Finsleroid-regular space: curvature tensor, continuation of gravitational Schwarzschild metric

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Abstract

The method of simple straightforward calculation of the curvature tensor of the Finsleroid–regular space is indicated. The Schwarzschild metric which underlines the gravitational field produced by static spherical-symmetric body is shown to be uniquely extended to the Finslerian domain upon a consistent treatment of the pseudo-Finsleroid axis vector field $b_i$ to be the field of the time variable.

*Keywords:* Finsler metrics, gravitational equations, curvature tensors.
1. Description of new conclusions

The regular Finsleroid-Finsler spaces \([1]\) of the positive-definite type \(\mathcal{FR}_{g;c}^{PD}\) and of the relativistic case \(\mathcal{FR}_{g;c}^{SR}\) are constructed with the help of the set \(\{g(x), b_i(x), a_{ij}(x), y\}\), where \(g(x)\) is a scalar which plays the role of the Finsleroid charge, \(b_i(x)\) is an involved vector field, and \(a_{ij}(x)\) is a metric tensor (which is positive-definite in the space \(\mathcal{FR}_{g;c}^{PD}\) and indefinite in the space \(\mathcal{FR}_{g;c}^{SR}\)). The norm

\[
|b| = c \equiv \sqrt{a_{ij}(x)b_i(x)b_j(x)}
\]  

(1.1)
of the input 1-form \(b = b_iy^i\) is taken to be an arbitrary positive scalar subject to the restrictions: \(0 < c(x) < 1\) in the positive-definite space \(\mathcal{FR}_{g;c}^{PD}\), and \(c(x) > 1\) in the indefinite space \(\mathcal{FR}_{g;c}^{SR}\). In the latter space, we assume the 1-form \(b\) to be time-like.

If we specify the Riemannian metric tensor according to

\[
a_{ij} = \frac{1}{c^2}b_ib_j + mu_{ij}
\]  

(1.2)
(with \(b^iu_{ij} = 0\) and \(b^i = a^{ij}b_j\); \(u_{ij}\) is the \((N - 1)\)-dimensional part of Euclidean metric tensor, in compliance with (A.1)-(A.7)) and assume the scalars \(c, m\) to be of the dependence \(c = c(r)\), \(m = m(r)\) on the radius variable \(r = \sqrt{u_{ij}x^ix^j}\) (assuming also \(c > 0\) and \(m \neq 0\)), we can readily conclude that the Riemannian covariant derivative

\[
\nabla_i b_j = \frac{\partial b_j}{\partial x^i} - b_n a^n_{ij}
\]  

(1.3)
\((a^n_{ij}\) stand for the Christoffel symbols constructed from the tensor \(a_{ij}(x)\)) fulfills the equality

\[
\nabla_i b_j = \frac{1}{c}(c_i b_j + c_j b_i)
\]  

(1.4)
(the respective calculations are presented in Appendix A); \(c_i = \partial c/\partial x^i\). The right-hand part of (1.4) does not involve the function \(m\). It follows that

\[
y^i y^j \nabla_i b_j = \frac{2}{c}b(yc),
\]  

(1.5)
where \((yc) = y^hc_h\). We have

\[
b_i c_i = 0.
\]  

(1.6)

Under the condition (1.2) the Finsleroid-regular spaces cannot be of neither Berwald case nor the Landsberg type. Obviously, the Landsberg characteristic condition \(\nabla_i b_j = kr_{ij}\) (derived and used in [2-5], and applied to cosmological consideration in [6]) is not realized when (1.4) takes place. The Berwald case would imply \(\nabla_i b_j = 0\) (see [1]), that is \(c_i = 0\).

At the same time, the Schwarzschild metric (which underlines the gravitational field produced by static spherical-symmetric body) belongs to the form (1.2). Therefore, the pseudo-Finsleroid regular space \(\mathcal{FR}_{g;c}^{SR}\) taken together with the representations (1.2)-(1.4) provides us with a straightforward continuation of the Riemannian Schwarzschild-metric framework into the pseudo-Finslerian domain with respect to the pseudo-Finsleroid charge \(g(x)\) (which thereby plays the role of the characteristic parameter of extension). The pseudo-Finsleroid axis vector field \(b_i(x)\) plays the role of the distribution of the time variable.
Key objects involved are explicitly calculated below in Appendix A upon using the metric tensor (1.2). Namely, calculating the associated Christoffel symbols $\gamma^{k}_{ij}$ leads to (A.31), thereafter the Riemannian curvature tensor $\sigma_{ik}^j$ (see the definition (A.32)) is obtained according to (A.39), and the entailed Ricci tensor $\sigma_{im}^i$ is given by (A.40)-(A.43). With these representations, we examine the particular case:

\[
c = \frac{1 + \frac{\xi}{4r}}{1 - \frac{\xi}{4r}}
\]  

(see (A.47)) and

\[
m = - \left(1 + \frac{\xi}{4r}\right)^4
\]  

(see (A.49)), obtaining

\[a_{ni^m} = 0 \quad \text{when} \quad N = 4.
\]

The respective curvature tensor $\sigma_{ik}^j$ is found to read (A.54). The vanishing (1.9) demonstrates that the metric tensor (1.2) with the choice (1.7) and (1.8) fulfills the gravitational field equations in vacuum. Raising forth the identification

\[
\xi = \text{r}_g,
\]

where $r_g$ is the gravitational radius of the massive static and spherical-symmetric body, we may conclude that in the dimension $N = 4$ the metric tensor (1.2) with the choice (1.7) and (1.8) is just the Schwarzschild metric tensor (cf. [8,9]).

It should be noted that $c < 1$ of the positive-definite space transforms to $c > 1$ in the relativistic counterpart with the timelike vector $b_i$. The reason is that the Riemannian metric

\[S^2 = b^2 + q^2\]

of the positive-definite space must be replaced by

\[S^2 = b^2 - q^2,
\]

when treating the relativistic case.

In Appendix B, using the spray coefficients found in the previous work [1], we evaluate the associated Finslerian curvature tensor $R^k_i$ under the only conditions that $\nabla_n b_m - \nabla_m b_n = 0$ and $g = \text{const}$, simultaneously keeping the general case $c = c(x)$ of the norm value $||b||$. The resultant representation is sufficiently simple and transparent and, therefore, opens up the operative possibility to evaluate the tensor

\[
\rho_{ij} := \frac{1}{2}(R^m_i m_j + R^m_{ijm}) - \frac{1}{2}g_{ij}R^m_{nm}
\]

and the current

\[
\rho^j := \rho^j_{ij}y^i
\]

(see [2,6]) which can be used to represent the energy-momentum distribution in the respective pseudo-Finsleroid-regular space-time. Various relativistic and other applications are possible.
Appendix A: Spherical-symmetric Riemannian background

Let us start with the (pseudo)Euclidean metric tensor $e_{ij}$, which inverse will be denoted by $e^{ij}$, so that $e_{ij}e^{jn} = \delta_i^n$, and introduce the decomposition

$$e_{ij} = e_i e_j + \epsilon u_{ij}, \quad \epsilon = \pm 1,$$  \hspace{1cm} (A.1)

where $e_i$ is a unit vector and

$$\text{rank}(u_{ij}) = N - 1, \quad y^i y^j u_{ij} \geq 0.$$  \hspace{1cm} (A.2)

The value $\epsilon = 1$ corresponds to the positive-definite case; in the relativistic case we must take $\epsilon = -1$. If we construct the contravariant vector

$$e^i = e^{ij} e_j,$$  \hspace{1cm} (A.3)

we obtain

$$e_i e^i = 1, \quad e^i u_{ij} = 0.$$  \hspace{1cm} (A.4)

We get also

$$u_i^j = u^{jm} u_{im}, \quad u^{ij} = e^{im} e^{jn} u_{mn},$$  \hspace{1cm} (A.5)

and

$$u^{ij} u_{jn} = \delta_i^i - e^i e^n, \quad u_i^j = \delta_i^j - e_i e^j.$$  \hspace{1cm} (A.6)

Let us set forth the identification

$$b_i = e_i,$$  \hspace{1cm} (A.7)

introduce two scalars $c > 0$ and $m \neq 0$, and specify the Riemannian metric tensor as follows:

$$a_{ij} := \frac{1}{c^2} b_i b_j + m u_{ij},$$  \hspace{1cm} (A.8)

which inverse reads

$$a^{ij} = \frac{1}{c^2} b^i b^j + \frac{1}{m} u^{ij},$$  \hspace{1cm} (A.9)

so that $a_{ij} a^{jn} = \delta_i^j$.

We use the contravariant vector

$$b^i = a^{ij} b_j,$$  \hspace{1cm} (A.10)

which entails

$$b^i u_{ij} = 0$$  \hspace{1cm} (A.11)

and

$$b^i b_i = c^2,$$  \hspace{1cm} (A.12)

together with

$$b^i = c^2 e^i.$$  \hspace{1cm} (A.13)

We introduce the radius variable

$$r = \sqrt{u_{ij} x^i x^j},$$  \hspace{1cm} (A.14)
obtaining

\[ n_i := \frac{1}{r} u_{ij} x^j = \frac{\partial r}{\partial x^i}, \quad (A.15) \]

\[ \frac{\partial n_i}{\partial x^j} = \frac{1}{r} \left( u_{ij} - n_i n_j \right), \quad (A.16) \]

\[ n^i = u^{ij} n_j = \epsilon^{ij} n_j, \quad (A.17) \]

and also

\[ n_i n_i = 1 \quad (A.18) \]

together with

\[ n_i b^i = 0. \quad (A.19) \]

We shall specify the scalars \( c, m \) to be of the dependence

\[ c = c(r), \quad m = m(r), \quad (A.20) \]

assuming \( c > 0 \) and \( m \neq 0 \). We shall also use the notation

\[ c_i = \frac{\partial c}{\partial x^i} \equiv c'i, \quad (A.21) \]

and

\[ c^i = a^{ij} c_j, \quad (A.22) \]

so that

\[ c^i = \frac{1}{m} c'n^i \quad (A.23) \]

and

\[ b^i c_i = 0. \quad (A.24) \]

Under these conditions, we obtain

\[ \frac{\partial a_{ij}}{\partial x^n} = n_n \left( -\frac{2c'}{c^3} b_i b_j + m' u_{ij} \right), \quad (A.25) \]

from which it just follows that

\[ b_n a^n_{ij} = \frac{1}{2} b^n \left[ n_i \left( -\frac{2c'}{c^3} b_n b_j + m' u_{nj} \right) + n_j \left( -\frac{2c'}{c^3} b_i b_n + m' u_{in} \right) + n_n \left( \frac{2c'}{c^3} c'b_i b_j - m' u_{ij} \right) \right], \]

or

\[ b_n a^n_{ij} = -\frac{1}{c} c'(n_i b_j + n_j b_i) = -\frac{1}{c} (c_i b_j + c_j b_i). \]

Thus for the Riemannian covariant derivative

\[ \nabla_i b_j = \frac{\partial b_j}{\partial x^i} - b_n a^n_{ij} \quad (A.26) \]

we obtain the simple expression

\[ \nabla_i b_j = \frac{1}{c} (c_i b_j + c_j b_i), \quad (A.27) \]
the right-hand part of which does not involve the function \( m \). It follows that
\[
y^i y^j \nabla_i b_j = \frac{2}{c} b(y c). \tag{A.28}
\]

We use the notation
\[
(y c) = y^h c_h. \tag{A.29}
\]

With (A.25), we also obtain
\[
c_n a^n_{ij} = \frac{1}{2} \frac{c}{c^3} \left[ n_i \left( -\frac{2}{c^3} c' b_n b_j + m' u_{nj} \right) + n_j \left( -\frac{2}{c^3} c' b_i b_n + m' u_{ni} \right) + n_n \left( \frac{2}{c^3} c' b_i b_j - m' u_{ij} \right) \right]
\]
and derive
\[
c_n a^n_{ij} = \frac{1}{2} \frac{1}{c} \left[ 2 m' n_i n_j + 2 \frac{c}{c^3} c' b_i b_j - m' u_{ij} \right],
\]
so that
\[
\nabla_i c_j = c'' n_i n_j + \frac{1}{c} \frac{1}{r} (u_{ij} - n_i n_j) - \frac{1}{2} \frac{1}{m} \left[ 2 m' n_i n_j + 2 \frac{c}{c^3} c' b_i b_j - m' u_{ij} \right]. \tag{A.30}
\]

Calculating the Christoffel symbols \( a^k_{ij} \) on the basis of (A.8) and (A.9) yields straightforwardly that
\[
a^k_{ij} = \frac{1}{2} a^{kn} \left[ n_i \left( -\frac{2}{c^3} c' b_n b_j + m' u_{nj} \right) + n_j \left( -\frac{2}{c^3} c' b_i b_n + m' u_{ni} \right) + n_n \left( \frac{2}{c^3} c' b_i b_j - m' u_{ij} \right) \right],
\]
or
\[
a^k_{ij} = -\frac{1}{c^3} c' b^k (n_i b_j + n_j b_i) + \frac{1}{2} \frac{1}{m} \left[ m' n_i u^k_j + m' n_j u^k_i + n^k \left( \frac{2}{c^3} c' b_i b_j - m' u_{ij} \right) \right]. \tag{A.31}
\]

We are aimed now to obtain the Riemannian curvature tensor
\[
a_{nm}^i = \frac{\partial a^i_{nm}}{\partial x^k} - \frac{\partial a^i_{nk}}{\partial x^m} + a^u_{nm} a^i_{uk} - a^u_{nk} a^i_{um}. \tag{A.32}
\]

To this end we write (A.31) in the form
\[
a_{nm}^i = -\frac{1}{c} c' e^i (n_i b_m + n_m b_n) + \frac{1}{2} \frac{1}{m} \left[ m' n_i u^i_m + m' n_m u^i_i + n^i \left( \frac{2}{c^3} c' b_i b_m - m' u_{nm} \right) \right] \tag{A.33}
\]
and get
\[
\frac{\partial a^i_{nm}}{\partial x^k} - \frac{\partial a^i_{nk}}{\partial x^m} = - \left( \frac{1}{c} c'' - \frac{1}{c^2} c' c' \right) e^i n_n (n_k b_m - n_m b_k)
\]
\[
- \frac{1}{c} e' \frac{1}{r} \left[ (u_{nk} - n_n n_k) b_m - (u_{nm} - n_n n_m) b_k \right] e^i.
\]
\[-\frac{1}{2} m' n_k \left[ m' n_m u_m - n^i \left( -\frac{2}{c^3} c' b_n b_m + m' u_{nm} \right) \right] \]

\[+ \frac{1}{2} m' n_k \left[ m' n_k u_k - n^i \left( -\frac{2}{c^3} c' b_n b_k + m' u_{nk} \right) \right] \]

\[+ \frac{1}{2} \left[ m'' n_k + m' \frac{1}{r} (u_{nk} - n_k n_k) \right] u_m^i \]

\[-\frac{1}{2} m \left[ m'' n_k + m' \frac{1}{r} (u_{nm} - n_k n_k) \right] u_k^i \]

\[+ \frac{1}{2} m \left( u_k^i - n^i n_k \right) \left( \frac{2}{c^3} c' b_n b_m - m' u_{nm} \right) \]

\[-\frac{1}{2} m \left( u_m^i - n^i n_m \right) \left( \frac{2}{c^3} c' b_n b_k - m' u_{nk} \right) \]

\[+ \frac{1}{m} n^i \left( \frac{1}{c^3} c'' - \frac{3}{c^4} c' c' \right) b_n (n_k b_m - n_m b_k) - \frac{1}{2} m'' (n_k u_{nm} - n_m u_{nk}) n^i, \]

or

\[\frac{\partial a_{nm}^i}{\partial x^k} - \frac{\partial a_{nk}^i}{\partial x^m} = -m' \frac{1}{m} \left( u_{nm} u_k^i - u_{kn} u_m^i \right) \]

\[-\frac{1}{c^3} \left( \frac{1}{c''} - \frac{1}{c^2} c' c' - \frac{11}{r c} c' \right) b_n (n_k b_m - n_m b_k) b^i \]

\[-\frac{1}{c^3} c' \frac{1}{r} (u_{nk} b_m - u_{nm} b_k) b^i \]

\[+ \frac{1}{2} \frac{m'}{m} \left[ n_k \left( -\frac{2}{c^3} c' b_n b_m + m' u_{nm} \right) - n_m \left( -\frac{2}{c^3} c' b_n b_k + m' u_{nk} \right) \right] n^i \]

\[+ \frac{1}{2} \left[ m'' - \frac{1}{r} \frac{m'}{m} \right] n (n_k u_m^i - n_m u_k^i) \]

\[+ \frac{1}{m} \frac{m'}{c^2} \left( \frac{1}{c''} - \frac{3}{c^2} c' c' - \frac{11}{r c} c' \right) b_n (n_k b_m - n_m b_k) n^i \]

\[-\frac{1}{2} \left( m'' - \frac{1}{r} \frac{m'}{m} \right) (n_k u_{nm} - n_m u_{nk}) n^i. \]
In this way we get

\[ \frac{\partial a^i_{nm}}{\partial x^k} - \frac{\partial a^i_{nk}}{\partial x^m} = -\frac{m'1}{m} \left( u_{mn}u_k^i - u_{kn}u_m^i \right) \]

\[ -\frac{1}{c^2} \left( \frac{1}{c} c'' - \frac{1}{c^2} c' c - \frac{11}{r} c' \right) \left[ n_n(n_kb_m - n_mb_k)b^i - \frac{1}{m} b_n(n_kb_m - n_mb_k)n^i \right] \]

\[ -\frac{1}{c^3} c' \frac{1}{r} (u_{nk}b_m - u_{nm}b_k)b^i - \frac{1}{m} m' \frac{1}{m} c'^3 b_n(n_kb_m - n_mb_k)n^i \]

\[ -\frac{11}{2} m \left[ \frac{m'' - 1}{r} m' - \frac{(m')^2}{m} \right] \left( n_n(n_m u_k^i - n_k u_m^i) - (n_m u_{nk} - n_k u_{nm}) n^i \right) \]

\[ + \frac{11}{m r c^3} c' b_n (b_m u_k^i - b_k u_m^i) - \frac{1}{m} \frac{1}{c^2} c' c_b(b_m u_k^i - b_k u_m^i) \]

\[ \frac{1}{m} c' c' c c \frac{1}{m} m' n_n(i n b_k - b_k i n) \]

\[ + \frac{1}{2} m m_n u_n \left[ -\frac{c'}{c} i (u_k b + n_k b) + \frac{1}{2} i \left( m_n u_k^i + m' n_k u_m^i + n^i \left( \frac{2c'}{c^3} b_n - m' u_{nk} \right) \right) \right] \]

\[ + \frac{1}{2} m \left( \frac{2}{c^3} c' b_n - m' u_{nm} \right) \left[ -\frac{1}{c} c' e b + \frac{1}{2} \frac{m'}{m} u_k^i \right] - [km] \]

\[ = \frac{1}{c^2} c' c_n b_m - b_k n_m) e^i - \frac{1}{c^4} c' c' \frac{1}{m} b_n(n_m b_k - b_k b_m) n^i \]

\[ -\frac{11}{2} c' c m' n_n(i n b_k - n_k b_m) e^i \]

\[ -\frac{11}{2} c' c m' n_n(n_m b_k - n_k b_m) e^i + \frac{1}{4} \left( \frac{m'}{m} \right)^2 \left( n_n(n_m u_k^i - n_k u_m^i) - (n_m u_{nk} - n_k u_{nm}) n^i \right) \]

\[ + \frac{1}{2} \frac{c'}{c} m' b_n(b_m u_k^i - b_k u_m^i) - \frac{1}{2} \frac{m'}{m} u_{nm} \left[ -\frac{c'}{c} e b + \frac{1}{2} \frac{m'}{m} u_k^i \right] + \frac{1}{2} \frac{m'}{m} u_{nk} \left[ -\frac{c'}{c} b + \frac{1}{2} \frac{m'}{m} u_m^i \right]. \]
Eventually,

\[ a_{nm} a_{uk}^i - a_{nk} a_{um}^i = \]

\[ = \frac{1}{c^4} c' c' \left[ n_n (b_m n_k - b_k n_m) b^i - b_n (n_m b_k - n_k b_m) \frac{1}{m} n^i \right] \]

\[ - \frac{1}{c^3} c' m' n_n (n_m b_k - n_k b_m) b^i \]

\[ + \frac{1}{4} \left( \frac{m'}{m} \right)^2 \left( n_n (n_m u_k^i - n_k u_m^i) - (n_m u_{nk} - n_k u_{nm}) n^i \right) \]

\[ + \frac{1}{2} c' m' n_n (b_m u_k^i - b_k u_m^i) \]

\[ - \frac{1}{2} m' u_{nm} \left[ - \frac{1}{c^3} c' b^i b_k + \frac{1}{2} m' u_k^i \right] + \frac{1}{2} m' u_{nk} \left[ - \frac{1}{c^3} c' b^i b_m + \frac{1}{2} m' u_m^i \right]. \quad \text{(A.35)} \]

Adding (A.35) to (A.34) we find the curvature tensor (A.32):

\[ a_{n}^{i \ km} = \]

\[ = - \frac{m'}{m} \left( \frac{1}{4} \frac{m'}{m} + \frac{1}{r} \right) \left( u_{mn} u_k^i - u_{kn} u_m^i \right) \]

\[ - \frac{1}{c^2} \left( \frac{1}{c^2 c'} - \frac{2}{c^2 c'} - 1 \right) \left( c' - c' m' \right) \left[ n_n (n_k b_m - n_m b_k) b^i - \frac{1}{m} b_n (n_k b_m - n_m b_k) n^i \right] \]

\[ - \frac{1}{2} m'' \left[ - \frac{1}{r} - \frac{1}{m'} \right] \left( n_n (n_m u_k^i - n_k u_m^i) - (n_m u_{nk} - n_k u_{nm}) n^i \right) \]

\[ + \frac{1}{2} c' \left( \frac{m'}{m} + \frac{2}{r} \right) \left[ \frac{1}{m} b_n (b_m u_k^i - b_k u_m^i) - (b_m u_{nk} - b_k u_{nm}) b^i \right]. \quad \text{(A.36)} \]

Inserting here

\[ m u_{mn} = a_{mn} - \frac{1}{c^2} b_m b_n \quad \text{(A.37)} \]

and

\[ u_i^k = \delta_i^k - \frac{1}{c^2} b_i b^k \quad \text{(A.38)} \]

yields the result

\[ a_{n}^{i \ km} = - \frac{1}{m} \frac{m'}{m} \left( \frac{1}{4} \frac{m'}{m} + \frac{1}{r} \right) \left( a_{mn} \delta_k^i - a_{kn} \delta_m^i \right) \]
\[
\begin{align*}
+ \frac{1}{c^2} \frac{m'}{m} \left( \frac{1}{4} \frac{m'}{m} + \frac{1}{r} \right) \left( \frac{1}{m} (a_{mn}b_k b^i - a_{kn}b_m b^i) + b_m b_n \delta^i_k - b_k b_n \delta^i_m \right) \\
- \frac{1}{c^2} \left( \frac{1}{c} c' - \frac{2}{c^2} c' c' - \frac{11}{r} c' - \frac{c' c'}{c m} \right) \left[ n_n (n_k b_m - n_m b_k) b^i - \frac{1}{m} b_n (n_k b_m - n_m b_k) n^i \right] \\
- \frac{1}{2} \left[ m'' - \frac{1}{r} m' - \frac{3 (m')^2}{2} \right] \left[ n_n (n_m u^i_k - n_k u^i_m) - (n_m u^i_k - n_k u^i_m) n^i \right] \\
+ \frac{1}{2} \left( \frac{m'}{m} + \frac{2}{r} \right) \left[ c^2 n_n n_m + \frac{1}{m} b_n b_m \right] \\
- \frac{1}{2} \left[ m'' - \frac{1}{r} m' - \frac{3 (m')^2}{2} \right] \left[ u_{nm} + (N - 3) n_n n_m \right] \\
+ \frac{1}{2} \left( \frac{m'}{m} + \frac{2}{r} \right) \left[ c^2 u_{nm} + (N - 1) \frac{1}{m} b_n b_m \right],
\end{align*}
\]

Contracting over two indices leads to the representation

\[
a_{n im} = - \frac{m'}{m} \left( \frac{1}{4} \frac{m'}{m} + \frac{1}{r} \right) (N - 2) u_{nm}
\]

\[
+ \frac{1}{c^2} \left( \frac{1}{c} c'' - \frac{2}{c^2} c' c' - \frac{11}{r} c' - \frac{c' c'}{c m} \right) \left[ c^2 n_n n_m + \frac{1}{m} b_n b_m \right]
\]

\[
- \frac{1}{2} \left[ m'' - \frac{1}{r} m' - \frac{3 (m')^2}{2} \right] \left[ u_{nm} + (N - 3) n_n n_m \right]
\]

\[
+ \frac{1}{2} \left( \frac{m'}{m} + \frac{2}{r} \right) \left[ c^2 u_{nm} + (N - 1) \frac{1}{m} b_n b_m \right],
\]

which can be written as

\[
a_{n im} = n_1 u_{nm} + \frac{1}{c^2} \frac{1}{m} n_2 b_m b_n + n_3 n_m n_n
\]

with

\[
n_1 = -(N - 2) \frac{m'}{m} \left( \frac{1}{4} \frac{m'}{m} + \frac{1}{r} \right) - \frac{1}{2} \left[ m'' - \frac{1}{r} m' - \frac{3 (m')^2}{2} \right] + \frac{1}{2} \left( \frac{m'}{m} + \frac{2}{r} \right),
\]

\[
n_2 = \left( \frac{1}{c} c'' - \frac{2}{c^2} c' c' - \frac{11}{r} c' - \frac{c' c'}{c m} \right) + \frac{1}{2} \frac{c'}{c} \left( \frac{m'}{m} + \frac{2}{r} \right) (N - 1),
\]

and

\[
n_3 = \left( \frac{1}{c} c'' - \frac{2}{c^2} c' c' - \frac{11}{r} c' - \frac{c' c'}{c m} \right) - \frac{1}{2} \frac{1}{m} \left[ m'' - \frac{1}{r} m' - \frac{3 (m')^2}{2} \right] (N - 3).
\]
Let us make the substitution
\[ c = y \left( \frac{\xi}{4r} \right), \quad m = z \left( \frac{\xi}{4r} \right), \quad \xi = \text{const}, \] (A.44)
so that
\[ c' = -\frac{\xi}{4r^2} y', \quad m' = -\frac{\xi}{4r^2} z', \] (A.45)
and
\[ c'' = \frac{\xi}{2r^3} y' + \frac{\xi^2}{16r^4} y'', \quad m'' = \frac{\xi}{2r^3} z' + \frac{\xi^2}{16r^4} z'', \] (A.46)
\[ \frac{1}{c} c'' - \frac{2}{c^2} c' c' - \frac{1}{r} c' = \frac{\xi}{2r^3} y' + \frac{\xi^2}{16r^4} y'' - \frac{2\xi}{4r^2} y' y' + \frac{\xi}{4r^3} y', \]
and examine the particular case
\[ c = 1 + \frac{\xi}{4r}, \quad \xi > 0. \] (A.47)
We have
\[ y' = 2 \frac{1}{\left( 1 - \frac{\xi}{4r} \right)^2}, \quad y'' = 4 \frac{1}{\left( 1 - \frac{\xi}{4r} \right)^3}, \] (A.48)
whence
\[ \frac{1}{c} c'' - \frac{2}{c^2} c' c' - \frac{1}{r} c' = 3 \frac{\xi}{4r^3} y' + \frac{\xi^2}{16r^4} y'' - \frac{2\xi}{4r^2} y' y' + \frac{\xi}{4r^3} y'. \]
\[ = 6 \frac{1}{r^2} \frac{\xi}{4r} \left( \frac{\xi}{4r} \right)^2 + 4 \frac{1}{r^2} \frac{\xi^2}{16r^4} \left( 1 - \frac{\xi}{4r} \right)^2 - 8 \frac{1}{r^2} \frac{\xi^2}{16r^4} \left( 1 - \frac{\xi}{4r} \right)^2 \]
\[ = 6 \frac{1}{r^2} \frac{\xi}{4r} \left( 1 + \frac{\xi}{4r} \right) \left( 1 - \frac{\xi}{4r} \right) - 4 \frac{1}{r^2} \frac{\xi^2}{16r^4} \left( 1 + \frac{\xi}{4r} \right)^2 \left( 1 - \frac{\xi}{4r} \right)^2 \]
\[ = -\frac{1}{r^2} \frac{\xi}{4r} \left( 1 + \frac{\xi}{4r} \right) \left( 1 - \frac{\xi}{4r} \right) \left[ -6 + 4 \frac{\xi^2}{4r} \left( 1 + \frac{\xi}{4r} \right) \right]. \]
We shall confine the treatment to the relativistic \( \epsilon = -1 \):
\[ m = - \left( 1 + \frac{\xi}{4r} \right)^4, \] (A.49)
obtaining
\[
\frac{m'}{m} + \frac{2}{r} = -4 \frac{\frac{\xi}{4r}}{r} \left(1 + \frac{\xi}{4r}\right) + \frac{2}{r}
\]
and
\[
\frac{1}{2} c' \left(\frac{m'}{m} + \frac{2}{r}\right) = -\frac{1}{r} \frac{\frac{\xi}{4r}}{r} \left(1 + \frac{\xi}{4r}\right) \left(1 - \frac{\xi}{4r}\right) \left[-4 \frac{\frac{\xi}{4r}}{r} \left(1 + \frac{\xi}{4r}\right) + \frac{2}{r}\right] = -2 \frac{\frac{\xi}{4r}}{r^2} \left(1 + \frac{\xi}{4r}\right)^2,
\]
\[
\frac{m'}{m} \left(\frac{1}{4} m' + \frac{1}{r}\right) = -4 \frac{\frac{\xi}{4r}}{r} \left(1 + \frac{\xi}{4r}\right) \left[-\frac{1}{r} \frac{\frac{\xi}{4r}}{r} + \frac{1}{r}\right] = -4 \frac{\frac{\xi}{4r}}{r^2} \left(1 + \frac{\xi}{4r}\right)^2,
\]
and also
\[
\left(\frac{m'}{m}\right)' = -\frac{2 m'}{r m} + \frac{\xi}{4 r^2 m} \frac{1}{1 + \frac{\xi}{4r}}.
\]
Therefore,
\[
\frac{1}{m} \left(m'' - \frac{1}{r} m' - \frac{3 (m')^2}{2 m}\right) = \left(\frac{m'}{m}\right)' - \frac{1}{r m} \frac{1}{1 - \frac{1}{2} (m')^2} = \frac{m'}{m} \left[-\frac{2}{r} + \frac{\xi}{4 r^2} \frac{1}{1 + \frac{\xi}{4r}} - \frac{1}{r} + 2 \frac{\xi}{4 r^2} \frac{1}{1 + \frac{\xi}{4r}}\right]
\]
\[
= \frac{m'}{m} 3 \frac{\xi}{4r} \frac{1}{1 + \frac{\xi}{4r}} = -\frac{m'}{m} 3 \frac{1}{r m} \frac{1}{1 + \frac{\xi}{4r}} = 12 \frac{\frac{\xi}{4r}}{r^2} \left(1 + \frac{\xi}{4r}\right)^2,
\]
whence we have
\[n_1 = 0.\]  \hfill (A.50)

Thereafter, we obtain
\[
\frac{1}{c' c''} - \frac{2}{c^2} c' c' - \frac{1}{r} c' c - \frac{c' m'}{c m} =
\]
\[
= -\frac{1}{r^2} \frac{\xi}{4r} \left(1 + \frac{\xi}{4r}\right) \left(1 - \frac{\xi}{4r}\right) \left[-6 + 4 \frac{\xi}{4r} \left(1 + \frac{\xi}{4r}\right)\right] - 2 \frac{\xi}{4r} \frac{1}{r} \frac{\xi}{4r} \frac{4}{r} \frac{\xi}{4r} \left(1 + \frac{\xi}{4r}\right)^2
\]
\[= 6 \frac{1}{r^2} \left( \frac{\xi}{4r} \right) \left[ 1 + \frac{\xi}{4r} \right] \left[ 1 - 2 \frac{\xi}{4r} \left( 1 + \frac{\xi}{4r} \right) \right] = 6 \frac{1}{r^2} \left( \frac{\xi}{4r} \right)^2,\]

whence

\[n_2 = n_3 = 0. \quad (A.51)\]

The conclusions (A.50) and (A.51) tell us that

\[a_{n \ i \ m} = 0 \quad \text{when} \quad N = 4. \quad (A.52)\]

Under these conditions, the tensor (A.39) reduces to

\[a_{n \ i \ km} = 2 \frac{1}{r^2} \left( \frac{\xi}{4r} \right)^2 \left[ 2 \left( u_{mn} u_k^i - u_{kn} u_m^i \right) - \frac{3}{c^2} \left( n_n b_m - n_m b_k \right) b^i - \frac{1}{m} b_n (n_k b_m - n_m b_k) n^i \right.\]

\[-3 \left( n_n (n_m u_k^i - n_k u_m^i) - (n_m u_{nk} - n_k u_{nm}) n^i \right)\]

\[-\frac{1}{c^2} \left( \frac{1}{m} b_n (b_m u_k^i - b_k u_m^i) - (b_m u_{nk} - b_k u_{nm}) b^i \right) \left( 1 + \frac{\xi}{4r} \right) \left( 1 + \frac{\xi}{4r} \right)^2\]

\[= \frac{2}{r^2} \left( \frac{\xi}{4r} \right)^2 \left[ 2 \left( u_{mn} u_k^i - u_{kn} u_m^i \right) - \frac{3}{c^2} \left( n_n b_m - n_m b_k \right) b^i - \frac{1}{m} b_n (n_k b_m - n_m b_k) n^i \right.\]

\[-3 \left( n_n (n_m u_k^i - n_k u_m^i) - (n_m u_{nk} - n_k u_{nm}) n^i \right)\]

\[-\frac{1}{c^2} \left( \frac{1}{m} b_n (b_m u_k^i - 3n_k n^i) - b_k (u_m^i - 3n_m n^i) \right) - \left( b_m (u_{nk} - 3n_n n_k) - b_k (u_{nm} - 3n_n n_m) \right)b^i \left( 1 + \frac{\xi}{4r} \right) \left( 1 + \frac{\xi}{4r} \right)^2.\]

Contracting the last tensor by the vector \( b \) yields the simple result, namely,

\[b^m a_{n \ i \ km} = 2 \frac{1}{r^2} \left( \frac{\xi}{4r} \right)^2 \left[ - \left( \frac{1}{m} b_n (u_k^i - 3n_k n^i) - (u_{nk} - 3n_n n_k) b^i \right) \right] \quad (A.55)\]
and
\[ b^n a^n_{ikm} = \frac{2}{r^2} \frac{\xi}{4r} \left( \frac{1}{1 + \frac{\xi}{4r}} \right)^2 \left[ -\frac{1}{m} \left( b_m (u^n_i - 3n_k n^i) - b_k (u^n_m - 3n_m n^i) \right) \right], \tag{A.56} \]

from which it follows that
\[ (b^m b^n b^p y^p - b^m y^n - b^n y^m) a^n_{ikm} = \]
\[ \frac{2}{r^2} \frac{\xi}{4r} \left( \frac{1}{1 + \frac{\xi}{4r}} \right)^2 \left[ -(u_{nk} - 3n_n n_k) b^n y^p + \frac{1}{m} b(u^n_i - 3n_k n^i) - \frac{1}{m} b_k (u^n_m - 3n_m n^i) y^m \right]. \tag{A.57} \]

**Appendix B: Curvature tensor of the space $\mathcal{F}_g : c$**

Below the evaluations are restricted by the particular conditions
\[ \nabla_n b_m - \nabla_m b_n = 0, \quad g = \text{const}, \tag{B.1} \]

simultaneously keeping the general case
\[ c = c(x) \tag{B.2} \]

of the norm value $||b||$.

In terms of the convenient notation
\[ (ys) = y^j y^h \nabla_j b_h \tag{B.3} \]

the spray coefficients of the space $\mathcal{F}_g : c$ read
\[ G^i = \frac{g}{\nu} (ys) v^i + 2 a^i_{km} y^m y^k \tag{B.4} \]

(see [1]), entailing
\[ G^i_k = -\frac{1}{\nu} \nu_k \frac{g}{\nu} (ys) v^i + 2 \frac{g}{\nu} s_k v^i + \frac{g}{\nu} (ys) v^i_k + 2 a^i_{km} y^m, \tag{B.5} \]

where $s_k = y^h \nabla_k b_h$ and
\[ \nu_k = \frac{1}{q} v_k + (1 - c^2) g b_k. \tag{B.6} \]

The derivative
\[ \frac{\partial \left( \frac{1}{\nu} \nu_k \right)}{\partial y^m} = -\frac{1}{\nu^2} \nu_k \nu_m + \frac{1}{\nu} \frac{1}{q} \eta_{km} \tag{B.7} \]
will be applied.

Differentiating (B.4) with respect to \( y^m \) yields the result

\[
G^i_{km} = -\frac{1}{\nu^2} \nu_k \left( -\frac{1}{\nu^2} \nu_m \frac{g(y^s)}{\nu} v^i + 2\frac{g}{\nu} s_m v^i + \frac{g}{\nu} (y^s) r^i_m \right) - \left( -\frac{1}{\nu^2} \nu_k \nu_m + \frac{11}{\nu^2 q} \eta_{km} \right) \frac{g(y^s)}{\nu} v^i \\
- 2\frac{g}{\nu^2} \nu_m s_k v^i + 2 \frac{g}{\nu} (\nabla_m b_k) v^i + 2 \frac{g}{\nu} s_k r^i_m \\
- \frac{g}{\nu^2} (y^s) \nu_m r^i_k + 2 \frac{g}{\nu} s_m r^i_k + \Delta,
\]

which can conveniently be written as follows:

\[
G^i_{km} = - \left( \frac{2}{\nu^2} \nu_k \nu_m + \frac{1}{\nu^2 q} \eta_{km} \right) \frac{g(y^s)}{\nu} v^i + 2 \frac{g}{\nu} (\nabla_m b_k) v^i \\
- 2 \frac{g}{\nu^2} (\nu_m s_k + \nu_k s_m) v^i + 2 \frac{g}{\nu} (s_k r^i_m + s_m r^i_k) - \frac{g}{\nu^2} (y^s) (\nu_m r^i_k + \nu_k r^i_m) + 2 a^i_{km}. \tag{B.8}
\]

Next, we perform required differentiation with respect to \( x^k \), obtaining

\[
2 \frac{\partial G^i}{\partial x^k} = \frac{\partial G^i}{\partial x^k} = - \frac{g}{\nu^2} (y^s) \left( q_{k} + g(1 - c^2)b_{j,k}y^j - 2gbcc_k \right) v^i \\
+ \frac{g}{\nu^2} y^m y^n \nabla_k \nabla_m b_n - \frac{g}{\nu} (y^s) (b_{j,k}y^j b^i + b b^i_k) + \Delta.
\]

Since

\[
q_{k} = -\frac{1}{q} bb_{j,k}y^j = -\frac{1}{q} bs_k + \Delta, \tag{B.9}
\]

we can write

\[
\frac{\partial G^i}{\partial x^k} = \frac{g}{q \nu^2} (y^s) \left( b - gq(1 - c^2) \right) s_k v^i + \frac{g}{\nu^2} (y^s) 2gbcc_k v^i \\
+ \frac{g}{\nu^2} y^m y^n \nabla_k \nabla_m b_n - \frac{g}{\nu} (y^s) (s_k b^i + b\nabla_k b^i) + \Delta. \tag{B.10}
\]

Also,

\[
y^j \frac{\partial G^i}{\partial x^j} = \frac{g}{q \nu^2} (y^s) \left( b - gq(1 - c^2) \right) (y^s) v^i + \frac{g}{\nu^2} (y^s) 2gbc(y^c) v^i \\
+ \frac{g}{\nu^2} y^m y^n \nabla_j \nabla_m b_n - \frac{g}{\nu} (y^s) ((y^s) b^i + b s^i) + \Delta
\]

with the vector

\[
e_k = \frac{b}{q^2} v_k - b_k, \tag{B.11}
\]

which obeys the equality

\[
\frac{\partial e_k}{\partial y^j} = \frac{b}{q^2} \eta_{kj} - \frac{1}{q^2} v_k e_j.
\]
After that, we derive the equality

\[ \nu^2 \frac{\partial \left( y^i \frac{\partial G^i}{\partial y^j} \right)}{\partial y^k} = -\frac{g}{q\nu} \nu_k(ys) \left( b - gq(1 - c^2) \right)(ys)v^i - \frac{g}{q^3} v_k(ys) \left( b - gq(1 - c^2) \right)(ys)v^i \]

\[ +2 \frac{g}{q}(ys) \left( b - gq(1 - c^2) \right) s_k v^i + \frac{g}{q} (ys) \left( b - gq(1 - c^2) \right)(ys)r^i_k \]

\[ + \frac{g}{q}(ys) \left( b_k - \frac{1}{q} v_k(1 - c^2) \right)(ys)v^i \]

\[ -4 \frac{1}{\nu} \nu_k g(ys)gbc(yc)v^i + g(ys)2gbc(yc)v^i + g(ys)2gbcc_k v^i \]

\[ + g(ys)2gbc(yc)r^i_k + gs_k 2gbc(yc)v^i \]

\[ - g\nu v^i y^j y^m y^n \nabla_j \nabla_m b_n + g\nu y^i y^m y^n \nabla_j \nabla_m b_n r^i_k \]

\[ + g\nu v^i \left[ \delta^i_k y^m y^n + 2y^j y^m \delta^{n}_k \right] \nabla_j \nabla_m b_n \]

\[ + g\nu(ys)(b^i + bs^i) - g\nu s_k ((ys)b^i + bs^i) - g\nu(ys)(s_k b^i + b_k s^i) + \Delta. \]

With these observations, we find

\[ \nu^2 \left[ 3 \frac{\partial G^i}{\partial x^k} - \frac{\partial \left( y^i \frac{\partial G^i}{\partial x^j} \right)}{\partial x^k} \right] = \]

\[ \frac{3g}{q}(ys) \left( b - gq(1 - c^2) \right)s_k v^i + 3g(ys)2gbcc_k v^i \]

\[ + 3g\nu v^i y^m y^n \nabla_k \nabla_m b_n - 3g\nu(ys)s_k b^i - 3g\nu(ys)b\nabla_k b^i \]

\[ + 2 \frac{g}{q\nu} \nu_k(ys) \left( b - gq(1 - c^2) \right)(ys)v^i + \frac{g}{q^3} v_k(ys) \left( b - gq(1 - c^2) \right)(ys)v^i \]

\[ - 2 \frac{g}{q}(ys) \left( b - gq(1 - c^2) \right)s_k v^i - \frac{g}{q} (ys) \left( b - gq(1 - c^2) \right)(ys)r^i_k \]

\[ - \frac{g}{q}(ys) \left( b_k - \frac{1}{q} v_k(1 - c^2) \right)(ys)v^i \]
\[ + \frac{1}{\nu} \nu_k g(y) g_{bc}(yc) v^i - g(y) 2g_{bc}(yc) v^i - g(y) 2g_{cc} v^i \]

\[ - g(y) 2g_{bc}(yc) r^i_k - gs_k 2g_{bc}(yc) v^i \]

\[ + g \nu_k v^i y^j y^m y^n \nabla_j \nabla_m b_n - g \nu y^j y^m y^n \nabla_j \nabla_m b_n r^i_k \]

\[ - g \nu v^i \left[ \delta^j_k y^m y^n + 2y^j y^m \delta^n_k \right] \nabla_j \nabla_m b_n \]

\[ - g \nu_k (y) (y) b^i + bs^i + g \nu s_k ((y) b^i + bs^i) + g \nu (y) (s_k b^i + b_k s^i) + \Delta \]

\[ = 2 g \frac{q}{q} (b - gq(1 - c^2)) s_k v^i + 4 g(y) g_{cc} v^i \]

\[ + 2 g \nu v^i y^m y^n \nabla_k \nabla_m b_n - g \nu (y) s_k b^i - 3 g \nu (y) b \nabla_k b^i \]

\[ + 2 g \frac{q}{q} \nu_k (y) \left( b - gq(1 - c^2) \right) (y) v^i + \frac{g}{q^3} \nu_k (y) \left( b - gq(1 - c^2) \right) (y) v^i \]

\[ - g \frac{q}{q} (y) \left( b - gq(1 - c^2) \right) (y) r^i_k \]

\[ - g \frac{q}{q} (y) \left( b_k - g \frac{1}{q} \nu_k (1 - c^2) \right) (y) v^i \]

\[ + \left( \frac{1}{\nu} \nu_k g(y) g_{cv} - 2 g(y) g_{bc} v^i - 2 g(y) g_{cr} v^i - 2 g s_k g_{cv} \right) (yc) \]

\[ - g \left( y^j y^m y^n \nabla_j \nabla_m b_n \right) (\nu k^i - \nu k v^i) - 2 g \nu \left( y^j y^m \nabla_j \nabla_m b_k \right) v^i \]

\[ - g \nu_k (y) (y) b^i + \left[ - g \nu v (y) + g \nu s_k + g \nu (y) b_k \right] s^i + \Delta. \]

Finally,

\[ \nu^2 \left[ 3 \frac{\partial G^i}{\partial x^k} - \frac{\partial \left( y^j \frac{\partial G^i}{\partial x^j} \right)}{\partial x^k} \right] = \]

\[ - g \left( y^j y^m y^n \nabla_j \nabla_m b_n \right) (\nu r^i_k - \nu_k v^i) + 2 g \nu y^j y^m \left( \nabla_k \nabla_m b_j - \nabla_j \nabla_m b_k \right) v^i \]
\[-3g\nu(y_s)b\nabla_k b^i + g \left[ \nu_k - \nu_k(y_s) + \frac{1}{b} \nu(y_s)b_k \right] \left( b_s^i - (y_s)b^i \right) + \frac{1}{b} g\nu(y_s)(y_s)b_kb^i \]

\[+ \left( 4 - \frac{1}{b} \nu_k(y_s)gbcv^i - 2g(y_s)gb_kcv^i - 2g(y_s)gbcr^i_k - 2gs_kgbcv^i \right)(yc) + 4g(y_s)gbcc_kv^i \]

\[+ 2g(y_s) \left( b - gq(1 - c^2) \right) s_kv^i \]

\[+ 2g(y_s) \left( b - gq(1 - c^2) \right)(ys)\nu_kv^i \]

\[- \frac{g}{q}(ys) \left( b - gq(1 - c^2) \right)(ys)\eta^i_k \]

\[- \frac{g}{q}(ys) \left( b_k - \frac{1}{q} \nu_k(1 - c^2) \right)(ys)v^i + \Delta. \tag{B.12} \]

After that, we evaluate

\[\nu^A G^i_j G^j_k = \left[ -\nu_j g(y_s)v^i + 2g\nu s_j v^i + g\nu(y_s)r^j_i \right] \left[ -\nu_k g(y_s)v^j + 2g\nu s_k v^j + g\nu(y_s)r^j_k \right] = \]

\[-\nu_j g(y_s)v^i \left[ -\nu_k g(y_s)v^j + 2g\nu s_k v^j + g\nu(y_s)r^j_k \right] \]

\[+ 2g\nu s_j v^i \left[ -\nu_k g(y_s)v^j + 2g\nu s_k v^j + g\nu(y_s)r^j_k \right] \]

\[+ g\nu(y_s)r^j_i \left[ -\nu_k g(y_s)v^j + 2g\nu s_k v^j + g\nu(y_s)r^j_k \right] \]

and arrive at

\[2\nu^A G^m G^i_{km} = 2g(y_s)v^m \left( 2\nu_k \nu_m - \frac{1}{q} \eta_{km} \right) g(y_s)v^i + 2g(y_s)v^m 2g^2(\nabla_m b_k)v^i \]

\[- 2g\nu(y_s)v^m 2g(\nu_m s_k + \nu_k s_m)v^i + 2g^2(y_s)v^m 2g(s_k r^i_m + s_m r^i_k) \]

\[- g\nu(y_s)v^m g(y_s)(\nu_m r^i_k + \nu_k r^i_m) + \Delta. \tag{B.13} \]

Therefore,

\[2\nu^A G^i_j G^j_k - \nu^A G^i_j G^j_k = \]
\begin{align*}
2g^2(ys)v^m \left( 2\nu_k \nu_m - \nu \frac{1}{q} \eta_{km} \right) (ys)v^i + 4g^2(ys)v^m \nu^2 (\nabla_m b_k)v^i \\
-4g^2(ys)v^m(\nu_m s_k + \nu_k s_m)v^i + 4g^2\nu^2(ys)v^m s_k r^i_m + 4g^2\nu^2(ys)v^m s_m r^i_k \\
- g^2\nu(ys)v^m(ys)(\nu_m r^i_k + \nu_k r^i_m) \\
+ \nu_j g^2(ys) \left[ -\nu_k(ys)v^j + 2\nu s_k v^j + \nu(ys)r^j_k \right] v^i \\
+ 2g^2\nu s_j \left[ \nu_k(ys)v^j - 2\nu s_k v^j - \nu(ys)r^j_k \right] v^i \\
+ g^2\nu(ys)r^i_j \left[ \nu_k(ys)v^j - 2\nu s_k v^j - \nu(ys)r^j_k \right] + \Delta,
\end{align*}

or, on reducing similar terms,

\begin{align*}
2\nu^4 \bar{G}^i \bar{G}^i_{kj} - \nu^4 \bar{G}^i_j \bar{G}^j_k =
\end{align*}

\begin{align*}
2g^2(ys)v^m \left( 2\nu_k \nu_m - \nu \frac{1}{q} \eta_{km} \right) (ys)v^i + 4g^2(ys)v^m \nu^2 (\nabla_m b_k)v^i \\
-2g^2\nu(ys)v^m(\nu_m s_k + \nu_k s_m)v^i + 2g^2\nu^2(ys)v^m s_k r^i_m + 4g^2\nu^2(ys)v^m s_m r^i_k \\
- g^2\nu(ys)v^m(ys)\nu_m r^i_k \\
+ \nu_j g^2(ys) \left[ -\nu_k(ys)v^j + \nu(ys)r^j_k \right] v^i \\
+ 2g^2\nu s_j \left[ -2\nu s_k v^j - \nu(ys)r^j_k \right] v^i \\
- g^2\nu(ys)r^i_j \nu(ys)r^j_k + \Delta.
\end{align*}

On applying the contractions

\begin{align*}
v^m s_m = (ys) - b \sigma, \quad v^i r^j_i = v^i - (1 - c^2)bb^j,
\end{align*}

(B.14)
\[ r^j_m r^i_j = r^i_m - (1 - c^2)b^j b_m, \quad v_j v^j = q^2 - (1 - c^2)b^2, \]  
(B.16)

and
\[ v_j v^j = \nu - (1 - c^2)\frac{1}{q}(b^2 + gc^2bq) \]  
(B.17)

(see [1]), the above expression reads
\[
2\nu^A G^i_{kj} - \nu^A G^i_j G^j_k = 4g^2(ys)\nu_k \left[ \nu - (1 - c^2)\frac{1}{q}(b^2 + gc^2bq) \right] (ys) v^i \\
-2g^2(ys)\nu \frac{1}{q} \left[ v_k - (1 - c^2)bb_k - \frac{1}{q^2}v_k(q^2 - (1 - c^2)b^2) \right] (ys) v^i \\
+4g^2(ys)\nu^2(s_k - b\sigma_k)v^j \\
-2g^2\nu(ys) \left[ \nu - (1 - c^2)\frac{1}{q}(b^2 + gc^2bq) \right] s_k + \nu_k \left( (ys) - b\sigma \right) v^i \\
+2g^2\nu^2(ys)s_k \left[ v^i - (1 - c^2)bb^i \right] \\
+4g^2\nu^2(ys) \left( (ys) - b\sigma \right) r^i_k \\
-g^2\nu(ys) (ys) \left[ \nu - (1 - c^2)\frac{1}{q}(b^2 + gc^2bq) \right] r^i_k \\
+\nu_j g^2(ys)(ys) v^i - 2g^2\nu s_j \left[ (ys)r^j_k + 2s_k v^j \right] v^i \\
-g^2\nu^2(ys) (ys) \left[ r^i_k - (1 - c^2)b^i b_k \right] + \Delta. \]  
(B.18)

For the $hh$-curvature tensor $R^i_k$ we may use the formula
\[
K^2 R^i_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2\overline{G}^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} \\
= 2\frac{\partial \overline{G}^i}{\partial x^k} - \overline{G}^i_j \overline{G}^j_k - y^j \frac{\partial \overline{G}^i_k}{\partial x^j} + 2\overline{G}^j \overline{G}^i_{kj} \]  
(B.19)

(which is tantamount to the definition (3.8.7) on p. 66 of the book [7]); we apply the notation
\[
\overline{G}^i = \frac{1}{2}G^i, \quad \overline{G}^i_k = \frac{1}{2}G^i_k, \quad \overline{G}^i_{km} = \frac{1}{2}G^i_{km}, \quad \overline{G}^i_{kmn} = \frac{1}{2}G^i_{kmn}. \]
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