four 2nd-order terms. Let us now calculate some four point correlation function (which are the main topic of our interest) using this formalism.

\(^2\)As in the R.H.S of the following equation, the extra \( i\epsilon \) will be suppressed with the understanding that the integration contours are appropriately deformed
Chapter 4: Cosmological correlators from bulk perspective

4.2 Contact interactions in de Sitter space

In this section we calculate the four point functions of conformally coupled scalars, $\varphi$ (for which $m_\varphi = \sqrt{2}H$) arising from contact interaction of the kind $\lambda \varphi^4$. For contact interactions the leading order contribution comes from the second term of (4.12). Our interaction Hamiltonian can be written in terms of the Lagrangian as,

$$H_{int} \equiv \int d^3 x a^3 \lambda \varphi^4,$$

where $a(\eta)$ is the scale factor for the background metric as a function of conformal time $\eta$.

We will work in Fourier space where the interaction Hamiltonian takes the form

$$H_{int}^I(\eta) = \frac{\lambda}{H^4 \eta^4} \int \prod_{i=1}^4 \frac{d^3 p_i}{(2\pi)^3} \varphi^I p_1^I \varphi^I p_2^I \varphi^I p_3^I \varphi^I p_4^I (2\pi)^3 \delta^3 (p_1 + p_2 + p_3 + p_4)$$

(4.13)

where $\varphi^I_p(\eta)$ are the fields in Fourier space. In quantized form, they can be decomposed as in (4.9) where the mode functions for conformally coupled scalars are given by (see Appendix B for its derivation)

$$\varphi_k(\eta) = (-H \eta) \frac{e^{-ik\eta}}{\sqrt{2k}}$$

(4.14)

Thus, in Fourier space, our four point function can be written as

$$\langle \varphi \varphi \varphi \varphi \rangle = -i \int_{-\infty}^{\eta_0} d\eta \langle [\varphi^I_{k_1}(\eta_0) \varphi^I_{k_2}(\eta_0) \varphi^I_{k_3}(\eta_0) \varphi^I_{k_4}(\eta_0), H_{int}^I(\eta)] \rangle$$

(4.15)

where the future boundary is taken to be $\eta_0$ (we take it to be very close to zero, instead of zero). This is to ensure that the mode-functions doesn’t vanishes. It can be the end of inflation. To evaluate the integral, we need to calculate the (anti-) time ordered products of the operators sandwiched between the vacuum. In contrast to the Minkowski space, in time-dependent background, we can’t use a single Feynman
propagator to take care of the time ordering. Thus we first simplify the integrands by something called contraction and leave the complication of time ordering to the final integration.

**Contraction:** A contraction between two fields is defined as the non-zero commutator between the following components of the two fields, $[\phi^+_A, \phi^-_B]$, where $\phi^+_A$ and $\phi^-_A$ are the first and second term on the right hand side of (4.9), respectively. Thus we have

$$\phi_{k,A}(\eta)\phi_{p,B}(\eta') = \langle \phi_{k,A}\phi_{p,B} \rangle = [\phi^+_k, \phi^-_p] = \phi_{k,A}(\eta)\phi^+_p(\eta') \left[ a_k, a^+_p \right]$$  \hspace{1cm} (4.16)

After normal ordering, namely shifting the annihilation operators (by repeated use of the commutation relations) to the right-most so that they give zeros hitting the vacuum, it is not difficult to convince oneself that the only contributing terms are those with all fields contracted. Feynman diagrams can be used to keep track of what contractions are necessary.

We show the explicit calculation for one of the terms in the commutator for one of the contractions.

$$\varphi_{k_1}(\eta_0)\varphi_{k_2}(\eta_0)\varphi_{k_3}(\eta_0)\varphi_{k_4}(\eta_0)\varphi_{p_1}(\eta)\varphi_{p_2}(\eta)\varphi_{p_3}(\eta)\varphi_{p_4}(\eta)$$

$$= \varphi_{k_1}(\eta_0)\varphi_{p_1}(\eta)\varphi_{k_2}(\eta_0)\varphi_{p_2}(\eta)\varphi_{k_3}(\eta_0)\varphi_{p_3}(\eta)\varphi_{k_4}(\eta_0)\varphi_{p_4}(\eta)$$

$$= \prod_{i=1}^4 (2\pi)^3\delta^3(k_i + p_i) \varphi_{k_1} \varphi_{p_1} \varphi_{k_2} \varphi_{p_2} \varphi_{k_3} \varphi_{p_3} \varphi_{k_4} \varphi_{p_4}$$  \hspace{1cm} (4.17)

$$= (H\eta_0)^4(H\eta)^4 \prod_{i=1}^4 (2\pi)^3\delta^3(k_i + p_i) e^{ik_1\eta} e^{ik_2\eta} e^{ik_3\eta} e^{ik_4\eta} \frac{e^{ik_1\eta} e^{ik_2\eta} e^{ik_3\eta} e^{ik_4\eta}}{2k_1 2k_2 2k_3 2k_4}$$

The delta functions in the third line after contraction are coming from the canonical commutation relations, $[a_k, a^+_p] = (2\pi)^3\delta^3(k + p)$. In the fourth line we have used the expression for mode functions and have taken the leading order term in $\eta_0$, neglecting the exponentials $e^{-ik_0\eta}$. Putting all this together and doing the momentum integrals in $H_{int}$, we get

$$\langle \varphi \varphi \varphi \rangle = \frac{i}{16k_1 k_2 k_3 k_4} \left[ \int_{-\infty}^{0} d\eta e^{ik_1\eta} - \int_{-\infty}^{0} d\eta e^{-ik_1\eta} \right] (2\pi)^3\delta^3 \left( \sum_{i=1}^4 k_i \right)$$

$$\langle \varphi \varphi \varphi \rangle' = \frac{(H\eta_0)^4}{8kk_2k_3k_4} \frac{\lambda}{k_i}$$  \hspace{1cm} (4.18)

In the second line, we have evaluated the integrals keeping in mind the $i\epsilon$ prescription and we have written the correlator stripping off the momentum-conserving delta function. It is evident form the above expression that we encounter a singularity when $k_i \to 0$, where $k_i = \sum_i k_i$. This can’t happen for real momenta values, but we can access the singularity by analytically continuing the momenta. We see that
the residue at this singularity is proportional to the flat space four point scattering amplitude, \( A_{\text{flat}} \sim \lambda \). Now, we turn to the four-point correlations for exchange interactions, which is a little bit more complicated.

### 4.3 Exchange interactions of scalars in de Sitter space

In this section, we will consider the contribution to the four point function of conformally coupled scalars from the exchange interaction \( \mathcal{L} = g \varphi^2 \sigma \). The mode function for the massive exchange scalar is given by (see Appendix B)

\[
\sigma_k(\eta) = \frac{H \sqrt{\pi}}{2} e^{i\pi/4} e^{-\pi \mu/4} (-\eta)^{3/2} H^{(1)}_{i\mu}(-k\eta) \rightarrow (-H\eta)^{\frac{e^{-ik\eta}}{\sqrt{2k}}} \quad (4.19)
\]

where \( H^{(1)}_{i\mu} \) is the Hankel function of first kind and \( \mu \) is essentially the mass of the exchange scalar. The leading order contribution for the exchange interaction comes from the second order terms in the perturbation. There are four of these. Let’s first look at the \( H(\eta)Q(\eta_0)H(\eta') \) term. We will follow the same procedure as done while calculating the contact term. On contraction, with Feynman diagram in mind, the integrand becomes

\[
\begin{align*}
\varphi^I_{p_3}(\eta) \varphi^I_{k_1}(\eta_0) \varphi^I_{p_2}(\eta) \varphi^I_{k_2}(\eta_0) \varphi^I_{q_3}(\eta') \varphi^I_{k_3}(\eta_0) \varphi^I_{q_4}(\eta') \sigma^I_{p_3}(\eta) \sigma^I_{q_3}(\eta') \\
= H^8 \eta_0^4 \eta^2 \eta'^2 \prod_{i=1}^{2} (2\pi)^3 \delta^3 (p_i + p_i) \prod_{i=1}^{2} (2\pi)^3 \delta^3 (k_i + q_i) \frac{e^{-i k \eta} e^{-i k \eta'}}{2k_1} \frac{e^{i k \eta} e^{i k \eta'}}{2k_2} \frac{e^{i k \eta} e^{i k \eta'}}{2k_3} \frac{e^{i k \eta} e^{i k \eta'}}{2k_4} \\
\times \sigma_{p_3}(\eta) \sigma_{q_3}^*(\eta')(2\pi)^3 \delta^3(p_3 + q_3)
\end{align*}
\]

Putting the above expression into the integral and doing some straightforward simplifications we get

\[
g^2 \left( -i \right) \left( +i \right) \frac{\eta_0^8}{16 k_1 k_2 k_3 k_4} \left[ \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{-i k \eta} \int_{-\infty}^{0} \frac{d\eta'}{\eta'^2} e^{i k \eta'} G_{++}(s, \eta, \eta') \right] \quad (4.21)
\]

where we defined, \( k_{12} \equiv k_1 + k_2, k_{34} \equiv k_3 + k_4, s \equiv |k_1 + k_2| = |k_3 + k_4| \) and \( G_{+-} \equiv \sigma_{p_3}(\eta) \sigma_{q_3}^*(\eta') \).

Similarly, we evaluate the other terms. For the \( QHH \) and \( HHQ \) terms we use the fact that

\[
\int_{-\infty}^{0} d\eta \int_{-\infty}^{0} d\eta' F(\eta, \eta') = \int_{-\infty}^{0} d\eta \int_{-\infty}^{0} d\eta' F(\eta, \eta') \Theta(\eta - \eta')
\]

After some algebraic heavy lifting, we arrive at the final expression for the four-point function for massive scalar (\( \sigma \)) exchange in de Sitter space

\[
\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle = \frac{\eta_0^4}{16 k_1 k_2 k_3 k_4} F(k_{12}, k_{34}, s) \quad (4.22)
\]
where we have defined

$$F \equiv F_{++} + F_{+-} + F_{-+} + F_{--}$$

(4.23)

$$F_{\pm \pm} \equiv g^2 \frac{(\pm i)(\pm i)}{2} \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{\pm i k_{12} \eta} \int_{-\infty}^{0} \frac{d\eta'}{\eta'^2} e^{\pm i k_{34} \eta'} G_{\pm \pm}(s, \eta, \eta')$$

(4.24)

where

$$G_{++}(s, \eta, \eta') = \sigma_s(\eta) \sigma_s^*(\eta') \Theta(\eta - \eta') + \sigma_s^*(\eta) \sigma_s(\eta') \Theta(\eta' - \eta)$$

(4.25)

$$G_{+-}(s, \eta, \eta') = \sigma_s(\eta) \sigma_s(\eta')$$

(4.26)

$$G_{-+}(s, \eta, \eta') = \sigma_s^*(\eta) \sigma_s^*(\eta')$$

(4.27)

$$G_{--}(s, \eta, \eta') = \sigma_s(\eta) \sigma_s(\eta') \Theta(\eta' - \eta) + \sigma_s^*(\eta) \sigma_s^*(\eta') \Theta(\eta - \eta')$$

(4.28)

The functions $G_{++}$ and $G_{--}$ satisfy the inhomogeneous equation

$$3 \left( \eta^2 \partial_\eta^2 - 2\eta \partial_\eta + s^2 \eta^2 + \frac{m^2}{H^2} \right) G_{\pm \pm}(s, \eta, \eta') = -H^2 \eta^2 \eta'^2 \delta(\eta - \eta')$$

(4.29)

and a similar equation for $\eta'$, while the $G_{\pm \mp}$ satisfy the corresponding homogeneous equation. Here $G_{\pm \pm}$ are thus Green's function involving the massive scalars. In contrast to our familiar QFT, where we could do with only one time-ordered Feynman propagator; in the in-in formalism we need all the four Green’s function. Using the equations of motion for the Green’s function, it can be shown that the function $F$ obeys the differential equation

$$\left( (k_{12}^2 - s^2) \partial_{k_{12}}^2 - 2k_{12} \partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) F = H^2 g^2 \frac{\eta^2 \eta'}{k_{12}}$$

(4.30)

and a similar one with $k_{34}$. Changing the variables to $u \equiv \frac{s}{k_{12}}$, $v \equiv \frac{s}{k_{34}}$ and $\hat{F} \equiv sF$, we get

$$\left[ \Delta_u + \mu^2 + \frac{1}{4} \right] \hat{F} = H^2 g^2 \frac{\eta^2 \eta'}{u + v}$$

(4.31)

where $\Delta_u \equiv u^2(1 - u^2) \partial_u^2 - 2u^3 \partial_u$. There is a similar equation like (4.31) with $u \leftrightarrow v$, coming from the equation for $\eta'$ corresponding to equation (4.29). We can think of (4.31) as tracking the evolution in $\eta$ purely in boundary terms, while the corresponding equation in terms of $v$ tracks the evolution in $\eta'$. The fact that this two histories are consistent, and give the same four-point function, is made manifest in the bulk picture, but is nonetheless a non-trivial property of the solution. Thus, we have

$$\hat{F}(u, v) = \hat{F}(v, u).$$

(4.32)

Later, we will again derive equation (4.31) purely from boundary perspective. Also, instead of doing the integrals, we will solve this differential equation in the next chapter to get the four-point exchange correlators.

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3This can be easily derived from the Klein-Gordon equation of motion in de Sitter spacetime. We have to use the Laplace-Beltrami operator instead of the usual Laplace operator.
Singularities at $u \to -1$

We will now examine the behaviour of $\hat{F}_{++}$ and $\hat{F}_{--}$ near $u = -1$ for $|v| < 1$. From (4.24), we get the expressions for both to be

$$
\hat{F}_{\pm\pm} = -\frac{s}{2} \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{\pm ik_{12} \eta} \sigma_s(\eta) \int_{-\infty}^{0} \frac{d\eta'}{\eta'^2} e^{\pm ik_{34} \eta'} \sigma_s^*(\eta') + (k_{12} \leftrightarrow k_{34}).
$$

(4.33)

From the above expression it is evident that we need to evaluate the basic integral:

$$
I_0(a, b) = \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{-ia\eta} \sigma_b(\eta).
$$

(4.34)

This integral is evaluated in Appendix B of [4] and the answer which is regular at $a = b$ is given by

$$
I_0(a, b) = -\frac{1}{\sqrt{2b} \cosh \pi \mu} \frac{1}{\Gamma(1/2 - i\mu) \Gamma(1/2 + i\mu)} \log \left( \frac{a + b}{2b} \right).
$$

(4.35)

As can be seen from the above expression the hypergeometric function is zero-balanced. We have $\frac{b - a}{2b} = 1 - \frac{a + b}{2b}$ and thus for $a + b \to 0$, the $z$-argument of the hypergeometric function approaches 1. From the properties of the zero-balanced hypergeometric functions, in the limit $a + b \to 0$, we can then write

$$
\lim_{a + b \to 0} I_0(a, b) = -\frac{1}{\sqrt{2b} \cosh \pi \mu} \frac{1}{\Gamma(1/2 - i\mu) \Gamma(1/2 + i\mu)} \log \left( \frac{a + b}{2b} \right).
$$

(4.36)

For $a = k_{12}$ and $b = s$ we get the singularity at $u \to -1$. Inspecting (4.33), we see that the first term is non-singular for generic $v \neq \pm 1$; while the inner integral of the second term picks up a singular contribution at $u = -1$, rendering the integrals in a factorized form. Noting $\Gamma(1/2 - i\mu) \Gamma(1/2 + i\mu) = \frac{\pi}{\cosh \pi \mu}$ and using (4.35) and (4.36) in (4.33) we get

$$
\lim_{u \to -1} \hat{F}_{++} = -i \left[ e^{i\mu} \hat{J}_+(v) + e^{-i\mu} \hat{J}_-(v) \right] \log(1 + u).
$$

(4.37)

In order to compare with the boundary calculations in the next chapter, we use the formulas relating the hypergeometric functions at different values of the argument [22] (see Appendix F of [4]) to change the $z$-argument from $v + 1/2v$ to $v^2$:

$$
\lim_{u \to -1} \hat{F}_{++} = -i \left[ e^{i\mu} \hat{J}_+(v) + e^{-i\mu} \hat{J}_-(v) \right] \log(1 + u)
$$

(4.38)

for $v > 0$ and where we defined

$$
\hat{J}_\pm(v) = \frac{\Gamma \left( \frac{1}{4} \pm i\mu \right) \Gamma(\mp i\mu)}{4\sqrt{\pi}} \left( \frac{v}{2} \right)^{\frac{1}{4} \pm i\mu} 2F_1 \left[ \frac{1}{4} \pm i\mu, \frac{3}{4} \pm i\mu; 1 \pm i\mu; v^2 \right].
$$

(4.39)

The singular contribution from $\hat{F}_{--}$ is just the complex conjugate of (4.38):

$$
\lim_{u \to -1} \hat{F}_{--} = i \left[ e^{-i\mu} \hat{J}_+(v) + e^{i\mu} \hat{J}_-(v) \right] \log(1 + u).
$$

(4.40)
The singular behaviour of $F_{\mp\mp}$ can be obtained from the results for $F_{\pm\pm}$ by the analytic continuation $v \to -v$. This gives

$$\lim_{u \to -1} \hat{F}_{\mp\mp} = -\frac{\pi}{4 \cosh \mu} {}_2F_1\left[\frac{1}{2} - i\mu, \frac{1}{2} + i\mu \mid \frac{v - 1}{2v}\right]$$

$$= -\left[\hat{J}_+(v) + \hat{J}_-(v)\right] \log(1 + u). \quad (4.41)$$

where again $v > 0$. Putting all these together we get

$$\lim_{u \to -1} \hat{F} = -2 \left[(1 + i \sinh \mu)\hat{J}_+(v) + (1 - i \sinh \mu)\hat{J}_-(v)\right] \log(1 + u). \quad (4.42)$$

In the next chapter, we will see how we can re-derive all the important expressions relating the four-point function using boundary perspective. While doing so, this bulk picture will offer fruitful insights, which behooves us to have a good understanding of the contents of this chapter.
Chapter 5

Cosmological correlators from boundary perspective

In the previous chapter we calculated correlation functions in de Sitter spacetime by tracking the explicit time evolution of the quantum fields by the bulk integrals. To the contrary, we will evaluate the correlators from purely boundary perspective in this chapter, without any reference to the time-evolution whatsoever. In chapter 3, we have seen that the isometries of four-dimensional dS space are nothing but the conformal Ward identities in three-dimension acting on the three dimensional operators $O(k)$ at the future boundary. We will use these symmetries and some previously studied singularities of the correlators to evaluate them completely at the boundary. We will see that the time dependence of the bulk integrals is encoded in the momenta dependences of the boundary correlators. To warm up a bit, let’s first calculate the two-pint function using these symmetries.

5.1 Two-point functions

It is well known from conformal field theory that the two-point functions are completely specified by the symmetries only. As mentioned in (3.8) we can write the two-point function in the form

$$\langle O_1 O_2 \rangle = M(k_1) \times (2\pi)^3 \delta^3(k_1 + k_2)$$

In what follows we will use the fact that

$$\sum_{n=1}^{N} k_n^i \frac{\partial}{\partial k_n^i} \delta^3 \left( \sum_{j=1}^{N} k_j \right)$$

$$= k_1^i \frac{\partial}{\partial k_1^i} \delta^3 \left( \sum_{j} k_j \right) + k_2^i \frac{\partial}{\partial k_2^i} \delta^3 \left( \sum_{j} k_j \right) + k_3^i \frac{\partial}{\partial k_3^i} \delta^3 \left( \sum_{j} k_j \right) = -3 \delta^3 \left( \sum_{j} k_j \right)$$

where in the second line we have defined $k_i^i \equiv \sum_n k_n^i$ and used $\partial k_1^i / \partial k_1^i = \cdots = \partial k_N^i / \partial k_N^i = 1$ and $x \delta(x) = -\delta(x)$. Thus, invariance under dilatations (3.16) and

$$\sum_{n=1}^{N} k_n^i \frac{\partial}{\partial k_n^i} \delta^3 \left( \sum_{j=1}^{N} k_j \right)$$

$$= k_1^i \frac{\partial}{\partial k_1^i} \delta^3 \left( \sum_{j} k_j \right) + k_2^i \frac{\partial}{\partial k_2^i} \delta^3 \left( \sum_{j} k_j \right) + k_3^i \frac{\partial}{\partial k_3^i} \delta^3 \left( \sum_{j} k_j \right) = -3 \delta^3 \left( \sum_{j} k_j \right)$$

$$\sum_{n=1}^{N} k_n^i \frac{\partial}{\partial k_n^i} \delta^3 \left( \sum_{j=1}^{N} k_j \right)$$

$$= k_1^i \frac{\partial}{\partial k_1^i} \delta^3 \left( \sum_{j} k_j \right) + k_2^i \frac{\partial}{\partial k_2^i} \delta^3 \left( \sum_{j} k_j \right) + k_3^i \frac{\partial}{\partial k_3^i} \delta^3 \left( \sum_{j} k_j \right) = -3 \delta^3 \left( \sum_{j} k_j \right)$$

$$\sum_{n=1}^{N} k_n^i \frac{\partial}{\partial k_n^i} \delta^3 \left( \sum_{j=1}^{N} k_j \right)$$

$$= k_1^i \frac{\partial}{\partial k_1^i} \delta^3 \left( \sum_{j} k_j \right) + k_2^i \frac{\partial}{\partial k_2^i} \delta^3 \left( \sum_{j} k_j \right) + k_3^i \frac{\partial}{\partial k_3^i} \delta^3 \left( \sum_{j} k_j \right) = -3 \delta^3 \left( \sum_{j} k_j \right)$$
special conformal transformations (3.17) impose the constraints \((-3 + \sum D_n) M = 0\) and \(K_n M = 0\), respectively. Hence, the dilation symmetry becomes

\[
(-3 + D_1 + D_2) M = 0
\implies k_1^i \frac{\partial}{\partial k_1^i} M = (\Delta_1 + \Delta_2 - 3) M \tag{5.3}
\]

\[
k_1^i \partial_{k_1^i} M = \frac{\Delta_1 + \Delta_2 - 3}{2} M
\]

\[
M(k_1) = c_O k_1^{\Delta_1 + \Delta_2 - 3}
\]

where in the second line we used \(\partial_{k_1^2} = \partial_{k_1^3}\). While, the SCT constraint becomes

\[
2 (\Delta_1 - 3) \partial_{k_1^2} M = (\Delta_1 - 3) (\Delta_1 + \Delta_2 - 3) k_1^i \frac{\partial}{\partial k_1^i} M
\]

\[
\implies k_1^i \left[ 2 (\Delta_1 - 3) \partial_{k_1^2} - 4 k_1^j \partial_{k_1^j} + \frac{1}{2} \partial_{k_1^i} \partial_{k_1^j} \right] M = 0 \tag{5.4}
\]

Using the form in (5.3) we evaluate the different terms of the above expression.

\[
2 (\Delta_1 - 3) \partial_{k_1^2} M = (\Delta_1 - 3) (\Delta_1 + \Delta_2 - 3) \frac{M}{k_1^2}
\]

\[
4 k_1^2 \partial_{k_1^2} = (\Delta_1 + \Delta_2 - 3) (\Delta_1 + \Delta_2 - 5) \frac{M}{k_1^2}
\]

\[
\frac{1}{2} \partial_{k_1^i} \partial_{k_1^j} = (\Delta_1 + \Delta_2 - 3) \left( \frac{\Delta_1 + \Delta_2 - 4}{2} \right) \frac{M}{k_1^2}
\]

Putting all this together, the SCT constraint becomes

\[
\frac{\Delta_1 + \Delta_2 - 3}{k_1^2} \left\{ (\Delta_1 - 3) - (\Delta_1 + \Delta_2 - 5) + \left( \frac{\Delta_1 + \Delta_2 - 4}{2} \right) \right\} M = 0 \tag{5.5}
\]

Finally we got the expression for the two-point function as

\[
\langle O_1 O_2 \rangle = c_O k_1^{2\Delta - 3} \times (2\pi)^3 \delta^3(k_1 + k_2) \tag{5.6}
\]

where we have defined \(\Delta \equiv \Delta_1 = \Delta_2\). Thus we see that two-point functions are completely determined from the Ward identities. A similar exercise will show that, this is true for three-point functions also (see [23] for a detailed discussion). But the case for four-point functions is little bit complicated. We will now turn to the analysis of four-point functions in de Sitter space which is our main topic of interest.

### 5.2 Four-point function

Four-point functions in conformal field theories are less constrained kinematically. The four-point function of the three-dimensional operators \(O(k)\) in general depends
on 12 variables. The translational and rotational invariance give 3 constraints each. Invariance under dilation and SCT’s give 1 and 3 constraint equations (the Ward identities), respectively. Thus, ultimately the four-point function is made up of only two independent variables. In position space, these are taken as two conformally invariant cross-ratios:

$$x_{12}x_{34}, \quad x_{12}x_{34} \over x_{13}x_{24}, \quad x_{12}x_{34} \over x_{23}x_{14}$$

where $x_{ij} = |x_i - x_j|$ and the four-point function is given by an arbitrary function of these cross-ratios. In momentum space, the analogues of these cross-ratio are

$$u \equiv \frac{s}{k_{12}}, \quad v \equiv \frac{s}{k_{34}}$$

where $k_{12} \equiv k_1 + k_2$, $k_{34} = k_3 + k_4$ and $s \equiv |k_1 + k_2|$ is the momentum exchange variable. As usual we can write the four-point functions in momentum space as

$$\langle O_1O_2O_3O_4 \rangle = F(u, v) \times (2\pi)^3 \delta^3(k_1 + \cdots + k_4)$$

Using the above form, the Ward identities give the constraint equations for $F$:

$$\left[ 9 - \sum_{n=1}^{4} (\Delta_n - k^n_i \partial k^n_i) \right] F = 0, \quad (5.9)$$

$$\sum_{n=1}^{4} \left[ 2(\Delta_n - 3) \partial k^n_i - 2k^n_i \partial k^n_i + k^n_i \partial k^n_i \partial k^n_i \right] F = 0. \quad (5.10)$$

Now, using $\partial s = \frac{k_{12}}{s}$, we change the variable of differentiation to $s$.

$$\left[ 9 - \Delta_t + \frac{(k_{12})^2}{s} \partial s \right] F = 0 \quad (5.11)$$

$$\Rightarrow s \partial_s F = (\Delta_t - 9) F$$

where in the second line we used $(k_{12})^2 = s^2$. Thus, defining $F = s^{\Delta_t - 9} \tilde{F}$, manifestly represents the dilation symmetry. The form of the dimensionless function $\tilde{F}(u, v)$ will be dictated by special conformal invariance.

The three differential equations of the SCT constraints can be rearranged into

$$\sum_{n=1}^{4} k^n_i T_n F = 0 \quad (5.12)$$

where we have defined

$$T_1 F \equiv \left[ \frac{\partial^2}{\partial k_1^2} + \frac{1}{s} \frac{\partial}{\partial s} \left( k_1 \frac{\partial}{\partial k_1} + k_2 \frac{\partial}{\partial k_2} \right) + \frac{1}{t} \frac{\partial}{\partial t} \left( k_1 \frac{\partial}{\partial k_1} + k_4 \frac{\partial}{\partial k_4} \right) - \frac{k_1^2}{s^2} \frac{\partial^2}{\partial s \partial t} \right. - \frac{2(\Delta_1 - 2)}{k_1} \frac{\partial}{\partial k_1} + \frac{\Delta_1 + \Delta_2}{s} \frac{\partial}{\partial s} + \frac{\Delta_1 + \Delta_4}{t} \frac{\partial}{\partial t} \left. \right] F. \quad (5.13)$$
and all the other $T'$s are determined by cyclic permutation of the indices (keeping in mind that $t \leftrightarrow s$ under a cyclic shift). For example to get $T_2$ we perform $1 \to 2 \to 3 \to 4 \to 1$. The operator $T_n$ is a combination of the SCT and the dilation operator. The only non-trivial way of satisfying (5.12) is to set all $T_n F'$s equal. Then the left hand side vanishes simply by momentum conservation ($\sum_n k_n = 0$). This provides us with 6 constraint equations

$$ (T_n - T_m) F = 0 \quad (5.14) $$

for $n, m = 1, \cdots, 4$.

But as SCT was originally comprised of 3 equations, only three out of the six equations are independent and we can have our convenient pick.

As motivated earlier, we will now work with the four-point functions of conformally coupled scalars, mediated by exchange of massive scalars. In this case ($\Delta = 8$), the s-channel contribution\(^1\) can be written as $F = s^{-1} \hat{F}(u, v)$. This ansatz automatically satisfies equations $T_{12} F = 0$ and $T_{34} F = 0$, where $T_{nm} \equiv T_n - T_m$. The remaining conformal invariance equation $T_{13} F = 0$ becomes

$$ (T_1 - T_3) F = 0 \quad (5.15) $$

Changing variables from $k_{12} \to u$ in the first expression and from $k_{34} \to v$ in the second expression, we get

$$ (\Delta_u - \Delta_v) \hat{F} = 0 \quad (5.16) $$

where

$$ \Delta_u \equiv u^2 (1 - u^2) \frac{\partial^2}{\partial u^2} - 2u^3 \partial_u. \quad (5.17) $$

Finally we have our conformal constraint equation in a very compact form. In the next section we will solve this equation for contact interactions and exchange interactions with correct boundary conditions coming from appropriate handling of the singularities of the solution.

### 5.3 Contact interactions

The simplest solution to (5.16) corresponds to four-point functions arising from contact interactions. These solutions, which we will denote by $F_C(u, v)$, are characterized by simplest singularity structure possible. For four scattering amplitude, simplest analytic structure corresponds to a polynomial in Mandelstam variables (as we have seen in chapter 2). But scaling ($F_C \propto \frac{1}{s}$) forces our correlator to have some sort of poles. The simplest choice of poles corresponds to contact terms in bulk, where (as we have seen in chapter 4) our correlator has poles in the total energy

---

\(^1\)The other channels can be included by cyclic permutations, e.g. by replacing $u$ with $|k_2 + k_3|/(k_2 + k_3)$ and $|k_2 + k_4|/(k_2 + k_4)$ for the $t-$ and $u-$ channels, respectively.
variable \( k_t = \sum k_n \). Note that this singularity can’t be achieved for real momenta, we need to analytically continue them to the region \( k_n < 0 \).

Let’s calculate the simplest of the contact interactions from our boundary conformal constraint equation. For this, we assume \( F_C \) only depends on the magnitudes of \( k_n \). Thus, there is no \( s \) and \( t \) dependence and the constraint equation (5.14) can be written as

\[
(\partial_{k_n}^2 - \partial_{k_m}^2) F_C = 0
\]

These wave equations are solved by the following ansatz

\[
F_C(k_n) = \frac{c_0}{k_t}
\]

where \( c_0 \) is an arbitrary constant. Recalling (3.13), the corresponding correlation function for \( O^+ \) is

\[
F_C^+(k_n) = \frac{1}{k_1 k_2 k_3 k_4} \frac{c_0}{k_t}
\]

Using (3.4), we find the correlation function for \( \varphi \) up to leading order

\[
\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle' = \frac{\eta_0^4}{k_1 k_2 k_3 k_4} \frac{c_0}{k_t}
\]

Comparing with (4.18) we see that (5.19) is the four-point function due to the bulk interaction \( \varphi^4 \). Changing to \( u, v \) variables we get

\[
c_0 \frac{1}{k_t} = s^{-1} \left[ c_0 \frac{uv}{u+v} \right]
\]

Thus, the simplest solution of (5.16) is

\[
\hat{F}_C(u, v) = c_0 \frac{uv}{u+v} \equiv c_0 \hat{C}_0
\]

All the higher derivative contact terms can be generated by acting \( \Delta_u \) on \( \hat{C}_0 \), \( \hat{C}_n(u, v) \equiv \Delta_n \hat{C}_0 \). We can see this trivially satisfies the constraint equation (5.16).

Thus the most general solution for contact interactions is

\[
\hat{F}_C(u, v) = \sum_{n=0}^{\infty} c_n \Delta_u^n \hat{C}_0
\]

where the dimensionless parameters \( c_n \) are coupling constants. Let’s highlight some features of this solution:

- As can be seen from (5.21), the solution is organized in powers of \( uv/(u+v) = s/k_t \), multiplied by factors whose form is dictated by conformal symmetry.

- The shapes produced by different bulk interactions correspond to linear combinations of contact terms \( \hat{C}_n \): for example \((c_0, c_1, c_2) = (1, 0, 0)\) for \( \varphi^4 \) and \((1, 1, 1/4)\) for \((\partial_\mu \varphi)^4\).

- The solution is symmetric under \( u \leftrightarrow v \) exchange, as expected. The explicit symmetry of the zeroth order term \( \hat{C}_0 \) and the conformal symmetry (5.16) ensures the symmetry for all the higher order terms.
5.4 Tree-level exchange interactions

Finally we consider the four-point functions involving the tree-level exchange of a massive scalar. In the OPE limit (see Appendix C), we expect the four-point function to factorize into a product of three-point functions. With this expectation we take the ansatz

\[ \hat{F}(u,v) = \hat{I}(u)\hat{J}(v), \]

(5.22)

where \( \hat{I}(u) \) and \( \hat{J}(v) \) satisfy the correct constraint equation for a three-point functions involving a massive scalar:

\[
\left[ \Delta_u + \left( \mu^2 + \frac{1}{4} \right) \right] \hat{I}(u) = 0, \quad \left[ \Delta_v + \left( \mu^2 + \frac{1}{4} \right) \right] \hat{J}(v) = 0 \tag{5.23}
\]

We see that, this ansatz trivially satisfies the constraint equation (5.16)

\[
(\Delta_u - \Delta_v) \hat{F} = \hat{J} \Delta_u \hat{I} - \hat{I} \Delta_v \hat{J} = 0
\]

where in the second equality we have used (5.23). Thus, \( \hat{F} = \hat{I}\hat{J} \) is the simplest solution of the constraint equation. For a more general solution (which is not of the factorized form), we add a source term:

\[
\left[ \Delta_u + \left( \mu^2 + \frac{1}{4} \right) \right] \hat{F} = \left[ \Delta_v + \left( \mu^2 + \frac{1}{4} \right) \right] \hat{F} = \hat{C}(u,v) \tag{5.24}
\]

Note that these are now differential equations in \( u \) and \( v \) separately. Actually, the solution corresponding to the product of three-point functions, corresponds to the homogeneous solutions of the equation (5.24). The \( \mu \)-dependent term dictates the mass of the exchanged particle. For consistency, the source functions must themselves satisfy \( (\Delta_u - \Delta_v)\hat{C}(u,v) = 0 \). For a better understanding of the properties of these source functions, we take the limit \( \mu \to \infty \). In this limit, the exchange particle can be integrated out and the theory reduces to contact terms, \( \hat{F} \to \hat{F}_C = (H/\mu)^2\hat{C}(u,v) \).

Hence, the source functions are nothing but the contact terms of the theory. Each contact terms gives a corresponding exchange solution.

**Homogeneous solution:** The differential operator \( \Delta_u \) has three singularities at \( u \to 0 \) and \( u \to \pm 1 \) and thus the homogeneous solutions of (5.24) can be expressed as hypergeometric functions. We will denote the solutions of \( \hat{I}(u) \) and \( \hat{J}(v) \) by \( \hat{F}_+ \) and \( \hat{F}_- \). We will use a normalization which gives the Wronskian as \( W[\hat{F}_+, \hat{F}_-] = 1/(1 - u^2) \). This would avoid unnecessary factors of \( \mu \) in the intermediate steps; obviously final solution will not depend on it. With this normalization, the homogeneous solutions can be written as (corresponding to \( \hat{I} \))

\[
\hat{F}_\pm(u) = \left( \frac{i u}{2\mu} \right)^{\frac{1}{2} \pm i \mu} \binom{1}{2} \left\{ \frac{1}{4} \pm \frac{i \mu}{2}, \frac{3}{4} \pm \frac{i \mu}{2} ; u^2 \right\} \tag{5.25}
\]

and a similar solution for the differential equation involving \( v \) (corresponding to \( \hat{J} \)).
We get the final explicit form of the homogeneous solution of (5.24) as:

\[
\hat{F}_h(u,v) = \hat{I}(u)\hat{J}(v) = (C_1\hat{F}_+(u) + C_2\hat{F}_-(u)) \times (C_3\hat{F}_+(v) + C_4\hat{F}_-(v))
\]

\[
= C \left[ (\hat{F}_+(u)\hat{F}_-(v) - \hat{F}_-(u)\hat{F}_+(v)) + \beta_+\hat{F}_+(u)\hat{F}_+(v) + \beta_-\hat{F}_-(u)\hat{F}_-(v) \right]
\]

\[
+ \beta_0 \left( \hat{F}_+(u)\hat{F}_-(v) + \hat{F}_-(u)\hat{F}_+(v) \right)
\]

(5.26)

where in the third equality we rearranged terms to exploit the \( u \leftrightarrow v \) symmetry, as we will show later. This is the most general form of the homogeneous solution. There are four arbitrary constant, which we will fix in the later sections by imposing boundary conditions.

From the properties of the zero-balanced hypergeometric functions (see [25]), we see that the solution also has singularities at \( u \to \pm 1 \) corresponding to the operator \( \Delta_u \):

\[
\lim_{u \to +1} \hat{F}_\pm(u) = \alpha_\pm \log(1 - u), \quad \lim_{u,v \to -1} \hat{F}_\pm(u) \propto \log(1 + u)\log(1 + v)
\]

(5.27)

where

\[
\alpha_\pm = -\left( \frac{i}{2\mu} \right)^{\frac{1}{2} + i\mu} \frac{\Gamma(1 \pm i\mu)}{\Gamma(\frac{1}{4} \pm \frac{i\mu}{2})\Gamma(\frac{3}{4} \pm \frac{i\mu}{2})}
\]

For concreteness, we will first find the particular solution of the inhomogeneous equation (5.24) corresponding to the simplest contact term \( \hat{C}_0 \):

\[
\left[ \Delta_u + \left( \mu^2 + \frac{1}{4} \right) \right] \hat{F} = \frac{uv}{u + v}
\]

(5.28)

where the coupling constant of the contact term has been absorbed in the normalization of \( \hat{F} \). Recall that we had derived the same differential equation from bulk perspective in chapter 4 (equation 4.31). This shows the equivalence of the two methods.

### 5.4.1 EFT expansion

A formal solution of (5.28) can be written as

\[
\hat{F} = \frac{H^2\hat{C}_0}{\Delta_u + (M/H)^2} = \sum_n \frac{1}{n!} \left( -\frac{H^2}{M^2}\Delta_u \right)^n \frac{H^4}{M^2} \hat{C}_0
\]

(5.29)

where we put back the factors of \( H \) following equation (4.31). This is the effective field theory (EFT) expansion of the correlation function. We see that the solution is a sum over the contact terms \( \Delta_u \hat{C}_0 \) (see figure (5.1), organised as an expansion in the inverse mass of the exchange particle (in units of the Hubble parameter). For \( M >> H \), the exchange particle can be integrated out and the interactions are local. In this case EFT expansion is an accurate solution. But for \( M \sim H \),
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Figure 5.1: Comparison of the EFT expansion part (the particular series solution) from (5.39) and sum of contact terms. We see that there is a fast convergence. We need to go at most till the second contact term.

exchange particles can’t be integrated out and they give rise to non-local effects (see section 5.5 for detailed discussion). Also, the EFT expansion does not possess the expected singularities at $u \to \pm 1$ of the operator $\Delta_u$ (see equation 5.27). Thus, this is not the exact solution in this case ($M \sim H$). We have seen earlier that the homogeneous solutions have those singularities. So, we expect nonperturbative corrections coming from the homogeneous solutions. As we will discuss later, these non-perturbative correction signals towards spontaneous particle production due to expanding universe.

5.4.2 Non-perturbative corrections

Before diving for an exact solution, let us first study the solution in the simple limit $v \to 0$. The solution in this limit has all the essential features of the full solution. Writing $e^t \equiv u/v \equiv \xi$ and $f \equiv (uv)^{-1/2} \hat{F}$, equation (5.28) becomes the equation of a forced harmonic oscillator, with the frequency of oscillations given by the mass of the exchanged particle:

$$\left[ \frac{d^2}{dt^2} + \mu^2 \right] f = \frac{1}{2 \cosh (\frac{1}{2} t)}.$$  \hspace{1cm} (5.30)

The homogeneous solutions are

$$f_\pm = e^{\pm i\mu t} = \xi^{\pm i\mu}.$$  \hspace{1cm} (5.31)

The inhomogeneous solution analytic around $\xi = 0$ is

$$f_<(\xi) = \sqrt{\xi} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^n}{(n + \frac{1}{2})^2 + \mu^2}.$$  \hspace{1cm} (5.32)
which is convergent for $\xi \leq 1$ and divergent for $\xi \geq 1$. While the inhomogeneous equation analytic around $\xi \to \infty$ is

$$f_>(\xi) = \frac{1}{\sqrt{\xi}} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{-n}}{(n + \frac{1}{2})^2 + \mu^2}, \quad (5.33)$$

which is convergent for $\xi \geq 1$ and divergent for $\xi \leq 1$. The above solutions corresponds to the EFT expansion. Both of them are analytic with their respective radius of convergences. To get a solution over the whole range, we need to analytically continue the solution around $\xi \to \infty$ ($f_>(\xi)$) into the region $\xi > 1$ by matching the solutions (5.32) and (5.33) at $\xi = 1$. For that we note that, since $f_<$ and $f_>$ satisfy the same differential equation, the difference $f_< - f_>$ must be a solution of the homogeneous equation:

$$f_< (\xi) - f_>(\xi) = \sum_{\pm} A_{\pm} \xi^{\pm i \mu}. \quad (5.34)$$

Continuity at $\xi = 1$ then implies that $A_+ = -A_- \equiv A$ and matching $\xi$-derivative at $\xi = 1$ we get

$$2i \mu A = \sum_{n=0}^{\infty} (-1)^n \frac{(2n + 1)}{(n + \frac{1}{2})^2 + \mu^2} = \frac{\pi}{\cosh \pi \mu}. \quad (5.35)$$

Therefore, we obtain an explicit solution which is analytic around the origin:

$$\tilde{F}_<(\xi) = \begin{cases} 
\sum_{n=0}^{\infty} (-1)^n \frac{\xi^{n+1}}{(n + \frac{1}{2})^2 + \mu^2} & \xi \leq 1 \\
\sum_{n=0}^{\infty} (-1)^n \frac{\xi^{-n}}{(n + \frac{1}{2})^2 + \mu^2} + \frac{\pi}{\cosh \pi \mu} \frac{\xi^{\frac{1}{2} - i \mu} - \xi^{\frac{1}{2} + i \mu}}{2i \mu} & \xi \geq 1 
\end{cases} \quad (5.36)$$

We observe that the above solution is analytic around $\xi = 0$, but non-analytic around $\xi \to \infty$, as opposed to the EFT expansion solutions. The non-analyticity comes form the nonperturbative correction terms which arises due to matching the solutions at $\xi = 1$. Thus at 'early times' ($\xi \to 0$), the solution vanishes; while at 'late times' ($\xi \to \infty$), it becomes oscillatory (due to the correction term). In the harmonic oscillator analog, we begin with a ball at rest at early times, and "kick" it with the forcing term, ending up with an oscillating ball at late times. In the cosmological context, the presence of these oscillatory terms can physically be attributed to particle production by the time-dependent inflationary background. We will discuss this in detail in the later sections.

### 5.4.3 An exact solution

Let us return to the full solution of the differential equation (5.28). Carrying out the exercise as done for the $v \to 0$ limit, we can easily get the general solution. We again have two solutions, $\tilde{F}_<(u, v)$ and $\tilde{F}_>(u, v)$, which are analytic around $u \to 0$ and $u \to \infty$, respectively. As earlier, we have convergent series expansion for $\tilde{F}_<(u, v)$ in $u < v$ region and for $\tilde{F}_>(u, v)$ in $u > v$ region. We again match the solutions at $u = v$ to get analytically continued expression for $\tilde{F}_<(u, v)$ in the $u > v$ region. In
the process, we incur a homogeneous correction term. The explicit solution for \( \hat{F}_< \) is

\[
\hat{F}_<(u, v) = \begin{cases} 
\sum_{m,n} c_{mn} u^{2m+1} \left( \frac{u}{v} \right)^n & \quad u \leq v \\
\sum_{m,n} c_{mn} v^{2m+1} \left( \frac{u}{v} \right)^n + \frac{\pi}{\cosh \pi \mu} \left( \hat{F}_+(v) \hat{F}_-(u) - \hat{F}_-(v) \hat{F}_+(u) \right) & \quad u \geq v
\end{cases}
\]

where the series coefficient are given by

\[
c_{mn} = \frac{(-1)^n (n + 1)(n + 2) \cdots (n + 2m)}{\left( n + \frac{1}{2} \right)^2 + \mu^2} \left( n + \frac{1}{2} + 2m \right)^2 + \mu^2
\]

But this solution still has some deficiencies. First of all, it is not symmetric in \( u \leftrightarrow v \), as required by consistency of the bulk evolution (equation 4.32) (and the symmetry of the conformally-invariant contact terms). To get around this, we start by writing down the most general solution to the differential equation \(^2\)

\[
\hat{F}(u, v) = \begin{cases} 
\sum_{m,n} c_{mn} u^{2m+1} \left( \frac{u}{v} \right)^n + \hat{g}(u, v) & \quad u \leq v \\
\sum_{m,n} c_{mn} v^{2m+1} \left( \frac{u}{v} \right)^n + \hat{h}(u, v) & \quad u \geq v
\end{cases}
\]

where the function \( \hat{g} \) and \( \hat{h} \) takes the form of the most general homogeneous solution (5.26). Demanding \( \hat{F} \) to be symmetric under \( u \leftrightarrow v \) exchange implies \( \hat{h}(u, v) = \hat{g}(v, u) \). Also, it can be easily seen from (5.37) that the matching condition gives

\[
\hat{g}(u, v) - \hat{g}(v, u) = \frac{\pi}{\cosh \pi \mu} \left( \hat{F}_+(u) \hat{F}_-(v) - \hat{F}_-(u) \hat{F}_+(v) \right).
\]

This implies that in (5.26), \( C = \pi / 2 \cosh \pi \mu \) and the function \( \hat{g} \) is fixed up to three parameters:

\[
\begin{align*}
\hat{g}(u, v) &= \frac{\pi}{2 \cosh \pi \mu} \left[ \left( \hat{F}_+(u) \hat{F}_-(v) - \hat{F}_-(u) \hat{F}_+(v) \right) + \beta_+ \hat{F}_+(u) \hat{F}_+(v) \\
&\quad + \beta_- \hat{F}_-(u) \hat{F}_-(v) + \beta_0 \left( \hat{F}_+(u) \hat{F}_-(v) + \hat{F}_-(u) \hat{F}_+(v) \right) \right]
\end{align*}
\]

Next, we will analyse the singularities of the solution (5.39) in different limits and use them as our boundary condition to fix \( \beta_+ \), \( \beta_- \) and \( \beta_0 \).

**Folded limit**

This is the limit where \( u, v \to 1 \) (or, equivalently \( k_1 + k_2 = |k_1 + k_2| \)).\(^3\) Taking the limit \( u \to 1 \), for \( v \neq 1 \), in (5.39) we get (the singularity comes from the homogeneous part only)

\[
\lim_{u \to 1} \hat{g}(v, u) = \left[ (\alpha_- \beta_+ + \alpha_+ (\beta_0 + 1)) \hat{J}_-(v) + (\alpha_+ \beta_+ + \alpha_- (\beta_0 - 1)) \hat{J}_+(v) \right] \log(1 - u)
\]

\(^2\)This form is inspired from our earlier discussions where we see that the solution came out to be: an EFT expansion (from particular series solution) + non-perturbative corrections (from homogeneous solutions)

\(^3\)For three-point functions, this is also called the collinear limit.
Chapter 5: Cosmological correlators from boundary perspective

(a) Folded limit: \( u \to +1 \)
(b) Factorization limit: \( u, v \to -1 \)
(c) Collapsed or OPE limit: \( u, v \to 0 \)

Figure 5.2: Illustration of the singularities of the solution \( \hat{F}(u, v) \). The singularity in the folded or collinear limit is due to excited initial states and should be absent for Bunch-Davis vacuum (see Appendix B). The singularity in the factorization limit is similar to the factorization of the residue for flat space scattering amplitude and therefore expected. The non-analyticity in the collapsed limit is due to the non-local effects of particle production.

where we have shown the leading singular behaviour. In Appendix D, we will show that this singularity in the folded configuration is a signature of excited initial states. But in this treatise we will make the standard choice of the initial vacuum state, which is the Bunch-Davis vacuum. Hence, these singularities are unphysical and must be removed. This gives us the conditions:

\[
\beta_+ = -(\beta_0 - 1) \frac{\alpha_-}{\alpha_+}, \quad \beta_- = -(\beta_0 + 1) \frac{\alpha_+}{\alpha_-}.
\]

(5.42)

We are then left with only one undetermined parameter \( \beta_0 \), which we will fix next.

Factorization limit

Although the fixing of \( \beta_0 \) can be done purely from boundary perspective; we will do it by taking help from bulk calculations. The way we do it is by taking the limit \( u, v \to -1 \) and then comparing the amplitude at that singularity with the bulk amplitude. This approach makes the singularity structure in the unphysical region \( (u, v < 0) \) more explicit. First, we take the limit \( u \to -1 \), for generic \( v > 0 \)

\[
\lim_{u \to -1} \hat{g}(u, v) = -2i \sinh \pi \mu \left[ (\beta_0 + 1) \hat{J}_+(v) + (\beta_0 - 1) \hat{J}_-(v) \right] \log(1 + u) \quad (5.43)
\]

Next we take the limit \( v \to -1 \). Using the expression of \( \hat{J}(v) \) from (4.39) and properties of zero-balanced hypergeometric functions from [25], we get

\[
\lim_{u, v \to -1} \hat{F}(u, v) = \frac{\beta_0 i \sinh \pi \mu}{2} \log(1 + u) \log(1 + v) \quad (5.44)
\]

where we have ignored the imaginary term on the basis that our correlator must be real and thus imaginary terms are not physically relevant. Up to an overall scaling, the homogeneous piece of the four-point function (from which these logarithmic singularities have their origin) satisfies the same equation as the three-point functions. Therefore, we can think of this limit as a factorization channel, where the four-point
function factorizes into the product of two three-point functions. For an appropriate normalization of this limit, we compare this to the bulk calculation (equation (4.42)) and we get

$$\beta_0 = \frac{1}{i \sinh \pi \mu}$$  \hspace{1cm} (5.45)

### 5.4.4 Comparison of the analytical and numerical solutions

In this section, we verify how good our analytical solution is, by comparing with the numerical estimates. In figure (5.3), we have compared the analytical solution (5.39) with the numerical solution of the differential equation (5.28). It is quite evident from the graphs that the analytical solution is highly convergent near \( u = 0 \) and we can get a very good approximation by keeping only a few terms. The convergence is fast for smaller \( \mu \). This is due to the reason that the shape of the analytical solution is dominated by non-perturbative part for smaller \( \mu \) which in turn can be attributed to the factor \( \pi / (2 \cosh \pi \mu) \) sitting in front of the non-perturbative part. We can see that when full convergence has not been reached, there is a kink at \( u = v \). This is because, although the full solution is smooth everywhere, the particular series solution and the non-perturbative part are not. In fact, the derivatives of \( \tilde{g}(u, v) \) and \( \tilde{g}(v, u) \) are not equal at \( u = v \). The fact that \( u = v \) lies at the junction of the two disks of convergence gives rise to its slow convergence.

### 5.5 Particle production in the collapsed limit

One of the special limit is the collapsed limit \((u, v \to 0)\) where the effects of particle production are most manifested. In this limit, the homogeneous part of the solution (5.39) dominates. Taking the limit and keeping only the terms which are non-analytic in \( s \), we get

$$\lim_{u,v \to 0} \tilde{F}(u, v) = \left( \frac{uv}{4} \right)^{\frac{1}{2} + \imath \mu} (1 + \imath \sinh \pi \mu) \frac{\Gamma \left( \frac{1}{2} + \imath \mu \right)^2 \Gamma(-i\mu)^2}{2\pi} + c.c.$$  \hspace{1cm} (5.46)

To see how non-analyticity encodes particle production and what other information we can extract out from this limit (for more details see [26, 27]), let us first discuss
some aspects of the two point functions in de Sitter space. From Appendix B, we see that the two point function of a massive scalar field $\sigma$ in de Sitter space can be written as

$$
\langle \sigma_k(\eta)\sigma_{k'}(\eta') \rangle' = \frac{\pi}{4} (\eta\eta')^{3/2} e^{-\pi\mu} H_{i\mu}(-k\eta)H^*_{i\mu}(-k\eta').
$$

(5.47)

Although this expression is valid for both real and imaginary $\mu$, we will focus our attention on massive particles with $m > \frac{3}{2}H$ for which $\mu$ is real. In the late time limit, we can split (5.47) into its local and non-local parts

$$
\lim_{\eta,\eta' \to 0} \langle \sigma_k(\eta)\sigma_{k'}(\eta') \rangle'_{\text{local}} = \frac{(\eta\eta')^{3/2}}{4\pi} \Gamma(-i\mu)\Gamma(i\mu) \left[ e^{\pi\mu} \left( \frac{\eta}{\eta'} \right)^{i\mu} + e^{-\pi\mu} \left( \frac{\eta}{\eta'} \right)^{-i\mu} \right]
$$

(5.48)

$$
\lim_{\eta,\eta' \to 0} \langle \sigma_k(\eta)\sigma_{k'}(\eta') \rangle'_{\text{non-local}} = \frac{(\eta\eta')^{3/2}}{4\pi} \left[ \Gamma(-i\mu)^2 \left( \frac{k^2\eta\eta'}{4} \right)^{i\mu} + \Gamma(i\mu)^2 \left( \frac{k^2\eta\eta'}{4} \right)^{-i\mu} \right]
$$

(5.49)

The local and non-local part can be distinguished by their long distance behaviour. The local part has its contribution only from coincident points in position space, while the non-local part describes correlation over long distances. In Fourier space, the condition of locality is encoded in the analyticity of the two-point function in $k$. The local and non-local parts are analytic and non-analytic in $k$, respectively. The $(\eta\eta')^{3/2}$ factor reflects the dilution of the massive particles due to the expansion of the universe. If we square the wavefunction, we will find that the probability of finding the particles goes as $1/a^3 \sim 1/\text{Volume}$ where $a \sim 1/\eta$. Note that $\Gamma(\pm i\mu) \to e^{-\pi\mu/2}$ for large $\mu$, resulting in an overall suppression of $e^{-\pi\mu}$ of the non-local contribution (5.49). If we square it, we get the Boltzmann factor of massive particles, $e^{-2\pi\mu}$, which gives the probability of creating a pair of massive particles. Using the formula for Fourier transform of $|x|$

$$
\int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} |x|^{-2a} = 8\pi^{3/2} 2^{-2a} a^{2a-3} \Gamma\left(\frac{3}{2} - a\right) \Gamma\left(a\right),
$$

(5.50)

we get the corresponding expression of (5.49) in position space

$$
\lim_{\eta,\eta' \to 0} \langle \sigma(\mathbf{x},\eta)\sigma(\mathbf{0},\eta') \rangle'_{\text{non-local}} = \frac{1}{4\pi^{5/2}} \left[ \Gamma(-i\mu)\Gamma\left(\frac{3}{2} + i\mu\right) \left( \frac{\eta\eta'}{|\mathbf{x}|^2} \right)^{\frac{3}{2} + i\mu} + \Gamma(i\mu)\Gamma\left(\frac{3}{2} - i\mu\right) \left( \frac{\eta\eta'}{|\mathbf{x}|^2} \right)^{\frac{3}{2} - i\mu} \right]
$$

(5.51)

We can contrast (5.51) to the corresponding long spatial distance expectation value in the flat Minkowski space. In that case the long distance correlation decreases exponentially as $e^{-M|x|}$, while in de Sitter space we get an oscillatory behaviour. This oscillatory behaviour can be traced back to the time dependence of the background metric. The expansion of the universe create pairs of particles (Appendix E) which mediate these long range interactions.
Chapter 5: Cosmological correlators from boundary perspective

Let us now see how the non-analyticity of the non-local part of the two point function of the exchange scalar $\sigma$ is reflected in cosmological correlators. It is not difficult to verify that (5.46) can also be derived from the bulk picture by replacing the correlators $G_{\pm \pm}(s,\eta,\eta')$ in (4.24) by their long distance expressions given in (5.49) (see Appendix C on OPE limit). Borrowing the bulk picture, the four-point function can thus be thought of as produced from spontaneous creation of a pair of massive particle $\sigma$ and their subsequent decay into conformal scalar fields through interactions such as $\lambda \varphi^2 \sigma$ (see figure 5.4). These conformal fields are correlated as they are the decay products of pair-produced particles. This produces spatial correlations between cosmological structures at late times. Thus, the non-analyticity in the collapsed limit is due to the non-local correlations which can be attributed to particle production in an expanding universe.

Just like the case of two-point functions, four-point correlators also has oscillatory behaviour as can be seen from (5.46). Here, we have oscillations in the logarithm of the ratio $s^2/k_{12}k_{34}$ (since $u \equiv s/k_{12}$ and $v \equiv s/k_{34}$). Since a given momentum $k$ is associated with a characteristic time which is the time when this mode crossed the horizon during inflation; the logarithm is essentially the number of e-folds for which the pair has existed, adding the e-fold for each member of the pair. We see that the frequency of oscillations gives a measure of the mass of the massive particle.

5.6 Comparison of de Sitter solutions with flat space scattering amplitudes

Our boundary approach is similar in spirit with the S-matrix bootstrap technique in the following ways:

- Lorentz invariance requires the flat space scattering amplitude to be only function of Mandelstam variables, $A_4(s_{\text{flat}}, t_{\text{flat}})$ (note we only need two of them as the remaining one is related to the other two). Similarly in de Sitter space, conformal constraints on the four-point function causes it to depend on two dimensionless variables which are related to Mandelstam variables by

$$k_{12}^2(1-u^2) = s_{\text{flat}}^2, \quad k_{34}^2(1-v^2) = s_{\text{flat}}^2$$

(5.52)
Chapter 5: Cosmological correlators from boundary perspective

- Locality restrains the singularity structure of tree-level amplitudes to simple poles in the Mandelstam variables, \((s_{\text{flat}} - M^2)A_4 = \text{analytic}(\text{and a similar one with } t_{\text{flat}})\), where \(M\) is the mass of the exchanged particle. In de Sitter space, tree-level exchange is described by a pair of differential equations

\[
(\Delta_u - M^2)\hat{F} = \hat{C}, \quad (\Delta_v - M^2)\hat{F} = \hat{C}
\]

(5.53)

where \(\hat{C}\) is one of the contact interactions.

- When the poles are approached, \(s_{\text{flat}} \to M^2\), the flat space four-particle amplitude factorizes into a product of three-particle amplitudes with positive coefficients. Similar factorization occurs for de Sitter four point functions in the limit \(u, v \to -1\). We have seen how the correct normalization of this singularity gives us a boundary condition for solving the differential equations (5.53).

Despite the similarities, due to the time-dependent background and the fact that we are calculating correlators instead of scattering amplitudes lead to some differences between flat space scattering amplitudes and de Sitter four-point functions:

- The contact terms in flat space are polynomials in Mandelstam variables and thus are purely analytic. While contact terms in de Sitter space have poles at \(k_t = 0\).

- We have also seen that the de Sitter correlation functions have a singularity at \(u \to 1\) which corresponds to excited initial states. The assumption of standard Bunch Davis vacuum as the initial state requires the singularity to be absent. This gives another boundary condition which along with the appropriate normalization of the factorization channel completely fixes the solution.
Chapter 6

Conclusion

In this excerpt, we evaluated correlation functions for conformally coupled scalars in de Sitter space by using boundary approach. In this approach we exploit the de Sitter isometries (whose action on the correlator is identical to that of conformal symmetries) and the singularities of the correlators to uniquely fix the correlation function. Note that, no reference to Lagrangian description has been made in this approach and thus, it is drastically different from the bulk approach where we start with the Lagrangian, compute the evolution of the fields and then explicitly evaluate the time-integrals of the fields to find the correlation functions. Not so surprisingly, the two approaches yield the same result. This is because the same information of time-evolution of the fields is encoded in the momentum dependence of the boundary correlators. Thus, the effects of time-dependent background, like spontaneous particle production also emerged for the solutions from boundary perspective.

Finally we address the question posed at the beginning of this excerpt which is that what information do these cosmological correlators (actually inflationary correlators) give about the physics of the early universe. As mentioned earlier, inflation act as a natural particle collider. In particle accelerator we look patterns of energy deposition on the detector. In much the same manner, in cosmology, we look for patterns or spatial correlations in the large scale structure (LSS) or in cosmic microwave background (CMB). In collider physics, different resonance peaks signal towards the existence of new particles and determine their properties. The position of the resonance peak gives the mass of the new particle while the its height and width give information about the lifetime of the particle. The angular dependence of the decay products constrains the spin of the exchange particle. Similarly, the structure of inflationary correlators probe the properties of particles in the inflationary universe.

We have seen that tree-level exchange of particles of mass comparable to Hubble’s constant produce distinctive structures in the collapsed limit (see equation) similar as in the collider physics. In the limit $u \to 0$, the signal oscillates with a frequency fixed by the mass of the exchanged particle (see figure (6.1)). The measurement of these oscillations is analogous to measuring the position of resonance peak in collider physics. This would prove the existence of new particles and determine their mass. As we move away from the collapsed limit, the particular solution starts to dominate over the homogeneous solution and in both cases a description with EFT expansion
Chapter 6: Conclusion

Figure 6.1: Scalar exchange solution, $u^{-1} \hat{F}(u, 0.5)$, for conformally coupled scalars as external particles and an internal particle with $\mu = 3$. Comparing with the EFT solution, it is evident that the particular series solution serves as a good solution away from the collapsed limit suffices. For correlators with spinning exchange there is also an angular dependence in terms of Legendre polynomials [26, 5, 27] which gives information about the spin of the intermediate particle.

Acting on these four-point functions of the conformally coupled scalars with spin-raising operators, we can get correlators involving exchange particles of general spin and acting with something called weight-shifting operators the solution for conformally coupled scalars can be mapped to correlators with massless external fields, which is the case relevant to inflation [4, 5] (see figure (6.2)). To obtain inflationary 3-pt functions from de Sitter four point functions, we evaluate one of the external legs on the time-dependent background (i.e., taking one of the momenta to zero) [4, 28, 29, 26]. Ultimately, we observe that the effects of massive particles during inflation are characterised in terms of just two basis functions, viz., scalar exchange interaction ($\hat{F}(u,v)$) and lowest order contact interaction ($\hat{C}_0(u,v)$). In general, the inflationary bispectrum ($B(k_1, k_2, k_3)$) can be written using these two basic building blocks:

$$B(k_1, k_2, k_3) = \mathcal{W}_L \left[ \sum_S a_S S^{(S)} F(u, 1) + \sum_n b_n \Delta^n C_0 \right]$$

(6.1)

where $S^{(S)}$ and $\mathcal{W}$ are the spin-raising and weight-shifting operators respectively.

These inflationary bispectrum can give us measurable quantities. One of the most important of them is non-gaussianity. The amplitude of non-gaussianity denoted by $f_{NL}$ can be measured from bispectrum by defining it to be [3]

$$f_{NL} \equiv \frac{5}{18} \frac{B_R(k, k, k)}{P_R^2(k)}$$

(6.2)

where $B_R$ and $P_R$ are the bispectrum and power spectrum for $\mathcal{R}(x, t)$ (scalar fluctuations), respectively. Thus, determining the four-point function of conformally coupled scalars plays a pivotal role in this theory from which all the other relevant correlators and hence observables can be derived.
Figure 6.2: Schematics of how all the relevant types of correlators can be constructed from the fundamental de Sitter four point functions of conformally coupled scalars.

It is evident that the structures of the four-point correlator have observational signatures in the measurement of non-gaussianity, as $f_{NL}$ is related to the four-point functions by (6.1) and (6.2). Thus, observational probes on non-gaussianity ($f_{NL}$) open a new window for exploring the physics in the early universe and constraining the myriads of inflationary theories. To that end, a far more profound and formal understanding of these cosmological correlators and the associated symmetries is indispensable.
Appendix A

Quasinormal mode behaviour in the static patch

In this appendix, we will derive (3.4) by analysing the quasi-normal modes of fields in de Sitter spacetime. We closely follow the arguments given in [30]

The metric of a four dimensional dS spacetime in static coordinate is given by

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2 \]  

(A.1)

where \( f(r) = 1 - \frac{r^2}{R^2} \) (\( R \) is the minimal radius of dS space) and \( r^2d\Omega^2 \) as usual represents the metric on a two dimensional sphere of radius \( r \). The region covered by the static coordinates is called the static patch or the causal patch. This is the region of de Sitter accessible to a single observer, in the sense that the observer can both send and receive signals to/from this entire region. Also, since the metric is time-independent, the frequency of oscillations does not depend on time.

Any massive free-field in de Sitter space satisfies the equation of motion

\[ \phi_{,\nu}^{;\nu} = m^2 \phi \]  

(A.2)

This equation can be separated by the ansatz \( \phi = (u(r)/r)e^{-i\omega t}Y_l(\Omega) \). Here the spherical harmonics \( Y_l \) is the eigenfunction of two dimensional Laplace-Beltrami operator \( \nabla^2 \) with the eigenvalue \(-l(l+1)\). Going to the coordinates, \( \rho = \int dr/f(r) = R\tanh^{-1}(r/R) \), we can write the radial mode into a Schrodinger-like equation

\[ -\frac{d^2u}{d\rho^2} + V(\rho)u = \omega^2 u \]  

(A.3)

where the effective potential is given by

\[ V(\rho) = -\frac{A}{R^2 \cosh^2(\rho/R)} + \frac{B}{R^2 \sinh^2(\rho/R)} \]  

(A.4)

where \( A \) and \( B \) are defined by

\[ A = 2 - m^2 R^2 \]

\[ B = l(l+1) \]
In the variable \( z = 1 / \cosh^2(\rho/R) \), the differential equation (A.3) becomes

\[
z(1 - z) u'' + \left( 1 - \frac{3}{2} z \right) u' + \frac{1}{4} \left[ \frac{\omega^2 R^2}{z} - \frac{B}{1 - z} + A \right] u = 0 \quad (A.5)
\]

Finally, using the ansatz. \( u = z^\alpha (1 - z)^\beta F(z) \) with arbitrary \( \alpha \) and \( \beta \), we have

\[
z(1 - z) F'' + \left[ 1 + 2\alpha - \left( 2\alpha + 2\beta + \frac{3}{2} \right) z \right] F' + \frac{1}{4} \left( \alpha^2 + \frac{\omega^2 R^2}{4} \right) F = 0 \quad (A.6)
\]

We choose \( \alpha \) ans \( \beta \) such that the terms \( 1/z \) and \( 1/(1-z) \) disappears and the equation (A.6) becomes a hypergeometric differential equation. It is not difficult to see that near the horizon, i.e. \( z \to 0 \), we have \( z^\alpha \sim \exp(\pm i\omega \rho) \), so that the two independent solutions corresponds to incoming and outgoing waves at the cosmological horizon. Solving for the mode functions \( u(z) \) with the boundary condition that these modes are purely outgoing waves at the cosmological horizon and vanishes at \( r = 0 \), we get

\[
u(z) = C z^\alpha (1 - z)^\beta 2F1(a, b; c; z) \quad (A.7)
\]

where \( C \) is an arbitrary constant and the remaining parameters are given by

\[
\begin{align*}
\alpha &= -i\omega R/2 \\
\beta &= \frac{l + 1}{2} \quad \text{or} \quad -\frac{l}{2} \\
c &= 2\alpha + 1 \\
a &= \alpha + \beta + \frac{1}{4}(1 + \sqrt{1 + 4A}) \\
b &= \alpha + \beta + \frac{1}{4}(1 - \sqrt{1 + 4A})
\end{align*}
\]

where

\[
\omega = \pm \frac{1}{R} \sqrt{m^2 R^2 - \frac{9}{4} - \frac{i}{R} (2n + l + \frac{3}{2})} \quad (A.8)
\]

\[or \quad \omega = \pm \frac{1}{R} \sqrt{m^2 R^2 - \frac{9}{4} - \frac{i}{R} (2n - l + \frac{1}{2})} \quad (A.9)\]

with \( n = 0, 1 \cdots \).

Throughout the text we are mostly interested in the late-time behaviour of the fields. Thus, the leading contribution comes from the \( n = 0 \) and \( l = 0 \) term. Still we have two choices. We take the one which gives the appropriate dilution factor (i.e.\( \eta^{3/2} \)). Thus late-time behaviour of the fields goes as (realising that \( R = 1/H \))

\[
\lim_{\eta \to 0} \phi \sim O^+(r, \Omega) (e^{-Ht})^{\Delta^+} + O^-(r, \Omega) (e^{-Ht})^{\Delta^-} \quad (A.10)
\]

where \( \Delta^\pm = \frac{3}{2} \pm i \sqrt{\frac{m^2}{R^2} - \frac{9}{4}} \).
One may think that all the derivation is done on the static coordinates, but throughout the thesis we were working in flat-slicing coordinates. Actually, at late times, both the time coordinates coincide. If we call the static coordinates to be \((t, r)\) and call the flat-slicing coordinates (in spherical folding) to be \((t_f, r_f)\), then relation between the two can be written as \([31]\) (the angles \(\Omega\) are same for both)

\[
r = r_f e^{t_f/R} \quad , \quad e^{-2t/R} = e^{-2t_f/R} - \frac{r_f^2}{R^2}
\]

(A.11)

It is evident from the above expressions that

\[
t \sim t_f \quad \text{when} \quad t_f \to \infty
\]

(A.12)
Appendix B

Mode functions in de Sitter space

In this appendix, we will motivate the form of the solution given in equation (4.9) and then go on to explicitly calculate the mode functions for massive free scalar field in de Sitter space. We will assume that the scalar field carries an insignificant amount of the total energy density and hence, doesn’t backreact on the de-Sitter geometry. Thus, we can ignore the coupling between the field and the background metric and don’t need to incorporate metric fluctuations.

Let us start with the action of a massive scalar field in de Sitter space

\[ S_0 = \frac{1}{2} \int d^4x \sqrt{-g} \left[ g^{\mu \nu} \partial_\mu \phi^I \partial_\nu \phi^I - m^2 (\phi^I)^2 \right] \]

where \( a(\eta) = -1/H\eta \). Here, we have carried the notation from chapter 4. Changing the variable to \( v \equiv a\phi^I \) (Mukhanov variable), the action changes to

\[ S_0 = \frac{1}{2} \int d\eta d^3x a^2 \left[ \dot{v}^2 - (\partial_i v)^2 - m^2 a^2 v^2 + \ddot{a} v^2 \right] \]

Transforming to Fourier space

\[ v(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} v_k(\eta) e^{ik \cdot \mathbf{x}} \]

we get the equation of motion for the Fourier modes of the field

\[ \ddot{v}_k + \left( k^2 + m^2 a^2 - \frac{\ddot{a}}{a} \right) v_k = 0 \]

This equation is similar to a SHO with time-dependent frequency. The expansion of the universe is encoded in the time-dependence of the effective mass \( m_{eff}^2 \equiv m^2 a^2 - \frac{\ddot{a}}{a} \). In de Sitter space we can write this as

\[ \ddot{v}_k + \left( k^2 - \frac{\nu^2 - 1/4}{\eta^2} \right) v_k = 0, \quad \text{where} \quad \nu^2 \equiv \frac{9}{4} - \frac{m^2}{H^2} \]

Next we will solve this second order differential equation using boundary conditions coming from canonical quantization and vacuum selection.
Chapter B: Mode functions in de Sitter space

Canonical quantization

From (B.2) we get the momentum conjugate to $v$

$$\pi \equiv \frac{\partial L}{\partial \dot{v}} = \dot{v} \tag{B.6}$$

For quantization, we promote the fields $v_k(\eta)$ and $\pi_k(\eta)$ to operators $\hat{v}_k(\eta)$ and $\hat{\pi}_k(\eta)$ respectively. These operators satisfies the equal time commutation relation (required for locality)

$$[\hat{v}_k(\eta), \hat{\pi}_{k'}(\eta)] = (2\pi)^3 \delta(k + k') \tag{B.7}$$

where we have taken $\hbar = 1$.

Note that we are working in the Heisenberg picture where the operators evolve in time while states are time independent. Following the canonical quantization of SHO, we can write the operator solution as

$$\hat{v}_k(\eta) = v_k(\eta) a_k + v_k^*(\eta) a_k^\dagger \tag{B.8}$$

where the mode functions $v_k(\eta)(v_k^*(\eta))$ satisfies the classical equation of motions (B.5). The time independent, non-Hermitian operators $a_k$ and $a_k^\dagger$ are the lowering and raising operators, respectively. To satisfy the standard commutation relation for these operators

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta(k + k'),$$

the mode functions must be normalized as

$$v_k \dot{v}_k^* - \dot{v}_k v_k^* = i \tag{B.9}$$

This gives us the first boundary condition for solving the differential equation for the mode functions. The next one comes from the choice of vacuum.

Vacuum selection

The vacuum state $|0\rangle$ is defined by

$$a_k |0\rangle = 0. \tag{B.10}$$

A change in $v_k(\eta)$ could be accompanied by a change in $\hat{a}_k$ that keeps the solution $\hat{v}_k(\eta)$ unchanged. Thus, each such solution corresponds to a different vacuum state. Hence, to proceed further we need a choice of vacuum. A conventional choice is the Bunch-Davies vacuum [32, 21] which is the Minkowski vacuum of a comoving observer in the far past, $\eta \to -\infty$. The standard way this state is chosen is by arguing that at very early times all the comoving scales are deep inside the horizon and thus, doesn’t feel the background curvature. They behave as they are in a flat space and their vacuum state would reflect this. In practice, the $\eta \to -\infty$ limit of the solutions to the mode equation is taken, and the linear combination of the solutions that approaches a positive energy plane wave in this limit (this amounts to demanding the vacuum to be the ground state of the Hamiltonian) is then chosen.
Now, for very early times the mode equation \((B.5)\) becomes the equation of a SHO with time-independent frequency
\[
\ddot{v}_k + k^2 v_k \approx 0. \tag{B.11}
\]
The positive energy solution for the above equation is \(e^{-ik\eta}/\sqrt{2k}\). Thus, selecting Bunch-Davis vacuum amounts to the boundary condition
\[
\lim_{\eta \to -\infty} v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}. \tag{B.12}
\]

**Exact solution**

The most general solution of \((B.5)\) is
\[
v_k(\eta) = \sqrt{-\eta} \left[ c_1 H^{(1)}_{\nu}(-k\eta) + c_2 H^{(2)}_{\nu}(-k\eta) \right] \tag{B.13}
\]
where \(H^{(1)}_{\nu}\) and \(H^{(2)}_{\nu}\) are Hankel functions of the first and second kind respectively.

Using the two boundary conditions, we fix the two arbitrary constants as
\[
c_1 = \frac{\sqrt{\pi}}{2} e^{i(2\nu+1)\pi/4}, \quad c_2 = 0 \tag{B.14}
\]
Writing \(\nu \equiv i\mu\), we can then finally write the massive free scalar field as
\[
\phi_k^f(t) = \phi_k(t)\hat{a}_k + \phi_k^\dagger(t)\hat{a}_k^\dagger \tag{B.15}
\]
where the mode functions \(\phi_k(\eta)\) are given by
\[
\phi_k(\eta) = \frac{H}{\sqrt{\pi}} \frac{e^{i\pi/4} e^{-\pi\mu/2}(-\eta)^{3/2}}{(-k\eta)} H^{(1)}_{\mu}(-k\eta) \tag{B.16}
\]
Two important special cases are
\[
m^2 = 0 : \quad \phi_k(\eta) = H(1 + i\eta\eta) \frac{e^{-ik\eta}}{\sqrt{2k}} \tag{B.17}
\]
\[
m^2 = 2H^2 : \quad \phi_k(\eta) = H(-\eta) \frac{e^{-ik\eta}}{\sqrt{2k}} \tag{B.18}
\]
Appendix C

OPE limit of the exchange interactions

In this appendix, we will analyse the OPE limit of the exchange interactions. This involves taking the $s \to 0$ limit of the four-point function, focusing on the non-analytic terms as a function of $s$. Now momentum going to zero means the separation between the corresponding spatial points is very large. In that case, we can replace the correlators $G_{\pm \pm}(s, \eta, \eta')$ in (4.24) by their long distance expressions given in (5.49). The two terms in (5.49) can be integrated separately. For each of these two terms, the integrals $F_{\pm \pm}$ factorize and hence can be written as

$$F_{\pm \pm}(k_{12}, k_{34}, s) \propto P_{\pm}^{\Delta^+}(k_{12})P_{\pm}^{\Delta^+}(k_{34}) + P_{\pm}^{\Delta^-}(k_{12})P_{\pm}^{\Delta^-}(k_{34})$$ (C.1)

where

$$P_{\pm}(k_{12}) = \pm i \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{\pm ik_{12}\eta}(-\eta)^{\Delta} = -(\mp i)^{\Delta}(s_{12})^{\Delta-1}\Gamma(\Delta-1)$$ (C.2)

Thus for the first term we can write

$$[P_{\pm}^{\Delta^+}(k_{12}) + P_{\pm}^{\Delta^-}(k_{12})][P_{\pm}^{\Delta^+}(k_{34}) + P_{\pm}^{\Delta^-}(k_{34})]$$

$$= 2(1 + i \sinh \pi \mu)(k_{12}k_{34})^{-\frac{1}{2} - i\mu}\Gamma\left(\frac{1}{2} + i\mu\right)^2$$ (C.3)

For the second term (with $\Delta^-$ instead of $\Delta^+$), we get the same result as above but with $\mu \to -\mu$. Putting all of this together, including the numerical factors, we get

$$\langle \phi_1 \cdots \phi_4 \rangle_{s \to 0} \sim \frac{\eta_0^4 \lambda^2}{4k_{12}k_{34}k_4} F$$ (C.4)

where

$$F \sim \frac{1}{\sqrt{k_{12}k_{34}}} \left[ \left( \frac{s^2}{4k_{12}k_{34}} \right)^{i\mu} (1 + i \sinh \pi \mu) \frac{\Gamma\left(\frac{1}{2} + i\mu\right)^2 \Gamma(-i\mu)^2}{2\pi} + c.c \right]$$ (C.5)

This is same as (5.46) in the $u, v$ variable. It is evident from (C.1) and (C.3) that there is some kind of factorization of the four-point function in this limit. To better understand this factorization let us analyse what the functions $P_{\pm}$ really are,
Chapter C: OPE limit of the exchange interactions

namely, what differential equation do they satisfy. We had done a similar evaluation in section 4.3 when we derived the the differential equation satisfied by the four-point function. Taking inspiration from there, we hit $P_\pm(k_{12})$ by the operator $\left( (k_{12}^2 - s^2) \partial_{k_{12}}^2 - 2k_{12}\partial_{k_{12}} + m^2 - 2 \right)$:

\[
\left( (k_{12}^2 - s^2) \partial_{k_{12}}^2 - 2k_{12}\partial_{k_{12}} + m^2 - 2 \right) P_\pm(k_{12}) \\
= \pm i \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{\pm ik_{12}\eta} \left( (\eta^2 - s^2) \partial_{\eta}^2 - 2\eta\partial_{\eta} + m^2 \right) (-\eta)^{-\Delta} \\
= \pm i \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{\pm ik_{12}\eta} \left( (\Delta - 1) - 2\Delta + m^2 \right) (-\eta)^{-\Delta}
\]  
\[\text{(C.6)}\]

where in the second line we have ignore the $s^2\eta^2$ term in the limit $s \to 0$. Putting $\Delta = \Delta^\pm = 3/2 \pm i\mu$ we get $\Delta(\Delta - 1) - 2\Delta + m^2 = 0$. Thus, going to the $u, v$ variable we see that the functions $P_\pm^\Delta(u)$ satisfies the differential equation

\[
\left[ \Delta_u + \left( \mu^2 + \frac{1}{4} \right) \right] P_\pm^\Delta(u) = 0 \quad \text{(C.7)}
\]

The function $P_\pm^\Delta(v)$ (corresponding to $k_{34}$) satisfies a similar equation with $u \to v$. Coincidentally, the three-point correlators satisfies the same differential equation [4], which can be easily derived by carrying out the steps done in the case of two and four point functions. Thus it is clear that in this OPE limit the four-point function factorizes into a product of two three-point functions.
Appendix D

Excited initial states

In this appendix, we will show how initial excited states produce singularities in the folded limit. Here, the phrase "initial state" means the quantum state at very early time or equivalently, at the beginning of the inflationary phase. As already explained earlier the standard choice for the initial state is the BD vacuum or adiabatic vacuum. For BD vacuum, the dominant contribution to the three point function, which measures directly three particle interactions, comes from when the modes cross the horizon. In inflationary context, there is a intuitive explanation for this. At sub-horizon scales, the BD vacuum corresponds to no-particle state. As there is no particle to interact, there is no contribution to the three-point functions. As the modes cross the horizon, the WKB approximation breaks down and particles are created (more on this in Appendix E). These new particles now interact and contribute to the three point functions. Once the modes are well outside the horizon, they get frozen and further interactions become irrelevant.

The situation changes when we have excited initial states. In this case, particles are present from the very beginning which can then interact to give contributions to the three-point functions. For bi-spectrum, these contributions dominate for the flattened triangles (also called the folded or collinear limit), which was explicitly shown in [33]. Following the same methodology we will show that the case is same for the folded limit of four-point functions (which is the case we are interested in).

We will take the initial excited state to be a Bogoliubov transform of the BD state (more physical intuition on this state will be developed in Appendix F)

\[
\tilde{\phi}_k(\eta) = \alpha_k \phi_k(\eta) + \beta_k \phi_k^*(\eta)
\]

where \(\tilde{\phi}_k\) are the Bogoliubov mode functions and as usual \(\phi_k\) are the BD modes. Note that this state reduces to the BD state for \(\alpha_k = 1\) and \(\beta_k = 0\). Thus we must have \(\beta_k = 0\) for excited states. Hence, initially we have a non-zero particle number density given by \(|\beta_k|^2\) for particles of momentum \(k\).

We will work with the four-point correlator for conformally coupled scalars as done in the main text. For exchange interaction we will consider a toy model \(\frac{1}{6} \lambda \phi^3\) for which the exchange scalar is also a conformally coupled scalar. We have taken this toy model because along with the simplicity of the calculations, it provides

\[\footnote{Here, we are talking about the adiabatic particle number, which will be discussed in Appendix E} \]
us with the singularity structure similar to the general case. Thus the four-point function for the excited initial state becomes (replace the BD modes with bogoliubov modes in (4.20))

\[
\langle \phi_1 \cdots \phi_4 \rangle' \propto \frac{\lambda^2}{\hbar^8 \eta^4 \eta'^4} (2\pi)^3 \delta^3 \left( \sum_i k_i \right) \tilde{\phi}_{k_1}(\eta) \tilde{\phi}_{k_1}^*(\eta_0) \tilde{\phi}_{k_2}(\eta) \tilde{\phi}_{k_2}^*(\eta_0) \times \tilde{\phi}_{k_3}(\eta_0) \tilde{\phi}_{k_3}^*(\eta') \tilde{\phi}_{k_4}(\eta) \tilde{\phi}_{k_4}^*(\eta') (D.2)
\]

Expanding the above equation with the help of (D.1), we find that there are corrections from the \( \beta \) terms which involve change in the argument of the oscillatory exponential. In the folded limit, we get singularities from the terms which have \( k_1 + k_2 - s = k_{12}(1-u) \) as argument of the exponentials. The integral then picks up contributions from \( \eta = \eta' = -\infty \) and we get

\[
\lim_{u \to 1} \langle \phi_1 \cdots \phi_4 \rangle' \propto \beta_s \int_{-\infty}^{0} \frac{1}{\eta} e^{-ik_{12}(1-u)\eta} \sim \beta_s \log(1-u) (D.3)
\]

We see that we get the same logarithmic singularity as in equation (5.41) which we now understand is an artefact of having initial excited states.
Appendix E

Particle production in a time-dependent background

In this appendix, we will discuss on how time dependent background gives rise to particle production. For greater details, the reader can go to [34, 35]. We will work in the cosmological context where we will consider particle production in de Sitter space [36]. We all know that for QFT in static background, the time-independent annihilation operators $a_k$ define the vacuum state of the theory (by $a_k |0⟩ = 0$) and creation operators $a_k^\dagger$ produces particles on hitting with this vacuum state. There is a particle number operator defined by $a_k^\dagger a_k$ whose expectation value with a given state gives the average number of particles in that state. The vacuum state has zero number of particles which is what we expect.

$$N_k = ⟨0| a_k^\dagger a_k |0⟩ = 0 \quad (E.1)$$

In a background evolving with time, the situation changes drastically. The initial vacuum state will not be the true vacuum at future times. In fact, there is no well-defined vacuum at intermediate times. The annihilation and creation operators will evolve with time. We will now see quantitatively how it works.

We will work in the cosmological context where we will consider particle production in de Sitter space. For this purpose, let’s recall our free scalar fields in de Sitter space-time given in Appendix B. The equation of motion for mode functions $φ_k(t)$ is the Klein-Gordon equation for de Sitter metric. In the suitable Mukhanov variable $v ≡ aφ$ the equation of motion can be written as (see (B.5))

$$\ddot{v}_k + ω_k(t)v_k = 0 \quad (E.2)$$

which is a SHO equation with time-dependent frequency $ω_k(t) = k^2 + m^2a^2 - \frac{\dot{a}}{a}$. We solved this equation once in Appendix B corresponding to the initial time-independent creation and annihilation operators. But to manifestly represent the process of particle production we need to solve the mode function equation corresponding to the time-dependent creation and annihilation operators $\tilde{a}_k^\dagger(t)$ and $\tilde{a}_k(t)$. These time-dependent operators can be written as a Bogoliubov transform of the time-independent creation and annihilation operators $a_k^\dagger$ and $a_k$, defined at initial time [37]

$$\begin{bmatrix} \tilde{a}_k(t) \\ \tilde{a}_k^{-1}(t) \end{bmatrix} = \begin{bmatrix} α_k(t) & β_k^*(t) \\ β_k(t) & α_k^*(t) \end{bmatrix} \begin{bmatrix} a_k \\ a_k^{-1} \end{bmatrix} \quad (E.3)$$

53
where unitarity\(^1\) requires \(|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1\), for all \(t\). For an equivalent decomposition of the scalar fields as given in (B.8), we define the reference mode functions \(\tilde{v}_k(t)\) corresponding to the time-dependent creation and annihilation operators as

\[
\tilde{v}_k(t) = \tilde{v}_k(t)\hat{a}_k(t) + \tilde{v}_k^*(t)\hat{a}_k^\dagger(t)
\]

(E.4)

where the reference mode functions can be defined as \(^2\)

\[
\tilde{v}_k(t) = \frac{1}{\sqrt{2W_k}}e^{-i\int W_k(t)dt} \frac{t \rightarrow \infty}{1/2\omega_k(-\infty)t}\]

(E.5)

where \(\omega_k(-\infty) = k\). Clearly, there is an infinite number of such reference mode functions, all having the same initial asymptotic behaviour. The problem is to find physically suitable set of mode functions for use at intermediate times. For the reference mode functions \(\tilde{v}_k(t)\) to be a solution of the Klein-Gordon equation of motion, the function \(W_k(t)\) must satisfy the differential equation

\[
W_k^2(t) = \omega_k^2(t) - \left[\frac{\dot{W}_k(t)}{2W_k(t)} - \frac{3}{4} \left(\frac{\dot{W}_k(t)}{W_k(t)}\right)^2\right].
\]

(E.6)

The Bogoliubov transformation can also be interpreted as a linear transformation between the exact mode functions \(v_k(t)\) and the reference mode functions \(\tilde{v}_k(t)\):

\[
v_k(t) = \alpha_k(t)\tilde{v}_k(t) + \beta_k(t)\tilde{v}_k^*(t).
\]

(E.7)

This form is used in Appendix D to describe the initial excited states. It is clear that \(\alpha_k(-\infty) = 1\) and \(\beta_k(-\infty) = 0\) corresponds to Bunch-Davies initial vacuum while \(\beta_k(-\infty) \neq 0\) gives excited initial states.

We also need to specify the transformation of the scalar field momentum operator \(\pi^\dagger_k = \phi_k\):

\[
\pi^\dagger_k = Q_k(t)\tilde{v}_k(t)\hat{a}_k(t) + Q_k^*(t)\tilde{v}_k^*(t)\hat{a}_k^\dagger(t)
\]

(E.8)

with a corresponding decomposition of the first derivative:

\[
\dot{v}_k(t) = Q_k(t)\tilde{v}_k(t)\alpha_k(t) + Q_k^*(t)\tilde{v}_k^*(t)\beta_k(t)
\]

(E.9)

where \(Q_k(t)\) is defined as

\[
Q_k(t) = -iW_k(t) + V_k(t)
\]

(E.10)

The inclusion of real time dependent function \(V_k(t)\), specified later, in (E.8) and (E.9) represents the most general decomposition of the exact mode functions \(v_k\) that is consistent with the unitarity (the preservation of bosonic commutation relations).

Corresponding to the static background case, the time-dependent adiabatic particle number, \(\tilde{N}_k(t)\), is defined for each mode \(k\) by the expectation value in the original vacuum state \(|0\rangle\) of the time-dependent number operator \(\hat{a}_k^\dagger(t)\hat{a}_k(t)\)

\[
\tilde{N}_k(t) = \langle 0| \hat{a}_k^\dagger(t)\hat{a}_k(t) |0\rangle = |\beta_k(t)|^2.
\]

(E.11)

\(^1\)This is essentially the condition for the transformation to be canonical.

\(^2\)We are just generalizing the plane wave mode structure in the initial time with a function \(W_k(t)\). See below for the differential equation which \(W_k(t)\) has to satisfy.
For Bunch-Davies vacuum this means at $t \to -\infty$ there are no particles (as $\beta_k(-\infty) = 0$) but with time, as the universe expands, particles are being created with the average particle number given by $|\beta_k(t)|^2$. In a time-dependent background field there is no unique separation into positive and negative energy states [34, 37]. The freedom in choice of $W_k(t)$ and $V_k(t)$ encodes this arbitrariness of specifying positive and negative energy states at intermediate times. The adiabatic particle number thus depends on the choice of the basis $(W_k(t), V_k(t))$. Finally, all that remains is to evaluate the time evolution of the Bogoliubov transformation parameters $\alpha_k(t)$ and $\beta_k(t)$.

Substituting (E.7) and (E.8) in (E.2), we get the evolution equation of the Bogoliubov coefficients for arbitrary $W_k(t)$ and $V_k(t)$ [38]:

$$
\begin{bmatrix}
\dot{\alpha}_k(t) \\
\dot{\beta}_k(t)
\end{bmatrix} =
\begin{bmatrix}
\delta_k & [\Delta_k + \delta_k]^* e^{2i\int^t W_k} \\
[\Delta_k + \delta_k^* e^{-2i\int^t W_k}] & \delta_k^*
\end{bmatrix}
\begin{bmatrix}
\alpha_k(t) \\
\beta_k(t)
\end{bmatrix}
$$

(E.12)

where

$$
\delta_k \equiv \frac{1}{2iW_k} \left(\omega_k^2 - W_k^2 + \left(\dot{V}_k + V_k^2\right)\right)
$$

(E.13)

$$
\Delta_k \equiv \frac{\dot{W}_k}{2W_k} + V_k.
$$

(E.14)

The conventional choices are based on a WKB approximation, taking $W_k(t) = \omega_k(t)$. Solving this coupled differential equation with the initial condition $\alpha_k(-\infty) = 1$ and $\beta_k(-\infty) = 0$ and a particular choice of basis, we can get explicit form of the Bogoliubov coefficients at intermediate times.
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