Joule heating in the normal-superconductor phase transition in a magnetic field

J. E. Hirsch

Department of Physics, University of California, San Diego, La Jolla, CA 92093-0319

Joule heating is a non-equilibrium dissipative process that occurs in a normal metal when an electric current flows, in an amount proportional to the metal’s resistance. When it is induced by eddy currents resulting from a change in magnetic flux, it is also proportional to the rate at which the magnetic flux changes. Here we show that in the phase transformation between normal and superconducting states of a metal in a magnetic field, the total amount of Joule heating is determined by the thermodynamic properties of the system and is independent of the resistivity of the normal metal. We also show that Joule heating only occurs in the normal region of the material. The conventional theory of superconductivity however predicts that Joule heating occurs also in the superconducting region within a London penetration depth of the phase boundary. This implies that there is a problem with the conventional theory of superconductivity.

PACS numbers:

I. THE PROBLEM

Consider a cylindrical type I superconductor in the presence of a uniform magnetic field along its axis undergoing a transition from the normal (N) to the superconducting (S) state as shown in Fig. 1, or from the superconducting to the normal state as shown in Fig. 2. For simplicity we assume cylindrical symmetry throughout the process. In the superconducting region within a London penetration depth $\lambda_L$ of the phase boundary,

![FIG. 1: N-S transition. Cylindrical superconductor of unit height $h$ and radius $a << h$ seen from the top. Magnetic field $H$ points out of the paper. The phase boundary denoted by $r_0(t)$ is moving outward. In the normal region (dark grey), and in the superconducting region within a London penetration depth $\lambda_L$ of the phase boundary, normal electrons become electrons superconducting.](image1)

![FIG. 2: S-N transition. The phase boundary denoted by $r_0(t)$ is moving inward. The Faraday electric field $E_F$ and normal current $J_n$ is in opposite direction to Fig. 1, the supercurrent $J_s$ is in the same direction as in Fig. 1. Of radius $r_0(t)$, a supercurrent $J_s$ flows that nullifies the magnetic field in the interior. We will call that region the ‘boundary layer’ in what follows. A Faraday electric field $E_F$ exists throughout the normal region $r \geq r_0(t)$ as well as in the boundary layer during the transition, that points counterclockwise in the N-S transition and clockwise in the S-N transition, as shown in the figures. The Faraday electric field induces a normal current in the normal region during the transition process, that will dissipate Joule heat.](image2)

In this paper we show that the total Joule heat dissipated is independent of the resistivity of the normal metal and of the rate at which the process occurs, and depends only on thermodynamic properties of the system. To our knowledge, this has not been pointed out in the literature before. In addition, we show that thermo-
dynamics requires that no Joule heat is dissipated in the boundary layer during the transition. We point out that the conventional theory of superconductivity [1] predicts the existence of such a normal current and associated Joule heat in the boundary layer. Therefore we conclude that there is a problem with the conventional theory of superconductivity. Instead, we point out that the problem does not arise within the alternative theory of hole superconductivity [2].

In recent work we have shown that thermodynamic considerations for a superconductor in a magnetic field in a process where the temperature is changed between temperatures, both below $T_c$, lead to the same conclusion regarding the conventional theory of superconductivity [3]. In other recent work we have shown that consideration of the entropy production associated with transfer of momentum between electrons and the body during the transition between normal and superconducting states also leads to the same conclusion [4].

II. THERMODYNAMICS

We consider first the N-S transition shown in Fig. 3, i.e. the Meissner effect, in an applied magnetic field $H_c(1-p)$. Here, $H_c \equiv H_c(T_2)$ and $H_c(1-p) \equiv H_c(T_1)$ are the critical magnetic fields at temperatures $T_2$ and $T_1 > T_2$, and $p > 0$. The system is initially in the normal state denoted by 1, at temperature infinitesimally above $T_1$. In the final state 2, at temperature $T_2$, the system is in the superconducting state with the magnetic field excluded from its interior. The state 1' shown in Fig. 3 denotes the system in the superconducting state at temperature infinitesimally below $T_1$ with the magnetic field

![FIG. 3: N-S transition: states 1 and 1' are normal and superconducting states on the coexistence curve at temperature $T_1$, state 2 is the superconducting state at lower temperature $T_2$, all in external field $H_c(T_1) \equiv H_c(1-p)$. The critical field at temperature $T_2$ is $H_c \equiv H_c(T_2)$. States 1, 1', and 2 are equilibrium states. The supercooled non-equilibrium state 2' is the normal state for the same temperature and magnetic field as the equilibrium superconducting state 2. The system interchanges heat with a heat reservoir at temperature $T_2$.](image)

![FIG. 4: Two routes for the transition between equilibrium states 1 and 2 in figure 3 (see text). $L$ denotes latent heat. The state 2' is a non-equilibrium state where the system is in the supercooled normal state at temperature $T_2$, magnetic field $H_c(1-p)$, excluded. The three states 1, 1' and 2 are equilibrium states of the system. Consider the two different routes between the same initial and final equilibrium states 1 and 2 shown in Fig. 4, denoted by route A and route B. In route A, the system undergoes the normal-superconductor (N-S) transition at the coexistence curve at temperature $T_1$ transferring latent heat $L(T_1)$ to a reservoir at temperature $T_2$. Then, it cools to temperature $T_2$ transferring additional heat to the reservoir and coming into thermal equilibrium with it. The transition proceeds infinitely slowly because it occurs on the coexistence curve, and no Joule heat is generated as the magnetic field is expelled. The total heat (per unit volume) transferred to the reservoir is

$$Q_A = L(T_1) + \int_{T_2}^{T_1} dT C_s(T).$$

(1)

Here $C_s \equiv C_s(T)$ is the heat capacity of the system in the superconducting state. The change in entropy of the universe (system plus reservoir) in this process results from the transfer of latent heat between the system and the reservoir at different temperatures during the transition as well as from the transfer of heat during the cooling of the system from $T_1$ to $T_2$:

$$\Delta S_{univ,A} = L(T_1)(\frac{1}{T_1} - \frac{1}{T_2}) + \int_{T_2}^{T_1} dT C_s(T)[\frac{1}{T_2} - \frac{1}{T_1}]$$

(2)

In route B, we assume the system in the normal state 1 at temperature $T_1$ is rapidly supercooled to temperature $T_2$ by contact with the heat reservoir while remaining in the normal state, then undergoes the transition to the superconducting state while at temperature $T_2$, expelling the magnetic field in a finite amount of time hence generating Joule heat in the process, transferring both the latent heat and the Joule heat to the reservoir at temperature $T_2$. 

The total heat transferred to the reservoir in route B is then

\[ Q_B = \int_{T_2}^{T_1} dT C_n(T) + L(T_2) + Q_J \] (3)

where \( C_n \) is the heat capacity of the system in the normal state and \( Q_J \) is the Joule heat generated in the transition from the supercooled normal state at temperature \( T_2 \) to the superconducting state at temperature \( T_2 \), which takes a finite time. The change of entropy of the universe in route B is due to the transfer of heat between the system and the reservoir during cooling of the system in the normal state, and the generation of Joule heat during the transition. Since the latent heat is transferred between system and reservoir at the same temperature \( T_2 \), this transfer does not change the entropy of the universe. Hence the change in entropy of the universe in route B is given by

\[ \Delta S_{univ,B} = \int_{T_2}^{T_1} dT C_n(T) \left( \frac{1}{T_2} - \frac{1}{T_1} \right) + \frac{Q_J}{T_2} \] (4)

It is important to understand that both the final states of the system and of the reservoir are the same in routes A and B, whether the ‘reservoir’ is infinite or finite. For an infinite reservoir its temperature \( T_2 \) is unchanged, as assumed here for simplicity. For a finite ‘reservoir’, it and the system will reach an equilibrium temperature \( T_3 \), with \( T_2 < T_3 < T_1 \). If the final equilibrium temperatures in the two routes were to be \( T_3^A \neq T_3^B \), it would imply by conservation of energy that either the system or the ‘reservoir’ have negative heat capacity which is of course impossible. Because the system and the ‘reservoir’ constitute our ‘universe’, their final equilibrium temperature and their final states are uniquely defined.

Therefore, since both energy and entropy are functions of state, we necessarily have that

\[ Q_A = Q_B \] (5a)

and

\[ \Delta S_{univ,A} = \Delta S_{univ,B} \] (5b)

From Eq. (5a), we learn that the Joule heat is given by

\[ Q_J = L(T_1) - L(T_2) + \int_{T_2}^{T_1} dT [C_s(T) - C_n(T)] \] (6a)

and from Eq. (5b) we obtain, using Eq. (6a), that

\[ \frac{L(T_2)}{T_2} - \frac{L(T_1)}{T_1} = \int_{T_2}^{T_1} dT \left[ C_s(T) - C_n(T) \right] \] (6b)

Since the latent heat is given by \( L(T) = T (S_n(T) - S_s(T)) \), where \( S_n \) and \( S_s \) are entropies in the normal and superconducting states, and \( C_{s,n}(T) = T (\partial S_{s,n}/\partial T) \), Eq. (6b) is true. This demonstrates the consistency of our approach.

From Eq. (6a) we obtain, using these definitions and integrating by parts that the Joule heat is simply given by

\[ Q_J = \int_{T_2}^{T_1} dT [S_n(T) - S_s(T)]. \] (7)

in a process where the system in the normal state is supercooled from the equilibrium transition temperature \( T_1 \) to a lower temperature \( T_2 \). One way to understand this result is that when the system is supercooled it accumulates extra entropy by staying in the normal state relative to what it would have in the superconducting state, and rids itself of this extra entropy when it undergoes the transition at the lower temperature by generation of Joule heat.

In a similar fashion we can analyze the processes in the superconductor to normal transition shown in Fig. 5: here, route A involves the equilibrium transition between superconducting state 2 and normal state 2’ at coexistence, and route B involves superheating the system in the superconducting state to a non-equilibrium state 1’ at temperature \( T_1 \), then undergoing the transition to the normal state 1 in a finite amount of time absorbing latent heat from a reservoir at temperature \( T_1 \) and generating Joule heat. The corresponding equations are (here, \( Q_{A,B} \) are the heats absorbed by the system):

\[ Q_A = L(T_2) + \int_{T_2}^{T_1} dT C_n(T). \] (8)

\[ \Delta S_{univ,A} = L(T_2) \left( \frac{1}{T_2} - \frac{1}{T_1} \right) + \int_{T_2}^{T_1} dT C_n(T) \left[ \frac{1}{T} - \frac{1}{T_1} \right] \] (9)
\[ Q_B = \int_{T_2}^{T_1} dTC_s(T) + L(T_1) - Q_J \]  
(10)

\[ \Delta S_{\text{univ,B}} = \int_{T_2}^{T_1} dTC_s(T) \left[ \frac{1}{T} - \frac{1}{T_1} \right] + \frac{Q_J}{T_1} \]  
(11)

leading through the conditions Eq. (5) to

\[ Q_J = L(T_1) - L(T_2) + \int_{T_2}^{T_1} dT[C_s(T) - C_n(T)] \]  
(12a)

and

\[ \frac{L(T_2)}{T_2} - \frac{L(T_1)}{T_1} = \int_{T_2}^{T_1} dT \frac{[C_s(T) - C_n(T)]}{T} \]  
(12b)

which are identical to Eqs. (6). Therefore, again Eq. (12b) demonstrates the consistency of our approach also for superheating. Eq. (12a) implies that also for superheating the Joule heat generated is given by the same Eq. (7)

\[ Q_J = \int_{T_2}^{T_1} dT[S_n(T) - S_s(T)] \]  
(7)

in a process where the system in the superconducting state is superheated to a temperature higher than the coexistence temperature and then undergoes the transition to the normal state. This is more difficult to understand intuitively than for the case of supercooling, but equally true.

### III. CLAUSIUS-CLAPEYRON RELATION AND JOULE HEAT

We can shed further light on these results by considering the equation for the coexistence curve \( H_c(T) \). The Clausius-Clapeyron equation for a system with thermodynamic variables \( T, V, P \) (\( V=\text{volume}, \ P=\text{pressure} \)) undergoing a first order phase transformation is well known:

\[ \frac{dP}{dT} = \frac{L(T)}{T \Delta V} \]  
(13)

where \( L \) is the latent heat and \( \Delta V \) the volume change. The analogous equation for a superconductor is [5]

\[ \frac{dH_c}{dT} = \frac{L(T)}{T(M_n - M_s)} \]  
(14)

where \( M_n = 0 \) is the magnetization in the normal state and \( M_s = -H_c/(4\pi) \) is the magnetization in the superconducting state. From Eq. (14) it follows that

\[ \frac{L(T)}{T} = -\frac{H_c dH_c}{4\pi \frac{dT}{dt}} \]  
(15)

hence

\[ S_n(T) - S_s(T) = -\frac{H_c dH_c}{4\pi} \frac{dT}{dt}. \]  
(16)

Replacing in Eq. (7) we find that for both supercooling and superheating the Joule heat generated in the transition is given by

\[ Q_J = \frac{H_c(T_2)^2}{8\pi} - \frac{H_c(T_1)^2}{8\pi} \]  
(17)

which is simply the difference in the condensation free energies at the lower and the higher temperature.

### IV. ELECTRODYNAMICS

To check our result Eq. (17) we consider the electromagnetic energy equation

\[ \frac{d}{dt}\left(\frac{H^2}{8\pi}\right) = -\vec{J} \cdot \vec{E} - \frac{c}{4\pi} \nabla \cdot (\vec{E} \times \vec{H}), \]  
(18)

first for the N-S transition, where the system makes the transition from normal to superconducting in an applied magnetic field \( H_c(1-p) = H_c(T_1) \) at the supercooled temperature \( T_2 \). The left side represents the change in energy of the electromagnetic field as the magnetic field is expelled from the body, the first term on the right side is the work done by the electromagnetic field on currents in this process, and the second term is the outflow of electromagnetic energy. Integrating over the volume of the body \( V \) and over time we find for the change in electromagnetic energy per unit volume

\[ \frac{1}{V} \int d^3r \int_0^\infty dt \frac{d}{dt}\left(\frac{H^2}{8\pi}\right) = -\frac{H_c(T_1)^2}{8\pi}. \]  
(19)

since at the end the initial magnetic field \( H_c(T_1) \) is completely excluded from the body. From Faraday’s law and assuming cylindrical symmetry we have for the electric field generated by the changing magnetic flux at the surface of the cylinder

\[ \vec{E}(a,t) = -\frac{1}{2\pi a c} \frac{d}{dt} \phi(t) \hat{\theta} \]  
(20)

where \( a \) is the radius of the cylinder and \( \phi(t) \) is the magnetic flux through the cylinder, with \( \phi(t=0) = \pi a^2 H_c(T_1) \), \( \phi(t=\infty) = 0 \). Integration of the second term on the right in Eq. (18), the energy outflow, over space and time, converting the volume integral to an integral over the surface of the cylinder, using that \( H = H_c(1-p) \) at the surface of the cylinder independent of time and Eq. (20) for the electric field at the surface yields

\[ \frac{1}{V} \int_0^\infty dt \int (-\frac{c}{4\pi})(\vec{E} \times \vec{H}) \cdot d\vec{S} = -\frac{H_c(T_1)^2}{4\pi}. \]  
(21)

This gives the total electromagnetic energy flowing out through the surface of the sample during the transition.

The current \( \vec{J} \) in Eq. (18) flows in the azimuthal direction and is given by the sum of superconducting and normal currents

\[ J(r) = J_s(r) + J_n(r) \]  
(22)
where $J_n(r)$ flows in the region $r \leq r_0(t)$ and is of appreciable magnitude only within $\lambda_L$ of the phase boundary, where $\lambda_L$ is the London penetration depth. $r_0(t)$ is the radius of the phase boundary at time $t$. It is important to note the fact that at the superconductor-normal phase boundary the magnetic field is given by $H_c(T_2) \equiv H_c$, as indicated in Fig. 1, since there is coexistence of superconducting and normal phases at that radius [6]. The extra magnetic field relative to the field at the cylinder surface is supplied by the current $J_n$ induced by the Faraday field flowing in the normal region $r \geq r_0(t)$ [6]. Integration of the second term in Eq. (18) over the superconducting current yields [7]

$$\frac{1}{V} \int d^3r \int_0^\infty dt(-\vec{J}_s \cdot \vec{E}) = \frac{H_c(T_2)^2}{8\pi}.$$  (23)

This is because the Faraday field decelerates the supercurrent [7] as the phase boundary moves out.

The Joule heat per unit volume generated during the transition is

$$Q_J = \frac{1}{V} \int d^3r \int_0^\infty dt \vec{J}_n \cdot \vec{E}$$  (24)

hence from integrating Eq. (18) over space and time using Eqs. (19), (21), (22) and (23) we have

$$-\frac{H_c(T_1)^2}{8\pi} = \frac{H_c(T_2)^2}{8\pi} - Q_J - \frac{H_c(T_1)^2}{4\pi}$$  (25)

which implies

$$Q_J = \frac{H_c(T_2)^2}{8\pi} - \frac{H_c(T_1)^2}{8\pi}$$  (26)

identical to the thermodynamic result Eq. (17).

We leave it as a simple exercise for the reader to derive the same equation for the case of superheating, where the electromagnetic energy flows into the system through the surface and the Faraday electric field speeds up rather than slows down the supercurrent.

V. DIRECT CALCULATION OF THE JOULE HEAT

We next calculate the Joule heat generated in the process shown in Fig. 1 directly, assuming it originates only from current in the normal region. Ampere’s law and Faraday’s law in cylindrical geometry yield

$$\frac{\partial H}{\partial r} = -\frac{4\pi}{c} J$$  (27a)

$$\frac{1}{r} \frac{\partial}{\partial r}(rE_F) = -\frac{1}{c} \frac{\partial H}{\partial t}$$  (27b)

with $\vec{H} = \vec{H}_z$, $\vec{J} = J \hat{\theta}$ and $\vec{E} = E_F \hat{\theta}$ the magnetic field in the $z$ direction, current density and electric field in the azimuthal direction respectively. The boundary conditions are

$$H(r = a) = H_c(1 - p)$$  (28a)

$$H(r = r_0) = H_c.$$  (28b)

As pointed out by Pippard in his seminal paper [6], these equations cannot be solved exactly but can be solved in a power series expansion in $p$. To lowest order in $p$ we may assume that the magnetic field is $H_c$ for all $r \geq r_0$, hence the Faraday electric field is given for $r \geq r_0$ by

$$E_F(r) = \frac{r_0}{r} H_c$$  (29)

In the normal region $r \geq r_0$ we assume the normal current $J_n$ obeys the constitutive relation

$$J_n(r) = \sigma_n(r) E_F(r).$$  (30)

We allow the normal conductivity $\sigma_n$ to depend on $r$ for generality. From Eqs. (30), (29) and (27a) we deduce

$$\frac{\partial H}{\partial r} = -\frac{4\pi}{c^2} \sigma_n(r) \frac{r_0}{r}$$  (31)

and integrating between $r = r_0$ and $r = a$ and using Eq. (28) we obtain

$$pH_c = \frac{4\pi}{c^2} r_0 H_c \int_{r_0}^{a} dr \frac{\sigma_n(r)}{r}.$$  (32)

The normal current generates Joule heat per unit volume at rate given by

$$\frac{\partial W}{\partial t} = J_n(r) E_F(r) = \sigma_n(r) E_F(r)^2.$$  (33)

Integrating over the volume of the normal metal the rate of Joule heat generation is, using Eq. (29)

$$\frac{\partial W}{\partial t} = \int d^3r \frac{\partial W}{\partial t} = 2\pi \int_{r_0}^{a} dr \sigma_n(r) \frac{r_0^2}{r^2} \frac{r^2}{a^2} H_c^2$$  (34)

and using Eq. (32) we obtain the simple result

$$\frac{\partial W}{\partial t} = \frac{1}{2} pH_c^2 r_0.$$  (35)

Finally, integrating Eq. (35) over time and dividing by the volume of the cylinder we obtain the Joule heat per unit volume generated in the normal region during the entire process:

$$Q_J = \frac{1}{\pi a^2} \int_0^\infty dt \frac{\partial W}{\partial t} = \frac{H_c^2}{4\pi}.$$  (36)

Now from Eq. (26) we have, with $H_c(T_2) = H_c$, $H_c(T_1) = H_c(1 - p)$

$$Q_J = \frac{H_c^2}{8\pi} - \left(\frac{H_c(1 - p)}{8\pi}\right)^2 = \frac{H_c^2}{4\pi} p + O(p^2).$$  (37)
in agreement with Eq. (36).

The equality of Eqs. (36), calculated using only the current in the normal region, with Eqs. (17) and (26) obtained from thermodynamics and electrodynamics, implies that no Joule heat was generated in the superconducting region in this process. The same result is obtained by considering the Joule heat generated in the normal region in the S-N transition in the presence of magnetic field $H_c(1 + p) = H_c(T_2)$ (Fig. 5). We discuss the significance of these results in what follows.

VI. THE MISSING JOULE HEAT

We consider for definiteness the process of supercooling, the same issues arise for superheating. The magnetic field does not drop to zero discontinuously at the phase boundary $r = r_0$, rather it decays smoothly as governed by the London penetration depth. For $r \leq r_0$ London’s equation applies:

$$\vec{J}_s(r, t) = -\frac{c}{4\pi \lambda_L} \vec{A}(r, t)$$  \hspace{1cm} (38)

with $\vec{A}$ the magnetic vector potential, given by (assuming $r_0 >> \lambda_L$) [7]

$$\vec{A}(r, t) = H_c \lambda_L e^{(r-r_0)/\lambda_L} \hat{\theta}.$$  \hspace{1cm} (39)

The magnetic field $\vec{H} = \vec{\nabla} \times \vec{A}$ is

$$\vec{H}(r) = H_c e^{(r-r_0)/\lambda_L} \hat{z}.$$  \hspace{1cm} (40)

and the Faraday electric field $\vec{E}(r, t) = -(1/c)\partial \vec{A}(r, t)/\partial t$ is

$$E(r) = \frac{\hat{r}_0}{c} H_c e^{(r-r_0)/\lambda_L}.$$  \hspace{1cm} (41)

Within the conventional theory of superconductivity, at finite temperatures a superconductor can be modeled approximately as a two-fluid model [8], with normal and superconducting electrons of density $n_n$, $n_s = n - n_n$, with $n$ the conduction electron density. $n_s$ and $n_n$ depend on temperature. A more detailed treatment using Bogoliubov quasiparticles as the normal state excitations would yield equivalent results. The electric field Eq. (41) will give rise to a normal current

$$\vec{J}_n'(r) = \sigma_0 E(r).$$  \hspace{1cm} (42)

where we have approximately $\sigma_0 = (n_n/n)\sigma_n$. The predicted rate of Joule heat generation in the superconducting region is then

$$\frac{\partial W_s}{\partial t} = \sigma_0 E(r)^2 = \frac{\sigma_0}{c^2} H_c^2 e^{2(r-r_0)/\lambda_L}.$$  \hspace{1cm} (43)

and performing the spatial integral

$$\frac{\partial W_s}{\partial t} = \int_{r \leq r_0} d^3 r \frac{\partial W}{\partial t} = \pi \sigma_0 \frac{\hat{r}_0^2}{c^2} r_0 \lambda_L H_c^2.$$  \hspace{1cm} (44)

Under the assumption that $\sigma_n$ is independent of $r$ we can integrate Eq. (31) over space and time to obtain

$$\left(\frac{r_0}{a}\right)^2 [1 + 2 ln \frac{a}{r_0}] = \frac{t}{t_0}.$$  \hspace{1cm} (45)

where

$$t_0 = \frac{\pi \sigma_n a^2}{p c^2}$$  \hspace{1cm} (46)

is the total time to expel the magnetic field. For simplicity we can assume $\gamma_0 \sim a/t_0$ in Eq. (44), and performing the time integral we find for the Joule heat per unit volume generated in the superconducting region predicted by the conventional theory:

$$q \equiv \int d^3 r \frac{\partial W_s}{\partial t} = \frac{H_c^2}{\pi} \frac{\sigma_0 \lambda_L}{\sigma_n a}$$  \hspace{1cm} (47)

or

$$q = \frac{4 \sigma_0 \lambda_L}{\sigma_n a} Q_J.$$  \hspace{1cm} (48)

As one would expect, this Joule heat $q$ is proportional to the London penetration depth $\lambda_L$. But we saw in Sects. II-IV that the total Joule heat depends only on thermodynamic properties and is independent of $\lambda_L$. And we saw in Sect. V that the Joule heat generated in the normal region accounts for the entire Joule heat predicted by thermodynamics and electrodynamics. Therefore we conclude that $q$ does not exist. Therefore there cannot be a normal azimuthal current induced by the Faraday field in the superconducting region, contrary to what Eq. (42) says.

VII. OTHER CONSIDERATIONS

There are in fact other reasons for why a normal current in the superconducting region cannot exist. The total current in the superconducting region is fixed by the fact that it has to nullify the magnetic field in the deep interior. The supercurrent $J_s$ in the superconducting region is given by

$$\vec{J}_s = -\frac{c}{4\pi \lambda_L} H_c e^{(r-r_0)/\lambda_L} \hat{\theta}.$$  \hspace{1cm} (49)

as follows from the London equation Eq. (38), in the absence of normal current. If a normal current $J'_n$ (Eq. (42)) were to be induced in the boundary layer by the Faraday field, it would require that an additional supercurrent

$$\vec{J}_s = -\vec{J}_n'$$  \hspace{1cm} (50)

be generated in order not to change the total current, as indicated schematically in Figs. 1 and 2, so as to keep the interior magnetic field equal to zero. The total
supercurrent \( \vec{J}_s + J'_s \) would no longer satisfy London’s equation Eq. (38). This would be in disagreement with the conventional theory of superconductivity where the London equation is a consequence of the fact that the canonical momentum of electrons in the supercurrent is zero in a simply connected geometry.

Furthermore, consider the energy of the currents. The kinetic energy of the supercurrent \( J_s \) per unit volume is given by

\[
K_s(r) = \frac{m_e}{2n_s e^2} J_s(r)^2
\]

where \( m_e \) is the electron mass (we ignore possible differences between bare and effective mass for simplicity [9]). The London penetration depth \( \lambda_L \) satisfies [1]

\[
\frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{m_e c^2}
\]

and Eqs. (49), (51) and (52) yield for the kinetic energy of the supercurrent at the phase boundary

\[
K_s(r_0) = \frac{H^2_{\text{c}}}{8\pi}
\]

which is the condition for phase equilibrium at the normal-superconductor boundary, first discussed by H. London [1]. As electrons condense into the superconducting state, their condensation energy \( H^2_{\text{c}}/8\pi \) provides precisely the kinetic energy necessary for the electrons to join the supercurrent at the phase boundary.

Instead, if there is the additional supercurrent Eq. (50), the total kinetic energy of the supercurrent at the phase boundary would be different. Consider first the N-S transition, Fig. 1. The extra supercurrent \( J'_s \) flows in the same direction as \( \vec{J}_s \), hence the kinetic energy of the supercurrent would be

\[
K_s(r_0) = \frac{m_e}{2n_s} \left( J_s(r_0) + J'_s(r_0) \right)^2 > \frac{H^2_{\text{c}}}{8\pi}
\]

In other words, electrons condensing into the superconducting state would have to acquire a kinetic energy larger than the condensation energy. That is impossible. There is no extra energy source to supply the kinetic energy associated with the extra supercurrent nor with the normal current \( J'_n \). For the S-N transition, the extra supercurrent \( J'_s \) flows in direction opposite to \( \vec{J}_s \), here the kinetic energy of the supercurrent would be

\[
K_s(r_0) = \frac{m_e}{2n_s} \left( J_s(r_0) - J'_s(r_0) \right)^2 < \frac{H^2_{\text{c}}}{8\pi}
\]

Still, the sum of it plus the kinetic energy associated with \( J'_n \) would not equal the available energy \( H^2_{\text{c}}/(8\pi) \). Also the lack of symmetry between the supercooled and superheated situations, contrary to the symmetry found in the previous sections, indicates that the term \( J'_n \) should not be in either Eq. (54) or (55), hence that \( J'_n = 0 \).

VIII. RELATION WITH EARLIER WORK

Note that in our discussion of route A in the cooling process, Figs. 3 and 4, we computed the entropy change \( \Delta S_{\text{univ},A} \), Eq. (2), under the assumption that no Joule heat is generated when the system is cooled in the superconducting state from temperature \( T_1 \) to \( T_2 \), states \( 1' \) to 2. We did not make any assumption about the rate at which this cooling occurs. During this process, the London penetration depth decreases and a Faraday electric field is induced within \( \lambda_L \) of the surface of the cylinder. We analyzed that process in ref. [3] and pointed out that no normal current can be induced by the Faraday field during that process because that would be incompatible with thermodynamics. That is consistent with what we assumed here in Eq. (2), and with what we concluded in Sects. VI and VII regarding normal current in the superconducting region within \( \lambda_L \) of the phase boundary where the Faraday field exists: there isn’t any.

IX. SUMMARY AND DISCUSSION

We have calculated the Joule heat that is generated when a normal metal expels a magnetic field in the transition to the superconducting state, and when a superconductor goes normal in the presence of a magnetic field. We have found from purely thermodynamic considerations that the Joule heat takes the same simple form in both cases, Eq. (7) or equivalently Eq. (17), independent of the normal state conductivity and of the time that the process takes. This result was corroborated by a calculation using purely electrodynamic considerations, Eq. (26). To our knowledge, the fact that the Joule heat in these transitions is simply related to thermodynamic properties has not been pointed out before.

At first sight the result may seem counterintuitive. It follows from the fact that the time the process takes is proportional to the normal state conductivity \( \sigma_n \), as given by Eq. (46). For large \( \sigma_n \) the process occurs very slowly and the Faraday field is very small, for small \( \sigma_n \) the process is fast and the Faraday field is large, but the total Joule heat generated is the same in all cases. The physical reason that the process is slow if the normal state conductivity is large is the following: what limits the speed of the process is that the magnetic field at the phase boundary is exactly \( H_c \). The role of the current in the normal region in the cooling process is to generate the extra magnetic field \( (pH_c) \) to increase the magnetic field from its value at the cylinder surface to its value at the phase boundary, or in the heating process to reduce the applied magnetic field from its value at the cylinder surface by \( (pH_c) \). If the process proceeds too fast, the induced normal current would produce a magnetic field at the phase boundary larger than \( H_c \) in the cooling process or smaller than \( H_c \) in the heating process, reversing the direction of the process.

We then showed by direct calculation that the Joule
heat predicted by thermodynamics results from the normal current induced by the Faraday electric field in the normal region only. However, an induced Faraday electric field necessarily exists close to the phase boundary in the superconducting region during these processes. Within the conventional theory of superconductivity this electric field will both affect superfluid and normal electrons (i.e. Bogoliubov quasiparticles). For the superfluid electrons, in the cooling process the Faraday field slows them down so that as the phase boundary moves further out and they become part of the interior their velocity slows to zero, in the heating process it speeds them up so that as the phase boundary moves in the velocity reaches the value necessary to generate the current \( J_s \) at the boundary, as shown in [7]. For the normal electrons, the Faraday field will generate a normal current and associated Joule heat Eq. (48) within the conventional theory. But this is impossible, since all the Joule heat allowed by thermodynamics as well as by electrodynamics is generated in the normal region.

Note that within the conventional theory, two superconductors with the same thermodynamic properties but different transport properties would dissipate different amounts of Joule heat in the same thermodynamic process. For example, if one of them had more impurities it could have a much larger London penetration depth [1] with little or no change in its thermodynamic properties, hence a much larger boundary layer where Joule heat would be dissipated according to the conventional theory.

We conclude from these considerations that there is a problem with the conventional theory of superconductivity. The Faraday electric field does not induce a normal current in the superconducting region in these processes, contrary to what the conventional theory of superconductivity predicts [1]. We reached the same conclusion in recent work where we considered processes where the temperature changes always below \( T_c \): in such processes a Faraday electric field exists because the London penetration depth and consequently the magnetic flux is changing. What these processes have in common with the ones discussed in this paper is that the density of superfluid electrons is changing. We conclude that in such situations the resulting electric field does not generate a normal current, contrary to situations where an electric field is produced by ac currents or electromagnetic waves, where normal current is known to be generated [8].

In contrast to the conventional theory, within the theory of hole superconductivity [2] there is radial motion of charge in the normal-superconductor transition in a magnetic field. We have argued that this is necessary to explain the process of magnetic field expulsion and how momentum is transferred from electrons to the body as a whole in a reversible way to account for momentum conservation [11]. This physics can also explain the absence of Joule heat in the boundary layer in the situation discussed in this paper, as well as the absence of Joule heat in a process where the temperature below \( T_c \) is changed [3]. Note that a radial normal current in the presence of an azimuthal electric field does not give rise to dissipation. This physics can also explain the absence of entropy generation forbidden by thermodynamics when momentum is transferred between electrons and the body as a whole as discussed in [4]. It requires the charge carriers in the normal state to be holes [12].

Acknowledgments

The author is grateful to Bert Halperin and Tony Leggett for helpful discussions.

[1] M. Tinkham, “Introduction to superconductivity”, McGraw Hill, New York, 1996.
[2] See references in http://physics.ucsd.edu/~jorge/hole.html.
[3] J.E. Hirsch, “Thermodynamic inconsistency of the conventional theory of superconductivity”, arXiv:1907.11273 (2019); arXiv:1909.12786 (2019).
[4] J.E. Hirsch, “Entropy generation and momentum transfer in the superconductor to normal phase transformation and the consistency of the conventional theory of superconductivity”, Int. J. Mod. Phys. B, Vol. 32, 1850158 (2018).
[5] F. Reif, “Fundamentals of statistical and thermal physics”, McGraw-Hill, New York, 1965, Chpt. 11.
[6] A. B. Pippard, “Kinetics of the phase transition in superconductors”, Phil. Mag. 41, 243 (1950).
[7] J. E. Hirsch, “Dynamics of the normal–superconductor phase transition and the puzzle of the Meissner effect”, Annals of Physics 362, 1 (2015).
[8] M. Tinkham, “Introduction to superconductivity”, McGraw Hill, New York, 1996, Sects. 2.5, 3.10.
[9] J. E. Hirsch, “What is the speed of the supercurrent in superconductors?”, arXiv:1605.09460 (2016).
[10] H. London, “Phase-Equilibrium of Superconductors in a Magnetic Field”, Proc. Roy. Soc. London A 152, 650 (1935).
[11] J.E. Hirsch, “Momentum of superconducting electrons and the explanation of the Meissner effect”, Phys. Rev. B 95, 014503 (2017).
[12] J. E. Hirsch, “Why only hole conductors can be superconductors”, Proc. SPIE 10105, Oxide-based Materials and Devices VIII, 101051V (2017).