Multipoint Lax operator algebras: almost-graded structure and central extensions

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Abstract. Recently, Lax operator algebras appeared as a new class of higher genus current-type algebras. Introduced by Krichever and Sheinman, they were based on Krichever’s theory of Lax operators on algebraic curves. These algebras are almost-graded Lie algebras of currents on Riemann surfaces with marked points (in-points, out-points and Tyurin points). In a previous joint article with Sheinman, the author classified the local cocycles and associated almost-graded central extensions in the case of one in-point and one out-point. It was shown that the almost-graded extension is essentially unique. In this article the general case of Lax operator algebras corresponding to several in- and out-points is considered. As a first step they are shown to be almost-graded. The grading is given by splitting the marked points which are non-Tyurin points into in- and out-points. Next, classification results both for local and bounded cocycles are proved. The uniqueness theorem for almost-graded central extensions follows. To obtain this generalization additional techniques are needed which are presented in this article.

Bibliography: 30 titles.

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§ 1. Introduction

Lax operator algebras are a new class of current type Lie algebras introduced recently. In their full generality they were introduced by Krichever and Sheinman in [1], where the concept of Lax operators on algebraic curves, as considered by Krichever in [2], was generalized to \( \mathfrak{g} \)-valued Lax operators, where \( \mathfrak{g} \) is a classical complex Lie algebra. Krichever [2] extended the conventional Lax operator representation with a rational parameter to the case of algebraic curves of arbitrary genus. Such generalizations of Lax operators appear in many fields. They are closely related to integrable systems (Krichever-Novikov equations on elliptic curves, elliptic Calogero-Moser systems, Baker-Akhieser functions); see [2] and [3]. Another important application appears in the context of moduli spaces of bundles. In particular, they are related to Tyurin’s result on the classification of framed semi-stable holomorphic vector bundles on algebraic curves [4]. The classification
uses Tyurin parameters of such bundles, consisting of points $\gamma_s, s = 1, \ldots, ng$, and associated elements $\alpha_s \in \mathbb{P}^{n-1}(\mathbb{C})$ (where $g$ denotes the genus of the Riemann surface $\Sigma$, and $n$ corresponds to the rank of the bundle). We shall not make any reference to these applications in what follows. In addition to the papers mentioned above the reader could refer to Sheinman [5], [6] for more background in the case of integrable systems.

Here we shall concentrate on the mathematical structure of these algebras. Lax operator algebras are infinite dimensional Lie algebras of geometric origin and are interesting mathematical objects. In contrast to the classical genus zero algebras, appearing in Conformal Field Theory, they are not graded. In this article we will introduce an almost-graded structure for them (see Definition 3.1). This almost-grading will be an indispensable tool. A crucial task for these infinite dimensional Lie algebras is to construct and classify the central extensions. We do this here. We shall concentrate on those central extensions for which the almost-grading can be extended.

In certain respects the Lax operator algebras can be considered as generalizations of the higher-genus Krichever-Novikov type current and affine algebras, see [7]–[12]. These in turn are generalizations of the classical affine Lie algebras as introduced by Kac [13], [14] and Moody [15], for example.

This article extends the results for the two-point case (the definition is given in the next paragraph) to the multi-point case. As far as the almost-grading in the two-point case is concerned, see Krichever and Sheinman [1]. For the central extensions in the two-point case see our joint work with Sheinman [16].

To describe the results we have obtained, we first need a rough description of the setup. Full details will be given in §2. Let $\mathfrak{g}$ be one of the classical Lie algebras $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(n)$ over $\mathbb{C}$, and let $\Sigma$ be a compact Riemann surface. Let $A$ be a finite set of points in $\Sigma$ divided into two disjoint nonempty subsets $I$ and $O$. Furthermore, let $W$ be another finite set of points (called weak singular points). Our Lax operator algebra consists of meromorphic functions $\Sigma \to \mathfrak{g}$, holomorphic outside $W \cup A$, possibly with poles of order 1 (of order 2 for $\mathfrak{sp}(n)$) at the points in $W$ and with certain additional conditions, depending on $\mathfrak{g}$, on the Laurent series expansion there (see, for instance, (2.6)). It turns out [1] that due to the additional condition this set of matrix-valued functions closes to a Lie algebra $\bar{\mathfrak{g}}$ under the pointwise commutator. If $W = \emptyset$ then $\bar{\mathfrak{g}}$ is a Krichever-Novikov type current algebra (associated with this special finite-dimensional Lie algebra). These were studied extensively by Krichever and Novikov, Sheinman, and Schlichenmaier; see, for instance, [7]–[12] and [18]. We should point out that Krichever-Novikov type algebras can be defined for all finite-dimensional Lie algebras $\mathfrak{g}$.

If, furthermore, the genus of the Riemann surface is zero and $A$ consists only of two points, which we can take to be $\{0\}$ and $\{\infty\}$, then the algebras will be the usual classical current algebras. These classical algebras are graded algebras. This type of grading is used, for example, to introduce highest weight representations, Verma modules and Fock spaces, and to classify these representations. Unfortunately, the algebras which we consider here will not be graded. But they admit an almost-grading, see Definition 3.1. As Krichever and Novikov realized [7], for most

\footnote{For $G_2$ see the recent preprint of Sheinman [17], and the remark at the end of the introduction.}
applications it is a valuable replacement for the grading. They also gave a method for introducing it for two-point algebras of Krichever-Novikov type.

This kind of almost-grading was given by the author for the multi-point case of Krichever-Novikov type algebras [12], [19]–[23] and also [24]. The crucial point is that the almost-grading depends on how \( A \) is split into \( I \) and \( O \). Different splittings will give different almost-gradings. Hence, the multi-point case is more involved than the two-point case.

For the Krichever-Novikov current algebra \( \mathfrak{g} \) the grading comes from the grading of the function algebra (to be found in our papers cited above). This is because they are tensor products. If \( W \neq \emptyset \) the Lax operator algebras are not tensor products anymore and their almost-grading has to be constructed directly. This has been done in the two point case by Krichever and Sheinman [1].

Our first result in this article is to introduce an almost-grading of \( \mathfrak{g} \) for the arbitrary multi-point case. As we mentioned above, it depends in an essential way on how we split \( A = I \cup O \). This has been done in §3. The construction is much more involved than in the two-point case.

Our second goal is to study central extensions \( \widehat{\mathfrak{g}} \) of the Lax operator algebras \( \mathfrak{g} \). It is well-known that central extensions are given by Lie algebra two-cocycles of \( \mathfrak{g} \) with values in the trivial module \( \mathbb{C} \). Equivalence classes of central extensions are in 1 : 1 correspondence with the elements of the Lie algebra cohomology space \( \mathfrak{g} \). Whereas the classical current algebras associated with a finite-dimensional simple Lie algebra \( \mathfrak{g} \) have a unique extension class, this is not the case for higher genus, nor even for genus zero in the multi-point case. But we are only interested in central extensions \( \widehat{\mathfrak{g}} \) which allow us to extend the almost-grading of \( \mathfrak{g} \). This reduces the possibilities. The condition that the cocycle defining the central extension satisfies turns out to be that it is local (see (5.13)) with respect to the almost-grading given by the splitting \( A = I \cup O \). Hence, which cocycles are local also depends on the splitting.

If \( \mathfrak{g} \) is simple then the space of local cohomology classes for \( \mathfrak{g} \) is one-dimensional. For \( \mathfrak{gl}(n) \) we have to add another natural property for the cocycle, which means it is invariant under the action of the vector field algebra \( \mathcal{L} \) (see (5.3)). In this case the space of local and \( \mathcal{L} \)-invariant cocycle classes will be two-dimensional.

The action of the vector field algebra \( \mathcal{L} \) on \( \mathfrak{g} \) is given in terms of a certain connection, \( \nabla^{(\omega)} \), see §4.2. With the help of this connection we can define geometric cocycles

\[
\gamma_{1,\omega,C}(L, L') = \frac{1}{2\pi i} \int_C \text{tr}(L \cdot \nabla^{(\omega)} L'), \tag{1.1}
\]

\[
\gamma_{2,\omega,C}(L, L') = \frac{1}{2\pi i} \int_C \text{tr}(L) \cdot \text{tr}(\nabla^{(\omega)} L'), \tag{1.2}
\]

where \( C \) is an arbitrary cycle on \( \Sigma \) avoiding the points where there might be singularities. The cocycle \( \gamma_{2,\omega,C} \) only differs from zero in the \( \mathfrak{gl}(n) \)-case.

Special integration paths are circles \( C_i \) around the points in \( I \), and around the points in \( O \), respectively, and a path \( C_S \) separating the points in \( I \) from the points in \( O \).

Our main result is Theorem 6.7, which is about uniqueness of local cocycle classes and the fact that the cocycles are given by integrating along \( C_S \). The proof
presented in §6 is based on Theorem 6.4 which gives the classification of cohomology classes which are bounded from above (see (5.12)). The bounded cohomology classes constitute a subspace of dimension \(N \leq 2N\) for \(\mathfrak{gl}(n)\), where \(N = \# I\) and the integration is taken over the \(C_i, i = 1, \ldots, N\).

The proof of Theorem 6.4 is given in §§7 and 8. We use recursive techniques as developed in [25] and [12]. Using the boundedness and \(L\)-invariance we show that a cocycle with these properties is given by its values at pairs of homogeneous elements such that the sum of their degrees is equal to zero. Furthermore, we show that an \(L\)-invariant and bounded cocycle will be uniquely fixed by a certain finite number of such cocycle values. A more detailed analysis shows that the cocycles are of the form claimed. In §8 we show that in the simple Lie algebra case in each bounded cohomology class there is a representing cocycle which is \(L\)-invariant. For this we use the internal structure of the Lie algebra \(\mathfrak{g}\) related to the root system of the underlying finite dimensional simple Lie algebra \(\mathfrak{g}\), and the almost-gradedness of \(\overline{g}\). Recall that in the classical case \(g \otimes \mathbb{C}[z, z^{-1}] \) the algebra is graded. In this very special case the sequence of arguments is simpler and is similar to the arguments in Garland [16].

I would like to thank Oleg Sheinman for extensive discussions which were very helpful during the writing of this article. After I had finished this work he succeeded [17] in giving a definition of a Lax operator algebra for the exceptional Lie algebra \(G_2\) in such a way that all the properties and statements presented here are also true in this case. Hence, there is now another element in the list of Lax operator algebras associated with simple Lie algebras.

§ 2. The algebras

2.1. Lax operator algebras. Let \(g\) be one of the classical matrix algebras \(\mathfrak{gl}(n)\), \(\mathfrak{sl}(n)\), \(\mathfrak{so}(n)\), \(\mathfrak{sp}(2n)\), or \(\mathfrak{s}(n)\), where the latter denotes the algebra of scalar matrices. Our algebras will consist of certain \(g\)-meromorphic functions, forms, etc., defined on Riemann surfaces with additional structures (marked points, vectors associated to this points, ...).

To be more precise, let \(\Sigma\) be a compact Riemann surface of genus \(g\) (\(g\) is arbitrary) and let \(A\) be a finite subset of points in \(\Sigma\) divided into two nonempty disjoint subsets

\[
I := \{P_1, P_2, \ldots, P_N\}, \quad O := \{Q_1, Q_2, \ldots, Q_M\},
\]

(2.1)

with \(\# A = N + M\). The points in \(I\) are called incoming points, the points in \(O\) outgoing points.

To define a Lax operator algebra we have to fix some additional data. Fix \(K \in \mathbb{N}_0\) and a collection of points

\[
W := \{\gamma_s \in \Sigma \setminus A \mid s = 1, \ldots, K\}.
\]

(2.2)

To every point \(\gamma_s\) we assign a vector \(\alpha_s \in \mathbb{C}^n\) (\(\alpha_s \in \mathbb{C}^n\) for \(\mathfrak{sp}(2n)\)). The system

\[
\mathcal{T} := \{ (\gamma_s, \alpha_s) \in \Sigma \times \mathbb{C}^n \mid s = 1, \ldots, K\}
\]

(2.3)

is called Tyurin data. We have greater generality than in our earlier joint paper [16] with Sheinman, not only in that we allow \(A\) to have more than two points, but we
also do not require \( K \) to be \( n \cdot g \). Even \( K = 0 \) is allowed. In the latter case the Tyurin data will be empty.

**Remark 2.1.** For \( K = n \cdot g \) and for generic values of \((\gamma_s, \alpha_s)\) with \( \alpha_s \neq 0 \) the tuples of pairs \((\gamma_s, [\alpha_s])\) with \([\alpha_s] \in \mathbb{P}^{n-1}(\mathbb{C})\) parameterize framed semi-stable rank \( n \) and degree \( ng \) holomorphic vector bundles as shown by Tyurin [4]. Hence the name Tyurin data.

We fix local coordinates \( z_l, l = 1, \ldots, N \), centred at the points \( P_l \in I \) and \( w_s \) centred at \( \gamma_s, s = 1, \ldots, K \). In fact nothing depends on the choice of \( w_s \). In essence, this is also true for \( z_l \). Only its first jet is used to normalize certain basis elements uniquely.

We consider \( \mathfrak{g} \)-valued meromorphic functions

\[
L : \Sigma \to \mathfrak{g},
\]

(2.4)

which are holomorphic outside \( W \cup A \), have poles of at most order one (of order two for \( \mathfrak{sp}(2n) \)) at the points in \( W \), and fulfil certain conditions on \( W \) depending on \( \mathcal{T} \), \( A \) and \( \mathfrak{g} \). These conditions will be described in the following. The singularities at \( W \) are called *weak singularities*. These objects were introduced by Krichever [2] for \( \mathfrak{gl}(n) \) in the context of Lax operators for algebraic curves, and further generalized by Krichever and Sheinman in [1]. The conditions are exactly the same as in [16]. But for the convenience of the reader we recall them here.

For \( \mathfrak{gl}(n) \) the conditions are as follows. For \( s = 1, \ldots, K \) we require that there exist \( \beta_s \in \mathbb{C}^n \) and \( \kappa_s \in \mathbb{C} \) such that the function \( L \) has the following expansion at \( \gamma_s \in W \):

\[
L(w_s) = \frac{L_{s,-1}}{w_s} + L_{s,0} + \sum_{k>0} L_{s,k} w^k_s,
\]

(2.5)

with

\[
L_{s,-1} = \alpha_s \beta_s^t, \quad \text{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s.
\]

(2.6)

In particular, if \( L_{s,-1} \) is nonvanishing then it is a rank 1 matrix, and if \( \alpha_s \neq 0 \) then it is an eigenvector of \( L_{s,0} \).

The requirements (2.6) are independent of the chosen coordinates \( w_s \) and the set of all such functions constitute an associative algebra under pointwise matrix multiplication, see [1]. The proof goes over unchanged to the multi-point case. Even so, for the convenience of the reader and to give an illustration, we shall recall the proof in §9. We denote this algebra by \( \overline{\mathfrak{g}}l(n) \). Of course, it depends on the Riemann surface \( \Sigma \), the finite set of points \( A \), and the Tyurin data \( \mathcal{T} \). To avoid confusion we prefer not to use cumbersome notation, and so we shall just use \( \overline{\mathfrak{g}}l(n) \). We do the same for the other Lie algebras.

Note that if one of the \( \alpha_s = 0 \) then the conditions at the point \( \gamma_s \) correspond to the fact that \( L \) has to be holomorphic there. We can remove the point from the Tyurin data. Also if \( \alpha_s \neq 0 \) and \( \lambda \in \mathbb{C}, \lambda \neq 0 \) then \( \alpha \) and \( \lambda \alpha \) induce the same conditions at the point \( \gamma_s \). Hence only the projective vector \([\alpha_s] \in \mathbb{P}^{n-1}(\mathbb{C})\) plays a role.

The splitting \( \mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n) \) given by

\[
X \mapsto \left( \frac{\text{tr}(X)}{n} I_n, X - \frac{\text{tr}(X)}{n} I_n \right),
\]

(2.7)
where $I_n$ is the $(n \times n)$-unit matrix, induces a corresponding splitting for the Lax operator algebra $\overline{\mathfrak{gl}}(n)$:

$$\overline{\mathfrak{gl}}(n) = \mathfrak{s}(n) \oplus \overline{\mathfrak{sl}}(n).$$

For $\overline{\mathfrak{sl}}(n)$ the only additional condition is that in (2.5) all matrices $L_{s,k}$ are traceless. The conditions (2.6) remain unchanged.

For $\mathfrak{s}(n)$ all the matrices in (2.5) are scalar matrices. This implies that the corresponding $L_{s,-1}$ vanish. In particular, the elements of $\mathfrak{s}(n)$ are holomorphic on $W$. Also, as a scalar matrix $L_{s,0}$ has every $\alpha_s$ as an eigenvector. This means that apart from holomorphicity there are no further conditions. And we get $\mathfrak{s}(n) \simeq \mathcal{A}$, where $\mathcal{A}$ is the (associative) algebra of meromorphic functions on $\Sigma$ holomorphic outside $A$. This is the (multi-point) Krichever-Novikov type function algebra. It will be discussed in §3.2 below.

For $\mathfrak{so}(n)$ we require that all $L_{s,k}$ in (2.5) are skew-symmetric. In particular, they are traceless. Following [1], the set-up has to be slightly modified. First, only those Tyurin parameters $\alpha_s$ which satisfy $\alpha_s^t \sigma \alpha_s = 0$ are allowed. Then the first requirement in (2.6) is changed to obtain

$$L_{s,-1} = \alpha_s \beta_s^t - \beta_s \alpha_s^t, \quad \text{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s. \quad (2.9)$$

For $\mathfrak{sp}(2n)$ we consider a symplectic form $\sigma$ for $\mathbb{C}^{2n}$ given by a nondegenerate skew-symmetric matrix $\sigma$. The Lie algebra $\mathfrak{sp}(2n)$ is the Lie algebra of matrices $X$ such that $X^t \sigma + \sigma X = 0$. The condition $\text{tr}(X) = 0$ will be automatic. At the weak singularities we have the expansion

$$L(z_s) = \frac{L_{s,-2}}{w_s^2} + \frac{L_{s,-1}}{w_s} + L_{s,0}w_s + \sum_{k>1} L_{s,k} w_s^k. \quad (2.10)$$

The condition (2.6) is modified as follows (see [1]): there exist $\beta_s \in \mathbb{C}^{2n}, \nu_s, \kappa_s \in \mathbb{C}$ such that

$$L_{s,-2} = \nu_s \alpha_s \alpha_s^t \sigma, \quad L_{s,-1} = (\alpha_s \beta_s^t + \beta_s \alpha_s^t) \sigma, \quad \beta_s^t \sigma \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s. \quad (2.11)$$

Moreover, we require

$$\alpha_s^t \sigma L_{s,1} \alpha_s = 0. \quad (2.12)$$

Again, under the pointwise matrix commutator the set of such maps constitutes a Lie algebra.

**Theorem 2.2.** Let $\mathfrak{g}$ be the space of Lax operators associated to $\mathfrak{g}$, one of the finite-dimensional classical Lie algebras introduced above. Then $\mathfrak{g}$ is a Lie algebra under the pointwise matrix commutator. For $\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$ it is an associative algebra under pointwise matrix multiplication.

The proof in [1] extends without problem to the multi-point situation (see §9 for an example).

These Lie algebras are called Lax operator algebras.
2.2. Krichever-Novikov algebras of current type. Let $A$ be the (associative) algebra of meromorphic functions on $\Sigma$ holomorphic outside $A$. Let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra. A Lie algebra structure on the tensor product $\mathfrak{g} \otimes A$ is given by

$$[x \otimes f, y \otimes g] := [x, y] \otimes (f \cdot g), \quad x, y \in \mathfrak{g}, \quad f, g \in A. \quad (2.13)$$

The elements of this Lie algebra can be considered as the set of those meromorphic maps $\Sigma \to \mathfrak{g}$ which are holomorphic outside $A$. These algebras are called (multi-point) Krichever-Novikov algebras of current type, see [7]–[9], [11], [12], [27] and [28].

If the genus of the surface is zero and if $A$ consists of two points, then the Krichever-Novikov current algebras are the classical current (or loop) algebras $\mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$.

In the case when there are no weak singularities in the defining data of the Lax operator algebra, or all $\alpha_s = 0$, respectively, the requirements for the $\mathfrak{g}$-valued meromorphic functions reduce to the condition that they be holomorphic outside $A$. Hence, (for these $\mathfrak{g}$) we obtain the Krichever-Novikov current-type algebra. But note that we do not have an extension of the notion of Krichever-Novikov current to a Lax operator algebra for all finite-dimensional $\mathfrak{g}$.

§3. The almost-graded structure

3.1. The statements. In the construction of certain important representations of infinite dimensional Lie algebras (Fock space representations, Verma modules, etc.) a graded structure is usually assumed and is heavily used. The algebras we are considering, for higher genus or even for genus zero with many marked points where poles are allowed, cannot be nontrivially graded. As Krichever and Novikov realized [7], the weaker concept of almost-grading is enough to allow us to make the above constructions.

Definition 3.1. A Lie algebra $V$ is called almost-graded (over $\mathbb{Z}$) if there exist finite-dimensional subspaces $V_m$ and constants $S_1, S_2 \in \mathbb{Z}$ such that

1) $V = \bigoplus_{m \in \mathbb{Z}} V_m$;
2) $\dim V_m < \infty \ \forall m \in \mathbb{Z}$;
3) $[V_n, V_m] \subseteq \sum_{h=n+m+S_2}^{n+m+S_1} V_h \ \forall n, m \in \mathbb{Z}$.

If there exists an $R$ such that $\dim V_m \leq R$ for all $m$ it is called strongly almost-graded.

Accordingly, an almost-grading can be defined for associative algebras and for modules over almost-graded algebras.

In the following, we will introduce such a (strong) almost-graded structure for our multi-point Lax operator. The almost-grading will be induced by splitting our set $A$ into $I$ and $O$. Recall that $I = \{P_1, P_2, \ldots, P_N\}$. In the Krichever-Novikov function, the vector field, and the current algebra case this was done by Krichever and Novikov [7] for the two-point situation. In the two-point Lax operator algebra it was done by Krichever and Sheinman [1]. In the two-point case there is only one possible splitting. This contrasts with the multi-point case which turns out to be
more difficult. Multi-point Krichever-Novikov algebras of different types were dealt with by Schlichenmaier [19]–[21]. We will recall this in §3.2.

In §3.3, for each \( m \in \mathbb{Z} \) we will single out a subspace \( \mathfrak{g}_m \) of \( \mathfrak{g} \), which is known as a (quasi-)homogeneous subspace of degree \( m \). The degree is essentially related to the order of the elements of \( \mathfrak{g} \) at the points in \( I \). We will prove the following

**Theorem 3.2.** Induced by the splitting \( A = I \cup O \) the (multi-point) Lax operator algebra \( \mathfrak{g} \) becomes a (strongly) almost-graded Lie algebra

\[
\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m, \quad \dim \mathfrak{g}_m = N \cdot \dim \mathfrak{g}, \quad [\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \bigoplus_{h = n + m}^{n + m + S} \mathfrak{g}_h, \tag{3.1}
\]

where the constant \( S \) is independent of \( n \) and \( m \).

In addition we will show

**Proposition 3.3.** Let \( X \) be an element of \( \mathfrak{g} \). For each \( (m, s) \), where \( m \in \mathbb{Z} \) and \( s = 1, \ldots, N \), there is a unique element \( X_{m,s} \) in \( \mathfrak{g}_m \) such that locally in a neighbourhood of the point \( P_p \in I \) we have\(^2\)

\[
X_{m,s}(z_p) = X z_p^m \cdot \delta_s^p + O(z_p^{m+1}) \quad \forall p = 1, \ldots, N. \tag{3.2}
\]

**Proposition 3.4.** Let \( \{X^u \mid u = 1, \ldots, \dim \mathfrak{g}\} \) be a basis of the finite dimensional Lie algebra \( \mathfrak{g} \). Then

\[
\mathcal{B}_m := \{X^u_{m,p} \mid u = 1, \ldots, \dim \mathfrak{g}, p = 1, \ldots, N\} \tag{3.3}
\]

is a basis in \( \mathfrak{g}_m \), and \( \mathcal{B} = \bigcup_{m \in \mathbb{Z}} \mathcal{B}_m \) is a basis in \( \mathfrak{g} \).

**Proof.** By (3.1) we know that \( \dim \mathfrak{g}_m = N \cdot \dim \mathfrak{g} \). The elements in \( \mathcal{B}_m \) are pairwise distinct. Hence, we have \( \# \mathcal{B}_m = N \cdot \dim \mathfrak{g} \) elements \( \{X^u_{m,p}\} \) in \( \mathfrak{g}_m \). To show it is a basis it suffices to show that these are linearly independent. Suppose that there is a linear combination \( \sum_u \sum_p \alpha^u_{m,p} X^u_{m,p} = 0 \). We consider the local expansions at the point \( P_s \), for \( s = 1, \ldots, N \). From (3.2) we obtain

\[
0 = \left( \sum_u \alpha^u_{m,s} X^u \right) z_s^m + O(z_s^{m+1}).
\]

Hence \( 0 = \sum_u \alpha^u_{m,s} X^u \). As the \( X^u \) form a basis in \( \mathfrak{g} \), this implies that \( \alpha^u_{m,s} = 0 \) for all \( u, s \). Now, it follows from the direct sum decomposition in (3.1) that \( \mathcal{B} \) is a basis of the full algebra \( \mathfrak{g} \).

It is very convenient to introduce the associated filtration

\[
\mathfrak{g}^{(k)} := \bigoplus_{m \geq k} \mathfrak{g}_m, \quad \mathfrak{g}^{(k)} \subseteq \mathfrak{g}^{(k')} \quad k \geq k'. \tag{3.4}
\]

**Proposition 3.5.** (a) \( \mathfrak{g} = \bigcup_{m \in \mathbb{Z}} \mathfrak{g}^{(m)} \).

(b) \( \mathfrak{g}^{(k)} \cap \mathfrak{g}^{(m)} \subseteq \mathfrak{g}^{(k+m)} \).

(c) \( \mathfrak{g}^{(m)}/\mathfrak{g}^{(m+1)} \cong \mathfrak{g}_m \).

(d) The equivalence classes of elements of the set \( \mathcal{B}_m \) (see (3.3)) constitute a basis for the quotient space \( \mathfrak{g}^{(m)}/\mathfrak{g}^{(m+1)} \).

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\(^2\)The symbol \( \delta^p_s \) denotes the Kronecker delta, which is equal to 1 if \( s = p \), and 0 otherwise.
Equation (3.1) implies (a), (b) and (c) directly. Part (d) follows from Proposition 3.4.

There is another filtration
\[ \tilde{g}'_{(m)} := \{ L \in \tilde{g} \mid \text{ord}_{P_s}(L) \geq m, s = 1, \ldots, N \}. \] (3.5)

Note that the elements \( L \) are meromorphic maps from \( \Sigma \) to \( g \), hence it makes sense to talk about the orders of the component functions with respect to a basis. The minimum of these orders is meant in (3.5).

**Proposition 3.6.** (a) \( \tilde{g} = \bigcup_{m \in \mathbb{Z}} \tilde{g}'_{(m)} \).

(b) The two filtrations coincide, that is,
\[ \tilde{g}_{(m)} = \tilde{g}'_{(m)} \quad \forall m \in \mathbb{Z}. \]

**Proof.** Let \( L \in \tilde{g} \). Then as \( g \)-valued meromorphic functions the pole orders of the component functions at the points \( P_s \) are individually bounded. As there are only finitely many, there is a bound \( k \) for the pole order, hence \( L \in \tilde{g}_{(-k)} \). This proves (a) and consequently that \( \{(g'_{(m)})\} \) is a filtration.

By Proposition 3.4 we know that \( B \) is a basis of \( \tilde{g} \). Let \( L \in \tilde{g}'_{(m)} \). Every element of \( L \in \tilde{g} \) is a finite linear combination of the basis elements. The elements of \( B_k \) have exact order \( k \) and are linearly independent. Moreover, with respect to a fixed basis element of the finite dimensional Lie algebra we have \( N \) basis elements in \( B_k \) with orders given by (3.2). Hence the individual orders at the points \( P_s \) cannot increase with nontrivial linear combinations. Hence only \( k \geq m \) can appear in the combination. This shows \( L \in \tilde{g}_{(m)} \). Conversely, all elements from \( B_k \) for \( k \geq m \) obviously lie in the set (3.5). Hence, we have equality.

The second description of the filtration has the big advantage that it is very naturally defined. The only data which enters is the splitting of the points \( A \) into \( I \cup O \). Hence, it is given canonically by \( I \). By contrast, it will turn out that in the multi-point case, if \( \#O > 1 \) we may need to make some choices to fix \( g_m \), such as numbering the points in \( O \), or even use some different rules for the points in \( O \). But via Proposition 3.6 we know that the induced filtration (3.4) does not depend on any of these choices.

Here we should remark that we have given a proof of Proposition 3.6 above. But it is based on results which we will not prove until §3.3, namely, Theorem 3.2 and Proposition 3.3. Our starting point there will be the filtration \( \tilde{g}'_{(m)} \), hence we cannot assume equality from the very beginning.

The following very important result holds.

**Proposition 3.7.** Let \( X_{k,s} \) and \( Y_{m,p} \) be elements in \( \tilde{g}_k \) and \( \tilde{g}_m \) corresponding to \( X, Y \in g \), respectively, then
\[ [X_{k,s}, Y_{m,p}] = [X, Y]_{k+m,s} \delta^p_s + L, \] (3.6)
where \([X, Y]\) is the bracket in \( g \) and \( L \in \tilde{g}_{(k+m+1)} \).

**Proof.** Using the expression (3.2) for \( X_{k,s} \) and \( Y_{m,p} \) we obtain
\[ [X_{k,s}, Y_{m,p}]|_{(z_t)} = [X, Y]z^{k+m}_s \delta^p_t \delta^s_t + O(z^{k+m+1}_t) \]
for every \( t \). Hence, the element
\[
[X_{k,s}, Y_{m,p}] - ([X, Y])_{k+m,s}\delta^p_s
\]
has order \( \geq k + m + 1 \) at all points in \( I \). Using (3.5) and Proposition 3.6 we see that it lies in \( \mathfrak{g}_{(k+m+1)} \), which proves out statement.

3.2. The function algebra \( \mathcal{A} \) and the vector field algebra \( \mathcal{L} \). Before we supply the proofs of the statements in §3.1 we want to introduce those Krichever-Novikov type algebras which are of relevance in the following. We start with the Krichever-Novikov function algebra \( \mathcal{A} \) and the Krichever-Novikov vector field algebra \( \mathcal{L} \). Both algebras are almost-graded algebras
\[
\mathcal{A} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m, \quad \mathcal{L} = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}_m, \quad (3.7)
\]
where the almost-grading is induced by the same splitting of \( A \) into \( I \cup O \) as we used to define the Lax operator algebras. Recall that \( I = \{P_1, \ldots, P_N\} \) and \( O = \{Q_1, \ldots, Q_M\} \).

Let \( \mathcal{A} \) and \( \mathcal{L} \), respectively, be the space of meromorphic functions and meromorphic vector fields on \( \Sigma \), holomorphic on \( \Sigma \setminus A \). In particular, they are also holomorphic at the points in \( W \). Obviously, \( \mathcal{A} \) is an associative algebra under the product of functions and \( \mathcal{L} \) is a Lie algebra under the Lie bracket of vector fields. In the two-point case their almost-graded structure was introduced by Krichever and Novikov [7]. In the multi-point case it was given by Schlichenmaier [20], [21]. The results will be described in the following.

The homogeneous spaces \( \mathcal{A}_m \) have as basis the set of functions \( \{A_{m,s}, s = 1, \ldots, N\} \) given by the condition
\[
\text{ord}_{P_i}(A_{m,s}) = (n+1) - \delta^s_i, \quad i = 1, \ldots, N, \quad (3.8)
\]
and certain compensating conditions at the points in \( O \) to make it unique up to multiplication by a scalar. For example, when \( \#O = M = 1 \), the genus is either 0 or \( > 2 \), and the points are in generic position, then the condition is (with the exception of finitely many \( m \))
\[
\text{ord}_{Q_M}(A_{m,s}) = -N \cdot (n+1) - g + 1. \quad (3.9)
\]
To make it unique we require that the local expansion at the \( P_s \) satisfies (with respect to the chosen local coordinate \( z_s \))
\[
A_{n,s} \big|_{z_s} = z_s^n + O(z_s^{n+1}). \quad (3.10)
\]

The vector field algebra \( \mathcal{L}_m \) has the basis \( \{e_{m,s} \mid s = 1, \ldots, N\} \), where the elements \( e_{m,s} \) are given by the condition
\[
\text{ord}_{P_i}(e_{m,s}) = (n+2) - \delta^s_i, \quad i = 1, \ldots, N, \quad (3.11)
\]
and corresponding compensating conditions at the points in \( O \) to make it unique up to multiplication by a scalar. In exactly the same special situation as above the condition is
\[
\text{ord}_{Q_M}(e_{m,s}) = -N \cdot (n+2) - 3(g - 1). \quad (3.12)
\]
The local expansion at $P_s$ is
\[ e_{n,s}(z_s) = (z_s^{n+1} + O(z_s^{n+2})) \frac{d}{dz_s}. \] (3.13)

There are constants $S_1$ and $S_2$ (not depending on $m, n$) such that
\[ \mathcal{A}_k \cdot \mathcal{A}_m \subseteq \bigoplus_{h=k+m}^{k+m+S_1} \mathcal{A}_h, \quad [\mathcal{L}_k, \mathcal{L}_m] \subseteq \bigoplus_{h=k+m}^{k+m+S_2} \mathcal{L}_h. \] (3.14)

This says that we have an almost-grading. In what follows we will need the fine structure of the almost-grading
\[ A_{k,s} \cdot A_{m,t} = A_{k+m,s} \delta^t_s + Y, \quad Y \in \bigoplus_{h=k+m+1}^{k+m+S_1} \mathcal{A}_h, \] (3.15)
\[ [e_{k,s}, e_{m,t}] = (m-k)e_{k+m,s} \delta^t_s + Z, \quad Z \in \bigoplus_{h=k+m+1}^{k+m+S_2} \mathcal{L}_h. \] (3.16)

Again we have the induced filtrations $\mathcal{A}_{(m)}$ and $\mathcal{L}_{(m)}$.

The elements of the Lie algebra $\mathcal{L}$ act on $\mathcal{A}$ as derivations. This makes the space $\mathcal{A}$ an almost-graded module over $\mathcal{L}$. In particular, we have
\[ e_{k,s} \cdot A_{m,r} = mA_{k+m,s} \delta^r_s + U, \quad U \in \bigoplus_{h=k+m+1}^{k+m+S_3} \mathcal{A}_h, \] (3.17)
where the constant $S_3$ is independent of $k$ and $m$.

Indexed by the almost-grading of $\mathcal{A} = \bigoplus_m \mathcal{A}_m$ we get an almost-grading for the Krichever-Novikov type algebra of current type by setting
\[ g \otimes \mathcal{A} = \bigoplus_m (g \otimes \mathcal{A})_m, \quad (g \otimes \mathcal{A})_m := g \otimes \mathcal{A}_m \quad \forall m \in \mathbb{Z}. \] (3.18)

3.3. The proofs. Readers who are in a hurry, or only interested in the results can skip this rather technical section (involving Riemann-Roch type arguments) during their first reading and jump directly to §4.

Recall the definition
\[ \bar{g}'_{(m)} := \{ L \in \bar{g} \mid \text{ord}_{P_s}(L) \geq m, \ s = 1, \ldots, N \}. \] (3.19)

of the filtration. We will only deal with this filtration in this section, hence for notational reasons we will drop the ' in the following. In the end, the primed and unprimed will coincide.

**Proposition 3.8.** Given $X \in g$, $X \neq 0$, $s = 1, \ldots, N$, $m \in \mathbb{Z}$, there exists at least one $X_{m,s}$ such that
\[ X_{m,s}(z_p) = X z_p^m \delta^s_p + O(z_p^{m+1}). \] (3.20)
The proof is based on the Riemann-Roch theorem. The technique will be used extensively in this section, and so we will introduce some notation before we proceed with the proof. For any $m \in \mathbb{Z}$ we consider certain divisors

$$D_m = (D_m)_I + D_W + (D_m)_O,$$

where

$$(D_m)_I = -m \sum_{s=1}^{N} P_s, \quad (D_m)_O = \sum_{s=1}^{M} a_{s,m} Q_s, \quad a_{s,m} \in \mathbb{Z},$$

$$D_W = \varepsilon \sum_{s=1}^{K} \gamma_s, \quad \varepsilon = 1 \text{ for } \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n), \quad \varepsilon = 2 \text{ for } \mathfrak{sp}(n).$$

Recall that the genus of $\Sigma$ is $g$. Denote a canonical divisor by $K$. Set $L(D)$ to be the space consisting of meromorphic functions $u$ on $\Sigma$ whose divisors satisfy $(u) > -D$. The Riemann-Roch theorem says that

$$\dim L(D) - \dim L(K - D) = \deg D - g + 1.$$  \hspace{1cm} (3.23)

In particular, we have

$$\dim L(D) \geq \deg D - g + 1.$$  \hspace{1cm} (3.24)

There are several cases we need in the following:

1) if $\deg D \geq 2g - 1$, then we have equality in (3.24);
2) if $D$ is a generic divisor then for $g \leq \deg D \leq 2g - 2$ we have equality;
3) if $D \geq 0$ and $D$ is generic we have $\dim L(D) = 1$ for $0 \leq \deg D \leq g - 1$;
4) if $D \not\geq 0$ (meaning that there is at least one point in the support of $D$ with negative multiplicity) and $D$ is generic we have $\dim L(D) = 0$ for $0 \leq \deg D \leq g - 1$;
5) for $g = 0$ every divisor is generic and we have equality in (3.24) as long as the right-hand side is $\geq 0$, that is, $\dim L(D) = \max(0, \deg D + 1)$.

See [29], for instance, for information on divisors, the Riemann-Roch theorem and applications of them; see also [30].

When $u = (u_1, u_2, \ldots, u_r)$ is a vector valued function we define $L(D)$ to be the vector space of vector valued functions with $(u) \geq -D$. This means that $(u_i) \geq -D$ for all $i = 1, \ldots, r$. Now all the dimension formulae have to be multiplied by $r$:

$$\dim L(D) \geq r(\deg D - g + 1).$$  \hspace{1cm} (3.25)

We apply this to our Lax operator algebra $\mathfrak{g}$ by considering the component functions $u_i, i = 1, \ldots, r = \dim \mathfrak{g}$ with respect to a fixed basis. We set

$$L'(D) := \{u \in L(D) \mid u \text{ gives an element of } \mathfrak{g} \} \subseteq L(D).$$  \hspace{1cm} (3.26)

In $L'(D_m)$ we have to take account of the fact that at the weak singular points $\gamma_s$ we have $H$ additional linear conditions which the elements of the solution space $L(D_m)$ must satisfy. These are formulated in terms of the corresponding $\alpha_s$ for some finite part of the Laurent series. In total there are finitely many conditions.
When the \( \alpha_s \) are generic they will exactly compensate for the possible poles at \( \gamma_s \) (see [1]). But for the moment we still consider them to be arbitrary.

By the very definition of the filtration we always have
\[
\mathfrak{g}(m) = L'((D_m)_I), \quad \mathfrak{g}(m) \Supset L'(D_m).
\] (3.27)

**Proof of Proposition 3.8.** We start with a divisor \( D_m \) by choosing the part \( (D_m)_O = T \) such that the degrees of the divisors \( D_m \) and \( D_m - \sum P_i \) are still big enough such that for both case 1) of the Riemann-Roch equality (3.25) is true and \( \dim L(D) = l \geq r(N + 1) + H \). Hence, after applying the \( H \) linear conditions we have \( \dim L'(D) \geq r(N + 1) \). Let \( P_s \) be a fixed point from \( I \). We consider
\[
D'_m = D_m - \sum_{i=1}^{N} P_i, \quad D''_m = D'_m + P_s.
\] (3.28)

This yields
\[
\dim L'(D'_m) = l - rN, \quad \dim L'(D''_m) = l - rN + r.
\] (3.29)

The elements in \( L'(D'_m) \) have orders \( \geq (m + 1) \) at all points in \( I \). The elements in \( L'(D''_m) \) have orders \( \geq (m + 1) \) at all points \( P_i, i \neq s \) and orders \( \geq m \) at \( P_s \). From the dimension formula (3.29) we conclude that there exist \( r \) elements which have the exact order \( m \) at \( P_s \) and orders \( \geq (m + 1) \) at the other points in \( I \). This says that for every basis element \( X^u \) in the Lie algebra \( \mathfrak{g} \) there is an element \( X^u_{m,s} \in \mathfrak{g} \) which has the exact order \( m \) at the point \( P_s \) and order greater than \( m \) at the other points in \( I \) and can be written there as required in (3.20). By linearity we get the statement for all \( X \in \mathfrak{g} \).

**Remark 3.9.**

1. By modifying the degree of the divisor \( T \) we can even show that there exist elements such that the orders of \( X_{m,s} \) at the points \( P_p, p \neq s \), are equal to \( m + 1 \).

2. We remark that we have not used genericity arguments in this proof, either with respect to the points \( A \) and \( W \) or with respect to the parameter \( \alpha_s \). Hence, the statement is true in all situations.

3. The local coordinate \( z_s \) enters into the very definition of \( X_{m,s} \). In fact it only depends on the first order jet of the coordinate, and two different elements will just differ by a rescaling.

4. The elements \( X_{m,s} \) are highly nonunique. To introduce the almost-grading we will have to make them essentially unique by trying to find a divisor \( T \) as small as possible but such that the statement is still true. Later on we will come back to this.

**Proposition 3.10.** Let \( X^u, u = 1, \ldots, \dim \mathfrak{g}, \) be a basis of \( \mathfrak{g} \) and let
\[
X^u_{m,s}, \quad u = 1, \ldots, \dim \mathfrak{g}, \quad s = 1, \ldots, N, \quad m \in \mathbb{Z},
\] (3.30)
be any fixed set of elements chosen according to Proposition 3.8. Then
(a) these elements are linearly independent;
(b) the set of classes \([X^u_{m,s}]), u = 1, \ldots, \dim \mathfrak{g}, s = 1, \ldots, N,\) will constitute a basis of the quotient \( \mathfrak{g}(m)/\mathfrak{g}(m+1); \)
(c) \( \dim \mathfrak{g}(m)/\mathfrak{g}(m+1) = N \cdot \dim \mathfrak{g}; \)
(d) the classes of the elements \( X^u_{m,s} \) will not depend on the elements chosen.
Proof. Using the local expansion it follows, as in the proof of Proposition 3.4, that
the elements (3.30) are linearly independent, hence (a). Furthermore, ignoring
higher orders, that is, elements from \( \mathfrak{g}_{(m+1)} \), they stay linearly independent. Hence
(b) and (c) follow. Part (d) is true by the very definition of the elements.

Given \( X \in \mathfrak{g} \) for the moment we will denote any element fulfilling the conditions
in Proposition 3.8 by \( X_{m,s} \).

As the proof of Proposition 3.7 also remains valid for these elements we have

**Proposition 3.11.** The algebra \( \mathfrak{g} \) is a filtered algebra with respect to the filtration
(\( \mathfrak{g}_{(m)} \)) introduced, that is,

\[
[\mathfrak{g}_{(m)}, \mathfrak{g}_{(k)}] \subseteq \mathfrak{g}_{(m+k)}.
\]

Moreover,

\[
[X_{k,s}, Y_{m,p}] = [X, Y]_{k+m,s}\delta_p^s + L, \quad L \in \mathfrak{g}_{(m+k+1)}.
\]

Our next goal is to introduce the homogeneous subspaces \( \mathfrak{g}_m \). One method would
be to take the linear span of a fixed set of elements (3.30) for \( \mathfrak{g}_m \), but this is too
naive. The condition of almost-gradedness with respect to the lower bound would
be fulfilled by \( m + k \), but not necessarily for the upper bound. To fix this we have
to place more strict conditions on the pole orders at \( O \), and we have to specify
the divisor \( (D_m)_O \) in a coherent manner (with respect to \( m \)). Using our recipe
the elements will become essentially unique in the generic situation, at least for
nearly all \( m \). For non-generic \( \alpha_s \) it might be necessary to modify the prescription
for individual component functions. But all these modifications will only change
the upper bound by a constant.

**Remark 3.12.** Before we move on, we recall that both \( \mathcal{A} \) and the usual Krichever-
Novikov current algebra \( \mathfrak{g} \otimes \mathcal{A} \) have an almost-graded structure.

1. As we explained in §2 we have the direct sum decomposition (2.8). Moreover,
\( \mathfrak{g}(n) \cong \mathcal{A} \). Hence the scalar part is almost-graded and fulfills Theorem 3.2 and
Proposition 3.3. If we can prove the statements for \( \mathfrak{g}(n) \) then they will follow for
\( \mathfrak{g}(n) \). Hence, in the following, it is enough to consider the case when \( \mathfrak{g} \) is simple.

2. Moreover, if the Tyurin data is empty (or all \( \alpha_s = 0 \)) then our Lax operator
algebras reduce to the Krichever-Novikov current algebras. For these the statements
hold. Hence, it is enough to consider Lax operator algebras with nonempty Tyurin
data. The reader might ask why we make this differentiation. In fact, for nonempty
Tyurin data the proof will need fewer case distinctions.

We shall now give a general description for the generic situation for \( \mathfrak{g} \) simple,
and prove the claim about almost-gradedness in detail. In the non-generic situation
we will show where things have to be modified.

Recall that for the divisor \( D_m \) we had the decomposition (3.21). The terms
\((D_m)_l \) and \( D_W \) stay as above. For \((D_m)_O \) we require that

\[
(D_m)_O = \sum_{i=1}^{M} (a_i m + b_{m,i}) Q_i,
\]
with \( a_i, b_{m,i} \in \mathbb{Q} \) such that \( a_i m + b_{m,i} \in \mathbb{Z} \), \( a_i > 0 \), and also that there exists a \( B \) such that \( |b_{m,i}| < B \) \( \forall m \in \mathbb{Z}, i = 1, \ldots, M \). Furthermore,

\[
\sum_{i=1}^{M} a_i = N, \quad \sum_{i=1}^{M} b_{m,i} = N + g - 1, \quad (D_{m+1})_O > (D_m)_O. \tag{3.34}
\]

For the degrees we calculate

\[
\deg((D_m)_O) = m \cdot N + (N + g - 1), \quad \deg((D_{m+1})_O) = \deg((D_m)_O) + N. \tag{3.35}
\]

**Example 3.13.**

1. For \( M = 1 \) we have the unique solution

\[ (D_m)_O = (N \cdot m + (N + g - 1))Q_M. \tag{3.36} \]

2. For \( N \geq M \) the prescription

\[ (D_m)_O = (m + 1) \sum_{j=1}^{M-1} Q_j + ((N - M + 1)(m + 1) + g - 1)Q_M \tag{3.37} \]

will do. Apart from the \( D_W \) the corresponding divisor \( D_m \) was introduced in [21] where the almost-grading in the case of multi-point Krichever-Novikov algebras and tensors was considered for the first time (see also [24]).

3. In [21] prescriptions for the case \( N < M \) were also given. We shall not reproduce it here.

Hence in all cases we can find appropriate divisors.

Now we set

\[ \bar{g}_m := \{ L \in \mathfrak{g} \mid (L) \geq -D_m \}. \tag{3.38} \]

**Proposition 3.14.**

(a) \( \dim \bar{g}_m = N \dim \mathfrak{g} \).

(b) A basis of \( \bar{g}_m \) is given by elements \( X_{m,s}^u \), \( u = 1, \ldots, \dim \mathfrak{g} \), \( s = 1, \ldots, N \), satisfying the conditions

\[ X_{m,s}^u (z_p) = X^u z^m s \delta_p + O(z^{m+1}). \tag{3.39} \]

**Proof.** We set \( r := \dim \mathfrak{g} \). First we deal with the generic situation. As we explained above, at the weak singular points we have just as many relations as we have parameters from the poles. Hence in the calculation of \( \dim L'(D) \) the contribution of the degree of \( D_W \) (which is \( \varepsilon \cdot K \)) will be cancelled by the relations (there are \( r \cdot \varepsilon \cdot K \) of them). Here, depending on \( \mathfrak{g} \), \( \varepsilon \) is equal to 1 or 2. For the degree of \( D_m \) we calculate

\[ \deg D_m = g + (N - 1) + \varepsilon K \geq g. \tag{3.40} \]

We stay in the region where equality for (3.25) is true and calculate

\[ \dim L'(D_m) = \dim L(D_m) - \varepsilon r K = r N + r \varepsilon K - \varepsilon r K = r N. \tag{3.41} \]

As, by definition, \( \bar{g}_m = L'(D_m) \), we obtain (a).
Next we consider \( D'_m = D_m - \sum_{i=1}^{N} P_i \). We calculate its degree to be
\[
\deg \left( D_m - \sum_{i=1}^{N} P_i \right) = g - 1 + \varepsilon K.
\]
As \( K \geq 1 \), we are still in the domain where we have equality for the Riemann-Roch theorem. Hence \( \dim L'(D'_m) = 0 \). Now for \( D''_m = D'_m + P_s \) we calculate \( \dim L'(D''_m) = r \). This shows that for every basis element \( X^u \) of \( \mathfrak{g} \), up to multiplication by a scalar, there exists a unique element \( X^u_{m,s} \in \mathfrak{g}_m \) which has the local expansion
\[
X^u_{m,s}(z_p) = X^u \delta_s^p z_p + O(z_p^{m+1}).
\] (3.42)
Hence, (b) holds.

In the non-generic case we have to change the pole orders in the definition of the divisor part \( (D_m)_O \) in a minimal way by adding or subtracting finitely many points to reach a situation where we obtain both the dimension formula and the existence of a basis of the required type. We have to take care that the maximal number of changes needed will be bounded independent of \( m \). In fact, this number is bounded by the number of points \( Q \) from \( O \) we need to add to the divisor \( D_m \) in the generic case (which is of degree \( N + g - 1 + \varepsilon K \)) to reach a divisor \( D'_m \) with \( \deg D'_m \geq 2g - 1 + H \), where \( H \) is the number of relations for the \( \alpha_s \).

**Proposition 3.15.**
\[
\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m.
\] (3.43)

**Proof.** The elements \( X^u_{m,s} \) introduced as the basis elements in \( \mathfrak{g}_m \) are elements of the type in Proposition 3.8 with respect to the grading. By Proposition 3.10 they remain linearly independent even if we consider all the \( m \)'s together, as their classes are linearly independent. Hence, the sum on the right-hand side of (3.43) is a direct sum.

To avoid having to take care of the special adjustments needed for the non-generic situations we consider \( m \gg 0 \) and the divisor
\[
E_m := -(D_m)_I + D_W + (D_m)_O = m \sum_{i=1}^{N} P_i + D_W + (D_m)_O,
\] (3.44)
where \( (D_m)_O \) is the divisor used for fixing the basis elements in \( \mathfrak{g}_m \); see (3.33). Its degree satisfies
\[
\deg E_m = 2mN + (N + g - 1) + \varepsilon K.
\] (3.45)
For \( m \gg 0 \) we are in the region where (3.25) is an equality. Hence, after subtracting the relations we get
\[
\dim L'(E_m) = \dim \mathfrak{g} \cdot ((2m + 1)N).
\] (3.46)

The basis elements
\[
X^u_{k,s}, \quad u = 1, \ldots, \dim \mathfrak{g}, \quad s = 1, \ldots, N, \quad -m \leq k \leq m,
\] (3.47)
are in $L'(E_m)$. This is shown by considering the orders at I and O. For I it is obvious. For O we have to use the fact that $(D_{(k+1)})_O > (D_k)_O$, which comes from (3.34). Hence, $- (D_m)_O$ is a lower bound for the O-part of the divisors for the element (3.47). But these consist of $(2m + 1) \cdot N \cdot \dim g$ linearly independent elements. Hence,

$$L'(E_m) = \bigoplus_{k=-m}^m \mathfrak{g}_k.$$ (3.48)

An arbitrary element $L \in \mathfrak{g}$ has only finite pole orders at the points in I and O. Hence, there exists an $m$ such that $L \in L'(E_m)$. This is again obvious for the points in I. For the points in O, using the conditions for $(D_m)_O$, see (3.33), for all $i = 1, \ldots, M$ we have that $a_i > 0$. Hence every pole order at O will be superseded by a $(D_m)_O$ with $m$ suitably large. This proves the statement.

**Proposition 3.16.** There exist a constant $S$ independent of $n$ and $m$ such that

$$[g_m, g] \subseteq m + k + S M h = m + k g h.$$ (3.49)

**Proof.** We will give the proof for the generic case (and $g$ simple) first and then point out the modification needed for the general situation. Let $L \in [\mathfrak{g}_m, \mathfrak{g}_k]$; then

$$(L) \geq - (D_m + D_k)_I - D_W - (D_m + D_k)_O$$ (3.50)

(observe that $D_W$ does not redouble here). We consider the divisors $D_h$. Recall formula (3.33). As $a_i > 0$ for all $i$, there exists an $h_0$ such that $\forall h \geq h_0$ we have

$$(D_h)_O \geq (D_m + D_k)_O.$$ (3.51)

Hence, there exists a smallest $h \in \mathbb{Z}$ such that (3.51) is still true. We call this $h_{\text{max}}$. Again by (3.34) $h_{\text{max}} \geq m + k$. Now we consider the divisor

$$E_m = (D_m + D_k)_I + D_W + (D_{h_{\text{max}}})_O.$$ (3.52)

From (3.35) we calculate

$$\deg((D_{h_{\text{max}}})_O) = \deg((D_m + D_k)_O) + (h_{\text{max}} - (m + k))N.$$ (3.53)

Hence,

$$\deg(E_m) = -(m + k)N + \varepsilon K + h_{\text{max}} \cdot N + (N + g - 1).$$ (3.54)

As $\deg(E_m) \geq g$, under the assumption of genericity we stay in the region where

$$\dim L'(E_m) = \deg g \cdot (h_{\text{max}} - (m + k) + 1)N.$$ (3.55)

As in the proof of Proposition 3.15 we find that the elements (3.47) for $m + k < h \leq h_{\text{max}}$ lie in $L'(E_m)$. They are linearly independent, hence

$$L'(E_m) = \bigoplus_{h=n+m}^{h_{\text{max}}} \mathfrak{g}_h.$$ (3.56)
By (3.50) the $L$ we started with also lies in $L'(E_m)$ and consequently is also in the right-hand side of (3.56).

To show almost-grading we have to show that there exists an $S$ (independent of $m$ and $k$ such that $h_{\text{max}} = m + k + S$. The relation (3.51) can be rewritten as

$$a_i h + b_{h,i} \geq a_i (m + k) + b_{m,i} + b_{k,i} \quad \forall i = 1, \ldots, M.$$  

(3.57)

This can be rewritten as

$$h \geq m + k + \frac{b_{m,i} + b_{k,i} - b_{h,i}}{a_i} \quad \forall i = 1, \ldots, M.$$  

(3.58)

The smallest $h$ for which this is true is

$$h_{\text{max}} = m + k + \min_{i=1,\ldots,M} \left[ \frac{b_{m,i} + b_{k,i} - b_{h,i}}{a_i} \right],$$  

(3.59)

where for any real number $x$, $\lceil x \rceil$ denotes the smallest integer $\geq x$. As our $|b_{m,i}|$ are bounded uniformly by $B$ the third term in (3.59) will also be uniformly bounded by a constant $S$. Hence, we get almost-grading. In the case of non-generic points and $\alpha_s$’s the divisors at $O$ have to be modified by finitely many modifications. Hence the constant $S$ has to be adapted by adding a finite constant to it. But everything still remains almost-graded.

From the proof we can even calculate $h_{\text{max}}$ if needed. As an example we give

**Corollary 3.17.** In the generic simple Lie algebra case for $N \geq M$ with the standard prescription (3.37) we have

$$h_{\text{max}} = n + m + S$$

with

$$S = \begin{cases} 
0, & g = 0, \quad N = M = 1, \\
1, & g = 0, \quad M > 1, \\
1, & g = 1, \\
1 + \left\lfloor \frac{g - 1}{N - M + 1} \right\rfloor, & g \geq 2.
\end{cases}$$  

(3.60)

**Proof.** For the standard prescription we have

$$a_i = 1, \quad i = 1, \ldots, M - 1, \quad a_M = N - M + 1,$$

$$b_i = b_{m,i} = 1, \quad i = 1, \ldots, M - 1, \quad b_M = b_{m,M} = N - M + g.$$  

(3.61)

Hence,

$$S = \max_{i=1,\ldots,M} \left[ \frac{b_i}{a_i} \right],$$  

(3.62)

which yields the result.

Now we are ready to collate the results of Propositions 3.14–3.16. The statements yield both the statements of Theorem 3.2 and Proposition 3.3. All the statements in §3.2 have now been shown to be true. In particular, now we know that the
filtrations (3.5) and (3.4) coincide. Hence, (3.4) is also canonically defined by the splitting of \(A\) into \(I\) and \(O\).

A Lie algebra \(\mathcal{V}\) is called perfect if \(\mathcal{V} = [\mathcal{V}, \mathcal{V}]\). Simple Lie algebras are of course perfect. The usual Krichever-Novikov current algebras \(\mathfrak{g}\) for \(\mathfrak{g}\) simple are perfect too ([12], Proposition 3.2). Lax operator algebras are not necessarily perfect (at least we have no proof that they are). Lemma 3.18 below might be considered as a weak analogue of this property.

**Lemma 3.18.** Let \(\mathfrak{g}\) be simple and let \(y \in \mathfrak{g}\); then for every \(m \in \mathbb{Z}\) there exist finitely many elements \(y^{(s,1)}, y^{(s,2)} \in \mathfrak{g}\), \(i = 1, \ldots, l = l(m)\), such that

\[
y - \sum_{s=1}^{l} [y^{(s,1)}, y^{(s,2)}] \in \mathfrak{g}_m. \tag{3.63}
\]

**Proof.** Let \(y\) be an element of \(\mathfrak{g}\). Hence there exists a \(k\) such that \(y \in \mathfrak{g}(k)\), but \(y \notin \mathfrak{g}(k+1)\). In particular, for every point \(P_i\) there exist elements \(X_{k,i}^i\) such that

\[
y - \sum_{i=1}^{N} X_{k,i}^i \in \mathfrak{g}_{k+1}, \tag{3.64}
\]

where \(X_{k,i}^i = (X^i)_{k,i}\) is the element corresponding to \(X^i \in \mathfrak{g}\). As \(\mathfrak{g}\) is perfect we have \(X^i = [Y^i, Z^i]\), where \(Y^i, Z^i \in \mathfrak{g}\). We calculate

\[
X_{k,i}^i = [Y_{0,i}^i, Z_{k,i}^i] + y^i, \quad y^{(i)} \in \mathfrak{g}_{k+1}. \tag{3.65}
\]

In total

\[
y^{(k)} = y - \sum_{i=1}^{N} [Y_{0,i}^i, Z_{k,i}^i] \in \mathfrak{g}_{k+1}. \tag{3.66}
\]

Using the same procedure for \(y^{(k)}\) etc., we can approximate \(y\) to every finite order by sums of commutators.

§4. Module structure

4.1. Lax operator algebras as modules over \(\mathfrak{a}\). The space \(\mathfrak{g}\) is an \(\mathfrak{a}\)-module with respect to pointwise multiplication. Obviously, the relations (2.5), (2.6), (2.9), (2.11) are not altered.

**Proposition 4.1.** (a) The Lax operator algebra \(\mathfrak{g}\) is an almost-graded module over \(\mathfrak{a}\), that is, there exists a constant \(S_4\) (not depending on \(k\) and \(m\)) such that

\[
\mathfrak{a}_k \cdot \mathfrak{g}_m \subseteq \bigoplus_{h=k+m}^{k+m+S_4} \mathfrak{g}_h. \tag{4.1}
\]

(b) For \(X \in \mathfrak{g}\)

\[
A_{m,s} \cdot X_{n,p} = X_{m+n,s} \delta_p^s + L, \quad L \in \mathfrak{g}_{m+n+1}. \tag{4.2}
\]
Proof. We consider the orders of the elements in $I$ and $O$. As in the proof of Proposition 3.16 the existence of a constant $S_4$ follows so that (4.1) is true. Hence (a).

We study the lowest order term of $A_{m,s} \cdot X_{n,r}$ at the points $P_i \in I$. Using (3.20), (3.8), (3.10) we see that if $s \neq r$ then $A_{m,s} \cdot X_{n,r} \in \mathfrak{g}_{m+n+1}$ as all orders are $\geq n + m + 1$. The same is true for $s = r$ for the element $A_{m,s} \cdot X_{n,s} - X_{m+n,s}$. This yields the required result.

Warning: in general we do not have $A_{m,s} \cdot X_{0,s} = X_{m,s}$ as the orders at $O$ do not coincide. Also, $A_{m,s} \cdot X$ does not necessarily belong to $\mathfrak{g}$.

4.2. Lax operator algebras as modules over $\mathcal{L}$. Now we introduce an action of $\mathcal{L}$ on $\mathfrak{g}$. This is done with the help of a certain connection $\nabla^{(\omega)}$, along the lines of [1]–[3], with the modification made in [16]. The connection form $\omega$ is a $\mathfrak{g}$-valued meromorphic 1-form, holomorphic outside $I$, $O$ and $W$, and has a certain prescribed behaviour at the points of $W$. For $\gamma_s \in W$ with $\alpha_s = 0$ the requirement is that $\omega$ is also regular there. For the points $\gamma_s$ with $\alpha_s \neq 0$ we require that it has an expansion of the form

$$\omega(z_s) = \left( \frac{\omega_{s,1}}{z_s} + \omega_{s,0} + \omega_{s,1} z_s + \sum_{k>1} \omega_{s,k} z_s^k \right) dz_s. \quad (4.3)$$

For $\mathfrak{gl}(n)$: there exist $\tilde{\beta}_s \in \mathbb{C}^n$ and $\tilde{\kappa}_s \in \mathbb{C}$ such that

$$\omega_{s,1} = \alpha_s \tilde{\beta}_s, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s \alpha_s, \quad \text{tr}(\omega_{s,1}) = \tilde{\beta}_s^t \alpha_s = 1. \quad (4.4)$$

For $\mathfrak{so}(n)$: there exist $\tilde{\beta}_s \in \mathbb{C}^n$ and $\tilde{\kappa}_s \in \mathbb{C}$ such that

$$\omega_{s,1} = \alpha_s \tilde{\beta}_s - \tilde{\beta}_s \alpha_s^t, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s \alpha_s, \quad \tilde{\beta}_s^t \alpha_s = 1. \quad (4.5)$$

For $\mathfrak{sp}(2n)$: there exist $\tilde{\beta}_s \in \mathbb{C}^{2n}$, $\tilde{\kappa}_s \in \mathbb{C}$ such that

$$\omega_{s,1} = (\alpha_s \tilde{\beta}_s^t + \tilde{\beta}_s \alpha_s^t) \sigma, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s \alpha_s, \quad \alpha_s^t \sigma \omega_{s,1} \alpha_s = 0, \quad \tilde{\beta}_s^t \sigma \alpha_s = 1. \quad (4.6)$$

The existence of nontrivial connection forms that satisfy the listed conditions is proved by Riemann-Roch type argument as in Proposition 3.8. We can even require, and actually always will, that the connection form is holomorphic at $I$. Note also that if all $\alpha_s = 0$ we can take $\omega = 0$.

The connection form $\omega$ induces the following connection $\nabla^{(\omega)}$ on $\mathfrak{g}$

$$\nabla^{(\omega)} = d + [\omega, \cdot]. \quad (4.7)$$

Let $e \in \mathcal{L}$ be a vector field. In a local coordinate $z$ the connection form and the vector field are represented as $\omega = \tilde{\omega} dz$ and $e = \tilde{e} \frac{d}{dz}$ with a local function $\tilde{e}$ and a local matrix valued function $\tilde{\omega}$. The covariant derivative in the direction of $e$ is given by

$$\nabla^{(\omega)}_e = dz(e) \frac{d}{dz} + [\omega(e), \cdot] = e + [\tilde{\omega} \tilde{e}, \cdot] = \tilde{e} \cdot \left( \frac{d}{dz} + [\tilde{\omega}, \cdot] \right). \quad (4.8)$$
Here the first term (e.) corresponds to taking the usual derivative of functions in each matrix element separately, whereas \( \widetilde{e} \) means multiplication by the local function \( \widetilde{e} \).

Using the last description, for \( L \in \mathfrak{g}, \ g \in \mathcal{A}, \ e, f \in \mathcal{L} \) we obtain
\[
\nabla_e^{(\omega)}(g \cdot L) = (e.g) \cdot L + g \cdot \nabla_e^{(\omega)}L, \quad \nabla_g^{(\omega)}L = g \cdot \nabla_e^{(\omega)}L
\]  
and
\[
\nabla_{[e,f]}^{(\omega)} = [\nabla_e^{(\omega)}, \nabla_f^{(\omega)}].
\]  
The proofs of the following statements are identical to the proofs for the two-point case presented in [16]. Hence, they are omitted here.

**Proposition 4.2.** (a) \( \nabla_e^{(\omega)} \) act as a derivation on the Lie algebra \( \mathfrak{g} \), that is,
\[
\nabla_e^{(\omega)}[L, L'] = [\nabla_e^{(\omega)}L, L'] + [L, \nabla_e^{(\omega)}L'].
\]  
(b) The covariant derivative makes \( \mathfrak{g} \) into a Lie module over \( \mathcal{L} \).
(c) The decomposition \( \mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n) \) is a decomposition into \( \mathcal{L} \)-modules, that is,
\[
\nabla_e^{(\omega)} : \mathfrak{s}(n) \to \mathfrak{s}(n), \quad \nabla_e^{(\omega)} : \mathfrak{sl}(n) \to \mathfrak{sl}(n).
\]  
Moreover, the \( \mathcal{L} \)-module \( \mathfrak{s}(n) \) is equivalent to the \( \mathcal{L} \)-module \( \mathcal{A} \).

**Proposition 4.3.** (a) \( \mathfrak{g} \) is an almost-graded \( \mathcal{L} \)-module.
(b) For the corresponding \( \mathcal{L} \)-action we have
\[
\nabla_{e_{k,s}}^{(\omega)} X_{m,r} = m \cdot X_{k+m,s} + L, \quad L \in \mathfrak{g}_{k+1}
\]  

**Proof.** (a) By Proposition 4.2 \( \mathfrak{g} \) is an \( \mathcal{L} \)-module. It remains to show that there is an upper bound for the order of the elements of the type \( n + m + S_5 \), with \( S_5 \) independent of \( n \) and \( m \) (but it may depend on \( \omega \)). We write out (4.8) for homogeneous elements and obtain
\[
\nabla_{e_{k,s}}^{(\omega)} X_{m,r} = e_{k,s} \cdot X_{m,r} + [\widehat{\omega}e_{k,s}, X_{m,r}].
\]  
The form \( \omega \) has fixed orders at \( I \) and at \( O \), the action of \( \mathcal{L} \) on \( \mathcal{A} \) is almost-graded, and the bracket corresponds to the commutator in the almost-graded \( \mathfrak{g} \). By considering the corresponding bounds for the order of poles at \( I \) and \( O \) we get the required universal bound.

(b) Locally at \( P_i, i = 1, \ldots, N \), we have
\[
X_{m,r}(z_i) = X z_i^n \delta_i^r + O(z_i^{n+1}), \quad e_{k}|(z_i) = z_i^{k+1} \delta_i^d \frac{d}{dz} + O(z_i^{k+2}).
\]  
This implies that
\[
e_{k,s} \cdot X_{m,r}(z_i) = mX z_i^{k+m} \delta_i^s \delta_i^r + O(z_i^{k+m+1}), \quad \widehat{\omega}e_{k}(z_i) = B z_i^{k+1} + O(z_i^{k+2}),
\]  
with \( B \in \mathfrak{gl}(n) \). Hence
\[
[\widehat{\omega}e_{k}, X_{m}] = O(z_i^{k+m+1}) \quad \forall i,
\]  
and the second term will only contribute to higher order. It remains to consider the first term in (4.14). If \( r \neq s \) then \( e_{k,s} \cdot X_{m,r}(z_i) \in O(z_i^{k+m+1}) \) for all \( i \). If \( r = s \) then \( e_{k,s} \cdot X_{m,s} - mX_{m+k,s}(z_i) \in O(z_i^{k+m+1}) \). Hence, (4.13) follows.
4.3. Module structure over $\mathcal{D}^1$ and the algebra $\mathcal{D}_g^1$. The Lie algebra $\mathcal{D}^1$ of meromorphic differential operators on $\Sigma$ of degree $\leq 1$, holomorphic outside $I \cup O$, is defined as the semi-direct sum of $\mathcal{A}$ and $\mathcal{L}$, where the commutator between them is given by the action of $\mathcal{L}$ on $\mathcal{A}$. It is the vector space direct sum $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{L}$, with Lie bracket
\[
[(g,e), (h,f)] := (e.h - f.g, [e,f]). \tag{4.18}
\]
In particular
\[
[e,h] = e.h. \tag{4.19}
\]
It is an almost graded Lie algebra \[25\].

Proposition 4.4. The Lax operator algebras $\bar{\mathcal{g}}$ are almost graded Lie modules over $\mathcal{D}^1$ via
\[
e.L := \nabla_e^{(\omega)} L, \quad h.L := h \cdot L. \tag{4.20}
\]

Proof. As $\bar{\mathcal{g}}$ is an almost graded $\mathcal{A}$- and $\mathcal{L}$-module it is enough to show that the relation (4.19) is satisfied. For $e \in L$, $h \in A$, $L \in \mathcal{g}$, using (4.8) we get
\[
e.(h.L) - h.(e.L) = \nabla_e^{(\omega)}(hL) - h\nabla_e^{(\omega)}(L)
= \frac{d(hL)}{dz} + [\bar{\omega}, hL] - h\frac{dL}{dz} + [\bar{\omega}, L] = \left(\frac{d}{dz}\right)L = (e.h)L = [e,h].L.
\]

The Lax operator algebra $\bar{\mathcal{g}}$ is a module over the Lie algebra $\mathcal{L}$ which acts on $\bar{\mathcal{g}}$ by derivations (according to Proposition 4.2). Hence, as above, we can consider the semi-direct sum $\mathcal{D}^1_{\bar{\mathcal{g}}} = \bar{\mathcal{g}} \oplus \mathcal{L}$, with Lie product given by
\[
[e,L] := e.L = \nabla_e^{(\omega)} L. \tag{4.21}
\]
for the mixed pairs. See \[12\] for the corresponding construction for the classical Krichever-Novikov algebras of affine type.

§ 5. Cocycles

In this section we will study 2-cocycles for the Lie algebra $\bar{\mathcal{g}}$ with values in $\mathbb{C}$. It is well-known that the corresponding cohomology space $\check{H}^2(\bar{\mathcal{g}}, \mathbb{C})$ classifies equivalence classes of (one-dimensional) central extensions of $\bar{\mathcal{g}}$.

For the convenience of the reader we recall that a 2-cocycle for $\bar{\mathcal{g}}$ is a bilinear form $\gamma: \bar{\mathcal{g}} \times \bar{\mathcal{g}} \to \mathbb{C}$ which is antisymmetric and satisfies
\[
\gamma([L,L'], L'') + \gamma([L',L''], L) + \gamma([L'', L], L') = 0, \quad L, L', L'' \in \bar{\mathcal{g}}. \tag{5.1}
\]
A 2-cocycle $\gamma$ is a coboundary if there exists a linear form $\phi$ on $\bar{\mathcal{g}}$ such that
\[
\gamma(L,L') = \phi([L,L']), \quad L, L' \in \bar{\mathcal{g}}. \tag{5.2}
\]

Given a 2-cocycle $\gamma$ for $\bar{\mathcal{g}}$, the associated central extension $\hat{\mathcal{g}}_\gamma$ is given as a vector space direct sum $\hat{\mathcal{g}}_\gamma = \bar{\mathcal{g}} \oplus \mathbb{C} \cdot t$, with Lie product
\[
[[\hat{L}, \hat{L}'], [\hat{L}, \hat{L}'] + \gamma(L,L') \cdot t, \quad [\hat{L}, t] = 0, \quad L, L' \in \bar{\mathcal{g}}. \tag{5.3}
\]
Here have we used \( \hat{L} := (L, 0) \) and \( t := (0, 1) \). Conversely, every central extension

\[
0 \longrightarrow \mathbb{C} \xrightarrow{i_2} \hat{\mathfrak{g}} \xrightarrow{p_1} \mathfrak{g} \longrightarrow 0
\]  

(5.4)
defines a 2-cocycle \( \gamma : \mathfrak{g} \rightarrow \mathbb{C} \) by choosing a section \( s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}} \).

Two central extensions \( \hat{\mathfrak{g}}_\gamma \) and \( \hat{\mathfrak{g}}_{\gamma'} \) are equivalent if and only if the defining cocycles \( \gamma \) and \( \gamma' \) are cohomologous.

### 5.1. Geometric cocycles

Next we introduce geometric 2-cocycles. Let \( \omega \) be a connection form as introduced in §4.2 for defining the connection (4.7). Furthermore, let \( C \) be a differentiable cycle on \( \Sigma \) (not necessarily connected), which does not meet the sets \( A = I \cup O \) and \( W \).

As in the two point situation considered in [16] we define the following bilinear forms on \( \mathfrak{g} \):

\[
\gamma_1,\omega,C(L, L') = \frac{1}{2\pi i} \int_C \text{tr}(L \cdot \nabla^{(\omega)} L'), \quad L, L' \in \mathfrak{g},
\]  

(5.5)

and

\[
\gamma_2,\omega,C(L, L') = \frac{1}{2\pi i} \int_C \text{tr}(L) \cdot \text{tr}(\nabla^{(\omega)} L'), \quad L, L' \in \mathfrak{g}.
\]  

(5.6)

The following propositions and their proofs remain the same as in [16] (but of course they now have to be interpreted in this more general context), and we will not repeat them.

**Proposition 5.1.** The bilinear forms \( \gamma_1,\omega,C \) and \( \gamma_2,\omega,C \) are cocycles.

**Proposition 5.2.**

(a) The cocycle \( \gamma_2,\omega,C \) does not depend on the choice of the connection form \( \omega \).

(b) The cohomology class \([\gamma_1,\omega,C]\) does not depend on the choice of the connection form \( \omega \). More precisely

\[
\gamma_1,\omega,C(L, L') - \gamma_1,\omega',C(L, L') = \frac{1}{2\pi i} \int_C \text{tr}((\omega - \omega') \cdot [L, L']).
\]  

(5.7)

As \( \gamma_2,\omega,C \) does not depend on \( \omega \) we will drop \( \omega \) from the notation. Note that \( \gamma_{2,C} \) vanishes on \( \mathfrak{g} \) for \( \mathfrak{g} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n) \). But it does not vanish on \( \mathfrak{g}(n) \), hence it does not vanish on \( \mathfrak{g}(n) \) either.

### 5.2. \( \mathcal{L} \)-invariant cocycles

As we explained in §4.2, after fixing a connection form \( \omega' \) the vector field algebra \( \mathcal{L} \) operates on \( \mathfrak{g} \) via the covariant derivative \( e \mapsto \nabla^{(\omega')}_e \).

**Definition 5.3.** A cocycle \( \gamma \) for \( \mathfrak{g} \) is called \( \mathcal{L} \)-invariant (with respect to \( \omega' \)) if

\[
\gamma(\nabla^{(\omega')}_e L, L') + \gamma(L, \nabla^{(\omega')}_e L') = 0 \quad \forall e \in \mathcal{L}, \quad \forall L, L' \in \mathfrak{g}.
\]  

(5.8)

**Proposition 5.4.**

(a) The cocycle \( \gamma_{2,C} \) is \( \mathcal{L} \)-invariant.

(b) If \( \omega = \omega' \) then the cocycle \( \gamma_{1,\omega,C} \) is \( \mathcal{L} \)-invariant.
The proof is the same as that given in [16] for the two point case.

We call a cohomology class $L$-invariant if it has a representing cocycle which is $L$-invariant. The reader should be warned that this does not mean that all representing cocycles are $L$-invariant. On the contrary, see Corollary 6.5. Clearly, the $L$-invariant classes constitute a subspace of $H^2(\mathfrak{g}, \mathbb{C})$, which we denote by $H^2_L(\mathfrak{g}, \mathbb{C})$.

**5.3. Some remarks on the cocycles on $\mathfrak{g}^1$.** In the following let $\omega = \omega'$. The property that a cocycle is $L$-invariant has a deeper meaning. In §4.3 we introduced the algebra $\mathfrak{g}^1$. The Lax operator algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}^1$. Given a 2-cocycle $\gamma$ for $\mathfrak{g}$ we might extend it to $\mathfrak{g}^1$ as a bilinear form by setting (here $L, L' \in \mathfrak{g}, e, f \in \mathfrak{L}$)

$$
\tilde{\gamma}(L, L') = \gamma(L, L'), \quad \tilde{\gamma}(e, L) = \tilde{\gamma}(L, e) = 0, \quad \tilde{\gamma}(e, f) = 0. \quad (5.9)
$$

**Proposition 5.5.** The extended bilinear form $\tilde{\gamma}$ is a cocycle for $\mathfrak{g}^1$ if and only if $\gamma$ is $L$-invariant.

*Proof.* The conditions defining a cocycle are obviously fulfilled for the triples of elements consisting either of currents or of vector fields. The only condition which does not follow automatically from (5.9) for $\tilde{\gamma}$ is

$$
\tilde{\gamma}([L, L'], e) + \tilde{\gamma}([L', e], L) + \tilde{\gamma}([e, L], L') = 0. \quad (5.10)
$$

Using (4.21) we see that (5.10) is true if and only if

$$
\gamma(\nabla^{(\omega)}_e L, L') + \gamma(L, \nabla^{(\omega)}_e L') = 0, \quad (5.11)
$$

which is $L$-invariance.

**5.4. Bounded and local cocycles.**

**Definition 5.6.** Given an almost graded Lie algebra

$$
\mathcal{V} = \bigoplus_{m \in \mathbb{Z}} \mathcal{V}_m,
$$

a cocycle $\gamma$ is called bounded (from above) if there exists a constant $R_1 \in \mathbb{Z}$ such that

$$
\gamma(\mathcal{V}_n, \mathcal{V}_m) \neq 0 \quad \implies \quad n + m \leq R_1. \quad (5.12)
$$

Bounded from below is defined similarly.

A cocycle is called local if and only if it is bounded from above and below. Equivalently, there exist $R_1, R_2 \in \mathbb{Z}$ such that

$$
\gamma(\mathcal{V}_n, \mathcal{V}_m) \neq 0 \quad \implies \quad R_2 \leq n + m \leq R_1. \quad (5.13)
$$

The almost-gradation of $\mathcal{V}$ can be extended from $\mathcal{V}$ to the corresponding central extension $\mathcal{V}_\gamma$ (5.3) by assigning a certain degree to the central element $t$ (for example, degree 0) if and only if the defining cocycle for the central extension is local.
We call a cohomology class bounded (local) if it contains a bounded (local) representing cocycle. Again, not every representing cocycle of a bounded (local) class is bounded (local). The set of bounded cohomology classes is a subspace of $H^2(\mathfrak{g}, \mathbb{C})$, which we denote by $H^2_b(\mathfrak{g}, \mathbb{C})$. It contains the subspace of local cohomology classes denoted by $H^2_{loc}(\mathfrak{g}, \mathbb{C})$. This space classifies the almost-graded central extensions of $\mathfrak{g}$ up to equivalence. Both spaces admit subspaces consisting of those cohomology classes admitting a representing cocycle which is both bounded (local) and $L$-invariant. The subspaces are denoted by $H^2_{b,L}(\mathfrak{g}, \mathbb{C})$ and $H^2_{loc,L}(\mathfrak{g}, \mathbb{C})$, respectively.

If we consider our geometric cocycles $\gamma_{1,\omega,C}$ and $\gamma_{2,C}$ obtained by integrating over an arbitrary cycle then they will neither be bounded nor local, nor will they define a bounded or local cohomology class.

Next we will consider special integration paths. Let $C_i$ be positively oriented (deformed) circles around the points $P_i$ in $I$, $i = 1, \ldots, N$, and $C_j^*$ positively oriented circles around the points $Q_j$ in $O$, $j = 1, \ldots, M$. If we integrate over such cycles, the cocycle values of $\gamma$ can be calculated via residues, for example,

$$\gamma_{1,\omega,C_i}(L, L') = \text{res}_{P_i}(\text{tr}(L \cdot \nabla^{(\omega)} L')), \quad i = 1, \ldots, N. \quad (5.14)$$

**Proposition 5.7.** (a) The 1-form $\text{tr}(L \cdot \nabla^{(\omega)} L')$ has no poles outside $A = I \cup O$. (b) The 1-form $\text{tr}(L) \cdot \text{tr}(dL')$ has no poles outside $A = I \cup O$.

For (a) see [1]. For (b) see [16].

A cycle $C_S$ is called a separating cycle if it is smooth, positively oriented of multiplicity one, separates the points in $I$ from the points in $O$, and does not meet $A$ or $W$. It might have multiple components. For our cocycles (5.5), (5.6) we integrate the forms of Proposition 5.7 over closed curves $C$. By this proposition the integrals will yield the same results if $[C] = [C']$ in $H^1(\Sigma \setminus A, \mathbb{Z})$. Note that the weak singular points will not show up in this context. In this sense for every separating cycle we can write

$$[C_S] = \sum_{i=1}^{K} [C_i] = - \sum_{j=1}^{M} [C_j^*]. \quad (5.15)$$

The minus sign appears due to the opposite orientation. In particular the cocycle values obtained by integrating over a $C_S$ can be obtained by calculating residues over either the points in $I$ or the points in $O$.

**Theorem 5.8.** Let $\omega$ coincide with the connection form $\omega'$ associated with the $\mathcal{L}$-action; then

(a) for $i = 1, \ldots, N$ the cocycles $\gamma_{1,\omega,C_i}$ and $\gamma_{2,C_i}$, where $C_i$ is a circle around $P_i$, will be bounded from above and $\mathcal{L}$-invariant;

(b) for $j = 1, \ldots, M$ the cocycles $\gamma_{1,\omega,C_j^*}$ and $\gamma_{2,C_j^*}$, where $C_j^*$ is a circle around $Q_j$, will be bounded from below and $\mathcal{L}$-invariant;

(c) the cocycles $\gamma_{1,\omega,C_S}$ and $\gamma_{2,C_S}$, where $C_S$ is a separating cycle, will be local and $\mathcal{L}$-invariant;

(d) in cases (a) and (c) the upper bound will be zero.
Proof. The statement about $\mathcal{L}$-invariance follows from Proposition 5.4. In fact, to do this we only need $\omega = \omega'$. As we explained above, if we integrate over $C_i$ (or over $C_i^\ast$) the cocycle calculation reduces to calculating residues. Let $L \in \mathfrak{g}_n$, $L' \in \mathfrak{g}_m$. Then $\text{ord}_{P_i}(L) \geq n$ and $\text{ord}_{P_i}(L') \geq m$. As $\omega$ is holomorphic at $P_i$ we obtain
\[
\text{ord}_{P_i}(dL') \geq m - 1, \quad \text{ord}_{P_i}(\nabla(\omega)L') \geq m - 1.
\]
Hence, if $n + m > 0$ neither of the 1-forms appearing in the cocycle definition has poles at $I$ and consequently there are no residues. This proves (a).

For (b) we have to consider the orders of the basis elements of $\mathfrak{g}_m$ at the points in $O$. By the prescriptions (3.33) and (3.34) and taking possible poles of $\omega$ at $O$ into account we find an $R_2$ such that if $n + m \leq R_2$ the integrands will not have poles anymore. This proves (b).

Using (5.15) we can obtain the values of the cocycles integrated over $C_S$ by adding up the values obtained by integration either over $I$ or over $O$. Hence boundedness from below and from above. Hence, they are local.

That zero is an upper bound follows from the proof.

§ 6. Classification results

Recall that we are in the multi-point situation $A = I \cup O$ with $\# I = N$. Further, $C_i$, $C_i^\ast$, and $C_S$ are the special cycles introduced in §5.4. When we use the word bounded for a cocycle we always mean bounded from above if nothing else is said.

**Proposition 6.1.** The cocycles $\gamma_{1,\omega,C_i}$, $i = 1, \ldots, N$ (and $\gamma_{2,C_i}$, $i = 1, \ldots, N$, for $\mathfrak{g}(n)$), are linearly independent.

**Proof.** Assume that the linear relation
\[
0 = \sum_{i=1}^{N} \alpha_i \gamma_{1,\omega,C_i} + \sum_{i=1}^{N} \beta_i \gamma_{2,C_i}, \quad \alpha_i, \beta_i \in \mathbb{C}, \tag{6.1}
\]
holds. The last sum will not appear in the case of a simple algebra. Recall that for a pair $L, L' \in \mathfrak{g}$ the above cocycles can be calculated by taking residues
\[
0 = \sum_{i=1}^{N} \alpha_i \text{res}_{P_i}(\text{tr}(L \cdot \nabla(\omega)L')) + \sum_{i=1}^{N} \beta_i \text{res}_{P_i}(\text{tr}(L) \cdot \text{tr}(\nabla(\omega)L')). \tag{6.2}
\]
In the first sum a Cartan-Killing form is present, which is non-degenerate. Hence there exist $X, Y \in \mathfrak{g}$ such that $\text{tr}(XY) \neq 0$ and $\text{tr}(X) = \text{tr}(Y) = 0$. For $k = 1, \ldots, N$, using the almost-graded structure and following Proposition 3.3 we take $L = X_{1,k}$ and $L' = Y_{-1,k}$. In a neighbourhood of the point $P_l$, $l = 1, \ldots, N$, we have
\[
L(z_l) = X z_l \delta_l^k + O(z_l^2), \quad L'(z_l) = Y z_l^{-1} \delta_l^k + O(z_l^0), \tag{6.3}
\]
\[
\nabla(\omega)L'(z_l) = -Y z_l^{-2} \delta_l^k + O(z_l^{-1}),
\]
as $\nabla(\omega)L' = dL' + [\omega, L']$. Hence,
\[
\text{res}_{P_l}(\text{tr}(L \cdot \nabla(\omega)L')) = -\text{tr}(XY) \delta_l^k. \tag{6.4}
\]
As \( \text{tr}(X) = 0 \) the second sum will vanish anyway and we conclude that \( \alpha_k = 0 \), for all \( k = 1, \ldots, N \). For the second sum we take \( X = Y \) to be a nonvanishing scalar matrix and chose \( L = X_{1,k} \) and \( L' = X_{-1,k} \). We obtain \( \beta_k = 0 \) for all \( k = 1, \ldots, N \).

**Proposition 6.2.** (\( \mathfrak{g} = \mathfrak{gl}(n) \)) If \( \gamma = \sum_{i=1}^{N} \beta_i \gamma_{2,C_i} \) is a nontrivial linear combination, then it is not a coboundary.

**Proof.** Recall from (2.8) that \( \mathfrak{g}(n) \) is an abelian subalgebra of \( \mathfrak{gl}(n) \). Hence, every coboundary restricted to it will be identically zero. If, as in the previous proof, we again take elements \( X_{1,k} \) and \( X_{-1,k} \) from the scalar subalgebra we obtain \( \beta_k = 0 \), as above.

**Proposition 6.3.** If \( \gamma = \sum_{i=1}^{N} \alpha_i \gamma_{1,\omega,C_i} \) is a nontrivial linear combination, then it is not a coboundary.

**Proof.** Assume that \( \gamma \) is a coboundary. This means that there exists a linear form \( \phi: \mathfrak{g} \to \mathbb{C} \) such that \( \forall L, L' \in \mathfrak{g} \)

\[
\gamma(L, L') = \sum_{i=1}^{N} \alpha_i \text{res}_i \text{tr}(L \cdot \nabla^{(\omega)} L') = \phi([L, L']).
\]

(6.5)

Assume that \( \gamma \neq 0 \), so that one of the coefficients \( \alpha_k \) is nonzero. Take \( H \in \mathfrak{h} \) with \( \kappa(H, H) \neq 0 \), where \( \mathfrak{h} \) is the Cartan subalgebra of the simple part of \( \mathfrak{g} \) and \( \kappa \) its Cartan-Killing form. Let \( H_{0,k} \in \mathfrak{g} \) be the element defined by (3.2). In particular, we have \( H_{0,k} = H + O(z_k) \). We set \( H(n,k) := H_{0,k} \cdot A_{n,k} \in \mathfrak{g} \) and hence \( H_{n,k} = H \cdot A_{n,k} + O(z_k^{n+1}) \) in a neighbourhood of the point \( P_k \). Recall that from the local forms (3.2) and (3.8) of our basis elements, in the neighbourhood of points \( P_l \) with \( l \neq k \) we have

\[
H_{n,k} = O(z_l^{n+1}), \quad A_{n,k} = O(z_l^{n+1}), \quad H_{n,k} = O(z_l^{n+1}).
\]

(6.6)

In the following, let \( n \neq 0 \). We have

\[
\nabla^{(\omega)} H_{n,k} = \nabla^{(\omega)} (H_{0,k} \cdot A_{n,k}) = \nabla^{(\omega)} (H_{0,k}) \cdot A_{n,k} + H_{0,k} dA_{n,k}.
\]

(6.7)

The expression \( \nabla^{(\omega)} H_{0,k} \) is of nonnegative order, \( A_{n,k} \) is of order \( n \), \( H_{0,k} \) of order 0 and \( dA_{n,k} \) of order \( n - 1 \) at the point \( P_k \). Hence

\[
\nabla^{(\omega)} H_{n,k} = H_{0,k} dA_{n,k} + O(z_k^n) dz_k.
\]

(6.8)

Now we compute

\[
\gamma(H_{(-1,k)}, H_{(1,k)}) = \sum_{i=1}^{N} \alpha_i \text{res}_i \text{tr}(H_{(-1,k)} \cdot \nabla^{(\omega)} H_{(1,k)}) = \alpha_k \text{res}_k \text{tr}(H_{(-1,k)} \cdot \nabla^{(\omega)} H_{(1,k)}).
\]

(6.9)

\(^3\)Notice that \( H_{(n,k)} \) and \( H_{n,k} \) are, in general, different but they coincide up to higher order.
The last equality follows from the fact that by (6.6) we do not have any poles at the points $P_l$ for $l \neq k$. From the above it follows

$$
(\alpha_k)^{-1}\gamma(H_{(-1,k)}, H_{(1,k)}) = \text{res}_{P_k} \text{tr}(H_{0,k}A_{-1,k}H_{0,k}dA_{1,k}) = \text{res}_{P_k} \text{tr}
\left.
\left( H_{0,k}^2 \frac{dz_k}{z_k} \right)
\right|_{(6.10)}
$$

As $H_{0,k}^2 = H^2 + O(z_k)$ we obtain

$$
(\alpha_k)^{-1}\gamma(H_{(-1,k)}, H_{(1,k)}) = \text{res}_{P_k} \left( \text{tr}(H^2) \frac{dz_k}{z_k} \right) = \text{tr}(H^2) = \beta \cdot \kappa(H, H) \neq 0,
$$

(6.11)

with a nonvanishing constant $\beta$ relating the trace form with the Cartan-Killing form. But

$$
[H_{(-1,k)}, H_{(1,k)}] = [H_{0,k}A_{-1,k}, H_{0,k}A_{1,k}] = [H_{0,k}, H_{0,k}]A_{-1,k}A_{1,k} = 0.
$$

(6.12)

The relations (6.11) and (6.12) give a contradiction to (6.5).

Now we are able to formulate the basic theorem.

**Theorem 6.4.** (a) For $\mathfrak{g}$ with $\mathfrak{g}$ simple (that is, $\mathfrak{g} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$) the space of bounded cohomology classes is $N$-dimensional. If $\omega$ is any fixed connection form then this space has the classes of $\gamma_{1,\omega,C_i}$, $i = 1, \ldots, N$, as basis. Every $\mathcal{L}$-invariant bounded cocycle with respect to the connection $\omega$ is a linear combination of the $\gamma_{1,\omega,C_i}$.

(b) For $\mathfrak{g} = \mathfrak{gl}(n)$ the space of local cohomology classes which are $\mathcal{L}$-invariant once they have been restricted to the scalar subalgebra is $2N$-dimensional. If $\omega$ is any fixed connection form then the space has the classes of the cocycles $\gamma_{1,\omega,C_i}$ and $\gamma_{2,C_i}$, $i = 1, \ldots, N$, as basis. Every $\mathcal{L}$-invariant local cocycle is a linear combination of the $\gamma_{1,\omega,C_i}$ and $\gamma_{2,C_i}$.

**Proof.** Here we only outline the proof. The technicalities are postponed until §§7 and 8.

By Propositions 7.8 and 7.10 it follows that $\mathcal{L}$-invariant and bounded cocycles are necessarily linear combinations of the chosen form. This proves the theorem for the cohomology space $H_{b,\mathcal{L}}(\mathfrak{g}, \mathbb{C})$. For the scalar subalgebra we are done since we included $\mathcal{L}$-invariance in the conditions of the theorem. For semi-simple algebras we have to show that there is an $\mathcal{L}$-invariant representative in each local cohomology class. But by Theorem 8.1 the space $H_b(\mathfrak{g}, \mathbb{C})$ is at most $N$-dimensional. Since by Proposition 6.3 no nontrivial linear combination of the cocycles $[\gamma_{1,\omega,C_i}]$ is a coboundary, this space is exactly $N$-dimensional and the $[\gamma_{1,\omega,C_i}]$ for $i = 1, \ldots, N$ constitute a basis.

**Corollary 6.5.** Let $\mathfrak{g}$ be a simple classical Lie algebra and $\mathfrak{g}$ the associated Lax operator algebra. Let $\omega$ be a fixed connection form. Then in each $[\gamma] \in H_b(\mathfrak{g}, \mathbb{C})$ there exists a unique representative $\gamma'$ which is bounded and $\mathcal{L}$-invariant (with respect to $\omega$). Moreover, $\gamma' = \sum_{i=1}^N a_i \gamma_{1,\omega,C_i}$ with $a_i \in \mathbb{C}$.

**Proposition 6.6.** (a) If $\gamma$ is a bounded $\mathcal{L}$-invariant cocycle which is a coboundary, then $\gamma = 0$.

(b) Let $\mathfrak{g}$ be simple, then the cocycle $\gamma_{1,\omega',C_i}$ is $\mathcal{L}$-invariant with respect to $\omega$ if and only if $\omega = \omega'$.
Proof. (a) By Theorem 6.4 we see that $\gamma = \sum_{i=1}^{N} (\alpha_i \gamma_{1,\omega,C_i} + \beta_i \gamma_{2,C_i})$, including all $\beta_i$ for the case $g$ is simple. The summands constitute a basis of the cohomology. Hence, $\gamma$ can only be a coboundary if all the coefficients vanish.

(b) As $\gamma_{1,\omega,C_i}$ and $\gamma_{1,\omega',C_i}$ are local and $\mathcal{L}$-invariant with respect to $\omega$ their difference $\gamma_{1,\omega,C_i} - \gamma_{1,\omega',C_i}$ is also local and $\mathcal{L}$-invariant. By Proposition 5.2 it is a coboundary. Hence by part (a) $\gamma_{1,\omega,C_i} - \gamma_{1,\omega',C_i} = 0$. The relation (5.7) gives the explicit expression for the left-hand side. Assume $\omega \not= \omega'$. Let $m$ be the order of the element

$$\theta = \omega - \omega' = (\theta_m z_i^m + O(z_i^m)) dz_i$$

(6.13)

at the point $P_i$. As $g$ is simple the trace form $\text{tr}(A \cdot B)$ is nondegenerate and we find

$$\hat{\theta} = \hat{\theta}_{-m} z_i^{-m-1} + O(z_i^{-m})$$

(6.14)

such that $\beta = \text{tr}(\theta_m \cdot \hat{\theta}_{-m-1}) \not= 0$. Be Lemma 3.18 we get $\hat{\theta} = [L, L'] + L''$ with $\text{ord}_{P_i}(L'') \geq -m$. Hence,

$$0 \not= \beta = \text{tr}(\theta_m \cdot \hat{\theta}_{-m-1}) = \frac{1}{2\pi i} \int_{C_i} \text{tr}((\omega - \omega') \cdot ([L, L'] + L''))$$

$$= \frac{1}{2\pi i} \int_{C_i} \text{tr}((\omega - \omega') \cdot [L, L']) = \gamma_{1,\omega,C_i}(L, L') - \gamma_{1,\omega',C_i}(L, L') = 0,$$

(6.15)

which is a contradiction.

After these results which are valid for bounded cocycles we will deduce the corresponding classification theorem for local cocycles. In some sense this is the main theorem of this article. It will show for example that for Lax operator algebras associated to simple Lie algebras there is up to rescaling and equivalence only one nontrivial almost-graded central extension.

Recall the relation for the separating cycle

$$[C_S] = \sum_{i=1}^{N} [C_i] = - \sum_{j=1}^{M} [C_j^*]$$

(6.16)

and the corresponding relation for the cocycle obtained by integration.

Theorem 6.7. (a) For $\bar{g}$ with $g$ simple (that is, $g = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$) the space of local cohomology classes is one-dimensional. For any fixed connection form $\omega$ this space will be generated by the class of $\gamma_{1,\omega,C_S}$. Every local cocycle that is $\mathcal{L}$-invariant with respect to $\omega$ is a scalar multiple of $\gamma_{1,\omega,C_S}$,

(b) For $\bar{g} = \mathfrak{gl}(n)$ the space of local cohomology classes which are $\mathcal{L}$-invariant when they are restricted to the scalar subalgebra is two-dimensional. For any fixed connection form $\omega$ the space will be generated by the classes of the cocycles $\gamma_{1,\omega,C_S}$ and $\gamma_{2,C_S}$. Every $\mathcal{L}$-invariant local cocycle is a linear combination of $\gamma_{1,\omega,C_S}$ and $\gamma_{2,C_S}$.

Proof. Let $\gamma$ be a local cocycle. This says it is bounded above and below. For simplicity we abbreviate the notation in this proof:

$$\gamma_{1,i} := \gamma_{1,\omega,C_i}, \quad \gamma_{2,i} := \gamma_{2,C_i}, \quad \gamma_{1,j} := \gamma_{1,\omega,C_j^*}, \quad \gamma_{2,j} := \gamma_{2,C_j^*}.$$  

(6.17)
If we switch the roles of $I$ and $O$ we get an inverted almost-grading. Every cocycle of the original grading which is bounded below will be bounded above with respect to the inverted grading. Hence we can employ Theorem 6.4 in both directions and obtain two representations for the same cocycle

$$\gamma = \sum_{i=1}^{N} a_i \gamma_{1,i} + \sum_{i=1}^{N} b_i \gamma_{2,i} = - \sum_{j=1}^{M} a_j^* \gamma_{1,j}^* - \sum_{j=1}^{M} b_j^* \gamma_{2,j}^* \quad \text{with} \quad a_i, a_j^*, b_i, b_j^* \in \mathbb{C}. \quad (6.18)$$

If either $N = 1$ or $M = 1$ then via (6.16) the cocycle is obtained via integration over a separating cycle. Hence the statement.

Otherwise both $N, M > 1$. By Proposition 6.1 the type (1) and type (2) cocycles are linearly independent, hence they can also be treated independently in this context. First consider type (1). Note that from (6.16) we find that

$$\sum_{i=1}^{N} \gamma_{1,i} = - \sum_{j=2}^{M} \gamma_{1,j}^*. \quad (6.19)$$

Hence,

$$\gamma_{1,1}^* = - \sum_{i=1}^{N} \gamma_{1,i} - \sum_{j=2}^{M} \gamma_{1,j}^*. \quad (6.20)$$

From (6.18) we get

$$0 = \sum_{i=1}^{N} a_i \gamma_{1,i} + \sum_{j=1}^{M} a_j^* \gamma_{1,j}^*. \quad (6.21)$$

If we plug (6.20) into this relation we obtain

$$0 = (a_1 - a_1^*) \sum_{i=1}^{N} \gamma_{1,i} + \sum_{i=2}^{N} (a_i - a_1) \gamma_{1,i} + \sum_{j=2}^{M} (a_k^* - a_1^*) \gamma_{1,j}^*. \quad (6.22)$$

Fix $k$. We take $X, Y \in \mathfrak{g}$ such that $\text{tr}(XY) \neq 0$. By the Riemann-Roch theorem there exist $L', L \in \mathfrak{g}$ such that, around the point $P_k$, both

$$L(z_k) = Xz_k + O(z_k^2) \quad \text{and} \quad L'(z_k) = Yz_k^{-1} + O(z_k^0) \quad (6.23)$$

are holomorphic at the points in $O$ and at $P_l, l \neq 1, k$. The elements might have pole orders of sufficiently high degree at $P_1$ to guarantee existence. The weak singularities will not interfere. By construction

$$\gamma_{1,k}(L, L') \neq 0, \quad \gamma_{1,l}(L, L') = 0, \quad l = 2, \ldots, N, \quad l \neq k,$$

$$\gamma_{1,j}^*(L, L') = 0, \quad j = 1, \ldots, M. \quad (6.24)$$

Hence

$$\sum_{i=1}^{N} \gamma_{1,i}(L, L') = - \sum_{j=1}^{M} \gamma_{1,j}^*(L, L') = 0. \quad (6.25)$$
If we plug \((L, L')\) into (6.22), all the terms in (6.22) will vanish; the only exception is
\[
0 = (a_k - a_1)\gamma_{1,k}(L, L').
\] (6.26)
This shows \(a_k - a_1\) for all \(k\). (In a similar way we get \(a^*_j - a^*_1\) for all \(j\).) In particular, the cocycle we started with (the \(\gamma_1\) part of it, respectively) is a multiple of the cocycle obtained by integration over the separating cycle. This was the claim. The proof for the \(\gamma_2\) part works in exactly the same way, if we take \(X = Y\) a nonzero scalar matrix.

As in the bounded case we also obtain the following corollary in the local case.

**Corollary 6.8.** Let \(g\) be a simple classical Lie algebra and \(\overline{g}\) the associated Lax operator algebra. Let \(\omega\) be a fixed connection form. Then in each \([\gamma] \in H_{\text{loc}}(\overline{g}, \mathbb{C})\) there exists a unique representative \(\gamma'\) which is local and \(L\)-invariant (with respect to \(\omega\)). Moreover, \(\gamma' = a\gamma_{1,\omega}\) with \(a \in \mathbb{C}\).

### §7. Uniqueness of \(\mathcal{L}\)-invariant cocycles

#### 7.1. The induction step.
Recall from §3 the almost graded structure of the Lax operator algebra \(\overline{g}\) and in particular the decomposition into subspaces of homogeneous elements of degree \(n\), \(\overline{g} = \bigoplus_{n \in \mathbb{Z}} \overline{g}_n\). Also the basis \(\{L_{u}^{n,p} \mid u = 1, \ldots, \dim g, p = 1, \ldots, N\}\) of the subspace \(\overline{g}_n\) was introduced there (see (3.3)).

Let \(\gamma\) be an \(\mathcal{L}\)-invariant cocycle for the algebra \(\overline{g}\) which is bounded above, that is, there exists an \(R\) (independent of \(n\) and \(m\)) such that \(\gamma(\overline{g}_n, \overline{g}_m) \neq 0\) implies \(n + m \leq R\). Furthermore, we recall that the connection \(\omega\) we need to define the action of \(\mathcal{L}\) on \(g\) is chosen to be holomorphic at the points in \(I\).

For a pair \((L_{n,k}^{u}, L_{m,t}^{v})\) of homogeneous elements we call \(n + m\) the level of the pair. We apply the technique developed in [25]. We consider cocycle values \(\gamma(L_{n,k}^{u}, L_{m,t}^{v})\) on pairs of level \(l = n + m\) and will proceed by induction over the level. As the cocycle is bounded above, the cocycle values will vanish at all pairs of sufficiently high level. It turns out that everything is fixed by the values of the cocycle at level zero. Finally, we will show that the cocycle is a linear combination of the \(N(2N, \text{respectively})\) basic cocycles as claimed in Theorem 6.4.

For a cocycle \(\gamma\) evaluated for pairs of elements of level \(l\) we will use the symbol \(\equiv\) to denote that the expressions are the same on both sides of an equation involving cocycle values up to values of \(\gamma\) at a higher level. This has to be understood in the following strong sense:

\[
\sum \alpha^{(n,p,t)}_{(u,v)} \gamma(L_{n,p}^{u}, L_{l-n,t}^{v}) \equiv 0, \quad \alpha^{(n,p,t)}_{(u,v)} \in \mathbb{C},
\] (7.1)
means congruence modulo a linear combination of values of \(\gamma\) at pairs of basis elements of level \(l' > l\). The coefficients in the linear combination, and also the \(\alpha^{(n,p,t)}_{(u,v)}\), depend only on the structure of the Lie algebra \(\overline{g}\) and do not depend on the cocycle \(\gamma\).

We will also use the same symbol \(\equiv\) for equalities in \(\overline{g}\) which are true modulo terms of higher degree compared to the terms under consideration.

Because of the \(\mathcal{L}\)-invariance we have
\[
\gamma((\nabla_{e_{k,r}}^{(\omega)} L_{m,p}^{u}, L_{n,s}^{v}) + \gamma(L_{m,p}^{u}, \nabla_{e_{k,r}}^{(\omega)} L_{n,s}^{v}) = 0.
\] (7.2)
Using the almost-graded structure (4.13) we obtain (up to order $(k + m + n)$)
\begin{equation}
m\gamma(L^u_{k+m,p}, L^v_{n,s})\delta^v_r + n\gamma(L^v_{m,p}, L^u_{n+k,s})\delta^u_r \equiv 0,
\end{equation}
valid for all $n, m, k \in \mathbb{Z}$.

If all three indices $r, p$ and $s$ in (7.3) are distinct then the term on the left-hand side vanishes. If $r = p \neq s$ then we obtain
\begin{equation}
m\gamma(L^u_{k+m,p}, L^v_{n,s}) \equiv 0
\end{equation}
which is true for every $m$. Hence
\begin{equation}
\gamma(L^u_{l,p}, L^v_{n,s}) \equiv 0 \text{ for } p \neq s.
\end{equation}
The case $r = p = s$ remains, and this yields
\begin{equation}
m\gamma(L^u_{k+m,s}, L^v_{n,s}) + n\gamma(L^v_{m,s}, L^u_{n+k,s}) \equiv 0.
\end{equation}

**Proposition 7.1.** Let $m + n \neq 0$; then at level $m + n$
\begin{equation}
\gamma(\mathfrak{g}_m, \mathfrak{g}_n) \equiv 0.
\end{equation}

**Proof.** From (7.5) we conclude that only elements with the same second index can contribute at level $m + n$. We put $k = 0$ in (7.6) and obtain
\begin{equation}
(m + n)\gamma(L^u_{m,s}, L^v_{n,s}) \equiv 0 \quad \forall u, v.
\end{equation}
Hence, if $(m + n) \neq 0$ the claim follows.

**Proposition 7.2.**
\begin{equation}
\gamma(\mathfrak{g}_m, \mathfrak{g}_0) \equiv 0 \quad \forall m \in \mathbb{Z}.
\end{equation}
We evaluate (7.6) for the values $m = 1$ and $n = 0$ and obtain the result.

**Proposition 7.3.** (a) If $n + m > 0$ then $\gamma(\mathfrak{g}_n, \mathfrak{g}_m) = 0$, that is, the cocycle is bounded above by zero.

(b) If $\gamma(\mathfrak{g}_n, \mathfrak{g}_{-n}) = 0$ then the cocycle $\gamma$ vanishes identically.

**Proof.** The proof is word for word the same as in [16]. But, as it is one of the central arguments, for the convenience of the reader we repeat the arguments.

If $\gamma = 0$ there is nothing to prove. Assume $\gamma \neq 0$. As $\gamma$ is bounded above, there will be a minimal upper bound $l$, such that above $l$ all cocycle values will vanish. Assume that $l > 0$, then by Proposition 7.1 the values at level $l$ are expressions of levels bigger than $l$. But the cocycle vanishes there. Hence it vanishes at level $l$ too. This is a contradiction which proves (a).

By induction, using again Proposition 7.1 we obtain that if the cocycle vanishes at level 0, it vanishes everywhere. This proves (b).

Combining Propositions 7.2 and 7.3 we obtain

**Corollary 7.4.**
\begin{equation}
\gamma(\mathfrak{g}_m, \mathfrak{g}_0) = 0 \quad \forall m \geq 0.
\end{equation}
Proposition 7.5.

\[ \gamma(L_{n,r}^u, L_{-n,s}^v) = n \cdot \gamma(L_{1,r}^u, L_{-1,s}^v) \delta_r^s, \]
\[ \gamma(L_{1,r}^u, L_{-1,s}^v) = \gamma(L_{1,s}^v, L_{-1,r}^u). \]  

(7.11) \hspace{0.5cm} (7.12)

**Proof.** Assume that \( s \neq r \). Then all the expressions are of positive level and vanish by Proposition 7.3, hence the statement is true. For \( r = s \) in (7.6) we take the values \( n = -p, m = 1 \) and \( k = p - 1 \). This yields the expression (7.11) up to higher level terms. But as the level is zero, the higher level terms vanish. Setting \( n = -1 \) in (7.11) we obtain (7.12).

Independent of the structure of the Lie algebra \( \mathfrak{g} \), we obtained the following results for every \( \mathcal{L} \)-invariant and bounded cocycle \( \gamma \):

1) the cocycle is bounded from above by zero;
2) the cocycle is uniquely given by its values at level zero;
3) at level zero the cocycle is uniquely fixed by its values \( \gamma(L_{1,s}^u, L_{-1,s}^v) \), for \( u, v = 1, \ldots, \dim \mathfrak{g} \) and \( s = 1, \ldots, N \);
4) the other cocycle values at level zero are given by \( \gamma(L_{n,s}^u, L_{-n,s}^v) = 0 \) if \( s \neq r \), \( \gamma(L_{0,s}^u, L_{0,s}^v) = 0 \), and \( \gamma(L_{n,s}^u, L_{-n,s}^v) \) given by (7.11).

Let \( X \in \mathfrak{g} \) then, as always, we let \( X_{n,s}, s = 1, \ldots, N \), denote the element in \( \overline{\mathfrak{g}} \) with leading term \( X z_s^n \) at \( P_s \) and higher orders at the other points in \( I \). For \( s = 1, \ldots, N \) we define the maps \( \psi_{\gamma,s} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C} \):

\[ \psi_{\gamma,s}(X, Y) := \gamma(X_{1,s}, Y_{-1,s}). \]

(7.13)

Obviously, \( \psi_{\gamma,s} \) is a bilinear form on \( \mathfrak{g} \).

**Proposition 7.6.** (a) \( \psi_{\gamma,s} \) is symmetric, that is,

\[ \psi_{\gamma,s}(X, Y) = \psi_{\gamma,s}(Y, X). \]

(b) \( \psi_{\gamma,s} \) is invariant, that is,

\[ \psi_{\gamma,s}([X, Y], Z) = \gamma(X_{1,s}, Y_{1,s}, Z_{-1,s}). \]

(7.14)

**Proof.** First, by (7.12) we have

\[ \psi_{\gamma,s}(X, Y) = \gamma(X_{1,s}, Y_{-1,s}) = \gamma(Y_{1,s}, X_{-1,s}) = \psi_{\gamma}(Y, X). \]

This is the symmetry. Furthermore, using \( [X_{1,s}, Y_{0,s}] \equiv [X, Y]_{1,s} \), the fact that the cocycle vanishes for positive levels, and by the cocycle condition we obtain

\[ \psi_{\gamma,s}([X, Y], Z) = \gamma([X, Y]_{1,s}, Z_{-1,s}) = \gamma([X_{1,s}, Y_{0,s}], Z_{-1,s}) \]
\[ = -\gamma([Y_{0,s}, Z_{-1,s}], X_{1,s}) - \gamma([Z_{-1,s}, X_{1,s}], Y_{0,s}). \]

The last term vanishes due to Corollary 7.4. Hence

\[ \psi_{\gamma,s}([X, Y], Z) = \gamma(X_{1,s}, Y_{0,s}, Z_{-1,s}) = \gamma(X_{1,s}, [Y, Z]_{-1,s}) = \psi_{\gamma,s}(X, [Y, Z]). \]

As the cocycle \( \gamma \) is fixed by the values \( \gamma(L_{1,s}^u, L_{-1,s}^v), s = 1, \ldots, N \), and they are fixed by the bilinear maps \( \psi_{\gamma,s} \) we have proved

**Theorem 7.7.** Let \( \gamma \) be an \( \mathcal{L} \)-invariant cocycle for \( \overline{\mathfrak{g}} \) which is bounded above. Then \( \gamma \) is bounded above by zero and is completely fixed by the associated symmetric and invariant bilinear forms \( \psi_{\gamma,s}, s = 1, \ldots, N \), on \( \mathfrak{g} \) defined via (7.13).
7.2. Simple Lie algebras $\mathfrak{g}$. By Theorem 7.7 the $\mathcal{L}$-invariant cocycle $\gamma$ is completely determined by fixing the $N$-tuple $(\psi_{\gamma,1}, \psi_{\gamma,2}, \ldots, \psi_{\gamma,N})$ of symmetric invariant bilinear forms $\psi_{\gamma,s}$. For a finite-dimensional simple Lie algebra every such form is a multiple of the Cartan-Killing form $\kappa$. Hence the space of bounded cocycles is at most $N$-dimensional. Our geometric cocycles $\gamma_{1,\omega,C_i}$, see (5.14), for $i = 1, \ldots, N$ are $\mathcal{L}$-invariant bounded cocycles. They are linearly independent by Proposition 6.1. Hence, we find that every bounded, $\mathcal{L}$-invariant cocycle is a linear combination of the $\gamma_{1,\omega,C_i}$. Moreover, they are a basis of the space of $\mathcal{L}$-invariant, bounded cocycles. By Proposition 6.3 they remain linearly independent after passing to cohomology and we obtain

**Proposition 7.8.** Let $\mathfrak{g}$ be simple, then

$$\dim H_{b,\mathcal{L}}(\bar{\mathfrak{g}}, \mathbb{C}) = N,$$

and this cohomology space is generated by the classes of $\gamma_{1,\omega,C_i}$, $i = 1, \ldots, N$.

7.3. The case of $\overline{\mathfrak{gl}}(n)$. We have a direct decomposition, as Lie algebras, given by $\overline{\mathfrak{gl}}(n) = \mathfrak{S}(n) \oplus \mathfrak{sl}(n)$. Let $\gamma$ be a cocycle of $\overline{\mathfrak{gl}}(n)$ and denote its restriction to $\mathfrak{S}(n)$ and $\mathfrak{sl}(n)$ by $\gamma'$ and $\gamma''$, respectively. As in [16], using Lemma 3.18 we obtain the following.

**Proposition 7.9.**

$$\gamma(x, y) = 0 \quad \forall x \in \mathfrak{S}(n), \quad y \in \mathfrak{sl}(n).$$

Hence we can decompose the cocycle as $\gamma = \gamma' \oplus \gamma''$. If $\gamma$ is bounded/local and/or $\mathcal{L}$-invariant the same is true for $\gamma'$ and $\gamma''$.

First we consider the algebra $\mathfrak{S}(n)$. It is isomorphic to $\mathfrak{A}$, and the isomorphism is given by

$$L \mapsto \frac{1}{n} \text{tr}(L).$$

In [25], Theorem 5.7 it was shown that the space of $\mathcal{L}$-invariant cocycles for $\mathfrak{A}$ which are bounded above is $N$-dimensional and a basis is given by

$$\gamma_i(f, g) = \frac{1}{2\pi i} \int_{C_i} f \, dg = \text{res}_{P_i}(f \, dg), \quad i = 1, \ldots, N.$$ 

Note that as $\mathfrak{A}$ is abelian there do not exist nontrivial coboundaries. We obtain

$$\gamma'(L, M) = \sum_{i=1}^{N} \alpha_i \text{res}_{P_i}(\text{tr}(L) \cdot \text{tr}(dM)) = \sum_{i=1}^{N} \alpha_i \gamma_{2,C_i}(L, M)$$

by definition (5.6).

For the cocycle $\gamma''$ of $\overline{\mathfrak{sl}}(n)$ we use Proposition 7.8 and obtain $\gamma'' = \sum_{i=1}^{N} \beta_i \gamma_{1,\omega,C_i}$. Altogether we have proved the following.

**Proposition 7.10.**

$$\dim H_{b,\mathcal{L}}(\overline{\mathfrak{gl}}(n), \mathbb{C}) = 2N.$$ 

A basis is given by the classes of $\gamma_{1,\omega,C_i}$ and $\gamma_{2,C_i}$, $i = 1, \ldots, N$. 

In this section we have proved those parts of Theorem 6.4 which deal with $\mathcal{L}$-invariant cocycles. In fact we have proved the complete theorem under the additional assumption that our cohomology classes are $\mathcal{L}$-invariant. For the scalar part this is the best that could be expected. Without $\mathcal{L}$-invariance there will be many more nontrivial cohomology classes for the scalar algebra; see [25] for more information. In the next section we will describe a way to get rid of this condition for simple Lie algebras.

§8. The simple case in general

In this section the Lax operator algebra $\mathfrak{g}$ is always based on a finite simple classical Lie algebra. As we explained in the previous section if we had included $\mathcal{L}$-invariance in the assumptions Theorem 6.4 would have been proved. One way to complete the general proof is to show that after cohomological changes every bounded cocycle has also a bounded $\mathcal{L}$-invariant representing it. In fact, we will do this. Unfortunately, we do not have a direct proof. Instead, using quite a different approach we will show that, in case of a simple Lie algebra, the space of bounded cohomology classes (of the Lax operator algebras) is at most $N$-dimensional, without assuming $\mathcal{L}$-invariance a priori. Combining this result with the result in the last section that the space of $\mathcal{L}$-invariant bounded cohomology classes is $N$-dimensional we see that in the simple case each bounded cohomology class is automatically $\mathcal{L}$-invariant. Moreover, in that result we showed that it has a unique $\mathcal{L}$-invariant representing cocycle which is given as a linear combination of $\gamma_1, \omega, C_i, i = 1, \ldots, N$.

The theorem we are aiming for is as follows.

**Theorem 8.1.** Let $\mathfrak{g}$ be a simple classical Lie algebra over $\mathbb{C}$ and $\mathfrak{g}$ the associated Lax operator algebra with its almost-grading. Every bounded cocycle on $\mathfrak{g}$ is cohomologous to a distinguished cocycle which is bounded above by zero. The space of distinguished cocycles is at most $N$-dimensional.

**Remark 8.2.** What we will show is the following. Every cocycle that is bounded above is cohomologous to a cocycle which is fixed by its value at $N$ special pairs of elements in $\mathfrak{g}$ (namely by $\gamma(H_{1,s}^\alpha, H_{-1,s}^\alpha)$ for one fixed simple root $\alpha$, see below for the notation). Besides the structure of $\gamma$ we only use the almost-gradedness of $\mathfrak{g}$ with leading terms given in (8.4).

The presentation is quite similar to [16]. Those proofs which are completely identical in structure will not be repeated here.

First we need to recall some facts about the Chevalley generators of $\gamma$. Choose a root space decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$. As usual $\Delta$ denotes the set of all roots $\alpha \in \mathfrak{h}^*$. Furthermore, let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a set of simple roots ($p = \dim \mathfrak{h}$). With respect to this basis, the root system splits into positive and negative roots, $\Delta_+$ and $\Delta_-$, respectively. If $\alpha$ is a positive root, $-\alpha$ is a negative root and vice versa. For $\alpha \in \Delta$ we have $\dim \mathfrak{g}^\alpha = 1$. Certain elements $E^\alpha \in \gamma^\alpha$ and $H^\alpha \in \mathfrak{h}$, $\alpha \in \Delta$ can be fixed so that for every positive root $\alpha$

$$[E^\alpha, E^{-\alpha}] = H^\alpha, \quad [H^\alpha, E^\alpha] = 2E^\alpha, \quad [H^\alpha, E^{-\alpha}] = -2E^{-\alpha}. \quad (8.1)$$

We use also $H^i := H^{\alpha_i}$, $i = 1, \ldots, p$, for the elements assigned to the simple roots. A vector space basis, the Chevalley basis, of $\mathfrak{g}$ is given by $\{E^\alpha, \alpha \in \Delta; \ H^i, 1 \leq i \leq p\}$. 

We denote the inner product on $\mathfrak{h}^*$ induced by the Cartan-Killing form of $\mathfrak{g}$ by $(\cdot, \cdot)$. The following relations hold

$$[H^\alpha, H^\beta] = 0, \quad [H^\alpha, E^{\pm\beta}] = \pm 2\frac{(\beta, \alpha)}{(\beta, \beta)}E^{\pm\alpha}, \quad [H, E^\alpha] = \alpha(H)E^\alpha, \quad H \in \mathfrak{h},$$

$$[E^\alpha, E^\beta] = \begin{cases} H^\alpha, & \alpha \in \Delta_+, \beta = -\alpha, \\ -H^\alpha, & \alpha \in \Delta_-, \beta = -\alpha, \\ \pm (r + 1)E^{\alpha+\beta}, & \alpha, \beta, \alpha + \beta \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

(8.2)

Here $r$ is the largest nonnegative integer such that $\alpha - r\beta$ is still a root.

As in the other parts of this article, we use $E^\alpha_{n,s}$ and $H^\alpha_{n,s}$ to denote the unique elements in $\mathfrak{g}_n$ (that is, of degree $n$) for which the expansions at $P_s$ start with $E^\alpha z^n_s$ and $H^\alpha z^n_s$, respectively, and which at the points $P_i \in I$, $i \neq s$, are of higher order.

The following elements form a basis of $\mathfrak{g}$:

$$\{E^\alpha_{n,s}, \alpha \in \Delta; H^i_{n,s}, 1 \leq i \leq p \mid n \in \mathbb{Z}, s = 1, \ldots, N\}. \quad (8.3)$$

The structure equations, up to higher degree terms, are

$$[H^\alpha_{n,s}, H^\beta_{m,r}] \equiv 0, \quad [H^\alpha_{n,s}, E^{\pm\beta}_{m,r}] \equiv \pm 2\frac{(\beta, \alpha)}{(\beta, \beta)}E^{\pm\alpha}_{n,m,r}\delta_r^{s},$$

$$[H_{n,s}, E^\alpha_{m,r}] \equiv \alpha(H)E^\alpha_{n+m,r}\delta_r^{s}, \quad H \in \mathfrak{h},$$

$$[E^\alpha_{n,s}, E^\beta_{m,r}] \equiv \begin{cases} H^\alpha_{n+m,s}\delta_r^{s}, & \alpha \in \Delta_+, \beta = -\alpha, \\ -H^\alpha_{n+m,s}\delta_r^{s}, & \alpha \in \Delta_-, \beta = -\alpha, \\ \pm (r + 1)E^{\alpha+\beta}_{n+m,s}\delta_r^{s}, & \alpha, \beta, \alpha + \beta \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

(8.4)

Recall that the symbol $\equiv$ denotes equality up to elements of degree higher than the sum of the degrees of the elements under consideration. Here, the elements we have not written down are elements of degree $> n + m$. Also recall that by the almost-gradedness there exists an $S$, independent of $n$ and $m$, such that only elements of degree $\leq n + m + S$ appear.

Let $\gamma'$ be a cocycle for $\mathfrak{g}$ which is bounded above. For the elements in $\mathfrak{g}$ we get

$$E^{\pm\alpha} = \pm \frac{1}{2}[H^\alpha, E^{\pm\alpha}], \quad H^i = [E^\alpha_i, E^{-\alpha_i}], \quad i = 1, \ldots, p. \quad (8.5)$$

Consequently, for $\mathfrak{g}$ we obtain

$$E^{\pm\alpha}_{n,s} = \pm \frac{1}{2}[H^\alpha_{0,s}, E^{\pm\alpha}_{n,s}] + Y(n, s, \alpha),$$

$$H^i_{n,s} = [E^\alpha_i, E^{-\alpha_i}] + Z(n, s, i), \quad i = 1, \ldots, p,$$

(8.6)

where $Y(n, s, \alpha)$ and $Z(n, s, i)$ are sums of elements of degree between $n + 1$ and $n + S$. Fix a number $M \in \mathbb{Z}$ such that the cocycle $\gamma'$ vanishes for all levels $\geq M$. 
We define a linear map $\Phi: \overline{g} \to \mathbb{C}$ by (descending) induction on the degree of the basis elements (8.3). First

$$\Phi(E_{n,s}^\alpha) := \Phi(H_{n,s}^i) := 0, \quad \alpha \in \Delta, \quad i = 1, \ldots, p, \quad s = 1, \ldots, N, \quad n \geq M.$$  

(8.7)

Next we define inductively ($\alpha \in \Delta_+, s = 1, \ldots, N$)

$$\Phi(E_{\pm}^\alpha n,s) := \pm \frac{1}{2} \gamma' \gamma_{0,s}(H_{\alpha}^0, E_{\pm}^\alpha n,s) + \Phi(Y(n, s, \pm \alpha)),$$

$$\Phi(H_{n,s}^i) := \gamma' \gamma_{0,s}(E_{\alpha}^i, E_{n,s}^{-\alpha_i}) + \Phi(Z(n, s, i)).$$

(8.8)

The cocycle $\gamma = \gamma' - \delta \Phi$ is cohomologous to the original cocycle $\gamma'$. As $\gamma'$ is bounded above, and, by definition, $\Phi$ is also bounded above, the cocycle $\gamma$ is bounded above too.

By the construction of $\Phi$ we have

$$\Phi([H_{0,s}^\alpha, E_{n,s}^{\pm \alpha}] = \gamma'(H_{0,s}^\alpha, E_{n,s}^{\pm \alpha}), \quad \Phi([E_{0,s}^{\alpha_i}, E_{n,s}^{-\alpha_i}]) = \gamma'(E_{0,s}^{\alpha_i}, E_{n,s}^{-\alpha_i}).$$

Hence

**Proposition 8.3.**

$$\gamma(H_{0,s}^\alpha, E_{n,s}^{\pm \alpha}) = 0, \quad \gamma(E_{0,s}^{\alpha_i}, E_{n,s}^{-\alpha_i}) = 0,$$

$$\alpha \in \Delta_+, \quad i = 1, \ldots, p, \quad s = 1, \ldots, N, \quad n \in \mathbb{Z}.$$  

(8.9)

**Definition 8.4.** A cocycle $\gamma$ is called **normalized** if it fulfills (8.9).

By the above construction we showed that every cocycle which is bounded above is cohomologous to a normalized one, which is also bounded above. In the following we assume that our cocycle is already normalized.

**Proposition 8.5.** Let $\alpha_1$ be a fixed simple root, $\alpha$ and $\beta$ arbitrary roots and $\gamma$ a normalized cocycle. Then for all $s, r = 1, \ldots, N, n, m \in \mathbb{Z}$,

$$\gamma(E_{m,s}^\alpha, H_{n,r}) \equiv 0, \quad H \in \mathfrak{h}, \quad \alpha \in \Delta,$$

$$\gamma(E_{m,s}^\alpha, E_{n,r}^\beta) \equiv 0, \quad \alpha, \beta \in \Delta, \quad \beta \neq -\alpha,$$

$$\gamma(E_{m,s}^\alpha, E_{n,s}^{-\alpha}) \equiv u \gamma(H_{m,r}^{\alpha_1}, H_{n,r}^{\alpha_1}) \delta_r^s, \quad \alpha \in \Delta,$$

$$\gamma(H_{m,s}^\alpha, H_{n,s}^\beta) \equiv t \gamma(H_{m,r}^{\alpha_1}, H_{n,r}^{\alpha_1}) \delta_r^s, \quad \alpha, \beta \in \Delta_+,$$

where $u, t \in \mathbb{C},$

$$\gamma(H_{n,r}^{\alpha_1}, H_{0,s}^{\alpha_1}) \equiv 0,$$

$$\gamma(H_{n+1,s}^{\alpha_1}, H_{l-(n+1),r}^{\alpha_1}) \equiv (\gamma(H_{n-1,s}^{\alpha_1}, H_{l-(n-1),r}^{\alpha_1}) + 2 \gamma(H_{1,s}^{\alpha_1}, H_{l-1,r}^{\alpha_1}) \delta_r^s.$$  

(8.10)

For a simple root $\alpha_1$ and for a level $l \neq 0$

$$\gamma(H_{n,r}^{\alpha_1}, H_{l-n,s}^{\alpha_1}) \equiv 0.$$  

(8.11)
Proof. In the two point case the statement of the proposition consists of a sequence of individual statements which were proved in [16]. In fact, the proofs presented there remain valid if, in all relations there for the Lie algebra elements \( Y_n \), we just add the second index to obtain \( Y_{n,s} \). By the almost-graded structure, its fine structure (3.6), respectively, for the expressions \([Y_{n,s}, Z_{m,r}]\) in the relations only terms involving \( s = r \) will contribute on the level under consideration. If \( s \neq r \) they will contribute only to higher levels. Hence, all the relations there can be read with respect to all the second indices as being the same up to higher level. Hence, the proof is completely analogous.

**Proposition 8.6.** Let \( \gamma \) be a normalized cocycle. Then
(a) it vanishes for levels greater than zero, that is,
\[
\gamma(\overline{g}_n, \overline{g}_n) = 0 \quad \text{for} \quad n + m > 0;
\] (8.14)
(b) all levels \( l < 0 \) are fixed by the level zero.

**Proof.** By the propositions above we showed that the expressions at level \( l \) of the cocycle can be reduced to expressions of levels \( > l \) and values \( \gamma(H_1^\alpha, s, H_{-1}^\alpha, s) \). As long as the level is \( \neq 0 \), by (8.13) these values can also be expressed by higher levels. Hence by induction, starting with the upper bound of the cocycle, we obtain that the upper bound for the level of the cocycle values is equal to zero. Also, it follows that the values at levels \( l < 0 \) are fixed by induction by the values at level zero.

Hence it remains to consider the level zero.

**Proposition 8.7.** Let \( \alpha \) be a simple root. At level \( l = 0 \) the cocycle values for \( s = 1, \ldots, N \) are given by the relations
\[
\gamma(H_{n,s}^\alpha, H_{-n,s}^\alpha) = n \cdot \gamma(H_{1,s}^\alpha, H_{-1,s}^\alpha) \delta_s^r, \quad \gamma(H_{0,r}^\alpha, H_{0,s}^\alpha) = 0.
\] (8.15)

**Proof.** If we set the value \( l = 0 \) in (8.12) we obtain the relation
\[
\gamma(H_{n+1,s}^\alpha, H_{-(n+1),s}^\alpha) \equiv (\gamma(H_{n-1,s}^\alpha, H_{-(n-1),s}^\alpha) + 2\gamma(H_{1,s}^\alpha, H_{-1,s}^\alpha)) \cdot \delta_s^r.
\] (8.16)
As all cocycle values of level \( l > 0 \) vanish we can replace \( \equiv \) by \( = \). Now the expression we require follows.

**Proof of Theorem 8.1.** After adding a suitable coboundary we might replace the given \( \gamma \) by a normalized one. Using Propositions 8.3, 8.5, 8.7 everything depends only on the values \( \gamma(H_{1,s}^\alpha, H_{-1,s}^\alpha), s = 1, \ldots, N \), for one simple root. This proves that there are at most \( N \) linearly independent normalized cocycles.

**Proposition 8.8.** If a normalized cocycle \( \gamma \) is a coboundary then it vanishes identically.

**Proof.** As we explained above, a normalized cocycle is fixed by the values \( \gamma(H_{1,s}^\alpha, H_{-1,s}^\alpha) \). We set
\[
H_{(1,s)}^\alpha := H_{0,s}^\alpha A_{1,s} \equiv H_{1,s}^\alpha \quad \text{and} \quad H_{(-1,s)}^\alpha := H_{0,s}^\alpha A_{-1,s} \equiv H_{-1,s}^\alpha.
\] (8.17)
Hence
\[
[H_{(1,s)}^\alpha, H_{(-1,s)}^\alpha] = [H_{0,s}^\alpha, H_{0,s}^\alpha] A_{1,s} A_{-1,s} = 0.
\] (8.18)
As the cocycle vanishes for positive levels, and as $\gamma = \delta \phi$ is assumed to be a coboundary we get
\[ \gamma(H_{1,s}^\alpha, H_{-1,s}^\alpha) = \gamma(H_{(1,s)}^\alpha, H_{(-1,s)}^\alpha) = \phi([H_{(1,s)}^\alpha, H_{(-1,s)}^\alpha]) = \phi(0) = 0. \] (8.19)
Hence, all cocycle values are zero, as claimed.

§ 9. Example $\mathfrak{gl}(n)$

In this section, to illustrate our results for the reader, we will reproduce the proof that the product of two Lax operators for the algebra $\mathfrak{gl}(n)$ is again a Lax operator. This means that equations (2.6) hold for this product. Hence, $\overline{\mathfrak{gl}}(n)$ will be closed under the commutator too. This result is due to Krichever and Sheinman [1]. The other cases are treated in a similar manner (but now only for the commutators). Furthermore, it is shown that the connection operators $\nabla_{c}^{(\omega)}$ do indeed act on $\overline{\mathfrak{gl}}(n)$. The original proofs (involving some rather tedious calculations) can be found in [1], [6], [16].

The singularities at the points in $A$ are not bounded. Hence, they will not create problems and the proofs need only consider the weak singular points. Consequently, the statements are also true in the multi-point case.

We start with two elements $L'$ and $L''$ with corresponding expansions (2.5) and examine their product $L = L'L''$. To do this we have to consider each point $\gamma_s$ (with local coordinate $w_s$) of the weak singularities with $\alpha_s \neq 0$ separately. Taking only those parts which might contribute into account we find $L$ satisfies
\[ L = \frac{L'_{s,-1}L''_{s,-1}}{w_s^2} + \frac{L'_{s,-1}L''_{s,0} + L'_{s,0}L''_{s,-1}}{w_s^4} + (L'_{s,-1}L''_{s,1} + L'_{s,0}L''_{s,0} + L'_{s,1}L''_{s,-1}) + O(w_s^4). \] (9.1)
By expanding the first numerator we get
\[ L'_{s,-1}L''_{s,-1} = \alpha_s \beta_s^t \alpha_s \beta_s^{nt} = 0 \]
(9.2) as $\beta_s^t \alpha_s = 0$ by (2.6). Hence, no pole of order two appears.

Next we consider the expression which comes with a pole of order one.
\[ L_{s,-1} = L'_{s,-1}L''_{s,0} + L'_{s,0}L''_{s,-1} = \alpha_s \beta_s^t L''_{s,0} + L'_{s,0} \alpha_s \beta_s^{nt}. \] (9.3)
As by hypothesis $L'_{s,0} \alpha_s = \kappa_s' \alpha_s$ we can write
\[ L_{s,-1} = \alpha_s \beta_s^t, \text{ with } \beta_s^t = \beta_s^t L''_{s,0} + \kappa_s' \beta_s^{nt}. \] (9.4)
For the trace condition we obtain
\[ \text{tr}(L_{s,-1}) = (\beta_s^t L''_{s,0} + \kappa_s' \beta_s^{nt}) \alpha_s = \kappa_s'' \beta_s^t \alpha_s + \kappa_s' \beta_s^{nt} \alpha_s = 0. \] (9.5)
Hence, we have the required form.

Finally we have to verify that $\alpha_s$ is an eigenvector of $L_{s,0}$. First we note that $L''_{s,-1} \alpha_s = 0$ and $L'_{s,0}L''_{s,0} \alpha_s = \kappa_s' \kappa_s'' \alpha_s$. Also
\[ L'_{s,-1}L''_{s,1} \alpha_s = \alpha_s (\beta_s^t L''_{s,1} \alpha_s) = (\beta_s^t L''_{s,1} \alpha_s) \alpha_s. \] (9.6)
Hence, $\alpha_s$ is indeed an eigenvector with eigenvalue $\beta_s^t L''_{s,1} \alpha_s + \kappa'_s \kappa''_s$. This proves our claim that $L \in \mathfrak{gl}(n)$.

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