Limit Theorems for Optimal Mass Transportation

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Abstract

The optimal mass transportation was introduced by Monge some 200 years ago and is, today, the source of large number of results in analysis, geometry and convexity. Here I investigate a new, surprising link between optimal transformations obtained by different Lagrangian actions on Riemannian manifolds. As a special case, for any pair of non-negative measures $\lambda^+, \lambda^-$ of equal mass

$$W_1(\lambda^-, \lambda^+) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \inf_{\mu} W_p(\mu + \varepsilon \lambda^-, \mu + \varepsilon \lambda^+)$$

where $W_p, p \geq 1$ is the Wasserstein distance and the infimum is over the set of probability measures in the ambient space.

1 Introduction

The Wasserstein metric $W_p (\infty > p \geq 1)$ is a useful distance on the set of positive Borel measures on metric spaces. Given a metric space $(M, D)$ and a pair of positive Borel measures $\lambda^\pm$ on $M$ satisfying $\int_M d\lambda^+ = \int_M d\lambda^-:

$$W_p(\lambda^+, \lambda^-) := \inf_{\pi} \left\{ \left[ \int_M \int_M D^p(x, y) d\pi(x, y) \right]^{1/p} ; \pi \in \mathcal{P}(\lambda^+, \lambda^-) \right\} , \tag{1.1}$$

where $\mathcal{P}(\lambda^+, \lambda^-)$ stands for the set of all positive Borel measures on $M \times M$ whose $M$--marginals are $\lambda^+, \lambda^-$. Under fairly general conditions (e.g if $M$ is compact), a minimizer $\pi^0 \in \mathcal{P}(\lambda^+, \lambda^-)$ of (1.1) exists. Such minimizers are called optimal plans. I'll assume in this paper that $M$ is a compact Riemannian manifold and $D$ is a metric related (but not necessarily identical) to the geodesic distance.

If in addition $\lambda^+$ satisfies certain regularity conditions, the optimal measure $\pi^0$ is supported on a graph of a Borel mapping $\Psi : M \to M$. By some abuse of notation we call a Borel map $\Psi$ an optimal plan if it is a minimizer of

$$W_p(\lambda^+, \lambda^-) = \inf_{\Phi} \left\{ \left[ \int_D^p(x, \Phi(x)) d\lambda^+ \right]^{1/p} ; \Phi \# \lambda^+ = \lambda^- \right\}$$

(see Section 1.2-4 for notation).

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1This research was supported by M. & M. Bank Mathematics Res. Fund and by the Israel Science Foundation
The metric \( W_p, p \geq 1 \) is a metrization of the weak topology \( C^*(M) \) on positive Borel measures. In particular, it is continuous in the weak topology. Thus, it is possible to approximate \( W_p(\lambda^+, \lambda^-) \) (and the corresponding optimal plan) by \( W_p(\lambda^+_N, \lambda^-_N) \) on the set of atomic measures

\[
\lambda^+_N, \lambda^-_N \in \mathcal{M}^{+,N} := \left\{ \mu = \sum_{i=1}^N m_i \delta(x_i), m_i \geq 0, \quad x_i \in M \right\}, \quad N \to \infty \tag{1.2}
\]

reducing (1.1) into a finite-dimensional linear programming on the set of non-negative \( N \times N \) matrices \( \{P_{i,j}\} \) subjected to linear constraints.

There is, however, a sharp distinction between the case \( p > 1 \) and \( p = 1 \). If \( p > 1 \) then the optimal plan \( \pi_0 \) is unique (for regular \( \lambda^+ \)). This is, in general, not the case for \( p = 1 \).

Another distinctive feature of the case \( p = 1 \) is its "pinning property": The distance \( W_1 \) depends only on the difference \( \lambda := \lambda^+ - \lambda^- \). This is manifested by the alternative, dual formulation of \( W_1 \):

\[
W_1(\lambda) = \sup_{\phi} \left\{ \int \phi d\lambda \; ; \; \|\phi\|_{Lip} \leq 1 \right\} \tag{1.3}
\]

where \( \|\phi\|_{Lip} := \sup_{x \neq y \in M} (\phi(x) - \phi(y)) / D(x, y) \).

The optimal potential \( \phi \) yields some partial information on the optimal plan \( \Psi \) (if exists). In particular, \( \nabla \phi(x) \), whenever exists, only indicates the direction of the optimal plan. For example, if the metric \( D \) is Euclidean, then \( \Psi(x) = x + t(x) \nabla \phi(x) \) for some unknown \( t(x) \in \mathbb{R}^+ \). This is in contrast to the case \( p > 1 \) where a dual variational formulation, analogous to (1.3), yields the complete information on the optimal plan \( \Psi \) in terms of the gradient of some potential \( \phi \).

In this paper I consider an object called the \( p \)-Wasserstein distance \( (p > 1) \) of \( \lambda^+ \) to \( \lambda^- \), conditioned on a probability measure \( \mu \):

\[
W^{(p)}(\lambda \| \mu) := \sup_{\phi} \left\{ \int \phi d\lambda \; ; \; \int |\nabla \phi|^q d\mu \leq 1 \right\} \tag{1.4}
\]

where \( q = p/(p - 1) \).

The first result is

\[
W_1(\lambda) = \min_{\mu} \left\{ W^{(p)}(\lambda \| \mu) \; ; \; \int d\mu = 1 \right\}, \quad (p > 1) \tag{1.5}
\]

The problem associated with (1.5) is related to shape optimization, see [7]. In addition, the minimizer \( \mu \) in (1.5) and the corresponding maximizer \( \phi \) in (1.4) or (1.3) play an important role in the \( L_1 \) theory of transport [12]. In fact, the optimal \( \phi \) is, in general, a Lipschitz function which is differentiable \( \mu \) a.e. and satisfies \( |\nabla \phi| = 1 \mu \) a.e. The minimal measure \( \mu \) is called a transport measure. It verifies the weak form of the continuity equation which, under the current notation, takes the form

\[
\nabla \cdot (\mu \nabla \phi) = \frac{\lambda}{W_1(\lambda)}.
\]
The transport measure yields an additional information on the optimal plan Ψ along the transport rays which completes the information included in $\nabla \phi$ [12]. In the context of shape optimization it is related to the optimal distribution of conducting material [7]. See also [19], [23], [24].

The evaluation of the transport measure $\mu$ is therefore an important object of study. It is tempting to approximate the transport measure as a minimizer of (1.5) on a restricted finite space, e.g. for $\mu \in \mathcal{M}^{+,N}$ as defined in (1.2).

However, this cannot be done. Unlike $W_p$, $W^{(p)}(\lambda\|\mu)$ is not continuous in the weak topology of $C^*$ on Borel measures with respect to both $\mu$ and $\lambda$. Indeed, it follows easily that $W^{(p)}(\lambda\|\mu) = \infty$ for any atomic measure $\mu$.

The second result of this paper is

$$W^{(p)}(\lambda\|\mu) = \lim_{n \to \infty} nW_p(\mu + \lambda^+ / n, \mu + \lambda^- / n)$$

(1.6)

Here the limit is in the sense of $\Gamma$ convergence. A somewhat stronger result is obtained if we take the infimum over all probability measures $\mu$:

$$W_1(\lambda) = \lim_{n \to \infty} n \inf_{\mu} W_p(\mu + \lambda^+ / n, \mu + \lambda^- / n)$$

(1.7)

where the convergence is, this time, pointwise in $\lambda$.

The importance of (1.6, 1.7) is that $W^{(p)}(\lambda\|\mu)$ can now be approximated by a weakly continuous function

$$W_n^{(p)}(\lambda^+, \lambda^-\|\mu) := nW_p(\mu + \lambda^+ / n, \mu + \lambda^- / n)$$

Suppose $\mu_0$ is a unique minimizer of (1.5). If $\mu_n$ is a minimizer of $W_n^{(p)}(\lambda^+, \lambda^-\|\mu)$ then the sequence $\{\mu_n\}$ must converge to the transport measure $\mu_0$. In contrast to $W^{(p)}$, $W_n^{(p)}$ is continuous in the $C^*$ topology with respect to $\mu$. Hence $\mu_n$ can be approximated by atomic measures $\mu_n^N \in \mathcal{M}^{+,N}$ (1.2). In particular a transport measure can be approximated by a finite points allocation obtained by minimizing $W_n^{(p)}$ on $\mathcal{M}^{+,N}$ for a sufficiently large $n$ and $N$.

The results (1.5-1.7) can be extended to the case where the cost $D^p$ on $M \times M$ is generalized into an action function on a Riemannian manifold $M \times M$, induced by a Lagrangian function $l : TM \to \mathbb{R}$. This point of view reveals some relations with the Weak KAM Theory dealing with invariant measures of Lagrangian flows on manifolds.

1.1 Overview

Section 2 review the necessary background for the Weak KAM and its relation to optimal transport. Section 3 state the main results (Theorems 1-4), which correspond to (1.5-1.7) for homogeneous Lagrangian on $M \times M$. Section 4 presents the proof of the first of the main results which generalizes (1.4). Finally, Section 5 contains the proofs of the other main results which generalize (1.6, 1.7).
1.2 Standing notations and assumptions

1. \((M, g)\) is a compact, Riemannian Manifold and \(D : M \times M \to \mathbb{R}^+\) is the geodesic distance.

2. \(TM\) (res. \(T^*M\)) the tangent (res. cotangent) bundle of \(M\). The duality between \(v \in T_x M\) and \(p \in T^*_x M\) is denoted by \(\langle \xi, v \rangle \in \mathbb{R}\). The projection \(\Pi : TM \to M\) is the trivialization \(\Pi(x, v) = x\). Likewise \(\Pi^* : T^*M \to M\) is the trivialization \(\Pi^*(x, \xi) = x\).

3. For any topological space \(X\), \(\mathcal{M}(X)\) is the set of Borel measures on \(X\), \(\mathcal{M}_0(X) \subset \mathcal{M}(X)\) the set of such measures which are perpendicular to the constants, \(\mathcal{M}^+(X) \subset \mathcal{M}(X)\) the set of all non-negative measures in \(\mathcal{M}\), and \(\mathcal{M}_1^+(X) \subset \mathcal{M}^+(X)\) the set of normalized (probability) measures. If \(X = M\), the parameter \(X\) is usually omitted.

4. A Borel map \(\Phi : X_1 \to X_2\) induces a mapping \(\Phi^# : \mathcal{M}^+(X_1) \to \mathcal{M}^+(X_2)\) via \(\Phi^#(\mu_1)(A) = \mu_1(\Phi^{-1}(A))\) for any Borel set \(A \subset X_2\).

5. For any \(x, y \in M\) let \(K_{x,y}^T\) be the set of all absolutely continuous paths \(z : [0, T] \to M\) connecting \(x\) to \(y\), that is, \(z(0) = x, z(T) = y\).

6. Given \(\mu_1, \mu_2 \in \mathcal{M}^+\), the set \(\mathcal{P}(\mu_1, \mu_2)\) is defined as all the measures \(\Lambda \in \mathcal{M}^+(M \times M)\) such that \(\pi_{1,\#}\Lambda = \mu_1\) and \(\pi_{2,\#}\Lambda = \mu_2\), where \(\pi_i : \mathcal{M} \times \mathcal{M} \to \mathcal{M}\) defined by \(\pi_1(x, y) = x, \pi_2(x, y) = y\).

7. An hamiltonian function \(h \in C^2(T^*M; \mathbb{R})\) is assumed to be strictly convex and superlinear in \(\xi\) on the fibers \(T^*_x M\), uniformly in \(x \in M\), that is \(h(x, \xi) \geq -C + \hat{h}(\xi)\) where \(\lim_{\|\xi\| \to \infty} \frac{\hat{h}(\xi)}{\|\xi\|} = \infty\).

The Lagrangian \(l : TM \to \mathbb{R}\) is obtained by Legendre duality

\[
l(x, v) = \sup_{\xi \in T^*_x M} \langle \xi, v \rangle - h(x, \xi)
\]

satisfies \(l \in C^2(TM; \mathbb{R})\), and is super linear on the fibers of \(T_x M\) uniformly in \(x\).

8. \(Exp(l) : TM \times \mathbb{R} \to M\) is the flow due to the Lagrangian \(l\) on \(M\), corresponding to the Euler-Lagrange equation

\[
\frac{d}{dt}l_v = l_x.
\]

For each \(t \in \mathbb{R}\), \(Exp(l)^t : TM \to M\) is the exponential map at time \(t\).
2 Background

The weak version of Mather’s theory [20] deals with minimal invariant measures of Lagrangians, and the corresponding Hamiltonians defined on a manifold $M$. In this theory the concept of an orbit $z = z(t) : \mathbb{R} \to M$ is replaced by that of a closed probability measure on $TM$:

$$\mathcal{M}_0^c := \left\{ \nu \in \mathcal{M}_1^+(TM) : \int_{TM} l(x, v) d\nu(x, v) < \infty, \int_{TM} \langle d\phi, v \rangle d\nu = 0 \text{ for any } \phi \in C^1(M) \right\}. \tag{2.1}$$

A minimal (or Mather) measure $\nu_M \in \mathcal{M}_0^c$ is a minimizer of

$$\inf_{\nu \in \mathcal{M}_0^c} \int_{TM} l(x, v) d\nu(x, v) := -E \tag{2.2}$$

It can be shown ([2], [18], [3]) that any minimizer of (2.2) is invariant under the flow induced by the Euler-Lagrange equation on $TM$:

$$\frac{d}{dt} \nabla_x l(x, \dot{x}) = \nabla_x l(x, \dot{x}) \tag{2.3}$$

There is also a dual formulation of (2.2) [17], [29]:

$$\sup_{\mu \in \mathcal{M}_1^+} \inf_{\phi \in C^1(M)} \int_M h(x, d\phi) d\mu = E \tag{2.4}$$

where the maximizer $\mu_M$ is the projection of a Mather measure $\nu_M$ on $M$. The ground energy level $E$, common to (2.2, 2.4), admits several equivalent definitions. Evans and Gomes ([11] [13] [14]) defined $E$ as the effective hamiltonian value

$$E := \inf_{\phi \in C^1(M)} \sup_{x \in M} h(x, d\phi),$$

while the PDE approach to the WKAM theory ([16], [17]) defines $E$ as the minimal $E \in \mathbb{R}$ for which the Hamilton-Jacobi equation $h(x, d\phi) = E$ admits a viscosity sub-solution on $M$. Alternatively $E$ is the only constant for which $h(x, d\phi) = E$ admits a viscosity solution [15]. There are other, equivalent definitions of $E$ known in the literature. We shall meet some of them below.

**Example 2.1.**

i) $l = l_K := |v|^p/(p - 1)$ where $p > 1$. Here $E = 0$ and $\mu_M$ is the volume induced by the metric $g$.

ii) $l(x, v) = (1/2)|v|^2 - V(x)$ where $V \in C^2(M)$ (mechanical Lagrangian). Then $E = \max_{x \in M} V(x)$ and $\mu_M$ of (2.4) is supported at the points of maxima of $V$.

iii) $l(x, v) = l_K(v - W(x))$ where $W$ is a section in $TM$. Then (2.2) implies $E \leq 0$. In fact, it can be shown that $E = 0$ for any choice of $W$. 

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iv) In general, if $P$ is in the first cohomology of $M$ ($H^1(M)$) then $l \mapsto l(x,v) - \langle P, v \rangle$ induced the hamiltonian $h \mapsto h(x, \xi + P)$ and $E = \alpha(P)$ corresponds to the celebrated Mather ($\alpha$) function [20] on the cohomology $H^1(M)$. See also [27].

The Monge problem of mass transportation, on the other hand, has a much longer history. Some years before the the French revolution, Monge (1781) proposed to consider the minimal cost of transporting a given mass distribution to another, where the cost of transporting a unit of mass from point $x$ to $y$ is prescribed by a function $C(x,y)$. In modern language, the Monge problem on a manifold $M$ is described as follows: Given a pair of Borel probability measures $\mu_0, \mu_1$ on $M$, consider the set $K(\mu_0, \mu_1)$ of all Borel mappings $\Phi : M \rightarrow M$ transporting $\mu_0$ to $\mu_1$, i.e

$$\Phi \in K(\mu_0, \mu_1) \iff \Phi \# \mu_0 = \mu_1$$

and look for the one which minimize the transportation cost

$$C(\mu_0, \mu_1) := \inf_{\Phi} \left\{ \int_M C(x, \Phi(x))d\mu_0(x) ; \Phi \in K(\mu_0, \mu_1) \right\} . \quad (2.5)$$

In this generality, the set $K(\mu_0, \mu_1)$ can be empty if, e.g., $\mu_0$ contains an atomic measure while $\mu_1$ does not, so $C(\mu_0, \mu_1) = \infty$ in that case. In 1942, Kantorovich proposed a relaxation of this deterministic definition of the Monge cost. Instead of the (very nonlinear) set $K(\mu_0, \mu_1)$, he suggested to consider the set $P(\mu_0, \mu_1)$ defined in section 1.2-(6). Then, the definition of the Monge metric is relaxed into the linear optimization

$$C(\mu_0, \mu_1) = \min_{\Lambda} \left\{ \int_{M \times M} C(x,y)d\Lambda(x,y) ; \Lambda \in P(\mu_0, \mu_1) \right\} . \quad (2.6)$$

Example 2.2. The Wasserstein distance $W_p$ ($p \geq 1$) is obtained by the power $p$ of the metric $D$ induced by the Riemannian structure:

$$W_p(\mu_0, \mu_1) = \min_{\Lambda} \left\{ \left[ \int_{M \times M} D^p(x,y)d\Lambda(x,y) \right]^{1/p} ; \Lambda \in P(\mu_0, \mu_1) \right\} \quad (2.7)$$

The advantage of this relaxed definition is that $C(\mu_0, \mu_1)$ is always finite, and that a minimizer of (2.6) always exists by the compactness of the set $P(\mu_0, \mu_1)$ in the weak topology $C^*(M \times M)$. If $\mu_0$ contains no atomic points then it can be shown that $C(\mu_0, \mu_1)$'s given by (2.5) and (2.6) coincide [1].

The theory of Monge-Kantorovich (M-K) was developed in the last few decades in a countless number of publications. For updated reference see [12], [28].

Returning now to WKAM, it was observed by Bernard and Buffoni ([4][5]- see also [29]) that the minimal measure and the ground energy can be expressed in terms of the M-K problem subjected to the cost function induced by the Lagrangian (recall section 1.2-5)

$$C_T(x,y) := \inf_{\zeta} \left\{ \int_0^T l(\zeta(s); \dot{\zeta}(s)) \, ds ; \zeta \in K_{x,y}^T \right\} , T > 0 . \quad (2.8)$$

\footnotetext[2]{By convention, the name "Monge problem" is reserved for the metric cost, while "Monge-Kantorovich problem" is usually referred to general cost functions}
Then
\[ C_T(\mu) := C_T(\mu, \mu) = \min_{\Lambda} \left\{ \int_{M \times M} C_T(x, y)d\Lambda(x, y) \ ; \ \Lambda \in \mathcal{P}(\mu, \mu) \right\} \]
and
\[ \min_{\mu} \{ C_T(\mu) : \mu \in \mathcal{M}_1^+ \} = -TE \]
where the minimizers of (2.9) coincide, for any \( T > 0 \), with the projected Mather measure \( \mu_M \) maximizing (2.4) [5]. The action \( C_T \) induces a metric on the manifold \( M \):
\[ (x, y) \in M \times M \mapsto D_E(x, y) = \inf_{T > 0} C_T(x, y) + TE. \]
Example 2.3.

i) For $l(x,v) = |v|^p/(p-1)$, $p > 1$ we get $C_T(x,y) = D(x,y)^p/(p-1)^{p-1}$ while $D_E(x,y) = pE^{1-1/p}D(x,y)/(p-1)$ if $E \geq 0$, $D_E(x,y) = -\infty$ if $E < 0$.

ii) $l(x,v) = (1/2)|v|^2 - V(x)$ where $V \in C^2(M)$ (mechanical Lagrangian). Then $D_E(x,y)$ is the geodesic distance induced by conformal equivalent metric $(M, (E - V)g)$ on $M$, where $E \geq E = \sup_M V$.

It is not difficult to see that either $D_E(x,x) = 0$ for any $x \in M$, or $D_E(x,y) = -\infty$ for any $x, y \in M$. In fact, it follows ([22], [10]) that $D_E(x,y) = -\infty$ for $E < E$ and $D_E(x,x) = 0$ for $E \geq E$ and any $x, y \in M$. Let now $\lambda^+, \lambda^- \in \mathcal{M}^+$ where $\lambda := \lambda^+ - \lambda^- \in \mathcal{M}_0$, that is $\int_M d\lambda = 0$. Let

$$D_E(\lambda) := D_E(\lambda^+, \lambda^-) = \min_{\Lambda} \left\{ \int_{M \times M} D_E(x,y)d\Lambda(x,y) : \Lambda \in \mathcal{P}(\lambda) \right\}$$

be the Monge distance of $\lambda^+$ and $\lambda^-$ with respect to the metric $D_E$. There is a dual formulation of $D_E$ as follows: Consider the set $\mathcal{L}_E$ of $D_E$ Lipschitz functions on $M$:

$$\mathcal{L}_E := \{ \phi \in C(M) : \phi(x) - \phi(y) \leq D_E(x,y) \quad \forall \quad x, y \in M \}$$

Then (see, e.g [12], [26])

$$D_E(\lambda) = \max_{\phi} \left\{ \int_M \phi d\lambda \right\}$$

3 Description of the main results

The object of this paper is to establish some relations between the action $C_T$ and a modified action $\hat{C}_T$ defined below.

3.1 Unconditional action

For given $\lambda \in \mathcal{M}_0$ we generalize (2.1) into

$$\mathcal{M}_\lambda := \left\{ \nu \in \mathcal{M}^+_1(TM) : \int_{TM} l(x,v)d\nu(x,v) < \infty ; \int_{TM} \langle d\phi, v \rangle d\nu = \int_M \phi d\lambda \quad \text{for any} \quad \phi \in C^1(M) \right\}$$

and define

$$\hat{C}(\lambda) := \inf_{\nu} \left\{ \int_{TM} l(x,v)d\nu(x,v) : \nu \in \mathcal{M}_\lambda \right\}.$$ 

The modified action $\hat{C}_T : \mathcal{M}_0 \to \mathbb{R} \cup \{\infty\}$, $T > 0$ have several equivalent definitions as given in Theorem 1 below:

**Theorem 1.** The following definitions are equivalent:

1. $\hat{C}_T(\lambda) := T\hat{C}\left(\frac{\lambda}{T}\right)$.
2. $\hat{C}_T(\lambda) := \min_{\mu} \sup_{\phi} \left\{ \int_M -Th(x,d\phi)d\mu + \phi d\lambda \ ; \ \mu \in M^+_1, \ \phi \in C^1(M) \right\}.$

3. $\hat{C}_T(\lambda) := \max_{E \geq E} [D_E(\lambda) - ET].$

In addition if $T_c := D'_E(\lambda) < \infty$ then for $T \geq T_c,$

$$\hat{C}_T(\lambda) = \hat{C}_{T_c}(\lambda) - T E.$$

In that case the minimizer $\mu^T_\lambda \in M^+_1$ of (3), $T > T_c$ is given by

$$\mu^T_\lambda = \frac{T_c}{T} \mu^T_{\lambda^+} + \left(1 - \frac{T_c}{T}\right) \mu_M,$$

where $\mu_M$ is the projected Mather measure.

**Remark 3.1.** Note that $D_E(\lambda)$ (2.11, 2.13) is a monotone non-decreasing and concave function of $E$ while $D_E(\lambda) > -\infty$ by definition. Hence the right-derivative of $D'_E(\lambda)$ as a function of $E$ is defined and positive (possibly $+\infty$ at $E = E_c$).

**Remark 3.2.** A special case of Theorem 1 was introduced in [30].

For the next result we need a two technical assumptions:

**H** $\text{H}_1$ There exists a sequence of smooth, positive mollifiers $\delta_\varepsilon : M \times M \to \mathbb{R}^+$ such that, for any $\phi \in C^0(M)$ (res. $\phi \in C^1(M)$)

$$\lim_{\varepsilon \to 0} \delta_\varepsilon * \phi = \phi$$

where the convergence is in $C^0(M)$ (res. $C^1(M)$) and for any $\varepsilon > 0$ and $\phi \in C^1(M)$

$$\delta_\varepsilon * d\phi = d(\delta_\varepsilon * \phi).$$

$\text{H}_2$ For any $(x,p) \in T^*M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $h(x,\xi) - h(y,\xi_y) \leq \varepsilon(h(x,\xi) + 1)$ provided $D(x,y) < \delta.$ Here $\xi_y$ is obtained by parallel translation of $(x,\xi)$ to $y.$

**Remark 3.3.** $\text{H}_1$ holds for homogeneous spaces, e.g the flat $d-$torus $\mathbb{R}^d/\mathbb{Z}^n$ or the sphere $S^{d-1} = SO(d)/SO(1).$

$\text{H}_2$ holds, in particular, for any mechanical hamiltonian with continuous potential.

**Theorem 2.** Assume $\text{H}_1 + \text{H}_2.$ For any $\lambda = \lambda^+ - \lambda^-$ where $\lambda^\pm \in M^+_1,$

$$\hat{C}_T(\lambda) = \lim_{\varepsilon \to 0} \min_{\mu \in M^+_1} \varepsilon^{-1} C_{\varepsilon T}(\mu + \varepsilon \lambda^-, \mu + \varepsilon \lambda^+).$$

As an application of Theorem 2 we may consider the case where the lagrangian $l$ is homogeneous with respect to a Riemannian metric $g(x):
Example 3.1. If \( l(x, v) = |v|^p/(p - 1) \) where \( p > 1 \). Then \( C_T(x, y) = \frac{D^p(x, y)}{(p-1)^{p-1}} \) while \( D_E(x, y) = \frac{p}{p-1} E^{(p-1)/p} D(x, y) \) and \( E = 0 \). It follows that

\[
\hat{C}_T(\lambda) = \frac{W_1^p(\lambda)}{(p-1)^{p-1}}, \quad \varepsilon^{-1} C_{\varepsilon T}(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) = \frac{W_1^p(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-)}{(p-1)^{p-1}} \quad (3.3)
\]

where the Wasserstein distance \( W_p \) is defined in (2.7). Hence, by Theorem 1 and Theorem 2

\[
W_1(\lambda) = \lim_{\varepsilon \to 0} \inf_{\mu \in \mathcal{M}_1^+} W_p(\mu + \varepsilon \lambda^-, \mu + \varepsilon \lambda^+) .
\]

Remark 3.4. The optimal transport description of the weak KAM theory (2.9) can be considered as a special case of Theorem 2 where \( \lambda = 0 \). Indeed \( \inf_{\mu \in \mathcal{M}_1^+} \varepsilon^{-1} C_{\varepsilon T}(\mu, \mu) = -TE \) by (2.9). On the other hand, since \( D_E(0) = 0 \) for any \( E \geq E \) it follows that \( T_c = 0 \), hence \( \hat{C}_T(0) = 0 \) so \( \hat{C}_T = -TE \) as well by the last part of Theorem 1.

3.2 Conditional action

There is also an interest in the definition of action (and metric distance) conditioned with a given probability measure \( \mu \in \mathcal{M}_0^+ \). We introduce these definitions and reformulate parts of the main results Theorems 1-2 in terms of these.

For a given \( \mu \in \mathcal{M}_1^+ \) and \( E \geq E \), let

\[
\mathcal{H}_E(\mu) := \left\{ \phi \in C^1(M) : \int_M h(x, d\phi) d\mu \leq E \right\} . \quad (3.4)
\]

In analogy with (2.13) we define the \( \mu \)-conditional metric on \( \lambda \in \mathcal{M}_0^+ \):

\[
\mathcal{D}_E(\lambda \| \mu) := \sup_{\phi} \left\{ \int_M \phi d\lambda \ : \ \phi \in \mathcal{H}_E(\mu) \right\} . \quad (3.5)
\]

The conditioned, modified action with respect to \( \mu \in \mathcal{M}_1^+ \) is defined in analogy with Theorem 1 (2, 3)

\[
\hat{C}_T(\lambda \| \mu) := \max_{E \geq E} \mathcal{D}_E(\lambda \| \mu) - ET \equiv \sup_{\phi \in C^1(M)} \int_M -Th(x, d\phi) d\mu + \phi d\lambda . \quad (3.6)
\]

Example 3.2. As in Example 3.1, \( l(x, v) = |v|^p/(p - 1) \) implies \( h(\xi) = q^{-q}|\xi|^q \) where \( q = p/(p - 1) \). Then (3.4, 3.5) is related to (1.4), that is \( W_1^{(p)}(\lambda \| \mu) = \mathcal{D}_E(\lambda \| \mu) \) where \( E = q^{-q} \) or

\[
\mathcal{D}_E(\lambda \| \mu) = qE^{1/q} W_1^{(p)}(\lambda \| \mu), \quad \hat{C}_T(\lambda \| \mu) = \frac{q - 1}{p/(q-1)} \left( W_1^{(p)}(\lambda \| \mu) \right)^p \quad (3.7)
\]

Remark 3.5. It seems there is a relation between this definition and the tangential gradient [6]. There are also possible applications to optimal network and irrigation theory, where one wishes to minimize \( D(\lambda \| \mu) \) over some constrained set of \( \mu \in \mathcal{M}_1^+ \) (the irrigation network) for a prescribed \( \lambda \) (representing the set of sources and targets). See, e.g. [8], [9] and the ref. within.
The next result is

**Theorem 3.** For any $\lambda \in \mathcal{M}$,

$$
\mathcal{D}_E(\lambda) = \min_{\mu \in \mathcal{M}_T} \mathcal{D}_E(\lambda \parallel \mu), \\
\hat{C}_T(\lambda) = \min_{\mu \in \mathcal{M}_T} \hat{C}_T(\lambda \parallel \mu).
$$

The analog of Theorem 2 holds for the conditional action as well. However, we can only prove the $\Gamma$-convergence in that case. Recall that a sequence of functionals $F_n : X_n \to \mathbb{R} \cup \{\infty\}$ is said to $\Gamma$-converge to $F : X \to \mathbb{R} \cup \{\infty\}$ ($\Gamma - \lim_{n \to \infty} F_n = F$) if and only if

(i) $X_n \subseteq X$ for any $n$.

(ii) For any sequence $x_n \in X_n$ converging to $x \in X$ in the topology of $X$,

$$
\liminf_{n \to \infty} F_n(x_n) \geq F(x).
$$

(iii) For any $x \in X$ there exists a sequence $\hat{x}_n \in X_n$ converging to $x \in X$ in the topology of $X$ for which

$$
\lim_{n \to \infty} F_n(\hat{x}_n) = F(x).
$$

In Theorem 4 below the $\Gamma$-convergence is related to the special case where $X_n = X$:

**Theorem 4.** Let $X_n = \mathcal{M}_0 \times \mathcal{M}_T^+ = X$ and $F_n(\lambda, \mu) := nC_T/n(\mu + \lambda^-/n, \mu + \lambda^+/n)$. Then

$$
\hat{C}_T(\cdot \parallel \cdot) = \Gamma - \lim_{n \to \infty} F_n.
$$

From Theorem 4 and Theorem 2 it follows immediately

**Corollary 3.1.** In addition, if $\mu_n$ is a minimizer of $F_n$ in $\mathcal{M}_T^+$ then any converging subsequence of $\mu_n$, $n \to \infty$, converges to a minimizer of $\hat{C}(\lambda \parallel \cdot)$ in $\mathcal{M}_T^+$.

Finally, we note that (1.7) is a special case of Theorem 4. Using Examples 3.1, 3.2 with $\varepsilon = 1/n$, recalling $(q - 1)^{-1} = p - 1$ we obtain

**Corollary 3.2.**

$$
W_1(\lambda) = \lim_{n \to \infty} n \min_{\mu \in \mathcal{M}_T^+} W_p(\mu + \lambda^+/n, \mu + \lambda^-/n).
$$

4 Proof of Theorems 1&3

We first show that $\hat{C}(\lambda) < \infty$ (recall (3.2)).

**Lemma 4.1.** For any $\lambda \in \mathcal{M}_0$, $\mathcal{M}_\lambda \neq \emptyset$. In particular, since the Lagrangian $l$ is bounded from below, $\hat{C}(\lambda) < \infty$. 

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Proof. It is enough to show that there exists a compact set \( K \subset TM \) and a sequence \( \{\lambda_n\} \subset \mathcal{M}_0 \) converging weakly to \( \lambda \) such that for each \( n \) there exists \( \nu_n \in \mathcal{M}_{\lambda_n} \) whose support is contained in \( K \). Indeed, such a set is compact and there exists a weak limit \( \nu = \lim_{n \to \infty} \nu_n \) which satisfies \( \lim_{n \to \infty} v\nu_n = v\nu \) as well. Hence, if \( \phi \in C^1(M) \) then

\[
\lim_{n \to \infty} \int_M \langle d\phi, v \rangle d\nu_n = \int_M \langle d\phi, v \rangle d\nu , \quad \lim_{n \to \infty} \int_M \phi d\lambda_n = \int_M \phi d\lambda .
\]

Since \( \nu_n \in \mathcal{M}_{\lambda_n} \) we get

\[
\int_M \langle d\phi, v \rangle d\nu_n = \int_M \phi d\lambda_n
\]

for any \( n \), so the same equality holds for \( \nu \) as well.

Now, we consider

\[
\lambda_n = \alpha_n \sum_{j=1}^n (\delta_{x_j} - \delta_{y_j})
\]

(4.1)

where \( x_j, y_j \in M \) and \( \alpha_n > 0 \). For any pair \( (x_j, y_j) \) consider a geodesic arc corresponding to the Riemannian metric which connect \( x \) to \( y \), parameterized by the arc length: \( z_j : [0, 1] \to M \) and \( |\dot{z}| = D(x_j, y_j) \) (recall section 1.2-(1)). Then

\[
\nu_n := \alpha_n \sum_{j=1}^n \int_0^1 \delta_{z_j(t), v-\dot{z}_j(t)} dt
\]

satisfies for any \( \phi \in C^1(M) \)

\[
\int_M \langle d\phi, v \rangle d\nu_n = \alpha_n \sum_{j=1}^n \int_0^1 \langle d\phi(z_j(s), \dot{z}_j(s)) \dot{z}_j(t) \rangle dt = \alpha_n \sum_{j=1}^n \int_0^1 \frac{d}{dt} \phi(z_j(s)) dt
\]

\[
= \alpha_n \sum_{j=1}^n [\phi(y_j) - \phi(x_j)] = \int_M \phi d\lambda_n
\]

(4.2)

hence \( \nu_n \in \mathcal{M}_{\lambda_n} \). Finally, we can certainly find such a sequence \( \lambda_n \) of the form (4.1) which converges weakly to \( \lambda \). \( \square \)

4.1 Point distances and Hamiltonians

For \( E \in \mathbb{R} \), let \( \sigma_E : TM \to \mathbb{R} \) the support function of the level surface \( h(x, \xi) \leq E \), that is:

\[
\sigma_E(x, v) := \sup_{\xi \in T^*_xM} \{ \langle \xi, v \rangle(x) : h(x, \xi) \leq E \} .
\]

(4.3)

It follows from our standing assumptions (Section 1.2-7) that \( \sigma_E \) is differentiable as a function of \( E \) for any \( (x, v) \in TM \). For the following Lemma see e.g. [25].

Recall that

\[
D_E(x, y) := \inf_{T > 0} C_T(x, y) + ET
\]

(4.4)

where \( C_T \) as defined in (2.8). Recall also section 1.2-5:
Lemma 4.2.

\[ D_E(x, y) = \inf_{z \in \mathcal{K}_{1, y}^1} \int_0^1 \sigma_E(z(s), \dot{z}(s)) \, ds . \]  

(4.5)

Given \( x \in M \), let

\[ E := \inf \{ E \in \mathbb{R} ; D_E(x, x) > -\infty \} \]

(4.6)

For the following Lemma see [21] (also [27]):

**Lemma 4.3.** \( E \) is independent of \( x \in M \). The definitions (4.6) and (2.2) and (2.4) are equivalent. If \( E \geq \hat{E} \) then \( D_E(x, y) > -\infty \) for any \( x, y \in M \) and, in addition

i) \( D_E(x, x) = 0 \) for any \( x \in M \).

ii) For any \( x, y, z \in M \), \( D_E(x, z) \leq D_E(x, y) + D_E(y, z) \)

From (4.4), Lemma 4.2 and the continuity of \( \sigma_E \) with respect to \( E \geq \hat{E} \) we get

**Corollary 4.1.** If \( E \geq \hat{E} \) then for any \( x, y \in M \), \( D_E(x, y) \) is continuous, monotone non-decreasing and concave as a function of \( E \).

Note that the differentiability of \( \sigma_E \) with respect to \( E \) does not imply that \( D_E(x, y) \) is differentiable for each \( x, y \in M \). However, since \( D_E(x, y) \) is a concave function of \( E \) for each \( x, y \in M \), it is differentiable for Lebesgue almost any \( E > \hat{E} \). We then obtain by differentiation

**Lemma 4.4.** If \( E \) is a point of differentiability of \( D_E(x, y) \) then there exists a geodesic arc \( z \in \mathcal{K}_{1, y}^1 \) realizing (4.5) such that the \( E \) derivative of \( D_E(x, y) \) is given by

\[ T_E(x, y) := \frac{d}{dE} D_E(x, y) = \int_0^1 \sigma_E'(z(s), \dot{z}(s)) \, ds , \]

(4.7)

where \( \sigma_E' \) is the \( E \) derivative of \( \sigma_E \). Moreover

\[ D_E(x, y) = C T_E(x, y) + E T_E(x, y) . \]

(4.8)

From (4.3) we get \( \sigma_E(x, v) \leq |v| \max \{|p| : h(x, \xi) \leq E \} \). From our standing assumption on \( h \) (section 1.2-(7)) and (4.5) we obtain

**Lemma 4.5.** For any \( x, y \in M \) and \( E \geq \hat{E} \)

\[ D_E(x, y) \leq \hat{h}^{-1}(E + C)D(x, y) \]

In particular

\[ \lim_{E \to \infty} E^{-1} D_E(x, y) = 0 \]

(4.9)

uniformly on \( M \times M \).

**Corollary 4.2.** For \( E \geq \hat{E} \), the set \( \mathcal{L}_E \) (2.12) is contained in the set of Lipschitz functions with respect to \( D \), and \( \mathcal{L}_E \) is locally compact in \( C(M) \).
Given $\phi \in C^1(M)$ let
\[ \mathcal{H}(\phi) := \sup_{x \in M} h(x, d\phi) . \]  
(4.10)

We extend the definition of $\mathcal{H}$ to the larger class of Lipschitz functions by the following

Lemma 4.6. If $\phi \in C^1(M)$ then
\[ \mathcal{H}(\phi) = \min_{E \geq E} \{ E; \phi \in \mathcal{L}_E \} , \]
where $\mathcal{L}_E$ as defined in (2.12).

Proof. First we show that if $\phi \in \mathcal{L}_E \cap C^1(M)$ then $h(x, d\phi) \leq E$ for all $x \in M$. Indeed, for any $x, y \in M$ and any curve $z(\cdot)$ connecting $x$ to $y$

\[ \phi(y) - \phi(x) = \int_0^1 d\phi(z(t)) \cdot \dot{z}(t) dt \leq D_E(x, y) \leq \int_0^1 \sigma_E(z(t), \dot{z}(t)) dt \]

hence $d\phi(x) \cdot v \leq \sigma_E(x, v)$ for any $v \in T_xM$. Then, by definition, $d\phi(x)$ is contained in any supporting half space which contains the set $Q_x(E) := \{ \xi \in T^*_xM; h(x, \xi) \leq E \}$. Since this set is convex by assumption, it follows that $d\phi \in Q_x(E)$, so $h(x, d\phi) \leq E$ for any $x \in M$. Hence $\mathcal{H}(\phi) \leq E$.

Next we show the opposite inequality $h(x, d\phi) \geq E$ for all $x \in M$. Recall (4.8). Then for any $\varepsilon > 0$ we can find $T_\varepsilon > 0$ and $z_\varepsilon \in K_{T_\varepsilon x,y}$ so
\[ D_E(x, y) \geq \int_0^{T_\varepsilon} l(z_\varepsilon(t), \dot{z}_\varepsilon(t)) dt + (E - \varepsilon) T_\varepsilon . \]  
(4.11)

Next, for $a.e t \in [0, T_\varepsilon]$

\[ h(z_\varepsilon(t), d\phi(z_\varepsilon(t))) \geq \dot{z}_\varepsilon(t) \cdot d\phi(z_\varepsilon(t)) - l(z_\varepsilon(t), \cdot z_\varepsilon(t)) . \]  
(4.12)

Integrate (4.12) from 0 to $T_\varepsilon$ and use $z_\varepsilon \in K_{T_\varepsilon x,y}$, (4.11, 4.12) and the definition of $\mathcal{L}_E$ to obtain

\[ T_\varepsilon^{-1} \int_0^{T_\varepsilon} h(z_\varepsilon(t), d\phi(z_\varepsilon(t))) dt \geq T_\varepsilon^{-1} [\phi(y) - \phi(x)] - T_\varepsilon^{-1} \int_0^{T_\varepsilon} l(z_\varepsilon(t), \cdot z_\varepsilon(t)) dt \geq E - \varepsilon . \]

Hence, the supremum of $h(x, d\phi)$ along the orbit of $z_\varepsilon$ is, at least, $E - \varepsilon$. Since $\varepsilon$ is arbitrary, then $\mathcal{H}(\phi) \geq E$.

4.2 Measure distances and Hamiltonians

From Lemma 4.6 and Corollary 4.2 we extend the definition of $\mathcal{H}$ to the space $\text{Lip}(M)$ of Lipschitz functions on $M$. Let now define for $\lambda \in \mathcal{M}_0$

\[ \mathcal{H}^*_T(\lambda) := \sup_{\phi \in \text{Lip}(M)} \left\{ -T \mathcal{H}(\phi) + \int_M \phi d\lambda \right\} \in \mathbb{R} \cup \{\infty\} . \]  
(4.13)
Proposition 4.1. For any $\lambda \in \mathcal{M}_0$
\[ \overline{H}^*_T(\lambda) = \sup_{E \geq E} \{ D_E(\lambda) - TE \} . \] (4.14)

Proof. By definition of $\overline{H}^*$ and Lemma 4.6,
\[ \overline{H}^*_T(\lambda) = \sup_{\phi \in \text{Lip}(\mathcal{M})} \left[ \int_M \phi d\lambda - T \overline{H}(\phi) \right] = \sup_{E \geq E} \left[ \int_M \phi d\lambda - TE ; \phi \in \mathcal{L}_E \right] = \sup_{E \geq E} \left[ \int_M \phi d\lambda - TE ; \phi \in \mathcal{L}_E \right] = \sup_{E \geq E} \{ D_E(\lambda) - TE \}, \] (4.15)
where we used the duality relation given by (2.13). \qed

Corollary 4.3. $\overline{H}_T^*$ is weakly continuous on $\mathcal{M}_0$.

Proof. For each $E \geq E$, the Monge-Kantorovich metric $D_E : \mathcal{M}_0 \to \mathbb{R}$ is continuous on $\mathcal{M}_0$ (under weak* topology). Indeed, it is u.s.c. by (2.11) and l.s.c. by the dual formulation (2.13).

Also, for each $\lambda \in \mathcal{M}^+_1$, $D_E(\lambda)$ is concave and finite in $E$ for $E \geq E$. It follows that $D$ is mutually continuous on $\mathcal{E}, \infty \times \mathcal{M}_0$. From (4.9) we also get that $D$ is coercive on $\mathcal{M}_0$, that is $\lim_{E \to \infty} E^{-1}D_E(\lambda) = 0$ locally uniformly on $\mathcal{M}_0$. These imply that $\overline{H}_T^*$ is continuous on $\mathcal{M}_0$ via (4.14). \qed

We return now to Corollary 4.1 and Lemma 4.4. It follows that for any countable dense set $A \subset M$ there exists a (possibly empty) set $N \subset \mathcal{E}, \infty \setminus A$ of zero Lebesgue measure such that $D_E(x, y)$ is differentiable in $E \in \mathcal{E}, \infty \setminus N$, for any $x, y \in A$. Let $\mathcal{M}(A) \subset \mathcal{M}_0$ be the set of all measures in $\mathcal{M}_0$ which are supported on a finite subset of $A$, and such that $\lambda(\{x\})$ is rational for any $x \in A$. Again, since $\mathcal{M}(A)$ is countable, it follows by Corollary 4.1 that $D_E(\lambda)$ is differentiable (as a function of $E$) for any $\lambda \in \mathcal{M}(A)$ and any $E \in \mathcal{E}, \infty \setminus -N$ for a (perhaps larger) set $N$ of zero Lebesgue measure. It is also evident that $\mathcal{M}_0$ is the weak closure of $\mathcal{M}(A)$.

Lemma 4.7. For any $\lambda^+ - \lambda^- \equiv \lambda \in \mathcal{M}(A)$ and $E \in \mathcal{E}, \infty \setminus -N$, there exists an optimal plan $\Lambda_E^\lambda \in \mathcal{P}(\lambda^+, \lambda^-)$ realizing
\[ \int_{M \times M} D_E(x, y)d\Lambda_E^\lambda(x, y) = \min_{\Lambda \in \mathcal{P}(\lambda^+, \lambda^-)} \int_{M \times M} D_E(x, y)d\Lambda(x, y) \equiv D_E(\lambda) \] (4.16)
for which
\[ \frac{d}{dE} D_E(\lambda) = \sum_{x, y \in A} \Lambda_E^\lambda(\{x, y\})T_E(x, y) . \] (4.17)

Proof. Let $E_n \searrow E$. For each $n$, set $\Lambda_{E_n}^\lambda$ be a minimizer of (4.16) subjected to $E = E_n$. We choose a subsequence so that the limit
\[ \Lambda_{E_n}^{\lambda^+}(\{x, y\}) := \lim_{n \to \infty} \Lambda_{E_n}^\lambda(\{x, y\}) \] (4.18)
exists for any \( x, y \in A \). Evidently, \( \Lambda_{E+}^\lambda \in \mathcal{P}(\lambda^+, \lambda^-) \) is an optimal plan for (4.16). Next,

\[
D_{E_n}(\lambda) - D_E(\lambda) \geq \sum_{x,y \in A} \Lambda_{E_n}^\lambda(\{x, y\}) (D_{E_n}(x, y) - D_E(x, y))
\]

Divide by \( E_n - E > 0 \) and let \( n \to \infty \), using (4.18) and (4.7) we get

\[
\frac{d}{dE} D_E(\lambda) \geq \sum_{x,y \in A} \Lambda_{E+}^\lambda(\{x, y\}) T_E(x, y) .
\] (4.19)

We repeat the same argument for a sequence \( E_n \not\to E \) for which \( \Lambda_{E-}^\lambda(\{x, y\}) := \lim_{n \to \infty} \Lambda_{E_n}^\lambda(\{x, y\}) \) and get

\[
\frac{d}{dE} D_E(\lambda) \leq \sum_{x,y \in A} \Lambda_{E-}^\lambda(\{x, y\}) T_E(x, y) .
\] (4.20)

Again \( \Lambda_{E-}^\lambda \) is an optimal plan as well. If \( \Lambda_{E-}^\lambda = \Lambda_{E+}^\lambda \) then we are done. Otherwise, define \( \Lambda_{E-}^\lambda \) as a convex combination of \( \Lambda_{E-}^\lambda \) and \( \Lambda_{E+}^\lambda \) for which the equality (4.17) holds due to (4.19, 4.20).

Given \( x, y \in M \), let \( E \) be a point of differentiability of \( D_E(x, y) \), and \( z_{x,y}^E : [0, 1] \to M \) a geodesic arc connecting \( x, y \) and realizing (4.7). Then \( d\tau_{x,y}^E := \sigma_E'(z_{x,y}^E, \dot{z}_{x,y}^E) \, ds \) is a non-negative measure on \([0, 1]\), and (4.7) reads \( T_E(x, y) = \int_0^1 d\tau_{x,y}^E \). Let \( \mu_{x,y}^E \) be the measure on \( M \) obtained by pushing \( \tau_{x,y}^E \) from \([0, 1]\) to \( M \) via \( z_{x,y}^E \):

\[
\mu_{x,y}^E := (z_{x,y}^E)_# \tau_{x,y}^E \in \mathcal{M}^+ ,
\]

that is, for any \( \phi \in C(M) \),

\[
\int_M \phi d\mu_{x,y}^E := \int_0^1 \phi (z_{x,y}^E(t)) \, d\tau_{x,y}^E .
\] (4.21)

**Definition 4.1.** For any \( \lambda \in \mathcal{M}(A) \) and \( E \in ]E, \infty[-N \) let

\[
\mu_{\lambda}^E := \sum_{x,y \in A} \Lambda_{\lambda}^E(\{x, y\}) \mu_{x,y}^E
\]

where \( \mu_{x,y}^E \) are as given in (4.21) and \( \Lambda_{\lambda}^E \) is the particular optimal plan given in Lemma 4.7.

**Remark 4.1.** Note that \( \int_M d\mu_{\lambda}^E = D'_E(\lambda) \) for any \( \lambda \in \mathcal{M}_0(A) \) and \( E \in ]E, \infty[-N \) by Lemma 4.7, where \( D'_E(\lambda) = (d/dE)D_E(\lambda) \).

**Definition 4.2.** For any \( \lambda \in \mathcal{M}_0, T > 0 \), \( E(\lambda, T) \) is the maximizer of (4.14), that is

\[
D_{E(\lambda, T)}(\lambda) - T E(\lambda, T) = \mathcal{H}_T^\lambda(\lambda) .
\]
By Corollary 4.1 (in particular, the concavity of \( D_E(\lambda) \) with \( E \)) we obtain

**Lemma 4.8.** If \( E(\lambda, T) > E \) then

\[
\frac{d^+}{dE} D_E(\lambda) \bigg|_{E=E(\lambda, T)} \leq T \leq \frac{d^-}{dE} D_E(\lambda, T) \bigg|_{E=E(\lambda, T)}
\]

where \( d^+ / dE \) (res. \( d^- / dE \)) stands for the right (res. left) derivative. If \( E(\lambda, T) = E \) then

\[
\frac{d^+}{dE} D_E(\lambda) \bigg|_{E=E} \leq T.
\]

### 4.3 Proof of Theorem 1 \((1 \iff 2)\)

First we note that it is enough to assume \( T = 1 \). Consider

\[
\mathcal{F}(\mu, \phi) := \int_M -h(x, d\phi) d\mu + \phi d\lambda
\]  

(4.22)

where \( \lambda \in \mathcal{M}_0 \) is prescribed. Evidently, \( \mathcal{F} \) is convex lower semi continuous (l.s.c) in \( \mu \) on \( \mathcal{M}^+_1 \) and concave upper semi continuous (u.s.c) in \( \phi \) on \( C^1(M) \). Since \( \mathcal{M}^+_1 \) is compact, the Minimax Theorem implies

\[
\sup_{\phi \in C^1(M)} \min_{\mu \in \mathcal{M}^+_1} \mathcal{F}(\mu, \phi) = \min_{\mu \in \mathcal{M}^+_1} \sup_{\phi \in C^1(M)} \mathcal{F}(\mu, \phi).
\]  

(4.23)

Next define

\[
\mathcal{G}(\nu, \phi) := \int_{TM} (l(x, v) - \langle d\phi, v \rangle) d\nu + \int_M \phi d\lambda.
\]

on \( \mathcal{M}^+_1(TM) \times C^1(M) \). Then (recall (3.1))

\[
\sup_{\phi \in C^1(M)} \inf_{\nu \in \mathcal{M}^+_1(TM)} \mathcal{G}(\nu, \phi) \leq \inf_{\nu \in \mathcal{M}^+_1} \int_{TM} l(x, v) d\nu \equiv \tilde{C}(\lambda).
\]  

(4.24)

Now

\[
\tilde{\mathcal{G}}(\nu) := \sup_{\phi \in C^1(M)} \mathcal{G}(\nu, \phi) = \begin{cases} \int_{TM} l(x, v) d\nu & \text{if } \nu \in \mathcal{M}^+_1 \\ \infty & \text{if } \nu \notin \mathcal{M}^+_1 \end{cases}.
\]

We recall, again, from the Minimax Theorem that the inequality in (4.24) turns into an equality provided the set \( \{ \nu \in \mathcal{M}^+_1(TM); \; \tilde{\mathcal{G}}(\nu) \leq \tilde{C}(\lambda) \} \) is compact. However \( \tilde{C}(\nu) < \infty \) by Lemma 4.1. Since \( l \) is super linear in \( v \) uniformly in \( x \) (see section 1.2-7) it follows that the sub-level set \( \{ \nu \in \mathcal{M}^+_1; \; \int_{TM} l(x, v) d\nu \leq C < \infty \} \) is tight for any constant \( C \), hence compact.

Next

\[
\int_{TM} (l(x, v) - \langle d\phi, v \rangle) d\nu(x, v) + \int_M \phi d\lambda
\]

\[
= \int_M \phi d\lambda - h(x, d\phi) d\mu + \int_{TM} (l(x, v) - \langle d\phi, v \rangle + h(x, d\phi)) d\nu(x, v).
\]  

(4.25)
where $\mu = \Pi_{\#} \nu$. By the Young inequality $l(x, v) + h(x, \xi) \geq \langle \xi, v \rangle_\pi$ for any $\xi \in T^*_x M$, $v \in T_x M$ with equality if and only if $v = h_\xi(x, d\phi(x))$. So, the second term on the right of (4.25) is non-negative, but, for any $\mu \in M^+_1$

\[
\inf_{\nu} \left\{ \int_{TM} (l(x, v) - \langle d\phi, v \rangle) \, dv(x, v) : \nu \in M^+_1(TM), \Pi_{\#} \nu = \mu \right\} = -\int_M h(x, d\phi) \, d\mu
\]

is realized for $\nu = \delta_{v-h_\xi(x, d\phi(x))} \oplus \mu \in M^+_1(TM)$. From this and (4.25) we obtain

\[
\inf_{\nu \in M^+_1(TM)} G(\nu, \phi) = \inf_{\mu \in M^+_1} F(\phi, \mu)
\]

and this part of the Theorem follows from (4.23).

□

4.4 Proof of Theorem 1: (2 $\leftrightarrow$ 3)

We now define, for any $\lambda \in M_0$, a measure $\mu_\lambda \in M^+_1$ in the following way:

Assume, for now, that $\lambda \in M(A)$. If $E(\lambda, T) \in ]E, \infty[-N$ then define $\mu_\lambda = \mu_{E(\lambda, T)}$ according to Definition 4.1. Otherwise, fix a sequence $E_n \in ]E, \infty[-N$ such that $E_n \nearrow E(\lambda, T)$. Similarly, let $E_n \in ]E, \infty[-N$ such that $E_n \searrow E(\lambda, T)$.

Then $\mu_{E_n}^+$ and $\mu_{E_n}^-$ are given by Definition 4.1 for any $n$. Let $\mu_\lambda^+$ be a weak limit of the sequence $\mu_{E_n}^+$, and, similarly, $\mu_\lambda^-$ be a weak limit of the sequence $\mu_{E_n}^-$. By Lemma 4.8 and Remark 4.1 we get

\[
\int_M d\mu_\lambda^+ \leq T \leq \int_M d\mu_\lambda^- . \tag{4.26}
\]

If $E(\lambda, T) = E$ then we can still define $\mu_\lambda^+$, and it satisfies the left inequality of (4.26).

**Definition 4.3.** For any $\lambda \in M_0$, let $\mu_\lambda$ defined in the following way:

i) If $\lambda \in M_0(A)$ then

- If $E(\lambda, T) > E$ then $\mu_\lambda$ is a convex combination of $T^{-1} \mu_\lambda^+, T^{-1} \mu_\lambda^-$ given by (4.26) such that $\mu_\lambda \in M^+_1$ (that is, $\int d\mu_\lambda, = 1$).

- If $E(\lambda, T) = E$ then

\[
\mu_\lambda = T^{-1} \mu_\lambda^+ + \left(1 - T^{-1} \int_M d\mu_\lambda^+ \right) \mu_M
\]

where $\mu_M$ is a projected Mather measure.

ii) For $\lambda \notin M_0(A)$, let $\lambda_n \in M_0(A)$ be a sequence converging weakly to $\lambda$. Then $\{\mu_\lambda\}$ is the set of weak limits of the sequence $\mu_{\lambda_n}$.

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Define
\[ Q(\lambda, \mu) := \sup_{\phi \in C^1(M)} \left\{ - \int_M h(x, d\phi) d\mu + \int_M \phi d\lambda \right\} \in \mathbb{R} \cup \{\infty\}, \quad Q_T(\lambda, \mu) := Q(\lambda, T\mu). \] (4.28)

Recall from 1⇒2 that
\[ \hat{\mathcal{C}}_T(\lambda) = \inf_{\mu \in \mathcal{M}_1^+} Q_T(\lambda, \mu) = \inf_{\mu \in \mathcal{M}_1^+} Q(\lambda, T\mu). \] (4.29)

Also, from (4.13), (4.10) and Proposition 4.1
\[ \mathcal{H}_T^*(\lambda) \leq Q_T(\lambda, \mu) \quad \forall \mu \in \mathcal{M}_1^+. \] (4.30)

We have to show that
\[ \mathcal{H}_T^*(\lambda) = \inf_{\mu \in \mathcal{M}_1^+} Q_T(\lambda, \mu) \] (4.31)

for any \( \lambda \in \mathcal{M}_0 \). It is enough to prove (4.31) for a dense set of in \( \mathcal{M}_0 \), say for any \( \lambda \in \mathcal{M}_0(A) \). Suppose (4.31) holds for a sequence \( \{\lambda_n\} \subset \mathcal{M}_0(A) \) converging weakly to \( \lambda \in \mathcal{M}_0 \), that is, \( \mathcal{H}_T(\lambda_n) = \hat{\mathcal{C}}_T(\lambda_n) \). Since \( \mathcal{H}_T \) is weakly continuous by Corollary 4.3 we get \( \mathcal{H}_T(\lambda) = \lim_{n \to \infty} \mathcal{H}_T(\lambda_n) \). On the other hand we recall that, according to definition 2 of Theorem 1, \( \hat{\mathcal{C}}_T : \mathcal{M}_0 \mapsto \mathbb{R} \) is l.s.c. So \( \lim_{n \to \infty} \hat{\mathcal{C}}_T(\lambda_n) \geq \hat{\mathcal{C}}_T(\lambda) \), hence \( \mathcal{H}_T(\lambda) \geq \hat{\mathcal{C}}_T(\lambda) \). By (4.29, 4.30) we get (4.31) for any \( \lambda \in \mathcal{M}_0 \).

The proof of 2 ⇒ 3 then follows from

**Lemma 4.9.** For any \( \lambda \in \mathcal{M}_0(A) \)
\[ Q_T(\lambda, \mu_\lambda) = \mathcal{H}_T^*(\lambda) \] (4.32)

holds where \( \mu_\lambda \in \mathcal{M}_1^+ \) is as given in Definition 4.3.

**Proof.** Let \( \lambda \in \mathcal{M}_0(A) \) and \( E \in ]E, \infty[ \cap \mathbb{N} \). Then we use (4.21) for any \( \phi \in C^1(M) \)
\[ - \int_M h(x, d\phi) \mu_\lambda^E = - \sum_{x, y \in A} \Lambda(\{x, y\}) \int_0^1 h \left( z_{x,y}^E(s), d\phi \left( z_{x,y}^E(s) \right) \right) ds \]
We now perform a change of variables \( ds \to dt = \sigma(E) \left( z_{x,y}^E(s), \dot{z}_{x,y}^E(s) \right) ds \) which transforms the interval \([0,1]\) into \([0, T_E(x,y)]\) (see (4.7)) and we get
\[ - \int_M h(x, d\phi) \mu_\lambda^E = - \sum_{x, y \in A} \Lambda(\{x, y\}) \int_0^{T_E(x,y)} h \left( \dot{z}_{x,y}^E(t), d\phi \left( \dot{z}_{x,y}^E(t) \right) \right) dt \]
where \( \dot{z}_{x,y}^E \) is the re-parametrization of \( z_{x,y}^E \), satisfying \( \dot{z}_{x,y}^E(0) = x, \dot{z}_{x,y}^E(T_E(x,y)) = y \). Next
\[ \int_M \phi d\lambda = \int_M d\Lambda_\lambda^E(x,y) [\phi(y) - \phi(x)] = \sum_{x, y \in A} \Lambda(\{x, y\}) \int_0^{T_E(x,y)} d\phi \left( \dot{z}_{x,y}^E(t) \right) \dot{z}_{x,y}^E(t) dt \]

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so \( \int_M \phi d\lambda - \int_M h(x, d\phi) d\mu^E_\lambda = \)

\[
\sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) \int_0^{T_E(x,y)} \left[ d\phi \left( \hat{z}^E_{x,y}(t) \right) \hat{z}^E_{x,y}(t) - h \left( \hat{z}^E_{x,y}(t), d\phi \left( \hat{z}^E_{x,y}(t) \right) \right) \right] dt \\
\leq \sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) \int_0^{T_E(x,y)} l \left( \hat{z}^E_{x,y}(t), \hat{z}^E_{x,y}(t) \right) dt = \sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) C_{T_E(x,y)}(x, y) \\
= \sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) D_E(x, y) - E \sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) T_E(x, y) = \sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) D_E(x, y) - E \sum_{x,y \in A} \Lambda^E_\lambda \left( (x, y) \right) T_E(x, y) = D_E(\lambda) - E D'_E(\lambda). \quad (4.33)
\]

To obtain (4.33) we used the Young inequality in the second line, (4.8) and (4.17) on the last line.

Since (4.33) is valid for any \( \phi \in C^1(M) \) we get from this and (4.30) that

\[
D_E(\lambda) - E D'_E(\lambda) \geq Q(\lambda, \mu^E_\lambda) \geq \overline{H}_T(\lambda) = \max_{E \geq E} D_E(\lambda) - T E. \quad (4.34)
\]

holds for any \( E \geq E_0 \). Now, if it so happens that the maximizer \( E(\lambda, T) \) on the right of (4.34) is on the complement of the set \( N \) in \( [E, \infty] \), then \( D'_E(\lambda) = T = \int_M d\mu^E_\lambda \) for \( E = E(\lambda, T) \) via Lemma 4.8 and the inequality in (4.34) turns into an equality. Otherwise, if \( E(\lambda, T) \in N - \{E_0\} \), we take the sequences \( E_n \nearrow E(\lambda, T), E_n \searrow E(\lambda, T) \) for \( E_n, E_0 \in [E, \infty] \) and the corresponding limits \( \mu^+_\lambda, \mu^-_\lambda \) defined in (4.26). Since \( Q_T \) is a convex, l.s.c as a function of \( \mu \) we get that the left inequality in (4.34) survives the limit, and

\[
D_{E(\lambda, T)}(\lambda) - E(\lambda, T) \frac{d^+}{dE} D_{E(\lambda, T)}(\lambda) \geq Q(\lambda, \mu^+_\lambda), \quad D_{E(\lambda, T)}(\lambda) - E(\lambda, T) \frac{d^+}{dE} D_{E(\lambda, T)}(\lambda) \geq Q(\lambda, \mu^-_\lambda),
\]

while \( \frac{d^+}{dE} D_{E(\lambda, T)}(\lambda) = \int d\mu^+_\lambda \) and \( \frac{d^-}{dE} D_{E(\lambda, T)}(\lambda) = \int d\mu^-_\lambda \). Then, upon taking a convex combination \( \mu_\lambda = \alpha T^{-1} \mu^+_\lambda + T^{-1} (1 - \alpha) \mu^-_\lambda \) such that, according to Definition 4.3,

\[
\alpha \frac{d^+}{dE} D_{E(\lambda, T)}(\lambda) + (1 - \alpha) \frac{d^-}{dE} D_{E(\lambda, T)}(\lambda) = T \int d\mu_\lambda = T
\]

and using the convexity of \( Q \) in \( \mu \) we get from (4.35, 4.36)

\[
D_{E(\lambda, T)}(\lambda) - T E(\lambda, T) \geq Q(\lambda, T \mu_\lambda) \equiv Q_T(\lambda, \mu_\lambda)
\]

This, with the right inequality of (4.32) yields the equality \( Q_T(\lambda, \mu_\lambda) = \overline{H}_T(\lambda) \).

Finally, if \( E(\lambda, T) = E_0 \) we proceed as follows: Let \( E_n \nearrow E \) and \( \mu^+_\lambda := \lim_{n \to \infty} \mu^E_\lambda \). It follows that

\[
\int_M d\mu^+_\lambda = \lim_{n \to \infty} \int_M d\mu^E_\lambda = \lim_{n \to \infty} D'_E(\lambda) = \frac{d^+}{dE} D_E(\lambda) \in (0, T]. \quad (4.37)
\]
Let $\mu_\lambda$ as in (4.27). From (4.28, 4.37) and (2.4) we get
\[
Q_T(\lambda, \mu_\lambda) \leq Q(\lambda, \mu_\lambda^+) + \left( T - \frac{d^+}{dE} D_E(\lambda) \right) Q(0, \mu_M) = Q(\lambda, \mu_\lambda^+) - \left( T - \frac{d^+}{dE} D_E(\lambda) \right) E
\]
while (2.4) and the left part of (4.35) for $E = E$ imply
\[
Q(\lambda, \mu_\lambda^+) \leq D_E(\lambda) - E \frac{d^+}{dE} D_E(\lambda).
\]
From (4.38) and (4.39) we get
\[
Q_T(\lambda, \mu_\lambda) \leq D_E(\lambda) - ET \leq H^*_T(\lambda)
\]
and the equality holds via (4.30). The last part of Theorem 1 follows from the equality in (4.30) as well.

4.5 Proof of Theorem 3

Theorem 1-(2) and (3.6) imply
\[
\hat{C}_T(\lambda) = \min_{\mu \in M^+_1} \hat{C}_T(\lambda, \mu).
\]
Next, we note that $D_E(\lambda, \mu)$ is a concave function of $E$ for $E \geq E$. In fact, from (3.4) and convexity of $h(x, \cdot)$ for each $x \in M$ we obtain
\[
\phi_i \in H_{E_i}, \ i = 1, 2 \implies \alpha \phi_1 + (1 - \alpha) \phi_2 \in H_{\alpha E_1 + (1 - \alpha) E_2}
\]
for $\alpha \in (0, 1)$ and $E_1, E_2 \geq E$. The concavity of $D_{\{1\}}(\lambda, \mu)$ follows from its definition (3.5). Then, by convex duality and (3.6)
\[
D_E(\lambda, \mu) = \min_{T > 0} \left[ \hat{C}_T(\lambda, \mu) + ET \right].
\]
By the same argument
\[
D_E(\lambda) = \min_{T > 0} \left[ \hat{C}_T(\lambda) + ET \right].
\]
Hence, (4.40) and Theorem 1-(3) imply
\[
\min_{\mu \in M^+_1} D_E(\lambda, \mu) = \min_{\mu \in M^+_1} \min_{T > 0} \left[ \hat{C}_T(\lambda, \mu) + ET \right]
= \min_{T > 0} \min_{\mu \in M^+_1} \left[ \hat{C}_T(\lambda, \mu) + ET \right] = \min_{T > 0} \left[ \hat{C}_T(\lambda) + ET \right] = D_E(\lambda).
\]
\[\square\]
5 Proof of Theorems 2&4

5.1 Auxiliary results

Lemma 5.1 follows from the surjectivity of $\text{Exp}^{(t)}_i(x)$ as a mapping from $T_x \mathcal{M}$ to $\mathcal{M}$, for any $x \in \mathcal{M}$ and any $t \neq 0$ (Recall definition at Section 1.2-8):

**Lemma 5.1.** Let $\Lambda \in \mathcal{M}^+(\mathcal{M} \times \mathcal{M})$. For any $t > 0$ there exists a Borel measure $\hat{\Lambda}^{(t)} \in \mathcal{M}^+(\mathcal{M})$ such that $\left(I \otimes \text{Exp}^{(t)}_i\right)\hat{\Lambda}^{(t)} = \Lambda$. Here $I \otimes \text{Exp}^{(t)}_i(x, v) := \left(x, \text{Exp}^{(t)}_i(x, v)\right)$.

The proof of Lemma 5.2 follows directly from the definition of the optimal plan:

**Lemma 5.2.** Let $\Lambda$ be a minimizer for (2.6), $B \subset \mathcal{M} \times \mathcal{M}$ a Borel subset and $\Lambda|_B$ the restriction of $\Lambda$ to $B$. Let $\mu^0_B$, $\mu^1_B$ the marginals of $\Lambda|_B$ on the factors of $\mathcal{M} \times \mathcal{M}$. Then $\Lambda|_B$ is an optimal plan for $\mathcal{C}(\mu^0_B, \mu^1_B)$. In addition, if $B_1, B_2 \subset \mathcal{M} \times \mathcal{M}$ are disjoint Borel sets then

$$\mathcal{C}(\mu^0_{B_1}, \mu^1_{B_1}) + \mathcal{C}(\mu^0_{B_2}, \mu^1_{B_2}) = \mathcal{C}(\mu^0_{B_1} + \mu^0_{B_2}, \mu^1_{B_1} + \mu^1_{B_2}),$$

and $\Lambda|_{B_1 \cup B_2}$ is the optimal plan with respect to $\mathcal{C}(\mu^0_{B_1}, \mu^1_{B_2})$.

Lemma 5.3 represents the time interpolation of optimal plans (see [28]):

**Lemma 5.3.** Given $t > 0$ and $\lambda = \lambda^+ - \lambda^- \in \mathcal{M}_0$. Let $\Lambda^t \in \mathcal{P}(\lambda^+, \lambda^-)$ be an optimal plan realizing

$$C_t^\Lambda(\lambda^+, \lambda^-) = \int \int C_t(x, y)\Lambda^t(xdy).$$

Let $\hat{\Lambda}^{(t)} \in \mathcal{M}^+(\mathcal{M})$ given in Lemma 5.1 for $\Lambda = \Lambda^t$. Let $\lambda_s := \left(\text{Exp}^{(s)}_i\right)\hat{\Lambda}^{(t)}$. Then, if

$$0 < s < t,$$ 

$$C_s(\lambda^+, \lambda^-) + C_{t-s}(\lambda^+, \lambda^-) = C_t(\lambda^+, \lambda^-).$$

**Lemma 5.4.** For any $\lambda^+, \lambda^- \in \mathcal{M}_1^+$ satisfying $\lambda = \lambda^+ - \lambda^- \in \mathcal{M}_1^+$,

$$\mathcal{C}_T(\lambda^+, \lambda^-) \geq \hat{\mathcal{C}}_T(\lambda).$$

**Lemma 5.5.** $\hat{\mathcal{C}}_T(\lambda\|\mu)$ is l.s.c in the weak-* topology of $\mathcal{M}_0 \times \mathcal{M}_1^+$. Assuming $H_1$ and $H_2$, for any $\lambda \in \mathcal{M}_0$, $\mu \in \mathcal{M}_1^+$ there exists a sequence $\{\hat{\mu}_n\} = \{\rho_n(x)dx\} \subset \mathcal{M}_1^+$, $\{\hat{\lambda}_n\} = \{\rho_n(q^+_n - q^-_n)dx\} \subset \mathcal{M}_0$ where $\rho_n \in C^\infty(\mathcal{M})$ are positive everywhere, $q^+_n \in C^\infty(\mathcal{M})$ non-negatives such that $\hat{\lambda}_n \rightharpoonup \lambda$, $\hat{\mu}_n \rightharpoonup \mu$ and

$$\lim_{n \to \infty} \hat{\mathcal{C}}_T(\hat{\lambda}_n\|\hat{\mu}_n) = \hat{\mathcal{C}}_T(\lambda\|\mu). (5.1)$$

**Lemma 5.6.** For any $\mu \in \mathcal{M}_1^+$, $\lambda = \lambda^+ - \lambda^- \in \mathcal{M}_0$,

$$\liminf_{\varepsilon \to 0} \varepsilon^{-1}\mathcal{C}_T(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \geq \hat{\mathcal{C}}_T(\lambda\|\mu).$$

**Lemma 5.7.** Assume $\mu = \rho(x)dx$ and $\lambda = \rho(q^+ - q^-)dx$ where $\rho, q^\pm$ are $C^\infty$ functions, $\rho$ positive everywhere on $\mathcal{M}$. Then

$$\limsup_{\varepsilon \to 0} \varepsilon^{-1}\mathcal{C}_T(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \leq \hat{\mathcal{C}}_T(\lambda\|\mu).$$
Lemma 5.8. For $T > 0$,
\[
\hat{C}_T(\lambda) \geq \limsup_{\varepsilon \to 0} \varepsilon^{-1} \inf_{\mu \in \mathcal{M}_1^+} C_\varepsilon T(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) .
\]

Proof. of Lemma 5.4: We use the duality representation of the Monge-Kantorovich functional [26] to obtain (recall $\lambda^\pm \in \mathcal{M}_1^+$)
\[
C_T(\lambda^+, \lambda^-) + ET = \sup_{\psi, \phi} \left\{ \int_M \psi d\lambda^- - \phi d\lambda^+ , \phi(y) - \psi(x) \leq C_T(x, y) + ET \right\}
\]

By (2.10) $C_T(x, y) + ET \geq D_E(x, y)$ for any $x, y \in M$ so, by (2.12, 2.13)
\[
\sup_{\psi, \phi} \left\{ \int_M \psi d\lambda^- - \phi d\lambda^+ , \phi(y) - \psi(x) \leq C_T(x, y) + ET \right\} \geq \sup_{\phi} \left\{ \int_M \phi d\lambda , \phi(y) - \phi(x) \leq D_E(x, y) \right\} = D_E(\lambda) \quad (5.2)
\]

so
\[
C_T(\lambda^+, \lambda^-) \geq D_E(\lambda) - ET
\]

for any $E \geq E$. By Theorem 1-(3)
\[
C_T(\lambda^+, \lambda^-) \geq \sup_{E \geq E} D_E(\lambda) - ET = \hat{C}_T(\lambda) .
\]

Proof. of Lemma 5.5: From (3.5, 3.6) we obtain
\[
\hat{C}_T(\lambda||\mu) = \sup_{\phi \in C^1(M)} \int_M \phi d\lambda - Th(x, d\phi) d\mu .
\]

In particular $\hat{C}_T$ is l.s.c (and convex) on $\mathcal{M}_0 \times \mathcal{M}_1^+$. Let $\varepsilon_n \to 0$ and $\lambda_n := \lambda_{\varepsilon_n} := \delta_{\varepsilon_n} * \lambda \in \mathcal{M}_0$ defined by
\[
\int_M \psi d\lambda_n := \lambda(\delta_{\varepsilon_n} * \psi) \quad \forall \psi \in C^0(M) . \quad (5.3)
\]

By $H_1$, $\lambda_n \to \lambda$ while $\lambda_n$ are smooth. First, we observe that $\lim_{n \to \infty} \lambda_n \to \lambda$. Indeed, for any $\psi \in C^1(M)$:
\[
\lim_{n \to \infty} \int_M \psi d\lambda_n = \lim_{n \to \infty} \lambda(\delta_{\varepsilon_n} * \psi) = \lambda(\psi) .
\]

Next, by Jensen’s Theorem and $H_2$
\[
\int_M h(x, d\delta_{\varepsilon} * \phi) d\mu = \int_M h(x, \delta_{\varepsilon} * d\phi) d\mu \leq \int_M h(x, d\phi(y)) d\delta_{\varepsilon} (x, y) d\mu(x) dy \equiv \int h(x, d\phi) d\delta_{\varepsilon} * \mu + \int h(x, d\phi) d\delta_{\varepsilon} (x, y) d\mu(x) dy \quad (5.4)
\]

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From section 1.2-(7) and using \( \delta_\varepsilon(x,y) = o(1) \) for \( D(x,y) > \delta \),
\[
\int_{M \times M} [h(x,d\phi(y)) - h(y,d\psi(y))] \delta_\varepsilon(x,y) d\mu dy \leq O(\varepsilon) + o(1) \int_M h(x,d\phi) d\delta_\varepsilon * \mu .
\]
Next, define \( \mu_n = \delta_\varepsilon * \mu \). Let \( \psi_n \) be the maximizer of \( \hat{C}(\lambda_n||\mu_n) \), that is
\[
\hat{C}_T(\lambda_n||\mu_n) = \int_M \psi_n d\lambda_n - T h(x,d\psi_n) d\mu_n
\]
By (5.3, 5.4)
\[
\hat{C}_T(\lambda_n||\mu_n) \leq \int_M \delta_\varepsilon * \psi_n d\lambda - (1-o(1)) \int_M T h(x,d\delta_\varepsilon * \psi_n) d\mu + O(\varepsilon_n) =
\]
\[
(1-o(1)) \left[ \int_M \delta_\varepsilon * \psi_n \frac{d\lambda}{1-o(1)} - \int_M T h(x,d\delta_\varepsilon * \psi_n) d\mu \right] + \varepsilon_n \leq (1-o(1)) \hat{C}(\frac{\lambda}{1-o(1)}||\mu) + \varepsilon_n \tag{5.5}
\]
We obtained
\[
\limsup_{n \to \infty} \hat{C}_T(\lambda_n||\mu_n) \leq \hat{C}_T(\lambda||\mu)
\]
which, together with the l.s.c of \( \hat{C}_T \), implies the result. \( \Box \)

**Proof. of Lemma 5.6:** Recall that the Lax-Oleinik Semigroup acting on \( \phi \in C^0(M) \)
\[
\psi(x,t) = LO(\phi)(t,x) := \sup_{y \in M} [\phi(y) - C_t(x,y)]
\]
is a viscosity solution of the Hamilton-Jacobi equation \( \partial_t \psi - h(x,d\psi) = 0 \) subjected to \( \psi_0 = \phi(x) \). If \( \phi \in C^1(M) \) then \( \psi \) is a classical solution on some neighborhood of \( t = 0 \), so
\[
\lim_{T \to 0} LO(\phi)(T,) = \phi ; \quad \lim_{T \to 0} T^{-1} [LO(\phi)(T,x) - \phi(x)] = h(x,d\phi).
\]
Then for any \( \mu_1, \mu_2 \in M_1^+ \)
\[
C_T(\mu_1,\mu_2) = \sup_{\phi,\psi \in C^1(M)} \left\{ \int_M \phi d\mu_2 - \psi d\mu_1 ; \quad \phi(x) - \psi(y) \leq C_T(x,y) \quad \forall x,y \in M \right\} =
\[
\sup_{\phi \in C^1(M)} \int_M \phi d\mu_2 - LO(\phi)(T,x)d\mu_1 \tag{5.6}
\]
Hence
\[
\liminf_{\varepsilon \to 0} \varepsilon^{-1} C_\varepsilon T(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) =
\]
\[
\liminf_{\varepsilon \to 0} \sup_{\phi \in C^1(M)} \int_M \varepsilon^{-1} [\phi(x) - LO(\phi)(\varepsilon T,x)] d\mu + \int_M \phi d\lambda^+ - LO(\phi)(\varepsilon T,x)d\lambda^-
\]
\[
\geq \sup_{\phi \in C^1(M)} \lim_{\varepsilon \to 0} \int_M \varepsilon^{-1} [\phi(x) - LO(\phi)(\varepsilon T,x)] d\mu + \int_M \phi d\lambda^+ - LO(\phi)(\varepsilon T,x)d\lambda^-
\]
\[
= \sup_{\phi,\psi \in C^1(M)} \int_M -Th(x,d\phi) d\mu + \phi d\lambda := \hat{C}_T(\lambda||\mu). \tag{5.7}
\]
\( \Box \)
Proof of Lemma 5.7: We may describe the optimal mapping $S_{\varepsilon T}: M \rightarrow M$ associated with $C_{\varepsilon T}(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-)$ in local coordinates on each chart. It is given by the solution to the Monge-Ampère equation

$$
det \nabla_x S_{\varepsilon T} = \frac{\rho(x)(1 + \varepsilon q^-(x))}{\rho(S_{\varepsilon T}(x))(1 + \varepsilon T q^+(S_{\varepsilon T}(x)))} \quad (5.8)
$$

where

$$
\nabla \psi = -\nabla_x C_{\varepsilon T}(x, S_{\varepsilon T}(x)) \quad (5.9)
$$

and

$$
C_{\varepsilon T}(\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) = \int_M C_{\varepsilon T}(x, S_{\varepsilon T}(x))\rho(1 + \varepsilon T q^-)dx \quad (5.10)
$$

We recall that the inverse of $\nabla_x C_{\varepsilon T}(x, \cdot)$ with respect to the second variable is $I_d + \varepsilon T \nabla \psi$, to leading order in $\varepsilon$. That is,

$$
\nabla_x C_{\varepsilon T}(x, x + \varepsilon T \partial_p h(x, \xi) + (\varepsilon T)^2 Q(x, \xi, \varepsilon)) = -\xi \quad (5.11)
$$

where (here and below) $Q$ is a generic smooth function of its arguments.

Hence, $S_{\varepsilon T}$ can be expanded in $\varepsilon$ in terms of $\psi$

$$
S_{\varepsilon T}(x) = x + \varepsilon T h_\xi(x, \nabla \psi) + (\varepsilon T)^2 Q(x, \nabla \psi, x, \varepsilon) \quad (5.12)
$$

We now expand the right side of (5.8) using (5.12) to obtain

$$
1 + \varepsilon T \left[ q^-(x) - q^+(x) - h_\xi(x, d\psi) \cdot \nabla_x \ln \rho(x) \right] + (\varepsilon T)^2 Q(x, \nabla \psi, x, \varepsilon) \quad (5.13)
$$

while the left hand side is

$$
det(\nabla_x S_{\varepsilon T}) = 1 + \varepsilon T \nabla \cdot h_\xi(x, d\psi) + (\varepsilon T)^2 Q(x, \nabla \psi, \nabla \nabla \psi, x, \varepsilon) \quad (5.14)
$$

Comparing (5.13, 5.14), divide by $\varepsilon T$ and multiply by $\rho$ to obtain

$$
T \nabla \cdot (\rho h_\xi(x, d\psi)) = \rho(q^- - q^+) + \varepsilon T \rho Q(x, \nabla \psi, \nabla \nabla \psi, x, \varepsilon) \quad . \quad (5.15)
$$

Now, we substitute $\varepsilon = 0$ and get a quasi-linear equation for $\psi_0$:

$$
T \nabla \cdot (\rho h_\xi(x, d\psi_0)) = \rho(q^- - q^+) \quad . \quad (5.16)
$$

$\psi_0$ is a maximizer of

$$
\hat{C}_T(\lambda||\mu) = \int_M \rho(q^+ - q^-)\psi_0 - \int_M \rho Th_\xi(x, d\psi_0)dx
$$

By elliptic regularity, $\psi_0 \in C^\infty(M)$. Multiply (5.16) by $\psi_0$ and integrate over $M$ to obtain

$$
\int_M \rho(q^+ - q^-) = \int_M \rho Th_\xi(x, d\psi_0) \cdot \nabla \psi_0
$$
Then by the Lagrangian/Hamiltonian duality

$$\tilde{\mathcal{G}}_T(\lambda\|\mu) = \int_M \rho T [\nabla \psi_0 \cdot h_\xi(x, d\psi_0) - h(x, d\psi_0)] \equiv T \int_M \rho l(x, h_\xi(x, d\psi_0)) \ .$$  \hspace{1cm} (5.17)

We observe \( l \left( \frac{y}{T}, \frac{x}{T} \right) \geq T^{-1} C_T(x, y) \). So, (5.10) with (5.12) imply

$$ (\varepsilon T)^{-1} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \leq \int_M \rho (1 + \varepsilon T q^-) l(x, h_\xi(x, \nabla \psi_0 + \varepsilon T q(x, \nabla \psi_0, \varepsilon)) \) \hspace{1cm} (5.18)

where \( \psi_\varepsilon \) is a solution of (5.15). Now, if we show that \( \lim_{\varepsilon \rightarrow 0} \psi_\varepsilon = \psi_0 \) in \( C^1(M) \) then, from (5.17, 5.18)

$$ \limsup_{\varepsilon \rightarrow 0} (\varepsilon T)^{-1} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \leq T \int_M \rho l(x, h_\xi(x, d\psi_0)) = \tilde{\mathcal{G}}(\lambda\|\mu) .$$

Next we show that, indeed, \( \lim_{\varepsilon \rightarrow 0} \psi_\varepsilon = \psi_0 \) in \( C^1(M) \).
Substitute \( \psi_\varepsilon = \psi_0 + \phi_\varepsilon \) in (5.15). We obtain

$$ \nabla \cdot (\sigma(x) \nabla \phi_\varepsilon) = \varepsilon Q(x, \nabla \phi_\varepsilon, \nabla \nabla \phi_\varepsilon, \varepsilon) + \nabla \cdot (\rho(\nabla^I \phi_\varepsilon, \tilde{Q}(x, \nabla \phi_\varepsilon, \varepsilon) \cdot \nabla \phi_\varepsilon)) \hspace{1cm} (5.19)$$

where \( \sigma := \frac{\partial h_\xi(x, \nabla \psi_0(x))}{\partial x} \) is a positive definite form, while \( \tilde{Q} \) is a smooth matrix valued functions in both \( x \) and \( \varepsilon \), determined by \( \nabla \psi_0 \) and \( Q \) as given in (5.15). A direct application of the implicit function theorem implies the existence of a branch \( (\lambda(\varepsilon), \eta_\varepsilon) \) of solutions for

$$ \nabla \cdot (\sigma(x) \nabla \eta_\varepsilon) = \varepsilon Q(x, \nabla \eta_\varepsilon, \nabla \nabla \eta_\varepsilon, \varepsilon) + \nabla \cdot \left( \rho(\nabla^I \eta_\varepsilon, \tilde{Q}(x, \nabla \eta_\varepsilon, \varepsilon) \circ \nabla \eta_\varepsilon) \right) + \lambda(\varepsilon) \hspace{1cm} (5.20)$$

where \( \eta_0 = \lambda(0) = 0 \) and \( \varepsilon \mapsto \eta_\varepsilon \) is (at least) continuous in \( C^1(M) \). Note that for \( \varepsilon \neq 0 \) we may have a non-zero \( \lambda(\varepsilon) \) which follows from projecting the right side on the equation to the Hilbert space perpendicular to constants (recall that \( M \) is a compact manifold without boundary, and the left side is surjective on this space). We now show that \( \eta_\varepsilon = \phi_\varepsilon \), i.e \( \lambda(\varepsilon) = 0 \) also for \( \varepsilon \neq 0 \). Indeed, (5.19) is equivalent to (5.8) multiplied by \( \rho(x)/\varepsilon \), so (5.20) is equivalent to

$$ \det \nabla_x S_{\varepsilon T} = \frac{\rho(x)(1 + \varepsilon q^-(x))}{\rho(\tilde{S}_{\varepsilon T}(x))(1 + \varepsilon q^+(\tilde{S}_{\varepsilon T}(x)))} + \varepsilon \rho^{-1}(x) \lambda(\varepsilon) $$

where \( \tilde{S}_{\varepsilon T}(x) \) obtained from (5.12) with \( \psi_\varepsilon := \psi_0 + \eta_\varepsilon \).

Hence

$$ \int_M (\rho(\tilde{S}_{\varepsilon T}(x))(1 + \varepsilon q^+(\tilde{S}_{\varepsilon T}(x))) \det(\nabla_x \tilde{S}_{\varepsilon T}) = \int_M (\rho(x)(1 + \varepsilon q^-(x))) $$

$$ + \varepsilon \lambda(\varepsilon) \int_M \frac{\rho(\tilde{S}_{\varepsilon T}(x))}{\rho(x)}(1 + \varepsilon q^+(\tilde{S}_{\varepsilon T}(x))) \hspace{1cm} (5.21)$$

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However, $\hat{S}_{\varepsilon T}(x) = x + O(\varepsilon)$ is a diffeomorphism on $M$, so

$$
\int_M \left( \rho(\hat{S}_{\varepsilon T}(x))(1 + \varepsilon q^+(\hat{S}_{\varepsilon T}(x))) \right) \det(\nabla_x \hat{S}_{\varepsilon T}) = \int_M \left( \rho(\hat{S}_{\varepsilon T}(x))(1 + T q^+(\hat{S}_{\varepsilon T}(x))) \right) |\det(\nabla_x \hat{S}_{\varepsilon T})| \equiv \int_M \rho(x)(1 + \varepsilon q^-(x)) \cdot (5.22)
$$

It follows that

$$
\varepsilon \lambda(\varepsilon) \int_M \frac{\rho(\hat{S}_{\varepsilon T}(x))}{\rho(x)} (1 + \varepsilon q^+(\hat{S}_{\varepsilon T}(x))) = 0.
$$

Since $\rho$ is positive everywhere it follows that $\lambda(\varepsilon) \equiv 0$ for $|\varepsilon|$ sufficiently small. We proved that $\eta_{\varepsilon} \equiv \phi_{\varepsilon}$ and, in particular, $\phi_{\varepsilon} \to 0$ as $\varepsilon \to 0$ in $C^1 \perp 1$, which implies the convergence of $\psi_{\varepsilon}$ to $\psi_0$ at $\varepsilon \to 0$ in $C^1 \perp 1$. \qed

**Proof.** (of Lemma 5.8) Given $\varepsilon > 0$ let

$$
D^\varepsilon_E(x, y) := \inf_{n \in \mathbb{N}} \left[ C_{\varepsilon n T}(x, y) + \varepsilon n ET \right].
$$

(5.23)

Evidently, $D^\varepsilon_E(x, y)$ is continuous on $M \times M$ locally uniformly in $E \geq E$. Moreover,

$$
\lim_{\varepsilon \to 0} D^\varepsilon_E = D_E
$$

(5.24)

uniformly on $M \times M$ and locally uniformly in $E \geq E$ as well.

We now decompose $M \times M$ into mutually disjoint Borel sets $Q_n$:

$$
M \times M = \bigcup_n Q^\varepsilon_n, \quad Q^\varepsilon_n \cap Q^\varepsilon_{n'} = \emptyset \text{ if } n \neq n'
$$

such that

$$
Q^\varepsilon_n \subset \{(x, y) \in M \times M ; \; D^\varepsilon_E(x, y) = C_{\varepsilon n T}(x, y) + \varepsilon n ET\}.
$$

Let $\Lambda^E_\varepsilon \in \mathcal{P} (\lambda^+, \lambda^-)$ be an optimal plan for

$$
D^\varepsilon_E(\lambda) = \int_{M \times M} D^\varepsilon_E(x, y) d\Lambda^E_\varepsilon = \min_{\Lambda \in \mathcal{P} (\lambda^+, \lambda^-)} \int_{M \times M} D^\varepsilon_E(x, y) d\Lambda,
$$

(5.25)

and $\Lambda^E_n = \Lambda^E_\varepsilon |_{Q^\varepsilon_n}$, the restriction of $\Lambda^E_\varepsilon$ to $Q^\varepsilon_n$. Set $\lambda^\pm_n$ to be the marginals of $\Lambda^E_n$ on the first and second factors of $M \times M$. Then

$$
\sum_{n=1}^{\infty} \lambda^\pm_n = \lambda^\pm
$$

(5.26)

**Remark 5.1.** Note that $Q^\varepsilon_n = \emptyset$ for all but a finite number of $n \in \mathbb{N}$. In particular, the sum (5.26) contains only a finite number of non-zero terms.
Let $|\lambda_n| := \int_M d\lambda_n^\pm \equiv \int_{M \times M} d\Lambda_n$. The averaged flight time is

$$\langle T \rangle^\varepsilon := \varepsilon T \sum_{n=1}^{\infty} n|\lambda_n|$$

(5.27)

We observe that $\langle T \rangle^\varepsilon \in \partial E D^\varepsilon_E(\lambda)$, where $\partial E$ is the super gradient as a function of $E$. At this stage we choose $E$ depending on $\varepsilon, T$ such that

$$\langle T \rangle^\varepsilon = T + 2\varepsilon T|\lambda^\pm|$$

(5.28)

We now apply Lemma 5.1: Recalling Section 1.2-8, let $\hat{\Lambda}_n^\varepsilon \in \mathcal{M}(T, M)$ satisfying

$$(I \oplus \text{Exp}(t=\varepsilon T)) \hat{\Lambda}_n^\varepsilon = \Lambda_n^\varepsilon.$$ 

Use $\hat{\Lambda}_n^\varepsilon$ to define $\lambda_n^\varepsilon := \text{Exp}(t=\varepsilon n T) \hat{\Lambda}_n^\varepsilon \in \mathcal{M}(M)$ for $j = 0, 1 \ldots n$. Note that

$$\lambda_n^\varepsilon = \lambda_n^+ + \varepsilon \lambda_n^-.$$ 

(5.29)

By Lemma 5.3

$$C_{enT}(\lambda_n^+, \lambda_n^-) + \varepsilon n ET|\lambda_n| = \sum_{j=0}^{n-1} \left[ C_{\varepsilon T}(\lambda_n^j, \lambda_n^{j+1}) + \varepsilon ET|\lambda_n| \right]$$

(5.30)

From (5.23, 5.25, 5.26, 5.30) and Lemma 5.2

$$D^\varepsilon_E(\lambda) = \sum_{n=1}^{\infty} D^\varepsilon_E(\lambda_n) = \sum_{n=1}^{\infty} \left[ C_{enT}(\lambda_n^+, \lambda_n^-) + \varepsilon n ET|\lambda_n| \right] = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \left( C_{\varepsilon T}(\lambda_n^j, \lambda_n^{j+1}) + \varepsilon ET|\lambda_n| \right).$$

(5.31)

Let now

$$\mu^\varepsilon,E = \varepsilon \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \lambda_n^j.$$ 

Note that

$$\mu^\varepsilon,E = \varepsilon \sum_{n=1}^{\infty} \sum_{j=0}^{n} \lambda_n^j - \varepsilon \sum_{n=1}^{\infty} \lambda_n^0 - \varepsilon \sum_{n=1}^{\infty} \lambda_n^n.$$ 

By (5.26, 5.29, 5.27) we obtain

$$|\mu^\varepsilon,E| = \varepsilon \sum_{n=1}^{\infty} (n+1)|\lambda_n^\pm| - 2\varepsilon|\lambda^\pm| = 1 \implies \mu^\varepsilon,E \in \mathcal{M}(1).$$

(5.32)

By (5.26, 5.29)

$$\sum_{n=1}^{\infty} \sum_{j=0}^{n-1} C_{\varepsilon T}(\lambda_n^j, \lambda_n^{j+1}) \geq C_{\varepsilon T} \left( \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \lambda_n^j, \sum_{n=1}^{\infty} \sum_{j=1}^{n} \lambda_n^{j+1} \right) = \varepsilon^{-1} C_{\varepsilon T} \left( \varepsilon \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \lambda_n^j, \varepsilon \sum_{n=1}^{\infty} \sum_{j=1}^{n} \lambda_n^{j+1} \right)$$

$$= \varepsilon^{-1} C_{\varepsilon T} \left( \mu^\varepsilon,E + \varepsilon \lambda^+, \mu^\varepsilon,E + \varepsilon \lambda^- \right).$$

(5.33)
From (5.27, 5.31, 5.33, 5.32)
\[ D_E^\varepsilon(\lambda) - \langle T \rangle \varepsilon E \geq \varepsilon^{-1} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \geq \varepsilon^{-1} \inf_{\mu \in \mathcal{M}_1^+} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) . \]

Finally, Theorem 1-3, (5.24, 5.28, 5.34) imply
\[ \hat{C}_T(\lambda) \geq D_E(\lambda) - TE = \lim_{\varepsilon \to 0} D_E^\varepsilon(\lambda) - \langle T \rangle \varepsilon E \geq \limsup_{\varepsilon \to 0} \varepsilon^{-1} \inf_{\mu \in \mathcal{M}_1^+} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) . \]

\[ \square \]

5.2 Proof of theorem 2

From Theorem 1- (1) we get
\[ \hat{C}_T(\varepsilon \lambda) = \varepsilon \hat{C}_T(\lambda) . \]

We now apply Lemma 5.4, adapted to the case where \(|\lambda^\pm| := \int \lambda^\pm \neq 1\). Then
\[ C_T(\lambda^+, \lambda^-) = |\lambda^+| C_T \left( \frac{\lambda^+}{|\lambda^+|}, \frac{\lambda^-}{|\lambda^-|} \right) \geq |\lambda^+| \hat{C}_T \left( \frac{\lambda^-}{|\lambda^-|} \right) = \hat{C}_{T/|\lambda^\pm|}(\lambda) . \]

Note that \( d\mu + \varepsilon d\lambda^\pm = 1 + O(\varepsilon) \), hence
\[ \varepsilon^{-1} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \geq \hat{C}_T(\lambda) \]

where \( T_\varepsilon \to T \) as \( \varepsilon \to 0 \). Hence
\[ \liminf_{\varepsilon \to 0} \inf_{\mathcal{M}_1^+} \varepsilon^{-1} C_{\varepsilon T} (\mu + \varepsilon \lambda^+, \mu + \varepsilon \lambda^-) \geq \hat{C}_T(\lambda) . \]

The Theorem follows from this and Lemma 5.8.

\[ \square \]

5.3 Proof of Theorem 4

We have to show that for any \((\mu, \lambda) \in \mathcal{M}_1^+ \times \mathcal{M}_0\) and any sequence \((\mu_n, \lambda_n) \rightharpoonup (\mu, \lambda)\) as \( n \to \infty \):
\[ \lim_{n \to \infty} n C_{T/n} \left( \mu_n + n^{-1} \lambda_n^+, \mu_n + n^{-1} \lambda_n^- \right) \geq \hat{C}(\lambda\|\mu) \]  \hfill (5.35)

and, in addition, there exists a sequence \((\hat{\mu}_n, \hat{\lambda}_n) \rightharpoonup (\mu, \lambda)\) for which
\[ \lim_{n \to \infty} n C_{T/n} \left( \hat{\mu}_n + n^{-1} \hat{\lambda}_n^+, \hat{\mu}_n + n^{-1} \hat{\lambda}_n^- \right) = \hat{C}(\lambda\|\mu) . \]  \hfill (5.36)

The inequality (5.35) follows directly from Lemma 5.6. To prove (5.36), we first consider the sequence \((\hat{\mu}_n, \hat{\lambda}_n)\) subjected to Lemma 5.5. From Lemma 5.7 and Lemma 5.5,
\[ \lim_{j \to \infty} \limsup_{n \to \infty} n C_{T/n} \left( \hat{\mu}_j + n^{-1} \hat{\lambda}_j^+, \hat{\mu}_j + n^{-1} \hat{\lambda}_j^- \right) \leq \lim_{j \to \infty} \hat{C}_T \left( \hat{\lambda}_j\|\hat{\mu}_j \right) = \hat{C}(\lambda\|\mu) . \]
So, there exists a subsequence $j_n$ along which
\[
\limsup_{n \to \infty} nC_{T/n} \left( \tilde{\mu}_{j_n} + n^{-1} \tilde{\lambda}_{j_n}^+ \cdot \tilde{\mu}_{j_n} + n^{-1} \tilde{\lambda}_{j_n}^- \right) \leq \hat{C}(\lambda \| \mu) .
\]
This, with (5.35), implies (5.36).

The second part of the theorem follows from (5.35) and Theorem 2.

□

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