Knowledge Consensus in complex networks: the role of learning

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ABSTRACT

To reach consensus among interacting agents is a problem of interest for social, economical, and political systems. A computational and mathematical framework to investigate consensus dynamics on complex networks is naming games. In general, naming is not an independent process but relies on perception and categorization. Existing works focus on consensus process of vocabulary evolution in a population of agents. However, in order to name an object, agents must first be able to distinguish objects according to their features. We articulate a likelihood category game model (LCGM) to integrate feature learning and the naming process. In the LCGM, self-organized agents can define category based on acquired knowledge through learning and use likelihood estimation to distinguish objects. The information communicated among the agents is no longer simply in some form of absolute answer, but involves one’s perception. Extensive simulations with LCGM reveal that a more complex knowledge makes it harder to reach consensus. We also find that agents with larger degree contribute more to the knowledge formation and are more likely to be intelligent. The proposed LCGM and the findings provide new insights into the emergence and evolution of consensus in complex systems in general.

1 Introduction

To reach consensus among a population of agents is a problem with significant applications in social, economical, and political systems. A mathematical and computational paradigm to describe, characterize, and understand consensus dynamics is naming game (NG) and its variants. In an NG, agents attempt to reach consensus about a certain object or event via interactions. Most existing works emphasize on the consensus of vocabulary among the agents. While vocabulary is often related to objects and plays an important role in the development of natural language, for intelligent agents, another determining factor in NG is the contents and features of the objects. In fact, a pivotal cognitive ability of human being is categorization that recognizes, characterizes, and eventually names objects according to their features. In our daily activities, majority of the naming actions are based on a category rather than a specific term or vocabulary. For example, the word “cat” refers to a category in which objects have same features rather than a specific cat “Tom” or “Kitty.” For NG to better describe consensus dynamics in the real world, it is necessary to incorporate catagorization through learning and knowledge growth into the model. While there were previous works on categorization games that deal with feature recognition and analysis, the process of categorization itself remains to be an outstanding topic of research, due to the complexity of the underlying process. It is thus a challenging task to incorporate perception and categorization into NGs. The aim of this paper is to introduce a new kind of category game model to address this problem.

In previous works on NGs, consensus is achieved through some learning processes solely governed
by interactions between the agents. Therefore, topological features of the agents’ interaction network, such as degree, clustering coefficient, and path distance, have significant influences, and it is concluded that higher degrees, lower clustering and shorter network distance tend to promote consensus. The knowledge transfer in NGs is not limited between two agents, for example, Li et al. suggested a model with multiple hearers while Gao et al. considered negotiation to take place between multiple indirectly connected agents. An agent can also play the roles of a speaker and a hearer simultaneously. The more general setting was also studied in which the agents possess different propensities such as commitment and stubbornness, with the finding that the learning behaviors of these agents can affect significantly the consensus dynamics over the whole population. In addition, other issues for more realistic agents, such as learning errors during interactions, memory loss and multi-word/language, have been investigated. These works provide great insights into consensus dynamics. However, since only vocabulary is focused, significant feature-based cognitive abilities such as categorization were not taken into account which, as natural intuition would suggest, may play a more significant role in the emergence and evolution of consensus.

There were a few proposals on categorization games. For example, discrimination and guessing games were designed to accomplish the task of categorization. In a discrimination game, a speaker is trained with ground truth so that it can relate an object to the actual category using a classifier. Guessing game is similar to a typical NG, however, the hearer not only acquires the name but also updates his/her classifier for the category as instructed by the speaker. Based on these two types of games, the problem of color categorization was studied, and a category game model (CGM) was developed. In a game defined by CGM, a pair of objects are presented to a speaker and a hearer, and the target topic (the object to be learned) is selected by the speaker. If both objects are distinguishable to the speaker, one is randomly chosen as the target topic. Otherwise, the speaker must discriminate the two objects by creating a new boundary between them before selecting one as the target topic. The interaction between the speaker and the hearer then makes it feasible to learn and name the target topic. In the CGM framework, factors such as language aging, persistence, and individual biases can be studied. A feature common to both discrimination game and CGM is that certain pre-requisites are needed. Specifically, in a discrimination game, it is necessary to relate the object to an actual category, while in CGM, the two presented objects are assumed to belong to two different categories. In the real world, it often occurs that a population can reach an agreement without being given any ground truth, and this has consequences. For example, different language systems can give different color categorization, and the consensus of opinions among practitioners in a financial market can trigger a herd behavior that leads to the fat-tail distributions in prices and returns.

To understand and exploit consensus dynamics as realistically as possible requires the development of a more comprehensive type of game models centered about features. The base of our model is the recently proposed domain learning naming game (DLNG) to solve the color categorization problem through elimination of pre-requisites in the learning process. The color perception process in DLNG follows a variant algorithm based on the majority rule. Agents in DLNG have numerous sensors uniformly distributed in the domain, and one sensor can be dominated by at most one category. Then, an object is deemed as in a category if the majority of sensors near the object belong to that category. However, the original DLNG is a kind of coarse-grained model where a determined category may contain distinct objects. For example, intelligent agents such as humans are generally capable of assessing that crimson is much redder than magenta, although both colors belong to the same category of red.

To overcome this difficulty, here we propose a likelihood category game model (LCGM), in which self-organized agents define category based on acquired knowledge and use likelihood estimation to distinguish the objects in the same category. The information communicated among the agents is no longer...
simply some absolute answer but involves agents’ perception. That is, the agents are able to classify objects in terms of distinct likelihoods as determined by their knowledge and update knowledge through learning new objects. For the special case where the learning domain is highly localized, our model reduces to a variant of the minimum NG. The primary contribution of this paper is the proposed LCGM, with a new concept of likelihood to allow agents assess an object via perception, which better reflect our real experiences. Moreover, conclusive remarks related to consensus characteristics are provided. Firstly, it is noticed that short distance and heterogeneity can facilitate the consensus. Secondly, LCGM clearly reflects a more complex knowledge makes it harder to reach consensus. Lastly, agents with larger degree contributes more to the knowledge formation and consensus and accordingly to be more “intelligent”. Our work in LCGM provides novel insights into the process of consensus, with potential applications to predicting and controlling consensus – problems with implications to social, economical, and even political systems where achieving consensus is often a desired objective.

2 Model

2.1 Agent model

An object \( o_i \) in our LCGM is represented by a point in an \( N \)-dimensional domain of \( x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,N}) \in D \subseteq \mathbb{R}^N \). Each agent \( j \) maintains a set of categories \( \mathcal{C}_j \) in its memory. Each category, \( c_k \in \mathcal{C}_j \), is described by a weight \( \omega_k \) and \( N \)'s normal distributions \( \mathcal{N}(\mu_{k,n}, \sigma_{k,n}^2) \) for the corresponding \( n \) dimension with \( n = 1, 2, \ldots, N \). An agent, \( Agent_j \), classifies the object \( o_i \) based on a likelihood score that characterizes how likely \( o_i \) belongs to category \( c_k \). The score can be calculated through

\[
LS_j(o_i, c_k) = g(\omega_k) \prod_{n=1}^{N} f\left( \frac{x_{i,n} - \mu_{k,n}}{\sigma_{k,n}} \right) / f(0)
\]  

(1)

where \( f(\cdot) \) is the probability density function (PDF) of the standardized normal distribution and \( g(\cdot) \) is a scale function defined as

\[
g(\omega) = \tanh(\alpha \omega),
\]  

(2)

where \( \alpha \) is a constant. To be concrete, we set \( \alpha = 2.5 \), so \( g(1) = 0.987 \approx 1.0 \). Equation (1) indicates that the likelihood score contains two components: prior and likelihood. The prior component, represented by \( g(\omega_k) \), indicates the probability that category \( c_k \) is selected for an arbitrary object. The likelihood component is specified by the product term in Eq. (1). The normalized value \( f\left( \frac{x_{i,n} - \mu_{k,n}}{\sigma_{k,n}} \right) / f(0) \) measures the likelihood of \( o_i \) belonging to category \( c_k \) along the \( n \)th dimension. In the absence of any correlation among different dimensions, the product term gives the joint probability value.

In our model, each agent can execute four possible activities: object identification, category updating, category creation, and category deletion, which are described, as follows.

Object identification activity (OIA). Given an object \( o_i \), Agent \( j \) returns \( LS_j(o_i, c_a) \) and the corresponding name \( a \), where \( c_a = \arg \max_{c_k \in \mathcal{C}_j} LS_j(o_i, c_k) \) is the category having the highest score. For \( \mathcal{C}_j = \emptyset \), Agent \( j \) will return a null name (name \( a = \) null) and a zero likelihood score \( [LS_j(o_i, c_a) = 0] \). In addition, if \( LS_j(o_i, c_a) \) is below a specified threshold, denoted as LST, then the agent will also return a null name and a zero likelihood score. An illustrative example of OIA for the one-dimensional case is given in Fig. 1.

Category updating activity (CUA). If \( o_i \) is assigned to category \( c_a \), the features \( (\mu_{a,1}, \sigma_{a,1}, \ldots, \mu_{a,N}, \sigma_{a,N} \) and \( \omega_a) \) of \( c_a \) will be updated by merging the information of \( o_i \). Since \( o_i \) is solely represented by the
Figure 1. An illustrative example of object identification in one dimension. Suppose the agent has two categories: “Blue” and “Red”. With regard to objects “a”, “b”, “c” and “d”, there are four cases (a, b, c and d, respectively). In case a, since \( LS(a, \text{Blue}) > LS(a, \text{Red}) \) and \( LS(a, \text{Blue}) > LST \), the agent returns the name of category “Blue” and \( LS(a, \text{Blue}) \). In case b, \( LS(b, \text{Red}) > LS(b, \text{Blue}) \) and \( LS(b, \text{Red}) > LST \), the name of category “Red” and \( LS(b, \text{Red}) \) are returned. In case c, \( LS(c, \text{Blue}) \) is higher but still lower than LST, so the agent returns name = null and \( LS = 0 \). In case d, \( LS(d, \text{Red}) = LS(d, \text{Blue}) > LST \), resulting in category “Red” or “Blue” being randomly picked. Case d is rare because it occurs only at the intersection point of the two curves.

Point \( x_i \) in the domain \( D \), it is reasonable to let the weight of \( x_i \) be one. As a result, \( \omega_a \) is increased by one, forming a new weight

\[
\omega_a^* = \omega_a + 1
\]  

For each \( n \)th dimension, the updated mean, \( \mu_{a,n}^* \), is the weighted average of the category \( c_a \) and the point \( x_i \), which can be obtained through

\[
\mu_{a,n}^* = \frac{\omega_a \times \mu_{a,n} + x_i,n}{\omega_a^*}
\]  

Finally, the updated \( \sigma_{a,n}^* \) can be obtained from the following equation:

\[
\sigma_{a,n}^* = \max \left( \frac{-|x_i,n - \mu_{a,n}^*|}{F\left( \frac{1}{2\sigma_a^*} \right)}, \frac{-|\mu_{a,n} - \mu_{a,n}^*|}{F\left( \frac{\omega_a}{2\sigma_a^*} \right)}, \sigma_{a,n} \right)
\]  

where \( F(\cdot) \) is the CDF of \( f(\cdot) \). An illustrative example is given in Fig. 2. Note that \( \sigma \) of the category distribution is non-decreasing because only exceptions will result in its changes while normal observations will have no effect on \( \sigma \).

Category creation activity (CCA). If an object \( o_i \) with name \( i \) concluded in a game cannot be related to any existing category associated with the agent, a new category \( c_a \) is created. The parameters \( \sigma_a \) and \( \omega_a \) are set at some default values, while \( \mu_a = x_i \) and name\(_a = \text{name}_i \).
In reality, if knowledge is not recalled, it will be forgotten gradually. Agent φσ should be a Dirac delta function (S) where name is the winner, one has name becomes larger than one \[ \text{Eq. (3)} \] via CUA. Similarly, in LCGM, category will be removed from one’s memory if there is no updating, i.e., it has not been learned for a long time. The removal of a category is based on its weight. For any time \( t \), the weight of category \( c_k \), \( \omega_k \) is updated as

\[
\omega_{k,t} = \omega_{k,t-1} \times e^{-\phi}
\]

where \( \phi \) is the “forgetting” factor that describes the “forgetting speed” of a category in the agent’s memory, \( \omega_{k,t} \) and \( \omega_{k,t-1} \) are the value of \( \omega_k \) at time \( t \) and \( t - 1 \), respectively.

As Eq. (1) indicates, a reduction in \( \omega_k \) will also affect the chance of category \( c_k \) being assigned in OIA. If \( \omega_{k,t} \) further reduces and becomes smaller than a pre-defined threshold (denoted as \( \omega_T, \phi \)), the category \( c_k \) will be removed from the memory of the agent. However, once an object falls into category \( c_k \), \( \omega_k \) becomes larger than one [Eq. (3)] via CUA.

### 2.2 Game rules.

Game participants are all agents in a population on a network. Initially, every agent has empty memory: \( \mathcal{G} = \emptyset \), \forall j. A pair of connected agents, Agent\( _A \) and Agent\( _B \), are randomly selected in a game that proceeds as follows.

1. An object \( o_i \) is presented to both agents. Via OIA, Agent\( _A \) returns name\( _a \) and \( LS(x_i, c_a) \), while Agent\( _B \) returns name\( _b \) and \( LS(x_i, c_b) \).

2. Agent\( _A \) and Agent\( _B \) conclude the name of \( o_i \), referred to as name\( _i \), based on the following rules: if \( LS(o_i, c_a) > LS(o_i, c_b) \), name\( _i = \text{name}_a \) and Agent\( _A \) wins the game; if \( LS(o_i, c_a) < LS(o_i, c_b) \), name\( _i = \text{name}_b \) and Agent\( _B \) wins the game; if \( LS(o_i, c_a) = LS(o_i, c_b) = 0 \), name\( _i \) is randomly generated and the game is a draw; if \( LS(o_i, c_a) = LS(o_i, c_b) \neq 0 \), a winner is randomly selected. For example, if Agent\( _A \) is the winner, one has name\( _i = \text{name}_a \). In addition, if name\( _a = \text{name}_b \neq \text{null} \), the game is successful and name\( _i = \text{name}_a \). Otherwise, it has failed.
(3) Agent\(_A\) and Agent\(_B\) update/create their categories according to a set of rules. The rules for Agent\(_A\) can be described as follows (similar for Agent\(_B\)). If name\(_i\) equals the name of a category, say \(c_k \in C_A\), CUA is operated on category \(c_k\). However, if name\(_i\) is not contained in any category in \(C_A\), CCA is carried out to create a new category with name\(_i\) and \(\mu_i = x_i\).

(4) All agents in the population delete expired categories via CDA.

Two illustrative examples of the game process are given in Fig. 3.

![Game I and Game II](image)

**Figure 3.** Two illustrative examples of game process in terms of color objects. In Game I, via OIA, Agent\(_A\) considers that the object is “Blue” with 50% certainty, while Agent\(_B\) regards it as “Green” with 70% certainty. As Agent\(_B\) is more confident, the object is concluded as “Green.” The two agents then update their memories based on the new knowledge that the object is named “Green.” In addition, both agents forget categories that have not been recalled for a long period of time. In Game II, neither agent can identify the object. The agents return name = null and \(L_S = 0\) via OIA. As a result, a new name, say “Green,” is assigned. The remaining steps [Steps (3) and (4)] are the same as those in Game I.

### 3 Methods

We focus on color categorization to demonstrate and analyze LCGM in the present study. Color objects are randomly generated in RGB form with sample size \(256 \times 256 \times 256\), and are then mapped into CIELab color space for agents. Specifically, the color objects and categories are represented in the CIELab color space which is three dimensional: \((L^*, a^*, b*)\), where \(L^*\) stands for the lightness, \(a^*\) and \(b^*\) are the two abstract opponent dimensions\(^{25}\). The CIELab color space is assumed to be homogeneous, i.e., the difference between two color objects is defined as the Euclidean distance between them, as specified by the
year 1976 version of \textit{CIELab} color space. (Remark: To better fit with human vision, $\Delta E$ was re-defined in the year 2000 version\textsuperscript{26} where the color space is no longer homogeneous. However, this does not affect our main results.)

LCGM is applied to a population of agents connected through a pre-defined network, either from the real world or from a specific model. Initially, each agent has an empty memory. Games are conducted as described in \textit{Game Rules}, and repeated until a predefined number of iterations is reached. For all the simulations in this paper, the iteration number is $10^7$.

Model parameters are set as follows.
1. Scaling factor in Eq. (2): $\alpha = 2.5$
2. Default value of $\sigma$ in CCA: $\sigma_{\text{default}} = 5.0$. Empirically, two colors with the value of $\Delta E$ between 1 and 10 appear similar in human vision, so we set $\sigma_{\text{default}} = 5.0$.
3. Default value of $\omega$ in CCA: $\omega_{\text{default}} = 1.0$.
4. The threshold parameter $\omega_{T_h}$: a category will be deleted during CDA if its weight $\omega$ is smaller than $\omega_{T_h}$. We set $\omega_{T_h} = 0.01$.
5. Likelihood score threshold (LST): If the LS computed from an object-category pair is smaller than LST, the object should not belong to that category. We choose different values for LST to investigate its impact on the learning process.
6. Forgetting factor $\phi$: it characterizes the speed at which a category is forgotten. We choose different values of $\phi$ to investigate its impact on the learning process.

We conduct simulations by applying LCGM to two social networks (a subgraph of Facebook network and a subgraph of E-mail network) and various heterogeneous networks. The results from the Facebook network and the E-mail network are averaged by 10 runs. To investigate the effect of network topology, artificial networks are constructed. For each setting, ten network realizations are used to calculate the various statistical averages.

\section*{4 Results}

To demonstrate LCGM, we test it on learning color categorization by a population of agents. We define the following performance metrics.

\textbf{Accuracy (ACC)} indicates the success rate of the games, which is computed for every $K$ games and defined as:

$$ACC = \frac{k_{\text{success}}}{K}$$  \hspace{1cm} (7)

where $k_{\text{success}}$ is the number of successful games in every $K$ iterations.

\textbf{Number of categories (NC)} specifies the number of categories maintained by each agent, which reflects the resolving power of agents after learning. The average number of categories ($NC_{\text{Avg}}$) is primarily concerned in this paper.

\textbf{Total number of distinct names (TDN)} records how many distinct names remained in the whole population.

\textbf{Consensus score (CS)} characterizes the consensus of the whole population. A set of sampled objects ($\mathcal{O}$) is firstly selected. In this work, $\mathcal{O}$ consists of 512 objects uniformly sampled from the domain $D$. These objects are presented to every pair of agents in turns (including agents who are not neighbors) and games are performed. For Agent $i$ and Agent $j$ with $i \neq j$, $CS_{i,j}$ is defined as

$$CS_{i,j} = \frac{\text{agm}_{i,j}}{|\mathcal{O}|}$$  \hspace{1cm} (8)
where \( agm_{i,j} \) is the total number of successful games that Agent\(_i\) and Agent\(_j\) made for all \( o_i \in \mathcal{O} \), and \(|\cdot|\) is the cardinality of a set. The CS of the entire population \( \mathcal{Y} \) is the average over all possible agent pairs, which can be computed as

\[
CS = \frac{\sum_i \sum_{j \neq i} CS_{i,j}}{|\mathcal{Y}| \times (|\mathcal{Y}| - 1)}
\]  

(9)

4.1 Application: consensus in social network

We apply our LCGM algorithm to two social networks, a subgraph of Facebook social network\(^{27}\) and a subgraph of an E-mail network\(^{28}\), collected by the SNAP laboratory of Stanford university. The Facebook subgraph has 1,034 nodes and 26,749 edges. Its clustering coefficient, average path length and degree-based Gini coefficient are 0.5, 2.9 and 0.48, respectively. The E-mail network has 986 nodes and 25,552 edges. Its clustering coefficient, average path length and degree-based Gini coefficient are 0.31, 2.6 and 0.56, respectively. The relatively large value of the clustering coefficient and short average path length are typical of small-world networks, while the relatively high values of the Gini coefficient suggest that the networks are inhomogeneous.

Typical simulations with \( LST = 0.1 \) and \( \phi = 0.00002 \) were conducted. On Facebook social network, ACC reaches 88.0%, the average NC and TDN are about 4.7 and 6.8, respectively. And on E-mail network, ACC reaches 88.6%, the average NC and TDN are 4.6 and 5.8, respectively.

However, the high accuracy does not guarantee full consistencies of agents under the framework of category game. Since accuracy only considers gaming actions occur between two mutual neighbors, consensus between agents who are not neighbors within the population is ignored. The consensus score (CS) is then used to characterize the consistency of the whole population. The average CS for Facebook social network and E-mail network are about 0.71 and 0.85, respectively.

The performance of CS was further investigated. The histograms of consensus score are plotted in Fig. 4 (a) and (b). The majority of agent pairs have CS higher than 0.8 for both networks. However, some agent pairs have relatively low CS, particular for the Facebook social network. Then, we analyzed the influence of distance of agent pairs and observed a negative correlation between CS and distance of agent pairs (See Fig. 4 (c) and (d)). It is because agents involved in the transmission path would incorporate their understandings to the information. As a result, a longer distance between two agents generally imposes a higher variation in the knowledge between them.

The adoption of LCGM on the two social networks revealed that agents basically reach the consensus through the mutual learning, with consensus score 0.71 (the Facebook network) and 0.85 (the E-mail network) respectively. While the CS of the E-mail network is much higher than that of the Facebook network even though they have similar ACC. The extended analysis on CS and distance implies that the shorter average path length of the E-mail networks may facilitate the consensus. Other topology properties, such as clustering and heterogeneity, will be discussed later in this section.

Based on LCGM, an interesting question is, among the agents, who more frequently takes the lead in the game by providing the answer? The leader is the winner of the game as specified by the game rules. To identify the leader, we define the following game score for each agent:

\[
GS(A) = \sum_i GS_i(A)
\]  

(10)
Figure 4. Consensus score (CS) of agent pairs. Shown are (a) the histogram of the consensus score of Facebook social network, (b) the histogram of the consensus score of E-mail network, (c) boxplot of the correlation between distance and CS of Facebook social network and (d) boxplot of the correlation between distance and CS of E-mail network.

where $GS_i(A)$ is the game score obtained by Agent$_A$ in $i$th game:

$$GS_i(A) = \begin{cases} 
1, & \text{if Agent}_A \text{ participated in the } i \text{th game and won} \\
0.5, & \text{if Agent}_A \text{ participated in the } i \text{th game and the result was a draw} \\
0, & \text{otherwise.} 
\end{cases} \quad (11)$$

Equation (11) indicates that our proposed learning game is effectively a positive-sum game. In each game, the increment of the total score is one. When an agent wins the game, it dominates the game solely and takes the whole score. When the game is a draw, both players score 0.5 as they contribute to knowledge formation equally. In this case, the loser receives no penalty for the reason that learning should not be discouraged. To eliminate the effect of opportunity earning (agents involved in more games likely will have higher scores), we further define the following game scoring rate:

$$GSR(A) = \frac{GS(A)}{\#\text{Games}(A)} \quad (12)$$

where $\#\text{Games}(A)$ is the total number of games involving Agent$_A$. 

9/15
As it is similar for both studied social networks, we only discuss the results of the Facebook social network here. Figure 5 (a) shows the GSR distribution, giving a bell-shape in which the majority (93.5%) of agents have GSR values between 0.4 and 0.6. To check whether the distribution is normal, we use the Quantile-Quantile plot (QQ plot). Specifically, a dot \((x, y)\) in the QQ plot means \(F_X(x) = F_Y(y)\), where \(F_X(x)\) is the cumulative density function of random variable \(X\). If two distributions are identical, all dots would be located along the diagonal. As shown in Fig. 5 (b), dots within \(x \in (-2, 2)\) are quite close to the diagonal, indicating that the GSR possesses the characteristic of a normal distribution in this range. Such a bell-shape distribution reflects most agents have moderate intelligence after sufficient learning and communications. Furthermore, the emergence of a light right tail and a heavy left tail observed in Fig. 5 (b) indicates that “geniuses are minority” and “dummies are more than you expected.”

![Figure 5. Statistical features of game scoring rate for the Facebook social network. Shown are (a) the histogram of the game scoring rate (GSR) and (b) the corresponding Quantile-Quantile plot (QQ plot) for \(\mu_{GSR} = 0.49\) and \(\sigma_{GSR} = 0.06\).](image)

The correlation between GS/GSR and degree is shown in Fig. 6. As defined in Eqs. (10) and (11), GS reflects the frequency of transmitting self-knowledge to others. Figure 6(a) shows that agents with large degrees are likely to have a high GS, implying that agents with larger degree contributed more to the consensus process. Different from GS, GSR measures the capability of scoring in a game [Eq. (12)]. Figure 6(b) reveals a positive correlation between GSR and degree, indicating that agents with larger degrees are more “intelligent” with respect to eventual knowledge formation through more active learning and communication.

### 4.2 Effects of Likelihood score threshold (LST) and forgetting factors (FF)

Then, the effects of likelihood score threshold (LST) and forgetting factors (FF) of categories were investigated. Again, the studies were based on the Facebook social network.

**Likelihood score threshold of agents.** As shown in Fig. 1, LST imposes the minimum requirement for classifying an object into a category. Accordingly, a high LST value encourages the generation of a new
Figure 6. Correlation between GS/GSR and degree centrality for the Facebook social network. The Pearson correlation coefficients are $\rho_{\log(d),GS} = 0.60$ and $\rho_{\log(d),GSR} = 0.40$.

category. To investigate the effect of LST, we simulate the learning process for different values of LST with fixed forgetting factor $\phi = 0.00002$ in Eq. (6).

As shown in Table 1 (left), agents with high LST (the “rigorous” agents) spontaneously possess more categories. And the more complex knowledge makes it harder to reach consensus, resulting a lower accuracy. This is further confirmed by the plot of consensus scores with different LST as shown in Fig. 7 (a), which clearly shows that the increment of LST would depreciate CS significantly.

Table 1. Summary of LGCM results (after $10^7$ iterations) for different values of LST/FF Each data point is the result of averaging over ten game realizations.

| LST ($\phi = 0.00002$) | FF (LST = 0.2) |
|------------------------|----------------|
| LST | ACC | Avg. NC | TDN | $\phi$ | ACC | Avg. NC | TDN |
|----------|------|---------|------|------|------|---------|------|
| 0.1      | 88.0%| 4.7     | 6.8  | 0.00001| 81.3%| 10.2    | 15.4 |
| 0.2      | 85.5%| 5.5     | 7.7  | 0.00002| 85.5%| 5.5     | 6.7  |
| 0.3      | 84.1%| 7.5     | 10.0 | 0.00004| 89.3%| 3.6     | 4.3  |
| 0.4      | 82.3%| 9.3     | 42.3 | 0.00008| 93.8%| 2.2     | 2.2  |
| 0.5      | 79.4%| 13.6    | 412.9| 0.00016| 98.0%| 1.3     | 1.3  |
| 0.6      | 71.9%| 27.6    | 6472.6| 0.00032| 99.9%| 1.0     | 1.0  |
| 0.7      | 38.6%| 154.6   | 67670.0| 0.00064| 79.8%| 3.55    | 1399.6|
| 0.8      | 4.2% | 394.3   | 174366.5| 0.00128| 0%   | 6.92    | 3549.2|

Forgetting factor of categories. We next study the mechanism responsible for categories to be gradually forgotten associated with category deletion activities. In general, the threshold for removal of a category, $\omega_{Tb}$, should be small because, if $\omega_{Tb}$ is large, even useful categories would be deleted, possibly resulting in a dramatic effect on the learning process. However, if the value of $\omega_{Tb}$ is too small, the category with small $\omega$ results small likelihood score, leading to a small probability of any update (learning). These empirical considerations lead to our choice of $\omega_{Tb} = 0.01$.

To be concrete, we fix LST = 0.2 and focus on the impact of varying the forgetting factor $\phi$. The results are shown in Table 1 (right) and Fig. 7 (b). When $\phi$ increases, it becomes hard for agents to remember information, eventually leading to a reduction in the number of categories. Apparently, when
there are fewer categories, it is easier to reach consensus, which is verified through the ACC and CS results. However, the correlation between the FF and CS is non-monotonous. For $\phi > 0.00032$, the CS decreases dramatically and even equals to 0 for $\phi = 0.00128$ through the process. That is, quite naturally, consensus can never be reached if agents forget things too fast.

It is remarked that, the average result of CS is given. For the case with $\phi = 0.00064$, some experiments reached consensus with only one category (similar to the case with $\phi = 0.00032$) while others had $CS = 0$ (similar to the case with $\phi = 0.00128$).

4.3 Effect of network topology.

Does the network topology have a significant effect on LCGM? To address this question, we considered four topological characteristics, namely the average degree, network size, heterogeneity, and clustering.

Based on original BA scale-free networks, the impacts of average degree (with fixed network size) and network size (with fixed average degree) onto the performance of LCGM were investigated. The simulation results show no observable influence on LCGM by varying the average degree (see Supplementary Table S1). While the network size affects the consensus of LCGM slightly (see Supplementary Table S2), network with smaller size can reach a better consensus, which meets most people’s intuition that consensus are easily to be reached among fewer agents.

To investigate the effects of heterogeneity and clustering, two types of heterogeneous networks were adopted: scale-free network with aging and scale-free network with clustering attachment. The first type of networks incorporates the aging effect of nodal attraction, and the connection probability is redefined as

$$\text{Prob}(i) = \frac{\text{degree}_i \times e^{-\lambda \times \text{age}_i}}{\sum_j \text{degree}_j \times e^{-\lambda \times \text{age}_j}}$$

(13)

where $\lambda$ is a tunable parameter reflecting the aging speed of nodal attraction and $\text{age}_i$ denotes the age of node $i$. When node $i$ is newly added, $\text{age}_i = 0$. After every nodal addition cycle, $\text{age}_i$ is incremented by one. Networks with distinct degrees of homogeneities can be generated by adjusting the value of $\lambda$.

We perform LCGM on the aging scale-free networks of size 1000 with average degree of 10 for different values of $\lambda$. As $\lambda$ is increased, the Gini coefficient is significantly reduced and the average path length increases, but the clustering coefficient is hardly affected. We find that an increase in the value of $\lambda$ has little effect on the accuracy (ACC) and the average number of categories ($NC_{avg}$) in LCGM.
However, the consensus score (CS) decreases significantly with $\lambda$ even though the accuracy remains high (see Supplementary Table S3). As discussed before, there is a negative correlation between the agent-pair distance and CS. Therefore, we further investigated average consensus scores of agent pairs with respect to distance and $\lambda$ (see Supplementary Table S4). It is observed that $\lambda$ has no observable influence on CS between agents with short distance ($\leq 2$), while for agent pairs with longer distance, CS decreases generally as the network becomes more homogeneous ($\lambda$ increases).

Then, the effect of clustering on LCGM is investigated by considering the conventional clustering coefficient. Scale-free networks with different clustering coefficients are obtained through the process of clustering attachment where an edge is added to connect a new node and one of its two-hop neighbors, forming a triangle. The probabilities of clustering and preferential attachment are $p$ and $1 - p$, respectively. If no two-hop neighbor is available for a new node, preferential attachment is adopted. We perform LCGM on networks of size 1000 (with average degree of 10) for different values of $p$ (See results in Supplementary Table S5). It can be easily observed that the clustering coefficient increases with the value of $p$. An increase in the value of $p$, however, has only limited effect on the Gini Coefficient and the average path length. From the results, it can be concluded that the clustering coefficient does not hurt the consensus. Since categories have weights in LCGM, several strong categories will survive through evolution. Different from traditional naming game models, LCGM allows these categories to coexist, and consequently, they will dominate the whole population together.

5 Discussion

To reach consensus among a population of diverse agents is a problem of great complexity but one with broad relevance. A computational and mathematical paradigm to investigate this problem is naming games. Most existing works in this area were based on some learning processes that solely rely on direct interactions among the agents without taking into account agents’ perception. The basic idea underlying our proposed likelihood category game model (LCGM) is that knowledge acquiring is essential to achieving consensus. In LCGM, self-organized agents rely on acquired knowledge to define category and employ statistical likelihood estimation to distinguish and “name” objects that belong to the same category. Particularly, the agents equipped with knowledge acquiring are capable of exploiting distinct likelihoods as determined by their knowledge to classify objects. Importantly, knowledge is not static but dynamic: agents update knowledge through learning. The agents in our LCGM are thus “smart,” more closely mimic those in the real social, economical, and political world.

The distinct features of our proposed LCGM are the following. Firstly, it is a truly autonomous category game model, eliminating the need for ground truth knowledge. Secondly, introducing the concept of likelihood in this context makes the game model more realistic because, in each game, agents not only return the names but also the likelihood scores that represent their confidence corresponding to the answer. The information communicated among the agents is “smart” in the sense that it is no longer restricted to some form of absolute answer but is more content or feature based. Thirdly, in a pairwise game, the more knowledgeable agent (with a higher likelihood score) becomes the “teacher,” a feature that fits with the interaction in the real world. Through the incorporation of learning, our model generalizes the existing naming game models and is closer to describing reality. (Our model reduces to a variant of the much studied minimum name game in the special case where the learning domain is singular in the sense that its extent is effectively zero, i.e., no learning.)

Our model provides novel insights into consensus dynamics. For example, there is a trade-off between the amount of knowledge and consensus, providing a quantitative explanation for the phenomenon that consensus is hard to be reached among serious agents with a high LST (who know more categories).
Another finding is that hubs contributed more to knowledge formation, which accordingly have a larger probability to become “smart” and to take lead in a game. By studying the effects of network structural characteristics on the consensus dynamics, we identify the impacts of distance and heterogeneity on consensus. While the findings are preliminary, they are useful for understanding the dynamical evolution of consensus and may even serve as the base for formulating control strategies to harness consensus dynamics, warranting further efforts in investigating learning and likelihood based games.

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W.K.S.T. and Z.Y.F. conceived research. Z.Y.F. performed computations. W.K.S.T. and Z.Y.F. analyzed the results. W.K.S.T., Z.Y.F., and Y.C.L. wrote the paper.