THE INVISCID LIMIT OF THE NAVIER-STOKES EQUATIONS WITH KINEMATIC AND NAVIER BOUNDARY CONDITIONS

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Abstract. We are concerned with the inviscid limit of the Navier-Stokes equations on bounded regular domains in $\mathbb{R}^3$ with the kinematic and Navier boundary conditions. We first establish the existence and uniqueness of strong solutions in the class $C([0, T_\star); H^r(\Omega; \mathbb{R}^3)) \cap C^1([0, T_\star); H^{r-2}(\Omega; \mathbb{R}^3))$ with some $T_\star > 0$ for the initial-boundary value problem with the kinematic and Navier boundary conditions on $\partial \Omega$ and divergence-free initial data in the Sobolev space $H^r(\Omega; \mathbb{R}^3)$ for $r \geq 2$. Then, for the strong solution with $H^{r+1}$-regularity in the spatial variables, we establish the inviscid limit in $H^r(\Omega; \mathbb{R}^3)$ uniformly on $[0, T_\star)$ for $r > \frac{5}{2}$. This shows that the boundary layers do not develop up to the highest order Sobolev norm in $H^r(\Omega; \mathbb{R}^3)$ in the inviscid limit. Furthermore, we present an intrinsic geometric proof for the failure of the strong inviscid limit under a non-Navier slip-type boundary condition.

1. Introduction

We are interested in the analysis of strong solutions in the Sobolev spaces $H^r$ of the incompressible Navier-Stokes equations with positive viscosity coefficient $\nu > 0$ in a bounded regular domain $\Omega \subset \mathbb{R}^3$ subject to the kinematic and Navier boundary conditions on $\partial \Omega$ and the divergence-free initial data at $t = 0$, and their convergence to the corresponding strong solution of the Euler equations in the inviscid limit as $\nu \to 0$. One of our main motivations for such an analysis is to examine whether the boundary layers would develop in some high-order Sobolev norm in the inviscid limit.

We assume that the boundary, $\partial \Omega$, of domain $\Omega$ is an embedded oriented 2-dimensional (2-D) manifold, i.e. a regular surface. The incompressible Navier-Stokes equations in $[0, T] \times \Omega$ take the following form:

\[
\begin{align*}
\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla p^\nu &= \nu \Delta u^\nu, \\
\nabla \cdot u^\nu &= 0.
\end{align*}
\]

In (1.1), the vector field $u^\nu : \Omega \to \mathbb{R}^3$ is the velocity of the fluid and the scalar field $p^\nu : \Omega \to \mathbb{R}$ is the pressure, both of which depend on the viscosity constant $\nu > 0$. The divergence-free condition of $u^\nu$ describes the incompressibility of the fluid. The existence, uniqueness, and regularity of weak and strong solutions of the Navier-Stokes equations (1.1) are an important research topic in nonlinear PDEs and mathematical hydrodynamics; cf. [30, 31, 35, 47, 44] and the references cited therein. In this paper, we focus on the Navier-Stokes equations (1.1) in a general bounded regular domain $\Omega$, for which the geometry of $\Omega$ plays an important role in our analysis.
Consider the initial condition:

$$u^\nu|_{t=0} = u_0 \quad \text{on } \Omega,$$

where $u_0$ satisfies the compatibility condition: $\nabla \cdot u_0 = 0$ in $\Omega$.

The **kinematic boundary condition** is

$$u^\nu \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega \times [0, T],$$

i.e. the normal component of the velocity on the boundary vanishes.

The **Navier boundary condition** is imposed as:

$$u \cdot \tau = -2\zeta \mathbb{D}u(\tau, \mathbf{n}) \quad \text{on } \partial \Omega \times [0, T],$$

for any $\tau \in T(\partial \Omega)$, where the rate-of-strain tensor is the $3 \times 3$ matrix defined by

$$\mathbb{D}u := \frac{1}{2}(\nabla u + (\nabla u)^\top),$$

and constant $\zeta > 0$ is known as the **slip length** of the fluid.

Traditionally, the Navier-Stokes equations (1.1) have been studied with the no-slip condition, i.e. the Dirichlet boundary condition $u = 0$ on $\partial \Omega$. However, this does not always match with the experimental data; cf. [24, 42]. First proposed by Navier [39] in 1816, the Navier boundary condition (1.4) requires that the tangential component of the velocity field is proportional to that of the normal vector field of the Cauchy stress tensor. The proportionality constant $\zeta > 0$ is known as the **slip length**. Physically, the Navier boundary condition (1.4) can be induced by the effects of free capillary boundaries, perforated boundaries, or the exterior electric fields; cf. Achdou-Pironneau-Valentin [1], Bänsch [4], Beavers-Joseph [7], Einzel-Panzer-Liu [24], Maxwell [38], Jäger-Mikelić [28, 29], and the references cited therein.

To analyze the initial-boundary value problem (1.2)–(1.4) for the Navier-Stokes equations (1.1), we adopt an equivalent geometric formulation, as shown in Chen-Qian [15], for the boundary conditions on $\partial \Omega \times [0, T]$:  

$$\begin{align*}
\omega^\nu \cdot \tau &= -\frac{\zeta}{2}(\mathbb{R}u^\nu) \cdot \tau + 2\mathbb{R}(S(u^\nu)) \cdot \tau \quad \text{on } \partial \Omega \times [0, T],
\end{align*}$$

where

$$\omega^\nu := \nabla \times u^\nu$$

is the vorticity of the fluid, $\tau \in T(\partial \Omega)$ is an arbitrary tangential vector field on boundary $\partial \Omega$, $\mathbb{S}$ is the shape operator of surface $\partial \Omega$, and $\mathbb{R}$ is the operator corresponds to the left multiplication by the matrix in the local coordinate on $\partial \Omega$:

$$\mathbb{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

i.e. the anti-clockwise rotation by $\frac{\pi}{2}$. In fact, $\mathbb{R}$ can be identified with the Hodge star operator $*$ defined for the differential forms on $\mathbb{R}^2$: For a 2-D vector field $V = (V^1, V^2)^\top$,

$$\mathbb{R}V = (V^2)^* = (-V^2, V^1)^\top,$$
in which \( \sharp \) is the canonical isomorphism between vector fields and differential 1-forms, and \( b \) is its inverse. The second equation in \( (1.6) \) (i.e. the Navier boundary condition in the geometric formulation) has the vorticity on the left-hand side, but it involves only the zero-th order operations on the velocity on the right-hand side.

The problem of inviscid limits has been a central topic in mathematical hydrodynamics (cf. Constantin \[18\]). In 1975, Swann \[46\] proved that, for \( \Omega = \mathbb{R}^3 \), when the initial vorticity is in \( H^{3+\delta} \), divergence-free, and vanishing at spatial infinity, and the right-hand side of the vorticity equation lies in \( C(\{0,T\}; H^2(\mathbb{R}^3)) \) for some small \( T \), then the initial-boundary value problem for the Navier–Stokes equations with zero boundary condition has a unique strong solution, and the vanishing viscosity limit holds in \( L^6 \cap H^1 \). In 1986, Constantin \[17\] showed that, for \( \Omega = \mathbb{R}^3 \), if the Cauchy problem for the Euler equations with initial data \( v_0 \in H^{m+2}(\mathbb{R}^3) \) for \( m \geq 3 \) has a strong solution in \( X = C(\{0,T\}; H^m(\mathbb{R}^3)) \) up to time \( T \), then there exists \( \nu_* = \nu_*(T, v_0) \) such that the Cauchy problem for the corresponding Navier-Stokes equations for any \( \nu \leq \nu_* \) also has a strong solution in \( X \), and the vanishing viscosity limit holds in \( H^m \). In fact, for \( \Omega = \mathbb{R}^d \) for \( d = 2 \) or \( 3 \), for any \( s > \frac{d}{2} + 1 \) and initial data \( v_0 \in H^s \), the convergence can be obtained in the \( H^s \)-norm; cf. Masmoudi \[39\]. Moreover, in Constantin-Wu \[20\], the vanishing viscosity limits were also proved on \( \Omega = \mathbb{R}^2 \) for the initial vorticity in \( L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \).

On the other hand, in the case that \( \Omega \) is a bounded domain with boundary, and the Navier-Stokes equations are equipped with the Dirichlet boundary condition, the vanishing viscosity limit fails in general: This is due to the formation of boundary layers, in which the Prandtl equations serve as a candidate for matching the Navier-Stokes and Euler equations; see e.g., Alexandre-Wang-Xu-Yang \[3\], Gérard-Varet-Dormy \[25\], and the references cited therein. In contrast, when the Navier and kinematic boundary conditions are imposed to the Navier-Stokes equations, the vanishing viscosity limit can be established in the affirmative. In 2007, Xiao-Xin \[49\] proved that, for the initial data in \( H^3 \) on a 3-D flat domain, there exists \( T_* > 0 \) such that the vanishing viscosity limit holds in \( C(\{0,T\}; H^2) \cap L^p(0,T_0; H^3) \), for all \( 1 \leq p < \infty \). Various convergence results of this kind for a non-Navier “slip-type boundary condition” (first proposed by Bardos \[5\], which agrees with the Navier condition if and only if the domain is a part of the flat half-space) have been established, in \( W^{k,p} \), \( H^s \), or \( L^p \) spaces and on 2-D or 3-D spatial domains; cf. Xiao-Xin \[49\], Beirão da Veiga-Crispo \[8,9\], Bellout-Neustupa-Penel \[12\], Berselli-Spirito \[13\], Chen-Osborne-Qian \[14\], Clopeau-Mikelić-Robert \[10\], Kelliher \[32\], Wang-Xin-Zang \[48\], Zhong \[50\], and the references cited therein.

Furthermore, for the Navier boundary conditions, Chen-Qian \[15\] and Ifimie-Planas \[20\] obtained the vanishing viscosity limit in \( L_t^\infty L_x^2 \) on smooth domains \( \Omega \subset \mathbb{R}^3 \) and \( \mathbb{R}^d \), \( d \geq 2 \), provided that strong solutions exist in \( H^2 \) and \( H^{d/2+1+\epsilon} \), respectively; see also the related results by Ifimie-Raugel-Sell \[27\] on a 3-D thin domain and by Lopes Filho-Nussenzveig Lopes-Planas \[34\] on 2-D domains, and the recent results by Drivas-Nguyen \[22\]. In addition, by computations in local coordinates, Neustupa-Penel \[10,11\] proved the convergence in \( L^\infty(0,T_*; H^4) \cap L^2(0,T_*; H^2) \), provided that the initial data is in \( H^4 \), where \( T_* > 0 \) is a constant depending only on \( \Omega \) and the initial data. Moreover, using the geometric vector field approach, Masmoudi-Rousset \[37\] established the existence of strong solutions in \( L^\infty(0,T; E^m(\Omega; \mathbb{R}^3)) \cap L^2(0,T; H^{m+1}(\Omega; \mathbb{R}^3)) \) for \( m > 6 \) and the inviscid limit in \( L_t^\infty L_x^2 \), where the anisotropic co-normal Sobolev space \( E^m \) is given by \( E^m := \{ u \in H^m_{co} : \nabla u \in H^{m-1}_{co} \} \), and \( u \in H^m_{co} \) whenever \( \sum_{0 \leq |l| \leq m} \| Z_l u \|_{L^2(\Omega)} < \infty \) with \( \{ Z_l \} \) spanning the space of vector fields tangential to \( \partial \Omega \).
In this paper, by performing the higher-order energy estimates for the weak solutions constructed in \cite{15}, we first establish the existence and uniqueness of the strong solution of the Navier-Stokes equations in $C(0,T_\star;H^r(\Omega;\mathbb{R}^3)) \cap C^1(0,T;H^{r-2}(\Omega;\mathbb{R}^3))$ for some $T_\star > 0$ and $r \geq 2$, subject to the kinematic and Navier boundary conditions. We assume that domain $\Omega$ is regular, with the smooth second fundamental form $\Pi$. In fact, in the estimates, we need $\|\Pi\|_{C^{r-1}(\partial\Omega)} < \infty$. Moreover, an explicit lower bound for $T_\star$ is obtained. This is achieved by employing more delicate energy estimates, which take into account the effects of the curvature (equivalently, the second fundamental form $\Pi$) of $\partial\Omega$ and the Navier boundary conditions. In addition, we study the inviscid limit (also known as the vanishing viscosity limit) of the Navier-Stokes equations \eqref{1.1}: We send $\nu \to 0^+$ and investigate whether the strong solutions $u^\nu$ converge, in suitable norms, to the corresponding solution of the Euler equations describing the motion of incompressible, inviscid fluids:

$$\begin{aligned}
&\frac{\partial}{\partial t} u + (u \cdot \nabla) u + \nabla p = 0 \quad \text{in } [0,T] \times \Omega, \\
&\nabla \cdot u = 0 \quad \text{in } [0,T] \times \Omega, \\
&u|_{t=0} = u_0 \quad \text{on } \Omega,
\end{aligned}$$

subject to the no-penetration boundary condition:

$$u \cdot n = 0 \quad \text{on } [0,T] \times \partial \Omega. \quad \text{(1.11)}$$

As discussed above, for the kinematic and Navier boundary conditions, the inviscid limit problem was answered in the affirmative for strong solutions on domains with flat boundaries (e.g. the half-space) by Xiao-Xin \cite{49} and Beirão da Veiga-Crispo \cite{8, 9}. This is achieved by analyzing the aforementioned simplified boundary condition in \cite{5, 45}, which agrees with the Navier boundary condition for flat boundaries. Similar affirmative results are also established for several modified versions of the slip-type boundary conditions in \cite{48, 50}. In addition, the inviscid limit for the strong solutions in $L^2$ or $H^1$ under the kinematic and Navier boundary conditions are proved by Chen-Qian \cite{15}, Intimie-Planas \cite{26}, and Neustupa-Penel \cite{40, 41} for bounded, regular, possibly non-flat domains in $\mathbb{R}^3$.

On the other hand, recently in \cite{10, 11}, Beirão da Veiga-Crispo proved that the inviscid limits in strong topologies of $W^{s,p}$ for $s > 1$ and $p > 1$ fails for general non-flat domains, with the Navier-Stokes equations equipped with the simplified boundary conditions as in \cite{5, 45}. In comparison, the inviscid limit in strong topologies always holds for regular domains in 2-D, when the Navier boundary condition is assumed. This is largely due to the fact that the vorticity is transported in 2-D; cf. \cite{16, 34, 19}.

In view of the discussions above, it is important to understand whether the inviscid limit holds for strong solutions in the higher-order Sobolev norms in $H^r(\Omega;\mathbb{R}^3)$ for $r > 1$ in a bounded, regular, generally non-flat domain $\Omega \subset \mathbb{R}^3$, when the Navier-Stokes equations \eqref{1.1} are equipped with the Navier boundary conditions (i.e. Eq. \eqref{1.6}). To the best of our knowledge, this problem is still largely open. In Theorem \ref{5.1}, we answer this question in the affirmative: If the strong solution exists in $H^{r+1}(\Omega;\mathbb{R}^3)$ for $r > \frac{5}{2}$, we establish its strong convergence in $H^r(\Omega;\mathbb{R}^3)$ as the viscosity constant $\nu \to 0$. This implies that the boundary layers do not develop up to the highest order Sobolev norm in $H^r(\Omega;\mathbb{R}^3)$ for $r > \frac{5}{2}$.

The rest of the paper is organized as follows: In §2 we briefly sketch the derivation of the boundary conditions in terms of geometric quantities. In §3, we prove a lemma which expresses
the $H^r$–norm of a divergence-free vector field by the $L^2$–norm of the iterated curls, subject to the kinematic and Navier boundary conditions. Next, in §4, we derive the a priori, higher-order energy estimates in $H^r(\Omega; \mathbb{R}^3)$ for $r \geq 2$ for the Navier-Stokes equations with kinematic and Navier boundary conditions. We also deduce the existence of strong solutions from the energy estimates. Then, in §5, the inviscid limit is established. Finally, in §6, we discuss the inviscid limit problem for other non-Navier slip-type boundary conditions.

Before concluding this introduction, we present some notations that will be used from now on in this paper. We denote $H^r(\Omega; \mathbb{R}^3) = W^{2,r}(\Omega; \mathbb{R}^3)$ as the Sobolev space of vector fields $\phi : \Omega \to \mathbb{R}^3$ with the norm in the multi-index notation:

$$\|\phi\|_{H^r(\Omega)} := \left( \sum_{0 \leq |\alpha| \leq r} \int_{\Omega} |\nabla^\alpha \phi|^2 \, dx \right)^{1/2} < \infty.$$  \hspace{1cm} (1.12)

We write $\nabla[s]$ to denote a generic differential operator $\nabla_i \nabla_j \cdots \nabla_s$ for any $s \geq 1$. The Einstein summation convention is used. For the indices, we write $i, j, k, \ldots \in \{1, 2, 3\}$ and $\alpha, \beta, \gamma, \delta, \ldots \in \{1, 2\}$. The angular bracket $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of two vectors in $\mathbb{R}^3$. Furthermore, we write $f \lesssim g$ if $|f| \leq C|g|$ for a generic constant $C$ depends only on $r$, $\|II\|_{C^{r-1}(\partial \Omega)}$, and $\zeta$: and write $f \simeq g$ whenever $f \lesssim g$ and $g \lesssim f$. Denote $\mathcal{H}^2$ as the 2-D Hausdorff measure. Finally, $\text{curl}^r := \text{curl} \circ \cdots \circ \text{curl}$ means the composition of $r$ curls.

## 2. The Navier Boundary Condition

In this section, we briefly sketch the derivation of the boundary conditions in terms of geometric quantities.

First of all, we justify that our geometric formulation of the Navier boundary condition (i.e. the second equation in 1.6, reproduced below):

$$\omega^r \cdot \tau = -\frac{1}{\zeta} (Ru^r) \cdot \tau + 2R(S(u^r)) \cdot \tau \quad \text{on } \partial \Omega \times [0, T] \text{ for any } \tau \in T(\partial \Omega)$$

is indeed equivalent to the one proposed by Navier in [39]. For simplicity, we drop superscript $\nu$ in this section.

We start by remarking on the geometric notations. Recall that the boundary of the domain of fluid, $\partial \Omega$, is a regular surface embedded in $\mathbb{R}^3$. We denote its second fundamental form by $\Pi : T(\partial \Omega) \times T(\partial \Omega) \to \mathbb{R}$, where $T(\partial \Omega)$ is the tangent bundle of $\partial \Omega$. Thus, writing $n \in T(\partial \Omega)^\perp$ as the outward unit normal (viewed as the Gauss map $n : \partial \Omega \to S^2$), we have

$$\Pi = -\nabla n.$$  \hspace{1cm} (2.1)

In addition, take $\{e_1, e_2, e_3\}$ to be an orthonormal frame such that $e_1, e_2 \in T(\partial \Omega)$ and $e_3 = n$. Then we have the local expression:

$$\Pi(u, v) = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \Pi_{\alpha\beta} u^\alpha v^\beta.$$  \hspace{1cm} (2.2)

The shape operator $S : T(\partial \Omega) \to T(\partial \Omega)$ is then defined as

$$S(u) := -\nabla_u n,$$  \hspace{1cm} (2.3)

where $\nabla_u$ means the directional derivative in the direction of $u$. 

Now, recall that the Navier boundary condition reads that, for any $\tau \in T(\partial \Omega)$,
\[ u \cdot \tau = -2\zeta \mathbb{D}u(\tau, n) \quad \text{on} \quad \partial \Omega \times [0, T], \]
where, in local coordinates, the rate-of-strain tensor is given by
\[ (\mathbb{D}u)_{ij} = \frac{1}{2}(\nabla_i u^j + \nabla_j u^i), \quad 1 \leq i, j \leq 3. \]
Suppose that $\{e_1, e_2, e_3\}$ is an orthonormal moving frame adapted to $\partial \Omega$, with $e_3 = n$. Then the Navier boundary condition is equivalent to the following:
\[ u^1 = -\zeta(\nabla_3 u^1 + \nabla_1 u^3), \quad u^2 = -\zeta(\nabla_3 u^2 + \nabla_2 u^3) \quad \text{on} \quad \partial \Omega \times [0, T]. \quad (2.4) \]

The main issue of this paper is to derive the higher-order energy estimates of velocity $u$. As shown in §3 below, the $H^r$–norm of $u$ is estimated purely by the $L^2$–norm of the $r$-th iterated curls of $u$ (cf. Theorem 3.1). We now seek for the boundary condition with respect to the vorticity: $\omega = \nabla \times u$. For this purpose, note that $\omega = \begin{bmatrix} \nabla_2 u^3 - \nabla_3 u^2 \\ \nabla_3 u^1 - \nabla_1 u^3 \\ \nabla_1 u^2 - \nabla_2 u^1 \end{bmatrix}$ in the local frame $\{e_1, e_2, e_3\}$. Then the Navier boundary condition (1.4) becomes
\[ \nabla_k u^3 + \nabla_3 u^k = -\frac{1}{\zeta} u^k \quad \text{for} \quad k \in \{1, 2\}. \quad (2.5) \]

On the other hand, $\nabla_k u^3$ can be computed as
\[ \nabla_k u^3 = \nabla_k(u \cdot n) = \partial_3(u \cdot n) + \sum_{j=1}^{3} \Gamma^3_{kj} u^j, \quad (2.6) \]
where $\Gamma^3_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ are the Christoffel symbols. Observe also that
\[ \Pi_{jk} = \Pi(e_j, e_k) \cdot n = -\nabla_j e_k \cdot n = -\sum_{l=1}^{3} \Gamma^l_{jk} e_l \cdot n = -\Gamma^3_{jk}. \quad (2.7) \]
Then, by collecting Eqs. (2.5)–(2.7), we have
\[ \nabla_k u^3 - \nabla_3 u^k = 2\partial_k(u \cdot n) - 2 \sum_{j=1}^{3} \Pi_{jk} u^j + \frac{1}{\zeta} u^k. \quad (2.8) \]

Finally, in view of the kinematic boundary condition (i.e. the first equation in (1.6)), $u \cdot n = 0$ on $\partial \Omega$. Then, by taking $k = 1, 2$, respectively, and recalling the definition of $\mathcal{R}$, we immediately recover the second equation in (1.6). Note that the term, $2\mathcal{R}(\mathcal{S}(u^\nu))$, reflects the geometry of the curvilinear fluid domain. It vanishes when the domain is flat, e.g. the half plane. In the rest of the paper, this is referred to as the Navier boundary condition.

3. A Div-Curl Estimate for Divergence-free Vector Fields

In this section, we show that the $H^{r+1}$–norm of a divergence-free vector field is equivalent to the sum of the $L^2$–norms of its iterated curls up to the $(r + 1)$-th order. It is a variant of the well-known div-curl estimate due to Caldéron-Zygmund for divergence-free vector fields.
Theorem 3.1. Let \( u \in H^{r+1}(\Omega; \mathbb{R}^3) \cap K_2(\Omega) \) for \( r \geq 0 \) satisfy the kinematic and Navier boundary conditions \([1, 0]\), where
\[
K_2(\Omega) := \{ u \in L^2(\Omega; \mathbb{R}^3) : \nabla \cdot u = 0 \}.
\]
Then there exists a universal constant \( M = M(r, \Omega) > 0 \) such that
\[
\| \nabla^{r+1} u \|^2_{L^2(\Omega)} \leq M \sum_{i=0}^{r+1} \| \text{curl}^i u \|^2_{L^2(\Omega)}.
\]

Here and in the sequel, the time variable \( t \) is always suppressed when only the spatial regularities are considered. The following Sobolev trace theorem is also frequently used:

Lemma 3.2 (Theorem 5.36 in [2]). Let \( \Omega \) be a domain in \( \mathbb{R}^n \) satisfying the uniform \( C^{m} \)-regularity condition. Assume that there exists a \((m, p)\)-extension operator for \( \Omega \). Suppose that
\[
mp < n, \quad p \leq q \leq p^* := \frac{(n-1)p}{n-mp}.
\]
Then the continuous embedding \( W^{m,p}(\Omega) \hookrightarrow L^q(\partial\Omega) \) holds.

Proof of Theorem 3.1. We prove the theorem by induction on \( r \). The arguments are divided into seven steps.

1. We first establish the base case \( r = 0 \). Indeed, in view of the following identity (see Eq. (3.3) in Chen-Qian [15]):
\[
\| \nabla u \|^2_{L^2(\Omega)} = \| \nabla \times u \|^2_{L^2(\Omega)} + \| \nabla \cdot u \|^2_{L^2(\Omega)} - \int_{\partial\Omega} (\nabla \cdot u)(u, n) \, d\mathcal{H}^2 - \int_{\partial\Omega} (u \cdot \nabla u, n) \, d\mathcal{H}^2,
\]
for the incompressible velocity field satisfying the kinematic boundary condition, we have
\[
\| \nabla u \|^2_{L^2(\Omega)} = \| \nabla \times u \|^2_{L^2(\Omega)} + \int_{\partial\Omega} \Pi(u, u) \, d\mathcal{H}^2,
\]
where we have utilized the definition of the second fundamental form \( \Pi := -\nabla n \). Since \( \| \Pi \|_{L^\infty(\partial\Omega)} < \infty \), we bound
\[
\left| \int_{\partial\Omega} \Pi(u, u) \, d\mathcal{H}^2 \right| \leq \| \Pi \|_{L^\infty(\partial\Omega)} \| u \|^2_{L^2(\partial\Omega)} \leq \epsilon \| \nabla u \|^2_{L^2(\Omega)} + \frac{C}{\epsilon} \| u \|^2_{L^2(\Omega)},
\]
thanks to the Sobolev trace inequality and Young’s inequality. Thus, the case for \( r = 0 \) follows immediately by choosing \( \epsilon \) suitably small.

2. We now assume the result for \( r \geq 0 \) and prove it for \( r + 1 \). First of all, we apply integration by parts twice to obtain
\[
\| \nabla^{r+1} u \|^2_{L^2(\Omega)} = \int_{\Omega} \left( \frac{\partial_{i_1} \cdots \partial_{i_{r+1}} u^k}{\partial_{i_1} \cdots \partial_{i_{r+1}} u^k} \right) \, dx
\]
\[
- \int_{\Omega} \frac{\partial_{i_2} \cdots \partial_{i_{r+1}} u^k}{\partial_{i_1} \cdots \partial_{i_{r+1}} u^k} \, dx
\]
\[
+ \int_{\Omega} \left( \Delta \frac{\partial_{i_3} \cdots \partial_{i_{r+1}} u^k}{\partial_{i_1} \cdots \partial_{i_{r+1}} u^k} \right) \, dx
\]
\[
=: I + J + K.
\]
Using the divergence theorem, the above three integrals are expressed as
\[
\begin{align*}
I &= \frac{1}{2} \int_{\partial \Omega} \partial_u |\nabla^r u|^2 \, d\mathcal{H}^2, \\
J &= \int_{\partial \Omega} (\partial_{i_2} \cdots \partial_{i_{r+1}} u^k) \left( \Delta \partial_{i_3} \cdots \partial_{i_{r+1}} u^k \right) (\nabla_{i_2}, u) \, d\mathcal{H}^2, \\
K &= \int_{\Omega} |\nabla^{r-1} \psi|^2 \, dx,
\end{align*}
\]
where \( \psi = \text{curl} \omega = -\Delta u \) is the stream function.

3. Now we bound the surface integral \( I \) in (3.7). For this purpose, we introduce a local moving frame \( \{e_1, e_2, e_3\} \) on surface \( \partial \Omega \) such that \( e_1, e_2 \in T(\partial \Omega) \) and \( e_3 = n \). Then
\[
I = \int_{\partial \Omega} \left( \nabla_{i_1} \cdots \nabla_{i_r} u^k \right) \left( \nabla_{i_1} \cdots \nabla_{i_r} \nabla_3 u^k \right) \, d\mathcal{H}^2 + \int_{\partial \Omega} \left( \nabla_{i_1} \cdots \nabla_{i_r} u^k \right) \left( \left[ \nabla_3, \nabla_{i_1} \cdots \nabla_{i_r} \right] u^k \right) \, d\mathcal{H}^2
\]
\[
=: I^1 + I^2,
\]
where \([\cdot, \cdot]\) denotes the commutator. Since the commutator is of lower order, the second term in the integrand of \( I^2 \) is schematically represented as \( \nabla^{[r-1]} u^k \). More precisely, by the Ricci identity:
\[
\nabla_i \nabla_j V^k - \nabla_j \nabla_i V^k = \sum_{l} C^{kl}_{ij} V_l
\]
for any vector field \( V \in T \mathbb{R}^3 \) and some constants \( C^{kl}_{ij} \), each time we exchange \( \nabla_3 \) with \( \nabla_{i_j} \), a zero-th order term is obtained. Then the Leibniz rule yields
\[
[\nabla_3, \nabla_{i_1} \cdots \nabla_{i_r}] u^k \simeq \nabla^{[r-1]} u^k.
\]
(3.10)
Then the Cauchy-Schwarz inequality leads to
\[
|I^2| \lesssim \|u\|^2_{H^r(\partial \Omega)} + \|u\|^2_{H^{r-1}(\partial \Omega)} \lesssim \epsilon \|\nabla^{r+1} u\|_{L^2(\Omega)}^2 + (1 + \frac{1}{\epsilon}) \|u\|^2_{H^r(\Omega)},
\]
where the second line follows from the Sobolev trace embedding \( H^{r+1}(\Omega) \hookrightarrow H^r(\partial \Omega) \) for \( r \geq 0 \), together with the interpolation inequalities.

4. To bound \( I^1 \), we make a crucial use of the kinematic and Navier boundary conditions (3.6). First, we rewrite it in the local frame \( \{e_1, e_2, e_3\} \) as
\[
\begin{align*}
u^3 &= 0, \\
\nabla_3 u^\beta &= 2I_{a\beta} u^a - \frac{1}{4} u^\beta \quad \text{for } \beta \in \{1, 2\},
\end{align*}
\]
(3.12)
where \( \nabla_a u^3 \equiv 0 \) so that \( \omega^1 = -\nabla_3 u^2 \) and \( \omega^2 = \nabla_1 u^3 \). Moreover, from the incompressibility condition: \( \nabla \cdot u = 0 \), the following identities hold:
\[
\begin{align*}
\nabla_3 \nabla_3 u^\alpha &= -\nabla^\alpha, \\
\nabla_3 \nabla_3 u^\alpha &= -\psi^\alpha - \nabla_\beta \nabla_\beta u^\alpha.
\end{align*}
\]
(3.13)
The key to Eqs. (3.12) – (3.13) is that the normal derivatives \( \nabla_3 \) of the normal components can be replaced by the tangential derivatives, and the normal derivatives of the tangential components can be replaced by the lower-order terms.

5. We now estimate \( I^1 \). For simplicity, we introduce the short-hand notations:
\[
\nabla^{(r-3)} A \cdot \nabla^{(r-3)} B := \left( \nabla_{i_1} \cdots \nabla_{i_{r-3}} A \right) \cdot \left( \nabla_{i_1} \cdots \nabla_{i_{r-3}} B \right),
\]
(3.14)
for any sufficiently regular functions \( A \) and \( B \). Then we split \( I^1 \) into six terms:
\[
I^1 := I^{1.1} + I^{1.2} + I^{1.3} + I^{1.4} + I^{1.5} + I^{1.6},
\]
8
In the sequel, we estimate these terms one by one.

First of all, $I^{1,1} = 0$, since $\nabla_3 u_3 \equiv 0$.

To estimate $I^{2,2}$, we first notice that

$$\left| \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 \right| = \left| \nabla^{(r-3)} \nabla_\alpha \nabla_\beta (2II_{\delta\gamma} u^\delta - \frac{1}{\zeta} u^\gamma) \right| = C|\nabla^{(r-1)} u| + \text{l.o.t.} ,$$

where $C$ depends on $||\Pi||_{C^{r-1}(\partial \Omega)}$ and $\zeta^{-1}$, and l.o.t. contains the derivatives of $u$ of order less than or equal to $r - 2$. Next, considering the two cases: $\alpha = \beta$ and $\alpha \neq \beta$ separately, we deduce

$$I^{1,2} = \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 \right) \cdot \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_3 u^3 \right) d\mathcal{H}^2,$nabla_3 u_3 \right) \cdot \left( \nabla^{(r-3)} \nabla_1 \nabla_2 u_2 \right) \cdot \left( \nabla^{(r-3)} \nabla_1 \nabla_2 \nabla_3 u_3 \right) d\mathcal{H}^2.$$

For the first term, again by the Ricci identity, we write

$$\nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 = \nabla^{(r-3)} \nabla_3 \nabla_\gamma \nabla_3 u^3 + \nabla^{(r-3)} \nabla_\gamma \nabla_3 u^3 = -\nabla^{(r-3)} \nabla_3 \nabla_\gamma u^3 - \nabla^{(r-1)} \nabla_\gamma u^3,$$

and treat the second term as in Eq. (3.13) above. Then we obtain

$$|I^{1,2}| \lesssim \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 \right) \cdot \left( \nabla^{(r-1)} u \right) d\mathcal{H}^2 + \int_{\partial \Omega} |\nabla^{(r-1)} u|^2 d\mathcal{H}^2$$

by the Cauchy-Schwarz inequality. By the trace and interpolation inequalities, we have

$$\|\nabla^{(r-3)}(\nabla \times \psi)\|_{L^2(\Omega)}^2 \lesssim \epsilon \|\nabla^{(r+1)} u\|_{H^{r+1}(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^{r}(\Omega)}^2.$$

Then

$$|I^{1,2}| \lesssim \epsilon \|\nabla^{(r+1)} u\|_{H^{r+1}(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^{r}(\Omega)}^2 .$$

For $I^{1,3}$, again by Eq. (3.13), the Ricci identity, the boundary condition (3.12), and the trace and interpolation inequalities, we have

$$|I^{1,3}| = \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta u^\beta \right) \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 \right) d\mathcal{H}^2 \right|$$

$$\leq \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta u^\beta \right) \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 + \nabla^{(r-1)} u \right) d\mathcal{H}^2 \right|$$

$$\leq \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta u^\beta \right) \left( \nabla^{(r-3)} \nabla_\alpha \nabla_\beta \nabla_3 u^3 \right) \left( 2II_{\delta\gamma} u^\delta - \frac{1}{\zeta} u^\gamma \right) + \nabla^{(r-1)} u \right) d\mathcal{H}^2 \right|$$

$$\lesssim \|u\|_{H^{r-1}(\partial \Omega)}^2 \lesssim \|u\|_{H^{r}(\Omega)}^2.$$

(3.21)
The treatment for $I^{1,4}$ is similar to the above for $I^{1,3}$:

$$|I^{1,4}| = \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_{\alpha} (2 \Pi_{\beta} u^\beta - \frac{1}{\zeta} u^\gamma) \right) \left( \nabla^{(r-3)} \nabla_{\alpha} (-\psi^\alpha - \nabla_\delta \nabla_\delta u^\gamma) \right) \, d\mathcal{H}^2 \right|$$

$$\lesssim \left| \int_{\partial \Omega} \left( \nabla^{(r-2)} u \right) \left( \nabla^{(r)} u \right) \, d\mathcal{H}^2 \right|$$

$$\lesssim \|u\|_{L^2(\partial \Omega)}^2 \lesssim \epsilon \|\nabla^{r+1} u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2.$$  \hfill (3.22)

For $I^{1,5}$, we first substitute in Eq. (3.12) to derive

$$I^{1,5} = \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\beta y^\beta \right) \left( \nabla^{(r-3)} \nabla_3 \nabla_\alpha u^\alpha \right) \, d\mathcal{H}^2.$$  \hfill (3.23)

Then, applying the Ricci identity once to the first term and twice to the second term in the integrand, we have

$$|I^{1,5}| \lesssim \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\beta \nabla_3 y^\beta + \nabla^{(r-2)} u \right) \left( \nabla^{(r-3)} \nabla_\alpha \nabla_3 \nabla_\beta y^\beta + \nabla^{(r-1)} u \right) \, d\mathcal{H}^2 \right|$$

$$\lesssim \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_\beta \left( 2 \Pi_{\gamma} y^\gamma - \frac{1}{\zeta} u^\gamma \right) + \nabla^{(r-2)} u \right) \left( \nabla^{(r-3)} \nabla_\alpha \left( -\psi^\alpha - \nabla_\delta \nabla_\delta u^\alpha \right) + \nabla^{(r-1)} u \right) \, d\mathcal{H}^2 \right|$$

$$\lesssim \left| \int_{\partial \Omega} \left( \nabla^{(r-2)} u \right) \left( \nabla^{(r)} u \right) \, d\mathcal{H}^2 \right| \lesssim \epsilon \|\nabla^{r+1} u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2.$$  \hfill (3.24)

where the last line follows analogously to the final inequality in Eq. (3.22).

Finally, for $I^{1,6}$, using Eqs. (3.23)–(3.24), we have

$$|I^{1,6}| = \left| \int_{\partial \Omega} \left( \nabla^{(r-3)} \left\{ -\psi^\alpha - \nabla_\beta y^\beta \right\} \right) \left( \nabla^{(r-3)} \nabla_3 \nabla_\alpha \left( 2 \Pi_{\gamma} y^\gamma - \frac{1}{\zeta} u^\gamma \right) \right) \, d\mathcal{H}^2 \right|$$

$$\lesssim \|\nabla^{(r-1)} u\|_{L^2(\partial \Omega)}^2 \lesssim \|u\|_{H^r(\Omega)}^2.$$  \hfill (3.25)

Therefore, combining Eqs. (3.22)–(3.25) all together, $I^1$ is estimated by

$$|I^1| \lesssim \epsilon \|\nabla^{r+1} u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2.$$  \hfill (3.26)

6. Now we derive the estimates for the $J$ term.

In fact, $J$ differs from $I$ only by the lower-order terms so that the estimates follow immediately. More precisely, notice that

$$J = \int_{\partial \Omega} \left( \nabla_{i_2} \nabla^{(r-3)} \nabla_{i_3+1} u^k \right) \left( \Delta \nabla^{(r-3)} \nabla_{i_3+1} u^k \right) \left( \nabla_{i_2}, n \right) \, d\mathcal{H}^2,$$  \hfill (3.26)

where we have relabelled $\nabla^{(r-3)} = \nabla_{i_3} \cdots \nabla_{i_r+1}$ as before. Then, invoking the Ricci identity again, it follows that

$$J \lesssim \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_{i_2} \nabla_{i_3+1} u^k + \nabla^{(r-2)} u \right) \left( \nabla^{(r-3)} \nabla_{i_3+1} \Delta u^k + \nabla^{(r-1)} u \right) \, d\mathcal{H}^2$$

$$= \int_{\partial \Omega} \left( \nabla^{(r-3)} \nabla_{i_2} \nabla_{i_3+1} u^k \right) \left( \nabla^{(r-3)} \nabla_{i_3+1} \Delta u^k \right) \, d\mathcal{H}^2 + \int_{\partial \Omega} \left( \nabla^{(r-2)} u \right) \left( \nabla^{(r-1)} u \right) \, d\mathcal{H}^2$$

$$+ \int_{\partial \Omega} \left( \nabla_{i_3+1} \nabla^{(r-3)} \Delta u^k \right) \left( \nabla^{(r-2)} u \right) \, d\mathcal{H}^2 + \int_{\partial \Omega} \left( \nabla^{(r-1)} u \right) \left( \nabla^{(r-1)} u \right) \, d\mathcal{H}^2$$

$$=: J^1 + J^2 + J^3 + J^4.$$  \hfill (3.27)
By the trace, interpolation, and Young’s inequalities, again we have

$$|J^2| + |J^4| \lesssim \|\nabla^{r-1}u\|_{L^2(\partial \Omega)}^2 \lesssim \epsilon \|\nabla^{r+1}u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2.$$  \hspace{1cm} (3.28)

Also, $J^1$ has the same decomposition as $I^1$ into $I^{1,1}, \ldots, I^{1,6}$ so that, by Step 4, we conclude

$$|J^1| \lesssim \epsilon \|\nabla^{r+1}u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2.$$  \hspace{1cm} (3.29)

In the end, $J^3$ is estimated via integration by parts again: Since $\partial \Omega$ is a 2-D surface without boundary, the divergence theorem yields

$$J^3 = -\int_{\partial \Omega} \left(\nabla^{(r-3)}u^k\right) \left(\nabla_{v+1} \nabla^{(r-2)}u\right) d\mathcal{H}^2 \approx \int_{\partial \Omega} \left(\nabla^{(r-1)}u\right) \left(\nabla^{(r-1)}u\right) d\mathcal{H}^2.$$  \hspace{1cm} (3.30)

Thus, this verifies the same estimate for $J^4$.

7. Finally, putting together all the estimates for $I$, $J$, and $K$ in Steps 1–6, we conclude

$$\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 \lesssim \epsilon \|\nabla^{r+1}u\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2 + \int_{\Omega} |\nabla^{r-1}\psi|^2 \, dx.$$  \hspace{1cm} (3.31)

Choose $\epsilon$ sufficiently small so that

$$\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 \lesssim \|u\|_{H^r(\Omega)}^2 + K,$$  \hspace{1cm} (3.32)

where $K := \int_{\Omega} |\nabla^{r-1}\psi|^2 \, dx$ as before. The first term on the right-hand side, $\|u\|_{H^r(\Omega)}^2$, is bounded by $\sum_{l=0}^{r} \|\text{curl}^l u\|_{L^2(\Omega)}^2$ up to a multiplicative constant, thanks to the induction hypothesis.

Now, it remains to show that $K$ is bounded by the $L^2$–norm of the iterated curls: This is achieved by iterating the constructions in Step 1. Indeed, relabelling the indices yields

$$K = \int_{\Omega} \left(\Delta \partial_{l_1} \cdots \partial_{l_{r-1}} u^k\right) \left(\Delta \partial_{l_1} \cdots \partial_{l_{r-1}} u^k\right) \, dx.$$  \hspace{1cm} 

Then, as in Step 1, we integrate by parts twice to compute as

$$K = \int_{\partial \Omega} \left(\Delta \partial_{l_2} \cdots \partial_{l_{r-1}} u^k\right) \left(\Delta \partial_{l_1} \cdots \partial_{l_{r-1}} u^k\right) \left(\partial_{l_1}, n\right) \, d\mathcal{H}^2$$

$$- \int_{\partial \Omega} \left(\Delta \partial_{l_2} \cdots \partial_{l_{r-1}} u^k\right) \left(\Delta \Delta \partial_{l_3} \cdots \partial_{l_{r-1}} u^k\right) \left(\partial_{l_2}, n\right) \, d\mathcal{H}^2$$

$$+ \int_{\Omega} \left(\Delta \Delta \partial_{l_1} \cdots \partial_{l_{r-1}} u^k\right) \left(\Delta \Delta \partial_{l_1} \cdots \partial_{l_{r-1}} u^k\right) \, dx$$

$$= \frac{1}{2} \int_{\partial \Omega} |\nabla^{r-2}u|^2 \, d\mathcal{H}^2 - \int_{\partial \Omega} \left(\Delta \partial_{l_2} \cdots \partial_{l_{r-1}} u^k\right) \left(\Delta \Delta \partial_{l_3} \cdots \partial_{l_{r-1}} u^k\right) \left(\partial_{l_2}, n\right) \, d\mathcal{H}^2$$

$$+ \int_{\Omega} |\Delta \Delta \nabla^{r-3}u|^2 \, dx =: \tilde{I} + \tilde{J} + \tilde{K}.$$  \hspace{1cm} (3.33)

It is crucial here that the flat gradient $\partial_{l_1}$ and flat Laplacian $\Delta$ on $\mathbb{R}^3$ commute. We notice that $\tilde{I}$ and $\tilde{J}$ are obtained from $I$ and $J$, respectively, by taking the trace over a pair of indices, so that they satisfy the same estimates, which are given in Step 5 above. Repeating this process for finitely many times, we refine estimate (3.32) as

$$\|\nabla^{r+1}u\|_{L^2(\Omega)}^2 \lesssim \sum_{l=0}^{r} \|\text{curl}^l u\|_{L^2(\Omega)}^2 + \begin{cases} \int_{\Omega} |\Delta \frac{\partial \nabla u}{\partial x_1}|^2 \, dx & \text{if } r \text{ is odd}, \\
\int_{\Omega} |\Delta \nabla u|^2 \, dx & \text{if } r \text{ is even}. \end{cases}$$  \hspace{1cm} (3.34)

To conclude the proof, we notice that, for the divergence-free vector field $u$, $\Delta u = -\text{curl}^2 u$. Thus, Eq. (3.34) gives the desired estimate for odd $r$. On the other hand, for even $r$, we apply
Eq. \(3.4\) in Step 1 of the same proof to the divergence-free vector field \(\Delta \vec{r} \cdot u\) to deduce
\[
\int_\Omega |\Delta \vec{r} \nabla u|^2 \, dx = \int_\Omega |\nabla \Delta \vec{r} u|^2 \, dx = \int_\Omega |\text{curl}(\Delta \vec{r} u)|^2 \, dx + \int_{\partial \Omega} I(\Delta \vec{r} u, \Delta \vec{r} u) \, d\mathcal{H}^2,
\] (3.35)
where we need the commutativity of divergence, gradient, and curl. For the first term on the right-hand side, \(\text{curl}(\Delta \vec{r} u) = (-1)^2 \text{curl}^{r+1} u\), while, for the second term,
\[
\int_{\partial \Omega} I(\Delta \vec{r} u, \Delta \vec{r} u) \, d\mathcal{H}^2 \lesssim \|\Delta \vec{r} u\|^2_{L^2(\partial \Omega)} \lesssim \|\nabla^{r+1} u\|^2_{L^2(\Omega)} + \frac{1}{c'} \|u\|^2_{H^{r}(\Omega)},
\] (3.36)
again by the boundedness of the second fundamental form, as well as the trace and interpolation inequalities. The proof is then completed by choosing \(c'\) sufficiently small.
\[\square\]

To conclude the section, we emphasize that Theorem 3.1 is independent of the Navier-Stokes equations \((1.1)\). It is a general property of divergence-free vector fields satisfying the kinematic and Navier boundary conditions \((1.6)\). For the Dirichlet boundary condition, Theorem 3.1 also holds, which follows from the divergence-free condition.

4. Energy Estimates in \(H^r\) for Strong Solutions

In this section, we derive the higher-order energy estimates. We show that the solution is in the spatial Sobolev space \(H^r\) for \(r \geq 2\), provided that the initial data lies in the same space. This allows us to prove the existence of strong solutions with spatial regularity \(H^r\).

For this purpose, our starting point is the existence of \textit{weak solutions} to the Navier-Stokes equations \((1.1)\) under the kinematic and Navier boundary conditions. This can be established, \textit{e.g.}, via the Galerkin approximation scheme in \(15\). We summarize it here for the subsequent developments. In this section, we drop superscript \(\nu\) in solution \(u^\nu\) of the Navier-Stokes equations \((1.1)\), since we do not deal with the inviscid limits here.

To begin with, consider the following vector space direct sum
\[
L^2(\Omega; \mathbb{R}^3) = K_2(\Omega) \bigoplus G_2(\Omega)
\] (4.1)
for the Hodge (or Helmholtz) decomposition, where \(K_2(\Omega)\) is defined in \((3.1)\). Next, for projection \(P_\infty\) onto the first factor, we introduce the \textit{Stokes operator}:
\[
S := P_\infty \circ \Delta,
\] (4.2)
where \(\Delta\) is the flat Laplacian on \(\mathbb{R}^3\). It is shown in \$4\ of \(15\) that \(S\) is densely defined on \(K_2(\Omega)\) with a compact resolvent. Thus, it has a discrete spectrum \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \downarrow -\infty\), and the corresponding eigenfunctions \(\{a_n\}\) form a complete orthonormal basis of \(K_2(\Omega)\). Now we look at the graded chain of finite-D Hilbert spaces:
\[
K_2(\Omega) \supset \ldots \supset V_N := \bigoplus_{j=1}^N \mathbb{R}a_j \supset V_{N-1} \supset \ldots \supset V_2 \supset V_1,
\] (4.3)
and denote by \(P_N : K_2(\Omega) \to V_N\) the canonical projection:
\[
(P_N u)(t, x) := \sum_{j=1}^N a_j(x) \int_\Omega \langle a_j(y), u(t, y) \rangle \, dy.
\] (4.4)
Thus, \(P_\infty\) is indeed the \(L^2\)-limit of \(P_N\), as \(N\) tends to \(\infty\).

In \$5\ of \(15\), the \textit{weak formulation} of Eq. \((1.1)\) has been introduced.
Definition 4.1. For $T > 0$, we say that $u \in L^2([0, T]; H^1(\Omega; \mathbb{R}^3))$ is a weak solution of the initial boundary problem (1.1)–(1.4), provided that

(i) $u(t, \cdot) \in K_2(\Omega)$ for each $t \in (0, T)$;
(ii) For each $\phi \in C^\infty((0, T] \times \Omega)$ with $\phi(t, \cdot) \in K_2(\Omega)$,

$$\int_\Omega \langle u(T, \cdot), \phi(T, \cdot) \rangle \, dx = \int_\Omega \langle u_0(x), \phi(0, x) \rangle \, dx + \int_0^T \int_\Omega (u, \partial_t \phi) \, dx \, dt - \int_0^T \int_\Omega \langle \text{curl} \, u, (u \times \phi + \nu \text{curl} \, \phi) \rangle \, dx \, dt \tag{4.5}$$

(iii) The energy inequality holds:

$$\|u(T, \cdot)\|_{L^2(\Omega)}^2 + 2\nu \int_0^T \|\nabla u(t, \cdot)\|_{L^2(\Omega)}^2 \, dt + 2\nu \int_\partial \Omega \Pi(u, \phi) \, d\mathcal{H}^2 \, dt \leq \|u_0\|_{L^2(\Omega)}^2 \tag{4.6}$$

Therefore, by solving the projected equations obtained via taking $P_N$ to Eq. (1.1) and deriving the a priori estimates for the finite-D approximate solutions $\{u_N\} \subset L^2([0, T]; H^1(\Omega; \mathbb{R}^3))$ uniformly in $N$, we are able to deduce the existence of weak solutions via a compactness argument. This method is known as the Galerkin approximation scheme, which relies crucially on the spectral analysis of the Stokes operator $S = P_\infty \circ \Delta$.

More precisely, the following result is obtained:

Lemma 4.2 (Theorem 5.1 in [15]). For any $u_0 \in K_2(\Omega)$ and $T > 0$, there exists a weak solution $u \in L^2([0, T]; K_2(\Omega))$ to the initial-boundary problem (1.1)–(1.4). Such a solution $u$ can be obtained as a weak subsequential limit of the family of finite-D approximate solutions $\{u_N\}$.

Now, taking the weak solution $u$ of the Navier-Stokes equations (1.1) constructed by the Galerkin approximation scheme in Lemma 4.2 above, we derive the a priori estimate for the higher-order energy of $u$ in the Sobolev spaces $H^r$ with $r \geq 2$. Indeed, the case, $r = 2$, has been proved in Theorem 5.3 of [15]. The higher-order energy estimate is proved by induction on $r$, for which purpose the reduction of order of differentiations in the boundary terms is essential. This is achieved by exploiting by the kinematic and Navier boundary conditions (1.6). In particular, we need to explore the role of the curl operator, the rotation matrix $\mathcal{R}$, and the shape operator $S$ (see §1).

Our main theorem of this section is the following:

Theorem 4.3. Let $u_0 \in H^r(\Omega; \mathbb{R}^3) \cap K_2(\Omega)$ for some $r \geq 2$. Then there exists some $T_*>0$ such that the weak solution $u \in L^2([0, T_*]; K_2(\Omega))$ of the initial-boundary problem (1.1)–(1.4) satisfies

$$\sup_{0 \leq t \leq T_*} \left( \|u(t, \cdot)\|_{H^r(\Omega)} + \|\partial_t u(t, \cdot)\|_{H^{r-2}(\Omega)} \right) \leq C, \tag{4.7}$$

where constant $C > 0$ depends only on $\zeta$, $\nu$, $\|\Pi\|_{C^{r-1}(\partial \Omega)}$, and $\|u_0\|_{H^r(\Omega)}$. As a consequence, there exists a unique strong solution $u \in C([0, T_*]; H^r(\Omega; \mathbb{R}^3)) \cap C^1((0, T_*); H^{r-2}(\Omega; \mathbb{R}^3))$.

Proof. We divide the arguments in six steps. In Step 1, we set up the equations for the energy estimate. Then, in Steps 2–5, we control $\|u(t, \cdot)\|_{H^r(\Omega)}$ and specify the lifespan, $T_*$. Finally, in Step 6, we derive the energy estimate for $\partial_t u$. 

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1. We first deduce the evolution equation for the iterated curls of the velocity field $u$. For this purpose, we apply the divergence-free projection $P_\infty : L^2(\Omega; \mathbb{R}^3) \to K_2(\Omega)$ to the Navier-Stokes equations (1.1) to obtain

$$
\partial_t u - \nu \Delta u + \nabla (u \cdot \nabla u) = 0.
$$

(4.8)

On the other hand, we have the following vectorial identity in 3-D:

$$
u \cdot \nabla u = \frac{1}{2} \nabla (|u|^2) - u \times \omega,
$$

so that the projected Navier-Stokes equations (4.8) are equivalent to

$$
\partial_t u - \nu \Delta u + P_\infty (u \times \omega) = 0.
$$

(4.9)

Here and in the sequel, we view $P_\infty$ as extended to the bounded projection operator from $H^r(\Omega; \mathbb{R}^3)$ to $H^r(\Omega; \mathbb{R}^3) \cap K_2(\Omega)$. This follows from the generalized Hodge decomposition theory on the manifolds with boundaries subject to the kinematic boundary condition; see Theorem 2.4.2 in Schwarz [43]. Then the Stokes’ operator:

$$
S := P_\infty \circ \Delta
$$

gives rise to a densely defined, closable, self-adjoint bilinear form on $H^r(\Omega; \mathbb{R}^3) \cap K_2(\Omega)$:

$$
E_r(u, w) := - \int_\Omega \sum_{0 \leq |\alpha| \leq r} \langle \nabla^\alpha Su, \nabla^\alpha w \rangle \, dx.
$$

(4.10)

In particular, the spectral analysis in Sections 4.1–4.2 in [15] also carries through in our setting to $H^r(\Omega; \mathbb{R}^3)$.

For simplicity of presentation, we use the following abbreviation:

$$
\Psi := P_\infty (u \times \omega).
$$

(4.11)

Then, taking the iterated curls to Eq. (4.9), we obtain the evolution equation:

$$
\partial_t q_r - \nu \Delta q_r + \text{curl}^r \Psi = 0,
$$

(4.12)

where and in the sequel, we denote

$$
q_r := \text{curl}^r u.
$$

(4.13)

To derive the energy estimate, we multiply $q_r$ to Eq. (4.12) and integrate over $\Omega$ to obtain

$$
0 = \frac{1}{2} \frac{d}{dt} \int_\Omega |q_r|^2 \, dx - \nu \int_\Omega \langle q_r, \Delta q_r \rangle \, dx + \int_\Omega \langle q_r, \text{curl}^r \Psi \rangle \, dx.
$$

(4.14)

We integrate the last two terms by parts. For the second term, we have

$$
\int_\Omega \langle q_r, \Delta q_r \rangle \, dx = \int_\partial \Omega \langle (q_r \cdot \nabla)q_r, n \rangle \, d\mathcal{H}^2 - \int_\Omega |\nabla q_r|^2 \, dx.
$$

For the final term, notice that, for any 3-D vector fields $V$ and $W$,

$$
\int_\Omega \langle V, \text{curl} W \rangle \, dx = \int_\Omega V^k \epsilon^{ijk} \partial_i W_j \, dx
$$

$$
= \int_\partial \Omega \epsilon^{ijk} V^k W^j \langle \partial_i, n \rangle \, d\mathcal{H}^2 - \int_\Omega \epsilon^{ijk} W^j \langle \partial_i V^k \rangle \, dx
$$

$$
= \int_\partial \Omega \langle W \times V, n \rangle \, d\mathcal{H}^2 + \int_\Omega \langle \text{curl} V, W \rangle \, dx.
$$

(4.15)
As a result,
\[
\int_{\Omega} \langle q_r, \mathbf{curl} \circ \mathbf{curl}^{-1} \Psi \rangle \, dx = \int_{\partial \Omega} \langle \mathbf{curl}^{-1} \Psi \times q_r, \mathbf{n} \rangle \, d\mathcal{H}^2 + \int_{\Omega} \langle \mathbf{curl} q_r, \mathbf{curl}^{-1} \Psi \rangle \, dx,
\]
so that Eq. (4.14) can be written as
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |q_r|^2 \, dx + \nu \int_{\Omega} |\nabla q_r|^2 \, dx = \nu \int_{\partial \Omega} \langle (q_r \cdot \nabla) q_r, \mathbf{n} \rangle \, d\mathcal{H}^2 - \int_{\partial \Omega} \langle \mathbf{curl}^{-1} \Psi \times q_r, \mathbf{n} \rangle \, d\mathcal{H}^2 - \int_{\Omega} \langle \mathbf{curl} q_r, \mathbf{curl}^{-1} \Psi \rangle \, dx,
\]
(4.16)

Our task is to estimate each of terms $I, J,$ and $K$. Since the case, $r = 2,$ has been established in Theorem 5.3 of [15], in the sequel, we assume the result for $r - 1$ and prove it for $r$ by induction, with $r \geq 3$.

2. For $I$ in Eq. (4.16), observe that
\[
I = \nu \int_{\partial \Omega} \partial_n |q_r|^2 \, d\mathcal{H}^2,
\]
(4.17)
which has been treated in the proof of Theorem 3.1. Indeed, it coincides with $I$ in Eq. (3.7) up to a constant $\nu$. Utilizing the estimates in Steps 2–4 of the proof therein, we have
\[
|I| \lesssim \epsilon \|\nabla q_r\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|u\|_{H^r(\Omega)}^2.
\]
(4.18)

3. To prove for term $K$ in Eq. (4.16), we first notice that, by the Cauchy-Schwarz inequality and Young’s inequality,
\[
|K| \leq \|\mathbf{curl} q_r\|_{L^2(\Omega)} \|\mathbf{curl}^{-1} \Psi\|_{L^2(\Omega)} \lesssim \epsilon \|\nabla q_r\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|\mathbf{curl}^{-1} (\mathbb{P}_\infty(u \times \omega))\|_{L^2(\Omega)}^2.
\]
(4.19)
Since $H^s(\Omega)$ for $s > \frac{3}{2}$ is a Banach algebra on $\mathbb{R}^3$ and $\mathbb{P}_\infty$ is a bounded linear operator, we have
\[
\|\mathbf{curl}^{-1} (\mathbb{P}_\infty(u \times \omega))\|_{L^2(\Omega)} \lesssim \|u\|_{H^r(\Omega)}^2.
\]
(4.20)
This gives us the estimate for $K$.

4. Now it remains to control the boundary term $J$ in Eq. (4.16):
\[
J := \int_{\partial \Omega} \langle \mathbf{curl}^{-1} (\mathbb{P}_\infty(u \times \omega)) \times q_r, \mathbf{n} \rangle \, d\mathcal{H}^2.
\]
(4.21)

It is crucial to reduce the order of differentiation by using the boundary conditions (1.10). For this purpose, we establish the following identity on $\partial \Omega$:
\[
\pi \{ \mathbf{curl}^k (\mathbb{P}_\infty(u \times \omega)) \} = -\frac{1}{\zeta} \mathcal{R} \circ \pi \{ \mathbf{curl}^{k-1} (\mathbb{P}_\infty(u \times \omega)) \} + 2 \mathcal{R} \circ \pi \{ \mathbf{curl}^{k-1} \mathcal{S} (\pi \circ \mathbb{P}_\infty(u \times \omega)) \}.
\]
(4.22)

We recall that $\mathcal{R}$ is the orthogonal matrix rotating in the $(x,y)$–plane anti-clockwise by 90 degrees, $\mathcal{S}$ is the shape operator corresponding to the second fundamental form $\Pi$, and operator $\pi$ denotes the projection onto the tangential components of a vector field,
\[
\pi(V) := V - \langle V, \mathbf{n} \rangle
\]
(4.23)
viewed either as a 2-D vector or a 3-D vector with zero \( x_3 \)-component. Here it suffices to consider the tangential components, since \( \langle n \times q_r, n \rangle \equiv 0 \).

The above identity is proved by induction. The base step \( k = 1 \) is shown in the computations preceding Eq. (5.21) in [15]. Now we assume the result for \( k \). Then, by the induction hypothesis,

\[
\pi \{ \text{curl}^{k+1}(P_\infty(u \times \omega)) \} = \pi \{ \text{curl} \circ \text{curl}^k(P_\infty(u \times \omega)) \}
\]

\[
= \pi \circ \text{curl} \left\{ -\frac{1}{\zeta} \mathcal{R} \circ \pi \circ \text{curl}^{k-1}(P_\infty(u \times \omega)) + 2\mathcal{R} \circ \pi \circ \text{curl}^{k-1}(S \circ P_\infty(u \times \omega)) \right\}
\]

\[
= \pi \circ \text{curl} \left\{ -\frac{1}{\zeta} \mathcal{R} \circ \pi \circ \text{curl}^{k-1}\Psi + 2\mathcal{R} \circ \pi \circ \text{curl}^{k-1}(S \circ \pi(\Psi)) \right\},
\]

(4.24)

where we recall the short-hand notation \( \Psi \) in Eq. (4.11). Here and throughout, for a 2-D vector field \( W = (W^1, W^2)^\top \) (e.g. \( W = S \circ \pi(\Psi) \)), we define its curl as \( \text{curl} W := \text{curl}(W^1, W^2, 0)^\top \). It suffices to check that

\[
\pi \circ \text{curl} \circ \mathcal{R} \circ \pi(V) = \mathcal{R} \circ \pi \circ \text{curl} V \quad \text{for any } V \in T(\partial \Omega).
\]

(4.25)

From here, Eq. (4.24) implies

\[
\pi \{ \text{curl}^{k+1}(P_\infty(u \times \omega)) \} = -\frac{1}{\zeta} \mathcal{R} \circ \pi \circ \text{curl}^k \Psi + 2\mathcal{R} \circ \pi \circ \text{curl}^k(S \circ \pi(\Psi)).
\]

(4.26)

Indeed, we observe

\[
\pi \circ \text{curl} \circ \mathcal{R} \circ \pi \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} = \pi \circ \text{curl} \begin{bmatrix} -V^2 \\ V^1 \\ 0 \end{bmatrix} = \pi \begin{bmatrix} -\nabla_3 V^1 \\ -\nabla_3 V^2 \\ -\nabla_3 V^3 \end{bmatrix},
\]

and

\[
\mathcal{R} \circ \pi \circ \text{curl} \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} = \mathcal{R} \circ \pi \begin{bmatrix} \nabla_2 V^3 - \nabla_3 V^2 \\ \nabla_3 V^1 - \nabla_1 V^3 \\ \nabla_1 V^2 - \nabla_2 V^1 \end{bmatrix} = \mathcal{R} \begin{bmatrix} -\nabla_3 V^2 \\ -\nabla_3 V^1 \\ -\nabla_3 V^3 \end{bmatrix},
\]

since \( V^3 = 0 \). Therefore, Eq. (4.24) is proved, and the identity in Eq. (4.22) follows by induction.

Now, in view of the above identity, \( J \) can be expressed as

\[
J = \int_{\partial \Omega} \left\langle \left\{ -\frac{1}{\zeta} \mathcal{R} \circ \pi(\text{curl}^{r-2}(P_\infty(u \times \omega)))
+ 2\mathcal{R} \circ \pi \circ \text{curl}^{r-2}(S \circ \pi \circ P_\infty(u \times \omega)) \right\} \times q_r, n \right\rangle dH^2.
\]

(4.27)

The crucial observation is that only the derivatives up to the \( (r-1) \)-th order of \( u \) are involved. This is because \( S \) has a bounded norm in \( C^{r-1} \) owing to the assumption of bounded extrinsic geometry, and \( P_\infty, \pi, \) and \( \mathcal{R} \) are all smooth operators with the operator norm bounded by a
universal constant. Then we arrive at the following estimates:

\[ |J| \lesssim \| \text{curl} r^2 (u \times \omega) \|_{L^2(\partial \Omega)} \| q_r \|_{L^2(\partial \Omega)} \]

\[ \lesssim \| u \|_{H^{r-1}(\partial \Omega)}^2 \| u \|_{H^r(\partial \Omega)} \]

\[ \lesssim \left( \| u \|_{H^r(\Omega)}^2 + \| u \|_{H^{r-1}(\Omega)}^2 \right) \left( \| u \|_{H^{r+1}(\Omega)} + \| u \|_{H^r(\Omega)} \right) \]

\[ \simeq \epsilon^2 \| u \|_{H^r(\Omega)}^2 \| u \|_{H^{r+1}(\Omega)} + \epsilon \| u \|_{H^{r+1}(\Omega)} + \epsilon^2 \| u \|_{H^{r-1}(\Omega)}^2 \| u \|_{H^{r+1}(\Omega)} + \| u \|_{H^r(\Omega)} \]

\[ \lesssim (\epsilon^2 + \epsilon) \| u \|_{H^{r+1}(\Omega)} + \| u \|_{H^r(\Omega)}^4. \]  

(4.28)

In the above, the first line follows from the Cauchy-Schwarz inequality, the second line follows from the argument as for Eq. (4.20), the third line holds by the Sobolev trace inequality, and the final line follows by the interpolation and Young’s inequalities.

5. Now, combining the estimates in Steps 2–4 for \( I, J, \) and \( K \) (especially Eqs. (4.18)–(4.20) and (4.28)), Eq. (4.16) becomes

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |q_r|^2 \, dx + \nu \int_{\Omega} |
abla q_r|^2 \, dx \]

\[ \lesssim \| \nabla r^{-1} u \|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \| u \|_{H^r(\Omega)}^2 + \epsilon \| q_r \|_{L^2(\Omega)}^2 + (\epsilon + \epsilon^2) \| u \|_{H^{r+1}(\Omega)}^2 + (1 + \frac{1}{\epsilon}) \| u \|_{H^r(\Omega)}^4, \]  

(4.29)

where, in light of Theorem 3.1, \( \| \nabla r^{-1} u \|_{L^2(\Omega)} \simeq \| q_r \|_{L^2(\Omega)} \simeq \| \text{curl} r^2 u \|_{L^2(\Omega)} \), and similarly \( \| q_r \|_{L^2(\Omega)} \lesssim \| u \|_{H^r(\Omega)} \). Then, choosing \( \epsilon \) suitably small in comparison with \( \nu \) and considering the energy at the \( r \)-th order:

\[ E_r := \| q_r \|_{L^2(\Omega)}^2, \]  

(4.30)

we obtain the following differential inequality:

\[ E_r'(t) \leq E_r'(t) + \nu E_{r+1} \leq M E_r(t) + ME_r(t)^2, \]  

(4.31)

where \( M \) depends on \( \nu, \zeta, \) and \( \| \Pi \|_{C^{r-1}(\Omega)} \).

To proceed, consider the auxiliary Cauchy problem for ODE:

\[ \begin{cases}
A'(t) = M \left( A(t) + A(t)^2 \right), \\
A(0) = E_r(0) + \eta
\end{cases} \]  

(4.32)

for arbitrary \( \eta > 0 \). It is solved explicitly by

\[ A(t) = \frac{(\eta + E_r(0)) e^{Mt}}{1 - (\eta + E_r(0)) (e^{Mt} - 1)}. \]

so that, for any \( t > 0 \) before the blowup time:

\[ T_* = \frac{1}{M} \log \left( 1 + \frac{1}{\eta + E_r(0)} \right) > 0, \]  

(4.33)

we see that \( A(t) < \infty \). Comparing the differential inequality (4.31) with the ODE in (4.32), we find that \( E_r(t) \leq A(t) \) for all \( 0 \leq t < T_* \). In particular, since \( \eta > 0 \) is arbitrary, the upper bound for \( A \) (hence for \( E_r \)) is controlled by \( E_r(0) := \| u_0 \|^2_{H^r(\Omega)} \) and \( M \). This implies

\[ \sup_{0 \leq t < T_*} E_r(t) \leq C. \]  

(4.34)

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6. It remains to derive the \(L^2\)-estimate for \(\partial_t q_r\). To this end, we take \(\partial_t\) to Eq. (4.12), multiply by \(\partial_t q_r\), and then integrate over \(\Omega\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\partial_t q_r|^2 \, dx - \nu \int_\Omega \langle \Delta \partial_t q_r, \partial_t q_r \rangle \, dx + \int_\Omega \langle \text{curl} \partial_t q_r, \partial_t q_r \rangle \, dx = 0. \tag{4.35}
\]
Applying integration by parts and the divergence theorem to the last two terms (cf. Eq. (4.15) for curl), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\partial_t q_r|^2 \, dx + \nu \int_\partial \Omega |\nabla \partial_t q_r|^2 \, ds
= \nu \int_{\partial \Omega} \langle (\nabla \partial_t q_r \cdot \partial_t q_r), n \rangle \, d\mathcal{H}^2 + \int_\Omega \langle \text{curl}^{-1} \partial_t q_r \times \partial_t q_r, n \rangle \, d\mathcal{H}^2
+ \int_\Omega \langle \text{curl}^{-1} \partial_t q_r, \text{curl} \partial_t q_r \rangle \, dx =: \tilde{I} + \tilde{J} + \tilde{K}. \tag{4.36}
\]
Now, it is crucial to observe the following: Eq. (4.16) differs from Eq. (4.36) only by the time derivatives. More precisely, if we change variables \((\partial_t q_r, \partial_t \Psi)\) in the terms, \(\tilde{I}, \tilde{J}, \text{and} \tilde{K}\), in Eq. (4.36) to \((q_r, \Psi)\), then \(I, J, \text{and} K\) in Eq. (4.16) are immediately recovered.

Furthermore, since the spatial derivatives commute with \(\partial_t\), the integration by parts arguments in Steps 2–5 above all carry through. Therefore, given that \(\|\partial_t u(t, \cdot)\|^2_{L^2(\Omega)} \leq C\) for \(u_0 \in H^2(\Omega; \mathbb{R}^3)\) which has been established in Theorem 5.3 of [15], we repeat the arguments above for \((\partial_t q_r, \partial_t \Psi)\) to deduce
\[
\sup_{0 \leq t < T_*} \|\partial_t u(t, \cdot)\|^2_{H^{-2}(\Omega)} \leq C, \tag{4.37}
\]
where \(C\) depends on \(\|u_0\|^2_{H^2(\Omega)}, \zeta, \nu, \|\Pi\|_{C^{-1}(\Omega)}, \text{and} T_*\) that is the same blowup time as in Step 5.

Therefore, combining the estimates in Eqs. (4.34) and (4.37), we conclude the proof. \(\square\)

As a corollary, if the initial energy \(E_r(0)\) of the fluid is uniformly bounded for all \(r \in \mathbb{N}\), Eq. (4.33) implies that there exists a uniform life span \(T_* > 0\) for all levels of the kinetic energy. Therefore, in view of \(C^\infty(\Omega; \mathbb{R}^3) = \bigcap_{r \in \mathbb{N}} H^r(\Omega; \mathbb{R}^3)\), we have

**Corollary 4.4.** Let \(u_0 \in C^\infty(\Omega; \mathbb{R}^3)\) be a divergence-free vector field, and let domain \(\Omega \subset \mathbb{R}^3\) be of smooth second fundamental form. Then there exists \(T_* > 0\) such that the initial-boundary value problem (1.2)–(1.4) for the Navier-Stokes equations (1.1) has a smooth solution \(u \in C^1([0, T_*); C^\infty(\Omega; \mathbb{R}^3))\) satisfying the kinematic and Navier boundary conditions (1.6).

5. INVISCID LIMIT

In this section, we establish the inviscid limit from the Navier-Stokes equations under the kinematic and Navier boundary conditions (1.10) to the Euler equations under the no-penetration condition (1.11). The existence and uniqueness of \(u\), the strong solution of the Euler equations (1.10) satisfying the no-penetration boundary condition (1.11), have been known (cf. Ebin-Marsden [23]). We obtain the convergence in the Sobolev spaces \(H^r, r > \frac{3}{2}\), via strong compactness arguments.

To this end, a priori estimates of the evolution equations for \(u^\nu - u\), i.e. the difference between the Navier-Stokes solution and the Euler solution, are required. In particular, technicalities are involved in the estimates for the higher-order Sobolev norms of the nonlinear terms,
for instance, the iterated curls of \((u^\nu - u) \cdot \nabla (u^\nu - u)\). To deal with the nonlinearities, we need to make full use of the incompressibility condition of \(u^\nu\) and \(u\), as well as the kinematic and Navier boundary conditions \((1.9)\).

The main theorem of this section is stated as follows:

**Theorem 5.1 (Inviscid Limit).** Let \(u \in C([0,T_*]; H^{r+1}(\Omega; \mathbb{R}^3) \cap K_2(\Omega))\) be the unique strong solution of the incompressible Euler equations \((1.10)\) subject to the no-penetration boundary condition \((1.11)\). Then there exists some \(\nu_* \in (0, \nu)\), such that, whenever \(\nu < \nu_*\), the strong solution of the Navier-Stokes equations \((1.1)\) with the kinematic and Navier boundary conditions \((1.6)\) exists in \(L^\infty((0,T_*); H^r(\Omega; \mathbb{R}^3))\) for \(r > 2\). Moreover, if \(r > \frac{5}{2}\), then there exists a constant \(C\) depending only on \(T_*, \|u\|_{L^2([0,T_*]; H^{r+1}(\Omega))}\), and \(\|u\|_{C^\nu(\partial \Omega)}\) such that

\[
\sup_{0 \leq t < T_*} \|u^\nu(t, \cdot) - u(t, \cdot)\|_{H^r(\Omega)} \leq C\nu^\frac{1}{r}.
\]

In particular, as \(\nu \to 0^+\), \(u^\nu\) converges to \(u\) in \(H^r(\Omega)\) uniformly in \(t \in (0,T_*)\).

**Proof.** We divide the arguments into seven steps.

1. First of all, define

\[
v^\nu := u^\nu - u, \quad P^\nu = p^\nu - p.
\]

Our goal is to show that \(v^\nu\) converges to zero in the \(H^r\)-norm, uniformly in \(\nu\). First, subtracting the Euler equations \((1.10)\) from the Navier-Stokes equations \((1.1)\) yields

\[
\begin{align*}
\partial_t v^\nu + \left((u + v^\nu) \cdot \nabla\right)v^\nu - \nu \Delta v^\nu + (v^\nu \cdot \nabla)u - \nu \Delta u + \nabla P^\nu &= 0, \\
\nabla \cdot v^\nu &= 0.
\end{align*}
\]

Next, for

\[
V^\nu_r := \text{curl}^r v^\nu \in H^1(\Omega; \mathbb{R}^3),
\]

noticing that \(\text{curl}(\nabla v^\nu) = 0\), we have

\[
\partial_t V^\nu_r + \text{curl}^r \left\{((u + v^\nu) \cdot \nabla)v^\nu\right\} + \text{curl}^r \left\{(v^\nu \cdot \nabla)u\right\} - \nu \Delta V^\nu_r - \nu \Delta \text{curl}^r u = 0.
\]

In view of Theorem 5.1 in order to bound \(v^\nu\) in \(H^r\), it suffices to bound the \(L^2\)-norm of \(V^\nu_r\). To do so, we multiply \(V^\nu_r\) to Eq. \((5.5)\) and integrate over \(\Omega\) to obtain

\[
0 = \frac{1}{2} \frac{d}{dt} \int_\Omega |V^\nu_r|^2 \, dx + \int_\Omega V^\nu_r \cdot \text{curl}^r \left\{(u + v^\nu) \cdot \nabla\right\} v^\nu \, dx + \int_\Omega V^\nu_r \cdot \text{curl}^r \left\{(v^\nu \cdot \nabla)u\right\} \, dx \\
- \nu \int_\Omega V^\nu_r \cdot \Delta V^\nu_r \, dx - \nu \int_\Omega V^\nu_r \cdot \Delta \text{curl}^r u \, dx.
\]

Using integration by parts and the divergence theorem, the fourth term on the right-hand side of Eq. \((5.6)\) becomes

\[
- \nu \int_\Omega V^\nu_r \cdot \Delta V^\nu_r \, dx = - \nu \int_{\partial \Omega} (V^\nu_r \cdot \nabla V^\nu_r) \cdot \mathbf{n} \, d\mathcal{H}^2 + \nu \int_\Omega |\nabla V^\nu_r|^2 \, dx
\]

\[
= - \nu \int_{\partial \Omega} \Pi(V^\nu_r, V^\nu_r) \, d\mathcal{H}^2 + \nu \int_\Omega |\nabla V^\nu_r|^2 \, dx,
\]

since \(\Pi = - \nabla \mathbf{n}\). Meanwhile, the fifth term on the right-hand side of Eq. \((5.6)\) becomes

\[
- \nu \int_\Omega V^\nu_r \cdot \Delta \text{curl}^r u \, dx = - \nu \int_{\partial \Omega} \langle V^\nu_r \cdot \nabla (\text{curl}^r u), \mathbf{n} \rangle \, d\mathcal{H}^2 + \nu \int_\Omega (\nabla V^\nu_r) : (\nabla \text{curl}^r u) \, dx,
\]
where \( A : B = \sum_{j=1}^{3} A_{ij} B_{ij} \) for \( 3 \times 3 \) matrices \( A \) and \( B \). Therefore, the following identity is derived from Eq. (5.6):

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |V^{r}_{\nu}|^2 \, dx + \nu \int_{\Omega} |\nabla V^{r}_{\nu}|^2 \, dx \\
= \nu \int_{\partial \Omega} \Pi(V^{r}_{\nu}, V^{r}_{\nu}) \, dH^2 - \nu \int_{\Omega} (\nabla V^{r}_{\nu} : (\nabla \text{curl}^{r} u) \, dx - \int_{\Omega} \langle V^{r}_{\nu}, \text{curl}^{r} \{((u + v^{r}) \cdot \nabla)v^{r}\} \rangle \, dx \\
+ \nu \int_{\partial \Omega} \langle V^{r}_{\nu}, \partial_{n}(\text{curl}^{r} u) \rangle \, dH^2 - \int_{\Omega} \langle V^{r}_{\nu}, \text{curl}^{r} \{(v^{r} \cdot \nabla)u\} \rangle \, dx \\
= \mathcal{V}^1 + \mathcal{V}^2 + \mathcal{V}^3 + \mathcal{V}^4 + \mathcal{V}^5. \tag{5.7}
\]

In the following steps, we estimate each of the five terms \( \mathcal{V}^1 \sim \mathcal{V}^5 \).

2. The bound for \( \mathcal{V}^1 \) is straightforward: As the second fundamental form is bounded in \( C^{r-1}(\partial \Omega) \), we have

\[
|\mathcal{V}^1| \lesssim \nu \| V^{r}_{\nu} \|_{L^2(\partial \Omega)}^2 \approx \nu \| v^{r} \|_{H^r(\partial \Omega)} \lesssim \epsilon \nu \| \text{curl}^{r+1} v^{r} \|_{L^2(\Omega)}^2 + \frac{L}{\epsilon} \| v^{r} \|_{H^r(\Omega)}^2,
\tag{5.8}
\]

thanks to the Cauchy-Schwarz inequality, Theorem 3.1, the Young inequality, and the Sobolev trace inequality. In addition, the bound for \( \mathcal{V}^2 \) is also immediate:

\[
|\mathcal{V}^2| \lesssim \nu \| \nabla V^{r}_{\nu} \|_{L^2(\Omega)} \| \nabla \text{curl}^{r} u \|_{L^2(\Omega)} \\
\approx \nu \| \text{curl}^{r+1} v^{r} \|_{L^2(\Omega)} \| u \|_{H^{r+1}(\Omega)} \\
\lesssim \epsilon \nu \| \text{curl}^{r+1} v^{r} \|_{L^2(\Omega)}^2 + \frac{L}{\epsilon} \| u \|_{H^{r+1}(\Omega)}^2,
\tag{5.9}
\]

by using the Cauchy-Schwarz and Young inequalities again.

3. Next, we bound \( \mathcal{V}^3 := -\int_{\Omega} \langle V^{r}_{\nu}, \text{curl}^{r} \{((u + v^{r}) \cdot \nabla)v^{r}\} \rangle \, dx \) by applying a commutator estimate in the spirit of Kato-Ponce \( \mathcal{H} \) to the nonlinear convective terms. For this purpose, we introduce the following abbreviation for the operator:

\[
\mathcal{T} = \mathcal{T}(u, v^{r}) := (u + v^{r}) \cdot \nabla. \tag{5.10}
\]

Then \( \mathcal{V}^3 \) can be written as

\[
\mathcal{V}^3 = -\int_{\Omega} \langle V^{r}_{\nu}, \text{curl}^{r} \circ \mathcal{T}(v^{r}) \rangle \, dx \\
= -\int_{\Omega} \langle V^{r}_{\nu}, \mathcal{T}(V^{r}_{\nu}) \rangle \, dx - \int_{\Omega} \langle V^{r}_{\nu}, [\text{curl}^{r}, \mathcal{T}]v^{r} \rangle \, dx, \tag{5.11}
\]

where

\[
[\text{curl}^{r}, \mathcal{T}] = \text{curl}^{r} \circ \mathcal{T} - \mathcal{T} \circ \text{curl}^{r}
\]

is the commutator of the differential operators. Then the first term on the right-hand side of Eq. (5.11) equals

\[
-\int_{\Omega} \langle V^{r}_{\nu}, \mathcal{T}(V^{r}_{\nu}) \rangle \, dx = -\frac{1}{2} \int_{\partial \Omega} |V^{r}_{\nu}|^2 (u + v^{r}) \cdot n \, dH^2 = 0,
\tag{5.12}
\]

to the divergence theorem, the incompressibility of \( u \) and \( v^{r} \), and the kinematic boundary conditions for \( u \) and \( v^{r} \).

For the second term on the right-hand side of Eq. (5.11), since \( \mathcal{T} \) is a first-order differential operator, \( [\text{curl}^{r}, \mathcal{T}] \) is of order less than or equal \( r \), with the coefficients involving the derivatives of \( u \) and \( v^{r} \). More precisely, we have

\[
| [\text{curl}^{r}, \mathcal{T}] v^{r} | \lesssim \sum_{l=1}^{r} |\nabla \lbrack (l) u + \nabla \lbrack (l) v^{r}) | |\nabla \lbrack (r+l-l) \lbrack v^{r} |, \tag{5.13}
\]
by directly applying the Leibniz rule, where the schematic symbol $\nabla^{[l]}$ denotes, as before, the derivatives up to the $l$-th order. Next, utilizing the interpolation inequalities, we obtain

$$|\mathcal{V}^3| = \left| \int_{\Omega} \langle \nabla v^{r}, [\text{curl}^{r} T] v^{r} \rangle \right| dx \lesssim \sum_{l=1}^{L} \int_{\Omega} \left\{ |\nabla^{[l]} v^{r}| |\nabla^{[l]} u||\nabla^{[l+1]} v^{r}| + |\nabla^{[l]} v^{r}| |\nabla^{[l]} u||\nabla^{[l+1]} v^{r}| \right\} dx \lesssim \|v^{r}\|_{L^2(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} + \|v^{r}\|_{H^r(\Omega)} \|\nabla v^{r}\|_{L^\infty} + \|v^{r}\|_{H^r(\Omega)} \|\nabla v^{r}\|_{L^\infty}. \quad (5.14)$$

Therefore, in view of the Sobolev embedding $H^{r+1} \hookrightarrow W^{1,\infty}$ in 3-D for $r > \frac{5}{2}$, we have

$$|\mathcal{V}^3| \lesssim (1 + \|u\|_{H^{r+1}(\Omega)}) \|v^{r}\|_{H^r(\Omega)}^2 + \|v^{r}\|_{H^r(\Omega)}^3. \quad (5.15)$$

In particular, we observe that the above upper bound for $\mathcal{V}^3$ involves only the derivatives up to the $r$-th order of $v^{r}$.

4. Next, to treat the fourth term

$$\mathcal{V}^4 := \nu \int_{\partial\Omega} \left( \mathcal{V}^{\nu, r}_\nu \partial_n (\text{curl}^{r} u) \right) d\mathcal{H}^2,$$

it is crucial to take into account the Navier boundary condition. Indeed, using Einstein’s summation convention and integrating by parts on $\partial\Omega$, we first obtain

$$\mathcal{V}^4 = \nu \int_{\partial\Omega} \langle \mathcal{V}^{\nu, r}_\nu \nabla j(\text{curl}^{r} u)^i n^j \rangle d\mathcal{H}^2$$

$$= -\nu \int_{\partial\Omega} \nabla j(\mathcal{V}^{\nu, r}_\nu)^i (\text{curl}^{r} u)^i n^j d\mathcal{H}^2 + \nu \int_{\partial\Omega} H(\mathcal{V}^{\nu, r}_\nu, \text{curl}^{r} u) d\mathcal{H}^2 =: \mathcal{V}^{4,1} + \mathcal{V}^{4,2}, \quad (5.16)$$

where $H = -\nabla j n^j$ is the mean curvature of boundary $\partial\Omega$. Here, by the boundedness of the second fundamental form, the second term is bounded by

$$|\mathcal{V}^{4,2}| \lesssim \nu \left( \|v^{r}\|_{H^r(\partial\Omega)}^2 + \|u\|_{H^r(\partial\Omega)}^2 \right) \lesssim \nu \left( \|v^{r}\|_{H^{r+1}(\Omega)}^2 + \|u\|_{H^{r+1}(\Omega)}^2 \right) + \frac{\nu}{\epsilon} \left( \|v^{r}\|_{H^r(\Omega)}^2 + \|u\|_{H^r(\Omega)}^2 \right). \quad (5.17)$$

For $\mathcal{V}^{4,1}$, we add and subtract $\nabla i(\mathcal{V}^{\nu, r}_\nu)^i$ from the integrand to obtain

$$\mathcal{V}^{4,1} = -\nu \int_{\partial\Omega} \left\{ \nabla j(\mathcal{V}^{\nu, r}_\nu)^i - \nabla i(\mathcal{V}^{\nu, r}_\nu)^i \right\} (\text{curl}^{r} u)^i n^j d\mathcal{H}^2$$

$$+ \nu \int_{\partial\Omega} \nabla i(\mathcal{V}^{\nu, r}_\nu)^j (\text{curl}^{r} u)^i n^j d\mathcal{H}^2 =: \mathcal{V}^{4,1}_a + \mathcal{V}^{4,1}_s. \quad (5.18)$$

By the incompressibility condition, the integral vanishes for $i = j$ so that it suffices to consider $i \neq j$ in the summation. In this case, denoting by $k$ the index in $\{1, 2, 3\}$ different from $i, j$, we have

$$\mathcal{V}^{4,1}_a = \sigma \nu \int_{\partial\Omega} (\text{curl}^{r+1} v^{r}) k (\text{curl}^{r} u)^i n^j d\mathcal{H}^2, \quad (5.19)$$

where $\sigma \in \{1, -1\}$ is a sign.

To proceed, we first establish the following claim: For the tangential projection $\pi : T\mathbb{R}^3 \rightarrow T(\partial\Omega)$ as before, the iterated curls of $v^{r}$ satisfy the following non-Navier slip-type boundary condition:

$$\pi \circ \text{curl}^{r+1} v^{r} = -\frac{1}{\zeta} \mathcal{R} \circ \pi(\text{curl}^{r} v^{r}) + 2\mathcal{R} \circ \pi \circ \text{curl}^{r} \circ \mathcal{S} \circ \pi(v^{r}) \quad \text{on } \partial\Omega. \quad (5.20)$$

The proof of the above claim goes by induction on $r$, similar to the arguments in Step 4 in the proof of Theorem 4.3. Indeed, for $r = 0$, it reduces to the Navier boundary condition.
This is because, thus, we assume the result first for \( r \) and then prove it for \( r + 1 \). Taking a moving frame \( \{\nabla_1, \nabla_2, \nabla_3\} \) adapted to surface \( \partial \Omega \) such that \( \nabla_1, \nabla_2 \in T(\partial \Omega) \) and \( \nabla_3 = n \), by the induction hypothesis, we have

\[
\pi(\text{curl}^{r+1} v^\nu) = \pi \circ \text{curl} \left( - \frac{1}{\zeta} R \circ \pi(\text{curl}^{r-1} v^\nu) + 2R \circ \pi \circ \text{curl}^{r-1} \circ S \circ \pi(v^\nu) \right)
\]

\[
= \begin{bmatrix}
-\nabla_3 \left( - \frac{1}{\zeta} R \circ \pi(\text{curl}^{r-1} v^\nu) + 2R \circ \pi \circ \text{curl}^{r-1} \circ S \circ \pi(v^\nu) \right) \\
\nabla_3 \left( - \frac{1}{\zeta} R \circ \pi(\text{curl}^{r-1} v^\nu) + 2R \circ \pi \circ \text{curl}^{r-1} \circ S \circ \pi(v^\nu) \right)^{1/2}
\end{bmatrix}.
\]

Since \( R(V^1, V^2)^\top = (-V^2, V^1)^\top \) for any 2-D vector \( V \), the above equalities yield

\[
\pi(\text{curl}^{r+1} v^\nu) = \begin{bmatrix}
-\nabla_3 \left( - \frac{1}{\zeta} (\text{curl}^{r-1} v^\nu) + 2(\text{curl}^{r-1} \circ S \circ \pi(v^\nu)) \right) \\
\nabla_3 \left( - \frac{1}{\zeta} (\text{curl}^{r-1} v^\nu) + 2(\text{curl}^{r-1} \circ S \circ \pi(v^\nu)) \right)^{1/2}
\end{bmatrix}.
\]

Thus, claim (5.20) is proved.

The above claim shows that the iterated curls of \( v^\nu \) can be expressed on boundary \( \partial \Omega \) by the derivatives of \( v^\nu \) up to the \( r \)-th order, together with the bounded operators \( R, S, \) and \( \pi \). Thus, Eq. (5.19) becomes

\[
\mathcal{V}_{a}^{4,1} = \sigma \nu \int_{\partial \Omega} \left\{ - \frac{1}{\zeta} \left[ R \circ \pi(\text{curl}^{r} v^\nu) \right]^k (\text{curl}^{r} u)^i n^i \right\} d\mathcal{H}^2
\]

\[
+ 2 \sigma \nu \int_{\partial \Omega} \left\{ \left[ R \circ \pi \circ \text{curl}^{r} \circ S \circ \pi(v^\nu) \right]^k (\text{curl}^{r} u)^i n^i \right\} d\mathcal{H}^2,
\]

where the non-zero contributions come only from the tangential components of \( \text{curl}^{r} v^\nu \), and \( n^i = 0 \) unless \( j = 3 \) in the moving frame \( \{\nabla_1, \nabla_2, \nabla_3\} \), which forces \( k \in \{1, 2\} \). We then obtain the following estimate:

\[
|\mathcal{V}_{a}^{4,1}| \lesssim \nu \left( \|v^\nu\|^2_{H^{r+1}(\partial \Omega)} + \|u\|^2_{H^{r+1}(\partial \Omega)} \right)
\]

\[
\lesssim \nu \left( \epsilon \|\text{curl}^{r+1} v^\nu\|^2_{H^{r+1}(\Omega)} + \|v^\nu\|^2_{H^{r+1}(\Omega)} + \epsilon \|u\|^2_{H^{r+1}(\Omega)} + \|u\|^2_{H^{r+1}(\Omega)} \right).
\]

On the other hand, for \( \mathcal{V}_{s}^{4,1} \), we integrate by part once more to obtain

\[
\mathcal{V}_{s}^{4,1} = -\nu \int_{\Omega} \left( \text{curl}^{r} u \right)^i \nabla^i n d\mathcal{H}^2 = \nu \int_{\partial \Omega} II(V^r, \text{curl}^{r} u) d\mathcal{H}^2.
\]

This is because \( \text{div} \circ \text{curl}^{r} u = 0 \), and \( II = -\nabla n \) by the definition of the second fundamental form. By assumption, \( ||II||_{L^\infty(\partial \Omega)} \leq C \) so that

\[
|\mathcal{V}_{s}^{4,1}| \lesssim \nu \|V^r\|_{L^2(\partial \Omega)} \|\text{curl}^{r} u\|_{L^2(\partial \Omega)}
\]

\[
\lesssim \nu \left( \|\text{curl}^{r+1}\|_{L^2(\Omega)}^2 + \|u\|^2_{H^{r+1}(\Omega)} \right) + \nu \left( \|v^\nu\|^2_{H^{r+1}(\Omega)} + \|u\|^2_{H^{r+1}(\Omega)} \right).
\]

Thus, putting together Eqs. (5.19), (5.22), (5.24), and (5.17), we conclude

\[
|\mathcal{V}^4| \lesssim \epsilon \nu \left( \|\text{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 + \|u\|^2_{H^{r+1}(\Omega)} \right) + \nu \left( \|v^\nu\|^2_{H^{r+1}(\Omega)} + \|u\|^2_{H^{r+1}(\Omega)} \right).
\]

(5.25)
5. The estimate for $\mathcal{V}^5 := -\int_\Omega \langle \text{curl}^r v^\nu, \text{curl}^l (v^\nu \cdot \nabla u) \rangle \, dx$ proceeds by the Leibniz rule:

$$|\mathcal{V}^5| \lesssim \int_\Omega |\mathcal{V}^r|^2 \times \left( \sum_{l=0}^{r} |\nabla^l v^\nu||\nabla^{r-l+1}u| \right) \, dx$$

$$\lesssim \int_\Omega |\nabla^{r+1} (v^\nu)|(|\nabla^{r} v^\nu||\nabla u| + |v^\nu||\nabla^{r+1}u|) \, dx$$

$$\lesssim \|v^\nu\|_{L^2(\Omega)}^2 \|\nabla u\|_{L^\infty(\Omega)} + \|v^\nu\|_{H^1(\Omega)} \|v^\nu\|_{L^\infty(\Omega)} \|u\|_{H^{r+1}(\Omega)} + \|v^\nu\|_{H^1(\Omega)} \|v^\nu\|_{L^2(\Omega)} \|u\|_{H^{r+1}(\Omega)},$$

where the last two lines hold by the interpolation inequality and the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Thus, we have

$$|\mathcal{V}^5| \lesssim \|v^\nu\|_{H^1(\Omega)}^2 \|u\|_{H^{r+1}(\Omega)}, \quad (5.26)$$

whenever $r \geq 2$.

6. Now, putting all the estimates for $\mathcal{V}^1 - \mathcal{V}^5$ in Eqs. (5.3)–(5.9), (5.15), and (5.24)–(5.26) together, we obtain:

$$\frac{d}{dt} \int_\Omega |\mathcal{V}^r|^2 \, dx + \nu \int_\Omega |\nabla \mathcal{V}^r|^2 \, dx \lesssim \nu \|\text{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 + \left( \nu + \frac{\nu^2}{\epsilon} \right) \|u\|_{H^{r+1}(\Omega)}^2,$$  

$$\lesssim \nu \|\text{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 + \left( \nu + \frac{\nu^2}{\epsilon} \right) \|u\|_{H^{r+1}(\Omega)}^2.$$  

Thus, we can choose $\epsilon > 0$ to be small so that

$$\frac{d}{dt} \int_\Omega |\mathcal{V}^r|^2 \, dx + \nu \|\text{curl}^{r+1} v^\nu\|_{L^2(\Omega)}^2 \leq C_1 (1 + \|u\|_{H^{r+1}(\Omega)}) \|v^\nu\|_{H^1(\Omega)}^2 + \|v^\nu\|_{H^1(\Omega)}^3 + C_2 \nu \|u\|_{H^{r+1}(\Omega)}^2,$$  

where constants $C_1$ and $C_2$ depend on $\nu$ and $\epsilon$. Here, it is crucial to choose $\epsilon$ that depends only on $\|\Pi\|_{C^\gamma(\partial \Omega)}$, independent of $\nu$.

As a consequence, the energy function $E(t) := \|v^\nu(t, \cdot)\|_{H^1(\Omega)}^2$ satisfies the differential inequality:

$$E'(t) \leq C_3 E(t)^{3/2} + C_4 \nu,$$  

where $C_3 = C_3(C_1, \|u\|_{H^{r+1}(\Omega)})$ and $C_4 = C_2 \|u\|_{H^{r+1}(\Omega)}^2$ by interpolation.

7. Finally, in order to show that the lifespan of strong solutions of the Navier-Stokes equations are no less than $T_*$ uniformly in $\nu \in (0, \nu_*)$ and to derive the inviscid limit, for any fixed $T \in (0, T_*)$, we define

$$\alpha(t) := \exp \{ -C_1 t - C_1 \int_0^t \|u(s, \cdot)\|_{H^{r+1}(\Omega)} \, ds \},$$

$$F(t) := C_2 \alpha(t) \|u(t, \cdot)\|_{H^{r+1}(\Omega)}^2,$$

$$G := C_1 \alpha(T)^{-1} = \text{const}.,$$

$$\Phi(t) := \alpha(t) \|v^\nu(t, \cdot)\|_{H^1(\Omega)}^2.$$  

Then $\Phi$ satisfies the ordinary differential inequality:

$$\Phi'(t) \leq \nu F(t) + G \Phi(t)^{3/2} \quad \text{on } [0, T].$$  

For some parameter $\beta > 0$ to be chosen later, we divide Eq. (5.31) by $(1 + \beta \Phi(t))^{3/2}$ to obtain

$$\frac{\Phi'(t)}{(1 + \beta \Phi(t))^{3/2}} \leq \nu F(t) + \frac{G}{\beta^{3/2}}.$$
Thus, by integrating from $0$ to $t \in [0, T]$, we obtain the estimate:

$$\frac{2}{\beta} \left( 1 - \frac{1}{\sqrt{1 + \beta \Phi(t)}} \right) \leq \nu \int_0^T F(t) dt + \frac{GT}{\beta^{3/2}},$$

that is,

$$\frac{1}{\sqrt{1 + \beta \Phi(t)}} \geq 1 - \frac{\beta \nu}{2} \int_0^T F(t) dt - \frac{GT}{2\sqrt{\beta}}. \quad (5.32)$$

Notice that the right-hand side of Eq. (5.32) is maximized when the two negative terms are equal:

$$\beta = \left( \frac{GT}{\nu \int_0^T F(t) dt} \right)^{2/3}. \quad (5.33)$$

Then the right-hand side equals

$$1 - (GT)^{2/3} \left( \nu \int_0^T F(t) dt \right)^{1/3}.$$

It is bigger than or equal to $\frac{1}{2}$ if and only if

$$\nu \leq \left( \frac{8(GT)^2}{\int_0^T F(t) dt} \right)^{-1}.$$

On the other hand, by Eq. (5.32), $\frac{1}{\sqrt{1 + \beta \Phi(t)}} \geq \frac{1}{2}$ for all $0 \leq t \leq T$; that is, $\Phi(t) \leq \frac{3}{\beta}$ on $[0, T]$.

In summary, we have established the following: If we set

$$\nu_* = \left( 8G^2T^2 \int_0^T F(t) dt \right)^{-1},$$

then, whenever $0 < \nu \leq \nu_*$,

$$\Phi(t) \leq 3 \left( \frac{\int_0^T F(t) dt}{GT} \right)^{2/3} \nu^{2/3}, \quad (5.34)$$

which is equivalent to

$$\sup_{t \in [0, T]} \| u'(t, \cdot) - u(t, \cdot) \|_{H^{r+1}(\Omega)}^2 \leq \frac{3}{\nu_\ast} \exp \left\{ C_1 T + C_1 \int_0^T \| u(s, \cdot) \|_{H^{r+1}(\Omega)} ds \right\} \left( \frac{\int_0^T F(t) dt}{GT} \right)^{2/3} \nu^{2/3}. \quad (5.35)$$

This holds for any $t \in [0, T]$, where $T$ is an arbitrary number in $(0, T_\ast)$. Therefore, the Navier-Stokes solution $u'$ does not blow up on $[0, T]$ in the $H^r$–norm in space, provided that $0 < \nu \leq \nu_*$. This completes the proof.

To conclude this section, we now give the following three remarks.

**Remark 5.2.** A key point of Theorem 5.1 is that the strong solutions $u'$ to the Navier-Stokes equations do not blow up before $T_\ast$ in $H^r$, where $T_\ast$ is the lifespan of the corresponding Euler equations in $H^{r+1}$ for $r > \frac{5}{2}$. The arguments (Step 7 of the proof) are adapted from §1 in Constantin [17], in which the case of periodic boundary conditions are treated. This does not directly follow from our proof of Theorem 4.3. In fact, constant $M$ is proportional to $\nu^{-1}$ in Eq. (4.31), so that the lifespan for the Navier-Stokes equations (in $H^{r+1}$ in space) is proportional to viscosity $\nu$, which goes to zero in the vanishing viscosity limit.

**Remark 5.3.** In Theorem 5.1, the rate of convergence in the inviscid limit is $O(\nu^{1/3})$. It can be improved to $O(\sqrt{\nu})$, provided that $\{ \nabla u' - \nabla u \}$ is uniformly bounded in space-time. Moreover, in this case, the $H^{r+1}$–norm of $u'$ is also close to that of $u$ in the average in time.
Proposition 5.4. Let \( v^\nu, u, v^0, T_* \), and \( r \) be as in Theorem 5.1. In addition, suppose that \( \{ \nabla v^\nu - \nabla u \} \) is uniformly bounded in \( L^\infty([0,T_*) \times \Omega; \mathbb{R}^3) \). Then there exists a constant \( C \), depending only on \( T_* \), \( \| u \|_{L^2([0,T_*]; H^{r+1}(\Omega))} \), and \( \| \Pi \|_{C^r(\partial \Omega)} \), such that
\[
\sup_{0 \leq t < T_*} \| u(t, \cdot) - u(t, \cdot) \|_{H^r(\Omega)} \leq C \sqrt{\nu}.
\] (5.36)

In particular, as \( \nu \to 0^+ \), \( v^\nu \) converges to \( u \) in \( H^r(\Omega) \) uniformly in time. In addition,
\[
\int_0^T \| u^\nu(t, \cdot) - u(t, \cdot) \|_{H^{r+1}(\Omega)} \leq C \quad \text{for any } T \in [0, T_*).
\] (5.37)

Proof. The proof follows essentially from the arguments in Theorem 5.1 above, i.e. by considering the evolution equation for \( v^\nu := u^\nu - u \). We only emphasize the differences.

Indeed, starting from Eq. (5.7), we estimate the terms, \( \mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^4 \), and \( \mathcal{V}^5 \), as in Steps 1–2 and 4–5 in the proof of Theorem 5.1. The only difference occurs in Step 3. Recall Eq. (5.14) therein:
\[
|\mathcal{V}^3| \lesssim \| v^\nu \|_{H^r(\Omega)}^2 \| \nabla u \|_{L^\infty(\Omega)} + \| v^\nu \|_{H^r(\Omega)} \| u \|_{H^r(\Omega)} \| \nabla v^\nu \|_{L^\infty} + \| v^\nu \|_{H^r(\Omega)}^2 \| \nabla v^\nu \|_{L^\infty}.
\]

Under the additional assumption, \( \| \nabla v^\nu \|_{L^\infty} \leq C \) so that
\[
|\mathcal{V}^3| \lesssim (1 + \| u \|_{H^r(\Omega)}) \| v^\nu \|_{H^r(\Omega)}^2.
\] (5.38)

As a consequence, by choosing \( \epsilon \) suitable small, estimate (5.28) in Step 6 can be improved to
\[
\frac{d}{dt} \int_\Omega |v^\nu|^2 dx + \nu \| \text{curl} v^\nu \|_{L^2(\Omega)}^2 \leq C_1 \left(1 + \| u \|_{H^r(\Omega)}^2 \right) \| v^\nu \|_{H^r(\Omega)}^2 + C_2 \nu \| u \|_{H^{r+1}(\Omega)}^2.
\] (5.39)

which does not contain the cubic terms in \( \| v^\nu \|_{H^r} \).

From here, the usual Gronwall inequality yields
\[
\| v^\nu(t, \cdot) \|_{H^r(\Omega)}^2 \leq \| v^\nu(0, \cdot) \|_{H^r(\Omega)}^2 \exp \left\{ C_1 t + \int_0^t \| u(s, \cdot) \|_{H^{r+1}(\Omega)} ds \right\}
\]
\[
+ C_2 \nu \int_0^t \left\{ \exp \left( C(t - s) + \int_s^t \| u(\tau, \cdot) \|_{H^{r+1}(\Omega)} d\tau \right) \| u(s, \cdot) \|_{H^{r+1}(\Omega)} \right\} ds
\]
\[
= C_2 \nu \int_0^t \left\{ \exp \left( C(t - s) + \int_s^t \| u(\tau, \cdot) \|_{H^{r+1}(\Omega)} d\tau \right) \| u(s, \cdot) \|_{H^{r+1}(\Omega)}^2 \right\} ds
\] (5.40)

for any \( t \in [0, T_] \). This is because \( v^\nu(0, \cdot) = 0 \), since the Navier-Stokes and the Euler solutions have the same initial data. Thus, for some constant \( C_0 = C_0(v^0, \| \Pi \|_{C^r(\Omega)}, T_*, \| u \|_{L^2(0,T_*;H^{r+1}(\Omega))}) \),
\[
\sup_{t \in [0, T_*)} \| v^\nu(t, \cdot) \|_{H^r(\Omega)} \leq C_0 \sqrt{\nu} \longrightarrow 0 \quad \text{as } \nu \to 0^+.
\] (5.41)

Finally, integrate Eq. (5.39) with Eq. (5.40) substituted into the right-hand side. In this way, we find a constant \( C_7 \) with the same dependence as \( C_0 \) such that
\[
\sup_{t \in [0, T_*)} \int_0^t \| \text{curl} v^\nu(s, \cdot) \|_{L^2(\Omega)}^2 ds \leq C_7.
\] (5.42)

Therefore, in view of Theorem 3.1 and Eq. (5.41), we have
\[
\sup_{t \in [0, T_*)} \int_0^t \| v^\nu(s, \cdot) \|_{H^{r+1}(\Omega)} ds \leq C_8 = C_8(v^0, \| \Pi \|_{C^r(\Omega)}, T_*, \| u \|_{L^2(0,T_*;H^{r+1}(\Omega))}),
\] (5.43)

which completes the proof.

\[\square\]
Remark 5.5. Combining the results in §4–§5 together, we have established the existence of strong solutions in $H^{r+1}$ for $r > \frac{5}{2}$ of the Navier-Stokes equations, while the inviscid limit has been proved in $H^r$. Therefore, it remains an open question whether the inviscid limit holds or fails (e.g., due to the development of boundary layers) in $H^{r+1}$, i.e. the highest order the spatial regularity of the strong solutions.

6. Remarks on the Non-Navier Slip-type Boundary Condition

In the introduction (§1), a modified version of the Navier boundary condition, which is originally introduced by Bardos [5] and Solonnikov-Ščadilov [45], has been briefly discussed. Physically, it describes the phenomenon that the tangentia part of the normal vector field of the Cauchy stress tensor is uniformly vanishing, and it agrees with the Navier boundary condition if and only if boundary $\partial \Omega$ is flat. Together with the kinematic boundary condition, we have
\begin{align*}
\nabla \cdot u &= 0, \\
\omega \times n &= 0
\end{align*}
on $\partial \Omega$. (6.1)

The second line is referred to as the non-Navier slip-type boundary condition.

In Beirão da Veiga-Crispo [10, 11], the inviscid limit problem is analyzed for the Navier-Stokes equations subject to the boundary conditions (6.1) and the Euler equations subject to the no-penetration boundary condition (1.11). In this section, we write $K$ for the Gauss curvature of surface $\partial \Omega$. We first introduce the following notions (see also Definitions 2.1 and 2.3 in [11]):

Definition 6.1. For the non-Navier slip-type boundary conditions (6.1), we say that
\begin{enumerate}
\item $u_0 \in C^\infty(\Omega; \mathbb{R}^3)$ is an admissible initial data if $\nabla \cdot u_0 = 0$ in the closure $\overline{\Omega}$, as well as $u_0 \cdot n = 0$ and $\omega_0 \times n = 0$ on $\partial \Omega$;
\item The inviscid limit $u^\nu \to u$ holds “strongly” in $L^p([0,T]; W^{s,q}(\Omega; \mathbb{R}^3))$ for some $T > 0$, $p,q \geq 1$, and $s > 1$, if the convergence holds with respect to the strong topology on $L^p([0,T]; W^{s,q}(\Omega; \mathbb{R}^3))$.
\end{enumerate}

In particular, we notice that, if $u^\nu \to u$ strongly, $\omega^\nu \times n = 0$ on $[0,T] \times \partial \Omega$ implies $\omega \times n = 0$ on $[0,T] \times \partial \Omega$. This is termed as the “persistence property” in [10, 11]. In the presence of such a property, the following non-convergence result is established by Beirão da Veiga-Crispo, first by considering a special example on $\mathbb{S}^2$ in [10] and then proved in full generality via computations of the principal curvatures on $\partial \Omega$ in local coordinates in [11]:

Theorem 6.2 (Beirão da Veiga-Crispo, [10, 11]). Let $\Omega \subset \mathbb{R}^3$ be a bounded regular domain. Let the admissible initial data $u_0$ be given for the initial-boundary value problem (1.11) and (6.1) and problem (1.10)–(1.11) such that the following condition holds:
\begin{equation}
\omega_0(x_0) \neq 0 \quad \text{for some } x_0 \in \partial \Omega \text{ such that } K(x_0) \neq 0.
\end{equation}

Then, for arbitrary $\delta > 0$, $p,q \geq 1$ and $s > 1$, the “strong” inviscid limit fails:
\begin{equation}
u^\nu \nrightarrow u \quad \text{in } L^p([0,\delta]; W^{s,q}(\Omega; \mathbb{R}^3)).
\end{equation}

This theorem says that, if the initial vorticity vanishes somewhere on the curved part of the boundary, i.e. the Gauss curvature is non-vanishing at this point, then the “strong” inviscid limit fails in an arbitrarily short time interval, so that the Prandtl boundary layers must be developed. Heuristically, this is due to the incompatibility of the vorticity directions of the slip-type boundary conditions in the limiting process $\nu \to 0^+$. 

Now we give an alternative proof of Theorem 6.2 which avoids the computations in local coordinates on \( \partial \Omega \) as in [10] [11]. This offers a new, global perspective for the above theorem, and the proof makes essential use of the properties of Lie derivatives in \( \mathbb{R}^3 \).

**Proof.** First of all, following the original proof in [10] [11], we consider the inviscid vorticity equation, which is obtained by taking the curl of the Euler equation (1.10):
\[
\partial_t \omega + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u = 0.
\]
Then taking the cross product with the outer unit normal \( n: \partial \Omega \to \mathbb{S}^2 \) leads to
\[
\partial_t (\omega \times n) + \{(u \cdot \nabla) \omega - (\omega \cdot \nabla) u \} \times n = 0. \tag{6.5}
\]
It is observed by Xiao-Xin (cf. Corollary 8.3 in [49]) that a necessary condition for the “strong” inviscid limit in the time interval \( (0, \delta) \) is:
\[
\omega \times n \equiv 0 \quad \text{on} \quad (0, \delta) \times \partial \Omega, \tag{6.6}
\]
that is, the persistence property is verified (cf. Definition 6.1). Thus, if the “strong” inviscid limit were valid, then Eq. (6.5) implies that \( \{(u \cdot \nabla) \omega - (\omega \cdot \nabla) u \} \times n = 0 \) for all time. In particular, sending \( t \to 0^+ \), the following condition must be fulfilled:
\[
\{(u_0 \cdot \nabla) \omega_0 - (\omega_0 \cdot \nabla) u_0 \} \times n = 0 \quad \text{on} \quad \partial \Omega. \tag{6.7}
\]

Our crucial observation is that the expression in the bracket in (6.7) coincides with the Lie bracket of the vector fields \( u_0, \omega_0 \in T \mathbb{R}^3 \):
\[
(u_0 \cdot \nabla) \omega_0 - (\omega_0 \cdot \nabla) u_0 = [u_0, \omega_0]. \tag{6.8}
\]

To prove Eq. (6.8), let \( V = \sum_{i=1}^3 V^i \partial_i \) and \( W = \sum_{j=1}^3 W^j \partial_j \) be two smooth vector fields in \( T \mathbb{R}^3 \), where \( \{\partial_1, \partial_2, \partial_3\} \) denotes the canonical Euclidean frame. We follow the convention in differential geometry to identify vector fields with first-order differential operators. Thus, the Lie bracket of \( V \) and \( W \) can be computed as
\[
[V, W] = VW - WV = \left( V^i \partial_i (W^j \partial_j) - W^j \partial_j (V^i \partial_i) \right) =: (V \cdot \nabla) W - (W \cdot \nabla) V,
\]
which verifies the above identity. In the above, the Einstein summation convention is adopted.

To proceed, we recall that the Lie bracket of two vector fields on a differentiable manifold equals the *Lie derivative* (denoted by \( \mathcal{L} \)) of one vector field along the other:
\[
[u_0, \omega_0] = \mathcal{L}_{u_0} \omega_0. \tag{6.9}
\]

This result is standard in the differentiable manifold theory, which can be found in the classical texts (cf. do Carmo [21]). Furthermore, the Lie derivative at a point \( x_0 \in \partial \Omega \) is given by
\[
\mathcal{L}_{u_0} \omega_0(x_0) := \lim_{s \to 0^+} \frac{\theta_s^* \{\omega_0(\theta_s(x_0))\} - \omega_0(x_0)}{s}, \tag{6.10}
\]
where \( \{\theta_s(x) : s \geq 0, x \in \mathbb{R}^3\} \) is the one-parameter subgroup defined by the following ODE:
\[
\begin{cases}
\frac{d}{ds} \theta_s(x) = u_0(\theta_s(x)) & \text{for all } s \geq 0, x \in \mathbb{R}^3, \\
\theta_0(x) = x & \text{for all } x \in \mathbb{R}^3.
\end{cases} \tag{6.11}
\]
In other words, trajectory \( \{\theta_s(x)\}_{s \geq 0} \) is the *integral curve* of the vector field \( u_0 \) emanating from point \( x \). Also, \( (\theta_s)^* \) denotes the pullback operation under map \( \theta_s \).
By the admissibility of the initial data, \( \omega_0 \in T(\partial \Omega) \perp \) is orthogonal to the boundary since \( \omega_0 \times n = 0 \), and \( u_0 \in T(\partial \Omega) \) is tangential to the boundary because of the kinematic boundary condition: \( u_0 \cdot n = 0 \). The expression on the right-hand side of Eq. (6.9) is well-defined since \( \omega_0 \) is a vector field defined along the manifold \( \partial \Omega \). In geometric terminologies, it means that \( \omega_0 \in \iota^* T\mathbb{R}^3 \), where \( \iota : \partial \Omega \rightarrow \mathbb{R}^3 \) is the embedding of Riemannian submanifold, and \( \iota^* T\mathbb{R}^3 \) is the pullback vector bundle. This enables us to take the Lie derivative on \( \omega_0 \) along any vector field (e.g. \( u_0 \)) tangent to \( \partial \Omega \).

Now, by the assumptions, there is a point \( x_0 \in \partial \Omega \) such that \( \omega_0(x_0) \neq 0 \) and \( K(x_0) \neq 0 \). Owing to the non-vanishing curvature, there exists some small neighbourhood \( U \subset \partial \Omega \) of \( x_0 \) such that every smooth curve \( \gamma : (-\delta, \delta) \rightarrow \partial \Omega \) satisfying \( \gamma(0) = x_0 \) is not a straight line segment in \( \mathbb{R}^3 \). In addition, as vorticity \( \omega_0 \) is non-vanishing at \( x_0 \), we have

\[ \langle u_0, \dot{\gamma} \rangle \neq 0 \quad \text{on} \ U. \quad (6.12) \]

On the other hand, using the definition of the Lie derivative in terms of the integral curve, i.e. \( (6.10) \), we have

\[ \mathcal{L}_{\dot{\gamma}} \omega_0 \neq 0 \quad \text{on} \ T(\partial \Omega). \quad (6.13) \]

This is because the parallel-transport of \( \omega_0 \) along \( \gamma \) cannot be obtained by a Euclidean translation, so that \( (\theta_s)^* \{ \omega_0(\theta_s(x_0)) \} - \omega_0(x_0) \neq 0 \) in Eq. \( (6.10) \). Therefore, we conclude that \( \mathcal{L}_{\langle u_0, \dot{\gamma} \rangle / \dot{\gamma}} \omega_0 \neq 0 \) on \( T(\partial \Omega) \), from which it follows

\[ \mathcal{L}_{u_0} \omega_0(x_0) \times n(x_0) \neq 0 \quad \text{in} \ \mathbb{R}^3. \quad (6.14) \]

This contradicts Eq. \( (6.7) \).

Acknowledgement. The research of Gui-Qiang G. Chen was supported in part by the UK Engineering and Physical Sciences Research Council Award EP/E035027/1 and EP/L015811/1, and the Royal Society–Wolfson Research Merit Award (UK). The research of Siran Li was supported in part by the UK EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1). The research of Zhongmin Qian was supported in part by the ERC grant (ESig ID291244).

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