TOPOLOGICAL EQUIVALENCE OF SMOOTH
FUNCTIONS WITH ISOLATED CRITICAL POINTS ON
A CLOSED SURFACE

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Abstract. We consider functions with isolated critical points on
a closed surface. We prove that in a neighborhood of a critical point
the function conjugates with Rez^k for the some nonnegative integer
k. The full topological invariant of such functions is constructed.

Let F be a closed smooth surface, f, g : F → R be a smooth functions
with a finite number of critical points. In view of the compactness of the
surface, this condition is equivalent that each critical point is isolated.

Functions f, g are called topologically equivalent if there are home-
omorphisms h : F → F and h' : R → R such that fh = h'g. We say
that the functions are topologically conjugate if they are topologically
equivalent and homeomorphism h' preserves the orientation. Homeo-
morphisms h and h' we call conjugate homeomorphisms.

In the theory of singularities the differentiable equivalence are stud-
ied. It is such a topological equivalence, in which the conjugate homeo-
morphisms are diffeomorphisms.

The purpose of the given paper is to give a topological classification
of smooth functions on a closed surface with isolated critical points.
For Morse function such classification was obtained by Sharko [5] and
Kulinich [1]. There is a classification for functions with unique critical
level (except minimum and maximum) that is imbedded graph [2].

In the section 1 it is proved that in a neighborhood of an isolated
critical point the function is topologically conjugated with the func-
tion Rez^k, where k is a nonnegative integer. In the section 2 the full
topological invariant of the function on a closed surface is constructed
and the criterions of topological conjugation of functions with isolated
singular points are proved. In the section 3 the numbers of minimal
non-conjugated functions and the numbers of minimal topologically
non-equivalent functions are calculated for closed oriented and non-
oriented surfaces of genus 0, 1, 2 and 3.

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1. **Local Conjugation of Functions in a Neighborhood of an Isolated Critical Point.**

Let $F$ be a smooth closed surface and $f : F \to \mathbb{R}$ be a smooth function.

**Lemma 1.1.** For each isolated critical point $x_0$ with the critical value $y_0 = f(x_0)$ there exists a neighborhood $U(x_0)$, in which

$$f^{-1}(y_0) \cap (U(x_0)) \simeq \text{Con}(\cup S^0).$$

Here $\text{Con}(\cup S^0)$ is a cone on the union of 0-dimensional spheres $S^0$, that is the union of the even number of the segments with a unique common point, which is an endpoint for each of the segments.

**Proof.** Since $x_0$ is an isolated singular point, there is a simply connected neighborhood $U$, which closure does not contain other critical points of the function and which boundary is a smooth closed curve. Let us assume $g$ is a restriction of function $f$ to the set $U \setminus \{x_0\}$. Then the levels of both functions coincide in this set. As $y_0$ is a regular value of the function $g$, the pre-image $g^{-1}(y_0)$ is a 1-dimensional submanifold in $U \setminus \{x_0\}$, that is a disjoint union of imbedded circles and open intervals.

Let $H$ be a connected component of the set $g^{-1}(y_0)$. If $H$ is homeomorphic to a circle, then it is the boundary of a closed set $D$. As function $f$ is not constant on $D$ the maximum or minimum value of it on $D$ is distinct from $y_0$, and the appropriate point of maximum or minimum is another, except for $x_0$, critical point in domain $U$. The obtained contradiction proves that $H$ is homeomorphic to an open interval.

Let us show that all accumulation points of $H$ in $U \setminus \{x_0\}$ belong to $H$. By contradiction, let there is such a point $z$ that does not belong to $H$. Then, according to the continuity of the function $f$: $z \in g^{-1}(y_0)$. As $z \notin H$, then $z$ belong other component $H_1$ of $g^{-1}(y_0)$. Since $g^{-1}(y_0)$ is a submanifold, there is a neighborhood of a point $z$ that does not contain other points from $g^{-1}(y_0)$, except the points of $H_1$. It contradicts assumption that $z$ is an accumulation point of $H$.

Using similar argument to the neighborhood of closure $\text{cl}(U)$, we obtain that the accumulation points of $H$ in $\text{cl}(U)$ belong the same component of $f^{-1}(y_0) \cap \text{cl}(U)$ as $H$. So in the boundary of $U$ there is no more than two accumulation points of interval $H$. Accumulation points of $H$ that belong to the boundary of $U$ or coincide with $x_0$ we call extremities of the interval. Three types of the interval $H$ are possible:

1) both extremities coincide with the point $x_0$,
2) both extremities lay on the boundary of $U$,
3) one of extremities coincides with a point \( x_0 \) and another lays on the boundary \( U \).

Let us prove that there doesn’t exit a component of the level \( g^{-1}(y_0) \) that form a loop with vertex in the point \( x_0 \) (i.e. both extremities coincide with \( x_0 \)). By contradiction, let such component exists, then it is a boundary of the two-dimensional disk. In this disk there is a critical point of minimum or maximum that distinct from \( x_0 \). It contradicts a choice of a neighborhood \( U(x_0) \). So there doesn’t exit an interval of type 1).

Note that only the finite number of extremities of intervals can lay on the boundary of \( U \). Otherwise, the condition that \( f^{-1}(y_0) \) is a submanifold in a limit point is false. So there are a finite number of intervals of type 2) and 3).

We contract a neighborhood \( U \) to such a neighborhood \( U(x_0) \), which closure is not intersected with interval of type 2) and such, that the boundary of \( U(x_0) \) is smooth and has transversal intersection with \( g^{-1}(y_0) \). Then \( f^{-1}(y_0) \cap cl(U(x_0)) \) consists of finite union of the closed intervals, one of which extremities coincides with a point \( x_0 \), and another lays on the boundary of \( U(x_0) \). Hence \( f^{-1}(y_0) \cap cl(U(x_0)) \) is a cone of the finite number of points.

Let us prove that number of extremities laying on the boundary \( \partial U \) is even. We call domains in \( U(x_0) \setminus f^{-1}(y_0) \) with \( f(x) > y_0 \) by positive and with \( f(x) < y_0 \) by negative. Each interval contains in the boundary both positive and negative domain. These domains alternate. Thus we have the same number of the domains of each type. Hence, total number of the domains, as well as intervals, is even. \( \square \)

*Remark 1.2.* The similar lemma takes place for three-dimensional manifolds [6].

**Theorem 1.3.** For each isolated critical point \( x_0 \) (except for the local minimum and maxima) of a smooth function on a surface there is a neighborhood, in which the function is topologically conjugated with the function \( Rez^k \) for some nonnegative integer \( k \).

**Proof.** Let \( U(x_0) \) be a neighborhood with the same properties as in the lemma and \( 2k \) be the number of arcs, which an extremity is the point \( x_0 \).

The critical level devides the neighborhood \( U(x_0) \) to domains. Let \( V \) be one of them and such that \( f(x) < y_0 \) for \( x \in V \). Then in the points of the intersection \( \partial U(x_0) \) with \( \partial V \) the vector field \( \text{grad} f \) is directed inside of the domain \( V \).
Let us prove that there is a trajectory that passes through $\partial U(x_0) \cap \partial V$ and tends to $x_0$. Really, let $h : \partial U(x_0) \cap \partial V \to f^{-1}(y_0)$ be a map given by the formula

$$h(x) = \begin{cases} x_0 & \text{if } \gamma(x) \cap f^{-1}(y_0) = \emptyset, \\ \gamma(x) \cap f^{-1}(y_0) & \text{if } \gamma(x) \cap f^{-1}(y_0) \neq \emptyset. \end{cases}$$

In view of the smoothness of the vector field $\nabla f$, the continuity of map $h$ follows for points, which image does not coincide with the point $x_0$. If the set of such points coincides with $\partial U(x_0) \cap \partial V$, then the image of this connected set by the continuous map is connected set. The obtained inconsistency proves the existence of a required trajectory.

Let $W$ be a neighborhood of a point $x_0$ in $f^{-1}(y_0)$. We define a neighborhood $W(\varepsilon)$ in $F$ for $\varepsilon > 0$ by formula

$$W(\varepsilon) = \{x \in F : |f(x) - y_0| < \varepsilon, \gamma(x) \cap W \neq \emptyset\}.$$ 

By the same formula, as above we define a map

$$h : W(\varepsilon) \to W.$$

Let us prove that for any connected neighborhood $W$ of the point $x_0$ in $f^{-1}(y_0)$, such that $cl(W) \cap U(x_0) \neq \emptyset$ there exists $\varepsilon > 0$ that $W(\varepsilon) \subset U(x_0)$. Let $z_i \in \partial W, i = 1, 2, \ldots, 2k$. Let $z_i', z_i'' \in \gamma(z_i) \cap \partial U(x_0)$ be such points that $f(z_i') < y_0 < f(z_i'')$ and between points $z_i', z_i''$ on the trajectory $\gamma(z_i)$ there don’t exist points of its intersection with boundary $\partial U(x_0)$. Let $\varepsilon_i = \min\{|y_0 - f(z_i')|, |y_0 - f(z_i'')|\}$. Denote by $T$ union of all arcs of the boundary $\partial U(x_0)$ that their endpoints belong to the set $\{z_1', z_1'', z_2', z_2'', \ldots, z_2k', z_2k''\}$ and which do not contain points of $f^{-1}(y_0)$. Let $\varepsilon < \min_{z \in T}\{\varepsilon_i, |y_0 - f(z)|\}$. Then it is easy to see that $W(\varepsilon) \subset U(x_0)$.

Let us construct an conjugated homeomorphism from this neighborhood to the appropriate neighborhood of the function $\text{Re}z^k$. We take a set

$$\{z : |z| < 1, arg z = (2n + 1)/2k, 0 \leq n \leq k - 1, n \in N\}$$

in the capacity of the neighborhood $W$ of the function $\text{Re}z^k$. Let us select one-to-one the correspondence of domains, into which level lines decompose neighborhoods $W(\varepsilon)$, so that to adjacent domains corresponded adjacent ones and positive domains to positive ones. We construct an conjugated homeomorphism between appropriate domains. Let for a determination, domain $V$ corresponds domain with a polar angle:

$$\pi/(2k) < \varphi < 3\pi/(2k).$$
Let \( x_1, x_1' = \partial V \cap \partial W, x_2 = \gamma(x_1) \cap f^{-1}(y_0 - \varepsilon) \), and \( x_3 \) be a nearest to \( x_2 \) on \( \partial V \cap f^{-1}(y_0 - \varepsilon) \) point, for which \( h(x_3) = x_0 \). We fix a homeomorphism \( p \) of the curve \( \theta \subset f^{-1}(y_0) \) with endpoints \( x_0 \) and \( x_1 \) to \([0, 1]\) so, that \( p(x_0) = 0, p(x_1) = 1 \). Let us define map \( H \) from the closed domain \( D \), limited by a curvilinear tetragon with vertex \( x_0, x_1, x_2, x_3 \) and sides laying on \( f^{-1}(y_0), \gamma(x_2), f^{-1}(y_0 - \varepsilon), \gamma(x_3) \) to the rectangle \([0, 1] \times [0, \varepsilon] \) by formula

\[
H(x) = (p(h(x)), f(x) - y_0).
\]

Thus, the level lines of function and integrated curves are mapped in segments parallel to the sides of a rectangle.

Let us prove that the map \( H \) is a homeomorphism. Since each point lays on an integrated trajectory and the function is monotonic at going on a trajectory in one direction, the uniqueness of map \( H \) follows.

For an arbitrary point \( z \in D \) and sequence of points \( \{z_i \in D\} \), it is necessary to prove that \( z_i \to z \) if and only if \( H(z_i) \to H(z) \). By contradiction, let \( z_i \to z \) and \( H(z_i) \) does not converge to \( H(z) \). Then there is such \( \varepsilon > 0 \) and subsubsequence \( z_{i_k} \) that \( |h(z_{i_k}) - h(z)| < \varepsilon \)

or \( |f(z_{i_k}) - f(z)| < \varepsilon \). It is equivalent to that

\[
z_{i_k} \in D \setminus (h^{-1}(h(z) - \varepsilon, h(z) + \varepsilon) \cap f^{-1}(f(z) - \varepsilon, f(z) + \varepsilon)).
\]

Thus, points \( z_{i_k} \) lay out of the neighborhood of a point \( z \), that are bounded by curves \( h^{-1}(h(z) - \varepsilon), f^{-1}(f(z) - \varepsilon), h^{-1}(h(z) + \varepsilon), f^{-1}(f(z) + \varepsilon) \). It contradicts with convergence of the sequence \( z_i \).

Let us assume now that the sequence \( H(z_i) \) converges to \( H(z) \), and the sequence \( z_i \) does not converge to \( z \). Since \( D \) is the compact set then there is a subsubsequence \( z_{i_k} \) that converges to \( z_0 \neq z \). Then \( H(z_{i_k}) \to H(z_0) \). It contradicts that \( H(z_i) \to H(z) \).

By analogy with above, we construct a curvilinear tetragon with vertex in points \( x_0, x_1', x_2', x_3' \). If \( x_3' = x_3 \) as well as above we create, a homeomorphism of this tetragon on a rectangle \([0, 1] \times [0, \varepsilon] \).

If \( x_3' \neq x_3 \) we denote by \( D' \) a curvilinear tetragon with vertex in the points \( x_0, x_1', x_2', x_3' \), and by \( D'' \) a curvilinear triangle with vertex in points \( x_0, x_3, x_3' \). Let us fix homeomorphisms \( p, q \) of arcs on level lines of the function between the points \( x_0, x_1' \) and \( x_3, x_3' \) to \([0, 1]\) and \([0, 1/2]\), accordingly. Let us set a map \( H : D' \cup D'' \to [0, 1] \times [0, \varepsilon] \) by the formula:

\[
H(x) = \begin{cases} 
(p(h(x)) - \frac{1-p(h(x))}{2\varepsilon} f(x), -f(x)) & \text{if } x \in D', \\
(-\frac{f(x)}{\varepsilon} q(\gamma(x) \cap f^{-1}(\varepsilon)), -f(x)) & \text{if } x \in D''.
\end{cases}
\]
As above, this map is a homeomorphism that maps a level lines to horizontal curves.

Similarly we build homeomorphisms of domain for function \( \text{Re}z^k \). Then, having taken appropriate compositions of homeomorphisms, we obtain homeomorphisms of domain of function \( f \) to domain of function \( \text{Re}z^k \). By the construction, these homeomorphisms coincide in the boundaries of domain and thus set a required homeomorphism of the neighborhood \( W(\varepsilon) \).

\[ \square \]

Remark 1.4. If \( k = 1 \), in a neighborhood of the critical point, the level line of function is same as well as in a neighborhood of a regular point (to within a homeomorphism).

Remark 1.5. Similarly to proof of the theorem one can show, that each local minimum (maxima) has neighborhood, in which the function conjugates with the functions \( g(x, y) = x^2 + y^2 \) (\( g(x, y) = -x^2 - y^2 \)).

Definition 1.6. We call by a Poincare index \( \text{ind}_xf \) of a critical point \( x \) of a function \( f \) a Poincare index of the gradient field \( \text{grad}f \) in some Riemannian metric.

Corollary 1.7. Let \( x \) be an isolated critical point of a function \( f \), and \( y \) be a same of a function \( g \). The functions \( f \) and \( g \) are topologically equivalent in some neighborhoods of these points if and only if

\[ \text{ind}_xf = \text{ind}_yg. \]

Remark 1.8. For the function \( f = \text{Re}z^k \), the Poincare index \( \text{ind}_0f = 1-k \). It is known [4] that for each critical point \( x \) on a surface \( \text{ind}_xf \leq 1 \).

2. Global conjugation of functions with isolated critical points on a closed surface.

2.1. Diagram of the function. Let \( F \) be a smooth closed surface and \( f : F \rightarrow \mathbb{R} \) be a smooth function with isolated critical points and critical values \( y_1, y_2, \ldots, y_n \) that \( y_i < y_j \), if \( i < j \). From the theorem 1 the pre-image \( f^{-1}(y_i) \) of each critical value \( y_i \) is homeomorphic to the graph united with circles. The edges of the graph and the circles are smoothly imbedded except of critical points that coincide with vertexes of the graph. On each circle we select one point and we consider the circle as loop with a vertex in the selected point.

Denote by \( G_i(f) \) the graph (probably disconnected) that coincide with a pre-image of a critical value \( y_i \) of the function \( f \). Vertexes of the graph are critical and selected points.
Definition 2.1. The surface $F$ together with the graphs $G_i(f)$ that are imbedded in it is called a diagram $D$ of the functions $f$. So $D = \{F, G_1(f), ..., G_n(f)\}$. Two diagrams are called isomorphic if there is a homeomorphism of surfaces, which maps the graphs to the graphs, and the vertexes to then vertexes and preserve the order of the graphs.

Theorem 2.2. Two functions with isolated critical points on closed surfaces are topologically equivalent if and only if their diagrams are isomorphic.

Proof. Necessity. If a conjugated homeomorphism of the surface $F$ is given, it maps critical levels to critical levels. It sets an isomorphism of the diagrams.

Sufficiency. Suppose that the diagrams of two functions are isomorphic. Then there is a homeomorphism $g$ of a surface $F$ that maps critical levels to critical levels. Let us cut the surface $F$ by the critical levels. We obtain surfaces $F_k$, which is homeomorphic to cylinders $S^1 \times [0, 1]$. Then the homeomorphism $g$ induces homeomorphisms of obtained cylinders. We replace these homeomorphisms with homeomorphisms, which map the levels of the function to the levels proportionally to value of function between two critical values and coincide with initial ones on each component of the boundary of the cylinders. As the constructed homeomorphisms coincide on the boundaries, they set an conjugated homeomorphism of the surface $F$. □

2.2. Distinguishing graph of the function. Let $D = \{F, G_1(f), ..., G_n(f)\}$ be a diagram of a function $f$ with isolated critical points on a surface $F$. Let cylinder $F_k$ has component of itself boundary on the graphs $G_i(f), G_{i+1}(f)$. These component of boundary form cycles on the graphs. The component of boundary and the cycle on $G_i(f)$ we call by upper and on $G_{i+1}(f)$ by lower. Thus to local minima and maximas there correspond cycles consisting of one vertex. Note that each edge contains in exactly one upper and one lower cycle.

If the surface $F$ is oriented then the orientation of surface induces the orientation on the upper boundary of each cylinder. This set the orientation of graphs $G_1, G_2, ..., G_n$. Then upper and lower cycles are oriented.

Definition 2.3. By a distinguishing graph, we call the graphs $G$, divided on not intersected subgraphs $G_1, G_2, ..., G_n$ with fixed upper and lower cycles and the given one-to-one correspondence between the lower cycles from $G_i$ and upper cycles from $G_{i+1}$, $i = 1, ..., n - 1$. Thus each edge contains exactly in one upper and in one lower cycle, and each isolated vertex form one upper or lower cycle.
The isomorphism of the distinguishing graphs is such an isomorphism of the graphs that the subgraphs are mapped to the subgraphs, the upper cycles to the upper ones, lower to lower and also the correspondence between the upper and lower cycles is preserved.

Up to isomorphism, each function $f$ with isolated singular points on a closed surface sets unique distinguishing graph, which we call the distinguishing graph of function $f$.

**Theorem 2.4.** Two functions are topologically conjugate if and only if there is an isomorphism of their distinguishing graphs that preserve the order of subgraphs.

**Proof.** Necessity. If the functions are topologically conjugated, there is an isomorphism of their diagrams. The restriction of this isomorphism to the graphs sets isomorphism of the distinguishing graphs.

Sufficiency. Let distinguishing graphs are isomorphic. We construct an isomorphism of the diagrams. By the construction, we have one-to-one correspondence of the cylinders $F_k$ and the pairs of the upper and lower cycles. Thus the correspondence between cylinders is given. Let us construct arbitrarily homeomorphisms between cylinders, so their restrictions on upper and lower foundation coincide with restrictions of isomorphisms of the graphs on appropriate cycles. As these homeomorphisms coincide on the boundaries, they set a homeomorphism of a surface $F$ and the isomorphism of the diagrams signifies.

**Remark 2.5.** If the surface $F$ is oriented and subgraphs have induced orientations then conjugated homeomorphism preserve the orientation of the surface only if the isomorphisms of distinguishing graphs preserve the orientation of cycles. Correspondent functions are called oriented conjugated.

It is obvious that functions $f$ and $g$ are topologically equivalent iff functions $f$ and $g$ or functions $f$ and $-g$ are topologically conjugated.

**Corollary 2.6.** Two functions are topologically equivalent if and only if their distinguishing graphs are isomorphic.

**Remark 2.7.** The distinguishing graphs can be used to classify imbeddings of graphs into surfaces [3].

2.3. **Realization.** We discuss problem when the distinguishing graphs sets function with isolated singular points on a closed surface.

If there is a loop in the graph we can fix a point on it that divide loop to two edges. We repeat this procedure for each loop. So we can suppose that there isn’t a loop in the graph.
Two edges $e_1, e_2$ that incident to the same vertex $v$ are called adjacent for vertex $v$, if in the distinguishing graph there is such a cycle, which contains a fragment $(e_1, v, e_2)$ or $(e_2, v, e_1)$.

**Proposition 2.8.** A distinguishing graphs sets function with isolated singular points on a closed oriented surface if and only if for any two edges that incident to the vertex there is a sequence of the edges, in which everyone consequent is adjacent in this vertex with previous.

**Proof.** The necessity is obvious. Let us prove sufficiency. The conditions of the theorem guarantee that after gluing of cylinders to the graphs we obtain a closed surface. Let us construct the function $f$. For this purpose we set $f(G_i) = i$, and for the cylinder $S^1 \times [0, 1]$ with boundary on $G_i$ and $G^{i+1}$: $f(S^1 \times \{t\}) = i + t$. Thus in vertexes of the graph the cylinders are pasted so, that the function $f$ coincides with function $i + \text{Rez}^k$ in an appropriate coordinate system. The constructed function is desired. 

**Definition 2.9.** A vertex of a distinguishing graph is called planar if the proposition holds for it. Otherwise it is called conic.

### 3. Examples

At first we consider function on closed oriented surfaces with minimal number of critical points (minimal functions) and conjugated homeomorphisms that preserve the orientation. On sphere $S^2$ minimal function have two critical points that are minimum and maximum. It is obvious that all such functions are conjugated.

On other surfaces minimal function have three critical points. Graphs $G_1, G_3$ are points and graph $G_2$ is a graph with unique vertex that is a bouquet of $2g + 1$ circle, where $g$ is the genus of surface. Since the cycles on graph $G_1$ and $G_3$ are trivial, the diagram of function consist of graph $G = G_2$ and two cycles on it. Let $v$ be unique vertex of $G$. We orient the edges of $G$ according to lower cycles and name edges by $2g + 1$ letters: $a, b, c, d, e, \ldots$ that lower cycles is given by the word $abcde\ldots$.

Then the upper cycle is an oriented cycle that is described by word $w$ consisting of the same letters. Thus the function up to conjugation can be given by this word. Since cyclic perturbation of the letters in the word lead to the same function, we can assume that the first letter of the word $w$ is $a$.

Two words correspond to the same function if and only if one can obtain from other by cyclic renaming of letters ($a \rightarrow b, b \rightarrow c, c \rightarrow d, \ldots$, last letter $\rightarrow a$).
If there is a fragment of two successive letters \((ab, bc, cd, \ldots)\) in the word then the vertex \(v\) is conic, because the end first edge and the ending of the second edge have only each other as adjective edges. Bellow we consider words without such fragment.

For torus the words consists of three letters \(a, b, c\). It is obviously there is a unique word \(w = acb\) that satisfy condition above and correspond to graph with planar vertex. Thus up to oriented conjugation there is a unique minimal function on the torus.

For surface of genus 2 word consist of letters \(a, b, c, d, e\). There are 4 words:

1) \(acbed\), 2) \(acebd\), 3) \(adbec\), 4) \(aecd\).

The cyclic renaming of letters in each of words 2), 3), 4) gives the same word and in word 1) all other possible words: \(acedb, adecb, aebdc, aeceb\). All of words 1) - 4) correspond to the graph with a planar vertex \(v\). Thus there are 4 non oriented conjugated minimal functions on surface of genus 2.

For surface of genus 3 the words consists of 7 letters \(a, b, c, d, e, f, g\). There are 37 words that satisfy the condition above:

1) \(acbedgf\), 2) \(acbedf\), 3) \(acbegdf\), 4) \(acbfedg\), 5) \(acbfegd\), 6) \(acbgedf\), 7) \(acbgfed\), 8) \(acbdgf\), 9) \(acebgdf\), 10) \(acebgfd\), 11) \(acegbdf\), 12) \(acfbdge\), 13) \(acfgedb\), 14) \(acfdbge\), 15) \(acfdgeb\), 16) \(acfgedb\), 17) \(acfegdb\), 18) \(acbgfde\), 19) \(acgdbfe\), 20) \(acgfbed\), 21) \(acgbed\), 22) \(adbcege\), 23) \(adbgcfe\), 24) \(adbgfec\), 25) \(adgbe\), 26) \(adgbe\), 27) \(acebgfd\), 28) \(agfbed\), 29) \(acbfegd\), 30) \(acfbged\), 31) \(acbdgf\), 32) \(acfbdge\), 33) \(acgebf\), 34) \(acgfdbe\), 35) \(acbfge\), 36) \(adcgbe\), 37) \(acgfd\).

The words 1)-30) set planar vertexes, others set conic ones. For example, if we denote the beginings of the vertexes by \(a_-, b_-, c_-, d_-, e_-, f_-, g_-\) and the ends by \(a_+, b_+, c_+, d_+, e_+, f_+, g_+\), then for word 1) we have a chain of adjacent edges: \(a_-c_b_e_d_g_f-a_g-f_e_-d_c_b_-a_-\). So arbitrary edges (their beginnings and ends) can be connected by consequence of adjacent. For word 31) we have three chain: \(a_-c_b_-d_-c_-b_-a_-\), \(d_-g_f-e_-d_-\) and \(g_-f_c_-a_a_-g_-\). The edges from other chain can’t be connected by consequence of adjacent. Thus there are 30 non oriented conjugated minimal functions on closed oriented surface of genus 4.

Let us admit that conjugated homeomorphism of surface reverse the orientation of surface. It correspond to the reverse orientation of cycles. Then the words \(w\) change the order of letters and change the letters on their reverse. For example, if word contain 7 letters we change \(b-g, c-f, d-e\). So word \(acbegfd\) is changed to \(acebdgf\). For surfaces of genus 1 and 2 each word is changed to itself. For surface of genus 3 we have following 5 pairs of word that are changed one to other: 3)
and 8), 2) and 16), 4) and 15), 7) and 21), 17) and 19). All other word is changed to itself. Denote \( k(g) \) the number of topologically non conjugated minimal functions on the closed oriented surface of the genus \( g \). So we have: \( k(0) = 1, k(1) = 1, k(2) = 4, k(3) = 25 \).

Let us consider topological equivalence of minimal functions. Having the word \( w \) for function \( f \), we rewrite word \( w' \) for function \(-f\). If the word \( w = a_1a_2a_3a_4... \), where \( a_1, a_2, a_3, ... (a_i \in \{a, b, c, d, ...\}) \) are letters, then the word \( w \) is obtained from word \( abcde \) by the following replacement of the letters:

\[
a_1 \rightarrow a, a_2 \rightarrow b, a_3 \rightarrow c, a_4 \rightarrow d, ... 
\]

For example in the word \( w = acebd \) we replace \( a \rightarrow a, c \rightarrow b, e \rightarrow c, b \rightarrow d, d \rightarrow e \) in the word \( abcde \). Then \( w' = adbec \). So functions given by words \( abcd \) and \( abdce \) are topologically equivalent. If \( w = acbed \) or \( w = aedbc \) then \( w' = w \). Hence there are 3 topologically non equivalent function on the oriented surface of the genus 2.

For the oriented surface of the genus 3, the following pairs of the words determinate topologically equivalent functions: 3) − 4), 5) − 6), 8) − 17), 9) − 18), 10) − 24), 12) − 13), 15) − 19), 16) − 22), 21) − 25). For all other words \( w' = w \). So there are 16 topologically non equivalent functions on the oriented surface of the genus 3.

Let \( f : F \rightarrow \mathbb{R} \) be a minimal function on a non-oriented surface. As above we denote edges and orient them in such a way that the lower word have the form \( abcd... \). Upper cycle is non-oriented. The letters that correspond to the edges with reverse orientation are rewritten with the negative degree. So functions are given by words consisting of letters with positive and negative degree.

For \( \mathbb{R}P^2 \) there is unique word \( ab^{-1} \). For Klein bottle all words \( ab^{-1}c^{-1}, acb^{-1}, ac^{-1}b \) are cyclic equivalent. So there is a unique (up to topological conjugations and up to topological equivalence) minimal function on \( \mathbb{R}P^2 \) and on Klein bottle.

For the non-oriented surface of the genus 3 there are 4 topologically non conjugated functions that are given by words 1)\( ab^{-1}c^{-1}d^{-1} \), 2)\( ab^{-1}dc \), 3)\( ab^{-1}d^{-1}c \), 4)\( ab^{-1}dc^{-1} \). The word 3) and 4) determinate topologically equivalent function. So there are 3 topologically non equivalent functions on the non-oriented surface of the genus 3.

The obtained result of the calculations can be summarized in the table (we write the genus for the oriented surfaces with " + " and for the non-oriented surfaces with " − "): 
equivalence \ genus | \ 0 \ 1 \ 2 \ 3 \ -1 \ -2 \ -3 \\
orient. conjugation | \ 1 \ 1 \ 4 \ 30 \ - \ - \ - \\
top. conjugation | \ 1 \ 1 \ 4 \ 25 \ 1 \ 1 \ 4 \\
top. equivalence | \ 1 \ 1 \ 3 \ 16 \ 1 \ 1 \ 3 \\
the number of non-equivalent minimal functions on the surface of genus $g$

If all critical points of function $f$, except of the minimums and the maximums, has Poincare index $−1$, then $f$ conjugate with a Morse function. In [1] the numbers $n(g)$ of topologically non equivalent Morse functions with one local minimum and one local maximum on the oriented surface of genus $g \leq 5$ was calculated:

$n(1) = 1, n(2) = 3, n(3) = 31, n(4) = 778, n(5) = 37998.$

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