Conics Touching a Quartic Surface with 13 Nodes

Ingo Hadan

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Abstract

For a given real quartic surface in complex $\mathbb{P}^3$ that has exactly 13 ordinary nodes as singularities the parameter space of those conics is investigated that have only even order contact with the given quartic. In particular, its irreducible components are described.

1 Introduction

The investigations of this article are motivated by problems arising in the study of twistor spaces over the connected sum of three complex projective planes. In [KK] it is shown that, under suitable conditions, such a twistor space is a small resolution of a Double Solid branched over a real quartic surface with exactly 13 ordinary nodes. (A Double Solid is a branched double cover of $\mathbb{P}^3$; see [C] for many aspects of these varieties. The properties of Double Solids used in this paper are contained in [K2].) However, it is not known for which quartics and for which resolutions twistor spaces occur. One approach to solving this problem is to study the family of twistor lines. These are smooth rational curves with normal bundle $\mathcal{O}(1)\oplus 2$ in the total space of the twistor fibration. Those curves will be called “lines” in the sequel. The base of the twistor fibration must be contained in the set of real points in the parameter space of all lines in the twistor space. Therefore, it seems to be promising to investigate the parameter space of all lines in small resolutions of Double Solids or at least something which is, hopefully, similar enough to this space.

There are some articles on Double Solids e.g. [C] or [I] but there is nothing (as far as known to the author) about lines in Double Solids. In [I] “lines” in Double Solids are studied but Tikhomirov uses an essentially different notion of lines. (His lines are mapped to double tangents of the branch locus – but cf. Proposition 1.1.) Furthermore, he assumes the branch locus to be smooth.

The following Proposition completes the motivation for the investigations announced in the abstract:

**Proposition 1.1** Let $\pi : Z \rightarrow \mathbb{P}^3$ be a Double Solid branched over a quartic $B$ that has at most ordinary nodes as singularities. Let $C \subset Z$ be a line (i.e., a smooth rational curve with normal bundle $\mathcal{O}(1)^{\oplus 2}$) outside the singular set of $Z$. Then the image $\pi(C)$ of $C$ is a conic that has only even order contact with $B$.

By this fact the study of the parameter space of all those “touching conics” is motivated. In this article we will study the parameter space of touching conics by examining its fibration over $\mathbb{P}^3$. (This fibration is given by assigning to each conic the unique plane in which it is contained). In
Section 3 the generic fibre of this fibration is described. Our approach to the description of the space of touching conics is the detailed study of the reducible touching conics which are complanar pairs of double tangents at the quartic $B$. Section 4 is devoted to the examination of the space of double tangents at $B$. These results are used in Section 5 to determine the monodromy of the space

$$Y_F := \{(\ell, H) \in \text{Grass}(2, 4) \times \mathbb{P}^3 \mid \ell \subset H, \ell \text{ is double tangent}\}$$

over $\mathbb{P}^3 \setminus \Delta$, where $\Delta$ denotes the closed subset of planes $H$ such that $B \cap H$ is singular. As a corollary we get the monodromy of the symmetric product $(Y_F \times^\alpha \mathbb{P}^3 Y_F) \setminus \text{Diag}$. The latter can be regarded as the parameter space of reducible touching conics. It serves as a “frame” within the parameter space of all touching conics which, finally, is described using the knowledge on this “frame”.

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2 General Preliminaries

First, for completeness, Proposition 1.1 is to be proven, now.

Proof of (1.1): Let $\omega_Z$ be the canonical sheaf of $Z$. It holds $\omega_Z \cong \pi^*\mathcal{O}_{\mathbb{P}^3}(-2)$ (cf. [BPV] Lemma I.17.1) and by adjunction formula we have

$$\omega_C \cong \omega_Z \otimes \det N_{C|Z}$$

hence

$$\mathcal{O}_C(-2) \cong \pi^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_C(2)$$

$$\mathcal{O}_C(-4) \cong \pi^*\mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_C$$

or for divisors on $C$

$$-4[pt] = \pi^*(-2H) \cdot [C] \quad (H \text{ a hyperplane section in } \mathbb{P}^3)$$

and therefore

$$\pi_*(-4[pt]) = \pi_*((2H) \cdot [C])$$

$$= (-2H) \cdot \pi_*([C]) \quad \text{by projection formula and finally}$$

$$2[pt] = H \cdot \pi_*([C]).$$

Thus we get

$$\pi_*([C]) = 2H^2.$$
Now, if \( \pi(C) \) were of degree one in \( \mathbb{P}^3 \) and \( \pi|_C : C \to \pi(C) \) of degree two we would have

\[
N_{C|Z} = N_{\pi^{-1}(\pi(C))|Z} \cong \pi^* \mathcal{N}_{\pi(C)\mathbb{F}^3} \cong \pi^* \left( \mathcal{O}_{\pi(C)}(1)^{\oplus 2} \right) \cong \mathcal{O}_{C}(2)^{\oplus 2}
\]

in contradiction to \( N_{C|Z} \cong \mathcal{O}_{C}(1)^{\oplus 2} \). Therefore \( \pi(C) \) is a rational curve of degree two and \( \pi|_C : C \to \pi(C) \) is an isomorphism, i.e., \( \pi(C) \) is a smooth conic. Furthermore, \( \pi^{-1}(\pi(C)) \) must split into two irreducible components if not \( \pi(C) \subset B \).

Now, let \( \pi(C) \not\subset B \). Spec \( A = U \subset \pi(C) \) a suitable open subset, \( f \) the equation of \( B \) restricted to \( U \), and \( A_Z := \mathcal{A}[T]/(T^2 - f) \). Then Spec \( A_Z = \pi^{-1}(U) \subset \pi^{-1}(\pi(C)) \) and Spec \( A_Z \) is reducible if and only if \( (T^2 - f) \) is reducible in \( \mathcal{A}[T] \), i.e., if and only if there is an \( g \in A \) with \( f = g^2 \). This proves the proposition. \( \square \)

### 2.1 Parameter space of conics

We will construct the parameter space of conics in \( \mathbb{P}^3 \). The space of “touching conics” will be contained within this parameter space. Let \( \mathcal{H}^3 \) be the space of planes in \( \mathbb{P}^3 \) and \( \mathcal{S} \) the universal subbundle over \( \mathcal{H}^3 = \text{Grass}(3,4) \). Then \( P := \mathbb{P}(\text{Sym}^2 \mathcal{S}^\vee) \) is the parameter space of all conics in \( \mathbb{P}^3 \). (Every conic can be given by a plane and a symmetric form of degree two in this plane. On the other hand, every conic determines a unique plane which it sits in and in that plane a symmetric 2-form which is unique up to multiplication by scalars. Even for a double line there is a unique plane in which it is contained. It is determined by the non-reduced subscheme structure of the double line.) The projection \( p : P \to \mathcal{H}^3 \) assigns to each conic the unique plane which it is contained in.

There is a universal family over \( P \), constructed as follows: Let \( H := \mathbb{P}(p^* \mathcal{S}) \) be the pull-back of the universal plane over \( \mathcal{H}^3 \), \( \tau : H \to P \) the projection, and \( \mathcal{O}_H(1) \) the relative tautological bundle of \( H \) over \( P \). Then there is a distinguished section in \( (\tau^* \mathcal{O}_{p|\mathcal{H}^3}(1))^\vee \otimes \mathcal{O}_H(2) \), for it is

\[
\left( \tau^* \mathcal{O}_{p|\mathcal{H}^3}(1) \right)^\vee \otimes \mathcal{O}_H(2) = \text{Hom} \left( \tau^* \mathcal{O}_{p|\mathcal{H}^3}(1), (\mathcal{O}_H(-1)^\vee)^{\oplus 2} \right)
\]

and there are canonical injections of vector bundles over \( H \):

\[
\mathcal{O}_H(-1) \hookrightarrow \tau^* p^* \mathcal{S}.
\]

\[
\tau^* \mathcal{O}_{p|\mathcal{H}^3}(-1) \hookrightarrow \tau^* p^* (\text{Sym}^2 \mathcal{S}^\vee) = \tau^* \text{Sym}^2 (p^* \mathcal{S})^\vee.
\]

The distinguished section is given by the composition

\[
\tau^* \mathcal{O}_{p|\mathcal{H}^3}(-1) \hookrightarrow \tau^* \text{Sym}^2 p^* \mathcal{S}^\vee \to \text{Sym}^2 \mathcal{O}_H(-1)^\vee.
\]

The universal family over \( P \) is the zero locus of this section.

The above construction shows that there is a natural projection from the parameter space of “touching conics” onto \( \mathcal{H}^3 \), assigning to each conic the plane which it sits in. Section 3 is devoted to the study of the fibres of this projection.
2.2 13-nodal quartics

As outlined in the introduction, the focus of the present investigations lies on the study of quartic surfaces with exactly 13 ordinary double points. Those quartics are extensively studied in \([K2]\). The results needed in the sequel are to be summarised here.

**Proposition 2.1** Every real\(^1\) quartic surface \(B\) with exactly one real point and 13 ordinary double points can be defined by an equation of the form

\[
F = x^2 f_2 + 2x_3 L_0 L_1 L_2 + f_2^2 - f_2 (L_0^2 + L_1^2 + L_2^2) + L_0^2 L_1^2 + L_0^2 L_2^2 + L_1^2 L_2^2
\]

where \(E_1 = x_3 - L_0 - L_1 - L_2, E_2 = x_3 + L_0 + L_1 - L_2, E_3 = x_3 + L_0 - L_1 + L_2, E_4 = x_3 - L_0 + L_1 + L_2, f_2 = x_3^2 + x_1^2 + x_2^2, Q = 2f_2 + x_3^2 - L_0^2 - L_1^2 - L_2^2\) and \(L_j = \sum_{i=0}^{2} a_{ij} x_i\) such that \(f_2 - L_j^2\) \((j = 0, 1, 2)\) defines three smooth conics with 12 different intersection points.

If, moreover, the planes \(E_i\) are real then the quadratic forms \(f_2 - L_j\) are positive definite. If the forms \(L_j\) are mutually linearly independent then \(F\) defines a real quartic with exactly 13 ordinary double points and exactly one real point \(P = (0 : 0 : 0 : 1)\). Each of the six lines \(E_i = E_j = 0\) intersects the quadric \(Q\) in two different points which form a conjugate pair of double points of the quartic.

**Remark:** Those quartics \(B\) satisfying all the conditions of the above proposition are just the quartics which generically occur in connection with twistor spaces as mentioned in the Introduction (cf. \([K2]\) and \([KR]\)).

**Lemma 2.2** The projection of \(B\) (as above – with real planes \(E_i\)) from \(P\) onto the plane \(x_3 = 0\) defines a double cover of \(\mathbb{P}^2\) which is branched along the sextic \(\tilde{S} = (f_2 - L_0^2)(f_2 - L_1^2)(f_2 - L_2^2)\) in \(\mathbb{P}^2\). The conic \(f_2 = 0\) touches \(S\) in six smooth points.

\[\square\]

3 Conics touching a plane quartic

In this section\(^2\), for a given quartic curve \(B\) in \(\mathbb{P}^2\), the set of conics that have only even order contact with \(B\) (but are different from double lines) is investigated. Those conics will be called touching conics in the sequel.

Throughout this section, let \(B \subset \mathbb{P}^2\) be an irreducible quartic curve given by a form \(F\) of degree four which has at most one ordinary node as its only singularity.

Then the following lemma holds.

\(^1\)with respect to the standard real structure of \(\mathbb{P}^3\)

\(^2\)Many ideas of this section are already contained in \([Sa]\).
Lemma 3.1 The set of all touching conics is the union of one-parameter-families. In each family the elements are mutually different. If $B$ is smooth, the families are pairwise disjoint. If $B$ is singular, two families can only intersect in a reducible conic which consists of two lines both containing the singular point of $B$.

Proof: The quadratic form $U$ defines a touching conic if and only if there exist two further quadratic forms $V$ and $W$ such that

$$F = UW - V^2.$$ 

Since

$$UW - V^2 = U(\lambda^2 U + 2\lambda V + W) - (\lambda U + V)^2 \quad \lambda \in \mathbb{C},$$
by $U$, $W$, and $V$ a whole one-parameter-family of touching conics is given:

$$\lambda^2 U + 2\lambda V + W \quad (\lambda : \mu) \in \mathbb{P}^1.$$

All conics in such a family are different from each other. Otherwise we would have

$$F = U(\lambda^2 U + 2\lambda V + W) - (\lambda U + V)^2 = U(\mu^2 U + 2\mu V + W) - (\mu U + V)^2$$
with $\lambda^2 U + 2\lambda V + W = \mu^2 U + 2\mu V + W$ and $\lambda \neq \mu$. Consequently

$$(\lambda U + V)^2 = (\mu U + V)^2$$
hence

$$\lambda U + V = -(\mu U + V) \quad \text{thus} \quad V = \frac{\lambda + \mu}{2} U$$
and finally

$$F = U \left( W - \left( \frac{\lambda + \mu}{2} \right)^2 U \right),$$
so that the quartic $B$ would be reducible in contradiction to the assumption.

Suppose $U$ is contained in two different families [I], i.e., there exist $V$, $W$, resp. $V'$ and $W'$ satisfying

$$F = UW - V^2 = UW' - V'^2$$
and therefore

$$U(W - W') = V^2 - V'^2 = (V + V')(V - V').$$

If $U$ is irreducible it follows $U | (V + V')$ or $U | (V - V')$ and consequently $V' = \pm(\lambda U - V)$ and $W' = \lambda^2 U - 2\lambda V + W$, i.e., $(U, V, W)$ and $(U, V', W')$ yield the same family. If $U$ is reducible and neither $U | (V + V')$ nor $U | (V - V')$ holds, then the intersection of the two lines of $U$ must be a point of $B$ which is necessarily singular. If, finally, $U$ were a double line, $B$ would have an equation of the form $F = L^2 W - V^2$ and therefore would have at least two singular points if the line $L = 0$ intersects $B$ in two points or a cusp in the intersection point of $B$ with $L = 0$. This completes the proof. \qed

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Lemma 3.2 Let $B$ be a quartic with exactly one ordinary node. A conic $U$ consisting of two lines through the node is contained in exactly two one-parameter-families \((3)\) that intersect in just this conic.

**Proof:** Let $F = f_1 x^4 + f_2 y^4 + \cdots$ be the equation of the quartic $B$ and let the node of $B$ have the coordinates \((0:0:1)\). In suitable coordinates $U$ is given by the equation $xy = 0$. By assumption, $F$ is of the form

$$F = xyW - V^2$$

with quadratic forms $V$ and $W$. By comparing the coefficients on both sides of \((3)\) one finds exactly two one-parameter-families $W_\lambda$ containing the conic $U$. \(\square\)

Now, the number of reducible conics in each family \((3)\) is to be determined. Knowing the number of double tangents at a quartic curve, this permits to count the one-parameter-families of touching conics.

Lemma 3.3 Every one-parameter-family \((3)\) contains at least five and at most six reducible elements. The family contains only five reducible conics if and only if it contains a conic which splits into two lines intersecting in a point of $B$ (which is necessarily singular).

**Proof:** A conic is reducible if and only if the determinant of its matrix of coefficients vanishes. Therefore one has to examine the roots of the polynomial in $\lambda$: $\det(\lambda^2U + 2\lambda V + W)$. This polynomial is of sixth degree and, therefore, has at most six roots. For proving the lemma it is necessary to investigate the conditions under which the equation has multiple roots. Since the two triples \((U,V,W)\) and \((\lambda^2U + 2\lambda V + W, \lambda U + V, U)\) of conics define the same one-parameter-family it is sufficient to study under which conditions $\lambda = 0$ is a multiple root. For $\lambda = 0$ to be a multiple root, conditions on the coefficients of $U$, $V$, and $W$ are posed. A simple calculation shows that a multiple root corresponds to a reducible conic the lines of which intersect in a point $P \in B$, and that no other conic of the one-parameter-family then can contain $P$. Therefore, at most one root of the considered equation can be of higher multiplicity since $B$ was supposed to have at most one singular point. If the equation had a root of multiplicity greater than two then the quartic $B$, given by the equation $F = UW - V^2$, would have a cusp. This proves the lemma. \(\square\)

The following proposition is proved e.g. in [Sa] or [B], the smooth case is also treated in [GH] Section 4.4.

**Proposition 3.4** Let $B$ be an irreducible quartic curve. If $B$ is smooth then $B$ has 28 double tangents\(^3\). If $B$ has one ordinary node as its only singularity then $B$ has 22 double tangents.

**Remark:** The Plücker formulas seem to contradict the second part of the Proposition. One must, however, take into account that lines passing through the node and touching the quartic in an other point are not counted by the Plücker formulas. There are six of those lines as one finds for instance in [Sa].

**Proposition 3.5** Let $B$ be an irreducible quartic curve. If $B$ is smooth then the set of touching conics splits into 63 mutually disjoint one-parameter-families \((4)\). If $B$ has an ordinary node as its only singular point then the set of touching conics is divided into 16 families \((4)\) each of which is disjoint from all other families, and 15 pairs of families that intersect in exactly one conic which consists of two lines intersecting in a point of $B$. 

\(^3\)resp. lines that have fourth order contact with $B$
Proof: A smooth quartic curve has 28 double tangents, hence, there are 378 pairs of double tangents, i.e., 378 reducible touching conics. Thus, according to Lemma 3.1, there are 63 families of touching conics and every touching conic is contained in one of these families.

In the singular case there are:

a) 15 pairs of double tangents intersecting in the node of \( B \)
b) 96 pairs one line of which contains the node
c) 120 pairs no line of which contains the node.

According to Lemma 3.2 each of the pairs "a)" is contained in two families each of which contains four pairs "c)" (cf. Lemma 3.3). Thus the pairs "a)" and "c)" spread over 30 families which contain exactly these reducible conics. These families intersect pairwise as stated. The remaining 96 pairs "b)" must be contained in some families of touching conics. Therefore the 96 pairs "b)" generate 16 further families.

Lemma 3.6 Let \( B \) be a smooth quartic curve.

a) The double tangents occurring in the reducible conics in one one-parameter-family are mutually different.
b) Let \( ab, cd, \) and \( eh \) be reducible conics contained in the same one-parameter-family (\( a, b, c, d, e, \) and \( h \) linear forms). Then the double tangent \( e \) does not occur in any reducible conic of that one-parameter-family in which the conics \( ac \) and \( bd \) are contained.

Proof: a) If there were reducible conics \( ab \) and \( ac \) in the same one-parameter-family then the equation \( F \) of \( B \) could be written in the form \( F = a^2bc - V^2 \) with a quadratic form \( V \). The intersection points of the conic \( V \) with the line \( a \) then would be singular points of \( F \).

b) The quartic \( F \) may be written in the form \( F = abcd - V^2 \) with a quadratic form \( V \). Since \( eh \) is contained in the one-parameter-family spanned by the conics \( ab \) and \( cd \) there is a \( \lambda \neq 0 \) such that \( ef = \lambda^2 ab + 2\lambda V + cd \). Now, by writing \( F \) in the form \( F = ac \cdot bd - V^2 \) one finds the one-parameter-family containing \( ac \) and \( bd \) to be \( \mu^2ac + 2\mu\lambda V + \lambda^2bd \). Suppose \( eg \) is contained in the one-parameter-family of \( ac \) and \( bd \), i.e. there is a \( \mu \neq 0 \) such that \( eg = \mu^2ac + 2\mu V + bd \). This yields

\[
V = \frac{1}{2\lambda} (ef - \lambda^2 ab - cd) = \frac{1}{2\mu} (eg - \mu^2 ac - bd)
\]

Hence \( e(\mu f - \lambda g) = (\lambda b - \mu c)(\lambda d - a) \), i.e. the linear form \( e \) divides one of the two forms \( (\lambda b - \mu c) \) or \( (\lambda d - a) \). Thus the line defined by \( e \) contains one of the intersection points of \( b \) and \( c \) or of \( a \) and \( d \). In both cases the common point of the three lines is a singular point of \( B \).
4 The parameter space of double tangents

Let $B \subset \mathbb{P}^3$ be a quartic surface with ordinary double points as its only singularities which is given by an equation of the form

$$g_1g_3 - g_2^2 = 0 \quad (3)$$

where $g_i$ are homogeneous of degree $i$. Let $(x_0 : \ldots : x_4)$ be homogeneous coordinates on $\mathbb{P}^4$ and let $\mathbb{P}^3 \subset \mathbb{P}^4$ as the hyperplane $x_4 = 0$. Then

$$x_4^2g_1 + 2x_4g_2 + g_3 = 0 \quad (4)$$

defines a cubic $K$ in $\mathbb{P}^4$ with $P := (0 : 0 : 0 : 1) \in K$. Consider the projection from $P$ onto the hyperplane $x_4 = 0$. The extension $\tilde{K} \rightarrow \mathbb{P}^3$ of this map to the blow-up $\tilde{K}$ of $K$ in $P$ is a partial small resolution of the double solid $Z_0$ branched over $B$ (cf. [K1]). The $\pi$-fibre of any point $x \in \mathbb{P}^3$ with $g_1(x) = g_2(x) = g_3(x) = 0$ (i.e. the singular points of $B$ in the plane $g_1 = 0$) is just the strict transform of the line through $P$ and $x$, all other fibres consist of at most two points. In particular, there are exactly six lines through $P$ in $K$ (if the three surfaces $g_i = 0$ intersect properly) namely the six lines that are contracted by $\pi$. The only singularities of $K$ are the preimages of double points of $B$ not contained in the plane $g_1 = 0$. These are ordinary double points, as well.

Lemma 4.1 $\pi : K \setminus \{P\} \rightarrow \mathbb{P}^3$ maps lines in $K \setminus \{P\}$ onto lines that have even intersection with $B$ or which are contained in $B$. If such a line is contained in $B$ the it passes through a singular point of $B$ which is contained in the plane $g_1 = 0$. Moreover, such a line has one point of higher order intersection with the cubic $g_3 = 0$ or the line is contained in this cubic.

Proof: Let

$$L : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4$$

be the parameter representation of a line $L$ in $\mathbb{P}^4$, which is contained in $K$. Thus the equation (3) restricted to $L$ vanishes, i.e.

$$g_3|_L = -l_4^2g_1|_L - 2l_4g_2|_L \quad \text{thus}$$

$$g_1|_L \cdot g_3|_L - g_2|_L^2 = -l_4^2g_1|_L^2 - 2l_4g_1|_L \cdot g_2|_L - g_2|_L^2 = -(l_4g_1|_L + g_2|_L)^2$$

Hence, equation (3) restricted to $\pi(L)$ is a complete square and therefore the image of $L$ in $\mathbb{P}^3$ is a line with even intersection with $B$ or is contained in $B$. If $\pi(L)$ is contained in $B$ then $L$ is a line with fourth order contact with $B$.

We will call those lines simply double tangents, i.e. lines with fourth order contact will be called double tangents, too.
contained in the ramification locus of $\pi$. Hence, the plane spanned by $L$ and $P$ intersects $K$ in a cubic curve which consists of $L$ counting twice and a line through $P$. The lines through $P$ in $K$ are the lines connecting $P$ with the singular points of $B$ in the plane $g_1 = 0$. Moreover, if $\pi(L) \subset B$ then $(-l_4 \cdot g_1|_L + g_2|_L)$ must vanish and, hence, $g_3|_L = l_3^2 \cdot g_1|_L$. □

**Remark:** In the case that we are particularly interested in – $-B$ is given by an equation as described in Proposition 2.1 and $g_3$ is the product of three of the linear forms $E_i$, say $g_3 = E_2E_3E_4$. $Q$ and $E_i$ cannot have common real zeros for there is only one real point on $B$ which is outside the quadric $Q = 0$. Therefore, the planes $E_i = 0$ intersect the quadric $g_2 := Q = 0$ along smooth conics. Consequently, there is no line which is contained in $B$ and in the cubic $g_3 = 0$.

A line $\ell$ (not contained in $g_3 = 0$) that has a point $P_0$ of higher order intersection with the cubic $g_3 = 0$ must meet the intersection of two of the three planes the cubic consists of. On the other hand, no common point of three of the four planes $E_i$ is a point of $B$ by Proposition 2.1. In particular, if $\ell \subset B$ then $P_0$ is not contained in the plane $g_1 = 0$. Hence, if $\ell$ is the image of a line in $K$ which is contained in $B$ then $\ell$ must pass through two singular points of $B$, namely $P_0$ and the the singular point $P_1$ of $B$ in the plane $g_1$ which $\ell$ must pass through by the above lemma.

Moreover, any line in $B$ (if any) can appear as images of lines in $K$ that pass through two double points of $B$. But lines through two double points are double tangents unless they are contained in $B$ and hence, lines in $B$ through two double points necessarily appear in the parameter space of double tangents.

**Lemma 4.2** Every double tangent of $B$ that is not contained in the plane $g_1 = 0$ is the image under $\pi$ of a line in $K$.

**Proof:** Let

$$L' : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$$

$$(s : t) \mapsto (l_0(s, t) : \ldots : l_3(s, t))$$

be the parameter representation of a line in $\mathbb{P}^3$. If $L'$ is contained in the surface $g_3 = 0$ then there is nothing to show; so, let $L'$ not be contained in $g_3 = 0$. If $L'$ is a double tangent then there exists a quadratic form $q$ on $L'$ such that

$$g_1|_{L'} \cdot g_3|_{L'} - g_2|_{L'}^2 = -q^2$$

and therefore

$$g_1|_{L'} \cdot g_3|_{L'} = (g_2|_{L'} + q)(g_2|_{L'} - q)$$

Hence there is a linear form $l_4(s, t)$ on $L'$ such that $l_4|_{L'} g_1|_{L'} = -(g_2|_{L'} \pm q)$. By eventually replacing $q$ with $-q$ we can assume that $l_4|_{L'} g_1|_{L'} = - (g_2|_{L'} + q)$. Then

$$g_1|_{L'} \cdot (l_3^2 g_1|_{L'} + 2l_4 g_2|_{L'} + g_3|_{L'}) =$$

$$= (g_2|_{L'} + q)^2 + 2l_4 g_1|_{L'} \cdot g_2|_{L'} + (g_2|_{L'} + q)(g_2|_{L'} - q)$$

$$= (g_2|_{L'} + q)(g_2|_{L'} + q + g_2|_{L'} - q) + 2l_4 g_1|_{L'} \cdot g_2|_{L'}$$

$$= 2 g_2|_{L'} (g_2|_{L'} + q + l_4 g_1|_{L'})$$

$$= 0$$
Therefore, since \( g_1|_{L'} \neq 0 \), \( l_3^2 g_1|_{L'} + 2l_4 g_2|_{L'} + g_3|_{L'} = 0 \) and hence \((l_1(s, t) : \ldots : l_4(s, t))\) \(((s : t) \in \mathbb{P}^1)\) defines a line in \( \mathbb{P}^4 \) which is contained in \( K \) and the image under \( \pi \) of which is \( L' \).

\[ \square \]

**Lemma 4.3** Let \( L \) be a double tangent of \( B \) which is not contained in \( g_1 = 0 \). If \( L' \) and \( L'' \) are different lines in \( K \), which are mapped onto \( L \) then \( L \) contains one of the points \( x \) with \( g_1(x) = g_2(x) = g_3(x) = 0 \). In this case \( L' \) and \( L'' \) are the only lines in \( K \) that are mapped to \( L \).

**Proof:** Consider the plane \( H \) in \( \mathbb{P}^4 \) spanned by \( L \) and \( P \). Then \( H \) is not contained in \( K \). If \( H \) were contained in \( K \) then \( L \) would be contained in the locus \( g_3 = 0 \) in \( \mathbb{P}^3 \). \( H \subset K \) then implies \( g_1 = g_2 = 0 \) on \( L \). But \( L \) was supposed to be not contained in \( g_1 = 0 \). Therefore \( K \cap H \) is a plane cubic curve that contains \( L' \) and \( L'' \). Hence, this plane cubic must split into three components all of which are lines. One of these lines must contain \( P \), but the lines through \( P \) are just the lines \( \overline{P \cdot x} \) with \( x \in \mathbb{P}^3 \) and \( g_1(x) = g_2(x) = g_3(x) = 0 \). On the other hand \( x \) is in \( L \) which proves the lemma.

The projection \( \pi \) induces a morphism \( \overline{\pi} \) between Grassmannians of lines of \( \mathbb{P}^4 \) and \( \mathbb{P}^3 \):

\[ \overline{\pi} : \text{Grass}(2, 5) \setminus \{ \text{lines through } P \} \cup \text{Fano}(K) \setminus \{ \text{lines through } P \} \rightarrow \text{Grass}(2, 4) \cup \{ \text{lines with even intersection with } B \} \]

where \( \text{Fano}(K) \) denotes the set of lines in \( K \). The above lemmata suggest that \( \overline{\pi} \) defines a birational map onto the set of bitangents of \( B \) in \( \text{Grass}(2, 4) \). This will be shown later.

### 4.1 Lines in a nodal cubic threefold and bisecants of a space curve

Let \( K \subset \mathbb{P}^4 \) be a cubic hypersurface with only ordinary double points as singularities. Let \( R_i \) be one of the double points, \( H \) a hyperplane not containing \( R_1 \), and let \( Q' \) be the intersection of the tangent cone of \( R_1 \) with the hyperplane \( H \). Finally, let \( S = K \cap Q' \). Assume the curve \( S \) to have only ordinary double points as singularities. (Under the above assumptions on \( K \) this is, in fact, always true, cf. [Finkelnberg].) The singularities of \( S \) are just the images of the double points of \( K \setminus \{ R_i \} \). Denote by \( p \) the projection \( p : K \setminus \{ R_i \} \rightarrow H \). Then the following proposition holds:

**Proposition 4.4** Let \( \ell \) be a line in \( K \). If \( R_i \in \ell \) then \( \ell \) is a line \( \overline{P \cdot x} \) with \( x \in S \) and, conversely, every \( x \in S \) defines a line in \( K \) through \( R_i \). A line \( \ell \) not containing \( R_i \) is mapped onto a line \( \overline{\ell} = p(\ell) \subset H \) not contained in \( Q' \) and either connecting two points of \( S \) or being a tangent of \( S \) at a smooth point of \( S \) or a tangent of \( Q' \) at a non-smooth point of \( S \). Conversely every such line \( \overline{\ell} \) is the image of a line in \( K \).

**Proof:** Here, a sketch of the proof of Finkelnberg (cf. [Finkelnberg]) is to be given. The case where \( R_i \in \ell \) is obvious. Let \( \overline{\ell} \) be an arbitrary line in \( H \) and let \( V \) be the plane in \( \mathbb{P}^4 \) spanned by \( \overline{\ell} \) and \( R_i \). Noting that \( Q' \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \) and \( S \in |O_{Q'}(3, 3)| \) the following cases occur:

- \( \overline{\ell} \cap S = \emptyset \) and \( \overline{\ell} \) intersects \( Q' \) transversally. Then \( V \cap K \) is an irreducible plane cubic with an ordinary double point in \( R_i \).
\[ \subset S = \emptyset \] and \( \tau \) is tangent to \( Q' \). Then \( V \cap K \) is an irreducible plane cubic with a cusp in \( R \).

\( \subset S = \{T\} \) and \( \tau \) intersects \( Q' \) transversally. Then \( V \cap K \) consists of the line \( \overline{RT} \) and a conic intersecting \( \overline{RT} \) transversally. The conic must pass through \( R \) and so can not split into two lines since one of the lines were a line through \( R \) and would intersect \( S \) in a point different from \( T \). If \( \overline{RT} \) were tangent to the conic then \( V \cap K \) had a cusp which is impossible since \( \tau \) intersects \( Q' \) in two points.

\( \subset S = \{T, T_2\} \) with \( T_1 \neq T_2 \) and \( \tau \) is not contained in \( Q' \). Then \( V \cap K \) contains the two lines \( \overline{RT_1} \) and \( \overline{RT_2} \). Even if one or both points \( T_i \) are singular points of \( S \) none of these lines counts twice since this is only possible if \( \tau \) is tangent to \( Q' \). Therefore \( V \cap K \) contains a third line not through \( R \).

\( \tau \) is tangent to \( S \) in a smooth point \( T \) of \( S \). Then \( V \cap K \) splits into the line \( \overline{RT} \) with multiplicity two and a second line that does not pass through any singular point of \( K \). (To see that the first line carries multiplicity two move \( \tau \) a bit.)

\( \tau \) is tangent to but not contained in \( Q' \) and meets \( S \) in a singular point \( T \). Then \( V \cap K \) consists of the line \( \overline{RT} \) with multiplicity two (move \( \tau \) a bit!) and a second line. The second line can not pass through \( R \) since then it would intersect \( S \) in a point different from \( T \).

\( \tau \) is contained in \( Q' \). Then \( \tau \) meets \( S \) in three (not necessarily different) points. Each of these points is the intersection point of a line through \( R \) in \( K \) with \( S \), multiple points corresponding to multiple lines.

Therefore, only those lines mentioned in the proposition can be images of lines in \( K \) not through \( R \) and each of them determines such a line in \( K \). This proves the proposition. \( \square \)

### 4.2 The Fano scheme of lines on the cubic threefold

From now on it becomes convenient to make use of the special type of the quartic \( B \) as described in Proposition 2.1. Let \( B \) be a real quartic with exactly 13 ordinary double points such that (using the notation of Proposition 2.1) the linear forms \( E_1, \ldots, E_4 \) are real (i.e. have real coefficients). Take \( g_1 = E_1, g_3 = E_2E_3E_4 \) and \( g_2 = Q \) to achieve the form \( g_1g_3 - g_2^2 \), i.e. \( B \) is given by an equation of the form

\[
F = x_3^2f_2 + 2x_3L_0L_1L_2 + f_2^2 - f_2(L_0^2 + L_1^2 + L_2^2) + L_0^2L_2^2 + L_0^2L_2^2 + L_1^2L_2^2
\]

\[
= \frac{1}{4}(Q^2 - E_3E_4E_4)
\]

\[
= \frac{1}{4}(g_2^2 - g_1g_3)
\]

With the notation of the previous paragraph an letting \( H \subset \mathbb{P}^4 \) be the plane \( x_4 = 0 \), the following lemma holds.
**Lemma 4.5**  The curve $S = K \cap Q'$ consists of three components, each a smooth conic in one of the planes $E_i = 0$ ($i = 2, 3, 4$). Each two components intersect in two points.

**Proof:** First, the equation of $Q'$ is determined: Let $T$ be a point in $H$. Then $T \in Q'$ if and only if the line $\overline{PT}$ is contained in the tangent cone of $K$ in $P_1$ which is the case if and only if this line has third order contact with $K$ in $P_1$. A simple calculation then shows that $T \in Q'$ if and only if $T$ is contained in the zero locus of the quadratic form $(x_3 + L_0 + L_1 + L_2)^2 - 4f_2$ (notation as in Proposition 2.3). Since $S = K \cap Q' = (K \cap H) \cap Q'$ and since $K \cap H$ is the union of the three planes $E_i = 0$ ($i = 2, 3, 4$), the components of $S$ are the three conics $Q' \cap \{E_i = 0\}$ (see also last remark in [K1]). These three conics are given by the equations

\[
\begin{align*}
 f_2 - L_2^2 & = 0 \quad \text{in the plane } E_2 = 0 \\
 f_2 - L_1^2 & = 0 \quad \text{in the plane } E_3 = 0 \\
 f_2 - L_0^2 & = 0 \quad \text{in the plane } E_4 = 0
\end{align*}
\]

In the plane $x_3 = 0$, these equations define smooth conics (by Proposition 2.1) and since none of the planes $E_i = 0$ contains the point $(0 : 0 : 0 : 1)$ they define smooth conics in the planes $E_i = 0$, as well. Another short calculation shows that the quadrics $Q$ and $Q'$ coincide on the lines $E_i = E_j = 0$. Therefore, by the assumptions on the quartic and Proposition 2.1, these lines intersect $Q'$ in two different points. \hfill \Box

Denote by $\text{Bisec}(S) \subset \text{Grass}(2,4)$ the closure of the set of bisecants of $S$. By the above lemma, $S$ splits into 3 irreducible components which will be denoted by $S_i$, $i = 1, 2, 3$. Let $B_{ij} \subset \text{Bisec}(S)$ $(1 \leq i \leq j \leq 3)$ be the closure of the set of bisecants that connect $S_i$ with $S_j$. $B_{ii}$ then consists of the bisecants and the tangents of $S_i$ whereas $B_{ij}$ contains the lines connecting different points of $S_i$ and $S_j$ together with all tangents at $Q'$ in the to intersection points of $S_i$ and $S_j$. The $B_{ij}$ are the six irreducible components of $\text{Bisec}(S)$, for they are irreducible, cover all of $\text{Bisec}(S)$ and contain open subsets that are disjoint from all other $B_{ij}$.

Now, using Proposition 4.3, morphisms from $B_{ij}$ to $\text{Fano}(K)$ are to be constructed which will turn out to induce a birational map between $\text{Bisec}(S)$ and $\text{Fano}(K)$. Let $\mathcal{U}_{B_{ij}} \subset \mathbb{P}^3 \times B_{ij}$ be the "universal line" over $B_{ij}$; $\mathcal{U}_{B_{ij}} = \{(x, \ell) \mid x \in \ell\}$. Let $C_{B_{ij}}$ be the cone from $\{P\} \times B_{ij} \subset \mathbb{P}^4 \times B_{ij}$ over $\mathcal{U}_{B_{ij}}$ (where $\mathbb{P}^3 \subset \mathbb{P}^4$ as the hyperplane $x_4 = 0$):

\[
C_{B_{ij}} := \left\{(x, \ell) \in \mathbb{P}^4 \times B_{ij} \mid x \text{ is contained in the plane spanned by } \ell \text{ and } P \text{ in } \mathbb{P}^4 \times \{\ell\}\right\}
\]

Finally let $K_{B_{ij}} := C_{B_{ij}} \cap (K \times B_{ij})$. Every fibre of $K_{B_{ij}}$ over $B_{ij}$ splits into lines at least two of
which contain $R_1$ (possibly one line counted twice if $\ell \in B_{ij}$ is a tangent at $S$).

$$\mathbb{P}^3 \times B_{ij} \cup \mathbb{P}^3 \times B_{ij} \rightarrow B_{ij}$$

$$\cup$$

$$\cup$$

$$\cup$$

$$\cup$$

$$\cup$$

Consider now $(S_i \cup S_j) \times B_{ij} \cap U_{B_{ij}}$ the fibre over $\ell \in B_{ij}$ of which consists of the intersection points of $\ell$ with $S_i \cup S_j$. Let $M_{B_{ij}} \subset \mathbb{P}^3 \times B_{ij}$ be the cone from $\{R\} \times B_{ij}$ over $(S_i \cup S_j) \cap U_{B_{ij}}$ which by construction is contained in $K_{B_{ij}}$. Since a line $\ell \in B_{ij}$ can meet $S$ in three different points if and only if it meets all three components of $S$ the fibre over $\ell$ consists of one or two lines through $R_1$ - one of them possibly counting twice. Let $G_{B_{ij}}$ be the closure in $\mathbb{P}^3 \times B_{ij}$ of $K_{B_{ij}} \setminus M_{B_{ij}}$.

By Proposition 4.2, $G_{B_{ij}}$ is a family of lines in $\mathbb{P}^3$ all lying in the cubic $K$. Let $\ell \in B_{ij}$ be an arbitrary line and let $V$ be the plane in $\mathbb{P}^3$ spanned by $\ell$ and $R_1$. By subtracting $M_{B_{ij}}$ from $K_{B_{ij}}$ just those lines in $K \cap V$ are removed that pass through $R_1$ and through the intersection points of $\ell$ with $S_i$ and $S_j$. For each line $\ell \in B_{ij}$ such that $V \cap K$ contains a line not through $R_1$ just this line remains in $G_{B_{ij}}$. These $\ell$ are just the lines in $\text{Bisec}(S)$ that do not meet all three components of $S$. But also for the lines $\ell$ that are contained in $Q'$ the fibre of $G_{B_{ij}}$ over $\ell$ consists of one single line. $\ell$ must meet all three components of $S$ and, therefore, must be contained in each $B_{ij}$ with $i \neq j$. Subtracting $M_{B_{ij}}$ from $K_{B_{ij}}$ removes just the lines in $V \cap K$ that intersect $S_i$ or $S_j$ leaving the the third line which intersects the third component of $S$ in $G_{B_{ij}}$. Since $G_{B_{ij}}$ is the closure of $K_{B_{ij}} \setminus M_{B_{ij}}$ this construction also yields just one line in the case that $\ell$ meets a singular point of $S$ which is always the intersection of two components of $S$.

Since $G_{B_{ij}} \rightarrow B_{ij}$ is a family of lines in $K$ there is a uniquely determined morphism $B_{ij} \rightarrow \text{Fano}(K)$. The images of all $B_{ij}$ cover $\text{Fano}(K)$. For lines in $K$ that do not contain $R_1$ this is clear from Proposition 4.4. But also the lines through $R_1$ have their representation by an element of one of the $B_{ij}$: Let $T$ be the intersection of such a line with $S$. There are two lines of the rulings of $Q'$ through $T$. Since $S_i \in |O_{Q'}(1,1)|$ for all $i$ non of them can be tangent to $S$. Let $\ell$ be one of these two lines. $\ell$ intersects all three components of $S$ and is therefore contained in each $B_{ij}$ with $i \neq j$. Therefore, we can find $i$ and $j$ such that $T$ is neither contained in $S_i$ nor in $S_j$ and $\ell \in B_{ij}$. (If $T \in S_m \cap S_n$ then e.g. $i = m$ and $j \neq n$ is a fitting choice.) As is clear from the above discussion, the morphism $B_{ij} \rightarrow \text{Fano}(K)$ then maps $\ell \in B_{ij}$ to the line $T^{\perp} \in \text{Fano}(K)$.

Now, each $B_{ij}$ contains an open subset (say $U_{ij}$) on which the above morphisms to $\text{Fano}(K)$ are injective and such that the images of different $U_{ij}$ do not intersect in $\text{Fano}(K)$. (The sets of lines that intersect $S$ in exactly two different nonsingular points will do.) Therefore the morphisms $G_{B_{ij}} \rightarrow B_{ij}$ induce a birational map $\text{Bisec}(S) \dashrightarrow \text{Fano}(K)$.

### 4.3 The components of the space of double tangents

Denote by $Y_0 \subset \text{Grass}(2,4)$ the closed subscheme of double tangents of the quartic $B$. Earlier in this section we constructed a morphism $\text{Fano}(K) \setminus \{\text{lines through } P\} \rightarrow Y_0$. By Lemma 4.3 this
morphism is injective outside the closed subset of lines in $K$ that meet one of the six lines through $P$. As this closed subset is not an irreducible component of $\text{Fano}(K)$ (which is clear from the map $\text{Bisec}(S) \longrightarrow \text{Fano}(K)$) its complement is open and dense. On the other hand, the image of $\text{Fano}(K) \setminus \{\text{lines through } P\}$ in $Y_0$ contains the set of all double tangents outside the plane $g_1 = 0$.

But the double tangents contained in this plane form an irreducible component of their own: Every line in that plane is a double tangent. Thus the set of these double tangents is closed in $Y_0$ and of dimension two. On the other hand the dimension of $Y_0$ is also two: Consider the variety

$$Y_F := \{(\ell, H) \in \text{Grass}(2, 4) \times \mathbb{P}^3 \mid \ell \subset H \text{ and } \ell \in Y_0\}$$

which is a subvariety of the flag variety $F(1, 2) := \{(\ell, H) \in \text{Grass}(2, 4) \times \mathbb{P}^3 \mid \ell \subset H\}$. This variety is fibre over $\mathbb{P}^3$ and the fibre over a general $H \in \mathbb{P}^3$ is zero dimensional by Proposition 4.6. Hence the dimension of $Y_F$ is at least three and since every line is contained in a pencil of planes the dimension of $Y_0$ is at least two. The dimension cannot be greater than two for there is a morphism $\text{Fano}(K) \setminus \{\text{lines through } P\} \longrightarrow Y_0$ (which is surjective onto the set of double tangents outside the plane $g_1 = 0$) and the dimension of $\text{Fano}(K)$ is two since it is birationally equivalent to $\text{Bisec}(S)$. Therefore $Y_0$ is the union of two-dimensional varieties and hence is of dimension two. So, the closed subvariety of lines in the plane $g_1 = 0$ has the same dimension as $Y_0$ and thus is an irreducible component.

This way we have found a rational map $\text{Bisec}(S) \longrightarrow Y_0$ which is birational onto those components of $Y_0$ which are different from the component of lines in the plane $g_1 = 0$. $Y_0$ must therefore have seven irreducible components. Four of them consist of all lines in a plane. These are the sets $B_{ij}$ in $K$ as well as in the hyperplane $x_4 = 0$ they keep fixed under the map $\text{Bisec}(S) \longrightarrow Y_0$ and therefore correspond to the lines in the $B_{ij} \subset \text{Bisec}(S)$.

Thus, the irreducible components of $Y_0$ are determined:

**Proposition 4.6** $Y_0$ consists of seven irreducible components, namely the four components with all lines in the planes $E_i$ ($i = 1, 2, 3, 4$) and three further components corresponding to the components $B_{ij} \subset \text{Bisec}(S)$ with $i \neq j$. \(\square\)

The following proposition will be needed in the next section.

**Proposition 4.7** Let $\Delta' \subset \mathbb{P}^3$ be the closed set of planes $H$ such that $B \cap H$ is a plane quartic which is not smooth and has more or other singularities than just one ordinary double point. Then $\varphi : Y_F|_{\mathbb{P}^3 \setminus \Delta'} \longrightarrow \mathbb{P}^3 \setminus \Delta'$ is flat. The ramification locus of $\varphi$ is just the set of those $(\ell, H)$ such that $B \cap H$ is a quartic with a node and $\ell$ is a double tangent containing the node. The ramification index in those points is two. Outside the ramification locus $\varphi$ is a smooth 28-fold cover.

**Proof:** $Y_F$ is contained in the Flag variety $F(1, 2) \subset \text{Grass}(2, 4) \times \mathbb{P}^3$ which is a $\mathbb{P}^{28}$-bundle over $\mathbb{P}^3$. For the proof of the flatness we will determine the Hilbert polynomial of the fibres of $Y_F|_{\mathbb{P}^3 \setminus \Delta'}$ over $\mathbb{P}^3 \setminus \Delta'$ considered as subsets of $\mathbb{P}^3$. Let $U \subset \mathbb{P}^3$ be a standard open subset (say $U = \{(y_0 : \ldots : y_3) \in \mathbb{P}^3 \mid y_3 = -1\}$) such that $F(1, 2)|_U \cong U \times \mathbb{P}^2$. In $\mathbb{P}^2$ choose a standard open set $U' = \{(l_0 : l_1 : l_2) \in \mathbb{P}^2 \mid l_0 = -1\}$. Then,
by substituting \( x_3 = x_0 y_0 + x_1 y_2 + x_2 y_2 \) and \( x_0 = x_1 l_1 + x_2 l_2 \) in the equation of \( B \) we get a map from \( U \times U' \) to \( \mathbb{P}^4 \) – the set of binary quartic forms – which associates to each pair \((\ell, H) \in U \times U'\) the equation of \( B \) restricted to \( \ell \). If \( C \subset \mathbb{P}^4 \) denotes the closed subset parametrising those quartic forms that are complete squares then \( Y_{\mathcal{F}}|_{U \times U'} \) is the preimage of \( C \) under the map \( U \times U' \rightarrow \mathbb{P}^4 \). Therefore, we can get equations for \( Y_{\mathcal{F}}|_{U \times U'} \) (with its reduced scheme structure) by pulling back equations defining \( C \).

Let \((a : b : c : d : e) \in \mathbb{P}^4 \) correspond to the quartic form \( ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \). Obviously we get the equations for \( C \) with the reduced scheme structure by eliminating \( \xi, \eta, \) and \( \zeta \) from the equations

\[
\begin{align*}
a &= \xi^2 \\
b &= 2\xi \eta \\
c &= 2\xi \zeta - \eta^2 \\
d &= 2\eta \zeta \\
e &= \zeta^2. \\
\end{align*}
\]

(6)

This is done by the technique of Gröbner bases (cf. [Co] for details): One has to compute a Gröbner basis for the above equations using the lexicographic monomial order induced by the ordering of variables \((\xi > \eta > \zeta > a > b > c > d > e)\). The equations defining \( C \in \mathbb{P}^4 \), then, are those which do not contain \( \xi, \eta, \) or \( \zeta \) (cf. [Co]). The following is the Gröbner basis computed by MAPLE V (the polynomials not containing \( \xi, \eta, \) or \( \zeta \) listed last):

\[
\begin{align*}
\xi^2 - a, \\
\xi b - 2a \eta, \\
4\xi c + \eta d - 2\zeta c, \\
4\xi e + \eta d - 2\zeta c, \\
bn^2 - cb + 2ad, \\
2\eta c - d, \\
4ac - ab\zeta - b^2, \\
4\eta^2 - c\eta^2 + \zeta d + 8ea\zeta, \\
2\eta c - \zeta d, \\
8a^2d - 4cba + b^3, \\
ad^2 - eb^2, \\
2a^2d - 4abc - 16ac^2 + 16a^2e, \\
bd^2 - 4ce + 8ead, \\
d^3 - 4cde + 8e^2b \\
\end{align*}
\]

(7)

One can easily verify that these polynomials form a Gröbner basis with respect to the above monomial order (e.g. by forming S-polynomials and reducing them with respect to the set of polynomials (6)). Even easier is it to check that the equations (6) and the polynomials (7) define the same ideal. (The equations (6) are among the polynomials (7) and the remaining polynomials are easily reduced to zero using the equations (6).)

Now, let \( H \in U \setminus \Delta' \) be a plane in \( \mathbb{P}^3 \). For any \((\ell, H)\) in the fibre \( \varphi^{-1}(H) \) of \( Y_{\mathcal{F}} \) over \( H \) we will compute the length of the local ring \( \mathcal{O}_{\varphi^{-1}(H),(\ell,H)} \). Let
be the polynomial defining $B \cap H \subset H \cong \mathbb{P}^2$. Suppose that $\ell \subset \mathbb{P}^2$ is the line $x_0 = 0$ contained in $U \times U' \subset F(1, 2)$. The equations of $Y_F$ near $(\ell, H)$ are obtained by substituting $x_0 = x_1 l_1 + x_2 l_2$ in $f$ and substituting the resulting coefficients in the seven last polynomials of $(\mathring{B})$. (The local ring $O_{x^{-1}(H), (\ell, H)}$ then is obtained as the factor of $\mathbb{C}[l_1, l_2](0, 0)$ by the ideal generated by the equations of $Y_F$ we got in this way.)

If $\ell \subset H$ is a double tangent at $B \cap H$ touching $B \cap H$ in the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ then the coefficients in $f$ have to satisfy: $a_0 = 0$, $a_{10} = 0$, $(\text{since } x = 0)$ is a tangent at $(0 : 0 : 1)$, $a_{40} = 0$, $a_{30} = 0$ $(\text{since } x = 0)$ is a tangent at $(0 : 1 : 0)$, and finally $a_{20}$ must not be zero since otherwise the line $x = 0$ would be contained in $B$ (we, thus, can set $a_{20} = 1$). Computing the Jacobian of the resulting seven equations in the point $(\ell, H)$ (i.e. for $l_1 = l_2 = 0$) yields the matrix

$$
\begin{pmatrix}
0 & 0 \\
-4a_{41} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & -4a_{11} \\
0 & 0
\end{pmatrix}
$$

Thus, $(\ell, H)$ is a singular point in its fibre if and only if $a_{41} = 0$ or $a_{11} = 0$, i.e., if and only if $\ell$ touches $B \cap H$ in singular points of $B \cap H$. We only have to compute the ramification index of the points $(\ell, H)$ which are singular in their fibre. Suppose, that the line $\ell$ (given by $x = 0$) contains the point $(0 : 0 : 1)$ which is assumed to be a singular point of $B \cap H$ and touches $B \cap H$ in the smooth point $(0 : 1 : 0)$. In particular we get the condition $a_{11} = 0$. We get seven polynomials in $l_1$ and $l_2$, the sixth of which has the form

$$(\cdots \text{terms containing } l_1 \text{ or } l_2 \cdots - 4a_{22} + a_{21}^2) l_2^2.
$$

But $4a_{22} - a_{21}^2$ must not vanish since otherwise $B \cap H$ would have a cusp in $(0 : 0 : 1)$. Thus $(\cdots \text{terms containing } l_1 \text{ or } l_2 \cdots - 4a_{22} + a_{21}^2)$ is a unit in $\mathbb{C}[l_1, l_2](0, 0)$. Therefore, this equation can be replaced by $l_2^2$ and the terms containing $l_2^2$ can be cancelled in the other equations without changing the ideal generated by the equations. Now, the second equation has the form

$$(\cdots \text{terms containing } l_1 \text{ or } l_2 \cdots - 4a_{41}) l_1.
$$

Since $a_{41}$ must not vanish (otherwise $(0 : 1 : 0)$ would be singular on $B \cap H$), we can replace this equation by $l_1$ and cancel all terms containing $l_1$ in the other equations. We obtain that the considered ideal in $\mathbb{C}[l_1, l_2](0, 0)$ is generated by the two elements $l_2^2$ and $l_1$. Hence, if $(\ell, H)$ is of
the kind that ℓ contains the only node of B ∩ H then the local ring of (ℓ, H) in its fibre has length two. All other points over \( \mathbb{P}^3 \setminus \Delta' \) are smooth in their fibres.

For \((\ell, H) \in \mathbb{P}^3 \setminus \Delta'\) the fibre \(\varphi^{-1}((\ell, H))\) consists, by Proposition \[3.4\], of 28 points if \(B \cap H\) is smooth, and of 22 points otherwise. By the Remark following Proposition \[3.4\] for a quartic \(B \cap H\) with one node, there are six double tangents through the node. The Hilbert polynomial of \(\varphi^{-1}((\ell, H)) \in \mathbb{P}^2\), hence, is 28 for \(B \cap H\) being smooth and \(16 + 6 \cdot 2 = 28\) otherwise. Therefore \(\varphi : Y_f|_{\mathbb{P}^3 \setminus \Delta'} \rightarrow \mathbb{P}^3 \setminus \Delta'\) is flat. From the length of the local rings computed above we see that the ramification behaviour is as stated. \(\square\)

5 The parameter space of touching conics

5.1 Double tangents and curves in double covers

Let \(B \subset \mathbb{P}^3\) be a real quartic surface with exactly 13 nodes, given by an equation of the form \(E_1E_2E_3E_4 - Q^2\), as described in Proposition \[2.1\]. Let \(Z \rightarrow \mathbb{P}^3\) be the double cover which is ramified over \(B\). It is constructed as follows (cf. [BPV], I.17.). Denote by \(p : L \rightarrow \mathbb{P}^3\) the total space of the line bundle \(\mathcal{O}_{\mathbb{P}^3}(2)\) and by \(s \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))\) the section who’s zero locus is \(B\). Finally, let \(y \in H^0(L, p^*\mathcal{O}(2))\) be the tautological section. Then \(Z \subset L\) is the zero locus of the section \(y^2 - p^*(s) \in H^0(L, p^*\mathcal{O}(4))\).

Each of the divisors given by \(p^*(E_i)\) splits into two components which are defined by the sections \(y^2 \pm \sqrt{-1}p^*(Q)\) of \(H^0(L, p^*\mathcal{O}(2))\). Denote these varieties by \(S_i^+\) and \(S_i^-\), corresponding to \(y^2 + \sqrt{-1}p^*(Q)\) and \(y^2 - \sqrt{-1}p^*(Q)\) respectively.

Now, let \(H \in \mathbb{P}^3\) be a general plane. In particular, let \(B_H := B \cap H\) be a smooth quartic curve. The restriction \(Z_H\) of \(Z\) to \(H\) then is the smooth double cover of \(H \cong \mathbb{P}^2\) branched along the nonsingular curve \(B_H\). Its canonical bundle is

\[
K_{Z_H} = p^*\mathcal{O}_H(-3) \otimes p^*\mathcal{O}_H(2) = p^*\mathcal{O}_H(-1),
\]

which implies that the morphism \(p|_H\) is induced by the linear system \(|-K_{Z_H}|\).

By [GH] Chapter 4.4, \(Z_H\) is isomorphic to the blow-up of \(\mathbb{P}^2\) in seven points. The 56 \((-1)\)-curves are: the seven exceptional divisors \(E_i\) (\(i = 1, \ldots, 7\)), the strict transforms of the cubics in \(\mathbb{P}^2\) through all seven points with a node in the \(i\)-th point \(K^i\) (\(i = 1, \ldots, 7\)), the strict transforms of the lines through the \(i\)-th and the \(j\)-th point \(G^{ij}\) (\(1 \leq i < j \leq 7\)), and the strict transforms of the conics through all but the \(i\)-th and the \(j\)-th point \(C^{ij}\) (\(1 \leq i < j \leq 7\)).

The projection \(p|_H\) maps the 56 \((-1)\)-curves onto the 28 double tangents of \(B_H\) in such a way that each double tangent has exactly two \((-1)\)-curves in its preimage.

The restriction of the \(S_i^+\) to \(Z_H\) yields eight curves which are denoted by \(D_i^\pm\). These are mapped onto double tangents of \(B_H\) and, hence, are \((-1)\)-curves.

**Lemma 5.1** \(Z_H\) can be realised as the blow-up of \(\mathbb{P}^2\) in such a way that \(D_1^+ = K^7, D_1^- = E^7, D_2^+ = G^{12}, D_2^- = C^{12}, D_3^+ = G^{34}, D_3^- = C^{34}, D_4^+ = G^{56}, \) and \(D_4^- = C^{56}\).
**Proof:** All we have to prove is that the $D^\pm_i$ intersect in the correct way, i.e., $D^+_i \cdot D^-_i = 2$ for $i = 1, \ldots, 4$ and $D^+_i \cdot D^+_j = D^-_i \cdot D^-_j = 1$ as well as $D^+_i \cdot D^-_j = 0$ for $i \neq j$.

From the fact that $p|_H$ is induced by the linear system of the anticanonical divisor we deduce that $D^+_i + D^-_i = -K_{Z_H}$. Hence

$$2 = (-K_{Z_H})^2 = (D^+_i + D^-_i)^2$$

and consequently $D^+_i \cdot D^-_i = 2$.

Next observe that the rational function

$$
\frac{y + \sqrt{-1}Q}{(p|_H)^*(E_i \cdot E_j)}
$$

corresponds to the principal divisor $D^+_k + D^+_i - D^-_l - D^-_j$, i.e. $[D^+_k + D^+_i] = [D^-_l + D^-_j]$ where $\{k, l\} = \{1, \ldots, 4\} \setminus \{i, j\}$. This yields

$$(D^+_k + D^+_i)^2 = (D^-_l + D^-_j)^2 = (D^+_k + D^+_i)(D^-_l + D^-_j)$$

$$2D^+_kD^+_i - 2 = 2D^-_lD^-_j - 2 = D^+_kD^-_l + D^+_lD^-_k + D^+_iD^-_j + D^+_jD^-_i \geq 0$$

as the product of different effective divisors is always non-negative. Hence $D^+_i \cdot D^+_i = D^-_i \cdot D^-_i = 1$ since two $(-1)$-curves have intersection product 2 if and only if their sum is an element of $-K_{Z_H}$. But then the sum $D^+_kD^-_l + D^+_lD^-_k + D^+_iD^-_j + D^+_jD^-_i$ must vanish and so $D^+_i \cdot D^-_j = 0$. \hfill $\Box$

Consider now $Z := Z \times F^3$ where $H \subset \mathbb{P}^3 \times \mathbb{P}^3$ is the universal (hyper-)plane. Via $Z \to H \to \mathbb{P}^3$, $Z$ is the family of all surfaces $Z_H$, $H \in \mathbb{P}^3$. Let $\Delta \subset \mathbb{P}^3$ be the set of all those planes that do not intersect $B$ transversally (i.e., who’s intersection with $B$ is not a smooth quartic curve). Then $\mathbb{Z}|_{\mathbb{P}^3 \setminus \Delta}$ is smooth and proper and therefore, by the Ehresmann–Fibration–Theorem (cf. [1]), locally trivial as a fibration of differentiable manifolds. For any $H_0 \in \mathbb{P}^3 \setminus \Delta$ the fundamental group $\pi_1(\mathbb{P}^3 \setminus \Delta, H_0)$ acts via monodromy on $H^2(Z|_{H_0}, \mathbb{Z}) = H^2(Z|_{H_0}, \mathbb{Z}) = \text{Pic}(Z|_{H_0})$ preserving the intersection pairing. In particular the fundamental group acts via monodromy on the set of $(-1)$-curves preserving their intersection behaviour.

On the other hand, this fundamental group $\pi_1(\mathbb{P}^3 \setminus \Delta, H_0)$ acts via monodromy of the finite unramified cover $Y_F|_{\mathbb{P}^3 \setminus \Delta} \to \mathbb{P}^3 \setminus \Delta$ on the set of double tangents of $B \cap H_0$. (Recall that $Y_F$ was defined as $Y_F := \{(\ell, H) \in \text{Grass}(2, 4) \times \mathbb{P}^3 | \ell \in H$ and $\ell$ is double tangent at $B\}$.) Obviously, this action is the same as the action that is induced from the action on $Z|_{H_0}$ by mapping a $(-1)$-curve to its corresponding double tangent.

The properties of the monodromy action on the $(-1)$-curves are to be examined in the sequel. First, observe that the eight curves $\mathbb{D}_m$ keep fixed under the monodromy action since they are restrictions of the globally defined $S^\pm_i$.

A pair of $(-1)$-curves lying over the same double tangent is mapped to a pair of $(-1)$-curves that again are mapped onto the same (maybe different) double tangent. This follows from the fact that the sum $[C + C']$ of such a pair $(C, C')$ is equal to the anti-canonical class. which for the $(-1)$-curves is equivalent to $C \cdot C' = 2$. 

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The set of \((-1)\)-curves that are different from, say, \(D_1^\pm\) can be split in two monodromy–invariant subsets each containing 27 curves: one subset consisting of all curves \(C\) with \(C \cdot D_1^- = 1\) and the other containing the curves \(C\) satisfying \(C \cdot D_1^- = 0\). I.e., these subsets are characterised by the intersection number of their elements with \(D_1^- = E^7\) which can take the values 0 and 1. As \(D_1^-\) is invariant under the monodromy action this condition is invariant and so are the subsets. Each pair \((C, C')\) of \((-1)\)-curves with \(C \cdot C' = 2\) (except the pair \((D_1^+, D_1^-)\)) has one member in each of the two sets. Therefore the monodromy action on the \((-1)\) curves is determined by the action on one of the invariant subsets.

We choose the set of curves that do not intersect \(E^7\). This set with its incidence relations is equivalent to the set of the 27 lines of a smooth cubic surface: We get the correspondence by blowing down \(E^7\). The \((-1)\)-curves not intersecting \(E^7\) are mapped onto the \((-1)\)-curves in the blown-down surface which is \(\mathbb{P}^2\) blown-up in six points.

### 5.2 The lines in a cubic surface

Let \(S\) be the cubic surface obtained by blowing down \(E^7 \subset Z_{H_0}\). Denote the lines in \(S\) by \(E^i\) \((i = 1, \ldots, 6)\) for the images of \(E^i \subset Z_{H_0}\); \(G^{ij}\) \((1 \leq i < j \leq 6)\) for the images of the corresponding curves \(G^{ij}\) in \(Z_{H_0}\); and \(C^i\) \((i = 1, \ldots, 6)\) for the images of the curves \(C^i \subset Z_{H_0}\). The monodromy action on the \((-1)\)-curves of \(Z_{H_0}\) induces an action of the fundamental group \(\pi_1(\mathbb{P}^3 \setminus \Delta, H_0)\) on the 27 lines of \(S\) and, hence, induces a morphism of \(\pi_1(\mathbb{P}^3 \setminus \Delta, H_0)\) into the group of symmetries of lines in the cubic surface \(S\) (i.e. the group of permutations of the 27 lines that respect their incidence relations).

Let \(G\) be the image of the fundamental group in this symmetry group. Clearly, \(G\) leaves the three lines \(G^{12} \ (\equiv D_2^+)\), \(G^{34} \ (\equiv D_3^+)\), and \(G^{56} \ (\equiv D_4^+)\) invariant. To each of them there are exactly 10 lines that intersect this line. Furthermore, to each pair \((C, C')\) of intersecting lines there is exactly one line that intersects them both. Thus, to each of the three lines \(G^{12}\), \(G^{34}\), and \(G^{56}\) there is associated a set of eight lines that intersect this curve and that are different from these three lines. The three sets of eight lines have to be disjoint since the three lines \(G^{12}\), \(G^{34}\), and \(G^{56}\) meet each other and so the unique line that intersects two of them is just the third.

**Proposition 5.2** \(G\) acts transitively on the three \(G\)-invariant sets of curves associated to the three curves \(G^{12}\), \(G^{34}\), and \(G^{56}\).

**Proof:** The variety \(Y_F\) is naturally fibred over \(Y_0\) – the parameter space of double tangents. The fibres are isomorphic to \(\mathbb{P}^1\) since each line in \(\mathbb{P}^3\) is contained in a pencil of planes. Therefore the irreducible components of \(Y_F\) are just the preimages of the irreducible components of \(Y_0\). The projection \(Y_F \rightarrow \mathbb{P}^3\) is a finite cover which is étale over \(\mathbb{P}^3 \setminus \Delta\) by Proposition 4.7 and, hence, \(Y_F\big|_{\mathbb{P}^3 \setminus \Delta}\) is smooth. Therefore its irreducible components are just its arc–connected components in the Euclidian topology. By Proposition 4.6, \(Y_F\) has seven components. A general fibre \(Y_F|_H\) over \(\mathbb{P}^3 \setminus \Delta\) contains 28 points by Proposition 3.4, four of them corresponding to the four double tangents in the planes \(E_i = 0\). (Recall that the closed subset in \(Y_0\) of lines in one of the planes \(E_i = 0\) were recognised to be irreducible components.) The remaining 24 points belong to double tangents of the other three components of \(Y_0\). We claim that any fibre \(Y_F|_H\) over \(\mathbb{P}^3\) must contain at least one point of each component.

For each component of \(Y_0\) there is a plane in \(\mathbb{P}^3 \setminus \Delta\) which contains a double tangent of this component: For a double tangent \(\ell\) that does not pass through one of the singular points of \(\mathbb{P}\), the
pencil of planes containing $\ell$ is not contained in $\Delta \subset \mathbb{P}^3$. The set of double tangents through a singular point $p \in B$ is one-dimensional as this set is parametrised by the ramification locus of the projection of $B$ from $p$. On the other hand, each component of $Y_0$ is two-dimensional and, hence, in any component of $Y_0$ there is an open set of double tangents $\ell$ not through any of the singular points of $B$. Now, choose a component of $Y_0$ and a double tangent $\ell$ of this component which does not contain singular points of $B$. Let $H \in \mathbb{P}^3 \setminus \Delta$ be a plane containing $\ell$, i.e. $(\ell, H) \in Y_F|_{\mathbb{P}^3 \setminus \Delta}$. Since $Y_F|_{\mathbb{P}^3 \setminus \Delta} \rightarrow \mathbb{P}^3 \setminus \Delta$ is an étale cover the component of $Y_F$ containing $(\ell, H)$ dominates $\mathbb{P}^3$. Therefore, every plane in $\mathbb{P}^3$ contains at least one double tangent of each component of $Y_0$.

As mentioned above, the fundamental group $\pi_1(\mathbb{P}^3 \setminus \Delta, H_0)$ acts on the fibre $Y_F|_{H_0}$ via monodromy and, as the irreducible components of $Y_F|_{\mathbb{P}^3 \setminus \Delta}$ are just its connected components, each orbit of the monodromy action contains all those points of the fibre that belong to the same component.

On the other hand, the monodromy action on $Y_F|_{H_0}$ is obtained from the monodromy action on $(-1)$-curves on $Z|_{H_0}$ by projecting the $(-1)$-curve onto the corresponding double tangent. So the action on the 27 lines of a cubic surface cannot have more than six orbits. (The seventh orbit in $Y_F|_{H_0}$ consisting of the point corresponding to the double tangent in the plane $E_1 = 0$ has no representation among the 27 lines.) The three sets $\{G^{12}\} \{G^{34}\} \{G^{56}\}$ are $G$-invariant and consequently the three $G$-invariant sets of lines intersecting $G^{12}, G^{34}$ or $G^{56}$ have to be $G$-orbits. \hfill $\square$

**Corollary 5.3** Let $H$ be a plane in $\mathbb{P}^3 \setminus \Delta$. There are four components of $Y_0$ with exactly one double tangent in $H$ — namely the four components parametrising the lines in the planes $E_i = 0$ ($i = 1, \ldots, 4$). Of each of the remaining three components of $Y_0$ there are eight double tangents in $H$.

**Proof:** The 24 double tangents in $H$ which are not contained in one of the four planes $E_i = 0$ correspond to the 24 lines in the cubic surface that are not fixed under the action of $G$. These 24 lines in the cubic split into three $G$-orbits, each orbit containing eight of them. The $G$-action on the lines in the cubic is equivalent to the monodromy action on the double tangents in $H$. (The double tangent in the plane $E_1 = 0$ is fixed by the monodromy.) Hence, the 24 double tangents are spread over the three components in such a way that each component contains eight of them. \hfill $\square$

**Proposition 5.4** $G$ is generated by elements $g$ with the following properties:

- $g$ is of order two.
- $g$ leaves at least 15 of the 27 $(-1)$-curves fixed.

In other words: There are at most 6 pairs of lines such that the lines in these pairs are swapped by $g$.

**Proof:** Let $L$ be a general line in $\mathbb{P}^3$. Without loss of generality one can assume that $H_0 \in L$. Then, due to [Section 7.4.1](#), $\pi_1(L \setminus \Delta, H_0) \rightarrow \pi_1(\mathbb{P}^3 \setminus \Delta, H_0)$ is surjective and therefore it is sufficient to study the monodromy of paths in $L \setminus \Delta$. The fundamental group $\pi_1(L \setminus \Delta, H_0)$ is generated by the homotopy classes of paths that loop once counter clockwise round one of the points in $L \cap \Delta$. 20
By a sufficiently general choice of $L$, we can achieve that for each $H \in L \cap \Delta$ the plane quartic $B \cap H$ has exactly one ordinary node as its only singularity. For a smooth surface $B$ a generic line in $\mathbb{P}^3$ will do (cf. [L] Section 1.6.4). The proof in [L] works as well in the case of hypersurfaces with isolated singularities. In the proof one only needs to replace the hypersurface by the open set of its regular points in all occurrences. Hence if for a plane $H$ of a generic pencil of planes the quartic curve $B \cap H$ has more or other singularities than one ordinary node then $H$ must contain a singular point of $B$ (which is an ordinary node in our case). But the set of those planes $H$, that contain a singular point of $B$ and for which $B \cap H$ is not a curve with exactly one ordinary node, has at least codimension two in $\mathbb{P}^3$. So we can choose $L$ in such a way that $L$ does not intersect this codimension-2-subset in $\mathbb{P}^3$.

Now, by Proposition 4.7, the fibre of $Y_F$ over any $H \in L \cap \Delta$ contains exactly six ramification points and the ramification index in each of them is two. The monodromy of a loop round $H$ can only interchange two sheets of $Y_F|_L$ which meet in one of the ramification points over $H$. Therefore, the monodromy of this loop can swap at most six couples of points in the fibre of $Y_F$ over $H_0$. Using the correspondence between the monodromy action on $Y_F|_{H_0}$ and the monodromy action on the lines in the cubic $S$, this proves the proposition.

The group of symmetries of the 27 lines in a cubic surface (i.e. the group of permutations that respect the incidence relations among the lines) has been intensively studied. (Cf. [He], [Se], [M] – to mention only a few.) The following theorem holds ([M] Ch. IV Theorem 1.9):

**Theorem 5.5** The group of symmetries of the 27 lines on a cubic surface is isomorphic to the Weyl–Group $E_6$.

In particular, the group $G$ is a subgroup of $E_6$. Using the Propositions 5.2 and 5.4 we will be able to determine $G \subset E_6$ as a subgroup of $E_6$. For this purpose, we first determine the elements of $E_6$ that admit the properties required in Proposition 5.4. In [M] (Ch. IV § 9) as well as in [Sw] a complete list of the conjugacy classes of $E_6$ can be found. Moreover, to each conjugacy class the action of its elements on the 27 lines is described. It turns out that the only conjugacy class who’s elements act on the 27 lines as postulated in Proposition 5.4 is the class which is denoted by $C_{16}$ in [M] and [Sw]. This class contains exactly 36 elements.

It is a classical result (cf. e.g. [Sc]) that the group of symmetries of the 27 lines in a cubic surface is generated by elements which swap the lines in one of Schl{" a}fli’s 36 “double six”. A “double six” consists of a pair of sextuples of lines such that the lines in each sextuple are mutually skew and each line of one of the tuples intersects exactly five lines of the other tuple. Associating to each line of one tuple the unique line of the other tuple that is skew to this line yields a one-to-one correspondence between the two sextuples of a double six. We identify a double six with the element of $E_6$ that exchanges the lines of the sextupels in such a way that each line is swapped with the unique line of the other tuple which is skew to this line. This transformation keeps the other 15 lines fixed. An example of a double six is the pair $[(E^1, \ldots, E^6), (C^1, \ldots, C^6)]$.

Obviously, a permutations of the 27 lines corresponding to a double six satisfies the conditions of Proposition 5.4, i.e. is in the conjugacy class $C_{16}$. As this class contains exactly 36 elements (cf. [M] or [Sw]) and as there are exactly 36 double sixes, $C_{16}$ consists just of these double six transformations. Consequently, the group $G$ has its generators among these special transformations.

Next, we will give a list of all double six transformations, that leave the three lines $G^{12}, G^{34},$ and $G^{56}$ invariant. For this purpose we will use the notation of [M] and give the transformations in
terms of reflections of a root system. The Picard group \( \text{Pic}(S) \) of a smooth cubic surface \( S \) is the free abelian group with generators \([H]\) – the pull-back of \( \mathcal{O}_{\mathbb{P}^2}(1) \) under the blow-up morphism \( S \to \mathbb{P}^2 \) – and the six classes \([-E^i]\) (where \( E^i \) \((i = 1, \ldots, 6)\) are the exceptional divisors). An element \( a[H] - b_1[E^1] - \cdots - b_6[E^6] \in \text{Pic}(S) \) will be denoted by \( (a; b_1, \ldots, b_6) \) in the sequel. The intersection pairing on the Picard group is given by

\[
(a; b_1, \ldots, b_6) \cdot (a'; b'_1, \ldots, b'_6) = aa' - \sum_{i=1}^{6} b_i b'_i
\]

Denote by \( \omega = -(3; 1, 1, 1, 1, 1, 1) \) the canonical line bundle of the cubic surface. Let \( \omega^\perp \subset \text{Pic}(S) \otimes \mathbb{R} \) be the orthogonal complement of \( \omega \) with respect to the scalar product induced by the intersection pairing. Note that on \( \omega^\perp \) the intersection pairing is negative definite (cf. [M] Proposition IV.3.3). There is a root system of type \( E_6 \) in \( \omega^\perp \) who’s reflections are in one-to-one correspondence with the double six transformations. The roots of this root system are the following:

- \([E^i] - [E^j] \in \text{Pic}(S) \subset \text{Pic}(S) \otimes \mathbb{R} \) \((1 \leq i, j \leq 6, i \neq j)\) (30 roots).
- \(\pm([H] - [E^i] - [E^j] - [E^k]), \) \((1 \leq i < j < k \leq 6)\) (40 roots).
- \(\pm(2; 1, 1, 1, 1, 1)\) (2 roots).

For a root \( x \) the corresponding reflection \( s_x \) is given by

\[
s_x(v) = v + (v, x) x
\]

(where the scalar product is the one given by the intersection pairing on \( \text{Pic}(S) \)). The restriction of any reflection to \( \text{Pic}(S) \subset \text{Pic}(S) \otimes \mathbb{R} \) induces an endomorphism of \( \text{Pic}(S) \) that respects the intersection pairing. In particular it induces a transformation of the 27 lines which is a double six transformation.

Now it is easy to find the double six transformations which fix the three lines \( G^{12}, G^{34}, \) and \( G^{56}. \) These correspond to the roots

\[
\begin{align*}
  x_1 &= (2; 1, 1, 1, 1, 1, 1), & x_2 &= (1; 1, 0, 1, 0, 0, 1), & x_3 &= (0; -1, 1, 0, 0, 0, 0), \\
  x_4 &= (1; 1, 0, 0, 1, 1, 0), & x_5 &= (1; 1, 0, 1, 0, 1, 1), & x_6 &= (1; 0, 1, 1, 0, 1, 1), \\
  x_7 &= (0; 0, 0, -1, 1, 0, 0), & x_8 &= (1; 0, 1, 1, 0, 0, 1), & x_9 &= (1; 0, 1, 0, 1, 1, 0), \\
  x_{10} &= (1; 0, 1, 0, 1, 0, 1), & x_{11} &= (0; 0, 0, 0, 0, -1, 1), & x_{12} &= (1; 1, 0, 1, 0, 1, 0).
\end{align*}
\]
The corresponding double sixes are:

\[
\begin{align*}
x_1 &\equiv \left( \begin{array}{c} E^1 \ E^2 \ E^3 \ E^4 \ E^5 \ E^6 \\ C_1 \ C_2 \ C_3 \ C_4 \ C_5 \ C_6 \end{array} \right), \\
x_2 &\equiv \left( \begin{array}{c} E^1 \ E^3 \ E^6 \ G^{45} \ G^{25} \ G^{24} \\ G^{36} \ G^{16} \ G^{13} \ C_2 \ C_4 \ C_5 \end{array} \right), \\
x_3 &\equiv \left( \begin{array}{c} E^1 \ C_1 \ G^{23} \ G^{24} \ G^{25} \ G^{26} \\ E^2 \ C_2 \ G^{13} \ G^{14} \ G^{15} \ G^{16} \end{array} \right), \\
x_4 &\equiv \left( \begin{array}{c} E^1 \ E^4 \ E^5 \ G^{36} \ G^{26} \ G^{23} \\ G^{45} \ G^{15} \ G^{14} \ C_2 \ C_3 \ C_6 \end{array} \right), \\
x_5 &\equiv \left( \begin{array}{c} E^1 \ E^4 \ E^6 \ G^{35} \ G^{25} \ G^{23} \\ G^{46} \ G^{16} \ G^{14} \ C_2 \ C_3 \ C_5 \end{array} \right), \\
x_6 &\equiv \left( \begin{array}{c} E^2 \ E^3 \ E^5 \ G^{36} \ G^{16} \ G^{14} \\ G^{35} \ G^{25} \ G^{23} \ C_1 \ C_4 \ C_6 \end{array} \right), \\
x_7 &\equiv \left( \begin{array}{c} E^3 \ C_3 \ G^{14} \ G^{24} \ G^{45} \ G^{46} \\ E^4 \ C_4 \ G^{13} \ G^{23} \ G^{35} \ G^{36} \end{array} \right), \\
x_8 &\equiv \left( \begin{array}{c} E^2 \ E^4 \ E^5 \ G^{36} \ G^{15} \ G^{14} \\ G^{35} \ G^{26} \ G^{23} \ C_1 \ C_4 \ C_5 \end{array} \right), \\
x_9 &\equiv \left( \begin{array}{c} E^2 \ E^4 \ E^5 \ G^{36} \ G^{16} \ G^{13} \\ G^{45} \ G^{25} \ G^{24} \ C_1 \ C_3 \ C_6 \end{array} \right), \\
x_{10} &\equiv \left( \begin{array}{c} E^2 \ E^4 \ E^5 \ G^{35} \ G^{15} \ G^{13} \\ G^{46} \ G^{26} \ G^{24} \ C_1 \ C_3 \ C_6 \end{array} \right), \\
x_{11} &\equiv \left( \begin{array}{c} E^5 \ C_5 \ G^{16} \ G^{36} \ G^{36} \ G^{26} \\ G^{46} \ G^{36} \ G^{25} \ G^{45} \end{array} \right), \\
x_{12} &\equiv \left( \begin{array}{c} E^1 \ E^3 \ E^5 \ G^{46} \ G^{26} \ G^{24} \\ G^{35} \ G^{15} \ G^{13} \ C_2 \ C_4 \ C_6 \end{array} \right).
\end{align*}
\]

The 12 roots \( \{x_i\} \) (together with their negatives) form a root system. The group \(G'\) which is generated by the double six transformations corresponding to these roots is the Weyl group to this root system. Obviously \(G \supset G'\) is a subgroup. To determine \(G'\) observe that the four roots \(x_3, x_7, x_{11}\) and \(x_{12}\) form a basis of the root system \(\{x_i\}\), i.e. any other root in the system is a linear combination of these four roots and the coefficients are either all non-negative or all non-positive. \(G'\) is, thus, uniquely determined by the corresponding Dynkin diagram:

![Diagram](https://via.placeholder.com/150)

This diagram clearly is the Dynkin diagram of the group \(D_4\). The group \(D_4\) is isomorphic to the semi-direct product of the permutation group \(S_4\) with \((\mathbb{Z}_2)^3\).

We are, now, going to show that \(G\) must be the whole group \(G'\). For this purpose consider the curve \(G^{12}\) which is fixed under the group action. As already mentioned, there are ten lines in the cubic \(S\) that intersect \(G^{12}\). Since for any two intersecting lines in a cubic surface there exists a unique line in this cubic which intersects them both the ten lines which intersect \(G^{12}\) are grouped into five pairs of intersecting lines. These are \(p_0 := (G^{34}, G^{56})\) (which is fixed under the monodromy action) and \(p_1 := (G^{46}, G^{35}), p_2 := (G^{36}, G^{45}), p_3 := (C^1, E^2), p_4 := (C^2, E^1)\).

As the group \(G'\) acts on the 27 lines preserving their incidence relations and keeping the line \(G^{12}\) fixed, each of the above pairs of intersecting lines must be moved to such a pair by the group action. Using the explicit description \(\{x_i\}\) of the action of \(G'\) one observes that any of the transformations corresponding to the \(x_i\) interchanges two of the four pairs \(p_1, \ldots, p_4\) – either preserving or reversing the order of the lines in the pairs. As \(G \supset G'\) is to act transitively on the lines of the pairs \(p_1, \ldots, p_4\) it must, in particular, act transitively on these four pairs. But a transitive subgroup of \(S_4\) that is generated by transpositions must be the group \(S_4\) itself. (Remember that \(G\) is to be generated by a subset of the transformations corresponding to the \(x_i\).)
Let $x_{i_1}$, $x_{i_2}$, and $x_{i_3}$ correspond to elements of $G$ that act by swapping the pairs $p_1 \leftrightarrow p_2$, $p_2 \leftrightarrow p_3$, and $p_3 \leftrightarrow p_4$ respectively. The subgroup of $G$ generated by these elements is isomorphic to $S_4$. The action of any of its elements does not map one line in a pair $p_i$ onto the other line of this pair. Therefore – as $G$ is to act transitively on the eight lines – there is an $x'_{i_4}$ such that the corresponding transformation is in $G$ and is not in the subgroup generated by the $x_{i_j}$ ($j = 1, 2, 3$). Let $x_{i_4}$ be the transformation that acts on the pairs in the same manner as $x'_{i_4}$ does and which is in the subgroup generated by the $x_{i_j}$ ($j = 1, 2, 3$). Then the composition of $x_{i_4}$ and $x'_{i_4}$ swaps the lines in the two pairs that are interchanged by the action of $x_{i_4}$ and $x'_{i_4}$.

By conjugating $x'_{i_4}$ with the elements of $\langle x_{i_1}, x_{i_2}, x_{i_3} \rangle \subset G$ on can interchange the lines in any two of the four pairs. Hence $G$ contains a subgroup (isomorphic to $(\mathbb{Z}_2)^3$) who’s elements interchange the lines in an even number of pairs. This proves that $G$ contains a subgroup which is the semi direct product of $S_4$ and $(\mathbb{Z}_2)^3$ and therefore $G = G'$.

Now, that we have a detailed knowledge of the action on the 27 lines (or on the 56 $(-1)$-curves on $Z_{H_0}$) induced by the monodromy, we will examine the induced action on the pairs of (different) lines formed out of these lines. Denote by $A$, $B$, $C$ the three orbits consisting of the eight lines intersecting the lines $G^{12}$, $G^{34}$, $G^{56}$ respectively. Then there are the following monodromy invariant subsets in the set of pairs of lines:

- The 9 sets $\{G_{12}\} \times A$, $\{G_{12}\} \times B$, $\{G_{12}\} \times C$, $\{G_{34}\} \times A$ etc.
- The 3 sets $\{(G_{12}, G_{34})\}$, $\{(G_{12}, G_{56})\}$, and $\{(G_{34}, G_{56})\}$.
- The 3 sets $\langle AA \rangle$ (which means the set of pairs consisting of two lines of $A$), $\langle BB \rangle$, $\langle CC \rangle$.
- The 3 sets $\langle AB \rangle$ (which means the set of pairs consisting of one line of $A$ and one line of $B$), $\langle AC \rangle$, $\langle BC \rangle$.

The sets of the first two items are obviously orbits under the monodromy action, whereas the sets of the last two items will turn out to be composite of two orbits.

One orbit in the set $\langle AA \rangle$ consists of the four pairs $p_1 := (G^{46}, G^{35})$, $p_2 := (G^{36}, G^{45})$, $p_3 := (C^1, E^2)$, $p_4 := (C^2, E^1)$ which are the four pairs which consist of intersecting lines. We claim that the remaining 24 elements in $\langle AA \rangle$ form an orbit of the monodromy action.

Let $(\ell_1, \ell_2) \in A$ be a pair of lines which do not intersect. We will calculate the number of elements in the orbit of this pair by finding its stabiliser. Embed $S_4 \subset G$ as the subgroup who’s elements permute the pairs $p_1, \ldots, p_4$ but leave the order of the lines in the pairs unchanged. If $(\mathbb{Z}_2)^3 \subset G$ is the subgroup of elements that leave the four pairs $p_1, \ldots, p_4$ fixed but swaps the lines in an even number of these pairs then $G$ is the semi-direct product of $S_4$ and $(\mathbb{Z}_2)^3$ (maybe in a different presentation as above). Let the lines $\ell_1$ and $\ell_2$ belong to the pairs $p_{i_1}$ and $p_{i_2}$ respectively. Then the stabiliser of $(\ell_1, \ell_2)$ consists of those elements of $G$ which leave the pairs $p_{i_1}$ and $p_{i_2}$ unchanged (i.e. those which interchange the other two pairs with or without changing the order of the lines in the pairs) together with those elements that interchange $p_{i_1}$ and $p_{i_2}$ in such a way that $\ell_1$ is mapped to $\ell_2$. The stabiliser is, therefore, $(S_2 \times S_2) \ltimes Z_2 \subset S_4 \ltimes (\mathbb{Z}_2)^3 \cong G$. The orbit of $(\ell_1, \ell_2)$, hence must contain 24 = 192/8 elements. Applying the same argument to $B$ and $C$ yields: $A$, $B$, and $C$ each split into two orbits – one with four and one with 24 elements.

In the same manner we will attack the sets $\langle AB \rangle$, $\langle AC \rangle$, and $\langle BC \rangle$ – e.g. the set $\langle AB \rangle$ (the two other sets being treated in an analogous way). Consider the line $E^1 \in A$. The stabiliser in $G$ of $E^1$ consists of exactly those elements which act only on the three pairs different from $(C^2, E^1)$. This subgroup is isomorphic to $S_3 \ltimes (\mathbb{Z}_2)^2$. It is generated by the elements corresponding to
the double six transformations \( x_6, \ldots, x_{11} \). Using the explicit description \((\mathcal{B})\) it is easy to check that the set \( \mathcal{B} \) has two orbits under the action of this subgroup – namely \( \{E^3, E^4, G^{25}, G^{26}\} \) and \( \{C^3, C^4, G^{15}, G^{16}\} \). (Note that the second orbit consists of the set of lines in \( \mathcal{B} \) that intersect \( E^1 \).) Thus, any pair \((\ell_1, \ell_2) \in (\mathcal{A}B)\) can be moved into a pair \((E^1, \ell)\) via monodromy action and this pair can be moved into each of the pairs \((E^1, \ell')\) with \( \ell' \in \{E^3, E^4, G^{25}, G^{26}\} \) or \( \ell' \in \{C^3, C^4, G^{15}, G^{16}\} \) depending on in which set \( \ell \) is contained. So \((\mathcal{A}B)\) can contain at most two orbits. On the other hand, a pair \((\ell_1, \ell_2)\) of intersecting lines cannot be moved into a pair of non-intersecting lines and vice versa since the monodromy action respects the incidence relations. Hence, \((\mathcal{A}B)\) must split in at least two orbits – one containing pairs of intersecting lines and the other containing pairs of non-intersecting lines. This proves that \((\mathcal{A}B)\) (as well as \((\mathcal{A}C)\) and \((\mathcal{B}C)\)) are the composite of two orbits of equal cardinality. Summing up, we have proved the following proposition.

**Proposition 5.6** The set of pairs formed out of the 27 lines has the following orbits under the action of \( G \) induced by the monodromy action on the lines.

1. Each of the three subsets \((\mathcal{A}A), (\mathcal{B}B), (\mathcal{C}C)\) contains two orbits – one with four and one with 24 pairs.

2. Each of the subsets \((\mathcal{A}B), (\mathcal{A}C), (\mathcal{B}C)\) contains two orbits with 32 pairs – one orbit with pairs of intersecting lines and one orbit with pairs of non-intersecting lines.

3. The 9 sets \(\{G_{12}\} \times \mathcal{A}, \{G_{12}\} \times \mathcal{B}, \{G_{12}\} \times \mathcal{C}, \{G_{34}\} \times \mathcal{A}\) etc. with 8 pairs each and

4. the 3 sets \(\{(G_{12}, G_{34})\}, \{(G_{12}, G_{56})\}, \text{ and } \{(G_{34}, G_{56})\}\) are orbits.

\( \square \)

### 5.3 One-parameter-families of touching conics and linear systems

We, now, want to use our knowledge on pairs of lines of a cubic surface and their monodromy to examine the irreducible components of the parameter space of touching conics. For this purpose return to the the double cover \( Z \) of \( \mathbb{P}^3 \) branched along our quartic \( B \).

Let \( H \subset \mathbb{P}^3 \) be an arbitrary plane that intersects \( B \) transversally and denote by \( Z_H \) the restriction of \( Z \) to \( H \) and by \( \pi : Z_H \rightarrow H \) the induced morphism. (Recall that \( Z_H \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) in seven points in general position.) We will establish a connection between one-parameter-families of touching conics of \( B \cap H \) and certain linear systems in \( Z_H \). Thereby the reducible elements of the linear systems will correspond to the reducible touching conics in the one-parameter-families.

Let \( C_1 \) and \( C_2 \) be two \((-1)\)-curves that have different images under \( \pi \) (i.e. \( \pi(C_1) \neq \pi(C_2) \)) which means that they are \((-1)\)-curves over different double tangents of \( B \cap H \). This is equivalent to \( [C_1 + C_2] \neq -K_{Z_H} \) where \( K_{Z_H} \) denotes the canonical class of \( Z_H \). Let \( C_1' \) and \( C_2' \) be the \((-1)\)-curves defined by \([C_1 + C_1'] = -K_{Z_H}\). Then these curves intersect as follows:

\[
C_1 \cdot C_2 = C_1' \cdot C_2' = 1 - C_1 \cdot C_2' = 1 - C_1' \cdot C_2.
\]

This follows from

\[
1 = C_1 \cdot (-K_{Z_H}) = C_1 \cdot (C_2 + C_2') = C_1 \cdot C_2 + C_1 \cdot C_2'
\]

and analogous identities. Therefore, by eventually interchanging \( C_1 \) and \( C_1' \), one can achieve that \( C_1 \cdot C_2 = 1 \) (the corresponding double tangents keeping unchanged).
Proposition 5.7 If $C_1$ and $C_2$ are chosen as above with $C_1 \cdot C_2 = 1$ then the linear system $|C_1 + C_2|$ is one-dimensional. Its generic element is a smooth rational curve that by the projection $\pi : Z_H \to H$ is mapped onto a touching conic.

Proof: For any of the 56 $(-1)$-curves $C$ of $Z_H$ consider the exact sequence

$$0 \to \mathcal{O}_{Z_H} \to \mathcal{O}_{Z_H}(C) \to \mathcal{O}_C(C) \to 0.$$ 

Noting that $C$ is a smooth rational curve with $\mathcal{O}_C(C) = \mathcal{O}_C(C \cdot C) = \mathcal{O}_C(-1)$ and that $Z_H$ is a smooth rational surface so that $0 = h^1(\mathcal{O}_{Z_H}) = h^1(\mathcal{O}_{Z_H}(C))$ we get $h^0(\mathcal{O}_{Z_H}) = h^0(\mathcal{O}_{Z_H}(C)) = 1$. Now, the exact sequence

$$0 \to \mathcal{O}_{Z_H}(C_2) \to \mathcal{O}_{Z_H}(C_1 + C_2) \to \mathcal{O}_{C_1}(C_1 + C_2) \to 0$$

and the fact that $\mathcal{O}_{C_1}(C_1 + C_2) = \mathcal{O}_{C_1}$ (since $C_1 \cdot (C_1 + C_2) = 0$) yield

$$h^0(\mathcal{O}_{Z_H}(C_1 + C_2)) = h^0(\mathcal{O}_{Z_H}(C_2)) + h^0(\mathcal{O}_{C_1}) = 2$$

and, hence, $\dim |C_1 + C_2| = 1$.

$|C_1 + C_2|$ cannot have a fixed component. Since $C_1$ and $C_2$ are irreducible this fixed component would have to be one of these two curves and then the linear system would only contain the divisor $C_1 + C_2$ in contradiction to the dimension of the system being 1. Hence, as $(C_1 + C_2)^2 = 0$, the system cannot have base points at all.

By Bertini’s Theorem the generic element of $|C_1 + C_2|$ is smooth away from the base locus. As the base locus is empty the generic element of the system is smooth everywhere. Let $C \in |C_1 + C_2|$ be general and $D_1, \ldots, D_n$ be its irreducible components. The $D_i$ cannot intersect each other for any intersection point would be a singular point of $C$. Thus $D_i \cdot D_j = 0$ for $i \neq j$ and consequently

$$0 = (C_1 + C_2)^2 = \left( \sum_{i=1}^{n} D_i \right)^2 = \sum_{i=1}^{n} D_i^2.$$ 

$|C_1 + C_2|$ has no fixed components – therefore

$$0 \leq D_i \cdot (C_1 + C_2) = D_i^2$$

and hence $D_i^2 = 0$ for all $i$. From the adjunction formula we get

$$g_i := \text{genus}(D_i) = \frac{K_{Z_H} \cdot D_i}{2} + 1.$$ 

On the other hand $K_{Z_H} \cdot D_i < 0$ since $-K_{Z_H}$ is ample. From $g_i \geq 0$ we then get $K_{Z_H} \cdot D_i = -2$ for all $i$. But

$$-K_{Z_H} \cdot \left( \sum_{i=1}^{n} D_i \right) = -K_{Z_H} \cdot (C_1 + C_2) = 2$$

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and hence \( n = 1, C = D_1, \) and genus(\( C \)) = 0, i.e. \( C \) is a smooth rational curve.

Now, let \( R \subset Z_H \) be the ramification divisor of the map \( Z_H \rightarrow H: \mathcal{O}_{Z_H}(R) = \pi^*(\mathcal{O}_H(2)) = -2K_{Z_H} \). Hence

\[
(C_1 + C_2) \cdot [R] = 2(C_1 + C_2) \cdot (-K_{Z_H}) = 4
\]

and by projection formula

\[
4[pt] = \pi_*((C_1 + C_2) \cdot [R]) = \pi_*((C_1 + C_2) \cdot \pi^*(2[1])) = \pi_* (C_1 + C_2) \cdot (2[1])
\]

where \([pt]\) denotes the class of a point and \([l]\) the class of a line in \( H \). Therefore \( \pi_* (C_1 + C_2) \in [2[1]] \).

Let \( C \in |C_1 + C_2| \) be a general element. If \( \pi(C) \) were a line then \( C \) was contained in the preimage of a line. As \( |C_1 + C_2| \neq K_{Z_H} \), there would exist an effective \( C' \) such that \([C] + [C'] = -K_{Z_H} \). But then \( C' \cdot (-K_{Z_H}) = (-K_{Z_H})^2 - C \cdot (-K_{Z_H}) = 0 \) which is impossible as \(-K_{Z_H} \) is ample. So the the image of \( C \) under \( \pi \) must be a smooth conic and \( \pi|_C \) is of degree one onto \( \pi(C) \).

Consider now the preimage \( \pi^{-1}(\pi(C)) \) in \( Z_H \). As \( \pi \) is a double cover and \( \pi|_C \) is only of degree one, \( \pi^{-1}(\pi(C)) \) must contain other components than \( C \) or \( \pi(C) \) must be contained in \( B \cap H \). The latter is not possible since \( B \cap H \) was supposed to be a smooth quartic curve. Obviously, \( \pi^{-1}(\pi(C)) \) is an element of \([-2K_{Z_H}] \) since \( \pi(C) \) is an element of \( |\mathcal{O}_{B^2}(2)| \) and \( \pi: Z_H \rightarrow B^2 \cong H \) is induced by the anticanonical linear system. Therefore, the sum of the other components of \( \pi^{-1}(\pi(C)) \) must be an element of \([-2K_{Z_H} - (C_1 + C_2)] = [C_1' + C_2'] \). \( C_1' \) was defined to be the \((-1)\)-curve in \( Z_H \) such that \( C_1' + C_1' = -K_{Z_H} \).

Let \( C' \in |C_1' + C_2'| \) be the divisor which is complementary to \( C \) in \( \pi^{-1}(\pi(C)) \). Note that \((C_1' + C_2')\), as well as \((C_1 + C_2)\), is the sum of two \((-1)\)-curves with intersection \( C_1' \cdot C_2' = 1 \) and consequently the above arguments equally apply to \( (C_1' + C_2') \). In particular, a general element of \( |C_1' + C_2'| \) is a smooth rational curve which by \( \pi \) is mapped onto a smooth conic in \( H \). So if \( C \in |C_1 + C_2| \) is sufficiently general then \( C' \), as well, is a smooth rational curve that is mapped onto a smooth conic. Therefore, \( \pi^{-1}(\pi(C)) \) splits into two components each of which is a smooth rational curve.

Now, one shows just like in the proof of Proposition 1.1 that the preimage in a double cover of a conic in \( H \) splits into two components if and only if it has even intersection with the ramification locus \( B \cap H \subset H \). Therefore \( C \) is mapped onto a touching conic and the proposition is proved. \( \square \)

By the above proposition \( \pi \) induces a morphism \( |C_1 + C_2| \cong B^1 \rightarrow B^5 \) where \( B^5 \) is the parameter space of conics in \( H \). This morphism is necessarily injective as the preimage of \( \pi(C) \) consists of \( C \) and an element of the linear system \([-K_{Z_H} - [C]] \) which is different from \([C_1 + C_2]\). There is an open subset in \([C_1 + C_2]\) which is mapped into the closed subset of touching conics in \( B^5 \). Therefore any element of \([C_1 + C_2]\) is mapped onto a (maybe reducible) touching conic. Moreover, an element of \([C_1 + C_2]\) is mapped to a reducible conic if and only if it is the sum of two \((-1)\)-curves.

We, thus, have constructed a correspondence between the linear systems \([C_1 + C_2]\) (with \((-1)\)-curves \( C_i \) satisfying \( C_1 \cdot C_2 = 1 \)) and one-parameter-families of conics in \( H \) touching \( B \cap H \): For each one parameter family there exist exactly two of these linear systems that are mapped to this family.

Let again \( Z := Z \times_{\mathbb{PP}^3} H \) where \( H \subset \mathbb{PP}^3 \times \mathbb{PP}^3 \) is the universal (hyper-)plane. We have seen that for any \( H_0 \in \mathbb{PP}^3 \setminus \Delta \) the fundamental group \( \pi_1(\mathbb{PP}^3 \setminus \Delta, H_0) \) acts via monodromy of \( Z|_{\mathbb{PP}^3 \setminus \Delta} \rightarrow \mathbb{PP}^3 \setminus \Delta \) on \( \text{Pic}(Z_{H_0}) \) preserving the intersection pairing. In particular, the fundamental group acts on the above linear systems.

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Denote by $X' \subset P$ ($P$ – the parameter space of all conics in $\mathbb{P}^3$ as constructed in Section 2.1) the closed subscheme of conics that have only even intersection with $B$ and let $X \subset X'$ be the union of all irreducible components that do not entirely consist of double lines. From $P X$ inherits a morphism to $\mathbb{P}^3$. By Proposition 3.3 the fibre of $X$ over any $H \in \mathbb{P}^3 \setminus \Delta$ consists of 63 disjoint smooth conic curves in the fibre of $P$ over $H \in \mathbb{P}^3$ which is isomorphic to $\mathbb{P}^5$. The fundamental group $\pi_1(\mathbb{P}^3 \setminus \Delta, H_0)$ acts on the set of the 63 one parameter families in the fibre $X_{H_0}$ in a natural way by monodromy: Any path $\gamma : \{0, 1\} \rightarrow \mathbb{P}^3 \setminus \Delta$ can be lifted (in a non-unique way) to a path in $X_{|\mathbb{P}^3 \setminus \Delta}$. Though the lift is not unique, the connected component of the fibre $X_{\gamma(t)}$ in which the lifted path is contained is well determined.

Obviously, the correspondence between the linear systems in the lifted path is contained is well determined. Denote by $P$ smooth conic curves in the fibre of $X$ and let $\Delta = \{H \in \mathbb{P}^3 \setminus \Delta \mid \text{the fibre of } H \text{ is flat over } ˇX \}$.

We are mainly interested in the connected components of the monodromy orbits would be to list all linear systems $|\mathbb{A} \cdot C| \cap \mathbb{P}^3 \setminus \Delta$ and then determining the monodromy orbits of these linear systems using our knowledge on the monodromy of $(-1)$-curves. The monodromy of $X_{|\mathbb{P}^3 \setminus \Delta}$ is then easily calculated. This approach is a bit cumbersome. So we modify this method using our knowledge about one-parameter-families of touching conics.

First, we consider the linear system $|G^{12} + G^{34}| = |\{2; 1, 1, 1, 0, 0, 0\}|$ (the elements of $\text{Pic}(\mathbb{A} \cdot C)$ are denoted similarly as on page 22 i.e. $(a; b_1, \ldots, b_7) \in \text{Pic}(\mathbb{A} \cdot C)$ denotes the element $a[H] - b_1[E^1] - \cdots - b_7[E^7]$). As the two $(-1)$-curves $G^{12}$ and $G^{34}$ are monodromy invariant the linear system must keep fixed under the monodromy action, as well. This system may be presented as the sum of two $(-1)$-curves in five further ways:

\[
(2; 1, 1, 1, 0, 0, 0) = C^{56} + E^7 = C^{57} + E^6 = C^{67} + E^5
= G^{13} + G^{24} = G^{14} + G^{23}.
\]

The two pairs $(G^{12}, G^{34})$ and $(C^{56}, E^7)$ correspond to the pairs of double tangents $(e_2, e_3)$ and $(e_4, e_1)$ where $e_i$ denotes the double tangent given by the equation $E_i = 0$ in $H_0$ ($E_i$ being the linear forms appearing in the equation $E_1E_2E_3E_4 - Q^2$ of $B$). The remaining four pairs are those of the set $(\mathbb{A} \cdot C)$ when identified with the corresponding lines in a cubic surface. The corresponding one-parameter-family of touching conics, thus, contains four pairs of double tangents all eight double tangents being in the same component of $Y_0$. Denote this component of $Y_0$ by $\mathbb{C}$.

Analogously (considering the systems $|G^{12} + G^{56}|$ and $|G^{34} + G^{56}|$ respectively) the one-parameter-family of touching conics containing the pairs $(e_2, e_4)$ and $(e_1, e_3)$ contains four pairs formed of double tangents of the component $\mathbb{B}$ of $Y_0$ and the family containing the pairs $(e_3, e_4)$ and $(e_1, e_2)$ contains four pairs formed of double tangents of the component $\mathbb{A}$. These three one-parameter-families are the ones which are obvious from the special form of the equation of $B$: $E_1E_2E_3E_4 - Q^2$ is of the form $UW - V^2$ by letting $V = Q$ and letting $U$ be one of $E_1E_2$, $E_3E_4$ or $E_1E_4$. So in any plane $H_0$ we get three one-parameter-families which we will call the “obvious” families.

In particular, each of these obvious one-parameter-families contains two reducible conics consisting of the four double tangents $e_1, \ldots, e_4$ and four reducible elements formed of the eight double tangents in $H_0$ that belong to the same component $\mathbb{A}$, $\mathbb{B}$ or $\mathbb{C}$ of $Y_0$. 28
Denote, for simplicity the eight double tangents in $H_0$ of the component $A$ by $a_1, \ldots, a_8$ and the double tangents of $B$ and $C$ by $b_1, \ldots, b_8$ and $c_1, \ldots, c_8$ respectively. We will examine how the pairs formed out of these double tangents can be distributed to the one-parameter-families. By Lemma 3.3, each one-parameter-family of touching conics in $H_0$ contains exactly six reducible conics. From the above arguments we already know the reducible elements of three families (given in terms of pairs of double tangents):

\[ e_1e_2, e_3e_4, a_1a_2, a_3a_4, a_5a_6, a_7a_8 \]
\[ e_1e_3, e_2e_4, b_1b_2, \ldots \]
\[ e_1e_4, e_2e_3, c_1c_2, \ldots \]

In particular, the equation of the plane quartic $B \cap H_0$ can be written in the form $e_1e_2 \cdot a_1a_2 - V^2$ where $V$ is a quadratic form. (Lines and the corresponding linear forms are denoted by the same letter.) By writing this equation as $e_1a_1 \cdot e_2a_2 - V^2$ we see that the couples $e_1a_1$ and $e_2a_2$ belong to the same one-parameter-family. By Lemma 3.3, in this family no further couple of double tangents is contained that has one of the double tangents $e_i$ or $a_i$ as an element. Couples of the form $b_i b_j$ or $c_i c_j$ also must not occur in this family. If, for instance, the couple $b_1 b_3$ were in one group together with $e_1 a_2$ then the equation of $B \cap H_0$ could be written as $e_1a_2 \cdot b_1b_3 - V^2 = e_1b_1 \cdot a_1b_3 - V^2$ and thus $e_1b_1$ and $a_1b_3$ would belong to the same family. But this is impossible by the above argument. Hence, only the couples $b_k c_l$ may occur in families together with couples $e_i a_j$. There are exactly 16 families each of which contains two couples $e_i a_j$. On the other hand, there are 64 couples $b_k c_l$ which is just the number of couples needed to complete these 16 families.

By the same argument, the couples $a_i b_j$ belong to families which contain two couples of the form $e_k c_l$ and four couples of the form $a_i b_j$ and, analogously, the couples $a_i c_j$ spread over families containing two couples $e_k b_l$ and four couples $a_i c_j$.

As we have just seen couples $b_i c_j$ and $b_k c_l$ have to occur in one family. Thus, there is a family containing both $b_i b_k$ and $c_j b_l$. Analogously, there are couples $a_i a_j$ and $b_k b_l$ as well as $a_i a_j$ and $c_k c_l$ in one group. There are $3 \cdot \binom{4}{2} − 4 = 72$ pairs of the form $a_i a_j$, $b_i b_j$ and $c_k c_l$ which do not occur in the “obvious” families. These spread over the remaining 12 one-parameter-families.

In the consequence of these considerations we can determine the connected components of $X|\mathcal{F}^3 \setminus \Delta$.

Let $Y_F^2 \subset X|\mathcal{F}^3 \setminus \Delta$ be the closed subscheme which parametrises the reducible touching conics. $Y_F^2$ is naturally isomorphic to the open subset in the symmetric product of $Y_F$ with itself:

$$ Y_F \times_{\mathcal{F}^3 \setminus \Delta} Y_F \setminus \text{Diag} \longrightarrow Y_F^2 $$

by simply associating to each pair of coplanar double tangents the corresponding reducible touching conic. As any one-parameter-family in any fibre of $X|\mathcal{F}^3 \setminus \Delta$ over $\mathcal{F}^3 \setminus \Delta$ contains reducible conics and as the monodromy action on the set of one-parameter-families in the fibre over $H_0 \in \mathcal{F}^3 \setminus \Delta$ is independent of the chosen lift of the path one can choose the lift of any path $\gamma \subset \mathcal{F}^3 \setminus \Delta$ in such a way that the lifted path is contained in $Y_F^2$.

But the connected components of $Y_F^2$ are already determined: The following proposition is just a corollary of Proposition 5.6.

\footnote{In fact, using Lemma 3.6 one could determine which pairs of double tangents pertain to which one-parameter-family. Those considerations can be found in [6].}
Proposition 5.8 The connected components of $Y^2_F \rightarrow \mathbb{P}^3 \setminus \Delta$ are the following:

1. Three components with 24 points in each fibre over $\mathbb{P}^3 \setminus \Delta$ – each component corresponding to one orbit (the one with 24 elements) in the sets $(AA)$, $(BB)$, and $(CC)$ of pairs of lines in a cubic.

2. Three components with four points in each fibre – each component corresponding to the other orbit in the sets $(AA)$, $(BB)$, and $(CC)$ of pairs of lines in a cubic.

3. Six components with 32 points in each fibre corresponding to the orbits in the sets $(AB)$, $(AC)$, and $(BC)$.

4. 12 components with eight points in each fibre: For each $i = 1, \ldots, 4$ and each component $A$, $B$ or $C$ of $Y_0$ there is an irreducible component of $Y^2_F$. The corresponding reducible conics in a plane $H$ consist of the double tangent $e_i$ given by the equation $E_i = 0$ in $H$ and one double tangent of the chosen component of $Y_0$.

5. Six components with just one point in each fibre, namely the six pairs of double tangents $e_i e_j$.

Consequently, $X|_{\mathbb{P}^3 \setminus \Delta}$ has the following connected components:

- The three “obvious” families in the fibre of $X|_{\mathbb{P}^3 \setminus \Delta}$ over $H_0 \in \mathbb{P}^3 \setminus \Delta$ are invariant under monodromy as they contain reducible conics consisting of the double tangents $e_i e_j$ which are fixed under monodromy. Each of the three families contains two of them. The other four reducible conics in each family are necessarily the four pairs of double tangents of one of the orbits in item 2 in the above Proposition.

- There are six connected components with eight one-parameter-families in any fibre of $X|_{\mathbb{P}^3 \setminus \Delta}$ over $\mathbb{P}^3 \setminus \Delta$: All one-parameter-families containing reducible conics of the type $e_i a_j$, $e_i b_j$ or $e_i c_j$ pertain to one of these components. As we have seen, any one-parameter-family that contains those pairs of double tangent contains two of them and four pairs of the form $a_i b_j$, $a_i c_j$, or $b_i c_j$. By Lemma 3.6 and the above discussion, the two pairs $e_i a_j$ and $e_i a_l$ are in the same one-parameter-family only if $i \neq k$ and $j \neq l$ (analogously for $e_i b_j$ and $e_i c_j$). Therefore each component of $Y^2_F$ in item 4 of Proposition 5.8 intersects a one-parameter-family in a fibre over $\mathbb{P}^3 \setminus \Delta$ in at most one point. Consequently, the monodromy orbit of such a one-parameter-family in the fibre of $X|_{\mathbb{P}^3 \setminus \Delta}$ over $H_0$ consists of exactly eight families.

- The remaining pairs of double tangents are those of item 5 of Proposition 5.8. The families containing these reducible conics belong to the same connected component: We have seen that there is a one-parameter-families that contains a pair $a_i a_j$, as well as a pair $b_i b_j$ and a family that contains some $a_i a_j$ together with a $c_k c_l$. So, one can connect the first family with any family containing a pair $a_i a_j$ or a pair $b_i b_j$ by a path lying in $Y^2_F$ and, analogously, the second family can be connected with any family containing a pair $a_i a_j$. Hence the twelve one-parameter-families belong to the same connected component.

As the connected components of $X|_{\mathbb{P}^3 \setminus \Delta}$ are just its irreducible components the following Theorem is proved by the above discussion.

Theorem 5.9 $X|_{\mathbb{P}^3 \setminus \Delta}$ has 10 irreducible components – namely
• three components each with one one-parameter-family in every fibre over $\mathbb{P}^3$,

• six components with eight families in every fibre, and

• one component with twelve families in each fibre over $\mathbb{P}^3 \setminus \Delta$.

The three types of irreducible components differ by the type of reducible conics that they contain. In each one-parameter-family there are reducible conics that contain two, one or none double tangent $e_i$ respectively for the three types.

**Remark:** The components of $X$ are not entirely determined by the above theorem. There are at least four components which are contained in $X|_{\Delta}$ (i.e. over $\Delta \subset \mathbb{P}^3$) namely the four sets consisting of all conics in the planes $E_i = 0$. All of them have to be touching conics since $B \cap \{E_i = 0\}$ is a non-reduced conic. But these four components are, conjecturally, all components which are contained in $X|_{\Delta}$.

**References**

[BPV] Barth, W.; Peters, C.; Van de Ven, A.: Compact Complex Surfaces. Springer–Verlag Berlin Heidelberg 1984

[B] Burau, Werner: Algebraische Kurven und Flächen. Band 1: Algebraische Kurven der Ebene; Walter de Gruyter & Co. Berlin 1962

[Cl] Clemens, C. Herbert: Double Solids. Advances in Mathematics 47 (1983); p. 107–230

[Co] Cox, David; Little, John; O’Shea, Donal: Ideals, Varieties, and Algorithms. Springer-Verlag New York 1992

[Fi] Finkelberg, H.: Fano surfaces of cubic threefolds with isolated singularities. Preprint Leiden (1987)

[GH] Griffiths, Phillip; Harris, Joseph: Principles of algebraic geometry. Wiley 1978.

[He] Henderson, Archibald: The Twenty-Seven Lines upon The Cubic Surface. Cambridge University Press, 1911

[K1] Kreußler, Bernd: Another description of quartic double solids. Mathematica Gottingensis 14/1991 (Göttinger Georg-August-Universität, Math. Inst. SFB 170 – Geometrie & Analysis 1991).

[K2] Kreußler, Bernd: Small Resolution of Double Solids, Branched over a 13-nodal Quartic. Annals of Global Analysis and Geometry 7 (1989), p.227–267

[KK] Kreußler, Bernd; Kurke, Herbert: Twistor Spaces over the Connected Sum of 3 Projective Planes. Compositio Mathematica 82 (1992), p.25–55

[L] Lamottke, Klaus: The Topology of complex projective varieties after S.Lefschetz. Topology 20 (1981) pp. 15–51.

[M] Manin, Juri Ivanović: Kubičeskie formy, algebra, geometrija, arifmetika. Izdatel’stvo Nauka, Moskau, 1972. English translation: Cubic forms, algebra, geometry, arithmetics. North-Holland, Amsterdam, 1972
[Sa] Salmon, George: Geometrie der höheren ebenen Kurven. Teubner, Leipzig 1873

[Se] Segre, B.: The Non-Singular Cubic Surfaces. Oxford University Press, 1942

[Sw] Swinnerton-Dyer, H. P. F.: The zeta function of a cubic surface over a finite field. Proceedings of the Cambridge Philosophical Society, Vol. 63 (1967), pp.55–71.

[T] Tikhomirov, A.S.: Geometrija poverchnosti fano dvojnogo prostranstva $\mathbb{P}^3$ s vetvleniem v kvartike. Izvestija Akademii Nauk SSSR, Serija matematičeskaya; Tom 44, No. 2, 1980; S. 415–442. English translation: The Geometry of the Fano Surface of the Double Cover of $\mathbb{P}^3$ branched in a Quartic. Math. USSR Izv. 16 1981 pp. 373–397.

Ingo Hadan
Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Ziegelstraße 13a
10099 Berlin
e-mail: hadan@mathematik.hu-berlin.de