Elementary equivalence of infinite-dimensional classical groups

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Abstract. Let $D$ be a division ring such that the number of conjugacy classes in the multiplicative group $D^*$ is equal to the power of $D^*$. Suppose that $H(V)$ is the group GL($V$) or PGL($V$), where $V$ is a vector space of infinite dimension $κ$ over $D$. We prove, in particular, that, uniformly in $κ$, and $D$, the first order theory of $H(V)$ is mutually syntactically interpretable with the theory of the two-sorted structure $⟨κ, D⟩$ (whose only relations are the division ring operations on $D$) in the second order logic with quantification over arbitrary relations of power $⩽κ$. A certain analogue of this result is proved for the groups $ΓL(V)$ and $PΓL(V)$. These results imply criteria of elementary equivalence for infinite-dimensional classical groups of types $H = ΓL, PTL, GL, PGL$ over division rings, and solve, for these groups, a problem posed by Felgner. It follows from the criteria that if $H(V_1) ≡ H(V_2)$ then $κ_1$ and $κ_2$ are second order equivalent as sets.

In the present paper we deal with the problem to what extent the first order theory of an infinite-dimensional classical group over a division ring determines the dimension of the group and the ring.

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In the case of finite dimension, for many types of classical groups, the problem can be easily reduced to the problem when two classical groups of the same type are isomorphic. Indeed, by the Keisler–Shelah theorem, structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent iff, for some ultrafilter $F$, the ultrapowers $\mathcal{M}^F$ and $\mathcal{N}^F$ are isomorphic. The following Isomorphism Theorem is known [8]. For $H = \text{GL}$, $\text{SL}$, $\text{PGL}$, $\text{PSL}$ and any division rings $D_1, D_2$, if $n_1, n_2 \geq 3$ then

$$H(n_1, D_1) \simeq H(n_2, D_2) \text{ if and only if } n_1 = n_2, \text{ and } D_1 \simeq D_2 \text{ or } D_1 \simeq D_2^{\text{op}}.$$ 

For $H = \text{GL}$, $\text{SL}$, $\text{PGL}$, the same holds even for $n_1, n_2 \geq 2$. (For $H = \text{PSL}$, in the case of dimension 2 there are some exceptional isomorphisms.) Taking into account $H(n, D)^F \simeq H(n, D^F)$, we have that, for $H, n_1, n_2$ satisfying the conditions of the Isomorphism Theorem,

$$H(n_1, D_1) \equiv H(n_2, D_2) \text{ if and only if } n_1 = n_2, \text{ and } D_1 \equiv D_2 \text{ or } D_1 \equiv D_2^{\text{op}}.$$ 

Maltsev [10] proved the latter result in the special case of groups over fields of characteristic 0; his proof was based on an interpretation of the field $D$ in the group $H(n, D)$.

In [7] Felgner suggested to study the problem of elementary equivalence for infinite-dimensional general linear groups and other classical groups over fields. In the present paper we solve Felgner’s problem for infinite-dimensional groups of types $\text{GL}$, $\text{PGL}$, $\Gamma L$, $\text{PGL}$ for a wide class of division rings.

In a more general setting, the subject of the paper can be described as a study of the expressive power of the first order logic for infinite-dimensional classical groups and related structures. The similar problem was considered in many papers, in particular, in the papers [20, 21] by Shelah on infinite symmetric groups, in his paper [22] devoted to endomorphism semi-groups of free algebras, in a series of papers on automorphism groups of Boolean algebras by Rubin and Shelah (e.g. [19]), in the paper [13] by Magidor, Rosental, Rubin and Srour on lattices of closed subsets of Steinitz exchange systems.

According to [22, 3], one can measure the expressive power of a first order theory by the richness of the fragment of set theory interpretable in it. In [20, 22, 13] this idea has been realized in the following way. With every structure $\mathcal{M}$ from a given class of structures a structure $\mathcal{M}^*$ is associated, so that the elementary equivalence of structures $\mathcal{M}$ and $\mathcal{N}$ from the class implies the $\mathcal{L}$-equivalence of $\mathcal{M}^*$ and $\mathcal{N}^*$ in a certain logic $\mathcal{L}$. The structures of the form $\mathcal{M}^*$ are chosen to be ‘algebra-free’ as much as possible, and the logic $\mathcal{L}$ is chosen to be as ‘strong’ as possible. A nice illustration of this method is given by a following version of Theorems 1.6 and 3.1 from [13]: if $\mathcal{K} = \langle K, +, \cdot \rangle$ is an uncountable algebraically closed field then the full second order theory of the set $K$ is syntactically interpretable (uniformly in $K$) in the first order theory of the lattice $L(K)$ of algebraically closed subfields of $K$. Hence for any uncountable algebraically closed fields $K_1$ and
the elementary equivalence of the lattices \( L(K_1) \) and \( L(K_2) \) implies the equivalence of the sets \( K_1 \) and \( K_2 \) in the full second order logic.

The paper is also concerned with the question when set theory is interpretable in the automorphism groups of algebras which are free in a variety \( V \) and of infinite rank. This question, as Shelah notes in his paper [22], is natural in view of the following result obtained in that paper: set theory is interpretable in the endomorphism semi-group of a free \( V \)-algebra which is of ‘large’ infinite rank. The answer in the automorphism group case essentially depends on the variety \( V \) (for example, one cannot interpret set theory in any infinite symmetric group – the automorphism group of an algebra in empty language [20, 21]).

Let \( V \) be a vector space of infinite dimension \( \kappa \) over a division ring \( D \).

The present paper can be divided into three parts. The aim of the first part (Sections 1–5) is to interpret the projective space of \( V \) (that is, \( P(V), \subseteq \), the lattice of subspaces of \( V \)) in the group \( \text{PGL}(V) \) (Theorem 5.1). The assumption \( \kappa \geq \aleph_0 \) is essential for the proof. As one of the key points of the proof we show the \( \emptyset \)-definability of the set of involutions of the first kind in \( \text{PGL}(V) \) (that is, involutions induced by involutions in \( \text{GL}(V) \)). This solves the problem of group-theoretic characterization of involutions of the first kind in \( \text{PGL}(V) \) posed by Rickart [18] and enables us to describe isomorphisms of infinite-dimensional groups of types \( \Gamma L \), \( P\Gamma L \), \( GL \), and \( PGL \) by classical methods. This description modulo the mentioned problem was known since the early fifties, but has been justified only in 1977 by O’Meara [15] who used non-classical techniques.

In the second part of the paper (Sections 6–8) we show that, uniformly in \( \kappa \) and \( D \),

\[
\text{Th(}\Gamma L(V)\text{)} \geq \text{Th(PTL(V))} \geq \text{Th(PGL(V))} \geq \text{Th(GL(V))}
\]

(Theorem 6.1). Here \( \geq \) means ‘syntactically interprets’ (for the definition, see Section 0). Moreover, we prove that \( \text{Th(PTL(V))} \geq \text{Th(GL(V))} \) (Theorem 8.1). Thus, the logical power does not drop under the transition to the projective image. Since the group \( \text{PGL}(V) \) is obviously interpretable in the group \( \text{GL}(V) \), we can reconstruct the projective space in all the groups \( \Gamma L(V) \), \( PTL(V) \), \( PGL(V) \), and \( GL(V) \).

Let \( \lambda \) be an infinite cardinal. We denote (in the manner of Shelah [20]) by \( L_2(\lambda) \) the second order logic, which allows to quantify over arbitrary relations of power \( < \lambda \). The monadic fragment of this logic, \( \text{Mon}(\lambda) \), allows to quantify over arbitrary subsets of power \( < \lambda \). We denote by \( (\kappa, D) \) the two-sorted structure, whose first sort is the cardinal \( \kappa \), the second one is the division ring \( D \), and the only relations of this structure are the standard ring operations on \( D \). Consider also the two-sorted structure \( (V, D) \), by combining the abelian group of \( V \) and the division ring \( D \) with their basic relations and the ternary relation for the action of \( D \) on \( V \).

Let \( D \) be division ring such that
the number of conjugacy classes of the multiplicative group $D^*$ is equal to the power of $D^*$.

In the third part of the paper (Sections 9–12) we demonstrate that various theories associated with the vector space $V$ are pairwise mutually syntactically interpretable, uniformly in $\kappa$ and $D$ (Theorem 11.4). In particular, we prove this for the first order theories of

- the projective space $P$,
- $\text{End}(V)$, the endomorphism semi-group of $V$,
- the groups $\text{PGL}(V)$ and $\text{GL}(V)$,

and the second order theories

- $\text{Th}(\langle \kappa, D \rangle, \text{Mon}(\kappa^+))$,
- $\text{Th}(\langle \kappa, D \rangle, L_2(\kappa^+))$.

(Note that the mentioned first order theories are mutually interpretable for arbitrary division rings.) As a consequence, $\text{Th}(\text{GL}(V)) \geq \text{Th}_2(\kappa)$, or, in other words, the automorphism groups of infinite-dimensional vector spaces interpret set theory. This provides a solution to the question from [22] mentioned above for any variety of vector spaces over a fixed division ring with $(\ast)$.

An early version of Theorem 11.4 without the elementary theories of classical groups in the list of mutually interpretable $V$-theories and under the additional assumption of commutativity of $D$, has been proved in the joint paper of the author and Belegradek [4].

Theorem 11.4 gives a solution to Felgner’s problem for infinite-dimensional linear groups of types $\text{GL}$ and $\text{PGL}$:

$$
\text{GL}(\kappa_1, D_1) \equiv \text{GL}(\kappa_2, D_2) \iff \\
\text{PGL}(\kappa_1, D_1) \equiv \text{PGL}(\kappa_2, D_2) \iff \\
\text{Th}(\langle \kappa_1, D_1 \rangle, L_2(\kappa_1^+)) = \text{Th}(\langle \kappa_2, D_2 \rangle, L_2(\kappa_2^+))
$$

(\ast\ast)

where $\kappa_1, \kappa_2$ are infinite cardinals, $D_1$ and $D_2$ are arbitrary division rings with $(\ast)$. Theorem 11.4 provides also more accurate estimate of the logical power of the elementary theory of projective space $\text{Th}(P)$: according to [13] Theorem 1.7] if $D$ is commutative, then the theory $\text{Th}(P)$ has the logical power at least that of second order logic on the cardinal $\min(\kappa, |D|)$.

To estimate the logical strength of the elementary theories $\text{Th}(\text{TL}(V))$ and $\text{Th}(\text{PTL}(V))$ we need a stronger logic than $L_2(\kappa^+)$ is. This logic is $\mathcal{L}_D(\kappa^+)$, extending expressive power of $L_2(\kappa^+)$ by a possibility to quantify over arbitrary automorphisms of the division ring $D$. Theorem 11.5 states that the theories $\text{Th}(\text{TL}(V))$, $\text{Th}(\text{PTL}(V))$, and $\text{Th}(\langle \kappa, D \rangle, \mathcal{L}_D(\kappa^+))$ are pairwise mutually syntactically interpretable, uniformly in $\kappa$ and $D$. This enables us to give a criterion of elementary equivalence of infinite-dimensional semi-linear groups similar to (\ast\ast).

We do not consider in this paper the classification of elementary types for the class of infinite-dimensional linear groups of types $E$ and $E_7$, which are natural infinite-dimensional analogues of finite-dimensional groups of the
type SL over fields (see \[8\], 1.2, 2.1 for details). We prove in \[24\] that all
infinite-dimensional groups of the types $E$ and $E_X$ over a fixed division ring
$D$ are elementary equivalent.

In Section 12 we examine the condition
\[
\text{Th}(\langle \kappa_1, D_1 \rangle, L_2(\kappa_1+1)) = \text{Th}(\langle \kappa_2, D_2 \rangle, L_2(\kappa_2+2)).
\]
(0.1)

In particular, this makes possible to prove that $GL(\aleph_0, R) \equiv GL(\kappa, D)$ iff
$\kappa = \aleph_0$ and $D \cong R$, and $GL(\aleph_0, C) \equiv GL(\kappa, D)$ iff $\kappa = \aleph_0$ and
$D \cong C$. This demonstrates that the condition (0.1) does not suffice for the
elementary equivalence of semi-linear groups.

In Section 0 we recall a number of basic facts of linear group theory and
a small portion of mathematical logic.

0. Basic concepts and notation

Let $V$ be always (throughout all the text) a left infinite-dimensional
vector space over a division ring $D$. The dimension of $V$ will be denoted by
$\kappa$. We denote the elements of $V$ by lower case Latin letters $a, b, c, \ldots$,
and the elements of $D$ by lower case Greek letters $l, \mu, \nu$. We shall use the letter
$W$ as the notation of an arbitrary left vector space over $D$.

The projective space $P(W)$ is treated as the set of all subspaces of $W$
\[2, 1, 14\]; $P^*(W)$ will denote the set of all proper non-zero subspaces of $W$.
$P^n(W)$ is the standard notation for the set of all $n$-dimensional subspaces of $W$ \[14\]. We denote by $P^{(n)}(W)$ the set of all subspaces in $W$ of dimension
or codimension $n$. The subspaces of $W$ of dimension one and codimension
one will be called lines and hyperplanes, respectively; the term ‘line’ will be
never used in the present paper in the sense of projective geometry. The
letter $N$ usually denotes a line of $W$ and the letter $M$ a hyperplane of $W$
(possibly with indices, primes, etc.).

Let $f$ be an isomorphism between division rings $D_1$ and $D_2$, and $W_1$,
$W_2$ be vector spaces over $D_1$ and $D_2$, respectively. Recall that a transfor-
mation $\sigma$ from $W_1$ to $W_2$ is a semi-linear transformation with respect to the
(associated) isomorphism $f$, if
\[
\sigma(a + b) = \sigma(a) + \sigma(b), \quad a, b \in W;
\]
\[
\sigma(\lambda a) = f(\lambda) \cdot \sigma(a), \quad a \in W, \lambda \in D_1.
\]
Since a semi-linear transformation determines uniquely its associated iso-
morphism, the action of the associated isomorphism is usually written in the
form $\lambda^\sigma$.

The group of all bijective semi-linear transformations from $W$ into itself
(collineations) is called the semi-linear (collinear) group of the space $W$ \[15\].
The standard notation is $\Gamma L(W)$. The subgroup of all linear transformations
from $\Gamma L(W)$ is the general linear group of the space $W$; it is written as
$GL(W)$. 

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Every collineation \( \sigma \in \Gamma L(W) \) induces in a natural way a permutation \( \hat{\sigma} \) of the set \( P(W) \). The transformation \( \hat{\sigma} \) is said to be the projective image of \( \sigma \). The set of all projective images of the elements of the group \( \Gamma L(W) \) with the composition law is the projective semi-linear group of \( W \). Notation: \( \text{PTL}(W) \). Clearly, the mapping \( \hat{\cdot} \) is a homomorphism from the group \( \Gamma L(W) \) to the group \( \text{PTL}(W) \). The projective general linear group of the space \( W \) is a subgroup of \( \text{PTL}(W) \), consisting of the projective images of linear transformations; it is written as \( \text{PGL}(W) \).

A transformation \( \tau : W \to W \) such that for some \( \lambda_\tau \in D^* \)

\[
\tau a = l_\tau \cdot a, \quad \forall a \in W,
\]

is called a radiation. We shall denote \( \tau \) by \( l_\cdot \text{id}(W) \). The set of all radiations of \( W \) with the composition law is obviously the group isomorphic to \( D^* \), multiplicative group of \( D \). Notation: \( \text{RL}(W) \).

**Proposition 0.1.** ([2] Chapter III, Section 3"). Let \( \dim W \geq 2 \), and \( \sigma_1, \sigma_2 \in \Gamma L(W) \). Then \( \hat{\sigma}_1 = \hat{\sigma}_2 \) if and only if \( \sigma_1 \sigma_2^{-1} \in \text{RL}(W) \).

The group \( \text{RL}(W) \) is clearly a normal subgroup of \( \Gamma L(W) \). It is easy to see that a radiation \( \tau \) lies in \( \text{GL}(W) \) iff \( l_\tau \in Z(D) \), where \( Z(D) \) is the center of \( D \). Furthermore, \( Z(\text{GL}(W)) = Z(\text{RL}(W)) = \text{GL}(W) \cap \text{RL}(W) \). Thus,

**Corollary 0.2.** Let \( \dim W \geq 2 \). Then

(a) The group \( \text{PTL}(W) \) is isomorphic to the quotient group \( \Gamma L(W)/\text{RL}(W) \).

(b) The group \( \text{PGL}(W) \) is isomorphic to the quotient group \( \text{GL}(W)/Z(\text{RL}(W)) \).

We describe now the involutions in the group \( \text{GL}(W) \). Let \( \sigma \) be an arbitrary involution of \( \text{GL}(W) \). There are two different cases: \( \text{char } D \neq 2 \) and \( \text{char } D = 2 \).

I. The characteristic of \( D \) is not 2. In the case we have a decomposition

\[
W = W^+ + W^-,
\]

(0.1)

where \( W^+ = \{a \in W : \sigma a = a\} \) and \( W^- = \{a \in W : \sigma a = -a\} \). The subspaces \( W^+ \) and \( W^- \) are called the subspaces of \( \sigma \). The decomposition (0.1) implies that there is a basis of \( W \) in which \( \sigma \) is diagonalized. Furthermore, one can easily prove the following

**Lemma 0.3.** Let \( \sigma_1, \ldots, \sigma_n \) be pairwise commuting involutions in \( \text{GL}(W) \). Then there is a basis of \( W \) in which all \( \sigma_1, \ldots, \sigma_n \) are diagonalized.

An involution \( \sigma \in \text{GL}(W) \) is called extremal if some its subspace is a line (or, equivalently, a hyperplane).

II. The characteristic of \( D \) is equal to 2. In this case we can also assign to an involution \( \sigma \in \text{GL}(W) \) two subspaces of \( W \). These subspaces are \( \text{Fix}(\sigma) = \{a \in W : \sigma a = a\} \) and \( \text{Rng}(\text{id}(W) + \sigma) \), where \( \text{Rng}(\pi) \) is the image of a transformation \( \pi \).

Choose a linearly independent set \( \{d_i : i \in I\} \) such that

\[
W = \text{Fix}(\sigma) \oplus \langle d_i : i \in I \rangle.
\]
Let \( e_i = d_i + \sigma d_i, i \in I \). The set \( \{ e_i : i \in I \} \) is obviously a linearly independent subset of \( \text{Fix}(\sigma) \).

Let \( \{ e_j : j \in J \} \) be a complement of \( \{ e_i : i \in I \} \) to a basis of \( \text{Fix}(\sigma) \). Thus, \( \sigma \) acts on the basis \( \{ e_i : i \in I \} \cup \{ e_j : j \in J \} \cup \{ d_i : i \in I \} \) of \( W \) as follows

\[
\begin{align*}
\sigma e_i &= e_i, \quad i \in I, \\
\sigma e_j &= e_j, \quad j \in J, \\
\sigma d_i &= d_i + e_i, \quad i \in I.
\end{align*}
\] (0.2)

On the other hand, any \( \sigma \in \text{GL}(W) \), which acts on some basis of \( W \) similar to (0.2), is an involution.

The definition of the extremal involutions remains the same: an involution of the group \( \text{GL}(W) \) is called extremal, if it has a subspace of dimension one. It is important that in the case when \( \text{char} \ D = 2 \), the extremal involutions are also transvections, that is, linear transformations of the form

\[
\sigma a = a + \delta(a)b,
\]

where \( \delta \) is a non-zero linear function from \( W \) to \( D \) such that \( \delta(b) = 0 \). Clearly, \( \text{Fix}(\sigma) = \ker(\delta) \) and \( \text{Rng}(\text{id}(W) + \sigma) = \langle b \rangle \). The line \( \langle b \rangle \) is called the line of \( \sigma \) and the hyperplane \( \ker(\delta) \) is called the hyperplane of \( \sigma \) (subspaces of \( \sigma \)).

**Lemma 0.4.** (a) Let \( M \) and \( N \) be a hyperplane and a line with \( N \subseteq M \). There is a transvection \( \sigma \) in \( \text{GL}(W) \) such that the subspaces of \( \sigma \) are identical to \( M \) and \( N \).

(b) Let \( \sigma_1 \) and \( \sigma_2 \) be two transvections in \( \text{GL}(W) \), and let \( N_k, M_k \) be the subspaces of \( \sigma_k \), where \( k = 1, 2 \). Then \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \) if and only if \( (N_1 \subseteq M_2 \& M_1 \supseteq N_2) \).

(c) Let \( \sigma_1 \) and \( \sigma_2 \) be two distinct transvections in \( \text{GL}(W) \). Then \( \sigma_1 \) and \( \sigma_2 \) have a mutual subspace if and only if \( \sigma_1 \sigma_2 \) is a transvection.

Lemma 0.4 is a well-known result, which can be found, for example, in [15, pp. 101-102], where it is formulated for arbitrary vector spaces over division rings (not only for finite-dimensional ones as in most of the works in linear group theory, but also for infinite-dimensional vector spaces).

**Remarks.** (a) It should be pointed out that both methods of assigning subspaces to an involution \( \sigma \) (whether characteristic is equal to 2, or not) could be treated in a uniform way, if we assign, in the style of O’Meara, to an involution of \( \text{GL}(W) \) its fixed and residual subspaces, where the residual one is the subspace \( \text{Rng}(\text{id}(W) - \sigma) \).

(b) Note also that in both cases we assign to each involution in \( \text{GL}(W) \) an unordered pair of subspaces of \( W \).

Let us describe the involutions in the group \( \text{PGL}(W) \) of dimension at least two. An involution \( \hat{\sigma} \in \text{PGL}(W) \) is said to be an involution of the first kind in the group \( \text{PGL}(W) \), if \( \hat{\sigma} \) is induced by an involution of \( \text{GL}(W) \) [5, p. 8]. The involutions that are not of the first kind are called involutions of the second kind. We denote the identity element of \( \text{PGL}(W) \) simply by \( 1 \).
By Proposition 0.1, $\sigma^2 = 1$ iff $\sigma^2 = l \cdot \text{id}(W)$, where $l \in Z(D)$. It is easy to see that if $\sigma^2 = l \cdot \text{id}(W)$, then $\sigma$ is an involution of the first kind iff $l$ is a square in $Z(D)$.

The $\text{PGL}(W)$-involutions induced by extremal $\text{GL}(W)$-involutions are called extremal, too. Let $\sigma$ be an involution of the first kind induced by a $\text{GL}(W)$-involution $\sigma$. The subspaces of $\sigma$ are surely chosen (well-defined) to be identical to the subspaces of $\sigma$. Let $\sigma$ be an involution of $\text{GL}(W)$ or a $\text{PGL}(V)$-involution of the first kind. We call $\sigma$ a $\gamma$-involution, if $\gamma$ is equal to $\min(\dim R, \dim S)$, where $R$ and $S$ are the subspaces of $\sigma$.

We discuss now the important notion of a minimal pair. The notion appeared in the paper of Mackey [12]. That paper has a section devoted to the groups of autohomeomorphisms of infinite-dimensional normed linear spaces over the field of reals. In [16, 17, 18] Rickart extended the methods of Mackey from the groups of autohomeomorphisms to some classical groups.

Let $\dim W \geq 3$. Minimal pairs interpret the elements of $P^{(1)}(W)$ as follows. An extremal involution determines two subspaces: a line $N$ and a hyperplane $M$. Then, if we have a pair $\langle \sigma_1, \sigma_2 \rangle$ such that

1. $\sigma_1, \sigma_2$ are extremal $\text{GL}(W)$-involutions;
2. $(N_1 = N_2 & M_1 \neq M_2)$ or $(N_1 \neq N_2 & M_1 = M_2)$,

then involutions $\sigma_1$ and $\sigma_2$ have the unique mutual subspace – namely, the subspace which is in both the pairs of subspaces associated with $\sigma_1$ and $\sigma_2$, and therefore the tuple $\langle \sigma_1, \sigma_2 \rangle$ codes some subspace of $W$. In the case, when char $D = 2$ it is technically convenient to add the condition

0. $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$.

So a pair of $\langle \sigma_1, \sigma_2 \rangle$ of $\text{GL}(W)$-involutions satisfying (1,2) in the case char $D \neq 2$ or the conditions (0,1,2) in the case char $D = 2$ is called a minimal pair of the group $\text{GL}(W)$.

Rickart in the above mentioned above papers [16, 17, 18] modifying the methods of Mackey showed that if char $D \neq 2$ then the property of being a $\text{GL}(W)$-minimal pair is group-theoretic. Rickart denoted by $c(I)$ the set of all involutions in the centralizer of a subset $I$ of the group $\text{GL}(W)$.

Theorem 0.5. ([16, Theorem 2.6]). Let char $D \neq 2$ and let $\dim W \geq 3$. Then the following properties are equivalent

(a) $\langle \sigma_1, \sigma_2 \rangle$ is a minimal pair in $\text{GL}(W)$;
(b) $c(c(\sigma_1, \sigma_2)) = c(c(\pi_1, \pi_2))$, where $\pi_1$ and $\pi_2$ are arbitrary non-commuting extremal involutions in $c(c(\sigma_1, \sigma_2))$.

Thus, modulo definability of the extremal involutions, the property of being a minimal pair is even first order.

In the case when char $D = 2$ we may obtain the first order characterization of $\text{GL}(W)$-minimal pairs (modulo definability of extremal involutions) by applying Lemma 0.4.

Proposition 0.6. Let char $D = 2$, $\dim W \geq 3$, and $\sigma_1, \sigma_2$ be commuting transvections in $\text{GL}(W)$. Then the following conditions are equivalent

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(a) \(\langle \sigma_1, \sigma_2 \rangle\) is a minimal pair in \(GL(W)\);
(b) \(\sigma_1 \sigma_2\) is a transvection and there is a transvection \(\sigma\), commuting with \(\sigma_1\), but not with \(\sigma_2\).

Proof. ⇒. Suppose that \(\sigma_k\) has the subspaces \(N_k, M_k\), where \(k = 1, 2\). We assume that \((N_1 \neq N_2 \& M_1 = M_2)\). Then we may take as \(\sigma\) a transvection with the line \(N_1\) and the hyperplane \(M'\), where \(M'\) is a hyperplane which does not contain \(N_2\). In the dual case \((N_1 = N_2 \& M_1 = M_2)\) one may choose a line \(N'\), the linear span of an element from \(M_1 \setminus M_2\), and then construct \(\sigma\) by the subspaces \(N'\) and \(M_1\).

⇐. If \(\sigma_1\) and \(\sigma_2\) commute, but the pair \(\langle \sigma_1, \sigma_2 \rangle\) is not minimal, then \((N_1 = N_2 \& M_1 = M_2)\) and the second condition in 0.6(b) is false by 0.4(b), since any transvection, commuting with \(\sigma_1\), must commute with \(\sigma_2\).

A pair \(\langle \sigma_1, \sigma_2 \rangle\) \(\in PGL(W)\) is called minimal, if there are involutions \(\sigma_1\) and \(\sigma_2\) in the preimages of \(\sigma_1\) and \(\sigma_2\), respectively, such that \(\langle \sigma_1, \sigma_2 \rangle\) is a \(GL(W)\)-minimal pair.

Using minimal pairs Rickart, Dieudonné [16, 17, 18, 5] and other authors described the groups of automorphisms for various types of classical groups.

Remark. Note that certain automorphisms of finite-dimensional group \(GL(W)\) (or \(PGL(W)\)) send minimal pairs with a mutual line to minimal pairs with a mutual hyperplane; this is impossible in the infinite-dimensional case (see [6, 8, 15] for details). Thus, roughly speaking, there is no hope to distinguish lines and hyperplanes coded by minimal pairs, and, moreover, the subspaces of dimension \(k\) and the subspaces of codimension \(k\) in the finite-dimensional case. On the other hand, we shall interpret in the infinite-dimensional linear group \(PGL(V)\) the elements of the projective space \(P(V)\), and then interpret the inclusion relation on \(P(V)\). This will demonstrate that the set of all minimal pairs with a mutual line is \(\varnothing\)-definable in the infinite-dimensional group \(PGL(V)\) in contrast to the finite-dimensional case.

We close this section with a portion of logic.

We shall denote by \(Th(M, L)\) the theory of a structure \(M\) in a logic \(L\).

Let \(\{T^0_i : i \in I\}\) and \(\{T^1_i : i \in I\}\) be two families of theories in logics \(L_0\) and \(L_1\), respectively. We say that the theory \(T^0_i\) is syntactically interpretable in \(T^1_i\) uniformly in \(i \in I\), in symbols \(T^0_i \preceq T^1_i\), if there is a mapping \(*\) from the set of all \(L_0\)-sentences to the set of \(L_1\)-sentences, such that, for every \(L_0\)-sentence \(\chi\) and for every \(i \in I\), \(T^0_i \vdash \chi\) iff \(T^1_i \vdash \chi^*\) [3, 9, Chapter V]. If, in addition, \(T^1_i \preceq T^0_i\) uniformly in \(i \in I\), the theories \(T^0_i, T^1_i\) are said to be mutually syntactically interpretable uniformly in \(i \in I\). The relation \(\preceq\) is clearly reflexive and transitive.

Quite informally, in cases when the class of indices is clear from the context, we shall often write interpretability results in the form \(T^0 \preceq T^1\).

We state now two sufficient conditions for uniform syntactical interpretation.
A structure $\mathcal{M}$ is said to be $\emptyset$-interpretable/reconstructible (without parameters) in a structure $\mathcal{N}$ by means of a logic $\mathcal{L}$, if there are a positive integer $n$, a $\emptyset$-definable by means of $\mathcal{L}$ set $X$ of $n$-tuples in $\mathcal{N}$, and a surjective mapping $f : X \to \mathcal{M}$ such that $f$-preimages of all the basic relations on $\mathcal{M}$ (including the equality relation) are $\emptyset$-definable by means of $\mathcal{L}$ in $\mathcal{N}$. From this definition, one can easily realize, what means ‘an interpretation with parameters’ or ‘an interpretation uniform in ...’.

A usual way to construct a syntactical interpretation is the following well-known result (see e.g. [9, Chapter V]).

**Theorem 0.7 (Reduction Theorem).** If, uniformly in $i \in I$, a structure $\mathcal{M}_i$ is interpretable without parameters in the structure $\mathcal{N}_i$ by means of logic $\mathcal{L}$, then, uniformly in $i$, the theory $\text{Th}(\mathcal{M}_i, \mathcal{L})$ is syntactically interpretable in the theory $\text{Th}(\mathcal{N}_i, \mathcal{L})$.

In some cases, to provide the conclusion of the Theorem above, one can use interpretations with parameters. Suppose, there is a $\mathcal{L}$-formula $\chi(x)$ such that for each tuple $\overline{a}_i$ in the domain of the structure $\mathcal{N}_i$, satisfying $\chi$, the structure $\mathcal{M}_i$ is reconstructible (uniformly) in the structure $\langle \mathcal{N}_i; \overline{a}_i \rangle$ by means of $\mathcal{L}$. Therefore the theory $\text{Th}(\mathcal{M}_i, \mathcal{L})$ is uniformly syntactically interpretable in the theory $\text{Th}(\langle \mathcal{N}_i; \overline{a}_i \rangle, \mathcal{L})$. Let $\theta \mapsto \theta^*(\overline{a})$ be the corresponding mapping of $\mathcal{L}$-sentences. Then the mapping $\theta \mapsto (\forall \overline{a})(\chi(\overline{a}) \to \theta^*(\overline{a}))$

provides a uniform syntactical interpretation $\text{Th}(\mathcal{M}_i, \mathcal{L})$ in $\text{Th}(\langle \mathcal{N}_i; \overline{a}_i \rangle, \mathcal{L})$ [9, Chapter V]. We shall often use such a trick below.

Most of our structures will be multi-sorted; they can be treated as ordinary ones, in a usual way.

1. **Relation Cov**

In his paper [5] Dieudonné used at times the binary relation ‘$y$ is the product of two commuting conjugates of $x$’:

$$\exists z_1 z_2 (y = x^{z_1} x^{z_2} = x^{z_2} x^{z_1}),$$

(1.1)

where $x^z = zxx^{-1}$.

In [23] the author applied the relation given by formula (1.1) in order to prove the following result: if $\text{GL}(\langle \mathcal{N}_\alpha; D_1 \rangle) \equiv \text{GL}(\langle \mathcal{N}_\beta; D_2 \rangle)$, where $D_1$ and $D_2$ are division rings of characteristic $\neq 2$, then $\langle \alpha; < \rangle \equiv \langle \beta; < \rangle$. Note that McKenzie [11] proved the similar result for the class of symmetric groups $S_\alpha = \text{Sym}(\mathcal{N}_\alpha)$.

We say that ‘$\sigma$ covers $\pi$’, if the pair $\langle \sigma, \pi \rangle$ satisfies the formula (1.1). So we denote this formula by $\text{Cov}(x, y)$.

Recall our convention from Section 0: $V$ is the notation for an arbitrary infinite-dimensional vector space over a division ring. The letters $D$ and $\times$ are always associated with $V$, and denote the underlying division ring of $V$ and the dimension of $V$, respectively.
We shall apply below the relation $Cov$ to solve the problem of group-theoretic (first order) characterization of involutions of the first kind in the group $\text{PGL}(V)$; the latter ones will be used later for a reconstruction of the projective space $<P(V); \subseteq>$ in the group $\text{PGL}(V)$. In this section we describe the behaviour of the relation $Cov$ on the set of all $\text{PGL}(V)$-involutions of the first kind.

Again, according to Section 0, a $\gamma$-involution of the group $\text{PGL}(V)$ is the projective image of a $\gamma$-involution of the group $\text{GL}(V)$. The conditions of being a $\gamma$-involution of the group $\text{PGL}(V)$ for some $\gamma$ and of being a $\text{PGL}(V)$-involution of the first kind are clearly equivalent.

We shall denote elements of the group $\text{PGL}(V)$ by lower case Greek letters $\sigma, \pi, \ldots$ and elements of the group $\text{GL}(V)$ by lower case Latin letters $s, p, \ldots$ so that $\sigma = \hat{s}$, $\pi = \hat{p}$, $\rho = \hat{r}$ and so on. It is convenient to agree that in the situation, when $\sigma = \hat{s}$ and $\sigma$ is a $\text{PGL}(V)$-involution of the first kind $s$ is also an involution (in $\text{GL}(V)$). Following this agreement, we state that

**Lemma 1.1.** If $\text{char } D \neq 2$, and $\sigma, \pi$ are involutions in the group $\text{PGL}(V)$, then $\sigma\pi = \pi\sigma$ if and only if $sp = \pm ps$.

**Proof.** Indeed, by 0.1 $\sigma\pi = \pi\sigma$ implies $sp = \mu ps$, where $\mu \in \mathbb{Z}(D)$. Since $p$ induces an involution, $p^2 = \lambda \cdot \text{id}(V)$, where $\lambda \in \mathbb{Z}(D)$. We have

$$(sp^{-1})^2 = (\mu p)^2 \Rightarrow \lambda \cdot \text{id}(V) = \mu^2 \lambda \cdot \text{id}(V).$$

So $\mu^2 = 1$ that is $\mu = \pm 1$. $\Box$

**Lemma 1.2.** Suppose that $\text{char } D \neq 2$, and $\gamma, \gamma'$ are cardinals $\leq \kappa = \text{dim } V$. Then

(a) a $\gamma$-involution in the group $\text{PGL}(V)$ with infinite $\gamma$ covers any $\gamma'$-involution if and only if $\gamma' \leq \gamma$;

(b) a $\gamma$-involution in $\text{PGL}(V)$ with finite $\gamma$ covers any $\gamma'$-involution if and only if $\gamma'$ is finite even cardinal and $\gamma' \leq 2\gamma$.

**Proof.** (a) Assume that a $\gamma$-involution $\sigma$ covers some $\gamma'$-involution $\pi$. This means that $\pi = \sigma_1\sigma_2 = \sigma_2\sigma_1$, where $\sigma_1, \sigma_2$ are conjugate to $\sigma$. Since $\sigma_1\sigma_2 = \sigma_2\sigma_1$, we have by 1.3 $s_1s_2 = \pm s_2s_1$, where $s_k = \hat{s}_k$, $k = 1, 2$. First suppose that $s_1$ and $s_2$ commute. We have $p = \mu s_1s_2$, where $\pi = \hat{p}, \mu \in \mathbb{Z}(D)$. Since $p$ is an involution, then $\mu = \pm 1$. Without loss of generality we can assume that $\mu = 1$, because $(-p)$ induces $\pi$, too.

For an involution $s$, inducing $\sigma$, we have $\kappa = \text{dim } V = \text{dim } V_s^- + \text{dim } V_s^+$. Therefore we can assume that $(\text{dim } V_s^- = \gamma \& \text{dim } V_s^+ = \kappa)$ (otherwise one can use $-s$).

By Lemma 0.3 there is a basis $\{e_i : i < \kappa\}$ of $V$, in which both $s_1$ and $s_2$ are diagonalized. Let $A_k = \{i : s_ke_i = -e_i\}, k = 1, 2$. Then

$$V_p^- = \langle e_i : i \in (A_1 \cup A_2) \setminus (A_1 \cap A_2) \rangle$$

(1.2)
Therefore $\dim V_p^- \leq \gamma$, because $|A_1| = |A_2| = \gamma$. So

$$\gamma' = \min(\dim V_p^-, \dim V_p^+) \leq \gamma.$$

Suppose now that $s_1s_2 = -s_2s_1$. Then $s_2$ is conjugate to $(-s_2)$. Hence the involution $s$ is a $\gamma$-involution. So $\gamma' \leq \gamma$.

Now we prove the converse. Suppose that $\gamma' \leq \gamma$. Choose a basis of $V$ in the form $\{e_i : i \in I_1\} \cup \{e_i : i \in I_2\} \cup \{e_i : i \in I\}$, where $I_1, I_2, I$ are disjoint index sets of powers $\gamma, \gamma'$, and $\kappa$, respectively. We define $s_1$ and $s_2$ as follows:

$$s_1e_i = -e_i, \quad i \in I_1, \quad s_2e_i = -e_i, \quad i \in I_1,$$

$$s_1e_i = e_i, \quad i \in I_2, \quad s_2e_i = -e_i, \quad i \in I_2,$$

$$s_1e_i = e_i, \quad i \in I, \quad s_2e_i = e_i, \quad i \in I.$$

Clearly, $s_1$ and $s_2$ are conjugate and commuting $\gamma$-involutions, and the involution $\widehat{s_1s_2}$ is a $\gamma'$-involution.

Now we prove (b). If a $\gamma$-involution covers some $\gamma'$-involution, then by (1.2) we have

$$\gamma' = \left|(A_1 \cup A_2) \setminus (A_1 \cap A_2)\right| = |A_1| + |A_2| - 2|A_1 \cap A_2| = 2(\gamma - |A_1 \cap A_2|).$$

The converse is obvious. □

**Corollary 1.3.** Suppose that $\operatorname{char} D \neq 2$. Then

(a) Every $\kappa$-involution of $\operatorname{PGL}(V)$ covers every involution of the first kind;

(b) if some $\gamma$-involution covers some $\kappa$-involution, then $\gamma = \kappa$.

**Proof.** By Lemma 1.2 □

2. ‘First kind’ and ‘first order’

In this section we solve the problem of a group-theoretic characterization of involution of the first kind in the group $\operatorname{PGL}(V)$. We show that the set of all involutions of the first kind is a $\emptyset$-definable subset of $\operatorname{PGL}(V)$.

In his book [5] pp. 8-13 Dieudonné showed that the extremal involutions in the projective general linear group $\operatorname{PGL}(W)$ of finite dimension at least three over a division ring of characteristic $\neq 2$ can be distinguished from other involutions of this group by group-theoretic methods.

He applied (if $\dim W \neq 6$) the following techniques. First of these is the use of maximal sets of pairwise commuting and pairwise conjugate involutions ($m$-sets, for short). Second one is (in our terms) the use of the relation $\operatorname{Cov}$. For example, the power of an $m$-set of extremal involutions is less than the power of an $m$-set of $\gamma$-involutions for any $\gamma \geq 2$. The involutions of the second kind either provide the greater powers of $m$-sets or cover (up to conjugacy) more involutions than the extremal ones. In the case $\dim W = 6$ more delicate methods are used.
Our task is more general: we have to distinguish the involutions of the first kind in $\text{PGL}(V)$ from involutions of the second kind by means of first order logic. The method of $m$-sets is (in the general case) not first order. Furthermore, in the infinite-dimensional case it even has not the algebraic efficiency, because, for example, the power of an $m$-set of extremal $\text{PGL}(V)$-involutions coincide with the power of an $m$-set of $n$-involutions for any natural $n$. 

I. Let $D$ be a division ring of characteristic $\neq 2$. The following formula is an obstacle for the involutions of the second kind, because there is no involution of the second kind satisfying it:

$$\text{Ob}(x) = (\exists y_1 y_2 y_3)\left( \bigwedge_{k \neq m} x \sim x^{y_k} x^{y_m} & x = x^{y_1} x^{y_2} x^{y_3} \right),$$

($\sim$ is the conjugacy relation).

**Proposition 2.1.** An involution $\sigma \in \text{PGL}(V)$ satisfies the formula $\text{Ob}$ if and only if $\sigma$ is a $\gamma$-involution, where $\gamma = 4\gamma'$ for some cardinal $\gamma'$.

**Remark.** Therefore $\gamma$ is either an infinite cardinal, or finite, which is a multiple of four.

**Proof.** Suppose for a contradiction that some involution of the second kind $\sigma$ satisfies the formula $\text{Ob}$. Let $\sigma = \hat{s}$, where $s^2 = \lambda \cdot \text{id}(V)$ and $\lambda$ is in $Z(D)$, but is not a square in $Z(D)$.

It is clear that each involution $\sigma'$ which is a conjugate of $\sigma$ is induced (in particular) by a transformation $s'$ such that $s'^2 = \lambda \cdot \text{id}(V)$.

If $\models \text{Ob}[,\sigma]$, then $\sigma$ covers itself, and hence $\sigma = \sigma_1 \sigma_2 = \sigma_2 \sigma_1$, where $\sigma_1$ and $\sigma_2$ are conjugates of $\sigma$ (the second equality holds, since all $\sigma, \sigma_1, \sigma_2$ are involutions). For some $s_1$ and $s_2$, which induce $\sigma_1$ and $\sigma_2$ we have

- $s = \nu \cdot s_1 s_2$, $\nu \in Z(D)$,
- $s_k^2 = \lambda \cdot \text{id}(V)$, $k = 1, 2$,
- $s_1 s_2 = \pm s_2 s_1$.

To verify the latter equality, one can use an argument similar to that used to prove Lemma 1.1.

Therefore

$$s^2 = \nu s_1 s_2 \nu s_1 s_2 = \nu^2 s_1 s_2 s_1 s_2 = \pm \nu^2 s_1^2 s_2^2 = \pm \nu^2 \lambda^2 \cdot \text{id}(V).$$

Thus, there exists $s$ with $\sigma = \hat{s}$ such that $s^2 = -\text{id}(V)$. The condition $\models \text{Ob}[,\sigma]$ also implies that $\sigma$ is a product of three conjugates of $\sigma$: $\sigma = \sigma_1 \sigma_2 \sigma_3$. As we have just observed, $s_k s_m = -s_m s_k$, where $k \neq m$. So for some $\mu \in Z(D)$

$$s^2 = \mu s_1 s_2 s_3 \mu s_1 s_2 s_3 = \mu^2 (-1)^2 s_1^2 s_2 s_3 s_2 s_3$$

$$= \mu^2 (-1)^{2+1} \cdot s_1^2 s_2 s_3^2 = \mu^2 (-1)^6 \text{id}(V) = \mu^2 \text{id}(V).$$

Hence $\sigma$ is an involution of the first kind, a contradiction.
Assume now that $|= Ob[\sigma]$, where $\sigma$ is an involution of the first kind. Therefore $\sigma$ covers itself. By Lemma 1.2 $\sigma$ is a $\gamma$-involution, where the cardinal $\gamma$ is even (in particular, infinite): $\gamma = 2\delta$. Clearly, we should only consider the case $\gamma < \aleph_0$. Let $\sigma = \sigma_1 \sigma_2 \sigma_3$ or $s = \mu_1 s_2 s_3$, where $\sigma_i$ is a conjugate of $\sigma$ and $\mu \in Z(D)$. It is easy to see that $\mu = \pm 1$:

$$s = s_1 s_2 s_3 \text{ or } s = -s_1 s_2 s_3. \quad (2.1)$$

By multiplying both sides of the equations in (2.1) by $-\text{id}(V)$ if necessary we may suppose that $\gamma = \dim V_s^- < \aleph_0$. Furthermore, $s_k s_m = s_m s_k, k \neq m$, because the equation $s_k s_m = -s_m s_k$ holds only for $\infty$-involutions.

Since $s_1, s_2, s_3$ are pairwise commuting, we may apply Lemma 0.3. Therefore the $V^+$-subspace of the involution ($-s_1 s_2 s_3$) has finite dimension, and it cannot be equal to $s$.

Thus, $s = s_1 s_2 s_3$. Consider a basis $\{e_i : i < \infty\}$ in which all $s_1, s_2, s_3$ are diagonalized. Construct (as in the proof of Lemma 1.2) the sets $A_k = \{i : s_k e_i = -e_i\}, k = 1, 2, 3$. It is obvious that the cardinal $\dim V_s^-$ is equal to

$$\begin{align*}
|A_1 \cap A_2 \cap A_3| + |A_1 \setminus (A_2 \cup A_3)| + |A_2 \setminus (A_1 \cup A_3)| + |A_3 \setminus (A_1 \cup A_2)|.
\end{align*}$$

Since $s \sim s_1 s_m k \neq m$, we have $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = \delta$, because, for example,

$$\gamma = \dim V_s^- = |A_1| + |A_2| - 2|A_1 \cap A_2| = 4\delta - 2|A_1 \cap A_2|$$

Therefore

$$\begin{align*}
|A_1 \setminus (A_2 \cup A_3)| &= |A_1| - (|A_1 \cap A_2| + |A_1 \cap A_3| - |A_1 \cap A_2 \cap A_3|) \\
&= |A_1 \cap A_2 \cap A_3|.
\end{align*}$$

Hence $\dim V_s^- = 4|A_1 \cap A_2 \cap A_3|$.

We prove the converse. Let

$$\{e_{i,n} : i < \gamma', n < 7\} \cup \{e_i : i < \infty\},$$

where $\gamma' \leq \infty$ is a cardinal, be a basis of $V$. Consider the involutions $s_1, s_2, s_3 \in \text{GL}(V)$ such that

1. $s_k e_i = e_i$, where $i < \infty$;
2. the following table is realized for all $i < \gamma'$:

$$\begin{array}{cccccccc}
s_1 & s_2 & s_3 & s_1 s_2 s_3 & s_1 s_2 & s_1 s_3 & s_2 s_3 \\
e_{i,0} & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
e_{i,1} & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
e_{i,2} & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
e_{i,3} & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
e_{i,4} & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
e_{i,5} & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
e_{i,6} & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\
\end{array}$$

(a column of the table demonstrates the behaviour of the corresponding transformation on the set $e_{i,0}, e_{i,1}, \ldots, e_{i,6}$). It is easy to check that $s_1$ (a $4\gamma'$-involution) satisfies the formula $Ob$. \hfill \square
Corollary 2.2. \(\kappa\)-involutions of the group \(\text{PGL}(V)\) over a division ring of characteristic \(\neq 2\) are exactly involutions satisfying the formula
\[
K(x) = \text{Ob}(x) \& (\forall y)(\text{Ob}(y) \rightarrow \text{Cov}(x, y)).
\]

Proof. By Corollary 1.3 and Proposition 2.1 \(\square\)

Proposition 2.3. Let \(\sigma\) be an involution of the group \(\text{PGL}(V)\) over a division ring of characteristic \(\neq 2\). Then the following statements are equivalent:
(a) \(\sigma\) is an involution of the first kind;
(b) \(\sigma\) satisfies the formula
\[
FK_1(x) = K(x) \lor (\exists y)(K(y) \& \text{Cov}(y, x) \& \neg \text{Cov}(x, y)).
\]

Proof. (a) \(\Rightarrow\) (b) is obvious by Corollaries 1.3 and 2.2.
(b) \(\Rightarrow\) (a). Assume that \(\models FK_1[\sigma]\) and \(\sigma\) is not a \(\kappa\)-involution. Let \(\pi\) be a \(\kappa\)-involution, covering \(\sigma\). Hence there is a transformation \(s \in \text{GL}(V)\) in the preimage of \(\sigma\), which is a product \(p_1p_2\) of two \(\kappa\)-GL(V)-involutions. We have \(p_1p_2 = \pm p_2p_1\). If \(p_1\) commutes with \(p_2\), then \(s\) is a GL(V)-involution.

So let us consider the case \(p_1p_2 = -p_2p_1\). Clearly \(p_1V_2^\pm = V_2^\pm\), where \(V_2^\pm\) are subspaces of \(p_2\). Choose two bases \(\{e_i : i < \kappa\}\) and \(\{e_{i^*} : i < \kappa\}\) of the subspaces \(V_2^+\) and \(V_2^-\), respectively, such that \(p_1e_i = e_{i^*}, i < \kappa\). So
\[
se_i = p_1p_2e_i = e_{i^*}, \quad i < \kappa,
\]
\[
se_{i^*} = p_1p_2e_{i^*} = -p_1e_i = -e_i.
\]

We shall show that \(\sigma\) covers \(\pi\). Partition the set \(\kappa\) into four subsets \(I_1, I_2, I_3, I_4\) of power \(\kappa\). We define the transformations \(s_1\) and \(s_2\) as follows
\[
s_1e_{i_1} = e_{i_3}, \quad s_2e_{i_1} = e_{i_2}, \quad i_k \in I_k, \quad k \in \{1, 2, 3, 4\},
\]
\[
s_1e_{i_2} = e_{i_4}, \quad s_2e_{i_2} = -e_{i_1},
\]
\[
s_1e_{i_3} = -e_{i_1}, \quad s_2e_{i_3} = e_{i_4},
\]
\[
s_1e_{i_4} = -e_{i_2}, \quad s_2e_{i_4} = -e_{i_3}.
\]

Clearly, \(\sigma_1\) and \(\sigma_2\) are conjugate to \(\sigma\). It is easy to verify that \(s_1s_2 = s_2s_1\).

II. \(D\) is a division ring of characteristic 2. In contrast to the case I, we may apply the methods of Dieudonné from [5]. As a byproduct of his classification of automorphisms of the groups \(\text{PGL}(n, D)\), for \(3 \leq n < \aleph_0\) and \text{char} \(D = 2\), he proved that in this group the conditions ‘to cover itself’ and ‘to be an involution of the first kind’ are equivalent. We transfer this result to infinite dimensions.

Proposition 2.4. Let \text{char} \(D = 2\). Then the set of all \(\text{PGL}(V)\)-involutions covering themselves coincides with the set of all involutions of the first kind.
Proof. If \(\sigma, \pi\) are PGL\((V)\)-involutions then \(\sigma\pi = \pi\sigma\) iff \(sp = ps\). Suppose that a PGL\((V)\)-involution \(\sigma\) of the second kind covers itself. Let \(\sigma = \hat{s}\) and \(s^2 = \lambda \cdot \text{id}(V)\), where \(\lambda\) is an element of \(Z(D)\) which is not a square in \(Z(D)\). If \(\sigma = \sigma_1\sigma_2\) for some \(\sigma_1\) and \(\sigma_2\), which are conjugate to \(\sigma\), then \(s = \mu s_1s_2\), where \(\mu \in Z(D)\) and \(s_k^2 = \lambda \cdot \text{id}(V), k = 1, 2\). Squaring the both parts of the equation \(s = \mu s_1s_2\), we obtain the equation \(\lambda = \mu^2\lambda^2\). So \(\lambda = \mu^{-2}\), but this is impossible. The argument is due to Dieudonné [5, p. 17].

Conversely, we show that every PGL\((V)\)-involution of the first kind covers itself. Let \(s\) be a GL\((V)\)-involution. According to Section 0, there exists a basis \(\{e_i : i \in I\} \cup \{e_j : j \in J\} \cup \{d_i : i \in I\}\) of \(V\) such that
\[
\begin{align*}
s e_i &= e_i, & i &\in I, \\
s e_j &= e_j, & j &\in J, \\
s d_i &= d_i + e_i, & i &\in I.
\end{align*}
\] (2.2)

Assume that \(|I| \leq |J|\). Then the cardinal \(|J|\) coincides with \(\dim V\), in particular, \(|J| \geq \aleph_0\). So one can partition \(J\) into two subsets \(J_0\) and \(J_1\) of powers \(|I|\) and \(|J|\), respectively. Consider \(s_0 \in \text{End}(V)\) such that
\[
\begin{align*}
s_0 e_i &= e_i, & i &\in I, \\
s_0 e_j &= e_j, & j &\in J, \\
s_0 d_i &= d_i + e_i + e_{f(i)}, & i &\in I.
\end{align*}
\]
where \(f\) is a bijection from \(I\) onto \(J_0\). The element \(s_0\) belongs to GL\((V)\), because the system \(\{e_i : i \in I\} \cup \{e_j : j \in J\} \cup \{d_i + e_i + e_{f(i)} : i \in I\}\) is a basis of \(V\). It is easy to check that \(s_0 \sim s\) and \(ss_0 \sim s\). Indeed, \(ss_0\) acts on the vectors of the basis \(\{e_i : i \in I\} \cup \{e_j : j \in J\} \cup \{d_i : i \in I\}\) as follows
\[
\begin{align*}
ss_0 e_i &= e_i, & i &\in I, \\
ss_0 e_j &= e_j, & j &\in J, \\
ss_0 d_i &= d_i + e_{f(i)}, & i &\in I.
\end{align*}
\] (2.3)

Equations (2.3) may be rewritten in the following form:
\[
\begin{align*}
ss_0 e_j &= e_j, & j &\in J_0, \\
ss_0 e_k &= e_k, & k &\in I \cup J_1, \\
ss_0 c_j &= c_j + e_j, & j &\in J_0.
\end{align*}
\]
where \(c_j = d_{f^{-1}(j)}\). The conjugacy of \(s\) and \(ss_0\) follows then from \(|I| = |J_0|\) and \(|I \cup J_1| = |J|\).

Suppose that \(|I| > |J|\). Clearly, \(|I| = \aleph \geq \aleph_0\). Partition \(I\) into two subsets \(I_0\) and \(I_1\) of power \(\aleph\). Consider the transformations \(s_0, s_1 \in \text{GL}(V)\)
such that

\[
\begin{array}{cccc}
  s_0 e_i &=& e_i, & i \in I, \\
  s_0 e_j &=& e_j, & j \in J, \\
  s_0 d_i &=& d_i + e_i + e_f(i), & i \in I_0, \\
  s_0 d_i &=& d_i + e_i, & i \in I_1,
\end{array}
\]

where \( f \) is a bijection from \( I_0 \) onto \( I_1 \). Both \( s_0 \) and \( s_1 \) are conjugates of \( s \) (since, for example, for the former one the system \( \{ e_i + e_f(i) : i \in I_0 \} \cup \{ e_i : i \in I_1 \} \) is a basis of the subspace \( \langle e_i : i \in I = I_0 \cup I_1 \rangle \)). Their product \( s_0 s_1 \) is also a conjugate of \( s \):

\[
\begin{array}{cccc}
  s_0 s_1 e_i &=& e_i, & i \in I, \\
  s_0 s_1 e_j &=& e_j, & j \in J, \\
  s_0 s_1 d_i &=& d_i + e_f(i), & i \in I_0, \\
  s_0 s_1 d_i &=& d_i + e_f^{-1}(i), & i \in I_1.
\end{array}
\]

\( \square \)

**Theorem 2.5.** There exists a first order formula \( FK(x) \) in the group-theoretic language such that \( \text{PGL}(V) \models FK[\sigma] \) if and only if \( \sigma \) is an involution of the first kind.

**Proof.** All we have to do is to construct a sentence \( \theta \) such that \( \text{PGL}(V) \models \theta \) if \( \text{char } D = 2 \). One may take as \( \theta \) the sentence

\[
(\forall x)(x^2 = 1 \rightarrow \text{Cov}(x,x)) \lor (\forall x)(x^2 = 1 \& \text{Cov}(x,x) \rightarrow \text{Ob}(x)) \quad (2.4)
\]

Let \( \text{char } D \neq 2 \). By Lemma [1.2] a 1-involution cannot cover itself. By Lemma [1.2] and Proposition [2.1] a 2-involution can cover itself, but does not satisfy the formula \( \text{Ob} \).

Let now the underlying division ring \( D \) be of characteristic 2 and of power \( \geq 5 \). Then there are two distinct elements \( \mu_1, \mu_2 \) in \( D \setminus \{0,1\} \) such that \( \mu_1 + \mu_2 + 1 \neq 0 \). Let \( \mu_0 = 1 \). So if \( s_0, s_1, s_2 \in \text{GL}(V) \) are defined by the following conditions where \( k \in \{0,1,2\} \)

\[
\begin{array}{cccc}
  s_k e_i &=& e_i, & i \in I, \\
  s_k e_j &=& e_j, & j \in J, \\
  s_k d_i &=& d_i + \mu_k e_i, & i \in I,
\end{array}
\]

then (a) \( s_k \sim s_0, k = 1,2 \), (b) \( s_0 \sim s_0 s_1 s_2 \), (c) any product of two distinct transformations in \( \{ s_0, s_1, s_2 \} \) is conjugate to \( s_0 \). Therefore \( \models \text{Ob}[s_0] \).

Let, finally, \( D \) have characteristic 2 and \( |D| \leq 4 \). By the Wedderburn theorem, \( D \) is a field. As any finite field is perfect, every element in \( D \) is a square. So the group \( \text{PGL}(V) \) has no involutions of the second kind. \( \square \)
3. The reconstruction of the betweenness relation (char $D \neq 2$)

In his book [2] Baer described the automorphisms of the groups $GL(W)$ over division rings of characteristic $\neq 2$. His methods were different from the methods by Mackey–Rickart–Dieudonné. Namely, instead of interpretation of the set $P^{(1)}(W)$, lines and hyperplanes of $W$, in the group $GL(W)$ he reconstructed (in logical terms) by means of monadic second order logic the structure $\langle P(W); B \rangle$; here $B$ is the ternary betweenness relation on the set $P(W)$: $B(L_0, L_1, L_2)$ iff $(L_0 \subseteq L_1 \subseteq L_2)$ or $(L_0 \supseteq L_1 \supseteq L_2)$, where $L_0, L_1, L_2$ are elements of $P(W)$, the projective space over $W$.

In this and in the next sections we show that the structure $P(V) = \langle P(V); \subseteq \rangle$ is reconstructible in the group $PGL(V)$ by means of first order logic. Since the group $PGL(V)$ is interpretable in $GL(V)$, the structure $P(V)$ is reconstructible in $GL(V)$ by means of first order logic. So the above-mentioned results from [2] are significantly strengthened.

First, we shall reconstruct in $PGL(V)$ the structure

$$P^{G'}(V) = \langle PGL(V), P^*(V); \circ, B, act \rangle,$$

where $P^*(V)$ is the set of proper non-zero subspaces of $V$, $\circ$ is the composition law on $PGL(V)$, $B$ is the restriction of the betweenness relation on $P^*(V)$, and $act$ is the ternary relation for the action of the group $PGL(V)$ on the set $P^*(V)$. Then we shall reconstruct $P(V)$ in $P^{G'}(V)$.

**Theorem 3.1.** The structure $P^{G'}(V)$ can be (uniformly in $\dim V$ and $D$) reconstructed without parameters in the group $PGL(V)$ by means of first order logic.

In this section we investigate the case char $D \neq 2$ and in the next section we prove Theorem 3.1 for the case char $D = 2$. In each case we shall realize the following strategy. The first step will be the reconstruction of the set $P^*(V)$. Let $C$ denote the two-placed relation on $P(V)$ defined by the condition $(L_0 \subseteq L_1) \lor (L_0 \supseteq L_1)$. The next step will be a reconstruction of the relation $C$. Having the reconstruction of $C$ one can easily reconstruct $B$, because $B(L_0, L_1, L_2)$ is true iff

$$C(L_0, L_2) \land (\forall L)(L \in P^{(1)}(V) \land C(L, L_0) \land C(L, L_2) \rightarrow C(L, L_1)). \quad (3.1)$$

We have proved the $\varnothing$-definability of the set of all $PGL(V)$-involutions of the first kind (Theorem 2.5). Thus, we may use the variables $x, y, z, \ldots$ only for the involutions of the first kind. Since we shall work later only with the involutions of the first kind, it is convenient to call them simply involutions.

From now on and to the end of this section we suppose that char $D \neq 2$. Convention: the letters $R$ and $S$ will be always used for the subspaces of an involution $\sigma$. Recall that the subspaces of $\sigma$ are identical to the subspaces $V_s^\pm$ of an involution $s \in GL(V)$ in the preimage of $\sigma$. The letter $\rho$ will be
used for the extremal involutions (1-involutions); we shall also always use
the letter \( N \) for the line of an extremal involution \( \rho \), and the letter \( M \) for
its hyperplane. The following fact is simple, but very useful.

**Lemma 3.2.** ([17 Lemma 2.2]). (a) Let \( r, s \) be involutions of the
group \( \text{GL}(W) \) over a division ring of characteristic \( \neq 2 \), and suppose that \( r \)
is extremal. Then \( rs = sr \) if and only if a subspace of \( s \) is contained in the
hyperplane of \( r \) and another subspace of \( s \) contains the line of \( r \);
(b) in particular, extremal \( \text{PGL}(V) \)-involutions \( \rho_1 \) and \( \rho_2 \) commute if
and only if \( (N_1 \subseteq M_2 \& N_2 \subseteq M_1) \) or \( (N_1 = N_2 \& M_1 = M_2) \).

In order to reconstruct \( P^\ast(V) \), we first actually reconstruct \( P^{(1)}(V) \),
interpreting its elements by minimal pairs in \( \text{PGL}(V) \). We start therefore
with

**Lemma 3.3.** The set of all minimal pairs is a \( \varnothing \)-definable subset of the
group \( \text{PGL}(V) \).

**Proof.** As we noted in Section 0, it follows from Theorem 0.5 by Rickart
that, modulo definability of extremal involutions, the set of minimal pairs
is a \( \varnothing \)-definable in \( \text{GL}(V) \). A careful analysis of the conditions of Theorem
0.5 shows that in order to obtain \( \varnothing \)-definability of minimal pairs in \( \text{PGL}(V) \)
only two things need to be done:

- we have to prove \( \varnothing \)-definability of \( \text{PGL}(V) \)-extremal involutions;
- and to construct a first order formula, say, \( \text{Com}(x, y) \) such that \( \models \text{Com}[\sigma, \pi] \), where \( \sigma, \pi \) are involutions (of the first kind) in \( \text{PGL}(V) \),
  if and only if \( \sigma \) and \( \pi \) commute iff they have preimages \( s, p \in \text{GL}(V) \)
  which commute (it may happen that \( sp = -ps \)).

By Lemma 1.2 extremal \( \text{PGL}(V) \)-involutions (1-involutions) cover only
2-involutions, or, in other words, involutions covered by extremal ones are
conjugate. The latter condition is obviously \( \varnothing \)-definable, and hence there
is a first order formula, which will be denoted by \( E_1(x) \), whose realizations
are exactly \( \text{PGL}(V) \)-extremal involutions. (We use the index 1 to specify
the case under consideration: char \( D \neq 2 \); for the same purpose in the next
section we shall use the index 2.)

Let us construct the formula \( \text{Com} \). The key technical step here is a proof
of \( \varnothing \)-definability of the set of all triples \( \langle \sigma; \rho_1, \rho_2 \rangle \) such that the subspaces
of involution \( \sigma \) and the subspaces of extremal involutions \( \rho_1, \rho_2 \) realize the
following configuration:

\[
\{(N_1 \subseteq S \& M_1 \supseteq R) \& (N_2 \subseteq R \& M_2 \supseteq S)\} \lor \quad (3.2)
\{(N_1 \subseteq R \& M_1 \supseteq S) \& (N_2 \subseteq S \& M_2 \supseteq R)\},
\]

that is, the lines of involutions \( \rho_1, \rho_2 \) lie in distinct subspaces of \( \sigma \), and,
dually, their hyperplanes contain distinct subspaces of \( \sigma \).

**Claim 3.4.** The set of all triples \( \langle \sigma, \rho_1, \rho_2 \rangle \) with (3.2) is \( \varnothing \)-definable in
\( \text{PGL}(V) \).
Let us deduce the conclusion of the lemma from the latter Claim. Suppose that a formula \( \vartheta(x; x_1, x_2) \) defines the triples with (3.2).

If involutions \( \sigma, \pi \in \text{PGL}(V) \) as well as some their preimages commute, then \( \pi \) preserves the subspaces of \( \sigma \). So for every pair of extremal involutions \( \langle \rho_1, \rho_2 \rangle \) realizing with \( \sigma \) the configuration (3.2), the triple \( \langle \sigma; \rho_1^\pi, \rho_2 \rangle \) also realizes (3.2).

If \( \sigma \) and \( \pi \) commute, but \( sp = -ps \), then \( \pi \) moves the subspace \( V_{s^-} \) to the subspace \( V_{s^+} \). Therefore if \( \langle \sigma; \rho_1, \rho_2 \rangle \) satisfies (3.2), then \( \langle \sigma; \rho_1^\pi, \rho_2 \rangle \) does not.

Thus, one may take as a formula Com the formula

\[
[x, y] = 1 \land (\forall x_1, x_2)(\vartheta(x; x_1, x_2) \rightarrow \vartheta(x; x_1^\pi, x_2))
\]

Now we prove the Claim. The following formula can serve as the formula \( \vartheta \) characterizing our triples:

\[
\vartheta(x; x_1, x_2) = \bigvee_{k=1}^2 E_1(x_k) \land [x_k, x] = 1 \land (x_1 \neq x_2) \land [x_1, x_2] = 1 \land \bigvee_{k=1}^2 [y, x_k] = 1.
\]

We demonstrate that a triple \( \langle \sigma; \rho_1, \rho_2 \rangle \) realizes (3.2) iff \( \models \vartheta[\sigma; \rho_1, \rho_2] \). The necessity part is easy by Corollary 3.2. Suppose now, towards a contradiction, that the configuration is not realized by a triple \( \langle \sigma; \rho_1, \rho_2 \rangle \), but \( \models \vartheta[\sigma; \rho_1, \rho_2] \). We claim that there exists an extremal involution \( \rho \) commuting with \( \sigma \), but not commuting with both \( \rho_1 \) and \( \rho_2 \). Since \( \rho_1 \neq \rho_2 \& [\rho_1, \rho_2] = 1 \), then \( N_1 \) and \( N_2 \) are distinct lines. Thus, \( [\rho_1, \rho_2] = 1 \) implies \( N_2 \subseteq M_1 \). The line \( N = \langle a_1 + a_2 \rangle \), where \( N_k = \langle a_k \rangle \) and \( k = 1, 2 \), does not lie in \( M_1 \). As (3.2) is not realized by \( \langle \sigma; \rho_1, \rho_2 \rangle \), one can find both \( N_1 \) and \( N_2 \) in a subspace \( S \) of \( \sigma \); another subspace \( R \) of \( \sigma \) is therefore contained in \( M_1 \). Thus, the involution \( \rho \), constructed by the subspaces \( N \) and \( M_1 \), commutes with \( \sigma \). On the other hand, \( \rho \) commutes neither with \( \rho_1 \), nor \( \rho_2 \), because \( (N \neq N_1 \& N \neq N_2) \) and \( (N \not\subseteq M_1 \& N \not\subseteq M_2) \).

The proof of the lemma is now completed. \( \square \)

Let \( MP_1(x_1, x_2) \) denote a first order formula defining the set of all minimal pairs in PGL(V).

A natural generalization of the latter Lemma is

**Theorem 3.5.** *The relation ‘involutions \( \sigma_1 \) and \( \sigma_2 \) have a unique mutual subspace’ is a \( \varnothing \)-definable (uniformly in \( V \)) relation on the group PGL(V).*

**Proof.** Fix a minimal pair \( \varpi_l \), which determines a line, and a pair \( \varpi_l \) non-conjugate to \( \varpi_h \) (which must determine a hyperplane). Let \( \rho \) be an extremal involution, commuting with a non-extremal involution \( \sigma \). Suppose that the line of \( \rho \) lies in a subspace \( S \) of \( \sigma \). Consider the set of extremal
of the pair \( N \) construct an extremal involution \( \rho \) that if \( M \) is the hyperplane of \( \rho \). Clearly, \( R = N_1 + (R \cap M_1) \). Assume that \( N_1 = \langle a_1 \rangle \). Let \( a \) be a non-zero element in \( R \cap M_1 \) and \( R \cap M_1 = \langle a \rangle \). We construct an extremal involution \( \rho_2 \) using the line \( N_1 \) and the hyperplane \( M_2 = \langle a + a_1 \rangle \). The pair \( \langle \rho_1, \rho_2 \rangle \) is obviously minimal \( (M_1 \neq M_2) \); as \( N_1 \subseteq R \) and \( S \subseteq M_2 \), then \( \rho_2 \) commutes with \( \sigma \). Hence \( \rho_1 \) satisfies the condition \( \chi \).

Therefore, if for a pair of involutions \( \langle \sigma_1, \rho_1 \rangle \), where \( \sigma_1 \) is non-extremal, we have \( \models (\forall x)(\chi_1(x; \sigma_1, \rho_1) \rightarrow \chi_1(x; \sigma, \rho)) \), then \( R_1 \subseteq R \). The replacement of the pair \( \pi_1 \) in \( \chi_1 \) by the pair \( \pi_2 \) gives us the condition \( \chi_2(x; \sigma_1, \rho_1) \) such that if \( \models (\forall x)(\chi_2(x; \sigma_1, \rho_1) \rightarrow \chi_2(x; \sigma_1, \rho_1)) \) then \( S_1 \supseteq S \) (the condition dual to \( R_1 \subseteq R \)).

Let us formalize our considerations.

**Claim 3.6.** The relation ‘a subspace of an involution \( \sigma_1 \) is in the relation \( C \) with a subspace of an involution \( \sigma_2 \)’ is a \( \varnothing \)-definable relation on the group \( \text{PGL}(V) \).

**Proof.** Consider the formula

\[
\chi(t; x, \overline{y}, z) = E_1(t) \& MP_1(\overline{y}) \& E_1(z) \& [x, z] = 1 \& (\exists z_1)(\vartheta(x; z, z_1) \& MP_1(z_1, t) \& \overline{y} \sim \langle z_1, t \rangle).
\]

It follows from the above arguments that the formula

\[
(\exists \overline{y}, z_1, z_2)(\forall t)(\chi(t; x, \overline{y}, z_1) \rightarrow \chi(t; x, \overline{y}, z_2))
\]

guarantees the conclusion of Claim 3.6 in the case, when both involutions \( \sigma_1, \sigma_2 \) are non-extremal. In the case, when exactly one involution in the pair \( \langle \sigma_1, \sigma_2 \rangle \) is extremal, we can use the formula

\[
(\exists \overline{y}, z_1, z_2)(\chi(x_2; x_1, \overline{y}, z_1) \vee \chi(x_1; x_2, \overline{y}, z_2)).
\]

Finally, if both involutions \( \sigma_1, \sigma_2 \) are extremal, the condition

\[
(\exists y)(E_1(y) \& ([x_1, y] = 1 \& MP_1(y, x_2)) \vee ([x_2, y] = 1 \& MP_1(y, x_1)))
\]

can be used.

Summing up all the cases, we can easily construct a formula \( \chi'(x_1, x_2) \), providing the definability of the relation specified in the claim. \( \square \)

Let us complete the proof of Theorem 3.5. Put

\[
\chi''(x_1, x_2) = (x_1 \neq x_2) \& (\exists \overline{y}, z_1, z_2)(\forall t)(\chi(t; x_1, \overline{y}, z_1) \leftrightarrow \chi(t; x_2, \overline{y}, z_2))
\]

Hence an arbitrary pair of distinct involutions \( \langle \sigma_1, \sigma_2 \rangle \) has a mutual subspace iff \( \models MS_1[\sigma_1, \sigma_2] \), where

\[
MS_1(x_1, x_2) = MP_1(\overline{x}) \vee (\neg E_1(x_1) \& \neg E_1(x_2) \& \chi''(\overline{x})).
\]
We are ready now to give a proof of Theorem 3.1 for the case \( \text{char } D \neq 2 \).

We shall code subspaces from \( P^*(V) \) by pairs of involutions, satisfying the formula

\[
MS_1^*(x_1, x_2) = MS_1(x_1, x_2) \land (\exists y)(E(y) \land \bigwedge_{k=1}^{2} [x_k, y] = 1).
\]

The formula \( MS_1^* \) is useful due to the following property: if \( \models MS_1^*[\overline{\sigma}] \) and \( S, R_1, R_2 \) are subspaces of \( \sigma_1 \) and \( \sigma_2 \), then there is no involution with the subspaces \( R_1 \) and \( R_2 \), because either they both lie in the same hyperplane or their intersection is at least one-dimensional (Lemma 3.2). Therefore, if \( \models MS_1^*[\overline{\sigma}] \) and \( \models MS_1[\sigma_1, \tau] \& MS_1[\sigma_2, \tau] \), then a subspace of \( \tau \) coincides with \( S \).

Let a pair \( \overline{\sigma} \) satisfy \( MS_1^* \) and \( S(\overline{\sigma}) \) denote by the mutual subspace of \( \sigma_1 \) and \( \sigma_2 \). If \( \theta(x_1, x_2) \) is the formula

\[
(x_1 = x_2) \lor MS_1(x_1, x_2),
\]

then a quadruple of involutions \( \langle \overline{\sigma}, \overline{\pi} \rangle \) satisfies the formula

\[
EP_1(\overline{\sigma}, \overline{\pi}) = MS_1^*(\overline{\sigma}) \& MS_1^*(\overline{\pi}) \& \bigwedge_{i,j=1}^{2} \theta(x_i, y_j),
\]

iff \( S(\overline{\sigma}) = S(\overline{\pi}) \). Indeed, assume that \( \models EP_1[\overline{\sigma}, \overline{\pi}] \), but \( S(\overline{\sigma}) \neq S(\overline{\pi}) \). Let \( (S, S_1) \) and \( (S, S_2) \) be the subspaces of \( \sigma_1 \) and \( \sigma_2 \), respectively; \( (P, P_1) \) and \( (P, P_2) \) be the subspaces of \( \pi_1, \pi_2 \). Without loss of generality we can assume that \( P = S_1 \), therefore \( P_1 = S \), because \( \models MS_1^*[\overline{\sigma}] \). By symmetry \( P_2 = S \). Hence \( \pi_1 = \pi_2 \), and \( \models MS_1^*[\overline{\pi}] \). The converse is easy.

We reconstruct now the relation \( C \). Let \( \overline{\sigma}, \overline{\pi} \) be two pairs of involutions, satisfying \( MS_1^* \). We claim that \( C(S(\overline{\sigma}), S(\overline{\pi})) \) holds iff \( \models C_1[\overline{\sigma}, \overline{\pi}] \), where \( C_1 \) denotes the formula

\[
C_1(\overline{\sigma}, \overline{\pi}) = (\forall \overline{\pi}')(EP_1(\overline{\sigma}, \overline{\pi}) \rightarrow \bigwedge_{i,j=1}^{2} \chi'(z_i, y_j)).
\]

The ‘only if’ part is obvious. To prove the converse, we can use the following fact: there exists a subspace \( R \), a direct complement of \( S(\overline{\sigma}) \) such that the subspace \( R \) is in the relation \( C \) neither with any subspace of \( \pi_1 \), nor with any subspace of \( \pi_2 \).

To reconstruct the betweenness relation \( B \) we may use 3.1.

Clearly, \( \varphi S(\overline{\sigma}) = S(\overline{\pi}) \), where \( \varphi \in \text{PGL}(V) \), holds iff the pairs \( \langle \sigma_1^\varphi, \sigma_2^\varphi \rangle \) and \( \langle \pi_1, \pi_2 \rangle \) satisfy the formula \( EP_1 \). This provides an interpretation of the action of \( \text{PGL}(V) \) on \( P^*(V) \). The proof of Theorem 3.1 for the case \( \text{char } D \neq 2 \) is completed.
4. The reconstruction of the betweenness relation $(\text{char } D = 2)$

*Throughout this section we assume, unless otherwise stated, that char $D = 2$. According to Section 0, one can assign two subspaces $R = R(\sigma)$ and $S = S(\sigma)$ of $V$ with $R \subseteq S$ to each involution $\sigma = \hat{s} \in \text{PGL}(V)$, namely $S = \text{Fix}(s)$ and $R = \text{Rng}(\text{id}(V) + s)$. As in Section 3 we shall call involutions of the first kind just involutions.*

We shall apply the strategy described in the previous section. Our first step is therefore

**Lemma 4.1.** The set of all minimal pairs is a $\emptyset$-definable in $\text{PGL}(V)$.

**Proof.** Proposition 0.6 from Section 0 says that the minimal pairs are $\emptyset$-definable in $\text{GL}(V)$. This time in order to transfer this result to the group $\text{PGL}(V)$ we need only to prove the $\emptyset$-definability of $\text{PGL}(V)$-extremal involutions, because in the case when char $D = 2$ commuting involutions (of the first kind) $\sigma, \pi \in \text{PGL}(V)$ have preimages $s, p \in \text{GL}(V)$ which also commute (and hence there is no need in a formula similar to $\text{Com}$ from the previous section).

The condition ‘there are exactly two conjugacy classes of involutions covered by $x$’ was used by Dieudonné for characterization of the extremal involutions in the projective general linear groups over division rings of characteristic 2 and of finite dimensions at least 6 [5, pp. 13-14]. Let $E_2(x)$ denote a first order sentence corresponding to the mentioned condition. One may prove a $\emptyset$-definability of the extremal involutions in the infinite-dimensional case by a slight modification of Dieudonné’s arguments.

Indeed, we know from Section 2 that any $\text{PGL}(V)$-involution of the first kind covers itself (Proposition 2.4). Any 1-involution can cover some 2-involution:

$$
\begin{align*}
    s_0e_i &= e_i, & s_1e_i &= e_i, & i < \lambda, \\
    s_0d_0 &= d_0 + e_0, & s_1d_0 &= d_0, \\
    s_0d_1 &= d_1, & s_1d_1 &= d_1 + e_1
\end{align*}
$$

($s_0$ and $s_1$ are commuting 1-involutions whose product $s_0s_1$ is a 2-involution).

On the other hand, if $s_0$ and $s_1$ are extremal involutions in $\text{GL}(V)$ then $R(s_0s_1) \subseteq R(s_0) + R(s_1)$ since for every $a \in V$

$$
    s_0s_1a + a = (s_0(s_1a) + s_1a) + (s_1a + a).
$$

Hence $\dim R(s_0s_1) \leq 2$ (we reproduce here an argument from [15, p. 100]). Thus, extremal involutions cannot cover $\gamma$-involutions with $\gamma > 2$, and therefore any extremal involution satisfies $E_2(x)$.

We claim now that every $\gamma$-involution, where $\gamma > 1$, covers involutions in at least three conjugacy classes of involutions, and hence does not satisfy $E_2(x)$.  

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Let first $\gamma$ be a cardinal $>2$. Then any $\gamma$-involution can cover some 1-involution:

(i) \[ s_0e_i = e_i, \quad s_1e_i = e_i, \quad i < \gamma, \]
\[ s_0e_j = e_j, \quad s_1e_j = e_j, \quad j \in J, \]
\[ s_0d_0 = d_0 + e_0, \quad s_1d_0 = d_0 + e_1, \]
\[ s_0d_1 = d_1 + e_1, \quad s_1d_1 = d_1 + e_0, \]
\[ s_0d_i = d_i + e_i, \quad s_1d_i = d_i + e_i, \quad i \in \gamma \setminus 2 \]
as well as some 2-involution:

(ii) \[ s_0e_i = e_i, \quad s_1e_i = e_i, \quad i < \gamma, \]
\[ s_0e_j = e_j, \quad s_1e_j = e_j, \quad j \in J, \]
\[ s_0d_0 = d_0 + e_0, \quad s_1d_0 = d_0 + e_0 + e_1, \]
\[ s_0d_1 = d_1 + e_1, \quad s_1d_1 = d_1 + e_0, \]
\[ s_0d_i = d_i + e_i, \quad s_1d_i = d_i + e_i, \quad i \in \gamma \setminus 2. \]

So any $\gamma$-involution covers 1-involutions, 2-involutions, and $\gamma$-involutions.

In the case when $\gamma = 2$ we show that any 2-involution can cover, for example, some 3-involution (and hence we again have elements from at least three conjugacy classes):

\[ s_0e_i = e_i, \quad s_1e_i = e_i, \quad i < \kappa, \]
\[ s_0e_j = e_j, \quad s_1e_j = e_j, \quad j \in J, \]
\[ s_0d_0 = d_0 + e_0, \quad s_1d_0 = d_0 + e_0 + e_1, \]
\[ s_0d_1 = d_1 + e_1, \quad s_1d_1 = d_1 + e_0, \]
\[ s_0d_2 = d_2, \quad s_1d_2 = d_2 + e_2. \]

□

It is easy to construct a formula (we denote it by $\bar{\pi} \equiv \bar{y}$), which is satisfied by a tuple $\langle \bar{\pi}_1, \bar{\pi}_2 \rangle$, where $\bar{\pi}_1$ and $\bar{\pi}_2$ are minimal pairs iff the subspace determined by $\bar{\pi}_1$ coincides with the subspace determined by $\bar{\pi}_2$. By 0.4(c) the desired formula may be chosen in the following form:

\[ \bigwedge_{i,j=1}^{2} (x_i y_j \sim x_i) \lor (x_i = y_j). \]

Let us fix, as in the previous section, a minimal pair $\bar{\pi}_l$ with a mutual line and a non-conjugate to $\bar{\pi}_l$ minimal pair $\bar{\pi}_h$. Consider the formula

\[ C'_2(x_1, x_2; \bar{y}) = (\forall \bar{x})(\bar{x} \sim \bar{y} \land \bar{x}^{\pi_1} \equiv \bar{x} \rightarrow \bar{x}^{\pi_2} \equiv \bar{x}), \]

where $\bar{y}$ is of length 2.

First step to the reconstruction of binary relation $C$ on $P^*(V)$ is

**Claim 4.2.** Let $\sigma_1$ and $\sigma_2$ be involutions. Then

(a) $\models C'_2[\sigma_1, \sigma_2; \bar{\pi}_l]$ iff $S_1 \subseteq S_2$;
(b) $\models C'_2[\sigma_1, \sigma_2; \bar{\pi}_h]$ iff $R_1 \supseteq R_2$. 

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Proof. Take an arbitrary involution \( \sigma \). If \( N \) is an arbitrary line, then \( \sigma N = N \) iff \( N \subseteq S \); if \( M \) is any hyperplane, then \( \sigma M = M \) iff \( R \subseteq M \). Let us consider the second ‘iff’ statement. Assume \( \sigma = \hat{s} \). Since \( R = \{ a + sa : a \in V \} \subseteq M \), then for every \( m \in M \) the element \( m + sm \) is in \( M \). On the other hand, take an element \( a \notin M \). If suffices to prove that \( a + sa \notin M \). The elements \( a \) and \( sa \) are linearly dependent over \( M \): \( \mu a + sa \in M \) for some non-zero \( \mu \in \mathbb{D} \). Due to the \( s \)-invariance of \( M \) we have that \( \mu sa + a \in M \), and hence \( (1 + \mu^2)a \in M \). As the underlying ring \( \mathbb{D} \) is of characteristic 2, then \( 1 + \mu^2 = (1 + \mu)^2 = 0 \) or \( \mu = 1 \). \( \square \)

So by Claim 4.2(a) a tuple \( \langle \sigma_1, \sigma_2; \overline{m} \rangle \) satisfies the formula

\[
MS_2(x_1, x_2; \overline{\gamma}) = C_2'(x_1, x_2; \overline{\gamma}) \land C_2'(x_2, x_1; \overline{\gamma})
\]

iff \( S_1 = S_2 \) and a tuple \( \langle \sigma_1, \sigma_2; \overline{m}_h \rangle \) satisfies this formula iff \( R_1 = R_2 \).

We have also to build formulæ \( C_2''(x_1, \overline{\gamma}_3; x_2, \overline{\gamma}_2) \) and \( C_2''(x_1, \overline{\gamma}_2; x_2, \overline{\gamma}_3) \) describing the situations of the form \( R_1 \subseteq S_2 \) and \( R_1 \supseteq S_2 \). Note that the situation of the form \( S_1 \subset R_2 \) (strict inclusion) is realized iff the dimension of the underlying vector space is infinite.

We use as a formula \( C_2'' \) the formula

\[
C_2''(x_1, \overline{\gamma}_3; x_2, \overline{\gamma}_2) = (\exists x_3, x_4)[MS_2(x_1, x_3; \overline{\gamma}_3) \land MS_2(x_2, x_4; \overline{\gamma}_2) \land
\{(C_2'(x_3, x_4; \overline{\gamma}_3) \land C_2'(x_3, x_4; \overline{\gamma}_2)) \lor
(C_2'(x_4, x_3; \overline{\gamma}_3) \land C_2'(x_4, x_3; \overline{\gamma}_2))\}]
\]

Claim 4.3. Let \( \sigma_1 \) and \( \sigma_2 \) be involutions. Then
(a) \( \models C_2''[\sigma_1, \overline{m}_3; \sigma_2, \overline{m}_h] \) iff \( S_1 \supseteq R_2 \);
(b) \( \models C_2''[\sigma_1, \overline{m}_h; \sigma_2, \overline{m}_3] \) iff \( R_1 \subseteq S_2 \).

Proof. (a) There are involutions \( \sigma_3, \sigma_4 \) such that \( S_1 = S_3 \) and \( R_2 = R_4 \). On the other hand, \( S_3 \subseteq S_4 \& R_3 \supseteq R_4 \) or \( S_3 \supseteq S_4 \& R_3 \subseteq R_4 \). Since \( S_3 \supseteq R_4 \) and \( S_4 \supseteq R_3 \), then in both cases \( S_1 \supseteq R_2 \).

We prove the converse. Suppose that \( \text{codim } S_1 > \text{dim } R_2 \). Hence we can find subspaces \( L_0 \) and \( L_1 \) such that \( V = S_1 \oplus L_0 \oplus L_1 \) and \( R_2 \cong L_1 \). Then there is an involution \( \sigma_4 \) such that \( S_4 = S_1 \oplus L_0 \) and \( R_4 = R_2 \). An involution \( \sigma_3 \) is constructed as follows: \( S_3 = S_1 \) and \( R_3 \) is a subspace in \( S_1 \) containing \( R_2 \) and isomorphic to \( L_0 \oplus L_1 \). In the case \( \text{codim } S_1 \leq \text{dim } R_2 \) we choose a subspace \( S_4 \) with \( S_4 \supseteq R_2 \) lying in \( S_1 \) such that \( \text{codim } S_4 = \text{dim } R_2 \). An involution \( \sigma_4 \) is constructed by the subspaces \( S_4 \) and \( R_2 \). We choose a subspace \( R_3 \) with \( R_3 \subseteq R_2 \), of dimension equal to \( \text{codim } S_1 \). Then we take as \( \sigma_3 \) an involution with the subspaces \( S_3 = S_1 \) and \( R_3 \).

Part (b) can be proved by analogy with (a). \( \square \)

Let us construct \( C_2'''(x_1, \overline{\gamma}_1; x_2, \overline{\gamma}_2) \):

\[
C_2'''(x_1, \overline{\gamma}_1; x_2, \overline{\gamma}_2) = (\forall x)(MS_2(x, x_1; \overline{\gamma}_1) \to C_2''(x_2, x; \overline{\gamma}_2)).
\]

Claim 4.4. Let \( \sigma_1 \) and \( \sigma_2 \) be involutions. Then
(a) \( \models C_2'''[\sigma_1, \overline{m}_3; \sigma_2, \overline{m}_h] \) iff \( S_1 \subseteq R_2 \);
(b) \( \models C_2'''[\sigma_1, \overline{m}_h; \sigma_2, \overline{m}_3] \) iff \( R_1 \supseteq S_2 \).
Proof. (a) Suppose $S_1 \subseteq R_2$. Consider an involution $\sigma$ such that $S = S_1$. Then $R_2 \supseteq R$, because $R_2 \supseteq S_1 = S \supseteq R$. So $\models C'''$. Conversely, if for each involution $\sigma$ such that $S = S_1$ we have $R_2 \supseteq R$, then $R_2 \supseteq S_1$. Indeed, the sum of all subspaces $R = R(\sigma)$ for such $\sigma$’s is equal to $S_1$.

(b) Suppose $R_1 \supseteq S_2$. We have then $S \supseteq S_2$ for an involution $\sigma$ with $R = R_1$. Hence $\models C'''$. Conversely, if for each involution $\sigma$, such that $R = R_1$ we have $S_2 \subseteq S$, then $S_2 \subseteq R_1$, because the intersection of all $S = S(\sigma)$ for such $\sigma$’s is equal to $R_1$. \hfill $\Box$

Let us now turn to a proof of Theorem 5.1 for the case $\text{char } D = 2$.

The elements of $P^*(V)$ will be interpreted by triples $\overline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1$ is a (non-identity) involution, and $\langle \sigma_2, \sigma_3 \rangle$ is a minimal pair.

In the case when $\langle \sigma_2, \sigma_3 \rangle$ is a minimal pair with a mutual line, we assign to the triple the subspace $S_1 = \text{Fix}(s_1), \sigma_1 = s_1$. Otherwise we assign to the triple $\overline{\sigma}$ the subspace $R_1 = \text{Rng}(\text{id}(V) + s_1)$.

Let $S(\overline{\sigma})$ denote the subspace, which corresponds to a triple of involutions $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$. By Claims 4.2–4.4 $C(S(\overline{\sigma}_1), S(\overline{\sigma}_2))$ iff $\models C_2[\overline{\sigma}_1, \overline{\sigma}_2]$, where $C_2$ denotes the formula

$$
C_2(x_1, \overline{y}_1; x_2, \overline{y}_2) = \begin{cases}
\overline{y}_1 \sim \overline{y}_2 \& C_2'(x_1, \overline{y}_1; x_2, \overline{y}_2) \| \\
\overline{y}_1 \not\sim \overline{y}_2 \& (C_2''(x_1, \overline{y}_1; x_2, \overline{y}_2) \lor C_2'''(x_1, \overline{y}_1; x_2, \overline{y}_2))
\end{cases}
$$

Using the equivalence

$$
L_0 = L_1 \iff (\forall L)(C(L, L_0) \leftrightarrow C(L, L_1))
$$

we may construct a formula $ET_2(x_1, \overline{y}_1; x_2, \overline{y}_2)$, where $|\overline{y}_k| = 2$, such that $\models ET_2[\overline{\sigma}_1; \overline{\sigma}_2]$ iff $S(\overline{\sigma}_1) = S(\overline{\sigma}_2)$. Then, applying (3.1), we may reconstruct the betweenness relation $B$. Having $ET_2$, we may easily reconstruct the action of $\text{PGL}(V)$ on $P^*(V)$. \hfill $\Box$

Thus, we have interpreted by means of first order logic the structure $\mathcal{P}'(V) = \langle \text{PGL}(V), P^*(V); \circ, B, \text{act} \rangle$ in the group $\text{PGL}(V)$ uniformly in $\dim V$ for the case, when char $D \neq 2$ and for the case char $D = 2$. Recall that these cases can be distinguished from one another by a suitable first order sentence (see the formula (2.1)). Using standard techniques, one can therefore build an interpretation of $\mathcal{P}'(V)$ in $\text{PGL}(V)$ which is also uniform in $D$.

5. The reconstruction of the inclusion relation

The following theorem is our crucial result.

**Theorem 5.1.** The projective space $\mathcal{P}(V) = \langle \text{P}(V); \subseteq \rangle$ can be reconstructed without parameters in the projective linear group $\text{PGL}(V)$ by means of first order logic.

In the previous section we reconstructed in $\text{PGL}(V)$ the structure

$$
\mathcal{P}'(V) = \langle \text{PGL}(V), P^*(V); \circ, B, \text{act} \rangle
$$
(one may surely use the corresponding binary relation $C$ instead of the betweenness relation $B$, but this is sometimes less technically convenient).

In this section we shall prove that the inclusion relation on $P^*(V)$ is a definable relation in the structure $\mathcal{P}G''(V)$, that is the structure

$$\mathcal{P}G''(V) = \langle \text{PGL}(V), P^*(V); \circ, \subseteq, \text{act} \rangle$$

can be reconstructed in $\text{PGL}(V)$. Therefore the projective space can be reconstructed in $\text{PGL}(V)$, too, due to

**Claim 5.2.** *The structure $\mathcal{P}(V)$ is $\emptyset$-interpretable in $(P^*(V), \subseteq)$.***

**Proof.** For $R_1, R_2$ in $P^*(V)$ put $f(R_1, R_2) = R_1$ if $R_1 = R_2$, put $f(R_1, R_2) = V$ if $R_1 \subset R_2$ and put $f(R_1, R_2) = \{0\}$ in other cases. Clearly, $f$ maps $P^*(V)^2$ onto $P(V)$, and the $f$-preimages of the equality and inclusion relations are $\emptyset$-definable in $(P^*(V), \subseteq)$.

So let us concentrate our efforts on the proof of following

**Proposition 5.3.** *The inclusion relation on $P^*(V)$ is a $\emptyset$-definable relation on the structure $\mathcal{P}G'(V)$ (uniformly in $V$).*

**Proof.** Clearly, the inclusion relation on $P^*(V)$ is definable in $\mathcal{P}G'(V)$ iff the condition $(\dim L = 1)$ is definable in this structure. Indeed, $L_0 \subseteq L_1$ is equivalent to the condition

$$(\forall L)(\dim L = 1 \& C(L, L_0) \rightarrow C(L, L_1)),$$

where $C(L_0, L_1)$ is an abbreviation for the formula $B(L_0, L_0, L_1)$.

Consider the following function from $P(V)$ to $\mathcal{P}$:

$$\text{dcd} L = \min(\dim L, \text{codim } L).$$

**Lemma 5.4.** *The following conditions are definable in $\mathcal{P}G'(V)$:

(a) $\text{dcd} L = 1$;
(b) $\text{dcd} L \geq \aleph_0$;
(c) $\text{dcd} L = \aleph_0$.*

**Proof.** (a) The condition $\text{dcd} L = 1$ is equivalent to

$$(\forall L_0, L_1)(B(L_0, L, L_1) \rightarrow \bigvee_{k=0}^1 L = L_k).$$

(b) It is easy to see that the condition is equivalent to

$$(\exists \varphi)(\varphi L \neq L \& C(L, \varphi L)).$$

The proof of (c) is based on the following property of each pair $(S_0, S_1) \in P^*(V)$ with

$$(\dim S_0 = \text{codim } S_1 = \aleph_0) \& (S_0 \subseteq S_1):$$

every subspace $S$ of infinite dimension and codimension can be transformed by an element of $\text{PGL}(V)$ to a subspace lying between $S_0$ and $S_1$. 

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Conversely, if a pair \((S_0, S_1)\), where \(\operatorname{dcd} S_0 \geq \aleph_0\) and \(\operatorname{dcd} S_1 \geq \aleph_0\), has the mentioned property, then \(\operatorname{dcd} S_0 = \operatorname{dcd} S_1 = \aleph_0\). Indeed, if a subspace \(S\) with \(\operatorname{dcd} S \geq \aleph_0\) lies between \(S_0\) and \(S_1\), then
\[
\dim S \geq \min\{\dim S_0, \dim S_1\} \geq \aleph_0,
\]
\[
\operatorname{codim} S \geq \min\{\operatorname{codim} S_0, \operatorname{codim} S_1\} \geq \aleph_0.
\]
Since we may choose \(S\) such that \(\dim S = \aleph_0\) or \(\operatorname{codim} S = \aleph_0\), then
\[
\min\{\dim S_0, \dim S_1\} = \min\{\operatorname{codim} S_0, \operatorname{codim} S_1\} = \aleph_0.
\]
Hence \(\operatorname{dcd} S_0 = \operatorname{dcd} S_1 = \aleph_0\). So the condition \(\operatorname{dcd} L = \aleph_0\) is definable, because the equivalent condition
\[
(\operatorname{dcd} L \geq \aleph_0) \& (\exists L_1)(\operatorname{dcd} L_1 \geq \aleph_0) \&
(\forall L_2)(\operatorname{dcd} L_2 \geq \aleph_0 \rightarrow (\exists \varphi)(B(L, \varphi L_2, L_1)))
\]
does.

Let \(\varphi \in \operatorname{PGL}(V)\) and \(\varphi = \hat{f}\). The least \(f\)-invariant subspace, containing a line \(N = \langle a \rangle\), is the subspace \(N_\varphi = \sum_{n \in \mathbb{Z}} \varphi^n N = \langle f^n a : n \in \mathbb{Z} \rangle\). Consider the dual version. Let \(M\) be an arbitrary hyperplane and \(M_\varphi\) denote the greatest \(f\)-invariant subspace, which is contained in \(M\). Clearly, \(M_\varphi\) equals to \(\bigcap_{n \in \mathbb{Z}} \varphi^n M\). One can verify that the subspaces \(N_\varphi\) and \(M_\varphi\) are definable with parameters \(\{N, \varphi\}\) and \(\{M, \varphi\}\), respectively (uniformly for lines and hyperplanes). Indeed, let
\[
\theta(L; L', \psi) = (\psi L = L) \& (C(L, L')) \&
(\forall L'')(\psi L'' = L'' \& C(L', L'')) \rightarrow B(L', L, L'').
\]
It is easy to see that if \(S\) is a line or a hyperplane then \(S_\varphi\) is the unique (if any) realization of the formula \(\theta(L; S, \varphi)\). Thus, we may use in our formulae the expressions of the form \(L_\varphi\) for \(L\) with \(\operatorname{dcd} L = 1\).

Consider the case \(\varkappa = \aleph_0\).

Claim 5.5. Let \(\dim V = \aleph_0\). Then the following conditions are equivalent:
(a) \(S\) is a line;
(b) \(\operatorname{dcd} S = 1\) and for each \(\varphi \in \operatorname{PGL}(V)\) if \(S_\varphi \not\subseteq P^*(V)\) then \(\varphi\) does not preserve any subspace isomorphic to \(S\).

Proof. Let \(S = \langle a \rangle\) be a line and \(\varphi = \hat{f}\). If \(S_\varphi \not\subseteq P^*(V)\), then \(S_\varphi = V = \langle f^n a : n \in \mathbb{Z} \rangle\). So there are no \(f\)-invariant lines in \(V\). Conversely, choose a basis of \(V\) in the form \(\{e\} \cup \{e_n : n \in \mathbb{Z}\}\). Consider \(f \in \operatorname{GL}(V)\) such that
\[
fe = e,
fe_n = e_{n+1}, \quad n \in \mathbb{Z}.
\]

The hyperplane \(\langle e_n : n \in \mathbb{Z} \rangle\) is \(f\)-invariant. Let the hyperplane \(M\) be the span of the set \(\{e_0 + e\} \cup \{e_n : n \neq 0\}\). It is easy to see that \(M_\varphi = \{0\} \not\subseteq P^*(V)\).

□
Clearly, the condition \[5.3\](b) is definable. So the conclusion of Proposition \[5.3\] is true if \(\kappa = \aleph_0\). Let \(\kappa\) be again an arbitrary infinite cardinal.

**Lemma 5.6.** There are \(\varphi \in \text{PGL}(V)\) and a hyperplane \(M \in \mathcal{P}^*(V)\) such that \(\text{dcd} M_\varphi \geqslant \kappa', \) where \(\kappa' = \min\{|D|^\aleph_0, \kappa\}\).

**Proof.** Let \(D\mathbf{Z}\) be the vector space of functions from \(\mathbf{Z}\) to \(D\). Clearly, \(\dim D\mathbf{Z} = |D|^\aleph_0\). Then we may find in \(D\mathbf{Z}\) a linearly independent set of functions \(\{F_j : j < \kappa'\}\) of power \(\kappa'\). Choose a basis \(V\) in the following form:

\[\{a\} \cup \{a_{j,n} : j < \kappa', n \in \mathbf{Z}\} \cup \{b_i : i < \kappa\}.

Consider \(f \in \text{GL}(V)\) acting on the basis as follows

(i) \(fa = a\);
(ii) \(f a_{j,n} = a_{j,n+1}, \ j < \kappa', n \in \mathbf{Z};\)
(iii) \(fb_i = b_i, \ i < \kappa\).

Let \(\varphi = \hat{f}\), and let \(M\) denote the hyperplane with a basis

\[\{a_{j,n} - F_j(-n)a : j < \kappa', n \in \mathbf{Z}\} \cup \{b_i : i < \kappa\}.

Let \(\delta_n : n \in \mathbf{Z}\) be linear functions from \(V\) to \(D\) such that the kernel of \(\delta_n\) is \(\varphi^n M\) and \(\delta_n(a) = 1\). We show that \(\delta_n(a_{j,0}) = F_j(n)\). Since \(f^n(a_{j,-n} - F_j(n)a) = a_{j,0} - F_j(n)a\),

then \(\delta_n(a_{j,0} - F_j(n)a) = 0\) and \(\delta_n(a_{j,0}) = F_j(n)\).

It is obvious that \(b \in M_\varphi\) iff, for all \(n \in \mathbf{Z}\), \(\delta_n(b) = 0\). Let \(b\) be a non-zero element in \(\langle a_{j,0} : j < \kappa'\rangle\). We prove that \(b \notin M_\varphi\). Indeed, if for all \(n \in \mathbf{Z}\), \(\delta_n(b) = 0\), where \(b = \sum_{j \in J} \mu_j a_{j,0}\), then we have for an arbitrary integer \(n\):

\[\delta_n(b) = \delta_n(\sum_{j \in J} \mu_j a_{j,0}) = \sum_{j \in J} \mu_j F_j(n) = 0\]

or \(\sum_{j \in J} \mu_j F_j = 0\). Hence the set \(\{F_j : j \in J\}\) is linearly dependent. \(\square\)

Suppose now that \(\kappa > \aleph_0\). As a consequence of the latter theorem we have that any line \(N\) satisfies the following definable condition:

\[(\text{dcd} L = 1) \& (\exists \varphi)(\exists L')(\text{dcd} L' = 1 \& L' \neq L \& \text{dcd} L'_\varphi > \aleph_0), \]

where \(L_0 \neq L_1\) is an abbreviation for the formula \(\neg(\exists \psi)(\psi L_0 = L_1)\). Conversely, assume \((5.1)\) is satisfied by a hyperplane \(S\). Then there are a line \(S' = \langle a \rangle\) and \(\varphi = f\) such that the cardinal \(\text{dcd} S'_\varphi = \text{dcd} \langle f^n a : n \in \mathbf{Z}\rangle\) is strictly greater than \(\aleph_0\). On the other hand, as \(\kappa > \aleph_0\), we have

\(\text{dcd} \langle f^n a : n \in \mathbf{Z}\rangle \leqslant \aleph_0\).

Therefore the condition \(\dim L = 1\) is a \(\emptyset\)-definable.
A final remark: \( \kappa = \aleph_0 \) iff the structure \( \mathcal{P}(V) \) satisfies the definable condition \( (\forall L)(\text{dec } L \leq \aleph_0) \).

\[ \square \]

6. Semi-linear groups

We devote this and two next sections to a proof of the following theorem.

**Theorem 6.1.** Uniformly in \( \dim V \) and \( D \),

\[
\text{Th}(\Gamma L(V)) \geq \text{Th}(PGL(V)) \geq \text{Th}(PGL(V)) \geq \text{Th}(GL(V)). 
\]

(Recall that \( \geq \) means ‘syntactically interprets’, see Section 0 for the definition).

**Remarks.** (a) We shall prove in Section 9 that the elementary theory of the projective space \( P(V) = \langle P(V); \subseteq \rangle \) syntactically interprets the second order theory \( \text{Th}(\langle \kappa, D \rangle, L_2(\kappa^+)) \) (see the Introduction for the definitions of the latter structure and the logic \( L_2(\kappa^+) \)). Therefore

\[
\text{Th}(PGL(V)) \geq \text{Th}(P(V)) \geq \text{Th}(\langle \kappa, D \rangle, L_2(\kappa^+)).
\]

Under the assumption \( \kappa = \dim V \geq |D| \) the theory \( \text{Th}(\langle \kappa, D \rangle, L_2(\kappa^+)) \) becomes the full second theory of the structure \( \langle \kappa, D \rangle \); one can interpret in this theory the elementary theory of the semi-linear (general linear) group of \( V \):

\[
\text{Th}(\langle \kappa, D \rangle, L_2(\kappa^+)) \geq \text{Th}_2(\langle \kappa, D \rangle) \geq \text{Th}(\Gamma L(V)) \geq \text{Th}(GL(V)).
\]

Then it follows from Theorem 6.1 that, under the assumption \( \dim V \geq |D| \), the elementary theories of the groups \( \Gamma L(V), PTL(V), PGL(V), \) and \( GL(V) \) are pairwise mutually syntactically interpretable, or, in other words, they have the same logical power.

We shall prove in Section 8 that both relations \( \text{Th}(PTL(V)) \geq \text{Th}(\Gamma L(V)) \) and \( \text{Th}(PGL(V)) \geq \text{Th}(GL(V)) \) hold without any assumptions on the dimension of \( V \). Note also that in the general case \( \text{Th}(GL(V)) \not\geq \text{Th}(\Gamma L(V)) \) (Section 12).

(b) The results, we shall consider in Sections 6–8, will not be used for a proof of mutual syntactical interpretability of the elementary theories of the groups \( PGL(V) \) and \( GL(V) \) with the second order theory \( \text{Th}(\langle \kappa, D \rangle, L_2(\kappa^+)) \). So a reader who is not interested in the semi-linear case may now move to Section 9.

We shall prove Theorem 6.1 in three steps following the sign \( \geq \) in (6.1).

To start a proof of the first relation, \( \text{Th}(\Gamma L(V)) \geq \text{Th}(PTL(V)) \), we need some facts on so-called semi-involutions. We shall also use this information in a proof of the second relation, \( \text{Th}(PTL(V)) \geq \text{Th}(PGL(V)) \).

According to Chapter I, Section 3, a transformation \( \sigma \in \Gamma L(W) \) is said to be a *semi-involution* if it induces an involution in the group \( PTL(W) \). Consider a semi-involution \( \sigma \). Clearly, the square of a semi-involution is a radiation: \( \sigma^2 = \lambda \cdot \text{id}(V) \). The semi-involution \( \sigma \) can induce the same
involution in \( \text{PGL}(W) \) as an \textit{involution} \( \pi \in \text{GL}(W) \), that is \( \hat{\sigma} = \hat{\pi} \). It is easy to see that this is possible iff the following condition holds

\[
(\exists \mu \in D)(\lambda = \mu^\sigma \mu).
\]  

(6.2)

(Recall from Section 0 that \( \mu^\sigma \) denotes the action of the associated automorphism of \( \sigma \) on a scalar \( \mu \).) Hence there is no involution in \( \text{GL}(W) \) which induces the involution \( \hat{\sigma} \) iff

\[
(\forall \mu \in D)(\lambda \neq \mu^\sigma \mu).
\]  

(6.3)

Notice the similarity between the concepts we introduce here and the concepts which produce the partition of the set of \( \text{PGL}(W) \)-involutions on involutions of the first kind and involutions of the second kind.

In the next section we shall discuss semi-involutions satisfying (6.3) in more detail. The key fact on semi-involutions satisfying (6.2) is the following

\textbf{Proposition 6.2.} ([6], Chapter I, Section 3). Assume that a semi-involution \( \sigma \in \text{GL}(W) \) satisfies (6.2) that is \( \sigma^2 = l \cdot \text{id}(W) \) and \( l = \mu^\sigma \mu \) for some \( \mu \in D \). Then there exists a basis of \( W \) on which \( \sigma \) acts as the radiation \( \mu \cdot \text{id}(W) \).

As an immediate consequence we have

\textbf{Corollary 6.3.} (a) Every non-linear involution in the group \( \text{GL}(W) \) has a basis of \( W \) which it pointwise fixes.

(b) Non-linear involutions in the group \( \text{GL}(W) \) over division ring \( D \) are conjugate if and only if their associated automorphisms are conjugate in the group \( \text{Aut}(D) \).

Proposition 6.2 is proved in [6] formally for finite-dimensions, but, in fact, the proof works for arbitrary vector space \( W \). The proof of Corollary 6.3 for semi-linear groups of characteristic \( \neq 2 \) can be also found in [2], Chapter VI, Section 6.

\textbf{Theorem 6.4.} \( \text{Th}(\text{GL}(V)) \geq \text{Th}(\text{PGL}(V)) \).

\textbf{Proof.} By Corollary 0.2 it suffices to show that the group \( \text{RL}(V) \) is a \( \emptyset \)-definable subgroup of \( \text{GL}(V) \). For semi-linear groups of characteristic \( \neq 2 \) it follows from the well-known result from [2] (Theorem 5.6 below), the group-theoretic description \( \text{RL}(W) \) in the group \( \text{GL}(W) \). The author has no information about a similar result for the semi-linear groups of characteristic 2. We shall realize in this case a natural geometrical plan which works without any assumptions on the characteristic. Suppose we have proved that the set of all \( \text{GL}(V) \)-minimal pairs is a definable subset in the group \( \text{GL}(V) \). Then the definability of \( \text{RL}(V) \) can be easily deduced from the fact that the radiations and only the radiations preserve all the subspaces in \( P^{(1)}(V) \).

We can use all the results on the behaviour of the relation \( \text{Cov} \) we obtained earlier due to the following simple fact:
Lemma 6.5. Let $\gamma, \gamma'$ be cardinals $\leq \dim V$. Then some $\gamma$-involution covers some $\gamma'$-involution in the group $\text{GL}(V)$ if and only if some $\gamma$-involution of the first kind covers some $\gamma'$-involution of the first kind in $\text{PGL}(V)$.

I. The characteristic of $D$ is not 2. The above mentioned result from [2] is the following

Theorem 6.6. ([2], Chapter VI, Section 6). Let $W$ be a vector space of dimension at least 3 over a division ring of characteristic $\neq 2$. Then the subgroup $\text{RL}(W)$ of the group $\text{GL}(W)$ is the centralizer of the set of all involutions $\sigma$ such that $\sigma \not\sim -\sigma$.

In particular, the group $\text{RL}(V)$ is a definable subgroup of the group $\text{GL}(V)$, because $-\text{id}(V)$ is the unique involution in the center of $\text{GL}(V)$.

The following claim will help us later to distinguish the cases $\text{char } D \neq 2$ and $\text{char } D = 2$.

Claim 6.7. Let $\text{char } D \neq 2$. Then every non-linear involution in $\text{GL}(V)$ covers every linear involution.

Proof. By Corollary 6.3(b), all non-linear involutions with the same associated automorphism are conjugate. Using Corollary 6.3(a), choose a basis $\{e_i : i < \kappa\}$, where $\kappa = \dim V$, which $\sigma$ pointwise fixes. Let $I_1 \cup I_2$ be a partition of $\kappa$. Consider an involution $\sigma_1$, having the same associated automorphism (of the order 2) as $\sigma$ has, and such that

\[ \sigma_1 e_i = -e_i, \quad i \in I_1, \]
\[ \sigma_1 e_i = e_i, \quad i \in I_2. \]

As the associated automorphism of the product of two transformations in the semi-linear group is the product of the associated automorphisms of the factors, the transformation $\sigma_1 \sigma (= \sigma \sigma_1)$ is in $\text{GL}(V)$. \hfill \Box

We could finish the consideration of the case here, but we shall do a little more. After completing a proof of Theorem 6.1 we are going to consider the structure of isomorphisms for infinite-dimensional linear groups of types $\text{GL}$, $\text{PGL}$, $\text{GL}$, and $\text{PGL}$. For this purpose we need a group-theoretic characterization of minimal pairs in these groups.

It is easy to see that $\text{GL}(V)$ is the centralizer of the subgroup $\text{RL}(V)$ in $\text{GL}(V)$. Therefore by Theorem 6.6 $\text{GL}(V)$ is also a definable subgroup of the group $\text{GL}(V)$. According to Lemma 6.5 $\text{GL}(V)$-extremal involutions are exactly those $\text{GL}(V)$-involutions which cover up to conjugacy just one involution, and hence they are $\emptyset$-definable in $\text{GL}(V)$. Then by applying Theorem 6.5 we conclude that

Claim 6.8. Let $\text{char } D \neq 2$. Then the set of all $\text{GL}(V)$-minimal pairs is $\emptyset$-definable in the group $\text{GL}(V)$.

II. The characteristic of $D$ is 2. We have demonstrated in Section 4 that $\text{PGL}(V)$-extremal involutions are exactly $\text{PGL}(V)$-involution of the
first kind that cover, up to conjugacy, only two conjugacy classes of involutions. Hence by Lemma 6.5, the extremal involutions are $\emptyset$-definable in $\text{GL}(V)$. By Proposition 0.6 if the set of extremal involutions is definable, the set of all minimal pairs is definable, too. Therefore,

**Claim 6.9.** If $\text{char } D = 2$, the set of all $\text{GL}(V)$-minimal pairs is $\emptyset$-definable in the group $\Gamma\text{L}(V)$.

Let us prove now a fact similar to Claim 6.7.

**Claim 6.10.** Let $\text{char } D = 2$. Then every non-linear involution in $\Gamma\text{L}(V)$ covers every linear involution.

**Proof.** Let $B = \{ e_i : i \in I \} \cup \{ e_j : j \in J \} \cup \{ d_i : i \in I \}$ be a basis of $V$, where $I \cup J$ is a partition of $V$. Choose non-linear involutions $\sigma_1, \sigma_2 \in \Gamma\text{L}(V)$, having the same associated automorphism such that

1. $\sigma_1$ pointwise fixes $B$;
2. $\sigma_2 e_i = e_i$, $i \in I$,
   $\sigma_2 e_j = e_j$, $j \in J$,
   $\sigma_2 d_i = d_i + e_i$, $i \in I$.

Since $\sigma_1$ and $\sigma_2$ have the same associated automorphism, then $\sigma_1 \sim \sigma_2$. It is easy to see that $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$. According to the description of $\text{GL}(V)$-involutions given in Section 0, an arbitrary involution in $\text{GL}(V)$ can be obtained in this way. □

The first order condition ‘there is an involution covering, up to conjugacy, just one involution’ holds for the group $\Gamma\text{L}(V)$ iff the characteristic of the division ring $D$ is not 2. Indeed, by Claims 6.7 and 6.10 non-linear involutions are out of play. Then we use Lemma 6.5. Hence the interpretation of the theory $\text{Th}(\text{PGL}(V))$ in the theory $\text{Th}(\Gamma\text{L}(V))$ can be done uniformly in $D$. □

**7. PGL(V) is a definable subgroup of PTL(V)**

**Theorem 7.1.** The subgroup $\text{PGL}(V)$ is $\emptyset$-definable in the group $\text{PTL}(V)$, and therefore the theory $\text{Th}(\text{PTL}(V))$ syntactically interprets the theory $\text{Th}(\text{PGL}(V))$.

**Proof.** We start with the well-known description of the subgroup $\text{PGL}(W)$ in the group $\text{PTL}(W)$ of dimension at least 3 [2 Chapter III, Section 2]. Let us discuss key points of this description. Let $\hat{\sigma}$ is an element of $\text{PTL}(V)$. It is first proved that $\hat{\sigma}$ is in the subgroup $\text{PGL}(W)$ of $\text{PTL}(V)$ iff the associated automorphism of $\sigma$ is inner. Then the family of transformations satisfying the condition

$$\exists L (\dim L = 2 \& \forall N (N \subset L \rightarrow \hat{\sigma}N = N)),$$

(7.1)

(the variable $N$ passes through the set of all lines) is considered; all such transformations belong to $\text{PGL}(W)$. The final point of the description of $\text{PGL}(W)$ in $\text{PTL}(W)$ is

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Theorem 7.2. (2 Chapter III, Section 1)] Let \( \dim W \geq 3 \). A transformation \( \hat{\sigma} \) in \( \text{PTL}(W) \) belongs to \( \text{PGL}(W) \) if and only if \( \hat{\sigma} \) is a product of (at most three) transformations satisfying the condition (7.1).

We shall prove that the group \( \text{PGL}(V) \) is a definable subgroup of \( \text{PTL}(V) \). We shall use a weaker version of (7.1). Namely, the second disjunctive term in (7.2) is the condition dual to the condition (7.1). Assume \( \dim W \geq 3 \). Let us show that in the group \( \text{PTL}(W) \) the condition (7.2) holds only for elements \( \hat{\sigma} \in \text{PGL}(W) \). Indeed, suppose there is a subspace \( L \) of codimension 2 such that every hyperplane containing \( L \) is a \( \hat{\sigma} \)-invariant. Let \( e_1, e_2 \) be a pair of linearly independent elements over \( L \). Since the hyperplane \( \langle e_1 \rangle \oplus L \) is \( \sigma \)-invariant, \( \sigma e_1 = \lambda_1 e_1 + m_1 \), where \( m_1 \in L \).

On the other hand, \( \sigma e_2 = \lambda_2 e_2 + m_2 \). The \( \sigma \)-invariance of \( \langle e_1 + e_2 \rangle \oplus L \) gives the equalities \( \lambda = \lambda_1 = \lambda_2 \). Furthermore, if \( m \) is any element of \( D \), we have \( \sigma(e_1 + m e_2) = \nu(e_1 + m e_2) + m \) for some \( \nu \in D \). Let us calculate \( \sigma(e_1 + m e_2) \) in another way: \( \sigma(e_1 + m e_2) = \lambda_1 e_1 + m_1 + \mu^\sigma(\lambda_2 e_2 + m_2) \). This implies \( \nu = \lambda \) and \( \nu \mu = \mu^\sigma \lambda \), and hence \( \lambda \mu \lambda^{-1} = \mu^\sigma \). The associated automorphism of \( \sigma \) is therefore inner, and \( \hat{\sigma} \) is in \( \text{PGL}(W) \).

So Theorem 7.2 remains true if we replace (7.1) with (7.2). The group \( \text{PGL}(V) \) can be therefore interpreted in the structure \( \langle \text{PTL}(V), P^2(V); \circ, B, \text{act} \rangle \), where \( P^2 \) is the set of all subspaces in \( P(V) \) of dimension or codimension \( \leq 2 \). Indeed, the condition of being a line or a hyperplane (or \( \text{dcd} L = 1 \) for short as in Section 5) is surely definable in the latter structure, and hence the following condition does:

\[
(\exists L_0, L_1)(\bigwedge_{k=0}^1 \text{dcd } L_k \neq 1 \& L_0 \neq L_1 \& B(L_0, L_1, L_1) \& (\forall L)(\text{dcd } L = 1 \& B(L, L_0, L_1) \rightarrow \varphi L = L)).
\]

Clearly, this condition is equivalent to the condition (7.2). Thus, we have to interpret the structure \( (P^2; B) \) in \( \text{PTL}(V) \). First we consider some facts on semi-involutions of \( \Gamma L(V) \) in order to distinguish later involutions they induce in \( \text{PTL}(V) \) from involutions of the first kind of \( \Gamma L(V) \).

Recall from the previous section that a semi-involution \( \sigma \in \Gamma L(W) \) with \( \sigma^2 = \lambda \cdot \text{id}(W) \) satisfies the condition

\[
(\forall \mu \in D)(\lambda \neq \mu^\sigma \mu)
\]

iff the involution \( \hat{\sigma} \) is not induced by any involution in the group \( \Gamma L(W) \). If the group \( \Gamma L(W) \) consists of such a semi-involution, then the cardinal \( \dim W \) is necessarily even that is \( \dim W = 2 \gamma \) and there is a basis

\[
\{e_i : i < \gamma\} \cup \{e_i^* : i < \gamma\}
\]
of $W$ such that

\[ \sigma e_i = e_{i^*}, \quad i < \gamma, \]
\[ \sigma e_{i^*} = \lambda e_i. \]

(7.3)

The latter can be proved using arguments from Dieudonné’s book [6, Chapter I, Section 3, A]. We give a simple proof of this fact for the group $\Gamma L(V)$, which does not use the mentioned arguments of Dieudonné.

Take an arbitrary non-zero element $e_0 \in V$. Let $e_0^*$ denote the element $\sigma e_0$. The vectors $e_0$ and $e_0^*$ are linearly independent, because $\sigma e_0 = \mu e_0$ implies $l = \mu^2 \mu$. Suppose now that the system \{ $e_i, e_{i^*} : i < \beta$\}, where $\beta < \kappa$, is linearly independent. As $|\beta| < \kappa = \dim V$, there is a vector $e_\beta \not\in \langle e_i, e_{i^*} : i < \beta \rangle$. Assume $e_\beta^* = \sigma e_\beta$ and show that the system \{ $e_i, e_{i^*} : i \leq \beta$\} is linearly independent. If not, we have

\[ \sigma e_\beta = \sum_{i < \beta} (\mu_i e_i + \nu_i e_{i^*}) + \mu_\beta e_\beta. \]

Applying $\sigma$ to both parts of this equation, we get

\[ \lambda e_\beta = \sum_{i < \beta} (\mu_i^2 e_{i^*} + \nu_i^2 \lambda e_i) + \mu_\beta^2 \sigma e_\beta. \]

Clearly, $\mu_\beta \neq 0$. Therefore

\[ \lambda e_\beta - \sum_{i < \beta} (\mu_i^2 e_{i^*} + \nu_i^2 \lambda e_i) = \sum_{i < \beta} (\mu_\beta^2 \mu_i e_i + \mu_\beta^2 \nu_i e_{i^*}) + \mu_\beta^2 \mu_\beta e_\beta \]

and $\lambda$ must be equal to $\mu_\beta^2 \mu_\beta$, a contradiction.

**Lemma 7.3.** Suppose $\sigma \in \Gamma L(V)$ is a semi-involution such that $\sigma^2 = \lambda \cdot \text{id}(V)$ and $\lambda$ satisfies the condition (6.3). Then the involution $\hat{\sigma}$ covers in $\text{PGL}(V)$ every $2\gamma$-involution of $\text{PGL}(V)$, where $\gamma \leq \kappa = \dim V$.

**Proof.** We use the same method as in the proof of Proposition 2.3. Consider a basis of $V$ of the form

\[ \bigcup_{k=1}^4 \{ e_{i_k} : i_k \in I_k \} \cup \{ e_i : i < \kappa \} \cup \{ e_{i^*} : i < \kappa \}, \]

where the index sets $I_1, I_2, I_3, I_4$, all of power $\gamma$, are disjoint from each other and from $\kappa$. Let $\sigma_1, \sigma_2$ be semi-involutions with the same associated automorphism as $\sigma$ has. Suppose $\sigma_1$ and $\sigma_2$ take the vectors from the basis as follows

\begin{align*}
\sigma_1 e_{i_1} &= e_{i_3}, & \sigma_2 e_{i_1} &= e_{i_2}, & i_k \in I_k, \quad k = 1, 2, 3, 4, \\
\sigma_1 e_{i_2} &= e_{i_4}, & \sigma_2 e_{i_2} &= \lambda e_{i_1}, \\
\sigma_1 e_{i_3} &= \lambda e_{i_1}, & \sigma_2 e_{i_3} &= e_{i_4}, \\
\sigma_1 e_{i_4} &= \lambda e_{i_2}, & \sigma_2 e_{i_4} &= \lambda e_{i_3}, \\
\sigma_1 e_i &= e_{i^*}, & \sigma_2 e_i &= e_{i^*}, & i < \kappa, \\
\sigma_1 e_{i^*} &= \lambda e_i, & \sigma_2 e_{i^*} &= \lambda e_i.
\end{align*}
The transformations $\sigma_1$ and $\sigma_2$ are conjugates of $\sigma$. To check the properties below, one should know that $\lambda^\sigma = \lambda (\sigma^3 = \sigma^2 \sigma = \lambda \cdot \sigma = \sigma \sigma^2 = \lambda^\sigma \sigma)$. The transformation $\tau = \lambda^{-1} \sigma_1 \sigma_2$ sends one to another: (a) $e_{i_1}$ and $\lambda^{-1} e_{i_4}$, (b) $e_{i_2}$ and $e_{i_3}$, where $i_k \in I_k$ and $k = 1, 2, 3, 4$. Check, for example, (a):

\[
\tau e_{i_1} = \lambda^{-1} \sigma_1 \sigma_2 e_{i_1} = \lambda^{-1} \sigma_1 e_{i_2} = \lambda^{-1} e_{i_4},
\]

\[
\tau (\lambda^{-1} e_{i_4}) = \lambda^{-1} \sigma_1 \sigma_2 (\lambda^{-1} e_{i_4}) = \lambda^{-1} \sigma_1 (\lambda^{-1} \cdot \lambda e_{i_4}) = e_{i_1}.
\]

Furthermore, (c) $\tau$ fixes each element in $\{e_i : i < \kappa\} \cup \{e_{i^*} : i < \kappa\}$. It is easy to see that $\sigma_1 \sigma_2 = \sigma_2 \sigma_1$. Since the square of the associated automorphism of $\sigma$ is the inner automorphism $\mu \mapsto \lambda \mu \lambda^{-1}$, and $\mu \mapsto \lambda^{-1} \mu \lambda$ is the associated automorphism of $\lambda^{-1} \id(V)$, then $\tau$ is a linear transformation. According to (a,b,c) above, $\tau$ (and $\hat{\tau}$ as well) is a $2\gamma$-involution. We therefore have $\tau = l^{-1} \sigma_1 \sigma_2 = l^{-1} \sigma_2 \sigma_1$, and then the involution $\hat{\sigma}$ covers $\hat{\tau}$ in $\PGL(V)$.

We shall reconstruct the structure $\langle P^{(2)}(V); B \rangle$ in $\PGL(V)$, using the methods of Section 3 and Section 4. Much the easier case here is the case when

**I. The characteristic of $D$ is 2.** Let us agree that the term ‘a $\gamma$-involution’ means ‘a $\gamma$-involution of the first kind in the subgroup $\PGL(V)$’. We use as the ‘building materials’ for $P^{(2)}(V)$ extremal involution and 2-involutions in $\PGL(V)$.

Our immediate task is therefore to prove the definability of $\PGL(V)$-extremal involutions in $\PGL(V)$. Then we show that 2-involutions are involutions, covered by extremal ones, but not extremal. By Proposition 0.6, if the extremal involutions are definable in $\PGL(V)$, then in $\PGL(V)$ the set of $\PGL(V)$-minimal pairs is definable. As in Section 4, we code the elements of $P^{(2)}(V)$ by triples $\langle \sigma, \sigma_1, \sigma_2 \rangle$, where $\sigma$ is a 1- or 2-involution, and $\langle \sigma_1, \sigma_2 \rangle$ is a minimal pair. The formulae $C_2', C_2^\prime\prime, C_2^\prime\prime\prime$ *mutatis mutandis* retain all needed properties.

The extremal involutions could be distinguished by familiar first order condition ‘to cover, up to conjugacy, only two involutions’ from Section 4 (see the proof of Proposition 4.11). The fact that all non-extremal involutions do not satisfy this condition follows from the results from Section 4 Proposition 6.10 and Proposition 7.3.

**II. The characteristic of $D$ is not 2.** Although the extremal involutions and 2-involutions are still needed, in contrast to the previous case, we have to show that the set of all $\gamma$-involutions of the first kind, or, in other words, the set of all $\gamma$-involutions, where $\gamma$ is an arbitrary cardinal $\leq \kappa$ is definable in the group $\PGL(V)$.

We have to do this, because in Section 3 the proof of definability of $\PGL(V)$-minimal pairs was based on Theorem 0.5. The first order formula $MP_1(x_1, x_2)$, where $x_1$ and $x_2$ represent extremal involutions, and which is a translation of the conditions of the Theorem into a first order logic, requires quantification over the elements of sets $c(x_1, x_2)$ and $c(c(x_1, x_2))$. Recall that
c(I), where $I \subseteq \text{GL}(W)$ is the set of all involutions in the centralizer (in $\text{GL}(W)$) of $I$. On the other hand, if $\sigma_1, \sigma_2$ are $\text{GL}(V)$-extremal involutions, the set $c(\sigma_1, \sigma_2)$ consists of $\gamma$-involutions for every $\gamma \geq 2$ \cite{17} Lemma 2.4. In particular, we cannot restrict ourselves to the work with extremal involutions as in the previous case.

We show the definability in $\text{PTL}(V)$ of the set of all extremal involutions from the subgroup $\text{PGL}(V)$. Since by \cite{12, 6, 7} and \cite{3} any non-extremal involutions covers elements in at least two conjugacy classes of involutions, we can again use the definable condition ‘to cover, up to conjugacy, just one involution’.

To prove the definability of the $\text{PGL}(V)$-involutions of the first kind in $\text{PTL}(V)$ we shall use the same idea as in Section 2. First we prove the definability of the set of $\kappa$-involutions, and then apply Proposition 2.3. Obviously, a $\kappa$-involution can cover only elements in the subgroup $\text{PGL}(V)$ of the group $\text{PTL}(V)$. According to Proposition 2.3 a $\text{PGL}(V)$-involution $\hat{\sigma}$ is of the first kind iff either it is a $\kappa$-involution or it is covered by some $\kappa$-involution $\hat{\pi}$, but does not cover $\hat{\pi}$.

So let us prove the definability of $\kappa$-involutions in $\text{PTL}(V)$. Let the formula $\chi_0(x) = (\exists y)(xy = yx & \neg \text{Com}^*(x, y))$

iff $\gamma = \kappa$.

There could be $\text{PGL}(V)$-involutions of the second kind in the set of all realizations of $\chi_0(x)$. We cut them off using the formula $\text{Ob}$ from Section 2

$$\chi_1(x) = \chi_0(x) & \text{Ob}(x).$$

Clearly, the formula $\chi_1$ are satisfied only by $\kappa$-involutions, and possibly by some involutions in $\text{PTL}(V) \setminus \text{PGL}(V)$. Any involution in $\text{PTL}(V) \setminus \text{PGL}(V)$ can cover all $\kappa$-involutions; any $\kappa$-involution covers itself. Hence the formula

$$\chi_2(x) = \chi_1(x) & (\forall y)(\chi_1(y) \rightarrow \text{Cov}(y, x))$$

is satisfied by every $\kappa$-involution. On the other hand, an involution in $\text{PTL}(V) \setminus \text{PGL}(V)$ cannot satisfy $\chi_2$, since $\kappa$-involutions cover only elements from $\text{PGL}(V)$.

It follows from the results in Section 3 that, having the set of all $\text{PGL}(V)$-involutions of the first kind definable in $\text{PTL}(V)$, we can reconstruct in this group the structure $\langle P^*(V); B \rangle$, and hence its definable reduct $\langle P^{(2)}(V); B \rangle$ (interpreting the structure $\langle P^*(V); B \rangle$ in the group $\text{PGL}(V)$, we worked
inside the set of all PGL(\(V\))-involutions of the first kind, and the only relations on this set we used were the relations ‘\(x\) commutes with \(y\)’ and ‘\(x\) is conjugate to \(y\)’).

The definable condition we used in the end of the previous section – ‘there is an involution covering, up to conjugacy, just one involution’ distinguishes between the cases \(\text{char } D \neq 2\) and \(\text{char } D = 2\). This completes the proof of Theorem 7.1. □

**Proposition 7.4.** The structure \((\text{PGL}(\mathcal{V}), P(\mathcal{V}); \circ, \subseteq, \text{act})\) can be interpreted without parameters in the group PGL(\(V\)) by means of first order logic (uniformly in \(V\)).

**Proof.** By Theorems 7.1 and 5.1 □

8. Overcoming the projectivity

The following theorem proves that the expressive power of first order logic for the infinite-dimensional classical groups is preserved under the taking of the projective image. In particular, \(\text{Th}(\text{PGL}(\mathcal{V})) \geq \text{Th}(\text{GL}(\mathcal{V}))\), and hence the proof of Theorem 6.1 will be completed.

**Theorem 8.1.** Let \(H(\mathcal{V})\) be the group GL(\(\mathcal{V}\)) or the group \(\Gamma \text{L}(\mathcal{V})\), and \(PH(\mathcal{V})\) the projective image of \(H(\mathcal{V})\). Then the theories \(\text{Th}(PH(\mathcal{V}))\) and \(\text{Th}(H(\mathcal{V}))\) are mutually syntactically interpretable.

**Proof.** It is obvious that \(\text{Th}(\text{GL}(\mathcal{V})) \geq \text{Th}(\text{PGL}(\mathcal{V}))\); the relation \(\text{Th}(\Gamma \text{L}(\mathcal{V})) \geq \text{Th}(\text{PGL}(\mathcal{V}))\) has been proved in Theorem 6.4. Then by Theorems 5.1 and 7.4 it suffices to prove the following proposition.

**Proposition 8.2.** Let \(H(\mathcal{V})\) be \(\Gamma \text{L}(\mathcal{V})\) or GL(\(\mathcal{V}\)). Then the theory \(\text{Th}(H(\mathcal{V}))\) is syntactically interpretable in the elementary theory of the structure \((PH(\mathcal{V}), P(\mathcal{V}); \circ, \subseteq, \text{act})\)

**Proof.** Let first \(H(\mathcal{V}) = \Gamma \text{L}(\mathcal{V})\). Suppose that \(L_1^*\) and \(L_2^*\) are elements of \(P(\mathcal{V})\) such that

\[
(\dim L_1^* = 1) \& (L_1^* \oplus L_2^* = V). \tag{8.1}
\]

We shall interpret the elements of \(\Gamma \text{L}(\mathcal{V})\) by transformations \(\varphi \in \text{PGL}(\mathcal{V})\) satisfying the \(\{L_1^*, L_2^*\}\)-definable condition

\[
(\varphi L_1^* = L_1^*) \& (\varphi L_2^* = L_2^*), \tag{8.2}
\]

Clearly, the set \(\Gamma\) of all elements \(\varphi \in \text{PGL}(\mathcal{V})\) satisfying (8.2) is a subgroup of PGL(\(V\)).

Let us construct the interpretation mapping \(\varepsilon\). Fix a non-zero element \(a \in L_1^*\). If some \(\varphi \in \text{PGL}(\mathcal{V})\) satisfies (8.2), then for a transformation \(f \in \Gamma \text{L}(\mathcal{V})\) inducing \(\varphi\), there is a scalar \(\lambda_f \in D\) such that \(fa = \lambda_f a\). Assume that \(\varepsilon(\varphi) = \lambda_f^{-1} f|_{L_2^*}\). It is easy to check that \(\varepsilon\) is well-defined. Indeed, for
any \( f' \) also inducing \( \varphi \), we have \( f' = \mu f \), where \( \mu \in D \). Then \( \lambda_{f'} = \mu \lambda_f \), and hence
\[
\lambda_{f'}^{-1} f'|_{L_2^*} = \lambda_f^{-1} \mu^{-1} \mu f|_{L_2^*} = \lambda_f^{-1} f|_{L_2^*}.
\]

The groups \( \Gamma L(L_2^*) \) and \( \Gamma L(V) \) are evidently isomorphic. On the other hand, we show that

**Claim.** The mapping \( \varepsilon \) is an isomorphism between the groups \( \Gamma \) and \( \Gamma L(L_2^*) \).

**Proof.** Clearly, \( \varepsilon \) is a surjective. Let now \( \varepsilon(\varphi_1) = \varepsilon(\varphi_2) \), where \( \varphi_1, \varphi_2 \in \Gamma \), \( \varphi_k = f_k \), and \( k = 1, 2 \). Suppose that \( f_k a = \lambda_k a \), where \( k = 1, 2 \). The associated automorphism of the transformation \( \lambda^{-1}_{f1} f1 \) is \( \mu \mapsto \lambda^{-1}_{f1} \mu \lambda_1 \).

Thus, if \( \lambda^{-1}_{f1} f1|_{L_2^*} = \lambda^{-1}_{f2} f2|_{L_2^*} \), then \( \lambda^{-1}_{f1} \mu \lambda_1 = \lambda^{-1}_{f2} \mu \lambda_2 \) for every \( \mu \in D \). Let \( m \) be an arbitrary element of \( L_2^* \). We have
\[
\lambda^{-1}_{f1} f1(\mu a + m) = \lambda^{-1}_{f1} \mu \lambda_1 f1(a) + \lambda^{-1}_{f1} f1(m) = \lambda^{-1}_{f1} \mu \lambda_1 a + \lambda^{-1}_{f1} f1(m) = \lambda^{-1}_{f2} \mu \lambda_2 a + \lambda^{-1}_{f2} f2(m) = \lambda^{-1}_{f2} \mu \lambda_2 f2(a) + \lambda^{-1}_{f2} f2(m) = \lambda^{-1}_{f2} f2(\mu a + m).
\]

Hence \( \lambda^{-1}_{f1} f1 = \lambda^{-1}_{f2} f2 \), or \( \varphi_1 = \varphi_2 \).

Assume further that \( \varphi = \varphi_1 \circ \varphi_2 \), where \( \varphi, \varphi_1, \varphi_2 \) are elements of \( \Gamma \). Then \( f = \varphi f_1 \circ f_2 \), where \( \varphi_k = f_k \) and \( k = 1, 2 \). The scalar \( \nu \) is determined by its behavior on \( a \):
\[
f a = \lambda a = \nu f_1 \circ f_2(a) = \nu f_1(\lambda_2 a) = \nu \lambda^{-1}_{f2} \lambda_2 a.
\]

So \( \nu = \lambda \lambda^{-1}_{f1}(\lambda_2^{-1} f1) \), and then \( \lambda^{-1} f = \lambda_1^{-1}(\lambda_2^{-1} f1) f1 \circ f2 \), or \( \varepsilon(\varphi) = \lambda^{-1}_{f1} f1 \circ \lambda^{-1}_{f2} f2 = \varepsilon(\varphi_1) \circ \varepsilon(\varphi_2) \).

Since the set of all pairs \( (L_1^*, L_2^*) \) satisfying (8.1) is \( \emptyset \)-definable, the proof of the Proposition in the case \( H(V) = \Gamma L(V) \) is completed.

Consider now the case \( H(V) = GL(V) \). The choice of parameters should be surely done in a different way. If \( V = N \oplus M \), and for some \( f \in GL(V) \) both subspaces \( N = \langle a \rangle \) and \( M \) are \( f \)-invariant, then it can happen that \( fa = \lambda a \), where the scalar \( \lambda \) is not necessarily in the center of \( D \). In this case the transformation \( \lambda^{-1} f|_M \) is not in \( GL(\mathbb{M}) \).

This difficulty is easily overcome, if we take as \( L_1^* \) and \( L_2^* \) a couple of subspaces satisfying the condition
\[
(\dim L_1^* = 2) \& (L_1^* \oplus L_2^* = V),
\]
and replace the condition (8.2) with
\[
(\forall N)(N \subseteq L_1^* \rightarrow \varphi N = N) \& (\varphi L_2^* = L_2^*), \hspace{1cm} (8.3)
\]
where the variable \( N \) passes as usual through \( P_1(V) \) (we have already used the first conjunctive term in Section 7). If \( \varphi \in PGL(V) \) satisfies (8.3), then \( \varphi \) is induced by an element \( f \in GL(V) \) such that
\[
f a = \lambda_f a, \hspace{1cm} \lambda_f \in Z(D) \text{ for all } a \text{ in } L_1^*.
\]
Hence $\varepsilon(\varphi) = \lambda^{-1}_{f,f} f|_{L^*_2}$ is an isomorphism from the group of all $\varphi$ with (8.3) onto $GL(L^*_2)$. This completes the proof of the Proposition, and hence the proof of Theorem 8.1. □

We close this section with two isomorphism theorems for infinite-dimensional linear groups (both theorems easily follow from general isomorphism theorems proved by O’Meara in [15] Theorem 5.10, Theorem 6.7). We explained in the Introduction the reason we consider these theorems: we prove them by classical methods basing on the machinery developed by Mackey, Dieudonné, and Rickart.

**Theorem 8.3.** Let $H(V)$ be the group $GL(V)$ or the group $\Gamma L(V)$, $V_1$ an infinite-dimensional vector space over a division ring $D_1$, and suppose that the group $H(V_1)$ is of the same type as $H(V)$ is. Then

(a) $H(V) \cong H(V_1)$ if and only if $\langle V, D \rangle \cong \langle V_1, D_1 \rangle$;

(b) every isomorphism $\Lambda$ between the groups $H(V)$ and $H(V_1)$ has the following form

$$\Lambda(\varphi) = \varepsilon(\varphi)g \circ \varphi \circ g^{-1}, \quad \varphi \in H(V), \quad (8.4)$$

where $\varepsilon$ is a homomorphism from $H(V)$ to $RL(V_1)$, and $g$ is a collineation from $V$ onto $V_1$.

**Proof.** Suppose that $\Lambda$ is an isomorphism from the group $H(V)$ onto the group $H(V_1)$. By Corollary 0.2 and Theorems 6.4, 7.1 the isomorphism $\Lambda$ induces, in a natural way, an isomorphism $\Lambda'$ of the groups $PGL(V)$ and $PGL(V_1)$.

We denote by $P\mathcal{G}(V)$ the two sorted-structure, whose first sort is the group $PGL(V)$, the second one is the projective space $\mathcal{P} = \langle P(V); \subseteq \rangle$ and the action of $PGL(V)$ on $P(V)$ is the only new relation added to the basic relations on the sorts. It follows from the proofs of Theorems 3.1 and 5.1 that $\Lambda'$ induces an isomorphism $\Lambda''$ between structures $P\mathcal{G}(V)$ and $P\mathcal{G}(V_1)$. Therefore by the Fundamental Theorem of Projective Geometry, we have (a).

As has been shown by Rickart [18, p. 444-448], $\Lambda$ has the form (8.4), if it sends any $GL(V)$-minimal pair determining a line to a $GL(V_1)$-minimal pair determining a line (but he had no proof that this always takes place; see also the remark below).

By Claims 6.8 and 6.9 the GL(V)-minimal pairs form a $\emptyset$-definable subset in $\Gamma L(V)$. By Theorem 0.5 and Proposition 0.6 such pairs are $\emptyset$-definable in $GL(V)$. Thus, $\Lambda$ preserves the GL-minimal pairs.

Since in the structure $P\mathcal{G}(V)$ the minimal pairs, which determine a line, form a $\emptyset$-definable subset, $\Lambda''$ takes any $PGL$-minimal pair with a mutual line to a $PGL$-minimal pair with a mutual line. Therefore, by the construction of $\Lambda''$, the isomorphism $\Lambda$ must preserve GL-minimal pairs which determine a line. □

**Remark.** Note that, if the underlying vector space is of finite dimension, the set of minimal pairs with a mutual line can be transformed into the set
of minimal pairs with a mutual hyperplane; this provides one more class of isomorphisms, which are not described by the formula (8.4), see, for example, [6, Chapter IV, Section 1]).

**Theorem 8.4.** Let $H(V)$ be the group $\operatorname{PGL}(V)$ or the group $\operatorname{PΓL}(V)$, $V_1$ an infinite-dimensional vector space over a division ring $D_1$, and suppose that the group $H(V_1)$ is of the same type as $H(V)$ is. Then

(a) $H(V) \cong H(V_1)$ if and only if $(V, D) \cong (V_1, D_1)$;

(b) every isomorphism $\Lambda$ between the groups $H(V)$ and $H(V_1)$ has the form

$$\Lambda(\varphi) = g \circ \varphi \circ g^{-1}, \quad \varphi \in H(V),$$

where $g$ is a projective collineation from $P(V)$ onto $P(V_1)$.

**Proof.** It is known that $\Lambda$ has the form (8.5), if it preserves the PGL-minimal pairs which determine a line (it easily follows from the arguments in [6, Chapter IV, Section 1, Section 6]). □

9. **Theories interpretable in** $\text{Th}(P)$

Let $\mathcal{E}$, $\mathcal{P}$, $\mathcal{V}$, and $\mathcal{D}$ denote the endomorphism ring of $V$, the projective space over $V$, the abelian group of vectors of the space $V$, and the division ring $D$, respectively (with their standard relations). We shall construct new multi-sorted structures, by gluing together the structures in the list $\mathcal{E}, \mathcal{P}, \mathcal{V}, \mathcal{D}$.

Thus, $\mathcal{PV}$ denotes the following two-sorted structure: its first sort consists of the elements of $P(V)$, and the second one consists of the elements of $V$; its basic relations are those of $\mathcal{P}$ and $\mathcal{V}$ together with membership relation $\in$ between the elements of $\mathcal{V}$ and $\mathcal{P}$. The elements of the structure $\mathcal{E}\mathcal{PV}$ are divided into three sorts: endomorphisms of $V$, subspaces of $V$, elements of $V$. Its basic relations are those of $\mathcal{E}$, $\mathcal{P}$, and $\mathcal{V}$ together with two ternary relations for the action of $\operatorname{End}(V)$ on $V$ and $P(V)$. We denote by $\mathcal{VD}$ the two-sorted structure whose sorts are $\mathcal{V}$ and $\mathcal{D}$, and the basic relations are those of $\mathcal{V}$ and $\mathcal{D}$ together with the ternary relation for the action of $\mathcal{D}$ on $V$.

The main personage of the remaining part of the paper, the two-sorted structure $(\kappa, D)$, has the following description: its first sort is the cardinal $\kappa$ with no relations, the second one is the division ring $D$ with standard relations, and there are no other relations.

Recall that the logic $L_2(\lambda)$, where $\lambda$ is a cardinal, is a second order logic with quantification over arbitrary relations of power $< \lambda$, and $\text{Mon}(\lambda)$ is its monadic fragment. The main result of this and the two next sections can be informally described as follows: the first order theories of the structures associated above with $V$ have the logical power at least that of the theory of the structure $(\kappa, D)$ (which is ‘algebra-free’ as much as possible) in the logic $L_2(\kappa^+)$ (as ‘strong’ as possible).

**Theorem 9.1.** $\text{Th}(\mathcal{P}) \supseteq \text{Th}(\mathcal{PV}) \supseteq \text{Th}(\mathcal{EPV})$. 
Let us prove the first $\geq$-statement.

**Proposition 9.2.** Th$(\mathcal{P}) \geq$ Th$(\mathcal{PV})$.

**Proof.** The result is essentially known for arbitrary dimensions $\geq 3$ (it follows from the well-known reconstruction the abelian group of vectors of $W$ in the projective space $P(W)$ over $W$ [2, Chapter III]), but we suggest an especially simple proof in the infinite-dimensional case.

Consider two parameters: a line $N^*$ and a hyperplane $M^*$ such that $N^* \oplus M^* = V$. We shall interpret the structure $\mathcal{V}$ in $\mathcal{P}$ with the parameters $N^*$ and $M^*$. Let $a^*$ be a non-zero element in $N^*$. If $a$ is an element of $M^*$, then we denote by $a'$ the line $\langle a + a^* \rangle$. Clearly, $a = 0$ iff $a' = N^*$. Let $\Lambda$ denote the set of all one-dimensional subspaces lying outside $M^*$. It is easy to see that the mapping $'$ is a bijection from $M^*$ onto $\Lambda$. The operation $+$ on $M^*$ induces a binary operation $+'$ on $\Lambda$. We show that $+'$ is $\{N^*, M^*\}$-definable.

Consider a pair $a_1, a_2$ of linearly independent elements of $M^*$. An element $a \in M^*$ coincides with the element $a_1 + a_2$ iff the following hold:

(i) $\{0\} \subset \langle a^*, a_1 \rangle \cap \langle a^* + a, a^* + a_2 \rangle \subseteq M^*$,

(ii) $\{0\} \subset \langle a^*, a_2 \rangle \cap \langle a^* + a, a^* + a_1 \rangle \subseteq M^*$

**Necessity:**

$$\langle a^*, a_1 \rangle \cap \langle a^* + a, a^* + a_2 \rangle = \langle a^*, a_1 \rangle \cap \langle a^* + a_1 + a_2, a^* + a_2 \rangle = \langle a^*, a_1 \rangle \cap \langle a^* + a_2, a_1 \rangle = \langle a_1 \rangle,$$

because of the linear independence of $\{a^*, a_1, a_2\}$.

**Sufficiency.** An element $\lambda_1 (a^* + a) + \lambda_2 (a^* + a_2)$ of the subspace $\langle a^* + a, a^* + a_2 \rangle$ is in $M^*$ iff $\lambda_1 = - \lambda_2$. Hence if (i) holds, then there exist $\lambda, \mu \in D$ such that $\lambda \neq 0$ and $\mu a_1 = \lambda a - \lambda a_2$. Since $\lambda \neq 0$, then $\nu a_1 = a - a_2$ for some $\nu \in D$. By analogy one deduces from (ii) that $\nu' a_2 = a - a_1$ for some $\nu' \in D$. We then have that $\nu a_1 + a_2 = a_1 + \nu' a_2$, and therefore $\nu = \nu' = 1$.

The linear independence of $a_1, a_2$ is equivalent to the following conditions: (a) both $a_1'$ and $a_2'$ are different from $N^*$, and (b) the plane $N^* + a_2'$ does not contain the line $a_2'$. On the other hand, the condition (i) is obviously equivalent to the condition

(i)' $(N^* + a_1') \cap \langle a + a_2 \rangle$ is different from $\{0\}$ and lies in $M^*$.

The condition (ii) can be rewritten in a similar way.

Suppose $a_1, a_2$ are linearly dependent non-zero elements of $M^*$. Then the condition $a = a_1 + a_2$ is equivalent to the following condition: there exist $b, c, d \in M^*$ such that

(a) each of the pairs $\{b, a_1\}, \{c, a_2\}, \{b, c\}$ is linearly independent, and

(b) $b + a_1 = c, c + a_2 = d, b + a = d$.

Thus, we can conclude that the operation $+'$ on $\Lambda$ is $\{N^*, M^*\}$-definable in $\mathcal{P}$.

Assign to every subspace $L \subseteq M^*$ the subspace $L' = L + N^*$. The mapping $L \mapsto L'$ is injective, and the condition $a \in L$ is equivalent to $a' \subseteq L'$. Let $P(M^*)$ be the image of the set $P(M^*)$. The $\{N^*, M^*\}$-definable structure $\langle \Lambda, P(M^*)^\prime; +', \subseteq \rangle$ is isomorphic to $\langle M^*, P(M^*); +, \in \rangle$, and the
latter one is isomorphic to $\mathcal{P}V$. Since the set of all pairs $(N^*, M^*)$, whose sum is $V$, is $\varnothing$-definable in $\mathcal{P}$, the result follows. \hfill $\Box$

**Proposition 9.3.** $\text{Th}(\mathcal{P}V) \supseteq \text{Th}(\mathcal{E}\mathcal{P}V)$.

**Proof.** Let us start with a preliminary remark. To each endomorphism $\varphi$ of a vector space $W$, assign the subspace

$$L_\varphi = \{(a, \varphi a) : a \in W\}$$

of the vector space $W^2$. On the other hand, each direct complement $L$ of the subspace $\{(0, c) : c \in W\}$ in $W^2$ determines some endomorphism $\varphi \in \text{End}(W)$: if a pair $(a, b)$ is in $L$, then put $\varphi a = b$. We check that $\varphi$ is well-defined. Indeed, if two pairs $(a, b)$ and $(a, b')$ are in $L$, then $(0, b - b') \in L$ and $b = b'$. The fact that for every $a \in W$ there exists an element $b \in W$ such that $(a, b) \in L$ follows from a decomposition

$$W^2 = L \oplus \{(0, c) : c \in W\}.$$ 

It is clear also that $\varphi$ is linear.

Since $V$ is infinite-dimensional, the Cartesian square of $V$ is isomorphic to $V$, and it makes sense to realize the above arguments for the reconstruction of $\mathcal{E}\mathcal{P}V$ in $\mathcal{P}V$.

We shall use three parameters: elements $L_1^*, L_2^*, L_3^* \in P(V)$, satisfying the $\varnothing$-definable condition

$$\bigwedge_{i \neq j} L_i \oplus L_j = V. \tag{9.1}$$

One easily verifies that $\dim L_1^* = \text{codim} L_i^* = \omega$, where $i = 1, 2, 3$.

Let $L$ be a direct complement of $L_2^*$ in $V$. The transformation $\sigma_L$ with the graph

$$\{(a, b) : a \in L_1^*, \ b \in L_2^*, \ a + b \in L\}$$

is a linear mapping from $L_1^*$ to $L_2^*$, as we have actually proved above. Moreover, every linear mapping from $L_1^*$ to $L_2^*$ can be constructed in such a way. By (9.1) the transformation $\sigma = \sigma_{L_3^*}$ is bijective. Hence $\varphi_L = \sigma^{-1} \circ \sigma_L$ is an element of $\text{End}(L_1^*)$. Formally, $\varphi_L a = b$ iff the following condition

$$(a, b \in L_1^*) \& (\exists a')(a' \in L_2^* \& a + a' \in L \& b + a' \in L_3^*).$$

is true. The transformations $\varphi_L$ and $\varphi_{L'}$ coincide iff

$$(\forall a)(a \in L_1^* \rightarrow \varphi_L a = \varphi_{L'} a).$$

The analogous arguments may be used for interpretations of the operations $\circ$ and $+$ on $\text{End}(L_1^*)$. Thus, we have reconstructed the first sort of the structure

$$\mathcal{M} = (\text{End}(L_1^*), P(L_1^*), L_1^*),$$

constructed from $L_1^*$ similarly to the construction of $\mathcal{E}\mathcal{P}V$ from $V$. Having the relation $\in$ in the language of $\mathcal{P}V$, we can reconstruct the relation $\subseteq$ on $P(L_1^*)$. We reconstructed $\text{End}(L_1^*)$ with its action on $L_1^*$. Having the action

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of \( \text{End}(L^*_1) \) on \( L^*_1 \), one can obtain the action of \( \text{End}(L^*_1) \) on \( P(L^*_1) \). And, finally, \( M \cong E\mathcal{P}V \), because \( \dim L^*_1 = \dim V \). \( \square \)

**Claim 9.4.** The division ring \( \langle D; +, \cdot \rangle \) can be reconstructed (with parameters from a \( \emptyset \)-definable set) in the structure \( \mathcal{P}V \) by means of first order logic.

**Proof.** Fix a non-zero element \( a^* \in V \). We identify the elements of \( D \) with the elements of the line \( \langle a^* \rangle \). Clearly, \( \langle D; + \rangle \cong \langle \langle a^* \rangle; + \rangle \). In [6, Chapter III, Section 1] Dieudonné proving the Fundamental Theorem of Projective Geometry interprets (algebraically) the division ring \( \langle D; +, \cdot \rangle \) in the projective space \( \langle P(W); \subseteq \rangle \), where \( \dim W \geq 3 \). For the reconstruction of the multiplication the following diagram is used:

![Diagram 1](image)

Let us reconstruct the multiplication on \( D \) basing on the Diagram 1.

We need one more parameter: an element \( b^* \notin \langle a^* \rangle \). Let \( a \in \langle a^* \rangle \) and \( \varepsilon(a) \) be the element of the division ring such that \( a = \varepsilon(a)a^* \). Consider non-zero elements \( a, a_1, a_2 \) of \( \langle a^* \rangle \). We claim that \( \varepsilon(a) = \varepsilon(a_1)\varepsilon(a_2) \) iff \( \{a^*, b^*\}\)-definable condition

\[
\begin{align*}
(\exists y)(y \in \langle b^* \rangle & \& \langle a^* + b^* \rangle = \langle a_1 + y \rangle \& \& \langle a_2 + b^* \rangle = \langle a + y \rangle) \\
(9.2)
\end{align*}
\]

is true. If \( \models (9.2) \), then \( \lambda(a^* + b^*) = \varepsilon(a_1)a^* + \mu b^* \) for some \( \lambda, \mu \in D \), and hence \( \mu = \varepsilon(a_1) \). For some \( \lambda' \in D \) we have \( \lambda'(\varepsilon(a_2)a^* + b^*) = \varepsilon(a)a^* + \varepsilon(a_1)b^* \), and hence \( \varepsilon(a) = \varepsilon(a_1)\varepsilon(a_2) \). The converse is easy. \( \square \)

**10. Recovering a basis**

In the remaining part of the paper we shall suppose that the underlying division ring \( D \) satisfies the following condition:

the number of conjugacy classes of the multiplicative group \( D^* \) is equal to the power of \( D^* \). \( \text{(10.1)} \)

Furthermore, everywhere below the term ‘division ring’ will be understood to mean only a division ring of the mentioned form. As the reader will see later in this section, the condition on \( D \) we introduce gives a natural
way of ‘increasing’ of the logical power of first order theories associated with $V$ in the case when the dimension of $V$ is ‘small’ (less or equal to $|D|$).

**Proposition 10.1.** There exist formulae $\chi(\overline{X}), B(\overline{x}; \overline{X})$ in the language of the structure $\mathcal{EPV}$ such that for every tuple $\overline{A}$ from the domain, satisfying $\chi$, the set $$\{a : \mathcal{EPV} \models B[a; \overline{A}]\}$$ is a basis of $V$.

**Theorem 10.2.** $\text{Th}(\mathcal{EPV}) \supseteq \text{Th}(\langle \kappa, D \rangle, \mathbf{L}_2(\kappa^+))$.

**Proof (assuming 10.1).** Let $\overline{A}$ satisfy $\chi$ in $\mathcal{EPV}$, and $B = B(\mathcal{EPV}, \overline{A})$. Then $B$ is a basis of $V$. Choose in $V$ linearly independent elements $a^*, b^*$ which lie outside $B$. Let $\overline{A}' = \overline{A} \cup \{a^*, b^*\}$. We identify $B$ with the set $\kappa$, and, using Claim 9.4, introduce on the line $\langle a^* \rangle$ a structure which is isomorphic to $D$. Put $B' = B \cup \langle a^* \rangle$; it will be a copy of the domain of the structure $\langle \kappa, D \rangle$.

It is a well-known fact that the logic with quantification over arbitrary partial functions and the full second order logic (which allows quantification over arbitrary relations) are mutually syntactically interpretable. Similarly, since in the case of the logic $\mathbf{L}_2(\kappa^+)$ quantification is allowed only over relations of power $\leq \kappa$, it suffices to interpret in $\mathcal{EPV}$ the set of all partial functions from $B'$ to $B'$ whose domains are of power less or equal to $\kappa$.

We shall interpret these partial functions by triples $\overline{\sigma} = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ of endomorphisms of $V$ such that

(a) $\sigma_0$ sends each element $b$ of $B$ either to $a^*$, or to $b^*$, or to $b$;
(b) $\sigma_1$ maps $B$ to $B$ and the set $B_d(\overline{\sigma}) = \{b \in B : \sigma_0 b = a^*\}$ (the preimage under $\sigma_2$ of the domain of a reconstructible partial function) onto the set $B_r(\overline{\sigma}) = \{b \in B : \sigma_0 b = b^*\}$;
(c) $\sigma_2(B) \subseteq B'$ and its restrictions on $B_d$ and $B_r$ both are injective.

Then, if we substitute any triple $\sigma_0, \sigma_1, \sigma_2$ satisfying (a,b,c) in the $\overline{A}'$-definable scheme $PF$ below for $\varphi_0, \varphi_1, \varphi_2$, we obtain a partial function $x_0 \mapsto x_1$ from $B'$ to $B'$ with the domain of power $\leq \kappa$:

$$PF(x_0, x_1; \varphi_0, \varphi_1, \varphi_2) = \{\exists y_0 \in B_d(\overline{\sigma}) \exists y_1 \in B_r(\overline{\sigma}) (\varphi_1 y_0 = y_1) \& (\varphi_2 y_0 = x_0) \& (\varphi_2 y_1 = x_1)\}.$$  

**Proof of 10.2.** We consider here two cases: $\kappa > |D|$ and $\kappa \leq |D|$.

I. $\kappa > |D|$. We shall use results from the deep paper [22] by Shelah, where he does, as the title of his paper says, ‘interpretation of set theory in the endomorphism semi-group of a free algebra’. Let $C$ be a variety of algebras in some language $\mathcal{L}$. Suppose that $\gamma$ is an infinite cardinal, and $F_\gamma$ is a free algebra with $\gamma$ free generators. Shelah builds a family $\mathcal{L}$-terms, which he calls beautiful terms, satisfying three special conditions (we describe them below) on free algebras of infinite rank in $C$; we just note that in some important cases (e.g. for the variety of abelian groups) the only beautiful
and reduced terms are the terms \( x_k \), where \( k \in \mathbb{N} \). We formulate one of the key technical results from [22] in the following form.

**Lemma 10.3.** ([22] Lemma 4.2). Let \( \gamma \) be an infinite cardinal, which is strictly greater than the power of the language of \( C \). Suppose that \( \mathcal{B} \) freely generates \( F_\gamma \), and write \( \mathcal{B} \) in the form

\[
\{ a_\alpha^\beta : \alpha < \gamma \} \cup \{ b_i : i < \gamma \}.
\]

Then there are a first order formula \( \vartheta[x; \overline{y}] \) in the semi-group language and a tuple \( \overline{a} \) of endomorphisms of \( F_\gamma \) such that \( \text{End}(F_\gamma) = \vartheta[\varphi; \overline{a}] \) if and only if there exist a beautiful term \( t(x_1, \ldots, x_n) \) and ordinals \( \alpha_1, \ldots, \alpha_n < \gamma \), so that

\[
\varphi(a_\alpha^\beta) = t(a_{\alpha_1}^\beta, \ldots, a_{\alpha_n}^\beta)
\]

for every ordinal \( \beta < \gamma \).

One deduces from the latter Lemma that

**Corollary 10.4.** If the only beautiful and reduced terms are the terms \( x_k \) then for any \( \beta < \gamma \) the set

\[
\{ \varphi(a_\alpha^\beta) \in \text{End}(F_\gamma) \mid \vartheta[\varphi; \overline{a}] \} = \{ a_\alpha^\beta : \alpha < \gamma \}
\]

is a subset of the basis \( \mathcal{B} \) of power and copower \( \gamma \).

Fortunately, we have such a very nice situation for the variety of vector spaces over the division ring \( D \). Here the language consists of a two-placed function symbol + and one-placed function symbols \( \{ h_\mu : \mu \in D \} \).

By Shelah’s definition a term \( t(x_1, x_2, \ldots, x_n) \) is said to be beautiful, if

(A) for every term \( q(x_1, x_2, \ldots, x_m) \)

\[
t(q(x_1^1, x_2^1, \ldots, x_m^1), q(x_1^2, x_2^2, \ldots, x_m^2), \ldots, q(x_1^n, x_2^n, \ldots, x_m^n)) =
\]

\[
q(t(x_1^1, x_2^1, \ldots, x_1^n), t(x_2^1, x_2^2, \ldots, x_2^n), \ldots, t(x_m^1, x_m^2, \ldots, x_m^n))
\]

is an identity of every free algebra \( F_\gamma \) of infinite rank in \( C \);

(B)

\[
t(t(x_1^1, x_2^1, \ldots, x_1^n), t(x_2^1, x_2^2, \ldots, x_2^n), \ldots, t(x_m^1, x_m^2, \ldots, x_m^n)) =
\]

\[
t(x_1^1, x_2^1, \ldots, x_n^n)
\]

is an identity of \( F_\gamma \);

(C) \( t(x, x, \ldots, x) = x \) is an identity of \( F_\gamma \).

In the variety of vector spaces over \( D \) every term is equivalent to a term of the form \( \sum_{i=1}^n \mu_i x_i \), where \( \mu_i \in D \).

Clearly, (C) is satisfied only by non-zero terms. Consider a linearly independent set \( \{ e_i^j : i, j \in \mathbb{N} \} \) of power \( \aleph_0 \) in \( V \). If for some non-zero term \( t(x_1, x_2, \ldots, x_n) \) (B) is true, we have

\[
\sum_{i=1}^n \mu_i \sum_{j=1}^n \mu_j e_i^j = \sum_{i=1}^n \mu_i e_i^i.
\]
Therefore \(\mu_i\mu_j = 0\) for \(i \neq j\) and \(\mu_i^2 = \mu_i\) for each \(i = 1, \ldots, n\). Hence the term \(t(x_1, x_2, \ldots, x_n)\) is \(x_k\) for some \(k\).

Thus, Corollary [10.3] and the above arguments imply that for suitable \(a^* \in V\) and \(\varphi^r = (\varphi_1^r, \ldots, \varphi_m^r) \in \text{End}(V)\) we have that the set of all realizations of the formula \(B_1(x) = B_1(x; a^*, \varphi^r) = \exists \varphi(\varphi; \varphi^r) \& x = \varphi a^*\) is a linearly independent set of power \(\kappa = \dim V\) such that the linear span of this set has dimension and codimension \(\kappa\). To explain that the set \(B_1(\mathcal{EPV})\) is linearly independent we write that

\[
(\forall x)\{(B_1(x) \rightarrow (\exists M)(\text{codim } M = 1 \& x \notin M \& (\forall y)((B_1(y) \& y \neq x) \rightarrow y \in M))\}
\]

The linear span \(L^*\) of \(B_1(\mathcal{EPV})\) is the unique realization of the formula

\[
(\forall x)(B_1(x) \rightarrow x \in L) \& (\forall L_1)\{(\forall x)(B_1(x) \rightarrow x \in L_1) \rightarrow (L \subseteq L_1)\}
\]

To explain further that \(\dim L^* = \text{codim } L^* = \kappa\) we need one parameter. It can be an invertible endomorphism \(\pi^* \in \text{End}(V)\) such that

\[\pi^* L^* \cap L^* = \{0\}\] and \((V = \pi^* L^* + L^*)\).

Therefore the set of all realizations of the formula

\[B(x; a^*, \varphi^r, \pi^*) = B_1(x) \lor (\exists y)(B_1(y) \& x = \pi^* y)\]

form a basis of \(V\).

Finally, the tuple of parameters \(\overline{A} = (a^*, \varphi^r, \pi^*)\) can be replaced by any tuple \((b^*, \varphi^r, \rho^*)\) in \(\mathcal{EPV}\) of the same length which satisfies the \(\emptyset\)-definable condition

\[\chi(X) = B(\mathcal{EPV}; \overline{X})\]

is a basis of \(V\).

II. \(\kappa \leq |D|\). Suppose that \(\{\lambda_i : i < \kappa\}\) is a set of pairwise non-conjugate elements of \(D^*\), the multiplicative group of \(D\), and \(B = \{a_i : i < \kappa\}\) is a basis of \(V\) (recall that \(D\) satisfies the condition (10.1)). Let us consider a diagonalizable transformation \(\varphi^* \in \text{End}(V)\) such that

\[\varphi^* a_i = \lambda_i a_i, \quad i < \kappa.\]

It is easy to see that \(\varphi^*\) preserves a line \(N = (\sum_{i \in I} \mu_i a_i)\) if and only if the elements \(\mu_i \lambda_i^{-1}\) are all equal. Hence, by the choice of the elements \(l\), the line \(N\) is of the form \(\langle a_j \rangle\) for a suitable \(j \in I\).

Taking an endomorphism \(\rho^*\) and a non-zero element \(a^* \in V\) such that

\[\rho^* a_i = a^*, \quad \forall i \in I,\]

we obtain that the realizations of the formula

\[(\exists N)\{(\varphi^* N = N) \& (x \in N) \& (\rho^* x = a^*)\}\]

are exactly elements of the basis \(B\). The proof of the case II can be now completed as the proof of the previous case. \(\square\)
11. Theorems on mutual interpretability

In this section we give a long list of pairwise mutually syntactically interpretable $V$-theories. At first we make an effort to close the ‘chain’ of $V$-theories, begun in Sections 9–10 and then add to the constructed ‘chain’ new elements.

Theorem 11.1.

$$\text{Th}(⟨x, D⟩, L_2(x^+)) \supseteq \text{Th}(V D, L_2(x^+)) \supseteq \text{Th}(V D, \text{Mon}(x^+)) \supseteq \text{Th}(P).$$  \hspace{1cm} (11.1)

Proof. The second $\supseteq$-statement in (11.1) is obvious.

Proposition 11.2. $\text{Th}(⟨x, D⟩, L_2(x^+)) \supseteq \text{Th}(V D, L_2(x^+))$.

Proof. $D_{<\omega}^\leq$ is the standard notation for the set of all finite partial functions from $x$ to $D$. Consider the structure $⟨x, D, D_{<\omega}^\leq⟩$ in the language of $⟨x, D⟩$ expanded by an additional predicate symbol to distinguish $D_{<\omega}^\leq$ and a ternary symbol $R$ such that $R(\alpha, \mu, f)$ if and only if $\alpha \in x$, $\mu \in D$, $f \in D_{<\omega}^\leq$, and $f(\alpha) = \mu$.

Taking into account $\forall \equiv \bigoplus_{i<\omega} D$, it is easy to reconstruct $V D$ in $⟨x, D, D_{<\omega}^\leq⟩$ by means of first order logic. Hence $\text{Th}(⟨x, D, D_{<\omega}^\leq⟩, L_2(x^+)) \supseteq \text{Th}(V D, L_2(x^+))$.

Let $\mathcal{M}$ be a structure. $\mathcal{M}^\Pi$ is (quite standard) notation for the structure, with the domain $\bigcup_{n<\omega} R_n(M)$, where $M$ is the domain of $\mathcal{M}$, and $R_n(M), n \in \omega$, is the set of all $n$-placed relations on $M$; here 0-placed relations represent the elements of $M$. The unique $n$-placed ($n \geq 1$) basic relation on $\mathcal{M}^\Pi$ says whether $R(a_1, \ldots, a_{n-1})$ is true or false for any tuple $a_1, \ldots, a_{n-1} \in M$ and an arbitrary element $R \in R_{n-1}(M)$. When we require that our $R_n$ are formed from the relations of power $\leq x$, we obtain the structure $\mathcal{M}_{\Pi}^x$. The elementary theory of the structure $\mathcal{M}_{\Pi}^x$ and the theory of the structure $\mathcal{M}$ in the logic $L_2(x^+)$ are obviously mutually syntactically interpretable.

We prove now that the $L_2(x^+)$-theory of $⟨x, D, D_{<\omega}^\leq⟩$ is syntactically interpretable in $\text{Th}(⟨x, D⟩, L_2(x^+))$. Let $\mathcal{M}^*$ denote the structure $\mathcal{M}_{\Pi}^x$.

We first show that $⟨x, D, D_{<\omega}^\leq⟩^*$ can be reconstructed in $⟨x, D⟩^*$. Since the conditions ‘$A$ is a finite set’ and ‘every injection from $A$ into itself is bijective’ are equivalent, then it is possible to reconstruct the set $D_{<\omega}^\leq$ in $⟨x, D⟩^*$. Every relation of power $\leq x$ on $x \cup D \cup D_{<\omega}^\leq$ can be represented as the image of a function with the domain in $x$. Hence the structure $⟨x, D, D_{<\omega}^\leq⟩^*$ is mutually interpretable with the structure $⟨x, D, D_{<\omega}^\leq, G(x)⟩$, where $G(x)$ is the set of all functions of the form

$$g : x \rightarrow A_1 \times \ldots \times A_n,$$

and $A_i$ is $x$, or $D$, or $D_{<\omega}^\leq$. It can be shown quite easily that the latter structure is bi-interpretable with the structure $⟨x, D, D_{<\omega}^\leq, G_1(x), G_2(x), G_3(x)⟩$, where
where \( G_1(\mathfrak{x}), G_2(\mathfrak{x}), G_3(\mathfrak{x}) \) are the sets of all functions from \( \mathfrak{x} \) to \( \mathfrak{x} \), from \( \mathfrak{x} \) to \( D \), and from \( \mathfrak{x} \) to \( D_{<\omega}^\kappa \), respectively.

So we have only to interpret the set \( G_3(\mathfrak{x}) \) in the structure \( \langle \mathfrak{x}, D \rangle^* \). This can be done as follows. To every function in \( G_3(\mathfrak{x}) \) there corresponds the set \( Q \) in \( \mathfrak{x} \times \mathfrak{x} \times D \) satisfying the definable condition

\[
\{((\beta, \mu) : (\alpha, \beta, \mu) \in Q) \in D_{<\omega}^\kappa \text{ for all } \alpha \in \mathfrak{x} \}.
\]

in \( \langle \mathfrak{x}, D \rangle^* \). Hence \( \text{Th}(\langle \mathfrak{x}, D \rangle, L_2(\mathfrak{x}^+)) \supseteq \text{Th}(\langle \mathfrak{x}, D, D_{<\omega}^\kappa, L_2(\mathfrak{x}^+) \rangle) \). \qed

**Proposition 11.3.** \( \text{Th}(V D, \text{Mon}(\mathfrak{x}^+)) \supseteq \text{Th}(P) \).

**Proof.** Let this time \( V D^* \) be the structure \( \langle V D, \text{Pow}_{\leq \mathfrak{x}}(V D) \rangle \), where \( \text{Pow}_{\leq \mathfrak{x}}(X) \) is the family of all subsets of \( X \) of power \( \leq \mathfrak{x} \). Clearly, the theories \( \text{Th}(V D, \text{Mon}(\mathfrak{x}^+)) \) and \( \text{Th}(V D^*) \) are mutually syntactically interpretable. Let us reconstruct \( P \) in the structure \( V D^* \).

For every subset \( A \subseteq V \) of cardinality \( \leq \mathfrak{x} \), there naturally corresponds the subspace \( \langle A \rangle \), the linear span of \( A \). Since \( \dim V \leq \mathfrak{x} \), all subspaces of \( V \) can be constructed in a such way. Thus, we should prove that the relation ‘\( a \in \langle A \rangle \)’ is definable in the structure \( V D^* \).

Fix a division subring \( K \) of power \( \leq \mathfrak{x} \) in \( D \). One can consider \( V \) as a vector space over \( K \) and, moreover, all the \( K \)-subspaces of \( V \) of dimension \( \leq \mathfrak{x} \) are definable in \( V D^* \) with the parameter \( K \). This therefore implies that the relation ‘\( a \in \langle A \rangle_K \)’ is definable with the parameter \( K \). Hence the relation ‘\( a \in \langle A \rangle \)’ is definable, too, because

\[
a \in \langle A \rangle \iff (\exists K)(K \text{ is a division subring of } D \& \vert K \vert \leq \mathfrak{x} \& a \in \langle A \rangle_K).
\]

Let \( \mathfrak{M} \) denote the structure \( V D \) and \( \text{Th}(\mathfrak{M}, \text{End}), \text{Th}(\mathfrak{M}, \text{Sub}) \) be the theories of this structure in the logics with quantifier over endomorphisms and subspaces of \( V \), respectively.

**Theorem 11.4.** The following theories are pairwise mutually syntactically interpretable: \( \text{Th}(P), \text{Th}(E), \text{Th}(H(V)), \text{Th}(\mathfrak{M}, \text{Sub}), \text{Th}(\mathfrak{M}, \text{End}), \text{Th}(\mathfrak{M}, \text{Mon}(\mathfrak{x}^+)), \text{Th}(\mathfrak{M}, L_2(\mathfrak{x}^+)), \text{Th}(\langle \mathfrak{x}, D \rangle, L_2(\mathfrak{x}^+)), \) where \( H = \text{GL}, \text{PGL}, \text{End}, \text{PEnd} \).

**Proof.** One readily checks that

\[
\text{Th}(P) \subseteq \text{Th}(\mathfrak{M}, \text{Sub}) \subseteq \text{Th}(\mathfrak{M}, \text{End}) \subseteq \text{Th}(P \mathcal{P}V).
\]

On the other hand, by Theorem 9.1

\[
\text{Th}(P) \supseteq \text{Th}(P \mathcal{P}V) \supseteq \text{Th}(\text{End}(V)) \supseteq \text{Th}(\text{PEnd}(V));
\]

using the fact that \( \text{PGL}(V) \) is the group of all invertible elements of \( \text{PEnd}(V) \), and applying then Theorem 5.1 we have that

\[
\text{Th}(\text{PEnd}(V)) \supseteq \text{Th}(\text{PGL}(V)) \supseteq \text{Th}(P).
\]

Finally,

\[
\text{Th}(P) \supseteq \text{Th}(\text{End}(V)) \supseteq \text{Th}(\text{GL}(V)) \supseteq \text{Th}(\text{PGL}(V)) \supseteq \text{Th}(P).
\]
Therefore each theory mentioned in the theorem and the elementary theory of the projective space \( P \) are mutually syntactically interpretable and the result follows. \( \square \)

Consider the logic \( \mathcal{L}(\kappa^+) \) the only difference of which from the logic \( L_2(\kappa^+) \) is an additional quantifier over arbitrary automorphisms of the division ring \( D \).

**Theorem 11.5.** The theories \( \text{Th}(\Gamma L(V)) \), \( \text{Th}(P\Gamma L(V)) \), \( \text{Th}(\langle \kappa, D \rangle, \mathcal{L}(\kappa^+)) \) are pairwise mutually syntactically interpretable.

**Proof.** By Theorem 8.1 \( \text{Th}(\Gamma L(V)) \) and \( \text{Th}(P\Gamma L(V)) \) are mutually interpretable. Let \( B \) be some basis of \( V \), and \( a \) a non-zero element of \( V \). By Theorems 5.1, 6.1, and 10.1 the elementary theory of the structure \( \langle \Gamma L(V), V, \langle a \rangle, B \rangle \) with natural relations is syntactically interpretable in \( \text{Th}(\Gamma L(V)) \).

The subgroup \( \Phi \) of \( \Gamma L(V) \) consisting of all elements of \( \Gamma L(V) \), which satisfies the definable condition

\[
(\forall b \in B)(\varphi b = b),
\]

is isomorphic to the group \( \text{Aut}(D) \). As in Theorems 9.4 and 10.2 we once more identify the set \( \langle B, \langle a \rangle \rangle \) with \( \langle \kappa, D \rangle \) and introduce on \( \langle a \rangle \) a structure isomorphic to \( D \). The subgroup \( \Phi \) acting on \( \langle a \rangle \), interprets the action of the group \( \text{Aut}(D) \) on \( D \). We then use Theorem 10.2 to interpret all the relations on the structure \( \langle \kappa, D \rangle \) of power \( \leq \kappa \).

Conversely, using 11.2 we can interpret in the theory \( \text{Th}(\langle \kappa, D \rangle, \mathcal{L}(\kappa^+)) \) the elementary theory of the structure \( \langle V D_{\Pi^+}^{\kappa^+}, \Phi \rangle \), where \( \Phi \cong \text{Aut}(D) \) and the action of \( \Phi \) on \( V \) is defined. Namely, we define an action of \( \text{Aut}(D) \) on \( D_{<\omega}^\kappa \), and the method of the proof of Proposition 11.2 gives an action of \( \text{Aut}(D) \) on \( \bigoplus_{i<\kappa} D \). Hence we get (a faithful) action of \( \text{Aut}(D) \) on \( V \). Thus, the group \( \text{Aut}(D) \) is now embedded into \( \Gamma L(V) \) and we denote the image under this embedding by \( \Phi \). It is easy to see that \( \Phi \) acts trivially on some basis \( \{e_i : i < \kappa \} \) of \( V \). The group \( \text{GL}(V) \) with its action on \( V \) may be also reconstructed in \( \langle V D_{\Pi^+}^{\kappa^+}, \Phi \rangle \), so we can work with the structure \( \langle V D_{\Pi^+}^{\kappa^+}, \text{GL}(V), \Phi \rangle \). One can build an arbitrary collineation as follows: let \( \{b_i : i < \kappa \} \) be any basis of \( V \) and \( \sigma \in \text{Aut}(D) \), then the transformation

\[
\sigma(\sum \mu_i e_i) = \sum \mu^\sigma_i b_i
\]

is an element of \( \Gamma L(V) \). Clearly, \( \sigma \) is the composition of some element of \( \Phi \) and a transformation of \( \text{GL}(V) \), taking the basis \( \{e_i : i < \kappa \} \) to the basis \( \{b_i : i < \kappa \} \). \( \square \)

Let Mon denote the monadic logic (with quantification over arbitrary subsets).

**Corollary 11.6.** Let \( \kappa \geq |D| \). Then the following theories are pairwise mutually syntactically interpretable: \( \text{Th}(\mathcal{P}) \), \( \text{Th}(\mathcal{E}) \), \( \text{Th}(H(V)) \), \( \text{Th}(\mathcal{M}, \text{Sub}) \), \( \text{Th}(\mathcal{Q}, \text{End}) \), \( \text{Th}(\mathcal{Q}, \text{Mon}) \), \( \text{Th}(\mathcal{W}, L_2) \), \( \text{Th}(\langle \kappa, D \rangle, L_2) \), where \( H = \Gamma L, \text{PGL}, \text{GL}, \text{PGL}, \text{End}, \text{PEnd} \).
Proof. If $\kappa \geq |D|$, then the power of $\langle \kappa, D \rangle$ is the cardinal $\kappa$. Hence
\[
\text{Th}(\langle \kappa, D \rangle, \mathcal{L}(\kappa^+)) \geq \text{Th}(\langle \kappa, D \rangle, \mathcal{L}_2(\kappa^+)) = \text{Th}(\langle \kappa, D \rangle, \mathcal{L}_2) \geq \text{Th}(\langle \kappa, D \rangle, \mathcal{L}(\kappa^+)).
\]
\[\square\]

Corollary 11.7. All first order theories mentioned in Theorems 11.4 and 11.5 are unstable and undecidable.

Consider two infinite-dimensional vector spaces $V_1$ and $V_2$ over division rings $D_1$ and $D_2$, respectively. Assume also that $\kappa_1 = \dim V_1$ and $\kappa_2 = \dim V_2$.

Theorem 11.8. (a) Let $H = \text{GL}$, PGL, End, or PEnd. Then the following conditions are equivalent:

(i) $H(V_1) \equiv H(V_2)$;
(ii) $(P(V_1), \subseteq) \equiv (P(V_2), \subseteq)$;
(iii) $\mathcal{E}(V_1) \equiv \mathcal{E}(V_2)$;

(iv) $\text{Th}(\langle \kappa_1, D_1 \rangle, \mathcal{L}_2(\kappa_1^+)) = \text{Th}(\langle \kappa_2, D_2 \rangle, \mathcal{L}_2(\kappa_2^+))$.

(b) Let $H = \Gamma \Lambda$, PGL. Then the condition $H(V_1) \equiv H(V_2)$ is equivalent to

\[
\text{Th}(\langle \kappa_1, D_1 \rangle, \mathcal{L}(\kappa_1^+)) = \text{Th}(\langle \kappa_2, D_2 \rangle, \mathcal{L}(\kappa_2^+));
\]
in particular, the condition $H(V_1) \equiv H(V_2)$ implies

\[
\text{Th}(\langle \kappa_1, D_1 \rangle, \mathcal{L}_2(\kappa_1^+)) = \text{Th}(\langle \kappa_2, D_2 \rangle, \mathcal{L}_2(\kappa_2^+))
\]

Proof. Use Theorem 11.4 for (a) and Theorem 11.5 for (b). \[\square\]

12. Examples

Throughout this section $\kappa$, $\kappa'$ are infinite cardinals, and $D, D'$ are division rings. Let $T(\kappa, D)$ denote by the theory $\text{Th}(\langle \kappa, D \rangle, \mathcal{L}_2(\kappa^+))$. In this section we discuss a number of natural conditions, necessary/sufficient for

\[T(\kappa, D) = T(\kappa', D').\] (12.1)

We shall also investigate the logical strength of the elementary theories of infinite-dimensional semi-linear groups over algebraically closed fields. This will enable us to prove that the condition $T(\kappa, D) = T(\kappa', D')$ – necessary and sufficient for the elementary equivalence of groups of types GL and PGL – is not sufficient for the elementary equivalence for groups of types $\Gamma \Lambda$ and PGL.

Claim 12.1. The following conditions are necessary for $T(\kappa, D) = T(\kappa', D')$:

(a) $\kappa = |D| \leftrightarrow \kappa' = |D'|$;
(b) $\kappa > |D| \leftrightarrow \kappa' > |D'|$;
(c) $\kappa < |D| \leftrightarrow \kappa' < |D'|$;
(d) $\kappa \equiv_{\mathcal{L}_2} \kappa'$;
(e) $\text{Th}(D, \mathcal{L}_2(\kappa^+)) = \text{Th}(D', \mathcal{L}_2(\kappa'^+))$. 51
that of a bijection between $\kappa$ and $D$. Similar arguments prove (b) and (c). (d) It follows from $T(\kappa, D) = T(\kappa', D')$ that $\text{Th}(\kappa, D) = \text{Th}(\kappa', D')$, that is $\text{Th}_2(\kappa) = \text{Th}_2(\kappa')$.  

Under the condition $\kappa \geq |D|$ the theory $T(\kappa, D)$ becomes the theory $\text{Th}_2((\kappa, D))$. Hence if $\kappa \geq |D|$ and $\kappa' \geq |D'|$, then (12.1) is equivalent to $(\kappa, D) \equiv_{L_2} (\kappa', D')$, whence we obtain $\kappa \equiv_{L_2} \kappa'$ and $D \equiv_{L_2} D'$. The converse is not true. Indeed, consider a couple of distinct $L_2$-equivalent cardinals $\kappa, \kappa'$. Let $D$ be a division ring of power $\kappa$. Then $(\kappa, D) \not\equiv_{L_2} (\kappa', D)$ by (12.1)a.

Some simplification can be also obtained in the case $\kappa \leq |D|$. We claim that

$$\text{Th}(D, L_2(\kappa^+)) \equiv \text{Th}(D, L_2(\kappa^+)) \equiv \text{Th}(|D|, D, L_2(\kappa^+)) \equiv \text{Th}(D, L_2(\kappa^+)).$$

The first sort of the structure $\langle |D|, D \rangle$ can be identified in $D$ with the set $D \times \{1\}$, and the second one with the set $D \times \{0\}$. Hence if (12.1) is true and $\kappa = |D|$, then $\kappa' = |D'|$ and $D \equiv_{L_2} D'$. The condition (12.1), along with the condition $\kappa < |D|$, is equivalent by Claim (12.1)c and the above arguments to the conditions $\kappa' < |D'|$ and $\text{Th}(D, L_2(\kappa^+)) = \text{Th}(D', L_2(\kappa'^+))$.

Note one important particular case: if a division ring $D$ is characterized up to isomorphism by a single sentence of the full second order logic, then under the condition $\kappa \geq |D|$, (12.1) is equivalent to conditions $\kappa' \geq |D'|$, $\kappa \equiv_{L_2} \kappa'$, and $D \equiv D'$. Examples of such $D$ are the fields $Q, R$ and $C$, countable algebraically closed fields and finite fields.

It can be deduced from (12.1)d that if one of the cardinals $\kappa, \kappa'$ is $L_2$-definable, then (12.1) $\Rightarrow \kappa = \kappa'$. Obvious examples are all cardinals $\aleph_n$, where $n \in \omega$.

Since the cardinal $\aleph_0$ is $L_2$-definable, and the field of reals $R$ can be described up to isomorphism by a single $L_2(\aleph_1)$-sentence (the field $Q$ is $L_2(\aleph_1)$-definable, and $R$ can be reconstructed from $Q$, using Dedekind cuts), then

$$T(\aleph_0, R) = T(\kappa, D) \iff \kappa = \aleph_0 \text{ and } D \cong R.$$  

In contrast, $C$ cannot be determined in a such way: $T(\aleph_0, C) = T(\kappa, D)$ iff $\kappa = \aleph_0$ and $D$ is an uncountable algebraically closed field of characteristic zero. Necessity: as $|C| > \aleph_0$, then $|D| > \aleph_0$ by Claim (12.1)b. It follows from $T(\aleph_0, C) = T(\kappa, D)$, that $\text{Th}(C) = \text{Th}(D)$; therefore $D$ is an algebraically closed field of characteristic zero. The sufficiency is an immediate consequence of the following lemma from the joint paper by Belegradek and the author [4].

**Lemma 12.2.** Let $T$ be an uncountably categorical first order theory and $\kappa \geq \aleph_0$. Then all models of $T$ of power $\kappa$ are $L_2(\kappa^+)$-equivalent.
Proof of 12.2 (using standard model-theoretic techniques) is based on the fact that if $M, N$ are two models of $T$ of power $> \kappa$ and $M \prec N$, then $M|_{\kappa^+} \prec N|_{\kappa^+}$.

In particular, if $V, V'$ are vector spaces of dimension $\aleph_0$ over uncountable algebraically closed fields of the same characteristic, then $GL(V) \equiv GL(V')$. We shall see now that this result is not true for groups of type $\Gamma L$.

Consider the vector space $\bigoplus_{i<\kappa} D$ over a division ring $D$. The group of type $H$ over this space is denoted by $H(\kappa, D)$. It follows from Theorem 11.5 that the logical power of the theory $Th(\Gamma L(\kappa, D))$ grows with the growth of $\kappa$. We shall now demonstrate that the growth of logical power of $Th(\Gamma L(\kappa, D))$ can be also achieved by exploiting the second ‘parameter’, the underlying division ring.

**Proposition 12.3.** Let $K$ be an algebraically closed field of infinite transcendence degree over the prime field. Assume that $\kappa$ is an infinite cardinal. Then $Th_2(K)$, the full second order theory of the field $K$, is syntactically interpretable in $Th(\Gamma L(\kappa, K))$ (in $Th(PTL(\kappa, K))$, uniformly in $\kappa$.

**Proof.** According to Theorems 1.6 and 3.1 from the paper 13], if a field $K$ satisfies the conditions of Theorem 12.3 then the full second order theory of the set $|K|$ can be syntactically interpreted in the elementary theory of the lattice of all algebraically closed subfields of $K$.

We need the following well-known fact.

**Lemma 12.4.** Every algebraically closed subfield $k$ of $K$ is the fixed field of some automorphism of $K$.

By Theorem 11.5 we can interpret in $Th(\Gamma L(\kappa, K))$ the elementary theory of the structure $K_{\kappa} = (\langle K, |\kappa^+\rangle, \text{Aut}(K))$. Let us build an interpretation of the lattice of algebraically closed subfields of $K$ in $K_{\kappa}$. The above remarks reduce our task to finding a definable condition $\chi$ such that $\models \chi[\sigma]$ iff $\sigma \in \text{Aut}(K)$ and the fixed field of $\sigma$ is algebraically closed. For this purpose, it is enough to model the situation ‘$\mu$ is a root of a polynomial $f(x) = \lambda_0 + \lambda_1 x + \ldots + \lambda_n x^n$’.

The ordered tuple $\langle \lambda_0, \lambda_1, \ldots, \lambda_n \rangle$ can be coded by the quadruple $\langle \lambda_0, \lambda_n, A, g \rangle$, where $A = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$, and $g$ is a partial function such that $g(\lambda_i) = \lambda_{i-1}$, $i = 1, \ldots, n$ (the condition ‘a finite set $A$ is an orbit of $g$’ can be described by a single sentence in the language of the structure $(\kappa, D)|_{\kappa^+}$).

An element $\mu \in K$ is a root of $f(x)$ iff the following definable condition holds

$$(\exists h)(h \text{ is a function from } K \text{ to } K \& h(\lambda_n) = \lambda_n \& (\forall \lambda', \lambda'' \in A)[\lambda' \neq \lambda_0 \& g(\lambda') = \lambda'' \rightarrow h(\lambda'') = h(\lambda')\mu + \lambda'' \& h(\lambda_0) = 0]).$$

Indeed, if the latter condition is satisfied, $h(\lambda_{n-1}) = \lambda_{n-1}\mu + \lambda_{n-1}, h(\lambda_{n-2}) = \lambda_{n-2}\mu^2 + \lambda_{n-1}\mu + \lambda_{n-2}$, and so on. Hence $h(\lambda_0) = f(\mu) = 0$.

So we can work with the theory $Th_2(|K|)$. We now interpret $Th_2(K)$ in this theory. Choose subsets $X_1, X_2$ of $|K|$ with $X_1 \subseteq X_2$, so that $X_2$ has

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the power $|K|$. Choose further binary relations $R_i, S_i$ on $X_i$, where $i = 1, 2$ such that

(i) the structure $\langle X_1; R_1, S_1 \rangle$ is isomorphic to the prime field of $K$;
(ii) the structure $\langle X_2; R_2, S_2 \rangle$ is an algebraically closed field;
(iii) $\langle X_1; R_1, S_1 \rangle$ is a substructure of $\langle X_2; R_2, S_2 \rangle$;
(iv) the transcendence degree of $\langle X_2; R_2, S_2 \rangle$ over $\langle X_1; R_1, S_1 \rangle$ is $|K|$.

Clearly, if ($\mathbb{L}_2$-definable) conditions (i-iv) are true, then the structure $\langle X_2; \ldots \rangle$ is isomorphic to $K$. □

Thus, we see that the logical power of the theory $\text{Th}(\GammaL(V))$ can be significantly higher than the logical power of $\text{Th}(\text{GL}(V))$.

We summarize some of our results.

**Proposition 12.5.** (a) $\text{GL}(\aleph_0, R) \equiv \text{GL}(\pi, D)$ if and only if $\pi = \aleph_0$ and $D \cong R$;
(b) $\text{GL}(\aleph_0, C) \equiv \text{GL}(\pi, D)$ if and only if $\pi = \aleph_0$ and $D$ is an uncountable algebraically closed field of characteristic zero;
(c) $\GammaL(\aleph_0, C) \equiv \GammaL(\pi, D)$ if and only if $\pi = \aleph_0$ and $D \cong C$;

**Proof.** By Lemma 12.2 and Proposition 12.3 □

So the condition $T(\pi, D) = T(\pi', D')$ is not sufficient for the elementary equivalence of groups $\GammaL(\pi, D)$ and $\GammaL(\pi', D')$.

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