Entanglement Entropy for Logarithmic Conformal Field Theory

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Abstract

We study holographic entanglement entropy for certain logarithmic conformal field theories by making use of their gravity descriptions. The corresponding gravity descriptions are provided by higher derivative gravity at critical points where the equations of motion degenerate leading to a log gravity. When a central charge of the dual theory is zero, the entanglement entropy has a new divergent term whose coefficient is given by the “new anomaly” of the logarithmic conformal field theory.
1 Introduction

In this paper using the holographic description of entanglement entropy [1,2] we study entanglement entropy for certain logarithmic conformal field theories (LCFT’s) [3]. To do so, we utilize gravity descriptions of LCFT’s which may be provided by higher derivative gravities at critical points. In three dimensions such theories are given by Topologically Massive Gravity (TMG) [4,5] or New Massive Gravity (NMG) [6] while for higher dimensions the theories are known as log gravity [7–10].

Since in a gravity with higher derivative terms the corresponding equations of motion are higher order differential equations, there is a possibility to have critical points where the equations of motion degenerate leading to a logarithmic solution [1]. In general the resultant logarithmic solutions have the following form [8]

\[ ds^2_{d+1} = \frac{L^2}{r^2} \left[ -F(x_+, r) dx_+^2 - 2 dx_+ dx_- + dr^2 + dv^2 \right], \]

where \( F(x_+, r) = \beta_1(x_+) + \beta_2(x_+) r^{d-1} + \left[ \beta_3(x_+) + \beta_4(x_+) r^{d-1} \right] \ln \left( \frac{r}{L} \right), \)

with \( \beta_i \)'s are arbitrary functions of \( x_+ \).

More precisely for \( \beta_3 \neq 0 \) at the critical points these higher derivative gravities admit a new vacuum solution [8,12,13] which is not asymptotically locally AdS. To accommodate this solution one needs to change the asymptotic behaviour of the AdS metric. Actually using the Fefferman-Graham coordinates for a \( d + 1 \) dimensional metric

\[ ds^2_{d+1} = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(\rho, \vec{x}) \, dx^i dx^j, \]

the equations of motion allow us to have a wider class of the boundary conditions for the metric as follows [14,16]

\[ g_{ij}(\rho, \vec{x}) = b_{(0)ij}(\vec{x}) \ln \rho + g_{(0)ij}(\vec{x}) + \cdots + \left( b_{(d)ij}(\vec{x}) \ln \rho + g_{(d)ij}(\vec{x}) \right) \rho^d + \cdots. \]

It is then obvious that for \( b_{(0)ij} \neq 0 \) the solution is not asymptotically locally AdS. Indeed in this case in order to maintain the variational principle well posed with the Dirichlet boundary condition one needs to modify the variational principle by imposing an additional boundary condition. In fact, from the above asymptotic expansion one has

\[ g_{(0)ij} = \lim_{\rho \to 0} (g_{ij} - \rho \partial_\rho g_{ij}), \quad b_{(0)ij} = \lim_{\rho \to 0} \rho \partial_\rho g_{ij}, \]

which shows that the boundary condition can be fixed not only by the value of the boundary metric but also by its radial derivative.

From AdS/CFT correspondence [17] point of view this means that in the dual CFT there are two operators which may be associated with the metric, one of them is sourced by \( b_{(0)ij} \) and the other by \( g_{(0)ij} \). This is, indeed, the reason why the corresponding dual field theory might be a LCFT [11,15] (see also [16,18,19]). It is important to mention that in this context adding log

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1 The fact that at the critical point there is a new logarithmic mode, has been first obtained for TMG in [11] where the authors have also conjectured that the dual field theory must be a LCFT.
term to an AdS solution may be holographically identified to a deformation of the dual CFT with an irrelevant operator. Therefore, adding this term would destroy the conformal symmetry at UV and it is not clear how to apply the AdS/CFT correspondence. Nevertheless, following [15] we will assume that the deformation is sufficiently small and this term may be treated perturbatively. To conclude, it is believed that the dual field theory of higher derivative gravity on the logarithmic solution (1) is a LCFT.

Having established gravity descriptions for LCFT's it is then natural to study different aspects of them by making use of AdS/CFT correspondence. In particular one may study entanglement entropy in a LCFT using the holographic description of the entanglement entropy [11][2]. We note, however, that since in the case of our interest the corresponding action of the dual gravity contains higher derivative terms the simple expression of the holographic entanglement entropy in terms of a minimal surface in the bulk is not applicable. Thus one needs to proceed with another procedure.

Of course unlike the Wald formula [21] for the black hole entropy, there is no such a generalization for the entanglement entropy for an arbitrary action with higher derivative terms. The only case where the minimization procedure is argued to work is for that of Lovelock theories [22][23]. We note, however, that neither TMG and NMG nor higher dimensional log gravities belong to this category.

Therefore to compute the holographic entanglement entropy one should proceed with another method. Actually using the method based on a regularization of squashed cones introduced in [24], the authors of [25] have computed the entanglement entropy of an AdS vacuum in NMG as well as a five dimensional higher derivative gravity away from the critical point where they have found that the regularization procedure leads to the expected universal terms in the entanglement entropy for spherical and cylindrical entangling surfaces [3]. In the present paper we will extend this consideration to the model at the critical points where the vacuum is not an AdS solution [4].

The paper is organized as follows. In the next section we will study holographic entanglement entropy for NMG model at the critical point. In section three we shall redo the same calculations for higher dimensional higher derivative log gravities. The last section is devoted to discussions.

2 Three dimensional log-gravity

In this section we shall consider two dimensional LCFT's whose gravity duals may be provided by NMG or TMG at the critical points. To explore the physical content of these models, it is worth mentioning that TMG model is parity odd while the NMG model is parity even. From dual field theory point of view this property leads to the fact that the LCFT dual to TMG has logarithmic behaviour in the left hand sector while the right hand sector is usual CFT, while for LCFT dual to NMG both sectors are logarithmic.

More precisely in the case of TMG the dual LCFT has central charges $c_L = 0, c_R = \frac{3L}{G_N}$ and new anomaly $b_R = 0, b_L = \frac{-3L}{G_N}$ while for NMG one has $c_L = c_R = 0$ and $b_R = b_L = \frac{-12L}{G_N}$.  

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2The importance of computing entanglement entropy for LCFT’s has also been mentioned in [20]. We would like to thank D. Grumiller and T. Zojer for bringing this paper to our attentions.

3An alternative way to compute the entanglement entropy for arbitrary gravitational model is the procedure presented in [26] where the authors introduced the generalized gravitational entropy.

4Holographic entanglement entropy for a solution which is not asymptotically AdS has been also considered in [27].
Here $G_N$ is the Newton constant and $L$ is the radius of space-time.

In what follows we shall study entanglement entropy of a two dimensional LCFT whose gravitational description is provided by NMG at the critical point. To do so, we will consider an interval with the width $\ell$ as the entangling region. We will back to the TMG model in the discussions section.

The action of NMG model is

$$S_{NMG} = -\frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left[ R - 2\Lambda - \frac{1}{m^2} \left( R_{\mu\nu}R^{\mu\nu} - \frac{3}{8} R^2 \right) \right].$$  

(6)

For generic $m$ this model admits several vacua, including an AdS$_3$ vacuum. It is believed that the NMG model on an asymptotically locally AdS$_3$ solution may have a dual CFT whose central charges are given by [28, 29]

$$c_L = c_R = \frac{3L}{2G_N} \left(1 - \frac{1}{2m^2L^2}\right),$$  

(7)

where $L$ is the radius of the AdS solution. At critical point where $m^2L^2 = \frac{1}{2}$ both central charges vanish and indeed the model exhibits a logarithmic solution.

Entanglement entropy for a CFT dual to an asymptotically locally AdS solution in NMG model has been recently studied in [25]. As we have already mentioned when the action contains higher derivative terms the holographic description of entanglement entropy in terms of a minimal surface does not work. Nevertheless to compute the entanglement entropy for the present case, based on the results of [24], the authors of [25] have considered the following entropy functional to be minimized to compute the entanglement entropy

$$S_{EE} = \frac{1}{4G_N} \int dx \sqrt{g_{\text{ind}}} \left\{ 1 - \frac{1}{m^2} \left[ \left( R_{\mu\nu}n^\mu_i n^{\nu}_i - \frac{1}{2} K_i^2 \right) - \frac{3}{4} R \right] \right\},$$  

(8)

where $i = 1, 2$ denotes two transverse directions to a co-dimension two hypersurface in the bulk, $n^\mu_i$ are two unit mutually orthogonal normal vectors on the co-dimension two hypersurface and $K_i$ is trace of the two extrinsic curvature tensors defined by

$$K^{(i)}_{\mu\nu} = \pi^\sigma_\mu \pi^\rho_\nu \nabla_\rho (n_i)_\sigma, \quad \text{with} \quad \pi^\sigma_\mu = \epsilon^\sigma_\mu + \xi \sum_{i=1,2} (n_i)^\sigma (n_i)_\mu ,$$  

(9)

where $\xi = -1$ for space-like and $\xi = 1$ for time-like vectors.

In our notation for the AdS geometry

$$ds^2 = \frac{L^2}{r^2} (-dt^2 + dx^2 + dr^2),$$  

(10)

the co-dimension two hypersurface in the bulk is fixed by $x = x(r)$ and $t = 0$. This parametrization can be used to compute the entanglement entropy of an interval with the width $\ell$ along $x$ direction in the dual conformal field theory. Indeed plugging this parametrization into the entropy functional [8] and minimizing it, one may find the profile of the co-dimension two hypersurface [25], $x(r) = \sqrt{\ell^2/4 - r^2}$. Then, this can be used to compute the entropy functional on this hypersurface which is indeed the entanglement entropy. Doing so, one finds the universal part of the entanglement entropy as follows [25]

$$S_{EE} = \frac{L}{4G_N} \left(1 - \frac{1}{2m^2L^2}\right) \ln \frac{\ell}{\epsilon},$$  

(11)
where $\epsilon$ is a UV cut off. From this expression it is evident that at the critical point where $m^2L^2 = \frac{1}{2}$ the universal term vanishes. Of course it was expected due to fact that both central charges are zero at the critical point. It is then interesting to study entanglement entropy at this point.

As we already mentioned at the critical point the model admits a new solution. It is then important to compute the holographic entanglement entropy for this background. The corresponding background is

$$ds^2 = \frac{L^2}{r^2}\left[-\beta \ln\left(\frac{r}{L}\right) dx_+^2 - 2dx_+dx_- + dr^2\right], \quad (12)$$

where $L$ is a UV cut off. From this expression it is evident that at the critical point where $m^2L^2 = \frac{1}{2}$ the universal term vanishes. Of course it was expected due to fact that both central charges are zero at the critical point. It is then interesting to study entanglement entropy at this point.

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where the light-cone coordinates are defined as $x_\pm = \frac{t \pm \sqrt{2} x}{\sqrt{2}}$.

Note that in this metric the constant parameter $\beta$ in the logarithmic term can be set to one by a rescaling of the coordinates. Nevertheless in order to trace effects of the logarithmic term we keep it in the metric. More importantly, as we already mentioned, the above metric is not asymptotically AdS. Actually, its deviation from an AdS geometry is holographically identified to deforming the dual CFT with an irrelevant operator. On the other hand in order to work within the framework of a CFT where one can benefit from the power of conformal symmetry, one needs to assume that the coefficient of the irrelevant operator is sufficiently small. From gravity point of view this, in turn, indicates that $\beta$ is very small. Therefore in what follows we will minimize the entropy functional (8) for the logarithmic metric (12) with the assumption that $\beta$ is very small.

To proceed we should consider a co-dimension two hypersurface in the bulk whose boundary coincides with the boundary of the entangling region. The corresponding co-dimension two hypersurface is parametrized as follows

$$x_+ + x_- = 0, \quad x_+ - x_- = \sqrt{2}f(r). \quad (13)$$

Then the induced metric reads

$$ds^2 = \frac{L^2}{r^2}\left[1 + f'^2(r) - \frac{\beta}{2} f'^2(r) \ln\left(\frac{r}{L}\right) dx_+^2 + dx_-^2 + dr^2\right]. \quad (14)$$

The corresponding two unit vectors are given by

$$x_+ + x_- = \text{const.} \quad n_1 = \frac{L}{r \sqrt{2 - \beta \ln\left(\frac{r}{L}\right)}}(0, 1, 1),$$

$$x_+ - x_- - \sqrt{2}f(r) = \text{const.} \quad n_2 = \frac{L}{r \sqrt{2 + \beta \ln\left(\frac{r}{L}\right)} + 2f'^2}(-\sqrt{2}f', 1, -1). \quad (15)$$

Using these expressions one can compute the extrinsic curvatures whose traces are found to be

$$K_1 = \frac{\beta (-(4 + (\beta - 4) \ln\left(\frac{r}{L}\right) + 2\beta \ln^2\left(\frac{r}{L}\right)) f')}{\sqrt{2L} \left(2\beta \ln\left(\frac{r}{L}\right) + 2f'^2\right)^{3/2}},$$

$$K_2 = \frac{\sqrt{2} \left(4 - 2(\beta + 4\beta \ln\left(\frac{r}{L}\right)) f' + (4 - 4 - \beta + 2\beta \ln\left(\frac{r}{L}\right)) f'^3 + z(4 - \beta^2 \ln^2\left(\frac{r}{L}\right)) f''\right)}{L \left(2\beta \ln\left(\frac{r}{L}\right) + 2f'^2\right)^{3/2}}. \quad (16)$$

Also the Ricci tensor and Ricci scalar are given by

$$R_{\mu\nu} = \frac{2}{r^2} \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & \beta \left(\ln\left(\frac{r}{L}\right) - \frac{1}{2}\right) & 0 \\ 0 & 1 & 0 \end{array}\right), \quad R = -\frac{6}{L^2}. \quad (17)$$
By making use of these expressions the entropy functional reads

$$S_A = \frac{2\pi L}{l_p} \int \frac{dr}{r} \sqrt{1 + f'^2 + \frac{\beta}{2} f'^2 \ln \frac{r}{L} (1 - 2L^2 F)},$$

where

$$F = 1 + \frac{\beta}{1 - \frac{\beta}{2} \ln r - f'^2} - \frac{\beta}{1 + \frac{\beta}{2} \ln r + f'^2}$$

$$+ \frac{1}{4 \left(1 - \frac{\beta}{2} \ln r \right)^2 \left(1 + \frac{\beta}{2} \ln r + f'^2\right)^2} \times \left[\frac{\beta^2 \left(1 + \ln r \left(1 - \frac{\beta}{2} - \frac{\beta}{2} \ln r\right)\right)^2 f'^2}{1 - \frac{\beta}{2} \ln r}\right]
- \frac{4 \left(1 + \frac{\beta}{2} \ln^2 r\right) f' + \left(1 + \frac{\beta}{2} - \frac{\beta}{2} \ln \frac{r}{L}\right) f'^3 - r \left(1 - \frac{\beta}{2} \ln \frac{r}{L}\right) f''}{1 + \frac{\beta}{2} \ln r + f'^2}. \tag{19}$$

Now the aim is to minimize $S_A$ to find a differential equation for $f$ whose solution is the profile of the co-dimension two hypersurface in the bulk. To do so, we may consider the above expression as an action for the dynamical field $f$. Since the action does not depend on $f$, the corresponding momentum is a constant of motion. It is then straightforward to write down the equation of the conservation law, though in general it is not obvious whether the corresponding equation can be solved exactly. Nevertheless since we are interested in small $\beta$ limit, one may solve the equation perturbatively to find $f$. Indeed at leading order one finds

$$f'(r) = \frac{r}{\sqrt{r_t^2 - r^2}} \left(1 + \frac{\beta}{4} \frac{r_t^2 \ln r}{r_t^2 - r^2} - \frac{r_t^2 \ln r}{r_t^2 - r^2}\right) + \mathcal{O}(\beta^2), \tag{20}$$

where $r_t$ is the turning point, which is related to the width of the entangling region $\ell$ via the following constraint

$$\ell = 2 \int_0^{r_t} dr f'(r). \tag{21}$$

In particular at leading order one finds

$$\ell = 2 r_t \left[1 - \frac{\beta}{4} \left(1 - \ln \frac{4r_t}{L}\right) + \mathcal{O}(\beta^2)\right]. \tag{22}$$

Putting everything together and keeping in our mind that at the leading order $\ell = 2r_t$, one can evaluate the holographic entanglement entropy as follows

$$S_{EE} = -\frac{\beta L}{4G_N} + \frac{\beta^2 L}{8G_N} \left[\ln^2 \left(\frac{\ell}{L}\right) - \frac{5}{6} \ln^2 \left(\frac{\ell}{L}\right) + \frac{8}{9} \ln \left(\frac{\ell}{L}\right) + c_0\right] + \mathcal{O}(\beta^3)$$

$$= \frac{\beta b}{48} - \frac{\beta^2 b}{96} \left[\ln^2 \left(\frac{L}{\epsilon}\right) - \frac{5}{6} \ln^2 \left(\frac{L}{\epsilon}\right) + \frac{8}{9} \ln \left(\frac{L}{\epsilon}\right) + c_0\right] + \mathcal{O}(\beta^3), \tag{23}$$

where $b = \frac{b_L + b_R}{2}$ is the new anomaly of the LCFT and $c_0$ is a numerical constant.
It is interesting to note that since the central charges of the NMG model at critical point are zero the leading universal log term associated to two dimensional CFT’s, $\ln \ell$, is absent. Therefore the universal divergent term should come from higher order corrections to the entanglement entropy which has a new form. From dual field theory point of view this corresponds to the fact that at the critical point the theory has new degrees of freedom. So, when we are computing the entanglement entropy, we are indeed measuring the entanglement between these new degrees of freedom. Therefore the short range behaviour of the entanglement entropy may be changed. Moreover the coefficient of the universal term is given by the new anomaly. This might indicate that the degrees of freedom of the new modes are controlled by the new anomaly.

3 Higher Dimensional Log-gravity

In this section we will study holographic entanglement entropy for certain higher derivative $d+1$ dimensional gravity. The corresponding action is \[ I = -\frac{1}{2l_p^{d-1}} \int d^{d+1}x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{m^2} \left( R_{\mu\nu} R_{\mu\nu} - \frac{d+1}{4d} R^2 \right) \right]. \] (24)

For generic values of the parameters $\Lambda$ and $m$ the model has an AdS vacuum solution whose radius of the curvature, setting $x = -\frac{d(d-1)}{2L^2}$, can be obtained from the roots of the following equation \[ d - 3 \frac{2dm}{x^2} x^2 + x - \Lambda = 0. \] (25)

Here $L$ is the radius of the AdS solution.

It can be shown that the model has a critical point where $m^2 = \frac{(d-1)^2}{2L^2}$ \[7\]. At this point the corresponding equations of motion degenerate leading to a logarithmic solution. The aim of this section is to compute the entanglement entropy for this solution.

Holographic entanglement entropy for a generic higher derivative terms in a five dimensional AdS solution has been studied in \[25\]. Here we use the same procedure, though our main concern is the logarithmic solution. In the present case following the results of \[24\], one needs to minimize the following entropy functional

\[ S_{EE} = \frac{2\pi}{l_p^{d-1}} \int d^{d-1}x \sqrt{g_{\text{ind}}} \left\{ 1 - \frac{1}{m^2} \left[ \left( R_{\mu\nu} n_\mu^n n_\nu^n - \frac{1}{2} K_i^2 \right) - \frac{d+1}{2d} R \right] \right\}, \] (26)

In what follows we will compute the entanglement entropy for the following generic solution

\[ ds^2 = \frac{L^2}{x^2} \left[ -\beta \ln \left( \frac{r}{L} \right) dx_+^2 - 2dx_+dx_- + d\rho^2 + \rho^2 d\Omega^2_{d-3} + d\nu^2 \right]. \] (27)

Note that in our computations we will consider both $m$ and $\beta$ as free parameters, though one should keep in our mind that $\beta$ is zero when $m^2 \neq (d-1)^2/2L^2$. In other words, for the logarithmic solution $m$ is fixed and is not a free parameter.

Let us consider an entangling region in the shape of cylinder at fixed time and $0 \leq \rho \leq \ell$. Then one needs to consider a co-dimension two hypersurface in the bulk whose boundary coincides with the boundary of the entangling region. The corresponding hypersurface is given by

\[ x_+ + x_- = 0, \quad \rho = f(r). \] (28)

\[5\] This action can be found by setting $\gamma = 0$ in the action given in \[7\].
In this case the induced metric reads
\[
 ds^2 = \frac{L^2}{r^2} \left( (1 + f'^2)dr^2 + (2 - \beta \log \frac{r}{L})dx_+^2 + f^2 d\Omega^2_{d-3} \right).
\] (29)

On the other hand two unit vectors normal to the co-dimension two hypersurfaces are
\[
 x_+ + x_- = \text{const.} \quad \quad n_1 = \frac{L}{\sqrt{r^2 - \beta \log \frac{r}{L}}} (0, 1, 1, 0, 0, \cdots),
\]
\[
 \rho - f(r) = \text{const.} \quad \quad n_2 = \frac{L}{\sqrt{1 + f'^2}} (-f', 0, 0, 1, 0, \cdots). \tag{30}
\]

It is then straightforward though tedious to compute the entropy functional. Indeed setting \( F = 2 - \beta \ln \frac{r}{L} \) the entropy functional (26) reads
\[
 S_A = \frac{2\pi \Omega_{d-3} H_+ L^{d-1}}{\sqrt{2d-1}} \int dr \sqrt{ (1 + f'^2) F } \left\{ 1 - \frac{1}{m^2} \left( \frac{(d-1)^2 F + d\beta}{2L^2 F} \right) - \frac{\left[ (2(d-3)rF + ff' \left( 2(d-1)F + \beta \right) \right)}{8L^2 f^2 F^2 (1 + f'^2)} \right\}^2, \tag{31}
\]
where \( \Omega_{d-3} \) is the volume of the \( S^{d-3} \) sphere in the metric.

Now following our procedure in the previous section, one should minimize the above entropy functional to find a differential equation for \( f \). Again, in general it is hard to solve the corresponding differential equation. Nevertheless one may solve the equation perturbatively in power of \( \beta \) near the boundary. Indeed at leading order one finds
\[
 f(r) = \ell - \frac{d - 3}{2(d - 2)} \frac{r^2}{\ell} + \mathcal{O}(\beta & r^4). \tag{32}
\]

We must emphasis that since we are solving the differential equation near the boundary, we can only extract information about the UV contributions or the most divergent terms of the entanglement entropy. Note also that the above equation is not valid for \( d = 3 \). Indeed in this case the entangling region is a strip with the width \( \ell \) and the equation of motion for \( f \) at leading order can be exactly solved. More precisely one finds (see also [2])
\[
 f'(r) = \frac{r^2}{\sqrt{r^4 - r^2}} + \mathcal{O}(\beta), \tag{33}
\]
where \( r_t \) is the turning point which can be fixed in terms of \( \ell \) by the constraint \( \ell = 2 \int_0^{r_t} dr f'(r) \).

Plugging the solution (32) (or (33) for \( d = 3 \)) into the entropy functional (31), one can read the UV behaviour of the entanglement entropy. Indeed for \( \beta = 0 \) where the solution is an AdS solution the most divergent term of the entanglement entropy for arbitrary \( m \) is
\[
 S_{EE} = -\frac{2\pi \Omega_{d-3} L^{d-1}}{\ell^d - 1} \left( 1 - \frac{(d-1)^2}{2m^2 L^2} \right) \left[ \frac{H_+ \ell^{d-3}}{(d-2)c^{d-2}} + \cdot \cdot \cdot + c_\alpha \frac{H_+}{\ell} \ln \frac{\epsilon}{\ell} \right], \tag{34}
\]
where $H_+$ is the height of the cylinder. Note that the logarithmic term whose coefficient is a universal constant is non-zero only for even $d$. More precisely, one has

$$
\tilde{c}_3 = \tilde{c}_5 = 0, \quad \tilde{c}_4 = \frac{1}{8}, \quad \tilde{c}_6 = -\frac{135}{2048}.
$$

(35)

Since the entangling region is a cylinder, the coefficient of the universal part of the entanglement entropy is related to the central charge of the dual CFT [30]. In the present case writing the universal part as $\frac{c_d H_+}{\ell} \ln \frac{\ell}{\tilde{c}}$, one finds

$$
c_4 = \frac{3\pi^2 L^3}{2l_p^3} \left(1 - \frac{9}{2m^2 L^2}\right), \quad c_6 = \frac{405\pi^3 L^5}{256 l_p^5} \left(1 - \frac{25}{2m^2 L^2}\right).
$$

(36)

From the above expressions it is evident that at the critical point where $m^2 = \frac{(d-1)^2}{2L^2}$ all divergent terms, including the logarithmic term (for even $d$), vanish. Of course at this point the model exhibits a new logarithmic solution and therefore it is natural to study the entanglement entropy of this solution. In fact the situation is very similar to that in NMG case. For this case, setting $m^2 = \frac{(d-1)^2}{2L^2}$, at leading order one finds

$$
S_{EE} = \beta \frac{d\pi d_{d-3} L^{d-1}}{(d-1)^2 l_p^{d-1}} \left[H_+ \ell^{d-3} + \cdots + \tilde{b}_d H_+ \ell^{d-3} \ln \frac{\ell}{\tilde{c}} \right],
$$

(37)

Again the logarithmic term is non-zero for even $d$. Indeed one has

$$
\tilde{b}_3 = \tilde{b}_5 = 0, \quad \tilde{b}_4 = \frac{1}{8}, \quad \tilde{b}_6 = -\frac{315}{2048}.
$$

(38)

Since at the critical point the dual theory is supposed to be a LCFT, it is then natural to identify the universal constant with the new anomaly of the dual LCFT. More precisely writing the universal term as $\frac{b_d H_+}{48 \ell} \ln \frac{\ell}{\tilde{c}}$, we have

$$
b_4 = -\frac{16\pi^2 L^3}{3l_p^3}, \quad b_6 = -\frac{567\pi^3 L^5}{80l_p^5}.
$$

(39)

It is worth recalling that for higher dimensional CFT’s there are several central charges and depending on the shape of the entangling region the universal part of the entanglement entropy could be proportional to different central charges. In particular in four dimensional CFT when the entangling region is a sphere the universal part is proportional to $a$ while for cylinder it is proportional to $c$ [30]. On the other hand taking a strip as the entangling region there would be no universal part.

In this section, due to the symmetry of the logarithmic metric, we have considered a cylinder entangling surface. So, we would expect to get a universal term in the expression of the entanglement entropy, at least for even dimensional CFT’s. It seems natural to take the coefficient of the universal part as a parameter of the dual LCFT. Therefore one may identify this parameter as the new anomaly in the dual LCFT.

We note, however, that in the four dimensions, as we have already mentioned, the entangling region is essentially a strip. Thus, the resultant entanglement entropy does not contain a universal term.

6The case of $d = 4$ has also been considered in [25].
part. Therefore from our computations one cannot read the new anomaly of the corresponding three dimensional LCFT, even if it is non-zero $[31]$. Indeed, in this case, what we have computed is the coefficient of the most divergent term when the central charge is zero. Of course eventually it might be related to the new anomaly.

As a final remark we note that in higher dimensions we could have also considered a co-dimension two hypersurface which is parametrized as that in the equation (13). In this case the entangling region is, indeed, a strip along the light like direction and the UV contribution to the entanglement entropy does not have a universal part, though the most divergent terms are still given by that in the equations (34) and (37). One also observes that in this case the divergent terms come from the order of $\beta^2$, which is consistent with the NMG case.

4 Discussions

In this paper we have studied entanglement entropy of certain LCFT’s by making use of their holographic descriptions. The gravity dual of the corresponding LCFT’s may be provided by higher derivative gravities at critical points where the corresponding equations of motion of the models degenerate leading to logarithmic solutions which are not asymptotically locally AdS.

Our considerations are based on the following assumptions. Since so far there is no a systematic procedure to compute holographic entanglement entropy when the action has arbitrary higher derivative terms, we have used the regularization of squashed cones method introduced in [24]. In this method one minimizes (extremizes) the corresponding entropy functional (see for example the equation (8)) to find the profile of the co-dimension two hypersurface in the bulk gravity. Then the entanglement entropy is given by the entropy functional evaluated on this hypersurface. We note, however, that to compute the entanglement entropy, a priori, it is not obvious whether the entropy functional should be evaluated on the hypersurface which minimizes the whole entropy functional $[8]$. Actually for using this procedure we are encouraged by the results of [25] where the authors have shown that the regularization procedure leads to the expected universal terms in the entanglement entropy for spherical and cylindrical entangling surfaces.

We note that since there is no a proof for the above procedure, it is important to examine how robust the results are. Indeed to see this, one may compute the entropy functional on a hypersurface which only minimizes the area. In this case one observes that although the finite terms of resultant entanglement entropy will be changed, the universal divergent terms remain unchanged. In other words it seems that the coefficient of the divergent terms are robust.

On the other hand the logarithmic solutions have a non-renormalizable mode which destroys the asymptotic behaviour from that of an AdS solution. Therefore it is not clear how to implement the holographic renormalization which is the main stone of AdS/CFT correspondence. Nevertheless it was argued in [15] that this non-renormalizable mode should be associated to an irrelevant operator in the boundary theory. Therefore for sufficiently small deformation one may still use the CFT tools. Of course the results should only be trusted at leading order in the perturbation in the coefficient of the irrelevant operator.

Following this procedure we have considered an ad hoc parameter in front of the log term in the metric. Then we have computed the holographic entanglement entropy in leading order of the parameter. Therefore it is important to keep in our mind that what we have really computed

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5 We would like to thank T. Zojer for a comment on this point.
6 We would like to thank the referee for his/her comment on this point.
is the entanglement entropy of a CFT which is slightly deformed by an irrelevant operator. So, the deformation parameter, $\beta$, appeared in the final expressions of the entanglement entropy. Thus one could only trust the results at the lowest orders. Indeed if we had been able to directly apply AdS/CFT for log gravity, the parameter $\beta$ would have not been appeared in the final expressions.

For a generic two dimensional CFT the entanglement entropy has the following form

$$S_{EE} = \frac{c_L + c_R}{6} \ln \frac{\ell}{\epsilon}, \quad (40)$$

On the other hand for the NMG model one has $c_L + c_R = \frac{3L}{G_N} (1 - \frac{1}{2m^2L^2})$ which shows that the $R^2$ terms contribute to the above universal term in the entanglement entropy. In particular at the critical point where $2m^2L^2 = 1$ it vanishes indicating the universal divergent term should come from higher order terms in $\beta$. Indeed this is what we have found in this paper.

From dual field theory point of view this corresponds to the fact that at the critical point the theory has new degrees of freedom. So, when we are computing the entanglement entropy, we are indeed measuring the entanglement between these new degrees of freedom. Therefore the short range behaviour of the entanglement entropy may be changed. Moreover the coefficient of the universal term is given by the new anomaly. This might indicate that the degrees of freedom of the new modes are controlled by the new anomaly (see also [32]).

In this paper we have also studied logarithmic CFT in higher dimensions using the higher dimensional logarithmic gravity. Actually the result has a similarity with that of NMG model. This, indeed, can be understood from fact that in both cases the corresponding actions have the same form.

In the context of the entanglement entropy the entropic c-function may be defined by derivative of the entanglement entropy with respect to the size of the entangling region [35]

$$c_e(\ell) = \ell \frac{\partial S_{EE}}{\partial \ell}, \quad (41)$$

which is universal, positive, and due to strong sub-additivity property of entanglement entropy satisfies

$$\frac{\partial c_e(\ell)}{\partial \ell} \leq 0. \quad (42)$$

Although the NMG model at the critical point is not a unitary theory and the c-function may not be applied, having found the corresponding entanglement entropy it might be useful to compute the entropic c-function for this model. Doing so, one finds that it is universal in the sense that it does not depend on the UV cut off, though it is positive for particular values of the entangling region. Nevertheless its first derivative is negative which in turn indicates that the resultant entanglement entropy satisfies strong sub-additivity [35].

By making use of the holographic description of entanglement entropy, it would be interesting to explore a possible analogue of c-theorem or perhaps b-theorem in LCFT’s. See [32] for recent discussions on this subject.

As a final remark let us make a comment on the holographic entanglement entropy of a two dimensional LCFT whose gravity dual is provided by TMG at the critical point. The action of the TMG model may be written as follows

$$S_{TMG} = \frac{1}{16\pi G_N} \int d^3x \left[ R + \frac{2}{L^2} + \frac{\epsilon^{\mu\nu\rho}}{4\mu} \left( R_{ab,\mu\nu} \omega^{ab,\rho} + \frac{2}{3} \omega^a_{b,\mu} \omega^b_{c,\nu} \omega^c_{a,\rho} \right) \right], \quad (43)$$
where $\omega_{a b, \mu}$ is the spin connection whose inner Lorentz indices are denoted by $a, b, \cdots$ while the space-time indices are denoted by $\mu, \nu, \cdots$.

This model admits a logarithmic solution as (12) at $\mu L = 1$ while for a generic value of $\mu$ the model has an AdS vacuum solution. It is conjectured that the TMG model on an asymptotically locally AdS solution with a proper boundary condition would provide a gravitational dual for a two dimensional CFT with the following central charges

$$c_L = \frac{3L}{2G_N} \left( 1 - \frac{1}{\mu L} \right), \quad c_R = \frac{3L}{2G_N} \left( 1 + \frac{1}{\mu L} \right).$$

(44)

Holographic entanglement entropy for this model has been studied in [34] where it was shown that for AdS geometry the contribution of the Chern-Simons term vanishes.

To compute the holographic entanglement entropy, inspired by the results of [33] where the author has evaluated the contribution of the Chern-Simons term to the entropy of BTZ black hole, we might naively consider the following entropy functional

$$S_A = \frac{1}{4G_N} \int dx \left[ \sqrt{g_{\text{ind}}} + \frac{L}{2} \omega_{ab, \mu} e^a_{\alpha} e^b_{\beta} e^{\mu \nu \alpha} (n^i_{\alpha} n^i_{\mu})(n^j_{\beta} n^j_{\nu}) \right], \quad (45)$$

where $e^a_{\alpha}$ is the vielbein and the normal vectors $n^i_{\mu}$ are the same as those in section two. It is then straightforward to minimize the above entropy functional to find the profile of the corresponding co-dimension two hypersurface and then the entanglement entropy. Doing so, for small $\beta$ at leading order one finds

$$S_{EE} = \frac{c_R + c_L}{6} \ln \frac{\ell}{\epsilon} + \frac{\beta b}{24} \ln 2 + \mathcal{O} (\beta^2). \quad (46)$$

Note that unlike the NMG model in the present case since $c_L + c_R$ is $\mu$ independent, the universal part remains unchanged, though one gets a correction due to logarithmic term in the action which is consistent with the results of [34], namely setting $\beta = 0$ the correction vanishes.

It is, however, important to note that although the result seems physically reasonable and also consistent with the literature, it might be misleading. The reason is as follows.

Actually the equation (45) has been obtained in the context of the BTZ black hole where the extrinsic curvature vanishes, though in the present case where the general co-dimension two hypersurface might have non-zero extrinsic curvature it is not clear how to modify the equation (45) to include the effects of the extrinsic curvature. From the equation (16) one observes that $K_1$ is non-zero and indeed is proportional to $\beta$. Since we are interested in the small $\beta$ limit, there might be a possibility to have $\mathcal{O}(\beta)$ corrections to the entanglement entropy due to non-zero extrinsic curvature. It would be interesting to explore this possibility.

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The non-zero components of the vielbeins and spin connections for the metric (12) are

$$e^1_r = \frac{L}{r}, \quad e^2_+ = \frac{L}{r}, \quad e^3_+ = \frac{L}{r}, \quad e^2_2 = \frac{\beta L}{2r} \log \left( \frac{r}{L} \right), \quad \omega_{+,21} = \frac{1}{r}, \quad \omega_{-,31} = \frac{1}{r}, \quad \omega_{+,13} = \frac{\beta}{2r} [1 - \log \left( \frac{r}{L} \right)].$$

10We would like to thank the referee for his/her comment on this point and in particular pointing out the importance of the extrinsic curvature.
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