New Bound on the Chebyshev function and the Riemann Hypothesis

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Abstract
Under the assumption that the Riemann hypothesis is true, von Koch deduced the improved asymptotic formula \( \theta(x) = x + O(\sqrt{x} \times \log^2 x) \), where \( \theta(x) \) is the Chebyshev function. We prove if there exists some real number \( x \geq 10^8 \) such that \( \theta(x) > x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x \), then the Riemann hypothesis should be false. In this way, we show that under the assumption that the Riemann hypothesis is true, then \( \theta(x) < x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x \) for all \( x \geq 10^8 \).

Keywords: Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers
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1. Introduction
The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part \( \frac{1}{2} \) [1]. The Riemann hypothesis belongs to the David Hilbert’s list of 23 unsolved problems [1]. Besides, it is one of the Clay Mathematics Institute’s Millennium Prize Problems [1]. This problem has remained unsolved for many years [1]. In mathematics, the Chebyshev function \( \theta(x) \) is given by
\[
\theta(x) = \sum_{p \leq x} \log p
\]
where \( p \leq x \) means all the prime numbers \( p \) that are less than or equal to \( x \). Say Nicolas\( (p_n) \) holds provided
\[
\prod_{q \leq p_n} \frac{q}{q - 1} > e^\gamma \times \log \theta(p_n).
\]
The constant \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant, \( \log \) is the natural logarithm, and \( p_n \) is the \( n^{th} \) prime number. The importance of this property is:

Theorem 1.1. [2], [3]. Nicolas\( (p_n) \) holds for all prime numbers \( p_n > 2 \) if and only if the Riemann hypothesis is true.

We know the following properties for the Chebyshev function:
Theorem 1.2. [4]. If the Riemann hypothesis holds, then
\[ \theta(x) = x + O(\sqrt{x} \times \log^2 x) \]
for all \( x \geq 10^8 \).

Theorem 1.3. [5]. For \( 2 \leq x \leq 10^8 \)
\[ \theta(x) < x. \]

We also know that

Theorem 1.4. [6]. If the Riemann hypothesis holds, then
\[ \left( e^{-\gamma} \times \prod_{q \leq x} \frac{1}{q - 1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \]
for all numbers \( x \geq 13.1 \).

Let’s define \( H = \gamma - B \) such that \( B \approx 0.2614972128 \) is the Meissel-Mertens constant [7]. We know from the constant \( H \), the following formula:

Theorem 1.5. [8].
\[ \sum_q \left( \log(\frac{q}{q-1}) - \frac{1}{q} \right) = \gamma - B = H. \]

For \( x \geq 2 \), the function \( u(x) \) is defined as follows
\[ u(x) = \sum_{q > x} \left( \log(\frac{q}{q-1}) - \frac{1}{q} \right) \]

We use the following theorems:

Theorem 1.6. [9]. For \( x > -1 \):
\[ \frac{x}{x + 1} \leq \log(1 + x). \]

Theorem 1.7. [10]. For \( x \geq 1 \):
\[ \log(1 + \frac{1}{x}) < \frac{1}{x + 0.4}. \]

Let’s define:
\[ \delta(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log x - B \right). \]

Definition 1.8. We define another function:
\[ \sigma(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right). \]

Putting all together yields the proof that the inequality \( \sigma(x) > u(x) \) is satisfied for a number \( x \geq 3 \) if and only if Nicolas(p) holds, where \( p \) is the greatest prime number such that \( p \leq x \). In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.
2. Results

**Theorem 2.1.** The Riemann hypothesis is true if and only if the inequality $\sigma(x) > \theta(x)$ is satisfied for all numbers $x \geq 3$.

*Proof.* In the paper [3] is defined the function:

$$f(x) = e^\gamma \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q - 1}{q}.$$ 

We know that $f(x)$ is lesser than 1 when Nicolas($p$) holds, where $p$ is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + \theta(x)$$

where $U(x) = -\sigma(x)$ [3]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\sigma(x) + \theta(x) < 0$$

which is the same as $\sigma(x) > \theta(x)$. Therefore, this is a consequence of the theorem 1.1. \qed

**Theorem 2.2.** If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1$$

for all numbers $x \geq 13.1$.

*Proof.* Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q - 1} < e^\gamma \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.4 for all numbers $x \geq 13.1$. If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log\left(\frac{q}{q - 1}\right) < \gamma + \log \log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} + 1.2 \times \log x + 2\right)$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log\left(\frac{q}{q - 1}\right) - \frac{1}{q}\right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) < \frac{1}{\log x + 0.4} + 0.4$$

$$= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)}$$

$$= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$
according to theorem 1.7 since $\frac{8x\pi\sqrt{x}}{3x\log x} \geq 1$ for all numbers $x \geq 13.1$. We use the theorem 1.5 to show that

$$\sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) = H - u(x)$$

and $\gamma = H + B$. So,

$$H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$ 

We eliminate the value of $H$ and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Under the assumption that the Riemann hypothesis is true, we know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$ 

Suppose that $\theta(x) = \epsilon \times x$ for some constant $\epsilon > 1$. Then,

$$\log \log \theta(x) - \log \log x = \log \log (\epsilon \times x) - \log \log x$$

$$= \log (\log x + \log \epsilon) - \log \log x$$

$$= \log \left( \log x \times (1 + \frac{\log \epsilon}{\log x}) \right) - \log \log x$$

$$= \log \log x + \log (1 + \frac{\log \epsilon}{\log x}) - \log \log x$$

$$= \log(1 + \frac{\log \epsilon}{\log x}).$$

In addition, we know that

$$\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.6 since $\frac{\log \epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log x + 1} = \frac{\log \epsilon}{\log x + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$
Thus, 
\[ 3 \times \log x + 5 \]
\[ \frac{\log x}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}. \]

If we add the following value of \( \frac{\log x}{\log \theta(x)} \) to the both sides of the inequality, then
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)}
\]
\[
= \frac{\log \epsilon + \log x}{\log \theta(x)}
\]
\[
= \frac{\log \theta(x)}{\log \theta(x)}
\]
\[
= 1.
\]

We know this inequality is satisfied when \( 0 < \epsilon \leq 1 \) since we would obtain that \( \frac{\log x}{\log \theta(x)} \geq 1 \). Therefore, the proof is done. \( \square \)

**Theorem 2.3.** If there exists some real number \( x \geq 10^8 \) such that
\[
\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x,
\]
then the Riemann hypothesis is false.

**Proof.** If the Riemann hypothesis holds, then
\[
\theta(x) = x + O(\sqrt{x} \times \log^2 x)
\]
for all \( x \geq 10^8 \) due to the theorem 1.2. Now, suppose there is a real number \( x \geq 10^8 \) such that
\[
\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x.
\]
That would be equivalent to
\[
\log \theta(x) > \log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x)
\]
and so,
\[
\frac{1}{\log \theta(x)} < \frac{1}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x)}
\]
for all numbers \( x \geq 10^8 \). Hence,
\[
\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x)}.
\]

If the Riemann hypothesis holds, then
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x)} > 1
\]
for those values of \( x \) that complies with
\[
\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x
\]
due to the theorem 2.2. By contraposition, if there exists some number \( y \geq 10^8 \) such that for all \( x \geq y \) the inequality

\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x)} \leq 1
\]

is satisfied, then the Riemann hypothesis should be false. Let’s define the function

\[
\nu(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x)} - 1.
\]

The Riemann hypothesis would be false when there exists some number \( y \geq 10^8 \) such that for all \( x \geq y \) the inequality \( \nu(x) \leq 0 \) is always satisfied. We ignore when \( 2 \leq x \leq 10^8 \) since \( \theta(x) < x \) according to the theorem 1.3. We know that the function \( \nu(x) \) is monotonically decreasing for every number \( x \geq 10^8 \). The derivative of \( \nu(x) \) is negative for all \( x \geq 10^8 \). Indeed, a function \( \nu(x) \) of a real variable \( x \) is monotonically decreasing in some interval if the derivative of \( \nu(x) \) is lesser than zero and the function \( \nu(x) \) is continuous over that interval [11]. It is enough to find a value of \( y \geq 10^8 \) such that \( \nu(y) \leq 0 \) since for all \( x \geq y \) we would have that \( \nu(x) \leq \nu(y) \leq 0 \), because of \( \nu(x) \) is monotonically decreasing. We found the value \( y = 10^8 \) complies with \( \nu(y) \leq 0 \). In this way, we obtain that \( \nu(x) \leq 0 \) for every number \( x \geq 10^8 \). Hence, the proof is complete.

**Theorem 2.4.** Under the assumption that the Riemann hypothesis is true, then

\[
\theta(x) < x + \frac{1}{\log \log x} \times \sqrt{x} \times \log^2 x
\]

for all \( x \geq 10^8 \).

**Proof.** This is a direct consequence of the theorem 2.3.

**Appendix**

We found the derivative of \( \nu(x) \) in the web site https://www.wolframalpha.com/input. Besides, we determine the sign of the function \( \nu(x) \) using the tool gp from the web site https://pari.math.u-bordeaux.fr. In the project PARI/GP, the method sign\((F(X))\) returns \(-1\) when the function \( F(X) \) is negative in the value of \( X \). We checked that is negative for \( X = 10^8 \) with a real precision of 1000016 significant digits when \( F(X) = \nu(x) \). We also checked that is still negative for \( X = 100000! \), where \((...)!\) means the factorial function.

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References

[1] P. B. Borwein, S. Choi, B. Rooney, A. Weirathmueller, The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike, Vol. 27, Springer Science & Business Media, 2008.
[2] J.-L. Nicolas, Petites valeurs de la fonction d’Euler et hypothese de Riemann, Séminaire de Théorie des nombres DPP, Paris 82 (1981) 207–218.
[3] J.-L. Nicolas, Petites valeurs de la fonction d’Euler, Journal of number theory 17 (3) (1983) 375–388. doi:10.1016/0022-314X(83)90055-0.
[4] H. Von Koch, Sur la distribution des nombres premiers, Acta Mathematica 24 (1) (1901) 159.
[5] J. B. Rosser, L. Schoenfeld, Approximate Formulas for Some Functions of Prime Numbers, Illinois Journal of Mathematics 6 (1) (1962) 64–94. doi:10.1215/jim/1255631807.
[6] J. B. Rosser, L. Schoenfeld, Sharper Bounds for the Chebyshev Functions \( \theta(x) \) and \( \phi(x) \), Mathematics of computation (1975) 243–269. doi:10.1090/S0025-5718-1975-0457373-7.
[7] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie., J. reine angew. Math. 1874 (78) (1874) 46–62. doi:10.1515/crll.1874.78.46. URL https://doi.org/10.1515/crll.1874.78.46
[8] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin’s criterion for the Riemann hypothesis, Journal de Théorie des Nombres de Bordeaux 19 (2) (2007) 357–372. doi:10.5802/jtnb.591.
[9] L. Kozma, Useful Inequalities, http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf, accessed on 2022-01-17 (2021).
[10] A. Ghosh, An Asymptotic Formula for the Chebyshev Theta Function, arXiv preprint arXiv:1902.09231.
[11] G. Anderson, M. Vamanamurthy, M. Vuorinen, Monotonicity Rules in Calculus, The American Mathematical Monthly 113 (9) (2006) 805–816. doi:10.1080/00029890.2006.11920367.