Strong Magnetic field effects on Neutron Stars within $f(T)$ theory of gravity

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Abstract

We investigate in this paper the structures of neutron stars under the strong magnetic field in the framework of $f(T)$ gravity where $T$ denotes the scalar torsion. The TOV equations in this theory of gravity have been considered and numerical resolution of these equations has been performed within perturbative approach taking into account the equation of state of neutron dense matter in magnetic field. We simplify the problem by considering the very strong magnetic field which affects considerably the dense matter; and for quadratic and cubic corrections to Teleparallel term, one finds that the mass of neutron stars can increase for different values of the perturbation parameter. The deviation from Teleparallel for different values of magnetic field is found out and this feature is very appreciable in the case of cubic correction. Our results are related to the hadronic particles description with very small hyperon contributions and the mass-radius evolution is consistency with the observational data.

Keywords: TOV, magnetic field, Teleparallel, mass-radius

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1 Introduction

The explanation of the present state of our Universe and the several structures appearing in it, has opened the gate to various reflexions. Indeed, the current acceleration of the Universe is widely accepted through several independent observational data, as supernovae Ia [1]-[2], the large scale structure of the Universe [3], cosmic shear through gravitational weak lensing surveys [4] and the Lyman alpha forest absorption lines [5]. Due to the need to understand the acceleration of the Universe, various theories have been introduced. The well known standard equivalent theories, the General Relativity (GR) and the Teleparallel Theory, are the first theories used for explaining the acceleration of the Universe, including the existence of the dark energy as a new component of the Universe [6]. We recall here that the Teleparallel Theory is an idea of Einstein in 1928 in his attempt to unify gravity and electromagnetism. This theory nowadays, is formulated as higher gauge theory [7]; furthermore it has already been established that GR can in fact be re-cast into Teleparallel language [8], known as the Teleparallel Equivalent of General Relativity (TEGR). The second attempt in theories of gravity for explaining this state of Universe consists to modify the GR and the Teleparallel Theory. Due to the fact that the Teleparallel theory does not yield consistent results according to the observational data, the immediate attempt to modify this theory of gravity is replacing the Teleparallel term, the torsion scalar $T$ by an algebraic function $f(T)$ based on the Weitzenböck connection instead of
Levi-Civita connection used in GR.

Furthermore, structure formations have also been investigated in this modified theory of gravity. The neutron stars structures have already been executed in several theories of gravity mainly in $f(R)$ [24, 25, 42] and $f(T)$ [23]. Indeed the structures of neutron and quark Stars have recently been investigated by [23] through the deviation of the mass-radius diagrams for power-law and exponential type correction from the Teleparallel Theory gravity by considering some equations of state. In this paper, we focus our attention on the study of these stars in particular the neutron stars by investigating the effects of strong magnetic fields in the framework of $f(T)$ gravity. Note that such studies have been performed in the so-called $f(R)$ theory of gravity where interesting result have been found [24, 32, 41, 42, 43, 44]). We emphasize here that neutron stars are observed as several classes of self-gravitating systems [24]. The structure of neutron stars and the relation between the mass and the radius are determined by equations of state (EOS) of dense matter. Moreover, the maximal mass of neutron star is still an open question. Recent observations allow to estimate this limit at least as $2M_\odot$. It exits some well-measured limits according to several observational data on the pulsars and massive neutron stars: there are 1.972$M_\odot$ for pulsars PSR J1614 – 2230 [9], 2.01$M_\odot$ for pulsars J0348 + 0432 [10], 1.972$M_\odot$ for Vela X-1 [11], 2$M_\odot$ for 4U1822 – 371[12], 2.7$M_\odot$ for J1748 – 2021B [46]. Several works with different approaches show that the mass limit of neutron star can increase: larger hyperon-vector couplings with stiffness of the EOS [14], model with chiral quark-meson coupling [15],

the quark-meson coupling model which naturally incorporates hyperons [16] and others. However, it has been shown in the literature that ultrastrong magnetic field affected considerably the equation of state (EOS) of neutron-star matter and consequently could lead to the increase of the maximal mass limit of these stars [40]. In order to explain this fact, various models of dense nuclear matter in presence of strong magnetic fields have been considered such as models with hyperons and quarks, and model with interacting $npe\mu$ gas. It has also been performed that the Landau quantization leads to the softening of the EoS for matter but account for contributions of magnetic field into pressure and density.

In this paper, we present the models of neutron star for simple EOS in the ultrastrong magnetic field via the $f(T)$ gravity. In Ref.[40], it is clearly shown that the structure of a magnetized neutron star will be mostly affected by contributions from the magnetic field stress $P_f = B^2/8\pi = 4.814 \times 10^{-8}B^2$ Mev fm$^{-3}$ which greatly exceeds the matter pressure $P_m$ at
all relevant densities for $B^* \geq 10^5$. Having in hand such fundamental result and through a perturbative approach, one obtains that the maximal mass limit of neutron stars can increase under an ultrastrong field magnetic. We consider two corrections of the Teleparallel Theory (quadratic and cubic corrections) and find that the parameter of correction plays an important role in this analysis. The deviation of mass-radius from Teleparallel Theory appears clearly in the case of cubic corrections. Our investigation through perturbative approach shows that in the case of null magnetic field, the mass of neutron stars can increase as it was predicted by [24]. The paper is organized as follows. In Sec.II, we make some reviews on the Teleparallel Theory via its equivalence to GR and its modified version, the so-called $f(T)$ gravity. In Sec.III, we present the TOV equations and its perturbative versions. Strong magnetic field effect on the dense matter in the framework of the relativistic mean field has been described in the Sec.IV. Our main results have been presented in the Sec.V and the conclusions in Sec.VI.

2 From Teleparallel equivalent of General Relativity to $f(T)$

In Teleparallel, equivalent of General Relativity, as in General Relativity, the structure of space-time is represented by a manifold $M$. At every point $p \in M$ in the local coordinate chart $\{x^1, ..., x^n\}$, the tangent space $T_pM$ at $p$ is spanned by the coordinate vector fields $\{\partial_1, ..., \partial_n\}$. The corresponding dual space is denoted by $T^*_pM$ and generated by $\{dx^1, ..., dx^n\}$. In addition, the tangent space is a 4 dimension space described by the Lorentzian metric $g$ with signature $(-, +, +, +)$. We will label all space-time coordinates by Greek subscripts that run from 0 to 3, with 0 denoting the time dimension, while all spatial coordinates will be labeled by $i, j, k, \ldots$ that run from 1 to 3.

Let us assume $\{e_A(x)\}$ as an arbitrary base of $T_pM$. We can express the total derivative covariant as $\nabla e_A$:

$$\nabla e_A(x) = \Gamma^B_A(x) e_B(x),$$

with $\Gamma^B_A(x)$, the 1-form connection satisfying

$$\Gamma^B_A(x) = \langle z^B(x), \nabla e_A(x) \rangle = \Gamma^B_{\nu A}(x) dx^\nu.$$  

In this last expression, $\{z^B(x)\}$ represents the dual of $\{e_B(x)\}$ where as $\{dx^\nu\}$ stayed for base dual of local base $\{\partial_\nu\}$. Then, it comes that $e_A = e^\nu_A \partial_\nu$ and $z^B = e^B_\nu dx^\nu$. In this paper, the
capital letters $A, B, C,...$ take the values running from 0 to 3 and $a, b, c,...$ take the values running from 1 to 3.

Assuming that the spacetime is parallelisable (i.e. there exist $n$ vector fields $\{v_1, ..., v_n\}$ such that at any point $p \in M$ the tangent vectors $v_i|_p$ provide a basis of the tangent space at $p$), the mapping between the bases in coordinate frame $\{\partial_\mu\}$ to that of non-coordinate frame $\{e_A\}$ is an isomorphism $TM \rightarrow M \times \mathbb{R}^4$. This also comes from the fact that all $n$ dimensional parallelisable manifold has a tangent space which can be decomposed in the direct product of $M$ and $\mathbb{R}^n$.

Note that the frame field depends only on the affine structure of the manifold and hence a priori has any relation with the metric. Actually, we describe the frames with the metric by equipping the $\mathbb{R}^4$ space with the minkowskian metric $\eta_{AB}$ such as

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu.$$  

This also means that on the manifold $M$, we arbitrary choose a frame $\{e_A\}$ at each point namely locally on some open chart $U \subset M$. This approach can be extended by parallelisability. One can define the metric $\eta$ on the open chart $U$ by

$$\eta(e_A, e_B) = \eta_{AB},$$

which shows the orthonormality on the tetrad. It is important to notify here that for the $(3 + 1)$ dimension gravity, the previous results hold according to the Steenrod theorem: all 3 dimensional and orientable manifold is parallelisable [34, 35]. Thus a 4-dimensional spacetime with orientable spatial section is also parallelisable. To put it slightly differently, if any spatial slice of spacetime is an orientable 3-manifold (and as such parallelizable) with initial data that can be propagated uniquely in time, in the manner of $3 + 1$ decomposition of $\text{ADM}$ [36], then the entire spacetime is parallelizable. We also emphasize here from Geroch theorem [37] that a non-compact 4-dimensional Lorentzian manifold $M$ admits a spin structure if and only if it is parallelisable.

Let introduce $\overset{\cdot}{\nabla}$, the Weitzenböck connection [38] defined by

$$\overset{\cdot}{\nabla}_X Y := (XY^A)e_A,$$

with $Y = Y^A e_A$. The teleparallelism condition on the tetrads imposes $\overset{\cdot}{\nabla} e_A = 0$ which allows...
to define the coefficients of the connection as
\[ \Gamma_{\mu\nu}^{\lambda} = e_{A}^{\lambda} \partial_{\nu} e_{A}^{\mu} = -e_{A}^{A} \partial_{\nu} e_{A}^{\lambda}. \]  
(6)
The coefficients defined in (6) is for an unique connection [39] which, at each vector field \(X\), gives rise to parallelisation on \(M\), assuming of course that \(M\) is parallelisable.

\[ T(X,Y) = \nabla_{X} Y - \nabla_{Y} X - [X,Y]. \]  
(7)
And then
\[ T(X,Y) = X^{A} Y^{B}[e_{A}, e_{B}]. \]  
(8)

The equation (8) shows that the torsion tensor associated to Weitzenböck connection is generically non zero because in general, the basis vectors \(\{e_{A}\}\) are not integrable. In the local coordinates, the torsion tensor components can be expressed by
\[ T_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda} = e_{A}^{\lambda}(\partial_{\mu} e_{A}^{\nu} - \partial_{\nu} e_{A}^{\mu}) \neq 0. \]  
(9)

We also define here the curvature tensor associated to the Weitzenböck connection through the Riemann curvature by
\[ R(X,Y)Z = (\nabla_{X} \nabla_{Y} - \nabla_{Y} \nabla_{X} - \nabla_{[X,Y]} )Z. \]  
(10)
From the fact that \(\nabla_{X} e_{A} = (X^{C} e_{A})e_{c} = 0\), one gets
\[ R(e_{A}, e_{B})e_{C} = \nabla_{e_{A}} (\nabla_{e_{B}} e_{C}) - \nabla_{e_{B}} (\nabla_{e_{A}} e_{C}) - \nabla_{[e_{A}, e_{B}]} e_{C} = 0. \]  
(11)
It follows that the curvature tensor associated to the Weitzenböck connection is equal to zero contrary to the case of the Levi-Civita connection in GR. We can thus say that the curvature is an intrinsic property of spacetime but it also only depends on the connection that is put on spacetime.

Another important tensor emerging from the use of the Weitzenböck connection is the contortion which shows the difference between the Weitzenböck connection and the Levi-Civita connection [18] according to
\[ K_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\lambda\mu}^{\nu} = \frac{1}{2}(T_{\nu}^{\lambda} \mu + T_{\mu}^{\lambda} \nu - T^{\lambda}_{\mu\nu}), \]  
(12)
\[ K_{\mu \nu \beta} = -\frac{1}{2} \left( T_{\mu \nu \beta} - T_{\nu \mu \beta} - T_{\beta \mu \nu} \right), \quad (13) \]

where \( \Gamma^\lambda_{\mu \nu} \) represents the coefficient of Levi-Civita connection. In order to show that the both connections make the same description of spacetime namely, prove that the Teleparallel Theory based on the Weitzenböck connection is equivalent to GR using the Levi-Civita connection, we recall here the action of Einstein-Hilbert GR as

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R, \quad (14) \]

With \( \kappa^2 = \frac{16\pi G}{c^4} \), and \( R \) is the Ricci scalar curvature coming from the Levi-Civita connection. The action (14) can be written with scalar torsion as

\[ S = \frac{1}{2\kappa^2} \int d^4x eT, \quad (15) \]

where \( e = |\text{det}(e^A_\mu)| \) is equivalent to \( \sqrt{-g} \) in GR and \( T \) the scalar torsion defined by

\[ T := S_\beta^{\mu \nu} T^{\beta \mu \nu}, \quad (16) \]

with

\[ S_\beta^{\mu \nu} = \frac{1}{2} \left( K_{\mu \nu \beta} + \delta^\mu_\beta T_{\alpha \nu \alpha} - \delta^\nu_\beta T_{\alpha \mu \alpha} \right). \quad (17) \]

By using the relations (12) and (17), the scalar torsion defined in (16) can be explicitly put in the following relations

\[ T = \frac{1}{4} T_{\beta \mu \nu} T_{\beta \mu \nu} + \frac{1}{2} T_{\beta \mu \nu} T_{\nu \mu \beta} - T_{\beta \nu \beta} T_{\nu \mu \mu}. \quad (18) \]

In the local base \( \{ \partial_\mu \} \), the components of Riemann tensor given in (10) can be obtained in the framework of Levi-Civita connection by

\[ R^\rho_{\mu \lambda \nu} = \partial_\lambda \Gamma^\rho_{\mu \nu} - \partial_\nu \Gamma^\rho_{\mu \lambda} + \Gamma^\rho_{\sigma \lambda} \Gamma^\sigma_{\mu \nu} - \Gamma^\rho_{\sigma \nu} \Gamma^\sigma_{\mu \lambda}. \quad (19) \]

By making using the relation (12) and after some contractions, one obtains the associated Ricci tensor as

\[ R_{\mu \nu} = \nabla_\mu K_{\rho \nu} - \nabla_\nu K_{\rho \mu} + K_{\rho \sigma \nu} K^{\sigma \mu \rho} - K_{\rho \sigma \mu} K^{\sigma \rho \nu}, \quad (20) \]
where $\nabla$ represents the covariant derivative in GR. By combining the relation (12) with the following relations

$$K^{\mu\nu}\sigma = T^\mu(\nu\sigma) = S^{\nu(\nu\sigma)} = 0$$

and by taking into consideration

$$S^{\mu\rho\nu} = 2 K^{\mu\rho\nu} = -2 T^{\mu\rho\nu},$$

we have

$$R_{\mu\nu} = -\nabla^\rho S_{\nu\rho\mu} - g_{\mu\nu} \nabla^\sigma T^\rho_{\rho\sigma\nu} - S^{\rho\sigma\mu} K_{\sigma\rho\nu},$$

whose total contraction gives

$$R = -T - 2\nabla^\mu T^\nu_{\mu\nu}.$$  

This last relation finds out the trivial equivalence of GR and Teleparallel under the actions defined in (14) and (15). It follows that these two relations are the Einstein-Hilbert action respectively in GR and Teleparallel. In the rest of this work, we will leave the strackrel bullet (•) on all quantities resulting from the Weitzenböck connection.

The action of the modified versions of TEGR (Teleparallel Equivalent of GR) is obtained by substituting the scalar torsion of the action (15) by an arbitrary function of scalar torsion giving then to modified theory $f(T)$. This approach is similar in spirit to the generalisation of Ricci scalar curvature of Einstein-Hilbert action (14) by a function of this scalar leading to the well known $f(R)$ theory. The action of $f(T)$ theory can be defined as

$$S = \frac{1}{2\kappa^2} \int [f(T) + \mathcal{L}_{\text{Matter}}] e^A e^B e^C e^D,$$

with $T$ the scalar torsion and $\mathcal{L}_{\text{Matter}}$ the matter density Lagrangian which only depends from the tetrads $e_A^\mu$ without their derivative. The variation of the action with respect to tetrads $e_A^\mu$ gives [22, 23]

$$\frac{1}{e} e^A_{\mu}(e S_A^{\mu\nu}) f_T(T) - e_A^{\lambda} T^\rho_{\mu\lambda} S^\rho_{\mu\nu} f_T(T) + S_A^{\mu\nu} \partial_\mu(T) f_{TT}(T) + \frac{1}{4} e_A^\nu f(T) = \Theta^\nu_A,$$

with $f_T(T) = df(T)/dT$, $f_{TT}(T) = d^2f(T)/dT^2$ and $\Theta^\nu_A$ the energy-momentum tensor naturally related to the matter.

### 3 The Generalised Tolman-Oppenheimer-Volkoff (TOV) equations in $f(T)$

TOV equations are often used in gravitational theories to study the structure of relativistic stars. Their determination requires the choice of a given theory of gravity, the precision on
the type of spacetime (metric) and the matter source in the Universe. In the framework of our present work, based on the $f(T)$ theory, we express the geometric Lagrangian density as:

$$f(T) = T + \xi g(T)$$

with $\xi$ a real constant such that $\xi = 0$ matches to the Teleparallel theory.

The equations of motion (24) become [23]

$$\frac{1}{c^4} \epsilon^A_{\sigma} \partial_\mu (\epsilon e_A^\rho S^\rho_{\mu \nu}) + T^{\rho}_{\mu \sigma} S^\nu_{\rho \mu \nu}(1 + \xi g_{TT}(T)) + \xi S^\nu_{\sigma \rho} \partial_\mu(T) g_{TT}(T) + \frac{1}{4} \delta^\nu_{\sigma}(T + \xi g(T)) = \frac{\kappa^2}{2} \Theta^\nu_{\sigma}, (25)$$

In order to obtain solutions that describe the stellar objects, we consider a spherical symmetric metric having two functions $\varphi$ and $\lambda$ depending on the radial coordinate by

$$ds^2 = -e^{2\varphi} dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (26)$$

which can be generated by the following table of tetrads

$$e^{A}_{\mu} = \text{diag}(e^\varphi, e^\lambda, r, r \sin \theta). \quad (27)$$

Let recall here that the most successful metric in the stellar objects description is the one of Schwarschild [23, 24, 26, 27] which is a particular case of metric described in (26) with the following fundamental relation

$$e^{-2\lambda} = 1 - \frac{2GM}{c^2 r}. \quad (28)$$

We also consider that the Universe described by this metric has as content, an isotropic fluid such that the associated energy-momentum tensor can be expressed by

$$\Theta^\nu_{\sigma} = -(\rho + P) u_{\sigma} u^{\nu} + P \delta^\nu_{\sigma}, \quad (29)$$

with $u^{\nu}$ the four-velocity, $\rho$ and $P$ the energy density and the pressure of the matter respectively.

Having the metric (26) and the energy-momentum (29) in hand, we establish the field equations from the general equation (25) via the following relations

$$\frac{1}{4} [T + \xi g(T)] - \frac{1}{2} [T - \frac{1}{r^2}] - 2 e^{-2\lambda} \frac{c^2}{r} (\varphi' + \lambda') (1 + \xi g_{TT}(T)) = -\frac{4\pi \rho}{c^4}, \quad (30)$$

$$-\frac{1}{4} [T + \xi g(T)] - \frac{1}{4} \left[ \frac{3T}{2} - \frac{1}{r^2} + e^{-2\lambda} \frac{c^2}{r} (\varphi'' + (\varphi' + \frac{1}{r})(\varphi' - \lambda')) \right] (1 + \xi g_{TT}(T)) = \frac{4\pi P}{c^4}, \quad (31)$$

$$\cot \theta \frac{\theta}{2r^2} \xi T' g_{TT}(T) = 0. \quad (32)$$
The “prime” in these relations means the derivative with respect to the radial coordinate \( r \). The scalar torsion can also be calculated as

\[
T = -\frac{2e^{-2\lambda}}{r^2} (2r\varphi' + 1).
\]  
(33)

However, from the conservation of the energy-momentum tensor, \( \nabla_\mu \Theta^\mu_{\sigma} = 0 \), the quantity \( \varphi' \) can be expressed in terms of the energy density and the pressure as

\[
\varphi' = - (\rho + P)^{-1} \frac{dP}{dr}.
\]  
(34)

In order to obtain the TOV equations, we introduce the following dimensionless variables:

\[
M = mM_\odot, \quad r \to r_g r, \quad \rho \to \rho M_\odot/r_g^3, \quad P \to pM_\odot c^2/r_g^3, \quad T \to T/r_g^2, \quad \xi r_g^2 g(T) \to \xi g(T), \quad \text{with} \quad r_g = GM_\odot/c^2 = 1.47473 \text{km}.
\]

Indeed, the relation (35) becomes

\[
e^{-2\lambda} = 1 - \frac{2m}{r},
\]  
(35)

and

\[
\lambda' = \left( \frac{m'}{r} - \frac{m}{r^2} \right) \left( 1 - \frac{2m}{r} \right)^{-1}.
\]  
(36)

The scalar torsion becomes (33)

\[
T(r) = -\frac{2}{r_g^2 r^2} \left( 1 - \frac{2m}{r} \right) \left[ 1 - 2rr_g(\rho + p)^{-1} \frac{dp}{dr} \right].
\]  
(37)

By using the field equations (30), (31) and the relations (34), (36) and (37), one gets the TOV
The relations (38), (39) and (40) can be numerically solved by having the algebraic function $g(T)$. But in this work, one can use the perturbative approach. In this approach, the terms containing $g(T)$ must be of first order of the small parameter $\xi$. The last equation, namely the equation (40), is the trace of the equations (30) and (31). We emphasize here that in order to obtain the stellar structures, it is important to require some asymptotic flatness as the radial coordinate evolves [23].
3.0.1 Perturbation TOV equations

In order to solve numerically the equations (38), (39) and (40), one proceeds by a perturbative approach. In the framework of a perturbative solution, the density, pressure, mass and scalar torsion can be expanded as [41, 42, 43]

\[ p = p^{(0)} + \xi p^{(0)}, \quad \rho = \rho^{(0)} + \xi \rho^{(0)}, \quad (42) \]
\[ m = m^{(0)} + \xi m^{(0)}, \quad T = T^{(0)} + \xi T^{(0)}, \quad (43) \]

where \( p^{(0)}, \rho^{(0)}, m^{(0)} \) and \( T^{(0)} \) satisfy the standard TOV equations (see [41] for the zeroth order mass). The scalar torsion at zeroth order can be specified as \( T^{(0)} = -16\pi \rho^{(0)} \). Finally, perturbative TOV equations can be established by [43]

\[ \frac{r_g}{r^2} \frac{dm}{dr} = \xi g_T^{(0)}(T) \left[ \frac{1}{2r^2}(1 - r_g) + \frac{1}{r^2 g^2} \left( 1 - \frac{2m(0)}{r} \right) - 8\pi r_g \rho^{(0)} \right] + \frac{1}{2r^2} \left( 1 - r_g \right) + \xi \rho^{(0)} \left( \rho + p \right)^{-1} \frac{dp}{dr} - \frac{1}{4} \xi g^{(0)}(T) - 4\pi \rho \]

\[ -\frac{1}{4} \left( 1 - \frac{2m(0)}{r} \right) \left( -\frac{3}{2r^2 g^2} + \frac{r_g}{4r^2} + \frac{\pi \rho^{(0)} r_g}{4r^2} \right) \left[ g_T^{(0)}(T) - \frac{1}{4} \xi g^{(0)}(T) - \frac{r_g}{16r^2} - \frac{1}{4r^2} - 4\pi p \right] = 0. \]

4 Strong magnetic field effect on the dense matter in the framework of the relativistic mean field: brief reviews

Our goal in this work consists to study the effect of strong magnetic field on the neutron stars in the framework of \( f(T) \) theory. In general, for nuclear matter containing baryon octet (\( b = p, n, \Lambda, \Sigma^{0, \pm}, \Xi^{0, -} \)) interacting with the following elements: a magnetic field \( B \) with quadrupotential \( A^{\mu\nu} = (0, 0, B_x, 0) \) and a scalar, isoscalar-vector \( \omega_{\mu} \) and isovector-vector \( \rho_{\mu} \), meson fields and leptons (\( l = e^-, \mu^- \)), the Lagrangian density is expressed as [28]:

\[ \mathcal{L} = \sum_b \bar{\psi}_b \left[ \gamma_\mu (i\partial^\mu - g_b A^\mu - g_{w b} \omega^\mu - \frac{1}{2} g_{\rho b} \tau^\mu \rho^b) \right] \psi_b + \sum_l \bar{\psi}_l \left[ \gamma_\mu (i\partial^\mu - q_l A^\mu) - m_l \right] \psi_l + \left( \frac{1}{2} \right) \left( \partial_\mu \sigma \right)^2 - m_\sigma^2 \sigma^2 - V(\sigma) - \frac{1}{4} F_{\sigma\lambda} F^{\sigma\lambda} + \frac{1}{2} m_\omega^2 \omega^2 - \frac{1}{4} \omega_{\sigma\lambda} \omega^{\sigma\lambda} - \frac{1}{4} \rho_{\sigma\lambda} \rho^{\sigma\lambda} + \frac{1}{2} m_\rho^2 \rho^{\mu\nu}, \right] \]
where the strong interaction couplings $g_{b\sigma}$, $g_{b\omega}$ and $g_{b\rho}$ depend on the density. Furthermore, the following relations $\rho_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ define the mesonic and electromagnetic field strength, respectively. We also assume the frozen-field configurations of electromagnetic field and neglect the anomalous magnetic moments of baryons and lepton because their effect is very small. The terms of strong interaction are parametrized by [24]

$$g_j(\rho) = a_j g_{j0} \frac{1 + b_j (x + d_j)^2}{1 + c_j (x + d_j)^2},$$

where $x = \rho / \rho_0$ and $a_j, b_j, c_j, d_j$ are constants (see [28]). The correspondent isovecteur field is also given by

$$g_{b\rho} = g_{b0} \exp[-a_\rho (x - 1)].$$

The mean field approximation constrains the mesonic fields in the following equations [24]

$$m_{\sigma}^2 \sigma + \frac{dV}{d\sigma} = \sum_b g_{b\rho} n_b^s, \quad m_{\omega}^2 \omega_0 = \sum_b g_{b\rho} n_b, \quad m_{\rho}^2 \rho_0 = \sum_b g_{\rho b} n_b.$$

With $\sigma$, $\omega_0$, $\rho_0$, the expectation values of meson fields in the uniform matter, $n_b^s$ and $n_b$ the associated scalar and vector baryon number densities, respectively. One recalls the very used scalar field potential

$$V(\sigma) = \frac{1}{3} p m_N (g_{\sigma N} \sigma)^3 + \frac{1}{4} q (g_{\sigma N} \sigma)^4,$$

where $p$ and $q$ are dimensionless constants and the values of nucleon-meson couplings and parameters $p$ and $q$ for different models are given in [24].

The energy spectra of charged and neutral leptons and baryons with effective mass $m_b^* = m_b - g_{\sigma b} \sigma$ results from Dirac equation and can be expressed by

$$E_T^b = (k_z^2 + m_b^* \gamma_z + 2 \gamma_T |q_b| B)^{1/2} + g_{\omega b} \omega_0 + \tau_{3b} g_{\rho b} \rho_0 + \Sigma_{-\lambda}^R,$$

$$E_b^b = (k_z^2 + m_b^* \gamma_z + 2 \gamma_l |q_l| B)^{1/2} + g_{\omega b} \omega_0 + \Sigma_{-\lambda}^R,$$

$$E_T^b = (k_z^2 + m_b^* \gamma_z + 2 \gamma_T |q_l| B)^{1/2}.$$
the spin degeneracy for lowest and others levels of Landau, respectively while the last term \( \Sigma^R_0 \) stays for the rearrangement self-energy term and is defined by

\[
\Sigma^R_0 = -\frac{\partial \ln g_{\sigma N}}{\partial n} m^2_{\sigma} \sigma^2 + \frac{\partial \ln g_{\omega N}}{\partial n} m^2_{\omega} \omega^2 + \frac{\partial \ln g_{\rho N}}{\partial n} m^2_{\rho} \rho^2,
\]

with \( n = \Sigma_b n_b \). We define the scalar densities for neutral and charged baryons [25] by

\[
n^{s,n}_b = \frac{m^*_b}{2\pi^2} \left( E^b_f k^b_f - m^*_b \ln \left| \frac{k^b_f + E^b_f}{m^*_b} \right| \right),
\]

for neutral baryons with Fermi energy \( E^b_f = (k^2_f + m^*_b)^{1/2} \) and

\[
n^{s,c}_b = \frac{|q_b| B}{2\pi^2} \sum_{\Upsilon} g_{\Upsilon} \ln \left| \frac{k^{b,\Upsilon}_f + E^b_f}{\sqrt{m^2_b + 2\Upsilon |q_b| B}} \right|,
\]

for charged baryons with Fermi energy \( (E^b_f = k^2_f + m^*_b + 2\Upsilon |q_b| B)^{1/2} \). The vector densities for both types of baryons are defined by

\[
n_b = \frac{1}{3\pi^2} k^{b,3}_f : \text{neutral baryons ,} \quad (57)
\]

\[
n_b = \frac{|q_b| B}{2\pi^2} \sum_{\Upsilon} g_{\Upsilon} k^{b,1}_{f,\Upsilon} : \text{for charged baryons.} \quad (58)
\]

We recall here that the summation on \( \Upsilon \) terminates at the values \( \Upsilon_{\text{max}} \) where the square is still positive. For the strong magnetic field \( B \sim 10^{18} \text{G} \), only few Landau levels are occupied. For hyperon-meson couplings there are no well-defined rule. One can use for these constants quark counting rule [29, 30]:

\[
g_{\omega \Lambda} = g_{\omega \Sigma} = \frac{2}{3} g_{\omega N}, \quad g_{\omega \Xi} = \frac{1}{2} g_{\omega N},
\]

and also

\[
g_{\rho \Sigma} = 2 g_{\rho N}, \quad g_{\rho \Xi} = g_{\rho N}.
\]

The hyperon-meson coupling are assumed to be fixed fraction of nucleon-meson couplings explicitly \( g_{iH} = x_{iH} g_{iN} \) where \( x_{\sigma H} = x_{\rho H} = 0.600 \) and \( x_{\omega H} = 0.653 \) (see [44]). The chemical potentials of baryons and leptons are defined by

\[
\mu_b = E^f_b + g_{\omega b} \omega_0 + g_{\rho b} \tau_3 \rho_0 + \Sigma^R_0, \quad \mu_l = E^f_l
\]
The following conditions should be imposed on the matter for obtaining the EOS:

(i) conservation of baryon number:
\[ n = \sum b n_b, \quad (62) \]

(ii) charge neutrality:
\[ \sum_j q_j n_j = 0, \quad j = b, l, \quad (63) \]

(iii) beta-equilibrium conditions:
\[ \mu_n = \mu_\Lambda = \mu_{\Xi^0} = \mu_{\Sigma^0}, \quad \mu_p = \mu_{\Sigma^+} = \mu_n - \mu_e, \quad \mu_{\Sigma^-} = \mu_{\Xi^-} = \mu_n + \mu_e, \quad \mu_\mu = \mu_e. \quad (64) \]

At given \( n \), equations (49) and (64) can be numerically solved and the Fermi energy for the particles and the meson fields can be found. One can also put the correspondingly energy density of the matter in the following form
\[ \varepsilon_m = \sum_b \varepsilon_b + \sum_l \varepsilon_l + \frac{1}{2} m_\sigma^2 + \frac{1}{2} m_\omega^2 + \frac{1}{2} m_\rho^2 + W(\sigma). \quad (65) \]

The energy densities of neutral and charged baryons are respectively given by
\[ \varepsilon^n_b = \frac{1}{4\pi^2} \left[ \frac{k_f^b(E_f^b)^3}{m_b^*} - \frac{1}{2} m_b^* \left( m_b^* k_f^b E_f^b + m_b^* I \ln \frac{k_f^b + E_f^b}{m_b^*} \right) \right], \quad (66) \]
\[ \varepsilon^c_b = \frac{|q_b| B}{4\pi^2} \sum_Y g_T \left( E_f^b k_f^b E_f^b + (m_b^* I^2 + 2Y|q_b|B) \ln \frac{k_f^b + E_f^b}{\sqrt{m_b^* I^2 + 2Y|q_b|B}} \right) \quad (67) \]

The last energy density can also stay for leptons by setting \( m_b^* \rightarrow m_l \). The matter pressure can be defined by
\[ p_m = \sum_b p_b + \sum_l p_l - \frac{1}{2} m_\sigma^2 + \frac{1}{2} m_\omega^2 + \frac{1}{2} m_\rho^2 - W(\sigma) + \Sigma_R \quad (68) \]

and as consequence, it can be established for neutral and charged baryons respectively as
\[ p^n_b = \frac{1}{12\pi^2} \left[ \frac{k_f^b(E_f^b)^3}{m_b^*} - \frac{3}{2} m_b^* \left( m_b^* k_f^b E_f^b + m_b^* I \ln \frac{k_f^b + E_f^b}{m_b^*} \right) \right], \quad (69) \]
\[ p^c_b = \frac{|q_b| B}{12\pi^2} \sum_Y g_T \left( E_f^b k_f^b E_f^b + (m_b^* I^2 + 2Y|q_b|B) \ln \frac{k_f^b + E_f^b}{\sqrt{m_b^* I^2 + 2Y|q_b|B}} \right) \quad (70) \]

Indeed, the equation of state taking into account the magnetic field is given by
\[ \varepsilon = \varepsilon_m + \frac{B^2}{8\pi}, \quad p = p_m + \frac{B^2}{8\pi}. \quad (71) \]
For the models where the magnetic field only depends on the baryon density, the parametrization given by [32, 33] is adopted

\[ B = B_s + B_0[1 - \exp(-\beta(n/n_s)^\gamma)] \]

(72)

\( B_s \) is the magnetic field at the surface taken equal to \( 10^{15} \) G in accordance with the values inferred from observations and \( B_0 \) represents the magnetic field for large densities. The parameters \( \beta \) and \( \gamma \) are chosen in such a way that the field decreases fast or slow with the density from the centre to the surface. In several works, it is used two sets of values: a slowly varying field with \( \beta = 0.05 \) and \( \gamma = 2 \), and a fast varying defined by \( \beta = 0.05 \) and \( \gamma = 2 \). Frequently the magnetic field is expressed in units of the critical field \( B_e = 4.414 \times 10^{13} \) G, so that \( B = B^* B_e \). But in Ref. [40], it is clearly shown that the structure of a magnetized neutron stars will be mostly affected by contributions from the magnetic field stress \( P_f = B^2/8\pi = 4.814 \times 10^{-8} B^2 \) Mev fm\(^{-3} \) which greatly exceeds the matter pressure \( P_m \) at all relevant densities for \( B^* \geq 10^5 \). We will use this result in our present work in order to investigate the effects of strong magnetic field on the mass-radius relation of neutron via the corrections of Teleparallel.

5 Main results

The effect of strong magnetic field has been developed in other kind of modified theories of gravity, namely the so called \( f(R) \) gravity [24, 25, 42]. In our case, we also investigate the effect of the strong magnetic field on some observables features of neutron stars such as the mass-radius relation. In order to reach our goal, we consider the perturbative TOV equations and the equation of state described in the previous section. For simplicity we make use of the result from Ref. [40] notified at the end of the previous section. For different values of magnetic field parameter \( B^* \), we plot the mass-radius relations for different corrections to Teleparallel term.

5.1 Quadratic scalar torsion corrections

Such gravity models are given by

\[ f(T) = T + \xi T^2. \]

(73)
It is one case of power-law action function \( f(T) = T + b_1 T^n \) studied by [23] to explore the mass-radius relation of neutron and quarks stars within \( f(T) \) theory. By considering this model, we numerically solve the perturbed TOV equations. From figure 1, we can note the following features for \( B^* = 0 \): the mass of the neutron stars is very affected by \( \xi \) and can differ from the one in Teleparallel. But for the two figures 2 and 3 which correspond to the large magnetic fields,

the maximal mass of neutron stars increases and can exceed the limit value \( 3M_\odot \) as it is notified in [24]. This can lead to stable configuration of neutron stars [40]. In this case of large magnetic field, the deviation from the Teleparallel is very appreciable in the case of figure 2 and it requires a large value of the coefficient of corrections.

5.2 Cubic scalar torsion corrections

The cubic corrections of Teleparallel can be expressed by the model:

\[
 f(T) = T + \xi T^3, \tag{74}
\]

which is also power-law function. The mass-radius relation according to this model has practically the same behaviours as in the case of quadratic corrections. The interesting feature which appears here is that the deviation from Teleparallel is very clearly appreciable (see figures 4 and 5). In figure 4, the mass of neutron stars increases as the correction parameter \( \xi \) evolves contrarily to the case of figure 5.

In general for the two models, we can note that the mass of neutron stars increases due to the very strong magnetic field which leads to the maximum mass configuration of stars as shown in [44]. These values should be able to describe highly massive compact stars, such as the one associated to the millisecond pulsars PSR B1516 + 02B [45], and the one in PSR J1748-2021B [46] if the existence of these stellar objects is confirmed. Due to the magnetic field pressure, the quarks are not favored, and stars with strong magnetic field are mainly hadronic with very small hyperon contributions. In addition, according to the considered magnetic field values, these curves can predict stars with very high masses and radius and masses of observed compact stars as it was mentioned in [32].
Figure 1: The figure shows the mass-radius relation in model $f(T) = T + \xi T^2$ for $B^* = 0$. The graphs are plotted for $T^{(0)} = -6 \times 10^2 \text{km}^2 \text{s}^{-2} \text{Mpc}^{-2}$ and $\xi = a \times 10^1$, with $a = -5, -4, -3, -2, 0, 2, 3, 4, 5$.

Figure 2: The figure shows the mass-radius relation in model $f(T) = T + \xi T^2$ for $B^* = 5 \times 10^{18}$. The graphs are plotted for $T^{(0)} = -6 \times 30^2 \text{km}^2 \text{s}^{-2} \text{Mpc}^{-2}$ and $\xi = a \times 10^{31}$, with $a = 0, 2, 3, 4, 5$.

Figure 3: The figure shows the mass-radius relation in model $f(T) = T + \xi T^3$ for $B^* = 3 \times 10^{18}$. The graphs are plotted for $T^{(0)} = -6 \times 30^2 \text{km}^2 \text{s}^{-2} \text{Mpc}^{-2}$ and $\xi = a \times 10^{31}$, with $a = 0, 2, 3, 4, 5$. 

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Figure 4: The figure shows the mass-radius relation in model $f(T) = T + \xi T^3$ for $B^* = 0$. The graphs are plotted for $T^{(0)} = -6 \times 30^2 \text{km}^2 \text{s}^{-2} \cdot \text{Mpc}^{-2}$ and $\xi = a \times 10^3$, with $a = -2, -1, 0, 1, 2$.

Figure 5: The figure shows the mass-radius relation in model $f(T) = T + \xi T^3$ for $B^* = 5 \times 10^{15}$. The graphs are plotted for $T^{(0)} = -6 \times 30^2 \text{km}^2 \text{s}^{-2} \cdot \text{Mpc}^{-2}$ and $\xi = a \times 10^{20}$, with $a = 0, 2, 4, 5$. 

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6 Conclusion

We have studied the influence of a static very strong magnetic field on the neutron stars through the power-law $f(T)$ gravity models. Our main goal consists to search how the strong magnetic field affects the mass-radius relation of neutron star namely the evolution of the neutron stars mass as it is done in most of GR modify theories of gravity such as $f(R)$ theory. In order to reach this goal, we have in the first time revisited the geometrical equivalence between Teleparallel Theory and GR. The second step of this work concerns the establishing of TOV equations in the framework of $f(T)$ gravity before proceeding to its perturbation. We then recall according to the literature, the equation of the state describing dense matter in magnetic field using a model with baryon octet interacting. This leads to the matter equation of state. We assume two cases of corrections to Teleparallel in order to solve numerically the perturbed equations and find out the evolution of the mass-radius relation and the deviation from Teleparallel by considering the very strong magnetic field. Our investigations show that the mass of neutron stars increase even in the case of null magnetic field where the correction parameter $\xi$ plays an important role. For the both quadratic and cubic corrections, the mass of neutron stars can increase and the deviation from Tele-Parallel is very appreciable in the case of cubic corrections. Moreover the evolution of the mass is very specified in this case: the mass increase as $\xi$ evolves (figure 4) and decreases in case of figure 5. Due to the consideration of ultrastrong magnetic, the magnetic pressure has only been considered and by consequence the quark star is not favored whereas the hadronic with very small hyperon contributions can be compatible with the results obtained in this work.

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