The Two-Loop Euler-Heisenberg Lagrangian in Dimensional Renormalization

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Abstract

We clarify a discrepancy between two previous calculations of the two-loop QED Euler-Heisenberg Lagrangian, both performed in proper-time regularization, by calculating this quantity in dimensional regularization.

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One of the earliest results in quantum electrodynamics was Euler and Heisenberg’s calculation of the one-loop effective Lagrangian induced by an electron loop for an electromagnetic background field with constant field strength tensor $F_{\mu\nu}$. Written in Schwinger’s proper-time representation, this Lagrangian reads

$$L^{(1)}_{\text{spin}}[F] = -\frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2T} \frac{(eaT)(ebT)}{\tan(eaT) \tanh(ebT)}. \quad (1)$$

Here $a$ and $b$ may be expressed in terms of the two invariants of the electromagnetic field,

$$a^2 = \frac{1}{2} \left[ E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \right]$$
$$b^2 = \frac{1}{2} \left[ -(E^2 - B^2) + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \right]. \quad (2)$$

Schwinger also supplied the corresponding result for scalar quantum electrodynamics,

$$L^{(1)}_{\text{scal}}[F] = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2T} \frac{(eaT)(ebT)}{\sin(eaT) \sinh(ebT)}. \quad (3)$$

Even the first radiative correction to the Euler-Heisenberg Lagrangian, due to the exchange of one internal photon in the electron loop, has been obtained many years ago by Ritus in terms of a two-parameter integral.

Dittrich and one of the authors later obtained a similar but simpler integral representation for the same quantity. They also verified the agreement of both representations in the strong field limit.

In a recent publication, three of the present authors showed that this type of calculation can be considerably simplified using the worldline path integral variant of the Bern-Kosower formalism. This third calculation led to exactly the same parameter integral for the regularized effective Lagrangian as the Dittrich-Reuter calculation, however in a more elegant way.

All three calculations were performed in four dimensions, proper-time regularization, and on-shell renormalization. This choice of regularization keeps the integrations simple, but at the two-loop level makes it already somewhat nontrivial to achieve a consistent on-shell renormalization. The difficulty is due to the non-universal nature of the proper-time cutoff, and somewhat similar to the problems encountered in multiloop Feynman diagrams calculations performed with a naive momentum space cutoff (see and references therein).

In fact, the calculation in was incomplete in so far as we were not able there to determine the finite part of the one-loop mass displacement appropriate to the present calculational scheme. Moreover, as a byproduct of this investigation we found that the formulas obtained by Ritus and Dittrich-Reuter for the renormalized effective Lagrangians are incompatible as they stand, and for the very same reason; if at all, they can be identified only after a certain finite mass renormalization. One had to conclude that in (at least) one of the two previous calculations the physical renormalized electron mass had been misidentified (the strong field limit checked in is not sensitive to this discrepancy).

In the present letter, we clarify this matter by recalculating this effective Lagrangian in dimensional regularization. This paper should thus be seen as supplementing, and the reader is referred to that publication for some of the details of the formalism used here.
As usual in applications of the “string-inspired” technique to QED [12], our calculation of the two-loop Euler-Heisenberg Lagrangian for spinor QED will yield the corresponding Lagrangian for scalar QED as a partial result. We therefore consider the scalar QED case first.

In scalar QED, our starting point is the following worldline path integral representation for the bare dimensionally regularized two-loop Euler-Heisenberg Lagrangian [5, 9, 10],

\[ \mathcal{L}_{\text{scal}}^{(2)}[A] = -\frac{e^2}{2} \frac{\Gamma(\frac{D}{2} - 1)}{4\pi^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} e^{-m^2T} \int_0^T d\tau_a \int_0^T d\tau_b \]
\[ \times \int \mathcal{D}y \frac{\dot{y}(\tau_a) \cdot \dot{y}(\tau_b)}{(y(\tau_a) - y(\tau_b))^2} \exp \left[ -\int_0^T d\tau \left( \frac{1}{4} \dot{y}^2 + ie \gamma^\mu A_\mu(x_0 + y) \right) \right] \]  

(4)

Here \( T \) denotes the usual Schwinger proper-time parameter for the scalar circulating in the loop. The path integral \( \mathcal{D}y \) is to be performed over the space of all closed loops in \( D \)-dimensional spacetime, with an arbitrary but fixed center of mass \( x_0 \), and traversed in the fixed proper-time \( T \). The parameters \( \tau_{a,b} \) parametrize the end-points of the photon exchanged in the scalar loop. This form of the photon insertion corresponds to Feynman gauge. We use euclidean conventions both on the worldline and in spacetime.

This representation of the effective action in terms of a first-quantized path integral goes essentially back to Feynman [11], except that we have dimensionally continued it to \( D \) dimensions.

It is useful to take the background field \( A \) in Fock-Schwinger gauge centered at the loop center of mass \( x_0 \), where one has \( A_\mu(x_0 + y) = \frac{1}{2} y^\mu F_{\mu\nu} \). Exponentiating the denominator of the photon insertion using a proper-time parameter \( \bar{T} \), one arrives at

\[ \mathcal{L}_{\text{scal}}^{(2)[F]} = -\frac{e^2}{2} \int_0^\infty \frac{dT}{T} e^{-m^2T} \int_0^\infty d\bar{T} (4\pi\bar{T})^{-\frac{D}{2}} \int_0^\bar{T} d\tau_a \int_0^\bar{T} d\tau_b \]
\[ \times \int \mathcal{D}y \hat{y}_a \cdot \hat{y}_b \exp \left[ -\int_0^\bar{T} d\tau \left( \frac{1}{4} \dot{y}^2 + i \frac{e}{2} y^\mu F_{\mu\nu} y^\nu \right) - \frac{(y_a - y_b)^2}{4 \bar{T}} \right] \]  

(5)

The new path integral is gaussian, so that its evaluation amounts to a single Wick contraction of \( \langle \hat{y}_a \cdot \hat{y}_b \rangle \). This leads to

\[ \mathcal{L}_{\text{scal}}^{(2)[F]} = -(4\pi)^{-\frac{D}{2}} \frac{e^2}{2} \int_0^\infty \frac{dT}{T} e^{-m^2T} T^{-\frac{D}{2}} \int_0^T d\bar{T} \int_0^T d\tau_a \int_0^T d\tau_b \]
\[ \times \det -\frac{i}{2} \left[ \sin(eF \bar{T}) \right] \det -\frac{i}{2} \left[ \bar{T} - \frac{1}{2} C_{ab} \right] \langle \hat{y}_a \cdot \hat{y}_b \rangle \]  

(6)

Of the (Lorentz) determinant factors appearing here, the first one yields just the one-loop Euler-Heisenberg-Schwinger integrand eq. [3]. It represents the dependence of the path integral determinant on the external field [3], while the second determinant factor represents its dependence on the photon insertion. The Wick contraction is given by

\[ \langle \hat{y}_a \cdot \hat{y}_b \rangle = \text{tr} \left[ \hat{G}_{Bab} + \frac{1}{2} \left( \hat{G}_{Baa} - \hat{G}_{Bab} (\hat{G}_{Bab} - \hat{G}_{Bbb}) \right) \right] \]  

(7)
where
\[ G_B(\tau_1, \tau_2) = \frac{1}{2(eF)^2} \left( \frac{eF}{\sin(eFT)} e^{-ieFT\dot{G}_{B12}} + ieF\dot{G}_{B12} - \frac{1}{T} \right) \] (8)
is the bosonic one-loop worldline Green’s function modified by the constant field \([5, 17, 18, 19]\), and \( C_{ab} \equiv G_{Baa} - G_{Bab} - G_{Bba} + G_{BBb} \). \( G_B \) generalizes the ordinary worldline Green’s function \( G_B \),
\[ G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}. \] (9)
We will often abbreviate \( G_{B12} \equiv G_B(\tau_1, \tau_2) \) etc., and a “dot” always denotes a derivative with respect to the first variable, e.g. \( \dot{G}_{B12} = \text{sign}(\tau_1 - \tau_2) - 2\frac{(\tau_1 - \tau_2)}{T} \).

Performing a partial integration with respect to \( \tau_a \) on the first term in eq. (7) one can derive the alternative parameter integral
\[
L^{(2)}_{\text{scal}}[F] = -(4\pi)^{-D} \int_0^\infty d\tau_1 \frac{d\tau_2}{T} e^{-m^2T} T^{-\frac{D}{2}} \int_0^\infty d\tau_a \int_0^T d\tau_b \left[ \det^{-\frac{1}{2}} \left[ \sin(eFT) \right] \det^{-\frac{1}{2}} \left[ \frac{\dot{G}_{Bab}}{T - \frac{1}{2} C_{ab}} \right] + \frac{1}{2} \left\{ \text{tr} \dot{G}_{Bab} \text{tr} \left[ \frac{\dot{G}_{Bab}}{T - \frac{1}{2} C_{ab}} \right] + \text{tr} \left[ \frac{(\dot{G}_{Baa} - \dot{G}_{Bab})(\dot{G}_{Bab} - \dot{G}_{BBb})}{T - \frac{1}{2} C_{ab}} \right] \right\} \right].
\] (10)
For our present purpose, we can restrict ourselves to the pure magnetic field case. This field we take along the z-axis, so that \( F^{12} = B, F^{21} = -B \) are the only non-vanishing components of the dimensionally continued field strength tensor.

We also introduce the following abbreviations,
\[
\begin{align*}
z & \equiv eBT \\
\gamma & \equiv (T + G_{Bab})^{-1} \\
\gamma^z & \equiv (T + G^z_{Bab})^{-1}
\end{align*}
\]
and the \( z \) – dependent Green’s function \( G^z_{Bab} \),
\[
\begin{align*}
G^z_{Bab} & = \frac{T}{2} \left[ \cosh(z) - \cosh(z\dot{G}_{ab}) \right] = G_{Bab} - \frac{1}{3T} G^2_{Bab} z^2 + O(z^4) \tag{11} \\
\dot{G}^z_{Bab} & = \frac{\sinh(z\dot{G}_{Bab})}{\sinh(z)} = \dot{G}_{Bab} - \frac{2}{3T} \dot{G}_{Bab} G_{Bab} z^2 + O(z^4) \tag{12}
\end{align*}
\]
(compare eq. (8)).

With these definitions, we can then rewrite the various traces and determinants appearing in eqs. (5), (10) as
\[ \det \left[ \frac{\sin(eFT)}{eFT} (T - \frac{1}{2} \delta_{ab}) \right] = \frac{z}{\sinh(z)} \gamma^2 - 1 \gamma^z \]

\[ \text{tr} \left[ \tilde{G}_{Bab} \right] = 2D \delta(\tau_a - \tau_b) - 2(D - 2) \frac{1}{T} - \frac{4}{T} \frac{z \cosh(z \tilde{G}_{Bab})}{\sinh(z)} \]

\[ \frac{1}{2} \text{tr} \tilde{G}_{Bab} \text{tr} \left[ \frac{\tilde{G}_{Bab}}{T - \frac{1}{2} \delta_{ab}} \right] = \frac{1}{2} \left[ (D - 2) \tilde{G}_{Bab} + 2 \tilde{G}_{Bab}^2 \right] \left[ (D - 2) \tilde{G}_{Bab} \gamma + 2 \tilde{G}_{Bab}^2 \gamma^2 \right] \]

\[ \frac{1}{2} \text{tr} \left[ \frac{\tilde{G}_{ab} - \tilde{G}_{ab}}{T - \frac{1}{2} \delta_{ab}} \right] = -\frac{1}{2} (D - 2) \tilde{G}_{Bab}^2 \gamma - \left[ \tilde{G}_{Bab}^2 + \frac{4}{T^2} \gamma^2 \right] \gamma^z. \] (13)

The term involving \( \delta(\tau_a - \tau_b) \), stemming from \( \tilde{G}_{Bab} \), can be omitted, since it will not contribute in dimensional regularization (it corresponds to a massless tadpole insertion in field theory).

Inserting these expressions into either eq. (6) or eq. (10), one finds that the resulting integrals suffer from two kinds of divergences:

1. An overall divergence of the scalar proper-time integral \( \int_0^\infty dT \) at the lower integration limit.

2. Divergences of \( \int_0^T d\tau_a \int_0^T d\tau_b \) at the point \( \tau_a = \tau_b \), where the photon end points become coincident.

The first one must be removed by one- and two-loop photon wave function renormalization, the second one by the one-loop renormalization of the scalar mass.

It turns out that this program is easier to carry out on a certain linear combination of eqs. (6) and (10), namely

\[ \mathcal{L}_{\text{scal}}^{(2)}[B] = \frac{D - 1}{D} \times \text{eq.(6)} + \frac{1}{D} \times \text{eq.(10)} \] (14)

(A similar simplification can be achieved by taking the photon insertion in Landau gauge, though the resulting parameter integrals are not identical). Moreover, we rescale to the unit circle, \( \tau_a, b = T u_a, b \), \( \hat{T} = T \hat{T} \), and use translation invariance in \( \tau \) to set \( \tau_b = 0 \). Thus in the following we have \( G_{Bab} = u_a(1 - u_a), \tilde{G}_{Bab} = 1 - 2u_a \).

The resulting integral we write

\[ \mathcal{L}_{\text{scal}}^{(2)}[B] = -\frac{(4\pi)^{-D}e^2}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} T^{2-D} \int_0^\infty d\hat{T} \int_0^1 du_a I(z, u_a, \hat{T}, D) \] (15)

where the rescaled integrand \( I(z, u_a, \hat{T}, D) \) depends on \( T \) only through \( z \).

In contrast to the calculation in proper-time regularization, the \( \hat{T} \) – integration is nontrivial in dimensional regularization. It will therefore be easier to extract all subdivergences before performing this integral. An analysis of the divergence structure shows that the integrand can be rewritten in the following way,

\[ K(z, u_a, D) \equiv \int_0^\infty d\hat{T} I(z, u_a, \hat{T}, D) = K_{02}(z, u_a, D) + f(z, D)G_{Bab}^{1 - \frac{D}{2}} + O(z^4, G_{Bab}^{2 - \frac{D}{2}}) \] (16)
with

\[ K_{02}(z, u_a, D) = -4 \frac{D - 1}{D - 2} G_{Bab}^{-1/2} + \frac{2}{3D(D - 2)} \left[ (D - 1)(D - 4) G_{Bab}^{1/2} + (-2D^2 + 18D - 4) G_{Bab}^{2 - 2/7} \right] z^2 \]

\[ f(z, D) = \frac{D - 1}{D(D - 2)} \left[ 4D - \frac{2}{3} (D - 4) z^2 + (8 - 4D) \frac{z}{\sinh(z)} - 8 \frac{z^2 \cosh(z)}{\sinh^2(z)} \right] = O(z^4). \] (17)

\[ K_{02} \] consists of the terms constant and quadratic in \( z \), which are the only ones causing a divergence at \( T = 0 \). The second term is \( O(z^4) \), so that its integral already converges at \( T = 0 \), however it diverges at \( u_a = 0 \).

After splitting off these two terms, the integral over the remainder is already finite, so that one can set \( D = 4 \) in its explicit computation. For \( D = 4 \) the \( \hat{T} \)-integral becomes elementary, and yields

\[
K(z, u_a, 4) = \frac{z}{\sinh(z)} \left\{ A_0 \ln \left( \frac{G_{Bab}/G_{Bab}^*}{G_{Bab} - G_{Bab}^*} \right) + A_1 \ln \left( \frac{G_{Bab}/G_{Bab}^*}{G_{Bab} - G_{Bab}^*} \right) \right. \\
+ \left. \frac{A_2}{G_{Bab}^*} \right( \frac{G_{Bab} - G_{Bab}^*}{G_{Bab}^*} \right) + \frac{A_3}{G_{Bab}^*} \right\},
\
A_0 = 3 \left[ 2z^2 G_{Bab}^2 - \frac{z}{\tanh(z)} - 1 \right],
A_1 = 4z^2 G_{Bab}^2 + \frac{1}{2} (\dot{G}_{Bab} - \dot{G}_{Bab}^*)
A_2 = -4z^2 G_{Bab}^2 + \frac{1}{2} \dot{G}_{Bab}^2 (\dot{G}_{Bab} - \dot{G}_{Bab}^*)
A_3 = \frac{1}{2} \dot{G}_{Bab} (\dot{G}_{Bab} - \dot{G}_{Bab}^*). \] (18)

The divergences will now be removed by mass and photon wave function renormalization,

\[
m^2 = m_0^2 + \delta m_0^2
\]
\[
e = e_0 Z_3^{\frac{1}{3}}
\]
\[
B = B_0 Z_3^{-\frac{1}{3}}. \] (19)

So far we have worked in the bare regularized theory, so that all our previous formulas should, for the following, be considered written in terms of \( m_0, e_0, B_0 \) instead of \( m, e, B \) (note that this leaves \( z \) unaffected).

Since we aim at a direct comparison with previous calculations, the renormalization will be done using on-shell rather than minimal subtraction. In on-shell subtraction, the photon wave function renormalization has the effect of simply removing the \( z^2 \)-part of \( K_{02} \), and the remaining \( z \)-independent term can, of course, be also discarded.

The removal of the divergence caused by the second term in eq. (16) takes more effort, and the mere possibility requires a little conspiration.

Let us denote the corresponding contribution to the effective Lagrangian by \( G_{scal}(z, D) \). For this term the \( u_a \)-integration factors out, yielding
\[
\int_0^1 du G_{\text{Bab}}^{1/2} = B \left(2 - \frac{D}{2}, 2 - \frac{D}{2}\right) = -\frac{4}{\epsilon} + O(\epsilon)
\]  
(20)

where \(B\) denotes the Euler Beta-function.

To proceed further, it is essential to note that the function \(f(z, D)\) can be related to the integrand of the scalar one-loop Euler-Heisenberg Lagrangian, eq. (3). In dimensional regularization, and with the two terms lowest order in \(z\) subtracted out via one-loop photon wave function renormalization, this Lagrangian reads

\[
\bar{L}^{(1)}_{\text{scal}}[B_0] = \int_0^\infty \frac{dT}{T} e^{-m_0^2 T} (4\pi T)^{-\frac{D}{2}} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right].
\]  
(21)

On the other hand, we can rewrite

\[
f(z, D) = 8 \left(\frac{D - 1}{D(D - 2)}\right) T^{D + 1} \frac{d}{dT} \left\{ T^{-\frac{D}{2}} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \right\}. 
\]  
(22)

By a partial integration over \(T\), we can therefore reexpress

\[
\int_0^\infty \frac{dT}{T} e^{-m_0^2 T} T^{2-D} f(z, D) = 8 \left(\frac{D - 1}{D(D - 2)}\right) m_0^2 \int_0^\infty \frac{dT}{T} e^{-m_0^2 T} T^{3-D} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right]
\]
\[+ \frac{D - 4}{2} \int_0^\infty \frac{dT}{T} e^{-m_0^2 T} T^{2-D} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \}
(23)

(there are no boundary terms since \(f(z) = O(z^4)\)).

At the two-loop level, the effect of mass renormalization consists in the following shift produced by the one-loop mass displacement \(\delta m_0^2\),

\[
\delta \mathcal{L}^{(2)}_{\text{scal}}[B_0] = \delta m_0^2 \frac{\partial}{\partial m_0^2} \bar{L}^{(1)}_{\text{scal}}[B_0].
\]  
(24)

\(\delta m_0^2\) is generated by the UV divergence of the one-loop scalar self energy in scalar QED. This quantity we have to take from standard field theory. In dimensional regularization one has

\[
\delta m_0^2 = m_0^2 \frac{\alpha_0}{4\pi} \left[ \frac{6}{\epsilon} + 7 - 3[\gamma - \ln(4\pi)] - 3\ln(m_0^2) \right] + O(\epsilon).
\]  
(25)

Here \(\epsilon = D - 4\), and \(\gamma\) denotes the Euler-Mascheroni constant. Expanding eqs. (20), (23), and (24) in \(\epsilon\) one finds that, up to terms of order \(O(\epsilon)\),

\[
G_{\text{scal}}(z, D) = \delta m_0^2 \frac{\partial}{\partial m_0^2} \bar{L}^{(1)}_{\text{scal}}[B_0] + m_0^2 \frac{\alpha_0}{(4\pi)^3} \int_0^\infty \frac{dT}{T^2} e^{-m_0^2 T} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right]
\]
\[\times \left[ -3\gamma - 3\ln(m_0^2 T) + \frac{3}{m_0^2 T} + \frac{9}{2} \right]. 
\]  
(26)

\(^1\)Note that this differs by a sign from \(\delta m^2\) as used in \([1]\). Here this denotes the mass displacement itself, there the corresponding counterterm.

\(^2\)In comparing with \([3, 4, 5, 20]\) note that there this constant had been denoted by \(\ln(\gamma)\).
Note that the whole divergence of $G_{\text{scal}}(z, D)$ for $D \to 4$ has now been absorbed into $\delta m^2_0$.

Our final answer for the two-loop contribution to the finite renormalized scalar QED Euler-Heisenberg thus becomes

\[ L_{\text{scal}}^{(2)}[B] = -\frac{\alpha}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \int_0^1 du \left[ K(z, u_a, 4) - K_02(z, u_a, 4) - \frac{f(z, 4)}{G_{\text{Bab}}} \right] \]

\[ + \frac{\alpha}{(4\pi)^3} m^2 \int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \left[ -3\gamma - 3\ln(m^2 T) + \frac{3}{m^2 T} + \frac{9}{2} \right]. \]

(27)

As far as is known to the present authors, the only previous calculation of the two-loop Euler-Heisenberg for scalar QED is the one in [20, 21]. The parameter integral given there is rather different from ours, and we have not succeeded in directly identifying both representations. However, we have used MAPLE to expand both formulas in a Taylor expansion in $B$ up to order $O(B^{20})$, and found exact agreement for the coefficients.

Let us just give the first few terms in this expansion,

\[ L_{\text{scal}}^{(2)}[B] = \frac{\alpha m^4}{(4\pi)^3} \left[ \frac{275}{8} B_{\text{cr}}^4 - \frac{5159}{200} B_{\text{cr}}^6 + \frac{2255019}{39200} B_{\text{cr}}^8 - \frac{931061}{3600} B_{\text{cr}}^{10} + \ldots \right]. \]

(28)

The expansion parameter has been rewritten in terms of $B_{\text{cr}} \equiv \frac{m^2}{e} \approx 4.4 \cdot 10^{13} G$.

The corresponding calculation for spinor QED is completely analogous. In the worldline superfield formalism of [3, 4, 22], the formulas (3), (7) generalize to the following integral representation for the two-loop effective action induced by the spinor loop,

\[ L_{\text{spin}}^{(2)}[F] = (-2)(4\pi)^{-D} \left( \frac{e^2}{2} \right) \int_0^\infty \frac{dT}{T} e^{-m^2 T} T^{-\frac{D}{2}} \int_0^\infty dT \int_0^T d\tau_a d\tau_b \int d\theta_a d\theta_b \]

\[ \times \det \left[ \frac{\tan(eFT)}{eFT} \right] \det \left[ \frac{T - \frac{1}{2} \hat{G}_{ab}}{T} \right] \langle -D_a y_a \cdot D_b y_b \rangle, \]

\[ \langle -D_a y_a \cdot D_b y_b \rangle = \text{tr} \left[ D_a D_b \hat{G}_{ab} + \frac{1}{2} D_a (\hat{G}_{aa} - \hat{G}_{ab}) D_b (\hat{G}_{ab} - \hat{G}_{bb}) \right] \left[ T - \frac{1}{2} \hat{G}_{ab} \right]. \]

(29)

The $\hat{G}$ appearing here is the constant field worldline superpropagator,

\[ \hat{G}(\tau_1, \theta_1; \tau_2, \theta_2) \equiv G_B(\tau_1, \tau_2) + \theta_1 \theta_2 G_F(\tau_1, \tau_2) \]

(30)

which besides the bosonic propagator eq. (8) also contains a fermionic piece,

\[ G_F(\tau_1, \tau_2) = \text{sign}(\tau_1 - \tau_2) \frac{e^{-i eFT \theta_{12}}}{\cos(eFT)}. \]

(31)

Our superfield conventions are $D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \tau}, \int d\theta = 1$.

Performing the Grassmann integrations, and removing $\hat{G}_{\text{Bab}}$ by partial integration over $\tau_a$, we obtain the equivalent of eq. (10),
\[ L_{\text{spin}}^{(2)}[F] = (4\pi)^{-D} e^2 \int_{0}^{\infty} \frac{dT}{T} e^{-m^2 T} T^{\frac{D-1}{2}} \int_{0}^{\infty} d\tilde{T} \int_{0}^{T} d\tau_a \int_{0}^{\tilde{T}} d\tau_b \]
\[ \times \det \left[ \frac{\tan(eFT)}{eFT} \left( \frac{\tilde{T}}{2} \right) \right] \left[ \frac{1}{2} \right] \left\{ \text{tr} \hat{G}_{Bab} \text{tr} \left[ \frac{\hat{G}_{Bab}}{T - \frac{1}{2} C_{ab}} \right] - \text{tr} \hat{G}_{Fab} \text{tr} \left[ \frac{\hat{G}_{Fab}}{T - \frac{1}{2} C_{ab}} \right] \right\} + \text{tr} \left[ \left( \hat{G}_{Baa} - \hat{G}_{Bab} \right) \left( \hat{G}_{Bab} - \hat{G}_{Bbb} + 2 \hat{G}_{Faa} + \hat{G}_{Fab} \hat{G}_{Fab} - \hat{G}_{Faa} \hat{G}_{Fbb} \right) \right]. \] (32)

Note that this formula reduces to eq. (10), if one replaces \( \tan(eFT) \) by \( \sin(eFT) \), and deletes all the \( \hat{G}_F \), as well as the global factor of \(-2\) (which accounts for the difference in statistics and degrees of freedom between the spin 0 and spin \( \frac{1}{2} \) loops).

Contrary to the scalar QED case, here the partially integrated integral is already a suitable starting point for renormalization (for more on this point see chapter 7 of [5]).

Specializing to the magnetic field case, it is again easy to calculate the Lorentz determinants and traces. After rescaling to the unit circle, one obtains a parameter integral

\[ L_{\text{spin}}^{(2)}[B] = (4\pi)^{-D} e^2 \int_{0}^{\infty} \frac{dT}{T} e^{-m^2 T} T^{2-D} \int_{0}^{\infty} d\tilde{T} \int_{0}^{1} du_a J(z, u_a, \tilde{T}, D). \] (33)

The extraction of the subdivergences yields

\[ L(z, u_a, D) \equiv \int_{0}^{\infty} d\tilde{T} J(z, u_a, \tilde{T}, D) = L_{02}(z, u_a, D) + g(z, D)G_{Bab}^{1-\frac{D}{2}} + O(z^4, G_{Bab}^{2-\frac{D}{2}}) \] (34)

with

\[ L_{02}(z, u_a, D) = -4(D-1)G_{Bab}^{1-\frac{D}{2}} - \frac{4}{3D} \left[ (D-1)(D-4)G_{Bab}^{1-\frac{D}{2}} + (D-2)(D-7)G_{Bab}^{2-\frac{D}{2}} \right] z^2 \]
\[ g(z, D) = - \frac{4}{3} \frac{D-1}{D} \left[ 6 \frac{z^2}{\sinh^2(z)} + 3(D-2)z \coth(z) - (D-4)z^2 - 3D \right] = O(z^4). \] (35)

\( L_{02} \) is again removed by photon wave function renormalization. Denoting the contribution of the second term by \( G_{\text{spin}}(z, D) \), we note that the \( u_a \) - integral is the same as in the scalar QED case, eq. (27). Using the following identity analogous to eq. (22),

\[ g(z, D) = 8 \frac{D-1}{D} T^{D-1} \frac{d}{dT} \left[ T^{-\frac{D}{2}} \left( \frac{z}{\tanh(z)} - \frac{z^2}{3} - 1 \right) \right] \] (36)

we partially integrate the remaining integral over \( T \). The \( \frac{1}{\epsilon} \) - part of \( G_{\text{spin}} \) is then again found to be just right for absorbing the shift induced by the one-loop mass displacement,

\[ \delta m_0 = m_0 \frac{\alpha_0}{4\pi} \left[ - \frac{6}{\epsilon} + 4 - 3[\gamma - \ln(4\pi)] - 3\ln(m_0^2) \right] + O(\epsilon). \] (37)

Up to terms of order \( \epsilon \) one obtains
\[ G_{\text{spin}}(z, D) = \delta m_0 \frac{\partial}{\partial m_0} L_{\text{spin}}^{(1)}[B_0] + m_0^2 \frac{\alpha_0}{(4\pi)^3} \int_0^\infty \frac{dT}{T^2} e^{-m_0^2 T} \left( \frac{z}{\tanh(z)} - \frac{z^2}{3} - 1 \right) \times \left[ 12\gamma + 12 \ln(m_0^2 T) - \frac{12}{m_0^2 T} - 18 \right]. \]  

(38)

Our final result for the on-shell renormalized two-loop spinor QED Euler-Heisenberg Lagrangian is

\[ L_{\text{spin}}^{(2)}[B] = \frac{\alpha}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \int_0^1 du_a \left[ L(z, u_a, 4) - L_{02}(z, u_a, 4) - \frac{g(z, 4)}{G_{Bab}} \right] \]

\[ - \frac{\alpha}{(4\pi)^3} m_0^2 \int_0^\infty \frac{dT}{T^2} e^{-m_0^2 T} \left[ \frac{z}{\tanh(z)} - \frac{z^2}{3} - 1 \right] \left[ 18 - 12\gamma - 12 \ln(m_0^2 T) + \frac{12}{m_0^2 T} \right] \]

(39)

with

\[ L(z, u_a, 4) = \frac{z}{\tanh(z)} \left\{ B_1 \frac{\ln(G_{Bab}/G_{Bab}^2)}{(G_{Bab} - G_{Bab}^2)} + \frac{B_2}{G_{Bab}(G_{Bab} - G_{Bab}^2)} + \frac{B_3}{G_{Bab}(G_{Bab} - G_{Bab}^2)} \right\} \]

\[ B_1 = 4z \left( \coth(z) - \tanh(z) \right) G_{Bab}^2 - 4G_{Bab} \]

\[ B_2 = 2G_{Bab} G_{Bab}^2 + z(8 \tanh(z) - 4 \coth(z)) G_{Bab}^2 - 2 \]

\[ B_3 = 4G_{Bab} - 2G_{Bab} G_{Bab}^2 - 4z \tanh(z) G_{Bab}^2 + 2 \]

\[ L_{02}(z, u_a, 4) = -\frac{12}{G_{Bab}} + 2z^2 \]

\[ g(z, 4) = -6 \left[ \frac{z^2}{\sinh(z)^2} + z \coth(z) - 2 \right]. \]

(40)

Comparing with the previous results by Ritus and Dittrich-Reuter, we have again not succeeded in a direct identification with the more complicated parameter integral given by Ritus [3]. However, as in the scalar QED case we have verified agreement between both formulas up to the order of \(O(B^{20})\) in the weak-field expansion in \(B\). The first few coefficients are

\[ L_{\text{spin}}^{(2)}[B] = \frac{\alpha m^4}{(4\pi)^3} \frac{1}{81} \left[ 64 \left( \frac{B}{B_{\text{cr}}} \right)^4 - \frac{1219}{25} \left( \frac{B}{B_{\text{cr}}} \right)^6 + \frac{135308}{1225} \left( \frac{B}{B_{\text{cr}}} \right)^8 - \frac{791384}{1575} \left( \frac{B}{B_{\text{cr}}} \right)^{10} + \ldots \right]. \]

(41)

On the other hand, our formula almost allows for a term by term identification with the result of Dittrich-Reuter [4], as given in eqs. (7.21),(7.22) there. This requires a rotation to Minkowskian proper-time, \(T \rightarrow is\), a transformation of variables from \(u_a\) to \(v := \tilde{G}_{Bab}\), the use of trigonometric identities, and another partial integration over \(T\) for the last two terms in eq. (39). The only discrepancy arises in the constant 18, which reads 10 in the Dittrich-Reuter formula.

Since this constant can be adjusted by a change of the finite constant appearing in \(\delta m_0\), we conclude that the two previous results for this effective Lagrangian differ precisely by a finite
mass renormalization. Moreover, it is clear that Ritus’ formula is the one which correctly identifies the physical electron mass.

Finally, let us mention that the one-loop Euler-Heisenberg Lagrangian, and perhaps even its two-loop correction considered here, may possibly be measured in optical experiments in the near future [23].

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