Proportional Cake-Cutting among Families

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Abstract This paper extends the classic cake-cutting problem to a situation in which the "cake" is divided among families. Each piece of cake is owned and used simultaneously by all members of the family. A typical example of such a cake is land. We examine three ways to assess the fairness of such a division, based on the classic proportionality criterion: (a) Average proportionality means that for each family, the value of its share averaged over all family members is at least 1/k (where k is the number of families); (b) Unanimous proportionality means that in each family, each member values the family’s share at least 1/k; (c) Democratic proportionality means that in each family, at least half the members value the family’s share at least 1/k. We compare these criteria based on the number of connected components in the resulting division.

Keywords fair division, cake-cutting, public good, club good, proportionality

This research was funded in part by the following institutions: The Doctoral Fellowships of Excellence Program at Bar-Ilan University, the Mordechai and Monique Katz Graduate Fellowship Program, and the Israel Science Fund grant 1083/13.

We are grateful to Yonatan Aumann, Avinatan Hassidim, Noga Alon, Christian Klamler, Ulle Endriss and Neill Cliff for helpful discussions.

This paper started with a discussion in the MathOverflow website at [http://mathoverflow.net/questions/203060/fair-cake-cutting-between-groups]. We are grateful to the members who participated in the discussion: Pietro Majer, Tony Huynh and Manfred Weis. Other members of the StackExchange network contributed useful answers and ideas: Alex Ravsky, Andrew D. Hwang, BKay, Christian Esholtz, Daniel Fischer, David K, D.W. Hurkyl, Ittay Weiss, Kitssil, Michael Albanese, Raphael Reitzig, Real, babou, Domotor Palvolgyi (domotorp), Ian Turton (iant) and Ivancho.

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1 Introduction

Fair division of land and other heterogeneous resources among agents with different preferences has been an important issue since Biblical times. Today it is an active area of research in the interface of computer science (Robertson and Webb 1998; Procaccia 2015) and economics (Moulin 2004). Its applications range from politics (Brams and Taylor 1996) to multi-agent systems (Chevaleyre et al. 2006).

In most fair division problems, the resource is divided among individual agents, and the fairness of a division is assessed based on the valuation of each agent. A common fairness criterion is proportionality: a division is called **proportional** if each agent values his own share as at least 1/n of the total value, where n is the number of agents (Steinhaus 1948).

However, in real life, goods are often owned and used by groups. As an example, consider a land-estate inherited by k families, or a nature reserve that has to be divided among k states. The land should be divided to k pieces, one piece per group. Each group’s share is then used by all members of the group simultaneously. The land-plot allotted to a family is inhabited by the entire family. The share of the nature-reserve allotted to a state becomes a national park open to all citizens of that state. In economic terms, the allotted piece becomes a “club good” (Buchanan 1965). The happiness of each group member depends on his/her valuation of the entire share of the group. But, in each group there are different people with different valuations. The same division can be considered proportional by some family members and not proportional by other members of the same family. How, then, should the fairness of a division be assessed?

One option that comes to mind is to aggregate the valuations in each family to a single family valuation (also known as: collective welfare function). Following the utilitarian tradition (Bentham 1789), the family-valuation can be defined as the sum or (equivalently) the arithmetic average of the valuations of all family members. We call a division **average-proportional** if every family receives a share with an average value (averaged over all family members) of at least 1/k of its average value of the entire cake. This definition makes the family-division problem easy, since each family can be regarded as a single agent, so the problem reduces to fair division among k agents. Classic results (Steinhaus 1948) imply that an average-proportional division always exists (Section 3).

The problem with average-proportionality is that it makes sense only when the numeric values of the agents’ valuations are meaningful and they are all measured in the same units, e.g. in dollars (see chapter 3 of Moulin (2004) for some real-life examples of such situations). However, if the valuations represent individual happiness measures that cannot be put on a common scale, then their sum is meaningless, and other fairness criteria should be used.

A second option is to require that all members of every family agree that the division is fair. We call a division **unanimous-proportional** if every agent values his/her family’s share as at least 1/k of the total value. The advantage
of this definition is that it does not need to assume that all valuations share a common scale. A unanimous-proportional division always exists (Section 4).

A disadvantage of unanimous-proportionality, compared to average-proportionality, is that unanimous-proportional divisions might be highly fractioned. As an illustration, if the cake is an interval, then there always exists an average-proportional division in which each family receives an interval. However, an unanimous-proportional division in which each family receives an interval might not exist, and moreover, in all unanimous-proportional divisions, the total number of intervals might have to be at least $n -$ the number of agents (Section 4). When the number of agents is large, as in the case of dividing land between states, such divisions might be impractical.

In democratic societies, decisions are almost never accepted unanimously. In fact, when the number of citizens is large, it may be impossible to attain unanimity on even the most trivial issue. The simplest decision rule in such societies is the majority rule. Inspired by this rule, we suggest a third fairness criterion. We call a division democratic-proportional if at least half the citizens in each family value their family’s share as at least $1/k$. This definition can be justified according to the following process. After a division is proposed, each group conducts a referendum in which each citizen approves the division if he/she feels that the division is proportional. The division is implemented only if, in every group, at least half of its members approve it. The democratic-proportionality criterion combines some advantages of the other two criteria. It is similar to unanimous-proportionality in that it does not need to assume that all valuations share a common scale. When there are $k = 2$ families with equal rights, it is similar to average-proportionality in that it can be satisfied with connected pieces - there always exists a democratic-proportional division in which each family receives a single connected piece. An additional advantage of democratic-proportionality in this case is that it can be found by an efficient division protocol (Section 5).

The present paper compares the three fairness criteria in different settings: the number of families can be two or more than two, and the entitlements of the families can be equal or different. In the common case when there are two families with equal entitlements, democratic-proportionality is apparently the most practical criterion, since it guarantees the existence of connected divisions without assuming a common utility scale. Although democratic-fairness might leave up to half the citizens unhappy, this may be unavoidable in real-life situations. This adds an aspect to Winston Churchill’s dictum: “democracy is the worst form of government, except all the others that have been tried”.

The rest of the paper is organized as follows. Section 2 formally presents the model. Sections 3, 4 and 5 analyze the average-proportionality, unanimous-proportionality and democratic-proportionality criteria respectively. Section 6 presents a table comparing the results for the three criteria and presents some

1 In contrast, average-proportional and unanimous-proportional allocations cannot be found by any finite protocol. We omit the details here since the present paper focuses on existence rather than computational efficiency. More details can be found in Segal-Halevi and Nitzan (2016).
open questions. Section 7 briefly discusses some alternative fairness criteria. Comparison to related work is left to Section 8.

2 Model and Notation

There is a resource $C$ ("cake") that has to be divided. As in many papers about fair cake-cutting, $C$ is assumed to be an interval in $\mathbb{R}$.

2.1 Agents and valuations

There are $n$ agents (in all families together).

Every agent $i \in \{1, \ldots, n\}$ has a value measure $V_i$, defined on the Borel subsets of $C$. The $V_i$ are assumed to be absolutely continuous with respect to the length measure (or simply continuous). This implies that all singular points have a value of 0 to all agents (a property often termed non-atomicity). The $V_i$ are also assumed to be additive - the value of a union of two disjoint pieces is the sum of the values of the pieces. Such value functions can be viewed as having a "constant marginal utility" property (Chambers 2005). The value measures are normalized such that $\forall i: V_i(\emptyset) = 0, V_i(C) = 1$. The continuity, additivity and normalization assumptions are common to most cake-cutting papers.

2.2 Families and entitlements

There are $k$ families, denoted by $F_j, j \in \{1, \ldots, k\}$.

The number of agents in $F_j$ is $n_j$. Each agent is a member of exactly one family, so $n = \sum_{j=1}^{k} n_j$.

For each family $j$, there is a positive weight $w_j$, representing the entitlement of this family. The sum of all weights is one: $\sum_{j=1}^{k} w_j = 1$.

In the simplest setting, the families have equal entitlements, i.e, for each $j \in \{1, \ldots, k\}: w_j = 1/k$. Equal entitlements make sense, for example, when $k$ siblings inherit their parents’ estate. While an heir will probably like to take his family’s preferences into account when selecting a share, each heir is entitled to $1/k$ of the estate regardless of the size of his/her family.

In general, each family may have a different entitlement. The entitlement of a family may depend on its size but may also depend on other factors. For example, when two states jointly discover a new island, they will probably want to divide the island between them in proportion to their investment and not in proportion their population.
2.3 Allocations and components

An *allocation* is a vector of \( k \) pieces, \( X = (X_1, \ldots, X_k) \), one piece per family, such that the \( X_j \) are pairwise-disjoint and \( \cup_j X_j = C \).

Each piece is a finite union of intervals. We denote by \( \text{Comp}(X_j) \) the number of connected components (intervals) in the piece \( X_j \), and by \( \text{Comp}(X) \) the total number of components in the allocation \( X \), i.e:

\[
\text{Comp}(X) = \sum_{j=1}^{k} \text{Comp}(X_j)
\]

Ideally, we would like that each piece be connected, i.e., \( \forall i : \text{Comp}(X_i) = 1 \) and \( \text{Comp}(X) = k \). This requirement is especially meaningful when the divided resource is land, since a contiguous piece of land is much easier to use than a collection of disconnected patches.

However, a division with connected pieces is not always possible. Several countries have a disconnected territory. A striking example is the India-Bangladesh border. According to Wikipedia\[^2\] “Within the main body of Bangladesh were 102 enclaves of Indian territory, which in turn contained 21 Bangladeshi counter-enclaves, one of which contained an Indian counter-counter-enclave... within the Indian mainland were 71 Bangladeshi enclaves, containing 3 Indian counter-enclaves”. Another example is Baarle-Hertog - a Belgian municipality made of 24 separate parcels of land, most of which are enclaves in the Netherlands\[^3\].

In case a division with connected pieces is not possible, it is still desirable that the number of connectivity components - \( \text{Comp}(X) \) - be as small as possible. This is a common requirement in the cake-cutting literature. When the cake is an interval, the components are sub-intervals and their number is one plus the number of cuts. Hence, the number of components is minimized by minimizing the number of cuts.\[^{Robertson and Webb 1995; Webb 1997; Shishido and Zeng 1999; Barbanel and Brams 2004, 2014}^\] In a realistic, 3-dimensional world, the additional dimensions can be used to connect the components, e.g., by bridges or tunnels. Still, it is desirable to minimize the number of components in the original division in order to reduce the number of required bridges/tunnels. The goal of minimizing the number of components is also pursued in real-life politics. Going back to India and Bangladesh, after many years of negotiations they finally started to exchange most of their enclaves during the years 2015-2016. This is expected to reduce the number of components from 200 to a more reasonable number.

\[^2\] Wikipedia page “India–Bangladesh enclaves”.
\[^3\] Wikipedia page “Baarle-Hertog”. Many other examples are listed in Wikipedia page “List of enclaves and exclaves”. We are grateful to Ian Turton for the references.
2.4 Three fairness criteria

To define the criterion of average-proportionality, consider the following family-valuation functions:

\[ W^\text{avg}_j(X_j) = \frac{\sum_{i \in F_j} V_i(X_j)}{n_j} \quad \text{for} \quad j \in \{1, \ldots, k\}. \]

An allocation \( X \) is called **average-proportional** if

\[ \forall j \in \{1, \ldots, k\} : W^\text{avg}_j(X_j) \geq w_j \]

An allocation \( X \) is called **unanimous-proportional** if:

\[ \forall j \in \{1, \ldots, k\} : \forall i \in F_j : V_i(X_j) \geq w_j \]

An allocation \( X \) is called **democratic-proportional** if for all \( j \in \{1, \ldots, k\} \), for at least half the members \( i \in F_j \):

\[ V_i(X_j) \geq w_j \]

where \( w_j \) is the entitlement of family \( j \).

Of these three fairness criteria, unanimous-proportionality is clearly the strongest: it implies both average-proportionality and democratic-proportionality. The other two definitions do not imply each other, as shown in the following example.

Consider a land-estate consisting of four districts. It has to be divided between two families: (1) \{Alice,Bob,Charlie\} and (2) \{David,Eva,Frankie\}. The families have equal entitlements, i.e, \( w_1 = w_2 = 1/2 \). Each member’s valuation of each district is shown in the table below:

|       | Alice | 60   | 30   | 3   | 3   |
|-------|-------|------|------|-----|-----|
|       | Bob   | 50   | 40   | 3   | 3   |
|       | Charlie | 10  | 80   | 3   | 3   |
|       | David | 3    | 3    | 60  | 30  |
|       | Eva   | 3    | 3    | 60  | 30  |
|       | Frankie | 3   | 3    | 0   | 90  |

Note that the value of the entire land is 96 according to all agents, so proportionality implies that each family should get at least 48.

If the two leftmost districts are given to family 1 and the two rightmost districts are given to family 2, then the division is **unanimous-proportional**, since each member of each family feels that his family’s share is worth 90. This division is also, of course, average-proportional and democratic-proportional.

If only the single leftmost district is given to family 1 and the other three districts are given to family 2, then the division is still **democratic-proportional**, since Alice and Bob feel that their family received more than 48. However, Charlie feels that his family received only 10, so the division is not unanimous-proportional. Moreover, the division is not average-proportional since the average valuation of family 1 is only \((60+50+10)/3=40\).
If the three leftmost districts are given to family 1 and only the rightmost district is given to family 2, then the division is average-proportional, since family 2’s average valuation of its share is \((30+30+90)/3=50\). However, it is not unanimous-proportional and not even democratic-proportional, since David and Eva feel that their share is worth only 30.

A property of cake partitions is called feasible if for every \(k\) families and \(n\) agents there exists an allocation satisfying this property. Otherwise, the property is called infeasible. In the following sections we study the feasibility of the three fairness criteria in turn.

3 Average fairness

Given any \(n\) additive value functions \(V_i\), the \(k\) family-valuations \(W^\text{avg}_i\) defined above are also additive. Therefore, the family cake-cutting problem can be reduced to the classic problem of cake-cutting among individuals: there are \(k\) individual agents, indexed by \(j \in \{1, \ldots, k\}\), and the valuation of agent \(j\) is the additive value measure \(W^\text{avg}_j\). This implies the following easy positive result:

**Theorem 1** When families have equal entitlements, average-proportionality with connected pieces is feasible.

**Proof** This follows from classic results proving the existence of connected proportional allocations for individual agents \(\text{Steinhaus 1948 Even and Paz 1984}\).

The situation is more difficult with different entitlements, as shown by the following negative result.

**Theorem 2** When families have different entitlements, average-proportionality with connected pieces may be infeasible. Moreover, at least \(2k - 1\) components may be required to attain an average-proportional allocation.

**Proof** Suppose there are \(k\) families, the entitlement of family 1 is \(\frac{s}{k}\) and the entitlement of each of the the other families is \(\frac{1}{k^2+1}\). The cake consists of \(2k - 1\) districts and the average family valuations in these districts are:

| Family 1 | 1 | 0 | 1 | 0 | 1 | 0 | ... | 1 | 0 | 1 |
|----------|---|---|---|---|---|---|-----|---|---|---|
| Family 2 | 0 | 1 | 0 | 0 | 0 | 0 | ... | 0 | 0 | 0 |
| Family 3 | 0 | 0 | 0 | 1 | 0 | 0 | ... | 0 | 0 | 0 |
| Family 4 | 0 | 0 | 0 | 0 | 1 | 0 | ... | 0 | 0 | 0 |
| ...      |   |   |   |   |   |   |     |   |   |   |
| Family \(k\) | 0 | 0 | 0 | 0 | 0 | 0 | ... | 0 | 1 | 0 |

Family 1 must receive more than \((k-1)/k\) of the cake, so it must receive a positive slice of each of its \(k\) positive districts. But, it cannot receive a single
interval that touches two of its positive districts, since such an interval will leave one of the other families with zero value. Therefore, family 1 must receive at least \( k \) components. Each of the other families must receive one component, so the total number of components is at least \( 2k - 1 \).

We do not know if the lower bound of \( 2k - 1 \) is tight even for individual agents. Interestingly, our results on unanimous-proportional division with different entitlements can be used to attain a non-trivial upper bound on the number of cuts required for dividing a cake among \( k \) individuals with different entitlements.

**Lemma 1** Given \( k \) agents with different entitlements, a proportional division with \( \lceil \log_2 k \rceil \cdot (2^k - 2) + 1 \) components is feasible.

**Proof** In Theorem 7 we will prove that, given \( n \) agents in \( k \) families with different entitlements, a unanimous-proportional division with \( \lceil \log_2 k \rceil \cdot (2n - 2) + 1 \) components is feasible. Now, suppose each family has a single member and let \( n = k \).

This immediately implies the same upper bound for average-proportionality:

**Theorem 3** Given \( k \) families with different entitlements, an average-proportional division with \( \lceil \log_2 k \rceil \cdot (2^k - 2) + 1 \) components is feasible.

This matches the lower bound of \( 2k - 1 \) for \( k = 2 \) families, but leaves a gap for \( k \geq 3 \) families.

### 4 Unanimous fairness

Before presenting our results, we note that unanimous-proportionality, like average-proportionality, can also be defined using family-valuation functions. Define:

\[
W_{j}^{\text{min}}(X_j) := \min_{i \in F_j} V_i(X_j) \quad \text{for } j \in \{1, \ldots, k\}.
\]

Then, a division is unanimous-proportional if-and-only-if:

\[
\forall j : W_{j}^{\text{min}}(X_j) \geq w_j
\]

However, in contrast to the functions \( W^{\text{avg}} \) defined in Section 3, the functions \( W^{\text{min}} \) are in general not additive. For example, consider a cake with three districts and a family with the following valuations:

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4 McAvaney et al (1992); Robertson and Webb (1997, 1998) discuss the computational aspect of this question - how many intermediate "cut" marks are required (mainly for two agents). But they do not discuss the existential question of how many cuts are needed in the final division.
|       | $C_1$ | $C_2$ | $C_3$ | $C_1 \cup C_2 \cup C_3$ |
|-------|-------|-------|-------|--------------------------|
| Alice | 1     | 1     | 1     | $3 = 1 + 1 + 1$         |
| Bob   | 0     | 2     | 1     | $3 = 0 + 2 + 1$         |
| Charlie | 0 | 1     | 2     | $3 = 0 + 1 + 2$         |
| $W^{\text{min}}$ | 0 | 1     | 1     | $3 > 0 + 1 + 1$         |

While the individual valuations are additive, $W^{\text{min}}$ is not additive (it is not even subadditive). Therefore, the classic cake-cutting results on proportional cake-cutting cannot be used, and different techniques are needed.

4.1 Exact division

Initially, we assume that the entitlements are equal, i.e: $w_j = 1/k$ for all $j$. We relate unanimous-proportionality to a classic cake-cutting problem of finding an exact division:

**Definition 1** Exact$(N, K)$ is the following problem. Given $N$ agents and an integer $K$, find a division of the cake to $K$ pieces, such that each of the $N$ agents assigns exactly the same value to all pieces:

$$\forall j = 1, \ldots, K : \forall i = 1, \ldots, N : V_i(X_j) = 1/K.$$ 

Exact division is a difficult problem, since it requires all agents to agree on the values of all pieces, not only their own piece. In this section we prove that finding a unanimous-proportional division is similarly difficult: we show a two-way reduction between the problem of unanimous-proportional division and the problem of exact division.

Denote by UnanimousProp$(n, k)$ the problem of finding a unanimous-proportional division when there are $n$ agents grouped in $k$ families with equal entitlements.

4.2 UnanimousProp $\implies$ Exact

**Lemma 2** For every pair of integers $N \geq 1, K \geq 1$, a solution to UnanimousProp $(N(K - 1) + 1, K)$ implies a solution to Exact $(N, K)$.

**Proof** Given an instance of Exact$(N, K)$ ($N$ agents and a number $K$ of required pieces), create $K$ families. Each of the first $K - 1$ families contains $N$ agents with the same valuations as the given $N$ agents. The $K$-th family contains a single agent whose valuation is the average of the $N$ given valuations:

$$V^* = \frac{1}{N} \sum_{i=1}^{N} V_i.$$
The total number of agents in all $K$ families is $N(K - 1) + 1$. Use UnanimousProp $(N(K - 1) + 1, K)$ to find a unanimous-proportional division, $X$. For each agent $i$ in family $j$: $V_i(X_j) \geq 1/K$.

By construction, each of the first $K - 1$ families has an agent with valuation $V_i$. Hence, all $N$ agents value each of the first $K - 1$ pieces as at least $1/K$ and:

$$\forall i = 1, ..., N : \sum_{j=1}^{K-1} V_i(X_j) \geq \frac{K-1}{K}.$$  

Hence, by additivity, every agent values the $K$-th piece as at most $1/K$:

$$\forall i = 1, ..., N : V_i(X_K) \leq 1/K.$$  

The piece $X_K$ is given to the agent with value measure $V^*$, so by proportionality: $V^*(X_K) \geq 1/K$. By construction, $V^*(X_K)$ is the average of the $V_i(X_K)$. Hence:

$$\forall i = 1, ..., N : V_i(X_K) = 1/K.$$  

Again by additivity:

$$\forall i = 1, ..., N : \sum_{j=1}^{K-1} V_i(X_j) = \frac{K-1}{K}.$$  

Hence, necessarily:

$$\forall i = 1, ..., N, \forall j = 1, ..., K - 1 : V_i(X_j) = 1/K.$$  

So we have found an exact division and solved Exact($N, K$) as required.  

Alon (1987) proved that for every $N$ and $K$, an Exact($N, K$) division might require at least $N(K - 1) + 1$ components. Combining this result with the above lemma implies the following negative result:

**Theorem 4** For every $N, K$, let $n = N(K - 1) + 1$. A unanimous-proportional division for $n$ agents in $K$ families might require at least $n$ components.  

This implies that, in particular, unanimous-proportionality with connected pieces is infeasible.

4.3 Exact $\implies$ UnanimousProp

**Lemma 3** For each $n, k$, a solution to Exact ($n - 1, k$) implies a solution to UnanimousProp $(n, k)$ for any grouping of the $n$ agents to $k$ families.
Proof Suppose we are given an instance of UnanimousProp($n, k$), i.e., $n$ agents in $k$ families. Select $n - 1$ agents arbitrarily. Use Exact($n - 1, k$) to find a partition of the cake to $k$ pieces, such that each of the $n - 1$ agents values each of these pieces as exactly $1/k$. Ask the $n$-th agent to choose a favorite piece; by the pigeonhole principle, this value is worth at least $1/k$ for that agent. Give that piece to the family of the $n$-th agent. Give the other $k - 1$ pieces arbitrarily to the remaining $k - 1$ families. The resulting division is unanimous-proportional. 

Alon (1987) proved that for every $N$ and $K$, Exact($N, K$) has a solution with at most $N(K - 1) + 1$ components (at most $N(K - 1)$ cuts). Combining this result with the above lemma implies the following positive result:

**Theorem 5** Given $n$ agents in $k$ families with equal entitlements, a unanimous-proportional division with $(n - 1) \cdot (k - 1) + 1$ components is feasible.

For $k = 2$ families, the positive result of Theorem 5 is $n$, which matches the lower bound of Theorem 4.

For $k > 2$ families, the number of components can be made smaller, as explained in the following subsections.

### 4.4 Less components: equal entitlements

We start with an example. Assume there are $k = 4$ families. By Theorem 5 using $3(n - 1)$ cuts, the cake can be divided to 4 subsets which are considered equal by all $n$ members. But for a unanimous-proportional division, it is not required that all members think that all pieces are equal, it is only required that all members believe that their family’s share is worth at least $1/4$. This can be achieved as follows:

- Divide the cake to two subsets which all $n$ agents value as exactly $1/2$. This is equivalent to solving Exact($n, 2$), which by Alon (1987), can be done with at most $n$ cuts. Call the two resulting subsets West and East.
- Assign arbitrary two families to West and the other two families to East. Mark by $n_W$ the total number of members in the families assigned to West and by $n_E$ the total number of members assigned to East.
- Divide the West to two pieces which all $n_W$ agents value as exactly $1/4$; this can be done with $n_W$ cuts. Give a piece to each family. Divide the East similarly using $n_E$ cuts.

The first step requires $n$ cuts and the second step requires $n_W + n_E = n$ cuts too. Hence the total number of cuts required is only $2n$, rather than $3n - 1$.

In fact, two cuts can be saved in each step by excluding two members (from two different families) from the exact division. These members will not think that the division is equal, but they will be allowed to choose the favorite piece for their family. Thus only $2(n - 2)$ cuts are required. A simple inductive argument shows that whenever $k$ is a power of 2, $(\log_2 k) \cdot (n - k/2)$ cuts are required.
When $k$ is not a power of 2, a result by Stromquist and Woodall (1985) can be used. They prove that, for every fraction $r \in [0, 1]$, it is possible to cut a piece of cake such that all $n$ agents agree that its value is exactly $r$ using at most $2n - 2$ cuts. This can be used as follows:

- Select integers $l_1, l_2 \in \{1, \ldots, k - 1\}$ such that $l_1 + l_2 = k$.
- Apply Stromquist and Woodall (1985) with $r = l_1/k$: using $2n - 4$ cuts, cut a piece $X_1$ that $n - 1$ agents value as exactly $l_1/k$. This means that these $n - 1$ agents value the other piece, $X_2$, as exactly $l_2/k$.
- Let the $n$-th agent choose a piece for his family; assign the other families arbitrarily such that $l_1$ families are assigned to piece $X_1$ and the other $l_2$ families to piece $X_2$.
- Recursively divide piece $X_1$ to its $l_1$ families and piece $X_2$ to its $l_2$ families.

After a finite number of recursion steps, the number of families assigned to each piece becomes 1 and the procedure ends. The number of cuts in each level of the recursion is at most $(2n - 4)$. The depth of recursion can be bounded by $\lceil \log_2 k \rceil$ by dividing $k$ to halves (if it is even) or to almost-halves (if it is odd; i.e. take $l_1 = (k - 1)/2$ and $l_2 = (k + 1)/2$). Hence:

**Theorem 6** Given $n$ agents in $k$ families with equal entitlements, a unanimous-proportional division with $\lceil \log_2 k \rceil \cdot (2n - 4) + 1$ components is feasible.

Note that Theorem 5 and Theorem 6 both give upper bounds on the number of components required for unanimous-proportionality. The bound of Theorem 5 is stronger when $k$ is small and the bound of Theorem 6 is stronger when $k$ is large.

### 4.5 Less components: different entitlements

When the families have different entitlements, the procedure of the previous subsection cannot be used. We cannot let the $n$-th agent select a piece for his family, since the pieces are different. For example, suppose there are two families with entitlements $w_1 = 1/3$, $w_2 = 2/3$. We can divide the cake to two pieces $X_1, X_2$ such that $n - 1$ agents value $X_1$ as 1/3 and $X_2$ as 2/3. So all of them agree that $X_1$ should be given to family 1 and $X_2$ should be given to family 2. But, the $n$-th agent might select the wrong piece for his family. Therefore, the procedure should be modified as follows.

- Select an integer $l \in \{1, \ldots, k\}$.
- Divide the families to two subsets: $F_1, \ldots, F_l$ and $F_{l+1}, \ldots, F_k$.
- Apply Stromquist and Woodall (1985) with $r = \sum_{j=1}^{l} w_j$: using $2n - 2$ cuts, cut a piece $X_1$ which all $n$ agents value as exactly $\sum_{j=1}^{l} w_j$. This means that all $n$ agents value the other piece, $X_2$, as exactly $\sum_{j=l+1}^{k} w_j$.

5 They prove that, if the cake is a circle, the number of connected components is $n - 1$. Hence, the number of cuts is $2n - 2$. This is also true when the cake is an interval, although the number of connected components in this case is $n$. 

5
– Recursively divide piece $X_1$ to $F_1, \ldots, F_l$ and piece $X_2$ to $F_{l+1}, \ldots, F_k$.

Here, the number of cuts in each level of the recursion is at most $(2n - 2)$. The depth of recursion can be bounded by $\lceil \log_2 k \rceil$ by choosing $l = k/2$ (if $k$ is even) or $l = (k - 1)/2$ (if $k$ is odd). Hence:

**Theorem 7** Given $n$ agents in $k$ families with different entitlements, a unanimous-proportional division with $\lceil \log_2 k \rceil \cdot (2n - 2) + 1$ components is feasible.

In concluding the analysis of unanimous-proportionality, recall that, even for $k = 2$ families, unanimous-proportionality is as difficult as exact division and might require the same number of components - $n$. In the worst case, we might need to give a disjoint component to each member, which negates the concept of division to families. Therefore we now turn to the analysis of an alternative fairness criterion that yields more useful results.

### 5 Democratic fairness

Like unanimous-proportionality (Section 4), democratic-proportionality can also be defined using family-valuation functions. Define:

$$W_{med}^j(X_j) := \frac{\text{median}_{i \in F_j} V_i(X_j)}{n_j} \text{ for } j \in \{1, \ldots, k\}.$$  

A division is democratic-proportional if-and-only-if:

$$\forall j : W_{med}^j(X_j) \geq w_j$$

However, the $W_{med}^j$ functions are not additive, so classic cake-cutting results cannot be used.

#### 5.1 Two families: a division procedure

We start with a positive result for two families with equal entitlements, which shows that democratic-proportionality is substantially easier than unanimous-proportionality.

**Theorem 8** When there are $k = 2$ families with equal entitlements, democratic-proportionality with connected pieces is feasible.

**Proof** Algorithm 1 finds a democratic-proportional division between two families. For each family, a location $M_j$ is calculated such that, if the cake is cut at $M_j$, half the members value the interval $[0, M_j]$ as at least $1/2$ and the other half value the interval $[M_j, 1]$ as at least $1/2$. Then, the cake is cut between the two family medians, and each family receives the piece containing

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*See the example in the beginning of Section 4. In that example $W_{med}$ is identical to $W_{min}$.*
its own median. By construction, at least half the members in each family value their family’s share as at least 1/2, so the division is democratic-envy-free. In contrast to the impossibility results of the previous sections, here each family receives a single connected piece.

Algorithm 1 Finding a democratic-envy-free division for two families

INPUT:
- A cake, which is assumed to be the unit interval [0, 1].
- \( n \) additive agents, all of whom value the cake as 1.
- A grouping of the agents to 2 families, \( F_1, F_2 \).

OUTPUT:
A democratic-envy-free division of the cake to 2 pieces.

ALGORITHM:
- Each agent \( i = 1, \ldots, n \) marks an \( x_i \in [0, 1] \) such that \( V_i([0, x_i]) = V_i([x_i, 1]) = 1/2 \).
- For each family \( j = 1, 2 \), find the median of its members’ marks: \( M_j = \text{median}_{i \in F_j} x_i \).
- Find the median of the family medians: \( M^* = (M_1 + M_2)/2 \).
- If \( M_1 < M_2 \) then give \([0, M^*]\) to \( F_1 \) and \([M^*, 1]\) to \( F_2 \).
- Otherwise give \([0, M^*]\) to \( F_2 \) and \([M^*, 1]\) to \( F_1 \).

Unfortunately, this positive result is not applicable when there are more than two families, as shown in the following subsection.

5.2 Three or more families: an impossibility result

Given a specific allocation of cake to families, define a zero agent as an agent who values his family’s share as 0 and a positive agent as an agent who believes his family received a share with a positive value. Note that positivity is a much weaker requirement than proportionality.

Lemma 4 Assume there are \( n = mk \) agents, divided into \( k \) families with \( m \) members in each family. To guarantee that at least \( q \) members in each family are positive, the total number of components may need to be at least:

\[
k \cdot \frac{kq - m}{k - 1}
\]

Proof Number the families by \( j = 0, \ldots, k - 1 \) and the members in each family by \( i = 0, \ldots, m - 1 \). Assume that the cake is the interval \([0, mk]\). In each family \( j \), each member \( i \) wants only the following interval: \((ik + j, ik + j + 1)\). Thus there is no overlap between desired pieces of different members. The table below illustrates the construction for \( k = 2, m = 3 \). The families are \{Alice, Bob, Charlie\} and \{David, Eva, Frankie\}:
Suppose the piece $X_j$ (the piece given to family $j$) is made of $l \geq 1$ components. We can make $l$ members of $F_j$ positive using $l$ intervals of positive length inside their desired areas. However, if $q > l$, we also have to make the remaining $q - l$ members positive. For this, we have to extend $q - l$ intervals to length $k$. Each such extension totally covers the desired area of one member in each of the other families. Overall, each family creates $q - l$ zero members in each of the other families. The number of zero members in each family is thus $(k - 1)(q - l)$. Adding the $q$ members which must be positive in each family, we get the following necessary condition: $(k - 1)(q - l) + q \leq m$. This is equivalent to:

$$l \geq \frac{kq - m}{k - 1}.$$ 

The total number of components is $k \cdot l$, which is at least equal to the expression stated in the Lemma.

In a unanimous-proportional division, all members in each family must be positive. Taking $q = m$ gives $l \geq m$ and the number of components is at least $km = n$, which coincides with the bound of Theorem 1. In a democratic-proportional division, at least half the members in each family must be positive. Taking $q = m/2$ yields the following negative result:

**Theorem 9** In a democratic-proportional division with $n$ agents grouped into $k$ families, the number of components may need to be at least

$$\frac{n \cdot k/2 - 1}{k - 1}$$

Note that for $k = 2$ the lower bound is 0, and indeed we already saw that in this case a connected allocation is feasible.

5.3 Three or more families: positive results

Suppose we do want a democratic-proportional division for three or more families. How many components are sufficient?

As a first positive result, we can use Theorem 9, substituting $n/2$ instead of $n$: select half of the members in each family arbitrarily, then find a division which is unanimous-proportional for them while ignoring all other members. This leads to:

|      | Alice | 1 | 0 | 0 | 0 | 0 | 0 |
|------|-------|---|---|---|---|---|---|
|      | Bob   | 0 | 0 | 1 | 0 | 0 | 0 |
|      | Charlie | 0 | 0 | 0 | 0 | 0 | 1 |
|      | David | 0 | 1 | 0 | 0 | 0 | 0 |
|      | Eva   | 0 | 0 | 0 | 1 | 0 | 0 |
|      | Frankie | 0 | 0 | 0 | 0 | 0 | 1 |
Theorem 10  Given \( n \) agents in \( k \) families with different entitlements, democratic-proportionality with \( \lceil \log_2 k \rceil \cdot (n - 2) + 1 \) components is feasible.

However, for families with equal entitlements we can do much better. Algorithm 2 generalizes Algorithm 1 for any number of families.

**Algorithm 2** Finding a democratic-proportional division for \( k \geq 2 \) families.

**INPUT:**
- A cake, which is assumed to be the unit interval \([0, 1]\).
- \( n \) additive agents, all of whom value the cake as 1.
- A grouping of the agents to \( k \) families, \( F_1, \ldots, F_k \).

**OUTPUT:**
A democratic-proportional division of the cake to \( k \) pieces.

**ALGORITHM:**
- Each agent \( i = 1, \ldots, n \) selects an \( x_i \in [0, 1] \) such that \( V_i([0, x_i]) = \frac{[k/2]}{k} \) (this means \( \frac{1}{2} \) if \( k \) is even and \( \frac{k+1}{2k} \) if \( k \) is odd). Note: \( V_i([x_i, 1]) = \frac{1-k/2}{k} \).
- For each family \( j = 1, \ldots, k \), find the median of its members’ selections: \( M_j = \text{median}_{i \in F_j} x_i \).
- Order the families in increasing order of their medians. Find the median of the family-mediants: \( M^* = M_{\lceil k/2 \rceil} \). Cut the cake at \( x = M^* \).
- Define the western families as the \( F_j \) with \( j = 1, \ldots, \lceil k/2 \rceil \). Let \( n_W \) be the total number of members in these families. Divide the interval \([0, M^*]\) among the western families using \( \text{UnanimousProp}(n_W/2, \lceil k/2 \rceil) \).
- Similarly, define the eastern families as the \( F_j \) with \( j = \lfloor k/2 \rfloor + 1, \ldots, k \). There are \( \lfloor k/2 \rfloor \) such families. Let \( n_E \) be their total number of members. Divide the interval \((M^*, 1]\) among the eastern families using \( \text{UnanimousProp}(n_E/2, \lfloor k/2 \rfloor) \).

The algorithm works in two steps.

**Step 1: Halving.** For each family, a location \( M_j \) is calculated such that, if the cake is cut at \( M_j \), half the family members value the interval \([0, M_j]\) as at least \( \frac{[k/2]}{k} \) and the other half value the interval \([M_j, 1]\) as at least \( \frac{[k/2]}{k} \).

Then, the cake is cut in \( M^* \) - the median of the family medians. The \([k/2]\) “western families” - for which \( M_j \leq M^* \) - are assigned to the western interval of the cake - \([0, M^*]\). By construction, at least half the members in each of the western families value \([0, M^*]\) as at least \( \frac{[k/2]}{k} \). We say that these members are “happy”. Similarly, the \([k/2]\) eastern families - for which \( M_j \geq M^* \) - are assigned to the eastern interval \((M^*, 1]\); at least half the members in each of these families are “happy”, i.e., value the interval \((M^*, 1]\) as at least \( \frac{[k/2]}{k} \).

If there are only two families (\( k = 2 \)), then we are done: there is exactly one western family and one eastern family \(([k/2] = \lfloor k/2 \rfloor = 1)\). For each family \( j \in \{1, 2\} \), at least half the members of each family value their family’s share as at least \( 1/2 \). Hence, the allocation of \( X_j \) to family \( j \) is democratic-proportional.

If there are more than two families (\( k > 2 \)), an additional step is required.

**Step 2: Sub-division.** Each of the two sub-intervals should be further divided to the families assigned to it. In each family \( F_j \), at least \( n_j/2 \) members are happy. So for each \( F_j \), select exactly \( n_j/2 \) members who are happy. Our
goal now is to make sure that these agents remain happy. This can be done using a unanimous-proportional allocation, where only \( n_j/2 \) happy members in each family (hence \( n/2 \) members overall) are counted. The unanimous-proportional allocation guarantees that every western-happy-member believes that his family’s share is worth at least \( \left\lceil \frac{k}{2} \right\rceil \cdot \frac{1}{\left\lceil k/2 \right\rceil} = \frac{1}{k} \). Similarly, every eastern-happy-member believes that his family’s share is worth at least \( \left\lfloor \frac{k}{2} \right\rfloor \cdot \frac{1}{\left\lfloor k/2 \right\rfloor} \). Hence, the resulting division is democratic-proportional.

We now calculate the number of components in the resulting division. One cut is required for the halving step. For the unanimous-proportional division of the western interval, the number of required cuts is at most \( \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right) \cdot (n_W/2 - 1) \) by Theorem\(^5\) and at most \( \log_2 k \cdot (n_W - 4) \) by Theorem\(^6\). Similarly, for the eastern interval the number of required cuts is at most the minimum of \( \left( \left\lfloor \frac{k}{2} \right\rfloor - 1 \right) \cdot (n_E/2 - 1) \) and \( \log_2 k \cdot (n_E - 4) \). The total number of cuts is thus at most \( 1 + \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right) \cdot (n/2 - 2) \) and at most \( 1 + \log_2 k \cdot (n - 8) \). The total number of components is larger by one. We obtain:

**Theorem 11** Given \( n \) agents in \( k \) families with equal entitlements, democratic-proportionality is feasible with at most

\[
\min \left( \left( 2 + \left\lceil \frac{k}{2} \right\rceil \cdot (n/2 - 2) \right), \left( 2 + \log_2 k \cdot (n - 8) \right) \right)
\]

components.

### 6 Comparison and Open Questions

The following table compares the three fairness criteria studied in the present paper, for families with equal entitlements. Recall that \( n \) is the total number of agents in all families.

| Proportionality        | #Families \((k)\) | #Connectivity Components | Lower | Upper |
|------------------------|------------------|--------------------------|-------|-------|
| Average (Sec.\(^3\))   | \( k \)          | \( k \)                   | \( k \) (connected) |
| Unanimous (Sec.\(^4\)) | 2                | \( n \)                   | \( n \)                  |
| Democratic (Sec.\(^5\))| \( k \)          | \( n \)                   | \( \min(1 + \log_2 k \cdot (2n - 4), (k - 1) \cdot (n - 1) + 1) \) |

The case of \( k = 2 \) families is well-understood. The results for all fairness criteria are tight: by all fairness definitions, we know that a fair division exists with the smallest possible number of connectivity components.

The case of \( k > 2 \) families opens some questions:
− Is unanimous-proportionality with $n$ components feasible for all $k$? (particularly, with $k = 3$ families, is the number of required components $n$ as in the lower bound, or $2n - 1$ as in the upper bound?).
− Is democratic-proportionality with $n \cdot \frac{k/2 - 1}{k - 1}$ components feasible for all $k$? (particularly, with $k = 3$ families, is the number of required components $n/4$ as in the lower bound, or $n/2$ as in the upper bound?).

The case of different entitlements is much less understood even for individual agents. As far as we know, it is an open question whether cake-cutting among $k$ individuals with $2k - 1$ components is feasible for $k > 2$. This has direct implications on the number of required components for average-proportionality.

7 Alternatives

Instead of proportionality, it is possible to use envy-freeness as the basic fairness criterion. Envy-freeness means that the valuation of each family in its share should be at least as large as the valuation of the family in another share. Then, average-envy-freeness means that the average value of each family in its allocated share (averaged over all family members) is at least as large as its average value in each of the other shares; unanimous-envy-freeness means that every agent values his family’s share at least as much as any other share; democratic-envy-freeness means that at least half the members in each family believe that their family received the best share. Note that this definition inherently assumes that the families have equal entitlements. Section 3 (the equal-entitlements case) holds as-is for average-envy-freeness. In Theorems 2 and 3, the recursive-halving procedure cannot be used, and the number of components in the positive results is $O(nk)$ instead of $O(n \log k)$. More details are available in Segal-Halevi and Nitzan (2016).

One could consider the following alternative fairness criterion: an allocation is individually-proportional if the allocation $X = (X_1, \ldots, X_k)$ admits a refinement $Y = (Y_1, \ldots, Y_n)$, where for each family $F_j$, $\cup_{i \in F_j} Y_i = X_j$, such that for each agent $i$, $V_i(Y_i) \geq 1/n$. Individually-proportional allocations always exist and can be found by using any classic proportional cake-cutting procedure on the individual agents, disregarding their families. The number of components is at most $n$. Individual-proportionality makes sense if, after the division of the land among the families, each family intends to further divide its share among its members. However, often this is not the case. When an inherited land-estate is divided between two families, the members of each family intend to live and use their entire share together, rather than dividing it among them. Therefore, the happiness of each family member depends on the entire value of his family’s share, rather than on the value of a potential private share he would get in a hypothetic sub-division.
8 Related Work

There are numerous papers about fair division in general and fair cake-cutting in particular. We mentioned some of them in the introduction. Here we survey some work that is more closely related to family-based fairness.

8.1 Group-envy-freeness and on-the-fly coalitions

Berliant et al (1992); Hüsseinov (2011) study the concept of group-envy-free cake-cutting. Their model is the standard cake-cutting model in which the cake is divided among individuals (and not among families as in our model). They define a group-envy-free division as a division in which no coalition of individuals can take the pieces allocated to another coalition with the same number of individuals and re-divide the pieces among its members such that all members are weakly better-off. Coalitions are also studied by Dall’Aglio et al (2009); Dall’Aglio and Di Luca (2012).

In our setting, the families are pre-determined and the agents do not form coalitions on-the-fly. In an alternative model, in which agents are allowed to form coalitions based on their preferences, the family-cake-cutting problem becomes easier. For instance, it is easy to achieve a unanimous-proportional division with connected pieces between two coalitions: ask each agent to mark its median line, find the median of all medians, then divide the agents to two coalitions according to whether their median line is to the left or to the right of the median-of-medians.

8.2 Fair division with public goods

In our setting, the piece given to each family is considered a "public good" in this specific family. The existence of fair allocations of homogeneous goods when some of the goods are public has been studied e.g. by Diamantaras (1992); Diamantaras and Wilkie (1994, 1996); Guth and Kliemt (2002). In these studies, each good is either private (consumed by a single agent) or public (consumed by all agents). In the present paper, each piece of land is consumed by all agents in a single family - a situation not captured by existing public-good models.

8.3 Family preferences in matching markets

Besides land division, family preferences are important in matching markets, too. For example, when matching doctors to hospitals, usually a husband and a wife want to be matched to the same hospital. This issue poses a substantial challenge to stable-matching mechanisms (Klaus and Klijn 2008; Kojima et al 2013; Ashlagi et al 2014).
8.4 Fairness in group decisions

The notion of fairness between groups has been studied empirically in the context of the well-known ultimatum game. In the standard version of this game, an individual agent (the proposer) suggests a division of a sum of money to another individual (the responder), which can either approve or reject it. In the group version, either the proposer or the responder or both are groups of agents. The groups have to decide together what division to propose and whether to accept a proposed division.

Experiments by Robert and Carnevale (1997); Bornstein and Yaniv (1998) show that, in general, groups tend to act more rationally by proposing and accepting divisions which are less fair. Messick et al (1997) studies the effect of different group-decision rules while Santos et al (2015) uses a threshold decision rule which is a generalized version of our majority rule (an allocation is accepted if at least $M$ agents in the responder group vote to accept it).

These studies are only tangentially relevant to the present paper, since they deal with a much simpler division problem in which the divided good is homogeneous (money) rather than heterogeneous (cake/land).

8.5 Non-additive utilities

As explained in Sections 4 and 5, the difficulty with unanimous-proportionality and democratic-proportionality is that the associated family-valuation functions are not additive. It is therefore interesting to compare our work to other works on cake-cutting with non-additive valuations.

Berliant et al (1992); Maccheroni and Marinacci (2003); Dall’Aglio and Maccheroni (2005) focus on sub-additive, or concave, valuations, in which the sum of the values of the parts is more than the value of the whole. These works are not applicable to the family-cake-cutting problem, because the family-valuations are not necessarily sub-additive - the sum of values of the parts might be less than the value of the whole (see the example in the beginning of Section 4).

Sagara and Vlach (2005); Dall’Aglio and Maccheroni (2009); Hüseminov and Sagara (2013) consider general non-additive value functions. They provide pure existence proofs and do not say much about the nature of the resulting divisions (e.g., the number of connectivity components), which we believe is important in practical division applications.

Su (1999) presents a protocol for envy-free division with connected pieces which does not assume additivity of valuations. However, when the valuations are non-additive, there are no guarantees about the value per agent. In particular, with non-additive valuations, the resulting division is not necessarily proportional.

Mirchandani (2013) suggests a division protocol for non-additive valuations using non-linear programming. However, the protocol is practical only when the cake is a collection of a small number of homogeneous components, where the only thing that matters is what fraction of each component is allocated.
to each agent. Our model is the standard, general model where the cake is a single heterogeneous good.

Finally, Berliant and Dunz [2004]; Caragiannis et al [2011]; Segal-Halevi et al [2015] study specific non-additive value functions which are motivated by geometric considerations (location, size and shape). The present paper contributes to this line of work by studying specific non-additive value functions which are motivated by a different consideration: handling the different preferences of family members. A possible future research topic is to find fair division rules that handle these considerations simultaneously, as both of them are important for fair division of land.

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