Solitary wave solutions for nonlinear partial differential equations containing monomials of odd and even grades with respect to participating derivatives

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Abstract

We apply the method of simplest equation for obtaining exact solitary traveling-wave solutions of nonlinear partial differential equations that contain monomials of odd and even grade with respect to participating derivatives. We consider first the general case of presence of monomials of the both (odd and even) grades and then turn to the two particular cases of nonlinear equations that contain only monomials of odd grade or only monomials of even grade. The methodology is illustrated by numerous examples.

Keywords: Nonlinear partial differential equations, Method of simplest equation, Solitary wave solutions.

1. Introduction

Traveling waves exist in many natural systems. Because of this traveling wave solutions of the nonlinear partial differential equations are studied much in the last decades [1]-[11] and effective methods for obtaining such solutions for integrable systems are developed [12], [13]. Our discussion below will be based on a specific approach for obtaining exact special solutions of nonlinear PDEs: the method of simplest equation and its version called modified method of simplest equation [14] - [17]. The method of simplest equation is based on a procedure analogous to the first step of the test for the Painleve property [18]-[20]. In the version of the method called modified method of the
simplest equation [16, 17] this procedure is substituted by the concept for the balance equation. This version of the method of simplest equation has been successfully applied for obtaining exact traveling wave solutions of numerous nonlinear PDEs such as versions of generalized Kuramoto - Sivashinsky equation, reaction - diffusion equation, reaction - telegraph equation [16], [21] generalized Swift - Hohenberg equation and generalized Rayleigh equation [17], extended Korteweg-de Vries equation [22], etc. [23] - [28].

The method of simplest equation works as follows. By means of an appropriate ansatz the solved nonlinear PDE is reduced to a nonlinear ODE

\[ P(u, u_\xi, u_{\xi\xi}, \ldots) = 0. \] (1)

Then the finite-series solution \( u(\xi) = \sum_{\mu=-\nu}^{\nu} p_\mu [f(\xi)]^\mu \) is substituted in (1). \( p_\mu \) are coefficients and \( f(\xi) \) is solution of simpler ordinary differential equation called simplest equation. Let the result of this substitution be a polynomial of \( f(\xi) \). \( u(\xi) \) is a solution of Eq.(1) if all coefficients of the obtained polynomial of \( f(\xi) \) are equal to 0. This condition leads to a system of nonlinear algebraic equations. Each non-trivial solution of this system corresponds to a solution of the studied nonlinear PDE.

Below we shall consider traveling-wave solutions \( u(x, t) = u(\xi) = u(\alpha x + \beta t) \) constructed on the basis of the simplest equation

\[ f^2_\xi = 4(f^2 - f^3), \] (2)

which solution is \( f(\xi) = \frac{1}{\cosh^2(\xi)} \). \( \alpha \) and \( \beta \) are parameters. In addition we shall use the following concept of grade of monomial with respect to participating derivatives. Let us consider polynomials that are linear combination of monomials where each monomial contains product of terms consisting of powers of derivatives of different orders. This product of terms can be multiplied by a polynomial of \( u \). Let a term from a monomial contain \( k \)-th power of a derivative of \( l \)-th order. We shall call the product \( kl \) grade of the term with respect to participating derivatives. The sum of these grades of all terms of a monomial will be called grade of the monomial with respect to participating derivatives. The general case is (1): The polynomial contains monomials that contain derivatives have odd or even grades with respect to participating derivatives. There are two particular cases: (1A): All monomials that contain derivatives have odd grades with respect to participating derivatives; (1B):All monomials that contain derivatives have even grades with respect to participating derivatives. We shall formulate our main result
for the general case (1) and then we shall demonstrate several applications for the cases (1), (1A), (1B).

2. Main result

Below we search for solitary wave solutions for the class of nonlinear PDEs that contain monomials of derivatives which order with respect to participating derivatives is even and monomials of derivatives which order with respect to participating derivatives is odd.

**Theorem:** Let $\mathcal{P}$ be a polynomial of the function $u(x, t)$ and its derivatives. $u(x, t)$ belongs to the differentiability class $C^k$, where $k$ is the highest order of derivative participating in $\mathcal{P}$. $\mathcal{P}$ can contain some or all of the following parts: (A) polynomial of $u$; (B) monomials that contain derivatives of $u$ with respect to $x$ and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of $u$; (C) monomials that contain derivatives of $u$ with respect to $t$ and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of $u$; (D) monomials that contain mixed derivatives of $u$ with respect to $x$ and derivatives of $u$ with respect to $t$. Each such monomial can be multiplied by a polynomial of $u$; (E) monomials that contain products of derivatives of $u$ with respect to $x$ and derivatives of $u$ with respect to $t$. Each such monomial can be multiplied by a polynomial of $u$; (F) monomials that contain products of derivatives of $u$ with respect to $x$ and mixed derivatives of $u$ with respect to $x$ and $t$. Each such monomial can be multiplied by a polynomial of $u$; (G) monomials that contain products of derivatives of $u$ with respect to $t$ and mixed derivatives of $u$ with respect to $x$ and $t$. Each such monomial can be multiplied by a polynomial of $u$.

Let us consider the nonlinear partial differential equation

$$\mathcal{P} = 0.$$  \hfill (3)

We search for solutions of this equation of the kind $u(\xi) = u(\alpha x + \beta t) = u(\xi) = \gamma f(\xi)$, where $\gamma$ is a parameter and $f(\xi)$ is solution of the simplest equation $f_\xi^2 = 4(f^2 - f^3)$. The substitution of this solution in Eq. (3) leads to
a relationship \( \mathcal{R} \) of the kind

\[
\mathcal{R} = \sum_{i=0}^{N} C_i f(\xi)^i + f_\xi \left( \sum_{j=0}^{M} D_j f(\xi)^j \right)
\]

(4)

where \( N \) and \( M \) are natural numbers depending on the form of the polynomial \( \mathcal{P} \). The coefficients \( C_i \) and \( D_j \) depend on the parameters of Eq. (3) and on \( \alpha, \beta \) and \( \gamma \). Then each nontrivial solution of the nonlinear algebraic system

\[
C_i = 0, \ i = 1, \ldots, N; \quad D_j = 0, \ j = 1, \ldots, M
\]

(5)

leads to solitary wave solution of the nonlinear partial differential equation (3).

Proof. First we shall prove the following:

Let \( f(\xi) \) be a solution of the nonlinear ordinary differential equation \( f_\xi^2 = 4(f^2 - f^3) \). Then the even derivatives of \( f(\xi) \) contain only a polynomial of \( f(\xi) \). The odd derivatives of \( f(\xi) \) contain a polynomial of \( f(\xi) \) multiplied by \( f_\xi \).

We shall use the method of induction. The above statement is true for the second, third, and the fourth derivative of \( f(\xi) \). Let the \( 2n \)-th derivative of \( f \) be a polynomial of \( f: \frac{d^{2n}f}{d\xi^{2n}} = F(f) \). Then \( \frac{d^{2n+1}f}{d\xi^{2n+1}} = \frac{dF}{df} \frac{df}{d\xi} \), and

\[
\frac{d^{2n+2}f}{d\xi^{2n+2}} = \frac{d^2F}{d\xi^2} \left( \frac{df}{d\xi} \right)^2 + \frac{dF}{df} \frac{d^2f}{d\xi^2} = F^*(f),
\]

because of the fact that \( \left( \frac{df}{d\xi} \right)^2 \) and \( \frac{d^2f}{d\xi^2} \) are polynomials of \( f \) (this follows from the definition of \( f \)). Thus each even derivative of \( f \) is a polynomial of \( f \) and each odd derivative of \( f \) is a polynomial of \( f \) multiplied by \( f_\xi \).

Let the nonlinear PDE \( \mathcal{P} = 0 \) be reduced by the traveling-wave ansatz to the nonlinear ODE \( \mathcal{Q} = 0 \) where each monomial has odd or even grades with respect to the participating derivatives. For the monomials with odd grades the chosen kind of solution reduces further the corresponding part of \( \mathcal{Q} \) to relationship of the kind \( \sum_{i=0}^{N} C_i f(\xi)^i \). For the monomials with even grades the chosen kind of solution reduces further the corresponding part of \( \mathcal{Q} \) to relationship of the kind \( f_\xi \left( \sum_{j=0}^{M} D_j f(\xi)^j \right) \). In such a way \( \mathcal{Q} \) is reduced to a relationship of the kind \( \mathcal{R} = \sum_{i=0}^{N} C_i f(\xi)^i + f_\xi \left( \sum_{j=0}^{M} D_j f(\xi)^j \right) \). Then each nontrivial solution of the nonlinear algebraic system \( C_i = 0; \ D_j = 0 \) (if such
solution exists) leads to a solitary traveling wave solution of the nonlinear PDE $P = 0$.

What follows are several examples. Let us consider the equation

$$\nu u_{ttt} + \pi uu_{xx} + \sigma u_x^2 + (\delta + \mu u)u_t + \omega u^3 = 0. \quad (6)$$

The substitution of the studied kind of solution in Eq.(6) leads to an equation that contains monomials of odd and even grade with respect to derivatives of $u(\xi)$. The application of the above theorem leads to the following system of nonlinear algebraic equations:

$$-12\beta^3\nu + \alpha \gamma \mu = 0; \quad 4\beta^3\nu + \alpha \delta = 0; \quad -6\pi \alpha^2 - 4\beta^2 \sigma + \gamma \omega = 0; \quad \pi \alpha^2 + \beta^2 \sigma = 0. \quad (7)$$

One solution of the system (7) is

$$\alpha = -\frac{1}{2} \frac{(-\sigma^2 \nu^2)^{3/4}}{\delta \nu^2 \pi^{3/4}}, \quad \beta = \frac{1}{2} \frac{(-\sigma \delta^2 \nu^2)^{1/4}}{\nu \pi^{1/4}}; \quad \gamma = \frac{3\delta}{\mu}; \quad \omega = \frac{1}{6} \frac{\sigma \nu (-\sigma \delta^2 \nu^2)^{1/2}}{\mu \delta \nu^2 \pi^{1/2}}, \quad (8)$$

and the corresponding solitary wave is

$$u(x, t) = -\frac{3\delta}{\mu \cosh^2 \left[ -\frac{1}{2} \frac{(-\sigma \delta^2 \nu^2)^{3/4}}{\delta \nu^2 \pi^{3/4}} x + \frac{1}{2} \frac{(-\sigma \delta^2 \nu^2)^{1/4}}{\nu \pi^{1/4}} t \right]]. \quad (9)$$

As second example we consider the equation

$$\pi u^2 u_{xxx} + \mu u u_{xtt} + \nu u_{xx} u_t^2 + +\sigma u^2 u_{xt} + \delta u_x u_t + (\epsilon u + \kappa u^2)u_t + \theta u^3 = 0. \quad (10)$$

The application of the theorem (here and in the examples below we let the calculations to the interested reader) leads to the solitary wave:

$$u(x, t) = -\frac{3\epsilon(\delta + 2\mu)}{[2\kappa(\delta + \mu)] \cosh^2 \left[ -\frac{5^{3/4}}{10} \frac{\epsilon^{1/2}(-\nu)^{1/4}}{\pi^{1/4}(\delta + \mu)^{1/2}} x + \frac{5^{1/4}}{2} \frac{\pi^{1/4} \epsilon^{1/2}}{(\delta + \mu)^{1/2}(-\nu)^{1/4}} t \right]]. \quad (11)$$
3. The two particular classes (1A) and (1B)

For the class of nonlinear PDEs that contain monomials of derivatives which order with respect to participating derivatives is odd only (particular class (1A)) the above theorem is valid when there are no terms of kind \((A)\) and \(D_i = 0\). Famous example of nonlinear PDE of this class is the Korteweg-de Vries equation

\[ u_t + K u u_x + u_{xxx} = 0, \]  

(12)

where \(K\) is a constant (in the original equation \(K = 6\)). Eq. (12) contains three monomials and all of them have odd grade with respect to the participating derivatives. The application of above theorem to Eq. (12) leads to the famous solitary-wave solution of the Korteweg-deVries equation.

Another nonlinear PDE that belongs to this class of equations is the generalized Degasperis-Procesi equation

\[ u_t + c_0 u_x + d u_{xxx} - h^2 u_{xxt} = \partial_x \left[ c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx} \right]. \]  

(13)

When \(c_0 = c_2 = c_3 = h = 0, d = 1\) and \(c_1 = -K/2\) Eq. (13) is reduced to the Korteweg-de Vries equation. When \(c_1 = -(3c_3)/(2h^2)\) and \(c_2 = c_3/2\) Eq. (13) is reduced to the Camassa-Holm equation. Finally when \(c_1 = -2c_3/h^2\) and \(c_2 = c_3\) Eq. (13) is reduced to the Degasperis-Procesi equation which is used as model equation for propagation of shallow water waves over a flat bed [29]. The application of the theorem to Eq. (13) leads to the solitary-wave solution

\[ u(x, t) = \frac{6\alpha^2(c_0h^2 + d)}{(4\alpha^2h^2 - 1)(c_1 - 2\alpha^2c_3) \cosh^2 \left[ \alpha x + \left( \frac{\alpha(4\alpha^2d + c_0)}{4\alpha^2h^2 - 1} \right) t \right]]. \]  

(14)

As a last example we consider the nonlinear PDE

\[ \sigma_1 u_{xxxx} + \sigma_2 u u_{xxx} + \sigma_3 u_{xxx} + \sigma_4 u_x u_{xx} + \sigma_5 u^2 u_x + \sigma_6 u u_x + u_t = 0. \]  

(15)

The application of the theorem leads to the solitary wave solution

\[ u(x, t) = \frac{12\alpha^2(20\alpha^2\sigma_1 + \sigma_3)}{(4\alpha^2\sigma_2 + 4\alpha^2\sigma_4 + \sigma_6) \cosh^2 \left[ \alpha x + \left( -16\alpha^5\sigma_1 - 4\alpha^3\sigma_3 \right) t \right]]. \]  

(16)
where
\[ \alpha = \frac{10^{1/2}}{20\sigma_2(10\sigma_1\sigma_5 - \sigma_2^2 - \sigma_2\sigma_4)} \left[ \sigma_1(10\sigma_1\sigma_5 - \sigma_2^2 - \sigma_2\sigma_4) \left( -20\sigma_1\sigma_3\sigma_5 - 5\sigma_1\sigma_4\sigma_6 + 2\sigma_2^2\sigma_3 + 3\sigma_2\sigma_3\sigma_4 + \sigma_3\sigma_4^2 + 5\sigma_1\sigma_6 - \sigma_3\sigma_4 \right)^{1/2} \right], \]

\[ \sigma_3\sigma_4 \left(-40\sigma_1\sigma_5 + 4\sigma_2^2 + 4\sigma_2\sigma_4 + \sigma_4^2 \right)^{1/2} \right]^{1/2}, \]

(17)

We would like to note the following. Let us consider the nonlinear equations of the Korteweg-de Vries hierarchy of PDEs [30]

\[ K^{(n)}_x + u_t = 0 \]  

(18)

where \( K^{(-1)} = -1 \) and \( K^{(n)}_x = \frac{1}{4}[K^{(n-1)}_{xxx} + 8uK^{(n-1)} + 4u_xK^{(n-1)}]. \) All the equations from the KdV hierarchy contain only monomials of odd grade with respect to the derivatives, i.e., are within the scope of the above theorem.

For the second particular class (1B) of nonlinear PDEs the above theorem is valid with \( C_i = 0. \) Famous representative of this class of equations is the Boussinesq equation

\[ u_{tt} - u_{xx} - u_{xxxx} - K(u^2)_{xx} = 0 \]  

(19)

that is nonlinear model equation for shallow water waves valid for weakly nonlinear long water waves. The application of the above theorem to Eq. (19) leads to the well known solitary wave solution of the Boussinesq equation.

As another example we consider the nonlinear PDE

\[ \mu u_{xx} + \rho u_t + \delta uu_{xx} + \epsilon u_{xxtt} + \pi u_x^2 + \sigma u_t^2 = 0. \]

(20)

The application of the theorem leads to the following solitary wave solution

\[ u(x, t) = -\frac{3\rho}{2\sigma} \left( \frac{5[3\delta \rho(4\pi \sigma + 3\delta \rho)]^{1/2} + 15\delta \rho + 4\pi \sigma}{[3\delta \rho(4\pi \sigma + 3\delta \rho)]^{1/2} + 12\delta \rho} \right) \times \]

\[ \cosh^2 \left\{ \frac{1}{2} \left( \frac{\rho}{\epsilon} \right)^{1/2} x + \frac{1}{2\sigma(6\delta)^{1/2}} \left\{ -\sigma \left[ 3\delta \rho(4\pi \sigma + 3\delta \rho) \right]^{1/2} + 3\delta \rho \right\}^{1/2} t \right\} \]  

(21)
4. Concluding remarks

In this article we formulate an approach for obtaining solitary wave solutions of a large class of nonlinear PDEs. The discussed solitary wave solutions have 3 parameters: $\alpha$, $\beta$, and $\gamma$. If the system of nonlinear algebraic equations contains 3 equations then these 3 parameters can be expressed by the parameters of the nonlinear PDE and there will be no need of relationships between the parameters of the nonlinear PDE. If the system of nonlinear algebraic equations contain more than 3 equations then there are also relationships among the parameters of the solved nonlinear PDE.

Finally we note that the discussed methodology is a simple and effective tool for obtaining exact solitary traveling wave solutions of numerous nonlinear partial differential equations.

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