A slice genus lower bound from $sl(n)$
Khovanov-Rozansky homology

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1 Introduction

1.1 $sl(2)$ Khovanov homology

Given the data of an oriented link diagram $D$, one can compute the HOMFLY polynomial $P(D)$ (a polynomial over $\mathbb{Z}$ in the variables $a^{\pm 1}$ and $b^{\pm 1}$) using the local skein relations in Figure 1 (up to an arbitrary choice for the HOMFLY polynomial $P(U)$ of the unknot $U$).

This polynomial is invariant under the oriented Reidemeister moves and hence defines an invariant of oriented links. If we specialize for $n \geq 2$ by defining

$$P_n(q) = P(a = q^n, b = q + q^{-1})$$

we obtain the $sl(n)$ quantum polynomial of the link given by the diagram $D$. We will be using the normalization $P_n(U) = \frac{q^n - q^{-n}}{q - q^{-1}}$. The polynomial $P_2(q)$ is known as the Jones polynomial.

In [4], Khovanov associated to a link diagram $D$ a bigraded chain complex $CKh_2^{i,j}$ with differential

$$d : CKh_2^{i,j}(D) \to CKh_2^{i+1,j}(D)$$

Figure 1: HOMFLY skein relation
(i is called the homological grading, j is called the quantum grading). To diagrams \( D \) and \( D' \) differing by a single Reidemeister move, Khovanov gave a quantum-degree 0 chain homotopy equivalence between \( CKh_2(D) \) and \( CKh_2(D') \), thus showing that the homology groups \( HKh_{i,j}^2(D) \) are knot invariants. Furthermore these homology groups provided a categorification of the Jones polynomial \( P_2(D) \), by which is meant

\[
P_2(q) = \sum_{i,j} (-1)^i q^j \dim(HK_h_{i,j}^2)
\]

A powerful facet of the \( HKh_2 \) theory was conjectured by Khovanov (and later proved by Jacobsson [3]), namely that the homology theory should be functorial for link cobordisms up to sign. More explicitly, suppose we start with a smooth embedding of a surface \( \Sigma \) with boundary

\[
\Sigma \hookrightarrow [0, 1] \times \mathbb{R}^3
\]

\[
\partial \Sigma = (L_0 \hookrightarrow \{0\} \times \mathbb{R}^3) \coprod (L_1 \hookrightarrow \{1\} \times \mathbb{R}^3)
\]

Choose link diagrams \( D_0, D_1 \) of the links \( L_0, L_1 \). Next we take a representation of the surface \( \Sigma \) as a product of elementary cobordisms. Elementary cobordisms consist of before-and-after link diagrams where we have made one local change in the before diagram to get to the after diagram. The local changes that we are allowed to make consist of each of the Reidemeister moves and the Morse moves which correspond to adding a handle to the surface as shown in Figure 2.

To diagrams \( D \) and \( D' \) differing by one elementary cobordism Khovanov associated a chain map \( CKh_2(D) \rightarrow CKh_2(D') \), inducing a map on homology. For the Reidemeister moves, these maps are just the chain homotopy equivalences referred to earlier. For the 0-handle and 2-handle moves these maps are graded of quantum-degree +1 and for the 1-handle move it is graded of degree -1. By composing the maps corresponding to these elementary cobordisms we get a map graded of quantum-degree \( \chi(\Sigma) \)

\[
HKh_2(\Sigma) : HKh_2(L_0) \rightarrow HKh_2(L_1)
\]

This map (up to multiplication by \(-1\)) is an invariant of \( \Sigma \), as the notation suggests. In other words, whichever decomposition of \( \Sigma \) into elementary cobordisms is chosen, the induced map \( HKh(\Sigma) \) stays the same up to sign.
1.2 Perturbed $sl(2)$ theory

In 2002, Lee gave a description of a perturbation $HKh'$ of the Khovanov homology $HKh_2$ [7]. This homology theory is no longer graded in the quantum grading direction, but filtered: $\ldots \subseteq F_j^{j+1} HKh'^{ji} \subseteq F_j^{j} HKh'^{ji} \subseteq \ldots$. $HKh'(L)$ has a particularly simple form: when we take the base ring to be $\mathbb{C}$ it consists of $2^l$ copies of $\mathbb{C}$ where $l$ is the number of components of the link $L$. Lee gave explicit chain representatives of these generators defined from any diagram presentation of $L$. As a formal consequence of the properties of filtered chain complexes, there is a spectral sequence with $E_2$ page $HKh_2^{i,j}$ converging to $E_\infty$ page $Gr^j HKh'^ja := F_j^{j} HKh'^ja / F_j^{j+1} HKh'^ja$ the associated graded vector space of $HKh'$ with respect to the filtration $F$.

Analysis of the behaviour of $HKh'$ under link cobordism was carried out by Rasmussen [9]. In this remarkable paper, Rasmussen showed that Lee’s generators provide a lower bound for the slice genus of a knot. Let us digress to define the slice genus.

Given a knot $K \hookrightarrow \mathbb{R}^3$, a classical knot invariant is the genus $g \geq 0$ of the knot. That is, the minimal genus of the surfaces-with-boundary $\Sigma$ for which there exists an embedding $\Sigma \hookrightarrow \mathbb{R}^3$ with $\partial \Sigma = K$.

If we believe that manifolds with boundary should rightfully live within other manifolds with boundary, we are motivated to consider surfaces $\Sigma$ smoothly embedded in $(-\infty,0] \times \mathbb{R}^3$ with $K = \partial \Sigma \hookrightarrow \{0\} \times \mathbb{R}^3$. The
minimal genus of such surfaces we call the *slice genus* $g_s(K)$ of the knot $K$.

By removing a neighbourhood of a point, such a surface provides a knot cobordism between $K$ and the unknot $U$. Rasmussen showed that the map associated to any presentation (as a composition of elementary cobordisms) of a connected knot cobordism $\Sigma$ between two (1-component) knots $K_0$ and $K_1$ preserves the generators of Lee’s homology $HKh$\(^l\). This map is filtered of quantum-degree $\chi(\Sigma)$.

$$\mathcal{F}^i HKh^0 \to \mathcal{F}^{i+\chi(\Sigma)} HKh^0$$

The homology of the unknot is computable as $Gr^1 HKh^0 = \mathbb{Q}$ and $Gr^{-1} HKh^0 = \mathbb{Q}$ with no homology in any other bigrading. Hence as an immediate corollary, $2g_s(K)$ is bounded below by one less than the highest degree in which $HKh^l(K)$ is non-zero.

To compute $HKh^l$ directly is difficult, but it can be carried out for positive knots (those knots that admit a diagram with only positive crossings). In this case we get a tight bound on the slice genus since we can explicitly construct a Seifert surface $\Sigma$ of the same genus as our lower bound, and then just push this surface into $(-\infty, 0] \times \mathbb{R}^3$ to get a slice surface. In particular we can apply this to torus knots $K_{p,q}$ to get a combinatorial proof of Milnor’s conjecture on the value of $g_s(K_{p,q})$.

### 1.3 $sl(n)$ Khovanov homology

In [5], Khovanov and Rozansky describe bigraded homology theories $HKh_n$ for $n \geq 3$ which categorify the $sl(n)$ polynomials $P_n$. Again, given a knot diagram $D$ Khovanov and Rozansky associate to it a chain complex $CKh_n(D)$ (in a much more complicated way than for the case $n = 2$) with differentials

$$d_n : CKh_n^{i,j}(D) \to CKh_n^{i+1,j}(D)$$

And also as before, if $D$ and $D'$ differ by a Reidemeister move, they give a chain homotopy equivalence (graded of quantum-degree 0) between $CKh_n(D)$ and $CKh_n(D')$, thus showing that $HKh_n^{i,j}(D)$ is an invariant of the knot and not just the diagram $D$. This homology theory is also functorial for knot cobordisms.

Gornik has carried out for $HKh_n$ a direct analogue of Lee’s work on $HKh_2$ [2], describing a perturbation with filtered homology $HKh'_n$, which can be interpreted as the $E_\infty$ page of a spectral sequence with $E_2$ page $HKh_n$. Given an $l$-component link diagram $D$, Gornik gives an explicit description of the $n^l$ generators of $HKh'_n(D)$. 
1.4 Statement of results

Our intention in this paper is to do for Gornik’s work something of what Rasmussen did for that of Lee’s. We start by generalizing Gornik’s $HKh'_n$ to a theory that we shall denote $HKh_w$ where $w \in \mathbb{C}[x]$ is a monic polynomial of degree $n+1$ such that $\partial_x w = dw/dx$ is a product of distinct linear factors. In this set up,

$$HKh'_n = HKh_{x^{n+1}+(n+1)\beta^n}$$

for $\beta \in \mathbb{C} - \{0\}$. $HKh_w$ is filtered in the quantum direction

$$\ldots \subseteq F^j HKh_w^i \subseteq F^{j+1} HKh_w^i \subseteq \ldots$$

Our procedure will essentially follow that of [9]. For diagrams $D, D'$ differing by a Reidemeister move we wish to construct a map $F^j HKh_w^i(D) \to F^j HKh_w^i(D')$ which preserves our analogues of the generators in [2]. If $D, D'$ differ by a 0- or 2-handle attachment we wish to give maps $F^j HKh_w^i(D) \to F^{j-n+1} HKh_w^i(D')$ and if $D, D'$ differ by a 1-handle we wish to give maps $F^j HKh_w^i(D) \to F^{j+n-1} HKh_w^i$. We would aim to do this so that we could then compute that any representation of a connected knot cobordism $\Sigma: D \to D'$ as a product of elementary knot cobordisms preserves the generators of $HKh_w(D)$.

Due to the complexity of the Reidemeister III move, we do not quite fulfill all of our wishes, however a topological argument (in which we only allow certain products of elementary cobordisms) is enough for us to get the desired analogue of the slice genus bound in [9].

We write $HKh_{w}^{i,j}(D) = F^j HKh_w^i / F^{j-1} HKh_w^i$ for the associated graded vector space to the quantum filtration.

Our main theorem is

**Theorem 1.1** Let $D$ be a knot diagram of a knot $K$. If $HKh_{w}^{0,j}(D) \neq 0$ and $HKh_{w}^{0,j'}(D) = 0$ for all $j' > j$ then

$$(n - 1)(2g^*(K) - 1) \geq -j$$

As a corollary we obtain another proof of Milnor’s conjecture on the slice genus of torus knots.

It is worth noting here that this paper does not contain a proof that

$$1. HKh_w(D) = HKh_w(D')$$
for two knot diagrams differing by a single Reidemeister move, even for Gornik’s $w$, nor that

\[
2. HKh_w(D) = HKh_{w'}(D)
\]

for two different degree $n+1$ polynomials $w, w'$. However we expect both of these statements to be true (for example because there is a spectral sequence with $E_2$ pages independent of choice of $w$ or $D$ converging to $HKh_w(D)$).

In the final stages of the preparation of this paper a preprint appeared on the arXiv [10], which contains, amongst others, results similar to those in this paper.

1.5 The Khovanov cube

In this section we shall outline the construction of $HKh_w$. This is similar to the construction of the original Khovanov-Rozansky homology $HKh_n$ - indeed, replacing any occurrence of the word “filtered” by the word “graded” in this section will give an outline of the construction of $HKh_n$.

To define $HKh_w$ of an oriented link diagram $D$ with $m$ crossings we start by forming the $2^m$ possible resolutions of $D$ and decorating each vertex of the cube $[0,1]^n$ with one of these resolutions. We form the resolutions as shown in Figure 3.

Choosing an (arbitrary) ordering of the crossings allows us to associate each resolution $\Gamma$ of $D$ with one of the corners of the cube. For a corner $v$
call the associated resolution $\Gamma_v$. In the following subsections we shall see how to associate to $\Gamma_v$ a filtered vector space $H(\Gamma_v)$. We shall often refer to the filtration as the \textit{quantum} filtration.

If two corners of the cube $v, v'$ are connected by an edge $e$ then we see that their associated resolutions $\Gamma_v$ and $\Gamma_{v'}$ differ only in the resolution of a single crossing of $D$. Suppose that $v$ is at the 0-coordinate of the edge and $v'$ is at the 1-coordinate of the edge. In the following subsections we shall define a map $\Phi_e : H(\Gamma_v) \to H(\Gamma_{v'})$ of quantum filtered degree 1. If $e_1, e_2, e_3, e_4$ are edges bounding a face of the cube then $\Phi_{e_1}\Phi_{e_2} = \Phi_{e_3}\Phi_{e_4}$ if the edges are ordered so the composition makes sense.

For a vertex $v$ write the sum of its coordinates as $s(v)$ (this will be an integer between 0 and $n$). We write $W$ for the \textit{writhe} of the knot diagram $D$: the signed number of crossings of $D$ as in Figure 4. We define the chain groups of a chain complex by

$$C^k = \sum_{s(v)=k+(1/2)(n+W)} H(\Gamma_v)\{(n-1)W - k\}$$

where by $\{-\}$ we mean a shift in the quantum filtration.

The differentials of the chain complex are defined by

$$d_k = \sum \pm \Phi_e : C^k \to C^{k+1}$$

where the sum is taken over all edges for which it makes sense. The $\pm$ signs are chosen so that each face of the cube has exactly 1 or 3 of its edges decorated with a minus sign; the commutivity of the maps associated the edges of any face will then ensure that $d_{k+1}d_k = 0$. The shift in the quantum filtrations of the $H(\Gamma_v)$’s ensures that the differentials are filtered of degree 0.
Taking homology we obtain a vector space graded in the homological direction and filtered in the quantum direction.

1.6 Introduction to matrix factorizations

The definition of the $sl(n)$ Khovanov homology, standard or perturbed, makes use of the notion of a matrix factorization. Here we first define ungraded matrix factorizations, in the next section we see how to associate filtered matrix factorizations to trivalent graphs.

A polynomial $p(x) \in \mathbb{C}[x_1,\ldots,x_n] = R$ may not admit a non-trivial factorization into polynomials. However if $M^0 = M^1$ is a free module over $R$ then we may be able to find $R$-module maps $f_0, f_1$

$$M^0 \xrightarrow{f_0} M^1 \xrightarrow{f_1} M^0$$

such that $f_0f_1 = f_1f_0 = p(x)$. Thus we would have factored $p(x)$ into a product of $R$-module maps - this is called a matrix factorization of $p(x)$.

If $M$ and $\tilde{M}$ are both matrix factorizations of $p(x) \in \mathbb{C}[x_1,\ldots,x_n] = R$ then a map of matrix factorizations $G : M \to \tilde{M}$ is a pair of $R$-module maps

$$G_0 : M^0 \to \tilde{M}^0, \quad G_1 : M^1 \to \tilde{M}^1$$

satisfying $\tilde{f}_0G_0 = G_1f_0$ and $\tilde{f}_1G_1 = G_0f_1$. Note that this echoes the definition of a map of chain complexes. Similarly, considering a matrix factorization as being something akin to a 2-periodic chain complex, we can define the notion of a tensor product:

Suppose $M$ and $\tilde{M}$ are matrix factorizations of $p(x), q(x) \in R$ respectively. Then we can define the matrix factorization $M \otimes \tilde{M}$ of $p(x) + q(x)$ by $(M \otimes \tilde{M})^0 = (M^0 \otimes \tilde{M}^0) \oplus (M^1 \otimes \tilde{M}^1)$, $(M \otimes \tilde{M})^1 = (M^1 \otimes \tilde{M}^0) \oplus (M^0 \otimes \tilde{M}^1)$ with maps

$$\left( \begin{array}{cc} f_0 & \tilde{g}_1 \\ -\tilde{g}_0 & f_1 \end{array} \right) : (M \otimes \tilde{M})^0 \to (M \otimes \tilde{M})^1$$

$$\left( \begin{array}{cc} \tilde{f}_1 & -\tilde{g}_1 \\ \tilde{g}_0 & f_0 \end{array} \right) : (M \otimes \tilde{M})^1 \to (M \otimes \tilde{M})^0$$

Since matrix factorizations have the look of 2-periodic complexes (but with $d^2$ now being a polynomial, not necessarily zero), we can define the obvious notions of homotopic maps of matrix factorizations and homotopy equivalent matrix factorizations. We will be working in the category of matrix factorizations and homotopy classes of maps.
1.7 Trivalent graphs

Khovanov and Rozansky describe a way of associating a matrix factorization to any finite trivalent graph with thick edges and labelled boundary components. An example of what we mean by a trivalent graph with thick edges is given in Figure 5 - here we have indicated the boundary as a dotted circle which shall hereafter be omitted from our diagrams of trivalent graphs.

Note that the thin edges are oriented and at a thick edge the orientations of the incident thin edges look like Figure 6. Each trivalent vertex has one thick edge and two thin edges incident to it. Only thin edges are allowed to end on the boundary of the graph. We allow closed thin loops.

Also note that we have included what appear to be crossings (which are circled) of thin edges. But a trivalent graph need not come with an embedding into $\mathbb{R}^2$ or even $\mathbb{R}^3$, and we are thinking of these thin edges as neither intersecting nor giving rise to an “overcrossing” or an “undercrossing”.

Figure 5: An example of a trivalent graph

Figure 6: The neighbourhood of a thick edge
To associate a matrix factorization to such a trivalent graph will first involve choosing a potential: a polynomial $w(x) \in \mathbb{C}[x]$ of degree $n + 1$. Khovanov’s and Rozansky’s original theory took $w = x^{n+1}$ and Gornik used the perturbation $w = x^{n+1} + (n + 1)\beta^n x$ for $\beta \in \mathbb{C} - 0$. In the course of this paper we will be using a general polynomial $w$ of degree $n + 1$, satisfying the condition that $\partial_x w = dw/dx$ factors as a product of distinct linear factors.

Every variable $x_i$ that appears in the polynomial rings used to define matrix factorizations is taken to have quantum degree 2 (we often use the word quantum so as to distinguish from homological). We will be using filtered matrix factorizations, which means each module $M$ is filtered ... $\subseteq \mathcal{F}^i M \subseteq \mathcal{F}^{i+1} M \subseteq ...$. The filtration on $M$ (which will always be a free module over $R$) shall be induced by the grading on $R$ with possibly an overall shift. To denote a shift of quantum degree $d$ in the filtration grading of a module $M$ we use the notation $M\{d\}$. The two differentials in the matrix factorization should have filtered degree $n + 1$ (note multiplication by $w(x)$ is filtered of degree $2(n+1)$). Maps of matrix factorizations, unless stated otherwise, are of filtered degree 0, homotopies are of filtered degree $-(n + 1)$.

We define two fundamental factorizations - that of an oriented thin line segment (Figure 7) and that of a neighbourhood of a thick edge (Figure 8). Then we can obtain the factorization corresponding to any trivalent graph by tensoring together these fundamental factorizations, identifying boundary variables where we have joined the fundamental factorizations together.

The fundamental factorization in Figure 7 is the factorization of $w(x_1) - w(x_2)$ over $\mathbb{C}[x_1, x_2]$ given by

$$R \xrightarrow{\pi x_1 + x_2} R\{1 - n\} \xrightarrow{x_1 - x_2} R$$
Figure 8: A thick edge with labelled boundary components

where $M^0$ and $M^1$ are rank 1 modules and $\pi_{x_1,x_2} = \frac{w(x_1) - w(x_2)}{x_1 - x_2}$. Remember the pair of curly brackets denotes a grading shift so that each of our differentials is of filtered degree $n + 1$.

We now define the factorization in Figure 8 over the ring $\mathbb{C}[x_1, x_2, x_3, x_4]$. This will be a factorization of the polynomial $w(x_1) + w(x_2) - w(x_3) - w(x_4)$.

Consider the unique polynomial $p$ in two variables such that $p(x + y, xy) = w(x) + w(y)$.

Then

$$w(x_1) + w(x_2) - w(x_3) - w(x_4)$$

$$= p(x_1 + x_2, x_1 x_2) - p(x_3 + x_4, x_3 x_4)$$

$$= (x_1 + x_2 - x_3 - x_4) u_1 + (x_1 x_2 - x_3 x_4) u_2$$

for some polynomials $u_1, u_2$.

We define the second fundamental factorization as the tensor product over $R(\Gamma)$ of the two factorizations

$$R\{-1\} \xrightarrow{u_1} R\{-n\} \xrightarrow{x_1 + x_2 - x_3 - x_4} R\{-1\}$$

$$R \xrightarrow{u_2} R\{3 - n\} \xrightarrow{x_1 x_2 - x_3 x_4} R$$

Note that the differentials of the resulting factorization have filtered degree $n + 1$.

Hence the factorization $C(\Gamma)$ (a tensor product of fundamental factorizations), associated to a trivalent graph $\Gamma$, also has differentials of this filtered degree. Let $R(\Gamma) = Q[X_e | e \in \text{edge endpoints}]$, then the graph $\Gamma$ defines a matrix factorization $C(\Gamma)$ over $R(\Gamma)$.
\[ M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^0 \]

where \( M^0 \) and \( M^1 \) are free modules over \( R(\Gamma) \) and

\[
d^1d^0 = d^0d^1 = \sum_{\text{edge endpoints } e \text{ pointing out of } \Gamma} w(X_e) - \sum_{\text{edge endpoints } e \text{ pointing into } \Gamma} w(X_e) \quad (1)
\]

### 1.8 Method of attack

We shall now give an overview of the organization of the remaining sections of this paper.

Section 2 contains definitions and theorems whose statements echo those to be found in [2] and [5] but adapted to our choice of potential. One purpose of the section is to show that the results of [2] hold for more general choices of potential than those considered in that paper. As a consequence of the results in this section it will follow that for \( D \) a diagram of an \( l \)-component link, \( HKh_w \) consists of \( n^l \) copies of \( \mathbb{C} \). We give explicit chain representatives of these \( n^l \) generators of the homology. When \( l = 1 \), \( HKh_w(D) \) is supported in homological degree 0.

Although conventional practice dictates that one should deal with invariance under Reidemeister moves before other considerations, issues of logical dependence encourage us to postpone this until Section 4.

Section 3 deals with Morse moves for knot cobordisms as in Figure 2. To link diagrams \( D, D' \) differing by a Morse move we associate an isomorphism \( HKh_w(D) \rightarrow HKh_w(D') \), induced by a chain map \( CKh_w(D) \rightarrow CKh_w(D') \), that is filtered of quantum degree 1 – \( n \) in the case of the 0-handle and 2-handle Morse moves and of degree \( n - 1 \) in the case of the 1-handle Morse move. We see that the maps we define have good properties in terms of preserving the generators defined in Section 4.

Section 4 is also computational. The purpose of this section is to show that for link diagrams \( D \) and \( D' \) differing by a Reidemeister move I or II there is a map, filtered of quantum-degree 0, \( HKh(D) \rightarrow HKh(D') \), induced from a chain map \( CKh_w(D) \rightarrow CKh_w(D). \) If we could show this also for Reidemeister move III, a consequence would be that the graded groups associated to the quantum filtration \( HKh^{i,j}_w(D) = \mathcal{F}^j HKh^i_w / \mathcal{F}^{j-1} HKh^i_w \) would be invariants of the link represented by the diagram \( D \). Although we do not do this for Reidemeister III, a topological trick detailed in a later section means that we still have enough to prove our main result on the slice genus.
Sections 3 and 4 taken together tell us that if we have a surface \( \Sigma \) with boundary

\[
\Sigma \hookrightarrow [0, 1] \times \mathbb{R}^3
\]

\[
\partial \Sigma = (K \hookrightarrow \{0\} \times \mathbb{R}^3) \coprod (U \hookrightarrow \{1\} \times \mathbb{R}^3)
\]

where \( K \) is a 1-component knot and \( U \) is the unknot and we have a representation of \( \Sigma \) as a product of elementary cobordisms (section 1.1), starting with a diagram \( D \) for \( K \) and ending with the 0-crossing diagram of the unknot, without using Reidemeister move III, then

\[
(n - 1)(2\text{genus}(\Sigma) + 3) > |j|
\]

where \( HKh^{0,j}_w(D) \neq 0 \).

Section 5 will detail a topological trick which shall mean that we can avoid doing for Reidemeister III a similar computation to those in Section 2 and yet still obtain our main theorem 1.1, the proof of which is completed in this section.

## 2 Homology of a trivalent graph

We have required that \( \partial_x w \) is a product of distinct linear factors. This condition is the main ingredient in many of results in [2], the proofs of which carry across very easily to our potential. We shall content ourselves, most often, with stating these results as they apply in our case without proof, but in a few cases we shall give a proof where things are more difficult for our potential than for that of Gornik’s or where we feel it is important for the rest of the paper to understand the proof.

Assume now that \( \Gamma \) is a closed graph so that \( C(\Gamma) \) (the matrix factorization associated to \( \Gamma \)) is actually a 2-periodic complex (see Equation 11), call the homology of this 2-complex \( H(\Gamma) \). Since \( C(\Gamma) \) is filtered \( H(\Gamma) \) is also filtered \( \ldots \subseteq F^i H(\Gamma) \subseteq F^{i+1} H(\Gamma) \subseteq \ldots \). Let \( R(\Gamma) \) be the polynomial ring over \( \mathbb{C} \) generated by variables \( X_e \) as \( e \) runs over the thin edges of \( \Gamma \) (note: not just the thin edges with boundary as in the previous section). Certainly the homology \( H(\Gamma) \) is a module over \( R(\Gamma) \), but in fact it is also a module over the ring \( \bar{R}(\Gamma) \) obtained from \( R(\Gamma) \) by quotienting out any polynomial in the \( X_e \)'s appearing in the definition of the fundamental factorizations making up \( C(\Gamma) \), (since these polynomials define 0-homotopic endomorphisms of \( C(\Gamma) \)).
Proposition 2.1 (See [2] 2.4) The algebra $\bar{R}(\Gamma)$ is generated by elements $X_e$ (we are using $X_e$ here to mean the image of $X_e \in R(\Gamma)$ in $\bar{R}(\Gamma)$) where $e$ runs over the thin edges of $\Gamma$, and each $X_e$ satisfies

$$\partial_x w(X_e) = 0$$

Writing $\Sigma_n$ for the roots of $\partial_x w$, we make an assignment

$$\phi : e(\Gamma) \rightarrow \Sigma_n$$

of elements of $\Sigma_n$ to the thin edges $e(\Gamma)$ of $\Gamma$ is called a state of $\Gamma$.

Proposition 2.2 (See [2] 2.5) Given a state $\phi$, define $Q_\phi \in \bar{R}(\Gamma)$ by

$$Q_\phi = \prod_{e \in e(\Gamma)} \left( \prod_{\alpha \in \Sigma_n \setminus \phi(e)} \frac{1}{\phi(e) - \alpha} \right) \frac{1}{n + 1} \frac{\partial_x w(X_e)}{X_e - \phi(e)}$$

where by $\partial_x w(X_e)/(X_e - \phi(e))$ we mean the result of substituting $X_e$ for $x$ in the polynomial $\partial_x w(x)/(x - \phi(e))$.

Then we have

$$Q_{\phi_1} Q_{\phi_2} = \begin{cases} Q_{\phi_1}, & \phi_1 = \phi_2 \\ 0, & \phi_1 \neq \phi_2 \end{cases}$$

$$\sum_{\phi} Q_\phi = 1$$

Proof It suffices to check that the relations hold in the ring

$$\bigotimes_{e \in e(\Gamma)} C[X_e]/\partial_x w(X_e)$$

since $\bar{R}(\Gamma)$ is a quotient of this ring.

Given an embedding of $\Gamma$ into $\mathbb{R}^2$ we can define the notion of admissability of states. In the construction of the Khovanov homology of a knot, we start with a diagram of the knot and all $\Gamma$’s appearing in the construction will come with an embedding into $\mathbb{R}^2$ induced by the knot diagram. A state $\phi$ is called admissible if the assignment $\phi : \Sigma_n \rightarrow e(\Gamma)$ looks like either of the possibilities in Figure 9 in a neighbourhood of each thick edge. We call the set of all states $S'(\Gamma)$ and the set of admissible states $S(\Gamma)$. 

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Theorem 2.3 (See [2] 3) For non-admissable states $\phi$ we have

$$Q_{\phi} = 0.$$ 

For admissable states $\phi$ we have

$$0 \neq \mathbb{C}Q_{\phi} = \bar{R}(\Gamma)Q_{\phi}$$

and so $\dim_{\mathbb{C}} \bar{R}(\Gamma)Q_{\phi} = 1$.

We have a decomposition as a $\mathbb{C}$-algebra

$$\bar{R}(\Gamma) = \bigoplus_{\phi \in \mathcal{S}(\Gamma)} \mathbb{C}Q_{\phi}$$

and

$$\dim_{\mathbb{C}} \bar{R}(\Gamma) = P_n(\Gamma)|_{q=1}.$$ 

Proposition 2.4 Let $k \in \mathbb{Z}$. Let $Kh^k(\Gamma)$ be the quantum degree $k$ piece of the classical $sl(n)$ Khovanov-Rozansky homology of the graph $\Gamma$. Then there is an isomorphism of vector spaces

$$\Phi : Kh^k(\Gamma) \to F^k H(\Gamma)/F^{k-1} H(\Gamma).$$

Corollary 2.5 Since the graded dimension of $Kh(\Gamma)$ is just $P_n(\Gamma)$ (the $sl(n)$ polynomial of $\Gamma$), the filtered dimension of $H(\Gamma)$ is $P_n(\Gamma)$. The number of admissable states is $P_n(\Gamma)(1)$ so that the dimension of $H(\Gamma)$ as a complex vector space agrees with the number of admissable states.

Here since the translation of Gornik’s proof is not completely straightforward, we give some details.
Proof (of Proposition) Khovanov and Rozansky have shown that the classical 2-periodic complex $CKh(\Gamma)$ has cohomology only in one of the two homological gradings. Suppose without loss of generality that it lies in grading 1.

$CKh(\Gamma)$ looks like

$$CKh^0(\Gamma) \xrightarrow{d_0} CKh^1(\Gamma) \xrightarrow{d_1} CKh^0(\Gamma)$$

where $d_0$ and $d_1$ are of graded degree $n + 1$. $C(\Gamma)$ looks like

$$CKh^0(\Gamma) \xrightarrow{d'_0} CKh^1(\Gamma) \xrightarrow{d'_1} CKh^0(\Gamma)$$

where $d'_0$ and $d'_1$ are filtered of degree $n + 1$. In fact we can decompose $d'_i$ as

$$d'_i = d_i^0 + d_i^1 + d_i^2 + \ldots$$

where $d_i^j$ is graded of degree $n + 1 - 2j$ and $d_i^0 = d_i$ for $i = 1, 2$. To define the $\Phi$ mentioned in the Proposition we give first a lift of $\Phi$

$$\phi : (\ker d_1)^k \to F^kH(\Gamma)/F^{k-1}H(\Gamma)$$

$$\phi : \alpha \mapsto \alpha + \alpha^1 + \alpha^2 + \ldots$$

where $\alpha^i$ has degree $k - 2i$. We define each $\alpha^i$ inductively. We wish to define each $\alpha^i$ so that

$$\sum_{i=0}^{k} d_{i}^1 (\alpha^{j-i}) = 0 \forall j \quad (2)$$

Writing $\alpha = \alpha^0$ gives us the root case. Suppose that we know (2) for $k \leq K$; we wish to define $\alpha^{K+1}$ so that we have (2) for $k \leq K + 1$.

If we can show that the following holds:

$$d_0^0 \left( \sum_{i=0}^{K} d_{i+1}^1 (\alpha^{K-i}) \right) = 0 \quad (3)$$

then we will be done since we know that $\ker d_0^0 = \text{im} d_1^0$ so we can find an $\alpha^{K+1}$ satisfying

$$d_1^0(-\alpha^{K+1}) = \sum_{i=0}^{K} d_{i+1}^1 (\alpha^{K-i}).$$
\[ \lambda_1 \lambda_2 \]

\[ \lambda_1 \lambda_2 \]

Figure 10: Type I

Now,

\[
\begin{align*}
    d_0^j \left( \sum_{i=0}^{K} d_1^{i+1}(\alpha^{K-i}) \right) \\
    = \sum_{i=0}^{K} d_0^j d_1^{i+1}(\alpha^{K-i}) = - \sum_{i}^{K} \sum_{j=0}^{d_1^{i+1}} d_0^j d_1^{i-j}(\alpha^{K-i}) \\
    = - \sum_{j=0}^{K} \sum_{i=j}^{K} d_0^j d_1^{i-j}(\alpha^{K-i}) = - \sum_{j=0}^{K} d_0^j \sum_{i=0}^{K-j} d_1^{i}(\alpha^{(K-j)-i}) = 0
\end{align*}
\]

It is easy to check that the map \( \phi \) so defined gives a well-defined isomorphism \( \Phi \).

**Proposition 2.6**

\[
H(\Gamma) = \bigoplus_{\phi \in S(\Gamma)} Q_\phi H(\Gamma)
\]

\[
\dim_{\mathbb{C}} Q_\phi H(\Gamma) = 1
\]

**Proof** For dimensional reasons, it is enough to show that for any \( \phi \in S(\Gamma) \) we can find a non-zero element of \( Q_\phi H(\Gamma) \). Such a non-zero element is described explicitly below.

Given \( \phi \in S(\Gamma) \) form a new graph \( \Gamma' \), with each component decorated with an element \( \lambda \in \Sigma \) by following the local rule in Figures 10 and 11 at all thick edges. We remind the reader that a trivalent graph describing a matrix need not come with an embedding into \( \mathbb{R}^2 \). The circled crossings in \( \Gamma' \) should not be thought of as undercrossings or overcrossings but just as the two strands not intersecting.
The state $\phi$ gives a corresponding state $\phi'$ of $\Gamma'$. Now since $\Gamma'$ is a union of circles, $H(\Gamma') = R(\Gamma')$ naturally so we can pick unambiguously a $Q_{\phi'} \in H(\Gamma)$.

We apply $\chi_0$ and $\eta_0$ repeatedly to $H(\Gamma')$ until we arrive in $H(\Gamma)$. ($\chi_0$ is defined in the remainder of this section and $\eta_0$ is defined in Appendix 6.2). We define $Q_{\phi}$ to be the image of $Q_{\phi'}$ under these map. Gornik shows that this element is non-zero.

To define $HKh_w(D)$ for a link diagram $D$ we have to give two maps of matrix factorizations

$$\chi_0 : C(\Gamma_0) \to C(\Gamma_1)$$

$$\chi_1 : C(\Gamma_1) \to C(\Gamma_0)$$

From these maps we build up the differentials in the chain complex of matrix factorization $CKh_w(D)$. Given a knot diagram $D$, the Khovanov
cube with closed trivalent graphs (giving matrix factorizations of 0 i.e. 2-periodic complexes) at the vertices and maps of matrix factorizations on each edge is given by taking the tensor product of

$$0 \rightarrow C(\Gamma_0)\{1 - n\} \xrightarrow{\chi_0} C(\Gamma_1)\{-n\} \rightarrow 0$$

for every positive crossing (the left of Figure 4) and

$$0 \rightarrow C(\Gamma_1)\{n\} \xrightarrow{\chi_1} C(\Gamma_0)\{n - 1\} \rightarrow 0$$

for every negative crossing (the right of Figure 4). This definition clearly agrees with that of Subsection 1.5.

We give an explicit description ($R$ is the ring $\mathbb{C}[x_1, x_2, x_3, x_4]$). The factorization $C(\Gamma_0)$ is

$$\begin{pmatrix} R \\ R\{2 - 2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R\{1 - n\} \\ R\{1 - n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R \\ R\{2 - 2n\} \end{pmatrix}$$

$$P_0 = \begin{pmatrix} \pi_{13} & x_2 - x_4 \\ \pi_{24} & x_3 - x_1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} x_1 - x_3 & x_2 - x_4 \\ \pi_{24} & -\pi_{13} \end{pmatrix}$$

and $C(\Gamma_1)$ is the factorization

$$\begin{pmatrix} R\{-1\} \\ R\{3 - 2n\} \end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R\{-n\} \\ R\{2 - n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R\{-1\} \\ R\{3 - 2n\} \end{pmatrix}$$

$$Q_0 = \begin{pmatrix} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_4 + x_3 - x - 1 - x_2 \end{pmatrix}$$
\[ Q_1 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1 x_2 - x_3 x_4 \\ u_2 & -u_1 \end{pmatrix} \]

The map \( \chi_0 : C(\Gamma_0) \rightarrow C(\Gamma_1) \) is defined by the pair of maps:

\[
U_0 = \begin{pmatrix} x_3 - x_2 & 0 \\ a & 1 \end{pmatrix}
\]

\[
U_1 = \begin{pmatrix} x_3 & -x_2 \\ -1 & 1 \end{pmatrix}
\]

Here

\[ a = \frac{u_1 + x_3 u_2 - \pi_{24}}{x_1 - x_3} \]

The map \( \chi_1 : C(\Gamma_1) \rightarrow C(\Gamma_0) \) is defined by the pair of maps:

\[
V_0 = \begin{pmatrix} 1 & 0 \\ -a & x_3 - x_2 \end{pmatrix}
\]

\[
V_1 = \begin{pmatrix} 1 & x_2 \\ 1 & x_3 \end{pmatrix}
\]

Note that \( \chi_0 \chi_1 \) and \( \chi_1 \chi_0 \) are homotopic to multiplication by \( x_2 - x_3 \) (we can just take the zero homotopies in both cases).

**Proposition 2.7** The Khovanov homology groups of an \( l \)-component link diagram \( D \) for our potential \( w \), \( Kh_w(D) \), have dimension

\[ \dim_C = n! \]

There is a canonical basis of generators, one generator for each assignment

\[ \psi : \text{components}(D) \rightarrow \Sigma_n \]

Each such \( \psi \) defines in an obvious way a type II admissible state of a resolution \( \Gamma_\psi \). The generator corresponding to \( \psi \) lives in the chain group summand \( H(\Gamma_\psi) \), and comes from the state \( \psi \) as described in (2.7).

It is these generators that will play the same roles in this paper as Lee’s generators [7] played in Rasmussen’s slice genus result [9].
3 Morse moves

A link cobordism can be written as a finite sequence of link diagrams, where successive diagrams differ either by a Reidemeister move or a Morse move. The Morse moves correspond to adding 0-, 1-, and 2-handles.

In this section we will assign to each Morse move a filtered chain map between the complexes of the diagrams that it connects. Then we shall compute what each Morse moves does to the canonical generators of the homology of a link diagram.

3.1 0-handle move

A 0-handle move is the creation of a simple loop as in Figure 14.

The 2-periodic complex associated to a circle is

$$A <1> = (0 \to A \to 0)$$

where $A = (\mathbb{C}[x]/\partial_x w(x))\{1 - n\}$ (the curly brackets indicate a shift in the filtration).

The unit map $i : \mathbb{C} \to A$ has filtered degree $1 - n$. To the 0-handle move above we associate the map of complexes

$$1 \otimes i : CKh_w(D) \to CKh_w(D \sqcup S^1)$$

The canonical generators of the cohomology of $A <1>$ are the chain elements

$$q_{\beta} = \frac{1}{n + 1} \left( \prod_{\alpha \in \Sigma_n \setminus \beta} \frac{1}{\beta - \alpha} \right) \frac{\partial_x w(x)}{x - \beta}$$
and they satisfy

\[ 1 = \sum_{\beta \in \Sigma_n} q_\beta \]

So our map \( 1 \otimes i \) takes a canonical generator \( g \in CKh_w(D) \) to the sum of canonical generators \( \sum_{\beta \in \Sigma_n} g \otimes q_\beta \).

### 3.2 2-handle move

The 2-handle move is the removal of a simple closed loop as in Figure 15.

The trace map \( \epsilon : A \to \mathbb{C} \) is defined by

\[ \epsilon(x^i) = \begin{cases} 1, & i = n - 1 \\ 0, & i < n - 1 \end{cases} \]

To the 2-handle move above we associate the map of complexes

\[ 1 \otimes \epsilon : CKh_w(D \sqcup S^1) \to CKh_w(D) \]

and this has filtered degree \( 1 - n \). Since the coefficient of \( x^{n-1} \) in \( q_\beta \) is just \( \prod_{\alpha \in \Sigma_n \setminus \beta} \frac{1}{\beta - \alpha} \) our map takes a canonical generator \( g \otimes q_\beta \in CKh_w(D \sqcup S^1) \) to

\[ \left( \prod_{\alpha \in \Sigma_n \setminus \beta} \frac{1}{\beta - \alpha} \right) g \]

### 3.3 1-handle move

The 1-handle move is the addition of a saddle as in Figure 16.

Here \( C(\Gamma_0) \) is the factorization
Figure 16: 1-handle addition

\[
\begin{pmatrix}
R & R\{2-2n\} \\
R\{1-n\} & R\{1-n\}
\end{pmatrix}
\xrightarrow{P_0}
\begin{pmatrix}
R & R\{1-n\} \\
R\{1-n\} & R\{2-2n\}
\end{pmatrix}
\xrightarrow{P_1}
\begin{pmatrix}
R & R\{1-n\} \\
R\{1-n\} & R\{2-2n\}
\end{pmatrix}
\]

\[P_0 = \begin{pmatrix}
\pi_{14} & -(x_3 - x_2) \\
\pi_{23} & x_1 - x_4
\end{pmatrix}
\]

\[P_1 = \begin{pmatrix}
x_1 - x_4 & x_3 - x_2 \\
-x_{23} & \pi_{14}
\end{pmatrix}
\]

And \(C(\Gamma_1) < 1\) is the factorization

\[
\begin{pmatrix}
R\{1-n\} \\
R\{1-n\}
\end{pmatrix}
\xrightarrow{Q_0}
\begin{pmatrix}
R & R\{2-2n\} \\
R\{1-n\} & R\{1-n\}
\end{pmatrix}
\xrightarrow{Q_1}
\begin{pmatrix}
R\{1-n\} \\
R\{1-n\}
\end{pmatrix}
\]

\[Q_0 = \begin{pmatrix}
x_1 - x_2 & x_3 - x_4 \\
-x_{34} & \pi_{12}
\end{pmatrix}
\]

\[Q_1 = \begin{pmatrix}
\pi_{12} & -(x_3 - x_4) \\
\pi_{34} & x_1 - x_2
\end{pmatrix}
\]

Ignoring the grading shift, the filtered \(\mathbb{C}[x_1, x_2, x_3, x_4]\)-module of filtered matrix factorization maps \(C(\Gamma_0) \to C(\Gamma_1) < 1\) is isomorphic to the quotient \(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 = x_2 = x_3 = x_4, \partial_2 w(x_1) = 0)\). A generator is given by the pair of matrices:

\[F_0 = \begin{pmatrix}
e_{123} & -1 \\
-e_{134} & -1
\end{pmatrix}
\]

\[F_1 = \begin{pmatrix}
-1 & 1 \\
e_{123} & e_{134}
\end{pmatrix}
\]

where
The map \( F \) is filtered of degree \( n - 1 \).

Suppose we have two knot diagrams \( D_1 \) and \( D_2 \) (of links \( L_1, L_2 \)) such that \( D_2 \) is got from \( D_1 \) by the addition of a 1-handle. We will compute the map induced by \( F \)

\[
\text{HKh}_w(D_1) \to \text{HKh}_w(D_2)
\]
in terms of the generators of \( \text{HKh}_w(D_1) \) and \( \text{HKh}_w(D_2) \) that were constructed in Section 1.

Recall that for each choice of assignment \( \phi : \\{\text{components of } L_1\} \to \Sigma_n \) we got a corresponding resolution \( \Gamma_\phi \) of \( D_1 \) and a state \( \phi : e(\Gamma_\phi) \to \Sigma_n \) giving us an element \( Q_\phi \in H(\Gamma_\phi) \) which represents one of our basis elements of \( C\text{Kh}_w(D_1) \).

To construct \( Q_\phi \) we followed the recipe which converted the resolution \( \Gamma_\phi \) into a disjoint union of circles \( \Gamma'_\phi \) and pushing forward an element (again determined by \( \phi \)) of \( H(\Gamma'_\phi) \) to \( H(\Gamma_\phi) \) via repeated use of \( \eta_0 \). Since \( F \) commutes with \( \eta_0 \), it is enough for us to determine the map induced by \( F \) acting on the homology of disjoint circles.

There are thus two possibilities: either the 1-handle move joins two components of the link as in Figure 17 or it splits a single component into two as in Figure 18.

3.3.1 First case

Performing the endpoint indentifications in Figure 17 we have
\[ C(\Gamma'_0) = \left( \begin{array}{c} R \\ R\{2 - 2n\} \end{array} \right) P'_0 \left( \begin{array}{c} R\{1 - n\} \\ R\{1 - n\} \end{array} \right) P'_1 \left( \begin{array}{c} R \\ R\{2 - 2n\} \end{array} \right) \]

\[ P'_0 = \left( \begin{array}{cc} \partial_x w(x_1) & 0 \\ \partial_x w(x_2) & 0 \end{array} \right) \]

\[ P'_1 = \left( \begin{array}{cc} 0 & 0 \\ -\partial_x w(x_2) & \partial_x w(x_1) \end{array} \right) \]

where \( R = \mathbb{C}[x_1, x_2] \) thought of as an infinite dimensional \( \mathbb{C} \)-module.

\[ C(\Gamma'_1) < 1 > = \left( \begin{array}{c} R\{1 - n\} \\ R\{1 - n\} \end{array} \right) Q'_0 \left( \begin{array}{c} R \\ R\{2 - 2n\} \end{array} \right) Q'_1 \left( \begin{array}{c} R\{1 - n\} \\ R\{1 - n\} \end{array} \right) \]

\[ Q'_0 = \left( \begin{array}{cc} (x_1 - x_2) & -(x_1 - x_2) \\ -\pi_{12} & \pi_{12} \end{array} \right) \]

\[ Q'_1 = \left( \begin{array}{cc} \pi_{12} & (x_1 - x_2) \\ \pi_{12} & (x_1 - x_2) \end{array} \right) \]

And the map \( F' \) is given by

\[ F'_0 = \left( \begin{array}{cc} e' & -1 \\ -f' & -1 \end{array} \right) \]

\[ F'_1 = \left( \begin{array}{cc} -1 & 1 \\ e' & f' \end{array} \right) \]

where

\[ e' = \frac{\pi_{12} - \partial_x w(x_2)}{x_1 - x_2}, \quad f' = \frac{\partial_x w(x_1) - \pi_{12}}{x_1 - x_2} \]

We observe that \( H(\Gamma'_0) = \ker(P'_0)/\text{im}(P'_1) \). The canonical generators indexed by pairs of roots \( \alpha, \beta \in \Sigma_n \) have representatives in the zeroeth chain group of \( C(\Gamma_0) \):

\[ \left( \begin{array}{c} 0 \\ q_1^{1} q_2^{2} \end{array} \right) \in \left( \begin{array}{c} R \\ R\{2 - 2n\} \end{array} \right) \]

Also, \( H(\Gamma'_1 < 1 >) = \ker(Q'_0)/\text{im}(Q'_1) \), and the canonical generators indexed by roots \( \alpha \in \Sigma_n \) have representatives in the 0th chain group of \( C(\Gamma_1) \):

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\[
\begin{pmatrix}
q^1_1 \\
q^2_2
\end{pmatrix} \in \begin{pmatrix}
R\{1 - n\} \\
R\{1 - n\}
\end{pmatrix}
\]

where

\[
q^i_\alpha = \frac{1}{n+1} \left( \prod_{\beta \in \Sigma \setminus \alpha} \frac{1}{\alpha - \beta} \right) \frac{\partial_x w(x_i)}{x_i - \alpha} \text{ for } i = 1, 2
\]

Now \( H(\Gamma'_1 < 1 >) \) is isomorphic to \( \mathbb{C}[x_1, x_2]/(x_1 = x_2, \partial_x w(x_1) = 0) \), and in this module

\[
q^1_\alpha q^2_\beta = \begin{cases} 
0, & \alpha \neq \beta \\
1, & \alpha = \beta
\end{cases}
\]

So it is easy to see what \( \Psi' \) does to the generators of \( H(\Gamma'_0) \):

\[
\begin{pmatrix}
e' & -1 \\
-f' & -1
\end{pmatrix} \begin{pmatrix} 
0 \\
q^2_\beta
\end{pmatrix} = \begin{pmatrix} 
-q^2_\beta \\
-q^2_\beta
\end{pmatrix}
\]

\[
= \begin{pmatrix} 
-q^1_\beta \\
-q^2_\alpha
\end{pmatrix} \text{ or } \begin{pmatrix} 
0 \\
0
\end{pmatrix}
\]

depending as \( \alpha = \beta \) or \( \alpha \neq \beta \).

### 3.3.2 Second case

Under the identifications in Figure 18 we have

\[
C(\Gamma''_0) = \begin{pmatrix} 
R \\
R\{2 - 2n\}
\end{pmatrix} P''_0 \begin{pmatrix} 
R\{1 - n\} \\
R\{1 - n\}
\end{pmatrix} P''_1 \begin{pmatrix} 
R \\
R\{2 - 2n\}
\end{pmatrix}
\]

\[
P''_0 = \begin{pmatrix} 
\pi_{13} & -(x_3 - x_1) \\
\pi_{13} & x_1 - x_3
\end{pmatrix}
\]

\[
P''_1 = \begin{pmatrix} 
x_1 - x_3 & x_3 - x_1 \\
-\pi_{13} & \pi_{13}
\end{pmatrix}
\]

where \( R = \mathbb{C}[x_1, x_3] \)

\[
C(\Gamma''_1 < 1 >) = \begin{pmatrix} 
R\{1 - n\} \\
R\{1 - n\}
\end{pmatrix} Q''_0 \begin{pmatrix} 
R \\
R\{2 - 2n\}
\end{pmatrix} Q''_1 \begin{pmatrix} 
R\{1 - n\} \\
R\{1 - n\}
\end{pmatrix}
\]

27
The map $F''$ is given by

$$F''_0 = \begin{pmatrix} e'' & -1 \\ -f'' & -1 \end{pmatrix}$$

$$F''_1 = \begin{pmatrix} -1 & 1 \\ e'' & f'' \end{pmatrix}$$

where

$$e'' = \frac{\partial_x w(x_1) - \pi_{13}}{x_1 - x_3}, \quad f'' = \frac{\pi_{13} - \partial_x w(x_3)}{x_1 - x_3}$$

We observe that $H(\Gamma''_0) = ker(P''_1)/im(P''_0)$. The canonical generators indexed by roots $\alpha \in \Sigma_n$ have representatives in the 0th chain group of $C(\Gamma''_0)$

$$\begin{pmatrix} q^-_{\alpha} \\ q^1_{\alpha} \end{pmatrix} \in \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix}$$
Also \( H(\Gamma''_1 < 1 >) = \ker(Q''_1)/\text{im}(Q''_0) \), and the canonical generators indexed by roots \( \alpha, \beta \in \Sigma_n \) have representatives in the 0th chain group of \( C(\Gamma''_1 < 1 >) \)

\[
\begin{pmatrix}
0 \\
q^1_{\alpha} q^3_{\beta}
\end{pmatrix} \in \begin{pmatrix}
R \\
R\{2 - 2n\}
\end{pmatrix}
\]

where

\[
q^i_{\alpha} = \frac{1}{n+1} \left( \prod_{\beta \in \Sigma_n \setminus \alpha} \frac{1}{\alpha - \beta} \right) \frac{\partial_{x_i} w(x)}{x_i - \alpha} \text{ for } i = 1, 3
\]

Now \( H(\Gamma''_1 < 1 >) \) is isomorphic to \( \mathbb{C}[x_1, x_3]/(\partial_{x_1} w(x_1), \partial_{x_3} w(x_3)) \) and in this module

\[
\left( \frac{\partial_{x_1} w(x_1) - \partial_{x_3} w(x_3)}{x_1 - x_3} \right) \left( \frac{\partial_{x_1} w(x_1)}{x_1 - \alpha} \right) = \left( \frac{\partial_{x_1} w(x_1)}{x_1 - \alpha} \right) \left( \frac{\partial_{x_3} w(x_3)}{x_3 - \alpha} \right)
\]

since

\[
\frac{\partial_{x_1} w(x_1) - \partial_{x_3} w(x_3)}{x_1 - x_3} - \frac{\partial_{x_3} w(x_3)}{x_3 - \alpha}
\]

has a factor of \( (x_1 - \alpha) \).

Now it is easy to see what \( F'' \) does to the generators:

\[
\begin{pmatrix}
-1 & 1 \\
e'' & f''
\end{pmatrix}
\begin{pmatrix}
q^1_{\alpha} \\
q^3_{\alpha}
\end{pmatrix} = \begin{pmatrix}
0 \\
(e'' + f'')q^1_{\alpha}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\partial_{x_1} w(x_1) - \partial_{x_3} w(x_3) q^1_{\alpha} \\
0
\end{pmatrix}
\]

\[
= \left( \prod_{\beta \in \Sigma_n \setminus \alpha} (\alpha - \beta) \right) \begin{pmatrix}
0 \\
q^3_{\alpha} q^3_{\alpha}
\end{pmatrix}
\]

4 Reidemeister moves I and II

In this section we prove invariance of \( HKh_w(D) \) under changing the (closed) link diagram \( D \) by an oriented Reidemeister I or II move.

The diagrams on either side of an oriented Reidemeister move each give a chain complex of matrix factorizations. We wish to define two chain maps
(one in each direction) between these two chain complexes. Each chain map shall be quantum-graded of degree 0. These chain maps will then induce degree 0 chain maps on the chain complexes associated to closed link diagrams differing locally by the oriented Reidemeister move. We have a description of chain representatives for the generators of the homology $HKh_w$ of each link diagram and we show that the maps preserve the generators up to multiplication by a non-zero number. Since the chain maps were of degree 0 this tells us that the graded vector space associated to the quantum filtration of $HKh_w$ is invariant under changing the input diagram by the Reidemeister move.

Every matrix in this section describes a map between free modules over polynomial rings. Each of these modules, as per the definition of $CKh_w$, is possibly subject to a shift in quantum filtration. In what is below, we suppress mention of the shifts in quantum filtration. For those readers wishing to check the validity of the following computations, this omission should not encumber them with any great difficulties, while explicitly to describe the shifts would thicken somewhat our exposition. In this section we write $\partial_i w$ for $\partial x_i(x_i)$.

4.1 Reidemeister move I

4.1.1 Reidemeister move I.1

The first case of the Reidemeister I move is shown in Figure 19. Either side of the move corresponds to a chain complex of matrix factorizations. Our first task is to define a chain maps between the left chain complex to and the right complex of degree 0. Then we wish to see that these chain maps induce maps on the homologies of two closed link diagrams, differing locally by the move, which preserve the generators of the homology.

The set-up is shown in Figure 20. The chain complex from the left of Figure 19 is above, and the one from the right is below. To give the two chain maps we need to define $F$ and $G$. First we shall write down the three factorizations $M$, $N$, and $P$.

The factorizations $M$, $N$, and $P$ are

$$M : R\{1 - n\} \xrightarrow{\partial x_2} R \xrightarrow{\pi x_3} R\{1 - n\}$$

$$N : \left( \begin{array}{c} S\{2 - 2n\} \\ S\{2 - 2n\} \end{array} \right) \xrightarrow{A^0} \left( \begin{array}{c} S\{1 - n\} \\ S\{3 - 3n\} \end{array} \right) \xrightarrow{A^1} \left( \begin{array}{c} S\{2 - 2n\} \\ S\{2 - 2n\} \end{array} \right)$$
Figure 19: Reidemeister move I.1

Figure 20: Reidemeister move I.1 chain maps
A^0 = \begin{pmatrix} 0 & x_2 - x_3 \\ -\pi_{23} & \partial w_1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} \partial w_1 & -(x_2 - x_3) \\ \pi_{23} & 0 \end{pmatrix}

P : \begin{pmatrix} S\{2 - 2n\} \\ S\{2 - 2n\} \end{pmatrix} \mapsto \begin{pmatrix} S\{-2n\} \\ S\{-2n\} \end{pmatrix} \mapsto \begin{pmatrix} S\{2 - 2n\} \\ S\{2 - 2n\} \end{pmatrix}

B^0 = \begin{pmatrix} x_2 - x_3 & x_1(x_2 - x_3) \\ -u_2 & u_1 \end{pmatrix}, \quad B^1 = \begin{pmatrix} u_1 & -x_1(x_2 - x_3) \\ u_2 & x_2 - x_3 \end{pmatrix}

where \( R = \mathbb{C}[x_2, x_3] \) and \( S = \mathbb{C}[x_1, x_2, x_3] \) considered as a module over \( R \).

Also

\[
\pi_{23} = \frac{w(x_2) - w(x_3)}{x_2 - x_3}, \quad \partial w_1 = w'(x_1)
\]

\[
u_1 = \frac{p(x_1 + x_2, x_1x_2) - p(x_1 + x_3, x_1x_2)}{x_2 - x_3}
\]

\[
u_2 = \frac{p(x_1 + x_3, x_1x_2) - p(x_1 + x_3, x_1x_3)}{x_1(x_2 - x_3)}
\]

As usual, \( p \) denotes the unique two-variable polynomial such that \( p(x + y, xy) = w(x) + w(y) \).

The eagle-eyed reader may spot that there has been a shift in the usual \( \mathbb{Z}/2 \)-grading of the matrix factorizations \( N \) and \( P \). This is due to the introduction of an extra component in the oriented resolutions of the diagram defining these factorizations. We shall not mention this kind of shift in the remainder of this paper where it may occur, but content ourselves most often with giving the factorizations explicitly on which we are focussed.

The map \( \chi_0 \) is given by the pair of matrices

\[
\begin{pmatrix} x_1 & -x_2 \\ -1 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} x_1 - x_2 & 0 \\ -a & 1 \end{pmatrix}
\]

Finally, we are in a position to define the first of our chain maps: \( F \). This comes as the pair of matrices

\[
\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}
\]
where \( \alpha = (p(x_1 + x_2, x_1x_2) - p(x_1 + x_2, x_1x_3))/(x_1(x_2 - x_3)) \).

It is straightforward to check that this a map of matrix factorizations, and it is clearly filtered of degree 0. However, it is less straightforward to check that this gives a map of chain complexes; to do so we need to see that \( \chi_0 F \) is homotopic to the zero map of chain complexes. The map \( \chi_0 F \) is given by the pair of matrices:

\[
\begin{pmatrix}
x_1 \alpha \\
-\alpha
\end{pmatrix}, \begin{pmatrix}
0 \\
-\alpha
\end{pmatrix}
\]

We define a homotopy \( H^0 : M_0 \to P_1, H^1 : M_1 \to P_0 \):

\[
H^0 = \begin{pmatrix}
-1 \\
x_2 - x_3
\end{pmatrix}, H^1 = \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

(a glance at the definitions of \( \alpha \) and \( u_2 \) assures us that the second entry of \( H^0 \) is a polynomial).

To see that \( H \) is a homotopy between \( \chi_0 F \) and 0 we need to compute that

\[
\begin{pmatrix}
x_1 \alpha \\
-\alpha
\end{pmatrix} = H^1(\pi_{23}) + B^1 H^0, \begin{pmatrix}
0 \\
-\alpha
\end{pmatrix} = H^0(x_2 - x_3) + B^0 H^1
\]

which is left as an exercise.

We have now shown that \( F \) is a degree 0 chain map. To complete the proof of one direction of the Reidemeister I.1 move, it remains to check that \( F \) induces a map that preserves the generators of the homology of a closed link diagram. Because of the way that the generators are defined (as the image of various elements of the homology of some disjoint circles under \( \eta_0 \) maps), it is enough to check that the generators are preserved by the map \( F \) when \( x_2 \) is identified with \( x_3 \) as in Figure 21.

We write \( p = p(s, t) \) where \( s = x_1 + x_2 \) and \( t = x_1x_2 \). Looking above at the definition of the polynomial \( \alpha \) that appears in the entries of the matrix \( F \), we see that result of substituting \( x_2 \) for \( x_3 \) in \( \alpha \) is

\[
\alpha_{3 \to 2} = \frac{\partial}{\partial t} p(s, t) = \frac{\partial}{\partial t} (w_1 + w_2)
\]

\[
= \frac{\partial x_1}{\partial t} \frac{\partial}{\partial t} w_1 + \frac{\partial x_2}{\partial t} \frac{\partial}{\partial t} w_2
\]

\[
= (x_2 - x_1)^{-1} \frac{\partial}{\partial t} w_1 - (x_2 - x_1)^{-1} \frac{\partial}{\partial t} w_2
\]

\[
= \frac{\partial w_1 - \partial w_2}{x_1 - x_2}
\]
Figure 21: $\tilde{F}$ induced by $F$ 

Taking homology of the graphs in Figure 21 $\tilde{F}$ becomes multiplication by $\alpha_{3 \to 2}$

$$\tilde{F} = \alpha_{3 \to 2} : \mathbb{C}[x_2]/(\partial w_2) \to \mathbb{C}[x_1, x_2]/(\partial w_1, \partial w_2)$$

A general basis element, as constructed in Section 2, of the homology of the lower circle in Figure 21 is written (up to non-zero multiple) as $\partial w_2/(x_2 - \psi)$ where $\psi$ is a root of $\partial w$. Under $\tilde{F}$ this gets mapped to

$$\alpha_{3 \to 2} \frac{\partial w_2}{x_2 - \psi} = -\left( \frac{\partial w_1 - \partial w_2}{x_1 - x_2} \right) \left( \frac{\partial w_2}{x_2 - \psi} \right)$$

$$= -\left( \frac{\partial w_1}{x_1 - \psi} \right) \left( \frac{\partial w_2}{x_2 - \psi} \right)$$

since

$$\frac{\partial w_1 - \partial w_2}{x_1 - x_2} - \frac{\partial w_1}{x_1 - \psi}$$

has factor of $(x_2 - \psi)$ and $\partial w_2 = 0$ in the image of $\tilde{F}$.

Write $D, D'$ for closed link diagrams which are the same except as they differ locally as the right and left parts of Figure 19 respectively. We have shown that if $g$ is a standard basis element of $HKh_w(D)$ then it gets taken by $\tilde{F}$ to (a non-zero multiple of) the basis element of $HKh_w(D')$ which is obtained by the same decoration of components of $D'$ with elements of $\Sigma_n$ as is $g$.

Next we need to define the map $G$ to give us a degree 0 chain map from the upper chain complex of matrix factorizations to the lower. In fact, we can see that $G$ just has to be a map of matrix factorizations of degree-0 and
it will define a chain map between the two chain complexes since the only commuting square that we need to worry about will be automatically 0 in both directions.

As a $\mathbb{C}[x_2, x_3]$-module, $\mathbb{C}[x_1, x_2, x_3]$ is isomorphic to the following direct sum

$$\mathbb{C}[x_1, x_2, x_3] = \bigoplus_{i=0}^{n-2} x_1^i \mathbb{C}[x_2, x_3] \oplus \bigoplus_{i \geq 0} x_1^{i-1}(\partial w_1 - c) \mathbb{C}[x_2, x_3]$$

where we have written $c$ for the constant term in the polynomial $\partial w$. Let $\beta$ be the $\mathbb{C}[x_2, x_3]$-module map

$$\beta : \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[x_2, x_3]$$

$1, x_1, x_1^2, \ldots, x_1^{n-2} \mapsto 0, x_1^{i-1}(\partial w_1 - c) \mapsto (x_2 - x_3)^i \forall i \geq 0$

And let $1_{1\to 2, 3}$ be the map

$$1_{1\to 2, 3} : \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[x_2, x_3]$$

$$x_1^i \mapsto (x_2 - x_3)^i \forall i \geq 0$$

We define the map of matrix factorizations $G$ by the pair of matrices

$$\begin{pmatrix} \beta & 0 \\ 1_{1\to 2, 3} & -\beta \end{pmatrix}$$

(it is straightforward to check that this defines a map of matrix factorizations filtered of degree 0).

The map $G$ automatically defines a map of chain complexes. Next, as we did for $\tilde{F}$, we need to compute what the induced map $\bar{G}$ does on the homologies of the closed-up graphs in Figure 21. Looking at the definition of $\beta$ we see that $\bar{G}$ will take

$$\begin{pmatrix} \partial w_1 \\ x_1 - \psi \end{pmatrix} \begin{pmatrix} \partial w_2 \\ x_2 - \psi \end{pmatrix} \in \mathbb{C}[x_1, x_2]/(\partial w_1, \partial w_2)$$

to

$$\frac{\partial w_2}{x_2 - \psi} \in \mathbb{C}[x_2]/(\partial w_2)$$

Again, write $D$, $D'$ for closed link diagrams which are the same except as they differ locally as the right and left parts of Figure 19 respectively.
We have shown that if $g$ is a standard basis element of $HKh_w(D')$ then it gets taken by $\tilde{G}$ to (a non-zero multiple of) the basis element of $HKh_w(D)$ which is obtained by the same decoration of components of $D$ with elements of $\Sigma_n$ as is $g$.

4.1.2 Reidemeister move I.2

Figure 22 decomposes the Reidemeister I.2 move into other elementary cobordisms. From other parts of this paper it follows that this gives degree 0 chain maps (by composition of chain maps corresponding to the elementary cobordisms used) between $CKh_w(D)$ and $CKh_w(D')$ where $D$ and $D'$ are closed link diagrams differing locally by the Reidemeister I.2 move. These chain maps will preserve generators of $HKh_w(D)$ and $HKh_w(D)$ up to non-zero multiples just as those corresponding to the Reidemeister I.1 move did.

4.2 Reidemeister move II

Since our link comes with an orientation, there are two cases to compute, the first is in Figure 23.
4.2.1 Reidemeister move II.1

We start with a note about ordering of bases. The trivalent graphs in Figure 12 and Figure 13 are associated to factorizations defined explicitly in Section 2. We write either of these factorizations as

\[
\begin{pmatrix}
  M_{00} \\
  M_{11}
\end{pmatrix} \rightarrow
\begin{pmatrix}
  M_{10} \\
  M_{01}
\end{pmatrix} \rightarrow
\begin{pmatrix}
  M_{00} \\
  M_{11}
\end{pmatrix}
\]

where each \( M_{ij} \) is a free module of rank 1 over the ring \( \mathbb{C}[x_1, x_2, x_3, x_4] \). The basis that we use for the tensor product of two of these factorizations, as appears in the lower half of Figure 24 for example, is

\[
\begin{pmatrix}
  M_{0000} \\
  M_{0011} \\
  M_{0101} \\
  M_{0110} \\
  M_{1001} \\
  M_{1010} \\
  M_{1100} \\
  M_{1111}
\end{pmatrix} \rightarrow
\begin{pmatrix}
  M_{0001} \\
  M_{0010} \\
  M_{0100} \\
  M_{0110} \\
  M_{1000} \\
  M_{1011} \\
  M_{1101} \\
  M_{1111}
\end{pmatrix} \rightarrow
\begin{pmatrix}
  M_{0000} \\
  M_{0011} \\
  M_{0101} \\
  M_{0110} \\
  M_{1001} \\
  M_{1010} \\
  M_{1100} \\
  M_{1111}
\end{pmatrix}
\]

where \( M_{ijkl} = M_{ij} \otimes M_{kl} \) and \( M_{ij} \) is a summand of the higher matrix factorization, \( M_{kl} \) a summand of the lower.

Each \( M_{ijkl} \) is a \( \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6] \) considered as a module over the ring \( \mathbb{C}[x_1, x_2, x_3, x_4] \). In what follows we have many \( 8 \times 8 \) matrices. Each matrix entry should be understood as a \( \mathbb{C}[x_1, x_2, x_3, x_4] \)-module map \( \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6] \rightarrow \)
$\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$. We will often abuse notation - for example $x_5$ appearing as a matrix entry will mean the “multiply by $x_5$” map, even though $x_5$ technically lies in the module and not the ring.

The complex of matrix factorizations coming from the left-hand side of Reidemeister move II.1 is in the lower half of Figure 24. In this diagram, as in the others in this section, where a point would normally be labelled with a variable $x_i$ we have suppressed the $x$ and just labelled with $i$.

Figure 24 shows the two complexes of matrix factorizations coming from either side of Figure 23. There are also some double-headed arrows. Each double-headed arrow is two maps of matrix factorizations (one in either direction); these maps of matrix factorizations are the component maps of two chain maps of matrix factorizations (one going each way between the higher and lower complexes). It is our first task in this section to define these maps of matrix factorizations so that we indeed do have two chain maps.

We also aim to define these maps so that the induced chain maps between two complexes coming from closed link diagrams differing by a single Reidemeister II.1 move preserve the generators of the homology $\text{HKh}_w$.

Two of the double-headed arrows are automatically the zero map. The double-headed arrow going between the two factorizations with no thick
edges shall just be homotopy equivalences of matrix factorizations as described in the Appendix. For the double-headed arrow which goes between the top factorization and the doubly thick-edged factorization, we shall write down two explicit maps of matrix factorizations.

First we shall consider the chain map from the lower chain complex of matrix factorizations to the higher. In Figure 25 we have isolated the relevant part of Figure 24 and factored the map from the doubly thick-edged factorization into the composition of two maps. The arrows marked $id$ are the identity in the homotopy category of matrix factorizations. We will have a chain map upwards if we can define the map $\phi$ so that Figure 25 is anti-commutative.

Using the basis conventions discussed earlier we define $\phi$ to be

$$
\Pi = \phi_0 \circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\Pi \circ = \phi_1 \circ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

where $\Pi$ is the map which on each module summand $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ looks like
The map $\phi$ is easily checked to be a map of matrix factorizations. Next we want to see that we have defined a chain map by seeing that Figure 25 commutes.

Let us compute the map in Figure 25 which runs from the leftmost singly thick-edged factorization to the singly thick-edged factorization which is the target of $\phi$ (this is a composition of three maps of matrix factorizations). We will denote this map by the pair of matrices

$$\begin{pmatrix} b_{01} & b_{02} \\ b_{03} & b_{04} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{13} & b_{14} \end{pmatrix}$$

We shall write $f : M^0 \to M^1$ and $g : M^1 \to M^0$ with suitable subscripts to stand for the matrix factorizations in Figure 26.

Let

$$f_{15} = (f_1 - f_5)/(x_1 - x_5), g_{15} = (g_1 - g_5)/(x_1 - x_5)$$

$$f_{26} = (f_2 - f_6)/(x_2 - x_6), g_{26} = (g_2 - g_6)/(x_2 - x_6)$$

Then we have
where \( a_1 \) is the usual polynomial appearing in the \( \chi_0 \) map in Figure 27. In our previous notation (with now \( x_5 + x_6 \) and \( x_5 x_6 \) replacing \( x_3 + x_4 \) and \( x_3 x_4 \) respectively in the definitions of \( u_1 \) and \( u_2 \)) we have
Figure 27: $\chi_0$ map

$$a_1 = \frac{u_1 + x_5u_2 - \pi_{26}}{x_1 - x_5} = -\frac{u_1 + x_2u_2 - \pi_{15}}{x_2 - x_6}$$

Multiplying out the matrices:

$$\begin{pmatrix} b_{01} & b_{02} \\ b_{03} & b_{04} \end{pmatrix} = \Pi \circ \begin{pmatrix} x_5 - x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \text{id}_{R^2} \\ f_{26} \\ -g_{15}f_{26} \\ f_{15} \end{pmatrix}$$

$$= \Pi \circ \left( \begin{pmatrix} x_5 - x_2 & 0 \\ -a_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_{15}f_{26} \right)$$

Let us write

$$f_2 = \begin{pmatrix} v_{12} & x_3x_4 - x_1x_2 \\ v_{22} & x_1 + x_2 - x_3 - x_4 \end{pmatrix}, f_6 = \begin{pmatrix} v_{16} & x_3x_4 - x_1x_2 \\ v_{26} & x_1 + x_2 - x_3 - x_4 \end{pmatrix}$$

so

$$f_{26} = \begin{pmatrix} \frac{v_{12} - v_{16}}{x_2 - x_6} & -x_1 \\ \frac{v_{22} - v_{26}}{x_2 - x_6} & 1 \end{pmatrix}$$

Now

$$g_{15} = \begin{pmatrix} 1 & x_6 \\ . & . \end{pmatrix}$$

so that we see immediately

$$b_{01} = 1, b_{02} = 0, b_{04} = 1$$
It remains to compute $b^{03}$:

$$b^{03} = \Pi \left( \frac{1}{x_2 - x_6} \left( (u_1 + x_2 u_2 - \pi_{15}) - (v_{12} - v_{16}) - (x_6 (v_{22} - v_{26})) \right) \right)$$

$$= \Pi \left( \frac{1}{x_2 - x_6} (u_1 + x_2 u_2 - \pi_{15} - v_{12} + v_{16} - x_2 v_{22} + x_6 v_{26}) + v_{22} \right)$$

$$= \Pi \left( \frac{1}{x_2 - x_6} (\pi_{15} + v_{16} + x_6 v_{26}) + v_{22} \right)$$

(the last equality follows from the action of $\Pi$)

Now $v_{22}$ certainly gets killed by $\Pi$ and

$$\frac{-\pi_{15} + v_{16} + x_6 v_{26}}{x_2 - x_6}$$

is a polynomial with no term involving $x_2$ in the numerator and hence also gets killed by $\Pi$. Thus we have shown that

$$b^{03} = 0$$

Now we work on $b^{11}, b^{12}, b^{13}, b^{14}$

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{13} & b^{14} \end{pmatrix} = \Pi \circ \begin{pmatrix} x_5 & 0 & 0 & 0 & -x_2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{15} \\ -f_{15} g_{26} \\ g_{26} \\ id_{R^2} \end{pmatrix}$$

$$= \Pi \circ \begin{pmatrix} x_5 & 0 \\ -1 & 0 \end{pmatrix} g_{15} + \begin{pmatrix} -x_2 & 0 \\ 1 & 0 \end{pmatrix} g_{26}$$

We compute that

$$g_{15} = \begin{pmatrix} 1 & x_6 \\ . & . \end{pmatrix}, g_{26} = \begin{pmatrix} 1 & x_1 \\ . & . \end{pmatrix}$$

Hence

$$\begin{pmatrix} b^{11} & b^{12} \\ b^{13} & b^{14} \end{pmatrix} = \Pi \circ \begin{pmatrix} x_5 - x_2 & x_5 x_6 - x_1 x_2 \\ -1 + 1 & -x_6 + x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now it is clear that Figure 25 commutes (see theorem 6.1 in the appendix if you need more justification). Hence we have defined a chain map.
It remains to check that the chain map we have defined preserves the generators of the homology. Recall that the first step to define a generator involves decorating the link components with roots of $\partial w$. We then resolve the diagram by adding a thick edge at each crossing where the roots disagree and adding the oriented resolution at each crossing where the roots agree.

If the decoration of the two strands of the Reidemeister II move is by the same root then we have nothing to show since the relevant component of the chain map is homotopic to the identity.

In the case that the decoration is by two different roots then we use the result in 6.2 and refer to Figure 28. Suppose that the strand with left endpoints is decorated by the root $\alpha_1$ and the strand with right endpoints is decorated by the root $\alpha_2$.

Recall that to produce the generator we have to start with a resolution which has the homology of a set of disjoint circles and then push forward a generator (specified by our choice of root decoration) of that homology by $\eta_0$ maps. The appendix 6.2 tells us that this pushing forward by $\eta_0$ maps gives us the same generator, up to sign, as pushing forward by two $\chi_0$ maps at the site of the Reidemeister II move and by $\eta_0$ maps elsewhere.

In other words, we can create the generator by pushing forward the relevant generator of the homology of the diagram looking locally like the far left resolution in Figure 28 by the two $\chi_0$ maps indicated. Since we have shown that this is a commutative diagram, the image of the generator (in the homology of the diagram looking locally like the top of Figure 28) under the chain map that we gave is the same as first applying $\chi_0$ and then $\chi_1$ to the corresponding generator of the homology of the diagram looking locally like the far left resolution. Earlier, we have shown that this is the same up
to homotopy as the map induced by multiplication by $x_6 - x_3$. This is the same as multiplying our generator by the non-zero number $\alpha_2 - \alpha_1$. Hence our generator is preserved up to non-zero multiple.

Now we shall consider the chain map going from the higher chain complex to the lower. In Figure 29 we have isolated the relevant part of the diagram and factored the map to the doubly thick-edged factorization into the composition of two maps. The arrows marked id are the identity in the homotopy category of matrix factorizations. We shall have a chain map if we can give the map $\phi$ so that the diagram anti-commutes.

Let $F$ and $G$ be the $2 \times 2$ matrices of the factorization in Figure 29. The entries of $F$ and $G$ are polynomials in $x_1 + x_2, x_3 + x_4, x_1x_2, x_3x_4$, and we shall write $F_{5+6}$ and $G_{5+6}$ for the matrices obtained from $F$ and $G$ by replacing $x_3 + x_4$ with $x_5 + x_6$ and $F_{5+6}^{56}$ and $G_{5+6}^{56}$ for the matrices obtained from $F$ and $G$ by replacing $x_3 + x_4$ with $x_5 + x_6$ and $x_3 x_4$ with $x_5 x_6$ respectively.

Writing $P(x + y, xy) = w(x) + w(y)$ we define

$$u_1(x, y, z) = \frac{P(x, z) - P(y, z)}{x - y}$$

Using the basis-ordering conventions given at the start of this section we define $\phi$ to be
\[ \phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id}_{R^2} \\ X_1 \\ X_2 \\ X_3 + X_4 \end{pmatrix} \]

\[ \phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \text{Id}_{R^2} \\ Y_1 \\ Y_2 \\ Y_3 + Y_4 \end{pmatrix} \]

where

\[ X_1 = \frac{F - F_{5+6}}{x_3 + x_4 - x_5 - x_6}, \quad X_2 = \frac{F_{5+6} - F_{5+6}^{56}}{x_3 x_4 - x_5 x_6}, \]

\[ X_3 = \frac{u_1(x_5 + x_6, x_3 + x_4, x_3 x_4) - u_1(x_5 + x_6, x_3 + x_4, x_5 x_6)}{x_3 x_4 - x_5 x_6}, \]

\[ X_4 = \frac{G_{5+6} - G_{5+6}^{56}}{x_3 x_4 - x_5 x_6 - x_3 + x_4 - x_5 - x_6}, \]

\[ Y_1 = \frac{G - G_{5+6}}{x_3 + x_4 - x_5 - x_6}, \quad Y_2 = \frac{G_{5+6} - G_{5+6}^{26}}{x_3 x_4 - x_5 x_6}, \]

\[ Y_3 = \frac{-u_1(x_5 + x_6, x_3 + x_4, x_3 x_1) - u_1(x_5 + x_6, x_3 + x_4, x_5 x_6)}{x_3 x_4 - x_5 x_6}, \]

\[ Y_4 = \frac{F_{5+6} - F_{5+6}^{56}}{x_3 x_4 - x_5 x_6 - x_3 + x_4 - x_5 - x_6}. \]

It is a simple, if somewhat laborious, matter to check that this is a well-defined map of matrix factorizations. We now need to check that we have a chain map by seeing that Figure 29 commutes.

If we let II be the map which on each module summand looks like
\[ \Pi : \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6] \to \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]/(x_5 - x_3, x_6 - x_4) = \mathbb{C}[x_1, x_2, x_3, x_4] \]

then the components of the map from the leftmost singly thick-edged factorization to the rightmost singly thick-edged factorization are

\[ \Pi \circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & a_3 \end{pmatrix} \]

and

\[ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ X_1 & X_2 \\ X_3 + X_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \]
Figure 30: Preservation of generators under Reidemeister 2.1

\[ \Pi \circ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} x_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 & 0 & a_3 \end{pmatrix} \]

Hence we see, just as before, that we have indeed defined a chain map.

We need now to check that this chain map, so defined, preserves the generators of the homology. Again, the only case that is not immediate is that when the roots of \( \partial w \) decorating the two strands of the Reidemeister move are distinct. Let us suppose that the strand with ends on the left is decorated by the root \( \alpha_1 \) and the strand with ends on the right is decorated by the root \( \alpha_2 \).
Suppose that $g$ is the corresponding generator in $H(\Gamma)$ where $\Gamma$ looks locally like the lowest trivalent graph in Figure 30 and $H(\Gamma)$ is a chain group summand of $CKh_w$. The generator $g$ was obtained by pushing forward $g' \in H(\Gamma')$ by $\eta_0$ maps where $\Gamma$ is a union of disjoint circles. In [2], Gornik shows that applying $\eta_1$ to each thick edge of $\Gamma$ results in a non-zero multiple of $g'$.

In Section 6.2 it is shown that applying two $\eta_1$ maps at the site of the Reidemeister II.1 move is the same up to sign as applying two $\chi_1$ maps. Since Figure 30 commutes we see that applying two $\eta_1$ maps to $g$ is the same as applying $\chi_1 \chi_0$ to $g'$. And $\chi_1 \chi_0$ is homotopic to the map induced by multiplication by $x_2 - x_5$. This acts the same on $g'$ as multiplication by $\alpha_2 - \alpha_1$ which is a non-zero number. Hence our generator is preserved up to non-zero multiple.

4.2.2 Reidemeister move II.2

Figure 31 decomposes the Reidemeister II.2 move into other elementary cobordisms. It follows that this gives degree 0 chain maps (by composition of chain maps corresponding to the elementary cobordisms used) between $CKh_w(D)$ and $CKh_w(D')$ where $D$ and $D'$ are closed link diagrams differing locally by the Reidemeister II.2 move. These chain maps will preserve generators of $Kh_w(D)$ and $Kh_w(D)$ up to non-zero multiples.

5 Computation for a slice surface of a knot

We start with a knot diagram and a presentation of a slice surface for the knot which begins with the given diagram. We would like to see that the map induced by the presentation between the homologies of the knot and of the unknot is an isomorphism of a filtered degree depending on the genus on the surface. This will then give us our main theorem.

This is not precisely what we do, however. First we perturb our presentation so that we avoid having to prove similar theorems for the Reidemeister III move as we have already done for the I and II moves and for the handle attachments. Essentially we show that it is possible to delay any Reidemeister III moves until the end of the presentation, which makes our degree computations very easy.
5.1 Avoiding Reidemeister III

In this section we detail a topological argument that we use to avoid having to do similar calculations to those in the previous sections. The idea in a nutshell is that in describing a smooth surface $\Sigma \hookrightarrow \mathbb{R}^4$ as a composition of elementary cobordisms we can postpone the Reidemeister III moves until after all the 0- and 1-handles have been added. This is not the whole story; we shall also be postponing some of the 1-handle moves, and some of Reidemeister I and II moves as well.

Suppose that $S$ is a composition of elementary cobordisms representing a connected surface $\Sigma$

$$\Sigma \hookrightarrow \mathbb{R}^3 \times [0, 1]$$

$$\partial \Sigma = K \cup U, K \hookrightarrow \mathbb{R}^3 \times \{0\}, U \hookrightarrow \mathbb{R}^3 \times \{1\}$$

where $S$ begins with a diagram $D$ of the knot $K$ and ends with the 0-crossing diagram of the unknot $U$.

We can assume that the 0-handle additions of $S$ take place before any other elementary cobordism (since a 0-handle addition at any point of the cobordism can be replaced by a 0-handle addition at a distance from the rest.
of the diagram at the start of the cobordism and then later sliding it over by Reidemeister II moves to the point where it is required). Similarly, we can assume that the 2-handle additions take place after every other elementary cobordism.

Suppose that $S$ has $h$ 0-handle additions at the beginning, so that immediately after the 0-handle additions we have a diagram of a $(h+1)$-component link. Since $\Sigma$ is connected we can make a list of $h$ “fusing” 1-handle additions in $S$ that connect all the components together. We shall be wanting to postpone all the Reidemeister III moves and all the other 1-handle additions until after these $h$ 1-handles have been added, giving us a (1-component) knot diagram.

The composition of elementary cobordisms $S$ consists of a list of diagrams $D_1, D_2, D_3, \ldots, D_d$ where each successive diagram differs from the previous diagram by a specified elementary cobordism. Forgetting about the 0- and 2-handle attachments, we can assume that $D_1$ is $D$ union $h$ 0-crossing unknots, and $D_d$ is the union of $k + 1$ 0-crossing unknots.

We shall specify a presentation of a surface which starts with the diagram $\tilde{D}_1 = D_1$, and to get from diagram $\tilde{D}_i$ to $\tilde{D}_{i+1}$ we perform a sequence of Reidemeister II moves (each of which introduces two new crossings) or a Reidemeister I move (introducing a crossing) or no moves. Note that we never add any handles in moving from $\tilde{D}_i$ to $\tilde{D}_{i+1}$. Each $\tilde{D}_i$ is a link diagram (of $h+1$ components), but we shall find it convenient to draw $\tilde{D}_i$ as a diagram consisting of normal edges (thin edges), thick edges (pipes), and balls. In Figure 32 we show what each $\tilde{D}_i$ can look like locally. Each pipe represents two uncrossed strands, each ball with three or more incident pipes represents some arbitrary tangle. If we delete all pipes and balls from the diagram $\tilde{D}_i$
we get the diagram $D_i$. Now we give some diagrams showing with what it is that we replace each elementary cobordism of the presentation $S$ (Figures 33, 34, 35, 36).

Now that we have reached diagram $\tilde{D}_d$, the next thing that we want to do in our surface presentation is to add some 1-handles, joining the strands together in certain pipes. Certain of the cobordisms described already have amalgamated previous pipes and balls into bigger balls. If we imagine that each ball is translucent, we can see all the pipes that we have introduced. We add the following 1-handles:

- In one of the Reidemeister I moves (Figure 33), a new pipe is introduced. Add a 1-handle joining the two strands in this pipe.
- In one of the Reidemeister II moves (Figure 34), two new pipes are introduced. Add two 1-handles, one in each pipe, joining the two strands together.
- In the Reidemeister III move (Figure 35), three new pipes are introduced. Add three 1-handles, one in each pipe, joining the two strands together.
- In the 1-handle move (Figure 36), a new pipe is introduced. Add a 1-handle joining the two strands in this pipe.

The only condition that we impose for the order of these 1-handle additions is that we want to add the $h$ chosen 1-handles that fuse together the $h+1$ components of $\tilde{D}_d$ before we add the rest of the 1-handles. After doing all the 1-handle additions we arrive at diagram $\tilde{D}_{d+1}$, in which some of the pipes have had 1-handles added. The idea now is to isotope the strands in each pipe away down the pipe, moving away from the site of the 1-handle addition. We are going to do this in the reverse order to that in which the pipes appeared in the cobordism.

We shall write $E_d = D_{d+1}$. Our cobordism will now pass through intermediary diagrams $E_d, E_{d-1}, \ldots, E_1$. We describe the cobordism that takes us from $E_{i+1}$ to $E_i$:

- If $D_i$ was taken to $D_{i+1}$ by a Reidemeister move I or II that increased the total number of crossings then $E_i = E_{i+1}$ with the trivial cobordism.
- If $D_i$ was taken to $D_{i+1}$ by a Reidemeister move I or II that decreased the total number of crossings, by a Reidemeister III move, or by a 1-handle addition, then $E_{i+1}$ is taken to $E_i$ by a number of Reidemeister
Figure 33: Reidemeister I
Figure 34: Reidemeister II
Figure 35: Reidemeister III
II moves, each reducing the total number of crossings, as illustrated in Figure 37.

Although only the Reidemeister III case is shown in Figure 37, the reader should be able to deduce what is intended for the other cases. The effect intended is that associated to trying eat a plate of spaghetti by putting a few strands in one’s mouth and sucking. The extra thin edge strands crossing the three pipes in Figure 37 which are not present in Figure 35 represent the possibility of a Reidemeister II or III move or 1-handle $D_j \rightarrow D_{j+1}$ for $i < j < d$ introducing new balls in $D_{j+1}$ where a thin edge has to cross some pipes as in the intermediate stages of Figures 34 and 35.

We arrive at diagram $E_1$ which is now free of pipes and balls. The components of $E_1$ are as shown in Figure 38, the areas of the components labelled by $\phi$ are, by construction, empty of other components. By Reidemeister I and II moves we arrive at diagram $E_0$, a diagram of the unlink consisting of 0-crossing unknots and 3-component unlinks as in the bottom right of Figure 38.

It can be shown that if we perform Reidemeister moves to convert our diagram to the 0-crossing diagram of the unlink and then add 2-handles to cap off all but one of the components then what we have described is a new presentation $S'$ of the surface $\Sigma$. However, we do not strictly need to know
Figure 37: Retracting pipes
this for the proof of our result; but the reader should convince herself that $S'$ is a presentation of a surface of the same genus as $\Sigma$ (by counting 1-handles and the number of components of our final unlink).

5.2 The image of a generator

Associated to the presentation $S'$ of the previous section is a map

$$HKh_w(S') : HKh_w(D) \rightarrow HKh_w(E_0)$$

where $E_0$ is the final diagram in the presentation $S'$ and is an unlink of, say, $e_0$ components.

By the work of the previous sections we know that $HKh_w(S')$ is injective. If $\alpha_\xi \in HKh_w(D)$ is the generator associated to decorating $K$ with the root $\xi$ of $\partial w$, then we know that

$$HKh_w(S')(\alpha_\xi) = \alpha_\xi \in HKh_w(E_0)$$

where we write $\alpha_\xi$ also for the element of $HKh_w(E_0)$ associated to decorating each component of $E_0$ with $\xi$ (here is where it is important to remember that we added the $h$ “fusing” 1-handles first.
We would like know in which graded degree of the grading associated to the quantum filtration of $HKh_w(E_0)$ $\alpha_\xi$ lies. Write $B$ for the faux Borromean rings unlink that appears in the bottom right of Figure 38 and in Figure 39. First we would like to compute in which degree $\alpha_\xi$ lies in $HKh_w(B)$. Figure 39 shows $B$ and the oriented resolution $B^o$ of $B$. The chain representative of $\alpha_\xi$ lies in $H(B^o)$ which is a summand of the 0th homological degree chain group.

The chain representative of $\alpha_\xi$ in $H(B^o)$ is, up to non-zero scalar multiplication, the module element

$$\prod_{i=1,2,3} \frac{\partial w_i}{x_i - \xi} \in \mathbb{C}[x_1, x_2, x_3]/(\partial w_1, \partial w_2, \partial w_3)[3 - 3n]$$

which is of top filtration grading $3n - 3$. *A priori*, of course, it does not follow that $[\alpha_\xi] \in HKh_w^{0, 3n-3}$, since we have not yet seen that $\alpha_\xi$ is not homologous to an element of $CKh_w$ which is of a lower filtered degree. We shall show this now.

The diagram $B$ is a diagram of the unlink, so for dimensional reasons the spectral sequence converging from $E_2 = HKh_n^{i,j}$ to $E_\infty = HKh_w^{i,j}$ collapses immediately ($E_2 = E_\infty$).

The reduction $\tilde{\alpha}_\xi \in \mathcal{F}^{3n-3}H(B^o)/\mathcal{F}^{3n-4}H(B^o)$ is a cycle in the page $E_1 = CKh_n^{i,j}$. If we can show that $\tilde{\alpha}_\xi$ is not a boundary with respect to the $E_1$ differential (which is just the standard Khovanov-Rozansky $sl(n)$ differential) then, since $E_2 = E_\infty$, $\tilde{\alpha}_\xi$ will represent a non-zero class on the $E_\infty$ page. The grading of this class (which is necessarily $j = 3n - 3$) will be the grading in which $[\alpha_\xi]$ lies in $HKh_w^{0,j}$.
So, in order to show that $[\alpha_\xi] \in HKh_n^{0,3n-3}$, we need to see that $\tilde{\alpha}_\xi$ represents a non-zero class in $HKh_n$. Khovanov and Rozansky [5] have provided quantum-degree 0 chain homotopy equivalences between the chain complexes $CKh_n$ corresponding to tangle diagrams differing by a single Reidemeister move. We shall change $B$ by certain of these Reidemeister moves to arrive at the 0-crossing 3-component unlink. The chain maps of Khovanov-Rozansky’s chain homotopy equivalences will act on $\tilde{\alpha}_\xi$, mapping it to a cycle of the chain complex corresponding to the 0-crossing 3-component unlink. This cycle represents a non-zero class in the homology, so $\tilde{\alpha}_\xi$ represents a non-zero class. This will complete our argument that $[\alpha_\xi] \in HKh_n^{0,3n-3}$.

In Figure 40 we have shown the sequence of Reidemeister moves that we use to convert $B$ to the 0-crossing 3-component unlink. Note that all the Reidemeister II moves are II.1 moves, that the Reidemeister III move involves three positive crossings, and that the number of circles in the oriented resolution never changes.

The cycle $\tilde{\alpha}_\xi$ in $CKh_n$ lies in the 0th chain group summand $H(B^n)$. Up to non-zero scalar multiplication:

$$\tilde{\alpha}_\xi = x_1^{n-1}x_2^{n-1}x_3^{n-1} \in \left(\mathbb{C}[x_1,x_2,x_3]/x_1^n x_2^n x_3^n\right)^{\{3-3n\}} = H(B^n)$$
Unpacking the proofs a little in [5], we see that (for the Reidemeister II.1 move which reduces the number of crossings and for the Reidemeister III move with all positive crossings) the maps induced from the chain group summands corresponding to the oriented resolutions are just the identity map of matrix factorizations. (The fact that, in section 4.2.1 of this paper, the component of this section’s chain map that runs from the oriented resolution of the two crossed strands to the uncrossed strands is the identity map, is a reflection of this fact).

This means that the element $\tilde{\alpha}_\xi$ always gets mapped to the top-degree element in the chain group summand that is the homology of the oriented resolution. In the final diagram of the 0-crossing 3-component unlink, the homology of the oriented resolution is the only chain group summand. So the image of $\tilde{\alpha}_\xi$ represents a non-zero element in homology and, since the chain maps are all chain homotopy equivalences, $\tilde{\alpha}_\xi$ also represents a non-zero element in homology.

### 5.3 Putting it all together

We have shown that given a presentation of a connected cobordism $\Sigma$ between a knot $K$ and the unknot (starting with diagram $D$ for the knot, and ending with the 0-crossing unknot diagram) we can convert this into a presentation of a surface with the same genus ending in a diagram of an $h$-component unlink $U^h$. Furthermore, this presentation induces a map that is filtered of degree $(1 - n)\chi$ (where $\chi$ is the Euler characteristic of the new multiply-punctured surface) from the homology $HKh_w(D)$ of $D$ to the homology of the $h$-component unlink.

If $\alpha_\xi$ is the element of $HKh_w(D)$ which is given by decorating $K$ with the root $\xi$ of $\partial w$, then we have seen that this element gets mapped to the top-degree part of $HKh_w(U^h)$, i.e. to a non-zero element of $HKh_w^{0,(n-1)h}$. Since the map was filtered of degree $(1 - n)\chi$, we see that $\alpha_\xi$ must have been in $HKh_w^{0,j}(D)$ where $j \geq (1 - n)(2g(\Sigma) - 1)$. This completes the proof of our main theorem [1].

### 5.4 Computation for positive knots

In this subsection we deduce Milnor’s conjecture on the slice genus of torus knots as a corollary to our main theorem.

Suppose that a knot $K$ has a diagram $D$ in which all the crossings are positive (i.e. look like the crossing on the left of Figure [4]). Suppose that $D$
has \( k \) crossings and \( l \) circles in its oriented resolution. We compute the top grading of \( HKh_w(D) \).

Since all the crossings are positive, the homology of the oriented resolution is the leftmost chain group in the Khovanov cube (also the chain group of homological degree 0). Hence the homology \( HKh_w(D) \) will just be the kernel of the differential coming from this chain group, and in particular the filtration grading of chain representatives of the homology will be the same taken in the chain group as taken in the homology since there is no boundary group by which to quotient.

Each basis element (got by decorating \( K \) with a root \( \xi \) of \( \partial w \)) of \( HKh_w(D) \) lies in \( HKh_w^{0,(1−n)(k−l)}(D) \). In particular, \( HKh_w^{0,(1−n)(k−l)}(D) \neq 0 \). Now, \( (1 − n)(k − l) \) is the highest filtration degree of the 0th chain group, so our theorem says that

\[
(n − 1)(2g^*(K) − 1) \geq (n − 1)(k − l)
\]

There exists an explicit description of a Seifert surface for \( K \) which consists of \( l \) disks (filling the circles of the oriented resolution of \( D \)), connected by \( k \) bands (where the crossings of \( D \) are). The Euler characteristic of this surface with boundary is \( l − k \), so the surface is of genus \( (k − l + 1)/2 \). Pushing this Seifert surface slightly into the 4-ball yields a slice surface for \( K \) of the same genus. Since we have seen that

\[
2g^*(K) \geq k − 1 + 1
\]

it follows that \( g^*(K) = k − l + 1 \). Performing this computation in the case of the standard diagram of a torus knot yields Milnor’s conjecture on the slice genus of torus knots.

6 Appendix

Here we provide a proof of basic results used (sometimes implicitly) throughout this paper.

6.1 Removal of Marks

In this section we omit mention of the various quantum grading shifts that occur, although readers friendly with matrix factorizations can easily assure themselves that the filtration gradings of each map is as expected.

To first explain Figure 41, the matrix factorization \( N \) (resp. \( N' \)) is obtained from \( Q \) (resp. \( Q' \)) by tensoring with the matrix factorization \( P \).
Also, the matrix factorization $M$ (resp. $M'$) is the same as $Q$ (resp. $Q'$) with the formal replacement of the variable $y$ by $x$.

**Theorem 6.1** Removal of marks

We prove that the matrix factorizations $M$ (resp. $M'$) and $N$ (resp. $N'$) (in which we intend the circles to contain the same arbitrary trivalent graph with thick edges) in Figure 11 are equal in the homotopy category of matrix factorizations.

Furthermore, suppose $M', N'$ are factorizations, maybe different from $M, N, Q$ but with the same boundary labels. If we then have a map of matrix factorizations $\alpha_x : M \to M'$ (and, replacing $x$ by $y$, $\alpha_y : Q \to Q'$), inducing a map of matrix factorizations $A : N \to N'$, then

$$\alpha_x = L \circ A \circ K$$

where $L : N \to M$ and $K : M \to N$ are the homotopy equivalences constructed.

**Proof** We exhibit filtered degree-0 maps $M \to N, N \to M$ which we show are homotopy equivalences.

The matrix factorizations $M$ and $N$ are both defined over the same ground ring $R = \mathbb{C}[x_1, x_2, ..., x_r, x]$. Let’s write $M$ as

$$M^0 \xrightarrow{f_x} M^1 \xrightarrow{g_y} M^0$$

$Q$ is the same factorization but with any occurrence of $x$ relabelled as $y$, we shall consequently write $Q$ as

$$Q^0 \xrightarrow{f_y} Q^1 \xrightarrow{g_y} Q^0$$

where the subscripts are to remind us that the difference between the factorizations $M$ and $Q$ is just the interchanging of $x$ with $y$.

Then if $P$ is the factorization

$$P^0 \xrightarrow{\pi_{xy}} P^1 \xrightarrow{x-y} P^0$$

where $P^0$ and $P^1$ are free rank-1 modules over $\mathbb{C}[x, y]$ we have that $N^0 = (Q^0 \otimes P^0) \oplus (Q^1 \otimes P^1)$, $N^1 = (Q^1 \otimes P^0) \oplus (Q^0 \otimes P^1)$ and $N$ can be written

$$N^0 \begin{pmatrix} f_y & -(x-y) \\ \pi_{xy} & g_y \end{pmatrix} \quad N^1 \begin{pmatrix} g_y & x-y \\ -\pi_{xy} & f_y \end{pmatrix}$$

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in which we are implicitly thinking of $N$ as a factorization over the polynomial ring $\mathbb{C}[x_1, x_2, ..., x_r, x, y]$ with degenerate potential.

Now we give the homotopy equivalence between $M$ and $N$. We define maps of matrix factorizations $K : M \to N$ and $L : N \to M$ by the following $\mathbb{C}[x_1, x_2, ..., x_r, x]$-module maps:

$$K_0 = \begin{pmatrix} 1 & -f_x & f_y \\ 0 & x-y \\ 0 & x-y \end{pmatrix}, K_1 = \begin{pmatrix} 1 & g_x & g_y \\ 0 & x-y \\ 0 & x-y \end{pmatrix}$$

$$L_0 = ( P_{y \to x} 0 ), L_1 = ( P_{y \to x} 0 )$$

where by $P_{y \to x}$ we mean the map which looks like the ring map

$$\mathbb{C}[x_1, x_2, ..., x_r, x, y] \to \mathbb{C}[x_1, x_2, ..., x_r, x, y]/(x-y) = \mathbb{C}[x_1, x_2, ..., x_r, x]$$

on each module summand $\mathbb{C}[x_1, x_2, ..., x_r, x, y]$.

Note that $L \circ K : M \to M$ is already the identity map.

We define an homotopy $H_0 : N^0 \to N^1, H_1 : N^1 \to N^0$ as follows:

$$H_0 = \begin{pmatrix} 0 & 0 \\ 1-P_{y \to x} & 0 \\ x-y & 0 \end{pmatrix}, H_1 = \begin{pmatrix} 0 & 0 \\ P_{y \to x}-1 & 0 \\ x-y & 0 \end{pmatrix}$$
It is now an exercise to see that

\[ id_{N^0} - K_0 \circ L_0 = H_1 \circ \begin{pmatrix} f_y & -(x - y) \\ \pi_{xy} & g_y \end{pmatrix} + \begin{pmatrix} g_y & x - y \\ -\pi_{xy} & f_y \end{pmatrix} \circ H_0 \]

and

\[ id_{N^1} - K_1 \circ L_1 = \begin{pmatrix} f_y & -(x - y) \\ \pi_{xy} & g_y \end{pmatrix} \circ H_1 + H_0 \circ \begin{pmatrix} g_y & x - y \\ -\pi_{xy} & f_y \end{pmatrix} \]

The second part of the theorem amounts to the observation that if \( \alpha_y,0 \) and \( \alpha_y,1 \) are the components of \( \alpha_y : Q \to Q' \) then the induced map \( A : N \to N' \) has components

\[ A_0 = \begin{pmatrix} \alpha_y,0 & 0 \\ 0 & \alpha_y,1 \end{pmatrix}, A_1 = \begin{pmatrix} \alpha_y,1 & 0 \\ 0 & \alpha_y,0 \end{pmatrix} \]

and then it is immediate that

\[ \alpha_{x,0} = \begin{pmatrix} P_{y \to x} & 0 \end{pmatrix} \begin{pmatrix} \alpha_y,0 & 0 \\ 0 & \alpha_y,1 \end{pmatrix} \begin{pmatrix} 1 \\ -f_x - f_y \end{pmatrix} \frac{x - y}{x - y} \]

and

\[ \alpha_{x,1} = \begin{pmatrix} P_{y \to x} & 0 \end{pmatrix} \begin{pmatrix} \alpha_y,1 & 0 \\ 0 & \alpha_y,0 \end{pmatrix} \begin{pmatrix} 1 \\ g_x - g_y \end{pmatrix} \frac{x - y}{x - y} \]

### 6.2 The \( \eta \) map

The \( \eta_0 \) and \( \eta_1 \) maps from Section 2 are defined as in Figure [12].

The equalities in Figure [12] are equalities as bare matrix factorizations. It is useful for us to know how this \( \eta \) map would be different if it were defined as conjugation of the \( \chi \) map by swapping 1 and 2 instead of 3 and 4. For example, at the end of our proof that the generators of \( HKh_w \) are preserved under the Reidemeister 2.1 move, we implicitly use the fact that the \( \eta \) map would merely change up to sign. It is a justification of this that we give here.

We can restate this result in terms of Figure [13] which we assert to be commutative, as is easy to verify.
Figure 42: Definition of the $\eta$ maps

Figure 43: A commutative diagram
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