On characteristic equations, trace identities and Casimir operators of simple Lie algebras

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Abstract

Two approaches are developed to exploit, for simple complex or compact real Lie algebras \( g \), the information that stems from the characteristic equations of representation matrices and Casimir operators. These approaches are selected so as to be viable not only for ‘small’ Lie algebras and suitable for treatment by computer algebra. A very large body of new results emerges in the forms, a) of identities of a tensorial nature, involving structure constants etc. of \( g \), b) of trace identities for powers of matrices of the adjoint and defining representations of \( g \), c) of expressions of non-primitive Casimir operators of \( g \) in terms of primitive ones. The methods are sufficiently tractable to allow not only explicit proof by hand of the non-primitive nature of the quartic Casimir of \( g_2, f_4, e_6 \), but also e.g. of that of the tenth order Casimir of \( f_4 \).

1 Introduction

This paper is concerned with two related matters. One is the analysis of higher order Casimir operators of a simple complex or compact real Lie algebra \( g \) in terms of primitive Casimirs. The other is the provision of identities involving the structure constants of \( g \) and related invariant tensors, such identities being often presented as expressions for the trace of the product of matrices from either the defining or the adjoint representation of \( g \).

Much of course is known about these matters, and we attend with care below in the last paragraphs of this introduction to the relationship of our work to that of previous authors. Our purpose however has been to develop and apply methods that remain viable for \( g \) of large rank or dimension, and that are amenable to treatment by computer algebra. In fact, we claim to have a large body of new results, for many Lie algebras, not only for \( g_2, f_4 \) and \( e_6 \), but also for the classical families, with even quite a few for \( a_2 = su(3) \).

It is well known (see [1,2], for example, for \( a_\ell = su(\ell+1) \)) that identities for ad-invariant tensors for \( g \) can be separated into two classes. The first class contains results readily available

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for all \( \mathfrak{g} \), and having a common appearance for all \( \mathfrak{g} \) in any of the four classical families, which arise (as section 2 below explains in detail) from use of Jacobi type identities. The second class of identities, which take on a form special for each Lie algebra \( \mathfrak{g} \), arise from the use in some way of the characteristic equations of representation matrices and Casimir operators. The paper aims to justify its existence by its approaches to identities in the second class, to the results that flow from them, and from their application to the treatment of non-primitive Casimir operators.

We follow here two approaches to class 2 identities and characteristic equations. One is based on the characteristic equation of the second order Casimir operator on \( V_{\text{adj}} \otimes V_{\text{adj}} \) where \( V_{\text{adj}} \) is the adjoint representation of \( \mathfrak{g} \). The other proceeds by diagonalization of representation matrices of the defining and adjoint representations of \( \mathfrak{g} \).

It is convenient to explain these methods first for \( a_2 = su(3) \) or, at some points, \( su(n) \). Thus our paper is organized as follows. After a few brief remarks in section 2 about notation, we take up our two methods for \( su(3) \) in sections 3.1 and 3.2. While often the results here are well-known, their derivations may well be simpler, and, as subsequent work indicates, more amenable to generalization than previous ones. Furthermore, one finds, even in this well-studied context, new results of clear importance (cf. (3.38a) and (3.38b)).

Section 4 then proceeds briefly through application of the first method to the four classical families of Lie algebras and to \( g_2, f_4 \) and \( e_6 \), displaying a wide variety of new results, especially for the three exceptional algebras. Our strength ran out, as did the calls made on information storage by our computer programs, during the study of \( e_6 \), so that \( e_7 \) and \( e_8 \) have not been treated. Since our second method requires little explanation beyond that given in section 3.2 for \( su(3) \), and yields useful output easily by hand for simple cases, we simply present the results in an appendix. Much here is new and we think valuable.

For what is merely a selection of interesting items, we refer to eqs. (A.2) and (A.3) for \( a_2 = su(3) \), to the situation surrounding the primitivity for \( a_4 = su(5) \) of the traces \( \text{tr} F^{2k} \), to (A.42) for \( e_6 \), (A.48) for \( f_4 \) and corresponding results for \( g_2 \). As an indication of results for which pen and paper derivation were viable ahead of their confirmation by algebraic computation, we cite, for example, (A.51a)–(A.51d) and (A.52a) and (A.52b), as well as (A.48b), yes, for \( f_4 \).

A C-program was used to produce data regarding representations of \( \mathfrak{g} \) and the reduction of tensor products with only the Cartan matrix as input. Maple was used to prepare the appendix.

Finally, attention must be given to placing our paper in the context of previous work. To begin this, we recall that work on \( su(3) \) tensorial quantities, the two classes of identities, and use of characteristic equations, began many years ago, see e.g. [1–5] for \( su(n) \). See also [6, 7] for class 1 identities. The study of Casimir operators [8] likewise has a long history, which may be further traced from 2: see references 2 to 8 in [3]. That paper, 3, contains a canonical definition via cocycles [9] of a set of primitive Casimir operators for \( \mathfrak{g} \). It also relates these to a set of totally symmetric isotropic tensors for \( \mathfrak{g} \). In the process, various identities for such tensors are formed, especially for \( su(n) \). The present paper reproduces many of these, by methods that are in general easier and more readily extended. Ref. 2 employed the identities featured in it in the reduction of non-primitive Casimir operators in terms of primitive ones, a topic also carried much further here.

Next, we comment on the non-primitive nature of the quartic Casimir of \( g_2 \), proved by Okubo [11] on the basis of identities whose proof was not displayed by him, although it needs only an easy calculation especially using the method of section 3.2. Meyberg [12] gave a proof
of a similar result for all exceptional Lie algebras, and Cvitanović indicated (only, as far as we know, via private communication to Okubo) that proof of the same result was available via ‘bird-track’ methods \[13\]. Our method of proof was easy enough to enable us to extend it by hand to show the non-primitive nature of the tenth order Casimir of \(f_4\).

It should be remarked also there is much common ground between the outlook of this paper and the work of Cvitanović \[13, 14\] even though the latter is devoted to the development of diagrammatic methods.

Next, we note that Meyberg \[15\] used the decomposition of \(V_{\text{adj}} \otimes V_{\text{adj}}\) in conjunction with trace identities, but only as far as fourth powers, and not in conjunction with tensorial methods. Another significant work on trace calculations is that of Mountain \[16\]. His method and the work of Cvitanović \[13, 14\] even though the latter is devoted to the development of diagrammatic methods.

Let \(g\) be a simple complex or compact real Lie algebra. For the general discussions we choose a basis \((X_j)\) of \(g\) such that the Cartan-Killing form is \(\kappa_{jk} = \text{tr}(\text{ad}_j \circ \text{ad}_k) = -\delta_{jk}\), and write the Lie product, with totally antisymmetric structure constants \(C_{jkl}\), in the form

\[
[X_j, X_k] = C_{jkl} X_l.
\]

(2.1)

In discussing specific examples we adhere to conventions commonly used in the physics literature. Thus, for \(su(n)\), we use the Gell-Mann matrices \(\lambda_j\). These are a set of \(n \times n\) traceless hermitean matrices, normalized according to \(\text{tr}(\lambda_j \lambda_k) = 2\delta_{jk}\). They have the multiplication law

\[
\lambda_j \lambda_k = \frac{2}{n} \delta_{jk} \mathbf{1} + (d_{jkl} + if_{jkl}) \lambda_l,
\]

(2.2)

where the completely symmetric \(d\)-tensor satisfies \(\mathbf{1} d_{jkl} = 0\). Since (2.2) implies \([\lambda_j, \lambda_k] = 2i f_{jkl} \lambda_l\), and the basis vectors of \(su(n)\) are defined by \(x_j = \lambda_j/2\), the \(f_{jkl}\) serve as structure constants for \(su(n)\). The relationship of this basis to that of the general discussion is seen via \(\text{tr}(\text{ad}_j \circ \text{ad}_k) = n \delta_{jk}\), and is given by \(X_j = i x_j / \sqrt{n}\) and therefore \(C_{jkl} = -f_{jkl} / \sqrt{n}\).

The well known class 1 identities which are valid in general for all \(n\) follow from Jacobi type identities

\[
0 \quad = \quad [[\lambda_j, \lambda_k], \lambda_l] + [[\lambda_k, \lambda_j], \lambda_j] + [[\lambda_j, \lambda_l], \lambda_k],
\]

(2.3a)

\[
0 \quad = \quad [[\lambda_j, \lambda_k], \lambda_l] + \{\lambda_j, [\lambda_k, \lambda_l], \lambda_j\} - \{\lambda_k, [\lambda_j, \lambda_l], \lambda_j\},
\]

(2.3b)

\[
0 \quad = \quad [[\lambda_j, \lambda_k], \lambda_l] + \{\lambda_k, [\lambda_j, \lambda_l], \lambda_j\} + \{\lambda_j, [\lambda_k, \lambda_l], \lambda_j\},
\]

(2.3c)

together with consequences based on trace properties and completeness relations for the \(\lambda_j\).

Class 2 identities emerge in forms different for each \(n\). Many of them are absent or degenerate for \(n = 2\), and, in general, exhibit a complexity that increases with \(n\). Such class 2 identities stem from the use of the characteristic equation \[1, 4\], e.g. of \(A = a_j \lambda_j, a_j \in \mathbb{C}\). The procedure is already cumbersome \[2, 3\] for modestly large \(n\).
3 Description of the methods

In this section we describe our methods and apply them to $su(n)$, $n = 3$ or $4$ for illustrative purpose.

3.1 Tensor products of the adjoint representation

Our first method uses the tensor product $V_{\text{adj}} \otimes V_{\text{adj}}$, where $V_{\text{adj}}$ denotes the adjoint representation of $\mathfrak{g}$, and is based on the characteristic equation of the second order Casimir operator. It offers a convenient way of deriving certain class 2 identities, and as a by-product yields explicit expressions (in tensorial form) for Clebsch-Gordan coefficients occurring in the reduction.

Consider the tensor product of a finite-dimensional irreducible representation $V$ of $\mathfrak{g}$ and its decomposition into irreducible components $W_i$

$$V \otimes V \simeq W_1 \oplus W_2 \oplus \cdots \oplus W_k. \quad (3.1)$$

For example let $V$ be the adjoint representation $[8]$ of $su(3)$, then

$$[8] \otimes [8] = [1] \oplus [8] \oplus [27] \oplus [8] \oplus [10] \oplus [10], \quad (3.2)$$

or in $su(4)$

$$[15] \otimes [15] = [1] \oplus [15] \oplus [20] \oplus [84] \oplus [15] \oplus [45] \oplus [45]. \quad (3.3)$$

These decompositions are always understood over the field of complex numbers. In the basis $(X_i)$, the second order Casimir operator on $V$ is given by

$$C_V = - \sum_{r=1}^n X_r^2. \quad (3.4a)$$

It acts on $V \otimes V$ as

$$C_{V \otimes V} = - \sum_{r=1}^n (X_r \otimes 1 + 1 \otimes X_r)^2 = C_V \otimes 1 + 1 \otimes C_V + 2 L, \quad (3.4b)$$

where we have defined

$$L = \frac{1}{2}(C_{V \otimes V} - C_V \otimes 1 - 1 \otimes C_V) = - \sum_{r=1}^n X_r \otimes X_r. \quad (3.4c)$$

$L$ has the same $\mathfrak{g}$-invariant subspaces as $C_{V \otimes V}$. Written with indices, it has the form

$$L_{jk,pq} = -C_{jpr}C_{kqr}. \quad (3.5)$$

The normalization of the Casimir operators obviously is important when their eigenvalues are considered. Our normalization of $C_V$ is such that the adjoint representation has eigenvalue 1 regardless of the algebra. The eigenvalue is furthermore equal to $\langle \Lambda, \Lambda + 2 \delta \rangle$ where $\Lambda$ is the
highest weight of a finite-dimensional irreducible representation, \( \delta \) denotes the half-sum of positive roots of \( g \) and \( \langle \cdot, \cdot \rangle \) is induced from the Cartan-Killing form on the space of weights.

When \( V \) denotes the adjoint representation of \( su(n) \), we have, for example, \( L_{jk,pq} = -\frac{1}{n} f_{rpj} f_{rqk} \). If we were working only with \( su(n) \), we would absorb the factor \( 1/n \) into the definition of \( L \) to make the eigenvalues integral, but this does not allow uniform treatment of algebras of different series.

Assume that in the decomposition \((3.1)\) \( m \) projectors \( P^{(1)}, \ldots, P^{(m)} \) onto the \( W_1, \ldots, W_m \) are explicitly known \((m < k)\). Then \( P^{(\text{others})} = \mathbf{1} - P^{(1)} - \cdots - P^{(m)} \) projects onto the sum \( W_{m+1} \oplus \cdots \oplus W_k \). The characteristic equation of \( L \) (which has the same structure as that of \( C_{V \otimes V} \)) implies

\[
(L - \ell_{m+1} \mathbf{1}) \cdots (L - \ell_k \mathbf{1}) P^{(\text{others})} = 0,
\]

where \( \ell_i \) are the eigenvalues of \( L \) on the components \( W_i \). Eq. \((3.6)\) implies a relation of the form

\[
L^{k-m} = c_1 L^{k-m-1} + \cdots + c_{k-m} \mathbf{1} + c_1' P^{(1)} + \cdots + c_m' P^{(m)},
\]

with coefficients \( c_i \) and \( c_i' \). Eq. \((3.7)\) is used in the following to reduce powers of \( L \).

In the decomposition \((3.2)\) for \( su(3) \), the following projectors have a very simple form:

\[
P_{[jk,pq]}^{(1)} = \frac{1}{8} \delta_{jk} \delta_{pq}, \quad P_{[jk,pq]}^{[8]} = \frac{3}{5} d_{jkr} d_{pq}, \quad P_{[jk,pq]}^{[8]} = \frac{1}{3} f_{jkr} f_{pq},
\]

where \([8]_S\) and \([8]_A\) denote the adjoint representation in the symmetric and antisymmetric part of the decomposition \((3.2)\), respectively.

The eigenvalues of \( C_{V \otimes V} \) on the representations \([1]_S, [8]_S\) and \([27]_S\) are 0, 1 and 8/3, respectively, in view of the normalization defined in eq. \((2.3)\). We thus have \( \ell_{[1]} = -1, \ell_{[8]} = -1/2 \) and \( \ell_{[27]} = 1/3 \). Eq. \((3.7)\) for the symmetric part of \((3.2)\) reads

\[
L [1]_S = \frac{1}{3} [1]_S - \frac{4}{3} P^{(1)} - \frac{5}{6} P^{[8]},
\]

where \([1]_S\) \(= \frac{1}{2}(\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) \) projects onto the symmetric part of the tensor product. In terms of the \( f- \) and \( d- \) tensors of \( su(3) \), eq. \((3.8)\) reads

\[
f_{jpr} f_{kq} + f_{jpq} f_{kpr} = -\delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp} + \delta_{jk} \delta_{pq} + 3 d_{jkr} d_{pq}.
\]

This is a well known identity. It coincides eq. \((2.23)\) in \([1]\).

A similar argument could be applied to the antisymmetric part of the decomposition \((3.2)\), but this would only give the Jacobi identity as a relation. Reversing the argument, we can use the Jacobi identity to determine the constituents of the antisymmetric part. If \( V_{\text{adj}} \) denotes the adjoint representation of \( g \), there is always an adjoint component in the antisymmetric part of \( V_{\text{adj}} \otimes V_{\text{adj}} \) determined by the projector

\[
P_{[jk,pq]}^{[\text{adj}]} = C_{jk} C_{pq},
\]

or, in the \( su(3) \) example, by the last entry in \((3.8)\). Because of \((3.3)\), the Jacobi identity is the same as

\[
L [1]_A = -\frac{1}{2} P^{[\text{adj}]}_A,
\]

where \([1]_A\) \(= \frac{1}{2}(\delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}) \) is the antisymmetrizer. This implies

\[
L^2 [1]_A = -\frac{1}{2} L [1]_A.
\]
Hence the minimal equation of $L$ on the antisymmetric subspace, for all $g$, takes the form

$$L(L + \frac{1}{2})\mathbb{I}_A = 0.$$  

(3.13)

We conclude that the only representations contained in the antisymmetric subspace have eigenvalues 0 or $-1/2$ of $L$, i.e. eigenvalues 2 or 1 of $C_{V\otimes V}$. We employ this fact in section 4.3 in the discussion of the exceptional simple Lie algebra $g_2$ to associate representations of the same dimension with the symmetric versus antisymmetric part. The relations (3.12) are furthermore useful to get rid of the symmetrizer in eq. (3.9).

Now we apply a Jacobi type identity twice to eq. (3.10) and derive

$$d_{r(jk)d_pq} = \frac{1}{3}\delta(jk)d_pq,$$  

(3.14)

obtained as eq. (2.22) in [1] from the characteristic equation of $A = a_j\lambda_j$, $a_j \in \mathbb{C}$. We see (3.14) easily allows us to calculate

$$\text{tr}(\lambda(i\lambda_j\lambda_k)\lambda_\ell) = 2\delta(i\delta_k\delta_\ell),$$  

(3.15)

and

$$\text{tr}A^4 = \frac{1}{2}(\text{tr}A^2)^2,$$  

(3.16)

which reflects the fact (trivial in this context) that the fourth order Casimir operator [11, 23] of $su(3)$ is not primitive. Our second method described in section 3.2 analyzes these dependencies systematically.

For $su(4)$, referring to the decomposition (3.13), we note that the projectors $P^{[1]}$ and $P^{[15s]}$ belonging to the symmetric part have a simple form, but the two remaining ones, $P^{[20]}$ and $P^{[84]}$, have not. The eigenvalues of $C_{V\otimes V}$ are 0, 1, 3/2 and 5/2. Therefore $\ell^{[1]} = -1$, $\ell^{[15]} = -1/2$, $\ell^{[20]} = -1/4$ and $\ell^{[84]} = 1/4$. Since there are two projectors which are not known immediately, we derive instead of (3.9) an equation for the square of $L$:

$$L^2\mathbb{I}_S = \frac{1}{16}\mathbb{I}_S + \frac{15}{16}P^{[1]} + \frac{3}{16}P^{[15s]},$$  

(3.17)

from which the term proportional to $L\mathbb{I}_S$ accidentally vanishes. This means

$$f_{jmr}f_{knr}(f_{mps}f_{nqs} + f_{mqs}f_{nps}) = \delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp} + 2\delta_{jk}\delta_{pq} + 2d_{jkr}d_{pqr}.$$  

(3.18)

Using the relations (3.12), we obtain

$$L^2 = \frac{1}{16}\mathbb{I}_S + \frac{15}{16}P^{[1]} + \frac{3}{16}P^{[15s]} + \frac{1}{4}P^{[15A]}$$  

(3.19)

or

$$2f_{jmr}f_{knr}f_{mps}f_{nqs} = \delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp} + 2\delta_{jk}\delta_{pq} + 2d_{jkr}d_{pqr} + 2f_{jkr}f_{pqr}.$$  

(3.20)

This is equivalent to eq. (A.5) in [2] and can be seen, by use of eq. (2.31), to imply eq. (A.11) in [4] in the case $n = 4$.

The method described above has the advantage that it does not rely on a well-developed $f$- and $d$-tensor technique but instead refers to the representation theory of the adjoint representation. It is therefore applicable even in cases where the tensor calculations are of significant difficulty (e.g. $g_2$, section 4.3) or not available at all (e.g. $f_4$, $e_6$, section 4.6 and 4.7).
As a by-product, it allows us to determine the remaining projectors, and thus Clebsch-Gordan coefficients, via

$$P^{(j)} = \prod_{i \neq j} \frac{L - \ell_i \mathbb{1}}{\ell_j - \ell_i}, \quad (3.21)$$

where relation (3.7) reduces the power of the operator $L$ appearing in the expression for $P^{(j)}$ to at most $L^{k-m-1}$.

In $su(3)$ there is, of course [1],

$$P^{[27]}_{jk,pq} = \left( \mathbb{1} - P^{[1]} - P^{[8]} \right)_{jk,pq} = \frac{1}{2} (\delta_{j,k} \delta_{p,q} + \delta_{j,q} \delta_{k,p}) - \frac{1}{8} \delta_{j,k} \delta_{p,q} - \frac{3}{5} d_{jk} d_{pq} r. \quad (3.22)$$

But in $su(4)$ we have the less obvious but clearly useful results

$$P^{[20]}_{jk,pq} = \frac{1}{4} (\delta_{j,p} \delta_{k,q} + \delta_{j,q} \delta_{k,p} + f_{jpr} f_{kqr} + f_{jqr} f_{kpr}) - \frac{1}{6} \delta_{j,k} \delta_{p,q} - \frac{1}{2} d_{jk} d_{pqr}, \quad (3.23a)$$

$$P^{[84]}_{jk,pq} = \frac{1}{4} (\delta_{j,p} \delta_{k,q} + \delta_{j,q} \delta_{k,p} - f_{jpr} f_{kqr} - f_{jqr} f_{kpr}) + \frac{1}{10} \delta_{j,k} \delta_{p,q} + \frac{1}{6} d_{jk} d_{pqr}. \quad (3.23b)$$

Since we use the characteristic (more precisely: minimal) polynomial of $L$ in (3.6), our method fails to separate projectors onto representations which have the same eigenvalues of the quadratic Casimir operator as e.g. two conjugate representations do have. Possible treatments of this point involve the use of higher order Casimir operators or the explicit consideration of a conjugation operation in the field of complex numbers [3].

While we have examined here only the decomposition of the adjoint representation (in order to derive relations involving the structure constants), the same method can be applied to other representations to obtain convenient expressions for some of the projectors occurring there.

### 3.2 Relations of trace polynomials

Our second method is based on the characteristic equation of representation matrices $A = a_j x_j$ of elements $a_j X_j \in \mathfrak{g}, a_j \in \mathbb{C}$. It enables us to derive explicit relations between invariant polynomials of $\mathfrak{g}$ and to express non-primitive polynomials explicitly in terms of primitive ones. This gives rise to further class 2 identities that generalize (3.16).

Let $G$ be any connected Lie group with Lie algebra $\mathfrak{g}$. Due to a theorem of Chevalley (see e.g. [24, 25]) the algebra $I_\mathfrak{g}$ of polynomials on $\mathfrak{g}$ that are invariant under the adjoint action of $G$ is isomorphic to a polynomial algebra in $\ell = \text{rank} \mathfrak{g}$ indeterminates. Since the generalized Casimir operators [1, 24, 26] and the centre of the universal enveloping algebra $U(\mathfrak{g})$ can be constructed from these polynomials, an explicit description of them is desired.

Let $A = a_j x_j$ be the matrix of an arbitrary element $a_j X_j \in \mathfrak{g}$ in the $d$-dimensional defining representation. One possible strategy [16] is to consider the characteristic polynomial of $A$

$$\chi_A(t) = \det(A - t \mathbb{1}) = \sum_{j=0}^{d} p_{d-j}(A) t^j, \quad (3.24)$$

whose coefficients $p_k(A)$ are homogeneous polynomials of degree $k$ in the elements of the matrix $A$. They are $G$-invariant by construction. Depending on the Lie algebra, a certain set of $\ell$ polynomials can be selected that generates $I_\mathfrak{g}$ freely [24, 23]. These generators are
therefore called primitive, and their degrees are a property of \( \mathfrak{g} \) itself. For easy reference, we include the table of their degrees for the simple Lie algebras in table 1.

Another way of constructing manifestly \( G \)-invariant polynomials uses the trace polynomials \( \text{tr}(A^k) \), \( k \in \mathbb{N} \), which we study below. Again it is desired to select a subset of algebraically independent polynomials which have the required degrees and therefore freely generate \( I_{\mathfrak{g}} \).

The fact that eq. (3.24) only gives primitive \( G \)-invariant polynomials up to a certain degree \( m \) (at least not higher than the dimension \( d \) of the defining representation) was exploited in \( [16] \) to derive relations expressing \( \text{tr}(A^k) \), \( k > m \), as polynomials in the lower degree traces. But the method in \( [16] \) needs additional input in certain situations. For example, the identities for \( \mathfrak{g}_2 \) that account for the non-primitive nature of the quartic Casimir operator of \( \mathfrak{g}_2 \) are not themselves generated by the method. Thus, we seek a systematic approach which expresses all non-primitive \( \text{tr}(A^j) \) in terms of primitive ones.

In order to perform explicit calculations involving the \( \text{tr}(A^j) \), we exploit their invariance under a change of basis in their representation space in order to diagonalize \( A \). The resulting matrix can be seen to belong to the Cartan subalgebra \( \mathfrak{g}_0 \) of \( \mathfrak{g} \) and therefore may be written

\[
A = \gamma_j h_j, \quad 1 \leq j \leq \ell = \text{rank} \, \mathfrak{g},
\]

where \( h_1, \ldots, h_\ell \) span \( \mathfrak{g}_0 \). They are determined by the weights of the representation. The expressions \( \text{tr}(A^k) \) are polynomials of degree \( k \) in the \( \ell \) indeterminates \( \gamma_1, \ldots, \gamma_\ell \). To express a non-primitive \( \text{tr}(A^k) \) in terms of the primitive ones means to write it as a linear combination of products of the primitive ones. The problem of finding the relations is reduced to a (very manageable) problem in linear algebra.

It follows from the construction that the resulting relations do not depend on the choice of basis of \( \mathfrak{g}_0 \) and therefore not on the normalization or orthogonality of the weights.

| simple Lie algebra \( \mathfrak{g} \) | degrees                  |
|----------------------------------|---------------------------|
| \( a_\ell, \ell \geq 1 \)        | 2, 3, \ldots, \ell + 1    |
| \( b_\ell, \ell \geq 2 \)        | 2, 4, \ldots, 2\ell       |
| \( c_\ell, \ell \geq 3 \)        | 2, 4, \ldots, 2\ell       |
| \( d_\ell, \ell \geq 4 \)        | 2, 4, \ldots, 2\ell - 2, \ell |
| \( e_6 \)                        | 2, 5, 6, 8, 9, 12        |
| \( e_7 \)                        | 2, 6, 8, 10, 12, 14, 18  |
| \( e_8 \)                        | 2, 8, 12, 14, 18, 20, 24, 30 |
| \( f_4 \)                        | 2, 6, 8, 12              |
| \( g_2 \)                        | 2, 6                      |

Table 1: Degrees of primitive invariant polynomials of the simple Lie algebras.

\[
\text{tr} A = 0, \quad (3.27a)
\]
\[ \text{tr } A^2 = 2(\alpha^2 + \alpha\beta + \beta^2), \quad (3.27b) \]
\[ \text{tr } A^3 = 3(\alpha^2\beta + \alpha\beta^2), \quad (3.27c) \]
\[ \text{tr } A^4 = 2(\alpha^4 + 2\alpha^3\beta + 3\alpha^2\beta^2 + 2\alpha\beta^3 + \beta^4), \quad (3.27d) \]

\[ \ldots \]

Because of the degrees of primitive invariant polynomials of \(su(3)\) (table 1) we can select \(\text{tr } A^2\) and \(\text{tr } A^3\) as generators of the algebra of invariant polynomials. The other \(\text{tr } A^k\) can be seen to satisfy the relations

\[ \text{tr } A^4 = \frac{1}{2}(\text{tr } A^2)^2, \quad (3.28a) \]
\[ \text{tr } A^5 = \frac{5}{6}(\text{tr } A^2)(\text{tr } A^3), \quad (3.28b) \]
\[ \text{tr } A^6 = \frac{1}{4}(\text{tr } A^2)^3 + \frac{1}{3}(\text{tr } A^3)^2, \quad (3.28c) \]

\[ \ldots \]

and many more (see appendix A.1.1). It is also easy to compute the characteristic polynomial of \(A\),

\[ \chi_A(t) = t^3 - \frac{1}{2}(\text{tr } A^2)t - \frac{1}{3}(\text{tr } A^3), \quad (3.29) \]

which has been explicitly known for a long time \([1]\). We are also able to reproduce eq. (6.5) and (6.7) from \([2]\) by translating the relations into the language of generalized \(d\)-tensors (see \([4]\)). We have, for example,

\[ \text{tr}(\lambda_{i_1\lambda_{i_2}\lambda_{i_3}}) = 2d_{(i_1i_2i_3)}; \quad (3.30a) \]
\[ \text{tr}(\lambda_{i_1\lambda_{i_2}\lambda_{i_3}\lambda_{i_4}}) = \frac{4}{n}d_{(i_1i_2i_3i_4)} + 2d_{(i_1i_2i_3i_4)}; \quad (3.30b) \]
\[ \text{tr}(\lambda_{i_1\cdots\lambda_{i_5}}) = \frac{2}{n}d_{(i_1i_2i_3i_4i_5)} + 2d_{(i_1i_2i_3i_4i_5)}; \quad (3.30c) \]

\[ \ldots \]

the first of which defines the third order invariant tensor already present in \(su(3)\). Whereas in \(su(3)\) the higher order \(d_{(i_1\cdots i_k)}^{(k)}\) can be reduced to this and to the second order \(\delta_{i_1i_2}\) tensor, for general \(su(n)\) there are in total \(n - 1\) primitive of them. Using the expressions (3.30), our method easily reproduces the symmetrized forms of eq. (27) to (29) in \([4]\) for \(su(3)\) and likewise for higher \(su(n)\).

A similar procedure works with the matrices \(F = a_jF_j; \ a_j \in \mathbb{C}, \ (F_j)_{k\ell} = iF_{j\ell k}\) of the adjoint representation of \(su(3)\). Here we have \(\text{tr}(F_jF_k) = 3\delta_{jk}\). Using the same basis of the Cartan subalgebra as in eq. (3.26), we write

\[ F = 2\alpha F_3 + \beta(F_3 + \sqrt{3}F_8) \]
\[ = \text{diag}(\alpha + 2\beta, \alpha - \beta, 2\alpha + \beta, 0, 0, -2\alpha - \beta, -\alpha + \beta, -\alpha - 2\beta), \quad (3.31) \]

so that \(\text{tr } F^{2k-1} = 0\) for all \(k \in \mathbb{N}\), and

\[ \text{tr } F^2 = 12(\alpha^2 + \alpha\beta + \beta^2), \quad (3.32a) \]
\[ \text{tr } F^4 = 36(\alpha^4 + 2\alpha^3\beta + 3\alpha^2\beta^2 + 2\alpha\beta^3 + \beta^4), \quad (3.32b) \]

\[ \ldots \]
Although these $\text{tr} F^{2k}$ have unsuitable degrees to generate the algebra of invariant polynomials freely, their relations are worth noting,

\begin{align}
\text{tr} F^4 & = \frac{1}{4}(\text{tr} F^2)^2, \\
\text{tr} F^8 & = -\frac{5}{192}(\text{tr} F^2)^4 + \frac{2}{3}(\text{tr} F^2)(\text{tr} F^6),
\end{align}

(3.33a)

(3.33b)

\begin{align*}
\text{as well as is the characteristic polynomial of } F, \\
\chi_F(t) = t^8 - \frac{1}{2}(\text{tr} F^2) t^6 + \frac{1}{16}(\text{tr} F^2)^2 t^4 + \left(\frac{1}{96}(\text{tr} F^2)^3 - \frac{1}{6}(\text{tr} F^6)\right) t^2.
\end{align*}

(3.34)

In eq. (3.33) we have listed only those relations which express a certain $\text{tr} F^{2k}$ in terms of lower degree traces. Since the $\text{tr} F^{2k}$ fail to define a third order invariant, the $\text{tr} F^{2k}$ are unable to generate the algebra of invariant polynomials freely. It turns out, further, that in $a_4 = su(5)$ (see appendix A.1.3) the $\text{tr} F^{2k}, 2k \in \{2, 4, 6, 8, 10\}$ cannot be written as polynomials in the lower degree traces. The fact that there are five of them while the rank of the algebra is only four, does not contradict any theorem because they do not generate the algebra freely. Instead they should satisfy more complicated (i.e. higher order) relations which we have not analyzed systematically.

Eq. (3.33a) is already of interest; it yields

\begin{align}
\text{tr}(F_i F_j F_k F_\ell) = \frac{9}{4}\delta_{(ij}\delta_{k\ell)},
\end{align}

(3.35)

which agrees (A.12) of [2], but has been more simply derived here. As a further check on our work, we confirmed that our result for $\text{tr} F^8$ agrees with relations derived in a different way in [16].

Since we are using the same basis of the Cartan subalgebra for the diagonal forms of both $A$ and $F$, we are able to express the $\text{tr} F^{2k}$ in terms of the primitive elements $\text{tr} A^2$ and $\text{tr} A^3$, namely

\begin{align}
\text{tr} F^2 & = 6(\text{tr} A^2), \\
\text{tr} F^4 & = 9(\text{tr} A^2)^2, \\
\text{tr} F^6 & = \frac{33}{2}(\text{tr} A^2)^3 - 18(\text{tr} A^3)^2, \\
\text{tr} F^8 & = \frac{129}{4}(\text{tr} A^2)^4 - 72(\text{tr} A^2)(\text{tr} A^3)^2, \\
\end{align}

(3.36a)

(3.36b)

(3.36c)

(3.36d)

\begin{align*}
\text{where eq. (3.36a) reflects the relative normalizations of the matrices } A \text{ and } F. \text{ The characteristic polynomial of } F \text{ can thus be given in a form more useful than (3.34) }
\chi_F(t) = t^8 - 3(\text{tr} A^2) t^6 + \frac{9}{4}(\text{tr} A^2)^2 t^4 + \left(-\frac{1}{2}(\text{tr} A^2)^3 + 3(\text{tr} A^3)^2\right) t^2.
\end{align*}

(3.37)

From eq. (3.36c) and eq. (3.36d), for example, we can show that

\begin{align}
\text{tr}(F_{(i_1 \cdots i_6)}) & = \frac{33}{16}\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} - \frac{9}{8}d_{(i_1 i_2 i_3}{d_{i_4 i_5 i_6)}} ,
\end{align}

(3.38a)

\begin{align}
\text{tr}(F_{(i_1 \cdots i_8)}) & = \frac{129}{64}\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6}\delta_{i_7 i_8} - \frac{9}{4}d_{(i_1 i_2}{d_{i_3 i_4 i_5}{d_{i_6 i_7 i_8)}} ,
\end{align}

(3.38b)
which are new results, the first being related to but not easily derived from (A.21) in [2]. The second could be also derived from eq. (3.33b) and eq. (3.38a).

Since this method of determining the explicit relations among invariant polynomials only relies on the properties of the relevant polynomials, it is well suitable for automatization using computer algebra systems. Applications to other rank 2 algebras including \( g_2 \) can still be easily performed by hand, whereas for higher rank the polynomials become quite lengthy. We performed many calculations, including these for low degrees in the rank 2 examples by hand, and used Maple to confirm our results and to handle the more complicated computations.

4 Applications and selected results

In this section we describe the most important results concerning other Lie algebras than those considered in the examples of section 3, and we comment on some aspects of them. A more comprehensive list of results obtained by our second method is contained in appendix A.

4.1 The simple Lie algebras \( a_{\ell}, \ell \geq 3 \)

In this paragraph we summarize our results for the simple Lie algebras \( a_{\ell}, \ell \geq 3 \) (or \( su(n), n = \ell + 1 \geq 4 \)). In order to write them in a fashion independent of \( n \), we characterize the representations by their highest weight which we specify in terms of the fundamental weights \( \Lambda_1, \ldots, \Lambda_\ell \) in standard form [24, 27]. The adjoint representation is therefore \((1,0,\ldots,0,1)\), and we find the decomposition into irreducible components

\[
(1,0,\ldots,0,1) \otimes (1,0,\ldots,0,1) = (0,0,\ldots,0) \oplus (1,0,\ldots,0,1) \oplus (0,1,0,\ldots,0,1,0) \oplus (2,0,\ldots,0,2)
\]

where, for example, \((0,1,0,\ldots,0,1,0)\) corresponds to the highest weight \( \Lambda = \Lambda_2 + \Lambda_{\ell-1} \). This reduces to e.g. \((0,2,0)\) if \( \ell = 3 \). The dimensions and the eigenvalues of the quadratic Casimir operator and of \( L \) can be directly computed from the Cartan matrix and are listed in table 2.

In the decomposition (4.1), the following projectors \( P^{(i)} \) (where \( i \) is used as shown in the first column of table 2) are initially known:

\[
P^{(1)}_{jk,pq} = \frac{1}{n^2-1} \delta_{jk} \delta_{pq},
\]

\[
P^{(2s)}_{jk,pq} = \frac{n}{n^2-4} d_{jkr} d_{pqr},
\]

\[
P^{(2A)}_{jk,pq} = \frac{1}{n} f_{jkr} f_{pqr}.
\]

Relation (3.7) gives in this case

\[
L^2 = \frac{1}{n^2-1} S + \frac{n^2-1}{n^2} P^{(1)} + \frac{n^2-4}{4n^2} P^{(2s)} + \frac{1}{2} L_A,
\]

\[
4 f_{jmr} f_{knr} f_{mps} f_{nqs} = 2(\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) + 4 \delta_{jk} \delta_{pq} + n( d_{jkr} d_{pqr} + f_{jkr} f_{qpr} ).
\]
Applying eq. (2.3b) we see that (4.3b) coincides with eq. (A.5) and implies eq. (A.11) in [2]. The remaining projectors
\[ P_{6,0}^{(0,\ldots,0)}, \quad P_{5,0}^{(1,0,\ldots,0,1)}, \quad P_{4,0}^{(0,1,0,\ldots,0,1,0)}, \quad P_{3,0}^{(2,0,\ldots,0,2)}, \quad P_{2,0}^{(2,0,\ldots,0,1,0)}, \quad P_{1,0}^{(0,1,0,\ldots,0,2)} \]
are also given. The number \( i \) is used to refer to them in the text.

The irreducible components occurring in the tensor product of the adjoint representation of \( a_\ell, \ell \geq 3 \), with itself. The representations are of dimension \( d \) and have the highest weight \( \Lambda \). The eigenvalues of the quadratic Casimir operator \( C \) and of \( L \) (see eq. (3.5)) are also given.

| \( i \) | representation of \( a_\ell \) | \( d \) | \( \Lambda \) | \( C \) | \( L \) |
|---|---|---|---|---|---|
| 1 | \((0,\ldots,0)\) | 1 | 0 | 0 | -1 |
| 2 | \((1,0,\ldots,0,1)\) | \(\ell(\ell+2)\) | \(\Lambda_1 + \Lambda_\ell\) | 1 | \(-\frac{1}{\ell+1}\) |
| 3 | \((0,1,0,\ldots,0,1,0)\) | \(\frac{1}{4}(\ell+1)^2(\ell+2)(\ell-2)\) | \(\Lambda_2 + \Lambda_{\ell-1}\) | \(\frac{2}{\ell+1}\) | \(-\frac{1}{\ell+1}\) |
| 4 | \((2,0,\ldots,0,2)\) | \(\frac{1}{4}(\ell+1)^2(\ell+4)\) | \(2\Lambda_1 + 2\Lambda_\ell\) | \(\frac{2\ell+2}{\ell+1}\) | \(-\frac{1}{\ell+1}\) |
| 5 | \((2,0,\ldots,0,1,0)\) | \(\frac{1}{4}(\ell+2)(\ell+3)(\ell-1)\) | \(2\Lambda + \Lambda_{\ell-1}\) | 2 | 0 |
| 6 | \((0,1,0,\ldots,0,2)\) | \(\ell(\ell+2)(\ell+3)(\ell-1)\) | \(\Lambda_2 + 2\Lambda_\ell\) | 2 | 0 |

Table 2: The irreducible components occurring in the tensor product of the adjoint representation of \( a_\ell, \ell \geq 3 \), with itself. The representations are of dimension \( d \) and have the highest weight \( \Lambda \). The eigenvalues of the quadratic Casimir operator \( C \) and of \( L \) (see eq. (3.5)) are also given. The number \( i \) is used to refer to them in the text.

Applying eq. (2.3b) we see that (4.3b) coincides with eq. (A.5) and implies eq. (A.11) in [2]. The remaining projectors \( P_{3}^{(3)} \) and \( P_{4}^{(4)} \) in the symmetric part can also be found:

\[
P_{jk,pq}^{(3)} = \frac{1}{4} \delta_{jk} \delta_{pq} + \frac{1}{2(n-1)} \delta_{jk} \delta_{pq} - \frac{n}{4(n-2)} d_{jkr} d_{pqr} \]

(4.4a)

\[
+ \frac{1}{4} (f_{jpq} f_{kqr} + f_{jqp} f_{kpr}),
\]

\[
P_{jk,pq}^{(4)} = \frac{1}{4} \delta_{jk} \delta_{pq} + \frac{1}{2(n+1)} \delta_{jk} \delta_{pq} + \frac{n}{4(n+2)} d_{jkr} d_{pqr} \]

(4.4b)

(4.4b)

\[
- \frac{1}{4} (f_{jpq} f_{kqr} + f_{jqp} f_{kpr}).
\]

The \( su(4) \) example presented in section 3.1 is a special case of these results. From eq. (3.3b) we derive

\[
\text{tr}(F_i F_j F_k F_\ell) = 2\delta_{ij} \delta_{k\ell} + \frac{n}{4} d_{r(ij) d_{k\ell)r}}
\]

(4.5)

which implies

\[
\text{tr} F^4 = 6(\text{tr} A^2)^2 + 2n(\text{tr} A^4).
\]

(4.6)

Note that because of our normalization conventions \( \text{tr}(F_j F_k) = f_{pqj} f_{pqk} = n\delta_{jk} \) and thus \( \text{tr} F^2 = 2n (\text{tr} A^2) \). From the relations obtained with our second method from eq. (A.6), we obtain e.g. for \( su(4) \)

\[
\text{tr}(F_{i_1 \cdots i_6}) = \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} + \frac{9}{4} \delta_{i_1 i_2} d_{i_3 i_4} d_{i_5 i_6} r - \frac{13}{12} d_{i_1 i_2 i_3} d_{i_4 i_5 i_6},
\]

(4.7)

and for \( su(5) \) (cf. eq. (A.10))

\[
\text{tr}(F_{i_1 \cdots i_6}) = \frac{65}{64} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} + \frac{75}{32} \delta_{i_1 i_2} d_{i_3 i_4} d_{i_5 i_6} r - \frac{25}{24} d_{i_1 i_2 i_3} d_{i_4 i_5 i_6}.
\]

(4.8)

4.2 The simple Lie algebras \( b_\ell, \ell \geq 2 \)

We describe the Lie algebras \( b_\ell \) (or \( so(n) \), \( n = 2\ell + 1 \)) by a basis \( (x_j) \) of \( n \times n \) antisymmetric hermitian matrices \( x_j \) in the defining representation. They are normalized such that
\[
\begin{array}{cccc}
\ell & \text{representation of } b_{\ell} & d & \Lambda \\
1 & (0,\ldots,0) & 1 & 0 \\
2 & (0,1,0,\ldots,0) & \ell(2\ell+1) & \Lambda_2 \\
3 & (0,2,0,\ldots,0) & \frac{1}{3}(\ell-1)(\ell+1)(2\ell+1)(2\ell+3) & 2\Lambda_2 \\
4 & (0,0,0,1,0,\ldots,0) & \frac{1}{6}\ell(\ell-1)(2\ell-1)(2\ell+1) & \Lambda_4 \\
5 & (2,0,\ldots,0) & \ell(2\ell+3) & 2\Lambda_1 \\
6 & (1,0,1,0,\ldots,0) & \frac{1}{2}\ell(\ell-1)(2\ell+1)(2\ell+3) & \Lambda_1 + \Lambda_3
\end{array}
\]

Table 3: The irreducible components occurring in the tensor product of the adjoint representation of \( b_{\ell} \), \( \ell \geq 5 \), with itself. Here \( i, d, \Lambda, C \) and \( L \) are as for table 2. The formulae for dimensions and Casimir eigenvalues are also valid for the corresponding representations in the cases \( \ell = 2, 3, 4 \). We comment on this fact in the text.

\[ \text{tr}(x_j x_k) = 2\delta_{jk} . \]

We define the structure constants \( c_{j k \ell} \) using \([x_i, x_j] = i c_{j k \ell} x_\ell \). Since the symmetrized product \( \{ x_j, x_k \} \) is a symmetric matrix, we have

\[ \{ x_j, x_k \} = \frac{4}{n} \delta_{jk} \mathbb{I} + d_{j k \alpha} y_{\alpha} , \quad (4.9) \]

where the \( y_{\alpha} \) span the space of traceless symmetric \( n \times n \) matrices and have the normalization \( \text{tr}(y_{\alpha} y_{\beta}) = 2\delta_{\alpha \beta} \). The coefficients \( d_{j k \alpha} \) form a tensor with the properties \( d_{j k \alpha} = d_{k j \alpha} \) and \( d_{j j \alpha} = 0 \). The collection of all \( x_i \) and \( y_{\alpha} \) can serve as a set of Gell-Mann matrices of \( su(n) \).

It is to be noted that here for \( b_{\ell} \) as well as below for \( c_{\ell} \) and for \( d_{\ell} \), we index the basis vectors \( x_j \) by a single letter rather than by the (perhaps more usual) method that uses an index pair, as in \( M_{ab} = E_{ab} - E_{ba} = -M_{ba} \), where \( (E_{ab})_{cd} = \delta_{ac}\delta_{bd} \). This is first suitably convenient for all our purposes and second allows a uniform treatment of all Lie algebras.

Since the Cartan-Killing form reads in our basis \( \text{tr}(\text{ad} x_j \circ \text{ad} x_k) = 2(2\ell-1)\delta_{jk} \), the structure constants \( c_{j k r} \) are related to the \( C_{j k r} \) used in our general discussion by \( C_{j k r} = -c_{j k r}/\sqrt{2(2\ell-1)} \).

Table 3 lists the representations which are relevant to the decomposition of the tensor product of the adjoint representation with itself for the cases \( \ell \geq 5 \)

\[
(0,1,0,\ldots,0) \otimes (0,1,0,\ldots,0) \\
= \underbrace{(0,\ldots,0)}_{\text{symmetric}} \oplus \underbrace{(2,0,\ldots,0)}_{\text{symmetric}} \oplus \underbrace{(0,0,0,1,0,\ldots,0)}_{\text{antisymmetric}} \oplus \underbrace{(0,2,0,\ldots,0)}_{\text{antisymmetric}}
\]  

The highest weights written in terms of the fundamental weights have special forms for \( \ell = 2, 3, 4 \). The above decomposition reads in these cases

\[ (0,2) \otimes (0,2) = \underbrace{(0,0) \oplus (2,0) \oplus (1,0) \oplus (0,4)}_{\text{symmetric}} \oplus \underbrace{(0,2) \oplus (1,2)}_{\text{antisymmetric}} \]

(4.10b)
\begin{align*}
(0, 1, 0) \otimes (0, 1, 0) &= \underbrace{(0, 0, 0) \oplus (2, 0, 0) \oplus (0, 0, 2) \oplus (0, 2, 0)}_{\text{symmetric}} \\
&\oplus \underbrace{(0, 1, 0) \oplus (1, 0, 2)}_{\text{antisymmetric}},
\end{align*}
\begin{align*}
(0, 1, 0, 0) \otimes (0, 1, 0, 0) &= \underbrace{(0, 0, 0, 0) \oplus (2, 0, 0, 0) \oplus (0, 0, 2, 0) \oplus (0, 2, 0, 0)}_{\text{symmetric}} \\
&\oplus \underbrace{(0, 1, 0, 0) \oplus (1, 0, 1, 0)}_{\text{antisymmetric}}.
\end{align*}

But nevertheless, the formulae in table 3 for dimensions and Casimir eigenvalues are valid for arbitrary \( \ell \geq 2 \).

The following projectors \( P^{(i)} \) (where \( i \) refers to the rows of table 3) are easily written down:

\begin{align*}
P^{(1)}_{jk,pq} &= \frac{2}{n(n-1)} \delta_{jk} \delta_{pq}, \quad (4.11a) \\
P^{(5)}_{jk,pq} &= \frac{1}{2(n-2)} d_{jka} d_{pqa}, \quad (4.11b) \\
P^{(2)}_{jk,pq} &= \frac{1}{2(n-2)} c_{jkr} c_{pqr}. \quad (4.11c)
\end{align*}

The projector \( P^{(5)} \) has been constructed using the tensor \( d_{jka} \) from eq. (4.9). A careful analysis of the tensors involved in relations like (4.9) shows that this is in fact a projector onto an irreducible component of the symmetric tensor product of the adjoint representation. A consideration of the relevant dimensions furthermore allows us to identify the representation to be \((2, 0, \ldots, 0)\).

Relation (3.7) gives in this case

\begin{align*}
L^2 &= \frac{2}{(n-2)^2} \mathbb{1}_S + \frac{(n-1)(n-4)}{(n-2)^2} P^{(1)} + \frac{n-8}{4(n-2)} P^{(5)} \quad (4.12a) \\
&\quad - \frac{1}{n-2} L \mathbb{1}_S - \frac{1}{2} L \mathbb{1}_A, \\
c_{jmr} c_{kns} c_{mps} c_{nqs} &= 4 (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) + \frac{8(n-4)}{n} \delta_{jk} \delta_{pq} + \frac{n-8}{2} d_{jka} d_{pqa} \quad (4.12b) \\
&\quad + \frac{n}{2} c_{jps} c_{kpq} - \frac{n-4}{2} c_{jqr} c_{kpq},
\end{align*}

which implies

\begin{align*}
\text{tr}(F_{i_1} F_{i_2} F_{i_3} F_{i_4}) &= \frac{16(n-2)}{n} \delta_{(i_1 i_2} \delta_{i_3 i_4)} + \frac{n-8}{2} d_{(i_1 i_2} \alpha d_{i_3 i_4)} \alpha. \quad (4.13)
\end{align*}

Since we have

\begin{align*}
\text{tr}(x_{i_1} x_{i_2} x_{i_3} x_{i_4}) &= \frac{4}{n} \delta_{(i_1 i_2} \delta_{i_3 i_4)} + \frac{1}{2} d_{(i_1 i_2} \alpha d_{i_3 i_4)} \alpha, \quad (4.14)
\end{align*}

we finally derive

\begin{equation}
\text{tr} F^4 = 3(\text{tr} A^2)^2 + \frac{1}{2} (n-8)(\text{tr} A^4) \quad (4.15)
\end{equation}
in agreement with the formulae listed in appendix A.2 which have been derived by our second method. The matrices \( F_j \) are normalized in such a way that \( \text{tr}(F_j F_k) = c_{pqj} c_{pqk} = 2(n-2) \delta_{jk} \).
Since the symmetrized product adding further basis vectors $y$,

$$\text{tr}(x_{(i_1 \cdots i_n)}) = \frac{1}{5} \delta_{(i_1 i_2 i_3 i_4 i_5 i_6)} + \frac{3}{4} \delta_{(i_1 i_2 d_{i_3 i_4}^{\alpha} d_{i_5 i_6})}$$

and

$$\text{tr}(F_{(i_1 \cdots i_6)}) = \frac{93}{5} \delta_{(i_1 i_2 i_3 i_4 i_5 i_6)} - \frac{21}{4} \delta_{(i_1 i_2 d_{i_3 i_4}^{\alpha} d_{i_5 i_6})}.$$  

The remaining projectors in the symmetric part can be found from eq. (4.12):

$$P_{jk,pq}^{(3)} = \frac{1}{3} (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) + \frac{2(n-4)}{3n(n-2)} \delta_{jk} \delta_{pq} + \frac{n-8}{12(n-2)} d_{jko} d_{pqo}$$

$$P_{jk,pq}^{(4)} = \frac{1}{6} (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) - \frac{2}{3n} \delta_{jk} \delta_{pq} - \frac{1}{12} d_{jko} d_{pqo} + \frac{1}{12} (c_{jpr} c_{kqr} + c_{jqr} c_{kpr}).$$

### 4.3 The simple Lie algebras $c_\ell$, $\ell \geq 2$

We describe the Lie algebras $c_\ell$ (or $sp(2n)$, $n = \ell$) by a basis $(x_j)$ of $2n \times 2n$ traceless hermitean matrices $x_j$ in the defining representation satisfying

$$J x_j J^{-1} = -x_j^T, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad (4.19)$$

They are normalized in such a way that $\text{tr}(x_j x_k) = 2 \delta_{jk}$. We write the structure constants as $[x_i, x_j] = i c_{ijk} x_k$. Given these $x_j$, the matrices $J x_j$ span the space of symmetric $2n \times 2n$ matrices (including the pure trace). If we extend the set of the $x_i$ to a basis of $su(2n)$ by adding further basis vectors $y_\alpha$, the $J y_\alpha$ are antisymmetric, and the $y_\alpha$ thus satisfy

$$J y_\alpha J^{-1} = y_\alpha^T.$$  

(4.20)

Since the symmetrized product $\{x_j, x_k\}$ also satisfies (4.20), we have

$$\{x_j, x_k\} = \frac{2}{n} \delta_{jk} + d_{jko} y_\alpha,$$

(4.21)

because the $y_\alpha$ together with the unit matrix span the space of solutions of eq. (4.20). The coefficients $d_{jko}$ occurring here form a tensor which satisfies $d_{jko} = d_{kjo}$ and $d_{jjo} = 0$.

Since the Cartan-Killing form reads in our basis $\text{tr}(\text{ad} x_j \circ \text{ad} x_k) = 4(\ell + 1) \delta_{jk}$, the structure constants $c_{jkr}$ are related to the $C_{jkr}$ used in our general discussion by $C_{jkr} = -c_{jkr}/(2\sqrt{\ell+1})$.

Table 4 lists the representations which are relevant to the decomposition of the tensor product of the adjoint representation with itself

$$\begin{align*}
(2,0,\ldots,0) \otimes (2,0,\ldots,0) \\
= (0,\ldots,0) \oplus (0,1,0,\ldots,0) \oplus (4,0,\ldots,0) \oplus (0,2,0,\ldots,0) \\
\oplus (2,0,\ldots,0) \oplus (2,1,0,\ldots,0).
\end{align*}$$

antisymmetric

symmetric
| $i$ | representation of $c_\ell$ | $d$ | $\Lambda$ | $C$ | $L$ |
|-----|-----------------|-----|-----|-----|-----|
| 1   | $(0, \ldots, 0)$ | 1   | 0   | 0   | $-1$ |
| 2   | $(2, 0, \ldots, 0)$ | $\ell(2\ell + 1)$ | $2\Lambda_1$ | 1 | $-\frac{1}{2}$ |
| 3   | $(4, 0, \ldots, 0)$ | $\frac{1}{6}\ell(\ell + 1)(2\ell + 1)(2\ell + 3)$ | $4\Lambda_1$ | $\frac{2(\ell+2)}{\ell+1}$ | $\frac{1}{\ell+1}$ |
| 4   | $(0, 2, 0, \ldots, 0)$ | $\frac{1}{3}\ell(\ell - 1)(2\ell - 1)(2\ell + 3)$ | $2\Lambda_2$ | $\frac{2\ell}{\ell+1}$ | $-\frac{2(\ell+1)}{2(\ell+1)}$ |
| 5   | $(0, 1, 0, \ldots, 0)$ | $(\ell - 1)(2\ell + 1)$ | $\Lambda_2$ | $\ell$ | $-\frac{2(\ell+2)}{2(\ell+1)}$ |
| 6   | $(2, 1, 0, \ldots, 0)$ | $\frac{1}{2}\ell(\ell - 1)(2\ell + 1)(2\ell + 3)$ | $2\Lambda_1 + \Lambda_2$ | 2 | 0 |

Table 4: The irreducible components occurring in the tensor product of the adjoint representation of $c_\ell$, $\ell \geq 2$, with itself. Here $i$, $d$, $\Lambda$, $C$ and $L$ are as for table 2.

The following projectors $P^{(i)}$ (where $i$ refers to the rows of table 4) are easily written down:

\begin{align}
P^{(1)}_{jk,pq} &= \frac{1}{n(2n + 1)} \delta_{jk} \delta_{pq}, \quad (4.23a) \\
P^{(5)}_{jk,pq} &= \frac{1}{4(n + 1)} d_{jk\alpha} d_{pq\alpha}, \quad (4.23b) \\
P^{(2)}_{jk,pq} &= \frac{1}{4(n + 1)} c_{jkr} c_{pqr}, \quad (4.23c)
\end{align}

where $P^{(5)}$ has been constructed using the tensor $d_{jk\alpha}$ from eq. (4.21).

Relation (3.7) reads in this case

\begin{align}
L^2 &= \frac{1}{2(n + 1)^2} \mathbb{1}_S + \frac{(2n + 1)(n + 2)}{2(n + 1)^2} P^{(1)} + \frac{n + 4}{4(n + 1)} P^{(5)} \quad (4.24a) \\
&+ \frac{1}{2(n + 1)} L \mathbb{1}_S - \frac{1}{2} L \mathbb{1}_A, \\
c_{jmr} c_{knr} c_{mps} c_{nqs} &= 4 (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) + \frac{8(n + 2)}{n} \delta_{jk} \delta_{pq} + (n + 4) d_{jk\alpha} d_{pq\alpha} \quad (4.24b) \\
&+ n c_{jqr} c_{kqr} - (n + 2) c_{jqr} c_{kpr},
\end{align}

which implies

\[ \text{tr}(F_{(i} F_{i_2} F_{i_3} F_{i_4}) = \frac{16(n + 1)}{n} \delta_{(i_1 i_2} \delta_{i_3 i_4)} + (n + 4) d_{(i_1 i_2} a_{i_3 i_4)\alpha} \quad (4.25) \]

and

\[ \text{tr} F^4 = 3(\text{tr} A^2)^2 + 2(n + 4)(\text{tr} A^4). \quad (4.26) \]

This is consistent with the results presented in appendix A.3. The matrices $F_j$ are normalized in such a way that $\text{tr}(F_j F_k) = c_{pqj} c_{pqk} = 4(n + 1) \delta_{jk}$ and thus $\text{tr} F^2 = 2(n + 1) \text{tr} A^2$. The remaining projectors in the symmetric part are

\begin{align}
P^{(3)}_{jk,pq} &= \frac{1}{6} (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) + \frac{1}{3n} \delta_{jk} \delta_{pq} + \frac{1}{12} d_{jk\alpha} d_{pq\alpha} \quad (4.27a)
\end{align}
Table 5: The irreducible components occurring in the tensor product of the adjoint representation of $d_\ell$, $\ell \geq 6$, with itself. We comment on the special cases $\ell = 3, 4, 5$ in the text.

$$P_{jk, pq}^{(4)} = \frac{1}{3} \left( \delta_{jk} \delta_{kp} + \delta_{jq} \delta_{kp} \right) - \frac{2(n+2)}{3n(2n+1)} \delta_{jk} \delta_{pq} - \frac{n+4}{12(n+1)} d_{j\alpha} d_{p\alpha} \quad (4.27b)$$

$$+ \frac{1}{12} \left( c_{jpr} c_{kqr} + c_{jqr} c_{kpr} \right).$$

4.4 The simple Lie algebras $d_\ell$, $\ell \geq 3$

In order to describe the Lie algebras $d_\ell$, $\ell \geq 3$ (or $so(n)$, $n = 2\ell$), we choose the same basis as for odd $n$ in section 4.2. Because of the different relation between $\ell$ and $n$, we find $\text{tr}(\text{ad} x_j \circ \text{ad} x_k) = 4(\ell - 1) \delta_{jk}$ and therefore $C_{jkr} = -c_{jkr}/(2\sqrt{\ell - 1})$.

Table 5 lists the representations which are relevant to the decomposition of the tensor product of the adjoint representation with itself in the cases $\ell \geq 6$:

$$(0, 1, 0, \ldots, 0) \otimes (0, 1, 0, \ldots, 0)$$

$$= \underbrace{(0, 0, 0, 1, 0, \ldots, 0)}_{\text{symmetric}} \oplus \underbrace{(0, 2, 0, \ldots, 0)}_{\text{antisymmetric}} \oplus \underbrace{(0, 0, 0, 0, 1, 0, \ldots, 0)}_{\text{antisymmetric}} \oplus (0, 2, 0, \ldots, 0).$$

The special cases for $\ell = 3, 4, 5$ are

$$(0, 1, 1) \otimes (0, 1, 1)$$

$$= \underbrace{(0, 0, 0)}_{\text{symmetric}} \oplus \underbrace{(0, 1, 1)}_{\text{symmetric}} \oplus \underbrace{(0, 1, 1)}_{\text{antisymmetric}} \oplus \underbrace{(1, 0, 2)}_{\text{antisymmetric}} \oplus \underbrace{(1, 2, 0)}_{\text{antisymmetric}} \quad (4.28b)$$

The special cases for $\ell = 3, 4, 5$ are

$$(0, 1, 0, 0) \otimes (0, 1, 0, 0)$$
\[
(0,0,0,0) \oplus (2,0,0,0) \oplus \left( (0,0,2,0) \oplus (0,0,0,2) \right) \oplus (0,2,0,0) \quad \text{(4.28c)}
\]

symmetric

\[
\oplus (0,1,0,0) \oplus (1,0,1,1),
\]

antisymmetric

\[
(0,1,0,0) \otimes (0,1,0,0)
\]

\[
= (0,0,0,0) \oplus (2,0,0,0) \oplus (0,2,0,0,0) \oplus (0,0,0,1,1) \oplus (0,2,0,0,0) \quad \text{(4.28d)}
\]

symmetric

\[
\oplus (0,1,0,0,0) \oplus (1,0,1,0,0).
\]

antisymmetric

The structure of the decomposition is somewhat exceptional for \( \ell = 3 \) (because \( d_3 \) is isomorphic with \( a_3 \)) and for \( \ell = 4 \) (because of the higher symmetry of the Dynkin diagram of \( d_4 \)). We have indicated by additional parentheses, in \( (4.28a) \) and \( (4.28b) \), the cases in which a representation corresponding to one piece in \( (4.28a) \) has decomposed into even smaller constituents. Nevertheless, the formulae in table 5 for dimensions and Casimir eigenvalues are useful for arbitrary \( \ell \geq 3 \). In the case of further decomposition, the table lists the sum of the dimensions, and both constituents turn out to have the same Casimir eigenvalue.

If the projectors \( P^{(i)} \) and the Casimir eigenvalues \( C \) from table 5 are written using \( n = 2\ell \), they have exactly the same form as for the algebras \( b_\ell \). Therefore we have the same results as eqs. \( (4.11) \) to \( (4.15) \) and \( (4.18) \).

### 4.5 The exceptional simple Lie algebra \( g_2 \)

For the structure and the construction of representations of \( g_2 \), see, for example, [28, 31]. We use for the defining representation of \( g_2 \) a suitable set of 14 traceless hermitean \( 7 \times 7 \) matrices \( x_j \) which have the additional properties

\[
x_j^T = -x_j, \quad \text{tr}(x_j x_k) = 2\delta_{jk}, \quad \text{(4.29)}
\]

and write the structure constants \( c_{jkt} \) such that \( [x_j, x_k] = i c_{jkt} x_k \). Note that they are related to the constants \( C_{jkt} \) from the general discussion by \( C_{jkt} = -c_{jkt}/\sqrt{8} \). The space of \( 7 \times 7 \) traceless hermitean matrices (the matrices of the defining representation of \( a_6 = su(7) \)) involves 21 antisymmetric matrices which span the \( b_3 = so(7) \) subalgebra of \( a_6 \), and 28 symmetric matrices \( y_\alpha \), \( 1 \leq \alpha \leq 28 \). We do not need to introduce here the 7 antisymmetric matrices \( z_\alpha \), \( 1 \leq \alpha \leq 7 \), which lie outside the \( g_2 \) subalgebra of \( b_3 \), but we do need the \( y_\alpha \). They satisfy

\[
y_\alpha^T = y_\alpha, \quad \text{tr}(y_\alpha y_\beta) = 2\delta_{\alpha\beta}, \quad \text{tr}(x_j y_\alpha) = 0, \quad \text{(4.30)}
\]

and are related to the \( x_j \) by

\[
x_j x_k = \frac{2}{7} \delta_{jk} + \frac{1}{2} ic_{jkt} x_k + \frac{1}{2} d_{jka} y_\alpha, \quad \text{(4.31)}
\]

where \( d_{jka} = d_{kja} \) and \( d_{jja} = 0 \). Of course, complete control of \( g_2 \) technology depends on consideration of \( x_i \), \( y_\alpha \) and \( z_\alpha \), and the various isotropic tensors that enter product laws like eq. \( (4.31) \). A full treatment of these matters will be presented elsewhere [32]. Here we quote class 1 results as needed and attend to our main purpose, that of deriving class 2 results.
is worth noting that the set of all $x_i$, $z_a$, $y_\alpha$ can be viewed as a set of 48 Gell-Mann type matrices $\lambda_A$ of $a_6 = su(7)$.

Table 6 contains information about the irreducible representations of $g_2$ relevant to the application of our first method. [7] is the defining representation, [14] the adjoint. The tensor product of the adjoint representation with itself decomposes into irreducible components as follows

$$ [14] \otimes [14] = [1] \oplus [27] \oplus [77] \oplus [14] \oplus [77'] \ . $$

In order to decide which of the [77] or [77'] representations occur in the symmetric versus antisymmetric part, we consider their $C_V \otimes V$ eigenvalues. Only [77'] gives the eigenvalue 2 and therefore, according to the discussion of eq. (3.13), belongs to the antisymmetric part.

The following projectors are initially known:

$$ P_{jk,pq}^{[1]} = \frac{1}{14} \delta_{jk} \delta_{pq} \ , \quad (4.33a) $$

$$ P_{jk,pq}^{[27]} = \frac{9}{32} d_{jk\alpha} d_{pq\alpha} \ , \quad (4.33b) $$

$$ P_{jk,pq}^{[14]} = \frac{1}{8} c_{jkr} c_{pqr} \ , \quad (4.33c) $$

in virtue of the identities

$$ c_{jkp} c_{jkq} = 8 \delta_{pq} \ , \quad (4.34a) $$

$$ d_{jk\alpha} d_{j\beta} = \frac{32}{9} \delta_{\alpha\beta} \ . \quad (4.34b) $$

As for $su(3)$ in section 3.1 we deal with the characteristic equation and, noting that in our basis we have $L_{jk,pq} = -\frac{1}{8} c_{jpr} c_{kqr}$, derive

$$ L \mathbb{I}_S = \frac{1}{4} \mathbb{I}_S - \frac{5}{4} P_{jk,pq}^{[1]} - \frac{2}{3} P_{jk,pq}^{[27]} \ , \quad (4.35a) $$

$$ c_{jpr} c_{kqr} + c_{jqr} c_{kpr} = -2(\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) + 10 \frac{2}{7} \delta_{jk} \delta_{pq} + 3 d_{jk\alpha} d_{pq\alpha} \ . \quad (4.35b) $$

By two applications of the Jacobi identity, (4.35b) yields

$$ d_{(jk)\alpha} d_{pq\alpha} = \frac{6}{7} \delta_{(jk) \delta_{pq}} \ . \quad (4.36) $$

| representation of $g_2$ | $d$ | $A$ | $C$ | $L$ |
|--------------------------|-----|-----|-----|-----|
| $(0, 0)$                 | 1   | 0   | 0   | $-1$ |
| $(0, 1)$                 | 7   | $A_2$ | $\frac{1}{2}$ | $-\frac{3}{6}$ |
| $(1, 0)$                 | 14  | $A_1$ | 1   | $-\frac{1}{3}$ |
| $(0, 2)$                 | 27  | $2A_2$ | $\frac{7}{6}$ | $-\frac{5}{12}$ |
| $(2, 0)$                 | 77  | $2A_1$ | $\frac{5}{2}$ | $\frac{1}{4}$ |
| $(0, 3)$                 | $77'$ | $3A_2$ | 2   | 0   |

Table 6: Irreducible representations of $g_2$. |
The simplification process used these class 1 identities

\[ c_{pqj}c_{qkr}c_{r\ell p} = -4c_{jk\ell}, \quad (4.37a) \]

\[ d_{jk\alpha}d_{\ell m\alpha}c_{kmp} = \frac{20}{7}c_{jfp}, \quad (4.37b) \]

\[ c_{pqj}c_{qkr}d_{jk\alpha} = \frac{10}{3}d_{p\alpha r}. \quad (4.37c) \]

The results (4.35) and (4.36) are new, as is their convenient and therefore important method of derivation. We note the check that (4.36) implies (4.34a). From (4.36), we can obtain

\[ \text{tr}(x_{i}x_{j}x_{k}x_{\ell}) = \delta_{(ij}\delta_{k\ell)} , \quad (4.38) \]

and hence if \( A = a_{j}x_{j}, a_{j} \in \mathbb{C}, \) the important result

\[ \text{tr} A^{4} = \frac{1}{4}(\text{tr} A^{2})^{2} , \quad (4.39) \]

which was quoted by Okubo [11] (without what he termed its rather involved proof).

If we had to perform our calculations in \( g_{2} \) without an explicit form of the projector \( P^{[27]} \) (which we do indeed know thanks to the \( d \)-tensor calculus available), the procedure would have been more like the treatment of the \( su(4) \) example given in section 3.1, and would have yielded the weaker results

\[ L^{2} = \frac{5}{48} \mathbb{I}_{S} + \frac{35}{48} P^{[1]} - \frac{1}{6} L \mathbb{I}_{S} - \frac{1}{2} L \mathbb{I}_{A}, \quad (4.40a) \]

\[ c_{jmr}c_{kmr}c_{mps}c_{nqs} = \frac{10}{3}(\delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp} + \delta_{jk}\delta_{pq}) + \frac{8}{3} c_{jqr}c_{kqr} - \frac{4}{3} c_{jqr}c_{kpr}. \quad (4.40b) \]

But with this information, we can still construct the projector in question, getting

\[ P^{[27]}_{jk,pq} = \frac{3}{16}(\delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp}) - \frac{15}{112}\delta_{jk}\delta_{pq} + \frac{3}{32}(c_{jqr}c_{kqr} + c_{jqr}c_{kpr}) , \quad (4.41) \]

which, of course, agrees eq. (4.33b) upon use of (4.35b). Also, from eq. (4.40b) we can derive

\[ \text{tr}(F_{j}F_{k}F_{\ell}) = 10\delta_{(ij}\delta_{k\ell)}, \quad (4.42) \]

where \( (F_{j})_{k\ell} = i c_{jtk} \) are the matrices in the adjoint representation. Since \( \text{tr}(F_{j}F_{k}) = 8\delta_{jk}, \) this leads to

\[ \text{tr} F^{4} = \frac{5}{32}(\text{tr} F^{2})^{2} = \frac{5}{2}(\text{tr} A^{2})^{2}. \quad (4.43) \]

This is consistent with the relations found more easily using our second method, and listed in appendix A.7 for \( g_{2}. \) From those relations, we obtain further identities, e.g.

\[ \text{tr}(x_{(i_{1}\cdots i_{8})}) = -\frac{5}{192}\delta_{(i_{1}i_{2}\delta_{i_{3}i_{4}\delta_{i_{5}i_{6}\delta_{i_{7}i_{8})}} + \frac{2}{3}\delta_{(i_{1}i_{2}d_{(6)}^{i_{3}\cdots i_{8})}} , \quad (4.44a) \]

\[ \text{tr}(x_{(i_{1}\cdots i_{10})}) = -\frac{1}{64}\delta_{(i_{1}i_{2}\delta_{i_{3}i_{4}\delta_{i_{5}i_{6}\delta_{i_{7}i_{8}\delta_{i_{9}i_{10})}} + \frac{5}{16}\delta_{(i_{1}i_{2}\delta_{i_{3}i_{4}}d_{(6)}^{i_{5}\cdots i_{10})} , \quad (4.44b) \]

where

\[ d_{(6)}^{i_{1}\cdots i_{8})} = \text{tr}(x_{i_{1}\cdots i_{8})} \quad (4.45) \]
denotes the sixth order invariant of $g_2$. In particular we observe that our identity for $\text{tr} A^8$ is consistent with the results obtained in [16]. Furthermore we can also reduce the symmetric traces in the adjoint representation:

$$\text{tr}(F_{(i_1 \cdots i_6)}) = \frac{15}{4} \delta_{(i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6)} - 26 d^{(6)}_{(i_1 \cdots i_6)}, \quad (4.46a)$$

$$\text{tr}(F_{(i_1 \cdots i_8)}) = \frac{515}{96} \delta_{(i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8)} - \frac{160}{3} d^{(6)}_{(i_1 i_2 i_3 \cdots i_8)}. \quad (4.46b)$$

Our discussion of $f_4$ in the next section is similar to that of $g_2$ without referring to explicit $d$-tensor formulae.

### 4.6 The exceptional simple Lie algebra $f_4$

For the analysis of $f_4$, we use the structure constants $C_{jk\ell}$ in the general notation defined in the beginning of section 2 because no analogue of the $f$- and $d$-tensor calculus is known. Table 7 contains information about the irreducible representations of $f_4$ relevant to the application of our first method. $[26]$ is the defining representation, $[52]$ the adjoint. The tensor product of the adjoint representation with itself decomposes into irreducible components as follows

$$[52] \otimes [52] = \underbrace{[1]} \oplus \underbrace{[324]} \oplus \underbrace{[1053]} \oplus \underbrace{[52]} \oplus \underbrace{[1274]}. \quad (4.47)$$

We have the relation

$$L^2 = \frac{5}{162} \mathbb{1}_S + \frac{65}{81} P^{[1]} - \frac{1}{6} L \mathbb{1}_S - \frac{1}{2} L \mathbb{1}_A, \quad (4.48a)$$

$$C_{jm}\n C_{kn} C_{mps} C_{nqs} = \frac{5}{324} \left( \delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp} + \delta_{jk} \delta_{pq} \right) + \frac{1}{3} C_{jpr} C_{kqr} - \frac{1}{6} C_{jqr} C_{kpr}. \quad (4.48b)$$

and the projectors

$$P^{[1]}_{jk, pq} = \frac{1}{52} \delta_{jk} \delta_{pq}, \quad (4.49a)$$

$$P^{[52]}_{jk, pq} = C_{jkr} C_{pqr}. \quad (4.49b)$$
Furthermore we derive from (4.48b) and therefore

\[ P[j,k,pq] = \frac{1}{7} (\delta_{jq}\delta_{kp} + \delta_{jq}\delta_{kp}) - \frac{5}{91} C_{jpr} C_{kqr} + \frac{9}{7} C_{jqr} C_{kpr} , \]  

(4.49c)

\[ P[j,k,pq] = \frac{5}{14} (\delta_{jq}\delta_{kp} + \delta_{jq}\delta_{kp}) + \frac{1}{28} C_{jpr} C_{kqr} - \frac{9}{7} C_{jqr} C_{kpr} , \]  

(4.49d)

\[ P[j,k,pq] = \frac{1}{2} (\delta_{jq}\delta_{kp} - \delta_{jq}\delta_{kp}) - C_{jkr} C_{pqr} . \]  

(4.49e)

Furthermore we derive from (4.48b)

\[ \text{tr}(F(i,F(j,F(k,F(\ell)))) = \frac{5}{108} \delta_{(ij}(\delta_{k\ell)} , \]  

(4.50)

and therefore

\[ \text{tr} F^4 = \frac{5}{108} (\text{tr} F^2)^2 , \]  

(4.51)

where \((F_j)_{k\ell} = C_{j\ell k}\) are the matrices of the adjoint representation, which obey \(\text{tr}(F_j F_k) = -\delta_{jk}\).

Since there is no generally accepted definition of the invariant tensors of \(f_4\), we use \(A^k, A = a_j x_j\), to define them in a totally symmetric form

\[ d_{i_1 \cdots i_k}^{(k)} := \text{tr}(x_{i_1} \cdots x_{i_k}), \quad k \in \{2, 6, 8, 12\} . \]  

(4.52)

Here the \(x_j\) are the matrices of the defining representation of \(f_4\) and \(d_{i_1 i_2}^{(2)} \sim \delta_{i_1 i_2}\). The relations that express the non-primitivity of trace polynomials (see appendix A.8) thus read

\[ \text{tr}(x_{i_1} \cdots x_{i_4}) = \frac{1}{2} d_{i_1 i_2}^{(2)} d_{i_3 i_4}^{(2)} , \]  

(4.53a)

\[ \text{tr}(x_{i_1} \cdots x_{i_{10}}) = \frac{7}{41472} d_{i_1 i_2}^{(2)} d_{i_3 i_4}^{(2)} \cdots d_{i_9 i_{10}}^{(2)} - \frac{7}{144} d_{i_1 i_2}^{(2)} d_{i_3 i_4}^{(2)} d_{i_5 i_6}^{(6)} \] 

\[ + \frac{3}{8} d_{i_1 i_2}^{(2)} d_{i_3 \cdots i_{10}}^{(8)} , \]  

(4.53b)

and so on. Furthermore we derive for the matrices \((F_j)_{k\ell}\) of the adjoint representation (in a suitable normalization)

\[ \text{tr}(F(i_1 \cdots F(i_4)) = \frac{5}{12} d_{i_1 i_2}^{(2)} d_{i_3 i_4}^{(2)} , \]  

(4.54a)

\[ \text{tr}(F(i_1 \cdots F(i_6)) = \frac{5}{36} d_{i_1 i_2}^{(2)} d_{i_3 i_4}^{(2)} d_{i_5 i_6}^{(2)} - 7 d_{i_1 i_2 i_3 i_4 i_5 i_6}^{(6)} . \]  

(4.54b)

**4.7 The exceptional simple Lie algebra \(e_6\)**

In \(e_6\) we use again the structure constants \(C_{j\ell k}\) in the general notation. The structure of the decomposition of the adjoint representation and therefore of the results is similar to what we found for \(f_4\). Table 8 contains information about the relevant representations. Either \([27]\) or \([27']\) can play the role of a defining representation, \([78]\) is the adjoint. The decomposition is

\[ [78] \otimes [78] = [1] \oplus [650] \oplus [2430] \oplus [78] \oplus [2925] . \]  

(4.55)
As before we obtain the relation

\[ L^2 = \frac{1}{48} S^2 + \frac{13}{16} P^{[1]} - \frac{1}{6} L S - \frac{1}{2} L A, \]  
(4.56a)

\[ C_{jmr} C_{knr} C_{mps} C_{nqs} = \frac{1}{96} (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp} + \delta_{jk} \delta_{pq}) + \frac{1}{3} C_{jpr} C_{kqr} - \frac{1}{6} C_{jpr} C_{kqr}. \]  
(4.56b)

The relevant projectors are

\[ P^{[1]}_{jk, pq} = \frac{1}{78} \delta_{jk} \delta_{pq}, \]  
(4.57a)

\[ P_{78}^{[1]}_{jk, pq} = C_{jkr} C_{pqr}, \]  
(4.57b)

\[ P^{[650]}_{jk, pq} = \frac{1}{8} (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) - \frac{1}{24} \delta_{jk} \delta_{pq} + \frac{3}{2} (C_{jpr} C_{kqr} + C_{jqr} C_{kpr}), \]  
(4.57c)

\[ P^{[2430]}_{jk, pq} = \frac{3}{8} (\delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) - \frac{3}{104} \delta_{jk} \delta_{pq} - \frac{3}{2} (C_{jpr} C_{kqr} + C_{jqr} C_{kpr}), \]  
(4.57d)

\[ P^{[2925]}_{jk, pq} = \frac{1}{2} (\delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}) - C_{jkr} C_{pqr}. \]  
(4.57e)

From the relation (4.56b) we derive

\[ \text{tr}(F_i F_j F_k F_\ell) = \frac{1}{32} \delta_{(ij} \delta_{k\ell)}, \]  
(4.58)

which implies

\[ \text{tr} F^4 = \frac{1}{32} (\text{tr} F^2)^2, \]  
(4.59)

where we have defined the matrices of the adjoint representation as \((F_j)_{k\ell} = C_{jtk}, i.e.\)

\[ \text{tr}(F_j F_k) = -\delta_{jk}. \]

Again there is no common choice for the invariant tensors of \(e_6\), so we define

\[ d_{i_1 \ldots i_k}^{(k)} = \text{tr}(x_{(i_1} \ldots x_{i_k)}), \quad k \in \{2, 5, 6, 8, 9, 12\}. \]  
(4.60)
Here the $x_j$ are the matrices of the defining representation of $\mathfrak{e}_6$ and $d^{(2)}_{i_1 i_2} \sim \delta_{i_1 i_2}$. The simplest non-trivial relations from appendix A.5 read

$$\text{tr}(x_{(i_1 \cdots x_{i_4})}) = \frac{1}{12} d^{(2)}_{i_1 i_2} d^{(2)}_{i_3 i_4}, \quad (4.61a)$$

$$\text{tr}(x_{(i_1 \cdots x_{i_7})}) = \frac{7}{24} d^{(2)}_{i_1 i_2} d^{(5)}_{i_3 i_4 i_5 i_6 i_7}, \quad (4.61b)$$
as well as

$$\text{tr}(F_{(i_1 \cdots F_{i_{11}})}) = \frac{1}{2} d^{(2)}_{i_1 i_2} d^{(2)}_{i_3 i_4}, \quad (4.62a)$$

$$\text{tr}(F_{(i_1 \cdots F_{i_{16}})}) = \frac{5}{36} d^{(2)}_{i_1 i_2} d^{(2)}_{i_3 i_4} d^{(2)}_{i_5 i_6} - 6 d^{(6)}_{(i_1 i_2 i_3 i_4 i_5 i_6)}, \quad (4.62b)$$

where $(F_j)_{k\ell}$ are the matrices of the adjoint representation (in a suitable normalization).

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## A Appendix: Relations of trace polynomials

In the appendix we list class 2 relations, specific to each $\mathfrak{g}$, obtained by the method described in section 3.2 in a systematic way. If required, our algorithm is able to deal with even higher order traces.

### A.1 The simple Lie algebras $\mathfrak{a}_\ell$

We define the matrices of the defining representations of the Lie algebras $\mathfrak{a}_\ell$ by the Gell-Mann matrices $A = \frac{1}{2} a_\lambda \lambda_j$, so that $\text{tr} A = 0$. The adjoint representations are given by $(F_j)_{k\ell} = i f_{j k\ell}$ (where $[\lambda_j, \lambda_k] = 2i f_{j k\ell} \lambda_\ell$). The odd traces $\text{tr} F^{2k-1}$, $k \in \mathbb{N}$, vanish.

#### A.1.1 The Lie algebra $\mathfrak{a}_2$

**The defining representation:** In the defining representation $(1,0)[3]$, the polynomials $\text{tr} A^2$ and $\text{tr} A^3$ can be taken as generators of the algebra of invariant polynomials. Some of the results have been given in section 3.3. We obviously do not repeat them. Thus, we have that the $\text{tr} A^k$ satisfy three relations, given as eq. (3.28) for $k = 4, 5, 6$, and also

$$\text{tr} A^7 = \frac{7}{12} (\text{tr} A^2)^2 (\text{tr} A^3), \quad (A.1a)$$

$$\text{tr} A^8 = \frac{1}{8} (\text{tr} A^2)^4 + \frac{4}{9} (\text{tr} A^2) (\text{tr} A^3)^2, \quad (A.1b)$$

$$\text{tr} A^9 = \frac{3}{8} (\text{tr} A^2)^3 (\text{tr} A^3) + \frac{1}{9} (\text{tr} A^3)^3, \quad (A.1c)$$

$$\text{tr} A^{10} = \frac{1}{16} (\text{tr} A^2)^5 + \frac{5}{12} (\text{tr} A^2)^2 (\text{tr} A^3)^2. \quad (A.1d)$$

The characteristic polynomial of the matrix $A$ is given by eq. (3.29).
The adjoint representation: The traces $\text{tr} F^k$, $k \in \mathbb{N}$, in the adjoint representation $(1,1)[8]$ can be expressed in terms of the primitive polynomials $\text{tr} A^2$ and $\text{tr} A^3$ via eq. (3.36) for $2k = 2,4,6,8$, and also

\[
\begin{align*}
\text{tr} F^{10} &= \frac{513}{8} (\text{tr} A^2)^5 - \frac{405}{2} (\text{tr} A^2)^2 (\text{tr} A^3)^2, \\
\text{tr} F^{12} &= \frac{2049}{16} (\text{tr} A^2)^6 - 504 (\text{tr} A^2)^3 (\text{tr} A^3)^2 + 54 (\text{tr} A^3)^4.
\end{align*}
\] (A.2a)

The simplest relations of the $\text{tr} F^{2k}$ are given by eq. (3.33) for $2k = 4,8$, and also

\[
\begin{align*}
\text{tr} F^{10} &= -\frac{1}{64} (\text{tr} F^2)^5 + \frac{5}{16} (\text{tr} F^2)^2 (\text{tr} F^6), \\
\text{tr} F^{12} &= -\frac{19}{3072} (\text{tr} F^2)^6 + \frac{5}{48} (\text{tr} F^2)^3 (\text{tr} F^6) + \frac{1}{6} (\text{tr} F^6)^2.
\end{align*}
\] (A.3a)

The characteristic polynomial of $F$ has been given above in eqs. (3.34) and (3.37).

A.1.2 The Lie algebra $a_3$

The defining representation: In the defining representation $(1,0,0)[4]$ of $a_3$, the polynomials $\text{tr} A^2$, $\text{tr} A^3$ and $\text{tr} A^4$ can be taken as generators of the algebra of invariant polynomials. The $\text{tr} A^k$ satisfy these relations:

\[
\begin{align*}
\text{tr} A^5 &= \frac{5}{6} (\text{tr} A^2)(\text{tr} A^3), \\
\text{tr} A^6 &= -\frac{1}{8} (\text{tr} A^2)^3 + \frac{1}{3} (\text{tr} A^3)^2 + \frac{3}{4} (\text{tr} A^2)(\text{tr} A^4), \\
\text{tr} A^7 &= \frac{7}{24} (\text{tr} A^2)^2 (\text{tr} A^3) + \frac{7}{12} (\text{tr} A^3)(\text{tr} A^4), \\
\text{tr} A^8 &= -\frac{1}{16} (\text{tr} A^2)^4 + \frac{4}{9} (\text{tr} A^2)(\text{tr} A^3)^2 + \frac{1}{4} (\text{tr} A^2)^2 (\text{tr} A^4) + \frac{1}{4} (\text{tr} A^4)^2, \\
\text{tr} A^9 &= \frac{1}{9} (\text{tr} A^3)^3 + \frac{3}{4} (\text{tr} A^2)(\text{tr} A^3)(\text{tr} A^4), \\
\text{tr} A^{10} &= -\frac{1}{64} (\text{tr} A^2)^5 + \frac{5}{18} (\text{tr} A^2)^2 (\text{tr} A^3)^2 + \frac{5}{18} (\text{tr} A^3)^2 (\text{tr} A^4) + \frac{5}{16} (\text{tr} A^2)(\text{tr} A^4)^2.
\end{align*}
\] (A.4a)

The characteristic polynomial of the matrix $A$ is

\[
\chi_A(t) = t^4 - \frac{1}{2} (\text{tr} A^2) t^2 - \frac{1}{3} (\text{tr} A^3) t + \left(\frac{1}{8} (\text{tr} A^2)^2 - \frac{1}{4} (\text{tr} A^4)\right).
\] (A.5)

The adjoint representation: The traces $\text{tr} F^k$, $k \in \mathbb{N}$, in the adjoint representation $(1,0,1)[15]$ can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr} F^2 &= 8(\text{tr} A^2), \\
\text{tr} F^4 &= 6(\text{tr} A^2)^2 + 8(\text{tr} A^4), \\
\text{tr} F^6 &= -(\text{tr} A^2)^3 - \frac{52}{3} (\text{tr} A^2)^2 + 36(\text{tr} A^2)(\text{tr} A^4),
\end{align*}
\] (A.6a)
\[ \text{tr} F^8 = -\frac{15}{2} (\text{tr} A^2)^4 - \frac{640}{9} (\text{tr} A^2)(\text{tr} A^3)^2 + 44(\text{tr} A^2)^2(\text{tr} A^4) + 72(\text{tr} A^4)^2, \quad (A.6d) \]
\[ \text{tr} F^{10} = -\frac{23}{4} (\text{tr} A^2)^5 - \frac{1825}{9} (\text{tr} A^2)^2 (\text{tr} A^3)^2 - 30(\text{tr} A^2)^3 (\text{tr} A^4) + \frac{20}{9} (\text{tr} A^3)^2 (\text{tr} A^4) + 340(\text{tr} A^2)(\text{tr} A^4)^2. \quad (A.6e) \]

The simplest relations of the tr \( F^{2k} \) are the following.
\[ \text{tr} F^8 = \frac{35}{1248} (\text{tr} F^2)^4 - \frac{43}{104} (\text{tr} F^2)^2 (\text{tr} F^4) + \frac{9}{8} (\text{tr} F^4)^2 + \frac{20}{39} (\text{tr} F^2)(\text{tr} F^6), \quad (A.7a) \]
\[ \text{tr} F^{10} = \frac{41}{2496} (\text{tr} F^2)^5 - \frac{295}{1248} (\text{tr} F^2)^3 (\text{tr} F^4) + \frac{35}{52} (\text{tr} F^2)(\text{tr} F^4)^2 + \frac{115}{624} (\text{tr} F^2)^2 (\text{tr} F^6) - \frac{5}{312} (\text{tr} F^4)(\text{tr} F^6). \quad (A.7b) \]

A.1.3 The Lie algebra \( a_4 \)

**The defining representation:** In the defining representation \((1, 0, 0, 0)[5]\), the polynomials \( \text{tr} A^k, k \in \{2, 3, 4, 5\} \) can be taken as generators of the algebra of invariant polynomials. The \( \text{tr} A^k \) satisfy these relations:
\[ \text{tr} A^6 = -\frac{1}{8} (\text{tr} A^2)^3 + \frac{1}{3} (\text{tr} A^3)^2 + \frac{3}{4} (\text{tr} A^2)(\text{tr} A^4), \quad (A.8a) \]
\[ \text{tr} A^7 = -\frac{7}{24} (\text{tr} A^2)^2 (\text{tr} A^3) + \frac{7}{12} (\text{tr} A^3)(\text{tr} A^4) + \frac{7}{10} (\text{tr} A^2)(\text{tr} A^5), \quad (A.8b) \]
\[ \text{tr} A^8 = -\frac{1}{16} (\text{tr} A^2)^4 + \frac{1}{4} (\text{tr} A^2)^3 (\text{tr} A^4) + \frac{1}{4} (\text{tr} A^4)^2 + \frac{8}{15} (\text{tr} A^3)(\text{tr} A^5). \quad (A.8c) \]

The characteristic polynomial of the matrix \( A \) is
\[ \chi_A(t) = t^5 - \frac{1}{2} (\text{tr} A^2) t^3 - \frac{1}{3} (\text{tr} A^3) t^2 + \left( \frac{1}{8} (\text{tr} A^2)^2 - \frac{1}{4} (\text{tr} A^4) \right) t + \left( \frac{1}{6} (\text{tr} A^2)(\text{tr} A^3) - \frac{1}{5} (\text{tr} A^5) \right). \quad (A.9) \]

**The adjoint representation:** The traces in the adjoint representation \((1, 0, 0, 1)[24]\) of \( a_4 \) can be expressed in terms of the primitive polynomials via
\[ \text{tr} F^2 = 10(\text{tr} A^2), \quad (A.10a) \]
\[ \text{tr} F^4 = 6(\text{tr} A^2)^2 + 10(\text{tr} A^4), \quad (A.10b) \]
\[ \text{tr} F^6 = -\frac{5}{4} (\text{tr} A^2)^3 - \frac{50}{3} (\text{tr} A^3)^2 + \frac{75}{2} (\text{tr} A^2)(\text{tr} A^4), \quad (A.10c) \]
\[ \text{tr} F^8 = -\frac{61}{8} (\text{tr} A^2)^4 + \frac{56}{3} (\text{tr} A^2)(\text{tr} A^3)^2 + \frac{89}{2} (\text{tr} A^2)^2(\text{tr} A^4) + \frac{145}{2} (\text{tr} A^4)^2 - \frac{320}{3} (\text{tr} A^3)(\text{tr} A^5). \quad (A.10d) \]

There are no relations expressing \( \text{tr} F^k, k \in \{2, 4, 6, 8, 10\} \) as polynomials of the lower degree ones. The first such relation involves \( \text{tr} F^{12} \). We discussed this fact in section 3.2.
\[ \text{tr} F^{12} = \frac{13799}{6144000} (\text{tr} F^2)^6 - \frac{8193}{1024000} (\text{tr} F^2)^4 (\text{tr} F^4). \quad (A.11) \]
A.2 The simple Lie algebras $b_\ell$

We define the matrices of the defining representations of the Lie algebras $b_\ell$ by the matrices $A_{jx}^j$ given in section 4.2. The adjoint representations are defined by $(F_j)_{k\ell} = i c_{j\ell k}$ (where $[x_i, x_j] = ic_{j\ell k} x_\ell$). The odd traces $\text{tr} A_{2k-1}$ and $\text{tr} F_{2k-1}$, $k \in \mathbb{N}$, vanish.

A.2.1 The Lie algebra $b_2$

The defining representation: In the defining representation $(1, 0)[5]$ of $b_2$, the polynomials $\text{tr} A^2$ and $\text{tr} A^4$ can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones as follows:

\[
\begin{align*}
\text{tr} A^6 &= -\frac{1}{8} \left( \text{tr} A^2 \right)^3 + \frac{3}{4} \text{tr} A^2 \left( \text{tr} A^4 \right), \\
\text{tr} A^8 &= -\frac{1}{16} \left( \text{tr} A^2 \right)^4 + \frac{1}{4} \text{tr} A^2 \left( \text{tr} A^4 \right)^2 + \frac{1}{4} \left( \text{tr} A^4 \right)^2, \\
\text{tr} A^{10} &= -\frac{1}{64} \left( \text{tr} A^2 \right)^5 + \frac{5}{16} \text{tr} A^2 \left( \text{tr} A^4 \right)^2, \\
\text{tr} A^{12} &= -\frac{3}{64} \left( \text{tr} A^2 \right)^4 \left( \text{tr} A^4 \right) + \frac{3}{16} \left( \text{tr} A^2 \right)^2 \left( \text{tr} A^4 \right)^2 + \frac{1}{16} \left( \text{tr} A^4 \right)^3.
\end{align*}
\]

The characteristic polynomial of the matrix $A$ is

\[
\chi(t) = t^5 - \frac{1}{2} (\text{tr} A^2)^3 + \left( \frac{1}{8} \left( \text{tr} A^2 \right)^2 - \frac{1}{4} \left( \text{tr} A^4 \right) \right) t.
\]

The adjoint representation: The traces in the adjoint representation $(0, 2)[10]$ can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr} F^2 &= 3 (\text{tr} A^2), \\
\text{tr} F^4 &= 3 (\text{tr} A^2)^2 - 3 (\text{tr} A^4), \\
\text{tr} F^6 &= \frac{27}{8} (\text{tr} A^2)^3 - \frac{21}{4} (\text{tr} A^2) (\text{tr} A^4), \\
\text{tr} F^8 &= \frac{67}{16} (\text{tr} A^2)^4 - \frac{39}{4} (\text{tr} A^2)^2 (\text{tr} A^4) + \frac{17}{4} (\text{tr} A^4)^2, \\
\text{tr} F^{10} &= \frac{327}{64} (\text{tr} A^2)^5 - 15 (\text{tr} A^2)^3 (\text{tr} A^4) + \frac{165}{16} (\text{tr} A^2) (\text{tr} A^4)^2, \\
\text{tr} F^{12} &= \frac{99}{16} (\text{tr} A^2)^6 - \frac{1395}{64} (\text{tr} A^2)^4 (\text{tr} A^4) + \frac{339}{16} (\text{tr} A^2)^2 (\text{tr} A^4)^2 + \frac{63}{16} (\text{tr} A^4)^3.
\end{align*}
\]
The simplest relations involving the \( \text{tr} F^{2k} \) alone are these:

\[
\begin{align*}
\text{tr} F^6 &= -\frac{5}{72} (\text{tr} F^2)^3 + \frac{7}{12} (\text{tr} F^2)(\text{tr} F^4), \\
\text{tr} F^8 &= -\frac{7}{432} (\text{tr} F^2)^4 + \frac{5}{108} (\text{tr} F^2)^2 (\text{tr} F^4) + \frac{17}{36} (\text{tr} F^4)^2, \\
\text{tr} F^{10} &= \frac{1}{576} (\text{tr} F^2)^5 - \frac{5}{72} (\text{tr} F^2)^3 (\text{tr} F^4) + \frac{55}{144} (\text{tr} F^2)(\text{tr} F^4)^2, \\
\text{tr} F^{12} &= \frac{35}{15552} (\text{tr} F^2)^6 - \frac{187}{5184} (\text{tr} F^2)^4 (\text{tr} F^4) + \frac{25}{216} (\text{tr} F^2)^2 (\text{tr} F^4)^2 \quad (A.15a) \\
&\quad + \frac{7}{48} (\text{tr} F^4)^3. 
\end{align*}
\]

A.2.2 The Lie algebra \( b_3 \)

The defining representation: In the defining representation \((1, 0, 0)[7]\), the polynomials \( \text{tr} A^2, \text{tr} A^4 \) and \( \text{tr} A^6 \) can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones as follows:

\[
\begin{align*}
\text{tr} A^8 &= \frac{1}{48} (\text{tr} A^2)^4 - \frac{1}{4} (\text{tr} A^2)^2 (\text{tr} A^4) + \frac{1}{4} (\text{tr} A^4)^2 + \frac{2}{3} (\text{tr} A^2)(\text{tr} A^6), \\
\text{tr} A^{10} &= \frac{1}{96} (\text{tr} A^2)^5 - \frac{5}{48} (\text{tr} A^2)^3 (\text{tr} A^4) + \frac{5}{24} (\text{tr} A^2)^2 (\text{tr} A^6) \\
&\quad + \frac{5}{12} (\text{tr} A^4)(\text{tr} A^6), \\
\text{tr} A^{12} &= \frac{1}{384} (\text{tr} A^2)^6 - \frac{1}{64} (\text{tr} A^2)^4 (\text{tr} A^4) - \frac{3}{32} (\text{tr} A^2)^2 (\text{tr} A^4)^2 \\
&\quad + \frac{1}{16} (\text{tr} A^4)^3 + \frac{1}{24} (\text{tr} A^2)^3 (\text{tr} A^6) + \frac{1}{4} (\text{tr} A^2)(\text{tr} A^4)(\text{tr} A^6) \\
&\quad + \frac{1}{6} (\text{tr} A^6)^2. 
\end{align*}
\]

The characteristic polynomial of the matrix \( A \) is

\[
\chi_A(t) = t^7 - \frac{1}{2} (\text{tr} A^2) t^5 + \left( \frac{1}{8} (\text{tr} A^2)^2 - \frac{1}{4} (\text{tr} A^4) \right) t^3 \\
&\quad + \left( -\frac{1}{48} (\text{tr} A^2)^3 + \frac{1}{8} (\text{tr} A^2)(\text{tr} A^4) - \frac{1}{6} (\text{tr} A^6) \right) t. 
\]

The adjoint representation: The traces in the adjoint representation \((0, 1, 0)[21]\) can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr} F^2 &= 5(\text{tr} A^2), \\
\text{tr} F^4 &= 3(\text{tr} A^2)^2 - (\text{tr} A^4), \\
\text{tr} F^6 &= 15(\text{tr} A^2)(\text{tr} A^4) - 25(\text{tr} A^6), \\
\text{tr} F^8 &= -\frac{121}{48} (\text{tr} A^2)^4 + \frac{121}{4} (\text{tr} A^2)^2 (\text{tr} A^4)^2 + \frac{19}{4} (\text{tr} A^4)^2 \\
&\quad - \frac{158}{3} (\text{tr} A^2)(\text{tr} A^6), \\
\text{tr} F^{10} &= -\frac{415}{96} (\text{tr} A^2)^5 + \frac{1985}{48} (\text{tr} A^2)^3 (\text{tr} A^4)^2 + \frac{45}{4} (\text{tr} A^2)(\text{tr} A^4)^2 \quad (A.18a) \\
&\quad + \frac{7}{48} (\text{tr} A^4)^3. 
\end{align*}
\]
The simplest relations involving the $\text{tr} F^2 k$ alone are these:

$$
\text{tr} F^8 = \frac{8683}{150000}(\text{tr} F^2)^4 - \frac{543}{500}(\text{tr} F^2)^2(\text{tr} F^4) + \frac{19}{4}(\text{tr} F^4)^2 + \frac{158}{375}(\text{tr} F^2)(\text{tr} F^6),
$$

(A.19a)

$$
\text{tr} F^{10} = \frac{8003}{300000}(\text{tr} F^2)^5 - \frac{2987}{6000}(\text{tr} F^2)^3(\text{tr} F^4) + \frac{11}{5}(\text{tr} F^2)^2(\text{tr} F^4)^2 + \frac{367}{3000}(\text{tr} F^2)^2(\text{tr} F^6) - \frac{1}{60}(\text{tr} F^4)(\text{tr} F^6).
$$

(A.19b)

### A.2.3 The Lie algebra $b_4$

#### The defining representation:

In the defining representation $(1,0,0,0)[9]$, the polynomials $\text{tr} A^k$, $k \in \{2,4,6,8\}$, can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones, e.g.:

$$
\text{tr} A^{10} = -\frac{1}{384}(\text{tr} A^2)^5 + \frac{5}{96}(\text{tr} A^2)^3(\text{tr} A^4) - \frac{5}{32}(\text{tr} A^2)(\text{tr} A^4)^2 - \frac{5}{24}(\text{tr} A^2)^2(\text{tr} A^6) + \frac{5}{12}(\text{tr} A^4)(\text{tr} A^6) + \frac{5}{8}(\text{tr} A^2)(\text{tr} A^8).
$$

(A.20)

The characteristic polynomial of the matrix $A$ is

$$
\chi_A(t) = t^9 - \frac{1}{2}(\text{tr} A^2) t^7 + \left(\frac{1}{8}(\text{tr} A^2)^2 - \frac{1}{4}(\text{tr} A^4)\right) t^5
+ \left(-\frac{1}{48}(\text{tr} A^2)^3 + \frac{1}{8}(\text{tr} A^2)(\text{tr} A^4) - \frac{1}{6}(\text{tr} A^6)\right) t^3
+ \left(\frac{1}{384}(\text{tr} A^2)^4 - \frac{1}{32}(\text{tr} A^2)^2(\text{tr} A^4) + \frac{1}{32}(\text{tr} A^4)^2 + \frac{1}{12}(\text{tr} A^2)(\text{tr} A^6)
- \frac{1}{8}(\text{tr} A^8)\right) t.
$$

(A.21)

#### The adjoint representation:

The traces in the adjoint representation $(0,1,0,0)[36]$ can be expressed in terms of the primitive polynomials via

$$
\text{tr} F^2 = 7(\text{tr} A^2),
$$

(A.22a)

$$
\text{tr} F^4 = 3(\text{tr} A^2)^2 + (\text{tr} A^4),
$$

(A.22b)

$$
\text{tr} F^6 = 15(\text{tr} A^2)(\text{tr} A^4) - 23(\text{tr} A^6),
$$

(A.22c)

$$
\text{tr} F^8 = 35(\text{tr} A^4)^2 + 28(\text{tr} A^2)(\text{tr} A^6) - 119(\text{tr} A^8),
$$

(A.22d)

$$
\text{tr} F^{10} = \frac{503}{384}(\text{tr} A^2)^5 - \frac{2515}{96}(\text{tr} A^2)^3(\text{tr} A^4) + \frac{2515}{32}(\text{tr} A^2)(\text{tr} A^4)^2 + \frac{2515}{24}(\text{tr} A^2)^2(\text{tr} A^6) + \frac{5}{12}(\text{tr} A^4)(\text{tr} A^6) - \frac{2155}{8}(\text{tr} A^2)(\text{tr} A^8).
$$

(A.22e)

The simplest relation involving the $\text{tr} F^{2k}$ alone is:

$$
\text{tr} F^{10} = -\frac{657127}{2523470208}(\text{tr} F^2)^5 + \frac{2285}{262752}(\text{tr} F^2)^3(\text{tr} F^4).
$$

(A.23)
\[-\frac{4535}{87584}(\text{tr } F^2)(\text{tr } F^4)^2 - \frac{16385}{459816}(\text{tr } F^2)^2(\text{tr } F^6) - \frac{5}{276}(\text{tr } F^4)(\text{tr } F^6) + \frac{2155}{664}(\text{tr } F^2)(\text{tr } F^8).\]

A.3 The simple Lie algebras $c_\ell$

We define the matrices of the defining representations of the Lie algebras $c_\ell$ by the matrices $A = a_j x_j$ given in section 4.3. The adjoint representations are defined by $(F_j)_{k\ell} = i c_{j\ell k}$ (where $[x_i, x_j] = i c_{j\ell k} x_\ell$). The odd traces $\text{tr } A_{2k-1}^2$ and $\text{tr } F_{2k-1}^2$, $k \in \mathbb{N}$, vanish.

A.3.1 The Lie algebra $c_2$

The defining representation: In the defining representation $(1,0)[4]$, the polynomials $\text{tr } A^2$ and $\text{tr } A^4$ can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones:

\[
\begin{align*}
\text{tr } A^6 &= -\frac{1}{8}(\text{tr } A^2)^3 + \frac{3}{4}(\text{tr } A^2)(\text{tr } A^4), \\
\text{tr } A^8 &= -\frac{1}{16}(\text{tr } A^2)^4 + \frac{1}{16}(\text{tr } A^2)^2(\text{tr } A^4) + \frac{1}{4}(\text{tr } A^4)^2, \\
\text{tr } A^{10} &= -\frac{1}{64}(\text{tr } A^2)^5 + 5(\text{tr } A^2)^2(\text{tr } A^4)^2, \\
\text{tr } A^{12} &= -\frac{3}{64}(\text{tr } A^2)^4(\text{tr } A^4) + \frac{3}{16}(\text{tr } A^2)^2(\text{tr } A^4)^2 + \frac{1}{16}(\text{tr } A^4)^3.
\end{align*}
\]

The characteristic polynomial of the matrix $A$ is

\[
\chi_A(t) = t^4 - \frac{1}{2}(\text{tr } A^2)t^2 + \left(\frac{1}{8}(\text{tr } A^2)^2 - \frac{1}{4}(\text{tr } A^4)\right). 
\]

The adjoint representation: The traces in the adjoint representation $(2,0)[10]$ can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr } F^2 &= 6(\text{tr } A^2), \\
\text{tr } F^4 &= 3(\text{tr } A^2)^2 + 12(\text{tr } A^4), \\
\text{tr } F^6 &= -\frac{9}{2}(\text{tr } A^2)^3 + 42(\text{tr } A^2)(\text{tr } A^4), \\
\text{tr } F^8 &= -\frac{47}{4}(\text{tr } A^2)^4 + 54(\text{tr } A^2)^2(\text{tr } A^4) + 68(\text{tr } A^4)^2, \\
\text{tr } F^{10} &= -\frac{87}{8}(\text{tr } A^2)^5 - 15(\text{tr } A^2)^3(\text{tr } A^4) + 330(\text{tr } A^2)(\text{tr } A^4)^2, \\
\text{tr } F^{12} &= \frac{99}{16}(\text{tr } A^2)^6 - \frac{855}{4}(\text{tr } A^2)^4(\text{tr } A^4) + 789(\text{tr } A^2)^2(\text{tr } A^4)^2 + 252(\text{tr } A^4)^3.
\end{align*}
\]

The simplest relations involving the $\text{tr } F^{2k}$ alone are these:

\[
\begin{align*}
\text{tr } F^6 &= -\frac{5}{72}(\text{tr } F^2)^3 + \frac{7}{12}(\text{tr } F^2)(\text{tr } F^4), \\
\text{tr } F^8 &= -\frac{1}{276}(\text{tr } F^2)^4 + \frac{1}{12}(\text{tr } F^2)^2(\text{tr } F^4) + \frac{1}{18}(\text{tr } F^4)^2, \\
\text{tr } F^{10} &= -\frac{1}{276}(\text{tr } F^2)^5 + \frac{5}{36}(\text{tr } F^2)^3(\text{tr } F^4) - \frac{3}{2}(\text{tr } F^2)(\text{tr } F^4)^2 + \frac{1}{2}(\text{tr } F^4)^3, \\
\text{tr } F^{12} &= -\frac{1}{162}(\text{tr } F^2)^6 + \frac{5}{12}(\text{tr } F^2)^4(\text{tr } F^4) - \frac{5}{6}(\text{tr } F^2)^2(\text{tr } F^4)^2 + \frac{1}{2}(\text{tr } F^4)^3.
\end{align*}
\]
\[
\begin{align*}
\text{tr} F^8 &= -\frac{7}{432} (\text{tr} F^2)^4 + \frac{5}{108} (\text{tr} F^2)^2 (\text{tr} F^4) + \frac{17}{36} (\text{tr} F^4)^2, \\
\text{tr} F^{10} &= \frac{1}{576} (\text{tr} F^2)^5 - \frac{5}{72} (\text{tr} F^2)^3 (\text{tr} F^4) + \frac{55}{144} (\text{tr} F^2)(\text{tr} F^4)^2, \\
\text{tr} F^{12} &= \frac{35}{15552} (\text{tr} F^2)^6 - \frac{187}{5184} (\text{tr} F^2)^4 (\text{tr} F^4) + \frac{25}{216} (\text{tr} F^2)^2 (\text{tr} F^4)^2 + \frac{7}{48} (\text{tr} F^4)^3.
\end{align*}
\]

A.3.2 The Lie algebra \( c_3 \)

The defining representation: In the defining representation \((1,0,0)[6]\), the polynomials \( \text{tr} A^2, \text{tr} A^4 \) and \( \text{tr} A^6 \) can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones:

\[
\begin{align*}
\text{tr} A^8 &= \frac{1}{48} (\text{tr} A^2)^4 - \frac{1}{4} (\text{tr} A^2)^2 (\text{tr} A^4) + \frac{1}{4} (\text{tr} A^4)^2 + \frac{2}{3} (\text{tr} A^2)(\text{tr} A^6), \\
\text{tr} A^{10} &= \frac{1}{96} (\text{tr} A^2)^5 - \frac{5}{48} (\text{tr} A^2)^3 (\text{tr} A^4) + \frac{5}{24} (\text{tr} A^2)^2 (\text{tr} A^6) + \frac{5}{12} (\text{tr} A^4)(\text{tr} A^6).
\end{align*}
\]

The characteristic polynomial of the matrix \( A \) is

\[
\chi_A(t) = t^6 - \frac{1}{2} (\text{tr} A^2) t^4 + \left( \frac{1}{8} (\text{tr} A^2)^2 - \frac{1}{4} (\text{tr} A^4) \right) t^2 + \left( -\frac{1}{48} (\text{tr} A^2)^3 + \frac{1}{8} (\text{tr} A^2)(\text{tr} A^4) - \frac{1}{6} (\text{tr} A^6) \right).
\]

The adjoint representation: The traces in the adjoint representation \((2,0,0)[21]\) can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr} F^2 &= 8(\text{tr} A^2), \\
\text{tr} F^4 &= 3(\text{tr} A^2)^2 + 14(\text{tr} A^4), \\
\text{tr} F^6 &= 15(\text{tr} A^2)(\text{tr} A^4) + 38(\text{tr} A^6), \\
\text{tr} F^8 &= \frac{67}{24} (\text{tr} A^2)^4 - \frac{67}{2} (\text{tr} A^2)^2 (\text{tr} A^4) + \frac{137}{2} (\text{tr} A^4)^2 \\
&\quad + \frac{352}{3} (\text{tr} A^2)(\text{tr} A^6), \\
\text{tr} F^{10} &= \frac{19}{3} (\text{tr} A^2)^5 - \frac{1565}{24} (\text{tr} A^2)^3 (\text{tr} A^4) + \frac{45}{4} (\text{tr} A^2)(\text{tr} A^4)^2 \\
&\quad + \frac{1655}{12} (\text{tr} A^2)^2 (\text{tr} A^6) + \frac{2555}{6} (\text{tr} A^4)(\text{tr} A^6).
\end{align*}
\]

The simplest relations involving the \( \text{tr} F^{2k} \) alone are these:

\[
\begin{align*}
\text{tr} F^8 &= \frac{2011}{357504} (\text{tr} F^2)^4 - \frac{1815}{14896} (\text{tr} F^2)^2 (\text{tr} F^4) + \frac{137}{392} (\text{tr} F^4)^2 \\
&\quad + \frac{22}{57} (\text{tr} F^2)(\text{tr} F^6), 
\end{align*}
\]
\[
\begin{align*}
\text{tr } F^{10} &= \frac{1081}{1430016} (\text{tr } F^2)^5 - \frac{2615}{357504} (\text{tr } F^2)^3 (\text{tr } F^4) - \frac{745}{7448} (\text{tr } F^2)(\text{tr } F^4)^2 \quad (A.31b) \\
&\quad + \frac{35}{1824} (\text{tr } F^2)^2 (\text{tr } F^6) + \frac{365}{456} (\text{tr } F^4)(\text{tr } F^6).
\end{align*}
\]

### A.3.3 The Lie algebra \( c_4 \)

**The defining representation:** In the defining representation \((1, 0, 0, 0)[8]\), the polynomials \( \text{tr } A^k, k \in \{2, 4, 6, 8\} \) can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones, e.g.

\[
\begin{align*}
\text{tr } A^{10} &= -\frac{1}{384} (\text{tr } A^2)^5 + \frac{5}{96} (\text{tr } A^2)^3 (\text{tr } A^4) - \frac{5}{32} (\text{tr } A^2)(\text{tr } A^4)^2 \\
&\quad - \frac{5}{24} (\text{tr } A^2)^2 (\text{tr } A^6) + \frac{5}{12} (\text{tr } A^4)(\text{tr } A^6) + \frac{5}{8} (\text{tr } A^2)(\text{tr } A^8).
\end{align*}
\]

The characteristic polynomial of the matrix \( A \) is

\[
\chi(t) = t^8 - \frac{1}{2} (\text{tr } A^2) t^6 + \left( \frac{1}{8} (\text{tr } A^2)^2 - \frac{1}{4} (\text{tr } A^4) \right) t^4 \\
+ \left( -\frac{1}{48} (\text{tr } A^2)^2 + \frac{1}{8} (\text{tr } A^2)(\text{tr } A^4) - \frac{1}{6} (\text{tr } A^6) \right) t^2 \\
+ \left( \frac{1}{384} (\text{tr } A^2)^4 - \frac{1}{32} (\text{tr } A^2)^2 (\text{tr } A^4) + \frac{1}{32} (\text{tr } A^4)^2 + \frac{1}{12} (\text{tr } A^2)(\text{tr } A^6) - \frac{1}{8} (\text{tr } A^8) \right).
\]

**The adjoint representation:** Let \((F_j)_{k\ell} = i c_{j\ell k}\) be the matrices of the adjoint representation \((2, 0, 0, 0)[36]\). The odd traces \( \text{tr } F^{2k-1}, k \in \mathbb{N} \), vanish. The others can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr } F^2 &= 10 (\text{tr } A^2), \\
\text{tr } F^4 &= 3 (\text{tr } A^2)^2 + 16 (\text{tr } A^4), \\
\text{tr } F^6 &= 15 (\text{tr } A^2)(\text{tr } A^4) + 40 (\text{tr } A^6), \\
\text{tr } F^8 &= 35 (\text{tr } A^2)^3 + 28 (\text{tr } A^2)(\text{tr } A^4) + 136 (\text{tr } A^8), \\
\text{tr } F^{10} &= -\frac{65}{48} (\text{tr } A^2)^5 + \frac{325}{12} (\text{tr } A^2)^3 (\text{tr } A^4) - \frac{325}{4} (\text{tr } A^2)(\text{tr } A^4)^2 \\
&\quad - \frac{325}{3} (\text{tr } A^2)^2 (\text{tr } A^6) + \frac{1280}{3} (\text{tr } A^4)(\text{tr } A^6) + 370 (\text{tr } A^2)(\text{tr } A^8).
\end{align*}
\]

The simplest relation involving the \( \text{tr } F^{2k} \) alone is:

\[
\begin{align*}
\text{tr } F^{10} &= -\frac{2039}{6528000} (\text{tr } F^2)^5 + \frac{2269}{163200} (\text{tr } F^2)^3 (\text{tr } F^4) - \frac{143}{1088} (\text{tr } F^2)(\text{tr } F^4)^2 \\
&\quad - \frac{1349}{20400} (\text{tr } F^2)^2 (\text{tr } F^6) + \frac{2}{3} (\text{tr } F^4)(\text{tr } F^6) + \frac{37}{136} (\text{tr } F^2)(\text{tr } F^8). \\
\end{align*}
\]

### A.4 The simple Lie algebras \( d_\ell \)

We define the matrices of the defining representations of the Lie algebras \( d_\ell \) by the matrices \( A = a_j x_j \) given in section [1.2]. In order to generate the invariant polynomials of a simple Lie algebra of type \( d_\ell \), we need a square root of \( \det(A) \) in addition to the trace polynomials. The square of this invariant naturally appears in [8, 24] and is related to a Pfaffian form [10, 33].

The adjoint representations are defined by \((F_j)_{k\ell} = i c_{j\ell k}\) (where \([x_i, x_j] = ic_{j\ell k} x_\ell\)). The odd traces \( \text{tr } A^{2k-1} \) and \( \text{tr } F^{2k-1}, k \in \mathbb{N}, \) vanish.
A.4.1 The Lie algebra $d_3$

**The defining representation:** In the defining representation $(1, 0, 0)[6]$, the polynomials $\text{tr} \, A^2$, $\text{tr} \, A^4$ and $\sqrt{\det A}$, which is of degree three, can be used as generators of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones, e.g.:

\[
\begin{align*}
\text{tr} \, A^6 & = -\frac{1}{8} (\text{tr} \, A^2)^3 - 6(\det A) + \frac{3}{4} (\text{tr} \, A^2)(\text{tr} \, A^4), \\
\text{tr} \, A^8 & = -\frac{1}{16} (\text{tr} \, A^2)^4 - 4(\text{tr} \, A^2)(\det A) + \frac{1}{4} (\text{tr} \, A^2)^2(\text{tr} \, A^4) + \frac{1}{4} (\text{tr} \, A^4)^2, \\
\text{tr} \, A^{10} & = -\frac{1}{64} (\text{tr} \, A^2)^5 - \frac{5}{4} (\text{tr} \, A^2)^2(\det A) - \frac{5}{2} (\det A)(\text{tr} \, A^4) \\
& \quad + \frac{5}{16} (\text{tr} \, A^2)(\text{tr} \, A^4)^2.
\end{align*}
\]

The characteristic polynomial of the matrix $A$ is

\[
\chi(t) = t^6 - \frac{1}{2} (\text{tr} \, A^2) t^4 + \left(\frac{1}{8}(\text{tr} \, A^2)^2 - \frac{1}{4}(\text{tr} \, A^4)\right) t^2 + (\det A). 
\]  

(A.37)

**The adjoint representation:** The traces in the adjoint representation $(0, 1, 1)[15]$ can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr} \, F^2 & = 4(\text{tr} \, A^2), \\
\text{tr} \, F^4 & = 3(\text{tr} \, A^2)^2 - 2(\text{tr} \, A^4), \\
\text{tr} \, F^6 & = \frac{13}{4} (\text{tr} \, A^2)^3 + 156(\det A) - \frac{9}{2} (\text{tr} \, A^2)(\text{tr} \, A^4), \\
\text{tr} \, F^8 & = \frac{33}{8} (\text{tr} \, A^2)^4 + 320(\text{tr} \, A^2)(\det A) - \frac{19}{2} (\text{tr} \, A^2)^2(\text{tr} \, A^4) + \frac{9}{2} (\text{tr} \, A^4)^2, \\
\text{tr} \, F^{10} & = \frac{163}{32} (\text{tr} \, A^2)^5 + \frac{905}{2}(\text{tr} \, A^2)^2(\det A) - 15(\text{tr} \, A^2)^3(\text{tr} \, A^4) \\
& \quad + 5(\det A)(\text{tr} \, A^4) + \frac{85}{8} (\text{tr} \, A^2)(\text{tr} \, A^4)^2.
\end{align*}
\]

(A.38)

The simplest relations involving the $\text{tr} \, F^{2k}$ alone are

\[
\begin{align*}
\text{tr} \, F^8 & = \frac{35}{1248} (\text{tr} \, F^2)^4 - \frac{43}{104} (\text{tr} \, F^2)^2(\text{tr} \, F^4) + \frac{9}{8} (\text{tr} \, F^4)^2 + \frac{20}{39} (\text{tr} \, F^2)(\text{tr} \, F^6), \\
\text{tr} \, F^{10} & = \frac{41}{2496} (\text{tr} \, F^2)^5 - \frac{295}{1248} (\text{tr} \, F^2)^3(\text{tr} \, F^4) + \frac{35}{52} (\text{tr} \, F^2)(\text{tr} \, F^4)^2 + \frac{115}{624} (\text{tr} \, F^2)^2(\text{tr} \, F^6) - \frac{5}{312} (\text{tr} \, F^4)(\text{tr} \, F^6).
\end{align*}
\]

(A.39)

A.4.2 The Lie algebra $d_4$

**The defining representation:** In the defining representation $(1, 0, 0, 0)[8]$, the polynomials $\text{tr} \, A^k$, $k \in \{2, 4, 6\}$, and $\sqrt{\det A}$, which, like $\text{tr} \, A^4$, is of degree four, can be used as generators
of the algebra of invariant polynomials. The others can be expressed in terms of the primitive ones, e.g.:

\[
\begin{align*}
\text{tr } A^8 &= \frac{1}{48}(\text{tr } A^2)^4 - 8(\det A) - \frac{1}{4}(\text{tr } A^2)^2(\text{tr } A^4) + \frac{1}{4}(\text{tr } A^4)^2 + \frac{2}{3}(\text{tr } A^2)(\text{tr } A^6), \\
\text{tr } A^{10} &= \frac{1}{96}(\text{tr } A^2)^5 - 5(\text{tr } A^2)(\det A) - \frac{5}{48}(\text{tr } A^2)^3(\text{tr } A^4) + \frac{5}{24}(\text{tr } A^2)^2(\text{tr } A^6) + \frac{5}{12}(\text{tr } A^4)(\text{tr } A^6).
\end{align*}
\]

The characteristic polynomial of the matrix \( A \) is

\[
\chi(t) = t^8 - \frac{1}{2}(\text{tr } A^2)t^6 + \left( \frac{1}{8}(\text{tr } A^2)^2 - \frac{1}{4}(\text{tr } A^4)^2 \right)t^4 + \left( -\frac{1}{48}(\text{tr } A^2)^3 + \frac{1}{8}(\text{tr } A^2)(\text{tr } A^4) - \frac{1}{6}(\text{tr } A^6) \right)t^2 + (\det A).
\]

The **adjoint representation**: The traces in the adjoint representation \((0,1,0,0,0)[28]\) can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr } F^2 &= 6(\text{tr } A^2), \\
\text{tr } F^4 &= 3(\text{tr } A^2)^2, \\
\text{tr } F^6 &= 15(\text{tr } A^2)(\text{tr } A^4) - 24(\text{tr } A^6), \\
\text{tr } F^8 &= -\frac{5}{2}(\text{tr } A^2)^4 + 960(\det A) + 30(\text{tr } A^2)^2(\text{tr } A^4) + 5(\text{tr } A^4)^2 - 52(\text{tr } A^2)(\text{tr } A^6), \\
\text{tr } F^{10} &= -\frac{69}{16}(\text{tr } A^2)^5 + 2160(\text{tr } A^2)(\det A) + \frac{165}{4}(\text{tr } A^2)^3(\text{tr } A^4) + \frac{45}{4}(\text{tr } A^2)(\text{tr } A^4)^2 - 75(\text{tr } A^2)^2(\text{tr } A^6).
\end{align*}
\]

The simplest relations involving the \( \text{tr } F^{2k} \) alone are

\[
\begin{align*}
\text{tr } F^4 &= \frac{1}{12}(\text{tr } F^2)^2, \\
\text{tr } F^{10} &= \frac{7}{41472}(\text{tr } F^2)^5 - \frac{7}{144}(\text{tr } F^2)^2(\text{tr } F^6) + \frac{3}{8}(\text{tr } F^2)(\text{tr } F^8), \\
\text{tr } F^{14} &= -\frac{2761}{179159040}(\text{tr } F^2)^7 + \frac{24409}{5598720}(\text{tr } F^2)^4(\text{tr } F^6) - \frac{1001}{19440}(\text{tr } F^2)(\text{tr } F^6)^2 - \frac{7931}{311040}(\text{tr } F^2)^3(\text{tr } F^8) + \frac{77}{360}(\text{tr } F^6)(\text{tr } F^8) + \frac{497}{1080}(\text{tr } F^2)(\text{tr } F^{12}).
\end{align*}
\]

**A.5 The simple Lie algebra \( e_6 \)**

**The defining representation**: We denote the matrices of the defining representation \((1,0,0,0,0,0)[27]\) of \( e_6 \) by \( A \). We have \( \text{tr } A = 0 \) and \( \text{tr } A^3 = 0 \), and the polynomials \( \text{tr } A^k \),
$k \in \{2, 5, 6, 8, 9, 12\}$ are suitable generators of the algebra of invariant polynomials. The relations are

$$
\begin{align*}
\text{tr} A^4 &= \frac{1}{12} (\text{tr} A^2)^2, & \text{(A.44a)} \\
\text{tr} A^7 &= \frac{7}{24} (\text{tr} A^2)(\text{tr} A^5), & \text{(A.44b)} \\
\text{tr} A^{10} &= \frac{7}{41472} (\text{tr} A^2)^5 + \frac{7}{40} (\text{tr} A^5)^2 - \frac{7}{144} (\text{tr} A^2)^2 (\text{tr} A^6) & \text{+ } \frac{3}{8} (\text{tr} A^2)(\text{tr} A^8), & \text{(A.44c)} \\
\text{tr} A^{11} &= -\frac{55}{3456} (\text{tr} A^2)^3 (\text{tr} A^5) + \frac{11}{36} (\text{tr} A^5)(\text{tr} A^6) + \frac{605}{1512} (\text{tr} A^2)(\text{tr} A^9), & \text{(A.44d)} \\
\text{tr} A^{13} &= -\frac{143}{27648} (\text{tr} A^2)^4 (\text{tr} A^5) + \frac{143}{2700} (\text{tr} A^2)(\text{tr} A^5)(\text{tr} A^6) & \text{+ } \frac{1859}{18144} (\text{tr} A^2)^2 (\text{tr} A^9). & \text{(A.44e)}
\end{align*}
$$

The adjoint representation: Let $(F_j)_{k\ell} = i c_{j\ell k}$ be the matrices of the adjoint representation $(0,0,0,0,1)[78]$. The odd traces $\text{tr} F^{2k-1}$, $k \in \mathbb{N}$, vanish. The others can be expressed in terms of the primitive polynomials via

$$
\begin{align*}
\text{tr} F^2 &= 4(\text{tr} A^2), & \text{(A.45a)} \\
\text{tr} F^4 &= 2(\text{tr} A^2)^2, & \text{(A.45b)} \\
\text{tr} F^6 &= \frac{5}{36} (\text{tr} A^2)^3 - 6(\text{tr} A^6), & \text{(A.45c)} \\
\text{tr} F^8 &= \frac{35}{432} (\text{tr} A^2)^4 - \frac{28}{3} (\text{tr} A^2)(\text{tr} A^6) + 18(\text{tr} A^8), & \text{(A.45d)} \\
\text{tr} F^{10} &= \frac{91}{2304} (\text{tr} A^2)^5 - \frac{21}{20} (\text{tr} A^5)^2 - \frac{133}{24} (\text{tr} A^2)^2 (\text{tr} A^6) + \frac{51}{4} (\text{tr} A^2)(\text{tr} A^8). & \text{(A.45e)}
\end{align*}
$$

A.6 The simple Lie algebra $f_4$

The defining representation: We denote the matrices of the defining representation $(0,0,0,1)[26]$ of $f_4$ by $A$. The polynomials $\text{tr} A^k$, $k \in \{2, 6, 8, 12\}$ are suitable generators of the algebra of invariant polynomials. The odd traces $\text{tr} A^{2k-1}$, $k \in \mathbb{N}$, vanish. The relations are

$$
\begin{align*}
\text{tr} A^4 &= \frac{1}{12} (\text{tr} A^2)^2, & \text{(A.46a)} \\
\text{tr} A^{10} &= \frac{7}{41472} (\text{tr} A^2)^5 - \frac{7}{144} (\text{tr} A^2)^2 (\text{tr} A^6) + \frac{3}{8} (\text{tr} A^2)(\text{tr} A^8), & \text{(A.46b)} \\
\text{tr} A^{14} &= -\frac{2761}{179159040} (\text{tr} A^2)^7 + \frac{24409}{5598720} (\text{tr} A^2)^4 (\text{tr} A^6) & \text{+ } \frac{1001}{19440} (\text{tr} A^2)(\text{tr} A^6)^2 - \frac{7931}{311040} (\text{tr} A^2)^3 (\text{tr} A^8) & \text{+ } \frac{77}{360} (\text{tr} A^6)(\text{tr} A^8) + \frac{497}{1080} (\text{tr} A^2)(\text{tr} A^{12}). & \text{(A.46c)}
\end{align*}
$$
The adjoint representation: Let \((F_j)_{k\ell} = i c_{j\ell k}\) be the matrices of the adjoint representation \((1, 0, 0, 0)[52]\) in a suitable normalization relative to those of the defining representation. The odd traces \(\text{tr} F^{2k-1}, k \in \mathbb{N}\), vanish. The others can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr} F^2 &= 3(\text{tr} A^2), \\
\text{tr} F^4 &= \frac{5}{12}(\text{tr} A^2)^2, \\
\text{tr} F^6 &= \frac{5}{36} (\text{tr} A^2)^3 - 7(\text{tr} A^6), \\
\text{tr} F^8 &= \frac{35}{432} (\text{tr} A^2)^4 - \frac{28}{3} (\text{tr} A^2)(\text{tr} A^6) + 17(\text{tr} A^8), \\
\text{tr} F^{10} &= \frac{1631}{41472} (\text{tr} A^2)^5 - \frac{791}{144} (\text{tr} A^2)^2 (\text{tr} A^6) + \frac{99}{8} (\text{tr} A^2)(\text{tr} A^8), \\
\text{tr} F^{12} &= \frac{1309}{62208} (\text{tr} A^2)^6 - \frac{2387}{648} (\text{tr} A^2)^3 (\text{tr} A^6) + \frac{154}{9} (\text{tr} A^6)^2 \\
&\quad + \frac{209}{18} (\text{tr} A^2)^2 (\text{tr} A^8) - 63(\text{tr} A^{12}).
\end{align*}
\]

The simplest relations of the \(\text{tr} F^{2k}\) are

\[
\begin{align*}
\text{tr} F^4 &= \frac{5}{108} (\text{tr} F^2)^2, \\
\text{tr} F^{10} &= \frac{161}{6345216} (\text{tr} F^2)^5 - \frac{455}{22032} (\text{tr} F^2)^2 (\text{tr} F^6) + \frac{33}{136} (\text{tr} F^2)(\text{tr} F^8). 
\end{align*}
\]

A.7 The simple Lie algebra \(g_2\)

The defining representation: We define the defining representation \((0, 1)[7]\) by the same matrices as in section 1.3, \(A = a_j x_j\). The polynomials \(\text{tr} A^2\) and \(\text{tr} A^6\) can be taken as generators of the algebra of invariant polynomials. The odd traces \(\text{tr} A^{2k-1}, k \in \mathbb{N}\), vanish. The \(\text{tr} A^k\) satisfy these relations:

\[
\begin{align*}
\text{tr} A^4 &= \frac{1}{4} (\text{tr} A^2)^2, \\
\text{tr} A^8 &= -\frac{5}{192} (\text{tr} A^2)^4 + \frac{2}{3} (\text{tr} A^2)(\text{tr} A^6), \\
\text{tr} A^{10} &= -\frac{1}{64} (\text{tr} A^2)^5 + \frac{5}{6} (\text{tr} A^2)^2 (\text{tr} A^6), \\
\text{tr} A^{12} &= -\frac{19}{3072} (\text{tr} A^2)^6 + \frac{5}{48} (\text{tr} A^2)^3 (\text{tr} A^6) + \frac{1}{6} (\text{tr} A^6)^2.
\end{align*}
\]

The characteristic polynomial of the matrix \(A\) is

\[
\chi_A(t) = t^7 - \frac{1}{2} (\text{tr} A^2) t^5 + \frac{1}{16} (\text{tr} A^2)^2 t^3 + \left(\frac{1}{96} (\text{tr} A^2)^3 - \frac{1}{6} (\text{tr} A^6)\right) t.
\]

The adjoint representation: Let \((F_j)_{k\ell} = i c_{j\ell k}\) be the matrices of the adjoint representation \((1, 0)[14]\) (where \([x_j, x_k] = i c_{j\ell k} x_\ell\)). The odd traces \(\text{tr} F^{2k-1}, k \in \mathbb{N}\), vanish. The others
can be expressed in terms of the primitive polynomials via

\[
\begin{align*}
\text{tr } F^2 &= 4(\text{tr } A^2), \\
\text{tr } F^4 &= \frac{5}{2}(\text{tr } A^2)^2, \\
\text{tr } F^6 &= \frac{15}{4}(\text{tr } A^2)^3 - 26(\text{tr } A^6), \\
\text{tr } F^8 &= \frac{515}{96}(\text{tr } A^2)^4 - \frac{160}{3}(\text{tr } A^2)(\text{tr } A^6), \\
\text{tr } F^{10} &= \frac{431}{64}(\text{tr } A^2)^5 - \frac{605}{8}(\text{tr } A^2)^2(\text{tr } A^6), \\
\text{tr } F^{12} &= \frac{12865}{1536}(\text{tr } A^2)^6 - \frac{1315}{12}(\text{tr } A^2)^3(\text{tr } A^6) + \frac{365}{3}(\text{tr } A^6)^2.
\end{align*}
\] (A.51)

The \( \text{tr } F^{2k} \) themselves satisfy these relations:

\[
\begin{align*}
\text{tr } F^4 &= \frac{5}{32}(\text{tr } F^2)^2, \\
\text{tr } F^8 &= -\frac{2905}{319488}(\text{tr } F^2)^4 + \frac{20}{39}(\text{tr } F^2)(\text{tr } F^6), \\
\text{tr } F^{10} &= -\frac{217}{53248}(\text{tr } F^2)^5 + \frac{605}{3328}(\text{tr } F^2)^2(\text{tr } F^6).
\end{align*}
\] (A.52)

The characteristic polynomial of the matrix \( F \) is

\[
\chi_F(t) = t^{14} - 2(\text{tr } A^2)t^{12} + \frac{11}{8}(\text{tr } A^2)^2t^{10} + \left( -\frac{17}{24}(\text{tr } A^2)^3 + \frac{13}{3}(\text{tr } A^6) \right)t^8 \\
+ \left( \frac{49}{256}(\text{tr } A^2)^4 - 2(\text{tr } A^2)(\text{tr } A^6) \right)t^6 \\
+ \left( -\frac{1}{64}(\text{tr } A^2)^5 + \frac{3}{16}(\text{tr } A^2)^2(\text{tr } A^6) \right)t^4 \\
+ \left( -\frac{11}{3072}(\text{tr } A^2)^6 + \frac{5}{48}(\text{tr } A^2)^3(\text{tr } A^6) - \frac{3}{4}(\text{tr } A^6)^2 \right)t^2 \\
= t^{14} - \frac{1}{2}(\text{tr } F^2)t^{12} + \frac{11}{128}(\text{tr } F^2)^2t^{10} \\
+ \left( -\frac{1}{768}(\text{tr } F^2)^3 - \frac{1}{6}(\text{tr } F^6) \right)t^8 \\
+ \left( -\frac{323}{85192}(\text{tr } F^2)^4 + \frac{1}{52}(\text{tr } F^2)(\text{tr } F^6) \right)t^6 \\
+ \left( \frac{19}{1703936}(\text{tr } F^2)^5 - \frac{3}{6656}(\text{tr } F^2)^2(\text{tr } F^6) \right)t^4 \\
+ \left( -\frac{2159}{2126512128}(\text{tr } F^2)^6 + \frac{35}{519168}(\text{tr } F^2)^3(\text{tr } F^6) - \frac{3}{2704}(\text{tr } F^6)^2 \right)t^2.
\] (A.53)

References

[1] A. J. Macfarlane, A. Sudbery, and P. H. Weisz, Comm. Math. Phys. 11, 77 (1968).

[2] J. A. de Azcárraga, A. J. Macfarlane, A. J. Mountain, and J. C. Pérez-Bueno, Nucl. Phys. B 510 [PM], 657 (1998).
[3] A. Sudbery, Ph.D. thesis, Corpus Christi College, Cambridge, 1969.
[4] A. Sudbery, J. Phys. A: Math. Gen. 23, L705 (1990).
[5] M. A. Rashid and Saifuddin, J. Math. Phys. 14, 630 (1973).
[6] G. Racah, Rend. Lincei, Ser. 8 VIII, 108 (1950).
[7] G. Racah, Group Theory and Spectroscopy, Institute for Advanced Studies Lectures (1951).
[8] G. Racah, in Ergebnisse der exakten Naturwissenschaften (Springer, Berlin, 1965), Vol. 37.
[9] J. A. de Azcárraga and J. C. Pérez-Bueno, Comm. Math. Phys. 184, 669 (1997).
[10] J. A. de Azcárraga and J. M. Izquierdo, Lie groups, Lie algebras, cohomology and some applications in physics (Cambridge University Press, Cambridge, 1995).
[11] S. Okubo, J. Math. Phys. 18, 2382 (1977).
[12] K. Meyberg, J. Algebra 84, 279 (1983).
[13] P. Cvitanović, Group theory (Nordita Classics, Kopenhagen, 1984).
[14] P. Cvitanović, Classical and exceptional Lie algebras as invariance algebras, Oxford University Preprint (1977).
[15] K. Meyberg, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 54, 177 (1984).
[16] A. J. Mountain, J. Math. Phys. 39, 5601 (1998).
[17] M. D. Gould, J. Austr. Math. Soc. B 26, 257 (1985).
[18] A. J. Bracken and H. S. Green, J. Math. Phys. 12, 2099 (1971).
[19] H. S. Green, J. Math. Phys. 12, 2106 (1971).
[20] R. Slansky, Phys. Rev. 79, 1 (1981).
[21] J. Fuchs and C. Schweigert, Symmetries, Lie algebras and representations (Cambridge University Press, Cambridge, 1997).
[22] S. Okubo, J. Math. Phys. 20, 586 (1979).
[23] S. Okubo, J. Math. Phys. 23, 8 (1982).
[24] J. E. Humphreys, Introduction to Lie algebras and representation theory, 3 ed. (Springer, New York, 1980).
[25] V. S. Varadarajan, Lie groups, Lie algebras, and their representations (Springer, New York, 1984).
[26] M. J. Englefield and R. King, J. Phys. A: Math. Gen. 13, 2297 (1980).
[27] J. F. Cornwell, *Group theory in physics* (Academic Press, London, 1984), Vol. II.

[28] F. Gürsey and C.-H. Tze, *On the role of division, Jordan and related algebras in particle physics* (World Scientific, Singapore, 1956).

[29] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. 34, 1 (1962).

[30] P. Ramond, Introduction to exceptional Lie groups and algebras, Caltech. preprint CALT-68-577 (1976).

[31] S. Okubo, *Introduction to octonion and other non-associative algebras in physics* (Cambridge University Press, Cambridge, 1995).

[32] A. J. Macfarlane, to be published.

[33] L. J. Boya, Rep. Math. Phys. 30, 149 (1991).