The Pseudoscalar Meson Mass to Two Loops in Three-Flavor Partially Quenched \( \chi \)PT

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This paper presents a first study of the pseudoscalar meson masses to two loops, or NNLO, within the supersymmetric formulation of partially quenched chiral perturbation theory (PQ\( \chi \)PT). The expression for the pseudoscalar meson mass in the case of three valence and three sea quarks with equal masses, but different from each other, is given to \( \mathcal{O}(p^6) \), along with a numerical analysis.

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INTRODUCTION

In the theory of the strong interaction, Quantum Chromodynamics (QCD), it has so far been impossible to derive hadron masses directly using analytical methods. Lattice QCD is an approach whereby the functional integral is evaluated numerically to obtain various physical quantities. One difficulty in this has been the inclusion of dynamical quark effects, but this is an area where much recent progress has been made. However, for computational reasons, the masses of the valence quarks can be much more easily varied than those of the sea quarks. This situation is referred to as partially quenched lattice QCD.

The quark masses which can be accessed by present simulations are, especially for the sea quarks, significantly larger than the physical up and down quark masses. In order to obtain the masses of the lightest hadrons, one thus needs to extrapolate from the quark masses used in the lattice calculations to the physical ones.

In this domain Chiral Perturbation Theory (\( \chi \)PT), which is an effective low-energy field theory approximation to QCD, should be valid. It provides the correct quark mass dependence of the pseudoscalar meson masses, including the nonanalytic dependences often referred to as chiral logarithms. The three flavour \( \chi \)PT, as formulated by Gasser and Leutwyler \(^1\), is valid in the QCD case with equal valence and sea quark masses, and cannot be used to extrapolate from partially quenched lattice QCD calculations. The extension of \( \chi \)PT to the quenched \(^2\) and partially quenched \(^3\) cases has been carried out by Bernard and Golterman. The most general expression at one loop for the pseudoscalar meson masses was worked out by Sharpe and Shoresh \(^4\). In this work we present the result for the meson masses at next-to-next-to-leading order (NNLO) in PQ\( \chi \)PT. It should be pointed out that the case of full QCD corresponds to the special case of partially quenched QCD where the sea and valence quark masses are put equal. The free parameters or low energy constants (LECs) of \( \chi \)PT, i.e. the QCD case, may thus be unambiguously determined from those of PQ\( \chi \)PT. These points are discussed in detail in \( \chi \)PT.

We work here in the version of PQ\( \chi \)PT without the supersinglet \( \Phi_0 \), discussed in detail at one loop in \( \chi \)PT. Here we present the expression for the charged, or off-diagonal, meson mass for the case of equal valence and equal sea quark masses, but these different from each other. The charged meson masses for the more general quark mass cases, as well as more details of the present calculation will be presented elsewhere \( \chi \)PT. Planned work includes the decay constants and the neutral, or diagonal, meson masses. Note that the presence of NNLO terms is already seen in the works of ref. \( \chi \)PT.

In the next sections we present the technical background, the full expression for the pseudoscalar meson mass, and finally a short discussion and numerical results as a function of the input quark masses.

TECHNICAL OVERVIEW

Most of the technical aspects required for calculations at the two-loop level in PQ\( \chi \)PT without \( \Phi_0 \) already exist, but they have not been mentioned as such. The full divergence structure at one loop was worked out in \( \chi \)PT. There, it was noted that the divergence structure is really identical to that of normal \( \chi \)PT with the number of flavours equal to the number of sea quark flavors, provided that the traces and matrices are replaced by the supertraces and matrices appropriate for quenched \( \chi \)PT. This is justified by the replica method. In ref. \( \chi \)PT it has been argued that the quenched approximation, i.e. no sea quarks, can also be obtained using the replica method. This entails a calculation with \( n_F \) flavors, setting \( n_F = 0 \) in the final answer. In ref. \( \chi \)PT this was proven explicitly at one-loop order. These arguments can be generalized to higher orders and to PQ\( \chi \)PT. This allows the use of
the known results for \( n_F \) flavours in normal \( \chi \)PT (valence and sea quark masses equal), in order to obtain the needed Lagrangians at \( \mathcal{O}(p^2) \) and \( \mathcal{O}(p^6) \) from ref. [11], as well as the full divergence structure from ref. [12]. In particular, the \( n_F \) flavor Lagrangian at \( \mathcal{O}(p^4) \) contains the term with \( L_0 \), see also [11]. In the context of PQ\( \chi \)PT, this extra term was pointed out in ref. [13]. We will use the notation of refs. [11, 12] for the Lagrangian at \( \mathcal{O}(p^4) \) and \( \mathcal{O}(p^6) \), \( L_i^r, i = 0, \ldots, 10 \) and \( H_i^r, i = 1, 2 \) for the renormalized \( \mathcal{O}(p^4) \) LECs, and \( K_i^r, i = 1, \ldots, 115 \) for the renormalized LECs at \( \mathcal{O}(p^6) \). The reason why these Lagrangians can be obtained by simply replacing the trace by the relevant supertrace [2] is that the structure of the equations of motion and all other identities used are the same as for the \( n_F \) flavor case of refs. [11, 12]. However, the Cayley-Hamilton identity used there to reduce the Lagrangians for the two and three flavour case is not valid in PQ\( \chi \)PT.

The last technical difficulty in PQ\( \chi \)PT as compared to the mass calculations at two loops in \( \chi \)PT [14, 15] are the extra two-loop integrals needed since the propagators in PQ\( \chi \)PT can have a double pole structure. These extra integrals can be evaluated using the same methods as was used for the integrals in ref. [14], provided that derivatives of those expressions w.r.t. the relevant masses in the propagators are taken.

### THE PSEUDOSCALAR MESON MASS TO \( \mathcal{O}(p^6) \)

The mass of the pseudoscalar meson is obtained by means of dimensional regularization from the diagrams in Fig. 1 and those at \( \mathcal{O}(p^2) \) and \( \mathcal{O}(p^4) \).

The result is expressed in terms of the valence quark mass, \( m_{qV} \), and the sea quark mass, \( m_{qS} \), via \( \chi_1 = 2B_0 m_{qV}, \chi_4 = 2B_0 m_{qS} \) and \( \chi_{14} = (\chi_1 + \chi_4)/2 \). These correspond to the lowest order charged meson masses. Other parameters include the decay constant in the chiral limit \( (F_0) \), the quark condensate in the chiral limit, via \( \langle \bar{q}q \rangle = -B_0 F_0^2 \), and the LECs of \( \mathcal{O}(p^4) \) and \( \mathcal{O}(p^6) \), the \( L_i^r \) and \( K_i^r \).

The finite parts of the loop integrals in the expressions below are

\[
\tilde{A}(\chi) = -\pi_{16} \chi \log(\chi/\mu^2),
\]
\[
\tilde{B}(\chi, \chi; 0) = -\pi_{16} \left( 1 + \log(\chi/\mu^2) \right),
\]
\[
\tilde{C}(\chi, \chi; 0) = -\pi_{16}/(2\chi),
\]

where the subtraction scale dependence has been moved into the loop integrals. We also define \( \pi_{16} = 1/(16\pi^2) \). The finite two-loop sunset integrals \( H^F, H^F_i, H^F_{ii} \) that appear in the top right diagram of Fig. 1 may be evaluated using the methods of [14]. The notation used for the integrals is the same as in ref. [14] except that an extra integer argument now indicates the needed propagator structure. Index (1) corresponds to the case of single propagators only, as in ref. [14], whereas index (2) indicates that the first propagator appears squared, index (3) that the second propagator appears squared, and finally index (5) that the first and second propagators appear squared. Explicit expressions can be found in ref. [14], and by taking derivatives w.r.t. the masses of the expressions there.

![Fig. 1: Feynman diagrams at \( \mathcal{O}(p^6) \) or two-loop for the pseudoscalar meson mass. Filled circles denote vertices of the \( \mathcal{L}_2 \) Lagrangian, whereas open squares and shaded diamonds denote vertices of the \( \mathcal{L}_4 \) and \( \mathcal{L}_6 \) Lagrangians, respectively.](image-url)
\[ \delta_{\text{loops}}^{(6)} = \pi_{16}^{2} [15/32 \chi_{1} \chi_{4} \chi_{1} - 3/32 \chi_{4}^{2} + 73/64 \chi_{4}^{2}] - \pi_{16} \pi_{12}^{2} [41/9 \chi_{1} \chi_{4} + 10/9 \chi_{4}^{2} + 17/4 \chi_{4}^{2}] + \pi_{16} [26/3 \chi_{1} \chi_{4} - \chi_{4}^{2} + 3 \chi_{4}^{2}] L_{0}^{2} + 4 \pi_{16} \chi_{1} L_{1}^{2} + \pi_{16} [2 \chi_{4}^{2} + 16 \chi_{4}^{2}] L_{2}^{2} + \pi_{16} [17/3 \chi_{1} \chi_{4} - 5/2 \chi_{4}^{2} + 3/2 \chi_{4}^{2}] L_{3}^{2} + 3 \pi_{16} \tilde{A} (\chi_{1}) \chi_{4} + 384 \chi_{1} \chi_{4} L_{5} L_{5} - 1152 \chi_{4}^{2} L_{5} L_{6} - 384 \chi_{1} \chi_{4} L_{5} L_{6}^{2} + 576 \chi_{4}^{2} L_{5}^{2} - 384 \chi_{1} \chi_{4} L_{5} L_{6}^{2} - 128 \chi_{4}^{2} L_{5} L_{6}^{2} + 64 \chi_{4}^{2} L_{5}^{2} - 8 \tilde{A} (\chi_{1}) [2 \chi_{1} - \chi_{4}] L_{0}^{2} + 8 \tilde{A} (\chi_{1}) \chi_{1} L_{2}^{2} - 8 \tilde{A} (\chi_{1}) [2 \chi_{1} - \chi_{4}] L_{2}^{2} + 16 \tilde{A} (\chi_{1}) \chi_{4} L_{2}^{2} + 16 \tilde{A} (\chi_{1}) [1 - 1/3 \chi_{4}] L_{0}^{2} + 16 \tilde{A} (\chi_{1}) [2 \chi_{1} - \chi_{4}] L_{0}^{2} + 32 \tilde{A} (\chi_{1}) [1 - 1/4 - 64/3 \tilde{A} (\chi_{1}) \chi_{1} L_{0}^{2} - \tilde{A} (\chi_{1})^{2} [13/72 - 5/18 \chi_{4}] - 2 \tilde{A} (\chi_{1}) \tilde{B} (\chi_{1}, \chi_{1}, 0) \chi_{4} + 128 \tilde{A} (\chi_{1}) \chi_{4} L_{1}^{2} + 32 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} - 128 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} + 128 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} - \tilde{A} (\chi_{1})^{2} [27/4 - 1 \chi_{4} \chi_{1}^{-1}] - 2 \tilde{A} (\chi_{1}) \tilde{B} (\chi_{1}, \chi_{1}, 0) \chi_{4} + 128 \tilde{A} (\chi_{1}) \chi_{4} L_{1}^{2} + 32 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} - 128 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} + 128 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} - \tilde{A} (\chi_{1})^{2} [27/4 - 1 \chi_{4} \chi_{1}^{-1}] - 2 \tilde{A} (\chi_{1}) \tilde{B} (\chi_{1}, \chi_{1}, 0) \chi_{4} + 128 \tilde{A} (\chi_{1}) \chi_{4} L_{1}^{2} + 32 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} - 128 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} + 128 \tilde{A} (\chi_{1}) \chi_{4} L_{0}^{2} - \tilde{A} (\chi_{1})^{2} [27/4 - 1 \chi_{4} \chi_{1}^{-1}]
\]

As expected, the divergences in these expressions have cancelled. Note that in both eq. 8 and eq. 9, the extra pole structure from the partially quenched nature disappears if we set \( \chi_{1} = \chi_{4} \). The remaining \( \tilde{B} (\chi_{1}, \chi_{1}, 0) \) terms at \( O(p^{4}) \) in that case are due to the fact that we have expressed the \( O(p^{4}) \) result in terms of the lowest order masses rather than the full physical masses, which was called unrenormalized in ref. [14]. Consequently, they do not produce unphysical logarithms.

**DISCUSSION AND CONCLUSIONS**

In the long run, all input parameters should of course be determined by a fit of the formulas of PQ\( \chi \)PT to lattice QCD data. At the present time, we can only present results using input parameters from the continuum work in \( \chi \)PT at the same order. We use as input parameters the fit to data presented in ref. [12], called fit 10, which had \( F_{0} = 87.7 \) MeV and \( \mu = 770 \) MeV. Of the parameters which were not determined there, we set \( K_{0}^{\pi} = L_{0}^{\pi} = L_{0}^{\rho} = 0 \). The last one cannot be determined from experimental data and some recent results on \( L_{0}^{\pi} \) and \( L_{0}^{\rho} \) can be found in ref. [16].

In Fig. 2 we show the lines in the \( \chi_{1}-\chi_{4} \) plane used in the following figures. The remaining figures show the relative correction to the meson mass. In Fig. 2 we plot \( M_{\rho_{\pi}}^{2}/\chi_{1} - 1 \) to \( O(p^{4}) \), and in Fig. 4 to \( O(p^{6}) \), in both cases with the LECs set to the fit 10 values of ref. [12]. Fig. 5 also shows the result at \( O(p^{6}) \), but with the LECs set to zero. Note that the effect of the unphysical logarithms is clearly visible along the line labelled \( A \) which
has a constant sea quark mass. That the $O(p^6)$ contributions to the pseudoscalar meson masses are sizable was expected. Similar results for the case of $\chi$PT were obtained in refs. [14, 15], but cancellations with the contributions from the $O(p^6)$ Lagrangian might change this.

In conclusion, we have calculated the off-diagonal pseudoscalar meson mass at NNLO order in PQ$\chi$PT, and have presented first results for the case of equal valence quark masses and equal sea quark masses. The $O(p^6)$ contributions are sizable, and the effects of the loop contributions are definitely nonnegligible at presently used quark masses in lattice QCD calculations.

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