Characterizations of a class of Pilipović spaces by powers of harmonic oscillator

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Abstract
We show that a smooth function $f$ on $\mathbb{R}^d$ belongs to the Pilipović space $\mathcal{H}_{b_\sigma}(\mathbb{R}^d)$ or the Pilipović space $\mathcal{H}_{0,b_\sigma}(\mathbb{R}^d)$, if and only if the $L^p$ norm of $H_d^N f$ for $N \geq 0$, satisfy certain types of estimates. Here $H_d = |x|^2 - \Delta_x$ is the harmonic oscillator.

Keywords Harmonic oscillator · Pilipović spaces

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0 Introduction

In the paper we characterize Pilipović spaces of the form $\mathcal{H}_{b_\sigma}(\mathbb{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbb{R}^d)$, considered in [3,11], in terms of estimates of powers of the harmonic oscillator, on the involved functions.

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The set of Pilipović spaces is a family of Fourier invariant spaces, containing any Fourier invariant (standard) Gelfand-Shilov space. The (standard) Pilipović spaces $\mathcal{H}_s(\mathbb{R}^d)$ and $\mathcal{H}_{0,s}(\mathbb{R}^d)$ with respect to $s \in \mathbb{R}_+$, are the sets of all formal Hermite series expansions

$$f(x) = \sum_{\alpha \in \mathbb{N}^d} c_{\alpha}(f)h_\alpha(x)$$

(0.1)

such that

$$|c_{\alpha}(f)| \lesssim e^{-r|\alpha|^{\frac{1}{2}}}$$

(0.2)

holds true for some $r > 0$ respective for every $r > 0$. Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ for some constant $c > 0$ which is independent of $\theta$ in the domain of $f$ and $g$ (see also [6] and Sect. 1 for notations). Evidently, $\mathcal{H}_s(\mathbb{R}^d)$ and $\mathcal{H}_{0,s}(\mathbb{R}^d)$ increase with $s$. It is proved in [7] that if $S_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are the Gelfand-Shilov spaces of Roumieu respective Beurling type of order $s$, then

$$\mathcal{H}_s(\mathbb{R}^d) = S_s(\mathbb{R}^d), \quad s \geq \frac{1}{2},$$

(0.3)

$$\mathcal{H}_{0,s}(\mathbb{R}^d) = \Sigma_s(\mathbb{R}^d), \quad s > \frac{1}{2},$$

(0.4)

and

$$\mathcal{H}_{0,s}(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}, \quad s = \frac{1}{2}.$$

It is also well-known that $S_s(\mathbb{R}^d) = \{0\}$ when $s < \frac{1}{2}$ and $\Sigma_s(\mathbb{R}^d) = \{0\}$ when $s \leq \frac{1}{2}$. These relationships are completed in [11] by the relations

$$\mathcal{H}_s(\mathbb{R}^d) \neq S_s(\mathbb{R}^d) = \{0\}, \quad s < \frac{1}{2}$$

and

$$\mathcal{H}_{0,s}(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}, \quad s \leq \frac{1}{2}.$$

In particular, each Pilipović space is contained in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

For $\mathcal{H}_s(\mathbb{R}^d)$ ($\mathcal{H}_{0,s}(\mathbb{R}^d)$) we also have the characterizations

$$f \in \mathcal{H}_s(\mathbb{R}^d) \quad (f \in \mathcal{H}_{0,s}(\mathbb{R}^d)) \iff \|H_d^Nf\|_{L^\infty} \lesssim r^N N!^{2s}$$

(0.5)

for some $r > 0$ (for every $r > 0$) concerning estimates of powers of the harmonic oscillator

$$H_d = |x|^2 - \Delta_x, \quad x \in \mathbb{R}^d,$$

acting on the involved functions. These relations were obtained in [7] for $s \geq \frac{1}{2}$, and in [11] in the general case $s > 0$.

In [3,11] characterizations of $\mathcal{H}_s(\mathbb{R}^d)$ and $\mathcal{H}_{0,s}(\mathbb{R}^d)$ were also obtained by certain spaces of analytic functions on $\mathbb{C}^d$, via the Bargmann transform. From these mapping properties it follows that near $s = \frac{1}{2}$ there is a jump concerning these Bargmann images. More precisely, if $s = \frac{1}{2}$, then the Bargmann image of $\mathcal{H}_s(\mathbb{R}^d)$ (of $\mathcal{H}_{0,s}(\mathbb{R}^d)$) is the set of all entire functions $F$ on $\mathbb{C}^d$ such that $F$ obeys the condition

$$|F(z)| \lesssim e^{\left(\frac{1}{2} - r\right)|z|^2} \quad (|F(z)| \lesssim e^r|z|^2)$$

(0.6)
for some \( r > 0 \) (for every \( r > 0 \)). For \( s < \frac{1}{2} \), this estimate is replaced by

\[
|F(z)| \lesssim e^{r(\log(1+|z|))^{1-\frac{1}{2s}}}
\]  
(0.7)

for some \( r > 0 \) (for every \( r > 0 \)), which is indeed a stronger condition compared to the case \( s = \frac{1}{2} \).

An important motivation for considering the spaces \( \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \) is to make this gap smaller. More precisely, \( \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \), which are Pilipović spaces of Roumieu respectively Beurling type, is a family of function spaces, which increases with \( \sigma \) and such that

\[
\mathcal{H}_{s_1}(\mathbb{R}^d) \subset \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \subset \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \subset \mathcal{H}_{0,b_2}(\mathbb{R}^d), \quad s_1 < \frac{1}{2}, \ s_2 \geq \frac{1}{2}.
\]

The spaces \( \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \) consist of all formal Hermite series expansions (0.1) such that

\[
|c_\alpha| \lesssim r^{|\alpha|} \alpha!^{-\frac{1}{2s}}
\]  
(0.8)

hold true for some \( r > 0 \) respectively for every \( r > 0 \). For the Bargmann images of \( \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \), the conditions (0.6) and (0.7) above are replaced by

\[
|F(z)| \lesssim e^{r|z|^\frac{2s}{2s+1}},
\]

for some \( r > 0 \) respectively for every \( r > 0 \). It follows that the gaps of the Bargmann images of \( \mathcal{H}_{s}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) between the cases \( s < \frac{1}{2} \) and \( s \geq \frac{1}{2} \) are drastically decreased by including the spaces \( \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \), \( \sigma > 0 \), in the family of Pilipović spaces.

In [3], characterizations of \( \mathcal{H}_{b_1}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_1}(\mathbb{R}^d) \) in terms of estimates of powers of the harmonic oscillator acting on the involved functions which corresponds to (0.5) are deduced. On the other hand, apart from the case \( \sigma = 1 \), it seems that no such characterizations for \( \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d) \) have been obtained so far.

In Sect. 2 we fill this gap in the theory, and deduce such characterizations. In particular, as a consequence of our main result, Theorem 2.1 in Sect. 2, we have

\[
f \in \mathcal{H}_{b_\sigma}(\mathbb{R}^d) \quad (f \in \mathcal{H}_{0,b_\sigma}(\mathbb{R}^d)) \quad \iff \quad \|H_0^N f\|_{L^\infty} \lesssim 2^N r^{\frac{N}{\log(N\sigma)}} \left(2N\sigma \frac{2}{\log(N\sigma)} \right)^{N(1-\frac{1}{\log(N\sigma)})}
\]

for some (every) \( r > 0 \). By choosing \( \sigma = 1 \) we regain the corresponding characterizations in [3] for \( \mathcal{H}_{b_1}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,b_1}(\mathbb{R}^d) \).

1 Preliminaries

In this section we recall some facts about Gelfand-Shilov spaces, Pilipović spaces and modulation spaces.

Let \( s > 0 \). Then the (Fourier invariant) Gelfand-Shilov spaces \( S_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \) of Roumieu and Beurling type, respectively, consist of all \( f \in C^\infty(\mathbb{R}^d) \) such that

\[
\|f\|_{S_{s,r}} \equiv \sup_{\alpha,\beta \in \mathbb{N}^d} \left(\|x^\alpha D^\beta f\|_{L^\infty}(\mathbb{R}^d)\right) r^{\alpha+\beta!(\alpha!\beta!)^s}
\]  
(1.1)
is finite, for some \( r > 0 \) respectively for every \( r > 0 \). The topologies of \( S_\varepsilon (\mathbb{R}^d) \) and \( \Sigma_\varepsilon (\mathbb{R}^d) \) are the inductive limit topology and the projective limit topology, respectively, supplied by the norms \((1.1)\). We refer to [1,5] for more facts about Gelfand-Shilov spaces.

For \( \mathcal{H}_s (\mathbb{R}^d) \) and \( \mathcal{H}_{0,s} (\mathbb{R}^d) \) we consider the norms

\[
\| f \|_{\mathcal{H}_{s,r}} = \sup_{\alpha \in \mathbb{N}^d} \left( |c_\alpha(f)| e^{r|\alpha|^{2r}} \right) \quad \text{when } s \in \mathbb{R},
\]

and

\[
\| f \|_{\mathcal{H}_{s,r}} = \sup_{\alpha \in \mathbb{N}^d} \left( |c_\alpha(f)| r^{-|\alpha|} \right) \quad \text{when } s = b_\sigma,
\]

when \( r > 0 \) is fixed. Then the set \( \mathcal{H}_{s,r} (\mathbb{R}^d) \) consists of all \( f \in C^\infty (\mathbb{R}^d) \) such that \( \| f \|_{\mathcal{H}_{s,r}} \) is finite. It follows that \( \mathcal{H}_{s,r} (\mathbb{R}^d) \) is a Banach space.

The Pilipović spaces \( \mathcal{H}_s (\mathbb{R}^d) \) and \( \mathcal{H}_{0,s} (\mathbb{R}^d) \) are the inductive limit and the projective limit, respectively, of \( \mathcal{H}_{s,r} (\mathbb{R}^d) \) with respect to \( r > 0 \). In particular,

\[
\mathcal{H}_s (\mathbb{R}^d) = \bigcup_{r > 0} \mathcal{H}_{s,r} (\mathbb{R}^d) \quad \text{and} \quad \mathcal{H}_{0,s} (\mathbb{R}^d) = \bigcap_{r > 0} \mathcal{H}_{s,r} (\mathbb{R}^d)
\]

and it follows that \( \mathcal{H}_s (\mathbb{R}^d) \) is complete, and that \( \mathcal{H}_{0,s} (\mathbb{R}^d) \) is a Fréchet space. It is well-known that the identities \((0.3)\) and \((0.4)\) also hold in topological sense (cf. [7]).

By extending \( R_+ \) into \( R_\# = R_+ \cup \{ b_\sigma \}_{\sigma > 0} \) and letting

\[
s_1 < b_\sigma_1 < b_\sigma_2 < s_2 \quad \text{when} \quad s_2 \geq \frac{1}{2}, \ s_1 < \frac{1}{2} \quad \text{and} \quad \sigma_1 < \sigma_2,
\]

we have

\[
\mathcal{H}_{s_1} (\mathbb{R}^d) \subseteq \mathcal{H}_{0,s_2} (\mathbb{R}^d) \subseteq \mathcal{H}_{s_2} (\mathbb{R}^d), \quad s_1, s_2 \in R_\# \text{ and } s_1 < s_2.
\]

We also need some facts about weights and modulation spaces, a family of (quasi-)Banach spaces, introduced by Feichtinger in [2]. A weight on \( \mathbb{R}^d \) is a function \( \omega \in L^\infty_{\text{loc}} (\mathbb{R}^d) \) such that \( \omega(x) > 0 \) for every \( x \in \mathbb{R}^d \) and \( 1/\omega \in L^\infty_{\text{loc}} (\mathbb{R}^d) \). The weight \( \omega \) on \( \mathbb{R}^d \) is called moderate of polynomial type, if there is an integer \( N \geq 0 \) such that

\[
\omega(x + y) \leq \omega(x)(1 + |y|)^N, \quad x, y \in \mathbb{R}^d.
\]

The set of moderate weights of polynomial type on \( \mathbb{R}^d \) is denoted by \( \mathcal{P}(\mathbb{R}^d) \).

Let \( p, q \in (0, \infty], \phi \in \mathcal{P}(\mathbb{R}^d) \setminus 0 \) and \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \) be fixed. Then the modulation space \( M_{\omega}^{p,q} (\mathbb{R}^d) \) consists of all \( f \in \mathcal{P}(\mathbb{R}^d) \) such that

\[
\| f \|_{M_{\omega}^{p,q}} = \| V_\phi f \cdot \omega \|_{L^p L^q}
\]

is finite. Here \( V_\phi f \) is the short-time Fourier transform of \( f \) with respect to \( \phi \), given by

\[
V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} (f, e^{i\langle \cdot, \xi \rangle} \phi(\cdot - x))
\]

and

\[
\| F \|_{L^p L^q} = \| F \|_{L^p L^q(\mathbb{R}^{2d})} = \| gF \|_{L^p L^q(\mathbb{R}^d)} \quad \text{when} \quad gF(\xi) = \| F(\cdot, \xi) \|_{L^p(\mathbb{R}^d)}
\]

and \( F \) is measurable on \( \mathbb{R}^{2d} \).

Modulation spaces possess several convenient properties. For example we have the following proposition (see [2,4] for proofs).

\[ \square \]
2 Characterizations of \( \mathcal{H}_{p,q} (\mathbb{R}^d) \) and \( \mathcal{H}_{0,b,q} (\mathbb{R}^d) \) in terms of powers of the harmonic oscillator

In this section we deduce characterizations of the test function spaces \( \mathcal{H}_{p,q} (\mathbb{R}^d) \) and \( \mathcal{H}_{0,b,q} (\mathbb{R}^d) \).

More precisely we have the following.

**Theorem 2.1** Let \( \sigma > 0, \ N, N_0 \in \mathbb{N} \) be such that \( N_0 \sigma > 1, \ p_0 \in [1, \infty), \ p, q \in (0, \infty], \ \omega \in \mathcal{P}(\mathbb{R}^{2d}) \) and let \( f \in C^\infty(\mathbb{R}^d) \) be given by (0.1). Then the following conditions are equivalent:

1. \( f \in \mathcal{H}_{p,q} (\mathbb{R}^d) \) (\( f \in \mathcal{H}_{0,b,q} (\mathbb{R}^d) \));
2. for some \( r > 0 \) (for every \( r > 0 \)) it holds
   \[
   \{ c_\alpha (f) r^{-|\alpha|} (\sigma \alpha!)^{1/\sigma} \}_{\alpha \in \mathbb{N}^d} \in \ell^q (\mathbb{N}^d);
   \]
3. for some \( r > 0 \) (for every \( r > 0 \)) it holds
   \[
   \| H_N^d f \|_{L^p (\mathbb{R}^d)} \lesssim 2^N r^{N/\log (N \sigma)} \left( \frac{2N \sigma}{\log (N \sigma)} \right)^N (1 - \frac{1}{\log (N \sigma)}), \quad N \geq N_0;
   \]  
   \[
   (2.1)
   \]
4. for some \( r > 0 \) (for every \( r > 0 \)) it holds
   \[
   \| H_N^d f \|_{M_{p,q} (\mathbb{R}^d)} \lesssim 2^N r^{N/\log (N \sigma)} \left( \frac{2N \sigma}{\log (N \sigma)} \right)^N (1 - \frac{1}{\log (N \sigma)}), \quad N \geq N_0.
   \]  
   \[
   (2.2)
   \]

We need some preparations for the proof. In the following proposition we treat separately the equivalence between (3) and (4) in Theorem 2.1.

**Proposition 2.2** Let \( p_0 \in [1, \infty], \ p, q \in (0, \infty], \ \sigma > 0 \ N_0 > \sigma^{-1} \) be an integer and let \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \). Then the following conditions are equivalent:

1. \( \text{(2.1) holds for some } r > 0 \) (for every \( r > 0 \));
2. \( \text{(2.2) holds for some } r > 0 \) (for every \( r > 0 \)).

We need the following lemma for the proof of Proposition 2.2.

**Lemma 2.3** Let \( R \geq e, I = (0, R) \),

\[
 g(r, t_1, t_2) \equiv \frac{r_{t_2}^2}{r_{t_1}^2} \quad \text{and} \quad h(t_1, t_2) \equiv \begin{cases} 2t_1 (1 - \frac{1}{\log t_2}) & \text{if } 2t_1 \geq 1 \\ \frac{2t_1}{\log t_1} & \text{if } 2t_1 < 1 \end{cases},
\]
when \( t_1, t_2 > e \) and \( r > 0 \). Then
\[
0 \leq g(r, t_1, t_2) \leq C \quad \text{and} \quad 0 \leq h(t_1, t_2) \leq \left( \frac{2t_1}{\log t_1} \right)^C
\]  \hspace{1cm} (2.3)

when
\[
t_1, t_2 > R, \ 0 \leq t_2 - t_1 \leq R, \ r \in I,
\]
for some constant \( C > 0 \) which only depends on \( R \).

**Proof** Since \( t \mapsto \frac{t}{\log t} \) is increasing when \( t \geq e \), \( g \) is upper bounded by one when \( r \leq 1 \), and the boundedness of \( g \) follows in this case.

If \( r \geq 1 \), \( t = t_1, u = t_2 - t_1 > 0 \) and \( \rho = \log r \), then
\[
0 \leq \log g(r, t_1, t_2) = \left( \frac{t + u}{\log(t + u)} - \frac{t}{\log t} \right) \rho
\]
\[
= \frac{t}{\log t} \left( \frac{1 + \frac{u}{t}}{1 + \frac{u}{\log t}} - 1 \right) \rho = \frac{t}{\log t} \left( \frac{\frac{u}{t} - \frac{1}{\log t}}{1 + \frac{1}{\log t}} \right) \rho
\]
\[
< \frac{t}{\log t} \cdot \frac{u}{t} \cdot \rho
\]
\[
= \frac{u \rho}{\log t} \leq C
\]
for some constant \( C \) which only depends on \( R \). This shows the boundedness of \( g \).

Next we show the estimates for \( h(t_1, t_2) \) in (2.3). By taking the logarithm of \( h(t_1, t_2) = h(t, t_2) \) we get
\[
\log h(t, t_2) = t_2 \log \left( \frac{2t_2}{\log t_2} \right) - t \log \left( \frac{2t}{\log t} \right) - b(t, t_2),
\]
where
\[
b(t, t_2) = \left( \frac{t_2}{\log t_2} \log \left( \frac{2t_2}{\log t_2} \right) - \frac{t}{\log t} \log \left( \frac{2t}{\log t} \right) \right).
\]
Since \( b(t, t_2) > 0 \) when \( t_2 > t \), we get
\[
\log h(t_1, t_2) < t_2 \log \left( \frac{2t_2}{\log t_2} \right) - t \log \left( \frac{2t}{\log t} \right)
\]
\[
= (t + u) \left( \log \left( \frac{2t}{\log t} \right) + \log \left( \frac{1 + \frac{u}{t}}{1 + \frac{1}{\log t}} \right) \right) - t \log \left( \frac{2t}{\log t} \right)
\]
\[
\leq u \log \left( \frac{2t}{\log t} \right) + t \log \left( 1 + \frac{u}{t} \right) + C
\]
\[
\leq u \log \left( \frac{2t}{\log t} \right) + u + C
\]
for some constant \( C \geq 0 \). Here we have used that \( t_1, t_2 > R \geq e \) and the fact that \( t \mapsto \frac{t}{\log t} \) increases for \( t \geq R \).
Proof of Proposition 2.2.} First we prove that (2.2) is independent of \( N_0 > \sigma^{-1} \) when \( p, q \geq 1 \). Evidently, if (2.2) is true for \( N_0 \), then it is true for any larger replacement of \( N_0 \). On the other hand, the map

\[
H_d^N : M_{(v_N,\omega)}^{p,q}(\mathbb{R}^d) \to M_{(\omega)}^{p,q}(\mathbb{R}^d), \quad v_N(x, \xi) = (1 + |x|^2 + |\xi|^2)^N,
\]

and its inverse are continuous and bijective (cf. e.g. [8, Theorem 3.10]). Hence, if \( \sigma^{-1} < N_1 \leq N_0 \), \( N_2 = N_0 - N_1 \geq 0 \) and (2.2) holds for \( N_0 \), then

\[
\| H_d^{N_1} f \|_{M_{(\omega)}^{p,q}} \lesssim \| H_d^{N_0} f \|_{M_{(\omega)}^{p,q}} \lesssim \| H_d^{N_0} f \|_{M_{(\omega)}^{p,q}} < \infty,
\]

and a straightforward combination of these estimates and (2.3) shows that (2.2) holds for \( N_1 \) in place of \( N_0 \). This implies that (2.2) is independent of \( N_0 > \sigma^{-1} \) when \( p, q \geq 1 \).

Next we prove that (2.2) is independent of the choice of \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \). By the first part of the proof, we may assume that \( N_0 \sigma > e \). For every \( \omega_1, \omega_2 \in \mathcal{P}(\mathbb{R}^{2d}) \), we may find an integer \( N_0 > \sigma^{-1}e \) such that

\[
\frac{1}{v_{N_0}} \lesssim \omega_1, \omega_2 \lesssim v_{N_0},
\]

and then

\[
\| f \|_{M_{(1/v_{N_0})}^{p,q}} \lesssim \| f \|_{M_{(1/v_{N_0})}^{p,q}}, \quad \| f \|_{M_{(1/v_{N_0})}^{p,q}} \lesssim \| f \|_{M_{(1/v_{N_0})}^{p,q}}.
\]

Hence the stated invariance follows if we prove that (2.2) holds for \( \omega = v_{N_0} \), if it is true for \( \omega = 1/v_{N_0} \).

Therefore, assume that (2.2) holds for \( \omega = 1/v_{N_0} \). Let \( f_N = H_d^N f, u = 2N_0 \sigma, t = t_1 = N \sigma, N_2 = N + 2N_0 \) and \( t_2 = t_1 + u = N_2 \sigma \). If \( N \geq 2N_0 \), then the bijectivity of (2.4) gives

\[
\begin{align*}
\| f_N \|_{M_{(1/v_{N_0})}^{p,q}}^{\sigma} & \lesssim \left( \frac{N_2}{\log(N_2)} \right)^{N_2} + t \left( \frac{t}{\log(t)} \right)^{t} \left( \frac{1}{1 - \log(t)} \right) \left( \frac{N_2}{\log(N_2)} \right)^{N_2} \left( \frac{t}{\log(t_2)} \right)^{t_2} \left( \frac{1}{1 - \log(t_2)} \right) \\
& = 2^u g(r, t_1, t_2) h(t_1, t_2), \quad \frac{\| f_N \|_{M_{(1/v_{N_0})}^{p,q}}^{\sigma}}{\| f_N \|_{M_{(1/v_{N_0})}^{p,q}}^{\sigma}} \lesssim \frac{\| f_N \|_{M_{(1/v_{N_0})}^{p,q}}^{\sigma}}{\| f_N \|_{M_{(1/v_{N_0})}^{p,q}}^{\sigma}},
\end{align*}
\]

where \( g(r, t_1, t_2) \) and \( h(t_1, t_2) \) are the same as in Lemma 2.3. A combination of Lemma 2.3, (2.6) and the fact that \( N \sigma > e \) shows that (2) is independent of \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \). For general \( p, q > 0 \), the invariance of (2.2) with respect to \( \omega, p \) and \( q \) is a consequence of the embeddings

\[
M_{(v_N,\omega)}^{\infty}(\mathbb{R}^d) \subseteq M_{(\omega)}^{p,q}(\mathbb{R}^d) \subseteq M_{(\omega)}^{\infty}(\mathbb{R}^d), \quad N > d \left( \frac{1}{p} + \frac{1}{q} \right)
\]

(see e.g. [4, Theorem 3.4] or [10, Proposition 3.5]).
The equivalence between (1) and (2) now follows from these invariance properties and the continuous embeddings
\[ M^{p_0,q_1} \subseteq L^{p_0} \subseteq M^{p_0,q_2}, \quad q_1 = \min(p_0, p'_0), \quad q_2 = \max(p_0, p'_0), \]
which can be found in e.g. [9, Proposition 1.7].

**Proposition 2.4** Let \( f \in C^\infty(\mathbb{R}^d) \) and \( \sigma > 0 \). If
\[
\| H_d^N f \|_{L^2} \lesssim 2^N r \frac{2N\sigma}{\log(N\sigma)} \left( \frac{N(1 - \frac{1}{\log(N\sigma)})}{\log(N\sigma)} \right)^N, \quad N \in \mathbb{N}, \ N\sigma \geq e, \tag{2.7}
\]
for some \( r > 0 \) (for every \( r > 0 \)), then
\[
|c_\alpha(f)| \lesssim r^{|\alpha|}|\alpha|^{-\frac{|\alpha|}{2r}}, \quad \alpha \in \mathbb{N}^d, \tag{2.8}
\]
for some \( r > 0 \) (for every \( r > 0 \)).

**Proposition 2.5** Let \( f \in C^\infty(\mathbb{R}^d) \) and \( \sigma > 0 \). If (2.8) holds for some \( r > 0 \) (for every \( r > 0 \)), then (2.7) holds for some \( r > 0 \) (for every \( r > 0 \)).

For the proofs we need some preparation lemmas.

**Lemma 2.6** Let \( \sigma > 0, \sigma_0 \in [0, \sigma] \) and let
\[
F(r, t) = \left( \frac{2t}{\log t} \right)^{t \left( 1 - \frac{1}{\log t} \right)} r^{\frac{t}{\log t}}, \quad r \geq 0, \quad t \geq e \cdot \max(1, \sigma).
\]
Then
\[
F(r, t) \leq F(r, t + \sigma_0), \quad r \in [1, \infty), \tag{2.9}
\]
and
\[
F(r, t) \leq F(r^{1 - \frac{1}{r}}, t + \sigma_0), \quad r \in (0, 1]. \tag{2.10}
\]

**Proof** If \( r \geq 1 \), then it follows by straight-forward tests with derivatives that \( F(r, t) \) is increasing with respect to \( t \geq e \). This gives (2.9).

In order to prove (2.10), let \( t_1 = t + \sigma_0 \) and
\[
h(t_1, \sigma_0) = \frac{1 - \frac{\sigma_0}{t_1}}{1 + \frac{\sigma_0}{\log(t_1)}},
\]
where \( 0 \leq \sigma_0 \leq \sigma \). Then
\[
\left( \frac{2t}{\log t} \right)^{t \left( 1 - \frac{1}{\log t} \right)} r^{\frac{t}{\log t}} \leq \left( \frac{2t_1}{\log t_1} \right)^{t_1 \left( 1 - \frac{1}{\log t_1} \right)} r^{\frac{t_1}{\log t_1}} \tag{2.11}
\]
and
\[
\frac{2t}{\log t} = h(t_1, \sigma_0) \cdot \frac{2t_1}{\log t_1}.
\]
Since
\[
0 \leq \frac{\sigma_0}{t_1} \leq \frac{1}{e} \quad \text{and} \quad -1 < \frac{\log \left(1 - \frac{\sigma_0}{\sigma} \right)}{\log t_1} \leq 0
\]
we get

\[ h(t_1, \sigma_0) \geq 1 - \frac{\sigma_0}{t_1} \geq 1 - \frac{1}{e}. \]

Hence the facts \( \frac{r}{\log t} \geq 1 \) and \( 0 < r \leq 1 \) give

\[ r \frac{r}{\log t} = r^{h(t_1, \sigma_0)} \frac{r}{\log t_1} \leq r^{(1 - \frac{1}{e})} \frac{r}{\log t_1}. \]

A combination of the latter inequality with (2.11) gives

\[ F(r, t) \leq \left( \frac{2t_1}{\log t_1} \right)^{t_1 \left( 1 - \frac{1}{\log t_1} \right)} \left( r^{1 - \frac{1}{e}} \right)^{\frac{r}{\log t_1}} = F \left( r^{1 - \frac{1}{e}}, t_1 \right). \]

\[ \Box \]

**Lemma 2.7** Let \( s \geq \sigma (e + 1) + e^2 \)

\[ \Omega_1 = [e, \infty) \cap (\sigma \cdot N) \quad \text{and} \quad \Omega_2 = [e, \infty). \]

Then the following is true:

1. For any \( r_2 > 0 \), there is an \( r_1 > 0 \) such that

\[ \inf_{t \in \Omega_j} \left( s - t \left( \frac{2t}{\log t} \right)^{t \left( 1 - \frac{1}{\log t} \right)} \left( r_1^{1 - \frac{1}{e}} \right)^{\frac{r}{\log t_1}} \right) \leq r_2 s^{-\frac{1}{e}}, \quad j = 1, 2; \]

(2.12)

2. For any \( r_1 > 0 \), there is an \( r_2 > 0 \) such that (2.13) holds.

**Proof** First prove the result for \( j = 2 \). Let

\[ x = \log t, \quad y = \log s \geq \log(\sigma (e + 1) + e^2) > 2, \quad \rho_j = \log r_j, \quad j = 1, 2. \]

By applying the logarithm on (2.12), the statements (1) and (2) follow if we prove:

1. For any \( \rho_2 \in \mathbb{R} \), there is a \( \rho_1 \in \mathbb{R} \) such that

\[ \inf_{x \geq x_0} F(x) \leq 0, \quad x_0 = \log(\sigma (e + 1) + e^2) \]

(2.13)

where

\[ F(x) = -e^x y + e^x \left( 1 - \frac{1}{x} \right) (x + \log 2 - \log x) + \rho_1 \frac{e^x}{x} - \rho_2 e^y + \frac{e^y y}{2} \]

(2.14)

2. For any \( \rho_1 \in \mathbb{R} \), there is a \( \rho_2 \in \mathbb{R} \) such that (2.13) holds.

We choose

\[ x = y + \log y - \log 2 \geq \log s \geq x_0 \quad \text{and let} \quad h = g(y), \]

where

\[ g(u) = \frac{\log u - \log 2}{u}. \]

Obviously, \( x \) increases with \( y \), and by function investigations it follows that

\[ 0 = g(2) < g(u) \leq g(2e) = \frac{1}{2e}, \quad u > 2, \]
Let $C$ for some large number $2.6$ can be applied since $\epsilon > 1$. Then (2.14) becomes

$$
e^{-\gamma} F(y + \log y - \log 2) = -\frac{y^2}{2} + \frac{y}{2} \left(1 - \frac{1}{y + \log \frac{y}{2}}\right) \left(y + \log y - \log \left(y + \log \frac{y}{2}\right)\right)$$

$$+ \frac{\rho_1 y}{2(y + \log \frac{y}{2})} - \rho_2 + \frac{y}{2}$$

$$= -\frac{y}{2} \log(1 + h) + \frac{\log y + \log(1 + h)}{2(1 + h)} + \frac{\rho_1 - \log 2}{2(1 + h)} - \rho_2.$$

If $\rho_1 \in \mathbb{R}$ is fixed, then we choose $\rho_2 \in \mathbb{R}$ such that

$$\rho_1 - \log 2 \leq -C_0$$

(2.15)

for some large number $C_0 > 0$. In the same way, if $\rho_2 \in \mathbb{R}$ is fixed, then we choose $\rho_1 \in \mathbb{R}$ such that (2.15) holds. For such choices and the fact that $0 < h < 1$, the inequalities

$$0 < h - \frac{h^2}{2} \leq \log(1 + h) \leq h$$

give

$$F(y + \log y - \log 2) \leq e^y \left(-\frac{y}{2} \log(1 + h) + \frac{\log y + \log(1 + h)}{2(1 + h)} - C_0\right)$$

$$\leq e^y \left(-\frac{y}{2} \log(1 + h) + \frac{\log y + \log(1 + h)}{2} - C_0\right)$$

$$\leq e^y \left(-\frac{\log y - \log 2}{2} + \frac{(\log y - \log 2)^2}{4y} + \frac{1}{2} (\log y + h) - C_0\right)$$

$$\leq e^y \left(\frac{1}{2} \log 2 + \frac{(\log y - \log 2)^2}{4y} + \frac{h}{2} - C_0\right) < 0,$$

provided $C_0$ was chosen large enough. This gives the result in the case $j = 2$.

Next we prove the result for $j = 1$. Let $t_2 > 0$. By the first part of the proof, there are $t_1 \geq e(\sigma + 1) + \sigma$ and $r_0 > 0$ such that

$$s^{-t_1} \left(\frac{2t_1}{\log t_1}\right)^{t_1} \left(1 - \frac{1}{\log t_1}\right) = r_0^{t_1} \leq r_2^{x-t} s^{-t}.$$

Let $r_1 = r_0$ if $r_0 \geq 1$ and $r_1 = r_0^{x-t}$ otherwise. By Lemma 2.6 it follows that

$$s^{-t} \left(\frac{2t}{\log t}\right)^{t} \left(1 - \frac{1}{\log t}\right) r_1^{t} \leq r_2^{x-t} s^{-t}$$

holds when $t = N\sigma$ and $N \in \mathbb{N}$ is chosen such that $0 \leq t_1 - N\sigma \leq \sigma$. Observe that Lemma 2.6 can be applied since $N\sigma \geq e(\sigma + 1)$. This gives (1) for $j = 1$.

By similar arguments, (2) for $j = 1$ follows from (2) in the case $j = 2$. The details are left for the reader. \hfill \Box

**Proof of Proposition 2.4.** Suppose that (2.7) holds for some $r = r_1 > 0$. By

$$c_{\alpha}(H^{N}_{\delta} f) = (2|\alpha| + d)N c_{\alpha}(f), \quad |c_{\alpha}(H^{N}_{\delta} f)| \leq \|H^{N}_{\delta} f\|_{L^2}$$

(2.16)
and (2.7) we get
\[
|c_\alpha(f)| = \frac{|c_\alpha(H_d^N f)|}{(2|\alpha| + d)^N} \\
\leq \left(|\alpha| + \frac{d}{2}\right)^{-N} r_1^{\frac{N}{\log(N\sigma)}} N^\left(1 - \frac{1}{\log(N\sigma)}\right) \\
\leq \left(|\alpha|^{-N\sigma} r_1^{\frac{N\sigma}{\log(N\sigma)}} N^\sigma \left(1 - \frac{1}{\log(N\sigma)}\right)\right)^{\frac{1}{\sigma}}.
\]

By taking the infimum over all \( N \geq 0 \), it follows from Lemma 2.7 (2) that
\[
|c_\alpha(f)| \lesssim \left(r_2^{|\alpha|} |\alpha|^{-\frac{|\alpha|}{2}}\right)^{\frac{1}{\sigma}} = r^{|\alpha|} |\alpha|^{-\frac{|\alpha|}{2\sigma}}, \quad |\alpha| \geq 2\sigma (e + 1) + e^2,
\]
for some \( r_2 > 0 \), where \( r = r_2^{\frac{1}{\sigma}} \). Hence (2.8) holds for some \( r > 0 \).

By similar arguments, using (1) instead of (2) in Lemma 2.7, it follows that if (2.7) holds for every \( r > 0 \), then (2.8) holds for every \( r > 0 \).

For the proof of Proposition 2.5 we will use the following result which is essentially a slight clarification of [3, Lemma 2]. The proof is therefore omitted.

**Lemma 2.8** Let \( r > 0 \) and
\[
f(s, t, r) = \frac{s^{2t} (2r e)^s}{s^s}, \quad s > 1, \ t \geq 0.
\]

Then there exist a positive increasing function \( \theta \) on \([0, \infty)\) and a constant \( t_0 = t_0(r) > e \) which only depends on \( r \) such that
\[
\max_{s > 0} f(s, t, r) \leq \left(\frac{2t}{\log t}\right)^{2t} (\theta(r))^\frac{2t}{\log t}, \quad t \geq t_0(r). \quad (2.17)
\]

**Remark 2.9** The constants \( s, t \) and \( t_0(r) \) in Lemma 2.8 are denoted by \( t, N \) and \( N_0(r) \), respectively in Lemmas 1 and 2 in [3]. In the latter results it is understood that \( N \) and \( N_0(r) \) are integers. On the other hand, it is evident from the proofs of these results that they also hold when \( N \) and \( N_0(r) \) are allowed to be in \( \mathbb{R}_+ \).

**Proof of Proposition 2.5.** Let \( \theta \) be as in Lemma 2.8 and let \( \rho \in (0, 1) \). Suppose that (2.8) holds for some \( r > 0 \) and let \( r_2 > r^\sigma \). From (2.8) and (2.16) we get
\[
\|H_d^N f\|_{L^2}^2 = \sum_{\alpha \in \mathbb{N}^d} |(2|\alpha| + d)^N c_\alpha(f)|^2 \\
\leq \sup_{|\alpha| \geq 1} \left(2|\alpha| + d\right)^{2N} r_2^{2|\alpha| |\alpha|^{-\frac{|\alpha|}{\sigma}}} \\
= \sup_{s \geq 1} \left(2^{2t} \left(s + \frac{d}{2}\right)^{2t} r_2^{2s} s^{-\frac{s}{\sigma}}\right)^{\frac{1}{\sigma}}.
\]
where $s = |\alpha|$ and $t = N\sigma$. Since $0 < \rho < 1$ we have

$$s^s = (s - \frac{d}{\rho})^{s - \frac{d}{\rho}} \leq \left( s - \frac{d}{2}\right)^{s - \frac{d}{2}} \left( 1 + \frac{d}{2s - d}\right)^{s - \frac{d}{2}}.$$ 

This gives

$$\|H_N^d f\|_2^2 \lesssim \sup_{s \geq 1} \left( 2^{2t} \left( s + \frac{d}{2}\right) 2^{4s} s^{-\frac{d}{2}} \right)\frac{1}{\sigma}$$

Using (2.18) and Lemma 2.8 we obtain

$$\|H_N^d f\|_2^2 \lesssim \sup_{s \geq 1 + \frac{d}{2}} \left( 2^{2t} s^{2t} \left( \frac{r_2}{\rho}\right) 2s - s^{\frac{d}{2}} \right)\frac{1}{\sigma}$$

when $c \in (0, 1)$, which shows that (2) is independent of the choice of $q$. The equivalence between (1) and (2) now follows by the definitions and choosing $q = \infty$ in (2).

**Proof of Theorem 2.1.** We have

$$\|\{c_\alpha(f)^r - |\alpha|^r(\alpha!)^{\frac{1}{\alpha'}}\}_{\alpha \in \mathbb{N}^d}\|_{\ell^\infty(\mathbb{N}^d)} \leq \|\{c_\alpha(f)^r - |\alpha|^r(\alpha!)^{\frac{1}{\alpha'}}\}_{\alpha \in \mathbb{N}^d}\|_{\ell^q(\mathbb{N}^d)}$$

when $c \in (0, 1)$, which shows that (2) is independent of the choice of $q$. The equivalence between (1) and (2) now follows by the definitions and choosing $q = \infty$ in (2).
By Proposition 2.2 we may assume that $p = 2$. The result now follows from Propositions 2.4 and 2.5, together with the fact that

$$(d \cdot e)^{-|\alpha|} |\alpha|^{|\alpha|} \leq \alpha! \leq |\alpha|^{|\alpha|}, \quad \alpha \in \mathbb{N}^d.$$ 

\[\square\]

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