Manin Triples and Differential Operators on Quantum Groups

To Jiro Sekiguchi on his 60th birthday

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Abstract. Let $G$ be a simple algebraic group over $\mathbb{C}$. By taking the quasi-classical limit of the ring of differential operators on the corresponding quantized algebraic group at roots of 1 we obtain a Poisson manifold $\Delta G \times K$, where $\Delta G$ is the subgroup of $G \times G$ consisting of the diagonal elements, and $K$ is a certain subgroup of $G \times G$. We show that this Poisson structure coincides with the one introduced by Semenov-Tyan-Shansky geometrically in the framework of Manin triples.

1. Introduction

In this paper we will explicitly compute the Poisson bracket of a certain Poisson manifold arising from the ring of differential operators on a quantized algebraic group at roots of 1. This result will be a foundation in the author’s recent works regarding the Beilinson-Bernstein type localization theorem for representations of quantized enveloping algebras at roots of 1 (see [16], [17]).

Let $G$ be a simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Take Borel subgroups $B^+$ and $B^-$ of $G$ such that $H = B^+ \cap B^-$ is a maximal torus of $G$. Set $N^\pm = [B^\pm, B^\pm]$. We define a subgroup $K$ of $G \times G$ by

$$K = \{(tx, t^{-1}y) \mid t \in H, x \in N^+, y \in N^- \} \subset B^+ \times B^- \subset G \times G.$$ 

Let $\zeta \in \mathbb{C}^\times$ be a primitive $\ell$-th root of 1, where $\ell$ is an odd positive integer satisfying certain conditions depending on $\mathfrak{g}$, and let $U_\zeta$ be the De Concini-Kac type quantized enveloping algebra of $\mathfrak{g}$ at $\zeta$. It is expected that there exists a certain correspondence between representations of $U_\zeta$ and modules over the ring $D_{B_\zeta}$ of differential operators on the quantized flag manifold $B_\zeta$. Since $D_{B_\zeta}$ is closely related to the ring $D_{G_\zeta}$ of differential operators on the quantized algebraic group $G_\zeta$, it is an important step in establishing the expected correspondence to investigate the ring $D_{G_\zeta}$ in detail. Note that $D_{G_\zeta}$ is nothing but the Heisenberg

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double $\mathcal{C}[G_\zeta] \otimes U_\zeta$ of the Hopf algebras $\mathcal{C}[G_\zeta]$ and $U_\zeta$, where $\mathcal{C}[G_\zeta]$ is the coordinate algebra of $G_\zeta$. We have natural central embeddings $\mathcal{C}[G] \subset \mathcal{C}[G_\zeta]$, $\mathcal{C}[K] \subset U_\zeta$ of Hopf algebras, and hence $G$ and $K$ become Poisson algebraic groups. By De Concini-Procesi [4] and De Concini-Lyubashenko [3] these Poisson algebraic group structures of $G$ and $K$ turn out to be the ones defined geometrically from the Manin triple $(G \times G, \Delta G, K)$, where $\Delta G$ is the subgroup of $G \times G$ consisting of diagonal elements. The aim of the present paper is to give a description of the Poisson algebra structure of $\mathcal{C}[G] \otimes \mathcal{C}[K]$ induced by the central embedding

\begin{equation}
\mathcal{C}[G] \otimes \mathcal{C}[K] \subset \mathcal{C}[G_\zeta] \otimes U_\zeta
\end{equation}

of algebras.

Let $(a, m, l)$ be a Manin triple over $\mathcal{C}$. Assume that we are given a connected algebraic group $A$ with Lie algebra $a$ and connected closed subgroups $M$ and $L$ of $A$ with Lie algebras $m$ and $l$ respectively. Then Semenov-Tyan-Shansky [13], [14] showed that $A$ has a natural structure of Poisson manifold. Hence by considering the pull-back with respect to the local isomorphism $M \times L \to A ((m, l) \mapsto ml)$ the manifold $M \times L$ also turns out to be a Poisson manifold.

**Theorem 1.1.** The Poisson structure of $G \times K$ induced from the central embedding (1.1) coincides with the one defined geometrically from the Manin triple $(G \times G, \Delta G, K)$.

As explained above, the coincidence of the two Poisson brackets

$$\mathcal{C}[G \times K] \times \mathcal{C}[G \times K] \to \mathcal{C}[G \times K]$$

is already known for the parts $\mathcal{C}[G] \times \mathcal{C}[G] \to \mathcal{C}[G]$ and $\mathcal{C}[K] \times \mathcal{C}[K] \to \mathcal{C}[K]$ by [4], [3]. Hence we will be only concerned with the mixed part of the Poisson bracket between $\mathcal{C}[G]$ and $\mathcal{C}[K]$. We point out that a closely related result in the case of $\zeta = 1$ for general Manin triples already appeared in [14].

In [14] it is noted that the Poisson manifold $L$ associated to a Manin triple $(a, m, l)$ can also be recovered as a Hamiltonian reduction with respect to the action of $M$ on $M \times L$. In order to pass from $D_{G_\zeta}$ to $D_{B_\zeta}$ we need to consider Hamiltonian reduction for more general situation. As a result we obtain the following.

**Proposition 1.2.** The varieties

$$\mathbf{\mathcal{V}} = \{(N^\gamma, (k_1, k_2)) \in (N^\gamma \setminus G) \times K \mid g k_1 k_2^{-1} g^{-1} \in H N^\gamma\},$$

$$\mathbf{\mathcal{V}}_t = \{(B^\gamma, (k_1, k_2)) \in (B^\gamma \setminus G) \times K \mid g k_1 k_2^{-1} g^{-1} \in t N^\gamma\} \quad (t \in H)$$

turn out to be Poisson manifolds with respect to the Poisson tensors induced from that of $G \times K$. Moreover, the Poisson tensors of $\mathbf{\mathcal{V}}$ and $\mathbf{\mathcal{V}}_t$ are non-degenerate. Hence they are symplectic manifolds.

In fact the Poisson manifold arising from the Poisson structure of the center of $D_{B_\zeta}$ coincides with $\mathbf{\mathcal{V}}$ above (see [16]). The non-degeneracy of the Poisson tensor plays a crucial
role in the argument of [16].

The contents of this paper is as follows. In Section 2 we recall the definition of the Poisson structure due to Semenov-Tyan-Shansky, and show that the technique of the Hamiltonian reduction works for certain cases. The case of the typical Manin triple \((g \oplus g, \Delta g, \mathfrak{k})\) is discussed in detail. In Section 3 we give a summary of some of the known results on quantized enveloping algebras at roots of 1 due to Lusztig [9], De Concini-Kac [2], De Concini-Lyubashenko [3], De Concini-Procesi [4], Gavarini [6]. In Section 4 we show that the Poisson structure arising from the algebra of differential operators acting on quantized coordinate algebra of \(G\) at roots of 1 coincides with the one coming from the typical Manin triple.

2. Poisson structures arising from Manin triples

2.1. Manin triples. We first recall standard facts on Poisson structures (see e.g., [5], [4]). A commutative associative algebra \(R\) over \(\mathbb{C}\) equipped with a bilinear map \(\{ , \} : R \times R \to R\) is called a Poisson algebra if it satisfies

- \(\{a, a\} = 0 \quad (a \in R)\),
- \(\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad (a, b, c \in R)\),
- \(\{a, bc\} = b\{a, c\} + \{a, b\}c \quad (a, b, c \in R)\).

A map \(F : R \to R'\) between Poisson algebras \(R, R'\) is called a homomorphism of Poisson algebras if it is a homomorphism of associative algebras and satisfies \(F(\{a_1, a_2\}) = \{F(a_1), F(a_2)\}\) for any \(a_1, a_2 \in R\). The tensor product \(R \otimes_{\mathbb{C}} R'\) of two Poisson algebras \(R, R'\) over \(\mathbb{C}\) is equipped with a canonical Poisson algebra structure given by

\[(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2,
\]

\[a_1 \otimes b_1, a_2 \otimes b_2 = \{a_1, a_2\} \otimes b_1b_2 + a_1a_2 \otimes \{b_1, b_2\}\]

for \(a_1, a_2 \in R, b_1, b_2 \in R'\). A commutative Hopf algebra \(R\) over a field \(\mathbb{C}\) equipped with a bilinear map \(\{ , \} : R \times R \to R\) is called a Poisson Hopf algebra if it is a Poisson algebra and the comultiplication \(\Delta : R \to R \otimes_{\mathbb{C}} R\) is a homomorphism of Poisson algebras (in this case the counit \(\Delta : R \to \mathbb{C}\) and the antipode \(\Delta : R \to R\) become automatically a homomorphism and an anti-homomorphism of Poisson algebras respectively).

For a smooth algebraic variety \(X\) over \(\mathbb{C}\) let \(\mathcal{O}_X\) (resp. \(\Theta_X, \Omega_X\)) be the sheaf of regular functions (resp. vector fields, 1-forms). We denote the tangent and the cotangent bundles of \(X\) by \(TX\) and \(T^*X\) respectively. A smooth affine variety \(X\) over \(\mathbb{C}\) is called a Poisson variety if we are given a bilinear map \(\{ , \} : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}[X]\) so that the \(\mathbb{C}\)-algebra \(\mathbb{C}[X]\) is a Poisson algebra. In this case \(\{f, g\}(x)\) for \(f, g \in \mathbb{C}[X]\) and \(x \in X\) depends only on \(df_x, dg_x\), and hence we have \(\delta \in \Gamma(X, \bigwedge^2 \Theta_X)\) (called the Poisson tensor of the Poisson variety \(X\)) such that

\[\{f, g\}(x) = \delta_x(df_x, dg_x).\]

Consequently we also have the notion of Poisson variety which is not necessarily affine.
Let $S$ be a linear algebraic group over $\mathbb{C}$ with Lie algebra $s$. For $a \in s$ we define vector fields $R_a, L_a \in \Gamma(S, \Theta S)$ by
\[
(R_a(f))(s) = \frac{d}{dt} (f(\exp(-ta)s))_{t=0} \quad (f \in \mathcal{O}_S, \ s \in S),
\]
\[
(L_a(f))(s) = \frac{d}{dt} (f(\exp(ta)))_{t=0} \quad (f \in \mathcal{O}_S, \ s \in S).
\]
For $\xi \in s^*$ we also define 1-forms $L_\xi^*, R_\xi^* \in \Gamma(S, \Omega_s)$ by
\[
(L_\xi^*, L_a) = (R_\xi^*, R_a) = \langle \xi, a \rangle \quad (a \in s).
\]
For $s \in S$ we define $\ell_s : S \to S$ by $\ell_s(x) = sx$.

A linear algebraic group $S$ over $\mathbb{C}$ is called a Poisson algebraic group if we are given a bilinear map $\{, \} : C[S] \times C[S] \to C[S]$ so that $C[S]$ is a Poisson Hopf algebra. Let $\delta$ be the Poisson tensor of $S$ as a Poisson variety, and define $\varepsilon : S \to \bigwedge^2 \mathfrak{s}$ by $(d\ell_s)(\varepsilon(s)) = \delta_s$ for $s \in S$. Here, we identify the tangent space $(TS)_1$ at the identity element $1 \in S$ with $s$ by $L_a \leftrightarrow a \ (a \in s)$. We sometimes identify $s$ and $s^*$ as vector spaces.

For each Manin triple we can associate a Poisson algebraic group by reversing the above process as follows. Let $(a, m, l)$ be a Manin triple with respect to a symmetric bilinear form $\rho$ on $a$ if

\begin{itemize}
  \item[(a)] $a$ is a finite-dimensional Lie algebra,
  \item[(b)] $\rho$ is $a$-invariant and non-degenerate,
  \item[(c)] $m$ and $l$ are subalgebras of $a$ such that $a = m \oplus l$ as a vector space,
  \item[(d)] $\rho(m, m) = \rho(l, l) = [0].$
\end{itemize}

Conversely, for each Manin triple we can associate a Poisson algebraic group by reversing the above process as follows. Let $(a, m, l)$ be a Manin triple with respect to a bilinear form $\rho$ on $a$ and let $M$ be a linear algebraic group with Lie algebra $m$. Denote by $\pi_m : a \to m$, $\pi_l : a \to l$ the projections with respect to the direct sum decomposition $a = m \oplus l$. We sometimes identify $m^*$ and $l^*$ with $l$ and $m$ respectively via the non-degenerate bilinear form $\rho_{|m \times l} : m \times l \to \mathbb{C}$. Hence we have also a natural identification
\[
2.1 \quad a^* = (m \oplus l)^* \cong m^* \oplus l^* \cong l \oplus m = a.
\]

For $m \in M$ we denote by $\text{Ad}(m) : a \to a$ the adjoint action. Then we have the following (see e.g., [5], [4]).

**Proposition 2.1.** The algebraic group $M$ is endowed with a structure of Poisson algebraic group whose Poisson tensor $\delta^M$ is given by
\[ \delta_m^M(L^*_\xi, L^*_\eta) = \rho(\pi_m(\text{Ad}(m)(\xi)), \text{Ad}(m)(\eta)) \quad (\xi, \eta \in l = m^*), \]
\[ \delta_m^M(R^*_\xi, R^*_\eta) = -\rho(\pi_m(\text{Ad}(m^{-1})(\xi)), \text{Ad}(m^{-1})(\eta)) \quad (\xi, \eta \in l = m^*). \]

for \( m \in M \).

2.2. Semenov-Tyan-Shansky Poisson structure. Let \((a, m, l)\) be a Manin triple over \( \mathbb{C} \) with respect to a bilinear form \( \rho \) on \( a \). We assume that we are given a connected algebraic group \( A \) and its closed connected subgroups \( M \) and \( L \) with Lie algebras \( a, m, l \) respectively. Define an alternating bilinear form \( \omega \) on \( a \) by
\[ \omega(a + b, a' + b') = \rho(a, b') - \rho(b, a') \quad (a, a' \in m, b, b' \in l). \]

Denote the adjoint action of \( A \) on \( a \) by \( \text{Ad} : A \to GL(a) \).

PROPOSITION 2.2 (Semenov-Tyan-Shansky [13], [14]). The smooth affine variety \( A \) is endowed with a structure of Poisson variety whose Poisson tensor \( \tilde{\delta} \) is given by
\[ \tilde{\delta}_g(L^*_\xi, L^*_\eta) = \frac{1}{2}(\omega(\text{Ad}(g)(\xi), \text{Ad}(g)(\eta)) + \omega(\xi, \eta)) \quad (\xi, \eta \in a^*, g \in A). \]

Here, we identify \( a \) with \( a^* \) via (2.1).

Note that we can rewrite \( \tilde{\delta} \) in terms of \( \rho \) as
\[ \tilde{\delta}_g(R^*_a, R^*_b) = \rho(a, (-\pi_m + \text{Ad}(g)(\pi_l \text{Ad}(g^{-1}))(b)) \]
\[ = \rho(a, (\pi_l - \text{Ad}(g)\pi_m \text{Ad}(g^{-1}))(b)), \]
\[ \tilde{\delta}_g(L^*_a, L^*_b) = \rho(a, (-\pi_m + \text{Ad}(g^{-1})(\pi_l \text{Ad}(g)))(b)) \]
\[ = \rho(a, (\pi_l - \text{Ad}(g^{-1})\pi_m \text{Ad}(g))(b)) \quad (g \in A, a, b \in a). \]

Consider the map
\[ (2.2) \quad \Phi : M \times L \to A \quad ((m, l) \mapsto ml). \]

Since \( \Phi \) is a local isomorphism, we obtain a Poisson structure of \( M \times L \) whose Poisson tensor \( \delta \) is the pull-back of \( \tilde{\delta} \) with respect to \( \Phi \). Let us give a concrete description of \( \delta \). By Proposition 2.1 \( M \) is endowed with a structure of Poisson algebraic group. By the symmetry of the notion of a Manin triple \( L \) is also a Poisson algebraic group whose Poisson tensor \( \delta_L \) is given by
\[ \delta^L_m(L^*_\xi, L^*_\eta) = \rho(\pi_l(\text{Ad}(l)(\xi)), \text{Ad}(l)(\eta)) \quad (l \in L, \xi, \eta \in m = l^*), \]
\[ \delta^L_m(R^*_\xi, R^*_\eta) = -\rho(\pi_l(\text{Ad}(l^{-1})(\xi)), \text{Ad}(l^{-1})(\eta)) \quad (l \in L, \xi, \eta \in m = l^*). \]

By a standard computation we have the following.

PROPOSITION 2.3. The Poisson tensor \( \delta \) is given by
\[ \delta_{(m, l)} : ((T^*M)_m \oplus (T^*L)_l) \times ((T^*M)_m \oplus (T^*L)_l) \to \mathbb{C} \]
for \((m, l) \in M \times L\) with

\[(2.3) \quad \delta_{(m,l)} |_{(T^*M)_m \times (T^*M)_m} = \delta^M_m,\]

\[(2.4) \quad \delta_{(m,l)} |_{(T^*L)_l \times (T^*L)_l} = \delta^L_I,\]

\[(2.5) \quad \delta_{(m,l)} (L^*_a, R^*_\xi) = \rho(a, \xi) \quad (a \in l = m^*, \xi \in m = l^*).\]

As noted in [14] the Poisson tensors \(\tilde{\delta}\) and \(\delta\) are non-degenerate at generic points, and hence some open subsets of \(A\) and \(M \times L\) turn out to be symplectic manifolds. We give below the condition on the point of \(A\) and \(M \times L\) so that the Poisson tensor is non-degenerate.

**LEMMA 2.4.** (i) Let \(g \in A\). Then \(\tilde{\delta}_g\) is non-degenerate if and only if

\[\text{Ad}(g)(l) \cap m = \text{Ad}(g)(m) \cap l = \{0\}.\]

(ii) Let \((m, l) \in M \times L\). Then we have

\[\dim \text{rad} \delta_{(m,l)} = \dim (l \cap \text{Ad}(ml)(m)).\]

Especially, \(\delta_{(m,l)}\) is non-degenerate if and only if

\[\text{Ad}(m^{-1})(l) \cap \text{Ad}(l)(m) = \{0\}.\]

**PROOF.** (i) Set \(F = -\pi_m + \text{Ad}(g)\pi_l \text{Ad}(g^{-1}) : a \to a\) for simplicity. By definition \(\tilde{\delta}_g\) is non-degenerate if and only if \(F\) is an isomorphism.

Assume that \(F\) is an isomorphism. Since \(F\) is surjective, we must have \(a = m + \text{Ad}(g)(l)\) by the definition of \(F\). By \(\dim a = \dim m + \dim l\) we have \(a = m \oplus \text{Ad}(g)(l)\) and \(m \cap \text{Ad}(g)(l) = 0\). Then

\[\ker F = \{a \in a \mid \pi_m(a) = \text{Ad}(g)\pi_l \text{Ad}(g^{-1})(a) = 0\} = l \cap \text{Ad}(g)(m).\]

Hence the injectivity of \(F\) implies \(l \cap \text{Ad}(g)(m) = \{0\}\).

Assume \(\text{Ad}(g)(l) \cap m = \text{Ad}(g)(m) \cap l = \{0\}\). By \(\text{Ad}(g)(l) \cap m = \{0\}\) we have \(a = m \oplus \text{Ad}(g)(l)\). Then \(\ker F = l \cap \text{Ad}(g)(m) = \{0\}\). Hence \(F\) is an isomorphism.

(ii) For \(g = ml\) we have

\[\text{Ad}(g)(l) \cap m = \text{Ad}(m)(\text{Ad}(l)(l) \cap \text{Ad}(m^{-1})(m)) = \text{Ad}(m)(l \cap m) = \{0\}.\]

Hence by the proof of (i) we obtain

\[\dim \text{rad} \delta_{(m,k)} = \dim \ker(-\pi_m + \text{Ad}(g)\pi_l \text{Ad}(g^{-1}))\]

\[= \dim(l \cap \text{Ad}(g)(m)).\]

\(\square\)

**COROLLARY 2.5.** (i) The Poisson structure of \(A\) induces a symplectic structure of the open subset

\[\hat{U} = \{g \in A \mid \text{Ad}(g)(l) \cap m = \text{Ad}(g)(m) \cap l = \{0\}\}.\]
of $A$

(ii) The Poisson structure of $M \times L$ induces a symplectic structure of the open subset

$$U := \{(m, l) \in M \times L \mid \text{Ad}(m^{-1})(l) \cap \text{Ad}(l)(m) = \{0\}\}$$

of $M \times L$.

2.3. A variant of Hamiltonian reduction. Let $X$ be a Poisson variety with Poisson tensor $\delta$ and let $S$ be a connected linear algebraic group acting on the algebraic variety $X$ (we do not assume that $S$ preserves the Poisson structure of $X$). Assume also that we are given an $S$-stable smooth subvariety $Y$ of $X$ on which $S$ acts locally freely. Denote by $\mathfrak{s}$ the Lie algebra of $S$.

For $y \in Y$ the linear map

$$\mathfrak{s} \ni a \mapsto \partial_a \in (TY)_y, \quad (\partial_a f)(y) = \frac{d}{dt} f(\exp(-ta)y)|_{t=0}$$

is injective by the assumption. Hence we may regard $\mathfrak{s} \subset (TY)_y$ for $y \in Y$. This gives an embedding

$$Y \times \mathfrak{s} \subset TY \subset (TX|_Y)$$

of vector bundles on $Y$. Correspondingly, we have

$$T^*_Y X \subset (Y \times \mathfrak{s})^\perp \subset (T^*X|_Y)$$

where

$$(Y \times \mathfrak{s})^\perp = \{v \in (T^*X|_Y) \mid \langle v, Y \times \mathfrak{s} \rangle = 0\},$$

and $T^*_Y X$ denotes the conormal bundle.

By restricting $\delta \in \Gamma(\bigwedge^2(TX))$ to $Y$ we obtain $\delta|_Y \in \Gamma(\bigwedge^2(TX|_Y))$. For $y \in Y$ restricting the anti-symmetric bilinear form $(\delta|_Y)_y$ on $(T^*X)_y$ to $((Y \times \mathfrak{s})^\perp)_y$, we obtain an anti-symmetric bilinear form $\hat{\delta}_y$ on $((Y \times \mathfrak{s})^\perp)_y$. Then we have $\hat{\delta} \in \Gamma(\bigwedge^2((TX|_Y)/(Y \times \mathfrak{s})))$.

Denote the action of $g \in S$ by $r_g : X \to X$. Then for $y \in Y$ the isomorphism $(dr_g)_Y : (TX)_y \to (TX)_g$, induces

$$(dr_g)_Y : (TY)_y \to (TY)_g, \quad (dr_g)_Y : s \ni a \mapsto \text{Ad}(g)(a) \in \mathfrak{s},$$

where $\mathfrak{s}$ is identified with subspaces of $(TY)_y$ and $(TY)_g$. In particular, $S$ naturally acts on $\Gamma(\bigwedge^2((TX|_Y)/(Y \times \mathfrak{s})))$.

Proposition 2.6. Assume that $\hat{\delta}$ is $S$-invariant and $(T^*_Y X)_y \subset \text{rad}(\hat{\delta}_y)$ for any $y \in Y$. Then the quotient space $S \backslash Y$ admits a natural structure of Poisson variety as follows. Let $\varphi$, $\psi$ be functions on $S \backslash Y$, and let $\hat{\varphi}$, $\hat{\psi}$ be the corresponding $S$-invariant functions on $Y$. Take extensions $\hat{\varphi}$, $\hat{\psi}$ of $\varphi$, $\psi$ to $X$ (not necessarily $S$-invariant). Then $\{\hat{\varphi}, \hat{\psi}\}|_Y$ is $S$-invariant and
does not depend on the choice of $\hat{\phi}, \hat{\psi}$. We define $[\hat{\phi}, \hat{\psi}]$ to be the function corresponding to $[\hat{\phi}, \hat{\psi}]_Y$.

Moreover, if we have $(T^*_Y X)_y = \operatorname{rad}(\hat{\delta}_y)$ for any $y \in Y$, then the Poisson tensor of $S \setminus Y$ is non-degenerate. Hence $S \setminus Y$ turns out to be a symplectic variety.

**Proof.** For $F \in \mathcal{O}_X$, $\partial \in \Theta_\mathcal{X}$, $y \in Y$ we have $\langle (dF)_\gamma, \partial \rangle = (\partial(F))(y)$, and hence $F|_Y$ is $S$-invariant (resp. $F|_Y$ is a locally constant function) if and only if $dF|_Y \in (Y \times s)^\perp$ (resp. $dF|_Y \in T^*_Y X$).

Take $\varphi, \psi$ and $\hat{\varphi}, \hat{\psi}$ as above. We first show that $[\hat{\varphi}, \hat{\psi}]_Y$ does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. For that it is sufficient to show that $[\hat{\varphi}, \hat{\psi}]_Y = 0$ if $\hat{\psi} = 0$. By $d\hat{\varphi}|_Y \in (Y \times s)^\perp$, $d\hat{\psi}|_Y \in T^*_Y X$ we have

$$\{\hat{\varphi}, \hat{\psi}\}(y) = \delta_y((d\hat{\varphi})_y, (d\hat{\psi})_y) = \delta_y((d\hat{\psi})_y, (d\hat{\psi})_y) = 0$$

by the assumption.

Let us show that $[\hat{\varphi}, \hat{\psi}]_Y$ is $S$-invariant. For $g \in S$, $y \in Y$ we have

$$\{\hat{\varphi}, \hat{\psi}\}(gy) = \hat{\delta}_{gy}((d\hat{\varphi})_{gy}, (d\hat{\psi})_{gy}) = \hat{\delta}_y(d(\hat{\varphi} \circ r_g)_y, d(\hat{\psi} \circ r_g)_y)$$

$$= \{\hat{\varphi} \circ r_g, \hat{\psi} \circ r_g\}(y)$$

by the $S$-invariance of $\hat{\delta}$. Since $\hat{\varphi}, \hat{\psi}$ are $S$-invariant, we have $\hat{\varphi} \circ r_g|_Y = \hat{\varphi}$ and $\hat{\psi} \circ r_g|_Y = \hat{\psi}$.

Hence the independence of $[\hat{\varphi}, \hat{\psi}]_Y$ on the choice of $\hat{\varphi}, \hat{\psi}$ implies

$$\{\hat{\varphi} \circ r_g, \hat{\psi} \circ r_g\}(y) = [\hat{\varphi}, \hat{\psi}](y)$$

for $g \in S$ and $y \in Y$.

The remaining assertions are now clear.

Now we apply the above general result to our Poisson varieties $M \times L$ and $A$.

Assume that we are given a connected closed subgroup $F$ of $M$. Let $\mathfrak{f}$ be the Lie algebra of $F$ and set $\mathfrak{f}^\perp = \{a \in \mathfrak{a} \mid \rho([a, a]) = 0\}$. The action $F \times A \ni (x, g) \mapsto xg \in A$ of $F$ on $A$ induces an injection

$$\mathfrak{f} \ni a \mapsto R_a \in (TA)_g \quad (g \in A).$$

Define a subbundle $(A \times \mathfrak{f})^\perp$ of $T^*A$ by

$$((A \times \mathfrak{f})^\perp)_g = \{R^*_c \mid c \in \mathfrak{f}^\perp\} \subset (T^*A)_g,$$

and set $\hat{\delta} = \hat{\delta}|_{((A \times \mathfrak{f})^\perp \times (A \times \mathfrak{f})^\perp)}$.

**Lemma 2.7.** If $\mathfrak{f}^\perp \cap \mathfrak{l}$ is a Lie subalgebra of $\mathfrak{l}$, then $\hat{\delta}$ is $F$-invariant.

**Proof.** By definition $\hat{\delta}_g$ for $g \in A$ is given by

$$\hat{\delta}_g(R^*_c, R^*_{c'}) = \rho(c, (-\pi_m + \operatorname{Ad}(g)\pi_1 \operatorname{Ad}(g^{-1}))(c')) \quad (c, c' \in \mathfrak{f}^\perp).$$
On the other hand for \( x \in F, g \in A \) the isomorphism \((T^*A)_g \cong (T^*A)_{xg}\) induced by the action of \( x \) is given by
\[
(T^*A)_g \cong (T^*A)_{xg} \quad (R^*_b \mapsto R^*_\Ad(x)(b)).
\]
Hence it is sufficient to show
\[
\rho(\Ad(x)(c), \pi_m \Ad(x)(c')) = \rho(c, \pi_m(c')) \quad (x \in F, c, c' \in \mathfrak{f}^\perp).
\]
Since \( F \) is connected, this is equivalent to its infinitesimal counterpart
\[
\rho([a, c], \pi_m(c')) + \rho(c, \pi_m([a, c'])) = 0 \quad (a \in \mathfrak{f}, c, c' \in \mathfrak{f}^\perp).
\]
Note that \( \mathfrak{f} = \mathfrak{m} \oplus (\mathfrak{f}^\perp \cap \mathfrak{l}) \). If \( c \in \mathfrak{m} \), then we have \([a, c] \in \mathfrak{m}\) and hence \( \rho([a, c], \pi_m(c')) = \rho(c, \pi_m([a, c'])) = 0 \). If \( c' \in \mathfrak{m} \), then
\[
\rho([a, c], \pi_m(c')) + \rho(c, \pi_m([a, c'])) = \rho([a, c], c') + \rho(c, [a, c']) = 0
\]
by the invariance of \( \rho \). Hence we may assume that \( c, c' \in \mathfrak{f}^\perp \cap \mathfrak{l} \). In this case we have
\[
\rho([a, c], \pi_m(c')) + \rho(c, \pi_m([a, c'])) = \rho(c, \pi_m([a, c'])) = \rho(c, [a, c'])
\]
\[
= -\rho([c', c], a) \in \rho(\mathfrak{f}^\perp \cap \mathfrak{l}, \mathfrak{f}) = 0.
\]
By Proposition 2.6 and Lemma 2.7 we have the following.

**Proposition 2.8.** Assume that \( \mathfrak{f}^\perp \cap \mathfrak{l} \) is a Lie subalgebra of \( \mathfrak{l} \). Let \( V \) be an \( F \)-stable smooth subvariety of \( A \) such that the action of \( F \) on \( V \) is locally free. Assume also that for \( g \in V \) we have
\[
\text{rad}(\hat{\delta}_g) \supset (T^*_V A)_g.
\]
Then \( F \setminus V \) has a structure of Poisson variety whose Poisson bracket is defined as follows: Let \( \varphi, \psi \) be functions on \( F \setminus V \), and denote by \( \hat{\varphi}, \hat{\psi} \) the corresponding \( F \)-stable functions on \( V \). Take extensions \( \hat{\varphi}, \hat{\psi} \) of \( \varphi, \psi \) respectively to \( A \). Then \( [\hat{\varphi}, \hat{\psi}]_V \) is \( F \)-stable and does not depend on the choice of \( \hat{\varphi}, \hat{\psi} \). We define \([\varphi, \psi]_V \) to be the function on \( F \setminus V \) corresponding to \([\hat{\varphi}, \hat{\psi}]_V \).

If, moreover,
\[
\text{rad}(\hat{\delta}_g) = (T^*_V A)_g
\]
holds for any \( g \in V \), then the Poisson tensor of \( F \setminus V \) is non-degenerate (hence \( F \setminus V \) turns out to be a symplectic variety).

**2.4. A special case.** Let \( G \) be a connected simple algebraic group over \( \mathbb{C} \), and let \( H \) be its maximal torus. We take Borel subgroups \( B^+, B^- \) of \( G \) such that \( H = B^+ \cap B^- \), and
Set \( N^\pm = [B^\pm, B^\pm] \). Denote the Lie algebras of \( G, H, B^\pm, N^\pm \) by \( g, h, b^\pm, n^\pm \). Define subalgebras \( \Delta g \) and \( \mathfrak{t} \) of \( g \oplus g \) by
\[
\Delta g = \{(a, a) \mid a \in g\}, \quad \mathfrak{t} = \{(h + x, -h + y) \mid h \in h, x \in n^+, y \in n^-\},
\]
and denote by \( \Delta G \) the connected closed subgroups of \( G \times G \) with Lie algebras \( \Delta g \) and \( \mathfrak{t} \) respectively. In particular, \( \Delta G = \{(g, g) \mid g \in G\} \). We fix an invariant non-degenerate symmetric bilinear form \( \kappa : g \times g \rightarrow \mathbb{C} \), and define a bilinear form \( \rho : (g \oplus g) \times (g \oplus g) \rightarrow \mathbb{C} \) by
\[
\rho((a, b), (a', b')) = \kappa(a, a') - \kappa(b, b').
\]
Then \( (g \oplus g, \Delta g, \mathfrak{t}) \) is a Manin triple with respect to the bilinear form \( \rho \).

By Proposition 2.2 (resp. Proposition 2.3) we have a Poisson structure of \( \Delta G \times K \) with Poisson tensor \( \delta \) (resp. \( \delta \)). Moreover, the Poisson structure of \( \Delta G \times K \) is the pull-back of that of \( G \times G \) with respect to
\[
\Phi : \Delta G \times K \rightarrow G \times G \quad (((g, g), (k_1, k_2)) \mapsto (gk_1, gk_2)).
\]

**Lemma 2.9.**
\[
\text{Im} \, \Phi = \{(g_1, g_2) \in G \times G \mid g_1^{-1}g_2 \in N^+HN^-\}.
\]

**Proof.** We have
\[
(gk_1)^{-1}(gk_2) = k_1^{-1}k_2 \in N^+HN^-.
\]
Assume \( g_1^{-1}g_2 \in N^+HN^- \). Then for \( (k_1, k_2) \in K \) with \( k_1^{-1}k_2 = g_1^{-1}g_2 \) we have
\[
(g_1, g_2) = (g_1k_1^{-1}, g_2k_2^{-1})(k_1, k_2) \in \text{Im} \, \Phi.
\]

**Proposition 2.10.** \( \delta_{((g,g),(k_1,k_2))} \) is non-degenerate if and only if we have
\[
gk_1k_2^{-1}g^{-1} \in N^+HN^-.
\]

**Proof.** Note that
\begin{equation}
\dim \text{rad}(\delta_{((g,g),(k_1,k_2))}) = \dim(\mathfrak{t} \cap \text{Ad}(gk_1, gk_2)(\Delta g))
\end{equation}
by Lemma 2.4. In general for \( (g_1, g_2) \in G \times G \) set \( d(g_1, g_2) := \dim(\mathfrak{t} \cap \text{Ad}(g_1, g_2)(\Delta g)) \). For \( (k_1, k_2) \in K \) and \( (g, g) \in \Delta G \) we have
\[
d((k_1, k_2)(g_1, g_2)(g, g)) = d(g_1, g_2),
\]
and hence \( d(g_1, g_2) \) is regarded as a function on \( K \setminus (G \times G)/\Delta G \). Denote by \( W = N_G(H)/H \) the Weyl group of \( G \). A standard fact on simple algebraic groups tells us that for any \( (g_1, g_2) \in \)
\[ G \times G \text{ there exists some } w \in W \text{ and } t \in H \text{ such that } K(g_1, g_2)\Delta G \ni (t \hat{w}, 1), \text{ where } \hat{w} \text{ is a representative of } w. \text{ By}
\]
\[
 d(t \hat{w}, 1) = \dim(\mathfrak{t} \cap \operatorname{Ad}(t \hat{w}, 1)(\Delta g)) = \dim(\operatorname{Ad}((t \hat{w}, 1)^{-1})(\mathfrak{t}) \cap \Delta g),
\]
\[
 \operatorname{Ad}((t \hat{w}, 1)^{-1})(\mathfrak{t}) = \{(w^{-1}h + \hat{w}^{-1}x, -h + y) \mid h \in \mathfrak{h}, x \in \mathfrak{n}^+, y \in \mathfrak{n}^-\}
\]
we see easily that \( d(t \hat{w}, 1) = 0 \) if and only if \( w = 1 \). The assertion follows from this easily. \( \square \)

**Corollary 2.11.** The Poisson structure of \( \Delta G \times K \) induces a symplectic structure of the open subset
\[
 U := \{((g, g), (k_1, k_2)) \in \Delta G \times K \mid gk_1k_2^{-1}g^{-1} \in N^+HN^-\}.
\]

Set
\[
 Y = \{((g, g), (k_1, k_2)) \in \Delta G \times K \mid gk_1k_2^{-1}g^{-1} \in B^-\} \subset U \subset \Delta G \times K,
\]
\[
 \tilde{Y} = \Phi(Y) \subset G \times G.
\]
Then we have
\[
 Y = \{(g_1, g_2) \in G \times G \mid g_1g_2^{-1} \in B^-, g_1^{-1}g_2 \in N^+HN^-\}.
\]
Moreover, setting
\[
 \tilde{Z} = \{(g, b) \in G \times B^- \mid g^{-1}b^{-1}g \in N^+HN^-\}
\]
we have
\[
 \tilde{Y} \cong \tilde{Z} \quad ((g_1, g_2) \leftrightarrow (g_1, g_1g_2^{-1}), (g, b^{-1}g) \leftrightarrow (g, b)).
\]
Since \( N^+HN^- \) is an open subset of \( G \), \( \tilde{Z} \) is open in \( G \times B^- \). In particular, \( \tilde{Z} \) is a smooth variety. Hence \( \tilde{Y} \) is also smooth. Define an action of \( N^- \) on \( G \times G \) by
\[
 x(g_1, g_2) = (xg_1, xg_2) \quad (x \in N^-, (g_1, g_2) \in G \times G).
\]
Then \( \tilde{Y} \) is \( N^- \)-invariant. Moreover, (2.8) preserves the action of \( N^- \), where the action of \( N^- \) on \( \tilde{Z} \) is given by
\[
 x(g, b) = (xg, xb^{-1}) \quad (x \in N^-, (g, b) \in \tilde{Z}).
\]
For \( C \subset G \) such that \( C \ni c \mapsto N^- c \in N^- \setminus G \) is an open embedding we have
\[
 \{(g, b) \in \tilde{Z} \mid g \in N^- C\}
\]
\[
 = \{(yc, yb^{-1}) \mid y \in N^-, c \in C, b \in B^-, c^{-1}b^{-1}c \in N^+HN^-\}
\]
\[
 \cong N^- \times \{(c, b) \in C \times B^- \mid c^{-1}b^{-1}c \in N^+HN^-\},
\]
and hence the action of \( N^- \) on \( \tilde{Z} \) is locally free. Hence we have the following.
Hence

\[ \delta(\phi) \mid \delta(\Delta G) \mid \delta(\Delta K) \]

In particular, \((\Delta n^-)^\perp \cap \mathfrak{k} \) is a Lie subalgebra of \( \mathfrak{k} \).

For \((g_1, g_2) \in \tilde{Y}\) we have

\[ T(g_1 \times g_2) = \{ R(a_1, a_2) \mid (a_1, a_2) \in \mathfrak{g} \oplus \mathfrak{g} \}, \]

\[ T^*(g_1 \times g_2) = \{ R^*_\mathfrak{g}(u_1, u_2) \mid (u_1, u_2) \in \mathfrak{g} \oplus \mathfrak{g} \}, \]

\[ \{ R(a_1, a_2), R^*_\mathfrak{g}(u_1, u_2) \} = \kappa(a_1, u_1) - \kappa(a_2, u_2). \]

By (2.8) we have also

\[ (T\tilde{Y}) = \{ R(a, \text{Ad}(g_2 g_1^{-1})a) \mid a \in \mathfrak{g} \} \oplus \{ R(b, b) \mid b \in \mathfrak{b}^- \} \]

for \((g_1, g_2) \in \tilde{Y}\). By Lemma 2.12 the natural map \( n^- \to (T\tilde{Y}) \) is injective and is given by

\[ n^- \ni c \mapsto R(c, c) \in (T\tilde{Y}). \]

Hence under the identification \( n^- \subset (T\tilde{Y}) \subset T(G \times G)(g_1, g_2) \) we have

\[ (n^-)^\perp = \{ R^*_\mathfrak{g}(u_1, u_2) \mid u_1 - u_2 \in \mathfrak{b}^- \} = \{ R^*_\mathfrak{g}(u, v) \mid u \in \mathfrak{g}, v \in \mathfrak{b}^- \}, \]

\[ ((T\tilde{Y}))^\perp = \{ R^*_\mathfrak{g}(\text{Ad}(g_2 g_1^{-1})y, y) \mid y \in n^- \}. \]

**Lemma 2.14.** For \((g_1, g_2) \in \tilde{Y}\) we have

\[ \text{rad} \tilde{\delta}(g_1, g_2)|_{(n^-)^\perp \times (n^-)^\perp} = ((T\tilde{Y})^\perp). \]

**Proof.** For \( a \in \mathfrak{g}, v \in \mathfrak{b}^- \) we have \( R^*_\mathfrak{g}(a, a) \in \text{rad} \tilde{\delta}(g_1, g_2)|_{(n^-)^\perp \times (n^-)^\perp} \) if and only if

\[ \tilde{\delta}(g_1, g_2)(R^*_\mathfrak{g}(a, a), R^*_\mathfrak{g}(a, a)) = 0 \]

for any \( a \in \mathfrak{g}, b \in \mathfrak{b}^- \). Setting

\[ (-\pi_{\Delta \mathfrak{g}} + \text{Ad}(g_1, g_2)\pi_{\mathfrak{k}} \text{Ad}(g_1^{-1}, g_2^{-1}))(u, u + v) = (x, y) \]

we have

\[ \tilde{\delta}(g_1, g_2)(R^*_\mathfrak{g}(a, a), R^*_\mathfrak{g}(a, a)) = \kappa(a, x) - \kappa(a + b, y) = \kappa(a, x - y) - \kappa(b, y). \]

Hence \( R^*_\mathfrak{g}(a, a) \in \text{rad} \tilde{\delta}(g_1, g_2)|_{(n^-)^\perp \times (n^-)^\perp} \) if and only if \( x = y \in n^- \). By \((g_1, g_2) \in \Phi(\Delta G \times K\} \) we have \( \mathfrak{g} \oplus \mathfrak{g} = \Delta \mathfrak{g} \oplus \text{Ad}(g_1, g_2)(\mathfrak{k}) \). Therefore,

\[ R^*_\mathfrak{g}(a, a) \in \text{rad} \tilde{\delta}(g_1, g_2)|_{(n^-)^\perp \times (n^-)^\perp} \]
\[ \iff \pi_{\Delta g}(u, u + v) = (y, y) (\exists y \in n^-), \quad \pi_{\varepsilon} \Ad(g_1^{-1}, g_2^{-1})(u, u + v) = 0 \]
\[ \iff u \in n^-. \quad \Ad(g_1^{-1}, g_2^{-1})(u, u + v) \in \Delta g \]
\[ \iff u \in n^-, \quad v = \Ad(g_2g_1^{-1})(u) - u. \]

It follows that
\[ \text{rad}(\tilde{h}_{(g_1, g_2)}|_{(n^-)^{\perp} \times (n^-)^{\perp}}) = \left\{ R^e_{(u, \Ad(g_2g_1^{-1})(u))} \mid u \in n^- \right\} = \left( (T\tilde{Y})_{(g_1, g_2)} \right)^{\perp}. \]

By Proposition 2.8 and the above argument we obtain the following.

**Proposition 2.15.** We have a natural Poisson structure of $N^+ \setminus \tilde{Y}$ whose Poisson tensor is non-degenerate and defined as follows (hence $N^+ \setminus \tilde{Y}$ turns out to be a symplectic variety): Let $\varphi, \psi$ be functions on $N^+ \setminus \tilde{Y}$, and let $\hat{\varphi}, \hat{\psi}$ be the corresponding $N^-$-invariant functions on $\tilde{Y}$. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\varphi, \psi$ to $G \times G$. Then $\{ \hat{\varphi}, \hat{\psi} \}_Y$ is $N^-$-invariant and does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. We define $\{ \varphi, \psi \}$ to be the function on $N^+ \setminus \tilde{Y}$ corresponding to $\{ \hat{\varphi}, \hat{\psi} \}_Y$.

By considering the pull-back to $Y$ via $\Phi$ we also obtain the following.

**Proposition 2.16.** Consider the action of $N^-$ on $Y$ given by
\[ x((g, g), (k_1, k_2)) = ((xg, xg), (k_1, k_2)) \quad (x \in N^-, (g, g), (k_1, k_2) \in Y). \]

Then we have a natural Poisson structure of $N^+ \setminus Y$ whose Poisson tensor is non-degenerate and defined as follows (hence $N^+ \setminus Y$ turns out to be a symplectic variety): Let $\varphi, \psi$ be functions on $N^+ \setminus Y$, and let $\tilde{\varphi}, \tilde{\psi}$ be the corresponding $N^-$-invariant functions on $Y$. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\varphi, \psi$ to $\Delta G \times K$. Then $\{ \hat{\varphi}, \hat{\psi} \}_Y$ is $N^-$-invariant and does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. We define $\{ \varphi, \psi \}$ to be the function on $N^+ \setminus Y$ corresponding to $\{ \hat{\varphi}, \hat{\psi} \}_Y$.

Note that
\[ N^+ \setminus Y \cong \{ ((N^- g, (k_1, k_2)) \in (N^- \setminus G) \times K \mid gk_1k_2^{-1}g^{-1} \in B^- \}. \]

Fix $t \in H$ and set
\[ Y_t = \{ ((g, g), (k_1, k_2)) \in \Delta G \times K \mid gk_1k_2^{-1}g^{-1} \in tN^- \} \subset U \subset \Delta G \times K. \]

Then by a similar argument we have the following.

**Proposition 2.17.** Consider the action of $B^- \setminus Y_t$ given by
\[ x((g, g), (k_1, k_2)) = ((xg, xg), (k_1, k_2)) \quad (x \in B^- \setminus Y_t, ((g, g), (k_1, k_2)) \in Y_t). \]

Then we have a natural Poisson structure of $B^- \setminus Y_t$, whose Poisson tensor is non-degenerate and defined as follows (hence $B^- \setminus Y_t$ turns out to be a symplectic variety): Let $\varphi, \psi$ be
functions on $B^- \setminus Y_t$, and let $\hat{\phi}, \hat{\psi}$ be the corresponding $B^-$-invariant functions on $Y_t$. Take extensions $\hat{\phi}, \hat{\psi}$ of $\tilde{\phi}, \tilde{\psi}$ to $\Delta G \times K$. Then $\{\hat{\phi}, \hat{\psi}\}|_{Y_t}$ is $B^-$-invariant and does not depend on the choice of $\hat{\phi}, \hat{\psi}$. We define $\{\phi, \psi\}$ to be the function on $B^- \setminus Y_t$ corresponding to $\{\hat{\phi}, \hat{\psi}\}|_{Y_t}$.

Note that we have

\[(2.10) \quad B^- \setminus Y_t \cong \{(B^- g, (k_1, k_2)) \in (B^- \setminus G) \times K \mid g k_1 k_2^{-1} g^{-1} \in t N^\perp\}.
\]

3. Quantized enveloping algebras

3.1. Lie algebras. In the rest of this paper we will use the notation of Section 2.4. In particular, $g$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, and $G$ is a connected algebraic group with Lie algebra $g$. We further assume that $G$ is simply-connected and the symmetric bilinear form\n
\[(3.1) \quad (\ , \ ) : h^* \times h^* \rightarrow \mathbb{C}
\]

induced by $\kappa$ satisfies $(\beta, \beta)/2 = 1$ for short roots $\beta$. We denote by $\Delta \subset h^*, Q \subset h^*, \Lambda \subset h^*$ and $W \subset GL(h^*)$ the set of roots, the root lattice $\sum_{\alpha \in \Delta} \mathbb{Z} \alpha$, the weight lattice and the Weyl group respectively. By our normalization of (3.1) we have

$$\frac{(\Lambda, Q)}{|Q/\Lambda|} \subset \mathbb{Z}.$$ 

For $\beta \in \Delta$ we set

$$g_{\beta} = \{x \in g \mid [h, x] = \beta(h) x \mid h \in h\}.$$ 

We choose a system of positive roots $\Delta^+ \subset h^*$ so that $n^+ = \bigoplus_{\beta \in \Delta^+} g_{\beta}$. Let $\{a_i\}_{i \in I}$, $\{s_i\}_{i \in I} \subset W$ be the corresponding sets of simple roots and simple reflections respectively. Set

$$Q^+ = \sum_{a \in \Delta^+} \mathbb{Z}_{\geq 0} a = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} a_i \subset h^*.$$ 

We denote the longest element of $W$ by $w_0$. For each $i \in I$ we take $e_i \in g_{a_i}, f_i \in g_{-a_i}, h_i \in h$ such that $[e_i, f_i] = h_i$ and $a_i(h_i) = 2$.

Define subalgebras $t^0, t^+, t^-$ of $t$ by

$$t^0 = \{(h, -h) \mid h \in h\}, \quad t^+ = \{(x, 0) \mid x \in n^+\}, \quad t^- = \{(0, y) \mid y \in n^-\}.$$ 

Then we have $t = t^+ \oplus t^0 \oplus t^-$. For $i \in I$ set

$$x_i = (e_i, 0) \in t^+, \quad y_i = (0, f_i) \in t^-, \quad t_i = (h_i, -h_i) \in t^0.$$ 

We denote by $K^0, K^\pm$ the connected closed subgroups of $K$ with Lie algebras $t^0, t^\pm$ respectively.
3.2. Quantized enveloping algebra of $\mathfrak{g}$. For $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ we set

$$[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \quad [m]_t! = [m]_t[m - 1]_t \cdots [2]_t[1]_t \in \mathbb{Z}[t, t^{-1}],$$

$$\left[\begin{array}{c} n \\ m \end{array}\right] = [n]_t[n - 1]_t \cdots [n - m + 1]_t/[m]_t! \in \mathbb{Z}[t, t^{-1}].$$

The quantized enveloping algebra $U = U_q(\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra over $\mathbb{F} = C(q^{1/|\Lambda'|})$ with identity element 1 generated by the elements $K_\lambda (\lambda \in \Lambda)$, $E_i, F_i (i \in I)$ satisfying the following defining relations:

\begin{align*}
(3.2) & \quad K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda + \mu} \\
(3.3) & \quad K_\lambda E_i K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_i \\
(3.4) & \quad K_\lambda F_i K_\lambda^{-1} = q^{-(\lambda, \alpha_i)} F_i \\
(3.5) & \quad E_i F_j - F_j E_i = \delta_{ij} K_i - K_i^{-1} \\
(3.6) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n E_i^{(1-a_{ij}+n)} E_j E_i^{(n)} = 0 \\
(3.7) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n F_i^{(1-a_{ij}+n)} F_j F_i^{(n)} = 0
\end{align*}

where $q_i = q^{(\alpha_i, \alpha_i)/2}$, $K_i = K_{\alpha_i}$, $a_{ij} = 2(\alpha_i, \alpha_j)/\langle \alpha_i, \alpha_i \rangle$ for $i, j \in I$, and

$$E_i^{(n)} = E_i^n/[n]_q!, \quad F_i^{(n)} = F_i^n/[n]_q!.$$

for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. Algebra homomorphisms $\Delta : U \to U \otimes U$, $\varepsilon : U \to \mathbb{F}$ and an algebra anti-automorphism $S : U \to U$ are defined by:

\begin{align*}
(3.8) & \quad \Delta(K_\lambda) = K_\lambda \otimes K_\lambda \\
(3.9) & \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \\
(3.10) & \quad \varepsilon(K_\lambda) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0 \\
(3.11) & \quad S(K_\lambda) = K_\lambda^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i
\end{align*}

and $U$ is endowed with a Hopf algebra structure with the comultiplication $\Delta$, the counit $\varepsilon$ and the antipode $S$.

We define subalgebras $U^0, U^\geq 0, U^\leq 0, U^+, U^-$ of $U$ by

\begin{align*}
(3.11) & \quad U^0 = \{ K_\lambda \mid \lambda \in \Lambda \} \\
(3.12) & \quad U^\geq 0 = \{ K_\lambda, E_i \mid \lambda \in \Lambda, i \in I \}
\end{align*}
\[ (3.13) \quad U_{\leq 0} = \langle K_\lambda, F_i \mid \lambda \in \Lambda, i \in I \rangle, \]
\[ (3.14) \quad U^+ = \langle E_i \mid i \in I \rangle, \]
\[ (3.15) \quad U^- = \langle F_i \mid i \in I \rangle. \]

The following result is standard.

**Proposition 3.1.** (i) \( \{ K_\lambda \mid \lambda \in \Lambda \} \) is an \( F \)-basis of \( U^0 \).

(ii) The linear maps
\[ U^- \otimes U^0 \otimes U^+ \to U^+ \otimes U^0 \otimes U^-, \]
\[ U^+ \otimes U^0 \to U_{\geq 0}^- \leftarrow U^0 \otimes U^+, \quad U^- \otimes U^0 \to U_{\leq 0}^- \leftarrow U^0 \otimes U^- \]
induced by the multiplication are all isomorphisms of vector spaces.

For \( \gamma \in \mathbb{Q} \) we set
\[ U^\pm = \{ x \in U^\pm \mid K_\lambda x K_\lambda^{-1} = q^{(\lambda, \gamma)} x \ (\lambda \in \Lambda) \}, \]
\[ U^\pm_{\pm \gamma} = \{ x \in U^\pm \mid K_\lambda x K_\lambda^{-1} = q^{(\lambda, \gamma)} x \ (\lambda \in \Lambda) \}. \]
We have \( U^\pm_{\pm \gamma} = \{0\} \) unless \( \gamma \in \mathbb{Q}^+ \), and
\[ U^\pm = \bigoplus_{\gamma \in \mathbb{Q}^+} U^\pm_{\pm \gamma}, \quad \dim U^\pm_{\pm \gamma} < \infty \ (\gamma \in \mathbb{Q}^+). \]

For \( i \in I \) we can define an algebra automorphism \( T_i \) of \( U \) by
\[ T_i(K_\mu) = K_{s_i \mu} \quad (\mu \in \Lambda), \]
\[ T_i(E_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (j \in I, j \neq i), \\ -F_i K_i & (j = i), \end{cases} \]
\[ T_i(F_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (j \in I, j \neq i), \\ -K_i^{-1} E_i & (j = i). \end{cases} \]

For \( w \in W \) we define an algebra automorphism \( T_w \) of \( U \) by \( T_w = T_{s_1} \cdots T_{s_n} \) where \( w = s_{i_1} \cdots s_{i_n} \) is a reduced expression. The automorphism \( T_w \) does not depend on the choice of a reduced expression (see Lusztig [10]).

We fix a reduced expression
\[ w_0 = s_{i_1} \cdots s_{i_N} \]
of \( w_0 \), and set
\[ \beta_k = s_{i_1} \cdots s_{i_{k-1}} (a_{ik}) \quad (1 \leq k \leq N). \]
Then we have \( \Delta^+ = \{ \beta_k \mid 1 \leq k \leq N \} \). For \( 1 \leq k \leq N \) set
\[ (3.16) \quad E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} (E_{i_k}), \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} (F_{i_k}). \]
Then \( \{E_{mN} \cdots E_{m1} | m_1, \ldots, m_N \geq 0 \} \) (resp. \( \{F_{mN} \cdots F_{m1} | m_1, \ldots, m_N \geq 0 \} \)) is an \( \mathbb{F} \)-basis of \( U^+ \) (resp. \( U^- \)), called the PBW-basis (see Lusztig [9]). For \( 1 \leq k \leq N, m \geq 0 \) we also set

\[
E_{\beta}^{(m)} = E_{\beta k}^{m} / [m]_{q_{\beta k} !}, \quad F_{\beta}^{(m)} = F_{\beta k}^{m} / [m]_{q_{\beta k} !},
\]

where \( q_{\beta} = q^{(\beta, \beta)/2} \) for \( \beta \in \Delta^+ \).

There exists a bilinear form

\[
\tau : U^0 \times U^0 \to \mathbb{F},
\]

called the Drinfeld paring, which is characterized by

\[
\tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in U^0, y_1, y_2 \in U^0),
\]
\[
\tau(x_1 x_2, y) = (\tau \otimes \tau)(x_1 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U^0, y \in U^0),
\]
\[
\tau(K_{\lambda}, K_{\mu}) = q^{-(\lambda, \mu)} \quad (\lambda, \mu \in \Lambda),
\]
\[
\tau(E_i, F_j) = \delta_{ij} / (q_{ij}^{-1} - q_i) \quad (i, j \in I).
\]

**Proposition 3.2** ([7], [8], [11]). We have

\[
\tau(E_{\beta}^{mN} \cdots E_{\beta1}^{m1} K_{\lambda}, F_{\beta}^{mN} \cdots F_{\beta1}^{m1} K_{\mu}) = q^{-(\lambda, \mu)} \prod_{k=1}^{N} \delta_{mk, nk} (-1)^m [m]_{q_{\beta k} !} q_{\beta k}^{m(m-1)/2} (q_{\beta k} - q_{\beta k}^{-1})^{-m_k}. \]

### 3.3. Quantized coordinate algebra of \( G \).

We denote by \( C \) the subspace of \( U^* = \text{Hom}_{\mathbb{F}}(U, \mathbb{F}) \) spanned by the matrix coefficients of finite dimensional \( U \)-modules \( E \) such that

\[
E = \bigoplus_{\lambda \in \Lambda} E_\lambda \quad \text{with} \quad E_\lambda = [v \in E | K_{\mu} v = q^{(\lambda, \mu)} v (\forall \mu \in \Lambda)].
\]

Then \( C \) is endowed with a structure of Hopf algebra via

\[
\langle \varphi \psi, u \rangle = \langle \varphi \otimes \psi, \Delta(u) \rangle \quad (\varphi, \psi \in C, \ u \in U),
\]
\[
\langle 1, u \rangle = \varepsilon(u) \quad (u \in U),
\]
\[
\langle \Delta(\varphi), u \otimes u' \rangle = \langle \varphi, uu' \rangle \quad (\varphi \in C, \ u, u' \in U),
\]
\[
\varepsilon(\varphi) = \langle \varphi, 1 \rangle, \quad (\varphi \in C),
\]
\[
\langle S(\varphi), u \rangle = \langle \varphi, S(u) \rangle \quad (\varphi \in C, \ u \in U),
\]

where \( \langle , , \rangle : C \times U \to \mathbb{F} \) is the canonical paring. \( C \) is also endowed with a structure of \( U \)-bimodule by

\[
\langle u' \varphi uu'', u \rangle = \langle \varphi, u'' uu' \rangle \quad (\varphi \in C, u, u', u'' \in U).
\]
The Hopf algebra $C$ is a $q$-analogue of the coordinate algebra $C[G]$ of $G$ (see [9], [15]).

Set

$$(U^\pm)^* = \bigoplus_{\gamma \in Q^+} \text{Hom}_F(U^\pm, F) \subset \text{Hom}_F(U, F).$$

For $\lambda \in A$ define an algebra homomorphism $\chi_\lambda : U^0 \to F$ by $\chi_\lambda(K_\mu) = q^{(\lambda,\mu)}$. Under the identification $U^{-} \otimes U^{0} \otimes U^{+} \cong U$ of vector spaces we have

$$(3.24) \quad C \subset (U^-)^* \otimes \left( \bigoplus_{\lambda \in A} F \chi_\lambda \right) \otimes (U^+) \subset U^*.$$

3.4. Ring of differential operators. In general for a Hopf algebra $H$ over $C$ we use the following notation for the comultiplication $\Delta : H \to H \otimes H$:

$$(\Delta(u) = \sum_{(u)} u(0) \otimes u(1)) \quad (u \in H).$$

We have an $F$-algebra structure of $D = C \otimes F U$, called the Heisenberg double of $C$ and $U$ (see e.g. [12]). It is given by

$$(\varphi \otimes u)(\varphi' \otimes u') = \sum_{(u)} \varphi(u(0)\varphi') \otimes u(1)u' \quad (\varphi, \varphi' \in C, u, u' \in U).$$

In our case the algebra $D$ is an analogue of the ring of differential operators on $G$. We will identify $U$ and $C$ with subalgebras of $D$ by the embeddings $U \ni u \mapsto 1 \otimes u \in D$ and $C \ni \varphi \mapsto \varphi \otimes 1 \in D$ respectively.

3.5. Quantized enveloping algebra of $\mathfrak{t}$. The quantized enveloping algebra $V = U_q(\mathfrak{t})$ of $\mathfrak{t}$ is an associative algebra over $F$ with identity element 1 generated by the elements $Z_\lambda (\lambda \in A), X_i, Y_i (i \in I)$ satisfying the following defining relations:

$$(3.25) \quad Z_0 = 1, \quad Z_\lambda Z_\mu = Z_{\lambda + \mu} \quad (\lambda, \mu \in A),$$

$$(3.26) \quad Z_\lambda X_i Z_\mu^{-1} = q^{(\lambda, \alpha_i)} X_i \quad (\lambda \in A, i \in I),$$

$$(3.27) \quad Z_\lambda Y_i Z_\mu^{-1} = q^{(\lambda, \alpha_i)} Y_i \quad (\lambda \in A, i \in I),$$

$$(3.28) \quad X_i Y_j - Y_j X_i = 0 \quad (i, j \in I),$$

$$(3.29) \quad \sum_{n=0}^{1-a_{ij}} (-1)^n X_i^{(1-a_{ij}-n)} X_j X_i^{(n)} = 0 \quad (i, j \in I, i \neq j),$$

$$(3.30) \quad \sum_{n=0}^{1-a_{ij}} (-1)^n Y_i^{(1-a_{ij}-n)} Y_j Y_i^{(n)} = 0 \quad (i, j \in I, i \neq j),$$
where
\[\hat{X}_{i}^{(n)} = X_{i}^{n} / [n]_{q_{i}}! \quad \hat{Y}_{i}^{(n)} = Y_{i}^{n} / [n]_{q_{i}}! ,\]

We define subalgebras \( V^{0}, V^{\geq 0}, V^{\leq 0}, V^{+}, V^{-} \) of \( V \) by

\[(3.31) \quad V^{0} = \{ Z_{\lambda} \mid \lambda \in \Lambda \}, \]
\[(3.32) \quad V^{\geq 0} = \{ Z_{\lambda}, X_{i} \mid \lambda \in \Lambda, i \in I \}, \]
\[(3.33) \quad V^{\leq 0} = \{ Z_{\lambda}, Y_{i} \mid \lambda \in \Lambda, i \in I \}, \]
\[(3.34) \quad V^{+} = \{ X_{i} \mid i \in I \}, \]
\[(3.35) \quad V^{-} = \{ Y_{i} \mid i \in I \}. \]

Similarly to Proposition 3.1 we have the following.

**Proposition 3.3.**

(i) \( \{ Z_{\lambda} \mid \lambda \in \Lambda \} \) is an \( F \)-basis of \( V^{0} \).

(ii) The linear maps
\[
V^{-} \otimes V^{0} \otimes V^{+} \to V \leftarrow V^{+} \otimes V^{0} \otimes V^{-},
\]
\[
V^{+} \otimes V^{0} \to V^{\geq 0} \leftarrow V^{0} \otimes V^{+},
\]
\[
V^{-} \otimes V^{0} \to V^{\leq 0} \leftarrow V^{0} \otimes V^{-}
\]
induced by the multiplication are all isomorphisms of vector spaces.

Moreover, we have algebra isomorphisms
\[
j^{\leq 0} : V^{\leq 0} \to U^{\leq 0} \quad (Y_{i} \mapsto F_{i}, Z_{\lambda} \mapsto K_{-\lambda}),
\]
\[
j^{\geq 0} : V^{\geq 0} \to U^{\geq 0} \quad (X_{i} \mapsto E_{i}, Z_{\lambda} \mapsto K_{\lambda}).
\]

We define a bilinear form
\[(3.36) \quad \sigma : U \times V \to F \]
by
\[
\sigma(u_{+}u_{0}S(u_{-}), v_{-}v_{0}v_{+}) = \tau(u_{+}, j^{\leq 0}(v_{-}))\tau(u_{0}, j^{\leq 0}(v_{0}))\tau(j^{\geq 0}(v_{+}), u_{-})
\]
\[
(u_{\pm} \in U^{\pm}, u_{0} \in U^{0}, v_{\pm} \in V^{\pm}, v_{0} \in V^{0}).
\]

The following result is a consequence of Gavarini [6, Theorem 6.2].

**Proposition 3.4.** We have
\[
\sigma(u, vv') = (\sigma \otimes \sigma)(\Delta(u), v \otimes v') \quad (u \in U, v, v' \in V).
\]

3.6. **A-forms.** We fix a subring \( A \) of \( F \) containing \( C[q^{\pm 1}/|\Lambda/Q|] \). We denote by \( U^{L}_{A} \) the Lusztig \( A \)-form of \( U \), i.e., \( U^{L}_{A} \) is the \( A \)-subalgebra of \( U \) generated by the elements
\[
E_{i}^{(m)}, F_{j}^{(m)}, K_{\lambda} \quad (i \in I, m \geq 0, \lambda \in \Lambda).
\]
Set

\[ U^L_{+\pm} = U^L_{+} \cap U^\pm, \quad U^L_{0} = U^L_{+} \cap U^0, \]
\[ U^L_{\geq 0} = U^L_{+} \cap U^{L, \geq 0}, \quad U^L_{\leq 0} = U^L_{+} \cap U^{L, \leq 0}. \]

Then \( U^L_{+}, U^L_{0}, U^L_{\geq 0}, U^L_{\leq 0} \) are endowed with structures of Hopf algebras over \( A \) via the Hopf algebra structure of \( U \), and the multiplication of \( U^L_{+} \) induces isomorphisms

\[ U^L_{+} \cong U^L_{-} \otimes U^L_{0} \otimes U^L_{+} \cong U^L_{+} \otimes U^L_{0} \otimes U^L_{-}, \]
\[ U^L_{\geq 0} \cong U^L_{0} \otimes U^L_{+} \cong U^L_{+} \otimes U^L_{0}, \]
\[ U^L_{\leq 0} \cong U^L_{0} \otimes U^L_{-} \cong U^L_{-} \otimes U^L_{0}. \]

of \( A \)-modules. Fix a subset \( \Lambda_0 \) of \( \Lambda \) such that \( \Lambda_0 \to \Lambda/2Q \) is bijective. Then \( U^L_{\geq 0}, U^L_{\leq 0}, U^L_{0} \) are free \( A \)-modules with bases

\[ \{ E^{(m_\alpha)}_{\beta\gamma} \cdots E^{(m_\beta)}_{\gamma\alpha} | m_1, \ldots, m_N \geq 0 \}, \]
\[ \{ F^{(m_\alpha)}_{\beta\gamma} \cdots F^{(m_\beta)}_{\gamma\alpha} | m_1, \ldots, m_N \geq 0 \}, \]
\[ \{ K_\lambda \prod_{i \in I} [K_i]_{n_i} | \lambda \in \Lambda_0, n_i \geq 0 \} \]

respectively, where

\[ [K_i]_m = \prod_{s=0}^{m-1} \frac{q_i^{-s} K_i - q_i^s K_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (m \geq 0). \]

We denote by \( V_A \) the \( A \)-subalgebra of \( V \) generated by the elements

\[ X^{(m)}_i, Y^{(m)}_i, Z_\lambda, [Z_i]_m \quad (i \in I, \ m \geq 0, \lambda \in \Lambda), \]

where \( Z_i = Z_{\alpha_i} \) for \( i \in I \) and

\[ [Z_i]_m = \prod_{s=0}^{m-1} \frac{q_i^{-s} Z_i - q_i^s Z_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (m \geq 0). \]

Set

\[ V^\pm = V_A \cap V^\pm, \quad V^0_A = V_A \cap V^0, \]
\[ V^{\geq 0} = V_A \cap V_{\geq 0}, \quad V^{\leq 0} = V_A \cap V_{\leq 0}. \]

Then the multiplication of \( V_A \) induces isomorphisms

\[ V_A \cong V_A^- \otimes V_A^0 \otimes V_A^+ \cong V_A^- \otimes V_A^0 \otimes V_A^-, \]
of $A$-modules, and we have

$$V_A^{\geq 0} \simeq V_A^0 \otimes V_A^+ \simeq V_A^+ \otimes V_A^0,$$

$$V_A^{\leq 0} \simeq V_A^0 \otimes V_A^- \simeq V_A^- \otimes V_A^0$$

of $A$-modules, and we have

$$j^{\geq 0} (V_A^{\geq 0}) = U_A^{L,\geq 0},$$

$$j^{\leq 0} (V_A^{\leq 0}) = U_A^{L,\leq 0}.$$
It follows that $U_A$ coincides with the $A$-form of $U$ considered in De Concini-Procesi [4]. In particular, we have the following.

**Proposition 3.6.** (i) $U_A^0, U_A^+, U_A^-, U_A^\pm, U_A^\pm_0, U_A$ are $A$-subalgebras of $U$.

(ii) $U_A^0, U_A^\pm_0, U_A^\pm, U_A$ are Hopf algebras over $A$.

Let $i: U_A \to U_A^L$ be the inclusion. We denote by

$$\sigma_A : U_A \times V_A \to A,$$

the bilinear form induced by $\sigma : U \times V \to F$.

We set

$$C_A = \{ \varphi \in C \mid \langle \varphi, U_A^L \rangle \subset A \},$$

$$D_A = C_A \otimes_A U_A \subset D.$$

Then $C_A$ is a Hopf algebra over $A$ as well as a $U_A^L$-bimodule, and $D_A$ is an $A$-subalgebra of $D$. It easily follows that

$$\left( \bigoplus_{\lambda \in \Lambda} \mathbf{F} \chi_\lambda \right) \cap \text{Hom}_A(U_A^{L,0}, A) = \bigoplus_{\lambda \in \Lambda} A \chi_\lambda.$$

Hence by (3.24) we have

$$C_A = (U_A^{L,-})^* \otimes \left( \bigoplus_{\lambda \in \Lambda} A \chi_\lambda \right) \otimes (U_A^{L,+})^* \cap C \subset \text{Hom}_A(U_A^L, A),$$

where $(U_A^{L,\pm})^* = \text{Hom}_A(U_A^{L,\pm}, A) \cap (U^{\pm})^*$.

**3.7. Specialization.** For $z \in C^\times$ set

$$A_z = \{ f/g \mid f, g \in C[q^{\pm 1}/|A^\times/Q], g(z) \neq 0 \} \subset F,$$

and define an algebra homomorphism

$$\pi_z : A_z \to C$$

by $\pi_z (q^{1/|A^\times/Q}) = z$. We set

$$U^L_z = C \otimes_{A_z} U^L_{A_z}, \quad V_z = C \otimes_{A_z} V_{A_z}, \quad U_z = C \otimes_{A_z} U_{A_z},$$

$$C_z = C \otimes_{A_z} C_{A_z}, \quad D_z = C \otimes_{A_z} D_{A_z},$$

with respect to $\pi_z$. Then $U^L_z, U_z, C_z$ are Hopf algebras over $C$, and $V_z, D_z$ are $C$-algebras.

We denote by

$$\pi_z^U : U^L_{A_z} \to U^L_z, \quad \pi_z^V : V_{A_z} \to V_z, \quad \pi_z^U : U_{A_z} \to U_z,$$
the natural homomorphisms. We also define $U^L_{\pm}, U^L_0, U^L_{\pm,0}, V^L_{\pm,0}, V^L_{\pm}, V^L_0, V^L_{\pm,0}$, $V^\leq_0, U^\pm_0, U^\pm_0, U^\pm_0$ similarly. The bilinear form $\sigma_{z}: U_z \times V_z \rightarrow A_z$ induces a bilinear form

\begin{equation}
\sigma_{z} : U_z \times V_z \rightarrow C.
\end{equation}

Set

\[ J_z = \{ v \in V_z \mid \sigma_z(U_z,v) = \{0\}\}, \quad J_z^n = J_z \cap V^n_z. \]

**Lemma 3.7.** $J_z$ is a two-sided ideal of $V_z$, and we have $J_z = V^+_z V^+_z J^n_z$. In particular, we have $J_z \cap V^\leq_0 = V^+_z J^n_z$.

**Proof.** By Proposition 3.4 $J_z$ is a two-sided ideal. Set $V'_z = V_z/V^+_z V^+_z J^n_z$. Since the multiplication of $V_z$ induces an algebra $V_z \simeq V^+_z \otimes V^+_z$, we have

\[ V'_z \simeq (V^-_z \otimes V^+_z \otimes V^+_z)/(V^-_z \otimes V^+_z \otimes J^n_z) \simeq V^+_z \otimes V^+_z \otimes (V^n_z/J^n_z). \]

Let $\sigma'_z : U_z \times V'_z \rightarrow C$ be the bilinear form induced by $\sigma_z$. Then we see easily from the definition of $\sigma$ and Proposition 3.2 that $\{ v \in V'_z \mid \sigma'_z(U_z,v) = \{0\}\} = \{0\}$. Hence $J_z = V^-_z V^+_z J^n_z$.

We define an algebra $\overline{V}_z$ by $\overline{V}_z = V_z/J_z$, and denote by $\overline{\sigma}_z : U_z \times \overline{V}_z \rightarrow C$ the canonical homomorphism. Let $\overline{\sigma}_z : U_z \times \overline{V}_z \rightarrow C$ be the bilinear form induced by (3.50). Denote the images of $V^0_z, V^\pm_0, V^\leq_0, V^\leq_0$ under $\overline{V}_z \rightarrow \overline{V}_z$ by $\overline{V}^0_z, \overline{V}^\pm_0, \overline{V}^\leq_0, \overline{V}^\leq_0$ respectively. Then the multiplication of $\overline{V}_z$ induces isomorphisms

\[ \overline{V}_z \simeq \overline{V}^+_z \otimes \overline{V}^+_z \otimes \overline{V}^0_z, \]

\[ \overline{V}^\leq_0 \simeq \overline{V}^+_z \otimes \overline{V}^0_z, \quad \overline{V}^\leq_0 \simeq \overline{V}^-_z \otimes \overline{V}^0_z. \]

Let $\lambda \in A$. By abuse of notation we also denote by $\chi_\lambda : U_z^{L,0} \rightarrow C$ the algebra homomorphism induced by $\chi_\lambda : U \rightarrow F$. We see easily the following

**Lemma 3.8.** $\{ \chi_\lambda \mid \lambda \in A \}$ is a linearly independent subset of $(U_z^{L,0})^*$. 

**Lemma 3.9.** The bilinear form $\overline{\sigma}_z$ is perfect in the sense that

\begin{align}
\tag{3.51}
& u \in U_z, \quad \overline{\sigma}_z(u, \overline{V}_z) = \{0\} \implies u = 0, \\
\tag{3.52}
& v \in \overline{V}_z, \quad \overline{\sigma}_z(U_z, v) = \{0\} \implies v = 0.
\end{align}
PROOF. (3.52) is clear from the definition. We see easily from the definition of \( \sigma \) and Proposition 3.2 that the proof of (3.51) is reduced to showing
\[
u \in U^0_z, \quad \sigma_z(\nu, V^0_z) = \{0\} \implies \nu = 0.
\]
This follows from Lemma 3.8 in view of
\[
U^0_z = \bigoplus_{\lambda \in \Lambda} C K_{\lambda}, \quad V^0_z \cong U^L_z.
\]
□

Set
\[
I^0_z = j^0_0(J^0_z) \subseteq U^L_z,
\]
\[
I^\geq_0 = U^L_z + I^0_z \subseteq U^L_z, \quad I^{\leq}_0 = U^{L,\leq}_z - I^0_z \subseteq U^{L,\leq}_z, \quad I^\leq_0 = U^{L,\leq}_z - U^{L,\geq}_z + I^0_z \subseteq U^L_z.
\]
The we have
\[
I^0_z = \{\nu \in U^L_z \mid \chi_{\lambda}(\nu) = 0 (\lambda \in \Lambda)\}.
\]

**LEMMA 3.10.** \( I^0_z, I^\geq_0, I^{\leq}_0, I^\leq_0 \) are Hopf ideals of \( U^L_z \) respectively.

**PROOF.** From (3.53) we see easily that \( I^0_z \) is a Hopf ideal of \( U^L_z \). It remains to show \( I^0_z U^{L,\pm}_z \subseteq U^{L,\pm}_z I^0_z \). Using \( J^\geq_0, J^{\leq}_0 \) we see that this is equivalent to \( J^0_z V^\pm_z \subseteq V^\pm_z J^0_z \). This follows from Lemma 3.7. □

We define a Hopf algebra \( \overline{U}^L_z \) by \( \overline{U}^L_z = U^L_z / I^\leq_0 \), and denote by \( j^\leq_0 : U^L_z \to \overline{U}^L_z \) the canonical homomorphism. Denote the images of \( U^L_0, U^{L,\pm}_z, U^{L,\leq}_z, U^{L,\geq}_z \) under \( U^L_z \to \overline{U}^L_z \) by \( \overline{U}^L_0, \overline{U}^{L,\pm}_z, \overline{U}^{L,\leq}_z, \overline{U}^{L,\geq}_z \) respectively. We also denote by
\[
j^\geq_0 : \overline{U}^{\geq}_z \to \overline{U}^{\geq}_z, \quad j^{\leq}_0 : \overline{U}^{\leq}_z \to \overline{U}^{\leq}_z
\]
the algebra isomorphisms induced by \( j^\geq_0 \) and \( j^{\leq}_0 \).

By (3.49) and Lemma 3.8 we have
\[
C_z \subset (U^{L,\pm}_z)^\star \left( \bigoplus_{\lambda \in \Lambda} C K_{\lambda} \right) \otimes (U^{L,\pm}_z)^\star \subset (U^L_z)^\star,
\]
where \( (U^{L,\pm}_z)^\star = C \otimes_A (U^{L,\pm}_z)^\star \subset \text{Hom}_C(U^{L,\pm}_z, C) \). Hence the natural paring \( (, ) : C_z \times U^L_z \to C \) descends to
\[
(, ) : C_z \times \overline{U}^L_z \to C,
\]
by which the canonical map \( C_z \rightarrow (UL_z)^* \) is injective. Moreover, \( C_z \) turns out to be a \( UL_z \)-bimodule.

### 3.8. Specialization to 1.

For an algebraic groups \( S \) over \( C \) with Lie algebra \( s \) we will identify the coordinate algebra \( C[S] \) of \( S \) with a subspace of the dual space \( U(s)^* \) of the enveloping algebra \( U(s) \) by the canonical Hopf paring

\[
(\cdot, \cdot) : C[S] \otimes U(s) \rightarrow C
\]
given by

\[
(\varphi, u) = (L_u(\varphi))(1) \quad (\varphi \in C[S], \, u \in U(s)).
\]

Here, \( U(s) \ni u \mapsto L_u \in \text{End}_C(C[S]) \) is the algebra homomorphism given by

\[
(L_u(\varphi))(g) = \frac{d}{dt}\varphi(g \exp(ta))|_{t=0} \quad (a \in g, \, g \in S, \, \varphi \in C[S]).
\]

We see easily that \( J_1 \) is generated by the elements \( \pi_{V_1}(Z_\lambda) - 1 \in V_1 \) for \( \lambda \in \Lambda \). From this we see easily the following.

**Lemma 3.11.** (i) We have an isomorphism \( V_1 \cong U(\mathfrak{t}) \) of algebras satisfying

\[
\pi_{V_1}(X_i) \leftrightarrow x_i, \quad \pi_{V_1}(Y_i) \leftrightarrow y_i,
\]

\[
\pi_{V_1}
\begin{bmatrix}
Z_i \\
m
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
t_i \\
m
\end{bmatrix}
:= t_i(t_i - 1) \cdots (t_i - m + 1)/m!.
\]

(ii) We have an isomorphism \( UL_1 \cong U(\mathfrak{g}) \) of Hopf algebras satisfying

\[
\pi_{UL_1}(E_i) \leftrightarrow e_i, \quad \pi_{UL_1}(F_i) \leftrightarrow f_i,
\]

\[
\pi_{UL_1}
\begin{bmatrix}
K_i \\
m
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
h_i \\
m
\end{bmatrix}
:= h_i(h_i - 1) \cdots (h_i - m + 1)/m!.
\]

In the rest of this paper we will occasionally identify \( V_1 \) and \( UL_1 \) with \( U(\mathfrak{t}) \) and \( U(\mathfrak{g}) \) respectively.

From the identification \( UL_1 = U(\mathfrak{g}) \) we have the following.

**Lemma 3.12.** The canonical paring

\[
(\cdot, \cdot) : C_1 \times UL_1 \rightarrow C
\]

induces an isomorphism

\[
C_1 \cong C[G] \subset U(\mathfrak{g})^* \cong (UL_1)^*
\]
of Hopf algebras.
In [4] De Concini-Procesi proved an isomorphism

\[(3.57) \quad U_1 \cong \mathbb{C}[K] \]

of Poisson Hopf algebras. They established (3.57) by giving a correspondence between generators of both sides and proving the compatibility after a lengthy calculation. Later Gavarini [6] gave a more natural approach to the isomorphism (3.57) using the Drinfeld paring. Namely we have the following.

**Proposition 3.13 (Gavarini [6]).** The bilinear form \(\sigma_1 : U_1 \times \overline{V}_1 \to \mathbb{C}\) induces a Hopf algebra isomorphism

\[(3.58) \quad \Upsilon : U_1 \to \mathbb{C}[K] \subset U(\mathfrak{k})^* \cong \overline{V}_1.\]

The enveloping algebra \(U(\mathfrak{k}^\pm)\) has the direct sum decomposition

\[U(\mathfrak{k}^\pm) = \bigoplus_{\beta \in \mathbb{Q}^+} U(\mathfrak{k}^\pm)_\beta,\]

where

\[U(\mathfrak{k}^\pm)_\beta = \{ x \in U(\mathfrak{k}^\pm) \mid [(h, -h), x] = \beta(h)x \ (h \in h) \}\]

for \(\beta \in \mathbb{Q}^+\) (note that we have an isomorphism \(h \ni h \leftrightarrow (h, -h) \in \mathfrak{h}_0\)). Then we have

\[\mathbb{C}[\mathfrak{k}^\pm] = \bigoplus_{\beta \in \mathbb{Q}^+} (U(\mathfrak{k}^\pm)_\beta)^* \subset U(\mathfrak{k}^\pm)^*.\]

Moreover, we have

\[\mathbb{C}[\mathfrak{k}^0] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}\hat{\chi}_\lambda \subset U(\mathfrak{k}_0)^*,\]

where \(\hat{\chi}_\lambda : U(\mathfrak{k}_0) \to \mathbb{C}\) is the algebra homomorphism given by \(\hat{\chi}_\lambda(h, -h) = \lambda(h) \) \((h \in h)\).

The isomorphism

\[K^+ \times K^- \times \mathbb{K}^0 \cong K \quad ((g_+, g_-, g_0) \leftrightarrow g_+ g_0)\]

of algebraic varieties induced by the product of the group \(K\) gives an identification

\[(3.59) \quad \mathbb{C}[K^+] \otimes \mathbb{C}[K^-] \otimes \mathbb{C}[K^0] \cong \mathbb{C}[K]\]

of vector spaces. On the other hand the multiplication of the algebra \(U(\mathfrak{k})\) induces an identification

\[U(\mathfrak{k}^+) \otimes U(\mathfrak{k}^-) \otimes U(\mathfrak{k}^0) \cong U(\mathfrak{k}).\]

Then the canonical embedding \(\mathbb{C}[K] \subset U(\mathfrak{k})^*\) is given by

\[\mathbb{C}[K] \cong \mathbb{C}[K^+] \otimes \mathbb{C}[K^-] \otimes \mathbb{C}[K^0] \subset U(\mathfrak{k}^+)^* \otimes U(\mathfrak{k}^-)^* \otimes U(\mathfrak{k}^0)^*.\]
\[ \subset (U(t^+) \otimes U(t^-) \otimes U(t^0))^* = U(t)^*. \]

For \( i \in I \) we define \( a_i \in C[K^-] \subset U(t^-)^* \), \( b_i \in C[K^+] \subset U(t^+)^* \) by

\[ \langle a_i, U(t^-)_\beta \rangle = 0 \ (\beta \neq \alpha_i), \quad \langle a_i, y_i \rangle = -1, \]

\[ \langle b_i, U(t^+)_\beta \rangle = 0 \ (\beta \neq \alpha_i), \quad \langle b_i, x_i \rangle = 1. \]

We identify \( C[K^\pm], C[K^0] \) with subalgebras of \( C[K] \) via (3.59), and regard \( a_i, b_i, \hat{\chi}_\lambda \ (i \in I, \lambda \in \Lambda) \) as elements of \( C[K] \). By the above argument we see easily the following.

**Lemma 3.14.** Under the identification (3.58) we have

\[ \pi U_1(A_i) \leftrightarrow a_i, \pi U_1(B_i) \leftrightarrow b_i, \pi U_1(K_\lambda) \leftrightarrow \hat{\chi}_\lambda \ (i \in I, \lambda \in \Lambda). \]

Let \( \iota_1 : U_1 \to UL_1 \) be the homomorphism induced by the inclusion \( \iota : U_{A_1} \to UL_{A_1} \). By Lemma 3.5 we see easily the following.

**Lemma 3.15.** For \( x \in U_1 \) we have \( \iota_1(x) = \epsilon(x) 1 \).

From this we obtain the following easily.

**Lemma 3.16.** \( D_1 \) is a commutative algebra. In particular, it is identified as an algebra with the coordinate algebra \( C[G] \otimes C[K] \) of \( G \times K \).

### 3.9. Specialization to roots of 1

From now on, we fix an integer \( \ell > 1 \) satisfying

(a) \( \ell \) is odd,

(b) \( \ell \) is prime to 3 if \( g \) is of type \( G_2 \),

(c) \( \ell \) is prime to \( |\Lambda/Q| \),

and a primitive \( \ell \)-th root \( \zeta \in C \) of 1. Note that \( \pi_\zeta : A_\zeta \to C \) sends \( q \) to \( \zeta^{1/|\Lambda/Q|} \), which is also a primitive \( \ell \)-th root of 1 by our assumption (c).

**Remark 3.17.** Denote by \( U^{DK}_{C[q^{\pm 1/|\Lambda/Q|}]} \) the De Concini-Kac \( C[q^{\pm 1/|\Lambda/Q|}] \)-form of \( U \) (see [2]). Namely \( U^{DK}_{C[q^{\pm 1/|\Lambda/Q|}]} \) is the \( C[q^{\pm 1/|\Lambda/Q|}] \)-subalgebra of \( U \) generated by \( \{K_\lambda, E_i, F_i \mid \lambda \in \Lambda, i \in I\} \). Then we have \( U_\zeta \simeq C \otimes_{C[q^{\pm 1/|\Lambda/Q|}]} U^{DK}_{C[q^{\pm 1/|\Lambda/Q|}]} \) with respect to \( q^{1/|\Lambda/Q|} \mapsto \zeta \).

We denote by \( \hat{\xi} : U^{L}_{\zeta} \to UL_1 \) Lusztig’s Frobenius morphism (see [9]). Namely, \( \hat{\xi} \) is an algebra homomorphism given by

\[ \hat{\xi}(\pi^{UL}_{\zeta}(E_i^{(n)})) = \begin{cases} \pi^{UL}_{1}(E_i^{(n/\ell)}) & (\ell | n) \\ 0 & (\ell \nmid n), \end{cases} \]

(3.60)

\[ \hat{\xi}(\pi^{UL}_{\zeta}(F_i^{(n)})) = \begin{cases} \pi^{UL}_{1}(F_i^{(n/\ell)}) & (\ell | n) \\ 0 & (\ell \nmid n), \end{cases} \]

(3.61)
(3.62) \[ \tilde{\xi} \left( \pi^U_L \left( \left[ K_i \right] \right) \right) = \begin{cases} \pi^U_L \left( \left[ K_i m/\ell \right] \right) & (\ell | m) \\ 0 & (\ell \nmid m) \end{cases}. \]

(3.63) \[ \tilde{\xi}(\pi^U_L(K_\lambda)) = \pi^U_L(K_\lambda) \quad (\lambda \in \Lambda). \]

It is a Hopf algebra homomorphism. Moreover, for any \( \beta \in \Delta^+ \) we have

(3.64) \[ \tilde{\xi}(\pi^U_L(E^{(\beta)}_\alpha)) = \begin{cases} \pi^U_L \left( E^{(\beta)(n/\ell)}_\alpha \right) & (\ell | n) \\ 0 & (\ell \nmid n) \end{cases}, \]

(3.65) \[ \tilde{\xi}(\pi^U_L(F^{(\beta)}_\alpha)) = \begin{cases} \pi^U_L \left( F^{(\beta)(n/\ell)}_\alpha \right) & (\ell | n) \\ 0 & (\ell \nmid n). \end{cases} \]

**Lemma 3.18.** We have \( \tilde{\xi}(I_\zeta) \subset I_1 \).

**Proof.** It is sufficient to show \( \tilde{\xi}(I^0_\zeta) \subset I^0_1 \). For \( \zeta \in \mathbb{C}^\times \), \( m = (m_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I \), and \( \nu \in A_0 \) set

\[ K_{m,\nu}(\zeta) = \pi^U_L \left( K_\nu \prod_{i \in I} \left[ K_i \right] m_i \right) \in U^L_\zeta \cdot \]

Any element \( u \) of \( U^L_{\zeta,0} \) is uniquely written as a finite sum

\[ u = \sum_{m,\nu} c_{m,\nu} K_{m,\nu}(\zeta) \quad (c_{m,\nu} \in \mathbb{C}). \]

Then we have \( u \in I^0_\zeta \) if and only if

\[ \sum_{m,\nu} c_{m,\nu} q^{(\lambda,\nu)} \left[ \left( \lambda, \alpha_i^\vee \right) m_i \right] \bigg|_{q^{1/|A_\zeta|}=\zeta} = 0 \quad (\forall \lambda \in \Lambda). \]

Hence it is sufficient to show that

(3.66) \[ \sum_{m,\nu} c_{m,\nu} q^{(\lambda,\nu)} \left[ \left( \lambda, \alpha_i^\vee \right) m_i \right] \bigg|_{q^{1/|A_\zeta|}=\zeta} = 0 \quad (\forall \lambda \in \Lambda) \]

implies

(3.67) \[ \sum_{m,\nu} c_{m,\nu} q^{(\mu,\nu)} \left[ \left( \mu, \alpha_i^\vee \right) m_i \right] = 0 \quad (\forall \mu \in \Lambda). \]

Indeed (3.67) follows by setting \( \lambda = \ell \mu \) in (3.66). \[ \square \]
We denote by
\[ \xi : \overline{U}_\zeta \to \overline{U}_1 (= U(g)) \]
the Hopf algebra homomorphism induced by \( \tilde{\xi} \). By Lusztig [9] we have the following.

**Proposition 3.19.** There exists a unique linear map
\[ \iota \xi : C_1 (= C[G]) \to C_\zeta \]
satisfying
\[ \langle \iota \xi(\varphi), v \rangle = \langle \varphi, \xi(v) \rangle \quad (\varphi \in C_1, \ v \in \overline{U}_\zeta). \]
It is an injective Hopf algebra homomorphism whose image is contained in the center of \( C_\zeta \).

**Lemma 3.20.** There exists an algebra homomorphism
\[ \eta : \overline{V}_\zeta \to \overline{V}_1 \]
such that
\[ \eta(v) = (j \geq 0)^{-1}(\xi(j \geq 0 \zeta (v))) (v \in \overline{V}_\zeta^0), \]
\[ \eta(v) = (j \leq 0)^{-1}(\xi(j \leq 0 \zeta (v))) (v \in \overline{V}_\zeta^0). \]

**Proof.** It is sufficient to show that the linear map \( \eta : \overline{V}_\zeta \to \overline{V}_1 \) defined by
\[ \eta(v_- v_\geq 0) = (j \leq 0)^{-1}(\xi(j \leq 0 \zeta (v_-))) (j \geq 0)^{-1}(\xi(j \geq 0 \zeta (v_\geq 0))) \]
for \( v_- \in \overline{V}_\zeta^-, v_\geq 0 \in \overline{V}_\zeta^0 \) is an algebra homomorphism. This follows easily from 
\[ [\overline{V}_1^+, \overline{V}_\zeta^-] = 0. \]

By Gavarini [6, Theorem 7.9] we have the following.

**Proposition 3.21.** There exists a unique linear map
\[ \iota \eta : U_1 \to U_\zeta \]
satisfying
\[ \overline{\sigma}_\zeta(\iota \eta(u), v) = \overline{\sigma}_1(u, \eta(v)) \quad (u \in U_1, \ v \in \overline{V}_\zeta). \]
It is an injective Hopf algebra homomorphism whose image is contained in the center of \( U_\zeta \).

Moreover, for any \( \beta \in \Delta^+ \) we have
\[ \iota \eta(\pi_1^U(A_\beta)) = \pi_\zeta^U(A^\beta_1), \quad \iota \eta(\pi_1^U(B_\beta)) = \pi_\zeta^U(B^\beta_1). \]
Let \( \iota_\zeta : U_\zeta \to U_\zeta^L \) be the homomorphisms induced by \( \iota : U_\zeta \to U_\zeta^L \). We see easily the following.

**Lemma 3.22.** (i) For \( x \in U_\zeta \) we have \( \xi(\iota_\zeta(x)) = \varepsilon(x)1 \).

(ii) For \( y \in U_1 \) we have \( \iota_\zeta(\iota(y)) = \varepsilon(y)1 \).

**Proposition 3.23.** The image of the linear map
\[ \iota_1 \otimes \iota_1 : D_1 (= C_1 \otimes U_1) \to D_\zeta (= C_\zeta \otimes U_\zeta) \]
is contained in the center of \( D_\zeta \). In particular, \( \iota_1 \otimes \iota_1 \) is an algebra homomorphism.

**Proof.** Let \( \varphi \in C_1 \) and \( x \in U_\zeta \). For \( u \in U_\zeta^L \) we have
\[
\sum_{(x)} (\iota_\zeta(x(0)) \cdot \iota_1(\varphi), u)x(1) = \sum_{(x)} \langle \iota_1(\varphi), u\iota_\zeta(x(0)) \rangle x(1)
\]
and hence \( x'\iota_1(\varphi) = \iota_1(\varphi)x \) in \( D_\zeta \). It follows that \( \iota_1(\varphi) \) is contained in the center for any \( \varphi \in C_1 \).

Let \( y \in U_1 \). For \( \psi \in C_\zeta \) we have
\[
\iota_1(\psi(y)) \psi = \sum_{(y)} (\iota_\zeta(\psi(y(0))), \cdot \psi)'(\psi(y(1))) = \sum_{(y)} \varepsilon(y(0))\psi'(\psi(y(1))) = \psi(\iota_1(\psi(y))),
\]
and hence \( \iota_1(\psi(y)) \) is contained in the center for any \( y \in U_1 \).

**4. Poisson structure arising from quantized enveloping algebras**

The following result is well-known (see [4]).

**Proposition 4.1.** Let \( B \) be a commutative algebra over \( C \). We assume that we are given \( \bar{h} \in B \) such that \( B/\bar{h}B \cong C \).

Let \( \mathcal{R} \) be a (not necessarily commutative) \( B \)-algebra such that \( \mathcal{R} : \mathcal{R} \to \mathcal{R} \) is injective. Then the center \( Z(\mathcal{R}/h\mathcal{R}) \) of \( \mathcal{R}/h\mathcal{R} \) is endowed with a structure of Poisson algebra by
\[
\{ \overline{b_1}, \overline{b_2} \} = \left( \frac{b_1b_2 - b_2b_1}{\bar{h}} \right) \quad (b_1, b_2 \in \mathcal{R}, \overline{b_1}, \overline{b_2} \in Z(\mathcal{R}/h\mathcal{R})).
\]

Assume moreover that \( \mathcal{R} \) is a Hopf algebra and that there exists a Hopf subalgebra \( H \) of \( \mathcal{R}/h\mathcal{R} \) such that \( H \subset Z(\mathcal{R}/h\mathcal{R}) \) and \( \{H, H\} \subset H \). Then \( H \) is naturally a Poisson Hopf algebra.
We will apply this fact to the situation $B = A_\zeta$, $h = \ell(q^\ell - q^{-\ell})$, and $R = C_{A_\zeta}, U_{A_\zeta}, D_{A_\zeta}$. Note that we have $A_\zeta/\ell(q^\ell - q^{-\ell})A_\zeta \cong C$ by

$$\text{Ker } \pi_\zeta = A_\zeta(q^{1/|Q|} - \zeta) = A_\zeta\ell(q^\ell - q^{-\ell}).$$

The cases $R = C_{A_\zeta}, U_{A_\zeta}$ is already known. Namely, we have the following.

**Theorem 4.2** ([3]). The Hopf subalgebra $\text{Im}^t \xi$ of $Z(C_\zeta)$ is closed under the Poisson bracket given in Proposition 4.1. Moreover, the isomorphism $\text{Im}^t \xi \cong C[G]$ is that of Poisson Hopf algebras, where the Poisson Hopf algebra structure of $C[G]$ is the one for $C[\Delta G] \cong C[G]$ attached to the Manin triple $(g \oplus g, \Delta_g, \ell)$.

**Theorem 4.3** ([4], [6]). The Hopf subalgebra $\text{Im}^t \eta$ of $Z(U_\zeta)$ is closed under the Poisson bracket given in Proposition 4.1. Moreover, the isomorphism $\text{Im}^t \eta \cong C[K]$ is that of Poisson Hopf algebras, where the Poisson Hopf algebra structure of $C[K]$ is the one attached to the Manin triple $(g \oplus \hat g, \Delta_{\hat g}, \ell)$.

In the rest of this paper we will deal with the case where $R = D_{A_\zeta}$. The following is the main result of this paper.

**Theorem 4.4.** The subalgebra $\text{Im}^t (\xi \otimes \eta)$ of $Z(D_\zeta)$ is closed under the Poisson bracket given in Proposition 4.1. Moreover, under the identification $\text{Im}^t (\xi \otimes \eta) \cong C[G] \otimes C[K] \cong C[\Delta G] \otimes C[K]$ this Poisson algebra structure coincides with the one attached to the Manin triple $(g \oplus \hat g, \Delta_{\hat g}, \ell)$ as in Proposition 2.3.

Set

$$J = \text{Ker}(\xi \circ \pi_U^{UL_\zeta}) \subset U_{A_\zeta}^L,$$

$$I = \{x \in U_{A_\zeta} \mid \langle x, V_{A_\zeta} \rangle \subset \ell(q^\ell - q^{-\ell})A_\zeta\}.$$

**Lemma 4.5.** Let $h \in \text{Im}^t \xi$ and $\varphi \in \text{Im}^t \eta$. Take $p \in C[G]$ and $\Phi \in U_{A_\zeta}$ such that $h = ^t \xi(p)$ and $\varphi = \pi_U^{UL_\zeta}(\Phi)$ respectively. Assume

$$\Phi \otimes 1 - \sum_{(\Phi)} \Phi^{(1)} \otimes \iota(\Phi^{(0)}) \in \ell(q^\ell - q^{-\ell}) \sum_r \Psi_r \otimes X_r + I \otimes J \subset U_{A_\zeta} \otimes U_{A_\zeta}.$$

with $\Psi_r \in U_{A_\zeta}, X_r \in U_{A_\zeta}^L$. Then we have

$$[h, \varphi] = \sum_r ^t \xi((\xi \circ \pi_U^{UL_\zeta})(X_r) \cdot p) \otimes \pi_U^{UL_\zeta}(\Psi_r)$$

with respect to the Poisson structure of $Z(D_\zeta)$ given in Proposition 4.1.
Take $H \in C_{A_t}$ such that $h = \pi^G_C(H)$. For $u \in U_{A_t}, v \in V_{A_t}$ we see easily that

$$\langle \{h, \varphi\}, \pi^{U_L}_{\zeta}(u) \otimes \pi^{V_L}_{\zeta}(v) \rangle = \pi_{\zeta}\left( (H, u)((\Phi, v)1 - \sum_{(\Phi)} (\Phi(1), v)\ell(\Phi(0))) / \ell(q^\ell - q^{-\ell}) \right).$$

Write

$$\Phi \otimes 1 - \sum_{(\Phi)} \Phi(1) \otimes \xi(\Phi(0)) = \ell(q^\ell - q^{-\ell}) \sum_r \Psi_r \otimes X_r + \sum_s \Xi_s \otimes Y_s,$$

where $\Xi_s \in I, Y_s \in J$. Then we have

$$\langle \{h, \varphi\}, \pi^{U_L}_{\zeta}(u) \otimes \pi^{V_L}_{\zeta}(v) \rangle = \sum_r \pi_{\zeta}(\langle \Psi_r, v \rangle) \pi_{\zeta}(\langle H, uX_r \rangle) + \sum_s \pi_{\zeta}\left( (\Xi_s, v) / \ell(q^\ell - q^{-\ell}) \right) \pi_{\zeta}(\langle H, uY_s \rangle).$$

By $h = t^\xi(p)$ we have

$$\langle h, \pi^{U_L}_{\zeta}(u) \pi^{V_L}_{\zeta}(X_r) \rangle = \langle p, (\xi \circ \pi^{U_L}_{\zeta})(u)(\xi \circ \pi^{V_L}_{\zeta})(X_r) \rangle = \langle (\xi \circ \pi^{U_L}_{\zeta})(X_r) \cdot p, (\xi \circ \pi^{V_L}_{\zeta})(u) \rangle = \ell^\xi((\xi \circ \pi^{U_L}_{\zeta})(X_r) \cdot p, \pi^{V_L}_{\zeta}(u)).$$

Similarly, we have

$$\langle h, \pi^{U_L}_{\zeta}(u) \pi^{V_L}_{\zeta}(Y_s) \rangle = \langle p \cdot (\xi \circ \pi^{V_L}_{\zeta})(u), (\xi \circ \pi^{V_L}_{\zeta})(Y_s) \rangle = 0.$$

Now the assertion is clear. \qed

Now let us show Theorem 4.4. By Theorem 4.2 and Theorem 4.3 it is sufficient to show that for $h \in \text{Im}(\xi), \varphi \in \text{Im}(\eta)$ our Poisson bracket $\{h, \varphi\}$ defined above coincides with the one coming from the Manin triple. In order to avoid confusion we denote by $\{\cdot, \cdot\}'$ the Poisson bracket of $C[G] \otimes C[K]$ coming from the Manin triple. We need to show

$$\{h, \varphi\}' = [h, \varphi]' \quad (\forall h \in \text{Im}(\xi))$$

for any $\varphi \in \text{Im}(\eta)$. If (4.1) holds for $\varphi \in \text{Im}(\eta)$, we have

$$[f, \varphi]' = [f, \varphi]' \quad (\forall f \in \text{Im}(\xi \otimes \eta)).$$
by

\[ \{ h \psi, \varphi \} = [h, \varphi] \psi + h[\psi, \varphi] = [h, \varphi] \psi + h[\psi, \varphi]' = [h \psi, \varphi]' \]

for \( h \in \mathrm{Im}(\ell \xi) \), \( \psi \in \mathrm{Im}(\ell \eta) \). Hence for each \( \varphi \in \mathrm{Im}(\ell \eta) \) (4.1) is equivalent to (4.2). Then it follows from the definition of the Poisson algebra that (4.1) for \( \varphi = \varphi_1, \varphi = \varphi_2 \) imply those for \( \varphi = \varphi_1 \varphi_2, \varphi = \{ \varphi_1, \varphi_2 \} \). Therefore it is sufficient to show (4.1) in the cases where \( \varphi \) belongs to a generator system of the Poisson algebra \( \mathrm{Im}(\ell \eta) \). By \( [4] \) the Poisson algebra \( \mathbb{C}[K] \) is generated by the elements of the form \( \hat{\chi}_\lambda, a_i, b_i \) for \( \lambda \in \Lambda, \, i \in I \). Under the isomorphism \( \mathbb{C}[K] \cong \mathrm{Im}(\ell \eta) \) of Poisson algebras we have

\[
\begin{align*}
\hat{\chi}_\lambda & \leftrightarrow \pi_U \zeta (K_\ell \lambda) \quad (\lambda \in \Lambda), \\
a_i \hat{\chi}_{-a_i} & \leftrightarrow \pi_U ((q_i - q_i^{-1})^\ell E_i K_i^{-\ell}) \quad (i \in I), \\
b_i \hat{\chi}_{-a_i} & \leftrightarrow \pi_U ((q_i - q_i^{-1})^\ell F_i) \quad (i \in I).
\end{align*}
\]

Hence we have only to show (4.1) in the cases

\[ \varphi = \pi_U (K_\ell \lambda), \quad \varphi = \pi_U ((q_i - q_i^{-1})^\ell E_i K_i^{-\ell}), \quad \varphi = \pi_U ((q_i - q_i^{-1})^\ell F_i) \]

for \( \lambda \in \Lambda, \, i \in I \).

For bases \( \{ X_r \} \) and \( \{ Y_r \} \) of \( g \) and \( k \) respectively such that \( \rho((X_r, X_r), Y_s) = \delta_{rs} \) we have

\[ \{ h, \varphi \}' = \sum_r L_{X_r}(h) R_{Y_r}(\varphi) \quad (h \in \mathbb{C}[G], \varphi \in \mathbb{C}[K]). \]

From this we can easily deduce

\[
\begin{align*}
(4.3) & \quad [h, \hat{\chi}_\lambda]' = -\frac{1}{2} L_{H_\lambda}(h) \hat{\chi}_\lambda \quad (\lambda \in \Lambda), \\
(4.4) & \quad [h, a_i \hat{\chi}_{-a_i}]' = -\frac{(a_i, a_i)}{2} L_{E_i}(h) \hat{\chi}_{-a_i} \quad (i \in I), \\
(4.5) & \quad [h, b_i \hat{\chi}_{-a_i}]' = -\frac{(a_i, a_i)}{2} L_{F_i}(h) \hat{\chi}_{-a_i} \quad (i \in I),
\end{align*}
\]

where \( H_\lambda \in \mathfrak{h} \) is given by \( \kappa(H_\lambda, H) = \lambda(H) \quad (H \in \mathfrak{h}) \).

Let us show (4.1) for \( \varphi = \pi_U ((q_i - q_i^{-1})^\ell F_i) \). For \( \Phi = (q_i - q_i^{-1})^\ell F_i \) we have

\[ \Phi \otimes 1 - \sum_{(\Phi)} \Phi(1) \otimes (\Phi(0)) \]

\[ = (q_i - q_i^{-1})^\ell \left( F_i \otimes 1 - \sum_{r=0}^{\ell} q_i^{r(\ell-r)} \ell! \left[ r \right]_q q_i F_i^{r(\ell-r)} K_i^{-\ell} \otimes F_i^{(\ell)} \right) \]

\[ \in (q_i - q_i^{-1})^\ell (F_i \otimes 1 + (F_i \otimes 1 + [\ell]_q K_i^{-\ell} \otimes F_i^{(\ell)})) + \mathcal{I} \otimes \mathcal{J} \]

\[ = \ell (q^\ell - q^{-\ell}) \left( \frac{(q_i - q_i^{-1})^\ell [\ell]_q}{\ell(q^\ell - q^{-\ell})} K_i^{-\ell} \otimes F_i^{(\ell)} \right) + \mathcal{I} \otimes \mathcal{J}. \]
Hence the assertion follows from
\[
\left. \frac{(q_1 - q_1^{-1})^\ell [\ell]}{\ell(q^\ell - q^{-\ell})} \right|_{q^{1/\Lambda}/Q = \zeta} = \frac{(\alpha_i, \alpha_i)}{2},
\]
which is easily checked. The verification of (4.1) for \( \varphi = \pi_{\xi}^{U}(q_1 - q_1^{-1})^\ell E_i K_i^{-\ell} \) is similar and omitted.

Let us finally show (4.1) for \( \varphi = \pi_{\xi}^{U}(K_{\xi}) \). We need to show
\[
\{t_\xi(p), \varphi\} = -\frac{1}{2} t_\xi(L_{H_\lambda}(p)) \otimes \varphi
\]
for \( p \in C[G] \). Take \( H \in C_{\Lambda_\xi} \) such that \( \pi_{\xi}^C(H) = t_\xi(p) \). For \( z \in C^\times \) and \( \nu \in \Lambda \) we set
\[
(C_{\Lambda_\xi})_{\nu} = \{ \varphi \in C_{\Lambda_\xi} \mid u \cdot \varphi = \chi_{\nu}(u) \varphi \ (u \in U_{\Lambda_\xi}(L_0)) \},
\]
\[
(C_{\xi_\nu})_{\nu} = \{ \varphi \in C_{\xi_\nu} \mid u \cdot \varphi = \chi_{\nu}(u) \varphi \ (u \in U_{\xi_\nu}(L_0)) \}.
\]
Then we have
\[
t_\xi(C[G]_{\nu}) \subset (C_{\xi_\nu})_{\ell\nu} = \pi_{\xi}^C((C_{\Lambda_\xi})_{\ell\nu}) \ (\nu \in \Lambda),
\]
and hence we may assume \( p \in C[G]_{\nu} \), and \( H \in (C_{\Lambda_\xi})_{\ell\nu} \). For \( \Phi = K_{\xi_\nu} \) we have
\[
\Phi \otimes 1 - \sum_{(\Phi)} \Phi(1) \otimes t(\Phi(0)) = \pi_{\xi}^C(1) = \pi_{\xi}^C((C_{\Lambda_\xi})_{\ell\nu}) \ (\nu \in \Lambda).
\]
Hence for \( u \in U_{\Lambda_\xi} \), \( v \in V_{\Lambda_\xi} \) we have
\[
\{t_\xi(p), \varphi\}, \pi_{\xi}^{U_\xi}(u) \otimes \pi_V^{V_\xi}(v)
\]
\[
\pi_{\xi} \left( H, u \left( (\Phi, v) \left( 1 - \sum_{(\Phi)} (\Phi(1), v) t(\Phi(0)) \right) \right) \right)/\ell(q^\ell - q^{-\ell})
\]
\[
= -\pi_{\xi}(1) (H, u) /\ell(q^\ell - q^{-\ell})
\]
\[
= -\pi_{\xi}((K_{\lambda_\xi}, v) (H, u) /\ell(q^\ell - q^{-\ell}))
\]
\[
= -\pi_{\xi}(\zeta((K_{\lambda_\xi}, v) (H, u) ) /\ell(q^\ell - q^{-\ell}))
\]
\[
= -\frac{\ell(\lambda, v)}{2 \ell} \pi_{\xi}(v) \left( \pi_V^{V_\xi}(v) \right) \langle t_\xi(p), \pi_{\xi}^{U_\xi}(u) \rangle
\]
\[
= -\frac{1}{2} t_\xi(L_{H_\lambda}(p)) \otimes \varphi, \pi_{\xi}^{U_\xi}(u) \otimes \pi_V^{V_\xi}(v)
\]
The proof of Theorem 4.4 is complete.
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