Abstract

We revisit a deformed Jackiw-Teitelboim model with a hyperbolic dilaton potential, constructed in the preceding work \[\text{arXiv:1701.06340}\]. Several solutions have been found in a series of the subsequent papers, but all of them are pathological because of a naked singularity intrinsic to the deformation. In this paper we perform a Weyl transformation to the original deformed model and discuss a Liouville type potential with a cosmological constant term. Then we can construct regular solutions with coupling to a conformal matter by employing $SL(2)$ transformations. For a black hole solution, the Bekenstein-Hawking entropy is computed. It is reproduced by evaluating the boundary stress tensor with an appropriate local counter-term (which is essentially provided by a Liouville type potential).
1 Introduction

A recent interest in the study of string theory is to establish a toy model of the AdS/CFT correspondence [1–3]. In particular, it is significant to understand quantum mechanical description of black hole, especially the holographic principle [4, 5]. Along this direction, Kitaev proposed a one-dimensional system composed of $N \gg 1$ fermions with a random, all-to-all quartic coupling [6]. This model is a variant of the Sachdev-Ye (SY) model [7], and so it is called the Sachdev-Ye-Kitaev (SYK) model. A remarkable point is that this system exhibits the maximal Lyapunov exponent in an out-of-time-order four-point function [6,8,9]. Hence the SYK model may have a gravity dual described by Einstein gravity [10,11]. There are a lot of developments. All point correlation functions have been computed in [14]. An extension to multi-flavor cases is discussed in [15]. Supersymmetric extensions are presented in [16]. A possible relation to 3D bulk dual has been discussed in [17]. Analytic spectral

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1 As another interesting direction, the conformal SYK model was proposed by Gross and Rosenhaus [12,13]. Then, the bulk dual may not be a gravitational theory but a scalar field theory on the rigid AdS$_2$. 
density is computed in [18]. The SYK model without disorder was proposed in [19,20] and the related tensor models are discussed in [21,22].

A naive candidate of the gravity dual for the SYK model is a particular 1+1 dimensional dilaton gravity model originally introduced by Jackiw [23] and Teitelboim [24] (called the Jackiw-Teitelboim (JT) model)\(^2\). This model has been revisited by Almheiri and Polchinski [26] from the point of view of holography. Hence this model is sometimes called the AP model. A lot of efforts have been made to clarify the relation between the JT model and the SYK model, but it would be fair to say that the two models coincide in a low-energy region as the Schwarzian theory [27–30].

In the preceding works [31,32], we have studied deformations of the JT model by applying the Yang-Baxter deformation technique [33–35]. The deformed model has a hyperbolic dilaton potential. We have already shown that solutions of the deformed model are constructed by a couple of Liouville’s solutions. By using this technique, we have found several solutions like the general vacuum solutions, the shock wave case and the deformed black hole. Typically, The region near the boundary is deformed to dS\(_2\) and a new naked singularity appears. It is known that these are common characters of the geometries generated by the Yang-Baxter deformations. However their physical interpretation has been totally unknown.

In this paper, we study the deformed JT model by employing a proper frame proposed by Frolov and Zelnikov [36]. This frame is realized by performing a particular Weyl transformation and in this frame the deformed model is described as a dilaton gravity model with a Liouville potential and a cosmological constant term. Then we find out regular solutions without singularities intrinsic to the deformations by employing \(SL(2)\) transformations. In particular, a black hole solution is also included. We investigate the thermodynamics properties by computing the Bekenstein-Hawking entropy. This entropy is reproduced by evaluating the boundary stress tensor with a particular Liouville-type counter-term. Notably, the counter-term in this proper frame is very simple in comparison to the ones in the preceding works [31,32].

This paper is organized as follows. Section 2 gives a brief review of the deformed dilaton gravity model. In section 3, we revisit the deformed model by employing a particular Weyl

\(^2\)For a nice review on 2D dilaton-gravity, see [25].
transformation. In this new frame, the deformed model is recaptured as a Liouville dilaton gravity system with a cosmological constant term. In section 4, we consider a black hole solution and its thermodynamic quantities. In particular, the Bekenstein-Hawking entropy is reproduced by evaluating the boundary stress tensor with an appropriate counter-term. Section 5 is devoted to conclusion and discussion. In Appendix A, we discuss the case with a free massless scalar field. In this case, a black hole solution can be constructed as well, but its thermodynamic property exhibits a different behavior in comparison to the case with the conformal coupling to the scalar curvature. The specific heat becomes negative due to the time dependence of the dilaton. In Appendix B, we will show the difference between the black hole solution newly obtained in this paper and the one previously found in the preceding works [31, 32].

2 A review of a deformed Jackiw-Teitelboim model

In this section, let us give a short review of a deformed Jackiw-Teitelboim (JT) model with a hyperbolic dilaton potential presented in the preceding works [31, 32].

2.1 The general vacuum solutions of the deformed model

We work here in the Lorentzian signature and the (1+1)-dimensional spacetime is described by the coordinates $x^\mu = (x^0, x^1) = (t, x)$. The basic ingredients of this system are the metric $g_{\mu\nu}$ and the dilaton $\Phi$.

The classical action for $g_{\mu\nu}$ and $\Phi$ is given by [31, 32]

$$S_\Phi = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \left[ \Phi^2 R + \frac{1}{\eta L^2} \sinh (2\eta \Phi^2) \right] + \frac{1}{8\pi G} \int dt \sqrt{-\gamma_{tt}} \Phi^2 K, \quad (2.1)$$

where $G$ is a two-dimensional Newton constant, $R$ and $g$ are Ricci scalar and determinant of $g_{\mu\nu}$, and $L$ is an AdS radius. The last term is the Gibbons-Hawking term that consists of extrinsic metric $\gamma_{tt}$ and extrinsic curvature $K$.

The real constant parameter $\eta$ measures the deformation. We assume that $\eta$ is positive. In the $\eta \to 0$ limit, the classical action (2.1) reduces to the original JT model [23, 24].

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3 The sinh-type potential is related to a $q$-deformed $sl(2)$ algebra via (4.1) in the work [37].
(without matter fields)

\[ S_\Phi^{(\eta=0)} = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi^2 \left[ R + \frac{2}{L^2} \right] + \frac{1}{8\pi G} \int dt \sqrt{-\gamma} \Phi^2 K. \tag{2.2} \]

Thus the model (2.1) can be regarded as a deformation of the JT model.

In the following, we will work with the metric in the conformal gauge:

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\omega} dx^+ dx^-, \tag{2.3} \]

where the light-cone coordinates \( x^\pm \) are defined as

\[ x^\pm \equiv t \pm z. \tag{2.4} \]

By taking variations of \( S_\Phi \), the equations of motion are obtained as follows:

\[ 4\partial_+ \partial_- \Phi^2 + \frac{e^{2\omega}}{\eta L^2} \sinh (2\eta \Phi^2) = 0, \tag{2.5} \]
\[ 4\partial_+ \partial_- \omega + \frac{e^{2\omega}}{L^2} \cosh (2\eta \Phi^2) = 0, \tag{2.6} \]
\[ -e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) = 0, \tag{2.7} \]
\[ -e^{2\omega} \partial_- (e^{-2\omega} \partial_- \Phi^2) = 0. \tag{2.8} \]

To solve the equations of motion (2.5)-(2.8) systematically, it is very useful to introduce a set of new variables \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \):

\[ \tilde{\omega}_1 = \omega + \eta \Phi^2, \quad \tilde{\omega}_2 = \omega - \eta \Phi^2. \tag{2.9} \]

From (2.5) and (2.6), one can obtain two Liouville equations for \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \):

\[ 4\partial_+ \partial_- \tilde{\omega}_1 + \frac{1}{L^2} e^{2\tilde{\omega}_1} = 0, \]
\[ 4\partial_+ \partial_- \tilde{\omega}_2 + \frac{1}{L^2} e^{2\tilde{\omega}_2} = 0. \tag{2.10} \]

It is known that the general solutions for the Liouville equation are given by arbitrary holomorphic functions \( X_i^+(x^+) \) and anti-holomorphic functions \( X_i^-(x^-) \):

\[ e^{2\tilde{\omega}_1} = \frac{4L^2 \partial_+ X_1^+ \partial_- X_1^-}{(X_1^+ - X_1^-)^2}, \]
\[ e^{2\tilde{\omega}_2} = \frac{4L^2 \partial_+ X_2^+ \partial_- X_2^-}{(X_2^+ - X_2^-)^2}. \tag{2.11} \]
By using $X_1^\pm$ and $X_2^\pm$, the constraint conditions (2.7) and (2.8) can be simplified into

$$\text{Sch}\{X_1^\pm, x^\pm\} - \text{Sch}\{X_2^\pm, x^\pm\} = 0.$$  \hspace{1cm} (2.12)

Here the Schwarzian derivative $\text{Sch}(X, x)$ is defined as

$$\text{Sch}\{X, x\} \equiv \frac{X'''}{X'} - \frac{3}{2} \left(\frac{X''}{X'}\right)^2.$$  \hspace{1cm} (2.13)

As a result, the metric $e^{2\omega}$ and the dilaton $\Phi^2$ are represented by two solutions of the Liouville equations:

$$e^{2\omega} = \sqrt{e^{2\tilde{\omega}_1} e^{2\tilde{\omega}_2}} = 4L^2 \sqrt{\frac{\partial_+ X_1^+ \partial_- X_1^- \partial_+ X_2^+ \partial_- X_2^-}{(X_1^+ - X_1^-)^2 (X_2^+ - X_2^-)^2}},$$  \hspace{1cm} (2.14)

$$\Phi^2 = \frac{\tilde{\omega}_1 - \tilde{\omega}_2}{2\eta} = \frac{1}{4\eta} \log \left| \frac{\partial_+ X_1^+ \partial_- X_1^-}{\partial_+ X_2^+ \partial_- X_2^-} \frac{(X_2^+ - X_2^-)^2}{(X_1^+ - X_1^-)^2} \right|.$$  \hspace{1cm} (2.15)

The general solutions of the deformed model (2.1) are discussed as Yang-Baxter deformations of AdS$_2$ [32].

### 2.2 Deformed black hole with conformal matters

Next, let us consider a conformal matter $\chi$ which couples to the Ricci scalar and the dilaton.

$$S_\chi = -\frac{N}{24\pi} \int d^2x \sqrt{-g} \left[ \chi(R - 2\eta \nabla^2 \Phi^2) + (\nabla \chi)^2 \right] - \frac{N}{12\pi} \int dt \sqrt{-\gamma_{tt}} \chi K.$$  \hspace{1cm} (2.16)

Here $N$ denotes the central charge of $\chi$.

Similarly to the derivation of (2.10), the equations of motion can be rewritten by using $\tilde{\omega}_1$ and $\tilde{\omega}_2$. As a result, the equations of motion are given by

$$\partial_+ \partial_-(\tilde{\omega}_1 + \chi) = 0,$$

$$4\partial_+ \partial_- \tilde{\omega}_1 + e^{2\tilde{\omega}_1} = \frac{16}{3} GN\eta \partial_+ \partial_- \chi,$$

$$4\partial_+ \partial_- \tilde{\omega}_2 + e^{2\tilde{\omega}_2} = 0,$$

$$e^{\tilde{\omega}_1} \partial_+ \partial_- e^{-\tilde{\omega}_1} - e^{\tilde{\omega}_2} \partial_+ \partial_- e^{-\tilde{\omega}_2} = \frac{2}{3} GN(-\partial_+ \partial_\pm \chi + \partial_\pm \chi \partial_\pm \chi + 2\partial_\pm \chi \partial_\pm \tilde{\omega}_1).$$  \hspace{1cm} (2.17)

Note here that the third equation still preserves the Liouville form, while the second one acquires the source term due to the matter contribution. The last one gives the constraint conditions for solutions.
Here, let us derive a black hole solution. Suppose that the solution is static. Then, by solving the first equation in the set of equations of motion (2.17), $\chi$ can be expressed as
\[ \chi = -\tilde{\omega}_1 - \sqrt{\mu} (x^+ - x^-). \] (2.18)

By eliminating $\chi$ from the other equations, one can derive a couple of Liouville equations and the constraint conditions:
\[
4 \left( 1 + \frac{4}{3} GN \right) \partial_+ \partial_+ e^{2\tilde{\omega}_1} + 4\partial_+ \partial_- e^{2\tilde{\omega}_2} = 0, \\
\left( 1 + \frac{2}{3} GN \right) e^{\tilde{\omega}_1} \partial_+ e^{x^-\tilde{\omega}_1} - e^{\tilde{\omega}_2} \partial_+ e^{x^-\tilde{\omega}_2} = \frac{2}{3} GN\mu. \] (2.19)

Still, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are represented by the general solutions of the Liouville equation:
\[
e^{2\tilde{\omega}_1} = \frac{4L^2}{(X_1^+ - X_1^-)^2} \partial_+ X_1^+ \partial_- X_1^-, \\
e^{2\tilde{\omega}_2} = \frac{4L^2}{(X_2^+ - X_2^-)^2} \partial_+ X_2^+ \partial_- X_2^- . \] (2.20)

By using $X_i^\pm$ and the Schwarzian derivative, the constraint conditions can be rewritten as
\[
\left( 1 + \frac{2}{3} GN \right) \text{Sch}\{X_1^+, x^+\} - \text{Sch}\{X_2^+, x^+\} = -\frac{4}{3} GN\mu, \\
\left( 1 + \frac{2}{3} GN \right) \text{Sch}\{X_1^-, x^-\} - \text{Sch}\{X_2^-, x^-\} = -\frac{4}{3} GN\mu. \] (2.21)

It is an easy task to see that
\[ X_{1,2}^\pm = \tanh(\sqrt{\mu} x^\pm) \]
satisfy the constraint conditions. In general, linear fractional transformations of them,
\[ X_{1,2}^\pm = \frac{a \tanh(\sqrt{\mu} x^\pm) + b}{c \tanh(\sqrt{\mu} x^\pm) + d}, \] (2.22)
also satisfy the conditions because of a property of the Schwarzian derivative.

By taking certain parameters, a deformed black hole solution \[31,32\] is given by
\[ e^{2\omega} = \frac{4\mu L^2 (1 - \eta^2 \mu) \sqrt{1 + \frac{4}{3} GN\eta}}{\sinh^2(2\sqrt{\mu} Z) - \eta^2 \mu \cosh^2(2\sqrt{\mu} Z)}, \] (2.23)

\footnote{The general solution is given by $\chi + \tilde{\omega}_1 = c_1 (x^+ - x^-) + c_0$, where $c_1$ and $c_0$ are arbitrary real constants. We have set that $c_0 = 0$ for simplicity and $c_1 = \sqrt{\mu}$ for later convenience, where $\mu$ is a real positive constant.}
\[ \Phi^2 = \frac{1}{2\eta} \log \left| \frac{1 + \eta \sqrt{\mu} \coth(2\sqrt{\mu} Z)}{1 - \eta \sqrt{\mu} \coth(2\sqrt{\mu} Z)} \right| + \frac{1}{4\eta} \log \left( 1 + \frac{4}{3} G N \eta \right), \]  

(2.24)

where due to the range of \( \sqrt{\mu} \) is restricted as

\[ 0 \leq \sqrt{\mu} \leq \frac{1}{\eta}, \]

so as to ensure the positivity of the exponential in (2.23). Notably, solutions in the deformed model have naked singularities intrinsic to the deformations. For the detail, see [31,32].

### 3 Moving to a proper frame: Weyl transformation

In this section, we shall introduce a new proper frame, which was originally utilized by Frolov and Zelnikov [36]. One can see that in this proper frame, the deformed JT model can be recaptured as a Liouville dilaton gravity model with a cosmological constant term, while solutions are still given by \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \).

The proper frame can be introduced through the following Weyl transformation:

\[ g_{\mu\nu} = e^{-2\eta \Phi^2} \tilde{g}_{\mu\nu}. \]  

(3.1)

In the conformal gauge, \( \tilde{\omega}_1 \) plays the role of the conformal factor in front of the metric:

\[ d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -e^{2\tilde{\omega}_1} dx^+ dx^- . \]  

(3.2)

In terms of the new metric \( \tilde{g}_{\mu\nu} \), the classical action of the deformed JT model (2.1) can be rewritten as

\[ \tilde{S}_\Phi = \frac{1}{16\pi G} \int d^2x \sqrt{-\tilde{g}} \left[ \Phi^2 \tilde{R} - 2\eta (\tilde{\nabla} \Phi^2)^2 - \frac{1}{2\eta L^2} \left( e^{-4\eta \Phi^2} - 1 \right) \right]. \]  

(3.3)

Note here that the kinematic term of \( \Phi^2 \) is well-defined because we have assumed that \( \eta \) is a positive real constant. The potential is now bounded from below, but it is the run-away type potential. Note here that \( \Phi^2 \) (instead of \( \Phi \)) appears in the classical action (3.3) and \( \Phi^2 \) should be definitely positive. Hence this is not the usual Liouville gravity but rather a \textit{constrained} Liouville gravity. Interestingly, this constrained system can also be derived from Einstein-Hilbert action with a cosmological constant [38].

It is remarkable that the equations of motion for \( \omega \) and \( \Phi^2 \) are equivalent to the equations for \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \). Thus, the solutions of (3.3) are obtained by \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) as (2.11). From \( \tilde{\omega}_1 \)
and \( \tilde{\omega}_2 \), the dilation is determined by (2.15) again. However, in the proper frame, the metric is given by \( \tilde{\omega}_1 \) directly.

In summary, the general vacuum solutions of (3.3) are given by

\[
e^{2\tilde{\omega}_1} = \frac{4L^2 \partial_+ X_1^+ \partial_- X_1^-}{(X_1^+ - X_1^-)^2},
\]

\[
\Phi^2 = \frac{1}{4\eta} \log \left| \frac{\partial_+ X_1^+ \partial_- X_1^-}{(X_1^+ - X_1^-)^2} \left( \frac{X_2^+ - X_2^-}{\partial_+ X_2^+ \partial_- X_2^-} \right) \right|. \tag{3.4}
\]

Note here that the metric is given by the solution of Liouville equation and the rigid AdS geometry is preserved in the new frame (i.e., proper frame). This result indicates that the Weyl transformation carried out here has undone the Yang-Baxter deformation from the metric, while the deformations effect has been encoded into only the dilaton part. It should be remarked that this is a rather natural result, noticing that the Yang-Baxter deformation effect has been factored out as shown in [31, 32].

4 A new black hole solution and its thermodynamics

In this section, we present a new black hole solution with a conformal matter. The proper frame (3.1) enables us to construct an AdS2 black hole solution (i.e., the metric is the same as the undeformed case [26]). After that, we compute the entropy of the black hole solution in two manners: 1) the Bekenstein-Hawking entropy and 2) the boundary stress tensor with a certain counter-term.

A new black hole solution

In the proper frame, the classical action of the matter (2.16) is given by

\[
\tilde{S}_\chi = -\frac{N}{24\pi} \int d^2 x \sqrt{-\tilde{g}} \left[ \chi \tilde{R} + (\nabla \chi)^2 \right] - \frac{N}{12\pi} \int dt \sqrt{-\gamma} \chi K. \tag{4.1}
\]

Note here that in the proper frame, \( \chi \) couples to only the Ricci scalar, while in the old frame (2.16), \( \chi \) coupled to both Ricci scalar and dilaton. This point is the same as in the undeformed case [26].
We have already known that solutions are given by (2.20). Hence, the remaining task is to determine how to choose parameters of linear fractional transformations (2.22) so as to realize a black hole solution.

In order to find out a black hole solution of the system (4.1), we first employ the black hole coordinates for $X^\pm_1$ like

$$X^\pm_1(x^\pm) = \frac{1}{\sqrt{\mu}} \tanh(\sqrt{\mu}x^\pm),$$

(4.2)

by following the argument in [26]. Then, for $X^\pm_2$, let us take the following linear fractional transformation.

$$X^+_2(x^+)=\frac{\tanh(\sqrt{\mu}x^+)+2\eta\sqrt{\mu}}{2\eta\mu\tanh(\sqrt{\mu}x^+)+\sqrt{\mu}},$$

$$X^-_2(x^-)=\frac{1}{\sqrt{\mu}}\tanh(\sqrt{\mu}x^-).$$

(4.3)

Thus, we have obtained the static solutions for $\tilde{\omega}_1, \tilde{\omega}_2$:

$$e^{2\tilde{\omega}_1} = \frac{4\mu(1 + \frac{4}{3}GN\eta)L^2}{\sinh^2(2\sqrt{\mu}Z)},$$

(4.4)

$$e^{2\tilde{\omega}_2} = \frac{4\mu(1 - 4\eta^2\mu)L^2}{(\sinh(2\sqrt{\mu}Z) + 2\eta\sqrt{\mu}\cosh(2\sqrt{\mu}Z))^2},$$

(4.5)

and hence a black hole solution with conformal matters has been derived as follows:

$$e^{2\tilde{\omega}_1} = \frac{4\mu(1 + \frac{4}{3}GN\eta)L^2}{\sinh^2(2\sqrt{\mu}Z)},$$

(4.6)

$$\Phi^2 = \frac{1}{2\eta} \log \left| 1 + 2\eta\sqrt{\mu} \coth(\sqrt{\mu}Z) \right| + \Phi^2_0.$$  

(4.7)

Here $\Phi^2_0$ is the constant part of the dilaton:

$$\Phi^2_0 = \frac{1}{2\eta} \log \left( 1 + \frac{4}{3}GN\eta \right).$$

(4.8)

The matter contribution just rescales the metric and shifts the dilaton by a constant. Note here that the allowed region of $\mu$ is restricted like

$$0 \leq \sqrt{\mu} \leq \frac{1}{2\eta}$$

(4.9)

so as to make the value of $\Phi^2_0$ well-defined and preserve the positivity of (4.5).
Note here that this solution is different from the previous black hole solution with (2.23) and (2.24), though the two solutions are quite similar but the \( \mu \)-dependence of the metric and the range of \( \sqrt{\mu} \) are different as explicitly shown in Appendix B.

By taking the undeformed limit \( \eta \to 0 \), this solution goes to the black hole solution with conformal matters in the undeformed case [20]:

\[
e^{2\omega} = \frac{4\mu L^2}{\sinh^2(2\sqrt{\mu}Z)}, \tag{4.10}
\]
\[
\Phi^2 = \sqrt{\mu} \coth(\sqrt{\mu}Z) + \frac{1}{3} GN. \tag{4.11}
\]

In the following, let us evaluate the black hole entropy associated with (4.6) and (4.7) in two manners.

1) Bekenstein-Hawking entropy

Let us first compute the Bekenstein-Hawking entropy. From the black hole metric (4.6), one can compute the Hawking temperature as

\[
T_H = \frac{\sqrt{\mu}}{\pi}. \tag{4.12}
\]

This is the same as in the undeformed case [20]. From the classical action, one can read off the effective Newton constant \( G_{\text{eff}} \) as

\[
\frac{1}{G_{\text{eff}}} = \frac{\Phi^2}{G} - \frac{2N\chi}{3}. \tag{4.13}
\]

Given that the horizon area \( A \) is 1, the Bekenstein-Hawking entropy \( S_{\text{BH}} \) is computed as

\[
S_{\text{BH}} = \left. \frac{A}{4G_{\text{eff}}} \right|_{Z \to \infty} = \frac{\text{arctanh}(2\pi T_H \eta)}{8G\eta} + \frac{N}{6} \log(T_H) + \text{constant}. \tag{4.14}
\]

The last term is a constant term independent of the Hawking temperature. Note here that the argument of arctanh should be less than 1. This means that

\[
0 \leq T_H \leq \frac{1}{2\pi \eta}.
\]

This range agrees with the possible values of \( \sqrt{\mu} \) given in [4.9].
2) Boundary stress tensor

In the conformal gauge, the total action, including the Gibbons-Hawking term, can be rewritten as

\[
\tilde{S}_\Phi = \frac{1}{8\pi G} \int d^2x \left[ -4 \partial_+ \Phi^2 \partial_- \tilde{\omega}_1 + 4\eta \partial_+ \Phi^2 \partial_- \Phi^2 - \frac{1}{4\eta L^2} e^{2\tilde{\omega}_1} (e^{-4\eta \Phi^2} - 1) \right], \\
\tilde{S}_\chi = \frac{N}{6\pi} \int d^2x \left[ \partial_+ \chi \partial_- \chi + 2\partial_+ \chi \partial_- \tilde{\omega}_1 \right].
\]

(4.15)

By using the explicit expression of the black hole solution, the on-shell bulk action can be evaluated on the boundary.

The on-shell action diverges at the boundary \( Z = 0 \), hence one needs to introduce a cut-off as \( Z = \epsilon \ (> 0) \), where \( \epsilon \) is an infinitesimal quantity. Then the on-shell action can be expanded with respect to \( \epsilon \) like

\[
\tilde{S}_\Phi + \tilde{S}_\chi = \int dt \left[ 1 + \frac{4}{3} \frac{G N \eta}{16\pi G \eta \epsilon} + O(\epsilon^1) \right].
\]

(4.16)

We have ignored the terms which vanish in the \( \epsilon \to 0 \) limit, and only the divergent term has explicitly been written down. To cancel out the divergence, it is necessary to add an appropriate counter-term.

Our proposal for the counter-term is the following\[7\]

\[
\tilde{S}_{ct} = \int dt \frac{\sqrt{-\gamma_{tt}}}{L'} \left[ \frac{-1}{16\pi G \eta} \left( 1 + (2\eta \Phi_0^2 - 1)e^{-2\eta(\Phi^2 - \Phi_0^2)} \right) - \frac{N}{24\pi} \right].
\]

(4.17)

Here \( \Phi_0 \) is the constant defined in (4.18) and \( L' \) is the rescaled AdS radius defined as

\[
L'^2 \equiv L^2 \left( 1 + \frac{4}{3} GN \eta \right).
\]

(4.18)

Then the extrinsic metric \( \tilde{\gamma}_{tt} \) on the boundary is defined as

\[
\tilde{\gamma}_{tt} \equiv -e^{2\tilde{\omega}_1} \bigg|_{Z=\epsilon}.
\]

In the undeformed limit \( \eta \to 0 \), the counter-term (4.17) reduces to

\[
\tilde{S}_{ct}^{(\eta=0)} = \int dt \frac{\sqrt{-\gamma_{tt}}}{L} \left( -\frac{\Phi^2}{8\pi G} - \frac{N}{24\pi} \right).
\]

(4.19)

\[7\]The uniqueness of the counter-term has not been confirmed. It is significant to revisit it by following the works [41–43].
This is nothing but the counter-term utilized in the undeformed case [26].

It is straightforward to check that the sum \( \tilde{S} = \tilde{S}_\Phi + \tilde{S}_\chi + \tilde{S}_{ct} \) becomes finite on the boundary by using the expanded form of the counter-term (4.17):

\[
\tilde{S}_{ct} = \int dt \left[ -\frac{1}{16\pi G\eta} + \frac{1}{16\pi G\eta^2} + O(\epsilon) \right].
\]

In a region near the boundary, the warped factor of the metric can be expanded as

\[
e^{2\tilde{\omega}_1} = \frac{L'^2}{\epsilon^2} + O(\epsilon^0).
\]

Hence, by normalizing the boundary metric as

\[
\hat{\gamma}_{tt} = \frac{\epsilon^2}{L'^2} \tilde{\gamma}_{tt},
\]

the boundary stress tensor is defined as

\[
\langle \hat{T}_{tt} \rangle = -\frac{2}{\sqrt{-\hat{\gamma}_{tt}}} \delta S = \lim_{\epsilon \to 0} \frac{\epsilon}{L'} \frac{2}{\sqrt{-\tilde{\gamma}_{tt}}} \frac{\delta S}{\delta \tilde{\gamma}_{tt}}.
\]

After all, \( \langle \hat{T}_{tt} \rangle \) has been evaluated as

\[
\langle \hat{T}_{tt} \rangle = -\frac{\log(1 - 4\eta^2\mu)}{32\pi G\eta^2} + \frac{\log (1 + \frac{4}{3}G\eta)}{32\pi G\eta^2} + \frac{N\sqrt{\mu}}{6\pi}.
\]

To compute the associated entropy, \( \langle \hat{T}_{tt} \rangle \) is identified with energy \( E \) like

\[
E = -\frac{\log(1 - 4\eta^2T_H^2\eta^2)}{32\pi G\eta^2} + \frac{\log (1 + \frac{4}{3}G\eta)}{32\pi G\eta^2} + \frac{N}{6} T_H,
\]

where we have used the expression of the Hawking temperature (4.12).

Then, by solving the thermodynamic relation,

\[
dE = \frac{dS}{T_H},
\]

the entropy is obtained as

\[
S = \frac{\arctanh(2\pi T_H\eta)}{8G\eta} + \frac{N}{6} \log(T_H) + S_{T_H=0}.
\]

Here \( S_{T_H=0} \) has appeared as an integration constant that measures the entropy at zero temperature. Thus the resulting entropy precisely agrees with the Bekenstein-Hawking entropy (4.14), up to the temperature-independent constant.
5 Conclusion and discussion

In this paper we have reconsidered a deformed Jackiw-Teitelboim model with a hyperbolic dilaton potential, constructed in the preceding work \cite{31,32}. By performing a Weyl transformation to the deformed model, we have discussed a Liouville type potential with a cosmological constant term. Then regular solutions coupled to a conformal matter have been presented with the help of $SL(2)$ transformations.

For a black hole solution, its thermodynamic behavior has been investigated. In particular, the Bekenstein-Hawking entropy has been reproduced by evaluating the boundary stress tensor with an appropriate counter-term. Notably, this counter-term is essentially provided by a Liouville type potential, and so very concise, in comparison to the previous ones utilized in \cite{31,32}, as in the discussion of the undeformed model \cite{26}.

It is significant to argue the implication of our result in the context of Yang-Baxter deformations. The point is that the deformation effect is factored out as the overall factor in the case of AdS$_2$. That is why the undeformed AdS$_2$ is realized by performing an appropriate Weyl transformation, then the vestiges of the deformation is encoded into the dilaton and other matter fields. Of course, in higher dimensions, it would not be possible to realize the undeformed metric completely.

In general, Yang-Baxter deformed backgrounds suffer from a naked singularity. A famous example is the $\eta$-deformed background \cite{40}. The $\eta$-deformed AdS$_2 \times S^2$ is discussed in \cite{44,45}. As an exercise, it would be nice to apply our argument for this case. We will report some results along this direction in the near future \cite{46}.

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Appendix

A Solutions with a massless scalar field

Let us consider solutions with a massless scalar field \( f(t, x) \) without the conformal coupling term (i.e., a free massless scalar field).

The classical action is given by the sum of the dilaton gravity part \( S_\Phi \) and the matter part \( S_f \) like

\[
S = S_\Phi + S_f, \\
S_f = \frac{-1}{32\pi G} \int \! d^2x \, \sqrt{-g} (\nabla f)^2. 
\]

(A.1)

By taking variations of \( S \), the equations of motion are obtained as follows:

\[
\partial_+ \partial_- f = 0, \\
4\partial_+ \partial_- \Phi^2 + \frac{e^{2\omega}}{\eta L^2} \sinh (2\eta \Phi^2) = 0, \\
4\partial_+ \partial_- \omega + \frac{e^{2\omega}}{L^2} \cosh (2\eta \Phi^2) = 0, \\
-e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) = 8\pi G T_{++}, \\
-e^{2\omega} \partial_- (e^{-2\omega} \partial_- \Phi^2) = 8\pi G T_{--}, 
\]

(A.2) \quad (A.3) \quad (A.4) \quad (A.5) \quad (A.6)

Here \( T_{\mu\nu} \) is the energy-momentum tensor defined as

\[
T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_f}{\delta g_{\mu\nu}}. 
\]

(A.7)

In the light-cone coordinates, the components of \( T_{\mu\nu} \) are given by

\[
T_{\pm\pm} = \frac{1}{16\pi G} (\partial_\pm f)^2, \quad T_{+-} = 0. 
\]

(A.8)

The matter \( f \) modifies only the constraint conditions (A.5) and (A.6). Thus, the new variables \( \tilde{\omega}_{1,2} \) introduced in (2.9) are still described as Liouville’s solutions (2.11). By using \( X_1^\pm \) and \( X_2^\pm \), the constraint conditions can be rewritten into

\[
\text{Sch}\{X_1^\pm, x^\pm\} - \text{Sch}\{X_2^\pm, x^\pm\} = -32\pi G \eta T_{\pm\pm}. 
\]

(A.9)

In order to solve the constraint conditions in (A.9), it is useful to use the chain rule for the Schwarzian derivative:

\[
\text{Sch}\{X_2(X_1), x\} = \text{Sch}\{X_2, X_1\} (\partial_x X_1)^2 + \text{Sch}\{X_1, x\}. 
\]

(A.10)
Then the constraint conditions in (A.9) can be rewritten as

\[ \text{Sch}\{X_2^\pm, X_1^\pm\}(\partial_\pm X_1)^2 = -32\pi G\eta T_{\pm\pm}. \quad \text{(A.11)} \]

**An example**

One may take \(X_2(X_1)\) as a hyperbolic-type function, for example, like

\[ X_2^\pm = \tanh(\alpha X_1^\pm), \quad \text{(A.12)} \]

where \(\alpha\) is a real constant. For this choice, the Schwarzian derivative takes a constant value:

\[ \text{Sch}\{X_2^\pm, X_1^\pm\} = -2\alpha^2. \quad \text{(A.13)} \]

Then, we obtain the simple form of the constraint conditions.

\[ (\partial_\pm X_1^\pm)^2 = \frac{16\pi G\eta}{\alpha^2} T_{\pm\pm}. \quad \text{(A.14)} \]

Let us next examine the matter contribution. From the equation of motion (A.2), \(f\) is separated into the holomorphic and anti-holomorphic parts like

\[ f(x^+, x^-) = f^+(x^+) + f^-(x^-). \quad \text{(A.15)} \]

The constraints in (A.14) are rewritten as

\[ (\partial_\pm X_1^\pm)^2 = \frac{\eta}{\alpha^2}(\partial_\pm f^\pm)^2. \quad \text{(A.16)} \]

By integrating them, the solutions \(X_1^\pm\) are expressed as

\[ X_1^+ = \pm \frac{\sqrt{\eta}}{\alpha} f^+ + \beta_1, \]
\[ X_1^- = \pm \frac{\sqrt{\eta}}{\alpha} f^- + \beta_2. \quad \text{(A.17)} \]

Here \(\beta_1\) and \(\beta_2\) are arbitrary constants.

By using (A.12), the final expression of the solution associated with the simple choice (A.12) is given by

\[ f = f^+(x^+) + f^-(x^-), \]
\[ e^{2\tilde{\omega}_1} = \frac{4L^2\partial_+ X_1^+\partial_- X_1^-}{(X_1^+ - X_1^-)^2}, \]

15
\[ e^{4\eta \Phi^2} = \frac{\alpha^2 (X_1^+ - X_1^-)^2}{\sinh^2(\alpha (X_1^+ - X_1^-))}. \]

However, the last expression should be problematic. On the left-hand side, the dilaton \( \Phi^2 \) is assumed to be positive and \( \eta \) is also supposed to be positive. Hence the left-hand side is more than 1. But the right-hand side should be less than 1, so this expression is contradictory. This result indicates that our simple choice (A.12) is not appropriate. Hence we should seek for another good solution.

**An improved example**

More generally, one may take an arbitrary linear fractional transformations of (A.12). For example, it is possible to take a shift by a constant \( \beta \) like

\[ X_2^+ = \tanh(\alpha X_1^- - \beta), \quad X_2^- = \tanh(\alpha X_1^-). \]  

The dilaton is modified as

\[ e^{4\eta \Phi^2} = \frac{\alpha^2 (X_1^+ - X_1^-)^2}{\sinh^2(\alpha (X_1^+ - X_1^-) - \beta)}. \]  

This solution is not conflict to the positivity of \( \Phi^2 \). Note here that the dilaton \( \Phi^2 \) diverges at \( X_1^+ - X_1^- = \beta/\alpha \). When \( X_1^+ - X_1^- \) is sufficiently small or large, the right-hand side of (A.20) becomes less than 1. In particular, the dilaton \( \Phi^2 \) vanishes at \( (\bar{X}_1^+, \bar{X}_1^-) \), satisfying the following condition:

\[ \alpha^2 (\bar{X}_1^+ - \bar{X}_1^-)^2 = \sinh^2(\alpha (\bar{X}_1^+ - \bar{X}_1^-) - \beta). \]  

Because of this property, the geometry is available only for the region with

\[ X_1^+ - X_1^- \sim \frac{\beta}{\alpha}. \]  

Notably, the boundary specified by \( X_1^+ - X_1^- = 0 \) is excluded and hence it would not be obvious to evaluate the boundary stress tensor in this case.

It is significant to see the undeformed limit \( \eta \to 0 \). So far, the value of constant \( \beta \) has been arbitrary. But so as to reproduce the correct undeformed result, we need to impose a certain condition to the value of \( \beta \). It is necessary to rescale \( \beta \) as

\[ \beta = 2\eta^3 \bar{\beta}. \]
For simplicity, we set
\[ X_1^+ = \frac{\sqrt{\eta}}{\alpha} f^+, \quad X_1^- = \frac{\sqrt{\eta}}{\alpha} f^- . \] (A.23)

Note that after that, the \( \alpha \)-dependence vanishes. Then, in the undeformed limit \( \eta \to 0 \), the metric and dilaton are evaluated as
\[ e^{2\omega_1} = \frac{4L^2 \partial_+ f_1^+ \partial_- f_1^-}{(f_1^+ - f_1^-)^2} , \]
\[ \Phi^2 \bigg|_{\eta \to 0} = \frac{\beta}{f^+ - f^-} - \frac{1}{12} \left( f^+ - f^- \right)^2 . \] (A.24)

By fixing \( \bar{\beta} = 1 \), the first term of the dilaton is equivalent to a RG flow solution of the JT model discussed in [26]. The second term is a matter contribution.

**A black hole with a dynamical dilaton**

In the following discussion, we will set \( \bar{\beta} = 1 \) and focus on the simple solution (A.23). It would be interesting to consider a black hole solution in this case.

Taking the black hole coordinates for \( f_1^\pm \) leads to the following black hole solution:
\[ e^{2\omega_1} = \frac{4\mu L^2}{\sinh^2 \left( \sqrt{\mu} (x^+ - x^-) \right)} , \]
\[ e^{4\eta \Phi^2} = \frac{\eta}{\mu} \frac{\left( \tanh(\sqrt{\mu} x^+) - \tanh(\sqrt{\mu} x^-) \right)^2}{\sinh^2 \left( \sqrt{\mu} \left( \tanh(\sqrt{\mu} x^+) - \tanh(\sqrt{\mu} x^-) - 2\eta^3 \right) \right)} . \] (A.25)

Note here that the dilaton is time-dependent.

Even though the dilaton is dynamical, the Wald entropy is applicable. For the dilaton gravity case, it is evaluated from the value of dilaton at the horizon like
\[ S_{\text{BH}} = \Phi^2 \bigg|_{x^\pm \to \pm \infty} = \frac{1}{8G\eta} \log \frac{2\sqrt{\eta}}{\pi T_H \sinh \left( \frac{2\sqrt{\eta}}{\pi T_H} - 2\eta^3 \right)} . \] (A.26)

Here \( T_H \) is the Hawking temperature given in (4.12).

Note here that there is no restriction for the possible values of the Hawking temperature, in comparison to the case with the conformal coupling. Hence the entropy becomes discontinuous at
\[ T_H = \frac{1}{\pi \eta} , \] (A.27)
as depicted in Fig. 1. The discontinuity occurs due to the presence of the curvature singularity. Hence one cannot trust the geometry around that point, while the condition (A.22) is satisfied.

![Figure 1: A plot of the entropy with $G = \eta = 1$. It exhibits discontinuity at $T_H = \frac{1}{\sqrt{\eta}}$.](image)

### Specific heat

Let us here examine the specific heat associated with the black hole solution.

The specific heat $C$ is evaluated as

$$C = T_H \frac{\partial S_{BH}}{\partial T_H} = -\frac{1}{8G\eta} + \frac{\coth\left(\frac{2\sqrt{\eta}}{\pi T_H} - 2\eta^{\frac{3}{2}}\right)}{4\pi GT_H\sqrt{\eta}}. \quad (A.28)$$

This is discontinuous at $T_H = \frac{1}{\pi \eta}$ as plotted in Fig. 2.

It is helpful to see the asymptotic behaviors in the high-temperature and low-temperature regions. The specific heat $C$ exhibits different behaviors as follows:

$$C \approx \begin{cases} -\frac{1}{8G\eta} - \frac{\coth 2\eta^{\frac{3}{2}}}{4\pi GT_H\sqrt{\eta}} & (T_H \gg 1) \\ \frac{1}{4\pi GT_H\sqrt{\eta}} & (T_H \ll 1) \end{cases}. \quad (A.29)$$

In particular, $C$ becomes negative in the high-temperature limit.
Figure 2: A plot of the specific heat with $G = \eta = 1$. It exhibits discontinuity. In the low-temperature region, it is definitely positive, but it becomes negative in the high-temperature region.

In order to compare with the undeformed case, let us evaluate the Bekenstein-Hawking entropy (A.26) and the specific heat (A.28) in the $\eta \rightarrow 0$ limit,

\begin{align*}
S_{\text{BH}}^{(\eta=0)} &= \frac{1}{4G} \left( \frac{\pi T_H}{2} - \frac{1}{3\pi^2 T_H^2} \right), \\
C^{(\eta=0)} &= \frac{1}{6\pi^2 G T_H^2} + \frac{\pi T_H}{8G}.
\end{align*}

The expression (A.30) can be reproduced from a renormalization-group flow solution of the JT model (A.24) by taking the black hole coordinates.

Note that in the undeformed case, $C^{(\eta=0)}$ is definitely positive. Thus, the negative specific heat is intrinsic to the deformed model.

B Mapping of the solution with (2.23) and (2.24)

In Sec. 4 we have considered a black hole solution. This is slightly different from the previous one presented in Sec. 2.2 (or equivalently found in [31, 32]). In the following, we shall show the difference explicitly by mapping the previous one to the proper frame.

B.1 The deformed black hole solution

First of all, let us revisit the black hole solution with (2.23) and (2.24). In the following, we will focus upon the case without matters for simplicity (i.e., the case with $N = 0$).
We have employed the following linear fractional transformation:

\[
X_1^+ (x^+) = \frac{(1 - \eta \beta)X^+(x^+) - 2\eta\alpha}{-2\eta\gamma X^+(x^+) + (1 + \eta \beta)}, \quad X_1^- (x^-) = X^-(x^-),
\]
\[
X_2^+ (x^+) = \frac{(1 + \eta \beta)X^+(x^+) + 2\eta\alpha}{2\eta\gamma X^+(x^+) + (1 - \eta \beta)}, \quad X_2^- (x^-) = X^-(x^-).
\] (B.1)

Then, \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) are given by

\[
e^{2\tilde{\omega}_1} = \frac{4(1 - \eta^2(\beta^2 + 4\alpha\gamma))}{(X^+ - X^- - \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+ X^-))^2} \partial_+ X^+ \partial_- X^-.
\]
\[
e^{2\tilde{\omega}_2} = \frac{4(1 - \eta^2(\beta^2 + 4\alpha\gamma))}{(X^+ - X^- + \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+ X^-))^2} \partial_+ X^+ \partial_- X^-.
\] (B.2)

In particular, by taking the following parameters

\[
\alpha = \frac{1}{2}, \quad \beta = 0, \quad \gamma = \frac{\mu}{2},
\] (B.3)

and the black hole coordinates

\[
X^\pm = \frac{1}{\sqrt{\mu}} \tanh \left[ \sqrt{\mu} (T \pm Z) \right],
\] (B.4)

the deformed black hole with (2.23) and (2.24) can be reproduced.

**B.2 Mapping to the proper frame**

The next task is to map the solution with (2.23) and (2.24) to the proper frame.

So as to realize the rigid \( \text{AdS}_2 \) metric in the proper frame, we need to perform the following coordinate transformation:

\[
X^+(x^+) = \frac{(1 - \eta \beta)\bar{X}^+ + 2\eta\alpha}{2\eta\gamma \bar{X}^+ + (1 - \eta \beta)}, \quad X^-(x^-) = \bar{X}^-.
\] (B.5)

Then the holomorphic functions are transformed to

\[
X_1^+ = \bar{X}^+, \quad X_2^+ = \frac{((1 + \eta \beta)^2 + 4\eta^2 \alpha \gamma)\bar{X}^+ + 4\eta\alpha}{4\eta\gamma \bar{X}^+ + ((1 - \eta \beta)^2 + 4\eta^2 \alpha \gamma)}.
\] (B.6)

Again, by choosing the parameters (B.3) and the black hole coordinates (B.4), the black hole solution in the proper frame can be obtained as

\[
e^{2\tilde{\omega}_1} = \frac{4\mu L^2}{\sinh^2(2\sqrt{\mu} Z)},
\]
\[ e^{2\omega_2} = \left( \frac{1 - \eta^2 \mu}{1 + \eta^2 \mu} \right)^2 \frac{4 \mu L^2}{\left( \sinh(2\sqrt{\mu} Z) + \frac{2\eta \sqrt{\mu}}{1 + \eta^2 \mu} \cosh(2\sqrt{\mu} Z) \right)^2}. \]  

(B.7)

The dilaton is given by

\[ \Phi^2 = \frac{1}{2\eta} \log \left| 1 + \frac{2\eta \sqrt{\mu}}{1 + \eta^2 \mu} \coth(2\sqrt{\mu} Z) \right| + \frac{1}{2\eta} \log \left( \frac{1 + \eta^2 \mu}{1 - \eta^2 \mu} \right). \]  

(B.8)

These expressions are clearly different from the solution with (4.6) and (4.7) for the \( N = 0 \) case.

Finally, the Bekenstein-Hawking entropy is evaluated as

\[ S_{\text{BH}} = \left. \frac{A}{4G_{\text{eff}}} \right|_{Z \to \infty} = \frac{\text{arctanh}(\pi T_H \eta)}{4G\eta}. \]  

(B.9)

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