Notes on semiclassical Weyl gravity

Claus Kiefer and Branišlav Nikolić

Abstract In any quantum theory of gravity, it is of the utmost importance to recover the limit of quantum theory in an external spacetime. In quantum geometrodynamics (quantization of general relativity in the Schrödinger picture), this leads in particular to the recovery of a semiclassical (WKB) time which governs the dynamics of non-gravitational fields in spacetime. Here, we first review this procedure with special emphasis on conceptual issues. We then turn to an alternative theory - Weyl (conformal) gravity, which is defined by a Lagrangian that is proportional to the square of the Weyl tensor. We present the canonical quantization of this theory and develop its semiclassical approximation. We discuss in particular the extent to which a semiclassical time can be recovered and contrast it with the situation in quantum geometrodynamics.

1 Notes on semiclassical Einstein gravity

Among Paddy’s many interests in physics was always the deep desire to understand the relationship between classical and quantum gravity. In his paper “Notes on semiclassical gravity”, written together with T. P. Singh in 1989, they write [29]:

In the course of our investigation we came across a variety of methods for defining classical and semiclassical limits, apparently different, and all of which were possibly applicable to a quantum gravity. It then became necessary to compare these methods and to settle, once and for all, the relation of semiclassical gravity to quantum gravity.

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The understanding of semiclassical gravity was also a long-term project by one of us, and we thus devote our festschrift contribution to this topic. More precisely, the topic is the recovery of quantum (field) theory in an external spacetime from canonical quantum gravity. We briefly review the standard procedure of obtaining this limit from the Wheeler-DeWitt equation of quantum general relativity (quantum geometrodynamics). In the next two sections, we then apply these methods to a different theory called Weyl gravity or conformal gravity. This is the main concern of our paper.

What is our motivation for doing so? Weyl gravity is a theory without intrinsic scale. It seems therefore not appropriate, by itself, to replace general relativity (GR) in the empirically tested macroscopic limit. It may, however, be appropriate to serve as a model for a fundamental conformally invariant theory, being of relevance in quantum gravity and its application to the very early universe. Many researchers entertain, in fact, the idea that Nature does not contain any scale at the most fundamental level; see, for example, [31] and [2]. In these following sections, we shall outline the procedure for classical and quantum canonical Weyl gravity and perform the semiclassical limit. We shall point out in detail the similarities to and the differences from quantum GR. We shall see, in particular, that while a semiclassical time can be recovered, this time is of a different nature than the one recovered from quantum GR.

In canonical GR, the configuration variable is the three-metric \( h_{ab}(x) \), while the canonical momentum \( p^{cd}(x) \) is a linear function of the extrinsic curvature (second fundamental form) \( K_{cd}(x) \). In the Dirac way of quantization, these variables are heuristically transformed into operators acting on wave functionals,

\[
\hat{h}_{ab}(x)\Psi[h_{ab}(x)] = h_{ab}(x) \cdot \Psi[h_{ab}(x)],
\]

\[
\hat{p}^{cd}(x)\Psi[h_{ab}(x)] = \frac{\hbar}{16\pi G} \delta^{cd} \Psi[h_{ab}(x)].
\]

The wave functionals are defined on the configuration space of all three-metrics (plus non-gravitational fields, which are not indicated here). In GR, one has four local constraints, the Hamiltonian constraint and the three diffeomorphism (momentum) constraints. They are implemented in the quantum theory as restrictions on physically allowed wave functionals [15],

\[
\hat{H}^{\perp}_{G} \Psi := \left( -16\pi G \hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} - \frac{\sqrt{\hbar}}{16\pi G} \left( \frac{3}{2} R - 2\Lambda \right) \right) \Psi = 0 ,
\]

\[
\hat{H}^{\perp}_{\mathcal{A}} \Psi := -2D_{b} h_{bc} \frac{\hbar}{16\pi G} \frac{\delta \Psi}{\delta h_{bc}} = 0 .
\]

The quantum Hamiltonian constraint (3) is called the Wheeler-DeWitt equation. The momentum constraints (4) guarantee that the wave functional remains unchanged (apart possibly from a phase) under a three-dimensional coordinate transformation. In the presence of non-gravitational fields, we need the corresponding contributions \( \hat{H}^{m}_{\perp} \) for (3) and \( \hat{H}^{m}_{\mathcal{A}} \) for (4), see below.
The coefficients $G_{abcd}$ in front of the kinetic term in (3) are the components of
the DeWitt metric, which is the metric on configuration space. One of its important
properties is its indefinite nature. Using instead of $h_{ab}$ its scale part $\sqrt{h}$ (where $h$
denotes its determinant) and the conformal part $\bar{h}_{ab} = h^{-1/3}h_{ab}$, the Wheeler-DeWitt
equation reads

\[
\left(6\pi G h^2 \frac{\delta^2}{\delta(\sqrt{h})^2} - \frac{16\pi G h^2 \bar{h}_{ac} \bar{h}_{bd}}{\delta \bar{h}_{ab}} \frac{\delta^2}{\delta \bar{h}_{cd}} \right) \Psi[\sqrt{h}, \bar{h}_{ab}] = 0 .
\]

One recognizes that the kinetic term connected with the local scale has a different
sign. For this reason, the Wheeler-DeWitt equation is of a (local) hyperbolic nature
and $\sqrt{h}$ can be interpreted as a local measure of intrinsic time. We shall introduce
the scale and conformal parts of the metric also for the Weyl theory below, but as we
shall see, the scale part (and thus the intrinsic time part) will be absent in the Weyl
version of the Wheeler-DeWitt equation.

An important step in understanding the semiclassical limit for the above quantum
equations is the WKB approximation [29]. One starts with the ansatz

\[
\Psi[h_{ab}] = C[h_{ab}] \exp \left( \frac{i}{\hbar} S[h_{ab}] \right)
\]

and assumes that $C[h_{ab}]$ is a ‘slowly varying amplitude’ and $S[h_{ab}]$ is a ‘rapidly
varying phase’. This corresponds to the substitution

\[
p^{ab} \rightarrow \frac{\delta S}{\delta h_{ab}},
\]

which is the classical relation for the canonical momentum. From (3) and (4)
one finds then for $S[h_{ab}]$ the equations

\[
16\pi G G_{abcd} \frac{\delta S}{\delta h_{ab}} \frac{\delta S}{\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G} \left( (3)R - 2\Lambda \right) = 0 ,
\]

\[
D_{a} \frac{\delta S}{\delta h_{ab}} = 0 .
\]

In the presence of matter one has additional terms. The first equation (7) is the
Hamilton-Jacobi equation for the gravitational field. One can prove that the four
local equations (7) and (8) are equivalent to all ten Einstein equations.

If non-gravitational fields are present, as we will now assume, a mixture of this
WKB ansatz with the Born-Oppenheimer ansatz from molecular physics is appro-
priate [15, 14, 29]. One writes instead of (6) now

\[
\Psi[h_{ab}, \phi] \equiv \exp \left( \frac{i}{\hbar} S[h_{ab}, \phi] \right),
\]
where $S[h_{ab}, \phi]$ here denotes a complex function that depends on both the three-metric $h_{ab}$ and the non-gravitational fields denoted by $\phi$ (usually taken to be a scalar field). Plugging this ansatz into the quantum constraints (3) and (4) and performing an expansion scheme with respect to the square of the Planck mass $m_p = \sqrt{\hbar/G}$,

$$S[h_{ab}, \phi] = m_p^2 S_0 + S_1 + m_p^{-2} S_2 + \ldots,$$

(10)

one finds at highest order ($m_p^2$) that $S_0$ depends only on the three-metric $h_{ab}$ and that it obeys the Hamilton-Jacobi equation (7) and Eq. (8) for the pure gravitational field.

The next order ($m_0$) gives a functional Schrödinger equation for a wave functional $\psi[h_{ab}, \phi]$ in the background spacetime defined from a solution $S_0$ to (7) and (8), where

$$\psi[h_{ab}, \phi] := D[h_{ab}] \exp \left( \frac{i}{\hbar} S_1[h_{ab}, \phi] \right),$$

(11)

and $D$ obeys the standard WKB prefactor equation (see e.g. Eq. (2.36) in [14]). This step yields a Tomonaga-Schwinger equation for $\psi[h_{ab}, \phi]$ with respect to a local time functional $\tau(x)$ that is defined from the solution $S_0$ by

$$\frac{\delta}{\delta \tau(x)} := G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}}.$$

(12)

In spite of its appearance, $\tau$ is not a scalar function [9]. The functional Schrödinger equation is obtained by evaluating $\psi[h_{ab}, \phi]$ along a solution of the classical Einstein equations, $h_{ab}(x,t)$, that corresponds to a solution, $S_0[h_{ab}]$, of the Hamilton-Jacobi equation, $\psi[h_{ab}(x,t), \phi]$. After a certain choice of lapse and shift functions, $N$ and $N^a$, has been made, this solution is obtained from

$$\dot{h}_{ab} = NG_{abcd} \frac{\delta S_0}{\delta h_{cd}} + 2D(a N_b).$$

(13)

Instead of $\psi[h_{ab}, \phi]$, we can write $|\psi[h_{ab}]\rangle$ to indicate (by the bra-ket notation) that one has a well-defined (standard) Hilbert space for the non-gravitational field $\phi$. Defining

$$\frac{\partial}{\partial \tau} \langle \psi(t) \rangle := \int d^3 x \dot{h}_{ab}(x,t) \frac{\delta}{\delta h_{ab}(x)} \langle \psi[h_{ab}] \rangle,$$

(14)

one finds the functional Schrödinger equation for quantized non-gravitational fields in the chosen external classical gravitational field,

$$i\hbar \frac{\partial}{\partial \tau} \langle \psi(t) \rangle = \hat{H}^m \langle \psi(t) \rangle,$$

$$\hat{H}^m := \int d^3 x \left\{ N(x) \dot{\hat{H}}^m(x) + N^a(x) \dot{\hat{H}}^m_a(x) \right\}.$$

(15)

Here, $\hat{H}^m$ is the non-gravitational Hamiltonian in the Schrödinger picture, parametrically depending on the metric coefficients of the curved spacetime background. This is the standard approach for obtaining the limit of quantum field theory in
curved spacetime from canonical quantum gravity. Extending this scheme to higher orders in $\hbar^2$, one arrives at quantum gravitational correction terms to this equation \[13\]. These terms can be used to calculate potentially observable effects such as corrections to the CMB anisotropy spectrum \[4\].

The Born-Oppenheimer approximation starting from (9) provides only part of the understanding why we observe a classical spacetime. The remaining part is provided by the process of decoherence. It was suggested in \[32\] and elaborated in \[13\] to using small inhomogeneities such as density perturbations or tiny gravitational waves as a “quantum environment” in configuration space, whose interaction with relevant degrees of freedom such as the global size of the universe gives rise to their classical appearance. Technically, this comes from tracing out these inhomogeneities in the globally entangled quantum states. In \[24\], Paddy has extended these investigations to more general situations and found that three-geometries with the same intrinsic metric but different size contribute decoherently to the reduced density matrix for the relevant degrees of freedom. He concludes his paper with the words

\[ \text{...the classical nature of the space-time will tend to disappear as we observe more and more matter modes. Probably, ignorance is bliss.} \]

The recovery of time in semiclassical gravity raises the question whether time in quantum gravity can be recovered from a general solution of the Wheeler-DeWitt solution. The idea was followed independently by Paddy \[25\] and Greensite \[10\]. This generalized time is recovered from the phase of the wavefunction and used to define a Schrödinger-type inner product where all variables are integrated over except for this time. A necessary prerequisite for this to work is that the wavefunction is complex and that its phase is not a constant. One can then prove that the first Ehrenfest theorem is valid if this time variable and the corresponding inner product is used. Unfortunately, only a restricted class of solutions fulfills all consistency conditions (including the validity of the second Ehrenfest theorem), so one either has to abandon this proposal as a solution to the time problem or to use it as a new type of boundary condition to select physically allowed solutions \[5\].

2 Quantization of conformal (Weyl) gravity

The role of conformal transformations and of conformal symmetry is of central interest for gravitational systems at least since Hermann Weyl’s pioneering work from 1918. Weyl suggested a theory in which not only the direction of a vector depends on the path along which the vector is transported through spacetime, but also its length. This means that space distances and time intervals depend on the path of rods and clocks through spacetime. In Weyl’s theory there exists the freedom to re-scale (“gauge”) rods and clocks; the metric can be multiplied by an arbitrary positive spacetime-dependent function,

\[ g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x). \]  \[16\]
This transformation is called *conformal transformation* (later also *Weyl transformation*) and is an invariance in Weyl’s theory. Connected with this freedom is a new quantity that Weyl identified with the electromagnetic four-potential, suggesting the idea of a unification between gravity and electromagnetism\(^1\).

Weyl’s theory is impressive, but empirically wrong, as soon noticed by others, in particular Einstein. If it were true, spectral lines, for example, would depend on the history of the atomic worldlines, because an atom can be understood as constituting a clock\(^2\). Quite generally, a particle with rest mass \(m\) can be taken as a clock with frequency

\[
\nu = \frac{m c^2}{h},
\]

so a path-dependent frequency would correspond to a path-dependent rest mass, since \(c\) and \(h\) are universal units. This is definitely empirically wrong.

Weyl thus had to give up his theory, but later used essential elements of his idea to provide the foundation of modern gauge theory. Einstein, however, was speculating about the existence of a theory that, while preserving the conformal invariance of Weyl’s theory, does not include a hypothesis about the transport of rods and clocks, thus avoiding the problems of Weyl’s theory. In a paper entitled “Über eine naheliegende Ergänzung des Fundamentes der allgemeinen Relativitätstheorie” [7],\(^3\) Einstein suggested to use the scalar

\[
C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}
\]

formed from the Weyl tensor \(C_{\mu\nu\lambda\rho}\) as the basis of this theory\(^4\). The Weyl tensor \(C^{\mu}_{\nu\lambda\rho}\) (with one upper component) is invariant under the conformal transformations\(^5\).

At the end of his article, Einstein intended to add the following short summary, which can be found in his hand-written manuscript, but which he deleted before submission. It reads (our translation from the German)\(^6\):

Short summary: it is shown that one can, following Weyl’s basic ideas, develop a theory of invariants on the objective existence of lightcones (invariance of the equation \(ds^2 = 0\)) alone, which does not, in contrast to Weyl’s theory, contain a hypothesis about transport of distances and in which the potentials \(\phi_\nu\) do not enter explicitly the equations. Later investigations must show whether the theory will be physically valid.

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1. For a review and reference to original articles, see [11].
2. Recall that the modern time standard is based on the hyperfine transitions in caesium-133.
3. English translation: “On a Natural Addition to the Foundation of the General Theory of Relativity”
4. In his paper, Einstein acknowledges the help of the Austrian mathematician Wilhelm Wirtinger in his attempt. In a letter to Einstein sent one day after Einstein’s academy talk on which [7] is based, Wirtinger suggested as one possibility to use an action principle based on (17), see [8], p. 117.
5. The \(\phi_\nu\) denote the components of the electromagnetic four-potential.
6. The original German reads [8], p. 416: “Kurze Zusammenfassung: Es wird gezeigt, dass man entsprechend dem Weyl’schen Grundgedanken auf die objektive Existenz der Lichtkegel (Invarianz der Gleichung \(ds^2 = 0\)) alleine eine Invarianten-theorie gründen kann, die jedoch im Gegensatz zu
In fact, even before Einstein, Rudolf Bach had considered an action based on (17) and derived and discussed the ensuing field equations [1]. When we talk here of conformal gravity or Weyl gravity, we do not mean Weyl’s original gravitational gauge theory from 1918, but a theory that is based on the action suggested by Bach, Einstein, and Wirtinger. We write the action in the following form:

\[ S^W := -\frac{\alpha_W h}{4} \int d^4x \sqrt{-g} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}, \]  

(18)

where \( \alpha_W \) is a dimensionless coupling constant. We have introduced Planck’s constant \( \hbar \) (which, of course, is irrelevant for the classical theory) for two reasons: first, it gives the correct dimensions for the action and renders the constant \( \alpha_W \) dimensionless and, second, \( S^W / \hbar \) is the relevant quantity in the quantum theory on which we will focus in our paper; in fact, this will suggest the semiclassical expansion scheme with respect to \( \alpha_W \) presented below.

The theory based on (18) was discussed at both the classical and quantum level [22]. At the classical level, it was used, for example, to explain galactic rotation curves without the need for dark matter, although this explanation has met severe criticism [26]. At the quantum level, it is a candidate for a renormalizable theory of quantum gravity, although it seems to violate unitarity. We adopt here the point of view to take (18) as the starting point for a conformally invariant gravity theory, for which a semiclassical expansion scheme can be applied to its canonically quantized version and compared with the scheme for quantized GR. We do not assume that (18) is a candidate for an alternative to GR. In the following, we shall study the canonical structure of this theory. Our treatment is based on the more general treatment presented in [16].

In order to deal with higher-derivative theories such as (18) it is convenient to reduce the order by introducing new independent variables. In our case, this is achieved by introducing the extrinsic curvature \( K_{ij} \), which in general relativity is a function of the time derivative of the three-metric \( h_{ij} \). This can be implemented in the canonical formalism by adding a constraint \( \lambda^{ij} (2K_{ij} - \mathcal{L}_n h_{ij}) \), where \( \lambda^{ij} \) is a Lagrange multiplier. There are also other methods to “hide” the second derivative of the three-metric [3, 6].

In order to manifestly reveal conformal invariance of the Weyl action, we will use here an irreducible decomposition of the 3-metric into its scale part:

\[ a = (\sqrt{\hbar})^{1/3} \]  

(19)

\[ \text{Weyl’s Theorie keine Hypothese über Streckenübertragung enthält und in welcher die Potentiale \( \phi \) nicht explizite in die Gleichungen eingehen. Ob die Theorie auf physikalische Gültigkeit Anspruch erheben kann, müssen spätere Untersuchungen ergeben.} \]

7 We write “seems”, because the ghosts connected with non-unitarity may be removable [21].

8 For the history of such theories, see for example [28].

9 In quantum GR, there exist attempts to quantize solely the conformal factor [23]. Paddy has derived from this the interesting conclusion that the Planck length provides a lower bound to measurable physical lengths. The situation will be different here, because Weyl gravity does not contain an intrinsic length scale.
and its conformally invariant (unimodular) part

\[ \bar{h}_{ij} = a^{-2} h_{ij}. \]  

(20)

In addition, we define the following variables \[16\]

\[ \bar{N}^i = N^i, \quad \bar{N}_i = a^{-2} N_i, \quad \bar{N} = a^{-1} N, \]  

(21)

\[ \bar{K}^T_{ij} = a^{-1} K^T_{ij}, \quad \bar{K} = a K, \]  

(22)

where \( \bar{K}^T_{ij} \) and \( \bar{K} \) are the rescaled traceless and trace parts of the extrinsic curvature, respectively; as a consequence of the decomposition of the three-metric (19) and (20) they are given by

\[ \bar{K}^T_{ij} = \frac{1}{\bar{N}} \left( \bar{h}_{ij} - 2 \left[ D_i (\bar{N}_j) \right]^T \right), \quad \bar{K} = \frac{1}{\bar{N}} \left( \frac{\dot{a}}{a} - \frac{1}{3} D_i N^i \right). \]  

(23)

It can be shown by direct calculation that \( \left[ D_i (\bar{N}_j) \right]^T \) is independent of \( a \) and that \( \bar{K}^T_{ij} \) is the conformally invariant part of the extrinsic curvature. We refer to the variables in (21) and (22) as unimodular-conformal variables, and we will formulate the canonical theory in terms of them. The advantage of using these variables is that only the scale \( a \) and the trace \( \bar{K} \) transform under conformal transformation,

\[ a \to \Omega a, \quad \bar{K} \to \bar{K} + \bar{n}^\mu \partial_\mu \log \Omega, \]  

(24)

where \( \bar{n}^\mu = a n^\mu \) and \( \bar{n}_\mu = a^{-1} n_\mu \). This significantly simplifies the canonical formulation and makes conformal invariance of the theory manifest, since the only two variables affected by conformal transformation completely vanish from the constraints, as will be shown below.

The canonical approach employs a 3 + 1 decomposition of spacetime quantities. For GR, this is the standard ADM approach \[15\]. In the present case, one has to perform a 3 + 1 decomposition of the Weyl tensor, which can be found, for example, in \[12, 20\]. The constrained 3+1-decomposed Lagrangian density of the Weyl action in terms of the unimodular-conformal variables introduced above then becomes

\[ \mathcal{L}_W^c = \bar{N} \left\{ -\frac{\alpha_0 \hbar^i \bar{h} \bar{h}^j \bar{C}^i_{ab} \bar{C}^T_{ij} + \alpha_0 \hbar^i \bar{C}^2_{ab} - a^5 \lambda^{ijT} \left[ 2 \bar{K}^T_{ij} - \frac{1}{\bar{N}} \left( \frac{\dot{a}}{a} - \frac{1}{3} D_i N^i \right) \right] \right\} \right. 

\left. -2a^3 \lambda \left[ \bar{K} - \frac{1}{\bar{N}} \left( \frac{\dot{a}}{a} - \frac{1}{3} D_a N^a \right) \right] \right\}, \]  

(25)

where \( \lambda^{ijT} \) and \( \lambda \) are traceless and trace parts of the Lagrange multiplier \( \lambda^{ij} \), and

\[ \bar{C}^T_{ij} = \mathcal{L}_W^T \bar{R}^T_{ij} - \frac{2}{3} \bar{h}_{ij} \bar{R}^T_{ab} \bar{h}^a \bar{R}^{bm} \bar{K}^T_{nm} - (\bar{R}^T_{ij} - \frac{1}{\bar{N}} [D_i D_j]^T \bar{N} \right. \]  

(26)

is the “electric part” of the Weyl tensor, containing only velocities of the traceless part of the extrinsic curvature, and
can write the total Hamiltonian as
\begin{equation}
\bar{C}_{ijk} = 2 \delta_i^d \left( \delta_j^e \delta_k^f - \bar{h}_{jk} \bar{h}^{ef} \right) D_a \bar{R}_{cf}, \quad \bar{C}_{ij}^2 = \bar{C}_{ijk} \bar{h}^{ia} \bar{h}^{ib} \bar{h}^{kc} \bar{C}_{abc} \tag{27}
\end{equation}
is related to the “magnetic part” of the Weyl tensor, as explained in [12]. The second expression in (27) should not contain any traces \( \bar{R} \) and therefore be conformally invariant, but we assume this without proof. Each object with superscript “\( \prime \)” is traceless. It can be shown easily that the trace of the sum of the first two terms in (26) vanishes, that is, that \( h^{ij} \bar{L}_a \bar{K}_{T}^{ij} = 2a^{-2} \bar{K}_a^{ij} \bar{h}^{en} \bar{h}^{bm} \bar{K}_{T}^{mn} \).

We now take unimodular-conformal variables (19), (20), (21), and (22) and derive their conjugate momenta in the standard way,
\begin{align*}
p_N &= \frac{\partial \mathcal{L}_w}{\partial \dot{N}} \approx 0, \quad p^j = \frac{\partial \mathcal{L}_w}{\partial \dot{h}_{ij}} \approx 0, \quad \bar{P} = \frac{\partial \mathcal{L}_w}{\partial \dot{T}} \approx 0, \tag{28} \\
\bar{p}^{ij} &= \frac{\partial \mathcal{L}_w}{\partial \dot{h}^{ij}} = a^{\prime} \lambda'^{ij}, \quad p_a = \frac{\partial \mathcal{L}_w}{\partial \dot{h}^{a}} = 2a^{\prime} \lambda, \tag{29} \\
\bar{P}^{ij} &= \frac{\partial \mathcal{L}_w}{\partial \dot{R}^{ij}} = -\alpha_0 \bar{h} \bar{h}^{ai} \bar{h}^{bj} C^{\tau}_{ab}, \tag{30}
\end{align*}
Note that the momenta \( \bar{p}^{ij} \) and \( \bar{P}^{ij} \) are traceless. The novelty with respect to GR is the emergence of another primary constraint, \( \bar{P} \approx 0 \); this suggests that \( \bar{K} \) is arbitrary, in the same manner as \( p_N \approx 0 \) and \( p_i \approx 0 \) suggest that \( N \) and \( N' \) are arbitrary.

It can easily be checked that the transformation from the original variables to the unimodular-conformal variables is a canonical one. The Poisson brackets (PBs) of the variables are
\begin{equation}
\{ q_{ij}(x), \Pi_B^{ab}(y) \} = \left( \delta_i^a \delta_j^b - \frac{1}{3} \delta_{ij} h^{ab} \right) \tilde{\delta}_B^A \delta(x, y), \quad \{ q^A(x), \Pi_B(y) \} = \delta_B^A \delta(x, y), \tag{31}
\end{equation}
where \( q_{ij} = (\bar{h}_{ij}, \bar{K}_{ij}) \), \( \Pi_B^{ab} = (\bar{p}^{ab}, \bar{P}^{ab}) \) are the variables in the conformally invariant subspace of phase space, and \( q^A = (a, \bar{K}) \), \( \Pi_B = (p_a, \bar{P}) \) is the scale-trace subspace of phase space (and similar for the lapse-shift sector). All other PBs vanish.

After performing the Legendre transformation (from which \( \bar{K} \bar{P} \) is absent, since \( \bar{K} \) does not appear in the Lagrangian) and investigating the emerging constraints, we can write the total Hamiltonian as
\begin{equation}
H^w = \int d^3 x \left( \bar{N} \mathcal{H}_L^W + N' \mathcal{H}_T^W + \left( \bar{N} \bar{K} - \frac{1}{3} D_a N' \right) \mathcal{Q}^w + \lambda_0 p_N + \lambda_i p_i + \lambda_a P \right) + H_{\text{surf}}, \tag{32}
\end{equation}
from which one finds the secondary constraints
\begin{align*}
\mathcal{H}_L^W &= -\frac{\dot{h}_{ab} \bar{p}^{ab}}{2 \omega_0 h} + \left( \varphi \bar{R}_T^{ij} + D_i D_j \right) \bar{p}^{ij} + 2 \bar{K}_a^{ij} \bar{p}^{ij} - \alpha_0 \bar{h} \bar{C}_{ijk} \approx 0, \tag{33} \\
\mathcal{H}_T^W &= -2 \partial_k (\bar{h}_{ij} \bar{p}^{ik}) + \partial_j \bar{h}_{jk} \bar{p}^{jk} - 2 \partial_k \left( \bar{K}_a^{ij} \bar{p}^{ik} \right) + \partial_i \bar{K}_a^{jk} \bar{p}^{jk} \approx 0, \tag{34} \\
\mathcal{Q}^w &= a p_a \approx 0. \tag{35}
\end{align*}
The first two are the Hamiltonian and momentum constraints, and they are analogous to (3) and (4), although the structure of the Hamiltonian constraint is significantly different. The new constraint (35) comes from the consistency condition for the primary constraint $\bar{P} \approx 0$,

$$\dot{\bar{P}} = \{\bar{P}, H^W\} = -\frac{\partial H^W}{\partial \bar{K}} = -Na p_a \approx 0.$$  \hfill (36)

A brief inspection of constraints reveals that the Hamiltonian and momentum constraints are manifestly conformally invariant, due to the use of the unimodular-conformal variables. The Hamiltonian and momentum constraints are independent of the scale $a$ and trace $\bar{K}$. The constraints $\bar{P}$ and $\bar{P}^W$ commute, and they also commute with the rest of the constraints. The Hamiltonian and the momentum constraints close the same hypersurface foliation algebra as in GR, see [6]. This is expected for any reparametrization invariant metric theory, see [30], p. 57. Hence, all constraints are first class.

Therefore, the Hamiltonian and momentum constraints have the same meaning as in GR. The momentum constraint is extended to include the extrinsic curvature sector, since the components of $K_{ij}$ are treated as independent variables in this higher-derivative theory. Thus the three-dimensional diffeomorphism invariance now includes changes of $\bar{K}^T_{ij}$. But what is the meaning of the $\bar{P}$ and $\bar{P}^W$ constraints? It can be shown that these constraints comprise a generator of conformal gauge transformation, as shown in [12] in terms of the original variables (which also include the lapse, prone to conformal transformation). In unimodular-conformal variables, a procedure similar to [12] leads to the following generator of conformal transformation [16]:

$$G^W_{a\omega}[\omega, \bar{\omega}] = \int d^3x \{ \bar{P}^W \omega + \bar{P} \mathcal{L}_\bar{\omega} \omega \} = \int d^3x (a p_a \omega + \bar{P} \mathcal{L}_\bar{\omega} \omega),$$  \hfill (37)

which generates here a transformation only for the scale $a$ and the trace $\bar{K}$. We emphasize that primary and secondary constraints have to appear together to ensure a correct transformation, as emphasized in particular by Pitts [27].

A closer look at the Hamiltonian constraint (33) reveals that the “intrinsic time” of GR contained in the scale part $a$ is absent. This is not surprising, because we are dealing here with a conformally invariant theory. The “problem of time” in quantum gravity [15] is for the Weyl theory thus of a different nature than for GR. This difference will also be relevant for the recovery of semiclassical time discussed below.

Let us now turn to configuration space. In analogy to the Hamilton-Jacobi function of GR, Eq. (7), one can define a Hamilton-Jacobi functional in Weyl gravity as well, which is defined on full configuration space,

$$S^W = S^W[\bar{h}_{ij}, a, \bar{K}^T_{ij}, \bar{K}].$$

It can be shown that terms in \( \{R^2_{ij} + D_i D_j\} P^{ij} \) which depend on $a$ cancel, making this expression conformally invariant.
We expect that the conjugate momenta \( \bar{p}^{ij} \) follow from this functional in the usual way,

\[
\bar{p}^{ij} = \frac{\delta S^w}{\delta h_{ij}}, \quad \bar{p}^i = \frac{\delta S^w}{\delta \bar{K}^i_j}, \quad p_a = \frac{\delta S^w}{\delta a}, \quad \bar{P} = \frac{\delta S^w}{\delta \bar{K}}.
\]  

Due to the primary-secondary pair of constraints \( \bar{P} \approx 0 \) and \( \bar{Q}^w \approx 0 \), we can conclude, however, that the functional \( S^w \) does not depend on \( a \) and \( \bar{K} \), since its infinitesimal conformal variations vanish,

\[
\frac{\delta S^w}{\delta a} = 0, \quad \frac{\delta S^w}{\delta \bar{K}} = 0 \quad \Rightarrow \quad \delta_w S^w = \int d^3x \left( \frac{\delta S^w}{\delta a} \delta a + \frac{\delta S^w}{\delta \bar{K}} \delta \bar{K} \right) = 0.
\]  

One can then interpret \( S^w \) as a conformally invariant functional solving the conformally invariant \textit{Weyl-Hamilton-Jacobi equation} (WHJ equation) obtained from (33),

\[
-\frac{1}{2\alpha_w h} \bar{h}_{ik} \bar{h}_{jl} \frac{\delta S^w}{\delta \bar{K}^i_j} \frac{\delta S^w}{\delta \bar{K}^k_l} + (3\bar{R}^i_j + D_i D_j) \frac{\delta S^w}{\delta \bar{K}^i_j} + 2\bar{K}^i_j \frac{\delta S^w}{\delta h_{ij}} - \alpha_w h \bar{C}^2_{ikj} = 0.
\]  

We expect that \( S^w \), as a solution to the above equation, gives a “classical trajectory” in the configuration subspace spanned by \( \{ \bar{h}_{ij}, \bar{R}^i_j \} \). Due to (39), a tangent to this trajectory does not have components in the \( a \) and \( \bar{K} \) directions of the configuration space. In other words, the classical state of this theory does not follow directions along changes of \( a \) and \( \bar{K} \) in configuration space.

Quantization is now performed in the sense of Dirac by implementing the classical constraints as restrictions on physically allowed wave functionals on the full configuration space [15],

\[
\Psi = \Psi[\bar{h}_{ij}, a, \bar{R}^i_j, \bar{K}].
\]

The canonical variables are promoted into operators in the standard way,

\[
\hat{\bar{h}}_{ij}(x)\Psi = \bar{h}_{ij}(x)\Psi, \quad \hat{\bar{p}}^{ij}(x)\Psi = -i\hbar \frac{\delta}{\delta \bar{h}_{ij}(x)}\Psi, \quad (41)
\]

\[
\hat{\bar{K}}^i_j(x)\Psi = \bar{K}^i_j(x)\Psi, \quad \hat{\bar{p}}^i(x)\Psi = -i\hbar \frac{\delta}{\delta \bar{K}^i_j(x)}\Psi, \quad (42)
\]

\[
\hat{a}(x)\Psi = a(x)\Psi, \quad \hat{\bar{p}}_a(x)\Psi = -i\hbar \frac{\delta}{\delta a(x)}\Psi, \quad (43)
\]

\[
\hat{\bar{K}}(x)\Psi = \bar{K}(x)\Psi, \quad \hat{\bar{P}}(x)\Psi = -i\hbar \frac{\delta}{\delta \bar{K}(x)}\Psi. \quad (44)
\]

The quantization of the constraints yields [17]

\[
\hat{\mathcal{H}}^{w}_+\Psi = 0, \quad \hat{\mathcal{H}}^{w}_-\Psi = 0, \quad \hat{\bar{P}}\Psi = 0, \quad \hat{\bar{Q}}^w\Psi = 0. \quad (45)
\]  

The first of these equations is the quantized Hamiltonian constraint, which replaces the WDW equation of quantum GR and which we will therefore call the “Weyl-Wheeler-DeWitt” (WWDW) equation. Neglecting here the ubiquitous factor ordering problem, it assumes the explicit form
\[
\left[ \frac{\hbar}{2\alpha_W} \tilde{h}_{ij} \tilde{h}^{ij} \frac{\delta^2}{\delta K^T_{ij} \delta K^T_{kl}} - i\hbar \left( (^{(3)}R^T_{ij} + D^T_{ij}) \frac{\delta}{\delta K^T_{ij}} - 2i\hbar \tilde{K}^T_{ij} \frac{\delta}{\delta h_{ij}} \right) - \alpha_W \hbar C^2_{ij} + \hat{H}_m \right] \Psi = 0.
\]

One recognizes that the WWDW equation is structurally different from the WDW equation, since the wave functional does not depend only on the three-metric, but also on its evolution (the second fundamental form). There is also no scale \(a\) present and therefore no intrinsic time in the sense of the WDW equation; there is no indefinite “DeWitt metric”. It is also interesting to see that \(\hbar\) drops out after dividing the whole equation by \(\hbar\). Formally this is due to our use of \(\alpha_W\) in the action instead of just \(\alpha_W\); re-scaling \(\alpha_W \rightarrow \alpha_W / \hbar\) would bring back \(\hbar\) at the places similar to the Wheeler-DeWitt equation \(3\), but the important point is that \(\hbar\) can be made to disappear by a simple re-scaling. This is, of course, a property of the vacuum theory. If we add a matter Hamiltonian density to the WWDW equation, as we shall do below, \(\hbar\) will not disappear when dividing the whole equation by \(\hbar\).

The quantum momentum constraints read

\[
\hat{H}_W^m \Psi = i\hbar \left[ 2\partial_k \left( \tilde{h}_{ij} \frac{\delta \Psi}{\delta \bar{h}_{jk}} \right) - \partial_i \bar{h}_{jk} \frac{\delta}{\delta \Psi} \bar{h}_{jk} + 2\partial_k \left( \tilde{K}^T_{ij} \frac{\delta \Psi}{\delta \bar{K}^T_{jk}(x)} \right) \right. \\
\left. - \partial_i \bar{K}^T_{jk} \frac{\delta \Psi}{\delta \bar{K}^T_{jk}} + \hat{H}_m \Psi \right] = 0, \tag{47}
\]

or alternatively, in a manifestly covariant version,

\[
\hat{H}_W^m \Psi = i\hbar \left[ 2D_k \left( \tilde{h}_{ij} \frac{\delta \Psi}{\delta \bar{h}_{jk}} \right) + 2D_k \left( \tilde{K}^T_{ij} \frac{\delta \Psi}{\delta \bar{K}^T_{jk}} \right) - D_i \bar{K}^T_{jk} \frac{\delta \Psi}{\delta \bar{K}^T_{jk}} + \hat{H}_m \Psi \right] = 0. \tag{48}
\]

Finally, the new quantum constraints read

\[
\frac{\delta \Psi}{\delta K} = 0, \quad \frac{\delta \Psi}{\delta a} = 0. \tag{49}
\]

The meaning of (49) is obvious: the wave functional does not depend on \(a\) and \(\bar{K}\); hence, it is conformally invariant (apart from a possible phase factor). This is a direct consequence of the first class nature of the constraints \(\hat{P} = 0\) and \(\hat{Q}^W = \alpha p_a = 0\). Thus, we have a conformally invariant canonical quantum gravity theory derived from the Weyl action. Equivalently, one could have started from a reduced phase space without \(a\) and \(K\) and ended up with (46) and (47) only, with \(\Psi\) depending on 10 (instead of 12) configuration variables from the start.

Looking at the whole picture, we conclude that solutions to the WWDW equation are conformally invariant (scale and trace independent), and are indistinguishable for two three-metrics that are conformal to each other.
3 Semiclassical Weyl gravity and the recovery of time

We consider quantum Weyl gravity with a conformally coupled matter field \( \phi \), for conformal matter does not spoil the first-class nature of constraints; it only modifies their explicit form. We can then quantize the theory while preserving its conformal invariance.

In the spirit of the semiclassical (Born-Oppenheimer type) expansion for the WDW equation, we make an ansatz for the wave functional in which the “heavy” part, being the pure gravitational part, is separated from the matter part \[17\]. We write for the full quantum state in analogy to (9)

\[
\Psi \equiv \exp \left( \frac{i}{\hbar} S [\bar{h}_{ij}, K^i_{jk}, \phi] \right).
\]  

Plugging (50) into the WWDW equation (46) gives

\[
\frac{i}{2\alpha} \bar{h}_{ik} \bar{h}_{jl} \frac{\delta^2 S}{\delta K^i_{jk} \delta K^j_{kl}} - \frac{1}{2\alpha \hbar} \bar{h}_{ik} \bar{h}_{jl} \frac{\delta S}{\delta K^i_{jk}} \frac{\delta S}{\delta K^j_{kl}} + (3R^i_{jk} + D^i_{jk}) \frac{\delta S}{\delta K^j_{kl}} + 2K^i_{ik} \frac{\delta S}{\delta h_{ij}} - \alpha \hbar \bar{C}^2_{ij} + \frac{(\bar{H}^m_{\perp} \Psi)}{\Psi} = 0.
\]  

The expansion can be performed with respect to \( \alpha^{-1} \), for this coupling constant appears at the same place (in the kinetic term and in part of the potential) as \( m_p^2 \) appears in the WDW equation. The functional \( S \) can then be expanded in powers of \( \alpha^{-1} \ll 1 \), assuming \( \alpha \) to be large; this is similar to the Planck-mass expansion for quantum GR, see \[10\] above. Note that \( \alpha \) is a dimensionless quantity, unlike the Planck mass in the case of the WDW equation. This is similar to the semiclassical expansion of quantum electrodynamics, with the (dimensionless) fine structure constant as the appropriate expansion parameter \[19\]. We thus write

\[
S = \alpha \sum_{n=0}^{\infty} \left( \frac{1}{\alpha} \right)^n S_n^W.
\]  

Note that \( (\bar{H}^m_{\perp} \Psi) / \Psi \), when expanded in powers of \( \alpha \), is at most of the order \( \alpha^2 \), since the highest derivative with respect to the matter field \( \phi \) in \( \bar{H}^m_{\perp} \) is the second order, which is the kinetic term (we assume it is the only one of that kind). We shall then denote with \( (\bar{H}^m_{\perp} \Psi) / \Psi)^{(n)} \), \( n \leq 2 \), terms proportional to \( \alpha^0 \).

Inserting the ansatz (52) into the WWDW equation and collecting the powers of \( \alpha^2 \), we find

\[
\alpha^2 : \quad (\bar{H}^m_{\perp} \Psi)^{(2)} = 0 \quad \Rightarrow \quad \frac{\delta S_n^W}{\delta \phi} = 0.
\]  

This is analogous to the situation in GR \[18\]. At the next order, \( \alpha \), we have
\[ \alpha_1^\alpha : - \frac{1}{2\hbar} \hbar \frac{\delta S^\alpha_0}{\delta K_{ij}} \frac{\delta^2 S^\alpha_0}{\delta K_{kl}} + \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta S^\alpha_0}{\delta K^i_{ij}} + 2 K^i_{ij} \frac{\delta S^\alpha_0}{\delta h_{ij}} - \hbar \hat{C}^2 = 0, \quad (54) \]

which is nothing else than the Weyl-HJ equation \( \text{[40]} \), with \( S^\alpha = \alpha \alpha^\alpha_0 S^\alpha_0 \).

At the next order, \( \alpha^0_0 \), we obtain

\[ \alpha^0_0 : \frac{i}{2\hbar} \hbar \frac{\delta^2 S^\alpha_0}{\delta K_{ij} \delta K_{kl}} - \frac{1}{2\hbar} \hbar \frac{\delta S^\alpha_0}{\delta K_{ij}} \frac{\delta S^\alpha_0}{\delta K_{kl}} + \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta S^\alpha_0}{\delta h_{ij}} + 2 K^i_{ij} \frac{\delta S^\alpha_0}{\delta h_{ij}} + \left( \hat{H}^m_m \Psi \right) \Psi^{(0)} = 0. \quad (55) \]

A procedure analogous to the one used to arrive at the functional Schrödinger equation in quantum GR motivates us to propose the following functional:

\[ f \equiv D[i \hbar_{ij}, \tilde{K}^T_{ij}] \exp \left( \frac{i}{\hbar} S^w_i \right), \quad (56) \]

with a condition on the “WKB prefactor” \( D \) that will be derived below. We first calculate the following functional derivatives:

\[ i \hbar_{ij} \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta f}{\delta K_{kl}} = i \hbar_{ij} \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta D}{\delta h_{ij}} f - \frac{1}{\hbar} \hbar \frac{\delta S^w_i}{\delta h_{ij}} \frac{\delta S^w_i}{\delta K_{ij}} f, \]

\[ -2i h \tilde{K}^T_{ij} \frac{\delta f}{\delta h_{ij}} = -2i h \tilde{K}^T_{ij} \frac{\delta D}{\delta h_{ij}} f + 2 K^i_{ij} \frac{\delta S^w_i}{\delta h_{ij}} f, \]

\[ -i h \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta f}{\delta K^i_{ij}} = -i h \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta D}{\delta h_{ij}} f + \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta S^w_i}{\delta h_{ij}} f. \]

These expressions are used in \( (55) \) to eliminate the second, third, and fourth terms, after multiplying with \( f \). As a result, one obtains

\[ \frac{i}{2} \hbar \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta f}{\delta K_{ij}} - i h \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta f}{\delta h_{ij}} - 2i h \tilde{K}^T_{ij} \frac{\delta S^w_i}{\delta h_{ij}} - \hat{H}^m_m f \]

\[ + \left( \frac{i}{2} \hbar \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta D}{\delta h_{ij}} \frac{1}{\delta h_{ij}} D - \frac{i}{2} \hbar \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta D}{\delta K_{kl}} \frac{1}{\delta K_{ij}} D \right) f = 0, \quad (57) \]

where \( \hat{H}^m_m f \) comes from \( \left( \left( \hat{H}^m_m \Psi \right) / \Psi \right)^{(0)} f \). We now choose \( D \) such that the term in the parenthesis vanishes. This gives us the equation that defines \( D \), in analogy to the situation in quantum GR \( \text{[14]} \):

\[ \frac{i}{2} \hbar \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta D}{\delta K_{ij}} \frac{1}{\delta h_{ij}} + \frac{i}{2} \hbar \frac{\delta S^w_i}{\delta K_{ij}} \frac{\delta D}{\delta K_{kl}} \frac{1}{\delta K_{ij}} D + i h \left( \frac{1}{2} R^i_{ij} + D^i_{ij} \right) \frac{\delta D}{\delta h_{ij}} \frac{1}{\delta h_{ij}} D + 2i h \tilde{K}^T_{ij} \frac{\delta D}{\delta h_{kl}} \frac{1}{\delta K_{ij}} D = 0. \]
With this condition, (57) reduces to

$$i\hbar \left[ -\frac{1}{2\hbar} \delta_{kl} \frac{\delta S_0}{\delta K_{ij}} + \left( \delta S_0^{(3)} + D_t^i \right) \frac{\delta}{\delta K_{ij}} + 2 R_{ij}^i \frac{\delta}{\delta h_{ij}} \right] f = \hat{H}_m f. \quad (58)$$

Introducing a local “bubble” (Tomonaga-Schwinger) time functional by

$$\frac{\delta}{\delta \tau_W(x)} := -\frac{1}{2\hbar} \delta_{kl} \frac{\delta S_0}{\delta K_{ij}} \frac{\delta}{\delta K_{ij}} + \left( \delta S_0^{(3)} + D_t^i \right) \frac{\delta}{\delta K_{ij}} + 2 R_{ij}^i \frac{\delta}{\delta h_{ij}}, \quad (59)$$

we arrive at the Tomonaga-Schwinger equation

$$i\hbar \frac{\delta f}{\delta \tau_W} = \hat{H}_m f. \quad (60)$$

Note that $\tau_W$ is, like its GR-counterpart (12), not a scalar function [9]. We emphasize that the wave function $f$ is conformally invariant.

At a formal level, the Tomonaga-Schwinger equation (60) resembles the corresponding equation in quantum GR. We see, however, from the explicit expression for the WKB time (59) that it is defined only from the semiclassical shape degrees of freedom, since the traces (especially $a$) are absent. A functional Schrödinger equation of the form (15) can be derived from the Tomonaga-Schwinger equation by a procedure similar to the one in GR. This will involve a time parameter that should be identical with the time parameter of the classical solutions of Weyl gravity.

Proceeding with the Born-Oppenheimer scheme to higher orders in $\alpha_W$, one arrives at quantum gravitational corrections terms proportional to $\alpha^{-1}_{W}$, in analogy to the higher orders proportional to $m_P^{-2}$ in quantum GR [18]. These may serve to study correction terms to the limit of quantum field theory in curved (Weyl) spacetime, but we will not discuss them here.

4 Outlook

Although there is not yet a consensus about the correct quantum theory of gravity, and about the need to quantize gravity, there exist several approaches within which concrete questions with potential observational relevance can be posed and answered. Among them is canonical quantum gravity in the metric formulation. If general relativity is quantized in this way, one arrives at the Wheeler-DeWitt equation and the momentum constraints. A semiclassical expansion leads to the recovery of quantum field theory in curved spacetime plus quantum gravitational corrections. The latter may be observationally tested, for example, in the CMB anisotropy spectrum.

Our concern here was to discuss canonical quantization and the semiclassical limit for an alternative theory based on the Weyl tensor. This “Weyl gravity” does not contain any scale, so it may be of empirical relevance only in the early Universe,
where scales may be unimportant. Independent of this possibility, it is of structural interest to compare this theory in its quantum version with quantum general relativity. We have seen here that a semiclassical limit can be performed by a well defined approximation scheme, although the emerging semiclassical time has properties different from standard semiclassical time. In future investigations, we plan to apply a theory based on the sum of Weyl and Einstein-Hilbert action to the early Universe and to the understanding of spacetime structure at a fundamental level, topics that are also at the centre of Paddy’s interest.

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