Abstract

This is a summary of a recursive construction of solutions of the $h$-dependent KP hierarchy. We give recursion relations for the coefficients $X_n$ of an $h$-expansion of the operator $X = X_0 + hX_1 + h^2X_2 + \cdots$ for which the dressing operator $W$ is expressed in the exponential form $W = \exp(X/h)$. The wave function $\Psi$ associated with $W$ turns out to have the WKB form $\Psi = \exp(S/h)$, and the coefficients $S_n$ of the $h$-expansion $S = S_0 + hS_1 + h^2S_2 + \cdots$, too, are determined by a set of recursion relations. This WKB form is used to show that the associated tau function has an $h$-expansion of the form $\log \tau = h^{-2}F_0 + h^{-1}F_1 + F_2 + \cdots$. 
0 Introduction

The $h$-dependent formulation of the KP hierarchy was introduced to study the dispersionless KP hierarchy [KG], [Kr], [TT1] as a classical limit (i.e., the lowest order of the $h$-expansion) of the KP hierarchy. This point of view turned out to be very useful for understanding various features of the dispersionless KP hierarchy. In this paper, we return to the $h$-dependent KP hierarchy itself, and consider all orders of the $h$-expansion.

We first address the issue of solving a Riemann-Hilbert problem for the pair $(L, M)$ of Lax and Orlov-Schulman operators [OS]. This is a kind of “quantisation” of a Riemann-Hilbert problem that solves the dispersionless KP hierarchy [TT1]. In this paper, we settle this issue by an $h$-expansion of the dressing operator $W$, which is assumed to have the exponential form $W = \exp(X/h)$ with an operator $X$ of negative order. Roughly speaking, the coefficients $X_n, n = 0, 1, 2, \ldots$, of the $h$-expansion of $X$ are shown to be determined recursively from the lowest order term $X_0$ (in other words, from a solution of the dispersionless KP hierarchy).

We next convert this result to the language of the wave function $\Psi$. Namely, given the dressing operator in the exponential form $W = \exp(X/h)$, we show that the associated wave function has the WKB form $\Psi = \exp(S/h)$ with a phase function $S$ expanded into nonnegative powers of $h$. Borrowing an idea from Aoki’s “exponential calculus” of microdifferential operators [A], we show that dressing operators of the form $W = \exp(X/h)$ and wave functions of the form $\Psi = \exp(S/h)$ are determined from each other by a set of recursion relations for the coefficients of their $h$-expansion. Con-
sequently, the wave function of the solution of the aforementioned
Riemann-Hilbert problem, too, are recursively determined by the
\( h \)-expansion.

Having the \( h \)-expansion of the wave function, we can readily
derive an \( h \)-expansion of the tau function as stated in our previous
work \[TT2\].

Details are found in \[TT3\] and shall be published elsewhere.

1 \( h \)-dependent KP hierarchy: review

In this section we recall several facts on the KP hierarchy depending
on a formal parameter \( h \) in \[TT2\], §1.7.

The \( h \)-dependent KP hierarchy is defined by the Lax representa-
tion

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n = (L^n)_{\geq 0}, \quad n = 1, 2, \ldots ,
\]

where the Lax operator \( L \) is a microdifferential operator of the form

\[
L = \hbar \partial + \sum_{n=1}^{\infty} u_{n+1}(\hbar, x, t)(\hbar \partial)^{-n}, \quad \partial = \frac{\partial}{\partial x},
\]

and “( )_{\geq 0}” stands for the projection onto a differential operator
dropping negative powers of \( \partial \). The coefficients \( u_n(\hbar, x, t) \) of \( L \) are
formally regular with respect to \( h \).

We introduce the notion of the \( h \)-order defined by

\[
\text{ord}^h \left( \sum a_{n,m}(x, t) h^n \partial^m \right) \overset{\text{def}}{=} \max \{ m - n \mid a_{n,m}(x, t) \neq 0 \}.
\]

In particular, \( \text{ord}^h h = -1 \), \( \text{ord}^h \partial = 1 \), \( \text{ord}^h \hbar \partial = 0 \). The regularity
condition which we imposed on the coefficients \( u_n(\hbar, x, t) \) can be
restated as \( \text{ord}_h(L) = 0 \). The \emph{principal symbol} of a microdifferential operator \( A = \sum a_{n,m}(x,t)\hbar^n\partial^m \) with respect to the \( \hbar \)-order is \( \sigma_h(A) \overset{\text{def}}{=} \sum_{m-n=\text{ord}(A)} a_{n,m}(x,t)\hbar^m \).

As in the usual KP theory, the Lax operator \( L \) is expressed by a \emph{dressing operator} \( W \):

\begin{equation}
L = \text{Ad} W(h\partial) = W(h\partial)W^{-1}
\end{equation}

The dressing operator \( W \) should have a specific form:

\begin{equation}
W = \exp(h^{-1}X(h,x,t,h\partial))(h\partial)^{\alpha(h)}/h,
\end{equation}

where \( X(h,x,t,h\partial) = \sum_{k=1}^{\infty} \chi_k(h,x,t)(h\partial)^{-k} \) is a 0-th order operator, \( \text{ord}_h(X) = 0 \), and and \( \alpha(h) \) is a constant with respect to \( x \) and \( t \) with \( \hbar \)-order 0, \( \text{ord}_h \alpha(h) = 0 \).

The \emph{wave function} \( \Psi(h,x,t;z) \) is defined by

\begin{equation}
\Psi(h,x,t;z) = We^{(xz+\zeta(t,z))/\hbar},
\end{equation}

where \( \zeta(t,z) = \sum_{n=1}^{\infty} t_n z^n \). It is a solution of linear equations \( L\Psi = z\Psi, \ h\partial \Psi = B_n\Psi \ (n = 1,2,\ldots) \) and has the WKB form as we shall show in Section [3]. Moreover it is expressed by means of the \emph{tau function} \( \tau(h,t) \) as follows:

\begin{equation}
\Psi(h,x,t;z) = \frac{\tau(t+x-h[z^{-1}])}{\tau(t)}e^{h^{-1}\zeta(t,z)},
\end{equation}

where \( t+x = (t_1+x,t_2,t_3,\ldots) \) and \( [z^{-1}] = (1/z,1/2z^2,1/3z^3,\ldots) \). We shall study the \( \hbar \)-expansion of the tau function in Section [4].

The \emph{Orlov-Schulman operator} \( M \) is defined by

\begin{equation}
M = \text{Ad} \left( W \exp \left( h^{-1}\zeta(t,h\partial) \right) \right)x = W \left( \sum_{n=1}^{\infty} nt_n(h\partial)^{n-1} + x \right)W^{-1}
\end{equation}
where \( \zeta(t, \hbar \partial) = \sum_{n=1}^{\infty} t_n (\hbar \partial)^n \). It is easy to see that \( M \) has a form

\[
M = \sum_{n=1}^{\infty} nt_n L^{n-1} + x + \alpha(\hbar) L^{-1} + \sum_{n=1}^{\infty} v_n(\hbar, x, t) L^{-n-1},
\]

and satisfies \( \text{ord}^\hbar(M) = 0 \), the canonical commutation relation \([L, M] = \hbar\) and the same Lax equations as \( L \): \( \hbar \partial M / \partial t_n = [B_n, M] \), \( n = 1, 2, \ldots \).

The following proposition (Proposition 1.7.11 of [TT2]) is a Riemann-Hilbert type construction of solutions of the \( \hbar \)-KP hierarchy.

**Proposition 1.1.** (i) Let \( f(\hbar, x, \hbar \partial) \) and \( g(\hbar, x, \hbar \partial) \) be 0-th order operators (\( \text{ord}^\hbar f = \text{ord}^\hbar g = 0 \)) and canonically commuting, \([f, g] = \hbar\), operators \( L \) and \( M \) have the form (1.2) and (1.8) respectively and commute canonically, \([L, M] = \hbar\). Suppose \( f(\hbar, M, L) \) and \( g(\hbar, M, L) \) are differential operators: \( (f(\hbar, M, L))_{<0} = (g(\hbar, M, L))_{<0} = 0 \), where \( (\ )_{<0} \) is the projection to the negative order part: \( P_{<0} := P - P_{\geq 0} \). Then \( L \) is a solution of the KP hierarchy (1.1) and \( M \) is the corresponding Orlov-Schulman operator.

(ii) Conversely, for any solution \((L, M)\) there exists a pair \((f, g)\) satisfying the conditions in (i).

The leading term of this system with respect to the \( \hbar \)-order gives the dispersionless KP hierarchy. Namely,

\[
\mathcal{L} := \sigma^\hbar(L) = \xi + \sum_{n=1}^{\infty} u_{0,n+1} \xi^{-n},
\]

satisfies the dispersionless Lax type equations

\[
\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \quad n = 1, 2, \ldots ,
\]
where $(\quad)_{\geq 0}$ is the truncation of Laurent series to its polynomial part and $\{,\}$ is the Poisson bracket defined by $\{a(x, \xi), b(x, \xi)\} = \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial \xi}$.

The dressing operation (1.3) for $L$ becomes the following dressing operation for $L$:

$$L = \exp(\text{ad}\{,\} X_0) \xi,$$

where $X_0 = \sigma \hbar (X)$ and

$$\text{ad}\{,\}(f)(g) := \{f, g\}.$$

The principal symbol of the Orlov-Schulman operator is

$$(1.11) \quad M = \sum_{n=1}^{\infty} nt_n L^{n-1} + x + \alpha_0 L^{-1} + \sum_{n=1}^{\infty} v_{0,n} L^{-n-1}$$

$(v_{0,n} = \sigma^h(v_n), \alpha_0 = \sigma^h(\alpha))$, which is equal to $M = \exp(\text{ad}\{,\} X_0) \exp(\text{ad}\{,\} \alpha_0 \log \xi) \exp(\text{ad}\{,\} \zeta(t, \xi)) x$, where $\zeta(t, \xi) = \sum_{n=1}^{\infty} t_n \xi^n$.

The series $M$ satisfies the canonical commutation relation $\{L, M\} = 1$, and the Lax type equations: $\frac{\partial M}{\partial t_n} = \{B_n, M\}$, $n = 1, 2, \ldots$. The Riemann-Hilbert type construction of the solution is essentially the same as Proposition 1.1.

**Proposition 1.2.** (i) Let $f_0(x, \xi)$ and $g_0(x, \xi)$ be functions canonically commuting, $\{f_0, g_0\} = 1$, functions $L$ and $M$ have the form (1.9) and (1.11) respectively. Suppose $f_0(M, L)$ and $g_0(M, L)$ do not contain negative powers of $\xi$, $(f_0(M, L))_{<0} = (g_0(M, L))_{<0} = 0$, where $(\quad)_{<0}$ is the projection to the negative degree part: $P_{<0} := P - P_{\geq 0}$. Then $L$ is a solution of the dispersionless KP hierarchy (1.10) and $M$ is the corresponding Orlov-Schulman function.

(ii) Conversely, for any solution $(L, M)$ there exists a pair $(f_0, g_0)$ satisfying the conditions in (i).

If $f$, $g$, $L$ and $M$ are as in Proposition 1.1 then $f_0 = \sigma^h(f)$, $g_0 = \sigma^h(g)$, $L = \sigma^h(L)$ and $M = \sigma^h(M)$ satisfy the conditions in
Proposition 1.2 In other words, \((f, g)\) and \((L, M)\) are quantisation of the canonical transformations \((f_0, g_0)\) and \((\mathcal{L}, \mathcal{M})\) respectively. (See, for example, [S] for quantised canonical transformations.)

2 Recursive construction of the dressing operator

In this section we prove that the solution of the KP hierarchy corresponding to the quantised canonical transformation \((f, g)\) is recursively constructed from its leading term, i.e., the solution of the dispersionless KP hierarchy corresponding to the Riemann-Hilbert data \((\sigma^h(f), \sigma^h(g))\).

Given the pair \((f, g)\), we have to construct the dressing operator \(W\), or \(X\) and \(\alpha\) in (1.4), such that operators

\[
\begin{align*}
\quad f(h, M, L) &= \text{Ad} \left( W \exp \left( h^{-1} \zeta(t, h\partial) \right) \right) f(h, x, h\partial) \\
\quad g(h, M, L) &= \text{Ad} \left( W \exp \left( h^{-1} \zeta(t, h\partial) \right) \right) g(h, x, h\partial)
\end{align*}
\]

are both differential operators (cf. Proposition 1.1). Let us expand \(X\) and \(\alpha\) with respect to the \(h\)-order as follows:

\[
\begin{align*}
\quad X(h, x, t, h\partial) &= \sum_{n=0}^{\infty} h^n X_n(x, t, h\partial), \\
\quad X_n(x, t, h\partial) &= \sum_{k=1}^{\infty} \chi_{n,k}(x, t)(h\partial)^{-k}, \\
\quad \alpha(h) &= \sum_{n=0}^{\infty} h^n \alpha_n, \text{ where } \chi_{n,k} \text{ and } \alpha_n \text{ do not depend on } h.
\end{align*}
\]

Assume that the solution \((\mathcal{L}, \mathcal{M})\) of the dispersionless KP hierarchy corresponding to \((\sigma^h(f), \sigma^h(g))\) is given. Namely, \((\sigma^h(f)(\mathcal{M}, \mathcal{L}), \sigma^h(g)(\mathcal{M}, \mathcal{L}))\) do not contain negative powers of \(\xi\):

\[
\begin{align*}
\quad (\sigma^h(f)(\mathcal{M}, \mathcal{L}))_{<0} &= (\sigma^h(g)(\mathcal{M}, \mathcal{L}))_{<0} = 0.
\end{align*}
\]

Let \((X_0, \alpha_0)\) be corresponding dressing functions.
We are to construct $X_n$ and $\alpha_n$ recursively, starting from $X_0$ and $\alpha_0$. The explicit procedure is as follows.

• (Step 0) Assume $X_0, \ldots, X_{i-1}$ and $\alpha_0, \ldots, \alpha_{i-1}$ are given. Set $X^{(i-1)} := \sum_{n=0}^{i-1} \hbar^n X_n, \alpha^{(i-1)} := \sum_{n=0}^{i-1} \hbar^n \alpha_n$.

• (Step 1) Set

\begin{align*}
P^{(i-1)} &= \text{Ad} \left( \exp(\hbar^{-1} X^{(i-1)}) (\hbar \partial)^{\alpha^{(i-1)}/\hbar} \right) f_t, \\
Q^{(i-1)} &= \text{Ad} \left( \exp(\hbar^{-1} X^{(i-1)}) (\hbar \partial)^{\alpha^{(i-1)}/\hbar} \right) g_t.
\end{align*}

Expand $P^{(i-1)}$ and $Q^{(i-1)}$ as $P^{(i-1)} = \sum_{k=0}^{\infty} \hbar^k P_k^{(i-1)}, Q^{(i-1)} = \sum_{k=0}^{\infty} \hbar^k Q_k^{(i-1)}$.

(2.4) \hspace{1cm} (ord $\hbar P_k^{(i-1)} = \text{ord} \hbar Q_k^{(i-1)} = 0$.)

• (Step 2) Put $\mathcal{P}_0 := \sigma^h(P_0^{(i-1)}), \mathcal{Q}_0 := \sigma^h(Q_0^{(i-1)}), \mathcal{P}_i^{(i-1)} := \sigma^h(P_i^{(i-1)}), \mathcal{Q}_i^{(i-1)} := \sigma^h(Q_i^{(i-1)})$ and define a constant $\alpha_i$ and a series $\tilde{X}_i(x, t, \xi) = \sum_{k=1}^{\infty} \tilde{X}_{i,k}(x, t) \xi^{-k}$ by

\begin{equation}
\alpha_i \log \xi + \tilde{X}_i := \int_{-1}^{\xi} \left( \frac{\partial \mathcal{Q}_0}{\partial \xi} P_i^{(i-1)} - \frac{\partial \mathcal{P}_0}{\partial \xi} Q_i^{(i-1)} \right) \xi^{-1} \, d\xi.
\end{equation}

The integral constant of the indefinite integral is fixed so that the right hand side agrees with the left hand side.

• (Step 3) Define a series $\mathcal{X}_i(x, t, \xi) = \sum_{k=1}^{\infty} \chi_{i,k}(x, t) \xi^{-k}$ by

\begin{equation}
\mathcal{X}_i = \tilde{X}'_i - \frac{1}{2} \{ \sigma^h(X_0), \tilde{X}'_i \} + \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} (\text{ad}_{\{,\}}(\sigma^h(X_0)))^{2p} \tilde{X}'_i,
\end{equation}

where $B_{2p}$'s are the Bernoulli numbers.

\begin{equation}
\tilde{X}'_i := \alpha_i \log \xi + \tilde{X}_i(x, \xi) - \exp(\text{ad}_{\{,\}} \sigma^h(X_0))(\alpha_i \log \xi).
\end{equation}

• (Step 4) The operator $X_i(x, t, \hbar \partial)$ is defined as the operator with the principal symbol $\mathcal{X}_i$: $X_i = \sum_{k=1}^{\infty} \chi_{i,k}(x, t)(\hbar \partial)^{-k}$.  

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The main theorem is the following:

**Theorem 2.1.** Assume that $X_0$ and $\alpha_0$ satisfy (2.3) and construct $X_i$’s and $\alpha_i$’s by the above procedure recursively. Then $W = \exp(X/\hbar)(\hbar\partial)^{\alpha/\hbar}$ is a dressing operator of the $\hbar$-dependent KP hierarchy corresponding to $(f, g)$ by Proposition 1.1.

3 Asymptotics of the wave function

In this section we prove that the dressing operator of the form $W(\hbar, x, t, \hbar\partial) = \exp(X(h, x, \hbar \partial)/\hbar)$, $\text{ord}\, X \leq 0$, $\text{ord}\, X \leq -1$, gives a wave function of the form $\Psi(h, x, t; z) = W e^{(xz+\zeta(t,z))/\hbar} = \exp(S(h, x, t; z)/\hbar)$, $S(h, x, t; z) = \sum_{n=0}^{\infty} \hbar^n S_n(x, t; z)+\zeta(t, z)$, $\zeta(t, z) := \sum_{n=1}^{\infty} t^n z^n$, and vice versa.

Since the time variables $t_n$ do not play any role in this section, we set them to zero. As the factor $(\hbar \partial)^{\alpha/\hbar}$ in (1.4) becomes a constant factor $z^{\alpha/\hbar}$ when it is applied to $e^{xz/\hbar}$, we also omit it here.

Let $A(h, x, \hbar \partial) = \sum_n a_n(h, x)(\hbar \partial)^n$ be a microdifferential operator. The total symbol of $A$ is a power series of $\xi$ defined by $\sigma_{\text{tot}}(A)(h, x, \xi) := \sum_n a_n(h, x)\xi^n$, or, equivalently defined by the formula $A e^{xz/\hbar} = \sigma_{\text{tot}}(A)(h, x, z)e^{xz/\hbar}$.

**Proposition 3.1.** Let $X = X(h, x, \hbar \partial)$ be a microdifferential operator such that $\text{ord}\, X = -1$ and $\text{ord}\, X = 0$. Then the total symbol of $e^{X/\hbar}$ has such a form as $\sigma_{\text{tot}}(\exp(\hbar^{-1}X(h, x, \hbar \partial)))) = e^{S(h, x, \xi)/\hbar}$, where $S(h, x, \xi)$ is a power series of $\xi^{-1}$ without non-negative powers of $\xi$ and has an $\hbar$-expansion $S(h, x, \xi) = \sum_{n=0}^{\infty} \hbar^n S_n(x, \xi)$.  

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Moreover, the coefficient $S_n$ is determined by $X_0, \ldots, X_n$ in the $\hbar$-expansion of $X = \sum_{n=0}^{\infty} \hbar^n X_n$.

We omit the explicit formula for $S_n$. (See [TT3].)

**Proposition 3.2.** Let $S = \sum_{n=0}^{\infty} \hbar^n S_n$ be a power series of $\xi^{-1}$ without non-negative powers of $\xi$. Then there exists a microdifferential operator $X(h, x, h\partial)$ such that $\text{ord} X \leq -1$, $\text{ord}^h X \leq 0$ and $\sigma_{\text{tot}}(\exp(h^{-1}X(h, x, h\partial))) = e^{S(h, x, \xi)}/\hbar$. Moreover, the coefficient $X_n(x, \xi)$ in the $\hbar$-expansion $X = \sum_{n=0}^{\infty} \hbar^n X_n$ of the total symbol $X = X(h, x, \xi)$ is determined by $S_0, \ldots, S_n$ in the $\hbar$-expansion of $S$.

We omit the explicit formula for $X_n$. (See [TT3].)

Combining these propositions with the results in Section 2, we can, in principle, make a recursion formula for $S_n$ ($n = 0, 1, 2, \ldots$) of the wave function of the solution of the KP hierarchy corresponding to the quantised canonical transformation $(f, g)$ as follows: let $S_0, \ldots, S_{i-1}$ be given.

1. By Proposition 3.2 we have $X_0, \ldots, X_{i-1}$.

2. We have a recursion formula for $X_i$ by Theorem 2.1.

3. Proposition 3.1 gives a formula for $S_i$.

If we take the factor $(h\partial)^{\alpha/h}$ into account, this process becomes a little bit complicated, but essentially the same.

**4 Asymptotics of the tau function**

In this section we derive an $\hbar$-expansion $\log \tau(h, t) = \sum_{n=0}^{\infty} \hbar^{n-2} F_n(t)$ of the tau function (cf. [1,4]). Note that we have suppressed the
variable $x$, which is understood to be absorbed in $t_1$.

The logarithmic derivation of (4.1) gives

$$-hD'(z) \log \tau(t) = \hbar^{-1} \left( \frac{\partial}{\partial z} + hD'(z) \right) \hat{S}(t; z),$$

where $\hat{S}(t; z) = S(t; z) - \zeta(t, z)$ and $D'(z) := -\sum_{j=1}^{\infty} z^{-j-1} \frac{\partial}{\partial t_j}$.

By substituting the $h$-expansions $\log \tau(t) = \sum_{n=0}^{\infty} h^n F_n(t)$, $\hat{S}(t; z) = \sum_{n=0}^{\infty} h^n S_n(t; z)$, and expanding $S_n(t; z)$ as $S_n(t; z) = -\sum_{k=1}^{\infty} \frac{z^{-k}}{k} v_{n,k}$, we have the equations

$$\frac{\partial F_n}{\partial t_j} = v_{n,j} + \sum_{k+l=j, k \geq 1, l \geq 1} \frac{1}{l} \frac{\partial v_{n-1,l}}{\partial t_k} \quad (v_{-1,j} = 0).$$

This system determines $F_n$ up to integration constants.

Combining this with the results in Section 2 and Section 3 if $F_0$, which determines a solution of the dispersionless KP hierarchy, and the corresponding quantised canonical transformation $(f, g)$ are given, we can construct $F_n$ recursively and consequently the tau function of a solution of the $h$-KP hierarchy.

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