Ray Singer Analytic Torsion of Calabi Yau manifolds I.

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Abstract. In this paper we generalized the variational formulas for the
determinants of the Laplacians on functions of CY metrics to forms of type (0,q) on
CY manifolds. We also computed the Ray Singer Analytic torsion on CY manifolds
we proved that it is bounded by a constant. In case of even dimensional CY manifolds
the Ray Singer Analytic torsion is zero. The interesting case is the odd dimensional
one.

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1. Introduction.
One of the most remarkable formula that I encounter is the Kronecker limit formula. It
states that if

\[ E(s) = \sum'_{n,m \in \mathbb{Z}} \frac{1}{|n + m\tau|^s} \]

where \( \tau \in \mathbb{C} \), \( \text{Im} \tau > 0 \) and ’ means that the sum is taken over all pair of integers
\((m,n) \neq (0,0)\), then \( E(s) \) has a meromorphic continuation in \( \mathbb{C} \) with only one pole at
\( s = 1 \) and \( \exp(-\frac{d}{ds}E(s)|_{s=0}) = (\text{Im} \tau)^2 |\eta|^4 \) where \( \eta \) is the Dedekind eta function.

It is a well know fact that in the case of elliptic curves \( \{ E_\tau = \mathbb{C}/(n + m\tau), \text{Im} \tau > 0 \} \),
\( E(s) \) is the zeta function of the Laplacian of the flat metric on the elliptic curves \( E_\tau \),
the regularized determinant of the Laplacian is \( \exp(-\frac{d}{ds}E(s)|_{s=0}) \) and \( \eta^{24} \) is equal to
the discriminant of the elliptic curve \( E_\tau \). \( \eta^{24} \) vanishes at \( \infty \), which corresponds to an
elliptic curve with the node. Thus the Kronecker limit formula is an explicit formula for
the determinant of the Laplacian of an elliptic curve and gives a relation between the
spectrum of the Laplacian and the discriminant of elliptic curves. The Kronecker limit
formula has a modern interpretation as the Quillen norm of a section of the determinant
line bundle.
There is a simple non formal explanation of the above mentioned fact. It is a well known fact that the spectrum of the Laplacian of a Riemannian metric on a compact manifold is discrete. When the manifold acquires singularities than the specter becomes continuous. This phenomenon suggests that when the metric "degenerates" together with the manifold, then the regularized determinant vanishes on the points that parametrize the singular varieties. The problem is how to relate the specter of the Laplacian with the discriminant locus. The relation is suggested by the theory of determinant line bundles on the moduli space, their Quillen metrics and the Ray-Singer torsion as developed recently by Quillen, Donaldson, Bismut, Gillet and Soulé and others.

The problem that we are going to study in a series of two papers is to find the generalization of the analogue of the Dedekind eta function for odd dimensional CY manifolds.

The idea on which these two papers are based is very simple. The Quillen metric is related to the spectral properties of the Laplacian acting on $(0,q)$ forms in case of Kähler manifolds. The main question is when the Ray Singer analytic torsion is the Quillen metric of some holomorphic section of the determinant line bundle. It is easy to prove that if the index of the $\overline{\partial}$ operator is zero, then one can construct a non vanishing $C^\infty$ section $\det(\overline{\partial})$ of the determinant line bundle $L$ up to a constant whose Quillen norm is exactly the analytic Ray Singer torsion. We will show that knowing the existence of the non vanishing section $\det(\overline{\partial})$ implies that there exists a holomorphic section $\eta^N$ of some power of the determinant line bundle which vanishes on $D_\infty = \mathcal{M}(M) \setminus \mathcal{M}(\overline{M})$, where $\mathcal{M}(M)$ is some projective compactification of $\mathcal{M}(M)$ such that $D_\infty = \mathcal{M}(M) \setminus \mathcal{M}(\overline{M})$ is a divisor with normal crossings. According to Viewheg $\mathcal{M}(M)$ is a quasi projective variety. See [25].

For this program we need the analogue of the variational formulas for the determinant of the Laplacian of a CY metric acting on $(0,q)$ forms. The variational formulas are very important in the construction of the holomorphic section $\eta^N$ mentioned before.

In this paper we will generalize our variational formulas that were proved for the Laplacian of a CY metric acting on functions to $(0,q)$ forms. See [7] and [11].

We discussed the problem of finding the relations between the spectral properties of the Laplacian of CY metric on K3 surfaces in a series of joint papers with J. Jorgenson. (See [1], [2], [3] and [4].) The results of these papers showed that the problem of relating the spectral properties of the Laplacian of CY metric on even dimensional CY manifold is very delicate one. For example, in the case of algebraic polarized K3 surfaces we showed that the determinant of the Laplacian defines the discriminant locus of polarized K3 surfaces for polarization classes $\epsilon$ such that the Baily Borel compactification of the moduli space of algebraic pseudo-polarized K3 surfaces $\Gamma_0 \setminus SO(2,19)/SO(2) \times SO(19)$ contains only one zero dimensional cusp. In the other cases the discriminant locus can not be recovered from the spectral properties of the Laplacians of CY metrics. The difficulties in the even dimensional case are based on the fact that Ray Singer Analytic torsion is zero and the index of the $\overline{\partial}$ is equal to 2. So we can not find a non vanishing canonical section of the determinant line bundle.

The analytic torsion for Enriques surfaces is discussed in [13] from point of view of string theory and in [20] from mathematical point of view. Based on these two papers one should consider the Enriques surfaces from the point of view of the spectral properties of the Laplacian of CY metric as an odd dimensional CY manifolds.

There are relations between the results our results in the series of the two papers and the results of in [4]. Some of these relations are discussed in [22]. The results and the

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1The mistakes that appeared in these papers are corrected in [14].
conjectures stated in these two papers are related to the results in [13], [14] and [16].

This article is organized as follows. In Section II we introduce the basic definition and some notations. In Section III we review Kodaira-Spencer-Kuranishi deformation theory of Calabi Yau manifolds following [21]. In Section IV we introduce some canonical identifications of different Hilbert spaces on a CY manifold. We prove that some operators are of trace class and compute their traces. In Section V we proved that

$$\text{Tr} \left( \exp \left( -t \Delta_q \right) \right) = \binom{n}{q} \text{Tr} \left( \exp \left( -t \Delta_0 \right) \right),$$

where $\Delta_q$ is the Laplacian of the Calabi Yau metric acting on $(0,q)$ forms. Our proof is based on Bochner type technique. In Section VI we formulate and prove the main Theorem of this article, namely that $\log(\det(\Delta_q))$ is a potential for the Weil-Petersson metric.

In Section VII we gave some applications of the technique and results that we used in the previous section. We prove that the coefficients $a_{-k}$ in the short term asymptotic expansion of the trace of the heat kernel of CY metric for $0 \leq k \leq \dim\mathbb{C}M$ are constants. In Section VIII show that Ray Singer analytic torsion of CY metric $I(M)$ is bounded.

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2. Some Remarks, Notations and Preliminary Results.

2.1. Definition of the Regularized Determinant. Let $(M,g)$ be an $n$ dimensional Riemannian manifold. Let $\Delta_q = dd^* + d^*d$ be the Laplacian acting on the space of $q$ forms on $M$. It is a well known fact that the spectrum of the Laplacian $\Delta_q$ is positive and discrete. This means that the non zero eigen values of $\Delta_q$ are $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$. We will define the zeta function of $\Delta_q$ as follows: $\zeta_q(s) = \sum_{i=1}^{\infty} \lambda_i^{-s}$. It is a well known fact that $\zeta_q(s)$ is a well defined analytic function for $\text{Re}(s) \gg C$, it has a meromorphic continuation in the complex plane and 0 is not a pole of $\zeta_q(s)$. Then we define $\det(\Delta_q) = \exp \left( -\frac{d}{ds} \zeta_q(s) \right|_{s=0} \right)$.

2.2. Definitions and Notations. Let $M$ be a $n$-dimensional Kähler manifold with a zero canonical class. Suppose that $H^k(M,\mathcal{O}_M) = 0$ for $1 \leq k < n$. Such manifolds are called Calabi-Yau manifolds. A pair $(M,L)$ will be called a polarized CY manifold if $M$ is a CY manifold and $L \in H^2(M,\mathbb{Z})$ is a fixed class such that it represents the imaginary part of a Kähler metric on $M$.

Yau’s celebrated theorem asserts the existence of a unique Ricci flat Kähler metric $g$ on $M$ such that the cohomology class $[\text{Im}(g)] = L$. From now on we will consider polarized CY manifolds of odd dimension. The polarization class $L$ determines the CY metric $g$ uniquely. We will denote by $\Delta_q = \overline{\partial} \circ \partial + \partial \circ \overline{\partial}$ the associated Laplacians that act on

\[H^1,1(M,\mathbb{R}) = H^2(M,\mathbb{R})\] since $H^2(M,\mathcal{O}_M) = 0$ for CY manifolds.
smooth (0, q) forms on M for 0 ≤ q ≤ n. \(\overline{\nabla}\) is the adjoint operator of \(\nabla\) with respect to the CY metric g. The determinant of these operators \(\Delta_q\), defined through zeta function regularization, will be denoted by \(\det(\Delta_q)\).

The Hodge decomposition theorem asserts that \(\Gamma(M, \Omega^{0,q}) = \text{Im}(\nabla) \oplus \text{Im}(\overline{\nabla'})\) for 1 ≤ q ≤ \(\dim_C M - 1\). The restriction of \(\Delta_q\) on \(\text{Im}(\nabla)\) will be denoted by \(\Delta_q^\nabla = \nabla \circ \overline{\nabla}\), and the restriction of \(\Delta_q\) on \(\text{Im}(\overline{\nabla'})\) will be denoted by \(\Delta_q^\overline{\nabla'} = \overline{\nabla'} \circ \nabla\). Hence we have \(\text{Tr}(\exp(-t\Delta_q)) = \text{Tr}(\exp(-t\Delta_q^\nabla)) + \text{Tr}(\exp(-t\Delta_q^\overline{\nabla'}))\). This implies that \(\zeta_q(s) = \sum_{k=1}^\infty \lambda_k^{-s} = \zeta_q'(s) + \zeta_q''(s)\), where \(\lambda_k > 0\) are the positive eigen values of \(\Delta_q\) and \(\zeta_q'(s) \& \zeta_q''(s)\) are the zeta functions of \(\Delta_q^\nabla\) and \(\Delta_q^\overline{\nabla'}\). From here and the definition of the regularized determinant we obtain that \(\log \det(\Delta_q) = \log \det(\Delta_q^\nabla) + \log \det(\Delta_q^\overline{\nabla'})\). It is a well known fact that the action of \(\Delta_q^\nabla\) is isospectral to the action of \(\Delta_q^\overline{\nabla'}(q+1)\) on \(\text{Im}(\nabla)\), which means that the spectrum of \(\Delta_q\) is equal to the spectrum of \(\Delta_q'(q+1)\). So we have the equality \(\det(\Delta_q^\nabla) = \det(\Delta_q'(q+1))\).

**Notation 2.** Let \(f\) be a map from a set \(A\) to a set \(B\) and let \(g\) be a map from the set \(B\) to the set \(C\), then the compositions of those two maps we will denote by \(f \circ g\).

3. Kodaira-Spencer-Kuranishi Theory for CY

3.1. Basic Definitions. In [20] and [21] was developed the local deformation theory of CY manifolds. We will review the results in [20] and [21] in this section.

Let \(M\) be an even dimensional \(C^\infty\) manifold. We will say that \(M\) has an almost complex structure if there exists a section \(\phi\in \text{Hom}(T^*, T^*)\) such that \(f^2 = -\text{id}\). \(T\) is the tangent bundle and \(T^*\) is the cotangent bundle on \(M\). This definition is equivalent to the following one: Let \(M\) be an even dimensional \(C^\infty\) manifold. Suppose that there exists a global splitting of the complexified cotangent bundle \(T^* \oplus C = \Omega^{1,0} \oplus \Omega^{0,1}\), where \(\Omega^{1,0} = \Omega^{1,0}_{\nabla}\). Then we will say that \(M\) has an almost complex structure. We will say that an almost complex structure is an integrable one, if for each point \(x \in M\) there exists an open set \(U \subset M\) such that we can find local coordinates \(z^1, ..., z^n\), such that \(dz^1, ..., dz^n\) are linearly independent in each point \(m \in U\) and they generate \(\Omega^{1,0}\) on \(U\).

**Definition 3.** Let \(M\) be a complex manifold. Let \(\phi \in \Gamma(M, \text{Hom}(\Omega^{0,1}, \Omega^{1,0}))\), then we will call \(\phi\) a Beltrami differential.

Since \(\Gamma(M, \text{Hom}(\Omega^{1,0}, \Omega^{0,1})) \simeq \Gamma(M, \Omega^{0,1} \otimes T^{1,0})\), we deduce that locally \(\phi\) can be written as follows: \(\phi|_U = \sum \phi^a_\nabla \overline{\partial z}^a \otimes \frac{\partial}{\partial x^a}\). From now on we will denote by

\[
A_\phi = \left( \begin{array}{cc} id & \phi(\tau) \\ \phi(\tau) & id \end{array} \right).
\]

We will consider only those Beltrami differentials \(\phi\) such that \(\det(A_\phi) \neq 0\). The Beltrami differential \(\phi\) defines an integrable complex structure on \(M\) if and only if the following equation holds: \(\overline{\nabla}\phi + \frac{i}{2} [\phi, \phi] = 0\), where

\[
[\phi, \phi]|_U := \sum_{\nu=1}^n \sum_{1 < a < b \leq n} \left( \sum_{\mu=1}^n \left( \phi^{a\nu}_\nabla \left( \partial_\nu \phi^{b\mu}_\nabla \right) - \phi^{b\mu}_\nabla \left( \partial_\nu \phi^{a\nu}_\nabla \right) \right) \overline{dz}^a \wedge \overline{dz}^b \otimes \frac{\partial}{\partial x^a}. \]
3.2. Kuranishi Space and Flat Local Coordinates. Kuranishi proved the following Theorem:

**Theorem 4.** Let \{\phi_i\} be a basis of harmonic (0, 1) forms of \(\mathbb{H}^1(M, T^{1,0})\) on a Hermitian manifold \(M\). Let \(G\) be the Green operator and let \(\phi(\tau^1, ..., \tau^N)\) be defined as follows:

\[
\phi(\tau^1, ..., \tau^N) = \sum_{i=1}^{N} \phi_i \tau^i + \frac{1}{2} \partial \bar{\partial} G[\phi(\tau^1, ..., \tau^N), \phi(\tau^1, ..., \tau^N)],
\]

then there exists \(\varepsilon > 0\) such that if \(\tau = (\tau^1, ..., \tau^N)\) satisfies \(|\tau_i| < \varepsilon\), then \(\phi(\tau^1, ..., \tau^N)\) is a global \(C^\infty\) section of the bundle \(\Omega^{0,1} \otimes T^{1,0}\). (See \[13]\.)

Based on the Theorem 4 we proved in \[21\] the following Theorem:

**Theorem 5.** Let \(M\) be a CY manifold and let \(\{\phi_i\}\) be a basis of harmonic \(0,1\) forms in \(T^{1,0}\) of \(\mathbb{H}^1(M, T^{1,0})\), then the equation: \(\bar{\partial}\phi + \frac{1}{2} |\phi, \phi| = 0\) has a solution in the form:

\[
\phi(\tau^1, ..., \tau^N) = \sum_{i=1}^{N} \phi_i \tau^i + \sum_{|I_N| \geq 2} \phi_{I_N} \tau^{I_N} = \sum_{i=1}^{N} \phi_i \tau^i + \frac{1}{2} \partial \bar{\partial} G[\phi(\tau^1, ..., \tau^N), \phi(\tau^1, ..., \tau^N)],
\]

\(\bar{\partial}\phi(\tau_1, ..., \tau_N) = 0\), where \(I_N = (i_1, ..., i_N)\) is a multi-index, \(\phi_{I_N} \in C^\infty(M, \Omega^{0,1} \otimes T^{1,0})\), \(\tau^{I_N} = (\tau^{i_1})^{i_1} ... (\tau^{i_N})^{i_N}\) and for some \(\varepsilon > 0\) \(\phi(\tau) \in C^\infty(M, \Omega^{0,1} \otimes T^{1,0})\) if \(|\tau^i| < \varepsilon\) and \(i = 1, ..., N\). See \[20\] and \[21\].

It is a standard fact from Kodaira-Spencer-Kuranishi deformation theory that for each \(\tau = (\tau^1, ..., \tau^N)\) as in Theorem 3 the Beltrami differential \(\phi(\tau^1, ..., \tau^N)\) defines a new integrable complex structure on \(M\), i.e. the points of \(K\), where \(K: \{\tau = (\tau^1, ..., \tau^N)\} | |\tau^i| < \varepsilon\) defines a family of operators \(\bar{\partial}_\tau\) on the \(C^\infty\) family \(K \times M \to M\), parametrized by \(K\) and \(\bar{\partial}_\tau\) are integrable in the sense of Newlander-Nirenberg. Moreover it was proved by Kodaira, Spencer and Kuranishi that we get a complex analytic family of CY manifolds \(\pi: X \to K\), where as \(C^\infty\) manifold \(X \simeq K \times M\). The family \(\pi: X \to K\) is called the Kuranishi family. The operators \(\bar{\partial}_\tau\) are defined as follows:

**Definition 6.** Let \(\{U_i\}\) be an open covering of \(M\), with local coordinate system in \(U_i\) given by \(\{z^k\}\) with \(k = 1, ..., n = \dim\mathbb{C}M\). Assume that: \(\phi(\tau^1, ..., \tau^N)|_{U_i}\) is given by:

\[
\phi(\tau^1, ..., \tau^N) = \sum_{j,k=1}^{n} (\phi(\tau^1, ..., \tau^N))_{j}^{k} \partial\bar{\partial} \tau^{j} \otimes \frac{\partial}{\partial \bar{z}^k}.
\]

Then we define \((\bar{\partial}_{\tau_j})_\tau = \frac{\partial}{\partial \tau_j} - \sum_{k=1}^{n} (\phi(\tau^1, ..., \tau^N))_{j}^{k} \frac{\partial}{\partial \bar{z}^k}.

**Definition 7.** The coordinates \(\tau = (\tau^1, ..., \tau^N)\) defined in Theorem 3 will be fixed from now on and will be called the flat coordinate system in \(K\).

3.3. Weil-Petersson Metric. It is a well known fact from Kodaira-Spencer-Kuranishi theory that the tangent space \(T_{\tau,K}\) at a point \(\tau \in K\) can be identified with the space of harmonic (0,1) forms with values in the holomorphic vector fields \(\mathbb{H}^1(M_r, T)\). We will view each element \(\phi \in \mathbb{H}^1(M_r, T)\) as a pointwise linear map from \(\Omega^{(1,0)}_{M_r}\) to \(\Omega^{(0,1)}_{M_r}\). Given \(\phi_1\) and \(\phi_2 \in \mathbb{H}^1(M_r, T)\), the trace of the map: \(\phi_1 \bar{\phi}_2 : \Omega^{(0,1)}_{M_r} \to \Omega^{(0,1)}_{M_r}\) at the point \(m \in M_r\) with respect to the metric \(g\) is simply:

\[
Tr(\phi_1 \bar{\phi}_2)(m) = \sum_{k,l,m,p=1}^{n} (\phi_1)_l^{k} (\bar{\phi}_2)^{\bar{k}}_{m} g^{\bar{p}p} g_{k,m}.
\]
Definition 8. We will define the Weil-Petersson metric on $\mathcal{K}$ via the scalar product:

$$<\phi_1, \phi_2> = \int_M Tr(\phi_1 \overline{\phi_2}) vol(g).$$

We proved in [21] that the coordinates $\tau = (\tau^1, ..., \tau^N)$ as defined in Definition 7 are flat in the sense that the Weil-Petersson metric is Kähler and in these coordinates we have that the components $g_{i\bar{j}}$ of the Weil Petersson metric are given by the following formulas in these coordinates:

$$g_{i\bar{j}} = \delta_{i\bar{j}} + R_{i\bar{j},l} \tau^l \overline{\tau}^k + O(\tau^3).$$

On page 332 of [21] the following results is proved:

Lemma 9. Let $\phi \in \mathcal{H}(M,T)$ be a harmonic form with respect to the CY metric $g$. Let

$$\phi|_U = \sum_{k,l=1}^n \phi_{k,l}(\tau) \frac{\partial}{\partial \tau^k} \otimes \frac{\partial}{\partial \overline{\tau}^l},$$

then $\phi|_{\mathcal{K}} = \sum_{j=1}^n g_{j\bar{j}} \phi^j = \sum_{j=1}^n g_{j\bar{j}} \phi_{j\bar{j}}$.

We will use Lemma 9 to prove the following theorem:

3.4. Infinitesimal Deformation of the Imaginary Part of the Weil-Petersson Metric.

Theorem 10. Near the point $\tau = 0$ of the Kuranishi space $\mathcal{K}$ the imaginary part $\Im(g)$ of the CY metric $g$ has the following expansion in the coordinates $\tau := (\tau^1, ..., \tau^N)$:

$$\Im(g) = \Im(g)(0) + O(\tau^2).$$

PROOF: In [21] we proved that the forms $\theta^k_r = dz^k + \sum_{l=1}^n \phi_r(\tau) dz^l \overline{\tau}^k \overline{T}^l$ ($k = 1, ..., n$) form a basis of $(1,0)$ forms relative to the complex structure defined by $\tau \in \mathcal{K}$ in $U \subset M$.

Let

$$\Im(g_r) = \sqrt{-1} \sum_{1 \leq k \leq l \leq n} g_{k\bar{l}}(\tau,\overline{\tau}) \theta^k_r \wedge \overline{\theta^l_r},$$

and

$$g_{k\bar{l}}(\tau,\overline{\tau}) = g_{k\bar{l}}(0) + \sum_{i=1}^N \left( (g_{k\bar{l}}(1)) \right)_{i} \tau^i + \left( g^l_{k\bar{l}}(1) \right)_{i} \overline{\tau}^j + O(2).$$

Substituting in the expression for $\Im(g_r)$ the expressions for $\theta^k_r$ we get the following formula:

$$\Im(g_r) = \sqrt{-1} \sum_{1 \leq k \leq l \leq n} g_{k\bar{l}}(\tau,\overline{\tau}) \theta^k_r \wedge \overline{\theta^l_r} = \sqrt{-1} \sum_{1 \leq k \leq l \leq n} g_{k\bar{l}}(0) dz^k \wedge dz^l +$$

$$+ \sum_{i=1}^N \tau^i \sqrt{-1} \left( \sum_{1 \leq k \leq l \leq n} \left( g_{k\bar{l}}(1) \right)_{i} dz^k \wedge dz^l + \sum_{m=1}^n (\overline{\phi^m_{k\bar{l}}(1)} - \phi^m_{k\bar{l}}(1)) dz^k \wedge dz^l \right) \wedge \overline{\theta^l_r} +$$

$$+ \overline{\theta^l_r} \wedge \sum_{i=1}^N \tau^i \sqrt{-1} \left( \sum_{1 \leq k \leq l \leq n} \phi^m_{k\bar{l}}(1) dz^k \wedge dz^l + \sum_{m=1}^n \left( \phi^m_{k\bar{l}}(1) - \overline{\phi^m_{k\bar{l}}(1)} \right) dz^k \wedge dz^l \right).$$

From Lemma 9 we conclude that

$$\sum_{m=1}^n (g_{k\bar{l}}(1) - g^m_{k\bar{l}}(1)) dz^k \wedge dz^l = 0$$

and so:

$$\Im(g_r) = \sqrt{-1} \sum_{1 \leq k \leq l \leq n} g_{k\bar{l}}(0) dz^k \wedge dz^l +$$

$$+ \sum_{i=1}^N \tau^i \sqrt{-1} \left( \sum_{1 \leq k \leq l \leq n} \left( g_{k\bar{l}}(1) \right)_{i} dz^k \wedge dz^l \right) +$$

$$+ \sum_{i=1}^N \tau^i \sqrt{-1} \sum_{1 \leq k \leq l \leq n} \phi^m_{k\bar{l}}(1) dz^k \wedge dz^l + O(2).$$
Let us define (1,1) forms $\psi_i$:

$$\psi_i = \sqrt{-1} \left( \sum_{1 \leq k \leq l \leq n} \left( g_{k,l}(1) \right) dz^k \wedge \overline{dz^l} \right).$$

Since

$$\text{Im}(g) = \text{Im}(g_0) + \sum_{i=1}^N \tau^i \psi_i + \sum_{i=1}^N \tau^i \psi_i + O(\tau^2) = \text{Im}(g_0)$$

we deduce that each $\psi_i$ is an exact form, i.e., $\psi_i = \sqrt{-1} \bar{\partial}f_i$, where $f_i$ are globally defined functions on $M$. If we prove that $\psi_i = 0$ our theorem will follow. In [2] we proved that: $\det(g_\tau) = \det(g_0) + O(2)$. From this result we deduce by direct computations that:

$$\det(g_\tau) = \det(g_0) + \sum_{i=1}^N \tau^i \left( \sqrt{-1} \sum_{k,l} g^{1,k} \partial_k \overline{\partial_l}(f_i) \right) + \sum_{i=1}^N \tau^i \text{(complex conjugate)} + O(2).$$

Hence we obtain that for each $i$ we have: $\sum_{k,l} g^{1,k} \partial_k \overline{\partial_l}(f_i) = \Delta(f_i) = 0$, where $\Delta$ is the Laplacian of the metric $g$. From the maximum principle, we deduce that all $f_i$ are constants. Theorem [10] is proved.

4. Hilbert Spaces and Operators of Trace Class.

4.1. Spectral Canonical Identifications of Some Hilbert Spaces.

**Definition 11.** We will denote by $L^2_{0,q-1}(\text{Im}(\partial))$ the Hilbert subspace in $L^2(M, \Omega^{0,q-1})$ which is the $L^2$ completion of the $\partial$ exact forms in $C^\infty(M, \Omega^{0,q-1})$ for $q \geq 1$. In the same manner we will denote by $L^2_{0,q}(\text{Im}(\partial))$ the Hilbert subspace in $L^2(\Omega^{0,q})$ which is the $L^2$ completion of the $\partial$ exact $(0,q)$ forms in $C^\infty(M, \Omega^{0,q})$ for $q \geq 0$ and by $L^2_{1,q-1}(\text{Im}(\partial))$ which is the $L^2$ completion of the $\overline{\partial}$ exact $(1,q-1)$ forms in $C^\infty(M, \Omega^{1,q-1})$. All the completions are with respect to the scalar product on the bundles $\Omega^{p,q}$ defined by the CY metric $g$.

Let $\phi(\tau^1, ..., \tau^N)$ be the solution of the equation $\overline{\partial}\phi(\tau^1, ..., \tau^N) = \frac{1}{2}[\phi(\tau^1, ..., \tau^N), \phi(\tau^1, ..., \tau^N)]$ established in Theorem [4]. From the Definition [3] of the Beltrami differential we know that $\phi(\tau^1, ..., \tau^N)$ defines a linear fibrewise map $\phi(\tau^1, ..., \tau^N) : \Omega^{1,0} \rightarrow \Omega^{0,1}$. So

$$\phi(\tau^1, ..., \tau^N) \in C^\infty(M, \text{Hom}(\Omega^{1,0}, \Omega^{0,1}).$$

We define the following linear map between the vector bundles $\phi \wedge id : \Omega^{1,q-1} \rightarrow \Omega^{0,q}$ as $\phi(dz^1 \wedge \alpha) = \phi(dz^1) \wedge \alpha$.

**Definition 12.** For each $1 \leq q \leq n$, $\phi \wedge id$ defines a natural operator $F(q, \phi)$ between the Hilbert spaces $L^2(M, \Omega^{1,q-1})$ and $L^2(M, \Omega^{0,q})$.

**Definition 13.** The restriction of the map $F(q, \phi)$ on the subspace $\text{Im}(\partial) \subset L^2(M, \Omega^{1,q-1})$ to $\text{Im}(\partial) \subset L^2(M, \Omega^{0,q})$ will be denoted by $F'(q, \phi)$.

**Lemma 14.** The Hilbert subspaces $L^2_{0,q-1}(\text{Im}(\partial))$, $L^2_{0,q}(\text{Im}(\partial))$ and $L^2_{1,q-1}(\text{Im}(\partial))$ are invariant with respect to the Laplacians.
Lemma 15. The forms \( \{\omega_i(0, q - 1)\}_{i = 1, \ldots, \infty} \) all the eigen forms of the Laplacian \( \Delta_{q - 1} \) in the Hilbert space \( L^2_{0, q - 1}(\text{Im}(\overline{\partial})) \) with norm equal to one. In the same way we will denote by \( \{\omega_i(0, q)\}_{i = 1, \ldots, \infty} \) all the eigen forms of the Laplacian \( \Delta_q \) with norm one in the Hilbert space \( L^2_{0, q}(\text{Im}(\overline{\partial})) \) and by \( \{\omega_i(1, q - 1)\}_{i = 1, \ldots, \infty} \) all the eigen forms of norm one of the Laplacian \( \Delta_q \) in the Hilbert space \( L^2_{1, q - 1}(\text{Im}(\overline{\partial})) \).

PROOF: The proof of this lemma is standard fact from Kähler geometry. The first two identities followed from directly from the definition of the Laplacian. The last equality follows from the fact that in Kähler geometry the Laplacians \( \Delta_{q - 1} = \overline{\partial}_{q - 1} \partial_q + \partial_{q - 1} \overline{\partial}_q \) and \( \Delta_q = \partial_q - \partial^*_q + \partial^*_q \partial_q \) coincide, \( \overline{\partial}_q = [\Lambda, \partial_q] \) and \( \partial^*_q = [\Lambda, \overline{\partial}_q] \). See [15] and [19]. Our lemma is proved.

Let us denote by \( \{\omega_i(0, q - 1)\}_{i = 1, \ldots, \infty} \) all the eigen forms of the Laplacian \( \Delta_{q - 1} \) in the Hilbert space \( L^2_{0, q - 1}(\text{Im}(\overline{\partial})) \) with \( \Delta_{q - 1} \omega_i(0, q - 1) = \lambda_i (\omega_i(0, q - 1)) \). Then

\[
\text{PROOF:} \quad \lambda_i (\omega_i(0, q - 1)) = \omega_i(0, q) \quad \text{if this trace exists}
\]

\[
\text{acting on the identified Hilbert spaces}
\]

Lemma 16. Let \((M, g)\) be a Kähler manifold with a Kähler metric \( g \). Let

\[
(L^2_{0, q - 1}(\text{Im}(\overline{\partial})), \{\omega_i(0, q - 1)\}, (L^2_{0, q}(\text{Im}(\overline{\partial})), \{\omega_i(0, q)\})
\]

and \((L^2_{1, q - 1}(\text{Im}(\partial)), \{\omega_i(1, q - 1)\})\) be the Hilbert spaces with orthonormal bases defined in Definition 11 for \( q \geq 1 \). Then

\[
\text{PROOF:} \quad \lambda_i (\omega_i(0, q - 1)) = \omega_i(0, q) \quad \text{if this trace exists}
\]

\[
\text{acting on the identified Hilbert spaces}
\]

Remark 17. Lemma 16 gives a natural identification of the Hilbert spaces \( L^2_{0, q - 1}(\text{Im}(\overline{\partial})), L^2_{0, q}(\text{Im}(\overline{\partial})) \) and \( L^2_{1, q - 1}(\text{Im}(\partial)) \) because we can choose natural bases of all these Hilbert spaces by choosing an orthonormal basis consisting of eigen forms of the Laplacians. We are using the following orthonormal bases to get the above identifications:

\[
\{\omega_i(0, q)\}, \{\frac{\partial (\omega_i(0, q))}{\|\partial (\omega_i(0, q))\|} := e_i\} \quad \text{and} \quad \{\frac{\overline{\partial} (\omega_i(0, q))}{\|\overline{\partial} (\omega_i(0, q))\|} := f_i\}.
\]

4.2. Trace Class Operators in Hilbert Spaces. We will define the trace of the operator \( F^1(q, \phi) : L^2_{0, q - 1}(\text{Im}(\overline{\partial})) \to L^2_{0, q}(\text{Im}(\overline{\partial})) \) if this trace exists on the identified Hilbert spaces as the usual trace of an operator acting on a Hilbert space. For example we define the trace of the operator \( F^1(q, \phi) \) with respect to the orthonormal bases

\[
\{\frac{\partial (\omega_i(0, q))}{\|\partial (\omega_i(0, q))\|} := \omega_i(1, q) := e_i\} \quad \text{and} \quad \{\frac{\overline{\partial} (\omega_i(0, q))}{\|\overline{\partial} (\omega_i(0, q))\|} := \omega_i(0, q + 1) := f_i\}.
\]
Theorem 18. Let $F'(q,\phi)$ be defined as in Definition 13 then $F'(q,\phi)$ are operators of trace class.

**PROOF:** From the Definition 13 of the operators $F'(q,\phi)$ we know that they are induced by the fibrewise linear maps $\phi \wedge \text{id} : \Omega^{1,q-1} \to \Omega^{0,q}$.

Since $M$ is a compact manifold we can choose $N_{1,q-1}$ global $C^\infty$ forms $\psi_i$ of type $(1,q-1)$ such that they span at each point $y \in M$, the space $\Omega_y^{1,q-1}$. In the same way we can find $N_{0,q}$ forms $\sigma_j$ of type $(0,q)$ such that they span at each point $y \in M$, the space $\Omega_y^{1,q-1}$. Without lost of generality we may assume that both $\psi_i$ and $\sigma_j$ are linearly independent vectors in the identified Hilbert spaces $L^2_{1,q-1}(\text{Im}(\partial))$ & $L^2_{0,q}(\text{Im}(\partial))$.

Then the maps $F(q,\phi): L^2_{1,q-1}(\text{Im}(\partial)) \to L^2_{0,q}(\text{Im}(\partial))$ are given by $N_{1,q-1} \times N_{0,q}$ matrix. So the maps $F(q,\phi)$ are linear operators between finite dimensional spaces therefore they are of trace class. Since $F'(q,\phi)$ are the restriction of the trace class operators $F(q,\phi)$, we deduce that $F'(q,\phi)$ are of trace class too. Theorem 18 is proved. $\blacksquare$.

Corollary 19. The operator $\overline{\partial}^{-1} \circ F'(q,\phi) \circ \partial$ is of trace class.

**PROOF:** We have identified the Hilbert spaces $L^2_{0,q-1}(\text{Im}(\overline{\partial})), L^2_{0,q}(\text{Im}(\overline{\partial}))$ and $L^2_{1,q-1}(\text{Im}(\partial))$ in Remark 17.

The operators $\overline{\partial}^{-1} \circ F'(q,\phi) \circ \partial$ which act on $L^2_{0,q-1}(\text{Im}(\overline{\partial}))$ can be considered as a composition of a differential operator, operators with a smooth kernel and integral operator by using the above identification. From Proposition 2.45 page 96 in the book [3] it follows directly that the operator $\overline{\partial}^{-1} \circ F'(q,\phi) \circ \partial$ is of trace class. Cor. 19 is proved. $\blacksquare$.

Theorem 20. For $t > 0$ and $q \geq 1$ the following equality holds

$$Tr \left( \exp \left( -t(\Delta_{q-1}) \right) \right) \circ F'(q,\phi) \circ \partial = Tr \left( \exp \left( -t(\Delta'_q) \right) \right) = \sum_{i=1}^\infty \exp(-t\lambda_i) a_{ii},$$

where $\lambda_i$ are eigen values of $\Delta_{q-1}$ and we have the following expression for the trace:

$$Tr \left( F'(q,\phi) \right) = \sum_{i=1}^\infty a_{ii}$$

in the orthonormal bases consisting of eigen vectors of the corresponding Laplacians as defined in Lemma 17.

**PROOF:** Theorem 18 and Corollary 19 imply that the operators $\overline{\partial}^{-1} \circ F'(q,\phi) \circ \partial$ and $F'(q,\phi)$ are of trace class. The proof of this theorem is based on the direct computation of the traces of the operators $\overline{\partial}^{-1} \circ F'(q,\phi) \circ \partial$ and $F'(q,\phi)$ with respect to the standard bases of orthonormal vectors

$$\left\{ \frac{\partial \omega_i(0,q-1)}{\| \partial \omega_i(0,q-1) \|^2} = \omega_i(1,q-1) \right\} \text{ and } \left\{ \frac{\overline{\partial} \omega_i(0,q-1)}{\| \overline{\partial} \omega_i(0,q-1) \|^2} = \omega_i(0,q) \right\},$$

where $\Delta_{q-1}(\omega_i(0,q-1)) = \lambda_i \omega_i(0,q-1)$, $\Delta_q \omega_i(1,q-1) = \lambda_i \omega_i(1,q-1)$ and $\Delta_q \omega_i(0,q) = \lambda_i \omega_i(0,q).$ (See Lemma 17). Let
\[ F'(q, \phi)(\omega_i(1, q - 1)) = \sum_{j=1}^{\infty} a_{ij}(\omega_j(0, q)). \]

**Lemma 21.** We have the following formula:

\[ \text{Tr } (\overline{\partial}^{-1} \circ F'(q, \phi) \circ \partial) = \text{Tr}(F'(q, \phi)) = \sum_{i=1}^{\infty} a_i \text{ and } q \geq 1. \]

**Proof:** The operator \( \overline{\partial}^{-1} \circ F'(q, \phi) \circ \partial \) act on the Hilbert space \( L^{2}_{0, q-1}(\text{Im}(\overline{\partial})) \) with an orthonormal basis of non zero eigen vectors of the Laplacian \( \Delta_{q-1} \{ \omega_i(0, q - 1) \} \). Recall that \( \| \partial \omega_i(0, q - 1) \| = \| \overline{\partial} \omega_i(0, q - 1) \| = \sqrt{\lambda_i} \). So we have

\[ \partial \omega_i(0, q - 1) = \sqrt{\lambda_i} \omega_i(1, q - 1) \text{ and } \overline{\partial} \omega_i(0, q - 1) = \sqrt{\lambda_i} \omega_i(0, q). \]

From the expression \( F'(q, \phi)(\omega_i(1, q - 1)) = \sum_{j=1}^{\infty} a_{ij}(\omega_j(0, q)) \) and above equalities we obtain the following formula for the matrix of the operator \( \overline{\partial}^{-1} \circ F'(q, \phi) \circ \partial \) in the basis \( \{ \omega_i(0, q - 1) \} \)

\[ \overline{\partial}^{-1} \circ F'(q, \phi)(\partial(\omega_i(0, q - 1))) = \sqrt{\lambda_i} \sum_{j=1}^{\infty} a_{ij}(\omega_j(0, q)) = (\sqrt{\lambda_i}) \sum_{j=1}^{\infty} a_{ij}(\overline{\partial}^{-1}(\overline{\partial} \omega_j(0, q))). \]

Substituting in the last formula the expression \( \overline{\partial} \omega_j(0, q - 1) = \omega_j(0, q) \) we obtain that

\[ \text{Tr } (\overline{\partial}^{-1} \circ F'(q, \phi) \circ \partial) = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \sqrt{\lambda_i} a_{ij}(\omega_j(0, q - 1), \omega_i(0, q - 1)) \right) = \sum_{i=1}^{\infty} a_i \sum_{j=1}^{\infty} \left( \text{Tr}(F'(q, \phi)(\omega_i(0, q)), \omega_j(0, q - 1)) \right). \]

On the other hand we know from Theorem 18 that the operator \( F'(q, \phi) \) is of trace class. From the canonical identifications of the Hilbert spaces \( L^{2}_{1, q-1}(\text{Im}(\partial)) \) and \( L^{2}_{0, q}(\text{Im}(\overline{\partial})) \) by the orthonormal eigen forms with a non zero eigen forms we deduce that

\[ \text{Tr}(F'(q, \phi)) = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \sqrt{\lambda_i} a_{ij}(\omega_j(0, q - 1), \omega_i(0, q)) \right) = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij}(\omega_j(0, q), \omega_i(0, q)) \right). \]

**Lemma 21** is proved. \( \square \)

**The End of the Proof of Theorem 20:** The formulas \( \triangle_{q-1}^\omega \omega_j(0, q - 1) = \lambda_i \omega_j(0, q - 1) \) imply that

\[ \exp(-t\lambda_j) \omega_j(0, q - 1) = \exp(-t\lambda_j) \omega_j(0, q - 1). \]

From the expression \( \overline{\partial}^{-1} \circ F'(q, \phi) \circ \partial(\omega_i(0, q - 1)) = \sum_{j=1}^{\infty} a_{ij} \frac{\sqrt{\lambda_j}}{\sqrt{\lambda_i}} \omega_j(0, q - 1) \) proved in Lemma 21 we deduce:
Theorem 24. We have the following formula:

\[
\sum_{i=1}^{\infty} \left( \exp(-t(\Delta_q-1) \circ \partial^{-1} \circ F'(q, \phi) \circ \partial) \right) = \\
\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \exp(-t(\lambda_j) a_{ij} \sqrt{\lambda_i} \omega_j(0, q-1)), \omega_i(0, q-1) \right) = \\
\sum_{i=1}^{\infty} a_{ii} \exp(-t\lambda_i).
\]

So we obtain \( Tr(\exp(-t(\Delta_q') \circ \partial^{-1} \circ F'(q, \phi) \circ \partial) \mid \tau=0) = \sum_{i=1}^{\infty} a_{ii} \exp(-t\lambda_i). \) From the expression \( \Delta_q'(\partial \omega_i(0, q-1)) = \lambda_i (\partial \omega_i(0, q-1)) \) we obtain that \( \exp(-t\Delta_q') \omega_i(1, q-1) = \exp(-\lambda_i t) \omega_i(1, q-1). \) From the formula \( F'(q, \phi)(\omega_i(1, q-1)) = \sum_{j=1}^{\infty} a_{ij} (\omega_j(0, q)) \) we conclude:

\[
\left( \exp(-t\Delta_q') \circ F'(q, \phi) \right) (\omega_i(1, q-1)) = \sum_{j=1}^{\infty} a_{ij} \exp(-\lambda_j t) \omega_i(0, q)
\]

and so

\[
Tr \left( \exp(-t\Delta_q') \circ F'(q, \phi) \right) = \sum_{j=1}^{\infty} a_{ii} \lambda_i \exp(-\lambda_i t) = \\
Tr \left( \exp(-t\Delta_q-1) \circ \partial^{-1} \circ F'(q, \phi) \circ \partial \right).
\]

Theorem 24 is proved. \( \blacksquare \)

Corollary 22. We have the following formula:

\[
Tr \left( \exp(-t(\Delta_q-1) \circ \partial^{-1} \circ F'(q, \phi) \circ \partial) \right) Tr \left( \exp(-t\Delta_q') \circ F'(q, \phi) \right) = \\
Tr \left( \exp(-t\Delta_q') \circ F'(q, \phi) \right) = \sum_{j=1}^{\infty} a_{ii} \exp(-\lambda_i t).
\]

Corollary 23. \( a_{ii} \) tends to zero with \( i \to \infty \) exponentially fast.

Repeating the arguments that we used to prove Theorem 20 we get the following Theorem:

Theorem 24. For \( t > 0 \) and \( q \geq 1 \) the following equality holds

\[
Tr \left( \exp \left( -t(\Delta_q-1) \circ \partial^{-1} \circ F'(q, \phi_j) \circ \partial \right) \right) = \\
Tr \left( \exp \left( -t\Delta_q' \circ F'(q, \phi_j) \circ \partial \right) \right) = \sum_{j=1}^{\infty} c_{ii} \exp(-\lambda_j t)
\]

where \( Tr \left( F'(q, \phi_j) \circ F'(q, \phi) \right) = \sum_{j=1}^{\infty} c_{ii} \).

5. Bochner’s Formulas for CY manifolds.

We will now give explicit expression for Ray-Singer torsion using 24. In order to use 27 we need to have information about the relations between \( Tr(\exp(-\Delta_q)) \) and \( Tr(\exp(-\Delta_0)). \) We will use Bochner technique to find these relations.

Theorem 25. \( Tr(\exp(-t\Delta_q)) = \binom{n}{q} Tr(\exp(-t\Delta_0)). \)
PROOF: In order to prove Theorem 25 we will use the following formulas proved in [15] on page 119: Let $M$ be a Kähler manifold and let $\Delta$ be the Laplacian of a Kähler metric defined on $(p,q)$ form
\[
\phi = \frac{1}{p!q!} \sum \phi_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q},
\]
then
\[
(\Delta \phi)_{i_1,\ldots,i_p,j_1,\ldots,j_q} = -\sum_{l,j} g^{i_l,j} \nabla_{i_l} \phi_{i_1,\ldots,i_{l-1},j_1,\ldots,j_{q-1}} +
\sum_k \sum_l \sum_{m,n} R_{i_k,j_l}^{\overline{m},\overline{n}} \phi_{i_1,\ldots,i_{k-1},j_1,\ldots,j_{q-1}} -
\sum_{k=1}^n \sum_{m,n} R_{\overline{m},\overline{n}} \phi_{i_1,\ldots,j_{k-1},\overline{m},\overline{n},j_{k+1},\ldots,j_q},
\]
where $R_{i,j,k,l}$ is the curvature of the Kähler metric $g$, $\nabla_j = \partial_j$ and $\nabla_i$ is the covariant derivative in the direction $\partial/\partial z^i$ and
\[
R_{\overline{m},\overline{n}} = \sum_{k=1}^n g^{m,k} R_{k,\overline{n}}.
\]
where $R_{k,\overline{n}}$ is the Ricci curvature. If $M$ is a CY manifold and $g$ is a CY metric, then $R_{k,\overline{n}} = 0$. When $\phi$ is a form of type $(0,q)$, then from the above mentioned formulas we obtain that:
\[
(\Delta \phi)_{j_1,\ldots,j_q} = -\sum_{m,n} g^{\overline{m},n} \nabla_n \phi_{j_1,\ldots,j_q}.
\]
On page 110 in [15] the following formula is proved:
\[
(\overline{\partial} \phi)_{j_1,\ldots,j_q} = -(-1)^p \sum_{m,n} g^{\overline{m},n} \nabla_n \phi_{j_1,\ldots,j_q}.
\]
Using all these formulas we get
\[
(\Delta \phi)_{j_1,\ldots,j_q} = \overline{\partial} \overline{\partial} (\phi_{j_1,\ldots,j_q}) = \Delta_0 (\phi_{j_1,\ldots,j_q}).
\]
From here Theorem 25 follows directly, i.e. $Tr(\exp(-\Delta_q)) = (\binom{n}{q}) Tr(\exp(-\Delta_0))$. Our Theorem is proved. ■

6. Variational Formulas.
Let $<\phi_i, \phi_j>$ be defined as in Definition 8, then

**Theorem 26.** The following variational formulas hold for CY manifolds of complex dimension $n \geq 2$:

i. $-\left(\frac{\partial^2}{\partial \tau_i \partial \tau_j} \log(\det(\Delta_q^0))(0) = \binom{n-1}{q-1} <\phi_i, \phi_j> \right)$ for $1 \leq q \leq n - 1$.

ii. $-\left(\frac{\partial^2}{\partial \tau_i \partial \tau_n} \log(\det(\Delta_q^0))(0) = <\phi_i, \phi_j> \right)$ for $q = 1$ or $n$. 
6.1. Ideas of the Proof. Let \( q \geq 1 \). The proof of Theorem 27 is based on the fact that \( \zeta_{q-1}(s) \) is the Mellin transform of \( Tr \exp(-t(\Delta_{q-1}^\mu)) \), i.e. we have

\[
\zeta_{q-1}(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Tr \exp(-t(\Delta_{q-1}^\mu))) t^{s-1} dt.
\]

The definition of \( det(\Delta_{q-1}^\mu) = -\frac{d}{ds} (\zeta_{q-1}(s))_{|s=0}, \) and the power series expansion of zeta function \( \zeta_{q-1}(s) = \zeta_{q-1}(0) + \frac{d}{ds} (\zeta_{q-1}(0)) s + \ldots \) suggest that in order to compute \( \frac{\partial^2}{\partial \tau_i \partial \tau_j} \left( \det \left( \log(\Delta_{q-1}) \right) \right) \big|_{\tau = 0 \atop s = 0} \) we need to compute

\[
\frac{d}{ds} \left( \frac{\partial^2}{\partial \tau_i \partial \tau_j} \frac{1}{\Gamma(s)} \int_0^\infty Tr \exp(-t(\Delta_{q-1}^\mu)) t^{s-1} dt \right) \big|_{s=0, \tau=0}.
\]

First we will compute \( \frac{d}{ds} \left( \frac{\partial^2}{\partial \tau_i \partial \tau_j} \frac{1}{\Gamma(s)} \int_0^\infty Tr \exp(-t(\Delta_{q-1}^\mu)) t^{s-1} dt \right) \big|_{s=0, \tau=0} \) and will prove that

\[
\frac{d}{ds} \left( \frac{\partial^2}{\partial \tau_i \partial \tau_j} \frac{1}{\Gamma(s)} \int_0^\infty Tr \exp(-t(\Delta_{q-1}^\mu)) t^{s-1} dt \right) \big|_{s=0, \tau=0} = \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty Tr \left( \exp(-t(\Delta_{q-1}^\mu) \circ \mathcal{D}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) t^{s-1} dt \right).
\]

From the last formula we will obtain that:

\[
\frac{d}{ds} \left( \frac{\partial^2}{\partial \tau_i \partial \tau_j} \frac{1}{\Gamma(s)} \int_0^\infty Tr \exp(-t(\Delta_{q-1}^\mu)) t^{s-1} dt \right) \big|_{s=0, \tau=0} = -\lim_{t \to 0} Tr \left( \exp(-t(\Delta_{q-1}^\mu) \circ \mathcal{D}^{-1} \circ F'(q, \phi_i) \circ \partial) \right).
\]

Direct computation will show that:

\[
\lim_{t \to 0} Tr \left( \exp(-t(\Delta_{q-1}^\mu) \circ \mathcal{D}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) = \lim_{t \to 0} Tr \left( \exp(-t(\Delta_{q-1}^\mu) \circ \mathcal{D}^{-1} \circ F'(q, \phi_j) \circ \partial) \right) = Tr \left( \overline{F'(q, \phi_j) \circ F'(q, \phi_i) \circ \partial} \right) = \langle \phi_i, \phi_j \rangle.
\]

6.2. Preliminary Results.

Lemma 27. The following formulas are true:

\[
\frac{\partial}{\partial \tau_i} (\overline{\partial}) \big|_{\tau=0} = -\overline{F'(q, \phi_i) \circ \partial}.
\]

**PROOF:** From the expression in Definition 27 we conclude that

\[
\delta'_i (\overline{\partial}) = \frac{\partial}{\partial \tau_i} (\overline{\partial}) = \frac{\partial}{\partial \tau_i} (\overline{\partial}) - \sum_{m=1}^{N} (\tau^m \sum_{k=1}^{N} (\phi_m \delta_{2^k} + O(\tau^2)).
\]

So

\[
\frac{\partial}{\partial \tau_i} (\overline{\partial}) \big|_{\tau=0} = -\sum_{k=1}^{N} (\phi_i \delta_{2^k}.
\]
Lemma 27 follows directly from this expression and the Definition 13 of $F'(q, \phi_i)$. ■

Lemma 28. \( \frac{\partial}{\partial \tau_i} (\overline{\mathcal{D}}) \big|_{\tau=0} = 0. \)

**PROOF:** We know from Kähler geometry that \((\partial \tau_i)^* = [\Lambda, \partial \tau_i]\), where \(\Lambda\) is the contraction with \((1,1)\) vector filed:

\[
\frac{\sqrt{-1}}{2} \sum_{k,l=1}^{n} e^{n} (\theta^k_i)^* \wedge (\overline{\theta^l})^*.
\]

on \(M_\tau\) and \((\theta^k_i)^*\) is \((1,0)\) vector field on \(M_\tau\) dual to the \((1,0)\) form \(\theta^k_i = dz^k + \sum_{j=1}^{N} \tau^j (\sum_{k=1}^{n} (\phi_j) \overline{\tau^k})\). Theorem 26 implies \(\frac{\partial}{\partial \tau_i} (\Lambda) \big|_{\tau=0} = 0\). On the other hand \(\partial \tau_i\) depends antiholomorphically on \(\tau\), i.e. it depends on \(\overline{\tau} = (\overline{\tau_1}, \ldots, \overline{\tau_N})\). So we deduce that:

\[
\frac{\partial}{\partial \tau_i} ((\overline{\mathcal{D}})^* \big|_{\tau=0} = \left( [\frac{\partial}{\partial \tau_i} (\Lambda), \partial \tau_i] + [\Lambda, \frac{\partial}{\partial \tau_i} (\partial \tau_i)] \right) \big|_{\tau=0} = 0.
\]

Lemma 28 is proved. ■

6.3. **Computation of holomorphic derivative.**

**Theorem 29.** The following formula is true

\[
\frac{\partial}{\partial \tau_i} (\text{Tr}(\exp (-t \Delta^\tau_{\tau,q-1})) \big|_{\tau=0} = -t \text{Tr} \left( \frac{d}{dt} (\exp (-t \Delta^\tau_{\tau,q-1})) \circ \overline{\mathcal{D}}^{-1} \circ F'(q, \phi_i) \circ \partial \tau \right) \big|_{\tau=0}.
\]

**PROOF:** Direct computations show that:

\[
\frac{\partial}{\partial \tau_i} (\text{Tr}(\exp (-t \Delta^\tau_{\tau,q-1})) = -t \exp \left( (-t \Delta^\tau_{\tau,q-1}) \circ \frac{d}{dt} (\Delta^\tau_{\tau,q-1}) \right) \big|_{\tau=0}.
\]

Lemma 27 and 28 imply that

\[
\frac{d}{d\tau_i} (\Delta^\tau_{\tau,q-1}) \big|_{\tau=0} = \left( \overline{\mathcal{D}}^{-1} \circ \frac{d}{d\tau_i} \overline{\mathcal{D}} \right) \big|_{\tau=0} = \left( \overline{\mathcal{D}} \circ F'(q, \phi_i) \circ \partial \right) \big|_{\tau=0}.
\]

Since for CY manifolds of complex dimension \(\geq 3\) the operators \(\overline{\mathcal{D}}\) give isomorphisms between the spaces of non constant functions on \(M_\tau\) \(C^\infty(M_\tau)/\mathbb{C}\) and the space of \(C^\infty\) \(\overline{\mathcal{D}}\) closed \((0,1)\) forms on \(M_\tau\). So we have that for \(\overline{\mathcal{D}}\) is well defined on the space of \(C^\infty\) \(\overline{\mathcal{D}}\) closed \((0,1)\) forms on \(M_\tau\) and we have the following formula on \(\text{Im} \overline{\mathcal{D}}\):

\[
\overline{\mathcal{D}}^{-1} = (\Delta^\tau_{\tau,q-1}) \circ \overline{\mathcal{D}}^{-1}.
\]

Using all this information we get by direct substitutions that:

\[
\frac{\partial}{\partial \tau_i} (\text{Tr}(\exp (-t \Delta^\tau_{\tau,q-1})) \big|_{\tau=0} = -t \text{Tr} \left( \exp \left( (-t \Delta^\tau_{\tau,q-1}) \circ \frac{d}{dt} (\overline{\mathcal{D}}^{-1} \circ \overline{\mathcal{D}}) \right) \right) \big|_{\tau=0} = -t \text{Tr} \left( \exp (-t \Delta^\tau_{q-1}) \circ \overline{\mathcal{D}}^{-1} \circ F'(q, \phi_i) \circ \partial \right) \big|_{\tau=0} = t \text{Tr} \left( (-t \Delta^\tau_{q-1}) \circ \overline{\mathcal{D}}^{-1} \circ F'(q, \phi_i) \circ \partial \right) \big|_{\tau=0}.
\]

Theorem 29 is proved. ■
Lemma 30. The following formula is true:
\[
\frac{\partial}{\partial s} \left( \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr} \exp(-t(\Delta^\tau_{q-1})t^{s-1}dt) \right) |_{\tau=0} = \frac{s}{\Gamma(s)} \int_0^\infty \text{Tr} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) t^{s-1}dt.
\]

PROOF: Theorem 29 imply that we have:
\[
\frac{\partial}{\partial s} \left( \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr} \exp(-t(\Delta^\tau_{q-1})t^{s-1}dt) \right) |_{\tau=0} = \frac{s}{\Gamma(s)} \int_0^\infty \text{Tr} \left( \frac{d}{dt} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) \right) t^{s-1}dt.
\]

Taking into account that Theorem 20 implies that
\[
\lim_{t \to 0} \left( \text{Tr} \left( \frac{d}{dt} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) \right) \right) = \lim_{t \to 0} (-t \sum_{i=1}^{\infty} \exp(-\lambda_i t) \lambda_i a_{ii}) = 0
\]

and by integrating by parts as in [5] we derived the formula stated in Lemma 30. Lemma 30 is proved. ■.

Corollary 31. The following formula is true:
\[
\frac{d}{ds} \left( \frac{\partial}{\partial \tau} (\zeta_{q-1}(s)) \right) |_{s=0} = \lim_{t \to 0} \text{Tr} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right).
\]

PROOF: Lemma 30 implies:
\[
\frac{d}{ds} (\zeta_{q-1}(s)) = \frac{s}{\Gamma(s)} \int_0^\infty \text{Tr} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) t^{s-1}dt.
\]

We already computed the trace of the operator \( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \) so we obtain:
\[
\lim_{t \to 0} \text{Tr} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) = \lim_{t \to 0} \sum \exp(-t \lambda_i) \lambda_i a_{ii} = \sum \lambda_i a_{ii} < \infty.
\]

From the fact that \( \frac{s}{\Gamma(s)} = s^2 + O(s^3) \) and direct easy computations we conclude that
\[
\frac{d}{ds} \left( \frac{\partial}{\partial \tau} (\zeta_{q-1}(s)) \right) |_{s=0} = \lim_{t \to 0} \text{Tr} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right).
\]

Corollary 31 is proved. ■.

6.4. Computation of the Antiholomorphic Derivative. Corollary 31 implies that we need to compute the antiholomorphic derivative \( \frac{\partial}{\partial \tau} \text{Tr} \left( \exp(-t(\Delta^\tau_{q-1}) \circ \overline{\Omega}^{-1} \circ F'(q, \phi_i) \circ \partial) \right) \) in order to finish the proof of Theorem 29. The computations of the antiholomorphic derivative are based on the arguments of Quillen as modified in [5].

Definition 32. We define the function \( k^\#_r(w, z, t) \) in a neighborhood of the diagonal \( M \) in \( MxM \) as follows: Let \( \rho_r \) be the injectivity radius on \( M_r \). Let \( d_r(w, z) \) be the distance between the points \( w \) and \( z \) on \( M_r \) with respect to CY metric \( g_r \) and let \( \mathcal{P}_r(w, z)(q) \) be the parallel transport of the bundle \( \Omega^{0,q+1}_{\mathcal{P}} \) along the minimal geodesic joining the point \( w \) and \( z \) with respect to the Levi Civita connection of the CY metric. We suppose that \( |r| < \varepsilon \). Let \( \delta \) be such that \( \delta > \rho_r \). Then we define the function \( k^\#_r(w, z, t) \) as a \( C^\infty \) function using partition of unity as follows:
Let $k_t^\#(w,z,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(w,z)}{4t}\right) \mathcal{P}_\tau(w,z)(q) & \text{if } d_\tau(w,z) < \rho_\tau \\ 0 & \text{if } d_\tau(w,z) > \delta. \end{cases}$

It was proved in [3] on page 87 that we can represent the operator $\exp(-t\Delta_{\tau,q})$ by an integral kernel $k_t(w,z,\tau)$ where

$$k_t(w,z,\tau) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(w,z)}{4t}\right) (\mathcal{P}_\tau(w,z)(q) + O(t)).$$

We will denote by $M_\Delta \subset M \times M$ the diagonal in $M \times M$. Following the arguments from page 258 of [3], we will prove the following theorem:

**Theorem 33.** The following formula holds:

$$\frac{\partial}{\partial \tau} \log (\det \Delta_{\tau,q}^\tau) |_{\tau=0} = \lim_{t \to 0} \int_M \left( \text{Tr} \left( k_t^\#(w,z,t) \big|_{\text{Im}(\partial)} \right) \circ F'(q,\phi_i) \right) |_{\tau=0} \text{vol}(g(0)).$$

**Proof:** An easy calculation, using the fact that \(\frac{\partial}{\partial \tau}(F'(\phi,q))|_{\tau=0} = 0\), Theorem 20, and the definition of $\varepsilon(w,z,t) = \exp(-\Delta_q) - k_0^\#(w,z,t)$, show that on the diagonal of $M \times M$ for $t > 0$ we have

$$\frac{\partial}{\partial \tau} \log (\det \Delta_{\tau,q}^\tau) |_{\tau=0} = \lim_{t \to 0} \left( \text{Tr} \left( \exp(-t(\Delta_q)) \big|_{\text{Im}(\partial)} \right) \circ F'(q,\phi_i) \right) \text{vol} = \lim_{t \to 0} \left( \int_M \left( \text{Tr} \left( k_t^\#(w,z,t) \big|_{\text{Im}(\partial)} \right) \circ F'(q,\phi_i) \right) \text{vol} \right) -$$

$$\lim_{t \to 0} \int_M \left( \exp(-\varepsilon_0(w,z,t)) \big|_{\text{Im}(\partial)} \circ F'(q,\phi_i) \right) \text{vol}.$$

On the other hand, the definition of $k_t^\#(w,z,t)$ implies that $\exp(-\Delta_q) - k_0^\#(w,z,t) = \varepsilon_0(w,z,t)$ is bounded and tends to zero away from the diagonal, as $t$ tends to zero. From here we deduce that

$$\lim_{t \to 0} \int_M \text{Tr} \left( \varepsilon_0(w,z,t) \big|_{\text{Im}(\partial)} \right) \circ F'(q,\phi_i) \text{vol} = 0.$$

uniformly in $z$. Thus, to calculate the limit

$$\lim_{t \to 0} \int_M \text{Tr} \left( \exp(-t(\Delta_q)) \big|_{\text{Im}(\partial)} \right) \circ F'(q,\phi_i) |_{\tau=0} \text{vol}$$

we may replace the Heat kernel $\exp(-t\Delta_{\tau,q})$ by its explicit approximation $k_t^\#(w,z,t)$. So we may deduce that

$$\frac{\partial}{\partial \tau} \log (\det \Delta_{\tau,q}) |_{\tau=0} = \lim_{t \to 0} \int_M \text{Tr} \left( k_t^\#(w,z,t) \big|_{\text{Im}(\partial)} \circ F'(q,\phi_i) \right) |_{\tau=0} \text{vol}.$$

This proves Theorem 33. $\blacksquare$.

**Corollary 34.** Let $\text{Pr}_q$ be the projection operator from $L^2(M, \Omega^0,q)$ to $L^2_{(0,q)}(\text{Im}(\partial))$, then we have the following formula:
\[ \lim_{t \to 0} \frac{1}{t} \int_M Tr \left( \left( \Pr_q \left( \exp - \frac{d^2_q (w, z)}{4t} \right) \circ \delta_j \left( \mathcal{P}_\tau (w, z) \big|_{\tau = 0} \right) \right) \circ F' (q, \phi_i) \right) \, vol. \]

**PROOF:** The proof of the corollary follows directly from Theorem 33 and the fact that computation of the trace of a kernel means to restrict the kernel to the diagonal. From here we deduce that: \( \frac{\partial}{\partial q_j} \left( d^2_q (w, z) \right) \big|_{w=z} = 0. \) Now Corollary 34 follows directly. ■

**Theorem 35.** \( \frac{\partial}{\partial q_j} \mathcal{P}_\tau (0, z)(q) = F'(q, \phi_j) + O(\tau), \) where \( F' (q, \phi_j) \) is defined in Definition 24.

**PROOF:** We will prove the theorem first for \( q = 1. \) In this case the operator \( \mathcal{P}_\tau (w, z) \) is the parallel transportation for the bundle \( \Omega^{0,1}_w \) and it defines a linear map: \( \mathcal{P}_\tau (w, a) : \Omega^{0,1}_w \to \Omega^{0,1}_z. \) Once we prove Theorem 35 for \( q = 1, \) the general case will follow directly from standard facts from linear algebra.

Since \( g_\tau \) is a Kähler metric, the parallel transport operator \( \mathcal{P}_\tau (w, z) \) preserves the splitting of the complexified cotangent bundle of \( M \) into \((1,0)\) and \((0,1)\) forms. So the operator \( \mathcal{P}_\tau (w, z) \) maps linearly \( \Omega^{0,1}_\tau = \Omega^{1,0}_\tau \) into \( \Omega^{0,1}_\tau. \) The parallel transport operators \( \mathcal{P}_\tau (w, z) \) are defined by the Levi-Civita connection \( \nabla_\tau \) of the metrics \( g_\tau. \) We are going to study the local expansion of \( \mathcal{P}_\tau (w, z) \) in terms of \( \tau. \) It is a standard fact that

\[ \nabla_\tau = \nabla^{1,0}_\tau + \nabla^{0,1}_\tau = (\partial_\tau - (g^{-1}_\tau \partial_\tau g_\tau)) + \mathcal{G}_\tau. \]

In order to define the parallel transportation between \( \Omega^{0,1}_w \) and \( \Omega^{0,1}_z \) we need to join the points \( w \) and \( z \) by geodesics. We will suppose that \( w \) and \( z \) are "close". This assumption can be made since we need to compute a trace of an operator given by some kernel. So our computations will be done on the diagonal \( M \subset M \times M. \) So from here it follows that we can join \( w \) and \( z \) with a unique geodesic. The parallel transportation of the \((0,1)\) form \( \eta \in \Omega^{0,1}_\tau = 0_{\tau} \in N, \) from a point \( w \) to a point \( z \) is given by solving the equations for fix \( \tau: \)

\[ \nabla^{0,1}_\tau \left( \pi^{(0,1)}_\tau \eta \right) := \partial_\tau (\pi^{(0,1)}_\tau \eta) = 0 \]

\[\nabla^{1,0}_\tau \left( \pi^{(0,1)}_\tau \eta \right) = (\partial_\tau - (g^{-1}_\tau \partial_\tau g_\tau)) \left( \pi^{(0,1)}_\tau \eta (t) \right) = 0 \& \eta(0) = \eta.\]

where \( \pi^{(1,0)}_\tau \) and \( \pi^{(0,1)}_\tau \) are the projection operators on \((1,0)\) and \((0,1)\) forms on the complex manifold \( M_\tau. \) Without loss of generality we can assume that \( w = 0. \) So we can write the following expression for the parallel transportation operator:

\[ \mathcal{P}_\tau (0, z)(\eta) = \pi^{(1,0)}_\tau (\eta) + \pi^{(0,1)}_\tau (\eta) + z(\mathcal{B}_\tau(\eta)) + O(z^2) \]

for point \( z \in M_\tau \) near the fix point \( 0 \in M_\tau \) and \( \mathcal{B}_\tau \) is a linear operator depending on \( \tau, \) i.e.: \( \mathcal{B}_\tau : \Omega^{0,1}_\tau \to \Omega^{0,1}_\tau. \) Kodaira-Spencer deformation theory implies that \( \Omega^{1,0}_\tau \) depends holomorphically on \( \tau. \) This fact implies that: \( \delta_j^{(1)} \left( \pi^{(1,0)}_\tau \right) = 0. \) So we obtain the following formula:

\[ \frac{\partial}{\partial q_j} \mathcal{P}_\tau (0, z)(\eta) = \delta_j^{(0,1)} (\pi^{(0,1)}_\tau (\eta)) + O(z) \]

for \( z \) near \( 0. \) It is easy to see from the definition of the tangent space to a point of the Grassmanian of \( \Omega^{0,1}_\tau \subset T_0^* \otimes \mathbb{C} \) that \( \delta_j^{(0,1)} \left( \pi^{(0,1)}_\tau \right) \big|_{\tau = 0} = \phi_j. \) This implies that for \( z = 0 \) we get
Theorem 35 is proved for \( q = 1 \).

In order to prove Theorem 35 for any \( q > 1 \), we notice that for \( n_1 \ldots < n_q \):

\[
\nabla \left( d\bar{z}^{n_1} \wedge \ldots \wedge d\bar{z}^{n_q} \right) = \sum_{j=1}^{n} (-1)^{j-1} \left( d\bar{z}^{n_1} \wedge \ldots \wedge \left( \nabla (d\bar{z}) \right) \right) \wedge \ldots \wedge d\bar{z}^{n_q}.
\]

From here the last formula in Theorem 35 follows directly once it is established for the case \( q = 1 \). Our theorem is proved.\(\Box\)

Theorem 36. The following formula is true for \( q > 0 \):

\[-\frac{d^2}{\partial \tau \partial \tau} \left( \log \det \Delta_{\tau,q-1}^{-1} \right) \big|_{\tau = 0} = Tr \left( F'(q, \phi_i) \circ F'(q, \phi_j) \right) .\]

PROOF: Corollary 34 and Theorem 35 implies that we have the following formula:

\[-\frac{d^2}{\partial \tau \partial \tau} \left( \log \det \Delta_{\tau,q-1}^{-1} \right) \big|_{\tau = 0} =
\lim_{t \to 0} \frac{1}{(4\pi t)^n} \int M \left( Tr \left(Pr_q \left( \exp - \frac{d^2(w,z)}{4t} \circ \delta (\tau) \right) \right) \big|_{\tau = 0} \right) \circ F'(q, \phi_i) \right) \circ F'(q, \phi_i) \right) \vol =
\lim_{t \to 0} \frac{1}{(4\pi t)^n} \int M \left( Tr \left(Pr_q \left( \exp - \frac{d^2(w,z)}{4t} \circ \delta (\tau) \right) \right) \big|_{\tau = 0} \right) \circ F'(q, \phi_i) \right) \circ F'(q, \phi_i) \right) \vol .
\]

Since

\[-\lim_{t \to 0} \frac{1}{(4\pi t)^n} Tr \left(Pr_q \left( \exp - \frac{d^2(w,z)}{4t} \right) \right) \big|_{w = z} = Dirac(\delta)(z),\]

where \(\text{Dirac}(\delta)\) is the Dirac delta function. So we obtain that

\[-\frac{d^2}{\partial \tau \partial \tau} \left( \log \left( \det_{\tau,q-1}^{-1}(0) \right) \right) \big|_{\tau = 0} = Tr \left( F'(q, \phi_i) \circ F'(q, \phi_j) \right) .\]

This proves Theorem 36.\(\Box\)

In order to end the proof of the Theorem 26 we will need the following lemma from linear algebra:

Lemma 37. Let \( F \) be a linear map of a vector space \( V \) of dimension \( n \). Then the linear operator \( F \wedge \text{id} \) on \( V \) for \( 1 < q \leq n \) has a trace given by the formula: \( \text{Tr}(F \wedge \text{id}) = \binom{n-1}{q-1} \text{Tr}(F) .\)

PROOF: The proof is obvious.\(\Box\)

Theorem 38. \( \text{Tr} \left( F'(q, \phi_i) \circ F'(q, \phi_j) \right) = \binom{n-1}{q-1} < \phi_i, \phi_j > .\)

PROOF: Applying the variational formula from Theorem 36 for \( q = n-1 \) we get that

\[-\frac{d^2}{\partial \tau \partial \tau} \left( \log \left( \det_{n-1}^{-1}(0) \right) \right) \big|_{\tau = 0} = Tr \left( F'(n, \phi_i) \circ F'(n, \phi_j) \right) = Tr \left( F'(n, \phi_i) \circ F'(n, \phi_j) \right) .\]

It is easy to see that the composition of the maps \( F'(n, \phi_i) \) and \( F'(n, \phi_j) \) is defined on the Hilbert space \( L^2(M, \Omega^{0,n}) \), i.e.
From Hodge theorem and the definition of CY manifold we deduce that: $L^2(M, \Omega^{0,n}) = \text{Im}(\bar{\partial}) \oplus \tilde{\mathcal{F}}$. From here and from Lemma 37 we obtain that
\[
-\frac{d^2}{\partial r_i \partial r_j} \left( \log \left( \det_{n-1}(0) \right) \right) |_{r=0} = \text{Tr} \left( F(1, \phi_i) \circ \overline{F(1, \phi_j)} \right).
\]

On the other hand, the Hodge star operator $*$ for CY metric on CY manifold $M$, gives us a spectral isomorphism between the Hilbert spaces
\[
* : L^2(M, \Omega^{0,0}) \rightarrow L^2(M, \Omega^{0,n}).
\]

Since the antiholomorphic form $\omega^\tau$ is a parallel form with respect to the Levi-Chevita connection of the CY metric we can deduce that the Hodge $*$ operator gives a spectral isometry between those two spaces. From this fact and the fact that
\[
\text{Tr} \left( F(1, \phi_i) \circ \overline{F(1, \phi_j)} \right) = \langle \phi_i, \phi_j \rangle,
\]
we conclude that:
\[
-\frac{d^2}{\partial r_i \partial r_j} \left( \log \left( \det_{n-1}(0) \right) \right) |_{r=0} = \text{Tr} \left( F(1, \phi_i) \circ \overline{F(1, \phi_j)} \right) = \langle \phi_i, \phi_j \rangle.
\]

This proves Theorem 26 when $q=0$ and $q=n-1$. The formula we just proved, i.e.
\[
\text{Tr} \left( F(1, \phi_i) \circ \overline{F(1, \phi_j)} \right) = \langle \phi_i, \phi_j \rangle
\]

Theorem 36 and Lemma 37 directly imply Theorem 26 for any $q$. ■

7. Some Applications.

7.1. Computation of the Analytic Torsion.

**Theorem 39.** Let $M$ be an odd dimensional CY manifold, then
\[
\log(I(M)) = -2 \log(\det(\Delta_0)).
\]

**Proof:** It is a standard fact that
\[
\log(I(M)) = \sum_{q=0}^{n-1} (-1)^q \log(\det(\Delta_q)) = \sum_{q=1}^{n+1} (-1)^q \log(\det(\Delta_q^{\prime})).
\]

From the formulas:
\[
\log(\det(\Delta_q)) = -\zeta_{\Delta_q}^{\prime}(0), \quad \zeta_{\Delta_q}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(\exp(-t\Delta_q)) t^{s-1} dt,
\]
\[
\zeta_{\Delta_q^{\prime}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(\exp(-t\Delta_q^{\prime})) t^{s-1} dt
\]

and Theorems 25 & 26 we deduce that $\log(\det(\Delta_q^{\prime})) = \binom{n-1}{q} \log(\det(\Delta_0))$. From here we obtain that
\[
\log(I(M)) = \sum_{q=1}^{n-1} (-1)^q \binom{n-1}{q} \log(\det(\Delta_0)) + (-1)^{n+1} \log(\det(\Delta_0)).
\]

From the equality:
\[
(1-1)^{n-1} = \left( \sum_{q=0}^{n-1} (-1)^q \binom{n-1}{q} \right) = 0
\]
we conclude that $\log(I(M)) = -2 \log(\det(\Delta_0))$. Theorem 39 is proved. ■
7.2. Some Invariants of the Short Term Asymptotic Expansion of the Heat Kernel. From the well know fact that for small \( t \) we have \( \text{Tr}(\exp(-t\Delta_0)) = \frac{\text{vol}(N)}{t^n} + \frac{k(g)}{t^{n-k}} + \ldots + a_0 + h(t, \tau, \tau) \), we will deduce that:

**Theorem 40.** Suppose that \( M \) is a CY manifold and \( g \) is a CY metric with a fixed class of cohomology, then the coefficients \( a_k \) for \( k=0,..,n \) in the expression defined above are constant which depends only on the CY manifolds and the fixed class of cohomology of the CY metric.

**Idea of the Proof.** We know that the moduli space of CY metrics \( g \) with fixed class of cohomology is the same as the moduli space of complex structures. This follows directly from the uniqueness and existence of the solution of the Calabi problem. See [27]. From here and results of Kodaira it follows that \( \text{Tr}(\exp(-t\Delta_0)) \) is a smooth function with respect the coordinates \( \tau = (\tau_1, .., \tau_N) \) of the Kuranishi space \( K(M) \). If we prove that

\[
\lim_{t \to 0} \frac{\partial}{\partial \tau} (\text{Tr}(\exp(-t\Delta_0)))|_{\tau=0} = t \sum_{i=1}^{\infty} \text{exp}(-t\lambda_i)\lambda_i a_{ii} < \infty,
\]

then this implies that \( a_k \) for \( k=0,..,n = \dim_{\mathbb{C}} M \) are constants on the moduli space.

**PROOF:** Let \( F'(1, \phi) \) be the operator defined in Definition [13] and let \( \sum a_{ii} \) be its trace, then we have:

**Lemma 41.** The following formula is true:

\[
\frac{\partial}{\partial \tau} (\text{Tr}(\exp(-t\Delta_{t,0})))|_{\tau=0} = t \sum_{i=1}^{\infty} \text{exp}(-t\lambda_i)\lambda_i a_{ii} < \infty,
\]

for all \( t \geq 0. \lambda_i \) are eigen values of the Laplacian \( \Delta_0 \) on \( M_0 \) and \( \partial \).

**PROOF:** According to Theorem [29] the following formula is true:

\[
\frac{\partial}{\partial \tau} (\text{Tr}(\exp(-t\Delta_{t,0})))|_{\tau=0} = -t \text{Tr} \left( \frac{\partial}{\partial \tau} (\text{Tr}(\exp(-t\Delta_{t,0}))) \circ \overline{\partial}^{-1} \circ F'(1, \phi) \circ \partial \right).
\]

According to Theorem [29] we have

\[
\text{Tr} \left( \frac{\partial}{\partial \tau} (\text{Tr}(\exp(-t\Delta_{t,0}))) \circ \overline{\partial}^{-1} \circ F'(1, \phi) \circ \partial \right) = -t \sum_{i=1}^{\infty} \lambda_i \text{exp}(-t\lambda_i) a_{ii} < \infty.
\]

Lemma [41] is proved. ●

Since \( \{\lambda_n\} \) for \( n \geq 1 \) are the eigen values of the Laplacian \( \Delta_0 \) then

\[
\lim_{n \to \infty} \lambda_n \left( (\text{dim}_{\mathbb{C}} M)^{-1} \right) = C > 0,
\]

where \( \dim_{\mathbb{C}} M \) is the complex dimension of \( M \). From here and the fact that the kernel of the operator \( \Phi \) is a matrix with \( C^\infty \) coefficients we derive that \( \sum_{i=1}^{\infty} \lambda_i a_{ii} < \infty \). So Lemma [41] implies that

\[
\lim_{t \to 0} \frac{\partial}{\partial \tau} (\text{Tr}(\exp(-t\Delta_0))) = \lim_{t \to 0} \sum_{i=1}^{\infty} \lambda_i \text{exp}(-t\lambda_i) a_{ii} = 0.
\]

This will imply that \( \frac{\partial}{\partial \tau} (a_k) = 0, \) for \( k=0,..,n \). Since \( \text{Tr}(\exp(-t\Delta_0)) \) is a real function when \( t \in \mathbb{R} \) and \( t > 0 \). So \( \frac{\partial}{\partial \tau} (a_k) = 0 \) implies Theorem [40] directly. Theorem [10] is proved. ●
8. The Analytic Torsion on CY Manifolds is Bounded.

**Theorem 42.** Let $M$ be any CY manifold, then $0 \leq \det(\triangle_q) \leq C_q$ for $0 \leq q \leq n = \dim M$.

8.1. Outline of the Proof that of the Ray Singer Torsion on CY Manifolds is bounded. **Theorem 23** implies that in order to prove Theorem 42 it is enough to bound $\det(\triangle_0)$. The bound of $\det(\triangle_0)$ is based on the following expression for the zeta function of the Laplacian acting on functions:

$$
\zeta_0(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\operatorname{Tr}(\exp(-t\triangle_0))) t^{s-1} dt = b_0 + b_1 s + O(s^2).
$$

From the definition of $\det(\triangle_0)$ it follows that $\det(\triangle_0) = \exp(-b_1)$. So if we bound $b_1$ Theorem 23 will be proved. The bound $b_1$ is based on two facts. The first one is the following asymptotic expansion of the $\operatorname{Tr}(\exp(-t\triangle_0))$,

$$
\operatorname{Tr}(\exp(-t\triangle_0)) = \left(\sum_{k=0}^n \frac{a_k}{t^k}\right) + O(t) = \frac{\operatorname{vol}(\Omega)}{t^n} + \frac{k(g)}{t^{n-r}} + \ldots + a_0 + O(t),
$$

where $n = \dim \mathbb{C} M$ and $k(g)$ is the scalar curvature of $g$. See [23] on page 79. The second one is the explicit formula for $b_1$ in [11]:

$$
b_1 = \gamma a_0 + \sum_{k=1}^n \frac{a_k}{k} + \int_0^1 \left(\operatorname{Tr}(\exp(-t\triangle_0)) - \sum_{k=0}^n \frac{a_k}{k} \right) \frac{dt}{t} + \int_1^\infty \operatorname{Tr}(\exp(-t\triangle_0)) \frac{dt}{t},
$$

where $\gamma$ is the Euler constant. We will show that $b_1 = \psi(t, \tau, \overline{\tau})$, where $\psi(t, \tau, \overline{\tau}) \geq 0$. From here we will obtain that $\det(\triangle_0) \leq \exp(-C)$.

8.2. Proof of that the Analytic Torsion is Bounded.

**Remark 43.** From now on we will consider the following situation: We will restrict our function $b(t, \tau, \overline{\tau})$ on an one dimensional disk in the Teichmüller space $\mathcal{T}(M)$ of $M$ and this disk is defined as follows: $\phi \in \mathbb{H}^1(M, T^{1,0})$ and let

$$
\phi(\tau) := \phi + \frac{1}{2\pi} \int_{\mathbb{H}} \mathcal{G}[\phi(\tau), \phi(\tau)].
$$

then we know that the Beltrami differential $\phi(\tau)$ is well defined $C^\infty$ section of $C^\infty(M_0, \operatorname{Hom}(\Omega^{1,0}, \Omega^{0,1}))$ in a small disk for $|\tau| < \varepsilon$ in the Teichmüller space $\mathcal{T}(M)$.

**Theorem 44.** $-\log(\det(\triangle_0)) = b_1(\tau, \overline{\tau}) = C + \psi(\tau, \overline{\tau})$, where $C$ is a constant and $\psi(\tau, \overline{\tau}) \geq 0$.

**PROOF:** Let us define $\psi_1(\tau, \overline{\tau}) := \frac{\partial^2}{\partial \tau \partial \overline{\tau}} b_1(\tau, \overline{\tau})$. **Theorem 23** implies that $\psi_1(\tau, \overline{\tau}) \geq 0$. Let us define

$$
\psi(\tau, \overline{\tau}) := \frac{1}{2\pi} \int_{|w - \tau| \leq 1} \psi_1(\tau, \overline{\tau}) G(\tau, w) d(w) \wedge d(w),
$$

where $G(\tau, w) = -\log |w - \tau|$ is the Green kernel of the Laplacian $\frac{\partial^2}{\partial \tau \partial \overline{\tau}}$. Clearly since $\psi_1(\tau, \overline{\tau}) \geq 0$ and $G(\tau, w) \geq 0$ for $|\tau - w| \leq 1$ we can conclude that $\psi(\tau, \overline{\tau}) \geq 0$. From the definition of the Green kernel we obtain that $\frac{\partial^2}{\partial \tau \partial \overline{\tau}} \psi(\tau, \overline{\tau}) = \frac{\partial^2}{\partial \tau \partial \overline{\tau}} b_1(\tau, \overline{\tau}) = \psi_1(\tau, \overline{\tau})$.

This fact implies that $b_1(\tau, \overline{\tau}) = \psi(\tau, \overline{\tau}) + g(\tau) + g(\overline{\tau})$, where $g(\tau)$ is a complex analytic function in the disk $D$ defined in Remark 43.

**Lemma 45.** $g(\tau) = \text{const.}$
PROOF: According to [18] we have the following expression for $Tr(\exp(-t\triangle_0))$:

$$Tr(\exp(-t\triangle_0)) = \left(\sum_{k=0}^{n} \frac{a_k}{t^k}\right) + O(t) = \frac{\text{vol}(\mathcal{M})}{t^n} + \frac{k(t)}{t^{n-1}} + ... + a_0 + b(t, \tau, \tau).$$

According to [1] we have the following formula for $b_1(\tau, \tau)$:

$$\left(\frac{d}{ds}\zeta_0(s)\right)|_{s=0} = b_1(\tau, \tau) = \gamma a_0 + \sum_{k=1}^{n} \frac{a_k}{s} + \int_0^1 (Tr(\exp(-t\triangle_0)) - \sum_{k=0}^{n} \frac{a_k}{t^k}) \frac{dt}{t} + \int_1^\infty Tr(\exp(-t\triangle_0)) \frac{dt}{t^2}.$$

Next we will compute $\frac{d}{ds}(\frac{d}{ds}\zeta_0(s))$. We have for large $s$ the following formula for $\zeta_0(s)$:

$$\zeta_0(s) := \left(\frac{1}{t(s)}\right)^{1/2} \int_0^1 (a_0 + \sum_{k=1}^{n} \frac{a_k}{t^k}) t^{-1} dt + \int_{t(s)}^{1/s} \left(\int_0^1 (Tr(\exp(-t\triangle_0)) - \sum_{k=0}^{n} \frac{a_k}{t^k}) t^{-1} dt + \int_1^\infty Tr(\exp(-t\triangle_0)) \frac{dt}{t^2}\right).$$

So direct computations and Theorem [10] show that we have

$$\frac{d}{ds}(\zeta_0(s)) = \frac{d}{ds} \left(\frac{1}{t(s)}\right)^{1/2} \int_0^1 (a_0 + \sum_{k=1}^{n} \frac{a_k}{t^k}) t^{-1} dt + \int_{t(s)}^{1/s} \left(\int_0^1 (Tr(\exp(-t\triangle_0)) t^{-1} dt + \int_1^\infty Tr(\exp(-t\triangle_0)) \frac{dt}{t^2}\right).$$

According to Lemma [11],

$$\frac{d}{ds}(Tr(\exp(-t\triangle_0))) = t \sum_{i=1}^{\infty} \lambda_i \exp(-t\lambda_i) a_{ii} < \infty.$$

where $Tr(F(1, \phi)) = \sum_{i=1}^{\infty} a_{ii}$. Combining these facts we conclude that

$$\frac{d}{ds}b_1(\tau, \tau) = \frac{d}{ds} \left(\zeta_0(s)\right) |_{s=0} = \frac{d}{ds} \left(\frac{1}{t(s)}\right)^{1/2} \left(Tr(\frac{d}{dt}\exp(-t\triangle_0)) t^{-1} dt\right) |_{s=0} = \frac{d}{ds} \left(\frac{1}{t(s)}\right)^{1/2} \left(t \sum_{i=1}^{\infty} \lambda_i \exp(-t\lambda_i) t^{-1} dt\right) |_{s=0} = \frac{d}{ds} \left(\sum_{i=1}^{\infty} \lambda_i a_{ii}\right) |_{s=0} = \sum_{i=1}^{\infty} \log(\lambda_i) a_{ii}.$$

Kodaira proved that the positive eigen values of the Laplacians $\overline{\partial}_e \circ \overline{\partial}_e$ depend on a $C^\infty$ manner in a small neighborhood of $\tau_0 \in D$. See [17]. From here and the formula:

$$\frac{d}{ds}b_1(\tau, \tau) = \frac{d}{ds} (\zeta_0(s)) |_{s=0} = \frac{d}{dt} (g(\tau)) + \frac{d}{d\tau} (\psi(t, \tau, \tau)) = \sum_{i=1}^{\infty} \log(\lambda_i) a_{ii}$$

we can conclude that $\frac{d}{d\tau} (g(\tau)) = 0$. Lemma [12] is proved. □. Lemma [12] implies Theorem [13] directly. □.

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