Anosov Flows, Surface Groups and Curves in Projective Space

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1 Introduction

In his beautiful paper [13], N. Hitchin studied the connected components of the space

\[ \text{Rep}(\pi_1(S), \text{PSL}(n, \mathbb{R})) = \text{Hom}^{\text{red}}(\pi_1(S), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R}), \]

of reducible representations of the fundamental group of a compact surface \( S \) into \( \text{PSL}(n, \mathbb{R}) \). By reducible, we mean representations that act as sum of irreducible representations on the Lie algebra by the adjoint representation. Using Higgs bundles techniques, he proved two remarkable results. The first one deals with the number of components of this space.

**Theorem 1.1 [Hitchin]** If \( n > 2 \), the space \( \text{Rep}(\pi_1(S), \text{PSL}(n, \mathbb{R})) \) has three connected components when \( n \) is odd, and six when \( n \) is even.

Notice that W. Goldman gave a complete description of these connected components in the case of finite covers of \( \text{PSL}(2, \mathbb{R}) \) in [8]. In the case of \( \text{PSL}(2, \mathbb{R}) \), two homeomorphic components, called Teichmüller spaces, play a central role. These two components are well known to be homeomorphic to a ball of dimension \( 6g - 6 \).

N. Hitchin has generalised this situation to \( \text{PSL}(n, \mathbb{R}) \). Indeed, one of these components when \( n \) is odd, and two when \( n \) is even, have a very simple topology. Let’s define a \( n \)-Fuchsian representation to be a representation \( \rho \) which can be written as \( \rho = \iota \circ \rho_0 \), where \( \rho_0 \) is a cocompact representation with values in \( \text{PSL}(2, \mathbb{R}) \) and \( \iota \) is the irreducible representation of \( \text{PSL}(2, \mathbb{R}) \) in \( \text{PSL}(n, \mathbb{R}) \). We denote by \( \text{Rep}_H(\pi_1(S), \text{PSL}(n, \mathbb{R})) \) a connected component that contains Fuchsian representations, and call it a **Hitchin’s component**. Actually there is one Hitchin’s component when \( n \) is odd and two isomorphic when \( n \) is even. N. Hitchin proved in [13].

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Theorem 1.2 [Hitchin] Each component \( \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R})) \) is homeomorphic to a ball of dimension \( \chi(S)(1 - n^2) \).

This last result actually extends to the case of adjoint groups of real split forms. Although Hitchin’s proof gives an explicit parametrisation of this component, the construction by itself sheds no light on the geometry underlying these representations. With Higgs bundle techniques, one can prove the representations in Hitchin’s component are irreducible (Lemma [H]), but it seems hard to detect with these techniques whether these representations are faithful, discrete, as well as to understand the action of the group of outer automorphisms of \( \pi_1(S) \) acts properly on this specific component.

Nevertheless, the geometric significance of this component is well known in dimension 2 and 3. For \( n = 2 \), it is Teichmüller component, corresponding to holonomies of hyperbolic structures on \( S \). For \( n = 3 \), S. Choi and W. Goldman proved in [3]

Theorem 1.3 [Choi-Goldman] For \( n = 3 \), Hitchin’s component consists of holonomies of convex real projective structures on \( S \). That is, for every representation \( \rho \) in \( \text{Rep}_H(\pi_1(S), PSL(3, \mathbb{R})) \), there exists an open set \( \Omega \) in \( \mathbb{P}(\mathbb{R}^3) \) such that \( \Omega/\rho(\pi_1(S)) \) is homeomorphic to \( S \).

As a consequence of this result, a representation \( \rho \) in Hitchin’s component when \( n = 3 \) preserves a \( C^1 \)-convex curve in \( \mathbb{P}(\mathbb{R}^3) \), namely the boundary of the open set \( \Omega \) obtained by the previous theorem.

Our first result generalises this last situation. Let’s introduce a definition. A curve \( \xi \) with values in \( \mathbb{P}(\mathbb{R}^n) \) is said to be hyperconvex if for any distinct points \( (x_1, \ldots, x_n) \) the following sum is direct

\[
\xi(x_1) + \ldots + \xi(x_n).
\]

Furthermore, we say a hyperconvex curve is a Frenet curve, if there exists a family of maps \( (\xi^1, \xi^2, \ldots, \xi^{n-1}) \) with \( \xi^p \subset \xi^{p+1} \), called the osculating flag of \( \xi \), such that

- \( \xi = \xi^1 \) and \( \xi^p \) takes values in the Grassmannian of \( p \)-planes,
- if \( (n_1, \ldots, n_l) \) are positive integers such that \( \sum_{i=1}^{l} n_i \leq n \), if \( (x_1, \ldots, x_l) \) are distinct points, then the following sum is direct
  \[
  \xi^{n_1}(x_1) + \ldots + \xi^{n_l}(x_l);
  \]
- finally, for every \( x \), let \( p = n_1 + \ldots + n_l \), then
  \[
  \lim_{(y_1, \ldots, y_l) \text{ all distinct}} \left( \bigoplus_{i=1}^{l} \xi^{n_i}(y_i) \right) = \xi^p(x).
  \]
One notices that for a Frenet hyperconvex curve, $\xi^1$ completely determines $\xi^p$. Also if $\xi^1$ is $C^\infty$, then $\xi^p(x)$ is generated by the derivatives at $x$ of $\xi^1$ up to order $p-1$. However, in general, a Frenet hyperconvex curve has no reason to be $C^\infty$ also its image is obviously a $C^1$-submanifold. Our main result is the following

**Theorem 1.4** For every representation $\rho$ in Hitchin’s component, there exists a $\rho$-equivariant hyperconvex Frenet curve from $\partial_\infty \pi_1(S)$ to $\mathbb{P}(\mathbb{R}^n)$.

Notice that the Veronese embedding from $\mathbb{P}(\mathbb{R}^2)$ to $\mathbb{P}(\mathbb{R}^n)$ is a $\text{SL}(2, \mathbb{R})$-equivariant hyperconvex Frenet curve; it corresponds to Fuchsian representations. Our main theorem therefore says that the Veronese embedding persists under eventually large deformation of the representation.

Recall that we say a representation $\rho$ of $\Gamma$ with values in a semi-simple Lie group $G$ is purely loxodromic, if for every $\gamma$ in $\Gamma$ different from the identity, $\rho(\gamma)$ is conjugate to an element in the interior of the Weyl chamber. For $G = \text{PSL}(n, \mathbb{R})$, this just means that the eigenvalues of $\rho(\gamma)$ are real and with multiplicity 1. For $G = \text{PSL}(2, \mathbb{C})$, we recover the classical notion. As a consequence of the techniques involved in the proof, we also obtain

**Theorem 1.5** Every representation in Hitchin’s component is discrete, faithful and purely loxodromic.

This theorem is a generalisation of the classical result for Teichmüller Space. It bears also some relations with a beautiful recent result of M. Burger, A. Iozzi and A. Wienhard [2], announced while the second draft of this paper was completed. They proved in particular that surface group representations with maximal Toledo invariant also have discrete images. Although the methods and the target groups are different (so that the results have non empty intersection, but none contains the other), it appears after a discussion together that dynamical ideas quite similar to those appearing in this paper can be applied to their situation, improving some geometrical aspects. It is also quite surprising that two classes of groups, namely isometry groups of Hermitian symmetric spaces on one hand, and $\text{SL}(n, \mathbb{R})$ (and quite plausibly all real split forms) on the other hand, have some common features, not shared for instance with $\text{PSL}(2, \mathbb{C})$.

We shall also prove in a following paper that the mapping class group acts properly on the Hitchin component [13].

We shall also state converse or refinements of these results in Section 4. We now describe more precisely the structure of this paper, then proceed to discussion and conjectures.

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1.1 Description of the article

We describe briefly the content of the main sections of this article.

• 2 Geometric Anosov flows. We introduce in this section a notion of “geometric structure”, called Anosov structure, related to flows. Our main aim in the article is to describe the representations in Hitchin’s component as holonomies of these structures. As a preliminary, we show these holonomies form an open set in the space of representations.

• 3 Quasi-Fuchsian and Anosov representations. We specify this geometric structure to study rank 1 subgroups of semi-simple Lie groups, and more specifically the irreducible $PSL(2, \mathbb{R})$ in $PSL(n, \mathbb{R})$; we introduce quasi-Fuchsian representations as deformations of Fuchsian representations. In quite a similar way as for classical quasi-Fuchsian representations in $PSL(2, \mathbb{C})$, limit curves appear; instead of taking values in $\mathbb{CP}^1$ as in the “classical” case, they take their values in the flag manifold. Their properties will play a central role in the sequel.

• 4 Statement of the main results. With all the basic notions in hand, we can state our main results. The first one is basically that the limit curve of a quasi-Fuchsian representation is built from a hyperconvex curve. The second one is that every representation in Hitchin’s component is quasi-Fuchsian. We also state converse results.

• 5 Hyperconvex curves. In this section, we study more specifically hyperconvex curves and prove in general they admit “left” and “right” osculating flags. This section is independent of the rest of the article.

• 6 Preserving a hyperconvex curve. We prove that a representation preserving a hyperconvex curve is the holonomy of an Anosov structure.

• 7 Curves and Anosov representations. This is the core of the the article: we prove the limit curve of certain Anosov representation is the osculating flag of a Frenet hyperconvex curve in Corollary 7.2.

• 8 Anosov representations, 3-hyperconvexity and Property (H) In the core of the proof of the previous result, certain properties to be satisfied by limit curves were introduced. We study here their relations with quasi-Fuchsian representations.

• 9 Closedness. We show the set of quasi-Fuchsian representations is closed in the space of all representations. This helps us to conclude the proof of our main results.

• 10 Appendix: some lemmas.
1.2 Further discussions and conjectures

This section is rather programmatic, it contains some announcements, precise conjectures. This section should be skipped by a reader interested in concrete results. It is a rather random collection of remarks which is aimed at suggesting many aspects of Teichmüller theory considered as a dictionary between various fields of mathematics should extend to these Hitchin’s components.

1.2.1 Crossratios: $n = \infty$

In a subsequent article, currently under preparation [17], we explain the relation between Hitchin’s component and crossratios on $\pi_1(S)$. We define a crossratio on $\pi_1(S)$ is a real Hölder function $b$ defined on $(\partial_{\infty} \pi_1(S))^4 \setminus \{(x, y, z, t)/x = w, z = y\}$ satisfying the following rules

\[
\begin{align*}
    b(x, y, z, t) &= \frac{b(x, y, z, w)}{b(x, w, z, t)}, \\
    b(x, y, z, t) &= b(x, t, z, y)^{-1}, \\
    b(x, y, z, t) &= b(z, t, x, y).
\end{align*}
\]

As an example of crossratio, one has the classical projective crossratio and the crossratio associated by J.-P. Otal to a negatively curved metric on $S$ [22]. They were extensively studied by U. Hamenstädt in [14] (Notice however our definition includes more general crossratios than those she defined and that some of her results are not true in our generality). For a complete description of various aspects of crossratio, one is advised to read F. Ledrappier’s presentation [19].

Associated to a crossratio are numbers called periods. If $\gamma$ is an element of $\pi_1(S)$, let $\gamma^+$ (resp. $\gamma^-$) be the attracting (resp. repelling) fixed point of $\gamma$ on $\partial_{\infty}(\pi_1(S))$. We define the period $l(\gamma)$ of $\gamma$ by

\[
\forall y \in \partial_{\infty}(\pi_1(S)), \quad l(\gamma) = \log |b(\gamma^+, \gamma^-, y, \gamma y)|.
\]

It turns out that a crossratio is completely determined by its set of periods which in the case of Otal’s crossratio is just the collection of lengths of the corresponding closed orbits.

The main result of our article [17] explains that there exists a correspondence between representations in Hitchin’s component and crossratios satisfying some functional relations, one for each $n$, which are completely explicit but technical to state. Under this correspondence, the period of $\gamma$ is equal to

\[
\log \left(\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right),
\]

where $\rho$ is the corresponding representation and $\lambda_{\max}(A)$ (resp. $\lambda_{\min}(A)$) is the largest (resp. smallest) real eigenvalue of the matrix $A$. According to this result, each component $Rep_H(\pi_1(S), \text{PSL}(n, \mathbb{R}))$ embeds in the space of all crossratios, which may be consider as a candidate for $Rep_H(\pi_1(S), \text{PSL}(\infty, \mathbb{R}))$. 


Notice also that the space of all crossratios is identified (cf. [19]) as the space of all Hölder Anosov flows on the unit tangent bundle of the surface. This is a rather mysterious picture, but is has the advantage of (almost) describing Hitchin’s component as a space of objects, crossratios or Hölder Anosov flows, that may be thought as “geometric structures” on the surface.

Talking with Hitchin, we also realised this picture is coherent with a conjectural picture of his. Namely, he suggested to consider the group $SL(\infty, \mathbb{R})$ as the group of symplectic diffeomorphisms of $\mathcal{G} = \mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \Delta$. On our dynamical side, a crossratio defines a measure, equivalent to Lebesgue, on $\mathcal{G}$. For instance, the choice of a negatively curved metric defines a symplectic form on the space of geodesics by symplectic reduction, and this space is identified with $\mathcal{G}$ via the identification of $S^1$ with the boundary at infinity. Of course, this measure is invariant under the action of $\pi_1(S)$. It follows that after the conjugation by a homeomorphism sending the measure associated to the crossratio to the “standard measure” on $\mathcal{G}$, we obtain a representation of $\pi_1(S)$ in the group of symplectic homeomorphisms of $\mathcal{G}$. It is striking that these two pictures coming from different areas of mathematics agree.

1.2.2 Universal Hitchin’s component: $g = \infty$

One should notice that Theorem 1.4 allows us to let the genus $g$ of the surface goes to infinity an thus provides an extension of the theory of universal Teichmüller space. Indeed, we may consider the space $\mathcal{T}(n)$ of all Frenet hyperconvex curves in $\mathbb{P}(\mathbb{R}^n)$, this is a natural candidate for the universal Hitchin component, generalising the group of quasi-symmetric homeomorphisms when $n = 2$. Here are some natural questions: how sit the various components in this space? Does it have a Kähler geometry?

1.2.3 Frenet curves and integrable systems

We may now hope to relate the subject with integrable systems. We strongly suggest to read G. Segal very clear exposition [26]. A way to build at least locally a hyperconvex Frenet curve is through differential equations. Namely, let’s consider a $n^{th}$-order linear differential operator - a Hill operator - of the following form

$$L(f) = f^{(n)} + a_2 f^{(n-2)} + a_3 f^{(n-3)} + \ldots a_n.$$  \hspace{1cm} (3)

If $(f_1, \ldots, f_n)$ are $n$ independent solutions of the equation $L(f) = 0$, the projective coordinates given by

$$[f_1, \ldots, f_n]$$

defines locally a hyperconvex Frenet curve. A different choice of $f_i$ yields the same curve up to a projective transformation. Since the curves in Theorem 1.4 have low regularity (they are usually only $C^1$), they cannot be related to smooth regular operators like the one in Formula (3). However one would like to know if they can described by some operator in a weak sense.
The motivation for this question is the following: the space of Hill’s operator is naturally a symplectic manifold and its Poisson algebra relates to the so-called $W(n)$-algebras, where $W(2)$ is the Virasoro Algebra (cf \[26\]).

Apparently, physicists tend to believe a Teichmüller theory should hold for these $W(n)$-algebras for which Hitchin’s component would play the role of Teichmüller space. Honestly, I have never understood in the papers that allude to this question what they really expect as a link between $W(n)$-algebras and Hitchin’s component. Apparently, the goal is rather to obtain Hitchin’s component as a “double quotient” of $W(n)$-algebras like it has been done by M. Kontsevich for Virasoro algebra \[13\], than to copy the relation of Virasoro algebra with the universal Teichmüller space as is provided by our previous discussion.

But at least Theorem 1.4 provides a relation between $W(n)$-algebras and Hitchin’s component which may well be coherent with the expected picture. Also, the fact that we still have a candidate to be a companion for $W(\infty)$ as discusses in the Paragraph 1.2.1 seems appealing.

### 1.2.4 Holomorphic differentials and the link with Hitchin’s theory

To prove his theorem, N. Hitchin gave an explicit parametrisation of Hitchin’s component. Namely after the choice of a complex structure $J$ on the compact surface $S$, he identified the component $\text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R}))$ with the vector space $Q(2, J) \oplus \ldots \oplus Q(n, J)$, where $Q(p, J)$ denotes the space of holomorphic $p$-differentials on the Riemann surface $S, J$. The main idea in the proof is to identify representations with harmonic mappings as in K. Corlette’s seminal paper \[4\], or in \[5\] and \[15\].

Now a harmonic mapping $f$ with values in a symmetric space gives rise to holomorphic differentials $q_2(f), \ldots$ in quite a similar fashion as a connexion gives rise to differential forms in Chern-Weil theory.

Can one improve this parametrisation, and in particular get rid of the choice of a complex structure and obtain a parametrisation by holomorphic objects invariant under the mapping class group? Here is a suggestion. A rather standard check shows that the quadratic differential part $q_2(f)$ vanishes exactly when $f$ is minimal. We may now wonder if, fixing the representation $\rho$, we can choose in a unique manner a complex structure on $S$ so that the associated harmonic is actually minimal. Another way to state this question is the following conjecture which I have discussed many times with Bill Goldman.

**Conjecture 1.6** Let $\rho$ be a representation in Hitchin’s component. For every complex structure $j$ in Teichmüller space $T$, let $e(j)$ be the energy of the corresponding harmonic mapping. Then $e$ has a unique minimum.

This conjecture is well known to be true for $n = 2$. For $n = 3$, one can prove it using the ideas linking real projective structures, affine spheres, Blaschke metric as in J. Loftin paper \[20\] or in the preprint \[10\]; in order to complete the circle of ideas contained in these papers, one has just to realise that, for an
affine sphere \( S \), the Blaschke metric, seen as a map from \( S \) to \( SL(3, \mathbb{R})/SO(3) \), is minimal. For a general \( n \), one can at least show that \( e \) is proper \( [18] \).

If the last conjecture is true, using our previous discussion, we would have proved the following result, which helps to understand the action of the mapping class group \( \mathcal{M}(S) \) on Hitchin’s component

**Conjecture 1.7** The quotient \( \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R}))/\mathcal{M}(S) \) is homeomorphic to the total space of the vector bundle \( E \) over Riemann moduli space, whose fibre at a point \( J \) is

\[
E_J = Q(3, J) \oplus \ldots \oplus Q(n, J).
\]

Again, by the previous discussion this result is true for \( n = 2 \) and \( n = 3 \). The fact that the energy is proper would say that the map we can define using Hitchin’s identification (described in the beginning of this paragraph) from \( E \) to \( \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R}))/\mathcal{M}(S) \) is at least surjective.

### 1.2.5 Compactification

W. Thurston \( [6] \) gave a compactification of Teichmüller space, which has been extended in many ways. More specifically, A. Parreau gave a compactification of the set of discrete representations in \( SL(n, \mathbb{R}) \) \( [23] \). Since all representations in Hitchin’s component are discrete, in particular her work gives a compactification of \( \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R}))/\mathcal{M}(S) \). It should be interesting to relate this compactification and Theorem 1.4.

### 1.2.6 Further extensions and questions

So far this article only deals with the group \( PSL(n, \mathbb{R}) \), although Hitchin’s Theorem 1.2 actually extends to adjoint groups of all real split forms. It is rather tempting to conjecture that at least Theorem 1.5 extends to this general context. It is actually obvious in cases like \( PSO(n, n+1) \) when the corresponding component is a subset of the component for \( PSL(n, \mathbb{R}) \).

Another natural extension is to consider surfaces with marked points, the holonomy around marked points being forced to preserve a full flag. To my knowledge, even the case \( n = 3 \) is not known, although Hitchin’s version has been worked out \( [1] \).

Notice however that in their remarkable paper \( [7] \), Volodya Fock and Sacha Goncharov gave a construction and a combinatorial description of a “Teichmüller space” for surfaces with punctures or boundary, as well as coordinates and Poisson structures. Actually their picture extends to the case of real split group. For the moment, it is not clear yet that their Teichmüller space is indeed is connected component. But it is quite believable.

### 2 Geometric Anosov Flows

Our starting point is to obtain representations in \( \text{Rep}(\pi_1(S), PSL(n, \mathbb{R})) \) as holonomies of “geometric structures” associated to flows. We prove for these
new geometric structures Proposition 2.1 which is an analog in our context of Ehresmann Theorem, sometimes called Thurston-Lok Holonomy Theorem \[21\], that states that the deformation of the holonomy representation for a compact manifold can be obtained through a deformation of the structure.

We then explain the example that arises when considering a rank 1 subgroup of a semisimple group, thus making sense of a notion of quasi-Fuchsian representation. We finally concentrate on the case which is the subject of this paper, associated to the irreducible $PSL(2, \mathbb{R})$ in $PSL(n, \mathbb{R})$.

As a motivation for our notion of geometric structure, we begin with a remark. When one defines a $(G, X)$-geometric structure on a manifold $M$ as an atlas modelled on $X$ with transition maps in $G$, one requires that $M$ and $X$ have the same dimension and that the charts are homeomorphisms (or at least submersions in the case of transverse structures to foliations), although this is not formally necessary. Indeed for instance, if $X$ is allowed to have a larger dimension, the corresponding “geometric structure” would not be rigid enough and too vague to have a useful meaning. The presence of a flow and a subsequent hyperbolic hypothesis will allow us to enlarge the definition in this direction, and still obtain “rigid” geometric structures.

Before proceeding to the definition, we recall the definition of a contracting (or dilating) bundle over a dynamical system.

Let $X$ be a topological space equipped with a flow $\phi_t$. Let $E$ be a topological vector bundle over $X$ such that the action of $\phi_t$ lifts to an action of a flow $\psi_t$ by bundle automorphisms. Assume $E$ is equipped with a metric $g$. The bundle $E$ is contracting (resp. dilating), if there exist positive constants $A$ and $B$, such that for every $u$ in $E$, for every $t$ such that $t > 0$ (resp. $t < 0$)

$$\|\psi_t(u)\| \leq Ae^{-B|t|}\|u\|.$$ 

It is useful and classical to remark that if $X$ is compact,

1. the metric $g$ plays no role;

2. the parametrisation of the flow plays no role either, that is if we change the parametrisation of the flow the bundle will remain contracting for this new flow.

Therefore to be contracting or dilating over a compact topological space $X$ is a property of the orbit lamination $\mathcal{L}$, the bundle $E$ and the “parallel transport” on $E$ along leaves of $\mathcal{L}$.

2.0.7 (M, G)-Anosov structure

Let $M$ be a manifold equipped with a pair of continuous foliations $\mathcal{E}^\pm$, whose tangential distributions are $E^\pm$, and such that

$$TM = E^+ \oplus E^-.$$ 

Let $G$ be a Lie group of diffeomorphisms preserving these foliations.
Let \( V \) be a manifold equipped with an Anosov flow \( \psi_t \). Let \( \mathcal{L} \) be the orbit foliation. Let \( \tilde{V} \) be a Galois covering with covering group \( \Gamma \).

We shall say \( V \) is \( A \)-modelled on \( M \) ("A" stands for Anosov), if there exists a representation \( \rho \) of \( \Gamma \) in \( G \), the holonomy representation, a continuous map \( F \) from \( \tilde{V} \) to \( M \), the developing map, enjoying the following properties

- \( \Gamma \)-equivariance:
  \[ \forall \gamma \in \Gamma, \quad F \circ \gamma = \rho(\gamma) \circ F, \]

- Flow invariance:
  \[ F \circ \psi_t(x) = F(x), \]

- Hyperbolicity: We consider the induced bundle \( F^\pm = F^* E^\pm \). By the flow invariance, these bundles are equipped with a parallel transport along the orbit of \( \psi_t \) (induced for instance by the pull back of any connection on \( E^\pm \)). By \( \Gamma \)-equivariance this parallel transport is invariant under \( \Gamma \). Our last hypothesis is that the corresponding lift of the action of \( \psi_t \) on \( F^+ \) (resp. on \( F^- \)) is contracting (resp. dilating).

We also say \((V, \mathcal{L})\) admits a \((M, G)\)-Anosov structure.

2.0.8 Remarks
1. The continuous map \( F \) will have in our examples a very low regularity. It will only be Hölder.

2. As we shall see in the proof of Proposition 2.1, it will turn out that the notion of being \( A \)-modelled is fairly rigid. In other words, if we fix the holonomy representation, the only allowed infinitesimal transformation of \( F \) are translates by \( \psi_t \).

3. One can link this notion to a very classical one. We first consider the associated \( M \)-bundle over \( V \) by \( \rho \), that is \( M_\rho = (M \times \tilde{V})/\Gamma \) where the action is the diagonal one. By construction, we have now a \( \Gamma \)-invariant flow \( \varphi_t \) on \( M \times \tilde{V} \) given by \( \varphi_t(m, v) = (m, \psi_t(v)) \). This flow gives rise to a flow \( \phi_t \) on \( N_\rho \) lifting \( \psi_t \). Notice now that \( F \) gives rise to a flow equivariant section of \( N_\rho \) which we call \( \sigma_F \). Now our hyperbolicity condition just means that \( \sigma_F(V) \) is a hyperbolic subset of \( M_\rho \) with respect to \( \phi_t \).

From this last observation and the stability of hyperbolic sets, we obtain the following Proposition

**Proposition 2.1** Let \( M \) be a manifold equipped with a pair of foliations as described above. Let \( G \) be the group of diffeomorphisms preserving these foliations. Let \( V \) be a compact manifold equipped with an Anosov flow \( \psi_t \). Let \( \tilde{V} \) be a Galois covering with covering group \( \Gamma \). Let \( O \) be the subset of homomorphisms \( \rho \) from \( \Gamma \) to \( G \) which are holonomy representations of \((M, G)\)-Anosov structures. Then \( O \) is open.
Proof: We use the notations of the previous paragraph. We first have to prove that $\sigma_F(V)$ is an isolated hyperbolic set of $N_\rho$. That is, we have to find an isolating neighbourhood $U$ characterised by the property that

$$\sigma_F(V) = \bigcap_{n \in \mathbb{Z}} \phi^n(U).$$

Remember that $M$ has a local product structure given by the two foliations $\mathcal{E}^\pm$.

Let’s denote by $\pi$ the fibration $M_\rho \to V$ described above. We fix for every $x$ in $V$ a complete Riemannian metric $g_x$ on $\pi^{-1}(x) \approx M$ depending continuously on $x$. If $\pi(y) = x$, we consider $d_y^\pm$ the associated distance on the leaves $\mathcal{E}_y^\pm$ through $y$ of the foliations $\mathcal{E}^\pm$. We denote by $B_y^\pm(\epsilon)$ the ball of radius $\epsilon$ on $\mathcal{E}^\pm$ centred at $y$.

Since $M$ has a local product structure, for every $y$, we can find a real positive number $\epsilon$, such that

- for every $x$ in $B_y^+(\epsilon)$, for every $t$ in $B_y^-(\epsilon)$, the leaves $\mathcal{E}_x^-$ and $\mathcal{E}_x^+$ have a unique intersection $G_y(x,z)$ in the ball of centre $y$ and radius $10\epsilon$,
- furthermore $G_y$ is a differentiable embedding.

We set

$$U_y(\epsilon) = G_y(B_y^+(\epsilon) \times B_y^-(\epsilon)).$$

Since $\sigma_F(V)$ is compact, we can find $\epsilon$ that satisfies the above conditions for all $y$ in $\sigma_F(V)$. We now consider

$$U(\epsilon) = \bigcup_{y \in \sigma_F(V)} U_y(\epsilon).$$

This last set is a neighbourhood of $\sigma_F(V)$.

For $\epsilon$ small enough, since $\sigma_F(V)$ is an hyperbolic set, there exists positive constants $A$ and $B$, such that we have

$$\forall z, w \in B_y^+(\epsilon), \forall t > 0, \quad d_y(\phi_{\pm t}(z), \phi_{\pm t}(w)) \leq d_y(z, w)Ae^{-Bt},$$

this last condition implies that $U$ is an isolating neighbourhood.

By Theorem 7.4 of C. Robinson’s book [25], which is stated for diffeomorphisms but whose proof extends to flows from the discussion of the next page, we deduce that $\sigma_F(V)$ is stable. In our case, this implies that after a small perturbation $\hat{\rho}$ of $\rho$, there exists a hyperbolic set $W$ of $N_\hat{\rho}$ a homeomorphism $h$ from $\sigma_F(V)$ to $W$ close to the identity and conjugating the flows up to a small time change.

Now, we prove there exists a section $\hat{\sigma}$ such that $W = \hat{\sigma}(V)$. Indeed, $H = \pi \circ h \circ \sigma_F$ is a mapping from $V$ to $V$, $C^0$-close to the identity and conjugating the flows up to a small time change. Since the flow of $\psi_t$ on $V$ is Anosov, we deduce that $H$ is an homeomorphism. It follows that $\pi : W \to V$ is a homeomorphism. Hence, $W$ is the image of a section $\hat{\sigma}$.

Finally, we know that $W$ is a hyperbolic set; recall that the tangent spaces to the foliations $\mathcal{E}^\pm$ are invariant by the flow, it follows that these tangent spaces remain contracting and dilating bundles after a small perturbation. Q.E.D.
3 Quasi-Fuchsian and Anosov representations

We now give concrete examples of the situation described above.

3.1 Rank 1 subgroups and geometric Anosov structures

Let $G$ be a semi-simple group and $G$ its Lie algebra. Let $H$ be a connected rank 1 semi-simple subgroup of $G$. Associated to this situation, we are going to describe geometric Anosov structures carried by the unit tangent bundle of the symmetric space associated to $H$ with its geodesic flow.

We introduce some notations.

- Let $A$ be the real split Cartan subgroup of $H$ and $Z(A)$ the centraliser of $A$ in $G$. Let $Z_0(A)$ be the connected component of $Z(A)$ containing the identity.
- Write $Z_0(A) = U$. Let $M = G/U$. Notice that $G$ acts on the left on $M$.
- Let $U \cap H = W \times A$, where the Lie algebra of $W$ is orthogonal to $A$.
- Notice that the right action of $A$ on $H/W$ is identified the geodesic flow of the unit tangent bundle of the symmetric space of $H$. Let $\mathcal{L}$ be the orbit foliation of this flow.
- Let $P^+$ (resp. $P^-$) be the parabolic subgroup whose Lie algebra is generated by the eigenvectors of non negative (resp. nonpositive) eigenvalues of $ad(A)$. Notice that $M$ is an open set in $G/P^+ \times G/P^-$. Let $\mathcal{E}^\pm$ be the pair of foliations coming from this product structure on $M$.

We are interested in $(M,G)$-Anosov structures, which we abusively call again $(H,G)$-Anosov structures.

3.1.1 Fuchsian representations.

Our initial result is the following

**Proposition 3.1** Let $\Gamma$ be a torsion free discrete subgroup of $H$. Let $V = \Gamma\backslash H/W$. Then $(V,\mathcal{L})$ admits a canonical $(H,G)$-Anosov structure. The developing map is the identification of $H/W$ with the left orbit of $H$ of the identity class in $M$. The corresponding holonomy representation is the injection of $\Gamma$ in $G$ through $H$. We call such a representation an $(H,G)$-Fuchsian representation.

**Proof:** We let $H$ act on the left on $M = G/U$. Let $m_0$ be the class of the identity in $M$. We define $F$ from $H/W$ to $M$ by

$$F(g) = gm_0.$$  

We consider the pulled back vector bundle $E$ on $H$ defined by

$$E = F^*TM.$$
We also consider the bundles $E^\pm$ that come from the product structure on $M$. We wish to prove that the right $A$ action on $E^\pm$ is contracting/dilating. Notice that $H$ acts on the left on $E$ by an action that lifts the standard left action of $H$ on $H/W$. We denote by $g_*$ the linear map from $E_{m_0}$ to $E_{g^m_0}$ associated to the action of an element $g$ of $H$.

Recall that $W$ is compact and that $W_{m_0} = m_0$. We can now choose a metric $q_{m_0}$ on $E_{m_0}$ invariant by the action of $W$. We equip now the bundle $E$ with the metric $q$ defined by $q_{g^m_0}((u,u)) = q_{m_0}(g^*-1(u),g^*-1(u)))$. This is a well defined metric.

Notice that this metric is invariant by the left action and hence by the action of any discrete subgroup. Furthermore, every left $H$-invariant metric on $E$ arises from this construction.

We finally have a right action of $A$ on $M$ commuting with the left $H$ action, this action of $A$ preserves globally the orbit $F(H/W)$; the corresponding action of $A$ on $H/W$ is the geodesic flow, whenever $H/W$ is identified with the unit tangent bundle of the symmetric space of $H$. We therefore obtain a right action of $A$ on $E$. If $a$ is an element of $A$ and $q$ a left $H$ invariant metric, $\tilde{q} = a^*q$ is also a $H$-invariant metric completely determined by $q$. By construction, we have $\tilde{q}_{m_0} = Ad(a)q_{m_0}$, it follows the action of $A$ on $E^\pm$ is contracting/dilating.

Q.E.D.

3.1.2 Anosov, quasi-Fuchsian representations and limit curves.

Assume now that $\Gamma$ is a cocompact lattice. We define a $(H,G)$-Anosov representation of $\Gamma$ in $G$ as the holonomy of a $(H,G)$-Anosov structure on $\Gamma\backslash H/W$ with its geodesic flow. We define a $(H,G)$-quasi-Fuchsian representation in $G$ as a representation in the connected component of Fuchsian representations in the set of $(H,G)$-Anosov representations in the space of all representations. From Proposition 2.1, the set of $(H,G)$-Anosov representations is open. One can check that $(PSL(2,\mathbb{R}),PSL(2,\mathbb{C}))$-quasi-Fuchsian representations coincides with quasi-Fuchsian representations in the classical sense. Recall that, in this classical case, a quasi-Fuchsian representation preserves a quasi circle on $\mathbb{C}P^1$. We explain now the counterpart of this fact.

Proposition 3.2 Let $\Gamma$ be a cocompact lattice in $H$. Let $\rho$ be a $(H,G)$-Anosov representation of $\Gamma$ in $G$. Let $\partial_\infty \Gamma$ be the boundary at infinity of $\Gamma$. Then, there exist Hölder $\rho$-equivariant mappings $\xi^\pm$ from $\partial_\infty \Gamma$ to $G/P^\pm$ called the positive and negative limit curves of $\rho$, such that

- if $x \neq y$, $\xi^+(x)$ and $\xi^-(y)$ are opposite parabolics,
- finally, if $\gamma^+$ is an attractive fixed point of $\gamma$ in $\partial_\infty \Gamma$, then $\xi^\pm(\gamma^+)$ is an attractive fixed point of $\rho(\gamma)$ in $G/P^\pm$.

Proof: By definition of an Anosov structure, the stable and unstable manifold of $\phi_t$ (along $F(V)$) are the right and left orbit foliations by $P^+$ and $P^-$. Furthermore, these foliations are well known to be Hölder (Theorem 19.1.6. of [12]).
We therefore have $\rho$ equivariant Hölder maps from $\tilde{V}$ to $G/P^+$ and $G/P^-$ constant along the central stable (resp. unstable) foliations of the geodesic flow of $H/W$. Since the space of these central stable leaves is identified with $\partial_\infty \Gamma$, we have proved Proposition 3.2. The final two statements are immediate.

Q.E.D.

3.2 Irreducible $PSL(2, \mathbb{R})$ in $PSL(n, \mathbb{R})$.

From now on we concentrate on the following example. First, we shall consider $V = US$ the unit tangent bundle of a compact hyperbolic surface, and $\tilde{V}$ to be the unit tangent bundle of its universal cover. We consider the lamination $\mathcal{L}$ given by the orbit foliation of the geodesic flow. We consider also $\mathcal{F}^\pm$ the central stable and unstable foliations of the geodesic flow. It is well known that this data just depends on the fundamental group $\pi_1(S)$ of the surface. Indeed we can describe this data the following way. Let

$$\Delta_3 = \{(x_1, x_2, x_3) \in (\partial_\infty \pi_1(S))^3/\exists i \neq j, x_i = x_j\}.$$

Let’s choose an arbitrary orientation on $\partial_\infty \pi_1(S)$ and let $\partial_\infty \pi_1(S)^{3+}$ be the space of positively ordered triples. Then the following identification holds

$$\tilde{V} = \partial_\infty \pi_1(S)^{3+} \setminus \Delta_3,$$
$$V = (\partial_\infty \pi_1(S)^{3+} \setminus \Delta_3)/\pi_1(S).$$

Furthermore, for every $x = (x_+, x_-, x_0)$ in $US$, the leaf $\mathcal{L}_x$ of $\mathcal{L}$ through $x$ in $\tilde{V}$ is

$$\mathcal{L}_x = \{(y_+, y_-, y_0)/y_+ = x_+, y_- = x_-\}.$$

Similarly

$$\mathcal{F}_x^\pm = \{y_+, y_-, y_0)/x_\pm = y_\pm\}.$$

We are going to model these flows on a specific situation, namely we consider

- $G = PSL(n, \mathbb{R})$,
- $H$ the image of the irreducible representation of $PSL(2, \mathbb{R})$.

In order to simplify our notation we shall speak of $n$-quasi-Fuchsian representations (resp. $n$-Anosov structures) or just quasi-Fuchsian representations if the context is clear, instead of $(H, PSL(n, \mathbb{R}))$-quasi-Fuchsian representations.

3.2.1 Description of the model

In this case, $A$ lies in the interior of the Weyl Chamber and $U = Z_0(A)$ is nothing else than the full Cartan subgroup of $G$, that is the subgroup of diagonal matrices in a given basis. It is useful to think of $M = G/U$ as an open set in $Flag \times Flag$, where $Flag$ is the space of flags.

Recall that a point of $M$ is a family of $n$ lines $L = \{L_i\}_{i \in \{1, \ldots, n\}}$ in a direct sum.
3.2.2 A vector bundle description of $n$-Anosov representations

We immediately have

**Proposition 3.3** Let $\rho$ be a $n$-Anosov representation of $\pi_1(S)$ in $PSL(n, \mathbb{R})$ which can be lifted to $SL(n, \mathbb{R})$. Let $E$ be the associated $\mathbb{R}^n$ bundle over $V = US$ with its flat connection $\nabla$. Then $E$ splits as the sum of $n$ continuous line bundles $V_i$ parallel along the leaves of $\mathcal{L}$. Furthermore, let $E^+ = (E_i^+)$ (resp. $E^- = (E_i^-)$) be the corresponding positive and negative flag bundles,

$$E_i^+ = \bigoplus_{j=i}^{n} V_j$$

$$E_i^- = \bigoplus_{j=n}^{n-i-1} V_j.$$

Then, $E_i^+$ (resp. $E_i^-$) is parallel along $\mathcal{F}^+$ (resp. $\mathcal{F}^-$). Finally, if we lift the action of $\mathcal{L}$ by the connection, this action is contracting on $V_i^* \otimes V_j$ for $i > j$.

Furthermore, if we lift the vector bundle $E$ over $\tilde{V}$ and identify this bundle with the trivial bundle $\mathbb{R}^n \times \tilde{V}$ using the flat connection, we have the following identification with the positive and negative limit curves of $\rho$

$$E_{(x_+, x_0, x_-)}^\pm = \xi^\pm(x_\pm).$$

Conversely, the holonomy of such a connection is an $n$-Anosov representation.

**Proof:** It suffices to remark that $L = (V_1, \ldots, V_n)$ is a section of the associated $M = PSL(n, \mathbb{R})/U$ bundle and the tangent spaces to the associated foliations are

$$E_+^i = \bigoplus_{i>j} (V_i^* \otimes V_j),$$

$$E_-^i = \bigoplus_{i<j} (V_i^* \otimes V_j).$$

Q.E.D.

3.2.3 Faithfulness and discreteness

Recall that an element of a semi-simple Lie group is *purely loxodromic* if it is conjugate to an element in the interior of the Weyl chamber. In the case of $PSL(n, \mathbb{R})$, this just means that it is real split with eigenvalues of multiplicity 1.

Some people may prefer to call purely loxodromic elements *strictly hyperbolic*. However, we feel this last terminology may be confusing from the dynamical systems point of view: purely loxodromic element may well have 1 as
an eigenvalue and this is a fact which is not felt to be compatible with strict hyperbolicity for a dynamicist.

We then have

**Proposition 3.4** Let $\rho$ be a $n$-Anosov representation. Then for each $\gamma$ in $\pi_1(S)$ different from the identity, $\rho(\gamma)$ is purely loxodromic. In particular $\rho$ is faithful. Furthermore if $\rho$ is $n$-quasi-Fuchsian, it is irreducible and discrete.

**Proof:** An element is purely loxodromic if it has an attractive fixed point in the space of flags. Therefore the first assertion of the proposition follows from Proposition 3.2. Next, $\rho$ is obviously faithful since a loxodromic element is not trivial. Irreducibility follows from Lemma 10.1 and discreteness from Lemma 10.4 which are both proved in an independent appendix. Q.E.D.

### 3.2.4 Basic properties of limit curves and 2-hyperconvexity.

If $\rho$ is a $n$-Anosov representation, according to Proposition 3.2 we deduce two Hölder mappings $\xi^+$ and $\xi^-$ from $\partial_\infty \pi_1(S)$ to the the corresponding homogeneous spaces $G/P^+$ and $G/P^-$ which in our case are both identified with the space of flags. Since for every attracting point $\gamma^+$ in $\partial_\infty \pi_1(S)$ of some element $\gamma$ in $\pi_1(S)$, $\xi^\pm$ is an attracting point of $\rho(\gamma)$ in the space of flags and such an attracting point is unique for a loxodromic element, we conclude that $\xi^+(\gamma^+) = \xi^-(\gamma^+)$ and hence, by density of the fixed points, that $\xi^+ = \xi^-$. From now on, we shall therefore write

$$\xi^\pm = \xi = (\xi^1, \xi^2, \ldots, \xi^{n-1}).$$

Here, $\xi^i$ takes values in the Grassmannian of $i$-planes in $E = \mathbb{R}^n$. By definition, we have

$$\forall x \in \partial_\infty \pi_1(S), \, \xi^i(x) \subset \xi^{i+1}(x).$$

The curve $\xi$ will be called the **limit curve** of $\rho$. Notice that for $x \neq y$, $\xi(x)$ and $\xi(y)$ are transverse flags since they correspond to opposite parabolics (cf. Proposition 3.2). Hence, we have the following property which we shall call in short **2-hyperconvexity**,\n
$$\forall x, y \in \partial_\infty \pi_1(S), x \neq y \implies \xi^p(x) \oplus \xi^{n-p}(y) = E.$$  \hspace{1cm} (5)

Certainly, the curve $\xi$ cannot be any curve; it has to have some properties. For instance, in the $(\text{PSL}(2, \mathbb{R}), \text{PSL}(2, \mathbb{C}))$ situation this is a quasi-circle. It also follows from S. Choi and W. Goldman’s work that in the $(\text{PSL}(2, \mathbb{R}), \text{PSL}(3, \mathbb{R}))$ case, the curve $\xi^1$ is $C^1$ and bounds a convex set $\xi$.

### 4 Statement of the main results

We state now our main theorem concerning the properties of the curve $\xi$, which generalises S.-Y. Choi and W. Goldman’s situation.
4.1 Quasi-Fuchsian representations, limit curves and Hitchin’s component

Our main Theorem is a slight refinement of Theorem 1.4.

**Theorem 4.1** Let \( \rho \) be a representation in Hitchin’s component. Then \( \rho \) is quasi-Fuchsian. Furthermore, let

\[
\xi = (\xi^1, \xi^2, \ldots, \xi^{n-1})
\]

be its limit curve. Then \( \xi^1 \) is a hyperconvex Frenet curve, and \( \xi \) is its osculating flag. Furthermore, for any triple of distinct points \( (x, y, z) \) of \( \partial_\infty \pi_1(S) \) the following sum is direct,

\[
(\xi^{k+1}(y) + \xi^{n-k-2}(x)) + (\xi^{k+1}(z) \cap \xi^{n-k}(x)) = E. \tag{6}
\]

We recall that saying \( \xi \) is the osculating flag of the hyperconvex Frenet curve \( \xi^1 \) means the following.

- Let \((x_1, \ldots, x_p)\) be pairwise distinct points of \( \partial_\infty \pi_1(S) \). Let \( p \) be an integer. Let \((n_1, \ldots, n_l)\) be positive integers such that

\[
l = \sum_{i=1}^{i=p} n_i \leq n.
\]

Then, the following sum is direct

\[
\sum_{i=1}^{i=l} \xi^{n_i}(x_i) \tag{7}
\]

- Furthermore for every \( x \in \partial_\infty \pi_1(S) \),

\[
\lim_{(y_1, \ldots, y_p) \to x} \left( \bigoplus_{i=1}^{i=p} \xi^{n_i}(y_i) \right) = \xi^1(x). \tag{8}
\]

As a consequence \( \xi \) is completely determined by \( \xi^1 \), and \( \xi^1 \) is a \( C^1 \) curve. Theorem 4.1 together with Proposition 3.4 give rise to Theorem 1.5. Theorem 1.4 is proved in Paragraph 9.1.1.

4.2 Converse results

It turns out that the curve \( \xi^1 \) contains all the information needed to reconstruct our geometry.

**Theorem 4.2** Let \( \rho \) be a representation of \( \pi_1(S) \) in \( SL(E) \). Let \( \xi^1 \) be a \( \rho \)-equivariant continuous map from \( \partial_\infty \pi_1(S) \) to \( \mathbb{P}(E) \). Assume that for all distinct points \( (x_1, \ldots, x_n) \), we have the following direct sum

\[
\xi^1(x_1) + \ldots + \xi^1(x_n) = E.
\]
Then $\rho$ is a $n$-Anosov representation and $\xi^1$ is the projection in $\mathbb{P}(E)$ of the limit curve $\xi$ of $\rho$. Finally, $\xi^1$ is a hyperconvex Frenet curve and $\xi$ is its osculating flag.

This Theorem is proved in Paragraph 6.2. Unfortunately, we cannot prove every $n$-Anosov representation is quasi-Fuchsian. To our present knowledge, the set of $n$-Anosov representation could well not be connected. However, Olivier Guichard recently proved the following result, which was conjectured in an earlier version of the present paper and which gives a complete geometric characterisation of Hitchin’s component [9].

**Theorem 4.3** [Guichard] Let $\rho$ be a representation of $\pi_1(S)$ in $SL(E)$. Let $\xi^1$ be a $\rho$-equivariant continuous map from $\partial_\infty \pi_1(S)$ to $\mathbb{P}(E)$. Assume that for all distinct points $(x_1, \ldots, x_n)$, we have the following direct sum

$$\xi^1(x_1) + \ldots + \xi^1(x_n) = E.$$ 

Then the representation $\rho$ is in Hitchin’s component.

These two theorems provide a converse result to Theorem 4.1.

### 5 Hyperconvex curves

#### 5.1 Definition and notations

Let $\xi$ be a map from an interval $J$ to $\mathbb{P}(E)$. We shall say $\xi$ is hyperconvex, if for all $n$-uples of distinct points $(x_1, \ldots, x_n)$ we have

$$\xi(x_1) + \ldots + \xi(x_n) = E.$$

As a notation if $p \leq n$, we write if $X = (x_1, \ldots, x_p)$ is a $p$-uple of distinct points

$$\xi^{(p)}(X) = \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_p).$$

We also say $X < x$ if for all $i$, $x_i < x$. We write $X \rightarrow x^+$ as a shorthand for $X \rightarrow x$, $X < x$. We have a similar convention for $X \rightarrow x^-$. We shall actually need a refinement of the notion of hyperconvexity in order to take in account non continuous maps. Let $\Omega$ be an orientation on $E$. Let $\xi$ be a map (not necessarily continuous) from an interval $J$ to $\mathbb{P}(E)$. We say that $\xi$ is $\ast$-hyperconvex if the following sum is direct,

$$\xi(\ast)(X) = \xi(x_1) + \ldots + \xi(x_p).$$

Furthermore, we require there exists a map $\hat{\xi}$, the lift of $\xi$, with values in $E \setminus \{0\}$ such that the following holds

1. for all $y$ in $J$, $\hat{\xi}(y) \in \xi(y)$,
2. for all \( n \)-uple of distinct increasing points \( X = (x_1, \ldots, x_n) \), we have
\[
\Omega(\hat{\xi}(x_1), \ldots, \hat{\xi}(x_n)) \geq 0.
\]
Notice that this last inequality and the first condition actually implies
\[
\Omega(\hat{\xi}(x_1), \ldots, \hat{\xi}(x_n)) > 0. \tag{9}
\]
The existence of this “coherent” lift should be understood in the following way: the map \( \xi \), though not being continuous, preserves some ordering. It is also obvious that such a lift exists whenever \( \xi \) is continuous. It follows that every hyperconvex curve defined on a (contractible) interval is in particular \( * \)-hyperconvex.

5.2 Left and right osculating flags

The main result of this section is the following

**Lemma 5.1** Let \( \xi \) be an \( * \)-hyperconvex map from \( J \) to \( P(E) \). Assume that the sequence \( \{X_m\}_{m \in \mathbb{N}} \) (resp. \( \{Y_m\}_{m \in \mathbb{N}} \)) converges to \( (x_1, \ldots, x_p) \) (resp. \( (y_1, \ldots, y_{n-p}) \)) with
\[
x_1 \leq \ldots \leq x_p < y_1 \leq y_2 \ldots \leq y_{n-p}.
\]
Assume also that \( \{\xi^{(p)}(X_m)\}_{m \in \mathbb{N}} \) (resp. \( \{\xi^{(n-p)}(Y_m)\}_{m \in \mathbb{N}} \)) converges to \( F \) (resp. \( G \)).
Then
\[
F \oplus G = E. \tag{10}
\]

Furthermore, for every \( p \), with \( n \geq p \geq 1 \), there exist maps \( \xi^p_+ \) and \( \xi^p_- \) from \( J \) to \( \text{Gr}(p, E) \) such that
\[
\lim_{X \to x^\pm} \xi^{(p)}_\pm(X) = \xi^p_\pm(x), \tag{11}
\]
\[
\lim_{(z,y) \to x^\pm y^\pm} (\xi(z) \oplus \xi^p_\pm(y)) = \xi^{p+1}_\pm(x), \tag{12}
\]
\[
\xi^p_\pm(x) \subset \xi^{p+1}_\pm(x). \tag{13}
\]

Finally, if \( \xi^p_+ = \xi^p_- \), then both maps are continuous and
\[
\lim_{X \to x} \xi^{(p)}_\pm(X) = \xi^p_\pm(x), \tag{14}
\]
In particular, if \( \xi^1_+ = \xi^1_- \), then both maps are equal to \( \xi \) and the latter is continuous.

We shall begin by some preliminaries concerning increasing maps, then prove the Lemma.
5.3 Increasing maps

We make precise some properties of increasing maps. Let $p$ be some integer. Let $I$ be an oriented interval. We define

$$I^{(p)} = \{(x_1, \ldots, x_p) \in I^p / x_i \leq x_{i+1}\}.$$

We define partial orderings on $I^{(p)}$ by

$$(x_1, \ldots, x_p) \leq (y_1, \ldots, y_p) \text{ iff } \forall i, x_i \leq y_i,$$

$$(x_1, \ldots, x_p) < (y_1, \ldots, y_p) \text{ iff } \forall i, x_i < y_i,$$

We also define

$$\hat{I}^{(p)} = \{(x_1, \ldots, x_p) \in I^p / x_i < x_{i+1}\}.$$

Let now $f$ be a map from $\hat{I}^{(p)}$ to $\mathbb{R}$. We say $f$ is increasing if for every $(x_1, \ldots, x_p)$ in $\hat{I}^{(p)}$ and for every $j$,

$$x_{j-1} < z < y < x_{j+1} \implies f(x_0, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_p) \leq f(x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_p).$$

Notice that this immediately implies

$$X \leq Y \implies f(X) \leq f(Y).$$

For an increasing map $f$ and $X$ in $I^{(p)}$, we define

$$f^+(X) = \inf_{Y \geq X} f(Y)$$

$$f^-(X) = \sup_{Y < X} f(Y).$$

The next proposition summarises the properties that we shall need in the sequel. All these properties are immediate.

**Proposition 5.2** Assume $f$ is increasing. Then

$$f^-(X) = \lim_{Y \uparrow \geq X} f(Y)$$

$$f^+(X) = \lim_{Y \downarrow \geq X} f(Y)$$

$$f^-(X) \leq f^+(X)$$

$$f^+(X) \leq f^-(Y), \text{ if } X < Y,$$

$$f^-(X) \geq f^+(Y), \text{ if } X > Y.$$

Finally, assume that $f^+ = f^-$ are equal everywhere, then they are both continuous and

$$\lim_{Y \rightarrow X} f(Y) = f^+(X).$$
5.4 *-Hyperconvex curves and increasing maps

We begin with the following observation which follows at once from hyperconvexity. Let \((y_1, \ldots, y_{n-1}, w_1, w_2)\) be distinct points of the one-dimensional manifold \(J\). Write \(Y = (y_1, \ldots, y_{n-1})\) and \(W = (w_1, w_2)\). Let \(\xi\) be a *-hyperconvex curve from \(J\) to \(\mathbb{P}(E)\). Then

\[
\dim \left( \xi^{(n-1)}(Y) \cap \xi^{(2)}(W) \right) = 1. \tag{20}
\]

Let \(I\) an interval contained in \(J \setminus \{w_1, w_2\}\). We now consider the map

\[
\begin{align*}
I & \rightarrow \mathbb{P}(\xi^{(2)}(W)) \setminus \{\xi(w_1)\} \\
Y = (y_1, \ldots, y_{n-1}) & \mapsto \xi^{(n-1)}(Y) \cap \xi^{(2)}(W).
\end{align*}
\]

Notice this map is well defined thanks to Assertion (20) and the fact that \(\xi(y_1) \oplus \cdots \oplus \xi(y_{n-1}) \oplus \xi(w_1) = E\).

We prove now

**Proposition 5.3** Assume that the lift \(\hat{\xi}\) of \(\xi\) is well defined on \(J\). Then, for a suitable choice of orientation on \(I\) and \(\mathbb{P}(\xi^{(2)}(W)) \setminus \{\xi(w_1)\}\), the map \(f_W\) is increasing.

**Proof:** Let \(u_1 = \hat{\xi}(w_1)\). If \(y_1 < y_2 < \ldots < y_{n-1}\) are in \(I\), we have by Inequality \(\mathbb{H}\)

\[
\Omega(u_1, \hat{\xi}(y_1), \ldots, \hat{\xi}(y_{n-1})) > 0.
\]

Finally, we choose the orientation on \(\xi^{(2)}(W)\) given by the form

\[
\omega(w, t) = \Omega(\hat{\xi}(y_1), \ldots, \hat{\xi}(y_{n-2}), w, t).
\]

Thanks to Inequality \(\mathbb{H}\), we notice this orientation is independent on the choice of \((y_1, \ldots, y_{n-2})\) in \(I\) provided that

\[
y_1 < \ldots < y_{n-2}.
\]

This choice gives an ordering of \(\mathbb{P}(\xi^{(2)}(W)) \setminus \{\xi(w_1)\}\) in the following way. For every \(L\) in \(\mathbb{P}(\xi^{(2)}(W)) \setminus \{\xi(w_1)\}\), we choose \(x(L)\) in \(L\) such that \(\omega(u_1, x(L)) > 0\). Finally, we say \(L < L'\) if

\[
\omega(x(L), x(L')) > 0.
\]

We can now prove that the map \(f_W\) is increasing. Let

\[
Q = \xi(y_1) \oplus \cdots \xi(y_{n-2}).
\]

Let \(z\) be such that

\[
y_1 < y_2 < \ldots < y_{j-1} < z < y_j < \ldots < y_{n-2}.
\]
Let \( L_z = f_W(y_1, \ldots, y_{j-1}, z, y_j, \ldots, y_{n-2}) \).

Let \( \hat{x}(z) \) in \( L_z \) be such that
\[
\hat{x}(z) = (-1)^{n-j-1}\hat{\xi}(z) + w(z), \quad w(z) \in Q.
\]

Then \( \omega(u_1, \hat{x}(z)) > 0. \) Assume now that
\[ y_1 < \ldots < y_{j-1} < z < t < y_j < \ldots < y_{n-2}. \]

Then we have,
\[
\omega(\hat{x}(z), \hat{x}(t)) = \Omega(\hat{\xi}(y_1), \ldots, \hat{\xi}(y_{j-1}), \hat{\xi}(z), \hat{\xi}(t), \hat{\xi}(y_j), \ldots, \hat{\xi}(y_{n-2})) > 0.
\]

We have just proved
\[
f_W(y_1, \ldots, y_{j-1}, z, y_j, \ldots, y_{n-2}) < f_W(y_1, \ldots, y_{j-1}, t, y_j, \ldots, y_{n-2}).
\]

Q.E.D.

5.5 Proof of Lemma 5.1

5.5.1 First step: Assertion (11)

PROOF: We use the notations of the previous paragraph. Let \( p \) be an integer less than \( n. \) Let \( x \) be a point in \( J, \) \( I \) a small neighbourhood of \( x \) and \( Z = (y_1, \ldots, y_{n-p-1}, w_1, w_2) \) be some tuple of cyclically oriented points in \( J \setminus \{x\}. \)

Write \( Y = (y_1, \ldots, y_{n-p-1}), \) \( W = (w_1, w_2). \) According to Propositions 5.2 and 5.3 we obtain that there exist maps \( F_{\pm}^{\pm} \) from \( I \) to \( P(\xi(2)(W)) \) such that
\[
\lim_{X \to x_{\pm}} (\xi(p)(X) \oplus \xi(n-p-1)(Y)) \cap \xi(2)(W) = F_{\pm}^{\pm}(x).
\]

Using the fact the choice of \( Z \) is arbitrary, we will now show that there exist maps \( \xi_{\pm}^{\pm} \) verifying Assertion (11) and characterised by
\[
(\xi_{\pm}^{\pm}(x) \oplus \xi(n-p-1)(Y)) \cap \xi(2)(W) = F_{\pm}^{\pm}(x).
\]

Let’s prove this last point in detail.

This is done in two steps. First, let’s fix \( Y. \) Let \( U \) be an interval of \( J \pi_1(S) \setminus I \cup Y. \) We consider now the subspace
\[
H_{Y}^{\pm}(x) = \sum_{W \in U^{(2)}} F_{p,Y,W}^{\pm}(x).
\]

Notice that,
\[
\sum_{W \in U^{(2)}} (\xi(p)(X) \oplus \xi(n-p-1)(Y)) \cap \xi(2)(W) \subset \xi(p)(X) \oplus \xi(n-p-1)(Y),
\]

22
hence,

\[
\dim \left( \sum_{W \in U(2)} \left( \xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y) \right) \cap \xi^{(2)}(W) \right) \leq n - 1. \tag{21}
\]

We deduce that \( \dim(H^\pm_Y) \leq n - 1 \). We will now prove that \( \dim(H^\pm_Y) = n - 1 \) and

\[
\lim_{X \to x^\pm} \left( \xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y) \right) = H^\pm_Y(x). \tag{22}
\]

Let \( \{X_n\}_{n \in \mathbb{N}} \) be a subsequence converging to (let’s say) \( x^+ \), such that

\[
P_n = \xi^{(p)}(X_n) \oplus \xi^{(n-p-1)}(Y),
\]

converges to some hyperplane \( H \). By hyperconvexity, we choose \( w_1 \) in \( U \) such that

\[
H \oplus \xi(w_1) = E.
\]

By hyperconvexity again, we choose \((w_2, \ldots, w_n)\) in \( U \) such that

\[
\xi(w_1) \oplus \ldots \oplus \xi(w_n) = E.
\]

Let \( W_i = (w_1, w_i) \). We notice then that

\[
H = \bigoplus_i (H \cap \xi^{(2)}(W_i)).
\]

Since

\[
H \cap \xi^{(2)}(W_i) = F^+_p,\xi^{(n-p-1)}(Y)(x).
\]

It follows that \( H \subset H^\pm_Y \), hence \( \dim(H^\pm_Y) \geq n - 1 \). Combining with Inequality \(21\), we obtain that \( H = H^\pm_Y \), hence Assertion \(22\).

Our next step follows a similar path. We now consider an interval \( U \) not containing \( x \), and set

\[
\xi^{(p)}_\pm(x) = \bigcap_{Y \in U(n-p-1)} H^\pm_{p,Y}(x).
\]

Since

\[
\xi^{(p)}(X) \subset \bigcap_{Y \in U(n-p-1)} \left( \xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y) \right),
\]

we get,

\[
\dim \xi^{(p)}_\pm(x) \geq p. \tag{23}
\]

We prove now that \( \dim \xi^{(p)}_\pm(x) \leq p \) and Assertion \(11\). Let again \( \{X_n\}_{n \in \mathbb{N}} \) be a subsequence converging to (let’s say) \( x^+ \), such that \( \xi^{(p)}(X_n) \oplus \xi^{(n-p-1)}(Y) \) converges to some \( p \)-plane \( P \). By hyperconvexity, we now choose \((y_1, \ldots, y_{n-p})\) in \( U \) such that

\[
P \oplus \xi(y_1) \oplus \ldots \oplus \xi(y_{n-p}) = E.
\]
Write $Y_i = (\ldots, y_j, \ldots)_{j \neq i}$, and notice that

$$P = \bigcap_i (P \oplus \xi^{(n-p-1)}(Y_i)).$$

In particular, since

$$P \oplus \xi^{(n-p-1)}(Y_i) = H_{\mu'},$$

we get that $\xi^m(x) \subset P$, hence $\xi^m(x) = P$ thanks to Inequality [23], hence Assertion [21], Q.E.D.

5.5.2 Second step: completion of the proof of Lemma 5.1

Proof: Assume that $\{X_m\}_{m \in \mathbb{N}}$ (resp. $\{Y_m\}_{m \in \mathbb{N}}$) converges to $(x_1, \ldots, x_p)$ (resp. $(y_1, \ldots, y_{n-p})$) with

$$x_1 \leq \ldots \leq x_p < y_1 \leq y_2 \ldots \leq y_{n-p}.$$

Assume that $\{\xi^{(p)}(X_m)\}_{m \in \mathbb{N}}$ (resp. $\{\xi^{(n-p)}(Y_m)\}_{m \in \mathbb{N}}$) converges to $F$ (resp. $G$). We want to show

$$F \oplus G = E. \quad (24)$$

Let’s assume this is not true. We consider now the smallest integer $m$ for which there exist integers $p$ and $q$, such that $p + q = m$, satisfying the following property: there exist sequences $\{X_m\}_{m \in \mathbb{N}}$ (resp. $\{Y_m\}_{m \in \mathbb{N}}$) converging to $(x_1, \ldots, x_p)$ (resp. $(y_1, \ldots, y_{n-p})$) with

$$x_1 \leq \ldots \leq x_p < y_1 \leq y_2 \ldots \leq y_{n-p}$$

such that

- $\{\xi^{(p)}(X_m)\}_{m \in \mathbb{N}}$ (resp. $\{\xi^{(q)}(Y_m)\}_{m \in \mathbb{N}}$) converges to $P$ (resp. $Q$);
- $P \cap Q \neq \{0\}$.

Let $H = P + Q$. Write $X_m = (x_1^m, \ldots, x_p^m)$ and $Y_m = (y_1^m, \ldots, y_q^m)$, for $m$ large enough with $x_i^m < x_{i+1}^m < y_j^m < y_{j+1}^m$. Let’s introduce

$$X^- = (x_1^m, \ldots, x_{p-1}^m), \quad Y^- = (y_2^m, \ldots, y_q^m).$$

We can assume, after extracting a subsequence, that $\{\xi^{(p-1)}(X_-)\}_{m \in \mathbb{N}}$ and $\{\xi^{(q-1)}(Y_-)\}_{m \in \mathbb{N}}$ converge respectively to $P^-$ and $Q^-$. By the minimality of $m = p + q$, we obtain that

$$P^- \oplus Q^- = P \oplus Q^- = P + Q = H. \quad (25)$$

Using hyperconvexity, we now choose $Z = (z_1, \ldots, z_{n-p-q})$ points in $U$, $W = (w_1, w_2)$ not in $U$ such that the following sums are direct

$$H + \xi^{(n-p-q)}(Z) + \xi(w_1) = E, \quad (26)$$

$$P^- + Q^- + \xi^{(n-p-q)}(Z) + \xi(2)(W) = E. \quad (27)$$
Using the notations of Paragraph 5.4, we consider now the family of maps $g_m$ defined by

$$g_m(t) = f_W(X_m, t, Y_m, Z).$$

By Proposition 5.3, all these maps are increasing. From (25) and (26), we deduce that

$$\lim_{m \to \infty} g_m(x_m^p) = \lim_{m \to \infty} g_m(y_1^m) = (H \oplus \xi^{(n-p-q)}(Z)) \cap \xi^{(2)}(W) := D.$$

Recall that

$$\lim_{m \to \infty} x_m^p = x_p, \quad \lim_{m \to \infty} y_1^m = y_1.$$  

Since all the maps $g_m$ are increasing, it follows that for all $t$ in the interval $I$ joining $x_p$ and $y_1$, we have

$$\lim_{m \to \infty} g_m(t) = D.$$  

On the other hand for all $t$ in $I$, from (27), we have

$$W_m(t) := \xi^{(p-1)}(X_m) \oplus \xi^{(q-1)}(Y_m) \oplus \xi(t) \oplus \xi^{(n-p-q)}(Z)$$

$$= \xi^{(p-1)}(X_m) \oplus \xi^{(q-1)}(Y_m) \oplus \xi^{(n-p-q)}(Z) \oplus g_m(t).$$

It follows that for all $t$ in $I$,

$$\xi(t) \subset \lim_{n \to \infty} W_m(t) = P^- \oplus Q^- \oplus \xi^{(m-n-2)}(Z) \oplus D \subset E.$$  

This last assertion contradicts hyperconvexity, hence finishes the proof of Assertion (10).

Assertions (10) and (11) imply trivially Assertion (13). We finally notice that the final assertion concerning the case where $\xi_p^+ = \xi_p^-$ is a consequence of the last statement of Proposition 5.2. Q.E.D.

6 Preserving a hyperconvex curve

We prove the following converse of Proposition 3.3.

**Theorem 6.1** Let $\rho$ be a representation of $\pi_1(S)$ in $SL(E)$. Let $\xi^1$ be a $\rho$-equivariant $*$-hyperconvex map from $\partial_\infty \pi_1(S)$ to $\mathbb{P}(E)$. Assume that there exist for all integers $p$ less than $n$, $\rho$-equivariant maps $\xi^p_\pm$ from $\partial_\infty \pi_1(S)$ to $Gr(p, E)$ such that

$$\lim_{y \to x^\pm} (\xi^1(y) \oplus \xi^p_\pm(x)) = \xi^{p+1}_\pm(x).$$  

We also assume that if

- $\{x_m\}_{m \in \mathbb{N}}$ (resp. $\{y_m\}_{m \in \mathbb{N}}$) converges to $x$, (resp. to $y$), with $x \neq y$,
• if \( p + q \leq n \) and \( Z = (z_1, \ldots, z_{n-p-q}) \) are \( n - p - q \) points pairwise distinct and different from \( x \) and \( y \),
• \( \{ \xi^p_+ (x_m) \}_{m \in \mathbb{N}} \) (resp. \( \{ \xi^p_- (x_m) \}_{m \in \mathbb{N}} \), \( \{ \xi^q_+(y_m) \}_{m \in \mathbb{N}} \)) converges to \( P^+ \) (resp. to \( P^-, Q \))

Then
\[
P^+ \oplus Q \oplus \xi^1(z_1) \oplus \ldots \oplus \xi^1(z_{n-p-q}) = E,
\]
(29)

As a conclusion, then
• \( \rho \) is \( n \)-Anosov,
• \( \xi^+_p = \xi^-_p \),
• \( (\xi^1, \xi^2, \ldots, \xi^{n-1}) \) is the limit curve of \( \rho \).

The following corollary is immediate

**Corollary 6.2** Let \( \rho \) be a representation of \( \pi_1(S) \) in \( SL(E) \). Let
\[ \xi = (\xi^1, \ldots, \xi^{n-1}) \]
be a \( \rho \)-equivariant continuous map from \( \partial_\infty \pi_1(S) \) to \( Flag(E) \). Assume \( \xi^1 \) is hyperconvex and that
\[
\forall x, y \in \partial_\infty \pi_1(S), \forall p, \ x \neq y \quad \Rightarrow \quad \xi^p(x) \oplus \xi^{n-p}(y) = E,
\]
\[
\lim_{y \to x} (\xi^1(y) \oplus \xi^p(x)) = \xi^{p+1}(x).
\]

Then \( \rho \) is \( n \)-Anosov and \( \xi \) is the limit curve of \( \rho \).

As a corollary proved in Paragraph 6.2 we obtain Theorem 4.2

6.1 **Proof of Theorem 6.1**

**Proof:** Let’s choose an orientation on \( \partial_\infty \pi_1(S) \). Let
\[
M = \{(x, y, w) \in \partial_\infty \pi_1(S)^3, \text{ distinct and cyclically oriented } \}.
\]

Notice that \( \pi_1(S) \) acts properly on \( M \), in such a way the quotient is compact and homeomorphic to the unit tangent bundle of the surface \( S \). We write the generic element \( x \) of \( M \) as \( x = (x_+, x_0, x_-) \). Consider on \( M \) the lamination whose leaves are
\[
L_{x_+, x_-} = \{(x_+, w, x_-) \in M/w \in \partial_\infty \pi_1(S)\}.
\]

This lamination is \( \pi_1(S) \) equivariant and its quotient is identified with the lamination by leaves of the geodesic flow. Consider also the following 2-dimensional laminations whose leaves are:
\[
F^+_{x_+} = \{(x_+, w, y) \in M/w, y \in \partial_\infty \pi_1(S)\}
\]
\[
F^-_{x_-} = \{(y, w, x_-) \in M/w, y \in \partial_\infty \pi_1(S)\}.
\]
Now consider the $E$-associated bundle on $M/\pi_1(S)$ to $\rho$, also denoted abusively by $E$. Consider the subbundles $E^+_i$ and $E^-_i$ of $E$ given by
\[
E^+_i(x_+,x_0,x_-) = \xi^i_+(x_+)
\]
\[
E^-_i(x_+,x_0,x_-) = \xi^i_-(x_-).
\]
Notice these bundles are not a priori continuous. The bundle $E^+_i$ (resp. $E^-_i$) are parallel along the leaves of $\mathcal{F}^+$ (resp. $\mathcal{F}^-$). Let $V^i = E^+_i \cap E^-_{n-i+1}$. It is a well defined 1-dimensional subbundle of $E$ (thanks to Hypothesis (29)), which is parallel along the leaves of $\mathcal{L}$. Notice that a subspace supplementary to $V^i$ is $E^+_{i-1} \oplus E^-_{n-i}$. Denote by $\alpha_i$ a 1-form whose kernel is that supplementary subspace.

We first define a metric on $((V^i)^* \otimes V^{i+1})_w$. Write $w = (x_+,x_0,x_-)$. Let $u$ be a nonzero element of $V^i$, $z(w)$ a nonzero element of $\xi^1(x_0)$. The metric on $((V^i)^* \otimes V^{i+1})_w$ is given by
\[
\|\phi\|_w = \frac{|\langle \alpha_{i+1}|\phi(u)\rangle \langle \alpha_i|z(w)\rangle|}{\langle \alpha_{i+1}|z(w)\rangle \langle \alpha_i|u\rangle}.
\]
This metric is well defined since by Hypothesis (29) the following sum is direct
\[
\xi^{j-1}_+(x_+) + \xi^{n-j}_-(x_-) + \xi^1(x_0),
\]
and in particular for all $j$,
\[
\langle \alpha_j|z(w)\rangle \neq 0.
\]
Obviously, this metric is independent of the choice of $u$ and $z(w)$. For the moment this metric is not obviously continuous (although with little effort, one could prove it is bounded).

We are going to prove now that $(V^i)^* \otimes V^{i+1}$ are weakly contracting bundles for this metric. By weakly contracting bundle, we mean the following: if $\sigma$ is a parallel section of $(V^i)^* \otimes V^{i+1}$ along a leaf of $\mathcal{L}$, then
\[
\lim_{x_0 \to x_+} \|\sigma\|(x_+,x_0,x_-) = 0
\]
\[
\lim_{x_0 \to x_-} \|\sigma\|(x_+,x_0,x_-) = \infty.
\]
Notice that a parallel section $\sigma$ of $(V^i)^* \otimes V^{i+1}$ along a leaf of $\mathcal{L}$, corresponds to a fixed element $\phi$ in $(V^i)^* \otimes V^{i+1}$. Then
\[
\frac{\|\phi\|_w}{\|\phi\|_t} = \frac{|\langle \alpha_{i+1}|z(t)\rangle \langle \alpha_i|z(w)\rangle|}{|\langle \alpha_{i+1}|z(w)\rangle \langle \alpha_i|z(t)\rangle|}.
\]
Let $w = (x_+,x_0,x_-)$ and imagine now that $x_0$ converges to $x_+$. By Hypothesis (28), we now may choose $z'(w) = \alpha + \beta$ in
\[
\xi^1(x_0) \oplus E^+_{i-1} = \xi^1(x_0) \oplus \xi^i_+(x_+),
\]
27
with \( 0 \neq \alpha \in \xi^i(x_0) \) and \( \beta \in \xi_i^{-1}(x_+) \) such that \( z'(w) \) converges to a nonzero vector \( u \in \xi_i(x_+) \cap \xi_i^{-1}(x_-) \) when \( x_0 \) converges to \( x_- \). Notice that
\[
\frac{\langle \alpha_{i+1} | z(t) \rangle \langle \alpha_{i} | z(w) \rangle}{\langle \alpha_i | z(t) \rangle \langle \alpha_{i+1} | z(w) \rangle} = \frac{\langle \alpha_{i+1} | z(t) \rangle \langle \alpha_{i} | z'(w) \rangle}{\langle \alpha_i | z(t) \rangle \langle \alpha_{i+1} | z'(w) \rangle}.
\]

To conclude, we remark that
\[
\lim_{x_0 \to x_+} \frac{\langle \alpha_i | z'(w) \rangle}{\langle \alpha_{i+1} | z'(w) \rangle} = \frac{\langle \alpha_i | u \rangle}{\langle \alpha_{i+1} | u \rangle} = \infty. \tag{30}
\]

A similar reasoning when \( x_0 \) tends to \( x_- \), implies the bundles are weakly contracting.

We can now show that \( \xi^p_+ = \xi^p \). First, by Hypothesis 20, \( \xi^p_+ (x_+) \) is the graph of a homomorphism \( \psi \) in \( \text{Hom}(E_0^p, E_0^p) = (E_0^p)^* \otimes E_0^p \). Since \( (E_0^p)^* \otimes E_0^p \) is a weakly contracting bundle and \( \psi \) is parallel, to prove \( \xi^p_+ = \xi^p \), it suffices to show \( \| \psi \| \) is uniformly bounded. Let’s prove that. Assume we have a sequence of points \( \{x_m\}_{m \in \mathbb{N}} \) in \( M \) such that \( \{\| \psi \|_{x_m}\}_{m \in \mathbb{N}} \) tends to \( +\infty \). Since \( \pi_1(S) \) acts cocompactly on \( M \) and the whole situation is invariant under \( \pi_1(S) \), we can as well assume that \( \{x_m\}_{m \in \mathbb{N}} \) converges to \( y \) in \( M \). We can extract a subsequence such that

- for all \( i \), \( \{(V_i)_{x_m}\}_{m \in \mathbb{N}} \) converges to \( W_i \) in \( E_y \).
- \( \{(E_\pm_i)_{x_m}\}_{m \in \mathbb{N}} \) converges to \( F_\pm_i \).
- \( \{\xi^p_+(x_m)\}_{m \in \mathbb{N}} \) converges to \( Q \).

By Hypothesis 20, \( Q \) is a graph of a map \( \phi \) from \( F^p_+ \) to \( F^p \). In particular, \( \| \phi \| \) is bounded. The contradiction now follows from
\[
\lim_{m \to +\infty} (\| \psi \|_{x_m}) = \| \phi \| \neq \infty.
\]

Now that we know \( \xi^p_+ = \xi^p \), we can conclude that both maps are continuous by Proposition 5.3. The metric we have defined previously is therefore continuous. We can repeat the argument above about the weakly contracting property, the limits that we obtain are now uniform, and therefore the argument shows the bundles are contracting. Let’s do it in more detail. To prove the bundles \( (V_i)^* \otimes V_{i+1} \) are contracting, we need to show that there exist constant \( t_0 > 0 \), such that if \( \Psi_t \) is the lift of the geodesic flow \( \psi_t \) on \( M \), then for every vector \( \sigma \) in \( (V_i)^* \otimes V_{i+1} \), we have
\[
\forall t > t_0, \quad \| \Psi_t(\sigma) \| \leq \frac{1}{2} \| \sigma \|.
\]

Let’s prove it by contradiction. If this is not true, we would have a sequence of points \( \{w_n\}_{n \in \mathbb{N}} \) of \( M/\pi_1(S) \) a sequence \( \{t_n\}_{n \in \mathbb{N}} \) converging to \( +\infty \), a sequence \( \{\sigma_n\}_{n \in \mathbb{N}} \) in \( ((V_i)^* \otimes V_{i+1})_{w_n} \) such that
\[
\frac{\| \Psi_{t_n}(\sigma_n) \|}{\| \sigma_n \|} \geq \frac{1}{2}. \tag{31}
\]
We may now lift the sequence \( w_n \) to \( M \) and assume, since the action of \( \pi_1(S) \) is cocompact, that the sequence converges to \( w_0 = (x_0^+, x_0^0, x_0^-) \). Let’s write

\[
\begin{align*}
  w_n &= (x_n^+, x_n^0, x_n^-), \\
  \phi_{t_n}(w_n) &= (x_n^+, y_n^0, x_n^-).
\end{align*}
\]

By assumption

\[
\begin{align*}
  \lim_{n \to \infty} (x_n^+) &= \lim_{n \to \infty} (y_n^0) = x_0^+, \\
  \lim_{n \to \infty} (x_n^0) &= x_0^0, \\
  \lim_{n \to \infty} (x_n^-) &= x_0^-.
\end{align*}
\]

Let’s denote by \( z(t) \) a nonzero element in \( \xi_1(t) \), we then have

\[
\| \Psi_{t_n}(\sigma_n) \| \geq 1/2.
\]

We now obtain the contradiction knowing that

\[
\lim_{n \to \infty} (\xi^1(y_0^n) \oplus \xi^i(x_n^+)) = \xi^{i+1}(x_0^+),
\]

which implies as above

\[
\lim_{n \to \infty} \left( \frac{\langle \alpha_{i+1} | z(x_n^+) \rangle \langle \alpha_i | z(y_0^n) \rangle}{\langle \alpha_{i+1} | z(y_0^n) \rangle \langle \alpha_i | z(x_n^+) \rangle} \right) = 0.
\]

From this it follows the bundle are indeed contracting. The conclusion finally follows from Proposition 3.3. Q.E.D.

### 6.2 Proof of Theorem 4.2

We begin with a lemma of independent interest that will be used in the sequel, then proceed to the proof.

#### 6.2.1 Direct sums and limits

Our first lemma is the following:

**Lemma 6.3** Let \( \xi \) be the limit curve of an Anosov representation. Assume, that for all distinct points \((x_1, \ldots, x_q)\) in \( \partial_infty \pi_1(S) \), and integers \((n_1, \ldots, n_q)\) with \( k = \sum n_i \leq n \), the following sum is direct

\[
\xi^{n_1}(x_1) + \ldots + \xi^{n_q}(x_q);
\]

and furthermore

\[
\lim_{(x_0, x_1, \ldots, x_i) \to x} \left( \xi^{n_1}(x_1) \oplus \ldots \oplus \xi^{n_q}(x_q) \right) = \xi^k(x).
\]

Then, for all \( y \) distinct from \((x_1, \ldots, x_q)\), the following sum is direct

\[
\xi^{n-k}(y) + \xi^{n_1}(x_1) + \ldots + \xi^{n_q}(x_q).
\]
Proof: If \((y, x_1, \ldots, x_q)\) is a collection of \(q + 1\) distinct points of \(\partial_\infty \pi_1(S)\), it is a classical fact that there exist two distinct points \(t\) and \(z\), a sequence \(\{\gamma_n\}_{n \in \mathbb{N}}\) of elements of \(\pi_1(S)\), such that
\[
\forall i \leq q, \lim_{n \to \infty} \gamma_n(x_i) = t, \quad \lim_{n \to \infty} \gamma_n(y) = z.
\]
Now by 2-hyperconvexity,
\[
\xi^k(z) \oplus \xi^{n-k}(t) = E,
\]
for \(n\) sufficiently large, by Hypothesis (33), we have
\[
\xi^{n-k}(\gamma_n(y)) \oplus \xi^{n_1}(\gamma_n(x_1)) \oplus \cdots \oplus \xi^{n_q}(\gamma_n(x_q)) = E.
\]
Hence the result since
\[
\xi^\ast(\gamma(w)) = \rho(\gamma) \xi^\ast(w).
\]
Q.E.D.

6.2.2 Proof of Theorem 4.2

From Lemma 5.1 and Theorem 6.1, we deduce immediately that \(\rho\) is \(n\)-Anosov. Furthermore, \(\xi^1\) is the projection in \(\mathbb{P}(E)\) of the limit curve \(\xi\) of \(\rho\), and we have, for every \(x \in \partial_\infty \pi_1(S)\),
\[
\lim_{(y_1, \ldots, y_l) \to x, y_i \text{ all distinct}} \left(\bigoplus_{i=1}^{l} \xi^1(y_i)\right) = \xi^l(x). \quad (34)
\]
To conclude the proof of Theorem 4.2, it suffices to show Assertions (1) and (2) of the definition of Frenet hyperconvex curve.

Let’s first prove Assertion (1). Let \((x_1, \ldots, x_p)\) be pairwise distinct points of \(\partial_\infty \pi_1(S)\). Let \(p\) be an integer. Let \((n_1, \ldots, n_p)\) be positive integers such that
\[
k = \sum_{i=1}^{p} n_i \leq n.
\]
We want to prove the following sum is direct
\[
\xi^{n_1}(x_1) \oplus \cdots \oplus \xi^{n_p}(x_p). \quad (35)
\]
We prove it by induction on \(p\). It is true for \(p = 2\), by 2-hyperconvexity. Assume it is true for \(p = q - 1\). Using Assertion (34), we deduce that
\[
\lim_{(x_1, \ldots, x_{q-1}) \to x} \left(\xi^{n_1}(x_1) \oplus \cdots \oplus \xi^{n_{q-1}}(x_{q-1})\right) = \xi^{k-n_q}(x). \quad (36)
\]
Now Lemma (33) finishes the induction. Finally, Assertion (2) is an immediate consequence of Assertions (34) and (1). Q.E.D.
7 Curves and Anosov representations.

7.1 Definitions

We introduce some definitions.

7.1.1 \((p,l)\)-direct.

Let \((p,l)\) some integers such that \(p + l \leq n\). We say the limit curve \(\xi\) (or the corresponding representation \(\rho\)) is \((p,l)\)-direct if for all distinct \((y,x_0,\ldots,x_l)\) the following sum is direct

\[
\xi^{n-p-l}(y) + \xi^p(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l).
\]

Notice that to say the representation is \((1,n-1)\)-direct is to say \(\xi^1\) is hyper-convex.

7.1.2 \((p,l)\)-convergent.

Let \((p,l)\) some integers such that \(p + l \leq n\). We say the limit curve \(\xi\) (or the corresponding representation \(\rho\)) is \((p,l)\)-convergent if for all distinct points \((x_0,\ldots,x_l)\) in \(\partial_\infty \pi_1(S)\) the following sum is direct

\[
\xi^p(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l);
\]

and if furthermore

\[
\lim_{(x_0,x_1,\ldots,x_l) \to x} (\xi^p(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)) = \xi^{p+l}(x).
\]

7.1.3 3-hyperconvexity

We say the limit curve (or the corresponding representation) is 3-hyperconvex, if \(k + p + l \leq n\) and \((x,y,z)\) are distinct, then the following sum is direct

\[
\xi^k(x) + \xi^p(y) + \xi^l(z).
\]

7.1.4 Property \((H)\)

We say the limit curve \(\xi\) (or the corresponding representation) satisfies Property \((H)\), if for every triple of distinct points \(x, y\) and \(z\) and integer \(k\), we have

\[
\xi^{k+1}(y) \oplus (\xi^{k+1}(z) \cap \xi^{n-k}(x)) \oplus \xi^{n-k-2}(x) = E.
\]

7.2 Main Lemma

As a main step in the proof of Theorem 4.1, we shall prove the following lemma.

Lemma 7.1 Let \(\xi\) be the limit curve of an Anosov representation. Assume
• it is 3-hyperconvex,
• it satisfies Property (H).

Then, the curve is \((k, l)\)-convergent for all integers with \(k + l \leq n\). In particular, \(\xi^1\) is hyperconvex.

We state a corollary that follows at once from Theorem 4.2 and which will be the way we shall use Lemma 7.1:

**Corollary 7.2** Let \(\xi\) be the limit curve of an Anosov representation. Assume

• it is 3-hyperconvex,
• and satisfies Property (H).

Then \(\xi^1\) is a hyperconvex Frenet curve, and \(\xi\) is its osculating flag.

We begin with an observation that is a consequence of Lemma 6.3:

**Lemma 7.3** Let \(\xi\) be the limit curve of an Anosov representation. Assume the representation is \((p, l)\)-convergent, then it is \((p, l)\)-direct.

### 7.3 Bundles

To prove our Lemma 7.1, we shall need at some point the vector bundle description of Proposition 3.3 and prove some preliminary results in a general situation.

#### 7.3.1 Hyperconvex rank 2 vector bundle

Here is the situation we wish to describe. First let’s start with some convention. Let \(M\) be a manifold. For any vector bundle \(F\) over \(M\), we shall denote by \(F_x\) the fibre at a point \(x\) of a vector bundle \(F\), for any foliation \(L\) we denote by \(L_x\) the leaf passing through \(x\).

We are first interested in actions of flow on a compact manifold which preserves a one dimensional foliation. Namely

• Let \(\phi_t\) be a flow of homeomorphisms of a compact topological manifold \(M\).
• Let \(F\) be a 1-dimensional lamination of \(M\) with no compact leaves. We assume this foliation is invariant under the flow of \(\phi_t\).

This is for instance satisfied when \(\phi_t\) is an Anosov flow and \(F^+\) is the stable (or unstable) foliation.

We are interested on bundles over \(M\) and actions on these bundles which lift the previous one. We shall say a vector bundle \(E\) over \(M\) admits a **flag action** if it satisfies the following assumptions:
• $E$ is a vector bundle of rank 2, equipped with a parallel transport along leaves of $\mathcal{F}$;

• the action of $\phi_t$ lift to an action $\psi_t$ by bundle automorphism on $E$, and this action preserves the parallel transport;

• $E$ admits a direct sum decomposition, invariant under $\psi_t$, into continuous oriented subbundles of rank 1
  \[ E = W^1 \oplus W^2, \]

• $W^2$ is parallel along leaves of $\mathcal{F}$.

• **Contraction assumption.** We assume $(W^1)^* \otimes W^2$ is a contracting vector bundle for $\psi_t$. It follows that if we take a 1-dimensional vector space $L$ in $W^2_x \oplus W^1_x$ different than $W^2_x$, then
  \[ \lim_{t \to \infty} d(\psi_t(L), W^1_{\phi_t(x)}) = 0. \]

We now introduce some notations. Assume $x$ and $y$ are on the same leaf of $\mathcal{F}$. We shall denote by $W^1_{x,y}$ the vector subspace of $\mathcal{F}_x$ which is the parallel transport of $W^1_y$ along the leaf. We say the bundle $E$ is hyperconvex if and only if, for all distinct $z$ and $y$ in the leaf $\mathcal{F}_x$
  \[ W^1_{x,z} \oplus W^1_{x,y} = E_x. \] (37)

Our main Lemma is the following

**Lemma 7.4** Assume the rank 2 vector bundle $E$ equipped with a flag action is hyperconvex. Then, the map $J_x$

\[
\begin{align*}
  \mathcal{F}_x & \quad \mapsto \quad \mathbb{P}(E_x) \setminus \{W^2_x\} \\
y & \quad \mapsto \quad W^1_y
\end{align*}
\]

is a homeomorphism; furthermore, for every $x$ in $M$

\[ \lim_{y \to \infty, y \in \mathcal{F}_x} (W^1_{x,y}) = W^2_x. \]

We begin by explaining the relations of this notion with our situation, then prove a preliminary lemma and finally conclude.

### 7.3.2 Hyperconvex bundles and Anosov representations

Rank 2 vector bundles with a flag action arise naturally from Anosov representations. Indeed using the notations of Proposition 3.3, we shall soon show the bundles $E^+_k/E^+_k$ are of this type. More precisely, let $E$ be the vector bundle associated to a $n$-Anosov representation by Proposition 3.3 with its flat connection. Let $F_k = E^+_k/E^+_k$. Notice that $F$ is equipped with a flat connection
along $F^+$. Obviously for this connection $W^2 = E_{k-1}^+/E_{k-2}^+$ is parallel. We can identify $F_k$ with $E_{n-k+2}^- \cap E_k^+$. In this interpretation, we have

$$W^2 = E_{n-k+2}^- \cap E_{k-1}^+ = V^{k-1}.$$ 

Let

$$W^1 = E_{n-k+1}^- \cap E_k^+ = V^k.$$ 

Then we have the following

**Proposition 7.5** Let $E$ be the vector bundle associated to a n-Anosov representation by Proposition 3.3. Then $F_k$ with the structure described above is a rank 2 vector bundle equipped with a flag action. Furthermore, using Identification (38) of Proposition 3.3, we have

$$W^1_{(x,x_0,w), (x,x_0,y)} = (\xi_{n-k+1}^- (y) \oplus \xi_{k-2}^-(x)) \cap \xi_{n-k+2}^-(w) \cap \xi_k^-(x).$$

Finally, the representation satisfies Property (H) if and only if the bundles $F_k$ are hyperconvex.

**Proof:** By definition, the bundle $F_k$ (with its flat connection) along the leaf of $F^+$ passing through $(x,x_0,w)$ is identified with the trivial bundle whose fibre is $\xi_k^-(x)/\xi_{k-2}^-(x)$.

We identify $\xi_k^-(x) \cap \xi_{n-k+2}^-(w)$ with this fibre using the projection along $\xi_{k-2}^-(x)$. Then we get

$$W^1_{(x,x_0,w), (x,x_0,y)} = ((\xi_{n-k+1}^- (y) \cap \xi_k^-(x)) \oplus \xi_{k-2}^-(x)) \cap \xi_{n-k+2}^-(w).$$

This in turn implies Identification (38). Therefore, hyperconvexity is equivalent to, for $y \neq t$

$$((\xi_{n-k+1}^- (y) \cap \xi_k^-(x)) \oplus \xi_{k-2}^-(x)) \cap \xi_{n-k+2}^-(w)$$

$$\oplus ((\xi_{n-k+1}^- (t) \cap \xi_k^-(x)) \oplus \xi_{k-2}^-(x)) \cap \xi_{n-k+2}^-(w)$$

$$= \xi_k^-(x) \cap \xi_{n-k+2}^-(w).$$

(39)

If we add $\xi_{k-2}^-(x)$ to both sides of Equality (39), since $\xi_{k-2}^-(x) \oplus \xi_{n-k+2}^-(w) = E$ by 2-hyperconvexity, we get

$$(\xi_{n-k+1}^- (y) \cap \xi_k^-(x)) + \xi_{k-2}^-(x) + (\xi_{n-k+1}^- (t) \cap \xi_k^-(x)) = \xi_k^-(x).$$

(40)

Adding $\xi_{n-k}^-(y)$ to both sides of Equality (40), we get

$$\xi_{n-k+1}^- (y) + \xi_{k-2}^-(x) + (\xi_{n-k+1}^- (t) \cap \xi_k^-(x)) = E.$$

(41)

Now for dimension reasons, the above is direct and we get the following equality which is nothing else than Property (H),

$$\xi_{n-k+1}^- (y) \oplus \xi_{k-2}^-(x) \oplus (\xi_{n-k+1}^- (t) \cap \xi_k^-(x)) = E.$$
Conversely, let’s assume Property (H). This implies after taking the intersection with $\xi^k(x)$ to

$$(\xi^{n-k+1}(y) \cap \xi^k(x)) \oplus \xi^{k-2}(x) \oplus (\xi^{n-k+1}(t) \cap \xi^k(x)) = \xi^k(x).$$

Let denote $\pi$ the projection along $A = \xi^{k-2}(x)$ on $B = \xi^{n-k-2}(w)$. Let $L(m)$ be the line $\xi^{n-k+1}(m) \cap \xi^k(x)$. The last assertion can be restated as

$$L(y) \oplus L(t) \oplus A = \xi^k(x).$$

Using $\pi$ we get

$$\pi(L(t)) \oplus \pi(L(y)) = \pi(\xi^k(x)) = \xi^k(x) \cap \xi^{n-k+2}(w),$$

which is precisely Assertion (39), we wanted to show. Q.E.D.

### 7.3.3 Invariant subbundle

Let’s start with a definition just used in the next proof. A subbundle $L$ of $E$ is said to be invariant if

$$\psi_t(L_x) = L_{\psi_t(x)}.$$ 

Notice that no regularity is assumed on $L$.

**Lemma 7.6** Let $L$ be an invariant subbundle of rank 1 of $E = W^1 \oplus W^2$ equipped with a flag action. Assume that, for some auxiliary (continuous) metric $d$ on the bundle $P(E)$, we have

$$\exists \epsilon, \forall x \in M, d(L_x, W^1_x) > \epsilon > 0.$$ 

Then

$$\forall x \in M, L_x = W^2_x.$$ 

We stress again that no regularity is assumed on $L$.

**Proof:** The lemma is an immediate consequence of our contraction Assumption (7). Indeed if the lemma is not true, for some $x$, $L_x \neq W^2_x$. Then, for some large positive $s$ by our contraction Assumption (7)

$$d(L_{\phi_s(x)}, W^1_x) = d(\psi_s(L_x), W^1_x) < \frac{\epsilon}{2}.$$ 

Q.E.D.

### 7.3.4 Proof of Lemma

**Proof:** By assumption, we have that for $y \neq z$, $W^1_{x,y} \oplus W^1_{z,z} = E_x$. Therefore the continuous maps $J_x$ are injective. Since $P(E_x)$ is of dimension 1, the following limits exist (after a choice of orientation on $\mathcal{F}$)

$$\lim_{y \to +\infty} J_x(y) = J^+_x$$

$$\lim_{y \to -\infty} J_x(y) = J^-_x$$
Notice also that the bundles $J^\pm$ are flow invariant, although not a priori continuous. To finish proving the lemma, we just have to show that

$$J^+ = J^- = W^2_x.$$  

We can assume that $F$ is the orbit lamination of a flow $\theta_t$. Let’s introduce the continuous bundles $L^\pm_x = W^1_{x,\theta_{\pm 1}(x)}$. We notice that for all $x$, $L^\pm_x \neq W^1_x$ by our assumption \[37\]. And furthermore, since the maps $J_x$ are monotone, $J^+_x$ and $W^1_x$ are not in the same connected component of $P(E_x) \setminus \{L^-_x, L^+_x\}$ (and the same holds for $J^-_x$). It follows there exists $\epsilon > 0$ such that

$$d(J^\pm_x, W^1_x) \geq d(L^\pm_x, W^1_x) \geq \epsilon.$$  

Lemma \[36\] implies that for all $x$, $J^\pm_x = W^2_x$. Q.E.D.

### 7.4 Property (H) and hyperconvex bundles

We recall that a $n$-Anosov representation, with limit curve $\xi$, satisfies Property (H), if for every triple of distinct points $x$, $y$ and $z$ and integer $k$, we have

$$\xi^{k+1}(y) \oplus (\xi^{k+1}(z) \cap \xi^{n-k}(x)) \oplus \xi^{n-k-2}(x) = E.$$  

We explain now various ways to check this property, and use its relation with hyperconvex bundles to deduce an important consequence in Proposition \[7.7\].

#### 7.4.1 Main property

Let $x$ and $z$ be two distinct points of $\partial_\infty \pi_1(S)$. Let

$$G_{p,x,z} = \xi^{n-p+2}(z) \cap \xi^p(x).$$  

We consider the map $\mathcal{Y}_{p,x,z}$ defined by

$$\begin{align*}
\{ \partial_\infty \pi_1(S) \setminus \{x\} \} & \rightarrow P(G_{p,x,z}) \setminus \{\xi^{p-1}(x) \cap \xi^{n-p+2}(z)\} \\
y & \mapsto (\xi^{n-p+1}(y) \oplus \xi^{p-2}(x)) \cap G_{p,x,z}.
\end{align*}$$

Proposition \[7.8\] explains that this application is well defined. Our main result in this paragraph is the following Proposition

**Proposition 7.7** Assume the representation satisfies Property (H). Then, the map $\mathcal{Y}_{p,x,z}$ is onto from $\partial_\infty \pi_1(S) \setminus \{x\}$ to $P(G_{p,x,z}) \setminus \{\xi^{p-1}(x) \cap \xi^{n-p+2}(z)\}$.

#### 7.4.2 Construction of the map $\mathcal{Y}$

We use the notation of the previous paragraph

**Proposition 7.8** We have

$$\dim(\mathcal{Y}_{p,x,z}(y)) = 1,$$  

$$\mathcal{Y}_{p,x,z}(y) \oplus (\xi^{p-1}(x) \cap \xi^{n-p+2}(z)) = G_{p,x,z}.$$  

Furthermore, Property (H) is equivalent to the following assertion

$$\mathcal{Y}_{p,x,z}(t) \neq \mathcal{Y}_{p,x,z}(y).$$  

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Proof: From Identification (38), we get that
\[ W^1_{(x,x_0,w),(x,x_0,y)} = Y_{k,x,w}(y). \]
All the above results follows from Paragraph 7.3.2. Q.E.D.

7.4.3 Proof of Proposition 7.7
Proof: From Proposition 7.5, we know the bundles
\[ F^k = E^k/E^{k-2} \]
are hyper-convex if the representation satisfies Property (H). Furthermore, from Identification (38), we get that
\[ W^1_{(x,x_0,w),(x,x_0,y)} = Y_{k,x,w}(y). \]
Then Proposition 7.7 follows from Lemma 7.4. Q.E.D.

7.5 Proof of Lemma 7.1
7.5.1 Preliminary facts: case \( l = 1 \).
Let \( \xi \) be the limit curve of the Anosov representation \( \rho \). Let \((z,x,x_0,x_1, y)\) be distinct points of \( \partial_\infty \pi_1(S) \). Write \( X = (z,x,x_0,x_1) \). We introduce
\[
U_{y,x_0} = (\xi^{n-k}(y) \oplus \xi^{k-1}(x_0) \cap \xi^{n-k+1}(z)), \\
Z_{x_0,x_1} = (\xi^{k-1}(x_0) \oplus \xi^1(x_1) \cap \xi^{n-k+1}(z)).
\]
We first notice that thanks to 2-hyperconvexity (cf. Assertion 38) the sums in the definition of \( Z_{x_0,x_1} \) and \( U_{y,x_0} \) are indeed direct. We shall now prove

Lemma 7.9 If \( \xi \) is a limit curve of a 3-hyperconvex quasi-Fuchsian representation, then
\[
\dim(Z_{x_0,x_1}) = 1, \\
\dim(U_{y,x_0}) = n - k.
\]
And furthermore,
\[
Z_{x_0,x_1} \oplus U_{y,x_0} = \xi^{n-k+1}(z).
\]
Proof: Let
\[
C_y = \xi^{n-k}(y) \oplus \xi^{k-1}(x_0).
\]
Let \( \pi \) be the projection on \( A = \xi^{n-k+1}(z) \) along \( B = \xi^{k-1}(x_0) \). Notice that \( A \oplus B = E \) thanks to 2-hyperconvexity. Recall that
\[
\pi(W) = (W + B) \cap A.
\]
In particular
\[
\pi(C_y) = C_y \cap A = U_{y,x_0} \\
\pi(\xi^1(x_1)) = Z_{x_0,x_1}.
\]
We begin by computing the dimensions of $Z_{x_0,x_1}$ and $U_{y,x_0}$. We first notice that 
\[ \dim Z_{x_0,x_1} = \dim(\pi(\xi^1(x_1))) \leq 1. \]

By 2-hyperconvexity, the next sum is direct 
\[ \xi^1(x_1) + \underbrace{\xi^{k-1}(x_0)}_{B}. \]

Hence 
\[ \xi^1(x_1) \not\subset B, \]
and 
\[ \dim(Z_{x_0,x_1}) = 1. \]

Let's consider now $U_{y,x_0}$. First, we know that 
\[ \dim(C_y) = n - 1. \]

Finally, since the curve is assumed to be 3-hyperconvex, 
\[ \underbrace{\xi^{n-k}(z)}_{\subset A} \oplus \underbrace{\xi^{k-1}(x_0)}_{\subset C_y} \oplus \xi^1(y) = E. \]

Hence 
\[ A + C_y = E, \]
and 
\[ A \not\subset C_y. \]

Hence, 
\[ \dim(U_{y,x_0}) = \dim(\pi(C_y)) = \dim(C_y \cap A) = \dim(A) - 1 = n - k. \]

Finally, since the curve is 3-hyperconvex, 
\[ \underbrace{(\xi^{n-k}(y) \oplus \xi^{k-1}(x_0))}_{C_y} \oplus \xi^1(x_1) = E. \quad (45) \]

Applying $\pi$ to both sides of Formula (45), we finally get 
\[ Z_{x_0,x_1} + U_{y,x_0} = \xi^{n-k+1}(z). \]

This concludes our proof. Q.E.D.

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7.5.2 Proof of Lemma 7.1 case \( l = 1 \)

We concentrate on the case \( l = 1 \) of Lemma 7.1. We prove Lemma 7.10

**Lemma 7.10** Assume the limit curve \( \xi \) of the Anosov representation \( \rho \) is

- 3-hyperconvex
- satisfies Property (H).

Then it is \((k, 1)\)-convergent for all \( k \).

**Proof:** We shall prove this Lemma by induction on \( n - k \) and use the following induction hypothesis.

\[
\lim_{(x_0, x_1) \to x} (\xi^k(x_0) \oplus \xi^1(x_1)) = \xi^{k+1}(x),
\]

(46)

We notice this is true for \( k = n - 1 \) thanks to 2-hyperconvexity. We want to prove

\[
\lim_{(x_0, x_1) \to x} (\xi^{k-1}(x_0) \oplus \xi^1(x_1)) = \xi^k(x).
\]

(47)

Recall that by 2-hyperconvexity, for \( z \neq x \),

\[
\xi^{k-1}(x) \oplus \xi^{n-k+1}(z) = E.
\]

(48)

Let’s introduce as in the previous paragraph

\[
Z_{x_0, x_1} = (\xi^{k-1}(x_0) \oplus \xi^1(x_1)) \cap \xi^{n-k+1}(z),
\]

\[
U_{y, x_0} = (\xi^{n-k}(y) \oplus \xi^{k-1}(x_0)) \cap \xi^{n-k+1}(z),
\]

\[
B = \xi^{n-k+1}(z),
\]

\[
G_{x, z} = B \cap \xi^{k+1}(x).
\]

Using Assertion (48) and the notations of the previous paragraph, since \( \xi^{k-1}(x_0) \) converges to \( \xi^{k-1}(x) \), we first notice Assertion (47), would follow from

\[
\lim_{(x_0, x_1) \to x} Z_{x_0, x_1} = \xi^{n-k+1}(z) \cap \xi^k(x).
\]

(49)

Our aim now is to prove this last assertion. We shall do that by “trapping” \( Z_{x_0, x_1} \) using \( U_{y, x_0} \).

Let \( V \) be a connected neighbourhood of \( x \) homeomorphic to the interval. Let’s choose an orientation on \( V \). We shall say \((x_0, x_1)\) tends to \( x_+ \), (resp. \( x_- \)) if \( x_0 > x_1 \) (resp. \( x_0 < x_1 \)). Let \( \alpha \in \{+, -\} \). Let \( \Lambda_x^\alpha \) be the set of value of accumulation of \( Z_{x_0, x_1} \) when \((x_0, x_1)\) tends to \( x_\alpha \). We note that \( \Lambda_x^\alpha \) is a connected subset of \( \mathbb{P}(B) \). Notice that,

\[
Z_{x_0, x_1} \subset (\xi^k(x_0) \oplus \xi^1(x_1)) \cap \xi^{n-k+1}(z).
\]
From Hypothesis (46), it follows that the sets $\Lambda_x^\alpha$ is actually a subset of the projective space $\mathbb{P}(G_{x,z})$. Therefore $\Lambda_x^\alpha$ is a closed interval of the 1-dimensional manifold $\mathbb{P}(G_{x,z})$.

We now choose an auxiliary metric $\langle , \rangle$ on $B$, a unit vector $z_{x_0,x_1}$ in $Z_{x_0,x_1}$ continuous in $(x_0, x_1)$, a normal vector $u_{y,x_0}$ to $U_{y,x_0}$ continuous in $x_0$ and $y$. Then we have (maybe after replacing $u$ by $-u$) for all $y$, for $X = (x_0, x_1)$ close to $x$, thanks to Lemma 4.9,

$$\langle z_{x_0,x_1}, u_{y,x_0} \rangle > 0.$$  

We consider now $\hat{\Lambda}_x^\alpha$ the set of value of accumulation of $z_{x_0,x_1}$ as $(x_0, x_1)$ goes to $\alpha$.

Notice then, that for all $y$, for all $w$ in $\hat{\Lambda}_x^\alpha$, we have

$$\langle w, u_{y,x_0} \rangle \geq 0.$$  

Hence, for all $y, t$, $\hat{\Lambda}_x^\alpha$ is contained in the closure of one connected component of

$$G_{x,z} \setminus ((U_{y,x} \cap G_{x,z}) \cup (U_{t,x} \cap G_{x,z})).$$

Since, by Definition 4.4.1

$$U_{y,x} \cap G_{x,z} = \mathcal{Y}_{k+1,x,z}(y),$$

It follows $\Lambda_x^\alpha$ is contained in the closure of one connected component of

$$\mathbb{P}(G_z) \setminus \{\mathcal{Y}_{k+1,x,z}(y), \mathcal{Y}_{k+1,x,z}(t)\}.$$  

From Proposition 7.7, we know that the map $\mathcal{Y}_{k+1,x,z}$ is onto from $\partial_\infty \pi_1(S) \{x\}$ to $\mathbb{P}(G_{k,x,z}) \setminus \{\xi^k(x) \cap \xi^{n-k+1}(z)\}$. Hence, we have that

$$\Lambda_x^\alpha = \{\xi^k(x) \cap \xi^{n-k+1}(z)\}.$$  

Since this is true for all the ends $\alpha$, we obtain the desired result. Q.E.D.

7.5.3 Main Lemma: case $l > 1$.

The Main Lemma 7.1 will follow by an induction proved in Paragraph 7.5.6 from the next statement combined with Lemma 7.10

Lemma 7.11 Let $\xi$ be the limit curve of a quasi-Fuchsian representation. Let $k$ and $l$ be some integers such that $k + l \leq n$ and $l > 2$. We assume furthermore that the curve is

1. $(k, l-1)$-convergent,
2. $(k, l)$-convergent,
3. $(k-1, l-1)$-convergent,

Then, the curve is $(k-1, l)$-convergent.
7.5.4 Preliminary facts: case \( l > 1 \)

We assume now that the limit curves \( \xi \) satisfy the hypothesis of Lemma 7.11. That is we assume the curve is

1. \((k,l-1)\)-convergent,
2. \((k,l)\)-convergent,
3. \((k-1,l-1)\)-convergent,

Notice that since the curve is \((k,l)\)-direct by Hypothesis (2) and Lemma 7.3, the next sum is direct for all \( m \) with \( m \leq k \),

\[
\xi^m(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l).
\]

Let \( z \) be a point of \( \partial \infty \pi_1(S) \), \( Y = (y_0, y_1, \ldots, y_{l-1} \) be a \( l \)-uple of cyclically ordered points of \( \partial \infty \pi_1(S) \setminus \{z\} \). We denote by

\[
C(z, Y) = \xi^{n-k-l}(z) \oplus \xi^k(y_0) \oplus \bigoplus_{i=1}^{l-1} \xi^1(x_i).
\]

The sum in the definition of \( C(z, Y) \) is direct thanks to Hypothesis 7.11. and Lemma 7.3. We also need to make some choices of orientation. Let \( I = \partial \infty \pi_1(S) \setminus \{z\} \). Let’s choose an orientation on \( \xi^p(w) \) for all \( p \) depending continuously on \( w \) in \( I \). Let’s also choose an arbitrary orientation on \( \xi^k(z) \) for all \( k \). It follows that there exists a family of 1-forms \( \alpha(z, Y) \) continuous in \( Y \) such that

\[
C(z, Y) = \ker(\alpha(z, Y)).
\]

Let now \( X = (z, x_0, x_1, \ldots, x_l) \) be a \( l+2 \)-uple of distinct points of \( \partial \infty \pi_1(S) \) cyclically oriented. Let

\[
X^+ = (x_0, \ldots, x_{l-1}) \quad X^- = (x_0, \ldots, x_{l-2}, x_l).
\]

We introduce

\[
U_X^+ = C(z, X^+) \cap \xi^{n-k-l+2}(z), \quad U_X^- = C(z, X^-) \cap \xi^{n-k-l+2}(z), \quad Z_X = (\xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)) \cap \xi^{n-k-l+2}(z).
\]

We first notice that thanks Assertion 49 the sum in the definition of \( Z_X \) is indeed direct.

We shall now prove
Proposition 7.12 If $\xi$ is a limit curve of a quasi-Fuchsian representation which satisfies the hypothesis of Lemma 7.11, then

$$\dim(Z_X) = 1$$
$$\dim(U^\pm_X) = n - k - l + 1.$$ 

Furthermore,

$$Z_X \oplus U^+_X = Z_X \oplus U^-_X = \xi^{n-k-l+2}(z),$$

and considering orientations

$$[\alpha(z, X^+)]_{Z_X} = -[\alpha(z, X^-)]_{Z_X}.$$  \hspace{1cm} (50)

**Proof:**

Let

$$C^+ = C(z, X^+) = \xi^{n-k-l}(z) \oplus \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_{l-1}),$$

$$B^+ = \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_{l-1}),$$

$$B^- = \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_{l-1}),$$

$$C^- = C(z, X^-) = \xi^{n-k-l}(z) \oplus \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_{l-1}).$$

Let $\pi^\pm$ be the projection on $A = \xi^{n-k-l+2}(z)$ along $B^\pm$. Notice that $A \oplus B^\pm = E$ thanks to Lemma 7.3 and Hypothesis 7.11.(3). Recall that

$$\pi^\pm(W) = (W + B^\pm) \cap A.$$ 

In particular

$$\pi^\pm(C^\pm) = C^\pm \cap A = U^\pm_X$$

$$\pi^+(\xi^1(x_l)) = Z_X,$$

$$\pi^-(\xi^1(x_{l-1})) = Z_X.$$ 

We first compute the dimensions of $Z_X$ and $U^\pm_X$. We first notice that

$$\dim Z_X = \dim(\pi^+(\xi^1(x_l))) \leq 1.$$ 

Finally since the following sum is direct (cf Hypothesis 7.11(2))

$$\xi^k(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_{l-1}) + \xi^1(x_l),$$

It follows the next one is direct

$$\xi^1(x_l) + \underbrace{\xi^{k-1}(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_{l-1})}_{B^+}.$$ 

Hence

$$\xi^1(x_l) \not\in B^+,$$

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and 
\[
\dim(Z_X) = 1.
\]
Let’s consider now \(U_X^+\), the proof for \(U_X^-\) being symmetric. First, we know that 
\[
\dim(C^+) = n - 1.
\]
According to Hypothesis 7.11.(3) and Lemma 7.3, 
\[
\xi^{n-k-l+2}(z) \oplus \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-1}) = E.
\]
Hence 
\[
A + C^+ = E,
\]
and 
\[
A \nsubseteq C^+.
\]
It follows that 
\[
\dim(U_X^+) = \dim(\pi^+(C^+)) = \dim(C^+ \cap A) = \dim(A) - 1 = n - k - l + 1.
\]
Finally, by Hypothesis 7.11.(4) and Lemma 7.3, 
\[
(\xi^{n-k-l}(z) \oplus \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-1}) \oplus \xi^1(x_l)) \in C^+.
\]
Applying \(\pi^+\) on both sides of Formula (52), we finally get 
\[
Z_X + U_X^+ = \xi^{n-k-l+2}(z).
\]
The same holds for \(U_X^-\). Finally, it remains to check the orientations on \(Z_X \oplus U_X^+\) and \(Z_X \oplus U_X^-\) are opposite. We shall denote by \(\overline{V}\) the opposite of the oriented vector space \(V\).

Since \((z,x_0,\ldots,x_l)\) are distinct and cyclically oriented, there exists an arc \(t \mapsto w_t\) joining \(x_l\) to \(x_{l-1}\), such that 
\[
\forall t, w_t \notin \{z,x_0,\ldots,x_{l-2}\}.
\]
Let 
\[
B_t = \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(w_t).
\]
Notice that \(B_t\), as before, satisfies 
\[
B_t \oplus \xi^{n-k-l+2}(z) = E.
\]
Let’s choose an orientation on \(E\) such that with respect to the orientation 
\[
E = C^+ \oplus \xi^1(x_l) = \ldots \oplus \xi^1(x_{l-1}) \oplus \xi^1(x_l).
\]
Recall $B_t$ is oriented. We choose now the orientation on $\xi^{n-k-l+2}(z)$ compatible with Equation (53). It follows that considered as oriented space we have
\[ U^+_X \oplus Z_X = \xi^{n-k-l+2}(z). \]

Conversely, since
\[ E = C^- \oplus \xi^1(x_{l-1}) = \ldots \oplus \xi^1(x_l) \oplus \xi^1(x_{l-1}) \]
we obtain that
\[ U^+_X \oplus Z_X = \xi^{n-k-l+2}(z). \]

Q.E.D.

7.5.5 Proof of Lemma 7.11

Let’s first state the following elementary lemma.

**Lemma 7.13** Let $E$ be a vector space. Let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of oriented lines converging to an oriented line $L_\infty$. Let $\{P_n\}_{n \in \mathbb{N}}$ be a sequence of oriented hyperplanes converging to an oriented hyperplane $P_\infty$. Assume that the following sums are direct and with opposite orientations
\[ L_n \oplus P_{2n} = L_n \oplus P_{2n+1}. \]

Then $L_\infty \subset P_\infty$.

**Proof:** Let’s choose an auxiliary metric $g$ on $E$. Let $u_n$ be the positive unit vector in $L_n$ and $v_n$ be the normal unit vector to $P_n$. From the hypothesis, we get that
\[ g(u_n, v_{2n}).g(u_n, v_{2n+1}) < 0. \]

Therefore, by passing to the limit we obtain that $g(u_\infty, v_\infty) = 0$. Q.E.D.

Let’s now proceed to the main proof. We shall always assume that
\[ (z, x_0, x_1, \ldots, x_l) \]
are distinct and cyclically positively oriented. We choose as before an orientation on $\xi^n(w)$ depending continuously on $w$ in $\partial_\infty \pi_1(S) \setminus \{z\}$. Here are the hypothesis we shall assume.

\[ \lim_{(x_0, \ldots, x_l) \to x} (\xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)) = \xi^{k+l}(x), \quad (54) \]
\[ \lim_{(x_0, \ldots, x_l) \to x} (\xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-1})) = \xi^{k+l-1}(x), \quad (55) \]
\[ \lim_{(x_0, \ldots, x_{l-2}) \to x} (\xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-1})) = \xi^{k+l-2}(x). \quad (56) \]
We can actually assume the limit in Assertion (55) is a limit as oriented vector spaces. We want to prove
\[
\lim_{(x_0, \ldots, x_l) \to x} (\xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)) = \xi^{k+l-1}(x). \tag{57}
\]
We shall use the notations and results of the preceding paragraph.

It follows that using Hypothesis (56), Assertion (57) reduces to
\[
\lim_{(x_0, \ldots, x_l) \to x} Z_X = \xi^{n-k-l+2}(z) \cap \xi^{k+l-1}(x).
\]

Our aim now is to prove this last assertion. We shall do that by “trapping” $Z_X$ using $U_X^\pm$.

Let $\Lambda_x$ be the set of value of accumulation of $Z_X$ when $(x_0, \ldots, x_l)$ tends to $x$. We note that $\Lambda_x$ is a subset of $P(\xi^{n-k-l+2}(z))$. Recall that $Z_X \subset \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)$.

From Hypothesis (54), we finally get the set $\Lambda_x$ is actually a subset of the projective space $P(D)$, for $D = \xi^{n-k-l+2}(z) \cap \xi^{k+l}(x)$.

Finally, from Hypothesis (54),
\[
\lim_{X \to x} (U_X^\pm) = (\xi^{n-k-l}(z) \oplus \xi^{k+l-1}(x)) \cap \xi^{n-k-l+2}(z).
\]

Notice that $\xi^{n-k-l+2}(z) + \xi^{k+1}(x) = E$ by 2-hyperconvexity hence $D$ is indeed a hyperplane of $\xi^{n-k-l+2}(z)$. We can choose the orientation on $D$ such the limit is to be considered for oriented vector spaces. From Equation (51) and Lemma 7.13, we obtain that $\Lambda_x \subset P(D)$.

The conclusion of the proof follows from
\[
D \cap W = (\xi^{n-k-l}(z) \oplus \xi^{k+l-1}(x)) \cap \xi^{n-k-l+2}(z) \cap \xi^{k+l}(x)
\]
\[
= \xi^{k+l-1}(x) \cap \xi^{n-k-l+2}(z).
\]
Q.E.D.

7.5.6 Final induction

PROOF: It remains to prove Main Lemma (7.1) using Lemma (7.10) and Lemma (7.11). This is done by induction. We say a limit curve is $l$-superconvergent, if it is $(k, l)$-convergent for all $k$.

From Lemma (7.10) the curve if 1-superconvergent. We assume by induction the curve is $l-1$-superconvergent. From Lemma (7.11) and an easy induction, to prove that the curve is $l$-superconvergent, it suffices to show that it is $(n-l, l)$-convergent. But to be $(n-l, l)$-convergent just means that the following sum is direct
\[
\xi^{n-l}(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l) = E.
\]
But the fact this sum is direct follows from the fact the curve is $(1, l-1)$-convergent, hence $(1, l-1)$-direct by Lemma (7.6). All these conditions are guaranteed by the induction assumption. Q.E.D.
8 Anosov representations, Property (H) and 3-hyperconvexity

We now clarify some relations between Property (H), 3-hyperconvexity, and Anosov representations. We first say a representation is $S$-irreducible if its restriction to all finite index subgroups is irreducible. By Lemma 10.1 every representation in Hitchin’s component is $S$-irreducible. We shall denote

- $\mathcal{A}$ (resp. $\mathcal{QF}$) the space of $n$-Anosov $S$-irreducible representations (resp. quasi-Fuchsian representation),
- $\mathcal{A}_H$ (resp. $\mathcal{QF}_H$) the space of $S$-irreducible Anosov (resp. quasi-Fuchsian) representations satisfying Property (H)
- $\mathcal{A}_3$ (resp. $\mathcal{QF}_3$) be the set of $S$-irreducible Anosov (resp. quasi-Fuchsian) which are 3-hyperconvex.

We summarise in the next Proposition the results of this section.

**Proposition 8.1** $A_3$ is open in $A$. $A_H$ is a connected subset of $A$. Furthermore $QF_H = QF$, and every Fuchsian representation is 3-hyperconvex.

The proof of Proposition 8.1 follows the following path: we will prove in the next paragraph that $A_H$ and $A_3$ are open in $A$, and in the next one that $A_H$ is closed in $A$; finally we prove that every Fuchsian representation is 3-hyperconvex and satisfies Property (H). This will complete the proof of Proposition 8.1.

### 8.1 Open

We first notice.

**Proposition 8.2** The sets $A_H$ and $A_3$ are open in $A$.

**Proof:** This follows at once from the fact $(\partial_{\infty} \pi_1(S)^3 \setminus \Delta)/\pi_1(S)$ is compact and that the conditions defining 3-hyperconvexity and Property (H) are open in the corresponding product of flag manifolds. Q.E.D.

### 8.2 Closed

The aim of this paragraph is to prove the following assertion.

**Proposition 8.3** The set $A_H$ is closed in $A$.

**Proof:** Let consider $\rho$ a $S$-irreducible Anosov representation limit of representations in $A_H$. Let $\xi$ be the associated curve, and $Y = Y_{k,x,z}$, the associated map defined in Paragraph 7.4.1 By Proposition 7.8 we wish to prove that $Y$ is injective. Since $Y$ is a limit of continuous injective maps of a 1-dimensional manifold into another, $Y$ is monotone.
Therefore, if \( \mathcal{Y} \) fails to be injective, there is an open set \( U \) in \( \partial_{\infty} \pi_{1}(S) \) on which it is constant. We will prove this last assertion leads to a contradiction. Indeed we will prove the next assertion that contradicts Lemma 10.2.

**Assertion.** There exists some \((n-k-1)\)-plane \( A \), such that
\[
\forall y \in U, \dim \xi^{k+1}(y) \cap A \geq 1.
\] (58)

Notice first that
\[
\mathcal{Y}(y) = (\xi^{k+1}(y) \oplus \xi^{n-k-2}(z)) \cap G_{k,x,z}
\]
\[
= (\xi^{k+1}(y) \oplus \xi^{n-k-2}(z)) \cap \xi^{n-k}(z) \cap \xi^{k+2}(x)
\]
\[
= ((\xi^{k+1}(y) \cap \xi^{n-k}(z)) \oplus \xi^{n-k-2}(z)) \cap \xi^{k+2}(x).
\]
Since \( \xi^{k+2}(x) \) is a supplementary of \( \xi^{n-k-2}(z) \),
\[
\mathcal{Y}(y) = \mathcal{Y}(t),
\]
implies
\[
P(y) = P(t),
\]
where
\[
P(y) = (\xi^{k+1}(y) \cap \xi^{n-k}(z)) \oplus \xi^{n-k-2}(z).
\]
Notice that \( P(y) \) has dimension \( n - k - 1 \). As a conclusion, we get Assertion (58). Q.E.D.

### 8.3 Back to Fuchsian representations

We prove now that \( QF_{H} \) is not empty. More specifically

**Lemma 8.4** Every Fuchsian representation satisfies Property (H).

**Proof:** First we notice that for every distinct points \( x \) and \( z \) the following sum is direct
\[
\xi^{k+1}(z) + \xi^{n-k-2}(x),
\]
hence, next one is also direct,
\[
(\xi^{k+1}(z) \cap \xi^{n-k}(x)) + \xi^{n-k-2}(x) = P(z, x).
\]
It follows that if a Fuchsian representation does not satisfy Property (H), then there exists a triple of distinct points \((x, y, z)\) such that
\[
\dim(\xi^{k+1}(y) \cap P(z, x)) > 0.
\]
In the case of a Fuchsian representation, the limit curve is the Veronese embedding and is equivariant under the whole action of \( SL(2, \mathbb{R}) \). Since \( SL(2, \mathbb{R}) \)
acts transitively on the set of triple of distinct points, we obtain there exists a
\(n - k - 1\)-plane \(P\) (namely \(P(z, x)\) for some \(x\) and \(z\)) a such that for every \(y\),
\[
\dim(\xi^{k+1}(y) \cap P) > 0.
\]
It follows that for every there exist a \(k + 1\)-plane \(Q\) such that \(A\) in \(SL(2, \mathbb{R})\),
\[
\dim(\rho(A)Q \cap P) > 0.
\]
This last assertion contradicts Proposition 10.3. Q.E.D.

Actually, one could prove the previous proposition by an explicit computation. Indeed, if we identify \(\partial_{\infty} \pi_1(S) \setminus \{x\}\) with \(\mathbb{R}P^1 \setminus \{\infty\} = \mathbb{R}\), and use the fact the irreducible representation of \(SL(2, \mathbb{R})\) of dimension \(n\) is the representation on homogeneous polynomials of degree \(n - 1\) in variables \(t\) and \(s\), one sees that
\[
\xi^k(x) = \{P(s, t)/\exists Q\text{ such that } = (s + tx)^{n-k}Q(s, t)\}.
\]
Then it is an exercise (left to the reader) to prove the previous proposition along
these lines.

Similar arguments show

**Proposition 8.5** Every Fuchsian representation is 3-hyperconvex.

## 9 Closedness

Our aim in this section is to prove the following result

**Lemma 9.1** The set
\[
\tilde{A} = \mathcal{A}_3 \cap \mathcal{A}_H
\]
of 3-hyperconvex Anosov representations satisfying Property (H) is closed in the
space of \(S\)-irreducible representations.

This Lemma will be deduced from Lemma 9.2. We first show that as corollaries, we obtain our Theorem 4.1.

### 9.1 Proof of Theorems 4.1

#### 9.1.1 Theorem 4.1

**Proof:** We just have to put the previous statements in the correct order. We first know by Lemma 10.1 that every representation in Hitchin’s component is \(S\)-irreducible. By Proposition 5.1 \(Q^F \cap \mathcal{Q}_{3}\) is open in \(Q^F\), hence in Hitchin’s component by Lemma 2.1. It is non empty by Proposition 5.1 again, since it contains all Fuchsian representations. It is closed by Lemma 9.1. Hence
\(Q^{\mathcal{F}}_H \cap Q^{\mathcal{F}}_3\) is equal to all of Hitchin’s component. Let \(\rho\) be a representation in this component. Let 
\[
\xi = (\xi^1, \xi^2, \ldots, \xi^{n-1})
\]
be its limit curve. By Corollary we know that if \(\rho \in Q^{\mathcal{F}}_H \cap Q^{\mathcal{F}}_3 = \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R}))\), \(\xi^1\) is an hyperconvex Frenet curve and \(\xi\) is its osculating flag Q.E.D.

### 9.2 Convergence of limit curves

Our aim is to prove the following lemma,

**Lemma 9.2** Let \(\{\rho_m\}_{m \in \mathbb{N}}\) be a sequence of Anosov representations satisfying Property (H) converging to a \(S\)-irreducible representation \(\rho\). Let \(\xi_m = (\xi_m^1, \ldots, \xi_m^{n-1})\) be the limit curve of \(\rho_m\). Then, there exists

- a sequence of homeomorphisms \(\phi_m\) of \(S^1\) with \(\partial_\infty \pi_1(S)\),
- a monotone map \(\pi\) from \(S^1\) to itself,
- an injective map \(\hat{\xi}^1\) from \(S^1\) to \(\mathbb{P}(E)\),
- an injective left continuous orientation preserving map \(\phi_0\) from \(\partial_\infty \pi_1(S)\) to \(S^1\)

such that

- after extracting a subsequence the mappings \(\{\xi_m^1 \circ \phi_m\}_{m \in \mathbb{N}}\) converges to \(\hat{\xi}^1 \circ \pi\),
- the map \(\hat{\xi}^1 \circ \phi_0\) is \(\rho\)-equivariant and \(*\)-hyperconvex.

We explain first this Lemma implies Lemma 9.1: indeed, thanks to Theorem the limit representation is Anosov and it satisfies Property (H) thanks to Proposition 8.1. The proof of the Lemma by itself follows several steps which we describe now shortly using the notations and the hypothesis of the Lemma.

1. **Convergence of the images** (Proposition 9.3): there exists a sequence of homeomorphisms \(\phi_m\) of \(S^1\) with \(\partial_\infty \pi_1(S)\) such that \(\{\xi_m^1 \circ \phi_m\}_{m \in \mathbb{N}}\) converges to a rectifiable curve \(\xi^1\)

2. **Preliminary facts**: we prove lemmas of independent interest concerning rectifiable curves invariant under actions of groups.

3. **The limit and the boundary at infinity**: this is the core of the proof, in particular Lemma 9.2 is a consequence of Proposition 8.1. We basically prove that \(\hat{\xi}^1 = \tilde{\xi}^1 \circ \pi\) where \(\tilde{\xi}^1\) is \(*\)-hyperconvex, \(\rho\)-equivariant, and \(\pi\) is monotone from \(S^1\) to \(\partial_\infty \pi_1(S)\).

From now on, we use the notation of the Lemma. That is we consider

- a sequence \(\{\rho_m\}_{m \in \mathbb{N}}\) of 3-hyperconvex Anosov representations satisfying Property (H) converging to a \(S\)-irreducible representation \(\rho\),
- \(\xi_m = (\xi_m^1, \ldots, \xi_m^{n-1})\), the limit curve of \(\rho_m\).
9.3 Convergence of the images

Proposition 9.3 After passing to a subsequence, there exists a sequence of homeomorphisms $\phi_m$ of $S^1$ with $\partial_\infty \pi_1(S)$ such that $\{\xi_m^1 \circ \phi_m\}_{m \in \mathbb{N}}$ converges to a rectifiable curve $\xi^1$.

Proof: By Ascoli Theorem, it suffices to show that there exists a constant $B$ such that $\xi^1_m$ is rectifiable and with length bounded by $B$. But this follows from the next remark: if $c$ is a curve in $P(E^*)$, then

$$\text{length}(c) \leq \int_{P(E^*)} \sharp(c \cap P)d\mu(P).$$

In our case, by Lemma 7.1, $\xi^1_m$ is hyperconvex, hence we get

$$\text{length}(\xi^1_m) \leq (\dim(E) - 1)\mu(P(E^*)).$$

Q.E.D.

9.4 Preliminary facts

9.4.1 Wormlike

Let $Z$ be a subset of $P(E)$. We define $\langle Z \rangle$ to be the vector subspace generated by all the elements of $Z$:

$$\langle Z \rangle = \sum_{u \in Z} u.$$

Finally, assume $\Gamma$ acts on $S^1$. Let $\rho$ be a faithful representation of $\Gamma$ in $SL(E)$. Let $\xi$ be a $\rho$-equivariant injective map from $S^1$ to $P(E)$. Let $\Gamma_\mathbb{R} = \{\gamma \in \Gamma, \gamma \neq id, \rho(\gamma) \text{ is diagonalisable over } \mathbb{R}\}$.

For every $\gamma$ in $\Gamma_\mathbb{R}$, let $Fix(\gamma)$ (resp. $Fix^+(\gamma)$) be the set of (resp. attractive) fixed points of $\rho(\gamma)$ in $P(E)$. We define

$$\Lambda_{\xi,\rho,\Gamma} = \{a \in S^1/\exists \gamma \in \Gamma_\mathbb{R}, \xi(a) \in Fix^+(\gamma)\}.$$

We prove the following lemma.

Lemma 9.4 Let $\Gamma$ be a group acting on $S^1$ by orientation preserving homeomorphisms. Let $\rho$ be a $S$-irreducible representation of $\pi_1(S)$ in $SL(E)$. Assume $\xi$ is a $\rho$-equivariant rectifiable injective map from $S^1$ to $P(E)$ with finite length. Then

(i) $\xi(S^1)$ is not included in a finite union of proper vector subspaces of $E$,

(ii) For every $\gamma$ in $\Gamma_\mathbb{R}$, $S^1 \setminus \text{Fix}(\gamma)$, has finitely many connected components.

(iii) For every $\gamma$ in $\Gamma_\mathbb{R}$, there exists a unique $\gamma^+$ in $S^1$ such that $\xi(\gamma^+) \in \text{Fix}^+(\gamma)$. In particular, $\Lambda_{\xi,\rho,\Gamma}$ is not empty if $\Gamma_\mathbb{R}$ is not empty.
(iv) Let \( U \) be a neighbourhood of a point \( c^+ \) in \( \Lambda_{\xi,\rho,\Gamma} \). Then
\[
\xi(U) \not\subset P_0 \cup P_1,
\]
for any proper vector subspaces \( P_0 \) and \( P_1 \) of \( E \). In particular, \( \langle \xi(U) \rangle = E \).

PROOF: Statement (i) is a consequence of the fact that the connected component of the identity of the Zariski closure of \( \rho(\Gamma) \) is irreducible. Indeed, let \( E_1, \ldots, E_p \) be proper vector subspaces such that
\[
\xi(S^1) \subset E_1 \cup \ldots \cup E_p.
\]
We can as well assume that \( \langle \xi(S^1) \cap E_i \rangle = E_i \). It follows that for every \( \gamma \) in \( \Gamma \), one has
\[
\gamma(E_i) \subset E_1 \cup \ldots \cup E_p.
\]
The same property holds for \( \gamma \) in the Zariski closure \( H \) of \( \rho(\gamma) \). Let \( E_k \), such that \( \dim(E_k) = \sup_i(\dim(E_i)) \). Then, for every element \( g \) in \( H \) close to the identity, \( g(E_k) = E_k \). It follows the identity component of \( H \) preserves \( E_k \), hence is not irreducible. This is the contradiction.

Let’s now describe the action on \( \mathbb{P}(E) \) of an element \( f \) of \( SL(E) \) diagonalisable over \( \mathbb{R} \). These are elementary facts whose proofs are left to the reader.

(a) The stable manifold \( W \) of a fixed point \( z \) of \( f \) in \( \mathbb{P}(E) \) is described in the following way. There exists a vector subspace \( \tilde{W} \) of \( E \), such that \( W \) is an open set in \( \mathbb{P}(\tilde{W}) \). Furthermore \( W \) is open in \( \mathbb{P}(E) \), if and only if \( z \) is an attractive fixed point.

(b) Every closed invariant set of \( f \) contains a fixed point.

(c) If \( x \) is such that \( f^n(x) \) converges to \( a \) and \( f^{-n}(x) \) converges to \( b \) when \( n \) goes to infinity with \( a \neq b \), then \( a \) and \( b \) belong to different connected components of the space of fixed points of \( f \).

(d) \( f \) has at most one attractive fixed point.

We can now prove (ii). Let \( I = [\alpha, \beta] \) be a connected component of \( S^1 \setminus \text{Fix}(\gamma) \). Then \( I \) is fixed by \( \gamma \) since \( \gamma \) is orientation preserving. Furthermore, by (c), \( \xi(\alpha) \) and \( \xi(\beta) \) belong to different connected components of the space of fixed points of \( \rho(\gamma) \). Let \( \mathcal{W} \) be the set of connected components of \( \text{Fix}(\rho(\gamma)) \). It follows that
\[
\text{length}(\xi(I)) \geq \epsilon_0 = \inf_{A,B \in \mathcal{W},A \neq B} d(A,B).
\]
Since \( \xi(S^1) \) has finite length by hypothesis, we deduce (ii).

Let’s proceed to (iii). Assume that \( \text{Fix}^+(\gamma) \) is empty. Notice that every connected component \( [\alpha, \beta] \) of \( S^1 \setminus \text{Fix}(\gamma) \) is mapped to the stable manifold of \( \xi(\alpha) \) and unstable manifold of \( \xi(\beta) \) (after a choice of orientation). Therefore by (a), if \( \text{Fix}^+(\gamma) \) is empty, then \( \xi(I) \) lies in a proper subspace of \( E \). Since \( \rho(\gamma) \neq id \), \( \xi(\text{Fix}(\gamma)) \) lies in a finite union of proper vector subspace. It follows
that $\xi(S^1)$ lies in a finite (by (ii)) union of proper vector subspaces of $E$ and this contradicts (i). Uniqueness follows from (d) and the injectivity of $\xi$.

We shall now prove (iv) by similar arguments. Let $c^+$ be the point which is mapped to the attractive fixed point of $\rho(\gamma)$, with $\gamma \in \Gamma_R$. Write the finite decomposition in connected components

$$S^1 \setminus \text{Fix}(\gamma) = \bigcup_i V_i.$$ 

By convention, we assume that $V_1$ and $V_2$ have $c^+$ in their closure. Let $i \geq 3$.

The the closure of $V_i$ contains an element which is mapped by $\xi$ to a fixed point $c$ which is neither attractive nor repulsive. The sets $V_i$ lie in the stable (or unstable) manifold of $c$. The same holds for $\xi(V_i)$. It follows from (a) that $\xi(V_i)$ lies in a proper subspace $E_i$ of $E$, for $i \geq 3$.

Assume now that there exist a neighbourhood $U$ of $c^+$, two proper vector subspaces $P_0$ and $P_1$ of $E$, such that

$$\xi(U) \subset P_0 \cup P_1.$$ 

We choose $U$ small enough so that $U \subset \gamma^{-1}(U)$. Then, if $Q_i$ are limits of $\rho(\gamma^n)(P_i)$ for some subsequence $n_q$, we get

$$\bigcup_{n \in \mathbb{N}} \xi(\gamma^{-n}(U)) \subset Q_0 \cup Q_1.$$ 

But

$$\bigcup_{n \in \mathbb{N}} \gamma^{-n}(U) = V_1 \cup V_2.$$ 

It follows that $\xi(S^1)$ lies in the union of $E_i$ for $i \geq 3$, $\text{Fix}(\rho(\gamma))$ and $Q_0 \cup Q_1$, hence the contradiction by (i). Q.E.D.

### 9.4.2 Weak worm

We prove now a weak version of the previous Lemma

**Lemma 9.5** Let $\xi$ be a rectifiable map parametrised by arc length from $S^1$ to $\mathbb{P}(E)$. Let $\rho$ be an representation of $\pi_1(S)$ in $\text{SL}(E)$. Assume $\rho$ is $S$-irreducible. Assume $\xi(S^1)$ is $\rho(\pi_1(S))$-invariant. Let $x, y$ be two distinct points of $S^1$, then one of the connected component $I$ of $S^1 \setminus \{x, y\}$ satisfies $\langle \xi(I) \rangle = E$.

The main point here is that $\xi$ is not assumed to be injective. If it were, it would be an homeomorphism, and we would have deduced an action of $\pi_1(S)$ on $S^1$ such that $\xi$ is $\rho$-equivariant and we could apply Lemma 9.4.

**Proof:** If both connected components $I_0$ and $I_1$ of $S^1 \setminus \{x, y\}$ satisfy

$$\langle \xi(I_i) \rangle = P_i \subsetneq E.$$ 

Then $\xi(S^1) \subset P_0 \cup P_1$, hence, $\rho$ would not be $S$-irreducible, by the same argument used in the proof of (i) of the previous lemma, which also apply in this more general context. Q.E.D.
9.5 The limit and the boundary at infinity

From Proposition 9.3, we can as well assume that \( \{ \xi^1_m \}_{m \in \mathbb{N}} \) with the arc-length parametrisation converges. Let in particular \( \xi^1 \) be the limit. A priori, by using the arc-length parametrisation, we have lost control over the action of \( \pi_1(S) \). We just know that \( \xi^1(S^1) \) is globally invariant by \( \rho(\pi_1(S)) \). Our aim now is to show this action is semi-conjugate to the action of \( \pi_1(S) \) on \( \partial_\infty \pi_1(S) \).

We begin by replacing \( \xi^1 \) by its arc-length parametrisation \( \hat{\xi}^1 \) so that we have \( \xi^1 = \hat{\xi}^1 \circ \pi \) with \( \pi \) monotone.

We wish to prove

**Proposition 9.6** There is an injective left continuous map preserving the orientation \( \varphi_0 \) from \( \partial_\infty \pi_1(S) \) to \( S^1 \), such that \( \hat{\xi}^1 \circ \varphi_0 \) is \( \rho \)-equivariant and \( \ast \)-hyperconvex.

Notice that Proposition 9.6 implies Lemma 9.2. The proof falls is several steps, we prove

1. \( \hat{\xi}^1 \) is “hyperconvex” when restricted to a certain (non empty) subset \( \Lambda \): Proposition 9.5.1

2. finally, we prove Lemma 9.9 which, combined with the propositions of the previous section, implies Proposition 9.6.

9.5.1 \( \Lambda \)-Hyperconvexity

We are going to prove the following two related propositions

**Proposition 9.7** The map \( \hat{\xi}^1 \) is injective.

As a consequence, it is an homeomorphism onto its image, and we deduce there exists an action of \( \pi_1(S) \) by homeomorphisms on \( S^1 \) such that \( \hat{\xi}^1 \circ \varphi_0 \) is \( \rho \)-equivariant. Let \( \Gamma \) the normal subgroup of index 2 of orientation preserving elements of \( \pi_1(S) \). Let

\[ \Lambda = \Lambda_{\xi, \rho, \Gamma_0}. \]

Notice that \( \Lambda \) is \( \pi_1(S) \) invariant. We shall also prove.

**Proposition 9.8** For any \( n \)-uple of distinct points \( (x_1, \ldots, x_n) \) of distinct points of the closed set \( \Lambda \) the following sum is direct

\[ \sum_{i=1}^{i=n} \hat{\xi}^1(x_i). \]

**Proof:** Let’s write \( \check{\xi}^1_m = \xi^1_m \circ \phi_m \), so that

\[ \lim_{m \to \infty} \check{\xi}^1_m = \xi^1 = \hat{\xi}^1 \circ \pi. \]

Let’s prove the propositions. We split the proof in two parts, which are going to use very similar ideas.
Injectivity: proof of Proposition 9.5.1. First we want to prove the map is injective. Assume therefore that \( \hat{\xi}^1(y) = \hat{\xi}^1(z) \). Since \( \rho \) is S-irreducible, by Lemma 9.5, one of the connected component \( J \) of \( \partial_\infty \pi_1(S) \setminus \{y, z\} \) is such that \( \langle \xi(J) \rangle = E \).

We can therefore find \( n \) points \((x_1, \ldots, x_n)\) in \( J \) such that the following sums are direct

\[
\hat{\xi}^1(x_1) + \ldots + \hat{\xi}^1(x_n) = E, \\
\forall i, \sum_{i \neq j} \hat{\xi}^1(x_j) + \hat{\xi}^1(y) = E
\]

Let \( I \) be an interval containing \( y \) and \( z \) and none of the \( x_i \). For any of the points \( t \in \{y, z, x_1, \ldots, x_n\} \), we denote by \( \hat{t} \) a point such that \( \pi(\hat{t}) = t \). For any distinct integers \( i, j \), let’s write

\[
W_{ij} = (\hat{x}_i, \hat{x}_j), \quad Y_{ij} = (\ldots, \hat{x}_l, \ldots)_{l \neq \{i, j\}}.
\]

We can as in Section 5.4 consider the maps \( F_{ij}^m \) defined by

\[
\begin{aligned}
I & \rightarrow \mathbb{P}(\hat{\xi}_{m}^{(2)}(W_{ij})) \setminus \{\hat{\xi}_m^1(\hat{x}_i)\} \\
t & \mapsto (\hat{\xi}_m^{(n-2)}(Y_{ij}) \oplus \hat{\xi}_m^1(t)) \cap \hat{\xi}_{m}^{(2)}(W_{ij}).
\end{aligned}
\]

By Assertion (59), we obtain that

\[
\lim_{m \to \infty} (F_{ij}^m(\hat{y})) = \left( \bigoplus_{k \neq i, j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(y) \right) \cap \left( \hat{\xi}^1(x_i) \oplus \hat{\xi}^1(x_j) \right) = \lim_{m \to \infty} (F_{ij}^m(\hat{z})).
\]

By Proposition 5.8, all maps \( F_{ij}^m \) are monotone. It follows that for all \( t \in [\hat{y}, \hat{z}] \), we have

\[
\lim_{m \to \infty} (F_{ij}^m(\hat{y})) = \lim_{m \to \infty} (F_{ij}^m(\hat{t})).
\]

But for \( t \) in a neighbourhood of \( y \), the following sums are direct

\[
\forall i, \sum_{i \neq j} \hat{\xi}^1(x_j) + \hat{\xi}^1(t) = E,
\]

This implies that

\[
\lim_{m \to \infty} (F_{ij}^m(t)) = \left( \bigoplus_{k \neq i, j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(\pi(t)) \right) \cap \left( \hat{\xi}^1(x_i) \oplus \hat{\xi}^1(x_j) \right)
\]

Hence, for all \( t \) in a non empty open set, we have

\[
\hat{\xi}^1(t) = \hat{\xi}^1(y).
\]

But this is impossible, since \( \hat{\xi}^1 \) is parametrised by arc length and, hence, not locally constant.

\( \Lambda \)-Hyperconvexity: proof of Proposition 9.8. In this proof, we shall only use the following property of the elements of \( \Lambda \): If \( U \) is a neighbourhood of
an element of \( \Lambda \), \( \xi(U) \) is not included in a union of two proper subspaces (cf. Lemma 9.4(iv)). This property is then true for any element in \( \overline{\Lambda} \), the closure of \( \Lambda \).

Let \( p \) be the smallest integer, less than \( n \), if it exists, such that there exist \( p \) points \((x_1, \ldots, x_{p-2}, y, z)\) cyclically oriented with \( x_i, y, z, \in \overline{\Lambda} \) such that the following sum is not direct

\[
H = \sum_{i=1}^{i=p-2} \hat{\xi}^1(x_i) + \hat{\xi}^1(y) + \hat{\xi}^1(z).
\]

Thanks to Proposition, \( p \geq 3 \). Notice that, by minimality of \( p \), the following sums are direct and equal

\[
\sum_{i=1}^{i=p-2} \hat{\xi}^1(x_i) + \hat{\xi}^1(y) = \sum_{i=1}^{i=p-2} \hat{\xi}^1(x_i) + \hat{\xi}^1(z) = H. \tag{60}
\]

By our initial remark, we can now choose \((x_{p-1}, \ldots, x_n)\) in an arbitrarily small neighbourhood \( J \) of \( x_1 \) in \( \overline{\Lambda} \), such that the following sums are direct

\[
\forall i \geq p-1, \sum_{j \neq i} \hat{\xi}^1(x_j) + \hat{\xi}^1(y) = E, \\
\forall i \geq p-1, \sum_{j \neq i} \hat{\xi}^1(x_j) + \hat{\xi}^1(z) = E. \tag{61}
\]

Let \( I \) be an interval containing \( y \) and \( z \) and none of the \( x_i \). Like in the previous proof, for any of the points \( t \in \{x_1, \ldots, x_n, y, z\} \), we denote by \( t \) a point such that \( \pi(t) = t \). For any distinct integers \( i, j \), let’s write

\[
W_{ij} = (\hat{x}_i, \hat{x}_j), \quad Y_{ij} = (\ldots, \hat{x}_l, \ldots)_{l \neq \{i, j\}}.
\]

We can as in Section 5.4 consider the maps \( F^m_{ij} \) defined for \( i, j \geq p-1 \), by

\[
\begin{align*}
I & \rightarrow \mathbb{P}(\tilde{\xi}^m(2)(W_{ij})) \setminus (\tilde{\xi}^m_n(\hat{x}_i)) \\
t & \mapsto (\tilde{\xi}^m_{n-2}(Y_{ij}) \oplus \tilde{\xi}^m_n(t)) \cap \tilde{\xi}^m(2)(W_{ij}).
\end{align*}
\]

By Assertions 60 and 61, we obtain that for all \( i, j \geq p-1 \),

\[
\lim_{m \to \infty} (F^m_{ij}(\hat{y})) = \left( \bigoplus_{k \neq i, j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(y) \right) \cap (\tilde{\xi}^1(x_i) \oplus \tilde{\xi}^1(x_j)) = \lim_{m \to \infty} (F^m_{ij}(\hat{z})).
\]

By Proposition 5.3, all maps \( F^m_{ij} \) are monotone. It follows that for all \( t \) in \([\hat{y}, \hat{z}]\), we have

\[
\lim_{m \to \infty} (F^m_{ij}(\hat{y})) = \lim_{m \to \infty} (F^m_{ij}(t)).
\]

But for \( t \) in a neighbourhood of \( y \), the following sums are direct

\[
\forall i, \quad \sum_{i \neq j} \hat{\xi}^1(x_j) + \xi^1(t) = E.
\]
This implies that
\[
\lim_{m \to \infty} (F^m_{ij}(t)) = \left( \bigoplus_{k \neq i,j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(\pi(t)) \right) \cap \left( \hat{\xi}^1(x_i) \oplus \hat{\xi}^1(x_j) \right)
\]

Hence, for all \( t \) in a right neighbourhood \( U^+ \) of \( y \), we have
\[
\bigoplus_{i=p-2}^{i=1} \hat{\xi}^1(x_i) \oplus \hat{\xi}^1(t) = \bigoplus_{i=1}^{i=p-2} \hat{\xi}^1(x_i) \oplus \hat{\xi}^1(y)
\]

Hence, there exists a proper subspace \( H^+ \) of \( E \), such that
\[
\forall t \in U^+, \quad \hat{\xi}^1(t) \subset H^+.
\]

A symmetric reasoning (after some cyclic permutation of \((x_1, \ldots, y, z)\)), shows there exist a left neighbourhood \( U^- \) of \( y \), a proper subspace \( H^- \) of \( E \) such that
\[
\forall t \in U^-, \quad \hat{\xi}^1(t) \subset H^-.
\]

As a conclusion we obtain a neighbourhood \( U \) of \( y \) and two proper subspaces \( H^+ \) and \( H^- \) of \( E \) such that
\[
\forall t \in U, \quad \hat{\xi}^1(t) \subset H^- \cup H^+.
\]

This contradicts our initial remark. Q.E.D.

9.5.2 Conjugating to the action on the boundary at infinity

We place ourselves in the situation described by Proposition 9.3. That is we consider

1. a sequence \( \{\rho_m\}_{m \in \mathbb{M}} \) of representations of \( \pi_1(S) \) in \( PSL(E) \), converging to \( \rho \), such that for all non trivial \( \gamma \) in \( \pi_1(S) \), \( \rho_m(\gamma) \) is purely loxodromic

2. a sequence of maps \( \{\xi^1_m\}_{m \in \mathbb{N}} \) from \( \partial_{\infty} \pi_1(S) \) to \( \mathbb{P}(E) \), such that each \( \xi^1_m \) is \( \rho_m \)-equivariant,

3. a sequence of homeomorphisms \( \phi_m \) of \( S^1 \) to \( \partial_{\infty} \pi_1(S) \) such that \( \{\xi^1_m \circ \phi_m\}_{m \in \mathbb{N}} \) converges to \( \hat{\xi}^1 \circ \pi \) such that \( \hat{\xi}^1 \) is an embedding and \( \pi \) is a monotone map from \( S^1 \) to \( \partial_{\infty} \pi_1(S) \).

We say in this situation that \((\hat{\xi}^1, \rho)\) is a good limit. Notice in this case, there exists an action of \( \pi_1(S) \) on \( S^1 \) so that \( \hat{\xi}^1 \) is \( \rho \)-equivariant.

The next lemma finishes the proof of Proposition 9.6.

**Lemma 9.9** Let \( \pi_1(S) \) be a surface group. Let \( \rho \) be a representation of \( \pi_1(S) \) in \( SL(E) \). Assume

- the restriction of \( \rho \) to every finite index subgroup is irreducible,
there exists a map $\hat{\xi}^1$ from $S^1$ to $\mathbb{P}(E)$ such that $(\hat{\xi}^1, \rho)$ is a good limit (in particular every element of $\rho(\pi_1(S))$ has real eigenvalues).

Then the induced action of $\pi_1(S)$ on $S^1$ for which $\hat{\xi}^1$ is $\rho$-equivariant is topologically semi-conjugate to the action on $\partial_{\infty}\pi_1(S)$ in the following sense: there exists an orientation preserving, left continuous, $\pi_1(S)$-equivariant map $\varphi_0$ from $\partial_{\infty}\pi_1(S)$ to $S^1$.

Finally, $\hat{\xi}^1 \circ \varphi_0$ is $*$-hyperconvex.

**Proof:** We first notice that $\Gamma_R$ is not empty. Indeed, the connected component of the Zariski closure of $\rho(\pi_1(S))$ is irreducible. It therefore contains a non trivial diagonalisable element. Hence so does $\rho(\pi_1(S))$, but since this element has only real eigenvalues, this shows $\Gamma_R$ is not empty. Let now $\Gamma_0$ be the subgroup of finite index of orientation preserving elements of $\pi_1(S)$ acting on $S^1$. Let (cf. Paragraph 9.4.1). Again $\Gamma_0 R$, is not empty and invariant by conjugation.

$$\Lambda = \Lambda_{\xi, \rho, \Gamma} \subset S^1.$$ The set $\Lambda$ is not empty by Lemma 9.4(iii) and invariant under the action of $\pi_1(S)$.

Let $\gamma$ be an element of $\Gamma_0 R$,

- let $\gamma^+$ be such that $\xi(\gamma^+)$ is the attractive fixed point of $\rho(\gamma)$; the point $\gamma^+$ is well defined by Lemma 9.4(iii); let

$$\Lambda = \{\gamma^+, \gamma \in \Gamma_0 R\},$$

- Let $\gamma_0^+$ be the attractive fixed point of $\rho_0(\gamma)$, and

$$\Lambda_0 = \{\gamma_0^+, \gamma \in \Gamma_0 R\}.$$ We know that

$$\gamma_0^+ = \lambda_0^+ \implies \exists p, q \neq 0, \lambda^p = \gamma^q \implies \gamma^+ = \lambda^+.$$

Therefore we have a well defined map $\varphi_0$, maybe not injective, defined from $\Lambda_0$ to $\Lambda$. We notice that the first set is dense by the minimality of the action of $\rho_0(\Gamma)$.

We now prove that $\varphi_0$ preserves the cyclic ordering. We are going to use our full hypothesis concerning the construction of $\xi^1$, in particular that $\xi^1$ is a good limit. We begin by some observation. By construction, $\xi_m^1(\gamma_0^+)$ is an attractive fixed point of $\rho_m(\gamma)$ (cf Hypothesis 11 and 12), hence

$$\lim_{m \to \infty} (\xi_m^1(\gamma_0^+)) = \hat{\xi}^1(\gamma^+)$$

(62)

We can now extract a subsequence so that $\{\gamma_m^+\}_{m \in \mathbb{N}} = \{\phi_m^{-1}(\gamma_0^+)\}_{m \in \mathbb{N}}$ converges to a point $\hat{\gamma}^+$. By Hypothesis 13,

$$\lim_{m \to \infty} (\xi_m^1(\phi_m(\gamma_0^+))) = \hat{\xi}^1 \circ \pi(\hat{\gamma}^+)$$

(63)
Combining Assertions (63) and (62), and using the injectivity of $\hat{\xi}_1$ we get that $\pi(\tilde{\gamma}^+) = \gamma^+$. It follows that $\varphi_0$ preserves the orientation. Indeed, let $\gamma$, $\lambda$ and $\delta$ be three elements of $\Gamma$ such that $(\gamma^+_0, \lambda^+_0, \delta^+_0)$ are cyclically oriented. Then so are $(\phi_n(\gamma^+_0), \phi_n(\lambda^+_0), \phi_n(\delta^+_0))$, hence $(\pi(\gamma^+_0), \pi(\lambda^+_0), \pi(\delta^+_0)) = (\gamma^+, \lambda^+, \delta^+)$. 

Now that we know that $\varphi_0$ preserves the orientation, we can extend it by left continuity to a $\Gamma$ equivariant orientation preserving map from $\partial_\infty \pi_1(S)$ to $S^1$ since $\Lambda_0$ is dense by the minimality.

Let’s finally prove $\varphi_0$ is injective. For that let $U = \{x \in S^1 / \varphi_0 \text{ is constant on a neighbourhood of } x\}$. Let’s show that $U = \emptyset$. We can remark that $U$ is an open $\rho_0(\pi_1(S))$-invariant strict subset of $\partial_\infty \pi_1(S)$. By minimality of the action of $\pi_1(S)$ on $\partial_\infty \pi_1(S)$ we conclude that $U = \emptyset$. This imply that $\varphi_0$ is strictly monotone, hence injective.

It remains to prove that $\hat{\xi}_1 \circ \varphi_0$ is $*$-hyperconvex. First, we notice that $\varphi_0(\partial_\infty \pi_1(S)) = \overline{X}$. By Proposition 9.5.1, we have that for $n$ distinct points $(x_1, \ldots, x_n)$ of $\overline{X}$, the following sum is direct

$$\sum_i \hat{\xi}_1(x_i).$$

This implies the first condition on $*$-hyperconvexity. The last condition on $*$-hyperconvexity is a closed condition and follows from the fact that $\xi^1 \pi$ is a limit of $\{\xi^1_m \circ \phi_m\}$ which is a sequence of hyperconvex maps. Q.E.D.

10 Appendix: some lemmas

We prove several lemmas that are used several times in the article. This appendix is independent of what precedes. Next lemma is obvious for Higgs field experts and certainly well known.

**Lemma 10.1** If $\rho$ belongs to Hitchin’s component, then the connected component of the Zariski closure of $\rho(\pi_1(S))$ is irreducible, or equivalently $\rho$ restricted to every finite index subgroup is irreducible.

**PROOF:** We have to recall a little bit of Hitchin’s construction. We consider a surface $S$, its canonical bundle $K$ and the holomorphic vector bundle

$$E = K^{-n} \oplus K^{(2-n)} \oplus \ldots \oplus K^n.$$

We consider the Higgs field $\phi$ which is a section of $\text{End}(E)$ associated to a companion matrix.

Every representation in Hitchin’s component comes from such a Higgs field. From Lemma 1.2 in [27], the parallel sections of the endomorphism bundle are exactly those holomorphic sections commuting with the Higgs field. Let $A$ be such a section. Since $A$ is holomorphic the first row of its matrix in the
decomposition of $E$ vanishes. Now it is easy to check that a matrix whose first row vanishes and that commutes with a companion matrix is zero.

We have just proved that the endomorphism bundle has no parallel sections. Hence the representation is irreducible.

Since the restriction to a finite index subgroup comes from a representation in Hitchin’s component on the corresponding finite cover, we get the second part of the statement. Q.E.D.

Next result is used several times.

**Lemma 10.2** Let $\Gamma$ be a surface group. Let $\rho$ be a representation of $\pi_1(S)$ to $\text{SL}(n, \mathbb{R})$. Assume there exists a continuous $\rho$-equivariant map $\xi^{k+1}$ from $\partial_{\infty} \pi_1(S)$ to the Grassmannian of $k+1$-planes in $\mathbb{R}^n$. Assume there exists some $(n-k)$-plane $A$, a non empty open set $U$ in $\partial_{\infty} \pi_1(S)$ such that

$$\forall y \in U, \dim (\xi^{k+1}(y) \cap A) \geq 1. \quad (64)$$

Then, the restriction of $\rho$ to a finite index subgroup is not irreducible, or equivalently the connected component of the identity of the Zariski closure of $\rho(\pi_1(S))$ is not irreducible.

**Proof:** First step. We show first that there exists some $(n-k)$-plane $B$, such that

$$\forall y \in \partial_{\infty} \pi_1(S), \dim (\xi^k(y) \cap B) \geq 1. \quad (65)$$

First, we can find a smaller open subset $O$ of $U$ and some element $\gamma \in \pi_1(S)$ such that

$$\gamma^i(O) \subset \gamma^{i+1}(O) \quad O_{\infty} = \bigcup_{i \in \mathbb{N}} \gamma^i(O) \text{ is dense in } \partial_{\infty} \pi_1(S).$$

Next notice that for $B^i = \rho(\gamma)^{-i}(A)$ we have

$$\forall y \in \gamma^i(O), \dim (\xi^k(y) \cap B^i) \geq 1.$$

Now we extract from $\{B^i\}_{i \in \mathbb{N}}$ a convergent subsequence to a $(n-k)$-plane $B$. It follows that

$$\forall y \in O_{\infty}, \dim (\xi^k(y) \cap B) \geq 1.$$ 

Hence, by density of $O_{\infty}$, we obtain Assertion (65). The proof finally follows from the next proposition applied to $G$ the Zariski closure of $\rho(\pi_1(S))$. Q.E.D.

**Proposition 10.3** Let $G$ be an algebraic subgroup of $\text{SL}(n, \mathbb{R})$. If there exist a $k$-plane $C$, a $(n-k)$-plane $B$ such that

$$\forall g \in G, \dim (g(C) \cap B) \geq 1. \quad (66)$$

Then the connected component of the identity of $G$ is not irreducible.
Proof: Let $G$, $C$, $B$ be as in the Proposition. Notice that $B + C$ is a proper subspace, and $B \cap C$ is not reduced to $\{0\}$. Let $g_0$ be an element of $G$ such that

$$\forall g \in G, \ p := \dim(g_0(C) \cap B) \leq \dim(g(C) \cap B).$$

Let now $D = g_0(C)$, $F$ be a codimension $p - 1$ vector subspace of $B$ such that,

$$\dim(D \cap F) = 1.$$ 

Notice now that

$$\forall g \in G, \ dim(g(D) \cap F) \geq 1. \quad (67)$$

Let $\mathcal{G}$ be the Lie algebra of $G$.

We will show the following assertion that contradicts the irreducibility of the connected component of the identity of $G$,

$$\forall \alpha \in \mathcal{G}, \ \alpha(D \cap F) \subset D + F. \quad (68)$$

Indeed, let $(e_0, \ldots, e_k)$ be a basis of $D$, let $(u_1, \ldots, u_l)$ be a basis of $F$, such that $(u_l)$ is a basis of $D \cap F$. Notice that

$$e_0 \wedge \ldots \wedge e_k \wedge u_1 \wedge \ldots \wedge u_{l-1} \neq 0.$$

Let $h$ be an element of $\mathcal{G}$. From Assertion (67), we obtain

$$e_0 \wedge \ldots \wedge e_k \wedge e^{t\alpha}(u_1 \wedge \ldots \wedge u_l) = 0.$$ 

We take now the first order term of the above series in $t$. Since

$$e_0 \wedge \ldots \wedge e_k \wedge u_l = 0,$$

we obtain,

$$e_0 \wedge \ldots \wedge e_k \wedge u_1 \wedge \ldots \wedge u_{l-1} \wedge \alpha(u_l) = 0.$$ 

This implies Assertion (68). Q.E.D.

Finally, the next lemma is of independent interest

Lemma 10.4 Let $\Gamma$ be a subgroup of $SL(n, \mathbb{R})$ whose elements are all real split. Assume every finite index subgroup of $\Gamma$ is irreducible. Then $\Gamma$ is discrete.

Proof: Let $G$ be the Zariski closure of $\Gamma$. From the irreducibility hypothesis it follows $G$ is semi-simple. If $\Gamma$ is not discrete, since it is Zariski dense, it is classical that its closure (for the usual topology) contains one of the non trivial factor $H$ of $G$. But the closure of $\Gamma$ consists of elements whose elements have only real eigenvalues. This would imply the maximal compact subgroup of $H$ is reduced to the identity and this never happens for a simple Lie group. Q.E.D.
References

[1] I. Biswas, P. Ars-Gastesi, S. Govindarajan Parabolic Higgs bundles and Teichmüller spaces for punctured surfaces. Trans. Amer. Math. Soc. 349 (1997), no. 4, 1551–1560.

[2] M. Burger, A. Iozzi, A. Wienhard Sur les représentations d’un groupe de surface compacte avec invariant de Toledo maximal. C.R. Acad. Sci. Paris, Ser. I 336 (2003), 387-390

[3] S. Choi, W. Goldman Convex real projective structures on closed surfaces are closed. Proc. Amer. Math. Soc. 118 (1993), no. 2, 657–661.

[4] K. Corlette Flat G-bundles with canonical metrics. J. Differential Geom. 28 (1988), no. 3, 361–382.

[5] S.K. Donaldson Twisted harmonic maps and the self-duality equations. Proc. London Math. Soc. (3) 55 (1987), 127-131

[6] A. Fathi, F. Laudenbach, V. Poenaru Travaux de Thurston sur les surfaces Astérisque, No.66-67. Paris: Société Mathématique de France. 284 p. (1979).

[7] V. Fock, A.B Goncharov Moduli spaces of local systems and higher Teichmüller theory .math.AG/0311114

[8] W. Goldman Topological components of spaces of representations.Invent. Math. 93 (1988), no. 3, 557–607.

[9] O. Guichard Sur les représentations des groupes de surface preprint

[10] W. Goldman, Geometric Structures and varieties of representation in “The Geometry of Group Representations”Summer Conference 1987, Bouldern Colorado, W. Goldman and A. Magid (eds)

[11] U. Hamenstäd t Cocycles, Hausdorff measures and cross ratios Ergodic Theory and Dynamical Systems (1997), 17 pp 1061-1081

[12] B. Hasselblatt, A.Katok Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[13] N. Hitchin Lie Groups and Teichmüller spaces. Topology 31 (1992), no. 3, 449–473.

[14] M. Kontsevich The Virasoro algebra and Teichmüller spaces. (Russian) Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 78–79

[15] F. Labourie Existence d’applications harmoniques tordues à valeurs dans les variétés à courbure négative. Proc. Amer. Math. Soc. 111 (1991), no. 3, 877–882.
[16] F. Labourie $\mathbb{RP}^2$-structures et différentielles cubiques holomorphes. Proceedings of the GARC Conference in Differential Geometry, Seoul National University, 1997.

[17] F. Labourie Crossratios, Surface Groups and $SL(n, \mathbb{R})$. preprint

[18] F. Labourie Crossratios, Anosov Representations and Quasi-Issometries in preparation

[19] F. Ledrappier Structure au bord des variétés à courbure négative. Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, 97–122, Sémin. Théor. Spectr. Géom., 13, Univ. Grenoble I, Saint-Martin-d’Hères, 1995.

[20] J. Loftin Affine spheres and convex $\mathbb{RP}^n$-manifolds. Amer. J. Math. 123 (2001), no. 2, 255–274.

[21] W. L. Lok, Deformation of locally homogeneous spaces and Kleinian groups Doctoral Thesis, Columbia University 1984

[22] J.-P. Otal Le spectre marqué des longueurs des surfaces à courbure négative. Ann. of Math. (2) 131 (1990), no. 1, 151–162.

[23] A. Parreau Dégénérescences de sous-groupes discrets de groupes de Lie semi-simples et actions de groupes sur des immeubles affines Thèse Université Paris-Sud (2000)

[24] R. Penner The decorated Teichmüller space of punctured surfaces. Comm. Math. Phys. 113 (1987), no. 2, 299–339.

[25] C. Robinson Stability, Symbolic Dynamics, and Chaos. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[26] G. Segal The geometry of the KdV equation. Topological methods in quantum field theory (Trieste, 1990). Internat. J. Modern Phys. A 6 (1991), no. 16, 2859–2869.

[27] C. Simpson Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math. No. 75, (1992), 5–95.

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