Asymptotics of exponential moments of a weighted local time of a Brownian motion with small variance

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Received: 9 December 2015, Revised: 22 February 2016, Accepted: 29 February 2016,  
Published online: 5 April 2016

Abstract  We prove a large deviation type estimate for the asymptotic behavior of a weighted local time of $\epsilon W$ as $\epsilon \to 0$.

Keywords  Local time, exponential moment, large deviations principle

2010 MSC  60J55, 60F10, 60H10

1 Introduction and the main result

Let $\{W_t, t \geq 0\}$ be a real-valued Wiener process, and $\mu$ be a $\sigma$-finite measure on $\mathbb{R}$ such that

$$ \sup_{x \in \mathbb{R}} \mu([x - 1, x + 1]) < \infty. $$

(1)

Recall that the \textit{local time} $L_t^\mu(W)$ of the process $W$ \textit{with the weight} $\mu$ can be defined as the limit of the integral functionals

$$ L_t^\mu(W) := \int_0^t k_n(W_s) \, ds, \quad k_n(x) := \frac{\mu_n(dx)}{dx}, \quad n \geq 1, $$

(2)

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where $\mu_n, n \geq 1$, is a sequence of absolutely continuous measures such that
\[
\int_{\mathbb{R}} f(x) \mu_n(dx) \to \int_{\mathbb{R}} f(x) \mu(dx)
\]
for all continuous $f$ with compact support, and (1) holds for $\mu_n, n \geq 1$, uniformly. The limit $L^\mu_t(W)$ exists in the mean square sense due to the general results from the theory of $W$-functionals; see [3], Chapter 6. This definition also applies to $\varepsilon W$ instead of $W$ for any positive $\varepsilon$. In what follows, we will treat $\varepsilon W$ as a Markov process whose initial value may vary, and with a slight abuse of notation, we denote by $P_x$ the law of $\varepsilon W$ with $\varepsilon W_0 = x$ and by $E_x$ the expectation w.r.t. this law.

In this note, we study the asymptotic behavior as $\varepsilon \to 0$ of the exponential moments of the family of weighted local times $L^\mu_t(\varepsilon W)$. Namely, we prove the following theorem.

**Theorem 1.** For arbitrary finite measure $\mu$ on $\mathbb{R}$,
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log E_x e^{L^\mu_t(\varepsilon W)} = \frac{t}{2} \sup_{y \in \mathbb{R}} \mu(\{y\})^2.
\]

For arbitrary $\sigma$-finite measure $\mu$ on $\mathbb{R}$ that satisfies (1),
\[
\sup_x \lim_{\varepsilon \to 0} \varepsilon^2 \log E_x e^{L^\mu_t(\varepsilon W)} = \frac{t}{2} \sup_{y \in \mathbb{R}} \mu(\{y\})^2.
\]

We note that in this statement the measure $\mu$ can be changed to a signed measure; in this case, in the right-hand side, only the atoms of the positive part of $\mu$ should appear. We also note that, in the $\sigma$-finite case, the uniform statement (3) may fail; one example of such a type is given in Section 3.

Let us briefly discuss the problem that was our initial motivation for the study of such exponential moments. Consider the one-dimensional SDE
\[
dX_t^\varepsilon = a(X_t^\varepsilon) \, dt + \varepsilon \sigma(X_t^\varepsilon) \, dW_t
\]
with discontinuous coefficients $a, \sigma$. In [7], a Wentzel–Freidlin-type large deviation principle (LDP) was established in the case $a \equiv 0$ under mild assumptions on the diffusion coefficient $\sigma$. In [8], this result was extended to the particular class of SDEs such that the function $a/\sigma^2$ has a bounded derivative. This limitation had appeared because of formula (7) in [8] for the rate transform of the family $X^\varepsilon$. This formula contains an integral functional with kernel $(a/\sigma^2)'$ of a certain diffusion process obtained from $\varepsilon W$ by the time change procedure. If $a/\sigma^2$ is not smooth but is a function of a bounded variation, this integral function still can be interpreted as a weighted local time with weight $\mu = (a/\sigma^2)'$. Thus, Theorem 1 can be used in order to study the LDP for the SDE (5) with discontinuous coefficients. One of such particular results can be derived immediately. Namely, if $\mu$ is a continuous measure, then by Theorem 1 the exponential moments of $L^\mu_t(\varepsilon \tilde{W})$ are negligible at the logarithmic scale with rate function $\varepsilon^2$. This, after simple rearrangements, allows us to neglect the corresponding term in (7) of [8] and to obtain the statement of Theorem 2.1 of [8] under the weaker condition that $a/\sigma^2$ is a continuous function of bounded variation. The problem how to describe in a more general situation the influence of the jumps of $a/\sigma^2$ on the LDP
for the solution to (5) still remains open and is the subject of our ongoing research. We just remark that due to Theorem 1 the respective integral term is no longer negligible, which well corresponds to the LDP results for piecewise smooth coefficients \(a, \sigma\) obtained in [1, 2, 6].

2 Proof of Theorem 1

2.1 Preliminaries

For a measure \(\nu\) satisfying (1), denote by

\[
f_{t}^{\nu, \varepsilon}(x) = \mathbb{E}_{x} L_{t}^{\nu}(\varepsilon W) = \int_{0}^{t} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s \varepsilon^{2}}} e^{-\frac{(y-x)^{2}}{2s\varepsilon^{2}}} \nu(dy) \, ds, \quad t \geq 0, \, x \in \mathbb{R},
\]

the characteristic of the local time \(L_{t}^{\nu}(\varepsilon W)\) considered as a \(W\)-functional of \(\varepsilon W\); see [3], Chapter 6.

The following statement is a version of Khas’minskii’s lemma; see [9], Section 1.2.

Lemma 1. Suppose that

\[
\sup_{x \in \mathbb{R}} f_{s}^{\nu, \varepsilon}(x) \leq \frac{1}{2}, \quad (7)
\]

Then

\[
\sup_{x \in \mathbb{R}} \mathbb{E}_{x} e^{L_{t}^{\nu}(\varepsilon W)} \leq 2, \quad (8)
\]

Using the Markov property, as a simple corollary, we obtain, for arbitrary \(t > 0\),

\[
\sup_{x \in \mathbb{R}} \mathbb{E}_{x} e^{L_{t}^{\nu}(\varepsilon W)} \leq 2^{1+\frac{t}{s}} = 2^{\log 2(t/s)}
\]

where \(s > 0\) is such that (7) holds. This inequality, combined with (6), leads to the following estimate.

Lemma 2. For a nonzero measure \(\nu\) satisfying (1), denote

\[
N(\nu, \gamma) = \sup_{x \in \mathbb{R}} \nu([x-\gamma, x+\gamma]), \quad \gamma > 0.
\]

For any \(\lambda \geq 1\) and \(\gamma > 0\), there exists \(\varepsilon_{\lambda, \gamma} > 0\) such that

\[
\sup_{x \in \mathbb{R}} \mathbb{E}_{x} e^{\lambda L_{t}^{\nu}(\varepsilon W)} \leq 2e^{(4\log 2)c_{0}N(\nu, \gamma)^{2}t\lambda^{2}\varepsilon^{-2}}, \quad \varepsilon \in (0, \varepsilon_{\lambda, \gamma}),
\]

with

\[
c_{0} = \frac{2}{\pi} \left(1 + 2 \sum_{k=1}^{\infty} e^{-\frac{(2k-1)^{2}}{2}}\right)^{2}.
\]

Proof. If \(\varepsilon \sqrt{s} \leq \gamma\), then we have

\[
f_{s}^{\nu, \varepsilon}(x) = \sum_{k \in \mathbb{Z}} \int_{0}^{s} \int_{|y-x-2k\gamma| \leq \gamma} \frac{1}{\sqrt{2\pi \varepsilon^{2}}} e^{-\frac{(y-x)^{2}}{2\varepsilon^{2}}} \nu(dy) \, dv
\]

\[
\leq \sqrt{c_{0}N(\nu, \gamma)} \sqrt{\frac{s}{\varepsilon^{2}}},
\]

Take

\[
s = (2N(\nu, \gamma))^{-2}(c_{0})^{-1}\lambda^{-2}\varepsilon^{2}.
\]
Then the inequality \( \varepsilon \sqrt{s} \leq \gamma \) holds, provided that
\[
\varepsilon \leq (\gamma (2N(\nu, \gamma))^2 c_0 \lambda^2)^{1/3} =: \varepsilon_{\lambda, \gamma}.
\]
Under this condition,
\[
f^\lambda_{\varepsilon}(x) = \lambda f^\nu_{\varepsilon}(x) \leq \frac{1}{2}.
\]
Now the required inequality follows immediately from (8).

In what follows, we will repeatedly decompose \( \mu \) into sums of two components and analyze separately the exponential moments of the local times that correspond to these components. We will combine these estimates and obtain an estimate for \( L^\mu_t(\varepsilon W) \) itself using the following simple inequality. Let \( \mu = \nu + \kappa \) and \( p, q > 1 \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[
L^\mu_t(\varepsilon W) = L^\nu_t(\varepsilon W) + L^\kappa_t(\varepsilon W) = (1/p)L^\nu_t(\varepsilon W) + (1/q)L^\kappa_t(\varepsilon W),
\]
and therefore by the Hölder inequality we get
\[
\mathbb{E}e^{L^\mu_t(\varepsilon W)} \leq \left( \mathbb{E}e^{L^\nu_t(\varepsilon W)} \right)^{1/p} \left( \mathbb{E}e^{L^\kappa_t(\varepsilon W)} \right)^{1/q}.
\]
(10)
We will also use another version of this upper bound, which has the form
\[
\mathbb{E}e^{L^\mu_t(\varepsilon W)} 1_A \leq \left( \mathbb{E}e^{L^\nu_t(\varepsilon W)} \right)^{1/p} \left( \mathbb{P}(A) \right)^{1/q}, \quad A \in \mathcal{F}.
\]
(11)
We denote
\[
\Delta = \sup_{x \in \mathbb{R}} \mu(\{x\}).
\]
We will prove Theorem 1 in several steps, in each of them extending the class of measures \( \mu \) for which the required statement holds.

2.2 Step I: \( \mu \) is a finite mixture of \( \delta \)-measures
If \( \mu = a \delta_z \) is a weighted \( \delta \)-measure at the point \( z \), then we have
\[
L^\mu_t(\varepsilon W) = a \varepsilon^{-1} L^z_t(W),
\]
where
\[
L^z_t(W) = \lim_{\eta \to 0} \frac{1}{2\eta} \int_0^t 1_{|W_s - z| \leq \eta} ds
\]
is the local time of a Wiener process at the point \( z \). The distribution of \( L^z_t(W) \) is well known; see, e.g., [5], Chapter 2.2 and expression (6) in Chapter 2.1. Hence, the required statement in the particular case \( \mu = a \delta_z \) is straightforward, and we have the following:
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sup_x \log \mathbb{E}_x e^{a \varepsilon^{-1} L^z_t(W)} = \frac{ta^2}{2}.
\]
(12)
Note that in this formula the supremum is attained at the point \( x = z \).
In this section, we will extend this result to the case where \( \mu \) is a finite mixture of \( \delta \)-measures, that is,

\[
\mu = \sum_{j=1}^{k} a_j \delta_{z_j}.
\]

Let \( j^* \) be the number of the maximal value in \( \{a_j\} \), that is, \( \Delta = a_{j^*} \). Then \( L^\mu_t(\varepsilon W) \geq \Delta \varepsilon^{-1} L_{t}^{(\varepsilon z_{j^*})}(W) \), and it follows directly from (12) that

\[
\lim \inf_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log \mathbb{E}_x e^{t \Delta^2} t \Delta \geq \frac{2}{\varepsilon}. \tag{13}
\]

In what follows, we prove the corresponding upper bound

\[
\lim \sup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log \mathbb{E}_x e^{t \Delta^2} t \Delta \leq \frac{2}{\varepsilon}, \tag{14}
\]

which, combined with this lower bound, proves (3).

Observe that, for \( \gamma > 0 \) small enough,

\[
N(\mu, \gamma) = \Delta.
\]

Then by Lemma 2, for any \( \lambda \geq 1 \),

\[
\lim \sup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log \mathbb{E}_x e^{t \Delta^2} t \Delta \leq c_1 \lambda \Delta \tag{15}
\]

with

\[
c_1 = (4 \log 2) c_0 = \frac{8 \log 2}{\pi} \left( 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{(2k-1)^2}{2}} \right)^2.
\]

In particular, taking \( \lambda = 1 \), we obtain an upper bound of the form (14), but with a worse constant \( c_1 \) instead of required \( 1/2 \). We will improve this bound by using the large deviations estimates for \( \varepsilon W \), the Markov property, and the “individual” identities (12).

Denote \( \mu_j = a_j \delta_{z_j}, j = 1, \ldots, k \). Then

\[
L^\mu_t(\varepsilon W) = \sum_{j=1}^{k} L^\mu_t(\varepsilon z_j). \tag{12}
\]

Fix some family of neighborhoods \( O_j \) of \( z_j, j = 1, \ldots, k \), such that the minimal distance between them equals \( \rho > 0 \), and denote

\[
O^j = \mathbb{R} \setminus \bigcup_{i \neq j} O_i.
\]

For some \( N \geq 1 \) whose particular value will be specified later, consider the partition \( t_n = t(n/N), n = 0, \ldots, N \), of the segment \([0, t]\) and denote

\[
B_{n,j} = \{ f \in C(0, t) : f_s \in O^j, s \in [t_{n-1}, t_n] \}, \quad j \in \{1, \ldots, k\}, n \in \{1, \ldots, N\},
\]
\[ C_{j_1, \ldots, j_N} = \bigcap_{n=1}^{N} B_{n, j_n}, \quad j_1, \ldots, j_N \in \{1, \ldots, k\}. \]

Observe that if the process \( \varepsilon W \) does not visit \( O_j \) on the time segment \([u, v]\), then \( L^{\mu_j}(\varepsilon W) \) on this segment stays constant. This means that, on the set \( \{\varepsilon W \in C_{j_1, \ldots, j_N}\} \), we have

\[
L^{\mu_j}(\varepsilon W) = \sum_{n=1}^{N} \left( L_{t_n}^{\mu_{j_n}}(\varepsilon W) - L_{t_{n-1}}^{\mu_{j_n}}(\varepsilon W) \right).
\]

Because \( L^{\mu_j}(\varepsilon W) \) is a time-homogeneous additive functional of the Markov process \( \varepsilon W \), we have

\[
E_x \left[ e^{L_{t_n}^{\mu_{j_n}}(\varepsilon W) - L_{t_{n-1}}^{\mu_{j_n}}(\varepsilon W)} \mid \mathcal{F}_{t_{n-1}} \right] = E_y e^{L_{t_n}^{\mu_{j_n}}(\varepsilon W)} \bigg|_{y=\varepsilon W_{t_{n-1}}}. 
\]

Then by (12), for any \( j_1, \ldots, j_N \in \{1, \ldots, k\} \),

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log E_x e^{L_{t_n}^{\mu_{j_n}}(\varepsilon W)} 1_{\varepsilon W \in C_{j_1, \ldots, j_N}} \leq \frac{t}{2N} \sum_{n=1}^{N} (a_{j_n})^2 \leq \frac{t\Delta^2}{2}.
\]

Because we have a fixed number of sets \( C_{j_1, \ldots, j_N} \), this immediately yields

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log E_x e^{L_{t_n}^{\mu_{j_n}}(\varepsilon W)} 1_{\varepsilon W \in C} \leq \frac{t\Delta^2}{2}
\]

with

\[
C = \bigcup_{j_1, \ldots, j_N \in \{1, \ldots, k\}} C_{j_1, \ldots, j_N}.
\]

Hence, to get the required upper bound (14), it suffices to prove an analogue of (16) with the set \( C \) replaced by its complement \( D = C(0, t) \setminus C \). Using (11) with \( p = 2 \), \( A = \{\varepsilon W \in D\} \), and (15) with \( \lambda = 2 \), we get

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log E_x e^{L_{t_n}^{\mu_{j_n}}(\varepsilon W)} 1_{\varepsilon W \in D} \leq 2c_1 t \Delta^2 + \frac{1}{2} \limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log P_x(\varepsilon W \in D).
\]

By the LDP for the Wiener process ([4], Chapter 3, §2),

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log P_x(\varepsilon W \in D) = -\inf_{f \in \text{closure}(D)} I(f),
\]

where

\[
I(f) = \begin{cases} (1/2) \int_0^t (f'_s)^2 \, ds, & f \text{ is absolutely continuous on } [0, t]; \\ +\infty & \text{otherwise}. \end{cases}
\]

For any trajectory \( f \in D \), there exists \( n \) such that \( f \) visits at least two sets \( O_j \) on the time segment \([t_{n-1}, t_n]\). Therefore, any trajectory \( f \in \text{closure}(D) \) exhibits an oscillation \( \geq \rho \) on this time segment. On the other hand, for an absolutely continuous \( f \),

\[
|f_u - f_v| = \left| \int_u^v f'_s \, ds \right| \leq |u - v|^{1/2} \left( \int_0^t (f'_s)^2 \, ds \right)^{1/2}.
\]
This means that, for any \( f \in \text{closure}(D) \),
\[
I(f) \geq \rho^2 N_2 t,
\]
which yields
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log \mathbb{E}_x e^{L^n_t(\varepsilon W)} 1_{\varepsilon W \in D} \leq 2c_1 t \Delta^2 - \frac{\rho^2 N_2}{2t}.
\]
If in this construction, \( N \) was chosen such that
\[
N \geq (4c_1 - 1)\rho^{-2} t^2 \Delta^2,
\]
then the latter inequality guarantees the analogue of (16) with \( D \) instead of \( C \). This completes the proof of (14).

2.3 Step II: \( \mu \) is finite
Exactly the same argument as that used in Section 2.2 provides the lower bound (13). In this section, we prove the upper bound (14) for a finite measure \( \mu \) and thus complete the proof of the first assertion of the theorem. For finite \( \mu \) and any \( \chi > 0 \), we can find \( \gamma > 0 \) and decompose \( \mu = \mu_0 + \nu \) in such a way that \( \mu_0 \) is a finite mixture of \( \delta \)-measures and \( N(\nu, \gamma) < \chi \). Let \( p, q > 1 \) be such that \( 1/p + 1/q = 1 \). The measure \( p\mu_0 \) has the maximal weight of an atom equal to \( p\Delta \). Since we have already proved the required statement for finite mixtures of \( \delta \)-measures, we have
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log \left( \mathbb{E}_x e^{L^{\mu_0}_t(\varepsilon W)} \right)^{1/p} \leq \frac{t}{2} p \Delta^2.
\]
(17)
On the other hand, we have \( N(\nu, \gamma) < \chi \) and then by Lemma 2
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log \left( \mathbb{E}_x e^{L^{\nu}_t(\varepsilon W)} \right)^{1/q} \leq c_1 q t \chi^2.
\]
Hence, by (10),
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \mathbb{E}_x e^{L^{\nu}_t(\varepsilon W)} \leq \frac{t}{2} p \Delta^2 + c_1 q t \chi^2.
\]
Now we can finalize the argument. Fix \( \Delta_1 > \Delta := \max_{x \in \mathbb{R}} \mu(\{x\})^2 \) and choose \( p, q > 1 \) such that \( 1/p + 1/q = 1 \) and \( p\Delta^2 < \Delta_1^2 \). Then there exists \( \chi > 0 \) small enough such that
\[
p \Delta^2 + 2c_1 q t \chi^2 < \Delta_1^2.
\]
Taking the decomposition \( \mu = \mu_0 + \nu \) that corresponds to this value of \( \chi \) and applying the previous calculations, we obtain an analogue of the upper bound (14) with \( \Delta \) replaced by \( \Delta_1 \). Since \( \Delta_1 > \Delta \) is arbitrary, the same inequality holds for \( \Delta \).

2.4 Step III: \( \mu \) is \( \sigma \)-finite
In this section, we prove the second assertion of the theorem. As before, the lower bound can be obtained directly from the case \( \mu = a\delta_z \), and hence we concentrate ourselves on the proof of the upper bound
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{E}_x e^{L^n_t(\varepsilon W)} \leq \frac{t \Delta^2}{2}, \quad x \in \mathbb{R}.
\]
(18)
We will use an argument similar to that from the previous section and decompose $\mu$ into a sum $\mu = \mu_0 + \nu$ with finite $\mu_0$ and $\nu$, which is negligible in a sense. However, such a decomposition relies on the initial value $x$, and this is the reason why we obtain an individual upper bound (18) instead of the uniform one (14).

Namely, for a given $x$, we define $\mu_0, \nu$ by restricting $\mu$ to $[x - R, x + R]$ and its complement, respectively. Without loss of generality, we assume that for each $R$, the corresponding $\nu$ is nonzero. Since we have already proved the required statement for finite measures, we get (17).

Next, denote $M = \sup_{x \in \bar{R}} \mu([x - 1, x + 1])$ and observe that $N(\nu, 1) \leq M$. Then by Lemma 2 with $\gamma = 1$ and the strong Markov property, for any stopping time $\tau$, the exponential moment of $L_0^q(\varepsilon W)$ conditioned by $\mathcal{F}_\tau$ is dominated by $2e^{c_1M^2\tau^2\varepsilon^{-2}}$. This holds for $\varepsilon \leq \varepsilon_{q,1}$, where we put the indices $x, R$ in order to emphasize that this constant depends on $\nu$, which, in turn, depends on $x, R$. Since we have assumed that, for any $x, R$, the respective $\nu$ is nonzero, the constants $\varepsilon_{q,R}$ are strictly positive.

Now we take by $\tau$ the first time moment when $|\varepsilon W_\tau - x| = R$. Observe that $L_\tau^q(\varepsilon W)$ equals 0 on the set $\{\tau > t\}$ and it is well known that
\[
P_x(\tau < t) \leq 4P_x(\varepsilon W_t > R) \leq Ce^{-tR^2\varepsilon^{-2}/2}.
\]
Summarizing the previous statements, we get
\[
\mathbb{E}_x e^{L_0^q(\varepsilon W)} \leq 1 + 2Ce^{t\varepsilon^{-2}(c_1M^2\varepsilon^{-2} - R^2/2)}, \quad \varepsilon \leq \varepsilon_{\lambda,1},
\]
which implies
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log(\mathbb{E}_x e^{L_0^q(\varepsilon W)})^{1/q} \leq t(c_1M^2R^2 - 2(2q))_+ , \quad (19)
\]
where we denote $a_+ = \max(a, 0)$. By (10) inequalities (17) and (19) yield
\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{E}_x e^{L_\tau^0(\varepsilon W)} \leq \frac{t}{2}p\Delta^2 + t(c_1M^2R^2 - 2(2q))_+ .
\]

Now we finalize the argument in the same way as we did in the previous section. Fix $\Delta_1 > \Delta$ and take $p > 1$ such that $p\Delta^2 \leq \Delta_1^2$. Then take $R$ large enough so that, for the corresponding $q$, $c_1M^2q - R^2/(2q) \leq 0$.

Under such a choice, the calculations made before yield (18) with $\Delta$ replaced by $\Delta_1$. Since $\Delta_1 > \Delta$ is arbitrary, the same inequality holds for $\Delta$.

3 Example

Let
\[
\mu = \sum_{k=1}^{\infty} (\delta_{k^2} + \delta_{k^2+2-k}).
\]
Then $\mu$ satisfies (1) and $\Delta = 1$. However, it is an easy observation that when the initial value $x$ is taken in the form $x_k = k^2$, the respective exponential moments satisfy
\[
\mathbb{E}_{x_k} e^{L_\tau^0(\varepsilon W)} \to \mathbb{E}_0 e^{L_\tau^0(\varepsilon W)}, \quad k \to \infty,
\]
with $\nu = 2\delta_0$. Then

$$
\liminf_{\varepsilon \to 0} \varepsilon^2 \sup_{x \in \mathbb{R}} \log E_x e^{L^\nu_t(\varepsilon W)} \geq \liminf_{\varepsilon \to 0} \varepsilon^2 \log E_0 e^{L^\nu_t(\varepsilon W)} = 2t > \frac{t}{2},
$$

and therefore (3) fails.

**Acknowledgments**

The first author gratefully acknowledges the DFG Grant Schi 419/8-1 and the joint DFFD-RFFI project No. 09-01-14.

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