ON THE DETERMINANT OF MULTIPLICATION MAP OF A MONOMIAL COMPLETE INTERSECTION RING

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Abstract. In this article, we consider the monomial complete intersection algebra \( \mathbb{K}[x, y]/(x^d, y^q) \) in two variables. For elements \( l_1, \ldots, l_{d+q-2k} \) of degree 1, we give a formula of the determinant of linear map from the homogeneous component of degree \( k \) to the homogeneous component of degree \( d + q - k \) defined by the multiplication of \( l_1 \cdot \cdots \cdot l_{d+q-2k} \).

1. Introduction

Roughly speaking, the strong Lefschetz property for an algebra is an analogue of the property for the cohomology ring of compact Kähler variety known as Hard Lefschetz Theorem (see also [4, 8]). We say that a graded Artinian Gorenstein algebra \( R = \bigoplus_{k=0}^s R_k \) with socle degree \( s \) has the strong Lefschetz property with a Lefschetz element \( l \in R_1 \) if the multiplication map \( \times l^{s-2k} : R_k \ni f \mapsto fl^{s-2k} \in R_{s-k} \) is bijective for any \( 0 \leq k \leq s/2 \). For some algebras with the strong Lefschetz property, characterization of Lefschetz elements is studied, e.g. [1, 7]. The determinant of the linear map \( \times l^{s-2k} : R_k \rightarrow R_{s-k} \) is also studied, e.g. [3, 14, 9].

Consider the monomial complete intersection ring

\[
R = \mathbb{K}[x_1, \ldots, x_n]/\langle x_1^{d_1}, \ldots, x_1^{d_n} \rangle.
\]

Then the algebra \( R \) has the strong Lefschetz property (see also [10, 11, 13]). For example, \( l = x_1 + \cdots + x_n \) is a Lefschetz element for the algebra. In [3], the determinant of multiplication map of \( l^{s-2k} \) is calculated in the case where \( d_1 = \cdots = d_n = 1 \). In the case where \( n = 2 \), the determinants are implicitly given in [5]. In this article, we generalize the problem to the multiplication map of a product of \( l_1, \ldots, l_{s-2k} \in R_1 \). We consider the determinant of the linear map \( \times l_1 \cdot \cdots \cdot l_{s-2k} : R_k \ni f \mapsto f \cdot (l_1 \cdot \cdots \cdot l_{s-2k}) \in R_{s-k} \) for \( l_1, \ldots, l_{s-2k} \in R_1 \). In the case where \( n = 2 \), the determinant can be written with Schur polynomials.

This article is organized as follows: In Section 2, we recall notation and facts of symmetric polynomials and determinants. In Section 3, we calculate the determinant of the multiplication map.

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2. Notation and Formulae

In this section, we recall notation and facts which will be used in Section 3.

We call a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of weakly decreasing nonnegative integers a partition of \( m \) if \( \sum \lambda_i = m \). For nonnegative integers \( r, l \in \mathbb{N}, (r^l) \) denotes the partition consisting of \( l \) copies of \( r \). For a partition \( \lambda \) of \( n \), define \( \tilde{\lambda}_j = \{ i | 1 \leq j \leq \lambda_i \} \). Then \( \lambda = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots) \) is also a partition of \( n \). We call \( \tilde{\lambda} \) the conjugate partition to \( \lambda \). For example, the partition \( (l^r) \) is the conjugate partition to \( (r^l) \). For partitions

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\( \mu \) and \( \lambda \), we write \( \mu \subset \lambda \) to denote that they satisfy \( \mu_i \leq \lambda_i \) for all \( i \). For \( \lambda \subset (r^t) \), we define \( (r^t) \setminus \lambda \) to be the partition \( (r - \lambda_1, r - \lambda_{t-1}, \ldots, r - \lambda_t, 0, \ldots) \).

For a partition \( \lambda \), we define the Schur polynomial \( s_\lambda(x) \) in \( n \) variables \( x = (x_1, \ldots, x_n) \) to be

\[
s_\lambda(x) = \frac{\det((x_j)^{\lambda_i + n - i})_{i=1,\ldots,n}}{\det((x_j)^{n-1})_{j=1,\ldots,n}}.
\]

For a partition \( \lambda \) of a nonnegative integer \( m \), the Schur polynomial \( s_\lambda(x) \) is a homogeneous symmetric polynomial of degree \( m \). For \( k \), the Schur polynomial \( s_{(1^k)}(x) \) is the \( k \)-th elementary symmetric polynomial \( e_k(x) \), i.e., the sum

\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}
\]

of all square-free monomials of degree \( k \). If \( k > n \) or \( k < 0 \), then \( e_k(x) = 0 \). If \( k = 0 \), then \( e_0(x) = 1 \). It is known that the Schur polynomial and elementary symmetric polynomial satisfies

\[
(1) \quad s_{\lambda}(x) = \det(e_{\lambda_i + j - i}(x))_{i=1,\ldots,l}, \quad j=1,\ldots,l,
\]

for a partition such that \( \lambda_{i+1} = 0 \). (See e.g. [6, Section I.3].)

Next we recall the Cauchy–Binet formula for determinants. Assume that \( m \geq n \).

Let \( X \) be an \( n \times m \) matrix, \( Y \) an \( m \times n \) matrix. For \( S \subset \{ 1, 2, \ldots, n \} \) with \( \#S = m \), \( X^S \) (resp. \( Y_S \)) denotes the \( m \times m \) submatrix whose rows (resp. columns) are the rows (resp. columns) of \( X \) (resp. \( Y \)) at indices from \( S \). If the entries of \( X \) and \( Y \) are elements of a commutative ring, then we have the equation

\[
(2) \quad \det(YX) = \sum_S \det(Y_S) \det(X^S),
\]

where the sum is over all subsets \( S \subset \{ 1, 2, \ldots, n \} \) such that \( \#S = m \). (See e.g. [2, Section 5.6].)

3. The determinant of representation matrices.

Let \( K \) be a field. For positive integers \( d, q \) with \( d \geq q \), we consider the algebra \( R = K[x,y]/(dx^q, y^{d+1}) \). The algebra \( R \) can be decompose into homogeneous spaces \( R_k \) as follows: \( R = \bigoplus_{k=0}^{d+q} R_k \). The set \( B_k = \{ x^iy^{d-i} \mid 0 \leq i \leq d, 0 \leq k-i \leq q \} \) is a \( K \)-basis for the homogeneous space \( R_k \). Hence we have

\[
\dim_K R_k = \begin{cases} 
  k+1 & (0 \leq k \leq q) \\
  q+1 & (k \leq d) \\
  s-k+1 & (q \leq k \leq d+q).
\end{cases}
\]

Let \( k \leq \frac{d+q}{2} \). Then \( \dim_K R_k = \dim_K R_{d+q-k} \). Let \( l_1 = a_1x + b_1y, l_2 = a_2x + b_2y, \ldots \in R_1 \setminus \{0\} \). For \( l_t \), we define a linear map

\[
\times l_t: R_k+1 \ni f \mapsto f \cdot l_t \in R_k+t.
\]

Then we obtain the following sequence of linear maps:

\[
R_k \times l_1 \rightarrow R_{k+1} \times l_2 \rightarrow \cdots \rightarrow \times l_{d+q-2k} R_{d+q-k}.
\]

First we calculate the representation matrix of the linear map

\[
\times l_1 \cdot l_2 \cdots l_u: R_k \ni f \mapsto f \cdot l_1 \cdot l_2 \cdots l_u \in R_{k+u},
\]

with respect to the bases \( B_k \) and \( B_{k+u} \).
Lemma 3.1. Assume that \( \beta = b_1 b_2 \cdots b_u \neq 0 \). Let \( l_1 = a_1 x + b_1 y, l_2 = a_2 x + b_2 y, \ldots, l_u = a_u x + b_u y \). Then the coefficient of \( x^{w−j} y^j \) in \( x^{w−j} y^j \cdot l_1 l_2 \cdots l_u \) is
\[
\beta e_{w+j−1}(\frac{a}{b}),
\]
where \( \frac{a}{b} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_u}{b_u}) \).

Proof. Since \( b_i \neq 0 \) for all \( i \), we have
\[
l_1 l_2 \cdots l_u = (a_1 x + b_1 y)(a_2 x + b_2 y) \cdots (a_u x + b_u y) = \beta \cdot \prod_{i=1}^{u} e_{w−j}(\frac{a}{b}) x^{w−j} y^j.
\]
Hence
\[
x^{w−j} y^j \cdot l_1 l_2 \cdots l_u = \beta \cdot \prod_{i=1}^{u} e_{w−j−i}(\frac{a}{b}) x^{w−j−i} y^j.
\]

Now we calculate the determinant \( D_{d,q}(a_1, \ldots, a_{d+q−2k}; b_1, \ldots, b_{d+q−2k}) \) of the linear map
\[
x_1 \cdot x_2 \cdot \cdots \cdot x_{d+q−2k} : R_k \ni f \mapsto f \cdot l_1 \cdot l_2 \cdots l_{d+q−2k} \in R_{d+q−k}.
\]

Theorem 3.2 (Main Theorem). Let \( l_1 = a_1 x + b_1 y, l_2 = a_2 x + b_2 y, \ldots, l_{d+q−2k} = a_{d+q−2k} x + b_{d+q−2k} y \in R_k \setminus \{0\} \). Let \( i \) be a permutation on \( \{1, 2, \ldots, d+q−2k\} \).
Assume that \( \beta = b_1 b_2 \cdots b_u \neq 0 \), and that \( \tilde{a} = a_{u+1} a_{u+2} \cdots a_{d+q−2k} \neq 0 \). Let
\[
\begin{align*}
a &= (a_1, \ldots, a_{d+q−2k}), & b &= (b_1, \ldots, b_{d+q−2k}), \\
\frac{\tilde{a}}{b} &= (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_u}{b_u}), & \frac{\tilde{b}}{\tilde{a}} &= (\frac{b_1}{a_{u+1}}, \frac{b_2}{a_{u+2}}, \ldots, \frac{b_u}{a_{d+q−2k}}).
\end{align*}
\]

If \( q \leq k \leq \frac{q+d}{2} \), then
\[
D_{d,q}(a; b) = \tilde{a}^{q+1} \tilde{b}^{q+1} \cdot s_{((q+1)\cdot \lambda)} \left( \frac{\tilde{a}}{\tilde{b}} \right) s_{((q+1)\cdot \lambda−2k−u)} \left( \frac{b}{\tilde{a}} \right).
\]

If \( 0 \leq k \leq k+u \leq q \), then
\[
D_{d,q}(a; b) = \tilde{a}^{k+1} \tilde{b}^{k+1} \cdot \sum_{\lambda \subseteq (u+1)} s_{(\lambda)} \left( \frac{\tilde{a}}{\tilde{b}} \right) s_{(\lambda−(k+1)\cdot \lambda)} \left( \frac{b}{\tilde{a}} \right).
\]

If \( 0 \leq k \leq q \leq d \leq k+u \), then
\[
D_{d,q}(a; b) = \tilde{a}^{k+1} \tilde{b}^{k+1} \cdot \sum_{\lambda \subseteq (u+1)} s_{(\lambda−(d+1)\cdot \lambda)} \left( \frac{\tilde{a}}{\tilde{b}} \right) s_{(\lambda−(k+1)\cdot \lambda)} \left( \frac{b}{\tilde{a}} \right).
\]

If \( k \leq q \leq k+u \leq d \), then
\[
D_{d,q}(a; b) = \tilde{a}^{k+1} \tilde{b}^{k+1} \cdot \sum_{\lambda \subseteq ((q−k)\cdot \lambda)} s_{(\lambda−(k+1)\cdot \lambda)} \left( \frac{\tilde{a}}{\tilde{b}} \right) s_{(\lambda−(k+1)\cdot \lambda)} \left( \frac{b}{\tilde{a}} \right).
\]
Proof. Since $R$ is a commutative algebra, we can assume $i_t = t$ without loss of generality. In this case,
\[
\frac{a}{b} = \left( \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_u}{b_u} \right), \quad \frac{\hat{a}}{\alpha} = \left( \frac{b_{u+1}}{a_{u+1}}, \frac{b_{u+2}}{a_{u+2}}, \ldots, \frac{b_{d+q-2k}}{a_{d+q-2k}} \right),
\]
\[
\hat{\beta} = b_1 b_2 \cdots b_u, \quad \hat{\alpha} = a_{u+1} a_{u+2} \cdots a_{d+q-2k}.
\]

In the case where $q ≤ k ≤ \frac{d+2}{2}$, the bases $B_{k+u}$ and $B_{d+q-k}$ are
\[
\{ x^k y^0, x^{k-1} y^1, \ldots, x^{k-q} y^q \},
\{ x^{k+u} y^0, x^{k+u-1} y^1, \ldots, x^{k+u-q} y^q \},
\{ x^{d+q-k} y^0, x^{d+q-k-1} y^1, \ldots, x^{d-k} y^q \},
\]
respectively. Hence the representation matrix $X$ for $x l_1 \cdots l_u : R_k \to R_{k+u}$ is
\[
\left( \hat{\beta} e_{u+j-i} \left( \frac{a}{b} \right) \right)_{i=0,1,\ldots,q, j=0,1,\ldots,q},
\]
and the representation matrix $Y$ for $x l_{u+1} \cdots l_{d+q-2k} : R_{k+u} \to R_{d+q-k}$ is
\[
\left( \hat{\alpha} e_{d+q-2k-u+j-i} \left( \frac{b}{a} \right) \right)_{i=0,1,\ldots,q, j=0,1,\ldots,q}.
\]

Hence
\[
D_{d,q}(a; b) = \det(Y) \det(X)
= \beta^{q+1} s_{((q+1)^u)} \left( \frac{a}{b} \right) \cdot \hat{\alpha}^{q+1} s_{((q+1)^{d+q-2k-u})} \left( \frac{b}{a} \right).
\]

Next we consider the case where $0 ≤ k ≤ k + u ≤ q$. In this case, the bases $B_k$, $B_{k+u}$ and $B_{d+q-k}$ are
\[
\{ x^k y^0, x^{k-1} y^1, \ldots, x^{k} y^k \},
\{ x^{k+u} y^0, x^{k+u-1} y^1, \ldots, x^{k+u} y^{k+u} \},
\{ x^{d+q-k} y^0, x^{d+q-k-1} y^1, \ldots, x^{d-k} y^{k} \},
\]
respectively. Hence the representation matrix $X$ for $x l_1 \cdots l_u : R_k \to R_{k+u}$ is
\[
\left( \hat{\beta} e_{u+j-i} \left( \frac{a}{b} \right) \right)_{i=0,1,\ldots,k+u, j=0,1,\ldots,k},
\]
and the representation matrix $Y$ for $x l_{u+1} \cdots l_{d+q-2k} : R_{k+u} \to R_{d+q-k}$ is
\[
\left( \hat{\alpha} e_{d+q-2k-u+j-i} \left( \frac{b}{a} \right) \right)_{i=q-k, q-k+1,\ldots,q, j=0,1,\ldots,k+u} = \left( \hat{\alpha} e_{d-k-u+j-i} \left( \frac{b}{a} \right) \right)_{i=0,1,\ldots,k, j=0,1,\ldots,k+u}.
\]

To calculate $\det(Y X)$, we consider minors $\det(X \{ \delta_0, \ldots, \delta_k \})$ and $\det(Y \{ \delta_0, \ldots, \delta_k \})$ for $0 ≤ \delta_0 < \delta_1 < \cdots < \delta_k ≤ k + u$. Let $\lambda_i = u - \delta_i + i$, and $\mu_{k-j} = d - u + \delta_j - j$. Since $0 ≤ \delta_0 < \delta_1 < \cdots < \delta_k ≤ k + u$, it follows that $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k, 0, \ldots)$ and $\mu = (\mu_0, \mu_1, \ldots, \mu_k, 0, \ldots)$ are partitions satisfying $\lambda \subset (u^{k+1})$ and $\mu = (u^{k+1}) \setminus \lambda$. The minor $\det(X \{ \delta_0, \ldots, \delta_k \})$ is equal to
\[
\det \left( \hat{\beta} e_{u+j-i} \left( \frac{a}{b} \right) \right)_{i=\delta_0, \delta_1, \ldots, \delta_k, j=0,1,\ldots,k} = \beta^{k+1} \det \left( e_{u+j-i} \left( \frac{a}{b} \right) \right)_{i=0,1,\ldots,k, j=0,1,\ldots,k}
= \beta^{k+1} \det \left( e_{\lambda_i+j-i} \left( \frac{\alpha}{\beta} \right) \right)_{i=0,1,\ldots,k}.
\]
It follows from (1) that
\[ \det(X\{\delta_0, \ldots, \delta_k\}) = \hat{\alpha}^{k+1} s_\lambda \left( \frac{a}{b} \right). \]

On the other hand, the minor matrix \( Y\{\delta_0, \ldots, \delta_k\} \) is
\[ \det\left( \hat{\alpha} e_d - x_{-u} + k - i \left( \frac{b}{a} \right) \right) = \hat{\alpha}^{k+1} \det\left( e_{d-u} - \delta_j - i \left( \frac{b}{a} \right) \right) \]
\[ = \hat{\alpha}^{k+1} \det\left( e_{\mu_{-j} + j - i} \left( \frac{b}{a} \right) \right) \]

By flipping vertically and horizontally, we have
\[ \det(Y\{\delta_0, \ldots, \delta_k\}) = \hat{\alpha}^{k+1} \det\left( \mu_{j+i} - j \left( \frac{b}{a} \right) \right) \]

It follows from (1) formula that
\[ \det(Y\{\delta_0, \ldots, \delta_k\}) = \hat{\alpha}^{k+1} s_\mu \left( \frac{b}{a} \right). \]

Therefore it follows from (2) formula that
\[ \det(YX) = \sum_{0 \leq \delta_0 < \delta_1 < \cdots < \delta_k \leq k+u} \det(Y\{\delta_0, \ldots, \delta_j\}) \det(X\{\delta_0, \ldots, \delta_j\}) \]
\[ = (\hat{\alpha} \hat{\beta})^{k+1} \sum_{\lambda \subset (u^{k+1})} s_\mu \left( \frac{b}{a} \right) s_\lambda \left( \frac{a}{b} \right), \]

where \( \mu = (d^{k+1}) \setminus \lambda \).

Next we consider the case where \( 0 \leq k \leq q \) and \( d \leq k + u \). Since \( D_{d,q}(a; b) = D_{q,d}(b; a) \), we can obtain the formula in this case from the result for the case where \( 0 \leq k \leq k + u \leq q \).

Finally we consider the case where \( k \leq q \leq k + u \leq d \). In this case, the bases \( B_k, B_{k+u} \) and \( B_{d+q-k} \) are
\[ \{ x^0 y^0, x^0 y^1, \ldots, x^0 y^p \}, \]
\[ \{ x^0 y^0, x^1 y^0, \ldots, x^0 y^{q-1} \}, \]
\[ \{ x^0 y^{q-k}, x^{d-1} y^{q-k+1}, \ldots, x^d y^0 \}, \]
respectively. Hence the representation matrix \( X \) for \( x_1 \cdots x_u : R_k \to R_{k+u} \) is
\[ \left( \hat{\beta} e_{u+j} \left( \frac{a}{b} \right) \right)_{i=0,1,\ldots,q, j=0,1,\ldots,k}, \]
and the representation matrix \( Y \) for \( x_{u+1} \cdots x_{d+q-2k} : R_{k+u} \to R_{d+q-k} \) is
\[ \left( \hat{\alpha} e_{d+q-2k-u+j} \left( \frac{b}{a} \right) \right)_{i=q-k, q-k+1, \ldots, q, j=0,1,\ldots,k} = \left( \hat{\alpha} e_{d-k-u+j} \left( \frac{b}{a} \right) \right)_{i=0,1,\ldots,k, j=0,1,\ldots,q}. \]

Hence we obtain a formula similar to the case where \( 0 \leq k \leq k + u \leq q \). For \( 0 \leq \delta_0 < \delta_1 < \cdots < \delta_k \leq q \), we obtain a partition \( \lambda \) by \( \lambda_i = u - \delta_i + i. \) In this case, \( \lambda \) is a partition contained by \( ((q-k)^{k+1}) \). Hence we obtain the formula.

As corollary to Theorem 3.2, we have an explicit formula for the determinant in a special case.
Corollary 3.3. Assume that $\beta = b_1 b_2 \cdots b_{d+q-2k} \neq 0$. Let
\[ a = (a_1, \ldots, a_{d+q-2k}), \quad b = (b_1, \ldots, b_{d+q-2k}), \quad \frac{a}{b} = \left( \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_{d+q-2k}}{b_{d+q-2k}} \right). \]
If $0 \leq k \leq q$, then
\[ D_{d,q}(a; b) = \beta^{k+1} s_{((k+1)d-k)} \left( \frac{a}{b} \right). \]
If $q \leq k \leq \frac{d+q}{2}$, then
\[ D_{d,q}(a; b) = \beta^{q+1} s_{((q+1)d+q-2k)} \left( \frac{a}{b} \right). \]

In the case where $d = q$, the defining ideal of the ring is symmetric. Hence we have the following formula:

Corollary 3.4. Assume that $\alpha = a_1 a_2 \cdots a_{2m} \neq 0$ and that $\beta = b_1 b_2 \cdots b_{2m} \neq 0$. Let
\[ \left( \frac{a}{b} \right) = \left( \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_{2m}}{b_{2m}} \right), \quad \left( \frac{b}{a} \right) = \left( \frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_{2m}}{a_{2m}} \right). \]

Then the following equation holds:
\[ \beta^r s_{(r^m)} \left( \frac{a}{b} \right) = \alpha^r s_{(r^m)} \left( \frac{b}{a} \right). \]

Proof. Consider the case where $d = q$. By Corollary 3.3, the determinant for $a_1 x + b_1 y, \ldots, a_{2d-2k} x + b_{2d-2k} y$ is
\[ D_{d,d}(a; b) = \beta^{k+1} s_{((k+1)d-k)} \left( \frac{a}{b} \right). \]
On the other hand, the determinant for $b_1 x + a_1 y, \ldots, b_{2d-2k} x + a_{2d-2k} y$ is
\[ D_{d,d}(b; a) = \alpha^{k+1} s_{((k+1)d-k)} \left( \frac{b}{a} \right). \]

Since the defining ideal $\langle x^{d+1}, y^{d+1} \rangle$ of the ring $R$ is symmetric, these determinants are the same. Hence we have
\[ \beta^{k+1} s_{((k+1)d-k)} \left( \frac{a}{b} \right) = \alpha^{k+1} s_{((k+1)d-k)} \left( \frac{b}{a} \right). \]

Let $2m = 2d - 2k$ and $r = k + 1$. Then we have the equation. \hfill $\square$

Remark 3.5. For $\lambda \subset (r^n)$ and $\mu = (r^n) \setminus \lambda$, the Schur polynomials satisfy the following equation (see [12, Exercise 7.41]):
\[ (y_1 y_2 \cdots y_n)^r \cdot s_{-\lambda} \left( \frac{x}{y} \right) = (x_1 x_2 \cdots x_n)^r \cdot s_{\mu} \left( \frac{y}{x} \right), \]
where $\frac{x}{y} = \left( \frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n} \right)$, and $\frac{y}{x} = \left( \frac{y_1}{x_1}, \frac{y_2}{x_2}, \ldots, \frac{y_n}{x_n} \right)$. Let $\lambda = \mu = (r^m)$ and $n = 2m$. Then $\mu = (r^n) \setminus \lambda$. In this case, Equation (3) is the equation in Corollary 3.4.

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