A fundamental distinction between many-body quantum states are those with short- and long-range entanglement (SRE and LRE). The latter cannot be created by finite-depth circuits, underscoring the nonlocal nature of Schrödinger cat states, topological order, and quantum criticality. Remarkably, examples are known where LRE is obtained by performing single-site measurements on SRE, such as the toric code from measuring a sublattice of a 2D cluster state. However, a systematic understanding of when and how measurements of SRE give rise to LRE is still lacking. Here, we establish that LRE appears upon performing measurements on symmetry-protected topological (SPT) phases—of which the cluster state is one example. For instance, we show how to implement the Kramers-Wannier transformation by adding a cluster SPT to an input state followed by measurement. This transformation naturally relates states with SRE and LRE. An application is the realization of double-semion order when the input state is the $\mathbb{Z}_2$ Levin-Gu SPT. Similarly, the addition of fermionic SPTs and measurement leads to an implementation of the Jordan-Wigner transformation of a general state. More generally, we argue that a large class of SPT phases protected by $G \times H$ symmetry gives rise to anomalous LRE upon measuring $G$-charges, and we prove that this persists for generic points in the SPT phase under certain conditions. Our work introduces a new practical tool for using SPT phases as resources for creating LRE, and uncovers the classification result that all states related by sequentially gauging Abelian groups or by Jordan-Wigner transformation are in the same equivalence class, once we augment finite-depth circuits with single-site measurements. In particular, any topological or fracton order with a solvable finite gauge group can be obtained from a product state in this way.
whereas the other classes have long-range entanglement (LRE)\(^1\). Even restricting to gapped phases, the latter contains interesting cases such as intrinsic topological [8–13] and fracton order [14–20]. States with SRE can also be subdivided into distinct phases of matter if one imposes symmetry constraints on the aforementioned unitaries, giving rise to the notion of symmetry-protected topological (SPT) phases\(^2\) [3, 4, 21–33].

Recently, there has been growing interest in explicitly incorporating measurements into the study of many-body quantum states. For instance, a multitude of works have studied entanglement reduction from measurements, giving rise to surprising new structures [34–50]. However, there are also examples where measurements increase the entanglement. For example, it is known that performing single-site measurements on a subset of sites of a cluster state (with SRE) can produce a Greenberger-Horne-Zeilinger (GHZ) cat state [51], the toric code [52–54], and certain fracton codes via a layered construction [55, 56]. In fact, it has been remarked that all states realized by CSS stabilizer codes [57, 58] (i.e., stabilizers that are of the form \(\prod_{i \in S} Z_i\) or \(\prod_{i \in S} X_i\)) can be obtained by measuring an appropriate cluster state [59].

The existence of these examples begs the following question: What is the general framework for when, how, and why one can create LRE from SRE states and single-site measurements? In this work, we argue that the essential fact in the above examples is that the cluster state is an SPT. This deeper understanding confers at least four advantages. First, in contrast to earlier studies, we argue that LRE states are obtained on measuring not just the fixed-point wave function of the SPT but any state within the same phase. Second, the origin of LRE under measurement is tied to a specific anomaly involving the symmetries—related to the anomaly living at the boundary of the original SPT phase—thereby constraining the nature of the resulting LRE. Third, it allows for the preparation of states that are not realized by stabilizer codes, such as topological order described by twisted gauge theories or non-Abelian fracton orders [60–69]. Fourth, we achieve a new perspective on Kramers-Wannier (KW) [18, 70–78] and Jordan-Wigner (JW) [79–87] transformations. Indeed, we show how these nonlocal transformations can be efficiently implemented in a finite time by adding SPT entanglers to arbitrary initial states\(^3\) and subsequently performing single-site measurements. In a companion work [88], we explain how this general understanding can be utilized to prepare, e.g., \(Z_3\), \(S_3\), and \(D_4\) topological order in quantum devices such as Rydberg atom arrays.

This work is structured as follows. In Sec. II, we set the stage by reviewing some known examples, explaining how the 1D GHZ and 2D toric code states can be obtained by measuring particular cluster states. In Sec. III, we generalize these cases by reinterpreting the act of measuring cluster states as effectively implementing a KW transformation. To give illustrative examples, we explain how this allows one to transform the nontrivial \(Z_3\) SPT in 2D to the double-semion topological order, and to transform the 1D XY chain into two decoupled critical Ising models by using finite-depth circuits and single-site measurements. Moreover, we discuss how certain types of non-Abelian topological order can be obtained by sequential applications of this scheme. Sec. IV generalizes this to the fermionic case, where a similar procedure implements the JW transformation, illustrated by creating the Kitaev chain from a trivial spin chain. Sec. V broadens our scope further: First, we argue that this procedure is a robust property of the SPT phase (which we exemplify by obtaining cat states via measuring the spin-1 Heisenberg chain), and second we argue that anomalous symmetries and LRE are generically obtained by measuring a broad class of SPT states (which we discuss in detail for the \(Z_2^2\) SPT in 2D). We conclude with directions for future research in Sec. VI.

II. MOTIVATING EXAMPLES

We begin by reviewing how measuring cluster states in 1D and 2D can produce GHZ states [51] and the toric code [52], respectively. Consider a 1D chain with 2\(N\) qubits. The cluster state \(|\psi\rangle\) on this chain is the unique state that satisfies \(Z_{n-1}X_nZ_{n+1}|\psi\rangle = |\psi\rangle\) for all \(n\), where \(X, Y, Z\) denote the Pauli matrices. It can be prepared from the product state in the \(X\) basis by applying controlled-\(Z\) gates on all nearest neighboring qubits:

\[
|\psi\rangle = \prod_n CZ_{n,n+1} |+\rangle^\otimes 2N =: U_{\text{CZ}} |+\rangle^\otimes 2N. \tag{1}
\]

We call the above unitary \(U_{\text{CZ}}\) the cluster state entangler. Now suppose we measure \(X\) on all odd sites, with outcomes \(X_{2n+1} = (-1)^{s_{2n+1}}\). Since \(Z_{2n-2}X_{2n-1}Z_{2n}\) commutes with the measurement, the state after the measurement \(|\psi_{\text{out}}\rangle\) satisfies \(Z_{2n-2}Z_{2n}|\psi_{\text{out}}\rangle = (-1)^{s_{2n-1}}|\psi_{\text{out}}\rangle\). On the other hand, the even stabilizers do not commute with the measurement; only their product \(\prod_n Z_{2n-1}X_{2n}Z_{2n+1} = \prod_n X_{2n}\) commutes, implying \(|\psi_{\text{out}}\rangle\) is \(Z_2\)-symmetric. If all the \(s_m = 0\), then \(|\psi_{\text{out}}\rangle\) is the GHZ state on the even qubits:

\[
|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \cdots \uparrow\rangle + |\downarrow \downarrow \cdots \downarrow\rangle). \tag{2}
\]

Otherwise, it is the GHZ state up to single-site spin flips conditioned on the measurement outcomes: |GHZ\rangle = |\psi_{\text{out}}\rangle.
\[ \prod_{n=1}^{N} X_{2n-1}^n Z_{2n-2}^{n+1} |\psi_{\text{out}}\rangle. \]

Thus, regardless of the outcome, \(|\psi_{\text{out}}\rangle\) has long-range entanglement, as can, for example, be quantified by quantum Fisher information \([89, 90]\) (see also Sec. VA 5).

In 2D, we can consider a cluster state on the vertices and edges of the square lattice \([52]\). The stabilizers of the cluster state for each vertex and edge are \(X_v \prod_{e \ni v} Z_e\) and \(X_e \prod_{v \in e} Z_v\) respectively, where \(e \ni v\) and \(v \in e\) denote edges \(e\) that contain the vertex \(v\), and vertices \(v\) that are contained in \(e\), respectively. Measuring \(X\) on all the edges will give a GHZ state on the vertices (up to spin flips that depend on measurement outcomes). On the other hand, measuring \(X\) on all the vertices gives a state of the toric code: We have the vertex term of the toric code, \(\prod_{e \ni v} Z_e = \pm 1\) depending on the measurement outcome, and we have the plaquette operator \(\prod_{e \ni v} X_e = 1\) coming from a product of four edge stabilizers around a plaquette, which commutes with the measurement. Note that while the topological order of this state is independent of the sign of the aforementioned stabilizers, one can always bring this to a state with \(\prod_{e \ni v} Z_e = +1\) by applying string operators that pair up the vertices with \(\prod_{e \ni v} Z_e = -1\).

III. KRAMERS-WANNIER TRANSFORMATION FROM MEASURING CLUSTER STATE SPT PHASES

We have seen that long-range entangled states can be obtained by performing single-site measurements on the cluster state. To explore a deeper reason for this finding, we will show how the cluster state secretly encodes the KW transformation. For simplicity, we will first discuss the 1D case, where the KW transformation is defined as the map \(X_v \rightarrow Z_v X_{v+1}^n Z_{v+1}^n X_v\); although this map preserves the locality of \(Z_2\)-symmetric operators, it is a nonlocal mapping, relating SRE to LRE.

A first hint of the connection between the cluster state and the KW transformation is the fact that \(Z_{v+1} X_{v-1}^n X_{v+1}^n Z_{v+1}^n X_{v+1}^n\) act the same way on the cluster state. Moreover, \(X_v U_{\text{CW}} = U_{\text{CW}} X_v X_{v+1}^n X_{v+1}^n X_{v+1}^n\), where \(U_{\text{CW}}\) is the cluster entangler, Eq. (1). Let us divide the sites into the odd and even sublattices, denoted \(A\) and \(B\), respectively, and define the states \(|+\rangle_{A,B}\) on these subspaces. We find that the operator \(\sigma := \langle+|_A U_{\text{CW}} |+\rangle_B : \mathcal{H}_A \rightarrow \mathcal{H}_B\) gives the KW transformation. For example, we show that \(X_A\) is correctly mapped to \(Z_B Z_B\), i.e., \(\sigma X_A = Z_B Z_B\sigma\):

\[\langle+|_A U_{\text{CW}} |+\rangle_B X_A = \langle+|_A U_{\text{CW}} X_A |+\rangle_B = \langle+|_B Z_B U_{\text{CW}} Z_B |+\rangle_B = \langle+|_B Z_B Z_B U_{\text{CW}} |+\rangle_B = Z_B Z_B \langle+|_A U_{\text{CW}} |+\rangle_B,\]

and vice versa. This example is depicted graphically in Fig. 1. Note that this method works on any bipartite graph using a suitably generalized cluster state in any dimension, in which case, the \(Z_B\)'s that appear act on the \(B\) vertices adjacent to where \(X_A\) acts and vice versa.

Eq. (3) suggests a method to apply KW by measurement. We begin with a state in \(\mathcal{H}_A\) and then introduce the ancillas \(|+\rangle_B\). We then apply \(U_{\text{CW}}\) to the combined system and measure the \(X\) spins on \(A\). If the measurement outcomes are all \(+\) spins, then we have exactly implemented the KW duality. Otherwise, we have instead implemented the closely related operator

\[M = \langle+|_A \left( \prod_{a \in A} Z_{s_a}^a \right) U_{\text{CW}} |+\rangle_B = \sigma \prod_{a \in A} Z_{s_a}^a\]

where \(s_a \in \{0, 1\}\) are the measurement outcomes of site \(a\). By pushing through the excess operators from the \(A\) sites to the \(B\) sites using \(\sigma\), we can rewrite this formula as

\[M = \left( \prod_{b \in B} X_{s_b}^b \right) \langle+|_A U_{\text{CW}} |+\rangle_B = \left( \prod_{b \in B} X_{s_b}^b \right) \sigma,\]

where the \(s_b\) are functions of the \(s_a\) that depend on the graph. For example, in 1D, where \(A\) and \(B\) are the odd and even sublattices of the chain respectively, we have \(s_b = \sum_{1 \leq a < b} s_a\). Thus, we see that further applying \((\prod_{b \in B} X_{s_b}^b)\) restores the exact KW mapping \(\sigma\). See Fig. 2.

This finding explains why the measured 1D cluster state has long-range order—it produces the KW dual of the trivial state \(|+\rangle_A\), which is a GHZ state. Likewise in
2D we obtain the KW dual of the trivial state which is a toric code state\textsuperscript{4}.

We later argue that the long-range order holds for any state in the same SPT phase as the cluster state. Indeed, this fact can be seen by symmetry fractionalization for the two $\mathbb{Z}_2$ symmetries $\prod_{a \in A} X_a$ and $\prod_{b \in B} X_b$ (acting on the odd and even sublattices, respectively) protecting the SPT phase. If we act on any state $|\psi\rangle$ in the same SPT phase by the $\mathbb{Z}_2^A$ symmetry in a region $R$, it will reduce to some $\mathbb{Z}_2^B$ charged operators at the boundary of the region: $\prod_{a \in R} X_a |\psi\rangle = O_L O_R |\psi\rangle$, where $O$ is some operator with finite support situated at the left and right boundaries of $R$, which anticommutes with $\mathbb{Z}_2^B$.

Intuitively, this means that $|\psi\rangle$ has the KW property, exchanging order operators and disorder operators, at long distances. See Sec. \ref{sec:twisted_kw}.

In higher dimensions, the cluster state is an SPT for higher form or subsystem symmetries that depend on the lattice. For example, if $A$ and $B$ are sites at the vertices and edges of the square lattice, then we have symmetries $\prod_{a \in A} X_a$ and $\prod_{b \in C} X_b$, where we have a symmetry for each closed curve $\gamma$ drawn along the edges of the direct lattice. The KW so constructed is the duality between the Ising model and Ising gauge theory in 2+1D.

A summary of examples that arise from the KW transformation of various symmetries is given in Table \ref{table:summary}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Symmetry & Example & Description \\
\hline
$Z_2$ & Cluster state & Protected by $\mathbb{Z}_2$ symmetries $\prod_{a \in A} X_a$ \\
\hline
$Z_2^A$ & SPT phase & Protecting the SPT phase \\
\hline
$Z_2^B$ & SPT phase & Protecting the SPT phase \\
\hline
\end{tabular}
\caption{Summary of examples that arise from the KW transformation of various symmetries.}
\end{table}

\textbf{A. Twisted gauge theory from measuring cluster + SPT phases}

As a first application, we discuss what happens when we apply this procedure to other states on the $A$ sublattice, such as an SPT. As in Fig. 2, we add $|+\rangle_B$ ancillas, couple $A$ and $B$ with the cluster state entangler, and then perform measurements on the $A$ sublattice. The result of this procedure is equivalent to gauging the SPT phase\textsuperscript{5}.

To illustrate this procedure, we discuss how beginning with the $A$ sublattice in the pure $\mathbb{Z}_2$ or “Levin-Gu” SPT state $|\psi\rangle$ we obtain the double semion topological order \cite{thorlacius2015} after entangling and measuring. The Levin-Gu SPT is defined on the vertices of the triangular lattice (A) and is an eigenstate of the following (non-Pauli) stabilizers:

$$X_v \prod_{\langle vuu'\rangle} e^{\frac{\pi}{4} Z_u Z_{u'}} = X_v$$

where $\langle vuu'\rangle$ are the six triangles around $v$, and the wavy lines denote $e^{\frac{\pi}{4} Z_u Z_{u'}}$ between vertices $u$ and $u'$. Note that this stabilizer is not simply a product of Pauli operators. Let us also stress that since this is an SPT phase, it is possible to prepare this state by a finite-depth circuit\textsuperscript{6}, following our procedure, we add the $B$ sublattice consisting of edges of the triangular lattice, supporting a product with the trivial stabilizer

$$X_e = -i X_e$$

Next, we couple the two sublattices with the cluster state

\textsuperscript{4} We note that as a by-product, we obtain explicit tensor network representations of these states. This offers an alternative derivation of the 3D toric and fracton code projected entangled pair states (PEPS) \cite{ge2020, ge2021} obtained in Refs. \cite{ge2022, ge2023}.

\textsuperscript{5} Alternatively, by viewing the SPT and the cluster state as a single state, performing the measurement on this combined SPT can be thought of as a different way of performing the KW duality on the product state. This choice of adding an extra SPT before gauging is also known discrete torsion\cite{dijkgraaf1990}, or defectification classes\cite{gaiotto2008} in the literature.

\textsuperscript{6} The unitary that creates the Levin-Gu SPT is given by $e^{\frac{\pi}{4} (\sum a_u Z_u Z_v - 2 \sum v Z_v)}$ where $\Delta_{uvw}$ denotes all triangles.
Before we perform the measurements on all $A$ sites (the vertices of the triangular lattice), we note that the vertex stabilizer does not commute with the measurement. Thus, it would not directly give us a useful condition on the postmeasurement state. However, using the fact that $Z_{u}Z_{u'}|\psi\rangle = X_{(uu')}|\psi\rangle$, where $(uu')$ is the edge with $u$ and $u'$ as end points, the following is an equally valid set of stabilizers of $|\psi\rangle$:

$$X_{v} \prod_{(uu')} e^{\frac{\pi}{4}Z_{u}Z_{u'}} \prod_{e \supset v} Z_{e} = \begin{array}{ccc}
Z & Z & Z \\
Z & X & Z \\
Z & Z & Z
\end{array}$$

(8)

$$X_{v} \prod_{v \in e} Z_{v} = \begin{array}{ccc}
Z & X & Z \\
Z & Z & Z
\end{array}.$$  

(9)

Before we perform the measurements on all $A$ sites (the vertices of the triangular lattice), we note that the vertex stabilizer does not commute with the measurement. However, the stabilizers in Eq. (10) do not commute for adjacent vertices. However, this problem is cured by restricting to the subspace:

$$\prod_{e \in \Delta} X_{e} = X_{X} X = 1.$$  

(12)

which is a projector into this subspace on each triangle. Finally, $|\psi\rangle$ is identified as the unique state that has eigenvalue +1 under the following operators:

$$X_{v} \prod_{(uu')} (R_{(uu')}O_{vuu'}) \prod_{e \supset v} Z_{e} = \begin{array}{ccc}
R & Z & Z \\
R & Z & Z \\
R & Z & R
\end{array},$$

(14)

$$X_{v} \prod_{v \in e} Z_{v} = \begin{array}{ccc}
Z & X & Z \\
Z & Z & Z
\end{array}.$$  

(15)

Performing the measurement with outcomes $X_{e} = (-1)^{s_{e}}$, the postmeasurement state is the unique state that has eigenvalue +1 under the operators:

$$(-1)^{s_{e}} \prod_{(uu')} (R_{(uu')}O_{vuu'}) \prod_{e \supset v} Z_{e} = (-1)^{s_{e}} \begin{array}{ccc}
R & Z & Z \\
R & Z & Z \\
R & Z & R
\end{array},$$  

(16)

$$\prod_{e \in \Delta} X_{e} = \begin{array}{c}
X \ X \ X
\end{array},$$  

(17)

which is the ground state of the double semion model [31] up to single site $X$-rotations on edges that pair up the vertices where $s_{e} = 1$ to remove the signs, and swapping $X_{e}$ with $Z_{e}$ to match the choice in Ref. [31].

Our implementation of gauging via combining measurements with a cluster state entangler (including $Z_{n}$ generalizations) implies that we can produce all twisted quantum double models of a finite Abelian gauge group via stacking general SPTs prior to measuring—which can be prepared by finite-depth circuits [27]. Note that these models already contain certain non-Abelian phases, e.g., $D_{4}$ topological order arises upon gauging the $Z_{4}$ symmetry of an SPT phase with a type-III cocycle [97, 98]. (For obtaining non-Abelian topological order associated with
any solvable group, see Sec. III C.) Similarly, our procedure allows for the creation of twisted fracton phases by gauging 3D subsystem SPT phases [65, 84, 85, 99, 100]. Thus a much wider class of states can be obtained from local unitary circuits and local operations and classical communications (LOCC) [54] than previously established.

B. Physically applying the Kramers-Wannier transformation to a gapless state

Here, we discuss an example where the input state \( |\psi\rangle \) (in Fig. 2) itself has long-range entanglement. In particular, we focus on a well-known example of how the \( XY \) chain—an example of a gapless state—can be transformed into two decoupled critical Ising chains by gauging particle-hole symmetry\(^7\). Here, we achieve this gauging by using a finite-depth circuit and single-site measurements.

We place the \( XY \) chain on the odd sites (\( A \)) and initialize with \(|+\rangle\) states on the even sites (\( B \)). The aforementioned state can be considered the ground state of the following Hamiltonian

\[
H = \sum_n X_{2n-1}X_{2n+1} + Y_{2n-1}Y_{2n+1} - X_{2n} \tag{18}
\]

Next, we gauge the \( Z_2 \) subgroup \( \prod_n X_{2n-1} \) of the full \( U(1) \) symmetry of the \( XY \) chain. To do so, we couple the even and odd sites with the cluster state entangler \( U = \prod_n CZ_{n,n+1} \), resulting in

\[
UHU^\dagger = \sum_n Z_{2n-2}(X_{2n-1}X_{2n+1} + Y_{2n-1}Y_{2n+1})Z_{2n+2} - Z_{2n-1}X_{2n}Z_{2n+1} \tag{19}
\]

Note that since \( Z_{2n-1}X_{2n}Z_{2n+1} \) is an integral of motion, the following Hamiltonian also has the same wave function as its ground state:

\[
\sum_n Z_{2n-2}(X_{2n-1}X_{2n+1} - X_{2n})Z_{2n+2} - Z_{2n-1}X_{2n}Z_{2n+1} \tag{20}
\]

Now, we perform a measurement on the odd sites with measurement outcomes \( X = (-1)^g \); the state after the measurement is the ground state of the Hamiltonian

\[
\sum_n (-1)^{x_{2n-1} + x_{2n+1}} Z_{2n-2}Z_{2n+2} - Z_{2n-2}X_{2n}Z_{2n+2} \tag{21}
\]

with the integral of motion \( \prod_n X_{2n} \) serving as a global \( Z_2 \) symmetry. After appropriate spin flips to remove the signs and the circuit \( \prod_n CZ_{2n,2n+2} \), the Hamiltonian reads

\[
\sum_n Z_{2n-2}Z_{2n+2} - X_{2n} \tag{22}
\]

which describes two decoupled critical Ising chains. We thus confirm that we have physically implemented the KW transform on a gapless state.

Let us remark that this procedure does not rely on free-fermion solvability of the \( XY \) chain and the Ising model. For example, the procedure still works in the presence of the \( XXZ \) deformation, which respects the \( Z_2 \) symmetry (albeit opening up a gap).

C. Non-Abelian topological order from sequentially gauging Abelian groups

Beyond cyclic groups \( \mathbb{Z}_n \), cluster states and the corresponding KW dualities have been generalized to arbitrary finite groups [102–104], giving the potential to gauge non-Abelian groups by unitaries and measurement. However, unlike the Abelian case, which produces Abelian anyons depending on the measurement outcome, gauging non-Abelian groups can produce non-Abelian anyons that can only be paired up using linear depth string operators\(^8\). The intuition for this is that the string operators for moving such anyons consist of noncommuting operators which hence cannot be applied all at once\(^9\).

Our implementation of the KW duality avoids this issue by a sequence of circuits and measurements, which can be interpreted as sequentially gauging Abelian groups. In such a method, the measurement outcomes in all intermediate states correspond to Abelian anyons, which can all be paired up in finite depth. In this way, all gauge theories whose gauge group is solvable (i.e., obtained by extending finite Abelian groups) can be constructed efficiently in this manner. For example, the \( S_3 \) quantum double can be obtained by gauging a \( Z_4 \) symmetry (i.e., measuring a \( Z_4 \) cluster state), which prepares a \( Z_4 \) toric code, followed by gauging the charge conjugation symmetry that permutes anyons \( e \leftrightarrow e^2 \) and \( m \leftrightarrow m^2 \).

We note that since \( S_3 \) is not nilpotent, it can be used for universal quantum computation [105]. As a second example, the \( D_4 \) topological order can be obtained by first

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\(^7\) Field theoretically, this maps the compact boson to two copies of the Ising CFT [101].

\(^8\) We thank David T. Stephen for pointing out this subtlety.

\(^9\) We note one potential loophole. If the group is nilpotent, two non-Abelian anyons can be annihilated by first nucleating a whole density of pairs of anyons of the same type along a path connecting the two anyons and subsequently fusing them all at once. This potentially leaves a density of residual anyons all along the path, but the nilpotent sequence ensures that by repeating this process, we obtain simpler and simpler anyons—eventually leading to Abelian anyons which can be efficiently removed. Unfortunately, the known ways of using non-Abelian states for universal quantum computation rely on the group being not nilpotent [105].
Preparing the 2D color code and gauging the Hadamard symmetry. In our companion paper we provide explicit finite-depth qubit-based circuits for these two examples [88].

We note that sequentially gauging Abelian groups can also give rise to states beyond quantum doubles. For instance, the doubled Ising anyon theory can be obtained from the product state. Described how, e.g., the usual toric code can be obtained by measuring all the spins with outcomes $\pm 1$. In other words, a fermion is hopped if the spin at site 2 is up. We also remark that because all gates mutually commute, it can be implemented as a finite-depth circuit. The resulting SPT (which we will call the Jordan-Wigner state) is the +1 eigenstate of the stabilizers

$$U \equiv \prod_{n=1}^{N} CS_{2n-1, 2n}$$

where the operator

$$CS_{2n-1, 2n} = |\uparrow\rangle \langle\uparrow|_{2n-1} + |\downarrow\rangle \langle\downarrow|_{2n-1} S_{2n}$$

is a hopping operator controlled by the qubit at 2n − 1. In other words, a fermion is hopped if the spin at site 2n − 1 is down. We also remark that because all gates mutually commute, it can be implemented as a finite-depth circuit. The resulting SPT (which we will call the Jordan-Wigner state) is the +1 eigenstate of the stabilizers

$$UX_{2n-1}U^\dagger = i\gamma'_{2n-1}X_{2n-1}Z_{2n},$$

$$UP_{2n}U^\dagger = Z_{2n-1}P_{2n}Z_{2n+1}.$$ 

Now, we measure all the spins with outcomes $X_{2n-1} = (-1)^{s_{2n-1}}$. The stabilizers of the measured state are $(-1)^{s_{2n-1} - 2 \gamma'_{2n-2} \gamma_{2n}}$ and $\prod_{n} Z_{2n-1}P_{2n}Z_{2n+1} = \prod_{n} P_{2n}$, which after applying $\prod_{n=1}^{N} P_{2n}$, gives the ground state of the Kitaev chain. We note that, alternatively, starting with the SPT, measuring the parity of all the fermions gives the GHZ state.

IV. JORDAN-WIGNER TRANSFORMATION FROM MEASURING FERMIONIC SPT PHASES

Analogous to the KW transformation, the Jordan-Wigner (JW) map is a nonlocal transformation which maps between fermionic and bosonic degrees of freedom [79, 80]. Similar to the KW transformation, here we can prepare and entangle bosonic and fermionic degrees of freedom as shown in Fig. 3. We can then perform either bosonization of an arbitrary input fermionic state by measuring the parity of all fermions, or fermionization of an arbitrary input bosonic state by measuring $X$ on all the spins after the entangling step.

A. 1+1D bosonization

Let us demonstrate this case explicitly by preparing the Kitaev Majorana chain, which cannot be done in finite time with only unitary evolution [6]. We start with $N$ qubits on odd sites initialized in the $|+\rangle$ state and $N$ fermions on even sites initialized in the empty state $P = -i\gamma \gamma'$, where $\gamma = c + c'$ and $\gamma' = -i(c - c')$ are Majorana operators. Furthermore, we define the hopping operator $S_{2n} = i\gamma'_{2n-2} \gamma_{2n}$, which hops a fermion from site 2n − 2 to 2n. We create a $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ SPT [108–111] with the following circuit:

$$U = \prod_{n=1}^{N} CS_{2n-1, 2n}$$

FIG. 3. Jordan-Wigner transformation from finite-depth circuit and measurements. We show the process of entangling fermionic (red) and bosonic (blue) degrees of freedom and its relation to the JW transformation. Here $|0\rangle$ corresponds to contracting with the empty state of fermions. We use Jordan-Wigner* to emphasize that this transformation differs from the usual JW by an additional KW transformation. Similar to Fig. 2, this can be utilized to implement the JW transformation via measurements (see main text).
Upon measuring the fermion parity of all fermions, the resulting state is described by the stabilizers

\[ \begin{array}{c}
\text{Z} \quad \text{Z} \\
\text{Z} \quad \text{Z} \\
\text{Z} \quad \gamma \\
\gamma' \quad \text{Z} \\
\end{array} \]

which, up to a sign given by measurement outcomes, describe the 2D toric code.

To discuss the circuit required to prepare this SPT, we first define the fermion hopping operator for each edge as

\[ S_e = i\gamma \quad \gamma' \quad i\gamma \]

Then, we may define the controlled operator

\[ CS_e = |\uparrow\rangle \langle \uparrow_e | + |\downarrow\rangle \langle \downarrow_e | S_e. \]

Here, the only novel subtlety—not present in the bosonic case or the 1D JW transformation—is that not all of the \(CS\) gates mutually commute and therefore must be applied sequentially. Nevertheless, it turns out that their ordering is irrelevant: Each choice of ordering gives a valid JW transformation [84, 85], and moreover these choices only differ by phase gates. Thus, a given choice determines the spatial anisotropy of the stabilizers.

To obtain the stabilizers of the SPT in Eq. 27, the unitary that prepares it can be written as

\[ U = \prod_e CZ_{e_N(v)} e_{E(v)} \prod_e CS_e \prod_e CS_{e_y}. \]

where \(e_{N(v)}\) and \(e_{E(v)}\) refer to the edges directly north and east of the vertex \(v\), respectively. In other words, we have chosen to apply the control gates on all vertical edges (which mutually commute) followed by those on the horizontal edges; lastly, we apply appropriate \(CZ\) gates to obtain the desired form of the stabilizers.

The JW state has the property that if we form the open string operator associated to the 1-form symmetry, the associated open string must always end on a fermion, and in that sense there is only one SPT phase, but it is not quite trivial because the 1-form symmetry generator we’ve chosen is not completely “on-site”.

Indeed, in Refs.[112, 113] it was stressed that the 1-form symmetry in 2+1D bosonization has an anomaly \(S^2 B\) (unlike in 1+1D bosonization where we obtain an anomaly-free \(Z_2\) symmetry upon bosonizing) and the kernel of the bosonization transformation gives a trivialization of this anomaly in the presence of fermions. In simple terms, the \(S^2 B\) anomaly says that the 1-form symmetry generator needs to end on a fermion (which is essentially the Gu-Wen equation), so while it looks like a nontrivial SPT, there is really only one option, in harmony with the classification.

Similarly to the KW transformation, we can now apply the JW transformation to arbitrary states by measurements. For example, we can consider preparing the fermions in a 2+1D topological \(p + ip\) superconducting state with chiral Majorana edge modes. After coupling to the JW state and measuring fermion parity, the remaining spins will describe a chiral Ising topological order. Similarly, coupling \(\nu\) stacks of \(p + ip\) superconductors to the SPT and performing the measurement can realize the topological orders in Kitaev’s 16-fold way [13].

The generalization to higher dimensions [82, 83] and to other types of fermionic gauge theories (including fracton models with fermionic statistics [84, 85]) is straightforward by taking a sequential product of \(CS\) operators that mutually commute within each layer.

V. GENERALIZATIONS

Thus far, we have focused on two illustrative cases, where measuring sublattices of the cluster and JW states leads to LRE. In this last section, we generalize this approach in two directions. First, we make the case that the ability to produce LRE from measurements is indeed a property of the whole SPT phase, being robust to tuning away from a fixed-point limit. Second, we show that LRE is naturally obtained by measuring a broad class of SPT phases, of which the cluster and JW states are but two examples.
A. LRE generation as stable property of SPT phase

1. Intuition away from fixed-point limit

Let us first consider the 1D cluster SPT phase and ask whether one obtains a cat state upon measuring one of the sublattices starting with an arbitrary state in this phase. We present an intuitive argument, which holds away from the fixed-point limit. A key property of the cluster SPT phase in 1D is that it generically has long-range order for the following string operator [116]:

\[ \lim_{|n-m|\to\infty} \langle Z_{2m} S_{2m,2n} Z_{2n} \rangle = C \neq 0, \]

where \( S_{2m,2n} := X_{2m+1} X_{2m+3} \cdots X_{2n-1} \) is a string operator consisting of the \( Z_2 \) symmetry of the odd sites. The SPT invariant [117] is encoded in the fact that the string operator for one of the \( Z_2 \) symmetries only has long-range order if one includes an end-point operator that is charged under the other \( Z_2 \) symmetry (in this case \( Z_{2m} \) which is odd under \( \prod_m X_{2m-1} \)). Indeed, in the nontrivial SPT phase, one finds that the undressed string does not have long-range order:

\[ \lim_{|n-m|\to\infty} \langle S_{2m,2n} \rangle = 0. \]

We would like to understand what happens if we measure all odd sites in the \( X \)-basis, which is a rather challenging many-body question, and Secs. VA2-V A 5 will be devoted to addressing this issue. However, as a first encounter, and to build some intuition, let us imagine that instead of measuring all odd sites, we measure a sin-}

We thus find that measuring the string leads to long-range cat-state-like entanglement between the two end-points! This result is consistent with the notion of SPT entanglement explored in Ref. [118], where the author showed that measuring a large connected block of sites leads to a Bell pair between the two end-points.

The above argument can be extended to higher dimensions. For instance, let us revisit the 2D case mentioned in Sec. II: the Lieb lattice with spins on the vertices (A sublattice) and bonds (B sublattice) of the square lattice. The cluster state on this lattice is an SPT phase protected by a global \( Z_2 \) symmetry \( U^A = \prod_{a \in A} X_a \), as well as a “1-form symmetry,” \( U^B = \prod_{b \in \gamma \subset B} X_b \), meaning a symmetry defined for each closed curve \( \gamma \) on the bonds of the square lattice [98, 119, 120].

In the SPT phase, we have long-range order for the membrane operator \( S_{\partial R} \prod_{a \in A \cap R} X_a \), where \( R \) is some region and \( S_{\partial R} \) is a string operator on the boundary which “braids” with \( U^B \gamma \), meaning \( U^B S_{\gamma}(U^B)^\dagger = S_{\gamma}(1)^{n_{\gamma \gamma}^\gamma} \), where the exponent is the number of intersection points between the curves \( \gamma \) and \( \gamma' \). For the fixed point cluster state, \( S_{\gamma} = \prod_{b \in \gamma} Z_b \).

Upon measuring the membrane, we are left with long-range order for \( S_{\gamma} \) (see Fig. 5). This quantity serves as an order parameter for spontaneously breaking the 1-form symmetry, thereby implying topological order. In fact, this point of view naturally generalizes to other SPT phases, as we will discuss in Sec. VB.

However, while the above is intuitive and encouraging, it does not actually prove that the LRE persists upon measuring all (or a finite density of) sites. In particular, in the 1D case, we have thus far only measured \( S_{2m,2n} \) and not yet all odd sites. This calculation does not automatically guarantee that the long-range order in Eq. (36) persists after performing the other measurements since measurements can reduce entanglement. We now argue that, generically, it does indeed persist.

2. Conjecture and theorem: LRE from SPT

Having gained the above intuition, let us now try to formalize how and when long-range entanglement is produced by measuring SPT phases. To this end, we state a general conjecture, for which we give plausibility arguments. In addition, we provide a rigorous theorem for a slightly more constrained setting.

We consider a (short-range entangled) wave function \(|\psi\rangle\) in a nontrivial SPT phase protected by an Abelian symmetry group \( G \times H \). Moreover, we presume that the SPT phase is mixed, which means that explicitly breaking either \( G \) or \( H \) would trivialize the SPT phase. Note

\[ |\langle \psi_i | Z_{2m} Z_{2n} | \psi \rangle| = |C| \neq 0. \]
that the notion of an on-site symmetry automatically implies the notion of a unit cell, whereby a global symmetry $U \in G \times H$ can be decomposed as a tensor product over the unit cells: $U = \prod_n U_n$. The physical act of measuring the $G$-charge (for a given unit cell $n$) means that, mathematically, we apply a projector

$$P_G(q)_n = \frac{1}{|G|} \sum_{g \in G} \chi_g(g) \ (U_g)_n,$$

where $q$ is a charge labeling the (random) measurement outcome, and $\chi_g$ is the corresponding character. For a given set of measurement outcomes $\{q_n\}_n$ (one for each unit cell), we thus obtain the postmeasurement state

$$|\psi\rangle_{\{q_n\}} \propto \prod_n P_G(q)_n |\psi\rangle.$$

The probability of obtaining a given measurement outcome (and thus the corresponding postmeasurement state) is, of course, given by Born’s rule. For each given outcome, one can ask whether the postmeasurement state is long-range entangled. We generally expect that this outcome, and thus the corresponding postmeasurement state being long-range entangled, is of course, given by Born’s rule. For each given outcome, one can ask whether the postmeasurement state is long-range entangled. We generally expect that this outcome, and thus the corresponding postmeasurement state being long-range entangled, is of course, given by Born’s rule.

Conjecture. If the premeasurement state $|\psi\rangle$ has a conventional Abelian $G \times H$ SPT string order parameter\(^{11}\) for a mixed Abelian $G \times H$ SPT phase, then the probability of the postmeasurement state being long-range entangled is unity.

We will give plausibility arguments for this conjecture in the next subsection. The above claim of unit probability for a ‘measure zero’ case where the postmeasurement state can be short-range entangled. Indeed, we will see examples of this in our numerical exploration in Sec. V A 5. However, if we slightly strengthen our assumptions, we can prove one always obtains long-range entanglement:

Theorem. Let $|\psi\rangle$ be in a nontrivial mixed SPT phase for Abelian symmetry group $G \times H$. If it admits a finite-bond dimension matrix product state (MPS) description, then there exists a choice of unit cell such that measuring the $G$-charge for each unit cell produces a state with long-range entanglement for any measurement outcome. More precisely, the postmeasurement state is a cat state for the (partial) spontaneous symmetry breaking of $H$.

To phrase and prove this result, we use the notion of matrix product states (MPS). In fact, this same framework will provide an intuitive justification for our more general conjecture. We thus turn to an MPS-based description of our set-up.

3. Proof using matrix product states

For a review of MPS, we point the reader to Refs. 92 or 121. The key idea of MPS is that a wave function is written in terms of finite-dimensional tensors:

$$|\psi\rangle = \sum_{i_1,i_2,\cdots,i_N} \text{tr} \left( \prod_{n=1}^N A_{i_n} \right) |i_1,i_2,\cdots,i_N\rangle$$

where $N$ labels the number of unit cells, $i = 1,\cdots,d$ labels the states in each unit cell, and $A_i$ is a $\chi \times \chi$ matrix. (For convenience, we work with translation-invariant states, where the tensor is identical for all sites.) Here $\chi \in \mathbb{N}$ is called the bond dimension, with $\chi = 1$ corresponding to a product-state wave function. It is known that up to exponentially small errors in local quantities, ground states of gapped Hamiltonians are well approximated by such an MPS [122, 123]. In what follows, we will use the graphical notation. For instance, Eq. (39) becomes

$$|\psi\rangle = \cdots A A A A \cdots$$

where we ignore boundary conditions, or equivalently, we work in the thermodynamic limit.

A key property that makes MPS such a useful framework, is that global symmetries, such as $U = \prod_n U_n$, imply nice local properties on the MPS tensor. In particular, one can ‘push’ physical symmetries through to the ‘virtual’ level\(^ {12}\) [3, 22, 25, 92, 124]: There exists an operator $V_g$ such that

$$U_g = e^{i\theta} V_g \ A \ A \ V_g^\dagger$$

---

\(^{11}\) In other words, the SPT wave function has long-range order in $\langle O_{n,1} U_{n+1} U_{n+2} \cdots U_{n-1} O_n \rangle \neq 0$ for a certain $U \in G$ and for a particular choice of end-point operator $\mathcal{O}$ that is supported on a single unit cell, or at the very least, that commutes with $G$ in each individual unit cell. The nontrivial (mixed) SPT class implies that $\mathcal{O}$ will carry nontrivial charge under $H$. Ref. 117 proved there always exists an $\mathcal{O}$ that gives long-range order, although it does not guarantee the additional local properties.

\(^{12}\) The internal indices of the $A$ tensor that are contracted with one another in the wave function are commonly called virtual legs, to distinguish them from the physical legs labeling the spins.
In other words, we see that the physical operator $U_g$ is equivalent to acting with $V_g$ and $V_g^\dagger$ at the virtual level. As a sanity check, we indeed see that if we apply $U_g$ on each site, then each $V_g$ is canceled by a $V_g^\dagger$, thereby confirming $\prod_n (U_g)_n$ is a global symmetry of $|\psi\rangle$.

An interesting property of these virtual symmetry actions $V_g$ is that they only need to form a projective representation of the symmetry group. Thus, for any $g, g' \in G \times H$, we have $V_g V_{g'} = \omega(g, g') V_{gg'}$ with a potentially nontrivial phase factor $\omega(g, g') \in U(1)$. A nontrivial SPT class is then equivalent to the statement that $[\omega] \in H^2(G \times H, U(1))$ is a nontrivial cocycle; the simplest example is when $G \times H = \mathbb{Z}_2 \times \mathbb{Z}_2$, where the nontrivial SPT phase corresponds to the projective representation where the two generators anticommute. More generally, a mixed SPT class implies that $\omega(g, h) \neq \omega(h, g)$ for a certain choice of $g \in G$ and $h \in H$, which we will use to derive long-range entanglement in the postmeasurement state.

As discussed, the act of measurement corresponds to applying a projector (37). The MPS tensor for the postmeasurement state (38) is simply:

$$
\begin{array}{c}
\begin{array}{c}
\hline
B \\
\hline
\end{array}
\end{array} :=
\begin{array}{c}
\begin{array}{c}
\hline
A \\
\hline
\end{array}
\end{array}
\begin{array}{c}
\hline
P_G
\end{array}
$$

Since for any $g \in G$ we have $U_gP_G = \chi_g(g)P_G$ (i.e., the symmetry operator acts like a number) we thus have the following local tensor properties:

$$
\begin{array}{c}
\begin{array}{c}
\hline
B \\
\hline
\end{array}
\end{array} = e^{i\theta_g} \chi_g(g)
$$

(43)

$$
\begin{array}{c}
\begin{array}{c}
\hline
B \\
\hline
\end{array}
\end{array} = e^{i\theta_h}
$$

(44)

for $g \in G$ and $h \in H$. Eq. (44) tells us that $H$ still acts like a physical symmetry on the postmeasurement state; however, Eq. (43) tells us that $G$ now only acts on the virtual degrees of freedom, which we can interpret as a sort of higher symmetry. More concretely, as we will now argue, $V_g$ acts as an order parameter for the spontaneous breaking of $H$ symmetry, such that the postmeasurement state is a long-range entangled cat state for symmetry breaking.

The key identity we will need is the ability to push $V_g$ from the virtual level to the physical level. In particular, the question is whether there exists an operator $O_g$ such that

$$
\begin{array}{c}
\begin{array}{c}
\hline
B \\
\hline
\end{array}
\end{array} \cong \begin{array}{c}
\begin{array}{c}
\hline
V_g \\
\hline
\end{array}
\end{array} \begin{array}{c}
\hline
O_g \\
\hline
\end{array}
\begin{array}{c}
\hline
B \\
\hline
\end{array}
$$

(45)

Let us temporarily earmark the question of whether $O_g$ exists and first explain how its existence is sufficient to prove that the postmeasurement state is long-range entangled.

From the projective group relations $V_g V_{g'} = \omega(g, g') V_{gg'}$, one can straightforwardly prove that if $O_g$ exists, it must carry charge under $H$. In particular, in Appendix D we prove that Eq. (45) implies

$$
U_h O_g U_h^\dagger = \omega(g, h) \omega(h, g) O_g.
$$

(46)

Since we are considering a mixed SPT phase, we know that this phase factor is nontrivial for certain $g \in G$ and $h \in H$; let us henceforth fix those elements, such that $\alpha_{g, h} \neq 1$.

One consequence of Eq. (45) is that in the postmeasurement state, the expectation value of $O_g$ must vanish. Indeed, taking the expectation value of both sides of Eq. (46) and using that $U_h$ is a symmetry, we obtain

$$
\langle O_g \rangle_{\text{postmeas}} = \alpha_{g, h} \langle O_g \rangle_{\text{postmeas}}.
$$

(47)

Since $\alpha_{g, h} \neq 1$, this implies that $\langle O_g \rangle_{\text{postmeas}} = 0$. However, the two-point correlation is nonzero. Indeed, combining Eq. (45) with Eq. (43) directly implies that

$$
\left| \langle (O_g)_m (O_g)_n \rangle_{\text{postmeas}} \right| = 1.
$$

(48)

We thus have long-range mutual information and thus long-range entanglement. In more physical terms, we see that the postmeasurement state can be interpreted as a cat state for the (partial) spontaneous symmetry breaking of $H$.

We thus have thus proven that the existence of $O_g$, as defined in Eq. (45), is sufficient to prove long-range entanglement. The final issue is when we expect this to hold. One scenario where we can show that $O_g$ exists is when the conditions of the theorem in Sec. V A 2 are met. Indeed, it is known that short-range entangled MPS satisfy a certain injectivity condition [92] which means that after potentially blocking sites a finite number of times, the MPS tensor defines an injective map where we consider the virtual legs to be its input and the physical leg its output. Equivalently, there exists a tensor$^{13}$ $C$ that

---

$^{13}$ If the wave function has zero correlation length, then $C$ is simply the complex conjugate of $A$. However, $C$ exists even for nonzero correlation length.
functions as an inverse for $A$:

$$A C = O$$  \hspace{1cm} (49)$$

where we will henceforth presume one has blocked the unit cell to achieve the injectivity condition. Using this, we can define the physical operator $O_g$ as follows:

$$O_g \equiv V_g C A$$  \hspace{1cm} (50)$$

Using Eq. (49), one sees that this operator satisfies Eq. (45) for the $A$ tensor. Moreover, one can prove that $O_g$ commutes with the $G$ symmetry, i.e., $U_g O_g U_g^\dagger = O_g$ for any $g \in G$ (see Appendix D). Hence, $O_g$ commutes with the projection $P_G$, such that we obtain Eq. (45) also for the $B$ tensor. This concludes the proof of the theorem in Sec. V A 2.

Thus, if we are willing to block unit cells a finite number of times, we can prove that LRE is obtained for any measurement outcome. In the absence of such blocking, we believe one can only make a probabilistic statement. To see this case, let us first remark that to make probabilistic arguments, one only needs a weaker version of Eq. (45), namely, that there exists an $O_g$ such that one has finite overlap with the virtual $V_g$ action, i.e.,

$$B = \lambda V_g B + \cdots$$  \hspace{1cm} (51)$$

for some $\lambda \neq 0$. Indeed, one can again show that this implies $O_g$ carries nontrivial charge under $H$. Moreover, the same argument as above still implies that one expects $O_g$ to have a long-range two-point function, since it picks up on the long-range order of $V_g$ (see Eq. (43)). The only way this case can fail is if the multiple terms on the right-hand side of Eq. (51) conspire to exactly cancel out the long-range contributions, which this certainly can happen (we will give an example in the next subsection); however this requires a delicate balancing of terms and is thus likely a measure zero case over the ensemble of all possible measurement outcomes. Lastly, we note that Eq. (51) can be expected to hold for SPT phases which admit a conventional SPT order parameter, as defined in footnote 11. Indeed, the very reason the string order parameters have nontrivial end-point operators is because they are able to cancel out the virtual $V_g$ action of the symmetry string or disorder operator $[117]$. Commonly used string order operators have an end-point $O_g$ supported on a single unit cell and commute with the corresponding symmetry generator $U_g$, such that if Eq. (51) applies to the $A$ tensor it also automatically carries over to the postmeasurement $B$ tensor. In conclusion, for these reasons, we conjecture that only a measure zero of measurement outcomes can fail to give long-range entanglement. It would be interesting to sharpen this intuition into a rigorous proof of our conjecture.

4. Analytics: Cat state from the deformed cluster state and AKLT state

Let us illustrate our general theorem with two MPS-based examples. Both examples will be SPT phases with nonzero correlation length, i.e., away from the simple fixed-point cases studied in the earlier sections of this work.

First, we consider a deformation of the cluster state:

$$|\psi(\beta)\rangle \propto e^{\beta \sum_n X_n} |\text{cluster}\rangle.$$  \hspace{1cm} (52)$$

Here $|\psi(0)\rangle$ is the cluster state of Eq. (1). For any $\beta$, this state admits a $\chi = 2$ MPS representation $[125]$ and one can show that for any finite $\beta$, this state is in the nontrivial SPT phase protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Its correlation length $\xi$ increases monotonically with $\beta$ and diverges as $\beta \to \infty$. The MPS tensor turns out to be injective without blocking, meaning that our theorem implies that measuring, say, $X_{2n+1}$ on odd sites produces a long-range entangled state on the remaining qubits— for any possible measurement outcome.

As a second example, we consider the paradigmatic spin-1 AKLT state $[126]$, which is known to be described by a $\chi = 2$ MPS and is an SPT phase protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of $\pi$-rotations. As generators, we can choose $R^x = \prod_n e^{i\pi S_n^x}$ and $R^z = \prod_n e^{i\pi S_n^z}$. If we block the spin-1s into two-site unit cells, then the MPS satisfies the aforementioned injectivity property. Hence, our MPS-based arguments prove that if one measures, say, $R_{2n-1}^z R_{2n}^z \in \{-1, 1\}$ charge on each two-site unit cell, then the postmeasurement state will always have long-range entanglement.

What if we did not block in the last example? If we measure $R_n^z \in \{-1, 1\}$ in each single-site unit cell, then there is a measure-zero chance that we obtain $R_n^z = 1$ for all sites. In this case, the postmeasurement state is simply the product state $|0\rangle^N$, where $|0\rangle$ is the unique $+1$ eigenstate of $R^z = e^{i\pi S_n^z}$. However, as long as a finite density of sites projects onto the $-1$ eigenstate of $R^z$, the

\footnote{We emphasize the finiteness since if one is willing to block an unbounded number of times, we can effectively appeal to an RG-based argument whereby one flows to the fixed-point state with zero correlation length, which would be less interesting.}
postmeasurement state is a long-range entangled of GHZ type, capturing the spontaneous symmetry-breaking of \( R^z \). This example is thus consistent with our conjecture and it illustrates the importance of making probabilistic statements in the cases where one does not block unit cells.

While both examples are illustrative, by definition they are analytically tractable. One might wonder about SPT phases of ground states that are not exactly solvable. For this reason, we now turn to a numerical exploration.

5. Numerics: Cat state from the spin-1 Heisenberg chain

To emphasize the generality of our claim that SPT phases can be used to generated LRE upon measurement, we consider the incarnation of the Haldane SPT phase in the spin-1 Heisenberg chain. Its Hamiltonian is a just nearest-neighbor antiferromagnetic coupling:
\[
H = \sum_n S_n \cdot S_{n+1}.
\]

This spin chain is known to be gapped [127], forming a nontrivial SPT phase for the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) group of \( \pi \)-rotations generated by \( R^\gamma = \prod_n e^{i\pi S_n^\gamma} \) with \( \gamma = x, y, z \) [21, 126, 128, 129]. Indeed, it has been argued to be in the same phase as the tractable AKLT state encountered in the previous section [126].

By our general proposal, we expect that measuring, say, the \( R^z \) charge for every site, should result in a cat state for the remaining \( \mathbb{Z}_2 \) symmetry. An interesting difference from the cluster chain is that the symmetries do not act on distinct sites. We thus measure \( R_n^z = e^{i\pi S_n^z} \) on every single site. Effectively, this process comes down to measuring whether \( (S_n^z)^2 = 0 \) or 1. For the first outcome, the site has no degree of freedom left, whereas for the latter, we still have a remaining qubit \( (S_n^z = \pm 1) \) which is toggled by \( R^x \). Hence, with the exception of there being no qubits left (which is of measure zero in the thermodynamic limit), we expect a cat state for the remaining chain of qubits. This is similar to the AKLT discussion in Sec. V A 4, although now we cannot rely on an exact solution.

To test this prediction, we numerically obtain the ground state of Eq. (53) using the density matrix renormalization group (DMRG) [121, 130, 131] for a variable system size \( L \) with periodic boundary conditions. We then project each site into \( (S_n^z)^2 = 0 \) with probability 1/3 or \( (S_n^z)^2 = 1 \) with probability 2/3. As a robust way of detecting whether the resulting state is a cat state, we calculate the Fisher information, which in this case is simply the variance of the total (staggered) magnetization:
\[
F = \left( \sum_{n=1}^{L} (-1)^{n} S_n^z \right)^2 - \left( \sum_{n=1}^{L} (-1)^{n} S_n^z \right)^2. \tag{54}
\]

This Fisher information is a quantitative measure for the use of the state for quantum metrology purposes [89, 90]. While SRE states obey a scaling \( F \sim L \), only nonlocal cat states have \( F \sim L^2 \). Our numerical results\(^\text{15} \) are shown in Fig. 4. While the original ground state has \( F \sim L \), we find that the postmeasurement state indeed has \( F \sim L^2 \), confirming that it is a cat state. In addition, it is interesting to see that \( F(L) \) varies relatively continuously with \( L \), despite each system size having a completely random measurement outcome (each red dot is computed for only a single measurement shot).

The above emergence of a cat state can actually be linked to the original interpretation of the Haldane SPT phase. Indeed, when the topological string order parameter was first introduced in 1989 [128], it was designed to pick up the ‘hidden symmetry-breaking’ of the state, where it was observed that if one imagines removing all \( S_n^z = 0 \) states, then the remaining \( S_n^z = \pm 1 \) states form long-range Néel order. However, since the \( S_n^z = 0 \) states are interspersed within the \( S_n^z = \pm 1 \) states and are allowed to have quantum fluctuations, they disorder this local order (which can now only be picked up with a string order parameter). Our above procedure can be interpreted as making this hidden order manifest: The measurement pins the location of \( S_n^z = 0 \), preventing them from disordering the Néel state.

\(^{15} \) We went up to system sizes of \( L = 100 \), where we found that \( \chi \approx 500 \) was sufficient to guarantee convergence of the Fisher information.

FIG. 4. Cat state from measuring the Haldane SPT phase. We consider the ground state of the spin-1 Heisenberg chain, which is in a nontrivial SPT phase for the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry of \( \pi \)-rotations. In accordance with its short-range entanglement, we find that the Fisher information scales linearly with system size (blue dots). In contrast, if we measure the \( R_n^z = e^{i\pi S_n^z} \)-charge on every site, the remaining state has Fisher information \( F \sim L^2 \) (red dots), signaling long-range entanglement in the post-measurement state (here we have chosen different random measurement outcomes for each \( L \)). This finding confirms that measuring one \( \mathbb{Z}_2 \) symmetry of the Haldane SPT phase creates a cat state for the remaining \( \mathbb{Z}_2 \) symmetry, even if one is not at a fine-tuned fixed-point limit.
B. Measuring general SPT phases

Here, we discuss how LRE arises upon measuring more general SPT states, even beyond 1D. As a natural starting point, we consider one of the simplest SPT phases (beyond 1D) which are protected by more than a single cyclic group—such that it is meaningful to measure one symmetry and preserve the other. Let us thus consider the $\mathbb{Z}_2^3$ “cubic” SPT in 2+1D. One model for this phase [98, 132] is given by placing spins on the sites of a triangular lattice, with each $\mathbb{Z}_2$ acting as $\prod_{j \in A,B,C} X_j$ on each of three triangular sublattices $A, B, C$. For each site $j$, there is a stabilizer given by

$$S_j = X_j \prod_{\langle j qq' \rangle} CZ_{q,q'},$$

where the product is over triangles $\langle j qq' \rangle$ with vertices $j, q, q'$. When we measure $X_j$ on the $A$ sublattice, we are left with a state on a honeycomb sublattice with

$$\prod_{\langle j qq' \rangle} CZ_{q,q'} = (-1)^{x_j}$$

around each hexagon, for some fixed signs (determined by our measurement outcome $s_j$).

The loop operators $\prod_{i,j \in A} CZ_{i,j}$ along a closed path $\gamma$ of vertices can be considered as a $\mathbb{Z}_2$ 1-form symmetry of this state. Note that this acts as the cluster SPT entangler for $\mathbb{Z}_2^{ABC}$ along $\gamma$, which implies there is a mixed anomaly; therefore, the resulting state obtained from measurement cannot be short-range entangled. Note that this anomaly can be realized on the boundary of a lattice model of a 3D SPT protected by $\mathbb{Z}_2^2 \times \mathbb{Z}_2^1$ as studied in Ref. 98.

We believe that a similar conclusion holds generally when we measure SPT states, at least when the corresponding topological term is linear in the gauge field associated with the measured charge. Let $G$ and $H$ be $(p - 1)$- and $(q - 1)$-form symmetries where $G$ and $H$ are onsite symmetries that act only on subsystems $A$ and $B$ respectively. Denote the background gauge fields of $G$ and $H$, $A_p$ and $B_q$, respectively. Now, consider an SPT associated with the cohomology class $A_p F(B_q) \in H^{d+1}(G \times H, U(1))$, where $d$ is the space dimension and $F(B_q) \in H^{d+1-p}(H, G^*)$ describes a topological G current made from $B_q$ where $G^* = \text{Hom}(G, U(1))$. Physically, $F(B_q)$ can be understood as an $H$ SPT in $d - p + 1$ spatial dimensions, and the SPT $A_p F(B_q)$ corresponds to decorating fluctuating G-domain walls in this $H$ SPT[132].

In this fixed-point model, if we now measure the $G$ charges, we essentially project out the topological current $F(B_q)$. Analogously to the CZ ring in Eq.(56), we similarly obtain a $p$-form symmetry, the remnant of the $G$ symmetry action by symmetry fractionalization of the parent SPT phase before measurement—applying the $G$ symmetry in a region is equivalent to acting on the boundary of that region with the entangler of the $H$-SPT (see Fig. 5).

This anomaly can also be seen from studying the topological response of the $G \times H$ SPT. Projecting out $G$ charges is equivalent to making the $G$ gauge field $A_p$ dynamical. Measuring the $G$ charges can be thought of as making $A_p$ dynamical with a charge background fixed by the measurement outcome. Since we began with a gapped phase, there are no fluctuating $G$ charges at low energies. As a result, there is an emergent $p$-form symmetry that acts as $A_p \rightarrow A_p + \lambda$, known as the center, or electric symmetry [119]. This symmetry is the same as the $p$-form symmetry we defined above. From the form of the topological response, assumed to be $A_p F(B_q)$, we see that this global symmetry is broken when there is a nontrivial $B_q$ since it produces a variation of the effective action, namely $\int \mathcal{L} F(B_q)$. This variation is characteristic of an anomaly associated with a $d + 1$-dimensional topological response $A_{p+1} F(B_q)$ [134], where $A_{p+1}$ is the background $p + 1$-form gauge field (note the shift) associated with the center symmetry.

When the SPT class is not linear in $A_p$, we will not be able to fractionalize the $G$ symmetry so that the boundary operator commutes with the $G$ charges [33]. However, if it is the form $F_1(A_p) F_2(B_q)$, where $F_1(A_p) \in H^j(G, K)$ and $F_2(B_q) \in H^{d+1-j}(G, K^*)$, for some Abelian group $K$, then there will be a codimension $j + 1$ defect Poincaré dual to $dF_1(A_p)$ that can factorize, defining a $j + 1$-form symmetry in the fixed point model postmeasurement corresponding to a field $C_{j+2}$. The anomaly will then be $C_{j+2} F_2(B_q) \in H^{d+2}(K[j + 1] \times H, U(1))$. For example, if we measure both $\mathbb{Z}_2^1$ and $\mathbb{Z}_2^2$ in the cubic SPT, then in this case, we identify $A_p = (A_1^{(1)}, A_1^{(2)})$. 

![Diagram](image-url)
\[ B_q = A^{(3)}_q, \quad F(A_p) = \frac{1}{2} A^{(1)}_1 A^{(2)}_3 \] and \[ F(B_q) = A^{(3)}_1. \] Thus, the anomaly postmeasurement is \[ \frac{1}{2} C_3 A^{(3)}_1 \] for a \( \mathbb{Z}_2 \) 2-form symmetry associated with \( C_3 \).

VI. OUTLOOK

In this work, we have presented a general framework for which performing measurements of short-range entangled states produces long-range entanglement. We have given some intuitive arguments that this is a stable property of the SPT phase, as well as proven that this always holds if the measurements are performed in an appropriately large enough unit cell. We would also like to determine the nature of the long-range entangled states which appear.

We have also described how non-local transformations including Kramers-Wannier and Jordan-Wigner arise from coupling an arbitrary state with a symmetry to a cluster-like SPT and performing measurements. It would be interesting to see whether other SPTs define useful transformations this way. If so, what family of MPOs do they define? We note that, given a general MPO, it is not obvious how to implement it from finite-depth unitaries and measurements.

Sequential applications of our procedure even lead to non-Abelian topological order, including quantum doubles for solvable groups. A natural question is to find an analogue for nonsolvable groups—or to prove a no-go theorem. We also argued that non-Abelian states beyond symmetric phases in one-dimensional spin systems, Phys. Rev. B 64, 050401 (2006).

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FIG. 6. The KW MPO is obtained by starting with the MPS of the 1D cluster state flipping the legs on the B (blue) sublattice. Generalized KW dualities can be similarly obtained by a cluster state which is a nontrivial SPT protected by the desired symmetries on a bipartite lattice.

Appendix A: Matrix product for the 1D cluster state and KW

Consider a one-dimensional lattice of $2N$ qubits. We identify two sublattices $A$ and $B$ corresponding to the odd and even sites of the lattice, respectively. The 1D cluster state can be expressed using a MPS as

$$|\psi\rangle = \sum_{\{s\}} \text{Tr}[C^{s_1}C^{s_2} \cdots C^{s_{2N}}]|s_1, s_2, \ldots, s_{2N}\rangle,$$  \hspace{1cm} (A1)

where $s_n = 0, 1$ are $Z$-basis states and the tensor $C$ is defined as

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle 0 | & \langle 1 | \\ \langle 1 | & -\langle 1 | \end{pmatrix}. \hspace{1cm} (A2)$$

To turn this into a matrix product operator (MPO), we first double the unit cell to get an MPS with double the physical legs,

$$C \otimes C = \begin{pmatrix} \langle 0+ | & \langle 1- | \\ \langle 0- | & \langle 1+ | \end{pmatrix}. \hspace{1cm} (A3)$$

Flipping the leg of the first entry upwards (see Fig. 6) yields the MPO

$$\sigma = \begin{pmatrix} 0 \langle +| & 1 \langle -| \\ 0 \langle -| & \langle +| \end{pmatrix}. \hspace{1cm} (A4)$$

This expression is exactly the Kramers-Wannier duality. For example, if we plug in the $|+\rangle$ product state, then we get the MPS for the GHZ state

$$\begin{pmatrix} \langle 0 | & 0 \\ 0 & \langle 1 | \end{pmatrix}. \hspace{1cm} (A5)$$

Appendix B: More examples

1. Wen plaquette model

Consider the following cluster state given by stabilizers,

$$Z \quad Z \quad Z \quad Z \quad Z \quad Z$$

$$Z \quad X \quad Z$$

This state is in fact the cluster state on the triangular lattice, although we have placed it on the square lattice. This cluster state is a (strong) $\mathbb{Z}_2$ subsystem SPT protected by line symmetries, given by flipping spins along the $x$ and $y$ lines of the square lattice [151]. In fact, gauging this subsystem SPT gives rise to the Wen plaquette model[152].
Based on this finding, we show how to prepare the Wen plaquette model via measuring an appropriate cluster state. The $A$ and $B$ are the vertices of the square (red) and dual square (blue) sublattices, respectively. We create the cluster state given by the stabilizers

\[
\begin{array}{c}
Z \quad Z \\
Z \quad X \\
Z \quad Z \\
Z \quad Z
\end{array} \quad \begin{array}{c}
Z \quad Z \\
Z \quad X \\
Z \quad Z
\end{array}
\]

(B2)

Note that because of the couplings within the $A$ sublattice, this cluster state is not bipartite. Now, let us measure the $X$ operators on the $A$ sublattice. The local product of stabilizers that commute with the measurements is

\[
\begin{array}{c}
Z \\
X \\
Y \\
Z
\end{array}
\]

(B3)

and the non-local products are $\prod X$ along each $x$ and $y$ lines.

Thus, with measurement outcomes $X = (-1)^{x_v}$ we have the stabilizers

\[
\begin{array}{c}
Z \\
Y \\
(-1)^{x_v+1} \quad Y \\
Z
\end{array}
\]

(B4)

which, up to single site rotations, are the stabilizers of the Wen plaquette model.

Although the Wen plaquette model is in the same topological phase as the toric code, it has the advantage of treating the $e$ and $m$ anyons on equal footing. In particular, it naturally has a dislocation defect which permutes the $e$ and $m$ anyons that encircles the defect [153]. In other words, the dislocation hosts a Majorana zero mode. Consider the cluster state given by the graph

which features a dislocation on the $B$ sublattice (dotted lines). Here the black lines connect $AA$ sites, while the red lines connect $AB$ sites. Performing measurements on the $A$ sublattice, the stabilizers for each plaquette on the blue sites are given by
2. Three-Fermion Walker-Wang model

It is argued that the three-Fermion Walker-Wang (3FWW) model\[154\] cannot be created from a circuit; it requires a quantum cellular automaton\[155\]. Here, we argue that we can alternatively create this state by measuring an appropriate 3D cluster state. The preparation of such a state can prove useful for measurement-based quantum computation using such Walker-Wang models\[156\] by effectively evolving the two-dimensional topological order on the boundary using measurements\[157, 158\].

The 3FWW model can be obtained by gauging a $\mathbb{Z}_2^2$ 1-form SPT\[159\]. The response of this SPT to background $\mathbb{Z}_2$ 2-form gauge fields $B_1$ and $B_2$ is given by $B_1^2 + B_2^2 + B_1B_2$. The physical interpretation of the three terms is that they statistically transmute the anyons on the boundary to become that of fermions.

Conveniently, the above SPT phase is itself a cluster state. Therefore, combining with the cluster state that implements the KW duality on each sublattice, the cluster state we would like to perform measurements on to obtain the 3FWW is a $\mathbb{Z}_4^2$ 1-form SPT. Its response to background gauge fields $B_i$ for $i = 1, 2, 3, 4$ is $B_1^2 + B_2^2 + B_1B_2 + B_1B_4 + B_2B_3$. The 3FWW is obtained by measuring the 1 and 2 sublattices.

Because it is a 1-form SPT, we define the cluster state on the edges of a cubic lattice, with four qubits placed per edge (i.e. 12 sites per unit cell). It is convenient to describe the cluster state using polynomials\[160\], which denote the connectivity of this cluster state.

As a stepping stone, we describe the stabilizers for the $B^2$ SPT,

$$
\begin{pmatrix}
0 & (y + \bar{z}\bar{x})(1 + z) & (z + \bar{x}\bar{y})(1 + y) & 0 \\
(\bar{x} + \bar{y}\bar{z})(1 + z) & 0 & (z + \bar{x}\bar{y})(1 + x) & 0 \\
(\bar{x} + \bar{y}\bar{z})(1 + y) & (y + \bar{z}\bar{x})(1 + x) & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

(B5)

Here, each column denotes a stabilizer, and the top and bottom rows denote the positions of the Pauli-Z and Pauli-X's,
respectively. Similarly, the $B_1B_2$ SPT (RBH cluster state) \cite{98,161,162} has stabilizers

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \bar{x}(1 + \bar{z}) & \bar{x}(1 + \bar{y}) \\
0 & 0 & 0 & \bar{y}(1 + \bar{z}) & 0 & \bar{y}(1 + \bar{x}) \\
0 & 0 & 0 & 0 & \bar{z}(1 + \bar{y}) & \bar{z}(1 + \bar{x}) \\
0 & \bar{y}(1 + z) & z(1 + y) & 0 & 0 & 0 \\
x(1 + z) & 0 & z(1 + x) & 0 & 0 & 0 \\
x(1 + y) & \bar{y}(1 + x) & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(B6)

Therefore, our desired cluster state is the $+1$ eigenstate of the stabilizers

\[
\begin{pmatrix}
0 & (y + xz)(1 + z) & (x + xy)(1 + y) & 0 & x(1 + z) & x(1 + y) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
0 & (x + yz)(1 + y) & (x + yx)(1 + x) & y(1 + x) & 0 & y(1 + x) & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Appendix C: Equality of long-range order for measurement outcomes in 1D

In Sec. V A 1 in the main text, we claimed that \( \langle \psi | Z_{2m} Z_{2n} | \psi \rangle = -\langle \psi | Z_{2m} Z_{2n} | \psi \rangle \). This claim can be derived using the notion of symmetry fractionalization \cite{24}. In particular, since we have a gapped phase with \( \prod_k X_{2k} \) symmetry, one can argue that \( X_{2p} X_{2p+2} \cdots X_{2q} | \psi \rangle = U_{L,R} | \psi \rangle \), where \( U_{L,R} \) are exponentially localized near the end points of the original string operator. Equivalently, if we define \( \hat{S}_{2p,2q} = X_{2p} X_{2p+2} \cdots X_{2q} \), then our state \( | \psi \rangle \) is an eigenstate of \( U_L \hat{S}_{2p,2q} U_R \). Since we are in a nontrivial SPT phase, \( U_{L,R} \) will anticommute with the other \( Z \) symmetry \( \prod_k X_{2k-1} \). Let us now revisit the situation studied in the main text, where \( n \) and \( m \) are separated far away from one another. Then we can choose \( m \ll p \ll n \ll q \) such that \( S_{2m,2n} \times U_L \hat{S}_{2p,2q} U_R = -U_L \hat{S}_{2p,2q} U_R \times S_{2m,2n} \). Note that since this operator leaves \( | \psi \rangle \) invariant and toggles \( S_{2m,2n} \), we have that \( U_L \hat{S}_{2p,2q} U_R | \psi \rangle = e^{i\alpha} | \psi \rangle \). Thus,

\[
\langle \psi | Z_{2m} Z_{2n} | \psi \rangle = e^{-i\alpha} \langle \psi | Z_{2m} Z_{2n} U_L \hat{S}_{2p,2q} U_R | \psi \rangle = -e^{-i\alpha} \langle \psi | U_L \hat{S}_{2p,2q} U_R Z_{2m} Z_{2n} | \psi \rangle = -\langle \psi | Z_{2m} Z_{2n} | \psi \rangle \,
\]

(C1)

where we used the fact that \( Z_{2n} \) is odd under the spin-flip symmetry on the even sites, and since \( m \ll p \ll n \ll q \) it is thus odd under \( \hat{S}_{2p,2q} \) (whereas \( Z_{2m} \) is not).
Appendix D: Symmetry charge of push-through operator

We consider the MPS-based arguments in Sec. V A 3. There, we encountered the projective group relations $V_g V_{g'} = \omega(g, g') V_{gg'}$, which together with the Abelian symmetry relations $gg' = g'g$, imply that $V_g V_{g'} = \frac{\omega(g, g')}{\omega(g', g)} V_{g'} V_g$. Let us introduce $\alpha_{g,h} = \frac{\omega(g, g')}{\omega(g', g)} \in U(1)$ as a convenient shorthand notation. We now prove that Eq. (45) implies that $U_h \mathcal{O}_g U_h^\dagger = \alpha_{g,h} \mathcal{O}_g$ or, equivalently, $U_h^\dagger \mathcal{O}_g U_h = \alpha_{g,h}^* \mathcal{O}_g$.

Here, we used the fact that $V_h V_g = \alpha_{g,h}^* V_g V_h$.

Note that the above also carries through for the $A$ tensor, such that $U_{g'}^\dagger \mathcal{O}_g U_{g'} = \alpha_{g,g'} \mathcal{O}_g$ for any $g, g' \in G$. Since we are considering a mixed $G \times H$ SPT phase, the SPT must, by definition, be trivial if we restrict to just $G$ symmetry. This implies that $\alpha_{g,g'} = 1$. Hence, $U_{g'}^\dagger \mathcal{O}_g U_{g'} = \mathcal{O}_g$ for any $g, g' \in G$. 

