Spin Squeezing in the Ising Model

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We analyze the collective spin noise in interacting spin systems. General expressions are derived for the short time behaviour of spin systems with general spin-spin interactions, and we suggest optimum experimental conditions for the detection of spin squeezing. For Ising models with site dependent nearest neighbour interactions general expressions are presented for the spin squeezing parameter for all times. The reduction of collective spin noise can be used to verify the entangling powers of quantum computer architectures based on interacting spins.

I. INTRODUCTION

Parallel with the study of non-classical squeezed states of electromagnetic radiation [1], increasing attention has been devoted to the study of atomic spin squeezed states [2–14]. In Ramsey spectroscopy a sample of $N$ two-level atoms is represented by the collective operators $J_\alpha = \sum_{i=1}^{N} j_{\alpha,i}$ ($\alpha = x, y, or z$), where $j_{\alpha,i} = \sigma_{\alpha,i}/2$ and $\sigma_{\alpha,i}$ are the Pauli operators for the $i^{th}$ atom. Atomic spin squeezed states are quantum correlated states with reduced fluctuations in one of the collective spin components, and they have possible applications in atomic interferometers and high precision atomic clocks. Wineland et al. [3] have shown that the frequency resolution in spectroscopy depends on the spin squeezing parameter

$$\xi^2 = \frac{N(\Delta J_\perp)^2}{\langle \vec{J} \rangle^2},$$

where $\perp$ denotes a direction perpendicular to the mean spin. The inequality $\xi^2 < 1$ indicates that the system is spin squeezed, and it has been shown that any state with $\xi^2 < 1$ is an entangled state [10,11].

In 1993 Kitagawa and Ueda showed that spin squeezing is produced by simple nonlinear spin Hamiltonians in analogy with the non-linear Hamiltonians leading to squeezed light [2]. A number of experimental proposals for atomic spin squeezing have appeared involving interaction of atoms with squeezed light [3–7], quantum non-demolition measurement of atomic spin states [8], and atomic collisional interactions [9,10], and recently the first experimental realizations of spin squeezing have been achieved [12–14].

There is a link between the theory of quantum computing and the theory of spin squeezing through the observation that the register of quantum bits in a quantum computer constitutes an ensemble of effective spins, and a general purpose quantum computer can obviously produce a spin squeezed state of its register qubits. Moreover, proposals for quantum computing may be partly implemented, e.g., with only a restricted class of operations, which do not suffice for general computing purposes, but which do lead to massive entanglement and possibly to spin squeezing. We have previously considered such reduced instruction set (RISQ) quantum computers with ions and atoms [15], and we have pointed out that apart from their ability to address particular physics problem such as anti-ferromagnetism, they can also be used to synthesize useful quantum states. In this paper we present a theory for the spin squeezing expected for different models of interacting spins. These models encompass a number of theoretical proposals for quantum computing, and although spin squeezing may not be a particularly relevant property in, e.g., a spintronic or a quantum dot realization of a quantum computer, we wish to point out the possibility of verifying the entangling powers of the permanent or controllable interactions in these systems by simple measurements on the entire system. The detection of spin squeezing may constitute a useful diagnostic tool in the early stages of the construction of a quantum computer.

The treatment of the most general case of interacting spin systems is a formidable task, dealt with by a number of ingenious approximations in the theory of magnetism in solid state physics. In section [1] of the paper, we consider the case of an initially known state of the spins, subject to an arbitrary interaction Hamiltonian. By application of Ehrenfest’s theorem, we can determine the short time behaviour of the system analytically, we can identify the spin squeezing signal, and we can devise the optimum conditions for detecting this signal. In Section [11], we consider a
general Ising type Hamiltonian with the spins constituting a chain with only nearest neighbour interactions of varying magnitude. We obtain an analytical expression for the spin squeezing of the system at any future time, and we provide different examples of spin chains with constant, alternating and random couplings.

II. SHORT TIME BEHAVIOUR FOR GENERAL PAIRWISE INTERACTIONS

Consider a collection of spin 1/2 particles which only interacts through pairwise interactions. The most general Hamiltonian describing this situation is given by

$$H = \sum_{k \neq l} \vec{J}_k \cdot m^{kl} \cdot \vec{J}_l,$$

(2)

where $\vec{J}$ denotes the transpose $\vec{J}_k = (j_{x,k}, j_{y,k}, j_{z,k})$, and where $m^{kl}$ are real 3 by 3 matrices. We assume that the particles are initially prepared in a state where all spins are pointing in a given direction prepared for instance by optical pumping techniques, and we compute the time evolution of the noise in a component perpendicular to the direction of the mean spin. A convenient representation of this situation is by a collective spin state $\vec{R}(\alpha, \beta, \gamma)|JJ\rangle$, where $\vec{R}$ is a rotation operator given by the three Euler angles $\alpha$, $\beta$, and $\gamma$, and where $|JJ\rangle$ denotes an eigenstate of $\vec{J}^2$ and $J_z$ with eigenvalues $J(J+1)$ and $M$. The noise in a direction perpendicular to the mean spin can be found by calculating the square of the rotated $J_x$ operator $J_x = \vec{R}J_x\vec{R}^\dagger$.

With the general Hamiltonian (3) it is not possible to calculate the full time evolution of the noise. It is however straightforward by Ehrenfest’s theorem to determine the short time evolution

$$\frac{d}{dt}(\Delta J_\perp)^2 = \frac{i}{\hbar} \langle JJ|\{\vec{H}, J_\perp\}|JJ\rangle,$$

(3)

where we have introduced the transformed Hamiltonian $\vec{H} = \vec{R}^\dagger \vec{H} \vec{R}$ and we have used $\langle J_\perp\rangle = 0$. By using $\vec{R}^\dagger \vec{J}\vec{R} = \vec{R} \cdot \vec{J}$, where $\vec{R}$ is the 3 by 3 matrix representation of the rotation in coordinate space, we may express the transformed Hamiltonian as

$$\vec{H} = \sum_{k \neq l} \vec{J}_k \cdot \vec{m}^{kl} \cdot \vec{J}_l$$

where $\vec{m}^{kl} = \vec{R} \cdot m^{kl} \cdot \vec{R}$. By calculating the commutator and evaluating the expectation value we find

$$\frac{d}{dt}(\Delta J_\perp)^2 = \frac{1}{2\hbar} (0, 1, 0) \cdot \vec{M} \cdot \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right).$$

(4)

In this expression we have introduced a matrix $\vec{M} = \sum_{k \neq l} \vec{m}^{kl}$, and transformed it with the rotation matrix $\vec{M} = \vec{R}^\dagger \cdot \vec{M} \cdot \vec{R}$

In Eq. (4) the terms involving the $k$th and $l$th particles are $\vec{J}_k \cdot \vec{m}^{kl} \cdot \vec{J}_l$ and $\vec{J}_l \cdot \vec{m}^{lk} \cdot \vec{J}_k$. The sum of these terms can also be written as $1/2 (\vec{J}^T \cdot \vec{m}^{kl} \cdot \vec{J}) \cdot \vec{J}_l + \vec{J}_l \cdot (\vec{m}^{kl} \cdot \vec{J}^T \cdot \vec{J}_k)$. From this expression we see that without loss of generality we can assume the matrix $\vec{M}$ to be symmetric and hence diagonalizable and in a convenient coordinate system we have

$$\vec{M} = \left( \begin{array}{ccc} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_z \end{array} \right).$$

(5)

By calculating $\vec{M}$ using the form (4) for the matrix $\vec{M}$ and the well known rotation matrices $\vec{R}$, we can find the time derivative of the noise perpendicular to the mean spin for any orientation of the spin by using Eq. (4). Since we start out in a state with $\xi^2 = 1$ and since $d/dt(\vec{J})^2 = 0$ at $t = 0$, the interaction produces spin squeezing if we can find any set of Euler angles $\alpha$, $\beta$, and $\gamma$ which gives a negative derivative in Eq. (4). The optimal orientation of the spin is found by minimizing the derivative (4) with respect to the angles $\alpha$, $\beta$, and $\gamma$. We find that the extrema of Eq. (4) are always with the mean spin along one of the eigenvectors of $\vec{M}$. If the spin is polarized along the $z$-axis the change in the noise is maximal for the perpendicular components $J_{\pm\pi/4} = 1/\sqrt{2}(J_x \pm J_y)$ and we find

$$\frac{d}{dt}(\Delta J_{\pm\pi/4})^2 = \pm \frac{1}{4\hbar} (M_y - M_x).$$

(6)
Similar expressions are found if the spin is oriented along the $x$ or $y$-axes.

Note that no assumptions about the values of the coupling matrices are made, they may vary randomly for any pair $(k,l)$ of spins. Only the sum needs to be specified to determine the short time spin squeezing. The above argument only states that some squeezing will be produced, it does not say anything about the maximum squeezing. In the following section we shall analyze special cases of the interaction (9) where the squeezing can be calculated exactly for all times.

III. ISING CHAIN WITH NEAREST NEIGHBOUR COUPLING

Consider now a general model with arbitrary coupling constants between nearest neighbours in an Ising spin chain. The chain consists of $N$ spins with the Hamiltonian

$$H = \hbar \sum_{i=1}^{N} \chi_{i} j_{x,i} j_{x,i+1},$$

where we identify the $N + 1^{st}$ spin with the first one in the chain. Depending on the value of $\chi_{N}$, the chain can be equipped with open or closed boundary conditions.

This Hamiltonian arises in recent proposals for quantum computation with atoms in optical lattices [16, 17]. In these proposals the atoms interacts with the nearest neighbours and it has been shown that the interaction can be put in the form (6) and that this interaction produces spin squeezing [9]. Other application of this Hamiltonian in optical lattices can be found in [18]. Spin squeezing is much less experimentally challenging to produce and to verify than the full quantum computer, and squeezing can provide a demonstration of the entangling capabilities of the setup as well as having practical application in atomic clocks. From the discussion in section II follows that for short times the optimal squeezing is produced by having the spin initially polarized along $z$ and by looking at the noise in one of the components $J_{x=\pm \pi/4} = 1/\sqrt{2}(J_{x} \pm J_{y})$, where $\Delta_{ij}(\Delta J_{x=\pm \pi/4})^2 = \frac{1}{2} \sum_{i} \chi_{i}$. For longer time intervals the optimal noise reduction is in a component $J_{y} = \cos(\theta)J_{x} + \sin(\theta)J_{y}$ with $\theta \neq \pm \pi/4$ but for simplicity we shall only consider $\theta = \pm \pi/4$.

The unitary time evolution operator for the Hamiltonian $H$ can be written

$$U(t) = e^{-iHt/\hbar} = \prod_{i=1}^{N} e^{-i\chi_{i} j_{x,i} j_{x,i+1}},$$

and the operators $j_{i}(t)$ in the Heisenberg picture are readily obtained

$$
\begin{pmatrix}
  j_{x,i}(t) \\
  j_{y,i}(t) \\
  j_{z,i}(t)
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & \cos(\chi_{i} t j_{x,i-1} + \chi_{i} t j_{x,i+1}) - \sin(\chi_{i} t j_{x,i-1} + \chi_{i} t j_{x,i+1}) & 0 \\
  0 & \sin(\chi_{i} t j_{x,i-1} + \chi_{i} t j_{x,i+1}) & \cos(\chi_{i} t j_{x,i-1} + \chi_{i} t j_{x,i+1})
\end{pmatrix} \cdot
\begin{pmatrix}
  j_{x,i}(0) \\
  j_{y,i}(0) \\
  j_{z,i}(0)
\end{pmatrix}.
$$

We assume an initial state where all spins are pointing up, i.e., the $j_{z,i} = \frac{1}{2}$ eigenstate for all $i$, and Eq. (9) then provides the mean values of the collective spin components at later times:

$$
\langle J_{x} \rangle = \langle J_{y} \rangle = 0,
\langle J_{z} \rangle = \frac{1}{2} \sum_{i=1}^{N} \cos(\chi_{i} t j_{x} / 2) \cos(\chi_{i+1} t j_{x} / 2).
$$

Since the collective operator $J_{z}$ commutes with $H$, it follows that $\langle J_{z}^{2} \rangle$ retains its original value at $t = 0$: $\langle J_{z}^{2} \rangle = \frac{N}{4}$. In order to calculate $\langle J_{y}^{2} \rangle$, we need to know the the expectation values $\langle j_{y,i}(t) j_{y,i+k}(t) \rangle$. We always have $\langle j_{y,i}(t)^2 \rangle = \frac{1}{4}$, and by direct calculation we see that $\langle j_{y,i}(t) j_{y,i+1}(t) \rangle = 0$, whereas for $k = 2$ we obtain

$$
\langle j_{y,i}(t) j_{y,i+2}(t) \rangle = \frac{1}{4} \cos(\chi_{i+1} t j_{x} / 2) \sin(\chi_{i+1} t j_{x} / 2) \sin(\chi_{i+1} t j_{x} / 2) \cos(\chi_{i+1} t j_{x} / 2).
$$

From Eq. (9) we see that $\langle j_{y,i}(t) j_{y,i+k}(t) \rangle$ vanishes for $k \geq 3$, and we thus obtain

$$
\langle J_{y}^{2} \rangle = \frac{N}{4} + \frac{1}{2} \sum_{i=1}^{N} \cos(\chi_{i} t j_{x} / 2) \sin(\chi_{i} t j_{x} / 2) \sin(\chi_{i+1} t j_{x} / 2) \cos(\chi_{i+1} t j_{x} / 2),
$$

(12)
and similarly we can determine the expectation value

$$\langle J_y J_x + J_x J_y \rangle = -\frac{1}{2} \sum_{i=1}^{N} \sin \left( \frac{\chi_i + \chi_{i+1}}{2} t \right).$$

(13)

The degree of spin squeezing can be characterized by the squeezing parameter $\xi^2$ for a particular direction of the noise reduction. With the mean spin in the $z$ direction, and considering the reduction of the noise in the component $J_\theta$ the squeezing parameter can be calculated by

$$\xi^2_\theta = \frac{\cos^2(\theta) \langle J_y^2 \rangle + \sin^2(\theta) \langle J_y \rangle + \sin(\theta) \cos(\theta) \langle J_x J_y + J_y J_x \rangle}{\langle J_x \rangle^2},$$

(14)

and for the particular component $J_{\theta=\pi/4}$ the resulting squeezing factor is

$$\xi^2_{\pi/4} = \frac{N^2 + N \sum_{i=1}^{N} \cos \left( \frac{\chi_i t}{2} \right) \sin \left( \frac{\chi_i + \chi_{i+1} t}{2} \right) \sin \left( \frac{\chi_i + \chi_{i+1} t}{2} \right) - N \sum_{i=1}^{N} \sin \left( \frac{\chi_i + \chi_{i+1} t}{2} \right)}{\left[ \sum_{i=1}^{N} \cos \left( \frac{\chi_i t}{2} \right) \cos \left( \frac{\chi_i + \chi_{i+1} t}{2} \right) \right]^2},$$

(15)

which is a general expression for any set of values for the coupling coefficients $\chi_i$. For a uniform closed chain (all $\chi_i$ identical), the above equation reduces to

$$\xi^2_{\pi/4} = \frac{1 + 0.25 \sin^2(\chi t) - \sin(\chi t)}{\cos^2(\chi t/2)}$$

(16)
as found in [9].

**A. Ising model with few spins**

In the above expressions, coefficients $\chi_{i+k}$ with $i + k > N$ assume in a cyclic manner the values of the coupling among the first spins in the chain. It should accordingly be noted that Eq.(14) is not valid if $N \leq 4$, for which special expressions apply.

In the Ising model with only two spins $H_2 = 2h\chi_{jx,1,jx,2}$, the related expectation values are obtained as $\langle J_x \rangle = \langle J_y \rangle = 0$, $\langle J_z \rangle = \cos(\chi t)$, $\langle J_x^2 \rangle = \langle J_y^2 \rangle = 1/2$, and $\langle J_x J_y + J_y J_x \rangle = -\sin(\chi t)$, from which we obtain the squeezing parameter

$$\xi^2_{\pi/4} = \frac{1 - \sin(\chi t)}{\cos^2(\chi t/2)}.$$  

(17)

The system can be squeezed, and the maximum squeezing $\xi^2_{\pi/4} = 0.5$ occurs when $t = \pi/(2\chi)$.

For the Ising model with three spins $H_3 = h \sum_{i=1}^{3} \chi_{i,jx,i,jx,i+1}$, the squeezing parameter is

$$\xi^2_{\pi/4} = \frac{9 + 3 \sum_{i=1}^{3} \sin \left( \frac{\chi_i t}{2} \right) \sin \left( \frac{\chi_i + \chi_{i+1} t}{2} \right) - 3 \sum_{i=1}^{3} \sin \left( \frac{\chi_i + \chi_{i+1} t}{2} \right)}{\left[ \sum_{i=1}^{3} \cos \left( \frac{\chi_i t}{2} \right) \cos \left( \frac{\chi_i + \chi_{i+1} t}{2} \right) \right]^2},$$

(18)

and for a uniform closed chain this equation reduces to

$$\xi^2_{\pi/4} = \frac{1 + \sin^2 \left( \frac{\chi t}{2} \right) - \sin(\chi t)}{\cos^4 \left( \frac{\chi t}{2} \right)},$$

(19)

which is different from Eq.(14).

Finally, in the Ising model with four spins $H_4 = h \sum_{i=1}^{4} \chi_{i,jx,i,jx,i+1}$. The squeezing parameter is given by

$$\xi^2_{\pi/4} = \frac{8 \langle J_x^2 \rangle + \langle J_y^2 \rangle - 4 \sum_{i=1}^{4} \sin \left( \frac{\chi_i + \chi_{i+1} t}{2} \right)}{\left[ \sum_{i=1}^{4} \cos \left( \frac{\chi_i t}{2} \right) \cos \left( \frac{\chi_i + \chi_{i+1} t}{2} \right) \right]^2},$$

(20)
where
\[ \langle J_x^2 \rangle + \langle J_y^2 \rangle = 2 + \frac{1}{2} \sum_{i=1}^{4} \sin \left( \frac{\chi_i t}{2} \right) \sin \left( \frac{\chi_{i+1} t}{2} \right) \cos \left( \frac{\chi_{i+2} t}{2} \right) \cos \left( \frac{\chi_{i+3} t}{2} \right). \]  
(21)

For a uniform chain Eq. (20) actually reduces to Eq.(16), but for general coupling constants $\chi_i$, Eq. (15) is only valid for $N \geq 5$.

Fig. 1 is a plot of the squeezing parameter as a function of time for few spins in a uniform chain. It shows that maximum squeezing occurs for $N = 2$ and that the degree and the temporal range of squeezing are both small for $N = 3$ in comparison with the other cases.

**B. Dimerized and random chains**

It is interesting to study a dimerized Ising chain with even number of spins $N = 2M$, where the coupling constants are chosen as $\chi_i = \chi[1 + (-1)^{i+1}\delta]$. We have two values of the couplings $\chi_o = \chi(1 + \delta)$ for odd $i$ and $\chi_e = \chi(1 - \delta)$ for even $i$. Such systems appear naturally in heterogeneous structures where every second site is occupied with one type of qubit/particle, and their mutual communication is provided through intermediate particles acting as short range data-bus elements in e.g. the quantum computer. Two overlapping optical lattices with two different kinds of atoms or atoms in two different ground states may be moved relative to each other in order to establish the dimerized chain Hamiltonian.

For the dimerized Ising chain, Eq.(13) reduces to
\[ \xi_{\pi/4}^2 = \frac{1 + 0.25 \sin (\chi_o t) \sin (\chi_e t) - \sin (\chi t)}{\cos^2 \left( \frac{\chi_o t}{2} \right) \cos^2 \left( \frac{\chi_e t}{2} \right)}, \]  
(22)
and we see that the spin squeezing does not depend on the number of atoms. Obviously Eq.(22) reduces to (16) in the limit $\delta \to 0$, and in the limit $\delta \to 1$ it gives Eq. (17). Fig. 2 is a plot of the squeezing parameter as a function of the parameter $\delta$. We see that both the squeezing range and degree of squeezing increases as $\delta$ approaches unity, i.e., the introduction of a dimerized coupling makes the squeezing better. For $\delta$ larger than unity, the denominator vanishes within the time interval displayed in the figure, as shown for $\delta = 1.1$ which causes a divergence of the squeezing parameter at $\chi t = \pi/2.1 = 1.496$. 
Let us finally study a random chain model in which spins interact with their neighbours with a fixed coupling constant $\chi$, but only with probability $p$, i.e., any coupling constant is $\chi$ with probability $p$ and zero with probability $1-p$. This model corresponds, e.g., to the atomic lattice system with a non-unit filling fraction, so that empty lattice sites appear. Different assumptions may be made about the correlations between different lattice sites, but let us for simplicity consider the case where the occupancies are completely uncorrelated. Simulations for such a model were presented in Ref. [9], but we can now present analytical results (for simplicity presented only for the $\theta = \pi/4$ direction).

We introduce $\mu = \chi t/2$, in terms of which we can write the mean values of trigonometric functions,

$$
\frac{\cos(\chi_i t/2)}{2} = p \cos \mu + (1-p)
$$

$$
\frac{\sin(\chi_i t/2)}{2} = p \sin \mu
$$

$$
\sin\left(\frac{\chi_i + \chi_{i+1}}{2} t\right) = p^2 \sin 2\mu + 2p(1-p) \sin \mu.
$$

(23)

We assume a very large number of particles, so that the mean value of Eq. (15) over different realizations of the spin lattice, is effectively obtained by considering only a single realization consisting of a very large number of particles. The numerator is obtained simply as products of terms like in (23). The denominator of Eq. (16) involves the mean value of the square of a sum of products of cos-functions, and in the limit of large $N$, it is dominated by terms which are products of four uncorrelated cos-functions, and we obtain

$$
\xi_{\pi/4}^2 = \frac{1 + (1-p(1-\cos \mu))^2 p^2 \sin^2 \mu - (p^2 \sin 2\mu + 2p(1-p) \sin \mu)}{(1-p(1-\cos \mu))^4}
$$

(24)

As shown in Fig. 3 and observed already in [9], spin squeezing is obtained also in the case of a random filling.

FIG. 2. Squeezing parameter as function of time in the Ising model with dimerized coupling coefficients. The curves are obtained with the following values of $\delta$: 0 (solid curve), 0.5 (dashed), 0.75 (short-dashed), 1 (dotted), and 1.1 (dot-dashed). The curves with $\delta = 0$ and $\delta = 1$ are recovered in Fig. 1.

FIG. 3. Squeezing parameter as function of time in the Ising model with random coupling coefficients. Every coupling coefficient $\chi_i$ is a stochastic variable, attaining the value $\chi$ with probability $p$ and the value 0 with probability $(1-p)$. The curves are obtained with the following values of $p$: 0.25 (solid curve), 0.5 (dashed), 0.75 (short-dashed), and 1 (dotted).
IV. CONCLUSION

We have obtained general expressions for squeezing in different models of interacting spins. In the general model considered in section II we showed that almost any kind of spin-spin interaction can be used to create squeezed states. As long as the coupling matrix in Eq. (5) is not proportional to the identity, spin squeezing can be produced. In the one-dimensional Ising model with arbitrary nearest neighbour coupling constants we obtained analytical results for all times, and we addressed the case of constant couplings and different kinds of departure from constant couplings of the spins. We found that the amount of spin squeezing was larger in a dimerized model with periodically varying coupling coefficients than in a homogeneous model.

Among many possible generalizations of the present work we wish to mention long-range interactions and interactions in higher dimensions, and, e.g., the question whether extensions of the dimerized model to these cases are also superior over homogeneous couplings.

As mentioned in the introduction, spin squeezing is both a useful property in itself and a signature of entanglement. It can be accomplished and detected in different physical systems without the need for experimental access to individual spins, and as such it can be used to test, e.g., spintronics proposals for quantum information processing.

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