Beyond Einstein–Cartan gravity: quadratic torsion and curvature invariants with even and odd parity including all boundary terms

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Abstract

Recently, gravitational gauge theories with torsion have been discussed by an increasing number of authors from a classical as well as from a quantum field theoretical point of view. The Einstein–Cartan(–Sciama–Kibble) Lagrangian has been enriched by the parity odd pseudoscalar curvature (Hojman, Mukku and Sayed) and by torsion square and curvature square pieces, likewise of even and odd parity. (i) We show that the inverse of the so-called Barbero–Immirzi parameter multiplying the pseudoscalar curvature, because of the topological Nieh–Yan form, can be appropriately discussed if torsion square pieces are included. (ii) The quadratic gauge Lagrangian with both parities, proposed by Obukhov et al and Baekler et al, emerges also in the framework of Diakonov et al. We establish the exact relations between both approaches by applying the topological Euler and Pontryagin forms in a Riemann–Cartan space expressed for the first time in terms of irreducible pieces of the curvature tensor. (iii) In a Riemann–Cartan spacetime, that is, in a spacetime with torsion, parity-violating terms can be brought into the gravitational Lagrangian in a straightforward and natural way. Accordingly, Riemann–Cartan spacetime is a natural habitat for chiral fermionic matter fields.

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1. Einstein–Cartan theory and weak gravity

In gauge-theoretical approaches to gravity (see [1–4]), we have the orthonormal coframe 1-form \(\vartheta^\alpha\) as the translational potential and the connection 1-form \(\Gamma^{\alpha\beta} = -\Gamma^\beta{}_{\alpha\gamma}\) as the Lorentz potential. The corresponding field strengths are the torsion 2-form \(T^\alpha\) and the curvature 2-form \(R^\alpha\)
\[ R^\beta{}^a = -R^a{}^\beta. \] The first-order gravitational theory in this framework is called the Poincaré gauge theory of gravity (PG).

The simplest model within PG is the Einstein–Cartan (EC) theory of gravity, see [5], with the twisted gauge Lagrangian \( \kappa = \text{gravitational and } \Lambda_0 = \text{cosmological constant}^6 \)

\[
V_{\text{EC}} := \frac{1}{2\kappa} (\eta_{\alpha\beta} \wedge R^{\alpha\beta} - 2\Lambda_0 \eta) \quad \text{and with } \quad L_{\text{tot}} = V_{\text{EC}} + L(\psi, D\psi),
\]

where \( L = \text{the matter Lagrangian depending on the minimally coupled fermionic/bosonic matter fields } \psi(x). \) This is a viable gravitational theory that deviates from general relativity at extremely high matter densities \( \rho \lesssim \rho_{\text{crit}}, \) with \( \rho_{\text{crit}} \approx m/(\lambda_{\text{Compton}}\xi_{\text{Planck}}) \), and \( m \) is the mass of the field, see also [10]. At the same time, it is clear that GR can alternatively be reformulated as a teleparallelism theory with torsion square pieces in the Lagrangian. If we call the Newton–Einstein type of gravity ‘weak’ gravity, then its general quadratic gauge Lagrangian reads \( (0) \) and \( (1) \) components (2), we recover the teleparallel equivalent of GR, toted differently, a fact often overlooked.

To link up with the experience of GR, we recall that the Riemann–Cartan curvature 2-form

\[
R^\alpha{}^\beta = R^\alpha{}^\beta + \eta_{\alpha\beta} \wedge T^a + \sum_{I=1}^3 a_I^* (T^a). \tag{2}
\]

Here \( (1)T_a \) denotes the irreducible pieces of the torsion, with \( (2)T_a := \partial_a \wedge (e^\beta \wedge T^\beta)/3 \) (vector, four independent components), \( 3)T_a := e^\alpha \wedge (T^\alpha \wedge \partial_a)/3 \) (axitor, 4) and \( (4)T_a := T_a - (3)T_a - (2)T_a \) (tensor, 16). For the special cases \( R^\alpha{}^\beta = 0 \), enforced by a corresponding Lagrange multiplier term in (2), we recover the teleparallel equivalent of GR, provided that local Lorentz invariance of the gravitational Lagrangian is implemented, see [11–17], and alternatively, for \( T^a = 0 \), we find GR directly. Thus, GR is hidden in (2) in two totally different ways, a fact often overlooked.

We marked Lagrangian (2) with a plus sign + for being, as a twisted 4-form, parity even. However, already in 1980, Hojman, Mukku and Sayed (HMS) [18] and Nelson [19] added the parity odd\(^4\) pseudoscalar curvature piece \( R_{\alpha\beta} \wedge \partial^\alpha \) to the EC Lagrangian, see also [21–25]. More recently, in the context of the Ashtekar formalism [26], see Kiefer [27], and in loop quantum gravity, see [28, 29], this has become popular, see [30–39]; for related

\(^4\) Following essentially Schouten [6], our conventions are as follows [7]: we have the coframe 1-form \( \partial^\alpha = \epsilon^\alpha dx^i \) and the frame vectors \( e_\beta = e^i \partial_i \), with \( e_\beta \partial^\alpha = \delta_\alpha^\beta \). The connection 1-form is \( \Gamma^\alpha{}^\beta = \Gamma^\alpha{}^\beta dx^i \). Greek indices are raised and lowered by means of the Minkowski metric \( g_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \). The volume 4-form is denoted by \( \eta \), and \( \eta_a = \partial_a \wedge (e^\beta \wedge T^\beta)/3 \) (vector, four independent components), \( \eta_{\alpha\beta} = e^\alpha \wedge (T^\alpha \wedge \partial_a)/3 \) (axitor, 4) and \( \eta_{\alpha\beta} = T_a - (3)T_a - (2)T_a \) (tensor, 16). For the special cases \( R^\alpha{}^\beta = 0 \), enforced by a corresponding Lagrange multiplier term in (2), we recover the teleparallel equivalent of GR, provided that local Lorentz invariance of the gravitational Lagrangian is implemented, see [11–17], and alternatively, for \( T^a = 0 \), we find GR directly. Thus, GR is hidden in (2) in two totally different ways, a fact often overlooked.

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cosmological models, see also [51–56]. Including additionally odd torsion square pieces, we have \((b_0, \sigma_1, \sigma_2)\) are constants and \((^{(1)}R_{ab})\) is the irreducible pseudoscalar curvature 2-form

\[
V_{\text{weak}}^- = - \frac{b_0}{2\kappa} \left( {^{(3)}R_{ab}} \wedge \vartheta^{ab} + \frac{1}{\kappa} \left( \sigma_1 {^{(1)}T^a} \wedge {^{(1)}T_a} + \sigma_2 {^{(2)}T^a} \wedge {^{(3)}T_a} \right) \right). 
\] (4)

The inverse of \(b_0\) is sometimes called the Barbero–Immirzi parameter \([35, 57, 58]\). The total gauge Lagrangian would then read \(V^+ + V_{\text{weak}}^-\). However, we should be aware that for weak gravity there exists a boundary term, the untwisted parity odd Nieh–Yan 4-form\([7]\)

\[
B_{TT}^- = dC_{TT}^- = \frac{1}{\kappa} \left( {^{(3)}T^a} \wedge {^{(1)}T_a} + {^{(2)}T^a} \wedge {^{(3)}T_a} + {^{(3)}R_{ab}} \wedge \vartheta^{ab} \right),
\] (6)

with \(C_{TT}^\pm := \frac{1}{\kappa} \vartheta^a \wedge {^{(3)}T_a}\) and \(X\) the curvature pseudoscalar, \(X = \eta_{a\beta\gamma\delta} R^{(a\beta\gamma\delta)} / 4!\). We add this form with a suitable constant \(f_1\) to our weak gravity Lagrangian:

\[
V_{\text{weak}}^- = V_{\text{weak}}(a_0; b_0; a_1, a_2, a_3; \sigma_1, \sigma_2; f_1) := V^+_{\text{weak}} + V^-_{\text{weak}} + \frac{f_1}{\kappa} B_{TT}^-.
\] (7)

It depends on the gravitational constant \(\kappa\) and the cosmological constant \(\lambda_0\) and, furthermore, on the eight constants specified in (7). By a suitable choice of \(f_1\), we can compensate either the HMS term \([18]\)(that is, \(b_0 = 0\)) or one tensor square term of the torsion (that is, either \(\sigma_1 = 0\) or \(\sigma_2 = 0\)). However, since \({^{(1)}T^a}\) depends on 16 independent components, the pseudoscalar curvature only on one component, it seems to simplify the Lagrangian to a greater extent, if we kick out the term with \({^{(1)}T^a}\). Thus, for the weak gravity Lagrangian, we are left with six unspecifed constants \((a_0, b_0; a_1, a_2, a_3; \sigma_2)\).\(^3\)

Looking back at equation (3), it could appear that we forgot the boundary term \(B_{TT}^+ = \frac{2\kappa}{\lambda_0} d(\vartheta^a \wedge * T_a)\) and that we could add it as \(\frac{2\kappa}{\lambda_0} B_{TT}^-\) to Lagrangian (7), see the procedure of Mielke \([60, 61]\). However, if we compare the Nieh–Yan and the ‘teleparallel’ formulas

\[
d(\vartheta^a \wedge * T_a) = {^{(1)}T^a} \wedge * T_a - {^{(2)}T^a} \wedge D^a * T_a \quad \text{and} \quad d(\vartheta^a \wedge * T_a) = {^{(1)}T^a} \wedge * T_a - \vartheta^a \wedge D^a * T_a,
\] (8)

respectively, then we recognize that in the former equation, \(DT_a\) can be eliminated via the first Bianchi identity \(DT_a = R_{a\beta} \wedge \vartheta^\beta\), whereas in the latter equation, such a trick is impossible. Therefore, we would trade in for torsion square pieces derivatives of the torsion and would mess up the first-order character of our Lagrangian\(^9\).

If one starts with the EC Lagrangian and adds only an HMS term, as numerous people do, then, because of the Nieh–Yan form, two torsion square terms of odd parity are induced. Hence, the Lagrangian can be reformulated as the EC Lagrangian with specific additional torsion square pieces:

\[
V_{\text{EC}} + \frac{b_0}{2\kappa} {^{(3)}R_{ab}} \wedge \vartheta^{ab} = V_{\text{EC}} - \frac{b_0}{2\kappa} \left( {^{(1)}T^a} \wedge {^{(1)}T_a} + {^{(2)}T^a} \wedge {^{(3)}T_a} \right) + d(\ldots). 
\] (10)

\(^7\) In a metric-affine spacetime \([7]\) with the distortion 1-form \(N^\alpha_\beta := \Gamma^\alpha_\beta - \tilde{\Gamma}^\alpha_\beta\), we can bring the Nieh–Yan identity in a very compact form, see \([59]\):

\[
d(T^a \wedge \vartheta^\alpha) = {^{(3)}R^\alpha_\beta} \wedge \vartheta^a \wedge \vartheta^\beta - T^a \wedge N^\beta_\beta \wedge \vartheta^\beta,
\] (5)

\(^8\) Diakonov et al \([8]\) found an equivalent result, but they eliminated the curvature pseudoscalar. Thus, they are left with the six unspecified constants \((a_0; a_1, a_2, a_3; \sigma_1, \sigma_2)\), that is, with the curvature scalar plus the five torsion-square pieces.

\(^9\) We can combine the two equations in (8). This yields

\[
d(\vartheta^a \wedge \xi^\pm_a) = T^a \wedge \xi^\pm_a - \xi^\pm_a \wedge D^a \xi^\pm_a \quad \text{with} \quad \xi^\pm_a := T_a \pm * T_a.
\] (9)

Mielke \([60, 61]\) built a similar linear combination but with the imaginary unit in front of *\(T_a\), but neither his nor our version seems to lead to firm conclusions up to now.
Then the question can hardly be circumvented: Why one should choose only these specific torsion square pieces with very specific constants and why the other torsion square pieces should be forbidden, that is, the torsion square pieces come into focus? Moreover, it is known that GR can be reformulated as a teleparallelism theory with torsion square pieces in the Lagrangian [11–17]. In other words, the addition of the HMS term opens the door wide for torsion square Lagrangians.

Classically, it is consistent to consider only the two additional specific terms in (10). However, it is not particularly plausible. If torsion is introduced as a new concept, why should one then introduce it in the highly constrained form of (10)? In loop quantum gravity [29], which is thought of as a fundamental theory of gravity, the truncated Lagrangian (10) is taken as a classical starting point, see [29], equation (34), with the argument that also in QCD a similar parity odd piece is used. However, then in the Lagrangian the internal color group $SU(3)$ with its potential $A$ is put in analogy with the local Poincaré group $R^4 \supset \times SO(1, 3)$ with its translation potential $\theta^\alpha$ and Lorentz potential $\Gamma^\alpha\beta$. Apart from the fact that QCD is quadratic in the field strength and (10) is only linear in the curvature, this argument is less than convincing to us.

If gravity is seen in a quantum field theoretical context, see Diakonov et al [8], for instance, then, as Date et al [40] have pointed out, Lagrangian (10) is insufficient anyway: ‘in a complete theory of gravity, besides the Nieh–Yan topological term, we need to include two other topological terms, the Pontryagin density and the Euler density. This introduces two additional topological parameters associated with such topological terms, besides the parameter $\eta$ [our $b_0$] we have discussed here. Any quantum theory of gravity should have all these three CP-violating topological couplings’. Actually, the Euler 4-form is CP even, see equation (19) or [49].

From a totally different point of view, from observational cosmology and from quantum chromodynamics, there are indications that we may live in a parity-violating Universe, see the review by Urban and Zhiltzisky [62]. All the more investigations in a parity odd PG model seem desirable.

2. Quadratic Poincaré gauge theory and strong gravity

If one wants the Lorentz connection as a propagating field, then one has to allow for ‘strong’ gravity of the Yang–Mills type by adding quadratic curvature 4-forms to the weak Lagrangian. The curvature $R^\alpha\beta$ of a Riemann–Cartan space has six irreducible pieces: $R^\alpha\beta = \sum_{I=1}^{6} (I)R^\alpha\beta$. We write symbolically, using the self-explanatory computer names for the irreducible terms, $\text{curv} (36 \text{ indep. comp.}) = \text{weyl} (10) + \text{paircom} (9) + \text{pscalar} (1) + \text{ricymf} (9) + \text{ricanti} (6) + \text{scalar} (1)$, see [7] for the exact definitions. In a Riemann space, only $(1)R^\alpha\beta$, $(5)R^\alpha\beta$ and $(6)R^\alpha\beta$ are left over. Hence, the most general parity even quadratic Lagrangian, with a new dimensionless coupling constant $\varrho$, reads

\begin{equation}
V^+_{\text{strong}} = -\frac{1}{2\varrho} R^\alpha\beta \wedge \sum_{I=1}^{6} w_I (I)R^\alpha\beta.
\end{equation}

Alerted by the corresponding case in weak gravity, we now search for parity odd terms. They were found to be (see [63, 64]) as

\begin{equation}
V^-_{\text{strong}} = -\frac{1}{2\varrho} \left( \mu_1(1)R^\alpha\beta \wedge (1)R^\alpha\beta + \mu_2(2)R^\alpha\beta \wedge (4)R^\alpha\beta + \mu_3(3)R^\alpha\beta \wedge (6)R^\alpha\beta + \mu_4(5)R^\alpha\beta \wedge (5)R^\alpha\beta \right),
\end{equation}

4
which are the only quadratic curvature square invariants of odd character in a 4D Riemann–Cartan space. Note that in a Riemann space, that is, when torsion vanishes, only the first piece up from the Weyl curvature \( R^{(1)} \) is left over.

Taking a lesson from the above, we can now search for boundary terms. As in any Yang–Mills theory, we can find an untwisted Pontryagin 4-form \( B_{RR}^* \). But in gravity, the (anholonomic) Lorentz indices of the curvature can be contracted with the help of the totally antisymmetric Levi-Civita tensor \( \eta^{\mu\nu\rho\sigma} \). Accordingly, we introduce the so-called Lie dual of the curvature with the Lie star operator \( \ast \) as

\[
R^{(\ast)\mu\nu} := \frac{1}{2} R_{\mu\nu} \eta^{\rho\sigma\ast\mu\ast\nu} .
\]

Like \( \eta^{\mu\nu\rho\sigma} \), the Lie star \( \ast \) is twisted. It gives rise to the twisted Euler 4-form \( B_{RR}^{*} \). Following [7], we have then the following two boundary terms:

\[
B_{RR}^{-} := dC_{RR}^{-} = \frac{1}{2} R_{\mu\nu} \wedge R^{\mu\nu} \quad \text{and} \quad B_{RR}^{*} := dC_{RR}^{*} = \frac{1}{2} R_{\mu\nu} \wedge R^{(\ast)\mu\nu} .
\]

Thus, the strong part of the gauge Lagrangian turns out to be

\[
V_{\text{strong}} := V_{\text{strong}}^{+} + V_{\text{strong}}^{-} = \frac{f_2}{\mathcal{Q}} B_{RR}^{-} + \frac{f_3}{\mathcal{Q}} B_{RR}^{*} .
\]

The total quadratic gauge Lagrangian including boundary terms is then

\[
V_{\text{gauge}} := V_{\text{weak}}^{+} + V_{\text{weak}}^{-} + V_{\text{strong}}^{+} + V_{\text{strong}}^{-} + \frac{f_1}{\kappa} B_{TT} + \frac{f_2}{\mathcal{Q}} B_{RR}^{-} + \frac{f_3}{\mathcal{Q}} B_{RR}^{*} .
\]

For vanishing torsion, \( V_{\text{weak}} \) reduces to \( V_{\text{GR}} \) with cosmological constant and \( V_{\text{strong}} \) has only \((w_1, w_4, w_6; \mu_1; f_2, f_3) \neq 0\). By a suitable choice of \( f_2, f_3 \), only the two terms with \( w_4, w_6 \) survive, that is, those with the tracefree Ricci tensor and the curvature scalar.

3. The role of the Lie dual of the curvature

Before we continue the investigation of (15), we will derive some rules for manipulating curvature square terms containing a Lie star. Using heavily the computer-algebra system Reduce\footnote{Nowadays Reduce is freely available for download; for details, see http://reduce-algebra.com/} with the Exacalc package, compare [65–68], we were able to convert completely the Lie star \( \ast \) into the Hodge star \( \ast \) according to the following rules: the expression \( (I)^{R_{\mu\nu} \wedge (J)^{R^{(\ast)\mu\nu}}} \) is diagonal, that is, \( \propto \delta^{IJ} \); only the diagonal pieces do not vanish, namely

\[
(I)^{R_{\mu\nu} \wedge (J)^{R^{(\ast)\mu\nu}}} = \pm (I)^{R_{\mu\nu} \wedge *((I)^{R_{\mu\nu}}} ,
\]

with + for \( I = 1, 3, 5, 6 \) and with − for \( I = 2, 4 \). Note that on the left-hand side of this equation, we have the Lie star \( \ast \) and on the right-hand side, however, the Hodge star \( * \). This implies the relation, derived here for the first time explicitly\footnote{By using some simple algebra, equation (18) can alternatively be derived in a straightforward way from equations (10.17)–(10.22) of Obukhov [4].},

\[
R_{\mu\nu} \wedge R_{\mu\nu}^{(\ast)} = (1)^{R_{\mu\nu} \wedge *((1)^{R_{\mu\nu}}} - (2)^{R_{\mu\nu} \wedge *((2)^{R_{\mu\nu}}} + (3)^{R_{\mu\nu} \wedge *((3)^{R_{\mu\nu}}} - (4)^{R_{\mu\nu} \wedge *((4)^{R_{\mu\nu}}} - (5)^{R_{\mu\nu} \wedge *((5)^{R_{\mu\nu}}} + (6)^{R_{\mu\nu} \wedge *((6)^{R_{\mu\nu}}} .
\]

In particular, this shows that the Lie star is superfluous in forming a quadratic Lagrangian; the Hodge star is sufficient.

Comparison with (14) allows us to rewrite the Euler 4-form with the help of the Hodge star as

\[
B_{RR}^{*} = \frac{1}{2} (R_{\mu\nu} \wedge *R_{\mu\nu} - 2 (4)^{R_{\mu\nu} \wedge *R_{\mu\nu}} - 2 (2)^{R_{\mu\nu} \wedge *R_{\mu\nu}} .
\]

The Pontryagin 4-form, also defined in (14), after some algebra, can be expressed in terms of the irreducible pieces of the curvature as (also this relation is new)

\[
B_{RR}^{-} = \frac{1}{2} (1)^{R_{\mu\nu}} + (5)^{R_{\mu\nu} \wedge *R_{\mu\nu}} + (3)^{R_{\mu\nu} \wedge 2 (2)^{R_{\mu\nu}} .
\]
4. Comparison with Diakonov et al [8]

Recently, in the framework of perturbative quantum field theory, new results were reported [8] on including torsion in a gravitational gauge theory for describing fermionic matter, for a review of some earlier results, see [69]. In this context, Diakonov et al investigated gravitational gauge Lagrangians containing quadratic terms in the gauge fields of even and of odd parity. That is, contributions of the weak and the strong gravity sector (Lorentz gauge bosons) were considered. Generally, those 4-fermion interaction terms will give additional contributions to the energy–momentum current of matter. This aspect might be relevant in the context of quantum cosmological models, see [52, 55, 56]. Other groups address only weak gravity, even though they include Euler and Pontryagin terms, which refer to strong gravity, see Benedetti et al [70].

In the following, we would like to compare the approach given in [8] with the results we already gave in [64].

4.1. Torsion square invariants

Let us compare our torsion-square invariants in (7), see also [64], with those in [8], equation (53). If we add a plus sign for parity even and a minus sign for parity odd terms, the invariants of Diakonov et al., multiplied by the volume form $\eta$, read

$$K_1^+ = 2(1)T^a \wedge \ast(1)T_a - 2(2)T^a \wedge \ast(2)T_a + 2(3)T^a \wedge \ast(3)T_a,$$

$$K_2^+ = 3(2)T^a \wedge \ast(2)T_a,$$

$$K_3^+ = (1)T^a \wedge \ast(1)T_a + (2)T^a \wedge \ast(2)T_a - 2(3)T^a \wedge \ast(3)T_a,$$

$$K_4^- = -2(1)T^a \wedge (1)T_a + 4(2)T^a \wedge (3)T_a,$$

$$K_5^- = (1)T^a \wedge (1)T_a + 4(2)T^a \wedge (3)T_a,$$

and the inverse relations are

$$\ast(1)T^a \wedge (1)T_a = \frac{1}{3}(3K_1^- + K_2^+ + 3K_3^-),$$

$$\ast(2)T^a \wedge (2)T_a = \frac{1}{3}K_2^+,$$

$$\ast(3)T^a \wedge (3)T_a = \frac{1}{12}(3K_1^- + 4K_2^+ - 6K_3^-),$$

$$\ast(1)T^a \wedge (1)T_a = -\frac{1}{3}(K_4^- - K_5^-),$$

$$\ast(2)T^a \wedge (3)T_a = \frac{1}{12}(K_4^- + 2K_5^-).$$

These five invariants agree with those given in [64], equations (30) and (55).

4.2. Curvature square invariants

Diakonov et al [8], equation (59), find six even and four odd independent quadratic invariants $G_i$. As we did with the torsion invariants, we translate their component representations into the language of exterior differential forms used by us. We find, after some messy computer checked algebra, the following curvature invariants (multiplied by $\eta$):

$$G_1^+ = R^2 \eta = 12(6)R_{\alpha\beta} \wedge \ast(6)R^{\alpha\beta}. $$

6
\[ G^+_2 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \eta = 2 R_{\alpha\beta} \wedge R_{\alpha\beta} = \sum_{i=1}^{6} (I) R_{\alpha\beta} \wedge (i) R_{\alpha\beta}, \quad (32) \]

\[ G^+_3 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \eta \\
= 2 \left( (1) R_{\alpha\beta} \wedge (1) R_{\alpha\beta} - (2) R_{\alpha\beta} \wedge (2) R_{\alpha\beta} + (3) R_{\alpha\beta} \wedge (3) R_{\alpha\beta} \right) \wedge (4) R_{\alpha\beta} \wedge (4) R_{\alpha\beta} - (5) R_{\alpha\beta} \wedge (5) R_{\alpha\beta} + (6) R_{\alpha\beta} \wedge (6) R_{\alpha\beta}, \quad (33) \]

\[ G^+_4 = \left( R^2 - 4 \text{Ric}_{\mu\nu} \text{Ric}^{\lambda\mu} + R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \right) \eta \\
= 2 \left( (1) R_{\alpha\beta} \wedge (1) R_{\alpha\beta} - (2) R_{\alpha\beta} \wedge (2) R_{\alpha\beta} + (3) R_{\alpha\beta} \wedge (3) R_{\alpha\beta} \right) \wedge (4) R_{\alpha\beta} \wedge (4) R_{\alpha\beta} - (5) R_{\alpha\beta} \wedge (5) R_{\alpha\beta} + (6) R_{\alpha\beta} \wedge (6) R_{\alpha\beta}, \quad (34) \]

\[ G^+_5 = \left( R^2 - 4 \text{Ric}_{\mu\nu} \text{Ric}^{\mu\nu} + R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \right) \eta \\
= 2 \left( (1) R_{\alpha\beta} \wedge (1) R_{\alpha\beta} - (2) R_{\alpha\beta} \wedge (2) R_{\alpha\beta} + (3) R_{\alpha\beta} \wedge (3) R_{\alpha\beta} \right) \wedge (4) R_{\alpha\beta} \wedge (4) R_{\alpha\beta} - (5) R_{\alpha\beta} \wedge (5) R_{\alpha\beta} + (6) R_{\alpha\beta} \wedge (6) R_{\alpha\beta}, \quad (35) \]

\[ G^+_6 = \left( \eta^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} \right)^2 \eta = -48 (3) R_{\alpha\beta} \wedge (3) R_{\alpha\beta}, \quad (36) \]

\[ G^-_7 = R \eta^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} \eta = -24 (3) R_{\alpha\beta} \wedge (6) R_{\alpha\beta}, \quad (37) \]

\[ G^-_8 = \eta^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} R_{\alpha\beta} \eta \\
= -4 (1) R_{\alpha\beta} \wedge (1) R_{\alpha\beta} - 4 (5) R_{\alpha\beta} \wedge (5) R_{\alpha\beta} \wedge (3) R_{\alpha\beta} \wedge (6) R_{\alpha\beta} - 8 (2) R_{\alpha\beta} \wedge (4) R_{\alpha\beta}, \quad (38) \]

\[ G^-_9 = \eta^{\rho\nu\mu\lambda} R_{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} \eta \\
= -4 (1) R_{\alpha\beta} \wedge (1) R_{\alpha\beta} - 4 (5) R_{\alpha\beta} \wedge (5) R_{\alpha\beta} \wedge (3) R_{\alpha\beta} \wedge (6) R_{\alpha\beta}, \quad (39) \]

\[ G^-_{10} = \eta^{\rho\nu\lambda\mu} R_{\mu\nu\rho\lambda}, \quad (40) \]

The inverse relations are convenient for a detailed comparison. They turn out to be

\[ (1) R_{\alpha\beta} \wedge (1) R_{\alpha\beta} = -\frac{3}{12} G^+_4 + \frac{1}{8} \left( G^+_2 + G^+_3 + G^+_4 + G^+_5 \right) + \frac{1}{28} G^+_6, \quad (41) \]

\[ (2) R_{\alpha\beta} \wedge (2) R_{\alpha\beta} = \frac{1}{8} \left( G^+_2 - G^+_3 - G^+_4 + G^+_5 \right), \quad (42) \]

\[ (3) R_{\alpha\beta} \wedge (3) R_{\alpha\beta} = -\frac{1}{28} G^+_6, \quad (43) \]

\[ (4) R_{\alpha\beta} \wedge (4) R_{\alpha\beta} = \frac{1}{8} \left( G^+_2 + G^+_3 - G^+_4 + G^+_5 \right), \quad (44) \]
\[(5) R^{\alpha\beta} \wedge \ast(5) R_{\alpha\beta} = \frac{1}{8} (G_2^+ - G_3^+ + G_4^+ - G_5^+) ,\]

\[(6) R^{\alpha\beta} \wedge \ast(6) R_{\alpha\beta} = \frac{1}{12} G_1^- ,\]

and

\[(1) R^{\alpha\beta} \wedge \ast(1) R_{\alpha\beta} = -\frac{1}{12} (G_8^- + G_9^- + 2G_{10}^-) + \frac{1}{12} G_7^- ,\]

\[(2) R^{\alpha\beta} \wedge \ast(2) R_{\alpha\beta} = -\frac{1}{12} (G_8^- - G_9^-) ,\]

\[(3) R^{\alpha\beta} \wedge \ast(3) R_{\alpha\beta} = -\frac{1}{24} G_7^- ,\]

\[(5) R^{\alpha\beta} \wedge \ast(5) R_{\alpha\beta} = -\frac{1}{12} (G_8^- + G_9^- - 2G_{10}^-) .\]  

It is now straightforward to express the Euler 4-form (19) and the Pontryagin 4-form (20) in terms of the GI s. We find

\[B^+_{RR} = \frac{1}{4} G_4^+ \quad \text{and} \quad B^-_{RR} = -\frac{1}{8} G_8^- ,\]  

respectively. This is what Diakonov et al stressed that their invariants \(G_4^+\) and \(G_8^-\) are boundary terms. These two boundary terms can also be found in our earlier work, see [64], equations (33) and (50).

Hence the results of Diakonov et al [8] with respect to the quadratic invariants of torsion and curvature coincide with those of [64]. This is also manifest in the Riemannian subcase, that is, for vanishing torsion \(T^\alpha = 0\). Then,

\[(2) R_{\alpha\beta} = \ast(3) R_{\alpha\beta} = \ast(5) R_{\alpha\beta} = 0 ,\]  

or, in terms of the GI s,

\[G_2^+ = G_3^+, \quad G_4^+ = G_5^+, \quad G_6^+ = G_7^+ = 0, \quad G_8^- = G_9^- = G_{10}^+ .\]  

which can be read off directly from equations (31)–(40). Under the condition of vanishing torsion, the boundary terms read

\[B^+_{RR} | T^\alpha = 0 = \frac{1}{2} \left( (1) R^{\alpha\beta} \wedge \ast(1) R_{\alpha\beta} - (4) R^{\alpha\beta} \wedge \ast(4) R_{\alpha\beta} + (5) R^{\alpha\beta} \wedge \ast(5) R_{\alpha\beta} \right) ,\]

\[B^-_{RR} | T^\alpha = 0 = \frac{1}{2} (1) R^{\alpha\beta} \wedge \ast(1) R_{\alpha\beta} .\]

5. The number of independent terms in the most general quadratic PG Lagrangian

For the strong gravitational Lagrangian \(V_{\text{strong}}\) in (15), we can enter in a similar discussion as for \(V_{\text{weak}}\) in (7): besides the strong gravitational coupling constant \(\rho\), we have 12 constants \((w_1, w_2, \ldots, w_6; \mu_1, \mu_2, \mu_3, \mu_4; f_2, f_3)\). By a suitable choice of \(f_2\) and \(f_3\), we can compensate the terms containing the Weyl 2-form \((1) R^{\alpha\beta}\) with ten independent components, as can be seen from (19) and (20). Consequently, we are left with eight constants \((w_2, \ldots, w_6; \mu_2, \mu_3, \mu_4)\). Diakonov et al [8] found the ten invariants \(G_i\), two of which, namely \(G_4^+\) and \(G_8^-\), are boundary terms. Hence, they also arrive at eight independent invariants. Accordingly, also for strong gravity our results match those of Diakonov et al.
Our final gravitational Lagrangian is then\(^{12}\)

\[ V = \frac{1}{2\kappa} \left[ (a_0 R - 2\Lambda_0) + b_0 X \right] \eta 
+ \frac{a_2}{3} V \wedge T^\alpha \wedge \star (T^\alpha) 
- \frac{1}{2\kappa} \left[ \left( \frac{w_6}{12} R^2 - \frac{w_3}{12} X^2 + \frac{\mu_3}{12} RX \right) \eta + w_4^{(4)} R^\alpha \wedge \star (R^\beta) 
+ (^{(2)}R^\alpha \wedge (w_2^{(2)} R^\beta + \mu_2^{(2)} R_{\alpha\beta}) + (^{(5)}R^\alpha \wedge (w_5^{(5)} R^\beta + \mu_4^{(5)} R_{\alpha\beta})) \right]. \] \hspace{1cm} (56)

The first two lines represent weak gravity, and the last two lines strong gravity. The parity odd pieces are those with the constants \(b_0, \sigma_2; \mu_2, \mu_3, \mu_4\). In a Riemann space (where \(X = 0\)), only two terms of the first line and likewise two terms of the third line survive. All these four terms are parity even, that is, only torsion brings in parity odd pieces into the gravitational Lagrangian.

Yo and Nester [71–73] found that only a small subclass of the Lagrangians (56) is consistent from a Hamiltonian point of view. They, together with Shie, presented such a Lagrangian [74] and found an accelerating cosmological Friedman-type model with propagating connection. Shortly afterward, Nester and his group, see Chen et al [75], generalized this model and introduced a consistent Lagrangian containing the parity odd pieces \(A\) and \(X\). Since these terms occur quadratically, their Lagrangian was still parity even:

\[ V_{\text{Chen et al}} = \frac{1}{2\kappa} \left( a_0 R - 2\Lambda_0 \right) \eta + \frac{1}{6\kappa} \left( a_2 V \wedge T^\alpha \wedge \star (T^\alpha) \right) - \frac{1}{24\kappa} (w_6 R^2 - w_3 X^2) \eta. \] \hspace{1cm} (57)

The next step was carried out by Baekler et al [64], see also [76, 77]. They investigated a Lagrangian with three additional pieces with odd parity (57), namely those carrying the constants \(b_0, \sigma_2, \mu_3\):

\[ V_{\text{BHN}} = \frac{1}{2\kappa} \left( a_0 R - 2\Lambda_0 + b_0 X \right) \eta + \frac{1}{6\kappa} \left( a_2 V \wedge T^\alpha \wedge \star (T^\alpha) \right) - \frac{1}{24\kappa} (w_6 R^2 - w_3 X^2 + \mu_3 RX) \eta. \] \hspace{1cm} (58)

Now one should analyze the particle content of Lagrangian (56), that is, to find out which modes are propagating decently. This has been done in [64] by the simple method of the diagonalization of the Lagrangian. The results turned out to be in agreement with those of the Hamiltonian approach.

It is manifest already by now, looking beyond the Einstein–Cartan theory including parity odd Lagrangians is a field with bright prospects.

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\(^{12}\) If we introduce the notations \(R\) and \(X\) for the curvature scalar and the curvature pseudoscalar, then we find \(a_6^{(2)} R^\beta = -R \theta_{\beta\mu}/12\) and \(a_5^{(5)} R^\alpha = -X \theta_{\alpha\beta}/12\), respectively; moreover, for the torsion we can define the 1-forms of \(A\) and \(V\) for the axial vector and the vector torsion \(^{(2)}T^\alpha = \star (A \wedge \theta^\alpha)/3\) and \(^{(5)}T^\alpha = -(V \wedge \theta^\alpha)/3\), respectively.
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References

[1] Gronwald F and Hehl F W 1996 Proc. Int. School of Cosmol. Grav. 14th Course: Quantum Gravity (Erice, Italy) ed P G Bergmann et al (Singapore: World Scientific) p 148 (arXiv:gr-qc/9602013)
[2] Blagojević M 2002 Gravitation and Gauge Symmetries (Bristol: Institute of Physics Publishing)
[3] Ortin T 2004 Gravity and Strings (Cambridge: Cambridge University Press)
[4] Obukhov Y N 2006 Int. J. Geom. Methods Mod. Phys. 3 95 (arXiv:gr-qc/0601090)
[5] Trautman A 2006 Encyclopaedia of Math. Physics ed J-P Francoise et al (Oxford: Elsevier) p 189 (arXiv:gr-qc/0606062)
[6] Schouten J A 1954 Ricci-Calculas 2nd edn (Berlin: Springer)
[7] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 Phys. Rep. 258 1
[8] Diakonov D, Tumanov A G and Vladimirov A A 2011 arXiv:1104.2432v2
[9] Landau L D and Lifshitz E M 1975 The Classical Theory of Fields (Course of Theoretical Physics vol 2) 4th ed. rev. English edn (Transl. from the 6th rev. Russian edn) (Amsterdam: Elsevier)
[10] Ni-WT 2010 Rep. Prog. Phys. 73 056901 (arXiv:0912.5057)
[11] Hehl F W 1980 6th Course of the School of Cosmology and Gravitation on Spin, Torsion, Rotation, and Supergravity (Erice, Italy, May 1979), ed P G Bergmann and V de Sabbata (New York: Plenum) p 5 (see http://aflb.ensmp.fr/AFLB-322/aflb322m596.htm)
[12] Mielke E W 1992 Ann. Phys., NY 219 78
[13] Blagojević M and Vasilic M 2000 Class. Quantum Grav. 17 3785 (arXiv:hep-th/0006080)
[14] Pereira J G, Vargas T and Zhang C M 2001 Found. Phys. 31 258
[15] Itin Y 2002 Class. Quantum Grav. 19 173 (arXiv:gr-qc/0111036)
[16] Obukhov Y N and Pereira J G 2003 Phys. Rev. D 67 044016 (arXiv:gr-qc/0212080)
[17] Itin Y 2008 Classical and Quantum Gravity Research Progress ed M N Christiansen and T K Rasmussen (Hauopauge, NY: Nova Science Publishers) (arXiv:0711.4209)
[18] Hojman R, Mukku C and Sengupta S 1999 Phys. Lett. B 458 1
[19] Nelson P C 1980 Phys. Lett. A 79 285
[20] Purcell A J 1978 Phys. Rev. D 18 2730
[21] Nieh H T and Yan M L 1982 J. Math. Phys. 23 373
[22] Hehl F W and McCrea J D 1986 Found. Phys. 16 267
[23] McCrea J D, Hehl F W and Mielke E W 1990 Int. J. Theor. Phys. 29 1185
[24] Baekler P, Mielke E W and Hehl F W 1992 Nuovo Cimento B 107 91
[25] Obukhov Y N and Hehl F W 1996 Acta Phys. Pol. B 27 2685 (arXiv:gr-qc/9602014)
[26] Ashtekar A (with invited contributions) 1988 New Perspectives in Canonical Gravity (Napoli: Bibliopolis)
[27] Kiefer C 2007 Quantum Gravity 2nd edn (Oxford: Oxford University Press)
[28] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[29] Rovelli C 2011 Class. Quantum Grav. 28 153002 (arXiv:1012.4707)
[30] Holst S 1996 Phys. Rev. D 53 5966 (arXiv:gr-qc/9511026)
[31] Freidel L, Minic D and Takeuchi T 2005 Phys. Rev. D 72 104002 (arXiv:hep-th/0507253)
[32] Mercuri S 2006 Phys. Rev. D 73 084016 (arXiv:gr-qc/0601013)
[33] Freidel L, Kowalski-Glikman J and Starodubtsev A 2006 Phys. Rev. D 74 084002 (arXiv:gr-qc/0607014)
[34] Mercuri S 2008 Phys. Rev. D 77 024036 (arXiv:0708.0037)
[35] Bojowald M and Das R 2008 Phys. Rev. D 78 064009 (arXiv:0710.5722)
[36] Mukhopadhyaya B and Sengupta S 1999 Phys. Lett. B 458 1 (arXiv:hep-th/9811012)
[37] Mukhopadhyaya B, Sen S, SenGupta S and Sur S 2004 Eur. Phys. J. C 35 129 (arXiv:hep-th/0207165)
[38] Mielke E W 2009 Phys. Rev. D 80 067502
[39] Cantcheff M B 2008 Phys. Rev. D 78 025002 (arXiv:0801.0067)
[40] Date G, Kaul R K and Sengupta S 2009 Phys. Rev. D 79 044008 (arXiv:0811.4496)
[41] Mercuri S and Taveras V 2009 Phys. Rev. D 80 104007 (arXiv:0903.4407)
[42] Ertem U 2009 arXiv:0912.1433
[43] Pfeifer W 2010 arXiv:1012.1738
[44] Daum J E and Reuter M 2010 arXiv:1012.4280
[45] Hanisch F, Pfaffle F and Stephan C A 2010 Commun. Math. Phys. 300 877 (arXiv:0911.5074)
[46] Pfaffle F and Stephan C A 2011 arXiv:1101.1424
[47] Pfaffle F and Stephan C A 2011 arXiv:1102.0954
[48] Tilquin A and Schucker T 2011 Gen. Rel. Grav. (at press) (arXiv:1104.0160)
[49] Kaul R K and Sengupta S 2011 arXiv:1106.3027
[50] Adak M 2011 arXiv:1107.0569
[51] Baekler P, Hehl F W and Nester J M 2011 Phys. Rev. D 83 024001 (arXiv:1009.5112)
[52] Hearn A C 1993 REDUCE: User’s Manual, Version 3.5, RAND Publication CP78 (Rev. 10/93) (Santa Monica, CA: The RAND Corporation)
[53] Benetti D and Speziale S 2011 J. High Energy Phys. JHEP06(2011)107 (arXiv:1104.4028)
[54] Yeh H J and Nester J M 2007 Mod. Phys. Lett. A 22 2057 (arXiv:astro-ph/0612738)
[55] Chen H, Ho F H, Nester J M, Wang C H and Yo H J 2009 J. Cosmol. Astropart. Phys. JCAP10(2009)027 (arXiv:0908.3323)
[56] Ho F H and Nester J M 2011 7th Biennial Conf. on Classical and Quantum Relativistic Dynamics of Particles and Fields: IARD 2010 (Hualien, Taiwan, 30 May–1 June 2010) J. Phys., Conf. Ser. (submitted) (arXiv:1105.5001)
[57] Ho F H and Nester J M 2011 Int. J. Mod. Phys. D (submitted) (arXiv:1106.0711)