NUMERICAL APPROACH TO
TWO-LOOP THREE POINT FUNCTIONS WITH MASSES

Junpei FUJIMOTO, Yoshimitsu SHIMIZU†
National Laboratory for High-Energy Physics (KEK)
Oho 1-1, Tsukuba, Ibaraki 305, Japan

Kiyoshi KATO§
Kogakuin University
Nishi-Shinjuku 1-24, Shinjuku, Tokyo 160, Japan

and

Toshiaki KANEKO¶
Laboratoire d’Annecy-le-Vieux de Physique des Particules
(LAPP), B.P. 110, F-74941 Annecy-le Vieux Cedex, FRANCE

Extending the method successful for one-loop integrals, the computation of two-loop diagrams with general internal masses is discussed. For the two-loop vertex of non-planar type, as an example, we show a calculation related to $Z^0 \rightarrow t \bar{t}$ vertex.

The calculation of loop integrals is essential to obtain the precise theoretical prediction for high-energy reactions. Loop integrals for one-loop diagrams can be expressed by logarithms and dilogarithms. For a class of higher order diagrams, compact analytic expressions are obtained. For instance, some two-loop two point functions with single mass($m$) are given by functions of $x = s/m^2$. As two-loop diagrams in the electro-weak theory include complicated mass combinations, an analytic formula seems not available. In general, for the case with more than two independent dimensionless variables, it seems to be impossible to obtain a compact analytic formula by polylogarithms and so forth. In scattering processes, we have two or more invariants and in the electroweak theory we encounter diagrams with many different

* Presented by K. Kato at the AI-HENP 95 workshop, Pisa, April 1995
†junpei@minami.kek.jp
‡shimiz@minami.kek.jp
§kato@cc.kogakuin.ac.jp
¶kaneko@minami.kek.jp
masses. Thus the theoretical prediction in the electroweak theory requires a method to compute the loop integrals for arbitrary mass scales. To meet this request, we have already developed numerical methods for two-loop integrals. There exist a few other approaches for this direction by momentum space integral, momentum expansion with Padé approximation, asymptotic expansion, and so forth.

Automatic generation of two-loop diagrams is possible in the standard model. From the generated diagrams, symbolic expression for the numerator and the loop integrand can be generated. The latter, the integrand in two-loop as a function of Feynman parameter, is treated numerically.

Our strategy is as follows:

- Any loop integral can be given by the definite integral in Feynman parameter space.

- The integrand can be singular where the denominator becomes zero. If there is no singularity, the value of integral can be obtained without any trouble. The numerical accuracy is only limited by available computer time.

- If there exists singularity, the numerical integration is done after either of the following:
  1. (Symmetric method) The integrand is replaced by a symmetric sum of the function so as to eliminate the singularity.
  2. (Hybrid method) The integral for a few of variables is done analytically until the singularity is replaced by an integrable function.

In this report, we present results for those integrals appearing in the non-planar vertex functions.

We consider the following scalar integral for vertex function

\[ I = \int \frac{d^n l_1 \cdot d^n l_2}{(2\pi)^n (2\pi)^n} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6} \]  

where \(1/D_j\)'s are propagator functions. Introduction of Feynman parameters transforms it in the following form.

\[ I = \frac{1}{(4\pi)^n} \int_0^1 dx_1 \cdots dx_6 \delta(1 - \sum x) \frac{1}{U^{n/2} (V - i\epsilon)^{6-n}} \]  

where \(U\) and \(V\) can be determined from the topology of the diagram. If we confine ourselves to a class of diagrams where there is no
ultraviolet divergence in a subgraph, we can take $n = 4$. The function $U$ is

$$U = \sum_{T} \prod_{x_j \in T} x_j$$

(3)

where the summation is taken over co-trees ($\bar{T}$) in the diagram. The function $V$ is given by

$$V = \sum_{j} x_j m_j^2 - \frac{1}{U} \sum_{S} W_{Sp_s^2}$$

(4)

where the second sum is taken over cutsets ($S$) in the diagram.

External momenta, $p_1, p_2, p_3$ are defined to flow inward into the vertex and they satisfy $p_1 + p_2 + p_3 = 0$. We use the notation $s_i = p_i^2$, $x_i + x_j + \cdots = x_{ij\ldots}$, and $\bar{x}_{i\ldots} = 1 - x_{i\ldots}$. In the following, we consider the integral

$$J = \int_0^1 dx_1 \cdots dx_6 \delta(1 - \sum x) \frac{1}{(D + i\epsilon)^2}$$

(5)

where

$$D = \sum_{S} W_{Sp_s^2} - U \sum_{j} x_j m_j^2$$

(6)

and

$$\sum_{S} W_{Sp_s^2} = f_1 s_1 + f_2 s_2 + f_3 s_3.$$ 

(7)

For the diagrams in Fig.1, we obtain the following results.

(a) Planar type

$$U = x_{12}x_{3456} + x_{3}x_{456}$$

(8)
Numerical Approach to Two-Loop Integrals with Masses

\[ f_1 = x_{123}x_4x_6 + x_1x_3x_5 \]
\[ f_2 = x_{123}x_5x_6 + x_2x_3x_4 \]
\[ f_3 = x_1x_2x_{3456} + x_{123}x_4x_5 + x_1x_3x_5 + x_2x_3x_4 \]  

(9)

(b) Non-planar type

\[ U = x_{12}x_{3456} + x_{34}x_{56} \]  

(10)

\[ f_1 = x_{1256}x_3x_4 + x_1x_3x_6 + x_2x_4x_5 \]
\[ f_2 = x_{1234}x_5x_6 + x_2x_3x_6 + x_1x_4x_5 \]
\[ f_3 = x_{3456}x_1x_2 + x_2x_4x_6 + x_1x_3x_5 \]  

(11)

For the first case, we have already reported the numerical results. Here, we report the results for the non-planar vertex. First, variables are transformed as follows.

\[ x_1 = z_3(1 + y_3)/2, \quad x_2 = z_3(1 - y_3)/2 \]
\[ x_3 = z_1(1 - y_3)/2, \quad x_4 = z_1(1 + y_1)/2 \]
\[ x_5 = z_2(1 - y_2)/2, \quad x_6 = z_2(1 + y_2)/2 \]  

(12)

After the transform, three-fold symmetry in the non-planar vertex can be seen clearly. The integral becomes

\[ J = \frac{1}{8} \int_0^1 dz_1dz_2dz_3\delta(1 - \sum z)\hat{z} \int_{-1}^1 dy_1 \int_{-1}^1 dy_2 \int_{-1}^1 dy_3 \frac{1}{(D + i\epsilon)^2} \]  

(13)

where \( \hat{z} = z_1z_2z_3 \), and the function in the denominator becomes

\[ D = '\bar{y}A\bar{y} + \vec{b} \cdot \bar{y} + c \]  

(14)

where

\[ \bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \]  

(15)

\[ A = \frac{1}{4} \begin{pmatrix} -z_1^2 \bar{z}_1s_1 & \hat{z}(-s_1 - s_2 + s_3)/2 & \hat{z}(-s_1 + s_2 - s_3)/2 \\ \hat{z}(-s_1 - s_2 + s_3)/2 & -z_2^2 \bar{z}_2s_2 & \hat{z}(s_1 - s_2 - s_3)/2 \\ \hat{z}(-s_1 + s_2 - s_3)/2 & \hat{z}(s_1 - s_2 - s_3)/2 & -z_3^2 \bar{z}_3s_3 \end{pmatrix}, \]  

(16)

\[ \vec{b} = \frac{1}{2} \begin{pmatrix} z_1(-m_3^2 + m_4^2) \\ z_2(-m_5^2 + m_6^2) \\ z_3(m_1^2 - m_2^2) \end{pmatrix}, \]  

(17)
c = \frac{1}{4} U \left[ z_1 s_1 + z_2 s_2 + z_3 s_3 - 2(m_3^2 + m_4^2) z_1 - 2(m_5^2 + m_6^2) z_2 - 2(m_1^2 + m_2^2) z_3 \right], \tag{18}

U = z_1 z_2 + z_2 z_3 + z_3 z_1. \tag{19}

So the problem is transformed into how to carry out the double integral of a function similar to that which appears in a box diagram in one-loop. The determinant of $A$ is given by

$$\det A = \frac{1}{4^2} z^2 U (z_1 s_1 + z_2 s_2 + z_3 s_3) (s_1^2 + s_2^2 + s_3^2 - 2 s_1 s_2 - 2 s_2 s_3 - 2 s_1 s_3).$$ \tag{20}

As an example, we try to calculate “$Z_0$ exchange for $t\bar{t}$ vertex” i.e.,

\begin{align*}
    p_1^2 &= p_2^2 = m^2, \\
    m_1 &= m_2 = m_4 = m_5 = m, \\
    m_3 &= m_6 = M, \\
    m &= 150(\text{GeV}), \\
    M &= 91.17(\text{GeV}).
\end{align*} \tag{21}

Below the threshold, $s < 4m^2$, the integral has no singularity and it can be done easily by adaptive Monte-Carlo integration program BASES. Above threshold, the singularity, $D = 0$, appears. It is hyperboloid of one sheet or that of two sheets in $\vec{y}$ space. The transition of the topology occurs at $c' = 0$ where $c'$ is defined by

$$D = \vec{t} \vec{y}' A \vec{y}' + c', \quad c' = c - \frac{1}{4} \vec{b} A^{-1} \vec{b}. \tag{22}$$

For the case in Eq.(21), the condition $c' = 0$ becomes

$$z_3 = t, \quad (M^2 - s)(s + 3M^2 - 4m^2)t^2 + 2M^2(s - M^2)t + M^2(4m^2 - M^2) = 0. \tag{23}$$

By use of these formulas, we can calculate the value of integral based on the hybrid method as is described in detail in references.\[\[\]

Figure 2: Cuts for non-planar vertex.
The calculation of the non-planar vertex at hand can be done easily by the dispersion integral. With respect to the variable $s$, we have a two-body cut as in Fig.2(a), and three-body cuts as in Fig.2(b) and the reversed one.

$$J(s) = \frac{1}{\pi} \int \frac{\Im\mathcal{T}(s')}{s' - s - i\epsilon} ds', \quad \Im\mathcal{T}(s') = \Im\mathcal{T}_{\text{two}} + \Im\mathcal{T}_{\text{three}}$$

(24)

Here

$$\Im\mathcal{T}_{\text{two}} = \int T_0(p_3 \rightarrow k_1 + k_2)d\Gamma_2(k_1, k_2)T_{\text{box}}(k_1 + k_2 \rightarrow p_1 + p_2)$$

(25)

and

$$\Im\mathcal{T}_{\text{three}} = \int T_0(p_3 \rightarrow k_1+k_3+k_5)d\Gamma_3(k_1, k_3, k_5)T_0(k_1+k_3+k_5 \rightarrow p_1+p_2)+(\text{reversed})$$

(26)

where tree amplitudes, $T_0$ are expressed just by product of propagator(s), and $d\Gamma_n$ stands for the $n$-body phase space. The one-loop box integral in Eq.(25) is

$$T_{\text{box}} = \int dxdydz \frac{1}{(-xyt - z(1 - x - y - z)u + (x + y)M^2 + (1 - x - y)^2m^2)^2}$$

(27)

and is free from singularity because the denominator of is non-negative. Hence the 4-dimensional integrals in Eq.(25) and Eq.(26) are well behaved. The singularity at $s' = s$ in Eq.(24) can be handled by casting the integral into the form

$$\Re J(s) = \frac{1}{2\pi} \int \frac{\Im\mathcal{T}(s + \sigma) - \Im\mathcal{T}(s - \sigma)}{\sigma} d\sigma.$$

(28)

By use of BASES, this is evaluated as 5-dimensional integral together all the integral variables, Feynman parameters and kinematical ones.

Figure 3: Real part of non-planar vertex.

In Fig.3, we present the final results by these methods. The calculation of the integrals in Feynman parameter space is done at the same $s$ repeatedly by changing the assignment of external momenta and masses in Eq.(21). This diagnostics test works well for the points in the figures.

In this report, we presented only one example for the real calculation. However, the methods given here work in principle for any mass
parameters and external momenta\footnote{One needs more sophisticated treatment if the infrared divergence exists.}. The inclusion of numerator and the extension to four-point function are interesting and will be studied in the coming research.

Acknowledgements

This work is supported in part by the Ministry of Education, Science and Culture, Japan under the Grant-in-Aid for International Scientific Research Program No.04044158, and the Special Research Fund of the Kogakuin University.

References

1. G.'tHooft and M.Veltman, Nucl.Phys. \textbf{B153}, 365 (1979).
2. D.J.Broadhurst, Z.Physik \textbf{C47}, 115 (1989).
3. D.Kreimer, Phys.Lett. \textbf{B292}, 341 (1992),
   A.Czarnecki, U.Killian and D.Kreimer, Nucl.Phys. \textbf{B433}, 259(1995),
   A.Czarnecki, TTP 94-21, to appear in the Proceedings of 1995. Cargese Summer Institute.
4. J.Fleischer and O.V.Tarasov, Z.Physik \textbf{C64}, 413 (1994).
5. A.I.Davydychev and J.B.Tausk, Nucl.Phys. \textbf{397}, 123 (1993).
6. T.Kaneko, the talk in this workshop.
7. J.Fujimoto \textit{et al.}, KEK preprint 92–213 (1993).
8. P.Cvitanović and T.Kinoshtia, Phys.Rev. \textbf{D10}, 3978 (1974).
9. J.Fujimoto \textit{et al.}, Progr.Theor.Phys. \textbf{87}, 1233 (1992).
10. S.Kawabata, Comput.Phys.Commun. \textbf{41}, 127 (1986).