Determination of the correlation dimension of an attractor in a pipe based on the theory of stochastic equations and equivalence of measures

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Abstract. Scientific discovery of the physical law of the equivalence of measures and the system of the stochastic equations for the determination onset of turbulence in isothermal and non-isothermal flows has created the new way in stochastic theory of turbulence. It was shown that derived new formulas for critical Reynolds numbers and critical point, for the velocity and temperature profiles, for the second-order correlation, heat transfer and friction coefficients give satisfactory agreement with the classical experimental data for these flow characteristics. Using these new dependences the new formula and result of calculation of the correlation dimension of the attractor for the flow in the tube are presented.

1. Introduction

The study of the experimental and mathematical basis of the phenomenon of turbulence [1–15] have allowed to create the new theory of equivalent measures and stochastic equations [16–31], which made possible using this theoretical basis to determine most important parameters of the different turbulent flows: critical numbers, profiles of regular and random characteristics, correlation functions, distribution functions in determining the formation of the corresponding spectrum for a continuous medium, and a stochastic correlation tensor between the regular and random components. Based on the experimental data and on the basis of the existing statistical concepts of turbulence and conclusions derived from the statistical theory, was done the conclusion there is an unconditional requirement, that continuous spectrum of turbulent energy is formed under the condition an infinite number of frequencies, but under condition, that the turbulent energy is finite.

We emphasize that the need for such a theoretical estimate and the approximate ratio of the estimate of the number of degrees of freedom, from dimension considerations, was determined even in the papers of Landau [32], but quantitative estimates were not obtained after. In addition, this estimate was not determined depending on the initial parameters in the region of turbulence generation.

Therefore, on the basis of the solutions obtained with stochastic consideration of the turbulence problem and on the basis of the theory of measure equivalence, it seems reasonable to give an estimate of the number of excited frequencies (degrees of freedom) when the equivalence of the measures occurs, that is, to determine the correlation dimension at the space-time point under study.
Theoretical and experimental methods for estimating the dimension of an attractor in dynamical systems are presented mainly in [33–43]. Along with this, the dimension is determined both as a result of the numerical integration of the differential equations used to describe the corresponding dynamical systems [11,42], and through theoretical studies of directly differential equations that give mathematical estimates of the dimension of the attractor [34–36]. A widely used experimental algorithm for determining the dimension of an attractor is presented in [37–41].

In the study of the turbulence process in hydrodynamic flows, experimental determination of the dimension was carried out, in particular, in [38–40]. On the basis of numerical integration of the system of Navier-Stokes equations, the calculation of the attractor dimension can be carried out in accordance with [43]. However, the existing methods and the resulting relationships for determining the dimension of the attractor, in essence, require in fact the repeated carrying out of the numerous and laborious experimental and theoretical studies of hydrodynamic turbulence that have been carried out in the last 70 years.

2. The dimension of an attractor

Basically, the works for this period are based on the provisions of statistical mechanics, in particular on the hypothesis of the ergodicity, and define along with the averaged fields of velocity, temperature and concentration the corresponding correlation functions and correlation moments of the second and higher order [29]. The theoretical-computational methods developed in this period, in particular RANS, LES, made it possible to predict, with sufficient accuracy, various types of turbulent flows [29,44–52]. In this connection, the estimation of the attractor dimension using the correlation moments which already obtained in the experimental and theoretical calculations, will allow to expand the possibilities for qualitative and quantitative representation of the dimension in various hydrodynamic flows. In [34, 35] an estimate is given for the dimension of the attractor for the case of the two-dimensional Navier-Stokes equation for zero or periodic boundary conditions. However, the resulting relations represent a dimensional bound on the bottom or on the top of the attractor in the case of Hausdorff’s dimension:

\[ d_H \leq K (\nu)^{-1/n} \] 

(1)

where K and n are the constant and exponent, respectively, having different values depending on the boundary conditions, and the values of K remain undefined; \( \nu \) is the kinematic viscosity. In this connection it is more expedient to consider the expressions obtained in [32,33]. These papers give estimates of the number of degrees of freedom \( N \), Hausdorff’s \( d_H \), and fractal \( d_M \) dimensions, for which inequality.

\[ N \leq d_M \leq d_H \] 

(2)

Therefore, estimates of the dimension of the attractor \( d_H \) and \( d_M \) are understood as estimates from above of the value of the number of degrees of freedom \( N \) of dynamical systems. It was shown in [32] that the fractal dimension of a dynamical system is described as

\[ d_f \leq C_H (L/L_0)^{n_0} \] 

(3)

where \( C_H \) is a universal constant whose value in [7, 8] is not defined. The quantities \( L_0 \) and \( L_d \) in (3) are the turbulence scales of the energy-containing vortices and the Kolmogorov turbulence scale, respectively, the values of \( n_0 = 2, n_0 = 3 \) for the two- and three-dimensional cases of the Navier-Stokes equations, respectively. In [35], an estimate of the dimension \( d_H \) is also given, depending on the newly introduced Reynolds numbers, in which the numerical definition of a quantity having a velocity dimension is not trivial, for example

\[ d_H \leq C_4 (Re)^{\alpha} \] 

(4)
Here $C_{4}$ is the universal constant, whose value in [33, 35] is also not defined, and the value of the newly introduced Reynolds number is, where $L_{0}$, $\nu$ and $n_{0}$ are already mentioned quantities; $u$ is the maximum value of the velocity modulus on the attractor of the dynamical system under study. An analysis of the expressions for estimating the dimension of the attractor shows that the most promising is a dependence of the type (3), although, as will be shown below, this type (3) requires further refinement. The point is that the value $L_{0}$-scale of turbulence of energy-containing vortices, as shown in a number of works [29], performed on the basis of statistical hydrodynamics, is equal to the scale of the mixing path $L_{m}$.

In turn, the expression for $L_{m}$ is determined in a number of experimental and theoretical studies [29], depending on the type of hydrodynamic flow, which allows give a sufficiently accurate quantitative determination of $L_{0} = L_{m}$. The determination of the $L_{0} = (\nu^{3}/|\varepsilon|)^{1/4}$ -- Kolmogorov turbulence scale requires knowledge of the quantity $\varepsilon$. The dissipation of the turbulent kinetic energy $\varepsilon = \nu \frac{\partial u_{i}}{\partial x_{j}} \frac{\partial u_{j}}{\partial x_{i}}$ here $u_{i}, u_{j}$ are the pulsational components of the velocity vector $u$ in $i, j$ directions, the bar above denotes averaging over time in accordance with the hypothesis of the ergodicity of the hydrodynamic turbulence process here, $\rho$ and $\mu$-density and dynamic viscosity. As a result, the scale ratio in (3) can be written as

$$L_{0}/L_{m} = (L_{m}^{4} \varepsilon / \nu^{3})^{1/4}$$

(5)

Such a notation is not always convenient, since the dissipation value of the turbulent kinetic energy has not been determined experimentally for all cases of hydrodynamic flows, and in addition its definition is "experimentally and theoretically" "secondary" than the definition of turbulent kinetic energy and, accordingly, pulsational velocity components $u_{i}, u_{j}$. In addition, in the statistical modeling of turbulent flows, models already developed for second-order correlation moments exclude the simultaneous computation of the scale of turbulence and the magnitude of the dissipation from the corresponding transport equations [29, 44–52].

At the same time, in some models of turbulence [29] the scale of the mixing path (or the integral scale $L_{mix}$, $L_{m} = 0.75 L_{mix}$) and the magnitude of the turbulent kinetic energy are used. In this connection, it is more appropriate to write expression (5) in terms of $L_{m}$ and turbulent kinetic energy $e$ (or one of the components $(u')^{2}$). The dependence for the turbulent Reynolds number is $Re_{t} = \sqrt{u_{i} u_{j} L_{m} / \nu}$.

As a result, the Haussdorf’s dimension can be defined as the new formula

$$d_{H} \leq 1 + 1.2 \left[ C_{1}(Re_{t})^{2} + C_{0}(Re_{t}) \right]^{1/4}$$

(6)

This dependence can be considered as the upper limit of the number of degrees of freedom $N$ (the number of frequencies) at the considered point of the hydrodynamic flow, in case of the two-dimensional $n = 2$ or three-dimensional $n = 3$ flow. In this connection, it seems necessary to give a number of examples on the calculation of the Haussdorf’s dimension and the estimation of the number of degrees of freedom in various types of hydrodynamic flows using the experimental data already obtained for them. First of all, we note that the relations (1) – (6) include "uncertainty" with respect to the number of dimension "n" - that is, two or three-dimensional Navier-Stokes equations can describe the flow of the medium. This uncertainty does not seem problematic, based on a priori knowledge of the three-dimensionality of hydrodynamic turbulence.

3. The attractor dimension as function of parameters of initial fluctuation
Solutions which were obtained for space-time areas 1) (the beginning of the generation (index 1.0 or 1)) and 2) (generation (index 1.1)) have allowed to derive the fractal equations [16–28] for the space-time area 3) (the space-time area of the diffusion of turbulence). One of the equations is
The solution may be written as

$$(E_a)_j = C \exp \left[ -\frac{(R_{\tau1})_{j(1, 0)}}{2} \left( \frac{|x|}{\tau_{cor1}} \right)^2 \right]. \quad (8)$$

Here $(E_a)$ is the field energy component, which is actually the stochastic one (index “st”), index $j=1$ refers to the space-time area of the diffusion of turbulence $3$, $\tau$ is a time, $\tau_{cor1}$ is a time interval necessary for forming of the continuous spectrum. For the fractal coefficient $(R_{\tau1})_{1(1, 0)} = 1$ the turbulent energy in the calculating point of the flow is

$$(E_a)_{\tau cor1} = (E_a)_{\tau cor0} \left| Re_a - \frac{1}{Re_{a0}} \right| \quad (9)$$

$(E_{a0})_{\tau cor}$ is an estimation of full turbulent energy transferred from the deterministic to random motion, $(E_a)_{\tau cor}$ is the energy of initial turbulence. $Re_{a0}$ is the turbulent Reynolds number calculated by the parameters of the initial perturbation in the flow. So the equation (6) in case of the flow in the pipe has the form

$$d_H \leq 1 + 0.36 \cdot \left[ C_1 (Re_{a0}^2)^2 + C_0 (Re_{a0}^2)^{3/4} \right]^{5/4} \quad (10)$$

4. The calculation of the attractor dimension in the round tube

One of the most experimentally studied shear turbulent flow at the bounded by a smooth surface, is the flow in the pipe. The distribution of the pulsation intensity $K = (u_x^2)^{0.5} / U_0$ and the scale of the turbulent mole $(L_m / r)$ are presented in paper of Lauffer [29, 44–53]. Here $(u_t')$ is the longitudinal velocity pulsation, $U_0$ is the velocity on the tube axis of radius $r$. The flow in the tube was fully turbulent and the thickness of the boundary layer is $\delta = r$, $x_2$ is the transverse coordinate, along the normal to the wall, $0 \leq x_2 \leq r$. The data of Lauffer [29, 44-53] are given for the Reynolds numbers $Re_d = 0.510^5 \div 5.10^5$, where $Re_d = (U_0 d / n)$, the pipe diameter is $d = 0.247m$, velocity on the pipe axis $U_0 = 3 \div 30m / s$, $v = 1.510^{-5} \ m^2 / s$. Calculations $Re_{a0}$ for the equation (10) give values $Re_{a0} = 33 \div 100$. So for data of Lauffer [29, 44–53] maximum of values of the number of degrees of freedom in the fixed point of the flow in the according to the equation (10) are $d_H = N = 1 \times 10^0 \div 1 \times 10^1$

5. Conclusions

Based on the analysis of works in the field of nonlinear dynamics of dissipative systems and the statistical theory of hydrodynamic turbulence, a new generalized dependence is presented for the correlation dimension of the attractor as a function of the turbulent Reynolds number in the calculating point of the turbulent flow. Moreover, on the basis of the theory of equivalent measures and stochastic equations for a continuous medium, new dependencies are obtained, allowing us to determine the dimension of the attractor as a function of the parameters of the initial perturbation in the flow.

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