Optimal Control and Stabilization for MFSDEs with Multiple Defaults*

Zhun Gou\textsuperscript{a}, Nan-jing Huang \textsuperscript{\textdagger b}, Ming-hui Wang\textsuperscript{c}, and Jian-hao Kang\textsuperscript{d}

\textsuperscript{a}College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, P.R.China
\textsuperscript{b}Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China
\textsuperscript{c}School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, Sichuan 610074, P.R. China
\textsuperscript{d}School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 610031, P.R. China

Abstract. In this paper, we investigate an optimal control problem governed by the mean-field stochastic differential equation with multiple defaults, which is motivated by the optimal investment problems. This global optimal control problem is transformed and divided into several optimal control subproblems governed by mean-field stochastic differential equations with single default. We derive both the sufficient and the necessary maximum principles for such subproblems and then give the existence and uniqueness of solutions to the mean-field stochastic differential equation with multiple defaults and the mean-field backward stochastic differential equations with multiple defaults, respectively. Moreover, the obtained results are applied to solve an optimal investment problem, in which the wealth process is governed by a mean-field stochastic differential equation with multiple defaults. Finally, we study both the mean square exponential stability and the almost sure exponential stability under some mild conditions for the solution to the mean-field stochastic differential equation with multiple defaults under optimal feedback control.

Keywords: Mean-field SDEs; Mean-field BSDEs; Optimal control; Multiple defaults; Enlargement of filtration; Stability.

2020 Mathematics Subject Classification: 60H07, 60H20, 60J76, 91G80, 93E20, 93D15.

1 Introduction

The theory of enlargement of filtration, started by Jacod \cite{24}, Jeulin \cite{26} and Yor \cite{61} in the 1980s, is a powerful tool to preserve the adaptedness of stochastic processes under a change of filtration. Some

\textsuperscript{*}This work was supported by the National Natural Science Foundation of China (12171339), the grant from Chongqing Technology and Business University (2056004) and the Fundamental Research Funds for the Central Universities (2682023CX071).

\textsuperscript{\textdagger}Corresponding author: nanjinghuang@hotmail.com; njhuang@scu.edu.cn
interest models have been developed in mathematical finance [9, 18, 25] to study the asymmetry of information, in which different agents have a different level of information and credit default events assumed to occur at a random time $\tau$ in the same financial market. The solutions to corresponding problems under the above models are all obtained by employing the theory of the enlargement of filtration.

Usually, when investors consider credit default events, they assume that these events arrive surprisingly, i.e., $\tau$ is independent of the reference filtration $\mathcal{F}$ (these filtrations always are generated by inside the systems). The standard approach to deal with these default events is mainly based on the theory of enlargement a reference filtration $\mathcal{F}$ by the information of $\tau$, which leads to a new global filtration $\mathcal{G}$. This approach is called the progressive enlargement of filtration. Besides, the usual hypothesis (H) that any $\mathcal{F}$-martingale remains an $\mathcal{G}$-martingale is required to ensure the martingale representation in $\mathcal{G}$. Based on the above approach and hypothesis, Bachir et al. [7] transformed the stochastic differential equation (SDE) with default into an SDE driven by an additional martingale under the new filtration. Then, they studied the optimal control problem governed by the new SDE system. Moreover, Peng and Xu [46] studied backwards stochastic differential equations (BSDEs) with the random default time and applied it to investigate default risk. For more details about the method of progressive enlargement of filtration, we refer to [4, 5, 12, 18, 20, 22, 51].

On the other hand, it is well known that mean-field stochastic differential equation (MFSDE) has attracted much interest from many areas including the field of control theory. The optimal control problem governed by MFSDEs has been studied by numerous researchers (see [1–3, 6, 21, 32, 37, 42, 43, 56, 62, 63]). Furthermore, the optimal control problems governed by mean-field stochastic differential equation with a single default (SMFSDE) and the ones of mean-field stochastic differential equation with multiple defaults (MMFSDE) were studied in [7, 23] and [47], respectively, in which both the SMFSDE and the MMFSDE are considered in the framework of progressive enlargement of filtration. We also note that the conditional MFSDEs system and its applications have attracted much attention recently (see [11, 16, 38]). This system plays an important role in mean-field optimal control problems with partial information [10], and in mean-field games (MFGs) under asymmetric information (especially in MFGs with major and minor players [14, 15]). Nevertheless, to the best of our knowledge, the existing results including [10, 47] are almost unapplicable to solve the following practical problem, which is inspired from [7].

Problem 1.1. Consider the wealth process with default

$$
\begin{cases}
    dX(t) = X(t-)[\alpha - u(t)]dt + \beta dB_t + \sum_{i=1}^2 \varepsilon_i d\mathbb{I}_{\tau_i \leq t} + \delta \overline{X}(t)dt, & t \in (0, T], \\
    X(0) = x_0 \in \mathbb{R},
\end{cases}
$$

where $\alpha, \beta, \varepsilon, \delta$ are all constants satisfying some mild conditions; $u(t)$ is the control; $\tau_i$ is a random variable representing the default time; $\mathbb{I}_{\tau_i \leq t}$ is the indicator process; $\delta \overline{X}(t)$ is the mean-field term. Moreover, the performance function is given by

$$
J(u) = \mathbb{E}\left[\int_0^T \theta_1 \ln(X(t)u(t))dt + \theta_2 \ln X(T)\right],
$$

where $\theta_1$ and $\theta_2$ are positive constants. For given admissible control set $\mathcal{U}$, we would like to find $\tilde{u}(\cdot) \in \mathcal{U}$ such that $J(\tilde{u}) = \max_{u(\cdot) \in \mathcal{U}} J(u)$.

Problem 1.1 is nothing but a finite horizon optimal control problem governed by the MMFSDE, of which the discipline is still not fully explored and much is desired to be done. Thus the first purpose of
current paper is to investigate the finite horizon optimal control problems governed by the systems of MMFSDEs under some mild conditions, and ensure the solvability of such systems.

After achieving the solvability of SDEs, one important issue of studying the well-posedness of SDEs is the stability of the solutions, in particular for the qualitative study and/or for the long time asymptotic behaviour of the solutions [50]. In general, a solution to an SDE is stable if it is insensitive for small changes of the initial value or the parameters of the SDE. Up to now, the stability of the solutions to SDEs has been very well developed since the seminal work [28], which extended the celebrating concept of stability of deterministic dynamic systems introduced by Lyapunov [35]. Various stability, were carried out in a series of works by Mao [39, 41], such as stochastic stability, stochastic asymptotical stability, moment exponential stability (including mean square exponential stability), almost sure exponential stability, mean square polynomial stability etc. There are also fruitful results for the stability of the solutions to various type SDEs, such as SDEs with delays [29, 30, 32], fractional SDEs [30, 41, 51, 52] and stochastic functional differential equations [13, 49]. Moreover, the stability of the solutions to controlled SDEs (respectively MFSDEs) under optimal feedback controls has been established in [31, 34, 64] (respectively [54, 58]). Very recently, the stability of the solutions to controlled MFSDEs under optimal feedback controls was established in [54, 58]. However, to the best of our knowledge, there are no papers investigating the stability of the solutions to MMFSDEs. Thus, the second purpose of the current paper is to investigate the stability of the solutions to the controlled MMFSDEs under optimal feedback controls.

The main contributions of this paper can be summarized as follows: (i) The MMFSDE is considered in the sense of conditional expectation, which is quite different from the classical MFSDEs [1, 59], and the optimal control problem governed by the MMFSDE is proposed; (ii) Backward induction method is derived for dividing our optimal control problems into several subproblems, while it is not available to solve traditional mean-field type (not conditional type) optimal control problems. (iii) Both sufficient and necessary maximum principles are obtained for such subproblems with random coefficients; (iv) The existence and uniqueness results of solutions are obtained for both MMFSDEs and mean-field backward stochastic differential equation with multiple defaults (MMFBSDEs) under some mild conditions; (v) Mean square exponential stability and almost sure exponential stability are established for the solutions to controlled MMFSDEs under optimal feedback controls.

The rest of this paper is structured as follows. The next section introduces some necessary preliminaries including the enlargement progressive of filtration. After that in Section 3, the backward induction method is given, which transforms the global optimal control problem into several optimal control subproblems, and both sufficient and necessary maximum principles are derived for these subproblems. In Section 4, the existence and uniqueness results of solutions are given for both MMFSDEs and MMFBSDEs, and the obtained results are applied to solve an optimal investment problem. Before we conclude, the mean square exponential stability and the almost sure exponential stability are established for controlled MMFSDEs under optimal feedback control in Section 5.
2 Problem Formulation

In this paper, we consider the following system of controlled MMFSDE in the complete probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\):

\[
\begin{aligned}
\begin{cases}
    dX(t) = b(t, X(t), M(X(t)), u(t), N(u(t)))dt + \sigma(t, X(t), M(X(t)), u(t), N(u(t)))dB_t \\
    \quad + h(t, X(t), M(X(t)), u(t), N(u(t)))d\mathcal{H}_t, & t \in (0,T], \\
    X(0) = x_0,
\end{cases}
\end{aligned}
\]

where \(X(\cdot)\) is the state variable, \(u(\cdot)\) denotes the control variable, and \(M(X(\cdot)), N(u(\cdot))\) are the conditional mean-field terms. Denote \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) the right continuous, increasing and complete filtration generated by the one-dimensional Brownian motion \((B_t)_{t \geq 0}\). Set \(\mathcal{I}_0 = \{0, 1, \cdots, n\}\) and \(\mathcal{I} = \{1, 2, \cdots, n\}\).

The default term

\[\mathbb{H}_t = \mathbb{H}(t) = \sum_{k=1}^{n} \mathbb{H}^k(t) = \sum_{k=1}^{n} \mathbb{I}_{\tau_k \leq t}\]

is generated by a sequence of ordered default times \(\{\tau_k\}_{k \in \mathcal{I}}\) with

\[0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_{n+1} = T \quad \text{a.s.}\]

For \(k \in \mathcal{I}\) and all \(t \geq 0\), we denote \(\mathcal{N}^k = \sigma(\mathbb{I}_{\tau_k \leq u}, u \in [0,t])\). The global information is then naturally defined as the progressive enlargement of filtration \(\mathfrak{F} = \mathcal{F} \vee \mathcal{N}^1 \vee \mathcal{N}^2 \vee \cdots \vee \mathcal{N}^n\). The filtration \(\mathfrak{F} = (\mathfrak{F}_t)_{t \geq 0}\) is the smallest right continuous extension of \(\mathcal{F}\) such that each \(\tau_k (k \in \mathcal{I})\) becomes an \(\mathfrak{F}\)-stopping time. \(x_0\) is an \(\mathfrak{F}_0\)-measurable and square integrable random variable, the coefficients \(b, \sigma\) and \(h\) will be introduced later. Furthermore, we give the following necessary assumptions.

**Assumption 2.1.**

(i) (Hypothesis (\(\mathcal{H}\))) Every càdlàg \(\mathcal{F}\)-martingale remains an \(\mathfrak{F}\)-martingale.

(ii) Suppose that there exists an \(\mathfrak{F}\)-predictable (respectively, \(\mathcal{F}\)-predictable) process \(\gamma^{\mathcal{F}_k}\) (respectively, \(\gamma^{\mathfrak{F}_k}\)) with \(\gamma^{\mathfrak{F}_k}(s) = \mathbb{I}_{s < \tau_k} \gamma^{\mathcal{F}_k}(s)\) such that

\[A^k(t) = \mathbb{H}^k(t) - \int_0^t \gamma^{\mathcal{F}_k}(s)ds = \mathbb{H}^k(t) - \int_0^{t \wedge \tau_k} \gamma^{\mathcal{F}_k}(s)ds \quad (t \geq 0)\]

is an \(\mathfrak{F}\)-martingale with jump time \(\tau_k\). The process \(\gamma^{\mathcal{F}_k}\) (respectively, \(\gamma^{\mathcal{F}_k}\)) is called the \(\mathfrak{F}\)-intensity (respectively, \(\mathcal{F}\)-intensity) of \(\tau_k\). In addition, assume that \(\gamma^{\mathcal{F}_k}\) is upper bounded.

**Remark 2.1.** Hypothesis (\(\mathcal{H}\)) in Assumption 2.1 is also called the immersion property, i.e., the filtration \(\mathcal{F}\) is immersed in \(\mathfrak{F}\). Under Assumption 2.1, for any \(\mathfrak{F}\)-optional process \(X(t)\) with \(\mathbb{E}[\int_0^T X(t)^2dt] < \infty\), the stochastic integral \(\int_0^T X(t)dB_t\) is well defined [27].

**Remark 2.2.** In the rest of the paper we set \(Y^{\mathfrak{F}} = \sum_{k=1}^{n} \gamma^{\mathfrak{F}_k}\) and \(A(t) = \mathbb{H}(t) - \int_0^t Y^{\mathfrak{F}} ds\). It is easy to see that

\[\mathbb{E}[A(t)|\mathfrak{F}_t] = \mathbb{E} \left[ \sum_{k=1}^{n} \left( \mathbb{H}^k(t) - \int_0^t \gamma^{\mathfrak{F}_k}(s)ds \right) | \mathfrak{F}_t \right] = \mathbb{E} \left[ \sum_{k=1}^{n} A^k(t) | \mathfrak{F}_t \right] = \sum_{k=1}^{n} \mathbb{E}[A^k(t)|\mathfrak{F}_t] = \sum_{k=1}^{n} A^k(t) = A(t).\]

Therefore, \(A(t)\) is an \(\mathfrak{F}\)-martingale.

Next, we recall some useful sets and spaces, and the definition of the Wasserstein metric.
Definition 2.1. (i) Let $\mathcal{F}(\mathcal{F})$ (respectively, $\mathcal{P}(\mathcal{F})$) denote the $\sigma$-algebra of $\mathcal{F}$-optional (respectively, $\mathcal{F}$-predictable) measurable subsets of $[0, T] \times \Omega$, i.e., $\mathcal{F}(\mathcal{F})$ (respectively, $\mathcal{P}(\mathcal{F})$) is the $\sigma$-algebra generated by the right continuous (respectively, left continuous) $\mathcal{F}$-adapted processes.

(ii) For a given Borel set $\mathbb{U} \subset \mathbb{R}$, let $\mathcal{O}(\mathcal{F}, \mathbb{U})$ denote the set of elements in $\mathcal{O}(\mathcal{F})$ valued in $\mathbb{U}$.

(iii) $L^2([0, T])$ is the space of all square integrable functions $f$ with
$$\|f\|_{L^2([0, T])}^2 = \int_0^T |f(t)|^2 \, dt < +\infty.$$

(iv) For $T_0 \in [0, T]$, $L^2_{T_0}(\mathbb{P})$ is the space of $\mathcal{F}_{T_0}$-measurable random variable $\xi$ such that
$$\|\xi\|_{L^2_{T_0}(\mathbb{P})}^2 = \mathbb{E}[\xi^2] < +\infty.$$

(v) $L^2_T$ is the set of all $\mathcal{O}(\mathcal{F})$-measurable processes $X(t)$ such that
$$\|X(\cdot)\|_{L^2_T}^2 := \mathbb{E}\left( \int_0^T |X(t)|^2 \, dt \right) < +\infty,$$

(vi) $L^2_{+, \infty}$ is the set of all $\mathcal{O}(\mathcal{F})$-measurable processes $X(t)$ such that
$$\|X(\cdot)\|_{L^2_{+, \infty}}^2 := \mathbb{E}\left( \int_0^{+\infty} e^{-\delta t} |X(t)|^2 \, dt \right) < +\infty,$$
where $\delta$ is a given constant, and $H^2_T$ is the set of all $\mathcal{P}(\mathcal{F})$-measurable processes $X(t)$ such that
$$\|X(\cdot)\|_{H^2_T}^2 := \mathbb{E}\left( \int_0^T |X(t)|^2 \, dt \right) < +\infty,$$

(vi) $L^2_{T}A^k$ is the set of all $\mathcal{O}(\mathcal{F})$-measurable processes $X(t)$ such that
$$\|X(\cdot)\|_{L^2_{T}A^k}^2 := \mathbb{E}\left( \int_0^T \gamma^k(t)|X(t)|^2 \, dt \right) < +\infty$$
and $H^2_{T}A^k$ is the set of all $\mathcal{P}(\mathcal{F})$-measurable processes $X(t)$ such that
$$\|X(\cdot)\|_{H^2_{T}A^k}^2 := \mathbb{E}\left( \int_0^T \gamma^k(t)|X(t)|^2 \, dt \right) < +\infty.$$

(vii) $\mathcal{M}^k$, $\mathcal{G}^k$, $\mathcal{S}^k$ are the subfiltrations of $\mathcal{F}$ such that $\mathcal{M}_t^k, \mathcal{G}_t^k, \mathcal{S}_t^k \subseteq \mathcal{F}_t$ for all $t \in [0, T]$ and $\mathcal{M}_{T_k}^k = \mathcal{S}_{T_k}^k = \mathcal{F}_{T_k}$.

Definition 2.2. Consider the space of probability measures $m$’s on $\mathbb{R}$ with $\int_\mathbb{R} x^2 \, dm(x) < +\infty$. For any such $m$ and $m'$, the 2-Wasserstein metric of them is defined by
$$W_2(m, m') = \inf_{\pi \in \Pi_2(m, m')} \left( \int_{\mathbb{R} \times \mathbb{R}} (x - x')^2 \pi(dx, dx') \right)^{\frac{1}{2}},$$
where $\Pi_2(m, m')$ represents the set of joint probability measures with respective marginals $m$ and $m'$. Denote $\mathcal{P}_2(\mathbb{R})$ the space equipped with the 2-Wasserstein metric. The infimum is attainable such that there are random variables $X_m$ and $X_{m'}$ associated with $m$ and $m'$ respectively so that
$$W_2(m, m') = \sqrt{\mathbb{E}[(X_m - X_{m'})^2]}.$$
Usually, we denote $S^k_t$ the observed information up to time $t$, and use $M^k_t$ and $G^k_t$ to represent the information available to the state and the controller at time $t$, respectively. For instance, for $t \in [\tau_k, \tau_{k+1}]$, $S^k_t = \mathcal{F}_{\tau_k + (t - \tau_k - \delta_0)}$ represents the delayed information flow compared to $\mathcal{F}_t$, where $\delta_0$ is a positive constant. And the mean-field term $N(u(t))$ can be removed when $G^k_t = S^k_t$. Moreover, for $X(\cdot), u(\cdot) \in L^2_T$, the conditional mean-field terms $M(X(\cdot))$ and $N(u(\cdot))$ are given as follows:

$$
M(X(t)) = \sum_{k=0}^{n-1} \mathbb{E} \left[ X^k(t) \mathbb{I}_{t \in [\tau_k, \tau_{k+1}]} \right] \mathbb{I}_{t \in [\tau_n, T]},
$$

$$
N(u(t)) = \sum_{k=0}^{n-1} \mathbb{E} \left[ u^k(t) \mathbb{I}_{t \in [\tau_k, \tau_{k+1}]} \right] \mathbb{I}_{t \in [\tau_n, T]},
$$

where $X^i(t)$ and $u^i(t)$ $(i \in I_0)$ are determined by

$$
X(t) = \sum_{k=0}^{n-1} X^k(t) \mathbb{I}_{t \in [\tau_k, \tau_{k+1}]} + X^n(t) \mathbb{I}_{t \in [\tau_n, T]},
$$

$$
u(t) = \sum_{k=0}^{n-1} u^k(t) \mathbb{I}_{t \in [\tau_k, \tau_{k+1}]} + u^n(t) \mathbb{I}_{t \in [\tau_n, T]}.
$$

The above decompositions can be ensured by Lemma 2.1 in [47]. Without ambiguity, we write $A(s) = A(s^-)$ and $H(s) = H(s^-)$ since there are only finite default jumps. In the sequel we regard the conditional mean-field terms as some elements in the metric space $\mathcal{P}_2(\mathbb{R})$, which is complete and separable. And so under mild conditions, the functionals of the conditional mean-field terms enjoys the Frechet derivatives defined as follows.

**Definition 2.3.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. A mapping $F : \mathcal{X} \to \mathcal{Y}$ is said to have a directional derivative (or Gâteaux derivative) at $v \in \mathcal{X}$ in the direction $\zeta \in \mathcal{X}$ if

$$
D_\zeta F(v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F(v + \varepsilon \zeta) - F(v))
$$

exists in $\mathcal{Y}$. Moreover, $F$ is Fréchet differentiable at $v \in \mathcal{X}$ if there exists a continuous linear map $A : \mathcal{X} \to \mathcal{Y}$ such that

$$
\lim_{\|\zeta\| \to 0} \frac{1}{\|\zeta\|_\mathcal{X}} \|F(v + \zeta) - F(v) - A(\zeta)\|_\mathcal{Y} = 0,
$$

where $A(h) = (A, h)$ is the action of the linear operator $A$ on $h$. In this case $A$ is called the gradient (or Fréchet derivative) of $F$ at $v$ which can be written as $A = \nabla_v F$. If $F$ is Fréchet differentiable at $v$ with Fréchet derivative $\nabla_v F$, then $F$ has a directional derivative in all directions $\zeta \in \mathcal{X}$ and $D_\zeta F(v) = \nabla_v F(\zeta) = (\nabla_v F, \zeta)$.

**Remark 2.3.** Clearly, if $F$ is a linear operator, then $\nabla_v F = F$ for all $v \in \mathcal{X}$. By choosing $\mathcal{X} = \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}$ and $\mathcal{Y} = \mathbb{L}_2(\mathbb{R})$, the Fréchet derivatives of mean-field terms can be well defined [3].

The admissible control set $\mathcal{U}^{ad}$ of the control variable is defined as follows:

$$
\mathcal{U}^{ad} = \left\{ u \left| u(t) = \sum_{k=0}^{n-1} u^k(t) \mathbb{I}_{t \in [\tau_k, \tau_{k+1}]} + u^n(t) \mathbb{I}_{t \in [\tau_n, \tau_{n+1}]}, u^k \in \mathcal{U}^k_{ad}, k \in I_0 \right. \right\}.
$$

Here $\mathcal{U}^k_{ad} (k \in I_0)$ is the separable metric subspace of $\mathcal{O}(S^k, U^k)$, where $(U^k)_{k \in I_0}$ is a given sequence of subsets in $\mathbb{R}$. We note here that the admissible control set is allowed to be variant. There exists an important distinction between the controlled system with the Poisson jumps and the one with defaults,
i.e., the classical formulation has to fix an admissible control set which is invariant during the time horizon, while the more general formulation can deal with different admissible control sets between two default times. For these relevant works, we refer to [147]. Furthermore, we study the optimal control problem in the sense of partial information, which is inspired by some interesting financial phenomena; for instance in some situations of real markets like insider trading, one investor may get more information than the others, and then, the investor can make a better decision than the others [21, 57]. For simplicity, we use the notation \( u = (u^0, u^1, \ldots, u^n) \) for \( u \in \mathcal{U}^{ad} \) in the sequel. Without ambiguity, we set \( u^k(t) = 0 \) for \( t \notin [\tau_k, \tau_{k+1}) \).

The following lemma will be frequently used in the sequel.

**Lemma 2.1.** Let \( X_i(t) (i = 1, 2) \) be an Itô’s jump-diffusion process given by

\[
dX_i(t) = b_i(t)dt + \sigma_i(t)dB_i + \sum_{k=1}^{n} h^k_i(t) d\Lambda^k_i,
\]

where \( b_i(\cdot), \sigma_i(\cdot) \in L^2_T \) and \( h^k_i(\cdot) \in L^3_T A^k \) \((k \in \mathcal{I})\). Then

\[
d(X_1(t)X_2(t)) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + [dX_1(t), dX_2(t)]
\]

\[
= X_1(t)dX_2(t) + X_2(t)dX_1(t) + \sigma_1(t)\sigma_2(t)dt + \sum_{k=1}^{n} h^1_i(t)h^2_i(t)d\mathbb{H}^k.
\]

Especially, if \( h^1_i(t) = h^1(t) \) for all \( k \in \mathcal{I} \), then

\[
d(X_1(t)X_2(t)) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + \sigma_1(t)\sigma_2(t)dt + h^1(t)h^2(t)d\mathbb{H}^1.
\]

The finite horizon optimal control problem considered in this paper is specified as follows.

**Problem 2.1.** The performance functional associated to the control \( u = (u^0, u^1, \ldots, u^n) \in \mathcal{U}^{ad} = \mathcal{U}^{ad}_0 \times \mathcal{U}^{ad}_1 \times \cdots \times \mathcal{U}^{ad}_n \) takes the following form

\[
J(x_0, \mathcal{L}(x_0), u) = \mathbb{E} \left[ \int_0^T f(t, X(t), M(X(t)), u(t), N(u(t))) dt + g(X(T), M(X(T))) \bigg| \mathcal{F}_0 \right].
\]

Here, \( X(t) \) is described by (1), \( \mathcal{L}(\cdot) \) represents the law of a random variable, and the running gain function \( f : [0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R} \) and the terminal gain function \( g : \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R} \) are progressively measurable functions satisfying

\[
|f(t, X, M, u, N)| + |g(X, M)| \leq C(1 + X^2 + |M|^2_{\mathcal{P}_2(\mathbb{R})} + u^2 + |N|^2_{\mathcal{P}_2(\mathbb{R})}) \tag{2}
\]

for any \((t, X, M, u, N) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})\) and for some constant \( C > 0 \), a.s.. Moreover, \( f \) can be decomposed as

\[
f(t, X(t), M(X(t)), u(t), N(u(t))) = \sum_{k=0}^{n-1} f^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t)))1_{t \in [\tau_k, \tau_{k+1})} + f^n(t, X^n(t), M(X^n(t)), u^n(t), N(u^n(t)))1_{t \in [\tau_n, T]},
\]

where \( f^k (k \in \mathcal{I}_0) \). The problem is to find the optimal control

\[
\hat{u}(t) = \sum_{k=0}^{n-1} \hat{u}^k(t)1_{t \in [\tau_k, \tau_{k+1})} + \hat{u}^n(t)1_{t \in [\tau_n, T]} \in \mathcal{U}^{ad}
\]
such that

\[ V(x_0, \mathcal{L}(x_0)) = J(x_0, \mathcal{L}(x_0), \bar{u}) = \text{ess} \sup_{u \in \mathcal{U}^+} J(x_0, \mathcal{L}(x_0), u), \]

\( \bar{u} \) where the optimal controlled pair \((\bar{X}, \bar{u})\) is governed by (1).

**Remark 2.4.** In the running gain, there is a change of regimes after each default time, which is in the spirit of the concept of forward or progressive utility functions introduced in [43].

### 3 Backward Induction Method and Maximum Principle

In this section, we first give the backward induction method for solving Problem 2.1. We transform the global optimal control problem into several subproblems. Then, we give both the sufficient and necessary conditions for these optimal control subproblems.

#### 3.1 Backward Induction Method

To begin with, we make the following assumption throughout this paper.

**Assumption 3.1.** Suppose that \( b, \sigma, h : [0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R} \) are all progressively measurable functions. Furthermore, assume that for any \( \pi \in \{b, \sigma, h\}, \pi \) admits the following composition

\[
\pi^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) = \pi(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) I_{t \in [\tau_k, \tau_{k+1}]}
\]

Under the above assumption, we are able to restrict equation (1) on the random time interval \( t \in [\tau_k, \tau_{k+1}] \) \((k \in \mathbb{Z}_0)\):

\[
\begin{align*}
dX^k(t) &= b^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) dt + \sigma^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) dB_t \\
&\quad + h^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) dB_{\tau}^{k+1}, \quad t \in (\tau_k, \tau_{k+1}], \\
X^k(\tau_k) &= X^{k-1}(\tau_k),
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
dX^k(t) &= b^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) dt + \sigma^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) dB_t \\
&\quad + h^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) dA_{t}^{k+1}, \quad t \in (\tau_k, \tau_{k+1}], \\
X^k(\tau_k) &= X^{k-1}(\tau_k).
\end{align*}
\]

Here

\[
b^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) = b^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) + \gamma^{k+1}(t) h(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))),
\]

\( X^0(0) = X^{-1}(0) = x_0 \in \mathbb{R}, X^n(T) = X(T) \) and \( h^n \equiv 0 \). We show in Subsection 4.1 that under some normal conditions, there exists a unique square integrable optional solution to (1). Since Lemma 2.1 in [47] (or Theorem 6.5 of [53]) provides a key decomposition of optional processes with respect to the progressive enlargement of filtration, we can see that the solution \( X(t) \) of (1) has the following decomposition:

\[
X(t) = \sum_{k=0}^{n-1} X^k(t) I_{t \in [\tau_k, \tau_{k+1}]} + X^n(t) I_{t \in [\tau_n, \tau_{n+1}]},
\]
where $X^k(t)$ is governed by $\mathbb{E}^k$.

Now we use the decomposition to derive the following backward induction method for the performance function.

**Proposition 3.1.** Consider the following backward induction sequence $(k \in \mathcal{I}_0)$

$$J^n(X^{n-1}(\tau_n), \mathcal{L}^{n-1}(X^{n-1}(\tau_n)), u^n) = \mathbb{E}\left[\int_{\tau_n}^{T} f^n(t, X^n(t), M(X^n(t)), u^n(t), N(u^n(t)))dt + \mathcal{L}^n(X^n(T), M(X^n(T)))\bigg| \mathcal{F}_{\tau_n}\right],$$

$$J^k(X^{k-1}(\tau_k), \mathcal{L}^{k-1}(X^{k-1}(\tau_k)), u^k, \ldots, u^n) = \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} f^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t)))dt + J^{k+1}(X^k(\tau_{k+1}), \mathcal{L}^k(X^k(\tau_{k+1})), u^{k+1}, \ldots, u^n)\bigg| \mathcal{F}_{\tau_k}\right]$$

with $\mathcal{L}^n = g$ and $\mathcal{L}^{-1} = \mathcal{L}$. Then $J^0(x_0, \mathcal{L}(x_0), u) = J^0(X^{-1}(0), \mathcal{L}^{-1}(X^{-1}(0)), u^0, \ldots, u^n) = J(x_0, \mathcal{L}(x_0), u)$.

**Proof.** Since $\mathbb{E}\left[ X^{k-1}(\tau_k) \big| \mathcal{M}_{\tau_k} \right] = \mathbb{E}\left[ X^{k-1}(\tau_k) \big| \mathcal{F}_{\tau_k} \right] = X^{k-1}(\tau_k)$, it follows that $J^k(X^{k-1}(\tau_k), u^k, \ldots, u^n)$ and $J^n(X^{n-1}(\tau_n), u^n)$ are well-defined. Clearly,

$$J^{n-1}(X^{n-2}(\tau_{n-1}), \mathcal{L}^{n-2}(X^{n-2}(\tau_{n-1})), u^{n-1}, u^n)$$

$$= \mathbb{E}\left[\int_{\tau_{n-1}}^{\tau_n} f^{n-1}(t, X^{n-1}(t), M(X^{n-1}(t)), u^{n-1}(t), N(u^{n-1}(t)))dt + J^n(X^{n-1}(\tau_n), u^n)\bigg| \mathcal{F}_{\tau_{n-1}}\right]$$

$$= \mathbb{E}\left[\int_{\tau_{n-1}}^{\tau_n} f(t, X^{n-1}(t), M(X^{n-1}(t)), u^{n-1}(t), N(u^{n-1}(t)))dt \bigg| \mathcal{F}_{\tau_{n-1}}\right]$$

$$+ \mathbb{E}\left[\mathbb{E}\left[\int_{\tau_n}^{T} f(t, X^n(t), M(X^n(t)), u^n(t), N(u^n(t)))dt + g(X^n(T), M(X^n(T)))\bigg| \mathcal{F}_{\tau_n}\right] \bigg| \mathcal{F}_{\tau_{n-1}}\right]$$

$$= \mathbb{E}\left[\int_{\tau_{n-1}}^{\tau_n} f^{n-1}(t, X^{n-1}(t), M(X^{n-1}(t)), u^{n-1}(t), N(u^{n-1}(t)))dt \bigg| \mathcal{F}_{\tau_{n-1}}\right]$$

$$+ \mathbb{E}\left[\int_{\tau_n}^{T} f(t, X^n(t), M(X^n(t)), u^n(t), N(u^n(t)))dt + g(X^n(T), M(X^n(T)))\bigg| \mathcal{F}_{\tau_{n-1}}\right]$$

$$= \mathbb{E}\left[\int_{\tau_{n-1}}^{\tau_n} f^{n-1}(t, X(t), M(X(t)), u(t), N(u(t)))I_{t \in [\tau_{n-1}, \tau_n]}dt + g(X(T), M(X(T)))\bigg| \mathcal{F}_{\tau_{n-1}}\right].$$

Noticing that $X^n(T) = X(T)$ and $\tau_0 = 0$, it follows from the backward induction that

$$J^0(x_0, \mathcal{L}(x_0), u) = J^0(X^{-1}(0), \mathcal{L}^{-1}(X^{-1}(0)), u^0, \ldots, u^n)$$

$$= \mathbb{E}\left[\int_{\tau_0}^{T} \sum_{i=0}^{n-1} f^i(t, X(t), M(X(t)), u(t), N(u(t)))I_{t \in [\tau_i, \tau_{i+1}]}dt + g(X^n(T), M(X^n(T)))\bigg| \mathcal{F}_{\tau_0}\right]$$

$$= \mathbb{E}\left[\int_{0}^{T} f(t, X(t), M(X(t)), u(t), N(u(t)))dt + g(X^n(T), M(X^n(T)))\bigg| \mathcal{F}_0\right].$$

This ends the proof.

The following theorem provides a decomposition for the optimal control of Problem 1.1 which is essentially a conditional mean-field type dynamic programming principle.
Theorem 3.1. The optimal control \( \tilde{u} = (\tilde{u}^0, \ldots, \tilde{u}^n) \) can be obtained by solving each optimal control \( u^k (k \in \mathcal{I}_0) \) of the following subproblems \((k \in \mathcal{I}_0)\):

\[
V^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k))) = J^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k, \tilde{u}^{k+1}, \ldots, \tilde{u}^n)
= \text{ess sup}_{u^k \in U^d_{\tau_k}} J^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k)), u^k, u^{k+1}, \ldots, u^n). \quad (5)
\]

**Proof.** Omitting the elements \( \tilde{u}^{k+1}, \ldots, \tilde{u}^n \) \((k \in \mathcal{I}_0)\), it follows that

\[
V^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k))) = J^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k)
= \text{ess sup}_{u^k \in U^d_{\tau_k}} J^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k)), u^k)
= \text{ess sup}_{u^k \in U^d_{\tau_k}} \left[ \int_{\tau_k}^{\tau_{k+1}} f^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t)))dt + J^{k+1}(X^k(\tau_{k+1}), L^k(X^k(\tau_{k+1})), \tilde{u}^{k+1}, \ldots, \tilde{u}^n) \bigg| \mathcal{F}_{\tau_k} \right]
= \text{ess sup}_{u^k \in U^d_{\tau_k}} \left[ \int_{\tau_k}^{\tau_{k+1}} f^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t)))dt + V^{k+1}(X^k(\tau_{k+1}), L^k(X^k(\tau_{k+1}))) \bigg| \mathcal{F}_{\tau_k} \right], \quad (6)
\]

where \( V^{n+1}(X^n(\tau_{n+1}), L^n(X^n(\tau_{n+1}))) = g(X(T), M(X(T))) \). Here \((X^k, u^k)\) is the solution to the \((k+1)\)-th equation of (3), \( X^k(\tau_{k+1}) \) is the terminal value of this equation and \( V^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k))) \) is an \( \mathcal{F}_{\tau_k} \)-measurable random variable. Thus, we can set

\[
\psi^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k))) = J^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k, \tilde{u}^{k+1}, \ldots, \tilde{u}^n)
= \text{ess sup}_{(u^k, \ldots, u^n) \in U^d_{\tau_k} \times \ldots \times U^d_{\tau_n}} J^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k)), u^k, u^{k+1}, \ldots, u^n).
\]

We now aim to prove that

\[
V^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k))) = \psi^k(X^{k-1}(\tau_k), L^{k-1}(X^{k-1}(\tau_k))), \quad k \in \mathcal{I}_0. \quad (7)
\]

Trivially, equation (7) holds for \( k = n \). By Proposition 3.1 for any \((u^{n-1}, u^n) \in U^d_{\tau_{n-1}} \times U^d_n \) with \( k \in \mathcal{I}_0 \), we have

\[
\psi^n(X^{n-2}(\tau_{n-1}), L^{n-2}(X^{n-2}(\tau_{n-1}))) \geq J^{n-1}(X^{n-2}(\tau_{n-1}), L^{n-2}(X^{n-2}(\tau_{n-1})), u^{n-1}, u^n)
= \mathbb{E} \left[ \int_{\tau_{n-1}}^{\tau_n} f^{n-1}(t, X^{n-1}(t), M(X^{n-1}(t)), u^{n-1}(t), N(u^{n-1}(t)))dt + J^n(X^n(\tau_n), L^{n-1}(X^n(\tau_n), u^n)) \bigg| \mathcal{F}_{\tau_{n-1}} \right].
\]

Taking \( u^n = \tilde{u}^n \), one has

\[
\psi^n(X^{n-2}(\tau_{n-1}), L^{n-2}(X^{n-2}(\tau_{n-1}))) \geq \mathbb{E} \left[ \int_{\tau_{n-1}}^{\tau_n} f^{n-1}(t, X^{n-1}(t), M(X^{n-1}(t)), u^{n-1}(t), N(u^{n-1}(t)))dt + V^n(X^n(\tau_n), L^{n-1}(X^n(\tau_n))) \bigg| \mathcal{F}_{\tau_{n-1}} \right]
= V^{n-1}(X^{n-2}(\tau_{n-1}), L^{n-2}(X^{n-2}(\tau_{n-1})) \quad (8)
\]

Recalling that the admissible control set is a separable metric space, the measurable selection result (see, for example, [18,33,60]) indicates that for arbitrary \( \varepsilon > 0 \), there exist \( \varepsilon \)-optimal controls \( u^{n-1} \in U_{\tau_{n-1}}^d \) of \( \psi^{n-1}(X^{n-2}(\tau_{n-1}), L^{n-2}(X^{n-2}(\tau_{n-1}))) \) and \( u^{n-1} \in U_{\tau_n}^d \) of \( \psi^n(X^{n-1}(\tau_n), L^{n-1}(X^{n-1}(\tau_n))) \) such that

\[
\psi^{n-1}(X^{n-2}(\tau_{n-1}), L^{n-2}(X^{n-2}(\tau_{n-1}))) - \varepsilon.
\]
Then (7) is easily derived by using the backward induction. Furthermore, combining (8) and (9), we can obtain

\[ V^{n-1}(X^{\varepsilon,n-1}(\tau_n), \mathcal{L}^{n-1}(X^{\varepsilon,n-1}(\tau_n)), u^{\varepsilon,n}) \leq \mathbb{E} \left[ \int_{\tau_n}^{\tau_n} f^{n-1}(t, X^{\varepsilon,n-1}(t), M(X^{\varepsilon,n-1}(t)), u^{\varepsilon,n-1}(t), N(u^{\varepsilon,n-1}(t)))dt \right. \]

\[ + \mathcal{J}^n(X^{\varepsilon,n-1}(\tau_n), \mathcal{L}^{n-1}(X^{\varepsilon,n-1}(\tau_n))) \left| \mathcal{F}_{\tau_n-1} \right. \]

\[ \leq V^{n-1}(X^{\varepsilon,n-1}(\tau_n), \mathcal{L}^{n-1}(X^{\varepsilon,n-1}(\tau_n))) \]

Taking \( \varepsilon \to 0 \), one has

\[ V^{n-1}(X^{n-2}(\tau_n-1), \mathcal{L}^{n-2}(X^{n-2}(\tau_n-1))) \leq V^{n-1}(X^{n-2}(\tau_n-1), \mathcal{L}^{n-2}(X^{n-2}(\tau_n-1))). \] (9)

Combining (8) and (9), we can obtain

\[ V^{n-1}(X^{n-2}(\tau_n-1), \mathcal{L}^{n-2}(X^{n-2}(\tau_n-1))) = V^{n-1}(X^{n-2}(\tau_n-1), \mathcal{L}^{n-2}(X^{n-2}(\tau_n-1))). \]

Then (7) is easily derived by using the backward induction. Furthermore,

\[ V^0(x_0, \mathcal{L}(x_0)) = \mathbb{V}^0(x_0, \mathcal{L}(x_0)) = V(x_0, \mathcal{L}(x_0)), \quad k = 0. \]

This completes the proof. \( \square \)

Theorem 3.1 shows that \( \tilde{u}^0, \ldots, \tilde{u}^n \) obtained by (5) is the global optimal control. Thus, solving problem (2.1) is equivalent to solve the following subproblem:

**Subproblem 3.1.** Find \( \tilde{u}^k \in \mathcal{U}^d_k \) such that

\[ J^k(X^{k-1}(\tau_k), \mathcal{L}^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k) = \operatorname{ess sup}_{u^k \in \mathcal{U}^d_k} J^k(X^{k-1}(\tau_k), \mathcal{L}^{k-1}(X^{k-1}(\tau_k)), u^k), \quad k \in \mathcal{I}_0. \]

### 3.2 A sufficient maximum principle

In this subsection, we derive the sufficient version of the maximum principle for Subproblem 3.1. The following lemma plays an important role in optimal controls problems of MFSDEs with random coefficients.

**Lemma 3.1.** For any \( X(\cdot), Y(\cdot) \in L^3_T \) and \( k \in \mathcal{I}_0 \), we have \( M(X(\cdot)), M(Y(\cdot)) \in L^3_T \) and

\[ |M(X(t)) - M(Y(t))|^2 \leq \mathbb{E} \left[ |X(t) - Y(t)|^2 |\mathcal{M}^k_T| \right]. \]

Moreover, for \( t \in [\tau_k, \tau_{k+1}] \) with \( k \in \mathcal{I}_0 \),

\[ \mathbb{E} \left[ M(X(t))Y(t) |\mathcal{F}_{\tau_k} \right] = \mathbb{E} \left[ X(t)M(Y(t)) |\mathcal{F}_{\tau_k} \right], \]

\[ \mathbb{E} \left[ Y^{\delta_k+1}(t)M(X(t))Y(t) |\mathcal{F}_{\tau_k} \right] = \mathbb{E} \left[ X(t)M(Y^{\delta_k+1}(t)Y(t)) |\mathcal{F}_{\tau_k} \right]. \]

**Proof.** The first inequality follows from Cauchy’s inequality immediately. Noticing that \( \mathcal{M}^k_{\tau_k} = \mathcal{F}_{\tau_k} \), one has

\[ \mathbb{E} \left[ M(X(t))Y(t) |\mathcal{F}_{\tau_k} \right] = \mathbb{E} \left[ \mathbb{E} \left[ X(t)M^k_{\tau_k} \cdot Y(t) |\mathcal{F}_{\tau_k} \right] |\mathcal{F}_{\tau_k} \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ X(t)M^k_{\tau_k} \right] \cdot \mathbb{E} \left[ Y(t) |\mathcal{F}_{\tau_k} \right] |\mathcal{F}_{\tau_k} \right] \]
The second equality follows similarly.

We also need to define following Hamiltonian functional:

$$H^k : [0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$H^k(t) = H^k(t, x^k, M^k, u^k, N^k, p^k, q^k, r^k)$$

$$= f^k(t, x^k, M^k, u^k, N^k) + b^k(t, x^k, M^k, u^k, N^k)p^k + \sigma^k(t, x^k, M^k, u^k, N^k)q^k$$

$$+ \gamma^{k+1} h(t, x^k, M^k, u^k, N^k)r^k. \quad (10)$$

In the sequel, we assume that the functions $b^k, \sigma^k, \gamma^k$ and $f^k$ all admit bounded Fréchet derivatives with respect to $x^k, M^k, u^k$ and $N^k$, respectively. In addition, $V^{k+1}(X^k(\tau_{k+1}), \mathcal{Q}^k(X^k(\tau_{k+1})))$ admits bounded Fréchet derivatives with respect to $X^k(\tau_{k+1})$ and $\mathcal{Q}^k(X^k(\tau_{k+1}))$.

Now we associate the adjoint BSDE to Hamiltonian (10) in the unknown triple $(p^k, q^k, r^k)$ ($k \in \mathcal{I}_0$) satisfying

$$\begin{cases}
    dp^k(t) = - \left( \frac{\partial H^k}{\partial x}(t) + M(\nabla M H^k(t)) \right) dt + q^k(t) dB_t + r^k(t) dA^k_{t+1}, & t \in [\tau_k, \tau_{k+1}], \\
    p^k(\tau_{k+1}) = \mathcal{Q}^{k+1}(X^k(\tau_{k+1}), \mathcal{Q}^k(X^k(\tau_{k+1}))) + \mathcal{\nabla}^k V^{k+1}(X^k(\tau_{k+1}), \mathcal{Q}^k(X^k(\tau_{k+1}))),
\end{cases}$$

where $M(\nabla M H^k(t)) = \mathbb{E} \left[ \nabla_M H^k(t) \right]$.  

**Example 3.1.** If the state process satisfies ($k \in \mathcal{I}_0$)

$$\begin{cases}
    dX^k(t) = u^k(t) dt + M(X^k(t)) dH^k_{t+1}, & t \in (\tau_k, \tau_{k+1}], \\
    X^k(\tau_k) = X^k(\tau_{k+1})
\end{cases}$$

and the performance functional takes the form

$$J(X^k, u^k) = - \frac{1}{2} \mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} |X^k(t)|^2 + |u^k(t)|^2 dt + |X^k(\tau_{k+1})|^2 \right],$$

then the Hamiltonian functional is

$$H^k = \frac{1}{2} [\|X^k\|^2 + \|u^k\|^2] + (u^k + M(X^k)\gamma^{k+1})p + \gamma^{k+1} M(X^k)r^k$$

and the adjoint BSDE is

$$\begin{cases}
    dp^k(t) = - \left( X^k(t) - \mathbb{E} \left[ \gamma^{k+1}(t)p^k(t) \right] \right) dt \\
    + r^k(t) dA^k_{t+1}, & t \in [\tau_k, \tau_{k+1}], \\
    p^k(\tau_{k+1}) = -X^k(\tau_{k+1}).
\end{cases}$$

**Theorem 3.2.** Assume that

(i) (Concavity) For $t \in [\tau_k, \tau_{k+1}]$ with $k \in \mathcal{I}_0$, the functions

$$(x^k, M^k, u^k, N^k) \rightarrow H(t, x^k, M^k, u^k, N^k, p^k, q^k, r^k),$$

$$(X^k(\tau_{k+1}), \mathcal{Q}^k(X^k(\tau_{k+1}))) \rightarrow V^{k+1}(X^k(\tau_{k+1}), \mathcal{Q}^k(X^k(\tau_{k+1})))$$

are concave a.s..
(ii) (Maximum condition) For \( t \in [\tau_k, \tau_{k+1}] \) with \( k \in \mathbb{N}_0 \),
\[
E \left[ \tilde{H}(t, \tilde{x}^k, \tilde{M}^k, \tilde{u}^k, \tilde{N}^k, \tilde{p}^k, \tilde{q}^k, \tilde{r}^k) | \xi^k \right] = \text{ess sup}_{u^k \in U_k^a} E \left[ \tilde{H}(t, \tilde{x}^k, \tilde{M}^k, u^k, N^k, \tilde{p}^k, \tilde{q}^k, \tilde{r}^k) | \xi^k \right].
\]

Then, \( \tilde{u}^k \) is an optimal control of Subproblem 3.1.

Proof. Let
\[
J^k(X^{k-1}(\tau_k), L(X^{k-1}(\tau_k)), u^k) - J^k(\tilde{X}^{k-1}(\tau_k), L(\tilde{X}^{k-1}(\tau_k)), \tilde{u}^k) = I_1 + I_2, \quad k \in \mathbb{N}_0,
\]
where
\[
I_1 = E \left[ \int_{\tau_k}^{\tau_{k+1}} f^k(t, X^k(t), M(X^k(t)), u^k(t), N(u^k(t))) - f^k(t, \tilde{X}^k(t), M(\tilde{X}^k(t)), \tilde{u}^k(t), \tilde{N}(u^k(t))) dt | \xi_{\tau_k} \right],
\]
\[
I_2 = E \left[ V^{k+1}(X^k(\tau_{k+1}), \mathbf{p}^k(X^k(\tau_{k+1}))) - V^{k+1}(\tilde{X}^k(\tau_{k+1}), \mathbf{p}^k(\tilde{X}^k(\tau_{k+1}))) | \xi_{\tau_k} \right].
\]

Setting \( \tilde{X}^k(\tau_{k+1}) = X^k(\tau_{k+1}) \) and applying Lemma 3.1 one has
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} (\nabla_M H^k(t), M(\tilde{X}^k(t))) dt | \xi_{\tau_k} \right] = E \left[ \int_{\tau_k}^{\tau_{k+1}} M(\nabla_M H^k(t), \tilde{X}^k(t)) dt | \xi_{\tau_k} \right].
\]

Thus, by Itô’s formula and the martingale property of \( \int_{\tau_k}^{\tau_{k+1}} \tilde{r}^k(t) \tilde{h}^k(t) dA^{k+1}(t) \), we have
\[
I_2 \leq E \left[ \left( \frac{\partial V^{k+1}}{\partial x^k} \left( X^k(\tau_{k+1}), \mathbf{p}^k(X^k(\tau_{k+1})) \right) + \nabla_{\mathbf{p}^k} V^{k+1}(X^k(\tau_{k+1}), \mathbf{p}^k(X^k(\tau_{k+1}))) \right) \cdot \tilde{X}^k(\tau_{k+1}) | \xi_{\tau_k} \right]
\]
\[
= E \left[ \tilde{p}^k(\tau_{k+1}) \tilde{X}^k(\tau_{k+1}) | \xi_{\tau_k} \right] = E \left[ \tilde{p}^k(\tau_{k+1}) \tilde{X}^k(\tau_{k+1}) - \tilde{p}^k(\tau_k) \tilde{X}(\tau_k) | \xi_{\tau_k} \right]
\]
\[
= E \left[ \int_{\tau_k}^{\tau_{k+1}} \tilde{p}^k(t) d\tilde{X}^k(t) + \tilde{X}^k(t) d\tilde{p}^k(t) + \tilde{q}^k(t) d\tilde{r}^k(t) dt + \tilde{r}^k(t) \tilde{h}^k(t) dt d\tilde{\xi}_{\tau_k}^{k+1} | \xi_{\tau_k} \right]
\]
\[
= E \left[ \int_{\tau_k}^{\tau_{k+1}} \tilde{p}^k(t) \tilde{b}^k(t) dt - \tilde{X}^k(t) \frac{\partial H^k}{\partial x^k}(t) - \langle \nabla_M H^k(t), M(\tilde{X}^k(t)) \rangle + \tilde{q}^k(t) \tilde{r}^k(t) \gamma \tilde{S}^k(t) dt | \xi_{\tau_k} \right].
\]

It follows from 100 that
\[
I_1 = E \left[ \int_{\tau_k}^{\tau_{k+1}} \tilde{H}^k(t) - \tilde{p}^k(t) \tilde{b}^k(t) - \tilde{q}^k(t) \tilde{r}^k(t) - \tilde{r}^k(t) \tilde{h}^k(t) \gamma \tilde{S}^k(t) dt | \xi_{\tau_k} \right].
\]

Since \( \xi_{\tau_k} = \tilde{\xi}_{\tau_k} \), the concave condition imposed on \( H^k(t) \) leads to
\[
I_1 + I_2 = J^k(X^{k-1}(\tau_k), L(X^{k-1}(\tau_k)), u^k) - J^k(\tilde{X}^{k-1}(\tau_k), L(\tilde{X}^{k-1}(\tau_k)), \tilde{u}^k)
\]
\[
\leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \tilde{H}^k(t) - \tilde{X}^k(t) \frac{\partial H^k}{\partial x^k}(t) - \langle \nabla_M H^k(t), M(\tilde{X}^k(t)) \rangle dt | \xi_{\tau_k} \right]
\]
\[
\leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \tilde{u}^k(t) \frac{\partial H^k}{\partial u^k}(t) + \langle N(\tilde{u}^k(t)), \nabla_N H^k(t) \rangle dt | \xi_{\tau_k} \right]
\]
\[
= E \left[ \int_{\tau_k}^{\tau_{k+1}} E \left[ \left( \tilde{u}^k(t) \frac{\partial H^k}{\partial u^k}(t) + \tilde{u}^k(t) M(\nabla_N H^k(t)) \right) 1_{\tau_k \leq t < \tau_{k+1}} | S_t \right] dt | \xi_{\tau_k} \right]
\]
\[
= E \left[ \int_{\tau_k}^{\tau_{k+1}} E \left[ \left( \frac{\partial H^k}{\partial u^k}(t) + M(\nabla_N H^k(t)) \right) 1_{\tau_k \leq t < \tau_{k+1}} | S_t \right] \tilde{u}^k(t) dt | \xi_{\tau_k} \right].
\]

This implies that \( \tilde{u}^k \) is the optimal control to Subproblem 3.1. \( \Box \)
3.3 A Necessary Maximum Principle

We now proceed to study the necessary version of the maximum principle for subproblems.

Assumption 3.2. For each \( t_0 \in [\tau_k, \tau_{k+1}] \) with \( k \in I_0 \) and all bounded \( S_0 \)-measurable random variable \( \eta \), the process \( \vartheta(t) = \eta_{\{t_0,\tau_{k+1}\}}(t) \) belongs to \( U^{ad}_k \).

Thanking to the definition of \( U^{ad}_k \) with \( k \in I_0 \), there exists a constant \( \epsilon_0 > 0 \) such that

\[
u^k = \tilde{\nu}^k + \epsilon u^k \in U^{ad}_k, \quad \epsilon \in [-\epsilon_0, \epsilon_0]
\]

for any \( u^k \in U^{ad}_k \) and bounded \( \tilde{\nu}^k \in U^{ad}_k \). To simplify notations, for \( \pi \in \{b^k, \sigma^k, h^k, f^k\} \) with \( k \in I_0 \), we write

\[
(\nabla \pi(t))^T (x^k(t), u^k(t)) = \frac{\partial \pi}{\partial x^k}(t)x^k(t) + (\nabla \pi(t), M(x^k(t))) + \frac{\partial \pi}{\partial u^k}(t)u^k(t) + (\nabla_N \pi(t), N(u^k(t))).
\]

Next we consider the process \( Z^k(t) \) obtained by differentiating \( X^k(t) \) with respect to \( \epsilon \) at \( \epsilon = 0 \). Clearly, \( Z^k(t) \) satisfies the following equation (\( k \in I_0 \)):

\[
\begin{aligned}
dZ^k(t) &= (\nabla b^k(t))^T (Z^k(t), u^k(t))dt + (\nabla \sigma^k(t))^T (Z^k(t), u^k(t))dB_t \\
&\quad + (\nabla h^k(t))^T (Z^k(t), u^k(t))dA_{t+1}, \quad t \in [\tau_k, \tau_{k+1}].
\end{aligned}
\]

Then, the necessary maximum principle is given as follows.

Theorem 3.3. Suppose that Assumption \( \mathcal{A} \) holds. Then the following equalities are equivalent.

(i) For all \( u^k \in U^{ad}_k \) with \( k \in I_0 \),

\[
0 = \frac{d}{d\epsilon} J(X^{k-1}(\tau_k), \Omega^{k-1}(X^{k-1}(\tau_k)), \tilde{\nu}^k + \epsilon u^k)\bigg|_{\epsilon=0}.
\]

(ii) For any \( k \in I_0 \),

\[
0 = \mathbb{E} \left[ \frac{\partial H^k}{\partial u^k}(t) + M(\nabla_N H^k(t)) \bigg| S^k_{\tau_k} \right]_{u^k=\tilde{\nu}^k}, \quad \forall t \in (\tau_k, \tau_{k+1}].
\]

Proof. Assume \( \mathcal{A} \) holds. Then

\[
\begin{aligned}
0 &= \frac{d}{d\epsilon} J(X^{k-1}(\tau_k), \Omega^{k-1}(X^{k-1}(\tau_k)), \tilde{\nu}^k + \epsilon u^k)\bigg|_{\epsilon=0} \\
&= \mathbb{E} \left[ \left\{ \frac{\partial V^{k+1}}{\partial x^k}(X^{k}(\tau_k+1), \Omega^k(X^{k}(\tau_k+1))) + \nabla_x V^{k+1}(X^{k}(\tau_k+1), \Omega^k(X^{k}(\tau_k+1))) \right\} \cdot \tilde{Z}^k(\tau_{k+1}) \\
&\quad + \int_{\tau_k}^{\tau_{k+1}} (\nabla f^k(t))^T (Z^k(t), u^k(t))dt \bigg| S_{\tau_k} \right],
\end{aligned}
\]

where \( \tilde{Z}^k(t) \) is the solution to \( \mathcal{A} \). By Itô’s formula and \( \mathcal{A} \), we have

\[
\begin{aligned}
&\mathbb{E} \left[ \left\{ \frac{\partial V^{k+1}}{\partial x^k}(X^{k}(\tau_k+1), \Omega^k(X^{k}(\tau_k+1))) + \nabla_x V^{k+1}(X^{k}(\tau_k+1), \Omega^k(X^{k}(\tau_k+1))) \right\} \cdot \tilde{Z}^k(\tau_{k+1}) \bigg| S_{\tau_k} \right] \\
&= \mathbb{E} \left[ \hat{p}^k(\tau_k) \tilde{Z}^k(\tau_{k+1}) \bigg| S_{\tau_k} \right] \\
&= \mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} \hat{p}^k(t)d\tilde{Z}^k(t) + \tilde{Z}^k(t)d\hat{p}^k(t) + d(\hat{p}^k(t), \tilde{Z}^k(t)) \bigg| S_{\tau_k} \right]
\end{aligned}
\]

14
= E \left[ \int_{\tau_k}^{\tau_{k+1}} \tilde{p}^k(t)(\nabla b^k(t))^T(\tilde{Z}^k(t), u^k(t)) - \frac{\partial \tilde{H}^k}{\partial x^k}(t)\tilde{Z}^k(t) - (\nabla_M \tilde{H}^k(t), M(\tilde{Z}^k(t))) \\
+ \tilde{q}(t)(\nabla \sigma^k(t))^T(\tilde{Z}^k(t), u^k(t)) + \gamma^k(t)(\nabla h^k(t))^T(\tilde{Z}^k(t), u^k(t))dt \right] \tilde{\mathcal{F}}_{\tau_k} \\
= E \left[ \int_{\tau_k}^{\tau_{k+1}} \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t) + M(\nabla_N \tilde{H}^k(t)) \right) u^k(t) - (\nabla f^k(t))^T(\tilde{Z}^k(t), u^k(t))dt \right] \tilde{\mathcal{F}}_{\tau_k}. \\
(15)

Combining (14) and (15), one has

\[ 0 = \frac{d}{d\epsilon} J(X^{k-1}(\tau_k), \xi^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k + \epsilon \nu^k) \bigg|_{\epsilon=0} = E \left[ \int_{\tau_k}^{\tau_{k+1}} \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t) + M(\nabla_N \tilde{H}^k(t)) \right) u^k(t)dt \right] \tilde{\mathcal{F}}_{\tau_k}. \]

Set \( u^k(t) = \eta_{(t_0, \tau_{k+1})}(t) (t_0 \in [\tau_k, \tau_{k+1}]) \), where \( \eta \) is a bounded \( \mathcal{S}_{\tau_0} \)-measurable random variable. Then,

\[ 0 = \frac{d}{d\epsilon} J(X^{k-1}(\tau_k), \xi^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k + \epsilon \nu^k) \bigg|_{\epsilon=0} = E \left[ \int_{t_0}^{\tau_{k+1}} \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t) + M(\nabla_N \tilde{H}^k(t)) \right) \eta dt \right] \tilde{\mathcal{F}}_{\tau_k} \]

\[ = E \left[ \int_{t_0}^{\tau_{k+1}} \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t) + M(\nabla_N \tilde{H}^k(t)) \right) \eta dt \right]. \]

Differentiating with respect to \( t_0 \), we have

\[ 0 = E \left[ \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t_0) + M(\nabla_N \tilde{H}^k(t_0)) \right) \eta \right] = E \left[ \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t_0) + M(\nabla_N \tilde{H}^k(t_0)) \right) \bigg| \mathcal{S}_{t_0} \right] \eta. \]

(16)

Since (16) holds for all such \( \mathcal{S}_{t_0} \)-measurable \( \eta \), it follows that

\[ 0 = E \left[ \left( \frac{\partial \tilde{H}^k}{\partial u^k}(t_0) + M(\nabla_N \tilde{H}^k(t_0)) \right) \bigg| \mathcal{S}_{t_0} \right], \quad \forall t_0 \in [\tau_k, \tau_{k+1}]. \]

The argument above is reversible. Thus, (14) and (15) are equivalent. \( \square \)

Remark 3.1. It is worth mentioning that Lemma 3.1 plays an important role for deriving the maximum principles. However, it loses for the optimal control problems by mean-field type (in the sense of \( E[\cdot] \)) SDEs. Therefore, our results differs from many existing works. Moreover, when there is no mean-field term \( M(X^k(t)) \) (resp. \( N(u^k(t)) \)) in (3) and the performance functional \( J^k(X^{k-1}(\tau_k), \xi^{k-1}(X^{k-1}(\tau_k)), \tilde{u}^k) \) (see (5)), it is easy to see that Theorems 3.2 and 3.3 hold without requiring \( \mathcal{M}^k_{\tau_k} = \mathcal{F}_{\tau_k} \) (resp. \( \mathcal{G}^k_{\tau_k} = \mathcal{F}_{\tau_k} \)).

4 Solvability of MMFSDEs and MMFBSDEs

In this section, we prove the existence and uniqueness of solutions to MMFSDEs and MMFBSDEs, respectively, and apply the obtained results to solve an optimal investment problem.

4.1 Solution to MMFSDEs

Omitting the fixed control \( u^k \in \mathcal{U}_k^{ad} (k \in \mathcal{I}_0) \), MMFSDE with general initial condition is of the form

\[
\begin{align*}
    dX(t) &= b(t, X(t), M(X(t)))dt + \sigma(t, X(t), M(X(t)))dB_t + h(t, X(t), M(X(t)))d\mathbb{H}_t, \quad t \in [0, T], \\
    X(0) &= \beta.
\end{align*}
\]

(17)
We first give some necessary assumptions for both the coefficients and the initial condition of equation (17) as follows.

**Assumption 4.1.** For any \( k \in \mathcal{I}_0 \), assume that

(i) \( \beta \in L^2_{\mathbb{P}}(\mathbb{P}) \);

(ii) The functions \( b^k(\cdot, X^k, M(X^k)) \), \( \sigma^k(\cdot, X^k, M(X^k)) \) are \( \mathfrak{F} \)-progressively measurable, and \( h^k(\cdot, X^k, M(X^k)) \) is \( \mathfrak{F} \)-predictable;

(iii) \( \mathbb{E} \left[ \int_0^T |b^k(t, 0, 0)|^2 + |\sigma^k(t, 0, 0)|^2 + \gamma^{\mathfrak{F}}|h^k(t, 0, 0)|^2 dt \right] < +\infty \);

(iv) \( |b^k(t, x_1, M_1) - b^k(t, x_2, M_2)|^2 + |\sigma^k(t, x_1, M_1) - \sigma^k(t, x_2, M_2)|^2 + |h^k(t, x_1, M_1) - h^k(t, x_2, M_2)|^2 \leq C \left( |x_1 - x_2|^2 + |M_1 - M_2|_{\mathbb{P}^2(R)}^2 \right) \).

**Theorem 4.1.** Under Assumption 4.1 there exists a unique solution \( X(\cdot) \in L^2_{\mathbb{F}} \) to (17).

**Proof.** Define a map \( \Psi \) as follows:

\[
\Psi(X(t)) = \beta + \int_0^t b(s, X(s), M(X(s)))ds + \int_0^t \sigma(s, X(s), M(X(s)))dB_s + \int_0^t h(s, X(s), M(X(s)))d\mathbb{H}_s
\]

\[
= \beta + \int_0^t b(s, X(s), M(X(s))) + \Upsilon(s)h(s, X(s), M(X(s)))ds
\]

\[
+ \int_0^t \sigma(s, X(s), M(X(s)))dB_s + \int_0^t h(s, X(s), M(X(s)))d\mathbb{H}_s.
\]

Consider the sequence \( \{X^{(m)}(t)\}_{m=1}^{+\infty} \) defined by \( X^{(m+1)}(t) = \Psi(X^{(m)}(t)) \) with \( X^0(t) = \eta \). We firstly show the existence of the solution. To alleviate notations, we write \( X^{(m+1)}(t) = X^{(m+1)}(t) - X^{(m)}(t) \) and \( \pi^{(m)}(t) = \pi(t, X^{(m)}(t), M(X^{(m)}(t))) (\pi \in \{b, \sigma, h\}) \). Following from \( \gamma^{\mathfrak{F}}(t) \leq C \), Cauchy’s inequality, Lemmas \([2.3, 3.1]\) for \( t \in [T_k, \tau_{k+1}] \) \( (k \in \mathcal{I}_0) \), we have

\[
\mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} (b^{k,m}(s) - b^{k,m-1}(s))^2 + \sigma^{k,m}(s)(h^{k,m}(s) - h^{k,m-1}(s)) + h^{k,m}(s) - h^{k,m-1}(s) \right] ds
\]

\[
\leq 4\mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} (b^{k,m}(s) - b^{k,m-1}(s))^2 |\mathfrak{F}_{\tau_k} \right] + 4\mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} (\sigma^{k,m}(s)(h^{k,m}(s) - h^{k,m-1}(s)) + h^{k,m}(s) - h^{k,m-1}(s) ds |\mathfrak{F}_{\tau_k} \right]
\]

\[
\leq 4\mathbb{E} \left[ \left( \int_{\tau_k}^{\tau_{k+1}} (b^{k,m}(s) - b^{k,m-1}(s) ds \right)^2 |\mathfrak{F}_{\tau_k} \right] + 4\mathbb{E} \left[ \left( \int_{\tau_k}^{\tau_{k+1}} \sigma^{k,m}(s)(h^{k,m}(s) - h^{k,m-1}(s))ds \right)^2 |\mathfrak{F}_{\tau_k} \right]
\]

\[
\leq 4\mathbb{E} \left[ \left( \int_{\tau_k}^{\tau_{k+1}} \sigma^{k,m}(s)(h^{k,m}(s) - h^{k,m-1}(s))ds \right)^2 |\mathfrak{F}_{\tau_k} \right] + 4\mathbb{E} \left[ \left( \int_{\tau_k}^{\tau_{k+1}} h^{k,m}(s) - h^{k,m-1}(s) ds \right)^2 |\mathfrak{F}_{\tau_k} \right]
\]

\[
\leq C \mathbb{E} \left[ \left. \int_{\tau_k}^{\tau_{k+1}} (b^{k,m}(s) - b^{k,m-1}(s))^2 ds \right| \mathfrak{F}_{\tau_k} \right] + C \mathbb{E} \left[ \left. \int_{\tau_k}^{\tau_{k+1}} (\sigma^{k,m}(s)(h^{k,m}(s) - h^{k,m-1}(s)) + h^{k,m}(s) - h^{k,m-1}(s) ds \right| \mathfrak{F}_{\tau_k} \right]
\]

\[
\leq C \mathbb{E} \left[ \left. \int_{\tau_k}^{\tau_{k+1}} (b^{k,m}(s))^2 ds \right| \mathfrak{F}_{\tau_k} \right].
\]
Set $X^{k,(m)}(s) = 0$ for $s \in [0, \tau_k]$. Then, \( (18) \) yields
\[
\mathbb{E} \left[ |X^{k,(m+1)}(t)|^2 \right] \leq C \mathbb{E} \left[ \int_{\tau_k}^t |X^{k,(m)}(s)|^2 ds \right] = C \mathbb{E} \left[ \int_{0}^t |X^{k,(m)}(s)|^2 ds \right].
\]
According to (iii) and (iv) in Assumption \( 4.1 \) it is easy to show that, for $t \in [\tau_k, \tau_{k+1}]$ ($k \in \mathbb{I}_0$),
\[
\mathbb{E} \left[ |X^{k,(1)}(t)|^2 \right] \leq C(1 + \mathbb{E}[|X^{k,(0)}(t)|^2]) = C(1 + \mathbb{E}(\beta^2)) = C.
\]
Again by the induction, for $t \in [\tau_k, \tau_{k+1}]$ and $m \geq 1$, we have
\[
\mathbb{E} \left[ |X^{k,(m+1)}(t)|^2 \right] \leq C^{m+1} \frac{t^m}{m!}.
\]
Integrating both sides from $\tau_k$ to $\tau_{k+1}$ and taking expectation, one has
\[
\mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} |X^{k,(m+1)}(t)|^2 dt \right] \leq C^{m+1} \mathbb{E} \left[ (\tau_{k+1} - \tau_k)^{m+1} \right] \leq \frac{(CT)^{m+1}}{(m+1)!}.
\]
Thus,
\[
\mathbb{E} \left[ \int_{0}^{T} |X^{k}(t)|^2 dt \right] = \mathbb{E} \left[ \int_{0}^{T} \sum_{k=0}^{n} |X^{k,(m)}(t)\|_{t \in (\tau_k, \tau_{k+1})}|^2 dt \right]
\leq (n+1) \sum_{k=0}^{n} \mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} |X^{k,(m)}(t)|^2 dt \right]
\leq n(n+1) \left( \frac{CT}{m+1} \right)^{m+1} \mathbb{E} \left[ 1 + \beta^2 \right],
\]
which implies that \( \{X^{(m)}(t)\}_{m=1}^{+\infty} \) forms a Cauchy sequence in $L^2_T$. Letting $\lim_{m \to +\infty} X^{(m)}(t) = X(t)$, we have
\[
0 = \lim_{m \to +\infty} \|b(\cdot, X^{(m)}(\cdot), M(X^{(m)}(\cdot))) - b(\cdot, X(\cdot), M(X(\cdot)))\|_{L^2_T}
= \lim_{m \to +\infty} \|\sigma(\cdot, X^{(m)}(\cdot), M(X^{(m)}(\cdot))) - \sigma(\cdot, X(\cdot), M(X(\cdot)))\|_{L^2_T}
= \lim_{m \to +\infty} \|h(\cdot, X^{(m)}(\cdot), M(X^{(m)}(\cdot))) - h(\cdot, X(\cdot), M(X(\cdot)))\|_{L^2_T}.
\]
This shows that $X(\cdot) \in L^2_T$ is a solution to \( (17) \).

Next we prove the uniqueness. Suppose that there are two solutions $X^{(1)}(\cdot)$ and $X^{(2)}(\cdot)$ to \( (17) \). Similar to the proof of the existence for solutions to \( (17) \), we can show that, for any $k \in \mathbb{I}_0$,
\[
\mathbb{E} \left[ |X^{k,(1)}(t) - X^{k,(2)}(t)|^2 \right] \leq C \int_{\tau_k}^{t} \mathbb{E} \left[ |X^{k,(1)}(s) - X^{k,(2)}(s)|^2 \right] ds, \quad t \in [\tau_k, \tau_{k+1}].
\]
By Gronwall’s inequality, we have $\mathbb{E} \left[ |X^{k,(1)}(t) - X^{k,(2)}(t)|^2 \right] = 0$ and so the uniqueness follows immediately.

\[ \square \]

\subsection*{4.2 Solution to MMFBSDEs}

In this subsection, we prove the existence and uniqueness of solutions to the general MMFBSDEs of the following form
\[
\begin{cases}
d p(t) = -F(t, p(t), \mathfrak{M}(p(t)), q(t), \mathfrak{N}(q(t)), r_1(t), \cdots, r_n(t), \mathfrak{Q}(r_1(t)), \cdots, \mathfrak{Q}(r_n(t))) dt \\
+ q(t) dB_t + \sum_{k=1}^{n} r_k(t) dA_t^k, \quad t \in [0, T], \\
p(T) = \zeta,
\end{cases}
\tag{19}
\]
Lemma 4.2. If there are \( W(t) = (W^0(t), \ldots, W^n(t)) \) is a given non-negative and upper bounded process, and \( F \) is an \( \mathcal{F} \)-progressively measurable \( C^1 \) function.

Firstly, we recall the predictable representation theorem (PRT).

Lemma 4.1. \([3]\) (Predictable Representation Theorem) Under Assumption 2.1, for any martingale \( Z(\cdot) \in H^2_T \), there exist \( \mathcal{F} \)-predictable processes \( \varphi_0 \in H^2_T \) and \( \varphi_k \in H^2_T A^k \) (\( k \in \mathcal{I} \)) such that

\[
Z(t) = Z(0) + \int_0^t \varphi_0(s)dB_s + \sum_{k=1}^n \int_0^t \varphi_k(s)dA^k_s.
\]

(20)

We also need the following lemmas.

Lemma 4.2. The decomposition \([21]\) is unique.

Proof. If there are \( (\varphi_0, \varphi_1, \ldots, \varphi_n), (\psi_0, \psi_1, \ldots, \psi_n) \in H^2_T \times H^2_T A^1 \times \cdots \times H^2_T A^n \) satisfying

\[
Z(t) = Z(0) + \int_0^t \varphi_0(s)dB_s + \sum_{k=1}^n \int_0^t \varphi_k(s)dA^k_s = Z(0) + \int_0^t \psi_0(s)dB_s + \sum_{k=1}^n \int_0^t \psi_k(s)dA^k_s, \quad t \in [0, T],
\]

then

\[
\int_0^t \varphi_0(s) - \psi_0(s)dB_s + \sum_{k=1}^n \int_0^t (\varphi_k(s) - \psi_k(s))dA^k_s = 0
\]

and so

\[
0 = \mathbb{E} \left[ \int_0^T \varphi_0(s) - \psi_0(s)dB_s + \sum_{k=1}^n \int_0^T (\varphi_k(s) - \psi_k(s))dA^k_s \right]^2
\]

\[
= \mathbb{E} \left[ \int_0^T \varphi_0(s) - \psi_0(s)dB_s \right]^2 + \sum_{k=1}^n \mathbb{E} \left[ \int_0^T (\varphi_k(s) - \psi_k(s))dA^k_s \right]^2
\]

\[
= \mathbb{E} \left[ \int_0^T |\varphi_0(s) - \psi_0(s)|^2ds \right] + \sum_{k=1}^n \mathbb{E} \left[ \int_0^T \gamma^k(s)|\varphi_k(s) - \psi_k(s)|^2ds \right]
\]

\[
= ||\varphi_0(\cdot) - \psi_0(\cdot)||_{H^2_T}^2 + \sum_{k=1}^n ||\varphi_k(s) - \psi_k(s)||_{H^2_T A^k}^2.
\]

This implies that \( \varphi_k(s) = \psi_k(s) \) (\( k \in \mathcal{I}_0 \)) a.s.. \( \square \)

Lemma 4.3. For \( F(\cdot) \in H_T^2 \) and \( \zeta \in L_0^2(\mathbb{P}) \), the following MMFBSDE

\[
\begin{cases}
dp(t) = -F(t)dt + q(t)dB_t + \sum_{k=1}^n r_k(t)dA^k_t, & t \in [0, T]; \\
p(T) = \zeta
\end{cases}
\]

has a unique solution \( (p, q, r_1, \ldots, r_n) \in L_T^2 \times H_T^2 \times H_T^2 A^1 \times \cdots \times H_T^2 A^n \).

Proof. Set \( C(t) = \mathbb{E} \left[ \zeta + \int_0^T F(t)dt \right]_{\mathcal{F}_t} \). Clearly, \( C(t) \) is an \( \mathcal{F}_t \)-martingale. Moreover, it follows from Cauchy’s inequality that

\[
C^2(t) = \mathbb{E} \left[ \left[ \zeta + \int_0^T F(t)dt \right]_{\mathcal{F}_t} \right]^2 \leq \mathbb{E} \left[ \left[ \zeta + \int_0^T F(t)dt \right]_{\mathcal{F}_t} \right]^2 \leq 2\mathbb{E} \left[ \zeta^2_{\mathcal{F}_t} \right] + 2T\mathbb{E} \left[ \int_0^T F^2(t)dt \right]_{\mathcal{F}_t}.
\]
This indicates $C(\cdot) \in H_T^3$. By Lemmas 4.1 and 4.2, there exists a unique $(q_1, \cdots, q_n) \in H_T^3 \times H_T^{3, A_1} \times \cdots \times H_T^{3, A_n}$ such that

$$C(t) = C(0) + \int_0^t q(s)dB_s + \sum_{k=1}^n \int_0^t r_k(s)dA_k$$

and so

$$C(T) - C(t) = \int_t^T q(s)dB_s + \sum_{k=1}^n \int_t^T r_k(s)dA_k.$$ 

Setting

$$p(t) = C(t) - \int_0^t F(s)ds = \mathbb{E}\left[\zeta + \int_t^T F(s)ds | \mathcal{F}_t\right],$$

we have $p(T) = \zeta$ and

$$p(T) - p(t) = \int_t^T q(s)dB_s + \sum_{k=1}^n \int_t^T r_k(s)dA_k - \int_t^T F(s)ds. \tag{22}$$

This ends the proof.

To ensure the existence and uniqueness result, we also need the following assumption.

**Assumption 4.2.** Suppose that the following assumptions hold:

(i) $\zeta \in L^2_\mathcal{F}(\mathbb{P})$;

(ii) $F(t, 0, 0, \cdots, 0) \in H_T^3$;

(iii) For any $t, p_1, q_1, r_1, p_2, q_2, r_2$, there is a constant $C > 0$ such that

$$\left|F(t, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{\Omega}_1, \cdots, \tilde{\Omega}_n) - F(t, p, q, r, \Omega_1, \cdots, \Omega_n)\right|^2 \leq C \left( |p - \tilde{p}|^2 + |\tilde{p} - \tilde{p}|^2 + |q - \tilde{q}|^2 + |\tilde{q} - \tilde{q}|^2 + \sum_{k=1}^n |\Omega_k - \tilde{\Omega}_k|^2 \right).$$

**Theorem 4.2.** Under Assumption 4.2, MMFBSDE (19) has a unique solution $(p, q_1, \cdots, q_n) \in L_T^2 \times H_T^3 \times H_T^{3, A_1} \times \cdots \times H_T^{3, A_n}$.

**Proof.** We decompose the proof into two steps.

**Step 1: Special case**

Assume that the driver $F$ is independent of $p$ and $\tilde{p}$ such that

$$\begin{cases}
    dp(t) = -F(t, q(t), \Omega(q(t)), r_1(t), \cdots, r_n(t), \Omega(r_1(t)), \cdots, \Omega(r_n(t))) dt \\
    + q(t)dB_t + \sum_{k=1}^n r_k(t)dA_k, & t \in [0, T], \\
    p(T) = \zeta.
\end{cases} \tag{23}$$

We first prove the uniqueness and existence of solutions to (23). By Lemma 4.3, for each $m \in \mathbb{N}$, there exists a unique solution $(p^{(m+1)}, q^{(m+1)}, r_1^{(m+1)}, \cdots, r_n^{(m+1)}) \in L_T^2 \times H_T^3 \times H_T^{3, A_1} \times \cdots \times H_T^{3, A_n}$ to the following MMFBSDE:
where \( q^0(t) = r^0(t) = 0 \) a.s. for all \( t \in [0, T] \).

For simplicity, we write

\[
F^{(m)}(t) = F(t, q^{(m)}(t), \mathfrak{H}(q^{(m)}(t)), r_1^{(m)}(t), \ldots, r_n^{(m)}(t), \Omega(r_1^{(m)}(t)), \ldots, \Omega(r_n^{(m)}(t))),
\]

\[
L^{(m+1)}(t) = E \left[ \int_t^T |q^{(m+1)}(s) - q^{(m)}(s)|^2 + \sum_{k=1}^n \gamma^k(s) |r_k^{(m+1)}(s) - r_k^{(m)}(s)|^2 ds \right].
\]

Next we show that \( (p^{(m)}, q^{(m)}, r_1^{(m)}, \ldots, r_n^{(m)}) \) forms a Cauchy sequence. In fact, it follows from Lemma 3.1 that

\[
E \left[ \int_t^T |F^{(m)}(s) - F^{(m-1)}(s)|^2 ds \right] \leq CL^{(m)}(t).
\]

Applying Lemma 2.1 to \( |p^{(m+1)}(t) - p^{(m)}(t)|^2 \) and taking expectation, we have

\[
E \left[ \int_t^T |p^{(m+1)}(t) - p^{(m)}(t)|^2 \right] = \frac{1}{\rho} E \left[ \int_t^T \rho |p^{(m+1)}(s) - p^{(m)}(s)|^2 + \rho |F^{(m)}(s) - F^{(m-1)}(s)|^2 ds \right] - L^{(m+1)}(t)
\]

\[
\leq \frac{1}{\rho} E \left[ \int_t^T |p^{(m+1)}(s) - p^{(m)}(s)|^2 ds \right] + \rho C L^{(m)}(t) - L^{(m+1)}(t),
\]

where \( \rho \) is any positive constant. Multiplying by \( e^{\frac{\gamma t}{2}} \) and integrating both sides in \([0, T]\), one has

\[
0 \leq E \left[ \int_0^T |p^{(m+1)}(s) - p^{(m)}(s)|^2 ds \right] \leq \rho C \int_0^T e^{\frac{\gamma t}{2}} L^{(m)}(t) dt - \int_0^T e^{\frac{\gamma t}{2}} L^{(m+1)}(t) dt.
\]

By choosing \( \rho = \frac{1}{2\gamma} \), we have

\[
\int_0^T e^{2\gamma t} L^{(m)}(t) dt \leq \frac{1}{2m-1} \int_0^T e^{2\gamma t} L^{(1)}(t) dt \leq \frac{C}{2m-1}
\]

and

\[
\|p^{(m+1)}(\cdot) - p^{(m)}(\cdot)\|_{H_T^2} \leq \frac{1}{2} \int_0^T e^{\frac{\gamma t}{2}} L^{(m)}(t) dt \leq \frac{C}{2m^2}.
\]

which implies \( \lim_{m \to +\infty} \|p^{(m+1)}(\cdot) - p^{(m)}(\cdot)\|_{H_T^2} = 0 \). Substituting (27) and (28) into (24), one has

\[
E \left[ L^{(m+1)}(t) \right] \leq \frac{C}{2m} + \frac{1}{2} E \left[ L^{(m)}(t) \right],
\]

which implies

\[
E \left[ L^{(m+1)}(t) \right] \leq \frac{C}{2m} + \frac{1}{2m} E \left[ L^{(0)}(t) \right].
\]

Combining this and (25) obtains that, for any \( t \in [0, T] \),

\[
0 = \lim_{m \to +\infty} \left( E \left[ \int_t^T |q^{(m+1)}(s) - q^{(m)}(s)|^2 ds \right] + E \left[ \int_t^T \sum_{k=1}^n \gamma^k(s) |r_k^{(m+1)}(s) - r_k^{(m)}(s)|^2 ds \right] \right).
\]
From (28) and (29), we can see that \( \left( p^{(m+1)}, q^{(m+1)}, r_1^{(m+1)}, \ldots, r_n^{(m+1)} \right) \) forms a Cauchy sequence in \( L_T^\beta \times H_T^\beta \times H_T^{\beta, A^1} \times \cdots \times H_T^{\beta, A^n} \). Thus, we know that \( \left( p^{(m)}, q^{(m)}, r_1^{(m)}, \ldots, r_n^{(m)} \right) \) converges to a limit \( (p, q, r_1, \ldots, r_n) \in L_T^\beta \times H_T^\beta \times H_T^{\beta, A^1} \times \cdots \times H_T^{\beta, A^n} \). Letting \( m \to +\infty \) in (28), we conclude that \( (p, q, r) \) is a solution to (28). This indicates the existence of solutions to (28).

Finally, we proceed to show the uniqueness of solutions to (28). Let \( (p, q, r()) \) and \( (p^{(0)}, q^{(0)}, r^{(0)}()) \) be two solutions of (28). As the same arguments in Step 1, we can show that

\[
E[|p(t) - p^{(0)}(t)|^2] - \frac{1}{\rho} E \left[ \int_t^T |p(s) - p^{(0)}(s)|^2 ds \right] \leq (\rho C - 1)E \left[ \int_t^T |q(s) - q^{(0)}(s)|^2 + \sum_{k=1}^n \gamma^{\beta_k} |r_k(s) - r_k^{(0)}(s)|^2 ds \right].
\]

By setting \( \rho = \frac{1}{2C} \), one has

\[
E[|p(t) - p^{(0)}(t)|^2] \leq 2CE \left[ \int_t^T |p(s) - p^{(0)}(s)|^2 ds \right].
\]

By Gronwall’s Lemma, we can deduce that \( E|p(t) - p^{(0)}(t)|^2 = 0 \) and \( p(t) = p^{(0)}(t) \) a.s. and so

\[
-\frac{1}{2} E \left[ \int_t^T |q(s) - q^{(0)}(s)|^2 + \sum_{k=1}^n \gamma^{\beta_k} |r_k(s) - r_k^{(0)}(s)|^2 ds \right] \geq 0,
\]

which implies that \( q(t) = q^{(0)}(t) \) and \( r_k(t) = r_k^{(0)}(t) \) \((k \in \mathcal{I})\) a.s. and so the uniqueness follows.

**Step 2: General case**

Consider the following iteration with general driver \( F \)

\[
\begin{aligned}
& dp^{(m+1)}(t) = -F \left( t, p^{(m)}(t), \mathfrak{M}(p^{(m)})(t), \mathfrak{M}(q^{(m+1)})(t), \mathfrak{M}(q^{(m+1)})(t), r_1^{(m)}(t), \ldots, r_n^{(m)}(t), \right. \\
& \left. \mathfrak{M}(r_1^{(m)})(t), \ldots, \mathfrak{M}(r_n^{(m)})(t) \right) dt + q^{(m+1)}(t) dB_t + \sum_{k=1}^n r_k^{(m+1)}(t) dB^k_t, \quad t \in [0, T], \tag{30}
\end{aligned}
\]

where \( p^{(0)}(t) = 0 \). It follows from Step 1 that, for each \( m \in \mathbb{N} \), there exists a unique solution

\[
\left( p^{(m)}, q^{(m)}, r_1^{(m)}, \ldots, r_n^{(m)} \right) \in L_T^\beta \times H_T^\beta \times H_T^{\beta, A^1} \times \cdots \times H_T^{\beta, A^n}
\]

satisfying the following inequality by choosing \( \rho = \frac{1}{2C} \),

\[
E \left[ |p^{(m+1)}(t) - p^{(m)}(t)|^2 \right] + \frac{1}{2} L^{(m+1)}(t) \leq 2CE \left[ \int_t^T |p^{(m+1)}(s) - p^{(m)}(s)|^2 ds \right] + \frac{1}{2} E \left[ \int_t^T |p^{(m)}(s) - p^{(m-1)}(s)|^2 ds \right].
\]

Hence,

\[
E \left[ |p^{(m+1)}(t) - p^{(m)}(t)|^2 \right] \leq 2CE \left[ \int_t^T |p^{(m+1)}(s) - p^{(m)}(s)|^2 ds \right] + \frac{1}{2} E \left[ \int_t^T |p^{(m)}(s) - p^{(m-1)}(s)|^2 ds \right].
\]

Multiplying by \( e^{2Ct} \) and integrating both sides in \([T_0, T] \) \((T_0 \in [\tau_k, \tau_{k+1}]\)), one has

\[
\int_{T_0}^T E|p^{(m+1)}(t) - p^{(m)}(t)|^2 dt \leq \frac{1}{2} \int_{T_0}^T e^{2Ct} E \left[ \int_t^T |p^{(m)}(s) - p^{(m-1)}(s)|^2 ds \right] dt.
\]
\[ \leq C \int_{T_0}^{T} \mathbb{E} \left[ \int_{t}^{T} |p^{(m)}(s) - p^{(m-1)}(s)|^2 ds \right] dt. \]

Noticing that \( \mathbb{E} \left[ \int_{t}^{T} |p^{(1)}(s) - p^{(0)}(s)|^2 ds \right] \leq C \), it follows from the induction that
\[ \int_{T_0}^{T} \mathbb{E} |p^{(m+1)}(t) - p^{(m)}(t)|^2 dt \leq \frac{C^{m+1}(T - T_0)^m}{m!} \leq \frac{C^{m+1}T^m}{m!}. \]

Moreover,
\[ \mathbb{E} \left[ L^{(m+1)}(T_0) \right] = \mathbb{E} \left[ \int_{T_0}^{T} |q^{(m+1)}(s) - q^{(m)}(s)|^2 + \sum_{k=1}^{n} \gamma^{(k)}(s) |r_k^{(m+1)}(s) - r_k^{(m)}(s)|^2 ds \right] \]
\[ \leq 4C\mathbb{E} \left[ \int_{T_0}^{T} |p^{(m+1)}(s) - p^{(m)}(s)|^2 ds \right] + \mathbb{E} \left[ \int_{T_0}^{T} |p^{(m)}(s) - p^{(m-1)}(s)|^2 ds \right] \]
\[ \leq \frac{4Cm^{2}T^{m}}{m!} + \frac{C^{m}T^{m-1}}{(m-1)!}. \]

Setting \( T_0 = 0 \), it is easy to see that \( \left( p^{(m)}, q^{(m)}, r_1^{(m)}, \ldots, r_n^{(m)} \right) \) forms a Cauchy sequence in \( L^3_T \times H^3_T \times H^3_T; A^1 \times \ldots \times H^3_T; A^n \). A similar argument as in Step 1 shows that \( (p, q, r) = \lim_{m \to +\infty} (p^{(m)}, q^{(m)}, r^{(m)}) \) is the unique solution to (19), which completes the proof.

### 4.3 An Optimal Investment Problem

In this subsection, we consider an optimal investment problem, in which the wealth process \( X(t) \) governed by the following MMFSDE:
\[
\begin{aligned}
\begin{cases}
    dX^{(i)}(t) = X^{(i)}(t) \left[ (S^{i,0}(t) - u^{(i)}(t)) + \gamma^{(i+1)}(t)S^{i,2}(t) \right] dt + S^{i,1}(t)dB_t + S^{i,2}(t)\od dB_t^{(i+1)} \\
    + X^{(i)}(t)S^{i,3}(t)dt, & t \in (\tau_i, \tau_{i+1}], \\
    X^{(i)}(\tau_i) = X^{(i-1)}(\tau), & X^{(i-1)}(\tau_0) = x_0 \in \mathbb{R}^+. 
\end{cases}
\end{aligned}
\] (31)

Here \( X^{(i)}(t) = M(X^{(i)}(t)) \) and

- \( S^{i,j}(\cdot) (i = 0, 1, 2, j = 0, 1, 2, 3) \) are all given deterministic, bounded and continuous functions on \([0, +\infty)\);
- \( S^{0,2}(t) > -1, S^{1,2}(t) > -1, S^{2,2}(t) = 0 \) and \( S^{i,3}(t) \geq 0 \) for all \( t \geq 0 \);
- \( \gamma^{(i+1)}(t) \) is deterministic;
- \( \mathcal{U}^{ad} = \left\{ u = (u^{(0)}, u^{(1)}, u^{(2)}) \middle| u^{(i)} \in \mathcal{O}(\mathcal{G}^i, \mathbb{R}) \text{ measurable with } \sum_{i=0}^{2} \mathbb{E} \left[ \int_{\tau_i}^{T+i+1} |u^{(i)}(t)|^2 dt < \infty \right] \right\} \); 
- \( \mathcal{M}^i_t = \mathcal{G}^i_t = \mathcal{F}_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1}), i = 0, 1, 2. \)

We note that when \( \gamma^3(t) (i = 1, 2, 3) \) is deterministic and Assumption 23 holds, \( \tau_i \) can be regarded as the default time in the Cox model 3. Moreover, \( \tau_i \) has a closed relationship with the Poisson process. If \( N^i_t \) is an inhomogeneous Poisson process with intensity \( \gamma^3(t) \) and first jump time \( \tau_i \), then \( \mathbb{I}_{\tau_i \leq t} = N^i_{t \wedge \tau_i} \), where \( \mathbb{I}_{\tau_i \leq t} \) is the indicator process.
According to Theorem 4.1, there exists a unique solution \( X(\cdot) \in L^2_T \) to (31) for any fixed \( T > 0 \). Taking conditional expectation on both sides in (31) derives

\[
\begin{align*}
\left\{ dX^{(i)}(t) &= X^{(i)}(t) \left[ S^{(i,0)}(t) - u^{(i)}(t) + \gamma^{(i+1)}(t)S^{(i,2)}(t) + S^{(i,3)}(t) \right] dt, \quad t \in (\tau_i, \tau_{i+1}), \\
X^{(i)}(\tau_i) &= X^{(i-1)}(\tau_i), \quad X^{(i-1)}(\tau_0) = x_0 \in \mathbb{R}^+.
\end{align*}
\]

Moreover, by comparison theorem, we know that \( X^{(i)}(t) > 0 \text{ a.s.} \). Then, for any \( t \in [\tau_i, \tau_{i+1}] \) with \( i = 0, 1, 2 \), considering the following SDE

\[
\begin{align*}
dG^{(i)}(t) &= G^{(i)}(t) \left[ -S^{(i,0)}(t) + u^{(i)}(t) - \gamma^{(i+1)}(t)S^{(i,2)}(t) + |S^{(i,1)}(t)|^2 + \frac{1}{2} \gamma^{(i+1)}(t)|S^{(i,2)}(t)|^2 \right] dt, \\
G^{(i)}(\tau_i) &= G^{(i-1)}(\tau_i), \quad G^{(i-1)}(\tau_0) = 1.
\end{align*}
\]

and applying Lemma 2.3 to \( X^{(i)}(t)G^{(i)}(t) \), one can easily check that

\[
d(X^{(i)}(t)G^{(i)}(t)) = X^{(i)}(t)G^{(i)}(t)S^{(i,3)}(t)dt.
\]

Then applying Theorem A.1 in [40] gives

\[
\ln X^{(i)}(t) = \ln X^{(i)}(\tau_i) + \ell(\tau_i, t, u^{(i)})
\]

\[
= \ln X^{(i)}(\tau_i) - \ln G^{(i)}(t) + \ln \left[ \int_{\tau_i}^t G^{(i)}(s)S^{(i,3)}(s)q_i(s, u^{(i)})ds + 1 \right],
\]

where

\[
G^{(i)}(t) = \exp \left\{ -\int_{\tau_i}^t S^{(i,1)}(s)dB_s - \int_{\tau_i}^t S^{(i,2)}(s)dB_s + \int_{\tau_i}^t \ln(S^{(i,2)}(s) + 1) - S^{(i,2)}(s) d\mathbb{H}^k(s) \\
+ \int_{\tau_i}^t \left[ -S^{(i,0)}(s) + u^{(i)}(s) - \frac{1}{2} \gamma^{(i+1)}(s)|S^{(i,1)}(s)|^2 + \frac{1}{2} \gamma^{(i+1)}(s)|S^{(i,2)}(s)|^2 \right] ds \right\},
q_i(t, u^{(i)}) = \exp \left\{ \int_{\tau_i}^t S^{(i,0)}(s) - u^{(i)}(s) + \gamma^{(i+1)}(s)S^{(i,2)}(s) + S^{(i,3)}(s)ds \right\}.
\]

Next, we define a recursive utility process \( \mathcal{P}(t) \) governed by the following MMFBSDE:

\[
\begin{align*}
d\mathcal{P}^{(k)}(t) &= -\left[ \nu_{k,0}(t)\mathbb{E}[\mathcal{P}^{(k)}(t)|\mathcal{F}_{\tau_k}] + \nu_{k,1}(t)Q^{(k)}(t) + \nu_{k,2}(t)Y^{(k)}(t)R^{(k)}(t) + \ln|X(t)|u(t) \right] dt \\
&\hspace{1cm} + Q^{(k)}(t)dB_t + R^{(k)}(t)dA^{(k+1)}_t, \quad t \in [\tau_k, \tau_{k+1}), \quad k = 0, 1, 2,
\mathcal{P}^{(k)}(\tau_k) = \mathcal{P}^{(k-1)}(\tau_k), \quad k = 1, 2, \quad \mathcal{P}^{(2)}(T) = \ln(X(T))
\end{align*}
\]

in the unknown process

\[
(\mathcal{P}, Q, R) = \sum_{k=0}^{1} (\mathcal{P}^{(k)}, Q^{(k)}, R^{(k)})(t)\mathbb{I}_{t \in [\tau_k, \tau_{k+1}]} + (\mathcal{P}^{(2)}, Q^{(2)}, R^{(2)})(t)\mathbb{I}_{t \in [\tau_2, T]}.
\]

Here \( \nu_{k,j}(\cdot) (k, j = 0, 1, 2) \) are given deterministic and continuous functions on \([0, +\infty)\) satisfying \( \nu_{k,0}(t) \geq 0, \nu_{2,1}(t) > -1 \) and \( \nu_{2,2}(t) = 0 \). It follows from Theorem 1.2 that (33) has a unique solution \( (\mathcal{P}, Q, R) \in L^3_T \times H^3_T \times \bigotimes_{i=1}^{2} H^3_T A_i^\cdot \). Some related works concerned with the recursive utility process, we refer the reader to [19, 30, 33].

Thus, the optimal investment problem can be described as follows.
Proposition 4.1. The recursive utility can be rewritten as follows:

\[ J(x_0, \mathcal{L}(x_0), u) = \mathbb{E}[\mathcal{P}(0)] = \mathbb{E}\left[ \int_0^T Y(t) \ln |X(t)|u(t)|dt + Y(T) \ln |X(T)| \right], \]  

(34)

where \( Y(t) = (Y^{(0)}(t), Y^{(1)}(t), Y^{(2)}(t)) \), and \( Y^{(k)}(t) \) \((k = 0, 1, 2)\) are given by

\[
\begin{aligned}
  dY^{(k)}(t) &= \nu_{k,0}(t)\mathbb{E}[Y^{(k)}(t)|\mathcal{F}_t]dt + \nu_{k,1}(t)Y^{(k)}(t)dB_t + \nu_{k,2}(t)Y^{(k)}(t)dA_t^{(k+1)}, \quad t \in (\tau_k, \tau_{k+1}],

  Y^{(k)}(\tau_k) &= Y^{(k-1)}(\tau_k), \quad Y^{(-1)}(0) = 1.
\end{aligned}
\]

(35)

Remark 4.1. For \( k = 0, 1, 2 \), one can easily check that condition (2) holds and by Itô’s formula that

\[ \mathbb{E}[Y^{(k)}(t)|\mathcal{F}_t] = Y^{(k)}(\tau_k)e^{\int_{\tau_k}^t \nu_{k,0}(s)ds} \]

and

\[ Y^{(k)}(t) = \frac{Y^{(k)}(\tau_k)}{\varphi_{3,i}(t)} + Y^{(k)}(\tau_k)\int_{\tau_k}^t \varphi_{3,i}(s)e^{\int_{\tau_k}^s \nu_{k,0}(s)ds}ds > 0 \ \text{a.s.}, \]  

(36)

where

\[
\varphi_{3,i}(t) = \exp\left\{ -\int_{\tau_k}^t \nu^{k,1}(s)dB_s - \int_{\tau_k}^t (\ln(\nu^{k,2}(s) + 1))dA_s 
  + \int_{\tau_k}^t \left[ \nu^{k,2}(s)\gamma^{(k+1)}(s) - \frac{1}{2} \left( \nu^{k,1}(s) \right)^2 - \ln(\nu^{k,2}(s) + 1)\gamma^{(k+1)}(s) \right]ds \right\}.
\]

Moreover, if all coefficients in (31) and (33) are constants, then Proposition 4.1 indicates that Problem 4.1 reduces to Problem 1.1.

Then by Theorem 3.1, we introduce the following subproblem.

Subproblem 4.1. Find an optimal investment strategy \( \hat{u}^{(2)}(\cdot) \in \mathcal{U}_2^{ad} \) such that

\[
J^{(2)}(X^{(1)}(\tau_2), \mathcal{L}^{(1)}(X^{(1)}(\tau_2)), \hat{u}^{(2)}) = \mathbb{E}\sup_{u^{(2)} \in \mathcal{U}_2^{ad}} \int_{\tau_2}^T Y(t) \ln |X^{(2)}(t)|u^{(2)}(t)|dt + Y(T) \ln |X^{(2)}(T)|\mathbb{E}[\mathcal{F}_{\tau_2}],
\]

where \( X^{(2)}(T) = X(T) \).

Obviously, the Hamiltonian functional to Subproblem 4.1 reads

\[
H^{(2)} = Y \ln |X^{(2)}u^{(2)}| + \left[ X^{(2)}(S^{2,0} - u^{(2)}) + \nabla^{(2)}S^{2,3} \right] p^{(2)} + S^{2,1}X^{(2)}q^{(2)},
\]
where \((p^{(2)}, q^{(2)}, r^{(2)})\) is given by

\[
\begin{cases}
  dp^{(2)}(t) = -\left[\frac{Y(t)}{X(t)} + (S^{2.0}(t) - u^{(2)}(t))p^{(2)}(t) + S^{2.3}(t)p^{(2)}(t) + S^{2.1}(t)q^{(2)}(t)\right]dt + q^{(2)}(t)dB_t, & t \in [\tau_2, T], \\
p^{(2)}(T) = \frac{Y(T)}{X(T)}.
\end{cases}
\]

(37)

It follows from Theorems 3.2 and 3.3 that

\[
\mathbb{E}\left[ Y(t) \bigg| \mathcal{F}_{\tau_2} \right] - \mathbb{E}\left[ \tilde{X}^{(2)}(t)\tilde{p}^{(2)}(t) \bigg| \mathcal{F}_{\tau_2} \right] = 0,
\]

where \(\tilde{X}^{(2)}(t)\) and \(\tilde{p}^{(2)}(t)\) are respectively the solutions to (31) and (37) under the optimal investment strategy

\[
\tilde{u}^{(2)}(t) = \frac{\mathbb{E}\left[ Y(t) \bigg| \mathcal{F}_{\tau_2} \right]}{\mathbb{E}\left[ \tilde{X}^{(2)}(t)\tilde{p}^{(2)}(t) \bigg| \mathcal{F}_{\tau_2} \right]}, \quad t \in [\tau_2, T].
\]

On the other hand, by Itô’s formula, we have

\[
\tilde{X}^{(2)}(t)\tilde{p}^{(2)}(t) = \mathbb{E}\left[ \tilde{X}^{(2)}(T)\tilde{p}^{(2)}(T) + \int_t^T Y(s)ds \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ Y(T) + \int_t^T Y(s)ds \bigg| \mathcal{F}_t \right]
\]

and so

\[
\begin{cases}
  \tilde{u}^{(2)}(t) = \frac{\mathbb{E}\left[ Y(t) \bigg| \mathcal{F}_{\tau_2} \right]}{\mathbb{E}\left[ \tilde{X}^{(2)}(t)\tilde{p}^{(2)}(t) \bigg| \mathcal{F}_{\tau_2} \right]}, \quad t \in [\tau_2, T] \\
J^{(2)}(X^{(1)}(\tau_2), 2^{(1)}(X^{(1)}(\tau_2)), \tilde{u}^{(2)}) = \mathbb{E}\left[ \int_{\tau_2}^T Y(t)[\ln\tilde{u}^{(2)}(t)] + \ell(\tau_2, t, \tilde{u}^{(2)})]dt + Y(T)\ell(\tau_2; T, \tilde{u}^{(2)}) \bigg| \mathcal{F}_{\tau_2} \right] \\
+ \mathbb{E}\left[ Y(T) + \int_{\tau_2}^T Y(s)ds \bigg| \mathcal{F}_{\tau_2} \right]\ln X^{(1)}(\tau_2).
\end{cases}
\]

Similarly, \(\tilde{u}^{(1)}(t), \tilde{u}^{(0)}(t)\) and related utility can be given as follows:

\[
\begin{cases}
  \tilde{u}^{(1)}(t) = \frac{\mathbb{E}\left[ Y(t) \bigg| \mathcal{F}_{\tau_1} \right]}{\mathbb{E}\left[ \tilde{X}^{(1)}(t)\tilde{p}^{(1)}(t) \bigg| \mathcal{F}_{\tau_1} \right]}, \quad t \in [\tau_1, \tau_2], \\
\tilde{u}^{(0)}(t) = \frac{\mathbb{E}\left[ Y(t) \bigg| \mathcal{F}_{\tau_0} \right]}{\mathbb{E}\left[ \tilde{X}^{(0)}(t)\tilde{p}^{(0)}(t) \bigg| \mathcal{F}_{\tau_0} \right]}, \quad t \in [0, \tau_1], \\
J^{(1)}(X^{(0)}(\tau_1), 2^{(0)}(X^{(0)}(\tau_1)), \tilde{u}^{(1)}) = \sum_{i=1}^2 \mathbb{E}\left[ \int_{\tau_i}^{\tau_{i+1}} Y(t)[\ln \tilde{u}^{(i)}(t)] + \ell(\tau_i, t, \tilde{u}^{(i)})]dt + Y(T)\ell(\tau_i; T, \tilde{u}^{(i)}) \bigg| \mathcal{F}_{\tau_i} \right] \\
+ \mathbb{E}\left[ Y(T) + \int_{\tau_i}^T Y(s)ds \bigg| \mathcal{F}_{\tau_i} \right]\ln X^{(0)}(\tau_i), \\
J^{(0)}(x_0, 2^{(0)}(x_0), \tilde{u}^{(0)}) = \sum_{i=0}^2 \mathbb{E}\left[ \int_{\tau_i}^{\tau_{i+1}} Y(t)[\ln \tilde{u}^{(i)}(t)] + \ell(\tau_i, t, \tilde{u}^{(i)})]dt + Y(T)\ell(\tau_i; T, \tilde{u}^{(i)}) \bigg| \mathcal{F}_{\tau_i} \right] \\
+ \mathbb{E}\left[ Y(T) + \int_{\tau_i}^T Y(s)ds \bigg| \mathcal{F}_{\tau_i} \right]\ln x_0.
\end{cases}
\]

Finally, by Theorem 3.3 \(\tilde{u} = (\tilde{u}^{(0)}, \tilde{u}^{(1)}, \tilde{u}^{(2)})\) is indeed the global optimal investment strategy. Combining (36), we can see that

\[
\mathbb{E}\left[ \frac{Y^{(k)}(t)}{\mathcal{F}_{\tau_k}} \bigg| \mathcal{F}_{\tau_k} \right] = \frac{1}{e^{\int_t^T \nu_{k,o}(s)ds} + \int_t^T e^{\int_t^s \nu_{k,o}(s)ds}ds}ds
\]

is the unique feedback optimal control, which is non-zero and bounded. Therefore, we can organize the above results as the following theorem, which is unexpected but remarkably interesting.
Theorem 4.3. The optimal investment strategy of Problem 4.1 is given by

\[
\hat{\alpha}(t) = \begin{cases} 
\mathbb{I}_{0 \leq t < \tau_1} e^{\int_t^{\tau_1} \nu_0(s)ds} + \mathbb{I}_{\tau_1 \leq t < \tau_2} e^{\int_t^{\tau_1} \nu_1(s)ds} + \mathbb{I}_{\tau_2 \leq t \leq T} e^{\int_t^{\tau_2} \nu_2(s)ds} 
\end{cases}
\]

5 Stability

This section studies the mean square exponential stability and the almost sure exponential stability for the solution to MMFSDEs under optimal feedback control, respectively. To this end, we suppose that the coefficients in (3) are all progressively measurable functions from \([0, +\infty) \times \Omega \times \mathbb{P}_2(\mathbb{R}) \times \mathbb{P}_2(\mathbb{R})\) to \(\mathbb{R}\), and that (3) admits a unique solution \(X(\cdot) \in L^\infty_\tau\).

5.1 Mean Square Exponential Stability

Let us recall the following mean square exponential stability and almost sure exponential stability.

**Definition 5.1.** (i) The state \(X(t)\) is said to be mean square exponentially stable if there exists a pair of positive constants \(M_0\) and \(\alpha_0\) such that

\[
\mathbb{E}\left[(X(t))^2\right] \leq x_0^2 \cdot M_0 e^{-\alpha_0 t}.
\]

for all \(t \geq 0\) and \(x_0 \in \mathbb{R}\). Namely,

\[
\limsup_{t \to +\infty} \frac{\ln \mathbb{E}[|X(t)|^2]}{t} < 0
\]

for all \(x_0 \in \mathbb{R}\).

(ii) The state \(X(t)\) is said to almost surely exponentially stable if

\[
P\left\{\limsup_{t \to +\infty} \frac{\ln |X(t)|}{t} < 0\right\} = 1
\]

for all \(x_0 \in \mathbb{R}\).

Based on (3), consider the following system of MMFSDE with initial value \(x_0 \in \mathbb{R}\) and optimal feedback control \(\hat{u}^k(t) = \phi^k(t, \hat{X}^k(t), M(\hat{X}^k(t))) \quad (k \in \mathbb{Z}_0)\):

\[
\begin{cases}
\frac{d\hat{X}^k(t)}{dt} = b^k(t, \hat{X}^k(t), M(\hat{X}^k(t)), \hat{u}^k(t), N(\hat{u}^k(t)))dt + \sigma^k(t, \hat{X}^k(t), M(\hat{X}^k(t)), \hat{u}^k(t), N(\hat{u}^k(t)))dB_t \\
\quad + h^k(t, \hat{X}^k(t), M(\hat{X}^k(t)), \hat{u}^k(t), N(\hat{u}^k(t)))dA^k_{t+1}, \quad t \in (\tau_k, \tau_{k+1}], \\
\hat{X}^k(\tau_k) = \hat{X}^{k-1}(\tau_k), \quad \hat{X}^{-1}(\tau_0) = x_0,
\end{cases}
\]

where the \((n + 1)\)-th equation and the optimal feedback control remain valid on \(t \in [T, +\infty)\). We have the following theorem for the mean square exponential stability.

**Theorem 5.1.** Let

\[
V_k(\hat{X}^k(t), M(\hat{X}^k(t))) = 2b^k(t, \hat{X}^k(t), M(\hat{X}^k(t)), \hat{u}^k(t), N(\hat{u}^k(t)))\hat{X}^k(t) \\
+ |\sigma^k(t, \hat{X}^k(t), M(\hat{X}^k(t)), \hat{u}^k(t), N(\hat{u}^k(t)))|^2 \\
+ \gamma^{(i+1)}(t)|h^k(t, \hat{X}^k(t), M(\hat{X}^k(t)), \hat{u}^k(t), N(\hat{u}^k(t)))|^2.
\]
If there exist constants $C_0, C_1, \ldots, C_n$ such that $\sum_{i=0}^{k} C_i > 0$ and

$$
\mathbb{E} \left[ V_k(\hat{X}^k(t), M(\hat{X}^k(t))) \right] \leq -C_k \mathbb{E} \left[ |\hat{X}^k(t)|^2 \right] \quad \text{a.s.}
$$

(39)

for all $k \in \mathcal{I}_0$ and $t \geq 0$, then the state $\hat{X}^k(t)$ is mean square exponentially stable.

Proof. Applying Itô’s formula to $e^{C_k(t-\tau_k)}(\hat{X}^k(t))^2$ and taking expectation, one can check that for any $t \in [\tau_k, \tau_{k+1}]$ or $t \in [\tau_n, +\infty)$,

$$
\mathbb{E} \left[ e^{C_k(t-T)}(\hat{X}^k(t))^2 \right] \\
\leq \mathbb{E} \left[ e^{C_k(t-\tau_k)}(\hat{X}^k(t))^2 \right] \\
= e^{0} \mathbb{E} \left[ (\hat{X}^k(\tau_k))^2 \right] + \mathbb{E} \left[ \int_{\tau_k}^{t} C_k e^{C_k(t-\tau_k)}(\hat{X}^k(s))^2 ds \right] \\
= \mathbb{E} \left[ (\hat{X}^k(\tau_k))^2 \right] + \mathbb{E} \left[ \int_{\tau_k}^{t} C_k e^{C_k(t-\tau_k)}(\hat{X}^k(s))^2 ds \right] \\
\leq \mathbb{E} \left[ (\hat{X}^k(\tau_k))^2 \right] + \mathbb{E} \left[ \int_{\tau_k}^{t} C_k e^{C_k(t-\tau_k)}(\hat{X}^k(s))^2 ds \right] \\
\leq \mathbb{E} \left[ (\hat{X}^k(\tau_k))^2 \right] + \mathbb{E} \left[ \int_{\tau_k}^{t} C_k e^{C_k(t-\tau_k)}(\hat{X}^k(s))^2 ds \right] \\
= \mathbb{E} \left[ (\hat{X}^k(\tau_k))^2 \right]
$$

which implies

$$
\mathbb{E} \left[ (\hat{X}^k(t))^2 \right] \leq e^{C_k(T-t)} \mathbb{E} \left[ (\hat{X}^k(\tau_k))^2 \right] \leq e^{(C_k+C_{k-1})(T-t)} \mathbb{E} \left[ (\hat{X}^{k-1}(\tau_{k-1}))^2 \right].
$$

Thus, by the backward induction, we can obtain

$$
\mathbb{E} \left[ (\hat{X}^k(t))^2 \right] \leq x_0^2 \cdot e^{(T-t) \sum_{i=0}^{k} C_i},
$$

which ends the proof with $M_0 = e \sum_{i=0}^{k} C_i$ and $a_0 = \sum_{i=0}^{k} C_i$. \hfill \Box

The following proposition will be useful to verify condition (33).

**Proposition 5.1.** For any given $t \in [\tau_k, \tau_{k+1}]$ or $t \in [\tau_n, +\infty)$, one has

$$
\mathbb{E} \left[ M(\hat{X}^k(t))(\hat{X}^k(t)) \right] \quad \text{a.s.}
$$

Proof. It follows from Cauchy’s inequality and the condition $M^k_{\tau_k} = \mathfrak{F}_{\tau_k}$ that

$$
\mathbb{E} \left[ M(\hat{X}^k(t))(\hat{X}^k(t)) \right] = \mathbb{E} \left[ \hat{X}^k(t) M^k_{\tau_k} \cdot (\hat{X}^k(t)) \right] \\
\leq \mathbb{E} \left[ \hat{X}^k(t) M^k_{\tau_k} \right] \cdot \mathbb{E} \left[ (\hat{X}^k(t))^2 \right] \\
\leq \mathbb{E} \left[ (\hat{X}^k(t))^2 \right] \cdot \mathbb{E} \left[ (\hat{X}^k(t))^2 \right] \\
= \mathbb{E} \left[ (\hat{X}^k(t))^2 \right] \quad \text{a.s.}
$$

and so the desired result is obtained. \hfill \Box
Next, concerning on the optimal controlled pair \((\tilde{X}(\cdot), \tilde{u}(\cdot))\) governed by (31), we further analyse the mean square exponential stability of the state \(\tilde{X}(t)\). Noticing that all the coefficients are bounded on \([0, +\infty)\), we denote

\[
\tilde{C}_i = \sup_{t \in [0, +\infty]} \left[ 2S^{i,0}(t) - 2\hat{\gamma}(^{(i)})_2(t) + 2\hat{\gamma}(^{(i+1)})_3(t) + 2S^{i,1}(t) + |S^{i,1}(t)|^2 + \hat{\gamma}(^{(i+1)})_1S^{i,2}(t)|^2 \right] .
\]

Furthermore, we need the following assumption.

**Assumption 5.1.** Suppose that

\[
\max \{ \tilde{C}_1, \tilde{C}_1 + \tilde{C}_2, \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 \} < 0.
\]

Then the result of mean square exponential stability for \(\tilde{X}(t)\) can be specified as follows.

**Theorem 5.2.** Under Assumption 5.1 the state \(\tilde{X}(t)\) is mean square exponentially stable.

**Proof.** It follows from Proposition 5.1 and Theorem 5.1 immediately. \(\square\)

**Remark 5.1.** Theorem 5.2 implies that \(E \left[ \int_0^{+\infty} e^{-\delta t} |X(t)|^2 dt \right] < +\infty \) for all \(\delta > \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3\).

### 5.2 Almost Sure Exponential Stability

In general, it is hard to obtain the almost sure exponential stability for the solutions to nonlinear system (31). Thus this subsection will discuss the almost sure exponential stability for the solution to the linear MMFSDE (31) under optimal feedback control. To this end, from (32), we focus on the state of the third stage \((i = 2)\) with

\[
\ln \hat{X}^{(2)}(t) = \ln \hat{X}^{(2)}(t_2) - \ln \hat{G}^{(2)}(t) + \ln \left[ \int_{t_2}^{t} \hat{G}^{(i)}(s)S^{2,2}(s)\delta_i(s, \hat{u}^{(2)}(s))ds + 1 \right] = \ln \hat{X}^{(2)}(t_2) - \ln \hat{G}^{(2)}(t) + \ln \left[ \int_{t_2}^{t} S^{2,2}(s)e^{\hat{\gamma}(^{(2)})(s)} - \int_{t_2}^{t} S^{2,1}(s)dB_s + \int_{t_2}^{t} \left[ \frac{1}{2} |S^{2,1}(\tilde{s})|^2 + S^{2,3}(\tilde{s}) \right] d\tilde{s} \right] ds + 1 ,
\]

where

\[
\hat{G}^{(2)}(t) = \exp \left\{ -\int_{t_2}^{t} S^{2,1}(s)dB_s - \int_{t_2}^{t} \left[ -S^{2,0}(s) + \hat{\gamma}^{(2)}(s) + \frac{1}{2} |S^{2,1}(s)|^2 \right] ds \right\},
\]

\[
\hat{g}_2(t, \hat{u}^{(2)}) = \exp \left\{ \int_{t_2}^{t} S^{2,0}(s) - \hat{\gamma}^{(2)}(s) + S^{2,3}(s)ds \right\}.
\]

Suppose that the optimal feedback control \(\hat{u}(t)\) given by (33) remains valid on \(t \in [T, +\infty)\), i.e.,

\[
\hat{u}(t) = \frac{\| \hat{u}_{0,0}(s) \|_{L^{2,0}(s)ds} + \int_{t_2}^{t} e^{\int_{t_2}^{s} \hat{\gamma}(^{(2)})(\tau_1)\delta_i(\tau_1, \hat{u}^{(2)}(\tau_1))d\tau_1} \| \hat{u}_{0,0}(s) \|_{L^{2,0}(s)ds}}{e^{\int_{t_2}^{t} \hat{\gamma}(^{(2)})(s)\delta_i(s, \hat{u}^{(2)}(s))d\delta_i(s, \hat{u}^{(2)}(s))} + \frac{1}{\delta_i} e^{\int_{t_2}^{t} \hat{\gamma}(^{(2)})(s)\delta_i(s, \hat{u}^{(2)}(s))d\delta_i(s, \hat{u}^{(2)}(s))} + \frac{1}{\delta_i} e^{\int_{t_2}^{t} \hat{\gamma}(^{(2)})(s)\delta_i(s, \hat{u}^{(2)}(s))d\delta_i(s, \hat{u}^{(2)}(s))} + \frac{1}{\delta_i} e^{\int_{t_2}^{t} \hat{\gamma}(^{(2)})(s)\delta_i(s, \hat{u}^{(2)}(s))d\delta_i(s, \hat{u}^{(2)}(s))},
\]

and the \((n + 1)\)-th equation in (31) still holds for \(t \in [T, +\infty)\). Then, one can easily deduce that (31) has a unique solution \(\hat{X}(\cdot) \in L^{2,0,\delta}(t, +\infty)\) for some \(\delta > 0\) under the optimal feedback control \(\hat{u}(t)\). Moreover, we need the following assumption for the related coefficients.

**Assumption 5.2.** Suppose that the following assumptions hold:

(i) \(\lim_{t \to +\infty} S^{2,1}(t)\) exists;

(ii) \(\lim_{t \to +\infty} \left[ |S^{2,0}(t) - \hat{\gamma}^{(2)}(t) - \frac{1}{2} |S^{2,1}(t)|^2 \right] < 0;\)
(iii) $2S^{2.3}(s) \leq |S^{2.1}(s)|^2$, $\forall t \geq 0$.

Then we have the following result.

**Theorem 5.3.** Under Assumption 5.3, the state $\hat{X}(t)$ is almost surely exponentially stable.

**Proof.** The proof is divided into two steps.

**Step1** The quadratic variation of $\int_{\tau_2}^{t} S^{2.1}(s) dB_s$ reads $\int_{\tau_2}^{t} |S^{2.1}(s)|^2 ds$. Since

$$
\lim_{t \to +\infty} \frac{\int_{\tau_2}^{t} |S^{2.1}(s)|^2 ds}{t} = \lim_{t \to +\infty} |S^{2.1}(t)|^2 < +\infty, \text{ a.s.,}
$$

it follows from Theorem 1.3.4 in [41] that

$$
Theorem 5.3.$$

the state $\hat{X}(t)$ is almost surely exponentially stable.

Combining (40), (41) and (42), we have

$$
\lim_{t \to +\infty} \frac{\int_{\tau_2}^{t} S^{2.1}(s) dB_s}{t} = 0, \text{ a.s.}
$$

Then combining

$$
\lim_{t \to +\infty} \frac{\int_{\tau_2}^{t} [S^{2,0}(s) - \tilde{u}^{(2)}(s) - \frac{1}{2} |S^{2.1}(s)|^2] ds}{t} = \lim_{t \to +\infty} \left[ S^{2,0}(t) - \tilde{u}^{(2)}(t) - \frac{1}{2} |S^{2.1}(t)|^2 \right] < 0 \text{ a.s.,}
$$

we obtain that

$$
\lim_{t \to +\infty} - \ln \hat{G}^{(2)}(t) < 0, \text{ a.s.} \quad (41)
$$

**Step2** Noticing that exp $\left\{ -2 \int_{\tau_2}^{t} S^{2.1}(s) dB_s + 2 \int_{\tau_2}^{t} |S^{2.1}(s)|^2 ds \right\}$ is an exponential martingale, one has

$$
0 \leq \lim_{t \to +\infty} \mathbb{E} \left[ \ln^2 \left[ \int_{\tau_2}^{t} S^{2.3}(s) \exp \left\{ -\int_{\tau_2}^{t} S^{2.1}(s) dB_s + \int_{\tau_2}^{t} \frac{1}{2} |S^{2.1}(s)|^2 + S^{2.3}(s) ds \right\} ds + 1 \right] \right]
$$

$$
\leq \lim_{t \to +\infty} \mathbb{E} \left[ \left( \int_{\tau_2}^{t} S^{2.3}(s) \exp \left\{ -\int_{\tau_2}^{t} S^{2.1}(s) dB_s + \int_{\tau_2}^{t} \frac{1}{2} |S^{2.1}(s)|^2 + S^{2.3}(s) ds \right\} ds \right)^2 \right]
$$

$$
\leq \lim_{t \to +\infty} \mathbb{E} \left[ \left( \int_{\tau_2}^{t} |S^{2.3}(s)|^2 \exp \left\{ -\int_{\tau_2}^{t} 2S^{2.1}(s) dB_s + \int_{\tau_2}^{t} \frac{1}{2} |S^{2.1}(s)|^2 + 2S^{2.3}(s) ds \right\} ds \right)^2 \right]
$$

$$
\leq \lim_{t \to +\infty} \mathbb{E} \left[ \left( \int_{0}^{t} |S^{2.3}(s)|^2 ds \right)^2 \right]
$$

$$
= \lim_{t \to +\infty} \frac{\int_{\tau_2}^{t} |S^{2.3}(s)|^2 ds}{t^2}
$$

$$
= \lim_{t \to +\infty} \frac{|S^{2.3}(t)|^2}{2t}
$$

$$
= 0,
$$

where we have used Cauchy's inequality. This implies that

$$
\lim_{t \to +\infty} \ln \left[ \int_{\tau_2}^{t} S^{2.3}(s) \exp \left\{ -\int_{\tau_2}^{t} S^{2.1}(s) dB_s + \int_{\tau_2}^{t} \frac{1}{2} |S^{2.1}(\delta)|^2 + S^{2.3}(\delta) ds \right\} ds + 1 \right] = 0, \text{ a.s.} \quad (42)
$$

Combining (40), (41) and (42), we have

$$
\lim_{t \to +\infty} \frac{\ln \hat{X}^{(2)}(t)}{t} < 0, \text{ a.s.}
$$

Thus, the state $\hat{X}(t)$ is almost surely exponentially stable. \qed
As is well known, both the mean square exponential stability and the almost sure exponential stability imply the globally asymptotic stability. Therefore, combining Theorems 5.2 and 5.3 we have the following result.

**Theorem 5.4.** Under Assumptions 5.1 and 5.2 the state $\hat{X}(t)$ is globally asymptotically stable.

### 6 Conclusions

This paper is devoted to the study of stochastic optimal control problems governed by a system of MMFSDEs. By deriving the backward induction formula, the original problem is decomposed into several subproblems. Then both sufficient and necessary maximum principles are given for these subproblems with random coefficients. Next, the existence and uniqueness results are obtained for both MMFSDEs and MMFBSDEs, and the obtained results are applied to solve an optimal investment problem. Finally, under suitable conditions, the mean square exponential stability and almost sure exponential stability are guaranteed for the solutions to the MMFSDEs under optimal feedback controls.

We would like to mention that our results are still valid for solving the linear-quadratic optimal control problem governed by the MMFSDE. However, many interesting problems should be considered in the future: (i) find some second or higher-order necessary conditions for optimal control problems governed by MMFSDEs; (ii) solve optimal control problems governed by MMFSDEs with time delays and/or regime switchings.

### References

[1] Nacira Agram, Yaozhong Hu, and Bernt Øksendal. Mean-field backward stochastic differential equations and applications. *Systems & Control Letters*, 162:105196, 2022.

[2] Nacira Agram, Yaozhong Hu, and Bernt Øksendal. Stochastic Fokker-Planck PIDE for conditional Mckean-Vlasov jump diffusions and applications to optimal control. *SIAM Journal on Control and Optimization*, 61(3):1472–1493, 2023.

[3] Nacira Agram and Bernt Øksendal. Mean-field stochastic control with elephant memory in finite and infinite time horizon. *Stochastics*, 91(7):1041–1066, 2019.

[4] Anna Aksamit and Monique Jeanblanc. *Enlargement of Filtration with Finance in View*. Springer-Briefs in Quantitative Finance. Springer, Cham, 2017.

[5] Anna Aksamit, Monique Jeanblanc, and Marek Rutkowski. Integral representations of martingales for progressive enlargements of filtrations. *Stochastic Processes and their Applications*, 129(4):1229–1258, 2019.

[6] Mouhcine Assouli and Badr Missaoui. Deep learning for mean field games with non-separable Hamiltonians. *Chaos, Solitons & Fractals*, 174:113802, 2023.

[7] Khalida Bachir Cherif, Nacira Agram, and Kristina Dahl. Stochastic maximum principle with default. *arXiv preprint arXiv:2001.01535*, 2020.

[8] Alain Bensoussan, Ho Man Tai, and Sheung Chi Phillip Yam. Mean Field Type Control Problems, some Hilbert-space-valued FBSDEs, and Related Equations. *arXiv preprint arXiv:2305.04019*, 2023.
[9] Lijun Bo, Yongjin Wang, and Xuewei Yang. Stochastic portfolio optimization with default risk. *Journal of Mathematical Analysis and Applications*, 397(2):467–480, 2013.

[10] Rainer Buckdahn, Juan Li, and Jin Ma. A mean-field stochastic control problem with partial observations. *The Annals of Applied Probability*, 27(5):3201–3245, 2017.

[11] Rainer Buckdahn, Juan Li, and Jin Ma. A general conditional McKean-Vlasov stochastic differential equation. *The Annals of Applied Probability*, 33(3):2004–2023, 2023.

[12] Alessandro Calvia and Emanuela Rosazza Gianin. Risk measures and progressive enlargement of filtration: a BSDE approach. *SIAM Journal on Financial Mathematics*, 11(3):815–848, 2020.

[13] Wenping Cao and Quanxin Zhu. Razumikhin-type theorem for pth exponential stability of impulsive stochastic functional differential equations based on vector Lyapunov function. *Nonlinear Analysis: Hybrid Systems*, 39:100983, 2021.

[14] René Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications. I*, volume 83 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean Field FBSDEs, Control, and Games.

[15] René Carmona and François Delarue. *Probabilistic theory of Mean Field Games with Applications. II*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. Mean Field Games with Common Noise and Master Equations.

[16] René Carmona and Xiumeng Zhu. A probabilistic approach to mean field games with major and minor players. *The Annals of Applied Probability*, 26(3):1535–1580, 2016.

[17] Hua Deng and Miroslav Krstić. Stochastic nonlinear stabilization I: A backstepping design. *Systems & Control Letters*, 32(3):143–150, 1997.

[18] Paolo Di Tella. On the weak representation property in progressively enlarged filtrations with an application in exponential utility maximization. *Stochastic Processes and their Applications*, 130(2):760–784, 2020.

[19] Darrell Duffie and Larry G Epstein. Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, 60(2):353–394, 1992.

[20] Roxana Dumitrescu, Miryana Grigorova, Marie-Claire Quenez, and Agnès Sulem. BSDEs with default jump. In *Computation and combinatorics in dynamics, stochastics and control*, volume 13 of *Abel Symp.*, pages 233–263. Springer, Cham, 2018.

[21] Roxana Dumitrescu, Bernt Øksendal, and Agnès Sulem. Stochastic control for mean-field stochastic partial differential equations with jumps. *Journal of Optimization Theory and Applications*, 176(3):559–584, 2018.

[22] Romuald Elie, Tomoyuki Ichiba, and Mathieu Laurière. Large banking systems with default and recovery: A mean field game model. *arXiv preprint arXiv:2001.10206*, 2020.

[23] Zhun Gou, Nanjing Huang, and Minghui Wang. A linear-quadratic mean-field stochastic Stackelberg differential game with random exit time. *International Journal of Control*, 96(3):731–745, 2023.
[24] Jean Jacod. Grossissement initial, hypothèse (H') et théorème de Girsanov. In *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*, pages 1982–1983. Springer, Berlin, 1987.

[25] Monique Jeanblanc and Libo Li. Characteristics and constructions of default times. *SIAM Journal on Financial Mathematics*, 11(3):720–749, 2020.

[26] Thierry Jeulin. *Semi-martingales et grossissement d’une filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[27] Idris Kharroubi and Thomas Lim. Progressive enlargement of filtrations and backward stochastic differential equations with jumps. *Journal of Theoretical Probability*, 27(3):683–724, 2014.

[28] Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66 of *Stochastic Modelling and Applied Probability*. Springer, Heidelberg, second edition, 2012. With contributions by G. N. Milstein and M. B. Nevelson.

[29] Yoshio Komori, Alexey Eremin, and Kevin Burrage. Stability analysis of numerical methods using a linear test sde with delay and non-delay in a diffusion term. In *AIP Conference Proceedings*, volume 2293, page 100005. AIP Publishing LLC, 2020.

[30] Holger Kraft, Thomas Seiferling, and Frank Thomas Seifried. Optimal consumption and investment with Epstein-Zin recursive utility. *Finance and Stochastics*, 21(1):187–226, 2017.

[31] Guo Lei. Feedback and uncertainty: Some basic problems and results. *Annual Reviews in Control*, 49:27–36, 2020.

[32] Juan Li and Hui Min. Controlled mean-field backward stochastic differential equations with jumps involving the value function. *Journal of Systems Science and Complexity*, 29(5):1238–1268, 2016.

[33] Na Li, Guangchen Wang, and Zhen Wu. Linear–quadratic optimal control for time-delay stochastic system with recursive utility under full and partial information. *Automatica*, 121:109169, 2020.

[34] Xiaoyue Li and Xuerong Mao. Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control. *Automatica*, 112:108657, 2020.

[35] Alexandre Liapounoff. Problème général de la stabilité du mouvement. In *Annales de la Faculté des Sciences de Toulouse: Mathématiques*, volume 9, pages 203–474, 1907.

[36] Danfeng Luo, Mengquan Tian, and Quanxin Zhu. Some results on finite-time stability of stochastic fractional-order delay differential equations. *Chaos, Solitons & Fractals*, 158:111996, 2022.

[37] Heping Ma. Infinite horizon optimal control of mean-field forwaard-backward delayed systems with poisson jumps. *European Journal of Control*, 46:14–22, 2019.

[38] Jin Ma, Rentao Sun, and Yonghui Zhou. Kyle-Back equilibrium models and linear conditional mean-field sdes. *SIAM Journal on Control and Optimization*, 56(2):1154–1180, 2018.

[39] Xuerong Mao. *Exponential stability of stochastic differential equations*, volume 182 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1994.

[40] Xuerong Mao. Stability of stochastic differential equations with Markovian switching. *Stochastic Processes and their Applications*, 79(1):45–67, 1999.
[41] Xuerong Mao. *Stochastic Differential Equations and Applications*. Horwood Publishing Limited, Chichester, UK, second edition, 2008.

[42] Qingxin Meng and Yang Shen. Optimal control of mean-field jump-diffusion systems with delay: A stochastic maximum principle approach. *Journal of Computational and Applied Mathematics*, 279:13–30, 2015.

[43] Jun Moon. Risk-sensitive maximum principle for stochastic optimal control of mean-field type Markov regime-switching jump-diffusion systems. *International Journal of Robust and Nonlinear Control*, 31(6):2141–2167, 2021.

[44] Marek Musiela and Thaleia Zariphopoulou. Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 9(2):161–170, 2009.

[45] McSylvester Ejighikeme Omaba. Growth moment, stability and asymptotic behaviours of solution to a class of time-fractal-fractional stochastic differential equation. *Chaos, Solitons & Fractals*, 147:110958, 2021.

[46] Shige Peng and Xiaoming Xu. BSDEs with random default time and related zero-sum stochastic differential games. *Comptes Rendus. Mathématique*, 348(3-4):193–198, 2010.

[47] Huyê˘n Pham. Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. *Stochastic Processes and their Applications*, 120(9):1795–1820, 2010.

[48] Huyê˘n Pham and Xiaoli Wei. Bellman equation and viscosity solutions for mean-field stochastic control problem. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(1):437–461, 2018.

[49] Guangjun Shen, Rathinasamy Sakthivel, Yong Ren, and Mengyu Li. Controllability and stability of fractional stochastic functional systems driven by Rosenblatt process. *Collectanea Mathematica*, 71:63–82, 2020.

[50] Guangjun Shen, Jiang-Lun Wu, Ruidong Xiao, and Weijun Zhan. Stability of a non-Lipschitz stochastic Riemann-Liouville type fractional differential equation driven by Levy noise. *Acta Applicandae Mathematicae*, 180(1):2, 2022.

[51] Yang Shen and Tak Kuen Siu. A stochastic maximum principle for backward control systems with random default time. *International Journal of Control*, 86(5):953–965, 2013.

[52] Ajeet Singh, Anurag Shukla, V Vijayakumar, and R Udhayakumar. Asymptotic stability of fractional order \(1,2\) stochastic delay differential equations in Banach spaces. *Chaos, Solitons & Fractals*, 150:111095, 2021.

[53] Shiqi Song. Optional splitting formula in a progressively enlarged filtration. *ESAIM: Probability and Statistics*, 18:829–853, 2014.

[54] Jingrui Sun and Jiongmin Yong. Turnpike properties for mean-field linear-quadratic optimal control problems. *SIAM Journal on Control and Optimization*, 62(1):752–775, 2024.

[55] Daniel H. Wagner. Survey of measurable selection theorems: an update. In *Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979)*, volume 794 of *Lecture Notes in Math.*, pages 176–219. Springer, Berlin, 1980.

33
[56] Guangchen Wang and Zhen Wu. A maximum principle for mean-field stochastic control system with noisy observation. *Automatica*, 137:110135, 2022.

[57] Guangchen Wang, Hua Xiao, and Jie Xiong. A kind of LQ non-zero sum differential game of backward stochastic differential equation with asymmetric information. *Automatica*, 97:346–352, 2018.

[58] Wencan Wang and Yu Wang. Optimal control and stabilization for linear mean-field system with indefinite quadratic cost functional. *Asian Journal of Control*, 26(2):645–656, 2024.

[59] Qingmeng Wei. Stochastic maximum principle for mean-field forward-backward stochastic control system with terminal state constraints. *Science China Mathematics*, 59:809–822, 2016.

[60] Jiongmin Yong. Stochastic optimal control-A concise introduction. *Mathematical Control and Related Fields*, 12(4):1039–1136, 2022.

[61] Marc Yor. Grossissement de filtrations et absolue continuité de noyaux. In *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*, pages 110–146. Springer, Berlin, 1987.

[62] Feng Zhang. Stochastic maximum principle of mean-field jump-diffusion systems with mixed delays. *Systems & Control Letters*, 149:104874, 2021.

[63] Shuhua Zhang, Xinyu Wang, and Aleksandr Shanain. Modeling and computation of mean field equilibria in producers’ game with emission permits trading. *Communications in Nonlinear Science and Numerical Simulation*, 37:238–248, 2016.

[64] Ying Zhao and Quanxin Zhu. Stabilization by delay feedback control for highly nonlinear switched stochastic systems with time delays. *International Journal of Robust and Nonlinear Control*, 31(8):3070–3089, 2021.