Noncommutative differential calculus structure on secondary Hochschild (co)homology

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ABSTRACT

Let $B$ be a commutative algebra and $A$ be a $B$-algebra (determined by an algebra homomorphism $\varepsilon : B \to A$). M. D. Staic introduced a Hochschild like cohomology $H^\ast((A, B, \varepsilon); A)$ called secondary Hochschild cohomology, to describe the non-trivial $B$-algebra deformations of $A$. J. Laubacher et al later obtained a natural construction of a new chain complex $C^\ast((A, B, \varepsilon))$ in the process of introducing the secondary cyclic (co)homology. In this paper, we establish a connection between the two (co)homology theories for $B$-algebra $A$. We show that the pair $(H^\ast((A, B, \varepsilon); A), HH^\ast((A, B, \varepsilon)))$ forms a noncommutative differential calculus, where $HH^\ast((A, B, \varepsilon))$ denotes the homology of the complex $C^\ast((A, B, \varepsilon))$.

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1. Introduction

G. Hochschild introduced Hochschild cohomology to study certain extensions of associative algebras. In 1963, a pioneer work of M. Gerstenhaber related Hochschild cohomology to deformations of associative algebras [3]. He further proved that the Hochschild cohomology of an associative algebra carries a rich algebraic structure which is now known as Gerstenhaber algebra (see [3] for details). These structures appear in the context of the exterior algebra of Lie algebras, multivector fields on smooth manifolds, and differential forms on Poisson manifolds. A Gerstenhaber algebra is a graded commutative associative algebra $(A = \bigoplus_{i \in \mathbb{Z}} A_i, \cdot)$ together with a degree $-1$ graded Lie bracket $[-, -]$ on $A$ satisfying the following compatibility condition.

$$[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|-1)|b|} b \cdot [a, c].$$

A Batalin–Vilkovisky (BV-) operator on a Gerstenhaber algebra $(A = \bigoplus_{i \in \mathbb{Z}} A_i, \cdot)$ is a square zero degree $-1$ map $\triangle : A \to A$ with

$$[a, b] = \pm (\triangle(a \cdot b) - (\triangle a) \cdot b - (-1)^{|a|} a \cdot (\triangle b)).$$

This shows that the bracket $[-, -]$ obstructs $\triangle$ to be a derivation with respect to the product $\cdot$. A Gerstenhaber algebra with a BV-operator is called a Batalin–Vilkovisky algebra (BV-algebra).
Let $A$ be an associative $k$-algebra, $B$ a commutative $k$-algebra and $\varepsilon : B \to A$ an algebra morphism with $\varepsilon(B) \subset Z(A)$, the center of $A$. In 2016, Staic [14, 15] introduced the secondary Hochschild cohomology $H^\bullet((A, B, \varepsilon); M)$ of the triple $(A, B, \varepsilon)$ with coefficients in a $B$-symmetric $A$-bimodule $M$. This cohomology, motivated by an algebraic version of the second Postnikov invariant [14], controls deformations of the $B$-algebra structures on $A[[t]]$. When $B$ is the underlying field $k$, the secondary Hochschild cohomology coincides with the classical Hochschild cohomology. Several results which are true for classical Hochschild cohomology theory for an associative algebra with coefficients in itself have analogues for the secondary Hochschild cohomology $H^\bullet((A, B, \varepsilon), A)$ of a triple $(A, B, \varepsilon)$ with coefficients in $A$. In particular, it is proved in ref. [16] that the secondary complex $C^\bullet((A, B, \varepsilon), A)$ is a multiplicative non-symmetric operad, which induces a natural homotopy Gerstenhaber algebra structure on it. Consequently, one obtains a natural Gerstenhaber algebra structure on the secondary Hochschild cohomology $H^\bullet((A, B, \varepsilon); A)$.

The simplicial structure of the complex $C^\bullet((A, B, \varepsilon); M)$ that defines the secondary cohomology $H^\bullet((A, B, \varepsilon), A)$ is discussed in ref. [9]. For this purpose, the authors introduce a simplicial module $B((A, B, \varepsilon)$ over a certain simplicial algebra, and call it the secondary bar simplicial module. This simplicial module $B((A, B, \varepsilon)$ is the analogue of the bar resolution associated with a $k$-algebra $A$. Using this simplicial module, the cohomology $H^\bullet((A, B, \varepsilon), M)$ of the triple $(A, B, \varepsilon)$ with coefficients in $M$ can be realized as the homology of the associated complex of co-simplicial module. This construction of the secondary bar simplicial module leads to natural constructions of the secondary Hochschild homology groups $H_\bullet((A, B, \varepsilon), M)$ of the triple $(A, B, \varepsilon)$ with coefficients in $M$. Also the authors in ref. [9] have given natural constructions of secondary cohomology groups $HH^\bullet(A, B, \varepsilon)$ and the secondary cyclic cohomology groups $HC^\bullet(A, B, \varepsilon)$ associated to a triple $(A, B, \varepsilon)$. This has also prompted the authors to define the corresponding homology groups $HH_\bullet(A, B, \varepsilon)$ and $HC_\bullet(A, B, \varepsilon)$ associated to the triple $(A, B, \varepsilon)$. We refer to [9] and [1] for more Hochschild-like results on the secondary Hochschild and secondary cyclic (co)homology groups, associated to the triple $(A, B, \varepsilon)$.

This article aims to establish a connection between the two (co)homology theories for a triple $(A, B, \varepsilon)$ introduced in ref. [9] and [15]. We show that the pair $(H^\bullet((A, B, \varepsilon); A), HH_\bullet(A, B, \varepsilon))$ forms a noncommutative differential calculus. A calculus $(A, \Omega)$ consists of a Gerstenhaber algebra $A$ and a graded space $\Omega$ such that

- $\Omega$ carries a Gerstenhaber module structure over $A$, and
- there exists a differential $B : \Omega_\bullet \to \Omega_{\bullet+1}$ satisfying the Cartan–Rinehart homotopy formula (see Definition 2.4).

In differential geometry and noncommutative geometry, there are several concrete examples of noncommutative differential calculi [2, 13, 17]. These examples include the classical calculus of multivector fields and differential forms, a calculus on the pair Hochschild cohomology and Hochschild homology of associative algebras, the calculus for Poisson structures, and the calculus for general left Hopf algebroids with respect to general coefficients (see Section 6, [7]). In ref. [7] Kowalzig proved that if a comp module structure [6] is cyclic over a multiplicative operad, then this cyclic comp module structure induces a noncommutative differential calculus (in the sense of refs. [13, 17]) on the pair of the associated homology of the cyclic $k$-module and the cohomology of the operad.

In this article, we consider the multiplicative non-symmetric operad structure on secondary cochain complex $C^\bullet((A, B, \varepsilon); A)$ from ref. [16]. The secondary Hochschild chain complex associated to a triple $(A, B, \varepsilon)$ is denoted by $C_\bullet(A, B, \varepsilon)$. We define comp module actions of the operad $C^\bullet((A, B, \varepsilon); A)$ on the complex $C_\bullet(A, B, \varepsilon)$. With these actions, we prove that the complex $C_\bullet(A, B, \varepsilon)$ is a cyclic comp module over the operad $C^\bullet((A, B, \varepsilon); A)$. Subsequently, it follows that
the pair of underlying homologies \((H^\bullet((A, B, \varepsilon); A), HH_\bullet(A, B, \varepsilon))\) forms a (noncommutative) differential calculus.

In Section 2, we recall the definitions and results related to multiplicative non-symmetric operads and cyclic unital comodule structures over multiplicative operads. In Section 3, we recall secondary Hochschild (co)homology for a triple \((A, B, \varepsilon)\). In particular, we recall the definitions of the complexes \(C^\bullet((A, B, \varepsilon); A)\) and \(\bar{C}_\bullet(A, B, \varepsilon)\) from refs. [9, 15]. We also recall the multiplicative operad structure on the complex \(C^\bullet((A, B, \varepsilon); A)\) from ref. [16]. In Section 4, we define comp module actions of \(C^\bullet((A, B, \varepsilon); A)\) on the complex \(\bar{C}_\bullet(A, B, \varepsilon)\) and obtain a comp module structure on \(\bar{C}_\bullet(A, B, \varepsilon)\). We further consider an operator \(t : \bar{C}_\bullet(A, B, \varepsilon) \to \bar{C}_\bullet(A, B, \varepsilon)\) to show that the comp module structure is cyclic. Finally, we conclude that there exists a (noncommutative) differential calculus on the pair \((H^\bullet((A, B, \varepsilon); A), HH_\bullet(A, B, \varepsilon))\). This provides a connection between the cohomology groups of the triple \((A, B, \varepsilon)\) and the homology groups associated to the triple \((A, B, \varepsilon)\).

2. Comp modules over (non-symmetric) operads

In this section, we recall the notion of cyclic comp modules over operads and the related results from ref. [7]. In particular, we recall that a cyclic comp module structure over a multiplicative operad induces a noncommutative differential calculus.

**Definition 2.1.** Let \(k\) be a field. We denote the tensor product \(\otimes_k\) by \(\otimes\). A non-symmetric (unital) operad \(O\) in the category of \(k\)-modules is a sequence of \(k\)-modules \(\{O^n\}_{n \geq 0}\) equipped with \(k\)-linear maps

\[
\alpha_i : O^n \otimes O^m \to O^{n+m-1}, \quad \text{for} \quad m, n \geq 0 \quad \text{and} \quad 0 \leq i \leq n,
\]

and an element \(1 \in O^1\) such that the following identities hold

\[
f \circ_i g = 0 \quad \text{if} \quad n < i \quad \text{or} \quad n = 0,
\]

\[
(f \circ_i g) \circ_j h = \begin{cases} 
(f \circ_j h) \circ_{i+p-1} g & \text{if} \quad j < i, \\
(f \circ_i (g \circ_{j+1} h)) & \text{if} \quad i \leq j < m + i, \\
(f \circ_{j-m+1} h) \circ_i f & \text{if} \quad j \geq m + i,
\end{cases}
\]

and

\[
f \circ_i 1 = f = 1 \circ_i f, \quad \text{for all} \quad i \leq n,
\]

where \(f \in O^n, g \in O^m, \) and \(h \in O^p\).

If \(O\) is an operad, then one can define a circle product \(\circ : O^n \otimes O^m \to O^{n+m-1}\) by

\[
f \circ g = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g, \quad \text{for} \quad f \in O^n \quad \text{and} \quad g \in O^m.
\]

Subsequently, a degree \(-1\) bracket is given by

\[
[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f, \quad \text{for} \quad f \in O^n \quad \text{and} \quad g \in O^m.
\]

We recall that the operad \(O\) is a multiplicative operad if there exists an element \(\mu \in O^2\) and an element \(e \in O^0\) such that \(\mu \circ_1 \mu = \mu \circ_2 \mu\) and \(\mu \circ_1 e = 1 = \mu \circ_2 e\). The multiplication \(\mu\) on the operad \(O\) induces a differential \(\delta_\mu : O^n \to O^{n+1}\), given by \(\delta_\mu(f) = [\mu, f]\) for \(f \in O^n\). Let us denote by \(H_\mu^\bullet(O)\), the cohomology space of the complex \((O, \delta_\mu)\). The multiplication \(\mu\) also induces a cup product \(\cup : O^n \otimes O^m \to O^{n+m}\) on the complex \(O\), which is given by
\[ f \triangleright g = (\mu \circ_2 f) \circ_1 g, \text{ for } f \in \mathcal{O}^m \text{ and } g \in \mathcal{O}^n. \]

The graded commutative product \( \triangleright \) and the Gerstenhaber bracket \([ , ]\) induce a Gerstenhaber algebra structure on the cohomology \( H^*_n(\mathcal{O})\) (see [4] for more details).

**Definition 2.2.** A unital (left) comp module \( \mathcal{M}_\bullet \) over an operad \( \mathcal{O}^\bullet := \{ \mathcal{O}^n \}_{n \geq 0} \) is a sequence of \( k \)-modules \( \{ \mathcal{M}_n \}_{n \geq 0} \) equipped with comp module maps \( \bullet_i : \mathcal{O}^m \otimes \mathcal{M}_p \to \mathcal{M}_{p-m+1} \), for \( 1 \leq i \leq p-m+1 \) and \( 0 \leq m \leq p \) such that

\[
\begin{align*}
\langle f \bullet_i \rangle \cdot (g \bullet_j x) = & \begin{cases} 
\langle g \bullet (f \circ_{i+n-1} x) \rangle & \text{if } j < i, \\
\langle f \circ_{j-i+1} g \rangle \bullet_{j-i} x & \text{if } j-m < i \leq j, \\
g \bullet_{j-m+1} (f \bullet_j x) & \text{if } 1 \leq i \leq j-m,
\end{cases} \\
1 \bullet_i x = x & \text{for } 1 \leq i \leq p,
\end{align*}
\]

(1)

where \( f \in \mathcal{O}^m, g \in \mathcal{O}^n \), and \( x \in \mathcal{M}_p \), for \( m \geq 0, n, p \geq 0 \). The maps \( \bullet_i \) are zero, for \( m > p \).

**Definition 2.3.** A unital comp module \( \mathcal{M}_\bullet \) is called para-cyclic if it is additionally equipped with

(i) an extra comp module maps \( \bullet_0 : \mathcal{O}^m \otimes \mathcal{M}_p \to \mathcal{M}_{p-m+1} \), for \( 0 \leq m \leq p+1 \) (the maps \( \bullet_0 \) are assumed to be zero, for \( m > p+1 \)) such that the relations (1)-(2) also hold true for \( i = 0 \), and

(ii) a \( k \)-linear map \( t : \mathcal{M}_p \to \mathcal{M}_p \), for \( p \geq 1 \), such that

\[ t(f \bullet_i x) = f \bullet_{i+1} t(x), \]

for \( f \in \mathcal{O}^m, x \in \mathcal{M}_p \), and \( 0 \leq i \leq p-m \). If the map \( t : \mathcal{M}_p \to \mathcal{M}_p \) satisfies the condition: \( t^{p+1} = \text{Id} \), then the para-cyclic comp module \( \mathcal{M}_\bullet \) is said to be ‘cyclic’ comp module over the operad \( \mathcal{O}^\bullet \).

Let \( (\mathcal{O}^\bullet, \mu) \) be a multiplicative non-symmetric operad and \( \mathcal{M}_\bullet \) be a cyclic comp module over \( \mathcal{O}^\bullet \). We refer to ref. [10] for definitions of simplicial and cyclic \( k \)-modules. Let us recall from [7] that there is a cyclic \( k \)-module structure on \( \mathcal{M}_\bullet \) with cyclic operator \( t : \mathcal{M}_p \to \mathcal{M}_p \), face maps \( d_i : \mathcal{M}_p \to \mathcal{M}_{p-1} \) and degeneracies \( s_j : \mathcal{M}_p \to \mathcal{M}_{p+1} \) defined as follows

\[
\begin{align*}
d_i(x) &= \mu \bullet_i x, \quad \text{for } i = 0, \ldots, p-1, \\
d_p(x) &= \mu \bullet_0 t(x), \\
s_j(x) &= 1 \bullet_{j+1} x, \quad \text{for } j = 0, \ldots, p,
\end{align*}
\]

where \( x \in \mathcal{M}_p \). Recall from ref. [7] that a simplicial boundary \( b \), a norm operator \( N \), an extra degeneracy \( s_{-1} \), and the cyclic differential \( B \) are defined as follows

\[
b := \sum_{i=0}^{p} (-1)^i d_i, \quad N := \sum_{i=0}^{p} (-1)^{ip} t^i, \quad s_{-1} := t s_p, \quad B := (id - t) s_{-1} N.
\]

The normalized complex \( \mathcal{R}(\mathcal{M}) \) is the quotient of the complex \( \mathcal{M}_\bullet \) by the subcomplex spanned by the images of degeneracy maps \( s_j \), for \( j = 0, 1, \ldots, p \). The map \( B \) on the normalized complex \( \mathcal{R}(\mathcal{M}) \) is given by

\[
B(x) = s_{-1} N(x) = \sum_{i=0}^{p} (-1)^i 1 \bullet_0 t^i(x), \quad \text{for } x \in \mathcal{R}(\mathcal{M})_p.
\]
2.1. The noncommutative differential calculus associated to a cyclic comp module over a multiplicative operad

Definition 2.4 ([7]). A graded k-module \( \Omega := \oplus \Omega_n \) is called a Gerstenhaber module over Gerstenhaber algebra \( A \) if there exist maps \( i : A_m \otimes \Omega_p \to \Omega_{p-m} \), and \( L : A_m \otimes \Omega_p \to \Omega_{p-m+1} \) such that

(i) the action \( i \) makes \( \Omega \) a graded module over the graded commutative associative algebra \((A, \wedge)\);
(ii) the action \( L \) makes \( \Omega \) a graded Lie module over the graded Lie algebra \((A[1], [-,-])\);
(iii) for any \( X \in A_m \) and \( Y \in A_{n+1} \), the following relation holds

\[ i_{[X,Y]} = i_X L_Y - (-1)^{mn} L_Y i_X. \]

The Gerstenhaber module \( \Omega = \oplus \Omega_n \) is called a Batalin-Vilkovisky module if there exists a \( k \)-linear map \( B : \Omega_n \to \Omega_{n+1} \) such that \( B^2 = 0 \), and it satisfies the following Cartan-Rinehart homotopy formula

\[ L_X = B \circ i_X - (-1)^{mn} i_X \circ B, \quad \text{for} \ X \in A_m. \]

A pair \((A, \Omega)\), where \( A \) is a Gerstenhaber algebra and \( \Omega \) is a Batalin–Vilkovisky module over \( A \), is called a noncommutative differential calculus.

Let \((O, \mu)\) be a multiplicative operad and \( M \) be a cyclic unital comp module over \((O, \mu)\).

Definition 2.5 ([7]). For \( f \in O^m \), the cap product and the Lie derivative are given as follows.

(i) The cap product \( i_f : M_p \to M_{p-m} \) is defined by

\[ i_f x = (\mu \circ 2 f) \bullet_0 x, \quad \text{for} \ x \in M_p. \]

(ii) The Lie derivative \( L_f : M_p \to M_{p-m+1} \) of \( x \in M_p \) along an element \( f \in O^m \) is defined by

\[ L_f(x) = \begin{cases} \sum_{i=1}^{n-p+1} (-1)^{(p-1)(i-1)} f \bullet_i x + \sum_{i=1}^{p} (-1)^{n(i-1)+p-1} f \bullet_0 t^{(i-1)}(x) & \text{if} \ m < p + 1, \\ (-1)^{p-1} f \bullet_0 N(x) & \text{if} \ m = p + 1, \\ 0 & \text{if} \ m > p + 1. \end{cases} \]

It is shown in ref. [7] that the cap product and the Lie derivative satisfy the following identities.

\[ i_{\delta f} = b \circ i_f - (-1)^m i_f \circ b, \quad \left[ b, L_f \right] + L_{\delta f} = 0. \]  \hspace{1cm} (4)

\[ i_f i_g = i_{f \cdot g}, \quad \left[ L_f, L_g \right] = L_{[f,g]}. \] \hspace{1cm} (5)

It was shown in ref. [7] that from Equation (4), it follows that the cap product and Lie derivative descend to well-defined operators on the homology \( H_\ast(M) \). Moreover, from Equation (5), the cap product makes \( H_\ast(M) \) a graded module over the algebra \((H^\ast(O^\ast), \wedge)\), and the Lie derivative makes \( H_\ast(M) \) a graded Lie module over the graded Lie algebra \((H^{*+1}(O^\ast), [ , ])\).

For any two cocycles \( f \in O^m \) and \( g \in O^n \), the induced operators

\[ L_f : H_\ast(M) \to H_{\ast-m+1}(M) \quad \text{and} \quad i_g : H_\ast(M) \to H_{\ast-n}(M) \]

satisfy the relation

\[ \left[ i_f, L_g \right] = i_{[f,g]}. \] \hspace{1cm} (6)

The identity (6) shows that the homology \( H_\ast(M) \) is a Gerstenhaber module over the Gerstenhaber algebra \( H^\ast_{\mu}(O) \).
In fact, this Gerstenhaber module structure extends to a Batalin–Vilkovisky module structure. Let us consider the normalized complexes \( R(M) \) and \( R(O)^* \). Recall that the (co)homology of the normalized (co)chain complex is the same as the (co)homology of the original complex. The induced norm operator on the normalized complex \( R(M) \) induces a well-defined \( k \)-linear map \( B : H_*(M) \to H_{*-1}(M) \) satisfying \( B^2 = 0 \). For any \( m \)-cocycle \( f \in R(O)^m \), the operators

\[
L_f : H_*(M) \to H_{*-m+1}(M) \quad \text{and} \quad i_f : H_*(M) \to H_{*-m}(M)
\]

satisfy the following Cartan–Rinehart homotopy formula

\[
L_f = [B, i_f] = B \circ i_f - (-1)^m i_f \circ B.
\]

that is, \( H_*(M) \) is a Batalin–Vilkovisky module over the Gerstenhaber algebra \( H_*(O) \). Thus, the pair \( (H_*(O), H_*(M)) \) forms a noncommutative differential calculus (see [7] for more details).

### 3. Secondary Hochschild (co)homology

Let \( A \) be an associative \( k \)-algebra and \( B \) be a commutative \( k \)-algebra. Suppose that there is a \( k \)-algebra morphism \( \varepsilon : B \to A \) such that \( \varepsilon(B) \subseteq Z(A) \) the center of \( A \). We denote the above structure by a triple \( (A, B, \varepsilon) \). In this section, we recall the notion of secondary Hochschild (co)homology for a triple \( (A, B, \varepsilon) \) introduced in refs. [9, 15, 16].

#### 3.1. Secondary Hochschild (co)homology of the triple \( (A, B, \varepsilon) \) with coefficients in a

Let \( (A, B, \varepsilon) \) be a triple. With the above notations, the secondary Hochschild cohomology of a triple \( (A, B, \varepsilon) \) with coefficients in \( A \) is given by considering the cochain complex \( C^*((A, B, \varepsilon); A) = \bigoplus_{n \geq 0} C^n((A, B, \varepsilon); A) \), where

\[
C^n((A, B, \varepsilon); A) := Hom_k(A^\otimes n \otimes B^\otimes (n-1)/2, A), \quad \text{for } n \geq 0,
\]

and the differential \( \delta^\varepsilon : C^{n-1}((A, B, \varepsilon); A) \to C^n((A, B, \varepsilon); A) \) is defined by

\[
\delta^\varepsilon(f) \otimes \left( \begin{array}{cccc}
    a_1 & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n} \\
    1 & a_2 & \cdots & b_{2,n-1} & b_{2,n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\
    1 & 1 & \cdots & 1 & a_n
  \end{array} \right)
\]

\[
:= a_1 \varepsilon(b_{1,2} \cdots b_{1,n})f \otimes \left( \begin{array}{cccc}
    a_2 & b_{2,3} & \cdots & b_{2,n} \\
    1 & a_3 & \cdots & b_{3,n} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & a_n
  \end{array} \right)
\]

\[
+ \sum_{i=1}^{n-1} (-1)^f \otimes \left( \begin{array}{cccc}
    a_1 & b_{1,2} & \cdots & b_{1,i} & b_{1,i+1} & \cdots & b_{1,n-1} & b_{1,n} \\
    1 & a_2 & \cdots & b_{2,i} & b_{2,i+1} & \cdots & b_{2,n-1} & b_{2,n} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
    1 & 1 & \cdots & a_{i} & a_{i+1} & \varepsilon(b_{i,i+1}) & \cdots & b_{i,n-1} & b_{i,n} & b_{i,n+1,n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
    1 & 1 & \cdots & 1 & a_{n-1} & b_{n-1,n} \\
    1 & 1 & \cdots & 1 & 1 & a_n
  \end{array} \right)
\]

\[
+ (-1)^f \otimes \left( \begin{array}{cccc}
    a_1 & b_{1,2} & \cdots & b_{1,n-1} \\
    1 & a_2 & \cdots & b_{2,n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & a_{n-1}
  \end{array} \right) \varepsilon(b_{1,n} \cdots b_{n-1,n})a_n.
\]

The cohomology of this cochain complex is denoted by \( H^*((A, B, \varepsilon); A) \) and it is called the secondary Hochschild cohomology of the triple \( (A, B, \varepsilon) \) with coefficients in \( A \).
Remark 3.1. If \( B = k \) and \( \varepsilon : k \to A \) defining the \( k \)-algebra structure on \( A \), the secondary Hochschild (co)homology coincides with the classical Hochschild (co)homology of the associative algebra \( A \).

### 3.2. Gerstenhaber algebra structure on the secondary Hochschild cohomology

Let us consider an operad \( \{\mathcal{O}^n\}_{n \geq 0} \), where
\[
\mathcal{O}^n = C^n((A, B, \varepsilon); A).
\]

The underlying \( k \)-bilinear maps
\[
\circ_i : C^n((A, B, \varepsilon); A) \otimes C^m((A, B, \varepsilon); A) \to C^{n+m-1}((A, B, \varepsilon); A), \quad \text{for } n, m \geq 0 \text{ and } 1 \leq i \leq n,
\]
are given as follows
\[
(f \circ_i g)(T^{i}_{n+m-1}) = f \otimes \begin{vmatrix}
 a_1 & \cdots & b_{1,i-1} & \prod_{j=i}^{m+i-1} b_{1,j} & b_{1,m+i} & \cdots & b_{1,n+m-1} \\
 1 & \cdots & b_{2,i-1} & \prod_{j=i}^{m+i-1} b_{2,j} & b_{2,m+i} & \cdots & b_{2,n+m-1} \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 1 & \cdots & a_{i-1} & \prod_{j=i}^{m+i-1} b_{i-1,j} & b_{i-1,m+i} & \cdots & b_{i-1,n+m-1} \\
 1 & \cdots & 1 & \prod_{j=i}^{m+i-1} b_{j,m+i} & \cdots & \prod_{j=i}^{m+i-1} b_{j,n+m-1} \\
 1 & \cdots & 1 & 1 & a_{m+i} & \cdots & b_{m+i,n+m-1} \\
 1 & \cdots & 1 & 1 & 1 & \cdots & a_{n+m-1}
\end{vmatrix}, \quad (7)
\]
where
\[
T^{i}_{k} := \otimes \begin{pmatrix}
 a_i & b_{1,i} & \cdots & b_{i,k-1} & b_{i,k} \\
 1 & a_{i+1} & \cdots & b_{i+1,k-1} & b_{i+1,k} \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 1 & 1 & \cdots & a_{k-1} & b_{k-1,k} \\
 1 & 1 & \cdots & 1 & a_k
\end{pmatrix}
\]

Moreover, consider an element \( \mu \in \mathcal{O}^2 = C^2((A, B, \varepsilon); A) \) given by
\[
\mu \left( \otimes \begin{pmatrix}
 a_1 & b \\
 1 & a_2
\end{pmatrix} \right) = \varepsilon(b)a_1a_2. \quad (8)
\]

Then, it follows that \( \{\mathcal{O}^n\}_{n \geq 0} \) is a multiplicative non-symmetric operad with the multiplication \( \mu \in \mathcal{O}^2 \) and \( 1 \in \mathcal{O}^0 = A \) (see [16] for more details).

The precise construction of Gerstenhaber algebra structure on \( H^*((A, B, \varepsilon); A) \) is described as follows: the pre-Lie bracket \( \circ \) is given by
\[
f \circ g = \sum_{i=1}^{n} (-1)^{i-1}(m-1) f \circ_i g, \quad \text{for } f \in C^n, \ g \in C^m;
\]
the graded Lie algebra structure on \( \bigoplus_{n \geq 1} C^n((A, B, \varepsilon); A) \) is given by
\[ [f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f, \quad \text{for } f \in C^n, \ g \in C^m; \]

the cup-product on \( C^\bullet((A, B, \varepsilon); A) \) is defined by

\[
(f \smile g) \left( \begin{array}{cccc}
 a_1 & b_{1,2} & \cdots & b_{1,n+m} \\
 1 & a_2 & \cdots & b_{2,n+m} \\
 1 & 1 & \cdots & a_{m+n}
\end{array} \right) = \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \varepsilon(b_{ij})g^{(T^1_m)}f^{(T^{m+1}_{m+n})}.
\]

The induced operations on the secondary Hochschild cohomology \( H^\bullet((A, B, \varepsilon); A) \) make it a Gerstenhaber algebra.

### 3.3. Secondary Hochschild (co)homology associated to the triple \((A, B, \varepsilon)\)

We recall the notion of secondary Hochschild homology associated to the triple \((A, B, \varepsilon)\) from ref. [15]. Let us consider the chain complex \( \widetilde{C}_\bullet(A, B, \varepsilon) := \oplus_{n \geq 0} \widetilde{C}_n(A, B, \varepsilon, \partial) \), where

\[
\widetilde{C}_n(A, B, \varepsilon) := A^{\otimes (n+1)} \otimes B^{(n+1)/2}, \quad n \geq 0
\]

and the differential \( \partial : \widetilde{C}_n \to \widetilde{C}_{n-1} \) is defined by

\[
\partial \left( \begin{array}{cccc}
 a_0 & b_{0,1} & \cdots & b_{0,n-1} & b_{0,n} \\
 1 & a_1 & \cdots & b_{1,n-1} & b_{1,n} \\
 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\
 1 & 1 & \cdots & 1 & a_n
\end{array} \right)
\]

\[
:= \sum_{i=0}^{n-1} (-1)^i \otimes \left( \begin{array}{ccccc}
 a_0 & \cdots & b_{0,i-1} & b_{0,i} & b_{0,i+2} & \cdots & b_{0,n} \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 1 & \cdots & a_{i-1} & b_{i-1,i-1} & b_{i-1,i+1} & b_{i-1,i+2} & \cdots & b_{i-1,n} \\
 1 & \cdots & 1 & \varepsilon(b_{i+1}) a_{i+1} & b_{i+1,i+1} & b_{i+1,i+2} & \cdots & b_{i+1,n} \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 1 & \cdots & 1 & 1 & a_{i+2} & \cdots & \cdots & a_{i+2} \\
 1 & \cdots & 1 & 1 & 1 & \cdots & \cdots & a_n
\end{array} \right)
\]

\[+ (-1)^n \otimes \left( \begin{array}{cccc}
 \varepsilon(b_{0,n}) a_n a_0 & b_{1,n} & b_{0,1} & \cdots & b_{n-1,n} b_{0,n-1} \\
 1 & a_1 & \cdots & b_{1,i} & \cdots & b_{1,n-1} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 1 & \cdots & a_i & \cdots & b_{i,n-1} \\
 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & a_{n-1}
\end{array} \right).\]

The homology of the complex \( (\widetilde{C}_\bullet(A, B, \varepsilon), \partial) \) is called the secondary Hochschild homology associated to the triple \((A, B, \varepsilon)\) and denoted by \( HH_\bullet(A, B, \varepsilon) \). The homology \( HH_\bullet(A, B, \varepsilon) \) is different from \( H_\bullet((A, B, \varepsilon), A) \), the secondary Hochschild homology of the triple \((A, B, \varepsilon)\) with coefficients in the module \( A \). For the definition of \( HH^\bullet(A, B, \varepsilon) \), the secondary Hochschild cohomology associated to a triple \((A, B, \varepsilon)\), one should see the detailed construction in Section 4 of [9].
4. Calculus structure on secondary Hochschild (co)homology

In this section, we define comp module action of $O^* = C^*(A, B; \varepsilon; A)$ on the secondary Hochschild complex $\mathcal{C}_{\varepsilon}(A, B, \varepsilon)$. We prove that this comp module action makes $\mathcal{C}_{\varepsilon}(A, B, \varepsilon)$ a cyclic comp module over the operad $C^*(A, B, \varepsilon; A)$. We conclude that $(H^*((A, B, \varepsilon); A), HH_{\varepsilon}(A, B, \varepsilon))$ forms a (noncommutative) differential calculus.

4.1. Cyclic comp module structure on $\mathcal{C}_{\varepsilon}(A, B, \varepsilon)$

We denote $M$ cyclic comp module structure on $C_{\varepsilon}(A, B, \varepsilon)$ as follows.

In this section, we define comp module action of

\[
\begin{pmatrix}
a_0 & b_{0,1} & \cdots & b_{0,p-1} & b_{0,p} \\
1 & a_1 & \cdots & b_{1,p-1} & b_{1,p} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & a_{p-1} & b_{p-1,p} \\
1 & 1 & \cdots & 1 & a_p
\end{pmatrix}
\]

as follows. Let $f \in O^m, 0 \leq m \leq p$, and $0 \leq i \leq p - m + 1$. Then, for $m \geq 0$, $n, p \geq 0$, we first show that

\[
f \bullet_i (g \bullet T) = \begin{cases}
g \bullet_i (f \bullet_{i+n-1} T) & \text{if } j < i, \\
(f \circ_{i-j+1} g) \bullet T & \text{if } j - m < i \leq j, \\
g \bullet_{i-m+1} (f \bullet T) & \text{if } 0 \leq i \leq j - m.
\end{cases}
\]

The left hand side is given by

\[
f \bullet_i (g \bullet T) = f \bullet_i \begin{pmatrix}
a_0 & b_{0,1} & \cdots & b_{0,p-1} & b_{0,p} \\
1 & a_1 & \cdots & b_{1,p-1} & b_{1,p} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & a_{p-1} & b_{p-1,p} \\
1 & 1 & \cdots & 1 & a_p
\end{pmatrix} \otimes \begin{pmatrix}
a_0 & \cdots & b_{0,j-1} & \prod_{t=j}^{n+j-1} b_{t,1} & b_{0,n+j} & \cdots & b_{0,p} \\
1 & \cdots & b_{1,j-1} & \prod_{t=j}^{n+j-1} b_{1,1} & b_{1,n+j} & \cdots & b_{1,p} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \cdots & a_{j-1} & \prod_{t=j}^{n+j-1} b_{j-1,1} & b_{j-1,n+j} & \cdots & b_{j-1,p} \\
1 & \cdots & 1 & g(T_{n+j-1}) & \prod_{t=j}^{n+j-1} b_{n+j,1} & \cdots & \prod_{t=j}^{n+j-1} b_{n+j,p} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
1 & \cdots & 1 & 1 & a_{n+j} & b_{n+j} & \cdots & a_p
\end{pmatrix}.
\]
Case 1: For \( j < i \), we have the following expression
\[
f \bullet_i(g \bullet_j T)
\]
\[
= \begin{pmatrix}
    a_0 & \cdots & b_{0,i-1} & \prod_{i=i}^{i+m+n-2} b_{0,t} & b_{0,i+m+n-1} & \cdots & b_{0,p} \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    1 & \cdots & a_{i-1} & \prod_{i-1}^{i+m+n-2} b_{i-1,t} & b_{i-1,i+m+n-1} & \cdots & b_{i-1,p} \\
\end{pmatrix}
\]
\[
= g \bullet_j \begin{pmatrix}
    a_0 & \cdots & b_{0,i-1} & \prod_{i=i}^{i+m+n-2} b_{0,t} & b_{0,i+m+n-1} & \cdots & b_{0,p} \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    1 & \cdots & a_{i-1} & \prod_{i-1}^{i+m+n-2} b_{i-1,t} & b_{i-1,i+m+n-1} & \cdots & b_{i-1,p} \\
\end{pmatrix}
\]
\[
= g \bullet_j \begin{pmatrix}
    a_0 & b_{0,1} & \cdots & b_{0,p-1} & b_{0,p} \\
    1 & a_1 & \cdots & b_{1,p-1} & b_{1,p} \\
    1 & \cdots & a_{p-1} & b_{p-1,p} \\
    1 & \cdots & 1 & a_p \\
\end{pmatrix}
\]
\[
= g \bullet_j(f \bullet_{i+n-1} T).
\]

Case 2: For \( j - m < i \leq j \),
\[
f \bullet_i(g \bullet_j T)
\]
\[
= \begin{pmatrix}
    a_0 & \cdots & b_{0,i-1} & \prod_{i=i}^{i+m+n-2} b_{0,t} & b_{0,i+m+n-1} & \cdots & b_{0,p} \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    1 & \cdots & a_{i-1} & \prod_{i-1}^{i+m+n-2} b_{i-1,t} & b_{i-1,i+m+n-1} & \cdots & b_{i-1,p} \\
\end{pmatrix}
\]
\[
= (f \circ_{j-i+1} g)(T_{i+m+n-2}^{i+m+n-2}) \prod_{i=m+n-1}^{i+m+n-2} b_{k,i+m+n-1} \cdots \prod_{k=i+m+n-1}^{i+m+n-2} b_{k,p}
\]
\[
= (f \circ_{j-i+1} g) \bullet_i T.
\]
**Case 3**: For $0 \leq i \leq j - m$,

$$f \cdot_i (g \cdot_j T) = \left( \begin{array}{cccccc}
    a_0 & \ldots & b_{0,i-1} & \prod_{t=i}^{m+i-1} b_{0,t} & \ldots & \prod_{t=j}^{j+n-1} b_{0,t} & \ldots & b_{0,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \ldots & a_{i-1} & \prod_{t=i}^{m+i-1} b_{i-1,t} & \ldots & \prod_{t=j}^{j+n-1} b_{i-1,t} & \ldots & b_{i-1,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \ldots & 1 & f(T_{m+i-1}^i) & \prod_{s=i}^{m+i-1} b_{s,t} & \ldots & \prod_{s=j}^{j+n-1} b_{s,t} & \ldots & b_{s,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & a_p
\end{array} \right)$$

$$= g \cdot_{j-m+1} \left( \begin{array}{cccccc}
    a_0 & \ldots & b_{0,i-1} & \prod_{t=i}^{i+m-1} b_{0,t} & \ldots & b_{0,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \ldots & a_{i-1} & \prod_{t=i}^{i+m-1} b_{i-1,t} & \ldots & b_{i-1,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \ldots & 1 & f(T_{i+m-1}^i) & \prod_{s=i}^{i+m-1} b_{s,t} & \ldots & b_{s,p} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & a_p
\end{array} \right)$$

$$= g \cdot_{j-m+1} (f \cdot_i T).$$

Therefore, the identity (10) holds true. Similarly, in the case when $m = 0$, the following identity holds

$$f \cdot_i (g \cdot_j T) = \begin{cases} 
    g \cdot_i (f \cdot_{i+n-1} T) & \text{if } j < i, \\
    g \cdot_{i+1} (f \cdot_i T) & \text{if } 0 \leq i \leq j.
\end{cases}$$

**Unitality condition**: For $1 = Id_A \in \mathcal{O}^1 = C^1((A,B,e); A)$, it is clear that

$$1 \cdot_i T = T, \quad \text{for } T \in \mathcal{M}_n, \quad \text{and } 0 \leq i \leq n.$$
We define a map $t : \mathcal{M}_p \rightarrow \mathcal{M}_p$ as follows

$$
\begin{pmatrix}
    a_0 & b_{0,1} & \cdots & b_{0,p-1} & b_{0,p} \\
    1 & a_1 & \cdots & b_{1,p-1} & b_{1,p} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 1 & \cdots & a_{p-1} & b_{p-1,p} \\
    1 & 1 & \cdots & 1 & a_p
\end{pmatrix}
\otimes
\begin{pmatrix}
    a_p & b_{0,p} & b_{1,p} & \cdots & b_{p-1,p} \\
    1 & a_0 & b_{0,1} & \cdots & b_{0,p-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 1 & \cdots & b_{1,p-1} & b_{1,p-1} \\
    1 & 1 & \cdots & 1 & a_{p-1}
\end{pmatrix}
$$

(12)

Here, $t$ is induced by the natural action of the cyclic group $\mathbb{Z}_{p+1}$ (generated by the $p+1$ cycle $(0 1 2 \ldots p)$) on $\mathcal{M}_p$. For further details, see ref. [9]. Next, we show that the map $t : \mathcal{M}_p \rightarrow \mathcal{M}_p$ makes the unital comodule $\mathcal{M}_p$, a cyclic comodule over the opeard $\mathcal{O}_\ast$.

For $T \in \mathcal{M}_p$ and $f \in \mathcal{O}_\ast$, we have the following expression

$$
t(f \cdot_i T) = t \otimes
\begin{pmatrix}
    a_0 & \cdots & b_{0,i-1} & \prod_{t=1}^{i+m-1} b_{0,t} & \cdots & b_{0,p} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    1 & \cdots & a_{i-1} & \prod_{t=1}^{i+m-1} b_{i-1,t} & \cdots & b_{i-1,p} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    1 & \cdots & 1 & f(T^i_{i+m-1}) & \cdots & \prod_{s=i}^{m+i-1} b_{s,p} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    1 & \cdots & 1 & 1 & \cdots & a_p
\end{pmatrix}
$$

$$
= \otimes
\begin{pmatrix}
    a_p & b_{0,p} & b_{1,p} & \cdots & b_{p-1,p} \\
    1 & a_0 & b_{0,1} & \cdots & b_{0,p-1} \\
    \vdots & \vdots & \ddots & \vdots & \ddots \\
    1 & 1 & \cdots & b_{1,p-1} & b_{1,p-1} \\
    1 & 1 & \cdots & 1 & a_{p-1}
\end{pmatrix}
$$

$$
= f \cdot_{i+1}
\begin{pmatrix}
    a_p & b_{0,p} & b_{1,p} & \cdots & b_{p-1,p} \\
    1 & a_0 & b_{0,1} & \cdots & b_{0,p-1} \\
    \vdots & \vdots & \ddots & \vdots & \ddots \\
    1 & 1 & a_1 & \cdots & b_{1,p-1} \\
    1 & 1 & \cdots & 1 & a_{p-1}
\end{pmatrix}
= f \cdot_{i+1} t(T).
$$
For $i \geq 1$, 
\[
\begin{pmatrix}
    a_{p-i+1} & b_{p-i+1,p-i+2} & \cdots & b_{p-i+1,p} & b_{0,p-i+1} & b_{1,p-i+1} & \cdots & b_{p-p-i+1} \\
    1 & a_{p-i+2} & \cdots & b_{p-i+2,p} & b_{0,p-i+2} & b_{1,p-i+2} & \cdots & b_{p-p-i+2} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & a_{p} & b_{0,i} & b_{1,i} & \cdots & b_{p-p,i} \\
    1 & 1 & \cdots & 1 & a_{0} & b_{0,1} & \cdots & b_{0,i} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & 1 & 1 & 1 & \cdots & a_{p-i}
\end{pmatrix}
\]

Clearly, it follows that the map $\imath : M_p \to M_p$ satisfies the identity $\imath^{p+1} = Id$. Thus,
\[
\imath^{p+1} = Id \quad \text{and} \quad \imath (f \cdot T) = f \cdot \imath_{i+1} T (T).
\]

Therefore, by the above discussion we have the following theorem. \hfill \Box

### 4.2. Cyclic k-module structure on $\mathcal{C}_* (A, B, \varnothing)$

The complex $\mathcal{M}_* = \mathcal{C}_* (A, B, \varnothing)$ is a cyclic comp module over the multiplicative operad $(\mathcal{O}^* = C^*((A, B, \varnothing); A), \mu)$. It yields the following cyclic $k$-module structure on $\mathcal{C}_* (A, B, \varnothing)$:

(i) for $i = 0, 1, \ldots, p$, the face maps $d_i : M_p \to M_{p-1}$ are given as follows
(a) if $0 \leq i \leq p - 1$,
\[
\begin{pmatrix}
    a_0 & \cdots & b_{0,i-1} & b_{0,i}b_{0,i+1} & \cdots & b_{0,p} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    1 & \cdots & a_{i-1} & b_{i-1,i}b_{i-1,i+1} & \cdots & b_{i-1,p} \\
    1 & \cdots & 1 & a_{i}a_{i+1}b_{i,i+1} & \cdots & b_{i,n}b_{i+1,n} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    1 & \cdots & 1 & a_{p-i+1} & \cdots & a_{p}
\end{pmatrix}
\]

(b) if $i = p$,
\[
\begin{pmatrix}
    a_0 & \cdots & b_{0,1} & b_{1,1}b_{1,2} & \cdots & b_{p-1,p}b_{0,p-1} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    1 & \cdots & a_{1} & b_{1,1} & \cdots & b_{1,p-1} \\
    1 & \cdots & 1 & b_{2,1} & \cdots & b_{p-2,p-1} \\
    1 & \cdots & 1 & a_{p-1} & \cdots & a_{p}
\end{pmatrix}
\]

(ii) for $j = 0, 1, \ldots, p$, the degeneracies $s_j : M_p \to M_{p+1}$ are given by
\[
\begin{pmatrix}
    a_0 & b_{0,1} & \cdots & b_{0,p} \\
    1 & a_1 & \cdots & b_{1,p} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & a_{p}
\end{pmatrix}
\]
\[
\begin{pmatrix}
    a_0 & b_{0,1} & \cdots & b_{0,j} & 1 & 1 & \cdots & b_{0,p} \\
    1 & a_1 & \cdots & b_{1,j} & 1 & 1 & \cdots & b_{1,p} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & a_{j} & 1 & 1 & \cdots & b_{j+1,p} \\
    1 & 1 & \cdots & 1 & a_{j+1} & \cdots & b_{j+1,p} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    1 & 1 & \cdots & 1 & 1 & 1 & \cdots & a_{p}
\end{pmatrix}
\]

(iii) The cyclic operator $\imath : M_p \to M_p$ is given by the Equation (12).
The simplicial boundary operator on the complex $M_\bullet$ is given by

$$b := \sum_{i=0}^{p} (-1)^i d_i.$$  

Note that the complex $(M_\bullet, b)$ is the same as the secondary Hochschild chain complex associated to the triple $(A, B, \varepsilon)$, defined in Section 3. Thus, the associated homology

$$H_\bullet(M) = HH_\bullet(A, B, \varepsilon).$$  

Next, let us recall that the extra degeneracy map, the norm operator and the cyclic differential are given by

$$s_{-1}(T) := \ell \ s_p(T) = 1_A \bullet_0 T, \quad N := \sum_{p=0}^{n} (-1)^p t^i, \quad \text{and} \quad B := (Id - t)s_{-1}N.$$  

The normalized complex $\mathcal{N}(M)_\bullet$ is the quotient of the complex $M_\bullet$ by the subcomplex spanned by $\{1_A \bullet_{j+1} \rightarrow , j = 0, 1, ..., p\}$. The cyclic differential $B$ induces the following map on the normalized complex:

$$B(T) = s_{-1}N(T) = \sum_{i=0}^{p} (-1)^i 1_A \bullet_0 t^i(T), \quad \text{for} \quad T \in \mathcal{N}(M)_p. \quad (13)$$

### 4.3. Cosimplicial k-module structure on the complex $C^*((A, B, \varepsilon); A)$

Now, we consider the cosimplicial $k$-module structure on the complex $C_\bullet$ associated to the multiplicative operad $(O^\bullet, \mu)$. Then the face maps $d^i : \mathcal{O}^p \rightarrow \mathcal{O}^{p+1}$ and $s^j : \mathcal{O}^p \rightarrow \mathcal{O}^{p-1}$ are given as follows

$$d^i(f) = \begin{cases} \mu \circ_i f & \text{if} \ i = 0, \\ f \circ_i \mu & \text{if} \ 1 \leq i \leq p, \\ \mu \circ_i f & \text{if} \ i = p + 1 \end{cases}$$

and

$$s^j(f) = f \circ_{i+1} 1_A, \quad \text{for} \ j = 0, 1, ..., p - 1.$$  

In particular,

$$s^j(f) \begin{pmatrix} a_1 & b_{1,2} & \cdots & b_{1,p-1} \\ 1 & a_2 & \cdots & b_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & b_{p-2,p-1} \\ 1 & 1 & \cdots & a_{p-1} \end{pmatrix} = f \begin{pmatrix} a_1 & \cdots & b_{1,j} & 1 & 1 & \cdots & b_{1,p-1} \\ 1 & \cdots & b_{1,j-1} & 1 & 1 & \cdots & b_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & a_{j-1} & 1 & 1 & \cdots & b_{j-1,p-1} \\ 1 & \cdots & 1 & 1_A & 1 & \cdots & 1 \\ 1 & \cdots & 1 & a_j & \cdots & b_{j,p-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & a_{p-1} \end{pmatrix}.$$  

The coboundary map $\delta^p : \mathcal{O}^p \rightarrow \mathcal{O}^{p+1}$ is given by $\delta^p := \sum_{i=0}^{p} d^i$. The cochain complex $(O^\bullet, \delta)$ coincides with the complex $(C^*((A, B, \varepsilon); A), \delta^p)$. The conormalised cochain complex is given by

$$\mathcal{N}(\mathcal{O})^p := \mathcal{N}(\mathcal{O})^{p-1} \ker(s^j).$$

Here, $\mathcal{N}(\mathcal{O})^p$ is the collection of maps in $\text{Hom}_A(A^{\mathcal{O}^p} \otimes B^{\mathcal{O}^{p-1}}, A)$, which vanish at the elements of the type...
Note that the cohomology $H^\bullet(\mathcal{R}(O) \circledast) = H^\bullet_\mu(O^\bullet) = H^\bullet((A, B, \varepsilon); A)$.

4.4. A calculus structure on $(H^\bullet((A, B, \varepsilon); A), HH_\bullet(A, B, \varepsilon))$

The complex $\mathcal{M}_\bullet := \mathcal{C}_\bullet(A, B, \varepsilon)$ is a cyclic comp module over the multiplicative operad $O = C^\bullet((A, B, \varepsilon); A)$. Let us recall from Section 2 that a cyclic (unital) comp module $\mathcal{M}_\bullet$ over a multiplicative operad $(O, \mu)$ induces a calculus structure on the pair $(H^\bullet_\mu(O), H^\bullet_\mu(M))$. From Subsection 4.2 and Subsection 4.3, it follows that

$$H_\bullet(M) = HH_\bullet(A, B, \varepsilon), \quad H^\bullet_\mu(O^\bullet) = H^\bullet((A, B, \varepsilon); A).$$

Therefore, we obtain the following result.

Theorem 4.1. The pair $(H^\bullet((A, B, \varepsilon); A), HH_\bullet(A, B, \varepsilon))$ forms a noncommutative differential calculus. \hfill $\square$

5. Conclusion

M. Staic first introduced secondary Hochschild (co)homology [15] in order to study the deformation theory of associative algebras over a commutative ring. The secondary Hochschild (co)homology behaves similar to the Hochschild (co)homology in several aspects. However, unlike Hochschild (co)homology, there is no natural cyclic action on the (co)chain complex. The absence of a natural cyclic action also explains the natural constructions of new complexes while introducing secondary cyclic (co)homology [9].

In the Hochschild case, there is a noncommutative differential calculus $(HH^\bullet(A, A), HH_\bullet(A, A))$. Under certain conditions [5], the isomorphism $HH^\bullet(A, A) \cong HH_\bullet(A, A)$ yields a BV algebra structure on the cohomology $HH^\bullet(A, A)$. In fact, the condition obtained by [5] (in the case of Calabi–Yau algebras) can be written in general for any calculus: if $(A, \Omega)$ is a calculus, then

$$i_{f,g} = [i_f, L_g]$$

$$= i_f \circ L_g - (-1)^{m(n+1)} L_g \circ i_f$$

$$= i_f \circ (B \circ i_g - (-1)^{n} i_g \circ B) - (-1)^{m(n+1)}(B \circ i_g - (-1)^{n} i_g \circ B) \circ i_f$$

$$= i_f \circ B \circ i_g - (-1)^{n} i_f \circ B - (-1)^{m(n+1)}(-1)^{m} B \circ i_f - i_f = B \circ i_g - (-1)^{m(n+1)+n} i_g \circ B \circ i_f \quad (14)$$

for $f \in A^m$, $g \in A^n$. Subsequently, we obtain the following result which gives a condition on the calculus such that the underlying Gerstenhaber algebra is a Batalin–Vilkovisky (BV) algebra.

Theorem 5.1. Let $(A, \Omega)$ be a calculus. If there exists an element $c \in \Omega_k$ such that $B(c) = 0$ and the map $\Theta : A^\bullet \to \Omega_{k-\bullet}$, given by $\Theta(f) = i_f c$ is an isomorphism, then the map $\Delta : A^\bullet \to A^{\bullet-1}$, defined by $i_{Af} c = B(i_f c)$ makes the Gerstenhaber algebra $A$ into a BV algebra.
Proof. Since \( B(c) = 0 \) and \( i_{\Delta f}c = B(i_f c) \), by Equation (14) it follows that

\[
i_{[f, g]}(c) = -(1)^m (i_{\Delta(f \smallfrown g)}c - i_{\Delta(f \smallfrown g)}c - (1)^m i_{\Delta(f \smallfrown g)}c).
\]

In turn the condition that \( \Theta \) is an isomorphism implies that the operator \( \Delta \) generates the Gerstenhaber bracket on \( A \), that is

\[
[f, g] = -(1)^m (\Delta(f \smallfrown g) - \Delta(f) \smallfrown g - (1)^m f \smallfrown \Delta(g)), \quad \text{for all } f \in A^m, \ g \in A^n.
\]

The above theorem was first proved by Lambre in ref. [8]. Theorem 5.1 gives a condition on the calculus \((H^*((A, B, \varepsilon); A), HH_\bullet(A, B, \varepsilon))\) such that the Gerstenhaber algebra structure on \( H^*((A, B, \varepsilon), A) \) becomes a BV algebra structure. More precisely, if we have an element \([T] \in HH_\bullet(A, B, \varepsilon)\) such that \( B[T] = 0 \) and the map

\[\Theta : H^*((A, B, \varepsilon), A) \to HH_k(A, B, \varepsilon), \text{ given by } \Theta(f) = i_f[T]\]

is an isomorphism, then \( H^*((A, B, \varepsilon), A) \) is a BV algebra.

In the Hochschild case, if \( A \) is symmetric algebra then the calculus \((HH^*(A, A), HH_\bullet(A, A))\) satisfies the conditions in the Theorem 5.1 and the Hochschild cohomology \( HH^*(A, A) \) carries a BV algebra structure (see [11, 12, 18] for details). It will be interesting to find conditions on the \( B \)-algebra \( A \) such that the conditions in Theorem 5.1 hold. It needs further investigation since

- there is no derived functor description for the secondary Hochschild (co)homology, and
- even in the finite-dimensional symmetric algebra case, the most canonical choice for the BV-operator does not lift to the secondary Hochschild cohomology [1].

More precisely, in ref. [1] the authors consider a finite-dimensional symmetric algebra \( A \) and define a BV-operator \( \Delta \) on the homotopy Gerstenhaber algebra \( C^*((A, B, \varepsilon), A) \), which due to proposition 9 and theorem 10 in ref. [1], is the most canonical choice for a square zero BV-differential operator on \( H^*((A, B, \varepsilon), A) \). Though \( \Delta \) determines the graded Lie bracket on \( H^*((A, B, \varepsilon), A) \), it is not a cochain map in general (see theorem 7, [1]).

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