Partial dynamical symmetries in quantum systems

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Abstract. We discuss the notion of a partial dynamical symmetry (PDS), for which a prescribed symmetry is obeyed by only a subset of solvable eigenstates, while other eigenstates are strongly mixed. We present an explicit construction of Hamiltonians with this property, including higher-order terms, and portray their significance for spectroscopy and shape-phase transitions in nuclei. The occurrence of both a single PDS, relevant to stable structures, and of several PDSs, relevant to coexistence phenomena, are considered.

1. Introduction
Models based on spectrum generating algebras form a convenient framework to examine underlying symmetries in dynamical systems, and have been used extensively in diverse areas of physics [1]. Notable examples in nuclear physics are Wigner’s spin-isospin SU(4) supermultiplets [2], SU(2) single-\(j\) pairing [3], Elliott’s SU(3) model [4], symplectic model [5], Ginocchio’s monopole and quadrupole pairing models [6], interacting boson models (IBM) for even-even nuclei [7] and boson-fermion models (IBFM) for odd-mass nuclei [8]. Similar algebraic techniques have proven to be useful in the structure of molecules [9, 10] and of hadrons [11]. In such models the Hamiltonian is expanded in elements of a Lie algebra, \(G_0\), called the spectrum generating algebra. A dynamical symmetry occurs if the Hamiltonian can be written in terms of the Casimir operators of a chain of nested algebras, \(G_0 \supset G_1 \supset \ldots \supset G_n\) [12]. The following properties are then observed. (i) All states are solvable and analytic expressions are available for energies and other observables. (ii) All states are classified by quantum numbers, \(|\alpha_0, \alpha_1, \ldots, \alpha_n\rangle\), which are the labels of the irreducible representations (irreps) of the algebras in the chain. (iii) The structure of wave functions is completely dictated by symmetry and is independent of the Hamiltonian’s parameters.

A dynamical symmetry provides clarifying insights into complex dynamics and its merits are self-evident. However, in most applications to realistic systems, the predictions of an exact dynamical symmetry are rarely fulfilled and one is compelled to break it. The breaking of the symmetry is required for a number of reasons. First, one often finds that the assumed symmetry is not obeyed uniformly, i.e., is fulfilled by only some of the states but not by others. Certain degeneracies implied by the assumed symmetry are not always realized, e.g., axially deformed nuclei rarely fulfill the IBM SU(3) requirement of degenerate \(\beta\) and \(\gamma\) bands [7]). Secondly, forcing the Hamiltonian to be invariant under a symmetry group may impose constraints which are too severe and incompatible with well-known features of the dynamics (e.g., the models of [6] require degenerate single-nucleon energies). Thirdly, in describing systems in-between two different structural phases, e.g., spherical and deformed nuclei, the Hamiltonian by necessity mixes terms with different symmetry character. In the models mentioned above, the required
symmetry breaking is achieved by including in the Hamiltonian terms associated with (two or more) different sub-algebra chains of the parent spectrum generating algebra. In general, under such circumstances, solvability is lost, there are no remaining non-trivial conserved quantum numbers and all eigenstates are expected to be mixed. A partial dynamical symmetry (PDS) [13] corresponds to a particular symmetry breaking for which some (but not all) of the virtues of a dynamical symmetry are retained. The essential idea is to relax the stringent conditions of complete solvability so that the properties (i)–(iii) are only partially satisfied. It is then possible to identify several types of partial dynamical symmetries. PDS of type I corresponds to a situation where some of the states have all the dynamical symmetry. In this case, properties (i)-(iii) are fulfilled exactly, but by only a subset of states. PDS of type II corresponds to a situation for which all the states preserve part of the dynamical symmetry. In this case, there are no analytic solutions, yet selected quantum numbers (of the conserved symmetries) are retained. PDS of type III has a hybrid character, for which some of the states preserve part of the dynamical symmetry.

In what follows we discuss algorithms for constructing Hamiltonians with partial dynamical symmetries and demonstrate their relevance to quantum systems. For that purpose, we employ the interacting boson model (IBM) [7], widely used in the description of low-lying quadrupole collective states in nuclei in terms of \( N \) interacting monopole (s) and quadrupole (d) bosons representing valence nucleon pairs. The bilinear combinations \( \{ s^i s, s^i d_m, d_m s, d_m d_m' \} \) span a U(6) algebra, which serves as the spectrum generating algebra. The IBM Hamiltonian is expanded in terms of these generators and consists of Hermitian, rotational-scalar interactions which conserve the total number of s- and d- bosons, \( \hat{N} = \hat{n}_s + \hat{n}_d = s^i s + \sum_m d_m s, d_m d_m' \). Three dynamical symmetry limits occur in the model with leading subalgebras U(5), SU(3), and O(6), corresponding to typical collective spectra observed in nuclei, vibrational, rotational, and \( \gamma \)-unstable, respectively. Relevant information on these algebras is collected in Table 1.

### Table 1

| Algebra | Generators | Casimir operator \( \hat{C}_k(G) \) | Eigenvalues \( \langle \hat{C}_k(G) \rangle \) |
|---------|------------|----------------------------------------|---------------------------------|
| O(3)    | \( U^{(1)} \) | \( \hat{L} \cdot \hat{\Lambda} \) | \( L(L+1) \) |
| O(5)    | \( U^{(1)}, U^{(3)} \) | \( 2(U^{(1)} \cdot U^{(1)} + U^{(3)} \cdot U^{(3)}) \) | \( \tau(\tau + 3) \) |
| O(6)    | \( U^{(1)}, U^{(3)}, \Pi^{(2)} \) | \( \hat{C}_2(O(5)) + \Pi^{(2)} \cdot \Pi^{(2)} \) | \( \Sigma(\Sigma + 4) \) |
| SU(3)   | \( U^{(1)}, \hat{Q} \) | \( -4 \sqrt{7\hat{Q} \cdot (\hat{Q} \times \hat{\Lambda})} - \frac{3}{2} \sqrt{3\hat{Q} \cdot (\hat{L} \times \hat{\Lambda})} \) | \( \lambda^2 + (\lambda + \mu)(\mu + 3) \) |
| U(5)    | \( U^{(\ell)} \ell = 0, ..., 4 \) | \( \hat{n}_d, \hat{n}_d^s(\hat{n}_d + 4) \) | \( n_d, n_d(n_d + 4) \) |
| U(6)    | \( U^{(\ell)} \ell = 0, ..., 4 \) | \( \hat{N}, \hat{N}(\hat{N} + 5) \) | \( N, N(N + 5) \) |
| \( \Pi^{(2)}, \Pi^{(2)} \), \( \hat{n}_s \) | | | |
A geometric visualization of the model is obtained by an energy surface

\[ E_N(\beta, \gamma) = \langle \beta, \gamma; N | \hat{H} | \beta, \gamma; N \rangle , \]

defined by the expectation value of the Hamiltonian in the coherent (intrinsic) state \([14, 15]\)

\[ |\beta, \gamma; N \rangle = (N!)^{-1/2}(b^\dagger c)^N |0\rangle , \]

\[ b^\dagger c = (1 + \beta^2)^{-1/2}[\beta \cos \gamma d^\dagger + \beta \sin \gamma (d^\dagger_2 + d^\dagger_{-2})]/\sqrt{1 + s^2} . \]

Here \((\beta, \gamma)\) are quadrupole shape parameters whose values, \((\beta_0, \gamma_0)\), at the global minimum of \(E_N(\beta, \gamma)\) define the equilibrium shape for a given Hamiltonian. The shape can be spherical \((\beta = 0)\) or deformed \((\beta > 0)\) with \(\gamma = 0\) (prolate), \(\gamma = \pi/3\) (oblate), \(0 < \gamma < \pi/3\) (triaxial), or \(\gamma\)-independent. The equilibrium deformations associated with the dynamical symmetry limits are \(\beta_0 = 0\) for \(U(5)\), \((\beta_0 = \sqrt{2}, \gamma_0 = 0)\) for \(SU(3)\) and \((\beta_0 = 1, \gamma_0\) arbitrary) for \(O(6)\).

2. Construction of Hamiltonians with partial dynamical symmetries

PDS of type I corresponds to a situation for which the defining properties of a dynamical symmetry (DS), namely, solvability, good quantum numbers, and symmetry-dicted structure are fulfilled exactly, but by only a subset of states. An algorithm for constructing Hamiltonians with PDS has been developed in [16] and further elaborated in [17]. The analysis starts from the chain of nested algebras

\[ G^\text{dyn} \supset G \supset \cdots \supset G^\text{sym} \]

\[ \downarrow [h] \quad \downarrow \langle \Sigma \rangle \quad \downarrow \Lambda \]

(3)

where, below each algebra, its associated labels of irreps are given. Eq. (3) implies that \(G^\text{dyn}\) is the dynamical (spectrum generating) algebra of the system such that operators of all physical observables can be written in terms of its generators; a single irrep of \(G^\text{dyn}\) contains all states of relevance in the problem. In contrast, \(G^\text{sym}\) is the symmetry algebra and a single of its irreps contains states that are degenerate in energy. Assuming, for simplicity, that particle number is conserved, then all states, and hence the representation \([h]\), can then be assigned a definite particle number \(N\). For \(N\) identical particles the representation \([h]\) of the dynamical algebra \(G^\text{dyn}\) is either symmetric \([N]\) (bosons) or antisymmetric \([1^N]\) (fermions) and will be denoted, in both cases, as \([h_N]\). The occurrence of a DS of the type (3) signifies that the Hamiltonian is written in terms of the Casimir operators of the algebras in the chain,

\[ \hat{H}_{DS} = \sum_G a_G \hat{C}(G) , \]

(4)

and its eigenstates can be labeled as \([|h_N\rangle \langle \Sigma \rangle \cdots \Lambda]\); additional labels (indicated by \(\cdots\)) are suppressed in the following. The eigenvalues of the Casimir operators in these basis states determine the eigenenergies \(E_{DS}([h_N]\langle \Sigma \rangle \Lambda)\) of \(\hat{H}_{DS}\). Likewise, operators can be classified according to their tensor character under (3) as \(\hat{T}^\Lambda_{[h_n]([\sigma]_\Lambda)}\).

Of specific interest in the construction of a PDS associated with the reduction (3), are the \(n\)-particle annihilation operators \(\hat{T}\) which satisfy the property

\[ \hat{T}^\Lambda_{[h_n]([\sigma]_\Lambda)}|[h_N]\langle \Sigma_0 \rangle \Lambda] = 0 , \]

(5)

for all possible values of \(\Lambda\) contained in a given irrep \(\langle \Sigma_0 \rangle\) of \(G\). Equivalently, this condition can be phrased in terms of the action on a lowest weight (LW) state of the G-irrep \(\langle \Sigma_0 \rangle\),

\[ \hat{T}^\Lambda_{[h_n]([\sigma]_\Lambda)}|LW; [h_N]\langle \Sigma_0 \rangle\rangle = 0 , \]

(6)
from which states of good Λ can be obtained by projection. Any $n$-body, number-conserving normal-ordered interaction written in terms of these annihilation operators and their Hermitian conjugates (which transform as the corresponding conjugate irreps),

$$\hat{H}' = \sum_{\alpha,\beta} A_{\alpha\beta} \hat{T}^\dagger_\alpha \hat{T}_\beta,$$

(7)

has a partial $G$-symmetry. This comes about since for arbitrary coefficients, $A_{\alpha\beta}$, $\hat{H}'$ is not a $G$-scalar, hence most of its eigenstates will be a mixture of irreps of $G$, yet relation (5) ensures that a subset of its eigenstates $[[h_N]|(\Sigma_0)\Lambda)$, are solvable and have good quantum numbers under the chain (3). An Hamiltonian with partial dynamical symmetry is obtained by adding to $\hat{H}'$ the dynamical symmetry Hamiltonian, $\hat{H}_{DS}$ (4), still preserving the solvability of states with $(\Sigma) = (\Sigma_0)$,

$$\hat{H}_{PDS} = \hat{H}_{DS} + \hat{H}'.$$

(8)

If the operators $\hat{T}_{[h_n]}(\sigma)_{\lambda}$ span the entire irrep $(\sigma)$ of $G$, then the annihilation condition (5) is satisfied for all Λ-states in $(\Sigma_0)$, if none of the $G$ irreps $(\Sigma)$ contained in the $G_{dyn}$ irrep $[h_{N-n}]$ belongs to the $G$ Kronecker product $(\sigma) \times (\Sigma_0)$. So the problem of finding interactions that preserve solvability for part of the states (3) is reduced to carrying out a Kronecker product.

The arguments for choosing the special irrep lies in Eq. (5), which contains the solvable states, are based on physical grounds. A frequently encountered choice is the irrep which contains some of the symmetries $G$ symmetry, and hence will have the ground state of the system. The above algorithm is applicable to any semisimple group.

PDS of type II corresponds to a situation for which all the states of the system preserve part of the dynamical symmetry, $G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n$. In this case, there are no analytic solutions, yet selected quantum numbers (of the conserved symmetries) are retained. This occurs, for example, when the Hamiltonian contains interaction terms from two different chains with a common symmetry subalgebra [18], e.g.,

$$G_0 \supset \{ \begin{array}{c} G_1 \\ G'_1 \end{array} \} \supset G_2 \supset \ldots \supset G_n.$$

(9)

If $G_1$ and $G'_1$ are incompatible, i.e., do not commute, then their irreps are mixed in the eigenstates of the Hamiltonian. On the other hand, since $G_2$ and its subalgebras are common to both chains, then the labels of their irreps remain as good quantum numbers.

An alternative situation where PDS of type II occurs is when the Hamiltonian preserves only some of the symmetries $G_i$ in the DS chain and only their irreps are unmixed. A systematic procedure for identifying interactions with such property was proposed in [19]. Let $G_1 \supset G_2 \supset G_3$ be a set of nested algebras which may occur anywhere in the chain, in-between the spectrum generating algebra $G_0$ and the invariant symmetry algebra $G_n$. The procedure is based on writing the Hamiltonian in terms of generators, $g_i$, of $G_1$, which do not belong to its subalgebra $G_2$. By construction, such Hamiltonian preserves the $G_1$ symmetry but, in general, not the $G_2$ symmetry, and hence will have the $G_1$ labels as good quantum numbers but will mix different irreps of $G_2$. The Hamiltonians can still conserve the $G_3$ labels e.g., by choosing it to be a scalar of $G_3$. The procedure involves the identification of the tensor character under $G_2$ and $G_3$ of the operators $g_i$ and their products, $g_ig_j \ldots g_k$. The Hamiltonians obtained in this manner belong to the integrity basis of $G_3$-scalar operators in the enveloping algebra of $G_1$ and, hence, their existence is correlated with their order.

PDS of type III combines properties of both PDS of type I and II. Such a generalized PDS [20] has a hybrid character, for which part of the states of the system under study preserve part of the dynamical symmetry. In relation to the dynamical symmetry chain of Eq. (3), with associated
basis, \(|h_N\rangle\langle\Sigma\Lambda\rangle\), this can be accomplished by relaxing the condition of Eq. (5), so that it holds only for selected states \(\Lambda\) contained in a given irrep \(\langle\Sigma_0\rangle\) of \(G\) and/or selected (combinations of) components \(\lambda\) of the tensor \(\hat{T}_{\delta_{\lambda\nu}}\). Under such circumstances, let \(G' \neq G_{\text{sym}}\) be a subalgebra of \(G\) in the aforementioned chain, \(G \supset G'\). In general, the Hamiltonians, constructed from these tensors, in the manner shown in Eq. (7), are not invariant under \(G\) nor \(G'\). Nevertheless, they do possess the subset of solvable states, \(|h_N\rangle\langle\Sigma_0\rangle\Lambda\), with good \(G\)-symmetry \(\langle\Sigma_0\rangle\) (which now span only part of the corresponding \(G\)-irrep), while other states are mixed. At the same time, the symmetry associated with the subalgebra \(G'\), is broken in all states (including the solvable ones). Thus, part of the eigenstates preserve part of the symmetry. These are precisely the requirements of PDS of type III.

3. SU(3) partial dynamical symmetry

The SU(3) DS chain of the IBM and related quantum numbers are given by [7]

\[ U(6) \supset SU(3) \supset O(3) \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ [N] \quad (\lambda, \mu) \quad K \quad L \]

(10)

For a given \(U(6)\) irrep \([N]\), the allowed SU(3) irreps are \((\lambda, \mu) = (2N - 4k - 6m, 2k)\) with \(k, m\) non-negative integers, such that, \(\lambda, \mu \geq 0\). The multiplicity label \(K\) is needed for complete classification and corresponds geometrically to the projection of the angular momentum on the symmetry axis. The values of \(L\) contained in a given SU(3) irrep \((\lambda, \mu)\), are obtained from the known \(SU(3) \supset O(3)\) reduction. The states \(|[N](\lambda, \mu)K\rangle\) form the (non-orthogonal) Elliott basis [4] and the Vergados basis \(|[N](\lambda, \mu)\Lambda\rangle\) [7] is obtained from it by a standard orthogonalization procedure. The two bases coincide in the large-\(N\) limit and both are eigenstates of a Hamiltonian with SU(3) DS. The latter, for one- and two-body interactions, can be transcribed in the form

\[ \hat{H}_{DS} = h_2 \left[ -\hat{C}_2(SU(3)) + 2\hat{N}(2\hat{N} + 3) \right] + C\hat{C}_2(O(3)) \]

(11)

where \(\hat{C}_2(G)\) is the quadratic Casimir operator of \(G\), as defined in Table 1. The spectrum of \(\hat{H}_{DS}\) is completely solvable with eigenenergies

\[ E_{DS} = h_2 \frac{N}{6} \{2N(k + 2m) - k(2k - 1) - 3m(2m - 1) - 6km\} + C\hat{L}\hat{L} + 1 \]

(12)

and \((\lambda, \mu) = (2N - 4k - 6m, 2k)\). The spectrum resembles that of an axially-deformed rotovibrator and the corresponding eigenstates are arranged in SU(3) multiplets. In a given SU(3) irrep \((\lambda, \mu)\), each \(K\)-value is associated with a rotational band and states with the same \(L\), in different \(K\)-bands, are degenerate. The lowest SU(3) irrep is \((2N, 0)\), which describes the ground band \(g(K = 0)\) of a prolate deformed nucleus. The first excited SU(3) irrep \((2N - 4, 2)\) contains both the \(\beta(K = 0)\) and \(\gamma(K = 2)\) bands. States in these bands with the same angular momentum are degenerate. This \(\beta\)-\(\gamma\) degeneracy is a characteristic feature of the SU(3) limit of the IBM which, however, is not commonly observed. In most deformed nuclei the \(\beta\) band lies above the \(\gamma\) band. In the IBM framework, with at most two-body interactions, one is therefore compelled to break SU(3) in order to conform with the experimental data.

The construction of Hamiltonians with SU(3)-PDS of type I is based on identification of \(n\)-boson operators which annihilate all states in a given SU(3) irrep \((\lambda, \mu)\), chosen here to be the ground band irrep \((2N, 0)\). For that purpose, we consider the following two-boson SU(3) tensors, \(B_{[n](\lambda, \mu)\ell;\ell}\), with \(n = 2\), \((\lambda, \mu) = (0, 2)\) and angular momentum \(\ell = 0, 2\)

\[ B_{[2](\lambda, \mu)\ell;\ell} \propto P_{\ell} = d^\dagger \cdot d^\dagger - 2(s^\dagger)^2 \]

(13a)

\[ B_{[2](\lambda, \mu)\ell;\ell} \propto P_{\ell} = 2d^\dagger m^\dagger s^\dagger + \sqrt{7} (d^\dagger d^\dagger)_m^{(2)} \]

(13b)
The corresponding Hermitian conjugate boson-pair annihilation operators, $P_0$ and $P_{2m}$, transform as $(2,0)$ under SU(3), and satisfy

\begin{align}
P_0 \vert (N)(2N,0)K = 0, L \rangle &= 0 , \\
P_{2m} \vert (N)(2N,0)K = 0, L \rangle &= 0 , \quad L = 0, 2, 4, \ldots, 2N .
\end{align}

Equivalently, these operators satisfy

\begin{align}
P_0 \vert \beta = \sqrt{2}, \gamma = 0; N \rangle &= 0 , \\
P_{2m} \vert \beta = \sqrt{2}, \gamma = 0; N \rangle &= 0 ,
\end{align}

where $|\beta = \sqrt{2}, \gamma = 0; N \rangle$, is the condensate of Eq. (2) with the SU(3) equilibrium deformations. It is the lowest-weight state in the SU(3) irrep $(\lambda, \mu)$, where $|0\rangle$ is obtained from it by O(3) projection, and spans the entire SU(3) irrep $(\lambda, \mu)$. The relations in Eqs. (14)-(15) follow from the fact that the action of the operators $P_{\ell m}$ leads to a state with $N-2$ bosons in the U(6) irrep $[N-2]$, which does not contain the SU(3) irreps obtained from the product $(2,0) \times (2N,0) = (2N + 2,0) \oplus (2N,1) \oplus (2N - 2,2)$. In addition, $P_0$ satisfies

\begin{align}
P_0 \vert (2N-4k,2k)K = 2k, L \rangle &= 0 , \quad L = K, K+1, \ldots, (2N-2k) .
\end{align}

For $k > 0$ the indicated $L$-states span only part of the SU(3) irreps $(\lambda, \mu) = (2N-4k,2k)$ and form the rotational members of excited $\gamma^k(K = 2k)$ bands. This result follows from the fact that $P_0$ annihilates the intrinsic states of these bands, $|\gamma^k(K = 2k)\rangle \propto (P_{2,2}^\dagger)^k|\beta = \sqrt{2}, \gamma = 0; N-2k\rangle$.

Following the general algorithm, a two-body Hamiltonian with partial SU(3) symmetry can now be constructed as in Eq. (7), $\hat{H}' = h_0 P_0^\dagger P_0 + h_2 P_2^\dagger \cdot \hat{P}_2$, where $\hat{P}_{2m} = (-)^m P_{2-m}$. For $h_2 = h_0$, this Hamiltonian is an SU(3) scalar, while for $h_0 = -5h_2$, it transforms as a $(2,2)$ SU(3) tensor component. The scalar part is related to the quadratic Casimir operator of SU(3)

\begin{align}
\hat{C}_2 &\equiv P_0^\dagger P_0 + P_2^\dagger \cdot \hat{P}_2 = -\hat{C}_2(SU(3)) + 2\hat{N}(2\hat{N} + 3) ,
\end{align}

and is simply the first term in $\hat{H}_{DS}$, Eq. (11). In accord with Eq. (8), the two-body SU(3)-PDS Hamiltonian is thus given by

\begin{align}
\hat{H}_{PDS} &= \hat{H}_{DS} + \eta P_0^\dagger P_0 .
\end{align}

The $P_0^\dagger P_0$ term is not diagonal in the SU(3) chain, however, Eqs. (14)-(16) ensure that $\hat{H}_{PDS}$ retains selected solvable states with good SU(3) symmetry. Specifically, the solvable states are members of the ground $g(K = 0)$

\begin{align}
|N, (2N,0)K = 0, L \rangle &\quad L = 0, 2, 4, \ldots, 2N \\
E_{PDS} &= CL(L+1)
\end{align}

and $\gamma^k(K = 2k)$ bands

\begin{align}
|N, (2N-4k,2k)K = 2k, L \rangle &\quad L = K, K+1, K+2, \ldots, (2N-2k) \\
E_{PDS} &= h_2 6k(N-2)(2N-2k + 1) + CL(L+1) \quad k > 0 .
\end{align}

The remaining eigenstates of $\hat{H}_{PDS}$ do not preserve SU(3) and, therefore, get mixed.
The empirical spectrum of $^{168}$Er is shown in Fig. 1 and compared with SU(3)-DS, SU(3)-PDS and broken SU(3) calculations [21]. The SU(3)-PDS spectrum shows an improvement over the schematic, exact SU(3) dynamical symmetry description, since the $\beta$-$\gamma$ degeneracy is lifted.

Table 2. B(E2) branching ratios from states in the $\gamma$ band in $^{168}$Er. The column EXP lists the experimental ratios, PDS is the SU(3) partial dynamical symmetry calculation and WCD is a broken SU(3) calculation [21].
The quality of the calculated PDS spectrum is similar to that obtained in the broken-SU(3) calculation, however, in the former the ground \(g(K = 0_1)\) and \(\gamma(K = 2_1)\) bands remain solvable with good SU(3) symmetry, \((\lambda, \mu) = (2N, 0)\) and \((2N - 4, 2)\) respectively. At the same time, the excited \(K = 0^+_2\) band involves about 13\% SU(3) admixtures into the dominant \((2N - 4, 2)\) irrep. Since the wave functions of the solvable states (19)-(20) are known, one can obtain analytic expressions for matrix elements of observables between them. For example, the most general cubic and quadratic Casimir operators of SU(3), defined in Table 1, where \(\hat{\Lambda}_{\lambda, \mu}\) symmetry Hamiltonian and eigenenergies for states with \((\lambda, \mu)\) excited \(K\) one-body \(E(2)\) operator reads expressions for matrix elements of observables between them. For example, the most general one-body \(E(2)\) operator reads \(T(E2) = \alpha \hat{Q} + \rho \Pi(2)\), in the notation of Table 1. Since \(\hat{Q}\) is an SU(3) generator, it cannot connect different SU(3) irreps, hence only \(\Pi(2)\), which is a \((2, 2)\) SU(3) tensor, contributes to \(\gamma \rightarrow g\) transitions. Accordingly, the calculated B(E2) ratios for \(\gamma \rightarrow g\) transitions involve ratios of known SU(3) isoscalar factors and lead to parameter-free predictions. The latter, as shown in Table 2, are in excellent agreement with experiment, thus confirming the relevance of SU(3)-PDS to the spectroscopy of 

\[\text{Er [21]}\] and is based on identification of \(n\)-boson operators which annihilate all states in the irrep \((2N, 0)\). For \(n = 3\), the following SU(3) tensors, \(B_{[\ell]}^{[\lambda(\lambda, \mu); k; m]}\),

\[
\begin{align*}
\hat{B}_{[3]}^{[0; 0]} &\rightarrow W_2^{[1]} = 5 P_0^{[0]} \hat{s} - P_2^{[1]} \cdot \hat{d}^\dagger, \\
\hat{B}_{[3]}^{[2; 2]} &\rightarrow W_2^{[2]} = 6 P_0^{[0]} \hat{d} - P_2^{[2]} \hat{s}, \\
\hat{B}_{[3]}^{[0; 0]} &\rightarrow \lambda = P_0^{[0]} \hat{s}^\dagger + P_2^{[1]} \cdot \hat{d}.
\end{align*}
\]

The operators \(W_2^{[1]}, W_2^{[2]}, W_2^{[3]}, W_2^{[3]}\) in Eq. (21a) span the irrep \((\lambda, \mu) = (2, 2)\), while \(\Lambda\) of Eq. (21b) transforms as \((\lambda, \mu) = (0, 0)\). The latter SU(3)-scalar operator is related to the cubic and quadratic Casimir operators of SU(3), defined in Table 1,

\[2\Lambda^\dagger A = C_3(SU(3)) - 2N(4N + 3)(2N + 3) + 3(2N + 3)\hat{\theta}_2,\]

where \(\hat{\theta}_2\) is given in Eq. (17). In the presence of two- and three-body terms, the dynamical-Symmetries Hamiltonian and eigenenergies for states with \((\lambda, \mu) = (2N - 4k - 6m, 2k)\), read

\[
\begin{align*}
\hat{H}_{DS} &= h_1 \Lambda^\dagger \Lambda + h_2 \hat{\theta}_2 + C \hat{L} \cdot \hat{L}, \\
E_{DS} &= + h_1 54m \left[ N(2k + 2m + 1) - k(2k - 1) - (2m - 1)(2m + 1) - 6km \right] \\
&+ h_2 6 \left[ 2N(k + 2m) - k(2k - 1) - 3m(2m - 1) - 6km \right] + C L(L + 1). \quad (23a)
\end{align*}
\]

The SU(3)-PDS Hamiltonian has the following SU(3)-tensor expansion

\[
\begin{align*}
\hat{H}_{PDS} &= \hat{H}_{DS} + \eta P_0^{[1]} P_0^\dagger + a_1 W_0^{[1]} W_0^\dagger + a_2 \left( W_0^{[1]} \Lambda + \Lambda W_0^\dagger \right) + a_3 W_2^{[1]} \cdot \hat{W}_2 + a_4 V_2^{[1]} \cdot \hat{V}_2 + a_5 \left( W_2^{[1]} \cdot \hat{V}_2 + V_2^{[1]} \cdot \hat{W}_2 \right) + a_6 W_3^{[1]} \cdot \hat{W}_3 + a_7 W_4^{[1]} \cdot \hat{W}_4.
\end{align*}
\]

The relations of Eq. (14) ensure that the operators \(W_\ell m, V_2\) and \(\Lambda\) of Eq. (21a) annihilate all states of the SU(3) ground band \(g(K = 0)\). The solvable eigenstates and eigenenergies of \(\hat{H}_{PDS}\) are those shown in Eq. (19). In addition, the operator \(\Lambda\) (21b) annihilates all states in the irreps \((2N - 4k, 2k)\)

\[
\Lambda \left[ |N \rangle (2N - 4k, 2k) K; L \right] = 0. \quad (25)
\]

This property follows from the fact that the U(6) irrep \([N - 3]\) does not contain SU(3) irreps obtained from the product \((0, 0) \times (2N - 4k, 2k)\). Using Eqs. (16) and (25), we can identify the following sub-class of SU(3)-PDS Hamiltonians

\[
\hat{H}_{PDS} - 1 = \hat{H}_{DS} + h_3 P_0^{[1]} P_0^\dagger + h_4 P_0^{[1]} s^\dagger s P_0^\dagger + h_5 \left( \Lambda^\dagger s P_0^\dagger + P_0^{[1]} s^\dagger \Lambda \right), \quad (26)
\]
with additional solvable states which are the members of the \( g(K = 0) \) and \( \gamma^h(K = 2k) \) bands listed in Eqs. (19)-(20).

A second sub-class of SU(3)-PDS Hamiltonian corresponds to the choice

\[
\hat{H}_{PDS-2} = \hat{H}_{DS} + h_0 W_2^\dagger \cdot \hat{W}_2 + h_7 W_3^\dagger \cdot \hat{W}_3 .
\]  

(27)

\( \hat{H}_{PDS-2} \) has a solvable ground band \( g(K = 0) \), Eq. (19), and a solvable \( \beta(K = 0) \) band

\[
\begin{align*}
\langle N, (2N - 2) | K = 0, L \rangle &= 0, 2, 4, \ldots, (2N - 4) \\
E_{PDS-2} &= h_2 6(N - 2)(2N - 1) + CL(L + 1).
\end{align*}
\]

(28a)

This result follows from the fact that \( W_{2m} \) and \( W_{3m} \) annihilate the intrinsic state of this band, \( \langle \beta(K = 0) \rangle \propto \sqrt{2} P_1^0 - P_2^0 \rangle | \beta = \sqrt{2}, \gamma = 0; N - 2 \rangle \), as well as the projected Elliott basis states

\[
W_{\ell m}(|N\rangle(2N - 2), K = 0, L) = 0 \quad \ell = 2, 3 .
\]

(29)

Three-body terms allow an additional solvable symmetry-conserving operator

\[
\hat{\Omega} = -4\sqrt{3} \hat{Q} \cdot (\hat{L} \times \hat{L})^{(2)}.
\]

(30)

This operator is constructed from SU(3) generators, hence is diagonal in \( (\lambda, \mu) \). It breaks, however, the aforementioned \( K \)-degeneracy. A well defined procedure exists for obtaining the eigenstates of \( \hat{\Omega} \) and corresponding eigenvalues \( \langle \hat{\Omega} \rangle \) [22,23]. For example, for the irreps \( (\lambda, 0) \) and \( (\lambda, 2) \) with \( \lambda \) even, we have

\[
\begin{align*}
(\lambda, 0) & K = 0, \quad L = 0, 2, 4, \ldots, \lambda : \\
(\lambda, 2) & K = 0, \quad L = 3, 5, 7, \ldots, \lambda + 1, \lambda + 2 : \\
(\lambda, 2) & K = 0, \quad L = 0 : \\
(\lambda, 2) & K = 0, \quad L = 2, 4, 6, \ldots, \lambda :
\end{align*}
\]

\[
\langle \hat{\Omega} \rangle = (2\lambda + 5)[L(L + 1) - 6] \pm 6\sqrt{(2\lambda + 5)^2 + L(L + 1)(L - 1)(L + 2)}.
\]

(31d)

Several works have examined the influence of the symmetry-conserving operator \( \hat{\Omega} \) (30) on nuclear spectra, within the IBM [23-25] and the symplectic shell model [26,27]. It is interesting to note that the operator \( \hat{\Omega} \) can be expressed in terms of \( \hat{H}_{PDS-1} \) (26) and \( \hat{H}_{PDS-2} \) (27) as

\[
\hat{\Omega} = -2(\hat{N} - 1)\hat{\theta}_2 + 2\Lambda^\dagger \Lambda + (4\hat{N} + 3)\hat{L} \cdot \hat{L}
\]

\[
+6(\hat{N} - 1)P_0^\dagger P_0 - 2 \left( \Lambda^\dagger sP_0 + P_0^\dagger s^\dagger \Lambda \right) + 2W_2^\dagger \cdot \hat{W}_2 + 4W_3^\dagger \cdot \hat{W}_3 .
\]

(32)

4. \( O(6) \) partial dynamical symmetry

The \( O(6) \) DS chain of the IBM and related quantum numbers are given by [7]

\[
U(6) \supset O(6) \supset O(5) \supset O(3)
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]

\[
[N] \quad (\Sigma) \quad (\tau) \quad n_\Delta \quad L
\]

(33)

For a given \( U(6) \) irrep \( [N] \), the allowed \( O(6) \) and \( O(5) \) irreps are \( \Sigma = N, N - 2, \ldots, 0 \) or 1, and \( \tau = 0, 1, \ldots, \Sigma \), respectively. The values of \( L \) contained in the \( O(5) \) irrep \( (\tau) \) are obtained from the known \( O(5) \supset O(3) \) reduction and \( n_\Delta \) is a multiplicity label. The eigenstates \( |N, (\Sigma)(\tau)n_\Delta L \rangle \)
are obtained with a Hamiltonian with O(6) DS which, for one- and two-body interactions, can be transcribed in the form

\[ H_{DS} = h_0 \left[ -\hat{C}_2(O(6)) + \hat{N}(\hat{N} + 4) \right] + B \hat{C}_2(O(5)) + C \hat{C}_2(O(3)) \, . \]  

(34)

Here the quadratic Casimir operators, \( \hat{C}_2(G) \), are defined in Table 1. The spectrum of \( H_{DS} \) is completely solvable with eigenenergies

\[ E_{DS} = 4h_0 (N - v + 2)v + B \tau (\tau + 3) + C L(L + 1) \, . \]  

(35)

The spectrum resembles that of a \( \gamma \)-unstable deformed rotovibrator, where states are arranged in bands with O(6) quantum number \( \Sigma = N - 2v, \) \( (v = 0, 1, 2, \ldots) \). The ground band \( (v = 0) \) corresponds to the O(6) irrep with \( \Sigma = N \) \( \). The O(5) and O(3) terms in \( H_{DS} \) govern the in-band rotational splitting. The lowest members in each band have quantum numbers \( (\tau = 0, L = 0), (\tau = 1, L = 2) \) and \( (\tau = 2, L = 2, 4) \).

The construction of Hamiltonians with O(6)-PDS of type I is based on identification of \( n \)-boson operators which annihilate all states in a given O(6) irrep, \( \langle \Sigma \rangle \), chosen here to be the ground band irrep \( \langle \Sigma \rangle = \langle N \rangle \). For that purpose, a relevant operator to consider is

\[ \hat{B}^\dagger_{\tau}(0)000 \propto P_0^\dagger = d^\dagger \cdot d^\dagger - (s^\dagger)^2 \, . \]  

(36)

The corresponding Hermitian conjugate boson-pair annihilation operator, \( P_0 \), transforms also as \( \langle \Sigma \rangle = \langle 0 \rangle \) under O(6) and satisfies

\[ P_0 |[N]\langle N\rangle\tau n\Delta L\rangle = 0 \, , \quad \tau = 0, 1, 2, \ldots, N \, . \]  

(37)

Equivalently, this operator satisfies

\[ P_0 |\beta = 1, \gamma; N\rangle = 0 \, , \]  

(38)

where \( |\beta = 1, \gamma; N\rangle \), is the condensate of Eq. (2) with the O(6) equilibrium deformations. It is the lowest-weight state in the O(6) irrep \( \langle \Sigma \rangle = \langle N \rangle \) and serves as an intrinsic state for the O(6) ground band. The rotational members of the band, \( |[N]\langle N\rangle\tau n\Delta L\rangle \), Eq. (37), are obtained from it by O(5) projection, and span the entire O(6) irrep \( \langle \Sigma \rangle = \langle N \rangle \). The relations in Eqs. (37)-(38) follow from the fact that the action of the operator \( P_0 \) leads to a state with \( N - 2 \) bosons in the U(6) irrep \( [N - 2] \), which does not contain the O(6) irrep \( \langle N \rangle \) obtained from the product of \( (0) \times \langle N \rangle \).

Since both \( P_0^\dagger \) and \( P_0 \) (36) are O(6) scalars, they give rise to the following interaction

\[ P_0^\dagger P_0 = -\hat{C}_{O(6)} + \hat{N}(\hat{N} + 4) \, , \]  

(39)

which is simply the O(6) term in \( H_{DS} \), Eq. (34), with an exact O(6) symmetry. Thus, in this case, unlike the situation encountered with SU(3)-PDS, the algorithm does not yield an O(6)-PDS of type I with two-body interactions. In the IBM framework, an Hamiltonian with a genuine O(6)-PDS of this class, requires higher-order terms.

Focusing on three-body interactions with O(6)-PDS, one considers the following two three-boson O(6) tensors, \( \hat{B}^\dagger_{n}[\sigma](\tau)\ell \), with \( n = 3 \), \( \sigma = 1 \) and \( \ell = 0, 2, 3 \),

\[ \hat{B}^\dagger_{[3](0)000} \propto P_0^\dagger s^\dagger \, , \]  

(40a)

\[ \hat{B}^\dagger_{[3](1)02m} \propto P_0^\dagger d^\dagger \, . \]  

(40b)
The relation of Eq. (37) ensures that $sP_0$ and $d_mP_0$ annihilate the states of the O(6) ground band. The only three-body interactions that are partially solvable in O(6) are thus $P^\dagger_0\hat{n}_aP_0$ and $P^\dagger_0\hat{n}_dP_0$. Since the combination $P^\dagger_0(\hat{n}_a + \hat{n}_d)P_0 = (\hat{N} - 2)P^\dagger_0P_0$ is completely solvable in O(6), we can transcribe the O(6)-PDS Hamiltonian in the form

$$\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + \eta P^\dagger_0\hat{n}_sP_0,$$

(41)

### Table 3

Observed (EXP) and calculated B(E2) values (in e^2b^2) for ^{196}\text{Pt}. For both the exact (DS) and partial (PDS) O(6) dynamical symmetry calculations, the E2 operator is $T(E2) = e_B[\Pi^{(2)} + \chi U^{(2)}]$ with $e_B = 0.151$ eb and $\chi = 0.29$. Only the state $0^+\text{a}$ has a mixed O(6) character [17].

| Transition | EXP | DS | PDS | Transition | EXP | DS | PDS |
|------------|-----|----|-----|------------|-----|----|-----|
| $2^+_1 \rightarrow 0^+_1$ | 0.274 (1) | 0.274 | 0.274 | $2^+_3 \rightarrow 0^+_2$ | 0.034 (34) | 0.119 | 0.119 |
| $2^+_2 \rightarrow 2^+_1$ | 0.368 (9) | 0.358 | 0.358 | $2^+_3 \rightarrow 4^+_1$ | 0.0009 (8) | 0.0004 | 0.0004 |
| $2^+_2 \rightarrow 0^+_1$ | 3.10^{-5} (3) | 0.0018 | 0.0018 | $2^+_3 \rightarrow 2^+_2$ | 0.0018 (16) | 0.0013 | 0.0013 |
| $4^+_1 \rightarrow 2^+_1$ | 0.405 (6) | 0.358 | 0.358 | $2^+_3 \rightarrow 0^+_1$ | 0.00002 (2) | 0.0 | 0.0 |
| $0^+_2 \rightarrow 2^+_2$ | 0.121 (67) | 0.365 | 0.365 | $6^+_2 \rightarrow 6^+_1$ | 0.108 (34) | 0.103 | 0.103 |
| $0^+_2 \rightarrow 2^+_1$ | 0.019 (10) | 0.003 | 0.003 | $6^+_2 \rightarrow 4^+_2$ | 0.331 (88) | 0.221 | 0.221 |
| $4^+_2 \rightarrow 4^+_1$ | 0.115 (40) | 0.174 | 0.174 | $6^+_2 \rightarrow 4^+_1$ | 0.0032 (9) | 0.0008 | 0.0008 |
| $4^+_2 \rightarrow 2^+_2$ | 0.196 (42) | 0.191 | 0.191 | $0^+_3 \rightarrow 2^+_2$ | < 0.0028 | 0.0037 | 0.0028 |
| $4^+_2 \rightarrow 2^+_1$ | 0.004 (1) | 0.001 | 0.001 | $0^+_3 \rightarrow 2^+_1$ | < 0.034 | 0.0 | 0.0 |
| $6^+_1 \rightarrow 4^+_1$ | 0.493 (32) | 0.365 | 0.365 |     |     |     |     |
Here the dynamical symmetry Hamiltonian, $\hat{H}_{DS}$, is that of Eq. (34), since no new terms are added to it at the level of three-body interactions. The solvable states are members of the $\gamma$-unstable deformed ground band

$$\| N \rangle \langle N | \tau n_{\Delta L} \rangle , \quad \tau = 0, 1, 2, \ldots, N \quad (42a)$$

$$E_{PDS} = B \tau (\tau + 3) + CL(L + 1) . \quad (42b)$$

The experimental spectrum and E2 rates of $^{196}$Pt are shown in Fig. 2 and Table 3. The O(6)-DS limit is seen to provide a good description for properties of states in the ground band ($\Sigma = N$). This observation was the basis of the claim [28] that the O(6)-DS is manifested empirically in $^{196}$Pt. However, the resulting fit to energies of excited bands is quite poor. The $0^+_1$, $0^+_2$, and $0^+_3$ levels of $^{196}$Pt at excitation energies 0, 1403, 1823 keV, respectively, are identified as the bandhead states of the ground ($v = 0$), first- ($v = 1$) and second- ($v = 2$) excited vibrational bands [28]. Their empirical anharmonicity, defined by the ratio $R = E(v = 2)/E(v = 1) - 2$, is found to be $R = -0.70$. In the O(6)-DS limit these bandhead states have $\tau = L = 0$ and $\Sigma = N, N - 2, N - 4$, respectively. The anharmonicity $R = -2/(N + 1)$, as calculated from Eq. (35), is fixed by $N$. For $N = 6$, which is the appropriate boson number for $^{196}$Pt, the O(6)-DS value is $R = -0.29$, which is in marked disagreement with the empirical value. A detailed study of double-phonon excitations within the IBM, has concluded that large anharmonicities can be incorporated only by the inclusion of at least cubic terms in the Hamiltonian [29]. In the IBM there are 17 possible three-body interactions [7]. One is thus confronted with the need to select suitable higher-order terms that can break the DS in excited bands but preserve it in the ground band. On the basis of the preceding discussion this can be accomplished by the O(6)-PDS Hamiltonian of Eq. (41). The spectrum of $\hat{H}_{PDS}$ is shown in Fig. 2. The states belonging to the $\Sigma = N = 6$ multiplet remain solvable with energies (42b), which obey the same DS expression, Eq. (35). States with $\Sigma < 6$ are generally admixed but agree better with the data than in the DS calculation. For example, the bandhead states of the first- (second-) excited bands have the O(6) decomposition $\Sigma = 4$: 76.5% (19.6%), $\Sigma = 2$: 16.1% (18.4%), and $\Sigma = 0$: 7.4% (62.0%). Thus, although the ground band is pure, the excited bands exhibit strong O(6) breaking. The calculated O(6)-PDS anharmonicity for these bands is $R = -0.63$, much closer to the empirical value, $R = -0.70$. It should be emphasized that not only the energies but also the wave functions of the $\Sigma = N$ states remain unchanged when the Hamiltonian is generalized from DS to PDS. Consequently, the E2 rates for transitions among this class of states are the same in the DS and PDS calculations. Thus, the additional three-body term in the Hamiltonian (41), does not spoil the good O(6)-DS description for this segment of the spectrum. This is evident in Table 3 where most of the E2 data concern transitions between $\Sigma = N = 6$ states.

5. Coexistence of partial dynamical symmetries

The examples considered in previous sections involved Hamiltonians with a single PDS, describing stable structures, e.g., well-deformed nuclei. Multiple partial dynamical symmetries can occur in systems undergoing quantum phase transitions (QPTs) [30]. The latter are structural changes induced by a variation of parameters in the Hamiltonian. Such ground-state phase transitions are a pervasive phenomenon observed in many branches of physics [31], and are realized empirically in nuclei as transitions between different shapes. In the IBM, the dynamical symmetry limits correspond to possible phases of the system and the relevant Hamiltonians for studying shape-phase transitions involve terms with from different DS chains [15]. The nature of the phase transition is governed by the topology of the surface $E_N(\beta, \gamma)$, Eq. (1), which serves as a Landau’s potential with the equilibrium deformations as order parameters. The surface at the critical-point of a first-order transition is required to have two degenerate minima, corresponding to the two coexisting phases. Specifically, the first-order critical surface is

$$E_N(\beta, \gamma = 0) = 2h_2 N(N - 1)(1 + \beta^2)^{-2} \beta^2 (\beta - \beta_0)^2 ,$$

(43)
Consequently, \( \hat{H}_{cr} (\beta_0) \) where the various operators are defined in Table 1. The spectrum of \( \hat{H}_{cr} (\beta_0) \) with \( k = 0 \) and \( k = 1 \), respectively. \( L = 0, 2, 3 \) are the solvable U(5) states of Eq. (49) [30].

Figure 3. Spectrum of \( \hat{H}_{cr} (\beta_0 = \sqrt{2}) \), Eq. (44), with \( b_2 = 0.05, N = 10 \). \( L(K = 0) \) and \( L(K = 2) \) are the solvable SU(3) states of Eqs. (19)-(20) with \( k = 0 \) and \( k = 1 \), respectively. \( L = 0, 2, 3 \) are the solvable U(5) states of Eq. (49) [30].

and has degenerate spherical and deformed minima at \( \beta = 0 \) and \( (\beta = \beta_0, \gamma = 0) \), corresponding to spherical and axially-deformed shapes. A barrier of height \( h = h_2 N(N - 1)(1 - \sqrt{1 + \beta_0^2})^2/2 \) separates the two minima. Such a surface can be obtained from Eq. (1) with the following critical-point Hamiltonian

\[
\hat{H}_{cr} (\beta_0) = h_2 P_{2m}^\dag (\beta_0) \cdot \hat{P}_2 (\beta_0), \quad P_{2m}^\dag (\beta_0) = \beta_0 \sqrt{2} d_k^m s^\dagger + \sqrt{7} (d^\dagger d^\dagger)^{(2)} \ . \quad (44)
\]

\( P_{2m} (\beta_0) \) annihilates the condensate of Eq. (2) with \( (\beta = \beta_0, \gamma = 0) \) as well as the states of good \( L, \beta_0; N, L \), projected from it

\[
P_{2m} (\beta_0) | \beta_0, \gamma = 0; N \rangle = 0 \quad (45a)
\]

\[
P_{2m} (\beta_0) | \beta_0; N, L \rangle = 0 \quad L = 0, 2, 4, \ldots, 2N \ . \quad (45b)
\]

Consequently, \( \hat{H}_{cr} (\beta_0) \) has a solvable zero-energy prolate-deformed ground band, composed of the \( L \)-projected states mentioned above

\[
| \beta_0; N, L \rangle \quad E = 0 \quad L = 0, 2, 4, \ldots, 2N \ . \quad (46)
\]

The multipole form of \( \hat{H}_{cr} (\beta_0) \) is given by

\[
\hat{H}_{cr} (\beta_0) = h_2 \left[ 2(\beta_0^2 N - 2) \hat{n}_d + 2(1 - \beta_0^2) \hat{n}_d^2 + 2 \hat{C}_2 (O(5)) - \hat{C}_2 (O(3)) + \sqrt{14} \beta_0 \Pi^{(2)} \cdot U^{(2)} \right] , \quad (47)
\]

where the various operators are defined in Table 1. The \( \hat{n}_d, \hat{n}_d^2, \hat{C}_2 (O(5)) \) and \( \hat{C}_2 (O(3)) \) terms in Eq. (47) belong to the dynamical symmetry Hamiltonian of the U(5) chain

\[
\begin{array}{cccc}
U(6) & U(5) & O(5) & O(3) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
[N] & \langle n_d \rangle & (\tau) & n_\Delta \\
\end{array} \quad L , \quad (48)
\]

and describe the dynamics of an anharmonic spherical vibrator. The \( \Pi^{(2)} \cdot U^{(2)} \) term can connect states with \( \Delta n_d = \pm 1 \) and \( \Delta \tau = \pm 1, \pm 3 \), hence breaks the U(5) DS. Nevertheless, \( \hat{H}_{cr} (\beta_0) \) has selected solvable eigenstates with good U(5) symmetry,

\[
| N, n_d = \tau = L = 0 \rangle \quad E = 0 \ , \quad (49a)
\]

\[
| N, n_d = \tau = L = 3 \rangle \quad E = 6 h_2 [\beta_0^2 (N - 3) + 5] \ , \quad (49b)
\]
and therefore, by construction, \( \hat{H}_{\text{cri}}(\beta_0) \) exhibits U(5)-PDS of type I.

For \( \beta_0 = \sqrt{2} \), the critical Hamiltonian of Eq. (44) is recognized to be a special case of the Hamiltonian of Eq. (18), shown to have SU(3)-PDS of type I. As such, it has a subset of solvable eigenstates, Eqs. (19)-(20), which are members of deformed ground \( g(K = 0) \) and \( \gamma^k(K = 2k) \) bands with good SU(3) symmetry, \( (\lambda, \mu) = (2N - 4k, 2k) \). In addition, \( \hat{H}_{\text{cri}}(\beta_0 = \sqrt{2}) \) has the spherical states of Eq. (49), with good U(5) symmetry, as eigenstates. The spherical \( L = 0 \) state, Eq. (49a), is exactly degenerate with the SU(3) ground band, Eq. (19), and the spherical \( L = 3 \) state, Eq. (49b), is degenerate with the SU(3) \( \gamma \)-band, Eq. (20) with \( k = 1 \). The remaining levels of \( \hat{H}_{\text{cri}}(\beta_0 = \sqrt{2}) \), shown in Fig. 3, are calculated numerically and their wave functions are spread over many U(5) and SU(3) irreps, as is evident from Fig. 4. This situation, where some states are solvable with good U(5) symmetry, some are solvable with good SU(3) symmetry and all other states are mixed with respect to both U(5) and SU(3), defines a U(5) PDS of type I coexisting with a SU(3) PDS of type I.

For \( \beta_0 = 1 \), the critical Hamiltonian of Eq. (47) involves the Casimir operators of O(5) and O(3) which are diagonal in the corresponding quantum numbers, \( \sigma \) and \( \tau \), of the O(6)-DS chain, Eq. (33). It also contains a term involving \( \hat{n}_d \) which is a scalar under O(5) but can connect states differing by \( \Delta \sigma = 0, \pm 2 \) and a \( \Pi^{(2)} \cdot U^{(2)} \) term which induces both O(6) and O(5) mixing subject to \( \Delta \sigma = 0, \pm 2 \) and \( \Delta \tau = \pm 1, \pm 3 \). Although \( \hat{H}_{\text{cri}}(\beta_0 = 1) \) is not invariant under O(6), it has a solvable prolate-deformed ground band, Eq. (46) with \( \beta_0 = 1 \), which has good O(6)
symmetry, $\langle \sigma \rangle = \langle N \rangle$, but broken $O(5)$ symmetry. In addition, $\hat{H}_{\text{cri}}(\beta_0 = 1)$ has the spherical states of Eq. (49), with good $U(5)$ symmetry, as eigenstates. The remaining eigenstates in Fig. 5 are mixed with respect to both $U(5)$ and $O(6)$, as is evident from the decomposition shown in Fig. 6. Apart from the solvable $U(5)$ states of Eq. (49), all eigenstates of $\hat{H}_{\text{cri}}(\beta_0 = 1)$ are mixed with respect to $O(5)$ [including the solvable $O(6)$ states of Eq. (46) with $\beta_0 = 1$, as shown in Fig. 7]. It follows that the Hamiltonian has a subset of states with good $U(5)$ symmetry and a subset of states with good $O(6)$ but broken $O(5)$ symmetry, and all other states are mixed with respect to both $U(5)$ and $O(6)$. These are precisely the required features of $U(5)$ PDS of type I coexisting with $O(6)$ PDS of type III.

Second-order quantum phase transitions between spherical and deformed $\gamma$-unstable nuclei, can be accommodated in the IBM, by mixing the DS Hamiltonians of the $U(5)$ chain, Eq. (48), and the $O(6)$ chain, Eq. (33). Since $U(5)$ and $O(6)$ are incompatible, yet both chains have a common $O(5) \supset O(3)$ segment, we encounter a situation similar to that described in Eq. (9), which gives rise to $O(5)$-PDS of type II [18].

6. Concluding remarks

The notion of partial dynamical symmetry generalizes the concepts of exact and dynamical symmetries. In making the transition from an exact to a dynamical symmetry, states which are degenerate in the former scheme are split but not mixed in the latter, and the block structure of the Hamiltonian is retained. Proceeding further to partial symmetry, some blocks or selected states in a block remain pure, while other states mix and lose the symmetry character. A partial dynamical symmetry lifts the remaining degeneracies, but preserves the symmetry-purity of the selected states.

Having at hand concrete algorithms for identifying and constructing Hamiltonians with PDS, is a valuable asset. It provides selection criteria for the a priori huge number of possible symmetry-breaking terms, accompanied by a rapid proliferation of free-parameters. This is particularly important in complicated environments when many degrees of freedom take part in the dynamics and upon inclusion of higher-order terms in the Hamiltonian. Futhermore, Hamiltonians with PDS break the dynamical symmetry (DS) but retain selected solvable eigenstates with good symmetry. The advantage of using interactions with a PDS is that they can be introduced, in a controlled manner, without destroying results previously obtained with a DS for a segment of the spectrum. These virtues greatly enhance the scope of applications of algebraic modeling of quantum many-body systems.

PDSs appear to be a common feature in algebraic descriptions of dynamical systems. They are not restricted to a specific model but can be applied to any quantal systems of interacting particles, bosons, as demonstrated in the present contribution, and fermions [13, 32, 33]. They are also relevant to the study of mixed systems with coexisting regularity and chaos, where they lead to a suppression of chaos [34, 35].

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