Abstract

We exhibit a computational type theory which combines the higher-dimensional structure of cartesian cubical type theory with the internal parametricity primitives of parametric type theory, drawing out the similarities and distinctions between the two along the way. The combined theory supports both univalence and its relational equivalent, which we call relativity. We demonstrate the use of the theory by analyzing polymorphic types, including functions between higher inductive types, and we show by example how relativity can be used to characterize the relational interpretation of inductive types.

1 Introduction

This paper brings together two closely-related varieties of “augmented” dependent type theory: cubical type theory [Bezem et al., 2013; Cohen et al., 2015; Angiuli et al., 2017a, 2018] and parametric type theory [Bernardy and Moulin, 2012, 2013; Bernardy et al., 2015; Nuyts et al., 2017]. Each of these theories serves to internalize a feature found in particular models of Martin-Löf’s dependent type theory, higher-dimensional structure in the former case and parametric polymorphism in the latter. Moreover, each does so by introducing a notion of dimension variable. A term which varies over a dimension variable expresses some relationship between its endpoints, its values at a fixed set of dimension constants. In cubical type theory, such a term is called a path; we follow Nuyts et al. [2017] in calling the parametric equivalent a bridge. The connection between higher-dimensional and parametric type theories is no secret; it was observed already by Bernardy and Moulin in 2012, and the two lines of work have continued to influence each other. We present a type theory which combines the central tools of both: univalence and higher inductive types on the cubical side, and what we will call relativity on the parametric side.

Parametric type theory grows out of a long line of work on parametric polymorphism. A polymorphic function—one whose type contains free type variables—is intuitively said to be polymorphic when its behavior is uniform in those variables. Reynolds [1983] observed that a type discipline could ensure all polymorphic functions are parametric, and that this could be proven using a relational interpretation of the theory. Such interpretations have since been designed for a variety of theories, including dependent type theories [Bernardy et al., 2010]. Recently, Bernardy and Moulin and Nuyts et al. have introduced type theories which internalize their own relational interpretation, making it possible to prove and exploit parametricity results within the theory. Beginning with Bernardy and Moulin [2013], these theories have relied on dimension variables to organize the iterated relational structure which arises thence. Roughly, a term of type $A$ in $n$ dimension variables—an $n$-dimensional bridge—represents a term in the $n$th iterated relational interpretation of $A$.

Cubical type theory, on the other hand, endows dependent type theory with a coarse, proof-relevant notion of equality, the aforementioned path. A path is again a term in a dimension variable, with two endpoints given by substituting one of two constants 0, 1. As in the relational case, the use of dimension variables serves to organize the structure of iterated path types. In contrast, however, cubical type theory also includes Kan operations which ensure that all types respect paths. A central feature of cubical type
theory is univalence, which for any two types establishes a correspondence between the paths between them and the equivalences (roughly, isomorphisms) between them. Cubical type theory also permits the definition of higher inductive types (HITs), types inductively defined by higher-dimensional path generators [Coquand et al., 2018; Cavallo and Harper, 2019]. Cubical type theory gives a constructive interpretation of homotopy type theory, an extension of dependent type theory with axioms asserting univalence and the existence of HITs in terms of the Martin-Löf identity type [Univalent Foundations Program, 2013].

Theory The development of the combined type theory and its semantics is largely straightforward, as interaction between the bridge and path structure is minimal; we only need a minor modification to cubical type theory’s Kan conditions to make bridge types Kan. As compared with the work of Bernardy et al., our parametric side hews closer to cubical type theory: our bridges have two endpoints instead of one, and we aim for an equivalence between bridges in the universe and relations (the aforementioned relativity) rather than an exact equality. The latter means that we do not need the technical device of I-sets employed by Bernardy et al. [2015]; instead, we rely on univalence. Throughout, we present the parametric aspects in a style meant to clarify the connection to cubical type theory, drawing attention to the essential differences as they arise. Foremost among these is the use of structural dimension variables on the path side (following Angiuli et al.) as opposed to substructural dimensions on the bridge side (following Bernardy et al.), which is reflected in the differences between paths and bridges at function and universe types.

Applications By adding cubical structure to parametric type theory, we obtain function extensionality, which is particularly convenient for working with Church encodings. For example, we can actually show that \((X \times I) \to X \to X\) is equivalent to unit (Section 11.1). In the opposite direction, by adding relational structure to cubical type theory, we are able to derive “free theorems” [Wadler, 1989] for polymorphic functions on HITs (Section 11.5). Properties like these are of particular interest because coherence obligations can seriously complicate proofs about functions between HITs. For example, proofs about the monoidal structure of the smash product [Univalent Foundations Program, 2013, §6.8] are notoriously difficult. We also develop the methodology of parametric type theory beyond that discussed in prior work: we introduce the essential notion of bridge-discrete type and show how to characterize the bridge type of data types such as \(\text{bool}\) using relativity (Section 11.3). These results should transfer in some form to the theory of Bernardy et al., but have not previously been explored. Finally, the combined theory is a witness to the consistency of homotopy type theory and cubical type theory with the negation of (some versions of) the law of the excluded middle (Section 11.4).

Outline We develop our type theory primarily in the form of a partial equivalence relation (PER) semantics for a programming language, following the work of Angiuli et al. [2017a] for cubical type theory. We introduce the language in Section 2, followed by the PER semantics (including Kan conditions) in Section 3. In Section 4, we recall the Path and V type formers of Angiuli et al., which translate into the extended theory without incident, together with a few standard definitions and theorems of higher-dimensional type theory. We come to the parametric side in Section 5, introducing Bridge-types and showing that these are Kan. We show how Bridge-types commute with the connectives of cubical type theory in Section 6; for function types, this requires the introduction of the extent operator. Finally, we introduce Gel-types, the parametric equivalent of V-types, in Section 7. The Bridge, extent, and Gel operators correspond to the \(A \ni a, \langle t, u\rangle\), and \((x : A) \times_i P\) constructs of Bernardy et al. [2015] respectively. (We include an extended translation dictionary in Section 12).

This completes the design of the type theory. In Section 8, we collect the inference rules we have established for each type into a makeshift proof theory; we use only these rules in the remainder of the paper. We begin by proving relativity in Section 9, establishing the correspondence between bridges in the universe and relations on their endpoints. Section 10 defines the sub-universe of bridge-discrete types. In Section 11, we give a series of examples illustrating the use of the theory.
2 Programming language

We begin by introducing the untyped programming language on which our type systems are based. The language has three sorts: *bridge dimensions*, *path dimensions*, and *terms*.

### 2.1 Dimension terms and contexts

The sorts of bridge and path dimensions are defined by the following grammar.

\[(\text{bridge dim}) \quad r \ ::= \ x \mid 0 \mid 1\]

\[(\text{path dim}) \quad r \ ::= \ x \mid 0 \mid 1\]

We use the letters \(x, y, z, \ldots\) for path dimension variables and \(r, s, \ldots\) for path dimension terms, with bridge dimensions using the same letters but written in **bold type**. We use \(\varepsilon\) to stand for 0 or 1, likewise \(\varepsilon\) for 0 or 1. We use \(\rho\) for lists of bridge dimensions.

**Definition 2.1.** A bridge-path context is a pair \((\Phi \mid \Psi)\) where \(\Phi = \{x_1, \ldots, x_n\}\) is a set of bridge dimension variables and \(\Psi = \{x_1, \ldots, x_n\}\) is a set of path dimension variables. We use the letter \(\mathcal{D}\) for bridge-path contexts. We write \(r \text{ bdim } [\Phi \mid \Psi]\) and \(r \text{ pdim } [\Phi \mid \Psi]\) to mean \(r \in \Phi \cup \{0, 1\}\) and \(r \in \Psi \cup \{0, 1\}\) respectively. Likewise, we write \(\rho \text{ bdim } [\Phi \mid \Psi]\) when \(\rho = r_i\) with \(r_i \text{ bdim } [\Phi \mid \Psi]\) for each \(i\).

**Definition 2.2.** A bridge-path substitution \(\psi : (\Phi' \mid \Psi') \rightarrow (\Phi \mid \Psi)\) is a function \(\psi\) taking \(x \in \Phi\) to \(\psi(x) \in \Phi \cup \{0, 1\}\) and \(x \in \Psi\) to \(\psi(x) \in \Psi \cup \{0, 1\}\), such that if \(\psi(x) = \psi(y)\) then either \(\psi(x) \in \{0, 1\}\) or \(x = y\). We write \(r\psi\) and \(r\psi\) for the action of \(\psi\) on some \(r\) or \(r\), which replaces a variable by its image under \(\psi\) and leaves the constants \(0, 1, 0, 1\) untouched.

The category of bridge-path contexts and their substitutions is thus the product of a category of bridge contexts and substitutions and a category of path contexts and substitutions. The former is adapted from Bernardy et al.; the only change is the addition of a second constant 1 (corresponding to a move from unary to binary relations). It is also the base category used in the model of homotopy type theory due to Bezem et al. [2013, 2017], as well as that used by Johann and Sojakova [2017] to construct parametric models of System F. In the terminology of Buchholz and Morehouse [2017], it is \(\mathcal{C}_{\text{sec.}}\): we have weakening and exchange for bridge variables, but not contraction. The category of path contexts is taken unchanged from Angiuli et al.; it is the cartesian cube category \(\mathcal{C}_{\text{sec.}}\), which additionally supports contraction. The choice of the latter category is not essential; we conjecture that cartesian cubical type theory could be replaced in this development with any other cubical type theory without much change. This could be the theory of Cohen et al. [2015], which is based on \(\mathcal{C}_{\text{sec.}, \wedge, \vee}\), or that of Bezem et al. (though the latter is problematic for higher inductive types, as discussed in Section 12). The choice of a category without contraction for bridge dimensions, however, is apparently forced, for reasons we will first encounter in Section 6.

**Definition 2.3.** Given \(\Phi\) and \(r \in \Phi \cup \{0, 1\}\), we write \(\Phi^{\backslash r} := \Phi \setminus \{r\}\). Given \(\mathcal{D} = (\Phi \mid \Psi)\), we write \(\mathcal{D}^{\backslash r}\) for \((\Phi^{\backslash r} \mid \Psi)\). For a list \(\rho = r_1, \ldots, r_n\), we write \(-\rho\) to mean \(-r_n \ldots -r_1\).

**Definition 2.4.** Given \(\psi : (\Phi' \mid \Psi') \rightarrow (\Phi \mid \Psi)\), we write \((\psi, x) : (\Phi', x \mid \Psi') \rightarrow (\Phi \mid \Psi)\) and \((\psi, x) : (\Phi' \mid \Psi', x) \rightarrow (\Phi \mid \Psi)\) for the result of weakening the substitution with \(x \notin \Phi'\) or \(x \notin \Psi'\) respectively. Given \(r \text{ bdim } [\Phi \mid \Psi]\), we write \(\psi^{\backslash r} : (\Phi^{\backslash r} \mid \Psi) \rightarrow (\Phi^{\backslash r} \mid \Psi)\) for the restriction of \(\psi\) to \((\Phi^{\backslash r} \mid \Psi)\). Finally, given \((\Phi_0 \mid \Psi_0)\) disjoint from \((\Phi \mid \Psi)\) and \((\Phi' \mid \Psi')\), we write \(\psi \times (\Phi_0 \mid \Psi_0) : (\Phi' \mid \Psi') \rightarrow (\Phi_0 \mid \Psi_0)\) for \(\psi\) extended by the identity on \((\Phi_0 \mid \Psi_0)\).
\[ M ::= (a:M) \to M \mid \lambda a. M \mid MM \mid (a:M) \times M \mid \langle M, M \rangle \mid \text{fst}(M) \mid \text{snd}(M) \mid \text{Path}_{x,M}(M, M) \mid \lambda x. M \mid M \oplus r \mid V_r(M, M, M) \mid \text{vin}_r(M, M) \mid \text{Vproj}_r(M, M) \mid \text{Bridge}_{x,M}(M, M) \mid \lambda I \to x. M \mid M @ r \mid V_r(M, x.M, M) \mid \text{Vin}_r(M, M) \mid Vproj_r(M, M) \mid \text{Bridge}_{x,M}(x, M) \mid \lambda /BE \to x. M \mid M @ r \mid \text{Gel}_r(M, x, M, M) \mid \text{gel}_r(M, x, M, M) \mid \text{ungel}(x, M) \mid \text{extent}_r(M, a.M, a.a.a.M) \mid \text{ext}(M, a.M, a.a.a.a.M) \mid hcom_{M, \xi}^\rightarrow(M; \xi \mapsto x.M) \mid \text{coe}_{x,M}^\rightarrow(M) \mid \ldots \]

Figure 1: The term language

2.2 Operational semantics

Definition 2.5. We write \( M, N, \ldots \) for terms, which are drawn from some fixed superset of the grammar shown in Figure 1. We write \( M \psi \) for the action of a path-bridge substitution on a term; such substitution instances are called the aspects of \( M \). We write \( M \varnothing [\Phi | \Psi] \) when every dimension variable occurring in \( M \) appears either in \( \Phi \) or in \( \Psi \).

Definition 2.6. We write \( M \langle r/x \rangle \) for the result of substituting \( r \) for \( x \) in \( M \). Likewise, we write \( M \langle r/x \rangle \) for the substitution of \( r \) for \( x \) in \( M \). We have \( \langle r/x \rangle : (\Phi | \Psi) \to (\Phi \setminus r, x | \Psi) \) and \( \langle r/x \rangle : (\Phi | \Psi) \to (\Phi | \Psi, x) \).

Definition 2.7. An evaluation system is a pair of judgments \( M \to M' \) and \( M \varnothing \) over terms \( M, M' \) which are

1. deterministic: if \( M \to M' \) and \( M \to M'' \) then \( M' = M'' \), and it is never the case that both \( M \to M' \) and \( M \varnothing \); and
2. context-preserving: if \( M \varnothing [\Phi | \Psi] \) and \( M \to M' \) then \( M' \varnothing [\Phi | \Psi] \).

We write \( M \Downarrow V \) when \( M \to^* V \) and \( V \varnothing \).

We fix an evaluation system for the remainder of the paper. We will require that the judgments \( M \to M' \) and \( M \varnothing \) satisfy various inference rules concerning the grammar in Figure 1, which we introduce as we discuss each operator. The results hold for any language and evaluation system which extend those we present.

3 Type systems and judgments

We next introduce \( D \)-PERs, which serve as the semantics of types in context \( D \), and path-bridge type systems, which define a partial equivalence relation on type names and associate a \( D \)-relation to each equivalence class of types. Finally, we define a notion of Kan type. These definitions constitute a straightforward extension of the Angiuli et al. semantics from path dimension contexts to path-bridge contexts.

3.1 \( D \)-relations

Definition 3.1. Let a bridge-path context \( D \) be given. A (value) \( D \)-relation \( \alpha \) is a mapping from substitutions \( \psi' : (\Phi' | \Psi') \to D \) to relations \( \alpha_{\psi'}(-, -) \) on terms (values) \( M, M' \varnothing [\Phi' | \Psi'] \). We say \( \alpha \) is a \( D \)-PER when each \( \alpha_{\psi} \) is a PER.

Value \( D \)-PERs will serve as the semantics of types in context \( D \). The \( D \)-PER \( \alpha \) assigned to a type name \( A \) gives, for each \( \psi : D' \to D \), the PER \( \alpha_{\psi} \) of values dubbed equal in the aspect \( A \psi \) of \( A \).
Notation 3.2. We abbreviate $\alpha_{id}(M, M')$ as $\alpha(M, M')$. When $\psi : D' \to D$ and $\alpha$ is a $D$-relation, we define a $D'$-relation $\alpha_{\psi}$ by $(\alpha\psi)\psi' := \alpha_{\psi\psi'}$.

**Definition 3.3.** Let $\alpha$ be a value $D$-relation. We define a $D$-relation $\text{TM}(\alpha)$ as follows: $\text{TM}(\alpha)(M, M')$ holds for $\psi : D' \to D$ when for every $\psi_1 : D_1 \to D'$ and $\psi_2 : D_2 \to D_1$, there exist terms $M_1, M_1' \text{ tm } [D_1]$ and $M_2, M_2', M_{12}, M_{12}' \text{ tm } [D_2]$ such that

$$M\psi_1 \downarrow M_1 \quad M_1\psi_2 \downarrow M_2 \quad M\psi_1\psi_2 \downarrow M_{12}$$

$$M'\psi_1 \downarrow M_1' \quad M_1'\psi_2 \downarrow M_2' \quad M'\psi_1\psi_2 \downarrow M_{12}'$$

with $\alpha_{\psi_1\psi_2}(V, V')$ for all $V \in \{M_2, M_{12}\}$ and $V' \in \{M_2', M_{12}'\}$.

When $\alpha$ is the value $D$-PER assigned to a type $A$, we use $\text{TM}(\alpha)$ as the $D$-PER of elements of $A$: the terms which evaluate to values in $A$ in a way that is coherent with dimension substitution.

**Remark 3.4.** The $D$-relation $\text{TM}(\alpha)$ is always stable under dimension substitution, in the sense that given $\psi : D' \to D$ and $M, M' \text{ tm } [D']$, $\text{TM}(\alpha)(M, M')$ implies $\text{TM}(\alpha)(M\psi, M'\psi')$ for every $\psi' : D'' \to D'$. Determinism of the operational semantics ensures that $\text{TM}(\alpha)$ is a PER whenever $\alpha$ is a PER.

**Definition 3.5.** A value $D$-relation $\alpha$ is value-coherent, written $\text{CoH}(\alpha)$, when $\alpha \subseteq \text{TM}(\alpha)$.

To prove theorems about $D$-relations, we use a toolbox of utility lemmas. These are minor variations on lemmas used by Angiuli et al.; we include proofs in Appendix A.

**Lemma A.1** (Introduction). Let $\alpha$ be a value $D$-relation. If for every $\psi : D' \to D$, either $\alpha_{\psi}(M\psi, M'\psi)$ or $\text{TM}(\alpha)(M\psi, M'\psi)$, then $\text{TM}(\alpha)(M, M')$.

**Lemma A.2** (Coherent expansion). Let $\alpha$ be a value $D$-PER and let $M, M' \text{ tm } [\Phi \mid \Psi]$. If for every $\psi : D' \to D$, there exists $M''$ such that $M\psi \mapsto^* M''$ and $\text{TM}(\alpha)(M'', M'\psi)$, then $\text{TM}(\alpha)(M, M')$.

**Lemma A.3** (Evaluation). Let $\alpha$ be a value-coherent $D$-PER and let $M, M' \text{ tm } [\Phi]$ with $\text{TM}(\alpha)(M, M')$. Then $M \downarrow V$ and $M' \downarrow V'$ where $\text{TM}(\alpha)(Q, Q')$ holds for all $Q \in \{M, V\}$ and $Q' \in \{M', V'\}$.

We defer a final lemma for proving elimination theorems to Section 3.2, as it will be simpler to state using notation we have not yet introduced.

### 3.2 Type systems

**Definition 3.6.** A candidate path-bridge type system is a four-place relation $\tau(D, A_0, A'_0, \varphi)$ on path-bridge contexts $D$, values $A_0, A'_0 \text{ tm } [D]$, and (ordinary) relations $\varphi$ on values $V, V' \text{ tm } [D]$.

**Definition 3.7.** Given a candidate path-bridge type system $\tau$, we say that $\text{PTY}(\tau)(D, A, A', \alpha)$ holds of a path-bridge context $D$, terms $A, A' \text{ tm } [D]$, and a value $D$-relation $\alpha$ when for all $\psi_1 : D_1 \to D$ and $\psi_2 : D_2 \to D_1$, there exist terms $A_1, A'_1 \text{ tm } [D_1]$ and $A_2, A'_2, A_{12}, A'_{12} \text{ tm } [D_2]$ such that

$$A\psi_1 \downarrow A_1 \quad A_1\psi_2 \downarrow A_2 \quad A\psi_1\psi_2 \downarrow A_{12}$$

$$A'\psi_1 \downarrow A'_1 \quad A'_1\psi_2 \downarrow A'_2 \quad A'\psi_1\psi_2 \downarrow A'_{12}$$

and $\tau(D_2, V, V', \alpha_{\psi_1\psi_2})$ for all $V \in \{A_2, A_{12}\}$ and $V' \in \{A'_2, A'_{12}\}$.

**Definition 3.8.** A candidate path-bridge type system $\tau$ is a path-bridge type system when

1. if $\tau(D, A_0, A'_0, \varphi)$ and $\tau(D, A_0, A'_0, \varphi')$ then $\varphi = \varphi'$,
2. if $\tau(D, A_0, A'_0, \varphi)$ then $\varphi$ is a PER,
3. $\tau(D, \varphi, \varphi, \varphi)$ is a PER for each $D, \varphi$,
4. if $\tau(D, A_0, A'_0, \varphi)$ then $\text{PTY}(\tau)(D, A_0, A'_0, \alpha)$ for some $\alpha$. 


Figure 2: Non-type-specific operational semantics of \( h\text{com}, \text{coe}, \text{and com} \)

**Definition 3.9.** Given a candidate path-bridge type system \( \tau \), we define the closed judgments of type theory as follows.

\[
\begin{align*}
\tau \models A \equiv A' & \text{ type}_{\text{pre}} [D] \iff \exists \alpha. \text{PTy}(\tau)(D, A, A', \alpha) \land \text{COH}(\alpha) \\
\tau \models M \equiv M' & \in A [D] \iff \exists \alpha. \text{PTy}(\tau)(D, A, A) \land \alpha(M, M')
\end{align*}
\]

We abbreviate \( \tau \models A \equiv A' \text{ type}_{\text{pre}} [D] \) as \( \tau \models A \equiv A' \text{ type}_{\text{pre}} [D] \) and \( \tau \models M \equiv M' \in A [D] \) as \( \tau \models M \in A [D] \). If \( \tau \) is a path-bridge type system and \( \tau \models A \text{ type}_{\text{pre}} [D] \), we write \([A]\) for the (necessarily unique) \( \alpha \) such that \( \text{PTy}(\tau)(D, A, A, \alpha) \) holds. For much of this paper, we work relative to a fixed ambient path-bridge type system, so will drop the prefix \( \tau \models = \) and superscript on \([\cdot]\).

As with \( D \)-relations, we have a pair of lemmas for proving that terms are related by \( \text{PTy}(\tau) \).

**Lemma A.4** (Formation). Let \( \tau \) be a bridge-path type system, let \( A, A' \text{ tm } [D] \), and let \( \alpha \) be a value \( D \)-relation. If for every \( \psi : D' \to D \), either \( \text{PTy}(\tau)(D', A\psi, A'\psi, \alpha\psi) \) holds or \( \tau(D', A\psi, A'\psi, \alpha) \) holds, then \( \text{PTy}(\tau)(D, A, A', \alpha) \).

**Lemma A.5** (Coherent type expansion). Let \( \tau \) be a bridge-path type system, let \( A, A' \text{ tm } [D] \), and let \( \alpha \) be a value \( D \)-relation. If for all \( \psi : D' \to D \), there exists \( A'' \) such that \( A\psi \rightsquigarrow A'' \) and \( \text{PTy}(\tau)(D', A'', A'\psi, \alpha\psi) \), then \( \text{PTy}(\tau)(D, A, A', \alpha) \).

Finally, we state the elimination lemma referred to in Section 3.1. We use this lemma to prove typing rules for operators which evaluate their arguments. Certain operators of bridge-path type theory will evaluate arguments under dimension binders or in a restricted path-bridge context, so we first introduce a notion of expression context.

**Definition 3.10.** An expression context \( C \) is a term with at most one hole, written \([\cdot]\); we write \( C[M] \) for the result of filling the hole with a term \( M \). We write \( C : D \models D_0 \) where \( D \) and \( D_0 \) are disjoint when \( C[M] \text{ tm } [D] \) for every \( M \text{ tm } [D] \). For example, we have \( \lambda x.\lambda y.[-] : (\Phi | \Psi) \models (x | y) \). Given \( C : D \models D_0 \) and \( \psi : D' \to D \) with \( D' \) disjoint from \( D_0 \), we have \( C\psi : D' \models D_0 \).

**Definition 3.11.** An expression context \( C : D \models D_0 \) is **eager** when for any \( M, M' \text{ tm } [D][D_0] \) with \( M \to \to M' \), we have \( C[M] \models C[M'] \).

**Lemma A.6** (Elimination). Let \( C, C', T : D \models D_0, \rho \text{ bdims } [D] \), and let \( \alpha \) be a value-coherent \((D)^\rho[D_0])\)-\text{PER}. Suppose that for every \( \psi : D' \to D \) with \( D' \) disjoint from \( D_0 \), we have

1. \( T\psi[M] \models T\psi[M'] \text{ type}_{\text{pre}} [D'] \) for all \( \text{Tm}(\alpha) \psi \omega_{\rho \times D_0}(M, M') \),
2. \( C\psi, C'\psi \text{ are eager and } C\psi[V] \models C'\psi[V'] \in T\psi[V] [D'] \) for all \( \alpha \psi \omega_{\rho \times D_0}(V, V') \).

Then \( C[M] \models C'[M'] \in T[M] [D] \) for every \( \text{Tm}(\alpha)(M, M') \).
3.3 Kan operations

We have so far explained what it means for a term $A \text{ tm } [\Phi \mid \Psi]$ to be a pretype; a type is a pretype which additionally supports the Kan operations $\text{coe}$ and $\text{hcom}$. The former, $\text{coe}$, ensures that $A$ respects paths: for any $x \in \Psi$, it gives a function from $A(r/x)$ to $A(s/x)$ for every $r$ and $s$. The latter, $\text{hcom}$, is necessary to ensure that the iterated path and bridge types of any Kan type are also Kan: it takes a term in $\text{co}_A$ and adjusts its lower-dimensional boundary by a given collection of paths. Both $\text{coe}$ and $\text{hcom}$ are fixed operators which take the type $A$ as a parameter and evaluate it, as shown in Figure 2. Once $A$ is in canonical form, the further evaluation of $\text{coe}$ and $\text{hcom}$ is dependent on its form; for example, $\text{coe}$ in a pair type steps to a pair of $\text{coes}$ in the component types. We will introduce the type-specific operational semantics of $\text{coe}$ and $\text{hcom}$ later on, in tandem with the types themselves.

The definition of $\text{coe}$ is exactly that given by Angiuli et al. for cubical type theory.

**Definition 3.12.** We say that $A \equiv A'$ type$_{pre}$ $[\Phi \mid \Psi]$ are equally $\text{coe}$-Kan when for all $\psi : (\Phi' \mid \Psi', y) \rightarrow (\Phi \mid \Psi)$, $r, s \text{ pdim } [\Phi' \mid \Psi']$, and $M \equiv M' \in A\psi(y/r)[\Phi' \mid \Psi']$, we have

1. $\text{coer}_{\psi,A}[M] \equiv \text{coer}_{\psi,A}[M'] \in A\psi(s/r)[\Phi' \mid \Psi']$,
2. $\text{coer}_{\psi,A}[M] \equiv M \in A\psi(s/r)[\Phi' \mid \Psi']$ if $r = s$.

For $\text{hcom}$, a minor change is necessary, as we need to ensure that not only the path types but also the bridge types of a Kan type are Kan. We do so by adding $r = 0$ and $r = 1$ in the following definition.

**Definition 3.13.** A constraint $\xi$ is an equation drawn from the following grammar.

$$\xi ::= r = 0 \mid r = 1 \mid r = r'$$

We write $\xi \text{ eq } [\Phi \mid \Psi]$ when $\xi$ is an equation on variables in $(\Phi \mid \Psi)$, $x \notin \xi$ to mean that $x$ does not occur in $\xi$, and $\models \xi$ to mean that $\xi$ is true (i.e., is a reflexive equation). We use $\Xi$ for lists of constraints and likewise write $\Xi \text{ eqs } [\Phi \mid \Psi]$. We write $\Xi^r$ for the result of removing all equations mentioning a given $r$ from $\Xi$.

**Definition 3.14.** For each of the closed judgments $J [\Phi \mid \Psi]$ defined in Definition 3.9, we define its restricted form $\mathcal{J} [\Phi \mid \Psi \mid \Xi]$ to hold when $\mathcal{J} \psi [\Phi' \mid \Psi']$ holds for every $\psi : (\Phi' \mid \Psi') \rightarrow (\Phi \mid \Psi)$ such that $\models \Xi\psi$.

**Definition 3.15.** We say that $A \equiv A'$ type$_{pre}$ $[\Phi \mid \Psi]$ are equally $\text{hcom}$-Kan when for all $\psi : (\Phi' \mid \Psi') \rightarrow (\Phi \mid \Psi)$, $r, s \text{ pdim } [\Phi' \mid \Psi']$, and $\xi_i \text{ eqs } [\Phi' \mid \Psi']$, if

1. $M \equiv M' \in A\psi[\Phi' \mid \Psi']$,
2. $N_i \equiv N_j \in A\psi[\Phi' \mid \Psi', y \mid \xi_i, \xi_j]$ for all $i, j$,
3. $N_i(r/y) \equiv M \in A\psi[\Phi' \mid \Psi']$ for all $i$,

then

1. $\text{hcom}_{\psi,A}^r(M; \xi_i \leftarrow y.N_i) \equiv \text{hcom}_{\psi,A}^r(M'; \xi_i \leftarrow y.N'_i) \in A\psi[\Phi' \mid \Psi']$,
2. $\text{hcom}_{\psi,A}^r(M; \xi_i \leftarrow y.N_i) \equiv N_i(s/y) \in A\psi[\Phi' \mid \Psi']$ for all $i$ with $\models \xi_i$,
3. $\text{hcom}_{\psi,A}^r(M; \xi_i \leftarrow y.N_i) \equiv M \in A\psi[\Phi' \mid \Psi']$ if $r = s$.

**Remark 3.16.** Angiuli et al. impose an additional validity condition on the list of constraints $\Xi\xi$, which enables a stronger canonicity result for higher inductive types. This choice is orthogonal to the addition of bridges, so for simplicity’s sake we will leave it out.
The $\text{hcom}$ operator takes a term $M$, a list of constraints $\xi_i$, and a list of paths $y.N_i$ each of which is defined on the corresponding constraint and matches $M$ on its $\langle r/y \rangle$ face. The output of $\text{hcom}$ is a term which matches the $\langle r'/y \rangle$ face of $N_i$ under $\xi_i$ for each $i$. For example, given terms $M \in A[\Phi,x|\Psi]$ and terms $N_0, N_1 \in A[\Phi,\Psi,y]$ which agree with $M$ on their $\langle 0/y \rangle$ faces, we have the following picture.

A type's support for $\text{hcom}$ implies that its paths compose, hence the name: given paths from $M$ to $N$ and $N$ to $P$ in $A$, $\text{hcom}$ can be used to construct a path from $M$ to $P$ in $A$. Likewise, $\text{hcom}$ can be used to compose bridges with paths. For example, we can combine a bridge from $N$ and $\text{hcom}$ into a bridge from $M$.

### Definition 3.17

When $A \triangleq A'$ type$_{\text{pre}} [\Phi,\Psi]$ are both equally hcom-Kan and coe-Kan, we say they are equally Kan and write $A \triangleq A'$ type$_{\text{coe}} [\Phi,\Psi]$. We will use $\kappa$ as a metavariable standing for either pre or Kan.

In any Kan type, we have a derived heterogeneous composition operation $\text{com}$ which combines the functions of $\text{hcom}$ and coe. This operator, which has operational semantics defined in Figure 2, satisfies the following typing rules.

### Proposition 3.18

Let $A \triangleq A'$ type$_{\text{coe}} [\Phi,\Psi]$. For every $\psi : (\Phi'|\Psi',y) \to (\Phi|\Psi)$, $r,s \ p\text{dim} [\Phi'|\Psi']$, and $\xi_i$ eqs $[\Phi'|\Psi']$, if

1. $M \triangleq M' \in A\psi(r/y) [\Phi'|\Psi']$,
2. $N_i \triangleq N_j' \in A\psi [\Phi'|\Psi',y|\xi_i,\xi_j]$ for all $i,j$,
3. $N_i(r/y) \triangleq M \in A\psi(r/y) [\Phi'|\Psi'|\xi_i]$ for all $i$,

then

1. $\text{com}_{y,A\psi}^{r,s}(M;\xi_i \mapsto y.N_i) = \text{com}_{y,A\psi}^{r,s}(M';\xi_i \mapsto y.N_i') \in A\psi(s/y) [\Phi'|\Psi']$,
2. $\text{com}_{y,A\psi}^{r,s}(M;\xi_i \mapsto y.N_i) = N_i(s/y) \in A\psi(s/y) [\Phi'|\Psi']$ for all $i$ with $\models \xi_i$,
3. $\text{com}_{y,A\psi}^{r,s}(M;\xi_i \mapsto y.N_i) \triangleq M \in A\psi(s/y) [\Phi'|\Psi']$ if $r = s$.

**Proof.** See [Angiuli et al., 2017b, Theorem 44].

### 3.4 Open judgments

Finally, we extend the closed judgments $A \triangleq A'$ type$_{\text{coe}} [\Phi,\Psi]$ and $M \triangleq M' \in A[\Phi,\Psi]$ to open judgments in a term context $\Gamma$. In order to properly reason with the substructural context $\Phi$, we also record the introduction of bridge dimensions in term contexts à la Cheney [2009, 2012]. A context of the form $(\Gamma,x,\Gamma')$ indicates that the dimension $x$ was introduced after the variables in $\Gamma$ but before those in $\Gamma'$; thus we may substitute terms mentioning $x$ for variables in $\Gamma'$, but not for variables in $\Gamma$.

### Definition 3.19

We define the judgments $\Gamma \text{ctx} [\Phi,\Psi], M \triangleq M' \in \Gamma[\Phi,\Psi], \Gamma \gg B \triangleq B'$ type$_{\kappa} [\Phi,\Psi]$, and $\Gamma \gg N \triangleq N' \in B[\Phi,\Psi]$ by mutual induction as follows.
A. The context judgment $\Gamma \text{ctx} [\Phi | \Psi]$ is defined inductively by the following rules:

| Rule | Description |
|------|-------------|
| $\emptyset \text{ctx} [\Phi | \Psi]$ | $\Gamma \text{ctx} [\Phi | \Psi]$ |
| $\Gamma \text{ctx} [\Phi | \Psi]$ | $\Gamma \text{ctx} [\Phi | \Psi]$ |
| $\Gamma \gg A \text{type}_{\text{pre}} [\Phi | \Psi]$ | $\Gamma, a : A \text{ctx} [\Phi | \Psi]$ |
| $r \text{ bdim} [\Phi | \Psi]$ | $\Gamma \text{ctx} [\Phi | \Psi]$ |

B. The context element equality judgment $\overline{M} = \overline{M}' \in \Gamma [\Phi | \Psi]$, which presupposes $\Gamma \text{ctx} [\Phi | \Psi]$, is defined inductively by the following rules:

| Rule | Description |
|------|-------------|
| $\emptyset \equiv \emptyset \in [\Phi | \Psi]$ | $\overline{M} = \overline{M}' \in \Gamma [\Phi | \Psi]$ |
| $N = N' \in A[\overline{M} / \gamma] [\Phi | \Psi]$ | $(\overline{M}, N) = (\overline{M}', N') \in (\Gamma, a : A) [\Phi | \Psi]$ |
| $r \text{ bdim} [\Phi | \Psi]$ | $\overline{M} = \overline{M}' \in (\Gamma, r) [\Phi | \Psi]$ |

C. The open type equality judgment $\Gamma \gg B = B' \text{type}_{\text{pre}} [\Phi | \Psi]$, which presupposes $\Gamma \text{ctx} [\Phi | \Psi]$, is defined to hold when for every $\psi : (\Phi' | \Psi') \to (\Phi | \Psi)$ and $\overline{M} = \overline{M}' \in \Gamma \psi [\Phi' | \Psi']$ we have $B \psi [\overline{M} / \gamma] = B' \psi [\overline{M}' / \gamma] \text{type}_{\text{pre}} [\Phi' | \Psi']$ (where $\gamma$ is the list of term variables hypothesized in $\Gamma$).

D. The open element equality judgment $\Gamma \gg N = N' \in B [\Phi | \Psi]$, which presupposes $\Gamma \text{ctx} [\Phi | \Psi]$ and $\Gamma \gg B \text{type}_{\text{pre}} [\Phi | \Psi]$, is defined to hold when for every $\psi : (\Phi' | \Psi') \to (\Phi | \Psi)$ and $\overline{M} = \overline{M}' \in \Gamma \psi [\Phi' | \Psi']$ we have $N \psi [\overline{M} / \gamma] = N' \psi [\overline{M}' / \gamma] \in B \psi [\overline{M} / \gamma] [\Phi' | \Psi']$ (where $\gamma$ is the list of term variables hypothesized in $\Gamma$).

**Definition 3.20.** For each of the judgments $\Gamma \gg J [\Phi | \Psi]$ defined above, we define its restricted form $\Gamma \gg J [\Phi | \Psi | \Xi]$ to hold when $\Gamma \psi \gg J \psi [\Phi' | \Psi']$ holds for every $\psi : (\Phi' | \Psi') \to (\Phi | \Psi)$ such that $\Xi \psi = \Xi \psi'$.

**Definition 3.21.** Given $\Gamma \text{ctx} [\Phi | \Psi]$ and $r \in \Phi \cup \{0, 1\}$, we define the context $\Gamma \backslash r \text{ctx} [\Phi \backslash r | \Psi]$ of term variables which cannot refer to $r$ (if it is a variable) by

$$\Gamma \backslash r := \begin{cases} \Gamma_1, & \text{if } r \in \Phi \text{ and } \Gamma = (\Gamma_1, r, \Gamma_2) \text{ for some } \Gamma_1, \Gamma_2 \\ \Gamma, & \text{otherwise} \end{cases}$$

In the following sections, we will describe various type constructors as operators on $\mathcal{D}$-relations and show that, when these are included in a type system, they satisfy appropriate introduction and elimination rules. These rules all take the following form, where each $\mathcal{J}$ may be a dimension, term, or type equality judgment.

$$\Gamma_{\rho_1} \Phi_1 \Gamma_1 \gg \mathcal{J}_1 \left[ [\Phi_{\rho_1} \Phi_1 | \Psi_{\rho_1} | \Xi_{\rho_1} \Xi_1] \right] \cdots \Gamma_{\rho_n} \Phi_n \Gamma_n \gg \mathcal{J}_n \left[ [\Phi_{\rho_n} \Phi_n | \Psi_{\rho_n} | \Xi_{\rho_n} \Xi_n] \right]$$

$$\Gamma \gg \mathcal{J} [\Phi | \Psi | \Xi]$$

To prove such a rule, we will first prove its restriction to closed instances holds, in the sense that the following rule is validated.

$$\Gamma_1 \gg \mathcal{J}_1 \left[ [\Phi_{\rho_1} \Phi_1 | \Psi_{\rho_1} | \Xi_1] \right] \cdots \Gamma_n \gg \mathcal{J}_n \left[ [\Phi_{\rho_n} \Phi_n | \Psi_{\rho_n} | \Xi_n] \right]$$

$$\mathcal{J} [\Phi | \Psi | \Xi]$$

We then observe that the validity of the latter implies the validity of the former. This follows by definition of the interpretation of open judgments; we apply the closed rule “pointwise” to validate the open rule. For suppose that we know the premises of the open rule, and want to show $\Gamma \gg \mathcal{J} [\Phi | \Psi | \Xi]$. This means we must show $\mathcal{J}_{\rho} [\overline{M}, \overline{M}'] [\Phi' | \Psi']$ for every $\psi : (\Phi' | \Psi') \to (\Phi | \Psi)$ such that $\Xi \psi = \Xi \psi'$ (introducing some impromptu notation for instantiating a binary open judgment). For each $i$, we have $\Gamma_{\rho_i} \Phi_i \Gamma_i \gg \mathcal{J}_i [\Phi_{\rho_i} \Phi_i | \Psi_{\rho_i} | \Xi_{\rho_i} \Xi_i]$. We can instantiate these hypothesis judgments with $\psi$ and the prefixes $\overline{M}_i = \overline{M}_i' \in \Gamma_{\rho_i} \psi [\Phi_{\rho_i} \psi | \Psi']$ of $\overline{M}$ and $\overline{M}'$ corresponding to the prefix $\Gamma_{\rho_i} \psi$ of $\Gamma_{\psi}$, which gives us $\Gamma_{\psi} [\overline{M}_i] \gg (\mathcal{J}_i)_{\rho_i} [\overline{M}_i, \overline{M}_i'] [\Phi_{\rho_i} \psi | \Psi_i | \Xi_i]$. Once we have done this for each premise, we are in a position to apply the closed rule, and so $\mathcal{J}_{\psi} [\overline{M}, \overline{M}'] [\Phi' | \Psi']$ follows.
4 Imports from cubical type theory

Our parametric type theory is built on the substrate of cubical type theory. The addition of bridge dimension variables does not disrupt the existing constructs, so we are able to import them wholesale. In this section, we briefly recall those results from cubical type theory (and homotopy type theory) which our development requires. Details on the cubical constructions can be found in Angiuli et al. [2017b]. For homotopy type theory, the standard reference is the “HoTT Book” [Univalent Foundations Program, 2013], which we will henceforth cite as [HoTT]. Cubical translations of many of the results we use from homotopy type theory can be found in the standard library of the redtt cubical proof assistant [The RedPRL Development Team, 2018].

**Recollection 4.1.** Cubical type theory supports dependent pair, dependent function, and universe types as in standard dependent type theory. Given a type $\type_{\text{Kan}}[\Phi \mid x, \Psi]$, and if $\lambda : A \rightarrow B$ is a type $\type_{\text{Kan}}[\Phi \mid \Psi]$ and $\lambda : A \rightarrow B$ is a type $\type_{\text{Kan}}[\Phi \mid \Psi]$ for their pair and function types respectively. For simplicity’s sake, we will only make use of one universe, which we write as $\mathcal{U}$ type $\type_{\text{Kan}}[\Phi \mid \Psi]$; if $A \in \mathcal{U}[\Phi \mid \Psi]$ then $A$ type $\type_{\text{Kan}}[\Phi \mid \Psi]$.

**Recollection 4.2.** Given a “line of types” $A$ type $\type_{\text{Kan}}[\Phi \mid x, \Psi]$ and endpoint elements $M_0 \in A(0/x) [\Phi \mid \Psi]$ and $M_1 \in A(1/x) [\Phi \mid \Psi]$, their path type $\Path_{x,A}(M_0, M_1)$ classifies values of the form $\lambda x.P$ where $P \in A[\Phi \mid x]$ satisfies $P(0/x) \equiv M_0 \in A(0/x) [\Phi \mid \Psi]$ and $P(1/x) \equiv M_1 \in A(1/x) [\Phi \mid \Psi]$. This type satisfies rules shown in Figure 3. When $A$ does not depend on $x$, we abbreviate $\Path_{x,A}(M_0, M_1)$ as $\Path_{A}(M_0, M_1)$.

**Recollection 4.3.** A type $A$ type $\type_{\text{Kan}}[\Phi \mid \Psi]$ is contractible when it contains an element to which all other elements are connected by a path.

\[ \text{isContr}(A) := (a : A) \times (a' : A) \rightarrow \Path_{A}(a', a) \]

Given $F : A \rightarrow B[\Phi \mid \Psi]$ and $N \in B[\Phi \mid \Psi]$, the (homotopy) fiber of $F$ at $N$ is the type of elements $a : A$ with a path from $Fa$ to $N$.

\[ \text{Fiber}(A, B, F; N) := (a : A) \times \Path_{B}(F(a), N) \]

A map $F : A \rightarrow B[\Phi \mid \Psi]$ is an equivalence if its fibers are contractible: if each element of $B$ is the image of an element of $A$ under $F$ in a unique way.

\[ \text{isEquiv}(A, B, F) := (b : B) \rightarrow \text{isContr}(\text{Fiber}(A, B, F; b)) \]

---

\[ ^{1} \text{There is one new condition that must be checked: that the existing types are closed under homogeneous compositions whose tubes contain equations on bridge dimensions. However, this requires only cosmetic changes to the existing proofs.} \]
Finally, we set \( \text{Equiv}(A, B) := (f : A \to B) \times \text{isEquiv}(A, B, f) \). We will abbreviate \( \text{Equiv}(A, B) \) as \( A \simeq B \).

**Recollection 4.4** (Singleton contractibility). For every \( A \text{ type}_{\text{Kan}} [\Phi | \Psi] \) and \( M \in A [\Phi | \Psi] \), the type \( (a : A) \times \text{Path}_A(a, M) \) is contractible.

**Recollection 4.5.** We say that \( A \text{ type}_{\text{Kan}} [\Phi | \Psi] \) is a proposition when \( \text{isProp}(A) := (a, a' : A) \to \text{Path}_A(a, a') \in B \) is inhabited.

**Recollection 4.6.** For any \( A, B \text{ type}_{\text{Kan}} [\Phi | \Psi] \) and \( f \in A \to B [\Phi | \Psi] \), the type \( \text{isEquiv}(A, B, F) \) is a proposition.

*Proof.* \([\text{HoTT}, \text{Lemma 4.4.4}]\). □

**Recollection 4.7.** Given \( A, B \text{ type}_{\text{Kan}} [\Phi | \Psi] \), define the type \( \text{QEq}(A, B) \) of quasi-equivalences from \( A \) to \( B \) as follows.

\[
\text{QEq}(A, B) := (f : A \to B) \times (g : B \to A) \times ((b : B) \mapsto \text{Path}_B(f(g(b)), b)) \times ((a : A) \mapsto \text{Path}_A(g(f(a)), a))
\]

For any \( A, B \text{ type}_{\text{Kan}} [\Phi | \Psi] \), there are functions \( \text{QEq}(A, B) \to \text{Eq}(A, B) \) and \( \text{Eq}(A, B) \to \text{QEq}(A, B) \) which preserve the underlying forward map \( A \to B \).

*Proof.* \([\text{HoTT, Theorems 4.4.5 and 4.2.3}]\). □

**Notation 4.1.** Given an equivalence \( E \), we write \( \text{fwd}(E) \), \( \text{bwd}(E) \), \( \text{fwd-bwd}(E) \), and \( \text{bwd-fwd}(E) \) for the four components of the quasi-equivalence it induces. We write \( \text{id} \) for the identity equivalence on \( A \).

**Recollection 4.8.** Suppose we have a path dimension \( r \) \( \text{pdim} [\Phi | \Psi] \), a type \( A \text{ type}_{\text{Kan}} [\Phi | \Psi | \Xi, r = 0] \) at its left endpoint, a type \( B \text{ type}_{\text{Kan}} [\Phi | \Psi] \), and an equivalence \( E \in A \simeq B [\Phi | \Psi | r = 0] \); pictorially, a V-shape:

\[
\begin{array}{c}
A \\
\downarrow \\
B_0 \\
r \rightarrow \\
B_1 \\
\end{array}
\]

Their V-type \( V_r(A, B, E) \) is a type which, viewed as a path in direction \( r \), connects \( A \) to \( B_1 \).

\[
\begin{array}{c}
V_0(A, B, E) := A \text{ type}_{\text{Kan}} [\Phi | \Psi] \\
V_1(A, B, E) := B \text{ type}_{\text{Kan}} [\Phi | \Psi]
\end{array}
\]

This type is used to validate the *univalence axiom*, which asserts that paths in the universe \( \mathcal{U} \) correspond to equivalences: for each pair of types \( A, B \in \mathcal{U} [\Phi | \Psi] \), there is an equivalence \( \text{Equiv}(A, B) \simeq \text{Path}_{\mathcal{U}}(A, B) \).

The forward map of this equivalence is given by the function \( \lambda e. \lambda x. \mathcal{V}_r(A, B, e) \). The reverse map takes \( P \in \text{Path}_{\mathcal{U}}(A, B) [\Phi | \Psi] \) to the equivalence given by coercion, i.e., that with forward map \( \lambda a. \text{coer}^{P_0 \rho} \rho_{0, r}(a) \). We will see that univalence has an analogue on the parametric, side which identifies bridges in the universe with binary relations.

Separately, fcom-types are used to implement composition in the universe. In parametric cubical type theory, these must be modified to accommodate compositions with constraints of the form \( r = \epsilon \). However, this does not require any significant modification to the definition of fcom-types or the implementation of their Kan operations, so the details are omitted.
5 Bridge-types

We now introduce Bridge-types, the analogue of Path-types for bridge dimension variables. Their operational semantics is shown in Figure 4. Below, we define their PER semantics and prove that this definition satisfies the expected introduction, elimination, and equality rules. We collect the rules in inference rule format as part of a proof theory in Section 8.3. While we include full proofs of the various rules for the sake of completeness, these are essentially the same proofs as those given for Path-types by Angiuli et al. [2017b, §5.3]. The only differences come from substructurality: a term \( Q \) of type \( \text{Bridge}_{x.A}(M_0, M_1) \) may only be applied to a dimension variable which is fresh for \( Q \).

### 5.1 Definition

**Definition 5.1.** Fix a candidate type system \( \tau \), and let \( \tau \models A \text{type}_{\text{Kan}} [\Phi, x | \Psi] \), \( \tau \models M_0 \in A(0/x) [\Phi | \Psi] \), and \( \tau \models M_1 \in A(1/x) [\Phi | \Psi] \) be given. We define a value \( (\Phi | \Psi) \)-PER \( \text{Bridge}^\tau_{x.A}(M_0, M_1) \): for each \( \psi : (\Phi' | \Psi') \rightarrow (\Phi | \Psi) \), \( \text{Bridge}^\tau_{x.A}(M_0, M_1)_{\psi}(V, V') \) is defined to hold iff \( V = \lambda^2 x.P \) and \( V' = \lambda^2 x.P' \) where

1. \( \tau \models P \triangleq P' \in A\psi [\Phi', x | \Psi'] \),
2. \( \tau \models P(0/x) \triangleq M_0\psi \in A\psi [\Phi' | \Psi'] \), and
3. \( \tau \models P(1/x) \triangleq M_1\psi \in A\psi [\Phi' | \Psi'] \).

We will drop the superscript \( \tau \) when it is inferable.

**Lemma 5.2.** \( \text{Bridge}^\tau_{x.A}(M_0, M_1) \) is value-coherent.

**Proof.** Let \( \text{Bridge}^\tau_{x.A}(M_0, M_1)_{\psi}(V, V') \) be given. By definition of Bridge and stability of the typing judgments under dimension substitution, this implies \( \text{Bridge}^\tau_{x.A}(M_0, M_1)_{\psi}\psi(V\psi, V'\psi) \) for every \( \psi' \). Thus \( \text{TM}(\text{Bridge}^\tau_{x.A}(M_0, M_1))_{\psi}(V, V') \) by Lemma A.1. \( \square \)

**Proposition 5.3.** There exists a type system \( \tau \) which, for every \( \tau \models A \triangleq A' \text{type}_{\text{Kan}} [\Phi, x | \Psi] \), \( \tau \models M_0 \triangleq M'_0 \in A(0/x) [\Phi | \Psi] \), and \( \tau \models M_1 \triangleq M'_1 \in A(1/x) [\Phi | \Psi] \), has

\[
\tau((\Phi | \Psi), \text{Bridge}^\tau_{x.A}(M_0, M_1), \text{Bridge}^\tau_{x.A}(M'_0, M'_1), \text{Bridge}^\tau_{x.A}(M_0, M_1)_{ia}),
\]

in addition to supporting the standard type formers of cubical type theory. Moreover, there exists such a type system in which the universe \( \mathcal{U} \) is also closed under Bridge-types.

**Proof.** See Appendix B. \( \square \)

For the remainder of this section, we assume we are working within such a type system.
5.2 Rules

Rule 5.4 (Bridge-F). Let $A \vdash A' : \text{type}_{\text{Kan}} [\Phi, x | \Psi]$, $M_0 \vdash M'_0 \in A(0/x) [\Phi | \Psi]$, and $M_1 \vdash M'_1 \in A(1/x) [\Phi | \Psi]$ be given. Then $\text{Bridge}_{x,A}(M_0, M_1) \vdash \text{Bridge}_{x,A}(M'_0, M'_1)$ type$_{\text{pre}} [\Phi | \Psi]$.

Proof. By Lemmas A.4 and 5.2.

Rule 5.5 (Bridge-I). Let $P \vdash P' \in A [\Phi, x | \Psi]$. Then $\lambda^{x} x \cdot P \vdash \lambda^{x} x \cdot P' \in \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$.

Proof. This is exactly Lemma 5.2.

Rule 5.6 (Bridge-β). Let $A \vdash A' : \text{type}_{\text{Kan}} [\Phi \cdot r, x | \Psi]$ and $P \vdash P' \in A [\Phi \cdot r, x | \Psi]$. Then $(\lambda^{x} x \cdot P)@r \vdash P'(r/x) \in A(r/x) [\Phi | \Psi]$.

Proof. By Lemma A.2, as $((\lambda^{x} x \cdot P)@r)\psi \mapsto P'(r/x)\psi$ and $P'(r/x)\psi \in A(r/x) [\Phi | \Psi]$ for all $\psi$.

Rule 5.7 (Bridge-E). Let $r \vdash \text{bdim} [\Phi | \Psi]$, $A \vdash [\Phi \cdot r, x | \Psi]$, and $Q \vdash Q' \in \text{Bridge}_{x,A}(M_0, M_1) [\Phi \cdot r | \Psi]$. Then $Q@r \vdash Q '@r \in A(r/x) [\Phi | \Psi]$.

Proof. By Lemma A.6 applied with the expression contexts $[-]@r, [-]@r, A(r/x) : (\Phi | \Psi) \Rightarrow \emptyset$ and the $(\Phi \cdot r | \Psi)\cdot\text{PER}$ $\text{Bridge}_{x,A}(M_0, M_1)$. We need to show that for every $\psi : (\Phi' | \Psi') : \to (\Phi | \Psi)$ and $\text{Bridge}_{x,A}(M_0, M_1) \psi (V, V)$, we have $V@r \psi \mapsto V'@r \psi \in A(r/x) \psi [\Phi' | \Psi']$. By definition of Bridge, we have $V = \lambda^{x} x \cdot P$ and $V' = \lambda^{x} x \cdot P'$ with $P \vdash P' \in A\psi [\Phi \cdot r \psi, x | \Psi]$. The latter implies $P(r/x)\psi \vdash P'(r/x)\psi \in A(r/x) \psi [\Phi' | \Psi']$ by stability of the typing judgments. By Rule 5.6, we also have $V@r \psi \mapsto P(r/x)\psi \in A(r/x) \psi [\Phi' | \Psi']$. The desired equation follows by transitivity.

Rule 5.8 (Bridge-βε). If $\varepsilon \in \{0, 1\}$ and $Q \in \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$, then $Q@\varepsilon \vdash M_{\varepsilon} \in A(\varepsilon/x) [\Phi | \Psi]$.

Proof. By Lemma A.3, we have $Q \vdash V$ with $V \equiv \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$. By Rule 5.7 it follows that $Q@\varepsilon \equiv V@\varepsilon \in A(\varepsilon/x) [\Phi | \Psi]$. We have $V = \lambda^{x} x \cdot P$ for some $P \in A [\Phi, x | \Psi]$ with $P(\varepsilon/x) \equiv M_{\varepsilon} \in A(\varepsilon/x) [\Phi | \Psi]$, so $V@\varepsilon \equiv P(\varepsilon/x) \in A(\varepsilon/x) [\Phi | \Psi]$ by Rule 5.6. Thus $Q@\varepsilon \equiv V@\varepsilon \equiv P(\varepsilon/x) \equiv M_{\varepsilon}$ in $A(\varepsilon/x)$.

Rule 5.9 (Bridge-η). If $Q \equiv \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$, then $Q \equiv \lambda^{x} y_0 Q@y \in \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$.

Proof. By Lemmas 5.2 and A.3, we have $Q \vdash V$ with $V \equiv \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$. We have $V = \lambda^{x} x \cdot P$ for some $P \in A [\Phi, x | \Psi]$, so $\lambda^{x} y_0 V@y = \lambda^{x} y_0 P(y/x) \in \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$ by Rules 5.6 and 5.5. The right-hand side of this equation is $\alpha$-equal to $V$, so $\lambda^{x} y_0 V@y \equiv V \equiv \text{Bridge}_{x,A}(M_0, M_1) [\Phi | \Psi]$ by transitivity. We now obtain the result by replacing $V$ with $Q$ everywhere, using Rules 5.7, 5.6 and 5.5 to do so on the left-hand side.

5.3 Kan conditions

For this section, fix $A \vdash A' : \text{type}_{\text{Kan}} [\Phi, x | \Psi]$, $M_0 \vdash M'_0 \in A(0/x) [\Phi | \Psi]$, and $M_1 \vdash M'_1 \in A(1/x) [\Phi | \Psi]$.

Theorem 5.10. $\text{Bridge}_{x,A}(M_0, M_1) \vdash \text{Bridge}_{x,A}(M'_0, M'_1)$ type$_{\text{pre}} [\Phi | \Psi]$ are equally coe-Kan.

Proof. Let $\psi : (\Phi' | \Psi') : \to (\Phi | \Psi)$, $r, s$ pdim $[\Phi' | \Psi']$, and $Q \equiv Q' \in \text{Bridge}_{x,A}(M_0, M_1) \psi (r/y) [\Phi' | \Psi']$ be given. Abbreviating $B := \text{Bridge}_{x,A}(M_0, M_1)$ and $B' := \text{Bridge}_{x,A}(M'_0, M'_1)$, we need to show that

1. $\text{coe}^{r-s}_{y, B\psi}(Q) \equiv \text{coe}^{r-s}_{y, B'\psi}(Q') \in B\psi(s/y) [\Phi' | \Psi']$.
2. $\text{coe}^{r-s}_{y, B\psi}(Q) \equiv Q \in B\psi(s/y) [\Phi' | \Psi']$ if $r = s$.

We prove these in turn.
We prove these in turn.

Abbreviating Theorem 5.11.

Bridge Proof. 

1. We apply Lemma A.2 on either side of the equation to reduce our goal to proving

$$\lambda^2 x. \text{com}_{-\text{hcom}}(Q @ x; x = 0 \leftrightarrow y. M_0 \psi, x = 1 \leftrightarrow y. M_1 \psi)$$

$$\vdash$$

$$\lambda^2 x. \text{com}_{-\text{hcom}}(Q' @ x; x = 0 \leftrightarrow y. M_0' \psi, x = 1 \leftrightarrow y. M_1' \psi)$$

in $B \psi(s/y)$. We have $Q @ x \doteq Q' @ x \in A \psi(r/y) [\Phi, x \mid \Psi']$ by Rule 5.7 and $Q @ x \doteq M \psi(r/y) \in A \psi(r/y) [\Phi, x \mid \Psi' \mid x = \varepsilon]$ for each $\varepsilon$ by Rule 5.8. The desired equality thus follows from Proposition 3.18 and Rule 5.5.

2. Suppose $|= r = s$. Again, it suffices by Lemma A.2 to show

$$\lambda^2 x. \text{com}_{-\text{hcom}}(Q @ x; x = 0 \leftrightarrow y. M_0 \psi, x = 1 \leftrightarrow y. M_1 \psi) \doteq Q \in B \psi(s/y) [\Phi' | \Psi'].$$

This follows from Proposition 3.18 and Rule 5.9.

\[ \square \]

Theorem 5.11. Bridge$_{x, A}(M_0, M_1) \doteq \text{Bridge}_{x, A}(M_0', M_1')$ type$_{\text{pre}} [\Phi \mid \Psi]$ are equally hcom-Kan.

**Proof.** Let $\psi : (\Phi' \mid \Psi') \to (\Phi \mid \Psi), r, s \, \text{pdim} [\Phi' \mid \Psi'], \xi_i \, \text{eqs} [\Phi' \mid \Psi']$ be given, and suppose we have

1. $M \doteq M' \in \text{Bridge}_{x, A}(M_0, M_1) \psi [\Phi' \mid \Psi'].$
2. $N_i \doteq N_j' \in \text{Bridge}_{x, A}(M_0, M_1) \psi [\Phi' \mid \Psi', y \mid \xi_i, \xi_j]$ for all $i, j$.
3. $N_i(r/y) \doteq M \in \text{Bridge}_{x, A}(M_0, M_1) \psi [\Phi' \mid \Psi' \mid \xi_i]$ for all $i$.

Abbreviating $B := \text{Bridge}_{x, A}(M_0, M_1)$ and $B' := \text{Bridge}_{x, A}(M_0', M_1')$, we need to show

1. $\text{hcom}_{-\text{hcom}}^\approx_B(M; \xi_i \leftrightarrow y. N_i) \doteq \text{hcom}_{-\text{hcom}}^\approx_B(M'; \xi_i \leftrightarrow y. N_i') \in B \psi [\Phi' \mid \Psi'].$
2. $\text{hcom}_{-\text{hcom}}^\approx_B(M; \xi_i \leftrightarrow y. N_i) \doteq N_i(s/y) \in B \psi [\Phi' \mid \Psi']$ for all $i$ with $|= \xi_i$.
3. $\text{hcom}_{-\text{hcom}}^\approx_B(M; \xi_i \leftrightarrow y. N_i) \doteq M \in B \psi [\Phi' \mid \Psi']$ if $r = s$.

We prove these in turn.

1. We apply Lemma A.2 on either side of the equation to reduce our goal to proving

$$\lambda^2 x. \text{hcom}_{-\text{hcom}}^\approx_B(M @ x; \xi_i \leftrightarrow y. N_i @ x, x = 0 \leftrightarrow \_M_0 \psi, x = 1 \leftrightarrow \_M_1 \psi)$$

$$\vdash$$

$$\lambda^2 x. \text{hcom}_{-\text{hcom}}^\approx_B(M' @ x; \xi_i \leftrightarrow y. N_i' @ x, x = 0 \leftrightarrow \_M_0' \psi, x = 1 \leftrightarrow \_M_1' \psi)$$

in $B \psi$. By Rule 5.7, we have

(a) $M @ x \doteq M' @ x \in A \psi [\Phi', x \mid \Psi'],$
(b) $N_i @ x \doteq N_j' @ x \in A \psi [\Phi', x \mid \Psi', y \mid \xi_i, \xi_j]$ for all $i, j$,
(c) $(N_i @ x)(r/y) \doteq M @ x \in A \psi [\Phi', x \mid \Psi' \mid \xi_i]$ for all $i$.

By Rule 5.8, we also have

(a) $N_i @ x \doteq M \psi \in A \psi [\Phi', x \mid \Psi', y \mid \xi_i, x = \varepsilon]$ for all $i$ and $\varepsilon \in 0, 1$,
(b) $M \psi \doteq M @ x \in A \psi [\Phi', x \mid \Psi' \mid x = \varepsilon]$ for $\varepsilon \in 0, 1$.
From the equations, it follows by the $\text{hcom-Kan}$ condition on $A \rightarrow A'$ type$_{\text{Kan}} \Phi \mid \Psi$ that
\[
\text{hcom}_{A \rightarrow}^{\aleph^x}(M \circ x; \xi_i \mapsto y.N_i \circ x, x = 0 \mapsto \varphi_{M_0}, x = 1 \mapsto \varphi_{M_1}) = \\
\text{hcom}_{A \rightarrow}^{\aleph^x}(M' \circ x; \xi_i \mapsto y.N_i' \circ x, x = 0 \mapsto \varphi_{M_0'}, x = 1 \mapsto \varphi_{M_1'})
\]
in $A\psi$ at $(\Phi', x \mid \Psi')$. It also follows that, for each $\varepsilon \in \{0, 1\}$, we have
\[
\text{hcom}_{A \rightarrow}^{\aleph^x}(M \circ x; \xi_i \mapsto y.N_i \circ x, x = 0 \mapsto \varphi_{M_0}, x = 1 \mapsto \varphi_{M_1}) \cdot (\varepsilon/x) \cong M_{\varepsilon} \in A\psi(\varepsilon/x) \Phi \mid \Psi).
\]
Thus we may apply Rule 5.5 to obtain the desired equation.

2. Suppose $\models \xi_i$. By again applying Lemma A.2, it suffices to show
\[
\lambda^x.\text{hcom}_{A \rightarrow}^{\aleph^x}(M \circ x; \xi_i \mapsto y.N_i \circ x, x = 0 \mapsto \varphi_{M_0}, x = 1 \mapsto \varphi_{M_1}) \cong N_i(s/y) \in B\psi(\varepsilon/y) \Phi \mid \Psi'
\]
This follows from the $\text{hcom-Kan}$ condition for $A \rightarrow A'$ type$_{\text{Kan}} \Phi \mid \Psi$ and Rules 5.5 and 5.9.

3. Analogous to 2. \hfill \Box

6 Bridges in compound types

We intend to think of a type $A$ type$_{\text{Kan}} \Phi \mid \Psi$ varying in a dimension variable $x$ as a type-valued binary relation on its endpoints $A(0/x)$ and $A(1/x)$. This point of view will be validated when we prove relativity in Section 9, but we can already give one direction of the correspondence: the relation corresponding to $A$ is the family of Bridge-types $\text{Bridge}_{x,A}(\cdot, \cdot)$. We therefore expect that for compound types, such as pair, path, and function types, we can show that their bridge types align with their standard "logical" relational interpretations. For example, a bridge in a pair type should correspond uniquely to a pair of bridges in its component types. For pairs and paths, this is indeed straightforward. For function types, on the other hand, we will need to introduce a new operator we call extent, previously introduced by Bernardy et al. under the name $\langle -i- \rangle$. This is the first place where the role of substructurality becomes evident; the second will be in Section 7.

**Theorem 6.1.** Let $A$ type$_{\text{Kan}} \Phi \mid \Psi$, $a : A \rightarrow B$ type$_{\text{Kan}} \Phi \mid \Psi$, $M_0 \in ((a:A) \times B)(0/x) \Phi \mid \Psi$, and $M_1 \in ((a:A) \times B)(1/x) \Phi \mid \Psi$. Then we have the following equivalence.

\[
\text{Bridge}_{x,(a:A) \times B}(M_0, M_1) \cong (p: \text{Bridge}_{x,A}(\text{fst}(M_0), \text{fst}(M_1))) \times \text{Bridge}_{x,B}(p @ x/y) \text{snd}(M_0), \text{snd}(M_1))
\]

*Proof.* For the forward map, we send $q$ to $\langle \lambda^x.x.\text{fst}(q @ x), \lambda^x.x.\text{snd}(q @ x) \rangle$. For the inverse, we send $t$ to $\langle \lambda^x.x.\text{fst}(q @ x), \lambda^x.x.\text{snd}(q @ x) \rangle$. It is simple to establish via Recollection 4.7 that these maps give rise to an equivalence. \hfill \Box

**Theorem 6.2.** Let $a$ type $A$ type$_{\text{Kan}} \Phi \mid \Psi$, $M_0 \in A(0/y) \Phi \mid \Psi$, $M_1 \in A(1/y) \Phi \mid \Psi$, $P_0 \in \text{Path}_{y,A}(M_0, M_1)(0/x) \Phi \mid \Psi$, and $P_1 \in \text{Path}_{y,A}(M_0, M_1)(1/x) \Phi \mid \Psi$ be given. Then we have the following equivalence.

\[
\text{Bridge}_{x, \text{Path}_{y,A}(M_0, M_1)}(P_0, P_1) \cong \text{Path}_{y, \text{Bridge}_{x,A}(P_0 @ y, P_1 @ y)}(\lambda^x.x. M_0, \lambda^x.x. M_1)
\]

*Proof.* For the forward map, we send $p$ to $\lambda^y.\lambda^x.x.p @ x @ y$. For the inverse, we send $q$ to $\lambda^x.x.\lambda^y.q @ y @ x$. It is again simple to see with Recollection 4.7 that these give rise to an equivalence. \hfill \Box
We now come to function types. Our expectation is that a term of type $\text{Bridge}_{x,A} \rightarrow_{B} (F,F')$ corresponds to a term of type $(a:A(0/x))(a':A(1/x))(c:\text{Bridge}_{x,A}(a,a')) \rightarrow \text{Bridge}_{x,B}(c@x/a)(Fa,F'a')$, a function taking each bridge over $A$ to a bridge over $B$ between the images of its endpoints under $F$ and $F'$. Indeed, it is easy to give the forward direction of this anticipated equivalence: we send $q$ in the former type to $\lambda a.\lambda c.\lambda^2 x.M.(q@x)(c@x)$ in the latter.

It is in the reverse direction that we run into trouble. Suppose we have $q$ in the latter type. Given any $x$ and $a : A$, we need to be able to construct an element of $B$, but $g$ expects a bridge over $A$, not an element of $A$ varying in $x$. Intuitively, we would like to write “$g(a(0/x))(a(1/x))(\lambda^2 x.a)$,” capturing the occurrences of $x$ in $a$. The ability to internally abstract a term over a variable in this way is a characteristic feature of nominal sets [Pitts, 2013]. These are equivalent to presheaves on the category of names and injective substitutions, the subcategory of our category of bridge contexts excluding morphisms which send variables to 0 or 1. Injectivity, which amounts to the exclusion of diagonal substitutions $(y/x)$, is essential, as the map $(x,M) \mapsto \lambda^2 x.M$ does not commute with such substitutions. For if $M$ mentions some $y$, abstracting $y$ after applying $(y/x)$ will cause the occurrences of $y$ in $M$ to be captured; if we abstract $x$ before applying $(y/x)$, then these occurrences will not be captured.

We also need to consider the case of substitutions $0/x$ and $1/x$, so in the end we will provide a kind of case operator for dimension terms. This operator takes the form $\text{extent}_r(M;a.N,a'.P,a.a'.c.Q)$, so named because it reveals the full “extent” of the term $M$ in the $r$ direction. If $r$ is 0, then $M$ is supplied to $N$; if $r$ is 1, then $M$ is supplied to $P$. If $r$ is some $x$, then $M(0/x), M(1/x)$, and the abstracted $\lambda^2 x.M$ are supplied to $Q$, which should be a bridge between $N[M(0/x)/a]$ and $P[M(1/x)/a']$. The operational semantics of extent, which do exactly this, are shown in Figure 5. Below, we prove well-typedness and computation rules for extent, which are collected as part of the proof theory in Section 8.4.

**Rule 6.3 (ex-β₀).** If $A \text{ type}_K \Phi | \Psi$, $d : A \Rightarrow B \text{ type}_K \Phi | \Psi$, and $a : A \Rightarrow N \in B[a/d] | \Phi | \Psi$, then $\text{extent}_0(M;a.N,a'.P,a.a'.c.Q) \equiv N[M/a] \in B[M/d] | \Phi | \Psi$.

*Proof.* By Lemma A.2, as $\text{extent}_0(M;a.N,a'.P,a.a'.c.Q) \psi \mapsto N[M/a] \psi$ for all $\psi$. □

**Rule 6.4 (ex-β₁).** If $A \text{ type}_K \Phi | \Psi$, $d : A \Rightarrow B \text{ type}_K \Phi | \Psi$, and $a' : A \Rightarrow P \in B[a'/d] | \Phi | \Psi$, then $\text{extent}_1(M;a.N,a'.P,a.a'.c.Q) \equiv P[M/a'] \in B[M/d] | \Phi | \Psi$.

*Proof.* By Lemma A.2, as $\text{extent}_1(M;a.N,a'.P,a.a'.c.Q) \psi \mapsto P[M/a'] \psi$ for all $\psi$. □

**Rule 6.5 (ex-β).** If $r \text{ bdim } \Phi | \Psi$ and

1. $A \text{ type}_K \Phi \uplus r, x | \Psi$,
2. $d : A \Rightarrow B \text{ type}_K \Phi \uplus r, x | \Psi$,
3. $M \in A \Phi \downarrow r, x | \Psi$,
4. $a : A(0/x) \Rightarrow N \in B(0/x)[a/d] \Phi \downarrow r | \Psi$,
5. $a' : A(1/x) \Rightarrow P \in B(1/x)[a'/d] \Phi \downarrow r | \Psi$,
6. $a : A(0/x), a' : A(1/x), c : \text{Bridge}_{x,A}(a,a') \Rightarrow Q \in \text{Bridge}_{x,B}(c@x/d)(N,P) \Phi \downarrow r | \Psi$,
then $\text{extent}_r(M\langle r/x \rangle; a.N, a'.P, a.a'.c.Q) \equiv Q[M\langle 0/x \rangle/a][M\langle 1/x \rangle/a'][\lambda^2 x.M/c]@r \in B[M/d]\langle r/x \rangle [\Phi | \Psi]$.

Proof. Via Lemma A.2. Let $\psi : (\Phi' | \Psi') \rightarrow (\Phi | \Psi)$ be given. We have three cases:

- $\vdash r\psi = 0$.
  Then $\text{extent}_r(M\langle r/x \rangle; a.N, a'.P, a.a'.c.Q)\psi \equiv N[M\langle r/x \rangle/a]\psi \in B[M/d]\langle r/x \rangle [\Phi' | \Psi']$ by Rule 6.3, and the right-hand side is equal to $(Q[M\langle 0/x \rangle/a][M\langle 1/x \rangle/a'][\lambda^2 x.M/c]@r)\psi$ by Rule 5.8.

- $\vdash r\psi = 1$.
  Analogous to the previous case.

- $\not\vdash r\psi = \varepsilon$ for all $\varepsilon \in \{0, 1\}$. Then $\text{extent}_r(M\langle r/x \rangle; a.N, a'.P, a.a'.c.Q)\psi \rightarrow (Q[M\langle 0/x \rangle/a][M\langle 1/x \rangle/a'][\lambda^2 x.M/c]@r)\psi$
  and the reduct is well-typed by Rules 5.5, 5.8 and 5.7.

Rule 6.6 (ex). If $r \text{ bdim } [\Phi | \Psi]$ and

1. $A \text{ type}_\text{Kan} [\Phi \setminus r, x | \Psi]$,
2. $d : A \Rightarrow B \text{ type}_\text{Kan} [\Phi \setminus r, x | \Psi]$,
3. $M = M' \in A\langle r/x \rangle [\Phi | \Psi]$,
4. $a : A(0/x) \Rightarrow N \Rightarrow N' \in B(0/x)[a/d] [\Phi \setminus r | \Psi]$,
5. $a' : A(1/x) \Rightarrow P \Rightarrow P' \in B(1/x)[a'/d] [\Phi \setminus r | \Psi]$,
6. $a : A(0/x), a' : A(1/x), c : \text{Bridge}_{x.A}(a, a') \Rightarrow Q \Rightarrow Q' \in \text{Bridge}_{x.B[c@x/d]}(N, P) [\Phi \setminus r | \Psi]$,

then $\text{extent}_r(M; a.N, a'.P, a.a'.c.Q) \equiv \text{extent}_r(M'; a.N', a'.P', a.a'.c.Q') \in B\langle r/x \rangle [M/d] [\Phi | \Psi]$.

Proof. We have two cases: either $\vdash r = \varepsilon$ for some $\varepsilon \in \{0, 1\}$ or not. In the former case, we reduce either side with Rules 6.3 and 6.4 and apply the typing assumptions to equate the reducts. In the latter case, $r$ must be some variable $y \in \Phi$. In that case, we can reduce either side with Rule 6.5 and then equate the reducts with the typing assumptions and Rules 5.5, 5.8 and 5.7.

Rule 6.7 (ex-η). If $r \text{ bdim } [\Phi | \Psi]$ and

1. $A \text{ type}_\text{Kan} [\Phi \setminus r, x | \Psi]$,
2. $d : A \Rightarrow B \text{ type}_\text{Kan} [\Phi \setminus r, x | \Psi]$,
3. $M \in A\langle r/x \rangle [\Phi | \Psi]$,
4. $d : A \Rightarrow N \in B [\Phi \setminus r, x | \Psi]$,

then $N(r/x)[M/a] \equiv \text{extent}_r(M; a.N(0/x)[a/d], a'.N(1/x)[a'/d], a.a'.c.\lambda^2 x.N[c@x/d])$
in $B(r/x)[M/d]$ at $(\Phi | \Psi)$.

Proof. By case analysis on $r$. If $r = 0$, this follows from Rule 6.3; if $r = 1$, it follows from Rule 6.4. If $r = y$, then $M = M(x/y)\langle r/x \rangle$, and we have

$N[M(x/y)/a](r/x) \equiv \text{extent}_r(M(x/y)\langle r/x \rangle; a.N(0/x)[a/d], a'.N(1/x)[a'/d], a.a'.c.\lambda^2 x.N[c@x/d])$

by Rules 5.6 and 6.5.
Remark 6.8. A weak version of the rule (ex-η), in which one obtains a path rather than an equality, is derivable from the other rules for extent (much as with η-principles for positive types). Given the hypotheses of Rule 6.7, we have

\[
\text{extent}_r(M; a.λy.N(0/x), a.N(1/x), a.a’.c.λx.λy.N[c@x/a])
\]

of type

\[
\text{Path}_{B(M/a)}[M/a](N(r/x)[M/a], \text{extent}_r(M; a.N(0/x)[a/d], a’.N(1/x)[a’/d], a.a’.c.λx.N[c@x/d])).
\]

The weaker rule suffices for the proof of Theorem 6.9 below, which is the only place we use (ex-η). Thus, the strict rule may safely be omitted from a proof theory.

This completes the set of rules we need for extent. Using these rules, we can now characterize the type of bridges over a function type.

**Theorem 6.9.** Let \( A \) be a type \( Φ, x \mid Ψ \), \( a : A \gg B \) be a type \( Φ, x \mid Ψ \), \( F \in (a:A(0/x)) \rightarrow B(0/x) \) \( [Φ \mid Ψ] \), and \( F’ \in (a:A(1/x)) \rightarrow B(1/x) \) \( [Φ \mid Ψ] \) be given. Then there is an equivalence

\[
\text{Bridge}_{x,(a:A) \rightarrow B}(F,F’) \simeq (a:A(0/x))(a’:A(1/x))(c:\text{Bridge}_{x,A}(a,a’)) \rightarrow \text{Bridge}_{x,B}[c@x/a](Fa,F’a).
\]

**Proof.** By Recollection 4.7. We define forward and backward functions as follows.

\[
\begin{align*}
\text{bfunapp} & := λy.λa.λa’.λc.λx.λy.(q@x)(c@x) \\
\text{bfunext}^{F,F’} & := λh.λx.λd.\text{extent}_x(d; a.Fa,a’.F’a,a.a’.c.haa’a)
\end{align*}
\]

We show that bfunapp and bfunext\(^{F,F’}\) are mutually inverse, then apply Recollection 4.7. In a context with \( q : \text{Bridge}_{x,(a:A) \rightarrow B}(F,F’) \), we have

\[
\begin{align*}
\text{bfunext}^{F,F’}(\text{bfunapp}(q)) & \equiv λx.λd.\text{extent}_x(d; a.Fa,a’.F’a,a.a’.c.λx.λy.(q@x)(c@x)) \\
& \equiv λx.λd.(q@x)a,a’.(q@1)a’,a.a’.c.λx.λy.(q@y)(c@y)) \\
& \equiv q
\end{align*}
\]

in \( \text{Bridge}_{x,(a:A) \rightarrow B}(F,F’) \), where the first equation is β for bridges and functions, the second is Rule 5.8, the third is Rule 6.7, and the fourth is η for bridges and functions. For the other inverse, in a context with

\[
\begin{align*}
h & : (a:A(0/x)) \rightarrow (a’:A(1/x)) \rightarrow (c:\text{Bridge}_{x,A}(a,a’)) \rightarrow \text{Bridge}_{x,B}[c@x/a](Fa,F’a),
\end{align*}
\]

we have

\[
\begin{align*}
\text{bfunapp}(\text{bfunext}^{F,F’}(h)) & \equiv λa.λa’.λc.λx.\text{extent}_x(c@x; a.Fa,a’.F’a,a.a’.k.haa’a’k) \\
& \equiv λa.λa’.λc.h(c@0)(c@1)(λx.λy.(c@y)@x) \\
& \equiv λa.λa’.λc.haa’(λx.λy.(c@y)@x) \\
& \equiv h
\end{align*}
\]

in the same type, where the first step is β for bridges and functions, the second is Rule 6.5, the third is Rule 5.8, and the fourth is η for bridges and functions. \( \square \)

One way to conceptualize the difference between structural and substructural dimensions is in terms of the different “function extensionality” principles they provide. For substructural dimensions, we have the theorem just proven. If bridges were structural, on the other hand, we would instead be able to prove the following incomparable principle.

\[
\text{Bridge}_{(a:A) \rightarrow B}(F,F’) \preceq (a:A) \rightarrow \text{Bridge}_{B}(Fa,F’a)
\]

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We would define this equivalence by taking $q$ in the former type to $\lambda a.\lambda^2 x.(q @ x)a$ in the latter and $h$ in the latter to $\lambda^2 x.\lambda a.ha @ x$ in the former. While the first map is still well-defined substructurally, the second is not: $x$ is not fresh for $a$, so $ha$ cannot be applied at $x$. Conversely, without the extent operator which substructurality enables, we would not be able to prove Theorem 6.9. Note that on the path side, both principles are provable; the equivalent of Theorem 6.9 follows from the structural principle using Kan operations not available on the bridge side. Likewise, the substructural cubical type theory of Bezem et al. [2013] enjoys the same functional extensionality principle as the structural cubical type theories. Without Kan operations, however, the principles are distinct, and it is the substructural version which matches the standard definition of a logical relation at a function type.

Finally, we observe that the function extensionality principle induces a corresponding “equivalence extensionality” principle, which we will use in the proof of relativity.

**Corollary 6.10.** Let $A, B \text{ type}_{\text{Kan}} [\Phi \rangle r, x \mid \Psi]$. Suppose we have

1. $E_0 \in A(0/x) \simeq B(0/x) [\Phi \rangle r \mid \Psi]$,
2. $E_1 \in A(1/x) \simeq B(1/x) [\Phi \rangle r \mid \Psi]$,
3. $a_0 : A(0/x), a_1 : A(1/x) \Rightarrow E \in \text{Bridge}_{x,A}(a_0, a_1) \simeq \text{Bridge}_{x,B}(\text{fwd}(E)(a_0), \text{fwd}(E_1)(a_1)) [\Phi \rangle r \mid \Psi]$.

Then there is a term

$$\text{extent-equiv}_{r}(E_0; E_1, E) \in A(r/x) \simeq B(r/x) [\Phi \mid \Psi]$$

that satisfies $\text{extent-equiv}_{r}(E_0; E_1, E) \equiv E_\varepsilon \in A(\varepsilon/x) \simeq B(\varepsilon/x) [\Phi \mid \Psi \mid r = \varepsilon]$ for $\varepsilon \in \{0, 1\}$.

**Proof.** The proof is lengthy but straightforward; we will give a sketch. We first construct a quasi-equivalence between $A(r/x)$ and $B(r/x)$. We define a forward map in by

$$\text{in}_r := \lambda a.\text{extent}_{r}(a; a_0.\text{fwd}(E_0)(a_0), a_1.\text{fwd}(E_1)(a_1), a_0.a_1.p.\text{fwd}(E)(p)).$$

To define the reverse map, we first derive a term

$$b_0 : B(0/x), b_1 : B(1/x) \Rightarrow F \in \text{Bridge}_{x,A}(\text{bwd}(E_0)(b_0), \text{bwd}(E_1)(b_1)) \simeq \text{Bridge}_{x,B}(b_0, b_1) [\Phi \rangle r \mid \Psi]$$

from $E$ using the fact that $E_0$ and $E_1$ are equivalences. We then set

$$\text{out}_r := \lambda b.\text{extent}_{r}(b; b_0.\text{bwd}(E_0)(b_0), b_1.\text{bwd}(E_1)(b_1), b_0.b_1.q.\text{bwd}(F)(q)).$$

Proofs that $\text{in}_r$ and $\text{out}_r$ are mutually inverse can again be constructed by using extent to case on $r$. By applying Recollection 4.7, this shows that $\text{in}_r$ is an equivalence. However, although we have ensured that $\text{in}_r \equiv \text{fst}(E_\varepsilon) \in A(\varepsilon/x) \rightarrow B(\varepsilon/x) [\Phi \mid \Psi \mid r = \varepsilon]$ by construction, we do not know that the proof that $\text{in}_r$ is an equivalence has the correct boundary. To fix this, we use Recollection 4.6, which implies that the boundary of our equivalence term is connected by a pair of paths to the desired boundary. We can then modify it using an $\text{hcom}$ in $A(r/x) \simeq B(r/x)$ (with tube faces at $r = 0$ and $r = 1$) to construct an equivalence which has the correct boundary up to exact equality. 

**7 Gel-types**

The final constructor we need to complete the type theory is Gel, which takes a relation between two types and produces a bridge between them. This gives the inverse to the operation $C \mapsto \text{Bridge}_{x,C \times x}(\cdot, \cdot)$ mentioned at the beginning of Section 6, making it possible to prove relativity (Section 9). The Gel operator is for the bridge side what $V$ is for the path side, but there are important differences which again derive from and motivate the substructurality of bridge dimensions.
Recall from Section 4 that the type $V_x(A, B, E)$ takes a type $A$ at $x = 0$, a type line $B$ in $x$, and an equivalence $E$ between $A$ and $B(0/x)$ at $x = 0$, and composes them to create a new type line in $x$ from $A$ to $B(1/x)$. By taking $B$ to be a constant path, which corresponds to an identity equivalence, we can use $V$ to convert any equivalence into a path. However, it appears necessary to take this indirect route via composition with a line: we cannot restrict the typing rule of $V$ to only allow types $B$ which are constant in $x$, as such an apartness criterion is incompatible with diagonal substitutions. (This is far from a complete justification of the shape of $V$, but we hope it gives the reader a sense of the situation.) This is a problem if we want to translate $V$-types to the bridge side. Unlike paths and equivalences, the constant bridge $B$ does not necessarily correspond to the identity relation on $B$, which is to say $\text{Path}_B(-, -)$; rather, it corresponds to the bridge relation $\text{Bridge}_B(-, -)$. This means we could only use a “bridge $V$” to construct bridges to $B$ corresponding to relations that factor through $\text{Bridge}_B$.

Instead of resembling $V$-types, a Gel-type thus takes the form $\text{Gel}_x(A, B, a.b.R)$ where both types $A, B$ as well as the relation $R$ on $A$ and $B$ must be apart from $x$. Besides being the binary analogue of Bernardy et al.’s $A \ni a$ types, Gel-types are essentially the $G$-types of Bezem et al. [2017] adapted to the relational case, hence the name. The definitions of the Kan operations for Gel are, however, much simpler than for $V$ or $G$: the principal direction of a coe or hcom is always a path dimension, so can never coincide with the direction $x$ of the Gel-type.

The operational semantics of Gel-types are shown in Figure 6. We define PER the semantics of Gel and prove typing rules in this section, the latter of which are collected in inference rule format in Section 8.5. The values of type $\text{Gel}_x(A, B, a.b.R)$ are triples $\text{gel}_y(M, N, P)$ where $M$ is in $A$, $N$ is in $B$, and $P$ is a proof that the two are related by $R$. The eliminator $\text{ungel}$ takes a bridge over $x. \text{Gel}_x(A, B, a.b.R)$ and produces a proof that its left and right endpoints (in $A$ and $B$ respectively) are related by $R$. This is a second point of departure from $V$ or $G$ types. For equivalence-to-path types, the eliminator is a projection function that takes (in the case of $V$, for example) an element of $V_r(A, B, E)$ and extracts an element of $B$. This is not possible with $\text{Gel}_x(A, B, a.b.R)$, as there is no way to produce such an element of $B$ when $r$ is 0. As such, $\text{ungel}$ takes not an element but a bridge over the Gel-type. In order to make use of $\text{ungel}$, the implementations of hcom and coe for $\text{Gel}_x$ capture occurrences of $x$ in their arguments, just as $\text{ext}_{\text{gel}}_x$ does.

7.1 Definition

Definition 7.1. Let $\tau$ be a candidate type system. Let $r \text{ bdim } [\Phi | \Psi]$ and

1. $\tau \vdash A \text{ type}_{\text{kan}} [\Phi \backslash \tau | \Psi]$,
2. $\tau \vdash B \text{ type}_{\text{kan}} [\Phi \backslash \tau | \Psi]$,
3. $\tau \vdash a : A, b : B \Rightarrow R \text{ type}_{\text{kan}} [\Phi \backslash \tau | \Psi]$

be given. We define a value $(\Phi | \Psi) \text{-PER } \text{Gel}_x^r(A, B, a.b.R)$ by saying that, for each $\psi : (\Phi | \Psi) \rightarrow (\Phi | \Psi)$, $\text{Gel}_x^r(A, B, a.b.R)_{\psi}((V, V'))$ holds iff one of the following holds:

- $r\psi = 0$ and $[A]_{\psi}(V, V')$,
- $r\psi = 1$ and $[B]_{\psi}(V, V')$,
- $r\psi = y$ and $V = \text{gel}_y(M, N, P)$, $V' = \text{gel}_y(M', N', P')$ with
  
  1. $\tau \vdash M \doteq M' \in A\psi [\Phi \backslash \tau | \Psi']$,
  2. $\tau \vdash N \doteq N' \in B\psi [\Phi \backslash \tau | \Psi']$,
  3. $\tau \vdash P \doteq P' \in R\psi[M/a][N/b] [\Phi \backslash \tau | \Psi']$.

We will drop the superscript $\tau$ when it is inferable.

Lemma 7.2. If $r\psi = 0$ and $M \in A\psi [\Phi | \Psi]$, then $\text{TM}(\text{Gel}_x^r(A, B, a.b.R))_{\psi}(\text{gel}_{\psi}(M, N, P), M)$. 20
| $\text{Gel}_0(A, B, a.b.R) \mapsto A$ | $\text{Gel}_1(A, B, a.b.R) \mapsto B$ | $\text{Gel}_x(A, B, a.b.R) \text{ val}$ |
|---|---|---|
| $\text{gel}_0(M, N, P) \mapsto M$ | $\text{gel}_1(M, N, P) \mapsto N$ | $\text{gel}_x(M, N, P) \text{ val}$ |

$M \mapsto M'$

$\text{ungel}(x.M) \mapsto \text{ungel}(x.M')$

$\text{ungel}(x.\text{gel}_x(M, N, P)) \mapsto P(0/x)^a$

\[
\begin{align*}
  M' &:= \text{hcom}^{\rightarrow s}(A, \text{gel}_x(M, N, P)) (\text{ungel}(x.O); (\xi_i \mapsto y.\text{ungel}(x.Q_i))_{x \notin \xi_i}) \\
  N' &:= \text{coe}^{-s}_{y.A}(O(1/x)) \\
  P &:= \text{com}^{-s}_{y.R[\text{gel}_x(M, N, P)](\text{ungel}(x.O); (\xi_i \mapsto y.\text{ungel}(x.Q_i))_{x \notin \xi_i})} \quad \text{coe}^{-s}_{y.Gel_x(A, B, a.b.R)}(O) \mapsto \text{gel}_x(M', N', P)
\end{align*}
\]

*In well-typed code, the substitution $\langle 0/x \rangle$ is always a no-op. We include it here only to ensure that the evaluation system is context-preserving.*

---

**Figure 6:** Operational semantics of Gel-types

**Proof.** By Lemma A.2, as $\text{gel}_0(M, N, P) \psi' \mapsto M \psi'$ for all $\psi'$.

**Lemma 7.3.** If $r \psi = 1$ and $N \in B \psi [\Phi' \mid \Psi']$, then $\text{TM}(\text{Gel}_r(A, B, a.b.R))(\text{gel}_{r\psi}(M, N, P), N)$.

**Proof.** By Lemma A.2, as $\text{gel}_1(M, N, P) \psi' \mapsto N \psi'$ for all $\psi'$.

**Lemma 7.4.** $\text{Gel}_r(A, B, a.b.R)$ is value-coherent.

**Proof.** Let $\text{Gel}_r(A, B, a.b.R)(\psi)(V, V')$ be given. If $r \psi = 0$, then $\text{Gel}_r(A, B, a.b.R)\psi = [A] \psi$, so by value-coherence of $A$ we have $\text{TM}(\text{Gel}_r(A, B, a.b.R)\psi)(V, V')$. Likewise, if $r \psi = 1$, then the same follows from value-coherence of $[B]$. Now suppose $r \psi = y$. Then we have $V = \text{gel}_y(M, N, P)$ and $V' = \text{gel}_y(M', N', P)$ satisfying the conditions in Definition 7.1. We go by Lemma A.1. For any $\psi : (\Phi' \mid \Psi') \mapsto (\Phi' \mid \Psi')$, we are in one of three cases.

- **$r \psi \psi' = 0$.** Then $\text{TM}(\text{Gel}_r(A, B, a.b.R))\psi(\psi', V' \psi')$ by Lemma 7.2, $M = M' \in A \psi [\Phi' \mid y \mid \Psi']$, and transitivity of $\text{TM}(\text{Gel}_r(A, B, a.b.R))$.

- **$r \psi \psi' = 1$.** Then we apply Lemma 7.3 analogously to the previous case.

- **$r \psi \psi' \notin \{0, 1\}$.** Then $\text{Gel}_r(A, B, a.b.R)\psi(\psi', V' \psi')$ by definition of $\text{Gel}$.

**Proposition 7.5.** There exists a type system $\tau$ which, for all $\tau \models A = A'$ in $\text{type}_{\text{Kan}}(\Phi \mid \Psi)$, $\tau \models B = B'$ in $\text{type}_{\text{Kan}}(\Phi \mid \Psi)$, $\tau \models a : A.b : B \Rightarrow R \equiv R'$ in $\text{type}_{\text{Kan}}(\Phi \mid \Psi)$, and $x \notin \Phi$, has $\tau((\Phi, x \mid \Psi), \text{Gel}_x(A, B, a.b.R), \text{Gel}_x(A', B', a.b.R'), \text{Gel}_x(A, B, a.b.R)_{\text{id}})$ in addition to Bridge-types and the standard constructs of cubical type theory. Moreover, there exists such a type system in which the universe $\mathcal{U}$ is also closed under Bridge- and Gel-types.
Proof. See Appendix B.

For the remainder of this section, we assume we are working in such a type system.

7.2 Rules

Rule 7.6 (Gel-F₀). If \( A \) type\(_{Kan} \ [\Phi \mid \Psi] \) then Gel₀(\( A, B, a.b.R \)) = A type\(_{pre} \ [\Phi \mid \Psi] \).

Proof. By Lemma A.5, as Gel₀(\( A, B, a.b.R \))ψ \( \longrightarrow \) Aψ for all ψ.

Rule 7.7 (Gel-F₁). If \( B \) type\(_{Kan} \ [\Phi \mid \Psi] \), then Gel₁(\( A, B, a.b.R \)) = B type\(_{pre} \ [\Phi \mid \Psi] \).

Proof. By Lemma A.5, as Gel₁(\( A, B, a.b.R \))ψ \( \longrightarrow \) Bψ for all ψ.

Rule 7.8 (Gel-G₁). If

1. \( A \equiv A' \) type\(_{Kan} \ [\Phi' \mid \Psi'] \),
2. \( B \equiv B' \) type\(_{Kan} \ [\Phi' \mid \Psi'] \),
3. \( a : A, b : B \triangleright \rightarrow R \equiv R' \) type\(_{Kan} \ [\Phi' \mid \Psi'] \),

then Gelᵣ(\( A, B, a.b.R \)) = Gelᵣ(\( A', B', a.b.R' \)) type\(_{pre} \ [\Phi \mid \Psi] \).

Proof. By Lemma A.4. Let \( \psi : (\Phi' \mid \Psi') \rightarrow (\Phi \mid \Psi) \) be given; we have three cases. If \( r \psi = 0 \), then we have

\[ \text{PTy}(\tau)((\Phi' \mid \Psi'), \text{Gel}_r(A, B, a.b.R) \psi, \text{Gel}_r(A', B', a.b.R') \psi, \text{Gel}_r(A, B, a.b.R) \psi) \]

by Rule 7.6, \( A \psi \equiv A' \psi \) type\(_{Kan} \ [\Phi' \mid \Psi'] \), and transitivity of \( \text{PTy}(\tau) \). If \( r \psi = 1 \), then the same follows by Rule 7.7, \( B \psi \equiv B' \psi \) type\(_{Kan} \ [\Phi' \mid \Psi'] \), and transitivity of \( \text{PTy}(\tau) \). Finally, if \( r \psi = y \), then we have

\[ \tau((\Phi' \mid \Psi'), \text{Gel}_r(A, B, a.b.R) \psi, \text{Gel}_r(A', B', a.b.R') \psi, \text{Gel}_r(A, B, a.b.R) \psi) \]

by our assumption on \( \tau \).

The following three rules are simply restatements of Lemmas 7.2 to 7.4.

Rule 7.9 (Gel-I₀). If \( M \in A \ [\Phi \mid \Psi] \), then gel₀(\( M, N, P \)) = M \in A \ [\Phi \mid \Psi] \).

Rule 7.10 (Gel-I₁). If \( N \in B \ [\Phi \mid \Psi] \), then gel₁(\( M, N, P \)) = N \in B \ [\Phi \mid \Psi] \).

Rule 7.11 (Gel-I). If

1. \( M \equiv M' \in A \ [\Phi' \mid \Psi] \),
2. \( N \equiv N' \in B \ [\Phi' \mid \Psi] \),
3. \( P \equiv P' \in R[M/a][N/b] \ [\Phi' \mid \Psi] \),

then gelᵣ(\( M, N, P \)) = gelᵣ(\( M', N', P' \)) \in Gelᵣ(A, B, a.b.R) \ [\Phi \mid \Psi] \).

Rule 7.12 (Gel-β). If

1. \( M \in A \ [\Phi \mid \Psi] \),
2. \( N \in B \ [\Phi \mid \Psi] \),
3. \( P \in R[M/a][N/b] \ [\Phi \mid \Psi] \),

then ungel(x.gelᵣ(M, N, P)) = P \in R[M/a][N/b] \ [\Phi \mid \Psi] \).

Proof. By Lemma A.2, as ungel(x.gelᵣ(M, N, P))ψ \( \longrightarrow \) P(\( 0/x \))ψ for all ψ and x does not occur in P.

Rule 7.13 (Gel-E). If \( A, B \) type\(_{Kan} \ [\Phi \mid \Psi] \), \( a : A, b : B \triangleright \rightarrow R \) type\(_{Kan} \ [\Phi \mid \Psi] \), and
1. \( Q \vdash Q' \in \text{Gel}_x(A, B, a.b.R) \ [\Phi, x | \Psi] \), then \( \text{ungel}(x.Q) = \text{ungel}(x.Q') \in R[Q(0/x)/a][Q(1/x)/b] [\Phi | \Psi] \).

Proof. By Lemma A.6 with \( \text{ungel}(x.[-]), \text{ungel}(x.[-]), R[(\lambda x.[-])@0/a][(\lambda x.[-])@1/b] : (\Phi | \Psi) \leftrightarrow (x | \emptyset) \). We need to check that the equations hold when \( Q \) and \( Q' \) are values in \([\text{Gel}_x(A, B, a.b.R)]\). In that case, each follows by reducing with Rule 7.12 on either side and then applying the assumptions given by \([\text{Gel}_x(A, B, a.b.R)](Q, Q')\).

\begin{rule}[\text{Gel-\eta}] \label{rule:gel-eta} \end{rule}

\begin{enumerate}
\item \( Q \in \text{Gel}_x(A, B, a.b.R) \ [\Phi, x | \Psi] \), then \( Q(r/x) = \text{gel}_x(Q(0/x), Q(1/x), \text{ungel}(x.Q)) \in \text{Gel}_x(A, B, a.b.R) \ [\Phi | \Psi] \).
\end{enumerate}

Proof. By Lemma A.3, we have \( Q \downarrow V \) with \( Q \vdash V \in \text{Gel}_x(A, B, a.b.R) \ [\Phi, x | \Psi] \). By Rules 7.11 and 7.13, it thus suffices to show 
\[
V(r/x) = \text{gel}_x(V(0/x), V(1/x), \text{ungel}(x.V)) \in \text{Gel}_x(A, B, a.b.R) \ [\Phi | \Psi].
\]

By definition of \( \text{Gel}_x \), we have \( V = \text{gel}_x(M, N, P) \) for some \( M \in A \ [\Phi | \Psi], N \in B \ [\Phi | \Psi] \), and \( P \in R[M/a][N/b] [\Phi | \Psi] \). We have
\begin{enumerate}
\item \( M \vdash V(0/x) \in A \ [\Phi | \Psi] \) by Rule 7.9.
\item \( N \vdash V(1/x) \in B \ [\Phi | \Psi] \) by Rule 7.10,
\item \( P \vdash \text{ungel}(x.V) \in R[M/a][N/b] [\Phi | \Psi] \) by Rule 7.12,
\end{enumerate}
so the equation follows by Rule 7.11.

\section{7.3 Kan conditions}

The definitions of the Kan operations for \( \text{Gel}_x(A, B, a.b.R) \) are actually quite simple: we extract the endpoints and relation data from each argument, apply the corresponding Kan conditions at \( A, B \), and \( R \) respectively, and then recombine them with gel. The endpoint data is extracted with the substitutions \( (0/x) \) and \( (1/x) \), while the relation data is extracted with \( \text{ungel}(x.\_\_). \) Note that in using the latter, we capture occurrences of \( x \) in the argument(s).

\begin{lemma}[\text{coe-\text{Gel}-\beta}] \label{lemma:coe-gel-beta} \end{lemma}

Let \( A, B \) type\_Kan \( [\Phi^r | \Psi, y] \) and \( a : A, b : B \gg R \) type\_Kan \( [\Phi^r | \Psi, y] \). Suppose we have \( r, s \) pdim \( [\Phi | \Psi] \) and \( O \in \text{Gel}_x(A, B, a.b.R)(r/y) \ [\Phi | \Psi] \). Then
\begin{enumerate}
\item if \( r = 0 \), then \( \text{coe}_{y.a}^{r \rightarrow s}(O, A(s/y) [\Phi | \Psi]) \), and
\item if \( r = 1 \), then \( \text{coe}_{y.b}^{r \rightarrow s}(O, B(s/y) [\Phi | \Psi]) \).
\end{enumerate}

Proof. By Lemma A.2, using the fact that \( A \) and \( B \) are coe\_Kan to type the reducts.

\begin{lemma}[\text{coe-\text{Gel}}] \label{lemma:coe-gel} \end{lemma}

Let \( A, B \) type\_Kan \( [\Phi | \Psi, y] \) and \( a : A, b : B \gg R \) type\_Kan \( [\Phi | \Psi, y] \). Suppose we have \( r, s \) pdim \( [\Phi, x | \Psi] \), and \( O \in \text{Gel}_x(A, B, a.b.R)(r/y) \ [\Phi, x | \Psi] \). Then
\[
\text{coe}_{y.a}^{r \rightarrow s}(O, M^s, N^s, P) \in \text{Gel}_x(A, B, a.b.R)(s/y) \ [\Phi, x | \Psi]
\]
where we define
\[
M^y := \text{coe}_{y.a}^{r \rightarrow y}(O(0/x)) \quad N^y := \text{coe}_{y.b}^{r \rightarrow y}(O(1/x)) \quad P := \text{coe}_{y,R(M^y/a)[N^y/b]}^{r \rightarrow s}(\text{ungel}(x.O)).
\]

Proof. Observe that the reduct is well-typed by the coe\_Kan conditions for \( A \) and \( B \), Rule 7.13, and the coe\_Kan condition for \( R \). We go by Lemma A.2. Let \( \psi : (\Phi' | \Psi') \rightarrow (\Phi, x | \Psi) \) be given; we have three cases.
• \( x\psi = 0 \).

Then \( \text{coe}^{r \rightarrow s}_y \hspace{1pt} \Gamma \hspace{1pt} \psi \rightarrow \text{coe}^{r \rightarrow s}_y \hspace{1pt} \psi \). We have \( M^s\psi \in A\psi[s/y] [\Phi' | \Psi'] \) by the coe-Kan condition for \( A \) and \( M^s\psi = \text{gel}_\omega(M^s, N^s, P)\psi \in A\psi [\Phi' | \Psi'] \) by Rule 7.9.

• \( x\psi = 1 \).

Analogous to the \( x\psi = 0 \) case.

• \( x\psi \notin \{0, 1\} \).

Then \( \text{coe}^{r \rightarrow s}_y \hspace{1pt} \Gamma \hspace{1pt} \psi \rightarrow \text{gel}_\omega(M^s, N^s, P^s)\psi \), and we have already shown that the reduct is well-typed.

\[ \square \]

**Theorem 7.17.** Let \( A \vdash A' \text{ type}_\text{Kan} [\Phi \rightarrow_r | \Psi] \), \( B \vdash B' \text{ type}_\text{Kan} [\Phi \rightarrow_r | \Psi] \), and \( a : A, b : B \gg R \rightarrow R' \text{ type}_\text{Kan} [\Phi \rightarrow_r | \Psi] \) be given. Then \( \text{gel}_r(A, B, a.b.R) \simeq \text{gel}_r(A', B', a.b.R') \text{ type}_\text{pre} [\Phi | \Psi] \) are equally coe-Kan.

**Proof.** Let \( \psi : (\Phi' | \Psi', y) \rightarrow (\Phi | \Psi), r, s \text{ pdim } [\Phi' | \Psi'] \), and \( O \vdash O' \in \text{gel}_r(A, B, a.b.R)\psi[r/y] [\Phi' | \Psi'] \) be given. We need to show

1. \( \text{coe}^{r \rightarrow s}_y \hspace{1pt} \Gamma \hspace{1pt} \psi \rightarrow \text{coe}^{r \rightarrow s}_y \hspace{1pt} \psi \rightarrow \text{gel}_\omega(A, B, a.b.R)\psi[s/y] [\Phi' | \Psi'] \), and

2. if \( r = s \), then \( \text{coe}^{r \rightarrow s}_y \hspace{1pt} \Gamma \hspace{1pt} \psi \rightarrow \text{gel}_\omega(A, B, a.b.R)\psi[s/y] [\Phi' | \Psi'] \).

We have three cases, depending on the status of \( r\psi \); we prove the two equations for each case in turn.

• \( r\psi = 0 \).

Then the equations follow from the assumption that \( A\psi \vdash A'\psi \text{ type}_\text{Kan} [\Phi' | \Psi'] \) by rewriting each coe term with Lemma 7.15.

• \( r\psi = 1 \).

Analogous to the \( r\psi = 0 \) case.

• \( r\psi = x \).

We apply Lemma 7.16, which gives

\[
\text{coe}^{r \rightarrow s}_y \hspace{1pt} \Gamma \hspace{1pt} \psi \rightarrow \text{gel}_\omega(A, B, a.b.R)\psi[s/y] [\Phi' | \Psi']
\]

where the reduct subterms are as defined there. We conclude that the first equation holds by a simple binary generalization of the well-typedness argument in the proof of that lemma.

For the second equation, suppose that \( r = s \). Then we have

\[
\text{gel}_\omega(M^s, N^s, P) \simeq \text{gel}_\omega(O(0/x), O(1/x), \text{ungel}(x.O)) \in \text{gel}_\omega(A, B, a.b.R)\psi[s/y] [\Phi' | \Psi']
\]

by the coe-Kan condition for \( A, B, \) and \( R, \) and the right-hand side is equal to \( O \) by Rule 7.14.

\[ \square \]

**Lemma 7.18 (hcom\text{Gel}_-\beta_x).** Let \( A, B \text{ type}_\text{Kan} [\Phi \rightarrow_r | \Psi] \) and \( a : A, b : B \gg R \text{ type}_\text{Kan} [\Phi \rightarrow_r | \Psi] \). Suppose we have \( r, s \text{ pdim } [\Phi | \Psi], \xi_i \text{ eqs } [\Phi | \Psi], \) and

1. \( O \in \text{gel}_r(A, B, a.b.R) [\Phi | \Psi], \)

2. \( Q_i \simeq Q_j \in \text{gel}_r(A, B, a.b.R) [\Phi | \Psi, y | \xi_i, \xi_j] \) for all \( i, j, \)

3. \( Q_i[r/y] \simeq O \in \text{gel}_r(A, B, a.b.R) [\Phi | \Psi | \xi_i] \) for all \( i. \)

Then
Then

\[ \text{Rule 7.13 gives cases.} \]

Proof. By Lemma A.2, using the fact that \( A \) and \( B \) are hcom-Kan to type the reducts.

Lemma 7.19 (hcom\(_{\text{GeL-\beta}}\)). Let \( A, B \) type\(_{\text{Kan}}\) \( [\Phi \mid \Psi] \) and \( a : A, b : B \Rightarrow R \) type\(_{\text{Kan}}\) \( [\Phi \mid \Psi] \). Suppose we have

\[ r, s \text{ pdim } [\Phi, x \mid \Psi], \xi_i \text{ eqs } [\Phi, x \mid \Psi], \text{ and} \]

1. \( O \in \text{GeL}_x(A, B, a.b.R) [\Phi, x \mid \Psi] \),
2. \( Q_i \equiv Q_j \in \text{GeL}_x(A, B, a.b.R) [\Phi, x \mid \Psi, y \mid \xi_i, \xi_j] \) for all \( i, j \),
3. \( Q_i(r/y) \equiv O \in \text{GeL}_x(A, B, a.b.R) [\Phi, x \mid \Psi] \).

Then

\[ \text{hcom}_{\text{GeL}}(A, a.b.R)(O; \xi_i \mapsto Q_i) \equiv \text{gel}_x(M^*, N^*, P) \in \text{GeL}_x(A, B, a.b.R) [\Phi, x \mid \Psi] \]

where we define

\[ M^y := \text{hcom}_{\text{A}}(O(0/x); \xi_i(0/x) \mapsto y.Q_i(0/x)) \quad N^y := \text{hcom}_{\text{B}}(O(1/x); \xi_i(1/x) \mapsto y.Q_i(1/x)) \]

\[ P := \text{com}_{\text{y},(M^y/a)[N^y/b]}[\text{ungel}(x, O); (\xi_i \mapsto y.\text{ungel}(x, Q_i))_{x \notin \xi_i}] \in \text{GeL}_x(A, B, a.b.R) [\Phi, x \mid \Psi] \]

Proof. We first argue that the reduct is well-typed. First, we have \( M^y \in A [\Phi, y] \) by Rule 7.6 and the hcom-Kan condition for \( A \), likewise \( N^y \in B [\Phi, y] \) by Rule 7.7 and the hcom-Kan condition for \( B \). Second, Rule 7.13 gives

1. \( \text{ungel}(x, O) \in R(O(0/x)/a)[O(1/x)/b] [\Phi, y] \),
2. \( \text{ungel}(x, Q_i) \equiv \text{ungel}(x, Q_j) \in R(Q_i(0/x)/a)[Q_i(1/x)/b] [\Phi, y, y \mid \xi_i, \xi_j] \) for all \( i, j \) with \( x \notin \xi_i, \xi_j \),
3. \( \text{ungel}(x, Q_i(r/y)) \equiv \text{ungel}(x, O) \in R(O(0/x)/a)[O(1/x)/b] [\Phi, y \mid \xi_i, \xi_i] \) for all \( i \) with \( x \notin \xi_i \).

Thus \( P \in R(M^y/a)[N^y/b] [\Phi, y] \) by Proposition 3.18, and so \( \text{gel}_x(M^*, N^*, P) \in \text{GeL}_x(A, B, a.b.R) [\Phi, x \mid \Psi] \) by Rule 7.11.

Now we prove the desired equation using Lemma A.2. Let \( \psi : (\Phi' \mid \Psi') \to (\Phi \mid \Psi) \) be given; we have three cases.

- \( x\psi = 0 \).
  Then \( \text{hcom}_{\text{GeL}}(A, a.b.R)(O; \xi_i \mapsto Q_i)\psi \to \text{hcom}_{\text{A}}(O; \xi_i \mapsto Q_i)\psi \). By the hcom-Kan condition for \( A \) we have \( \text{hcom}_{\text{A}}(O; \xi_i \mapsto Q_i)\psi \equiv M^x\psi \in A \psi [\Phi' \mid \Psi'] \), and then \( M^x\psi = \text{gel}_x(M^*, N^*, P)\psi \in A \psi [\Phi' \mid \Psi'] \) by Rule 7.9.

- \( x\psi = 1 \).
  Analogous to the \( \Xi' \models x\psi = 0 \) case.

- \( x\psi \notin \{0, 1\} \).
  Then \( \text{hcom}_{\text{GeL}}(A, a.b.R)(O; \xi_i \mapsto Q_i)\psi \to \text{gel}_x(M^*, N^*, P^x)\psi \), which is well-typed by the argument above.

\[ \square \]

Theorem 7.20. Let \( A \equiv A' \text{ type}_{\text{Kan}} [\Phi \mid \Psi], B \equiv B' \text{ type}_{\text{Kan}} [\Phi' \mid \Psi] \), and \( a : A, b : B \Rightarrow R \equiv R' \text{ type}_{\text{Kan}} [\Phi \mid \Psi] \) be given. Then \( \text{GeL}_x(A, B, a.b.R) \equiv \text{GeL}_x(A', B', a.b.R') \text{ type}_{\text{pre}} [\Phi \mid \Psi] \) are equally hcom-Kan.

Proof. Let \( \psi : (\Phi' \mid \Psi') \to (\Phi \mid \Psi), r, s \text{ pdim } [\Phi', \Psi'], \xi_i \text{ eqs } [\Phi' \mid \Psi'] \) be given, and suppose we have
Abbreviating the remainder of the paper, in which we prove various results within the proof theory without reference to rules which concern the treatment of bridge dimensions. Rather, it is intended to be a sufficient basis for or definitive set of rules: we omit rules for constructs already introduced by Angiuli et al. [2017b], do not include the (notationally burdensome) rules for \texttt{hcom} and \texttt{coe} at specific types, and only list those structural rules which concern the treatment of bridge dimensions. Rather, it is intended to be a sufficient basis for the remainder of the paper, in which we prove various results within the proof theory without reference to the operational semantics.

1. \( O \triangleq O' \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi'] \),
2. \( Q_i \triangleq Q_j' \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi' \mid \xi_i, \xi_j] \) for all \( i, j \),
3. \( Q_i(r/y) \triangleq O \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi' \mid \xi_i] \) for all \( i \).

Abbreviating \( C \triangleq \text{Gel}_x(A, B, a.b.R) \) and \( C' \triangleq \text{Gel}_x(A', B', a.b.R') \), we need to show

1. \( \text{hcom}_{C' \psi}^C(O; \xi_i \mapsto y.Q_i) \triangleq \text{hcom}_{C' \psi}^C(O'; \xi_i \mapsto y.Q_i') \in C' \psi [\Phi' | \Psi'] \),
2. \( \text{hcom}_{C' \psi}^C(O; \xi_i \mapsto y.Q_i) \triangleq Q_i(s/y) \in C' \psi [\Phi' | \Psi'] \) for all \( i \) with \( \models \xi_i \),
3. \( \text{hcom}_{C' \psi}^C(O; \xi_i \mapsto y.Q_i) \triangleq O \in C' \psi [\Phi' | \Psi'] \) if \( r = s \).

We have three cases, depending on the status of \( r \psi \); we prove the three equations for each case.

- \( r \psi = 0 \).

Then the equations follow from the assumption that \( A \psi \triangleq A' \psi \) type\texttt{pre} \( [\Phi' | \Psi'] \) are equally \texttt{hcom-Kan} by rewriting each \texttt{hcom} term with Lemma 7.18.

- \( r \psi = 1 \).

Analogous to the \( r \psi = 0 \) case.

- \( r \psi = x \).

We apply Lemma 7.19, which gives

\[
\text{hcom}_{C' \psi}^C(O; \xi_i \mapsto Q_i) \triangleq \text{gel}_x(M^*, N^*, P) \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi']
\]

\[
\text{hcom}_{C' \psi}^C(O'; \xi_i \mapsto Q_i') \triangleq \text{gel}_x(M'^*, N'^*, P') \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi']
\]

where the reduct subterms are as defined there. We conclude that the first equation holds by a simple binary generalization of the well-typedness argument in the proof of that lemma.

For the second equation, let \( i \) be given with \( \models \xi_i \). If \( \xi_i = (x = 0) \), then we have \( \text{gel}_x(M^*, N^*, P) \triangleq M^* \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi'] \) by Rule 7.9, and \( M^* \triangleq Q_i(0/x) \triangleq Q_i \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi' \mid \xi_i] \) by the \texttt{hcom-Kan} condition for \( A \). The \( \xi_i = (x = 1) \) case is similar. Finally, if \( x \not\models \xi_i \), we have

\[
\text{gel}_x(M^*, N^*, P) \triangleq \text{gel}_x(Q_i(0/x), Q_i(1/x), \text{ungel}(x.Q_i)) \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi' \mid \xi_i]
\]

by the \texttt{hcom-Kan} condition for \( A, B \) and Proposition 3.18 for \( R \). The right-hand side is then equal to \( Q_i \) by Rule 7.14.

For the third equation, suppose \( r = s \). As above, we have

\[
\text{gel}_x(M^*, N^*, P) \triangleq \text{gel}_x(O(0/x), O(1/x), \text{ungel}(x.O)) \in \text{Gel}_x(A, B, a.b.R) \psi [\Phi' | \Psi']
\]

by the \texttt{hcom-Kan} condition for \( A, B \) and Proposition 3.18 for \( R \), and the right-hand side is equal to \( O \) by Rule 7.14.

\section{Proof theory}

Having introduced all the operators we will need (\texttt{Bridge}, \texttt{extent}, and \texttt{Gel}), we now collect the rules we have proven into a makeshift proof theory for parametric cubical type theory. This is not meant to be a complete or definitive set of rules: we omit rules for constructs already introduced by Angiuli et al. [2017b], do not include the (notationally burdensome) rules for \texttt{hcom} and \texttt{coe} at specific types, and only list those structural rules which concern the treatment of bridge dimensions. Rather, it is intended to be a sufficient basis for the remainder of the paper, in which we prove various results within the proof theory without reference to the operational semantics.
8.1 Structural

\[ \Gamma', \varepsilon, \Gamma' \Rightarrow J \Phi | \Psi | \Xi \quad \varepsilon \in \{0, 1\} \]

\[ \Gamma, r, \Gamma' \Rightarrow J \Phi | \Psi | \Xi \]

8.2 Kan operations

\[ A \doteq A' \quad \text{type}_{Kan} \quad [\Phi | \Psi, z | \Xi] \]

\[ M \doteq M' \in A(r/z) \quad [\Phi | \Psi | \Xi] \]

\[ \operatorname{co}_x^\varepsilon x (M) \doteq \operatorname{co}_x^\varepsilon x (M') \in A(s/z) \quad [\Phi | \Psi | \Xi] \]

\[ A \text{ type}_{Kan} \quad [\Phi | \Psi, z | \Xi] \quad M \in A(r/z) \quad [\Phi | \Psi | \Xi] \]

\[ \operatorname{co}_x^\varepsilon x (M) \doteq M \in A(r/z) \quad [\Phi | \Psi | \Xi] \]

\[ \forall (i, j) \quad N_i \doteq N_j \in A \Phi | \Psi, y \Xi, \xi, \xi_j \quad \forall (i) \quad N_i(y) \doteq M \in A \Phi | \Psi | \Xi, \xi_i \]

\[ \text{hcom}_{\varepsilon x}^\varepsilon x (M; \xi_i \mapsto y N_i) \doteq \text{hcom}_{\varepsilon x}^\varepsilon x (M'; \xi_i \mapsto y N_i') \in A \Phi | \Psi | \Xi \]

\[ \forall (i, j) \quad N_i \doteq N_j \in A \Phi | \Psi, y \Xi, \xi, \xi_j \quad \forall (i) \quad N_i(y) \doteq M \in A \Phi | \Psi | \Xi, \xi_i \]

\[ \text{hcom}_{\varepsilon x}^\varepsilon x (M; \xi_i \mapsto y N_i) \doteq N_i(y) \in A \Phi | \Psi | \Xi \]

8.3 Bridge-types

\[ \Gamma, x \gg A \doteq A' \quad \text{type}_{Kan} \quad [\Phi, x | \Psi | \Xi] \]

\[ \Gamma \gg M_0 \doteq M'_0 \in A(0/x) \quad [\Phi | \Psi | \Xi] \]

\[ \Gamma \gg M_1 \doteq M'_1 \in A(1/x) \quad [\Phi | \Psi | \Xi] \]

\[ \Gamma \gg \text{Bridge}_{x, A}(M_0, M_1) \doteq \text{Bridge}_{x, A'}(M'_0, M'_1) \quad \text{type}_{Kan} \quad [\Phi | \Psi | \Xi] \quad \text{(Bridge-F)} \]

\[ \Gamma, x \gg P \doteq P' \in A \Phi, x | \Psi | \Xi \]

\[ \Gamma \gg P(0/x) \doteq M_0 \in A \Phi | \Psi | \Xi \quad \Gamma \gg P(1/x) \doteq M_1 \in A \Phi | \Psi | \Xi \]

\[ \Gamma \gg \lambda^2 x. P \doteq \lambda^2 x. P' \in \text{Bridge}_{x, A}(M_0, M_1) \quad [\Phi | \Psi | \Xi] \quad \text{(Bridge-I)} \]

\[ \Gamma \gg Q \doteq Q' \in \text{Bridge}_{x, A}(M_0, M_1) \quad [\Phi^r | \Psi | \Xi] \]

\[ \Gamma \gg Q \circ r \doteq Q' \circ r \in A(r/x) \quad [\Phi | \Psi | \Xi] \quad \text{(Bridge-E)} \]

\[ \Gamma \gg \lambda^2 x. P \circ r \doteq \lambda^2 x. P' \circ r \in A(r/x) \quad [\Phi | \Psi | \Xi] \quad \text{(Bridge-\beta)} \]

\[ \Gamma \gg Q \in \text{Bridge}_{x, A}(M_0, M_1) \quad [\Phi | \Psi | \Xi] \]

\[ \Gamma \gg Q \circ \varepsilon \doteq M_\varepsilon \in A(\varepsilon/x) \quad [\Phi | \Psi | \Xi] \quad \text{(Bridge-\beta_\varepsilon)} \]

\[ \Gamma \gg Q \doteq \lambda^2 y. Q \circ y \in \text{Bridge}_{x, A}(M_0, M_1) \quad [\Phi | \Psi | \Xi] \quad \text{(Bridge-\eta)} \]
8.4 Extent

\[ \Gamma^r, x \gg A \text{ type}_{\text{Kan}} [\Phi^r, x] \quad \Gamma^r, x, d: A \gg B \text{ type}_{\text{Kan}} [\Phi^r, x] \]
\[ \Gamma \gg M \gg M' \in A(0/x) [\Phi \mid \Xi] \quad \Gamma^r, a: A(0/x) \gg N \gg N' \in B(0/x) [\Phi^r \mid \Xi] \]
\[ \Gamma^r, a': A(1/x) \gg P \gg P' \in B(1/x) [\Phi^r \mid \Xi] \]
\[ \Gamma^r, a: A(0/x), a': A(1/x), c: \text{Bridge}_{x,A}(a, a') \gg Q \gg Q' \in \text{Bridge}_{x,B}(a, a') [\Phi^r \mid \Xi] \]
\[ \Gamma \gg \text{ext}_{\Gamma}(M; a, N, a', P, a.a'.c.Q) \cong \text{ext}_{\Gamma}(M'; a, N', a'.P, a.a'.c.Q') \in B(0/x)/M/d [\Phi \mid \Xi] \quad (\text{ex}) \]

\[ \Gamma \gg A \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \quad \Gamma, d: A \gg B \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \]
\[ \Gamma \gg M \gg M = A \gg B \text{ type}_{\text{Kan}} \]
\[ \Gamma \gg \text{ext}_{\Gamma}(M; a, N, a', P, a.a'.c.Q) \cong N/M/a \in B[M/d] [\Phi \mid \Xi] \quad (\text{ex}-\beta_0) \]
\[ \Gamma \gg A \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \quad \Gamma, d: A \gg B \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \]
\[ \Gamma \gg M \gg M = A \gg B \text{ type}_{\text{Kan}} \]
\[ \Gamma \gg \text{ext}_{\Gamma}(M; a, N, a', P, a.a'.c.Q) \cong P[M/a'] \in B[M/d] [\Phi \mid \Xi] \quad (\text{ex}-\beta_1) \]

\[ \Gamma \gg A \text{ type}_{\text{Kan}} [\Phi^r, x] \quad \Gamma \gg A(0/x), a': A(1/x), c: \text{Bridge}_{x,A}(a, a') \gg Q \gg Q' \in \text{Bridge}_{x,B}(a, a') [\Phi^r \mid \Xi] \]
\[ \Gamma^r, a: A(0/x) \gg N \gg N' \in B(0/x) [\Phi^r \mid \Xi] \]
\[ \Gamma^r, a': A(1/x) \gg P \gg P' \in B(1/x) [\Phi^r \mid \Xi] \]
\[ \Gamma \gg \text{ext}_{\Gamma}(M(0/x); a, N, a', P, a.a'.c.Q) = T \in B[M/d](0/x) [\Phi \mid \Xi] \]
\[ \text{where } T := Q[M(0/x)/a][M(1/x)/a'][\lambda^2 x.M/c@r] \quad (\text{ex}-\eta) \]
\[ \Gamma \gg N(0/x)[M/a] = E \in B(0/x)[M/d] [\Phi \mid \Xi] \]
\[ \text{where } E := \text{ext}_{\Gamma}(M; a, N(0/x)[a/d], a'.N(1/x)[a'/d], a.a'.c.N[c@x/d]) \]

8.5 Gel-types

Type

\[ \Gamma^r \gg A \gg A' \text{ type}_{\text{Kan}} [\Phi^r \mid \Xi] \quad \Gamma^r \gg B \gg B' \text{ type}_{\text{Kan}} [\Phi^r \mid \Xi] \]
\[ \Gamma \gg \text{Gel}_{\Gamma}(A, B, a.b.R) \cong \text{Gel}_{\Gamma}(A', B', a.b.R') \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \quad (\text{Gel-F}) \]
\[ \Gamma \gg \text{Gel}_0(A, B, a.b.R) \cong A \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \quad (\text{Gel-F}_0) \]
\[ \Gamma \gg \text{Gel}_1(A, B, a.b.R) \cong B \text{ type}_{\text{Kan}} [\Phi \mid \Xi] \quad (\text{Gel-F}_1) \]

Introduction

\[ \Gamma^r \gg M \gg M' \in A [\Phi^r \mid \Xi] \quad \Gamma^r \gg M \gg M' \in A [\Phi^r \mid \Xi] \]
\[ \Gamma \gg \text{gel}_{\Gamma}(M, N, P) \cong \text{gel}_{\Gamma}(M', N', P') \in \text{Gel}_{\Gamma}(A, B, a.b.R) [\Phi \mid \Xi] \quad (\text{Gel-I}) \]
\[ \Gamma \gg \text{gel}_0(M, N, P) \cong M \in A [\Phi \mid \Xi] \quad (\text{Gel-I}_0) \]
\[ \Gamma \gg \text{gel}_1(M, N, P) \cong N \in B [\Phi \mid \Xi] \quad (\text{Gel-I}_1) \]
Elimination

\[
\Gamma \triangleright A, B \text{ type}_{\text{Kan}} [\Phi \mid \Psi \mid \Xi] \quad \Gamma^n, a : A, b : B \triangleright R \text{ type}_{\text{Kan}} [\Phi^n \mid \Psi^n \mid \Xi^n] \\
\Gamma, x \triangleright Q \equiv Q' \in \text{Gel}_{x}(A, B, a.b.R) [\Phi, x \mid \Psi \mid \Xi] \\
\Gamma \triangleright \text{ungel}(x.Q) \equiv \text{ungel}(x.Q') \in R[Q(0/x)/a][Q(1/x)/b] [\Phi \mid \Psi \mid \Xi] \quad \text{(Gel-E)}
\]

\[
\Gamma \triangleright M \in A [\Phi \mid \Psi \mid \Xi] \quad \Gamma \triangleright N \in B [\Phi \mid \Psi \mid \Xi] \quad \Gamma \triangleright P \in R[M/a][N/b] [\Phi \mid \Psi \mid \Xi] \\
\Gamma \triangleright \text{ungel}(x.\text{gel}_x(M, N, P)) \equiv P \in R[M/a][N/b] [\Phi \mid \Psi \mid \Xi] \quad \text{(Gel-\beta)}
\]

\[
\Gamma, x \triangleright Q \in \text{Gel}_x(A, B, a.b.R) [\Phi, x \mid \Psi \mid \Xi] \\
\Gamma \triangleright Q(r/x) \equiv \text{gel}_x(Q(0/x), Q(1/x), \text{ungel}(x.Q)) \in \text{Gel}_{x}(A, B, a.b.R) [\Phi \mid \Psi \mid \Xi] \quad \text{(Gel-\eta)}
\]

9 Relativity

With all the pieces in place, we can prove the long-awaited correspondence between bridges in the universe \(U\) and type-valued relations. For the remainder of this section, we fix \(A, B \in U [\Phi \mid \Psi]\).

Notation 9.1. For \(R \in A \times B \to U [\Phi \mid \Psi]\), we abbreviate \(\text{Gel}_r(A, B, a.b.R(a, b))\) as \(\text{Gel}_r(A, B, R)\). We will also avail ourselves of pattern-matching notation for products: \(\lambda(a, b).M\) is short for \(\lambda t. M[fst(t)/a][snd(t)/b]\).

Lemma 9.2. For any \(R \in A \times B \to U [\Phi \mid \Psi]\), \(M \in A [\Phi \mid \Psi]\), and \(N \in B [\Phi \mid \Psi]\), the types \(R(M, N)\) and \(\text{Bridge}_{x.\text{Gel}_x(A, B, R)}(M, N)\) are equivalent.

Proof. By Recollection 4.7. In the forward direction we send \(p\) to \(\lambda x. \text{gel}_x(M, N, p)\), while in the reverse direction we send \(q\) to \(\text{ungel}(x.q@x)\). These are mutual inverses (up to exact equality, in fact) by the \(\beta\) and \(\eta\) rules for Gel-types (Rules 7.12 and 7.14).

\[\text{Theorem 9.3 (Relativity). The map} \quad \text{bridge-rel} := \lambda C. \lambda(a, b). \text{Bridge}_{x.\text{C}@x}(a, b) \in \text{Bridge}_{U}(A, B) \to (A \times B \to U) [\Phi \mid \Psi] \]

is an equivalence.

Proof. By Recollection 4.7. For our candidate inverse, we take \(ra(A, B; -)\) defined by

\[ra(A, B; R) := \lambda x. \text{Gel}_x(A, B, R) \in \text{Bridge}_{U}(A, B) [\Phi \mid \Psi].\]

For the first inverse condition, we have \(R : A \times B \to U\) and want a path from \(\text{bridge-rel}(ra(A, B; R))\) to \(R\) in \(A \times B \to U\). The former is equal to \(\lambda(a, b). \text{Bridge}_{x.\text{Gel}_x(A, B, R)}(a, b)\) by \(\beta\)-reduction for Bridge-types. For all \(a : A\) and \(b : B\), the type \(\text{Bridge}_{x.\text{Gel}_x(A, B, R)}(a, b)\) is equivalent to \(R(a, b)\) by Lemma 9.2. By univalence and function extensionality for paths, we thus have a path between \(\lambda(a, b). \text{Bridge}_{x.\text{Gel}_x(A, B, R)}(a, b)\) and \(\lambda(a, b). R(a, b)\), the latter of which is equal to \(R\) by the \(\eta\)-rules for product and function types.

For the second inverse condition, we have \(C : \text{Bridge}_{U}(A, B)\) and want a path from \(ra(A, B; \text{bridge-rel}(C))\) to \(C\) in \(\text{Bridge}_{U}(A, B)\). The former is equal to \(\lambda x. \text{Gel}_x(A, B, a.b.\text{Bridge}_{x.\text{C}@x}(a, b))\), so we want to construct a term of type

\[\text{Path}_{\text{Bridge}_{U}(A, B)}(\lambda x. \text{Gel}_x(A, B, a.b.\text{Bridge}_{x.\text{C}@x}(a, b)), \lambda x. \text{C}@x).\]

By swapping the bridge and path binders (Theorem 6.2), this is the same as constructing a term of type

\[\text{Bridge}_{x, \text{Path}_{U}(\text{Gel}_x(A, B, a.b.\text{Bridge}_{x.\text{C}@x}(a, b)), \text{C}@x)}(\lambda^1 \rightarrow A, \lambda^1 \rightarrow B).\]

By the univalence axiom and the fact that identity equivalences correspond to constant paths under univalence (see [HoTT, §2.10]), this is equivalent to constructing a path of type

\[\text{Bridge}_{x. \text{Equiv}(\text{Gel}_x(A, B, a.b.\text{Bridge}_{x.\text{C}@x}(a, b)), \text{C}@x)}(\text{ideq}(A), \text{ideq}(B)).\]
By applying Corollary 6.10, we can further reduce this to constructing a term of the following type.

\[(a:A) \to (b:B) \to (\text{Bridge}_{x,Gel_{x}(A,B,a,b)}(a,b) \simeq \text{Bridge}_{x,C@x}(a,b)).\]

Finally, we have such a term by Lemma 9.2.

**Remark 9.4.** Beyond its use in the proof of relativity (via Corollary 6.10), we observe that extent is also necessary to derive its higher-dimensional instances. For example, consider a two-dimensional bridge in the universe as shown below.

```
\[
\begin{array}{c}
\text{by relativity, the type } \text{Bridge}_{y,\text{Bridge}_{x}(Q_{0},Q_{1})}(\lambda^{2}x.P_{0},\lambda^{2}x.P_{1}) \text{ of fillers for this square is equivalent to the following type.}
\text{Bridge}_{y,Q_{0} \times Q_{1} \to U}(\lambda q_{0}.\lambda q_{1}. \text{Bridge}_{x,P_{0}}(q_{0},q_{1}),\lambda q_{0}.\lambda q_{1}. \text{Bridge}_{x,P_{1}}(q_{0},q_{1}))
\end{array}
```

To simplify further, we need a characterization of bridges in a function type. With Theorem 6.9 and a second application of relativity, we see that the type is indeed equivalent to the following type of two-dimensional relations on the boundary of the square.

\[
\begin{array}{c}
(a_{00}:A_{00})(a_{01}:A_{01})(a_{10}:A_{10})(a_{11}:A_{11})
\to \text{Bridge}_{x,P_{0}}(a_{00},a_{10}) \to \text{Bridge}_{x,P_{1}}(a_{01},a_{11})
\to \text{Bridge}_{y,Q_{1}}(a_{00},a_{01}) \to \text{Bridge}_{y,Q_{1}}(a_{10},a_{11})
\to U
\end{array}
\]

We will make use of such a two-dimensional relation (in a slightly massaged form) in our proof that the booleans are bridge-discrete (Section 11.3).

## 10 Bridge-discrete types

Part of the standard parametricity toolkit is the *identity extension lemma* [Reynolds, 1983], which implies in particular that the relational interpretation of a closed type is the identity relation. In our setting, the analogous result would be to have \(\text{Bridge}_{x,A}(M,M') \simeq \text{Path}_{x,A}(M,M')\) whenever \(x\) does not occur in \(A\). However, we follow Bernardy et al. in imposing no such condition on types. The condition is of course violated by the universe \(U\), where bridges are relations but paths are equivalences; as the theory stands, we could consistently impose it on types in \(U\) (i.e., small types), but it is debatable whether this is desirable. For example, Nuyts et al. [2017] productively use a type \(\text{Size}\) which has discrete \(\text{Path}\) but codiscrete \(\text{Bridge}\) structure. Moreover, we would need to give a computational interpretation of such an axiom in any case. We will therefore take a conservative approach: we internally define a sub-universe \(U_{\text{BDisc}}\) of *bridge-discrete types*, show it is closed under various type formers, and use it in place of \(U\) when the condition is necessary for a proof.

There is a canonical family of maps taking paths in \(A\) to bridges in \(A\). We will say that \(A\) is bridge-discrete when this map is an equivalence. This choice of definition ensures that bridge-discreteness is a *proposition*, that is, that any two proofs that \(A\) is bridge-discrete are equal up to a path. However, we observe below that it suffices to construct any family of equivalences between \(\text{Bridge}_{A}(-,-)\) and \(\text{Path}_{A}(-,-)\); indeed, it is enough to show that the former is a retract of the latter. (This is a consequence of a standard lemma used to characterize path spaces in homotopy type theory.)
Definition 10.1. Given a type $\text{type}_{Kan} [\Phi | \Psi]$, $M, N \in A [\Phi | \Psi]$, and $P \in \text{Path}_A(M, N) [\Phi | \Psi]$, we define a term

$$\text{loosen}_A(P) \in \text{Bridge}_A(M, N) [\Phi | \Psi]$$

by

$$\text{loosen}_A(P) := \text{coe}_{\text{Bridge}_A}^{\text{loosen}_A(P)}_{P}(\lambda z.P @ 0).$$

Lemma 10.2. For any $A \text{type}_{Kan} [\Phi | \Psi]$ and $M \in A [\Phi | \Psi]$, we have a term

$$\text{loosen-refl}_A(M) \in \text{Path}_{\text{Bridge}_A(M, M)}(\text{loosen}_A(\lambda \lambda z.\text{Bridge}_A(M, M)), \lambda^2 z.\text{Bridge}_A(M, M)) [\Phi | \Psi].$$

Proof. Take $\text{loosen-refl}_A(M) := \lambda^z.\text{coe}_{\text{Bridge}_A}^{\text{loosen}_A(\lambda \lambda z.\text{Bridge}_A(M, M)), \lambda^2 z.\text{Bridge}_A(M, M)}(\lambda^2 z.\text{Bridge}_A(M, M)) [\Phi | \Psi].$

Definition 10.3. We say that $A \text{type}_{Kan} [\Phi | \Psi]$ is bridge-discrete when the type

$$\text{isBDisc}(A) := (a, a' : A) \to \text{isEquiv}(\text{Path}_A(a, a'), \text{Bridge}_A(a, a'), \lambda p.\text{loosen}_A(p))$$

is inhabited.

Lemma 10.4. For any $A \text{type}_{Kan} [\Phi | \Psi]$, $\text{isBDisc}(A)$ is a proposition.

Proof. By Recollection 4.6 and the fact that a function type is a proposition when its codomain is a proposition [HoTT, Example 3.6.2].

Definition 10.5. Given $A, B \text{type}_{Kan} [\Phi | \Psi]$ and $G \in B \to A [\Phi | \Psi]$ we define the type

$$\text{Retract}(A, B) := (f : A \to B) \times (g : B \to A) \times (a : A) \to \text{Path}_A(g(f(a)), a).$$

When $\text{Retract}(A, B)$ is inhabited, we say that $A$ is a retract of $B$.

Lemma 10.6. Let $A \in U [\Phi | \Psi]$ and $a, a' : A \gg R \text{type}_{Kan} [\Phi | \Psi]$ be given. If there exists a family of retracts $S \in (a, a' : A) \to \text{Retract}(\text{Bridge}_A(a, a'), \text{Path}_A(a, a')) [\Phi | \Psi]$, then $\text{fst}(S(a'))$ is an equivalence for all $a, a' : A$.

Proof. This is a known result in homotopy type theory, but we sketch the proof for lack of a convenient reference. The term $S$ straightforwardly gives rise to a family of retracts of total spaces, a term of the following type.

$$(a : A) \to \text{Retract}((a' : A) \times \text{Bridge}_A(a, a'), (a' : A) \times \text{Path}_A(a, a'))$$

For each $a : A$, the type $(a' : A) \times \text{Path}_A(a, a')$ is contractible by Recollection 4.4: any retract of a contractible type is also contractible [HoTT, Lemma 3.11.7], so $(a' : A) \times \text{Bridge}_A(a, a')$ is as well. Any function between contractible types is an equivalence, so in particular the function

$$a : A \gg \lambda (a', p).((a', \text{loosen}_A(p)) \in ((a' : A) \times \text{Path}_A(a, a')) \to ((a' : A) \times \text{Bridge}_A(a, a')) [\Phi | \Psi]$$

is an equivalence. This equivalence on total spaces implies a fiberwise equivalence: for every $a' : A$, the function $\lambda p.\text{loosen}_A(p)$ is an equivalence [HoTT, Theorem 4.7.7].

Corollary 10.7. Let $A \in U [\Phi | \Psi]$. If $\text{Bridge}_A(a, a')$ is a retract of $\text{Path}_A(a, a')$ for all $a, a' : A$, then $A$ is bridge-discrete. In particular, if $\text{Bridge}_A(a, a')$ and $\text{Path}_A(a, a')$ are equivalent for all $a, a' : A$, then $A$ is bridge-discrete.

Definition 10.8. We define the sub-universe of bridge-discrete types by

$$U_{\text{BDisc}} := (X U) \times \text{isBDisc}(X).$$

Lemma 10.9. $U_{\text{BDisc}}$ is closed under pair, function, Path-, and Bridge-types, in the sense that each compound type is bridge-discrete when its component types are bridge-discrete.

Proof. These are straightforward corollaries of Theorems 6.1, 6.2 and 6.9.

With a little more work, we can show that $U_{\text{BDisc}}$ is also relativistic, in the sense that bridges in $U_{\text{BDisc}}$ correspond to $U_{\text{BDisc}}$-valued relations on the first components of their endpoints.

---

2See https://github.com/HoTT/book/issues/718.
Lemma 10.10. Let $A$ type$_{\text{kan}}$ $[\Phi, x \mid \Psi]$ and $P \in \text{isProp}(A) [\Phi, x \mid \Psi]$ be given. For any $M_0 \in A(0/x) [\Phi \mid \Psi]$ and $M_1 \in A(1/x) [\Phi \mid \Psi]$, the type $\text{Bridge}_{\text{A}}(M_0, M_1)$ is a proposition.

Proof. Let $q_0, q_1 : \text{Bridge}_{\text{A}}(M_0, M_1)$. Then we have

$$
\lambda y. x. \text{hcom}_A^{0 \rightarrow 1} \left( \begin{array}{ll}
q_0 \otimes x; \\
x = 0 & \rightarrow z. P M_0 M_0 \otimes z \\
x = 1 & \rightarrow z. P M_1 M_1 \otimes z \\
y = 0 & \rightarrow z. P (q_0 \otimes x) (q_0 \otimes x) \otimes z \\
y = 1 & \rightarrow z. P (q_0 \otimes x) (q_1 \otimes x) \otimes z 
\end{array} \right)
$$

of type $\text{Path}_{\text{Bridge}_{\text{A}}(M_0, M_1)}(q_0, q_1)$.

Lemma 10.11. For any $A$ type$_{\text{kan}}$ $[\Phi, x \mid \Psi]$, $D_0 \in \text{isBDisc}(A(0/x)) [\Phi \mid \Psi]$, $D_1 \in \text{isBDisc}(A(1/x)) [\Phi \mid \Psi]$, there is an equivalence $\text{Bridge}_{\text{A}}(D_0, D_1) \simeq (a_0 : A(0/x)) (a_1 : A(1/x)) \rightarrow \text{isBDisc}(\text{Bridge}_{\text{A}}(a_0, a_1))$.

Proof. The left-hand type is a proposition by Lemma 10.4 and Lemma 10.10, while the right-hand type is a proposition by Lemma 10.4 and the fact that a function type with propositional codomain is propositional [HoTT, Example 3.6.2]. Using Recollection 4.7, it therefore suffices to construct any pair of functions between the two.

In the forward direction, we are given $t : \text{Bridge}_{\text{A}}(D_0, D_1)$, $a_0 : A(0/x)$, $a_1 : A(1/x)$, and we need to show $\text{Bridge}_{\text{A}}(a_0, a_1)$ is bridge-discrete. By Corollary 10.7, it suffices to show that for any $q, q' : \text{Bridge}_{\text{A}}(a_0, a_1)$ we have an equivalence $\text{Bridge}_{\text{A}}(a_0, a_1)(q, q') \simeq \text{Path}_{\text{Bridge}_{\text{A}}(a_0, a_1)}(q, q')$. This follows from the chain of equivalences

$\text{Bridge}_{\text{A}}(a_0, a_1)(q, q') \simeq \text{Bridge}_{\text{A}}\left(\lambda a. a, \lambda a'. a' \mid \lambda a. a', \lambda a'. a' \mid \text{loosen}_A, D_0, a, a' \mid \text{loosen}_A, D_1, a, a' \right)$

where the center equivalence is given by $t \otimes x$ and Lemma 10.2, while the first and third are simply rearranging binders.

In the reverse direction, we are given $u : (a_0 : A(0/x)) (a_1 : A(1/x)) \rightarrow \text{isBDisc}(\text{Bridge}_{\text{A}}(a_0, a_1))$. We can construct a term of the desired type $\text{Bridge}_{\text{A}}(D_0, D_1)$ from a term of type $\text{Bridge}_{\text{A}}(a_0, a_1)$ by Corollary 10.7 (using Recollection 4.6 to adjust endpoints if necessary). By Theorem 6.9 applied twice and Corollary 6.10, we can in turn construct such a term from a term of the following type

$\text{Bridge}_{\text{A}}(a_0, a' : D) \rightarrow (\text{Path}_{\text{A}}(a_0, a')) (\lambda a, \lambda a' \mid \text{loosen}_A, D_0, a \mid \text{loosen}_A, D_1, a')$

By Corollary 10.7 (using Recollection 4.6 to adjust endpoints if necessary), we can in turn construct such a term from a term of the following type

$\lambda a. (a_0 : A(0/x)) (a_1 : A(1/x)) (\pi : \text{Bridge}_{\text{A}}(a_0, a_1))$

Finally, by rearranging the binders on either side of the equivalence, the codomain of this type is equivalent to the type $\text{Path}_{\text{Bridge}_{\text{A}}(a_0, a_1)}(\pi, \pi') \simeq \text{Bridge}_{\text{A}}(a_0, a_1)(\pi, \pi')$, which is inhabited by $u a_0 a_1$. \hfill \Box

Theorem 10.12. For any $\tilde{A}, \tilde{B} \in \text{UBDisc} [\Phi \mid \Psi]$, we have $\text{Bridge}_{\text{UBDisc}}(\tilde{A}, \tilde{B}) \simeq \text{fst}(\tilde{A}) \times \text{fst}(\tilde{B}) \rightarrow \text{UBDisc}$. 32
Remark 10.13. Although we will not give a proof here, one can show by a similar argument that the sub-universe $\mathcal{U}_{\text{prop}} := (X:U) \times \text{isProp}(X)$ of propositions is relativistic (has bridges corresponding to $\mathcal{U}_{\text{prop}}$-valued relations), and more generally that the sub-universe of $n$-types is relativistic for all $n \geq -2$. It is not clear to us whether there is a general theorem which has these results and Theorem 10.12 as corollaries.

11 Examples

In this final section, we give five examples of results which can be proven within parametric cubical type theory. As a warm-up, we prove in Section 11.1 that all terms of type $(X:U) \rightarrow X \rightarrow X$ are path-equal to the polymorphic identity function $\lambda X.\lambda a.a$. The proof is essentially that given by Bernardy et al. [2015, Example 3.3] in their system, although we obtain a slightly stronger result thanks to function extensionality for paths. In Section 11.2, we show that Leibniz equality is equivalent to path equality for bridge-discrete types, a more involved result of a similar kind. In Section 11.3, we show that the type of booleans with its standard typing rules is bridge-discrete. This proof includes an argument using so-called “iterated parametricity,” i.e., nested $\text{Gel}$-types. As a corollary, we see that there is no bridge between $\text{true}$ and $\text{false}$ in $\text{bool}$, which allows us to refute the law of the excluded middle for propositions in Section 11.4. Finally, in Section 11.5, we give an example of a parametricity result for functions between higher inductive types. Apart from uses of function extensionality at the edges, we expect that the proofs in Sections 11.2 to 11.4 could be repeated in a binary version of the system of Bernardy et al. [2015]; Section 11.5 of course requires a system with higher inductive types.

11.1 Polymorphic identity function

In this section, we show that any term inhabiting the type $(X:U) \rightarrow X \rightarrow X$ is equal to the polymorphic identity function up to a path. As mentioned above, this proof is not novel, but it will serve to introduce the methodology of internal parametricity proofs. We assume the existence of a unit type with a single element $\ast$.

Theorem 11.1. Let $F \in (X:U) \rightarrow X \rightarrow X [\Phi \mid \Psi]$ be given. Then there is a path from $F$ to $\lambda X.\lambda a.a$.

Proof. Set $\Gamma := (X:U, a:X)$. Define the relation $\Gamma \gg R := \lambda (a',\underline{\_}).\text{Path}_X(a',a) \in X \times \text{unit} \rightarrow U [\Phi \mid \Psi]$. Abstracting over a bridge dimension variable $x$, we can apply $F$ first at the $\text{Gel}$-type given by this relation, then at its inhabitant $\text{gel}_x(a,\ast,\lambda X.a)$ expressing that $R$ relates $a$ and $\ast$.

$$F(\text{gel}_x(X, \text{unit}, R))(\text{gel}_x(a,\ast,\lambda X.a)) \in \text{Gel}_x(X, \text{unit}, R)$$

This is a bridge over $x.\text{Gel}_x(X, \text{unit}, R)$ whose endpoints reduce to $F X a$ (at $x = 0$) and $F(\text{unit})(\ast)$ (at $x = 1$). By applying $\text{ungel}$, we turn this into a proof that the two endpoints are related.

$$\text{ungel}(x.F(\text{gel}_x(X, \text{unit}, R))(\text{gel}_x(a,\ast,\lambda X.a))) \in R(F X a, F(\text{unit})(\ast))$$

By definition of $R$, this means we have a term of type $\text{Path}_X(F X a, a)$ in context $\Gamma$. Function extensionality for paths now gives the desired result. \qed
We note that, despite the theorem above, not every term of type \((X:U) \to X \to X\) is exactly equal to \(\lambda a.a\); indeed, there are functions such as \(\lambda X.\lambda a.\coe^{0\to 1}_{X}(a)\) which are not exactly equal to the identity function. The theorem above shows that such functions are the same as \(\lambda a.a\) up to a path. The example of \(\lambda X.\lambda a.\coe^{0\to 1}_{X}(a)\) exposes a notable feature of parametric cubical type theory: we can prove uniformity theorems up to a path even in the presence of Kan operations which evaluate by case analysis on their type arguments.

As a simple application, we can use Theorem 11.1 to prove some contractibility results for polymorphic operators on paths.

**Corollary 11.2.** The types

1. \((X:U)(a:X) \to \Path_{X}(a,a),\)
2. \((X:U)(a,b:X) \to \Path_{X}(a,b) \to \Path_{X}(b,a),\)
3. \((X:U)(a,b,c:X) \to \Path_{X}(a,b) \to \Path_{X}(b,c) \to \Path_{X}(a,c),\)

are all contractible.

**Proof.** These three types are pairwise equivalent by way of Recollection 4.4, so it suffices to prove that the first is contractible. By function extensionality for paths, we have the following equivalence.

\[ ((X:U) \to (a:X) \to \Path_{X}(a,a)) \simeq \Path((X:U) \to X \to X)(\lambda a.a, \lambda a.a) \]

The path type of a contractible type is contractible [HoTT, Lemmas 3.11.3 and 3.11.10], so it follows from Theorem 11.1 that the right-hand type is contractible, whence the left-hand type is as well.

---

### 11.2 Leibniz equality

As a second example of characterizing a polymorphic function type, we show that the type \(\Path_{A}(a,a')\) of paths between elements \(a,a'\) of a bridge-discrete type \(A\) is equivalent to the type \((X:A \to U) \to Xa \to Xa'\) of proofs that they are Leibniz equal. This example highlights the role of bridge-discreteness, an assumption that will generally appear in parametricity theorems involving a fixed type (here \(A\)). In short, bridge-discreteness of \(A\) ensures that the relational interpretation \(\Bridge_{a,A \to U}(P,Q)\) is equivalent to the type \((a:A) \to Pa \times Qa \to U\) of pointwise relations on \(P\) and \(Q\), not only to the type \((a,a':A) \to \Bridge_{A}(a,a') \to Pa \times Qa' \to U\) as it is in the general case.

**Theorem 11.3.** Let \(A \in U [\Phi | \Psi]\) be a bridge-discrete type. Then we have a family of equivalences \((a,a':A) \to ((X:A \to U) \to Xa \to Xa' \simeq \Path_{A}(a,a')).\)

**Proof.** As \(A\) is bridge-discrete, we have a family of functions \(T \in (a,a':A) \to \Bridge_{A}(a,a') \to \Path_{A}(a,a')\) which are form an inverse to \(\loosen_{A}\). Abbreviate \(E_{A}(a,a') := (X:A \to U) \to Xa \to Xa'.\) By Lemma 10.6, it suffices to show that \(E_{A}(a,a')\) is a retract of \(\Path_{A}(a,a')\) for every \(a,a' : A\). In the forward direction, we have the following family of functions.

\[ \text{in}_{A} := \lambda a.\lambda a'.\lambda f.(\lambda b.\Path_{A}(a,b))(\lambda^{1}a) \in (a,a':A) \to E_{A}(a,a') \to \Path_{A}(a,a') \]

Conversely, the following gives us a family of candidate inverses.

\[ \text{out}_{A} := \lambda a.\lambda a'.\lambda q.\lambda X.\lambda u.\coe^{0\to 1}_{X(q@z)}(u) \in (a,a':A) \to \Path_{A}(a,a') \to E_{A}(a,a') \]

To show that these form a retract, we use a parametricity argument. Set \(\Gamma := (a,a' : A, X : A \to U, u : Xa).\) We define a family of relations \(R \in (b,b':A) \to \Bridge_{A}(b,b') \to (\Path_{A}(a,b) \times Xb') \to U\) in this context as follows.

\[ R := \lambda b.\lambda b'.\lambda q.\lambda (p,v).\Path_{X}(X(Tb'b'q@z))(\text{out}_{A}aa'pXu,v) \]
We have some \( L \in \Raa(\lambda^2 a)(\lambda^1 a, u) \), as we have a path \( \lambda^2 y.\coce^{\text{path}}_{X_X}(u) \in \Path_{\lambda^1 a}(\out_{\lambda^2 a}(\lambda^1 a)Xu, u) \) and know that \( Xa \) is path-equal to \( X(T\lambda^2 a)(\lambda^1 a)@x \) by Lemma 10.2.

Let \( f : E_A(a, a') \), and choose \( x \) fresh. We apply \( f \) to the relation \( R \) and its inhabitant \( L \) in the following way, using \( \text{extent} \) to create a bridge in the function type \( A \to U \).

\[
\begin{align*}
  f(\lambda x.\text{extent}_x(t; b, \Path_A(a, b), b'.Xb', b.b'.q.\lambda x.\text{gel}_{x}(\Path_A(a, b'), Xb, Rbb'q)))(\text{gel}_{x}(\lambda^1 a, u, L))
\end{align*}
\]

This is a bridge over \( x \) \( \text{gel}_{x}(\Path_A(a, b'), Xa', R'a'(\lambda^2 a')) \) when \( \in_{\lambda^2 a'} f \) and \( fXu \). Applying \( \text{ungel} \) thus gives a term of type \( \Path_{\lambda^2 a'}X(T\lambda' a'(\lambda^2 a')@x)(\out_{\lambda^2 a'}(\in_{\lambda^2 a'} f)Xu, fXu) \). We recall that \( T\lambda' a'(\lambda^2 a')@x \) is path-equal to \( a' \) by Lemma 10.2, so this gives a term of type \( \Path_{\lambda^2 a'}(\out_{\lambda^2 a'}(\in_{\lambda^2 a'} f)Xu, fXu) \). Finally, we obtain a term of type \( \Path_{E_A(a, a')}(\out_{\lambda^2 a'}(\in_{\lambda^2 a'} f), f) \) by function extensionality for paths.

\[\Box\]

### 11.3 Bridges of booleans

For this example, we assume the existence of a boolean type \( \text{bool} \) with elements \( \text{true}, \text{false} \) and an eliminator \( \text{if} \). For convenience, we assume an exact \( \eta \)-principle \( b : \text{bool} \Rightarrow b \equiv \text{if} \text{bool}(b; \text{true}, \text{false}) \in \text{bool} [\Phi \mid \Psi] \). This principle is derivable up to a path in any case, but having an exact equality simplifies the proofs. (It holds for the boolean type defined in [Angiuli et al., 2017b], but not for the “weak boolean” type also defined there.)

A standard argument characterizes the type \( \Path_{\text{bool}}(b, b') \) for all \( b, b' \): it is contractible when \( b \equiv b' \equiv \text{true} \) or \( b \equiv b' \equiv \text{false} \), and it is empty when \( b \equiv \text{true} \) and \( b' \equiv \text{false} \) or vice versa. In this section, we will show that \( \text{bool} \) is bridge-discrete, meaning that \( \text{Bridge}_{\text{bool}}(b, b') \) satisfies the same characterization. Where the calculation of \( \Path_{\text{bool}} \) uses a universe, our calculation of \( \text{Bridge}_{\text{bool}} \) will also use the fact that the universe is relativistic, i.e., that it is closed under \( \text{Gel} \)-types. This situation has an interesting parallel in the case of higher inductive types, where univalence is used to calculate path types.

Before proceeding with the proof, we can give an intuitive argument why relativity generally enables the characterization of bridges in inductive types. As Section 11.1 effectively does for the unit type, we can use relativity to prove that an inductive type is equivalent to its Church encoding; in the case of \( \text{bool} \), this is \( (XU) \to X \to X \to X \). When the Church encoding is composed of types whose bridges we already understand (such as \( U \) and \( \to \) ), we can calculate its bridge type; for \( \text{bool} \), we see that \( \text{Bridge}_{(XU) \to X \to X \to X}(G_0, G_1) \) is equivalent to the following type.

\[
\begin{align*}
  (X_0, X_1; U)(\overline{X}:X_0 \times X_1 \to U)(t_0, f_0:X_0)(t_1, f_1:X_1)\langle f(X(t_0, t_1)), \overline{X}(f_0, f_1) \rangle \to \overline{X}(G_0 X_0 t_0 f_0, G_1 X_1 t_1 f_1)
\end{align*}
\]

With a second parametricity argument, we can then show that this type is an encoding for the (appropriate index of) \( \Path_{\text{bool}} \). While our proof of Theorem 11.5 is somewhat more direct (we do not explicitly use the Church encoding), its conceptual structure follows this outline.

**Definition 11.4.** For any \( A \in U \ [\Phi \mid \Psi] \), define \( P_r(A) := \text{Gel}_{r}(A, A, a.a'.\Path_A(a, a')) \).

**Theorem 11.5.** \( \text{bool} \) is bridge-discrete.

**Proof.** By Corollary 10.7, it suffices to show that \( \text{Bridge}_{\text{bool}}(M_0, M_1) \) is a retract of \( \Path_{\text{bool}}(M_0, M_1) \) for every \( M_0, M_1 \). For any \( Q \in \text{Bridge}_{\text{bool}}(M_0, M_1) [\Phi \mid \Psi] \), we first define \( \text{tighten}_a(Q) \in P_{a}(\text{bool}) [\Phi, x \mid \Psi] \) by

\[
\text{tighten}_a(Q) := \text{if} \text{p}_{a}(\text{bool})(Q \otimes x; \text{gel}_{x}(\text{true}, \text{true}, \lambda^1 \text{true}), \text{gel}_{x}(\text{false}, \text{false}, \lambda^1 \text{false})).
\]

Observe that \( \text{tighten}_a(Q) \) has endpoints \( \text{tighten}_a(Q) \equiv \text{if} \text{bool}(M; \text{true}, \text{false}) \equiv M \) for \( \varepsilon \in \{0, 1\} \). Thus we may set \( \text{tighten}(Q) := \text{ungel}(x; \text{tighten}_a(Q)) \in \Path_{\text{bool}}(M_0, M_1) (\Phi \mid \Psi) \). For the map in the other direction, we take the previously-defined \( \text{loosen}_{\text{bool}} \). It remains to construct a path from \( \text{loosen}_{\text{bool}}(\text{tighten}(Q)) \) to \( Q \) for every \( Q \in \text{Bridge}_{\text{bool}}(M_0, M_1) (\Phi \mid \Psi) \).

As a preliminary, we observe that \( \text{tighten} \) takes reflexive bridges to reflexive paths. For \( M \in \text{bool} [\Phi \mid \Psi] \) we have a term \( \text{tighten-refl}(M) \in \Path_{\text{Path}_{\text{bool}}}(M, M)(\text{tighten}(\lambda^1 M), \lambda^1 M) (\Phi \mid \Psi) \) defined by case analysis:

\[
\text{tighten-refl}(M) := \text{if} \text{b}_{\text{Path}_{\text{Path}_{\text{bool}}}(b, b)}(\text{tighten}(\lambda^1 b), \lambda^1 b)(M; \lambda^1 \lambda^1 \text{true}, \lambda^1 \lambda^1 \text{false}).
\]
Next, we define a two-dimensional relation \( R \) of type
\[(b_0, b_1, b_10, b_11 : \text{bool}) \rightarrow \text{Path}_{\text{bool}}(b_0, b_10) \times \text{Bridge}_{\text{bool}}(b_10, b_11) \rightarrow \text{Path}_{\text{bool}}(b_10, b_11) \rightarrow U \]
by
\[
R := \lambda b_0. \lambda b_1. \lambda b_10. \lambda b_11. \lambda (p, q) . \lambda (p_0, p_1) . \text{Path}_{\text{bool}}(p_0 \circ q, p_1 \circ q) \left(\text{loosen}_{\text{bool}}(p), q\right).
\]
Pictorially, an inhabitant of \( \text{R}_{b_0} \langle p, q \rangle \rangle (p_0, p_1) \) is a filler for the following square,

We now convert this relation into a two-dimensional bridge in \( U \) using iterated \( \text{Gel} \)-types. As a first step, we define a one-dimensional bridge of relations \( R_x \in \text{P}_x(\text{bool}) \times \text{bool} \rightarrow U [\Phi, x | \Psi] \) as the following term.

\[
R_x := \lambda t . \text{extent}_x \left(\begin{array}{cc}
(b_0, b_1) . \text{Path}_{\text{bool}}(b_0, b_1), t ; (b_10, b_11) . \text{Path}_{\text{bool}}(b_10, b_11), u.
& \\
\text{Gel}_x(\text{Path}_{\text{bool}}(b_0, b_1), \text{Path}_{\text{bool}}(b_10, b_11), \text{R}_{b_0} \langle \text{ungel}(x . \text{fst}(u @ x)), \lambda^2 x . \text{snd}(u @ x) \rangle) &
\end{array}\right)
\]

Now we apply a second \( \text{Gel} \), defining \( R_{x,y} := \text{Gel}_y(\text{P}_x(\text{bool}), \text{bool}, R_x) \in U [\Phi, x, y | \Psi] \). Observe that this type square satisfies the following boundary conditions.

\[
R_{0,y} \doteq P_y(\text{bool}) \quad R_{1,y} \doteq P_y(\text{bool}) \quad R_{x,0} \doteq P_x(\text{bool}) \quad R_{x,1} \doteq \text{bool}
\]

We next define two terms \( T_{x,y} \in R_{x,y} [\Phi, x, y | \Psi] \) as follows.

\[
T_{x,y} := \text{gel}_y(\text{gel}_x(\text{true}, \text{true}, \lambda^1 \_ \text{true}), \text{true}, \text{gel}_x(\lambda^1 \_ \text{true}, \lambda^1 \_ \text{true}, \text{loosen-refl}_{\text{bool}}(\text{true})))
\]

\[
F_{x,y} := \text{gel}_y(\text{gel}_x(\text{false}, \text{false}, \lambda^1 \_ \text{false}), \text{false}, \text{gel}_x(\lambda^1 \_ \text{false}, \lambda^1 \_ \text{false}, \text{loosen-refl}_{\text{bool}}(\text{false})))
\]

These terms serve to witness the truth of \( R(\text{true})(\text{true})(\text{true})(\lambda^1 \_ \text{true}, \lambda^1 \_ \text{true})(\lambda^1 \_ \text{true}, \lambda^1 \_ \text{true}) \) and \( R(\text{false})(\text{false})(\text{false})(\lambda^1 \_ \text{false}, \lambda^1 \_ \text{false})(\lambda^1 \_ \text{false}, \lambda^1 \_ \text{false}) \) respectively.

Now, given \( Q \in \text{Bridge}_{\text{bool}}(M_0, M_1) [\Phi | \Psi] \), we define \( I_{x,y} := \text{if}_{\text{R}_{x,y}}(Q @ x; T_{x,y}, F_{x,y}) \in R_{x,y} [\Phi, x, y | \Psi] \).

By inspection of the definition of \( \text{tighten}_x \), we see that this term has the following boundary.

\[
\begin{array}{c}
y \downarrow \quad \downarrow \\
M_0 & M_0 \\
\text{tighten}_y(\lambda^2 \_ M_0) & \text{if}_{\text{R}_{x,y}}(Q @ x; T_{x,y}, F_{x,y}) \\
M_1 & M_1 \\
\text{Q @ x} & \text{Q @ x}
\end{array}
\]

Thus we have \( \text{ungel}(y . I_{x,y}) \in R_x(\text{tighten}_x(Q), Q @ x) [\Phi, x | \Psi] \) with \( \text{ungel}(y . I_{x,y}) \doteq \text{tighten}(\lambda^2 \_ M_\varepsilon) \) for \( \varepsilon \in \{0, 1\} \). In turn, we have

\[
\text{ungel}(x . \text{ungel}(y . I_{x,y})) \in \text{Path}_{\text{z}}(\text{Bridge}_{\text{bool}}(\text{tighten}(\lambda^2 \_ M_0) @ z; \text{tighten}(\lambda^2 \_ M_1) @ z) (\text{loosen}_{\text{bool}}(\text{tighten}(Q)), Q) [\Phi | \Psi].
\]

Finally, we can transform this term into a term of type \( \text{Path}_{\text{z}}(\text{Bridge}_{\text{bool}}(M_0, M_1) (\text{loosen}_{\text{bool}}(\text{tighten}(Q)), Q) \) by rewriting along \( \text{tighten-refl}(M_0) \) and \( \text{tighten-refl}(M_1) \) with \( \text{coe} \).

Assuming the existence of an empty type \( \text{void} \), we have the following corollary.

**Corollary 11.6.** There is a term of type \( \text{Bridge}_{\text{bool}}(\text{true}, \text{false}) \rightarrow \text{void} \).
11.4 Excluding an excluded middle

Exploiting the lack of a bridge between true and false, we can refute the law of excluded middle for propositions as formulated in [HoTT, §3.4]. For other results which follow from the refutation of this principle, see [Booij et al., 2017].

Lemma 11.7. Let $A : \text{type}_{\text{Kan}}[\Phi | \Psi]$ be bridge-discrete. Every function $f : \mathcal{U} \to A$ is constant, in the sense that there exists some $M \in A$ such that $(X : \mathcal{U}) \to \text{Path}_A(fX, M)$ is inhabited.

Proof. Let $f : \mathcal{U} \to A$. Set $M := fA$ (the choice of $A$ here is immaterial). For any $X : \mathcal{U}$, we have a bridge $\lambda^2 x. f(\text{Gel}_x(A, X, \_\_ \_ \_ A)) \in \text{Bridge}_A(fA, fX) \ [\Phi | \Psi]$, which gives rise to a path of type $\text{Path}_A(M, fX)$ by the assumption that $A$ is bridge-discrete.

Theorem 11.8. Define the weak law of the excluded middle $\text{WLEM} : \text{type}_{\text{Kan}}[\Phi | \Psi]$ by

$$\text{WLEM} := (X : \mathcal{U}) \to (b : \text{bool}) \times \text{if}_\mathcal{U}(b; \neg X, \neg \neg X).$$

There is a term of type $\text{WLEM} \to \text{void}$.

Proof. Suppose we have $f : \text{WLEM}$. Then $\lambda X. \text{fst}(\text{WLEM}(X)) \in \mathcal{U} \to \text{bool} \ [\Phi | \Psi]$. By Lemma 11.7 and Theorem 11.5 this function must be constant. On the other hand, it is easy to see that $\text{fst}(f(\text{void}))$ must be true and $\text{fst}(f(\text{unit}))$ must be false. Thus we have a contradiction.

Corollary 11.9. Define the law of the excluded middle $\text{LEM} : \text{type}_{\text{Kan}}[\Phi | \Psi]$ by

$$\text{LEM} := (X : \mathcal{U}) \to \text{isProp}(X) \to (b : \text{bool}) \times \text{if}_\mathcal{U}(b; X, \neg X).$$

Then there is a term of type $\text{LEM} \to \text{void}$.

Proof. $\text{LEM}$ implies $\text{WLEM}$, as any type of the form $\neg A$ is a proposition.

We note that the stronger principle $\text{LEM}_\infty := (X : \mathcal{U}) \to (b : \text{bool}) \times \text{if}_\mathcal{U}(b; X, \neg X)$ is already refutable in homotopy type theory using univalence [HoTT, Corollary 3.2.7]. As with $\text{LEM}_\infty$ in homotopy type theory, the refutation of $\text{LEM}$ need not be taken as a sign that parametric cubical type theory is philosophically anti-classical, merely as a sign that propositionality is not a sufficiently restrictive notion of irrelevance in this setting. In a system with a propositional truncation $\| \_ \|$ which identifies all terms up to path equality, $\text{LEM}$ is equivalent to the principle $(X : \mathcal{U}) \to \| (b : \text{bool}) \times \text{if}_\mathcal{U}(b; X, \neg X) \|$. While this principle is refutable, replacing $\| \_ \|$ with an operator which erases computational content, such as NuPrl’s squash type [Constable et al., 1986, §10.3], gives a principle which is perfectly consistent with parametric cubical type theory.

11.5 Polymorphic functions on higher inductive types

As a final example, we return to the problem of characterizing polymorphic functions, this time between higher inductive types. As our test case, we take the suspension type constructor. Given $A : \text{type}_{\text{Kan}}[\Phi | \Psi]$, its suspension type $\text{susp}(A)$ is generated by three constructors: $\text{north} \in \text{susp}(A)$, $\text{south} \in \text{susp}(A)$, and for every $a : A$ a path $\text{merid}_a(a) \in \text{susp}(A)$ with $\text{merid}_a(a) \cong \text{north}$ and $\text{merid}_a(a) \cong \text{south}$ [HoTT, §6.5]. We write $\text{susp}$-$\text{elim}$ for its eliminator. We refer to Cavallo and Harper [2019] and Coquand et al. [2018] for more complete accounts of higher inductive types in cubical type theory.

We aim to characterize the type $(X : \mathcal{U}) \to \text{susp}(X) \to \text{susp}(X)$ of polymorphic endofunctions on suspensions. Intuitively, such a function is completely determined by where it sends the poles $\text{north}$ and $\text{south}$. If it sends both to $\text{north}$ or both to $\text{south}$, then it must be a constant function. If it sends $\text{north}$ to $\text{north}$ and $\text{south}$ to $\text{south}$, then it must be the identity function. Finally, if it sends $\text{north}$ to $\text{south}$ and $\text{south}$ to $\text{north}$, then it must send each meridian $\lambda^1 y. \text{merid}_y(a)$ to its inverse path (defined using $\text{hcom}$).

Before we prove the main theorem, we first prove a general lemma which pushes suspension past Gel types given by functional relations. This is a result of a kind with the main theorem of Section 11.3, though in this case we need only one direction of the equivalence.
Definition 11.10. Given $F \in A \to B [\Phi']$, define $Gr_r(A, B, F) := \text{Gel}_r(A, B, a.b.\text{Path}_B(Fa,b))$.

Definition 11.11. Given $A, B$ type$_{\text{Kan}} [\Phi | \Psi]$ and $F \in A \to B [\Phi | \Psi]$, define

$$\text{susp-map}_{A,B}(F) := \lambda t.\text{susp-elim}_{\text{susp}(B)}(t; \text{north}, \text{south}, x.a.\text{merid}^x(Fa)) \in \text{susp}(A) \to \text{susp}(B) [\Phi | \Psi].$$

Lemma 11.12. Given $A, B$ type$_{\text{Kan}} [\Phi | \Psi]$ and $T \in \text{susp}(A) [\Phi | \Psi]$, there is a term

$$\text{susp-}η_A(T) \in \text{Path}_{\text{susp}(A)}(T, \text{susp-map}_{A,A}(λa.a)(T)) [\Phi | \Psi].$$

Proof. Set $\text{susp-}η_A(T) := \text{susp-elim}_{\text{Path}_{\text{susp}(A)}(T, \text{susp-map}_{A,A}(λa.a)(T))}(λt.t, λu.u) [\Phi | \Psi]$. □

Lemma 11.13. Given $A, B \in \mathcal{U} [\Phi | \Psi]$ and $F \in A \to B [\Phi | \Psi]$, there is a term

$$\text{get}_{A,B,F} \in \text{Bridge}_x.\text{susp}(Gr_x(A,B,F)) \to \text{Gr}_x(\text{susp}(A),\text{susp}(B),\text{susp-map}_{A,B}(F))(λt,t,λu.u) [\Phi | \Psi].$$

Proof. Define

$$\text{get}_{A,B,F} := \lambda x.\lambda t.\text{hcom}_{C}^{1-0}\left(\text{susp-elim}_{C}(t; N, S, y.g.M); \begin{array}{c} x = 0 \mapsto \text{w-}susp-η_A(t)@w \\ x = 1 \mapsto \text{w-}susp-η_B(t)@w \end{array}\right)$$

where

$$C := \text{Gr}_x(\text{susp}(A),\text{susp}(B),\text{susp-map}_{A,B}(F))$$

$$N := \text{gel}_x(\text{north}, \text{north}, \lambda x.\text{north})$$

$$S := \text{gel}_x(\text{south}, \text{south}, \lambda x.\text{south})$$

$$M := \text{extent}_x(g; a.\text{merid}^g(a), b.\text{merid}^g(b), a.b.u.\text{gel}_x(\text{merid}^g(a), \text{merid}^g(b), \lambda x.\text{merid}^g(\text{ungel}(x.u@x)@z))). □$$

We use the following lemma to see that for any $f : (X \mathcal{U}) \to \text{susp}(X) \to \text{susp}(X)$, the image of a meridian of $\text{susp}(X)$ by $f$ is uniquely determined.

Lemma 11.14. The type $(X \mathcal{U}) \to X \to \text{susp}(X)$ is contractible.

Proof. It suffices to show that $(X \mathcal{U}) \to X \to \text{susp}(X)$ is a retract of $\text{susp}($unit$)$ [HoTT, Lemma 3.11.7]; the latter is contractible by a standard argument. We have maps in each direction as follows.

$$\lambda m.m(\text{unit})(*) \in ((X \mathcal{U}) \to X \to \text{susp}(X)) \to \text{susp}(\text{unit})$$

$$\lambda t.\lambda X.\lambda a.\text{susp-map}_{\text{unit},X}(λa.a)(t) \in \text{susp}(\text{unit}) \to ((X \mathcal{U}) \to X \to \text{susp}(X))$$

To establish that these constitute a retract, we need to show that, for every $m : (X \mathcal{U}) \to X \to \text{susp}(X)$, $X : \mathcal{U}$, and $a : X$, we have a path from $\text{susp-map}_{\text{unit},X}(λa.a)(m(\text{unit})(*))$ to $mA$. We construct such a path with a parametricity argument, using get to extract a path from a suspended Gr-type:

$$\text{ungel}(x.\text{get}_{\text{unit},X,λa.a}\@x(m(\text{Gr}_x(\text{unit}, X, λa.a)))(\text{gel}_x(*,a,λa.a)))$$

has type $\text{Path}_{\text{susp}(X)}(\text{susp-map}_{\text{unit},X}(λa.a)(m(\text{unit})(*)), mA)$. □

Theorem 11.15. There is an equivalence $((X \mathcal{U}) \to \text{susp}(X) \to \text{susp}(X)) ≃ \text{bool} \times \text{bool}$.

Proof. We will construct an equivalence $((X \mathcal{U}) \to \text{susp}(X) \to \text{susp}(X)) ≃ \text{susp}(\text{void}) \times \text{susp}(\text{void})$; it is straightforward to check that $\text{susp}(\text{void})$ is equivalent to bool. As usual, we go by Recollection 4.7. We will construct an inverse to the following map.

$$F := \lambda k.(k(\text{void})(\text{north}), k(\text{void})(\text{south})) \in ((X \mathcal{U}) \to \text{susp}(X) \to \text{susp}(X)) \to \text{susp}(\text{void}) \times \text{susp}(\text{void})$$
| Parametric cubical type theory | Bernardy et al. [2015] |
|-------------------------------|------------------------|
| \(\text{Bridge}_{x,A}(a_0, a_1)\) | \(A \ni_a a\) |
| \(\lambda^2 x.a\) | \(a \cdot x\) |
| \(p \otimes x\) | \((a \cdot x) p\) |
| \(\text{extent}_{x}(b; a_0, t_0, a_1, t_1, a_0, a_1, c, u)\) | \((\lambda a.t_x \cdot \lambda a.\lambda c.u)(b)\) |
| \(\text{Gel}_{x}(A_0, A_1, a_0, a_1, R)\) | \((a : A) \times_{x} R\) |
| \(\text{gel}_{x}(a, 0, a_1, c)\) | \((a, a, c)\) |
| \(\text{ungel}(x.a)\) | \(a \cdot x\) |

Figure 7: Translation dictionary for parametric type theory.

Set \(I := \lambda X. \text{map}_{\text{void}, X}(\lambda v. \text{void-elim}_{\text{map}_{\text{void}, X}(v)}(x) \in (X \mathcal{U}) \rightarrow \text{susp}(\text{void}) \rightarrow \text{susp}(X)\). By Lemma 11.14, we have a term \(C \in (X \mathcal{U})(n, s: \text{susp}(\text{void})) \rightarrow X \rightarrow \text{Path}_{\text{susp}(X)}(IXn, IXs)\), as this type is equivalent to \((n, s: \text{susp}(\text{void})) \rightarrow \text{Path}(X \mathcal{U}) \rightarrow X \rightarrow \text{susp}(X)\) and every path type of a contractible type is contractible [HoTT, Lemmas 3.11.3 and 3.11.10]. We these in hand, we define the candidate inverse map as follows:

\[
G := \lambda (n, s). \lambda X. \lambda t. \text{susp-elim}_{\text{map}_{\text{void}, X}}(t; IXn, IXs, y. a. CXn s a @ y)
\]

It is straightforward to check that for any \(d: \text{susp}(\text{void}) \times \text{susp}(\text{void})\), \(F(Gd)\) is connected by a path to \(d\). For the other inverse condition, we use \(\text{Gel}\)-types. Let \(k : (X \mathcal{U}) \rightarrow \text{susp}(X) \rightarrow \text{susp}(X)\) be given. We can define

\[
P_{\text{north}} := \lambda X. \text{ungel}(x. k(\text{Gr}_{x}(\text{void}, X, \lambda v. \text{void-elim}_{\text{map}_{\text{void}, X}(v)}(x))) (\text{gel}_{x}(\text{north}, \text{north}, \lambda^2 \lambda \text{north})))
\]

which has type \((X \mathcal{U}) \rightarrow \text{Path}_{\text{susp}(X)}(G(Fk)X(\text{north}), kX(\text{north}))\) and an analogous term \(P_{\text{south}}\) of type \((X \mathcal{U}) \rightarrow \text{Path}_{\text{susp}(X)}(G(Fk)X(\text{south}), kX(\text{south}))\). Finally, we have a term

\[
P_{\text{merid}} \in (X \mathcal{U})(a: X) \rightarrow \text{Path}_{y. \text{Path}_{\text{susp}(X)}(G(Fk)X(\text{merid}^y(a)), kX(\text{merid}^y(a))))(P_{\text{north}} X, P_{\text{south}} X),
\]

because this type is an iterated path type of \((X \mathcal{U}) \rightarrow X \rightarrow \text{susp}(X)\) and therefore contractible by Lemma 11.14. We assemble these three cases to prove the inverse condition:

\[
\lambda X. \lambda t. \text{susp-elim}_{t. \text{Path}_{\text{susp}(X)}(G(Fk)Xt, kXt)}(t; P_{\text{north}} X, P_{\text{south}} X, y. a. P_{\text{merid}} X a @ y)
\]

has type \((X \mathcal{U})(t: \text{susp}(X)) \rightarrow \text{Path}_{\text{susp}(X)}(G(Fk)Xt, kXt)\). □

12 Related and future work

**Parametric type theory** The concept of parametricity originates with Reynolds [1983], who gave a relational interpretation of simply-typed \(\lambda\)-calculus with type variables in order to show that polymorphic functions treat their type arguments parametrically. Parametricity and relational interpretations have since been used for myriad purposes; we will keep our attention on the road to internal parametricity. Plotkin and Abadi [1993] define an external relational logic for proving properties of terms in the polymorphic \(\lambda\)-calculus, with axioms providing access to parametricity. Bernardy et al. [2010] observe that for a sufficiently expressive theory, such as dependent type theory, the relational interpretation can be defined in the same theory. However, their interpretation function remains external. Krishnaswami and Dreyer [2013] define a relational model of extensional dependent type theory and observe that its parametricity theorems can be internalized as axioms with computational content. However, each use of parametricity requires a new modification to the theory. Finally, Bernardy and Moulin [2012] complete the internalization of parametricity, adding internal operators.
- ∈ [−] and [−] which compute the relational interpretations of types and terms respectively. Notably, these operators have computational content, so the extended theory remains constructive. Later work substantially simplifies their original theory by using dimension variables [Bernardy and Moulin, 2013; Bernardy et al., 2015; Moulin, 2016]. Beyond being internal, their parametricity is also higher-dimensional: as relations are internal to the theory, they themselves have relational structure. Higher-dimensional (or proof-relevant) logical relations and parametricity have also been explored by Benton et al. [2014], Ghani et al. [2015], and Sojakova and Johann [2018].

Our theory is, for the most part, a direct extension of that of Bernardy et al. However, we do make a few changes beyond the obvious addition of cubical structure. First and most superficially, we use different notation for its parametricity constructs in order to match the cubical constructs; we include a translation dictionary in Figure 7. Our category of dimension contexts has two constants (0, 1) where theirs has one (0), giving a theory of binary rather than unary parametricity. (As such, pairs of terms in our notation correspond to single terms in theirs). The analogue of the equivalence Bridge_{A,B,R}(M,N) ≃ R(M,N), which we prove using the rules for Gel (Lemma 9.2), is a judgmental equality called Pair-Pred. With this equality, the bridge abstraction and application operators can do double duty as ungel and gel, as shown in Figure 7. However, validating this equality apparently requires a change to the usual presheaf semantics of type theory with dimensions, replacing sets with I-sets [Moulin, 2016, Chapter 3, §5.2]. Although we have not presented a presheaf semantics here, translating operational definitions to denotational definitions is straightforward enough that we are confident I-sets are not necessary to model our proof theory. As Nuyts et al. [2017] observe, Bernardy et al. do not address the lack of an identity extension lemma. Our introduction of bridge-discreteness, proofs that various types are bridge-discrete (Lemma 10.9 and Theorem 11.5), and observation that the bridge-discrete sub-universe is relativistic (Theorem 10.12) go some way towards addressing this lacuna. Our approach is maximally noncommittal: no types are made bridge-discrete by fiat, but one may always work in the bridge-discrete fragment.

A second line of related work is the parametric type theory of Nuyts et al. [2017], which builds on the work of Bernardy et al. as well as cubical type theory. Like us, they extend dependent type theory with contexts of bridge and path dimensions. However, their work centers around around modalities which mediate between the two contexts; in particular, types and elements are checked under different modalities. Thus, the idea that parametrically polymorphic functions do not inspect the type they are supplied plays a central role, though they decouple the type/element and parametric/continuous distinctions to some extent. In contrast, our work (like that of Bernardy et al.) focuses to the relational interpretation aspect of parametricity. Although we use the same “bridge” and “path” terminology, their category of contexts differs substantially from ours: bridge variables can be substituted for path variables, and both bridge and path variables are structural. The former means that the equivalent of loosen exists on the level of the base category. This map gives rise to a parametric modality, among others; a function is parametric in its input when it takes bridges to paths. We can simulate non-dependent parametric functions in our system given a propositional truncation [−]: a function is parametric when its image [HoTT, Definition 7.6.3] is bridge-discrete. (Simulating dependent functions is also possible, but more involved.)

\[
\text{isParametric}(A,B,F) := \text{isBDisc}((b:B) \times \text{Fiber}(A,B,F;b))
\]

Their notion of path is also distinct from ours, although they play a similar role. Paths do not support any kind of Kan operations in general; they are used by way of a path degeneracy axiom which turns homogeneous paths (of the form Path_{A}(M_0,M_1), where in general A depends on a variable) into elements of the Martin-Löf identity type Id_{A}(M_0,M_1). Due to this and other axioms (such as function extensionality for Id), the theory is not computational. Bridges in the universe can be defined using one of two operators, \text{Glue} and \text{Weld}. The former originates with Cohen et al. [2015] and is a more general form of V; the latter is its dual. The issues sketched in Section 7 with using a V-like operator for bridges are sidestepped by checking the relation argument under a modality. Unfortunately, doing so precludes higher-dimensional parametricity. To rectify this, Nuyts and Devriese [2018] introduce a system with an infinite ladder of increasingly permissive relations, paths and bridges being the first two rungs.

As part of an application, Nuyts et al. introduce a type \text{Size} which has natural numbers for elements but
codiscrete bridge structure. This raises the question of inductive types with bridge constructors: higher inductive types for the bridge direction. Unfortunately, here the use of substructural dimensions is an obstacle, a fact which originally motivated the use of structural dimensions in cubical type theory. The problem may be seen by considering a type \( 2 \) generated by points \( \text{zero}, \text{one} \) and a bridge \( \text{seg} \in \text{Bridge}_2(\text{zero}, \text{one}) \). For any type \( A \), we would expect an equivalence fitting into the following diagram.

\[
\begin{array}{ccc}
(2 \to A) & \xrightarrow{\cong} & (a_0, a_1 : A) \times \text{Bridge}_A(a_0, a_1) \\
\lambda b.(b(\text{zero}), b(\text{one})) & & \lambda a_0, a_1, q. (a_0, a_1) \\
A \times A & \xrightarrow{} & A \times A
\end{array}
\]

Observe that on the left hand side, we have a diagonal map \( \delta := \lambda b. \lambda i. bii \in (2 \to 2 \to A) \to (2 \to A) \) such that \( \delta b(\text{zero}) \equiv b(\text{zero})(\text{zero}) \) and \( \delta b(\text{one}) \equiv b(\text{one})(\text{one}) \). But no such map exists on the right hand side, precisely because bridge dimensions are substructural. In a substructural cubical type theory like that of Bezem et al. [2013], there is still some hope: although there is no diagonal map in the base category, one can define a diagonal map on the level of paths using the Kan operations. For bridges, however, this is impossible. It remains to be seen whether some restricted class of “bridge inductive types” can be given a usable proof theory.

**Cubical type theory** The cubical side of the theory, as well as the operational presentation, is adopted from Angiuli et al. [2018] practically without change. Bezem et al. [2013] and Cohen et al. [2015] have also developed cubical type theories, which differ in choice of cube category and formulation of the Kan operations. We believe that the parametric additions we make to cartesian cubical type theory could easily be replayed on top of any of these theories: one simply needs to add (a) a substructural context of bridge dimensions and (b) \( r = \varepsilon \) equations in the language of composition constraints (i.e., generating cofibrations). The latter addition does not interfere with the definition of the \( \forall x \) operator on constraints used by Cohen et al.: \( \forall x.(r = \varepsilon) \) can be defined as \( r = \varepsilon \). (Note that a \( \forall x \) operator is also necessary in order to define \( \text{hcom} \) in Gel-types.)

The bridge side of our theory is similar to the cubical type theory of Bezem et al. [2013], mostly insofar as that theory is (not coincidentally) similar to Bernardy et al.’s parametric type theory. As there is no composition or coercion along bridges, definitions are generally simpler in our setting than in theirs, especially where the universe is concerned.

**Directed type theory** Parametric cubical type theory bears a close resemblance to the directed type theory of Riehl and Shulman [2017]. Like our theory, it has two directions of higher structure. One direction is given by identity types with the axioms of homotopy type theory; this corresponds to our path direction. The other is given by dimension variables (and a language of inequality constraints) and corresponds to our bridge direction. The intended model of this theory is the category of Reedy fibrant presheaves over \( \Delta \times \Delta \), which parallels our anticipated model in suitably fibrant presheaves over \( C_{(\text{we}, \cdot)} \times C_{(\text{we}, \cdot)} \) (in the terminology of Buchholtz and Morehouse [2017]). In the case of directed type theory, the intent is to carve out the sub-universe of Segal types, those that support a directed composition operation in the “bridge” direction. The question of relativity arises under the name of directed univalence, but the most naive formulation appears to be false [Riehl, 2018].

**Future work** As mentioned in the introduction, one motivation for this work is to prove coherence theorems for functions on higher inductive types. From Section 11.5, we see that the behavior of a polymorphic function \( X \mathcal{U} \to \text{sus}p(X) \to \text{sus}p(X) \) can be analyzed by checking its behavior on zero-dimensional constructors. As such, one can avoid the higher-dimensional constructions that are necessary to prove properties of a function \( \text{sus}p(A) \to \text{sus}p(A) \) at a particular \( A \). A more complicated case of interest is that of the smash product, a certain binary operator on pointed types (elements of \( \mathcal{U}_* := (X \mathcal{U}) \times X \)). To show that the
smash product is commutative and associative is difficult, to show that the commutator and associator satisfy coherence laws even more so [Brumerie, 2018]. However, once the commutator and associator have been constructed, the other theorems can be posed in terms of characterizing polymorphic pointed functions between smash products, specifically terms of type \((X_1, \ldots, X_n, \mathcal{M}) \rightarrow \bigwedge_{i \leq n} X_i \rightarrow \bigwedge_{i \leq n} X_i\) for various \(n\). We conjecture that such terms can be characterized in parametric cubical type theory by their behavior on zero-dimensional constructors, and that this can be proven for all \(n\) uniformly.

Of course, we would like to have such results not only for parametric cubical type theory but for ordinary cubical type theory. Not every theorem of parametric cubical type theory is true in cubical type theory, but it is reasonable to suspect that some class of statements is transferable. To our knowledge, the only work in this vein is the interpretation in used in [Bernardy and Moulin, 2012] to prove strong normalization for their parametric type theory, which is also discussed in [Moulin, 2016, Chapter 1, §3.6].

## Appendix

\section{D-relation interface}

\textbf{Lemma A.1} (Introduction). \textit{Let \(\alpha\) be a value D-relation. If for every \(\psi : \mathcal{D}' \rightarrow \mathcal{D}\), either \(\alpha_\psi(M\psi, M'\psi)\) or \(\text{TM}(\alpha)\psi(M\psi, M'\psi)\), then \(\text{TM}(\alpha)(M, M')\).}

\textit{Proof.} Let \(\psi_1 : \mathcal{D}_1 \rightarrow \mathcal{D}\) and \(\psi_2 : \mathcal{D}_2 \rightarrow \mathcal{D}_1\) be given. We divide into three cases.

\textit{(a)} \(\alpha_{\psi_1}(M\psi_1, M'\psi_1)\) and \(\alpha_{\psi_2}(M\psi_2, M'\psi_2)\).

Then we have
\[
\begin{align*}
M\psi_1 &\downarrow M\psi_1 & M\psi_2 &\downarrow M\psi_2 & M\psi_2 &\downarrow M\psi_2 \\
M\psi_1 &\downarrow M'\psi_1 & M'\psi_2 &\downarrow M'\psi_2 & M'\psi_2 &\downarrow M'\psi_2
\end{align*}
\]
with \(\alpha_{\psi_2}(M\psi_2, M'\psi_2)\).

\textit{(b)} \(\alpha_{\psi_1}(M\psi_1, M'\psi_1)\) and \(\text{TM}(\alpha)\psi_2(M\psi_1, M'\psi_2)\).

By \(\text{TM}(\alpha)\psi_2(M\psi_1, M'\psi_2)\), we have \(M\psi_1 \downarrow M_{12}\) and \(M'\psi_2 \downarrow M'_{12}\) with \(\alpha_{\psi_2}(M_{12}, M'_{12})\). Thus
\[
\begin{align*}
M\psi_1 &\downarrow M_{12} & M\psi_2 &\downarrow M_{12} & M\psi_2 &\downarrow M_{12} \\
M'\psi_1 &\downarrow M'_{12} & M'\psi_2 &\downarrow M'_{12} & M'\psi_2 &\downarrow M'_{12}
\end{align*}
\]
with \(\text{TM}(\alpha)\psi_2(M_{12}, M'_{12})\).

\textit{(b*)} \(\text{TM}(\alpha)\psi_1(M\psi_1, M'\psi_1)\).

By \(\text{TM}(\alpha)\psi_1(M\psi_1, M'\psi_1)\), we have
\[
\begin{align*}
M\psi_1 &\downarrow M_1 & M\psi_2 &\downarrow M_2 & M\psi_2 &\downarrow M_{12} \\
M'\psi_1 &\downarrow M'_{1} & M'\psi_2 &\downarrow M'_{2} & M'\psi_2 &\downarrow M'_{12}
\end{align*}
\]
with \(\alpha_{\psi_2}(V, V')\) for all \(V \in \{M_2, M_{12}\}\) and \(V' \in \{M'_{2}, M'_{12}\}\). \(\square\)

\textbf{Lemma A.2} (Coherent expansion). \textit{Let \(\alpha\) be a value D-PER and let \(M, M'\) tm \(|\Phi|\Psi\). If for every \(\psi : \mathcal{D}' \rightarrow \mathcal{D}\), there exists \(M''\) such that \(M\psi \rightarrow\rightarrow M''\) and \(\text{TM}(\alpha)_\psi(M'', M'\psi)\), then \(\text{TM}(\alpha)(M, M')\).}

\textit{Proof.} Let \(\psi_1 : \mathcal{D}_1 \rightarrow \mathcal{D}\) and \(\psi_2 : \mathcal{D}_2 \rightarrow \mathcal{D}_1\) be given. By assumption, there exists \(M''_1\) such that \(M\psi_1 \rightarrow\rightarrow M''_1\) and \(\text{TM}(\alpha)\psi_1(M''_1, M'\psi_1)\). By \(\text{TM}(\alpha)\psi_1(M''_1, M'\psi_1)\) applied at the substitutions \(\text{id}\) and \(\psi_2\), we see that
\[
\begin{align*}
M''_1 &\downarrow M_1 & M\psi_2 &\downarrow M_2 & M\psi_2 &\downarrow M_{12} \\
M'\psi_1 &\downarrow M'_{1} & M'\psi_2 &\downarrow M'_{2} & M'\psi_2 &\downarrow M'_{12}
\end{align*}
\]
with \(\alpha_{\psi_2}(V, V')\) for \(V \in \{M_2, M_{12}\}\) and \(V' \in \{M'_{2}, M'_{12}\}\). Likewise, we have some \(M''_{12}\) such that \(M\psi_1 \psi_2 \rightarrow\rightarrow M''_{12}\) and \(\text{TM}(\alpha)\psi_1\psi_2(M''_{12}, M'\psi_1\psi_2)\). By the latter, we have \(M''_{12} \downarrow N_{12}\) and \(M'\psi_1\psi_2 \downarrow N'_{12}\) with \(\alpha_{\psi_1\psi_2}(N_{12}, N'_{12})\). Note that \(M'_{12} = N'_{12}\) by determinism of the operational semantics. As \(\alpha\) is a PER, we thus have \(\alpha_{\psi_1\psi_2}(N_{12}, M'_{12})\).
Combining this data, we have

\[ M \downarrow \psi_1 \downarrow M_1 \quad M_1 \psi_2 \downarrow M_2 \quad M \psi_1 \psi_2 \downarrow N_{12} \]
\[ M' \downarrow \psi_1 \downarrow M'_1 \quad M'_1 \psi_2 \downarrow M'_2 \quad M' \psi_1 \psi_2 \downarrow M'_{12} \]

with \( \alpha_{\psi_1 \psi_2}(V, M'_{12}) \) for all \( V \in \{ M_2, N_{12} \} \) and \( V' \in \{ M'_2, M'_{12} \} \). \( \square \)

**Lemma A.3** (Evaluation). Let \( \alpha \) be a value-coherent \( \mathcal{D} \text{-PER} \) and let \( M, M' \text{ tm } \mathcal{D} \) with \( \text{TM}(\alpha)(M, M') \). Then \( M \downarrow V \) and \( M' \downarrow V' \) with \( \text{TM}(\alpha)(Q, Q') \) holds for all \( Q \in \{ M, V \} \) and \( Q' \in \{ M', V' \} \).

**Proof.** By \( \text{TM}(\alpha)(M, M') \), we have \( M \downarrow V \) and \( M' \downarrow V' \) with \( \alpha(V, V') \), which implies \( \text{TM}(\alpha)(V, V') \) by value-coherence of \( \alpha \). We show \( \text{TM}(\alpha)(M, V') \); the proof of \( \text{TM}(\alpha)(V, M') \) is symmetric. Let \( \psi_1 : \mathcal{D}_1 \to \mathcal{D} \) and \( \psi_2 : \mathcal{D}_2 \to \mathcal{D}_1 \). By \( \text{TM}(\alpha)(M, M') \) applied at the substitutions id and \( \psi_1 \), we have

\[ M \downarrow V \quad V \psi_1 \downarrow V_1 \quad M \psi_1 \downarrow M_1 \]
\[ M' \downarrow V' \quad V' \psi_1 \downarrow V'_1 \quad M' \psi_1 \downarrow M'_1 \]

with, in particular, \( \alpha_{\psi_1}(M_1, V'_1) \) and \( \alpha_{\psi_1}(V_1, M'_1) \). By value-coherence, we then have \( \text{TM}(\alpha)(\psi_1)(M'_1, V'_1) \), which implies that \( M'_1 \psi_2 \downarrow M_2 \) and \( V'_1 \psi_2 \downarrow V_2 \) with \( \alpha_{\psi_1 \psi_2}(M_2, V_2) \); symmetrically, we have \( V_1 \psi_2 \downarrow V_2 \) and \( M'_1 \psi_2 \downarrow M'_2 \) with \( \alpha_{\psi_1 \psi_2}(M_2, V_2) \). We now apply \( \text{TM}(\alpha)(M, M') \) at the substitutions id and \( \psi_1 \psi_2 \) to obtain

\[ M \downarrow V \quad V \psi_1 \psi_2 \downarrow V_{12} \quad M \psi_1 \psi_2 \downarrow M_{12} \]
\[ M' \downarrow V' \quad V' \psi_1 \psi_2 \downarrow V'_{12} \quad M' \psi_1 \psi_2 \downarrow M'_{12} \]

with, in particular, \( \alpha_{\psi_1 \psi_2}(M_{12}, V'_{12}) \). Finally, applying \( \text{TM}(\alpha)(M, M') \) and \( \text{TM}(\alpha)(V, V') \) at \( \psi_1 \) and \( \psi_2 \) gives us \( \alpha_{\psi_1 \psi_2}(M_{12}, M'_{12}), \alpha_{\psi_1 \psi_2}(M_2, M'_{12}), \alpha_{\psi_1 \psi_2}(M_2, V'_{12}) \) and \( \alpha_{\psi_1 \psi_2}(V_2, V'_{12}) \). We now have

\[ \alpha_{\psi_1 \psi_2}(M_{12}, V'_{12}) \quad \alpha_{\psi_1 \psi_2}(M_2, M'_{12}) \quad \alpha_{\psi_1 \psi_2}(V_2, V'_{12}) \]

Combining these via transitivity of \( \alpha \) gives the desired result: \( \alpha_{\psi_1 \psi_2}(W, W') \) for all \( W \in \{ M_2, M_{12} \} \) and \( W' \in \{ V'_2, V'_{12} \} \). \( \square \)

**Lemma A.4** (Formation). Let \( \tau \) be a bridge-path type system, let \( A, A' \text{ tm } \mathcal{D} \), and let \( \alpha \) be a value \( \mathcal{D} \text{-relation} \). If for every \( \psi : \mathcal{D}' \to \mathcal{D} \), either \( \text{PT}(\tau)(\mathcal{D}', A\psi, A'\psi, \alpha \psi) \) holds or \( \tau(\mathcal{D}', A\psi, A'\psi, \alpha \psi) \) holds, then \( \text{PT}(\tau)(\mathcal{D}, A, A', \alpha) \).

**Proof.** A straightforward adaptation of the proof of Lemma A.1. \( \square \)

**Lemma A.5** (Coherent type expansion). Let \( \tau \) be a bridge-path type system, let \( A, A' \text{ tm } \mathcal{D} \), and let \( \alpha \) be a value \( \mathcal{D} \text{-relation} \). If for all \( \psi : \mathcal{D}' \to \mathcal{D} \), there exists \( A'' \) such that \( A\psi \to A'' \) and \( \text{PT}(\tau)(\mathcal{D}', A'', A'\psi, \alpha \psi) \), then \( \text{PT}(\tau)(\mathcal{D}, A, A', \alpha) \).

**Proof.** A straightforward adaptation of the proof of Lemma A.2. \( \square \)

**Lemma A.6** (Elimination). Let \( C, C' : \mathcal{D} \subseteq \mathcal{D}_0, \rho \text{ bdims } \mathcal{D} \), and let \( \alpha \) be a value-coherent \( (\mathcal{D} \setminus \rho \mathcal{D}_0) \text{-PER} \). Suppose that for every \( \psi : \mathcal{D}' \to \mathcal{D} \) with \( \mathcal{D}' \) disjoint from \( \mathcal{D}_0 \), we have

1. \( T\psi[M] \equiv T\psi[M'] \text{ type pre } \mathcal{D}' \) for all \( \text{TM}(\alpha \psi)_{\mathcal{D}_0}(M, M') \),
2. \( C\psi, C\psi' \text{ are eager and } C\psi[V] \equiv C\psi'[V'] \in T\psi[V] \text{ for } \alpha \psi_{\mathcal{D}_0}(V, V') \).

Then \( C[M] \equiv C[M'] \in T[M] \text{ for } \text{TM}(\alpha)(M, M') \).

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We assume the existence of an analogous operator \( \alpha_{(\psi_1, \psi_2)} \) for \( V \in \{ M_2, M_{12} \} \) and \( V' \in \{ M'_2, M'_{12} \} \); by applying it at \( \psi_1 \& D_0 \) and \( \psi_2 \& D_1 \), we also know \( \alpha_{(\psi_1, \psi_2)} \) for \( (M_1, M'_1) \). Thus \( C_{\psi_1}[M_1] \subseteq C'_{\psi_1}[M'_1] \) for \( C \in T[M] \psi_1 [D_1] \); applying this at id and \( \psi_2 \) gives

\[
C_{\psi_1}[M_1] \subseteq N_1 \quad N_1 \psi_2 \subseteq N_2 \quad C_{\psi_1}[M_1] \psi_2 \subseteq N_{12}
\]

where \( \alpha_{(\psi_1, \psi_2)} \) for \( V \in \{ M_2, M_{12} \} \) and \( V' \in \{ M'_2, M'_{12} \} \). For any \( t, t' \in \{ 2, 12 \} \) we have \( C_{\psi_1}[M_1] \subseteq C'_{\psi_1}[M'_1] \subseteq T[M] \psi_1 [D_2] \), which gives

\[
C_{\psi_1}[M_1] \psi_2[M_1] \subseteq P_t \quad C'_{\psi_1}[M'_1] \psi_2[M'_1] \subseteq P'_t
\]

with \( [\mathcal{T}[M]]_{\psi_1, \psi_2} \) for \( V \in \{ N_2, N_{12} \} \) and \( V' \in \{ N'_2, N'_{12} \} \). For any \( t, t' \in \{ 2, 12 \} \) we have \( C_{\psi_1}[M_1] \subseteq C'_{\psi_1}[M'_1] \subseteq T[M] \psi_1 [D_2] \), which gives

\[
C_{\psi_1}[M_1] \psi_2[M_1] \subseteq P_t \quad C'_{\psi_1}[M'_1] \psi_2[M'_1] \subseteq P'_t
\]

Now, observe that we have \( C_{\psi_1}[M_1] \psi_2[M_1] \subseteq N_{12} \) as well as \( C_{\psi_1}[M_2] \subseteq P_2 \); as \( M_1 \psi_2 \subseteq M_{12} \) and \( C_{\psi_1} \psi_2 \) is eager, this means \( N_{12} = P_2 \) by determinism. Likewise \( C_{\psi_1} \psi_2[M_2] \subseteq P'_2 \). Using transitivity of \( [\mathcal{T}[M]] \), we can therefore conclude that \( [\mathcal{T}[M]]_{\psi_1, \psi_2}(V, V') \) for \( V \in \{ N_2, P_2 \} \) and \( V' \in \{ N'_2, P'_{12} \} \).

Note that throughout this proof, we have implicitly identified indices of \( T \); for example, we use the fact that \( [\mathcal{T}[M]]_{\psi_1} \) and \( [\mathcal{T}[M]]_{\psi_1} \) are equal. Here we use the assumption that \( \alpha \) is value-coherent, which gives \( \text{TM}(\alpha_{\psi_1, \psi_2}) \) by Lemma A.3.

B Fixed-point construction

To construct concrete examples of bridge-path type systems, we use a minor variation on the fixed-point construction introduced by Angiuli et al. [2017b] following Allen [1987]. We sketch here the additions necessary to accommodate bridge variables and the Bridge and Gel types. As mentioned above, we satisfy ourselves with a single universe, but it is not significantly more difficult to construct a type system with an infinite hierarchy.

Definition B.1. Define an operator \( \mathcal{F}_B \) on candidate bridge-path type systems as follows.

\[
\mathcal{F}_B(\tau) := \{ (\Phi, \Psi), \text{Bridge}_{x.A}(M_0, M_1), \text{Bridge}_{x.A}(M_0, M_1), \text{Bridge}_{x.A}(M_0, M_1) \mid \tau \models A \vdash B \text{ type}_\text{kant} \} \cup \{ (\Phi, x : \psi), \text{Gel}_{x.(A, B, a.b.R)}, \text{Gel}_{x.(A', B', a.b.R')}, \text{Gel}_{x.(A, B, a.b.R)} \mid \tau \models A \vdash B \text{ type}_\text{kant} \} \quad \text{and}
\]

\[
\tau \models a : A, b : B \Rightarrow R \vdash R \text{ type}_\text{kant} \}
\]

We assume the existence of an analogous operator \( \mathcal{F}_C \) on candidate bridge-path type systems with clauses for each type former of cubical type theory: pair, function, Path-, V-, and fcom-types [Angiuli et al., 2017b, §3.1]. These can all be defined uniformly in the bridge context \( \Phi \).

Definition B.2. Define the candidate bridge-path type system \( \tau_0 := \mu \tau. \mathcal{F}_B(\tau) \cup \mathcal{F}_C(\tau) \), the least fixed-point of \( \tau \rightarrow \mathcal{F}_B(\tau) \cup \mathcal{F}_C(\tau) \) in the lattice of candidate bridge-path type systems ordered by inclusion (regarded as quaternary relations).
Definition B.3. Define the candidate bridge-path type system $\tau_1$ as follows.

$$
\tau_1 := \mu \tau. \mathcal{F}_B(\tau) \cup \mathcal{F}_C(\tau) \cup \{(\Phi \mid \Psi), \mathcal{U}, \mathcal{U}, \varphi \mid \varphi(A, A') \iff \tau_0 \models A \doteq A' \text{ type}_{Kan} [\Phi \mid \Psi]\}
$$

Proposition B.4. $\tau_0$ and $\tau_1$ are bridge-path type systems.

Proof. See [Angiuli et al., 2017b, Theorem 16].

Proposition B.5. $\tau_0$ and $\tau_1$ is closed under Bridge- and Gel-types and the constructs of cubical type theory. Moreover, $\tau_1$ contains a univalent universe $\mathcal{U}$ which is closed under these same type formers.

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