The effect of nuclear deformation on level statistics

A Al-Sayed

Physics Department, Faculty of Science, Zagazig University, Zagazig, Egypt
E-mail: a.alsayed@zu.edu.eg

Received 14 December 2008
Accepted 8 January 2009
Published 26 February 2009

Online at stacks.iop.org/JSTAT/2009/P02062
doi:10.1088/1742-5468/2009/02/P02062

Abstract. We analyze the nearest neighbor spacing distributions of low-lying $2^+$ levels of even–even nuclei. We grouped the nuclei into classes defined according to the quadrupole deformation parameter ($\beta_2$). We calculate the nearest neighbor spacing distributions for each class. Then, we determine the chaoticity parameter for each class with the help of the Bayesian inference method. We compare these distributions against a formula that describes the transition to chaos by varying a tuning parameter. This parameter appears to depend in a non-trivial way on the nuclear deformation, and takes small values indicating regularity in strongly deformed nuclei and especially in those having an oblate deformation.

Keywords: random matrix theory and extensions, connections between chaos and statistical physics
1. Introduction

There are two main ways to investigate nuclear structure. One of them is through the detailed study of individual nuclei. This method tests different nuclear models, and gives us the internal structure of the nucleus under investigation. However, this method does not allow constructing a correlation relation between different nuclei. In addition, we are in need of investigating a large number of studies to understand nearly 2500 discovered nuclei. The second way is by cumulative study, in which one attempts to classify nuclei in terms of a small number of parameters that can be related to nuclear structure. This method finds the correlations between different nuclei and helps us to understand the nuclear structure evolution through the nuclear chart.

Several papers have been published (see, e.g., [1]–[5]), whose purpose is to find the extent of chaos in nuclear dynamics. Random matrix theory (RMT) has achieved great success in the description of the fluctuation properties of spectra of quantum systems. It models a chaotic system using an ensemble of random matrices subject only to symmetry constraints. Systems conserving time reversal, such as the atomic nucleus, are described using the Gaussian orthogonal ensemble (GOE) [6].

The nearest neighbor spacing (NNS) distribution occupies an important place in the statistical theory of spectra. Unfortunately, RMT does not provide a closed form expression for the NNS distribution, which is suitable for the analysis of experimental data. However, Wigner proposed an approximate expression, which is exact in the case of $2 \times 2$ matrices. The NNS distribution of nuclear spectra at neutron threshold energies is in good agreement with the Wigner distribution [7]. The study of level statistics at low-lying excitation energies requires complete (few or no missing levels) and pure (few or no unknown spin parities) level schemes. Unluckily complete and pure level schemes are only available for a limited number of nuclides. Therefore, to improve the statistical significance of the study, spacing distributions for several sequences of levels (each sequence has the same spin parity) from different nuclides have been combined.

Such statistical studies concluded that the NNS distribution of nuclei in the ground state region is intermediate between the Poisson distribution expected for regular systems and the Wigner distribution for chaotic ones. In addition, light nuclides showed a behavior...
close to chaotic near the ground state, while heavier nuclides seemed to be more regular. This can be understood from the fact that the fluctuation properties of the spectra given by the nuclear shell model (which describes light nuclei well) agree with the GOE.

In the present study, we investigate the effect of nuclear deformation—as a measure of nuclear structure—on the chaoticity of even–even nuclei. Due to the serious statistical limitation of the experimental data basis, earlier results yielded only qualitative information concerning the effect of deformation. Some efforts have focused on the symmetries of the IBA, but again the statistics were too limited for reaching definite conclusions. In this paper we wish to establish a direct relation between deformation and nuclear level statistics by parameterizing the available experimental data via the quadrupole deformation parameter.

We focus on the $2^+$ states because of their abundance in even–even nuclei. The data set and the deformation parameter used to classify the nuclei are described in section 2. Section 3 describes the technique of the analysis, while the results are given in section 4.

2. Data set

In this section, we describe our choice of levels in even–even nuclei. We recall that the collective motion is interpreted as vibrations and rotations of the nuclear surface in the geometric collective model first proposed by Bohr and Mottelson [8], where the nucleus is modeled as a charged liquid drop. The moving nuclear surface may be described using an expansion in spherical harmonics with time-dependent shape parameters as coefficients. The quadrupole deformation seems to be the most important collective excitation of the nucleus. For axially symmetric nuclei, the nuclear radius can be written as

$$R(\theta, \phi) = R_{av}[1 + \beta_2 Y_{20}(\theta, \phi)],$$

where $\beta_2$ is the quadrupole deformation parameter. Positive and negative $\beta_2$ values correspond to prolate and oblate shapes respectively. The deformation parameter suffers from the difficulty in distinguishing between static deformation and the dynamic amplitude of the quadrupole vibration in soft spherical nuclei. It can be obtained only by a detailed analysis of spectra and transition probabilities. Despite this difficulty, we draw a crude picture of the dependence of nuclear level statistics on nuclear deformation in a direct way.

The quadrupole deformation parameters $\beta_2$ are taken from macroscopic–microscopic calculations [9]. In order to obtain a sufficient number of energy levels within each interval, we take the absolute value of $\beta_2$. The $Z$ dependence of the deformation is not taken into account. We compare results by analyzing the data set in terms of the experimental $\beta_2$ values, as deduced from the B(E2; $0^+ \rightarrow 2^+$) values [10].

The data on low-lying $2^+$ levels of even–even nuclei are taken from Endt [11] for $22 \leq A \leq 44$, and from the nuclear data sheets up to March 2005 for heavier nuclei. We consider nuclei in which the spin parity $J^\pi$ assignments of at least five consecutive levels are definite. In cases where the spin parity assignments are uncertain and where the most probable value appeared in brackets, we accept this value. We terminate the sequence for each nucleus when we arrive at a level with unassigned $J^\pi$, or when an ambiguous assignment involved a spin parity lying among several possibilities, e.g. $J^\pi = (2^+, 4^+)$. We chose one of the suggested assignments when only one such level occurred in the
sequence, and was followed by several definitely assigned levels containing at least two levels of the same spin parity, provided that the ambiguous level is found in a similar position in the spectrum of a neighboring nucleus. However, this situation occurred for less than 5% of the levels considered. In this way, we obtained 1132 levels of spin parity $2^+$ belonging to 150 nuclei.

3. Method of analysis

For statistical studies using RMT, one should have a spectrum of unit mean level spacing. This is obtained by fitting a theoretical expression to the number $N(E)$ of levels below excitation energy $E$; this process is called unfolding. The expression used here is the constant-temperature formula

$$N(E) = N_0 + \exp\left(\frac{E - E_0}{T}\right).$$

(2)

The three parameters $N_0$, $E_0$ and $T$ obtained for each nucleus vary considerably with mass number. Nevertheless, all three show a clear tendency to decrease with increasing mass number. A detailed account of the method of analysis used in the present work has been given in [12, 13], and references therein.

The nuclear states are characterized by their invariance under time reversal and space rotation, which can be represented by the Gaussian orthogonal ensemble (GOE) of random matrices. The NNS distribution of levels of the GOE is well approximated by Wigner’s distribution

$$P_w(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right).$$

(3)

Here, $s$ is the spacing of neighboring levels in units of the mean level spacing. For integrable systems, the NNS distribution is generically given by the Poisson distribution,

$$P_p(s) = \exp(-s).$$

(4)

The key ingredient of the present analysis is the assumption that the deviation of the NNS distribution of low-lying nuclear levels from the GOE statistics is caused by the neglect of possibly existing conserved quantum numbers other than energy, spin, and parity. A given sequence $S$ of levels can then be represented as a superposition of $m$ independent sequences $S_j$ each having fractional level density $f_j$, with $j = 1, \ldots, m$, and with $0 < f_j \leq 1$ and $\sum_{j=1}^m f_j = 1$. We assume that the NNS distribution $P_j(s)$ of $S_j$ obeys GOE statistics. The exact NNS distribution $P(s)$ of this superposition has been given in [6]. It depends on the $(m - 1)$ parameters $f_j, j = 1, \ldots, m - 1$. In [14], this expression has been simplified by observing that $P(s)$ is mainly determined by short-range level correlations. This reduces the number of parameters to unity and the proposed NNS distribution of the spectrum is

$$P(s, f) = \left[1 - f + f(0.7 + 0.3f)\frac{\pi s^2}{2}\right] \times \exp\left(- (1 - f) s - f(0.7 + 0.3f)\frac{\pi s^2}{4}\right),$$

(5)

which depends on only one parameter, the mean fractional level number $f = \sum_{j=1}^m f_j^2$ for the superimposed sub-spectra. This quantity will eventually be used as a fit parameter. For a large number $m$ of sub-spectra, $f$ is of the order of $1/m$. In this limit, $P(s, f)$

do:10.1088/1742-5468/2009/02/P02062
Figure 1. A relation between parameters $f$ and $\omega$ of the Brody distribution.

approaches the Poisson distribution $P(s, 0) = P_p(s)$. This expresses the well-known fact that the superposition of many GOE level sequences produces a Poissonian sequence. On the other hand, when $f \to 1$ the spectrum approaches the GOE behavior. Indeed, $P(s, 1)$ coincides with the Wigner distribution (3); this is expected as the system then consists of a single GOE sequence. This is why $f$ is called the chaoticity parameter.

In this study, we use the parameter $f$ instead of the empirical Brody interpolation formula $P(s, \omega) = \omega(\gamma + 1)s^\gamma \exp(-\omega s^{\gamma+1})$, where $\gamma$ is a fitting parameter and $\omega = [\Gamma((\gamma + 2)/(\gamma + 1))]^{\gamma+1}$ [15], since formula (5) has a theoretical basis because $f$ represents the area of phase space of the wavefunction occupied by the chaotic dynamics. We plot a relation in figure 1 between the variation of parameter $f$ and the parameter $\omega$, by calculating the chi squared fit between the NNS distribution in equation (5), and the Brody distribution $\chi^2 = \int_0^5 (P(s, f) - P(s, \omega))^2 ds$. By defining the parameter $f$, we calculate the $\omega$ parameter. We observe a nearly linear relationship between the two parameters, that strengthens our choice of parameter $f$.

A common method for determining the parameter $f$ is via the best fit obtained from chi squared. But working with histograms of experimental data, there is always the issue of selection of bin size. Different choices of the bin width, particularly when the widths vary, can lead to quite different visual conclusions. In order to avoid this difficulty, Bayesian inference [16] is used to determine the parameter $f$. The Bayesian analysis is independent of the bin size; it deals with the spacings directly [13]. The Bayesian analysis yields the best-fit value of the chaoticity parameter $f$ and its error for each NNS distribution. For a given sequence of spacings $s = (s_1, s_2, \ldots, s_N)$, the joint probability distribution $P(s|f)$ of these spacings, conditioned by the parameter $f$, is given by

$$P(s|f) = \prod_{i=1}^{N} P(s_i, f),$$

(6)

with $P(s_i, f)$ given by equation (5). Bayes’ theorem provides the posterior distribution

$$P(f|s) = \frac{P(s|f)\mu(f)}{M(s)}$$

(7)

doi:10.1088/1742-5468/2009/02/P02062
The nuclear deformation effect on level statistics of the parameter \( f \) given the events \( s \). Here, \( \mu(f) \) is the prior distribution and

\[
M(s) = \int_0^1 P(s|f)\mu(f)\,df
\]  

(8)
is the normalization.

The prior distribution is found from Jeffreys’ rule [17]

\[
\mu(f) \propto \left| \int P(s|f)[\partial \ln P(s|f)/\partial f]^2\,ds \right|^{1/2}.
\]  

(9)

By evaluating numerically the prior distribution (9), and substituting equation (6) into equation (9), and approximating the result by the polynomial of sixth order in \( f \), we obtain

\[
\mu(f) = 1.975 - 10.07f + 48.96f^2 - 135.6f^3 + 205.6f^4 - 158.6f^5 + 48.63f^6.
\]  

(10)

The distribution \( P(s|f) \) takes very small values even for only moderately large values of \( N \). Because of this, the accurate calculation of the posterior distribution becomes a formidable task. In order to simplify the calculation, equation (6) may be rewritten in the form

\[
P(s|f) = \exp(-N\phi(f)),
\]  

(11)

where

\[
\phi(f) = (1 - f)\langle s \rangle + \frac{\pi}{4}f(0.7 + 0.3f)\langle s^2 \rangle - \left\langle \ln\left[1 - f + \frac{\pi}{2}f(0.7 + 0.3f)s\right]\right\rangle.
\]  

(12)

Here, the notation

\[
\langle x \rangle = \frac{1}{N}\sum_{i=1}^{N}x_i
\]  

(13)

has been used. One finds the function \( \phi(f) \) to have a deep minimum, say at \( f = f_0 \). One can therefore represent the numerical results in analytical form by parameterizing \( \phi \) as

\[
\phi(f) = A + B(f - f_0)^2 + C(f - f_0)^3,
\]  

(14)

where the parameters \( A, B, C \) and \( f_0 \) are implicitly defined by (12).

We then obtain

\[
P(f|s) = c\mu(f)\exp(-N[B(f - f_0)^2 + C(f - f_0)^3]),
\]  

(15)

where \( c = \exp(-NA)/M(s) \) is the new normalization constant. When \( P(f|s) \) is not Gaussian, the best-fit value of \( f \) cannot be taken as the most probable value. Rather we take the best-fit value to be the mean value \( \bar{f} \) and measure the error in terms of the standard deviation \( \sigma \) of the posterior distribution, i.e.

\[
\bar{f} = \int_0^1 fP(f|s)\,df, \quad \text{and} \quad \sigma^2 = \int_0^1 (f - \bar{f})^2P(f|s)\,df.
\]  

(16)

The chaoticity parameter \( \bar{f} \) and the standard deviation \( \sigma \) are calculated for each group of nuclei. The results are given in section 4.
4. Results

The search for a phenomenological ‘control parameter’ for describing the evolution of the stochastic nature of nuclear dynamics became an area of nuclear structure research in the last two decades.

In the present work, we examine the use of the quadrupole deformation parameter as a probe of nuclear structure. In spite of the absence of a complete theoretical study defining certain fixed values of $\beta_2$ corresponding to critical points of shape/structural transitional regions, we tend to classify nuclei into fixed intervals of $\beta_2$ to allow qualitative study. We recall that the analysis of many short sequences of levels tends to overestimate the degree of chaoticity measured by parameter $f$. We focus our attention not on the absolute values of $f$ but on the way $f$ changes with $\beta_2$.

I follow the same method of analysis as was used in [12]. By grouping nuclei according to their $\beta_2$ deformation parameter, instead of the ratio $R_{4/2}$ (the ratio between lowest $4^+$ and $2^+$ level states). The work done in this paper would be a thankless task if there was a simple linear relationship between $\beta_2$ and $R_{4/2}$. But on producing a direct plot between them we get figure 2, which shows an almost uncorrelated relation. We start to classify the available even–even nuclei in terms of $\beta_2$ to find what can we get. Although the capability of the usage of $R_{4/2}$ to identify the collective excitation of nuclei has been confirmed in many published papers, depending on both theoretical (e.g., [18, 19]) and empirical [20, 21] calculations, the $R_{4/2}$ did not provide an acceptable method for differentiating between deformed nuclei, whether they are oblate or prolate deformed, and hence establishing their degree of chaoticity, while the $\beta_2$ parameter does provide one.

Figure 3 shows a comparison of the spacing distributions that depends on the parameter $f$ and the histograms for nuclei divided into classes according to the deformation parameter $\beta_2$. In view of the small number of spacings within each class, the agreement seems satisfactory. By plotting the best-fit values obtained by Bayesian analysis of the chaoticity parameters against $\beta_2$, we get figure 4. From this figure we see the apparent chaoticity of nuclei having $\beta_2$ equal or nearly equal to zero. These are
Figure 3. NNS distribution of even–even nuclei classified according to deformation parameter $\beta_2$.

spherical (magic or semi-magic) nuclei which are expected to have shell model spectra, and thus agree with GOE.

An observed minimum of chaoticity is centered at $\beta_2 \approx 0.20$. Although the statistical errors are not good enough for drawing a conclusion, this may indicate that something interesting is happening through that interval. Two remarkable minima are nearly centered at $\beta_2 \approx 0.26$ and 0.34; these two intervals have nearly the same chaoticity parameter, that may relate to well deformed nuclei.

Finally, there is a minimum at $\beta_2 \approx 0.14$. In order to understand this regular behavior we will not study this interval individually, since we also observe a regular behavior through its neighbor interval centered at $\beta_2 \approx 0.095$. As these two intervals are statistically significant from their neighbors, we will extend our study to cover the two intervals of $\beta_2 \approx 0.06$–0.16. In a detailed study of nuclei contributing to that interval, we notice nearly equal numbers of prolate and oblate nuclei. So, which are responsible for...
Figure 4. The chaoticity parameter $f$ and standard deviation $\sigma$ (error bars) against theoretical deformation parameter $\beta_2$.

Figure 5. Chaoticity of oblate and prolate deformed even–even nuclei in the range of $\beta_2 = 0.06$–0.16.

this regular behavior? We neglected the negative sign of deformation of oblate nuclei, but now the role for testing the correctness of such proposal comes to surface. We obtained 110 levels of spin parity $2^+$ belonging to 15 nuclei of oblate shape $^{62}$Ni, $^{124}$Te, $^{140}$Sm, $^{190}$Pt, $^{192}$Pt, $^{194}$Pt, $^{196}$Pt, $^{198}$Pt, $^{200}$Pt, $^{192}$Hg, $^{194}$Hg, $^{196}$Hg, $^{200}$Hg, $^{202}$Hg, and $^{204}$Hg while there were 123 levels belonging to 16 nuclei of prolate shape $^{82}$Kr, $^{84}$Kr, $^{86}$Kr, $^{76}$Ge, $^{80}$Se, $^{82}$Se, $^{92}$Sr, $^{96}$Mo, $^{106}$Cd, $^{108}$Cd, $^{110}$Cd, $^{112}$Cd, $^{132}$Ba, $^{136}$Ce, $^{144}$Ce, and $^{192}$Os. The NNS distribution is given in figure 5; it is interesting to find that the oblate deformed nuclei have more regular spectra than prolate ones. This may help us to understand the apparent regularity. Indeed, we show [22] that this is not just a special case for that interval, but it is the general rule: oblate nuclei are more regular than prolate ones. This may be interpreted as follows: the degree of interaction between single-particle motion which is chaotic and collective motion of whole nucleons which is believed to be more regular is weaker in the case of oblate deformed nuclei than for prolate ones.
Finally, we compare our results obtained on the basis of theoretically calculated $\beta_2$ with those from experimental $\beta_2$ values, as deduced from the $\text{B(E2; 0}^+ \rightarrow 2^+\text{)}$ values [10]. Two disadvantages of using experimental $\beta_2$ arise. One of them is that all nuclei have non-zero values, so it is difficult to differentiate between oblate and prolate deformed nuclei. On the other hand, $\beta_2$ values are not available for all nuclei under investigation in this study. We obtained 1050 levels of spin parity $2^+$ belonging to 137 nuclei. Figure 6 shows the trend of dependence of the chaoticity parameter on experimental $\beta_2$. In spite of the error bars being statistically insignificant, the figure shows nearly the same trend as was observed in figure 4.

5. Conclusion

We study the nearest neighbor spacing distribution of even–even nuclei classified according to the quadrupole deformation parameter using the available experimental energy levels. We use a model that interpolates between the Poisson (regular) and Wigner (chaotic) distributions by varying the chaoticity parameter from 0 to 1 respectively. The observed regular behavior of nuclei having certain values of the deformation parameter indicate that there must be a lot of work to do to qualify the usage of such a parameter as an indicator of variation in nuclear structure, and this work may be seen as a step along this road.

Acknowledgment

I am grateful to Professor A Y Abul-Magd, Faculty of Science, Zagazig University, for helpful discussions and scientific support.

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doi:10.1088/1742-5468/2009/02/P02062