Complex Surfaces and Null Conformal Killing Vector Fields

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Abstract
We study the relation between the existence of null conformal Killing vector fields and the existence of compatible complex and para-hypercomplex structures on a pseudo-Riemannian manifold with metric of signature \((2, 2)\). We establish first the topological types of the pseudo-Hermitian surfaces admitting a nowhere vanishing null vector field. Then we show that a pair of orthogonal, pointwise linearly independent, null, conformal Killing vector fields defines a para-hyperhermitian structure and use this fact for a classification of the smooth compact four-manifolds admitting such a pair of vector fields. We also provide examples of neutral metrics with two orthogonal, pointwise linearly independent, null Killing vector fields on most of these manifolds.

Keywords
Null conformal Killing vector fields · Compact pseudo-Hermitian surfaces · Para-hyperhermitian structures

Mathematics Subject Classification
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1 Introduction
The notion of self-duality for metrics of split signature \((2, 2)\) called also neutral metrics, has been studied for a long time and related to various fields like integrable systems [24] and superstring with \(N = 2\) supersymmetry [18, 19]. Many results can directly be extended from the positive definite to the split signature case but there are also some
significant differences. Most of the similarities, as well as the differences, are based on the fact that the metrics of signature (2, 2) are related to the split-quaternions

$$\mathbb{H}' = \left\{ q = a + bi + cs + dt \in \mathbb{R}^4 \mid s^2 = t^2 = 1 = -i^2, t = is, is = -si \right\}$$

in the same way as the positive definite metrics to the quaternions. For example, the similarities include a twistor space construction [3] and a spinor approach [9], while the differences come from the fact that the self-duality equations in split signature are not elliptic but ultra-hyperbolic. Most of the research has been focused on local studies in view of the fact that for ultra-hyperbolic equations the global properties are harder to understand.

There has also been some interest in compact 4-manifolds admitting neutral metrics and compatible complex or para-hypercomplex structures since topological information like the Kodaira classification of compact complex surfaces allows one to study global properties. Important results in this direction are the classifications of compact pseudo-Kähler Einstein and para-hyperkähler surfaces obtained by Petean [20] and Kamada [14], respectively. These structures appear in [18] as models for superstring theory with $N = 2$ supersymmetry and [11] in relation to deformation spaces of harmonic maps from Riemann surfaces into Lie groups. Note also that such structures have been used recently by Klingler [15] in his proof of the Chern conjecture for affine manifolds.

In our previous papers [6, 7] we have initiated the study of compact para-hyperhermitian surfaces, which are the neutral analog of the hyperhermitian 4-manifolds. These surfaces are anti-self-dual as in the positive definite case, but in contrast to the well-known classification of compact hyperhermitian surfaces [4], there are many more compact examples of para-hyperhermitian surfaces. In our study, we have also noticed that any two pointwise independent, null, orthogonal and parallel vector fields on a 4-manifold with a neutral metric define a para-hyperkähler structure and conversely, any compact para-hyperkähler surface admits two vector fields with the above properties [14]. In the present note, we weaken these conditions and study the neutral Hermitian surfaces admitting non-vanishing null conformal Killing vector fields. We observe first that the existence of a non-vanishing null vector field on a compact neutral Hermitian surface leads to a topological restriction which implies a rough classification of these surfaces. Next, we show that the existence of two pointwise independent, null and orthogonal conformal Killing vector fields leads to the existence of a para-hyperhermitian structure. These surfaces have been studied in [6, 7] and here we consider the problem for existence of two Killing vector fields having the above properties with respect to the corresponding metrics.

Now, we describe shortly the content of the paper. In Sect. 2, some known facts about almost para-hyperhermitian structures are collected and in Lemma 2 they are characterized in terms of 2-forms. Then in Sect. 3 we study the complex surfaces admitting neutral Hermitian metrics with non-vanishing null vector fields. After proving some useful local properties of these surfaces, we provide in Theorem 1, a classification in the compact case. In Theorem 2, we notice a property analogous to the positive definite case—if a neutral Hermitian surface admits complete null Killing vector field which
is not real holomorphic, then the metric is anti-self-dual. Section 4 is devoted to the study of the neutral 4-manifolds admitting two orthogonal and null conformal Killing vector fields which are linearly independent at each point. We show that any such a manifold admits a para-hypercomplex structure which is compatible with the given metric (Theorem 3) and provide a classification of these manifolds in the compact case (Theorem 4). In Theorem 4, we also exhibit explicit examples of neutral metrics having two Killing vector fields with the required properties on most of the possible smooth 4-manifolds.

2 Para-Hyperhermitian Structures

Recall that an almost para-hypercomplex structure on a smooth manifold \( M \) is a triple \((I, S, T)\) of anti-commuting endomorphisms of its tangent bundle \( TM \) with \( I^2 = -Id \) and \( S^2 = T^2 = Id, T = IS \). Such a structure is called para-hypercomplex if \( I, S, T \) satisfy the integrability condition \( N_I = N_S = N_T = 0 \), where

\[
N_A(X, Y) = A^2[X, Y] + [AX, AY] - A[A, Y] - A[X, AY]
\]

is the Nijenhuis tensor of \( A = I, S, T \). Note that if two of the structures \( I, S, T \) are integrable, the third one is also integrable [14].

If there exists a pseudo-Riemannian metric \( g \) for which the endomorphisms \( I, S, T \) are skew-symmetric, \((g, I, S, T)\) is called an almost para-hyperhermitian structure. Such a metric is necessarily neutral, i.e. has split signature. We also say that the metric \( g \) is compatible with the almost para-hypercomplex structure \((I, S, T)\) as well as that \((I, S, T)\) is compatible with \( g \). Every almost para-hypercomplex structure on a 4-manifold locally admits a compatible metric but a globally defined one may not exist. More precisely, such a structure determines a conformal class of compatible metrics up to a double cover of \( M \) [6]. Examples of almost para-hypercomplex structures that do not admit compatible metrics are given in [6, 7].

Let \((g, I, S, T)\) be an almost para-hyperhermitian structure on a 4-manifold. Then one can define three fundamental 2-forms \( \Omega_i, i = 1, 2, 3 \), setting

\[
\Omega_1(X, Y) = g(I X, Y), \quad \Omega_2(X, Y) = g(S X, Y), \quad \Omega_3(X, Y) = g(T X, Y).
\]

Note that the form \( \Omega = \Omega_2 + i\Omega_3 \) is of type \((2, 0)\) with respect to \( I \). As in the definite case, the corresponding Lee forms are defined by

\[
\theta_1 = -\delta\Omega_1 \circ I, \quad \theta_2 = -\delta\Omega_2 \circ S, \quad \theta_3 = -\delta\Omega_3 \circ T,
\]

where \( \delta \) is the co-differential with respect to \( g \).

It is well-known [4, 10, 13, 14] that the structures \( I, S, T \) are intergrable if and only if \( \theta_1 = \theta_2 = \theta_3 \). Thus, for a para-hyperhermitian structure, we have just one Lee form \( \theta \); it satisfies the identities

\[
d\Omega_i = \theta \wedge \Omega_i, \quad i = 1, 2, 3.
\]
When additionally the three 2-forms $\Omega_i$ are closed, i.e. $\theta = 0$, the para-hyperhermitian structure is \textit{para-hyperkähler} (also hypersymplectic or neutral hyperkähler). When $d\theta = 0$ the structure is \textit{locally conformally para-hyperkähler}. We note that, in dimension 4, the para-hyperhermitian metrics are self-dual and the para-hyperkähler metrics are self-dual and Ricci-flat [14]. It is well-known that every hyperhermitian structure on a 4-dimensional compact manifold is locally conformally hyperkähler [4], but it has been shown in [7, Theorem 9] that this is not true in the indefinite case.

An almost para-hyperhermitian structure on a 4-manifold can be characterized by means of the forms $\Omega_1$ in the following way [14].

**Proposition 1** Every almost para-hyperhermitian structure on a 4-manifold is uniquely determined by three non-degenerate 2-forms $(\Omega_1, \Omega_2, \Omega_3)$ such that

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \quad \Omega_l \wedge \Omega_m = 0, \quad 1 \leq l \neq m \leq 3.$$ 

This structure is para-hyperhermitian if and only if there is a 1-form $\theta$ such that $d\Omega_l = \theta \wedge \Omega_l$, $l = 1, 2, 3$.

Note that given 2-forms $(\Omega_1, \Omega_2, \Omega_3)$ with the above properties, the endomorphisms $I, S, T$ and the metric $g$ of the almost para-hyperhermitian structure they determine are defined by

$$\Omega_3(I X, Y) = \Omega_2(X, Y), \quad \Omega_1(S X, Y) = \Omega_3(X, Y), \quad \Omega_2(T X, Y) = -\Omega_1(X, Y),$$

and

$$g(X, Y) = \Omega_1(X, I Y), \quad X, Y \in TM.$$ 

One can show that

$$\Omega_1(X, I Y) = -\Omega_2(X, SY) = -\Omega_3(X, TY),$$

so $\Omega_1, \Omega_2, \Omega_3$ are the fundamental 2-forms of $I, S, T$.

We use later on the following characterization of the almost para-hyperhermitian structures with a given almost complex structure.

**Lemma 1** An almost complex 4-manifold $(M, I)$ admits an almost para-hyperhermitian structure with almost complex structure $I$ if and only if there is a $(2, 0)$-form $\Omega$ and a non-degenerate real 2-form $\omega$ such that $\Omega \wedge \overline{\Omega} = -2\omega^2$ and $\Omega \wedge \omega = 0$. If $\Omega$ and $\omega$ satisfy these conditions, $\omega, Re \Omega, Im \Omega$ are the fundamental 2-forms of the almost para-hyperhermitian structure.

**Proof** Suppose we are given 2-forms $\Omega$ and $\omega$ satisfying the conditions of the lemma. Set $\Omega_1 = \omega, \Omega_2 = Re \Omega, \Omega_3 = Im \Omega$. We have $\Omega \wedge \Omega = 0$ since $\Omega$ is a $(2, 0)$-form and $dim M = 4$. This implies $\Omega_2^2 = \Omega_3^2$. Thus, $2\Omega_2^2 = 2\Omega_3^2 = \Omega \wedge \overline{\Omega} = -2\omega^2 \neq 0$. Hence the 2-forms $\Omega_1, \Omega_2, \Omega_3$ are non-degenerate. The identities $\Omega \wedge \Omega = 0$ and $\Omega \wedge \overline{\Omega} = -2\omega^2$ give $-\Omega_1^2 = \Omega_2^2 = \Omega_3^2$ and $\Omega_2 \wedge \Omega_3 = 0$. Moreover, the identity

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\( \Omega \land \omega = 0 \) is equivalent to \( \Omega_2 \land \Omega_1 = \Omega_3 \land \Omega_1 = 0 \). By Proposition 1, there exists an almost para-hyperhermitian structure \((g, I', S, T)\) such that \( \Omega_1, \Omega_2, \Omega_3 \) are the respective fundamental 2-forms. The form \( \Omega = \Omega_2 + i \Omega_3 \) is of type \((2, 0)\) with respect to both \( I \) and \( I' \). It follows that \( \Omega_2(X, Y) = \Omega_3(X, IY) \) and \( \Omega_2(X, Y) = \Omega_3(X, I'Y) \). Hence \( I = I' \) since the form \( \Omega_3(X, Y) = g(X, TY) \) is non-degenerate.

The rest part of the lemma is obvious.

**Remark** If \( \Omega \) and \( \omega \) satisfy the conditions of Lemma 1, then \( \omega \) is of type \((1, 1)\) with respect to \( I \) since \( \omega(X, Y) = g(IX, Y) \) for a neutral metric \( g \) compatible with the almost complex structure \( I \).

### 3 Neutral Hermitian Surfaces with a Null Vector Field

We begin with preliminary algebraic considerations.

Let \( M \) be an oriented 4-manifold with a neutral metric \( g \) and let \((E_1, E_2, E_3, E_4)\) be an oriented orthonormal frame such that \( g(E_1, E_1) = g(E_2, E_2) = 1 \) and \( g(E_3, E_3) = g(E_4, E_4) = -1 \). The Hodge star operator of \( g \) acts as an involution on the bundle of 2-vectors \( \Lambda^2 TM \) and is given by

\[
*E_1 \land E_2 = E_3 \land E_4, \quad *E_1 \land E_3 = E_2 \land E_4, \quad *E_1 \land E_4 = -E_2 \land E_3.
\]

Let \( \Lambda^2_{\pm} TM \) be the subbundles of \( \Lambda^2 TM \) corresponding the the eigenvalues \( \pm 1 \) of the Hodge operator. Set

\[
s_1^+ = E_1 \land E_2 + E_3 \land E_4, \quad s_1^- = E_1 \land E_2 - E_3 \land E_4, \\
s_2^+ = E_1 \land E_3 - E_4 \land E_2, \quad s_2^- = E_1 \land E_3 + E_4 \land E_2, \\
s_3^+ = E_1 \land E_4 - E_2 \land E_3, \quad s_3^- = E_1 \land E_4 + E_2 \land E_3.
\]

Then \( \{s_1^\pm, s_2^\pm, s_3^\pm\} \) is an orthonormal frame of \( \Lambda^2_{\pm} TM \).

Recall that a 2-plane in \( TM \) is called self-dual or an \( \alpha \) plane if there is a basis \( a, b \) of the plane such that \(*a \land b = a \land b\). This condition does not depend on the choice of the basis \( a, b \). Similarly, a 2-plane in \( TM \) is called anti-self-dual or \( \beta \)-plane if there is a basis \( a, b \) with \(*a \land b = -a \land b\).

If \( \Pi \) is an isotropic 2-plane in a tangent space \( T_p M \), one can find a complementary isotropic 2-plane \( \Pi' \) and bases \( a_1, a_2 \) of \( \Pi \) and \( a'_1, a'_2 \) of \( \Pi' \) such that \( g(a_i, a'_j) = \frac{1}{2} \delta_{ij} \). Then \( e_1 = a_1 + a'_1, e_2 = a_2 + a'_2, e_3 = a_1 - a'_1, e_4 = a_2 - a'_2 \) is an orthonormal basis of \( T_p M \) such that \( ||e_1||^2 = ||e_2||^2 = 1, ||e_3||^2 = ||e_4||^2 = -1 \). We have \( 4a_1 \land a_2 = (e_1 + e_3) \land (e_2 + e_4) = (e_1 \land e_2 + e_3 \land e_4) + (e_1 \land e_4 - e_2 \land e_3) \). This shows that \( \Pi \) is either an \( \alpha \) or a \( \beta \)-plane depending on whether the basis \( (e_1, e_2, e_3, e_4) \) determines the given orientation of \( T_p M \) or the opposite one.

**Lemma 2** [7] Let \( M \) be a 4-manifold with a neutral metric \( g \) and let \( X \) and \( Y \) be orthogonal null vector fields which are linearly independent at every point of \( M \). Then the triple \((g, X, Y)\) determines an orientation and a unique orientation and \( g \)-compatible almost complex structure \( J \) on \( M \) such that \( JX = Y \).
In the proof of this lemma, it has been shown that in a neighbourhood of every point of $M$, there exist vector fields $Z$, $T$ such that:

(i) $(X, Y, Z, T)$ is a local frame of the tangent bundle $TM$;
(ii) $g(X, Z) = 1$, $g(X, T) = 0$, $g(Y, Z) = 0$, $g(Y, T) = 1$.

Also, the orientation determined by $(X, Y, Z, T)$ does not depend on the choice of the vector fields $Z, T$ and we shall say that it is determined by the triple $(g, X, Y)$. Set $a = g(Z, Z), b = g(T, T), c = g(Z, T)$ and

\[
E_1 = \frac{1-a}{2} X + Z, \quad E_2 = \frac{1-b}{2} Y + T - cX, \\
E_3 = -\frac{1+a}{2} X + Z, \quad E_4 = -\frac{1+b}{2} Y + T - cX.
\]

Then $(E_1, E_2, E_3, E_4)$ is an orthogonal frame, positively oriented with respect to the orientation determined by $(g, X, Y)$ and such that $g(E_1, E_1) = g(E_2, E_2) = 1, g(E_3, E_3) = g(E_4, E_4) = -1$. The almost complex structure $J$ for which $JE_1 = E_2, JE_3 = E_4$ has the required properties [7].

Now, by (2)

\[
X \wedge Y = (E_1 - E_3) \wedge (E_2 - E_4) = s_1^+ - s_3^+.
\]

Thus, $span\{X, Y\}$ is an $\alpha$-plane. Set

\[
U = X - bY + 2T.
\]

This vector field is null and orthogonal to $X$. Moreover

\[
X \wedge U = -bX \wedge Y \\
+ X \wedge (E_2 + E_4 + bY + 2cX) = (E_1 - E_3) \wedge (E_2 + E_4) = s_1^- + s_3^-.
\]

Hence $span\{X, U\}$ is a $\beta$-plane. Next, $JU$ is a null vector field and

\[
JX \wedge JU = Y \wedge (bX + 2JT) = bY \wedge X + Y \wedge (JE_2 + JE_4 - bX + 2cY) \\
= -(E_2 - E_4) \wedge (E_1 + E_3) = s_1^- - s_3^-.
\]

Thus $span\{JX, JU\}$ is also a $\beta$-plane. The vectors $X, JX = Y, U, JU$ are linearly independent at each point since

\[
X \wedge U \wedge JX \wedge JU = -4E_1 \wedge E_2 \wedge E_3 \wedge E_4.
\]

We sum up these observations in the following:

**Lemma 3** In the notation above:

(i) The vectors $\{X, JX = Y, U, JU\}$ are linearly independent at every point.
(ii) $g(X, U) = 0$, $g(JX, U) = 2$, $g(X, X) = g(U, U) = 0$. 

\[ Springer \]
Proof We have only to prove (iv),(v) and (vi). Suppose that $\Pi$ is a 2-plane containing $X$, and let $\widetilde{X} = \gamma X + \lambda Y + \mu U + vJU$ be a vector such that $\{X, \widetilde{X}\}$ is a basis of $\Pi$. We have by (2)

$$
X \wedge JU = X \wedge (Y - bX + 2JT) = (1 + 2c)(X \wedge Y) + X \wedge (JE_2 + JE_4)
$$

$$
= (1 + 2c)(s_1^+ - s_3^+) - (E_1 - E_3) \wedge (E_1 + E_3)
$$

$$
= (1 + 2c)(s_1^+ - s_3^+) - (s_3^+ + s_2^-).
$$

Then

$$
X \wedge \widetilde{X} = [\lambda + v(1 + 2c)](s_1^+ - s_3^+) - vs_2^+ + \mu(s_1^- + s_3^-) - vs_2^-.
$$

It follows that $\Pi$ is a $\beta$-plane if and only if $\lambda = v = 0$, i.e. $\widetilde{X} = \gamma X + \mu U$ where $\mu \neq 0$. Hence $\Pi$ is a $\beta$-plane if and only if $\Pi = \text{span}\{X, U\}$. Also, $\Pi$ is an $\alpha$-plane exactly when $\mu = v = 0$, i.e. $\widetilde{X} = \gamma X + \lambda Y$ where $\lambda \neq 0$ and $Y = JX$. This proves (vi).

The proof of (v) is similar.

The following lemma shows that a fixed null-vector in a complex vector space with a neutral signature Hermitian metric determines a unique para-hyperhermitian structure.

Lemma 4 Let $V$ be a 4-dimensional real vector space with a neutral scalar product $g$, and let $I$ be a compatible complex structure. Then for any null vector $X \neq 0$ in $V$ there is a unique skew-symmetric endomorphism $S$ of $V$ such that $S^2 = Id, IS = -SI, SX = X$.

Proof Note first that the complex structure determined by the triple $(g, X, Y = IX)$ as in Lemma 2 coincides with $I$.

Now, we apply Lemma 3, and define $S$ to be the identity on the unique null $\beta$-plane containing $X$, and minus the identity on the unique null $\beta$-plane containing $IX$. Then it is easy to check that $S$ has the required properties.

Conversely, suppose that $S$ is an endomorphism with these properties. Then $S$ is an involution different from $\pm Id$. Denote the eigenspace of $S$ corresponding to the eigenvalues $+1$ and $-1$ by $V^+$ and $V^-$. Clearly, the complex structure $I$ is an isomorphism of $V^+$ onto $V^-$ since it anti-commutes with $S$. In particular, $\text{dim} V^+ = \text{dim} V^- = 2$. Note also that the spaces $V^+$ and $V^-$ are isotropic since $S$ is skew-symmetric. Hence $V^+$ is either an $\alpha$ or a $\beta$-plane, and similarly for $V^-$. The space $V^+$ does not contain $IX$, hence it is a $\beta$-plane by Lemma 3. Similarly, $V^-$ is also a $\beta$-plane. This proves the uniqueness.

$\square$
Using the above algebraic observations, we provide a classification of the compact neutral Hermitian surfaces admitting a nowhere vanishing null vector field.

**Theorem 1** Let $(M, g, I)$ be a compact neutral Hermitian surface with nowhere vanishing null vector field $X$. Then the complex surface $(M, I)$ is one of the following:

(i) a complex torus;
(ii) a primary Kodaira surface;
(iii) a minimal properly elliptic surface of odd first Betti number;
(iv) an Inoue surface of type $S^0$ or $S^\pm$ without curves;
(v) a Hopf surface;

Conversely, each of the smooth manifolds underlying the complex surfaces (i)–(iii), an Inoue surface of type $S^+$ or a primary Hopf surface carries a neutral Hermitian metric with a nowhere vanishing null vector field.

**Proof** Let $S$ be the endomorphism of $TM$ defined by means of $I$ and $X$ as in Lemma 4. Setting $T = IS$, we get an almost para-hyperhermitian structure $(g, I, S, T)$ on $M$. Let $\Omega_S$ and $\Omega_T$ be the corresponding fundamental 2-forms defined by $\Omega_S(X, Y) = g(SX, Y)$ and similarly for $\Omega_T$. Then the 2-form $\Omega_S + i\Omega_T$ is non-vanishing and of type $(2, 0)$ with respect to $I$ which implies the vanishing of the first Chern class of $(M, I)$. Now, the classification of compact complex surfaces with vanishing first Chern class provides the types (i)–(v) in the theorem and also the $K3$-surfaces, see, e.g., [7, Theorem 8]. But we exclude the latter from the list since they do not admit non-vanishing global vector fields due to the fact that their Euler characteristic does not vanish.

That the manifolds underlying the surfaces (i)–(iii), an Inoue surface of type $S^+$ or a primary Hopf surface admit neutral Hermitian metrics with a null vector field follows from Theorem 4 proved in the next section. \(\Box\)

**Remark** If $X$ is Killing and $\mathcal{L}_X I = 0$, then $\mathcal{L}_X S = 0$, where $S$ is the endomorphism of $TM$ defined by means of $X$ as in Lemma 4, hence $X$ preserves the almost para-hypercomplex structure. Indeed, notice first that the $(+1)$-eigenspace of the involution $S$ is integrable, see [5, 8]. Hence there is a vector field $Y$ in a neighbourhood of each point of $M$ such that $\text{span}\{X, Y\} = S^+$ and $[X, Y] = 0$. Then $\text{span}\{IX, IY\}$ is the $(-1)$-eigenspace of the involution $S$. Clearly $[X, SX] = S[X, X] = 0$ and $[X, SY] = S[X, Y] = 0$, so $\mathcal{L}_X S(Z) = 0$ for $Z \in S^+$. By assumption, $\mathcal{L}_X I = 0$, i.e. $[X, IZ] = I[X, Z]$ for every $Z$, so $[X, IX] = I[X, X] = 0$. Then $[X, SIX] = -[X, IX] = 0$ and $[S, SY] = -[X, IY] = -I[X, Y] = 0$, hence $\mathcal{L}_X S(Z) = 0$ for $Z \in S^-$. This is well known that on a compact Kähler manifold every Killing vector field is real holomorphic. On general Hermitian manifolds, the existence of a Killing vector field which is not real holomorphic has significant consequences for the curvature. We adapt here the relevant considerations for the split-signature 4-dimensional case.

**Theorem 2** Let $(M, g, I)$ be a neutral Hermitian surface and $X$ a complete null Killing vector field, which is not real holomorphic. Then the metric $g$ is anti-self-dual.

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Proof  Let $\varphi_t$ be the global flow of isometries of $M$ generated by $X$. Then $I_t = (\varphi_t)_* \circ I \circ (\varphi_{-t})_*$, $t \in \mathbb{R}$, are compatible complex structures on $M$. Suppose that the Lie derivative $\mathcal{L}_X I \neq 0$ at a point $p \in M$. Then $\mathcal{L}_X I \neq 0$ in a neighborhood $U$ of $p$. For a point $x \in U$, we can take three different real numbers $t_1, t_2, t_3$ closed enough to 0 such that $I_t(x) \neq \pm I_t(x)$. It follows that the self-dual component $W_+$ of the Weyl tensor vanishes at the point $x$. This fact has been proved in [22] in the positive definite case, but the same proof using spinors works in the split signature case too. Hence $W_+$ vanishes on $U$ and by the proof of [21, Corollary 1.6], $W_+$ vanishes identically. Note that this result is based on [21, Proposition 1.3] which, although stated in the positive definite case, holds true in the split signature case with the same proof replacing the usual twistor space by the so-called hyperbolic twistor space [3].

Remark  As is well-known [17], on a complete pseudo-Riemannian manifold, every Killing vector field is complete.

4 Neutral 4-Manifolds with Two Null Conformal Killing Vector Fields

Recall that a vector field $K$ on a pseudo-Riemannian manifold $(M, g)$ is called conformal Killing if it satisfies

$$g(\nabla_A K, B) + g(\nabla_B K, A) = \frac{2}{\text{dim} M} \text{div}(K)g(A, B), \quad A, B \in TM,$$

where $\nabla$ is the Levi–Civita connection.

The following lemma is an extension of a result by Dunajsky and West [8, Lemma 1] and Calderbank [5, Lemma 4.1].

Lemma 5  Let $K$ and $L$ be two orthogonal null vector fields that are linearly independent at every point.

(i) If $K$ is conformal Killing, then $\nabla_A B \in \text{span}\{K, L\}$ for $A, B \in \text{span}\{K, L\}$.

Suppose that both $K$ and $L$ are conformal Killing vector fields. Then:

(ii) If $K$ and $L$ commute, the distribution $\text{span}\{K, L\}$ is parallel and $\nabla_L K = \nabla_K L = 0$.

(iii) If the distribution $\text{span}\{K, L\}$ is parallel, the vector fields $K$ and $L$ are Killing and commute.

Proof  (i) At every point, the null-plane $\Pi = \text{span}\{K, L\}$ is a Lagrangian subspace, so $\Pi = \Pi^\perp$. Hence $A \in \Pi$ if and only if $A \perp \Pi$. We have

$$g(\nabla_K K, K) = g(\nabla_K L, L) = g(\nabla_L K, K) = g(\nabla_L L, L) = 0$$

because $g(K, K) = g(L, L) = 0$. Moreover,

$$0 = g(K, \nabla_L K) = -g(\nabla_K K, L) = g(K, \nabla_K L)$$

since $K$ is conformal Killing and orthogonal to $L$. Thus, $\nabla_K K$ and $\nabla_K L$ are orthogonal to $\text{span}\{K, L\}$. Moreover, we have
\[ g(\nabla_L K, L) = -g(\nabla_L K, L), \]

hence
\[ g(\nabla_L L, K) = -g(L, \nabla_L K) = 0. \]

Therefore \( \nabla_L K \) and \( \nabla_L L \) are orthogonal to \( \text{span}\{K, L\} \).

(ii) For every tangent vector \( Z \)
\[ g(\nabla_Z K, L) = -g(K, \nabla_Z L) = g(\nabla_K L, Z) = g(\nabla_L K, Z) = -g(\nabla_Z K, L). \]

Hence
\[ g(\nabla_Z K, L) = 0. \]

Therefore \( \nabla_Z K \in \text{span}\{K, L\} \). Similarly, \( \nabla_Z L \in \text{span}\{K, L\} \). Moreover,
\[ g(\nabla_K L, Z) = -g(\nabla_Z L, K) = g(L, \nabla_Z K) = 0. \]

Hence \( \nabla_K L = 0 \) and \( \nabla_L K = 0 \).

(iii) If the distribution \( \text{span}\{K, L\} \) is parallel, then for every \( Z \in TM \)
\[ g(\nabla_L K, Z) = -g(\nabla_Z K, L) = 0. \]

Thus, \( \nabla_L K = 0 \). Similarly, \( \nabla_K L = 0 \). Hence \( [K, L] = 0 \). Also,
\[ 0 = g(\nabla_Z K, L) = -g(\nabla_Z L, K) = -\frac{1}{2} \text{div}(L)g(Z, K) + g(\nabla_K L, Z) \]
\[ = -\frac{1}{2} \text{div}(L)g(Z, K) + \frac{1}{2} \text{div}(K)g(Z, L) - g(\nabla_Z K, L) \]
\[ = -\frac{1}{2} \text{div}(L)g(Z, K) + \frac{1}{2} \text{div}(K)g(Z, L). \]

Since \( K, L \) are linearly independent at each point, it follows that \( \text{div}(L) = \text{div}(K) = 0 \). Thus, \( K \) and \( L \) are Killing.

It follows from Lemmas 2–4 that a neutral 4-manifold with two orthogonal, pointwise linearly independent, null vector fields admits an almost para-hyperhermitian structure. When the vector fields are conformal Killing we have a stronger result.

**Theorem 3** Let \((M, g)\) be a neutral 4-manifold with two orthogonal, pointwise linearly independent, null conformal Killing vector fields \( X \) and \( Y \). Then \( M \) admits a para-hyperhermitian structure \((g, I, S, T)\), where \( I \) is the structure determined by \( \{g, X, Y\} \) as in Lemma 2.

Moreover, if the vector fields \( X \) and \( Y \) commute, then they are real holomorphic and Killing.
**Proof** Consider $M$ with the orientation determined by $(g, X, Y)$ and let $I$ be the compatible almost complex structure from Lemma 2, so $Y = IX$. By Lemma 3, in the vicinity of every point, there is a null vector field $U$ orthogonal to $X$ such that the vector fields $X, IX, U, IU$ have the properties $(i) - (v)$ of this lemma. Then the $(1, 0)$-vector fields $X - iIX$ and $U - iIU$ constitute a frame for $(1, 0)$-vectors. Hence, for the integrability of $I$, it is enough to check that $g(\nabla Z W, W) = 0$ for any choice of $Z, V, W$ among $2X^{1,0} = X - iIX$ and $2U^{1,0} = U - iIU$. Now, since $(1,0)$-vectors are isotropic, for any $(1,0)$-vector fields $Z, U, W$, we have the following:

1. $g(\nabla_Z V, W) = -g(V, \nabla_Z W)$, and, as a consequence, $g(\nabla_Z V, V) = 0$.

Since $X, IX$ are conformal Killing, it is easy to see that
2. $g(\nabla_Z (X - iIX), V) = -g(\nabla_V (X - iIX), Z)$, and, as a consequence, $g(\nabla_Z (X - iIX), Z) = 0$.

From here we get the property $g(\nabla_Z V, W) = 0$ for any choice of $Z, V, W$ among $2X^{1,0} = X - iIX$ and $2U^{1,0} = U - iIU$ by a case by case argument. The details are as follows.

(a) $g(\nabla_{U^{1,0}} X^{1,0}, X^{1,0}) = 0$ (by 1.) and $g(\nabla_{U^{1,0}} X^{1,0}, U^{1,0}) = 0$ (by 2.)

(b) $g(\nabla_{X^{1,0}} U^{1,0}, X^{1,0}) = 0$ (by 2.) and $g(\nabla_{X^{1,0}} U^{1,0}, U^{1,0}) = 0$ (by 1.)

(c) $g(\nabla_{X^{1,0}} U^{1,0}, X^{1,0}) = 0$ (by 1.) and $g(\nabla_{X^{1,0}} U^{1,0}, U^{1,0}) = 0$ (by 1. and then b)

(d) $g(\nabla_{U^{1,0}} U^{1,0}, U^{1,0}) = 0$ (by 1.) and $g(\nabla_{U^{1,0}} U^{1,0}, X^{1,0}) = 0$ (by 1. and a)

Now, we define an endomorphism $S$ of $TM$ as $S = \text{Id on span}\{X, U\}$ and $S = -\text{Id on span}\{IX, IU\}$. Then Lemma 5 implies that the distributions span$\{X, U\}$ and span$\{IX = Y, IU\}$ are integrable. It follows that $S$ is integrable. At the end, we set $T = IS$. Then $T$ is integrable since $I$ and $S$ are integrable.

Suppose that the vector fields $X$ and $Y = IX$ commute. Then, by Lemma 5, the vector fields $X$ and $Y$ are Killing. If $\nabla$ is the Levi-Civita connection of $(M, g)$, we have $\nabla_X IX = \nabla_{IX} X$. It follows that

$$g(\nabla_X Z, V) = g(\nabla_{IX} Z, V) = 0$$

for $Z, V \in \text{span}\{X, IX\}$. Moreover

$$g(\nabla_{IX} U, X) = -g(\nabla_U IX, X) = g(\nabla_U X, IX)$$

and

$$g(\nabla_{IX} U, X) = g(\nabla_{IX} U, X) = -g(\nabla_U X, IX) + div(X)$$

Hence

$$g(\nabla_{IX} U, X) = \frac{1}{2} div(X)$$

Then

$$g([U, X], IX) = g(\nabla_U X, IX) - g(\nabla_X U, IX) = -g(\nabla_U IX, X) - g(\nabla_X U, IX) = 2g(\nabla_X IX, U) = div(X)$$
since the vector field $IX$ is conformal Killing, $g(U, X) = 0$ and $g(U, IX) = \text{const}$. Moreover

$$g(I[IU, X], IX) = g(\nabla_{IU} X, X) - g(\nabla_{X} IU, X) = g(\nabla_{X}, IU)$$

$$= -g(\nabla_{IU} X, X) + \frac{1}{2} \text{div}(X) g(IU, X) = -\text{div}(X).$$

Therefore

$$g([U, X] + I[IU, X], IX) = 0.$$

Next,

$$g([U, X], X) = g(\nabla_{U} X, X) + g(U, \nabla_{X} X) = g(\nabla_{U} X, X) - g(\nabla_{U} X, X) = 0.$$

Also,

$$g(I[IU, X]) = -g(\nabla_{IU} X, IX) + g(\nabla_{X} IU, IX)$$

$$= g(\nabla_{IX} X, IU) - g(\nabla_{IX} X, IU)$$

$$= g([X, IX], IU) = 0.$$

Hence

$$g([U, X] + I[IU, X], X) = 0.$$

It is easy to check that

$$g([U, X] + I[IU, X], U) = g([U, X] + I[IU, X], IU) = 0.$$

It follows that $L_{X} I = L_{IX} I = 0.$

Now, we determine the compact 4-manifolds admitting neutral metrics with two independent null conformal Killing vector fields.

**Theorem 4** (A) Let $(M, g)$ be a compact neutral 4-manifold with two orthogonal null conformal Killing vector fields $X$ and $Y$ which are linearly independent at each point. Then $M$ is the underlying smooth manifold of one of the following complex surfaces:

(i) a complex torus
(ii) a primary Kodaira surface
(iii) a minimal properly elliptic surface of odd first Betti number
(iv) an Inoue surface of type $S^{0}$ or $S^{\pm}$ without curves
(v) a Hopf surface
If in addition the vector fields \(X\) and \(Y\) commute, then in (iv) only \(S^+\) is allowed. (B) Conversely, the smooth manifolds underlying the complex surfaces (i)–(iii), an Inoue surface of type \(S^+\) and a primary Hopf surface admit neutral metrics with two orthogonal null Killing vector fields which are linearly independent at each point.

**Proof**  
(A) By Theorem 3, \(M\) carries a para-hyperhermitian structure \((g, I, S, T)\) such that \(Y = IX\). It follows from Theorem 1, applied to the compact Hermitian surface \((M, I, g)\), that \(M\) is one of the manifolds (i)–(v).

The claim for the commuting Killing fields follows from Theorem 3 and the fact that in (iv) only the Inoue surfaces of type \(S^+\) admit holomorphic vector fields [12].

(B) The proof for the cases (i)–(iii) follows from the proof of the second part of [7, Theorem 7]. Here are the details.

(i) Let \(M\) be the quotient of \(\mathbb{C}^2\) by a lattice \(< a_1, a_2, a_3, a_4 >\). Take a smooth function \(\alpha\) on \(\mathbb{C}\) such that \(\alpha(z + a_3) = \alpha(z), \alpha(z + a_4) = \alpha(z)\). Then the metric

\[
\tilde{g} = \alpha \, dz d\overline{z} + 2Re(dzd\overline{w})
\]

on \(\mathbb{C}^2\) descends to a split signature Ricci flat, Kähler metric \(g\) on the torus \(M\) [20]. For this metric, the global vector field \(W\) on \(M\) given in the standard local coordinates \((z, w)\) on the torus by \(\frac{\partial}{\partial w}\) is holomorphic, parallel and null. Hence if \(K = ReW\), the vector fields \(K\) and \(IK = ImW\) have the properties stated in the theorem.

(ii) A primary Kodaira surface [16] can be obtained in the following way. Consider the affine transformations \(\varphi_k(z, w)\) of \(\mathbb{C}^2\) given by

\[
\varphi_k(z, w) = (z + a_k, w + \overline{a}_k z + b_k),
\]

where \(a_k, b_k, k = 1, 2, 3, 4\), are complex numbers such that

\[
a_1 = a_2 = 0, \quad Im(a_3\overline{a}_4) = b_1 \neq 0, \quad b_2 \neq 0.
\]

They generate a group \(G\) of affine transformations acting freely and properly discontinuously on \(\mathbb{C}^2\) and \(M\) is the quotient space \(\mathbb{C}^2/G\) for a suitable choice of \(a_k\) and \(b_k\) [16, p.786]. Taking into account the identities

\[
\varphi_k \left( \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial z} + \frac{a_k}{\partial w}, \quad \varphi_k \left( \frac{\partial}{\partial w} \right) = \frac{\partial}{\partial w},
\]

we see that every holomorphic vector field on \(M\) is proportional to the vector field \(W\) given in the local coordinates \((z, w)\) as \(\frac{\partial}{\partial w}\). As in [20], set \(\alpha(z) = f(z) - \gamma z - \overline{\gamma} \overline{z}\), where \(f(z)\) is a smooth function on \(\mathbb{C}\) satisfying the identities \(f(z + a_3) = f(z), f(z + a_4) = f(z)\). Then the metric \(\tilde{g} = \alpha \, dz d\overline{z} + 2Re(dzd\overline{w})\) on \(\mathbb{C}^2\) descends to a neutral Kähler, Ricci flat metric \(g\) on \(M\) for which the holomorphic vector field \(W\) is parallel and null (the metric \(g\) is flat if \(f = const\)). Then the vector fields \(K = ReW\) and \(IK = ImW\) have the required properties.
Suppose $M$ is a minimal properly elliptic surface of odd first Betti number. Let $\tilde{SL}(2, R)$ be the universal covering group of $SL(2, R)$. By [23, Theorem 7.4], $M$ is the quotient of $SL(2, R) \times \mathbb{R}$ by a co-compact lattice. Let $A', B', C'$ be the frame of left-invariant vector fields on $SL(2, R)$ determined by the matrices $A' = 1/2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B' = 1/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $C' = 1/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Denote the lifts of $A', B', C'$ to vector fields on $\tilde{SL}(2, R)$ by $A, B, C$. These vector fields satisfy the following commutation relations

$$[A, B] = C, \quad [B, C] = -A, \quad [C, A] = B.$$ 

Hence, if $(\alpha, \beta, \gamma)$ is the dual frame of $(A, B, C)$,

$$d\alpha = \beta \wedge \gamma, \quad d\beta = \alpha \wedge \gamma, \quad d\gamma = -\alpha \wedge \beta.$$

Set $V = \frac{d}{dx}$, the standard vector field on $\mathbb{R}$, and let $I$ be the invariant almost complex structure on $\tilde{SL}(2, R) \times \mathbb{R}$ given by $IV = C, IA = B$. If $\theta$ is dual to $V$, $d\theta = 0$ and a basis of $(1, 0)$-forms for the invariant almost complex structure $I$ is given by $\theta + i\alpha, \beta + i\gamma$. Notice also that

$$d(\beta + i\gamma) = -i\alpha \wedge (\beta + i\gamma), \quad d(\theta + i\alpha) = \beta \wedge (\beta + i\gamma),$$

so $I$ is integrable. Now, consider the form

$$\Omega = (\theta + i\alpha) \wedge (\beta + i\gamma).$$

This is a $(2, 0)$-form and from above it follows $d\Omega = -\theta \wedge \Omega$. Note that $\omega = \theta \wedge \alpha - \beta \wedge \gamma$ satisfies $d\omega = -\theta \wedge \omega$, so the pair $(\Omega, \omega)$ defines a para-hyperhermitian structure by Lemma 1, which is left invariant (and locally conformally para-hyperkähler). In fact, the metric defined by $\omega$ and $I$ is the bi-invariant metric $g = \theta^2 + \alpha^2 - \beta^2 - \gamma^2$. One can directly check that

$$\mathcal{L}_V g = \mathcal{L}_A g = \mathcal{L}_B g = \mathcal{L}_C g = 0,$$

which also follows from the fact that $V, A, B, C$ are left invariant. Then the neutral Hermitian structure $(g, I)$ and the vector fields $K = V + B$ and $IK = A + C$ descend to $M$ giving a neutral Hermitian structure with two orthogonal null Killing vector fields which are linearly independent at each point.

(iv) In order to discuss the Inoue surfaces $S^+$, we recall first their definition [12]. Take a matrix $N = (n_{ij}) \in GL(2, \mathbb{Z})$ with $\det N = 1$ having two real eigenvalues $\alpha > 1$ and $\alpha^{-1}$. Note that $\alpha$ is an irrational number. Choose real eigenvectors $(a_1, a_2)$ and $(b_1, b_2)$ corresponding to $\alpha$ and $\alpha^{-1}$, respectively. Take integers $p, q, r, r \neq 0$
and a complex number \( t \). Let \((c_1, c_2)\) be the solution of the equation

\[
\varepsilon(c_1, c_2) = (c_1, c_2)N^{tr} + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q),
\]

where \( N^{tr} \) is the transpose matrix of \( N \) and

\[
e_k = \frac{1}{2}n_{k1}(n_{k1} - 1)a_1b_1 + \frac{1}{2}n_{k2}(n_{k2} - 1)a_2b_2 + n_{k1}n_{k2}b_1a_2, \quad k = 1, 2.
\]

Let \( G^+ = G^+_{N, p, q, r, t} \) be the group generated by the following automorphisms of \( \mathbb{C} \times \mathbb{H}, \mathbb{H} \) being the upper half-plane:

\[
g_0(z, w) = (\varepsilon z + \frac{1}{2}(1 + \varepsilon)t, \alpha w),
\]

\[
g_k(z, w) = (z + b_kw + c_k, w + a_k), \quad k = 1, 2,
\]

\[
g_3(z, w) = \left(z + \frac{b_1a_2 - b_2a_1}{r}, w\right).
\]

The group \( G^+ \) acts properly discontinuously and without fixed points in view of (3) and the fact that \((a_1, b_1)\) and \((a_2, b_2)\) are linearly independent vectors. The quotient \( S^+_{N, p, q, r, t} = (\mathbb{C} \times \mathbb{H})/G^+ \) is a compact complex surface, known as an Inoue surface of type \( S^+ \).

Set \( t_2 = \text{Im} \, t \) and

\[
\alpha_1 = dx - \frac{1}{v}(y - t_2 \ln v)du, \quad \alpha_2 = dy - \frac{1}{v}(y - t_2 \ln v)dv, \quad \alpha_3 = \frac{du}{v}, \quad \alpha_4 = \frac{dv}{v},
\]

where \( z = x + iy \) and \( w = u + iv \). These forms are linearly independent and invariant under the action of the group \( G^+ \). Note also that \( \alpha_1 + i\alpha_2 \) and \( \alpha_3 + i\alpha_4 \) constitute a frame of \((1, 0)\)-forms. Moreover,

\[
d\alpha_1 = \alpha_3 \land \alpha_2 - \frac{t_2}{\ln \alpha} \alpha_3 \land \alpha_4, \quad d\alpha_2 = \alpha_4 \land \alpha_2, \quad d\alpha_3 = \alpha_3 \land \alpha_4, \quad d\alpha_4 = 0.
\]

Set

\[
\Omega_1 = \alpha_1 \land \alpha_3 + \alpha_2 \land \alpha_4, \quad \Omega_2 = \alpha_1 \land \alpha_3 - \alpha_2 \land \alpha_4, \quad \Omega_3 = \alpha_1 \land \alpha_4 + \alpha_2 \land \alpha_3.
\]

Then

\[
-\Omega_1^2 = \Omega_2^2 = \Omega_3^2 = 2\alpha_1 \land \alpha_2 \land \alpha_3 \land \alpha_4,
\]

\[
\Omega_l \land \Omega_m = 0, \quad 1 \leq l, m \leq 3, \quad d\Omega_l = -\alpha_4 \land \Omega_l.
\]

Therefore, by Proposition 1, \( \Omega_1, \Omega_2, \Omega_3 \) define an \( G^+ \)-invariant para-hyperhermitian structure on \( \mathbb{C} \times \mathbb{H} \) with the neutral metric \( g(X, Y) = \Omega_1(X, JY) \) which is locally
conformally para-hyperkähler since its Lie form \( \theta = -\alpha_4 \) is closed. This structure descends to a para-hyperhermitian structure on the Inoue surface \( S^+ \).

The frame \((X_1, \ldots, X_4)\) of \( G^+ \)-invariant vector fields dual to the frame \((\alpha_1, \ldots, \alpha_4)\) is given by

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (y - t_2 \ln \alpha) \frac{\partial}{\partial x} + v \frac{\partial}{\partial u}, \quad X_4 = (y - t_2 \ln \alpha) \frac{\partial}{\partial y} + v \frac{\partial}{\partial v}.
\]

The non-zero Lie brackets of these vector fields are

\[
[X_2, X_3] = X_1, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = -X_3.
\]

If \( I \) is the standard complex structure on \( \mathbb{C} \times \mathbb{H} \), clearly \( IX_1 = X_2, \ IX_3 = X_4 \). Set \( K = X_1 \). Then \( L_K I = L_K \Omega_1 = 0 \), so \( K \) is Killing and real holomorphic vector field. One can check that \( IK = X_2 \) is also Killing and real holomorphic with \([K, IK] = 0\).

For any non-negative smooth function \( f \) on \( M \) with \( X_1(f) = X_2(f) = 0 \) consider the non-degenerate real \((1, 1)\)-form \( \omega_f = \Omega_1 + f \alpha_3 \wedge \alpha_4 \). It defines a neutral metric compatible with \( I \) and one can check as above that the vector fields \( K, IK \) are null and Killing with respect to this metric. It is easy to check that the \((2,0)\)-form \( \Omega = (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4) \) and the real \((1, 1)\)-form \( \omega_f \) satisfy the identities

\[
\Omega \wedge \overline{\Omega} = -2\omega_f^2, \quad \Omega \wedge \omega_f = 0, \quad d\Omega = -\alpha_4 \wedge \Omega, \quad d\omega_f = -\alpha_4 \wedge \omega_f.
\]

Hence, by Lemma 1, \( \Omega \) and \( \omega_f \) define a para-hyperhermitian structure on \( M \).

(v) Recall that every primary Hopf surface is diffeomorphic to \( S^1 \times S^3 \). Consider \( S^3 \) as the Lie group \( SU(2) \) and let \( X_2, X_3, X_4 \) be left-invariant vector fields defining its Lie algebra with the commutator relations

\[
[X_2, X_3] = X_4, \quad [X_3, X_4] = X_2, \quad [X_4, X_2] = X_3.
\]

Set \( X_1 = \frac{\partial}{\partial x} \) where \( x \) is the standard coordinate \( e^{it} \to t \) on \( S^1 \). Denote by \( g \) the metric on \( S^1 \times S^3 \) for which the frame \( X_1, \ldots, X_4 \) is orthogonal and \( ||X_1||^2 = ||X_2||^2 = 1, \ ||X_3||^2 = ||X_4||^2 = -1 \). This metric is Hermitian with respect to the complex structure defined by \( IX_1 = X_2, \ IX_3 = X_4 \). It is easy to check that \( L_{X_k} g = 0, \ k = 1, 2, 3, 4 \). Then \( X = X_1 + X_3 \) and \( IX = X_2 + X_4 \) are two orthogonal null Killing vector fields.

**Remark** We show that the Hopf surface \( S^1 \times S^3 = S^1 \times SU(2) \) does not admit any left-invariant metric with two non-collinear, left-invariant isotropic and orthogonal Killing vector fields that commute. To do this, we will use the notations above. Suppose that \( g \) is a left-invariant metric (of arbitrary signature) on \( S^1 \times SU(2) \) and let \( X = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3 \) and \( Y = y_0X_0 + y_1X_1 + y_2X_2 + x_3X_3 \) be left-invariant vector fields satisfying the above properties. Then

\[
0 = [X, Y] = (x_2y_3 - x_3y_2)X_1 + (x_3y_1 - x_1y_3)X_2 + (x_1y_2 - x_2y_1)X_3
\]
and therefore the vectors $U = x_1X_1 + x_2X_2 + x_3X_3$ and $V = y_1X_1 + y_2X_2 + x_3X_3$ are collinear. Without loss of generality we may assume that $V = aU$ and since $X = x_0X_0 + U$ and $Y = y_0X_0 + aU$ are isotropic and orthogonal, we get

\[
\begin{align*}
    x_0^2 \parallel X_0 \parallel^2 &+ 2x_0g(X_0, U) + \parallel U \parallel^2 = 0 \\
y_0^2 \parallel X_0 \parallel^2 &+ 2ay_0g(X_0, U) + a^2 \parallel U \parallel^2 = 0 \\
x_0y_0 \parallel X_0 \parallel^2 &+ (ax_0 + y_0)g(X_0, U) + a \parallel U \parallel^2 = 0.
\end{align*}
\]

Having in mind that $y_0 \neq ax_0$ ($X$ and $Y$ are non-collinear), it follows, respectively, from the first and second, and from the first and third equation in (6) that

\[
\begin{align*}
    (y_0 + ax_0) \parallel X_0 \parallel^2 &+ 2ag(X_0, U) = 0 \\
x_0 \parallel X_0 \parallel^2 &+ g(X_0, U) = 0.
\end{align*}
\]

Hence

\[
(y_0 - ax_0) \parallel X_0 \parallel^2 = 0 \Rightarrow \parallel X_0 \parallel^2 = 0 \Rightarrow g(X_0, U) = \parallel U \parallel^2 = 0.
\]

Note that a left-invariant vector field $A$ is Killing with respect to the left-invariant metric $g$ if and only if

\[
g([A, B], C) + g([A, C], B) = 0
\]

for all left-invariant vector fields $B$ and $C$. In particular, the vector field $X_0$ is Killing since it commutes with all left-invariant vector fields. Hence $U = x_1X_1 + x_2X_2 + x_3X_3$ is a Killing vector field too and, substituting $A = U, B = X_0$ in (8), we get $g([U, C], X_0) = 0$ for $C = X_1, X_2, X_3$. Set $a_i = g(X_0, X_i), i = 1, 2, 3$. Then, using the commutation relations for $X_1, X_2, X_3$, we obtain

\[
x_2a_1 - x_1a_2 = x_3a_1 - x_1a_3 = x_3a_2 - x_2a_3 = 0.
\]

These identities together with $0 = g(U, X_0) = x_1a_1 + x_2a_2 + x_3a_3$ imply that either $x_1 = x_2 = x_3 = 0$, i.e. $U = 0$ or $a_1 = a_2 = a_3 = 0$, i.e. $X_0 = 0$ since $g(X_0, X_0) = 0$. In both cases it follows that the vector fields $X$ and $Y$ are collinear which is a contradiction.

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