SURPRISING SPECTRAL GAP AND ENTROPY DECAY ESTIMATES IN OPEN QUANTUM SYSTEMS WITH A LARGE NUMBER OF QUBITS

YIDONG CHEN AND MARIUS JUNGE

Abstract. One of the major challenges in quantum information science is to control systems with a large number of qubits. Since any realistic quantum system interacts with the environment, it is important to have quantitative estimates on decoherence. The time evolution of an open quantum system can be modeled by a Lindbladian obtained by tracing out the environment degrees of freedom and performing a Born-Markov approximation. In this paper we study the spectral gap and modified logarithmic Sobolev constant of some very simple open systems given by a representation of \( su(2) \) on \( N \)-qubits. Our examples fall into the class of Lindbladians admissible to the dissipative quantum Church-Turing thesis \([\text{KBG}^+ 11]\). In addition, our examples can also be written as Davies generators \([\text{Dav80}]\). Moreover, the main example has a dimension-dependent spectral gap at finite temperature. This is complementary to the class of Davies generators in \([\text{KB16}][\text{BCR22}]\), where local spectral estimates automatically imply global ones.

1. Introduction

Understanding and controlling open quantum systems is a challenging problem for both experimentalists and theorists. In this paper, we study some very basic examples of open systems with a large number of qubits. Our work is motivated by

- the work of Kliesch et.al. \([\text{KBG}^+ 11]\) on the quantum Church-Turing thesis and efficient simulation of Lindbladians using quantum circuits;
- the theory of Davies generators in many-body systems; and
- the theory and experiments of atomic systems placed in a dispersing and absorbing environment \([\text{DKW02}][\text{KSW00}][\text{GW96}][\text{MAG22}][\text{SMAG22}]\), going back to Dicke’s original work \([\text{Dic54}]\).

It is well-known that the dissipative dynamics of an open quantum system can be described by a Lindbladian \([\text{GKS76}][\text{Lin76}]\). In \([\text{KBG}^+ 11]\) the authors introduced the notion of k-locality to generalize the notion of geometric locality typically used in many-body systems. Specifically, on a discrete system with \( N \) sites, a Lindbladian \( \mathcal{L} \) is (strictly) k-local if it acts nontrivially on at most \( k \) subsets of \( \{1, ..., N\} \). It was shown that the class of k-local Lindbladians can be efficiently simulated using quantum circuits. Here our main focus is a simple 1-local Lindbladian acting on \( N \) qubits:

\[
\mathcal{L}^\beta_N = e^{\beta/2} L_{\pi_N(a)} + e^{-\beta/2} L_{\pi_N(a^*)}
\]

where \( L_{\pi_N(a)}x = 2\pi_N(a^*)x\pi_N(a) - \pi_N(a^*)\pi_N(a)x - x\pi_N(a^*)\pi_N(a) \) and

\[
\pi_N(a) = \sum_{1 \leq j \leq N} \pi_{(j)}(a) = \sum_{1 \leq j \leq N} a \otimes ... \otimes a \otimes ... \otimes 1, a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

MJ was partially supported by NSF Grant DMS 1800872 and NSF RAISE-TAQS 1839177.
The parameter $\beta > 0$ is the inverse temperature.

In addition, $L_N^\beta$ has the form of a Davies generator with $\pi_N(h) := \pi_N\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ as the system’s Hamiltonian. The corresponding Gibbs state is given by

$$d_N := N_\beta \exp(-\frac{\beta}{2} \pi_N(h))$$

where $N_\beta$ is the normalization constant. The state $d_N$ is an invariant (i.e. equilibrium) state of the open system, i.e. $L_N^\beta(d_N) = 0$. The state can also be completely factorized in terms of a tensor product:

$$d_N := d_\otimes N = \begin{pmatrix} e^{-\beta/2} & 0 \\ 0 & e^{\beta/2} \end{pmatrix}^N$$

It turns out that $d_N$ satisfies the so-called strong clustering condition [KB16 BCG+21a BCG+21b BCL+21 BCPH22 BCR22]. Thus the results of [KB16] seem to indicate that $L_N^\beta$ has a dimension-independent spectral gap. Surprisingly, in this paper we will show that the opposite is true. 5.2

**Theorem 1.1.** For the Lindbladian $L_N^\beta$, there exists a constant $C(\beta) > 0$ such that

$$\lambda_2(L_N^\beta) \leq \frac{C(\beta)}{N}$$

where $\lambda_2(L_N^\beta)$ is the spectral gap of $L_N^\beta$.

This apparent contradiction is resolved when we realize that $L_N^\beta$ has multiple equilibrium states (non-primitive). The theorems in [KB16] are true only for Lindbladians with a unique equilibrium state (primitive). In addition, for $\beta = 0$, previous result [BGJ0a GR22] gives a uniform lower bound on the spectral gap. This phase transition from $\beta = 0$ to $\beta > 0$ was not expected by the authors and their collaborators. It shows that the properties of non-primitive Davies generators depend on both the geometry and the interactions. This class of non-primitive Davies generators is yet to be explored.

From an experimental perspective, our main example arises naturally from atomic systems placed in a dispersing and absorbing environment. These systems are important for studying quantum information devices (e.g. cavity/circuit QED [BGO20]). In order to better understand how these systems interact with electromagnetic fields, it is necessary to study medium-assisted quantum electrodynamics coupled with a matter system. However, quantization is not straightforward when the system is placed in a absorbing dielectric environment (it is not obvious how to preserve equal-time canonical commutation relation). This problem was solved by [DKW02] by performing a semi-classical analysis based on classical Maxwell’s equations and a quantized noise source. The key idea is to express field operators in terms of the quantized modes of the noise source. After tracing out the field degrees of freedom, the matter system is effectively an open system. It turns out that the dynamics of this open quantum system can be described by a Markovian master equation. Written in the Lindblad form, the time evolution of density matrix is governed by a Lindbladian. In the model of [DKW02], under a Born-Markov approximation, the effective time evolution of the
A Lindbladian $\sigma$ for the fixed reference state $\sigma$ with respect to the reference state $\sigma$ of $\sigma$ will be discussed later. We refer to the seminal papers [CM17] [CM20] for the general theory of detailed balance Lindbladians and the crucial inner product (KMS-inner product with a two-way heat exchange between the system and the bath, the corresponding Lindbladian for each eigenvalue $\Gamma_\nu$ has the following form:

\begin{equation}
\frac{\Gamma_\nu}{2}(2O_\nu^*O_\nu - \rho O_\nu^*O_\nu - O_\nu^*O_\nu^*\rho) + \frac{\Gamma_\nu^{-1}}{2}(2O_\nu\rho O_\nu^* - \rho O_\nu O_\nu^* - O_\nu O_\nu^*\rho)
\end{equation}

where $O_\nu$'s act on the $N$-qubits. Hence the Lindbladian $\mathcal{L}_N^\beta$ can be understood as a building block of the noisy dynamics.

For every inverse temperature $\beta$, the Lindbladian $\mathcal{L}_N^\beta$ satisfies the $d_N$-detailed balance condition. This is a direct consequence of the underlying $\mathfrak{su}(2)$ representation theory, which we will discuss later. We refer to the seminal papers [CM17] [CM20] for the general theory of $\sigma$-detailed balanced Lindbladians and the crucial inner product (KMS-inner product with respect to the reference state $\sigma$):

$$\langle x, y \rangle_\sigma := \text{tr}(\sigma^{1/2}x^*\sigma^{1/2}y)$$

For the fixed reference state $\sigma$, a Lindbladian $\mathcal{L}$ is KMS-symmetric if

$$\langle \mathcal{L}x, y \rangle_\sigma = \langle x, \mathcal{L}u \rangle_\sigma$$

A Lindbladian $\mathcal{L}$ admits a spectral gap $\lambda_2(\mathcal{L}) \geq c$ if there exists a constant $c > 0$ such that

$$c\langle x - E_{fix}x, x - E_{fix}x \rangle_\sigma \leq \langle \mathcal{L}x, x \rangle_\sigma$$

where $E_{fix}$ projects onto the kernel of $\mathcal{L}$. For a KMS-symmetric $\mathcal{L}$, its kernel is a closed $*$-algebra, and $E_{fix}$ becomes the so-called conditional expectation. Beyond the spectral gap, the decay to equilibrium is measured with the help of the relative entropy:

$$D(\rho|\sigma) := \text{tr}(\rho(\log \rho - \log \sigma))$$

The constant $\text{MLSI}(\mathcal{L})$ is the largest constant $c > 0$ such that for all $\rho$

$$cD(\rho|E_{fix}^*\rho) \leq \text{tr}(\mathcal{L}^*(\rho)(\log \rho - \log E_{fix}^*\rho))$$

The right hand side is called the entropy production [Spo78]. Here the (trace-)adjoint $E_{fix}^*$ and $\mathcal{L}^*$ acts on the densities. Equivalently, we have exponential decay

$$D(e^{-t\mathcal{L}}\rho|E_{fix}^*\rho) \leq e^{\text{MLSI}(\mathcal{L})t}D(\rho|E_{fix}^*\rho)$$

The complete version $\text{CLSI}(\mathcal{L}) := \inf_{\beta \geq 0} \text{MLSI}(\mathcal{L} \otimes \text{id}_d)$ was introduced in [GJL0a] in order to ensure tensor stability. Recently, a wealth of results [GJL0a] [BGJ0a] [GR22] [GJL21] [BCL+21] [GG22] provide lower bounds for the CLSI constant. For $\beta = 0$ (the infinite temperature limit), tools from harmonic analysis, Lie groups and foliation can be used to obtain a uniform spectral gap and CLSI constant. [GR22] [BGJ0a]
Theorem 1.2. The family of Lindbladians $\mathcal{L}_N^\beta$ admits a uniform lower bound on the spectral gaps and the CLSI constants.

Since $\mathcal{L}_N^\beta$ depends continuously on $\beta$, one expects a similar result to hold at finite temperature (at least for high temperature). Indeed, the change of measure argument [JLR19] provides such a link. However, the constants obtained by the Hooley-Stroock argument in [JLR19] are not uniform in dimension. Despite an extensive effort by the second named author, the uniformity in $N$ simply could not be achieved. This leads to a polynomial bound:

Theorem 1.3. Suppose for $\beta > 0$. There exists constants $C_1(\beta), C_2(\beta) > 0$ such that

$$
\frac{C_1(\beta)}{N^2} \leq \text{CLSI}(\mathcal{L}_N^\beta) \leq 2\lambda_2(\mathcal{L}_N^\beta) \leq \frac{C_2(\beta)}{N}
$$

It is an open problem to calculate the precise orders of the spectral gap and the CLSI constant. The proof has two parts. The first part relies on the representation theory of $\mathfrak{su}(2)$, and the second part follows essentially from estimates of modified logarithmic Sobolev constants on finite graphs. In addition, the upper bound of the spectral gaps can be achieved by tuning the off-diagonal coefficients between quantum numbers $N$ and $N-2$. The new dimension-dependent bounds show that there are certain densities embedded in a system of $N$ qubits which remain stable on a time scale of $O(N)$. It would be interesting to know how these states can be prepared and used for specific quantum tasks.

Lastly, the same argument gives us estimates on the CLSI constants of other Lindbladians as well. In particular, we have

Theorem 1.4. Let $a_N(\gamma) := \pi_N(a)(\pi_N(a^*)\pi_N(a))^\gamma$ where the parameter $\gamma \geq 1, \gamma \in \mathbb{Z}$. Consider the Lindbladian:

$$
\mathcal{L}_{N,\gamma}^\beta := e^{\beta/2}L_{a_N(\gamma)} + e^{-\beta/2}L_{a_N(\gamma)'}
$$

where $L_{a_N(\gamma)x} := 2a_N(\gamma)x - a_N(\gamma)^*a_N(\gamma)x - a_N(\gamma)x - a_N(\gamma)^*a_N(\gamma)x$. In addition, fix again the tensor product state $d_N = \pi_N^{\otimes N}$ as the reference state. Then there exist constants $C_3(\beta, \gamma) > 0$ such that

$$
2\lambda_2(\mathcal{L}_{N,\gamma}^\beta) \geq \text{CLSI}(\mathcal{L}_{N,\gamma}^\beta) \geq C_3(\beta, \gamma)
$$

We will show how such Davies generators can arise by coupling the system to a heat bath. Since the CLSI constant is independent of $N$, the lower bound is still valid for $N \to \infty$. The limit algebra is the type $III_\lambda$ ($\lambda = e^{-\beta}$) factor. These examples give a new class of manageable Davies generators that include non-local interactions. By tensoring two different type $III_\lambda$ factors, we also obtain a Lindbladian on the type $III_1$ factor [Tak03]. Since the type $III_1$ factor can be used to model a second quantized system (for example see the work of Witten [Wit18]), it is interesting to see how our result can be used to study entropy decay in quantum field theory.

2. CLSI of a Single Self-Adjoint Operator

In this section, we prove CLSI for Lindbladian generated by a single self-adjoint operator:

$$
\mathcal{L}_A(x) := 2AxA - A^2x - xA^2
$$

where $A$ is a self-adjoint operator with discrete spectrum and each eigenvalue has multiplicity 1. For simplicity, we first restrict ourselves to the case where $\mathcal{L}_A$ acts on a matrix algebra
\[ L_A : \mathbb{M}_d \rightarrow \mathbb{M}_d \]

In this case, the CLSI constant will be related to the spectral distribution of \( A \). Later, we will use a transference principle to generalize this result to finite von Neumann algebras.

The key to prove this result is the connection between complete return time and CLSI constant established in [BGJ0a]. More precisely, for a symmetric quantum Markov semigroup, it is proved that a lower Ricci curvature bound along with a complete return time estimate implies the semigroup satisfies CLSI.

To apply this result, we first make the simple observation that \( L_A \) can be represented as a Schur multiplier. Recall the definition of a Schur multiplier on a matrix algebra:

**Definition 2.1.** Let \( \mathbb{M}_n \) be a \( n \times n \) matrix algebra and \( a := (a_{ij})_{i,j=1}^n \in \mathbb{M}_n \) be a matrix. Then the Schur multiplier \( T_a \) associated with \( a \) is

\[ T_a : \mathbb{M}_n \rightarrow \mathbb{M}_n : T_a((x_{ij})) = (a_{ij}x_{ij}) \]

**Lemma 2.2.** Let \( A = \sum \lambda_i e_i \) be the spectral decomposition of \( A \) with distinct eigenvalues. Then the Lindbladian \( L_A : \mathbb{M}_d \rightarrow \mathbb{M}_d \) is given by a Schur multiplier.

\[ L_A(x) = -\sum_{i,j} (\lambda_i - \lambda_j)^2 e_i x e_j \]

The fixed point algebra is the subalgebra of diagonal matrices: \( \mathbb{M}_{\text{diag}} := \text{span}\{e_i : 1 \leq i \leq d\} \). In particular, the quantum Markov semigroup generated by \( L_A \) is not primitive.

**Proof.** By a direction computation we have:

\[ L_A(x) = 2Ax - A^2x - xA^2 = -\sum_i \lambda_i^2 e_i x - \sum_i \lambda_i^2 x e_i + 2 \sum_{i,j} \lambda_i \lambda_j e_i x e_j \]

\[ = -\sum_{i,j} (\lambda_i^2 e_i x e_j + \lambda_j^2 e_i x e_j - 2 \lambda_i \lambda_j e_i x e_j) = -\sum_{i,j} (\lambda_i - \lambda_j)^2 e_i x e_j \]

where in the third equation we used partition of unity \( \sum_i e_i = 1 \). From the Schur multiplier representation of \( L_A \), it is clear that \( L_A(x) = 0 \) if and only if \( x \in \mathbb{M}_{\text{diag}} \). Hence the fixed point algebra is the subalgebra of diagonal matrices.

It is shown in [BGJ0a] that the Ricci curvature of the Lindbladian associated with a Schur multiplier is bounded below by 0. In this case, the paper showed that the CLSI constant is inversely proportional to the complete return time. Therefore in the remainder of this section, we will calculate the complete return time of \( L_A \) in terms of the spectral distribution of \( A \).

It turns out that the correct definition of return time for a nonprimitive quantum Markov semigroup \( T_t : \mathcal{M} \rightarrow \mathcal{M} \) is given by [GJL20a] [GJL20b] [JP10]

**Definition 2.3.** For a nonprimitive quantum Markov semigroup \( T_t : \mathcal{M} \rightarrow \mathcal{M} \) where \( \mathcal{M} \) is a finite von Neumann algebra and let \( E : \mathcal{M} \rightarrow \mathcal{N} \) be the conditional expectation onto the fixed point algebra \( \mathcal{N} \subset \mathcal{M} \), the complete bounded return time of \( T_t \) is given by:

\[ t_{cb}(\epsilon) := \inf\{t \geq 0 : \|T_t - E : L^1(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty(\mathcal{M})\|_{cb} \leq \epsilon\} \]

where \( 0 < \epsilon < 1 \), and \( L^1(\mathcal{N} \subset \mathcal{M}) \) is the amalgamated \( L_\infty \) space equipped with the norm:

\[ \|x\|_{L_\infty^1} := \sup_{\|a\|_2 = \|b\|_2 = 1} ||axb||_1 \]
For $\epsilon = 1/2$, we write $t_{cb} := t_{cb}(1/2)$.

For the motivation of this definition, see [GJL20a] and [BGJ0a]. Below we present the key lemma to calculate the complete return time in the finite dimensional case [GJL20a]. It extends the classical result that relates cb-norm of a quantum channel with the norm of its Choi matrix. Fix a pair of unital algebras $\mathcal{N} \subset \mathcal{M}$, recall a map $T : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{N}$-bimodule map if for all $a, b \in \mathcal{N}$ and $x \in \mathcal{M}$ we have $T(axb) = aT(x)b$.

**Lemma 2.4.** Let $\mathcal{M}_d$ be a matrix algebra, and fix a set of matrix units $\{e_{ij}\}_{1 \leq i, j \leq d}$ of $\mathcal{M}_d$. Let $\mathcal{N} \subset \mathcal{M}_d$ be a subalgebra. Then for a completely bounded $\mathcal{N}$-bimodule map: $T : L_1(\mathcal{M}_d) \to L_\infty(\mathcal{M}_d)$ we have
\[
(2.6) \quad \|T : L_1^\infty(\mathcal{N} \subset \mathcal{M}_d) \to L_\infty(\mathcal{M}_d)\|_{cb} \leq \|\chi_T\|_{\mathcal{M}_d \otimes \mathcal{M}_d}
\]
where $\chi_T := \sum_{i,j} e_{ij} \otimes T(e_{ij})$ is the Choi matrix of $T$.

For a proof, see [GJL20a]. After applying this lemma to the Schur multipliers, we have

**Lemma 2.5.** With the same set-up as the previous lemma, let $T((e_{ij})) = (t_{ij}e_{ij})$ be a Schur multiplier that fixes the diagonal subalgebra $\mathcal{M}_d$. Assume $T$ is a $\mathcal{M}_d$-bimodule map. Then
\[
(2.7) \quad \|T\|_{1 \to \infty, cb} := \|T : L_1(\mathcal{M}_d) \to L_\infty(\mathcal{M}_d)\|_{cb} \leq \|(t_{ij})\|_{\mathcal{M}_d}
\]

**Proof.** Consider the following map:
\[
(2.8) \quad \varphi : \mathcal{M}_d \to \mathcal{M}_d \otimes \mathcal{M}_d
\]
\[
\varphi(e_{ij}) = e_{ij} \otimes e_{ij}
\]
$\varphi$ is a non-unital *-homomorphism. Since $\varphi(1) = \sum_i e_{ii} \otimes e_{ii}$, $\varphi$ is also an isometry. By the previous lemma, $\|T\|_{1 \to \infty, cb} \leq \|\varphi\|_{\mathcal{M}_d \otimes \mathcal{M}_d}$. Since $\varphi = \sum_{i,j} e_{ij} \otimes T(e_{ij}) = \sum_{i,j} e_{ij} \otimes t_{ij}e_{ij} = \varphi((t_{ij}))$, we have:
\[
(2.9) \quad \|T\|_{1 \to \infty, cb} \leq \|\chi_T\|_{\mathcal{M}_d \otimes \mathcal{M}_d} = \|(t_{ij})\|_{\mathcal{M}_d}
\]

Let’s go back to the complete return time of the quantum Markov semigroup $\{T_t := e^{-t\mathcal{L}_A} : \mathcal{M}_d \to \mathcal{M}_d\}_{t \geq 0}$. Let $E : \mathcal{M}_d \to \mathcal{M}_d$ be the conditional expectation onto the fixed point algebra, the previous lemma gives the following complete return time estimate.

**Corollary 2.6.** Let $\delta = \min_{i \neq j} |\lambda_i - \lambda_j|$ be the smallest gap in the spectral distribution of $A$, then the complete return time satisfies the following bound:
\[
(2.10) \quad t_{cb}(\epsilon) \leq \frac{\epsilon^2 \pi}{\delta^2}
\]
In particular, $t_{cb} = t_{cb}(1/2) \leq \frac{4\pi}{\delta^2}$.

**Proof.** Since $(T_t - E)x = \sum_{i,j} e^{-t(\lambda_i - \lambda_j)^2}x_{ij} - \sum_i x_{ii} = \sum_{i \neq j} e^{-t(\lambda_i - \lambda_j)^2}x_{ij}$, by the previous lemma
\[
(2.11) \quad \|T_t - E\|_{1 \to \infty, cb} = \|(e^{-t(\lambda_i - \lambda_j)^2})_{i \neq j}\|_{\mathcal{M}_d} = \|(\sum_{k=1}^{d-1} + \sum_{k=-1}^{-d+1}) \sum_i e^{-t(\lambda_i - \lambda_{i+k})}e_{i,i+k}\|
\]
\[
\leq 2 \sum_{k=1}^{d-1} \left| \sum_i e^{-t\delta^2k^2}e_{i,i+k} \right| \leq 2 \sum_{k=1}^{d-1} e^{-t\delta^2k^2} \leq \int_{-\infty}^{\infty} dx e^{-t\delta^2x^2} = \frac{\sqrt{\pi}}{\delta \sqrt{t}}
\]
Hence $t_{cb}(\epsilon) \leq \frac{\epsilon^2 \pi}{\delta^2}$ and $t_{cb} \leq \frac{4\pi}{\delta^2}$.\]
Theorem 2.8. Let $\text{tr}$ of multiplicity 1. Then the CLSI bounded below by 0 (for a definition of geometric Ricci curvature see [BGJ0a]) then

$$A$$

Definition 2.7. For two variables $F, G$ (1) $F \preceq G$ if and only if there exists an absolute constant $C > 0$ such that $F \leq CG$. (2) $F \succeq G$ if and only if there exists an absolute constant $c > 0$ such that $F \geq cG$. (3) $F \sim G$ if and only if there exists absolute constants $c, C > 0$ such that $cG \leq F \leq CG$.

Theorem 2.8. Let $\mathcal{L}_A(x) = 2Ax - A^2x - xA^2 : \mathbb{M}_d \to \mathbb{M}_d$ be the generator of a non-primitive quantum Markov semigroup where $A$ is a self-adjoint operator with discrete spectrum of multiplicity 1. Then the CLSI constant of $\mathcal{L}_A$ is proportional to the spectral gap of $\mathcal{L}_A$:

$$\text{CLSI}(\mathcal{L}_A) \sim \lambda(\mathcal{L}_A)$$

Proof. Let $\delta = \min_{i \neq j} |\lambda_i - \lambda_j|$. Since $\mathcal{L}_A(x) = -\sum_{i,j} (\lambda_i - \lambda_j)^2 e_i x e_j$, the spectral gap $\lambda(\mathcal{L}_A) \sim \delta^2$.

By the CLSI estimate from [BGJ0a], since Schur multiplier’s geometric Ricci curvature is $\geq \text{const}$, we have

$$\text{CLSI}(\mathcal{L}_A) \succeq \lambda(\mathcal{L}_A).$$

Hence CLSI($\mathcal{L}_A) \succeq \lambda(\mathcal{L}_A)$. By a well known result [KT13], we have $\text{CLSI}(\mathcal{L}_A) \leq 2\lambda(\mathcal{L})$. Hence $\text{CLSI}(\mathcal{L}_A) \sim \lambda(\mathcal{L}_A)$. 

Corollary 2.9. Let $\mathcal{L}_A(x) := 2Ax - A^2x - xA^2 : \mathcal{M} \to \mathcal{M}$ be the generator of a quantum Markov semigroup (not necessarily primitive) on a finite von Neumann algebra, and let $A$ be a self-adjoint operator with spectral decomposition $A = \sum_{1 \leq i \leq d} \lambda_i e_i$. Then the same conclusion of the last theorem holds.

Proof. By the same calculation as lemma 2.2, we have $\mathcal{L}_A(x) = -\sum_{1 \leq i,j \leq d} (\lambda_i - \lambda_j)^2 e_i x e_j$.

Let $\mathcal{L}_d$ be the Lindbladian on $\mathbb{M}_d$ given by: $\mathcal{L}_d(x) := -\sum_{1 \leq i,j \leq n} (\lambda_i - \lambda_j)^2 e_i x e_j$. Consider the non-unital trace preserving map:

$$\varphi : \mathcal{M} \to \mathbb{M}_d \otimes \mathcal{M}$$

$$\varphi(e_i x e_j) := e_{ij} \otimes e_i x e_j$$

Then we have

$$e^{-t\mathcal{L}_d} \otimes \text{id}_{\mathcal{M}} \circ \varphi(x) = \sum_{i,j} e^{-t(\lambda_i - \lambda_j)^2} e_{ij} \otimes e_i x e_j$$

$$= \sum_{i,j} e_{ij} \otimes e^{-t(\lambda_i - \lambda_j)^2} e_i x e_j = \sum_{i,j} e_{ij} \otimes e^{-t\mathcal{L}_A} x$$

$$= \varphi(e^{-t\mathcal{L}_A} x)$$

In addition, $\varphi$ is compatible with the projection onto the fixed point algebra. Hence by the definition of the CLSI constant, we have $\text{CLSI}(\mathcal{L}_A^M) \sim \lambda(\mathcal{L}_A^M)$.

Remark. In [GR22], Gao and Rouzé have shown that $\frac{\lambda(\mathcal{L}_A)}{C_{cb}(\mathcal{M} : \mathcal{N})} \leq \text{CLSI}(\mathcal{L}_A) \leq 2\lambda(\mathcal{L}_A)$ where $C_{cb}(\mathcal{M} : \mathcal{N})$ is the cb index of the pair $\mathcal{N} \subset \mathcal{M}$. Our result gives a tighter bound because the cb index of the diagonal subalgebra is the dimension of the underlying Hilbert space.
3. CLSI of KMS-Symmetric QMS with Single Bohr Frequency

Following [CM17] [CM20], we consider a Lindbladian associated with a single Bohr frequency in this section.

\[ \mathcal{L}^\beta : \mathcal{M} \to \mathcal{M} \]
\[ \mathcal{L}^\beta x := e^{\beta/2}L_ax + e^{-\beta/2}L_a^*x \]

where \( L_a x := 2a^*xa - a^*ax - xa^*a \) and \( \mathcal{M} \) is a finite von Neumann algebra. We assume \( \beta > 0 \). Fix a faithful state \( \phi \) on \( \mathcal{M} \) such that its density \( d_\phi \) is an fixed point state under the predual \( \mathcal{L}^\beta_* : L_1(\mathcal{M}) \to L_1(\mathcal{M}) \) (i.e. \( \mathcal{L}^\beta_*(d_\phi) = 0 \)). The KMS-inner product on \( \mathcal{M} \) with respect to \( d_\phi \) is given by:

\[ \langle x, y \rangle_\phi := \tau(d_\phi^{1/2}x^*d_\phi^{1/2}y) \]

where \( \tau \) is the canonical trace on \( \mathcal{M} \). We assume \( \mathcal{L}^\beta \) is self-adjoint with respect to this KMS-inner product:

\[ \langle \mathcal{L}(x), y \rangle_\phi = \langle x, \mathcal{L}(y) \rangle_\phi \]

We make the crucial assumption that the modular automorphism associated with \( \phi \) satisfies the following equation

\[ \sigma_t^\phi(a) = d_\phi^{it}ad_\phi^{-it} = e^{i\beta t}a \]

The modular automorphism assumption implies that the Lindbladian commutes with modular operator: \([\mathcal{L}^\beta, \Delta_\phi] = 0\). By a result from [CM17], this implies that the Lindbladian satisfies \( \phi \)-detailed balance condition. The state \( \phi \) induces a nontrivial modular automorphism group. In the following, we will use a 2 \( \times \) 2-matrix trick to cancel this modular automorphism group. Using this technical tool, we will provide a direct estimate of the CLSI constant.

Before stating the next proposition, recall the definition of the difference quotient:

**Definition 3.1.** Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be a continuously differentiable function, and let \( \rho \) be a faithful state with spectral decomposition \( \rho = \int \lambda dE_\lambda \). Then the difference quotient with respect to \( \rho \) is given by the double operator integral

\[ J^f_\rho(x) := \int \int \frac{f(\lambda) - f(\mu)}{\lambda - \mu}dE_\lambda xdE_\mu \]

where the ratio is defined to be \( f'(\lambda) \) if \( \lambda = \mu \).

If the spectral decomposition is discrete i.e. \( \rho = \sum_\lambda \lambda e_\lambda \), then

\[ J^f_\rho(x) = \sum_{\lambda,\mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}e_\lambda xe_\mu \]

**Proposition 3.1.** Let \( \tau \) be the canonical trace on the finite von Neumann algebra \( \mathcal{M} \). Fix \( \beta \) as above, consider the density matrix

\[ d_\beta := \begin{pmatrix} e^{-\beta/2} & 0 \\ e^{\beta/2}e^{-\beta/2} & e^{\beta/2} \end{pmatrix} \]

and consider the generator

\[ A := e_{12} \otimes a \in \mathcal{M}_2 \otimes \mathcal{M} \]
Then we have i)
\[ [d_\beta \otimes d_\phi, A] = 0 \]

ii) For state \( \rho_\beta := (d_\beta \otimes d_\phi)^{1/2}(1 \otimes x)(d_\beta \otimes d_\phi)^{1/2} \), we have
\[ EP_A(\rho_\beta) \leq C(\beta)EP_\phi(d_\phi^{1/2}x_d^{1/2}), \quad EP_A^-(\rho_\beta) \leq C(\beta)EP_{\phi^*}(d_\phi^{1/2}x_d^{1/2}) \]
where \( EP_A(\rho) := \langle [A, \rho], J^0_P[A, \rho] \rangle \) is the entropy production of state \( \rho \) under generator \( A \).

**Proof.** i) For the first claim, from direct calculation we have
\[ (d_\beta^{it} \otimes d_\phi^{it})(e_{12} \otimes a)(d_\beta^{-it} \otimes d_\phi^{-it}) = (d_\beta^{it}e_{12}d_\beta^{-it}) \otimes (d_\phi^{it}ad_\phi^{-it}) = e^{-i\beta t}e_{12} \otimes e^{i\beta a} = e_{12} \otimes a = A \]

Therefore \([d_\beta \otimes d_\phi, A] = 0\).

ii) For the second claim, we need the following technical result:

**Claim:**
\[ \log \frac{\lambda - \log \mu}{\lambda - \mu} \geq e^{-\beta/2} \log(\lambda e^{-\beta/2}) - \log(\mu e^{\beta/2}) \]

We postpone the proof of this inequality and proceed with the main proof.

Let \( d_\phi^{1/2}x_d^{1/2} = \sum_\lambda \lambda e_\lambda \) be the spectral decomposition of the state. We have
\[ (d_\beta^{1/2} \otimes d_\phi^{1/2})(e_{12} \otimes [a, x])(d_\beta^{1/2} \otimes d_\phi^{1/2}) = \frac{1}{e^{\beta/2} + e^{-\beta/2}}e_{12} \otimes (d_\phi^{1/2}[a, x]d_\phi^{1/2}) . \]

And the spectral decomposition of \( \rho_\beta \) is given by
\[ \rho_\beta = \sum_\lambda \frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}e_{11} \otimes e_\lambda + \sum_\mu \frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}e_{12} \otimes e_\mu . \]

Combining these two calculations, we have
\[ J^0_P[\rho_\beta[A, \rho_\beta]] = \frac{1}{e^{\beta/2} + e^{-\beta/2}} \sum_{\lambda, \mu} \log(\frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}) - \log(\frac{\lambda \mu e^{\beta/2}}{e^{\beta/2} + e^{-\beta/2}})(e_{12} \otimes e_\lambda d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\mu) \]
\[ = \sum_{\lambda, \mu} \log(\frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}) - \log(\frac{\lambda \mu e^{\beta/2}}{e^{\beta/2} + e^{-\beta/2}})(e_{12} \otimes e_\lambda d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\mu) . \]

Hence the entropy production with respect to the trace on \( \mathcal{M}_2 \otimes \mathcal{M} \) is given by
\[ EP_A(\rho_\beta) = \langle [A, \rho_\beta], J^0_P[\rho_\beta[A, \rho_\beta]] \rangle \]
\[ = \frac{1}{e^{\beta/2} + e^{-\beta/2}}e_{12} \otimes d_\phi^{1/2}[a, x]d_\phi^{1/2}e^{\beta/2} \sum_{\lambda, \mu} \log(\frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}) - \log(\frac{\lambda \mu e^{\beta/2}}{e^{\beta/2} + e^{-\beta/2}})e_{12} \otimes e_\lambda d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\mu) \]
\[ = \frac{e^{\beta/2}}{2(e^{\beta/2} + e^{-\beta/2})} \sum_{\lambda, \mu} \log(\frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}) - \log(\frac{\lambda \mu e^{\beta/2}}{e^{\beta/2} + e^{-\beta/2}})\tau(d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\lambda d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\mu) \]
\[ \leq \frac{e^{\beta}}{2(e^{\beta/2} + e^{-\beta/2})} \sum_{\lambda, \mu} \log(\frac{\lambda e^{-\beta/2}}{e^{\beta/2} + e^{-\beta/2}}) - \log(\frac{\lambda \mu e^{\beta/2}}{e^{\beta/2} + e^{-\beta/2}})\tau(d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\lambda d_\phi^{1/2}[a, x]d_\phi^{1/2}e_\mu) \]
\[ \frac{\log \lambda - \log \mu}{\lambda - \mu} \geq e^{-\beta/2} \frac{\log(\lambda e^{-\beta/2}) - \log(\mu e^{\beta/2})}{\lambda e^{-\beta/2} - \mu e^{\beta/2}} \]

**Proof.** Consider the following integral representation:

\[\frac{\lambda e^{-\beta/2} - \mu e^{\beta/2}}{\log(\lambda e^{-\beta/2}) - \log(\mu e^{\beta/2})} = \int_0^1 (\lambda e^{-\beta/2})^t (\mu e^{\beta/2})^{1-t} dt = \int_0^1 e^{\beta/2 - \beta t} \lambda^t \mu^{1-t} dt \]

\[\geq e^{-\beta/2} \int_0^1 \lambda^t \mu^{1-t} dt = e^{-\beta/2} \frac{\lambda - \mu}{\log \lambda - \log \mu} \]

Therefore we have \( \frac{\log \lambda - \log \mu}{\lambda - \mu} \geq e^{-\beta/2} \frac{\log(\lambda e^{-\beta/2}) - \log(\mu e^{\beta/2})}{\lambda e^{-\beta/2} - \mu e^{\beta/2}} \).

The previous proposition relates the entropy production of the augmented system (i.e. \( \mathbb{M}_2 \otimes \mathcal{M} \)) with generator \( A \) and state \( \rho \), to the entropy production of the original system (i.e. \( \mathcal{M} \)) with generator \( a \) and state \( d_\phi^{1/2} xd_\phi^{1/2} \).

For the next proposition, we first make the following observation. Let \( a = u|a| \) be the polar decomposition of \( a \), then \( a^* = |a|^* u^* \). Moreover, in \( \mathbb{M}_2 \otimes \mathcal{M} \), we have

\[ A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & |a| \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}. \]

Similarly, we have

\[ A^* = \begin{pmatrix} 0 & 0 \\ a^* & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ |a| & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}. \]

In the following, we shall denote \( U := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \). A final observation is that:

\[ H := A + A^* = U \begin{pmatrix} 0 & |a| \\ |a| & 0 \end{pmatrix} U^* = U(X \otimes |a|)U^* \]

where \( X \in \mathbb{M}_2 \) is the Pauli matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Let \( \sigma(|a|) \) be the spectrum of \( |a| \), then \( \{ |\lambda \pm \mu| : \lambda \neq \mu, \lambda, \mu \in \sigma(|a|) \} \) is the set of spectral difference of \( X \otimes |a| \).

**Proposition 3.2.** Let \( \delta = \min \{|\lambda \pm \mu| : \lambda \neq \mu, \lambda, \mu \in \sigma(|a|)\} \). Let \( C^*\{|a|\} \) be the \( C^* \)-algebra generated by \( |a| \) and let \( E_a : \mathcal{M} \to C^*\{|a|\} \) be the conditional expectation onto its commutant. Similarly define \( C^*\{|a^*|\} \) with the conditional expectation \( E_{a^*} \). Then for any state \( \rho = d_\phi^{1/2} x d_\phi^{1/2} \) on \( \mathcal{M} \), we have

\[ D(\rho|E_a^* \rho) \leq \frac{C(\beta)}{\delta^2} (EP_a(\rho) + EP_{a^*}(\rho)) \]

where the constant \( C(\beta) \) only depends on \( \beta \). The same holds for \( a^* \):

\[ D(\rho|E_{a^*} \rho) \leq \frac{C(\beta)}{\delta^2} (EP_a(\rho) + EP_{a^*}(\rho)) \]
Proof. Let $N := C^*(X \otimes |a|)$ be the $C^*$-subalgebra of $M_2 \otimes \mathcal{M}$ generated by $X \otimes |a|$, and let $E_{N'}$ be the conditional expectation onto the commutant $N'$. Since $X \otimes |a|$ is a self-adjoint operator with discrete spectrum

$$\sigma(X \otimes |a|) = \{ |\lambda \pm \mu| : \lambda, \mu \in \sigma(|a|) \}$$

Theorem 2.8 shows that there exists an absolute constant $C > 0$ such that for any state $\tilde{\rho} \in S_1(M_2 \otimes \mathcal{M})$

$$D(\tilde{\rho}) \leq \frac{C}{\delta^2} E_{X \otimes |a|}(\tilde{\rho})$$

where $E_{X \otimes |a|}(\tilde{\rho}) = \langle [X \otimes |a|, \tilde{\rho}], J^\log_{\tilde{\rho}}[X \otimes |a|, \tilde{\rho}] \rangle$ is the entropy production with generator $X \otimes |a|$. By the triangle inequality

$$E_{X \otimes |a|}(\tilde{\rho}) = \langle [e_{12} \otimes |a| + e_{21} \otimes |a|, \tilde{\rho}], J^\log_{\tilde{\rho}}[e_{12} \otimes |a| + e_{21} \otimes |a|, \tilde{\rho}] \rangle$$

$$\leq 2(\langle [e_{12} \otimes |a|, \tilde{\rho}], J^\log_{\tilde{\rho}}[e_{12} \otimes |a|, \tilde{\rho}] \rangle + \langle [e_{21} \otimes |a|, \tilde{\rho}], J^\log_{\tilde{\rho}}[e_{21} \otimes |a|, \tilde{\rho}] \rangle)$$

Using the notation introduced immediately before the proposition, we apply the inequality for the state $Ad_{U^*}(\rho_\beta)$ where

$$\rho_\beta = (d_\beta \otimes d_\phi)^{1/2}(1 \otimes x)(d_\beta \otimes d_\phi)^{1/2}$$

and

$$d_\beta := \left( \begin{array}{cc} e^{-\beta/2}/(e^{\beta/2} + e^{-\beta/2}) & 0 \\ 0 & e^{\beta/2}/(e^{\beta/2} + e^{-\beta/2}) \end{array} \right)$$

Then we have

$$E_{X \otimes |a|}(Ad_{U^*}(\rho_\beta)) \leq 2(\langle [e_{12} \otimes |a|, Ad_{U^*}(\rho_\beta)], J^\log_{Ad_{U^*}(\rho_\beta)}[e_{12} \otimes |a|, Ad_{U^*}(\rho_\beta)] \rangle + \langle [e_{21} \otimes |a|, Ad_{U^*}(\rho_\beta)], J^\log_{Ad_{U^*}(\rho_\beta)}[e_{21} \otimes |a|, Ad_{U^*}(\rho_\beta)] \rangle)$$

$$= 2(\langle [A, \rho_\beta], J^\log_{\rho_\beta}[A, \rho_\beta] \rangle + \langle [A^*, \rho_\beta], J^\log_{\rho_\beta}[A^*, \rho_\beta] \rangle)$$

$$= 2(EP_A(\rho_\beta) + EP_{A^*}(\rho_\beta))$$

$$\leq C(\beta)(EP_A(\rho) + EP_{A^*}(\rho))$$

where the last inequality follows from Proposition 3.1 and $C(\beta)$ is a constant depending only on $\beta$. This inequality provides an upper bound on the entropy production term.

On the other hand, since adjoint action $Ad_{U^*}$ preserves the diagonal subalgebra, we have

$$Ad_{U^*}(N') \cap \ell^2_\infty(\mathcal{M}) = \{ \begin{pmatrix} u xx^* u^* & 0 \\ 0 & y \end{pmatrix} : x, y \in C^*([a]') \}$$

In addition we deduce from the diagonal structure:

$$D(Ad_{U^*}(\rho_\beta)|E_{Ad_{U^*}(N')}Ad_{U^*}(\rho_\beta)) = \frac{e^{\beta}}{2(1 + e^{\beta})} D(u^* \rho u|E_{a^*}^* u^* \rho u) + \frac{1}{2(1 + e^{\beta})} D(\rho|E_{a^*}^* \rho)$$

$$\geq \frac{1}{2(1 + e^{\beta})} D(\rho|E_{a^*}^* \rho)$$
Combining these two inequalities, we find a constant $C(\beta)$ such that

$$D(\rho|E_a^*\rho) \leq \frac{C(\beta)}{\delta^2}(EP_a(\rho) + EP_{a^*}(\rho)) \tag{3.20}$$

Apply the same argument to $a^*$, we have

$$D(\rho|E_{a^*}\rho) \leq \frac{C(\beta)}{\delta^2}(EP_a(\rho) + EP_{a^*}(\rho)) \tag{3.21}$$

\[\square\]

**Corollary 3.3.** Using the same notation as above, suppose $\Omega := C^*|a|' \cap C^*|a^*|'$ and the conditional expectations form a commuting square, i.e.

$$E_\Omega = E_a E_{a^*} = E_{a^*} E_a \tag{3.22}$$

Then there exists a constant $C(\beta)$ depending only on $\beta$ such that

$$D(\rho|E_{\Omega}\rho) \leq \frac{C(\beta)}{\delta^2}(EP_a(\rho) + EP_{a^*}(\rho)) \tag{3.23}$$

**Proof.** By a well-known approximate tensorization result [Pet91], if a pair of conditional expectations satisfies the commuting square condition, we have

$$D(\rho|E_{\Omega}\rho) \leq D(\rho|E_a^*\rho) + D(\rho|E_{a^*}\rho)$$

Hence by proposition 3.2, we get

$$D(\rho|E_{\Omega}\rho) \leq \frac{C(\beta)}{\delta^2}(EP_a(\rho) + EP_{a^*}(\rho)) \tag{3.24}$$

for some constant $C(\beta)$.

\[\square\]

4. **APPLICATION TO SU(2)-LINDBLADIANs**

In this section, we apply results from last section to Lindbladians generated by tensor product representation of $\mathfrak{su}(2)$. Such Lindbladians can be used to describe dynamics of open physical systems with $\mathfrak{su}(2)$-symmetry (for example, a chain of qubits). The main goal of this section is to estimate the CLSI constant of these $\mathfrak{su}(2)$-Lindbladians.

4.1. **Structure and Representation Theory of $\mathfrak{su}(2)$**. Recall that the Lie algebra $\mathfrak{su}(2)$ is a 3-dimensional real Lie algebra. [FH04] Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ form a basis of $\mathfrak{su}(2)$. For our purpose, we need to consider the complexification of $\mathfrak{su}(2)$. Recall the complexification of $\mathfrak{su}(2)$ is given by:

$$\mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C}) := \{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \text{tr}(A) = 0, A \in M_2(\mathbb{C}) \}$$

As a complex Lie algebra, $\text{dim}_\mathbb{C}(\mathfrak{sl}(2, \mathbb{C})) = 3$. There exists another canonical basis for $\mathfrak{sl}(2, \mathbb{C})$ (Cartan-Weyl basis). The transformation from Pauli matrices to Cartan-Weyl basis is given by:

$$a = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
In physics literature, $a$ is more commonly known as creation operator, $a^*$ the annihilation operators and $h$ the Hamiltonian. Cartan-Weyl basis satisfies the following relation:

$$[h, a] = 2a, \ [h, a^*] = -2a^*, \ [a, a^*] = h$$

Recall the fundamental representation of $\mathfrak{su}(2)$ acts on the two-dimensional vector space $V := \mathbb{C}^2$ by matrix multiplication. The fundamental representation is the irreducible representation of $\mathfrak{su}(2)$ of the smallest dimension. All irreducible representations of $\mathfrak{su}(2)$ are completely known. For each dimension $n \geq 1$, there exists a unique irreducible representation:

$$\pi(n) : \mathfrak{su}(2) \to \mathbb{M}_{n+1}(\mathbb{C})$$

Weight theory shows that there exists a unique highest weight vector $|n, 0\rangle \in \mathbb{C}^{n+1}$ such that

$$a|n, 0\rangle = 0 \quad \quad h|n, 0\rangle = n|n, 0\rangle$$

From the highest weight vector, a complete basis for $\mathbb{C}^{n+1}$ is given by $\{|n, j\rangle\}_{0 \leq j \leq n}$ such that

$$h|n, j\rangle = (n - 2j)|n, j\rangle$$

$$a^*|n, j\rangle = \sqrt{(j + 1)(n - j)}|n, j + 1\rangle$$

$$a|n, j\rangle = \sqrt{j(n - j + 1)}|n, j - 1\rangle$$

Since the coefficients will be used repeatedly later, we denote $\alpha_{n,j} := \sqrt{j(n - j + 1)}$.

The $N$-fold tensor product $V^\otimes N := (\mathbb{C}^2)^\otimes N$ admits a tensor product representation:

$$\pi_N := \sum_{1 \leq j \leq N} \pi^j = \sum_{1 \leq j \leq N} 1 \otimes \ldots \otimes \pi_{(1)} \otimes \ldots \otimes 1 : \mathfrak{su}(2) \to \mathbb{M}_{2^N}(\mathbb{C})$$

$$\pi_N(g)(v_1 \otimes \ldots \otimes v_N) = \sum_{1 \leq j \leq N} v_1 \otimes \ldots \otimes (\pi_{(j)}v_j) \otimes \ldots \otimes v_N$$

The tensor product representation $\pi_N = \pi_N^{(1)}$ is not irreducible. Decomposition of $\pi_N$ is given by the Schur-Weyl duality. The basic observation of the Schur-Weyl duality is that, in addition to left action of $\mathfrak{su}(2)$, the symmetric group $S_N$ acts on $V^\otimes N$ from the right:

$$\forall \sigma \in S_N \text{ and } v_1 \otimes \ldots \otimes v_N \in V^\otimes N$$

$$(v_1 \otimes \ldots \otimes v_N) \cdot \sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(N)}$$

Since the left action by $\mathfrak{su}(2)$ commute with the right action by $S_N$, by double centralizer theorem $V^\otimes N$ has the following decomposition as a $\mathfrak{su}(2) - S_N$ bimodule:

$$V^\otimes N \cong \bigotimes_{\lambda \in \mathcal{P}(N)} V_\lambda \otimes W_\lambda$$

where $\lambda := (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l)$ is an ordered partition of $N$, $W_\lambda$ is the irreducible representation of $S_N$ labeled by the Young’s diagram with $l$-rows where the $i$-th row has $\lambda_i$ boxes. In addition, $V_\lambda \cong \text{Hom}_{\mathbb{C}[S_N]}(W_\lambda, V^\otimes N)$ is the $\mathbb{C}[S_N]$-module map from $W_\lambda$ to $V^\otimes N$.

Under the left $\mathfrak{su}(2)$ action, $V_\lambda$ is an irreducible representation. It is a fact that $V_\lambda \neq 0$ if the Young diagram $\lambda$ has no more than $\text{dim}(V)$-rows. Hence for fundamental representation of $\mathfrak{su}(2)$, $V_\lambda \neq 0$ if $\lambda$ has at most two rows.
As \( \mathfrak{su}(2) \)-modules, we obtain an irreducible decomposition:

\[
V \otimes N \cong \bigoplus_{\lambda \in \mathcal{P}_2(N)} V_\lambda^{\otimes \dim(W_\lambda)}
\]

where \( \mathcal{P}_2(N) \) is the set of partitions of \( N \) with at most 2 parts. Note that the multiplicity of each irreducible component is given by the dimension of \( W_\lambda \). It is well-known that the dimension of \( W_\lambda \) is given by the hook length formula of Young diagrams associated with \( \lambda \).

The Young diagram with only one row corresponds to the irreducible component with the largest dimension. We denote this subrepresentation by \( V_{(N)} \) since it corresponds to the trivial partition of \( N \). The following fact is well known \cite{FH04}.

**Lemma 4.1.**

\[
\dim(V_{(N)}) = N + 1
\]

In addition, let \( \{e_0, e_1\} \) be a basis of the fundamental representation where \( e_0 \) is the highest weight vector and \( e_1 \) is the lowest weight vector, then the highest weight vector of \( V_{(N)} \) is given by

\[
|N, 0\rangle = e_0 \otimes N
\]

Moreover, the weight space decomposition gives \( V_{(N)} \) a basis of totally symmetric vectors \( (v_k)_{0 \leq k \leq N} \), where

\[
v_k := \sum_{A \subset [[N]],|A| = k} \frac{1}{\binom{N}{k}} v_A = \sum_{A \subset [[N]],|A| = k} \frac{1}{\binom{N}{k}} (\bigotimes_{j \in A} e_1) \otimes (\bigotimes_{j \notin A} e_0)
\]

where \( A \) is a cardinality \( k \) subset of \( [[N]] := \{1, ..., N\} \), and \( v_A \) is a simple tensor product of \( e_0, e_1 \) where the \( j \)-th tensor factor is \( e_1 \) if and only if \( j \in A \).

To apply results from last section, we need the following lemma.

**Lemma 4.2.** Let \( \mathcal{H} := \bigoplus_{n=1}^{\infty} \mathbb{C}^{n+1} \) and consider the direct sum representation \( \pi := \bigoplus_{n=1}^{N} \pi_n \). Then we have:

i) The commutant of \( |\pi(a)| \) in \( \mathbb{B}(\mathcal{H}) \) is given by

\[
C^*(|\pi(a)|)' = \text{span}\{|n, j\rangle\langle m, k| : \alpha_{n,j} = \alpha_{m,k}\}
\]

ii) Let \( E_{\pi(a)} \) be the conditional expectation onto \( C^*(|\pi(a)|)' \) and \( E_{\pi(a^*)} \) be the conditional expectation onto \( C^*(|\pi(a^*)|)' \). Then

\[
E_{\pi(a)}E_{\pi(a^*)} = E_{\pi(a^*)}E_{\pi(a)}
\]

iii) The intersection of \( C^*(|\pi(a)|)' \cap C^*(|\pi(a^*)|)' \) is contained in \( \bigoplus_{n=1}^{N} \mathbb{C}^{n+1} \).

**Proof.** i) Since \( \pi(a) = \sum_{1 \leq n \leq N} \sum_{0 < j \leq n} \alpha_{n,j} |n, j-1\rangle \langle n, j| \), then

\[
|\pi(a)| = \sum_{1 \leq n \leq N} \sum_{0 \leq j \leq n} \alpha_{n,j} |n, j\rangle \langle n, j|
\]

(Note \( \alpha_{n,j} = 0 \) when \( j = 0 \).) Therefore a matrix

\[
x := \sum_{(n,j), (m,k)} x_{(n,j),(m,k)} |n, j\rangle \langle m, k| \in \mathbb{B}(\mathcal{H})
\]

commutes with \( |\pi(a)| \) if and only if \( x_{(n,j),(m,k)} = 0 \) whenever \( \alpha_{n,j} \neq \alpha_{m,k} \).
Since $\pi(a^*) = \sum_{1 \leq n \leq N} \sum_{0 \leq j < n} \alpha_{n,j+1}|n, j+1\rangle\langle n, j|$, then

$$|\pi(a^*)| = \sum_{1 \leq n \leq N} \sum_{0 \leq j < n} \alpha_{n,j+1}|n, j\rangle\langle n, j|$$

(Note $\alpha_{n,j+1} = 0$ when $j = n$.) Therefore a matrix $x$ commutes with $|\pi(a^*)|$ if and only if $x_{(n,j),(m,k)} = 0$ whenever $\alpha_{n,j+1} \neq \alpha_{m,k+1}$.

i) By computation above, both conditional expectations $E_{\pi(a)}$ and $E_{\pi(a^*)}$ are given by Schur multipliers in the basis of $|n, j\rangle$. Therefore they commute.

ii) By computation above, both conditional expectations $E_{\pi(a)}$ and $E_{\pi(a^*)}$ are given by Schur multipliers in the basis of $|n, j\rangle$. Therefore they commute.

iii) Moreover, we can give a concrete description of $\Omega := C^*|\pi(a)||C^*|\pi(a^*)|$' (Note $\alpha_{n,j+1} = 0$ when $j = n$.) Therefore a matrix $x$ commutes with $|\pi(a^*)|$ if and only if $x_{(n,j),(m,k)} = 0$ whenever $\alpha_{n,j+1} \neq \alpha_{m,k+1}$.

Then by an explicit computation (see the lemma below), we have $n = m, j = k$. Hence $|n, j\rangle\langle m, k| \in \Omega$ if and only if $|n, j\rangle\langle m, k|$ belongs to the diagonal subalgebra $\bigoplus_{n=1}^N \ell^1_{n+1}$.

**Lemma 4.3.** $\alpha_{n,j} = \alpha_{m,k}$ and $\alpha_{n,j+1} = \alpha_{m,k+1}$ hold simultaneously if and only if $n = m$ and $j = k$

**Proof.** One direction is obvious. For the other direction, denote $\delta := n - 2j$. Since $n - 2j = \alpha_{n,j+1} - \alpha_{n,j} = \alpha_{m,k+1} - \alpha_{m,k} = m - 2k$, then $m - 2k = \delta$. Then we have

$$\alpha_{n,j}^2 - \alpha_{m,k}^2 = j(n - j + 1) - k(m - k + 1) = j(\delta + j + 1) - k(\delta + k + 1) = (j - k)(\delta + 1 + j + k) = (j - k)(j + 1 + m - k)$$

By assumption, this equals to 0. If $j \neq k$, then $j + 1 + m - k = 0$. Since $j \geq 0$ and $k \leq m$, $j + 1 + m - k \geq 1$. Hence $j = k$. Then $n = \delta + 2j = \delta + 2k = m$. \hspace{1cm} \blacksquare

We are mainly interested in Lindbladian associated with the representation $\pi_N$. To apply the result of the last section, we need to check whether $|\pi_N(a)|$ has a uniform spectral gap.

**Lemma 4.4.** Using the notation above and fix the weight basis $(|n, j\rangle)_{0 \leq j \leq n}$ for each irreducible component of $\pi_N$. Then for different $\alpha_{n,j}, \alpha_{m,k}$, the uniform spectral gap is bounded below by

$$(4.10) \quad |\alpha_{n,j} - \alpha_{m,k}| \gtrsim \frac{1}{N}$$

In particular, this lower bound can be achieved by $n = m = N$. And $j = \frac{N+1}{2}, |j - k| = 1$ if $N + 1$ is even, and $j = \frac{N}{2}, k = \frac{N-2}{2}$ if $N + 1$ is odd.

**Proof.** If either $j = 0$ or $k = 0$, then without loss of generality we have

$$|\alpha_{n,j} - \alpha_{m,k}| = |\alpha_{n,j}| \gtrsim \sqrt{n} \gtrsim 1$$

The last equation is achieved for $n \sim 1$.

Assume from now on $j > 0$ and $k > 0$. Without loss of generality, assume $n \geq m$. If $n = m$, then we have

$$\alpha_{n,j} - \alpha_{n,k} = \sqrt{j(n - j + 1)} - \sqrt{k(n - k + 1)}$$

Define $f(\epsilon) := \sqrt{\left(\frac{n+1}{2} + \epsilon\right)\left(\frac{n+1}{2} - \epsilon\right)}$ where $-\frac{n+1}{2} < \epsilon \leq \frac{n-1}{2}$ and $\frac{n+1}{2} + \epsilon \in \mathbb{Z}$. By symmetry, we can restrict to $\epsilon \geq 0$. Then optimization with respect to $\epsilon$ shows that the unit difference
\[ f(\epsilon) - f(\epsilon + 1) \text{ achieves minimum when } \epsilon \text{ is roughly 0. More precisely, if } n + 1 \text{ is even, the minimum is achieved by } \epsilon = 0. \text{ In this case,} \]

\[ (4.11) \quad f(\epsilon) - f(\epsilon + 1) = \frac{n + 1}{2} - \sqrt{\left(\frac{n + 1}{2}\right)^2 - 1} \geq \frac{n + 1}{2} \left(1 - 1 + \frac{1}{2} \left(\frac{2}{n + 1}\right)^2\right) = \frac{1}{n + 1} \]

If \( n + 1 \) is odd, the minimum is achieved by \( \epsilon = \frac{1}{2} \). In this case,

\[ f(\epsilon) - f(\epsilon + 1) = \sqrt{\frac{n(n + 2)}{4}} (1 - \sqrt{\frac{8}{n(n + 2)}}) \geq \sqrt{\frac{4}{n(n + 2)}} \]

Therefore in this case, \( \alpha_{n,j} - \alpha_{n,k} \) achieves minimum at \( |j - k| = 1 \) and \( j \sim \frac{n+1}{2} \).

If \( n > m \), then we have

\[ \alpha_{n,j} - \alpha_{m,k} = \alpha_{n,j} - \alpha_{n,k} + \alpha_{n,k} - \alpha_{m,k} \]

Since \( \alpha_{n,k} - \alpha_{m,k} = \sqrt{n + m} - \sqrt{n - m} \) and the right hand side as a function of \( n - m \) is monotone decreasing, then \( m = n - 2 \). (In the case of tensor product representation, \( |n - m| \geq 2 \).) Now we consider two cases. If \( j = k \), then \( \alpha_{n,j} - \alpha_{m,k} = \alpha_{n,k} - \alpha_{n-2,k} \). Since \( \alpha_{n,k} - \alpha_{n-2,k} \) as a function of \( k \) is monotone decreasing, the minimum is achieved when \( k = 1 \) and the minimum is of the order \( \frac{1}{\sqrt{n}} \).

The remaining case is \( j \neq k \) and \( n > m \). This is the most delicate case. We first make the following observation.

**Claim:** Let \( a, b, c, d > 0 \) be four parameters and let \( 0 < \eta < \min\{1, \frac{a}{b}, \frac{c}{d}\} \). If \( \sqrt{a + b\eta} - \sqrt{c + d\eta} = O(\eta^2) \), then \( a = c \) and \( b = d \) and \( \sqrt{a + b\eta} - \sqrt{c + d\eta} = 0 \)

**Proof:**

\[ \sqrt{a + b\eta} - \sqrt{c + d\eta} = \sqrt{a}(1 + \frac{b\eta}{2a} + O(\eta^2)) - \sqrt{c}(1 + \frac{d\eta}{2c} + O(\eta^2)) \]

Then by assumption, the following two polynomials in \( \eta \) are equal: \( \sqrt{a}(1 + \frac{b\eta}{2a}) = \sqrt{c}(1 + \frac{d\eta}{2c}) \).

Thus \( a = c \) and \( b = d \), and \( \sqrt{a + b\eta} = \sqrt{c + d\eta} \).

Back to the main proof, the case we are interested in is \( j \neq k \) and \( n > m \). In addition, we assumed \( \alpha_{n,j} \neq \alpha_{m,k} \). Consider the following cases:

1. If \( \frac{m}{n} \ll 1 \), then \( 0 < k \leq m \sim 1 \). Hence \( \alpha_{m,k} \sim 1 \). In this case, \( \alpha_{n,j} - \alpha_{m,k} \sim \alpha_{n,j} - 1 \). Since \( \sqrt{n} \lesssim \alpha_{n,j} \lesssim n \) (the lower bound is achieved by \( j \sim 1 \) or \( j \sim n \), and the upper bound is achieved by \( j \sim \frac{n+1}{2} \)), then \( |\alpha_{n,j} - \alpha_{m,k}| \gtrsim \sqrt{n} \).

2. If \( \frac{m}{n} \sim 1 \), then there exists a constant \( \gamma \sim 1 \) such that \( m = \gamma n \). If \( n \sim 1 \), then in this case \( |\alpha_{n,j} - \alpha_{m,k}| \sim 1 \). Assume from now on \( n \gg 1 \). We consider two cases.

   i) Suppose \( \frac{k}{m} \ll 1 \), then \( k \sim 1 \ll n \). Then in this case, there exists constants \( a, b > 0 \) of order 1, such that \( \alpha_{m,k} = \sqrt{an + b} \). For \( \alpha_{n,j} \), by symmetry assume \( 0 < j \leq \frac{n+1}{2} \). For \( j \lesssim n \), \( \alpha_{n,j} \sim n \gg \sqrt{an + b} = \alpha_{m,k} \). Hence in this case, \( |\alpha_{n,j} - \alpha_{m,k}| = n \). For \( j = \sim 1 \ll n \), then there exists constants \( c, d > 0 \) of order 1, such that \( \alpha_{n,j} = \sqrt{cn + d} \). Then by the claim and the assumption that \( \alpha_{n,j} \neq \alpha_{m,k} \), we have that in this case \( |\alpha_{n,j} - \alpha_{m,k}| \gtrsim \frac{1}{\sqrt{n}} \).

   ii) Now suppose \( k \lesssim m \sim n \). There exists a constant \( \beta \) of order 1 such that \( k = \beta n \). Hence there exists constants \( a, b > 0 \) of order 1 such that \( \alpha_{m,k} = \sqrt{an^2 + bn} \). For \( j \lesssim n \), there exists constants \( c, d > 0 \) of order 1 such that \( \alpha_{n,j} = \sqrt{cn^2 + dn} \). Then
by the claim and the assumption that \( \alpha_{n,j} \neq \alpha_{m,k}, |\alpha_{n,j} - \alpha_{m,k}| \geq 1 \). For \( j \sim 1 \ll n \),
\[
\alpha_{n,j} \sim \sqrt{n} \ll \sqrt{an^2 + bn} = \alpha_{m,k}.
\]
Hence in this case, \( |\alpha_{n,j} - \alpha_{m,k}| \sim n \).

This concludes the analysis of the last case.

Combining all the cases, we see that the minimum is of the order \( \frac{1}{n} \). Since \( 1 \leq n \leq N \), the minimum is of the order \( \frac{1}{N} \). And this is achieved by \( n = m = N \) and \( j = \frac{N+1}{2} \), \( |j - k| = 1 \) if \( N + 1 \) is even, and \( j = \frac{N+1}{2}, k = \frac{N-1}{2} \) if \( N + 1 \) is odd. \( \blacksquare \)

**Corollary 4.5.** Using the same notation as [4.4] for integer \( \gamma \geq 2 \) and \( \alpha_{n,j} \neq \alpha_{m,k} \)

\[
|\alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma| \geq \max\{n, m\}^{(\gamma-2)/2}
\]

In particular, the lower bound is of the order \( 1 \) for \( 1 \leq n \leq N \).

**Proof.** Consider two cases. First assume \( n = m \). Then by mean value theorem, we have

\[
|\alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma| \geq \frac{\gamma}{2} \min\{\alpha_{n,j}^{\gamma-2} |n + 1 - 2j|, \alpha_{m,k}^{\gamma-2} |n + 1 - 2k|\}
\]

By symmetry, assume \( 0 \leq j, k \leq \frac{n+1}{2} \). Then we have

\[
|\alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma| \geq \min\{n \gamma/2, n \gamma/2\}
\]

Therefore for \( \gamma = 2, 3 \), the difference is bounded below by \( n^{\gamma/2} \). For \( \gamma \geq 4 \), the difference is bounded below by \( n^{\gamma/2} \).

Now consider the case \( n \neq m \). Without loss of generality, assume \( \alpha_{n,j} > \alpha_{m,k} \). We proceed by induction.

For \( \gamma = 2 \), by the proof of lemma [4.4], \( \alpha_{n,j} - \alpha_{m,k} \geq \frac{1}{\sqrt{\max\{n, m\}}} \). In this case, \( \alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma \geq \frac{1}{\sqrt{\max\{n, m\}}} (\alpha_{n,j} + \alpha_{m,k}) \). Since \( \alpha_{n,j} \geq \sqrt{n} \), then we have \( \alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma \geq 1 \).

Now assume the claim holds up to \( \gamma \). For \( \gamma + 1 \), we have

\[
\alpha_{n,j}^{\gamma+1} - \alpha_{m,k}^{\gamma+1} = (\alpha_{n,j} - \alpha_{m,k})\alpha_{n,j}^\gamma + \alpha_{m,k}(\alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma)
\]

\[
\geq (\alpha_{n,j} - \alpha_{m,k})\alpha_{n,j}^\gamma + \sqrt{\max\{n, m\}} \max\{n, m\}^{(\gamma-2)/2}
\]

By the proof of [4.4], \( \alpha_{n,j} - \alpha_{m,k} \geq 1/\sqrt{n} \) and \( \alpha_{n,j} \geq \sqrt{n} \). Therefore \( \alpha_{n,j}^{\gamma+1} - \alpha_{m,k}^{\gamma+1} \geq \max\{n, m\}^{(\gamma-1)/2} \).

Combining the cases, we see that for \( \gamma \geq 2 \)

\[
|\alpha_{n,j}^\gamma - \alpha_{m,k}^\gamma| \geq \max\{m, n\}^{(\gamma-2)/2}
\] \( \blacksquare \)

**Remark 4.6.** Since the dimension of \( \pi_N \) is of the order \( 2^N \), the uniform spectral gap of \( |\pi_N(a)| \) is of the order \( \Theta\left(\frac{1}{\sqrt{\log \dim(\pi_N)}}\right) \).

4.2. CLSI Constant of su(2)-Lindbladian. Recall that the Lindbladian we are interested in is given by:

\[
(4.12) \quad \mathcal{L}_N^\beta(x) = \beta^{1/2} L_{\pi_N(a)}(x) + \beta^{-1/2} L_{\pi_N(a^*)}(x).
\]

The next lemma shows that we may restrict this Lindbladian to any subrepresentation.

**Lemma 4.7.** Each irreducible component \( V_\lambda \) of \( \mathcal{H} := V^\otimes N \) is invariant under the Lindbladian \( \mathcal{L}_N^\beta \). Hence given any subset of partitions \( A \subseteq \mathcal{P}_2(N) \), if an operator \( x \in \mathbb{B}(\bigoplus_{\lambda \in A} V_\lambda) \), then \( \mathcal{L}_N^\beta(x) \in \mathbb{B}(\bigoplus_{\lambda \in A} V_\lambda) \).
Proof. The proof is obvious. For any $x \in \mathbb{B}(V_{\lambda})$, we can expand it in terms a basis:

\[
    x = \sum_{i,j=1}^{\dim(V_{\lambda})} x_{ij} |i\rangle \langle j|
\]

Since $\pi_N(a) |i\rangle \in V_{\lambda}$, $\pi_N(a^*) |j\rangle \in V_{\lambda}$, it is clear that each term in the Lindbladian maps $x$ to an operator on $V_{\lambda}$. The statement about direct sum of irreducible components is obvious. 

Recall Lie algebra relation $[\pi_N(a^*) , \pi_N(a)] = \pi_N(h)$. We consider the density

\[
    d_N := N' \exp\left(-\frac{\beta}{2} \pi_N(h)\right)
\]

where $N'$ is a normalization constant depending on dimension only. By simple computation, we have

**Lemma 4.8.** Following two equations hold:

\[
    d_N^t \pi_N(a) d_N^{-t} = e^{i\beta t} \pi_N(a) \\
    d_N^t \pi_N(a^*) d_N^{-t} = e^{-i\beta t} \pi_N(a^*)
\]

Proof. Since $[\pi_N(h), \pi_N(a)] = -2\pi_N(a)$, we have $[\pi_N^n(h), \pi_N(a)] = (-2)^n \pi_N(a)$. Hence

\[
    d_N^t \pi_N(a) d_N^{-t} = \exp\left(-\frac{i\beta t}{2} \pi_N(h)\right) \pi_N(a) \exp\left(i\beta t \pi_N(h)\right) \\
    = \sum_{n \geq 0} \frac{(-i\beta t)^n}{2^n n!} [\pi_N^n(h), \pi_N(a)] \\
    = \sum_{n \geq 0} \frac{(-i\beta t)^n}{2^n n!} (-2)^n \pi_N(a) = e^{i\beta t} \pi_N(a)
\]

Similar calculation holds for the other equation. 

Before applying results from section 3, we make the following simplification.

**Lemma 4.9.** Let $V^\otimes N = \bigoplus_{\lambda \in P_{2}(N)} V_\lambda^\otimes \dim(W_{\lambda})$ be the irreducible decomposition of $\pi_N$-representation. Let $n_0 = \max_{\lambda \in P_{2}(N)} \{\dim(W_{\lambda})\}$, then the inclusion map:

\[
    \iota : \mathbb{B}(V^\otimes N) \to \mathbb{B}\left(\bigoplus_{\lambda} V_{\lambda} \otimes \mathbb{M}_{n_0}\right)
\]

satisfies the following intertwining relation:

\[
    \iota \circ \mathcal{L}_N^{\beta} = (\mathcal{L}_n^{\beta} \otimes id) \circ \iota
\]

where $\mathcal{L}_n^{\beta}$ is the Lindbladian associated with the direct sum representation $\pi = \bigoplus_{\lambda} \pi_{\lambda}$.

Proof. Since $\mathcal{L}_N^{\beta}$ preserves irreducible decomposition, the intertwining relation is obvious. 

Therefore by tensorization property of the CLSI constant, to estimate $\text{CLSI}(\mathcal{L}_N^{\beta})$ it suffices to consider $\mathcal{L}_n^{\beta}$. Then by results of the last section and lemma above, we have the following estimate.

Recall the intermediate subalgebra is defined by $\Omega = C^*(|\pi_N(a)|') \cap C^*(|\pi_N(a^*)|')$. 

Proposition 4.1. Let $\rho_N := d_N^{1/2} x d_N^{1/2}$ be a state in $S_1(V^\otimes N)$ and $x \in \mathcal{B}(V^\otimes N)$. Then there exists a constant $C(\beta)$ depending only on $\beta$ such that

$$D(\rho_N | E_\Omega^* \rho_N) \leq C(\beta) N^2 (EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N))$$

where $E_\Omega$ is the conditional expectation onto $\Omega$.

Proof. By commutativity of the conditional expectations: $E_\Omega = E_{\pi_N(a)} E_{\pi_N(a^*)} = E_{\pi_N(a^*)} E_{\pi_N(a)}$, the relative entropy factorizes

$$D(\rho_N | E_\Omega^* \rho_N) = D(\rho_N | E_{\pi_N(a)}^* \rho_N) + D(\rho_N | E_{\pi_N(a^*)}^* \rho_N)$$

Since uniform spectral gap of $|\pi_N(a)|$ is of the order $\Theta(1/N)$, by proposition 3.2 there exists a constant $C(\beta)$ such that

$$D(\rho_N | E_{\pi_N(a)}^* \rho_N) \leq C(\beta) N^2 (EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N))$$

$$D(\rho_N | E_{\pi_N(a^*)}^* \rho_N) \leq C(\beta) N^2 (EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N))$$

Combing these two inequalities, we have the desired result. \hfill \blacksquare

Lemma 4.10. The fixed point algebra $\Omega_{fix}$ of $L_N^\beta$ in $V^\otimes N$ is $\bigoplus_\lambda \mathcal{B}(W_\lambda)$. In particular, $\Omega_{fix} \subset \Omega$.

Proof. Given the faithful density $d_N$, we obtain an embedding:

$$\iota_N : \mathcal{B}(V^\otimes N) \to S_2(V^\otimes N) : x \mapsto d_N^{1/4} x d_N^{1/4}$$

Recall that on the Schatten 2-space, the Lindbladian $L_N^\beta$ uniquely defines a Dirichlet form. Restricted to the dense subspace $\iota_N(\mathcal{B}(V^\otimes N))$, the Dirichlet form is given by the formula:

$$\langle \xi, \xi \rangle_L = \langle \iota_N(x), \iota_N(x) \rangle_L$$

$$= ||d_N^{1/4}(\pi_N(a)x - x\pi_N(a))d_N^{1/4}||^2_2 + ||d_N^{1/4}(\pi_N(a^*)x - x\pi_N(a^*))d_N^{1/4}||^2_2$$

where $|| \cdot ||^2_2$ is the Hilbert-Schmidt inner product. Therefore $x \in \Omega_{fix}$ if and only if $[\pi_N(a), x] = [\pi_N(a^*), x] = 0$. In particular, $x$ must commute with $|\pi_N(a)|$ and $|\pi_N(a^*)|$, and thus $\Omega_{fix} \subset \Omega = \bigoplus_\lambda \ell_n^\infty \otimes \mathcal{B}(W_\lambda)$, where $n_\lambda := \dim(W_\lambda)$ and the equation follows from the irreducible decomposition and lemma 4.2.

For an operator in one irreducible component $x \in \ell_n^\infty \otimes \mathcal{B}(W_\lambda)$, by irreducibility $x$ must have the form $C \otimes \mathcal{B}(W_\lambda)$. Therefore $\Omega_{fix} = \bigoplus_\lambda \mathcal{B}(W_\lambda)$. \hfill \blacksquare

Corollary 4.11.

$$D(\rho_N | E_{fix}^* \rho_N) = D(\rho_N | E_\Omega^* \rho_N) + D(\rho_N | E_{fix}^* \rho_N)$$

Proof. This is an immediate consequence of the previous lemma and the chain rule of relative entropy. \hfill \blacksquare

Remark 4.12. Lemma 4.10 gives a complete description of the fixed point algebra. We stress that the quantum Markov semigroup generated by $L_N^\beta$ is non-primitive.

To clarify the relation between the various algebras used in our proof, we present the following commutative diagram:
The commuting square of the conditional expectations is a special feature of $\mathfrak{su}(2)$ representation (more generally any representation of simple Lie algebra). It may not be true for general Lindbladian generators.

So far, we have discussed the left half of the diagram. To calculate $\text{CLSI} (\mathcal{L}_N^\beta)$, it remains to bound $D(E^*_\Omega \rho_N | E^*_\Omega \rho_N)$ from above by an entropy production term. This calculation will be done in two parts. We first bound $E\pi_{\pi_N(a)}(E^*_\Omega \rho_N)$ from above by $E\pi_{\pi_N(a)}(\rho_N)$.

**Lemma 4.13.** Given a state $\rho_N = d_N^{1/2} x a d_N^{1/2}$ we have

\[
E\pi_{\pi_N(a)}(E^*_\Omega \rho_N) + E\pi_{\pi_N(a^*)}(E^*_\Omega \rho_N) \leq E\pi_{\pi_N(a)}(\rho_N) + E\pi_{\pi_N(a^*)}(\rho_N)
\]

**Proof.** The key calculation is the following commutation relation: for any $a \in \mathfrak{B}(V^\otimes N)$

\[
[E_\Omega, \mathcal{L}_N^\beta](x) = E_\Omega(\pi_N(a^*) \pi_N(a) x + x \pi_N(a^*) \pi_N(a) - 2 \pi_N(a^*) x \pi_N(a) - 2 \pi_N(a^*) E_\Omega(x) \pi_N(a))
\]

Since $\pi_N(a^*) \pi_N(a) = \pi_N(a^*a) \in \Omega$, the equation above equals to

\[
[E_\Omega, \mathcal{L}_N^\beta](x) = -2E_\Omega(\pi_N(a^*) x \pi_N(a)) + 2\pi_N(a^*) E_\Omega(x) \pi_N(a)
\]

In addition, under the irreducible decomposition $\pi_N(a) = \sum_\lambda \pi_\lambda(a)$. Then we have

\[
\pi_N(a) = \sum_\lambda \sum_{n=\dim(V_\lambda)} \sum_{0 \leq j \leq n} \alpha_{n,j} |n, j - 1, \lambda\rangle \langle n, j, \lambda|
\]

where $\lambda$ in the vector labels which irreducible component that vector belongs to. For $x \in \mathfrak{B}(V^\otimes N)$, it can be written as

\[
x = \sum_{\lambda, \mu} \sum_{n=\dim(V_\lambda)} \sum_{0 \leq j \leq n} x^{\lambda\mu}_{(n,j),(m,k)} |n, j, \lambda\rangle \langle m, k, \mu|
\]

And we can write $E_\Omega(x) = \sum_\lambda \sum_{n=\dim(V_\lambda)} x^{\lambda\mu}_{(n,j),(m,k)} |n, j, \lambda\rangle \langle n, j, \lambda|$. Then we have

\[
E_\Omega(\pi_N(a^*) x \pi_N(a))
\]

\[
= E_\Omega(\sum_{\lambda, \mu} \sum_{n=\dim(V_\lambda)} \sum_{0 \leq j \leq n} \sum_{0 \leq k \leq m} \sum_{0 \leq j' \leq n} \sum_{0 \leq k' \leq m} \alpha_{n,j+1} \alpha_{m,k+1} x^{\lambda\mu}_{(n,j),(m,k)} |n, j+1, \lambda\rangle \langle m, k+1, \mu|)
\]

\[
= \sum_\lambda \sum_{n=\dim(V_\lambda)} \sum_{0 \leq j \leq n} \alpha_{n,j+1}^2 x^{\lambda\mu}_{(n,j),(n,j)} |n, j+1, \lambda\rangle \langle n, j+1, \lambda|
\]
On the other hand, we have

\[ \pi_N(a^*)E_\Omega(x)\pi_N(a) = \sum_{\lambda} \sum_{n=\dim(V_\lambda)} \sum_{0 \leq j \leq n} x_{(n,j),(n,j)}^{\lambda} \pi_N(a^*)|n, j, \lambda\rangle \langle n, j, \lambda| \pi_N(a) \]

\[ = \sum_{\lambda} \sum_{n=\dim(V_\lambda)} \sum_{0 \leq j \leq n} x_{(n,j),(n,j)}^{\lambda} \alpha_{n,j+1}^2 |n, j + 1, \lambda\rangle \langle n, j + 1, \lambda| \]

Hence \([E_\Omega, L_\beta^N](x) = 0\).

Given this commutation relation, we calculate the entropy production term on \(E_{\Omega}^*\rho_N = d_N^{-1/2}E_\Omega(x)d_N^{1/2}\):

\[
EP_{\pi_N(a)}(E_{\Omega}^*\rho_N) + EP_{\pi_N(a^*)}(E_{\Omega}^*\rho_N) = \tau_N\left( (L_\beta^N(E_\Omega x)) \left( \log(E_\Omega x) - \log(E_{fix} x) \right) \right)
\]

\[ = \tau_N\left( (E_\Omega(L_\beta^N x)) \left( \log(E_\Omega x) - \log(E_{fix} x) \right) \right)
\]

\[ = \tau_N\left( (L_\beta^N x) \left( \log(E_\Omega x) - \log(E_{fix} x) \right) \right)
\]

where the second equation holds because of the commutation relation and the third equation holds because of the module property of conditional expectation.

By operator monotonicity of \(\log\) and \(E_\Omega x \leq x\), we have \(0 \leq \log(E_\Omega x) - \log(E_{fix} x) \leq \log(x) - \log(E_{fix} x)\). Therefore

\[
EP_{\pi_N(a)}(E_{\Omega}^*\rho_N) + EP_{\pi_N(a^*)}(E_{\Omega}^*\rho_N) \leq \tau_N\left( (L_\beta^N x) \left( \log(x) - \log(E_{fix} x) \right) \right)
\]

\[ = EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N)
\]

The remaining part of this section will be devoted to prove the following CLSI inequality

**Proposition 4.2.** Let \(\rho\) be a state in \(\Omega\) then there exists a constant \(C(\beta)\) depending only on \(\beta\) such that

\[
D(\rho|E_{fix}^*\rho) \leq C(\beta)\left( EP_{\pi_N(a)}(\rho) + EP_{\pi_N(a^*)}(\rho) \right)
\]

Before giving the proof, recall the embedding:

\[ \iota : \mathcal{B}(V^\otimes N) \rightarrow \mathcal{B}(\bigoplus_\lambda V_\lambda) \otimes \mathcal{M}_{n_0} \]

where \(V^\otimes N = \bigoplus_{\lambda \in \mathcal{P}(N)} V_\lambda^\oplus \mathcal{W}(W_\lambda)\) is the irreducible decomposition and \(n_0 = \max\{\dim(W_\lambda)\}\). The direct sum representation \(\pi = \bigoplus_\lambda \pi_\lambda\) is multiplicity free. Since this embedding intertwines the Lindbladian \(L_\beta^N\) and \(L_\beta^\pi\), we have

\[ \text{CLSI}(L_\beta^N) \geq \text{CLSI}(L_\beta^\pi) \]

Therefore to prove the proposition, it suffices to prove that there exists a constant \(C(\beta)\) such that

\[
D(\rho|E_{fix}^*\rho) \leq C(\beta)\left( EP_{\pi(a)}(\rho) + EP_{\pi(a^*)}(\rho) \right)
\]
which we formally define as follows. Restricted to \(L^\beta_\pi\),

In particular, \(C\) is independent of dimension \(n_\lambda\).

Lemma 4.14. Let \(V_\lambda\) be the irreducible component of highest dimension in \(V^{\otimes N}\) (tensor product of fundamental representation), let \(\ell^{n_\lambda}_\infty\) be the diagonal subalgebra of \(\mathbb{B}(V_\lambda)\) and let \(E_{fix}: \ell^{n_\lambda}_\infty \rightarrow \mathbb{C}\) be the projection onto \(\mathbb{C}\). Then for any state \(\rho\) on \(\ell^{n_\lambda}_\infty\), there exists a constant \(C(\beta)\) depending only on \(\beta\) such that

\[
D(\rho|E_{fix}\rho) \leq C(\beta)(EP_{\pi_\lambda(a)}(\rho) + EP_{\pi_\lambda(a^*)}(\rho))
\]

In particular, \(C(\beta)\) is independent of dimension \(n_\lambda\).

The key idea of this proof is to study the relation between \(L^\beta_{\pi_\lambda}\) and the tensor product of \(L^\beta_{\pi(1)}\) (the Lindbladian associated with a single fundamental representation \(\pi(1)\)). When restricted to \(\ell^{n_\lambda}_\infty\), both Lindbladians have the structure of nearest-neighbor Lindbladian, which we formally define as follows.

Definition 4.15. Given a finite sequence of real scalars \((\gamma_j)_{0 \leq j \leq n}\), consider the following operator in \(\mathbb{M}_{n+1}\):

\[
S_\gamma := \sum_j \gamma_j |j - 1\rangle \langle j|
\]

Then the nearest-neighbor Lindbladian associated with \((\gamma_j)\) is given by:

\[
L^\beta_{\pi_\lambda} x := \beta^{1/2} L_{S_\gamma} x + \beta^{-1/2} L_{S_\gamma} x
\]

where

\[
L_{S_\gamma} x := 2S_\gamma^* x S_\gamma - S_\gamma^* S_\gamma x - x S_\gamma^* S_\gamma
\]

For \(x = \sum_{0 \leq j \leq n} x_j |j\rangle \langle j| \in \ell^{n+1}_\infty\), we have

\[
L_{S_\gamma} x = \sum_{0 \leq j \leq n} 2(\gamma_j^2 x_j |j - 1\rangle \langle j - 1| - \gamma_j^2 x_j |j\rangle \langle j|)
\]

As an example, for \(L^\beta_{\pi_\lambda}\) restricted to \(\ell^{n_\lambda}_\infty\), the corresponding coefficients are given by \(\gamma^2_k = \alpha_{n_\lambda-1,k}\) for \(0 \leq k \leq n_\lambda - 1\).

We will explicitly calculate the difference between the two Lindbladians on \(\ell^{n_\lambda}_\infty\). The difference turns out to be another nearest-neighbor Lindbladian. This calculation depends on the explicit description of \(V_\lambda\) as the totally symmetric subspace of \(V^{\otimes N}\).
Proof. Let \( \pi^j := 1 \otimes \ldots \otimes \pi_1 \otimes \ldots \otimes 1 \) be the \( j \)-th component in the Lie algebra tensor product representation \( \pi^{\otimes N}_{(j)} \), then the tensor product Lindbladian is given by

\[
L^\beta_j x := \sum_{1 \leq j \leq N} \pi^j (L^\beta_{\pi^{(j)}}) x = \sum_{1 \leq j \leq N} \beta^{1/2} \pi^j (L_{\pi^{(j)}_\pi^j}) x + \beta^{-1/2} \pi^j (L_{\pi^{(j)}})^* x
\]

\[
= \sum_{1 \leq j \leq N} \beta^{1/2} (2 \pi^j (a^*) x \pi^j (a) - \pi^j (a^* a) x - x \pi^j (a^* a))
\]

\[
+ \beta^{-1/2} (2 \pi^j (a) x \pi^j (a^*) - \pi^j (aa^*) x - x \pi^j (aa^*))
\]

By the tensor stability of CLSI constant, we know

\[
(4.27) \quad \text{CLSI}(L^\beta_{\otimes j}) \geq \text{CLSI}(L^\beta_{\pi^j})
\]

In particular, this constant is independent of \( N \).

Let \( E_d \) be the diagonal conditional expectation and let \( \widehat{L}_a := \sum_{1 \leq j \leq N} \pi^j (L_{\pi^{(j)}_\pi^j}) \), then we claim:

Claim: There exists coefficients \( (\tilde{\gamma}_k)_{0 \leq k \leq n_{\lambda} - 1} \) and \( (\tilde{\gamma}_k)_{0 \leq k \leq n_{\lambda} - 1} \) such that \( \alpha_{n_{\lambda} - 1,k} = \tilde{\gamma}_k^2 + \tilde{\gamma}_k^2 \) and

\[
(4.28) \quad E_d \widehat{L}_a E_d = L_{S_\tilde{\gamma}}
\]

\[
E_d L_{\pi^{(j)}_\pi^j} E_d = L_{S_\gamma} + L_{S_{\tilde{\gamma}}} = E_d \widehat{L}_a E_d + L_{S_{\tilde{\gamma}}}
\]

Proof: Recall \( V_\lambda \) has basis \( |n_\lambda - 1,k\rangle \) where \( |n_\lambda - 1,k\rangle = \sum_{A \in \{n_\lambda - 1\}} \frac{1}{2^n_{\lambda - 1}} v_A \) and \( v_A = (\bigotimes_{j \in A} e_1) \otimes (\bigotimes_{j \notin A} e_0) \). \( e_0, e_1 \) are the weight vectors of the fundamental representation. Subscript \( A \) indicates the position where the tensor factor is \( e_1 \). We calculate \( E_d \widehat{L}_a E_d \) on the diagonal basis \( |n_\lambda - 1,k\rangle \langle n_\lambda - 1,k| \).

Since \( \pi^j (a^* a) v_A = \mathbb{1}_{j \in A} v_A \), we have

\[
\sum_j \pi^j (a^* a) v_A = |A| v_A
\]

Hence

\[
\sum_j \pi^j (a^* a) |n_\lambda - 1,k\rangle \langle n_\lambda - 1,k| = k |n_\lambda - 1,k\rangle \langle n_\lambda - 1,k|
\]

In addition since \( \pi^j (a^*) v_A = \mathbb{1}_{j \notin A} v_{A \cup \{1\}} \), we have

\[
\sum_j \pi^j (a^*) |n_\lambda - 1,k\rangle \langle n_\lambda - 1,k| \pi^j (a) = (k - 1) |n_\lambda - 1,k - 1\rangle \langle n_\lambda - 1,k - 1|
\]

Therefore \( \tilde{\gamma}_k^2 = k \) and \( \tilde{\gamma}_k^2 = \alpha_{n_{\lambda} - 1,k} - \tilde{\gamma}_k^2 = k(n_{\lambda} - 1 - k) \).

Back to the main proof. Recall the density matrix \( d_N := N_\beta \exp(-\frac{\beta}{2} \pi_N (h)) \). In terms of tensor product decomposition, it can be written as

\[
d_N = N_\beta \exp(-\frac{\beta}{2} \pi_N (h))^{\otimes N} = d_\beta^{\otimes N}
\]

where \( d_\beta \) is the corresponding density for the fundamental representation. In particular this density is diagonal.

In addition, restricted to the commutative subalgebra \( \ell^a_\infty \), both \( L^\beta_{\otimes} \) and \( L^\beta_{\pi_\lambda} \) are \( d_\beta \)-self-adjoint. Hence the remainder Lindbladian \( L^\beta_{\otimes} \) is self-adjoint with respect to the \( d_N \)-KMS inner produce restricted to \( \ell^a_\infty \). Moreover, the fixed point algebra of the tensor product
Since reducible representation.

Therefore for any state \( \rho = d_1^{1/2} x d_1^{1/2} \) on \( \ell^n \) (\( x \) is diagonal), by CLSI on the two-point space \( (\ell^2) \) we have

\[
D(\rho | E_{\text{fix}}^{*} \rho) \leq \text{CLSI}(\mathcal{L}^\beta) \tau_N((\mathcal{L}^\beta x) \log(x))
\]

where \( \tau_N \) is the state associated with the diagonal tensor product density \( d_N \).

Since we have

\[
\tau_N((\mathcal{L}^\beta x) \log(x)) = \tau_N\left(\left((\mathcal{L}^\beta - \mathcal{L}^\gamma) x\right) \log(x)\right)
\]

and \( \mathcal{L}^\beta \geq 0 \) on \( \ell^n \), then we have

\[
\tau_N\left(\left((\mathcal{L}^\beta x) \log(x)\right) \leq \tau_N\left(\left((\mathcal{L}^\beta x) \log(x)\right) = EP_{\pi_\Lambda(a)}(\rho) + EP_{\pi_\Lambda(a^*)}(\rho)
\]

The last equation is true because the density is diagonal. Since the CLSI constant is given by the two-point space CLSI constant, it is independent of \( N \).

This completes the proof of the lemma. \( \square \)

**Proof of the Proposition.** For each irreducible representation \( V_n := \mathbb{C}^{n+1} \) of \( SU(2) \), there exists a tensor product of fundamental representation \( V^\otimes n \) such that \( V_n \) is the irreducible component of \( V^\otimes n \) with the highest dimension. Therefore we can apply (4.14) to each irreducible representation.

By the discussion before the lemma, for any state \( \rho \) in \( \Omega \), we have

\[
D(\rho | E_{\text{fix}}^{*} \rho) \leq \max_{\lambda} \{C(\beta, \lambda))\} \left(EP_{\pi_\Lambda(a)}(\rho) + EP_{\pi_\Lambda(a^*)}(\rho)\right)
\]

Since \( C(\beta, \lambda) \) are independent of \( \lambda \), the CLSI constant \( \max_{\lambda} C(\beta, \lambda) \) only depends on \( \beta \). \( \square \)

Now we can state the main theorem of this section.

**Theorem 4.16.** Let \( V^\otimes N \) be the tensor product of fundamental representation of \( \mathfrak{su}(2) \), and let \( \rho_N = d_1^{1/2} x d_1^{1/2} \) be a state on \( V^\otimes N \) with \( x \in \mathbb{B}(V^\otimes N) \). Consider the \( \mathfrak{d}_N \)-KMS self-adjoint Lindbladian \( \mathcal{L}^\beta \) associated with the tensor product representation \( \pi_N \). Let \( E_{\text{fix}} \) be the conditional expectation onto its fixed point algebra. Then there exists a constant \( C(\beta) \) depending only \( \beta \) such that

\[
D(\rho_N | E_{\text{fix}}^{*} \rho_N) \leq C(\beta) N^2 (EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N))
\]

(4.29)

**Proof.** Using the notation as above, \( \Omega = C^*([\pi_N(a)]') \cap C^*([\pi_N(a^*)']) \). Then by chain rule of relative entropy we have

\[
D(\rho_N | E_{\text{fix}}^{*} \rho_N) = D(\rho_N | E_{\text{fix}}^{*} \rho_N) + D(E_{\text{fix}}^{*} \rho_N | E_{\text{fix}}^{*} \rho_N)
\]

The first term is bounded above by \( C(\beta) N^2 (EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N)) \), and the second term is bounded above by

\[
(EP_{\pi_N(a)}(E_{\text{fix}}^{*} \rho_N) + EP_{\pi_N(a^*)}(E_{\text{fix}}^{*} \rho_N)) \leq C(\beta) (EP_{\pi_N(a)}(\rho_N) + EP_{\pi_N(a^*)}(\rho_N))
\]

Therefore we have the claimed upper bound. \( \square \)

**Corollary 4.17.** CLSI constant of \( \mathcal{L}^\beta_N \) is bounded below by \( \frac{C(\beta)}{N^2} \) for some constant \( C(\beta) \).

**Proof.** This is immediate from the theorem. \( \square \)
In the next section, we will give an upper bound on the spectral gap of $L^β_N$ by an explicit construction. It turns out that the spectral gap $\lambda(L^β_N) \lesssim \frac{1}{N}$. Combining the results from these two section, we have

$$\frac{1}{N^2} \lesssim \text{CLSI}(L^β_N) \leq 2\lambda(L^β_N) \lesssim \frac{1}{N}$$

In particular when $N$ goes to infinity, both the CLSI constant and the spectral gap go to 0.

Using the same framework and lemma 4.5, we obtain the following CLSI estimate.

**Theorem 4.18.** Consider the Lindbladian $L^{(3)}_N$ generated by $a_{(3)} := \pi_N(a^*a^*\pi_N(a)$.

(4.30) $$L^{(3)}_N x := e^{\beta/2}L_{a_{(3)}} x + e^{-\beta/2}L_{a^*_a{3}} x$$

where $L_{a_{(3)}} x := 2a_{(3^*)}a_{(3)} x - a^*_{a{3}}a_{(3^*)} a_{(3)}$. Then there exists a constant $C_3(\beta)$ such that

(4.31) $$\text{CLSI}(L^{(3)}_N) \geq C_3(\beta)$$

**Proof.** By corollary 4.5, the uniform spectral gap of $|a_{(3)}|$ is bounded by a constant of order 1. In addition, since $\pi_N(a^*a^*\pi_N(a)$ is diagonal, it is easy to see that the fixed point algebra of $L^{(3)}_N$ is the same as the fixed point algebra of $L^β_N$. Moreover, the same calculation as lemma 4.2 shows that the conditional expectations $E_{a_{(3)}} : \mathbb{B}(V^{\otimes N}) \to C^* (\{|a_{(3)}||, \beta\})'$ and $E_{{a^*_a{3}}} : \mathbb{B}(V^{\otimes N}) \to C^* (\{|a^*_{a{3}}||, \beta\})'$ commute and their intersection $\Omega^{(3)} := C^* (\{|a_{(3)}||, \beta\}) \cap C^* (\{|a^*_{a{3}}||, \beta\})$ equals $\Omega$. Therefore repeating the same arguments as in section 3 we have the lower bound on CLSI($L^{(3)}_N$).

5. **Upper Bound on Spectral Gap**

The main purpose of this section is to construct a state $\rho \in S_1(V^{\otimes N})$ that gives an upper bound of the order $\frac{1}{N}$ for the spectral gap of $L^β_N$. First we will construct a vector $\xi$ (not self-adjoint and not positive) in the Schatten 2-class such that the Dirichlet form on $\xi$ is bounded below by $\gtrsim ||\xi||^2_2 \frac{1}{N}$. Then the state $\rho$ will be a perturbation around the stationary state by $\xi + \xi^*$.

Recall from the proof of lemma 4.10, we considered the embedding:

$$\iota_N : \mathbb{B}(V^{\otimes N}) \to S_2(V^{\otimes N}) : x \mapsto d^{\iota_N}_N x d^{\iota_N}_N$$

And the Dirichlet form determined by the Lindbladian $L^β_N$ is given by:

$$\langle \xi, \xi \rangle_L = \langle \iota_N(x), \iota_N(x) \rangle_L = \frac{1}{2} \left( ||d^{\iota_N}_N (\pi_N(a)x - x\pi_N(a))d^{\iota_N}_N||_2^2 + ||d^{\iota_N}_N (\pi_N(a^*x - x\pi_N(a^*))d^{\iota_N}_N||_2^2 \right)$$

where $|| \cdot ||^2_2$ is the Hilbert-Schmidt inner product.

Let $V^{\otimes N} \cong \bigoplus_{\lambda \in \mathbb{P}_2(N)} V^{\otimes \dim(W_\lambda)}_\lambda$ be the irreducible decomposition. We will construct our example in the multiplicity free subspace $V := \bigoplus_{\lambda} V_\lambda$. Since $(|n,j\rangle)_{0 \leq j \leq n = \dim(V_\lambda)}$ form a basis of $V$, a generic element $\xi$ in $S_2(V)$ can be written as

(5.1) $$\xi := \sum_{n, m} \sum_{0 \leq j \leq n, 0 \leq k \leq m} \xi_{(n, j), (m, k)} |n, j\rangle \langle m, k|$$

Before stating the main result of this section, we first prove the following lemma.
Lemma 5.1. Suppose the coefficients of $\xi \in S_2(V)$ satisfy the relations: $\xi_{(N,1),(N-2,1)} = e^{-(j-1)2/3} \xi_{(N,1),(N-2,1)}$ and $\xi_{n,j} = 0$ for $n < N - 2$, then

$$\langle \xi, \xi \rangle_L \geq \frac{2e^{\beta/2}||\xi||_2^2}{N}.$$  

In addition, the symmetrization $\xi^* + \xi$ satisfies the same inequality.

Proof. By rescaling on both sides, we can take the trace in the Hilbert-Schmidt norm to be the standard trace on $B(V)$. The proof is based on direct computations of the Dirichlet form. Using the definition of the Dirichlet form, we have:

$$\langle \xi, \xi \rangle_L = ||e^{\beta/4} \pi_N(a)\xi - e^{-\beta/4} \pi_N(a^*)\xi - e^{\beta/4} \pi_N(a^*)\xi - e^{-\beta/4} \pi_N(a)\xi||_2^2 + ||e^{\beta/4} \pi_N(a)\xi - e^{-\beta/4} \pi_N(a^*)\xi - e^{\beta/4} \pi_N(a^*)\xi - e^{-\beta/4} \pi_N(a)\xi||_2^2$$

$$- \langle \pi_N(a)\xi, \pi_N(a)\xi \rangle - \langle \pi_N(a)\xi, \pi_N(a)\xi \rangle - \langle \pi_N(a)\xi, \pi_N(a)\xi \rangle - \langle \pi_N(a^*)\xi, \pi_N(a^*)\xi \rangle - \langle \pi_N(a)\xi, \pi_N(a^*)\xi \rangle - \langle \pi_N(a^*)\xi, \pi_N(a^*)\xi \rangle$$

$$+ 2||\pi_N(a)\xi||_2^2 - 2||\pi_N(a)\xi||_2^2 - 2||\pi_N(a)\xi||_2^2 + 2||\pi_N(a)\xi||_2^2$$

Expanding $\xi$ in terms of the canonical basis, we have:

$$e^{\beta/2} \tau(\xi[a^*, a]\xi^*) + e^{-\beta/2} \tau(\xi^*[a, a^*]\xi) = \sum \langle \xi_{(n,j), (m,k)} \rangle^2 (e^{\beta/2}(2k - m) - e^{-\beta/2}(2j - n))$$

Summation is over $n, m, j, k$. A direct computation involving this derivative term shows:

$$2||\pi_N(a)\xi||_2^2 - 2||\pi_N(a)\xi||_2^2 - 2||\pi_N(a)\xi||_2^2$$

$$= -2\tau(\xi^* a^* a) - 2\tau(\xi^* a^* a)$$

$$= -2 \sum \xi_{(n,j+1),(m,k+1)}^2 \xi_{(n,j+1),(m,k+1)}^2 \alpha_{n,j+1} \alpha_{m,k+1}$$

$$- 2 \sum \xi_{(n,j-1),(m,k-1)}^2 \xi_{(n,j-1),(m,k-1)}^2 \alpha_{n,j} \alpha_{m,k}$$

$$= -4 \sum \Re(\xi_{(n,j+1),(m,k+1)}^2 \xi_{(n,j),(m,k)}^2 \alpha_{n,j+1} \alpha_{m,k+1})$$

In addition we have:

$$(e^{\beta/2} + e^{-\beta/2})(||\pi_N(a)\xi||_2^2 + ||\pi_N(a)\xi||_2^2) = e^{\beta/2} \tau(\xi^* \pi_N([a^*, a]\xi^*) + e^{-\beta/2} \tau(\xi^* \pi_N([a, a^*]\xi)$$

$$= \sum \xi_{(n,j),(m,k)}^2 (e^{\beta/2}^2 \alpha_{n,j}^2 + e^{-\beta/2}^2 \alpha_{m,k+1}^1 + e^{\beta/2}^2 \alpha_{m,k+1}^2 + e^{-\beta/2}^2 \alpha_{n,j}^2)$$

$$= \sum \xi_{(n,j),(m,k)}^2 (e^{\beta/2}^2 \alpha_{n,j+1}^2 + e^{-\beta/2}^2 \alpha_{m,k+1}^2)$$

$$+ \sum \xi_{(n,j),(m,k)}^2 (e^{\beta/2}^2 \alpha_{n,j+1}^2 + e^{-\beta/2}^2 \alpha_{m,k+1}^2)$$

Combining the two computations, we have:

$$\langle \xi, \xi \rangle_L = \sum (e^{\beta/2}^2 \xi_{(n,j+1),(m,k+1)}^2 + e^{-\beta/2}^2 \xi_{(n,j),(m,k)}^2) \alpha_{n,j+1} \alpha_{m,k+1}$$

$$+ 4 \sum \Re(\xi_{(n,j+1),(m,k+1)}^2 \xi_{(n,j),(m,k)}^2 \alpha_{n,j+1} \alpha_{m,k+1}).$$
By the assumptions on the coefficients of $\xi$, we have:

$$\langle \xi, \xi \rangle_L = 2e^{\beta/2} \sum_{1 \leq j \leq N-2} e^{-(j-1)\beta} |\xi_{(N,1),(N-2,1)}|^2 (\alpha_{N,j} - \alpha_{N-2,j})^2$$

$$= 8e^{3\beta/2} |\xi_{(N,1),(N-2,1)}|^2 \sum_{1 \leq j \leq N-2} \frac{j e^{-j\beta}}{(\sqrt{N-j+1} + \sqrt{N-j-1})^2}$$

$$> \frac{2e^{\beta/2} |\xi_{(N,1),(N-2,1)}|^2}{N} \sum_{1 \leq j \leq N-2} e^{-(j-1)\beta}$$

In the last step, we used $\sqrt{N-j+1} + \sqrt{N-j-1} < 2\sqrt{N}$ for $1 \leq j \leq N-2$ and $\sum_{1 \leq j \leq N-2} j e^{-j\beta} > \sum_{1 \leq j \leq N-2} e^{-j\beta}$. Since

$$||\xi||_2 = |\xi_{(N,1),(N-2,1)}|^2 \sum_{1 \leq j \leq N-2} e^{-(j-1)\beta}$$

then we have

$$\langle \xi, \xi \rangle_L \geq \frac{2e^{\beta/2} \langle \xi, \xi \rangle_N}{N}.$$ 

For the symmetrized version, by equation 5.3 we have:

$$\langle \xi^* + \xi, \xi^* + \xi \rangle_L = \langle \xi^*, \xi^* \rangle_L + \langle \xi, \xi \rangle_L.$$

In addition, since $\xi$ is completely off-diagonal we have:

$$\langle \xi^* + \xi, \xi^* + \xi \rangle = \langle \xi^*, \xi^* \rangle + \langle \xi, \xi \rangle.$$ 

Therefore $\xi^* + \xi$ satisfies equation 5.2 with the same constant.

**Proposition 5.1.** There exists a positive matrix $x \in \mathbb{B}(V)$ such that

$$\langle t_N(x), t_N(x) \rangle_L \geq \frac{C(\beta) \langle x - E_{fix} x, x - E_{fix} x \rangle_N}{N}$$

where $\langle \cdot, \cdot \rangle_N$ is the KMS-inner product associated with $d_N$, $C(\beta)$ is a constant depending only on $\beta$ and $E_{fix}$ is the conditional expectation onto the fixed point algebra $\bigoplus \Lambda \mathbb{C}$.

**Proof.** We consider the candidate matrix $\bar{x}$ such that $t_N(\bar{x}) = \xi$, where $\xi$ is constructed in Lemma 5.1. Up to now we have left $\xi_{(N,1),(N-2,1)}$ as a free parameter. To fix this parameter, we calculate the operator $\bar{x}$ explicitly. Since $t_N(\bar{x}) = \xi_{(N,1),(N-2,1)} \sum_{1 \leq j \leq N-2} e^{-(j-1)\beta/2} |N,j\rangle \langle N-2,j|$, then we have:

$$\bar{x} = d_N^{-1/4} \xi_{(N,1),(N-2,1)} = N^{1/2}_\beta \xi_{(N,1),(N-2,1)} \sum_{1 \leq j \leq N-2} e^{-(j-1)\beta/2} e^{\frac{\beta\pi(h)}{8}} |N,j\rangle \langle N-2,j| e^{\frac{\beta\pi(h)}{8}}$$

$$= N^{1/2}_\beta \xi_{(N,1),(N-2,1)} \sum_{1 \leq j \leq N-2} e^{\frac{2N}{\pi} \frac{\beta}{4} e^{-j\beta}} |N,j\rangle \langle N-2,j|.$$ 

We choose $\xi_{(N,1),(N-2,1)} := N^{1/2}_\beta e^{-\beta N/4+5\beta/4}$, then $\bar{x}^* + \bar{x} = \sum_{1 \leq j \leq N-2} e^{-(j-1)\beta} (|N,j\rangle \langle N-2,j| + |N-2,j\rangle \langle N,j|)$. The spectral radius of $\bar{x}^* + \bar{x}$ is bounded above by 2. Therefore $x := 2 - (\bar{x}^* + \bar{x}) \geq 0$ is a positive matrix. In addition, since the fixed point algebra is diagonal, we have:

$$x - E_{fix}(x) = x - 2 = - (\bar{x}^* + \bar{x}).$$
And since the identity operator is in the fixed point algebra, we have:
\[ \langle \iota_N(x), \iota_N(x) \rangle_L = \langle \iota_N(x^* + x), \iota_N(x^* + x) \rangle_L \]
Therefore we have:
\[ \langle \iota_N(x), \iota_N(x) \rangle_L \geq \frac{2e^{\beta/2}(x - E_{fix}x, x - E_{fix}x)}{N} . \]

Corollary 5.2. For the \( \text{su}(2) \)-Lindbladian \( \mathcal{L}_N^\beta \), there exist constants \( C_1(\beta), C_2(\beta) > 0 \) such that
\[ \frac{C_1(\beta)}{N^2} \leq CLSI(\mathcal{L}_N^\beta) \leq 2\lambda(\mathcal{L}_N^\beta) \leq \frac{C_2(\beta)}{N} \]

6. Clustering and Spectral Gap

As mentioned in the introduction, Corollary 5.2 emphasizes the difference between primitive quantum Markov semigroups and non-primitive ones \([KB16][BCR22]\). In this section, we elaborate on this apparent contradiction. Our main example has the form of a Davies generator. Recall a Davies generator associated with a Hamiltonian \( \mathcal{H} \) is defined as \([Dav80]\):

**Definition 6.1.** Consider an open system \( S \) coupled with a heat bath \( B \), and the total system Hamiltonian is given by \( \mathcal{H}_S + \mathcal{H}_B + \sum_{\alpha} S_\alpha \otimes B_\alpha \) where the interaction term is written generically as a sum of tensor product coupling. Assume the heat bath is in thermal equilibrium and assume the system-bath coupling is weak, then by taking a Born-Markov approximation, the effective dissipative dynamics on system \( S \) is given by the Davies generator of the form:
\[ \mathcal{L}^Dx = \sum_{\omega,\alpha} \chi_{\omega,\alpha}(2S^*_{\omega,\alpha}xS_{\omega,\alpha} - S^*_{\omega,\alpha}S_{\omega,\alpha}x - xS^*_{\omega,\alpha}S_{\omega,\alpha}) \]
where \( \omega \)'s are the Bohr frequencies, \( \chi_{\omega,\alpha} \)'s come from the Fourier coefficients of the two-point correlation function of the bath, and \( S_{\omega,\alpha} \)'s are the Fourier modes of the coupling operator \( S_\alpha \):
\[ e^{-i\mathcal{H}_S} S_\omega e^{i\mathcal{H}_S} = \sum_{\omega} S_{\omega,\alpha} e^{i\omega} \]

In particular, let \( \rho_S \) be the Gibbs state associated with \( \mathcal{H}_S \), then the generators \( S_{\omega,\alpha} \) satisfy:
\[ \rho_S^{it} S_{\omega,\alpha} \rho_S^{-it} = e^{it\beta \omega} S_{\omega,\alpha} \]
where \( \beta \) is the inverse temperature of the heat bath. And by the KMS condition, coefficients \( \chi_{\omega,\alpha} \) satisfy:
\[ \chi_{-\omega,\alpha} = e^{-\beta \omega} \chi_{\omega,\alpha} \]

**Lemma 6.2.** \( \mathcal{L}_N^\beta \) is a Davies generator if we take \( \pi_N(h) \) to be the system Hamiltonian.

**Proof.** Since \( \pi_N(h) \) is taken to be the system Hamiltonian, the corresponding Gibbs state is exactly \( d_N \) with inverse temperature \( \beta \). In addition, since \( d_N^{it}\pi_N(a)d_N^{-it} = e^{it\beta} \pi_N(a) \), \( \pi_N(a) \) is the interaction operator corresponding to Bohr frequency \( \omega = 1 \). Therefore \( \mathcal{L}_N^\beta \) is a Davies generator. \( \blacksquare \)

Recall the definitions of minimal conditional expectation and local projection in \([KB16][BCR22]\). Here we directly apply the definitions to the Gibbs state \( d_N \).
Definition 6.3. Let \( A \subseteq [N] \) and fix the Gibbs state \( d_N \) as before. The minimal conditional expectation of \( d_N \) on \( A \) is given by:

\[
E_A(x) := \tau_A(d_A^{1/2}xd_A^{1/2})
\]

where \( \tau_A := \bigotimes_{i \in A} \tau \otimes \bigotimes_{i \notin A} \text{id} \) is a composition of partial trace on \( V^A \otimes V^\Omega \) and tensoring with the identity matrix, and \( d_A := \bigotimes_{i \in A} d_\beta \otimes \bigotimes_{i \notin A} \text{id} \).

The local projection of the Davies generator \( \mathcal{L}_N^\beta \) is given by

\[
E_A^\xi := \lim_{k \to \infty} E_A^k
\]

Compared with the original definitions, the definitions here are simplified because \( d_N \) is a factor state.

Lemma 6.4. Let \( A, B \subseteq [N] \) be two subsets with nontrivial overlap \( A \cap B \neq \emptyset \). Then

\[
E_A \circ E_B = E_B \circ E_A = E_{A \cup B}
\]

In addition, the local projections also commute:

\[
E_A^\xi \circ E_B^\xi = E_B^\xi \circ E_A^\xi
\]

Proof. Since \( d_N \) is a factor state, we have

\[
d_A = d_{A \cap B} \otimes d_{A - B}, d_B = d_{A \cap B} \otimes d_{B - A}, d_{A \cup B} = d_A \otimes d_{B - A} = d_{A - B} \otimes d_B
\]

Hence we have

\[
E_A \circ E_B(x) = \tau_A(d_A^{1/2} \tau_B(d_B^{1/2}xd_B^{1/2})d_A^{1/2}) = \tau_{A \cap B} \otimes \tau_{A - B} \otimes \tau_B(d_{A \cap B}^{1/2} \otimes d_{A - B}^{1/2} \otimes \tau_B(d_A^{1/2}xd_B^{1/2})d_{A \cap B}^{1/2} \otimes d_{A - B}^{1/2})
\]

\[
= \tau_{A \cap B} \otimes \tau_{A - B} \otimes \tau_B(d_{A \cap B}^{1/2} \otimes \tau_B(d_{A - B}^{1/2} \otimes \tau_B(d_A^{1/2}xd_B^{1/2})d_{A \cap B}^{1/2} \otimes d_{A - B}^{1/2}))
\]

\[
= \tau_{A \cap B} \otimes \tau_{A - B} \otimes \tau_B(d_{A \cap B}^{1/2} \otimes \tau_B(d_{A \cup B}^{1/2}xd_{A \cup B}^{1/2})d_{A \cup B}^{1/2}) = \tau_{A \cup B}(d_{A \cup B}^{1/2}xd_{A \cup B}^{1/2}) = E_{A \cup B}(x)
\]

Here in the third equation, we used the bimodule property of minimal conditional expectation. By the same calculation, we also have \( E_B \circ E_A = E_{A \cup B} \).

For the local projections, we have

\[
E_A^\xi \circ E_B^\xi = \lim_{m \to \infty} \lim_{n \to \infty} E_A^{m \xi} E_B^n = \lim_{m \to \infty} \lim_{n \to \infty} E_B^n E_A^m = E_B^\xi \circ E_A^\xi
\]

Recall the definition of conditional covariance \([KB16, BCR22]\).

Definition 6.5. Fix a faithful state \( \rho \) and fix \( A \subseteq [N] \) then conditional covariance (with respect to \( E_A^\xi \)) on \( A \) is given by

\[
\text{Cov}_A^\rho(x, y) := \langle x - E_A^\xi(x), y - E_A^\xi(y) \rangle_\rho
\]

One can also define conditional covariance with respect to the local projections by replacing \( E_A^\xi \) with \( E_A^\xi \). It turns out that the two definitions are equivalent for our purpose \([KB16]\).

Corollary 6.6. Let \( A, B \subseteq [N] \) be two subsets, and fix the Gibbs state \( d_N \). Then

\[
\text{Cov}_{A \cup B}^n(E_A^\xi(x), E_B^\xi(x)) = 0
\]

Proof.

\[
\text{Cov}_{A \cup B}^n(E_A^\xi(x), E_B^\xi(x)) = \langle E_A^\xi(x) - E_{A \cup B}^\xi(x), E_B^\xi(x) - E_{A \cup B}^\xi(x) \rangle
\]

\[
= \langle E_A^\xi(x) - E_B^\xi(x), E_B^\xi(x) - E_A^\xi(x) \rangle
\]

\[
= \langle x - E_B^\xi(x), E_A^\xi(x) - E_B^\xi(x) \rangle
\]
Recall the definition of strong clustering [KB16].

**Definition 6.7.** Let $A, B \subset [|N|]$ be two subsets such that $A \cap B \neq \emptyset$ and let $d_N$ be the Gibbs state. Then $d_N$ satisfies strong clustering (with respect to the minimal conditional expectation) if there exist constants $c, \xi > 0$ such that for any $x \in \mathbb{B}(V^\otimes N)$

$$\text{Cov}^N_{A \cup B}(E_A^c(x), E_B^c(x)) \leq c \langle x, x \rangle_N e^{-d(B-A,A-B)/\xi}$$

where $d(X, Y) := \min_{x \in X, y \in Y} |x - y|$ is the distance between the two subsets $X, Y \subset [|N|]$. From Corollary 6.6, the factor state $d_N$ satisfies strong clustering. By [KB16, Thm 23], we should expect that the spectral gap of $L^\beta_N$ is independent of the system size. Compare with the corollary 5.2, we see that the non-primitivity essentially closes the spectral gap.

7. Discussion

In this paper, we studied Lindbladians in Davies form which are not necessarily geometrically local. These Lindbladians are local in the sense of [KBG +11] and can be efficiently simulated by quantum circuits. When these Lindbladians are generated by $\mathfrak{su}(2)$-representations, we gave a general framework to estimate their CLSI constants from the uniform spectral gap of the absolute values of the $\mathfrak{su}(2)$-generators. These new examples of Lindbladians in Davies form are not primitive for any system size larger than 1. Therefore strong clustering of the equilibrium Gibbs state does not imply that the spectral gap is independent of system size. Although we focused on $\mathfrak{su}(2)$-representations, a careful analysis using representation theory of Lie algebras can generalize our results to more general simple Lie algebras. But this is beyond the scope of the current work.

Using this general framework, we also found non-local Lindbladians that have dimension-independent CLSI constant. These Lindbladians are generated by $\pi_N(a)\pi_N(a^*)\pi_N(a)$ [BCG +21a]. To see how these Lindbladians may arise in physics, consider the following system-bath coupling:

$$\pi_N(a^*)\pi_N(a)\pi_N(a^*) \otimes \pi_{M-N}(a) + h.c.$$

where the system is embedded in a larger 1-dimensional lattice: $|N| \subset |M|$, and h.c. stands for Hermitian conjugate. Then if we take $\pi_N(h)$ to be the system Hamiltonian, following the derivation of [Dav80] the Davies generator is given by $L^{(3)}_N$. This system-bath interaction can describe the scattering of a pair of particles in system $N$ with one of the particles escaping the system. Our analysis shows that even in the infinite size limit, regardless of the initial state, the system eventually decays to a stationary state. This final state depends on the initial state, but the decay rate is of order 1. Since (ultra-weak closure of) the limit $\lim_{N \to \infty} M^\otimes N_2$ with respect to the reference state $d_N$ is a type $III_\lambda$ factor where $\lambda := e^{-\beta}$, it is interesting to see how our result can be used to study entropy decay in quantum field theory (for example, black hole evaporation).

References

[Bar17] Ivan Bardet. Estimating the decoherence time using non-commutative functional inequalities. *arXiv preprint arXiv:1710.01039*, 2017.

[BCG +21a] Ivan Bardet, Angela Capel, Li Gao, Angelo Lucia, David Perez-Garcia, and Cambyse Rouze. Entropy decay for davies semigroups of a one dimensional quantum lattice. *arXiv preprint arXiv:2112.00601*, 2021.
[BCG+21b] Ivan Bardet, Angela Capel, Li Gao, Angelo Lucia, David Perez-Garcia, and Cambyse Rouze. Rapid thermalization of spin chain commuting hamiltonians. arXiv preprint arXiv:2112.00593, 2021.

[BCL+21] Ivan Bardet, Angela Capel, Angelo Lucia, David Perez-Garcia, and Cambyse Rouze. On the modified logarithmic sobolev inequality for the heat-bath dynamics for 1d systems. Journal of Mathematical Physics, 62(6):061901, 2021.

[BCPH22] Andreas Bluhm, Angela Capel, and Antonio Perez-Hernandez. Exponential decay of mutual information for gibbs states of local hamiltonians. Quantum, 6:650, feb 2022.

[BCR22] Michael Brannan, Li Gao, and Marius Junge. Complete logarithmic sobolev inequality via ricci curvature bounded below ii. Journal of Topology and Analysis, 0(0):1–54, 0.

[BGO20] Alexandre Blais, Steven M. Girvin, and William D. Oliver. Quantum information processing and quantum optics with circuit quantum electrodynamics. Nature Physics, 16:248–256, 2020.

[BJL+21] Ivan Bardet, Marius Junge, Nicholas Laracuente, Cambyse Rouzé, and Daniel Stilck França. Group transference techniques for the estimation of the decoherence times and capacities of quantum markov semigroups. IEEE Transactions on Information Theory, 67(5):2878–2909, 2021.

[BT03] Sergey Bobkov and Prasad Tetali. Modified log-sobolev inequalities, mixing and hypercontractivity. In Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, STOC ’03, page 287–296, New York, NY, USA, 2003. Association for Computing Machinery.

[Cip97] Fabio Cipriani. Dirichlet forms and markovian semigroups on standard forms of von neumann algebras. Journal of Functional Analysis, 147(2):259–300, 1997.

[CM17] Eric A. Carlen and Jan Maas. Gradient flow and entropy inequalities for quantum markov semigroups with detailed balance. Journal of Functional Analysis, 273(5):1810–1869, 2017.

[CM20] Eric A. Carlen and Jan Maas. Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems. J. Stat. Phys., 178:319–378, 2020.

[CRF20] Angela Capel, Cambyse Rouzé, and Daniel Stilck França. The modified logarithmic sobolev inequality for quantum spin systems: classical and commuting nearest neighbour interactions. arXiv preprint arXiv:2009.11817, 2020.

[CS03] Fabio Cipriani and Jean-Luc Sauvageot. Derivations as square roots of dirichlet forms. Journal of Functional Analysis, 201(1):78–120, 2003.

[Dav80] Edward Brian Davies. One-parameter semigroups. Academic Press, 1980.

[Dic54] R. H. Dicke. Coherence in spontaneous radiation processes. Phys. Rev., 93:99–110, Jan 1954.

[DKW02] Ho Trung Dung, Ludwig Kaôll, and Dirk-Gunnar Welsch. Resonant dipole-dipole interaction in the presence of dispersing and absorbing surroundings. Phys. Rev. A, 66:063810, Dec 2002.

[DL92] Edward Brian Davies and J. Martin Lindsay. Non-commutative symmetric markov semigroups. Mathematische Zeitschrift, 210(3):379–412, 1992.

[dPWS02] B. de Pagter, H. Witvliet, and F.A. Sukochev. Double operator integrals. Journal of Functional Analysis, 192(1):52–111, 2002.

[DR19] Nilanjana Datta and Cambyse Rouzé. Concentration of quantum states from quantum functional and transportation cost inequalities. Journal of Mathematical Physics, 60(1):012202, 2019.

[DR20] Nilanjana Datta and Cambyse Rouzé. Relating relative entropy, optimal transport and fisher information: A quantum hwi inequality. Ann. Henri Poincaré, 21:2115–2150, 2020.

[DSC96] P. Diaconis and L. Saloff-Coste. Logarithmic Sobolev inequalities for finite Markov chains. The Annals of Applied Probability, 6(3):695 – 750, 1996.

[Eis21] Jens Eisert. Entangling power and quantum circuit complexity. Physical Review Letters, 127(2), jul 2021.
[FH04] William Fulton and Joe Harris. *Representation Theory - A First Course*. Springer New York, NY, 2004.

[GG22] Li Gao and Maria Gordina. Complete modified logarithmic sobolev inequality for sub-laplacian on su(2). *arXiv preprint arXiv:2203.12731*, 2022.

[GJL20a] Li Gao, Marius Junge, and Nicholas LaRacuente. Fisher information and logarithmic sobolev inequality for matrix-valued functionals. *Ann. Henri Poincaré*, 21:3409 – 3478, 2020.

[GJL20b] Li Gao, Marius Junge, and Nicholas LaRacuente. Relative entropy for von neumann subalgebras. *International Journal of Mathematics*, 31(06):2050046, 2020.

[GJL21] Li Gao, Marius Junge, and Haojian Li. Geometric approach towards complete logarithmic sobolev inequalities. *arXiv preprint arXiv:2102.04434*, 2021.

[GKS76] Vittorio Gorini, Andrzej Kossakowski, and E.C.G. Sudarshan. Completely positive dynamical semigroups of n-level systems. *Journal of Mathematical Physics*, 17(5):821–825, 1976.

[GR22] Li Gao and Cambyse Rouzé. Complete entropic inequalities for quantum markov chains. *Archive for Rational Mechanics and Analysis*, 245:183–238, 2022.

[Gro75a] Leonard Gross. Hypercontractivity and logarithmic sobolev inequalities for the clifford-dirichlet form. *Duke Mathematical Journal*, 42(3):383–396, 1975.

[Gro75b] Leonard Gross. Logarithmic sobolev inequalities. *American Journal of Mathematics*, 97(4):1061–1083, 1975.

[Gro14] Leonard Gross. Hypercontractivity, logarithmic sobolev inequalities, and applications: a survey of surveys. *Diffusion, quantum theory, and radically elementary mathematics*, 47:45–73, 2014.

[GW96] T. Gruner and D.-G. Welsch. Green-function approach to the radiation-field quantization for homogeneous and inhomogeneous kramers-kronig dielectrics. *Phys. Rev. A*, 53:1818–1829, Mar 1996.

[HS87] Richard Holley and Daniel Stroock. Logarithmic sobolev inequalities and stochastic ising models. *J. Stat. Phys.*, 46:1159–1194, 1987.

[JLR19] Marius Junge, Nicholas LaRacuente, and Cambyse Rouzé. Stability of logarithmic sobolev inequalities under a noncommutative change of measure. *arXiv preprint arXiv:1911.08533*, 2019.

[JP10] Marius Junge and Javier Parcet. *Mixed-norm inequalities and operator space $L_p$ embedding theory*. American Mathematical Soc., 2010.

[KB16] Michael J. Kastoryano and Fernando G.S.L. Brandão. Quantum gibbs samplers: The commuting case. *Commun. Math. Phys.*, 344:915–957, 2016.

[KBG$^+$11] M. Kliesch, T. Barthel, C. Gogolin, M. Kastoryano, and J. Eisert. Dissipative quantum church-turing theorem. *Phys. Rev. Lett.*, 107:120501, Sep 2011.

[KE13] Michael J. Kastoryano and Jens Eisert. Rapid mixing implies exponential decay of correlations. *Journal of Mathematical Physics*, 54(10):102201, 2013.

[KSW00] Ludwig Knoll, Stefan Scheel, and Dirk-Gunnar Welsch. Qed in dispersing and absorbing media. *arXiv preprint arXiv:quant-ph/0006121*, 2000.

[KT13] Michael J. Kastoryano and Kristan Temme. Quantum logarithmic sobolev inequalities and rapid mixing. *Journal of Mathematical Physics*, 54(5):052202, 2013.

[Lin76] G. Lindblad. On the generators of quantum dynamical semigroups. *Communications in Mathematical Physics*, 48(2):119–130, 1976.

[LJJL20] Haojian Li, Marius Junge, and Nicholas LaRacuente. Graph hörmander systems. *arXiv preprint arXiv:2006.14578*, 2020.

[MAG22] Stuart J. Masson and Ana Asenjo-Garcia. Universality of dicke superradiance in arrays of quantum emitters. *Nature Communications*, 13(1), apr 2022.

[MFBO$^+$20] Stuart J. Masson, Igor Ferrier-Barbut, Luis A. Orozco, Antoine Browaeys, and Ana Asenjo-Garcia. Many-body signatures of collective decay in atomic chains. *Phys. Rev. Lett.*, 125:263601, Dec 2020.

[Pet91] Dénes Petz. On certain properties of the relative entropy of states of operator algebras. *Mathematische Zeitschrift*, 206(1):351 – 361, 1991.

[PP86] Mihai Pimsner and Sorin Popa. Entropy and index for subfactors. *Annales scientifiques de l’École Normale Supérieure*, Ser. 4, 19(1):57–106, 1986.

[SCG13] Heike Schwager, J. Ignacio Cirac, and Géza Giedke. Dissipative spin chains: Implementation with cold atoms and steady-state properties. *Physical Review A*, 87(2), feb 2013.
[SMAG22] Eric Sierra, Stuart J. Masson, and Ana Asenjo-Garcia. Dicke superradiance in ordered lattices: Dimensionality matters. *Phys. Rev. Research*, 4:023207, Jun 2022.

[Spo78] Herbert Spohn. Entropy production for quantum dynamical semigroups. *Journal of Mathematical Physics*, 19(5):1227–1230, 1978.

[Tak03] Masamichi Takesaki. *Theory of operator algebras. II*. Encyclopaedia of Mathematical Sciences, vol. 127. Springer-Verlag, Berlin, 2003.

[Wir18] Melchior Wirth. A noncommutative transport metric and symmetric quantum markov semigroups as gradient flows of the entropy. *arXiv preprint arXiv:1808.05419*, 2018.

[Wit18] Edward Witten. APS medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory. *Reviews of Modern Physics*, 90(4), oct 2018.

[WZ21] Melchior Wirth and Haonan Zhang. Complete gradient estimates of quantum markov semigroups. *Commun. Math. Phys.*, 387:761–791, 2021.