Uniformization of four manifolds

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Abstract

We characterize those Einstein four manifolds which are locally symmetric spaces of noncompact type. Namely they are four manifolds which admit solutions to the (non-Abelian) Seiberg Witten equations and satisfy certain characteristic number equality.

1 Introduction

The main theorem in this paper is:

**Theorem 1** Let $X$ be an Einstein four manifold with $\sigma(X) = 0$. Suppose for some Spin$^c$ structure $L$ on $X$, the non-Abelian Seiberg Witten equations

$$
F_A^+ = \tau(\phi) \\
\nabla_A \phi = 0,
$$

on $S^+_L$ has a solution, then the universal covering of $X$ must be $\mathbb{R}^4$, $B^2 \times B^2$ or $B^4$ with a locally symmetric Riemannian metric.

Combining with the work of Yau [Y] in the Kähler surface case and LeBrun [LB] in the general case for the complex ball quotient $\Gamma \backslash B^2_\mathbb{C}$, we can characterize every four dimensional Einstein manifold $X$ which is locally symmetric of noncompact type by using (non-Abelian) Seiberg Witten equations.

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Recall that in four dimension, the following is a complete list of symmetric spaces of noncompact type:

\[ B^4, \ B^2_\mathbb{C}, \ B^2 \times B^2. \]

Therefore we obtain:

**Theorem 2** Let \( X \) be an Einstein four manifold with \( \chi(X) > 0 \). Then \( X \) is a locally symmetric space of noncompact type iff one of the followings holds

1. the non-Abelian Seiberg Witten equation for \( S^+_L \) has a solution and \( \sigma(X) = 0 \).
2. the Seiberg Witten equation for \( L \) has a solution and \( 3\sigma(X) = \chi(X) \).

The universal cover of \( X \) is isometric to \( B^2 \times B^2 \) or \( B^4 \) in (1) and \( B^2_{\mathbb{C}} \) in (2).

**Remark 1** Kotschick points out to the author that any Einstein metric on a four manifold with \( \chi(X) = 0 \) must be flat.

**Remark 2** It is desirable to define Donaldson type invariant using non-Abelian Seiberg Witten equations. Because one would be able to replace the existence of Seiberg Witten solution by a (differential) topological condition. In particular, it will imply that Einstein metric on four dimension locally symmetric space of noncompact type is unique. To the author’s knowledge, uniqueness of Einstein metric on \( S^4 \) is still open.

In dimension two and three, Einstein metric has constant sectional curvature. Namely their universal covers are isometric to \( S^n, \mathbb{R}^n \) or \( B^n \) where \( B^n = \{ x \in \mathbb{R}^n : |x| < 1 \} \) is the unit ball with the hyperbolic metric. In particular they coincide with locally symmetric spaces in these dimensions.

In dimension four Einstein metric may not be locally symmetric, for instance the K3 surface with Yau’s metric [Y] is not locally symmetric. Hitchin [H] showed that there is topological obstruction to the existence of Einstein metric in this dimension. We combine Hitchin’s original ideas and the use of (non-Abelian) Seiberg Witten equations to determine which Einstein metric is locally symmetric of noncompact type. Recall that in dimension four, there are only three symmetric spaces \( G/K \) of noncompact type, namely
(i) the real four ball

\[ SO^0(4, 1) / SO(4) = B^4, \]

with the real hyperbolic metric;

(ii) the complex two ball

\[ SU(2, 1) / U(2) = B^2_C = \{ x \in \mathbb{C}^k : |x| < 1 \}, \]

with the complex hyperbolic metric and

(iii) the reducible case

\[ SO^0(2, 1) \times SO^0(2, 1) / SO(2) \times SO(2) = B^2 \times B^2, \]

which is the product of the unit disks.

Each of these spaces carries canonical (non-unitary) flat connection which provides a solution to the (non-Abelian) Seiberg Witten equations. We will reverse the process and use a solution of the (non-Abelian) Seiberg Witten equations to construct a perturbed non-unitary anti-self-dual connection over \( X \). Existence of such connection implies certain characteristic number inequality for \( X \). When the equality holds (which depends only on the homotopy structure of \( X \)) then such connection is in fact projectively flat. By analyzing the flatness condition, we will prove in section four that these informations are sufficient to show that \( X \) must be a locally symmetric space of noncompact type.

The pioneer work in this direction is due to Hitchin \( \text{[H]} \) who showed that an Einstein four manifold satisfies the following characteristic number inequality

\[ |\sigma(X)| \leq \frac{2}{3} \chi(X), \]

where \( \sigma(X) = b^2_+(X) - b^2_-(X) \) is the signature of \( X \) and \( \chi(X) \) is the Euler characteristic of \( X \). These are the only characteristic numbers in dimension four and they depend solely on the homotopic type of the space \( X \).

Moreover Hitchin showed that \( |\sigma(X)| = \frac{2}{3} \chi(X) \) if and only if \( X \) is flat or its universal cover is a K3 surface. They can be distinguished by their Euler characteristics being zero or not.

When the Einstein metric is also Kähler then the above characteristic number inequality can be strengthened to

\[ \sigma(X) \leq \frac{1}{3} \chi(X), \]
or equivalently
\[ c_1^2 (X) \leq 3c_2 (X). \]

In [Y] Yau showed that any Kähler surface with negative Ricci curvature carries a unique Kähler Einstein metric. Therefore we have \( c_1^2 (X) \leq 3c_2 (X) \) for any such Kähler surface and when the equality sign holds then the universal cover of \( X \) is the complex two ball \( B^2_C \).

After the introduction of Seiberg Witten theory [W], LeBrun [LB] showed that in order to get the characteristic number inequality \( \sigma (X) \leq 1/3 \chi (X) \) the Kählerian condition can be replaced by the existence of solution to the Seiberg Witten equation. Therefore Einstein four manifold with non-trivial Seiberg Witten invariant and \( \sigma (X) = 1/3 \chi (X) \) is covered by \( B^2_C \). In [L] the author interpreted LeBrun work as a construction of certain perturbed \( U (2, 1) \) anti-self-dual connection on the bundle \( S_L \oplus L \) (where \( L \) defines the \( \text{Spin}^c \) structure on \( X \)) which will be projectively flat if \( \sigma (X) = 1/3 \chi (X) \).

Notice that the number of solutions to the Seiberg Witten equations counted with multiplicity is a differentiable invariant [W]. As a result of it, we can show that Einstein metric on \( X = \Gamma \backslash B^2_C \) is unique up to diffeomorphisms.

In [L] the author showed that if \( X \) has a Kähler Einstein metric with \( c_1 (X) < 0 \) such that both orientations has non-trivial Seiberg Witten invariant then
\[ \sigma (X) \geq 0. \]

Moreover the equality sign holds if and only if \( X \) is covered by \( B^2 \times B^2 \). Later [K] generalized this result to any complex surfaces of general type.

We are going to further develop these ideas to study four dimensional locally symmetric spaces in general which include the quotient of the real ball \( B^4 = SO^0 (4, 1)/SO (4) \). Instead of trying to reconstruct the \( SO^0 (4, 1) \) flat connection, we shall construct a \( Sp (1, 1) \) connection on some rank four complex vector bundle \( (Sp (1, 1) \text{ is the universal (double) cover of } SO^0 (4, 1)) \).

It turns out that the correct bundle to look at is \( S^+ \bigoplus S^- \) and we need to study the non-Abelian Seiberg Witten equations on \( E = S^+ \). If \( X \) does not have a spin structure, then we shall use a \( \text{Spin}^c \) structure which always exists on a four manifold. This approach can be used to deal with both \( B^4 \) and \( B^2 \times B^2 \) cases together. For \( B^2 \times B^2 \), the canonical flat connection will live in a rank four complex vector bundle via the natural embedding \( U (1, 1) \times U (1, 1) \subset U (2, 2) \). The rank four bundle in question is again \( S^+ \bigoplus S^- \). In this approach, we do not need to impose any Kählerian assumption on \( X \).
2 Locally symmetric space in dimension four

As explained in last section that there are only three kinds of locally symmetric space of noncompact type in dimension four, namely their universal covering $\tilde{X}$ is $B^4, B^2_\mathbb{C}$ or $B^2 \times B^2$.

When $\tilde{X} = B^4$ or $B^2 \times B^2$ we have $\sigma (X) = 0$ by Hirzebruch proportionality principle because their compact duals are $S^4$ and $S^2 \times S^2$ and they both have zero signature. We also have $\chi (X) > 0$.

When $\tilde{X} = B^2_\mathbb{C}$ we have $3\sigma (X) = \chi (X) > 0$ and there is a canonical representation $\pi_1 (X) \to \mathbb{P}U (2, 1)$ up to conjugation.

When $\tilde{X} = B^2 \times B^2$ then we have a projective representation of $\pi_1 (X)$ to $\mathbb{P}U (1, 1) \times \mathbb{P}U (1, 1)$. When $\tilde{X} = B^4$ we have a projective representation of $\pi_1 (X)$ to $SO^0 (4, 1)$. Instead of finding a flat connection on a real rank five vector bundle over $X$, we will construct a complex rank four vector bundle with a projectively $U (2, 2)$ connection and show that the image of its holonomy does lie inside $Sp (1, 1) \subset U (2, 2)$.

Observe that we have a canonical isomorphism of Lie groups:

$$Spin^0 (4, 1) = Sp (1, 1)$$

extending the well-known isomorphism

$$Spin (4) = Sp (1) \times Sp (1).$$

They give an isomorphism between symmetric spaces:

$$SO^0 (4, 1) / SO (4) = Spin^0 (4, 1) / Spin (4) = Sp (1, 1) / Sp (1) \times Sp (1),$$

which is the real four ball $B^4$.

On $B^4$ there is a flat $SO^0 (4, 1)$ bundle $T \oplus \mathbb{R}$. The corresponding $Sp (1, 1)$ flat bundle is $S^+ \oplus S^-$. In particular there is an anti-self-dual connection on $S^+ \oplus S^-$ which restricts to the Levi-Civita connection on $S^-$. For compact $X = \Gamma \backslash B^4$ with $\Gamma$ a torsion free subgroup of $SO^0 (4, 1)$. We have $\pi_1 (X) = \Gamma$ and the above mentioned $SO^0 (4, 1)$ flat connection will descend to give a $SO^0 (4, 1)$ flat bundle over $X$.

Instead of the representation of $\pi_1 (X)$ into $SO^0 (4, 1)$ or $SO^0 (2, 1) \times SO^0 (2, 1)$, we want to lift these representations to $Spin^0 (4, 1)$ and $Spin^0 (2, 1) \times Spin^0 (2, 1)$ and use the identifications $Spin^0 (4, 1) = Sp (1, 1)$ and $Spin^0 (2, 1) = SU (1, 1)$. Both $Sp (1, 1)$ and $SU (1, 1) \times SU (1, 1)$ are closed subgroup of
$U(2,2)$ and therefore they give us a flat connection on a rank four complex vector bundle over $X$. The bundle in question is $S^+ \bigoplus S^-$. In general we may not be able to lift it to a $Spin^0$ flat connection. To resolve this problem, we introduce a $Spin^c$ structure on $X$ and study projectively flat $U(2,2)$ connection over $X$. Recall that $Spin^c(4) = Spin(4) \times \mathbb{Z}_2 \times U(1)$ and there is a short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin^c(4) \to SO(4) \times U(1) \to 0.$$ 

Therefore any $Spin^c$ structure on $X$ induces a $U(1)$ line bundle $L$ over $X$. The homomorphism to $SO(4)$ just gives us back the tangent bundle of $X$. Using the fact that $Spin^c(4) = SU(2) \times SU(2)$, we have two homomorphisms $Spin^c(4) \to U(2)$. They induce two rank two complex vector bundles over $X$ which are called the positive and negative spinor bundles and we denote them by $S^+_L$ and $S^-_L$. Formally they are $S^+ \otimes L^{1/2}$ and $S^- \otimes L^{1/2}$.

Now we assume that the universal cover $\tilde{X}$ of $X$ is either $B^4$ or $B^2 \times B^2$. Then $\tilde{X}$ has a canonical flat connection on $S^+ \bigoplus S^-$ which may not descend to one on $X$. After choosing a $Spin^c$ structure on $X$, we consider the rank four complex vector bundle $S^+_L \bigoplus S^-_L$ over $X$ and we fix any Hermitian connection $D_L$ on $L$, then the canonical flat connection on $S^+ \bigoplus S^-$ twisted by $D_L$ can be descended to $X$ and give a projectively flat connection over $X$. From our later discussions we shall see that such a projectively flat $U(2,2)$ connection provides a perturbed anti-self-dual connection over $X$ up to gauge transformations. This perturbed anti-self-dual connection will be constructed from a solution to the Seiberg Witten equations and using the Einstein metric on $X$.

3 (non-Abelian) Seiberg Witten equations

In this section we assume that $X$ is a four manifold with a Riemannian metric $g$ and a $Spin^c$ structure $L$, that is $c_1(L) \equiv w_2(X) \pmod{2}$. We fix a connection $D_L$ on $L$ throughout our discussions. Before introducing the non-Abelian Seiberg Witten equations, let us first review some well-known properties of curvature decomposition in dimension four.

From the choice of a $Spin^c$ structure, we have the corresponding positive and negative spinor bundle $S^+_L$ and $S^-_L$, they have canonical connections induced from the Levi-Civita connection of $X$ and $D_L$ on $L$. We now explain the relationship between the curvature of $\Lambda^2_{\pm}, S^+_L$ and the Riemann curvature

\begin{align*}
\end{align*}
tensor of $X$. The Riemann curvature tensor of $X$ defines a curvature operator $Rm$ on the space of two forms $\Lambda^2$ on $X$ and the Hodge star operator $*$ of $X$ decomposes $\Lambda^2$ into $\pm 1$ eigenspaces, namely the space of self-dual and anti-self-dual forms:

$$\Lambda^2 = \Lambda^2_+ \bigoplus \Lambda^2_-.$$

With respect to this decomposition $Rm$ has the following blocks decomposition:

$$Rm = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix},$$

where $A = A^* \in \text{End} (\Lambda^2_+)$, $D = D^* \in \text{End} (\Lambda^2_-)$ and $B \in \text{Hom} (\Lambda^2_-, \Lambda^2_+)$. If we denote the scalar curvature of $X$ by $s$ then $s = 4TrA = 4TrD$. Moreover $W^+ = A - \frac{i}{2}TrA$ and $W^- = D - \frac{i}{2}TrD$ is called the self-dual and anti-self-dual Weyl curvature of $X$. The sum $W = W^+ + W^-$ is called the Weyl curvature which depends only on the conformal class of $g$. In fact $W = 0$ if and only if $X$ is a conformally flat manifold.

We also have $B = Rc^0$ the trace free part of the Ricci curvature tensor.

By the Hirzebruch signature theorem, we have

$$\sigma (X) = \frac{1}{12\pi^2} \int_M \left( |W^+|^2 - |W^-|^2 \right) dv_g.$$

In particular, if $X$ has zero signature, then $W^- = 0$ if and only if $W = 0$. That is $X$ is a conformally flat manifold.

There are also natural isomorphisms

$$\Lambda^2_+ = \text{su} (S_L^+) \quad \text{and} \quad \Lambda^2_- = \text{su} (S_L^-).$$

Atiyah, Hitchin and Singer [AHS] observed that $X$ is Einstein if and only if the Levi-Civita connection on $\Lambda^2_-$ is anti-self-dual connection. They also showed that $X$ is Einstein if and only if the trace free part of the curvature on $S_L^-$ is anti-self-dual. Moreover $W^- = 0 = s$ if and only if the trace free part of the curvature on $S_L^-$ is self-dual. Corresponding statements for $\Lambda^2_+$ and $S_L^+$ also hold if we interchange self-dual by anti-self-dual.

In particular if $X$ is an Einstein manifold with $\sigma (X) = 0$ such that the trace free part of the curvature on $S_L^-$ is self-dual then the Riemann curvature tensor of $g$ vanishes identically. That is $X$ is a flat manifold.
Now we are going to introduce the (non-Abelian) Seiberg Witten equation: let $E$ be a complex Hermitian vector bundle over $X$ of rank $r$. Suppose $D_A$ is any Hermitian connection on $E$ and $\phi \in \Gamma (X, Hom (S^+_L, E)) = \Gamma (X, S^+_L \otimes E)$. We consider $\phi \otimes \bar{\phi} \in \Gamma (X, u(S^+_L) \otimes End (E))$. Under the identification $\text{su} (S^+_L) = \Lambda^2$, we project $\phi \otimes \bar{\phi}$ to the trace free part of $u(S^+_L)$ and denote its image by $\tau (\phi) \in \Omega^2_+ (X, \text{End} (E))$.

We consider the following non-Abelian Seiberg Witten equations

$$F^+_A = \tau (\phi)$$

$$D_A \phi = 0.$$ 

Here $\mathcal{D}_A$ is the Dirac operator on $S^+_{L-1}$ twisted by the connection $D_A$ on $E$.

These equations are natural generalization of Seiberg Witten equations $[W]$ to higher rank bundles. The Seiberg Witten equations have had numerous applications in four manifold geometry, for examples $[FS], [FM], [KM], [LL], [MST], [T]$ and many others. Their higher rank analog has also been studied extensively. For example, Pidstrigach, Tyurin $[P], [PT]$ and also Feehan, Leness $[FL]$ use these equations in an attempt to prove the equivalency of the Donaldson invariant and the Seiberg Witten invariant with much progress. In the context of Kahler surfaces, they are studied by Bradlow, Garcia-Prada $[BG]$ and Okonek, Teleman $[OT]$.

In this paper we generalize the arguments in $[L]$ to relate solutions to these equations and perturbed anti-self-dual connection. This is crucial for producing non-unitary flat connections over $X$. Most of our discussions for the rest of this section are based on methods used in section four of $[L]$. Even though almost everything is the same for general rank $r$ vector bundle, we shall restrict our attention to the case when $r = 2$ and $\det E = L$ for our purposes.

Now we suppose that $(D_A, \phi)$ is a solution to the non-Abelian Seiberg Witten equations. Via the Clifford multiplication, there is a canonical homomorphism

$$\Gamma (X, Hom (S^+_L, E)) \to \Omega^1 (X, Hom (S^-_L, E))$$

$$\phi \to \tilde{\phi}.$$ 

From $[L]$, we have

$$P_- (D_A \tilde{\phi}) = 0.$$
for any harmonic spinor $\phi$. Here $P_-$ is the orthogonal projection to the anti-self-dual two forms. Moreover by projecting to the self-dual two forms part we have

\[
P_+ \begin{pmatrix} \tilde{\phi}^\star \\ \tilde{\phi} \end{pmatrix} = Tr \tau_+ (\phi) I_{S_L_-} \in \Omega^2_+ \left( u \left( S_{L_-} \right) \right),
\]
\[
P_+ \begin{pmatrix} \tilde{\phi}^\star \\ \tilde{\phi} \end{pmatrix} = -2 \tau (\phi) \in \Omega^2_+ \left( u \left( E \right) \right).
\]

We consider the connection $D_{(A,\phi)} = \begin{pmatrix} \nabla^{LC}_L \tilde{\phi} \\ -\tau^\star \phi \\ \phi^\star D_A \end{pmatrix}$ on $E = S_{L}^- \bigoplus E$. $D_{(A,\phi)}$ preserves the Hermitian form $h_{S_{L}^-} - h_E$ on $E$ and therefore is a $U (2, 2)$ connection. If we extend the Hodge star operator $\ast$ on $X$ to $\Omega^2 (X, \text{End} \ (E))$ by

\[
\ast_E \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \ast A & -\ast B \\ -\ast C & \ast D \end{pmatrix}.
\]

Using $\ast_E$ we can decompose $\Omega^2 (X, \text{End} \ (E))$ into self-dual and anti-self-dual parts.

Now if we decompose the curvature $F_{(A,\phi)}$ of $D_{(A,\phi)}$ into self-dual and anti-self-dual accordingly, we obtain

\[
F_{(A,\phi)}^+ = \begin{pmatrix} Rc^0 - F_L^+ & 0 \\ 0 & -F_L^+ \end{pmatrix}.
\]

Here $F_L$ is the curvature of $D_L$. The proof of this formula is by direct computations and it is identical to the rank one case treated in section four of \cite{L}.

Hence we have shown the following result:

**Proposition 3** Let $X$ be an Einstein four manifold with Spin$^c$ structure $L$ on $X$. Suppose that the non-Abelian Seiberg Witten equations

\[
\begin{align*}
F_A^+ &= \tau (\phi) \\
D_A \phi &= 0,
\end{align*}
\]

on $E$ has a solution $(D_A, \phi)$, then $D_{(A,\phi)} = \begin{pmatrix} \nabla^{LC}_L \tilde{\phi} \\ \tilde{\phi} \\ D_A \end{pmatrix}$ is a perturbed anti-self-dual $U (2, 2)$ connection on $E = S_{L}^- \bigoplus E$. 

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Perturbed anti-self-dual bundle satisfies certain topological constraint, namely the Chern number inequality. When the equality holds, the connection under considerations is in fact a projectively flat connection.

**Proposition 4** Let $E$ be any complex vector bundle of rank $r$ over a Riemannian four manifold $X$. Suppose that $E$ has a perturbed anti-self-dual $U(p, q)$ connection, then we have

$$c^2_1(E) \leq \frac{2r}{r-1} c_2(E).$$

Moreover equality sign holds if and only if $E$ is a projectively flat $U(p, q)$ bundle.

The proof is the same as the Chern number inequality in the theory of Hermitian-Einstein-Higgs bundle [S] and this proposition is also stated in [L].

Now if we take $E$ to be $S^+_L \bigoplus S^+_L$, then the above Chern number inequality is equivalent to

$$\sigma(X) \leq 0.$$ 

Notice that the Chern class of $L$ does not show up in this inequality. Combining these two propositions we have the following theorem:

**Theorem 5** Let $X$ be an Einstein four manifold with Spin$^c$ structure $L$ on $X$. Suppose that the non-Abelian Seiberg Witten equations

$$F^+_A = \tau(\phi)$$
$$\nabla_A \phi = 0,$$

on $E = S^+_L$ has a solution $(D_A, \phi)$, then $\mathbb{D}_{(A, \phi)} = \left( \begin{array}{c} \nabla^L_C \phi \\ D_{A,L} \end{array} \right)$ is a perturbed anti-self-dual $U(2, 2)$ connection on $E = S^+_L \bigoplus E$.

In particular we have characteristic number inequality

$$\sigma(X) \leq 0,$$

Moreover, equality sign holds if and only if $\mathbb{D}_{(A, \phi)}$ is a projectively flat $U(2, 2)$ connection over $X$. 

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4 Proof of the main theorem

In this section we prove our main theorem:

**Theorem 6** Let $X$ be an Einstein four manifold with $\sigma(X) = 0$. Suppose that for some Spin$^c$ structure $L$ on $X$, the non-Abelian Seiberg Witten equations

\[
F_A^+ = \tau(\phi) \\
\mathcal{D}_A \phi = 0,
\]

on $E = S^+_L$ has a solution, then the universal covering of $X$ must be $\mathbb{R}^4, B^2 \times B^2$ or $B^4$ with a locally symmetric Riemannian metric.

Proof of theorem: From the previous section we know that $(D_A, \phi)$ induces a projectively flat connection on $S^-_L \oplus E = S^-_L \oplus S^+_L$.

From the flatness equation we have $D_A \phi = 0$. It follows that $\phi$ is a parallel spinor, $D_A \phi = 0$. When regarding $\phi$ as a homomorphism from $S^+_L$ to $E$, the rank of $\text{Ker} (\phi)$ is constant and $\text{Ker} (\phi)$ is a vector bundle over $M$. We divide into the following three cases: (i) $\text{rank} (\text{Ker} \phi) = 2$ (ii) $\text{rank} (\text{Ker} \phi) = 0$ and (iii) $\text{rank} (\text{Ker} \phi) = 1$.

(i) First case: $\text{rank} (\text{Ker} \phi) = 2$, that is $\phi$ is a trivial homomorphism. Hence the second fundamental form of the connection $\mathcal{D}_{(A, \phi)}$ on $S^-_L \oplus E$ is zero. Now the projectively flatness of $S^-_L \oplus E$ implies that $S^-_L$ is a projectively flat $U(2)$ bundle with respect to the Levi-Civita connection twisted by $D_L$.

Using curvature decomposition for $X$ and its associated bundles, we have $W^- = 0 = s$. Applying the equality

\[
\sigma(X) = \frac{1}{12\pi^2} \int_X \left( |W^+|^2 - |W^-|^2 \right) dv,
\]

we obtain $W^+ = 0$ because $\sigma(X) = 0$. Together with the fact that $Rc^0 = 0$ we know that $X$ has vanishing Riemann curvature tensor. That is the universal covering of $X$ is $\mathbb{R}^4$ with the flat metric, in particular $\chi(X) = 0$.

(ii) Second case: $\text{Ker} (\phi) = \{0\}$. That is $\phi$ is a parallel isomorphism $\phi : S^+_L \xrightarrow{\cong} E_L$. 

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Consider \( g = (\phi \phi^*)^{-1/2} \phi \). Then \( g \) is a unitary gauge transformation of \( E \). Moreover it carries \( \nabla_L^{LC} \) to \( D_A \), namely

\[
D_A = g \circ \nabla_L^{LC} \circ g^{-1}.
\]

Therefore by composing \( E \) with a unitary transformation if necessary, we can assume that \( D_A = \nabla_L^{LC} \) and \( g \) is the identity transformation. From the projective flatness for the twisted Levi-Civita connection on \( S_L^- \bigoplus S_L^+ \) and the curvature decomposition for four manifolds, we know that \( W^+, W^- \) and \( s \) are all parallel. Together with the vanishing of \( Rc^0 \), the whole Riemann curvature tensor is parallel: \( \nabla (Rm) = 0 \). This means that \( g \) is a locally symmetric metric. We now determine the locally symmetric space explicitly.

Since \( D_A = \nabla_L^{LC} \), the trace free part of \( \phi \) is a parallel self-dual (complex) two form on \( X \). If this self-dual two form is zero, then \( \phi = \frac{1}{2} (Tr \phi) I_{S^+} \) where \( Tr \phi \) is a constant function on \( X \). Using the identification \( \Lambda^2_- = \text{su} \left( S_L^- \right) \) we have

\[
\Omega^2_- \left( X, u \left( S_L^- \right) \right) = \Omega^2_- \left( X, \text{su} \left( S_L^- \right) \bigoplus \mathbb{R} \right) = \Gamma \left( X, \text{End} \left( \Lambda^2_- \right) \bigoplus \Lambda^2_+ \right).
\]

It is easy to check that the component of \( P_- \left( \tilde{\phi}^* \tilde{\phi} \right) \) in \( \Gamma \left( X, \text{End} \left( \Lambda^2_- \right) \bigoplus \Lambda^2_+ \right) \) equals \( cI_{\Lambda^2_-} \) for some constant \( c \) (moreover the component of \( P_- \left( \tilde{\phi}^* \tilde{\phi} \right) \) in \( \Gamma \left( X, \Lambda^2_- \right) \) is proportional to \( F^-_L \), even though we will not use this fact).

Applying the flatness condition, we have the trace free part of \( F^-_L \) equals \( cI_{\Lambda^2_-} \). This implies \( W^- = 0 \) and the scalar curvature \( s \) of \( X \) is a constant function. Using

\[
0 = \sigma (X) = \frac{1}{12 \pi^2} \int_X \left( |W^+|^2 - |W^-|^2 \right) dv;
\]

we also have \( W^+ = 0 \). As a result \( X \) has constant sectional curvature. By checking the sign of \( c \) we know \( s \) is negative and \( X \) has constant negative sectional curvature. Hence \( X = \Gamma \backslash SO(4, 1) / SO(4) = \Gamma \backslash B^4 \).

Now if trace free part of \( \phi \) is nonzero, then we have a nontrivial parallel self-dual two form on \( X \). As a result, the holonomy group of \((X, g)\) will be reduced to \( U(2) \). That is \( g \) is a Kähler metric on \( X \) and the above parallel self-dual two form is proportional to its Kähler form \( \omega \). In particular \((X, g)\) is a Kähler Einstein surfaces.
Recall that the Levi-Civita connection on $X$ twisted by $D_L$ induces a projectively flat connection on $S_L^- \oplus S_L^+$. Reversal of orientation will interchange $S_L^+$ and $S_L^-$, but their direct sum $S_L^- \oplus S_L^+$ remains unchanged. Moreover we just showed that the projectively flat connection on $S_L^- \oplus S_L^+$ is induced from the Levi-Civita connection twisted by $D_L$ and therefore also remain unchanged after the reversal of orientations. Unlike the anti-self-duality condition, flatness is insensitive to the orientation of the manifold.

Changing the orientation of $X$, the Riemannian metric on $X$ remain Einstein because the metric tensor is independent of the orientation on $X$. Even though the signature will change sign under the reversal of orientation, we still have $\sigma (X) = 0$. Repeat the same arguments as above, we show that $X$ is Kähler Einstein with respect to both orientations. Therefore the Seiberg Witten invariant for $X$ with both orientations are non-trivial. By the result of [L], we concludes that $X = \Gamma \backslash B^2 \times B^2$.

(iii) Third case: $\text{rank}(\text{Ker}\phi) = 1$. We shall see that this case cannot happen. Let $L_1 = \text{Ker}\phi \subset S_L^+$ and $L_2 = (L_1)^\perp \subset S_L^+$. Then we have $L_2 = \text{Im}\phi$ and an orthogonal decomposition $S_L^+ = L_1 \oplus L_2$. Since $\phi$ is parallel and $E$ and $S_L^+$ are isometric to each other, we also have an orthogonal decomposition of $E$: $E = L_1 \oplus L_2$. With respect these decompositions, we have

$$\phi = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix},$$

for some non-zero constant $\beta$. Therefore $\text{Tr}_{E^\tau} (\phi) \in \Omega^2_+ \cap \Omega^2_- \cap \Omega^2 \subset \Omega^2_-$ is a nonzero parallel self-dual two form on $X$. This implies that the holonomy group of $(X, g)$ is reduced to $U(2)$ or equivalently $g$ is a Kähler metric on $X$ with respect to some integrable complex structure. If we denote the Kähler form by $\omega$, then it is proportional to $\text{Tr}_{E^\tau} (\phi)$.

Then the component of $P_-$ ($\bar{\phi} \bar{\phi}$) in $\Gamma (X, \text{End}_0 (\Lambda^2_-)) \subset \Omega^2_-$ is zero by direct computation. Using the projectively flatness condition and curvature decomposition, we have $W^- = 0$ and in fact the universal covering of $X$ must be $B^2_\mathbb{C}$ [L]. In particular we have the Chern number equality $c_1^2 (X) = 3c_2 (X) > 0$. This violate our assumption that $\sigma (X) = 0$ because $\sigma (X) = (c_1^2 (X) - 2c_2 (X)) / 3$. Therefore $\text{rank}(\text{Ker}\phi)$ can never be one.$\square$

Combining these results and the work of Yau [Y] and LeBrun [LB], we obtain the following characterization of four dimensional locally symmetric spaces of noncompact type.

**Theorem 7** Let $X$ be an Einstein four manifold with $\chi (X) > 0$. Then $X$ is
a locally symmetric space of noncompact type iff one of the followings holds

(1) the non-Abelian Seiberg Witten equation for $S^+_L$ has a solution and
    $\sigma(X) = 0$.

(2) the Seiberg Witten equation for $L$ has a solution and
    $3\sigma(X) = \chi(X)$.

The universal cover of $X$ is isometric to $B^2 \times B^2$ or $B^4$ in (1) and $B^2_C$ in (2).

Notice that when $\tilde{X} = B^2_C$, Seiberg Witten equation has a unique solution provided by the Kähler Einstein metric on $X$. Moreover the number of solutions to the Seiberg Witten equation counted with multiplicity is a differentiable invariant. As a result, as LeBrun pointed out in [LB], Einstein metric on such manifolds is unique. Therefore an important question is to use the space of solutions to the non-Abelian Seiberg Witten equations to define Donaldson type differentiable invariant for $X$. The author is informed by P.Feehan that this might be possible in some situations, modulo certain technical difficulties. A positive solution to the above question will tell us that the above characterization of locally symmetric space depends only on Einstein metric and the differentiable structure of $X$. In particular, Einstein metric on locally symmetric four manifolds of noncompact type would be unique.

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