ALMOST OPTIMAL LOCAL WELL-POSEDNESS OF THE CHERN-SIMONS-DIRAC
SYSTEM IN THE COULOMB GAUGE

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ABSTRACT. In this paper, we consider the Cauchy problem of regularity and uniqueness of the Chern-Simons-Dirac system in the Coulomb gauge for initial data in $B^0_{2,1}$. The novelty of this paper is on proving almost critical regularity by using the fully localization of space-time Fourier side and bilinear estimates given by Selberg \[17\]. We also prove the Dirac spinor flow of Chern-Simons-Dirac system cannot be $C^3$ at the origin in $H^s$ if $s < 0$.

1. INTRODUCTION

We consider the Cauchy problem for Chern-Simons-Dirac system (CSD) under Coulomb gauge condition, namely,

\[ i\gamma^\mu \partial_\mu \psi = m\psi - N(\psi, \psi)\psi, \]

\[ \psi(0) = \psi_0 \in B^0_{2,1}(\mathbb{R}^2) \]

where $N$ is the bilinear operator given by

\[ N(\psi_1, \psi_2) = \frac{1}{\Delta} \left( \gamma^0 \left[ \partial_1(\overline{\psi_1}\gamma^2 \psi_2) - \partial_2(\overline{\psi_1}\gamma^1 \psi_2) \right] + \gamma^1 \partial_2(\overline{\psi_1}\gamma^0 \psi_2) - \gamma^2 \partial_1(\overline{\psi_1}\gamma^0 \psi_2) \right), \]

on the Minkowski space $\mathbb{R}^{1+2}$ equipped with the Minkowski metric of signature $(+, -, -)$. We adopt the Einstein summation convention, where Greek indices refer to 0, 1, 2 and Latin indices refer to 1, 2. Here $\psi$ is the spinor field represented by a column vector with two complex components. $\psi^\dagger$ is the complex conjugate transpose of $\psi$, whereas $\overline{\psi}$ is defined by $\overline{\psi} = \psi^\dagger\gamma^0$. The totally skew-symmetric tensor $\epsilon_{\mu\nu\lambda}$ is characterized by $\epsilon_{012} = 1$ and the nonnegative constant $m$ is the mass of the spinor field $\psi$.

The (CSD) system \[1.1\] is rewritten in the Coulomb gauge $\partial_1A_1 + \partial_2A_2 = 0$ from the system of curvature form, introduced in Li-Bhaduri \[13\] and Cho, Kim, and Park \[5\] to consider $(2 + 1)$-dimensional Chern-Simons-Dirac electrodynamics which gives a good description on the fractional quantum Hall effect and superfluid. We refer to \[11, 9\] for more physical issue. The (CSD) system has the conservation of charge $Q(t) = \int_{\mathbb{R}^2} |\psi(t, x)|^2 dx = Q(0)$ and $L^2$-scaling invariance in the case of $m = 0$, so the (CSD) system is charge critical. In this viewpoint, it is very interesting to lower the regularity of solutions to \[1.1\] to $L^2$ initial data.

In this paper we study the local well-posedness (LWP) of \[1.1\] in the Besov space $B^0_{2,1}$. The Besov space $B^0_{2,1}$ is defined by $\{ f \in L^2 : \| f \|_{B^0_{2,1}} := \sum_{N : \text{dyadic}} \| P_N f \|_{L^2_2} < \infty \}$, where $P_N$ is the Littlewood-Paley projection on $\{ \xi \in \mathbb{R}^2 : |\xi| \sim N \}$. Note that $H^s \not\subseteq B^0_{2,1} \subseteq L^2$ for $s > 0$, where the Sobolev space $H^s$ is defined by $\{ f \in L^2 : \| f \|_{H^s} := (\sum N^{2s} \| P_N f \|_{L^2_2})^{1/2} < \infty \}$. Our main result is stated as follows.

**Theorem 1.1.** Suppose that $\psi_0 \in B^0_{2,1}$. Then there exists $T = T(\| \psi_0 \|_{B^0_{2,1}}, m) > 0$ such that there exists unique solution $\psi \in C((-T, T); B^0_{2,1})$ of \[1.1\], which depends continuously on the initial data.
The Cauchy problem for Chern-Simons-Dirac system has been studied by Huh [11], in the several gauge, namely, temporal gauge, Coulomb gauge, and Lorenz gauge. After [11], Huh and Oh [12] proved that (CSD) system under the Lorenz gauge is locally well-posed in $H^s$ for $s > \frac{1}{4}$. Also Okamoto [15] proved the same regularity. Pecher [16] studied (CSD) in Fourier-Lebesgue space $\hat{H}^{s,r}$ for $1 < r \leq 2$ and $s > \frac{3}{2r} - \frac{1}{2}$. In [4] the authors proved the local well-posedness in $B^{1, \frac{4}{2}, 2}_1$, Under the Coulomb gauge, Okamoto [15] showed the LWP of (1.1) in $H^s$ for $s > \frac{1}{4}$, using global estimates in [6]. Also, Bournaveas, Candy, and Machihara [3] showed LWP of (1.1) for the same regularity condition, using the elliptic structure in the Coulomb gauge without exploiting the null structure of (1.1).

The novelty of Theorem 1.1 is to attain almost optimal regularity, in viewpoint of the conservation of charge and $L^2$-scaling invariance and ill-posedness in $H^s$ for $s < 0$. (See Theorem 1.2.) The key point of the proof of Theorem 1.1 is based on the argument in [4]. That is, we make use of duality in Besov type $X^{s,b}$ space. Since (1.1) is Dirac equation with cubic nonlinear, one may encounter quadralinear terms. In order to handle the quadralinear terms, we reveal the null structures introduced in [2], [7], [12] and exploit the 2D bilinear estimates of Selberg [17]. A dyadic decomposition of space-time Fourier side force us to treat the low-low-high modulation, which is the most serious case. In [4], the authors proved this case does not appear in the summation. In this paper, we follow the argument in [4]. In other words, we deduce that the low-low-high modulation with high-high-low frequency and high-low-high frequency cases do not occur in the summation by exploiting the support condition and apply angle Whitney decomposition for remaining cases. (See Remark 5.1.) By using $l^1$ summation on the modulation and frequency, we attain the critical index $s = 0$.

From Theorem 1.1 with $L^2$-scale invariance, one may expect the ill-posedness of (1.1) in $H^s$ for $s < 0$. Indeed, we prove the failure of smoothness of the flow map of (1.1).

**Theorem 1.2.** Let $s < 0$ and $T > 0$. Then the flow map of $\psi_0 \mapsto \psi$ from $H^s(\mathbb{R}^2)$ to $C([-T, T]; H^s(\mathbb{R}^2))$ cannot be $C^3$ at the origin.

Under the Lorenz gauge, in [15] the author showed that the flow of vector potential $A$ is not $C^2$ in $H^s$ for $s < \frac{1}{4}$. Also the authors in [4] proved that the Dirac spinor flow $\psi$ cannot be $C^2$. The Theorem 1.2 is the first result on the failure of smoothness of (CSD) under the Coulomb gauge condition. We prove Theorem 1.2 by following the argument in [10] and [14].

Our paper is organized as follows. In Section 2, we give some preliminaries on Dirac projection operator and Besov type $X^{s,b}$ space. In Section 3, we introduce 2D bilinear estimates of wave type and give some discussion on null structure. In Section 4, we construct Picard’s iterates and give a sketch of proof of Theorem 1.1. Then Section 5,6 are devoted to the proof of crucial parts for LWP. Finally, in the last section, we prove the failure of smoothness of (CSD) system.

Notations.

- Since we often use $L^2_{t,x}$ norm, we abbreviate $\|F\|_{L^2_{t,x}}$ by $\|F\|$.
- The spatial Fourier transform and space-time Fourier transform on $\mathbb{R}^2$ and $\mathbb{R}^{1+2}$ are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx, \quad \hat{u}(X) = \int_{\mathbb{R}^{1+2}} e^{-i(t \tau + x \cdot \xi)} u(t, x) dt dx,$$
where \( \tau \in \mathbb{R}, \xi \in \mathbb{R}^2, \) and \( X = (\tau, \xi) \in \mathbb{R}^{1+2}. \) Also we denote \( \mathcal{F}(u) = \tilde{u}. \)

- We denote \( D := |\nabla| \) whose symbol is \( |\xi|. \)
- \( N_{\min}^{012}, N_{\med}^{012}, \) and \( N_{\max}^{012} \) stand for the minimum, median and maximum of \( \{N_1, N_2, N_3\}, \) respectively.
- For any \( E \subset \mathbb{R}^{1+2} \) the projection operator \( P_E \) is defined by \( P_E u(\tau, \xi) = \chi_E \tilde{u}(\tau, \xi). \)
- As usual different positive constants depending only on \( N \) are denoted by the same letter \( C, \) if not specified. \( A \lesssim B \) and \( A \gtrsim B \) means that \( A \leq CB \) and \( A \geq C^{-1}B, \) respectively for some \( C > 0. \) \( A \sim B \) means that \( A \lesssim B \) and \( A \gtrsim B. \)

2. Preliminaries

2.1. Dirac operator. Let \( \eta_{\mu\nu} \) be the Minkowski metric on \( \mathbb{R}^{1+2} \) with signature \((+, -, -).\) We define the gamma matrices \( \gamma^\mu(\mu = 0, 1, 2) \) as follows:

\[
\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^2, \quad \gamma^2 = -i\sigma^1,
\]

where \( \sigma^j(j = 1, 2, 3) \) are the Pauli matrices given by

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then we have the following algebraic properties:

\[
\frac{1}{2}(\sigma^i \sigma^k + \sigma^k \sigma^i) = \delta^{jk} I_{2 \times 2}, \quad \sigma^1 \sigma^2 \sigma^3 = iI_{2 \times 2}.
\]

Using this identity, we note that \( \gamma^\mu \) satisfies the following multiplication law:

\[
\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \eta^{\mu\nu} I_{2 \times 2}.
\]

From now on, for the convenience, we use the original \( \beta, \alpha^i \) formulation of the Dirac operator, which is first used by P. M. Dirac. To do this, we multiply the gamma matrices by \( \gamma^0 \) on the left and define

\[
\beta = \gamma^0, \quad \alpha^0 = I_{2 \times 2}, \quad \alpha^1 = \gamma^0 \gamma^1 = \sigma^1, \quad \alpha^2 = \gamma^0 \gamma^2 = \sigma^2.
\]

Then we have the following relations:

\[
\beta \alpha^j + \alpha^j \beta = 0, \quad \frac{1}{2}(\alpha^j \alpha^j + \alpha^j \alpha^i) = \delta^{ij} I_{2 \times 2}.
\]

Thus using the notation \( \beta, \alpha^i, \) (1.1) is rewritten as

\[
(2.1) \quad i(\partial_0 + \alpha^j \partial_j)\psi = m\beta \psi + \mathcal{N}(\psi, \psi)\psi,
\]

where

\[
\mathcal{N}(\psi_1, \psi_2) = \frac{1}{2} \left( \epsilon_{ijk} \alpha^0 \partial_j (\psi_1^i \alpha^k \psi_2) + \epsilon_{lm0} \alpha^l \partial_m (\psi_1^i \alpha^0 \psi_2) \right).
\]

Now we define Dirac projection operator by

\[
\Pi_{\pm}(\xi) = \frac{1}{2} \left( I_{2 \times 2} \pm \frac{\xi \alpha^j}{|\xi|} \right).
\]

This projection operator has the following algebraic relations:

\[
\Pi_{\pm}(\xi) \Pi_{\pm}(\xi) = \Pi_{\pm}(\xi), \quad \Pi_{\pm}(\xi) \Pi_{\mp}(\xi) = 0.
\]

We note the useful identity

\[
\alpha^i \Pi_{\pm}(\xi) = \Pi_{\mp}(\xi) \alpha^i + \frac{\xi^i}{|\xi|} I_{2 \times 2}.
\]
To reveal the null structures in the next section, it is convenient to introduce the modified Riesz transform $R^\mu$, which are self-adjoint operators defined as follows.

$$R^\mu_\pm = -1, \quad R^\mu_\pm = -R^\mu_{\pm,j} = \mp \frac{\partial_j}{iD}.$$  

We shall use the notation $\Pi_\pm := \Pi_\pm (-i\nabla)$. Using the above identities, we get

$$\alpha^\mu \Pi_\pm = \Pi_\pm \alpha^\mu \Pi_\pm - R^\mu_\pm \Pi_\pm.$$  

We let $\psi_\pm := \Pi_\pm \psi$ and $\psi_\pm^{\text{hom}} = e^{\mp iT D} \psi_{0,\pm}$. By applying Dirac projection operator $\Pi_\pm$, (1.1) becomes

(2.2) \quad - (i\partial_0 \pm D) \psi_\pm = m\beta \psi_\mp + \Pi_\pm (\mathcal{N}(\psi, \psi))\psi,$$

and finally, by Duhamel’s principle, we have the following integral equation:

(2.3) \quad \psi_\pm (t, x) = e^{\mp iT D} \psi_{0,\pm} - i \int_0^t e^{\mp i(t-t')D} m\beta \psi_\mp (t') dt' + i \int_0^t e^{\mp i(t-t')D} \Pi_\pm (\mathcal{N}(\psi, \psi)) (t') dt'$$

2.2. Function spaces. In this subsection we introduce the Besov type $X^{s,b}$ space and its energy estimate lemma. Instead of considering square-sum ($l^2$) on the space-time frequencies, we can attain the critical regularity and uniqueness directly by using fully localization in the space-time Fourier side and ($l^1$) summation. Now we define function spaces.

**Definition 2.1.** Let $s, b \in \mathbb{R}$. Let $N, L \geq 1$ be dyadic number. We define Besov type $X^{s,b}$ space by

$$\|u\|_{B^{s,b}_{+,1}} = \sum_{N,L \geq 1} N^s L^b \|P_{K^{s,b}_{N,L}} u\|$$

and

$$\|u\|_{B^{s,b}_{+,\infty}} = \sup_{N,L \geq 1} N^s L^b \|P_{K^{s,b}_{N,L}} u\|,$$

where

$$K^{s,b}_{N,L} = \{(\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \sim N, \ |\tau + \pm|\xi|| \sim L\}.$$  

Then the $B^{s,b,1}_{\pm}$ norm can be recovered as follows:

(2.4) \quad \|u\|_{B^{s,b,1}_{\pm}} = \sup_{\|v\|_{B^{s,b,\infty}_{\pm,-b,\infty}} = 1} \int u \bar{v} dt dx.$$

Now we consider the time-slab

$$S_T = (-T, T) \times \mathbb{R}^2.$$  

Define

$$\|u\|_{B^{s,b,1}_{\pm}(S_T)} = \inf_{v = u \text{ on } S_T} \|v\|_{B^{s,b,1}_{\pm}}.$$  

This becomes a semi-norm on $B^{s,b,1}_{\pm}$, but is a norm if we identify elements which agree on $S_T$ and the resulting space is denoted $B^{s,b,1}_{\pm}(S_T)$. In other words, $B^{s,b,1}_{\pm}(S_T)$ is the quotient space $B^{s,b,p}_{\pm} / X$, where $X = \{v \in B^{s,b,p}_{\pm} : v = 0 \text{ on } S_T\}$. Since $X$ is a closed subspace in $B^{s,b,1}_{\pm}$, we conclude that the quotient $B^{s,b,1}_{\pm}(S_T)$ is a Banach space.

To control the nonlinear terms of (2.2) in $B^{s,b,1}_{\pm}(S_T)$, we introduce the energy estimate lemma.
Lemma 2.2. Let us consider the integral equation:
\[ v(t) = e^{i\langle t \rangle D} f + \int_0^t e^{i\langle t' \rangle D} F(t') dt'. \]
with sufficiently smooth \( f \) and \( F \). If \( T \leq 1 \), then for any \( s \in \mathbb{R} \) we have
\[ \|v\|_{B^s_{r,1}(S_T)} \lesssim \|f\|_{B^s_{r,1}} + \|F\|_{B^s_{r,-1}(S_T)}. \]
The proof is straightforward. The readers can find the details in the Appendix in [4].

3. Bilinear estimates and Null structure

3.1. Bilinear estimates. For dyadic \( N, L \geq 1 \), let us invoke that
\[ K^\pm_{N, L} = \{ (\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \sim N, \ |\tau \pm |\xi|| \sim L \}. \]
To handle the nonlinear terms in (2.3), we utilize the 2-dimensional bilinear estimates of wave type shown by Selberg.

Theorem 3.1 (Theorem 2.1 of [17]). For all \( u_1, u_2 \in L^2_{t,x}(\mathbb{R}^{1+2}) \) such that \( \tilde{u}_j \) is supported in \( K^\pm_{N_j, L_j} \), the estimate
\[ \|P_{K^\pm_{N_0, L_0}}(u_1 \overline{u_2})\| \leq C \|u_1\| \|u_2\| \]
holds with
\[ C \sim (N_{\min}^{0j}L_{\min}^{12})^{1/2}(N_{\max}^{0j}L_{\max}^{12})^{1/4}, \]
\[ C \sim (N_{\min}^{0j}L_{\min}^{0j})^{1/2}(N_{\max}^{0j}L_{\max}^{0j})^{1/4}, \quad j = 1, 2, \]
regardless of the choices of signs \( \pm_j \).

In the proof of Theorem 3.1 we must encounter the low-low-high modulation. In this case, since the summation on modulation is not obvious, we cannot get the required estimates with using only Theorem 3.1. To overcome this case, we first apply the angular Whitney decomposition of [17] as follows: For \( \gamma, r > 0 \) and \( \omega \in S^1 \), where \( S^1 \subset \mathbb{R}^2 \) is the unit circle, we define
\[ \Gamma_\gamma(\omega) = \{ \xi \in \mathbb{R}^2 : \angle(\xi, \omega) \leq \gamma \}, \]
\[ T_r(\omega) = \{ \xi \in \mathbb{R}^2 : |P_{\omega \perp} \xi| \lesssim r \}, \]
where \( P_{\omega \perp} \) is the projection onto the orthogonal complement \( \omega \perp \) of \( \omega \) in \( \mathbb{R}^2 \). Also we let \( \Omega(\gamma) \) denote a maximal \( \gamma \)-separated subset of the unit circle. Then for \( 0 < \gamma < 1 \) and \( k \in \mathbb{N} \), we have
\[ \chi_{\angle(\xi_1, \xi_2) \leq k \gamma} \lesssim \sum_{\omega_1, \omega_2 \in \Omega(\gamma)} \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) \chi_{\Gamma_\gamma(\omega_2)}(\xi_2), \]
for all \( \xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\} \) with \( \angle(\omega_1, \omega_2) > 0 \).

Second, we apply the following null form estimate.

Theorem 3.2 (Theorem 2.3 of [17]). Let \( r > 0 \) and \( \omega \in S^1 \). Then for all \( u_1, u_2 \in L^2_{t,x}(\mathbb{R}^{1+2}) \) such that \( \tilde{u}_j \) is supported in \( K^\pm_{N_j, L_j} \), we have
\[ \|B_{\tilde{u}_j}(P_{T_r}(\omega)u_1, u_2)\| \lesssim (rL_1L_2)^{1/2}\|u_1\|\|u_2\|. \]
Here, the bilinear form \( B_{\theta_{12}}(u_1, u_2) \) is defined on the Fourier side by inserting the angle \( \theta_{12} = \angle(\pm_1 \xi_1, \pm_2 \xi_2) \) in the convolution of \( u_1 \) and \( u_2 \); that is,

\[
\mathcal{F}B_{\theta_{12}}(u_1, u_2)(X_0) = \int_{X_0 = X_1 + X_2} \angle(\pm_1 \xi_1, \pm_2 \xi_2) \tilde{u}_1(X_1) \tilde{u}_2(X_2) dX_1 dX_2.
\]

3.2. Bilinear interaction. The space-time Fourier transform of the product \( \psi_1^\dagger \psi_2 \) of two spinor fields \( \psi_1 \) and \( \psi_2 \) is given by

\[
\tilde{\psi}_1^\dagger \psi_1(X_0) = \int_{X_0 = X_1 - X_2} \tilde{\psi}_2^\dagger (X_2) \tilde{\psi}_1(X_1) dX_1 dX_2,
\]

where \( \psi^\dagger \) is the transpose of complex conjugate of \( \psi \). Here the relation between \( X_1 \) and \( X_2 \) in the convolution integral of spinor fields is given by \( X_0 = X_1 - X_2 \) so called bilinear interaction. This is also the case for the product of two complex scalar fields. The following lemma is on the bilinear interaction.

**Lemma 3.3** (Lemma 2.2 of [17]). Given a bilinear interaction \( (X_0, X_1, X_2) \) with \( \xi_j \neq 0 \), and signs \( (\pm_0, \pm_1, \pm_2) \), let \( h_j = \tau_j \pm_j |\xi_j| \) and \( \theta_{12} = |\angle(\pm_1 \xi_1, \pm_2 \xi_2)| \). Then we have

\[
\text{max}(|h_0|, |h_1|, |h_2|) \gtrsim \text{min}(|\xi_1|, |\xi_2|) \theta_{12}^2.
\]

Moreover, we either have

\[|\xi_0| \ll |\xi_1| \sim |\xi_2| \quad \text{and} \quad \pm_1 \neq \pm_2\]

in which case

\[\theta_{12} \sim 1 \quad \text{and} \quad \text{max}(|h_0|, |h_1|, |h_2|) \gtrsim \text{min}(|\xi_1|, |\xi_2|),\]

or else we have

\[\text{max}(|h_0|, |h_1|, |h_2|) \gtrsim \frac{|\xi_1| |\xi_2|}{|\xi_0|} \theta_{12}^2.\]

Note that using interpolation with (3.4) and the trivial inequality \( \theta_{12} \leq 1 \), one can obtain

\[
\theta_{12} \lesssim \left( \frac{\text{max}(|h_0|, |h_1|, |h_2|)}{\text{min}(|\xi_1|, |\xi_2|)} \right)^p, \quad 0 \leq p \leq \frac{1}{2}.
\]

3.3. Null structure. In viewing the integral equation (2.3), we must encounter bilinear forms of wave type. The worst interaction resulting in resonance happens when two waves are collinear. But if bilinear forms have a cancellation property, then we expect it to yield better estimates. For this purpose, we reveal null structures hidden in (2.3).

The following two lemmas states that bilinear forms of two spinors have a null structure.

**Lemma 3.4.** For \( \xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\} \), the following holds.

\[
\Pi_{\pm_1} (\xi_1) \Pi_{\pm_2} (\xi_2) = O(\angle(\pm_1 \xi_1, \pm_2 \xi_2))
\]

The readers can find the proof in [2], [7].

**Lemma 3.5** (Lemma 2.6 of [12]). Let \( \psi_1, \psi_2 \) be Schwartz spinor fields. Then we have

\[
|\mathcal{F}[(\Pi_{\pm_1} \psi_1)^\dagger (\Pi_{\pm_2} \alpha \Pi_{\pm_2} \psi_2)](X_0)| \lesssim \int_{X_0 = X_1 + X_2} \theta_{12} |\tilde{\psi}_1(X_1)||\tilde{\psi}_2(X_2)| dX_1 dX_2.
\]
We shall discuss why the above lemmas represent the null structures. First, we consider Lemma 3.3. Given two spinors $\psi_1, \psi_2$, we consider $\psi_1^\dagger \psi_2$. Using projection operator, we have

$$\psi_1^\dagger \psi_2 = \sum_{\pm_1, \pm_2} \psi_{1, \pm_1}^\dagger \psi_{2, \pm_2},$$

where $\psi_{1, \pm_1} = \Pi_{\pm_1} \psi_1$ and $\psi_{2, \pm_2} = \Pi_{\pm_2} \psi_2$. By taking the spatial Fourier transform we get

$$\mathcal{F}_x \left[ \psi_{1, \pm_1}^\dagger \psi_{2, \pm_2} \right] (\xi) = \int_{\xi_0 = \xi_1 - \xi_2} \bar{\psi}_{1, \pm_1}^\dagger (\xi_1) \bar{\psi}_{2, \pm_2} (\xi_2) d\xi_1 d\xi_2 = \int_{\xi_0 = \xi_1 - \xi_2} (\Pi_{\pm_1, \xi_1} \bar{\psi}_{1, \pm_1} (\xi_1)) (\Pi_{\pm_2} \bar{\psi}_{2, \pm_2} (\xi_2)) d\xi_1 d\xi_2 = \int_{\xi_0 = \xi_1 - \xi_2} \bar{\psi}_{1, \pm_1}^\dagger (\xi_1) \Pi_{\pm_1} (\xi_1) \Pi_{\pm_2} \bar{\psi}_{2, \pm_2} (\xi_2) d\xi_1 d\xi_2,$$

where we used the self-adjointness of $\Pi_{\pm} (\xi)$. Now we see that in the worst interaction (when two spinors are collinear; $\angle(\pm_1 \xi_1, \pm_2 \xi_2) = 0$), the integral vanishes and we gain better estimates.

Also, in the case that $\alpha^\mu$ is inserted between two spinors, i.e., $\psi_{1, \pm_1} \alpha^\mu \psi_{2, \pm_2}$, we use the commutator identity of projection $\Pi_{\pm}$ to get

$$\psi_{1, \pm_1}^\dagger \alpha^\mu \psi_{2, \pm_2} = \psi_{1, \pm_1}^\dagger \alpha^\mu \Pi_{\pm} \psi_{2, \pm_2} = \psi_{1, \pm_1} \psi_{1, \pm_1}^\dagger (\Pi_{\mp_2} \alpha^\mu \Pi_{\pm_2} - \mathcal{R}^\mu_{\pm_2} \Pi_{\pm_2}) \psi_{2, \pm_2}$$

and we have the same conclusion by taking Fourier transform and (3.7), so we are left to treat bilinear form $\psi_{1}^\dagger \mathcal{R}_{\pm}^\mu \psi_2$. See the proof of (4.4).

4. Sketch of Proof of Theorem 1.1

We construct Picard’s iterates for the equation (2.2) in the standard way. That is, we set $\psi^{(0)}_{\pm} = \psi^\text{hom}_{\pm}$ and in the general inductive step, $\psi^{(n)}_{\pm}$ is obtained by solving (2.2) on $S_T$ with the previous iterate $\psi^{(n-1)}_{\pm}$ inserted on the right-hand side. Explicitly, by Duhamel’s principle, $\psi^{(n)}_{\pm}$ can be written by

$$\psi^{(n)}_{\pm}(t, x) = \psi^{(0)}_{\pm} - i \int_0^t e^{\mp i(t-t')D} m \beta \psi^{(n-1)}_{\mp} (t') dt' + i \int_0^t e^{\mp i(t-t')D} \Pi_{\mp} \left( \mathcal{N}(\psi^{(n-1)}_{\pm}, \psi^{(n-1)}_{\pm}) (t') \right) dt'.$$

Then we show that the Picard’s iterates converges. For this, we need to prove that $(\psi^{(n)}_{\pm}, \psi^{(n)}_{\pm})$ is a Cauchy sequence in the space $B^{\theta_1;1}_{\pm} \times B^{\theta_1;1}_{\pm}$. In fact, it suffices to show that the following estimates holds.

$$\begin{align}
(4.1) & \quad \|\psi^\text{hom}_{\pm}\|_{B^{\theta_1;1}_{\pm}(S_T)} \lesssim \|\psi_0\|_{B^{\theta_1;1}_{\pm}}, \\
(4.2) & \quad \|m \beta \psi_{\mp}\|_{B^{\theta_1;1}_{\mp}(S_T)} \lesssim \|\psi_{\mp}\|_{B^{\theta_1;1}_{\mp}}, \\
(4.3) & \quad \|\mathcal{N}_1(\psi_{1, \pm_1}, \psi_{2, \pm_2})\psi_{3, \pm_3}\|_{B^{\theta_1;1}_{\pm_1}} \lesssim \|\psi_{1, \pm_1}\|_{B^{\theta_1;1}_{\pm_1}} \|\psi_{2, \pm_2}\|_{B^{\theta_1;1}_{\pm_2}} \|\psi_{3, \pm_3}\|_{B^{\theta_1;1}_{\pm_3}}, \\
(4.4) & \quad \|\mathcal{N}_2(\psi_{1, \pm_1}, \psi_{2, \pm_2})\psi_{3, \pm_3}\|_{B^{\theta_1;1}_{\pm_1}} \lesssim \|\psi_{1, \pm_1}\|_{B^{\theta_1;1}_{\pm_1}} \|\psi_{2, \pm_2}\|_{B^{\theta_1;1}_{\pm_2}} \|\psi_{3, \pm_3}\|_{B^{\theta_1;1}_{\pm_3}},
\end{align}$$

where

$$\mathcal{N}_1(\psi_{1, \pm_1}, \psi_{2, \pm_2}) = -\frac{1}{\Delta} \xi_{0j} \alpha^0 \partial_j (\psi_{1}^\dagger \alpha^k \psi_2), \quad \mathcal{N}_2(\psi_{1, \pm_1}, \psi_{2, \pm_2}) = -\frac{1}{\Delta} \xi_{0j} \alpha^0 \partial_j (\psi_{1}^\dagger \alpha^0 \psi_2).$$

Now the local well-posedness in $B^{\theta_1;1}_{\pm}$ follows from the standard argument. We omit the details. The estimates (4.1) and (4.2) are trivial. In the following two sections, we will focus on the estimates (4.3) and (4.4).
5. Estimates of $N_1$: Proof of (5.1)

Now we exploit the nonlinear term to reveal its null structure.

$$N_1(\psi_1, \psi_2) = \sum_{\pm, \pm} N_1(\psi_{1, \pm}, \psi_{2, \pm})$$

$$= \sum_{\pm, \pm} \left( -\frac{1}{\Delta} \right) \epsilon_{0jk} \alpha^0 \partial_j \left[ (\Pi_{\pm, 1} \psi_1)^\dagger \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} \psi_2 \right]$$

$$- \sum_{\pm, \pm} \left( -\frac{1}{\Delta} \right) \epsilon_{0jk} \alpha^0 \partial_j \left[ (\Pi_{\pm, 1} \psi_1)^\dagger R^k_{\pm, 2} \Pi_{\pm, 2} \psi_2 \right]$$

$$:= \sum_{\pm, \pm} N_1^1(\psi_{1, \pm}, \psi_{2, \pm}) - \sum_{\pm, \pm} N_1^2(\psi_{1, \pm}, \psi_{2, \pm}) .$$

We shall prove the following estimates:

(5.1) $\| \Pi_\pm N_1^1(\psi_{1, \pm}, \psi_{2, \pm}) \|_{B^0_2, \frac{3}{4}, 1} \lesssim \| \psi_{1, \pm} \|_{B^0_2, \frac{3}{4}, 1} \| \psi_{2, \pm} \|_{B^0_2, \frac{3}{4}, 1} \| \psi_{3, \pm} \|_{B^0_2, \frac{3}{4}, 1}$

(5.2) $\| \Pi_\pm N_1^2(\psi_{1, \pm}, \psi_{2, \pm}) \|_{B^0_2, \frac{3}{4}, 1} \lesssim \| \psi_{1, \pm} \|_{B^0_2, \frac{3}{4}, 1} \| \psi_{2, \pm} \|_{B^0_2, \frac{3}{4}, 1} \| \psi_{3, \pm} \|_{B^0_2, \frac{3}{4}, 1}$

5.1. Proof of (5.1). By duality,

$$\left\| \Pi_\pm \left( -\frac{1}{\Delta} \epsilon_{0jk} \alpha^0 \partial_j \left[ (\Pi_{\pm, 1} \psi_1)^\dagger \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} \psi_2 \right] \right) \psi_{3, \pm} \right\|_{B^0_2, \frac{3}{4}, 1}$$

$$= \sup_{\| \psi_{4, \pm} \|_{B^0_2, \frac{3}{4}, 1} = 1} \left| \int \frac{1}{B^0_2, \frac{3}{4}, 1} \epsilon_{0jk} \alpha^0 \left( (\psi_{1, \pm}, 1 \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} \psi_{2, \pm}) \right) \mathcal{F}_{\pm, 3} (\psi_{4, \pm}, 3) dt dx \right|$$

$$= \sup_{\| \psi_{4, \pm} \|_{B^0_2, \frac{3}{4}, 1} = 1} \left| \int \frac{1}{[\xi_0]} \mathcal{F}(\psi_{1, \pm}, 1 \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} \psi_{2, \pm}) \mathcal{F}_{\pm, 3} (\psi_{4, \pm}, 3) dX_0 \right| .$$

Let us set

$$J^1 := \int \frac{1}{[\xi_0]} \mathcal{F}(\psi_{1, \pm}, 1 \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} \psi_{2, \pm}) \mathcal{F}_{\pm, 3} (\psi_{4, \pm}, 3) dX_0 .$$

A dyadic decomposition of space-time Fourier side give us $|J^1| \leq \sum_{N,L} |J^1_{N,L}|$, where

$$|J^1_{N,L}| = \int \frac{1}{[\xi_0]} \mathcal{F}(P_{K_{N_0, L_0}}^{\pm, 1} \psi_{1, \pm, 1} \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} P_{K_{N_2, L_2}}^{\pm, 2} \psi_{2, \pm, 2})$$

$$\times \mathcal{F}_{\pm, 3} (P_{K_{N_3, L_3}}^{\pm, 4} \psi_{4, \pm, 4} P_{K_{N_4, L_4}}^{\pm, 3} \psi_{3, \pm, 3}) dX_0 ,$$

and

$$N = (N_0, N_1, N_2, N_3, N_4), \quad L = (L_0, L_1, L_2, L_3, L_4) .$$

The following remark says that we can exclude the special case on the modulation and frequency by the support condition. The readers can find the similar remark in [4].

Remark 5.1. Here, we claim that the low-low-high modulation with high-high-low frequency, especially, $L_{max}^{12} \ll L_0 \ll N_0 \ll 1 \sim N_2$ is excluded in $J^1$. Indeed, we write $\varphi = P_{K_{N_3, L_3}}^{\pm, 3} \psi_{4, \pm, 4} P_{K_{N_3, L_3}}^{\pm, 3} \psi_{3, \pm, 3}$. Since the space-time Fourier support of $\varphi$ is contained in $K_{N_0, L_0}$, it is not harmful to write $\varphi = P_{K_{N_0, L_0}}^{\pm} \varphi$. Now we rewrite $J^1_{N,L}$ as

$$J^1_{N,L} = \int \frac{1}{[\xi_0]} \mathcal{F}(P_{K_{N_0, L_0}}^{\pm, 1} \psi_{1, \pm, 1} \Pi_{\mp, 2} \alpha^k \Pi_{\pm, 2} P_{K_{N_2, L_2}}^{\pm, 2} \psi_{2, \pm, 2}) \mathcal{F}_{\pm, 3} (P_{K_{N_0, L_0}}^{\pm} \varphi(-X_0) dX_0 .$$
On the other hand, keeping in mind the bilinear interaction $X_0 = X_1 - X_2$, $\mathbf{J}^1_{N,L}$ can be rewritten as

$$\mathbf{J}^1_{N,L} = \frac{1}{|\xi_0|} \langle \mathcal{F} P_{K_{N,L}} \psi_{1,\pm} \rangle \langle \mathcal{F} P_{K_{N,L}} \psi_{2,\pm} \rangle \langle \mathcal{F} P_{K_{N,L}} \psi_{3,\pm} \rangle \langle \mathcal{F} P_{K_{N,L}} \psi_{4,\pm} \rangle = \mathcal{F} P_{K_{N,L}} \psi_{1,\pm}(X_1) dX_1.$$

From the first representation of $\mathbf{J}^1_{N,L}$, we see that $\pm = \pm_2$ is excluded, because $\psi_{1,\pm}$ and $\psi_{2,\pm}$ have up and down cones or down and up cones, respectively. We also exclude the case $\pm \neq \pm_2$ in view of second representation. This gives the low-low-high modulation with high-low-high frequency does not appear in the summation of $\mathbf{J}^1_{N,L}$. Appealing the same argument, we can exclude the case low-low-high modulation and high-low-high frequency: $L_{1,2}^\text{max} \ll L_0 \ll N_1 \ll N_0 \sim N_2$.

As we argue the exclusion of the low-low-high modulation in $\psi_{1,\pm}$ and $\psi_{2,\pm}$, we exclude the high-high-low modulation in $\psi_{3,\pm}$ and $\psi_{4,\pm}$ by putting $\varphi = P_{K_{N,L}} \psi_{1,\pm} \Pi \frac{1}{2} \alpha \Pi \frac{1}{2} P_{K_{N,L}} \psi_{2,\pm}$, and then we get the exclusion on the case $L_{3,4}^\text{max} \ll N_0 \ll N_3 \sim N_4$, $L_{3,4}^\text{max} \ll N_0 \sim N_1$, and $L_{3,4}^\text{max} \ll N_0 \sim N_3$. Furthermore, we exclude the modulation $L_3 \ll L_4 \ll L_0$.

We can decompose the integrand of $\mathbf{J}^1$ with the combination of positive and negative parts of $\mathcal{F} P_{K_{N,L}} \psi_{1,\pm}$, $\mathcal{F} P_{K_{N,L}} \psi_{2,\pm}$, $\mathcal{F} P_{K_{N,L}} \psi_{3,\pm}$, and $\mathcal{F} P_{K_{N,L}} \psi_{4,\pm}$. Thus without loss of generality, we can assume that $\mathcal{F} P_{K_{N,L}} \psi_{1,\pm}$, $\mathcal{F} P_{K_{N,L}} \psi_{2,\pm}$, $\mathcal{F} P_{K_{N,L}} \psi_{3,\pm}$, and $\mathcal{F} P_{K_{N,L}} \psi_{4,\pm}$ are nonnegative real-valued functions.

By Cauchy-Schwarz inequality,

$$|\mathbf{J}^0| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_{1,2}}{N_1} \right)^{1/2} \left( (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} (N_0 L_2)^{1/2} (N_0 L_3)^{1/4} \right) \left\| P_{K_{N,L}} \psi_{1,\pm} \right\| \left\| P_{K_{N,L}} \psi_{2,\pm} \right\| \left\| P_{K_{N,L}} \psi_{3,\pm} \right\| \left\| P_{K_{N,L}} \psi_{4,\pm} \right\|$$

5.1.1. Case 1: $L_0 \leq L_1 \leq L_2$, $L_0 \leq L_3 \leq L_4$.

This case can be treated by using only bilinear estimates (1) and (2). Indeed, if $N_0 \ll N_1 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, by (1) with $j = 1$, we get

$$|\mathbf{J}^0| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_{1,2}}{N_1} \right)^{1/2} \left( (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} (N_0 L_2)^{1/2} (N_0 L_3)^{1/4} \right) \left\| P_{K_{N,L}} \psi_{1,\pm} \right\| \left\| P_{K_{N,L}} \psi_{2,\pm} \right\| \left\| P_{K_{N,L}} \psi_{3,\pm} \right\| \left\| P_{K_{N,L}} \psi_{4,\pm} \right\|$$

$$\lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_{1,2}}{N_1} \right)^{1/2} \left( L_0^{1/4} L_2^{1/2} L_3^{-1/4} \right) \left\| P_{K_{N,L}} \psi_{1,\pm} \right\| \left\| P_{K_{N,L}} \psi_{2,\pm} \right\| \left\| P_{K_{N,L}} \psi_{3,\pm} \right\| \left\| P_{K_{N,L}} \psi_{4,\pm} \right\|$$

$$\lesssim \sum_{N,L} \left( \frac{N_0}{N_1} \right)^{1/2} \left( L_0^{1/4} L_1^{1/2} L_2^{1/2} L_3^{-1/4} \right) \left\| P_{K_{N,L}} \psi_{1,\pm} \right\| \left\| P_{K_{N,L}} \psi_{2,\pm} \right\| \left\| P_{K_{N,L}} \psi_{3,\pm} \right\| \left\| P_{K_{N,L}} \psi_{4,\pm} \right\|$$

$$\lesssim \sum_{N,L} \left( \frac{N_0}{N_1} \right)^{1/2} \left( L_0^{1/4} L_1^{1/2} L_2^{1/2} L_3^{-1/4} \right) \left\| P_{K_{N,L}} \psi_{1,\pm} \right\| \left\| P_{K_{N,L}} \psi_{2,\pm} \right\| \left\| P_{K_{N,L}} \psi_{3,\pm} \right\| \left\| P_{K_{N,L}} \psi_{4,\pm} \right\|$$

$$\leq \left\| \psi_{1,\pm} \right\|_{B_{\frac{3}{2},+}^0} \left\| \psi_{2,\pm} \right\|_{B_{\frac{3}{2},+}^0} \left\| \psi_{3,\pm} \right\|_{B_{\frac{3}{2},+}^0} \left\| \psi_{4,\pm} \right\|_{B_{\frac{3}{2},+}^0} \left( \vdots L_0^{1/4} \leq L_1^{1/4} \right)$$
For $N_0 \ll N_1 \sim N_2$ and $N_3 \lesssim N_0 \sim N_4$, we use $\psi_2$ with $j = 1$ to get

$$|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L^3}{N_1} \right)^{1/2} (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} (N_3 L_0)^{1/2} (N_3 L_3)^{1/4} \| P_{K_{N_1,L_1}^\pm} \psi_{1,\pm} \|$$

$$\times \| P_{K_{N_2,L_2}^\pm} \psi_{2,\pm} \| \| P_{K_{N_3,L_3}^\pm} \psi_{3,\pm} \| \| P_{K_{N_4,L_4}^\pm} \psi_{4,\pm} \|$$

$$= \sum_{N,L} \left( \frac{N_0}{N_1} \right)^{1/2} \left( \frac{N_3}{N_0} \right)^{3/4} \left( \frac{L_0}{L_1} \right)^{1/4} \left( \frac{L_1}{L_2} \right)^{1/2} \left( \frac{L_2}{L_3} \right)^{1/4} \| \psi_{4,\pm} \| \| \psi_{3,\pm} \|$$

$$\lesssim \| \psi_{1,\pm} \| \| \psi_{2,\pm} \| \| \psi_{3,\pm} \| \| \psi_{4,\pm} \|$$

If $N_0 \ll N_1 \sim N_2$, $N_0 \ll N_3 \sim N_4$, similarly, we get

$$|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} \left( \frac{N_0 L_0}{N_1 L_1} \right)^{1/2} (N_0 L_0)^{1/4} (N_0 L_0)^{1/2} (N_3 L_0)^{1/2} (N_3 L_3)^{1/4} \| P_{K_{N_1,L_1}^\pm} \psi_{1,\pm} \|$$

$$\times \| P_{K_{N_2,L_2}^\pm} \psi_{2,\pm} \| \| P_{K_{N_3,L_3}^\pm} \psi_{3,\pm} \| \| P_{K_{N_4,L_4}^\pm} \psi_{4,\pm} \|$$

$$= \sum_{N,L} \left( \frac{N_4}{N_1} \right)^{1/4} \left( \frac{N_0}{N_0} \right)^{3/4} \left( \frac{L_0}{L_1} \right)^{1/4} \left( \frac{L_1}{L_2} \right)^{1/2} \left( \frac{L_2}{L_3} \right)^{1/4} \| \psi_{4,\pm} \| \| \psi_{3,\pm} \|$$

$$\lesssim \| \psi_{1,\pm} \| \| \psi_{2,\pm} \| \| \psi_{3,\pm} \| \| \psi_{4,\pm} \|$$

If $N_1 \lesssim N_0 \sim N_2$, $N_0 \ll N_3 \sim N_4$, then

$$|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_1 L_0)^{1/2} (N_1 L_1)^{1/2} (N_1 L_1)^{1/4} (N_0 L_0)^{1/2} (N_0 L_3)^{1/4} \| P_{K_{N_1,L_1}^\pm} \psi_{1,\pm} \|$$

$$\times \| P_{K_{N_2,L_2}^\pm} \psi_{2,\pm} \| \| P_{K_{N_3,L_3}^\pm} \psi_{3,\pm} \| \| P_{K_{N_4,L_4}^\pm} \psi_{4,\pm} \|$$

$$= \sum_{N,L} \left( \frac{N_1}{N_0} \right)^{1/2} \left( \frac{N_4}{N_0} \right)^{3/4} \left( \frac{L_0}{L_1} \right)^{1/4} \left( \frac{L_1}{L_2} \right)^{1/2} \left( \frac{L_2}{L_3} \right)^{1/4} \| \psi_{4,\pm} \| \| \psi_{3,\pm} \|$$

$$\lesssim \| \psi_{1,\pm} \| \| \psi_{2,\pm} \| \| \psi_{3,\pm} \| \| \psi_{4,\pm} \|$$

For $N_1 \lesssim N_0 \sim N_2$, $N_3 \lesssim N_0 \sim N_4$, we get

$$|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_1 L_0)^{1/2} (N_1 L_1)^{1/2} (N_1 L_1)^{1/4} (N_3 L_0)^{1/2} (N_3 L_3)^{1/4} \| P_{K_{N_1,L_1}^\pm} \psi_{1,\pm} \|$$

$$\times \| P_{K_{N_2,L_2}^\pm} \psi_{2,\pm} \| \| P_{K_{N_3,L_3}^\pm} \psi_{3,\pm} \| \| P_{K_{N_4,L_4}^\pm} \psi_{4,\pm} \|$$

$$\lesssim \sum_{N,L} \left( \frac{N_1}{N_0} \right)^{1/2} \left( \frac{N_3}{N_0} \right)^{3/4} L_0 L_1^{1/4} L_2^{1/2} L_3^{1/4} \| \psi_{4,\pm} \|$$

$$\lesssim \| \psi_{1,\pm} \| \| \psi_{2,\pm} \| \| \psi_{3,\pm} \| \| \psi_{4,\pm} \|$$
If $N_1 \lesssim N_0 \sim N_2$, $N_4 \lesssim N_0 \sim N_3$, similarly,

$$|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_1 L_0)^{1/2} (N_1 L_1)^{1/4} (N_4 L_0)^{1/2} (N_3 L_3)^{1/4} ||P_{K_{N_1 \cdot 1}^\pm} \psi_{1, \pm 1}||$$

$$\times ||P_{K_{N_2 \cdot 2}^\pm} \psi_{2, \pm 2}|| ||P_{K_{N_3 \cdot 3}^\pm} \psi_{3, \pm 3}|| ||P_{K_{N_4 \cdot 4}^\pm} \psi_{4, \pm 4}||$$

$$= \sum_{N,L} \left( \frac{N_1}{N_0} \right)^{1/4} \left( \frac{N_4}{N_0} \right)^{1/2} \left( \frac{N_3}{N_0} \right)^{1/4} L_0 L_1^{1/4} L_2^{1/2} L_3^{1/4} L_4^{-1/2} ||\psi_{4, \pm 4}||_{B^0_4 \pm \infty}$$

$$\times ||P_{K_{N_1 \cdot 1}^\pm} \psi_{1, \pm 1}|| ||P_{K_{N_2 \cdot 2}^\pm} \psi_{2, \pm 2}|| ||P_{K_{N_3 \cdot 3}^\pm} \psi_{3, \pm 3}||$$

$$\lesssim \left( \frac{L_0}{N_1 N_2} \right)^{1/2} \left( \frac{L_1}{N_1 N_2} \right)^{1/2} \left( \frac{L_2}{N_1 N_2} \right)^{1/2} \left( \frac{L_4}{N_1 N_2} \right)^{1/4} ||\psi_{4, \pm 4}||_{B^0_4 \pm \infty}$$

5.1.2. Case 2: $L_1 \leq L_2 \leq L_0$, $L_3 \leq L_4 \leq L_0$.

By Remark 5.1 in the case $L_{\max}^{12} \ll L_0 \ll N_0^{12} \min$ and $L_3 \sim L_4 \ll L_0 \ll N_0^{34} \min$, it suffices to treat the case $L_0 \ll N_0 \sim N_2 \sim N_3 \sim N_4$. The case $L_3 \ll L_4 \ll L_0$ is already excluded.) Here, we note that $\theta_{12} \sim \theta_{13}$ and $\theta_{24} \sim \theta_{23}$.

We pair up $\psi_1$ with $\psi_3$ and $\psi_2$ with $\psi_4$ by applying change of variables:

$$\tilde{\tau}_0 = \tau_1 + \tau_3 = \tau_2 + \tau_4, \quad \tilde{\xi}_0 = \xi_1 + \xi_3 = \xi_2 + \xi_4.$$

This relation is reasonable, since we have bilinear interactions: $X_0 = X_1 - X_2$ and $X_0 = X_4 - X_3$.

Then we obtain

$$|J^1| \lesssim \sum_{N,L} \int \frac{1}{|\xi_0|} F B_{\theta_{13}}(P_{K_{N_1 \cdot 1}^\pm} \psi_{1, \pm 1}, P_{K_{N_3 \cdot 3}^\pm} \psi_{3, \pm 3})(\tilde{X}_0)$$

$$\times F B_{\theta_{24}}(P_{K_{N_2 \cdot 2}^\pm} \psi_{2, \pm 2}, P_{K_{N_4 \cdot 4}^\pm} \psi_{4, \pm 4})(\tilde{X}_0) d\tilde{X}_0.$$

Recall the angle Whitney decomposition to get

$$|J^1| \lesssim \sum_{N,L} \sum_{\omega_1, \omega_2} \int \frac{1}{|\xi_0|} F B_{\theta_{13}}(P_{K_{N_1 \cdot 1}^\pm}^{\gamma_1 \omega_1} \psi_{1, \pm 1}, P_{K_{N_3 \cdot 3}^\pm}^{\gamma_3 \omega_3} \psi_{3, \pm 3})(\tilde{X}_0)$$

$$\times F B_{\theta_{24}}(P_{K_{N_2 \cdot 2}^\pm}^{\gamma_2 \omega_2} \psi_{2, \pm 2}, P_{K_{N_4 \cdot 4}^\pm}^{\gamma_4 \omega_4} \psi_{4, \pm 4})(\tilde{X}_0) d\tilde{X}_0,$$

where the summation is taken over $\omega_1, \omega_2 \in \Omega(\gamma_{12})$ with $\langle \omega_1, \omega_2 \rangle \leq 4 \gamma_{12}$ and $\psi_{j, \pm j}^{\theta, \omega_j} = P_{\pm, \xi_j \in \mu_\omega} \psi_{j, \pm j}$ and $\gamma_{12} \lesssim \left( \frac{N_0 L_0}{N_1 N_2} \right)^{1/2}$.

Since the spatial Fourier support of $\psi_{j, \pm j}^{\gamma \omega_j}$ is contained in a strip of radius comparable to $N_j \gamma_{12}$ about $\Re \omega_j$, we deduce that

$$|J^1| \lesssim \sum_{N,L} \sum_{\omega_1, \omega_2} \frac{1}{N_0} (N_1 \gamma_{12} L_1 L_3)^{1/2} (N_2 \gamma_{12} L_2 L_4)^{1/2} L_4^{-1/2} ||\psi_{4, \pm 4}||_{B^0_4 \pm \infty}$$

$$\times ||P_{K_{N_1 \cdot 1}^\pm}^{\gamma_1 \omega_1} \psi_{1, \pm 1}|| ||P_{K_{N_2 \cdot 2}^\pm}^{\gamma_2 \omega_2} \psi_{2, \pm 2}|| ||P_{K_{N_3 \cdot 3}^\pm}^{\gamma_3 \omega_3} \psi_{3, \pm 3}||$$

$$\lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_0}{N_1 N_2} \right)^{1/2} \left( \frac{L_1 L_2 L_3}{N_1 N_2} \right)^{1/2} ||\psi_{1, \pm 1}|| ||P_{K_{N_2 \cdot 2}^\pm} \psi_{2, \pm 2}|| ||P_{K_{N_3 \cdot 3}^\pm} \psi_{3, \pm 3}|| ||\psi_{4, \pm 4}||_{B^0_4 \pm \infty}$$

$$\lesssim \sum_{N,L} \left( \frac{L_0}{N_0} \right)^{1/2} \left( \frac{L_1 L_2 L_3}{N_0} \right)^{1/2} P_{K_{N_1 \cdot 1}^\pm} \psi_{1, \pm 1} ||P_{K_{N_2 \cdot 2}^\pm} \psi_{2, \pm 2}|| ||P_{K_{N_3 \cdot 3}^\pm} \psi_{3, \pm 3}|| ||\psi_{4, \pm 4}||_{B^0_4 \pm \infty}$$

$$\lesssim \left( \frac{L_0}{N_0} \right)^{1/2} \left( \frac{L_1 L_2 L_3}{N_0} \right)^{1/2} ||\psi_{1, \pm 1}|| ||\psi_{2, \pm 2}|| ||\psi_{3, \pm 3}|| ||\psi_{4, \pm 4}||_{B^0_4 \pm \infty}.$$
where we used the summation by $L_0; L_0 \ll N_0$ and $L_4; L_4 \sim L_3$ and then $N_0 \sim N_j, j = 1, 2, 3, 4$.

We turn our attention to the case: $N_{0 \text{min}}^{0j2} \lesssim L_0$ and $N_{0 \text{min}}^{034} \lesssim L_0$. From the Littlewood-Paley trichotomy we have the relation $N_{0 \text{min}}^{0j2} \lesssim N_{0 \text{med}}^{0j2} \sim N_{0 \text{max}}^{0j2}$, where $\{j, k\} = \{1, 2\}$ or $\{3, 4\}$, then $\text{Rem. 5.1}$. 

Thus we see the exclusion of the case $L_0 \ll N_{0 \text{med}}^{0j2} \sim N_{0 \text{max}}^{0j2}$. Furthermore, similar argument gives the exclusion of the case $N_{0 \text{min}}^{0j2}, N_{0 \text{min}}^{034} \ll L_0$ and hence we have $N_j \sim L_0, j = 0, 1, 2, 3, 4$.

For the sake of simplicity, we set

$$\gamma_{12} \lesssim \left( \frac{L_{0 \text{max}}^{0j2} N_0}{N_1 N_2} \right)^{1/2}, \quad \gamma_{34} \lesssim \left( \frac{L_{0 \text{max}}^{0j2} N_0}{N_3 N_4} \right)^{1/2}.$$

Then using (7.2), we write

$$|J| \lesssim \sum_{N, L, \omega_1, \omega_2} \int \frac{1}{|x_0|} \mathcal{F} B_{\theta_{12}} (P_{K_{N_1 L_1}}^{x_1}, P_{K_{N_2 L_2}}^{x_2}) dB_{\theta_{34}} (P_{K_{N_1 L_4}}^{x_4}, P_{K_{N_3 L_3}}^{x_3}) d\omega_1,$$

where $\psi_{j, \pm}^{k, \omega_j} = P_{x, \pm} \in \mathcal{G}_j (\omega_j) \psi_{j, \pm}$. Here the spatial Fourier support of $P_{K_{N_1 L_1}}^{x_1}, P_{K_{N_2 L_2}}^{x_2}$ is contained in a strip of radius compatible to $N_{0 \text{max}}^{12} \gamma_{12}$ around $\mathbb{R} \omega_1$. Then using Theorem 8.2 with $r \sim N_{0 \text{max}}^{12} \gamma_{12}$, we get

$$|J| \lesssim \sum_{N, L, \omega_1, \omega_2} \sum_{\omega_3, \omega_4} \left| \mathcal{F} B_{\theta_{12}} (P_{K_{N_1 L_1}}^{x_1}, P_{K_{N_2 L_2}}^{x_2}) \right| \left| \mathcal{F} B_{\theta_{34}} (P_{K_{N_1 L_4}}^{x_4}, P_{K_{N_3 L_3}}^{x_3}) \right|$$

$$\lesssim \left( \frac{L_4}{N_0} \right)^{1/2} \left( \frac{L_1 L_2}{L_4} \right)^{1/2} \left( \frac{L_1 L_2}{L_4} \right)^{1/2} \left( \frac{L_1 L_2}{L_4} \right)^{1/2} \left\| \psi_{4, \pm} \right\|_{B^0_{\frac{1}{4}, \infty}} \left\| \psi_{3, \pm} \right\|_{B^0_{\frac{1}{4}, \infty}}$$

where the summation by $\omega_1, \omega_2$ and $\omega_3, \omega_4$ is taken over $\mathcal{G}(\omega_1, \omega_2) \lesssim 4 \gamma_{12}$ and $\mathcal{G}(\omega_4, \omega_3) \lesssim 4 \gamma_{12}$.

Here, we treat the case $L_0 \sim L_2 \sim L_4$. This is very straightforward.
If $N_0 \ll N_1 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, by \((3.1)\) with $j = 1, 3$ we get

$$|J^1| \lesssim \sum_{N \leq L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} \left( (N_0 L_1)^{1/2} (N_0 L_0)^{1/4} (N_0 L_3)^{1/2} (N_0 L_0)^{1/4} \right) \left\| P_{K_{N_1 N_1}}^{\pm 1} \psi_{1, \pm 1} \right\|$$

$$\times \left\| P_{K_{N_2 N_2}}^{\pm 2} \psi_{2, \pm 2} \right\| \left\| P_{K_{N_3 N_3}}^{\pm 3} \psi_{3, \pm 3} \right\| \left\| P_{K_{N_4 N_4}}^{\pm 4} \psi_{4, \pm 4} \right\|$$

$$= \sum_{N \leq L} \left( \frac{N_0}{N_1} \right)^{1/2} \left( \frac{N_3}{N_0} \right)^{1/2} \left( \frac{L_0}{L_1} \right)^{1/2} \left( \frac{L_3}{L_2} \right)^{1/4} \left( \frac{L_2}{L_3} \right)^{1/2} (N_3 L_4)^{1/4} \left\| P_{K_{N_1 N_1}}^{\pm 1} \psi_{1, \pm 1} \right\|$$

$$\times \left\| P_{K_{N_2 N_2}}^{\pm 2} \psi_{2, \pm 2} \right\| \left\| P_{K_{N_3 N_3}}^{\pm 3} \psi_{3, \pm 3} \right\| \left\| P_{K_{N_4 N_4}}^{\pm 4} \psi_{4, \pm 4} \right\|$$

$$\lesssim \left\| \psi_{1, \pm 1} \right\|_{B^0_{\frac{3}{2}}} \left\| \psi_{2, \pm 2} \right\|_{B^0_{\frac{3}{2}}} \left\| \psi_{3, \pm 3} \right\|_{B^0_{\frac{3}{2}}} \left\| \psi_{4, \pm 4} \right\|_{B^0_{\frac{3}{2}}} \left( \cdot L_0 \sim L_2 \sim L_4 \right)$$

For $N_0 \ll N_1 \sim N_2$ and $N_3 \lesssim N_0 \sim N_3$, then using \((3.1)\), we get

$$|J^1| \lesssim \sum_{N \leq L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} \left( (N_0 L_1)^{1/2} (N_1 L_2)^{1/4} (N_0 L_3)^{1/2} (N_3 L_4)^{1/4} \right) \left\| P_{K_{N_1 N_1}}^{\pm 1} \psi_{1, \pm 1} \right\|$$

$$\times \left\| P_{K_{N_2 N_2}}^{\pm 2} \psi_{2, \pm 2} \right\| \left\| P_{K_{N_3 N_3}}^{\pm 3} \psi_{3, \pm 3} \right\| \left\| P_{K_{N_4 N_4}}^{\pm 4} \psi_{4, \pm 4} \right\|$$

$$= \sum_{N \leq L} \left( \frac{N_0}{N_1} \right)^{1/2} \left( \frac{N_3}{N_0} \right)^{1/4} \left( \frac{L_0}{L_1} \right)^{1/2} \left( \frac{L_3}{L_1} \right)^{1/4} \left( \frac{L_1}{L_3} \right)^{1/2} \left( \frac{L_3}{L_4} \right)^{1/4} \left\| P_{K_{N_1 N_1}}^{\pm 1} \psi_{1, \pm 1} \right\|$$

$$\times \left\| P_{K_{N_2 N_2}}^{\pm 2} \psi_{2, \pm 2} \right\| \left\| P_{K_{N_3 N_3}}^{\pm 3} \psi_{3, \pm 3} \right\| \left\| P_{K_{N_4 N_4}}^{\pm 4} \psi_{4, \pm 4} \right\|$$

$$\lesssim \left\| \psi_{1, \pm 1} \right\|_{B^0_{\frac{1}{4}}} \left\| \psi_{2, \pm 2} \right\|_{B^0_{\frac{1}{4}}} \left\| \psi_{3, \pm 3} \right\|_{B^0_{\frac{1}{4}}} \left\| \psi_{4, \pm 4} \right\|_{B^0_{\frac{1}{4}}} \left( \cdot L_0 \sim L_2 \sim L_4 \right)$$

For $N_1 \lesssim N_0 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, by \((3.2)_3\), we get

$$|J^1| \lesssim \sum_{N \leq L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} \left( (N_1 L_1)^{1/2} (N_1 L_0)^{1/4} \left( N_0 L_3 \right)^{1/2} (N_0 L_0)^{1/4} \right) \left\| P_{K_{N_1 N_1}}^{\pm 1} \psi_{1, \pm 1} \right\|$$

$$\times \left\| P_{K_{N_2 N_2}}^{\pm 2} \psi_{2, \pm 2} \right\| \left\| P_{K_{N_3 N_3}}^{\pm 3} \psi_{3, \pm 3} \right\| \left\| P_{K_{N_4 N_4}}^{\pm 4} \psi_{4, \pm 4} \right\|$$

$$= \sum_{N \leq L} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{L_0}{L_1} \right)^{1/2} \left( \frac{L_3}{L_1} \right)^{1/4} \left( \frac{L_1}{L_3} \right)^{1/2} \left( \frac{L_3}{L_4} \right)^{1/4} \left\| P_{K_{N_1 N_1}}^{\pm 1} \psi_{1, \pm 1} \right\|$$

$$\times \left\| P_{K_{N_2 N_2}}^{\pm 2} \psi_{2, \pm 2} \right\| \left\| P_{K_{N_3 N_3}}^{\pm 3} \psi_{3, \pm 3} \right\| \left\| P_{K_{N_4 N_4}}^{\pm 4} \psi_{4, \pm 4} \right\|$$

$$\lesssim \left\| \psi_{1, \pm 1} \right\|_{B^0_{\frac{1}{4}}} \left\| \psi_{2, \pm 2} \right\|_{B^0_{\frac{1}{4}}} \left\| \psi_{3, \pm 3} \right\|_{B^0_{\frac{1}{4}}} \left\| \psi_{4, \pm 4} \right\|_{B^0_{\frac{1}{4}}} \left( \cdot L_0 \sim L_2 \sim L_4 \right)$$. 
If \( N_1 \lesssim N_0 \sim N_2 \) and \( N_3 \lesssim N_0 \sim N_4 \), then using (3.31), we get

\[
|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_1 L_1)^{1/2} (N_1 L_2)^{1/4} (N_3 L_3)^{1/2} (N_4 L_4)^{1/4} \| P_{K_{N_1, L_1}}^{\pm_1} \psi_{1, \pm_1} \| \times \| P_{K_{N_2, L_2}}^{\pm_2} \psi_{2, \pm_2} \| \| P_{K_{N_3, L_3}}^{\pm_3} \psi_{3, \pm_3} \| \| P_{K_{N_4, L_4}}^{\pm_4} \psi_{4, \pm_4} \|
\]

\[
\lesssim \sum_{N,L} \frac{N_1^{3/4} N_3^{3/4}}{N_0} \frac{L_0}{L_1} \frac{L_1}{L_2} \frac{1/4}{L_2} \frac{1/4}{L_3} \frac{1/4}{L_4} \| \psi_{4, \pm_4} \|_{B_{2/4}^{0, \infty}}
\]

\[
\lesssim \ \| \psi_{1, \pm_1} \|_{B_{2/4}^{0, \infty}} \| \psi_{2, \pm_2} \|_{B_{2/4}^{0, \infty}} \| \psi_{3, \pm_3} \|_{B_{2/4}^{0, \infty}} \| \psi_{4, \pm_4} \|_{B_{2/4}^{0, \infty}}.
\]

For \( N_1 \lesssim N_0 \sim N_2 \) and \( N_3 \lesssim N_0 \sim N_3 \), similarly,

\[
|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_1 L_1)^{1/2} (N_1 L_2)^{1/4} (N_4 L_3)^{1/2} (N_4 L_4)^{1/4} \| P_{K_{N_1, L_1}}^{\pm_1} \psi_{1, \pm_1} \| \times \| P_{K_{N_2, L_2}}^{\pm_2} \psi_{2, \pm_2} \| \| P_{K_{N_3, L_3}}^{\pm_3} \psi_{3, \pm_3} \| \| P_{K_{N_4, L_4}}^{\pm_4} \psi_{4, \pm_4} \|
\]

\[
\lesssim \sum_{N,L} \frac{N_1^{3/4} N_4^{3/4}}{N_0} \frac{L_0}{L_1} \frac{1/2}{L_1} \frac{1/4}{L_2} \frac{1/4}{L_3} \frac{1/4}{L_4} \| \psi_{4, \pm_4} \|_{B_{2/4}^{0, \infty}}
\]

\[
\lesssim \ \| \psi_{1, \pm_1} \|_{B_{2/4}^{0, \infty}} \| \psi_{2, \pm_2} \|_{B_{2/4}^{0, \infty}} \| \psi_{3, \pm_3} \|_{B_{2/4}^{0, \infty}} \| \psi_{4, \pm_4} \|_{B_{2/4}^{0, \infty}}.
\]

5.1.3. Case 3: \( L_1 \leq L_2 \leq L_0, L_0 \leq L_3 \leq L_4 \).

We already excluded the case \( L_{\text{max}}^{12} \ll L_0 \ll N_0^{12} \). Thus we consider the case \( L_0 \sim N_k, k = 0, 1, 2 \) and \( L_{\text{max}}^{03} \ll L_4 \ll N_0^{03} \).

By the exclusion as stated in Remark 5.7, we only have to consider the case \( L_{\text{max}}^{03} \ll L_4 \ll N_4 \sim N_0 \sim N_3 \). Then we get \( L_4 \sim N_0 \sim L_0 \), which contradicts to the assumption \( L_{\text{max}}^{03} \ll L_4 \) and hence this case is excluded.

Now we assume \( L_4 \gtrsim N_0^{03} \). It follows that \( L_0 \sim L_4 \sim N_j, j = 1, 2, 3, 4 \). Then by applying (3.3) to \( \psi_{1, \pm_1} \) and \( \psi_{2, \pm_2} \) and using (3.31), we get

\[
|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{N_0 L_0}{N_1 L_2} \right)^{1/2} (N_0 L_3)^{1/2} (N_0 L_4)^{1/4} \| P_{K_{N_1, L_1}}^{\pm_1} \psi_{1, \pm_1} \| \times \| P_{K_{N_2, L_2}}^{\pm_2} \psi_{2, \pm_2} \| \| P_{K_{N_3, L_3}}^{\pm_3} \psi_{3, \pm_3} \| \| P_{K_{N_4, L_4}}^{\pm_4} \psi_{4, \pm_4} \|
\]

\[
\lesssim \sum_{N,L} \frac{L_0^{3/4} (L_1 L_2 L_3)^{1/2}}{L_4^{1/4}} \frac{1/4}{L_4} \frac{1/4}{L_3} \frac{1/4}{L_2} \frac{1/4}{L_1} \| \psi_{4, \pm_4} \|_{B_{2/4}^{0, \infty}}
\]

\[
\lesssim \ \| \psi_{1, \pm_1} \|_{B_{2/4}^{0, \infty}} \| \psi_{2, \pm_2} \|_{B_{2/4}^{0, \infty}} \| \psi_{3, \pm_3} \|_{B_{2/4}^{0, \infty}} \| \psi_{4, \pm_4} \|_{B_{2/4}^{0, \infty}}.
\]

Finally we treat the case \( L_0 \sim L_2 \) and \( L_3 \sim L_4 \). This is also straightforward.
If $N_0 \ll N_1 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, by (5.2), we get

$$| 1 | \lesssim \sum_{N_0} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_0 L_1)^{1/4} (N_0 L_0)^{1/4} (N_0 L_3)^{1/4} \left\| P_{K_{N_1 L_1}} \psi_{1, \pm, 1} \right\|$$

$$\times \left\| P_{K_{N_2 L_2}} \psi_{2, \pm, 2} \right\| \left\| P_{K_{N_3 L_3}} \psi_{3, \pm, 3} \right\| \left\| P_{K_{N_4 L_4}} \psi_{4, \pm, 4} \right\|$$

$$= \sum_{N_0} \left( \frac{N_0}{N_1} \right)^{1/2} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}$$

$$\times \left\| \psi_{1, \pm, 1} \right\|_{B_{L_1}^{0, 1/2}} \left\| \psi_{2, \pm, 2} \right\|_{B_{L_2}^{0, 1/2}} \left\| \psi_{3, \pm, 3} \right\|_{B_{L_3}^{0, 1/2}} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}.$$

For $N_0 \ll N_1 \sim N_2$ and $N_3 \lesssim N_0 \sim N_4$, then using (5.1) and (5.2) with $j = 3$, we get

$$| 1 | \lesssim \sum_{N_0} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_0 L_1)^{1/2} (N_1 L_2)^{1/4} (N_3 L_3)^{1/4} \left\| P_{K_{N_1 L_1}} \psi_{1, \pm, 1} \right\|$$

$$\times \left\| P_{K_{N_2 L_2}} \psi_{2, \pm, 2} \right\| \left\| P_{K_{N_3 L_3}} \psi_{3, \pm, 3} \right\| \left\| P_{K_{N_4 L_4}} \psi_{4, \pm, 4} \right\|$$

$$= \sum_{N_0} \left( \frac{N_3}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}$$

$$\times \left\| \psi_{1, \pm, 1} \right\|_{B_{L_1}^{0, 1/2}} \left\| \psi_{2, \pm, 2} \right\|_{B_{L_2}^{0, 1/2}} \left\| \psi_{3, \pm, 3} \right\|_{B_{L_3}^{0, 1/2}} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}.$$

For $N_0 \ll N_1 \sim N_2$ and $N_3 \lesssim N_0 \sim N_3$, similarly,

$$| 1 | \lesssim \sum_{N_0} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_1 L_1)^{1/2} (N_1 L_2)^{1/4} (N_4 L_0)^{1/4} (N_3 L_3)^{1/4} \left\| P_{K_{N_1 L_1}} \psi_{1, \pm, 1} \right\|$$

$$\times \left\| P_{K_{N_2 L_2}} \psi_{2, \pm, 2} \right\| \left\| P_{K_{N_3 L_3}} \psi_{3, \pm, 3} \right\| \left\| P_{K_{N_4 L_4}} \psi_{4, \pm, 4} \right\|$$

$$= \sum_{N_0} \left( \frac{N_4}{N_0} \right)^{1/2} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}$$

$$\times \left\| \psi_{1, \pm, 1} \right\|_{B_{L_1}^{0, 1/2}} \left\| \psi_{2, \pm, 2} \right\|_{B_{L_2}^{0, 1/2}} \left\| \psi_{3, \pm, 3} \right\|_{B_{L_3}^{0, 1/2}} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}.$$

If $N_1 \lesssim N_0 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, then

$$| 1 | \lesssim \sum_{N_0} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_1 L_1)^{1/2} (N_1 L_2)^{1/4} (N_0 L_0)^{1/4} (N_0 L_3)^{1/4} \left\| P_{K_{N_1 L_1}} \psi_{1, \pm, 1} \right\|$$

$$\times \left\| P_{K_{N_2 L_2}} \psi_{2, \pm, 2} \right\| \left\| P_{K_{N_3 L_3}} \psi_{3, \pm, 3} \right\| \left\| P_{K_{N_4 L_4}} \psi_{4, \pm, 4} \right\|$$

$$= \sum_{N_0} \left( \frac{N_1}{N_0} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}$$

$$\times \left\| \psi_{1, \pm, 1} \right\|_{B_{L_1}^{0, 1/2}} \left\| \psi_{2, \pm, 2} \right\|_{B_{L_2}^{0, 1/2}} \left\| \psi_{3, \pm, 3} \right\|_{B_{L_3}^{0, 1/2}} \left\| \psi_{4, \pm, 4} \right\|_{B_{L_4}^{0, 1/2}}.$$
If $N_1 \lesssim N_0 \sim N_2$ and $N_3 \lesssim N_0 \sim N_4$, then

$$
|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_1 L_1)^{1/2} (N_1 L_2)^{1/4} (N_3 L_0)^{1/2} (N_3 L_3)^{1/2} ||P_{K_{N_1},L_1}^\pm \psi_{1,\pm1}||
$$

$$
\times ||P_{K_{N_2},L_2}^\pm \psi_{2,\pm2}|| ||P_{K_{N_3},L_3}^\pm \psi_{3,\pm3}|| ||P_{K_{N_4},L_4}^\pm \psi_{4,\pm4}||
$$

$$
= \sum_{N,L} \frac{1}{N_0} \left( \frac{N_3}{N_0} \right)^{1/4} \left( \frac{N_3}{N_0} \right)^{3/4} \left( \frac{N_3}{N_0} \right)^{1/2} L_0 L_1^{1/2} L_2^{1/4} L_3^{1/4} L_4^{-1/2} ||\psi_{4,\pm4}||_{L^2_{\theta,\eta}}
$$

$$
\times ||P_{K_{N_1},L_1}^\pm \psi_{1,\pm1}|| ||P_{K_{N_2},L_2}^\pm \psi_{2,\pm2}|| ||P_{K_{N_3},L_3}^\pm \psi_{3,\pm3}||
$$

$$
\lesssim ||\psi_{1,\pm1}||_{B^{0,1}_{\theta,\eta}} ||\psi_{2,\pm2}||_{B^{0,1}_{\theta,\eta}} ||\psi_{3,\pm3}||_{B^{0,1}_{\theta,\eta}} ||\psi_{4,\pm4}||_{B^{0,1}_{\theta,\eta}}.
$$

If $N_1 \lesssim N_0 \sim N_2$ and $N_4 \lesssim N_0 \sim N_3$, then

$$
|J^1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_0}{N_1} \right)^{1/2} (N_1 L_1)^{1/2} (N_1 L_2)^{1/4} (N_4 L_0)^{1/2} (N_3 L_3)^{1/2} ||P_{K_{N_1},L_1}^\pm \psi_{1,\pm1}||
$$

$$
\times ||P_{K_{N_2},L_2}^\pm \psi_{2,\pm2}|| ||P_{K_{N_3},L_3}^\pm \psi_{3,\pm3}|| ||P_{K_{N_4},L_4}^\pm \psi_{4,\pm4}||
$$

$$
= \sum_{N,L} \frac{1}{N_0} \left( \frac{N_3}{N_0} \right)^{1/4} \left( \frac{N_3}{N_0} \right)^{1/4} \left( \frac{N_3}{N_0} \right)^{1/2} L_0 L_1^{1/2} L_2^{1/4} L_3^{1/4} L_4^{-1/2} ||\psi_{4,\pm4}||_{L^2_{\theta,\eta}}
$$

$$
\times ||P_{K_{N_1},L_1}^\pm \psi_{1,\pm1}|| ||P_{K_{N_2},L_2}^\pm \psi_{2,\pm2}|| ||P_{K_{N_3},L_3}^\pm \psi_{3,\pm3}||
$$

$$
\lesssim ||\psi_{1,\pm1}||_{B^{0,1}_{\theta,\eta}} ||\psi_{2,\pm2}||_{B^{0,1}_{\theta,\eta}} ||\psi_{3,\pm3}||_{B^{0,1}_{\theta,\eta}} ||\psi_{4,\pm4}||_{B^{0,1}_{\theta,\eta}}.
$$

5.1.4. Case 4: $L_0 \leq L_1 \leq L_2 , L_3 \leq L_4 \leq L_0$.

From Remark 5.1 and the observation given by the previous subsections, we only need to consider the case $N_0^{012} \lesssim L_2 \sim N_0^{102} \sim N_0^{012}$ and $L_0 \sim N_j$, $j = 0, 3, 4$. If $N_0 \lesssim N_1 \sim N_2$, then we have $N_0 \sim L_0 \lesssim L_2 \sim N_1$ and it leads to $N_0 \ll L_2$, which is impossible. Also $N_1 \ll N_0 \sim N_2$ cannot appear. Hence we are left to consider the case $L_2 \sim L_0 \sim N_j$, $j = 0, 1, 2, 3, 4$. To do this, we use (3.3) to $\psi_{3,\pm3}$ and $\psi_{4,\pm4}$ to get

$$
|J^1| \lesssim \sum_{N,L} \sum_{\omega \in \omega_0} \frac{1}{N_0} (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} \left( \frac{N_0 L_0}{N_3 N_4} \right)^{1/2} L_3 L_4^{1/2}
$$

$$
\times ||P_{K_{N_1},L_1}^\pm \psi_{1,\pm1}|| ||P_{K_{N_2},L_2}^\pm \psi_{2,\pm2}|| ||P_{K_{N_3},L_3}^\pm \psi_{3,\pm3}^{2\gamma_{34,\omega_0}}|| ||P_{K_{N_4},L_4}^\pm \psi_{4,\pm4}^{2\gamma_{34,\omega_0}}||
$$

$$
\lesssim \sum_{N,L} \frac{L_0^{3/4} L_1^{1/4} L_3^{1/2} P_{K_{N_1},L_1}^\pm \psi_{1,\pm1} ||P_{K_{N_2},L_2}^\pm \psi_{2,\pm2}|| ||P_{K_{N_3},L_3}^\pm \psi_{3,\pm3}|| ||P_{K_{N_4},L_4}^\pm \psi_{4,\pm4}||_{L^{0,1}_{\theta,\eta}}
$$

$$
\lesssim ||\psi_{1,\pm1}||_{L^{0,1}_{\theta,\eta}} ||\psi_{2,\pm2}||_{L^{0,1}_{\theta,\eta}} ||\psi_{3,\pm3}||_{L^{0,1}_{\theta,\eta}} ||\psi_{4,\pm4}||_{L^{0,1}_{\theta,\eta}}.
$$

Now we consider $L_1 \sim L_2$ and $L_4 \sim L_0$. The required estimates is followed by straightforward calculus.
If $N_0 \ll N_1 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, then using (3.2), we get

$$|J^1| \lesssim \sum_{N \leq N_0} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} (N_0 L_3)^{1/2} (N_0 L_0)^{1/4} \| P_{K_{N_1,L_1}}^\pm \psi_{1, \pm 1} \| \times \| P_{K_N^+} \psi_{2, \pm 2} \| \| P_{K_{N_3,L_3}}^\pm \psi_{3, \pm 3} \| \| P_{K_{N_4,L_4}}^\pm \psi_{4, \pm 4} \|$$

$$= \sum_{N \leq N_0} \left( \frac{N_0}{N_1} \right)^{1/2} L_0^{3/4} L_1^{1/4} L_2^{1/2} L_3^{1/2} L_4^{-1/2} \| \psi_{4, \pm 4} \|_{B^{1/2}_{\pm 4} \infty} \times \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \lesssim \| \psi_{1, \pm 1} \|_{B^{3/4}_{\pm 1} \infty} \| \psi_{2, \pm 2} \|_{B^{3/4}_{\pm 2} \infty} \| \psi_{3, \pm 3} \|_{B^{3/4}_{\pm 3} \infty} \| \psi_{4, \pm 4} \|_{B^{3/4}_{\pm 4} \infty}.$$

For $N_0 \ll N_1 \sim N_2$ and $N_3 \lesssim N_0 \sim N_4$, then using (3.2) with $j = 1$ and (3.1), we get

$$|J^1| \lesssim \sum_{N \leq N_0} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} (N_0 L_3)^{1/2} (N_0 L_4)^{1/4} \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \times \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \| P_{K_{N_4,L_4}^+} \psi_{4, \pm 4} \|$$

$$= \sum_{N \leq N_0} \left( \frac{N_0}{N_1} \right)^{1/2} \left( \frac{N_0}{N_1} \right)^{1/4} \left( \frac{N_0}{N_1} \right)^{1/4} \frac{L_0^{3/4} L_1^{1/4} L_2^{1/2} L_3^{1/2} L_4^{-1/4} \| \psi_{4, \pm 4} \|_{B^{1/2}_{\pm 4} \infty} \times \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \lesssim \| \psi_{1, \pm 1} \|_{B^{3/4}_{\pm 1} \infty} \| \psi_{2, \pm 2} \|_{B^{3/4}_{\pm 2} \infty} \| \psi_{3, \pm 3} \|_{B^{3/4}_{\pm 3} \infty} \| \psi_{4, \pm 4} \|_{B^{3/4}_{\pm 4} \infty}.$$

For $N_0 \ll N_1 \sim N_2$ and $N_4 \lesssim N_0 \sim N_3$,

$$|J^1| \lesssim \sum_{N \leq N_0} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_0 L_0)^{1/2} (N_0 L_1)^{1/4} (N_0 L_3)^{1/2} (N_0 L_4)^{1/4} \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \times \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \| P_{K_{N_4,L_4}^+} \psi_{4, \pm 4} \|$$

$$= \sum_{N \leq N_0} \left( \frac{N_0}{N_1} \right)^{1/4} \frac{L_0^{3/4} L_1^{1/4} L_2^{1/2} L_3^{1/2} L_4^{-1/4} \| \psi_{4, \pm 4} \|_{B^{1/2}_{\pm 4} \infty} \times \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \lesssim \| \psi_{1, \pm 1} \|_{B^{3/4}_{\pm 1} \infty} \| \psi_{2, \pm 2} \|_{B^{3/4}_{\pm 2} \infty} \| \psi_{3, \pm 3} \|_{B^{3/4}_{\pm 3} \infty} \| \psi_{4, \pm 4} \|_{B^{3/4}_{\pm 4} \infty}.$$

If $N_1 \lesssim N_0 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, then by (3.2), we get

$$|J^1| \lesssim \sum_{N \leq N_0} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} (N_1 L_0)^{1/2} (N_1 L_1)^{1/4} (N_0 L_3)^{1/2} (N_0 L_0)^{1/4} \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \times \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \| P_{K_{N_4,L_4}^+} \psi_{4, \pm 4} \|$$

$$= \sum_{N \leq N_0} \left( \frac{N_1}{N_0} \right)^{1/4} \frac{L_0^{3/4} L_1^{1/4} L_2^{1/2} L_3^{1/2} L_4^{-1/4} \| \psi_{4, \pm 4} \|_{B^{1/2}_{\pm 4} \infty} \times \| P_{K_{N_1,L_1}^+} \psi_{1, \pm 1} \| \| P_{K_{N_3,L_3}^+} \psi_{2, \pm 2} \| \| P_{K_{N_4,L_4}^+} \psi_{3, \pm 3} \| \lesssim \| \psi_{1, \pm 1} \|_{B^{3/4}_{\pm 1} \infty} \| \psi_{2, \pm 2} \|_{B^{3/4}_{\pm 2} \infty} \| \psi_{3, \pm 3} \|_{B^{3/4}_{\pm 3} \infty} \| \psi_{4, \pm 4} \|_{B^{3/4}_{\pm 4} \infty}.$$
For $N_1 \lesssim N_0 \sim N_2$ and $N_3 \lesssim N_0 \sim N_4$,

$$|\mathbf{J}_1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} \left( N_1 L_0 \right)^{1/2} \left( N_1 L_1 \right)^{1/4} \left( N_3 L_3 \right)^{1/2} \left( N_4 L_4 \right)^{1/4} \left\| P_{K_{N_1,L_1}^{\pm}} \psi_{1,\pm} \right\| \times \left\| P_{K_{N_2,L_2}^{\pm}} \psi_{2,\pm} \right\| \left\| P_{K_{N_3,L_3}^{\pm}} \psi_{3,\pm} \right\| \left\| P_{K_{N_4,L_4}^{\pm}} \psi_{4,\pm} \right\|$$

$$= \sum_{N,L} \left( \frac{N_1}{N_0} \right)^{1/4} \left( \frac{N_3}{N_0} \right)^{3/4} \left( L_0 \right)^{1/2} \left( L_1 \right)^{1/4} \left( L_2 \right)^{1/2} \left( L_3 \right)^{1/4} \left( L_4 \right)^{1/4} \left\| \psi_{4,\pm} \right\|_{B_{4}^{0,\frac{1}{2}}} \times \left\| P_{K_{N_1,L_1}^{\pm}} \psi_{1,\pm} \right\| \left\| P_{K_{N_2,L_2}^{\pm}} \psi_{2,\pm} \right\| \left\| P_{K_{N_3,L_3}^{\pm}} \psi_{3,\pm} \right\|$$

$$\lesssim \left\| \psi_{1,\pm} \right\|_{B_{1}^{0,\frac{1}{2}}} \left\| \psi_{2,\pm} \right\|_{B_{2}^{0,\frac{1}{2}}} \left\| \psi_{3,\pm} \right\|_{B_{3}^{0,\frac{1}{2}}} \left\| \psi_{4,\pm} \right\|_{B_{4}^{0,\frac{1}{2}}}.$$

For $N_1 \lesssim N_0 \sim N_2$ and $N_4 \lesssim N_0 \sim N_3$, similarly,

$$|\mathbf{J}_1| \lesssim \sum_{N,L} \frac{1}{N_0} \left( \frac{L_2}{N_1} \right)^{1/2} \left( N_1 L_0 \right)^{1/2} \left( N_1 L_1 \right)^{1/4} \left( N_4 L_3 \right)^{1/2} \left( N_3 L_4 \right)^{1/4} \left\| P_{K_{N_1,L_1}^{\pm}} \psi_{1,\pm} \right\| \times \left\| P_{K_{N_2,L_2}^{\pm}} \psi_{2,\pm} \right\| \left\| P_{K_{N_3,L_3}^{\pm}} \psi_{3,\pm} \right\| \left\| P_{K_{N_4,L_4}^{\pm}} \psi_{4,\pm} \right\|$$

$$= \sum_{N,L} \left( \frac{N_1}{N_0} \right)^{1/4} \left( \frac{N_4}{N_0} \right)^{1/2} \left( \frac{N_3}{N_0} \right)^{1/4} \left( L_0 \right)^{1/2} \left( L_1 \right)^{1/4} \left( L_2 \right)^{1/2} \left( L_3 \right)^{1/2} \left\| \psi_{4,\pm} \right\|_{B_{4}^{0,\frac{1}{2}}} \times \left\| P_{K_{N_1,L_1}^{\pm}} \psi_{1,\pm} \right\| \left\| P_{K_{N_2,L_2}^{\pm}} \psi_{2,\pm} \right\| \left\| P_{K_{N_3,L_3}^{\pm}} \psi_{3,\pm} \right\|$$

$$\lesssim \left\| \psi_{1,\pm} \right\|_{B_{1}^{0,\frac{1}{2}}} \left\| \psi_{2,\pm} \right\|_{B_{2}^{0,\frac{1}{2}}} \left\| \psi_{3,\pm} \right\|_{B_{3}^{0,\frac{1}{2}}} \left\| \psi_{4,\pm} \right\|_{B_{4}^{0,\frac{1}{2}}}.$$

There are several cases left, but the proof is essentially same. This completes the proof of (5.1).

5.2. Proof of (5.2). To reveal the null structure of $N_1^2$, first we note that

$$\left| \mathcal{F}_x \left[ -\frac{1}{\Delta} \epsilon_{ijk} \alpha^0 \partial_j (\psi_{1,\pm,1}^\dagger \mathfrak{R}_{k_{\pm,2}}^j \psi_{2,\pm,2}) \right] \xi_0 \right| = \mathcal{F}_x \left[ \pm \frac{1}{iD} \epsilon_{ijk} \mathfrak{R}_{k_{\pm,2}}^j (\psi_{1,\pm,1}^\dagger \mathfrak{R}_{k_{\pm,2}}^k \psi_{2,\pm,2}) \right]$$

$$= \frac{1}{|\xi_0|} \epsilon_{ijk} \xi_j \int_{\xi_0=\xi_1-\xi_2} \psi_{1,\pm,1}^\dagger (\xi_1) \left( \pm \frac{\xi_k}{|\xi_0|} \right) \psi_{2,\pm,2} (\xi_2) d\xi_1 d\xi_2$$

$$= \frac{1}{|\xi_0|} \epsilon_{ijk} \xi_j \int_{\xi_0=\xi_1-\xi_2} \psi_{1,\pm,1}^\dagger (\xi_1) \left( \pm \frac{\xi_k}{|\xi_0|} \right) \Pi_{1} (\xi_1) \Pi_{2} (\xi_2) \psi_{2,\pm,2} (\xi_2) d\xi_1 d\xi_2$$

$$\lesssim \int_{\xi_0=\xi_1-\xi_2} \theta_{12} \left| \psi_{1,\pm,1} (\xi_1) \right| \left| \psi_{2,\pm,2} (\xi_2) \right| d\xi_1 d\xi_2,$$

where we use (3.6).

Then the right-hand side of (5.2) is

$$\left\| \Pi_{\pm,4} \left( -\frac{1}{\Delta} \epsilon_{ijk} \partial_j \left( (\Pi_{\pm,1} \psi_{1}^\dagger) \mathfrak{R}_{k_{\pm,2}}^k \Pi_{\pm,2} \psi_{2} \right) \psi_{3,\pm,3} \right) \right\|_{B_{4}^{0,\frac{1}{2}}}$$

$$= \sup_{\| \psi_{4,\pm,4} \|_{B_{4}^{0,\frac{1}{2}}} = 1} \left| \int \frac{1}{iD} \epsilon_{ijk} \mathfrak{R}_{k_{\pm,2}}^j (\psi_{1,\pm,1}^\dagger \mathfrak{R}_{k_{\pm,2}}^k \psi_{2,\pm,2}) \psi_{4,\pm,4} \psi_{3,\pm,3} dtdx \right|$$

$$\leq \sup_{\| \psi_{4,\pm,4} \|_{B_{4}^{0,\frac{1}{2}}} = 1} \left| \int \frac{1}{|\xi_0|} \mathcal{F}_x \left( \mathfrak{R}_{k_{\pm,2}}^j (\psi_{1,\pm,1}^\dagger \mathfrak{R}_{k_{\pm,2}}^k \psi_{2,\pm,2}) \right) \mathcal{F}_x (\psi_{4,\pm,4} \psi_{3,\pm,3}) dX_0 \right|.$$
Set
\[ J^2 := \int \frac{1}{|\xi|} F(\mathcal{N}_0^l((\psi^\dagger_{1,1} \mathcal{P}_{1,1}^3 \psi^\dagger_{2,2}) F(\psi^\dagger_{3,3} \psi^3_{3,3}))dX_0. \]

By a dyadic decomposition, we get
\[
J^2 = \sum_{N,L} \int \frac{1}{|\xi|} F P_{K_{N,L}}^L \mathcal{R}_0^l \left( \mathcal{P}_{1,1}^3 \psi^\dagger_{1,1} \mathcal{P}_{2,2}^3 \psi^\dagger_{2,2} \right) \times F P_{K_{N,L}}^L \mathcal{R}_0^l \left( \mathcal{P}_{3,3}^3 \psi^\dagger_{3,3} \psi^3_{3,3} \right) dX_0,
\]
where
\[
S = (N_0, N_1, N_2, N_3, N_4), \quad L = (L_0, L_1, L_2, L_3, L_4).
\]

Similarly, by decomposing the integrand of \( J^2 \) suitably, we may assume that \( F P_{K_{N_1,L_1}}^L \psi^\dagger_{1,1} \), \( F P_{K_{N_2,L_2}}^L \psi^\dagger_{2,2} \), \( F P_{K_{N_3,L_3}}^L \psi^\dagger_{3,3} \), and \( F P_{K_{N_4,L_4}}^L \psi^\dagger_{3,3} \) are nonnegative real-valued functions. Then by Cauchy-Schwarz inequality followed by (3.6) and Lemma 5.3, we obtain
\[
|J^2| \leq \sum_{N,L} \frac{1}{N_0} \left( \frac{\epsilon_{\text{max}}}{N_0 \text{min}} \right)^{1/2} \left\| P_{K_{N_0,L_0}}^L \left( \mathcal{P}_{1,1}^3 \psi^\dagger_{1,1} P_{K_{N_2,L_2}}^L \psi^\dagger_{2,2} \right) \right\| \left\| P_{K_{N_0,L_0}}^L \left( \mathcal{P}_{3,3}^3 \psi^\dagger_{3,3} P_{K_{N_4,L_4}}^L \psi^\dagger_{3,3} \right) \right\|.
\]

We see that \( J^2 \) can be treated as \( J^1 \), so we omit explicit proof. This completes the proof of Theorem 1.1.

6. Estimates of \( N_2 \): Proof of (4.4)

We shall show that \( N_2 \) has the same null structure as \( N_0^2 \) using (3.6). Indeed,
\[
F_x \left[ \frac{1}{\Delta} \epsilon^{\alpha_0} \partial_m (\psi^\dagger_{1,1} \psi^\dagger_{2,2}) \right] (\xi_0) \leq \frac{1}{|\xi_0|} \frac{\epsilon_{\text{max}}}{|\xi_0| \text{min}} \int_{\xi_0 = \xi_1 - \xi_2} \psi^\dagger_{1,1} (\xi_1) \psi^\dagger_{2,2} (\xi_2) d\xi_1 d\xi_2
\]
\[
= \frac{1}{|\xi_0|} \frac{\epsilon_{\text{max}}}{|\xi_0| \text{min}} \int_{\xi_0 = \xi_1 - \xi_2} \psi^\dagger_{1,1} (\xi_1) \Pi \psi^\dagger_{2,2} (\xi_2) d\xi_1 d\xi_2
\]
\[
\leq \frac{1}{|\xi_0|} \int_{\xi_0 = \xi_1 - \xi_2} \theta_{\text{max}} \psi^\dagger_{1,1} (\xi_1) \psi^\dagger_{2,2} (\xi_2) d\xi_1 d\xi_2.
\]

Thus the proof of (4.4) is exactly same as Section 5. This completes the proof of Theorem 1.1.

7. Failure of the smoothness

The aim of this section is to show the smoothness failure of flow of (1.1) in \( H^s \) for \( s < 0 \). The Cauchy problem (1.2) is equivalent to solving the integral equation. To show the failure of smoothness we adopt the argument of [10, 14]. Let us consider the system of equation:
\[
\begin{cases}
ig^\mu \partial_\mu \psi = m \psi - N(\psi, \psi) \psi, \\
\psi(0) = \delta \psi_0 \in H^{s}(\mathbb{R}^2).
\end{cases}
\]

If the flow is \( C_3 \) at the origin in \( H^s \), then it follows that
\[
\partial^3_3 \psi(0, t, \cdot) = 6i \sum_{\pm, j=1,2,3,4} \int_0^t S_{\pm j} (t - t') \Pi \psi(0) \mathcal{N}(\mathcal{S}_{\pm j}(t') \psi(0), S_{\pm j}(t') \psi(0)) S_{\pm 3}(t') \psi(0) dt'
\]
where \( S_{\pm}(t) = e^{-itD}. \) From the \( C_3 \) smoothness we have that
\[
(7.1) \sup_{0 \leq t \leq T} \left\| \sum_{\pm, j=1,2,3,4} \int_0^t S_{\pm j} (t - t') \Pi \psi(0) \mathcal{N}(\mathcal{S}_{\pm j}(t') \psi(0), S_{\pm j}(t') \psi(0)) S_{\pm 3}(t') \psi(0) dt' \right\|_{H^s} \lesssim \| \psi_0 \|_{H^s}^3.
\]
for a local existence time $T$. However, we will show that (7.1) fails for $s < 0$. The explicit statement is as follows.

**Proposition 7.1.** Let $s < 0$. Then the estimate

$$\sup_{0 \leq t \leq T} \| \mathcal{L}(\varphi)(t) \|_{H^s} \lesssim \| \varphi \|_{H^s}^3.$$ 

fails to hold for all $\varphi \in H^s$, where $\mathcal{L}(\varphi)(t) = \sum_{\pm, j = 1, \ldots, 4} \mathcal{L}_{1, \ldots, 4}(\varphi)(t)$ with

$$\mathcal{L}_{1, \ldots, 4}(\varphi)(t) = \int_0^t S_{\pm, 1}(t - t') \Pi_{\pm, 1}(N(S_{\pm, 2}(t'') \varphi, S_{\pm, 3}(t'') \varphi) S_{\pm, 4}(t') \varphi)(t') dt'.$$

**Proof.** Fix $1 \leq \mu \ll \lambda$. We first choose $\mu = \lambda^{1-\varepsilon}$ for fixed $0 < \varepsilon < 1$. Let us define the boxes

$$R^+ = \{ \xi = (\xi_1, \xi_2) : |\xi_1| + |\xi_2| \ll \mu, |\xi_2| \lesssim \mu \},
R^0 = \{ \xi = (\xi_1, \xi_2) : |\xi_1| \lesssim \mu, i = 1, 2 \},$$

and consider $\varphi = \left( \frac{F_{\xi} \chi_{R^+}}{F_{\xi} \chi_{R^0}} \right)$. Then we have $\| \varphi \|_{H^s} \sim \mu \lambda^s$.

The definition of a bilinear operator

$$N(\psi_1, \psi_2) = \frac{1}{\Delta} \left( \gamma^0 \left[ \partial_1 (\overline{\psi_1} \gamma^0 \psi_2) - \partial_2 (\overline{\psi_1} \gamma^1 \psi_2) \right] + \gamma^1 \partial_2 (\overline{\psi_1} \gamma^0 \psi_2) - \gamma^2 \partial_1 (\overline{\psi_1} \gamma^0 \psi_2) \right)$$

derives that

$$F_x[N(\psi, \psi)](\eta) = -|\eta|^{-2} \begin{pmatrix} \eta \mathcal{F}_x(\overline{\psi_1} \psi_2)(\eta) - \eta \mathcal{F}_x(\overline{\psi_1} \psi_2)(\eta) & -\eta \mathcal{F}_x(|\psi_1|^2 + |\psi_2|^2)(\eta) \\ -\lambda \mathcal{F}_x(|\psi_1|^2 + |\psi_2|^2)(\eta) & -\eta \mathcal{F}_x(\overline{\psi_1} \psi_2)(\eta) + \eta \mathcal{F}_x(\psi_1 \overline{\psi_2})(\eta) \end{pmatrix}$$

for $\psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)$. So we get

$$F_x(L_{1, \ldots, 4}(\varphi)(t))(\xi)$$

$$= \left( I_{2 \times 2} \pm \frac{\xi_1 \alpha_3}{|\xi|} \right) \left( C_{1, \ldots, 4}(\xi) \right) \left( e^{-\frac{i}{2} c^3 \xi_1 t} \right)$$

$$= \left( I_{2 \times 2} \pm \frac{\xi_1 \alpha_3}{|\xi|} \right) \left( C_{1, \ldots, 4}(\xi) \right) \left( e^{-\frac{i}{2} c^3 \xi_1 t} \right)$$

where

$$C_{1, \ldots, 4}(\xi) \pm \frac{\xi_1 \alpha_3}{|\xi|} \left( C_{3, \ldots, 4}(\xi) \right)$$

$$C_{1, \ldots, 4}(\xi) \pm \frac{\xi_1 \alpha_3}{|\xi|} \left( C_{3, \ldots, 4}(\xi) \right)$$
\[ C_{1 \ldots 4}^1(\xi) = \int_{2R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{\tilde{\eta}}{|\eta|^2} \chi_{R^\mu} (-\zeta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta \]

\[ + \int_{2R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{-\tilde{\eta}}{|\eta|^2} \chi_{R^\mu} (-\zeta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta, \]

\[ C_{1 \ldots 4}^2(\xi) = \int_{R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{-\tilde{\eta}}{|\eta|^2} \chi_{R^\mu} (-\zeta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta \]

\[ + \int_{R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{-\tilde{\eta}}{|\eta|^2} \chi_{R^\mu} (-\zeta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta, \]

\[ C_{1 \ldots 4}^3(\xi) = \int_{R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{-\tilde{\eta}}{|\eta|^2} \chi_{R^\mu} (-\zeta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta \]

\[ + \int_{R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{-\tilde{\eta}}{|\eta|^2} \chi_{R^\mu} (-\zeta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta, \]

\[ C_{1 \ldots 4}^4(\xi) = \int_{2R^\mu} \int_{R^\mu} \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) \frac{\eta}{|\eta|^2} \chi_{R^\mu} (\xi - \eta) \chi_{R^\mu} (\eta - \zeta) \chi_{R^\mu} (\xi - \eta) d\zeta d\eta \]

and

\[ \mathbf{p}_{1 \ldots 4}(t, \xi, \eta, \zeta) := \int_0^t e^{-(\pm_1 (t-t')|\xi| \pm_2 |\zeta| \pm_3 |\eta - \zeta| \pm_4 |\eta|) dt'} \]

\[ = \frac{e^{-\pm_1 \omega |\xi| (e^{t \omega} - 1)}}{t \omega} \]

with

\[ \omega = \pm_1 |\xi| \pm_2 |\zeta| \pm_3 |\eta - \zeta| \pm_4 |\eta|. \]

By symmetry of \( \eta \) in \( R^\mu \), we have \( C_{1 \ldots 4}^2(\xi) = C_{1 \ldots 4}^3(\xi) = 0 \). This follows that

\[ \mathcal{F}_x \mathcal{L}_{1 \ldots 4}(\varphi)(t)(\xi) = \begin{pmatrix} C_{1 \ldots 4}(\xi) \pm_1 \frac{\tilde{\xi}}{|\xi|} C_{1 \ldots 4}(\xi) \\ C_{1 \ldots 4}(\xi) \pm_1 \frac{\tilde{\xi}}{|\xi|} C_{1 \ldots 4}(\xi) \end{pmatrix}. \]

Then the failure of (7.1) is reduced to the one of the following:

(7.2) \[ \left\| \sum_{\pm_1, j=1,2,3,4} \left( C_{1 \ldots 4}(\xi) \pm_1 \frac{\tilde{\xi}}{|\xi|} C_{1 \ldots 4}(\xi) \right) \right\|_{H^s} \lesssim \left\| \varphi \right\|_{H^s}. \]

For this, we will show that

(7.3) \[ \left\| \sum_{\pm_1, j=1,2,3,4} \left( C_{1 \ldots 4}(\xi) \pm_1 \frac{\tilde{\xi}}{|\xi|} C_{1 \ldots 4}(\xi) \right) \right\| \gtrsim \mu^4 \lambda^{-1}. \]

We now assume that (7.2) holds. Then (7.3) yields

(7.4) \[ t \mu^5 \lambda^{s-1} \lesssim \left\| \varphi \right\|_{L^2} \left\| \sum_{\pm_1, j=1,2,3,4} \left( C_{1 \ldots 4}(\xi) \pm_1 \frac{\tilde{\xi}}{|\xi|} C_{1 \ldots 4}(\xi) \right) \right\| \lesssim \mu^3 \lambda^{3s}. \]
Now we show (7.3). From the support condition it follows that \( \eta \in 2R_{3\mu}, 2R_{3\mu}^+, \) provided \( \xi \in R_{3\mu}, 3R_{3\mu}^+, \) respectively. If \( \pm_1 = \pm_2 = \pm_3 = \pm_4, \) then \( |\omega| \sim \lambda. \) If \( \pm_1 = \pm_3, \pm_2 = \pm_4, \) and \( \pm_1 \neq \pm_2, \) then \( |\omega| \lesssim \frac{\mu^2}{\chi}. \) If \( \pm_1 \cdot \pm_2 \cdot \pm_3 \cdot \pm_4 = -1, \) then \( |\omega| \lesssim \lambda. \)

Now we set \( t = \delta \lambda^{-1-\varepsilon} \) for fixed \( 0 < \delta \ll 1. \) Since \( |t\omega| \ll 1 \) for \( \lambda \) large enough, we get

\[
\sum_{\pm, j=1,2,3,4} p_{1-4}(t, \eta, \xi, \zeta) = \sum_{\pm, j=1,2,3,4} te^{-\pm_it|\xi|} \left( \frac{\cos(t\omega) - 1}{it\omega} + i \frac{\sin(t\omega)}{it\omega} \right)
\]

\[
= \sum_{\pm, j=1,2,3,4} te^{-\pm_it|\xi|}(O_{\pm}(\delta) + i)
\]

\[
= \sum_{\pm, j=1,2,3,4} te^{-\pm_it|\xi|}O_{\pm}(\delta) + i \sum_{\pm, j=1,2,3,4} te^{-\pm_it|\xi|}
\]

\[
= \sum_{\pm, j=1,2,3,4} te^{-\pm_it|\xi|}O_{\pm}(\delta) + 8it\cos(t|\xi|)
\]

\[
= \sum_{\pm, j=1,2,3,4} te^{-\pm_it|\xi|}O_{\pm}(\delta) + 8it(1 + O(\delta)).
\]

Hence we obtain

\[
\sum_{\pm, j=1,2,3,4} \left( C_{1-4}(\xi) \pm \frac{\overline{\xi}}{|\xi|} C_{1-4}(\xi) \right) \sim it \int_{2R_{3\mu}} \int_{R_{3\mu}^+} \frac{\eta}{|\eta|^2} x_{R_{3\mu}^+}(-\zeta) x_{R_{3\mu}^+}(\eta - \zeta) x_{R_{3\mu}^+}(\xi - \eta) d\eta d\zeta
\]

\[
- it \int_{2R_{3\mu}} \int_{R_{3\mu}^+} \frac{\eta}{|\eta|^2} x_{R_{3\mu}^+}(\eta - \zeta) x_{R_{3\mu}^+}(\xi - \eta) d\eta d\zeta
\]

\[
- \frac{\xi}{|\xi|} \int_{2R_{3\mu}} \int_{R_{3\mu}^+} \frac{\eta}{|\eta|^2} x_{R_{3\mu}^+}(\eta - \xi) x_{R_{3\mu}^+}(\xi - \eta) d\eta d\zeta
\]

\[
= it(\mathcal{H}_1(\xi) - \mathcal{H}_2(\xi) - \mathcal{H}_3(\xi) + \mathcal{H}_4(\xi)).
\]

Let \( S_1 := R_{3\mu}, S_2 := 3R_{3\mu}^+, S_3 := 3R_{3\mu}^+, S_4 := R_{3\mu}^+ \). Then we get \( S_i \cap S_j = \emptyset (i \neq j) \) and \( \mathcal{H}_i(\xi) = 0 \) for \( \xi \notin S_i. \)

This gives us that

\[
\sum_{\pm, j=1,2,3,4} \left( C_{1-4}(\xi) \pm \frac{\overline{\xi}}{|\xi|} C_{1-4}(\xi) \right) \sim it \mathcal{H}_i(\xi)
\]

for \( \xi \in S_i. \) This follows that

\[
\left| \sum_{\pm, j=1,2,3,4} \left( C_{1-4}(\xi) \pm \frac{\overline{\xi}}{|\xi|} C_{1-4}(\xi) \right) \right| = \sum_{i=1}^{4} |\mathcal{H}_i(\xi)| \gtrsim \mu^4 \lambda^{-1}.
\]

Therefore, by (7.4) and \( t = \delta \lambda^{-1-\varepsilon}, \) we finally have

\[
\delta \lesssim \lambda^{2s+2+\varepsilon} \mu^{-2} = \lambda^{2s+3\varepsilon}.
\]

This leads to failure of (7.2) for \( s < 0. \) This completes the proof of Proposition 7.1 \( \square \)

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