TOWARDS AN EFFECTIVISATION OF THE RIEMANN THEOREM

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Abstract

Let $Q$ be a connected and simply connected domain on the Riemann sphere, not coinciding with the Riemann sphere and with the whole complex plane $\mathbb{C}$. Then, according to the Riemann Theorem, there exists a conformal bijection between $Q$ and the exterior of the unit disk. In this paper we find an explicit form of this map for a broad class of domains with analytic boundaries.

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1. Introduction.

Let $Q$ be a connected and simply connected domain on the Riemann sphere, not coinciding with the Riemann sphere and with the whole complex plane $\mathbb{C}$. Then, according to Riemann Theorem, there exists a conformal bijection between $Q$ and the exterior of the unit disk $U = \{u \in \mathbb{C}||u| > 1\}$. In this paper we find an explicit form of this map for a broad class of domains with analytic boundaries.

Without loss of generally it is possible to assume that $\infty \in Q - \partial Q$ and $0 \in Q_{\circ}$, where $Q_{\circ} = \mathbb{C} - (Q \cup \partial Q)$ and $\partial Q$ is the boundary of $Q$. Let $\mathbb{H}$ be the set of all such domains. The harmonic moments

$$t_0 = \frac{1}{\pi} \int_{Q_{\circ}} dx dy, \quad t_k = -\frac{1}{\pi k} \int_{Q} z^{-k} dxdy \quad (k = 1, 2, \ldots) \quad (1.1)$$

form a coordinate system $t = (t_0, t_1, t_2, \ldots)$ on a considerable part of $\mathbb{H}$ [2, 14]. Note that the evolution of $Q_{\circ}$ as $t_0 \to 0$ and $t_i = const$ for $i > 1$ describes the evolution of a gas bubble surrounded by liquid [1].

Let $Q^t \in \mathbb{H}$ be the domain corresponding to $t$. The conditions $\partial_z w_{Q^t}(\infty) \in \mathbb{R}$, $\partial_z w_{Q^t}(\infty) > 0$ uniquely define the conformal bijection from $Q^t$ to $U$: $w(z, t) = w_{Q^t}(z) = p(t)z + \sum_{j=0}^{\infty} p_j(t)z^{-j}, \quad p(t) \in \mathbb{R}, p(t) > 0. \quad (1.2)$
Our goal is to represent \( p_j(t) \) as a Taylor series in \( t \) and \( \bar{t} \).

Put

\[
\partial_i = \frac{\partial}{\partial t_i}, \quad \bar{\partial}_i = \frac{\partial}{\partial \bar{t}_i}, \quad D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k, \quad D(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k \tag{1.3}
\]

In §1 we reproduce, following [3,15,16], an important result [4]: the Riemann maps have a potential \( F: \mathbb{H} \to \mathbb{C} \) (termed the tau-function of the curves in [11]) such that

\[
w(z,t) = zexp(-\frac{1}{2}\partial_0^2 - \partial_0 D(z))F(t) \tag{1.4}
\]

and, moreover, \( F \) satisfies the differential equations

\[
(z - \xi)e^{D(z)D(\xi)F} = ze^{-\partial_0 D(z)F} - \xi e^{-\partial_0 D(\xi)F}, \tag{1.5(a)}
\]

\[
(\bar{z} - \bar{\xi})e^{\bar{D}(\bar{z})\bar{D}(\bar{\xi})F} = \bar{z}e^{-\partial_0 \bar{D}(\bar{z})F} - \bar{\xi}e^{-\partial_0 \bar{D}(\bar{\xi})F}, \tag{1.5(b)}
\]

\[
1 - e^{-D(z)\bar{D}(\bar{\xi})F} = \frac{1}{z\xi}e^{\partial_0 (\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}. \tag{1.5(c)}
\]

This system of nonlinear differential equations is well known in mathematical physics and in theory of integrable systems as the dispersionless limit of the 2D Toda hierarchy [5]. The solution \( F(t) \) satisfies some additional equation, and it appears in string theory as the ”string solution” [6]. The string solution of the dispersionless limit of the 2D Toda hierarchy appears also in matrix models and in some other problems of mathematical physics [7,14]. Thus, a description of it has an independent interest.

In §2 we find, following [8], recursive formulas for coefficients of the Taylor series for the potential

\[
F = \sum N(i|_{i_1}\ldots,i_k|_{\bar{i}_1}\ldots,\bar{i}_k)t_{i_1}\ldots t_{i_k}\bar{t}_{i_1}\ldots \bar{t}_{i_k}. \tag{1.6}
\]

These formulas together with (1.4) allow one to find the Riemann maps via harmonic moments of the domains.

In §3 we find, following [9], a sufficient condition for the convergence of the Taylor series (1.6).

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2. The potential for Riemann maps.

The bijection \( w: Q \to U \) generates the Dirichlet Green function on \( Q \times Q \),

\[
G_Q(z,\xi) = \log \left| \frac{w(z) - w(\xi)}{w(z)\bar{w}(\xi) - 1} \right|. \tag{2.1}
\]

It solves the Dirichlet problem in \( Q \): the formula

\[
u(z) = -\frac{1}{2\pi} \oint_{\partial Q} u_0(\xi) \partial_n G_Q(z,\xi) d\xi \tag{2.2}
\]
restores a harmonic function from its boundary values \( u_0 = u|_{\partial D} \). Here \( \partial_n \) denotes the derivative along the internal normal to the boundary \( \partial Q \) with respect to the second variable, and \( |d\xi| \) is an infinitesimal element of length along \( \partial Q \).

The Dirichlet Green function is uniquely determined by the following properties [10].

\[ G_Q(z, \xi) \] is symmetric and harmonic in both arguments except the line \( z = \xi \)

everywhere in \( Q \), where \( G_Q(z, \xi) = \log |z - \xi| + O(1) \) as \( z \to \xi \);

\[ G_Q(z, \xi) = 0 \] if both variables \( z, \xi \) belong to the boundary. \tag{2.3(a)}

Denote by \( \mathbb{H}_z \) the set of all domains \( Q \in \mathbb{H} \) containing \( z \). The infinitesimal shift of the boundary \( \partial Q = \{ \xi \} \) on \(-\frac{\varepsilon}{2} \partial_n G_Q(z, \xi) \) \((\varepsilon \to 0)\) generates a vector field \( \delta_z \) on \( \mathbb{H}_z \). In particular,

\[ \delta_z(t_k) = -\frac{1}{\varepsilon \pi k} \oint_{\partial Q} \xi^{-k}(-\frac{\varepsilon}{2} \partial_n G_Q(z, \xi))|d\xi| = \frac{z^k}{k} \quad (k > 0) \quad \delta_z(t_0) = 1. \tag{2.4} \]

Thus, for any functional \( X : \mathbb{H}_z \to \mathbb{C} \), we have

\[ \delta_z X = \frac{\partial X}{\partial t_0} \delta_z t_0 + \sum \frac{\partial X}{\partial t_k} \delta_z t_k + \sum \frac{\partial X}{\partial \bar{t}_k} \delta_z \bar{t}_k = \nabla(z)X, \tag{2.5} \]

where \( \nabla(z) = \partial_0 + D(z) + \bar{D}(\bar{z}) \).

Put \( v_0 = \frac{2}{\pi} \int_{Q^t} \log |z| dxdy \) and \( v_k = \frac{1}{\pi} \int_{Q^t} z^k dxdy \) for \( k > 0 \).

**Theorem 2.1.** [4,11,16]. The function \( F(t) = -\frac{1}{\pi} \int_{Q^t} \log |\eta^{-1} - \nu^{-1}| d\eta d\nu \)
(where \( Q^t = \bar{C} - Q^t \)) satisfies the equations (1.4) and (1.5). Moreover, \( \partial_k F = v_k \).

Proof: It follows from (2.2) and from the definition of \( \delta_z \) that the function

\[ \tilde{G}_Q(z, \xi) = \log \left| \frac{1}{z} - \frac{1}{\xi} \right| + \frac{1}{2} \delta_z \delta_\xi F \tag{2.6} \]

satisfies (2.3). Thus, \( \tilde{G}_Q(z, \xi) = G_Q(z, \xi) \). Using (2.1) and (2.5), we find that

\[ \log \left| \frac{w(z) - w(\xi)}{w(z)\bar{w}(\xi) - 1} \right| = \log \left| \frac{1}{z} - \frac{1}{\xi} \right| + \frac{1}{2} \nabla(z) \nabla(\xi) F \tag{2.7} \]

This equation implies equations (1.4) and (1.5). Indeed, it implies

\[ h = \log \left| \frac{w(z) - w(\xi)}{w(z)\bar{w}(\xi) - 1} \right|^2 - \log \left| \frac{1}{z} - \frac{1}{\xi} \right|^2 - \nabla(z) \nabla(\xi) F = 0. \tag{2.8} \]
Furthermore, we have $h = h_1 + h_2$ where
\[ h_1 = \log \left( \frac{w(z) - w(\xi)}{w(z)\bar{w}(\xi) - 1} \right) - \log \left( \frac{1}{z} - \frac{1}{\xi} \right) - \left( \frac{1}{2} \partial_0 + D(z) \right) \nabla(\xi) F \] (2.9)
is a holomorphic function of $z$ while
\[ h_2 = \log \left( \frac{\bar{w}(z) - \bar{w}(\xi)}{\bar{w}(z)w(\xi) - 1} \right) - \log \left( \frac{1}{\bar{z}} - \frac{1}{\bar{\xi}} \right) - \left( \frac{1}{2} \partial_0 + \bar{D}(\bar{z}) \right) \nabla(\xi) F \] (2.10)
is an antiholomorphic function of $z$. Thus, $h_1$ is independent of $z$. Passing on to the limit as $z \to \infty$, we obtain
\[ h_1 = \log \left( \frac{1}{\bar{w}(\xi)} \right) - \log \left( -\frac{1}{\xi} \right) - \frac{1}{2} \partial_0 \nabla(\xi) F. \] (2.11)
Thus,
\[ \log \left( \frac{w(z) - w(\xi)}{z - \xi} \right) = D(z)\nabla(\xi) F. \] (2.12)
Passing on to the limit as $\xi \to \infty$ in (2.12), we obtain $\log \left( \frac{p\bar{z}}{w(z)} \right) = D(z)\partial_0 F$. Substituting (1.2) in (2.7) and passing on to the limit as $z \to \infty, \xi \to \infty$, we obtain $\log(p) = -\frac{1}{2} \partial_0^2 F$. The comparison of these formulas yields
\[ \log \left( \frac{z}{w(z)} \right) = D(z)\partial_0 F + \frac{1}{2} \partial_0^2 F, \] (2.13)
which is equivalent to (1.4).
Using (2.13), we transform the holomorphic parts of (2.12) into
\[ \log \left( \frac{w(z) - w(\xi)}{z - \xi} \right) = -\frac{1}{2} \partial_0^2 F + D(z)D(\xi) F. \] (2.14)
Substituting (1.4) into (2.14) leads to (1.5(a)). Changing $(z, \xi)$ to $(\bar{z}, \bar{\xi})$ in the proof we obtain (1.5(b)).

The antiholomorphic part of (2.12) is
\[ -\log \left( 1 - \frac{1}{w(z)\bar{w}(\xi)} \right) = D(z)\bar{D}(\xi) F. \] (2.15)
After substituting (1.4) into (2.15) we obtain (1.5(c)).

Moreover, if $z \in \partial Q$, then
\[ \nabla(z) F = \delta_z F = -\frac{2}{\pi} \int_{Q^c_\z(t)} \log |\nu^{-1} - z^{-1}|d\nu d\bar{\nu} = v_0 + 2Re \sum_{k \geq 1} \frac{v_k}{k} \bar{z}^{-k}. \]
Thus, $\partial_k F = v_k$. □
3. The Taylor series for the potential.

In this section we find, following [8], recursive formulas for the coefficients \( N \) of the Taylor series \( F = \sum N(i_1, \ldots, i_k) t_0 \cdot \cdots \cdot t_k \cdot \bar{t}_1 \cdots \bar{t}_k \) for the potential \( F \).

The formulas for \( N \) are found according to the following scheme. At first, using some combinatorial calculations, we transform Equation (1.5(a)) into an infinite system of equations

\[
\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} F = \sum_{m=1}^{\infty} \sum_{s_1+\cdots+s_m=i_1+\cdots+i_k, s_1+\cdots+s_m=m+k-2} \frac{i_1 \cdots i_k}{s_1 \cdots s_m} T_{i_1 \cdots i_k} \left( \frac{s_1 \cdots s_m}{\ell_1 \cdots \ell_m} \right) \partial_0 \partial_{s_1} F \cdots \partial_0 \partial_{s_m} F. \tag{3.1}
\]

During this process we find some recursive formulas for \( T \).

Then, using the definition of \( F \) as a function on the space of analytic curves, we find that \( \partial_0 F |_{t_0} = -t_0 + t_0 \ln t_0 + \text{const} \) and \( \partial_k F |_{t_0} = 0 \), if \( k > 0 \), where here and later \( |_{t_0} \) means the restriction of a function to the straight line \( t_1 = \bar{t}_1 = t_2 = \bar{t}_2 = \cdots = 0 \).

Because of this formula and from Equation (1.5(c)) it follows that

\[
\partial_i \partial_j F |_{t_0} = \begin{cases} 0, & \text{if } i \neq j, \\ it_0^i, & \text{if } i = j. \end{cases}
\]

Later, using (3.1) and the symmetry of the equations (1.5) we find, that

\[
\partial_i \partial_{i_1} \cdots \partial_{i_k} F |_{t_0} = \partial_i \partial_{i_1} \cdots \partial_{i_k} F |_{t=0} = \begin{cases} 0, & \text{if } i_1 + \cdots + i_k \neq i, \\ t_0^{i_1} \cdots t_0^{i_k} (i-1)!, & \text{if } i_1 + \cdots + i_k = i. \end{cases}
\]

This condition and Equation (3.1) give some recursive formulas for the coefficients \( N \). As the final result we get

**Theorem 3.1.** Suppose the formal series (1.6) satisfies in its domain of convergence Equations (1.4) and (1.5). Then it has the form, up to a linear summand,

\[
F = \frac{1}{2} t_0^2 \ln t_0 - \frac{3}{4} t_0^2 + \sum_{k, k', n, \bar{n}, \bar{n} \geq 1, 0 < i_1 < \cdots < i_k} \frac{\tilde{i}_1^{n_1} \cdots \tilde{i}_k^{n_k}}{n_1! \cdots n_k!} \bar{t}_1^{\bar{n}_1} \cdots \bar{t}_k^{\bar{n}_k},
\]

\[
\cdot N_i^2 \left( \begin{array}{cccc} i_1, & \ldots, & i_k \bar{t}_1, & \ldots, & \bar{i}_1, \\
-n, & \ldots, & n_k \bar{n}_1, & \ldots, & \bar{n}_k \end{array} \right) t_0^{-(n_1 + \cdots + n_k + \bar{n}_1 + \cdots + \bar{n}_k) + 2} \bar{t}_1^{\bar{i}_1} \cdots \bar{t}_k^{\bar{i}_k}
\]

where the coefficients \( N_i^2 \) can be found by the following recursive rules:
1) $P_{i,j}(s_1, \ldots, s_m) = \# \{(i_1, \ldots, i_m) \mid i = i_1 + \cdots + i_m, 1 \leq i_r \leq s_r - 1\}$, where $\#V$ is the cardinality of the set $V$;

2) $T_{i,j}^{1}(s_1, \ldots, s_m) = \sum_{k \geq 1} \frac{1}{kn_1! \cdots n_k!} P_{i,j} \left( \underbrace{s_1 + \cdots + s_{n_1}}_{n_1}, \ldots, \underbrace{s_{n_1} + \cdots + n_{k-1} + 1 + \cdots + s_{n_1} + \cdots + n_k}_{n_k} \right)$;

$T_{i_1, i_2}^{2}(s_1, \ldots, s_m) = T_{i_1, i_2}^{1}(s_1, \ldots, s_m)$;

$T_{i_1, \ldots, i_k}^{2}(s_1, \ldots, s_m, l_1, \ldots, l_m) = \sum_{1 \leq i \leq j \leq m} l T_{s, i}^{1}(s, l, s) T_{i_1, \ldots, i_{k-1}}^{2}(s_1, \ldots, s_{i-1}, l, s, s_j+1, \ldots, s_m, l, \ldots, l_m)$, where $s = s_i + \cdots + s_j - i_k$, $l = (l_i - 1) + \cdots + (l_j - 1)$;

3) $S_{i_1, \ldots, i_k}(s_1, \ldots, s_m, l_1, \ldots, l_m)$

$= \sum_{\{t_1^1, \ldots, t_k^n\} \sqcup \{\tilde{t}_m^1, \ldots, \tilde{t}_m^n\} = \{\tilde{i}_1, \ldots, \tilde{i}_k\}} \frac{1}{(s_1 - n_1 - l_1 + 1)! (l_1 - 1)!} \times \cdots \times \frac{1}{(s_m - n_m - l_m + 1)! (l_m - 1)!}$;

4) $N_{i}^{l}(i_1, \ldots, i_k | \tilde{i}_1, \ldots, \tilde{i}_k) = 0$, if $i \neq i_1 + \cdots + i_k$ or $i \neq \tilde{i}_1 + \cdots + \tilde{i}_k$;

in the other cases

$N_{i}^{l}(i_1, \ldots, i_k | \tilde{i}_1, \ldots, \tilde{i}_k) = \frac{(i-1)!}{(i-k+1)!} ;$ \quad $N_{i}^{l}(i_1, \ldots, i_k | \tilde{i}) = \frac{(i-1)!}{(i-k+1)!}$ ;

$N_{i}^{l}(i_1, \ldots, i_k | \tilde{i}_1, \ldots, \tilde{i}_k) = \sum_{m \geq 1} (-1)^{m+1} S_{i_1, \ldots, i_k}(s_1, \ldots, s_m, l_1, \ldots, l_m) T_{i_1, \ldots, i_k}^{2}(s_1, \ldots, s_m, l_1, \ldots, l_m)$, if $k, \tilde{k} > 1$;

$N_{i}^{l}(i_1, \ldots, i_k | \tilde{i}_1, \ldots, \tilde{i}_k)$

$= N_{i}^{l}(i_1, \ldots, i_k | \tilde{i}_1, \ldots, \tilde{i}_k, \tilde{i})$.
The theorem follows from the lemmas below.

**Lemma 3.1.** The following relations hold

\[
z - \sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_1 \partial_j F = ze^{-\partial_0 D(z) F}
\]

\[
\partial_1 \partial_j F = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!} \frac{\sum_{j_1+\cdots+j_m=j+1} j}{k_1 \cdots k_m} \partial_0 \partial_{k_1} \cdots \partial_0 \partial_{k_m} F.
\]

Proof: According to (1.5(a)), \((z - \xi)e^{D(z)D(\xi)F} = (z - \xi)(1 + (D(z)D(\xi)F) + \frac{1}{2}(D(z)D(\xi)F)^2 + \cdots) = (z - \xi)(1 + z^{-1}\xi^{-1}\partial_1^2 F + z^{-1} \sum_{j=2}^{\infty} \frac{1}{j} \xi^{-j} \partial_1 \partial_j F + \xi^{-1} \sum_{j=2}^{\infty} \frac{1}{j^2} \partial_0 \partial_1 \partial_j F + z^{-2}\xi^{-2} f) = (z - \xi) + \xi^{-1}\partial_1^2 F - z^{-1}\partial_1^2 F + \sum_{j=2}^{\infty} \frac{1}{j} \xi^{-j} \partial_1 \partial_j F - \sum_{j=2}^{\infty} \frac{1}{j^2} \partial_0 \partial_1 \partial_j F + z^{-1}\xi^{-1} f.

On the other hand, according to (1.5(a)) the function \((z - \xi)e^{D(z)D(\xi)F}\) is a sum of two functions \(f_1(z) + f_2(\xi)\). Thus, \(f = 0\) and \(ze^{-\partial_0 D(z) F} = z - \sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_1 \partial_j F\). Therefore,

\[
\sum_{j=1}^{\infty} \frac{1}{j} z^{-(j+1)} \partial_1 \partial_j F = 1 - e^{-\partial_0 D(z) F} = 1 - (1 + \sum_{m=1}^{\infty} \frac{(-\partial_0 D(z) F)^{m}}{m!}) = -\sum_{m=1}^{\infty} \frac{(-1)^{m}}{m!} \left( \sum_{k=1}^{\infty} \frac{z^{-k}}{k} \partial_0 \partial_k F \right)^{m} = -\sum_{m=1}^{\infty} \frac{(-1)^{m}}{m!} \left( \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \sum_{k_1+\cdots+k_m=n} \frac{1}{k_1 \cdots k_m} \partial_0 \partial_{k_1} \cdots \partial_0 \partial_{k_m} F \right) = -\sum_{n=1}^{\infty} z^{-n} \left( \sum_{m=1}^{\infty} \frac{(-1)^{m}}{m!} \sum_{k_1+\cdots+k_m=n} \frac{1}{k_1 \cdots k_m} \partial_0 \partial_{k_1} \cdots \partial_0 \partial_{k_m} F \right).
\]

Thus,

\[
\frac{1}{j} \partial_1 \partial_j F = -\sum_{m=1}^{\infty} \frac{(-1)^{m}}{m!} \sum_{k_1+\cdots+k_m=j+1} \frac{1}{k_1 \cdots k_m} \partial_0 \partial_{k_1} \cdots \partial_0 \partial_{k_m} F.
\]

**Lemma 3.2.** The following relation holds

\[
\partial_1 \partial_j F = \sum_{m=1}^{\infty} \sum_{s_1 + \cdots + s_m = j + 1} \frac{1}{s_1 \cdots s_m} \partial_0 \partial_{s_1} \cdots \partial_0 \partial_{s_m} F.
\]
\[
\frac{(-1)^{m+1}}{m} \frac{ij}{(s_1 - 1) \cdots (s_m - 1)} P_{ij}(s_1, \ldots, s_m - 1) \cdot \partial_1 \partial_{s_1 - 1} F \cdots \partial_1 \partial_{s_m - 1} F,
\]

where \( P_{ij}(s_1, \ldots, s_m - 1) \) is the number of representations of the form \( \{i = i_1 + \cdots + i_m | 1 \leq i_k \leq s_k - 1, k = 1, \ldots, m \} \) for the number \( i \).

Proof: According to Lemma 3.1 and Equation (1.5(a)),
\[
(z - \xi) e^{D(z)D(\xi)F} = \]
\[
= z - \sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_1 \partial_j F - (\xi - \sum_{j=1}^{\infty} \frac{1}{j} \xi^{-j} \partial_1 \partial_j F) = (z - \xi) - \sum_{j=1}^{\infty} \frac{1}{j} (z^{-j} - \xi^{-j}) \partial_1 \partial_j F.
\]

Thus,
\[
e^{D(z)D(\xi)} = 1 + z^{-1} \xi^{-1} \sum_{j=1}^{\infty} \frac{1}{j} \left( \sum_{s+t=j-1}^{\infty} z^{-s} \xi^{-t} \right) \partial_1 \partial_j F =
\]
\[
= 1 + \sum_{j=1}^{\infty} \frac{1}{j} \left( \sum_{s+t=j+1}^{\infty} z^{-s} \xi^{-t} \right) \partial_1 \partial_j F.
\]

Therefore,
\[
D(z)D(\xi)F = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \left( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{s+t=n+1}^{\infty} z^{-s} \xi^{-t} \right)^m \]
\[
= \sum_{j=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{i,j \geq 1} z^{-i} \xi^{-j}.
\]

\[
\left( \sum_{i_1 + \cdots + i_m = i, j_1 + \cdots + j_m = j, i_k, j_k \geq 1} \frac{1}{i_1 + j_1 - 1} \partial_1 \partial_{i_1 + j_1 - 1} F \cdots \frac{1}{i_m + j_m - 1} \partial_1 \partial_{i_m + j_m - 1} F \right),
\]
that is
\[
\partial_1 \partial_j F = \sum_{m=1}^{\infty} \sum_{s_1 + \cdots + s_m = i+j} (-1)^{m+1} \frac{ij}{(s_1 - 1) \cdots (s_m - 1)} P_{ij}(s_1, \ldots, s_m - 1) \cdot \partial_1 \partial_{s_1 - 1} F \cdots \partial_1 \partial_{s_m - 1} F.
\]
Remark. The equations

\[ \partial_i \partial_j F = \sum_{m=1}^{\infty} \sum_{s_1, \ldots, s_m = i+j} \frac{(-1)^{m+1}}{m} \frac{ij}{(s_1-1) \cdots (s_m-1)} \cdot P_{ij}(s_1-1, \ldots, s_m-1) \partial_1 \partial_{s_1-1} F \cdots \partial_1 \partial_{s_m-1} F \]

describe the dispersionless limit of the KP equation. Another description of this hierarchy is presented in [12]. A comparison of these descriptions gives some nontrivial combinatorial identity for \( P_{ij} \).

Lemma 3.3. The following relation holds

\[ \partial_i \partial_j F = \sum_{m=1}^{\infty} \sum_{p_1, \ldots, p_m = j+i} \frac{ij}{p_1 \cdots p_m} T_{ij}(p_1 \cdots p_m) \partial_0 \partial_{p_1} F \cdots \partial_0 \partial_{p_m} F, \]

where

\[ T_{ij}(p_1 \cdots p_m) = \sum_{n_1 + \cdots + n_k = m \atop n_i > 0} \frac{(-1)^{m+1}}{k! n_1! \cdots n_k!} P_{ij}(p_1 + \cdots + p_{n_1} - 1, \ldots, p_{n_k-1} + 1 + \cdots + p_{q_k-1},) \] and \( q_j = \sum_{i=1}^{j} n_i \).

Proof: According to Lemmas 3.1 and 3.2,

\[ \partial_i \partial_j F = \sum_{m=1}^{\infty} \sum_{s_1, \ldots, s_m = i+j} \frac{(-1)^{m+1}}{m} \frac{ij}{(s_1-1) \cdots (s_m-1)} P_{ij}(s_1-1, \ldots, s_m-1). \]

By induction, we get from Lemma 3.3
Lemma 3.4. The following relation holds

\[ \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} F = \sum_{m=1}^{\infty} \left( \sum_{s_1 + \cdots + s_m = i_1 + \cdots + i_k} \sum_{\ell_1 + \cdots + \ell_m = m + k - 2} \frac{i_1 \cdots i_k}{s_1 \cdots s_m} \right) \]

\[ \cdot T_{i_1 \cdots i_k} \left( \begin{array}{c} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{array} \right) \partial_{0}^{\ell_1} \partial_{s_1} F \cdots \partial_{0}^{\ell_m} \partial_{s_m} F, \]

where

\[ T_{i_1, i_2} \left( \begin{array}{c} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{array} \right) = \begin{cases} T_{i_1, i_2} (s_1 \cdots s_m), & \text{if } \ell_1 = \cdots = \ell_m = 1, \\ 0, & \text{otherwise} \end{cases} \]

\[ T_{i_1 \cdots i_k} \left( \begin{array}{c} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{array} \right) = \sum_{1 \leq i \leq j \leq m \atop s, \ell > 0} T_{i_1 \cdots i_{k-1}} \left( \begin{array}{c} s_1 \cdots s_{i-1} \\ \ell_1 \cdots \ell_{i-1} \end{array} \right) \left( \begin{array}{c} s_j \cdots s_m \\ \ell_{i} \cdots \ell_m \end{array} \right). \]

\[ \cdot T_{s, i_k} (s_i, s_{i+1}, \ldots, s_j) \frac{\ell!}{(\ell_i - 1)! \cdots (\ell_j - 1)!}, \]

and \( s = s_i + s_{i+1} + \cdots + s_j - i_k, \ \ell = (\ell_i - 1) + \cdots + (\ell_j - 1). \) □

Define now the Cauchy data for \( F. \)

Lemma 3.5. The following relations hold

\[ \partial_{0} F|_{t_0} = -t_0 + t_0 \ln t_0 + \text{const} \quad \text{and} \quad \partial_{k} F|_{t_0} = 0 \quad \text{for } k > 0. \]

Proof: If \( t_1 = t_2 = \cdots = 0, \) then \( Q_{t_0}^0 = \{ w \in \mathbb{C} | w < t_0^2 \} \) and thus \( w(z, t)|_{t_0} = Rz = t_0^2 z. \) Now it follows from Theorem 2.1 that

\[ \partial_{0} F|_{t_0} = \frac{2}{\pi} \int \ln |z| dxdy = \frac{2}{\pi} \int_{0}^{\infty} dr \int_{0}^{2\pi} d\varphi \cdot r \cdot \ln |r| = \]

\[ = \int_{0}^{R} \ln |r| dr = 2(r^2 \ln r)|_{0}^{R} - \int_{0}^{R} r dr = \]

\[ = R^2 \ln R^2 - R^2 = -t_0 + t_0 \ln t_0. \]

\[ \partial_{k} F|_{t_0} = \frac{1}{\pi} \int Q_{t_0}^k dxdy = 0 \quad \text{for } k > 0. \] □
Lemma 3.6. The following relations hold

\[ \partial_i \bar{\partial}_j F \big|_{t_0} = \begin{cases} 0 & \text{for } i \neq j; \\ it_0^i & \text{for } i = j. \end{cases} \]

Proof: It follows from Lemma 3.5 that \( \partial_0 \partial_k F \big|_{t_0} = 0 \) for \( k > 0 \) and \( \partial_0^2 F \big|_{t_0} = \ln t_0 \). Thus,

\[ e^{\partial_0(\partial_0 + D(z) + \bar{D}(\xi))} F \big|_{t_0} = t_0. \]

Moreover, according to (1.5(c)),

\[ 1 - e^{-D(z)\bar{D}(\xi)F} = z^{-1}e^{-1} e^{\partial_0(\partial_0 + D(z) + \bar{D}(\xi))} F, \]

and thus

\[ -D(z)\bar{D}(\xi)F \big|_{t_0} = \ln(1 - z^{-1}e^{-1} t_0) = -\sum_{k=1}^{\infty} k z^{-k} \xi^{-k} t_0^k. \]

Therefore, \( \partial_i \bar{\partial}_j F \big|_{t_0} = 0 \) for \( i \neq j \) and \( \partial_i \bar{\partial}_i F \big|_{t_0} = it_0^i \). □

Lemma 3.7. The following relations hold

\[ \partial_i \bar{\partial}_{i_1} \cdots \bar{\partial}_{i_k} F \big|_{t_0} = \bar{\partial}_i \partial_{i_1} \cdots \partial_{i_k} F \big|_{t_0} = \begin{cases} 0, & \text{if } i_1 + \cdots + i_k \neq i \\
 i_1 \cdots i_k \frac{it_0^{i-1}}{(i-k+1)!}, & \text{if } i = i_1 + \cdots + i_k. \end{cases} \]

Proof: The differentials \( \partial \) and \( \bar{\partial} \) enter Equation (1.5) in a symmetric way. This gives the first equality. Moreover, according to Lemmas 3.3 and 3.4, we have

\[ \partial_i \partial_{i_1} \cdots \partial_{i_k} F = \frac{i_1 \cdots i_k}{k} T_{i_1 \cdots i_k} \left( \begin{array}{c} i \\ k-1 \end{array} \right) \partial_0^{k-1} \partial_i F + \right.

\[ + \sum_{m=2}^{\infty} \sum_{s_1 + \cdots + s_m = i_1 + \cdots + i_k} \frac{i_1 \cdots i_k}{s_1 \cdots s_m} T_{s_1 \cdots s_m} \left( \begin{array}{c} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{array} \right) \left( \begin{array}{c} i_1 \cdots i_k \\ \ell_1 \cdots \ell_m \end{array} \right). \]

\[ - \partial_0^{\ell_j} \partial_{s_1} \cdots \partial_0^{\ell_m} \partial_{s_m} F \big|_{t_0} = \frac{i_1 \cdots i_k}{i} t_0^{i-1} \partial_i F + \right.

\[ + \sum_{m=2}^{\infty} \sum_{s_1 + \cdots + s_m = \ell_1 + \cdots + \ell_m} \frac{i_1 \cdots i_k}{s_1 \cdots s_m} T_{s_1 \cdots s_m} \left( \begin{array}{c} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{array} \right). \]
where \( i = i_1 + \cdots + i_k \). This equality and Lemmas 3.5 and 3.6 together give the second equality in the assertion of Lemma 3.7. □

**Lemma 3.8.** The following relation holds

\[
\left. \partial_{i_1} \cdots \partial_{i_k} \partial_{\overline{i_1}} \cdots \partial_{\overline{i_k}} F \right|_{t_0} = \sum_{m=1}^{\infty} \bar{N}_i(i_1 \cdots i_k | \overline{i_1} \cdots \overline{i_k})^2 \cdot \ell_0^{i-(k+\bar{k})+2},
\]

where \( \bar{N}_i(i_1 \cdots i_k | \overline{i_1} \cdots \overline{i_k}) = 0 \) if \( i - (k + \bar{k}) + 2 < 0 \), \( \sum_{j=1}^{k} i_j \neq i \) or \( \sum_{j=1}^{\bar{k}} \overline{i_j} \neq i \). In opposite case, \( \bar{N}_i(i_1 \cdots i_k | \overline{i_1} \cdots \overline{i_k}) = \sum_{m=1}^{\infty} \sum_{s_1 + \cdots + s_m = i_1 + \cdots + i_k} \sum_{\ell_1 + \cdots + \ell_m = m + k - 2} \sum_{s_j, \ell_j \geq 1} \frac{(s_1 - 1)! \cdots (s_m - 1)!}{(s_1 - N_1 + 1 - \ell_1)! \cdots (s_m - N_m + 1 - \ell_m)!} \cdot T_{i_1 \cdots i_k} \left( s_1 \cdots s_m \right) S_{i_1 \cdots \overline{i_k}} \left( \ell_1 \cdots \ell_m \right), \]

where

\[
S_{i_1 \cdots \overline{i_k}} \left( s_1 \cdots s_m \right) = \sum_{n_1 + \cdots + n_m = k} \sum_{s_i - n_i + 1 - \ell_i \geq 0} \frac{(s_1 - 1)! \cdots (s_m - 1)!}{(s_1 - N_1 + 1 - \ell_1)! \cdots (s_m - N_m + 1 - \ell_m)!} \cdot T_{i_1 \cdots \overline{i_k}} \left( s_1 \cdots s_m \right) S_{i_1 \cdots \overline{i_k}} \left( \ell_1 \cdots \ell_m \right),
\]

and the second summation is carried over all partitions of the set \( \{\overline{i_1}, \ldots, \overline{i_k}\} \) into subsets \( \{j_1^p, \ldots, j_{n_p}^p\} \) (\( p = 1, \ldots, m \)) such that \( \sum_{\alpha=1}^{n_p} j_{\alpha}^p = s_p \). □

Proof: According to Lemma 3.4,

\[
\partial_{i_1} \cdots \partial_{i_k} \partial_{\overline{i_1}} \cdots \partial_{\overline{i_k}} F = \partial_{\overline{i_1}} \cdots \partial_{\overline{i_k}} \left( \sum_{m=1}^{\infty} \sum_{s_1 + \cdots + s_m = i_1 + \cdots + i_k} \sum_{\ell_1 + \cdots + \ell_m = m + k - 2} \sum_{s_j, \ell_j \geq 1} \frac{(s_1 - 1)! \cdots (s_m - 1)!}{(s_1 - N_1 + 1 - \ell_1)! \cdots (s_m - N_m + 1 - \ell_m)!} \cdot T_{i_1 \cdots i_k} \left( s_1 \cdots s_m \right) \partial_{s_1} \cdots \partial_{s_m} \right) F \right|_{t_0} = \sum_{m=1}^{\infty} \sum_{s_1 + \cdots + s_m = i_1 + \cdots + i_k} \sum_{\ell_1 + \cdots + \ell_m = m + k - 2} \sum_{s_j, \ell_j \geq 1} \frac{(s_1 - 1)! \cdots (s_m - 1)!}{(s_1 - N_1 + 1 - \ell_1)! \cdots (s_m - N_m + 1 - \ell_m)!} \cdot T_{i_1 \cdots i_k} \left( s_1 \cdots s_m \right) \partial_{s_1} \cdots \partial_{s_m} F \right|_{t_0}.
\]
\[
\cdot \partial_{0}^{k} \partial_{i_{1}} \partial_{j_{1}} \ldots \partial_{j_{n_{1}}} F \ldots \partial_{0}^{m} \partial_{s_{m}} \partial_{j_{1}} \ldots \partial_{j_{n_{m}}} F \right) \),
\]
where the first summation is carried over all partitions of \( \{ i_{1}, \ldots, i_{k} \} \) into subsets \( \{ (j_{1}^{1}, \ldots, j_{1}^{n_{1}}) \ldots (j_{m}^{1}, \ldots, j_{m}^{n_{m}}) \} \). According to Lemma 3.7, this gives the assertion of Lemma 3.8 \( \Box \).

Lemma 3.8 yields

**Lemma 3.9** The following relation holds

\[
F = \frac{1}{2} t_{0}^{2} \log t_{0} - \frac{3}{4} t_{0}^{2} + \sum_{i_{1} < \cdots < i_{k}} \sum_{\bar{n}_{j}=1}^{\infty} \frac{1}{n_{1}! \cdots n_{k}! \cdots \bar{n}_{k}!} \tilde{N}_{i} \left( \frac{i_{1} \ldots i_{k}}{n_{1} \ldots n_{k}} | \frac{\bar{i}_{1} \ldots \bar{i}_{k}}{\bar{n}_{1} \ldots \bar{n}_{k}} \right),
\]

where

\[
\tilde{N}_{i} \left( \frac{i_{1} \ldots i_{k}}{n_{1} \ldots n_{k}} | \frac{\bar{i}_{1} \ldots \bar{i}_{k}}{\bar{n}_{1} \ldots \bar{n}_{k}} \right) = 0,
\]

if \( i - (\sum_{i=1}^{k} n_{i} + \sum_{i=1}^{k} \bar{n}_{i}) + 2 < 0, \ i \neq \sum_{j=1}^{k} n_{j} i_{j} \) or \( i \neq \sum_{j=1}^{k} \bar{n}_{j} \bar{i}_{j} \), and

\[
\tilde{N}_{i} \left( \frac{i_{1} \ldots i_{k}}{n_{1} \ldots n_{k}} | \frac{\bar{i}_{1} \ldots \bar{i}_{k}}{\bar{n}_{1} \ldots \bar{n}_{k}} \right) = \tilde{N}_{i} \left( \frac{i_{1} \ldots i_{k} \bar{i}_{2} \ldots \bar{i}_{k} \bar{i}_{1} \bar{i}_{2} \ldots \bar{i}_{k}}{n_{1} \ldots n_{k} \bar{n}_{1} \ldots \bar{n}_{k}} \right),
\]

otherwise; here in the last parentheses each \( i_{j} \) (respectively, each \( \bar{i}_{j} \)) occurs \( n_{j} \) (respectively, \( \bar{n}_{j} \)) times.

Theorem 3.1 is equivalent to Lemma 3.9.

Using the algorithm of the present paper, Yu. Klimov and A. Korzh created a computer program for calculating the coefficients of the Taylor series \( F \).

**4. Convergence conditions of the Taylor series for the potential.**

The combinatorial coefficients \( N_{i}^{2}(\ldots) \) have some remarkable properties. For example,

**Theorem 4.1.** \( N_{i}^{2} \left( \frac{i_{1} \ldots i_{k}}{n_{1} \ldots n_{k}} | \frac{1}{\bar{n}_{1}} \right) = \begin{cases} (i - 1)!, & \text{if } k = n_{1} = 1, \ i = i_{1} = \bar{n}_{1}, \\ 0, & \text{otherwise} \end{cases} \)

**Proof.** According to our definition,

\[
\sum_{\{ i_{1}^{1} \ldots i_{s_{r}}^{1} \} \sqcup \cdots \sqcup \{ i_{1}^{m} \ldots i_{s_{r}}^{m} \} = \{ 1, \ldots, 1 \}} \binom{s_{1}, \ldots, s_{m}}{l_{1}, \ldots, l_{m}} =
\]

\[
\sum_{s_{r} - n_{r} - \ell_{r} + 1 \geq 0} \frac{(s_{1} - 1)!}{(s_{1} - n_{1} - l_{1} + 1)!} (l_{1} - 1)! \times \cdots \times
\]
\[
\times \cdots \times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)! (l_m - 1)!} = \delta_{\ell_1,1} \cdots \delta_{\ell_m,1} \frac{k!}{s_1 \cdots s_m}.
\]

Thus if \( k > 2 \), then

\[
N^1_i (i_1, \ldots, i_k | 1, \ldots, 1) = \sum_{m \geq 1} (-1)^{m+1} S_{1,\ldots,1}^{1,\ldots,1} \left( \frac{s_1, \ldots, s_m}{l_1, \ldots, l_m} \right) \times \frac{k!}{s_1 \cdots s_m} \times T^2_{i_1,\ldots,i_k} \left( \frac{s_1, \ldots, s_m}{l_1, \ldots, l_m} \right) = 0.
\]

Let now \( k = 2 (i_1, i_2 \geq 1) \). Then

\[
N^1_i (i_1, i_2 | 1, \ldots, 1) = \sum_{m \geq 1} (-1)^{m+1} S_{1,\ldots,1}^{1,\ldots,1} \left( \frac{s_1, \ldots, s_m}{l_1, \ldots, l_m} \right) \times \frac{k!}{s_1 \cdots s_m} \times T^2_{i_1,i_2} \left( \frac{s_1, \ldots, s_m}{l_1, \ldots, l_m} \right) =\]

\[
= \sum_{m \geq 1} (-1)^{m+1} \frac{k!}{s_1 \cdots s_m} T^2_{i_1,i_2} \left( \frac{s_1, \ldots, s_m}{l_1, \ldots, l_m} \right) = \]

\[
= \sum_{m \geq 1} (-1)^{m+1} \frac{k!}{s_1 \cdots s_m} T^1_{i_1,i_2} (s_1, \ldots, s_m) = \]

\[
= \sum_{m \geq 1} (-1)^{m+1} \frac{k!}{s_1 \cdots s_m} \sum_{k \geq 1} \frac{1}{k n_1! \cdots n_k!} \times \]

\[
\times P_{i,j} \left( \frac{s_1 + \cdots + s_n}{n_1}, \ldots, \frac{s_1 + \cdots + s_n + k - 1 + 1 + \cdots + s_1 + \cdots + s_n}{n_k} \right) =\]
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\[
= \sum_{m \geq 1, s_1 + \cdots + s_m = i_1 + i_2} \frac{(-1)^{m+1}}{kn_1! \cdots n_k!} \frac{\bar{k}!}{s_1 \cdots s_m} \times
\]

\[
\times P_{i,j}\left(\underbrace{s_1 + \cdots + s_{n_1}}_{n_1}, \underbrace{s_{n_1} + \cdots + s_{n_k+1}}_{n_k}\right) = \sum_{k \geq 1} P_{i,j}(\tilde{s}_1, \ldots, \tilde{s}_k) \times
\]

\[
= \sum_{k \geq 1, \tilde{s}_1 + \cdots + \tilde{s}_k = i_1 + i_2} (-1)^{m+1} \frac{\bar{k}!}{kn_1! \cdots n_k!} \cdot
\]

\[
\times \prod_{1 \leq r \leq k} \sum_{n_r \geq 1, s_{n_1} + \cdots + s_{n_r} = \tilde{s}_r} \frac{(-1)^{n_r}}{n_r! s_1 \cdots s_{n_r}}.
\]

In addition, if \(s > 1\), then

\[
\sum_{n \geq 1, s_1 + \cdots + s_n = s} \frac{(-1)^n}{n! s_1 \cdots s_n} = \frac{1}{s!} \frac{\partial^s}{\partial x^s} \sum_{n \geq 1} \frac{(-1)^n}{n!} \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \right)^n \bigg|_{x=0} =
\]

\[
= \frac{1}{s!} \frac{\partial^s}{\partial x^s} \sum_{n \geq 1} \frac{(-1)^n}{n!} (-\log(1 - x))^n \bigg|_{x=0} =
\]

\[
= \frac{1}{s!} \frac{\partial^s}{\partial x^s} \sum_{n \geq 1} \frac{(\log(1 - x))^n}{n!} \bigg|_{x=0} = \frac{1}{s!} \frac{\partial^s}{\partial x^s} (\exp(\log(1 - x)) - 1) \bigg|_{x=0} = 0.
\]

Thus

\[
N_i^1(i_1, i_2 | 1, \ldots, 1) = \sum_{k \geq 1} \frac{-\bar{k}!}{k} P_{i,j}(\tilde{s}_1, \ldots, \tilde{s}_k) \times
\]
Proof. Lemma 4.1. Here are the estimates:

\[ \sum_{s_1 + \cdots + s_{n_r} = s_r} \frac{(-1)^{n_r}}{n_r! s_1 \cdots s_{n_r}} = 0, \]

since \( P_{i,j}(1, \ldots, 1) = 0. \)

If \( k = 1, \) then it follows from our definition that

\[ N^2_i \left( \frac{i}{n_1}, 1 \right) = \begin{cases} \frac{(i - 1)!}, & \text{if } n_1 = 1, \ i = i_1 = \bar{n}_1, \\ 0 & \text{otherwise}. \end{cases} \]

If \( t_i, \bar{t}_i = 0 \) for \( i > 2, \) then \( \partial Q \) is an ellipse. In this case Theorem 3.1 gives

\[ F = -\frac{3}{4} t_0^2 + \frac{1}{2} t_0^2 \log \left( \frac{t_0}{1 - 4|t_2|^2} \right) + \frac{t_0}{1 - 4|t_2|^2} (|t_1|^2 + t_1^2 \bar{t}_2 + \bar{t}_1^2). \]

This formula was obtained first in [4] by using formulas for conformal maps from an ellipse to the circle.

The recursive formulas for coefficients of the Taylor series \( F \) give a possibility to estimate the coefficients and to find sufficient convergence conditions for \( F \) provided \( t_i, \bar{t}_i = 0 \) for \( i > n. \)

Theorem 4.2. Let \( \bar{t} = (t_0, t_1, \bar{t}_1, t_2, \bar{t}_2, \ldots) \) be such that \( t_i, \bar{t}_i = 0 \) for \( i > n, 0 < t_0 < 1 \) and \( |t_i|, |\bar{t}_i| \leq (4n^3 2^n e^n)^{-1}. \) Then the series \( F(\bar{t}) \) converges.

The proof is based on a sequence of estimations of all values used in the definition of \( N^2. \) Here are the estimates:

Lemma 4.1. Let \( i + j = s_1 + \cdots + s_m. \) Then \( P_{ij}(s_1, \ldots, s_m) \leq \min(C_{i-1}^{m-1}, C_{j-1}^{m-1}). \)

Proof.

\[ P_{j,i}(s_1, \ldots, s_m) = P_{i,j}(s_1, \ldots, s_m) = \#\{(i_1, \ldots, i_m) \mid i = i_1 + \cdots + i_m, 1 \leq i_r \leq s_r - 1\} \leq \#\{(i_1, \ldots, i_m) \mid i = i_1 + \cdots + i_m, 1 \leq i_r \} = C_{i-1}^{m-1}. \]

Lemma 4.2. Let \( i + j = s_1 + \cdots + s_m. \) Then \( T_{ij}^1(s_1, \ldots, s_m) \leq \frac{l^{m-1}}{m!}, \) where \( \ell = \min(i, j). \)

Proof.

\[ T_{j,i}^1(s_1, \ldots, s_m) = T_{i,j}^1(s_1, \ldots, s_m) = \sum_{\substack{k \geq 1 \\ n_1 + \cdots + n_k = m \\ n_r \geq 1 \\ \frac{n_1! \cdots n_k!}{n_r!}} \frac{1}{kn_1! \cdots n_k!} \times \]

\[ \times P_{i,j} \left( \underbrace{s_1 + \cdots + s_{n_1}}_{n_1}, \ldots, \underbrace{s_{n_1 + \cdots + n_{k-1} + 1} + \cdots + s_{n_1 + \cdots + n_k}}_{n_k} \right) \leq \]
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\[ \leq \sum_{k \geq 1}^{n_1 + \cdots + n_k = m} \frac{C_{i-1}^{k-1}}{kn_1! \cdots n_k!} = \]

\[ = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \sum_{k \geq 1} C_i^k \sum_{n_1 + \cdots + n_k = m} \frac{1}{n_1! \cdots n_k!} = \]

\[ = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \sum_{k \geq 1} C_i^k \left( x + \frac{x^2}{2!} + \ldots \right)^k \bigg|_{x=0} = \]

\[ = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \left( \sum_{k \geq 1} C_i^k (e^x - 1)^k \right) \bigg|_{x=0} = \]

\[ = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} ((1 + (e^x - 1))^i - 1)\bigg|_{x=0} = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} (e^{ix} - 1)\bigg|_{x=0} = \frac{i^{m-1}}{m!}. \]

\[ \square \]

Lemma 4.3. Let \( i_1 + \cdots + i_k = s_1 + \cdots + s_m \) and \( (\ell_1 - 1) + \cdots + (\ell_m - 1) = k - 2 \). Then

\[ T_{i_1,\ldots,i_k}^{2} \left( s_1 \cdots s_m \right) \leq \frac{I^{m-1}(k-1)^m(k-2)!}{m!} \], where \( I = \max(i_r) \).

Proof. We use induction by \( k \). If \( k = 2 \), then

\[ T_{i_1,\ldots,i_2}^{2} \left( s_1, \ldots, s_m \right) = T_{i_1,\ldots,i_2}^{1} \left( s_1, \ldots, s_m \right) \leq \]

\[ \leq \frac{I^{m-1}}{m!} = \frac{I^{m-1}(k-1)^m(k-2)!}{m!} \bigg|_{k=2}. \]

Let \( k > 2 \). Note firstly that if \( l = (l_1 - 1) + \cdots + (l_j - 1) \), then

\[ \sum_{1 \leq i \leq j \leq m, j-i=d} \frac{l}{(k-2)(d+1)} = \sum_{1 \leq i \leq j \leq m, j-i=d} \frac{(l_1 - 1) + \cdots + (l_j - 1)}{(k-2)(d+1)} \leq \]

\[ \leq \frac{\sum_{1 \leq l \leq m} (l-1)(d+1)}{(k-2)(d+1)} = 1. \]

Thus

\[ T_{i_1,\ldots,i_k}^{2} \left( s_1, \ldots, s_m \right) = \sum_{1 \leq i \leq j \leq m, s, l \geq 1} \frac{l}{(k-2)(d+1)} \times \]

\[ \times T_{i_1,\ldots,i_{k-1}}^{2} \left( s_1, \ldots, s_{i-1}, s, s_{j+1}, \ldots, s_m \right) \leq \]

\[ \frac{I^{m-1}(k-1)^m(k-2)!}{m!}. \]
Lemma 4.4. Let $\bar{I} = \max(\bar{i}_r)$ and

$$
\bar{S}_{\bar{i}_1, \ldots, \bar{i}_k}(m, k) = \sum_{\{\bar{i}_1 \leq \ldots \leq \bar{i}_n\}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)!((l_1 - 1)! - 1)!} \times \cdots \times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!((l_m - 1)! - 1)!}.
$$

Then $\bar{S}_{\bar{i}_1, \ldots, \bar{i}_k}(m, k) \leq m(k - 1)!C_{\bar{I}k - \bar{I}}^{k - 2} \bar{C}_{\bar{I}k}^{\bar{k} - m}$.

Proof. We use the equality

$$
\sum_{\bar{n}_1 + \cdots + \bar{n}_m = \bar{k} - m} C_{\bar{I}\bar{n}_1 + \bar{I}}^{\bar{n}_1} \times \cdots \times C_{\bar{I}\bar{n}_m + \bar{I}}^{\bar{n}_m} \frac{\bar{I}}{\bar{I}\bar{n}_1 + \bar{I}} \times \cdots \times \frac{\bar{I}}{\bar{I}\bar{n}_m + \bar{I}} = C_{\bar{I}(k - m) + m\bar{I}}^{\bar{k} - m} \bar{I}(k - m) + m\bar{I}
$$

which follows from [13]. Then

$$
\bar{S}_{\bar{i}_1, \ldots, \bar{i}_k}(m, k) =
$$
\[
\sum_{\{\tilde{i}_1, \ldots, \tilde{i}_1\} \sqcup \cdots \sqcup \{\bar{i}_m, \ldots, \bar{i}_m\} = \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)!(l_1 - 1)!} \times \cdots \times \\
\sum_{s_r = \tilde{i}_r + \cdots + \bar{i}_r, n_r \geq 1} \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \\
\sum_{\{\tilde{i}_1, \ldots, \tilde{i}_1\} \sqcup \cdots \sqcup \{\bar{i}_m, \ldots, \bar{i}_m\} = \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \cdots \times \\
\sum_{s_r = \tilde{i}_r + \cdots + \bar{i}_r, n_r \geq 1} \frac{(s_m - 1)!}{(s_m - n_m)!} \\
\times \frac{(s_m - n_m)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \\
\sum_{\{\tilde{i}_1, \ldots, \tilde{i}_1\} \sqcup \cdots \sqcup \{\bar{i}_m, \ldots, \bar{i}_m\} = \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \cdots \times \\
\sum_{s_r = \tilde{i}_r + \cdots + \bar{i}_r, n_r \geq 1} \frac{(s_m - 1)!}{(s_m - n_m)!} \times \sum_{\bar{l}_1 + \cdots + l_m = k - 2} C_{s_1 - n_1} \cdots C_{s_m - n_m} = \\
\sum_{\{\tilde{i}_1, \ldots, \tilde{i}_1\} \sqcup \cdots \sqcup \{\bar{i}_m, \ldots, \bar{i}_m\} = \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \cdots \times \\
\sum_{s_r = \tilde{i}_r + \cdots + \bar{i}_r, n_r \geq 1} \frac{(s_m - 1)!}{(s_m - n_m)!} \times \sum_{\bar{l}_1 + \cdots + l_m = k - 2} C_{s_1 - n_1} \cdots C_{s_m - n_m} = \\
C_{\bar{i}_1 + \cdots + \bar{i}_k - \bar{k}}^{k - 2} \sum_{\{\tilde{i}_1, \ldots, \tilde{i}_1\} \sqcup \cdots \sqcup \{\bar{i}_m, \ldots, \bar{i}_m\} = \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \cdots \times \\
\sum_{s_r = \tilde{i}_r + \cdots + \bar{i}_r, n_r \geq 1} \frac{(s_m - 1)!}{(s_m - n_m)!} \leq \ \bar{S}_{\tilde{i}_1, \ldots, \tilde{i}_k}(m, k) = C_{\bar{i}_1 + \cdots + \bar{i}_k - \bar{k}}^{k - 2} \sum_{\{\tilde{i}_1, \ldots, \tilde{i}_1\} \sqcup \cdots \sqcup \{\bar{i}_m, \ldots, \bar{i}_m\} = \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \cdots \times \\
\sum_{s_r = \tilde{i}_r + \cdots + \bar{i}_r, n_r \geq 1} \frac{(s_m - 1)!}{(s_m - n_m)!} =
\]
\[
\begin{align*}
&= C_{\frac{I}{k-k}}^{k-2} \sum_{n_1 + \cdots + n_m = k} \frac{\bar{k}!}{n_1! \cdots n_m!} \frac{(\bar{I}n_1 - 1)!}{(\bar{I}n_1 - n_1)!} \times \cdots \times \frac{(\bar{I}n_m - 1)!}{(\bar{I}n_m - n_m)!} = \\
&= \bar{k}! C_{\frac{I}{k-k}}^{k-2} \sum_{\tilde{n}_1 + \cdots + \tilde{n}_m = k-m} \frac{(\bar{I}\tilde{n}_1 + \bar{I} - 1)!}{(\bar{I}\tilde{n}_1 + 1)!} \times \cdots \times \frac{(\bar{I}\tilde{n}_m + \bar{I} - 1)!}{(\bar{I}\tilde{n}_m + 1)!} = \\
&= \bar{k}! C_{\frac{I}{k-k}}^{k-2} \sum_{\tilde{n}_1 + \cdots + \tilde{n}_m = k-m} \frac{\bar{I}\tilde{n}_1 + \bar{I} - \tilde{n}_1}{(\bar{I}\tilde{n}_1 + I)(\tilde{n}_1 + 1)} \times \cdots \times \frac{(\bar{I}\tilde{n}_m + \bar{I} - \tilde{n}_m)!}{(\bar{I}\tilde{n}_m + I)(\tilde{n}_m + 1)} \leq \\
&= \bar{k}! C_{\frac{I}{k-k}}^{k-2} \sum_{\tilde{n}_1 + \cdots + \tilde{n}_m = k-m} \frac{\bar{I}\tilde{n}_1 + \bar{I} - \tilde{n}_1}{(\bar{I}\tilde{n}_1 + I)(\tilde{n}_1 + 1)} \times \cdots \times \frac{(\bar{I}\tilde{n}_m + \bar{I} - \tilde{n}_m)!}{(\bar{I}\tilde{n}_m + I)(\tilde{n}_m + 1)} \leq m(k-1)! C_{\frac{I}{ik}}^{k-2} C_{\frac{I}{ik}}^{k-m}.
\end{align*}
\]

Lemma 4.5. \(N_i^1(i_1, \ldots, i_k | \bar{i}_1, \ldots, \bar{i}_k) \leq (k-1)! (k-1)! e^{l(k-1)/2} e^{-k-2} \).
\[
\times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \sum_{m \geq 1} \frac{I_m^m(k-1)^{m-1}(k-2)!}{m!} \times \sum_{\{\tilde{i}_1, \ldots, \tilde{i}_m\} \leq \{i_1, \ldots, i_k\}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)!(l_1 - 1)!} \times \ldots \times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \sum_{m \geq 1} \frac{I_m^m(k-1)^{m-1}(k-2)!}{m!} \tilde{S}_{\tilde{i}_1, \ldots, \tilde{i}_k}(m, k) \leq \sum_{m \geq 1} \frac{I_m^m(k-1)^{m-1}(k-2)!}{m!} \tilde{S}_{\tilde{i}_1, \ldots, \tilde{i}_k}(m, k) \leq \sum_{m \geq 1} \frac{I_m^m(k-1)^{m-1}(k-2)!}{m!} \bar{m}(\bar{k}-1)!C_{\bar{k}-k}^{k-2}C_{\bar{k}}^{k-m} = (k-1)!((\bar{k}-1))! \sum_{m \geq 1} \frac{I_m^m(k-1)^{m-1}(k-2)!}{m!} C_{\bar{k}}^{k-2}C_{\bar{k}}^{k-m} \leq (k-1)!((\bar{k}-1))!e^{J(k-1)2\bar{k}-\bar{k}}. \quad \square
\]

**Proof of Theorem 4.2.** The coefficient of \(t_0 t_{i_1}^{n_1} \ldots t_{i_l}^{n_l} t_{i_1}^{\tilde{n}_1} \ldots t_{i_{\bar{l}}}^{\tilde{n}_{\bar{l}}} \) is equal to

\[
\frac{i_{1}^{n_{1}} \ldots i_{l}^{n_{l}} \tilde{i}_{1}^{\tilde{n}_{1}} \ldots \tilde{i}_{\bar{l}}^{\tilde{n}_{\bar{l}}}}{n_{1}! \ldots n_{l}! \tilde{n}_{1}! \ldots \tilde{n}_{\bar{l}}!} N_{i}^{2} \left( \begin{array}{c}
(\delta_{1}, \ldots, \delta_{l}) \vline \begin{array}{c}
\tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{\bar{l}}
\end{array}
\end{array} \right) = \frac{i_{1}^{n_{1}} \ldots i_{l}^{n_{l}} \tilde{i}_{1}^{\tilde{n}_{1}} \ldots \tilde{i}_{\bar{l}}^{\tilde{n}_{\bar{l}}}}{n_{1}! \ldots n_{l}! \tilde{n}_{1}! \ldots \tilde{n}_{\bar{l}}!} N_{i}^{2} \left( \begin{array}{c}
(\delta_{1}, \ldots, \delta_{l}) \vline \begin{array}{c}
\tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{\bar{l}}
\end{array}
\end{array} \right) \leq \frac{i_{1}^{n_{1}} \ldots i_{l}^{n_{l}} \tilde{i}_{1}^{\tilde{n}_{1}} \ldots \tilde{i}_{\bar{l}}^{\tilde{n}_{\bar{l}}}}{n_{1}! \ldots n_{l}! \tilde{n}_{1}! \ldots \tilde{n}_{\bar{l}}!} k!k!e^{I(k-1)2\bar{k}} \leq I^{K}e^{J}K2^{\bar{K}2}\frac{k!k!}{n_{1}! \ldots n_{l}! \tilde{n}_{1}! \ldots \tilde{n}_{\bar{l}}!} \leq I^{K}e^{J}K2^{\bar{K}2}\frac{k!k!}{\bar{n}_{1}! \ldots \bar{n}_{\bar{l}}!} \leq (\bar{n}2e^{\bar{J}2})^{K}.
\]
where \( k = n_1 + \cdots + n_I, \bar{k} = \bar{n}_1 + \cdots + \bar{n}_I, K = k + \bar{k} \) and \( \bar{I} = \max(I, \bar{I}) \).

Now consider monomials of degree \( K \) in \( t_0, t_1, \bar{t}_1, \ldots, t_n, \bar{t}_n \). The number of such monomial is \((2n)^K\). Thus, their sum in the series is at most

\[
(n^22^n e^n)^K (2n)^K (4n^32^n e^n)^{-K} \leq 2^{-K}.
\]

This implies the convergence of the series \( F(\bar{t}) \). □

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