A UNIVERSALITY RESULT FOR SUBCRITICAL COMPLEX
GAUSSIAN MULTIPLICATIVE CHAOS

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Abstract. In the present paper, we show that (under some minor technical assumption)
Complex Gaussian Multiplicative Chaos defined as the complex exponential of a log-
correlated Gaussian field can be obtained by taking the limit of the exponential of the
field convoluted with a smoothing Kernel. We consider two types of chaos:
\( e^{\gamma X} \) for a log

correlated field \( X \) and \( e^{\alpha \cdot X + i \beta Y} \) for \( X \) and \( Y \) two independent
fields with \( \alpha, \beta \in \mathbb{R} \).
Our result is valid in the range
\[ \mathcal{O}_{\text{sub}} := \{ \alpha^2 + \beta^2 < d \} \cup \{ \alpha \in (\sqrt{d/2}, \sqrt{2d}) \text{ and } |\beta| < \sqrt{2d} - |\alpha| \}, \]
which, up to boundary, is conjectured to be optimal.

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1. Introduction

1.1. Real Gaussian Multiplicative Chaos and the question of universality. The
theory of Gaussian multiplicative chaos (GMC) developped by Kahane [15] was developped
with the objective of giving a rigourous meaning to random measures of the type
\[ e^{\gamma X(x)} - \frac{\gamma^2}{2} \mathbb{E}[(X(x))^2] \nu(dx) \quad (1.1) \]
where \( X \) is a log-correlated Gaussian field, that is, a Gaussian field with a covariance
function of the form
\[ K(x, y) = \log \frac{1}{|x - y|} + L(x, y) \quad (1.2) \]
where \( L \) is continuous function and \( \nu(dx) \) is a finite measure, both defined in a bounded
measurable set \( D \subset \mathbb{R}^d \), and \( \gamma \) is a positive real number. For the sake of simplicity, we
assume in our discussion that \( \nu \) is absolutely continuous with respect to Lebesgue and with
bounded density \( \nu(dx) = \rho(x)dx \) where \( \rho \) is a positive bounded function and \( dx \) denotes
Lebesgue measure). Motivations to define a random distribution corresponding to (1.1)
are numerous and come from various fields such as fluid mechanics (study of turbulence),
quantitative finance and mathematical physics (Conformal Field Theory), we refer to [21]
for a detailed account of applications.

Let us quickly expose the reasons why giving a meaning to (1.1) poses a mathematical
challenge. As the kernel \( K \) diverges on the diagonal, the field \( X \) can be defined only as a
random distribution (see Section 2.1 below): the quantity \( X(x) \) is not well defined, and
one can only make sense of \( X \) integrated along suitable test functions. To give a meaning
to (1.1), a possibility (and this is the original idea of Kahane’s construction in [15]) is
to consider a sequence \( (X_n(x))_{x \in D} \) of functional approximations converging to \( X \) and to
consider the limit
\[ \lim_{n \to \infty} e^{\gamma X_n(x)} - \frac{\gamma^2}{2} \mathbb{E}[(X_n(x))^2] \nu(dx), \quad (1.3) \]
as the definition of GMC.

In [15] this approximation approach is successfully applied with the additional assumption that $K$ can be written in the form $K(x, y) = \sum_{k=1}^{\infty} Q_k(x, y)$ where $Q_k$ is a sequence of bounded definite positive function satisfying $Q_k(x, y) \geq 0$ for every $x, y$. This assumption allows in particular to approximate $X$ by a martingale sequence, by defining $X_n = \sum_{k=1}^{n} Y_k$ where $Y_k$ is a sequence of independent fields, each of covariance $Q_k(x, y)$. Under this assumption, it is shown in [15] that the limit (1.3) exists for all $\gamma \geq 0$, is non trivial when $\gamma \in (0, \sqrt{2d})$ (this range of parameter has been refered since as the subcritical phase of the GMC) and is equal to 0 when $\gamma \geq \sqrt{2d}$. The result of Kahane yields a couple of natural questions:

(A) Is the limit obtained a function of $X$ or does it depend on the extra information which is present in the sequence $(X_n)_{n \geq 1}$?
(B) Would one obtain the same limit for some other kind of approximation of $X$ (e.g. considering convolution of $X$ by a smooth kernel)?

A positive answer to both question is necessary to establish without a doubt that the construction in [15] as the natural definition of (1.1).

Let us focus on (B) which is the question of universality and has been the object of studies through several decades (an extensive account on this is given in [4]). A statement concerning universality in law was proved in [22]. More precisely, it was shown that if one approximates $X$ with convolution by a smooth kernel, then the sequence (1.1) converges in law and that the law of the limiting object is independent of the convolution kernel.

More recent works [4, 24] (see also [10]) gave a full answer to the universality question. In [24] an axiomatic definition of Gaussian Multiplicative Chaos which allows to uniquely define (1.1) without the need of approximation is given (in a setup which is much more general than the one considered here), and it is furthermore shown that for any reasonable notion of approximating sequence $X_n$ (1.3) converges in probability to the object given by this axiomatic definition. In [4], it is established via elementary computations that every convolution approximation of the field yields the same limit in probability (and that this limit is identical to the one from the martingale approximation by Kahane).

Note that this positive answer to (B) also entails that the Gaussian multiplicative chaos is indeed only a function of $X$, thus providing an answer to (A).

1.2. Complex Gaussian Multiplicative Chaos. More recently, Gaussian Multiplicative Chaos has been considered in a complex setup, the idea being to give a rigourous meaning to $e^{\gamma X(x)} - \frac{\partial}{\partial z} E((X(x))^2) \nu(dx)$ for complex values of $\gamma$ [4, 3, 13] (see also [2, 7, 8, 9] where hierarchical versions of the model are considered). A variant of this problem considered in [16] is to consider two independent log-correlated Gaussian fields $X$ and $Y$ and consider the measure

$$e^{\alpha X(x) + i\beta Y(x)} - \frac{\partial}{\partial z} E((X(x))^2) + \frac{\partial}{\partial z} E((Y(x))^2) \nu(dx).$$

(1.4)

Complex Gaussian multiplicative chaos found applications in random Geometry [20], in the study of log-gases [17], and connections with the Ising model [12], the Riemann Zeta function and random matrices [23]. We refer to the references mentionned above for further details and motivation.

The main objective of this work is to establish a result similar to the one of [4] for complex Gaussian multiplicative chaos. In the case of complex $\gamma$ it has been shown in [13]
under some regularity assumption for $L$ in (1.2) (more details are given below) that the real Gaussian multiplicative chaos admits an analytic continuation in an open domain which includes the real segment $(-\sqrt{2d}, \sqrt{2d})$. The domain is explicit (given by (2.13) see also Figure 1) and is optimal, in the sense that there are very strong heuristic evidences that convergence to a non trivial limit cannot hold outside of the closure of this open set. What we establish in the present paper is that under the same assumption, the approximation obtained by a convolution the field with a smooth kernel converges to this universal object.

Concerning the case of independent real and imaginary part (1.4), the existence of the measure has been proved for some martingale approximation under some restriction on the kernel $K$ (existence of an integral decomposition). In the present work, we prove convergence of the approximation by convolution with no additional assumption on $K$ besides the fact that it is log-correlated.

Before introducing our results in more details, we provide a short and comprehensive technical introduction to GMC in the real and complex case.

### 2. Setup and results

#### 2.1. Log-correlated fields and their regular convolutions.

Given an open set $\mathcal{D} \subset \mathbb{R}^d$. Consider $K$ a positive definite Kernel defined on $\mathcal{D}^2$ of the form

$$K(x, y) = \log \frac{1}{|x - y|} + L(x, y)$$

(2.1)

where $L$ is continuous function on $\mathcal{D}$. By positive definite, we mean that

$$\int_{\mathcal{D}^2} K(x, y) f(x) f(y) dx dy \geq 0$$

(2.2)

for every continuous $f$ with compact support. Using the same formalism as in [4], we define the field $X$ with covariance function $K$ as a random process indexed by a set of signed measure. We define $\mathcal{M}_K^+$ to be the set of positive borel measures on $\mathcal{D}$ such that

$$\int_{\mathcal{D}^2} |K(x, y)| \mu(dx) \mu(dy) < \infty$$

(2.3)

and let $\mathcal{M}_K$ be the space of signed measure spanned by $\mathcal{M}_K^+$

$$\mathcal{M}_K := \{\mu_+ - \mu_- : \mu_+, \mu_- \in \mathcal{M}_K^+\}.$$  

(2.4)

We define $\tilde{K}$ as the following quadratic form on $\mathcal{M}_K$

$$\tilde{K}(\mu, \mu') = \int_{\mathcal{D}^2} K(x, y) \mu(dx) \mu'(dy).$$

(2.5)

The assumption (2.2) ensures that $\tilde{K}$ is positive and definite. Finally let $X = (\langle X, \mu \rangle)_{\mu \in \mathcal{M}_K}$ be the centered Gaussian process indexed by $\mathcal{M}_K$ with covariance function given by $\tilde{K}$. Note that from (2.1), $\mathcal{M}_K$ contains all compactly supported continuous functions (with some abuse of notation we identify the measure $m(x)dx$ with function $m(x)$). We use the improper notation

$$\int_{\mathcal{D}} X(x)m(x)dx := \langle X, m \rangle$$

(2.6)

We want to consider now an approximation of $X$ obtained by convolution with a smooth Kernel. Consider $\theta$ a non-negative $C^\infty$ function whose compact support is included in
the Euclidean ball of radius one, and such that $\int_{B(0,1)} \theta \, dx = 1$. We define for $\varepsilon > 0$, 
$$
\theta_\varepsilon := \frac{1}{\varepsilon} \theta(\varepsilon^{-1} \cdot).
$$

Given $D \subset \mathcal{D}$ measurable with compact closure $\overline{D}$ and $\varepsilon < \varepsilon_0$ where

$$
\varepsilon_0(D, \mathcal{D}) := 1 \wedge \max_{x \in D, y \notin \mathcal{D}} |x - y| 
$$

we define the convoluted version of $X$ by setting

$$
X_\varepsilon(x) := \int_D X(y) \theta_\varepsilon(x - y) \, dy. \quad (2.8)
$$

With this definition one can check that $X_\varepsilon(x)$ is a centered Gaussian field indexed by $(0, \varepsilon_0) \times \mathcal{D}$ with covariance function

$$
K_{\varepsilon, \varepsilon'}(x, y) := \mathbb{E}[X_\varepsilon(x)X_\varepsilon'(y)] = \int_{(\mathbb{R}^d)^2} \theta_\varepsilon(x - z_1) \theta_\varepsilon'(y - z_2) K(z_1, z_2) \, dz_1 \, dz_2. \quad (2.9)
$$

We simply write $K_\varepsilon$ when $\varepsilon = \varepsilon'$, and $K_{\varepsilon}(x)$ when $x = y$. Finally $K_{\varepsilon, \varepsilon'}(x, y)$ is sufficiently regular (that is, both Hölder continuous in $x$ and $\varepsilon$) to apply Kolmogorov criterion (see e.g. [4, Theorem 2.9]). Thus, in particular, there exists a version of the field which is jointly continuous in $\varepsilon$ and $x$. In what follows we will always be considering this continuous version of the field.

### 2.2. Gaussian multiplicative chaos in the complex case.

Given $K$ satisfying (2.1), $X$ a Gaussian field with covariance $K$, $D \subset \mathcal{D}$ measurable with compact closure, and $(X_\varepsilon(x))$ a continuous version in $\varepsilon$ and $x$ of the mollified field and $\mu$ a finite Borel measure on $D$. We define for $\varepsilon < \varepsilon_0$ (recall that with our notation $K_\varepsilon(x) = \mathbb{E}[(X_\varepsilon(x))^2]$) we define the $\varepsilon$-mollified the Gaussian Multiplicative chaos associated with $X$ and reference measure $\nu$ and with parameter $\gamma = \alpha + i\beta \in \mathbb{C}$ by

$$
M_\varepsilon^{(\gamma)} = \int_D e^{\gamma X_\varepsilon(x)} - \frac{\gamma^2}{2} K_\varepsilon(x) \nu(dx). \quad (2.10)
$$

A variant of this problem with independent real and imaginary parts of the field in the exponential can also be considered. Given $\alpha$ and $\beta$ two real numbers, $X$ and $Y$ two independent fields with covariance $K$ we set

$$
M_\varepsilon^{(\alpha, \beta)} := \int_D e^{\alpha X_\varepsilon(x) + i\beta Y_\varepsilon(x) + \frac{\beta^2 - \alpha^2}{2} K_\varepsilon(x)} \nu(dx). \quad (2.11)
$$

We are interested in the limit when $\varepsilon$ tends to zero of the quantities defined above. More specifically we want to show that, within some range for the parameters $\alpha$ and $\beta$, $M_\varepsilon$ converges to a non-trivial limit which does not depend on the convolution kernel $\theta$. As mentioned in the introduction above such a result has been proved in the real case (when $\beta = 0$ since $\gamma = \alpha$ simply write $M_\varepsilon^{(\alpha)}$). Let us mention this result as found in [4]. For the rest of the paper we will assume that $\nu(dx) = \rho(x) \, dx$ where $\rho$ is a non-negative measurable function on $D$ (note that [4] allows for some flexibility on the choice of the measure $\nu(dx)$ but we have chosen to keep the setup as simple as possible here).

**Theorem A.** Let $\alpha \in (-\sqrt{2d}, \sqrt{2d})$ be a real number. Then for $M_\varepsilon^{(\alpha)}$ defined as in (2.10) we have the following convergence in probability and in $\mathbb{L}_1$

$$
\lim_{\varepsilon \to 0} M_\varepsilon^{(\alpha)} = M_0^{(\alpha)}. \quad (2.12)
$$
where \( M^{(\alpha)}_\epsilon > 0 \) almost surely. Furthermore the limit does not depend on the choice of of the smoothing kernel \( \theta \).

Note that the range of parameter \( \alpha \) considered above is optimal since it is known that when \( |\alpha| \geq \sqrt{2d} \) we have \( \lim_{\epsilon \to 0} M^{(\alpha)}_\epsilon = 0 \), in probability (see e.g. [22]).

In the complex setup presented above, we are focusing on the so-called subcritical case which corresponds to the following range for the parameter \( \alpha \) and \( \beta \)

\[
\mathcal{O}_{\text{sub}} := \{ \alpha^2 + \beta^2 < d \} \cup \{ |\alpha| \in (\sqrt{d}/2, \sqrt{2d}) \text{ and } |\beta| < \sqrt{2d} - |\alpha| \}. \tag{2.13}
\]

In words, \( \mathcal{O}_{\text{sub}} \) is the convex envelope of the union of the ball of radius \( \sqrt{d} \) and the segment \((-\sqrt{2d}, \sqrt{2d}) \times \{0\} \) (see Figure 1). Our aim is to extend Theorem A to the complex setup, in the subcritical case.

Let us mention that when \((\alpha, \beta) \in \mathcal{O}_{\text{sub}} \) (and under some additional assumption on the Kernel \( K \)) the existence of a random distribution corresponding the formal expressions

\[
e^{\alpha X(x) + i\beta Y(x) + \frac{\sigma^2 - \alpha^2}{2} \mathbb{E}[(X(x))^2] \nu(dx)} \quad \text{and} \quad e^{\gamma X(x) - \frac{\sigma^2}{2} \mathbb{E}[(X(x))^2] \nu(dx)} \tag{2.14}
\]

was established in in [16] and [13] respectively. In both cases, the construction relies on a martingale approximation of the field \( X \) similar to Kahane’s construction. What we establish in the present paper is that any convolution approximation of the field yields the same object in the limit.

2.3. Results. Our first result concerns the case where real and imaginary part are independent.

**Theorem 2.1.** If \((\alpha, \beta) \in \mathcal{O}_{\text{sub}} \) and \( M^{(\alpha, \beta)}_\epsilon \) is defined as in (2.11), then the following limit exists in probability and in \( L_1 \)
\[ \lim_{\varepsilon \to 0} M^{(\alpha,\beta)}_\varepsilon = M^{(\alpha,\beta)}_0. \]  

Furthermore the limit does not depend on the choice of the smoothing Kernel \( \theta \).

Note that the convergence in \( L_1 \) implies that
\[ E \left[ M^{(\alpha,\beta)}_0 \right] = \lim_{\varepsilon \to 0} E \left[ M^{(\alpha,\beta)}_\varepsilon \right] = \int_D \rho(x)dx, \]  

which indicates that the limit is non trivial.

In the case of a single complex parameter \( \gamma \), we require and extra regularity assumption on \( K \) (which comes from [13]). More specifically we are going to assume that \( K \) can be written in the form (2.1) where the function \( L \) belong to the local Sobolev space \( H^s_{\text{loc}}(D \times D) \) for some \( s > d \), where for an open set \( U \) the Sobolev space with index \( s \) is defined by
\[
H^s(U) := \left\{ \varphi : U \to \mathbb{R} \mid \int (1 + |\xi|^2)\gamma^2 \varphi(\xi)^2d\xi < \infty \right\}
\]

and \( H^s_{\text{loc}}(U) \) denotes the function which belongs to \( H^s(U) \) after multiplication by an arbitrary smooth function with compact support
\[
H^s_{\text{loc}} := \{ \varphi : U \to \mathbb{R} : \rho \varphi \in H^s(U) \text{ for all } \rho \in C^\infty_c(U) \}.
\]

**Theorem 2.2.** Assuming that \( K \) is of the form (2.1) for a function \( L \in H^s_{\text{loc}}, s > d \). If \( (\alpha, \beta) \in \mathcal{O}_{\text{sub}}, \gamma = \alpha + i\beta \) and \( M^{(\gamma)}_\varepsilon \) is defined as in (2.10), then the following limit exists in probability and in \( L_1 \)
\[ \lim_{\varepsilon \to 0} M^{(\gamma)}_\varepsilon = M^{(\gamma)}_0. \]  

Furthermore the limit does not depend on the choice of the smoothing Kernel \( \theta \).

**Remark 2.3.** While in [10], the complex GMC is not obtained using smoothing kernels, it is worthwhile mentionning that the limit defined above coincides with the complex GMC constructed in [10] (this follows uniqueness of the limit on the real line and analyticity in \( \gamma \), we refer to [10] for details on how to prove analyticity). In the same manner, the limit presented in Theorem 2.1 coincides with the one defined [16, Theorem 3.1]. Some details about this last point are given in Section 4.3.

### 2.4. Possible extensions of the result and open problems

We have chosen to keep the setup as simple as can be for the ease of the exposition but let us mention here some small extension that can be obtained with only minor modifications in the proof.

**Correlated real and imaginary part.** In [8, 9, 14] the case of multiplicative cascades with correlated real and imaginary part is also considered. In our context this corresponds to considering \( X \) and \( Y \) with covariance \( K \) and such that the covariance between \( X \) and \( Y \) is given by \( pK \) in the sense that
\[
E \left[ \langle X, \mu \rangle \langle Y, \mu' \rangle \right] := p \int K(x, y)d\mu(dx)d\mu'(dy).
\]

In that case, the techniques we develop for the proof of Theorem 2.2 (in Section 3) fully adapts (without any need for change) under the same assumption for \( K \) (that is \( L \in H^s_{\text{loc}} \) for some \( s > d \)).
More general reference measures \( \nu \). We restricted our study to measures which are absolutely continuous with respect to Lebesgue. This assumption can be relaxed, and we can adapt our proof to a setup as general as the one considered in the real case \([4]\). More precisely, considering \( d' \in (0, d] \) and assuming that the measure \( \nu \) satisfies \( \int_{D \times D} \frac{1}{|x-y|^d} \nu(dx) \nu(dy) \), then, we have convergence of \( M_\varepsilon^{(\alpha, \beta)} \) and \( M_\varepsilon^{(\gamma)} \) as soon as

\[
\alpha^2 + \beta^2 < d' \quad \text{or} \quad |\alpha| \in (\sqrt{d'/2}, \sqrt{2d'}) \quad \text{and} \quad |\beta| < \sqrt{2d} - |\alpha|.
\]

Convergence of \( M^{(\gamma)} \) as a distribution. Note that \( M_\varepsilon^{(\gamma)} \) can be defined as a distribution by setting for any bounded continuous \( f \)

\[
M_\varepsilon^{(\gamma)}(f) := \int_D e^{\gamma X_\varepsilon(x)} \frac{|\beta|^2}{2} K_\varepsilon(x) f(x) \nu(dx).
\]

Our results implies that, given \( f \), in the subcritical phase \( M_\varepsilon^{(\gamma)}(f) \) converges almost surely. Some additional effort is needed to show that \( M_\varepsilon^{(\gamma)}(\cdot) \) converges in an adequate space of distribution. Some results have been obtained concerning the regularity of the limiting object (see \([16, \text{Theorem 3.1}] \) and \([11]\)) and it is reasonable to expect convergence in the spaces corresponding regularity, but the proof may present some technical difficulties.

Convergence on a part of the boundary of \( \Omega_{\text{sub}} \). As mentioned in the introduction, the range of parameter \( \Omega_{\text{sub}} \) is almost optimal for the convergence problem. Indeed, from the phase diagram presented in \([16]\) (which was discovered earlier in \([7]\) for the hierarchical version of the model, see also \([14, 9]\)) indicates that the limit of \( M_\varepsilon^{(\alpha, \beta)} \) (and by analogy also \( M^{(\gamma)} \)) does not exists or is degenerate on the complement of the closure of \( \Omega_{\text{sub}} \). The boundary case is more delicate but \([16]\) indicates that \( M_\varepsilon^{(\alpha, \beta)} \) and \( M_\varepsilon^{(\gamma)} \) should converge to a non-trivial limit only when \( |\beta| = \sqrt{2d} - |\alpha| \), \( |\alpha| \in (\sqrt{d/2}, \sqrt{d}) \), the other boundary cases corresponding to either convergence to 0 or no convergence. Proving this rigourously and in full generality remains a challenging task.

2.5. Organization of the paper. In the short Section 3 we expose the argument which entails convergence in the case \( \alpha^2 + \beta^2 < d \) (the so called \( L_2 \) region). The argument is not new, but we include it since it is very short and yield some information about the proof strategy in the other cases. In Section 4 we prove Theorem 2.1 and in Section 5 we prove Theorem 2.2. The two proof are are partially inspired by the method used in \([4]\), though they present significant novelty, and share some common ideas, but the case of complex \( \gamma \) requires some more advanced strategy. The sections are placed in increasing order of technical difficulty and should be read in that order.

3. The \( L_2 \) convergence when \( |\gamma| < \sqrt{d} \)

Let us display in this section the full proof of the convergence of \( M_\varepsilon^{(\gamma)} \) when \( |\gamma| < \sqrt{d} \) (the same proof also applies to \( M_\varepsilon^{(\alpha, \beta)} \) in the same range of parameters). While this is not a new result (or proof), we have not seen it written up in details elsewhere in this context, and it may provide to the reader some insight for the techniques used in the next sections. For notational ease we write all the proof only in the case \( \rho \equiv 1 \), but the reader can check that the adaptation to the case of general bounded \( \rho \) is completely straight-forward.
Proposition 3.1. If \( \gamma \in \mathbb{C} \) satisfies \( |\gamma| < \sqrt{d} \), then the following limit exists in \( L_2 \)
\[
\lim_{\varepsilon \to 0} M_\varepsilon^{(\gamma)} = M_0^{(\gamma)}.
\]
Furthermore the limit does not depend on the choice of the smoothing Kernel \( \theta \).

Our proof is going to rely on the following estimate for correlation kernel. The proof is standard and left to the reader (note that since \( L \) is bounded and continuous, it is sufficient to prove (3.2) for \( K(x, y) = \log \frac{1}{|x - y|} \)).

Lemma 3.2. Given a correlation kernel of the form (2.1) \( D \in \mathcal{D} \) and a convolution kernel \( \theta \), there exists a constant such that for any \( \varepsilon, \varepsilon' \leq \varepsilon_0/2(D, \mathcal{D}) \) and any \( x, y \in D \)
\[
\left| K_{\varepsilon,\varepsilon'}(x, y) - \log \frac{1}{|x - y|} \right| \leq C
\]
and we have furthermore for \( x \neq y \)
\[
\lim_{\varepsilon,\varepsilon' \to 0} K_{\varepsilon,\varepsilon'}(x, y) = K(x, y).
\]

Proof of Proposition 3.1. It is sufficient to prove that the sequence is Cauchy in \( L_2 \). We have
\[
\mathbb{E} \left[ |M_\varepsilon - M_{\varepsilon'}|^2 \right] = \mathbb{E} \left[ |M_\varepsilon|^2 \right] + \mathbb{E} \left[ |M_{\varepsilon'}|^2 \right] - \mathbb{E} \left[ M_\varepsilon \overline{M}_{\varepsilon'} \right] - \mathbb{E} \left[ \overline{M}_\varepsilon M_{\varepsilon'} \right].
\]
Hence it is sufficient to show that \( \mathbb{E} \left[ M_\varepsilon \overline{M}_{\varepsilon'} \right] \) converges when \( \varepsilon \) and \( \varepsilon' \) both go to zero (this implies that the four terms in the r.h.s. of (3.4) cancel out in the limit). Now we have
\[
\mathbb{E} \left[ M_\varepsilon \overline{M}_{\varepsilon'} \right] = \int_{D^2} \mathbb{E} \left[ e^{\gamma X_\varepsilon(x)} \overline{e^{\gamma X_\varepsilon(y)}} - \frac{\gamma^2 K_{\varepsilon}(x,y) \overline{K}_{\varepsilon'}(x,y)}{2} \right] \, dx \, dy = \int_{D^2} e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)} \, dx \, dy.
\]
And thus from Lemma 3.2 we have \( e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)} \leq C|x - y|^{-|\gamma|^2} \) and thus we obtain by dominated convergence
\[
\lim_{\varepsilon,\varepsilon' \to 0} \int_{D^2} e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)} \, dx \, dy = \int_{D^2} e^{\gamma^2 K(x,y)} \, dx \, dy.
\]

4. Proof of Theorem 2.1

4.1. The strategy of proof. In this section we prove Theorem 2.1. The proof builds on the ideas developped [4] to prove Theorem [A] the main one being to consider a “truncated” partition function \( M_{\varepsilon}^{(\alpha,\beta)} \) by discarding the contribution of excessively high values of \( X_\varepsilon \). However, there is a key difference here. In [4], it is shown that the difference between the truncated partition function and the original one is small in \( L_1 \). This is not possible to show this in the complex case and we have to make sure that our truncated partition function exactly coincides with the original one with a probability which tends to one when the truncation level goes to infinity.

Our result is proved by showing that:

(A) The truncated version of the partition function converges in \( L_2 \),

(B) With a large probability the truncated and non-truncated version of the partition function coincide.
Note that without loss of generality we can assume that $\alpha$ and $\beta$ are both non-negative.

Let us assume that $p \alpha, \beta \in \mathcal{O}_{\text{sub}}$ with $\alpha \in (\sqrt{d/2}, \sqrt{2d})$ and $\beta > 0$ (the other case can be treated with the $L_2$ method). We choose a parameter $\lambda$ that satisfies

$$\sqrt{2d} < \lambda < 2\alpha \quad \text{and} \quad \frac{d + (2\alpha - \lambda)^2}{2} > \alpha^2 + \beta^2. \quad (4.1)$$

The reader can check that the existence of such a $\lambda$ follows from our assumptions. For $k \geq 1$ we define (with some minor abuse of notation) $X_k := X_{\varepsilon_k}$ where $\varepsilon_k = e^{-k}$. For any integer $q \geq 1$, we define the events, $A_{q,\lambda}$ and $\lambda q \in \mathbb{P}_{\text{sub}}$ with $\alpha p, \beta \in (\sqrt{d/2}, \sqrt{2d}) q$ and $\beta > 0$ (the other case can be treated with the $L_2$ method). We choose a parameter $\lambda$ that satisfies

$$\lambda \geq \lambda q \in \mathbb{P}_{\text{sub}}$$

$$\lambda q \in \mathbb{P}_{\text{sub}}$$

$$(4.1)$$

Now we define $M^{(\alpha,\beta)}_{\varepsilon,q}$ (we will omit the dependence in $\alpha$ and $\beta$ most of the time to alleviate the notation) by

$$M^{(\alpha,\beta)}_{\varepsilon,q} := \mathbb{E}_{\varepsilon,q} 1_{A_{\lambda q \in \mathbb{P}_{\text{sub}}} \lambda q \in \mathbb{P}_{\text{sub}}}$$

$$M^{(\alpha,\beta)}_{\varepsilon,q} := \mathbb{E}_{\varepsilon,q} 1_{A_{\lambda q \in \mathbb{P}_{\text{sub}}} \lambda q \in \mathbb{P}_{\text{sub}}}$$

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Now we define $M^{(\alpha,\beta)}_{\varepsilon,q}$ (we will omit the dependence in $\alpha$ and $\beta$ most of the time to alleviate the notation) by

$$M^{(\alpha,\beta)}_{\varepsilon,q} := \mathbb{E}_{\varepsilon,q} 1_{A_{\lambda q \in \mathbb{P}_{\text{sub}}} \lambda q \in \mathbb{P}_{\text{sub}}}$$

The convergence of $M^{(\alpha,\beta)}_{\varepsilon,q}$ is deduced from the two following statements.

**Proposition 4.1.** For every $q$ the sequence $(M^{(\alpha,\beta)}_{\varepsilon,q})_{\varepsilon \in (0,\varepsilon_0)}$ is Cauchy in $L_2$. In particular the following limit exists

$$\lim_{\varepsilon \to 0} M^{(\alpha,\beta)}_{\varepsilon,q} := M^{(\alpha,\beta)}_{\varepsilon,q}. \quad (4.4)$$

Furthermore the limit does not depend on the choice of $\theta$.

**Proposition 4.2.** We have for any $\delta > 0$

$$\lim_{q \to \infty} \mathbb{P}[A_{q,\lambda}] = 1. \quad (4.5)$$

The proof of Proposition 4.1 is detailed in the next subsection. The study maximum of the Gaussian Free Field has given rise to a rich literature and Proposition 4.2 is a much weaker statement than state of the art results found in [19, 6, 5] among other references, however since we could not find a reference that matches our setup we include a proof in Appendix A.

**Proof of Theorem 2.1 from Proposition 4.1 and 4.2.** Let $q_0$ be the smallest value of $q$ such that $A_{q,\lambda}$ holds. Proposition 4.2 implies that $q_0$ is finite almost surely. We have for every $\varepsilon$, $M^{(\alpha,\beta)}_{\varepsilon,q} = M^{(\alpha,\beta)}_{\varepsilon,q_0}$, and thus as consequence of Proposition 4.1, $M^{(\alpha,\beta)}_{\varepsilon,q}$ converges in probability towards $M^{(\alpha,\beta)}_{\varepsilon,q_0}$ ($q_0$ is a random variable but the convergence in probability can be obtained by decomposing on all its possible values since there are only countably many).

4.2. **Proof of Proposition 4.1.** We first prove the convergence result for a fixed $\theta$ and discuss the dependence in $\theta$ (which turns out to be direct consequence of the proof) in Section 4.3.
4.2.1. The convergence of $M_\varepsilon$. For the same reason as in (3.2), we only need to prove the convergence of $E\left[M_{\varepsilon,q}^{(\alpha,\beta)}M_{\epsilon',q}^{(\alpha,\beta)}\right]$ towards a finite limit. Let us assume that $\varepsilon' \leq \varepsilon$.

Averaging first we respect to $Y$, and setting $A_{q}(x,y) = A_{q,\lambda}(x) \cap A_{q,\lambda}(y)$ we obtain

$$E\left[M_{\varepsilon,q}M_{\epsilon',q}\right] = \int_{D^2} e^{\beta K_{\varepsilon,\epsilon'}(x,y)} E\left[e^{\alpha(X\epsilon(x) + X\epsilon(y))} - \frac{2\alpha^2}{\lambda^2}(K_{\varepsilon}(x) + K_{\epsilon}(y))\chi_{A_{q}(x,y)}\right] \, dx \, dy$$

(4.6)

where $\tilde{P}_{\varepsilon,\epsilon',x,y}$ is defined by its density with respect to $P$ which is equal to

$$\frac{d\tilde{P}_{\varepsilon,\epsilon',x,y}}{dP} = e^{\varepsilon X_{\varepsilon}(x) + \varepsilon X_{\epsilon}(y) - \frac{2\alpha^2}{\lambda^2}(K_{\varepsilon}(x) + K_{\epsilon}(y)) + 2K_{\varepsilon',x,y}}.$$

(4.7)

We conclude from (4.6) using dominated convergence theorem and the following estimate for $\tilde{P}_{\varepsilon,\epsilon',x,y}(A_{q}(x,y))$.

**Lemma 4.3.** The following domination and convergence results holds

(A) There exists a constant $C_q > 0$ such that for every $\varepsilon' \leq \varepsilon < \varepsilon_0$ and $x,y \in D$,

$$\tilde{P}_{\varepsilon,\epsilon',x,y}(A_{q}(x,y)) \leq C_q(\varepsilon' - \varepsilon)^{2(\alpha - \lambda)^2}.\tag{4.8}$$

(B) We have $\lim_{\varepsilon' \to 0} \tilde{P}_{\varepsilon,\epsilon',x,y}(A_{q}(x,y)) = P[\tilde{A}_{q}(x,y)]$, where

$$\tilde{A}_{q}(x,y) := \bigcap_{k \geq q} \{X_k(x) \leq k\lambda - \alpha H_k(x,y) ; \; X_k(y) \leq k\lambda - \alpha H_k(y,x)\},$$

and $H_k(x,y) := K_{\varepsilon,k,0}(x,x) + K_{\varepsilon,k,0}(y,y) \ (\varepsilon_k = e^{-k}$ and $K_{\varepsilon,0}$ is defined by (2.9))

By Lemma 3.2 and Lemma 4.3 the integrand in the r.h.s. of (4.6) satisfies

$$e^{(\alpha^2 + \beta^2)K_{\varepsilon',x,y}}\tilde{P}_{\varepsilon,\epsilon',x,y}(A_{q}(x,y)) \leq (\varepsilon' - \varepsilon)^{2(\alpha^2 + \beta^2)},$$

which is integrable due to the assumption (4.1). Hence using dominated convergence we obtain that

$$\lim_{\varepsilon' \to 0} E\left[M_{\varepsilon,q}M_{\epsilon',q}\right] = \int_{D^2} e^{(\alpha^2 + \beta^2)K_{\varepsilon',x,y}}P[\tilde{A}_{q}(x,y)] \, dx \, dy < \infty.\tag{4.9}$$

Proof of Lemma 4.3. The change of measure given by (4.7) is simply a Cameron-Martin shift. It does not change the mean of the field $X_k$ but it modifies its mean, we have

$$E_{\varepsilon,\epsilon',x,y}[X_k(z)] = \alpha \left(K_{\varepsilon,\varepsilon'}(z,x) + K_{\varepsilon,\epsilon'}(z,y)\right) =: \alpha J_{\varepsilon,\epsilon'}(k,z).\tag{4.10}$$

Hence we have

$$\tilde{P}_{\varepsilon,\epsilon',x,y}(A_{q}(x,y)) = P\left[\forall k \geq q, \forall \varepsilon \in \{x,y\}, \; X_k(z) \leq k\lambda - \alpha J_{\varepsilon,\epsilon'}(k,z)\right].\tag{4.11}$$

To obtain the domination (4.8) it is sufficient to evaluate the probability for the restriction of $X_{k_0}$, with $k_0(\varepsilon, x, y) := \log \frac{1}{|x-y|}\varepsilon$. We have from (3.2) for some adequate constant $C$

$$J_{\varepsilon,\epsilon'}(k,x) \geq 2k_0 - C/\alpha,$$

$$\text{Var}(X_{k_0}(x)) \leq k_0 + C,\tag{4.12}$$
Assuming that $k_0(\lambda - 2\alpha) + C$ is negative and that $k_0 \geq q$ (which we can, all other cases can be treated by taking $C_q$ large since a probability is always smaller than one) the probability we wish to bound is smaller than

$$P[X_{k_0}(x) \leq k_0(\lambda - 2\alpha) + C] \leq 2e^{-\frac{(k_0(2\alpha - \lambda) - C)^2}{2(4k_0 + \varepsilon)}} \leq C'(|x - y| \vee \varepsilon)^{2(\lambda - 2\alpha)^2} \quad (4.13)$$

where we have used (4.12) and the following simple Gaussian bound valid for all $u \geq 0$

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx \leq 2e^{-\frac{u^2}{2}}. \quad (4.14)$$

The convergence for fixed distinct values of $x$ and $y$ is simply a consequence of the convergence of $J_{\varepsilon,\varepsilon}'(k, x)$ and $J_{\varepsilon,\varepsilon}'(k, x)$ to $H_k(x, y)$ and $H_k(y, x)$ respectively. Some care is needed here since we are dealing with countably many $X_k$s. Let us define

$$B_q^\ell(\varepsilon, \varepsilon') := \{\forall k \in [q, \ell], \forall z \in \{y, z\}, X_k(z) \leq k \lambda - \alpha J_{\varepsilon,\varepsilon'}(k, z)\},$$

$$C_q^\ell(\varepsilon, \varepsilon') := \{\exists k \geq \ell + 1, \exists z \in \{y, z\}, X_k(z) > k \lambda - \alpha J_{\varepsilon,\varepsilon'}(k, z)\}. \quad (4.15)$$

We use the notation $B_q^\ell(0)$ the event corresponding to $\varepsilon, \varepsilon' = 0$. We have from (4.11)

$$\tilde{P}_{\varepsilon,\varepsilon', x, y}(A_q(x, y)) = P[B_q^\ell(\varepsilon, \varepsilon')] - P[B_q^\ell(\varepsilon, \varepsilon') \cap C_q^\ell(\varepsilon, \varepsilon')]. \quad (4.16)$$

Hence we have

$$|\tilde{P}_{\varepsilon,\varepsilon', x, y}(A_q(x, y)) - P[\overline{A}_q(x, y)]|$$

$$\leq |P[B_q^\ell(\varepsilon, \varepsilon')] - P[B_q^\ell(0)]| + |P[B_q^\ell(0)] - P[\overline{A}_q(x, y)]| + P[C_q^\ell(\varepsilon, \varepsilon')]. \quad (4.17)$$

We are first going to show that for $\ell = \ell_0(\delta, x, y)$ sufficiently large, each of the two last terms are smaller than $\delta/3$, and then conclude using the fact that since for fixed $\ell_0$ we have

$$\lim_{\varepsilon,\varepsilon' \to 0} P[B_q^{\ell_0}(\varepsilon, \varepsilon')] = P[B_q^{\ell_0}(0)],$$

the first term can also be made smaller than $\delta/3$ by choosing $\varepsilon$ and $\varepsilon'$ small. Since $\cap_{\ell \geq q} B_q^\ell(0) = \overline{A}_q(x, y)$, the second term is indeed small if $\ell_0$ sufficiently large. Now from (3.2) we have for every $\varepsilon, \varepsilon'$ and $z \in \{y, z\}$

$$J_{\varepsilon,\varepsilon'}(k, z) \leq k + \log \frac{1}{|x - y|} + C.$$

Using the Gaussian bound (4.14) and making the value of $\ell_0$ large if necessary, this implies that for some constant $C'$ (allowed to depend on $x$ and $y$)

$$P[C_q^{\ell_0}(\varepsilon, \varepsilon')] \leq P[\exists k \geq \ell_0 + 1, \exists z \in \{y, z\}, X_k(z) > k(\lambda - \alpha) - C'].$$  

The above probability can be bounded from above something arbitrarily small is $\ell_0$ is large by using union bound and the Gaussian tail bound (4.14) (here we are using that $\alpha < \lambda$ and the fact that the variance of $X_k$ is of order $k$).
4.3. **The limit does not depend on θ.** Given θ’ another smoothing kernel we let \( X'_\varepsilon \) be the regularized field obtained by convolution with \( \theta'_\varepsilon \) and \( M'_{q,\varepsilon} \) be the corresponding truncated partition function (based on the event \( A'_{q,\lambda} \) defined as in (4.2) with \( X \) replaced by \( X' \)). We show that \( \lim_{\varepsilon \to 0} E[|M_{q,\varepsilon} - M'_{q,\varepsilon}|^2] = 0 \) converges by showing that

\[
\lim_{\varepsilon \to 0} E[|M_{q,\varepsilon}|^2] = \lim_{\varepsilon \to 0} E[|M'_{q,\varepsilon}|^2] = \lim_{\varepsilon \to 0} E[M_{q,\varepsilon}M'_{q,\varepsilon}]
= \int_{D^2} e^{(\alpha^2 + \beta^2)K(x,y)} P[A_q(x,y)] dx dy. \tag{4.18}
\]

The two first convergence statements are special cases of (4.9). For \( E[|M_{q,\varepsilon} - M'_{q,\varepsilon}|^2] \), we just have to prove a variant of Lemma 4.3 for the adequate tilting measure, which can be done without difficulty by reproducing the exact same proof.

**About Remark 5.2** In [16], instead of being approximated by convolutions, \( X \) is given a martingale approximation (see [16, Equation (2.2)]) which we denote here by \( \tilde{X}_\varepsilon \). If similarly to what is done above, we replace \( M_{q,\varepsilon} \) by \( \tilde{M}_{q,\varepsilon} \) which is defined by replacing \( X \) by \( \tilde{X} \) in every definition, we can also prove in the same manner (and under the assumption of regularity given in [16] for the covariance kernel of \( \tilde{X}_\varepsilon \)) that

\[
\lim_{\varepsilon \to 0} E[|M_{q,\varepsilon} - \tilde{M}_{q,\varepsilon}|^2] = 0,
\]

and hence that our limit coincides with the chaos defined in [16].

5. **The case of complex \( \gamma \)**

The previously developed strategy does not adapt to the complex case, but we can nonetheless use some of the ingredients of the previous section. It requires some additional technical assumption on the covariance function, which ensures that the field \( X \) can be written as a sum of independent functional increments. Luckily enough, it has been recently proved in [13] that this assumption is satisfied locally as soon as our function \( L \) in (2.1) is sufficiently regular.

5.1. **The case of decomposable kernels.** We are going to prove the result with an additional assumption on the covariance kernel. We assume that \( K \) can be written in the form

\[
K(x,y) = Q_0(x,y) + \sum_{n \geq 1} Q_n(x,y), \tag{5.1}
\]

where \( Q_0(x,y) \) is a positive definite and Hölder continuous (in both variable \( x \) and \( y \)). The functions \((Q_n)_{n \geq 1}\) are continuous positive definite function on \( D \) satisfying

\[
\begin{align*}
Q_n(x,y) &\geq 0, \\
Q_n(x,x) &= 1, \\
Q_n(x,y) &= 0 \quad \text{if } |x - y| \geq e^{-n}, \\
|Q_n(x,y) - Q_n(x',y')| &\leq C e^n (|x - x'| + |y - y'|)
\end{align*}
\tag{5.2}
\]

for every \( x, x', y, y' \in D \). It is not very difficult to check that these assumptions implies in particular that (5.1) hold. Our main task will be to prove convergence of \( M_{q,\gamma} \) in this setup.
Proposition 5.1. Let us assume that $K$ satisfies assumption (5.1)-(5.2). If $(\alpha, \beta) \in \mathcal{O}_{\text{sub}}$, $\gamma = \alpha + i\beta$ and $M_{\varepsilon}^{(\gamma)}$ is defined as in (2.10), then the following limit exists in probability and in $L_1$

$$\lim_{\varepsilon \to 0} M_{\varepsilon}^{(\gamma)} = M_0^{(\gamma)}. \quad \text{(5.3)}$$

Furthermore the limit does not depend on the choice of the smoothing Kernel $\theta$.

Remark 5.2. Our assumptions on $Q$ are not all necessary. For instance the assumptions $Q_n(x,y) \geq 0$ could be suppressed. Some mild assumptions on the decay of correlation could replace the one about compact support and $Q_n(x,x) = 1$ could be replaced by $|Q_n(x,x) - 1| \leq r(n)$ for a summable function $n$. As we felt that this would not present a significant extension of Theorem 2.2 in any case, we preferred to keep stronger assumptions in order to keep the proof as readable as possible.

5.2. Deducing Theorem 2.2 from the decomposable case. To prove Theorem 2.2 building on the case of decomposable kernels, we crucially rely on a result in [13] which allows us to keep the proof as readable as possible.

We present only a simple consequence of this result which is sufficient to our purpose. Let us define for $r \geq 0$ (in the equation below $\cdot$ denotes the Lebesgue measure)

$$\kappa(r) := \frac{|B(0,1) \cap B(0,2re_1)|}{|B(0,1)|} \quad \text{(5.5)}$$

where $B(x,R)$ denote the open Euclidean ball of radius $R$ and $e_1$ is the vector $(1,0,\ldots,0)$. The reader can check that $q(x,y) := \kappa(|x-y|)$ defines a Lipshitz positive definite kernel with range 1 correlation. The following proposition is a particular case of [13] Theorem 4.5].

Proposition 5.3. If $K$ is of the form (2.1) with $L \in H^s_{\text{loc}}(D \times D)$, $s > d$, then for any $z \in D$ there exists $\delta(z) > 0$ and $t_0(z) > 0$ which are such that the function (extended by continuity on the diagonal)

$$Q_0(x,y) := L(x,y) - \int_{t_0}^\infty \kappa(e^t|x-y|)dt + \log \frac{1}{|x-y|}, \quad \text{(5.4)}$$

is a positive definite function on $B(z,\delta(z))$.

Deducing Theorem 2.2 from Propositions 5.1 and 5.3. Note that from Sobolev and Morrey’s inequality, the assumption $L \in H^s_{\text{loc}}(D \times D)$, $s > d$, implies that $L(x,y)$ is locally Hölder continuous for some positive Hölder exponent $\eta$ and thus so is $Q_0$ (the reader can check that $Q_0 - L$ is Lipshitz). Now defining

$$Q_n(x,y) := \int_{t_0+n}^{t_0+n+1} \kappa(e^t|x-y|)dt, \quad \text{(5.5)}$$

it is easy to check that (5.1)-(5.2) is satisfied on $B(z,\delta(z))$. Now since $D$ is compact, we can cover it by a finite collection of balls $B(z_i,\delta_i)_{i=1}^k$ obtained with Proposition 5.3. We can write $D$ as a disjoint union of measurable sets $\bigcup_{i=1}^k D_i$ where $D_i \subset B(z_i,\delta_i)$ for all $i = 1,\ldots,k$. Then we establish the convergence of

$$M_{\varepsilon}^{(\gamma)} := \sum_{i=1}^k \int_{D_i} e^{\gamma X_\varepsilon(x) - \frac{\varepsilon^2}{2} K_\varepsilon(x)} dx, \quad \text{(5.6)}$$
simply using Proposition 5.1 for each term of the sum.

5.3. Extending the probability space and truncating the partition function. To prove Proposition 5.1 we are going to work in an extended probability space. Together with the Gaussian process $X$ indexed by $\mathcal{M}_K$ (recall Section 2.1) we define a process $(Y_n(x))_{n \geq 1, x \in D}$ such that $(X, Y)$ is jointly Gaussian and centered. The covariance function of $Y$ is given by

$$E[Y_n(x)Y_m(y)] = \sum_{k=1}^{m \wedge n} Q_k(x, y) =: K_{n \wedge m}(x, y), \quad (5.7)$$

and the covariance with $X$ is given by $E[Y_n(x)X(x)] = \int_D K_n(x, z)\mu(dz)$, for $\mu \in \mathcal{M}_K$ in particular we have for $y \in D$ and $\varepsilon \leq \varepsilon_0$

$$K_{n,\varepsilon}(x, y) := E[Y_n(x)X_\varepsilon(x)] = \int_D K_n(x, z)\Theta_\varepsilon(z-x)dz. \quad (5.8)$$

We consider for every $n$ a continuous version of the field $Y_n(\cdot)$ (which exists since $K_n$ is Lipshitz).

We assume (without loss of generality) that both $\alpha$ and $\beta$ are positive, that $(\alpha, \beta) \in \mathcal{O}_{\text{sub}}$ with $\alpha > \sqrt{d}/2$ (the other cases belong to the $L_2$ region), and consider $\lambda$ satisfying (4.1).

We can now introduce a truncated version of the partition function, similar to the one considered in the previous section, but with $X_{s^{-k}}$ replaced by $Y_k$. We recycle the notation of the previous section, and redefine the events $A_{q,\lambda}(x)$, $A_{q,\lambda}$ by setting

$$A_{q,\lambda}(x) := \{\forall k \geq q, Y_k(x) \leq k\lambda\},$$

$${\mathcal{A}}_{q,\lambda} := \bigcap_{x \in D} A_{q,\lambda}(x) = \left\{\forall k \geq q, \sup_{x \in D} Y_k(x) \leq k\lambda\right\}, \quad (5.9)$$

and then set

$$M^{(\gamma)}_{q,\lambda} = \int_D e^{\gamma x_\varepsilon(x)} - \frac{\varepsilon^2}{2} K_\varepsilon(x) 1_{A_{q,\lambda}(x)} dx. \quad (5.10)$$

Then Proposition 5.1 is an immediate consequence of the two following results (see Section 4.1 above for details). Proposition 5.4 is the more important statement and the remainder of the section is dedicated to its proof. Proposition 5.5 is in spirit very similar to Proposition 4.2 and has a similar proof (see Appendix A).

**Proposition 5.4.** For every $q$ the sequence $(M^{(\gamma)}_{q,\lambda})_{\varepsilon \in (0,\varepsilon_0)}$ is Cauchy in $L_2$. In particular we the existence of the following limit

$$\lim_{\varepsilon \to 0} M^{(\gamma)}_{q,\lambda} := M^{(\gamma)}_{0,\lambda}. \quad (5.11)$$

Furthermore the limit does not depend on the choice of $\Theta$.

**Proposition 5.5.** We have for any $\delta > 0$

$$\lim_{q \to \infty} \mathbb{P}[A_{q,\lambda}] = 1. \quad (5.12)$$
5.4. Reducing the proof of Proposition [5.4] to a domination statement. As in previous cases, we only need to prove that $\mathbb{E} \left[ M_{\varepsilon,q}^{(\gamma)} M_{\varepsilon',q}^{(\gamma)} \right]$ converges to a finite limit when both $\varepsilon$ and $\varepsilon'$ go to zero (the fact that the limit does not depend on $\theta$ follows from the argument developed in Section [4.3] which also applies to the present case). We have, setting $A_q^{(\gamma)}(x,y) := A_{q,\lambda}(x) \cap A_{q,\lambda}(y)$ and interpreting the real part of the exponential tilt as a change of measure (we use the definition (4.7) for $\hat{\mathbb{P}}_{\varepsilon,x,y}$)

$$
\mathbb{E} \left[ M_{\varepsilon}^{(\gamma)} M_{\varepsilon'}^{(\gamma)} \right] = \int_{D^2} \mathbb{E} \left[ e^{\gamma \mathcal{W}(x) + \gamma \mathcal{W}(y) - \gamma^2 \frac{K_{\varepsilon}(x) + K_{\varepsilon'}(y)}{2}} 1_{A_q^{(\gamma)}(x,y)} \right] \, dx \, dy
$$

$$
= \int_{D^2} e^{\alpha^2 K_{\varepsilon,\varepsilon'}(x,z) + \beta^2 K_{\varepsilon,\varepsilon'}(z,y) - \alpha \beta K_{\varepsilon,\varepsilon'}(x,z) - \beta \alpha K_{\varepsilon,\varepsilon'}(y,z)} \mathbb{E}_{x,y,\varepsilon,\varepsilon'} \left[ e^{i\beta (X_{\varepsilon}(x) - X_{\varepsilon'}(y))} 1_{A_q^{(\gamma)}(x,y)} \right] \, dx \, dy
$$

(5.13)

As noticed before, under $\hat{\mathbb{P}}_{\varepsilon,x,y}$, the mean of the field is shifted and its covariance is preserved. More precisely we have

$$
\hat{\mathbb{E}}_{\varepsilon,\varepsilon',x,y} [X_{\eta}(z)] = \alpha \left( K_{\varepsilon,\eta}(x,z) + K_{\varepsilon',\eta}(x,z) \right),
$$

$$
\hat{\mathbb{E}}_{\varepsilon,\varepsilon',x,y} [Y_{\eta}(z)] = \alpha \left( K_{\varepsilon,\eta}(z,x) + K_{\varepsilon',\eta}(z,y) \right).
$$

(5.14)

We introduce the functions $L_{n,\varepsilon,\varepsilon'}$ and $L_n$ defined by (these functions depend also on $x$ and $y$ but we want to keep the notation as light as possible)

$$
L_{n,\varepsilon,\varepsilon'}(z) := K_{n,\varepsilon}(z,x) + K_{n,\varepsilon'}(z,y) \text{ and } L_n(z) = K_n(z,x) + K_n(z,y).
$$

(5.15)

Then setting

$$
\hat{A}_{q,\varepsilon,\varepsilon'}(x,y) := \{ \forall n \geq q, \forall z \in (x,y), \ Y_n(z) \leq \lambda n \} \quad \text{and} \quad \hat{A}_{q,\varepsilon,\varepsilon'}(x,y) := \forall z \in (x,y), \ Y_n(z) \leq \lambda n
$$

(5.16)

the quantity $\mathbb{E}_{x,y,\varepsilon,\varepsilon'} \left[ e^{i\beta (X_{\varepsilon}(x) - X_{\varepsilon'}(y))} 1_{\hat{A}_{q,\varepsilon,\varepsilon'}(x,y)} \right]$ can be rewritten as

$$
\int_{D^2} e^{(\alpha^2 + \beta^2)K_{\varepsilon,\varepsilon'}(x,y)} \mathbb{E} \left[ e^{i\beta (X_{\varepsilon}(x) - X_{\varepsilon'}(y))} : 1_{\hat{A}_{q,\varepsilon,\varepsilon'}(x,y)} \right] \, dx \, dy,
$$

(5.17)

where we have used the Wick exponential notation for a centered Gaussian variable $Z$

$$
: e^u Z : = e^{u Z - \frac{u^2}{2} \mathbb{E}[Z^2]}.
$$

(5.18)

To conclude we introduce statement, which is analogous to Lemma [4.3] and use dominated convergence.

**Proposition 5.6.** There exists a constant $C_q > 0$ such that for every $\varepsilon' \leq \varepsilon < \varepsilon_0$ and $x, y \in D$,

$$
\mathbb{E} \left[ e^{i\beta (X_{\varepsilon}(x) - X_{\varepsilon'}(y))} : 1_{\hat{A}_{q,\varepsilon,\varepsilon'}(x,y)} \right] \leq C_q (|x - y| \vee \varepsilon) \frac{(2\lambda - \lambda')^2}{2}
$$

(5.19)

Furthermore the above expectation admits a limit when $\varepsilon$ and $\varepsilon'$ both go to zero.

Indeed the integrand in (5.17) is dominated by $C (|x - y| \frac{(2\lambda - \lambda')^2}{2} - (\alpha^2 + \beta^2))$ which is integrable from assumption (4.11). The proof of Proposition [5.6] is slightly more involved than that of Lemma [4.3] and requires a new method. We develop it in the following subsection.
5.5. Proof of Proposition 5.6. Our main idea is to decompose $1_{\tilde{A}_{q,\varepsilon,\varepsilon}(x,y)}$ into an algebraic sums of indicator functions of events in $\mathcal{F}_n$ for some finite $n$ where $\mathcal{F}_n := \sigma(Y_k(\cdot), k \leq n)$. To underline the advantage of dealing with events in $\mathcal{F}_n$, let us perform a few Gaussian computations. Note that we have

$$E[X_{q}(z) \mid \mathcal{F}_n] = \int_D \theta(\varepsilon(z - z_1)Y_{n}(z_1)dz_1 =: Y_{n,q}(z)$$

hence using the fact that in a Gaussian space the conditional expectation of the Wick exponential coincides with the Wick exponential of the conditional expectation, if $B_{n,\varepsilon,\varepsilon'} \in \mathcal{F}_n$ we have

$$E\left[ e^{i\beta(\xi_{q}(x) - \xi_{q}(y))} : 1_{B_{n,\varepsilon,\varepsilon'}} \right] = E\left[ e^{i\beta(\xi_{q}(x) - \xi_{q}(y))} : 1_{B_{n,\varepsilon,\varepsilon'}} \right],$$

and since $\lim_{\varepsilon \to 0} Y_{n,\varepsilon} = Y_n$ the convergence of the right hand side can be proved using dominance argument provided $B_{n,\varepsilon,\varepsilon'}$ is suitably chosen. We set

$$n_0(\varepsilon, x, y) := \left\lfloor \log \frac{1}{|x - y| \vee \varepsilon} \right\rfloor$$

and

$$n_0^*(x, y) := \left\lfloor \log \frac{1}{|x - y|} \right\rfloor.$$  

We let $A^{(\varepsilon,\varepsilon')}_{n_0}$ denote the event that the upper bound constraint in $\tilde{A}_{q,\varepsilon,\varepsilon'}$ is satisfied for all $n \leq n_0$

$$A^{(\varepsilon,\varepsilon')}_{n_0} := \{ n \in [q, n_0], \forall z \in \{ x, y \}, Y_n(z) \leq n\lambda - \alpha L_{n,\varepsilon,\varepsilon'}(z) \}.$$  

Now we set for $n \geq n_0 + 1$ and define $B^{(\varepsilon,\varepsilon')}_{n,1}$ (resp. $B^{(\varepsilon,\varepsilon')}_{n,2}$) the events that $A^{(\varepsilon,\varepsilon')}_{n_0}$ is satisfied and that $n$ is the first index for which $Y_n(x)$ (resp. $Y_n(y)$) violates the upper constraint in $\tilde{A}_{q,\varepsilon,\varepsilon'}$.

$$B^{(\varepsilon,\varepsilon')}_{n,1} := A^{(\varepsilon,\varepsilon')}_{n_0} \cap \left\{ \inf \{ m \geq n_0 : Y_m(x) > m\lambda - \alpha L_{n,\varepsilon,\varepsilon'}(x) \} = n \right\},$$

$$B^{(\varepsilon,\varepsilon')}_{n,2} := A^{(\varepsilon,\varepsilon')}_{n_0} \cap \left\{ \inf \{ m \geq n_0 : Y_m(y) > m\lambda - \alpha L_{n,\varepsilon,\varepsilon'}(y) \} = n \right\}.$$  

Finally we define $C^{(\varepsilon,\varepsilon')}_{n,m} := B^{(\varepsilon,\varepsilon')}_{n,1} \cap B^{(\varepsilon,\varepsilon')}_{n,2}$. The reader can then quickly check that

$$1_{\tilde{A}_{q,\varepsilon,\varepsilon'}} = 1_{A^{(\varepsilon,\varepsilon')}_{n_0}} - \sum_{n > n_0 + 1} \left( 1_{B^{(\varepsilon,\varepsilon')}_{n,1}} + 1_{B^{(\varepsilon,\varepsilon')}_{n,2}} \right) + \sum_{n, m \geq n_0 + 1} 1_{C^{(\varepsilon,\varepsilon')}_{n,m}}.$$  

Note that the event remain well defined in the limit when $\varepsilon, \varepsilon'$ tend to 0. We let and we let $A^{*}_{n_0}$, $B^{*}_{n,j}$ and $C^{*}_{n,m}$ (for $n, m \geq n_0 + 1$) denote the event obtained in the $\varepsilon, \varepsilon' \to 0$ limit, replacing $n_0$ by $n_0^*$ and $L_{n,\varepsilon,\varepsilon'}$ by $L_n$. We are going to deduce Proposition 4.3 from the following estimates.

**Proposition 5.7.** Assume that $\varepsilon' \leq \varepsilon$. The following statement holds for a sufficiently large constant $C$ which may depend only on $\alpha, \beta, q$ and the kernel $K$.

(A) We have

$$\left| E\left[ e^{i\beta(\xi_{q}(x) - \xi_{q}(y))} : 1_{A^{(\varepsilon',\varepsilon')}_{n_0}} \right] \right| \leq C(|x - y| \vee \varepsilon)^{(2\alpha - \lambda)^2 / 2},$$

and

$$\lim_{\varepsilon, \varepsilon' \to 0} E\left[ e^{i\beta(\xi_{q}(x) - \xi_{q}(y))} : 1_{A^{(\varepsilon,\varepsilon')}_{n_0}} \right] = E\left[ e^{i\beta(\xi_{q}(x) - \xi_{q}(y))} : 1_{A^{*}_{n_0}} \right].$$
(B) We have for every $n \geq n_0 + 1$ and $j = 1, 2$
\[
\left| \mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{B_{n,j}^{(e,e')}} \right] \right| \leq C(|x - y| \vee \varepsilon) \frac{(2\alpha - \lambda)^2}{2} e^{\frac{1}{2}[(\lambda - \alpha)^2 + \beta^2](n - n_0)}. \tag{5.28}
\]
and for $n \geq n_0^* + 1$ we have
\[
\lim_{\varepsilon, \varepsilon' \to 0} \mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{B_{n,j}^{(e,e')}} \right] = \mathbb{E} \left[ : e^{i\beta(Y_n(x) - Y_n(y))} : 1_{B_{n,j}^*} \right]. \tag{5.29}
\]

(C) We have for every $n, m$
\[
\left| \mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{C_{n,m}} \right] \right| \leq C(|x - y| \vee \varepsilon) \frac{(2\alpha - \lambda)^2}{2} e^{\frac{1}{2}[(\lambda - \alpha)^2 + \beta^2](n \vee m - n_0)}. \tag{5.30}
\]
and for $n, m \geq n_0^* + 1$ we have
\[
\lim_{\varepsilon, \varepsilon' \to 0} \mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{C_{n,m}} \right] = \mathbb{E} \left[ : e^{i\beta(Y_n(x) - Y_n(y))} : 1_{C_{n,m}} \right]. \tag{5.31}
\]

Proof of Proposition 5.4 Looking at (5.25) and using the triangle inequality we deduce from (5.26), (5.28) and (5.30)
\[
\left| \mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{\tilde{A}_{q,e,e'}(x,y)} \right] \right| \leq C(|x - y| \vee \varepsilon) \frac{(2\alpha - \lambda)^2}{2} \left( 1 + \sum_{n \geq n_0 + 1} e^{(\lambda - \alpha)^2 + \beta^2}(n - n_0) + \sum_{n, m \geq n_0 + 1} e^{(\lambda - \alpha)^2 + \beta^2}(n + m - 2n_0) \right). \tag{5.32}
\]
Moreover, by dominated convergence (our bounds are uniformly summable in $n$ and $m$) and (5.27), (5.29), (5.31) we have
\[
\lim_{\varepsilon, \varepsilon' \to 0} \mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{\tilde{A}_{q,e,e'}(x,y)} \right] = \mathbb{E} \left[ : e^{i\beta(Y_n(x) - Y_n(y))} : 1_{\tilde{A}_{n,0}^*} \right] + \sum_{j=1}^{n_0 + 1} \mathbb{E} \left[ : e^{i\beta(Y_n(x) - Y_n(y))} : 1_{B_{n,j}^*} \right]
+ \sum_{n, m \geq n_0 + 1} \mathbb{E} \left[ : e^{i\beta(Y_n(x) - Y_n(y))} : 1_{C_{n,m}^*} \right]. \tag{5.33}
\]

5.6. Proof of Proposition 5.7 The proof of convergence statement is the easier part and is identical for (5.27), (5.29), (5.31) (recall that $n_0 = n_0^*$ when $\varepsilon$ and $\varepsilon'$ are sufficiently small). Let us fully detail the case (5.31) for the sake of completeness. Since the event $\mathcal{F}_{n,m} = \mathcal{F}_{n,m}$ measurable we have from (5.21)
\[
\mathbb{E} \left[ : e^{i\beta(X(x) - X(y))} : 1_{C_{n,m}^{(e,e')}} \right] = \mathbb{E} \left[ : e^{i\beta(Y_n(x) - Y_n(y))} : 1_{C_{n,m}^{(e,e')}} \right]. \tag{5.34}
\]
and we can conclude (using dominated convergence) by observing that the quantity inside the expectation converges in probability towards $: e^{i\beta(Y_n(x) - Y_n(y))} : 1_{C_{n,m}^*}$ and is bounded above by $e^{2\beta^2 \varepsilon}$.

To prove the domination part, we are going to rely on the following probability estimates for the events involved in the expectation. The proof of these estimates is postponed to the end of the section.
Lemma 5.8. The following inequalities are valid for all \( x, y, \varepsilon' \leq \varepsilon \), \( n \in \left[ n_0 + 1, \log 1/\varepsilon \right] \), \( m \in \left[ n_0 + 1, \log 1/\varepsilon \right] \) for a constant C that does not depend of the afore mentioned parameters
\[
\mathbb{P}[A_{n_0}^{(\varepsilon, \varepsilon')}] \leq C(|x - y| \vee \varepsilon)^{\frac{1}{2}(2\alpha - \lambda)^2}.
\]
(5.35)
\[
\mathbb{P}[B_{n,j}^{(\varepsilon, \varepsilon')}] \leq C(|x - y| \vee \varepsilon)^{\frac{1}{2}(2\alpha - \lambda)^2} e^{-\frac{(n-n_0)(\lambda-\alpha)^2}{2}},
\]
(5.36)
\[
\mathbb{P}[C_{n,m}^{(\varepsilon, \varepsilon')}] \begin{cases} 
C|x - y|^{\frac{1}{2}(2\alpha - \lambda)^2} e^{-\frac{(n+m-2n_0)(\lambda-\alpha)^2}{2}}, & \text{if } |x - y| \leq \varepsilon, \\
C\varepsilon^{\frac{1}{2}(2\alpha - \lambda)^2} e^{-\frac{(n+m-n_0)(\lambda-\alpha)^2}{2}}, & \text{if } |x - y| > \varepsilon.
\end{cases}
\]
(5.37)

We are also going to rely on an estimate for the covariance of \( Y_{n,\varepsilon} \). We set
\[
K_{n,\varepsilon}(x, y) := \mathbb{E}[Y_{n,\varepsilon}(x)Y_{n,\varepsilon}(y)].
\]
The following estimate follows from assumption (5.2) (we include a proof in Appendix B for completeness).

Lemma 5.9. We have for any \( x, y \in D \) any \( n \geq 1 \) and \( \varepsilon' \leq \varepsilon \)
\[
\left| K_{n,\varepsilon\varepsilon'}(x, y) - \min \left( \log \frac{1}{|x - y|}, \log \frac{1}{\varepsilon}, n \right) \right| \leq C.
\]
(5.38)

We now have all the ingredients to prove the domination statements

Proof of (5.20). We have
\[
\mathbb{E} \left[ e^{i\beta (Y_{n_0,\varepsilon}(x) - Y_{n_0,\varepsilon'}(y))} : 1_{A_{n_0}^{(\varepsilon, \varepsilon')}} \right] \leq e^{\frac{\beta^2}{2} \text{Var}(Y_{n_0,\varepsilon}(x) - Y_{n_0,\varepsilon'}(y))} \mathbb{P}[A_{n_0}^{(\varepsilon, \varepsilon')}].
\]
(5.39)

As a consequence of (5.35), and of the choice for \( n_0 \), the variance
\[
\text{Var}(Y_{n_0,\varepsilon}(x) - Y_{n_0,\varepsilon'}(y)) = K_{n_0,\varepsilon,\varepsilon'}(x, x) + K_{n_0,\varepsilon,\varepsilon'}(y, y) - 2K_{n_0,\varepsilon,\varepsilon'}(\varepsilon, \varepsilon')
\]
is uniformly bounded in \( x, y, \varepsilon \) and \( \varepsilon' \) and we can conclude using (5.35). 
\

Proof of (5.28) and (5.30). The idea is the same for (5.28) and (5.30). We treat only the latter, which is the more delicate, in details. The inequality we prove differs according to the value of \( \varepsilon \). When \( |x - y| > \varepsilon \) we prove (5.30) while if \( |x - y| \leq \varepsilon \) we prove the stricter inequality
\[
\left| \mathbb{E} \left[ e^{i\beta (X_\varepsilon(x) - X_\varepsilon'(y))} : 1_{C_{n,m}} \right] \right| \leq C|x - y|^{\frac{1}{2}(2\alpha - \lambda)^2} e^{-\frac{3}{2}[(\lambda-\alpha)^2 + \beta^2](n+m-n_0)}
\]
(5.40)

The reader can check here that simply repeating the proof of (5.20) replacing \( n_0 \) by \( n \vee m \) (case (C)), does not yield a satisfactory result (we obtain a factor \( \beta^2/2 \) instead of the desired \( \beta^2/2 \) in the exponential). We need thus some refinement in the conditioning. For simplicity, let us assume that \( n \leq m \) (strictly speaking, since we already assumed \( \varepsilon \leq \varepsilon' \), there is a loss of generality here but this is of no consequence). We define the \( \sigma \)-algebra \( \mathcal{G}_{n,m} \) as
\[
\mathcal{G}_{n,m} = \mathcal{G}_{n,m}(x, y) := \mathcal{F}_n \vee \sigma \left( Y_l(y), l \in [n + 1, m] \right).
\]
(5.41)

Clearly we have \( C_{n,m}^{(\varepsilon, \varepsilon')} \in \mathcal{G}_{n,m} \). Hence similarly to (5.21) we have
\[
\mathbb{E} \left[ e^{i\beta (X_\varepsilon(x) - X_\varepsilon'(y))} : 1_{C_{n,m}^{(\varepsilon, \varepsilon')}} \right] = \mathbb{E} \left[ e^{i\beta (\mathbb{E}[X_\varepsilon(x) - X_\varepsilon'(y) | \mathcal{G}_{n,m}] - X_\varepsilon'(y))} : 1_{C_{n,m}^{(\varepsilon, \varepsilon')}} \right]
\leq e^{\frac{\beta^2}{2} \text{Var}(\mathbb{E}[X_\varepsilon(x) - X_\varepsilon'(y) | \mathcal{G}_{n,m}])} \mathbb{P}[C_{n,m}^{(\varepsilon, \varepsilon')}].
\]
(5.42)
We can conclude using (5.37) provided that one can show that
\[
\text{Var} \left( \mathbb{E} \left[ X_\varepsilon(x) - X_\varepsilon'(y) \mid G_{n,m} \right] \right) \leq \begin{cases} 
 n + m - 2n_0, & \text{if } |x - y| \leq \varepsilon, \\
 m - n_0, & \text{if } |x - y| > \varepsilon.
\end{cases}
\] (5.43)

We perform a decomposition of \( \mathbb{E} \left[ X_\varepsilon(x) - X_\varepsilon'(y) \mid G_{n,m} \right] \) into a sum of orthogonal Gaussian variables. We let \( Z_n := Y_n - Y_{n-1} \) denote the \( n \)-th increment of \( Y_n \). Using independence of the increments we obtain
\[
\mathbb{E} \left[ X_\varepsilon(x) - X_\varepsilon'(y) \mid G_{n,m} \right] = Y_{n,x}(x) - Y_{n,x'}(y) + \sum_{k=n+1}^{m} \mathbb{E} \left[ X_\varepsilon(x) - X_\varepsilon'(y) \mid Z_n \right].
\] (5.44)

We have
\[
\text{Var}(Y_{n,x}(x) - Y_{n,x'}(y)) = K_{n,\varepsilon,\varepsilon}(x, x) + K_{n,\varepsilon',\varepsilon'}(y, y) - 2K_{n,\varepsilon,\varepsilon'}(x, y)
\] (5.45)

and thus, as a consequence of (5.38) we have
\[
\text{Var}(Y_{n,x}(x) - Y_{n,x'}(y)) \leq \begin{cases} 
 n - n_0 + C & \text{if } |x - y| > \varepsilon, \\
 2(n - n_0) + C & \text{if } |x - y| \leq \varepsilon.
\end{cases}
\] (5.46)

On the other hand, a simple Gaussian computation yields
\[
\text{Var}(\mathbb{E} \left[ X_\varepsilon(x) - X_\varepsilon'(y) \mid Z_k(y) \right]) = \mathbb{E} \left[ (X_\varepsilon(x) - X_\varepsilon'(y))Z_k(y) \right]^2 \mathbb{E}[Z_k(y)]^{-1}.
\] (5.47)

Similarly to (5.20) we have
\[
\mathbb{E} \left[ (X_\eta(z) \mid (Z_k(z')))_{z' \in \mathcal{D}} \right] = \int \theta_\eta(z_1 - z)Z_k(z_1)dz
\]
This allow to compute the covariance and we have
\[
\mathbb{E} \left[ (X_\varepsilon'(y) - X_\varepsilon(x))Z_k(y) \right] = \int_D (\theta_\varepsilon'(y - z) - \theta_\varepsilon(x - z)) Q_k(z, y)dz.
\] (5.48)

Now since \( 0 \leq Q_k(z, y) \leq 1 \) we have
\[
\left| \int_D (\theta_\varepsilon'(y - z) - \theta_\varepsilon(x - z)) Q_k(z, y)dz \right| \leq 1.
\] (5.49)

and thus (recall that \( \mathbb{E}[Z_k(y)]^{-1} = Q_k(y, y) = 1 \))
\[
\text{Var} \left( \sum_{k=n+1}^{m} \mathbb{E} \left[ X_\varepsilon(x) - X_\varepsilon'(y) \mid Z_n \right] \right) \leq m - n,
\] (5.50)

which together with (5.43) concludes the proof of (5.16). The case (5.28) is dealt similarly but with a conditioning with respect to \( G_{n_0,n}(y, x) \) (for \( j = 1 \)) or \( G_{n_0,n}(x, y) \) (for \( j = 2 \)). \( \square \)

**Proof of Lemma 5.8** The proof of (5.35) is identical to that of (4.38) in Lemma 4.3. It is sufficient to observe that
\[
\mathbb{P} \left( \mathcal{A}_{n_0}^{(\varepsilon,\varepsilon')} \right) \leq \mathbb{P}(Y_{n_0}(x) \leq \lambda n_0 + \alpha L_{n,\varepsilon,\varepsilon'}(x)) \leq \mathbb{P}(Y_{n_0}(x) \leq (\lambda - 2\alpha)n_0 + C).
\] (5.51)

For (5.36)-(5.37) we use the same idea and restrict the event to a single inequality. Let us give the details for (5.37), the case (5.36) being similar but simpler. We assume here also
for simplicity that \( m \geq n \). Let us start with the case \(|x - y| \leq \varepsilon\). Note that if \( C_{n,m}^{(\varepsilon, \varepsilon')}(x, y) \) is satisfied then we have

\[
Y_{n_0, n}(x) + Y_{n_0, m}(y) > \lambda(n + m - 2n_0) - \alpha((L_{n, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(x) + (L_{m, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(y)).
\]

(5.52)

where we used the short-hand notation \( Y_{n_1, n_2} := Y_{n_2} - Y_{n_1} \). If we denote \( \mathcal{C}_{n,m}^{(\varepsilon, \varepsilon')} \) denote the event in (5.52), as \( \mathcal{C}_{n,m}^{(\varepsilon, \varepsilon')} \) is independent from \( \mathcal{F}_{n_0} \) and hence of \( A_{n_0}^{(\varepsilon, \varepsilon')} \) (since \( Y_{n_0, n} \) and \( Y_{n_0, m} \) are), with the bound already proved for \( A_{n_0}^{(\varepsilon, \varepsilon')} \), we only need to show that

\[
P(\mathcal{C}_{n,m}^{(\varepsilon, \varepsilon')}) \leq e^{-\frac{(\alpha \lambda)^2}{2}(n + m - 2n_0)}.
\]

(5.53)

Hence we need an upper bound on the variance of \( Y_{n_0, n}(x) + Y_{n_0, m}(y) \) and on \((L_{n, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(x) + (L_{m, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(y)\). We have

\[
\text{Var}(Y_{n_0, n}(x) + Y_{n_0, m}(y)) = n + m - 2n_0,
\]

(5.54)

\[
(L_{n, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(x) + (L_{m, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(y) \leq n + m - 2n_0 + C,
\]

where the first line comes from the fact \( Y_{n_0, n}(x) \) and \( Y_{n_0, m}(y) \) are independent (due to Assumption (5.2) and the fact that \(|x - y| \leq \varepsilon\)). The second line comes from Lemma 5.2.

Then (5.53) is a consequence of (4.14) and (5.54).

When \(|x - y| \leq \varepsilon \) we observe that \( C_{n,m}^{(\varepsilon, \varepsilon')} \) implies

\[
Y_{n_0, m}(y) > \lambda(m - n_0) - (L_{m, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(y),
\]

(5.55)

and we conclude similarly using (4.14) together with the following estimates to conclude

\[
\text{Var}(Y_{n_0, m}(y)) = m - n_0,
\]

(5.56)

\[
(L_{m, \varepsilon, \varepsilon'} - L_{n_0, \varepsilon, \varepsilon'})(y) \leq m - n_0 + C.
\]

\[
\square
\]

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**Appendix A. Proof of Propositions 4.2-5.5**

The proof is almost identical for the two results so let us give the full details for the field \( X_k \) and explain briefly how to adapt the argument for \( Y_k \). We are going to prove that there exists \( c > 0 \) such that for all \( k \) sufficiently large

\[
P(B_k) := P \left( \max_{x \in D} X_k(x) > \lambda k \right) \leq \frac{1}{c} e^{-ck}.
\]

(A.1)

We first prove give a bound for the maximum on a dyadic grid of mesh \( e^{-(1+\delta)k} \) simply by using a union bound. Then we show that local fluctuation within a distance \( e^{-(1+\delta)k} \) are very small in amplitude. This second step of the proof is simply based on a quantitative version of the argument used to prove continuity of Gaussian processes from Kolmogorov-Chentsov criterion (see e.g. [18] Section 2.2) for the classic proof of continuity of Brownian Motion). Recalling the definition of \( \varepsilon_0 \), we extend a bit our domain \( D \) to make it more regular. We set

\[
D_+ := \{ x \in D : \exists y \in D, |x - y| \leq \varepsilon_0/2 \}.
\]
We assume that \( e^{-k} \leq \varepsilon_0 / 4 \) so that \( X_k \) is well defined in \( \mathcal{D} \). Let us consider for \( p \geq 1 \), \( \mathbb{D}_p \) the set of points in \( \mathcal{D} \) whose coordinates are integer multiple of \( 2^{-p} \) (the cardinality of \( \mathbb{D}_p \) is of order \( 2^{np} \)). We set \( p_0^{(k)} := \left\lfloor \frac{k}{\log 2} \right\rfloor \).

Let us fix \( \delta \) such that \( 2d(1 + \delta) < \lambda^2 \). From (3.2) the variance of \( X_k \) uniformly larger than \( k - C \) and we have thus from Gaussian tail bound (1.14), for some constant \( C' \)

\[
\mathbb{P} \left[ \max_{x \in \mathbb{D}_{p_0}} X_k(x) \geq \lambda k - 1 \right] \leq 2|\mathbb{D}_{p_0}| e^{-\frac{(\lambda k - 1)^2}{2(k - C)}} \leq C' e^{-\frac{(\lambda^2 - 2d(1 + \delta))}{2}}. \tag{A.2}
\]

Note that for every point in \( x \in \mathcal{D} \) and \( p \geq p_0 \) there exists a sequence \( (x_p)_{p \geq p_0} \), converging to \( x \) such that for every \( p \), \( x_p \in \mathbb{D}_p \) and \( |x_p - x_{p-1}| \leq \sqrt{d}2^{-p} \). What we are going to show is that with probability larger that \( 1 - e^{-ck} \) we have

\[
\forall p \geq p_0 + 1, \forall y, z \in \mathbb{D}_p, \quad \{|y - z| \leq \sqrt{d}2^{-p}\} \Rightarrow \left\{ |X_k(y) - X_k(z)| \leq \frac{1}{p(p - 1)} \right\} \tag{A.3}
\]

and we can conclude using continuity that

\[
|X_k(x) - X_k(x_{p_0})| = | \sum_{p \geq p_0 + 1} X_k(x_p) - X_k(x_{p-1}) | \leq \frac{1}{p_0} \leq 1.
\]

In order to control local fluctuation, first note that a simple computation allows to deduce from (2.9) that the Lipshitz constant of \( K_\varepsilon(x, y) \) is at most \( C\varepsilon^{-1} \log \varepsilon \) (and hence \( Ck\varepsilon k \) for \( \varepsilon_k \)). Hence the variance of \( (X_k(x) - X_k(y)) \) is at most \( k \varepsilon^k |x - y| \). Now taking into account that the number of pair of close-by vertices below is of order \( 2^p \), we have for any

\[
\mathbb{P} \left( \max_{|x - y| \leq \sqrt{d}2^{-p}} |X_k(x) - X_k(y)| \leq \frac{1}{p(p + 1)} \right) \leq C2^p e^{-\frac{2p}{2Ck\varepsilon k(p + 1)^2}}, \tag{A.4}
\]

and it only remains to check that the sum over \( p \geq p_0 \) indeed yields something exponentially small in \( k \). The field \( Y_k \) possesses the same kind of regularity as \( X_k \) so that the argument exposed above adapts verbatim to that case.

**Appendix B. Proof of Lemma 5.9**

Let us start with the case \( \varepsilon, \varepsilon' = 0 \) and prove

\[
\left| K_n(x, y) - \min \left( n, \log \frac{1}{|x - y|} \right) \right| \leq C. \tag{B.1}
\]

The assumptions \( Q_n(x, y) = 0 \) if \( |x - y| \geq e^{-n} \) and \( Q_n(x, y) \leq \sqrt{Q(x, x)} \sqrt{Q(y, y)} = 1 \) immediately yields the upper bound. For the lower bound, we have, using the positivity and Lipshitz constant for \( Q_k \)

\[
K_n(x, y) \geq \sum_{k=1}^{\min \left( n, \log \frac{1}{|x - y|} \right)} Q_k(x, y) \geq \sum_{k=1}^{\min \left( n, \log \frac{1}{|x - y|} \right)} \left[ Q_k(x, x) - C \varepsilon^k |x - y| \right], \tag{B.2}
\]

and conclude from the fact that \( \sum_{k=1}^{\log \frac{1}{|x - y|}} \varepsilon^k |x - y| \) is bounded. From the definition of \( Y_{n, \varepsilon} \) in Equation (5.20) we have

\[
K_{n, \varepsilon, \varepsilon'}(x, y) = \int_{\mathbb{R}^d} K_n(z_1, z_2) \theta_{\varepsilon}(z_1 - x) \theta_{\varepsilon'}(z_2 - y) dz_1 dz_2. \tag{B.3}
\]
From (B.1) we can replace $K_n(z_1, z_2)$ by $\min\left(n, \log \frac{1}{|z_1-z_2|}\right)$ and the results then follows from standard computations.

□

References

[1] Kari Astala, Peter Jones, Antti Kupiainen, and Eero Saksman. Random conformal weldings. *Acta Math.*, 207(2):203–254, 2011.
[2] Julien Barral, Xiong Jin, and Benoît Mandelbrot. Convergence of complex multiplicative cascades. *Ann. Appl. Probab.*, 20(4):1219–1252, 2010.
[3] Julien Barral, Xiong Jin, and Benoît Mandelbrot. Uniform convergence for complex [0,1]-martingales. *Ann. Appl. Probab.*, 20(4):1205–1218, 2010.
[4] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22:Paper No. 27, 12, 2017.
[5] Marek Biskup and Oren Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.*, 345(1):271–304, 2016.
[6] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 69(1):62–123, 2016.
[7] B. Derrida, M. R. Evans, and E. R. Speer. Mean field theory of directed polymers with random complex weights. *Comm. Math. Phys.*, 156(2):221–244, 1993.
[8] Lisa Hartung and Anton Klimovsky. The glassy phase of the complex branching Brownian motion energy model. *Electron. Commun. Probab.*, 20:no. 78, 15, 2015.
[9] Lisa Hartung and Anton Klimovsky. The phase diagram of the complex branching Brownian motion energy model. *Electron. J. Probab.*, 23:Paper No. 127, 27, 2018.
[10] Janne Junnila and Eero Saksman. The uniqueness of the gaussian multiplicative chaos revisited, 2015.
[11] Janne Junnila, Eero Saksman, and Lauri Viitasaari. On the regularity of complex multiplicative chaos, 2019.
[12] Janne Junnila, Eero Saksman, and Christian Webb. Imaginary multiplicative chaos: Moments, regularity and connections to the ising model, 2018.
[13] Janne Junnila, Eero Saksman, and Christian Webb. Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.*, 29(6):3786–3820, 2019.
[14] Zakhar Kabluchko and Anton Klimovsky. Complex random energy model: zeros and fluctuations. *Probab. Theory Related Fields*, 158(1-2):159–196, 2014.
[15] Jean-Pierre Kahane. Sur le chaos multiplicatif. (On multiplicative chaos). *Ann. Sci. Math. Qué.*, 9:105–150, 1985.
[16] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Complex Gaussian multiplicative chaos. *Comm. Math. Phys.*, 337(2):569–632, 2015.
[17] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. A probabilistic approach of ultraviolet renormalisation in the boundary sine-gordon model, 2019.
[18] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of Graduate Texts in Mathematics. Springer, [Cham], french edition, 2016.
[19] Thomas Madaule. Maximum of a log-correlated Gaussian field. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(4):1369–1431, 2015.
[20] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting SLEs. *Probab. Theory Related Fields*, 164(3-4):553–705, 2016.
[21] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probab. Surv.*, 11:315–392, 2014.
[22] Raoul Robert and Vincent Vargas. Gaussian multiplicative chaos revisited. *Ann. Probab.*, 38(2):605–631, 2010.
[23] Eero Saksman and Christian Webb. The riemann zeta function and gaussian multiplicative chaos: statistics on the critical line, 2016.
[24] Alexander Shamov. On Gaussian multiplicative chaos. *J. Funct. Anal.*, 270(9):3224–3261, 2016.