FOUR GAMES ON BOOLEAN ALGEBRAS

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Abstract
The games $G_2$ and $G_3$ are played on a complete Boolean algebra $B$ in $\omega$-many
moves. At the beginning White picks a non-zero element $p$ of $B$ and, in the
$n$-th move, White picks a positive $p_n < p$ and Black chooses an $i_n \in \{0, 1\}$. 
White wins $G_2$ iff $\lim \inf p_{i_n}^n = 0$ and wins $G_3$ iff $\bigvee_{A \in [\omega]} \bigwedge_{n \in A} p_{i_n}^n = 0$. It is shown that White has a winning strategy in the game $G_2$ iff White has a winning strategy in the cut-and-choose game $G_{c&c}$ introduced by Jech. Also, White has a winning strategy in the game $G_3$ iff forcing by $B$ produces a subset $R$ of the tree $\mathcal{L}_2$ containing either $\varphi \downarrow 0$ or $\varphi \downarrow 1$, for each $\varphi \in \mathcal{L}_2$, and having unsupported intersection with each branch of the tree $\mathcal{L}_2$ belonging to $V$. On the other hand, if forcing by $B$ produces independent (splitting) reals then White has a winning strategy in the game $G_3$ played on $B$. It is shown that $\diamond$ implies the existence of an algebra on which these games are undetermined.

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1. Introduction
In [3] Jech introduced the cut-and-choose game $G_{c&c}$, played by two players, White and Black, in $\omega$-many moves on a complete Boolean algebra $B$ in the following way. At the beginning, White picks a non-zero element $p \in B$ and, in the $n$-th move, White picks a non-zero element $p_n < p$ and Black chooses an $i_n \in \{0, 1\}$. In this way two players build a sequence $\langle p, p_0, i_0, p_1, i_1, \ldots \rangle$ and White wins iff $\bigwedge_{n \in \omega} p_{i_n}^n = 0$ (see Definition[1]).

A winning strategy for a player, for example White, is a function which, on the basis of the previous moves of both players, provides “good” moves for White such that White always wins. So, for a complete Boolean algebra $B$ there are three possibilities: 1) White has a winning strategy; 2) Black has a winning strategy or 3) none of the players has a winning strategy. In the third case the game is said to be undetermined on $B$.

The game-theoretic properties of Boolean algebras have interesting algebraic and forcing translations. For example, according to [3] and well-known facts concerning infinite distributive laws we have the following results.

1
Theorem 1. (Jech) For a complete Boolean algebra $B$ the following conditions are equivalent:

(a) White has a winning strategy in the game $G_{c\&c}$;
(b) The algebra $B$ does not satisfy the $(\omega_2, 2)$-distributive law;
(c) Forcing by $B$ produces new reals in some generic extension;
(d) There is a countable family of 2-partitions of the unity having no common refinement.

Also, Jech investigated the existence of a winning strategy for Black and using ♦ constructed a Suslin algebra in which the game $G_{c\&c}$ is undetermined. Moreover in [6] Zapletal gave a ZFC example of a complete Boolean algebra in which the game $G_{c\&c}$ is undetermined.

Several generalizations of the game $G_{c\&c}$ were considered. Firstly, instead of cutting of $p$ into two pieces, White can cut into $\lambda$ pieces and Black can choose more than one piece (see [3]). Secondly, the game can be of uncountable length so Dobrinen in [11] and [2] investigated the game $G^{\leq \mu}_k(\lambda)$ played in $\kappa$-many steps in which White cuts into $\lambda$ pieces and Black chooses less then $\mu$ of them.

In this paper we consider three games $G_2$, $G_3$ and $G_4$ obtained from the game $G_{c\&c}$ (here denoted by $G_1$) by changing the winning criterion in the following way.

Definition 1. The games $G_k$, $k \in \{1, 2, 3, 4\}$, are played by two players, White and Black, on a complete Boolean algebra $B$ in $\omega$-many moves. At the beginning White chooses a non-zero element $p \in B$. In the $n$-th move White chooses a $p_n \in (0, p)_B$ and Black responds choosing $p_n$ or $p \setminus p_n$ or, equivalently, picking an $i_n \in \{0, 1\}$ chooses $p_i^n$, where, by definition, $p_0^n = p_n$ and $p_1^n = p \setminus p_n$. White wins the play $\langle p, p_0, i_0, p_1, i_1, \ldots \rangle$ in the game

$G_1$ if and only if $\bigwedge_{n \in \omega} p_i^n = 0$;
$G_2$ if and only if $\bigvee_{k \in \omega} \bigwedge_{n \geq k} p_i^n = 0$, that is $\liminf p_i^n = 0$;
$G_3$ if and only if $\bigvee_{A \in [\omega]^{<\omega}} \bigwedge_{n \in A} p_i^n = 0$;
$G_4$ if and only if $\bigwedge_{k \in \omega} \bigvee_{n \geq k} p_i^n = 0$, that is $\limsup p_i^n = 0$.

In the following theorem we list some results concerning the game $G_4$ which are contained in [5].

Theorem 2. (a) White has a winning strategy in the game $G_4$ played on a complete Boolean algebra $B$ iff forcing by $B$ collapses $\epsilon$ to $\omega$ in some generic extension.
(b) If $B$ is the Cohen algebra r.o.($\langle \omega, 2 \rangle$) or a Maharam algebra (i.e. carries a positive Maharam submeasure) then Black has a winning strategy in the game $G_4$ played on $B$.
(c) ♦ implies the existence of a Suslin algebra on which the game $G_4$ is undetermined.
The aim of the paper is to investigate the game-theoretic properties of complete Boolean algebras related to the games \( G_2 \) and \( G_3 \). So, Section 2 contains some technical results, in Section 3 we consider the game \( G_2 \), Section 4 is devoted to the game \( G_3 \) and Section 5 to the algebras on which these games are undetermined.

Our notation is standard and follows [4]. A subset of \( \omega \) belonging to a generic extension will be called supported iff it contains an infinite subset of \( \omega \) belonging to the ground model. In particular, finite subsets of \( \omega \) are unsupported.

2. Winning a play, winning all plays

Using the elementary properties of Boolean values and forcing it is easy to prove the following two statements.

**Lemma 1.** Let \( B \) be a complete Boolean algebra, \( \langle b_n : n \in \omega \rangle \) a sequence in \( B \) and \( \sigma = \{ \langle \bar{n}, b_n \rangle : n \in \omega \} \) the corresponding name for a subset of \( \omega \). Then

(a) \( \bigwedge_{n \in \omega} b_n = \| \sigma = \bar{\omega} \| \);
(b) \( \liminf b_n = \| \sigma \text{ is cofinite}\| \);
(c) \( \bigvee_{A \in [\omega]^{\omega}} \bigwedge_{n \in A} b_n = \| \sigma \text{ is supported}\| \);
(d) \( \limsup b_n = \| \sigma \text{ is infinite}\| \).

**Lemma 2.** Let \( B \) be a complete Boolean algebra, \( p \in B^+, \langle p_n : n \in \omega \rangle \) a sequence in \( (0, p)_B \) and \( \langle i_n : n \in \omega \rangle \in \omega^2 \). For \( k \in \{0, 1\} \) let \( S_k = \{ n \in \omega : i_n = k \} \) and let the names \( \tau \) and \( \sigma \) be defined by \( \tau = \{ \langle \bar{n}, p_n \rangle : n \in \omega \} \) and \( \sigma = \{ \langle \bar{n}, p_n^i \rangle : n \in \omega \} \). Then

(a) \( p' \models \tau = \sigma = \bar{0} \);
(b) \( p \models \tau = \sigma \triangle \bar{S}_1 \);
(c) \( p \models \sigma = \tau \triangle \bar{S}_1 \);
(d) \( p \models \sigma = \bar{\omega} \iff \tau = \bar{S}_0 \);
(e) \( p \models \sigma = \bar{\omega} \iff \tau = \bar{S}_0 \);
(f) \( p \models | \sigma| < \bar{\omega} \iff \tau = \bar{S}_1 \).

**Theorem 3.** Under the assumptions of Lemma 2, White wins the play \( \langle p, p_0, i_0, p_1, i_1, \ldots \rangle \) in the game

- \( G_1 \) iff \( \| \sigma \text{ is not equal to } \bar{\omega} \| = 1 \) iff \( p \models \tau \neq \bar{S}_0 \);
- \( G_2 \) iff \( \| \sigma \text{ is not cofinite} \| = 1 \) iff \( p \models \tau \neq \bar{S}_0 \);
- \( G_3 \) iff \( \| \sigma \text{ is not supported} \| = 1 \) iff \( p \models "\tau \cap \bar{S}_0 \text{ and } \bar{S}_1 \setminus \tau \text{ are unsupported}" \);
- \( G_4 \) iff \( \| \sigma \text{ is not infinite} \| = 1 \) iff \( p \models \tau = \bar{S}_1 \).

**Proof.** We will prove the statement concerning the game \( G_3 \) and leave the rest to the reader. So, White wins \( G_3 \) iff \( \bigvee_{A \in [\omega]^{\omega}} \bigwedge_{n \in A} p_n^i = 0 \), that is, by Lemma [1] \( \| \sigma \text{ is not supported} \| = 1 \) and the first equivalence is proved.
Let $1 \models \text{“} \sigma \text{ is not supported} \text{”}$ and let $G$ be a $\mathbb{B}$-generic filter over $V$ containing $p$. Suppose $\tau_G \cap S_0$ or $S_1 \setminus \tau_G$ contains a subset $A \in [\omega]^{\omega} \cap V$. Then $A \subseteq \sigma_G$, which is impossible.

On the other hand, let $p \models \text{“} \tau \cap \check{S}_0 \text{ and } \check{S}_1 \setminus \tau \text{ are unsupported} \text{”}$ and let $G$ be a $\mathbb{B}$-generic filter over $V$. If $p' \in G$ then, by Lemma 2(a), $\sigma_G = \emptyset$ so $\sigma_G$ is unsupported. Otherwise $p \in G$ and by the assumption the sets $\tau_G \cap S_0$ and $S_1 \setminus \tau_G$ are unsupported. Suppose $A \subseteq \sigma_G$ for some $A \in [\omega]^{\omega} \cap V$. Then $A = A_0 \cup A_1$, where $A_0 = A \cap S_0 \cap \tau_G$ and $A_1 = A \cap S_1 \setminus \tau_G$, and at least one of these sets is infinite. But from Lemma 2(c) we have $A_0 = A \cap S_0$ and $A_1 = A \cap S_1$, so $A_0, A_1 \in V$. Thus either $S_0 \cap \tau_G$ or $S_1 \setminus \tau_G$ is a supported subset of $\omega$, which is impossible. So $\sigma_G$ is unsupported and we are done. $\blacksquare$

In the same way one can prove the following statement concerning Black.

**Theorem 4.** Under the assumptions of Lemma 2 Black wins the play $\langle p, p_0, i_0, p_1, i_1, \ldots \rangle$ in the game $G_1$ iff $\| \sigma \text{ is equal to } \check{\omega} \| > 0$ iff $\exists q \leq p \ q \models \tau = \check{S}_0$;
$\mathcal{G}_2$ iff $\| \sigma \text{ is cofinite } \| > 0$ iff $\exists q \leq p \ q \models \tau = \check{S}_0$;
$\mathcal{G}_3$ iff $\| \sigma \text{ is supported } \| > 0$ iff $\exists q \leq p \ q \models \text{“} \tau \cap \check{S}_0 \text{ or } \check{S}_1 \setminus \tau \text{ is supported} \text{”}$;
$\mathcal{G}_4$ iff $\| \sigma \text{ is infinite } \| > 0$ iff $\exists q \leq p \ q \models \tau \neq \check{S}_1$.

Since for each sequence $\langle b_n \rangle$ in a c.B.a. $\mathbb{B}$

$$\bigwedge_{n \in \omega} b_n \leq \liminf b_n \leq \bigvee_{A \in [\omega]^{\omega}} \bigwedge_{n \in A} b_n \leq \limsup b_n,$$

we have

**Proposition 1.** Let $\mathbb{B}$ be a complete Boolean algebra. Then

(a) White has a w.s. in $\mathcal{G}_4$ $\Rightarrow$ White has a w.s. in $\mathcal{G}_3$ $\Rightarrow$ White has a w.s. in $\mathcal{G}_2$ $\Rightarrow$ White has a w.s. in $\mathcal{G}_1$.

(b) Black has a w.s. in $\mathcal{G}_1$ $\Rightarrow$ Black has a w.s. in $\mathcal{G}_2$ $\Rightarrow$ Black has a w.s. in $\mathcal{G}_3$ $\Rightarrow$ Black has a w.s. in $\mathcal{G}_4$.

### 3. The game $\mathcal{G}_2$

**Theorem 5.** For each complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:

(a) $\mathbb{B}$ is not $(\omega, 2)$-distributive;

(b) White has a winning strategy in the game $\mathcal{G}_1$;

(c) White has a winning strategy in the game $\mathcal{G}_2$. 

Proof. (a)$\Leftrightarrow$(b) is proved in [3] and (c)$\Rightarrow$(b) holds by Proposition 1. In order to prove (a)$\Rightarrow$(c) we suppose $\mathcal{B}$ is not $(\omega,2)$-distributive. Then $p := \|\exists x \subseteq \omega \; x \notin V\| > 0$ and by The Maximum Principle there is a name $\pi \in V^{\mathcal{B}}$ such that

$$p \models \pi \subseteq \omega \wedge \pi \notin V. \quad (2)$$

Clearly $\omega = A_0 \cup A \cup A_p$, where $A_0 = \{ n \in \omega : \|\bar{n} \in \pi\| \wedge p = 0 \}$, $A = \{ n \in \omega : \|\bar{n} \in \pi\| \wedge p \in (0,p)_{\mathcal{B}} \}$ and $A_p = \{ n \in \omega : \|\bar{n} \in \pi\| \wedge p = p \}$. We also have $A_0,A,A_p \in V$ and

$$p \models \pi = (\pi \cap \bar{A}) \cup \bar{A}_p. \quad (3)$$

Let $f : \omega \to A$ be a bijection belonging to $V$ and $\tau = \{ (\bar{n},\|f(n)\| \in \pi\| \wedge p) : n \in \omega \}$. We prove

$$p \models f[\tau] = \pi \cap \bar{A}. \quad (4)$$

Let $G$ be a $\mathcal{B}$-generic filter over $V$ containing $p$. If $n \in f[\tau_G]$ then $n = f(m)$ for some $m \in \tau_G$, so $\|f(m)\| \in \pi\| \wedge p \in G$ which implies $\|f(m)\| \in \pi\| \wedge p \in G$ and consequently $n \in \pi_G$. Clearly $n \in A$. Conversely, if $n \in \pi_G \cap A$, since $f$ is a surjection there is $m \in \omega$ such that $n = f(m)$. Thus $f(m) \in \pi_G$ which implies $\|f(m)\| \in \pi\| \wedge p \in G$ and hence $m \in \tau_G$ and $n \in f[\tau_G]$.

According to (2), (3) and (4) we have $p \models \pi = f[\tau] \cup \bar{A}_p \notin V$ so, since $A_p \in V$, we have $p \models f[\tau] \notin V$ which implies $p \models \tau \notin V$. Let $p_n = \|f(n)\| \in \pi\| \wedge p$, $n \in \omega$. Then, by the construction, $p_n \in (0,p)_{\mathcal{B}}$ for all $n \in \omega$.

We define a strategy $\Sigma$ for White: at the beginning White plays $p$ and, in the $n$-th move, plays $p_n$. Let us prove $\Sigma$ is a winning strategy for White in the game $G_2$. Let $\langle i_n : n \in \omega \rangle \in \omega^2$ be an arbitrary play of Black. According to Theorem 3 we prove $p \models \tau \neq^* S_0$. But this follows from $p \models \tau \notin V$ and $S_0 \in V$ and we are done.

4. The game $G_3$

Firstly we give some characterizations of complete Boolean algebras on which White has a winning strategy in the game $G_3$. To make the formulas more readable, we will write $w_\varphi$ for $w(\varphi)$. Also, for $i : \omega \to 2$ we will denote $g^i = \{ i \upharpoonright n : n \in \omega \}$, the corresponding branch of the tree $^{<\omega}2$.

Theorem 6. For a complete Boolean algebra $\mathcal{B}$ the following conditions are equivalent:

(a) White has a winning strategy in the game $G_3$ on $\mathcal{B}$;

(b) There are $p \in \mathcal{B}^+$ and $w : ^{<\omega}2 \to (0,p)_{\mathcal{B}}$ such that

$$\forall i : \omega \to 2 \; \bigvee_{A \in [\omega]^{\omega}} A \wedge_{n \in A} w^{i(n)}_n = 0; \quad (5)$$
(c) There are \( p \in \mathbb{B}^+ \) and \( w : <\omega 2 \rightarrow [0, p]_\mathbb{B} \) such that (5) holds.

(d) There are \( p \in \mathbb{B}^+ \) and \( \rho \in V^\mathbb{B} \) such that

\[
p \vDash \rho \subseteq <\omega 2)^- \land \forall \varphi \in (<\omega 2)^-(\varphi^- \bar{0} \in \rho \vee \varphi^- \bar{1} \in \rho) \land \forall i \in (\omega 2)^-(\rho \cap g^i \text{ is unsupported}).
\]

(e) In some generic extension, \( V_\mathbb{B}[G] \), there is a subset \( R \) of the tree \( <\omega 2 \) containing either \( \varphi^- 0 \) or \( \varphi^- 1 \), for each \( \varphi \in <\omega 2 \), and having unsupported intersection with each branch of the tree \( <\omega 2 \) belonging to \( V \).

**Proof.** (a)⇒(c). Let \( \Sigma \) be a winning strategy for White. \( \Sigma \) is a function adjoining to each sequence of the form \( \langle p, p_0, i_0, \ldots, p_{n-1}, i_{n-1} \rangle \), where \( p, p_0, \ldots, p_{n-1} \in \mathbb{B}^+ \) are obtained by \( \Sigma \) and \( i_0, i_1, \ldots, i_{n-1} \) are arbitrary elements of \( \{0, 1\} \), an element \( p_n = \Sigma((p, p_0, i_0, \ldots, p_{n-1}, i_{n-1}) \) of \( (0, p)_\mathbb{B} \) such that White playing in accordance with \( \Sigma \) always wins. In general, \( \Sigma \) can be a multi-valued function, offering more “good” moves for White, but according to The Axiom of Choice, without loss of generality we suppose \( \Sigma \) is a single-valued function, which is sufficient for the following definition of \( p \) and \( w : <\omega 2 \rightarrow [0, p]_\mathbb{B} \).

At the beginning \( \Sigma \) gives \( \Sigma(\emptyset) = p \in \mathbb{B}^+ \) and, in the first move, \( \Sigma(\langle p \rangle) \in (0, p)_\mathbb{B} \). Let \( w_0 = \Sigma(\langle p \rangle) \).

Let \( \varphi \in n+1 \) and let \( w_{\varphi|k} \) be defined for \( k \leq n \). Then we define \( w_{\varphi} = \Sigma(\langle p, w_{\varphi|0}, \varphi(0), \ldots, w_{\varphi|n}, \varphi(n) \rangle) \).

In order to prove (5) we pick an \( i : \omega \rightarrow 2 \). Using induction it is easy to show that in the match in which Black plays \( i(0), i(1), \ldots, \), White, following \( \Sigma \) plays \( p, w_2|0, w_3|1, \ldots \). Thus, since White wins \( G_3 \), we have \( \bigvee_{A \in [\omega]^\omega} \bigwedge_{n \in A} w_{\varphi|n} = 0 \) and (5) is proved.

(c)⇒(b). Let \( p \in \mathbb{B}^+ \) and \( w : <\omega 2 \rightarrow [0, p]_\mathbb{B} \) satisfy (5). Suppose the set \( S = \{ \varphi \in <\omega 2 : w_{\varphi} \in \{0, p\} \} \) is dense in the ordering \( <\omega 2, \supseteq \). Using recursion we define \( \varphi_k \in S \) for \( k \in \omega \) as follows. Firstly, we choose \( \varphi_0 \in S \) arbitrarily. Let \( \varphi_k \) be defined and let \( i_k \in 2 \) satisfy \( i_k = 0 \) iff \( w_{\varphi_k} = p \). Then we choose \( \varphi_{k+1} \in S \) such that \( \varphi_{k+1} \supseteq \varphi_k \). Clearly the integers \( n_k = \text{dom}(\varphi_k), k \in \omega \), form an increasing sequence, so \( i = \bigcup_{k \in \omega} \varphi_k : \omega \rightarrow 2 \). Besides, \( i \upharpoonright n_k = \varphi_k \) and \( i(n_k) = i_k \). Consequently, for each \( k \in \omega \) we have \( w_{\varphi | n_k} = w_{\varphi_k | n_k} = p \). Now \( A_0 = \{ n_k : k \in \omega \} \subseteq [\omega]^\omega \) and \( \bigwedge_{n \in A_0} w_{\varphi | n} = p > 0 \). A contradiction to (5).

So there is \( \psi \in <\omega 2 \) such that \( w_{\varphi} \in (0, p)_\mathbb{B} \), for all \( \varphi \supseteq \psi \). Let \( m = \text{dom}(\psi) \) and let \( v_\varphi \) for \( \varphi \in <\omega 2 \) be defined by

\[
v_\varphi = \begin{cases} w_\psi & \text{if } |\varphi| < m, \\ w_\psi(\varphi(\text{dom}(\varphi)\setminus m)) & \text{if } |\varphi| \geq m. \end{cases}
\]
Clearly $v : <_2 \omega \rightarrow \langle 0, p \rangle_B$ and we prove that $v$ satisfies (5). Let $i : \omega \rightarrow 2$ and let $j = \psi^+(i \uparrow (\omega \setminus m))$. Then for $n \geq m$ we have $v^{i(n)}_{i(n)} = w^{i(n)}_{\psi^+(i(n) \uparrow m)} = w^{j(n)}_{j(n)}$. Let $A \in [\omega]^{\omega}$. Then $A \setminus m \in [\omega]^{\omega}$ and, since $w$ satisfies (5), for the function $j$ defined above we have $\bigwedge_{n \in A \setminus m} w_{j(n)}^{j(n)} = 0$, that is $\bigwedge_{n \in A \setminus m} v_{i(n)}^{i(n)} = 0$, which implies $\bigwedge_{n \in A} v_{i(n)}^{i(n)} = 0$ and (b) is proved.

(b)⇒(a). Assuming (b) we define a strategy $\Sigma$ for White. Firstly White plays $p$ and $p_0 = w_0$. In the $n$-th step, if $\varphi = (i_0, \ldots, i_{n-1})$ is the sequence of Black’s previous moves, White plays $p_n = w_{\varphi}$. We prove that $\Sigma$ is a winning strategy for White. Let $i : \omega \rightarrow 2$ code an arbitrary play of Black. Since White follows $\Sigma$, in the $n$-th move White plays $p_n = w_{i(n)}$, so according to (5) we have $\bigvee_{A \in [\omega]^n} \bigwedge_{n \in A} p^n_A = 0$ and White wins the game.

(b)⇒(d). Let $p \in B^+$ and $w : <_2 \omega \rightarrow \langle 0, p \rangle_B$ be the objects provided by (b). Let us define $v_0 = p$ and, for $\varphi \in <_2 \omega$ and $k \in 2$, let $v_{\varphi^{-k}} = w^k_\varphi$. Then $\rho = \{ (\varphi, v_\varphi) : \varphi \in <_2 \omega \}$ is a name for a subset of $<_2 \omega$. If $i : \omega \rightarrow 2$, then $\sigma^i = \{ ((i \uparrow n)^-, v_{i(n)}) : n \in \omega \}$ is a name for a subset of $g^i$ and, clearly,

$$1 \vdash \sigma^i = \rho \cap g^i.$$  

(7)

Let us prove

$$\forall i : \omega \rightarrow 2 \ 1 \vdash \rho \cap g^i \text{ is unsupported.}$$  

(8)

Let $i : \omega \rightarrow 2$. According to the definition of $v$, for $n \in \omega$ we have $w_{i(n)}^{i(n)} = v_{i(n+1)}^{i(n+1)}$ so, by (5), $\bigvee_{A \in [\omega]^n} \bigwedge_{n \in A} v_{i(n+1)}^{i(n+1)} = 0$. By (7) we have $v_{i(n+1)}^{i(n+1)} = ||(i \uparrow (n+1))^+ \in \rho \cap g^i||$ and we have $||\exists A \in (([\omega]^n)^-) \forall n \in A (i \uparrow (n+1))^+ \in \rho \cap g^i|| = 0$ that is $||\exists B \subseteq (([\omega]^n)^-) \bigwedge B \subseteq \rho \cap g^i|| = 1$ and (8) is proved. Now we prove

$$\forall \varphi \in <_2 \omega \ 2 \vdash \varphi^{-0} 0 \in \rho \lor \varphi^{-0} \in \rho.$$  

(9)

If $p \in G$, where $G$ is a $\mathcal{B}$-generic filter over $V$, then clearly $|G \cap \{ w_{\varphi}, p \setminus w_{\varphi} \}| = 1$. But $w_{\varphi} = w_{\varphi}^0 = w_{\varphi^{-0}} = ||\varphi^{-0} 0 \in \rho||$ and $p \setminus w_\varphi = w_\varphi = w_{\varphi^{-0}}^0 = ||\varphi^{-0} 0 \in \rho||$ and (9) is proved.

(d)⇒(c). Let $p \in B^+$ and $\rho \in V^B$ satisfy (9). In $V$ for each $\varphi \in <_2 \omega$ we define $w_\varphi = ||(\varphi^{-0})^- 0 \in \rho|| \land p$ and check condition (c). So for an arbitrary $i : \omega \rightarrow 2$ we prove

$$\bigvee_{A \in [\omega]^n} \bigwedge_{n \in A} v_{i(n)}^{i(n)} = 0.$$  

(10)

According to (6) for each $n \in \omega$ we have $p \vdash ((i \uparrow n)^- 0)^- 0 \in \rho \lor ((i \uparrow n)^- 1)^- 0 \in \rho$, that is $p \leq a_0 \lor a_1$ and $p \land a_0 \land a_1 = 0$, where $a_k = ||((i \uparrow n)^- k)^- 0 \in \rho||$, $k \in \{0, 1\}$, which clearly implies $p \land a_0 = p \land a_1$, i.e.

$$p \land ||((i \uparrow n)^- 0)^- 0 \in \rho||' = p \land ||((i \uparrow n)^- 1)^- 0 \in \rho||.$$  

(11)
Let us prove
\[ w_{i(n)}^i = \| (i \uparrow (n + 1))^* \| \wedge p. \] (12)

If \( i(n) = 0 \), then \( w_{i(n)}^i = \| ((i \uparrow n)^* \wedge p = \| ((i \uparrow n)^* i(n))^* \| \wedge p \) and (12) holds. If \( i(n) = 1 \), then according to (11) \( w_{i(n)}^i = p \setminus w_{i(n)} = p \wedge \| (i \uparrow n)^* \| \) holds again.

Now \( \bigwedge_{A \in [\omega]^\omega} \bigwedge_{n \in A} w_{i(n)}^i = p \wedge \| \exists A \in (\{\omega\}^\omega)^* \forall n \in A \ | (i \uparrow n + 1) \in p \| = p \wedge \| p \cap g^i \| \) is supported \( = 0 \), since by (6) \( p \leq \| p \cap g^i \| \) is unsupported \( = 0 \). Thus (10) is proved.

(d)\( \Rightarrow \) (e) is obvious and (e)\( \Rightarrow \) (d) follows from The Maximum Principle. \( \square \)

Concerning condition (e) of the previous theorem we note that in [5] the following characterization is obtained.

**Theorem 7.** White has a winning strategy in the game \( \mathcal{G}_4 \) on a c.B.a. \( \mathbb{B} \) if and only if in some generic extension, \( V_\mathbb{B}[\mathcal{G}] \), there is a subset \( R \) of the tree \( \prec \omega^2 \) containing either \( \varphi^*0 \) or \( \varphi^*1 \), for each \( \varphi \in \prec \omega^2 \), and having finite intersection with each branch of the tree \( \prec \omega^2 \) belonging to \( V \).

**Theorem 8.** Let \( \mathbb{B} \) be a complete Boolean algebra. If forcing by \( \mathbb{B} \) produces an independent real in some generic extension, then White has a winning strategy in the game \( \mathcal{G}_3 \) played on \( \mathbb{B} \).

**Proof.** Let \( p = \| \exists x \subseteq \check{\omega} \ x \text{ is independent} \| > 0 \). Then, by The Maximum Principle there is a name \( \tau \in V_\mathbb{B} \) such that
\[ p \models \tau \subseteq \check{\omega} \wedge \forall A \in (\{\omega\}^\omega)^* (|A \cap \tau| = \check{\omega} \wedge |A \setminus \tau| = \check{\omega}). \] (13)

Let us prove that \( K = \{ n \in \omega : \| \check{n} \in \tau \| \wedge p \in \{0, p\} \} \) is a finite set. Clearly \( K = K_0 \cup K_p \), where \( K_0 = \{ n \in \omega : p \models \check{n} \notin \tau \} \) and \( K_p = \{ n \in \omega : p \models \check{n} \in \tau \} \). Since \( p \models \check{K}_0 \subseteq \check{\omega} \setminus \tau \wedge \check{K}_p \subseteq \tau \), according to (13) the sets \( K_0 \) and \( K_p \) are finite, thus \( |K| < \omega \).

Let \( q \in (0, p)_\mathbb{B} \) and let \( p_n, n \in \omega \), be defined by
\[ p_n = \begin{cases} q & \text{if } n \in K, \\ \| \check{n} \in \tau \| \wedge p & \text{if } n \in \omega \setminus K. \end{cases} \]

Then for \( \tau_1 = \{ \langle \check{n}, p_n \rangle : n \in \omega \} \) we have \( p \models \tau_1 =^* \tau \) so according to (13)
\[ p \models \tau_1 \subseteq \check{\omega} \wedge \forall A \in (\{\omega\}^\omega)^* (|A \cap \tau_1| = \check{\omega} \wedge |A \setminus \tau_1| = \check{\omega}). \] (14)
Four games on Boolean algebras

Then \( p_n = \| \bar{n} \in \tau_1 \| \in (0, p) \) and we define a strategy \( \Sigma \) for White: at the beginning White plays \( p \) and, in the \( n \)-th move, White plays \( p_n \).

We prove \( \Sigma \) is a winning strategy for White. Let \( \langle p, p_0, i_0, p_1, i_1, \ldots \rangle \) be an arbitrary play in which White follows \( \Sigma \) and let \( S_k = \{ n \in \omega : i_n = k \} \), for \( k \in \{0, 1\} \). Suppose \( q = \bigvee_{A \in [\omega]^{\omega}} \bigwedge_{n \in A} P^n_n > 0 \). Now \( q \leq p \) and \( q = \bigvee_{A \in [\omega]^{\omega}} \bigwedge_{n \in A \cap S_0} \| \bar{n} \in \tau_1 \| \wedge \bigwedge_{n \in A \cap S_1} (p \wedge \| \bar{n} \notin \tau_1 \|) = p \wedge \bigvee_{A \in [\omega]^{\omega}} \| \bar{n} \in \tau_1 \| \wedge \bigwedge_{n \in A} (p \wedge \| \bar{n} \notin \tau_1 \|) \).

Theorem 9. Let \( \mathcal{B} \) be an \((\omega, 2)\)-distributive complete Boolean algebra. Then

(a) If \( \langle p, p_0, i_0, p_1, i_1, \ldots \rangle \) is a play satisfying the rules given in Definition 1 then Black wins the game \( \mathcal{G}_3 \) iff Black wins the game \( \mathcal{G}_4 \).

(b) Black has a winning strategy in the game \( \mathcal{G}_3 \) iff Black has a winning strategy in the game \( \mathcal{G}_4 \).

Proof. (a) The implication “\( \Rightarrow \)” follows from the proof of Proposition 1(b). For the proof of “\( \Leftarrow \)” suppose Black wins the play \( \langle p, p_0, i_0, p_1, i_1, \ldots \rangle \) in the game \( \mathcal{G}_4 \). Then, by Theorem 4 there exists \( q \in \mathcal{B}^+ \) such that \( q \Vdash \sigma \) is infinite”. Since the algebra \( \mathcal{B} \) is \((\omega, 2)\)-distributive we have \( 1 \Vdash \sigma \in V \), thus \( q \Vdash \sigma \in (([\omega]^{\omega})^V)^{\omega} \) and hence \( \sim 1 \Vdash \sigma \) is not supported” so, by Theorem 4 Black wins \( \mathcal{G}_3 \).

(b) follows from (a).

5. Indeterminacy, problems

Theorem 10. \( \Diamond \) implies the existence of a Suslin algebra on which the games \( \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \) and \( \mathcal{G}_4 \) are undetermined.

Proof. Let \( \mathcal{B} \) be the Suslin algebra mentioned in (c) of Theorem 2. According to Proposition 1(b) and since Black does not have a winning strategy in the game \( \mathcal{G}_4 \), Black does not have a winning strategy in the games \( \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \) as well. On the other hand, since the algebra \( \mathcal{B} \) is \((\omega, 2)\)-distributive, White does not have a winning strategy in the game \( \mathcal{G}_1 \) and, by Proposition 1(a), White does not have a winning strategy in the games \( \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4 \) played on \( \mathcal{B} \).

Problem 1. According to Theorem 8 Proposition 1 and Theorem 5 for each complete Boolean algebra \( \mathcal{B} \) we have:
\[ \mathbb{B} \text{ is } \omega\text{-independent} \Rightarrow \text{White has a winning strategy in } \mathcal{G}_3 \Rightarrow \mathbb{B} \text{ is not } (\omega, 2)\text{-distributive.} \]

Can one of the implications be reversed?

**Problem 2.** According to Proposition 1(b), for each complete Boolean algebra \( \mathbb{B} \) we have:

\[ \text{Black has a winning strategy in } \mathcal{G}_1 \Rightarrow \text{Black has a winning strategy in } \mathcal{G}_2 \Rightarrow \text{Black has a winning strategy in } \mathcal{G}_3. \]

Can some of the implications be reversed?

We note that the third implication from Proposition 1(b) cannot be replaced by the equivalence, since if \( \mathbb{B} \) is the Cohen or the random algebra, then Black has a winning strategy in the game \( \mathcal{G}_4 \) (Theorem 2(b)) while Black does not have a winning strategy in the game \( \mathcal{G}_3 \), because White has one (the Cohen and the random forcing produce independent reals and Theorem 8 holds).

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