Convergence of Star Product: From Examples to a General Framework*

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Abstract

We recall some of the fundamental achievements of formal deformation quantization to argue that one of the most important remaining problems is the question of convergence. Here we discuss different approaches found in the literature so far. The recent developments of finding convergence conditions are then outlined in three basic examples: the Weyl star product for constant Poisson structures, the Gutt star product for linear Poisson structures, and the Wick type star product on the Poincaré disc.

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1 Introduction: Formal Deformation Quantization

In the by now classical paper [3], Bayen, Frønsdal, Flato, Lichnerowicz, and Sternheimer introduced the notion of a formal star product on a Poisson manifold as a general method to pass from a classical mechanical system encoded by the Poisson manifold to its corresponding quantum system. The main difference compared to other, and more ad-hoc, quantization schemes is the emphasis on the role of the observable algebra. The observable algebra is constructed not as particular operators on a Hilbert space as this is usually done. Instead, one stays with the same vector space of smooth functions on the Poisson manifold and just changes the commutative pointwise product into a new noncommutative product using Planck's constant ℏ as a deformation parameter. This way one tries to model the quantum mechanical commutation relations.

While in [3] it is shown that this leads to the same results as expected in quantum mechanics for those systems where alternative quantization schemes are available, the proposed scheme of deformation quantization has several conceptual advantages: the first is of course its vast generality concerning the formulation. While other quantization schemes make much more use of specific features of the classical system, deformation quantization can be seen as almost universal in the sense that its requirements are virtually minimal. Only a Poisson algebra of classical observables is needed to formulate the program of deformation quantization. Of course, the hard part of the work consists then in actually proving the existence (and possible classifications) of star products, but in any case, the conceptual framework is fixed from the beginning. A second advantage is that the physical interpretation of the observables is fixed from the beginning: the observables simply stay the same elements of the same underlying vector space. Thus it is immediately clear which quantum observable is the Hamiltonian, the momentum etc. since they are the same as classical. It is only the product law which changes, the correspondence of classical and quantum observables is implemented trivially. A third advantage of this approach to focus on the algebra first is shared also by other formulations of quantum theory, most notably by axiomatic quantum field theory, see e.g. [100]: having the focus on the algebra one can now study different representations on Hilbert spaces which might be needed to encode different physical situations the system is exposed to. In quantum field theory this is a well-known feature and difficulty, but now the same options are available in quantum mechanical systems as well. It turns out that this is not just a mathematical game to play but required by physical reality. As argued in e.g. [24] inequivalent, representations are needed to encode the Aharonov-Bohm effect [1] in this framework.

There is of course a price to be paid in order to achieve this generality in the formulation of the quantization problem. To understand the difficulties we recall the precise definition of a formal star product: let \((M, \pi)\) be a Poisson manifold, i.e. a smooth manifold with a Poisson tensor \(\pi \in \Gamma^\infty(\Lambda^2 TM)\) such that we have a Poisson bracket \(\{f, g\} = \pi(df, dg)\) for smooth functions \(f, g \in \mathcal{C}^\infty(M)\). With other words, the commutative associative algebra \(\mathcal{C}^\infty(M)\) of complex-valued smooth functions becomes a Poisson algebra when equipped with this bracket \(\{ \cdot, \cdot \}\). Then a formal star product \(\star\) for \((M, \pi)\) consists in a product law

\[
\star : \mathcal{C}^\infty(M)[[\lambda]] \times \mathcal{C}^\infty(M)[[\lambda]] \ni (f, g) \mapsto f \star g \in \mathcal{C}^\infty(M)[[\lambda]]
\]

defined on the space of formal power series in a formal parameter \(\lambda\) with coefficients in the smooth functions such that the following properties hold: first, \(\star\) is \(\mathbb{C}[[\lambda]]\)-bilinear. This already implies that there are uniquely determined \(\mathbb{C}\)-bilinear maps \(C_r : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)\) such that

\[
f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g),
\]

where the \(C_r\) are extended to formal series by requiring \(\mathbb{C}[[\lambda]]\)-bilinearity. The second requirement is that \(\star\) should be associative. This is in fact a truly nontrivial feature as it results in a infinite chain
of quadratic equations for the coefficient operators $C_r$. The first orders of the associativity condition are the associativity of $C_0$ in zeroth order, then

$$C_0(C_1(f,g),h) + C_1(C_0(f,g),h) = C_0(f,C_1(g,h)) + C_1(f,C_0(g,h))$$

(1.3)

in first order and so on. The third requirement is the compatibility with the semi-classical limit $\lambda = 0$. One wants

$$C_0(f,g) = fg \quad \text{and} \quad C_1(f,g) - C_1(g,f) = i\{f,g\}$$

(1.4)

for all $f,g \in \mathcal{C}^\infty(M)$. This way, one implements the actual quantization condition that the product $\star$ deforms its zeroth order $C_0$ which should be the usual product and second, the commutator with respect to $\star$ deforms the Poisson bracket in the next nontrivial order. Including $i = \sqrt{-1}$ in the requirement as in (1.4) then allows for the identification

$$\lambda \sim h,$$

(1.5)

e.i. the formal deformation parameter $\lambda$ should correspond to the physical Planck constant $h$. The last requirements are more of a technical nature and not too essential for the physical interpretation. One requires $1 \star f = f = f \star 1$ for the constant function 1 and one requires the $C_r$ to be bidifferential operators. It turns out that in all known constructions these two features are automatically satisfied and will therefore not pose any additional difficulties.

With this definition, a star product $\star$ becomes a particular example of a formal deformation of an associative algebra in the sense of Gerstenhaber [90–94]. The main difficulty of formal deformation quantization is now that the deformation parameter $\lambda$ corresponds to the constant $h$ which is one of the very fundamental constants of nature. In particular, this constant is non-zero and not dimensionless. Thus it will make no sense whatsoever to speak of smallness of $h$ in an intrinsic way. In fact, one can always choose a unit system where $h$ has the numerical value $h = 1$ or, say, $h = 42$. Thus the convergence of the series (1.2) becomes an inevitable issue which has to be solve in order to have a physically sound model. This is the convergence problem in deformation quantization.

Before entering the details on the convergence issue let us first continue with some positive results in deformation quantization illustrating that it is worth to pursue the proposed path. Indeed, on the level of formal star products, deformation quantization is extraordinarily successful: after several classes of examples [50–52, 67, 69, 98] the existence of star products on general symplectic manifolds was shown by de Wilde and Lecomte [68]. Shortly later, Fedosov gave a different and geometrically very explicit construction [84–86], see also his book [87]. Finally, Omori, Maeda and Yoshioka found yet another construction for symplectic manifolds [127]. The classification of formal star products on symplectic manifolds was obtained be several groups in slightly different formulations in [11, 66, 87, 97, 111, 128, 145]. The existence (and classification) on general Poisson manifolds remained a hard problem for quite some time. Beside the remarkable case of linear Poisson structures [96] not much progress was achieved until the seminal work of Kontsevich proving his formality conjecture from [106] in the preprint [105], later published in [107]. Tamarkin gave another interpretation of Kontsevich’s formality theorem using the language of operads in [140]. A different approach to globalization of Kontsevich’s local explicit formula is due to Dolgushev [75, 76]. An interpretation of the formality in terms of the Poisson sigma-model [137] can be found in the work of Cattaneo, Felder and Tomassini [57, 60, 63, 64].

In addition to the existence and classification results many more details on deformation quantization have been discussed over the years. To mention just a few achievements we would like to point out the construction of star products on Kähler manifolds taking advantage of the additional complex structure. Here Cahen, Gutt, and Rawnsley showed in [53, 50] how the quantization scheme based on Berezin-Toeplitz operators [7] can be used to obtain formal star products by asymptotic expansions of the integrals encoding the Berezin-Toeplitz operators. In [24] this was refined to incorporate detailed operator norm estimates ultimately resulting in a continuous field of $C^*$-algebras.
The star products obtained this way turn out to be of Wick type in the sense that one function is differentiated in holomorphic directions while the other is differentiated in anti-holomorphic directions only. Such star products have been shown to exist in general by Karabegov [104], see also [29] for an explicit construction based on Fedosov’s approach. Beside Kähler manifolds, cotangent bundles are perhaps the most important classical phase spaces. Deformation quantizations adapted to this class of examples were discussed in [25, 27, 128, 129].

A large activity over the years was on the understanding of star products with symmetries: here one is given a classical symmetry in form of a Lie group action or a Lie algebra action compatible with the classical Poisson structure. Then the question is whether there is a star product such that the Lie group acts by automorphisms of it, or, infinitesimally, the Lie algebra acts by derivations. It turns out that the existence of an invariant covariant derivative is sufficient for this in general, see e.g. [2, 10, 75, 87]. Classically, a symmetry becomes most interesting if it is implemented by means of a momentum map. Again, the question is whether one also has a corresponding momentum map on the quantum side. Here one has several positive answers [98, 116, 117] including a complete classification in the symplectic case [131]. One important construction for classical mechanical systems with symmetry is the Marsden-Weinstein reduction [112] which allows to reduce the dimension by fixing the values of conserved quantities build out of the momentum map. In deformation quantization several quantum analogs of reduction are constructed by Fedosov [88], by Bordemann, Herbig, and Waldmann [23] based on a BRST formulation as well as by Gutt and Waldmann [99] including a careful discussion of the involutions. Recently, Reichert computed the characteristic classes of reduced star products in terms of the equivariant classes on the original phase space in [130]. On the more conceptual side, Cattaneo and Felder investigated the compatibility of formality morphisms with coisotropic submanifolds in general [59, 61, 62].

Finally, a last important aspect of deformation quantization is the development of a physically reasonable notion of states, considered as positive linear functionals on the algebra of observables [30]. Here positivity is understood in the sense of the canonical ring ordering of \( \mathbb{R}[\lambda] \subseteq \mathbb{C}[\lambda] \). In fact, large parts of operator algebraic theory on states and representations can be transferred to this purely algebraic framework based on *-algebras over ordered rings, see e.g. [28, 38, 39, 141] as well as the book [142, Chap. 7] and the lecture notes [144]. Beside studying the existence of positive functionals as deformations of classical ones [37, 43] one can establish a notion of strong Morita equivalence yielding the equivalence of categories of *-representations [32–34, 36, 39, 41, 42, 44–46] culminating in a complete and geometrically simple classification of star products up to Morita equivalence in [40, 49], first in the symplectic and then in the general Poisson case. Conversely, the investigations of the Picard group of star products triggered an analogous program also for the semi-classical counterpart, the Picard group(oid) in Poisson geometry [35, 47, 48]. Again, one has adapted versions including classical symmetries [102, 103].

Finally, it should be mentioned that star products found their way also to field-theoretic applications: while the original intention was to provide a quantization scheme for geometrically non-trivial but finite-dimensional phase spaces the notion of star products and hence deformation quantization is applicable also to classical systems with infinitely many degrees of freedom, i.e. classical field theories. The main difficulty is to find a suitable Hamiltonian formulation first with a Poisson algebra of observables. Once this is achieved, the very definition of a star product clearly makes sense and can be explored. In [70, 72, 74] Dito investigated this possibility for the Klein-Gordon field theory and studied the effects of renormalization needed for interacting fields. Fredenhagen and Dütsch continued and extended this considerably [78, 82] and ever since star products became an important tool in more conceptual approaches to quantum field theory on globally hyperbolic spacetimes, see e.g. [89] as well as the monograph [132] for further references.
The Quest for Convergence

All the above achievements will not allow to hide the open question on the convergence of the formal star products. It is clear that for a physically applicable quantization one cannot treat $\hbar$ as a formal parameter but needs to paste in a numerical value determined in a given system of units. If not on the level of observable algebras, the latest moment where this definitely has to be done is for expectation values and spectral values: ultimately, they are measurable quantities accessible by experiments and here we have no formal series anymore.

The problem becomes now manifest since in all known examples of star products the operators $C_r$ are bidifferential operators of order at least $r$ in each argument. In particular, the star product $f \ast g$ of two smooth functions will typically see and use essentially the whole infinite jet information of $f$ and $g$. This shows that one can find immediately two functions $f, g \in \mathcal{C}^\infty(M)$ in such a way that their (formal) star product $f \ast g$ at a given point $p \in M$ will have radius of convergence in $\lambda$ being zero. The reasons is simply that with the classical Borel lemma we can adjust the Taylor coefficients of $f$ and $g$ at a point $p$ arbitrarily bad, leading to a divergent series $f \ast g$ unless $\lambda = 0$. Thus the class of all smooth functions is not suitable for a naive (pointwise) convergence of $f \ast g$. There are now several ways considered to circumvent this fundamental problem.

The perhaps first interpretation of the convergence problem is that the formal star product $f \ast g$ should be interpreted as an asymptotic expansion of some honest product $f \circ \hbar g$ for $\hbar \to 0^+$, at least for a suitably interesting class of functions. In fact, this seems not to be completely hopeless as in many examples the formal star products where exactly constructed that way. One investigated certain integral formulas for operators and compositions of integral operators depending on a positive parameter $\hbar$ and showed that in the limit $\hbar \to 0$ one obtains a smooth dependence on $\hbar$ (from the right). Thus the resulting formal Taylor expansion in $\hbar$ of these integral expressions yielded a composition law $f \ast g$ which then turned out to be a formal star product.

This has been a successful approach in several situations, most notably in the early constructions of star products as in [24, 53–56]. The tricky question in these approaches is, however, whether one actually has an honest algebra of functions, i.e. whether one can specify a subspace of functions which is indeed closed under the integral formulas for the product. One might be able to circumvent this by starting with a small class of functions for which the integral formulas are easily defined, say with suitable support conditions, and define the algebra implicitly as all what is generated by the products of such nice functions. It remains a nontrivial question whether this is really a well-defined algebra as the integral formulas might produce functions for which a further application of the integration is no longer valid.

On a more conceptual level, the very few integral formulas which are known are used to obtain universal deformation formulas. The most prominent example is the integral formula for the Weyl-Moyal star product. It can be viewed as an integral kernel on the abelian Lie group $\mathbb{R}^{2n}$. Whenever one has now an action of this group on a suitable topological algebra, say a Banach algebra or a locally convex algebra, subject to certain continuity requirements as e.g. strongly continuous and polynomially bounded, then one can use the action to induce a deformation of the algebra. This way one obtains a new product, still being continuous, which depends on $\hbar$ in a very controlled way. The indicated procedure can be traced back to Rieffel’s seminal work [136], see also his works [133, 134] where the main focus is on the case of $C^*$-algebras. Here one wants to construct in addition a $C^*$-norm on the deformed algebra such that it becomes a $C^*$-algebra after completion. Ultimately, this results in a continuous field of $C^*$-algebras parametrized by $\hbar \in [0, \infty)$. Many more details on this construction and its application in quantization theory can be found in Landsman’s monograph [109], a generalization to general locally convex algebras and their modules under very mild assumptions on the group action is discussed in [110].

However, the above class of constructions only works if the abelian group $\mathbb{R}^{2n}$ acts. There are many interesting situations in symplectic geometry where one immediately finds obstructions that
the Poisson structure comes from an action of $\mathbb{R}^{2n}$, or, even worse, from any kind of Lie group action, see [17]. It took some considerable effort to find integral formulas beyond the abelian case. Here Bieliavsky and Gayral found a vast generalization of Rieffel’s original ideas and proposed universal deformation formulas in [18], see also the earlier works [12,13,15,19,20]. These ideas lead ultimately to a quantization of Riemann surfaces of higher genus [14]. Yet a different approach to quantum surfaces in a $C^*$-algebraic formulation is due to Natsume, Nest, and Peters [118–120] providing strict quantizations, too.

It is now a second option we would like to advocate for: instead of replacing the formal power series by integral formulas and re-interpreting the star product in such terms, we would like to take the series description serious and investigate the actual convergence of the formal power series in $\hbar$ for certain classes of functions. To achieve this goal, one has to master several steps:

1. First a class of function has to be determined on which the formal star product is known to converge. In many examples this is not too bad and one has good candidates. In fact, in the examples we discuss below this class of functions is always more or less related to polynomials and thus the formal power series simply will *terminate* after finitely many contributions. However, this naive class of functions is typically very small and too non-interesting for serious applications.

2. Thus the second step consists in finding a suitable locally convex topology on the above naive class of functions for which the product is continuous. This is typically a real extra piece of information and it seems that there is no canonical way to achieve this. In the examples one has several possibilities and it is not completely clear how to characterize an optimal solution among many.

3. The third step then consists in extending the product by continuity to the completion of the naive class of functions. Here one has hopefully found a coarse enough topology such that the completion is interesting enough.

4. In a last step one can now try to examine the completion and determine whether its elements can still be considered as functions on the original phase space. This essentially means that the evaluation functionals at points $p \in M$ are continuous with respect to the found locally convex topology. In addition one wants to determine things like positive functionals etc.

While there seems to be no general theory in sight which would allow to make conceptual statements about this program, it nevertheless turns out to work well in several important classes of examples. What is perhaps more important is the perspective towards infinite-dimensional situations. Here the integral formulas clearly stop to make sense and one needs a replacement. The question about convergence, however, still can be asked and, in many examples, be answered to the positive. Thus the above program might not only be interesting within the realm of deformation quantization of classical mechanical systems but also beyond in field-theoretic situations.

### 3 The Weyl Star Product

We start with the perhaps most important example in deformation quantization, the Weyl star product and its relatives. These are star products quantizing the constant (symplectic) Poisson structure on a vector space. In this section we follow closely the construction from [143] and [139]. The previous constructions from [1] only work in the finite-dimensional case and for a particular case, the Wick star product, where they produce a slightly finer topology.

To set the stage we consider a real vector space $V$ with a bilinear form $\Lambda: V \times V \rightarrow \mathbb{C}$. In many cases the values are real but we leave the option to have complex values at the moment. The Weyl
product will then be defined on polynomials. Now in finite dimensions, the real-valued polynomials on the dual space \( V^* \) are just the symmetric algebra \( S_\mathbb{R}^*(V) \) while for infinite-dimensional vector spaces this is of course no longer true. For technical reasons it is convenient to work with the symmetric algebra \( S_\mathbb{R}^*(V) \) over \( V \) as a replacement. They can be viewed as the polynomials on the pre-dual of \( V \) if such a pre-dual exists: we do not require this but consider \( S_\mathbb{R}^*(V) \) only.

We then consider the complexified symmetric algebra \( S_\mathbb{C}^*(V) = S_\mathbb{R}^*(V) \) of \( V \) as replacement for the complex-valued polynomials. Out of \( \Lambda \) we obtain the operator

\[
P_\Lambda : S_\mathbb{C}^*(V) \otimes S_\mathbb{C}^*(V) \to S_\mathbb{C}^*(V) \otimes S_\mathbb{C}^*(V)
\]

(3.1)
defined on factorizing symmetric tensors \( v_1 \cdots v_n \in S^n(V) \) and \( w_1 \cdots w_m \in S^m(V) \) by

\[
P_\Lambda(v_1 \cdots v_n \otimes w_1 \cdots w_m) = \sum_{k=1}^{n} \sum_{\ell=1}^{m} \Lambda(v_k, w_\ell)v_1 \cdots \widehat{v}_k \cdots \ell \cdots v_n \otimes w_1 \cdots \ell \cdots w_m
\]

(3.2)
and extended linearly. Here \( k \) means to omit the \( k \)-th factor and the empty products are defined to be \( 1 \in S^0(V) = \mathbb{C} \) and finally we set \( P_\Lambda(1 \otimes w) = 0 = P_\Lambda(v \otimes 1) \) for all \( v, w \in S_\mathbb{C}^*(V) \).

**Lemma 3.1** The operator \( P_\Lambda \) satisfies a Leibniz rule in each tensor factor and gives a Poisson structure by

\[
\{ v, w \} = \mu \circ (P_\Lambda - \tau \circ P_\Lambda \circ \tau)(v \otimes w)
\]

(3.3)
for \( v, w \in S_\mathbb{C}^*(V) \).

Here \( \mu \) denotes the symmetric tensor product and \( \tau \) is the canonical flip of two tensor factors. The proof consists in considering the following operators

\[
P_{12}, P_{13}, P_{23} : S_\mathbb{C}^*(V) \otimes S_\mathbb{C}^*(V) \otimes S_\mathbb{C}^*(V) \to S_\mathbb{C}^*(V) \otimes S_\mathbb{C}^*(V) \otimes S_\mathbb{C}^*(V)
\]

(3.4)
defined by \( P_{12} = P_\Lambda \otimes \text{id} \), \( P_{23} = \text{id} \otimes P_\Lambda \) and \( P_{13} = (\text{id} \otimes \tau) \circ (P_\Lambda \otimes \text{id}) \circ (\text{id} \otimes \tau) \). It can now be shown by a straightforward computation that they pairwise commute and satisfy the Leibniz rules

\[
P_\Lambda \circ (\mu \otimes \text{id}) = (\mu \otimes \text{id}) \circ (P_{13} + P_{23}) \quad \text{and} \quad P_\Lambda \circ (\text{id} \otimes \mu) = (\text{id} \otimes \mu) \circ (P_{13} + P_{12}).
\]

(3.5)
From this the properties of a Poisson bracket follow immediately. Moreover, they allow to obtain an associative deformation at once. Note that the operator \( P_\Lambda \) lowers the symmetric degrees in both tensor factors by one and hence the following exponential series

\[
v *_{z, \Lambda} w = \mu \circ e^{2P_\Lambda}(v \otimes w)
\]

(3.6)
is well-defined for \( v, w \in S_\mathbb{C}^*(V) \) and for all \( z \in \mathbb{C} \). This is the Weyl star product:

**Lemma 3.2** For all \( z \in \mathbb{C} \) the product \( *_{z, \Lambda} \) is an associative product for \( S_\mathbb{C}^*(V) \). In zeroth order of \( z \), i.e. for \( z = 0 \), it yields the symmetric tensor product and the commutator in first order of \( z \) is the Poisson bracket \((3.3)\).

Strictly speaking, the Weyl product would be obtained from an antisymmetric \( \Lambda \) taking only real values and \( z = i\hbar \). Allowing for symmetric contributions of \( \Lambda \) and possible real values will give other star products quantizing the constant Poisson structure like a Wick star product or a standard-ordered star product. Nevertheless, we will simply speak of the Weyl product in the following.

The important point is that the product \( *_{z, \Lambda} \) converges for trivial reasons on the polynomials, i.e. on \( S_\mathbb{C}^*(V) \), since the exponential series simply terminates after finitely many contributions.
It is now the second step of the program which requires some more effort. We need to find a suitable topology on $S^k(V)$ which will turn $*_\Lambda$ into a continuous product.

Here we have to require a locally convex topology for $V$ and continuity properties for $\Lambda$. In the finite-dimensional case this is automatic as then $V$ has a unique Hausdorff locally convex topology (even coming from Hilbert space structures) and every bilinear form is continuous. In the infinite-dimensional case this is an additional information we have to invest. Thus assume $V$ is locally convex and $\Lambda$ is continuous: we require continuity and not just separate continuity.

For each tensor power $T^k(V)$ and hence for each symmetric power $S^k(V) \subseteq T^k(V)$ of $V$ we can then use the projective topology: this is the locally convex topology determined by the seminorms

$$p^n = p \otimes \cdots \otimes p$$

(3.7)

for all continuous seminorms $p$ of $V$. As usual, it suffices to consider a defining system of continuous seminorms $p$ of $V$ to obtain a defining system $p^n$ for the projective topology of $T^k(V)$ and hence for $S^k(V)$. For $n = 0$ we define $p^0$ to be the usual absolute value on $\mathbb{R}$ and $\mathbb{C}$.

To generate a topology on the tensor algebra and the symmetric algebra is now less canonical. Here we fix a parameter $R \in \mathbb{R}$ to parametrize different possibilities for locally convex topologies [143 Def. 3.5].

**Definition 3.3 (T$_R$- and S$_R$-topology)** Let $R \in \mathbb{R}$ and let $V$ be a locally convex vector space. On the tensor algebra $T^*(V)$ of $V$ one defines the seminorm

$$p_R(v) = \sum_{n=0}^{\infty} n!^R p^n(v_n)$$

(3.8)

for a continuous seminorm $p$ on $V$ where $v = \sum_{n \in \mathbb{N}_0} v_n$ are the homogeneous components of $v \in T^*(V)$. The locally convex topology generated by all these seminorms on $T^*(V)$ for $p$ varying through all continuous seminorms on $V$ is called the $T_R$-topology. The subspace topology induced on $S^*(V)$ is called the $S_R$-topology.

The tensor algebra (symmetric algebra) equipped with the $T_R$-topology ($S_R$-topology) will be denoted by $T_R(V)$ and $S_R(V)$, respectively.

**Remark 3.4 (Properties of the T$_R$- and S$_R$-topology)** Let $V$ be a locally convex vector space. The following properties of the $T_R$- and $S_R$-topology were obtained in [143]:

i.) For every $R \in \mathbb{R}$ and every $k \geq 0$ the induced topology on $T^k(V) \subseteq T^*(V)$ is the projective topology. The same holds for the symmetric version. Conversely, the inclusion $T^k(V) \subseteq T^*(V)$ is continuous with respect to the projective topology and the $T_R$-topology, respectively.

ii.) For every $R \in \mathbb{R}$ the $T_R$-topology is Hausdorff iff $V$ is Hausdorff. In this case also the $S_R$-topology is Hausdorff.

iii.) An equivalent system of seminorms for the $T_R$-topology is obtained by taking all the seminorms

$$p_{R,\infty}(v) = \sup_{n \in \mathbb{N}_0} n!^R p^n(v_n)$$

(3.9)

for all continuous seminorms $p$ on $V$.

iv.) Let $R' > R$. Then the $T_R$-topology is coarser than the $T_{R'}$-topology.

v.) The completion of $T^*(V)$ with respect to the $T_R$ topology is explicitly given by

$$\hat{T}_R(V) = \left\{ v = \sum_{n=0}^{\infty} v_n \mid \sum_{n=0}^{\infty} n!^R p^n(v_n) < \infty \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} V^{\otimes n},$$

(3.10)
where $V^\otimes n$ denotes the completion of the $n$-th tensor power in the projective topology. Analogously, one obtains the completion $\hat{S}_R^\bullet(V)$ of $S_R^\bullet(V)$ as those series in $\hat{T}_R^\bullet(V)$ consisting of symmetric tensors in each degree $n \in \mathbb{N}_0$.

vi.) The tensor product as well as the symmetric tensor product are continuous products for $R \geq 0$. For $R = 0$ the tensor algebra $T_R^\bullet(V)$ is the free locally multiplicatively convex algebra generated by $V$ and $S_0(V)$ is the free locally multiplicatively convex commutative algebra generated by $V$ as used e.g. by Cuntz in [63]. For $R > 0$ the topologies are not locally multiplicatively for the (symmetric) tensor product anymore.

vii.) The $T_R^\bullet$- and the $S_R^\bullet$-topology are first countable iff the original topology of $V$ is first countable.

viii.) Let $R \geq 0$. Then for every continuous linear functional $\varphi \in V'$ on $V$, the corresponding evaluation functional

$$\delta_\varphi : T_R^\bullet(V) \ni v = \sum_{n=0}^{\infty} v_n \mapsto \delta_\varphi(v) = \sum_{n=0}^{\infty} \varphi^\otimes n(v_n) \in \mathbb{C}$$

(3.11)

is a continuous algebra homomorphism with respect to the tensor product. In particular, $\delta_\varphi : S_R^\bullet(V) \to \mathbb{C}$ is a character and hence we can identify $S^\bullet(V)$ with certain polynomial functions on $V'$.

All these properties are rather straightforward to check. Note that the statement about the completion shows that we indeed get an interesting completion the smaller $R$ becomes. Already for $R < 1$ we have e.g. the exponential series $\exp(v)$ for $v \in V$ in the completion.

Slightly more important are the next two properties which show that these topologies preserve interesting properties of the underlying vector space $V$. Note that in the finite-dimensional case we are in the situation to apply the following results [143 Thm. 4.10]:

Theorem 3.5 Let $V$ be a Hausdorff locally convex space and let $R \geq 0$.

i.) The space $V$ admits an absolute Schauder basis iff $T_R(V)$ admits an absolute Schauder basis iff $S_R(V)$ admits an absolute Schauder basis.

ii.) The space $V$ is nuclear iff $T_R(V)$ is nuclear iff $S_R(V)$ is nuclear.

We come now to the main result of this section: the continuity of the Weyl product. Here we have the following statement first formulated in this generality in [143 Thm. 3.17]:

Theorem 3.6 (Weyl product) Let $R \geq \frac{1}{2}$. Then the Weyl product $*_z^\Lambda$ is continuous with respect to the $S_R^\bullet$-topology. Moreover, for all elements $v, w \in \hat{S}_R(V)$ in the completion the series

$$v *_z^\Lambda w = \sum_{n=0}^{\infty} \frac{z^n}{n!} \mu(P^\Lambda_n(v \otimes w))$$

(3.12)

converges absolutely in the $S_R^\bullet$-topology. The dependence on $z \in \mathbb{C}$ is entire.

This settles the questions raised in Section 2 for the Weyl product $*_z^\Lambda$ completely. Note that the continuity of the characters allows to interpret the elements in the completion $\hat{S}_R^\bullet(V)$. We conclude this section now with a few remarks:

Remark 3.7 (Grassmann version) In [143] a slightly more general situation was considered: the vector space $V$ was allowed to carry a $\mathbb{Z}_2$-grading which then was used to incorporate additional signs in the definition of the symmetric tensor product and $P^\Lambda$. Then the graded symmetric algebra is the usual symmetric algebra for even vectors but the Grassmann algebra for odd vectors. In total one can obtain a combination of both, thereby allowing also a deformation quantization of both. The even
part is then quantized by the above Weyl product, for the Grassmann part one obtains a quantization by a Clifford algebra. Needless to say that the analysis and continuity estimates will ignore the signs and are thus valid also in this slightly more general situation.

Remark 3.8 (Finite dimensions) In finite dimensions [124–126] considered already a particular case of the Weyl product and discussed the convergence issues. They found a classical functional space of entire functions subject to certain growth conditions at infinity for which the Weyl product converges. It turns out that this functional space coincides with the above completion $\hat{S}_R(V)$ for an appropriate choice of $R$.

Remark 3.9 (Pro-Hilbert case) A more recent development is proposed in [139] where the locally convex space $V$ is considered to be a projective limit of Hilbert spaces. This includes the case where $V$ is actually a Hilbert space. But also every nuclear space is projectively Hilbert. In this situation one can use the inner products defining the Hilbert seminorms to establish a different topology on each tensor power $V^\otimes n$: one simply extends the inner products and computes the corresponding seminorm afterwards. This results in a slightly coarser topology than the projective topology on $V^\otimes n$. In some sense this corresponds more to a Hilbert-Schmidt like topology rather than a trace-class topology. Now the completion becomes more interesting already for fixed $n$. The continuity of the Weyl product can now be shown also in this case with an extension of the topology to the whole symmetric algebra as before: in [139] we focused on the case $R = \frac{1}{2}$ directly, yielding the coarsest possible topology. Many additional feature including a detailed description of the Gel’fand transform in the classical case can be done. This allows to determine the completion very explicitly as certain real-analytic functions on the topological dual $V'$. Finally, in the nuclear case we can benefit from both versions as here the two competing topologies on the tensor powers simply coincide. Thus all properties obtained above as well as those from [139] become available. Luckily, the physically relevant cases seem all to be of that type: finite-dimensional spaces as well as the usual test function spaces used in (quantum) field theory are all nuclear.

4 The Gutt Star Product

As a next important class of examples one can consider the linear Poisson structures on the dual of a Lie algebra. This has been investigated in Poisson geometry from many points of view and provides one of the most important examples as it shows that Lie algebra theory becomes accessible via Poisson geometric techniques. In this section we discuss the continuity and convergence results form [83].

To define the linear Poisson structure we consider a real Lie algebra $\mathfrak{g}$ and instead of complex-valued polynomial functions on its dual $\mathfrak{g}^*$ we focus on the symmetric algebra $S^*(\mathfrak{g})$ for the same reasons as already in the Weyl product case. In particular, we allow for infinite-dimensional Lie algebras as well.

The star product we want to consider originates from the Poincaré-Birkhoff-Witt isomorphism between the (complexified) symmetric algebra and the (complexified) universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$. In some more detail one considers the $n$-th symmetrization map defined on factorizing symmetric tensors by

$$q_n : S^n(\mathfrak{g}) \ni \xi_1 \cdots \xi_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \odot \cdots \odot \xi_{\sigma(n)} \in \mathfrak{u}(\mathfrak{g}), \quad (4.1)$$

and extended linearly. Here $\odot$ denotes the product of the universal enveloping algebra. Then the Poincaré-Birkhoff-Witt theorem says that the direct sum $q = \sum_{n=0}^\infty q_n$ provides a vector space isomorphism between $S^*(\mathfrak{g})$ and $\mathfrak{u}(\mathfrak{g})$. 
Denote the projection onto the $n$-th homogeneous component in $S^\bullet (\mathfrak{g})$ by $\text{pr}_n : S^\bullet (\mathfrak{g}) \rightarrow S^n (\mathfrak{g})$. Then for $z \in \mathbb{C}$ and homogeneous elements $x \in S^k (\mathfrak{g})$ and $y \in S^\ell (\mathfrak{g})$ one defines the product

$$x \star_z y = \sum_{n=0}^{k+\ell-1} z^n \text{pr}_{k+\ell-n} \left( q^{-1}(q(x) \circ q(y)) \right)$$

(4.2)

and extends this bilinearly to $S^\bullet (\mathfrak{g})$. Again, we see that this is a well-defined product for all $z \in \mathbb{C}$ which turns out to be associative. In fact, the case $z = 1$ is clear as there it becomes isomorphic to the universal enveloping algebra directly. The other values of $z$ can be understood as a rescaling of the Lie bracket by $z$. In particular, for $z = 0$ one is back at the symmetric algebra. The first order term in $z$ is then the usual linear Poisson bracket for $S^\bullet (\mathfrak{g})$ obtained from the Lie bracket of $\mathfrak{g}$. This can equivalently be obtained by extending $[\cdot , \cdot ]_g$ to $S^\bullet (\mathfrak{g})$ enforcing the Leibniz rule as usual.

**Definition 4.1 (Gutt star product)** The star product $\star_z$ is called the Gutt star product for $\mathfrak{g}$.

In fact, Gutt introduced this star product in [96] as an intermediate step to obtain a star product for the cotangent bundle of a Lie group $G$ integrating $\mathfrak{g}$. Alternatively and independently, Drinfel’d introduced this star product in [77] in the context of quantum group theory.

The first important observation is the relation of the Gutt star product with the Baker-Campbell-Hausdorff series. In fact, one has the following statement:

**Lemma 4.2** Viewing $\exp(\xi)$ as a formal series in $\prod_{n=0}^\infty S^n (\mathfrak{g})$ for $\xi \in \mathfrak{g}$ the Gutt star product is determined by

$$\exp(\xi) \star_z \exp(\eta) = \exp \left( z^2 \text{BCH}(z\xi, z\eta) \right),$$

(4.3)

where BCH is the usual Baker-Campbell-Hausdorff series of $\mathfrak{g}$.

Here the idea is that by differentiation of the left hand side one can obtain all monomials $\xi^k \eta^\ell$ with $k, \ell \in \mathbb{N}_0$. Then the corresponding right hand side gives an explicit formula for the Gutt star products of such monomials. By polarization, this determines $\star_z$ also for $\xi_1, \ldots, \xi_k \in S^k (\mathfrak{g})$ and $\eta_1, \ldots, \eta_\ell \in S^\ell (\mathfrak{g})$ for arbitrary $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. Hence the Gutt star product is completely encoded in the Baker-Campbell-Hausdorff series of $\mathfrak{g}$.

**Remark 4.3** The Gutt star product can also be formulated using integral formulas as this has (implicitly) been done already by Berezin [6] and later on by Rieffel in [135]. However, for the analysis we have in mind the above version based on the Baker-Campbell-Hausdorff series is the most suitable. In particular, the integral formulas only make sense in finite dimensions while (4.3) holds algebraically in general.

Now we consider a locally convex topology on $\mathfrak{g}$ in addition. As in the constant case of the Weyl product we want to use the $S_K$-topology also for the linear case. And, again as in the constant case, we need some continuity for the Lie bracket. It turns out that mere continuity of the bilinear map $[\cdot , \cdot ]_g : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is unfortunately not sufficient. Instead, we have to require a slightly stronger condition. To this end we recall the definition of an asymptotic estimate algebra, see [83, Def. 1.1] as well as [31].

**Definition 4.4 (Asymptotic estimate algebra)** Let $\mathcal{A}$ be a Hausdorff locally convex algebra (not necessarily associative) with multiplication $\cdot$.

i.) A continuous seminorm $q$ is called an asymptotic estimate for the continuous seminorm $p$ if for all $n \in \mathbb{N}$ and all words $w_n (x_1, \ldots, x_n)$ of $n - 1$ products of the $n$ elements $x_1, \ldots, x_n \in \mathcal{A}$ with arbitrary position of placing brackets one has

$$p(w_n (x_1, \ldots, x_n)) \leq q(x_1) \cdots q(x_n).$$

(4.4)
ii.) The algebra $\mathcal{A}$ is called an asymptotic estimate algebra (AE-algebra) if every continuous semi-norm has an asymptotic estimate.

Clearly, the case $n = 2$ already shows that the bilinear product is continuous. Thus we have a stronger form of continuity in an AE-algebra.

**Remark 4.5 (Finite-dimensional algebras are AE)** If $\mathcal{A}$ is an associative AE-algebra then $\mathcal{A}$ becomes an AE-Lie algebra with respect to the commutator. Indeed, this is a trivial estimate. Moreover, if $\mathcal{A}$ is locally multiplicatively convex then it is AE for trivial reasons. In particular, every finite-dimensional algebra is AE since it is locally multiplicatively convex. Thus the requirement is always fulfilled in finite dimensions but it becomes interesting in the infinite-dimensional case.

Having now an AE-Lie algebra one arrives at the following continuity statement for the Gutt star product, where we have to use the $S_R$-topology for $R \geq 1$ to obtain continuity [83, Thm. 1.2]:

**Theorem 4.6 (Gutt star product)** Let $R \geq 1$ and let $\mathfrak{g}$ be an AE-Lie algebra.

i.) The Gutt star product $\star_z$ is continuous with respect to the $S_R$-topology for all $z \in \mathbb{C}$.

ii.) The completion $\hat{S}_R(\mathfrak{g})$ becomes a locally convex Hopf algebra with respect to $\star_z$ and the undeformed coproduct, antipode, and counit.

iii.) The map $\mathbb{C} \ni z \mapsto x \star_z y \in \hat{S}_R(\mathfrak{g})$ is holomorphic for all $x, y \in \hat{S}_R(\mathfrak{g})$.

iv.) The construction is functorial for continuous Lie algebra morphisms.

Here we use the fact that the universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra in the usual way. Transferring this Hopf algebra structure back to $S(\mathfrak{g})$ gives the usual cocommutative coproduct $\Delta$ with corresponding antipode and counit being the projection $\epsilon = pr_0$. In particular, $\Delta$ does not depend on the Lie algebra structure at all. For a locally convex Hopf algebra, the coproduct is then allowed to take values in the completed tensor product instead of just the algebraic tensor product.

The proof requires a rather technical and detailed analysis of the Baker-Campbell-Hausdorff series. The idea is to find estimates for the homogeneous parts $\text{BCH}_n(\xi, \eta)$ of $\text{BCH}(\xi, \eta)$ where we have exactly $n$ letters $\xi$ or $\eta$ and hence $n - 1$ Lie brackets. The $n - 1$ brackets can be estimated by the asymptotic estimates (4.4) and one is left with the numerical constants in front. Here a result of Goldberg [95] shows that

$$\text{BCH}_n(\xi, \eta) = \sum_{\text{Lie words } w \text{ in } \xi, \eta \text{ of length } n} g_w \frac{w}{n}$$

with universal coefficients $g_w \in \mathbb{Q}$ satisfying the estimate

$$\sum_{\text{Lie words } w \text{ of length } n} \left| g_w \right| \frac{n}{n} \leq \frac{2}{n}.$$  \hspace{1cm} (4.6)

Using now the asymptotic estimate one finds for all Lie words $w$ containing $a$ times the letter $\xi$ and $b$ times the letter $\eta$ the estimate

$$p(w) \leq q(\xi)^a q(\eta)^b.$$  \hspace{1cm} (4.7)

Form these two estimates one concludes now the continuity estimate for the Gutt star product, once the correct combinatorics from (4.3) is taken into account, detail can be found in [83].

The result is somewhat disappointing as the condition $R \geq 1$ is sharp: e.g. for the three-dimensional Heisenberg Lie algebra with basis $Q, P, E$ and only nontrivial Lie bracket $[P, Q] = E$ one can show that for $R < 1$ the product is discontinuous. The bound $R \geq 1$ then forbids exponential series $\exp(\xi)$ to be in the completion. Of course, this would have been too nice to have as then the Lie group corresponding to $\mathfrak{g}$ would have become part of the completed algebra. On the other hand,
this is of course not to be expected as it would ultimately imply that the Baker-Campbell-Hausdorff series would have radius of convergence being infinite. And this one knows to be false in general.

Nevertheless, in the case of a nilpotent Lie algebra one can refine the above analysis in the following way. Instead of the $S_{\mathcal{R}}$-topology for $R \geq 1$ one can consider the projective limit of all $S_{\mathcal{R}}$-topologies for $R < 1$. It turns out that for a locally convex nilpotent Lie algebra the Gutt star product is also continuous with respect to this projective limit topology $S_{1}$ and all the above results stay valid, see [3]. Note that a nilpotent Lie algebra is trivially AE as soon as the Lie bracket is continuous at all.

5 The Wick Type Star Product on the Poincaré Disc

The last example we want to discuss involves still a topologically trivial phase space. However, now the underlying symplectic structure comes from a curved Kähler structure. We consider the Poincaré disc and its higher dimensional analogs. The star product of Wick type on the Poincaré disc has a long history and was re-discovered many times, see in particular the early contributions in [113–115] and [50] as well as the second part [54]. Later on, the first explicit formula was found in [21, 22] together with some first considerations on the convergence. We will build on this construction of the star product by a phase space reduction form.

Moreover, we make use of the summation convention throughout. The function $g$ use an indefinite metric specified by the matrix $g = \text{diag}(-1, 1, \ldots, 1) \in M_{n+1}(\mathbb{C})$ which we identify with the corresponding quadratic function

$$g = g_{\mu \nu} \mathrm{d}z^\mu \wedge \mathrm{d}z^\nu \in \mathcal{C}^\infty(\mathbb{C}^{n+1}).$$

(5.1)

Here and in the following Greek indices run from 0 to $n$ while Latin indices will vary over 1 to $n$ only. Moreover, we make use of the summation convention throughout. The function $g$ is then invariant under the canonical and linear $U(1, n)$-action on $\mathbb{C}^{n+1}$. This allows to consider the submanifold

$$Z = g^{-1}(\{1\})$$

(5.2)

of $\mathbb{C}^{n+1}$. Note that 1 is indeed a regular value of $g$ and hence $Z$ is a submanifold. Moreover, $Z$ is clearly $U(1, n)$-invariant.

In a next step we consider the central diagonal $U(1) \subseteq U(1, n)$. Since this subgroup acts freely (and properly for trivial reasons) on $Z$, the quotient $D_n = Z/U(1)$ is a smooth manifold again and pr: $Z \rightarrow D_n$ is a (U(1))-principal fiber bundle. The complementary subgroup $SU(1, n) \subseteq U(1, n)$ still acts on $D_n$ and provides a transitive smooth action. The manifold $D_n$ can now be embedded into $\mathbb{CP}^n$ as follows. Note that $U(1, n)$ smoothly acts on $\mathbb{CP}^n$ by holomorphic diffeomorphisms with three orbits, determined by the quadratic function $g$. The orbits are characterized by the values of $g$: either positive or negative or 0 on the complex lines in $\mathbb{C}^{n+1}$ representing the point in $\mathbb{CP}^n$. Since $g$ is homogeneous, this is clearly well-defined. Then an equivalence class $\text{pr}(p) \in D_n$ is mapped to the equivalence class of the complex line through a representative $p \in Z$. This gives a well-defined and $U(1, n)$-equivariant embedding and identifies $D_n$ with the orbit of $U(1, n)$ where $g$ is positive. It follows that $D_n$ is an open subset of $\mathbb{CP}^n$ and as such it inherits the structure of a complex manifold.

In a last step we need the symplectic structure for $D_n$. This is now comparably easy as we can use the Marsden-Weinstein reduction from $\mathbb{C}^{n+1}$. We take the constant symplectic structure $\omega = \frac{1}{2}g_{\mu \nu} \mathrm{d}z^\mu \wedge \mathrm{d}z^\nu$. For this, the function $g$ is a momentum map for the $U(1)$ action. Hence $Z$ corresponds to momentum level zero and the subsequent quotient yields the Marsden-Weinstein
reduced phase space $D_n$. It is now a final but straightforward check that the symplectic structure obtained this way is compatible with the complex structure so that we finally end up with a Kähler manifold $D_n$. Note that also the group $\text{SU}(1, n)$ acts in a Hamiltonian fashion on $D_n$.

We can now construct the star product for the disc as follows: we start with the Wick star product on $\mathbb{C}^{n+1}$ with respect to the pseudo Kähler structure defined by $g$ explicitly given by

$$f \star_{\text{Wick}} g = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} g^{\mu_1 \nu_1} \ldots g^{\mu_r \nu_r} \frac{\partial^r f}{\partial z^{\mu_1} \ldots \partial z^{\mu_r}} \frac{\partial^r g}{\partial \bar{z}^{\nu_1} \ldots \partial \bar{z}^{\nu_r}}, \quad (5.3)$$

where $g^{\mu\nu} = g_{\mu\nu}$ and summation of the $\mu_1, \ldots, \nu_r$ understood as usual. This is a particular case of the star product for constant Poisson structures on a vector space as discussed in Section 3. In particular, we have trivial convergence on polynomials.

To construct the star product on the disc $D_n$ one uses the $U(1,n)$-invariance of $\star_{\text{Wick}}$ and, in particular, the resulting $U(1)$-invariance. Classically, the functions $\mathcal{C}^\infty(D_n)$ can be obtained as the $U(1)$-invariant functions on $Z$ and hence as quotient of the $U(1)$-invariant functions on $\mathbb{C}^{n+1}$ modulo those which vanish on $Z$. The functions vanishing on $Z$ can now be obtained as the ideal generated by $g-1$ since 1 is a regular value of $g$. This gives the classical restriction map $\Psi_0 : \mathcal{C}^\infty(\mathbb{C}^{n+1})^U(1) \rightarrow \mathcal{C}^\infty(D_n)$ which turns then out to be a $SU(1,n)$-equivariant Poisson homomorphism. In a second step, this classical restriction is deformed into a quantum restriction inducing the star product on the disc.

Instead of formulating this in general as done in [21],[22] we give the explicit formula for this quantum restriction on $U(1)$-invariant polynomials.

For multiindices $P, Q \in \mathbb{N}_0^{n+1}$ we write

$$d_{P,Q} = z^P z^Q = (z^0)^{P_0} \ldots (z^n)^{P_n}(\bar{z}^0)^{Q_0} \ldots (\bar{z}^n)^{Q_n} \quad (5.4)$$

for the usual basis of monomials. Then $d_{P,Q}$ is $U(1)$-invariant iff $|P| = |Q|$. We denote their images in $\mathcal{C}^\infty(D_n)$ by

$$f_{P,Q} = \Psi_0(d_{P,Q}). \quad (5.5)$$

The difficulty is now that the functions $f_{P,Q}$ are no longer linearly independent. Instead, we get relations between them of the following form. First we specify for multiindices $P, Q \in \mathbb{N}_0^n$ without zeroth component the special functions

$$f_{r,P,Q} = \begin{cases} f_{(|Q|-|P|,P_1,\ldots,P_n),(0,0,Q_1,\ldots,Q_n)} & \text{for } |Q| \geq |P| \\ f_{(0,0,\ldots,0),(|P|-|Q|,Q_1,\ldots,Q_n)} & \text{for } |Q| \geq |P|. \end{cases} \quad (5.6)$$

They turn out to be linearly independent and we can write every image $f_{P,Q}$ as linear combination

$$f_{P,Q} = \sum_{T \in \mathbb{N}_0^n \atop |T| \leq \min(|P_0,Q_0|)} \left( \min\{P_0,Q_0\} \over |T| \right)! f_{r,P',T,Q',T}, \quad (5.7)$$

where $P' = (P_1,\ldots,P_n) \in \mathbb{N}_0^n$ and analogously for $Q$. This shows that we obtain a basis for the span of the images of $U(1)$-invariant polynomials.

In order to define the quantum restriction map we have to specify the possible values of $\hbar$ first. While for the ($U(1)$-invariant) monomials on $\mathbb{C}^{n+1}$ we have convergence for all $\hbar \in \mathbb{C}$, the quantum restriction will turn out to be defined only for the following values of $\hbar$. We define the admissible values for $\hbar$ to be the set

$$H = \mathbb{C} \setminus \left\{ 0, -\frac{1}{2m} \mid m \in \mathbb{N} \right\}. \quad (5.8)$$

In particular, the classical limit $\hbar = 0$ is not part of this open subset but a boundary point. Moreover, we have excluded the isolated points at $-\frac{1}{2m}$ for $m \in \mathbb{N}$. Then we can define the quantum restriction
\[ \Psi_h(d_{P,Q}) = (2\hbar)^{|P|} \left( \frac{1}{2\hbar i} \right)_{|P|} \Psi_0(d_{P,Q}) = (2\hbar)^{|P|} \left( \frac{1}{2\hbar i} \right)_{|P|} f_{P,Q}, \]  
where \( \hbar \in H \) and \((z)_m = z(z+1) \cdots (z+m-1)\) is the Pochhammer symbol. The relevance of this definition is now the following:

**Lemma 5.1** The kernel of the linear map \( \Psi_h \) coincides with the two-sided \(*_{\text{Wick}}\)-ideal inside the \( U(1) \)-invariant polynomials generated by \( g - 1 \).

The original construction of \([21,22]\) did not use the polynomial functions to construct the reduction, resulting in a slightly more complicated definition. However, there the ultimate formula was derived from a more conceptual point of view instead of our short-cut by guessing (5.9) and verifying its properties afterwards. When working only with polynomials as we do here, the verification of the lemma is a fairly simple computation.

Thanks to this observation we can define a product on the span of the image of \( \Psi_0 \), i.e. for the span of the functions \( f_{P,Q} \) simply by pushing forward \(*_{\text{Wick}}\). The algebra obtained this way is the quotient of the \( U(1) \)-invariant polynomials modulo the \(*_{\text{Wick}}\)-ideal generated by \( g - 1 \). Explicitly, this results in the formula

\[ f_{P,Q} \ast_{\text{red}} f_{R,S} = \sum_{T=0}^{\min\{P,S\}} (-1)^T \left( \frac{\hbar}{2} \right)^{|P+S-T|} \left( \frac{1}{2\hbar i} \right)^{|S|} \left( \frac{P}{T} \right)_{|P|} \left( \frac{S}{T} \right)_{|S|} f_{P+R-T,Q+S-T} \]  
for all \( P,Q,R,S \in \mathbb{N}_0^{1+n} \) with \(|P| = |Q|\) and \(|R| = |S|\). Note however, that the functions \( f_{P,Q} \) do not yet form a basis, one has to expand them according to (5.7) using the basis of the functions \( f_{r,P,Q} \) with \( P,Q \in \mathbb{N}_0^n \) instead.

Now we can use the continuity estimates for \(*_{\text{Wick}}\) from Section [3]. Since we have here a finite-dimensional situation it is convenient to describe the seminorms \( p_R \) defining the \( S_R \)-topology more explicitly. We focus on the limiting case \( R = \frac{1}{2} \) directly. Let \( \rho > 0 \). Then we define for a polynomial \( a \) on \( \mathbb{C}^{n+1} \) the norm

\[ \left\| \sum_{P,Q \in \mathbb{N}_0^{n+1}} a_{P,Q} d_{P,Q} \right\|_{\mathbb{C}^{n+1},\rho} = \sum_{P,Q \in \mathbb{N}_0^{n+1}} |a_{P,Q}| \rho^{|P+Q|} \sqrt{|P+Q|!}. \]  

Note that for a polynomial the series is in fact a finite sum.

**Lemma 5.2** The collection \( \left\| \cdot \right\|_{\mathbb{C}^{n+1},\rho} \) of norms for \( \rho > 0 \) defines the \( S_R \)-topology for the polynomials on \( \mathbb{C}^{n+1} \) for \( R = \frac{1}{2} \).

For the functions on the disc we consider the basis \( f_{r,P,Q} \) and specify norms on the span of these functions, i.e. on the quotient algebra. We set

\[ \left\| \sum_{P,Q \in \mathbb{N}_0^n} a_{P,Q} f_{r,P,Q} \right\|_{D_{\rho}} = \sum_{P,Q \in \mathbb{N}_0^n} |a_{P,Q}| \rho^{|P+Q|}. \]  

for \( \rho > 0 \). By a direct estimate one can then show that the resulting locally convex topology for the span of the \( f_{r,P,Q} \) coincides with the quotient topology induced by \( \Psi_h \):

**Lemma 5.3** The locally convex topology defined by the norms \( \left\| \cdot \right\|_{D_{\rho}} \) for \( \rho > 0 \) coincides with the quotient topology induced by \( \Psi_h \).
In particular, the star product \( \ast_{\text{red}} \) becomes continuous with respect to this topology. One can find explicit norm estimates for \( \ast_{\text{red}} \) but the next theorem already follows already from the previous two lemmas and our investigations concerning \( \ast_{\text{Wick}} \) in Section 3.

**Theorem 5.4** Let \( h \in H \). Then the star product \( \ast_{\text{red}} \) for the span of the functions \( f_{r,P,Q} \) is continuous with respect to the quotient topology explicitly described by the norms \( (5.12) \).

In [3] a slightly finer topology was used, the above one seems to be more appropriate as it will allow for a larger completion. In fact, the completion in [3] was identified with a certain nuclear Köthe space. Nevertheless, the interpretation as functions was still not satisfactory. The above topology will now allow to determine the functions in the completion very explicitly and geometrically.

To describe the completion we first note that all the functions \( d_{P,Q} \) as well as \( f_{P,Q} \) are real-analytic of a very particular form. Being real-analytic means that they can be extended to an open neighbourhood of the diagonal in the Cartesian product of the underlying Kähler manifold with itself. Now the polynomials as well as the \( f_{r,P,Q} \) can be extended holomorphically to a much larger complex manifold than an open neighbourhood of the diagonal. To describe these new complex manifolds we have to double all previous ones as follows:

First we consider \( \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \) with the \( U(1,n) \)-action defined by \( U \triangleright (x,y) = (Ux,\overline{U}y) \). This way, we have the \( U(1,n) \)-equivariant diagonal map \( \Delta(p) = (p,\overline{p}) \). Let \( \tau: \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \) be the anti-holomorphic flip diffeomorphism defined by \( \tau(x,y) = (\overline{y},\overline{x}) \). Then the image of \( \Delta \) coincides with those points \((x,y)\) with \( \tau(x,y) = (x,y) \), i.e. the fixed points of \( \tau \). We can now extend the function \( g \) to a holomorphic function

\[
\hat{g} = g_{\mu\nu}x^\mu y^\nu' \in \mathcal{O}(\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}).
\]  

(5.13)

Then \( \hat{g} \circ \Delta = g \) and thus \( \hat{g} \) is the unique holomorphic extension of the real-analytic function \( g \). Using \( \hat{g} \) we can define the complex submanifold

\[
\hat{Z} = \hat{g}^{-1}(\{1\}) \subseteq \mathbb{C}^{n+1} \times \mathbb{C}^{n+1},
\]  

(5.14)

which is indeed a complex submanifold as 1 is a regular value of \( \hat{g} \) which is invariant under \( \tau \). Clearly, \( U(1,n) \) acts on \( \hat{Z} \). Moreover, the diagonal \( \Delta \) maps \( Z \) into \( \hat{Z} \) and the image consists of the fixed points in \( \hat{Z} \) under \( \tau \).

In a next step we extend the \( U(1) \) action on \( Z \) to an action of the multiplicative Lie group \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) on \( \hat{Z} \) as follows. For \( z \in \mathbb{C}^* \) we define \( z \triangleright (x,y) = (zx,\frac{1}{z}y) \) resulting in a holomorphic action. The action commutes with the \( U(1,n) \) action and provides a free and proper action on \( \hat{Z} \). Thus the quotient

\[
\hat{D}_n = \hat{Z} / \mathbb{C}^*,
\]  

(5.15)

becomes a complex manifold which inherits the \( U(1,n) \) action. Of course, only the subgroup \( SU(1,n) \) acts non-trivially. The anti-holomorphic involution \( \tau \) also descends to an anti-holomorphic involution \( \tau \) on \( \hat{D}_n \). The original disc \( D_n \) can now be included into \( \hat{D}_n \) using the diagonal \( \Delta \). Indeed, the diagonal inclusion of \( Z \) into \( \hat{Z} \) descends to a smooth inclusion \( \Delta_D: D_n \hookrightarrow \hat{D}_n \) as a real submanifold, compatible with the \( U(1,n) \) actions. Note that \( \hat{D}_n \) can also be seen as an open subset of \( \mathbb{CP}^n \times \mathbb{CP}^n \).

It will be this doubled version of the disc encoding the completion. First we note the following observation:

**Lemma 5.5** A holomorphic function \( \hat{a} \in \mathcal{O}(\hat{D}_n) \) is uniquely determined by its restriction \( \Delta_D^* \hat{a} \in \mathcal{O}(D_n) \).

This allows to consider those (necessarily real-analytic) functions on \( D_n \) which are restrictions of holomorphic functions on \( \hat{D}_n \), i.e. those which allow for a (necessarily unique) holomorphic extension to \( \hat{D}_n \). We define

\[
\mathcal{O}(D_n) = \left\{ a \in \mathcal{O}(\hat{D}_n) \mid a = \Delta_D^* \hat{a} \text{ for some } \hat{a} \in \mathcal{O}(\hat{D}_n) \right\}.
\]  

(5.16)
Then $\Delta^*_p : \Theta(D_n) \to \mathcal{A}(D_n)$ becomes an algebra isomorphism with respect to the commutative product. This allows to pull the standard locally convex topology of holomorphic functions back to $\mathcal{A}(D_n)$. Explicitly, we define seminorms of functions in $\mathcal{A}(D_n)$ by

$$
\|a\|_{D_n,K} = \sup_{u \in K} |\hat{a}(u)|
$$

(5.17)

for a compact subset $K \subseteq \hat{D}_n$, where $\hat{a}$ is the unique holomorphic function with $\Delta^*_p \hat{a} = a$ as before. This way, $\Delta^*_p$ becomes an isomorphism of Fréchet spaces. The topology is the Fréchet topology of locally uniform convergence on $\hat{D}_n$.

Since the polynomials $d_{P,Q}$ can be extended to holomorphic polynomials on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ it follows fairly easy that the functions $\hat{f}_{P,Q}$ can be extended holomorphically to $\hat{D}_n$, i.e. we have

$$
\hat{f}_{P,Q} = \Delta^*_p \hat{f}_{P,Q} \in \mathcal{A}(D_n)
$$

(5.18)

with some unique $\hat{f}_{P,Q} \in \Theta(\hat{D}_n)$. In particular, this applies to the basis $f_{r,P,Q}$ itself. It requires now an adaption of the Cauchy integral formula to see that the corresponding functions $\hat{f}_{r,P,Q} \in \Theta(\hat{D}_n)$ play the role of monomials: more precisely, they form an absolute Schauder basis with coefficient

$$
\begin{align*}
\hat{a} = \sum_{P,Q \in \mathbb{N}_0^n} \hat{a}_{P,Q} \hat{f}_{r,P,Q} & \quad \text{with} \quad a_{P,Q} = \frac{1}{(-4\pi)^n} \int \cdots \int \hat{a} \frac{(1 - uv)^{\max(|P|,|Q|)}}{v^{P+1}u^{Q+1}} d^n u d^n v,
\end{align*}
$$

(5.19)

with the canonical holomorphic coordinates $(u, v)$ of $\hat{D}_n$ inherited from the standard chart of $\mathbb{C} \mathbb{P}^n \times \mathbb{C} \mathbb{P}^n$ where we just divide by the zeroth component as usual. The contour integrals are around $(0,0)$ in this chart. Transferring this back to $\mathcal{A}(D_n)$ gives the same expansion for every $a \in \mathcal{A}(D_n)$ with the very same coefficients.

The last step consists now in recognizing that the topology on the polynomial functions for which $\star_{\text{red}}$ was shown to be continuous simply coincides with the canonical topology of the holomorphic functions. This ultimately yields the following description of the completion:

**Theorem 5.6** Let $h \in H$. The completion of the span of the functions $f_{r,P,Q}$ with respect to the quotient topology, i.e. the seminorm system given by (5.12), coincides with $\mathcal{A}(D_n)$ and the resulting Fréchet topology is the inherited one from $\Theta(\hat{D}_n)$. Thus $\star_{\text{red}}$ extends to $\mathcal{A}(D_n)$.

**Remark 5.7 (Further properties)** Let $h \in H$.

i.) First we note that the expansion (5.19) converges in the $\mathcal{A}(D_n)$-topology. Since the product is continuous, the product of two elements $a, b \in \mathcal{A}(D_n)$ in the completion can be computed by the absolutely convergent seres

$$
a \star_{\text{red}} b = \sum_{P,Q,R,S \in \mathbb{N}_0^n} a_{P,Q} b_{R,S} \hat{f}_{r,P,Q} \star_{\text{red}} \hat{f}_{r,R,S}
$$

(5.20)

with the remaining product of the basis functions given by (5.10). This gives a very explicit way to compute the star product on the disc.

ii.) Since the original Wick star product had the pointwise complex conjugation as $\star$-involution if $h = \overline{h}$ is real, it follows easily that also $\star_{\text{red}}$ has the pointwise complex conjugation as (continuous) $\star$-involution. This turns $(\mathcal{A}(D_n), \star_{\text{red}})$ into a Fréchet $\ast$-algebra.
iii.) For any two elements $a, b \in \mathcal{A}(D_n)$ the map

$$H \ni h \mapsto a \ast_{\text{red}} h b \in \mathcal{A}(D_n)$$

is a holomorphic map with respect to the Fréchet topology of $\mathcal{A}(D_n)$. Here we write explicitly $\ast_{\text{red}}$ to emphasize the $h$-dependence of the reduced star product. One can construct explicit elements such that this statement can not be improved: there are $a$ and $b$ such that $a \ast_{\text{red}} b$ has poles at every critical value $h = -\frac{1}{2\pi m}$. Note that for the original polynomials only finitely many poles occurred in a given product, depending on the maximal degree.

It is the last remark which makes the analysis of the semi-classical limit more complicated as the nice cases discussed in Section 3 and Section 4. Now the classical limit $h = 0$ is not an interior point of the domain where the product depends holomorphically on $h$. It is a boundary point with poles accumulating on the negative axis. Thus the limit $h \to 0$ has to be taken with care from the right, i.e. we only can expect a reasonable limit for $h \to 0^+$. The following theorem shows that this can be done.

**Theorem 5.8** Let $a, b \in \mathcal{A}(D_n)$. Then one has

$$\lim_{h \to 0^+} a \ast_{\text{red}} h b = ab \quad \text{and} \quad \lim_{h \to 0^+} \frac{i}{h} (a \ast_{\text{red}} h b - b \ast_{\text{red}} h a) = \{a, b\}.$$  

The above statement becomes trivial for finite linear combinations in the basis functions $f_{r,P,Q}$ as there the reduced star product is still holomorphic around $h = 0$ and there are only finitely many poles on the negative axis. However, the above example shows that for general $a$ and $b$ the argument is more subtle. In fact, its proof is fairly technical and requires a considerable effort.

We conclude this section with a few remarks on further properties of the star product $\ast_{\text{red}}$. In [118] it was shown that even though the topology is not locally multiplicatively convex for the star product, many interesting transcendental functions can be defined. In particular, the algebra $\mathcal{A}(D_n)$ contains many bounded functions even though the span of the Schauder basis only contains constant bounded functions. The reason is simply that we have real functions $a \in \mathcal{A}(D_n)$ which can then be exponentiated to yield bounded functions $\Delta^a \exp(i\hat{a}) \in \mathcal{A}(D_n)$. This gives some hope that we also have periodic functions with respect to Fuchsian groups. In this case, one would obtain immediately a construction of convergent star products on Riemann surfaces of higher genus. Ultimately, one can then compare this approach with the one of Bialysavsky [11].

For $h > 0$ it was shown that every classically positive linear functional of $C^\infty(D_n)$, i.e. a positive Radon measure on $D_n$, is also positive for the $^*$-algebra $(\mathcal{A}(D_n), \ast_{\text{red}})$. In particular, all evaluation functionals at the point of the disc $D_n$ are positive functionals. This shows that the resulting star product algebra has sufficiently many positive linear functionals to separate elements. In particular, it has faithful $^*$-representations on pre-Hilbert spaces, see [118] for more details on the $^*$-representation theory of unbounded operator algebras and [12, 43, 111, 114] for some algebraic background.

The components of the momentum map for the $SU(1,n)$ are contained in $\mathcal{A}(D_n)$: they are linear combinations of the Schauder basis $f_{r,P,Q}$ for small $P$ and $Q$. However, their $\ast_{\text{red}}$-exponentials are not defined as elements of $\mathcal{A}(D_n)$. Nevertheless, suppose that one has a $^*$-representation $\pi$ of $\mathcal{A}(D_n)$ for $h > 0$ on some pre-Hilbert space $\mathcal{H}$ such that $a \mapsto \langle \phi, \pi(a)\psi \rangle$ is continuous for all $\phi, \psi \in \mathcal{H}$. Here every GNS representation for a continuous positive linear functional gives an example. Then one can show that the representing operators for the components of the momentum map are essentially self-adjoint and integrate to a strongly continuous unitary representation of $SU(1,n)$ on the Hilbert space completion of $\mathcal{H}$. This will allow to use the star product algebra to explicitly construct well-behaved representations of $SU(1,n)$.
6 Open Questions and Outlook

Let us conclude this review with some open questions and hints on a further development for the theory of convergence of star products.

1. The three classes of examples indicate that the proposed way to convergence in Section 2 is at least not completely hopeless. Of course, the examples are the most simple ones but already here we see a rich and nontrivial analytic structure when discussing convergence. Thus it is reasonable to stick with the proposal and investigate further examples.

2. The current situation of convergence of star products based on a detailed analysis of the formal power series should of course be compared to the previous approaches based on integral formulas. Here the situation is not completely easy as the functional spaces are typically rather complementary. On the one hand one poses conditions on the growth of the Taylor coefficients but allows fairly unbounded behaviour at infinity, on the other side one can abandon real-analytic functions and allow for more general smooth functions as long as the growth at infinity is either bounded or at least very moderate. Nevertheless, a comparison should be possible. In particular, when passing to suitable *-representations then a comparison by means of spectral measures is one option.

3. The formal power series approach is perhaps the only possibility to include infinite-dimensional phase spaces. The applications to (quantum) field theory are by far not yet explored. Here already the constant Poisson structures provide interesting scenarios, see e.g. [143, Sect. 6].

4. The above approaches are always based on the assumption to have a continuous product. As long as one is heading for Fréchet topologies this is not a big restriction. However, in various field theoretic models one might expect that already the Poisson bracket has less regularity: it can be only separately continuous or even only sequentially separately continuous. It remains an open question to develop appropriate tools to discuss the convergence for formal star products quantizing such Poisson structures.

5. At the moment it is perhaps too early to propose general definitions for convergence schemes of formal star products. In particular, the example of the disc shows that the dependence on \( \hbar \) might be more delicate than a first guess suggest. It seems that more examples have to be investigated. Here the Wick type star products might turn out to be very handy as they provide the better positivity properties. Having a real-analytic context for the manifold directions will also help to understand the convergence of the formal power series in \( \hbar \). Thus non-compact Hermitian symmetric spaces will be natural candidates for further investigations.

6. The convergence properties of the above examples can be studied further and will lead to applications beyond deformation quantization. In particular, the Gutt star product suggest immediately next steps in direction of representation theory as the self-adjointness of Lie algebra representations can now be studied with the help of the completed algebra. E.g. in GNS representations one can use the completed and therefore rich algebra to obtain analytic vectors and hence self-adjointness. But also within differential geometry there will be applications when thinking of the above Fréchet algebras as algebras of particular pseudo-differential operators. With the help of *-representations, one has the possibilities for spectral analysis in these algebras.

7. Finally, in the theory of locally convex algebras much effort is spend on the cases of locally multiplicatively convex algebras. Here one can use Banach algebra techniques and obtains a closely related extension compared to the Banach algebra case. However, all the above examples indicate that there is a non-trivial and interesting world of locally convex algebras far beyond
the locally multiplicatively convex case. Here one needs to develop new techniques also from a more conceptual point of view. The above examples can then be seen as a guideline to formulate such an extension: they should provide ideal testing grounds as they show already a fairly complicated nature but are still manageable and concrete enough to test general ideas.

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