Nodal Domain Theorems à la Courant.

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Abstract

Let $H(\Omega_0) = -\Delta + V$ be a Schrödinger operator on a bounded domain $\Omega_0 \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Suppose that the $\Omega_\ell$ ($\ell \in \{1, \ldots, k\}$) are some pairwise disjoint subsets of $\Omega_0$ and that $H(\Omega_\ell)$ are the corresponding Schrödinger operators again with Dirichlet boundary conditions. We investigate the relations between the spectrum of $H(\Omega_0)$ and the spectra of the $H(\Omega_\ell)$. In particular, we derive some inequalities for the associated spectral counting functions which can be interpreted as generalizations of Courant’s nodal Domain Theorem. For the case that equality is achieved we prove converse results. In particular, we use potential theoretic methods to relate the $\Omega_\ell$ to the nodal domains of some eigenfunction of $H(\Omega_0)$.

1 Introduction

Consider a Schrödinger operator

$$H = -\Delta + V$$

(1.1)

on a bounded domain $\Omega_0 \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Further we assume that $V$ is real valued and satisfies $V \in L^\infty(\Omega_0)$. (We could relax this condition and extend our results to the case $V \in L^\beta(\Omega_0)$ for some $\beta > d/2$ [11].)

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$H$ is selfadjoint if viewed as the Friedrichs extension of the quadratic forms of $H$ with form domain $W^{1,2}_0(\Omega_0)$ and form core $C_0^\infty(\Omega_0)$ and we denote it by $H(\Omega_0)$. Further $H(\Omega_0)$ has compact resolvent. So the spectrum of $H(\Omega_0)$, $\sigma(H(\Omega_0))$, can be described by an increasing sequence of eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \ldots$$

(1.2)

tending to $+\infty$, such that the associated eigenfunctions $u_k$ form an orthonormal basis of $L^2(\Omega_0)$. $\lambda_1$ is simple and the corresponding eigenfunction $u_1$ can be chosen to satisfy, see e.g. \cite{17},

$$u_1 > 0 \text{ for all } x \in \Omega_0.$$  

(1.3)

We can assume that the eigenfunctions $u_k$ are real valued and by elliptic regularity, \cite{9} (Corollary 8.36), $u_k$ belongs to $C^{1,\alpha}(\Omega_0)$ for every $\alpha < 1$. We shall often call for a bounded domain $D$, $H(D)$, the corresponding self-adjointed operator with Dirichlet boundary conditions on $\partial D$. Its lowest eigenvalue will be denoted by $\lambda(D)$.

We denote the zero set of an eigenvector $u$ by

$$N(u) = \{x \in \Omega_0 \mid u(x) = 0\}.$$  

(1.4)

The nodal domains of $u$, which are by definition the connected components of $\Omega_0 \setminus N(u)$, will be denoted by $D_j, j = 1, \ldots, \mu(u)$, where $\mu(u)$ denotes the number of nodal domains of $u$.

Suppose that the $\Omega_\ell$ ($\ell = 1, 2, \ldots, k$) are $k$ open pairwise disjoint subsets of $\Omega_0$. In this paper we shall investigate relations between the spectrum of $H(\Omega_0)$ and the spectra of the $H(\Omega_\ell)$. Roughly speaking, we shall derive an inequality between the counting function of $H(\Omega_0)$ and those of the $H(\Omega_\ell)$. This inequality can be interpreted as a generalization of Courant’s classical nodal domain theorem. For the case that equality is achieved this will lead to a partial characterization of the $\Omega_\ell$ which will turn out to be related to the nodal domains of one of the eigenfunction of $H(\Omega_0)$.

These results will be given in sections 2 and 3. From these results some natural questions of potential theoretic nature arise which will be analyzed and answered in section 7.

The proofs of the results stated in sections 2 and 3 are given in sections 4 and 5. In section 6 some illustrative explicit examples are given.
2 Main results

We start with a result which will turn out to be a generalization of Courant’s nodal Theorem. We consider again (1.1) on a bounded domain $\Omega_0$ and the corresponding eigenfunctions and eigenvalues. We first introduce

$$n(\lambda, \Omega_0) = \# \{ j \mid \lambda_j(\Omega_0) \leq \lambda \} , \quad (2.1)$$

where $\lambda_j(\Omega_0)$ is the $j$-th eigenvalue of $H(\Omega_0)$.

We also define

$$\underline{n}(\lambda, \Omega_0) = \# \{ j \mid \lambda_j(\Omega_0) < \lambda \} . \quad (2.2)$$

and

$$n(\lambda, \Omega_0) = \begin{cases} n(\lambda, \Omega_0) & \text{if } \lambda \not\in \sigma(H(\Omega_0)) \\ n(\lambda, \Omega_0) + 1 & \text{if } \lambda \in \sigma(H(\Omega_0)). \end{cases} \quad (2.3)$$

So we always have :

$$\mathbb{n}(\lambda, \Omega_0) \leq n(\lambda, \Omega_0) \leq \mathbb{n}(\lambda, \Omega_0) . \quad (2.4)$$

with equality when $\lambda$ is not an eigenvalue. Note that $\mathbb{n}(\lambda, \Omega_0) - \underline{n}(\lambda, \Omega_0)$ is the multiplicity of $\lambda$ when $\lambda$ is an eigenvalue of $H(\Omega_0)$, i.e. the dimension of the eigenspace associated to $\lambda$. We shall consider a family of $k$ open sets $\Omega_\ell (\ell = 1, \ldots, k)$ contained in $\Omega_0$ and the corresponding Dirichlet realizations $H(\Omega_\ell)$. For each $H(\Omega_\ell)$ the corresponding eigenvalues counted with multiplicity are denoted by $(\lambda_\ell^k)_{k \in \mathbb{N} \setminus \{0\}}$ (with $\lambda_\ell^k \leq \lambda_\ell^{k+1}$). When counting the eigenvalues less than some given $\lambda$, we shall for simplicity write

$$n_\ell = n_\ell(\lambda) = n(\lambda, \Omega_\ell) \quad (2.5)$$

and analogously for the quantities with over-, respectively, underbars.

**Theorem 2.1**

Suppose $\Omega_0$ is connected and that $\lambda \in \sigma(H(\Omega_0))$. Suppose that the sets $\Omega_\ell (\ell = 1, \ldots, k)$ are pairwise disjoint open subsets of $\Omega_0$. Then

$$\sum_{\ell=1}^{k} \pi_\ell \leq n_0 + \min_{\ell \geq 0} (\pi_\ell - n_\ell) \quad (2.6)$$

A direct weaker consequence of (2.6) is the more standard
Corollary 2.2
Under the assumptions of Theorem 2.1 we have
\[ \sum_{\ell=1}^{k} \bar{n}_\ell \leq \bar{n}_0. \] (2.7)

This corollary is actually present in the proofs of the asymptotics of the counting function (see for example the Dirichlet-Neumann bracketing in Lieb-Simon [14]).

Remark 2.3
Inequality (2.6) is also true if \( \lambda \not\in \sigma(H(\Omega_0)) \). The statement becomes more simply
\[ \sum_{\ell=1}^{k} n_\ell \leq n_0. \]
and is proved essentially in the same way.

Remark 2.4
The assumption that \( \Omega_0 \) is connected is necessary. Indeed, suppose \( \Omega_1 \) and \( \Omega_2 \) are connected and assume that \( \Omega_0 = \Omega_1 \cup \Omega_2 \) with \( \Omega_1 \cap \Omega_2 = \emptyset \) and that \( \lambda = \lambda_1(\Omega_1) = \lambda_1(\Omega_2) \). Then \( \lambda_1(\Omega_0) = \lambda_2(\Omega_0) \) and we deduce \( n(\lambda, \Omega_0) = 1 \). In general we just have, if we no longer assume the connectedness of \( \Omega_0 \), Corollary 2.2.

Finally we show that Courant’s nodal Theorem is an easy corollary of Theorem 2.1.

Corollary 2.5 : Courant’s nodal Theorem
If \( \Omega_0 \) is connected and if \( u \) is an eigenvector of \( H(\Omega_0) \) associated to some eigenvalue \( \lambda \), then
\[ \mu(u) \leq n(\lambda, \Omega_0). \]

Proof.
We now simply apply Theorem 2.1 by taking \( \Omega_1, \ldots, \Omega_{\mu(u)} \) as the nodal domains associated to \( u \). We just have to use (1.3) for each \( \Omega_\ell, \ell = 1, \ldots, \mu(u) \), which gives \( \bar{n}_\ell = n_\ell = 1 \). \( \square \)

Remark 2.6
Courant’s nodal Theorem is one of the basic results in spectral theory of Schrödinger type operators. It is the natural generalization of Sturm’s oscillation theorem for second order ODE’s. For recent investigations see for instance [11] and [4].
3 Converse results.

In this section we consider some results converse to Theorem 2.1.

**Theorem 3.1**
Suppose that the $\Omega_\ell$, $\ell \geq 1$, are pairwise disjoint open subsets of $\Omega_0$. If $\lambda \in \sigma(H(\Omega_0))$ and
\[
\sum_{\ell=1}^{k} n_\ell \geq n_0 ,
\]
then, for each $\Omega_\ell$, $\lambda \in \sigma(H(\Omega_\ell))$. If $U_\ell(\lambda)$ denotes the eigenspace of $H(\Omega_\ell)$ associated to the eigenvalue $\lambda$, then there is an eigenfunction $u$ of $H(\Omega_0)$ with eigenvalue $\lambda$ such that
\[
u = \sum_{\ell=1}^{k} \varphi_\ell \text{ in } W^{1,2}_0(\Omega_0) ,
\]
where each $\varphi_\ell$ belongs to $U_\ell(\lambda) \setminus \{0\}$ and is identified with its extension by 0 outside $\Omega_\ell$.

**Remark 3.2**
One can naturally think that formula (3.2) has immediate consequences on the family $\Omega_\ell$, which should have for example some covering property. The question is a bit more subtle because we do not want to assume a priori strong regularity properties for the boundary of the $\Omega_\ell$. We shall discuss this point in detail in the last section.

Another consequence of equalities in Theorems 2.1 or 3.1 is given by the following results.

**Theorem 3.3**
Suppose that, for some open subset $\Omega_0$ in $\mathbb{R}^d$, some $\lambda \in \sigma(H(\Omega_0))$ and some family of pairwise disjoint open sets $\Omega_\ell \subset \Omega_0$, $0 < \ell \leq k$, we have
\[
\sum_{\ell=1}^{k} n_\ell = n_0 + \min_{\ell \geq 0} (n_\ell - n_\ell) .
\]
Then, for any subset $L \subset \{1, 2, \ldots, k\}$, such that $\Omega^*_L = \text{Int} \left( \bigcup_{\ell \in L} \Omega_\ell \right) \setminus \partial \Omega_0$ is connected, we have
\[
\sum_{\ell \in L} n_\ell = n(\lambda, \Omega^*_L) + \min \left( \min_{\ell \in L} (n_\ell - n_\ell), n(\lambda, \Omega^*_L) - n(\lambda, \Omega^*_L) \right) .
\]
A simpler variant is the following:

**Theorem 3.4**

Suppose (3.1) holds and that $\Omega^*_L$ is defined as above. Then we have the inequality:

$$\sum_{i \in L} \pi_i \geq n(\lambda, \Omega^*_L).$$

(3.5)

**On the sharpness of Courant’s nodal Theorem**

It is well known that Courant’s nodal Theorem is sharp only for finitely many $k$’s [15].

Let $\Omega_0$ be connected. We will say that an eigenvector $u$ attached to an eigenvalue $\lambda$ of $H(\Omega_0)$ is **Courant-sharp** if $\mu(u) = n(\lambda, \Omega_0)$. Theorem 3.3 implies now:

**Corollary 3.5**

i) Let $u$ be a **Courant-sharp** eigenvector of $H(\Omega_0)$ with $\mu(u) = k$. Let $D^{(k)} = \{D_i\}_{i \in \{1, \ldots, k\}}$ be the family of the nodal domains associated to $u$. Let $L$ be a subset of $\{1, \ldots, k\}$ with $\#L = \ell$ and let $D_L$ be the subfamily $\{D_i\}_{i \in L}$. Let $\Omega_L = \text{Int} \left( \bigcup_{i \in L} D_i \right) \setminus \partial \Omega_0$. Then

$$\lambda_{\ell}(\Omega_L) = \lambda_k$$

(3.6)

where $\lambda_j(\Omega_L)$ are the eigenvalues of $H(\Omega_L)$.

ii) Moreover, when $\Omega_L$ is connected, and if $\ell < k$, $u|_{\Omega_L}$ is **Courant-sharp** and $\lambda_{\ell}(\Omega_L)$ is simple.

**4 Basic tools**

Let us first recall some basic tools (see e.g. [17]) which were already vital for the proof of Courant’s classical result.

**4.1 Variational characterization**

Let us first recall the variational characterization of eigenvalues.

**Proposition 4.1**

Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ and $V$ real in $L^\infty(\Omega)$. Suppose $\lambda \in$
\(\sigma(H(\Omega))\) and let \(U_\pm = \text{span} \langle u_1, \ldots, u_{k_\pm}\rangle\) where
\[
k_-=n(\lambda, \Omega) \text{ and } k_+=\pi(\lambda, \Omega).
\] (4.1)

Then
\[
\lambda = \inf_{\varphi \perp U_-, \varphi \in W^{1,2}_0(\Omega)} \frac{\langle \varphi, H(\Omega)\varphi \rangle}{\|\varphi\|^2}
\] (4.2)

and
\[
\lambda < \lambda_{\pi(\lambda, \Omega)+1} = \inf_{\varphi \perp U_+, \varphi \in W^{1,2}_0(\Omega)} \frac{\langle \varphi, H(\Omega)\varphi \rangle}{\|\varphi\|^2}.
\] (4.3)

If in (4.2) equality is achieved for some \(\Phi \neq 0\), then \(\Phi\) is an eigenfunction in the eigenspace of \(\lambda\).

Note that actually (4.2) and (4.3) are the same statement. We just state them separately for further reference. Note that we have not assumed that \(\Omega\) is connected.

4.2 Unique continuation

Next we restate a weak form of the unique continuation property:

**Theorem 4.2**

Let \(\Omega\) be an open set in \(\mathbb{R}^d\) and \(V\) real in \(L^\infty_{\text{loc}}(\Omega)\). Then any distributional solution solution in \(\Omega\) to \((-\Delta + V)u = \lambda u\) which vanishes on an open subset \(\omega\) of \(\Omega\) is identically zero in the connected component of \(\Omega\) containing \(\omega\).

There are stronger results of this type under weaker assumptions on the potential, see [11].

4.3 A consequence of Harnack’s Inequality

The standard Harnack Inequality (see e.g. Theorem 8.20 in [9]), together with the unique continuation theorem leads to the

**Theorem 4.3**

If \(u\) is an eigenvector of \(H(\Omega)\), then for any \(x\) in \(N(u) \cap \Omega\) and any ball \(B(x, r)\) \((r > 0)\), there exists \(y_\pm \in B(x, r) \cap \Omega\) such that \(\pm u(y_\pm) > 0\).
5 Proof of the main theorems.

5.1 Proof of Theorem 2.1

Assume first for contradiction that

\[ \sum_{\ell \geq 1} \pi_{\ell} > n_0 + \min_{\ell} (\pi_{\ell} - n_\ell) \]  \hspace{1cm} (5.1)

and recall that we assume \( \lambda \in \sigma (H(\Omega_0)) \). Pick some \( \ell_0 \) such that

\[ \pi_{\ell_0} - n_{\ell_0} = \min_{\ell} (\pi_{\ell} - n_{\ell}) \, . \]

Suppose first \( \ell_0 \geq 1 \).

We can rewrite (5.1) to obtain

\[ \sum_{\ell \neq \ell_0, \ell \geq 1} \pi_{\ell} + n_{\ell_0} > n_0 \, . \]  \hspace{1cm} (5.2)

Let \( \varphi_{\ell_0}^i, i = 1, \ldots, n_{\ell_0} \) denote the first \( n_{\ell_0} \) eigenfunctions of \( H(\Omega_{\ell_0}) \). The corresponding eigenvalues are strictly smaller than \( \lambda \). These functions and the remaining \( \sum_{\ell \neq \ell_0} \pi_{\ell} \) eigenfunctions associated to the other \( H(\Omega_{\ell}) \) span a space of dimension at least \( n_0 \). We can pick a linear combination \( \Phi \not\equiv 0 \) of these functions which is orthogonal to the \( n_0 \) eigenfunctions of \( H(\Omega_0) \). By assumption

\[ \frac{\langle \Phi, H(\Omega_0) \Phi \rangle}{\| \Phi \|^2} \leq \lambda, \]  \hspace{1cm} (5.3)

hence \( \Phi \) must by the variational principle be an eigenfunction and there must be equality in (5.3).

There are two possibilities: either some \( \varphi_{\ell_0}^i, i < n_{\ell_0} \) contributes to the linear combination which makes up \( \Phi \) or not. In the first case this means that the left hand side of (5.3) is strictly smaller than \( \lambda \), contradicting the variational characterization of \( \lambda \). In the other case we obtain a contradiction to unique continuation, since then \( \Phi \equiv 0 \) in \( \Omega_{\ell_0} \) and hence vanishes identically in all of \( \Omega_0 \).

Consider now the case when \( \ell_0 = 0 \).

We have to show that the assumption

\[ \sum_{\ell} \pi_{\ell} > \pi_0 \]  \hspace{1cm} (5.4)
leads to a contradiction. To this end it suffices to apply (4.3). Indeed, we can find a linear combination \( \Phi \) of the eigenfunctions \( \phi_j^\ell \), \( j \leq n_\ell \), corresponding to the different \( H(\Omega_\ell) \) such that \( 0 \not\equiv \Phi \perp U_+ \), but satisfies

\[
\frac{\langle \Phi, H(\Omega_0) \Phi \rangle}{\| \Phi \|^2} \leq \lambda = \lambda_0 ,
\]

and this contradicts (4.3). This proves (2.6).

5.2 Proof of Theorem 3.1

The inequality (3.1) implies that we can find a non zero \( u \perp U_- \) in the span of the eigenfunctions \( \phi_j^\ell \), \( j = 1, \ldots, K \), of the different \( H(\Omega_\ell) \). Again by the variational characterization, (4.2) and (5.5) hold and hence \( u \) must be an eigenfunction. \( \square \)

5.3 Proof of Theorem 3.3

We assume (3.3). Without loss we might assume that we have labeled the \( \Omega_\ell \) such that \( L = \{1, \ldots, K\} \), with \( K \leq k \). Let \( n_\ast = n(\lambda, \Omega_\ast_L) \). We apply Theorem 2.1 to the family \( \Omega_\ell \) (\( \ell \in L \)) and replace \( \Omega_0 \) by \( \Omega_\ast_L \) and obtain :

\[
\sum_{1 \leq \ell \leq K} n_\ell \leq n_\ast + \min \left( n_\ast - n_\ast, \min_{1 \leq \ell \leq K} \left( n_\ell - n_\ell \right) \right) . \tag{5.5}
\]

We assume for contradiction that

\[
\sum_{1 \leq \ell \leq K} n_\ell < n_\ast + \min \left( n_\ast - n_\ast, \min_{1 \leq \ell \leq K} \left( n_\ell - n_\ell \right) \right) . \tag{5.6}
\]

This implies

\[
\sum_{1 \leq \ell \leq K} n_\ell < n_\ast , \tag{5.7}
\]

and

\[
\sum_{1 \leq \ell \leq K} n_\ell < n_\ast + \min_{1 \leq \ell \leq K} \left( n_\ell - n_\ell \right) . \tag{5.8}
\]

Theorem 2.1 applied to the family \( \Omega_\ast_L, \Omega_\ell \) (\( \ell > K \)), implies that

\[
\pi_\ast + \sum_{K < \ell \leq k} n_\ell \leq n_0 + \min_{K < \ell \leq k} \left( n_0 - n_0, \min_{K < \ell \leq k} \left( n_\ell - n_\ell \right) \right) , \tag{5.9}
\]

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and
\[ n_* + \sum_{K < \ell \leq k} \pi_\ell \leq n_0 . \] (5.10)

By adding (5.7) and (5.9), we get:
\[ \sum_{1 \leq \ell \leq k} \pi_\ell < n_0 + \min \left( n_0 - n_0, \min_{K < \ell \leq k} (\pi_\ell - n_\ell) \right) . \] (5.11)

By adding (5.8) and (5.10), we obtain
\[ \sum_{1 \leq \ell \leq k} \pi_\ell < n_0 + \min_{1 \leq \ell \leq K} (\pi_\ell - n_\ell) . \] (5.12)

The combination of (5.11) and (5.12) is in contradiction with (3.3).

### 5.4 Proof of Theorem 3.4

For the case that (3.1) holds (3.5) can be shown similarly. (3.1) reads
\[ \sum_{1 \leq \ell \leq k} \pi_\ell \geq n_0 . \]

We assume for contradiction that
\[ \sum_{1 \leq \ell \leq K} \pi_\ell < n_* , \] (5.13)

where \( n_* \) is defined as above. The addition of (5.10) and (5.13) leads to a contradiction. \( \square \)

### 6 Illustrative examples

#### 6.1 Examples for a rectangle

We illustrate Theorem 2.1 by the analysis of various examples in rectangles. Pick a rectangle \( \Omega_0 = (0, 2\pi) \times (0, \pi) \) and take \( \Omega_1 = (0, \pi) \times (0, \pi) \) and consequently \( \Omega_2 = (\pi, 2\pi) \times (0, \pi) \). The eigenvalues corresponding to \( \Omega_0 \) for \(-\Delta\) with Dirichlet boundary conditions are given by
\[
\sigma(H(\Omega_0)) = \left\{ \lambda \in \mathbb{R} \mid \lambda = m^2/4 + n^2, (m, n) \in \mathbb{Z}^2, m, n > 0 \right\} ,
\] (6.1)
while those for $\Omega_1$ and hence for $\Omega_2$ which can be obtained by a translation of $\Omega_1$, are given by

$$\sigma(H(\Omega_1)) = \sigma(H(\Omega_2)) = \left\{ \lambda \in \mathbb{R} \mid \lambda = m^2 + n^2, \ (m, n) \in \mathbb{Z}^2, \ m, n > 0 \right\}. \hspace{1cm} (6.2)$$

Denote the eigenvalues associated to $\Omega_0$ by $\{\lambda_i\}$ and those to $\Omega_1$ by $\{\nu_i\}$. We easily check that $\lambda_5 = \lambda_6 = \nu_2 = \nu_3 = 5, \ \lambda_{11} = \lambda_{12} = \nu_5 = \nu_6 = 10$ so that for these cases Theorem 2.1 is sharp.

One could ask whether there are arbitrarily high eigenvalues cases for which we have equality in (2.6). This is not the case, as can be seen from the following standard number theoretical considerations. We have (see [18] and for more recent contributions [16] and [2]) the following asymptotic estimate for the number of lattice points in an ellipse. Let $a, b > 0$, then

$$A(\lambda) := \# \left\{ (m, n) \in \mathbb{Z}^2 \mid am^2 + bn^2 \leq \lambda \right\} \hspace{1cm} (6.3)$$

has the following asymptotics as $\lambda$ tends to infinity:

$$A(\lambda) = \frac{\pi}{\sqrt{ab}} \lambda + \mathcal{O}(\lambda^{1/3}). \hspace{1cm} (6.4)$$

We have not to consider $A(\lambda)$ but rather

$$A^+ = \# \left\{ (m, n) \subset \mathbb{Z}^2, \ m, n > 0 \mid am^2 + bn^2 \leq \lambda \right\}. \hspace{1cm} (6.5)$$

Hence we get

$$A(\lambda) = 4A^+(\lambda) + 2\# \left\{ m \in \mathbb{N}, \ m > 0 \mid m \leq \left\lfloor (\lambda/a)^{1/2} \right\rfloor \right\} + 1. \hspace{1cm} (6.6)$$

If we apply this to $A^+$ with $a = 1/4, b = 1$ (in this case denoted by $A^+_0$) and to $A^+$ with $a = 1, b = 1$ (in this case denoted by $A^+_1$), we get asymptotically

$$A^+_0(\lambda) - 2A^+_1(\lambda) = \frac{1}{2} \sqrt{\lambda} + o(\sqrt{\lambda}). \hspace{1cm} (6.7)$$

Note that

$$\mu_i(\lambda) = A^+_i(\lambda), \ i = 0, 1.$$
In order to control $n_i(\lambda)$, we observe that, for any $\epsilon > 0$:
\[ \pi_i(\lambda - \epsilon) \leq n_i(\lambda) \leq \pi_i(\lambda) . \]
This implies
\[ \pi_i(\lambda) - n_i(\lambda) = O(\lambda^{\frac{1}{3}}) . \tag{6.8} \]
The asymptotic formula (6.4) implies
\[ \pi_i(\lambda) - n_i(\lambda) = o(\sqrt{\lambda}) , \tag{6.9} \]
and this shows that for large $\lambda$ (2.6) is never sharp.

6.2 About Corollary 3.5.

One can ask whether there is a converse to Corollary 3.5 in the following sense. Suppose we have an eigenfunction $u$ with $k$ nodal domains and eigenvalue $\lambda$. For each pair of neighboring nodal domains of $u$, say, $D_i$ and $D_j$, let $\Omega_{i,j} = \text{Int} (D_i \cup D_j)$ and suppose that $\lambda = \lambda_2(\Omega_{i,j})$. Does this imply that $\lambda = \lambda_k$? The answer to the question is negative, as the following easy example shows:

Consider the rectangle $Q = (0, a) \times (0, 1) \subset \mathbb{R}^2$ and consider $H_0(Q)$. We can work out the eigenvalues explicitly as
\[ \{ \pi^2 \left( \frac{m^2}{a^2} + n^2 \right) \}, \text{ for } m, n \in \mathbb{N} \setminus 0, \tag{6.10} \]
with corresponding eigenvectors $(x, y) \mapsto \sin(\pi m \frac{x}{a})\sin(\pi n y)$. If
\[ a^2 \in \left( \frac{9}{4}, \frac{8}{3} \right) , \tag{6.11} \]
then
\[ \lambda_3(Q) = \pi^2 \left( \frac{1}{a^2} + 4 \right) < \lambda_4(Q) = \pi^2 \left( \frac{9}{a^2} + 1 \right) , \]
and the zero set of $u_4$ is given by $\{(x, y) \in Q \mid x = a/3, x = 2a/3\}$. For $u_4$ we have $\Omega_{1,2} = Q \cap \{0 < x < 2a/3\}$. If $2a/3 > 1$ (which is the case under assumption (6.11)), then $\lambda_2(\Omega_{1,2}) = \lambda_4(Q)$. We have consequently an example with $k = 3$. 

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7 Converse theorems in the case of regular open sets

7.1 Preliminary discussion about regularity

Before we present what we think would be the right notion of regular open set adapted to our problem, let us discuss briefly other possible notions.

As a consequence of Theorem 3.1 and using (1.3), we get that each nodal domain $D_{k\ell}$ of $\varphi_{\ell}$ is included in a nodal domain $D_{j0}$ of $u$. Using a result of Gesztesy and Zhao ([8], Theorem 1), this implies also that the capacity (see next subsection) of $D_{j0} \setminus D_{k\ell}$ (hence the measure) is 0.

At the “regular” points of the boundary of $\Omega_\ell$ one can get additional information.

Let us say that, if $\Omega$ is an open set and if $\partial \Omega = \overline{\Omega} \setminus \Omega$, that a point of $\partial \Omega$ is $C^{1,\alpha}$-regular if there exists a neighborhood $V(x)$ of $\partial \Omega$ such that $\partial \Omega \cap V(x)$ is a $C^{1,\alpha}$-hypersurface. We denote by $\partial \Omega^{c1reg}$ the subset of the regular points.

Then we get, from (3.2), using the property of the restriction map from $W^{1,2}(\Omega_\ell)$ into $W^{1,2}(\partial \Omega^{c1reg}_\ell)$, that, under the assumptions of Theorem 3.1, we have the inclusion

$$\bigcup_{\ell \geq 1} (\partial \Omega^{c1reg}_\ell) \subset N(u) \cup \partial \Omega_0.$$  (7.1)

One could say that an open set $D$ is topologically regular if the subset of regular points is dense in $\partial D$.

In this case, it is easy to see that if some function $f$ belongs to $C^0(\overline{D} \cap \Omega_0) \cap W_0^{1,2}(D)$, then $f$ vanishes on the boundary (we first get it at the regular points and then conclude by continuity).

More generally, one could ask under which weakest condition on a point $x$ of $\partial D \cap \Omega_0$ any function $f$ in $C^0(\overline{D} \cap \Omega_0) \cap W_0^{1,2}(D)$ satisfies $f(x) = 0$. This is what we will discuss in the next subsections.

7.2 Capacity

There are various equivalent definitions of polar sets and capacity (see e.g. [3], [7], [10], [13]). If $U$ is a bounded open subset of $\mathbb{R}^d$, we denote by
∥·∥_{W^{1,2}_0(U)} the Hilbert norm on $W^{1,2}_0(U)$:

$$u \mapsto \|u\|_{W^{1,2}_0(U)} := \left( \int_U |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

The capacity in $U$ of $A \subset U$ can be defined\(^\dagger\) as

$$\text{Cap}_U(A) := \inf \{ \|s\|^2_{W^{1,2}_0(U)} ; s \in W^{1,2}_0(U) \text{ and } s \geq 1 \text{ a.e. in some neighborhood of } A \}.$$

It is easily checked that if $K$ is compact and $K \subset U \cap V$, where $V$ is also open and bounded in $\mathbb{R}^d$, there is a $c = c(K,U,V)$ such that $\text{Cap}_U(A) \leq c \text{ Cap}_V(A)$ for $A \subset K$. So $\text{Cap}_U(A) = 0$ for some bounded open $U \supset A$ iff for each $a \in A$ there exists an $r > 0$ and a bounded region $V$ such that $V \supset B(a,r)$ and $\text{Cap}_V(B(a,r) \cap A) = 0$. In this case we may simply write $\text{Cap}(A) = 0$ without referring to $U$.

### 7.3 Converse theorem

We are now able to formulate our definition of regular point.

**Definition 7.1**

Let $D$ be an open set in $\mathbb{R}^d$. We shall say that a point $x \in \partial D$ is (capacity)-regular (for $D$) if, for any $r > 0$, the capacity of $B(x,r) \cap \complement D$ is strictly positive.

**Theorem 7.2**

Under the assumptions of Theorem 3.1, any point $x \in \partial \Omega_\ell \cap \Omega_0$ which is (capacity)-regular with respect to $\Omega_\ell$ (for some $\ell$) is in the nodal set of $u$.

This theorem admits the

**Corollary 7.3**

Under the assumptions of Theorem 3.1 and if, for all $\ell$, every point in $(\partial \Omega_\ell) \cap \Omega_0$ is (capacity)-regular for $\Omega_\ell$, then the family of the nodal domains of $u$ coincides with the union over $\ell$ of the family of the nodal domains of the $\varphi_\ell$, where $u$ and $\varphi_\ell$ are introduced in (3.2).

**Proof of corollary**

It is clear that any nodal domain of $\varphi_\ell$ is contained in a unique

\(^\dagger\)For $d \geq 3$ the restriction that $U$ is bounded can be removed and one may take $U = \mathbb{R}^d$. 

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nodal domain of $u$. Conversely, let $D$ be a nodal domain of $u$ and let $\ell \in \{1, \ldots, k\}$. Then, by combining the assumption on $\partial \Omega_\ell$, Proposition 7.4 and (3.2), we obtain the property:

$$\partial \Omega_\ell \cap D = \emptyset.$$ 

Now, $D$ being connected, either $\Omega_\ell \cap D = \emptyset$ or $D \subset \Omega_\ell$. Moreover the second case should occur for at least one $\ell$, say $\ell = \ell_0$. Coming back to the definition of a nodal set and (3.2), we observe that $D$ is necessarily contained in a nodal domain $D_{\ell_0}^0$ of $\varphi_{\ell_0}$.

Combining the two parts of the proof gives that any nodal set of $u$ is a nodal set of $\varphi_\ell$ and vice-versa.

### 7.4 Proof of Theorem 7.2

According to the discussion of Subsection 7.1, the proof will be a consequence of the following proposition.

**Proposition 7.4**

Let us consider two open subsets $D$ and $\Omega$ of $\mathbb{R}^d$ such that $D \subset \Omega$ and a point $x_0$ in $\partial D \cap \Omega$. Assume that, for some given $r_0 > 0$ such that $B(x_0, r_0) \subset \Omega$, there exists $u \in W^{1,2}_0(D)$ and $v \in C^0(B(x_0, r_0))$ such that:

$$u|_{D \cap B(x_0, r_0)} = v|_{D \cap B(x_0, r_0)} \text{ a.e. in } D \cap B(x_0, r_0).$$

Then if $v(x_0) \neq 0$, there exists a ball $B(x_0, r_1)$ ($r_1 > 0$), such that $B(x_0, r_1) \setminus D$ is polar, that is of capacity 0.

**Remark 7.5**

Using some standard potential theoretic arguments, Proposition 7.4 can be deduced from Théorème 5.1 in [6] which characterizes those $u \in W^{1,2}(\Omega)$ that belong to $W^{1,2}_0(\Omega)$. The proof below should be more elementary in character.

**Remark 7.6**

Given a region $D \subset \mathbb{R}^n$ and a ball $B = B(x, r)$, $x \in \partial D$, the difference set $B(x, r) \setminus D$ is polar if and only if $B(x, r) \cap \partial D$ is polar. This follows from the fact that a polar subset of $B = B(x, r)$ does not disconnect $B$ [3].
Remark 7.7
If $D$ is a nodal domain of an eigenfunction $u$ of $H(\Omega)$, then any point of $\partial D \cap \Omega$ is capacity-regular for $D$. This is an immediate consequence of Theorem 4.3 (it also follows from the preceding remark). Indeed, if $x$ is in $\partial D \cap \Omega$, then for any $r > 0$, one can find a ball $B(y, r')$ in $\overline{D} \cap B(x, r)$.

To prove Proposition 7.4 we require some well-known facts stated in the next three lemmas.

Lemma 7.8
Let $U$ be a bounded convex domain in $\mathbb{R}^d$ and let $B(a, \rho)$, $\rho > 0$ be a ball such that $\overline{B}(a, \rho) \subset U$. There exists a positive constant $c = c(a, \rho, U)$ such that, for every $f \in W^{1,2}(U)$ vanishing a.e. in $B(a, \rho)$,

$$\|f\|_{W^{1,2}(U)} \leq c \|\nabla f\|_{L^2(U)}.$$  

Proof.
We assume as we may that $a = 0$ and let $U' = U \setminus B(0, \rho)$. Fix $R$ so large that $U \subset B(0, R)$. By approximating $f$ by smooth functions (e.g. regularize $f((1 - \delta)z)$ for $\delta > 0$ and small to get $f_1 \in C^\infty(U)$), we may restrict to functions $f \in C^\infty(U)$ vanishing in $B(0, \rho)$. Then, since

$$|f(x)|^2 = \int_0^1 \nabla f(sx).x ds \leq R^2 \int_0^1 |\nabla f(sx)|^2 ds \text{ for } x \in U',$$

we have

$$\int_{U'} |f(x)|^2 dx \leq R^2 \int_{x \in U', \rho \leq |x| \leq 1} |\nabla f(sx)|^2 dx ds \leq R^2 \int_{z \in \mathbb{R}^d, \rho \leq |z|, s \leq 1} |\nabla f(z)|^2 \frac{ds}{s} \quad (7.2)$$

and the lemma follows.

Lemma 7.9
Let $U$ be a domain in $\mathbb{R}^d$. For every $f \in W^{1,2}(U)$ the function $g = f_+$ is also in $W^{1,2}(U)$, with $\|g\|_{W^{1,2}(U)} \leq \|f\|_{W^{1,2}(U)}$. Moreover the map $f \mapsto g$ from $W^{1,2}(U)$ into itself is continuous (in the norm topology).
Remark 7.10
Since \(\inf\{f_n, 1\} = 1 - (1 - f_n)_+\), it follows from the lemma that \(\inf\{f_n, 1\} \to \inf\{f, 1\}\) in \(W^{1,2}(U)\) whenever \(f_n \to f\) in \(W^{1,2}(U)\).

Proof.
For the first two facts we refer to [12] or [13], where it is moreover shown that the weak partial derivatives \(\partial_j f_+\) and \(\partial_j f\) satisfy

\[
\partial_j f_+ = 1_{\{f > 0\}} \partial_j f = 1_{\{f \geq 0\}} \partial_j f \quad \text{a.e. in } U.
\]

Therefore, for any \(\delta > 0\), we have:

\[
||\nabla [f_n]_+ - \nabla f_+||_{L^2} = ||1_{\{f_n > 0\}} \nabla f_n - 1_{\{f > 0\}} \nabla f||_{L^2} \\
\leq ||1_{\{f_n > 0\}} (\nabla f_n - \nabla f)||_{L^2} + ||(1_{\{f > 0\}} - 1_{\{f_n > 0\}}) \nabla f||_{L^2} \\
\leq ||\nabla f_n - \nabla f||_{L^2} + ||(1_{\{f > 0, f_n \leq 0\}} + 1_{\{f_0, f_n > 0\}}) \nabla f||_{L^2} \\
\leq ||\nabla f_n - \nabla f||_{L^2} + ||1_{\{0 \leq |f| \leq \delta\}} \nabla f||_{L^2} + 2||1_{\{|f_n - f| \geq \delta\}} \nabla f||_{L^2}.
\]

(7.3)

Given \(\varepsilon > 0\), fix \(\delta > 0\) so that \(||1_{\{0 \leq |f| \leq \delta\}} \nabla f||_{L^2} \leq \varepsilon\) (recall that \(\nabla f = 0\) a.e. in \(\{f = 0\}\)). Since \(\nabla f \in L^2(U)\) and \(||1_{\{|f - f_n| \geq \delta\}}||_{L^1} \leq \frac{||f - f_n||_{L^2}^2}{\delta^2}\), it follows that \(\lim_{n \to \infty} ||1_{\{|f - f_n| \geq \delta\}} \nabla f||_{L^2} = 0\). Therefore \(\limsup_{n \to \infty} ||\nabla [f_n]_+ - \nabla f_+||_{L^2} \leq \varepsilon\), which proves that \([f_n]_+ \to f_+\) in \(W^{1,2}(U)\), if \(f_n \to f\) in \(W^{1,2}(U)\).

Lemma 7.11
Let \(\omega\) be open in \(\mathbb{R}^d\) and let \(\{f_n\}\) be a sequence of continuous functions in \(\omega\) such that \(f_n \in W^{1,2}(\omega)\) for each \(n \geq 1\) and \(\lim_{n \to \infty} ||f_n||_{W^{1,2}(\omega)} = 0\).

Then the set \(F = \{x \in \omega; \liminf_{n \to \infty} |f_n(x)| > 0\}\) is polar.

Proof.
It suffices to show that \(\text{cap}_\omega(F \cap K) = 0\) for any compact subset \(K\) of \(\omega\). Let \(\varphi \in C^\infty_0(\mathbb{R}^d)\) be such that \(0 \leq \varphi \leq 1\) in \(\mathbb{R}^d\), \(\varphi = 1\) in \(K\) and \(\text{supp}(\varphi) \subset \omega\). Then \(g_n = f_n \varphi \to 0\) in \(W^{1,2}_0(\omega)\) and \(g_n = f_n\) in \(K\).

Set \(F_\nu = \{x \in \omega; |g_n(x)| \geq 2^{-\nu}\text{ for all } n \geq \nu\}\). By the definition of the capacity, we have \(\text{Cap}_\omega(F_\nu) \leq 2^{2\nu} ||\nabla g_n||_{L^2}^2\text{ for all } n \geq \nu\) and \(\text{cap}(F_\nu) = 0\). Therefore \(\text{cap}_\omega(\bigcup_{\nu \geq 1} F_\nu) = 0\) and \(\text{cap}_\omega(F \cap K) = 0\), since \(F \cap K \subset \bigcup_{\nu \geq 1} F_\nu\).

Proposition 7.12
Let \(U\) be a non-empty open subset of the ball \(B = B(a, r)\) in \(\mathbb{R}^d\). Suppose there exist a continuous function \(f\) in \(U\) and a sequence \(\{f_n\}\) of continuous
functions in $B$ such that
(i) $f \geq 1$ in $U$ and $f \in W^{1,2}(U)$,
(ii) $f_n = 0$ in a neighborhood of $B \setminus U$ and $f_n \in W^{1,2}(U)$ for each $n \geq 1$,
(iii) $\lim_{n \to \infty} \|f - f_n\|_{W^{1,2}(U)} = 0$.
Then the set $F := B \setminus U$ is polar.

Proof of Proposition 7.12
Replacing $f$ by $\inf\{f, 1\}$ and $f_n$ by $\inf\{f_n, 1\}$, we see from Lemma 7.9 that we may assume that $f = 1$ in $U$.
So $\lim_{n \to \infty} \|\nabla f_n\|_{L^2(U)} = 0$ and $\lim_{n \to \infty} \|1 - f_n\|_{L^2(U)} = 0$.

Fix a ball $B(z_0, 2\rho) \subset U$, $\rho > 0$, and a cut-off function $\alpha \in C_\infty(\mathbb{R}^d)$ such that $\alpha = 1$ in $B(z_0, \rho)$, $\alpha = 0$ in $\mathbb{R}^d \setminus B(z_0, 2\rho)$. Set $g = 1 - \alpha$, $g_n = (1 - \alpha) f_n$.
Then $g, g_n$ belong to $W^{1,2}(B)$, $\nabla g = \nabla g_n = 0$ a.e. in $F$ and
$$\lim_{n \to \infty} \|\nabla (g - g_n)\|_{L^2(B)} = \lim_{n \to \infty} \|\nabla (g - g_n)\|_{L^2(U)} = 0.$$  
So, by Lemma 7.8, $\lim_{n \to \infty} \|g - g_n\|_{W^{1,2}(B)} = 0$. But $g - g_n \geq 1$ in $F$ and it follows from Lemma 7.11 that $F$ is polar.

Proof of Proposition 7.4
Choose $r_1 > 0$ so small that $v \geq c_0 := \frac{1}{2}v(x_0)$ in $B(x_0, r_1)$. Since $u \in W^{1,2}_0(D)$, there is a sequence $\{u_n\}$ in $C_\infty(\mathbb{R}^d)$ such that $\text{supp}(u_n) \subset D$ and $u_n \to u$ in $W^{1,2}(\mathbb{R}^d)$. Applying Proposition 7.12 to the ball $B(x_0, r_1)$ and the functions $f = c_0^{-1} u|_{B(x_0, r_1)}$, $f_n = c_0^{-1} u_n|_{B(x_0, r_1)}$, we see that $B(x_0, r_1) \setminus D$ is polar.

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\footnote{The weak convergence $\inf\{f_n, 1\} \rightharpoonup \inf\{f, 1\}$ suffices here. It allows the approximation of $1 = \inf\{f, 1\}$ in the norm topology in $W^{1,2}(U)$ by finite convex combination of the $\inf\{f_n, 1\}$. So we are again left with the case when $f = 1$ in $U$.}
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