Spontaneous localization for a charged particle in non-relativistic quantum mechanics

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Abstract. The interaction of a moving charged particle with its coherent electromagnetic field is analysed in the framework of non-relativistic quantum mechanics. It is shown that, when this interaction is taken into account, a spatially localized state may have a mean energy lower than the one corresponding to a delocalized state.

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1 Introduction

We know, from the study of the infra-red divergence problem in relativistic QED, that a physical system cannot be separated from its classical (i.e. coherent) electromagnetic field. On the other hand it is a well-known fact, in classical physics, that the motion of a charged particle cannot be described correctly, unless the radiation-reaction force is taken into account properly. On the contrary, in our opinion, in non-relativistic quantum mechanics the effects due to the interaction of a system with the self-generated coherent field have not been analysed in a satisfactory way.

The aim of this paper consists of an analysis of the effects due to the interaction of a moving charged particle with its coherent field. Uncoherent-photon emission will not be considered here. Radiation damping for a non-relativistic quantum system has been studied e.g. in Ref. Decoherence effects due to soft-bremsstrahlung emission have been analysed by Breuer and Petruccione in the interesting papers of Ref.s and .

We will show that, for a moving particle, the interaction with the coherent field gives origin to a mechanism which favours, from the energetical point of view, a spontaneous localization in space. The effect is due essentially to the attractive force which acts among parallel currents. The Coulomb gauge is assumed in this paper. It is well known that, in the Coulomb gauge, the Coulomb interaction is described as an instantaneous action at a distance among different charged particles. The repulsive Coulomb self-interaction energy concerning a single charged particle, once the mass-renormalization contribution (which is independent of the wave function) has been subtracted, can be estimated of the order of the Lamb-shift energies.

The analysis presented in this paper is based on the fact that, when the residual interaction with the quantum field is neglected (i.e. when uncoherent photon emission is neglected), a rather simple expression can be given for the conserved total energy of the system (particle plus classical field), in terms of the classical vector potential in the Coulomb gauge and of the wave function of the particle. We will show that, for a freely moving electron with e.g. a convective velocity \( v_c \sim 10^{-1} c \), a minimum for the energy is attained for a radius of localization \( b \) of the order of \( 10^{-8} \) m, with a binding energy \( E_b \) of \( 10^{-4} \div 10^{-5} \) eV. For a proton with the same \( v_c \) the result would be \( b \sim 10^{-11} \) m and \( E_b \sim 10^{-1} \) eV. It will be shown that similar results hold for neutral atoms also.

2 Classical field and self-interaction

Let us consider an electron interacting with the transverse electromagnetic field. For the field we assume the Coulomb gauge, as well as the Schrödinger picture. Let \( A \) and \( E_\perp \) be quantum field operators and let \( A_c \) and \( E_\perp c \) be the corresponding classical fields. The Hamiltonian for the total system, consisting of the electron and the quantum field, is given by

\[
H_T = H_M + H_F + V,
\]

where \( H_M \) describes the free electron, \( H_F \) is the free-field Hamiltonian, given by the following normal-ordered expression

\[
H_F = \frac{\epsilon_0}{2} \int d\mathbf{r} \left( |\mathbf{E}_\perp|^2 + c^2 |\mathbf{B}|^2 \right),
\]

and the interaction \( V \) is given by
\[ V = -\int d\mathbf{r} (\mathbf{A} \cdot \mathbf{j}_p - \frac{e^2}{2m} \hat{n} |\mathbf{A}|^2 + \mathbf{B} \cdot \mathbf{M}). \] (3)

In eq. (3) \( \mathbf{j}_p \), given by
\[ \mathbf{j}_p (\mathbf{r}) = \frac{i e \hbar}{2m} [\delta (\mathbf{r} - \mathbf{r}_e) \nabla \mathbf{e} + \nabla \mathbf{e} \delta (\mathbf{r} - \mathbf{r}_e)], \] (4)
represents the canonical current (i.e. the current due to the canonical momentum), \( \hat{n} \) represents the electron-density operator and finally \( \mathbf{M} \) represents the spin magnetization-density operator.

The following time-independent commutation relations
\[ [A_i (\mathbf{r}), E_{\perp j} (\mathbf{r}')] = -i e \hbar \frac{1}{2m} \mathcal{P}_{ij} (\mathbf{r} - \mathbf{r}') \] (5)
hold, where
\[ \mathcal{P}_{ij} (\mathbf{r}) = \left( \frac{1}{2\pi} \right)^3 \int d\mathbf{q} e^{i \mathbf{q} \cdot \mathbf{r}} (\delta_{ij} - \frac{q_i q_j}{q^2}) \]
\[ = \frac{2}{3} \delta_{ij} \delta (\mathbf{r}) + \frac{3}{4\pi r^3} \left( \frac{x_i x_j}{r^2} - \frac{1}{3} \delta_{ij} \right). \] (6)
is the projector onto the transverse components of the field \( \mathbf{E}. \)

The Schrödinger equation describing the total system can be derived from the variational functional
\[ S_Q = \int dt \langle \Psi_T | D^+ (-i \hbar \partial_t + H_T) D | \Psi_T \rangle, \] (7)
where the displacement unitary operator \( D \), given by
\[ D (t) = \exp \left\{ \frac{i e \hbar}{\hbar} \int d\mathbf{r} [\mathbf{A}_c (t, \mathbf{r}) \cdot \mathbf{E}_{\perp} (\mathbf{r}) \right. \]
\[ \left. - \mathbf{E}_{\perp c} (t, \mathbf{r}) \cdot \mathbf{A} (\mathbf{r})] \right\}, \] (8)
has been introduced. The effect of the displacement operator \( D \) on the field operators consists of
\[ D^{-1} \mathbf{A} D = \mathbf{A} + \mathbf{A}_c \] (9)
\[ D^{-1} \mathbf{E}_{\perp} D = \mathbf{E}_{\perp} + \mathbf{E}_{\perp c}. \] (10)

By independent variations of \( S_Q \) with respect to \( \mathbf{E}_{\perp c} \) and \( \mathbf{A}_c \) (taking into account the transversality of both) we obtain the relation
\[ \mathbf{E}_{\perp c} = -\partial_t \mathbf{A}_c \] (11)
and the Maxwell equation
\[ \epsilon_0 c^2 \nabla \times \mathbf{B}_c - \epsilon_0 \partial_t \mathbf{E}_{\perp c} = \mathbf{j}_{\perp}, \] (12)
where we have assumed \( \langle \mathbf{A} \rangle = 0 \) as well as \( \langle \mathbf{E}_{\perp} \rangle = 0 \). The classical current \( \mathbf{j}_c \) is given by
\[ \mathbf{j}_c (\mathbf{r}) = \langle \mathbf{j}_p (\mathbf{r}) \rangle \] (13)
\[ - \frac{e^2}{m} [\hat{n} (\mathbf{r}) \mathbf{A}_c (\mathbf{r}) + \langle \hat{n} (\mathbf{r}) \mathbf{A} (\mathbf{r}) \rangle] + \nabla \langle \mathbf{M} (\mathbf{r}) \rangle. \]

We notice that the last term in square brackets represents an unconventional contribution to the classical current, that might be observable in cases of matter-field entanglement.

The Schrödinger equation for the total system can be derived from the functional \( S_Q \) of eq. (7), by variation with respect to \( |\Psi_T (t)\rangle \). Using eqs (11), (12) and dropping a global time-dependent phase-factor, one obtains
\[ i \hbar \partial_t |\Psi_T \rangle = (H_0 + H_{int}) |\Psi_T \rangle, \] (14)
where \( H_0 \), given by
\[ H_0 = H_M + H_F \]
\[ - \int d\mathbf{r} [\mathbf{A}_c \cdot \mathbf{j}_p - \frac{e^2}{2m} \hat{n} |\mathbf{A}_c|^2 + \mathbf{B}_c \cdot \mathbf{M}], \] (15)
will be assumed as unperturbed Hamiltonian. We remark that the self-interaction due to the coherent field is included in the unperturbed Hamiltonian \( H_0 \). Assuming an unperturbed state of the form
\[ |\Psi_T \rangle_0 = |\psi \rangle \otimes |0 \rangle_F, \] (16)
where \( |0 \rangle_F \) is the vacuum state for the field, we can write the unperturbed Schrödinger equation in the form
\[ i \hbar \partial_t \psi = (2m)^{-1} |(2m)^{-1} \mathbf{e} \mathbf{A}_c |^2 \psi + \mathbf{e} \mathbf{B}_c \cdot \mathbf{\sigma} \psi. \] (17)
The non-linearity of the Schrödinger equation (17) is evident if the eqs. (11) and (12) are taken into account.

The perturbative Hamiltonian \( H_{int} \) of eq.(14) is given by
\[ H_{int} = - \int d\mathbf{r} \mathbf{A} \cdot [\langle \mathbf{j}_p \rangle - \langle \mathbf{j}_p \rangle] - \frac{e^2}{m} (\hat{n} - \langle \hat{n} \rangle) \mathbf{A}_c \]
\[ + \frac{e^2}{2m} \int d\mathbf{r} \hat{n} \mathbf{A}_c |^2 - \int d\mathbf{r} \mathbf{B} \cdot (\mathbf{M} - \langle \mathbf{M} \rangle). \] (18)

The Hamiltonian \( H_{int} \) contains the residual uncoherent interaction with the quantum field. It describes processes like uncoherent real photon emission as well as virtual photon emission and reabsorption (Lamb shift). Such processes will not be analysed in this paper. A justification for this will be given at the end of Sec.V. Furthermore, for the sake of simplicity, what follows the spin interaction will be neglected.

Finally we observe that the Schrödinger equation (17) and the Maxwell equation (12) can be derived from the following Lagrangian density
\[ \mathcal{L} = i \hbar \psi^* \partial_t \psi - (2m)^{-1} |(-i \hbar \nabla + e \mathbf{A}_c) \psi |^2 \]
\[ + \frac{e^2}{2m} (|\partial \mathbf{A}_c / \partial t |^2 - c^2 |\mathbf{B}_c |^2), \] (19)
where the Coulomb gauge is understood and the spin term has been neglected, for the sake of simplicity.
3 Conserved energy.

By the Noether theorem we can obtain, from the Lagrangian density (19), expressions for the conserved mean energy of the system

$$\mathcal{E} = \int d^3x \psi^* \left( -\frac{\hbar^2}{2m} \nabla^2 - \frac{ie\hbar}{m} \mathbf{A}_c \cdot \nabla + \frac{e^2}{2m} A_c^2 \right) \psi + \frac{\epsilon_0}{2} \int d^3r (E_{\perp c}^2 + c^2 B_c^2)$$

as well as for the total momentum

$$\mathbf{P} = -i\hbar \int d^3x \psi^* \nabla \psi + \epsilon_0 \sum_j \int d^3r E_{\perp c,j} \nabla A_{c,j}. \quad (21)$$

For a localized wave-packet the conservation of $\mathcal{E}$ can be verified directly. In fact, by use of eq.(17), we obtain

$$\frac{d\mathcal{E}}{dt} = \int d^3x \frac{\partial A_c}{\partial t} \cdot \left( -\frac{ie\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right) + \frac{\epsilon_0}{2} \int d^3r (E_{\perp c}^2 + c^2 B_c^2)$$

where the r.h.s. represents the ingoing flux, and the l.h.s. the time derivative of the energy.

$$= \epsilon_0 \sum_j \int d^3r E_{\perp c,j} \nabla A_{c,j}. \quad (22)$$

$$= -\epsilon_0 c^2 \int d^3s \cdot (\mathbf{E}_{\perp c} \times \mathbf{B}_c),$$

where the l.h.s. represents the ingoing flux, and the r.h.s. the time derivative of the energy.

The last term in eq.(20), which represents the energy of the classical field, can be cast in the form:

$$\mathcal{E}_F = \frac{\epsilon_0}{2} \int d^3x (E_{\perp c}^2 + c^2 B_c^2)$$

$$= \int d^3x \psi^* \left( \frac{ie\hbar}{2m} \mathbf{A}_c \cdot \nabla + \frac{e^2}{2m} A_c^2 \right) \psi + \epsilon_0 \int d^3r E_{\perp c}^2$$

$$- \frac{\epsilon_0 c^2}{4} \int d^3x A_c^2 + \frac{1}{2} \epsilon_0 c^2 \int d^3s \cdot (\mathbf{A}_c \times \mathbf{B}_c). \quad (23)$$

In the absence of a significant emission of coherent radiation, the last term can be neglected for a finite system.

By use of this result, we obtain from eq.(20)

$$\mathcal{E} = -\int d^3x \psi^* \left( \frac{\hbar^2}{2m} \nabla^2 \psi - \frac{1}{2} \int d^3x \mathbf{A}_c \cdot \psi^* \frac{ie\hbar}{m} \nabla \psi \right) + \epsilon_0 \int d^3x E_{\perp c}^2 + \frac{\epsilon_0}{2} \frac{d^2}{dt^2} \int d^3x A_c^2. \quad (24)$$

Two features of eq.(24) are remarkable. The first one is the absence of $A_c^2$ from the interaction term. The second one is the factor of 1/2 appearing in the interaction term (a proper result for a self-interaction energy).

4 Self-interaction energy for a gaussian wave-packet.

In order to simplify the mathematical analysis of the problem, we assume a wave function of the form (a sort of de Broglie's double solution)

$$\psi(t, \mathbf{r}) = \exp\{i\hbar^{-1}[\mathbf{p}_c(t) \cdot \mathbf{r} - \mathbf{E}(t)] \} \phi(t, \mathbf{r} - \mathbf{r}_c(t)),$$

In eq.(25) the function $\phi$ is chosen in such a way that the following relations

$$\mathbf{p}_c = -i\hbar (\psi \nabla \psi) \quad (26)$$

and

$$\mathbf{r}_c = \langle \psi | \mathbf{r} | \psi \rangle \quad (27)$$

hold. With these assumptions $\mathbf{p}_c$ and $\mathbf{r}_c$ can be interpreted as classical momentum and position of the particle, respectively. In order to simplify the calculations, we refer to a Gaussian wave function

$$\phi(r) = \left( \frac{1}{4\pi b^2} \right)^{3/4} \exp\left[ -\frac{r^2}{8b^2} \right],$$

as a model.

The time evolution for both $\mathbf{p}_c$ and $\mathbf{r}_c$ can be obtained by the Ehrenfest theorem (for the sequel, while $\mathbf{p}_c$ see eq.(35) in the sequel, while $\mathbf{p}_c$ is approximately conserved, according to eq.(44) ).

The probability density $\rho$, corresponding to the wave function (25), is given by

$$\rho(t, \mathbf{r}) = \rho_0(t, \mathbf{r} - \mathbf{r}_c) = |\phi(t, \mathbf{r} - \mathbf{r}_c)|^2,$$

or, in Fourier representation, by

$$\hat{\rho}(t, \mathbf{q}) = e^{-i\mathbf{q} \cdot \mathbf{r}_c} \rho_0(t, \mathbf{q}) = e^{-i\mathbf{q} \cdot \mathbf{r}} e^{-b^2 q^2}, \quad (30)$$

where the last expression refers to the Gaussian wave function of eq.(28).

The solution to eq.(12) is given by the retarded potential

$$\mathbf{A}_c(t, \mathbf{r}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\tau \frac{\mathbf{J}_{\perp c}((t - \tau, \mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|} \quad (31)$$

where we have assumed a density $\rho_0$ sufficiently localized in space, in order that the effects depending on the retardation time

$$\tau = c^{-1} |\mathbf{r} - \mathbf{r}'| \quad (32)$$

be negligible. We recall that, in the classical limit, the first order contribution in $\tau$ (actually neglected) is responsible...
In the last equality we have assumed a classical momentum $p_e$ sufficiently large, in order that the current due to the internal motion be negligible, compared with the convective one.

Let us define $A_0^e(r) = A_e(r + r_e)$. From eq.s (30), (31) and (33) we obtain

$$\hat{A}_c^0 \simeq -\frac{e}{mc^2\epsilon_0 q^2} \rho^0(q)[p_e - q^{-2}q(q \cdot p_e)].$$

(34)

According to the Ehrenfest theorem, the classical velocity of the electron is given by

$$v_c \equiv d\mathbf{r}_e/dt = m^{-1}(p_e + e(A_0^e)).$$

(35)

where the average is taken over the internal wave function $\phi$. We obtain

$$e\langle A_0^e \rangle = e \int d\rho \rho^0(r) A_0^e(r)$$

$$= e(2\pi)^{-3} \int dq \rho^0(q) A_0^e(q)$$

$$= -\frac{e^2}{mc^2\epsilon_0}(2\pi)^{-3} \int dq q^{-2} |\rho^0(q)|^2 [p_e - q^{-2}q(q \cdot p_e)]$$

$$= -\frac{e^2}{mc^2\epsilon_0} \frac{8\pi}{3} (2\pi)^{-3} \int dq |\rho^0(q)|^2$$

$$= -\frac{4}{3} \tilde{E}_{el} p_e,$$

where the electrostatic energy

$$\tilde{E}_{el} = \frac{e^2}{8\pi\epsilon_0}(2\pi)^{-3} \int dq q^{-2} |\rho^0(q)|^2$$

$$= \frac{e^2}{4\pi^2\epsilon_0} \int dq |\rho^0(q)|^2 = \frac{e^2}{8\sqrt{2\pi}3/2\epsilon_0 b}$$

has been introduced. The last expression refers to the Gaussian model. We obtain in this way

$$p_e \simeq m(1 + \frac{4}{3} \frac{\tilde{E}_{el}}{mc^2}) v_c.$$  

(38)

Eq.(38) shows that, as a consequence of the self-interaction, a mass renormalization takes place. Exactly the same result holds for an extended classical particle (notice the famous factor of 4/3 in the r.h.s. of eq.(38)).

The classical transverse electric field is given by

$$\hat{E}_{\perp e}(t, q) = -\partial_t \hat{A}_c(t, q)$$

$$= \frac{e}{\epsilon_0 mc^2 q^2} \rho^0(t, q)[p_e - q^{-2}q(q \cdot p_e)][iq \cdot v_e - \partial_t \ln \rho^0].$$

$$\simeq \frac{i e}{\epsilon_0 mc^2 q^2} \rho^0(t, q)[p_e - q^{-2}q(q \cdot p_e)]q \cdot v_e,$$

where the contribution due to the time variation of ln $\rho^0$ has been neglected, with respect to the contribution due to the convective motion.

In what follows the r.h.s. of eq.(24) will be calculated up to the second order in the electron charge $e$. Moreover any power of $\beta = v_e/c$ higher than the second will be neglected with respect to unity.

Let us proceed to calculate the r.h.s. of eq.(24). The first term is given by

$$-\frac{\hbar^2}{2m} \int d\psi^* \nabla^2 \psi$$

$$= \frac{p_e^2}{2m} - \frac{\hbar^2}{2m} \int d\psi^* \nabla^2 \psi = \frac{p_e^2}{2m} + \frac{3\hbar^2}{16mb^2}.$$

The next term is given by

$$-\frac{1}{2} \int d\mathbf{r}_e \cdot A_e = -\frac{1}{16\pi^3} \int d\mathbf{q} \chi_e(q) \cdot \hat{A}_e(q)$$

$$= -\frac{e^2}{16\pi^3\epsilon_0 c^2} \int d\mathbf{q} q^{-2} |\rho_0(q)|^2 [1 - \frac{q \cdot p_e^2}{q^2 p_e^2}] \simeq -\frac{2}{3} \beta^2 \tilde{E}_{el}.$$  

(41)

Next let us calculate

$$\epsilon_0 \int d\mathbf{r} E_{\perp e}^2 \simeq \frac{4}{15} \beta^4 \tilde{E}_{el}.$$  

(42)

This term, of the order of $\beta^4$, is negligible. The last term consists of

$$-\frac{e_0}{4} \frac{d^2}{dt^2} \int d\mathbf{r} A_e^2 \simeq \beta^2 \frac{d}{dt} \left( \frac{4}{3} \tilde{E}_{el} \frac{db}{dt} \right)$$

$$\simeq \frac{8}{3} \sqrt{\frac{2}{\pi}} \beta^3 \tilde{E}_{el} \frac{db}{dt},$$

(43)

and is negligible also.

Finally let us calculate the total momentum $P$ of eq.(21), which is conserved according to the Nöther theorem. We obtain

$$P = p_e + i\epsilon_0 (2\pi)^{-3} \sum_j \int d\mathbf{q} \hat{E}_{\perp c}^* \hat{A}_{cj}(q)$$

$$\simeq p_e[1 + \frac{4}{15} \beta^2 \tilde{E}_{el}].$$

Using this result we obtain for the first term of eq.(40)
\[
\frac{p_e^2}{2m} \simeq \frac{P^2}{2m} \left[1 - \frac{8}{15} \beta^2 \frac{E_{el}}{mc^2} \right] \quad (45)
\]

This means that the first term in the r.h.s. of eq.(40) is approximately a constant of the motion.

Finally the total energy of eq.(24) is given by the following expression
\[
E \simeq \frac{1}{2} mv_e^2 - \frac{h^2}{2m} \int d\phi \phi^* \nabla^2 \phi - \frac{4}{3} \beta E_{el}. \quad (46)
\]

### 5 Localization and binding energy

From eq.(46) we obtain, for the Gaussian wave function of eq.(28),
\[
E \simeq \frac{1}{2}mv_e^2 + \frac{3h^2}{16mb^2} - \frac{e^2 \beta^2}{6\sqrt{2} \pi \epsilon_0 b} \quad (47)
\]

A minimum for \(E\) is attained for \(b\) given by
\[
b \simeq \frac{9\pi^{3/2} \epsilon_0 h^2 \beta^{-2}}{\sqrt{2}me^2} = \frac{9\sqrt{\pi}}{4\sqrt{2}} \beta^{-2} a_B \simeq 2.8 \beta^{-2} a_B, \quad (48)
\]

where \(a_B = 4\pi \epsilon_0 h^2/m e^2\) is the Bohr radius. The corresponding binding energy is given by
\[
E_{el}^b \simeq (4/27\pi) \beta^4 E_R, \quad (49)
\]

where \(E_R = me^4/(24\pi \epsilon_0 h^2)\) is the Rydberg energy. Indicatively, for \(\beta \sim 10^{-1}\) we obtain \(b \sim 1.5 \times 10^{-8} \text{ m}\) and \(E_{el}^b \sim 6.4 \times 10^{-5} \text{ eV}\).

For a particle with charge \(\pm Ze\) and mass \(M\) eq.s (48) and (49) read
\[
b \simeq (m/M)Z^{-2}b_{el}, \quad (50)
\]

and
\[
E_{el} \simeq (M/m)Z^4 E_{el}^b. \quad (51)
\]

Indicatively, for a proton with \(\beta \sim 10^{-1}\) one obtains \(b \sim 8.1 \times 10^{-12} \text{ m}\) and \(E_{el} \sim 1.2 \times 10^{-1} \text{ eV}\).

We observe that, for fixed \(\beta\), the ratio between the localization radius \(b\) and the de Broglie wave-length \(\lambda\) is independent of the mass, according to
\[
b/\lambda \simeq (2.8 a_B / 2\pi \lambda_e) \beta^{-1}Z^{-2} \simeq 62 \beta^{-1}Z^{-2}, \quad (52)
\]

where \(\lambda_e\) is the Compton wave-length for the electron.

All of these results have been obtained by assuming an isotropic Gaussian wave function \(\phi\). It can be expected that the lowest-energy configuration would not correspond to a spherically symmetric wave function \(\phi\), but rather to a cylindrically symmetric one.

We recall that the interaction of the system with the residual quantum-field, described by \(H_{int}\) of eq.(18), has been neglected in this paper. This amounts to neglect the emission of real uncoherent photons by the system, as well as the emission and reabsorption of virtual photons. In both processes the field is coupled essentially to the internal motion and not to the convective one, as shown by eq.s (18) and (33). We remark that the emission of uncoherent radiation is a dissipative effect. Presumably it plays a rôle in the relaxation of the system toward a bound state, but it cannot increase the internal energy or destroy the bound state itself. On the other hand, the emission and reabsorption of virtual photons can be expected to introduce a very small correction to the unperturbed binding energy, of the order of the Lamb shift energy for a system, whose space dimension is of the order of \(b\) given by eq.(48) or by eq.(50).

### 6 Neutral atom.

The charge density for a neutral atom is given by
\[
\rho_{ch}(r) = Ze \rho_{cm}(r) - Ze \int d\mathbf{r}' \rho_{cm}(\mathbf{r}') \rho_{el}(\mathbf{r} - \mathbf{r}'), \quad (53)
\]

where \(\rho_{cm}\) represents the probability density for the centre-of-mass co-ordinate (coinciding approximately with the nuclear co-ordinate) and \(Z \rho_{el}\) is the electron density referred to the nuclear position. In Fourier representation eq.(53) reads
\[
\hat{\rho}_{ch}(\mathbf{q}) = Ze \hat{\rho}_{cm}(\mathbf{q}) \left[1 - \hat{\rho}_{el}(\mathbf{q})\right]. \quad (54)
\]

If, for the sake of simplicity, we assume a Gaussian form for \(\hat{\rho}_{cm}\) and \(\rho_{el}\), the electrostatic energy \(E_{el}\) is given by
\[
E_{el} \simeq \frac{Z^2 e^2}{4\pi^2 \epsilon_0} \int_0^\infty dq |\hat{\rho}_{cm}|^2 |1 - \hat{\rho}_{el}|^2 \quad (55)
\]

\[
= \frac{Z^2 e^2}{4\pi^2 \epsilon_0} \int_0^\infty dq e^{-2b^2 q^2} \left[1 - e^{-\gamma^2 q^2}\right]^2
\]

\[
= \frac{Z^2 e^2}{8\sqrt{2\pi^3} \epsilon_0} \left[\frac{1}{b} - \frac{\sqrt{2}}{\sqrt{b^2 + \gamma^2}} + \frac{1}{\sqrt{b^2 + \gamma^2}}\right].
\]

For a delocalized centre-of-mass, i.e. for \(b \gg \gamma\), one obtains \(E_{el} \simeq 0\), as for a neutral particle. On the contrary, for a strong localization of the centre-of-mass, i.e. for \(b \ll \gamma\), one obtains
\[
E \simeq \frac{Z^2 e^2}{8\sqrt{2\pi^3} \epsilon_0 b}, \quad (56)
\]

as for a bare nucleus. In this case eq.s (50) and (51) as well as (52) hold.
7 Conclusion

We have shown the existence of a mechanism which favours a spontaneous localization in space for a moving charged particle. The effect is due essentially to the attractive force acting among parallel currents. It is well known that in quantum mechanics this force, for a single particle, is not contrasted by a corresponding repulsion due to the Coulomb field (as it happens, on the contrary, for an extended classical particle). In fact, the interaction of the electromagnetic field with non-relativistic matter is described in a simple way assuming the Coulomb gauge. In such a description, the Coulomb interaction is viewed as a direct instantaneous action among different particles, as e.g. in the hydrogen atom. No electrostatic self-interaction is assumed for a single particle. It is known, from the very beginning of wave mechanics, that such a kind of interaction, if present, would shift the energy eigenvalues of the hydrogen atom to physically wrong values [9]. Furthermore one can convince himself easily that, if absurdly such a kind of self-interaction should exist, the average energy of an atom would depend on the localization of its centre-of-mass. As a consequence of this fact e.g. the Van der Waals crystals could not exist.

Some comments are due about the symmetry-breaking processes involved in the localization effect. First of all a break-down of the translational invariance is involved. Physically it can be explained by the following argument. A physical electron, in a given state of momentum \( p \) and helicity \( s \), is represented by the minimal-energy state belonging to the sector corresponding to charge \(-e\), total momentum \( p \) and helicity \( s \) of the Hilbert space describing the total system (consisting of bare electron plus field). From this point of view it is obvious that, as a consequence of the interaction, the momentum must be shared between the field and the bare electron. One can say in other words that the momentum density does not coincide with the charge density, since part of the momentum is due to the field, which is neutral. This implies a localization for the charge.

From the point of view of special relativity, a more intriguing consequence stems from the velocity dependence of the localization effect. In fact a non-relativistic model, like the one analysed in this paper, should be imagined as a low-energy limit of some hypothetical Lorentz invariant one. From such a point of view it is evident that, as the spontaneous localization effect brakes down the translational invariance, so the velocity dependence of the effect can be interpreted as an indication for a spontaneous breakdown of the Lorentz invariance. On the other hand it is a known result, in relativistic QED, that the Lorentz invariance is broken in any charged sector of the Hilbert space (see e.g. Ref. [10]). In this context the result obtained in this paper appears as a confirmation, in a non-relativistic situation, of the theorem quoted above. The question arises if this result represents a mathematical strangeness only, or if it can lead really to possible physical observations conflicting with special relativity. A possible loophole to save the Lorentz invariance may be the following one. It may be possible that repeated preparations of electrons would in practice prepare different states, in which (partially) localized electrons are randomly distributed over the space region which supports the usual (non-localized) wave-function (a similar situation is assumed in Bohmian mechanics). In such a case the Lorentz invariance would not be violated by direct physical observations. However we remark that simple energetical considerations, like the ones developed in this paper, are inadequate to clarify this point completely.

Nevertheless, effects due to the localization, or more generally to the interaction with the self-generated coherent field, could be observed indirectly, through dynamical effects like e.g. spin dynamics, or more directly in condensed matter, where a preferred reference frame exists [11].

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