STOCHASTIC HOMOGENIZATION AND EFFECTIVE HAMILTONIANS OF HJ EQUATIONS IN ONE SPACE DIMENSION: THE DOUBLE-WELL CASE

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Abstract. We consider Hamilton-Jacobi equations in one space dimension with Hamiltonians of the form
\[ H(p, x, \omega) = G(p) + \beta V(x, \omega) \]
where \( V(\cdot, \omega) \) is a stationary & ergodic potential of unit amplitude. The homogenization of such equations is established in a 2016 paper of Armstrong, Tran and Yu for all continuous and coercive \( G \). Under the extra condition that \( G \) is a double-well function (i.e., it has precisely two local minima), we give a new and fully constructive proof of homogenization which yields a formula for the effective Hamiltonian \( \overline{H} \). We use this formula to provide a complete list of the heights at which the graph of \( \overline{H} \) has a flat piece. We illustrate our results by analyzing basic classes of examples, highlight some corollaries that clarify the dependence of \( \overline{H} \) on \( G, \beta \) and the law of \( V(\cdot, \omega) \), and discuss a generalization to even-symmetric triple-well Hamiltonians.

1. Introduction

Let \((\Omega, F, \mathbb{P})\) be a probability space equipped with a group of measure-preserving transformations \( \tau_x : \Omega \to \Omega, x \in \mathbb{R} \), such that \( \tau_0 = \text{id} \) and \( \tau_x \circ \tau_y = \tau_{x+y} \) for every \( x, y \in \mathbb{R} \). Assume that \( \mathbb{P} \) is ergodic under this group, i.e.,
\[ \mathbb{P}(\cap_{x \in \mathbb{R}} \tau_x A) \in \{0, 1\} \quad \text{for every } A \in F. \]
We will use \( \mathbb{E} \) to denote expectation with respect to \( \mathbb{P} \).

Consider a Hamilton-Jacobi (HJ) equation of the form
\[ \partial_t u(t, x, \omega) + G(\partial_x u(t, x, \omega)) + \beta V(x/\epsilon, \omega) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}, \]
with the following ingredients: (i) \( \beta, \epsilon > 0 \); (ii) \( \omega \in \Omega \); (iii) the function \( G : \mathbb{R} \to [0, +\infty) \) satisfies
\[ G \in C(\mathbb{R}) \quad \text{and} \quad \lim_{p \to \pm\infty} G(p) = +\infty, \]
i.e., it is continuous and coercive on \( \mathbb{R} \); (iv) the so-called potential \( V : \mathbb{R} \times \Omega \to \mathbb{R} \) is nonconstant and measurable,
\[ V(x, \omega) = V(0, \tau_x \omega) \quad \text{for every } (x, \omega) \in \mathbb{R} \times \Omega, \]
and the map
\[ x \mapsto V(x, \omega) \text{ is in } BUC(\mathbb{R}) \text{ for every } \omega \in \Omega, \]
i.e., it is bounded and uniformly continuous on \( \mathbb{R} \).

The parameter \( \beta \) adjusts the amplitude of the potential. Since adding a constant to \( V \) corresponds to adding a linear (in time) term to any solution of (1.1), we will assume without further loss of generality that
\[ \mathbb{P}(\inf \{V(x, \omega) : x \in \mathbb{R}\} = 0 < 1 = \sup \{V(x, \omega) : x \in \mathbb{R}\}) = 1. \]
The parameter $\epsilon$ adjusts the length scale of the potential. As $\epsilon \to 0$, the HJ equation in (1.1) homogenizes to a deterministic HJ equation of the form

\[
(1.6) \quad \partial_t \bar{u}(t, x) + \mathcal{H} (\partial_x \bar{u}(t, x)) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R},
\]

with a coercive $\mathcal{H} \in C(\mathbb{R})$, referred to as the effective Hamiltonian. Precisely, for every $g \in UC(\mathbb{R})$, the space of uniformly continuous functions on $\mathbb{R}$, let $u^\epsilon_g(\cdot, \cdot, \omega)$ and $\bar{u}_g$ be the unique viscosity solutions of (1.1) and (1.6) that satisfy $u^\epsilon_g(0, \cdot, \omega) = \bar{u}_g(0, \cdot) = g$. With this notation, there exists an $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$ and $g \in UC(\mathbb{R})$, $u^\epsilon_g(\cdot, \cdot, \omega)$ converges locally uniformly on $[0, +\infty) \times \mathbb{R}$ as $\epsilon \to 0$ to $\bar{u}_g$. This homogenization result (without any assumptions beyond (1.2)–(1.5)) is due to Armstrong, Tran and Yu [4].

In the course of their proof in [4], Armstrong, Tran and Yu establish several key properties of the effective Hamiltonian. Most notably:

- $\mathcal{H}$ is level-set convex (also called quasiconvex) if and only if the parameter $\beta$ is greater than or equal to an explicit threshold (which depends on the local extreme values of $G$);
- the graph $\{(\theta, \mathcal{H}(\theta)) : \theta \in \mathbb{R}\}$ has some flat pieces.

Their approach relies on showing that, for every $\theta$ outside the flat pieces, there exists a corrector that is sublinear in $x$ as $x \to \pm \infty$.

The main goal of this paper is to provide a more detailed picture of the effective Hamiltonian. For the sake of clarity and convenience, we choose to restrict our attention to the case where $G$ has precisely two local minima, i.e., it is a double-well function. In this special but representative case, we give a new proof of the homogenization result in [4]. Moreover:

- we derive a formula for $\mathcal{H}$ on the whole real line that is easy to analyze;
- we provide a complete list of the values of $\mathcal{H}$ where its graph has a flat piece (which depends on the parameter $\beta$, the local extreme values of $G$ and whether certain explicit events involving the oscillations of $V(\cdot, \omega)$ are null sets under $P$).

The distinguishing feature of our approach is that, for every $\theta$ outside the flat pieces, we construct a sublinear corrector “by hand”, i.e., without using any existence results or limit procedures.

Here is an outline of the rest of the paper.

**Section 2.** We start by surveying the literature on the homogenization of HJ equations in the periodic and the stationary & ergodic settings with convex, level-set convex and nonconvex Hamiltonians. We say a few words about some of the existing proof strategies and introduce (sublinear) correctors in a nontechnical way. After that, we focus on the aforementioned work of Armstrong, Tran and Yu [4], outline their proof, and present their findings regarding the effective Hamiltonian in order to put our results into context.

**Section 3.** For a general class of double-well functions $G$ with local extreme values $0 \leq m < M$, we state our homogenization result and identify three qualitatively distinct cases depending on $\beta$, $m$ and $M$ (see Theorem 3.2). In each of these cases, we provide a piecewise formula for the effective Hamiltonian $\mathcal{H}$. Then, we present our results that determine the set $\mathcal{L}(\mathcal{H})$ of heights at which the graph of $\mathcal{H}$ has a flat piece (see Theorems 3.4 and 3.5). Finally, we discuss some features, corollaries and generalizations of our results, in particular to even-symmetric triple-well Hamiltonians.

**Section 4.** After citing fundamental theorems on the existence & uniqueness of viscosity solutions and on reducing homogenization to the almost sure existence of a pointwise limit, we present elementary comparison-based results on how this pointwise limit follows from the existence of correctors (see Proposition 4.3) or certain one-sided variants of them (see Proposition 4.5). All statements are customized to our setting and purposes.
Sections 2–3. In the first three of these sections, we consider the three qualitatively distinct cases in the statement of Theorem 3.2. In each case, to obtain the nonflat pieces of the graph of \( \overline{H} \), we construct correctors and use Proposition 4.3. Then, we carefully concatenate some of these correctors and use Proposition 4.5 to obtain the flat pieces. For the convenience of the reader, each piece has a designated subsection. The most important one is Subsection 6.2 where we construct and work with correctors that are merely piecewise continuously differentiable. Finally, in Section 8 we put everything together and conclude the proof of Theorem 3.2.

Section 11. The piecewise formula in Theorem 3.2 identifies all intervals on which \( \overline{H} \) is constant. However, not all of these intervals necessarily have positive length. In the proof of Theorem 3.4 we show that the strict inequality between the two endpoints of some of these intervals is characterized by the potential almost surely attaining its supremum or infimum. In the proof of Theorem 3.5 in order to characterize the same strict inequality for the remaining intervals in question, we introduce two natural events regarding the upcrossings and the downcrossings of the potential. Both proofs involve elementary probability bounds and ultimately rely on the ergodic theorem. When combined, they determine the aforementioned set \( \mathcal{L}(\overline{H}) \) of heights at which the graph of \( \overline{H} \) has a flat piece.

Section 11. We illustrate our results by writing down the set \( \mathcal{L}(\overline{H}) \) for two basic classes of examples. In the first class, the potential is the periodic extension of a single-well or double-well function \( v_0 \). We demonstrate how \( \beta, m, M \) and the local extreme values of \( v_0 \) precisely interact to yield \( \mathcal{L}(\overline{H}) \). In the second class, the potential is constructed by linearly interpolating a stationary & ergodic process with index set \( \mathbb{Z} \) and law \( \mathbb{Q} \). Under additional structural assumptions, we show that \( \mathcal{L}(\overline{H}) \) is determined by \( \beta, m, M \) and the atoms of one-dimensional marginals or conditionals of \( \mathbb{Q} \).

2. Previous results

2.1. Homogenization of HJ equations: General. In this subsection, we review the literature on the homogenization of HJ equations that is directly relevant to our results (which we will present in Section 3). For this purpose, we temporarily generalize the setting in Section 1 (which we will promptly return to in Subsection 2.2).

Fix any \( d \geq 1 \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space equipped with a group of measure-preserving transformations \( \tau_x : \Omega \to \Omega \), \( x \in \mathbb{R}^d \), such that \( \tau_0 = \text{id} \) and \( \tau_x \circ \tau_y = \tau_{x+y} \) for every \( x, y \in \mathbb{R}^d \). Assume that \( \mathbb{P} \) is ergodic under this group. For every \( \epsilon > 0 \) and \( \omega \in \Omega \), consider the HJ equation

\[
\partial_t u^\epsilon(t,x,\omega) + H(\nabla_x u^\epsilon(t,x,\omega), x/\epsilon, \omega) = 0, \quad (t,x) \in (0, +\infty) \times \mathbb{R}^d,
\]

where the function \( H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to [0, +\infty) \) is measurable and it satisfies the following properties:

\[
(p, x) \mapsto H(p, x, \omega) \quad \text{is in } \text{BUC}(K \times \mathbb{R}^d) \quad \text{for every bounded } K \subset \mathbb{R}^d \quad \text{and } \omega \in \Omega,
\]

\[
\lim_{|p| \to +\infty} H(p, x, \omega) = +\infty \quad \text{uniformly in } (x, \omega) \in \mathbb{R}^d \times \Omega, \quad \text{and}
\]

\[
H(p, x, \omega) = H(p, 0, \tau_x \omega) \quad \text{for every } (p, x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega.
\]

The first homogenization result for the HJ equation in (2.1) is due to Lions, Papanicolaou and Varadhan [10]. They focus on the special case where \( x = (x_1, \ldots, x_d) \mapsto H(p, x, \omega) \) is 1-periodic in \( x_i \) for every \( i \in \{1, \ldots, d\} \) and use the compactness of \([0,1]^d\) to solve the following auxiliary problem (which is formulated by them and referred to as the cell problem): for every \( \theta \in \mathbb{R}^d \), find a \( \lambda = \lambda(\theta) \in \mathbb{R} \) and a periodic function \( h = h_\theta : \mathbb{R}^d \times \Omega \to \mathbb{R} \) such that

\[
H(\theta + \nabla_x h(x, \omega), x, \omega) = \lambda, \quad x \in \mathbb{R}^d,
\]

in the viscosity sense. The solutions of this cell problem are called correctors. Their existence readily implies that the HJ equation in (2.1) homogenizes to

\[
\partial_t \overline{u}(t,x) + \overline{H}(\nabla_x \overline{u}(t,x)) = 0, \quad (t,x) \in (0, +\infty) \times \mathbb{R}^d,
\]
with $\mathcal{H}(\theta) = \lambda(\theta)$ for every $\theta \in \mathbb{R}$, albeit when a priori restricted to affine initial conditions. Then, Lions, Papanicolaou and Varadhan extend the class of initial conditions to all uniformly continuous functions by identifying and using some key properties of the semigroup induced by (2.1).

In the stationary & ergodic setting, on top of (2.2)–(2.4) and some additional mild conditions, if one assumes that $p \mapsto H(p, x, \omega)$ is convex, then the HJ equation in (2.1) homogenizes. This was shown by Souganidis [24] (in the $\mathbb{P}$-a.s. sense that we described in Section 1) and independently by Rezakhanlou and Tarver [23] (in an $L^1(\mathbb{P})$ sense). In both papers, the main idea is to apply the subadditive ergodic theorem to a variational representation for the viscosity solutions of (2.1) that involves the Legendre transform of $p \mapsto H(p, x, \omega)$.

In their paper [23] mentioned above, Rezakhanlou and Tarver also describe how one can try to adapt the approach of Lions, Papanicolaou and Varadhan [19] to the stationary & ergodic setting. In this direction, following the work of Evans [11] on periodic Hamiltonians, they show that the desired homogenization result for (2.1) would be established if one could solve the following generalization of the cell problem: for every $\theta \in \mathbb{R}^d$, find a $\lambda = \lambda(\theta) \in \mathbb{R}$ and a Lipschitz continuous viscosity solution $h(\cdot, \omega) = h_\theta(\cdot, \omega) : \mathbb{R}^d \to \mathbb{R}$ of (2.5) that satisfies

$$\lim_{|x| \to +\infty} \frac{h(x, \omega)}{|x|} = 0$$

for $\mathbb{P}$-a.e. $\omega$. However, in a later paper [20], Lions and Souganidis demonstrate (by providing a convex counterexample in one space dimension) that such a sublinear corrector does not exist in general (at least if it is required to be bounded). Furthermore, whenever homogenization holds, they give a variational formula for the effective Hamiltonian.

The convexity assumption can be somewhat relaxed by assuming level-set convexity instead, i.e., for every $a \in [0, 1]$, $p, q, x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$H(ap + (1 - a)q, x, \omega) \leq \max\{H(p, x, \omega), H(q, x, \omega)\}.$$  

In this case, when $d = 1$, Davini and Siconolfi [9] prove homogenization by solving the following relaxation of the generalized cell problem: for every $\theta \in \mathbb{R}$ and $\delta > 0$, find a $\lambda = \lambda(\theta) \in \mathbb{R}$ and a Lipschitz continuous function $h(\cdot, \omega) = h_{\theta, \delta}(\cdot, \omega) : \mathbb{R} \to \mathbb{R}$ such that

$$\lambda - \delta \leq H(\theta + h(\cdot, \omega), x, \omega) \leq \lambda + \delta, \quad x \in \mathbb{R},$$

in the viscosity sense and (2.6) holds for $\mathbb{P}$-a.e. $\omega$. Such functions are called $\delta$-approximate correctors. When $d \geq 1$, Armstrong and Souganidis [2] bypass the existence of $\delta$-approximate correctors and instead prove homogenization by applying the subadditive ergodic theorem to the maximal solutions of (2.5) when the condition $x \in \mathbb{R}^d$ there is replaced by $x \in \mathbb{R}^d \setminus \{y\}$ for any $y \in \mathbb{R}^d$. Since these maximal solutions give an intrinsic distance between $x$ and $y$, they are referred to as the solutions of the metric problem.

The first homogenization result for a genuinely nonconvex example in the stationary & ergodic setting is due to Armstrong, Tran and Yu [3] who consider separable Hamiltonians which are of the form

$$H(p, x, \omega) = G(p) + V(x, \omega).$$

They assume the following: (i) $G(p) = (|p|^2 - 1)^2$; (ii) $x \mapsto V(x, \omega)$ is in $\text{BUC}(\mathbb{R}^d)$. Their special choice of $G$ (most notably its radial symmetry) allows them to construct a family of static HJ equations that involves a free parameter $\sigma \in [-1, 1]$ and generalizes (2.5) (in the sense that the latter is formally obtained when $\sigma = \pm 1$). They establish the desired homogenization result by applying the subadditive ergodic theorem to the maximal subsolutions of these equations and using comparison arguments.

When $d = 1$, the function $G(p) = (|p|^2 - 1)^2$ considered above is an even-symmetric double-well function. In their subsequent paper [4] in one space dimension, Armstrong, Tran and Yu generalize this homogenization result to essentially all separable Hamiltonians with a coercive $G$. (This is the work we cited in Section 1 and we will present it in Subsection 2.2 below.) Finally, Gao [15] removes
the separability assumption in [4] and thereby proves that, when \( d = 1 \), the HJ equation in (2.1) homogenizes under the assumptions (2.2)–(2.4).

In a relatively recent paper, Qian, Tran and Yu [22] go back to the periodic setting and consider a class of separable Hamiltonians with a coercive and nonconvex \( G \in C(\mathbb{R}^d) \) that is, in particular, even-symmetric (i.e., \( G(p) = G(-p) \) for every \( p \in \mathbb{R}^d \)). They introduce a decomposition method that is tailored to this class and provide a min-max formula for the effective Hamiltonian \( \overline{H} \). Moreover, using analytical and numerical methods, they partially answer the following interesting questions regarding the dependence of \( \overline{H} \) on the potential \( V \) (which we will revisit in Subsection 3.4):

\[
\begin{align*}
\text{When is } \overline{H} \text{ level-set convex (despite the fact that } G \text{ is not)?} \\
\text{When is } \overline{H} \text{ not even-symmetric (despite the fact that } G \text{ is)?}
\end{align*}
\]

Their approach generalizes to the stationary \& ergodic setting with the same class of Hamiltonians, yielding a simpler proof of homogenization that covers the aforementioned result of Armstrong, Tran and Yu [3] with \( G(p) = (|p|^2 - 1)^2 \).

For other (positive and negative) results on the homogenization of HJ equations with nonconvex Hamiltonians, see [14, 16, 26].

We end this review by mentioning that there are closely related works on the homogenization of second-order HJ equations, starting with [17, 21] under the assumption of convexity. However, the literature on nonconvex Hamiltonians is relatively sparse. In particular, when \( d = 1 \), homogenization is established in [7, 8, 18, 25] for certain classes of nonconvex Hamiltonians, but the picture is far from being complete. See also [1, 6, 13] for some (positive and negative) results in higher dimensions.

2.2. Homogenization of HJ equations: Separable Hamiltonians in one space dimension.

In this subsection, we present the aforementioned work of Armstrong, Tran and Yu [4] in more detail. For this purpose, we return to the setting in Section 1. In particular, we assume that the Hamiltonian is separable, \( d = 1 \), and (1.2)–(1.5) hold.

By a stability argument and a gluing procedure (see [4, Lemmas 2.3 and 4.1]), it suffices to prove the desired homogenization result for the HJ equation in (1.1) under several additional assumptions, mainly (when translated to our setting):

- \( G \) has a unique absolute minimum at 0 and \( G(0) = 0 \),
- \( G \) has \( L \) local minima (and hence \( L \) local maxima) in \( (-\infty, 0) \) for some \( L \geq 0 \),
- \( G \) has no local minima in \( (0, +\infty) \), and
- \( x \mapsto V(x, \omega) \) is a smooth function whose level sets have no cluster points.

Whenever \( L \geq 1 \), denote the local minimum and the local maximum values of \( G \) in \( (-\infty, 0) \) by \( m_1, \ldots, m_L \) and \( M_1, \ldots, M_L \), respectively. Let

\[
m_{\min} = \min\{m_1, \ldots, m_L\} \quad \text{and} \quad M_{\max} = \max\{M_1, \ldots, M_L\}.
\]

If \( \beta < M_{\max} - m_{\min} \), then two further gluing procedures (see [4, Lemmas 4.2 and 4.3]) imply that the desired result follows from that for Hamiltonians with strictly less number of local minima. This is a strong induction argument with two base cases.

**Base case 1: \( L = 0 \).** This case is covered by [2, 9]. The effective Hamiltonian \( \overline{H} \) inherits the level-set convexity of \( G \). Moreover, there exist \( \theta_{\ell}, \theta_r \in \mathbb{R} \) such that \( \theta_{\ell} < 0 < \theta_r \) and

\[
\overline{H} \begin{cases} 
\text{strictly decreasing on } (-\infty, \theta_{\ell}], \\
\text{identically equal to } \beta \text{ on } (\theta_{\ell}, \theta_r), \\
\text{strictly increasing on } [\theta_r, +\infty).
\end{cases}
\]
Base case 2: \( L \geq 1 \) and \( \beta \geq M_{\text{max}} - m_{\text{min}} \). This case is treated in [3] Section 3. The effective Hamiltonian \( \mathcal{H} \) is level-set convex despite the fact that \( G \) is not. Moreover, there exist \( \theta_\ell, \theta_r \in \mathbb{R} \) such that \( \theta_\ell < 0 < \theta_r \) and

\[
\mathcal{H} = \begin{cases}
\text{strictly decreasing on the semi-infinite interval } \{ \theta \leq \theta_\ell : \mathcal{H}(\theta) \geq M_{\text{max}} + \beta \}, \\
\text{nonincreasing on the nonempty interval } \{ \theta \leq \theta_\ell : m_{\text{min}} < \mathcal{H}(\theta) < M_{\text{max}} + \beta \}, \\
\text{strictly decreasing on the possibly empty interval } \{ \theta \leq \theta_\ell : \mathcal{H}(\theta) \leq m_{\text{min}} \}, \\
\text{strictly increasing on } [\theta_r, +\infty).
\end{cases}
\]

(2.9) 

In each of these two base cases, for every \( \theta \notin (\theta_\ell, \theta_r) \), there is a sublinear corrector. In particular, for every \( \theta \) in the intervals on which \( \mathcal{H} \) is stated to be strictly monotone, it is easy to directly write down a sublinear corrector and obtain a simple formula for \( \mathcal{H}(\theta) \) (which we omit here because we will provide such formulas in Subsection 3.2 after introducing some notation). In fact, the strict monotonicity of \( \mathcal{H} \) on those intervals can be deduced from this formula. On the other hand, in the second base case, when \( \theta \leq \theta_\ell \) and \( m_{\text{min}} < \mathcal{H}(\theta) < M_{\text{max}} + \beta \), the proof of the existence of a sublinear corrector is not constructive (see [3] Lemma 3.5) and it does not yield a formula for \( \mathcal{H}(\theta) \).

3. Our results

3.1. Double-well Hamiltonians. Recall the setting we described in Section 1. On top of the basic assumptions we have adopted there (which we will implicitly accept in the rest of the paper), we will state and prove our results under the additional assumption that \( G \) is a double-well function. Here is the precise formulation.

**Condition 3.1.** There exist \( m, M, p_m, p_M \in \mathbb{R} \) such that

\[ p_m < p_M < 0, \quad G(0) = 0 \leq G(p_m) = m < G(p_M) = M, \]

and \( G \) is of the form

\[
G(p) = \begin{cases}
G_1(p) & \text{if } p \in (-\infty, p_m], \\
G_2(p) & \text{if } p \in [p_m, p_M], \\
G_3(p) & \text{if } p \in [p_M, 0], \\
G_4(p) & \text{if } p \in [0, +\infty),
\end{cases}
\]

where \( G_1 \) and \( G_3 \) (resp. \( G_2 \) and \( G_4 \)) are defined on the intervals indicated next to them and they are strictly decreasing (resp. strictly increasing). See Figure 7.

Note that, if \( G \) satisfies Condition 3.1 then it has exactly three extreme points: an absolute minimum at 0, a local minimum at \( p_m \) (which is an absolute minimum when \( m = 0 \)) and a local maximum at \( p_M \). Conversely, if \( G \) has exactly two local minima, then it satisfies Condition 3.1 up to translation and reflection (if necessary) which would correspond to adding a linear (in \( t \) and \( x \)) term to any viscosity solution of (1.1) and substituting \(-x\) for \( x \) (see Subsection 3.4).

3.2. Homogenization. Define

\[
\theta_1 : [m + \beta, +\infty) \to (-\infty, p_m), \quad \theta_2 : [m + \beta, M] \to (p_m, p_M),
\]

\[
\theta_3 : [\beta, M] \to (p_M, 0), \quad \theta_4 : [\beta, +\infty) \to (0, +\infty)
\]

by

(3.1) \( \theta_i(\lambda) = \mathbb{E} \left[ G_i^{-1}(\lambda - \beta V(0, \omega)) \right], \quad i \in \{1, 2, 3, 4\}, \)

whenever their domains are nonempty. It follows that \( \theta_1 \) and \( \theta_3 \) (resp. \( \theta_2 \) and \( \theta_4 \)) are strictly decreasing (resp. strictly increasing) and continuous. With this notation, here is our first result.
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Theorem 3.2. Recall the (stationary & ergodic) setting in Section 1. In particular, assume that (1.2) – (1.5) hold. Moreover, impose Condition 3.1 on $G$. Then, the HJ equation in (1.1) homogenizes to the HJ equation in (1.6) with a coercive $\mathcal{H} \in C(\mathbb{R})$. Precisely, there is an $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$ and $g \in UC(\mathbb{R})$, the unique viscosity solution $u^\varepsilon(\cdot, \cdot, \omega)$ of (1.1) with initial condition $g$ converges locally uniformly on $[0, +\infty) \times \mathbb{R}$ as $\varepsilon \to 0$ to the unique viscosity solution $u^g$ of (1.6) with the same initial condition.

The effective Hamiltonian $\mathcal{H}$ has the following piecewise description.

**Case I:** If $\beta \leq m + \beta < M$ (henceforth referred to as weak potential), then

$$\mathcal{H}(\theta) = \begin{cases} 
\theta_1^{-1}(\theta) & \text{on } (-\infty, \theta_1(m + \beta)] \\
m + \beta & \text{on } (\theta_1(m + \beta), \theta_2(m + \beta)) \\
\theta_2^{-1}(\theta) & \text{on } [\theta_2(m + \beta), \theta_2(M)] \\
M & \text{on } (\theta_2(M), \theta_3(M)) \\
\theta_3^{-1}(\theta) & \text{on } [\theta_3(M), \theta_3(\beta)] \\
\beta & \text{on } (\theta_3(\beta), \theta_4(\beta)) \\
\theta_4^{-1}(\theta) & \text{on } [\theta_4(\beta), +\infty)
\end{cases}$$

(3.2)

In particular, $\mathcal{H}$ is not level-set convex.

**Case II:** If $\beta < M \leq m + \beta$ (henceforth referred to as medium potential), then

$$\mathcal{H}(\theta) = \begin{cases} 
\theta_1^{-1}(\theta) & \text{on } (-\infty, \theta_1(m + \beta)] \\
\overline{\lambda}(\theta) & \text{on } (\theta_1(m + \beta), \theta_3(M)) \\
\theta_3^{-1}(\theta) & \text{on } [\theta_3(M), \theta_3(\beta)] \\
\beta & \text{on } (\theta_3(\beta), \theta_4(\beta)) \\
\theta_4^{-1}(\theta) & \text{on } [\theta_4(\beta), +\infty)
\end{cases}$$

(3.3)

The nonincreasing function $\overline{\lambda} : (\theta_1(m + \beta), \theta_3(M)) \to [M, m + \beta]$ is given in Definition 8.1. (See Remark 3.3 below for a description.) In particular, $\mathcal{H}$ is level-set convex.
Case III: If $M \leq \beta \leq m + \beta$ (henceforth referred to as strong potential), then

\[
\mathcal{H}(\theta) = \begin{cases} 
\theta^{-1}_1(\theta) \text{ on } (-\infty, \theta_1(m + \beta)] & \text{(strictly decreasing),} \\
\mathcal{X}(\theta) \text{ on } (\theta_1(m + \beta), \theta_4(\beta)) & \text{(nonincreasing),} \\
\theta^{-1}_4(\theta) \text{ on } [\theta_4(\beta), +\infty) & \text{(strictly increasing).}
\end{cases}
\]

(3.4)

The nonincreasing function $\mathcal{X} : (\theta_1(m + \beta), \theta_4(\beta)) \rightarrow [\beta, m + \beta]$ is given in Definition 8.1. (See Remark 3.3 below for a description.) In particular, $\mathcal{H}$ is level-set convex.

Remark 3.3. Theorem 3.2 is mostly covered by the homogenization result of Armstrong, Tran and Yu in [4]. Specifically, it corresponds to taking $L = 1$ (so that $m_{\text{min}} = m$ and $M_{\text{max}} = M$) with the notation in Subsection 2.2 (assuming $m > 0$). Therefore, Case I (where $\beta < M - m$) reduces to the first base case of their induction argument and $\mathcal{H}$ is obtained by gluing effective Hamiltonians of the form in (2.8). (See [4, Section 5].) Similarly, Cases II and III (where $\beta \geq M - m$) are covered by the second base case of their induction argument and $\mathcal{H}$ satisfies (2.9). Moreover, in all cases, the continuous functions $\theta_1, \theta_2, \theta_3, \theta_4$ (defined in (3.1)) whose inverses give the effective Hamiltonian on the intervals where it is stated to be strictly monotone are already provided in [4]. As we will see in Section 5, these four functions are associated to correctors in $C^1(\mathbb{R})$.

The novelty of Theorem 3.2 lies in the identification of the function $\mathcal{X}$ in Cases II and III, i.e., when $\max\{\beta, M\} \leq m + \beta$.

• If $\max\{\beta, M\} = m + \beta$, then $\mathcal{X}$ is identically equal to $\max\{\beta, M\}$ (see Definition 8.1(a,c)).

• If $\max\{\beta, M\} < m + \beta$, then $\mathcal{X}$ is the unique nonincreasing generalized inverse (see (6.14)) of a strictly decreasing and right-continuous function $\theta_{1,3}$ (defined in (6.10)), possibly extended by adding flat pieces at the ends of its domain (see Definition 8.1(b,d)). The function $\theta_{1,3}$ is associated to correctors that are piecewise continuously differentiable. Since the construction of those correctors involves several steps and more notation (see Subsection 6.2), the formal definition of $\mathcal{X}$ is postponed to Section 8 where the proof of Theorem 3.2 is completed.

3.3. Flat pieces within the nonincreasing piece. Theorem 3.2 identifies intervals on which the effective Hamiltonian is either constant or strictly monotone, but it does not provide a complete list of such intervals. Precisely, in the case of medium potential (i.e., when $\beta < M \leq m + \beta$), on the interval $[\theta_1(m + \beta), \theta_3(M)]$, it merely states that $\mathcal{H}$ is nonincreasing,

$$\mathcal{H}(\theta_1(m + \beta)) = m + \beta \quad \text{and} \quad \mathcal{H}(\theta_3(M)) = M.$$

Similarly, in the case of strong potential (i.e., when $M \leq \beta \leq m + \beta$), on the interval $[\theta_1(m + \beta), \theta_4(\beta)]$, it merely states that $\mathcal{H}$ is nonincreasing,

$$\mathcal{H}(\theta_1(m + \beta)) = m + \beta \quad \text{and} \quad \mathcal{H}(\theta_4(\beta)) = \beta.$$

The following result characterizes the existence of flat pieces within these nonincreasing pieces that share one or two endpoints with them (i.e., at heights $\max\{\beta, M\}$ and $m + \beta$). It involves the events

$$\{\omega \in \Omega : V(x, \omega) = 0 \text{ for some } x \in \mathbb{R}\} \quad \text{and} \quad \{\omega \in \Omega : V(x, \omega) = 1 \text{ for some } x \in \mathbb{R}\}$$

whose $\mathbb{P}$-probabilities are in $\{0, 1\}$ by our ergodicity assumption.

Theorem 3.4. Under the assumptions in Theorem 3.2, the following are true.

(a) If $\beta < M = m + \beta$ (henceforth referred to as the easy subcase of medium potential), then $\mathcal{H}$ is identically equal to $\max\{\beta, M\} = M$ on the whole interval $[\theta_1(M), \theta_3(M)]$.

(b) If $\beta < M < m + \beta$ (i.e., medium potential, excluding the easy subcase above), then:

(i) the graph of $\mathcal{H}$ has a flat piece at height $\max\{\beta, M\} = M$ iff

$$\mathbb{P}(V(x, \omega) = 0 \text{ for some } x \in \mathbb{R}) = 1; \text{ and}$$

(3.5)

(ii) the graph of $\mathcal{H}$ has a flat piece at height $m + \beta$ iff

$$\mathbb{P}(V(x, \omega) = 1 \text{ for some } x \in \mathbb{R}) = 1.$$

(3.6)
(c) If $M \leq \beta = m + \beta$ (henceforth referred to as the easy subcase of strong potential), then $H$ is identically equal to $\max\{\beta, M\} = \beta$ on the whole interval $[\theta_1(\beta), \theta_2(\beta)]$.

(d) If $M \leq \beta < m + \beta$ (i.e., strong potential, excluding the easy subcase above), then:

(i) the graph of $H$ always has a flat piece at height $\max\{\beta, M\} = \beta$; and

(ii) the graph of $H$ has a flat piece at height $m + \beta$ iff (3.6) holds.

It remains to characterize the flat pieces in the graph of $H$ that are fully in the interior of the aforementioned nonincreasing piece. To this end, for every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$, consider two bi-infinite sequences $(x_i(\lambda, \omega))_{i \in \mathbb{Z}}$ and $(y_i(\lambda, \omega))_{i \in \mathbb{Z}}$ that satisfy the coupled recursion

\begin{equation}
(3.7)
\begin{align*}
x_i(\lambda, \omega) &= \inf\{x \geq y_{i-1}(\lambda, \omega) : \lambda - \beta V(x, \omega) \geq M\}, \\
y_i(\lambda, \omega) &= \inf\{x \geq x_i(\lambda, \omega) : \lambda - \beta V(x, \omega) < m\}.
\end{align*}
\end{equation}

Similarly, consider $(\pi_i(\lambda, \omega))_{i \in \mathbb{Z}}$ and $(\eta_i(\lambda, \omega))_{i \in \mathbb{Z}}$ that satisfy

\begin{equation}
(3.8)
\begin{align*}
\pi_i(\lambda, \omega) &= \inf\{x \geq \eta_{i-1}(\lambda, \omega) : \lambda - \beta V(x, \omega) > M\}, \\
\eta_i(\lambda, \omega) &= \inf\{x \geq \pi_i(\lambda, \omega) : \lambda - \beta V(x, \omega) \leq m\}.
\end{align*}
\end{equation}

Both pairs of sequences are well-defined for $\mathbb{P}$-a.e. $\omega$ under suitable “anchoring” conditions (see Subsection 6.2). With this notation, we introduce the events

\begin{equation}
(3.9)
\begin{align*}
U_\lambda &= \{\omega \in \Omega : \exists j \in \mathbb{Z} \text{ s.t. } x_0(\lambda, \omega) = \pi_j(\lambda, \omega)\} \\
&= \{\omega \in \Omega : \inf\{x \geq x_0(\lambda, \omega) : \lambda - \beta V(x, \omega) > M\} = x_0(\lambda, \omega)\} \\
&= \{\omega \in \Omega : \lambda - \beta V(\cdot, \omega) \text{ does not have a local maximum at } x_0(\lambda, \omega)\}
\end{align*}
\end{equation}

and

\begin{equation}
(3.10)
\begin{align*}
D_\lambda &= \{\omega \in \Omega : \exists k \in \mathbb{Z} \text{ s.t. } y_0(\lambda, \omega) = \eta_k(\lambda, \omega)\} \\
&= \{\omega \in \Omega : \inf\{x \geq y_0(\lambda, \omega) : \lambda - \beta V(x, \omega) < m\} = y_0(\lambda, \omega)\} \\
&= \{\omega \in \Omega : \lambda - \beta V(\cdot, \omega) \text{ does not have a local minimum at } y_0(\lambda, \omega)\}
\end{align*}
\end{equation}

which, respectively, involve the upcrossings and the downcrossings of the function $x \mapsto \lambda - \beta V(x, \omega)$. In each of the displays (3.9) and (3.10), the second and the third descriptions of the event are equivalent to the first one (see Section 10) and they are provided here to convey some intuition.

**Theorem 3.5.** Under the assumptions in Theorem 3.2 when $\max\{\beta, M\} < m + \beta$ (i.e., medium or strong potential, excluding their easy subcases), for every $\lambda \in (\max\{\beta, M\}, m + \beta)$, the graph of the effective Hamiltonian $H$ has a flat piece at height $\lambda$ if and only if

$$\mathbb{P}(U_\lambda \cap D_\lambda) < 1.$$  

### 3.4. Some remarks and generalizations.

#### 3.4.1. The heights of all flat pieces.

It follows readily from Theorems 3.2, 3.4 and 3.5 that, under Condition 3.3, the set

\begin{equation}
(3.11)
\mathcal{L}(H) = \{\lambda \in \mathbb{R} : \text{the graph of } H \text{ has a flat piece height } \lambda\}
\end{equation}

does not depend on the exact shape of the graphs of the strictly monotone functions $G_1, G_2, G_3, G_4$. Indeed:

- when $\max\{\beta, M\} \geq m + \beta$ (which covers weak potential as well as the easy subcases of medium and strong potential), $\mathcal{L}(H)$ is a subset of $\{\beta\} \cup \{m + \beta\} \cup \{M\}$ containing $\beta$ and it depends only on $\beta, m, M$;
- when $\max\{\beta, M\} < m + \beta$ (which covers medium and strong potential, except their easy subcases), $\mathcal{L}(H)$ is a subset of $\{\beta\} \cup [\max\{\beta, M\}, m + \beta]$ containing $\beta$ and it depends on $\beta, m, M$ plus the support of the law of $x \mapsto V(x, \omega)$ under $\mathbb{P}$.  


Moreover, in the latter regime, random potentials can be constructed to make \( \mathcal{L}(H) \) equal to any desired finite or countable subset of \( \{\beta\} \cup [\max\{\beta, M\}, m + \beta] \) containing \( \beta \). See Example 11.4 and Remark 11.5 for such a construction.

3.4.2. No reduction to smooth potentials, etc. As we mentioned in Subsection 2.3, Armstrong, Tran and Yu \[4\] use a stability lemma to argue that, in order to prove homogenization, it suffices to consider smooth potentials whose level sets have no cluster points. However, since our results are mainly concerned with the fine properties of the effective Hamiltonian and their dependence on the potential, we do not make such reductions. Indeed, in all our constructions and proofs (except in Section 11 where we analyze examples), we impose no conditions on the potential beyond (1.3)–(1.5).

3.4.3. Effective Hamiltonians under reflection. Given any double-well function \( G^- \) that satisfies (1.2) and Condition 3.1, consider its mirror image \( G^+ \) defined by

\[
G^+(p) = G^-(\theta p) \quad \text{for every } p \in \mathbb{R}.
\]

For any \( \theta \in \mathbb{R} \), consider the HJ equation in (1.1) with \( G = G^+ \) and initial condition \( u^0(0, x, \omega) = \theta x \).

Substituting \(-x\) for \( x \) corresponds to replacing \( \theta \) with \(-\theta\), the function \( G^+ \) with \( G^- \), and the potential \( x \mapsto V(x, \omega) \) with \( x \mapsto V(-x, \omega) \). Therefore, Theorem 3.2 applies to both \( G^- \) and \( G^+ \).

Denote the corresponding effective Hamiltonians by \( H^- \) and \( H^+ \), respectively. It is natural to ask the following question about them:

\[
\text{Is } H^+(\theta) = H^-(\theta) \quad \text{for every } \theta \in \mathbb{R}?
\]

The answer is a corollary of Theorem 3.2:

- When \( \max\{\beta, M\} \geq m + \beta \), it is shown in (3.3)–(3.4) and Definition 8.1(a,c) that the effective Hamiltonian depends not only on some of the functions \( \theta_1, \theta_2, \theta_3, \theta_4 \) (defined in (3.1)). Since the values of these functions do not change if we replace \( x \mapsto V(x, \omega) \) with \( x \mapsto V(-x, \omega) \), the answer to (3.12) is yes.

- When \( \max\{\beta, M\} < m + \beta \), it is shown in (3.3)–(3.4) and Definition 8.1(b,d) that the effective Hamiltonian depends not only on some of the functions \( \theta_1, \theta_2, \theta_3, \theta_4 \), but also on another function \( \theta_{1,3} \) (defined in (6.10)). Recall from Remark 3.3 that this function is associated to correctors that are piecewise continuously differentiable. As we will see in Subsection 6.2, the construction of those correctors involves the bi-infinite sequences introduced in (3.7)–(3.8). Therefore, in general, the values of \( \theta_{1,3} \) change if we replace \( x \mapsto V(x, \omega) \) with \( x \mapsto V(-x, \omega) \), and the answer to (3.12) is no.

Moreover, in the second parameter regime above, it follows from Theorem 3.5 that, even the sets \( \mathcal{L}(H^-) \) and \( \mathcal{L}(H^+) \) (with the definition in (3.11)) are not generally equal.

3.4.4. A generalization to even-symmetric triple-well Hamiltonians. Given any double-well function \( G^- \) that satisfies (1.2) and Condition 3.1, define \( G^s \) by

\[
G^s(p) = \begin{cases} 
G^-(p) & \text{if } p \leq 0, \\
G^+(p) & \text{if } p > 0,
\end{cases}
\]

where \( G^+ \) is the mirror image of \( G^- \) as in the previous remark. Note that \( G^s \) is a continuous, coercive and even-symmetric triple-well function with an absolute minimum at 0 where \( G^s(0) = 0 \). The superscript \( s \) stands for the word symmetric.

It follows immediately from Theorem 3.2 and a gluing result (see [4] Lemma 4.1) that the HJ equation in (1.1) with \( G = G^s \) homogenizes (which is of course covered by [4]) and the effective
Hamiltonian $\mathcal{H}^s$ is given by
\[
\mathcal{H}^s(\theta) = \begin{cases} 
\mathcal{H}^-(\theta) & \text{if } \theta \leq 0, \\
\mathcal{H}^+(\theta) & \text{if } \theta > 0,
\end{cases}
\]
with our notation in the previous remark. Recalling our answer to (3.12) there, we deduce the following properties of $\mathcal{H}^s$:

- if $\max\{\beta, M\} > m + \beta$, then $\mathcal{H}^s$ is even-symmetric but not level-set convex;
- if $\max\{\beta, M\} = m + \beta$, then $\mathcal{H}^s$ is even-symmetric and level-set convex;
- if $\max\{\beta, M\} < m + \beta$, then $\mathcal{H}^s$ is level-set convex but not generally even-symmetric.

This trichotomy rigorously answers the questions in (2.7) in the context of triple-well Hamiltonians when $d = 1$. To the best of our knowledge, it was first given by Qian, Tran and Yu [22]. Precisely, in the third parameter regime above, they consider a periodic potential and briefly mention how one can show that $\mathcal{H}^s$ is not even-symmetric (see [22, Remark 4]). Additionally, for a periodic example with a piecewise linear and even-symmetric triple-well function $G^s$, they use a Lax-Friedrichs based numerical method to plot $\mathcal{H}$ which is then observed to be not even-symmetric (see [22, Numerical Example 2]). We believe that our results completely clarify this connection between the emergence of level-set convexity and the loss of even-symmetry in the 1-d stationary & ergodic setting.

3.4.5. Removing Condition 3.1 The fully constructive approach that we take in this paper under Condition 3.1 can be adopted in the more general setting of the homogenization result of Armstrong, Tran and Yu [4] to obtain a formula for the effective Hamiltonian $\mathcal{H}$ on the whole real line and to identify the set $\mathcal{L}(\mathcal{H})$ (defined in (3.11)). Recalling their proof which is outlined in Subsection 2.2, it essentially suffices to consider the second base case of their induction argument and focus on the interval where $\mathcal{H}$ is stated to be nonincreasing in (2.9). We plan to include this generalization in a future paper.

4. Preliminaries

For an introduction to viscosity solutions of first-order HJ equations, see, e.g., [3, 12]. Solutions, subsolutions and supersolutions of all HJ equations considered in this paper are to be understood in the viscosity sense unless noted otherwise. We henceforth drop the word viscosity for the sake of brevity.

We start by stating an existence & uniqueness result that is tailored to our setting and purposes. It is covered by, e.g., the version in [10, Theorem A.1].

**Theorem 4.1.** If (1.2) and (1.4) hold, then for every $g \in \text{UC}(\mathbb{R})$, the HJ equations in (1.1) (with any fixed $\omega \in \Omega$) and (1.6) (with a coercive $\mathcal{H} \in \text{C}(\mathbb{R})$ to be determined) have unique solutions $u^s_g(\cdot, \cdot, \omega), \pi_g \in \text{UC}([0, +\infty) \times \mathbb{R})$ that satisfy
\[
u^s_g(0, x, \omega) = \pi_g(0, x) = g(x), \quad x \in \mathbb{R}.
\]
Moreover, if $g \in \text{Lip}(\mathbb{R})$, the space of Lipschitz continuous functions on $\mathbb{R}$ with a uniform Lipschitz constant, then $u^s_g(\cdot, \cdot, \omega), \pi_g \in \text{Lip}([0, +\infty) \times \mathbb{R})$.

When $g(x) = \theta x$ for some $\theta \in \mathbb{R}$, we will write $u^s_\theta$ and $\pi_\theta$ instead of $u^s_g$ and $\pi_g$, respectively. Moreover, when $\epsilon = 1$ and (1.1) becomes
\[
\partial_t u(t, x, \omega) + G(\partial_x u(t, x, \omega)) + \beta \mathcal{H}(x, \omega) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R},
\]
we will drop the superscript of $u^s_\theta$ and simply write $u_\theta$. With this notation,
\[
u^s_\theta(t, x, \omega) = \epsilon u_\theta(t/\epsilon, x/\epsilon, \omega),
\]
\[u_\theta(t, x, \tau_\omega \omega) = u_\theta(t, x + y, \omega) - \theta y \quad \text{and}
\]
\[\pi_\theta(t, x) = -\mathcal{H}(\theta)t + \theta x
\]
for every $t \geq 0$ and $x, y \in \mathbb{R}$. Indeed, in each line, the functions on the left- and the right-hand sides are solutions of the same HJ equation with the same linear initial condition, so they are equal by uniqueness.

If the HJ equation in (1.1) homogenizes to the HJ equation in (1.6) (as claimed in Theorem 3.2 with general $g \in \text{UC}(\mathbb{R})$), then we can choose to restrict our attention to linear initial conditions and deduce that, for every $\theta \in \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega$, $u_\theta^\epsilon(\cdot, \cdot, \omega)$ converges locally uniformly on $[0, +\infty) \times \mathbb{R}$ as $\epsilon \to 0$ to $\pi_\theta$. In particular, we have pointwise convergence at $(t, x) = (1, 0)$, which is equivalent to

$$
\mathbb{P} \left( \lim_{\epsilon \to 0} \epsilon u_\theta(1/\epsilon, 0, \omega) = -\mathbb{H}(\theta) \right) = 1
$$

by (4.2) and (4.4). In short, homogenization implies the almost sure limit in (4.5). The following result (Theorem 4.2) states that the converse is also true. It involves the quantities

$$
\mathbb{H}^I(\theta) = \liminf_{t \to +\infty} \frac{-u_\theta(t, 0, \omega)}{t} \quad \text{and} \quad \mathbb{H}^U(\theta) = \limsup_{t \to +\infty} \frac{-u_\theta(t, 0, \omega)}{t}.
$$

Since $u_\theta(\cdot, \cdot, \omega) \in \text{Lip}([0, +\infty) \times \mathbb{R})$ by Theorem 4.1 and it satisfies (4.3), our ergodicity assumption ensures that $\mathbb{H}^I(\theta)$ and $\mathbb{H}^U(\theta)$ are $\mathbb{P}$-essentially constant. Whenever $\mathbb{H}^U(\theta)$ and $\mathbb{H}^I(\theta)$ are equal, we set

$$
\mathbb{H}(\theta) = \mathbb{H}^I(\theta) = \mathbb{H}^U(\theta).
$$

**Theorem 4.2.** Recall the (stationary & ergodic) setting in Section 1. In particular, assume that (1.2)-(1.4) hold. If $\mathbb{H}^I(\theta) = \mathbb{H}^U(\theta)$ for every $\theta \in \mathbb{R}$, then the HJ equation in (1.1) homogenizes to a HJ equation of the form in (1.6) as in the statement of Theorem 3.2. Moreover, the effective Hamiltonian $\mathbb{H}$ is given by (4.6).

**Proof.** It follows from (4.3) and our ergodicity assumption that the uniform Lipschitz constant of $u_\theta(\cdot, \cdot, \omega)$ is $\mathbb{P}$-essentially constant. Hence, the desired result is a special case of [7, Lemma 4.1].

For every $\lambda \in [\beta, +\infty)$ and $\omega \in \Omega$, consider the static HJ equation

$$
G(f'(x, \omega)) + \beta V(x, \omega) = \lambda, \quad x \in \mathbb{R}.
$$

Recall that a function $f = f(\cdot, \omega) \in \text{C}(\mathbb{R})$ is a subsolution of (4.7) if

$$
G(p) + \beta V(x, \omega) \leq \lambda \quad \text{for every } p \in D^+ f(x, \omega)
$$

and every $x \in \mathbb{R}$. Similarly, a function $f = f(\cdot, \omega) \in \text{C}(\mathbb{R})$ is a supersolution of (4.7) if

$$
G(p) + \beta V(x, \omega) \geq \lambda \quad \text{for every } p \in D^- f(x, \omega)
$$

and every $x \in \mathbb{R}$. Finally, a function $f = f(\cdot, \omega) \in \text{C}(\mathbb{R})$ is a solution of (4.7) if both of (4.8) and (4.9) are satisfied at every $x \in \mathbb{R}$. Here, $D^+ f(x, \omega)$ and $D^- f(x, \omega)$ denote the (Fréchet) superdifferential and subdifferential of $f$ at $x$, respectively. In particular, if $f$ is left and right differentiable at $x$, then

$$
D^+ f(x, \omega) = [f'_+(x, \omega), f'_-(x, \omega)] \quad \text{and} \quad D^- f(x, \omega) = [f'_-(x, \omega), f'_+(x, \omega)].
$$

There is an elementary connection between the static HJ equation in (4.7) and the evolutionary HJ equation in (1.1). Namely, if $f(\cdot, \omega) \in \text{C}(\mathbb{R})$ is a subsolution (resp. supersolution) of (4.7), then $u(\cdot, \cdot, \omega) \in \text{C}([0, +\infty) \times \mathbb{R})$, defined by

$$
u = -\lambda t + f(x, \omega),$$

is a subsolution (resp. supersolution) of (1.1). In Sections 5-7 we will construct subsolutions and supersolutions of (1.1) that are of this form. Moreover, whenever applicable (which will turn out to be outside the flat pieces of the graph of the effective Hamiltonian), we will refer to the following proposition to establish the almost sure limit in (4.5). It is a customized version of a classical result (see, e.g., [23, Theorem 4.1]).
Proposition 4.3. Assume \([1.2]\) and \([1.4]\). If there exist a constant \(\theta \in \mathbb{R}\), a function \(f : \mathbb{R} \times \Omega \to \mathbb{R}\) and an event \(\Omega_0 \in \mathcal{F}\) with \(\mathbb{P}(\Omega_0) = 1\) such that
\[
(4.10) \quad f'(\cdot, \omega), f'_+ (\cdot, \omega) \text{ exist and they are uniformly bounded on } \mathbb{R},
\]
then for every \(\omega \in \Omega_0\),
\[
\overline{\Pi}(\theta) = \overline{\Pi}^L (\theta) = \overline{\Pi}^U (\theta) = \lambda. \tag{4.7}
\]

Proof. Let \(\varphi(x) = \sqrt{1 + x^2}\). For every \(\epsilon > 0\) and \(\omega \in \Omega_0\), define \(v = v(\cdot, \cdot, \omega) \in \text{Lip}(\mathbb{R})\) by
\[
v(t, x, \omega) = -(\lambda + \epsilon)t + f(x, \omega) - \delta \varphi(x) - K,
\]
where \(\delta, K > 0\) are to be determined. We will show that \(v\) is a subsolution of \((4.1)\). To this end, fix any
\[
p \in [\partial_x^+ v(t, x, \omega), \partial_x^- v(t, x, \omega)] = [f'_+(x, \omega) - \delta \varphi'(x), f'_-(x, \omega) - \delta \varphi'(x)].
\]
If this interval is empty, then we are done.) Since
\[
p(x, \omega) \geq f(x, \omega) - \delta \varphi(x) - K = \theta x + o(x) - \delta \varphi(x) - K \leq \theta x = u_b(0, x, \omega)
\]
for every \(x \in \mathbb{R}\). By the comparison principle in Proposition 4.4 (see below),
\[
\overline{\Pi}^U (\theta) = \lim_{t \to +\infty} -\frac{u_b (t, 0, \omega)}{t} = \lim_{t \to +\infty} -\frac{v(t, 0, \omega)}{t} = \lambda + \epsilon.
\]
Similarly, for every \(\epsilon > 0\) and \(\omega \in \Omega_0\), there exist \(\delta, K > 0\) such that
\[
w(t, x, \omega) = -(\lambda - \epsilon)t + f(x, \omega) + \delta \varphi(x) + K
\]
defines a supersolution \(w = w(\cdot, \cdot, \omega) \in \text{Lip}(\mathbb{R})\) of \((4.1)\) that satisfies
\[
w(0, x, \omega) \geq \theta x = u_b(0, x, \omega)
\]
for every \(x \in \mathbb{R}\). By the comparison principle in Proposition 4.4 (see below),
\[
\overline{\Pi}^L (\theta) = \lim_{t \to +\infty} -\frac{u_b (t, 0, \omega)}{t} = \lim_{t \to +\infty} -\frac{w(t, 0, \omega)}{t} = \lambda - \epsilon.
\]
Since \(\epsilon > 0\) is arbitrary, the desired equalities follow. \(\square\)

Proposition 4.4. Assume \([1.2]\) and \([1.4]\). For any \(\omega \in \Omega\), if \(u_1(\cdot, \cdot, \omega), u_2(\cdot, \cdot, \omega) \in \text{Lip}(\mathbb{R})\) are, respectively, a subsolution and a supersolution of the HJ equation in \((4.1)\), then
\[
u_1(t, x, \omega) - u_2(t, x, \omega) \leq \sup \{u_1(0, y, \omega) - u_2(0, y, \omega) : y \in \mathbb{R}\} \quad \text{for every } (t, x) \in [0, +\infty) \times \mathbb{R}.
\]
Proof. This comparison principle follows from, e.g., the more general one in [10 Proposition A.2]. \(\square\)

We will use the following one-sided variant of Proposition 4.3 to obtain the flat pieces of the graph of the effective Hamiltonian.
Proposition 4.5. Assume \([1.2]\) and \([1.4]\). Suppose there exist constants \(\theta_-, \theta_+ \in \mathbb{R}\), a function 
\(f : \mathbb{R} \times \Omega \to \mathbb{R}\) and an event \(\Omega_0 \in \mathcal{F}\) with \(\mathbb{P}(\Omega_0) = 1\) such that \([4.10]\) holds,
\[
\lim_{x \to -\infty} \frac{f(x, \omega)}{x} = \theta_- \quad \text{and} \quad \lim_{x \to +\infty} \frac{f(x, \omega)}{x} = \theta_+
\]
for every \(\omega \in \Omega_0\).

(a) If \(\theta_+ < \theta_-\) and \(f(\cdot, \omega)\) is a subsolution of \([4.7]\) with some \(\lambda \in [\beta, +\infty)\) for every \(\omega \in \Omega_0\), then
\[
\overline{H}'(\theta) \leq \lambda \quad \text{for every} \quad \theta \in (\theta_+, \theta_-).
\]

(b) If \(\theta_- < \theta_+\) and \(f(\cdot, \omega)\) is a supersolution of \([4.7]\) with some \(\lambda \in [\beta, +\infty)\) for every \(\omega \in \Omega_0\), then
\[
\underline{H}'(\theta) \geq \lambda \quad \text{for every} \quad \theta \in (\theta_-, \theta_+).
\]

Proof. Under the conditions that apply to part (a), for any \(\theta \in (\theta_+, \theta_-)\), it is easy to check that
\[
v(t, x, \omega) = -\lambda t + f(x, \omega) - K\]
gives a subsolution of \([4.1]\) and \(v(0, x, \omega) \leq \theta x = u_\theta(0, x, \omega)\) when \(K = K(\theta, \omega)\) is sufficiently large. By the comparison principle in Proposition \([4.4]\)
\[
\overline{H}'(\theta) = \limsup_{t \to +\infty} \frac{-u_\theta(t, 0, \omega)}{t} \leq \lim_{t \to +\infty} \frac{-v(t, 0, \omega)}{t} = \lambda.
\]
The proof of part (b) is similar. \(\square\)

It is evident from the proofs above that the conditions in Propositions \([4.3]\) and \([4.5]\) are not optimal (e.g., \([4.10]\) can be replaced with \(f \in \text{Lip}(\mathbb{R})\)), but they are sufficient for our setting and purposes. Recall Subsection \([2.1]\) and note that, for any \(\theta \in \mathbb{R}\) and \(f : \mathbb{R} \times \Omega \to \mathbb{R}\) as in Proposition \([4.3]\)
\[
(x, \omega) \mapsto f(x, \omega) - \theta x
\]
is a sublinear corrector. However, in the rest of the paper, we will not use this terminology because we will directly construct and work with functions as in Propositions \([4.3]\) and \([4.5]\) rather than their sublinearized versions.

5. Case I: Weak potential \((\beta \leq m + \beta < M)\)

5.1. Strictly decreasing piece involving \(G_1\). In this subsection, we will simultaneously consider
Cases I, II and III of Theorem \([3.2]\).

For every \(\lambda \in [m + \beta, +\infty)\) and \(\omega \in \Omega\), construct \(f_1^\lambda(\cdot, \omega) \in C^1(\mathbb{R})\) by setting \(f_1^\lambda(0, \omega) = 0\) and
\[
( f_1^\lambda )'(x, \omega) = G_1^{-1}(\lambda - \beta V(x, \omega)) \in [G_1^{-1}(\lambda), G_1^{-1}(\lambda - \beta)] \subset (-\infty, p_m].
\]
\(f_1^\lambda(\cdot, \omega)\) is a (classical) solution of \([4.7]\). Moreover, the ergodic theorem ensures that, for \(\mathbb{P}\)-a.e. \(\omega\),
\[
\lim_{x \to \pm \infty} \frac{f_1^\lambda(x, \omega)}{x} = \lim_{x \to \pm \infty} \frac{1}{x} \int_0^x G_1^{-1}(\lambda - \beta V(y, \omega))dy = \mathbb{E} \left[ G_1^{-1}(\lambda - \beta V(0, \omega)) \right] = \theta_1(\lambda) \in (-\infty, p_m)
\]
with the definition in \([3.1]\). Therefore,
\[
\overline{H}(\theta_1(\lambda)) = \overline{H}'(\theta_1(\lambda)) = \overline{H}''(\theta_1(\lambda)) = \lambda
\]
by Proposition \([4.3]\). Finally, since \(\theta_1 : [m + \beta, +\infty) \to (-\infty, \theta_1(m + \beta)]\) is a strictly decreasing bijection, we deduce that \(\overline{H}\) is strictly decreasing on \((-\infty, \theta_1(m + \beta)]\).
5.2. **Strictly increasing piece involving** $G_2$. For every $\lambda \in [m + \beta, M]$ and $\omega \in \Omega$, construct $f_2^4(\cdot, \omega) \in C^1(\mathbb{R})$ by setting $f_2^4(0, \omega) = 0$ and

\begin{equation}
(f_2^4)'(x, \omega) = G_2^{-1}(\lambda - \beta V(x, \omega)) \in [G_2^{-1}(\lambda - \beta), G_2^{-1}(\lambda)] \subset [p_m, p_M].
\end{equation}

$f_2^4(\cdot, \omega)$ is a (classical) solution of (4.7). Moreover, the ergodic theorem ensures that, for $\mathbb{P}$-a.e. $\omega$,

\[
\lim_{x \to \pm \infty} \frac{f_2^4(x, \omega)}{x} = \lim_{x \to \pm \infty} \frac{1}{x} \int_0^x G_2^{-1}(\lambda - \beta V(y, \omega))dy = E\left[G_2^{-1}(\lambda - \beta V(0, \omega))\right] = \theta_2(\lambda) \in (p_m, p_M)
\]

with the definition in (3.1). Therefore,

\[
\overline{H}(\theta_2(\lambda)) = \overline{H}^{L}(\theta_2(\lambda)) = \overline{H}^{J}(\theta_2(\lambda)) = \lambda
\]

by Proposition 4.3. Finally, since $\theta_2 : [m + \beta, M] \to [\theta_2(m + \beta), \theta_2(M)]$ is a strictly increasing bijection, we deduce that $\overline{H}$ is strictly increasing on $[\theta_2(m + \beta), \theta_2(M)]$.

5.3. **Strictly decreasing piece involving** $G_3$. In this subsection, we will simultaneously consider Cases I and II of Theorem 3.2.

Assume that $\beta < M$. For every $\lambda \in [\beta, M]$ and $\omega \in \Omega$, construct $f_3^4(\cdot, \omega) \in C^1(\mathbb{R})$ by setting $f_3^4(0, \omega) = 0$ and

\begin{equation}
(f_3^4)'(x, \omega) = G_3^{-1}(\lambda - \beta V(x, \omega)) \in [G_3^{-1}(\lambda), G_3^{-1}(\lambda - \beta)] \subset [p_M, 0],
\end{equation}

$f_3^4(\cdot, \omega)$ is a (classical) solution of (4.7). Moreover, the ergodic theorem ensures that, for $\mathbb{P}$-a.e. $\omega$,

\[
\lim_{x \to \pm \infty} \frac{f_3^4(x, \omega)}{x} = \lim_{x \to \pm \infty} \frac{1}{x} \int_0^x G_3^{-1}(\lambda - \beta V(y, \omega))dy = E\left[G_3^{-1}(\lambda - \beta V(0, \omega))\right] = \theta_3(\lambda) \in (p_M, 0)
\]

with the definition in (3.1). Therefore,

\[
\overline{H}(\theta_3(\lambda)) = \overline{H}^{L}(\theta_3(\lambda)) = \overline{H}^{J}(\theta_3(\lambda)) = \lambda
\]

by Proposition 4.3. Finally, since $\theta_3 : [\beta, M] \to [\theta_3(M), \theta_3(\beta)]$ is a strictly decreasing bijection, we deduce that $\overline{H}$ is strictly decreasing on $[\theta_3(M), \theta_3(\beta)]$.

5.4. **Strictly increasing piece involving** $G_4$. In this subsection, we will simultaneously consider Cases I, II and III of Theorem 3.2.

For every $\lambda \in [\beta, +\infty)$ and $\omega \in \Omega$, construct $f_4^4(\cdot, \omega) \in C^1(\mathbb{R})$ by setting $f_4^4(0, \omega) = 0$ and

\begin{equation}
(f_4^4)'(x, \omega) = G_4^{-1}(\lambda - \beta V(x, \omega)) \in [G_4^{-1}(\lambda - \beta), G_4^{-1}(\lambda)] \subset [0, +\infty).
\end{equation}

$f_4^4(\cdot, \omega)$ is a (classical) solution of (4.7). Moreover, the ergodic theorem ensures that, for $\mathbb{P}$-a.e. $\omega$,

\[
\lim_{x \to \pm \infty} \frac{f_4^4(x, \omega)}{x} = \lim_{x \to \pm \infty} \frac{1}{x} \int_0^x G_4^{-1}(\lambda - \beta V(y, \omega))dy = E\left[G_4^{-1}(\lambda - \beta V(0, \omega))\right] = \theta_4(\lambda) \in (0, +\infty)
\]

with the definition in (3.1). Therefore,

\[
\overline{H}(\theta_4(\lambda)) = \overline{H}^{L}(\theta_4(\lambda)) = \overline{H}^{J}(\theta_4(\lambda)) = \lambda
\]

by Proposition 4.3. Finally, since $\theta_4 : [\beta, +\infty) \to [\theta_4(\beta), +\infty)$ is a strictly increasing bijection, we deduce that $\overline{H}$ is strictly increasing on $[\theta_4(\beta), +\infty)$. 
5.5. Flat piece at height $\beta$. In this subsection, we will simultaneously consider Cases I and II of Theorem 3.2.

Assume that $\beta < M$. For every $\epsilon \in (0, \beta)$ and $\mathbb{P}$-a.e. $\omega$, there exists a $y_0 = y_0(\omega) \in \mathbb{R}$ such that $\beta V(y_0, \omega) = \beta - \epsilon$. Construct $f_{43}^{\beta}(\cdot, \omega), f_{34}^{\beta}(\cdot, \omega) \in \text{Lip}(\mathbb{R})$ by setting $f_{43}^{\beta}(0, \omega) = f_{34}^{\beta}(0, \omega) = 0$, (1) \[ (f_{43}^{\beta})'(x, \omega) = \begin{cases} (f_{43}^{\beta})'(x, \omega) & \text{if } x < y_0, \\ (f_{34}^{\beta})'(x, \omega) & \text{if } x > y_0, \end{cases} \]
and (2) \[ (f_{34}^{\beta})'(x, \omega) = \begin{cases} (f_{34}^{\beta})'(x, \omega) & \text{if } x < y_0, \\ (f_{34}^{\beta})'(x, \omega) & \text{if } x > y_0, \end{cases} \]
with the notation in (5.3) and (5.4). Observe that, at every $x \in \mathbb{R} \setminus \{y_0\}$, the functions $f_{43}^{\beta}(\cdot, \omega)$ and $f_{34}^{\beta}(\cdot, \omega)$ satisfy both (4.8) and (4.9) with $\lambda = \beta$. In addition, (1) \[ (f_{43}^{\beta})'(y_0, \omega) = (f_{34}^{\beta})'(y_0, \omega) = (f_{34}^{\beta})'(y_0, \omega) = G_{3}^{-1}(\beta - \beta V(y_0, \omega)) = G_{3}^{-1}(\epsilon) < 0 \]
and (2) \[ (f_{34}^{\beta})'(y_0, \omega) = (f_{34}^{\beta})'(y_0, \omega) = (f_{34}^{\beta})'(y_0, \omega) = G_{4}^{-1}(\beta - \beta V(y_0, \omega)) = G_{4}^{-1}(\epsilon) > 0. \]
It is clear from Condition 3.1 that $\beta - \epsilon \leq G(p) + \beta V(y_0, \omega) = G(p) + \beta - \epsilon \leq \beta$
for every $p \in [G_{3}^{-1}(\epsilon), G_{4}^{-1}(\epsilon)]$, i.e., $f_{43}^{\beta}(\cdot, \omega)$ satisfies (4.8) at $x = y_0$ with $\lambda = \beta$ and $f_{34}^{\beta}(\cdot, \omega)$ satisfies (4.9) at $x = y_0$ with $\lambda = \beta - \epsilon$. We deduce that $f_{43}^{\beta}(\cdot, \omega)$ is a subsolution of (4.7) with $\lambda = \beta$ and $f_{34}^{\beta}(\cdot, \omega)$ is a supersolution of (4.7) with $\lambda = \beta - \epsilon$. Moreover,
\[ \lim_{x \to +\infty} \frac{f_{43}^{\beta}(x, \omega)}{x} = \lim_{x \to -\infty} \frac{f_{43}^{\beta}(x, \omega)}{x} = \theta(\beta) < 0 < \theta(\beta) = \lim_{x \to +\infty} \frac{f_{34}^{\beta}(x, \omega)}{x} = \lim_{x \to +\infty} \frac{f_{34}^{\beta}(x, \omega)}{x} \]
for $\mathbb{P}$-a.e. $\omega$. Therefore, $\Pi^U(\theta) \leq \beta$ and $\Pi^L(\theta) \geq \beta - \epsilon$ for every $\theta \in (\theta(\beta), \theta(\beta))$.

by Proposition 4.5. Since $\epsilon \in (0, \beta)$ is arbitrary, we conclude that $\Pi(\theta) = \Pi^L(\theta) = \Pi^U(\theta) = \beta$ for every $\theta \in (\theta(\beta), \theta(\beta))$.

5.6. Flat piece at height $m + \beta$. For every $\epsilon \in (0, \beta)$ and $\mathbb{P}$-a.e. $\omega$, there exists a $y_0 = y_0(\omega) \in \mathbb{R}$ such that $\beta V(y_0, \omega) = \beta - \epsilon$. Construct $f_{21}^{m+\beta}(\cdot, \omega), f_{12}^{m+\beta}(\cdot, \omega) \in \text{Lip}(\mathbb{R})$ by setting $f_{21}^{m+\beta}(0, \omega) = f_{12}^{m+\beta}(0, \omega) = 0$, (1) \[ (f_{21}^{m+\beta})'(x, \omega) = \begin{cases} (f_{21}^{m+\beta})'(x, \omega) & \text{if } x < y_0, \\ (f_{12}^{m+\beta})'(x, \omega) & \text{if } x > y_0, \end{cases} \]
and (2) \[ (f_{12}^{m+\beta})'(x, \omega) = \begin{cases} (f_{12}^{m+\beta})'(x, \omega) & \text{if } x < y_0, \\ (f_{21}^{m+\beta})'(x, \omega) & \text{if } x > y_0, \end{cases} \]
with the notation in (5.1) and (5.2). Observe that, at every $x \in \mathbb{R} \setminus \{y_0\}$, the functions $f_{21}^{m+\beta}(\cdot, \omega)$ and $f_{12}^{m+\beta}(\cdot, \omega)$ satisfy both (4.8) and (4.9) with $\lambda = m + \beta$. In addition, (1) \[ (f_{21}^{m+\beta})'(y_0, \omega) = (f_{12}^{m+\beta})'(y_0, \omega) = (f_{12}^{m+\beta})'(y_0, \omega) = G_{1}^{-1}(m + \beta - \beta V(y_0, \omega)) \]
and (2) \[ = G_{1}^{-1}(m + \epsilon) < p \] and (3) \[ (f_{21}^{m+\beta})'(y_0, \omega) = (f_{12}^{m+\beta})'(y_0, \omega) = (f_{12}^{m+\beta})'(y_0, \omega) = G_{2}^{-1}(m + \beta - \beta V(y_0, \omega)) \]
and (4) \[ = G_{2}^{-1}(m + \epsilon) > p \].

It is clear from Condition 3.1 that $m + \beta - \epsilon \leq G(p) + \beta V(y_0, \omega) = G(p) + \beta - \epsilon \leq m + \beta$
for every $p \in [G_{1}^{-1}(m + \epsilon), G_{2}^{-1}(m + \epsilon)]$, i.e., $f_{21}^{m+\beta}(\cdot, \omega)$ satisfies (4.8) at $x = y_0$ with $\lambda = m + \beta$ and $f_{12}^{m+\beta}(\cdot, \omega)$ satisfies (4.9) at $x = y_0$ with $\lambda = m + \beta - \epsilon$. We deduce that $f_{21}^{m+\beta}(\cdot, \omega)$ is a subsolution
of (4.7) with $\lambda = m + \beta$ and $f_{\lambda,2}^{m+\beta}(\cdot,\omega)$ is a supersolution of (4.7) with $\lambda = m + \beta - \epsilon$. Moreover,

$$\lim_{x \to +\infty} \frac{f_{\lambda,2}^{m+\beta}(x,\omega)}{x} = \lim_{x \to -\infty} \frac{f_{\lambda,2}^{m+\beta}(x,\omega)}{x} = \theta_1(m + \beta) < p_m$$

$$\lim_{x \to -\infty} \frac{f_{\lambda,2}^{m+\beta}(x,\omega)}{x} = \lim_{x \to +\infty} \frac{f_{\lambda,2}^{m+\beta}(x,\omega)}{x}$$

for $\mathbb{P}$-a.e. $\omega$. Therefore,

$$\overline{H}^L(\theta) \leq m + \beta \quad \text{and} \quad \overline{H}^U(\theta) \geq m + \beta - \epsilon \quad \text{for every} \ \theta \in (\theta_1(m + \beta), \theta_2(m + \beta))$$

by Proposition 4.5. Since $\epsilon \in (0, \beta)$ is arbitrary, we conclude that

$$\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta) = m + \beta \quad \text{for every} \ \theta \in (\theta_1(m + \beta), \theta_2(m + \beta)).$$

5.7. Flat piece at height $M$. For every $\epsilon \in (0, \beta)$ and $\mathbb{P}$-a.e. $\omega$, there exists an $x_0 = x_0(\omega) \in \mathbb{R}$ such that $\beta V(x_0, \omega) = \epsilon$. Construct $f_{M,3}^M(\cdot,\omega)$, $f_{M,3}^M(\cdot,\omega) \in \text{Lip}(\mathbb{R})$ by setting $f_{M,3}^M(0,\omega) = f_{M}^M(0,\omega) = 0$,

$$(f_{M,3}^M)'(x,\omega) = \begin{cases} (f_{M}^M)'(x,\omega) & \text{if } x < x_0, \\
(f_{M}^M)'(x,\omega) & \text{if } x > x_0 \end{cases}$$

and

$$(f_{M,3}^M)'(x,\omega) = \begin{cases} (f_{M}^M)'(x,\omega) & \text{if } x < x_0, \\
(f_{M}^M)'(x,\omega) & \text{if } x > x_0 \end{cases}$$

with the notation in [5.2] and [5.3]. Observe that, at every $x \in \mathbb{R} \setminus \{x_0\}$, the functions $f_{M,3}^M(\cdot,\omega)$ and $f_{M,3}^M(\cdot,\omega)$ satisfy both (4.8) and (4.9) with $\lambda = M$. In addition,

$$(f_{M,3}^M)'_+(x_0,\omega) = (f_{M,3}^M)'_+(x_0,\omega) = \frac{1}{\gamma}(x_0,\omega) = G_2^1(M - \beta V(x_0,\omega)) = G_2^1(M - \epsilon) < p_M$$

and

$$(f_{M,3}^M)'_-(x_0,\omega) = (f_{M,3}^M)'_+(x_0,\omega) = (f_{M}^M)'(x_0,\omega) = G_3^1(M - \beta V(x_0,\omega)) = G_3^1(M - \epsilon) > p_M.$$
6.2. Nonincreasing piece involving both $G_1$ and $G_3$ (excluding the easy subcase). In this subsection, we will simultaneously consider Cases II and III of Theorem 3.2 (except their easy subcases which are deferred to Subsections 6.6 and 7.4).

Assume that $\max\{\beta, M\} < m + \beta$. For every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$, define two bi-infinite sequences $(\mathcal{X}_i)_{i \in \mathbb{Z}} = (\mathcal{X}_i(\lambda, \omega))_{i \in \mathbb{Z}}$ and $(\mathcal{Y}_i)_{i \in \mathbb{Z}} = (\mathcal{Y}_i(\lambda, \omega))_{i \in \mathbb{Z}}$ by the coupled recursion

\[
\mathcal{X}_i(\lambda, \omega) = \inf\{x \geq \mathcal{Y}_{i-1}(\lambda, \omega) : \lambda - \beta V(x, \omega) \geq M\},
\]

\[
\mathcal{Y}_i(\lambda, \omega) = \inf\{x \geq \mathcal{X}_i(\lambda, \omega) : \lambda - \beta V(x, \omega) < m\}
\]

and the following anchoring condition:

\[
\mathcal{X}_{-1}(\lambda, \omega) \leq 0 < \mathcal{X}_0(\lambda, \omega).
\]

Similarly, define $(\mathcal{T}_i)_{i \in \mathbb{Z}} = (\mathcal{T}_i(\lambda, \omega))_{i \in \mathbb{Z}}$ and $(\mathcal{Z}_i)_{i \in \mathbb{Z}} = (\mathcal{Z}_i(\lambda, \omega))_{i \in \mathbb{Z}}$ by the coupled recursion

\[
\mathcal{T}_i(\lambda, \omega) = \inf\{x \geq \mathcal{Z}_{i-1}(\lambda, \omega) : \lambda - \beta V(x, \omega) > M\},
\]

\[
\mathcal{Z}_i(\lambda, \omega) = \inf\{x \geq \mathcal{T}_i(\lambda, \omega) : \lambda - \beta V(x, \omega) \leq m\}
\]

and the following anchoring condition:

\[
\mathcal{T}_{-1}(\lambda, \omega) < 0 \leq \mathcal{T}_0(\lambda, \omega).
\]

Note that the set

\[
\Omega_0 := \bigcap_{\lambda \in [\beta, +\infty) \cap (M, m + \beta)} \{\omega \in \Omega : \mathcal{X}_i(\lambda, \omega), \mathcal{Y}_i(\lambda, \omega), \mathcal{T}_i(\lambda, \omega), \mathcal{Z}_i(\lambda, \omega) \in (-\infty, +\infty) \text{ for every } i \in \mathbb{Z}\}
\]

satisfies $\mathbb{P}(\Omega_0) = 1$ by our ergodicity assumption. For every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$ and $\omega \in \Omega_0$, it is easy to see that

\[
\lambda - \beta V(x, \omega) \geq m \quad \text{if } x \in [\mathcal{X}_i, \mathcal{Y}_i) \quad \text{and}
\]

\[
\lambda - \beta V(x, \omega) < M \quad \text{if } x \in [\mathcal{Y}_{i+1}, \mathcal{X}_i).
\]

In particular, whenever $\lambda - \beta V(x, \omega) = M$, we know that $x \in [\mathcal{X}_i, \mathcal{Y}_i)$ for some $i \in \mathbb{Z}$. Consequently, $(\mathcal{X}_i)_{i \in \mathbb{Z}}$ and $(\mathcal{Y}_i)_{i \in \mathbb{Z}}$ are well-defined by (6.1)–(6.2). Similarly,

\[
\lambda - \beta V(x, \omega) > m \quad \text{if } x \in [\mathcal{T}_i, \mathcal{Z}_i) \quad \text{and}
\]

\[
\lambda - \beta V(x, \omega) \leq M \quad \text{if } x \in [\mathcal{Z}_{i+1}, \mathcal{T}_i).
\]

In particular, whenever $\lambda - \beta V(x, \omega) = m$, we know that $x \in [\mathcal{Y}_i, \mathcal{T}_{i+1})$ for some $i \in \mathbb{Z}$. Consequently, $(\mathcal{T}_i)_{i \in \mathbb{Z}}$ and $(\mathcal{Z}_i)_{i \in \mathbb{Z}}$ are well-defined by (6.3)–(6.4). See Figure 2.
For every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$ and $\omega \in \Omega_0$, construct $f^\lambda_{1,3}(\cdot, \omega), \overline{f}^\lambda_{1,3}(\cdot, \omega) \in \text{Lip}(\mathbb{R})$ by setting $f^\lambda_{1,3}(0, \omega) = \overline{f}^\lambda_{1,3}(0, \omega) = 0,$

\begin{equation}
(\underline{f}^\lambda_{1,3})'(x, \omega) = \begin{cases} 
G_1^{-1}(\lambda - \beta V(x, \omega)) \in (-\infty, p_m) & \text{if } x \in (\overline{x}_i, \overline{y}_i), \\
G_3^{-1}(\lambda - \beta V(x, \omega)) \in (p_M, 0) & \text{if } x \in (\overline{y}_i, \overline{x}_{i+1})
\end{cases}
\end{equation}

and

\begin{equation}
(\overline{f}^\lambda_{1,3})'(x, \omega) = \begin{cases} 
G_1^{-1}(\lambda - \beta V(x, \omega)) \in (-\infty, p_m) & \text{if } x \in (\underline{x}_i, \underline{y}_i), \\
G_3^{-1}(\lambda - \beta V(x, \omega)) \in [p_M, 0) & \text{if } x \in (\underline{y}_i, \underline{x}_{i+1}).
\end{cases}
\end{equation}

Observe that, at every $x \in \mathbb{R} \setminus \{\ldots, \underline{x}_{-1}, \underline{y}_0, \overline{y}_0, \underline{x}_1, \ldots\}$, the function $f^\lambda_{1,3}(\cdot, \omega)$ satisfies both (4.8) and (4.9). In addition,

\begin{equation}
(\underline{f}^\lambda_{1,3})'_\pm(x, \omega) = G_1^{-1}(M) < p_m < p_M = G_3^{-1}(M) = (\underline{f}^\lambda_{1,3})'_\pm(x, \omega)
\end{equation}

and

\begin{equation}
(\overline{f}^\lambda_{1,3})'_\pm(\underline{y}_i, \omega) = G_1^{-1}(m) = p_m < p_M = G_3^{-1}(m) = (\overline{f}^\lambda_{1,3})'_\pm(\overline{y}_i, \omega).
\end{equation}

It is clear from Condition 3.1 that

\[ G(p) + \beta V(x, \omega) \leq M + \beta V(x, \omega) = \lambda \]

for every $p \in [G_1^{-1}(M), G_3^{-1}(M)]$, i.e., $f^\lambda_{1,3}(\cdot, \omega)$ satisfies (4.8) at $x = \underline{x}_1$, and

\[ G(p) + \beta V(y, \omega) \geq m + \beta V(y, \omega) = \lambda \]

for every $p \in [G_1^{-1}(m), G_3^{-1}(m)]$, i.e., $f^\lambda_{1,3}(\cdot, \omega)$ satisfies (4.9) at $x = \overline{y}_i$. We deduce that $f^\lambda_{1,3}(\cdot, \omega)$ is a solution of (4.7). Similarly for $\overline{f}^\lambda_{1,3}(\cdot, \omega)$.

It follows easily from (1.3), (6.1), (6.3), (6.8) and (6.9) that

\begin{equation}
(\underline{f}^\lambda_{1,3})'_\pm(x, \omega) = (\underline{f}^\lambda_{1,3})'_\pm(0, \tau_x \omega) \quad \text{and} \quad (\overline{f}^\lambda_{1,3})'_\pm(x, \omega) = (\overline{f}^\lambda_{1,3})'_\pm(0, \tau_x \omega)
\end{equation}

for every $(x, \omega) \in \mathbb{R} \times \Omega_0$. By the ergodic theorem,

\begin{equation}
\theta_{1,3}(\lambda) := \lim_{x \to \pm \infty} \frac{\underline{f}^\lambda_{1,3}(x, \omega)}{x} = \frac{1}{x} \int_0^x (\underline{f}^\lambda_{1,3})'(y, \omega)dy = \mathbb{E} \left[(\underline{f}^\lambda_{1,3})'(0, \omega)\right]
\end{equation}

and

\begin{equation}
\overline{\theta}_{1,3}(\lambda) := \lim_{x \to \pm \infty} \frac{\overline{f}^\lambda_{1,3}(x, \omega)}{x} = \frac{1}{x} \int_0^x (\overline{f}^\lambda_{1,3})'(y, \omega)dy = \mathbb{E} \left[(\overline{f}^\lambda_{1,3})'(0, \omega)\right]
\end{equation}

for $\mathbb{P}$-a.e. $\omega$. Therefore,

\begin{equation}
\mathcal{H}(\theta_{1,3}(\lambda)) = \mathcal{H}(\theta_{1,3}(\lambda)) = \mathcal{H}(\theta_{1,3}(\lambda)) = \lambda \quad \text{and}
\end{equation}

\begin{equation}
\mathcal{H}(\overline{\theta}_{1,3}(\lambda)) = \mathcal{H}(\overline{\theta}_{1,3}(\lambda)) = \mathcal{H}(\overline{\theta}_{1,3}(\lambda)) = \lambda
\end{equation}

by Proposition 4.3.

Examining the definitions of $(\underline{x}_i(\lambda, \omega))_{i \in \mathbb{Z}}$, $(\underline{y}_i(\lambda, \omega))_{i \in \mathbb{Z}}$, $(\overline{y}_i(\lambda, \omega))_{i \in \mathbb{Z}}$ and $(\overline{y}_i(\lambda, \omega))_{i \in \mathbb{Z}}$ reveals that they are not interlaced (i.e., $\underline{x}_i(\lambda, \omega) \leq \overline{x}_i(\lambda, \omega) \leq \overline{y}_i(\lambda, \omega) \leq \underline{y}_i(\lambda, \omega) \leq \overline{x}_{i+1}(\lambda, \omega)$ does not hold) in general. (See Figure 2 for an example.) However, they satisfy the following weaker property.

**Lemma 6.1.** For every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$, $i \in \mathbb{Z}$ and $\omega \in \Omega_0$, there exist $j, k \in \mathbb{Z}$ such that

\[ [\overline{x}_i(\lambda, \omega), \overline{y}_i(\lambda, \omega)] \subset [\underline{x}_j(\lambda, \omega), \overline{y}_j(\lambda, \omega)] \quad \text{and} \quad [\overline{y}_i(\lambda, \omega), \overline{x}_{i+1}(\lambda, \omega)] \subset [\overline{y}_k(\lambda, \omega), \overline{x}_{k+1}(\lambda, \omega)]. \]

**Proof.** For every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$, $i \in \mathbb{Z}$ and $\omega \in \Omega_0$, $\lambda - \beta V(\overline{x}_i(\lambda, \omega), \omega) = M$. Therefore, $\overline{x}_i(\lambda, \omega) \in [\underline{x}_j(\lambda, \omega), \overline{y}_j(\lambda, \omega)]$ for some $j \in \mathbb{Z}$. Since $\lambda - \beta V(\overline{y}_j(\lambda, \omega), \omega) = m$, 

\[ \lambda - \beta V(\overline{y}_j(\lambda, \omega), \omega) = M. \]
we deduce that
\[ y_0(\lambda, \omega) = \inf \{ x \geq x_{i+1}(\lambda, \omega) : \lambda - \beta V(x, \omega) \leq m \} \leq x_{i+1}(\lambda, \omega). \]
This proves the first set inclusion. The second set inclusion is proved similarly. \hfill \Box

**Lemma 6.2.** \( \theta_{1,3}(\lambda) \leq \overline{\theta}_{1,3}(\lambda) \) for every \( \lambda \in [\beta, +\infty) \cap (M, m + \beta) \).

**Proof.** For every \( \lambda \in [\beta, +\infty) \cap (M, m + \beta) \) and \( \omega \in \Omega_0 \), on any interval of the form \((x_i(\lambda, \omega), \overline{x}(\lambda, \omega))\),
\[
(f_{1,3}^{\lambda})'(x, \omega) = G_1^{-1}(\lambda - \beta V(x, \omega)) \leq (\overline{f}_{1,3}^{\lambda})'(x, \omega).
\]
Moreover, on any interval of the form \((\overline{y}_i(\lambda, \omega), \overline{x}_{i+1}(\lambda, \omega))\),
\[
(f_{1,3}^{\lambda})'(x, \omega) = G_3^{-1}(\lambda - \beta V(x, \omega)) = (\overline{f}_{1,3}^{\lambda})'(x, \omega).
\]
The last equality follows from Lemma 6.1. In short, \((f_{1,3}^{\lambda})'(x, \omega) \leq (\overline{f}_{1,3}^{\lambda})'(x, \omega)\) for a.e. \(x \in \mathbb{R}\) (with respect to Lebesgue measure). The desired inequality is now evident from (6.10) and (6.11). \hfill \Box

**Lemma 6.3.** For every \( \lambda_1, \lambda_2 \in [\beta, +\infty) \cap (M, m + \beta) \), if \( \lambda_1 < \lambda_2 \), then
\[
\theta_{1,3}(\lambda_1) > \overline{\theta}_{1,3}(\lambda_2).
\]

**Proof.** For every \( \omega \in \Omega_0 \), on any interval of the form \((\overline{y}_i(\lambda_1, \omega), \overline{x}_{i+1}(\lambda_1, \omega))\),
\[
(f_{1,3}^{\lambda_1})'(x, \omega) = G_3^{-1}(\lambda_1 - \beta V(x, \omega)) > G_3^{-1}(\lambda_2 - \beta V(x, \omega)) \geq (\overline{f}_{1,3}^{\lambda_2})'(x, \omega).
\]
Similarly, on any interval of the form \((x_j(\lambda_2, \omega), y_j(\lambda_2, \omega))\),
\[
(f_{1,3}^{\lambda_2})'_x(x, \omega) = G_1^{-1}(\lambda_1 - \beta V(x, \omega)) > G_1^{-1}(\lambda_2 - \beta V(x, \omega)) = (\overline{f}_{1,3}^{\lambda_2})'_x(x, \omega).
\]
Since the closures of such intervals cover \(\mathbb{R}\) by Lemma 6.4 below, the desired inequality is evident from (6.10) and (6.11). \hfill \Box

**Lemma 6.4.** For every \( \lambda_1, \lambda_2 \in [\beta, +\infty) \cap (M, m + \beta) \) and \( \omega \in \Omega_0 \), if \( \lambda_1 < \lambda_2 \), then
\[
\bigcup_{i \in \mathbb{Z}} \overline{\theta}_i(\lambda_1, \omega), \overline{x}_{i+1}(\lambda_1, \omega) \bigcup \bigcup_{j \in \mathbb{Z}} x_j(\lambda_2, \omega), y_j(\lambda_2, \omega) = \mathbb{R}.
\]

**Proof.** For every \( j \in \mathbb{Z} \) and \( \omega \in \Omega_0 \),
\[
\lambda_1 - \beta V(y_j(\lambda_2, \omega), \omega) = (\lambda_1 - \lambda_2) + (\lambda_2 - \beta V(y_j(\lambda_2, \omega), \omega) = (\lambda_1 - \lambda_2) + m < m.
\]
Therefore, \( y_j(\lambda_2, \omega) \in \overline{\theta}_i(\lambda_1, \omega), \overline{x}_{i+1}(\lambda_1, \omega) \) for some \( i \in \mathbb{Z} \). Since
\[
\lambda_2 - \beta V(\overline{x}_{i+1}(\lambda_1, \omega), \omega) = (\lambda_2 - \lambda_1) + (\lambda_2 - \beta V(\overline{x}_{i+1}(\lambda_1, \omega), \omega) = (\lambda_2 - \lambda_1) + M > M,
\]
we deduce that
\[
\overline{x}_{i+1}(\lambda_2, \omega) = \inf \{ x \geq y_j(\lambda_2, \omega) : \lambda_2 - \beta V(x, \omega) > M \} < \overline{x}_{i+1}(\lambda_1, \omega).
\]
In short, for every \( j \in \mathbb{Z} \) and \( \omega \in \Omega_0 \), there exists an \( i \in \mathbb{Z} \) such that
\[
[y_j(\lambda_2, \omega), \overline{x}_{j+1}(\lambda_2, \omega)] \subset [\overline{y}_i(\lambda_1, \omega), \overline{x}_{i+1}(\lambda_1, \omega)],
\]
which readily implies the desired result. \hfill \Box

**Lemma 6.5.** The maps \( \lambda \mapsto \theta_{1,3}(\lambda) \) and \( \lambda \mapsto \overline{\theta}_{1,3}(\lambda) \) are right-continuous and left-continuous, respectively, on \( [\beta, +\infty) \cap (M, m + \beta) \).

**Proof.** It is easy to see from (6.1)–(6.2) that \( \lambda \mapsto \overline{x}_i(\lambda, \omega) \) and \( \lambda \mapsto \overline{y}_i(\lambda, \omega) \) are right-continuous for every \( i \in \mathbb{Z} \) and \( \omega \in \Omega_0 \). Hence, the right continuity of \( \lambda \mapsto \theta_{1,3}(\lambda) \) follows from (6.11) and (6.10). There is a similar argument for the left continuity of \( \lambda \mapsto \overline{\theta}_{1,3}(\lambda) \). \hfill \Box
Define the quantities
\[
\theta_{1,3}(\max\{\beta, M\}) := \bar{\theta}_{1,3}(\max\{\beta, M\}) = \bar{\theta}_{1,3}(\max\{\beta, M\}+) \quad \text{and}
\]
\[
\theta_{1,3}(m + \beta) := \bar{\theta}_{1,3}(m + \beta) = \bar{\theta}_{1,3}(m + \beta-).
\]

(6.13)

Note that the second equality in each line follows from Lemmas 6.2 and 6.3.

**Lemma 6.6.** The collection
\[
C = \{(\bar{\theta}_{1,3}(\lambda), \bar{\theta}_{1,3}(\lambda)) : \lambda \in (\max\{\beta, M\}, m + \beta)\}
\]
is a partition of \((\theta_{1,3}(m + \beta), \theta_{1,3}(\max\{\beta, M\}))\). In other words, for every \(\theta\) in this open interval, there is a unique \(\lambda(\theta) \in (\max\{\beta, M\}, m + \beta)\) such that \(\theta \in [\bar{\theta}_{1,3}(\lambda(\theta)), \bar{\theta}_{1,3}(\lambda(\theta))]\).

**Proof.** The intervals in the collection \(C\) are disjoint by Lemma 6.3. It remains to show that their union is \((\theta_{1,3}(m + \beta), \theta_{1,3}(\max\{\beta, M\}))\). To this end, fix an arbitrary \(\theta\) in this open interval and let
\[
(6.14) \quad \Lambda(\theta) = \inf\{\lambda \in (\max\{\beta, M\}, m + \beta) : \theta_{1,3}(\lambda) \leq \theta\}.
\]

It follows from Lemmas 6.2, 6.3 and 6.5 that
\[
\theta_{1,3}(\Lambda(\theta)) = \theta_{1,3}(\Lambda(\theta)+) \leq \theta \leq \theta_{1,3}(\Lambda(\theta)-) = \bar{\theta}_{1,3}(\Lambda(\theta)-) = \bar{\theta}_{1,3}(\Lambda(\theta)). \quad \square
\]

**Theorem 6.7.** Under the assumptions in Theorem 7.2 if \(\max\{\beta, M\} < m + \beta\), then
\[
\mathcal{H}(\theta) = \mathcal{H}^L(\theta) = \mathcal{H}^U(\theta) = \Lambda(\theta) \quad \text{for every } \theta \in (\theta_{1,3}(m + \beta), \theta_{1,3}(\max\{\beta, M\}))
\]
with the definitions in (6.13)–(6.14). Consequently, \(\mathcal{H}\) is nonincreasing on this interval.

**Proof.** Fix \(\theta \in (\theta_{1,3}(m + \beta), \theta_{1,3}(\max\{\beta, M\}))\). Note that \(\theta \in [\theta_{1,3}(\Lambda(\theta)), \bar{\theta}_{1,3}(\Lambda(\theta))]\) by Lemma 6.6. If \(\theta \in \{\theta_{1,3}(\Lambda(\theta)), \bar{\theta}_{1,3}(\Lambda(\theta))\}\), then the desired equalities follow from (6.12).

It remains to consider the case \(\theta \in \theta_{1,3}(\Lambda(\theta)), \bar{\theta}_{1,3}(\Lambda(\theta))) \neq \emptyset\). (We will characterize the nonemptiness of this interval in the proof of Theorem 7.5.) See Section 10.) For every \(\omega \in \Omega_0\), construct \(f^{A(\theta)}(\cdot, \omega), f^{A(\theta)}(\cdot, \omega) \in \text{Lip}(\mathbb{R})\) by setting \(f^{A(\theta)}(0, \omega) = f^{A(\theta)}(0, \omega) = 0\),
\[
(f^{A(\theta)}(x, \omega), (f^{A(\theta)}(x, \omega) \quad \text{if } x < \alpha_0(\Lambda(\theta), \omega),
\]
\[
(f^{A(\theta)}(x, \omega) \quad \text{if } x > \alpha_0(\Lambda(\theta), \omega)
\]
and
\[
(f^{A(\theta)}(x, \omega), (f^{A(\theta)}(x, \omega) \quad \text{if } x < \beta_0(\Lambda(\theta), \omega),
\]
\[
(f^{A(\theta)}(x, \omega) \quad \text{if } x > \beta_0(\Lambda(\theta), \omega).
\]

Repeating the analysis we carried out earlier for \(f^{A(\theta)}(\cdot, \omega), \) we deduce that \(f^{A(\theta)}(\cdot, \omega), f^{A(\theta)}(\cdot, \omega)\) are solutions of (4.7) with \(\lambda = \Lambda(\theta)\). Moreover,
\[
\lim_{x \to -\infty} \frac{f^{A(\theta)}(x, \omega)}{x} = \lim_{x \to +\infty} \frac{f^{A(\theta)}(x, \omega)}{x} = \theta_{1,3}(\Lambda(\theta))
\]
\[
< \bar{\theta}_{1,3}(\Lambda(\theta)) = \lim_{x \to -\infty} \frac{f^{A(\theta)}(x, \omega)}{x} = \lim_{x \to +\infty} \frac{f^{A(\theta)}(x, \omega)}{x}
\]
for \(P\)-a.e. \(\omega\) by (6.10)–(6.11). Therefore,
\[
(6.15) \quad \mathcal{H}^U(\theta) = \Lambda(\theta) \quad \text{and } \mathcal{H}^L(\theta) = \Lambda(\theta) \quad \text{for every } \theta \in (\theta_{1,3}(\Lambda(\theta)), \bar{\theta}_{1,3}(\Lambda(\theta)))
\]
by Proposition 4.5. \(\square\)

6.3. **Strictly decreasing piece involving** \(G_3\). We proved in Subsection 5.3 that, in Cases I and II, for every \(\lambda \in [\beta, M]\),
\[
\mathcal{H}(\theta_3(\lambda)) = \mathcal{H}^L(\theta_3(\lambda)) = \mathcal{H}^U(\theta_3(\lambda)) = \lambda
\]
with the definition in (3.1). We added that \(\mathcal{H}\) is strictly decreasing on \([\theta_3(M), \theta_3(\beta)]\).
6.4. Strictly increasing piece involving $G_4$. We proved in Subsection 5.4 that, in Cases I, II and III, for every $\lambda \in [\beta, +\infty)$,
\[
\overline{H}(\theta_4(\lambda)) = \overline{H}^L(\theta_4(\lambda)) = \overline{H}^U(\theta_4(\lambda)) = \lambda
\]
with the definition in (3.1). We added that $\overline{H}$ is strictly increasing on $[\theta_4(\beta), +\infty)$.

6.5. Flat piece at height $\beta$. We proved in Subsection 5.5 that, in Cases I and II,
\[
\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta) = \beta \quad \text{for every } \theta \in (\theta_3(\beta), \theta_4(\beta)).
\]

6.6. Flat piece at height $M = m + \beta$ (in the easy subcase). Assume that $\beta < M = m + \beta$. For every $\epsilon \in (0, \beta)$ and $\mathbb{P}$-a.e. $\omega$, there exist $x_0 = x_0(\omega) \in \mathbb{R}$ and $y_0 = y_0(\omega) \in \mathbb{R}$ such that $\beta V(x_0, \omega) = \epsilon$ and $\beta V(y_0, \omega) = \beta - \epsilon$. Construct $f_{3,1}^M(\cdot, \omega), f_{1,3}^M(\cdot, \omega) \in \text{Lip}(\mathbb{R})$ by setting $f_{3,1}^M(0, \omega) = f_{1,3}^M(0, \omega) = 0,$
\[
(f_{3,1}^M)'(x, \omega) = \begin{cases} (f_{3,1}^M)'(x, \omega) & \text{if } x < x_0, \\ (f_{3,1}^M)'(x, \omega) & \text{if } x > x_0 \end{cases} \quad \text{and} \quad (f_{1,3}^M)'(x, \omega) = \begin{cases} (f_{1,3}^M)'(x, \omega) & \text{if } x < y_0, \\ (f_{1,3}^M)'(x, \omega) & \text{if } x > y_0 \end{cases}
\]
with the notation in (5.1) and (5.3). Observe that, at every $x \in \mathbb{R} \setminus \{x_0\}$, the function $f_{3,1}^M(\cdot, \omega)$ satisfies both (4.8) and (4.9) with $\lambda = M$. In addition,
\[
(f_{3,1}^M)'_+(x_0, \omega) = (f_{1,3}^M)'_+(x_0, \omega) = G_1^{-1}(M - \beta V(x_0, \omega)) = G_1^{-1}(M - \epsilon) \leq p_m < p_M \quad \text{and} \quad (f_{3,1}^M)'_-(y_0, \omega) = (f_{1,3}^M)'_-(y_0, \omega) = G_3^{-1}(M - \beta V(x_0, \omega)) = G_3^{-1}(M - \epsilon) > p_M.
\]
It is clear from Condition 3.1 that
\[
G(p) + \beta V(x_0, \omega) = G(p) + \epsilon \leq M + \epsilon
\]
for every $p \in [G_1^{-1}(M - \epsilon), G_3^{-1}(M - \epsilon)]$, i.e., $f_{3,1}^M(\cdot, \omega)$ satisfies (4.8) at $x = x_0$ with $\lambda = M + \epsilon$. We deduce that $f_{3,1}^M(\cdot, \omega)$ is a subsolution of (4.7) with $\lambda = M + \epsilon$. Similarly, at every $x \in \mathbb{R} \setminus \{y_0\}$, the function $f_{1,3}^M(\cdot, \omega)$ satisfies both (4.8) and (4.9) with $\lambda = M$. In addition,
\[
(f_{3,1}^M)'_-(y_0, \omega) = (f_{1,3}^M)'_-(y_0, \omega) = G_1^{-1}(m + \beta - \beta V(y_0, \omega)) = G_1^{-1}(m + \epsilon) \leq p_m < p_M \quad \text{and} \quad (f_{3,1}^M)'_+(y_0, \omega) = (f_{1,3}^M)'_+(y_0, \omega) = G_3^{-1}(m + \beta - \beta V(y_0, \omega)) = G_3^{-1}(m + \epsilon) > p_M.
\]
It is clear from Condition 3.1 that
\[
G(p) + \beta V(y_0, \omega) = G(p) + \beta - \epsilon \geq m + \beta - \epsilon = M - \epsilon
\]
for every $p \in [G_1^{-1}(m + \epsilon), G_3^{-1}(m + \epsilon)]$, i.e., $f_{3,1}^M(\cdot, \omega)$ satisfies (4.9) at $x = y_0$ with $\lambda = M - \epsilon$. We deduce that $f_{1,3}^M(\cdot, \omega)$ is a supersolution of (4.7) with $\lambda = M - \epsilon$. Moreover,
\[
\lim_{x \to +\infty} \frac{f_{3,1}^M(x, \omega)}{x} = \lim_{x \to -\infty} \frac{f_{3,1}^M(x, \omega)}{x} = \theta_1(M) < p_m < p_M
\]
\[
< \theta_3(M) = \lim_{x \to -\infty} \frac{f_{1,3}^M(x, \omega)}{x} = \lim_{x \to +\infty} \frac{f_{1,3}^M(x, \omega)}{x}
\]
for $\mathbb{P}$-a.e. $\omega$. Therefore,
\[
\overline{H}^U(\theta) \leq M + \epsilon \quad \text{and} \quad \overline{H}^L(\theta) \geq M - \epsilon \quad \text{for every } \theta \in (\theta_1(M), \theta_3(M))
\]
by Proposition 4.5. Since $\epsilon \in (0, \beta)$ is arbitrary, we conclude that
\[
\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta) = M \quad \text{for every } \theta \in (\theta_1(M), \theta_3(M)).
\]
6.7. Possible flat piece at height $M$ (excluding the easy subcase). Assume that $\beta < M < m + \beta$. Fix an arbitrary $\epsilon \in (0, m + \beta - M)$. Recall $\bar{\omega} = \bar{\omega}(M + \epsilon, \omega)$ and $\bar{y}_0 = \bar{y}_0(M + \epsilon, \omega)$ from Subsection 6.2. Note that

$$M + \epsilon - \beta V(\bar{\omega}, \omega) = M \quad \text{and} \quad M + \epsilon - \beta V(\bar{y}_0, \omega) = m$$

for $\mathbb{P}$-a.e. $\omega$. Construct $f_{3,1,3}^M(\cdot, \omega), f_{1,3,3}^M(\cdot, \omega) \in \text{Lip}(\mathbb{R})$ by setting $f_{3,1,3}^M(0, \omega) = f_{1,3,3}^M(0, \omega) = 0$, 

$$(f_{3,1,3}^M)'(x, \omega) = \begin{cases} (f_{3,1,3}^M)'(x, \omega) & \text{if } x < \bar{\omega}, \\ (f_{1,3,3}^M)'(x, \omega) & \text{if } x > \bar{\omega}, \end{cases}$$

with the notation in (6.8). Observe that, at every $x \in \mathbb{R} \setminus \{ \bar{\omega}, \bar{y}_0 \}$, the function $f_{3,1,3}^M(\cdot, \omega)$ satisfies (4.8) with $\lambda = M + \epsilon$. In addition,

$$(f_{3,1,3}^M)'(\bar{\omega}, \omega) = (f_{1,3,3}^M)'(\bar{\omega}, \omega) = G^{-1}_1(M + \epsilon - \beta V(\bar{\omega}, \omega)) = G^{-1}_1(M) < p_m < p_M \quad \text{and}$$

$$(f_{3,1,3}^M)'(\bar{y}_0, \omega) = (f_{1,3,3}^M)'(\bar{y}_0, \omega) = G^{-1}_3(M - \beta V(\bar{y}_0, \omega)) = G^{-1}_3(M - \epsilon) > p_M.$$
for \( \mathbb{P}\)-a.e. \( \omega \). Construct \( f^{m+\beta}_{1,1,3}(\cdot,\omega), f^{m+\beta}_{1,1,3}(\cdot,\omega) \in \text{Lip}(\mathbb{R}) \) by setting \( f^{m+\beta}_{1,1,3}(0,\omega), f^{m+\beta}_{1,1,3}(0,\omega) = 0 \),

\[
(f^{m+\beta}_{1,1,3})'(x,\omega) = \begin{cases} \frac{f^{m+\beta}_{1,1,3}}{f^{m+\beta}_{1,1,3}}(x,\omega) & \text{if } x < \bar{x}_0, \\
\frac{f^{m+\beta}_{1,1,3}}{f^{m+\beta}_{1,1,3}}(x,\omega) & \text{if } x > \bar{x}_0 \end{cases}
\]

with the notation in (5.1) and (6.9). Observe that, at every \( x \in \mathbb{R} \setminus \{\bar{x}_0\} \), the function \( f^{m+\beta}_{1,1,3}(\cdot,\omega) \) satisfies (4.8) with \( \lambda = m + \beta \). In addition,

\[
(f^{m+\beta}_{1,1,3})'_-(\bar{x}_0,\omega) = (f^{m+\beta}_{1,1,3})'_-(\bar{x}_0,\omega) = G_1^{-1}(m + \beta - \beta V(\bar{x}_0,\omega)) = G_1^{-1}(M + \epsilon) < p_m < p_M \quad \text{and} \quad (f^{m+\beta}_{1,1,3})'_+(\bar{x}_0,\omega) = G_3^{-1}(m + \beta - \epsilon - \beta V(\bar{x}_0,\omega)) = G_3^{-1}(M) = p_M.
\]

It is clear from Condition 3.1 that

\[
G(p) + \beta V(\bar{x}_0,\omega) = G(p) + m + \beta - \epsilon - M \leq m + \beta
\]

for every \( p \in [G_1^{-1}(M + \epsilon), G_3^{-1}(M)] \), i.e., \( f^{m+\beta}_{1,1,3}(\cdot,\omega) \) satisfies (4.8) at \( x = \bar{x}_0 \) with \( \lambda = m + \beta \). We deduce that \( f^{m+\beta}_{1,1,3}(\cdot,\omega) \) is a subsolution of (4.7) with \( \lambda = m + \beta \). Similarly, at every \( x \in \mathbb{R} \setminus \{y_0\} \), the function \( f^{m+\beta}_{1,1,3}(\cdot,\omega) \) satisfies (4.9) with \( \lambda = m + \beta - \epsilon \). In addition,

\[
(f^{m+\beta}_{1,1,3})'_+(y_0,\omega) = (f^{m+\beta}_{1,1,3})'_+(y_0,\omega) = G_1^{-1}(m + \beta - \beta V(y_0,\omega)) = G_1^{-1}(m + \beta - \epsilon - \beta V(y_0,\omega)) = G_3^{-1}(m) > p_M.
\]

It is clear from Condition 3.1 that

\[
G(p) + \beta V(y_0,\omega) = G(p) + \beta - \epsilon \geq m + \beta - \epsilon
\]

for every \( p \in [G_1^{-1}(m + \epsilon), G_3^{-1}(m)] \), i.e., \( f^{m+\beta}_{1,1,3}(\cdot,\omega) \) satisfies (4.9) at \( x = y_0 \) with \( \lambda = m + \beta - \epsilon \). We deduce that \( f^{m+\beta}_{1,1,3}(\cdot,\omega) \) is a supersolution of (4.7) with \( \lambda = m + \beta - \epsilon \). Moreover,

\[
\lim_{x \to +\infty} \frac{f^{m+\beta}_{1,1,3}(x,\omega)}{x} = \lim_{x \to -\infty} \frac{f^{m+\beta}_{1,1,3}(x,\omega)}{x} = \theta_1(m + \beta)
\]

for \( \mathbb{P}\)-a.e. \( \omega \). Therefore,

\[
\bar{H}^U(\theta) \leq m + \beta \quad \text{and} \quad \bar{H}^L(\theta) \geq m + \beta - \epsilon \quad \text{for every } \theta \in (\theta_1(m + \beta), \bar{\theta}_{1,3}(m + \beta - \epsilon))
\]

by Proposition 4.5. Since \( \epsilon \in (0, m + \beta - \max\{\beta, M\}) \) is arbitrary, we recall (6.13) and conclude that

\[
\bar{H}(\theta) = \bar{H}^L(\theta) = \bar{H}^U(\theta) = m + \beta \quad \text{for every } \theta \in (\theta_1(m + \beta), \bar{\theta}_{1,3}(m + \beta))
\]

We hereby showed that the graph of \( \bar{H} \) has a flat piece at height \( m + \beta \) if and only if \( \theta_1(m + \beta) < \theta_{1,3}(m + \beta) \). We will characterize this strict inequality in the proof of Theorem 3.4 (see Section 9).

7. Case III: Strong potential \( (M \leq \beta \leq m + \beta) \)

7.1. Strictly decreasing piece involving \( G_1 \). We proved in Subsection 5.1 that, in Cases I, II and III, for every \( \lambda \in [m + \beta, +\infty) \),

\[
\bar{H}(\theta_1(\lambda)) = \bar{H}^L(\theta_1(\lambda)) = \bar{H}^U(\theta_1(\lambda)) = \lambda
\]

with the definition in (3.1). We added that \( \bar{H} \) is strictly decreasing on \((-\infty, \theta_1(m + \beta)) \).
We proved in Subsection 6.2 that, in Cases II and III (except their easy subcases),
\[ H(\theta) = H_L(\theta) = H_U(\theta) = \Lambda(\theta) \quad \text{for every } \theta \in (\theta_{1,3}(m + \beta), \theta_{1,3}(\max\{\beta, M\})). \]
We added that \( H \) is nonincreasing on this interval. See Theorem 6.7.

### 7.3. Strictly increasing piece involving \( G_4 \)
We proved in Subsection 5.4 that, in Cases I, II and III, for every \( \lambda \in [\beta, +\infty) \),
\[ H(\theta(\lambda)) = H_L(\theta(\lambda)) = H_U(\theta(\lambda)) = \lambda \]
with the definition in (3.1). We added that \( H \) is strictly increasing on \( [\theta_4(\beta), +\infty) \).

### 7.4. Flat piece at height \( \beta = m + \beta \) (in the easy subcase).
Assume that \( M \leq \beta = m + \beta \). For every \( \epsilon \in (0, \beta) \) and \( \mathbb{P} \)-a.e. \( \omega \), there exist \( x_0 = x_0(\omega) \in \mathbb{R} \) and \( y_0 = y_0(\omega) \in \mathbb{R} \) such that \( \beta V(x_0, \omega) = \epsilon \) and \( \beta V(y_0, \omega) = \beta - \epsilon \). Construct \( f^\beta_{4,1}(\cdot, \omega), f^\beta_{4,1}(\cdot, \omega) \in \mathrm{Lip}(\mathbb{R}) \) by setting \( f^\beta_{4,1}(0, \omega) = f^\beta_{1,4}(0, \omega) = 0 \),
\[ (f^\beta_{4,1})'(x, \omega) = \begin{cases} (f^\beta_{4,1})'(x, \omega) & \text{if } x < x_0, \\ (f^\beta_{4,1})'(x, \omega) & \text{if } x > x_0 \end{cases} \quad \text{and} \quad (f^\beta_{4,1})(x, \omega) = \begin{cases} (f^\beta_{4,1})(x, \omega) & \text{if } x < y_0, \\ (f^\beta_{4,1})(x, \omega) & \text{if } x > y_0 \end{cases} \]
with the notation in (5.1) and (5.4). Observe that, at every \( x \in \mathbb{R} \setminus \{x_0\} \), the function \( f^\beta_{4,1}(\cdot, \omega) \) satisfies both (4.1) and (4.9) with \( \lambda = \beta \). In addition,
\[ (f^\beta_{4,1})_+(x_0, \omega) = (f^\beta_{4,1})_+(x_0, \omega) = G_{1}^{-1}(\beta - \beta V(x_0, \omega)) = G_{1}^{-1}(\beta - \epsilon) < p_m < 0 \quad \text{and} \quad (f^\beta_{4,1})_-(x_0, \omega) = (f^\beta_{4,1})_-(x_0, \omega) = G_{4}^{-1}(\beta - \beta V(x_0, \omega)) = G_{4}^{-1}(\beta - \epsilon) > 0. \]

It is clear from Condition 3.1 that
\[ G(p) + \beta V(x_0, \omega) = G(p) + \epsilon \leq \max\{\beta - \epsilon, M\} + \epsilon \leq \beta + \epsilon \]
for every \( p \in [G_1^{-1}(\beta - \epsilon), G_4^{-1}(\beta - \epsilon)] \), i.e., \( f^\beta_{4,1}(\cdot, \omega) \) satisfies (4.7) at \( x = x_0 \) with \( \lambda = \beta + \epsilon \). We deduce that \( f^\beta_{4,1}(\cdot, \omega) \) is a subsolution of (4.7) with \( \lambda = \beta + \epsilon \). Similarly, at every \( x \in \mathbb{R} \setminus \{y_0\} \), the function \( f^\beta_{4,1}(\cdot, \omega) \) satisfies both (4.8) and (4.9) with \( \lambda = \beta \). In addition,
\[ (f^\beta_{4,1})_-(y_0, \omega) = (f^\beta_{4,1})_-(y_0, \omega) = G_{1}^{-1}(\beta - \beta V(y_0, \omega)) = G_{1}^{-1}(\epsilon) < p_m < 0 \quad \text{and} \quad (f^\beta_{4,1})_+(y_0, \omega) = (f^\beta_{4,1})_+(y_0, \omega) = G_{4}^{-1}(\beta - \beta V(y_0, \omega)) = G_{4}^{-1}(\epsilon) > 0. \]

It is clear from Condition 3.1 that
\[ G(p) + \beta V(y_0, \omega) = G(p) + \beta - \epsilon \geq \beta - \epsilon \]
for every \( p \in [G_1^{-1}(\epsilon), G_4^{-1}(\epsilon)] \), i.e., \( f^\beta_{4,1}(\cdot, \omega) \) satisfies (4.9) at \( x = y_0 \) with \( \lambda = \beta - \epsilon \). We deduce that \( f^\beta_{4,1}(\cdot, \omega) \) is a supersolution of (4.7) with \( \lambda = \beta - \epsilon \). Moreover,
\[ \lim_{x \to +\infty} \frac{f^\beta_{4,1}(x, \omega)}{x} = \lim_{x \to -\infty} \frac{f^\beta_{4,1}(x, \omega)}{x} = \theta_1(\beta) < p_m < 0 \]
\[ < \theta_4(\beta) = \lim_{x \to -\infty} \frac{f^\beta_{4,1}(x, \omega)}{x} = \lim_{x \to +\infty} \frac{f^\beta_{4,1}(x, \omega)}{x} \]
for \( \mathbb{P} \)-a.e. \( \omega \). Therefore,
\[ H_U(\theta) \leq \beta + \epsilon \quad \text{and} \quad H_L(\theta) \geq \beta - \epsilon \quad \text{for every } \theta \in (\theta_1(\beta), \theta_4(\beta)) \]
by Proposition 4.3. Since \( \epsilon \in (0, \beta) \) is arbitrary, we conclude that
\[ H(\theta) = H_L(\theta) = H_U(\theta) = \beta \quad \text{for every } \theta \in (\theta_1(\beta), \theta_4(\beta)). \]
7.5. Flat piece at height $\beta$ (excluding the easy subcase). Assume that $M \leq \beta < m + \beta$. For every $\epsilon \in (0, m/2)$ and $\mathbb{P}$-a.e. $\omega$, there exists a $z_0 = z_0(\omega) \in \mathbb{R}$ such that $\beta V(z_0, \omega) = \beta - \epsilon$. Construct $f_{4,1,3}^\beta(\cdot, \omega), f_{1,3,4}^\beta(\cdot, \omega) \in \text{Lip}(\mathbb{R})$ by setting $f_{4,1,3}^\beta(0, \omega) = f_{1,3,4}^\beta(0, \omega) = 0$,

$$(f_{4,1,3}^\beta(x, \omega), f_{1,3,4}^\beta(x, \omega)) = \begin{cases} (f_{1,3}^\beta(x, \omega), f_{1,3}^\beta(x, \omega)) & \text{if } x < 0, \\ (f_{1,3}^\beta(x, \omega), f_{1,3}^\beta(x, \omega)) & \text{if } x > z_0, \\
(2f_{1,3}^\beta(x, \omega), f_{1,3}^\beta(x, \omega)) & \text{if } x > 0,
\end{cases}$$

with the notation in (5.4) and (5.8). Observe that, at every $x \in \mathbb{R} \setminus \{z_0\}$, the function $f_{4,1,3}^\beta(\cdot, \omega)$ (resp. $f_{1,3,4}^\beta(\cdot, \omega)$) satisfies (4.8) (resp. (4.9)) with $\lambda = \beta + \epsilon$ (resp. $\lambda = \beta$). In addition,

$$(f_{4,1,3}^\beta(0, \omega), f_{1,3,4}^\beta(0, \omega)) = (f_{4,1,3}^\beta(z_0, \omega), f_{1,3,4}^\beta(z_0, \omega)) = (f_{3,1}^\beta(x, \omega) + \epsilon G_3^{-1}(\beta + \epsilon - \beta V(z_0, \omega)), 0) < (2\epsilon, 0)$$

and $$(f_{4,1,3}^\beta(z_0, \omega), f_{1,3,4}^\beta(z_0, \omega)) = (f_{4,1,3}^\beta(z_0, \omega), f_{1,3,4}^\beta(z_0, \omega)) = (f_{3,1}^\beta(x, \omega) + \epsilon G_3^{-1}(\beta - \beta V(z_0, \omega)), 0) > (2\epsilon, 0).$$

The third equality in the first line of this display follows from the observation that $z_0(\omega) \in \{\bar{\gamma}_i(\beta + \epsilon, \omega), x_{i+1}(\beta + \epsilon, \omega)\}$ for some $i \in \mathbb{Z}$ by (6.6) since $\beta + \epsilon - \beta V(z_0, \omega) = 2\epsilon < m$. It is clear from Condition 3.1 that

$$\beta - \epsilon \leq G(p) + \beta V(z_0, \omega) = G(p) + \epsilon \leq \beta + \epsilon$$

for every $p \in [G_3^{-1}(2\epsilon), G_4^{-1}(1\epsilon)]$, i.e., $f_{4,1,3}^\beta(\cdot, \omega)$ satisfies (4.8) at $x = z_0$ with $\lambda = \beta + \epsilon$ and $f_{1,3,4}^\beta(\cdot, \omega)$ satisfies (4.9) at $x = z_0$ with $\lambda = \beta - \epsilon$. We deduce that $f_{4,1,3}^\beta(\cdot, \omega)$ is a subsolution of (4.7) with $\lambda = \beta + \epsilon$ and $f_{1,3,4}^\beta(\cdot, \omega)$ is a supersolution of (4.7) with $\lambda = \beta - \epsilon$. Moreover,

$$\lim_{x \to -\infty} \frac{f_{4,1,3}^\beta(x, \omega)}{x} = \lim_{x \to -\infty} \frac{f_{1,3,4}^\beta(x, \omega)}{x} = \frac{\theta_3(\beta + \epsilon)}{\theta_4(\beta)} < 0$$

and $\lim_{x \to -\infty} \frac{f_{4,1,3}^\beta(x, \omega)}{x} = \lim_{x \to +\infty} \frac{f_{1,3,4}^\beta(x, \omega)}{x}$ for $\mathbb{P}$-a.e. $\omega$. Therefore,

$$\mathbb{H}^{\text{L}}(\theta) \leq \beta + \epsilon \quad \text{and} \quad \mathbb{H}^{\text{L}}(\theta) \geq \beta - \epsilon$$

for every $\theta \in (\theta_4(\beta), \theta_4(\beta))$ by Proposition 4.3. Since $\epsilon \in (0, m/2)$ is arbitrary, we recall (6.13) and conclude that $\mathbb{H}(\theta) = \mathbb{H}^{\text{L}}(\theta) = \mathbb{H}^{\text{U}}(\theta) = \beta$ for every $\theta \in (\theta_4(\beta), \theta_4(\beta))$.

7.6. Possible flat piece at height $m + \beta$ (excluding the easy subcase). We proved in Subsection 6.8 that, in Cases II and III (except their easy subcases),

$$\mathbb{H}(\theta) = \mathbb{H}^{\text{L}}(\theta) = \mathbb{H}^{\text{U}}(\theta) = m + \beta$$

for every $\theta \in (\theta_1(m + \beta), \theta_3(m + \beta)]$. In words, the graph of $\mathbb{H}$ has a flat piece at height $m + \beta$ if and only if $\theta_1(m + \beta) < \theta_3(m + \beta)$. We will characterize this strict inequality in the proof of Theorem 3.4 (see Section 9).

8. Completing the proof of homogenization

**Definition 8.1.** Recall (3.1), (6.10), (6.11), (6.13) and (6.14).

(a) When $\beta < M = m + \beta$ (i.e., the easy subcase of medium potential), let

$$\bar{\Lambda}(\theta) = M \text{ on } (\theta_1(M), \theta_3(M)).$$

(b) When $\beta < M < m + \beta$ (i.e., medium potential, excluding the easy subcase above), let

$$\bar{\Lambda}(\theta) = \begin{cases} m + \beta \text{ on } (\theta_1(m + \beta), \theta_3(m + \beta)] \quad \text{(possible flat piece)}, \\
\Lambda(\theta) \text{ on } (\theta_3(m + \beta), \theta_3(M)) \quad \text{(nonincreasing)}, \\
M \text{ on } [\theta_3(M), \theta_3(M)] \quad \text{(possible flat piece)}.
\end{cases}$$
(c) When $M \leq \beta = m + \beta$ (i.e., the easy subcase of strong potential), let

$\overline{X}(\theta) = \beta$ on $(\theta_1(\beta), \theta_4(\beta))$.

(d) When $M \leq \beta < m + \beta$ (i.e., strong potential, excluding the easy subcase above), let

$\overline{X}(\theta) = \begin{cases} 
  m + \beta \text{ on } (\theta_1(m + \beta), \theta_1,3(m + \beta)] & \text{(possible flat piece)}, \\
  \Lambda(\theta) \text{ on } (\theta_1,3(m + \beta), \theta_1,3(\beta)) & \text{(nonincreasing)}, \\
  \beta \text{ on } [\theta_1,3(\beta), \theta_4(\beta)] & \text{(flat piece)}.
\end{cases}$

Proof of Theorem 3.2. Under the assumptions in Theorem 3.2 we showed in Sections 5, 6 and 7 that, in Cases I, II and III, respectively,

$\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta)$

for every $\theta \in \mathbb{R}$. Moreover, in each case, we verified that $\overline{H}$ has the piecewise description in the statement of Theorem 3.2 which, in Cases II and III, involves the function $\overline{X}$ in Definition 8.1. Therefore, the desired conclusions follow from Theorem 4.2. \hfill \Box

9. Flat pieces of the graph of $\overline{X}$ at its extreme values

We break down the proof of Theorem 3.4 into several lemmas involving the quantity

$q_1 = \mathbb{P}(V(x, \omega) = 1 \text{ for some } x \in [0, 1])$.

Lemma 9.1. Assume that $\max\{\beta, M\} < m + \beta$. Recall the bi-infinite sequence defined by (6.3) – (6.4). If $q_1 > 0$, then for every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$, there exist $K, \delta > 0$ such that the event

$A_{\lambda', K, \delta} = \{0 \leq y_i'(\lambda', \omega) < y_i(\lambda', \omega) + \delta \leq \pi_{i+1}(\lambda', \omega) \leq K \text{ for some } i \in \mathbb{Z}\}$

satisfies $\mathbb{P}(A_{\lambda', K, \delta}) \geq \frac{1}{2}$ whenever $\lambda' \in (\lambda, m + \beta)$.

Proof. For every $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$, our ergodicity assumption ensures that

$\mathbb{P}(\lambda - \beta V(x, \omega) > M \text{ for some } x \geq 0) = 1$.

Similarly, if $q_1 > 0$, then $\mathbb{P}(V(x, \omega) = 1 \text{ for some } x \geq 0) = 1$. Therefore, $\exists K' > 0$ such that

$\mathbb{P}(\lambda - \beta V(x, \omega) > M \text{ for some } x \in [0, K']) \geq \frac{7}{8}$ and $\mathbb{P}(V(x, \omega) = 1 \text{ for some } x \in [0, K']) \geq \frac{7}{8}$.

Consider the event

$B_{\lambda, K'} = \{\exists z_1, z_2, z_3 \in \mathbb{R} \text{ s.t. } 0 \leq z_1 \leq K' \leq z_2 \leq 2K' \leq z_3 \leq 3K', \lambda - \beta V(z_1, \omega) > M,$

$V(z_2, \omega) = 1 \text{ and } \lambda - \beta V(z_3, \omega) > M\}.$

It follows from (1.3) and a union bound that

$\mathbb{P}((B_{\lambda, K'})^c) \leq (1 - \frac{7}{8}) + (1 - \frac{7}{8}) + (1 - \frac{7}{8}) = \frac{3}{8}$.

The continuity of the potential implies the existence of a $\delta \in (0, K')$ such that the event

$B_{\lambda, K', \delta} = \{\exists z_1, z_2, z_3 \in \mathbb{R} \text{ s.t. } 0 \leq z_1 \leq K' \leq z_2 \leq 2K' \leq z_3 \leq 3K', \lambda - \beta V(z_1, \omega) > M,$

$V(z_2, \omega) = 1, m + \beta - \beta V(z, \omega) > M \text{ for all } z \in [z_2, z_2 + \delta] \text{ and } \lambda - \beta V(z_3, \omega) > M\}$

satisfies

$\mathbb{P}(B_{\lambda, K', \delta} \geq \mathbb{P}(B_{\lambda, K'}) - \frac{1}{8} \geq \frac{1}{2}$. 
Whenever \( \lambda' \in (\lambda, m + \beta) \) and \( K \geq 3K' \),

\[
B_{\lambda, K', \delta} = \{ \exists z_1, z_2, z_3 \in \mathbb{R} \text{ s.t. } 0 \leq z_1 < K' < z_2 < 2K' < z_3 < 3K', \ \lambda' - \beta V(z_1, \omega) > m, \ \lambda' - \beta V(z_2, \omega) < m, \ \lambda' - \beta V(z_3, \omega) < M \text{ for all } z \in [z_2, z_2 + \delta] \text{ and } \lambda' - \beta V(z_3, \omega) > M \}
\]

\[
\subset \{ \exists z_1, z_2, z_3 \in \mathbb{R} \text{ and } i \in \mathbb{Z} \text{ s.t. } 0 \leq z_1 < y_i(\lambda', \omega) < z_2 < z_2 + \delta < \varpi_{i+1}(\lambda', \omega) < z_3 < 3K' \}
\subset A_{\lambda', K, \delta}
\]

by (6.7). We conclude that \( \mathbb{P}(A_{\lambda', K, \delta}) \geq \mathbb{P}(B_{\lambda, K', \delta}) \geq \frac{1}{2} \).

**Lemma 9.2.** Assume that \( \max\{\beta, M\} < m + \beta \). Recall the definitions in (3.1) and (6.13). If \( q_1 > 0 \), then \( \theta_1(m + \beta) < \theta_{1,3}(m + \beta) \).

**Proof.** Fix a \( \lambda \in [\beta, +\infty) \cap (M, m + \beta) \). If \( q_1 > 0 \), then the ergodic theorem and Lemma 9.1 ensure the existence of \( K, \delta > 0 \) such that

\[
\lim_{L \to +\infty} \frac{1}{L} \int_0^L \mathbb{I}_{\{x, \omega \in A_{\lambda', K, \delta}\}}(x, \omega) dx = \mathbb{P}(A_{\lambda', K, \delta}) \geq \frac{1}{2}
\]

for \( \mathbb{P}\text{-a.e. } \omega \) and every \( \lambda' \in (\lambda, m + \beta) \). Let

\[
I_{\lambda', \delta} = \{ i \in \mathbb{Z} : y_i(\lambda', \omega) + \delta \leq \varpi_{i+1}(\lambda', \omega) \}
\]

and observe that

\[
\int_0^L \mathbb{I}_{\{x, \omega \in A_{\lambda', K, \delta}\}}(x, \omega) dx = \int_0^L \mathbb{I}_{\{x, \omega \in A_{\lambda', K, \delta}\}}(x, \omega) dx \leq \sum_{i \in I_{\lambda', \delta}} \int_0^L \mathbb{I}_{\{x, \omega \in A_{\lambda', K, \delta}\}}(x, \omega) dx \leq (K - \delta) \# \{ i \in I_{\lambda', \delta} : y_i(\lambda', \omega) \leq \varpi_{i+1}(\lambda', \omega) \leq L + K \}.
\]

Recall (5.1) and (6.9). For every \( \lambda' \in (\lambda, m + \beta) \), \( x \in \mathbb{R} \) and \( \mathbb{P}\text{-a.e. } \omega \),

\[
(f_1^{m+\beta})'(x, \omega) = G_1^{-1}(m + \beta - \beta V(x, \omega)) < G_1^{-1}(\lambda' - \beta V(x, \omega)) \leq (\overline{f}_{1,3})'(x, \omega).
\]

Moreover, on any interval of the form \( [y_i(\lambda', \omega), \varpi_{i+1}(\lambda', \omega)] \),

\[
(f_1^{m+\beta})'(x, \omega) = G_1^{-1}(m + \beta - \beta V(x, \omega)) \leq p_m < p_M \leq G_3^{-1}(\lambda' - \beta V(x, \omega)) = (\overline{f}_{1,3})'(x, \omega).
\]

Therefore,

\[
\overline{f}_{1,3}(\lambda') - \theta_1(m + \beta) = \lim_{L \to +\infty} \frac{1}{L} \left( \int_0^{L+K} (\overline{f}_{1,3})'(x, \omega) dx - \int_0^{L+K} (f_1^{m+\beta})'(x, \omega) dx \right) \geq \lim_{L \to +\infty} \frac{(p_M - p_m)\delta}{L} \# \{ i \in I_{\lambda', \delta} : 0 \leq y_i(\lambda', \omega) \leq \varpi_{i+1}(\lambda', \omega) \leq L + K \} \geq \frac{(p_M - p_m)\delta}{(K - \delta)L} \int_0^L \mathbb{I}_{\{x, \omega \in A_{\lambda', K, \delta}\}}(x, \omega) dx \geq \frac{(p_M - p_m)\delta}{2K} > 0.
\]

by (9.1) and (9.2). We send \( \lambda' \uparrow m + \beta \) and conclude that

\[
\theta_{1,3}(m + \beta) - \theta_1(m + \beta) \geq \frac{(p_M - p_m)\delta}{2K} > 0.
\]

Here is the converse of Lemma 9.2.

**Lemma 9.3.** Assume that \( \max\{\beta, M\} < m + \beta \). If \( q_1 = 0 \), then \( \theta_1(m + \beta) = \theta_{1,3}(m + \beta) \).
Proof. Fix a $\lambda \in [\beta, +\infty) \cap (M, m + \beta)$. For every $\eta > 0$, the event
\begin{equation}
A_{\lambda, K} = \{ 0 \leq \overline{y}_j(\lambda, \omega) \leq \underline{z}_{j+1}(\lambda, \omega) \leq K \text{ for some } j \in \mathbb{Z} \}
\end{equation}
satisfies
\begin{equation}
P(A_{\lambda, K}) > 1 - \eta
\end{equation}
for sufficiently large $K > 0$. If $q_1 = 0$, then
\begin{equation}
P(\lambda' - \beta V(y, \omega) \leq m \text{ for some } y \in [-2K, 2K]) < \eta
\end{equation}
by continuity when $\lambda' \in (\lambda, m + \beta)$ is sufficiently large.

Recall (5.1) and (6.9). Observe that, on any interval of the form $(\pi_i(\lambda', \omega), \pi_{i+1}(\lambda', \omega))$,
\begin{equation}
(\overline{f}_{i,3}^\lambda)'(x, \omega) - (f_{1}^{m+\beta})'(x, \omega) \leq \max\{G^{-1}_i(\lambda' - \alpha) - G^{-1}_i(m + \beta - \alpha) : 0 \leq \alpha \leq \beta\} < \eta
\end{equation}
when $\lambda' \in (\lambda, m + \beta)$ is sufficiently large. Similarly, on any interval of the form $(\pi_i(\lambda', \omega), \pi_{i+1}(\lambda', \omega))$,
\begin{equation}
(\overline{f}_{i,3}^\lambda)'(x, \omega) - (f_{1}^{m+\beta})'(x, \omega) \leq G^{-1}_3(\lambda' - \beta) - G^{-1}_3(m + \beta) \leq G^{-1}_3(m) - G^{-1}(m + \beta) + \eta
\end{equation}
when $\lambda' \in (\lambda, m + \beta)$ is sufficiently large. If we let $C = G^{-1}_3(m) - G^{-1}(m + \beta)$, then
\begin{equation}
0 \leq \theta_{1,3}(m + \beta) - \theta_1(m + \beta) \leq \overline{y}_{1,3}(\lambda') - \theta_1(m + \beta)
\end{equation}

\begin{align*}
&= \lim_{L \to +\infty} 1 \frac{1}{L} \left( \int_{0}^{L} (\overline{f}_{i,3}^\lambda)'(x, \omega)dx \right) - \int_{0}^{L} (f_{1}^{m+\beta})'(x, \omega)dx \\
&\leq \eta + \lim_{L \to +\infty} C \frac{1}{L} \int_{0}^{L} \#(y(\lambda', \omega) \leq \pi_{i+1}(\lambda', \omega) \text{ for some } i \in \mathbb{Z}) \, dx
\end{align*}

\begin{equation}
\leq \eta + \lim_{L \to +\infty} \sum_{j \in J_{\lambda, \lambda'}} \frac{C}{L} \int_{0}^{L} \#(y_j(\lambda, \omega) \leq \pi_{i+1}(\lambda, \omega) \text{ for some } j \in J_{\lambda, \lambda'}) \, dx
\end{equation}

\begin{align*}
&= \eta + \lim_{L \to +\infty} \sum_{j \in J_{\lambda, \lambda'} \cap N_{2K}} \frac{C}{L} \int_{0}^{L} \#(y_j(\lambda, \omega) \leq \pi_{i+1}(\lambda, \omega) \text{ for some } j \in J_{\lambda, \lambda'} \cap N_{2K}) \, dx
\end{align*}

Here, (9.5) follows from Lemma 6.4 with $\lambda_1 = \lambda$ and $\lambda_2 = \lambda'$,

\[ J_{\lambda, \lambda'} = \{ j \in \mathbb{Z} : \lambda' - \beta V(y, \omega) \leq m \text{ for some } y \in [\overline{y}_j(\lambda, \omega), \underline{z}_{j+1}(\lambda, \omega)] \} \]

and

\[ N_{2K} = \{ j \in \mathbb{Z} : \underline{z}_{j+1}(\lambda, \omega) - \overline{y}_j(\lambda, \omega) \leq 2K \}. \]

The limit in (9.6) is controlled as follows:

\begin{align*}
&\lim_{L \to +\infty} \sum_{j \in J_{\lambda, \lambda'} \cap N_{2K}} \frac{C}{L} \int_{0}^{L} \#(y_j(\lambda, \omega) \leq \pi_{i+1}(\lambda, \omega) \text{ for some } j \in J_{\lambda, \lambda'} \cap N_{2K}) \, dx \\
&= \lim_{L \to +\infty} C \frac{1}{L} \int_{0}^{L} \#(y_j(\lambda, \omega) \leq \pi_{i+1}(\lambda, \omega) \text{ for some } j \in J_{\lambda, \lambda'} \cap N_{2K}) \, dx
\end{align*}

\begin{align*}
&\leq \lim_{L \to +\infty} C \frac{1}{L} \int_{0}^{L} \#(\lambda' - \beta V(y, \omega) \leq m \text{ for some } y \in [x-2K, x+2K]) \, dx
\end{align*}

\begin{equation}
= C \mathbb{P}(\lambda' - \beta V(y, \omega) \leq m \text{ for some } y \in [-2K, 2K]) < C \eta
\end{equation}
by (9.4) and the ergodic theorem. To control the limit in (9.7), observe that
\[
\int_0^L \mathbb{I}_{\{\tau_x \in (A_{\lambda,K})^c\}} \, dx = \int_0^L \mathbb{I}_{\{\text{There is no } j \in \mathbb{Z} \text{ such that } x \leq \bar{\eta}_j(\lambda,\omega) \leq \bar{\eta}_{j+1}(\lambda,\omega) \leq x + K\}} \, dx
\]
\[
\geq \sum_{j \in (N_{2K})^c} \frac{1}{2} \mathbb{I}_{\{\bar{\eta}_j(\lambda,\omega) \leq \xi_{j+1}(\lambda,\omega) \cap [0,L] \geq 2K, \xi_j(\lambda,\omega) \leq L - 2K\}} \times \mathbb{I}_{\{\xi_{j+1}(\lambda,\omega) \geq 2K, \xi_j(\lambda,\omega) \leq L - 2K\}}
\]
\[
\geq \frac{1}{2} \int_{2K}^{L-2K} \mathbb{I}_{\{\bar{\eta}_j(\lambda,\omega) \leq x \leq \xi_{j+1}(\lambda,\omega) \text{ for some } j \in (N_{2K})^c\}} \, dx
\]
\[
\geq \frac{1}{2} \int_0^L \mathbb{I}_{\{\bar{\eta}_j(\lambda,\omega) \leq x \leq \xi_{j+1}(\lambda,\omega) \text{ for some } j \in (N_{2K})^c\}} \, dx - 2K
\]
for \( L \geq 4K \). Therefore,
\[
\lim_{L \to +\infty} \sum_{j \in (N_{2K})^c} \frac{C}{L} \int_0^L \mathbb{I}_{\{\bar{\eta}_j(\lambda,\omega) \leq x \leq \xi_{j+1}(\lambda,\omega)\}} \, dx \leq \lim_{L \to +\infty} \frac{2C}{L} \int_0^L \mathbb{I}_{\{\tau_x \in (A_{\lambda,K})^c\}} \, dx
\]
\[
= 2CP((A_{\lambda,K})^c) < 2C\eta
\]
by (9.3) and the ergodic theorem. We plug these bounds in (9.6)–(9.7) and deduce that
\[0 \leq \theta_{1,3}(m + \beta) - \theta_1(m + \beta) \leq (3C + 1)\eta.\]
Since \( \eta > 0 \) is arbitrary, we conclude that \( \theta_1(m + \beta) = \theta_{1,3}(m + \beta) \). \( \square \)

**Proof of Theorem 3.4**: We treat the four parts of the theorem in order.

(a) When \( \beta < M = m + \beta \), it follows readily from Theorem 3.2 and Definition 8.1(a) that
\[ \overline{H}(\theta) = M \text{ on } [\theta_1(M), \theta_3(M)]. \]

(b) When \( \beta < M < m + \beta \), it follows from Theorem 3.2 and Definition 8.1(b) that the graph of \( \overline{H} \) has flat pieces at heights \( M \) and \( m + \beta \) iff
\[ \theta_{1,3}(M) < \theta_3(M) \quad \text{and} \quad \theta_1(m + \beta) < \theta_{1,3}(m + \beta), \]
respectively. The second inequality in (9.8) is equivalent to \( q_1 > 0 \) by Lemmas 9.2 and 9.3 which is in turn equivalent to (3.6) by ergodicity. This proves item (ii). Similarly, the proof of item (i) reduces to showing that the first inequality in (9.8) is equivalent to
\[ q_0 = P(V(x,\omega) = 0 \text{ for some } x \in [0,1]) > 0. \]
This is a routine adaptation of Lemmas 9.1–9.3 We leave the details to the reader.

(c) When \( M \leq \beta = m + \beta \), it follows readily from Theorem 3.2 and Definition 8.1(c) that
\[ \overline{H}(\theta) = \beta \text{ on } [\theta_1(\beta), \theta_4(\beta)]. \]

(d) When \( M \leq \beta < m + \beta \), it follows from Theorem 3.2 and Definition 8.1(d) that the graph of \( \overline{H} \) has flat pieces at heights \( \beta \) and \( m + \beta \) iff
\[ \theta_{1,3}(\beta) < \theta_4(\beta) \quad \text{and} \quad \theta_1(m + \beta) < \theta_{1,3}(m + \beta), \]
respectively. The second inequality in (9.9) is equivalent to (3.6) precisely as in part (b). This proves item (ii). The proof of item (i) is equivalent to showing that the first inequality in (9.9) always holds. Indeed,
\[ \theta_{1,3}(\beta) \leq \theta_3(\beta) < 0 < \theta_4(\beta). \] \( \square \)
10. Flat pieces of the graph of $\mathbf{K}$ at its intermediate values

We break down the proof of Theorem 3.5 into several lemmas.

**Lemma 10.1.** Assume that $\max\{\beta, M\} < m + \beta$. For any $\lambda \in \{\max\{\beta, M\}, m + \beta\}$, if

$$P(\exists j \in \mathbb{Z} \text{ s.t. } x_0(\lambda, \omega) = \pi_j(\lambda, \omega)) = 1,$$

then $P(\forall i \in \mathbb{Z} \exists j \in \mathbb{Z} \text{ s.t. } x_i(\lambda, \omega) = \pi_j(\lambda, \omega)) = 1$. Similarly, if

$$P(\exists k \in \mathbb{Z} \text{ s.t. } y_0(\lambda, \omega) = \eta_k(\lambda, \omega)) = 1,$$

then $P(\forall i \in \mathbb{Z} \exists k \in \mathbb{Z} \text{ s.t. } y_i(\lambda, \omega) = \eta_k(\lambda, \omega)) = 1$.

**Proof.** Fix any $\lambda \in (\max\{\beta, M\}, m + \beta)$. For every $z \in \mathbb{R}$ and $\omega \in \Omega_0$ (defined in (6.5)), let

$$\hat{x}_1(\lambda, \omega, z) = \sup\{x \leq z : \lambda - \beta V(x, \omega) \geq M\} \leq z,$$

$$\hat{x}_2(\lambda, \omega, z) = \sup\{x \leq \hat{x}_1(\lambda, \omega, z) : \lambda - \beta V(x, \omega) < m\} < \hat{x}_1(\lambda, \omega, z),$$

$$\hat{x}_3(\lambda, \omega, z) = \inf\{x \geq \hat{x}_2(\lambda, \omega, z) : \lambda - V(x, \omega) \geq M\} \in (\hat{x}_2(\lambda, \omega, z), \hat{x}_1(\lambda, \omega, z)],$$

$$\hat{x}_4(\lambda, \omega, z) = \inf\{x \geq \hat{x}_3(\lambda, \omega, z) : \lambda - \beta V(x, \omega) < m\} > \hat{x}_1(\lambda, \omega, z),$$

$$\hat{x}_5(\lambda, \omega, z) = \inf\{x \geq \hat{x}_4(\lambda, \omega, z) : \lambda - \beta V(x, \omega) \geq M\} > z \quad \text{and}$$

$$\hat{x}_6(\lambda, \omega, z) = \inf\{x \geq \hat{x}_5(\lambda, \omega, z) : \lambda - \beta V(x, \omega) \geq M\} \geq \hat{x}_5(\lambda, \omega, z).$$

The inequalities and inclusions that are stated above regarding these quantities can be easily checked by considering the cases $\lambda - \beta V(z, \omega) < M$ and $\lambda - \beta V(z, \omega) \geq M$. Note also that they enjoy the following property which is due to stationarity:

$$\hat{x}_\ell(\lambda, \omega, z) = \hat{x}_\ell(\lambda, \tau_z \omega, 0) + z \quad \text{for every } \ell \in \{1, 2, 3, 4, 5, 6\} \text{ and } z \in \mathbb{R}.$$  

It follows from the coupled recursion in (6.1) and the anchoring condition in (6.2) that

$$\bar{x}_{-1}(\lambda, \omega) = \hat{x}_3(\lambda, \omega, 0), \quad \bar{y}_{-1}(\lambda, \omega) = \hat{x}_4(\lambda, \omega, 0) \quad \text{and} \quad \bar{x}_0(\lambda, \omega) = \hat{x}_5(\lambda, \omega, 0).$$

The last equality generalizes: For every $i \in \mathbb{Z}$ and $z \in \mathbb{R},$

$$B_{\lambda, i, z} := \{\omega \in \Omega_0 : \bar{x}_i(\lambda, \omega) = \hat{x}_5(\lambda, \omega, z)\} = \{\omega \in \Omega_0 : \bar{x}_{i-1}(\lambda, \omega) \leq z < \bar{x}_i(\lambda, \omega)\}.$$  

It is clear from the second set representation of $B_{\lambda, i, z}$ that

$$P(\bigcup_{z \in \mathbb{Q}} B_{\lambda, i, z}) = 1 \quad \text{for every } i \in \mathbb{Z}.$$  

Recall from Lemma 6.1 that, for every $i \in \mathbb{Z}$ and $\omega \in \Omega_0$, there exists a $j \in \mathbb{Z}$ such that

$$y_{i-1}(\lambda, \omega) < \bar{x}_i(\lambda, \omega) \leq \bar{x}_j(\lambda, \omega).$$

It follows easily that

$$B_{\lambda, i, z} \cap \{\exists j \in \mathbb{Z} \text{ s.t. } \bar{x}_i(\lambda, \omega) = \bar{x}_j(\lambda, \omega)\} = \{\omega \in \Omega_0 : \bar{x}_5(\lambda, \omega, z) = \hat{x}_6(\lambda, \omega, z)\}$$

for every $i \in \mathbb{Z}$ and $z \in \mathbb{R}$.

Finally, suppose (10.1) is true. For every $z \in \mathbb{R}$, define

$$\hat{\Omega}_z = \{\omega \in \Omega_0 : \hat{x}_5(\lambda, \tau_z \omega, 0) = \hat{x}_6(\lambda, \tau_z \omega, 0)\} = \{\omega \in \Omega_0 : \hat{x}_5(\lambda, \tau_z \omega, 0) = \hat{x}_6(\lambda, \tau_z \omega, 0)\},$$

where the second equality is due to (10.3). Since $P$ is invariant under $\tau_z$ and $P(B_{\lambda, 0, 0}) = 1$ by (6.2), we use the set equality in (10.6) to deduce that

$$P(\hat{\Omega}_z) = P(\hat{x}_5(\lambda, \tau_z \omega, 0) = \hat{x}_6(\lambda, \tau_z \omega, 0)) = P(\hat{x}_5(\lambda, \omega, 0) = \hat{x}_6(\lambda, \omega, 0))$$

$$= P(B_{\lambda, 0, 0} \cap \{\bar{x}_5(\lambda, \omega, 0) = \hat{x}_6(\lambda, \omega, 0)\}) = P(B_{\lambda, 0, 0} \cap \{\exists j \in \mathbb{Z} \text{ s.t. } \bar{x}_0(\lambda, \omega) = \bar{x}_j(\lambda, \omega)\})$$

$$= P(\exists j \in \mathbb{Z} \text{ s.t. } \bar{x}_0(\lambda, \omega) = \bar{x}_j(\lambda, \omega)) = 1.$$
Performing two separate inductions on the index sets

\[ \exists s \in \{0, 1\} \text{ s.t. } (\bar{x}_i(\lambda, \omega))_{i \in \mathbb{Z}} = (x_{i+s}(\lambda, \omega))_{i \in \mathbb{Z}} \text{ and } (\bar{y}_i(\lambda, \omega))_{i \in \mathbb{Z}} = (y_{i+s}(\lambda, \omega))_{i \in \mathbb{Z}) = 1. \]

Moreover,

\[
s = s(\omega) = \begin{cases} 0 & \text{if } \bar{x}_0(\lambda, \omega) > 0, \\ 1 & \text{if } \bar{x}_0(\lambda, \omega) = 0. \end{cases}
\]

Proof. Fix any \( \lambda \in \{\max \{\beta, M\}, m + \beta\} \). Recall from Lemma 6.1 that, for every \( i \in \mathbb{Z} \), there exists a \( j \in \mathbb{Z} \) such that

\[ \bar{x}_j(\lambda, \omega) \leq \bar{x}_i(\lambda, \omega) < y_i(\lambda, \omega) \leq y_j(\lambda, \omega). \]

If (10.7) holds, then, for \( \mathbb{P}\text{-a.e. } \omega \), there exist \( k, \ell \in \mathbb{Z} \) such that

\[ \bar{x}_j(\lambda, \omega) = \bar{x}_k(\lambda, \omega) \leq \bar{x}_\ell(\lambda, \omega) < y_i(\lambda, \omega) = y_{\ell}(\lambda, \omega) \leq y_j(\lambda, \omega). \]

It follows that \( k = i \). (Otherwise, for \( \mathbb{P}\text{-a.e. } \omega \), there would exist an \( m \in \mathbb{Z} \) such that

\[ \bar{x}_j(\lambda, \omega) = \bar{x}_k(\lambda, \omega) \leq \bar{x}_m(\lambda, \omega) < \bar{x}_i(\lambda, \omega) < y_j(\lambda, \omega), \]

which would be a contradiction.) Moreover, \( \ell = j \) since \( \bar{x}_j(\lambda, \omega) < y_i(\lambda, \omega) \leq y_j(\lambda, \omega) \). Therefore,

\[ \bar{x}_j(\lambda, \omega) = \bar{x}_i(\lambda, \omega) < y_i(\lambda, \omega) = y_j(\lambda, \omega). \]

Similarly, for every \( i \in \mathbb{Z} \) and \( \mathbb{P}\text{-a.e. } \omega \), there exists a \( k \in \mathbb{Z} \) such that

\[ y_k(\lambda, \omega) = y_i(\lambda, \omega) < y_{i+1}(\lambda, \omega) = \bar{x}_{i+1}(\lambda, \omega). \]

Performing two separate inductions on the index sets \{\ldots, -2, -1, 0\} and \{0, 1, 2\ldots\} (and using (10.9) and (10.10) at each induction step), we easily see that (10.8) is true, except that it remains to determine the possible values of \( s = s(\omega) \) defined there. To this end, recall (6.2) and (6.4). In particular, note that \( \bar{x}_0(\lambda, \omega) \geq 0 \). We have the following dichotomy:

- If \( \bar{x}_0(\lambda, \omega) > 0 \), then \( \bar{x}_{-s}(\lambda, \omega) = \bar{x}_0(\lambda, \omega) > 0 \) and \( s \leq 0 \). Moreover, \( 0 < \bar{x}_0(\lambda, \omega) = \bar{x}_s(\lambda, \omega) \) and \( s \geq 0 \). We deduce that \( s = 0 \).
- If \( \bar{x}_0(\lambda, \omega) = 0 \), then \( \bar{x}_{-s}(\lambda, \omega) = \bar{x}_0(\lambda, \omega) = 0 \) and \( s = 1 \).
Proof. For any $\lambda \in (\max\{\beta, M\}, m + \beta)$, if (10.8) holds, then the functions $f_{1,3}^\lambda(\cdot, \omega)$ and $\tilde{f}_{1,3}^\lambda(\cdot, \omega)$ (satisfying (6.8) and (6.9)) are identically equal on $\mathbb{R}$ for $\mathbb{P}$-a.e. $\omega$. The equality of $\theta_{1,3}(\lambda)$ and $\tilde{\theta}_{1,3}(\lambda)$ now follows from their definitions in (6.10) and (6.11). □

Next, we prove the converse of what we have shown in Lemmas 10.1, 10.2 and 10.3 (all combined).

Lemma 10.4. Assume that $\max\{\beta, M\} < m + \beta$. For any $\lambda \in (\max\{\beta, M\}, m + \beta)$, if

- $\mathbb{P}(\exists j \in \mathbb{Z} \text{ s.t. } x_0(\lambda, \omega) = \pi_j(\lambda, \omega)) < 1$ or
- $\mathbb{P}(\exists k \in \mathbb{Z} \text{ s.t. } y_0(\lambda, \omega) = \bar{y}_k(\lambda, \omega)) < 1$,

equivalently

- $\mathbb{P}(\inf\{x \geq x_0(\lambda, \omega) : \lambda - \beta V(x, \omega) > M\} = x_0(\lambda, \omega)) < 1$ or
- $\mathbb{P}(\inf\{y \geq y_0(\lambda, \omega) : \lambda - \beta V(x, \omega) < m\} = y_0(\lambda, \omega)) < 1$,

then $\theta_{1,3}(\lambda) < \tilde{\theta}_{1,3}(\lambda)$.

Proof. Fix any $\lambda \in (\max\{\beta, M\}, m + \beta)$. Recall the proof of Lemma 10.1 and note that

$\exists j \in \mathbb{Z} \text{ s.t. } x_0(\lambda, \omega) = \pi_j(\lambda, \omega) \iff \hat{x}_5(\lambda, \omega, 0) = \hat{x}_6(\lambda, \omega, 0)$

$\iff \inf\{x \geq x_0(\lambda, \omega) : \lambda - \beta V(x, \omega) > M\} = x_0(\lambda, \omega)$

for every $\omega \in \Omega_0$. Hence, (10.11) and (10.13) are equivalent. Similarly for (10.12) and (10.14).

Add the following definition to the list in (10.2):

$\hat{x}_7(\lambda, \omega, z) = \inf\{x \geq \hat{x}_5(\lambda, \omega, z) : \lambda - \beta V(x, \omega) < m\} > \hat{x}_5(\lambda, \omega, z)$.

If (10.11) holds, then there exist $K, \delta, \eta > 0$ such that the event

$S_{\lambda, K, \delta} = \{\hat{x}_5(\lambda, \omega, 0) - \hat{x}_3(\lambda, \omega, 0) \leq K \text{ and min}\{\hat{x}_6(\lambda, \omega, 0), \hat{x}_7(\lambda, \omega, 0)\} - \hat{x}_5(\lambda, \omega, 0) \geq \delta\}$

satisfies $\mathbb{P}(S_{\lambda, K, \delta}) \geq \eta$. By the ergodic theorem, for $\mathbb{P}$-a.e. $\omega$,

$$\lim_{L \to +\infty} \frac{1}{L} \int_0^L \mathbb{I}_{\{\tau_\omega \in S_{\lambda, K, \delta}\}} d\omega = \mathbb{P}(S_{\lambda, K, \delta}) \geq \eta.$$ 

For every $z \in \mathbb{R}$, the sets $(B_{\lambda,i,z})_{i \in \mathbb{Z}}$ (defined in (10.4)) are disjoint and

$$\bigcup_{i \in \mathbb{Z}} B_{\lambda,i,z} = \{x_i(\lambda, \omega) \in (-\infty, +\infty) \text{ for every } i \in \mathbb{Z}\} \supset \Omega_0.$$

Moreover, it follows from (10.3) that

$$B_{\lambda,i,z} \cap \{\tau \omega \in S_{\lambda, K, \delta}\}$$

$$= B_{\lambda,i,z} \cap \{\hat{x}_5(\lambda, \omega, z) - \hat{x}_3(\lambda, \omega, z) \leq K \text{ and min}\{\hat{x}_6(\lambda, \omega, z), \hat{x}_7(\lambda, \omega, z)\} - \hat{x}_5(\lambda, \omega, z) \geq \delta\}$$

$$= B_{\lambda,i,z} \cap \{\pi_i(\lambda, \omega) - \pi_{i-1}(\lambda, \omega) \leq K \text{ and min}\{\pi_j(i)(\lambda, \omega), \pi_j(i)(\lambda, \omega)\} - \pi_i(\lambda, \omega) \geq \delta\},$$

where $j(i) = j(i, \omega) = \inf\{j \in \mathbb{Z} : \pi_j(\lambda, \omega) \geq \pi_i(\lambda, \omega)\}$. Let

$I_{\lambda, K, \delta} = \{i \in \mathbb{Z} : x_i(\lambda, \omega) - x_{i-1}(\lambda, \omega) \leq K \text{ and min}\{\pi_j(i)(\lambda, \omega), \pi_j(i)(\lambda, \omega)\} - x_i(\lambda, \omega) \geq \delta\}$. With this notation, for every $L > 0$,

$$\int_0^L \mathbb{I}_{\{\tau_\omega \in S_{\lambda, K, \delta}\}} d\omega = \sum_{i \in \mathbb{Z}} \int_0^L \mathbb{I}_{B_{\lambda,i,z} \cap \{\tau_\omega \in S_{\lambda, K, \delta}\}} d\omega = \sum_{i \in I_{\lambda, K, \delta}} \int_0^L \mathbb{I}_{B_{\lambda,i,z}} d\omega$$

(10.16)

$$= \sum_{i \in I_{\lambda, K, \delta}} \int_0^L \mathbb{I}_{\{x_{i-1}(\lambda, \omega) \leq \pi_i(\lambda, \omega)\}} d\omega$$

$$\leq K \#\{i \in I_{\lambda, K, \delta} : 0 \leq x_i(\lambda, \omega) \text{ and } x_{i-1}(\lambda, \omega) \leq L\}.$$
Recall from the proof of Lemma 6.2 that \((f^\lambda_{1,3})'(x,\omega) \leq \left( f^\lambda_{1,3} \right)'(x,\omega)\) for every \(\omega \in \Omega_0\) and a.e. \(x \in \mathbb{R}\) (with respect to Lebesgue measure). Moreover, for every \(i \in I_{\lambda,K,\delta}\),

\[
\{ \varepsilon_i(\lambda,\omega), \varepsilon_i(\lambda,\omega) + \delta \} \subset (\varepsilon_i(\lambda,\omega), y_i(\lambda,\omega)) \cap (y_{j(i)-1}(\lambda,\omega), y_{j(i)}(\lambda,\omega)).
\]

Therefore,

\[
(f^\lambda_{1,3})'(x,\omega) = G_1^{-1}(\lambda - \beta V(x,\omega)) \leq p_m < p_M \leq G_3^{-1}(\lambda - \beta V(x,\omega)) = (f^\lambda_{1,3})'(x,\omega)
\]

whenever \(x \in (\varepsilon_i(\lambda,\omega), \varepsilon_i(\lambda,\omega) + \delta)\). We conclude that

\[
\begin{align*}
\overline{\theta}_{1,3}(\lambda) - \underline{\theta}_{1,3}(\lambda) &= \lim_{L \to +\infty} \frac{1}{L} \left( \int_0^{L+K+\delta} (f^\lambda_{1,3})'(x,\omega)dx - \int_0^{L+K+\delta} \left( f^\lambda_{1,3} \right)'(x,\omega)dx \right) \\
&\geq \liminf_{L \to +\infty} \frac{(p_M - p_m)\delta}{L} \# \{ i \in I_{\lambda,K,\delta} : 0 \leq \varepsilon_i(\lambda,\omega) \text{ and } y_{i-1}(\lambda,\omega) \leq L \} \\
&\geq \lim_{L \to +\infty} \frac{(p_M - p_m)\delta}{KL} \int_0^L \mathbb{1}_{(\tau,\omega \in S_{\lambda,K,\delta})}d\tau \geq \frac{(p_M - p_m)\delta \eta}{K} > 0
\end{align*}
\]

by \((10.15)\) and \((10.16)\).

The proof of \(\underline{\theta}_{1,3}(\lambda) < \overline{\theta}_{1,3}(\lambda)\) under the conditions \((10.12)\) and \((10.14)\) (which are equivalent to each other) is similar.

**Proof of Theorem 3.5.** It follows from Theorem 6.3 and display \((6.15)\) at the end of its proof that, for every \(\lambda \in (\max\{\beta,M\}, m + \beta)\), the graph of the effective Hamiltonian \(\overline{H}\) has a flat piece at height \(\lambda\) if and only if

\[
(10.17) \quad \underline{\theta}_{1,3}(\lambda) < \overline{\theta}_{1,3}(\lambda).
\]

Recall the events \(U_\lambda\) and \(D_\lambda\) which are defined in \((3.9)\) and \((3.10)\), respectively. Lemmas 10.1, 10.2, 10.3 and 10.4 readily imply that \(\mathbb{P}(U_\lambda \cap D_\lambda) < 1\) if and only if \((10.17)\) holds. This establishes the desired characterization.

**11. Examples**

Theorems 3.2, 3.4 and 3.5 enable us to identify the set

\[
\mathcal{L}(\overline{H}) = \{ \lambda \in \mathbb{R} : \text{the graph of } \overline{H} \text{ has a flat piece height } \lambda \}.
\]

In particular, in the following cases, this set depends only on the parameters \(\beta, m, M\).

- When \(\beta \leq m + \beta < M\) (i.e., weak potential),
  \[
  \mathcal{L}(\overline{H}) = \{ \beta \} \cup \{ m + \beta, M \}.
  \]

- When \(\beta < M = m + \beta\) (i.e., the easy subcase of medium potential),
  \[
  \mathcal{L}(\overline{H}) = \{ \beta, M \}.
  \]

- When \(M \leq \beta = m + \beta\) (i.e., the easy subcase of strong potential),
  \[
  \mathcal{L}(\overline{H}) = \{ \beta \}.
  \]

In the remaining parameter regime, i.e., when \(\max\{\beta,M\} < m + \beta\),

\[
\mathcal{L}(\overline{H}) \subset \{ \beta \} \cup [\max\{\beta,M\}, m + \beta]
\]

and it depends also on the law of \(V(\cdot,\omega)\) under \(\mathbb{P}\). Subsections 11.1 and 11.2 provide more insight into this latter dependence by fully determining \(\mathcal{L}(\overline{H})\) in two basic classes of examples.
11.1. Periodic potentials. Take any $G : \mathbb{R} \to [0, +\infty)$ that satisfies (1.2) and Condition 3.1. Assume that $\max(\beta, M) < m + \beta$ (i.e., medium or strong potential, excluding their easy subcases).

Let $\{\Omega, \mathcal{F}, P\}$ be the interval $[0, 1]$ equipped with the Lebesgue measure. Denote a generic element of $[0, 1)$ by $w$ (instead of the usual $\omega$). For every $x \in \mathbb{R}$, define $\tau_x : [0, 1) \to [0, 1)$ by
\[
\tau_x w = \frac{x + w}{1 + w} = (x + w) - \lfloor x + w \rfloor.
\]

It is well known that $P$ is stationary & ergodic under $(\tau_x)_{x \in \mathbb{R}}$. Given a 1-periodic, continuous and surjective function $v_0 : \mathbb{R} \to [0, 1]$ that satisfies $v_0(0) = 1$, introduce $V : \mathbb{R} \times [0, 1) \to [0, 1)$ by setting
\[
V(x, w) = v_0(x + w) = v_0(\tau_x w).
\]

It is clear that $V$ satisfies (1.3), (1.4) and (1.5). Therefore, Theorems 3.2, 3.4 and 3.5 are applicable.

In this periodic setting, since (3.5) and (3.6) always hold, we have the following results.

- When $\beta < M < m + \beta$ (i.e., medium potential, excluding its easy subcase),

\[
\mathcal{L}(H) \supset \{\beta, M, m + \beta\}.
\]

- When $M \leq \beta < m + \beta$ (i.e., strong potential, excluding its easy subcase),

\[
\mathcal{L}(H) \supset \{\beta, m + \beta\}.
\]

It remains to identify all elements of the set
\[
\mathcal{L}(\Lambda) := \mathcal{L}(H) \cap (\max(\beta, M), m + \beta).
\]

We carry out this task under additional assumptions on the graph of the function $v_0$ which we impose for the sake of concreteness.

**Example 11.1.** Assume that the function $v_0$ is strictly decreasing on $[0, z_1]$ and strictly increasing on $[z_1, 1]$ for some $z_1 \in (0, 1)$. It follows from our surjectivity assumption that $v_0(z_1) = 0$.

For every $\lambda \in (\max(\beta, M), m + \beta)$ and $w \in [0, 1)$, the function $\lambda - \beta V(\cdot, w) : \mathbb{R} \to [\lambda - \beta, \lambda]$ is a surjection and $0 < \lambda - \beta < m < M < \lambda$. It is easy to see that
\[
0 < z_0(\lambda, 0) = x_0(\lambda, 0) < z_1 < y_0(\lambda, 0) = y(\lambda, 0) < 1,
\]
\[
\frac{x_0(\lambda, w)}{y_0(\lambda, w)} = \frac{x_0(\lambda, 0) - w}{y_0(\lambda, 0) - w} \quad \text{and} \quad \frac{y_0(\lambda, w)}{x_0(\lambda, w)} = \frac{y(\lambda, 0) - w}{x(\lambda, 0) - w}.
\]

Therefore, $\lambda - \beta V(\cdot, w)$ is locally invertible at $x_0(\lambda, w)$ and $y_0(\lambda, w)$. In particular, $P(U_{\lambda} \cap D_{\lambda}) = 1$. By Theorem 3.5, there is no flat piece at height $\lambda$. We conclude that
\[
\mathcal{L}(\Lambda) = \mathcal{L}(H) \cap (\max(\beta, M), m + \beta) = \emptyset.
\]

**Example 11.2.** Assume that the function $v_0$ is strictly decreasing on $[0, z_1]$, strictly increasing on $[z_1, z_2]$, strictly decreasing on $[z_2, z_3]$ and strictly increasing on $[z_3, 1]$ for some $z_1, z_2, z_3 \in (0, 1)$ such that $z_1 < z_2 < z_3$. It follows from our surjectivity assumption that $\min\{v_0(z_1), v_0(z_3)\} = 0$.

Let $\lambda_1 = M + \beta v_0(z_1)$. If $\lambda_1 \in (\max(\beta, M), m + \beta)$, then $x_0(\lambda_1, 0) = z_1$. Moreover, if $w \in [0, z_1)$, then $x_0(\lambda_1, w) = z_1 - w$. Therefore, $\lambda_1 - \beta V(\cdot, w)$ has a local maximum at $x_0(\lambda_1, w)$. In particular, $P(U_{\lambda_1}) < 1$. By Theorem 3.5, there is a flat piece at height $\lambda_1$.

Let $\lambda_2 = m + \beta v_0(z_2)$. If $\lambda_2 \in (\max(\beta, M), m + \beta)$, then we consider three subcases.

- If $\lambda_2 = \lambda_1$, then it is already covered above.

- If $\lambda_2 > \lambda_1$, then

\[
0 < x_0(\lambda_2, 0) = x_0(\lambda_2, 0) < z_1 < y_0(\lambda_2, 0) = z_2.
\]

Moreover, if $w \in [0, x_0(\lambda_2, 0)]$, then $y_0(\lambda_2, w) = z_2 - w$. Therefore, $\lambda_2 - \beta V(\cdot, w)$ has a local minimum at $y_0(\lambda_2, w)$. In particular, $P(D_{\lambda_2}) < 1$. By Theorem 3.5, there is a flat piece at height $\lambda_2$. 


• If $\lambda_2 < \lambda_1$, then
\[
  z_2 < z_0(\lambda_2, 0) = x_0(\lambda_2, 0) < z_3 < y_0(\lambda_2, 0) = y_0(\lambda_2, 0) < 1.
\]

Moreover, for every $w \in [0, 1)$,
\[
  \frac{\beta v}{\lambda} \frac{z_0(\lambda_2, 0) - w}{\lambda} \quad \text{and} \quad \frac{y_0(\lambda_2, 0) - w}{\lambda}.
\]

Therefore, $\lambda_2 - \beta V(\cdot, w)$ is locally invertible at $x_0(\lambda_2, w)$ and $y_0(\lambda_2, w)$. In particular, $P(U_{\lambda_2} \cap D_{\lambda_2}) = 1$. By Theorem 3.5, there is no flat piece at height $\lambda_2$.

Let $\lambda_3 = M + \beta v_0(z_3)$. If $\lambda_3 \in (\max\{\beta, M\}, m + \beta)$, then $v_0(z_3) > v_0(z_1) = 0$ and $\lambda_3 > \lambda_1$. We consider three subcases.

• If $\lambda_3 = \lambda_2$, then it is already covered above.

• If $\lambda_3 < \lambda_2$, then
\[
  0 < x_0(\lambda_3, 0) = x_0(\lambda_3, 0) < z_1 < y_0(\lambda_3, 0) = y_0(\lambda_3, 0) < z_2 < x_1(\lambda_3, 0) = z_3.
\]

Moreover, for every $w \in [x_0(\lambda_3, 0), z_3)$, then $x_0(\lambda_3, w) = x_1(\lambda_3, 0) - w = z_3 - w$. Therefore, $\lambda_3 - \beta V(\cdot, w)$ has a local maximum at $x_0(\lambda_3, w)$. In particular, $P(U_{\lambda_3}) < 1$. By Theorem 3.5, there is a flat piece at height $\lambda_3$.

• If $\lambda_3 > \lambda_2$, then
\[
  0 < x_0(\lambda_3, 0) = x_0(\lambda_3, 0) < z_1 < z_3 < y_0(\lambda_3, 0) = y_0(\lambda_3, 0) < 1.
\]

Moreover, for every $w \in [0, 1)$,
\[
  \frac{\beta v}{\lambda} \frac{z_0(\lambda_3, 0) - w}{\lambda} \quad \text{and} \quad \frac{y_0(\lambda_3, 0) - w}{\lambda}.
\]

Therefore, $\lambda_3 - \beta V(\cdot, w)$ is locally invertible at $x_0(\lambda_3, w)$ and $y_0(\lambda_3, w)$. In particular, $P(U_{\lambda_3} \cap D_{\lambda_3}) = 1$. By Theorem 3.5, there is no flat piece at height $\lambda_3$.

For every $\lambda \in (\max\{\beta, M\}, m + \beta)$, if $\lambda - \beta V(\cdot, w)$ has a local maximum at $x_i(\lambda, w)$ for some $i \in \mathbb{Z}$ and $w \in [0, 1)$, then
\[
  \frac{\lambda_i(\lambda, w) + w}{\lambda} \in \{z_1, z_3\}
\]

and $\lambda = M + \beta v(x_i(\lambda, w), w) = M + \beta v_0(x_i(\lambda, w) + w) \in \{\lambda_1, \lambda_3\}$. Similarly, if $\lambda - \beta V(\cdot, w)$ has a local minimum at $y_i(\lambda, w)$ for some $i \in \mathbb{Z}$ and $w \in [0, 1)$, then $\lambda = m + \beta v_0(z_2) = \lambda_2$. In particular, if $\lambda \notin \{\lambda_1, \lambda_2, \lambda_3\}$, then $P(U_{\lambda} \cap D_{\lambda}) = 1$. Therefore, by Theorem 3.5, there is no flat piece at any height $\lambda \in (\max\{\beta, M\}, m + \beta) \setminus \{\lambda_1, \lambda_2, \lambda_3\}$.

Putting everything together, we conclude that the set $L(\Lambda) = L(\overline{\Lambda}) \cap (\max\{\beta, M\}, m + \beta)$ is given by

\[
  L(\Lambda) = \begin{cases} 
    \{\lambda_1\} & \text{if } \lambda_3 = M \leq \max\{\beta, M\} < \max\{\lambda_1, \lambda_2\} = \lambda_1 < m + \beta \\
    \{\lambda_2\} & \text{if } \lambda_1 = M \leq \max\{\beta, M\} < \lambda_1 < m + \beta = \lambda_2, \\
    \{\lambda_3\} & \text{if } \lambda_1 = M \leq \max\{\beta, M\} < \lambda_3 < m + \beta = \lambda_2, \\
    \{\lambda_1, \lambda_2\} & \text{if } \lambda_3 = M \leq \max\{\beta, M\} < \lambda_1 < \lambda_2 < m + \beta, \\
    \{\lambda_2, \lambda_3\} & \text{if } \lambda_1 = M \leq \max\{\beta, M\} < \lambda_3 < \lambda_2 < m + \beta, \\
    \emptyset & \text{otherwise},
  \end{cases}
\]

where $\lambda_1 = M + \beta v_0(z_1)$, $\lambda_2 = m + \beta v_0(z_2)$ and $\lambda_3 = M + \beta v_0(z_3)$. 
11.2. Piecewise linear random potentials. Take any $G : \mathbb{R} \to [0, +\infty)$ that satisfies (1.2) and Condition 3.1. Assume that $\max\{\beta, M\} < m + \beta$ (i.e., medium or strong potential, excluding their easy subcases).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the product space $[0, 1) \times [0, 1]^2$ equipped with $\mathbb{P} = \ell \times \mathbb{Q}$, where $\ell$ is the Lebesgue measure on $[0, 1)$ and $\mathbb{Q}$ is the law of a discrete-time stationary & ergodic stochastic process taking values in $[0, 1]$. Assume that the one-dimensional marginal distribution of $\mathbb{Q}$ is equal to some Borel probability measure $\mu$ on $[0, 1]$ such that

\begin{equation}
\{0, 1\} \subset \text{supp}(\mu).
\end{equation}

Denote a generic element of $\Omega$ by $\omega = (w, (v_i)_{i \in \mathbb{Z}})$. For every $x \in \mathbb{R}$, define $\tau_x : \Omega \to \Omega$ by

$$
\tau_x(w, (v_i)_{i \in \mathbb{Z}}) = (\text{frac}(x + w), (v_{i+w})_{i \in \mathbb{Z}}).
$$

We leave it to the reader to check that $\mathbb{P}$ is stationary & ergodic under $(\tau_x)_{x \in \mathbb{R}}$.

Introduce $V : \mathbb{R} \times \Omega \to [0, 1]$ by setting

$$
V(x, \omega) = (1 - \text{frac}(x + w))v_{x+w} + \text{frac}(x + w)v_{x+w+1}.
$$

This is a linear interpolation. In particular, $V(0, \omega) = (1 - w)v_0 + wv_1$.

It is easy to see that $V$ satisfies (1.3), (1.4) and (1.5). Therefore, Theorems 3.2, 3.4 and 3.5 are applicable.

Let $\mathcal{A}(\mu)$ denote the set of atoms of $\mu$. Note that (3.5) (resp. (3.6)) holds if and only if $\mu$ has an atom at 0 (resp. 1). Therefore, in this piecewise linear setting, we have the following results.

- When $\beta < M < m + \beta$ (i.e., medium potential, excluding its easy subcase), $\beta \in \mathcal{L}(\mathcal{H})$.

Moreover,

\begin{equation}
M \in \mathcal{L}(\mathcal{H}) \iff 0 \in \mathcal{A}(\mu) \quad \text{and} \quad m + \beta \in \mathcal{L}(\mathcal{H}) \iff 1 \in \mathcal{A}(\mu).
\end{equation}

- When $M \leq \beta < m + \beta$ (i.e., strong potential, excluding its easy subcase), $\beta \in \mathcal{L}(\mathcal{H})$.

Moreover,

\begin{equation}
m + \beta \in \mathcal{L}(\mathcal{H}) \iff 1 \in \mathcal{A}(\mu).
\end{equation}

It remains to identify all elements of the set

$$
\mathcal{L}(\Lambda) := \mathcal{L}(\mathcal{H}) \cap (\max\{\beta, M\}, m + \beta).
$$

We carry out this task under additional structural assumptions on the probability measure $\mathbb{Q}$.

**Example 11.3.** Assume that $\mathbb{Q} = \mu^2$, i.e., it is the law of a bi-infinite sequence of independent and identically distributed (i.i.d.) random variables with common distribution $\mu$. (See Remark 11.6 for a generalization.)

For every atom $a \in \mathcal{A}(\mu)$, if $\lambda_1(a) = M + \beta a \in (\max\{\beta, M\}, m + \beta)$, then the event

$$
\{\omega = (w, (v_i)_{i \in \mathbb{Z}}) \in \Omega : \lambda_1(a) - \beta v_0 < m, \lambda_1(a) - \beta v_1 = M, \lambda_1(a) - \beta v_2 < m\}
$$

has positive $\mathbb{P}$-probability by (11.1). On this event,

$$
\mathbb{P}_{-1}(\lambda_1(a), \omega) < -w \leq 0 < \mathbb{P}_0(\lambda_1(a), \omega) = 1 - w < \mathbb{P}_0(\lambda_1(a), \omega) < 2 - w
$$

and $\lambda_1(a) - \beta V(\cdot, \omega)$ has a local maximum at $x_0(\lambda_1(a), \omega)$. We deduce that $\mathbb{P}(U_{\lambda_1(a)}) < 1$. Therefore, by Theorem 3.5 there is a flat piece at height $\lambda_1(a)$. 

Remark 11.5

Let \( v \) be used at the index \( i \) of i.i.d. random variables with common distributions \( \lambda \).

4. **Markov Property**: The transition kernel \( \pi \) for some \( \lambda \) is given by

\[
(11.7) \quad \lambda, \omega \to \pi(v \to \omega) = \begin{cases} 
\mu(v) & \text{if } v \in \text{supp}(\mu), \\
\mu(\omega) & \text{if } v \in \text{supp}(\mu_2), 
\end{cases}
\]

and \( \lambda = \lambda_2(a) \) has a local minimum at \( y_0(\lambda, \omega) \). We recall Lemma 10.1 and deduce that \( \mathbb{P}(D_{\lambda(a)}) < 1 \). Therefore, by Theorem 3.5, there is a flat piece at height \( \lambda(a) \).

For every \( \lambda \in \{\max\{\beta, M\}, m + \beta\} \setminus \{\lambda(a) : a \in \mathcal{A}(\mu)\} \) and \( \mathbb{P}\)-a.e. \( \omega \), the piecewise linear function \( \lambda - \beta V(\cdot, \omega) \) is locally invertible at \( y_0(\lambda, \omega) \). In particular, \( \mathbb{P}(U_\lambda \cap D_{\lambda}) = 1 \).

By Theorem 3.5 there is no flat piece at height \( \lambda \).

Putting everything together, we conclude that the set \( \mathcal{L}(\Lambda) = \mathcal{L}(\overline{H}) \cap (\max\{\beta, M\}, m + \beta) \) is given by

\[
(11.6) \quad \mathcal{L}(\Lambda) = \left(\{\lambda_1(a) : a \in \mathcal{A}(\mu)\} \cup \{\lambda_2(a) : a \in \mathcal{A}(\mu)\}\right) \cap (\max\{\beta, M\}, m + \beta)
\]

Example 11.4. Let \( \mu_1 \) and \( \mu_2 \) be two Borel probability measures on \([0, 1]\) such that

\[
(11.7) \quad 0 \in \text{supp}(\mu_1) \subset [0, c) \quad \text{and} \quad 1 \in \text{supp}(\mu_2) \subset (c, 1]
\]

for some \( c \in (0, 1) \). Assume that \( \mathcal{Q} \) is the law of the discrete-time stationary Markov process whose transition kernel \( \pi \) and invariant probability distribution \( \mu \) are given by

\[
\pi(v, \cdot) = \begin{cases} 
\mu_2(\cdot) & \text{if } v \in \text{supp}(\mu_1), \\
\mu_1(\cdot) & \text{if } v \in \text{supp}(\mu_2), 
\end{cases}
\]

and \( \mu = \frac{1}{2}(\mu_1 + \mu_2) \). In words, \( \mathcal{Q} \) is the law of two independent and interlaced bi-infinite sequences of i.i.d. random variables with common distributions \( \mu_1 \) and \( \mu_2 \), and which of these two distributions is used at the index \( i = 0 \) is determined by a fair coin toss. (See Remark 11.6 for a generalization.)

For every atom \( a \in \mathcal{A}(\mu_1) \), if \( \lambda_1(a) = M + \beta a \in (\max\{\beta, M\}, m + \beta) \), then the event in \((11.4)\) has positive \( \mathbb{P}\)-probability by \((11.7)\). Therefore, by the corresponding argument in Example 11.3 there is a flat piece at height \( \lambda_1(a) \).

For every atom \( a \in \mathcal{A}(\mu_2) \), if \( \lambda_2(a) = m + \beta a \in (\max\{\beta, M\}, m + \beta) \), then the event in \((11.5)\) has positive \( \mathbb{P}\)-probability by \((11.7)\). Therefore, by the corresponding argument in Example 11.3 there is a flat piece at height \( \lambda_2(a) \).

For every \( \lambda \in (\max\{\beta, M\}, m + \beta) \) and \( \omega = (w, (v_i)_{i \in \mathbb{Z}}) \in \Omega \), it follows from \((11.7)\) that the set of local maxima of the piecewise linear function \( \lambda - \beta V(\cdot, \omega) \) is equal to \( \{2i + w : i \in \mathbb{Z}\} \) if \( v_0 \in \text{supp}(\mu_1) \) and \( \{2i + 1 - w : i \in \mathbb{Z}\} \) if \( v_0 \in \text{supp}(\mu_2) \). Therefore, if the event

\[
(U_\lambda)^c = \{\omega \in \Omega : \lambda - \beta V(\cdot, \omega) \text{ has a local maximum at } \omega \}
\]

has positive \( \mathbb{P}\)-probability, then \( \lambda = \lambda_1(a) \) for some \( a \in \mathcal{A}(\mu_1) \). Similarly, if \( \mathbb{P}(U_\lambda^c) > 0 \), then \( \lambda = \lambda_2(a) \) for some \( a \in \mathcal{A}(\mu_2) \).

Putting everything together, we conclude that the set \( \mathcal{L}(\Lambda) = \mathcal{L}(\overline{H}) \cap (\max\{\beta, M\}, m + \beta) \) is given by

\[
(11.8) \quad \mathcal{L}(\Lambda) = \left(\{\lambda_1(a) : a \in \mathcal{A}(\mu_1)\} \cup \{\lambda_2(a) : a \in \mathcal{A}(\mu_2)\}\right) \cap (\max\{\beta, M\}, m + \beta)
\]

Remark 11.5. In Example 11.4 it is clear from the description in \((11.8)\) that, for any finite or countable \( S \subset (\max\{\beta, M\}, m + \beta) \), there exist a \( c \in (0, 1) \) and measures \( \mu_1, \mu_2 \) satisfying \((11.7)\) such that \( \mathcal{L}(\Lambda) = S \). (Note that there is not enough degree of freedom in the description in \((11.6)\) to realize this in the setting of Example 11.3) We recall the characterizations in \((11.2)-(11.3)\) and conclude...
that we can make $L(H)$ equal to any desired finite or countable subset of $\{\beta\} \cup [\max\{\beta, M\}, m + \beta]$ containing $\beta$.

**Remark 11.6.** In Example 11.3 (resp. Example 11.4), it is clear from the arguments we gave that the description of the set $L(\Lambda)$ in (11.6) (resp. (11.8)) would not change if we replace $Q$ with the law $Q'$ of any discrete-time stationary & ergodic stochastic process taking values in $[0, 1]$ such that the finite-dimensional marginals of $Q$ and $Q'$ are mutually absolute continuous.

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