AN APPLICATION OF BRASCAMP-LIEB’S INEQUALITY

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Abstract. We use Brascamp-Lieb’s inequality to obtain new decoupling inequalities for general Gaussian vectors, and for stationary cyclic Gaussian processes. In the second case, we use a version by Bump and Diaconis of the strong Szego limit theorem. This extends results of Klein, Landau and Shucker.

1. Introduction-Results.

Let \( X = \{ X_j, j \in \mathbb{Z} \} \) be a centered Gaussian stationary sequence, and let \( \gamma(n) = \mathbb{E} X_0 X_n, n \in \mathbb{Z} \). We assume that \( X \) is strongly mixing or equivalently, that \( \lim_{n \to \infty} \gamma(n) = 0 \). When \( \gamma(n) \) tends sufficiently quickly to 0, more independence is naturally gained in the structure of \( X \). This can be quantified under the form of a decoupling inequality. For instance, if

\[
p(X) = \sum_{n \geq 0} \frac{\gamma(n)}{\gamma(0)} < \infty,
\]

then for any finite collection \( \{ f_j, j \in J \} \) of complex-valued Borel-measurable functions,

\[
\left| \mathbb{E} \prod_{j \in J} f_j(X_j) \right| \leq \prod_{j \in J} \| f_j(X_0) \|_{p(X)}.
\]

This remarkable inequality, which so nicely condenses the independence properties of these Gaussian sequences, is Theorem 3 (\( d = 1 \)) in Klein, Landau and Shucker [7].

Clearly \( |\mathbb{E} \prod_{j \in J} f_j(X_j)| \) measures the degree of independence between the random variables \( f_j(X_j) \). An immediate consequence of (1.2) and of well-known Katrhi-Sidák’s inequality, is that under assumption (1.1), we have the following sharp two-sided estimate,

\[
\prod_{j \in J} \mathbb{P}\{|X_j| \leq x\} \leq \mathbb{P}\{ \sup_{j \in J} |X_j| \leq x\} \leq \prod_{j \in J} \mathbb{P}\{|X_j| \leq x\}^{1/p(X)},
\]

where \( J \) is any finite index and \( x \) any non-negative real.

However, one is often faced with probabilistic questions where \( p(X) = \infty \), or simply, the process \( X \) is not stationary in the sense required in [7]. At our knowledge, no extension of (1.2) beyond condition (1.1) exists in the literature, and it is naturally interesting to search what form could take a decoupling inequality when (1.1) fails to be satisfied.

This is the question we address and study in this work, which is also somehow developing the recent paper [10]. We clarify at this stage that our goal is to obtain results valid for a broad range of Gaussian processes, and thus not (possibly) quite sharp estimates concerning specific cases, which is another problematic. Our aim is also to link the question considered

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with the general theory of Toeplitz forms, and draw the attention of the reader to the interest of
this connection. This is in that sense continuing the study made in section 5 of [10].

The role of the stationarity assumption of $X$ in [7] is crucial. The proof of (1.2) much relies
on an analytic inequality due to Brascamp and Lieb, which is of relevance in the present work.

A first natural question can be stated as follows. What form can take the decoupling
inequality (1.2) for an arbitrary Gaussian vector? As the law of a Gaussian vector, or more
generally of a Gaussian process is completely characterized by its covariance function, one can
make the question more consistent by asking which characteristics of the covariance matrix
$\{\mathbb{E} X_i X_j\}_{i,j=1}^n$ of $X$ should be involved (and are to be evaluated): a particular function of its
eigenvalues, or simply its determinant? It turns out that only the determinant suffices. More
precisely, we prove a general decoupling inequality, free of stationarity assumption.

Before stating it, we first extend the notion of decoupling coefficient introduced in [7] to
arbitrary Gaussian vectors.

**Definition 1.1.** Let $X = \{X_i, 1 \leq i \leq n\}$ be a centered Gaussian vector with non-degenerated
components. The decoupling coefficient $p(X)$ of $X$ is defined by

$$p(X) = \max_{i=1}^n \sum_{1 \leq j \leq n} \frac{|\mathbb{E} X_i X_j|}{\mathbb{E} X_i^2}.$$ 

This is a natural characteristic of $X$. When $X$ is stationary,

$$p(X) = \max_{i=1}^n \sum_{1 \leq j \leq n} \frac{|\gamma(i-j)|}{\gamma(0)},$$

and so

$$\sum_{1 \leq h \leq n-1} \frac{|\gamma(h)|}{\gamma(0)} \leq p(X) \leq 2 \sum_{1 \leq h \leq n-1} \frac{|\gamma(h)|}{\gamma(0)}.$$

Further $p(X) = 1$ if and only if $X$ has independent components. Some classes of examples
with $p(X) \ll n$ or $p(X) \asymp n$ are given in section 4.

Our first main result states as follows.

**Theorem 1.2.** Let $X = \{X_i, 1 \leq i \leq n\}$ be a centered Gaussian vector such that $\mathbb{E} X_i^2 = \sigma_i^2 > 0$ for each $1 \leq i \leq n$, and with positive definite covariance matrix $C$. Let $p$ be such that

$$p \geq 2 p(X).$$

Then for any complex-valued measurable functions $f_1, \ldots, f_n$ such that $f_i \in L^p(\mathbb{R})$, for all $1 \leq i \leq n$, the following inequality holds true,

$$\left| \mathbb{E} \left( \prod_{i=1}^n f_i(X_i) \right) \right| \leq \frac{2^n (1 - \frac{1}{p}) (\prod_{i=1}^n \sigma_i)^\frac{1}{p}}{\det(C)^{\frac{1}{2p}}} \prod_{i=1}^n \left( \mathbb{E} f_i(X_i)^p \right)^\frac{1}{p}.$$ 

From Theorem 1.2 and Kathri-Sidák’s inequality we also get,

**Corollary 1.3.** Let $X = \{X_i, 1 \leq i \leq n\}$ be a centered Gaussian vector such that $\mathbb{E} X_i^2 = \sigma_i > 0$, $1 \leq i \leq n$, and with positive definite covariance matrix $C$. Assume that assumption (1.4) is fulfilled for some $p \geq 2$. Then for any $\varepsilon_i > 0$, $i = 1, \ldots, n$,

$$\prod_{i=1}^n \mathbb{P}\{ |X_i| \leq \varepsilon_i \} \leq \mathbb{P}\left\{ \sup_{i=1}^n \frac{|X_i|}{\varepsilon_i} \leq 1 \right\} \leq \frac{2^n (1 - \frac{1}{p}) (\prod_{i=1}^n \sigma_i)^\frac{1}{p}}{\det(C)^{\frac{1}{2p}}} \prod_{i=1}^n \left( \mathbb{E} \frac{f_i(X_i)^p}{\varepsilon_i} \right)^\frac{1}{p}.$$
For estimating $\det(C)$, we place ourselves in the setting of Toeplitz matrices theory where this important question has been and is still much investigated. A salient aspect of this theory is that $\det(C)$ can be computed, sometimes with high degree of accuracy. We refer to the nice book of Grenander and Szegö [3], we also refer to [10] for a general presentation of the methods used, except for the Laplace transform method, essentially in the setting of stationary Gaussian processes. Let $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ be a function on the unit circle $T$. Let $T_{n-1}(f)$ be the Toeplitz matrix defined by $T_{n-1}(f) = \{d_{j-i}\}_{i,j=0}^{n-1}$ and let $D_{n-1}(f) = \det(T_{n-1}(f))$.

This corresponds to the case when $X$ has a spectral density function $f(t)$, summable over $[-\pi, \pi]$, and is thus of relevance in our setting. Indeed, as

\begin{equation}
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} f(t) dt, \quad n \in \mathbb{Z},
\end{equation}

$T_{n-1}(f)$ is just the $n$-th finite section of the infinite Toeplitz matrix given by the covariance matrix of the process $X$. Further as $f \in L^1([-\pi, \pi])$, by the Riemann-Lebesgue lemma, we have $\lim_{n \to \infty} d_n = 0$.

In the considerable literature on Toeplitz operators and determinants, $f$ is usually called a symbol or a generating function (generating $(T_{n-1}(f))_n$) and $f$ needs not being a density function. Toeplitz determinants with rational symbols occur for instance in statistical mechanics and quantum mechanics, see [2]. They can be calculated using a formula obtained by Day [5].

For Toeplitz matrices generated by a density function, $D_{n-1}(f)$ can also be expressed as an integral over the unitary group $U(n)$, by means of the Heine-Szegö identity,

\begin{equation}
D_{n-1}(f) = \int_{U(n)} \Phi_{n,f}(g) dg.
\end{equation}

Here the integration path is taken with respect to the normalized Haar measure on $U(n)$, and $\Phi_{n,f}(g)$ is defined by $\Phi_{n,f}(g) = f(t_1) \ldots f(t_n)$, where $t_1, \ldots, t_n$ are the eigenvalues of $g$. This identity is the starting point of the proof of a nice form of the strong Szegö limit theorem established in [4] by Bump and Diaconis.

Using their result we also prove

**Theorem 1.4.** Let $X = \{X_j, j \in \mathbb{Z}\}$ be a centered Gaussian stationary sequence with unit variance and spectral density function $f(t)$. Let $\log(f(t)) = \sum_{k \in \mathbb{Z}} c_k e^{ikt}$ where the $c_k$ satisfy the following conditions

\begin{align}
\sum_{k \in \mathbb{Z}} |c_k| &< \infty, \\
\sum_{k \in \mathbb{Z}} |k||c_k|^2 &< \infty.
\end{align}

Then there exist reals $\delta_n \downarrow 0$, such that for any integer $n \geq 2$, any complex-valued measurable functions $f_1, \ldots, f_n$ with $f_i \in L^p(\mathbb{R})$, for all $1 \leq i \leq n$, where

\begin{equation}
p \geq 2p(X),
\end{equation}

the following inequality holds true,

\[ \left| \mathbb{E}\left( \prod_{i=1}^{n} f_i(X_i) \right) \right| \leq \frac{(1 + \delta_n)^2}{(2b(f)G(f))^{\frac{p}{2}}} \prod_{i=1}^{n} \left( \mathbb{E}|f_i(X_i)|^p \right)^{\frac{1}{p}}, \]
where \( b(f) = \exp \left\{ \sum_{k=1}^{\infty} k c_k c_{-k} \right\} \) and \( G(f) \) is the geometric mean of \( f \), namely
\[
(1.11) \quad G(f) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(t)dt \right\}.
\]

**Corollary 1.5.** Let \( X = \{X_j, j \in \mathbb{Z}\} \) be a centered Gaussian stationary sequence with unit variance and spectral density function \( f(t) \) satisfying conditions \((1.8)\) and \((1.9)\).

Then there exist reals \( \delta_n \downarrow 0 \), such that for any integer \( n \geq 2 \), any \( p \) satisfying \((1.10)\), we have for any \( \varepsilon_i > 0 \), \( i = 1, \ldots, n \),
\[
\mathbb{P}\left\{ \sup_{i=1}^{n} |X_i| \leq \varepsilon_i \right\} \leq \frac{(1 + \delta_n) 2^\pi}{(2b(f)G(f))^{\frac{1}{p}}} \prod_{i=1}^{n} (\mathbb{P}\{|X_0| \leq \varepsilon_i\})^\frac{1}{p}.
\]

2. Proof of Theorem 1.2

We first state the proposition below which follows from Theorem 6 in Brascamp and Lieb [3]. We also refer to [7]. It should be indicated here that all that is required for the application of this Theorem, is that the matrix be positive definite. This one is written in terms of its eigenvectors and eigenvalues, and the eigenvectors are \( a^j, j = k + 1, \ldots, k + m \), which have nothing to do with the vectors \( a^j, j = 1, 2, \ldots, k \) of their Theorem 1. This point was clarified to the author by Abel Klein [3]. Introduce some notation. Let \( I \) be the \( n \times n \) identity matrix and let \( \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n \). Then \( I(B) \) will denote throughout the diagonal matrix whose values on the diagonal are the corresponding values of \( \mathbf{b} \). Also, when \( b_i \neq 0 \) for each \( i = 1, \ldots, n \), we will use the notation \( b_i^{-1} = (b_1^{-1}, \ldots, b_n^{-1}) \).

**Proposition 2.1.** Let \( 1 \leq p < \infty \). Let \( B \) be a positive definite \( n \times n \) matrix. Then for any measurable functions \( g_1, \ldots, g_n \) such that \( g_i \geq 0 \) and \( g_i \in L^p(\mathbb{R}) \), \( 1 \leq i \leq n \), the following inequality holds true,
\[
(2.1) \quad \int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} g_i(x_i) \right) \exp \left\{ -\frac{1}{2} \langle x, Bx \rangle \right\} dx \leq E_B \prod_{i=1}^{n} \left( \int_{\mathbb{R}} g_i(x)^p dx \right)^{\frac{1}{p}},
\]
where
\[
(2.2) \quad E_B = (2\pi)^{\frac{n}{2} (1/p)} p^\frac{n}{2p} \sup_{b_1, \ldots, b_n \geq 0} \frac{\prod_{i=1}^{n} b_i^{\frac{1}{p}}}{\det(B + I(B))^{\frac{1}{2}}},
\]

**Remark 2.2.** The constant \( E_B \) is defined in Theorem 6 by
\[
E_B = \sup_{b_1, \ldots, b_n \geq 0} \frac{\int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} \exp\left\{ -\frac{1}{2} b_i x_i^2 \right\} \right) \exp\left\{ -\frac{1}{2} \langle x, Bx \rangle \right\} dx}{\prod_{i=1}^{n} \left( \int_{\mathbb{R}} \exp\left\{ -\frac{1}{2} b_i x^2 \right\} dx \right)^{\frac{1}{p}}} \prod_{i=1}^{n} \left( \int_{\mathbb{R}} \exp\left\{ -\frac{1}{2} b_i x^2 \right\} dx \right)^{\frac{1}{p}},
\]

namely inequality \((2.1)\) is maximal when the \( g_i \)’s are Gaussian.

But it is elementary that (see also [7], p. 705, after (4))
\[
\frac{\int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} \exp\left\{ -\frac{1}{2} b_i x_i^2 \right\} \right) \exp\left\{ -\frac{1}{2} \langle x, Bx \rangle \right\} dx}{\prod_{i=1}^{n} \left( \int_{\mathbb{R}} \exp\left\{ -\frac{1}{2} b_i x^2 \right\} dx \right)^{\frac{1}{p}}} = \frac{\int_{\mathbb{R}^n} \exp\left\{ -\frac{1}{2} \langle x, (B + I(B))x \rangle \right\} dx}{\prod_{i=1}^{n} \left( \int_{\mathbb{R}} \exp\left\{ -\frac{1}{2} b_i x^2 \right\} dx \right)^{\frac{1}{p}}} = (2\pi)^{\frac{n}{2} (1/p)} p^\frac{n}{2p} \frac{\prod_{i=1}^{n} b_i^{\frac{1}{p}}}{\det(B + I(B))^{\frac{1}{2}}}.
\]
So that
\[ E_B = (2\pi)^{\frac{1}{2}(1-\frac{1}{p})} p^{\frac{1}{2p}} \sup_{\substack{b_i > 0 \\ i=1,\ldots,n}} \frac{\prod_{i=1}^{n} b_i^{\frac{1}{2p}}}{\det(B + I(b))^{\frac{1}{2}}}. \]

**Remark 2.3.** In the proof of Theorem 3 in [7], Klein, Landau and Shucker apply Theorem 6 under the form of that Proposition, p. 705, with the choice \( B = C^{-1} - \frac{1}{pc} I \), where \( C \) is the covariance matrix of the process \( X \), \( p \) is the decoupling coefficient of \( X \), \( c = \mathbb{E} X_0^2 \). Further \( g_i(x) = f_i(x) e^{-\frac{1}{2pc} x^2} \) for \( i = 1, \ldots, n \).

This requires that \( B \) is positive definite, or equivalently that \( pcI - C \) is positive definite. This is ensured by the choice of \( p \) made in [7].

In the next lemma, we establish a general bound of \( E_B \).

**Lemma 2.4.**
\[ E_B \leq \frac{(2\pi)^{\frac{1}{2}(1-\frac{1}{p})}}{\det(B)^{\frac{1}{2}(1-\frac{1}{p})}}. \]

**Proof.** We use the following Lemma.

**Lemma 2.5** ([1], Th. 4, p. 128). If \( U \) and \( V \) are positive definite matrices, then
\[ \det(\lambda U + (1-\lambda)V) \geq \det(U)^\lambda \det(V)^{1-\lambda}, \]
for any \( 0 \leq \lambda \leq 1 \).

Therefore
\[ \det(U + \left(\frac{1-\lambda}{\lambda}\right)V) \geq \lambda^{-n} \det(U)^\lambda \det(V)^{1-\lambda}, \]
if \( 0 < \lambda \leq 1 \). We apply this with the choice \( U = B, V = \left(\frac{\lambda}{1-\lambda}\right) I(b) \). We get
\[
\begin{align*}
\det(B + I(b)) &\geq \lambda^{-n} \det(B)^\lambda \det(\left(\frac{\lambda}{1-\lambda}\right) I(b))^{1-\lambda} \\
&= \lambda^{-n} \det(B)^\lambda \left(\frac{\lambda}{1-\lambda}\right)^{n(1-\lambda)} \det(I(b))^{1-\lambda} \\
&= \left(\frac{1}{\lambda} \left(\frac{\lambda}{1-\lambda}\right)^{1-\lambda}\right)^n \det(B)^\lambda \prod_{i=1}^{n} b_i^{1-\lambda}.
\end{align*}
\]

Consequently,
\[
\begin{align*}
\frac{\prod_{i=1}^{n} b_i^{\frac{1}{2p}}}{\det(B + I(b))^{\frac{1}{2}}} &\leq \frac{1}{\det(B)^{\frac{1}{2}}} \left(\frac{1}{\lambda} \left(\frac{\lambda}{1-\lambda}\right)^{1-\lambda}\right)^{\frac{n}{2}} \prod_{i=1}^{n} b_i^{\frac{1}{2p}(1-(1-\lambda))} \\
&= \frac{1}{\det(B)^{\frac{1}{2}}} \left(\lambda^\lambda (1-\lambda)^{1-\lambda}\right)^{\frac{n}{2}} \prod_{i=1}^{n} b_i^{\frac{1}{2p}(1-(1-\lambda))}.
\end{align*}
\]

Take \( \lambda = 1 - \frac{1}{p} \) and note that \((1-\lambda)^{1-\lambda} = p^{-\frac{1}{p}} \). We obtain
\[
\begin{align*}
\frac{\prod_{i=1}^{n} b_i^{\frac{1}{2p}}}{\det(B + I(b))^{\frac{1}{2}}} &\leq \frac{1}{\det(B)^{\frac{1}{2}(1-\frac{1}{p})}} p^{-\frac{1}{2p}} \left(1 - \frac{1}{p}\right)^{\frac{n}{2} (1-\frac{1}{p})}.
\end{align*}
\]
Therefore
\[(2\pi)^{\frac{n}{2}}(1 - \frac{1}{p}) p^{\frac{1}{2p}} \prod_{i=1}^{n} b_i^{\frac{1}{2p}} \leq \frac{(2\pi)^{\frac{n}{2}}(1 - \frac{1}{p})}{\det(B)^{\frac{1}{2}(1 - \frac{1}{p})}} p^{\frac{1}{2p}} \cdot p^{\frac{1}{2p}} (1 - \frac{1}{p})^{\frac{n}{2}(1 - \frac{1}{p})}\]
\[= \frac{(2\pi)^{\frac{n}{2}}(1 - \frac{1}{p})}{\det(B)^{\frac{1}{2}(1 - \frac{1}{p})}} (1 - \frac{1}{p})^{\frac{n}{2}(1 - \frac{1}{p})}\].

Whence
\[E_B \leq \frac{(2\pi)^{\frac{n}{2}}(1 - \frac{1}{p})}{\det(B)^{\frac{1}{2}(1 - \frac{1}{p})}} \cdot \]

\[\square\]

Proof of Theorem 1.2. It suffices to prove inequality (1.5) when \(f_i\) are real-valued and non-negative, for all \(i = 1, \ldots, n\). Let \(\gamma = (\sigma_1, \ldots, \sigma_n)\). We apply Proposition 2.1 with \(B = C^{-1} - \frac{1}{p} I(\gamma^{-1})\), \(g_i(x) = |f_i(x)| e^{-x^2/(2\sigma_i^2)}\), \(i = 1, \ldots, n\).

We get by using also Lemma 2.4
\[\int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} g_i(x_i) \right) \exp \left\{ - \frac{1}{2} \langle x, B x \rangle \right\} dx \leq E_B \prod_{i=1}^{n} \left( \int_{\mathbb{R}} g_i(x)^p dx \right)^{\frac{1}{p}} \]
\[\leq \frac{(2\pi)^{\frac{n}{2}}(1 - \frac{1}{p})}{\det(B)^{\frac{1}{2}(1 - \frac{1}{p})}} \prod_{i=1}^{n} \left( \int_{\mathbb{R}} g_i(x)^p dx \right)^{\frac{1}{p}}.\]

This is equivalently rewritten as
\[\int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} f_i(x_i) \right) \exp \left\{ - \frac{1}{2} \langle x, C^{-1} x \rangle \right\} \frac{dx}{(2\pi)^{\frac{n}{2}} \det(C)^{\frac{1}{2}}} \]
\[\leq \frac{\left( \prod_{i=1}^{n} \sigma_i \right)^{\frac{1}{p}}}{\det(B)^{\frac{1}{2}(1 - \frac{1}{p})} \det(C)^{\frac{1}{2}}} \prod_{i=1}^{n} \left( \int_{\mathbb{R}} g_i(x)^p e^{-x^2/(2\sigma_i^2)} dx \right)^{\frac{1}{p}},\]

namely
\[\mathbb{E} \left( \prod_{i=1}^{n} f_i(X_i) \right) \leq \frac{\left( \prod_{i=1}^{n} \sigma_i \right)^{\frac{1}{p}}}{\det(B)^{\frac{1}{2}(1 - \frac{1}{p})} \det(C)^{\frac{1}{2}}} \prod_{i=1}^{n} \left( \mathbb{E} f_i(X_i)^p \right)^{\frac{1}{p}}.\]

Writing
\[B = C^{-1} - \frac{1}{p} I(\gamma^{-1}) = \frac{1}{p} C^{-1} I(\gamma^{-1}) \left( p I(\gamma) - C \right),\]
we have
\[\det(B) = \frac{1}{p^n} \det \left( C^{-1} I(\gamma^{-1}) \right) \det \left( p I(\gamma) - C \right) = \frac{\det \left( p I(\gamma) - C \right)}{p^n \det(C) \prod_{i=1}^{n} \sigma_i^2}.\]

So that,
\[\mathbb{E} \left( \prod_{i=1}^{n} f_i(X_i) \right) \leq \frac{\left( \prod_{i=1}^{n} \sigma_i \right)^{\frac{1}{p}}}{\left( p^n \det(C) \prod_{i=1}^{n} \sigma_i^2 \right)^{\frac{1}{2}(1 - \frac{1}{p})} \det(C)^{\frac{1}{2}}} \prod_{i=1}^{n} \left( \mathbb{E} f_i(X_i)^p \right)^{\frac{1}{p}}.\]
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\[ \begin{align*}
\frac{p^{\frac{n}{2p}(1-\frac{1}{p})}}{\det \left( pI(\gamma) - C \right)^{\frac{1}{2p}} \det(C)^{\frac{1}{2p}}} \prod_{i=1}^{n} \left( \mathbb{E} f_i(X_i)^p \right)^{\frac{1}{p}}.
\end{align*} \]

In order to estimate \( \det \left( pI(\gamma) - C \right) \), we recall a well-known result on Hadamard matrices.

**Lemma 2.6** ([9], (2)). Let \( A = \{a_{i,j}, 1 \leq i, j \leq n \} \) and assume that

\[ \sum_{j=1, j \neq i}^{n} |a_{i,j}| < |a_{i,i}|, \quad i = 1, \ldots, n. \]

Then

\[ \det(A) \geq \prod_{i=1}^{n} \left( |a_{i,i}| - \sum_{j=1, j \neq i}^{n} |a_{i,j}| \right). \]

By assumption 1.4,

\[ \sum_{1 \leq j \leq n, j \neq i} |\mathbb{E} X_i X_j| \leq \left( \frac{p}{2} - 1 \right) \sigma_i^2, \quad i = 1, \ldots, n. \]

Letting \( pI(\gamma) - C = \{d_{i,j}, 1 \leq i, j \leq n \} \), we have \( |d_{i,i}| = (p-1)\sigma_i^2 \) and

\[ \sum_{j=1, j \neq i}^{n} |d_{i,j}| = \sum_{j=1, j \neq i}^{n} |\mathbb{E} X_i X_j| < \left( \frac{p}{2} - 1 \right) \sigma_i^2 < |d_{i,i}|, \quad i = 1, \ldots, n. \]

Thus

\[ |d_{i,i}| - \sum_{j=1, j \neq i}^{n} |d_{i,j}| \geq (p-1)\sigma_i^2 - \left( \frac{p}{2} - 1 \right) \sigma_i^2 = \frac{p\sigma_i^2}{2}. \]

By Lemma 2.6 it follows that

\[ \det \left( pI(\gamma) - C \right) \geq \prod_{i=1}^{n} \left( p\sigma_i^2 - \sum_{1 \leq j \leq n, j \neq i} |\mathbb{E} X_i X_j| \right) \geq \left( \frac{p}{2} \right)^n \prod_{i=1}^{n} \sigma_i^2. \]

Thus

\[ \mathbb{E} \left( \prod_{i=1}^{n} f_i(X_i) \right) \leq \frac{p^{\frac{n}{2p}(1-\frac{1}{p})}}{\det \left( pI(\gamma) - C \right)^{\frac{1}{2p}} \det(C)^{\frac{1}{2p}}} \prod_{i=1}^{n} \left( \mathbb{E} f_i(X_i)^p \right)^{\frac{1}{p}} \]

\[ \leq \frac{p^{\frac{n}{2p}(1-\frac{1}{p})}}{\left( \frac{p}{2} \right)^n \det(C)^{\frac{1}{2p}}} \prod_{i=1}^{n} \left( \mathbb{E} f_i(X_i)^p \right)^{\frac{1}{p}} \]

\[ = \frac{2^{\frac{n}{2p}(1-\frac{1}{p})}}{\det(C)^{\frac{1}{2p}}} \prod_{i=1}^{n} \left( \mathbb{E} f_i(X_i)^p \right)^{\frac{1}{p}}. \]

\qed
3. Proof of Theorem 1.4

Recall Bump and Diaconis [4, Th. 4] strong Szegő limit theorem. Let \((c_k)_{k \in \mathbb{Z}}\) satisfy conditions (1.8) and (1.9). Let \(\sigma(t) = \exp\{\sum_{k \in \mathbb{Z}} c_k e^{ikt}\}\). Then

\[
D_{n-1}(\sigma) \sim \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_k e^{-k}\right).
\]

Let \(\Gamma_n = \text{Cov}(X_1, \ldots, X_n)\). By applying (3.1) with the choice \(\sigma(t) = \exp\{\log f(t)\}\), we have

\[
\det(\Gamma_n) \sim \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_k e^{-k}\right).
\]

As \(c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(t)dt = G(f)\), it follows that

\[
\det(\Gamma_n) \geq \left(1 + \delta_n\right) \exp\left\{nG(f) + \sum_{k=1}^{\infty} kc_k e^{-k}\right\},
\]

where \(\delta_n \downarrow 0\). Whence

\[
\left|\mathbb{E}\left(\prod_{i=1}^{n} f_i(X_i)\right)\right| \leq (1 + \delta_n) 2^{\frac{p}{2}(1 - \frac{1}{p})} \exp\left\{-\frac{1}{2p}(nG(f) + \sum_{k=1}^{\infty} kc_k e^{-k})\right\} \prod_{i=1}^{n} \left(\mathbb{E}|f_i(X_i)|^p\right)^{\frac{1}{p}}
\]

\[
= (1 + \delta_n) b(f)^{-\frac{1}{2p}} 2^{\frac{n}{2}(1 - \frac{1}{p})} G(f)^{-\frac{n}{2p}} \prod_{i=1}^{n} \left(\mathbb{E}|f_i(X_i)|^p\right)^\frac{1}{p},
\]

recalling that \(b(f) = \exp\left\{\sum_{k=1}^{\infty} kc_k e^{-k}\right\}\) and \(G(f)\) is the geometric mean of \(f\).

4. Examples.

We list some remarkable classes of examples.

4.1. Let \(X = \{X_j, j \in \mathbb{Z}\}\) be a centered Gaussian stationary sequence, and assume that \(X\) has spectral density. Recall that we have the following representation

\[
X_k = \sum_{m \in \mathbb{Z}} c_m \xi_{k-m},
\]

where \((c_m) \in \ell_2(\mathbb{Z})\) and \((\xi_j)\) are i.i.d. standard Gaussian. Note that

\[
\mathbb{E}X_k X_\ell = \sum_{m \in \mathbb{Z}} c_m c_{m-k+\ell}.
\]

Consider the sections \(X^n = \{X_1, \ldots, X_n\}, n \geq 1\).
4.1.1. Let $c_m = |m|^{-1}$, $m \in \mathbb{Z} \setminus \{0\}$, $c_0 = 0$.

**Proposition 4.1.** We have

$$p(X^n) \leq 4 \log n + O(\log n).$$

The bound of $p(X^n)$ is in fact optimal, up to some numerical constant. As $p(X^n) \ll n$, this is making inequality (1.5) effective.

We first prove a lemma.

**Lemma 4.2.** We have

$$E X_k X_\ell = 4 \log \frac{|k - \ell|}{|k - \ell|} + O\left(\frac{1}{|k - \ell|}\right).$$

**Proof.** We note that

$$E X_k X_\ell = \sum_{m \in \mathbb{Z}} \frac{1}{m|m - |k - \ell||},$$

if $k \neq \ell$, and $E X_k^2 = \frac{\pi^2}{3}$. Let $\mu$ be some positive integer. Then,

$$\sum_{m \in \mathbb{Z}} \frac{1}{m|m - |k - \ell||} = \sum_{m \geq \mu + 1} \frac{1}{m(m - \mu)} + \sum_{m < \mu - 1} \frac{1}{m(|m - \mu|)} + \sum_{|m - \mu| \leq 1} \frac{1}{m(m - \mu)}$$

Recall that $\sum_{1 \leq m \leq x} \frac{1}{m} = \log x + \gamma + O(x^{-1})$, where $\gamma$ is Euler’s constant. At first,

$$\sum_{m = 1}^{\mu - 1} \frac{1}{m(m - \mu)} = \frac{1}{\mu} \sum_{m = 1}^{\mu - 1} \left(\frac{1}{m} + \frac{1}{\mu - m}\right) = \frac{1}{\mu} \sum_{m = 1}^{\mu - 1} \frac{1}{m} = 2 \frac{\log \mu}{\mu} + O(\mu^{-1}).$$

Next

$$\sum_{\nu \geq 1} \frac{1}{(\nu + \mu)\nu} = \sum_{1 \leq \nu \leq \mu - 1} \frac{1}{(\nu + \mu)\nu} + \sum_{\nu \geq \mu} \frac{1}{(\nu + \mu)\nu}$$

$$= \sum_{1 \leq \nu \leq \mu - 1} \frac{1}{(\nu + \mu)\nu} + O(\mu^{-1})$$

$$= \frac{1}{\mu} \sum_{1 \leq \nu \leq \mu - 1} \left(\frac{1}{\nu} - \frac{1}{\nu + \mu}\right) + O(\mu^{-1})$$

$$= \frac{\log \mu}{\mu} - \frac{1}{\mu} \sum_{\mu \leq h \leq 2\mu - 1} \frac{1}{h} + O(\mu^{-1})$$

$$= \frac{\log \mu}{\mu} + O(\mu^{-1}).$$
Consequently,
\[
\sum_{m \in \mathbb{Z}, m \neq \mu} \frac{1}{|m| |m - \mu|} = 2 \sum_{\nu \geq 1} \frac{1}{\nu(\nu + \mu)} + \sum_{m=1}^{\mu-1} \frac{1}{m(\mu - m)}
\]
\[
= \frac{4 \log \mu}{\mu} + O(\mu^{-1}).
\]
And so,
\[
\mathbb{E} X_k X_\ell = 4 \frac{\log |k - \ell|}{|k - \ell|} + O\left(\frac{1}{|k - \ell|}\right).
\]
\[\square\]

**Proof of Proposition 4.1.** It follows that
\[
\sum_{\ell=1}^{n} |\mathbb{E} X_k X_\ell| = 4 \sum_{\ell=1}^{n} \frac{\log |k - \ell|}{|k - \ell|} + O\left(\sum_{\ell=1}^{n} \frac{1}{|k - \ell|}\right).
\]
Now,
\[
\sum_{\ell \neq k} \frac{\log |k - \ell|}{|k - \ell|} = \sum_{\ell=1}^{k-1} \frac{\log(k - \ell)}{k - \ell} + \sum_{\ell=k+1}^{n} \frac{\log(\ell - k)}{\ell - k}
\]
\[
\leq \int_{1}^{k-1} \frac{\log t}{t} dt + \int_{1}^{n-k} \frac{\log t}{t} dt
\]
\[
= \frac{(\log(k - 1))^2 + ((\log(n - k))^2}{2}.
\]
Similarly,
\[
\sum_{\ell \neq k} \frac{\log |k - \ell|}{|k - \ell|} \geq \frac{(\log(k - 1))^2 + ((\log(n - k))^2}{2} - (\log 2)^2.
\]
Therefore
\[
\sum_{\ell \neq k} \frac{\log |k - \ell|}{|k - \ell|} = \frac{(\log(k - 1))^2 + ((\log(n - k))^2}{2} + O(1).
\]
Also,
\[
\sum_{\ell \neq k} \frac{1}{|k - \ell|} \leq \int_{1}^{k-1} \frac{dt}{t} + \int_{1}^{n-k} \frac{dt}{t} = \log(k - 1) + \log(n - k).
\]
Whence,
\[
\sum_{\ell \neq k} |\mathbb{E} X_k X_\ell| = 2 \left( (\log(k - 1))^2 + ((\log(n - k))^2 \right) + O(\log k + \log(n - k)).
\]
We consequently get the estimate
\[
p(X) \leq 4 (\log n)^2 + O(\log n).
\]
4.1.2. Now let \( c_m = |m|^{-r}, m \in \mathbb{Z}\setminus\{0\} \) where \( r \geq 1, c_0 = 0 \). Then naturally \( X \) has stronger asymptotical independence properties. In fact \( p(X) < \infty \), as soon as \( r \geq 2 \).

Indeed, by Hölder’s inequality, next Proposition 4.1

\[
0 \leq \mathbb{E} X_k X_\ell = \sum_{m \notin \mathbb{Z}} \frac{1}{|m|^r |m - |k - \ell||^r}
\leq \left( \sum_{m \notin \mathbb{Z}} \frac{1}{|m||m - |k - \ell||} \right)^r
\leq C_r \left( \frac{\log |k - \ell|}{|k - \ell|} \right)^r,
\]
a bound from which easily follows that \( p(X) < \infty \). So, this is an instance where Klein, Landau and Shucker’s inequality (1.2) directly applies.

**Remark 4.3 (A pathological example).** Gaussian stationary sequences with spectral density form a huge class, and may in particular exhibit pathological covariance functions. We provide here a simple example of which the study relies on additive Number Theory.

Let \( A \subset \mathbb{N} \) and \( (b_j) \in l^2(\mathbb{N}) \). Consider the following special case of (4.1),

\[
(4.2)
X_k = \sum_{|m| \in A} b_{|m|}|\xi_{k - m}|, \quad k \in \mathbb{Z}.
\]

Then

\[
(4.3)
\mathbb{E} X_k X_\ell = \begin{cases} 0 & \text{if } k - \ell \notin A - A, \\
\sum_{m \in A} b_{|m|}b_{|m - k + \ell|} & \text{otherwise.}
\end{cases}
\]

Thus the covariance function is supported on the difference set \( A - A \), making the study of this example depending on additive properties of the set \( A \).

4.2. **Hilbert type covariance matrices.** Now consider non stationary Gaussian sequences having Hilbert type matrices. More precisely, let \( C_n \) be the symmetric matrix defined by

\[
(4.4)
C_n = \left\{ \frac{1}{a_k + a_\ell} ; k, \ell = 1, \ldots, n \right\},
\]

where \( A = (a_i)_{i \geq 1} \) is a sequence of positive real numbers. That \( C_n \) is positive definite (and so is a Gram matrix) follows from the fact that

\[
\int_0^\infty \left| \sum_{k=1}^n x_ke^{-a_k t} \right|^2 dt \geq 0.
\]

Thus \( C_n \) is the covariance matrix of a Gaussian vector, which can be described explicitly. Indeed, there exist in \( \mathbb{R}^n \) vectors \( u^1, \ldots, u^n \) with Gram matrix \( C_n \), for instance the rows of \( C_n^{1/2} \). Let \( \{g_k\}_{1 \leq i \leq n} \) be independent Gaussian standard random variables, and form the Gaussian vector \( X^n = \{X_i\}_{1 \leq i \leq n} \) where

\[
X_i = \sum_{k=1}^n g_k u_k^i, \quad i = 1, \ldots, n.
\]
We immediately see that $X^n$ has covariance matrix $C_n$. Assume that the sequence $A$ is increasing. One easily to check that $p(X^n) \asymp n$.

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