ON PRISMS, MÖBIUS LADDERS AND THE CYCLE SPACE OF DENSE GRAPHS

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ABSTRACT. For a graph $X$, let $f_0(X)$ denote its number of vertices, $\delta(X)$ its minimum degree and $Z_1(X;\mathbb{Z}/2)$ its cycle space in the standard graph-theoretical sense (i.e. 1-dimensional cycle group in the sense of simplicial homology theory with $\mathbb{Z}/2$-coefficients). Call a graph Hamilton-generated if and only if the set of all Hamilton circuits is a $\mathbb{Z}/2$-generating system for $Z_1(X;\mathbb{Z}/2)$. The main purpose of this paper is to prove the following: for every $\gamma > 0$ there exists $n_0 \in \mathbb{Z}$ such that for every graph $X$ with $f_0(X) \geq n_0$ vertices,

1. if $\delta(X) \geq \left(1 + \gamma\right)f_0(X)$ and $f_0(X)$ is odd, then $X$ is Hamilton-generated,
2. if $\delta(X) \geq \left(1 + \gamma\right)f_0(X)$ and $f_0(X)$ is even, then the set of all Hamilton circuits of $X$ generates a codimension-one subspace of $Z_1(X;\mathbb{Z}/2)$, and the set of all circuits of $X$ having length either $f_0(X) - 1$ or $f_0(X)$ generates all of $Z_1(X;\mathbb{Z}/2)$,
3. if $\delta(X) \geq \left(1 + \gamma\right)f_0(X)$ and $X$ is square bipartite, then $X$ is Hamilton-generated.

All these degree-conditions are essentially best-possible. The implications in (1) and (2) give an asymptotic affirmative answer to a special case of an open conjecture which according to [European J. Combin. 4 (1983), no. 3, p. 246] originates with A. Bondy.

Keywords: Cayley graph, cycle group, cycle space, finite-dimensional vector spaces, Hamilton circuit, Hamilton-connected, Hamilton-laceable, prism graph, Möbius ladder, monotone graph property, spanning subgraphs

1. Introduction

There exist investigations in which the set underlying a finite-dimensional vector space is not forgotten, but made to play a central part. One such investigation was begun thirty years ago by I. B.-A. Hartman and concerns the cycle space $Z_1(X;\mathbb{Z}/2)$ of a finite graph $X$ (whose vectors are the Eulerian subgraphs of $X$): under what conditions does $Z_1(X;\mathbb{Z}/2)$ admit a basis over $\mathbb{Z}/2$ consisting of long graph-theoretical circuits only? Hartman proved [37, Theorem 1] a theorem which guarantees that—barring the sole exception of $X$ being a complete graph with an even number of vertices—for every 2-connected finite graph $X$, the set of all circuits of length at least $\delta(X) + 1$ generates $Z_1(X;\mathbb{Z}/2)$.

The lower the minimum degree $\delta(X)$, the larger the set of cycle-lengths one has to allow in order to be guaranteed a generating system by Hartman’s theorem. In particular, statements guaranteeing a generating system consisting entirely of Hamilton circuits (a natural thing to ask for once the topic of long circuits has been broached) remain almost inaccessible via this theorem: one has to set $\delta(X) := f_0(X) - 1$, hence $X \cong K^{f_0(X)}$, and what remains of Hartman’s general theorem is a rather special (albeit still non-obvious) statement about the complete graph.

The property of $Z_1(X;\mathbb{Z}/2)$ being generated by the Hamilton circuits of $X$ seems to have been first studied by B. Alspach, S. C. Locke and D. Witte [5]. They proved that $X$ has the property if $X$ is a connected Cayley graph on a finite abelian group and is either bipartite or has odd order (these hypotheses being mutually exclusive for connected Cayley graphs on finite abelian groups).

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Here, we will for the first time prove minimum degree conditions guaranteeing this property (Section 5 contains a short survey of the relevant literature.) We will accomplish this by way of a two-layered strategy which first harnesses theorems from extremal graph theory to prove the existence of certain spanning subgraphs which can be used to transfer the property to the entire ambient graph in a second step. The main purpose of the present paper is to prove the following previously unknown implications:

Theorem 1 (sufficient conditions for a cycle space generated by Hamilton circuits; (I3) had already been announced in [16]). For every $\gamma > 0$ there exists $n_0 \in \mathbb{Z}$ such that for every graph $X$ with $f_0(X) \geq n_0$, the following is true:

1. if $\delta(X) \geq (\frac{3}{4} + \gamma)f_0(X)$ and $f_0(X)$ is odd, then $X$ is Hamilton-generated,
2. if $\delta(X) \geq (\frac{1}{2} + \gamma)f_0(X)$ and $f_0(X)$ is even, then the set of all Hamilton-circuits of $X$ generates a codimension-one subspace of $Z_1(X; \mathbb{Z}/2)$ and the set of all circuits of $X$ with lengths either $f_0(X) - 1$ or $f_0(X)$ generates all of $Z_1(X; \mathbb{Z}/2)$,
3. if $\delta(X) \geq (\frac{3}{4} + \gamma)f_0(X)$ and $X$ is square bipartite, then $X$ is Hamilton-generated,
4. if in (1) and (2) the condition $\delta(X) \geq (\frac{1}{2} + \gamma)f_0(X)$ is replaced by $\delta(X) \geq \frac{3}{4}f_0(X)$, then without further change to (1) or (2) it suffices to take $n_0 := 2 \cdot 10^8$.

Implication (1) becomes false if ‘$(\frac{3}{4} + \gamma)f_0(X)$’ is replaced by ‘$\delta(X)$ and $X$ Hamilton-connected’.

Implication (3) becomes false if ‘$(\frac{1}{2} + \gamma)f_0(X)$’ is replaced by ‘$\frac{3}{4}f_0(X)$ and $X$ hamiltonian’.

A purely combinatorial way of phrasing the conclusions in Theorem 1 is to say that ‘every circuit in $X$ can be realized as a symmetric difference of some Hamilton circuits of $X$’. In this variant phrasing, talking only about graph-theoretical circuits (and not more generally about cycles in the sense of homology theory) does not lose any generality since for any graph $X$ and any cycle $c \in Z_1(X; \mathbb{Z}/2)$, the support $\text{Supp}(c)$ is an edge-disjoint union of graph-theoretical circuits [29, Proposition 1.9.2]. Let us note in passing that the latter fact generalizes to locally-finite infinite graphs [30, Theorem 7.2, equivalence (i) $\iff$ (iii)], that it has been given a precise sense for arbitrary compact metric spaces [34, Theorem 1.4], and, last but not least, that linear-algebraic properties of Hamilton circles (in the sense of [19]) in infinite graphs—i.e. the role of infinite Hamilton circles vis-à-vis the cycle space (in the sense of [27] [28] [30] [31])—is an unexplored research topic.

Theorem 1, the main result of the present paper, adds to the growing corpus of knowledge about the following phenomenon: when studying the set of Hamilton circuits as a function of the minimum degree $\delta(X)$, it pursues if it rains—slightly below a sufficient threshold there still exist graphs which do not have any Hamilton circuit, slightly above the threshold suddenly every graph contains not merely one but rather a plethora of Hamilton circuits satisfying many additional requirements.

This line of investigation appears to begin with C. St. J. A. Nash-Williams’ proof [54, Theorem 2] [55, Theorem 3] that for every graph $X$ with $\delta(X) \geq \frac{3}{2}f_0(X)$ there exists not only one (Dirac’s theorem [32, Theorem 3] [29, Theorem 10.1.1]) but at least $\left\lfloor \frac{2f_0(X)}{\overline{d}} \right\rfloor$ edge-disjoint Hamilton circuits. For sufficiently large graphs $X$ with $\delta(X)$ a little larger than $\frac{3}{2}f_0(X)$, Nash-Williams’ theorem was improved by D. Christofides, D. Kühn and D. Osthus [22, Theorem 2] to the guarantee that there are at least $\frac{2}{3}n$ edge-disjoint Hamilton circuits—this being an asymptotically best-possible result in view of examples [54, p. 818] which show that in graphs $X$ with $\delta(X) \geq \frac{3}{2}f_0(X)$ and having a slightly irregular degree sequence, the number of edge-disjoint Hamilton circuits is bounded by $\frac{n}{2}$. More can be achieved if besides a high minimum-degree, additional requirements are imposed on the host graph. Two aspects of this are (1) a regular degree sequence, (2) a random host graph.

As to (1), if the host graph is required to be regular in advance, a still unsettled conjecture of B. Jackson [40, p. 13, l. 17] posits that a $d$-regular graph with $d \geq \frac{f_0(X) - 1}{2}$ actually realizes the
Aspects of Hamilton circuits | Literature
---|---
efficient algorithms for finding a copy | [15, Section 4], [56]
number of all copies | [57], [24], [23]
number of mutually edge-disjoint copies | [54], [55]
host graph is (in some sense) random | [12], [44], [11], [43], [47], [46]
linear algebraic properties | this paper

Table 1. Some aspects of Hamilton circuits in graphs with high-minimum degree.

obvious upper bound \( \frac{1}{2}d \) for the number of edge-disjoint Hamilton circuits. Christofides, Kühn and Osthus proved a theorem which in a sense comes arbitrarily close to the conjecture [22, Theorem 5].

As to (2), A. Frieze and M. Krivelevich conjectured [33, p. 222] that for any \( 0 \leq p_n \leq 1 \) an Erdős–Rényi random graph \( G_{n,p_n} \) a.a.s. attains the a priori maximum of \( \frac{\log n}{n} \) edge-disjoint Hamilton-circuits. (For \( p_n \) which are low enough to a.a.s. imply \( \delta \leq 1 \) the conjecture claims nothing.) They proved [33, Theorem 1] the conjecture for \( p_n \leq (1 + o(1)) \frac{\log n}{n} \). In [43, Theorem 2] F. Knox, D. Kühn and D. Osthus proved the conjecture for a class of functions \( p_n \) that sweeps a huge portion of the range \( \frac{\log n}{n} < p_n < 1 \). A remaining gap (starting at \( \frac{\log n}{n} \)) in the probability range heretofore covered was recently closed by M. Krivelevich and W. Samotij [46]. According to [46, p. 2] the conjecture now remains open only for \( p_n \geq 1 - (\log(n))^9 \cdot n^{-\frac{1}{4}} \), i.e. for unusually dense Erdős–Rényi random graphs.

One way to look at these results is as providing ‘extremely orthogonal’ (i.e. no additive cancelation is involved in the vanishing of the standard bilinear form) sets of Hamilton circuits. As they stand, these theorems are far from providing ‘orthogonal’ Hamilton-circuit-bases for \( Z_1(X; Z/2) \); the relevant minimum degrees, the dimension of \( Z_1(X; Z/2) \) is much higher than \( \delta(X)/2 \) (roughly, one has \( \text{dim}_{Z/2} Z_1(X; Z/2) \in \Theta_{f_0(X) \to \infty} (\delta(X)^2) \)), so the sets of mutually disjoint Hamilton circuits are—while ‘very’ orthogonal—far from being generating sets of \( Z_1(X; Z/2) \). Yet it does not seem unlikely that the above-mentioned theorems can be extended in a more algebraic vein by devising generalizations of ‘edge-disjoint’ (e.g. ‘size of the intersection of the supports even’) and thus be made to resonate with results like Theorem 1.

Further context for Theorem 1 is provided by the following open conjecture (thirty years ago, S. C. Locke proved [49, Theorem 2 and Corollary 4] that Bondy’s conjecture is true under the additional assumption of ‘X non-hamiltonian or \( f_0(X) \geq 4d - 5 \)’):

**Conjecture 2** (J. A. Bondy 1979: [37, p. 246] [49, Conjecture 1] [50, p. 256] [51, Conjecture 1] [10, Conjecture A] [2, p. 21] [3, p. 12]). For every \( d \in Z \), in every vertex-3-connected graph \( X \) with \( f_0(X) \geq 2d \) and \( \delta(X) \geq d \), the set of all circuits of length at least \( 2d - 1 \) is a \( Z/2 \)-generating system of \( Z_1(X; Z/2) \).

The present paper gives an asymptotic answer for two special cases of Conjecture 2: If \( \delta(X) \geq d' \) is replaced by \( \delta(X) \geq (1 + \gamma)d \) for an arbitrary \( \gamma > 0 \), and if \( f_0(X) \) is sufficiently large, then (12) in Theorem 1 below says that in the case of \( f_0(X) \geq 2d' \) holding as \( f_0(X) = 2d' \), Bondy’s conclusion is true; in case that \( f_0(X) \geq 2d' \) holds as \( f_0(X) = 2d + 1 \), then (11) in Theorem 1 says that of the three circuit lengths \( f_0(X) - 2, f_0(X) - 1 \) and \( f_0(X) \) which Bondy allows as lengths of the generating circuits, \( f_0(X) \) alone is enough. It seems likely that with the techniques of this paper it will be possible to make further inroads towards the full Conjecture 2.

**Structure of the paper.** There are four sections after the Introduction 1. In Section 2 we develop a plan for proving Theorem 1, in the process introducing all the auxiliary statements that
we will later draw upon. In Section 3, the plan is carried out in detail, in particular by giving proofs for all the auxiliary statements. Section 4 is logically superfluous but provides an alternative argumentation for a part of the proof of (I3) in Theorem 1. Section 5 surveys the literature relevant to Theorem 1 and mentions open problems.

2. Main results

Let us first introduce terminology. We adopt the common convention that a 2-set \( \{v', v''\} \) can be abbreviated as \( v'v'' \). By ‘graph’ we will mean ‘finite simple undirected graph’, equivalently ‘1-dimensional simplicial complex’. If \( X \) and \( Y \) are graphs, then \( Y \hookrightarrow X \) means that there exists an injective graph homomorphism \( Y \to X \) (here there is a subgraph of \( X \) isomorphic to \( Y \)). A path of length \( \ell \) (i.e. number of its edges) will be denoted by \( P_\ell \) and a circuit of length \( \ell \) by \( C_\ell \). (As is done in e.g. [54] and [14] we reserve the word ‘cycle’ for the homological meaning and use the more specific term ‘circuit’ for ‘2-regular connected graph’.) For a graph \( X \) we will write \( V(X) \) for its vertex set, \( E(X) \) for its edge set, \( f_0(X) := |V(X)| \) and \( f_1(X) := |E(X)| \). If \( C \) is a circuit with \( V(C) = \{v_0, v_1, v_2, \ldots, v_{\ell-1}\} \) and \( E(C) = \{v_0v_1, v_1v_2, \ldots, v_{\ell-1}v_0\} \), then we abbreviate \( v_0v_1v_2 \ldots v_{\ell-1}v_0 := E(C) \). A subgraph \( Y \) of a graph \( X \) is called non-separating if and only if the graph \( X - Y := (V(X)\setminus V(Y), E(X)\setminus \{e \in E(X): e \cap V(Y) \neq \emptyset\}) \) is connected. A circuit \( C \) in a graph \( X \) is called non-separating induced if and only if \( C \) is non-separating and \( C \) has no chords in \( X \) (i.e. \( \{e \in E(X): e \subseteq V(C)\} = E(C) \)). We write \( \mathbb{Z}/2 := \mathbb{Z}/2 \mathbb{Z} \) and \( c_e \in (\mathbb{Z}/2)^{E(X)} \) for the unique map with \( c_e(e) = 1 \in \mathbb{Z}/2 \) and \( c_e(e') = 0 \in \mathbb{Z}/2 \) for every \( e \neq e' \in E(X) \). As usual, \( C_1(X; \mathbb{Z}/2) := \bigwedge^2(V(X))_{\mathbb{Z}/2} \) (second exterior power) denotes the 1-dimensional chain group, where \( \langle V(X)\rangle_{\mathbb{Z}/2} \) is the \( \mathbb{Z}/2 \)-vector space freely generated by \( V(X) \), and \( Z_1(X; \mathbb{Z}/2) := \ker(\partial; C_1(X; \mathbb{Z}/2) \to C_0(X; \mathbb{Z}/2)) \) (standard boundary operator of simplicial homology theory) denotes the 1-dimensional cycle group in the sense of simplicial homology with \( \mathbb{Z}/2 \)-coefficients. This is the standard graph-theoretical cycle space of a graph. It is a vector space over \( \mathbb{Z}/2 \) with \( \dim_{\mathbb{Z}/2} Z_1(X; \mathbb{Z}/2) = f_1(X) - f_0(X) + 1 = \beta_1(X) \), the 1-dimensional Betti number of \( X \). Since \( -1 = 1 \in \mathbb{Z}/2 \), in \( C_1(X; \mathbb{Z}/2) \) we have \( v_i \wedge v_j = v_j \wedge v_i \), hence we can write the standard basis of \( C_1(X; \mathbb{Z}/2) \) as \( \{v_i \wedge v_j : v_iv_j \in E(X)\} \), the latter notation being well-defined despite \( v_i \wedge v_j = v_j \wedge v_i \). The notation \( \mathcal{H}(X) \) denotes the set of all Hamilton circuits in \( X \). For any set \( \mathcal{M} \) of circuits in \( X \) we say that \( \mathcal{M} \) generates \( Z_1(X; \mathbb{Z}/2) \) if and only if \( \{C \in \mathcal{M}: C \in \mathcal{M}\} \) is a \( \mathbb{Z}/2 \)-generating system of \( Z_1(X; \mathbb{Z}/2) \), where \( C \) is defined as the element of \( C_1(X; \mathbb{Z}/2) \) having its support equal to \( E(C) \). A bipartite graph is called square if and only if its bipartition classes have equal size. If \( X \) and \( Y \) are graphs, we denote by \( X \circ Y \) the cartesian product of \( X \) and \( Y \) (see e.g. [39, Section 1.4]). Moreover, if \( X \) is a graph, then we write \( \delta(X) := \min_{v \in V(X)} |X(v)| \) for the minimum degree, \( \Delta(X) := \max_{v \in V(X)} |X(v)| \) for the maximum degree of \( X \), and \( N_X(v) := \{w \in V(X): \{v, w\} \in E(X)\} \) for every \( v \in V(X) \). By \( k \)-connected we mean the standard graph-theoretical notion of being ‘vertex-\( k \)-connected’ (cf. e.g. [29, Section 1.4]).

2.1. Plan of the proof of Theorem 1. The proof of Theorem 1 will be broken into the following steps (the strategy is the same for (I1)–(I4), but the auxiliary spanning subgraphs used are different):

(St1) Prove the existence of suitably chosen spanning subgraphs \( Y \hookrightarrow X \); for (I1) and (I2) by using Theorem 3, for (I3) by using Theorem 4, and for (I4) by using Theorem 5 below. These graphs \( Y \) serve as ‘scaffolds’ in step (St3) which help confer the desired properties to the ambient graph \( X \).

(St2) Prove that in each case the subgraph \( Y \) itself has its cycle space generated by its Hamilton circuits, and moreover that \( Y \) is Hamilton connected.\(^1\)

\(^1\) The weaker property ‘any two non-adjacent vertices are connected by a Hamilton path’ would suffice here, but we will work with the better-known property of being Hamilton-connected.
(St3) By adapting a lemma of S. C. Locke [48, Lemma 1] argue that the properties proved in
(St2) transfer from the subgraph $Y$ to the ambient graph $X$, thereby proving Theorem 1.

We now explain (St1)—(St3) in more detail.

2.1.1. Explanation of step (St1). The theorems mentioned in (St1) are the following. As to terminology,
the \textit{square} $Y^2$ of a graph $Y$ is the graph obtained from $Y$ by adding an edge between
any two vertices having distance two in $Y$. A graph $Y$ has \textit{bandwidth at most} $b$ if and only if
there exists a bijection $b: V(Y) \rightarrow \{1, \ldots, f_0(Y)\}$ such that if $vv' \in E(Y)$, then $|b(v) - b(v')| \leq b$;
any such bijection $b$ is called a \textit{bandwidth-$b$-labelling} of $Y$. Moreover, if $Y$ is a graph, $b: V(Y) \rightarrow
\{1, \ldots, f_0(Y)\}$ is a bijection and if $(c_1, c_2) \in \mathbb{Z}^2_{\geq 1}$ and $\rho \in \mathbb{Z}_{\geq 1}$, then a map $h: V(Y) \rightarrow \{0, \ldots, \rho\}$
is called $(c_1, c_2)$-\textit{zero-free} w.r.t. $b$ (cf. [18, p. 178]) if and only if for for every $v' \in V(Y)$ there
exists a $v'' \in b^{-1}(\{b(v'), b(v') + 1, \ldots, \min(f_0(Y), b(v') + c_1)\})$ such that $b(v'') \neq 0$ for every
$v'' \in b^{-1}(\{b(v'), b(v') + 1, \ldots, \min(f_0(Y), b(v') + c_2)\})$. As a tool for proving Theorem 1 we use:

\begin{theorem}\label{thm:main}
(Böttcher–Schacht–Taraz [18, Theorem 2]). For every $\gamma > 0$ and arbitrary $\rho \in \mathbb{Z}_{\geq 2}$ and
$\Delta \in \mathbb{Z}_{\geq 2}$ there exist numbers $\beta = \beta(\gamma, \Delta) > 0$ and $n_0 = n_0(\gamma, \Delta)$ such that the following is true: for
every graph $X$ with $f_0(X) \geq n_0$ and $\delta(X) \geq (\gamma + \frac{1}{\rho}) f_0(X)$, and for every graph $Y$ having $f_0(X) = f_0(Y)$, $\Delta(Y) \leq \Delta$ and $bw(Y) \leq \beta f_0(X)$, and admitting a bandwidth-$\beta f_0(Y)$-labelling $b: V(Y) \rightarrow
\{1, \ldots, f_0(Y)\}$ and a $(\rho + 1)$-colouring $h: V(Y) \rightarrow \{0, 1, \ldots, \rho\}$ which is $(\rho \beta f_0(Y), 4 \rho \beta f_0(Y))$-zero-free
w.r.t. $b$ and has $|h^{-1}(0)| \leq \beta f_0(Y)$, there is an embedding $Y \hookrightarrow X$. \hfill \qed
\end{theorem}

\begin{theorem}\label{thm:main2}
(Böttcher–Heinig–Taraz [16, Theorem 3]). For every $\gamma > 0$ and every $\Delta \in \mathbb{Z}$ there
exist numbers $\beta = \beta(\gamma, \Delta) > 0$ and $n_0 = n_0(\gamma, \Delta) \in \mathbb{Z}$ such that the following is true: for every
square bipartite graph $X$ with $f_0(X) \geq n_0$ and $\delta(X) \geq (\gamma + \frac{1}{\rho}) f_0(X)$, and for every square bipartite
graph $Y$ with $f_0(Y) = f_0(X)$, $\Delta(Y) \leq \Delta$ and $bw(Y) \leq \beta f_0(Y)$, there is an embedding $Y \hookrightarrow X$. \hfill \qed
\end{theorem}

Moreover, the lower bound of terrestrial magnitude that is provided in (I4) depends on a very
recent theorem of P. Cháu, L. DeBiasio and H. A. Kierstead (who say [20, p. 17, Section 5, l. 5] that
by optimizing their proof one may not push the bound further down than to about $n_0 = 10^5$, but
who nevertheless express optimism as to the possibility of getting rid of the $f_0$-condition altogether
by some new graph-theoretical methods):

\begin{theorem}\label{thm:main3}
(Komlós–Sárközy–Szemerédi [45, Theorem 1], Jamshed [41, Chapter 3]; explicit lower
bound on $f_0$ proved by Cháu–DeBiasio–Kierstead [20, Theorem 7]). For every graph $X$ with
$f_0(X) \geq 2 \cdot 10^8$ and $\delta(X) \geq \frac{2}{3} f_0(X)$ there exists an embedding $C^2_{f_0(X)} \hookrightarrow X$. \hfill \qed
\end{theorem}

Whereas for (I4) our use of Theorem 5 dictates employing $C^2_{f_0(\cdot)}$ as the auxiliary subgraph, there
are choices to be made as to what subgraph to employ from the set of spanning subgraphs offered
by the Theorems 3 and 4. We will choose to use the following graphs (in Definition 6 let $b_r := b_0$):

\begin{definition}\label{def:bipartite}
(Bipartite cyclic ladder). For $r \in \mathbb{Z}_{\geq 3}$ let $CL_r$ be the bipartite graph with $V(CL_r) :=\{a_0, \ldots, a_{r-1}\} \cup \{b_0, b_{r-1}\}$ and $E(CL_r) :=\bigsqcup_{i=0}^{r-1} \{a_i, b_{i+1}\} \cup \bigsqcup_{i=0}^{r-1} \{a_i, b_i\} \cup \bigsqcup_{i=0}^{r-1} \{a_i, b_{i+1}\}$.
\end{definition}

\begin{definition}\label{def:prism}
(prism, Möbius ladder). For every $n \geq 3$ and $r \geq 3$ let (where $v_0 := v_0, x_r := x_0$ and
$y_r := y_0$) the prism $Pr_r$ be defined by $V(Pr_r) := \{x_0, \ldots, x_{r-1}, y_0, \ldots, y_{r-1}\}$ and $E(Pr_r) :=\bigsqcup_{i=0}^{r-1} \{ x_i, x_{i+1}\} \cup \bigsqcup_{i=0}^{r-1} \{ y_i, y_{i+1}\} \cup \bigsqcup_{i=0}^{r-1} \{ x_i, y_i\}$, and the Möbius ladder $M_r$ be defined by $V(M_r) := V(Pr_r)$ and $E(M_r) := \{ E(Pr_r) \setminus \{ x_0 x_{r-1}, y_0 y_{r-1}\} \} \cup \{ x_0 y_{r-1}, y_0 x_{r-1}\}$. \hfill \qed
\end{definition}

\begin{definition}\label{def:meb}
(ME). For every $r \geq 3$ let $ME_r$ be defined by $V(ME_r) := V(Pr_r) \cup \{z\}$, with $z$ some new element, and $E(ME_r) := E(Pr_r) \cup \{ z x_0, z y_0, z x_1, z y_1\}$. Let $ME_r$ be defined by $V(ME_r) := V(ME_r)$ and $E(ME_r) := \{ E(ME_r) \setminus \{ z x_0 y_{r-1}, z y_0 x_{r-1}\} \} \cup \{ z x_0 y_{r-1}, z y_0 x_{r-1}\}$. \hfill \qed
\end{definition}
Figure 1. The graphs $M^o_r$ and $M^e_r$ for odd $r$, and $Pr^o_r$ and $Pr^e_r$ for even $r$ play a key role in the proof. These are bounded-degree, bounded-bandwidth and 3-chromatic graphs admitting a 3-colouring with a constant-sized third colour class. The Böttcher–Schacht–Taraz-theorem in its full form [18, Theorem 2] is sufficiently general to guarantee the existence of embeddings of these graphs as spanning subgraphs into graphs $X$ with $\delta(X) \geq (\frac{1}{2} + \gamma)f_0(X)$. If $M^o_r$ or $Pr^o_r$ spannily embed into $X$, this implies that $Z_1(X; \mathbb{Z}/2)$ is generated by Hamilton circuits. If $M^e_r$ or $Pr^e_r$ spannily embed into $X$ this implies that $Z_1(X; \mathbb{Z}/2)$ is generated by the circuits having lengths in $\{ f_0(X) - 1, f_0(X) \}$. If the edge $x_0z''$ were omitted from $M^e_r$ or $Pr^e_r$, the remaining graph could no longer serve the purpose these graphs have in the present paper.

Definition 9 ($Pr^e_r$ and $M^e_r$). For every $r \geq 3$ let $Pr^e_r$ be defined by $V(Pr^e_r) := V(Pr_r) \cup \{ z', z'' \}$ with $z'$ and $z''$ two new elements, $E(Pr^e_r) := E(Pr_r) \cup \{ x_0z', y_0z', x_0z'', x_1z'', y_1z'', z'z'' \}$. Let $M^e_r$ be defined by $V(M^e_r) := V(Pr^e_r)$ and $E(M^e_r) := (E(Pr^e_r) \setminus \{ x_0y_0, y_0x_0 \}) \cup \{ x_0y_0, y_0x_0 \}$.

Justifying that $CL_r$ is indeed one of the subgraphs guaranteed by Theorem 4 will pose no difficulty and can be done uniformly for every $r \in \mathbb{Z}_{\geq 3}$. Matters are being complicated by parity issues when it comes to step (St2). We will later make essential use of the following sets.

Definition 10. For every even $r \geq 4$ we define the sets of edge sets

\[(P.ES.1)\] \(CE^{[1]}_{Pr^e_r} := \{ C_{ev,r,1} := x_1x_2x_3 \ldots x_{r-2}x_{r-1}y_1y_2, \ldots \}; \]
\(C_{ev,r,2} := x_1x_2x_3 \ldots x_{r-2}y_{r-1}y_0x_0, \ldots \}; \)
\(C_{ev,r,3} := x_1x_2y_3 \ldots x_{r-2}x_{r-1}y_{r-1}y_0x_0, \ldots \}; \)
\(C_{ev,r,4} := x_1x_2y_3 \ldots y_{r-3}y_{r-2}y_{r-1}y_0x_0, \ldots \}; \)
\(C_{ev,r,5} := x_1x_2y_3 \ldots x_{r-2}x_{r-1}y_{r-1}y_0x_0x_1, \ldots \}; \)

\[(P.ES.2)\] \(CE^{[2]}_{Pr^e_r} := \{ C_{ev,r,1} := x_0x_{r-1}x_{r-2} \ldots x_2y_1y_2, \ldots \}; \)
\(C_{ev,r,2} := x_0x_{r-1}x_{r-2} \ldots x_3y_3y_4, \ldots \}; \)
\(C_{ev,r,3} := x_0x_{r-1}y_1y_2, \ldots \}; \)
\(C_{ev,r,4} := x_0x_{r-1}y_1y_2, \ldots \}; \)
\(C_{ev,r,5} := x_0x_{r-1}x_{r-2} \ldots y_0z, \ldots \}; \)

The set $C_{ev,r,1}^{x_{r-1}y_{r-1}}$ does not follow the pattern to be found in $C_{ev,r,1}^{y_1y_2}, \ldots, C_{ev,r}^{x_{r-2}y_{r-2}}$.

Definition 11. For every odd $r \geq 5$ we define the sets of edge sets

\[(M.ES.1)\] \(CE^{[1]}_{M^e_r} := \{ C_{od,r,1} := x_1x_2x_3, \ldots \}; \)
\(C_{od,r,2} := x_1x_2y_3, \ldots \}; \)
\(C_{od,r,3} := x_1x_2y_3, \ldots \}; \)
\(C_{od,r,4} := x_1x_2y_3, \ldots \}; \)
\(C_{od,r,5} := x_1x_2y_3, \ldots \}; \)

\[(M.ES.2)\] \(CE^{[2]}_{M^e_r} := \{ C_{od,r,1} := x_1x_2x_3, \ldots \}; \)
\(C_{od,r,2} := x_1x_2y_3, \ldots \}; \)
\(C_{od,r,3} := x_1x_2y_3, \ldots \}; \)
\(C_{od,r,4} := x_1x_2y_3, \ldots \}; \)
\(C_{od,r,5} := x_1x_2y_3, \ldots \}; \)
Theorem 15 (Alspach–Locke–Witte [5, Theorem 2.1 and Corollary 2.3]). For every finite abelian group \( G \) and every \( 0 \neq S \subseteq G \) with \( -S = S \) the graph \( X := \text{Cay}(\langle S \rangle; S) \) has the following properties:

1. \( X \) is not bipartite.
2. If \( X \) is Hamilton-connected, it is Hamilton-laceable in the case \( X \) is bipartite.
3. If \( X \) is Hamilton-connected, it is Hamilton-laceable in the case \( X \) is bipartite.

---

2The bipartite case appears to be susceptible to analogous arguments as in [21]. The author does not know of any published proof of the bipartite case. Nevertheless, it is mentioned in [6, Theorem 1.4], [4, Theorem 1.7], [53, Introductory Remarks and Proposition 2.1] and [52, Proposition 3]. Moreover, what little we need of the general bipartite case, namely Lemma 17(a,14), can be easily shown directly.
Lemma 17

(1) if $X$ is bipartite, then $\mathcal{H}(X)$ generates $\mathbb{Z}_1(X; \mathbb{Z}/2)$,
(2) if $|X| = |(S)|$ is odd, then $\mathcal{H}(X)$ generates $\mathbb{Z}_1(X; \mathbb{Z}/2)$,
(3) if $|X| = |(S)|$ is even and $X$ is not bipartite and not a prism over any circuit of odd length, then $\dim_{\mathbb{Z}/2}(\mathbb{Z}_1(X; \mathbb{Z}/2)/\langle \mathcal{H}(X) \rangle_{\mathbb{Z}/2}) = 1$.

To efficiently formulate properties of the auxiliary substructures, we have to agree upon some further terminology:

Definition 16. Let $\mathfrak{L}$ be a map from graphs to subsets of $\mathbb{Z}_{\geq 1}$, let $\mathfrak{L} = \{l - 1 : l \in \mathfrak{L}\}$ and let $\xi \in \mathbb{Z}_{\geq 0}$. We define:

1. a graph $X$ to be $\mathfrak{L}$-path-connected (if $\mathfrak{L} = \{f_0(\cdot) - 1\}$ we speak of being Hamilton-connected) if and only if for every $\{v, w\} \in (V(X))$ there exists in $X$ at least one $v$-$w$-path having its length in the set $\mathfrak{L}(X)$ (we denote the collection of all such graphs by $\mathfrak{CO}_{\mathfrak{L}}$),
2. a variant of $\mathfrak{CO}_{\mathfrak{L}}$ for bipartite graphs: adopting a by now widespread usage dating back at least to work of G. J. Simmons [9], a bipartite graph $X$ will be called $\mathfrak{L}$-laceable (if $\mathfrak{L} = \{f_0(\cdot) - 1\}$ also Hamilton-laceable) if and only if for any two $v, w \in V(X)$ not in the same bipartition class there exists at least one $v$-$w$-path having its length in the set $\mathfrak{L}(X)$ (we denote the collection of all such graphs by $\mathfrak{LA}_{\mathfrak{L}}$),
3. for a graph $X$ the set $\mathcal{C}_2(X)$ as the set of all graph-theoretical circuits in $X$ whose length is an element of $\mathfrak{L}$. (In particular, $\mathcal{C}_2(X) = \mathcal{H}(X)$.)
4. $\mathfrak{cd}_2 \mathcal{C}_2$ as the collection of graphs $X$ with $\dim_{\mathbb{Z}/2}(\langle \mathcal{C}_2(X) \rangle_{\mathbb{Z}/2}) = \beta_1(X) - \xi$, 
5. $\mathfrak{bcd}_2 \mathcal{C}_2 \subseteq \mathfrak{cd}_2 \mathcal{C}_2$ as the collection of all the bipartite elements of $\mathfrak{cd}_2 \mathcal{C}_2$,
6. $\mathcal{M}_{\mathfrak{cd}_2 \mathcal{L}} := \mathfrak{cd}_2 \mathcal{C}_2 \cap \mathfrak{CO}_{\mathfrak{L}, -1}$ and $\mathcal{bM}_{\mathfrak{cd}_2 \mathcal{L}} := \mathfrak{bcd}_2 \mathcal{C}_2 \cap \mathfrak{LA}_{\mathfrak{L}, -1}$.

The condition in (4) is equivalent to $\dim_{\mathbb{Z}/2}(\mathbb{Z}_1(X; \mathbb{Z}/2)/\langle \mathcal{C}_2(X) \rangle_{\mathbb{Z}/2}) = \xi$, in other words, $\mathfrak{cd}_2 \mathcal{C}_2(X)$ is the set of all graphs for which $\langle \mathcal{C}_2(X) \rangle_{\mathbb{Z}/2}$ has codimension $\xi$ in $\mathbb{Z}_1(X; \mathbb{Z}/2)$. In particular $\mathfrak{cd}_2 \mathcal{C}_2(\mathfrak{f}_0(\cdot)) (X) = \mathcal{H}(X)$.

Lemma 17 (properties of the auxiliary structures). For every $n \geq 5$ and every $r \in \mathbb{Z}_{\geq 4}$,

(a1) $C_n^r \cong \text{Cay}(\mathbb{Z}/n; \{\overline{T}, \overline{0}, \overline{n - T}, \overline{n - 1}\})$,
(a2) $C_n^r$ is not a prism over a graph (i.e. there does not exist a graph $Y$ with $C_n^r \cong \text{Cay}(\mathbb{Z}/n; \{P_1\})$),
(a3) if $n$ is even, then $C_n^r \in \mathcal{M}_{\mathfrak{f}_0(\cdot) - 1}$,
(a4) if $n$ is odd, then $C_n^r \in \mathcal{M}_{\mathfrak{f}_0(\cdot) - 1}$,
(a5) if $n$ is even, then $C_n^r \in \mathcal{M}_{\mathfrak{f}_0(\cdot) - 1}$,
(a6) $\text{Pr}_r \cong \text{Cay}(\mathbb{Z}/2 \oplus \mathbb{Z}/r; \{(\overline{T}, 0), (0, \overline{T}), (0, \overline{r - T})\})$,
(a7) $\text{M}_r \cong \text{Cay}(\mathbb{Z}/(2r); \{\overline{T}, \overline{2r - T}\})$,
(a8) if $r$ is even, then $\text{Pr}_r \in \mathcal{LA}_{\mathfrak{f}_0(\cdot) - 1}$,
(a9) if $r$ is odd, then $\text{M}_r \in \mathcal{LA}_{\mathfrak{f}_0(\cdot) - 1}$,
(a10) if $r$ is even, then $\text{Pr}_r \in \mathcal{bM}_{\mathfrak{f}_0(\cdot) - 1}$,
(a11) if $r$ is odd, then $\text{M}_r \in \mathcal{bM}_{\mathfrak{f}_0(\cdot) - 1}$,
(a12) if $r$ is even, then $\mathcal{CL}_r \cong \mathcal{Pr}_r$,
(a13) if $r$ is odd, then $\mathcal{CL}_r \cong \mathcal{M}_r$,
(a14) $\mathcal{CL}_r \in \mathcal{LA}_{\mathfrak{f}_0(\cdot) - 1}$,
(a15) $\mathcal{CL}_r \in \mathcal{bM}_{\mathfrak{f}_0(\cdot) - 1}$,
(a16) if $r$ is even, then $\mathfrak{Pr}_r \in \mathfrak{CO}_{\mathfrak{f}_0(\cdot) - 1}$,
(a17) if $r$ is odd, then $\mathfrak{M}_r \in \mathfrak{CO}_{\mathfrak{f}_0(\cdot) - 1}$,
(a18) if $r$ is even, then $\mathfrak{Pr}_r \in \mathfrak{CO}_{\mathfrak{f}_0(\cdot) - 1}$,
(a19) if $r$ is odd, then $\mathfrak{M}_r \in \mathfrak{CO}_{\mathfrak{f}_0(\cdot) - 1}$,
(a20) concerning $\mathfrak{Pr}_r$ and $\mathfrak{Pr}_r$ for even $r$, and concerning $\mathfrak{M}_r$ and $\mathfrak{M}_r$ for odd $r$, the set $\{c_\mathfrak{c}: C \in \mathcal{C}_1^{(\cdot)}\}$ is a linearly independent subset of $\mathbb{Z}_1(X; \mathbb{Z}/2)$ for all $X \in \{\mathfrak{Pr}_r, \mathfrak{Pr}_r, \mathfrak{M}_r, \mathfrak{M}_r\}$,
(a21) Concerning $\mathfrak{Pr}_r$ and $\mathfrak{Pr}_r$ for even $r$, and concerning $\mathfrak{M}_r$ and $\mathfrak{M}_r$ for odd $r$, the set $\{c_\mathfrak{c}: C \in \mathcal{C}_2^{(\cdot)}\}$ is a linearly independent subset of $\mathbb{Z}_1(X; \mathbb{Z}/2)$ for all $X \in \{\mathfrak{Pr}_r, \mathfrak{Pr}_r, \mathfrak{M}_r, \mathfrak{M}_r\}$,
(a22) Concerning $\mathfrak{Pr}_r$ and $\mathfrak{Pr}_r$ for even $r \geq 4$, and concerning $\mathfrak{M}_r$ and $\mathfrak{M}_r$ for odd $r \geq 5$, the sum $\langle \mathcal{C}_1^{(\cdot)} \rangle_{\mathbb{Z}/2} + \langle \mathcal{C}_2^{(\cdot)} \rangle_{\mathbb{Z}/2} \subseteq C_1(X; \mathbb{Z}/2)$ is direct for all $X \in \{\mathfrak{Pr}_r, \mathfrak{Pr}_r, \mathfrak{M}_r, \mathfrak{M}_r\}$,
(a23) Concerning \( Pr_0 \) and \( Pr_r \) for even \( r \), and concerning \( M_0 \) and \( M_r \) for odd \( r \),

\[
\begin{align*}
& (\mathbb{E} (0)) \quad \langle H(Pr_0^r) \rangle_{Z/2} = Z_1(Pr_0^r; Z/2) , \\
& (\mathbb{E} (1)) \quad \langle H(M_0^r) \rangle_{Z/2} = Z_1(M_0^r; Z/2) , \\
& (\mathbb{E} (0)) \quad \dim_{Z/2}(Z_1(Pr_0^r; Z/2)/\langle H(Pr_0^r) \rangle_{Z/2}) = 1 , \\
& (\mathbb{E} (1)) \quad \dim_{Z/2}(Z_1(M_0^r; Z/2)/\langle H(M_0^r) \rangle_{Z/2}) = 1 .
\end{align*}
\]

(a24) if \( r \) is even, then \( Pr_0^r \in M_{1(0)} \); (a25) if \( r \) is odd, then \( M_0^r \in M_{1(0)} \); (a26) if \( r \) is even, then \( Pr_0^r \in M_{1(1)} \); (a27) if \( r \) is odd, then \( M_0^r \in M_{1(1)} \); (a28) if \( r \) is even, then \( Pr_0^r \in M_{1(0)} \); (a29) if \( r \) is odd, then \( M_0^r \in M_{1(0)} \); (a30) for every \( \beta > 0 \) there exists \( n_0 = n_0(\beta) \in \mathbb{Z} \) such that—in case of \( Pr_0^r \) and \( Pr_0^r \) for even \( r \)
while in case of \( M_0^r \) and \( M_0^r \) for odd \( r \)—if \( Y \in \{ C_n^2, CL_r, Pr_0^r, Pr_1^r, M_r^r, M_r^r \} \) and \( f_0(Y) \not\equiv n_0 \),
the following is true: the bandwidth satisfies \( bw(Y) \leq \beta \cdot f_0(Y) \), and moreover for each \( Y \in \{ Pr_0^r, Pr_1^r, M_r^r, M_r^r \} \) there exists a bijection \( b_Y : V(Y) \to \{ 1, \ldots, f_0(Y) \} \) and a map \( h_Y : V(Y) \to \{ 0, 1, 2 \} \) such that \( b_Y(Y) \) is \( \beta \)-labelling and \( h_Y \) is a 3-colouring of \( Y \), and \( h_Y \) has \( |h_Y^{-1}(0)| \leq 3 \beta f_0(Y) \) and is \( (8 \cdot 2 \cdot \beta \cdot f_0(Y), 4 \cdot 2 \cdot \beta \cdot f_0(Y)) \)-zero-free w.r.t. \( b_Y \).

There are arbitrary choices to be made when proving Lemma 17. Let us especially mention that there are three different strategies for proving (a15):

(A1) Realize \( CL_r \) as a Cayley graph on a finite abelian group. Then cite a theorem of B. Aspach, S. C. Locke and D. Witte which implies that \( Z_1(Pr_0^r; Z/2) \) is generated by Hamilton circuits.

(A2) Determine the full set of non-separating induced circuits of \( CL_r \), then realize every single
such circuit as a \( Z/2 \)-sum of Hamilton circuits of \( CL_r \) and then appeal to a theorem of W. T. Tutte ([58, Statement (2.5) [29, Theorem 3.2.3]]) which states that in a 3-connected graph \( X \) the cycle space \( Z_1(X; Z/2) \) is generated by the set of all non-separating induced circuits.

(A3) Exhibit sufficiently many explicit Hamilton circuits of \( CL_r \), so that after choosing some basis
the matrix of these circuits has \( Z/2 \)-rank equal to \( \dim_{Z/2}(Z_1(Pr_0^r; Z/2)) \). It then follows that \( Z_1(Pr_0^r; Z/2) = \langle H(Pr_0^r) \rangle_{Z/2} \), since in a vector space, a maximal linearly independent subset is a generating system.

Each of (A1)–(A3) demands attention to the parity of \( r \), for despite a superficial similarity, the
sets of circuits in \( CL_r \) for odd and even \( r \) turn out to be quite different. A positive way to look at this is as helping to decide which of (A1)–(A3) to choose. While each argument can be used for each parity of \( r \), there are some reasons to choose (A1) for odd \( r \) and (A2) for even \( r \). The reason is a trade-off between being a circulant graph (i.e. a Cayley graph on a finite cyclic group) and being a planar graph: if \( r \) is even, then it can be shown that \( CL_r \) is not isomorphic to any Cayley graph on a cyclic group, whereas when \( r \) is odd, \( CL_r \) is a circulant graph. In return, \( CL_r \) is planar if and only if \( r \) is even, and this facilitates (A2): when it comes to proving that no non-separating induced circuits of \( CL_r \) have been overlooked, the planarity of \( CL_r \) for even \( r \) opens up a shortcut via a theorem of A. Kemans. For odd \( r \), however, the non-planarity of \( CL_r \) (easy to prove via Kuratowski's theorem [36, p. 494]), makes this shortcut disappear.3 For these reasons, (A2) takes considerably more work when \( r \) is odd than when \( r \) is even.

In the proofs in Section 3.2 we will opt for the shortest route (A1). However, since an argument via non-separating induced circuits appears to have some value for auxiliary structures not realizable as Cayley-graphs, we will give an example for the constructive argumentation (A2) in the special Section 4—but only for even \( r \). Strategy (A3), the most arbitrary of all three (usually there is

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3And the non-separating induced circuits of \( CL_r \) are more numerous to boot. While an argument by including each non-separating induced circuit as a \( Z/2 \)-sum of Hamilton circuits is of course still possible due to 3-connectedness of \( CL_r \), the point is that carrying out this argument suddenly becomes more laborious for the double reason that the convenient criterion for completeness of the list of all such circuits loses its validity and at the same time the number of such circuits is even larger.
no overriding justification for choosing a particular set of linearly-independent Hamilton circuits except that it works) will be used for proving (a23), i.e. for dealing with the rather ad-hoc auxiliary structures $P_r^1$, $P_r^2$, $M_r^1$ and $M_r^2$.

2.1.3. Explanation of (St3). A set of graphs is called a graph property if and only if it is fixed (as a set) by graph isomorphisms. A graph property $\mathcal{X}$ is called monotone increasing if and only if for every $X \in \mathcal{X}$, adding to $X$ an arbitrary edge again results in an element of $\mathcal{X}$. A graph property $\mathcal{X}$ consisting of bipartite graphs only is called a monotone increasing property of bipartite graphs if and only if for every $X \in \mathcal{X}$, adding to $X$ an arbitrary edge which does not create an odd circuit again results in an element of $\mathcal{X}$.

**Lemma 18.** For any function $\mathcal{L}$ mapping graphs to subsets of $\mathbb{Z}_{\geq 1}$ and any $\xi \in \mathbb{Z}_{\geq 0}$,

1. the set $\mathcal{M}_{\mathcal{L}, \xi}$ is a monotone increasing graph property 
2. the set $b\mathcal{M}_{\mathcal{L}, \xi}$ is a monotone increasing property of bipartite graphs .

Lemma 18 can serve to elevate theorems guaranteeing the existence of spanning subgraphs with a certain property to theorems guaranteeing this property for the entire ambient graph:

**Corollary 19** (lifting properties from spanning subgraphs to host graphs). Let $\mathcal{L}$ be a function mapping graphs to subsets of $\mathbb{Z}_{\geq 1}$, let $\xi \in \mathbb{Z}_{\geq 0}$, let $\mathcal{X}$ be a set of graphs and let $b\mathcal{X}$ be a set of bipartite graphs. Then:

1. if $X \in \mathcal{X}$, then $Y \in \mathcal{M}_{\mathcal{L}, \xi}$ with 
   \[
   f_0(Y) = f_0(X) \text{ and } Y \leftarrow X \quad \implies \quad (\text{if } X \in \mathcal{X}, \text{ then } X \in \mathcal{M}_{\mathcal{L}, \xi}).
   \]
2. if $X \in b\mathcal{X}$, then $Y \in b\mathcal{M}_{\mathcal{L}, \xi}$ with 
   \[
   f_0(Y) = f_0(X) \text{ and } Y \leftarrow X \quad \implies \quad (\text{if } X \in b\mathcal{X}, \text{ then } X \in b\mathcal{M}_{\mathcal{L}, \xi}).
   \]

Lemma 18 is what makes (St3) tick. It is very similar to a lemma of S. C. Locke [48, Lemma 1], but we will re-prove Lemma 18 in Section 3.2, for three reasons: first, Locke’s assumption of 2-connectedness and the attendant appeal to Menger’s theorem [48, p. 253, last line] were appropriate while being concerned with a (possibly small) subgraph of special nature within a larger 2-connected graph but seem out of place when dealing with spanning subgraphs. It feels more to the point to explicitly name a rank-one direct summand which is acquired as a result of the added edge.

Second, we will need a version of Locke’s lemma especially phrased for bipartite graphs, and this is not to be found in (but easily obtained by a small modification of) [48].

Third, there is a simple algebraic lemma underlying Lemma 18, and for this lemma it appears that free modules over principal ideal domains provide the natural generality. With a view towards possible future research on the role of $\mathcal{H}(X)$ vis-à-vis the $\mathbb{Z}$-module $\mathcal{Z}_1(X; \mathbb{Z})$ (for $X$ with high $\delta(X)$), let us opt for this generality right-away, at negligible additional cost, but with more insight into the underlying mechanism. If $R$ is a commutative ring, $M$ a free $R$-module and $B \subseteq M$ an $R$-basis of $M$, then for every $v \in M$ we write $(\lambda_{B, v, b})_{b \in B} \in R^B$ for the unique element of $R^B$ (count-finitely-many components zero) with $v = \sum_{b \in B} \lambda_{B, v, b} b$. Then for every $b \in B$ the map $\lambda_{B, v, b} \colon v \mapsto \lambda_{B, v, b}$ is an element of $\text{Hom}_R(M, R)$ (which elsewhere is often denoted by $b^*$). Moreover, we define $\text{Supp}_B(v) := \{ b \in B : \lambda_{B, v, b} \neq 0 \} \subseteq B$. We can now formulate a slight generalization of [48, Lemma 1] and [5, Corollary 3.2], which is the algebraic mechanism underlying Lemma 18:

**Lemma 20.** If $R$ is a principal ideal domain, $R^\times$ its group of units, $M$ a finitely-generated free $R$-module, $B \subseteq M$ an $R$-basis of $M$, $b_0 \in B$ an arbitrary element, $U \subseteq M$ an arbitrary sub-$R$-module, and $u_0 \in U$ an arbitrary element with $\lambda_{B, u_0, b_0} \in R^\times$, then

\[
U = \langle \{ u \in U : b_0 \notin \text{Supp}_B(u) \} \rangle_R \oplus \langle u_0 \rangle_R .
\]
3. Proofs

3.1. Proofs of the main results.

3.1.1. Proofs of the implications in Theorem 1. As to (I1), let \( \gamma > 0 \) be given and invoke Theorem 3
with this \( \gamma, \rho := 2 \) and \( \Delta := 4 \) to get a \( \beta > 0 \) and an \( n_0 \), here denoted by \( n_0' \), with the property
stated there. Give this \( \beta \) to Lemma 17.(a30) to get an \( n_0 = n_0(\beta) \), here denoted by \( n_0'' \), with the
properties stated there. We now argue that with \( n_0 := \max(n_0', n_0'') \) the claim in (I1) is true.
Let \( X \) be the set of all graphs \( X \) with odd \( f_0(X) \geq n_0 \) and \( \delta(X) \geq \left( \frac{1}{2} + \gamma \right)f_0(X) \). Let \( X \in X \) be arbitrary, \( r := \frac{1}{2}(f_0(X) - 1) \) and \( Y := Pr_r^X \) in case \( f_0(X) \equiv 1 \) (mod 4), resp. \( Y := M_r^X \) in case \( f_0(X) \equiv 3 \) (mod 4). Then \( Y \in M_{(f_0(\cdot))} \) in view of Lemmas 17.(a24) and 17.(a25), moreover \( f_0(Y) = f_0(X) \) and also \( Y \leftrightarrow X \) since \( \Delta(Y) = 4 \leq \Delta \) and Lemma 17.(a30) in the case \( 'Y = Pr_r^X' \) (resp. \( 'Y = M_r^X' \)) allows us to apply Theorem 3—-with the \( \gamma, \rho, \Delta, \beta, n_0 \) we already fixed—to the
graphs \( X \) and \( Y \). Therefore, by Corollary 19.(I) it follows that \( X \in M_{(f_0(\cdot))} \), in particular \( X \in \text{cd}(\mathcal{C}_{(f_0(\cdot))}) \), which is what is claimed in (I1).

As to (I2), if throughout the preceding paragraph we replace \( '(I1)' \) by \( '(I2)' \), ‘odd’ by ‘even’, \( r := \frac{1}{2}(f_0(X) - 1) \) by \( r := \frac{1}{2}f_0(X) \)' by \( 'Pr_r^X' \) by \( 'Pr_r^X' \), \( 'M_r^X' \) by \( 'M_r^X' \), \( 'M_{(f_0(\cdot))} \) by \( 'M_{(f_0(\cdot))} \)' \( (a24)' \) by \( '\text{Lemma 17.(a26)}' \)' \( (a25)' \) by \( '\text{Lemma 17.(a27)}' \)' \( (a26)' \) by \( '\text{Lemma 17.(a27)}' \)' \( (a27)' \) by \( '\text{Lemma 17.(a29)}' \)' \( (a28)' \) by \( '\text{Lemma 17.(a28)}' \)' \( (a29)' \) by \( '\text{Lemma 17.(a29)}' \)' \( (a30)' \) by \( '\text{Lemma 17.(a30)}' \)' \( (a31)' \) by \( '\text{Lemma 17.(a31)}' \)' \( (a32)' \) by \( '\text{Lemma 17.(a32)}' \)' \( (a33)' \) by \( '\text{Lemma 17.(a33)}' \)' \( (a34)' \) by \( '\text{Lemma 17.(a34)}' \)' \( (a35)' \) by \( '\text{Lemma 17.(a35)}' \)' \( (a36)' \) by \( '\text{Lemma 17.(a36)}' \)’ \( (a37)' \) by \( '\text{Lemma 17.(a37)}' \)” and then apply the new instructions once more to the first paragraph, we obtain a proof of the second claim in (I2).

As to (I3), let \( \gamma > 0 \) be given and invoke Theorem 4 with this \( \gamma \) and \( \Delta := 3 \) to get a \( \beta > 0 \)
and an \( n_0 \), here denoted by \( n_0' \), with the property stated there. Give this \( \beta \) to Lemma 17.(a30) to
get an \( n_0 = n_0(\beta) \), here denoted by \( n_0'' \), with the properties stated there. We now argue that with \( n_0 := \max(n_0', n_0'') \) the claim in (I3) is true. Let \( bX \) be the set of all square bipartite graphs \( X \) with \( f_0(X) \geq n_0 \) and \( \delta(X) \geq \left( \frac{1}{2} + \gamma \right)f_0(X) \). Let \( X \in X \) be arbitrary and set \( r := \frac{1}{2}f_0(X) \) and \( Y := CL_r(Y) \). Then \( Y \in bM_{(f_0(\cdot))} \) in view of Lemma 17.(a15), moreover \( f_0(Y) = f_0(X) \) and also \( Y \leftrightarrow X \) since \( \Delta(Y) = 3 \leq \Delta \) and Lemma 17.(a30) in the case \( 'Y = CL_r' \) allows us to apply Theorem 4—-with the \( \gamma, \rho, \Delta, \beta, n_0 \) we already fixed—to the
graphs \( X \) and \( Y \). Therefore, by Corollary 19.(I) it follows that \( X \in bM_{(f_0(\cdot))} \) in particular \( X \in \text{cd}(\mathcal{C}_{(f_0(\cdot))}) \), which is what is claimed in (I3).

As to (I4), let \( X \) be the set of all graphs \( X \) with \( f_0(X) \geq 2 \cdot 10^8 \) and \( \delta(X) \geq \frac{3}{2}f_0(X) \). Let \( X \in X \) be arbitrary. Then Theorem 5 guarantees that \( C_{f_0(\cdot)}^2 \to X \). If \( f_0(X) \) is odd, then by combining Corollary 19.(I) and Lemma 17.(a4), it follows that \( X \in M_{(f_0(\cdot))} \) in particular \( X \in \text{cd}(\mathcal{C}_{(f_0(\cdot))}) \), which proves (I4) in the case of odd \( f_0 \). If \( f_0(X) \) is even, then (I4) follows by combining Corollary 19.(I) with Lemma 17.(a3), resp. Lemma 17.(a5). All the implications in Theorem 1 have now been proved.

3.1.2. Proof of the claim about weakening the hypothesis of (II) in Theorem 1. Let \( CE_{(11)} \) denote the
seven-vertex graph with \( V(CE_{(11)}) := \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7 \} \) and \( E(CE_{(11)}) := \{ \{ v_2, v_1 \}, \{ v_2, v_4 \}, \{ v_2, v_5 \}, \{ v_3, v_4 \}, \{ v_3, v_6 \}, \{ v_3, v_7 \}, \{ v_6, v_7 \} \} \) (This is the graph underlying Figure 2.) Then \( \frac{1}{2}f_0(CE_{(11)}) = 3.5 \leq \delta(CE_{(11)}) \), i.e. \( CE_{(11)} \) barely misses the Dirac threshold. The graph \( CE_{(11)} \) has odd \( f_0 \), is 3-vertex-connected, pancyclic (i.e. contains at least one circuit of each of all possible lengths 3, ..., \( f_0(X) \)), Hamilton-connected and has each of its edges contained in a Hamilton circuit. Therefore the following fact (which proves the claim made in Theorem 1 about weakening (II)) also shows that the open question (Q1) in Section 5 can easily acquire a negative answer if its hypotheses are slightly weakened:

**Proposition 21.** \( \dim_{\mathbb{Z}/2}(Z_{1}(CE_{(11)}; \mathbb{Z}/2)/\langle \mathcal{H}(CE_{(11)}) \rangle_{\mathbb{Z}/2}) = 1 \)
We will now prove by a short argument that $f_0(\text{CE}_{(11)})$, is 3-vertex-connected, only barely fails to satisfy the Dirac condition, is pancyclic (despite with $f_1 = 12$, $\delta = 12.25$ narrowly missing Bondy’s sufficient size-condition for the pancyclicity of a hamiltonian graph [13, p. 81]), is Hamilton-connected and has each of its edges contained in a Hamilton circuit. And yet it has its cycle space not generated by its Hamilton circuits (all of which are shown in the figure).

Proof. The smallness of $\text{CE}_{(11)}$, makes it easy to check that $\mathcal{H}(\text{CE}_{(11)})$ consists precisely of the six circuits (shown in Figure 2) $C_1 := \{v_1v_7v_2v_5v_6v_3v_4v_1, C_2 := \{v_1v_7v_6v_3v_5v_2v_4v_1, C_3 := \{v_1v_7v_5v_2v_4v_3v_6v_1, C_4 := \{v_1v_6v_7v_2v_5v_3v_4v_1, C_5 := \{v_1v_6v_3v_7v_2v_4v_1, C_6 := \{v_1v_6v_5v_3v_4v_2v_7v_1. If the standard basis of $\text{CE}_{(11)} \otimes \mathbb{Z}/2$ is labelled $e_1 := c_{v_1v_4}, e_2 := c_{v_1v_6}, e_3 := c_{v_2v_4}, e_4 := c_{v_2v_5}, e_5 := c_{v_3v_6}, e_6 := c_{v_4v_7}, e_7 := c_{v_5v_4}, e_8 := c_{v_5v_6}, e_9 := c_{v_6v_6}, e_{10} := c_{v_6v_6}, e_{11} := c_{v_6v_6}, e_{12} := c_{v_6v_6}$, then w.r.t. to this basis the Hamilton circuits $C_1, \ldots, C_6$ give rise to the matrix shown in (2), which has $\mathbb{Z}/2$-rank 5.

$$
\begin{align*}
C_1 & := \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1
\end{pmatrix} \\
& \quad \text{(2)}
\end{align*}
$$

Therefore $\langle H(\text{CE}_{(11)}) \rangle_{\mathbb{Z}/2}$ is a 5-dimensional subspace of $\mathbb{Z}_1(\text{CE}_{(11)}; \mathbb{Z}/2)$, which has dimension $\beta_1(\text{CE}_{(11)}) = f_1(\text{CE}_{(11)}) - f_0(\text{CE}_{(11)}) + 1 = 12 - 7 + 1 = 6$. This proves Proposition 21. □

3.1.3. Proof of the claim about weakening the hypothesis of (3) in Theorem 1. Let $\text{CE}_{(13)}$ denote the six by six square bipartite graph with $V(\text{CE}_{(13)}) := \{v_1, \ldots, v_6\} \sqcup \{v_7, \ldots, v_{12}\}$ (bipartition classes indicated) and $E(\text{CE}_{(13)}) := \{v_1v_7, v_1v_8, v_1v_9, v_1v_{12}, v_2v_7, v_2v_8, v_2v_9, v_3v_7, v_3v_8, v_3v_9, v_4v_8, v_4v_{10}, v_4v_{11}, v_5v_7, v_5v_8, v_5v_9, v_5v_{11}, v_6v_12, v_6v_{11}, v_6v_{12}\}$. (This is the graph in Figure 3.) Then $\frac{1}{2}f_0(\text{CE}_{(13)}) = \delta(\text{CE}_{(13)}) = 3$ and $\text{CE}_{(13)}$ is hamiltonian. We will now prove by a short argument that $\langle H(\text{CE}_{(13)}) \rangle_{\mathbb{Z}/2}$ has at least codimension one in $\mathbb{Z}_1(\text{CE}_{(13)}; \mathbb{Z}/2)$, which is enough to establish $\text{CE}_{(13)}$ as a counterexample of the claimed kind. (By determining all 16 Hamilton circuits of $\text{CE}_{(13)}$ and subsequently computing the $\mathbb{Z}/2$-rank of a 12 by 16 matrix with zero-one entries it is possible to show that $\dim_{\mathbb{Z}/2}\langle H(\text{CE}_{(13)}) \rangle_{\mathbb{Z}/2} = 7 = \dim_{\mathbb{Z}/2}\mathbb{Z}_1(\text{CE}_{(13)}; \mathbb{Z}/2) - 1$, i.e. the codimension actually is equal to 1.)
Figure 3. A counterexample which proves that if in (I3) the hypothesis \( \delta(X) \geq (\frac{1}{2} + \gamma)f_0(X) \) is weakened to \( \delta(X) \geq \frac{1}{3}f_0(X) \) and \( X \) hamiltonian the implication becomes false: the graph \( CE_{(I3)} \) has \( \delta = 3 = \frac{1}{3}f_0 \) and is hamiltonian, yet \( \langle H(\cdot) \rangle_{Z/2} \) has codimension one in \( Z_1(\cdot; Z/2) \). If the edge \( \{v_1, v_9\} \) were omitted, we would have \( \langle H(\cdot) \rangle_{Z/2} = Z_1(\cdot; Z/2) \), hence the resulting graph \( CE_{(I3)} - \{v_1, v_9\} \) would—while still satisfying the weakened hypotheses with respect to which \( CE_{(I3)} \) is a counterexample—cease to be a counterexample. (This does not contradict the fact that ‘Hamilton-laceable and Hamilton-generated’ is a monotone property of bipartite graphs: \( CE_{(I3)} - \{v_1, v_9\} \) is not Hamilton-laceable.) The author could not find a counterexample showing that (I3) would become false were \( \delta(X) \geq (\frac{1}{2} + \gamma)f_0(X) \) weakened only to \( \delta(X) \geq \frac{1}{3}f_0(X) \) and \( X \) Hamilton-laceable.

**Proposition 22.** \( \dim_{Z/2} \langle Z_1(CE_{(I3)}; Z/2) / \langle H(CE_{(I3)}) \rangle_{Z/2} \rangle \geq 1 \)

**Proof.** It is enough to make the following simple observation: since \( \{v_1, v_9\} \) is a separator of \( CE_{(I3)} \), the edge \( \{v_1, v_9\} \) cannot be an edge of any Hamilton circuit of \( CE_{(I3)} \). Therefore the set of all Hamilton circuits of \( CE_{(I3)} \) equals the set of all Hamilton circuits of the graph \( CE_{(I3)} - \{v_1, v_9\} \) obtained after deleting \( \{v_1, v_9\} \) from \( CE_{(I3)} \). This in particular implies the first equality in the calculation \( \dim_{Z/2} \langle H(CE_{(I3)}) \rangle_{Z/2} = \dim_{Z/2} \langle H(CE_{(I3)} - \{v_1, v_9\}) \rangle_{Z/2} \leq \) (since the dimension of a subspace of a vector space is bounded by the dimension of the latter’s dimension) \( \dim_{Z/2} Z_1(CE_{(I3)} - \{v_1, v_9\}; Z/2) = \) (by the Euler–Poincaré relation) \( \dim_{Z/2} Z_1(CE_{(I3)}; Z/2) - 1 \), which is just what is claimed in Proposition 22.

### 3.2. Proofs of the auxiliary results.

**Proof of Lemma 18.** First note that for both \( M_{L, \xi} \) and \( bM_{L, \xi} \), it is obvious that the sets are fixed (as sets) under any graph isomorphism, i.e. both are graph properties.

As to the monotonicity claim in (1), if \( M_{L, \xi} = \emptyset \), the claim is vacuously true. Otherwise, let \( X \in M_{L, \xi} \) be an arbitrary element and let \( e \in \langle V(X) \rangle_{\pi}(E(X) \cup \{e\}) \). We will use the abbreviation \( X + e := (V(X), E(X) \cup \{e\}) \). We have to prove \( X + e \in M_{L, \xi} \). Trivially, \( X + e \in CO_{E-1} \). What has to be justified is that \( X + e \in cd_{E}C_{L} \). Since \( X \in CO_{E-1} \), there exists in \( X \) a path \( P \) with length in \( \{l - 1: l \in \mathbb{Z}\} \) linking the endvertices of \( e \) and we have \( e \notin E(P) \) since \( e \notin E(X) \). Choose any such \( P \). We now use Lemma 20 twice: let \( R := Z/2, \ M := C_1(X + e; Z/2), \ B := \{e \in E(X + e) \} \) (the standard basis of \( C_1(X + e; Z/2) \) and \( b_0 := e \). Since (with \( \{u, v\} := e \)) the circuit \( C := uPvu \) satisfies both \( C \in C_{L}(X + e) \) and \( C \in Z_1(X + e; Z/2) \), it follows that whether we define \( U := \langle C_{L}(X + e) \rangle_{Z/2} \) or \( U := Z_1(X + e; Z/2) \), in both cases we have \( u_0 := c_{C} \in U \), and therefore Lemma 20 gives us

\[
\text{(ds1)} \langle C_{L}(X + e) \rangle_{Z/2} = \langle C_{L}(X) \rangle_{Z/2} \oplus \langle C_{L} \rangle_{Z/2}, \quad \text{(ds2)} \ Z_1(X + e; Z/2) = Z_1(X; Z/2) \oplus \langle C_{L} \rangle_{Z/2}.
\]

The direct sum decompositions (ds1) and (ds2) imply \( \dim_{Z/2} Z_1(X + e; Z/2) / \langle C_{L}(X + e) \rangle_{Z/2} = \dim_{Z/2} (Z_1(X; Z/2) / \langle C_{L}(X) \rangle_{Z/2}) = 1 \) and therefore \( X + e \in cd_{E}C_{L} \), completing the proof of statement (1). As to (2), it suffices to note that the proof of (1) may be repeated to yield a proof of (2), the only change required being to restrict \( e \) to be an edge whose addition keeps the graph bipartite and to replace \( CO_{E-1} \) by \( L_{A_{E-1}} \).

**Proof of Lemma 20.** The sum is obviously direct: \( b_0 \in Supp_{E}(u_0) \) while \( b_0 \notin Supp_{E}(v) \) for every \( v \in \{u \in U: b_0 \notin Supp_{E}(u)\} \), hence the intersection of the summands is \( \{0\} \). What is to be
justified is that \( U \subseteq \langle \{u \in U : b_0 \notin \text{Supp}_G(u)\} \rangle_R + \langle u_0 \rangle_R \). So let \( v \in U \) be arbitrary. By a well-known theorem (e.g. [35, Theorem 6.1]), since \( M \) is a free module over a principal ideal domain, so is \( U \), and there exists a finite \( R \)-basis \( \mathcal{E} \subseteq (U_{rk_R(U)}) \) of \( U \). Let \( \mathcal{E}_0 := \{ e \in \mathcal{E} : b_0 \notin \text{Supp}_G(e) \} \). Since \( \lambda_{B,v,b_0} \in \text{Hom}_R(M,R) \), we have \( \lambda_{B,v,b_0} \left( \left( \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} e \right) + \left( \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} (e - \lambda_{B,v,b_0} (\lambda_{B,v,b_0})^{-1} u_0) \right) \right) = 0 \), and therefore \( b_0 \) is not an element of \( \text{Supp}_G(\cdot) \) of \( v - \left( \lambda_{B,v,b_0} \right)^{-1} \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} \lambda_{B,v,b_0} \left( \left( \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} e \right) + \left( \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} (e - \lambda_{B,v,b_0} (\lambda_{B,v,b_0})^{-1} u_0) \right) \right) \). (3)

Thus, writing \( v = \left( \lambda_{B,v,b_0} \right)^{-1} \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} \lambda_{B,v,b_0} \left( \left( \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} e \right) + \left( \sum_{e \in \mathcal{E}_0} \lambda_{e,v,e} (e - \lambda_{B,v,b_0} (\lambda_{B,v,b_0})^{-1} u_0) \right) \right) \) shows that \( v \in \langle \{u \in U : b_0 \notin \text{Supp}_G(u)\} \rangle_R + \langle u_0 \rangle_R \), completing the proof of \( U \subseteq \langle \{u \in U : b_0 \notin \text{Supp}_G(u)\} \rangle_R + \langle u_0 \rangle_R \). □

The above proof of Lemma 20 does not work if the assumption of \( M \) being finitely generated is dropped: while [35, Theorem 6.1] remains applicable, i.e. \( U \) then still admits a basis, there is no general reason why \( \mathcal{E}_0 \) should then still be a finite set, hence the sums in (3) may not be defined. This obstacle to adapting the monotonicity argument to an infinite setting may be a point of interest (possibly one to start from) in the unknown territory of linear-algebraic properties of Hamilton of circuits in infinite graphs. There is also the issue of how to adapt the monotonicity argument in order to include one additional infinitely-many edges.

**Proof of Lemma 17.** As to (a1), an easy verification shows that the map \( \{v_0, \ldots, v_{n-1}\} \rightarrow \mathbb{Z}/n \), \( v_i \mapsto \bar{i} \) is a graph isomorphism \( C_n^2 \rightarrow \text{Cay}(\mathbb{Z}/n; \{\bar{1}, \bar{n} - \bar{1}, \bar{n} - \bar{2}, \bar{n} - \bar{1}\}) \). (Both for this verification and for the ones required in (a6), (a7), (a12) and (a13), it is recommendable to use an obvious and known [38, Section 1.5, first paragraph] characterization of graph isomorphisms: *every injective graph homomorphism between two graphs with equal f-vectors is a graph isomorphism.* This relieves one of the responsibility to explicitly show that non-edges are mapped to non-edges.)

As to (a2), the definition of \( \triangle \) implies that for every graph \( X \), every vertex of the graph \( X \triangle P_1 \) has odd degree. But for every \( n \geq 5 \) the graph \( C_n^2 \) is regular with vertex degree four.

As to (a3) and (a4), first note that \( C_n^2 \) is non-bipartite, for both parities of \( n \), and therefore (a1) and Theorem 14 combined imply that \( C_n^2 \in \mathcal{C}_{\{f_0(\cdot)\}} \), for every \( n \). It remains to justify that \( C_n^2 \in \text{cd}_1C_{\{f_0(\cdot)\}} \) for even \( n \), resp. \( C_n^2 \in \text{cd}_0C_{\{f_0(\cdot)\}} \) for odd \( n \). Both these statements follows from combining (a1) and (a2) with Theorem 15.(2) and Theorem 15.(3).

As to (a5), first note that \( C_n^2 \) does indeed contain circuits of length \( f_0(C_n^2) - 1 \) (in fact, \( f_0(C_n^2) \) different ones), and then arbitrarily choose one such circuit \( \mathcal{C} \). Since \( n \) is even, \( C \) has odd length, and therefore \( \mathcal{C} \notin \langle \mathcal{H}(C_n^2) \rangle_{\mathbb{Z}/2} \). Moreover, \( \dim_{\mathbb{Z}/2}(\mathcal{H}(C_n^2))_{\mathbb{Z}/2} = \dim_{\mathbb{Z}/2} Z_1(C_n^2; \mathbb{Z}/2) - 1 \) by (a3), hence \( \dim_{\mathbb{Z}/2}(\mathcal{H}(C_n^2))_{\mathbb{Z}/2} \geq \dim_{\mathbb{Z}/2} Z_1(C_n^2; \mathbb{Z}/2) \) and due to \( \langle \mathcal{C} \rangle \subseteq \mathcal{H}(C_n^2)_{\mathbb{Z}/2} \) being a \( 2 \)-linear subspace of \( Z_1(C_n^2; \mathbb{Z}/2) \), this must hold with equality, proving (a5).

As to (a6), an easy verification shows that the map \( \{x_0, \ldots, x_{r-1}, y_0, \ldots, y_{r-1}\} \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/r \), \( x_i \mapsto (\bar{i}, \bar{0}), y_i \mapsto (\bar{i}, \bar{1}) \) is a graph isomorphism \( Pr_r \rightarrow \text{Cay}(\mathbb{Z}/2 \oplus \mathbb{Z}/r; (\bar{0}, \bar{1}), (\bar{0}, \bar{r} - \bar{1})) \).

As to (a7), an easy verification shows that the map \( V(M_r) = \{x_0, \ldots, x_{r-1}, y_0, \ldots, y_{r-1}\} \rightarrow \mathbb{Z}/(2r) \), \( x_i \mapsto \bar{i}, y_i \mapsto \bar{i} + \bar{r} \) is a graph isomorphism \( M_r \rightarrow \text{Cay}(\mathbb{Z}/(2r); (\bar{0}, \bar{1}), (\bar{0}, \bar{r} - \bar{1})) \).

As to (a8), it is easy to check that \( r \) being even implies that \( Pr_r \) is bipartite. Therefore (a8) follows from (a6) combined with Theorem 14. Moreover, (a8) is straightforward to prove directly.

As to (a9), it is easy to check that \( r \) being odd implies that \( M_r \) is bipartite. Therefore (a9) follows from (a7) combined with Theorem 14. Moreover, (a9) is straightforward to prove directly.

As to (a10), it is easy to check that \( r \) being even implies that \( Pr_r \) is bipartite. Therefore, combining (a6) with Theorem 14 yields that \( Pr_r \in \mathcal{L}A_{\{f_0(\cdot)\}}(\cdot) \), and combining (a6) with Theorem 15.(1) yields \( Pr_r \in \text{cd}_0C_{\{f_0(\cdot)\}} \), completing the proof of (a10).
As to (a11), it is easy to check that $r$ being odd implies that $M_r$ is bipartite. Therefore, combining (a7) with Theorem 14 yields that $M_r \in \mathcal{L}A_{f_r(-)}(-1)$, and combining (a7) with Theorem 15.(1) yields $M_r \in \mathcal{L}A_{f_r(0)}(0)$, completing the proof of (a11).

As to (a12) and (a13), an easy verification shows that the map $V(C_{L_r}) \to V(Pr_r) = V(M_r)$ defined by $a_i \to x_i$ for every even $0 \leq i \leq r - 1$, $a_i \to y_i$ for every odd $0 \leq i \leq r - 1$, $b_i \to x_i$ for every even $0 \leq i \leq r - 1$, $b_i \to y_i$ for every odd $0 \leq i \leq r - 1$, is a graph isomorphism $C_{L_r} \to Pr_r$ for every even $r \geq 4$ and a graph isomorphism $C_{L_r} \to M_r$ for every odd $r \geq 4$.

As to (a14), this follows by combining (a8) and (a9) with (a12) and (a13).

As to (a15), this follows by combining (a10) and (a11) with (a12) and (a13).

As to (a16) and (a17), the literature apparently does not contain a sufficient criterion for Hamilton-connectedness which would apply to either $Pr_r^\otimes$ or $M_r^\otimes$. Therefore a direct proof by distinguishing cases and providing explicit Hamilton paths appears to be unavoidable. 4 Let $\{v, w\} \subseteq V(M_r^\otimes) = V(Pr_r^\otimes)$ be arbitrary distinct vertices.

We will repeatedly reduce the work to be done by making use of symmetries. The automorphism group of both $Pr_r^\otimes$ and $M_r^\otimes$ is the group generated by the two unique homomorphic extensions of the maps \( \{1, x, y, x_1, y_1\} \) and \( \{1, x, y, x_1, y_1\} \) to all of $V(Pr_r^\otimes) = V(M_r^\otimes)$ (thus both $\text{Aut}(Pr_r^\otimes)$ and $\text{Aut}(M_r^\otimes)$ are isomorphic to the Klein four-group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$). These extensions are involutions on $V(Pr_r^\otimes) = V(M_r^\otimes)$ and will be denoted by $\Psi_{xy}$ (the map $z \mapsto z$ and $x_i \mapsto y_i$ for every $0 \leq i \leq r - 1$) and $\Psi_{xx}$ (the map $z \mapsto z$ and, for $u \in \{x, y\}$, by $u_1 \mapsto u_0$, $u_2 \mapsto u_{r-1}$, $u_3 \mapsto u_{r-2}$, $\ldots$, $u_{\mid u\mid} \mapsto u_{\mid u\mid}$). Both $\Psi_{xy}$ and $\Psi_{xx}$ are automorphisms of both $M_r^\otimes$ (for every $r \geq 5$) and $Pr_r^\otimes$ (for every $r \geq 4$).

Case 1. $z \in \{v, w\}$. In the absence of information distinguishing $v$ from $w$ we may assume $z = v$.

Case 1.1. $w \in \{x_0, y_0, x_1, y_1\}$. Since $\text{Aut}(Pr_r^\otimes)$ acts transitively on the set $\{x_0, y_0, x_1, y_1\}$ while keeping $z$ fixed, we may assume that $w = x_0$. Then $x_0 x_1 \ldots x_{r-1} y_{r-1} x_{r-1} x_{r-1} y_{r-1} x_{r-1} y_{r-1} \ldots y_0 z$ in both $Pr_r^\otimes$ and $M_r^\otimes$ is a Hamilton path linking $v$ and $w$. This proves both (a16) and (a17) in the Case 1.1.

Case 1.2. $w \notin \{x_0, y_0, x_1, y_1\}$. Due to $\Psi_{xy}$ we may assume that $w = x_i$ with $2 \leq i \leq r - 1$. Now consider the expressions:

\[
\begin{align*}
&\text{(Pr.1.2.0)} \quad x_0 y_0 x_1 x_2 x_3 x_4 x_5 x_6 \ldots x_{r-2} y_{r-2} x_{r-1} x_r x_{r+1} x_{r+2} \ldots x_{2r-2} x_{2r-1} x_0 x_1 x_2 \ldots x_{r-1} y_{r-1} y_{r-2} y_{r-3} \ldots y_0 z, \\
&\text{(Pr.1.2.1)} \quad x_0 y_0 x_1 x_2 x_3 x_4 x_5 x_6 \ldots x_{r-2} y_{r-2} x_{r-1} x_r x_{r+1} x_{r+2} \ldots x_{2r-2} x_{2r-1} x_0 x_1 x_2 \ldots x_{r-1} y_{r-1} y_{r-2} y_{r-3} \ldots y_0 z, \\
&\text{(M.1.2.0)} \quad x_0 y_0 x_1 x_2 x_3 x_4 x_5 x_6 \ldots x_{r-2} y_{r-2} x_{r-1} x_r x_{r+1} x_{r+2} \ldots x_{2r-2} x_{2r-1} x_0 x_1 x_2 \ldots x_{r-1} y_{r-1} y_{r-2} y_{r-3} \ldots y_0 z, \\
&\text{(M.1.2.1)} \quad x_0 y_0 x_1 x_2 x_3 x_4 x_5 x_6 \ldots x_{r-2} y_{r-2} x_{r-1} x_r x_{r+1} x_{r+2} \ldots x_{2r-2} x_{2r-1} x_0 x_1 x_2 \ldots x_{r-1} y_{r-1} y_{r-2} y_{r-3} \ldots y_0 z.
\end{align*}
\]

If $i$ is even, then (Pr.1.2.0), and if $i$ is odd then (Pr.1.2.1) is a Hamilton path of $Pr_r$ linking $v$ and $w$, for every even $r \geq 4$. If $i$ is even, then (M.1.2.0), and if $i$ is odd then (M.1.2.1) is a Hamilton path of $M_r$ linking $v$ and $w$, for every odd $r \geq 5$. This proves both (a16) and (a17) in the Case 1.2.

Case 2. $z \notin \{v, w\}$.

Case 2.1. $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\}$ or $\{v, w\} \subseteq \{y_0, \ldots, y_{r-1}\}$. In view of $\Phi_{xy}$ we may assume that $\{v, w\} \subseteq \{x_0, \ldots, x_{r-1}\}$.

Case 2.1.1. $\{v, w\} \cap \{x_0, x_1\} \neq \emptyset$. In the absence of information distinguishing $v$ from $w$ we may assume that $v \in \{x_0, x_1\}$. In view of the transitivity of both $\text{Aut}(Pr_r^\otimes)$ and $\text{Aut}(M_r^\otimes)$ on $\{x_0, x_1, y_0, y_1\}$ we may further assume that $v = x_0$. Then $w = x_i$ for some $i \in \{1, r - 1\}$. We can now reduce the claim we are currently proving to claims about a cartesian product of the form $P_1 \circ P_l$ (for some $l$) which is obtained after deleting certain vertices. The reduction is made possible

\footnote{It might be possible to economize somewhat by putting more emphasis on the known Hamilton-laceability of cartesian products of the form $P_1 \circ P_l$ (which opens up the possibility to argue by dividing the graph into appropriate pieces and subsequently glue Hamilton paths together). But even then one has to pay attention to parities, making the gain in brevity over explicitly exhibiting Hamilton paths seem small. To give a short example of this, Case 2.1.1 (where there is not much gluing to do) has been treated in that manner.}
by making—depending on the parity of the $i$ in $x_i$—the right choice of a 3-path or a 4-path within the graph induced by \{ $x_i, x_{i+1}, y_0, y_1$ \}.

If $i$ is even (hence in particular $i \geq 2$), then starting out with the 4-path $x_0y_0z_1z_1y_1$ leaves us facing the task of connecting $y_2$ with $x_i$ (which lies in the opposite colour class compared to $y_2$) via a Hamilton path of the graph remaining after deletion of \{ $x_0, y_0, x_1, y_1, z$ \}. This remaining graph is—regardless of whether we are currently speaking about $M^{(2)}$ or $P^{(2)}$—isomorphic to the cartesian product $P_2 \circ P_{r-3}$, of which the vertex $y_2$ is a ‘corner vertex’ in the sense of [21, Section 2]. Therefore this task can be accomplished according to [21, Lemma 1].

If on the contrary $i$ is odd, then starting out with the 3-path $x_0y_0z_1y_1$ leaves us facing the task of connecting $y_1$ with $x_i$ (which lies in the opposite colour class compared to $y_1$) by a Hamilton path of the graph remaining after deletion of \{ $x_0, y_0, z$ \}. This remaining graph is—regardless of whether we are currently speaking about $M^{(2)}$ or $P^{(2)}$—isomorphic to the cartesian product $P_2 \circ P_{r-2}$, of which the vertex ‘$y_1$’ is a corner vertex. Therefore this task, too, can be accomplished according to [21, Lemma 1]. This proves both (a16) and (a17) in the Case 2.1.1.

Case 2.2.1. $\{ v, w \} \cap \{ x_0, x_1 \} = \emptyset$. Then $v = x_i$ and $w = x_j$ for some \{ $i, j$ \} $\in \{(2,3,\ldots,r-1)\}$. In the absence of information distinguishing $v$ from $w$ we may assume that $2 \leq i < j \leq r - 1$.

Now consider the expressions

(Pr.2.1.2.(1)) $x_ix_{i+1} \ldots x_{j-1}y_1y_j \ldots y_{i-1}z \ldots x_2y_2y_1y_0z_0 \ldots x_1y_1y_jy_{j-1} \ldots x_{r-1}y_{r-2} \ldots x_{j+1}y_jy_{j-1} \ldots y_iy_{i-1}z \ldots x_{2r-3}y_{2r-2} \ldots y_{j+1}y_jy_{j-1} \ldots x_{i+1}y_{i+1}y_iy_{i-1} \ldots y_0y_{r-2} \ldots x_1y_1y_0z_0$.

(Pr.2.1.2.(2)) $x_ix_{i+1} \ldots x_{j-1}y_1y_j \ldots y_{i-1}z \ldots x_2y_2y_1y_0z_0 \ldots x_1y_1y_jy_{j-1} \ldots x_{r-1}y_{r-2} \ldots x_{j+1}y_jy_{j-1} \ldots y_iy_{i-1}z \ldots x_{2r-3}y_{2r-2} \ldots y_{j+1}y_jy_{j-1} \ldots x_{i+1}y_{i+1}y_iy_{i-1} \ldots y_0y_{r-2} \ldots x_1y_1y_0z_0$.

(Pr.2.1.2.(3)) $x_ix_{i+1} \ldots x_{j-1}y_1y_j \ldots y_{i-1}z \ldots x_2y_2y_1y_0z_0 \ldots x_1y_1y_jy_{j-1} \ldots x_{r-1}y_{r-2} \ldots x_{j+1}y_jy_{j-1} \ldots y_iy_{i-1}z \ldots x_{2r-3}y_{2r-2} \ldots y_{j+1}y_jy_{j-1} \ldots x_{i+1}y_{i+1}y_iy_{i-1} \ldots y_0y_{r-2} \ldots x_1y_1y_0z_0$.

(Pr.2.1.2.(4)) $x_ix_{i+1} \ldots x_{j-1}y_1y_j \ldots y_{i-1}z \ldots x_2y_2y_1y_0z_0 \ldots x_1y_1y_jy_{j-1} \ldots x_{r-1}y_{r-2} \ldots x_{j+1}y_jy_{j-1} \ldots y_iy_{i-1}z \ldots x_{2r-3}y_{2r-2} \ldots y_{j+1}y_jy_{j-1} \ldots x_{i+1}y_{i+1}y_iy_{i-1} \ldots y_0y_{r-2} \ldots x_1y_1y_0z_0$.

If $i$ is even and $j$ is even, then (Pr.2.1.2.(1)) for even $r$ is a Hamilton path of $Pr^{(2)}$ linking $v$ and $w$ and (M.2.1.2.(1)) for odd $r$ is one of $M^{(2)}$, while if $i$ is even and $j$ is odd, then (Pr.2.1.2.(2)) for even $r$ is a Hamilton path of $Pr^{(2)}$ linking $v$ and $w$ and (M.2.1.2.(2)) for odd $r$ is one of $M^{(2)}$, while if $i$ is odd and $j$ is even, then (Pr.2.1.2.(3)) for even $r$ is a Hamilton path of $Pr^{(2)}$ linking $v$ and $w$ and (M.2.1.2.(3)) for odd $r$ is one of $M^{(2)}$, while if $i$ is odd and $j$ is odd, then (Pr.2.1.2.(4)) for even $r$ is a Hamilton path of $Pr^{(2)}$ linking $v$ and $w$ and (M.2.1.2.(4)) for odd $r$ is one of $M^{(2)}$. This proves both (a16) and (a17) in the Case 2.1.2.

Case 2.2. $\{ v, w \} \cap \{ x_0, \ldots, x_{r-1} \} \neq \emptyset$ and $\{ v, w \} \cap \{ y_0, \ldots, y_{r-1} \} \neq \emptyset$. Since we are within Case 2 we know that $\{ v, w \} \subseteq \{ x_0, \ldots, x_{r-1} \} \cup \{ y_0, \ldots, y_{r-1} \}$. Therefore the statement defining Case 2.2 is the negation of the one defining Case 2.1. Due to $\Phi_{xy}$ we may assume $v = x_i$ with $0 \leq i \leq r - 1$ and $w = y_j$ with $0 \leq j \leq r - 1$. Due to $\Phi_{xz}$ we may further assume that $i < j$.

Case 2.2.1. $i \in \{0, 1\}$. Not only do both $\text{Aut}(P^{(2)})$ and $\text{Aut}(M^{(2)})$ act transitively on $\{x_0, x_1, y_0, y_1\}$, but it is possible to use this symmetry while still preserving the assumption $i \leq j$ that we already made: namely, if $i = 1$, hence $v = x_1$ and $w = y_j$ with $1 \leq i \leq j$, then $\Psi_{xz}(v) = x_0$ and $\Psi_{xz}(w) = y_{r+1-i}$ (with $y_r := y_0$) and still $0 \leq i < j = r - 1 + i$. Therefore we may further assume that $i = 0$, i.e. $v = x_0$. Now consider the expressions

(Pr.2.2.1.(0)) $x_0x_1x_2 \ldots x_j+1y_j+1y_{j+1}x_{j+2} \ldots x_{r-2}x_{r-1}y_{r-3} \ldots y_1y_0$.

(Pr.2.2.1.(1)) $x_0x_1x_2 \ldots x_{j+1}y_{j+1}x_{j+2} \ldots x_{r-2}x_{r-1}y_{r-3} \ldots y_1y_0$.

(Pr.2.2.1.(0)) $x_0x_1x_2 \ldots x_j+1y_j+1y_{j+1}x_{j+2} \ldots y_r \ldots y_1y_0$.

(Pr.2.2.1.(1)) $x_0x_1x_2 \ldots x_{j+1}y_{j+1}x_{j+2} \ldots y_r \ldots y_1y_0$.

If $j$ is even, then (Pr.2.2.1.(0)), and if $j$ is odd then (Pr.2.2.1.(1)) is a Hamilton path of $Pr^{(2)}$ linking $v$ and $w$, for every even $r \geq 4$. If $j$ is even, then (M.2.2.1.(0)), and if $j$ is odd then (M.2.2.1.(1)) is a Hamilton path of $M^{(2)}$ linking $v$ and $w$, for every odd $r \geq 4$. This proves (a16) in the Case 2.2.1.

Case 2.2.2. $i \notin \{0, 1\}$. Now consider the expressions

(Pr.2.2.2.(0)) $x_1x_2 \ldots x_{j+1}y_{j+1}x_{j+2} \ldots x_{r-2}x_{r-1}y_{r-3} \ldots y_1y_0$.

(Pr.2.2.2.(1)) $x_1x_2 \ldots x_{j+1}y_{j+1}x_{j+2} \ldots x_{r-2}x_{r-1}y_{r-3} \ldots y_1y_0$.
Since the automorphism $\Psi_{xx}$ changes the parity of the index of an $x_i$, and since (as explained in Case 2.2.1) the relation $i \leq j$ is preserved when applying $\Psi_{xx}$, we may assume that $i$ is even.

If $j$ is even, then (Pr.2.2.2.(0)), and if $j$ is odd then (Pr.2.2.2.(1)) is a Hamilton path of $P_{r^2}$ linking $v$ and $w$, for every even $r \geq 4$. If $j$ is even, then (M.2.2.2.(0)), and if $j$ is odd then (M.2.2.2.(1)) is a Hamilton path of $M_{r^2}$ linking $v$ and $w$, for every odd $r \geq 5$, completing the Case 2.2.2.

Since at each level of the case distinction the property defining the preceding level was partitioned into mutually exclusive properties, both (a16) and (a17) have now been proved.

As to (a18) and (a19), let $\{v, w\} \subseteq V(P_{r^2})$ be arbitrary distinct vertices. For a large part (i.e. for a large majority of instances of the property of being Hamilton connected) it is possible to deduce the Hamilton-connectedness of $Pr_{r^2}$ and $M_{r^2}$ from (the proof of) (a16) in Lemma 17: if $\{v, w\} \cap \{z', z''\} = \emptyset$, then we have $\{v, w\} \subseteq V(Pr_r)\backslash\{z\}$ and therefore each Hamilton path $P$ in $Pr_r$ or $M_r$ linking $v$ and $w$ contains $z$ as a vertex of degree two. This implies that $P$ can be extended to a Hamilton path in $Pr_{r^2}$ linking $v$ and $w$.

If on the contrary $\{v, w\} \cap \{z', z''\} \neq \emptyset$, then there are subcases: if $\{v, w\} = \{z', z''\}$, then $z'x_0y_0y_1\ldots y_{r-1}x_{r-1}x_{r-2}\ldots x_1z''$ is—in $Pr_r$ and in $M_r$ as well—a Hamilton path linking $v$ and $w$.

We are left with the case $|\{v, w\} \cap \{z', z''\}| = 1$. In the absence of information distinguishing $v$ from $w$ we may assume that $u \in \{z', z''\}$ and $w \notin \{z', z''\}$. One may treat this case, too, by re-using Hamilton paths in $Pr_r$ or $M_r$, now it can make a difference (for the extendability) how such Hamilton path looks like around the ‘special’ subgraph induced on the vertices $\{z, x_0, y_0, x_1, y_1\}$ and it therefore seems quicker to treat this case directly. Since the property $u \in \{z', z''\}$ and $w \notin \{z', z''\}$, at face value, still comprises several cases, we should reduce their number via automorphisms. However—essentially due to $x_0z''$ and the unique degree-5 vertex $x_0$ caused by it—both $Aut(Pr_{r^2})$ and $Aut(M_{r^2})$ are trivial. But since Hamilton-connectedness is a monotone graph property, it suffices to prove that $Pr_{r^2}^{-} := Pr_{r^2} - x_0z''$ and $M_{r^2}^{-} := M_{r^2} - x_0z''$ are Hamilton-connected, and these graphs do have symmetries again, essentially the same as $Pr_{r^2}$ and $M_{r^2}$.

The automorphism group of both $Pr_{r^2}^{-}$ and $M_{r^2}^{-}$ is the group generated by the two unique homomorphic extensions of $(\{z', z'' \to z', z'' \to y_0, x_0, y_0, x_1, y_1\})$ and $(\{z', z'' \to z', z'' \to x_0, y_0, x_1, y_1\})$ to all of $V(Pr_{r^2}^{-}) = V(M_{r^2}^{-})$ (thus both $Aut(Pr_{r^2}^{-})$ and $Aut(M_{r^2}^{-})$ are isomorphic to the Klein four-group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$). These extensions are involutions on $V(Pr_{r^2}^{-}) = V(M_{r^2}^{-})$ and will be denoted by $\Xi_{xy}$ (the map with $z' \leftrightarrow z'$, $z'' \leftrightarrow z''$ and $x_1 \leftrightarrow y_1$ for every $0 \leq i \leq r - 1$) and $\Xi_{xx}$ (the map with $z' \leftrightarrow z'$ and, for $u \in \{x, y\}$, $u_1 \leftrightarrow u_0$, $u_2 \leftrightarrow u_{r-2}$, $u_3 \leftrightarrow u_{r-1}$, $u_i \leftrightarrow u_{i+1}$ for $0 \leq i \leq r-1$). Both $\Xi_{xy}$ and $\Xi_{xx}$ are automorphisms of both $M_{r^2}^{-}$ (for every $r \geq 5$) and $Pr_{r^2}^{-}$ (for every $r \geq 4$).

Since $\Xi_{xx}$ interchanges $z'$ and $z''$, we may assume that $v = z'$. Then there are two cases left: $w \in \{x_0, y_0, x_1, y_1\}$ and its negation $w \in \{x_2, y_2, x_3, y_3, \ldots, x_{r-1}, y_{r-1}\}$ (keep in mind that we already assumed $u \notin \{z', z''\}$ and therefore this indeed is the negation).

Case 1. $w \in \{x_0, y_0, x_1, y_1\}$. Then since $\Xi_{xy}$ maps $x_0 \leftrightarrow y_0$ and $x_1 \leftrightarrow y_1$ while keeping $z'$ fixed, we may assume that $w \in \{x_0, x_1\}$ and are left with two cases.

Case 1.1. If $w = x_0$, then $z'y_0y_1z''x_1x_2x_3y_3x_4\ldots x_{r-2}y_{r-1}x_{r-1}y_0$ is a Hamilton path linking $v$ and $w$ in $Pr_{r^2}^{-}$ for every even $r \geq 4$, and $z'y_0y_1z''x_1x_2y_3y_4x_5\ldots x_{r-2}x_{r-1}y_{r-1}x_0$ is one in $M_{r^2}^{-}$ for every odd $r \geq 5$.

Case 1.2. If $w = x_1$, then $z'y_0y_1y_2y_3y_4\ldots y_2y_1z''x_1$ is a Hamilton path linking $v$ and $w$ in $Pr_{r^2}^{-}$ for every even $r \geq 4$, and $z'y_0y_1y_2y_3y_4\ldots y_2y_1z''x_1$ is one in $M_{r^2}^{-}$ for every odd $r \geq 5$. 

(Pr.2.2.2.(1)) $x_ix_{i+1}\ldots x_jy_j+1y_j+2x_j+2\ldots x_{r-2}y_{r-1}x_{r-1}y_0z_0z_1y_1y_2x_2x_3y_3x_4\ldots x_{r-2}x_{r-1}y_{r-1}y_{r+1}\ldots y_j$

(M.2.2.2.(0)) $x_ix_{i+1}\ldots x_jy_j+1y_j+2x_j+2\ldots x_{r-2}y_{r-1}x_{r-1}y_0z_0z_1y_1y_2x_2x_3y_3x_4\ldots x_{r-2}x_{r-1}y_{r-1}y_{r+1}\ldots y_j$

(M.2.2.2.(1)) $x_ix_{i+1}\ldots x_jy_j+1y_j+2x_j+2\ldots x_{r-2}x_{r-1}y_{r-1}x_0y_0z_0z_1y_1y_2x_2x_3y_3x_4\ldots x_{r-2}x_{r-1}y_{r-1}y_{r+1}\ldots y_j$
Case 2. \( w \in \{x_2, y_2, x_3, y_3, \ldots, x_{r-1}, y_{r-1}\} \). Then since \( \Xi_{xy} \) interchanges the sets \( \{x_0, \ldots, x_{r-1}\} \) and \( \{y_0, \ldots, y_{r-1}\} \) while fixing \( z' \), we may assume that \( w = x_i \) with \( 2 \leq i \leq r - 1 \). If \( i \geq 3 \), then \( z'x_0y_0y_1y_2y_3y_4x_iy_{i+1}y_{i+2}y_{i+3}x_{i+4}y_{i+4}x_{i+5}y_{i+6}x_{i+7}y_{i+7}y_{i+8}y_{i+9}x_iy_0 \), regardless of whether \( i \) is odd or even—a Hamilton path linking \( v \) and \( w \) in both \( \Pr_r^{P(\Xi)} \) and \( \Pr_r^{M(\Xi)} \). In the case that \( i = 2 \), the path \( z'y_0x_0y_1x_1y_2x_2y_3x_3y_4y_5y_6x_7y_7y_8y_9x_0y_0 \) is a Hamilton path linking \( v \) and \( w \) in \( \Pr_r^{P(\Xi)} \), and \( z'y_0x_0y_1y_2x_2y_3x_3y_4y_5y_6x_7y_7y_8y_9x_0y_0 \) is one in \( \Pr_r^{M(\Xi)} \), completing Case 2, and also the proof of both (a18) and (a19).

As to (a20) in the case \( X = \Pr_r^{P(\Xi)} \), for every even \( r \geq 4 \), the \((5 \times 5)\)-minor indexed by \( x_0y_0, x_1y_1, x_2y_2, x_3y_3, x_4y_4 \) of the \( (f_1(\Pr_r^{P(\Xi)}) \times 5) \)-matrix which represents the elements of \( \{c_C : C \in \mathcal{CB}(\Pr_r^{P(\Xi)}) \} \) as elements of \( C_1(\Pr_r^{P(\Xi)}; \mathbb{Z}/2) \), is the one shown in (4).

\[
\begin{pmatrix}
\begin{array}{lllll}
x_0 \wedge y_0 & 1 & 1 & 0 & 1 \\
x_1 \wedge y_1 & 1 & 0 & 1 & 1 \\
z \wedge x_1 & 1 & 0 & 1 & 0 \\
z \wedge y_1 & 0 & 1 & 1 & 0 \\
y_0 \wedge y_{r-1} & 0 & 0 & 1 & 1 \\
\end{array}
\end{pmatrix}
\]

The matrix in (4) is a nonsingular element of \((\mathbb{Z}/2)[5]_2\), its inverse being \( \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \). As to (a20) in the case \( X = \Pr_r^{M(\Xi)} \), for every odd \( r \geq 5 \), the \((5 \times 5)\)-minor indexed by \( x_0y_0, x_1y_1, x_2y_2, x_3y_3, x_4y_4 \) of the \( (f_1(\Pr_r^{M(\Xi)}) \times 5) \)-matrix which represents the elements of \( \{c_C : C \in \mathcal{CB}(\Pr_r^{M(\Xi)}) \} \) as elements of \( C_1(\Pr_r^{M(\Xi)}; \mathbb{Z}/2) \) w.r.t the standard basis of \( C_1(\Pr_r^{M(\Xi)}; \mathbb{Z}/2) \), is the one shown in (5).

\[
\begin{pmatrix}
\begin{array}{lllll}
x_0 \wedge y_0 & 1 & 1 & 0 & 1 \\
x_1 \wedge y_1 & 1 & 0 & 1 & 1 \\
z \wedge x_1 & 1 & 0 & 1 & 0 \\
z \wedge y_1 & 1 & 1 & 0 & 0 \\
x_0 \wedge y_{r-1} & 1 & 1 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]

The matrix in (5) is a nonsingular element of \((\mathbb{Z}/2)[5]_2\), its inverse being \( \begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \). The existence of one such minor by itself proves (a20) in the case \( X = \Pr_r^{P(\Xi)} \). As to (a20) in the case \( X = \Pr_r^{M(\Xi)} \), for every even \( r \geq 4 \), the \((5 \times 5)\)-minor indexed by \( x_0y_0, x_1y_1, z'x_0, z'y_0 \) and \( x_0x_{r-1} \) of the \( (f_1(\Pr_r^{M(\Xi)}) \times 5) \)-matrix which represents the elements of \( \{c_C : C \in \mathcal{CB}(\Pr_r^{M(\Xi)}) \} \) as elements of \( C_1(\Pr_r^{M(\Xi)}; \mathbb{Z}/2) \) w.r.t. the standard basis of \( C_1(\Pr_r^{M(\Xi)}; \mathbb{Z}/2) \), is the one shown in (6).

\[
\begin{pmatrix}
\begin{array}{lllll}
x_0 \wedge y_0 & 0 & 0 & 0 & 1 \\
x_1 \wedge y_1 & 0 & 1 & 1 & 0 \\
z' \wedge x_1 & 1 & 0 & 1 & 1 \\
z' \wedge y_1 & 0 & 0 & 0 & 1 \\
x_0 \wedge x_{r-1} & 0 & 1 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]

The matrix in (6) is a nonsingular element of \((\mathbb{Z}/2)[5]_2\) with inverse \( \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \). The existence of one such minor by itself proves (a20) in the case \( X = \Pr_r^{P(\Xi)} \). As to (a20) in the case \( X = \Pr_r^{M(\Xi)} \), due to the similar definitions in \( \Pr_r^{P(\Xi);\mathbb{ES}} \) and \( \Pr_r^{M(\Xi);\mathbb{ES}} \), it suffices to note that if in the preceding paragraph \( \Pr_r^{P(\Xi)} \) is replaced by \( \Pr_r^{M(\Xi)} \), ‘even \( r \geq 4 \)’ by ‘odd \( r \geq 5 \)’ and ‘\( x_0 \wedge x_{r-1} \)’ by ‘\( x_0 \wedge y_{r-1} \)’, then the matrix obtained is exactly the one in (6). This completes the proof of (a20) in its entirety.
(i, j) \in \prod_{i,j \leq r-1} \{ (i, t-1), (i, t) \} \) and \( A[x_t y_t, C_{ev,r}] := 0 \) for every other \((i, j) \in \{1, \ldots, r - 1\}^2\). This is a band matrix which in particular is lower triangular with its main diagonal filled entirely with ones, hence nonsingular. The existence of one such minor alone implies the claim in the case \( X = Pr_r \). As to the case \( X = Pr_r \), due to the similar definition of \( CE^{(2)}_{Pr_r} \) compared to \( CE^{(2)}_{Pr_5} \), a proof in this case is obtained if in the first paragraph \( Pr_r \) is replaced by \( Pr_5 \), \( C_{ev,r}^{(1)} \) by \( C_{ev,r}^{(1)} \), and \( C_{ev,r}^{(1,1)} \) by \( C_{ev,r}^{(1,1)} \). As to (a12) in the cases \( X = M_r \) (respectively, \( X = M_5 \)), due to the similar definition of \( CE^{(2)}_{M_r} \) compared to \( CE^{(2)}_{Pr_5} \), a proof of these two cases is obtained if in the first paragraph ‘even \( r \geq 4 \)’ is replaced by ‘odd \( r \geq 5 \)’, \( Pr_r \) by \( M_r \), \( C_{ev,r}^{(1,1)} \) by \( C_{ev,r}^{(1,1)} \), and \( C_{ev,r}^{(1,1)} \) by \( C_{ev,r}^{(1,1)} \) (respectively, \( C_{ev,r}^{(1,1)} \)). This completes the proof of (a21) in its entirety.

As to (a22) in the case \( X = Pr_r \), for an arbitrary even \( r \geq 4 \) let \( c \in \langle CE^{(1)}_{Pr_5} / Z_2 \rangle \cap \langle CE^{(2)}_{Pr_5} / Z_2 \rangle \) be arbitrary. Then there exist \( (\lambda_r^{(1)}) \in (Z/2)^{|5]} \) and \( (\lambda_r^{(2)}) \in (Z/2)[r-1] \) such that

\[
\sum_{1 \leq i \leq 5} \lambda_r^{(1)} c_{ev,r,i} = \sum_{1 \leq i \leq r-1} \lambda_r^{(2)} c_{ev,r,i}
\]

where \( c_M \) for some set of edges \( M \) denotes the element \( c \in C_1(Pr_r, Z/2) \) with \( Supp(c) = M \). We now show by contradiction that \( \lambda_r^{(2)} = \cdots = \lambda_r^{(2)} = 0 \), hence \( \langle CE^{(1)}_{Pr_5} / Z_2 \rangle \cap \langle CE^{(2)}_{Pr_5} / Z_2 \rangle = \{ 0 \} \).

To this end, we make the assumption that, on the contrary,

\[
\lambda_r^{(2)} = 1 \quad \text{for at least one} \quad i \leq r - 1.
\]

(7)

Drawing on the facts (straightforward to check using the definitions (P.ES.1) and (P.ES.2)),

\[
\begin{align*}
(F1) \quad & \{x_2 y_2, x_3 y_3, \ldots, x_{r-1} y_{r-1}\} = C_{ev,r,1} \cap C_{ev,r,2} \cap C_{ev,r,3} \cap C_{ev,r,4} \cap C_{ev,r,5}, \\
(F2) \quad & x_0 x_{r-1} \notin C_{ev,r,1} \cap C_{ev,r,2}, \quad x_0 x_{r-1} \notin C_{ev,r,3} \cap C_{ev,r,4} \cap C_{ev,r,5}, \\
(F3) \quad & y_0 y_{r-1} \notin C_{ev,r,1} \cap C_{ev,r,2}, \quad y_0 y_{r-1} \notin C_{ev,r,3} \cap C_{ev,r,4} \cap C_{ev,r,5}, \\
(F4) \quad & \{x_2 y_2, x_3 y_3, \ldots, x_{r-1} y_{r-1}\} \cap C_{ev,r,i} \neq \emptyset \quad \text{for every} \quad 1 \leq i \leq r - 1, \\
(F5) \quad & \{i \in \{1, 2, \ldots, r - 1\} : x_1 y_1 \in C_{ev,r,i}\} = \{1\}, \\
(F6) \quad & \{i \in \{1, 2, \ldots, r - 1\} : x_1 \in C_{ev,r,i}\} = \{1, \ldots, r - 2\}, \\
(F7) \quad & \{i \in \{1, 2, \ldots, r - 1\} : x_1 y_1 \in C_{ev,r,i}\} = \{i - 1, i\} \quad \text{for every} \quad 2 \leq i \leq r - 1, \\
(F8) \quad & \{x_1, x_0 y_0\} \cap C_{ev,r,i} = \emptyset \quad \text{for every} \quad 1 \leq i \leq r - 1, \\
(F9) \quad & \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C_{ev,r,5} \quad \text{for every} \quad 1 \leq i \leq r - 2, \\
(F10) \quad & \{x_0 x_{r-1}, y_0 y_{r-1}\} \cap C_{ev,r,1} = \emptyset
\end{align*}
\]

we can now reason as follows, distinguishing whether \( x_2 y_2 \in Supp(c) \) or not:

Case 1. \( x_2 y_2 \in Supp(c) \). Then (P.Su 1) together with (F1) implies that \( |\{i \in \{1, \ldots, 5\} : \lambda_r^{(1)} = 1\}| \) is odd, and this implies that exactly one of the two numbers \( |\{i \in \{1, 2\} : \lambda_r^{(1)} = 1\}| \) and \( |\{i \in \{3, 4, 5\} : \lambda_r^{(1)} = 1\}| \) is odd, which combined with (P.Su 1), (F2) and (F3) implies that \( |\{x_0 x_{r-1}, y_0 y_{r-1}\} \cap Supp(c)| = 1 \). But this contradicts (P.Su 2), (F9) and (F10), which when taken together imply that \( \{x_0 x_{r-1}, y_0 y_{r-1}\} \cap Supp(c) \neq \emptyset \). This contradiction proves that Case 1 cannot occur (and we have not used our assumption (7) to arrive at this conclusion).

Case 2. \( x_2 y_2 \notin Supp(c) \). From this we deduce

\[
\begin{align*}
(Co 1) \quad & y_1 \notin Supp(c), \\
(Co 2) \quad & |\{i \in \{1, \ldots, 5\} : \lambda_r^{(1)} = 1\}| = 1 \quad \text{is even}, \\
(Co 3) \quad & \{x_2 y_2, x_3 y_3, \ldots, x_{r-1} y_{r-1}\} \subseteq Supp(c) \neq \emptyset, \\
(Co 4) \quad & \lambda_r^{(2)} = \cdots = \lambda_r^{(2)} = 1, \\
(Co 5) \quad & \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq Supp(c) \neq \emptyset, \\
(Co 6) \quad & x_1 \notin Supp(c), \\
(Co 7) \quad & x_1 y_1 \in Supp(c), \\
(Co 8) \quad & x_0 y_0 \notin Supp(c)
\end{align*}
\]

These claims can be justified thus: (Co 1) follows from (P.Su 2) and (F8). (Co 2) follows from combining \( x_2 y_2 \notin Supp(c) \) with (P.Su 1) and (F1). (Co 3) follows from (Co 2), (P.Su 1) and (F1). (Co 4) follows from (Co 3), (P.Su 2), (F4) and (F7), together with our assumption (7). (At this instance we have learned that in (7)—if it is true—the existential quantifier must necessarily hold as a universal quantifier.) (Co 5) follows from (Co 4), (P.Su 2), (F9), (F10) and the evenness of
r = 2. (Co 6) follows from (Co 4), (F6), and the evenness of r = 2 = |{1, ..., r − 2}|. (Co 7) follows from (EE su 2) and (F5). (Co 8) follows from (EE su 2) and (F8).

Now from (Co 5) combined with (F2) and (F3), it follows that (Co 2) cannot be true with both $n_{1,2} := \{|i \in \{1, 2\} : \lambda_{1}^{(1)} = 1\}$ and $n_{3,4,5} := \{|i \in \{3, 4, 5\} : \lambda_{1}^{(1)} = 1\}$ being odd. Therefore both $n_{1,2}$ and $n_{3,4,5}$ must be even. To finish the proof, we use the abbreviations $S_{1,2} := \text{Supp}(\lambda_{1}^{(1)} \cdot c_{c_{ev,r,1}} + \lambda_{2}^{(1)} \cdot c_{c_{ev,r,2}})$ and $S_{3,4,5} := \text{Supp}(\lambda_{3}^{(1)} \cdot c_{c_{ev,r,3}} + \lambda_{4}^{(1)} \cdot c_{c_{ev,r,4}} + \lambda_{5}^{(1)} \cdot c_{c_{ev,r,5}})$, with which we have

$$\text{Supp}(c) = S_{1,2} \triangle S_{3,4,5} \quad \text{(symmetric difference)} \ , \ (8)$$

and distinguish cases according to the value of $n_{1,2} \in \{0, 2\}$.

Case 2.1. $n_{1,2} = 0$. Then in particular $x_{1}y_{1} \notin S_{1,2}$, $zx_{1} \notin S_{1,2}$ and $zy_{1} \notin S_{1,2}$.

Case 2.1.1. $n_{3,4,5} = 0$. This implies that $S_{3,4,5} = \emptyset$, and this together with $x_{1}y_{1} \notin S_{1,2}$ and (8) in particular implies $x_{1}y_{1} \notin \text{Supp}(c)$, contradicting (Co 7) and proving Case 2.1.1 to be impossible.

Case 2.1.2. $n_{3,4,5} = 2$. Let us distinguish whether $\lambda_{5}^{(1)} \in \mathbb{Z}/2$ is 0 or 1 (the motivation for this being that $zy_{1} \notin S_{1,2}$ and among $c_{c_{ev,r,3}}, c_{c_{ev,r,4}}, c_{c_{ev,r,5}}$ only $c_{c_{ev,r,5}}$ contains $y_{1}z_{1}$, making it possible to draw a conclusion from the value of $\lambda_{5}^{(1)}$). If $\lambda_{5}^{(1)} = 1$, then $zy_{1} \in \text{Supp}(\lambda_{3}^{(1)} \cdot c_{c_{ev,r,3}})$ and moreover exactly one of $\lambda_{5}^{(1)}$ and $\lambda_{4}^{(1)}$ is $1$. Whichever it is, due to $zy_{1} \notin \text{Supp}(\lambda_{1}^{(1)} \cdot c_{c_{ev,r,1}})$ and $zy_{1} \notin \text{Supp}(\lambda_{5}^{(1)} \cdot c_{c_{ev,r,5}})$ it follows that $zy_{1} \in S_{3,4,5}$, which combined with $zy_{1} \notin S_{1,2}$ and (8) implies $zy_{1} \in \text{Supp}(c)$, contradicting (Co 1) and proving $\lambda_{5}^{(1)} = 1$ to be impossible. If on the contrary $\lambda_{5}^{(1)} = 0$, then $\lambda_{4}^{(1)} = \lambda_{1}^{(1)} = 1$ and it follows that $x_{1}y_{1} \in S_{3,4,5}$. Being within Case 2.1 we know that $x_{1}y_{1} \notin S_{1,2}$, hence in view of (8) we may conclude that $x_{1}y_{1} \in \text{Supp}(c)$, contradicting (Co 6), proving Case 2.1.2, and therefore Case 2.1 as a whole, to be impossible.

Case 2.2. $n_{1,2} = 2$. This implies $x_{0}y_{0} \notin S_{1,2}$, $x_{1}y_{1} \in S_{1,2}$ and $zx_{1} \in S_{1,2}$. Again it remains to consider the possibilities for $n_{3,4,5} \in \{0, 1, 2, 3\}$ to be even.

Case 2.2.1. $n_{3,4,5} = 0$. Then $S_{3,4,5} = \emptyset$, and this together with $zx_{1} \in S_{1,2}$ and (8) in particular implies $zx_{1} \in \text{Supp}(c)$, contradicting (Co 6) and proving Case 2.2.1 to be impossible.

Case 2.2.2. $n_{3,4,5} = 2$. Again we analyse this case by distinguishing whether $\lambda_{5}^{(1)} \in \mathbb{Z}/2$ is 0 or 1. If $\lambda_{5}^{(1)} = 1$, then exactly one of $\lambda_{5}^{(1)}$ and $\lambda_{4}^{(1)}$ is $1$ and, whichever it is, it follows that $zx_{1} \in S_{3,4,5}$. Being within Case 2.2, we know $x_{1}y_{1} \in S_{1,2}$, hence in view of (8) it follows that $x_{1}y_{1} \notin \text{Supp}(c)$, contradicting (Co 7) and proving $\lambda_{5}^{(1)} = 1$ to be impossible. If on the contrary $\lambda_{5}^{(1)} = 0$, then $\lambda_{4}^{(1)} = \lambda_{1}^{(1)} = 1$ and it follows that $x_{0}y_{0} \in S_{3,4,5}$. Being within Case 2.2 we know that $x_{0}y_{0} \in S_{1,2}$ which in view of (8) implies $x_{0}y_{0} \in \text{Supp}(c)$, contradicting (Co 8) and proving $\lambda_{5}^{(1)} = 0$ to be impossible. This proves Case 2.2.2, and therefore also Case 2.2 and the entire Case 2, to be impossible. Since the mutually exclusive Cases 1 and 2 both lead to contradictions, the assumption (7) is false, completing the proof of (a22) for $X = \text{Pr}_{\mathbb{G}}^{\mathbb{E}}$.

As to (a22) in the case $X = \text{M}_{\mathbb{G}}^{\mathbb{E}}$, the proof given for the case $X = \text{Pr}_{\mathbb{G}}^{\mathbb{E}}$ can be repeated with the appropriate minor changes to obtain a proof in the case $X = \text{M}_{\mathbb{G}}^{\mathbb{E}}$, these changes being the following: first of all, the statements (F1)–(F10) have been chosen in such a way that each of (F1)–(F10) becomes a true statement about the set $\mathcal{B}_{\mathbb{G}}^{\mathbb{E}}$, and in fact the following changes are made in (F1)–(F10): ‘ev’ is to be replaced by ‘od’; ‘$x_{0}x_{r-1}$’ is to be replaced by ‘$y_{0}y_{r-1}$’ (all occurrences, i.e. in (F2), in (F9) and in (F10)), ‘$y_{0}y_{r-1}$’ is to be replaced by ‘$y_{0}y_{r-1}$’ (all occurrences, i.e. in (F3), in (F9) and in (F10)). With the references to (F1)–(F10) now referring to the statements thus modified, the only thing to be done in the entire remaining proof of the case $X = \text{Pr}_{\mathbb{G}}^{\mathbb{E}}$ (in order to arrive at a proof of the case $X = \text{M}_{\mathbb{G}}^{\mathbb{E}}$) is to replace ‘$x_{0}x_{r-1}$’ by ‘$x_{0}y_{r-1}$’ and ‘$y_{0}y_{r-1}$’ by ‘$y_{0}x_{r-1}$’ at all three occurrences of these edges (twice in Case 1, once in (Co 5)), and moreover to replace ‘ev’ by ‘od’. This completes the proof of (a22) for $X = \text{M}_{\mathbb{G}}^{\mathbb{E}}$.
As to (a22) in the case $X = \text{Pr}_r^2$, for an arbitrary even $r \geq 4$ let $c \in \langle \CB^{(1)}_{\text{Pr}_r^2} \rangle_{Z/2} \cap \langle \CB^{(2)}_{\text{Pr}_r^2} \rangle_{Z/2}$ be arbitrary. Then there are $(\lambda^{(1)}) \in (Z/2)^{[5]}$ and $(\lambda^{(2)}) \in (Z/2)^{[r-1]}$ such that

\[
\text{(Su 1) } c = \sum_{i \in [5]} \lambda^{(1)}_i \cdot c_{\text{CB}_{\text{ev},r-1}} \quad \text{and} \quad \text{(Su 2) } c = \sum_{i \in [r-1]} \lambda^{(2)}_i \cdot c_{\text{CB}_{\text{ev},r}} \,
\]

where $C_M$ for some set of edges $M$ denotes the unique element $c \in C_M(\text{Pr}_r^2, Z/2)$ with $\text{Supp}(c) = M$. We will show directly (this time we will not have any use for making the assumption (7)) that $c = 0$, hence $\langle \CB^{(1)}_{\text{Pr}_r^2} \rangle_{Z/2} \cap \langle \CB^{(2)}_{\text{Pr}_r^2} \rangle_{Z/2} = \{0\}$. We can now use the evident facts

\[
\begin{align*}
\text{(F1)} & \ z^z \in \bigcap_{1 \leq i \leq r-1} C_{\text{ev},r}^{[i]} \quad \text{for every } 1 \leq i \leq r-2 \,, \\
\text{(F2)} & \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C_{\text{ev},r}^{[r]} \quad \text{for every } 1 \leq i \leq r-2 \,, \\
\text{(F3)} & \ z^z \not\in C_{\text{ev},r}, z^z \not\in C_{\text{ev},r-2}, z^z \not\in C_{\text{ev},r-3}, z^z \not\in C_{\text{ev},r+4}, z^z \not\in C_{\text{ev},r-5} \,, \\
\text{(F4)} & \text{for even } r \geq 4, \text{ the only circuit among the circuits in } C_{\text{ev},r}^{[r]} \text{ to contain } x_0 y_0 \text{ is } C_{\text{ev},r}^{[r-1]} \,, \\
\text{(F5)} & \text{for even } r \geq 4, \text{ the only circuit among the circuits in } C_{\text{ev},r}^{[r]} \cup C_{\text{ev},r}^{[r+2]} \text{ to contain } y_0 y_0 \text{ is } C_{\text{ev},r}^{[r-5]} \,, \\
\text{(F6)} & \text{for even } r \geq 4, \text{ the only circuit among the circuits in } C_{\text{ev},r}^{[r]} \cup C_{\text{ev},r}^{[r+2]} \text{ to contain } x_0 y_0 \text{ is } C_{\text{ev},r}^{[r+4]} \,, \\
\text{(F7)} & \text{for even } r \geq 4, \text{ the only circuits among the circuits in } C_{\text{ev},r}^{[r]} \cup C_{\text{ev},r}^{[r+2]} \text{ to contain an odd number of the two edges } x_0 x_{r-1} \text{ and } y_0 y_{r-1} \text{ are the two circuits } C_{\text{ev},r-3} \text{ and } C_{\text{ev},r+3} \,. 
\end{align*}
\]

to argue as follows. First of all, we immediately conclude that

\[
\begin{align*}
\text{(Co 1) } & \lambda^{(1)}_4 = 0 \text{ because of (Su 1) and (Su 2) combined with (F6)} \,, \\
\text{(Co 2) } & \lambda^{(1)}_5 = 0 \text{ because of (Su 1) and (Su 2) combined with (F5)} \,. 
\end{align*}
\]

Case 1. \(|i \in \{1, \ldots, r-1\}: \lambda^{(2)}_i = 1\}| \text{ is odd. Then (Su 2) together with (F1) implies } z^z \in \text{Supp}(c). \text{ Therefore, and because of (F3), it follows that exactly one of } \lambda^{(1)}_2 \text{ and } \lambda^{(1)}_4 \text{ is equal to 1, hence } \lambda^{(1)}_2 = 1 \text{ because of (Co 1). Now let us consider } \lambda^{(1)}_1 \text{. It cannot be true that } \lambda^{(1)}_1 = 1 \text{, since then (F7) implies } \lambda^{(1)}_5 = 1 \text{, contradicting (Co 2). Thus we may assume that } \lambda^{(1)}_1 = 0 \text{. This implies } x_1 y_1 \in \text{Supp}(c) \text{ due to (Su 1), } \lambda^{(2)}_2 = 1, \text{ (Co 1) and the fact that . for every even } r \geq 4, \text{ the only circuits among the circuits in } C_{\text{ev},r}^{[r]} \text{ to contain } x_1 y_1 \text{ are } C_{\text{ev},r}^{[r-3]} \text{ and } C_{\text{ev},r+3} \text{. Among the coefficients } \lambda^{(1)}_i, 1 \leq i \leq 5, \text{ only the value of } \lambda^{(1)}_1 \text{ is not yet known to us.} \\
\text{Case 1.1. } & \lambda^{(1)}_1 = 0 \text{. Then } z^z \not\in C_{\text{ev},r}^{[r-1]} \text{, and } \lambda^{(1)}_1 = \lambda^{(1)}_3 = \lambda^{(1)}_5 = 0 \text{ together with (Su 1) imply that } z^z \not\in \text{Supp}(c). \text{ Since for every even } r \geq 4, \text{ the only circuit among the circuits in } C_{\text{ev},r}^{[r]} \text{ to contain } y_0 z^z \text{ is } C_{\text{ev},r}^{[r-1]} \text{, from } z^z \not\in \text{Supp}(c) \text{ it follows that } \lambda^{(2)}_{r-1} = 1. \text{ Being within Case 1, this implies that } |i \in \{1, \ldots, r-2\}: \lambda^{(2)}_i = 1\}| \text{ is even, which by (F2) implies that } \{x_0 x_{r-1}, y_0 y_{r-1}\} \cap \text{Supp}(c) = \emptyset; \text{ but } \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C_{\text{ev},r}^{[r-2]} \text{ together with (Su 1), } \lambda^{(1)}_4 = \lambda^{(1)}_4 = \lambda^{(1)}_5 = 0 \text{ and } \lambda^{(1)}_2 = 1 \text{ implies that, on the contrary, } \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq \text{Supp}(c). \text{ This contradiction proves Case 1.1 to be impossible.} \\
\text{Case 1.2. } & \lambda^{(1)}_1 = 1. \text{ Then } \lambda^{(1)}_{2} = \lambda^{(1)}_4 = \lambda^{(1)}_5 = 0, \lambda^{(1)}_1 = \lambda^{(1)}_2 = 1 \text{ and (Su 1) together imply } x_0 z^z \not\in \text{Supp}(c). \text{ Of course, from (F4), this implies } \lambda^{(2)}_{r-1} = 0. \text{ Being within Case 1, it follows that } |i \in \{1, \ldots, r-2\}: \lambda^{(2)}_i = 1\}| \text{ is even, hence (F2) together with (Su 2) implies that } \{x_0 x_{r-1}, y_0 y_{r-1}\} \cap \text{Supp}(c) = \emptyset; \text{ but } \lambda^{(1)}_1 = \lambda^{(1)}_2 = \lambda^{(1)}_5 = 0, \lambda^{(1)}_1 = \lambda^{(1)}_2 = 1, \text{ and (Su 2), together with the facts that } \{x_0 x_{r-1}, y_0 y_{r-1}\} \cap C_{\text{ev},r+1} = \emptyset \text{ and } \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq C_{\text{ev},r+2} \text{ imply } \{x_0 x_{r-1}, y_0 y_{r-1}\} \subseteq \text{Supp}(c), \text{ contradiction. Therefore Case 1.2 is impossible, too.} \\
\text{This proves the entire Case 1 to be impossible.} \\
\text{Case 2. } & |i \in \{1, \ldots, r-1\}: \lambda^{(2)}_i = 1\}| \text{ is even. Then (Su 2) together with (F1) imply } z^z \not\in \text{Supp}(c), \text{ hence in view of (F3) it follows that either } \lambda^{(2)}_2 = \lambda^{(1)}_1 = 0 \text{ or } \lambda^{(2)}_1 = \lambda^{(1)}_4 = 1, \text{ the latter being impossible because of (Co 1). Therefore, } \lambda^{(2)}_1 = \lambda^{(1)}_4 = 0.\]
Case 2.1. \( \lambda_3^{(1)} = 1 \). This, together with (Su 1), (Su 2), (F7) and the fact that every \( C \in \{C_{\emptyset}^{x_1y_1}, \ldots, C_{\emptyset}^{x_ry_r} : 1 \leq i \leq r-1 \} \) contains an even number of the edges \( x_0x_{r-1} \) and \( y_0y_{r-1} \), implies that we must have \( \lambda_3^{(1)} = 1 \), contradicting (Co 2).

Case 2.2. \( \lambda_3^{(1)} = 0 \). Then (Su 1), \( \lambda_2^{(1)} = 0 \) and the fact that \( C_{\emptyset}^{r} \) and \( C_{\emptyset}^{r+} \) are the only circuits among \( C_{\emptyset}^{r}, \ldots, C_{\emptyset}^{r+} \) to contain \( x_1y_1 \) imply that \( x_1y_1 \notin \text{Supp}(c) \). Hence from (Su 2), together with the fact that for every even \( r \geq 4 \), the only circuit among the circuits in \( \mathcal{CB}_{Pr}^{(2)} \) to contain \( x_1y_1 \) is \( C_{\emptyset}^{x_1y_1} \), it follows that \( \lambda_1^{(2)} = 0 \). Now let us consider \( \lambda_1^{(2)} \). If we would have \( \lambda_1^{(2)} = 1 \), then—being within Case 2—the number \( \{i \in \{2, \ldots, r-2\} : \lambda_{2i}^{(2)} = 1 \} \) is odd, hence \( \{x_0x_{r-1}, y_0y_{r-1}\} \subseteq \text{Supp}(c) \) by (Su 2) and (F2), but this contradicts (Su 1), \( \lambda_2^{(1)} = \lambda_3^{(1)} = 0 \). \( \{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\emptyset}^{r} = \emptyset \), \( \{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\emptyset}^{r+} = \emptyset \) and \( \{x_0x_{r-1}, y_0y_{r-1}\} \cap C_{\emptyset}^{r+} = \emptyset \), which when taken together imply \( \{x_0x_{r-1}, y_0y_{r-1}\} \subseteq \text{Supp}(c) \in \{G, \{x_0, x_{r-1}\}\} \).

Therefore we may assume \( \lambda_1^{(2)} = 0 \). Then—being within Case 2—the number \( \{i \in \{2, \ldots, r-2\} : \lambda_{2i}^{(2)} = 1 \} \) is even, hence (Su 2) and (F2) imply that \( \{x_0x_{r-1}, y_0y_{r-1}\} \subseteq \text{Supp}(c) \). Since among \( C_{\emptyset}^{r}, \ldots, C_{\emptyset}^{r+} \) only \( C_{\emptyset}^{r+} \) contains \( x_0x_{r-1} \), this implies \( \lambda_1^{(3)} = 0 \). We now know that \( \lambda_2^{(2)} = \lambda_3^{(2)} = \lambda_3^{(2)} = 0 \). Therefore, if we would have \( \lambda_1^{(1)} = 1 \), then \( x_1z^p \in \text{Supp}(c) \), contradicting the fact that (Su 2), \( \lambda_2^{(2)} = 0 \), the evenness of \( \{i \in \{2, \ldots, r-2\} : \lambda_{2i}^{(2)} = 1 \} \) and the property \( x_1z^p \in C_{\emptyset}^{x_1y_1} \) for every \( 1 \leq i \leq r \) together imply \( x_1z^p \notin \text{Supp}(c) \). Thus, \( \lambda_1^{(1)} = \lambda_2^{(2)} = \lambda_3^{(2)} = 0 \), \( \lambda_1^{(3)} = \lambda_2^{(3)} = \lambda_3^{(3)} = 0 \), hence \( c = 0 \) by (Su 1), completing the proof of \( \langle \mathcal{CB}_{Pr}^{(1)} \rangle_{Z/2} \cup \langle \mathcal{CB}_{Pr}^{(2)} \rangle_{Z/2} \cap \langle \mathcal{CB}_{Pr}^{(2)} \rangle_{Z/2} \rangle_{Z/2} \) in Case 2. This completes the proof of (a22) in the case \( X = Pr_{r} \). As to (a22) in the case \( X = M_{2r} \), again the proof of the case \( X = Pr_{r} \) can be repeated with the necessary small changes, namely: throughout, ‘Pr.’ is to be replaced by ‘M_{2r},’ ‘ev’ by ‘od,’ \( x_0x_{r-1} \) by \( x_0y_{r-1} \), ‘y_0y_{r-1}’ by ‘y_0x_{r-1}.’ Afterwards, (F1) and (F2) are still true and the proof given for the case \( X = Pr_{r} \) has become a proof for the case \( X = M_{2r} \). The proof of Lemma (a22) is now complete.

As to (a23), \( \emptyset \), note that \( \dim_{Z/2} Z_1(Pr_{r}^{(0)}, Z/2) = (3r + 4) - (2r + 1) + 1 = r + 4 \), and that (a20), (a21) and (a22) in the case \( X = Pr_{r}^{(2)} \) together imply that for even \( r \geq 4 \) we have \( \dim_{Z/2} \left( \langle \mathcal{CB}_{Pr}^{(1)} \rangle_{Z/2} + \langle \mathcal{CB}_{Pr}^{(2)} \rangle_{Z/2} \right) = r + 4 \). Therefore the set \( \langle \mathcal{CB}_{Pr}^{(1)} \rangle_{Z/2} + \langle \mathcal{CB}_{Pr}^{(2)} \rangle_{Z/2} \) is a \( Z/2 \)-linear subspace of \( Z_1(Pr_{r}^{(0)}, Z/2) \) having the same dimension as the ambient space. In a vector space this implies equality as a set. This proves (b)(0). An entirely analogous argument proves (a23), (a)(1).

As to (a23), \( \emptyset \), note that \( \dim_{Z/2} Z_1(Pr_{r}^{(0)}, Z/2) = (3r + 6) - (2r + 2) + 1 = r + 5 \) and that (a20), (a21) and (a22) in the case \( X = Pr_{r}^{(2)} \) together imply that for even \( r \geq 4 \) we have \( \dim_{Z/2} \left( \langle \mathcal{CB}_{Pr}^{(1)} \rangle_{Z/2} + \langle \mathcal{CB}_{Pr}^{(2)} \rangle_{Z/2} \right) = r + 4 \). Since \( \dim K(V / U) = \dim K(V) - \dim K(U) \) for finite-dimensional \( K \)-vectors spaces \( U \subseteq V \), this implies (a)(0). An entirely analogous argument proves (a23), (a)(1).

As to (a23), \( \emptyset \), \( f_0(\cdot) - 1 \), (0), this claim follows quickly from (b)(0): it suffices to note that in \( Pr_{r}^{(2)} \) there actually exists a circuit of length \( f_0(\cdot) - 1 \). Since \( f_0(Pr_{r}^{(2)}) = f_0(Pr_{r}^{(2)}) = r + 4 \) is even for even \( r \), and since the support of the sum of two circuits of even length is an edge-disjoint union of circuits of even length, any circuit of length \( f_0(\cdot) - 1 \) in \( Pr_{r}^{(2)} \) is not contained in \( \langle \mathcal{CB}_{Pr}^{(1)} \rangle_{Z/2} + \langle \mathcal{CB}_{Pr}^{(2)} \rangle_{Z/2} \), hence after adding this circuit to the set \( \mathcal{CB}_{Pr}^{(1)} \cup \mathcal{CB}_{Pr}^{(2)} \), the \( Z/2 \)-linear span has
dimension \((r + 4) + 1 = r + 5 = \dim_{\mathbb{Z}/2}\mathbb{Z}_1(\text{Pr}_r^\square; \mathbb{Z}/2)\), proving \((\mathbb{E}, f_0(\cdot) - 1(0))\), since that finite-dimensional vector spaces do not contain proper subspace of the same dimension. An entirely analogous argumentation proves \((\mathbb{E}, f_0(\cdot) - 1(1))\), this time using \((\mathbb{E}, 1)\).

We have now proved \((a24)-(a29)\): property \((a24)\) follows from \((\mathbb{E}, 0))\) (which is equivalent to \(\text{Pr}_r^\square \in cd_0\mathcal{C}_{f_0(\cdot)}(i))\), \((a16)\) and Definition 16.6); property \((a25)\) follows from \((\mathbb{E}, 1))\) (which is equivalent to \(\mathcal{M}_m^\square \in cd_0\mathcal{C}_{f_0(\cdot))}\), \((a17)\) and Definition 16.6); property \((a26)\) follows from \((\mathbb{E}, 0))\) (which is equivalent to \(\text{Pr}_r^\square \in cd_0\mathcal{C}_{f_0(\cdot)}(i))\), \((a18)\) and Definition 16.6); property \((a27)\) follows from \((\mathbb{E}, 1))\) (which is equivalent to \(\text{Pr}_r^\square \in cd_0\mathcal{C}_{f_0(\cdot)}(i))\), \((a19)\) and Definition 16.6); property \((a28)\) follows from \((\mathbb{E}, 0))\) (which is equivalent to \(\text{Pr}_r^\square \in cd_0\mathcal{C}_{f_0(\cdot)}(i))\), \((a18)\) and Definition 16.6); property \((a29)\) follows from \((\mathbb{E}, f_0(\cdot) - 1(1))\) (which is equivalent to \(\text{Pr}_r^\square \in cd_0\mathcal{C}_{f_0(\cdot)}(i))\), \((a19)\) and Definition 16.6).

As to \((a30)\), the bandwidth of any of \(\mathcal{C}_n^2, \mathcal{CL}_r, \text{Pr}_r^\square, \mathcal{M}_r^\square, \mathcal{M}_r^\square, \mathcal{M}_m^\square\) is constant, i.e. does not grow with \(r\) or \(n\). Therefore \((a30)\) is true in stronger form than is stated here. Since knowing the exact bandwidths would profit us nothing given the proof technology that is available at present, knowing the statement \((a30)\) is enough. To prove it, we employ a general characterization [17, Theorem 8] of low-bandwidth graphs due to J. Böttcher, K. P. Pruessmann, A. Taraz and A. Würfl. This characterization allows us to prove the smallness of the bandwidth for each of the rather different graphs \(\mathcal{C}_n^2, \mathcal{CL}_r, \text{Pr}_r^\square, \mathcal{M}_r^\square, \mathcal{M}_r^\square\) and \(\mathcal{M}_m^\square\) without any close attention to the specifics of these graphs—simply by exhibiting small separators: in \(\mathcal{C}_n^2\) there does not exist any edge between the two sets \(A := \{0, 1, \ldots, (\frac{n}{2}) - 2\}\) and \(B := \{(\frac{n}{2}) + 1, \ldots, n - 3\}\), and since both \(|A|\) and \(|B|\) are \(\leq \frac{2}{3}f_0(\mathcal{C}_n^2)\), the existence of the separator \(S := \{(\frac{n}{2}) - 1, \frac{n}{2}, n - 2, n - 1\}\) implies that the separation number (in the sense of [17, Definition 2]) of \(\mathcal{C}_n^2\) is at most 4. The claim \((a30)\) in the case of \(\mathcal{C}_n^2\) now follows by [17, Theorem 8, equivalence (2) \(\Leftrightarrow (4))\]. To prove the case \(X = \mathcal{CL}_r\) of \((a30)\), in the first sentence of this paragraph use ‘\(A := \bigcup_{i \leq \frac{n}{2} - 1} \{a_i, b_i\}\); ‘\(B := \bigcup_{\frac{n}{2} + 1 \leq i \leq r - 2} \{a_i, b_i\}\); ‘\(S := \{a_0, b_0, a_{\frac{n}{2}}, b_{\frac{n}{2}}\}\).’ To prove the cases \(X \in \{\text{Pr}_r^\square, \mathcal{M}_r^\square\}\) of \((a30)\), in the first sentence of this paragraph use ‘\(A := \{z\} \cup \bigcup_{\frac{n}{2} \leq i \leq \frac{n}{2} - 1} \{x_i, y_i\}\); ‘\(B := \bigcup_{\frac{n}{2} + 1 \leq i \leq r - 1} \{x_i, y_i\}\)’ and ‘\(S := \{a_0, b_0, a_{\frac{n}{2}}, b_{\frac{n}{2}}\}\).’ To prove the cases \(X \in \{\text{Pr}_r^\square, \mathcal{M}_r^\square\}\) of \((a30)\), use \(B\) and \(S\) as in the preceding sentence but ‘\(A := \{z', y'\} \cup \bigcup_{\frac{n}{2} \leq i \leq \frac{n}{2} - 1} \{x_i, y_i\}\).’ This proves the statement about the bandwidth in \((a30)\), for every \(Y \in \{\mathcal{C}_n^2, \mathcal{CL}_r, \text{Pr}_r^\square, \mathcal{M}_r^\square, \mathcal{M}_r^\square, \mathcal{M}_m^\square, \mathcal{M}_m^\square\}\).

As to the additional claims concerning \(Y \in \{\text{Pr}_r^\square, \mathcal{M}_r^\square, \mathcal{M}_m^\square, \mathcal{M}_m^\square\}\), we explicitly give suitable maps \(b_Y\) and \(h_Y\) (thus for \(\text{Pr}_r^\square, \mathcal{M}_m^\square, \mathcal{M}_m^\square, \mathcal{M}_m^\square\) giving another proof of the small bandwidth).

As to \(Y = \text{Pr}_r^\square\), for every even \(r \geq 4\), the map \(b_Y\) defined by \(z' \mapsto 1, x_0 \mapsto 2, x_i \mapsto 4i\) for \(1 \leq i \leq \frac{n}{2}\), \(x_i \mapsto 4(r - i) + 2\) for \(\frac{n}{2} + 1 \leq i \leq r - 1\), \(y_0 \mapsto 3, y_i \mapsto 4i + 1\) for \(1 \leq i \leq \frac{n}{2}\), and \(y_i \mapsto 4(r - i) + 3\) for \(\frac{n}{2} + 1 \leq i \leq r - 1\) is a bandwidth-4-labelling of \(\text{Pr}_r^\square\). Moreover, the map \(h_Y\) defined by \(z \mapsto 0, x_i \mapsto 1\) and \(y_i \mapsto 2\) for even \(0 \leq i \leq r - 1\), \(x_i \mapsto 2\) and \(y_i \mapsto 1\) for odd \(0 \leq i \leq r - 1\), is a 3-colouring of \(\mathcal{M}_r^\square\) which for every \(r\) large enough to have simultaneously \(\beta f_0(Y) = \beta(2r + 1) \geq 1 = h_Y^{-1}(0)\) and \(8 \cdot 2 \cdot \beta \cdot f_0(Y) = 16 \beta(2r + 1) \geq 2\) obviously satisfies the requirement in Theorem 3 of being \((8 \cdot 2 \cdot \beta \cdot f_0(Y), 4 \cdot 2 \cdot \beta \cdot f_0(Y))\)-zero-free w.r.t. \(b_Y\) and having \(h_Y^{-1}(0) \leq \beta f_0(Y)\). This proves \((a30)\) for \(Y = \text{Pr}_r^\square\).

As to \(Y = \mathcal{M}_m^\square\), the same map \(b_Y\) that was defined at the beginning of the preceding paragraph is (this being the reason for having used \([\cdot]\) despite even \(r\)) a bandwidth-5-labelling of \(\mathcal{M}_m^\square\) (which has bandwidth 4, by the way), for every odd \(r \geq 5\). Likewise, the same map \(h_Y\) defined in the preceding paragraph is a 3-colouring of \(\mathcal{M}_r^\square\) for which concerning \(h_Y^{-1}(0)\) and zero-freeness w.r.t. \(b_Y\) exactly the same can be said as in the previous paragraph. This proves \((a30)\) for \(Y = \mathcal{M}_r^\square\).

As to \(Y = \mathcal{M}_m^\square\), for every even \(r \geq 4\), the map \(b_Y\) defined by \(z' \mapsto 1, z'' \mapsto 2, x_0 \mapsto 3, y_0 \mapsto 4, x_i \mapsto 4i + 1\) and \(y_i \mapsto 4i + 2\) for \(1 \leq i \leq \frac{n}{2}\); \(x_i \mapsto 4(r - i) + 3\) and \(y_i \mapsto 4(r - i) + 4\) for
in the case shows that keeps adding Hamilton circuits to the current generating system—but never adds 1 and 1 (cf. [Theorem 23]) Hamilton-generated cycle space, then non-separating induced circuits will play a role in them.

proving Theorem 3.2, for proving Theorem 1 keeps adding Hamilton circuits to the current generating system—but never adds a circuit of length 0 to it—this also implies that in Theorem 1, a single circuit of length 0 succeeds in a generating set. Secondly, with $Pr_r^{2} := Pr_r^{2} - x_0z^n$ and $M_r^{2} := M_r^{2} - x_0z^n$, the study of the special cases $r = 4$ and $r = 6$ strongly suggests that for every even $r ≥ 4$, $\dim_{\mathbb{Z}/2}(Pr_r^{2})/\mathcal{H}(Pr_r^{2})_{\mathbb{Z}/2}) = 2$, but we will not prove this in this paper. The statements $\dim_{\mathbb{Z}/2}(Pr_r^{2})/\mathcal{H}(Pr_r^{2})_{\mathbb{Z}/2}) = 2$, if true in general, provide a justification for employing the symmetry-destroying edge $x_0z^n$: because of these two codimensions, the graphs $Pr_r^{2}$ and $M_r^{2}$—while spanning—are unsuitable as auxiliary substructures for proving (12) in Theorem 1; for when adding an edge, the codimension of the span of Hamilton circuits within the cycle space can at most stay the same, never decrease.

4. AN ALTERNATIVE ARGUMENTATION FOR STEP (St2)

The entire Section 4 is logically superfluous for our proof of Theorem 1. It is included here for two reasons. First, to provide readers with an alternative way of arguing. Secondly, it seems conceivable that if there should ever exist graph-theoretical characterizations of the property of a Hamilton-generated cycle space, then non-separating induced circuits will play a role in them.

The following theorem proved by A. Kelmans will save us work in proving Lemma 24:

**Theorem 23** (cf. [29, Theorem 4.5.2] and [42, p. 264]). If $X$ is a 3-connected graph, it is planar if and only if each $e ∈ E(X)$ is contained in at most two non-separating induced circuits of $X$. □

**Lemma 24.** For every $r ∈ \mathbb{Z}_{≥4}$, the set of nonseparating induced circuits in $Pr_r$ equals

$$N_{\text{nsip}} := \{C_{r,1}\} ∪ \{C_{r,2}\} ∪ \bigsqcup_{0 ≤ i ≤ r-1} \{C_{4,i}\}$$

where $C_{r,1} := x_0x_1 \ldots x_{r-1}x_0$, $C_{r,2} := y_0y_1 \ldots y_{r-1}y_0$ and $C_{4,i} := x_ix_{i+1}y_i+y_{i+1}y_ix_i$. In particular there are exactly $r + 2$ non-separating induced circuits in $Pr_r$.

**Proof.** Inclusion $⊇$ is easy to check. What we have to justify is that (9) is the complete list of non-separating induced circuits in $Pr_r$. This can be done by working directly from the definitions and...
distinguishing cases but we will take a shortcut via Kelmans’ characterization of planar 3-connected graphs: let $C$ be an arbitrary non-separating induced circuit in $\text{Pr}_r$. Suppose that $C$ is missing from (9). Let $e$ be an arbitrary edge of $C$. Since $E(C) \subseteq E(\text{Pr}_r)$, Definition 7 implies that there is $0 \leq j \leq r - 1$ with $e \in \{x_j x_{j+1}, x_j y_j, y_j y_{j+1}\}$. By swapping the symbols $x$ and $y$ if necessary we may assume that there are only the two alternatives $e \in \{x_j x_{j+1}, x_j y_j\}$. If $e = x_j x_{j+1}$, then $C_{r,1}$ and $C_{4,1}$ are two distinct non-separating induced circuits in $\text{Pr}_r$ which contain the edge $e$. Since by assumption $C$ does not appear in (9), $C$ is a third non-separating induced circuit containing $e$. This is where Kelmans’ theorem comes in: it is evident that $\text{Pr}_r$ is planar and also (using Menger’s theorem) that $\text{Pr}_r$ is 3-connected for every $r \in \mathbb{Z}_{\geq 3}$, and therefore Theorem 23 implies that every $e \in E(\text{Pr}_r)$ lies in at most two non-separating induced circuits of $\text{Pr}_r$, a contradiction. Similarly, for the alternative $e = x_j y_j$, the circuits $C_{4,1}$ and $C_{4,2}$ are two distinct non-separating induced circuits in $\text{Pr}_r$ containing $e$. Again, $C$ being a third one is a contradiction to Theorem 23. This proves that none of the non-separating induced circuits of $\text{Pr}_r$ has been forgotten in (9). 

Lemma 25. Let $r \in \mathbb{Z}_{\geq 3}$ be even and

$$H_{w,1} := x_0 x_1 y_1 y_2 x_2 x_3 \ldots x_r x_{r-1} y_{r-1} y_0 x_0 \ ,$$

$$H_{w,2} := y_0 x_1 y_1 x_2 y_2 y_3 \ldots y_{r-1} x_{r-1} x_0 y_0 \ ,$$

$$H_i := x_i x_{i+1} \ldots x_{i+r-1} y_{i+r-1} y_i \ldots y_1 x_i \ .$$

Then every $C \in \text{Ns}_r$ can be expressed as a symmetric difference of some of the Hamilton circuits in \{\{H_{w,1}\} \cup \{H_{w,2}\} \cup \bigsqcup_{0 \leq i \leq r-1} \{H_i\}\}. One way to do this is the following (for the definition of $C_{4,r}$, $C_{r,1}$ and $C_{r,2}$ see (9), and for the notation ‘$c_X$’ see Section 2). Regardless of the value of $r \mod 4$, for every $0 \leq i \leq r-1$,

$$c_{C_{4,1}} = c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}} \ .$$

Moreover, with the abbreviation $\Sigma := \sum_{0 \leq i \leq r-1} c_{H_{2i}}$, if $r \equiv 0 \ (\mod 4)$, then

$$c_{C_{r,1}} = c_{H_{w,1}} + \Sigma \quad \text{and} \quad c_{C_{r,2}} = c_{H_{w,2}} + \Sigma \ ,$$

while if $r \equiv 2 \ (\mod 4)$, then

$$c_{C_{r,1}} = c_{H_{w,2}} + \Sigma \quad \text{and} \quad c_{C_{r,2}} = c_{H_{w,1}} + \Sigma \ .$$

Proof. Among all non-trivial coefficient rings, $\mathbb{Z}/2$ is the only one which has the convenient property that two chains are equal if and only if their supports are. We will make use of this without further mention. We first prove (13) by showing $\text{Supp}(c_{C_{4,1}}) = c_{E(C_{4,1})} = \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$.

As to $E(C_{4,i}) \subseteq \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$ one may argue as follows. There are only three types of $e \in E(C_{4,i})$, namely $e = x_i y_i$, $e = x_i x_{i+1}$ and $e = y_i y_{i+1}$. Regardless of the parity of $i$, an $e = x_i y_i$ is simultaneously in $E(H_{w,1})$, in $E(H_{w,2})$ and in $E(H_{i+1})$. Thus, such an $e$ is in the support of each of the three summands $c_{H_{w,1}}$, $c_{H_{w,2}}$ and $c_{H_{i+1}}$ in (13), and this implies $e \in \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$. For the other two types of $e \in E(C_{4,i})$, we have to pay attention to the parity of $i$: if $i$ is even, then $x_i x_{i+1} \in E(H_{w,1})$, $x_i x_{i+1} \notin E(H_{w,2})$, $y_i y_{i+1} \notin E(H_{w,1})$ and $y_i y_{i+1} \in E(H_{w,2})$, while if $i$ is odd, the latter four statements are true with $\notin$ and $\notin$ interchanged. This shows that, for whatever parity of $i$, both $x_i x_{i+1}$ and $y_i y_{i+1}$ are contained in the support of exactly one of the two summands $c_{H_{w,1}}$ and $c_{H_{w,2}}$. Concerning the third summand $c_{H_{i+1}}$ we see from its definition that regardless of the parity of $i$ the edges $x_i x_{i+1}$ and $y_i y_{i+1}$ are precisely those two edges of the two circuits $x_0 x_1 \ldots x_{r-1} x_0$ and $y_0 y_1 \ldots y_{r-1} y_0$ which are missing from $E(H_{i+1})$. Therefore, for both parities of $i$, for both $e \in \{x_i x_{i+1}, y_i y_{i+1}\}$ we know that $e$ is contained in exactly one support of the three summands $c_{H_{w,1}}$, $c_{H_{w,2}}$ and $c_{H_{i+1}}$ in (13), and therefore $e \in \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$. This completes the proof of $E(C_{4,i}) \subseteq \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$.
As to $E(C_{4,4}) \supseteq \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{w,4}})$ we prove the equivalence inclusion $E(Pr_i) \setminus E(C_{4,4}) \subseteq E(Pr_i) \setminus \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{w,4}})$, thus taking advantage of a less complex description of the left-hand side of the inclusion: the set $E(Pr_i) \setminus E(C_{4,4})$ can be classified into three types of edges, namely $x_iy_i$ for every $i \in [0, r - 1] \setminus \{i, i + 1\}$, and the types $x_i x_{i+1}$ and $y_i y_{i+1}$ for every $i \in [0, r - 1] \setminus \{i, i + 1\}$. As to the type $x_i y_i$, by definition of $H_{i+1}$ we have $x_i y_i \in E(H_{w,1}) \cap E(H_{w,2})$ but $x_i y_i \notin E(H_{i+1})$, and therefore $x_i y_i \notin \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$, for every $i \in [0, r - 1] \setminus \{i, i + 1\}$. As to the types $x_i x_{i+1}$ and $y_i y_{i+1}$ our inspection of the $u \in [0, r - 1] \setminus \{i\}$ has to pay attention to the parity of $u$: for every even $u \in [0, r - 1] \setminus \{i\}$ we have $x_u x_{u+1} \in E(H_{w,1}) \cap E(H_{w,2})$ but $x_u x_{u+1} \notin E(H_{i+1})$, and therefore $x_u x_{u+1} \notin \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$, for every $u \in [0, r - 1] \setminus \{i\}$ we have $y_u y_{u+1} \in E(H_{w,1}) \cap E(H_{w,2})$ but $y_u y_{u+1} \notin E(H_{i+1})$ and therefore $y_u y_{u+1} \notin \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$; for every odd $u \in [0, r - 1] \setminus \{i\}$ we have $x_u x_{u+1} \in E(H_{w,2}) \cap E(H_{i+1})$ but $x_u x_{u+1} \notin E(H_{w,1})$ and therefore $x_u x_{u+1} \notin \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$, while $y_u y_{u+1} \in E(H_{w,1}) \cap E(H_{i+1})$ but $y_u y_{u+1} \notin E(H_{w,2})$ and therefore $y_u y_{u+1} \notin \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$.

All told, none of the edges of the stated types is in $\text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$ and this completes the proof of $E(C_{4,4}) \supseteq \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$.

By the two preceding paragraphs, $\text{Supp}(c_{C_{4,4}}) = E(C_{4,4}) = \text{Supp}(c_{H_{w,1}} + c_{H_{w,2}} + c_{H_{i+1}})$. This completes the proof of (13).

To prove (14) and (15) we will—since both equations involve this sum—first analyse the sum $\Sigma$ by itself, proving five claims which will combine to a proof of (14) and (15).

Claim 1: For every even $r$ and every $i \in [0, r - 1]$ the edge $x_{i_0} y_{i_0}$ lies in exactly one of the following $\frac{r}{2}$ summands ($H_{2i}: 0 \leq i \leq \frac{r}{2} - 1$) comprising $\Sigma$. Proof of Claim 1: Let $i_0 \in [0, r - 1]$ be given. For every $0 \leq i \leq \frac{r}{2} - 1$, there are exactly two edges of type $x_i y_i$ in $H_{2i}$, namely $x_{2i} y_{2i}$ and $x_{2i+1} y_{2i+1}$. Since $2i$ is even and $r$ is even by assumption, $(2i + r - 1)$ mod $r$ is odd. Therefore, if $i_0$ is even, only the edges $x_{2i} y_{2i}$ have the potential to be equal to $x_{i_0} y_{i_0}$, and if $i_0$ is odd, the unique solution $i = \frac{r - 1}{2}$ for the equation $x_{2i} y_{2i} = x_{i_0} y_{i_0}$, while if $i_0$ is odd, only the edges $x_{2i+1} y_{2i+1}$ qualify, and for $i \in [0, r - 1]$, there is the unique solution $i = \frac{r - 1}{2}$ for $x_{2i+1} y_{2i+1} = x_{i_0} y_{i_0}$. This proves Claim 1.

Claim 2: For every even $r \in \mathbb{Z}_{>3}$ and every even $i_0 \in [0, r - 1]$, both $x_{i_0} x_{i_0+1}$ and $y_{i_0} y_{i_0+1}$ lie in each of the $\{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\}$. In particular, they both lie in exactly $\frac{r}{2}$ (supports of) summands of $\Sigma$. In particular, if $r \equiv 0$ (mod 4), then both $x_{i_0} x_{i_0+1}$ and $y_{i_0} y_{i_0+1}$ lie in an even number of supports, and if $r = 2$ (mod 4), they both lie in an odd number of supports. Proof of Claim 2: Let an even $i_0 \in [0, r - 1]$ be given. It has to be shown that both $x_{i_0} x_{i_0+1}$ and $y_{i_0} y_{i_0+1}$ are edges of $H_{2i}$ for every $i \in [0, \frac{r}{2} - 1]$. Since $H_{2i} = x_{2i} x_{2i+1} \ldots x_{2i+r-1} y_{2i+r-1} y_{2i+r-2} \ldots y_{2i} x_{2i}$ it is evident that the only edge of type $x_{i_0} x_{i_0+1}$ which is missing from $E(H_{2i})$ is $x_{i_0} x_{i_0+1}$. If $x_{i_0} x_{i_0+1}$ were equal to $x_{i_0} x_{i_0+1}$, then the evenness of $i_0$ implies $2i = i_0$ and therefore $i_0 + 1 = 2i + r - 1 = i_0 + r - 1$ is even, to be interpreted as an equation in the group $\mathbb{Z}/r$, which because of $r \geq 5$ is a contradiction. Therefore, indeed $x_{i_0} x_{i_0+1} \in E(H_{2i})$. An entirely analogous argument proves that $y_{i_0} y_{i_0+1} \in E(H_{2i})$. Since the two other statements in Claim 2 are mere specializations of the first, the proof of Claim 2 is complete.

Claim 3: For every even $r \in \mathbb{Z}_{>3}$ and every odd $i_0 \in [0, r - 1]$, both $x_{i_0} x_{i_0+1}$ and $y_{i_0} y_{i_0+1}$ lie in each of the $\{H_{2i}: i \in [0, \frac{r}{2} - 1] \setminus \{\frac{r}{2}\}\}$. However, for $i = \frac{2n+1}{2}$ both $x_{i_0} x_{i_0+1} \notin E(H_{2i})$ and $y_{i_0} y_{i_0+1} \notin E(H_{2i})$. In particular, each of $x_{i_0} x_{i_0+1}$ and $y_{i_0} y_{i_0+1}$ lies in exactly $\frac{r}{2} - 1$ (supports of) summands of $\Sigma$. In particular, if $r \equiv 0$ (mod 4), then both $x_{i_0} x_{i_0+1}$ and $y_{i_0} y_{i_0+1}$ lie in an odd number of supports, and if $r = 2$ (mod 4), they both lie in an even number of supports. Proof of Claim 3: Retrace the steps of the proof of Claim 2. Now the equations $x_{2i+r-1} y_{2i} = x_{i_0} x_{i_0+1}$ and $y_{2i+r-1} y_{2i} = y_{i_0} y_{i_0+1}$ do have a (unique) solution $i = \frac{r + 1}{2}$, and this fact is responsible for the exceptional case mentioned in the claim. Since again the other statements are merely specializations of the first, this proves Claim 3.
The motivation for formulating the following statements is that the summands on the right-hand sides have mutually disjoint supports.

Claim 4. For every even \( r \in \mathbb{Z}_{\geq 3} \), if \( r \equiv 0 \pmod{4} \), then \( \Sigma = \sum_{0 \leq i \leq \frac{r}{2} - 1} c_{C_{2i+1}} \), and if \( r \equiv 2 \pmod{4} \), then \( \Sigma = \sum_{0 \leq i \leq \frac{r}{2} - 1} c_{C_{2i+1}} \). Proof of Claim 4. It is evident from the definition of \( C_{2i} \) that the \( \{C_{2i+1}: 0 \leq i \leq \frac{r}{2} - 1\} \) have pairwise disjoint supports. Therefore the support of \( \sum_{0 \leq i \leq \frac{r}{2} - 1} c_{C_{2i+1}} \) is the (disjoint) union of the supports of the \( C_{2i+1} \). Analogously for \( \sum_{0 \leq i \leq \frac{r}{2} - 1} c_{C_{2i+1}} \). Therefore, directly from the definition of \( C_{2i} \) in (9) it follows that

\[
\text{Supp}(\sum_{0 \leq i \leq \frac{r}{2} - 1} c_{C_{2i+1}}) = S_e \quad \text{and} \quad \text{Supp}(\sum_{0 \leq i \leq \frac{r}{2} - 1} c_{C_{2i+1}}) = S_o \text{ where}
\]

\[
S_e := \bigcup_{0 \leq i < \frac{r}{2} - 1} \{x_iy_i^e \} \bigcup \bigcup_{0 \leq i < \frac{r}{2} - 1} \{x_{2i+1}x_{2i+2}, y_{2i+1}y_{2i+2}\},
\]

\[
S_o := \bigcup_{0 \leq i < \frac{r}{2} - 1} \{x_iy_i^o \} \bigcup \bigcup_{0 \leq i < \frac{r}{2} - 1} \{x_{2i+1}x_{2i+2}, y_{2i+1}y_{2i+2}\}.
\]

Hence Claim 4 is equivalent to the statement that

1. if \( r \equiv 0 \pmod{4} \), then \( \text{Supp}(\sum_{0 \leq i < \frac{r}{2} - 1} c_{H_{2i}}) = S_e \).
2. if \( r \equiv 2 \pmod{4} \), then \( \text{Supp}(\sum_{0 \leq i < \frac{r}{2} - 1} c_{H_{2i}}) = S_o \).

That this is so can be deduced from the claims above as follows: As to (1), let \( r \in \mathbb{Z}_{\geq 3} \) with \( r \equiv 0 \pmod{4} \). To prove \( \exists \) in (1), note that by Claim 1 every \( x_0y_0 \) with \( \theta_0 \in \{0, r - 1\} \) is in exactly one of \( \{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\} \), hence \( x_0y_0 \in \text{Supp}(\sum_{0 \leq i \leq \frac{r}{2} - 1} c_{H_{2i}}) \). Moreover, Claim 3 invoked with \( \theta_0 := 2i + 1 \) for \( i \in [0, \frac{r}{2} - 1] \) guarantees that for every \( i \in [0, \frac{r}{2} - 1] \) both \( x_{2i+1}x_{2i+2} \) and \( y_{2i+1}y_{2i+2} \) are in an odd number of \( \{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\} \), hence \( \{x_{2i+1}x_{2i+2}, y_{2i+1}y_{2i+2}\} \subseteq \text{Supp}(\sum_{0 \leq i \leq \frac{r}{2} - 1} c_{H_{2i}}) \) for every \( i \in [0, \frac{r}{2} - 1] \). This proves \( \exists \) in (1). To prove \( \forall \) in (1), we prove \( \text{E}(Pr_{r}) \backslash S_e \subseteq \text{E}(Pr_{r}) \backslash \text{Supp}(\sum_{0 \leq j \leq \frac{r}{2} - 1} c_{H_{2j}}) \). By definition of \( S_e \) we have \( \text{E}(Pr_{r}) \backslash S_e = \{x_2x_{2i+1}: 0 \leq i \leq \frac{r}{2} - 1\} \cup \{y_{2i+1}y_{2i+2}: 0 \leq i \leq \frac{r}{2} - 1\} \), so these are the types of edges whose inclusion in \( \text{E}(Pr_{r}) \backslash \text{Supp}(\sum_{0 \leq j \leq \frac{r}{2} - 1} c_{H_{2j}}) \) we have to justify. Invoking Claim 2 successively with \( \theta_0 := 2i \) for \( i \in [0, \frac{r}{2} - 1] \) it follows that both \( x_{2i+1}x_{2i+2} \) and \( y_{2i+1}y_{2i+2} \) lie in an even number of \( \{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\} \), hence both are missing from \( \text{Supp}(\sum_{0 \leq j \leq \frac{r}{2} - 1} c_{H_{2j}}) \). Since this accounts for all the above mentioned types of edges in \( \text{E}(Pr_{r}) \backslash S_e \), we have proved \( \forall \) in (1). This completes the proof of (1).

As to (2), let \( r \in \mathbb{Z}_{\geq 3} \) with \( r \equiv 2 \pmod{4} \). To prove \( \exists \) in (2), note that Claim 1 guarantees that every \( x_0y_0 \) with \( \theta_0 \in \{0, r - 1\} \) is in exactly one of \( \{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\} \), hence \( x_0y_0 \in \text{Supp}(\sum_{0 \leq i \leq \frac{r}{2} - 1} c_{H_{2i}}) \), too. Moreover, Claim 2 invoked successively with \( \theta_0 := 2i \) for \( i \in [0, \frac{r}{2} - 1] \) now guarantees that both \( x_{2i+1}x_{2i+2} \) and \( y_{2i+1}y_{2i+2} \) lie in an odd number of \( \{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\} \), hence \( \{x_{2i+1}x_{2i+2}, y_{2i+1}y_{2i+2}\} \subseteq \text{Supp}(\sum_{0 \leq i \leq \frac{r}{2} - 1} c_{H_{2i}}) \) for every \( i \in [0, \frac{r}{2} - 1] \), proving \( \exists \) in (2). In order to prove \( \forall \) in (2) we again resort to proving the equivalent inclusion \( \text{E}(Pr_{r}) \backslash S_e \subseteq \text{E}(Pr_{r}) \backslash \text{Supp}(\sum_{0 \leq j \leq \frac{r}{2} - 1} c_{H_{2j}}) \). By definition of \( S_e \) shows that \( \text{E}(Pr_{r}) \backslash S_e = \{x_{2i+1}x_{2i+2}: 0 \leq i \leq \frac{r}{2} - 1\} \cup \{y_{2i+1}y_{2i+2}: 0 \leq i \leq \frac{r}{2} - 1\} \). Appealing to Claim 3 with \( \theta_0 := 2i + 1 \) for every \( i \in [0, \frac{r}{2} - 1] \) we deduce that both \( x_{2i+1}x_{2i+2} \) and \( y_{2i+1}y_{2i+2} \) lie in an even number of the \( \{H_{2i}: 0 \leq i \leq \frac{r}{2} - 1\} \), hence both are missing from \( \text{Supp}(\sum_{0 \leq j \leq \frac{r}{2} - 1} c_{H_{2j}}) \). This accounts for every edge in \( \text{E}(Pr_{r}) \backslash S_e \) and therefore \( \text{E}(Pr_{r}) \backslash S_e \subseteq \text{E}(Pr_{r}) \backslash \text{Supp}(\sum_{0 \leq j \leq \frac{r}{2} - 1} c_{H_{2j}}) \) is true. This completes the proof of \( \forall \) in (2).

The proof of (2) is now complete, as is the proof of Claim 4 as a whole.

Claim 5. For every even \( r \in \mathbb{Z}_{\geq 3} \) we have \( \{x_iy_i: 0 \leq i \leq r - 1\} \cap \text{Supp}(c_{H_{w,1}}) = \emptyset \) and \( \{x_iy_i: 0 \leq i \leq r - 1\} \cap \text{Supp}(c_{H_{w,2}}) = \emptyset \). Proof of Claim 5: By definition of \( H_{w,1} \) and \( H_{w,2} \), both \( \text{Supp}(c_{H_{w,1}}) \) and \( \text{Supp}(c_{H_{w,2}}) \) contain each of the edges \( \{x_iy_i: 0 \leq i \leq r - 1\} \). It follows from the mutual edge-disjointness of the (supports of) the summands on the right-hand sides of the sums
in Claim 4 that \( \text{Supp}(\Sigma) \) for both values of \( r \) mod 4 contains all of these edges, too, and this proves
the emptiness of the intersections in Claim 5.

We finally prove equations (14) and (15). First note that Lemma 25 demands \( r \) to be even from
the outset, hence all appeals to the claims above (all require even \( r \)) are valid.

As to (14), assume that \( r \equiv 0 \pmod 4 \), hence \( \frac{r}{2} \) is even. We first prove \( \text{Supp}(\mathbf{c}_{C_r,1}) = \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \). We begin with \( \text{Supp}(\mathbf{c}_{C_r,1}) \subseteq \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \). In \( E(C_{r,1}) \), there are only
edges of the form \( x_i x_{i+1} \). Of these, we distinguish the types of edges \( x_i x_{i+1} \) with even \( i \) from those
with odd \( i \) and argue as follows: for every \( \text{even } i \in [0, r-1] \), we know by Claim 2 that \( x_i x_{i+1} \)
lies in an even number of \( \{H_{2i} : 0 \leq i \leq \frac{r}{2} - 1\} \), hence \( x_i x_{i+1} \notin \text{Supp}(\Sigma) \), while directly from the
definition of \( H_{w,1} \) we see that \( x_i x_{i+1} \in E(H_{w,1}) \), hence \( x_i x_{i+1} \in \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \). For every \( \text{odd } i \not\equiv 0 \pmod 4 \), we know by Claim 3 that \( x_i x_{i+1} \)
lies in an odd number of \( \{H_{2i} : 0 \leq i \leq \frac{r}{2} - 1\} \), hence \( x_i x_{i+1} \in \text{Supp}(\Sigma) \), while directly from the definition of \( H_{w,1} \) we see that \( x_i x_{i+1} \notin E(H_{w,1}) \),
hence again \( x_i x_{i+1} \in \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \). Since now all edges of \( C_{r,1} \) have been found to lie in
\( \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \), this proves \( \text{Supp}(\mathbf{c}_{C_r,1}) \subseteq \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \).

We now prove \( \text{Supp}(\mathbf{c}_{C_r,1}) \equiv \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \), yet again by proving the equivalent inclusion
\( E(\text{Pr}_r) \setminus \text{Supp}(\mathbf{c}_{C_r,1}) \subseteq E(\text{Pr}_r) \setminus \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \). The only types of edges in \( E(\text{Pr}_r) \setminus \text{Supp}(\mathbf{c}_{C_r,1}) \) are
\( y_i y_{i+1} \) and \( x_i y_i \). As to the former, to justify why \( y_i y_{i+1} \notin \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \) for every \( i \in [0, r-1] \), we may repeat the preceding paragraph verbatim except for interchanging \( x \) and \( y \) and changing
\( \{x_i x_{i+1} \in E(H_{w,1}) \} \) to \( \{y_i y_{i+1} \notin E(H_{w,1}) \} \) and \( \{x_i x_{i+1} \notin E(H_{w,1}) \} \) to \( \{y_i y_{i+1} \in E(H_{w,1}) \} \) to find the
parities work out as they should. As to the type \( x_i y_i \), note that Claim 5 gives just what we need,
namely \( x_i y_i \notin \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \) for every \( i \in [0, r-1] \). The proof of \( \text{Supp}(\mathbf{c}_{C_r,1}) \equiv \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \)
is now complete, as is the proof of \( \text{Supp}(\mathbf{c}_{C_r,1}) = \text{Supp}(\mathbf{c}_{H_{w,1}} + \Sigma) \).

To justify \( \text{Supp}(\mathbf{c}_{C_{r,2}}) \equiv \text{Supp}(\mathbf{c}_{H_{w,2}} + \Sigma) \) in (14) it suffices to change \( \text{Claim } 1 \) into \( \text{Claim } 2 \), \( \text{Claim } 3 \)
and \( \text{Claim } 2 \) into \( \text{Claim } 3 \) and vice versa in the preceding two paragraphs. This completes the proof of (14).

As to (15), assume that \( r \equiv 2 \pmod 4 \), hence \( \frac{r}{2} \) is odd (which affects what Claims 2 and 3 will tell us about the parities of the number of containing supports). A proof for (15) can now be obtained by making obvious modifications in the preceding three paragraphs. The proof of 25 is now complete. \( \square \)

\section{5. Concluding Remarks}

\subsection{5.1. Two open questions and the state of contemporary knowledge.}

The formulation of Theorem 1 suggests further improvements (e.g. eliminating the lower bound on \( f_0 \), proving non-asymptotic minimum-degree thresholds, and finding an infinite set of counter-examples disproving the weakened implications for every \( f_0 \), instead of only for \( f_0 = 7 \) and \( f_0 = 12 \) as was done in the sections 3.1.2 and 3.1.3 above).

In particular, the author does not know whether the threshold in (11) can be lowered to the
Dirac threshold itself. Two noteworthy open questions in that regard are:

\begin{itemize}
  \item \textbf{(Q1)} Let \( X \) be a graph with \( f_0(X) \) odd and \( \delta(X) \geq \frac{1}{2} f_0(X) \). Does it follow that its cycle space is
generated by its Hamilton circuit? \( ? \)
  \item \textbf{(Q2)} Let \( X \) be square bipartite with \( \delta(X) \geq \frac{1}{2} f_0(X) + 1 \). Does it follow that \( \text{CL}_{\frac{1}{2} f_0(X)} \mapsto X \)?
\end{itemize}

If (Q2) has a positive answer then by arguments entirely analogous to those that were used to
prove Theorem 1, it would follow immediately that (B3) in Theorem 1 remains true when \( \delta(X) \geq \frac{1}{2} f_0(X) \) is replaced by \( \delta(X) \geq \frac{1}{2} f_0(X) + 1 \). There is a theorem of A. Czygrinow and
H. A. Kierstead [25, Theorem 1] which proves that if \( X \) is a sufficiently large square bipartite graph,
then \( \delta(X) \geq \frac{1}{2} f_0(X) + 1 \) implies that \( X \) contains a spanning copy of the non-cyclic ladder \( NC_{L_r} \) (i.e.
\( CL_r \) with the two edges \( \{u_{r-1}, b_0\} \) and \( \{a_0, b_{r-1}\} \) removed). This small defect is enough to render
this spanning subgraph unsuitable for serving as an auxiliary substructure in the same way \( CL_r \) has
done in the present paper: while the non-cyclic ladder still is Hamilton-laceable, the loss of the two
edges causes a drastic drop in the dimension of $\langle H(\cdot) \rangle_{\mathbb{Z}/2}$: whereas $\text{CL}_r \in \text{cd}_0 C_{\langle f_0(\cdot) \rangle}$ by (a15), it can be checked that $\text{NCL}_r$ contains only one Hamilton circuit, hence $\text{NCL}_r \in \text{cd}_{\delta_1(\text{NCL}_r)} C_{\langle f_0(\cdot) \rangle}$.

In the pursuit of Question (Q1) one should be aware of the following (probably known) implication (which without requiring $f_0(X)$ to be odd would be false):  

**Lemma 26** (above the Dirac threshold, a graph with odd order is Hamilton connected). Every graph $X$ with $f_0(X)$ odd and $\delta(X) \geq \frac{1}{2} f_0(X)$ is Hamilton-connected.

*Proof.* This is an immediate corollary of a theorem of A. S. Asratian, O. A. Ambartsumian and G. V. Sarkisian [8] which states that every connected graph $X$ with $f_0(X) \geq 3$ and the property $|N_X(u)| + |N_X(w)| \geq |N_X(u) \cup N_X(v) \cup N_X(w)| + 1$ for each of those $(u, v, w) \in V(X)^3$ which have $\emptyset \neq N_X(u) \cap N_X(v) \ni w$, is Hamilton-connected. It is evident that the present hypothesis of $\delta(X) \geq \frac{1}{2} f_0(X)$ and odd $f_0(X)$ makes the assumptions of this theorem true (in fact, our hypothesis makes them true for every $(u, v, w) \in V(X)^3$). \hfill $\square$

Question (Q1) seems not to have been explicitly asked in the literature. There is, however, the aforementioned Conjecture 2, which according to [49, Reference 1] [51, Reference 3] dates back to 1979 and apparently is still open. For $n := f_0(X) = 2d$, Conjecture 2 asks for a generating system consisting of Hamilton circuits together with all circuits shorter by one. For the case of even $n = 2d$, these additional circuits are clearly necessary, but the point of Question (Q1) is that for odd $n := 2d + 1$ it seems quite possible to make do solely with Hamilton circuits (instead of the three lengths $2d - 1$, $2d$ and $2d + 1 = f_0(X)$ allowed by Bondy’s conjecture), all the more so as Theorem 1 of the present paper gives an asymptotic affirmative answer to (Q1). The only papers explicitly addressing Bondy’s conjecture apparently are [37] [49] [50] [51] [10] [2] [3]. We will briefly consider each of them. In [37, p. 246], Conjecture 2 is merely mentioned at the end as a related open conjecture. In [49, Theorem 2 and Corollary 4] it is proved that for every $d \in \mathbb{Z}$, if $X$ is a 3-connected graph with $\delta(X) \geq d$ which is either non-hamiltonian or has $f_0(X) \geq 4d - 5$, then $Z_4(X; \mathbb{Z}/2)$ is generated by its circuits of length at least $2d - 1$ (note that if $f_0(X) \geq 4d - 5$, the conclusion in Bondy’s conjecture is far from generashed by Hamilton circuits). The paper [50] does not have the cycle space as its main concern but announces the results of [49] at the very end. Moreover, the concern of [51] is the question if and when there are inclusions $CO_{\pi'} \subseteq \text{cd}_0 C_{\pi'}$ for different sets of lengths $\pi'$ and $\pi''$; consequently the paper is not concerned with minimum-degree conditions and Conjecture 2 is mentioned merely in passing [51, p. 77]. In [10] the assumption about non-hamiltonicity appears in a different role, but it can be proved that the results of [10] do not answer (Q1):  

**Theorem 27** (Barovich–Locke [10, Theorem 2.2]). Let $d \in \mathbb{Z}$, let $X$ be a finite hamiltonian graph, let $X$ be $3$-connected, $\delta(X) \geq d$ and $f_0(X) \geq 2d - 1$. If $f_0(X) \in \{9, \ldots, 4d - 8\}$, and if there exists at least one $v \in V(X)$ such that $X - v$ is not hamiltonian, and if another condition holds (which to spell out would be irrelevant here), then $Z_4(X; \mathbb{Z}/2)$ is generated by the set of all circuits of length at least $2d - 1$.

The point to be made is that if $f_0(X)$ is odd and $\delta(X) \geq \frac{f_0(X)}{2}$, and if the theorem of Barovich–Locke is to yield generashed by Hamilton circuits, then necessarily we must set $2d - 1 = f_0(X)$. While this automatically makes the hypothesis $f_0(X) \in \{9, \ldots, 4d - 8\}$ true, and while $\delta(X) \geq \frac{f_0(X)}{2}$ ensures (by Dirac’s theorem) that $X$ is hamiltonian and also that $X$ is $3$-connected, the remaining hypothesis of Theorem 27 above cannot possibly be true in the setting of Question (Q1): for every $v \in V(X)$ we have $\delta(X - v) \geq \delta(X) - 1 \geq (\text{since } \delta(X) \text{ is an integer}) \geq \frac{f_0(X)}{2} - 1 = \frac{f_0(X)}{2} - \frac{1}{2} = \frac{1}{2} f_0(X - v)$, hence $X - v$ is still hamiltonian by Dirac’s theorem. Hence Theorem 27, as

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5 The author could not access this article and takes the statement of the theorem from [7, Theorem 3].
it stands, does not answer Question (Q1). Furthermore, in [2] the phrase “in the presence of a long cycle every $k$-path-connected graph is $(k+1)$-generated” [2, Introduction, last paragraph] cannot be construed so as to answer Question (Q1): each of the slightly different ways in which this phrase is made precise by the authors (cf. [2, Corollary 5, Lemmas 9 and 10]) involves additional assumptions one of which always is that there exists a circuit of length $2k - 2$ or $2k - 3$. The existence of such a circuit implies that $(k + 1)$-generated is far from meaning ‘generated by Hamilton circuits’. Finally, [3] is concerned with the same type of question as [51] and Conjecture 2 is again only mentioned in passing [3, p. 12].

5.2. A positive example for Question (Q1). We will now analyse a small yet relevant example which is a positive instance for Question (Q1). It provides an explicit illustration for how a minimum degree just barely satisfying the Dirac threshold can endow a non-Cayley graph with the property of having its cycle space generated by its Hamilton circuits.

Definition 28 (The graph $X$; this is the graph underlying Figure 4.). Let $X$ be the graph defined by $V(X) := \{v_1, \ldots, v_7\}$ and $E(X) := \{v_1v_2, v_1v_3, v_1v_6, v_1v_7, v_2v_3, v_2v_6, v_2v_7, v_3v_4, v_3v_5, v_4v_5, v_4v_6, v_4v_7, v_5v_6, v_5v_7\}$.

Obviously $X$ satisfies the hypotheses in Question (Q1) (but only barely so), and $\dim_{\mathbb{Z}/2}(X; \mathbb{Z}/2) = \beta_1(X) = f_1(X) - f_0(X) + 1 = 14 - 7 + 1 = 8$. Furthermore, because of the following fact we cannot prove that $X$ is a positive instance for Question (Q1) just by appealing to Theorem 15.(2):

Proposition 29. The graph $X$ is not a Cayley graph.

Proof. The order $f_0(X) = 7$ being prime, the only possible underlying group is $\mathbb{Z}/7$ with addition. Now suppose that $X$ were a Cayley graph on $\mathbb{Z}/7$. Since the spectrum of the adjacency matrix of $X$ is $(4, 1, -1, -1, 0, 0, -3) \in \mathbb{Z}^7$, the graph $X$ would then be a quartic connected Cayley graph on an abelian group having only integer adjacency-eigenvalues. But this would contradict a classification theorem due to A. Abdollahi and E. Vatandoost [1, Theorem 1.1] according to which the set of all orders of such graphs is a finite set which does not contain 7. □

Proposition 30 (X is Hamilton-generated). $\langle H(X) \rangle_{\mathbb{Z}/2} = Z_1(X; \mathbb{Z}/2)$.

Proof. Let us explicitly give a $\mathbb{Z}/2$-basis (shown in Figure 4) for $Z_1(X; \mathbb{Z}/2)$ consisting of Hamilton circuits only (there is no particular reason why we choose this basis among several others). Let $C_1^X := v_1v_2v_3v_4v_5v_7v_1$, $C_2^X := v_1v_2v_3v_4v_5v_6v_7$, $C_3^X := v_1v_2v_3v_4v_5v_7v_4v_1v_7$, $C_4^X := v_1v_2v_3v_4v_5v_7v_4v_2v_7$, $C_5^X := v_1v_2v_3v_4v_5v_7v_4v_3v_1$, $C_6^X := v_1v_2v_3v_4v_5v_7v_4v_6v_5v_3v_1$, $C_7^X := v_1v_2v_3v_4v_5v_7v_4v_6v_5v_3v_1$. 

Figure 4. An example of a $\mathbb{Z}/2$-basis for $Z_1(X; \mathbb{Z}/2)$ consisting only of Hamilton circuits in a situation where the underlying graph $X$ is not a Cayley graph and presumably owes its being Hamilton-generated to the Dirac condition (which it satisfies just barely).
$C^X_r := v_1v_7v_2v_6v_5v_4v_3v_1$. All these circuits are Hamilton circuits of $X$. With respect to the standard basis of $C_1(X;\mathbb{Z}/2)$ the (chains of) the Hamilton circuits $C^X_1, \ldots, C^X_8$ give rise to the matrix shown in (17), which has $\mathbb{Z}/2$-rank equal to 8 = dim$\mathbb{Z}/2(Z_1(X;\mathbb{Z}/2))$.

\[
\begin{array}{cccccccc}
  c^X_1 & c^X_2 & c^X_3 & c^X_4 & c^X_5 & c^X_6 & c^X_7 & c^X_8 \\
  v_1 \rightarrow v_2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
  v_2 \rightarrow v_3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
  v_3 \rightarrow v_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  v_4 \rightarrow v_5 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
  v_5 \rightarrow v_6 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
  v_6 \rightarrow v_7 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
  v_7 \rightarrow v_8 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
  v_8 \rightarrow v_1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
  v_1 \rightarrow v_8 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
  v_2 \rightarrow v_7 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
  v_3 \rightarrow v_6 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
  v_4 \rightarrow v_5 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
  v_5 \rightarrow v_4 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
  v_6 \rightarrow v_3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
  v_7 \rightarrow v_2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
  v_8 \rightarrow v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Therefore the span of (the chains of) $C^X_1, \ldots, C^X_8$ is an 8-dimensional subspace of the 8-dimensional $\mathbb{Z}/2$-vector space $Z_1(X;\mathbb{Z}/2)$, hence (this reasoning would not be valid over a general principal ideal domain) is equal to $Z_1(X;\mathbb{Z}/2)$, completing the proof of Proposition 30. \hfill \square

5.3. A group-theoretical question. Let us close by pointing out something else: the graph Pr$_r$ can also be realized as a Cayley graph on the semi-direct product $\mathbb{Z}/2 \times \mathbb{Z}/r$ with $\mathbb{Z}/2$ acting on $\mathbb{Z}/r$ by inversion (this is the usual dihedral group). Therefore, Pr$_r$ is an example of a graph which can simultaneously be realized as a Cayley graph on an abelian and on a non-abelian group. There seems to be nothing known in general about such graphs, and it does not seem hopeless to attempt a classification: Which graphs are simultaneously Cayley graphs on a finite abelian group and on a finite non-abelian group? And what can be deduced in general about the non-abelian groups which admit such a constellation? While for Cayley graphs on infinite non-abelian groups the prospects of reaching a complete classification of those 2-sets of (group,generator)-pairs with isomorphic Cayley graphs seem bleak (a point of departure to this topic can be [26, Section IV.A.9]), the very strong assumption of requiring one of the two groups to be finite abelian might mean that a complete classification of such graphs and such groups can be found.

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