EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS FOR THE SEMILINEAR COMPLEX GINZBURG–LANDAU TYPE EQUATION IN HOMOGENEOUS AND ISOTROPIC SPACETIME

Makoto NAKAMURA and Yuya SATO

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Abstract. The Cauchy problem for the semilinear complex Ginzburg–Landau type equation is considered in homogeneous and isotropic spacetime. Global solutions and their asymptotic behaviours for small initial data are obtained. The non-existence of non-trivial global solutions is also shown. The effects of spatial expansion and contraction are studied through the problem.

1. Introduction

In this paper, we consider the global solution for the Cauchy problem for the semilinear complex Ginzburg–Landau type equation in homogeneous and isotropic spacetime. This spacetime is the solution of the Einstein equations with the cosmological constant under the cosmological principle. We consider the spacetime with spatially flat curvature whose metric is given by $-c^2(d\tau)^2 = -c^2(dt)^2 + a_*(t)^2 \sum_{j=1}^n (dx^j)^2$, where $c$ is the speed of the light, $n$ is the spatial dimension, $\tau$ denotes the proper time, and $a_*(\cdot)$ denotes the scale-function defined as follows. Let $\sigma \in \mathbb{R}$, $a_0 > 0$ and $a_1 \in \mathbb{R}$. We put

$$ T_0 := \begin{cases} \infty & \text{if } (1 + \sigma)a_1 \geq 0, \\ -\frac{2a_0}{n(1 + \sigma)a_1} (> 0) & \text{if } (1 + \sigma)a_1 < 0. \end{cases} $$

We put $H := a_1/a_0$, which is called the Hubble constant. We define the scale-function $a_*(t)$ for $t \in [0, T_0)$ by

$$ a_*(t) := \begin{cases} a_0 \left( 1 + \frac{n(1 + \sigma)a_1 t}{2a_0} \right)^{(2/(n+1))} & \text{if } \sigma \neq -1, \\ a_0 \exp \left( \frac{a_1 t}{a_0} \right) & \text{if } \sigma = -1, \end{cases} \quad (1.1) $$

for $0 \leq t < T_0$, and we define a weight function

$$ w_*(t) := \left( \frac{a_0}{a_*(t)} \right)^{n/2}. \quad (1.2) $$

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This scale-function $a_\sigma(\cdot)$ describes Minkowski spacetime when $a_1 = 0$, expanding space when $a_1 > 0$ with $\sigma \geq 1$, blowing-up space when $a_1 > 0$ with $\sigma < 1$ (the ‘Big Rip’ in cosmology), contracting space when $a_1 < 0$ with $\sigma \leq 1$, and vanishing space when $a_1 < 0$ with $\sigma > 1$ (the ‘Big Crunch’ in cosmology). In particular, when $a_1 \neq 0$ with $\sigma = 1$, $a_\sigma(\cdot)$ describes de Sitter spacetime (see e.g. [2, 5]).

Let $\hbar > 0$ be the Planck constant, and let $m > 0$ be a mass constant. We put the gradient $\nabla := (\partial_1, \ldots, \partial_n)$ and the Laplacian $\Delta := \sum_{j=1}^n \partial_j^2$. We consider the Cauchy problem given by

$$
\left\{ \begin{array}{ll}
\frac{2m}{\hbar} \frac{\partial u}{\partial t}(t, x) + \frac{1}{a_\sigma(t)^2} \Delta u(t, x) - \frac{\lambda_\sigma}{e^{2i\omega} \omega_\sigma(t)} f(u \omega_\sigma(t), x) = 0,

u(0, x) = u_0(x),
\end{array} \right.
$$

for $(t, x) \in [0, \mathcal{T}_0) \times \mathbb{R}^n$, where $-\pi/2 < \omega \leq \pi/2$, $\lambda_\sigma \in \mathbb{C} \setminus \{0\}$, $1 < p < \infty,$

$$
f(v) := |v|^{p-1}v \text{ or } |v|^p
$$

and $u = u(t, x)$ is a complex-valued unknown function. The derivation of the first equation in (1.3) by the non-relativistic limit of a semilinear field equation has been shown in [12]. Concerning this Cauchy problem, the case $0 \leq \omega \leq \pi/2$ is considered in the framework of the Sobolev space $H^{\mu_0}(\mathbb{R}^n)$ with $\mu_0 \geq 0$ in [11, 13].

Let us consider the case $a_\sigma(\cdot) = 1$ (i.e. Minkowski spacetime) in (1.3) for simplicity. The differential equation in (1.3) reduces to the Schrödinger equation

$$
i \frac{2m}{\hbar} \frac{\partial u}{\partial t} + \Delta u - \lambda_\sigma f(u) = 0 \text{ or } i \frac{2m}{\hbar} \frac{\partial u}{\partial t} - \Delta u + \lambda_\sigma f(u) = 0,
$$

when $\omega = 0$ or $\omega = \pi/2$, respectively. It also reduces to the diffusion equation

$$
i \frac{2m}{\hbar} \frac{\partial u}{\partial t} - \Delta u + \lambda_\sigma f(u) = 0,
$$

when $\omega = \pi/4$. When $0 < \omega < \pi/2$, it is called the complex Ginzburg–Landau type equation. The differential equation in (1.3) is the generalization of these equations in homogeneous and isotropic spacetime whose scale-function is given by (1.1).

On the Cauchy problem of the semilinear diffusion equation in Minkowski spacetime given by

$$
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) = u^p(t, x),

u(0, x) = u_0(x),
\end{array} \right.
$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^n$, there exists a non-negative global solution for small data when $p > 1 + 2/n$, while any non-trivial and non-negative solution for (1.5) blows up in finite time when $p \leq 1 + 2/n$ (see Theorem 1 in [8], or Theorem 2 in [19]). This critical exponent $1 + 2/n$ is called the Fujita exponent. In [14], the Cauchy problem given by

$$
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t}(t, x) - e^{-2\mathcal{H}t} \Delta u(t, x) = \lambda e^{n\mathcal{H}t/2} f(u e^{-n\mathcal{H}t/2})(t, x),

u(0, x) = u_0(x),
\end{array} \right.
$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^n$ and $\lambda \in \mathbb{C} \setminus \{0\}$, namely (1.3) with $\sigma = -1$ (i.e. de Sitter spacetime) and $\omega = \pi/4$, is considered, and the authors have shown that global solutions are obtained for
any $p > 1$ for small data in Lebesgue space if $H \neq 0$. Thus, the Fujita exponent disappears in the case of de Sitter spacetime with $H \neq 0$, and spatial expansion and contraction have a positive effect on the existence of global solutions for small data. In this paper, we consider the Cauchy problem (1.3) which describes more general spatial expansion and contraction than the de Sitter spacetime. We give the critical exponent $p_0$ defined by (1.19), below, as the Fujita exponent in several spacetimes when $a_1 \neq 0$ (see Remark 1.7).

To transform the differential equation in (1.3) with a variable coefficient on the Laplacian into an equation with a constant coefficient, we introduce a new time variable $s$ defined by

$$s(s) := \int_0^t \frac{1}{a_\tau(t^2)} d\tau.$$  \hspace{1cm} (1.7)

We put $S_0 := s(T_0)$, $a(s) := a_\tau(s)$ and $u(s) := w_\tau(s)$ for $s \in [0, S_0)$. Direct computations show that

$$a(s) = \begin{cases} a_0 & \text{if } a_1 = 0 \text{ and } \sigma \in \mathbb{R}, \\ a_0 \left(1 - s/S_1\right)^{2/[n(1+\sigma)-4]} & \text{if } a_1 \neq 0 \text{ and } \sigma \neq -1 + 4/n, \\ a_0 e^{a_0 a_1 s} & \text{if } a_1 \neq 0 \text{ and } \sigma = -1 + 4/n, \end{cases}$$  \hspace{1cm} (1.8)

and

$$S_0 = \begin{cases} S_1 & \text{if } a_1 \{4 - n(1 + \sigma)\} > 0, \\ \infty & \text{if } a_1 \{4 - n(1 + \sigma)\} \leq 0, \end{cases}$$  \hspace{1cm} (1.9)

where we have put

$$S_1 := \frac{2}{a_0 a_1 \{4 - n(1 + \sigma)\}}$$  \hspace{1cm} (1.10)

when $a_1 \{4 - n(1 + \sigma)\} \neq 0$.

We have the following cases from (i) to (vii) for $S_0$:

(i) $a_1 = 0$, $\sigma \in \mathbb{R}$, $S_0 = \infty$;
(ii) $a_1 > 0$, $\sigma > -1 + 4/n$, $S_0 = \infty$;
(iii) $a_1 > 0$, $\sigma = -1 + 4/n$, $S_0 = \infty$;
(iv) $a_1 > 0$, $\sigma < -1 + 4/n$, $S_0 = S_1 (> 0)$;
(v) $a_1 < 0$, $\sigma > -1 + 4/n$, $S_0 = S_1 (> 0)$;
(vi) $a_1 < 0$, $\sigma = -1 + 4/n$, $S_0 = \infty$;
(vii) $a_1 < 0$, $\sigma < -1 + 4/n$, $S_0 = \infty$.

The Cauchy problem (1.3) is rewritten as

$$\begin{cases} 2m \frac{\partial u}{\partial s}(s, x) + \frac{1}{e^{2\omega} w(s)} \Delta u(s, x) - \frac{\lambda a(s)^2}{e^{2\omega} w(s)} f(u,v)(s, x) = 0, \\ u(0, x) = u_0(x), \end{cases}$$  \hspace{1cm} (1.12)

for $(s, x) \in [0, S_0) \times \mathbb{R}^n$. We rewrite (1.12) as

$$\begin{cases} \frac{\partial u}{\partial s}(s, x) - \kappa \Delta u(s, x) + \lambda a(s)^{2-\gamma(p-1)/2} f(u)(s, x) = 0, \\ u(0, x) = u_0(x), \end{cases}$$  \hspace{1cm} (1.13)
for \((s, x) \in [0, S_0) \times \mathbb{R}^n\), where we have put

\[
\kappa := \frac{hi}{2me^{2i\omega}} \quad \text{and} \quad \lambda := \frac{h_a (p-1)/2 i \lambda_*}{2me^{2i\omega}}.
\] (1.14)

In the following, we consider the Cauchy problem (1.13) under the assumptions

\[
\kappa \in \mathbb{C}, \quad \text{Re} \kappa > 0, \quad \lambda \in \mathbb{C},
\] (1.15)

which follow from the condition \(0 < \omega < \pi/2\). Namely, we consider the semilinear complex Ginzburg–Landau type equation in homogeneous and isotropic spacetimes with the scale-function (1.1). We say that \(u\) is the global solution for the Cauchy problem (1.13) if \(u\) exists on \([0, S_0)\) since the spacetime does not exist after the time \(S_0\).

We remark that the differential equation in (1.13) with \(f(u) = |u|^{p-1}u\) is rewritten as

\[
\frac{\partial v}{\partial s}(s, x) - \kappa \Delta v(s, x) + \lambda f(v)(s, x) - \gamma v(s, x) = 0
\] (1.16)

by the change of the unknown functions \(v := ue^{\gamma s}\) under the conditions \(\sigma = -1 + 4/n\), \(a_0 := 1\) and

\[
\begin{cases}
  a_1 \in \mathbb{R}, & \gamma = 0 \quad \text{if} \quad p = 1 + 4/n, \\
  a_1 = \frac{2(p-1)\gamma}{4-n(p-1)}, & \gamma \in \mathbb{R} \quad \text{if} \quad p \neq 1 + 4/n.
\end{cases}
\]

Equation (1.16) is the well-known generalized complex Ginzburg–Landau equation with linear driving. Among the large literature on the Cauchy problem for (1.16), we refer to the results by Bu [1] and Okazawa and Yokota [16] on the existence of the global solution, by Ginibre and Velo [10] on the uniqueness of the solution, by Cazenave, Dias and Figueira [3] on the blowing-up solution, by Doering, Gibbon and Levermore [6] and Doering, Gibbon, Holm and Nicolaenko [7] on the solution under the periodic boundary condition, and by Zhang, Li and Su [21] on the Cauchy problem for the time-fractional equation.

We now introduce some notation used in this paper. The convolution for functions \(f\) and \(g\) by the spatial variables is denoted by \(f \ast g\). The notation \(A \lesssim B\) denotes the inequality \(A \leq CB\) for a positive constant \(C > 0\) that is independent of \(A\) and \(B\) and is not essential in the argument. We denote the conjugate number of \(r\) with \(1 \leq r \leq \infty\) by \(r'\), which is defined by \(1/r + 1/r' = 1\). We put the Gauss kernel

\[
G(t, x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}
\] (1.17)

for \(t > 0\) and \(x \in \mathbb{R}^n\). We denote the Fourier transform of \(\phi\) by

\[
\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i x \cdot \xi} \phi(x) \, dx
\]

for \(\xi \in \mathbb{R}^n\), and the inverse Fourier transform by

\[
\mathcal{F}^{-1}\phi(x) = \hat{\phi}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i x \cdot \xi} \phi(\xi) \, d\xi.
\]

We define the heat operator \(e^{t\Delta}\) by

\[
e^{t\Delta} := \mathcal{F}^{-1} e^{-t|\xi|^2} \mathcal{F}.
\]
We denote the Lebesgue space by $L^r(\mathbb{R}^n) := \{ \phi \mid \| \phi \|_{L^r(\mathbb{R}^n)} < \infty \}$ for $1 \leq r \leq \infty$, where the norm $\| \cdot \|_{L^r(\mathbb{R}^n)}$ is defined by

$$\| \phi \|_{L^r(\mathbb{R}^n)} := \begin{cases} \left( \int_{\mathbb{R}^n} |\phi(x)|^r \, dx \right)^{1/r} & \text{for } 1 \leq r < \infty, \\ \text{ess.sup}_{x \in \mathbb{R}^n} |\phi(x)| & \text{for } r = \infty. \end{cases}$$

We define the space $L^q(I, L^r(\mathbb{R}^n))$ on the time interval $I$ by the norm

$$\| \phi \|_{L^q(I, L^r(\mathbb{R}^n))} := \| \| \phi \|_{L^r(\mathbb{R}^n)} \|_{L^q(I)}$$

for $1 \leq q, r \leq \infty$. We define a function space $X$ by

$$X := \begin{cases} L^\infty((0, T) \times \mathbb{R}^n) & \text{if } a_1 \{4 - n(1 + \sigma)\} > 0, \\ \{ u \mid \| u \|_X < \infty \} & \text{if } a_1 \{4 - n(1 + \sigma)\} \leq 0, \end{cases}$$

where we define the norm $\| \cdot \|_X$ by

$$\| u \|_X := \max \left\{ \| u \|_{L^\infty((0, T), L^1(\mathbb{R}^n))}, \sup_{0 < s < T} (1 + s)^{n/2} \| u(s) \|_{L^\infty(\mathbb{R}^n)} \right\}$$

when $a_1 \{4 - n(1 + \sigma)\} \leq 0$. For $R > 0$, we define a metric space $X(R)$ by

$$X(R) := \{ u \in X \mid \| u \|_X \leq R \}$$

with a metric $d(u, v) := \| u - v \|_X$.

We show the existence of global solutions for the Cauchy problem (1.13) for small data and asymptotic behaviours of them. We put

$$p_0 := 1 + \frac{2(1 + \sigma)}{n(1 + \sigma) - 2},$$

$$D := (\sin 2\omega)^{-np/2(p-1)},$$

$$M_1 := \begin{cases} 1 & \text{if } p \leq 1 + 4/n, \quad |S_1| \geq 1, \\ 1 & \text{if } p > 1 + 4/n, \quad |S_1| < 1, \\ (-S_1)^{n(p-1)-4/[n(1+\sigma)-4]} & \text{if } p \leq 1 + 4/n, \quad |S_1| < 1, \\ (-S_1)^{n(p-1)-4/[n(1+\sigma)-4]} & \text{if } p > 1 + 4/n, \quad |S_1| \geq 1, \end{cases}$$

and

$$M_2 := \begin{cases} (-S_1)^{n(p-1)-4/[n(1+\sigma)-4]} & \text{if } p \leq 1 + 4/n, \quad |S_1| \geq 1, \\ (-S_1)^{n(p-1)-4/[n(1+\sigma)-4]} & \text{if } p > 1 + 4/n, \quad |S_1| < 1, \\ 1 & \text{if } p \leq 1 + 4/n, \quad |S_1| < 1, \\ 1 & \text{if } p > 1 + 4/n, \quad |S_1| \geq 1. \end{cases}$$

For $a > 0$, we denote by $a \ll 1$ that $a$ is sufficiently small. We have the following results corresponding to each case in (1.11).

**Theorem 1.1.** (Small global solutions) *Let $n \geq 1$ and $1 < p < \infty$. Assume (1.15). Assume that one of the following cases from (1) to (7) holds:*

1. $a_1 = 0, \sigma \in \mathbb{R}, p > 1 + 2/n, D\| u_0 \|_{L^1(\mathbb{R}^n)} \ll 1; \quad$
(ii), (iii), (vi) and (vii) in (1.11). We consider the gauge-variant nonlinear term (namely,
\(v(4)\) and (5) in Theorem 1.1. If \(u(1), (2), (3), (6)\) and (7) in Theorem 1.1. Then, we have
\[ p > \begin{cases} 
2 + \sigma & \text{if } -1 < \sigma < -1 + 4/n, \\
1 & \text{if } \sigma \leq -1; 
\end{cases} \]  
(1.22)

(5) \(a_1 < 0, \sigma > -1 + 4/n, 1 < p < 2 + \sigma, S_1^{1/(p-1)} D\|u_0\|_{L^\infty(R^n)} \ll 1; \)

(6) \(a_1 < 0, \sigma = -1 + 4/n, 1 < p \leq 1 + 4/n, D\|u_0\|_{L^1(R^n)} \ll 1; \)

(7) \(a_1 < 0, \sigma < -1 + 4/n, M_2^{1/(p-1)} D\|u_0\|_{L^1(R^n)} \ll 1, \)
\[ p = \begin{cases} 
< p_0 & \text{if } -1 + 2/n < \sigma < -1 + 4/n, \\
> 1 & \text{if } -1 \leq \sigma \leq -1 + 2/n, \\
> p_0 & \text{if } \sigma < -1. 
\end{cases} \]  
(1.23)

Then there exists a unique global solution \(u\) for (1.13) in \(X(R)\) for some \(R > 0.\)

On the global solutions obtained in Theorem 1.1, we have the following results on asymptotic behaviours. The first result is on the case \(S_0 = \infty\) in (1.9).

**Theorem 1.2. (Asymptotic behaviours)** Let \(u\) be the global solution obtained in the cases (1), (2), (3), (6) and (7) in Theorem 1.1. Then, we have
\[ \|u(s, \cdot) - \theta G(\kappa s, \cdot)\|_{L^r(R^n)} = o(s^{-n(1-1/r)/2}) \text{ as } s \to \infty \]
for any \(1 \leq r \leq \infty.\) Here, \(G\) is defined by (1.17), and
\[ \theta := \int_{R^n} u_0(y) dy - \lambda \int_0^{\infty} \int_{R^n} a(\tau) 2^{-n(p-1)/2} f(u(\tau, y)) dy d\tau. \]  
(1.24)

The second result is on the case \(S_0 = S_1 < \infty\) in (1.9).

**Theorem 1.3. (Asymptotic behaviours)** Let \(u\) be the global solution obtained in the cases (4) and (5) in Theorem 1.1. If \(u_0 \in W^{1, \infty}(R^n),\) then we have
\[ \|u(s) - v\|_{L^\infty} = o(1) \text{ as } s \to S_0, \]
where \(v\) is a function in \(W^{1, \infty}(R^n)\) defined by
\[ v := e^{S_1} u_0 - \lambda \int_0^{S_1} e^{S_1(\tau - \sigma)} a(\tau) 2^{-n(p-1)/2} f(v(\tau)) d\tau. \]  
(1.25)

We show the non-existence of non-trivial global solutions for (1.12) in the cases (i), (ii), (iii), (vi) and (vii) in (1.11). We consider the gauge-variant nonlinear term (namely, \(f(v) := |v|^p (p > 1)\)) in (1.4). We rewrite (1.12) as
\[ \begin{cases} 
i \frac{2m}{h} \frac{\partial u}{\partial s}(s, x) + v \Delta u(s, x) - \mu a(s)^{2-n(p-1)/2} |u|^p(s, x) = 0, \\
u(0, x) = u_0(x), \end{cases} \]  
(1.26)
for \((s, x) \in [0, \infty) \times R^n,\) where we have put \(v := e^{-2i\omega}\) and \(\mu = \lambda e^{-2i\omega} a_0^{n(p-1)/2}.\) We define the weak solution for (1.26) as follows.
Definition 1.4. (Weak solutions) Let $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$. The function $u \in L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^n)$ is called the weak solution of (1.26) if $u$ satisfies the equation

$$-i \frac{2m}{\hbar} \int_{\mathbb{R}^n} u_0(x) \psi(0, x) \, dx + \int_0^\infty \int_{\mathbb{R}^n} u(s, x) \left(-i \frac{2m}{\hbar} \partial_s + \nu \Delta\right) \psi(s, x) \, dx \, ds$$

$$- \mu \int_0^\infty \int_{\mathbb{R}^n} a(s) 2^{-n(p-1)/2} |u(s, x)|^p \psi(s, x) \, dx \, ds = 0,$$

for any $\psi \in C^2_0((0, \infty) \times \mathbb{R}^n)$.

Remark 1.5. Equation (1.27) follows from the differential equation in (1.26) as follows. By multiplying the test function $\psi \in C^2_0((0, \infty) \times \mathbb{R}^n)$ to both sides of the equation, we have

$$\mu \int_{(0, \infty) \times \mathbb{R}^n} a(s) 2^{-n(p-1)/2} |u(s, x)|^p \psi(s, x) \, dx \, ds$$

$$= \int_{(0, \infty) \times \mathbb{R}^n} \left(\frac{2m}{\hbar} \partial_s + \nu \Delta\right) u(s, x) \psi(s, x) \, dx \, ds.$$

Since we have

$$\int_{(0, \infty) \times \mathbb{R}^n} i \frac{2m}{\hbar} \partial_s u(s, x) \psi(s, x) \, dx \, ds$$

$$= -i \frac{2m}{\hbar} \int_{\mathbb{R}^n} u_0(x) \psi(0, x) \, dx - i \frac{2m}{\hbar} \int_{(0, \infty) \times \mathbb{R}^n} u(s, x) \partial_s \psi(s, x) \, dx \, ds$$

and

$$\int_{(0, \infty) \times \mathbb{R}^n} \Delta u(s, x) \psi(s, x) \, dx \, ds = \int_{(0, \infty) \times \mathbb{R}^n} u(s, x) \Delta \psi(s, x) \, dx \, ds$$

by integration by parts, we obtain equation (1.27) for $u \in C^2((0, \infty) \times \mathbb{R}^n)$.

We have the following results, where $p_0$ is defined by (1.19), and $p_1$ and $p_2$ are defined by

$$p_1 := 1 + \frac{2}{n-2} \quad \text{and} \quad p_2 := 1 + \frac{4}{n(2+\sigma) - 4}.$$  

(1.28)

Theorem 1.6. (Non-existence of non-trivial global weak solutions) Let $\nu, \mu, n, p$ and $\sigma$ satisfy

$$\begin{cases}
\nu \in \mathbb{C}, \mu \in \mathbb{C}, \text{ Re } \mu > 0, \\
n \geq 1, \ 1 < p < \infty, \ \sigma \in \mathbb{R}.
\end{cases}$$

(1.29)

Assume that

$$\int_{\mathbb{R}^n} \text{Im} u_0(x) \, dx < 0.$$  

(1.30)

Assume one of the following conditions from (1) to (4):

(1) $n \geq 1, a_1 = 0, \sigma \in \mathbb{R}, 1 < p \leq 1 + 2/n.$

(2) $n \geq 1, a_1 > 0, \sigma = -1 + 4/n, 1 < p < 1 + 4/n.$ Moreover, $p \leq 1 + 2/(n-2)$ if $n \geq 3.$

(3) $1 \leq n \leq 3, a_1 < 0, \sigma = -1 + 4/n, 1 + 4/n < p < \infty.$ Moreover, $p \leq 3$ if $n = 3.$

(4) $a_1(\sigma + 1 - 4/n) > 0.$ Moreover, assume one of the following conditions from (i) to (iii)′:
(i) \( n = 1, 2, \sigma > -2 + 4/n, \) and
\[
\begin{align*}
1 < p < p_2 & \quad \text{if } \sigma > -1 + 4/n, \\
p_2 < p & \quad \text{if } -2 + 4/n < \sigma < -1 + 4/n.
\end{align*}
\]

(i)' \( n = 3, \sigma > 0, \) and
\[
\begin{align*}
1 < p < p_2 & \quad \text{if } \sigma > -1 + 4/n, \\
p_2 < p \leq p_1 & \quad \text{if } 0 < \sigma < -1 + 4/n.
\end{align*}
\]

(i)'' \( n = 4, \sigma > 0, \) and \( 1 < p < p_2. \)

(i)''' \( n \geq 5, \sigma > -1 + 4/n, \)
\[
\begin{align*}
1 < p \leq p_1 & \quad \text{if } \sigma < 0, \\
1 < p < p_1 & \quad \text{if } \sigma = 0, \\
1 < p < p_2 & \quad \text{if } \sigma > 0.
\end{align*}
\]

(ii) \( n = 1, -1 + 2/n < \sigma \) or \( \sigma < -1, \) and
\[
\begin{align*}
p_2 < p \leq p_0 & \quad \text{if } \sigma > -1 + 4/n, \\
p_0 \leq p < p_2 & \quad \text{if } -2 + 4/n < \sigma < -1 + 4/n, \\
p_0 \leq p & \quad \text{if } -1 + 2/n < \sigma \leq -2 + 4/n, \\
1 < p \leq p_0 & \quad \text{if } \sigma < -1.
\end{align*}
\]

(ii)' \( n = 2, -1 + 2/n < \sigma \) or \( \sigma < -1, \) and
\[
\begin{align*}
p_2 < p \leq p_0 & \quad \text{if } \sigma > -1 + 4/n, \\
p_0 \leq p < p_2 & \quad \text{if } -1 + 2/n < \sigma < -1 + 4/n, \\
1 < p \leq p_0 & \quad \text{if } \sigma < -1.
\end{align*}
\]

(ii)'' \( n = 3, \sigma > 0 \) or \( \sigma < -1, \) and
\[
\begin{align*}
p_2 < p \leq p_0 & \quad \text{if } \sigma > -1 + 4/n, \\
p_0 \leq p < p_2 & \quad \text{if } 0 < \sigma < -1 + 4/n, \\
1 < p \leq p_0 & \quad \text{if } \sigma < -1.
\end{align*}
\]

(ii)''' \( n = 4, \sigma > -1 + 4/n \) or \( \sigma < -2 + 4/n, \) and
\[
\begin{align*}
p_2 < p \leq p_0 & \quad \text{if } \sigma > -1 + 4/n, \\
1 < p \leq p_0 & \quad \text{if } \sigma < -2 + 4/n.
\end{align*}
\]

(ii)'''' \( n \geq 5, \sigma > 0 \) or \( \sigma < -1, \) and
\[
\begin{align*}
1 < p < p_2 & \quad \text{if } -2 + 4/n < \sigma < -1, \\
1 < p \leq p_0 & \quad \text{if } \sigma \leq -2 + 4/n, \\
p_2 < p \leq p_0 & \quad \text{if } \sigma > 0.
\end{align*}
\]

(iii) \( n = 1, 2, \sigma > -2 + 4/n, \) and \( p = p_2. \)
(iii)' \( n = 3, \sigma > 0, \text{ and } p = p_2. \)

(iii)'' \( n \geq 4, \sigma > 0, \text{ and } p = p_2. \)

Then the global weak solution \( u \) of (1.26) must satisfy \( u(s, x) = 0 \) for almost every \((s, x) \in [0, \infty) \times \mathbb{R}^n. \) Namely, any non-trivial weak solution must blow up in finite time.

Remark 1.7. For each case from (1) to (4) in Theorem 1.6, we have global solutions if \( p \) satisfies the following conditions by Theorem 1.1:

1. \( p > 1 + 2/n. \)
2. \( p \geq 1 + 4/n. \)
3. \( p \leq 1 + 4/n. \)

(4)(i) \[
\begin{cases}
p > p_0 (> p_2) & \text{if } \sigma > -1 + 4/n, \\
p < p_0 (< p_2) & \text{if } -2 + 4/n < \sigma < -1 + 4/n.
\end{cases}
\]

(4)(i)' \[
\begin{cases}
p > p_0 (> p_2) & \text{if } \sigma > -1 + 4/n, \\
p < p_0 (< p_2) & \text{if } 0 < \sigma < -1 + 4/n.
\end{cases}
\]

(4)(i)'' \[
\begin{cases}
p > p_0 (> p_2) & \text{if } \sigma < 0, \\
p > p_0 (= p_1 = p_2) & \text{if } \sigma = 0, \\
p > p_0 (> p_2) & \text{if } \sigma > 0.
\end{cases}
\]

(4)(ii) \[
\begin{cases}
p > p_0 & \text{if } \sigma > -1 + 4/n \text{ or } \sigma < -1, \\
p < p_0 & \text{if } -1 + 2/n < \sigma < -1 + 4/n.
\end{cases}
\]

(4)(ii)' \[
\begin{cases}
p > p_0 & \text{if } \sigma > -1 + 4/n \text{ or } \sigma < -1, \\
p < p_0 & \text{if } -1 + 2/n < \sigma < -1 + 4/n.
\end{cases}
\]

(4)(ii)'' \[
\begin{cases}
p > p_0 & \text{if } \sigma > -1 + 4/n \text{ or } \sigma < -1, \\
p < p_0 & \text{if } 0 < \sigma < -1 + 4/n.
\end{cases}
\]

(4)(ii)''' \[
\begin{cases}
p > p_0 & \text{if } \sigma > -1 + 4/n \text{ or } \sigma < -1, \\
p > p_0 (> p_2) & \text{if } -2 + 4/n < \sigma < -1 + 4/n, \\
p > p_0 & \text{if } \sigma \leq -2 + 4/n \text{ or } \sigma > 0.
\end{cases}
\]

(4)(iii) \[
\begin{cases}
p < p_0 (< p_2) & \text{if } -2 + 4/n < \sigma < -1 + 4/n, \\
p > p_0 (> p_2) & \text{if } \sigma > -1 + 4/n.
\end{cases}
\]

(4)(iii)' \[
\begin{cases}
p < p_0 (< p_2) & \text{if } 0 < \sigma < -1 + 4/n, \\
p > p_0 (> p_2) & \text{if } \sigma > -1 + 4/n.
\end{cases}
\]

(4)(iii)'' \[
\begin{cases}
p > p_0 (> p_2) & \text{if } \sigma > -1 + 4/n.
\end{cases}
\]

Especially, the following exponents are critical for the existence and the non-existence for global weak solutions:

\[
p = \begin{cases}
1 + 2/n & \text{for (1)}, \\
1 + 4/n & \text{for (2) with } n = 1, 2, \text{ or (3)}, \\
p_0 & \text{for (I) or (II) or (III)},
\end{cases}
\]

where (I), (II) and (III) denote the cases defined by
Finally, we show the blowing-up of solutions in finite time for the gauge-invariant nonlinear term (namely, \( f(u) := |u|^{p-1}u \) \( p > 1 \)) in (1.4). We rewrite (1.12) as

\[
\begin{aligned}
&\left\{
\begin{array}{l}
\frac{2m}{\hbar}\frac{\partial u}{\partial s}(s, x) + \nu \Delta u(s, x) - \mu a(s)^{2-n(p-1)/2} |u|^{p-1}u(s, x) = 0, \\
u(0, x) = u_0(x),
\end{array}
\right.
\end{aligned}
\]

for \( (s, x) \in [0, S_0) \times \mathbb{R}^n \), where we have put \( \nu := e^{-2i\omega} \) and \( \mu = \lambda e^{-2i\omega} a_0^{n(p-1)/2} \).

**Theorem 1.8.** Let \( \nu, \mu \in \mathbb{C} \) satisfy \( \text{Im} \nu < 0, \mu/\nu < 0 \). Let \( p \) satisfy \( 1 < p < \infty \) when \( n = 1, 2, \) and \( 1 < p \leq 1 + 4/(n - 2) \) when \( n \geq 3 \). Moreover, let \( p \) satisfy

\[
p \begin{cases} 
\leq 1 + 4/n \quad \text{if } a_1 \geq 0, \\
\geq 1 + 4/n \quad \text{if } a_1 < 0.
\end{cases}
\]

Let \( u_0 \in H^1(\mathbb{R}^n) \) satisfy

\[
E_0 := \int_{\mathbb{R}^n} |\nabla u_0(x)|^2 + \frac{2\mu a_0^{2-n(p-1)/2}}{\nu(p+1)} |u_0(x)|^{p+1} \, dx < 0.
\]

Put

\[
S := \frac{4m}{(p^2 - 1) \hbar (\text{Im} \nu) E_0},
\]

which is a positive number by \( \text{Im} \nu < 0 \) and \( E_0 < 0 \). Assume \( S < S_1 \) when \( a_1(4 - n(1 + \sigma)) > 0 \). Then the solution \( u \) of (1.31) blows up at time \( S \) in \( L^2(\mathbb{R}^n) \). Namely, \( \|u(s)\|_{L^2(\mathbb{R}^n)} \to \infty \) as \( s \not\to S \).

The results in Theorems 1.1, 1.2 and 1.6 have been shown in [8, 15, 17, 19] in the case of Minkowski spacetime (i.e. \( a_1 = 0 \)) for the diffusion equation (i.e. \( \kappa > 0 \)) in (1.13)). Our results extend these results to homogeneous and isotropic spacetime for the complex Ginzburg–Landau type equation with \( \kappa \in \mathbb{C} \) and \( \text{Re} \kappa > 0 \). By Theorems 1.1 and 1.6, we have obtained the critical exponent \( p_0 \) for the existence and non-existence of the global solution in several spacetimes, which works as the Fujita exponent. We note that \( p_0 = 1 + 4/n \) when \( \sigma = -1 + 4/n \) by (1.19). By Theorems 1.2 and 1.3, the global solution converges to the free solution when \( S_0 = \infty \) in (1.9), while it converges to the explicit function \( v \) in (1.25) when \( S_0 < \infty \). Theorem 1.8 is a generalization of [13, Corollary 3.8], which requires additional conditions \( 2/(\sin 2\omega)^2 - 1 < p, a_1(p - 1 - 4/n) \leq 0 \) and \( a_1(4 - n(1 + \sigma)) \leq 0 \). The condition \( 2/(\sin 2\omega)^2 - 1 < p \) is removed, and the cases \( a_1(p - 1 - 4/n) > 0 \) and \( a_1(4 - n(1 + \sigma)) > 0 \) are also considered in Theorem 1.8. The proof is based on the arguments in [3, Theorem 1.1] and [4, Theorem 1.1]. The proofs of the above theorems are given in the following sections from Section 2 to Section 6. We collect some fundamental results for the complex Ginzburg–Landau type equation which are used to prove the theorems in Appendix A.
2. Proof of Theorem 1.1

We use the following $L^p - L^q$ estimate for the complex Ginzburg–Landau type equation (see e.g. [3, (2.1)]).

**Proposition 2.1.** ($L^p - L^q$ estimate) Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Let $1 \leq q \leq p \leq \infty$. Then $u(t, x) := e^{(\alpha + i\beta)t} \Delta u_0(x) = (G((\alpha + i\beta)t, \cdot) * u_0)(x)$ satisfies the inequality

$$
\|u(t)\|_{L^p(\mathbb{R}^n)} \leq (4\pi t)^{n(1/p - 1/q)/2} \alpha^{-n(1+1/p - 1/q)/2} |\alpha + i\beta|^{n(1+2(1/p - 1/q))/2} \|u_0\|_{L^q(\mathbb{R}^n)}
$$

for any $t > 0$.

We divide the proof of Theorem 1.1 into the two cases $a_1 \{4 - n(1 + \sigma)\} \leq 0$ and $a_1 \{4 - n(1 + \sigma)\} > 0$.

2.1. The case $a_1 \{4 - n(1 + \sigma)\} \leq 0$

We prove Theorem 1.1 in the case $a_1 \{4 - n(1 + \sigma)\} \leq 0$ (namely, the cases (i), (ii), (iii), (vi) and (vii) in (1.11)). We regard the solution of the Cauchy problem (1.13) as the fixed point of the operator $\Phi$ defined by

$$
\Phi(u)(s) := e^{\kappa s \Delta} u_0 - \lambda \int_0^s e^{\kappa (s - \tau) \Delta} a(\tau)^{2 - n(p - 1)/2} f(u)(\tau) \, d\tau.
$$

(2.1)

Firstly, we derive the constant $M_1$ defined by (1.20) in the case (2) in Theorem 1.1, where $a_1 > 0$, $\sigma > -1 + 4/n$ and $S_1 < 0$. Putting $g(\tau) := (1 - \tau/S_1)/(1 + \tau)$, we have

$$
g'(\tau) = -\frac{1 + 1/S_1}{(1 + \tau)^2}.
$$

Thus, $g(\tau)$ is monotonically increasing if $|S_1| < 1$, and it is monotonically decreasing if $|S_1| \geq 1$. By

$$
\lim_{\tau \to \infty} \frac{1 - \tau/S_1}{1 + \tau} = -\frac{1}{S_1},
$$

we obtain

$$
g(\tau) \begin{cases} < -\frac{1}{S_1} & \text{if } |S_1| < 1, \\ \leq 1 & \text{if } |S_1| \geq 1. \end{cases}
$$
Thus we have

$$
\sup_{0 < \tau < s} \left( g(\tau) \right)^{2[2-n(p-1)/2]/\lfloor n(1+\sigma) - 4 \rfloor}
\begin{cases}
\sup_{0 < \tau < s} g(\tau) & \text{if } p \leq 1 + 4/n, \\
\inf_{0 < \tau < s} g(\tau) & \text{if } p > 1 + 4/n,
\end{cases}
\leq
\begin{cases}
(-S_1)^{-2[2-n(p-1)/2]/\lfloor n(1+\sigma) - 4 \rfloor} & \text{if } p \leq 1 + 4/n, \ |S_1| < 1, \\
1 & \text{if } p \leq 1 + 4/n, \ |S_1| \geq 1, \\
1 & \text{if } p > 1 + 4/n, \ |S_1| < 1, \\
(-S_1)^{-2[2-n(p-1)/2]/\lfloor n(1+\sigma) - 4 \rfloor} & \text{if } p > 1 + 4/n, \ |S_1| \geq 1,
\end{cases}
= M_1.
$$

Similarly, we can derive the constant $M_2$ defined by (1.21) in the case (7), where $a_1 < 0$, $\sigma < -1 + 4/n$ and $S_1 < 0$. Indeed, we have

$$
\sup_{0 < \tau < s} \left( g(\tau) \right)^{2[2-n(p-1)/2]/\lfloor n(1+\sigma) - 4 \rfloor}
\begin{cases}
\sup_{0 < \tau < s} g(\tau) & \text{if } p > 1 + 4/n, \\
\inf_{0 < \tau < s} g(\tau) & \text{if } p \leq 1 + 4/n,
\end{cases}
\leq
\begin{cases}
(-S_1)^{-2[2-n(p-1)/2]/\lfloor n(1+\sigma) - 4 \rfloor} & \text{if } p > 1 + 4/n, \ |S_1| < 1, \\
1 & \text{if } p > 1 + 4/n, \ |S_1| \geq 1, \\
1 & \text{if } p \leq 1 + 4/n, \ |S_1| < 1, \\
(-S_1)^{-2[2-n(p-1)/2]/\lfloor n(1+\sigma) - 4 \rfloor} & \text{if } p \leq 1 + 4/n, \ |S_1| \geq 1,
\end{cases}
= M_2.
$$

We show that $\Phi$ becomes the contraction mapping from $X(R)$ to $X(R)$ for some $R > 0$, which is determined later. By the $L^1 - L^1$ estimate (Proposition 2.1, below), we have

$$
\|\Phi(u)(s)\|_{L^1} \leq \|e^{\epsilon s\Delta} u_0\|_{L^1} + |\lambda| \int_0^s a(\tau)^{2-n(p-1)/2} \|e^{\epsilon (s-\tau)\Delta} f(u)(\tau)\|_{L^1} \, d\tau
\lesssim \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} \left\{ \|u_0\|_{L^1} + |\lambda| \int_0^s a(\tau)^{2-n(p-1)/2} \|f(u)(\tau)\|_{L^1} \, d\tau \right\}
\lesssim \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} \left\{ \|u_0\|_{L^1} + |\lambda| \int_0^\infty F(\tau) \, d\tau \right\},
$$

where we have used the estimate

$$
\|f(u)(\tau)\|_{L^1} \leq \|u(\tau)\|_{L^\infty}^{p-1} \|u(\tau)\|_{L^1} \leq (1 + \tau)^{-n(p-1)/2} \|u\|_{X}^p,
$$

(2.2)
the fact that $|\alpha + i\beta| = h/2m$ by (1.14), and we have put

$$F(\tau) := a(\tau)^{2-(p-1)/2}(1 + \tau)^{-n(p-1)/2}$$

\begin{align*}
\lesssim \begin{cases} 
(1 + \tau)^{-n(p-1)/2} & \text{for (1)}, \\
M_1(1 + \tau)^{p_\ast} & \text{for (2)}, \\
(1 + \tau)^{-n(p-1)/2} & \text{for (3) with } p \geq 1 + 4/n, \\
(1 + \tau)^{-n(p-1)/2} & \text{for (6) with } p \leq 1 + 4/n, \\
M_2(1 + \tau)^{p_\ast} & \text{for (7)},
\end{cases}
\end{align*}

and

$$p_\ast := \frac{4 - n(p - 1)}{n(1 + \sigma) - 4} - \frac{n(p - 1)}{2}.$$ 

Thus, we obtain

$$\|\Phi(u)(s)\|_{L^1} \leq \alpha^{-n/2} \left( \frac{2m}{h} \right)^{n/2} (C_0 \|u_0\|_{L^1} + |\lambda| B \|u\|^p_X), \quad (2.4)$$

where we have put

$$B := \int_0^\infty F(\tau) \, d\tau = \begin{cases} 
C & \text{for (1) with } p > 1 + 2/n, \\
CM_1 & \text{for (2) with } p > p_0, \\
C & \text{for (3) with } p \geq 1 + 4/n, \\
C & \text{for (6) with } 1 + 2/n < p \leq 1 + 4/n, \\
CM_2 & \text{for (7) with (1.23)},
\end{cases} \quad (2.5)$$

and $C_0$ and $C$ are some positive constants. The condition on $p$ in (2.5) yields the integrability of $F(\cdot)$ on $[0, \infty)$.

Next, we estimate $\|\Phi(u)(s)\|_{L^\infty}$. We have

$$\|\Phi(u)(s)\|_{L^\infty} \leq \|e^{ks \Delta} u_0\|_{L^\infty} + |\lambda| \int_0^s a(\tau)^{2-(p-1)/2} \|e^{k(s-\tau)\Delta} f(u)(\tau)\|_{L^\infty} \, d\tau. \quad (2.6)$$

By the $L^\infty - L^1$ and $L^\infty - L^\infty$ estimates given by

$$\|e^{ks \Delta} u_0\|_{L^\infty} \leq (4\pi s)^{-n/2} |\alpha + i\beta|^{-n/2} \|u_0\|_{L^1} \leq (4\pi s)^{-n/2} \alpha^{-n/2} \|u_0\|_{L^1}$$

and

$$\|e^{ks \Delta} u_0\|_{L^\infty} \leq \alpha^{-n/2} |\alpha + i\beta|^{n/2} \|u_0\|_{L^\infty},$$

(see Proposition 2.1, below), we have

$$\|e^{ks \Delta} u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim \alpha^{-n/2} (1 + s)^{-n/2} \|u_0\|_{L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)}.$$ 

We put

$$I := |\lambda| \int_0^s a(\tau)^{2-(p-1)/2} \|e^{k(s-\tau)\Delta} f(u)(\tau)\|_{L^\infty} \, d\tau. \quad (2.7)$$

We have

$$\|f(u)(\tau)\|_{L^\infty} \leq \|u(\tau)\|_{L^\infty} \leq (1 + \tau)^{-np/2} \|u\|^p_X. \quad (2.8)$$
When $0 \leq s \leq 1$, we have
\[
I \leq |\lambda| \alpha^{-n/2} |\alpha + i \beta|^{n/2} \int_0^{s} a(\tau)^{2^n (p - 1)/2} (1 + \tau)^{-n \beta/2} \, d\tau \|u\|_X^p
\]
\[
\lesssim |\lambda| \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (1 + s)^{-n/2} \int_0^{\infty} F(\tau) \, d\tau \|u\|_X^p
\]
\[
\lesssim |\lambda| B \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (1 + s)^{-n/2} \|u\|_X^p, \tag{2.9}
\]
by $|\alpha + i \beta| = h/2m$, (2.3) and (2.5). When $s \geq 1$, we divide $I$ as
\[
I = |\lambda| \int_0^{s/2} a(\tau)^{2^n (p - 1)/2} e^{\kappa(s - \tau) \Delta} f(u)(\tau) \parallel L^\infty \, d\tau
\]
\[
+ |\lambda| \int_{s/2}^{s} a(\tau)^{2^n (p - 1)/2} e^{\kappa(s - \tau) \Delta} f(u)(\tau) \parallel L^\infty \, d\tau
\]
\[
=: II + III, \tag{2.10}
\]
where we have put the terms on the right-hand side as $II$ and $III$. First, we estimate $II$. By the $L^\infty - L^1$ estimate, we have
\[
II \leq |\lambda| \int_0^{s/2} a(\tau)^{2^n (p - 1)/2} \{4\pi(s - \tau)^{n/2} |\alpha + i \beta|^{-n/2} \|f(u)(\tau)\|_L^1 \, d\tau
\]
\[
\lesssim |\lambda| \alpha^{-n/2} (1 + s)^{-n/2} \int_0^{\infty} F(\tau) \, d\tau \|u\|_X^p
\]
\[
\lesssim |\lambda| B \alpha^{-n/2} (1 + s)^{-n/2} \|u\|_X^p \quad \text{for } s \geq 1. \tag{2.11}
\]
Next, we estimate $III$. By the $L^\infty - L^\infty$ estimate, we have
\[
III \leq |\lambda| \int_{s/2}^{s} a(\tau)^{2^n (p - 1)/2} \alpha^{-n/2} |\alpha + i \beta|^{n/2} \|f(u)(\tau)\|_L^\infty \, d\tau
\]
\[
\lesssim |\lambda| \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (1 + s)^{-n/2} \int_0^{\infty} F(\tau) \, d\tau \|u\|_X^p
\]
\[
\lesssim |\lambda| B \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (1 + s)^{-n/2} \|u\|_X^p \quad \text{for } s \geq 1. \tag{2.12}
\]
By (2.9), (2.10), (2.11) and (2.12), we obtain
\[
I \lesssim |\lambda| B \alpha^{-n/2} \left\{ \left( \frac{h}{2m} \right)^{n/2} + 1 \right\} (1 + s)^{-n/2} \|u\|_X^p \quad \text{for } 0 \leq s < \infty. \tag{2.13}
\]
Therefore, by (2.6) and (2.13), we have
\[
(1 + s)^{n/2} \|\Phi(u)(s)\|_L^\infty \leq \alpha^{-n/2} \left\{ \left( \frac{h}{2m} \right)^{n/2} + 1 \right\} (C_0 \|u_0\|_{L^1 \cap L^\infty} + |\lambda| B \|u\|_X^p) \tag{2.14}
\]
for $0 \leq s < \infty$. By (2.4) and (2.14), we obtain
\[
\|\Phi(u)\|_X \leq \alpha^{-n/2} (C_0 \|u_0\|_{L^1 \cap L^\infty} + |\lambda| B \|u\|_X^p). \tag{2.15}
\]
Now, we assume $u, v \in X(R)$. By
\[ \Phi(u)(s) - \Phi(v)(s) = -\lambda \int_0^s e^{\kappa(s-\tau)\Delta} a(\tau) 2^{-n(p-1)/2} \{ f(u)(\tau) - f(v)(\tau) \} \, d\tau \]
and the estimates
\[ \| f(u)(\tau) - f(v)(\tau) \|_{L^1} \lesssim (\| u(\tau) \|_{L^\infty}^{p-1} + \| v(\tau) \|_{L^\infty}^{p-1}) \| u(\tau) - v(\tau) \|_{L^1} \]
\[ \lesssim (1 + \tau)^{-n(p-1)/2} \max_{w=u,v} \| w \|_{L^\infty}^{p-1} \| u - v \|_X \]
and
\[ \| f(u)(\tau) - f(v)(\tau) \|_{L^\infty} \lesssim (\| u(\tau) \|_{L^\infty}^{p-1} + \| v(\tau) \|_{L^\infty}^{p-1}) \| u(\tau) - v(\tau) \|_{L^\infty} \]
\[ \lesssim (1 + \tau)^{-np/2} \max_{w=u,v} \| w \|_{L^\infty}^{p-1} \| u - v \|_X, \]
which follow similarly to (2.2) and (2.8), we obtain
\[ \| \Phi(u)(s) - \Phi(v)(s) \|_{L^1} \]
\[ \leq |\lambda| \int_0^s a(\tau) 2^{-n(p-1)/2} \| e^{\kappa(s-\tau)\Delta} \{ f(u)(\tau) - f(v)(\tau) \} \|_{L^1} \, d\tau \]
\[ \leq |\lambda| |\alpha|^{-n/2} |\alpha + i\beta|^{n/2} \int_0^s a(\tau) 2^{-n(p-1)/2} \| f(u)(\tau) - f(v)(\tau) \|_{L^1} \, d\tau \]
\[ \lesssim |\lambda| |B\alpha|^{-n/2} \left( \frac{h}{2m} \right)^{n/2} \max_{w=u,v} \| w \|_{L^\infty}^{p-1} \| u - v \|_X \]
\[ (2.16) \]
for $0 \leq s < \infty$. We have
\[ (1 + s)^{n/2} \| \Phi(u)(s) - \Phi(v)(s) \|_{L^\infty} \]
\[ \leq |\lambda|(1 + s)^{n/2} \int_0^s a(\tau) 2^{-n(p-1)/2} \| e^{\kappa(s-\tau)\Delta} \{ f(u)(\tau) - f(v)(\tau) \} \|_{L^\infty} \, d\tau \]
\[ \leq |\lambda| |\alpha|^{-n/2} |\alpha + i\beta|^{n/2} (1 + s)^{n/2} \int_0^s a(\tau) 2^{-n(p-1)/2} \| f(u)(\tau) - f(v)(\tau) \|_{L^\infty} \, d\tau \]
\[ \lesssim |\lambda| |B\alpha|^{-n/2} \left( \frac{h}{2m} \right)^{n/2} \max_{w=u,v} \| w \|_{L^\infty}^{p-1} \| u - v \|_X \]
\[ (2.17) \]
for $0 \leq s \leq 1$. For $s \geq 1$, we have
\[ \| \Phi(u)(s) - \Phi(v)(s) \|_{L^\infty(\mathbb{R}^n)} \]
\[ \leq |\lambda| \int_0^{s/2} a(\tau) 2^{-n(p-1)/2} \| e^{\kappa(s-\tau)\Delta} \{ f(u)(\tau) - f(v)(\tau) \} \|_{L^\infty(\mathbb{R}^n)} \, d\tau \]
\[ + |\lambda| \int_{s/2}^s a(\tau) 2^{-n(p-1)/2} \| e^{\kappa(s-\tau)\Delta} \{ f(u)(\tau) - f(v)(\tau) \} \|_{L^\infty(\mathbb{R}^n)} \, d\tau \]
\[ =: IV + V. \]
Similarly to (2.11) and (2.12), we have
\[ IV \lesssim |\lambda| B a^{-n/2} (1 + s)^{-n/2} \max_{w = u, v} \| w \|_X^{p-1} \| u - v \|_X \]
and
\[ V \lesssim |\lambda| B a^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (1 + s)^{-n/2} \max_{w = u, v} \| w \|_X^{p-1} \| u - v \|_X \]
for \( s \geq 1 \). Thus, we have
\begin{align*}
(1 + s)^{n/2} \| \Phi(u)(x) - \Phi(v)(x) \|_{L^\infty(\mathbb{R}^n)} & \lesssim |\lambda| B a^{-n/2} \left( \frac{h}{2m} \right)^{n/2} + 1 \max_{w = u, v} \| w \|_X^{p-1} \| u - v \|_X \\
& \quad \text{for } s \geq 1.
\end{align*}

By (2.16), (2.17) and (2.18), we obtain
\[ \| \Phi(u)(s) - \Phi(v)(s) \|_X \leq |\lambda| B a^{-n/2} \max_{w = u, v} \| w \|_X^{p-1} \| u - v \|_X. \quad (2.19) \]

By (2.15) and (2.19), \( \Phi \) becomes a contraction mapping from \( X(R) \) to \( X(R) \) under the conditions
\[ a^{-n/2} (C_0 \| u_0 \|_{L^1 \cap L^\infty} + |\lambda| B R^p) \leq R, \quad 2|\lambda| B a^{-n/2} R^{n-1} \leq 1. \]
So that, if \( \| u_0 \|_{L^1 \cap L^\infty}, B \) and \( R \) satisfy
\[ 2C_0 a^{-n/2} \| u_0 \|_{L^1 \cap L^\infty} \leq R \leq \left( \frac{a^{n/2}}{2|\lambda| B} \right)^{1/(p-1)}, \]
then there exists a unique solution \( u = \Phi(u) \) for (1.13) in \( X(R) \).

2.2. The case \( a_1[4 - n(1 + \sigma)] > 0 \)

We prove Theorem 1.1 in the case \( a_1[4 - n(1 + \sigma)] > 0 \) (namely, the cases (iv) and (v) in (1.11)). By the \( L^\infty - L^\infty \) estimate, we have
\begin{align*}
\| \Phi(u)(s) \|_{L^\infty} & \leq e^{ks\Delta} u_0 \|_{L^\infty} + |\lambda| \int_0^s a(\tau)^{2-n(p-1)/2} e^{k(s-\tau)\Delta} f(u)(\tau) \|_{L^\infty} \, d\tau \\
& \leq a^{-n/2} \left( \frac{h}{2m} \right)^{n/2} \left( \| u_0 \|_{L^\infty} + |\lambda| \int_0^{S_1} a(\tau)^{2-n(p-1)/2} \| f(u)(\tau) \|_{L^\infty} \, d\tau \right).
\end{align*}

By
\[ \| f(u)(\tau) \|_{L^\infty} \lesssim (1 + \tau)^{-np/2} \| u \|_X^p \lesssim \| u \|_X^p \]
for \( \tau \in [0, S_1] \),
\[ \int_0^{S_1} a(\tau)^{2-n(p-1)/2} \, d\tau = S_1 \left\{ \frac{2}{n(1 + \sigma) - 4} \left( 2 - \frac{n(p-1)}{2} \right) + 1 \right\}^{-1}. \quad (2.20) \]
the condition (1.22) for the case (iv), and the condition $p < 2 + \sigma$ for the case (v), we obtain

$$\|\Phi(u)\|_X \leq \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (\|u_0\|_{L^\infty} + C|\lambda| S_1 \|u\|_X^p)$$  \hspace{1cm} (2.21)

for some constant $C > 0$. Similarly, we also have

$$\|\Phi(u(s)) - \Phi(v(s))\|_{L^\infty} \leq |\lambda| \int_0^s a(\tau)^{2-n(p-1)/2} \|e^{\kappa (s-\tau)\Delta} \{ f(u(\tau)) - f(v(\tau)) \}\|_{L^\infty} d\tau$$

$$\leq C|\lambda| \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} S_1 \max_{w=u,v} \|w\|_X^{p-1} \|u - v\|_X$$  \hspace{1cm} (2.22)

for any $u, v \in X(R)$. By (2.21) and (2.22), $\Phi$ becomes the contraction mapping from $X(R)$ to $X(R)$ under the conditions

$$\begin{cases}
\alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} (\|u_0\|_{L^\infty} + C|\lambda| S_1 R^p) \leq R, \\
2C|\lambda| \alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} S_1 R^{p-1} \leq 1.
\end{cases}$$

So that, if $\|u_0\|_{L^\infty}$ and $R$ satisfy

$$2\alpha^{-n/2} \left( \frac{h}{2m} \right)^{n/2} \|u_0\|_{L^\infty} \leq R \leq \left( \frac{\alpha^{n/2}}{2C|\lambda| S_1} \right)^{1/(p-1)} \left( \frac{h}{2m} \right)^{-n/2(p-1)},$$

then there exists a unique solution $u = \Phi(u)$ for (1.13) in $X(R)$.

3. Proof of Theorem 1.2

In this section, we prove the asymptotic behaviour of the global solutions that were obtained in the cases (1), (2), (3), (6) and (7) in Theorem 1.1. The proof is based on $[9, 15]$. We put

$$\theta_0 := \int_{\mathbb{R}^n} u_0(y) \, dy, \quad \theta_1 := \lambda \int_0^\infty \int_{\mathbb{R}^n} a(\tau)^{2-n(p-1)/2} f(u(\tau, y)) \, dy \, d\tau.$$

We note that

$$u(s, x) - \theta G(\kappa s, x) = e^{\kappa s\Delta} u_0 - \theta_0 G(\kappa s, x) + \theta_1 G(\kappa s, x)$$

$$- \lambda \int_0^s e^{\kappa (s-\tau)\Delta} a(\tau)^{2-n(p-1)/2} f(u(\tau)) \, d\tau$$  \hspace{1cm} (3.1)

by $\theta = \theta_0 - \theta_1$.

3.1. Convergence in $L^\infty(\mathbb{R}^n)$

First, we show that

$$\|e^{\kappa s\Delta} u_0 - \theta_0 G(\kappa s, \cdot)\|_{L^\infty} = o(s^{-n/2}) \quad \text{as} \ s \to \infty.$$


We use the estimate
\[ s^{n/2} |e^{\kappa s \Delta} u_0(x) - \theta_0 G(\kappa s, x)| \]
\[ \leq s^{n/2} \int_{\mathbb{R}^n} |G(\kappa s, x - y) - G(\kappa s, x)| |u_0(y)| \, dy \]
\[ \leq (4\pi |\kappa|)^{-n/2} \int_{\mathbb{R}^n} |e^{-|x-y|^2/4\kappa s} - e^{-|x|^2/4\kappa s}| |u_0(y)| \, dy \]
\[ \lesssim \int_{|y| < s^{1/4}} |e^{-|x-y|^2/4\kappa s} - e^{-|x|^2/4\kappa s}| |u_0(y)| \, dy + \int_{|y| > s^{1/4}} |u_0(y)| \, dy. \] (3.2)

By \( u_0 \in L^1(\mathbb{R}^n) \), the second term on the right-hand side in (3.2) converges to 0 as \( s \to \infty \). The first term also converges to 0 as \( s \to \infty \). Indeed, we have
\[ |e^{-|x-y|^2/4\kappa s} - e^{-|x|^2/4\kappa s}| \leq |y| \int_0^1 |(\nabla e^{-|x|^2/4\kappa s})(x - (1-\theta)y)| \, d\theta. \]

Since we have
\[ |\nabla e^{-|x|^2/4\kappa s}| \leq |x| e^{-|x|^2} \text{Re} \kappa / (4|x|^2 s) / (2|\kappa| s) \]
\[ = (\text{Re} \kappa)^{-1/2} z e^{-z^2} s^{-1/2} \]
\[ \lesssim s^{-1/2}, \]
where we have put \( z := (\text{Re} \kappa)^{1/2}|x|/2|\kappa| s^{-1/2} \), we obtain
\[ \int_{|y| < s^{1/4}} |e^{-|x-y|^2/4\kappa s} - e^{-|x|^2/4\kappa s}| |u_0(y)| \, dy \]
\[ \lesssim s^{-1/2} \int_{|y| < s^{1/4}} |y| |u_0(y)| \, dy \]
\[ \lesssim s^{-1/4} \int_{|y| < s^{1/4}} |u_0(y)| \, dy \]
\[ \to 0 \quad \text{as} \ s \to \infty. \]

Thus we obtain
\[ s^{n/2} \|e^{\kappa s \Delta} u_0(\cdot) - \theta_0 G(\kappa s, \cdot)\|_{L^\infty} \to 0 \quad \text{as} \ s \to \infty. \] (3.3)

Next, we have
\[ s^{n/2} \left| \lambda \int_0^s e^{\kappa (s-\tau) \Delta} a(\tau)^{2-n(p-1)/2} f(u)(\tau) \, d\tau - \theta_1 G(\kappa s, x) \right| \]
\[ \leq |\lambda| s^{n/2} \left| \int_0^s a(\tau)^{2-n(p-1)/2} \int_{\mathbb{R}^n} G(\kappa(s-\tau), x - y) f(u)(\tau, y) \, dy \, d\tau \right| \]
\[ - \int_0^\infty a(\tau)^{2-n(p-1)/2} \int_{\mathbb{R}^n} G(\kappa s, x) f(u)(\tau, y) \, dy \, d\tau \]
\[ \lesssim s^{n/2} \int_0^{s/2} a(\tau)^{2-n(p-1)/2} \int_{\mathbb{R}^n} |G(\kappa(s - \tau), x - y) - G(\kappa s, x)| \cdot |f(u)(\tau, y)| \, dy \, d\tau \]
+ \[ s^{n/2} \int_{s/2}^{s} a(\tau)^{2-n(p-1)/2} \int_{\mathbb{R}^n} |G(\kappa(s - \tau), x - y)| \cdot |f(u)(\tau, y)| \, dy \, d\tau \]
+ \[ \int_{s/2}^{\infty} \int_{\mathbb{R}^n} a(\tau)^{2-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \]
\[ =: I + II + III. \]  

(3.4)

(3.5)

where we have put the terms on the right-hand side as \( I, II \) and \( III \). For \( II \), by the Hölder inequality, we have

\[ II \lesssim s^{n/2} \int_{s/2}^{s} a(\tau)^{2-n(p-1)/2} \|G(\kappa(s - \tau), \cdot)\|_{L^1} \|u(\tau, \cdot)\|^p_{L^\infty} \, d\tau \]
\[ \lesssim s^{n/2} \int_{s/2}^{s} a(\tau)^{2-n(p-1)/2} (1 + \tau)^{-n(p-1)/2} (1 + \tau)^{-n/2} \|u\|^p_X \]
\[ \lesssim \int_{s/2}^{s} F(\tau) \, d\tau \|u\|^p_X \]
\[ \to 0 \quad \text{as} \quad s \to \infty, \]  

(3.6)

where \( F \) is the function defined by (2.3) and \( F \) is integrable on \([0, \infty)\). For the term \( III \), we obtain

\[ III \lesssim \int_{s/2}^{\infty} a(\tau)^{2-n(p-1)/2} (1 + \tau)^{-n(p-1)/2} \, d\tau \|u\|^p_X \]
\[ \lesssim \int_{s/2}^{\infty} F(\tau) \, d\tau \|u\|^p_X \]
\[ \to 0 \quad \text{as} \quad s \to \infty \]  

(3.7)

by

\[ \int_{\mathbb{R}^n} |f(u)(\tau, y)| \, dy \lesssim \|u(\tau)\|^{p-1}_{L^\infty} \|u(\tau)\|_{L^1} \lesssim (1 + \tau)^{-n(p-1)/2} \|u\|^p_X. \]

Next, we estimate \( I \). We have

\[ I = s^{n/2} \left( \int_{\Omega_1} + \int_{\Omega_2} \right) a(\tau)^{2-n(p-1)/2} \]
\[ \times |G(\kappa(s - \tau), x - y) - G(\kappa s, x)| \cdot |f(u)(\tau, y)| \, dy \, d\tau, \]

where the domains \( \Omega_1 \) and \( \Omega_2 \) are defined by

\[ \Omega_1 := [0, \delta s] \times \{ y \in \mathbb{R}^n \mid |y| \leq \delta s^{1/2} \}, \quad \Omega_2 := ([0, s/2] \times \mathbb{R}^n) \setminus \Omega_1 \]  

(3.8)
for arbitrarily fixed $0 < \delta < 1/2$. On $\Omega_2$, by $\int_0^\infty \int_{\mathbb{R}^n} a(\tau)^{2-n(p-1)/2} |u(\tau, y)|^p \, dy \, d\tau < \infty$, we obtain
\[
\begin{align*}
&\frac{s^{n/2}}{n^{1/2}} \int_{\Omega_2} a(\tau)^{2-n(p-1)/2} \left| G(\kappa(s - \tau), x - y) - G(\kappa s, x) \right| \cdot |f(u)(\tau, y)| \, dy \, d\tau \\
\leq& \frac{s^{n/2}}{n^{1/2}} \int_{\Omega_2} a(\tau)^{2-n(p-1)/2} (s - \tau)^{-n/2} |f(u)(\tau, y)| \, dy \, d\tau \\
&+ \int_{\Omega_2} a(\tau)^{2-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \\
\leq& \int_{\Omega_2} a(\tau)^{2-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \\
\rightarrow& 0 \quad \text{as} \quad s \rightarrow \infty
\end{align*}
\] (3.9)
by the Lebesgue convergence theorem, where we have used
\[
\begin{align*}
\int_0^\infty \int_{\mathbb{R}^n} a(\tau)^{2-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \\
\leq& \int_0^\infty a(\tau)^{2-n(p-1)/2} (1 + \tau)^{-n(p-1)/2} d\tau \cdot \|u\|_X^p \\
=& \int_0^\infty F(\tau) \, d\tau \cdot \|u\|_X^p \\
=& B \|u\|_X^p < \infty
\end{align*}
\] (3.10)
by (2.5).

Let us consider $\Omega_1$. Putting $\tau = st'$ and $y = \sqrt{s}z$, we have
\[
(\tau, y) \in \Omega_1 \iff (t', z) \in [0, \delta] \times \{z \in \mathbb{R}^n \mid |z| \leq \delta\}.
\]
We have
\[
\begin{align*}
G(\kappa(s - \tau), x - y) &= G(\kappa s, x) \\
&= s^{-n/2} \{G(\kappa(1 - t'), x/s^{1/2} - z) - G(\kappa, x/s^{1/2})\} \\
&= s^{-n/2} \int_0^1 \frac{d}{d\theta} G(\kappa(1 - \theta t'), x/s^{1/2} - \theta z) \, d\theta 
\end{align*}
\] (3.11)
and
\[
\frac{d}{d\theta} G(\kappa(1 - \theta t'), x/s^{1/2} - \theta z) = (-\kappa t' \partial_r G - z \cdot \nabla G)(\kappa(1 - \theta t'), x/s^{1/2} - \theta z).
\]
Since we have
\[
\partial_r G(\kappa(1 - \theta t'), x/s^{1/2} - \theta z) = \left( \frac{n}{2(1 - \theta t')} + \frac{|x/s^{1/2} - \theta z|^2}{4\pi(1 - \theta t')^2} \right) G(\kappa(1 - \theta t'), x/s^{1/2} - \theta z)
\]
and
\[
\nabla G(\kappa(1 - \theta t'), x/s^{1/2} - \theta z) = -\frac{x/s^{1/2} - \theta z}{2\kappa(1 - \theta t')} G(\kappa(1 - \theta t'), x/s^{1/2} - \theta z),
\]
by Lemma A.2, we have
\[ \left| \frac{d}{d\theta} G(\kappa(1 - \theta^t), x/s^{1/2} - \theta z) \right| \]
\[ \leq \left\{ \left( \frac{n}{2(1 - \theta^t)} + \frac{|x/s^{1/2} - \theta z|^2}{4\pi(1 - \theta^t)^2} \right) |\kappa^t| + \frac{|x/s^{1/2} - \theta z||z|}{2|\kappa|(1 - \theta^t)} \right\} \]
\[ \times |G(\kappa(1 - \theta^t), x/s^{1/2} - \theta z)| \]
\[ \lesssim \delta (x/s^{1/2} - \theta z)^2 e^{-|x/s^{1/2} - \theta z|^2/\alpha/2} |\kappa|^2 \]
\[ \lesssim \delta (x/s^{1/2} - \theta z)^2 e^{-|x/s^{1/2} - \theta z|^2/\alpha/2} |\kappa|^2 \]
\[ (3.12) \]
by (7). Thus we obtain
\[ |G(\kappa(s - \tau), x - y) - G(\kappa s, x)| \]
\[ \lesssim \delta s^{-n/2} \int_0^1 (x/s^{1/2} - \theta z)^2 e^{-|x/s^{1/2} - \theta z|^2/\alpha/2} \, d\theta \]
\[ \leq \delta s^{-n/2} \sup_{w \in \mathbb{R}^n} (w)^2 e^{-|w|^2/\alpha/2} |\kappa|^2 \]
\[ (3.13) \]
for \((\tau, y) \in \Omega_1\) by (3.11) and (3.12).
By (3.13), we have
\[ s^{n/2} |G(\kappa(s - \tau), x - y) - G(\kappa s, x)| \lesssim \delta \]
for \((\tau, y) \in \Omega_1\). Thus we obtain
\[ s^{n/2} \int_{\Omega_1} a(\tau)^{2-n(p-1)/2} |G(\kappa(s - \tau), x - y) - G(\kappa s, x)| \, dy \, d\tau \]
\[ \lesssim \delta \int_0^\infty \int_{\mathbb{R}^n} a(\tau)^{2-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau. \]
\[ (3.14) \]
Since \(\delta > 0\) is arbitrary, we obtain
\[ I \to 0 \quad \text{as } s \to \infty \]
\[ (3.15) \]
by (3.9) and (3.14). By (3.5), (3.6), (3.7) and (3.15), we have
\[ s^{n/2} \left\| \int_0^s e^{k(s - \tau)\Delta} a(\tau)^{2-n(p-1)/2} f(u)(\tau) \, d\tau - \theta_1 G(\kappa s, x) \right\|_{L^\infty} \]
\[ \to 0 \quad \text{as } s \to \infty. \]
\[ (3.16) \]
So that, by (3.3) and (3.16), we obtain
\[ \|u(s, \cdot) - \theta G(\kappa s, \cdot)\|_{L^\infty} = o(s^{-n/2}) \quad \text{as } s \to \infty. \]
\[ (3.17) \]

3.2. Convergence in \(L^1(\mathbb{R}^n)\)
We show that
\[ \|u(s, \cdot) - \theta G(\kappa s, \cdot)\|_{L^1} = o(1) \quad \text{as } s \to \infty. \]
We put
\[ h(s, x, y) := G(\kappa s, x - y) - G(\kappa s, x). \]
Let us consider the line $l$ connecting $0$ with $(2x \cdot y - |y|^2)/4\kappa s$ on the plane $\mathbb{C}$. Put $g(z) := e^z$ for $z \in \mathbb{C}$. By the Wolff–Noshiro theorem, there exist points $\Gamma_1, \Gamma_2 \in l \setminus \{0, (2x \cdot y - |y|^2)/4\kappa s\}$ such that

$$g\left(\frac{2x \cdot y - |y|^2}{4\kappa s}\right) - g(0) = \frac{2x \cdot y - |y|^2}{4\kappa s} (\text{Re } g'(\Gamma_1) + i \text{ Im } g'(\Gamma_2)).$$

Since there exist points $\gamma_1, \gamma_2 \in (0, 1)$ such that

$$\Gamma_1 = \frac{\gamma_1(2x \cdot y - |y|^2)}{4\kappa s}, \quad \Gamma_2 = \frac{\gamma_2(2x \cdot y - |y|^2)}{4\kappa s},$$

we have

$$e^{\frac{(2x \cdot y - |y|^2)^2}{4\kappa s}} - 1 = \frac{2x \cdot y - |y|^2}{4\kappa s} \left\{ e^{\frac{(2x \cdot y - |y|^2)^2}{4\kappa s}} \cos \left(\frac{2x \cdot y - |y|^2}{4\kappa s}\right) + e^{\frac{(2x \cdot y - |y|^2)^2}{4\kappa s}} \sin \left(\frac{2x \cdot y - |y|^2}{4\kappa s}\right) \right\}. \quad (3.18)$$

Thus, we obtain

$$|(4\pi \kappa s)^{1/2} h(s, x, y)|$$

$$= e^{-\alpha|x|^2/4\kappa s} \left| e^{\frac{(2x \cdot y - |y|^2)^2}{4\kappa s}} - 1 \right|$$

$$\leq e^{-\alpha|x|^2/4\kappa s} \left| \frac{2x \cdot y - |y|^2}{4\kappa s} \left( e^{\frac{(2x \cdot y - |y|^2)^2}{4\kappa s}} + e^{\frac{(2x \cdot y - |y|^2)^2}{4\kappa s}} \right) \right.$$ 

$$= \frac{2x \cdot y - |y|^2}{4\kappa s} \left( e^{\alpha(-|x|^2 + \xi_1)/4\kappa s} + e^{\alpha(-|x|^2 + \xi_2)/4\kappa s} \right). \quad (3.19)$$

where we have put $\xi_1 := \gamma_1(2x \cdot y - |y|^2)$ and $\xi_2 := \gamma_2(2x \cdot y - |y|^2)$. If $2x \cdot y - |y|^2 \geq 0$, then (3.19) yields

$$\frac{|4\kappa (4\pi \kappa s)^{1/2} h(s, x, y)|}{s}$$

$$\leq \frac{2(2|x - y|(|y| + |y|^2))}{s} e^{-\alpha|x-y|^2/4\kappa s}$$

$$\lesssim \frac{|y|s^{-1/2}}{s^{1/2}} e^{-\alpha|x-y|^2/4\kappa s}s^{1/2} \lesssim \left| A_{\eta}s^{-1/2} |y|^2 \right. e^{-\alpha|x-y|^2/4\kappa s}$$

by $2x \cdot y - |y|^2 = 2(x - y) \cdot y + |y|^2, \alpha > 0, |x - y|^2 \leq 2(x - y) \cdot y + |y|^2 \leq x^2 - \xi_1$ and $|x - y|^2 \leq |x|^2 - \xi_2$, where we have put $\mu := 1/(1 - \eta)$ for $0 < \eta < 1$, and $A_{\eta} := \sup_{z \geq 0} z e^{-\eta z^2/4} < \infty$. Thus, we obtain

$$|h(s, x, y)| \lesssim \left( A_{\eta}s^{-1/2} |y| + s^{-1} |y|^2 \right) G\left( \frac{|y|^2}{\alpha s}, x - y \right). \quad (3.20)$$
If $2x \cdot y - |y|^2 < 0$, then (3.19) yields

$$|4\kappa (4\pi \kappa s)^{n/2} h(s, x, y)|$$

$$\leq \frac{2|2x \cdot y - |y|^2|}{\eta_s} e^{-\alpha|x|^2/4\kappa^2 s}$$

$$\leq 2|y|^{-1/2} |x|^{1/2} e^{-\eta |x|^2/4\kappa^2 s - (1-\eta)\alpha|x|^2/4\kappa^2 s} + |y|^2 s^{-1} e^{-\alpha|x|^2/4\kappa^2 s}$$

$$\lesssim (A_\eta |y|^{-1/2} + |y|^2 s^{-1}) e^{-\alpha|x|^2/4\kappa^2 s}$$

by $\zeta_1 < 0, \zeta_2 < 0$ and $\mu > 1$. Thus, the inequality

$$|h(s, x, y)| \lesssim (A_\eta |y|^{-1/2} + |y|^2 s^{-1}) G\left(\frac{|x|^2 \mu}{\alpha} , s \right)$$

holds. By (3.20) and (3.21), we obtain

$$\|h(s, x, y)\|_{L^1} \lesssim A_\eta |y|^{-1/2} + |y|^2 s^{-1}.$$  \hspace{1cm} (3.22)

By

$$e^{ksA} u_0(x) - \theta_0 G(\kappa s, x) = \int_{\mathbb{R}^n} h(s, x, y) u_0(y) \ dy,$$

we have

$$\|e^{ksA} u_0(x) - \theta_0 G(\kappa s, x)\|_{L^1}$$

$$\leq \int_{|y| < s^{-1/4}} \|h(s, x, y)\|_{L^1} |u_0(y)| \ dy + \int_{|y| > s^{-1/4}} \|h(s, x, y)\|_{L^1} |u_0(y)| \ dy.$$

First, we estimate the first term on the right-hand side. By $u_0 \in L^1(\mathbb{R}^n)$ and the estimate (3.22), we have

$$\int_{|y| < s^{-1/4}} \|h(s, x, y)\|_{L^1} |u_0(y)| \ dy$$

$$\leq (s^{-1/4} + s^{-1/2}) \int_{\mathbb{R}^n} |u_0(y)| \ dy \to 0 \quad \text{as} \ s \to \infty.$$

For the second term, we have

$$\int_{|y| > s^{-1/4}} \|h(s, x, y)\|_{L^1} |u_0(y)| \ dy \lesssim \int_{|y| > s^{-1/4}} |u_0(y)| \ dy \to 0 \quad \text{as} \ s \to \infty$$

by $\|h(s, x, y)\|_{L^1} \leq \|G(\kappa s, x - y)\|_{L^1} + \|G(\kappa s, x)\|_{L^1} \lesssim 1$ due to Lemma A.1. Thus, we obtain

$$\|e^{ksA} u_0(\cdot) - \theta_0 G(\kappa s, \cdot)\|_{L^1} \to 0 \quad \text{as} \ s \to \infty. \hspace{1cm} (3.23)$$
Next, we consider the inhomogeneous part. We have
\[ \left\| \lambda \int_0^s e^{k(s-\tau)} a(\tau) 2^{-n(p-1)/2} f(u)(\tau) \, d\tau - \theta_1 G(\kappa s, x) \right\|_{L^1_s} \]
\[ \lesssim \int_0^{s/2} \int_{\mathbb{R}^n} \|G(\kappa(s - \tau), x - y) - G(\kappa s, x)\|_{L^1_s} a(\tau) 2^{-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \]
\[ + \int_{s/2}^s \int_{\mathbb{R}^n} a(\tau) 2^{-n(p-1)/2} |f(u)(\tau, y)| \cdot \|G(\kappa(s - \tau), x - y)\|_{L^1_s} \, dy \, d\tau \]
\[ + \int_{s/2}^\infty \int_{\mathbb{R}^n} a(\tau) 2^{-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \cdot \|G(\kappa s, x)\|_{L^1_s} \]
\[ \text{(3.24)} \]
by the definition of \( \theta_1 \). The last two terms converge to 0 as \( s \to \infty \) by \( \|G(\kappa s, x)\|_{L^1_s} \lesssim 1 \) and (3.10). Let us consider the first term on the right-hand side of (3.24). We decompose the integral region as
\[ I' := \int_0^{s/2} \int_{\mathbb{R}^n} \|G(\kappa(s - \tau), x - y) - G(\kappa s, x)\|_{L^1_s} a(\tau) 2^{-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \]
\[ = \left( \int_{\Omega_1} + \int_{\Omega_2} \right) a(\tau) 2^{-n(p-1)/2} \|G(\kappa(s - \tau), x - y) - G(\kappa s, x)\|_{L^1_s} \]
\[ \times |f(u)(\tau, y)| \, dy \, d\tau, \]
where \( \Omega_1 \) and \( \Omega_2 \) are defined by (3.8). On \( \Omega_2 \), we have
\[ \int_{\Omega_2} a(\tau) 2^{-n(p-1)/2} \|G(\kappa(s - \tau), x - y) - G(\kappa s, x)\|_{L^1_s} \cdot |f(u)(\tau, y)| \, dy \, d\tau \]
\[ \leq 2 \left( \frac{|\kappa|}{\alpha} \right)^{n/2} \int_{\Omega_2} a(\tau) 2^{-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \]
\[ \to 0 \quad \text{as} \quad s \to \infty, \quad \text{(3.25)} \]
where we have used Lemma A.1 and (3.10). On \( \Omega_1 \), we have
\[ \int_{\Omega_1} a(\tau) 2^{-n(p-1)/2} \|G(\kappa(s - \tau), x - y) - G(\kappa s, x)\|_{L^1_s} |f(u)(\tau, y)| \, dy \, d\tau \]
\[ \leq \delta \int_{\Omega_1} a(\tau) 2^{-n(p-1)/2} |f(u)(\tau, y)| \, dy \, d\tau \]
since we have
\[ \|G(\kappa(s - \tau), x - y) - G(\kappa s, x)\|_{L^1_s} \]
\[ \lesssim \delta s^{-n/2} \int_0^1 \left\| x/\delta^{1/2} - \theta z \right\|^2 e^{-|x/\delta^{1/2} - \theta z|^2/2a^2} \, d\theta \]
\[ \lesssim \delta \]
by (3.13). Since \( \delta > 0 \) is arbitrary, we obtain \( I' \to 0 \) as \( s \to \infty \). Thus we obtain
\[ \left\| \lambda \int_0^s e^{k(s-\tau)} a(\tau) 2^{-n(p-1)/2} f(u)(\tau) \, d\tau - \theta_1 G(\kappa s, x) \right\|_{L^1_s} \]
\[ \to 0 \quad \text{as} \quad s \to \infty. \quad \text{(3.26)} \]
3.3. Convergence in $L^r(\mathbb{R}^n)$

Finally, by (3.17), (3.27) and the Hölder inequality, we obtain
\[
\|u(s, \cdot) - \theta G(ks, \cdot)\|_{L^r} \leq \|u(s, \cdot) - \theta G(ks, \cdot)\|_{L^r}^{1/r} \|u(s, \cdot) - \theta G(ks, \cdot)\|_{L^1}^{1/r} \\
\lesssim o(s^{-n(1-1/r)/2}) \quad \text{as } s \to \infty
\]
for $1 \leq r \leq \infty$, as required.

4. Proof of Theorem 1.3

In this section, we consider the asymptotic behaviour of the solution that is obtained in the cases (4) and (5) in Theorem 1.1. In these cases, $a_1(4 - n(1 + \sigma)) > 0$, $S_0 = S_1$ and $X = L^\infty((0, S_1) \times \mathbb{R}^n)$ by (1.18). We put the function space as follows:
\[
Y := L^\infty((0, S_1), W^{1,\infty}(\mathbb{R}^n)), \quad Y(R) := \{u \in Y \mid \|u\|_Y \leq R\}
\]
for $R > 0$ with the metric in $X$. Since we have
\[
\nabla \Phi(u)(s) = e^{ks\Delta} \nabla u_0 - \lambda \int_0^s e^{k(s-\tau)\Delta} a(\tau)^{2-n(p-1)/2} \nabla f(u(\tau)) \, d\tau
\]
by (2.1), we obtain
\[
\|\nabla \Phi(u)(s)\|_{L^\infty} \\
\leq e^{ks\Delta} \nabla u_0 + |\lambda| \int_0^s a(\tau)^{2-n(p-1)/2} e^{k(s-\tau)\Delta} \nabla f(u(\tau)) \|_{L^\infty} d\tau \\
\leq \left( \frac{|\lambda|}{\alpha} \right)^{n/2} \left( \|\nabla u_0\|_{L^\infty} + |\lambda| \int_0^s a(\tau)^{2-n(p-1)/2} \|\nabla f(u(\tau))\|_{L^\infty} \, d\tau \right) \\
\leq \left( \frac{|\lambda|}{\alpha} \right)^{n/2} \left( \|\nabla u_0\|_{L^\infty} + p|\lambda| \int_0^{S_1} a(\tau)^{2-n(p-1)/2} d\tau \|u\|_{X}^{p-1} \|\nabla u\|_{X} \right) \\
\leq \left( \frac{|\lambda|}{\alpha} \right)^{n/2} \left( \|\nabla u_0\|_{L^\infty} + C S_1 \|u\|_{X}^{p-1} \|\nabla u\|_{X} \right) \quad \text{(4.1)}
\]
by Proposition 2.1, (2.20) and the condition $p \neq 2 + \sigma$. Thus, by (4.1), (2.21), $u_0 \in W^{1,\infty}(\mathbb{R}^n)$ and $u \in L^\infty((0, S_1), W^{1,\infty}(\mathbb{R}^n))$, we obtain
\[
\Phi(u) \in L^\infty((0, S_1), W^{1,\infty}(\mathbb{R}^n)).
\]
Therefore, by (2.21), (2.22) and (4.1), there exists a unique solution $u$ of (1.13) in $L^\infty((0, S_1), W^{1,\infty}(\mathbb{R}^n))$. Especially, $u(s)$ and $f(u)(s)$ are uniformly continuous on $\mathbb{R}^n$ for almost every $s \in [0, S_1]$ by $u \in L^\infty((0, S_1), W^{1,\infty}(\mathbb{R}^n))$. 
We show the asymptotic behaviour of the solution. Let \( v \) be the function defined by (1.25). We have

\[
\begin{aligned}
    u(s) - v &= e^{\kappa s} (1 - e^{\kappa(S_1 - s)}) u_0 \\
    &\quad - \lambda \int_0^s a(\tau)^2 - n(p-1)/2 e^{\kappa(s-\tau)} (1 - e^{\kappa(S_1 - s)}) f(u)(\tau) d\tau \\
    &\quad + \lambda \int_s^{S_1} a(\tau)^2 - n(p-1)/2 e^{\kappa(S_1 - \tau)} f(u)(\tau) d\tau \\
    &=: I - II + III.
\end{aligned}
\]

By \( u_0 \in W^{1,\infty}(\mathbb{R}^n) \) and Lemma A.4, we have

\[
\|I\|_{L^\infty} \leq \left( \frac{|\kappa|}{\alpha} \right)^{n/2} \| (e^{\kappa(S_1 - s)} - 1) u_0 \|_{L^\infty}
\]

\[
\leq \left( \frac{|\kappa|}{\alpha} \right)^{n/2} \| G(\kappa(S_1 - s), \cdot) * u_0 - u_0 \|_{L^\infty}
\]

\[
\to 0 \quad \text{as} \quad s \nearrow S_1.
\]

By \( f(u)(s) \in W^{1,\infty}(\mathbb{R}^n) \) for almost every \( 0 \leq s < S_1 \), (2.20), the Lebesgue convergence theorem and Lemma A.4, we have

\[
\|II\|_{L^\infty} \lesssim \int_0^{S_1} a(\tau)^2 - n(p-1)/2 \| (e^{\kappa(S_1 - s)} - 1) f(u)(\tau) \|_{L^\infty} d\tau
\]

\[
\to 0 \quad \text{as} \quad s \nearrow S_1.
\]

Since \( a(\cdot)^2 - n(p-1)/2 \in L^1((0, S_1)) \) by (2.20), we obtain

\[
\|III\|_{L^\infty} \lesssim \int_s^{S_1} a(\tau)^2 - n(p-1)/2 \| u \|^p_X \to 0 \quad \text{as} \quad s \nearrow S_1.
\]

Therefore, we obtain

\[
\|u(s) - v\|_{L^\infty} = o(1) \quad \text{as} \quad s \nearrow S_1.
\]

5. Proof of Theorem 1.6

The proof of Theorem 1.6 is based on the test function method in [15, 20] for a semilinear damped wave equation, where both results are considered in Minkowski spacetime. We have applied their methods to the complex Ginzburg–Landau type equation in homogeneous and isotropic spacetimes. We prepare the following lemma to prove Theorem 1.6.

LEMMA 5.1. Assume (1.29) and (1.30). Let \( a_1 \) and \( \sigma \) satisfy

\[
a_1(4 - n(1 + \sigma)) \leq 0. \tag{5.1}
\]

Put

\[
B(s) := a(s)^{2-n(p-1)/2}. \tag{5.2}
\]
If there exists a constant \( C > 0 \) such that the inequality
\[
\int_0^{R^2} B(s)^{-p'/p} \, ds \leq CR^{2-p-n}
\] (5.3)
holds for any \( R \geq 1 \), then the global weak solution \( u \) of (1.26) must satisfy \( u = 0 \) on \([0, \infty) \times \mathbb{R}^n\).

**Proof.** We define a non-negative function \( \eta \in C_0^\infty([0, \infty)) \) by
\[
\eta(s) := \begin{cases} 
1 & \text{if } 0 \leq s \leq \frac{1}{2}, \\
1 - \eta_0 \int_{1/2}^s e^{-(t-1/2)^{-1}(1-t)^{-1}} \, dt & \text{if } \frac{1}{2} < s < 1, \\
0 & \text{if } s \geq 1,
\end{cases}
\] (5.4)
where \( \eta_0 := (\int_{1/2}^1 e^{-(t-1/2)^{-1}(1-t)^{-1}} \, dt)^{-1} \). We have
\[
\frac{\partial s \eta}{\eta} \in C_0(\mathbb{R}).
\] (5.5)

Put \( \phi(x) := \eta(|x|) \) (5.6) for \( x \in \mathbb{R}^n \). Then we have
\[
\phi(x) = \begin{cases} 
1 & \text{if } |x| \leq \frac{1}{2}, \\
0 & \text{if } |x| \geq 1,
\end{cases}
\]
and
\[
\frac{|
abla \phi|^2}{\phi} \left( \frac{|\eta|^2}{\eta} \right) \in C_0(\mathbb{R}^n).
\]

For \( R > 0 \), we put
\[
\eta_R(s) := \eta \left( \frac{s}{R^2} \right), \quad \phi_R(x) := \phi \left( \frac{x}{R} \right), \quad \psi_R(s, x) := \eta_R(s) \phi_R(x)
\] (5.7)
for \( 0 \leq s < \infty \) and \( x \in \mathbb{R}^n \). Then for \( 1 < p \leq \infty \), we have
\[
\psi_R \in C_0^1(\mathbb{R}^{1+n}),
\] (5.8)

\[
|\partial_s \psi_R(s, x)| \lesssim \frac{1}{R^2} \chi_{|s| \leq 1|s|^2/2|s| \leq R^2}(s) \chi_{\{|y| \leq R\}}(x)
\] (5.9)
and
\[
|\Delta \psi_R \psi_R^{-1}(s, x)| \lesssim \frac{1}{R^2} |\psi_R^{-1}(s, x)| \chi_{\{|y| \leq R\}}(x)
\] (5.10)
for \( 0 \leq s < \infty \) and \( x \in \mathbb{R}^n \), where \( \chi_S \) denotes the characteristic function on the set \( S \).

Let \( u \) be an arbitrary fixed global weak solution of (1.26). By (5.8), we have
\[
-i \frac{2m}{\hbar} \int_{\mathbb{R}^n} u_0(x) \phi_R \, dx + \int_0^\infty \int_{\mathbb{R}^n} u(s, x) \left( -i \frac{2m}{\hbar} \partial_x + v \Delta \right) \psi_R \, dx \, ds
\]
\[
- \mu \int_0^\infty \int_{\mathbb{R}^n} B(s)|u(s, x)|^p \psi_R \, dx \, ds = 0,
\] (5.11)
replacing $\psi$ with $\psi^p_R$ in (1.27). For the last term on the left-hand side of this equation, we put
\[
I_R := \int_0^\infty \int_{\mathbb{R}^n} B(s)|u(s, x)|^p \psi^p_R(s, x) \, dx \, ds.
\]
Taking the real part of both sides of (5.11) and noting that $\psi^p_R$ is a real-valued function, we have
\[
(\text{Re } \mu) I_R = -\frac{2m}{\hbar} \text{Re} \left( i \int_{\mathbb{R}^n} u_0(x) \phi^p_R(x) \, dx \right)
+ \int_0^\infty \int_{\mathbb{R}^n} \text{Re} \left\{ u(s, x) \left( -i \frac{2m}{\hbar} \partial_s + \nu \Delta \right) \psi^p_R(s, x) \right\} \, dx \, ds
= \frac{2m}{\hbar} \int_{\mathbb{R}^n} (\text{Im } u_0(x)) \phi^p_R(x) \, dx
+ \frac{2m}{\hbar} \int_0^\infty \int_{\mathbb{R}^n} (\text{Im } u(s, x)) \partial_s \psi^p_R(s, x) \, dx \, ds
+ \int_0^\infty \int_{\mathbb{R}^n} (\text{Re } (\nu u(s, x))) \Delta \psi^p_R(s, x) \, dx \, ds.
\]
Since we have
\[
\frac{2m}{\hbar} \int_{\mathbb{R}^n} (\text{Im } u_0(x)) \phi^p_R(x) \, dx \to \frac{2m}{\hbar} \int_{\mathbb{R}^n} \text{Im } u_0(x) \, dx < 0
\]
as $R \to \infty$ by (1.30), we have
\[
(\text{Re } \mu) I_R \lesssim \frac{2m}{\hbar} \int_0^\infty \int_{\mathbb{R}^n} (\text{Im } u(s, x)) \partial_s \psi^p_R(s, x) \, dx \, ds
+ \int_0^\infty \int_{\mathbb{R}^n} (\text{Re } (\nu u(s, x))) \Delta \psi^p_R(s, x) \, dx \, ds
\leq \frac{2m}{\hbar} \int_0^\infty \int_{\mathbb{R}^n} |u(s, x)| \partial_s \psi^p_R(s, x) | \, dx \, ds
+ |\nu| \int_0^\infty \int_{\mathbb{R}^n} |u(s, x)| \Delta \psi^p_R(s, x) | \, dx \, ds
\lesssim J_{1,R} + J_{2,R} \tag{5.12}
\]
for sufficiently large $R > 0$, where we have put
\[
J_{1,R} := \int_0^\infty \int_{\mathbb{R}^n} |u(s, x)| \partial_s \psi^p_R(s, x) | \, dx \, ds,
J_{2,R} := \int_0^\infty \int_{\mathbb{R}^n} |u(s, x)| \Delta \psi^p_R(s, x) | \, dx \, ds.
\]
By $\partial_s \psi^p_R = p' \psi^p_R \partial_s \psi_R$ and (5.9), we have
\[
J_{1,R} \lesssim \frac{1}{R^2} \int_{R^2/2}^{R^2} \int_{|x| < R} |u(s, x)| \psi^p_{R}^{-1}(s, x) \, dx \, ds.
\]
By $|u|_{p-1}^{p'} = |u|_{p}^{p'-1} B^{1/p} \cdot B^{-1/p}$ and the Hölder inequality, we have
\[
\int_{R^2/2}^{R^2} \int_{|x|<R} |u(s, x)| \psi_{R}^{p'-1}(s, x) \, dx \, ds \leq \tilde{J}_{1,R}^{1/p} K_{1}^{1/p'},
\]
where we have put
\[
\tilde{J}_{1,R} := \int_{R^2/2}^{R^2} \int_{|x|<R} |u(s, x)|^{p} \psi_{R}^{p'}(s, x) B(s) \, dx \, ds,
\]
\[
K_{1} := \int_{R^2/2}^{R^2} \int_{|x|<R} B(s)^{-p'/p} \, dx \, ds.
\]
Thus, we obtain
\[
J_{1,R} \lesssim \frac{K_{1}^{1/p'}}{R^2} \tilde{J}_{1,R}^{1/p}. \tag{5.13}
\]
Similarly, by (5.10), we have
\[
J_{2,R} \lesssim \frac{1}{R^2} \int_{0}^{R^2} \int_{R/2<|x|<R} |u(s, x)| \psi_{R}^{p'-1}(s, x) \, dx \, ds.
\]
By $|u|_{p-1}^{p'} = |u|_{p}^{p'-1} B^{1/p} \cdot B^{-1/p}$ and the Hölder inequality, we have
\[
\int_{0}^{R^2} \int_{R/2<|x|<R} |u(s, x)| \psi_{R}^{p'-1}(s, x) \, dx \, ds \leq \tilde{J}_{2,R}^{1/p} K_{2}^{1/p'},
\]
where we have put
\[
\tilde{J}_{2,R} := \int_{0}^{R^2} \int_{R/2<|x|<R} |u(s, x)|^{p} \psi_{R}^{p'}(s, x) B(s) \, dx \, ds
\]
and
\[
K_{2} := \int_{0}^{R^2} \int_{R/2<|x|<R} B(s)^{-p'/p} \, dx \, ds.
\]
Thus, we obtain
\[
J_{2,R} \lesssim \frac{K_{2}^{1/p'}}{R^2} \tilde{J}_{2,R}^{1/p}. \tag{5.14}
\]
Since we have $\tilde{J}_{1,R} \leq I_{R}$ and $\tilde{J}_{2,R} \leq I_{R}$ by the definitions of $\tilde{J}_{1,R}$, $\tilde{J}_{2,R}$ and $I_{R}$, we obtain
\[
(\operatorname{Re} \mu) I_{R} \lesssim \tilde{J}_{1,R} + \tilde{J}_{2,R}
\]
\[
\lesssim \frac{K_{1}^{1/p'}}{R^2} \tilde{J}_{1,R}^{1/p} + \frac{K_{2}^{1/p'}}{R^2} \tilde{J}_{2,R}^{1/p} \tag{5.15}
\]
\[
\lesssim \left( \frac{K_{1}^{1/p'}}{R^2} + \frac{K_{2}^{1/p'}}{R^2} \right) \cdot I_{R}^{1/p} \tag{5.16}
\]
by (5.12), (5.13) and (5.14). Thus, we have
\[
I_{R} \lesssim 1 \tag{5.17}
\]
if $K_1, K_2$ and $R$ satisfy

$$\frac{K_1^{1/p}}{R^2} \lesssim 1 \quad \text{and} \quad \frac{K_2^{1/p}}{R^2} \lesssim 1 \quad \text{for } R \geq 1. \tag{5.18}$$

Taking $R \to \infty$ in (5.17), we have

$$\int_0^\infty \int_{\mathbb{R}^n} B(s)|u(s, x)|^p \, dx \, ds = \lim_{R \to \infty} I_R < \infty \tag{5.19}$$

by the definition of $I_R$ under the assumption (5.18). By (5.19) and $\psi_R^{p'} \leq 1$, we have $J_{1, R} \to 0$ and $J_{2, R} \to 0$ as $R \to \infty$. Thus, we have $I_R \to 0$ as $R \to \infty$ by (5.15), and we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} B(s)|u(s, x)|^p \, dx \, ds = 0$$

by (5.19), which shows that $u = 0$ on $[0, \infty) \times \mathbb{R}^n$. Now, we consider the assumption (5.18).

Since we have

$$K_1 \lesssim R^n \int_{R^2/2} B(s)^{-p'/p} \, ds \quad \text{and} \quad K_2 \lesssim R^n \int_0^{R^2} B(s)^{-p'/p} \, ds$$

by direct calculations, the assumption (5.18) is satisfied by the condition (5.3). \hfill \square

Now, we prove Theorem 1.6. We show that the condition (5.3) is satisfied by the conditions in the theorem. Then we obtain the required result by Lemma 5.1. We recall the definitions of $p_0$, $p_1$ and $p_2$ in (1.19) and (1.28).

1. When $a_1 = 0$, we have $a(\cdot) = a_0$ and $B(\cdot) = a_0^{2-n(p-1)/2}$. Since we have

$$\int_0^{R^2} B(s)^{-p'/p} \, ds = (a_0^{2-n(p-1)/2})^{-p'/p} R^2,$$

the condition (5.3) is rewritten as $R^{2-2p'+n} \lesssim 1$ for $R \geq 1$, which is satisfied if $p \leq 1 + 2/n$.

2. Let $a_1 \neq 0$. When $\sigma = -1 + 4/n$, we have $a(s) = a_0 e^{a_0 a_1 s}$, $S_0 = \infty$ and

$$B(s) = a_0^{2-n(p-1)/2} e^{a_0 a_1(2-n(p-1)/2) s}.$$

Since we have

$$\int_0^{R^2} B(s)^{-p'/p} \, ds$$

$$= a_0^{n/2-2/(p-1)} \int_0^{R^2} e^{a_0 a_1(n/2-2/(p-1)) s} \, ds$$

$$= a_0^{n/2-2/(p-1)}$$

$$\times \begin{cases} R^2 & \text{if } n/2 - 2/(p-1) = 0, \\ \frac{1}{a_0 a_1(n/2-2/(p-1))} (e^{a_0 a_1(n/2-2/(p-1)) R^2} - 1) & \text{if } n/2 - 2/(p-1) \neq 0, \end{cases}$$

the condition (5.3) for $R \geq 1$ is rewritten as

$$\begin{cases} R^{2-2p'+n} \lesssim 1 & \text{if } n/2 - 2/(p-1) = 0, \\ R^{-2p'+n} \frac{a_0 a_1(n/2-2/(p-1)) (e^{a_0 a_1(n/2-2/(p-1)) R^2} - 1)}{a_0 a_1(n/2-2/(p-1))} \lesssim 1 & \text{if } n/2 - 2/(p-1) \neq 0. \end{cases}$$
which is satisfied by

\[
\begin{cases}
2 - 2p' + n \leq 0 & \text{if } n/2 - 2/(p - 1) = 0, \\
-2p' + n \leq 0 & \text{if } a_1(n/2 - 2/(p - 1)) < 0.
\end{cases}
\] (5.20)

We note that \(2 - 2p' + n \leq 0\) (i.e. \(p \leq 1 + 2/n\)) does not hold under \(n/2 - 2/(p - 1) = 0\) (i.e. \(p = 1 + 4/n\)). Since \(-2p' + n \leq 0\) is rewritten as \(1 < p < \infty\) if \(n = 1, 2\), and \(1 < p \leq p_1\) if \(n \geq 3\), the conditions (5.20) are rewritten as

\[
a_1(p - 1 - \frac{4}{n}) < 0 \quad \text{and} \quad 1 < p \begin{cases} < \infty & \text{if } n = 1, 2, \\
\leq p_1 & \text{if } n \geq 3.
\end{cases}
\] (5.21)

When \(a_1 > 0\), the conditions (5.21) are rewritten as

\[
1 < p < 1 + 4/n \quad \text{for } n \geq 1,
\]

with

\[
p \leq p_1 \quad \text{for } n \geq 3
\]
since the condition \(a_1(p - 1 - 4/n) < 0\) is equivalent to \(p < 1 + 4/n\) by \(a_1 > 0\).

(3) When \(a_1 < 0\), the proof is the same as the proof of (2) until (5.21), which is satisfied by \(n = 1, 2, 3\), and

\[
1 + 4/n < p < \infty \quad \text{for } n = 1, 2, 3,
\]

with \(p \leq 3\) (\(= p_1\)) for \(n = 3\) since \(a_1(p - 1 - 4/n) < 0\) is equivalent to \(p > 1 + 4/n\) by \(a_1 < 0\), and \(1 + 4/n < p_1\) requires \(n = 1, 2, 3\).

(4) Assume \(a_1(\sigma + 1 - 4/n) > 0\). We have \(S_1 = 2/(a_0a_1[4 - n(1 + \sigma)]) < 0\), \(S_0 = \infty\), \(a(s) = a_0(1 - s/S_1)^{2/[n(1 + \sigma) - 4]}\)

and

\[
B(s) = a_0^{-n(p-1-4/n)/2}(1 - \frac{s}{S_1})^{-n(p-1-4/n)/[n(1+\sigma)-4]}
\]

by (1.8), (1.9), (1.10) and (5.2). We have

\[
\int_0^{R^2} B(s)^{-\gamma / p} ds = a_0^{(1-4/n(p-1))/2} \times \begin{cases}
\frac{S_1}{1 + \gamma} \left(1 - \left(1 - \frac{R^2}{S_1}\right)^{\gamma + 1}\right) & \text{if } \gamma < -1, \\
-\frac{S_1}{1 + \gamma} \left(1 - \left(1 - \frac{R^2}{S_1}\right)^{\gamma + 1}\right) & \text{if } \gamma > -1, \\
-S_1 \log \left(1 - \frac{R^2}{S_1}\right) & \text{if } \gamma = -1,
\end{cases}
\]

where we have put

\[
\gamma := \frac{n}{n(1 + \sigma) - 4} \left(1 - \frac{4}{n(p - 1)}\right).
\]

Thus, the condition (5.3) is rewritten as

\[
1 \lesssim R^{2\gamma - n} \quad \text{if } \gamma < -1, \\
R^{2(\gamma + 1)} \lesssim R^{2\gamma - n} \quad \text{if } \gamma > -1, \\
-S_1 \log \left(1 - \frac{R^2}{S_1}\right) \lesssim R^{2\gamma - n} \quad \text{if } \gamma = -1.
\]
which are satisfied by
\[ 0 \leq 2p' - n \quad \text{if } \gamma < -1, \quad (5.22) \]
\[ 2(\gamma + 1) \leq 2p' - n \quad \text{if } \gamma > -1, \quad (5.23) \]
\[ 0 < 2p' - n \quad \text{if } \gamma = -1. \quad (5.24) \]

We rewrite each inequality in (5.22), (5.23) and (5.24) as follows. The inequality
\[ 0 \leq 2p' - n \] is rewritten as
\[ 1 < p \begin{cases} < \infty & \text{if } n = 1, 2, \\ \leq p_1 & \text{if } n \geq 3. \end{cases} \quad (5.25) \]
The inequality \( \gamma < -1 \) is rewritten as
\[ \frac{1}{n(1 + \sigma) - 4} \left\{ (p - 1) \left( \sigma + 2 - \frac{4}{n} \right) - \frac{4}{n} \right\} < 0, \]
which is equivalent to
\[ \sigma > -2 + \frac{4}{n}, \quad \sigma \neq -1 + \frac{4}{n}, \quad \begin{cases} 1 < p < p_2 \quad \text{if } \sigma > -1 + 4/n, \\ p > p_2 \quad \text{if } -2 + 4/n < \sigma < -1 + 4/n. \end{cases} \quad (5.26) \]
The inequality \( 2(\gamma + 1) \leq 2p' - n \) is rewritten as
\[ \frac{1}{n(1 + \sigma) - 4} \left\{ (p - 1) \{n(1 + \sigma) - 2\} - 2(1 + \sigma) \right\} \leq 0, \]
which is equivalent to
\[ \sigma \neq -1 + 4/n, \quad \sigma \notin [-1, -1 + 2/n] \quad (5.27) \]
and
\[ \begin{cases} 1 < p \leq p_0 \quad \text{if } \sigma > -1 + 4/n \text{ or } \sigma < -1, \\ p \geq p_0 \quad \text{if } -1 + 2/n < \sigma < -1 + 4/n. \end{cases} \quad (5.28) \]
The inequality \( \gamma > -1 \) is rewritten as
\[ \frac{1}{n(1 + \sigma) - 4} \left\{ (p - 1) \left( \sigma + 2 - \frac{4}{n} \right) - \frac{4}{n} \right\} > 0, \]
which is equivalent to
\[ \sigma \neq -1 + 4/n, \quad \begin{cases} p > p_2 \quad \text{if } \sigma > -1 + 4/n, \\ 1 < p < p_2 \quad \text{if } -2 + 4/n < \sigma < -1 + 4/n, \\ 1 < p < \infty \quad \text{if } \sigma \leq -2 + 4/n. \end{cases} \quad (5.29) \]
The inequality \( 0 < 2p' - n \) is rewritten as
\[ 1 < p \begin{cases} \infty & \text{if } n = 1, 2, \\ p_1 & \text{if } n \geq 3. \end{cases} \quad (5.30) \]
The inequality $\gamma = -1$ is rewritten as
\[
\frac{1}{n(1 + \sigma)} - \frac{4}{n} \left\{ (p - 1) \left( \sigma + 2 - \frac{4}{n} \right) - \frac{4}{n} \right\} = 0,
\]
which is equivalent to
\[
\sigma > -2 + \frac{4}{n}, \quad \sigma \neq -1 + \frac{4}{n}, \quad p = p_2. \tag{5.31}
\]

We note that $p_2 < p_1$ is rewritten as $\sigma > 0$ when $n \geq 3$ and $\sigma > -2 + 4/n$. The inequality $p_1 < p_0$ is rewritten as $\sigma < 0$ when $n \geq 3$ and $\sigma > -1 + 2/n$. The inequality $p_2 < p_0$ is rewritten as $\sigma (\sigma + 1 - 4/n) > 0$, namely,
\[
\begin{cases} 
\sigma < 0 \text{ or } \sigma > -1 + 4/n & \text{if } n = 1, 2, 3, 4, \\
\sigma > 0 \text{ or } \sigma < -1 + 4/n & \text{if } n \geq 5
\end{cases} \tag{5.32}
\]
when $\sigma > -2 + 4/n$ and $\sigma > -1 + 2/n$. The inequality $p_2 > p_0$ is rewritten as $\sigma (\sigma + 1 - 4/n) < 0$, namely,
\[
\begin{cases} 
n \neq 4, \\
0 < \sigma < -1 + 4/n & \text{if } n = 1, 2, 3, \\
-1 + 4/n < \sigma < 0 & \text{if } n \geq 5
\end{cases} \tag{5.33}
\]
when $\sigma > -2 + 4/n$ and $\sigma > -1 + 2/n$.

The condition (5.22) is rewritten as (5.25) and (5.26), which are rewritten as
\[
\begin{cases} 
\sigma > -2 + 4/n, \quad \sigma \neq -1 + 4/n, \\
1 < p < p_2 \quad \text{if } \sigma > -1 + 4/n, \\
(p_0 \leq) p_2 < p \quad \text{if } (0 \leq) -2 + 4/n < \sigma < -1 + 4/n
\end{cases} \tag{5.34}
\]
when $n = 1, 2$,
\[
\begin{cases} 
\sigma > 0 \quad (\sigma > -2 + 4/n), \quad \sigma \neq -1 + 4/n, \\
1 < p < p_2 \quad (p_0 < p_1) \quad \text{if } \sigma > -1 + 4/n \ (\equiv \frac{1}{3}), \\
p_2 < p \leq p_1 \quad \text{if } 0 < \sigma < -1 + 4/n
\end{cases} \tag{5.35}
\]
when $n = 3$, where we note that $p_2 < p_1$ in (5.35) does not hold for $\sigma \leq 0$. When $n = 4$, (5.25) and (5.26) are rewritten as
\[
\begin{cases} 
\sigma > -2 + 4/n, \quad \sigma \neq -1 + 4/n, \\
1 < p < p_2 \quad (p_0 < p_1) \quad \text{if } \sigma > -1 + 4/n \ (\equiv 0), \\
p_2 < p \leq p_1 \quad \text{if } -2 + 4/n \ (\equiv -1) < \sigma < -1 + 4/n \ (\equiv 0),
\end{cases}
\]
which are rewritten as
\[
\sigma > 0, \quad 1 < p < p_2 \quad (p_0 < p_1), \quad \sigma \neq -1 + 4/n, \quad 1 < p \leq p_1, \tag{5.36}
\]
since $p_2 < p_1$ does not hold for $\sigma < 0$. When $n \geq 5$, (5.25) and (5.26) are rewritten as
\[
\begin{cases} 
\sigma > -2 + 4/n, \quad \sigma \neq -1 + 4/n, \\
p < p_2 \quad \text{if } \sigma > -1 + 4/n, \\
p_2 < p \leq p_1 \quad \text{if } -2 + 4/n < \sigma < -1 + 4/n \ (< 0),
\end{cases}
\]
which are rewritten as

\[
\sigma > -1 + 4/n, \quad 1 < p \begin{cases} \leq p_1 (< p_0 < p_2) & \text{if } \sigma < 0, \\ < p_1 (= p_0 = p_2) & \text{if } \sigma = 0, \\ < p_2 (< p_0 < p_1) & \text{if } \sigma > 0, \end{cases}
\]

(5.37)

since \( p_2 < p_1 \) does not hold for \( \sigma > -2 + 4/n \), where we have used \( p_2 < p_1 \) for \( \sigma > 0 \), \( p_2 = p_1 \) for \( \sigma = 0 \), and \( p_2 > p_1 \) for \( \sigma < 0 \). The conditions (5.34), (5.35), (5.36) and (5.37) are the conditions (i), (i)', (i)'' and (i)''' in the theorem, respectively.

To consider the condition (5.23), we put

\[
\sigma_1 := -1 + 4/n, \quad \sigma_2 := -1 + 2/n, \quad \sigma_3 := -2 + 4/n
\]

for \( n \geq 1 \), which satisfy

\[
\begin{align*}
\sigma_2 &= 1 < \sigma_3 = 2 < \sigma_1 = 3 & \text{for } n = 1, \\
\sigma_2 &= \sigma_3 = 0 < \sigma_1 = 1 & \text{for } n = 2, \\
\sigma_3 &= -2/3 < \sigma_2 = -1/3 < \sigma_1 = 1/3 & \text{for } n = 3, \\
\sigma_3 &\leq -1 < \sigma_2 < \sigma_1 \leq 0 & \text{for } n \geq 4.
\end{align*}
\]

The condition (5.23) is rewritten as (5.27), (5.28) and (5.29), which are rewritten as

\[
\sigma \notin [-1, -1 + 2/n (= 1)], \quad \sigma \neq -1 + 4/n (= 3),
\]

(5.38)

and

\[
\begin{align*}
p_2 < p &\leq p_0 & \text{if } \sigma > -1 + 4/n (= 3), \\
p_0 \leq p &< p_2 & \text{if } -2 + 4/n (= 2) < \sigma < -1 + 4/n, \\
p_0 \leq p &\leq p_0 & \text{if } -1 + 2/n (= 1) < \sigma \leq -2 + 4/n, \\
1 < p &\leq p_0 & \text{if } \sigma < -1
\end{align*}
\]

(5.39)

when \( n = 1 \),

\[
\sigma \notin [-1, -1 + 2/n (= 0)], \quad \sigma \neq -1 + 4/n (= 1),
\]

(5.40)

and

\[
\begin{align*}
p_2 < p &\leq p_0 & \text{if } \sigma > -1 + 4/n (= 1), \\
p_0 \leq p &< p_2 & \text{if } -1 + 2/n (= -2 + 4/n = 0) < \sigma < -1 + 4/n, \\
1 < p &\leq p_0 & \text{if } \sigma < -1 (< -2 + 4/n)
\end{align*}
\]

(5.41)

when \( n = 2 \),

\[
\sigma \notin [-1, -1 + 2/n (= -1/3)], \quad \sigma \neq -1 + 4/n (= 1/3),
\]

(5.42)

and

\[
\begin{align*}
p_2 < p &\leq p_0 & \text{if } \sigma > -1 + 4/n, \\
p_0 \leq p &< p_2 & \text{if } -1 + 2/n < \sigma < -1 + 4/n, \\
1 < p &\leq p_0 & \text{if } \sigma < -1
\end{align*}
\]

(5.43)

when \( n = 3 \), which are rewritten as

\[
\sigma \notin [-1, 0], \quad \sigma \neq -1 + 4/n (= 1/3),
\]

(5.44)
and
\[
\begin{cases}
p_2 < p \leq p_0 & \text{if } \sigma > -1 + 4/n, \\
p_0 \leq p < p_2 & \text{if } 0 < \sigma < -1 + 4/n, \\
1 < p \leq p_0 & \text{if } \sigma < -1
\end{cases}
\tag{5.45}
\]
when \( n = 3 \) since \( p_0 < p_2 \) in (5.43) requires \( \sigma > 0 \). The conditions (5.27), (5.28) and (5.29) are rewritten as
\[
\sigma \notin [-1, -1 + 2/n (=-\frac{1}{2})], \quad \sigma \neq -1 + 4/n (=0)
\]
and
\[
\begin{cases}
p_2 < p \leq p_0 & \text{if } \sigma > -1 + 4/n (=0), \\
p_0 \leq p < p_2 & \text{if } -1 + 2/n < \sigma < -1 + 4/n, \\
1 < p \leq p_0 & \text{if } \sigma < -2 + 4/n (= -1)
\end{cases}
\tag{5.46}
\]
which are rewritten as
\[
\sigma \notin [-1, -1 + 2/n (=0)]
\tag{5.47}
\]
and
\[
\begin{cases}
p_2 < p \leq p_0 & \text{if } \sigma > -1 + 4/n (=0), \\
1 < p \leq p_0 & \text{if } \sigma < -2 + 4/n (= -1)
\end{cases}
\tag{5.48}
\]
when \( n = 4 \), since \( p_0 < p_2 \) in (5.46) does not hold for \( -1 + 2/n < \sigma < -1 + 4/n \) by (5.33). The conditions (5.27), (5.28) and (5.29) are rewritten as
\[
\sigma \notin [-1, -1 + 2/n], \quad \sigma \neq -1 + 4/n
\]
and
\[
\begin{cases}
p_2 < p \leq p_0 & \text{if } \sigma > -1 + 4/n, \\
p_0 \leq p < p_2 & \text{if } -1 + 2/n < \sigma < -1 + 4/n, \\
1 < p < p_2 & \text{if } -2 + 4/n < \sigma < -1, \\
1 < p \leq p_0 & \text{if } \sigma \leq -2 + 4/n (< -1)
\end{cases}
\tag{5.49}
\]
which are rewritten as
\[
\sigma \notin [-1, 0]
\tag{5.50}
\]
and
\[
\begin{cases}
p_2 < p \leq p_0 & \text{if } \sigma > 0, \\
1 < p < p_2 (\leq p_0) & \text{if } -2 + 4/n < \sigma < -1, \\
1 < p \leq p_0 & \text{if } \sigma \leq -2 + 4/n
\end{cases}
\tag{5.51}
\]
when \( n \geq 5 \), since \( p_2 < p_0 \) in (5.49) does not hold for \( -1 + 4/n < \sigma < 0 \) by (5.33), and \( p_0 < p_2 \) does not hold for \( -1 + 2/n < \sigma < -1 + 4/n \) by (5.33). The conditions (5.38) with (5.39), (5.40) with (5.41), (5.44) with (5.45), (5.47) with (5.48), and (5.50) with (5.51) are the conditions (ii), (ii)', (ii)'', (ii)''' and (ii)'''' in the theorem. The condition (5.24) is rewritten as (5.30) and (5.31), which are rewritten as
\[
\sigma > -2 + 4/n (> 0), \quad \sigma \neq -1 + 4/n, \quad p = p_2
\tag{5.52}
\]
when $n = 1, 2$,

$$\sigma > 0 \ (\geq -2 + 4/n), \quad \sigma \neq -1 + 4/n \ (= \frac{1}{2}), \quad p = p_2 \ (< p_1) \quad (5.53)$$

when $n = 3$,

$$\sigma > 0 \ (\geq -1 + 4/n > -2 + 4/n), \quad p = p_2 \ (< p_1) \quad (5.54)$$

when $n \geq 4$. The conditions (5.52), (5.53) and (5.54) are the conditions (iii), (iii)$'$ and (iii)$''$ in the theorem.

6. Proof of Theorem 1.8

It suffices to show the following lemma since the result (9) in it shows the required blowing-up of the solution. Put $\partial_0 := \partial_s$.

**Lemma 6.1.** Under the assumption in Theorem 1.8, the following results hold.

1. $\partial_0 e^\sigma + e^{\sigma+1} = 0$, where

$$A := a^{2-n(p-1)/2}, \quad e^0 := |\nabla u|^2 + \frac{2\mu}{v(p+1)}A|u|^{p+1},$$

$$e^j := -2 \text{Re} (\overline{\partial_0 u} \partial_j u), \quad e^{\alpha+1} := -\frac{4(\text{Im} \nu)m}{|\nu|^2h}|\partial_\alpha u|^2 - \frac{2\mu}{v(p+1)} \partial_\alpha A|u|^{p+1}.$$

2. Put $E := \int_{\mathbb{R}^n} e^0 \text{d}x$. Then $e^{\alpha+1} \geq 0$, $\partial_0 E = -\int_{\mathbb{R}^n} e^{\alpha+1} \text{d}x$ and $E(s) \leq E(0)$ for $s \geq 0$.

3. $\partial_\alpha q^\alpha - h(\text{Im} \nu) q^{\alpha+1}/m = 0$, where

$$q^0 := |u|^2, \quad q^j := \frac{h}{m} \text{Im}(\nu \overline{\partial_\alpha u}),$$

$$q^{\alpha+1} := |\nabla u|^2 + \frac{\mu}{v} A|u|^{p+1}.$$

4. Put $Q := \int_{\mathbb{R}^n} q^0 \text{d}x$ and $N := \int_{\mathbb{R}^n} q^{\alpha+1} \text{d}x$. Then $\partial_0 Q = h(\text{Im} \nu) N/m$.

5. $N \leq (p+1)E/2 < 0$.

6. $(p+1)(\partial_0 Q)E - 2Q\partial_0 E \geq 0$.

7. $E(s) \leq E(0)(Q(s)/Q(0))^{(p+1)/2}$ for $s \geq 0$.

8. $Q(s) \geq Q(0)(1 \quad \frac{s}{S})^{-2/(p-1)}$ for $0 \leq s < S$, where $S$ is the positive number defined by (1.33).

9. $Q(s) \to \infty$ as $s \nearrow S$, provided that $S < S_0$ when $a_1 \{4-n(1+\sigma)\} > 0$.

**Proof.** (1) The first equation in (1.31) is rewritten as

$$i \frac{2m\overline{\nu}}{h|\nu|^2} \partial_\alpha u + \Delta u - \frac{\mu}{v} A|u|^{p-1}u = 0.$$

Multiplying $\overline{\partial_0 u}$ to both sides and taking its real part, we have the required result, where we use $\mu/v \in \mathbb{R}$:

$$2 \text{Re}(\overline{\partial_0 u} \Delta u) = 2 \text{Re}(\overline{\partial_0 u} \nabla u) - \partial_0 |\nabla u|^2,$$

$$A|u|^{p-1} 2 \text{Re}(\overline{\partial_0 uu}) = \frac{2}{p+1}(\partial_0 (A|u|^{p+1}) - \partial_0 A|u|^{p+1}).$$
(2) Integrating the equation in (1) on $\mathbb{R}^n$, we have
$$\partial_0 E = - \int_{\mathbb{R}^n} e^{n+1} \, dx.$$ We note that $\partial_0 A \geq 0$ is equivalent to (1.32) by $\partial_0 A = (2 - n(p - 1)/2) \partial_0 a A/a$ and (1.8). Thus, we have $e^{n+1} \geq 0$ by $\text{Im} \, \nu < 0$, $\mu/\nu < 0$ and $\partial_0 A \geq 0$, and we obtain $E(s) \leq E(0)$ for $s \geq 0.$

(3) The first equation in (1.31) is rewritten as
$$\partial_s u - i \frac{h \nu}{2m} \Delta u + i \frac{h \mu}{2m} A |u|^{p-1} u = 0.$$ Multiplying $\overline{u}$ to both sides, we have
$$\overline{u} \partial_0 u - i \frac{h \nu}{2m} \nabla \cdot (\overline{u} \nabla u) + i \frac{h \nu}{2m} q^{n+1} = 0. \quad (6.1)$$ Taking its real part, we obtain the required result.

(4) Integrating the equation in (3) on $\mathbb{R}^n$, we obtain the required result.

(5) Since we have $q^{n+1} \leq (p + 1)e^{0}/2$ by $p > 1$, we obtain
$$N(s) = \int_{\mathbb{R}^n} q^{n+1}(s) \, dx \leq \frac{p + 1}{2} \int_{\mathbb{R}^n} e^{0}(s) \, dx = \frac{p + 1}{2} E(s) \leq \frac{p + 1}{2} E(0) < 0$$ by (1) and the assumption $E(0) < 0$, as required.

(6) Integrating (6.1) on $\mathbb{R}^n$, we have
$$i \frac{h \nu}{2m} N = - \int_{\mathbb{R}^n} \overline{u} \partial_0 u \, dx,$$ which yields
$$\left( \frac{h |\nu|}{2m} \right)^2 N^2 \leq Q \int_{\mathbb{R}^n} |\partial_0 u|^2 \, dx$$ by the Cauchy–Schwarz inequality. Since we have
$$\int_{\mathbb{R}^n} |\partial_0 u|^2 \, dx \leq - \frac{h |\nu|^2}{4m \text{Im} \, \nu} \int_{\mathbb{R}^n} e^{n+1} \, dx = \frac{h |\nu|^2}{4m \text{Im} \, \nu} \partial_0 E$$ by the definition of $e^{n+1}$ and (2), we obtain
$$\frac{h}{m} N^2 \leq \frac{1}{\text{Im} \, \nu} Q \partial_0 E,$$ which yields the required result
$$Q \partial_0 E \leq \frac{h \text{Im} \, \nu}{m} N^2 \leq \frac{p + 1}{2} (\partial_0 Q) E$$ by (4) and (5), where we note that $\text{Im} \, \nu < 0$ and $N < 0$.

(7) The required result follows from integration of $\partial_0 (Q^{-(p+1)/2} E) \leq 0$, which is derived from (6).

(8) By (4), (5) and (7), we have
$$\partial_0 Q = \frac{h \text{Im} \, \nu}{m} N \geq \frac{(p + 1)h \text{Im} \, \nu}{2m} E \geq \frac{(p + 1)h (\text{Im} \, \nu) E(0)}{2m Q(0)^{(p+1)/2}} Q^{(p+1)/2},$$ which yields the required result by integration. The result (9) follows from (8). \qed
Appendix A

In this appendix, we collect several fundamental results for the complex Ginzburg–Landau type equation to prove our results. We introduce the following well-known result, which is proved by the Fresnel type integral (see e.g. [18, Chapter 2, p. 64]). For $\kappa \neq 0 \in \mathbb{C}$ which satisfies $-\pi/2 \leq \arg \kappa \leq \pi/2$, we have

$$
\int_{\mathbb{R}} e^{-\kappa x^2} \, dx = \sqrt{\frac{\pi}{\kappa}}. \tag{A.1}
$$

From this result, for $\kappa = \alpha + i \beta \in \mathbb{C}$, $\kappa \neq 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $-\pi/2 \leq \arg \kappa \leq \pi/2$, the function $G$ defined by (1.17) satisfies

$$
\int_{\mathbb{R}^n} G(\kappa t, x) \, dx = 1 \tag{A.2}
$$

for any $t > 0$ since $G$ is rewritten as

$$
G(\kappa t, x) = (4\pi \kappa t)^{-n/2} e^{-|x|^2 (\alpha - i \beta)/4|\kappa|^2 t}. \tag{A.3}
$$

We have the following results for the kernel $G$. We give their proofs for the completeness of the paper.

**Lemma A.1.** Let $\alpha > 0$, $\beta \in \mathbb{R}$, $1 \leq r \leq \infty$ and $t > 0$. Put $\kappa := \alpha + i \beta$. Then

$$
\|G(\kappa t, \cdot)\|_{L^r(\mathbb{R}^n)} = \begin{cases} 
(4\pi |\kappa| t)^{-n(1-1/r)/2} \left(\frac{|\kappa|}{r \alpha}\right)^{n/2} & \text{if } r \neq \infty, \\
(4\pi |\kappa| t)^{-n/2} & \text{if } r = \infty.
\end{cases}
$$

*Proof.* We have

$$
|G(\kappa t, x)| = (4\pi |\kappa| t)^{-n/2} e^{-|x|^2 \alpha/4|\kappa|^2 t}
$$

by (A.3). From this, we have $\|G(\kappa t, \cdot)\|_{L^\infty(\mathbb{R}^n)} = (4\pi |\kappa| t)^{-n/2}$ by $\alpha > 0$ and $t > 0$. We also have

$$
\|G(\kappa t, \cdot)\|_{L^r(\mathbb{R}^n)}^r = (4\pi |\kappa| t)^{-nr/2} \int_{\mathbb{R}^n} e^{-r|x|^2 \alpha/4|\kappa|^2 t} \, dx
$$

$$
= (4\pi |\kappa| t)^{-n(r-1)/2} \left(\frac{|\kappa|}{r \alpha}\right)^{n/2}
$$

for $1 \leq r < \infty$ by the change of variables $y = (r \alpha/4|\kappa|^2 t)^{1/2} x$ and $\int_{\mathbb{R}^n} e^{-|y|^2} \, dy = \pi^{n/2}$. Thus we obtain the required result. \hfill \Box

**Lemma A.2.** Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $t > 0$. Put $\kappa := \alpha + i \beta$. Then the following results hold.

1. $\nabla G(\kappa t, x) = -\frac{x}{2\kappa t} G(\kappa t, x)$.
2. $\|\nabla G(\kappa t, x)\|_{L^1(\mathbb{R}^n)} \leq \frac{1}{\sqrt{\alpha \pi}} \left(\frac{|\kappa|}{\alpha \pi}\right)^{n/2} \|y e^{-|y|^2} \|_{L^1(\mathbb{R}^n)}$.
3. $\|x^2 G(\kappa t, x)\|_{L^1(\mathbb{R}^n)} = \frac{4|\kappa|^2 |t|}{\alpha} \left(\frac{|\kappa|}{\alpha \pi}\right)^{n/2} \|y^2 e^{-|y|^2} \|_{L^1(\mathbb{R}^n)}$.
4. $\partial_t G(\kappa t, x) = \left(-\frac{n}{2t} + \frac{|x|^2}{4\kappa t^2}\right) G(\kappa t, x)$. 
Proof. The result (1) follows from a direct calculation and (1.17). The result (2) follows from (1) and
\[
\left\| \frac{|x|}{2\sqrt{t}} e^{-|x|^2/4\kappa t} \right\|_{L^1_x(\mathbb{R}^n)} = \left( \frac{|\alpha|}{\alpha t} \right)^{n/2} (4|\kappa|t)^{n/2} \left\| y e^{-|y|^2} \right\|_{L^1_y(\mathbb{R}^n)}
\]
by the change of variables \( y = x \alpha^{1/2}/2|\kappa|t^{1/2} \). The result (3) follows from
\[
\left\| |x|^2 G(\kappa t, x) \right\|_{L^1_x(\mathbb{R}^n)} = (4\pi |\kappa|t)^{-n/2} \left\| |x|^2 e^{-|x|^2/4\kappa t} \right\|_{L^1_x(\mathbb{R}^n)}
\]
and
\[
\left\| |x|^2 e^{-|x|^2/4\kappa t} \right\|_{L^1_x(\mathbb{R}^n)} = \left( \frac{4|\kappa|^2 t^2}{\alpha} \right)^{n/2+1} \left\| y e^{-|y|^2} \right\|_{L^1_y(\mathbb{R}^n)}.
\]
The result (4) follows from a direct calculation. The result (5) follows from (3), (4) and Lemma A.1. \( \square \)

**Lemma A.3.** Let \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). Then
\[
\lim_{s \searrow 0} \int_{|x| > \delta} |G((\alpha + i\beta)s, x)| \, dx = 0
\]
holds for any \( \delta > 0 \).

**Proof.** Put \( \kappa := \alpha + i\beta \). We have
\[
\int_{|x| > \delta} |G(\kappa s, x)| \, dx = \int_{|x| > \delta} (4\pi |\kappa|s)^{-n/2} e^{-|x|^2/4|\kappa|^2s^2} \, dx.
\]
By the change of variables \( y = (4s)^{-1/2}|\kappa|^{-1}x \), we obtain
\[
\int_{|x| > \delta} |G(\kappa s, x)| \, dx = (4\pi |\kappa|s)^{-n/2} (4s)^{n/2}|\kappa|^n \int_{|y| > (4s)^{-1/2}|\kappa|^{-1}\delta} e^{-\alpha|y|^2} \, dy
\]
\[
= |\kappa|^{n/2}\pi^{-n/2} \int_{|y| > (4s)^{-1/2}|\kappa|^{-1}\delta} e^{-\alpha|y|^2} \, dy
\]
\[
\rightarrow 0
\]
as \( s \searrow 0 \). \( \square \)

**Lemma A.4.** Let \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). Any bounded and uniformly continuous function \( f \) satisfies
\[
\lim_{s \searrow 0} \| G((\alpha + i\beta)s, \cdot) \ast f - f \|_{L^\infty(\mathbb{R}^n)} = 0.
\]

**Proof.** Put \( \kappa := \alpha + i\beta \). We have
\[
G(\kappa s, \cdot) \ast f(x) - f(x) = \int_{\mathbb{R}^n} G(\kappa s, y) f(x-y) \, dy - \int_{\mathbb{R}^n} G(\kappa s, y) f(x) \, dy
\]
\[
= \int_{\mathbb{R}^n} G(\kappa s, y) \{f(x-y) - f(x)\} \, dy
\]
for $s > 0$ by (A.2). Thus, we have

$$
|G(\kappa s, \cdot) \ast f(x) - f(x)| \leq \int_{|y| < \delta} |G(\kappa s, y)| |f(x - y) - f(x)| \, dy
+ \int_{|y| > \delta} |G(\kappa s, y)| |f(x - y) - f(x)| \, dy
=: I + II.
$$

(A.4)

For any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|y| \leq \delta$, then

$$
|f(x - y) - f(x)| < \varepsilon
$$

holds for any $x \in \mathbb{R}^n$, since $f$ is uniformly continuous on $\mathbb{R}^n$. We have

$$
I < \varepsilon \int_{|y| < \delta} |G(\kappa s, y)| \, dy < \varepsilon \left(\frac{|\kappa|}{\alpha}\right)^{n/2}
$$

(A.5)

and

$$
II \leq 2\|f\|_{L^\infty} \int_{|y| > \delta} |G(\kappa s, y)| \, dy
$$

(A.6)

by Lemma A.1. By Lemma A.3, there exists $s_0 > 0$ such that

$$
|II| \leq \varepsilon
$$

(A.7)

for $0 < s < s_0$. Therefore, by (A.4), (A.5) and (A.7), we obtain

$$
\|G(\kappa s, \cdot) \ast f - f\|_{L^\infty} \lesssim \varepsilon
$$

for $0 < s < s_0$, which shows the required result since $\varepsilon (> 0)$ is arbitrarily small. \hfill \square

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Makoto Nakamura  
Faculty of Science  
Yamagata University  
Kojirakawa-machi 1-4-12  
Yamagata 990-8560  
Japan  
(E-mail: nakamura@sci.kj.yamagata-u.ac.jp)

Yuya Sato  
DNP Digital Solutions Co. Ltd.  
Nishigotanda 3-5-20  
Shinagawa-ku  
Tokyo 141-0031  
Japan  
(E-mail: Satou-Y132@mail.dnp.co.jp)