TOPOLOGICAL ENTROPY OF FREE SEMIGROUP ACTIONS
FOR NONCOMPACT SETS

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Abstract. In this paper we introduce the topological entropy and lower and upper capacity topological entropies of a free semigroup action, which extends the notion of the topological entropy of a free semigroup action defined by Bufetov [10], by using the Carathéodory-Pesin structure (C-P structure). We provide some properties of these notions and give three main results. The first is the relationship between the upper capacity topological entropy of a skew-product transformation and the upper capacity topological entropy of a free semigroup action with respect to arbitrary subset. The second are a lower and an upper estimations of the topological entropy of a free semigroup action by local entropies. The third is that for any free semigroup action with $m$ generators of Lipschitz maps, topological entropy for any subset is upper bounded by the Hausdorff dimension of the subset multiplied by the maximum logarithm of the Lipschitz constants. The results of this paper generalize results of Bufetov [10], Ma et al. [26], and Misiurewicz [27].

1. Introduction. Topological entropy was first introduced by Adler, Konheim and McAndrew [1]. Later, Bowen [7] and Dinaburg [17] defined topological entropy for a uniformly continuous map on metric space and proved that for a compact metric space, they coincide with that defined by Adler et al. Since the topological entropy appeared to be a very useful invariant in ergodic theory and dynamical systems, there were several attempts to find its suitable generalizations for other systems such as groups, pseudogroups, graphs, foliations, nonautonomous dynamical systems and so on [3, 4, 5, 6, 10, 11, 12, 13, 19, 20, 21, 22, 23, 25, 34, 35]. Bowen [8] extended the concept of topological entropy for non-compact sets in a way which resembles the Hausdorff dimension. Pesin [28] gave a new characterization of topological entropy and topological pressure for non-compact sets by Carathéodory structure, which we call Carathéodory-Pesin structure or C-P structure for short. Topological entropy for non-compact sets can be used to investigate multifractal spectra and saturated
sets of dynamical systems, see, for example, [2, 14, 15, 18, 28, 29, 31, 32, 33]. Biš [3] and Bufetov [10] gave the definition of the topological entropy of free semigroup actions on a compact metric space, respectively. Related studies include [6, 11, 12, 13, 22, 23, 25, 30, 34, 35], etc.

Similar to [25], the main purpose of this paper is to introduce the topological entropy, lower and upper capacity topological entropies of a free semigroup action generated by \( m \) generators of continuous maps by using the C-P structure, which extends the notion of the topological entropy of the free semigroup action defined by Bufetov [10]. Some important properties of topological entropy and lower and upper capacity topological entropies are proved. We give some applications of these entropies.

This paper is organized as follows. In section 2, we give some preliminaries. In section 3, by using the C-P structure we give the definitions of the topological entropy, lower and upper capacity topological entropies of a free semigroup action. Several of their properties are provided. In section 4, two equivalent definitions of topological entropy are given. In section 5, with respect to arbitrary subset of a compact metric space, we give a theorem which illustrates the relationship between the upper capacity topological entropy of a free semigroup action and the upper capacity topological entropy of a skew-product transformation. In section 6, we estimate the topological entropy of a free semigroup action by local entropies. In section 7, we prove that for any free semigroup action with \( m \) generators of Lipschitz maps, topological entropy for any subset is upper bounded by the Hausdorff dimension of the subset multiplied by the maximum logarithm of the Lipschitz constants. These give generalizations of results of Bufetov [10], Ma et al. [26] and Misiurewicz [27].

2. Preliminaries.

2.1. Carathéodory-Pesin structure. Let \( X \) and \( S \) be arbitrary sets and \( \mathcal{F} = \{ U_s : s \in S \} \) a collection of subsets in \( X \). Following Pesin [28], we assume that there exist two functions \( \eta, \psi : S \to \mathbb{R}^+ \) satisfying the following conditions:

1. there exists \( s_0 \in S \) such that \( U_{s_0} = \emptyset \); if \( U_s = \emptyset \) then \( \eta(s) = \psi(s) = 0 \); if \( U_s \neq \emptyset \) then \( \eta(s) > 0 \) and \( \psi(s) > 0 \);
2. for any \( \delta > 0 \) one can find \( \varepsilon > 0 \) such that \( \eta(s) \leq \delta \) for any \( s \in S \) with \( \psi(s) \leq \varepsilon \);
3. for any \( \varepsilon > 0 \) there exists a finite or countable subcollection \( \mathcal{G} \subset S \) which covers \( X \) (i.e., \( \cup_{s \in \mathcal{G}} U_s \supseteq X \)) and \( \psi(\mathcal{G}) := \sup \{ \psi(s) : s \in \mathcal{G} \} \leq \varepsilon \).

Let \( \xi : S \to \mathbb{R}^+ \) be a function, we say that the set \( S \), collection of subsets \( \mathcal{F} \), and the set functions \( \xi, \eta, \psi \) satisfying Conditions (1), (2) and (3), introduce the Carathéodory-Pesin structure or C-P structure \( \tau \) on \( X \) and write \( \tau = (S, \mathcal{F}, \xi, \eta, \psi) \).

Given a subset \( Z \) of \( X \), \( \alpha \in \mathbb{R} \), and \( \varepsilon > 0 \), we define

\[
M(Z, \alpha, \varepsilon) := \inf \left\{ \sum_{s \in \mathcal{G}} \xi(s)\eta(s)^\alpha : \mathcal{G} \subset S \text{ covering } Z \text{ with } \psi(\mathcal{G}) \leq \varepsilon \right\},
\]

where the infimum is taken over all finite or countable subcollections \( \mathcal{G} \subset S \) covering \( Z \) with \( \psi(\mathcal{G}) \leq \varepsilon \). By Condition (3) the function \( M(Z, \alpha, \varepsilon) \) is correctly defined. It is non-decreasing as \( \varepsilon \) decreases. Therefore, the following limit exists:

\[
m(Z, \alpha) = \lim_{\varepsilon \to 0} M(Z, \alpha, \varepsilon).
\]
It was shown in [28] that there exists a critical value $\alpha_C \in [-\infty, \infty]$ such that
\[ m(Z, \alpha) = 0, \quad \alpha > \alpha_C, \]
\[ m(Z, \alpha) = \infty, \quad \alpha < \alpha_C. \]

The number $\alpha_C$ is called the Carathéodory-Pesin dimension of the set $Z$.

Now we assume that the following condition holds:
(3') there exists $\epsilon > 0$ such that for any $0 < \epsilon \leq \epsilon$ there exists a finite or countable subcollection $\mathcal{G} \subset \mathcal{S}$ covering $X$ such that $\psi(s) = \epsilon$ for any $s \in \mathcal{G}$.

Given $\alpha \in \mathbb{R}$ and $\epsilon > 0$, for any subset $Z \subset X$, define
\[ R(Z, \alpha, \epsilon) = \inf_{\mathcal{G}} \left\{ \sum_{s \in \mathcal{G}} \left( \xi(s) \eta(s) \right) \alpha \right\}, \]
where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{S}$ covering $Z$ such that $\psi(s) = \epsilon$ for any $s \in \mathcal{G}$. Set
\[ r(Z, \alpha) = \liminf_{\epsilon \to 0} R(Z, \alpha, \epsilon), \quad \mathcal{T}(Z, \alpha) = \limsup_{\epsilon \to 0} R(Z, \alpha, \epsilon). \]

It was shown in [28] that there exists $\underline{\alpha}_C, \overline{\alpha}_C \in \mathbb{R}$ such that
\[ r(Z, \alpha) = \infty, \quad \alpha < \underline{\alpha}_C; \]
\[ \underline{\alpha}(Z, \alpha) = \infty, \quad \alpha < \overline{\alpha}_C; \]
\[ r(Z, \alpha) = 0, \quad \alpha > \overline{\alpha}_C; \]
\[ \underline{\alpha}(Z, \alpha) = 0, \quad \alpha > \overline{\alpha}_C. \]

The numbers $\underline{\alpha}_C$ and $\overline{\alpha}_C$ are called the lower and upper Carathéodory-Pesin capacities of the set $Z$ respectively.

For any $\epsilon > 0$ and subset $Z \subset X$, put
\[ \Lambda(Z, \epsilon) := \inf_{\mathcal{G}} \left\{ \sum_{s \in \mathcal{G}} \xi(s) \right\}, \]
where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{S}$ covering $Z$ such that $\psi(s) = \epsilon$ for any $s \in \mathcal{G}$.

Assume that the function $\eta$ satisfies the following condition:
(4) $\eta(s_1) = \eta(s_2)$ for any $s_1, s_2 \in \mathcal{S}$ satisfying $\psi(s_1) = \psi(s_2)$.

It was shown in [28] that if the function $\eta$ satisfies Condition (4) then for any subset $Z \subset X$,
\[ \underline{\alpha}_C = \liminf_{\epsilon \to 0} \frac{\log \Lambda(Z, \epsilon)}{\log (1/\eta(\epsilon))}, \quad \overline{\alpha}_C = \limsup_{\epsilon \to 0} \frac{\log \Lambda(Z, \epsilon)}{\log (1/\eta(\epsilon))}. \]

Example 1. Let $(X, d)$ be a metric space, $Z \subset X$. Setting $\mathcal{F}$ the collection of the balls $\{B(x, \epsilon) : x \in X, \epsilon > 0\}$, $\mathcal{S} = \{(x, \epsilon) : x \in X, \epsilon > 0\}$, $\xi(x, \epsilon) \equiv 1$, and $\eta(x, \epsilon) = \psi(x, \epsilon) = \text{diam} B(x, \epsilon)$, we can get
\[ M(Z, \alpha, \epsilon) = \inf_{\mathcal{G}} \left\{ \sum_{(x_i, \epsilon_i) \in \mathcal{G}} \left( \text{diam} B(x_i, \epsilon_i) \right)^\alpha \right\}, \]
where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{S}$ covering $Z$ with $\epsilon_i < \epsilon$. Then $\alpha_c$ equals to the Hausdorff dimension of $Z$. Moreover, the lower and upper Carathéodory-Pesin capacities of $Z$ also equal to the lower and upper box dimensions of the set $Z$ respectively.
2.2. Words and sequences. Let $F_m^+$ be the set of all finite words of symbols $0, 1, \ldots, m-1$. For any $w \in F_m^+$, $|w|$ stands for the length of $w$, that is, the digits of symbols in $w$. Obviously, $F_m^+$ with respect to the law of composition is a free semigroup with $m$ generators. We write $w' \leq w$ if there exists a word $w'' \in F_m^*$ such that $w = w''w'$. For $w = i_1 \ldots i_k \in F_m^+$, denote $\overline{w} = i_k \ldots i_1$.

Let $\Sigma_m$ be the set of all two-side infinite sequences of symbols $0, 1, \ldots, m-1$, that is
\[
\Sigma_m = \{ \omega = (\ldots, i_{-1}, i_0, i_1, \ldots) : i_j = 0, 1, \ldots, m-1 \text{ for all integer } j \}.
\]
The metric on $\Sigma_m$ is defined by
\[
d'(\omega, \omega') = 1/2^k, \text{ where } k = \inf\{|n| : i_n \neq \omega_n'\}.
\]
Obviously, $\Sigma_m$ is compact with respect to this metric. The bernoulli shift $\sigma_m : \Sigma_m \to \Sigma_m$ is a homeomorphism of $\Sigma_m$ given by the formula
\[
(\sigma_m \omega)_k = i_{k+1}.
\]
Suppose that $\omega \in \Sigma_m$, $w \in F_m^+$, $a, b$ are integers, and $a \leq b$. We write $\omega|_{[a, b]} = w$ if $w = i_a i_{a+1} \ldots i_{b-1} i_b$.

Let $\Sigma_m^+$ be the set of all one-side infinite sequences of symbols $0, 1, \ldots, m-1$:
\[
\Sigma_m^+ = \{ \omega = (i_0, i_1, \ldots) : i_j = 0, 1, \ldots, m-1 \text{ for all integer } j \}.
\]

2.3. Topological entropy for a free semigroup action in [10]. In this section, we recall the topological entropy for a free semigroup action. Our presentation follows Bufetov [10].

Let $X$ be a compact metric space with metric $d$. Suppose a free semigroup with $m$ generators acts on $X$. Denote the maps corresponding to the generators by $f_0, f_1, \ldots, f_{m-1}$. Assume that these maps are continuous.

Let $w = i_1 i_2 \ldots i_k \in F_m^+$, where $i_j = 0, 1, \ldots, m-1$ for all $j = 1, \ldots, k$. Let $f_w = f_{i_1} f_{i_2} \ldots f_{i_k}, f_w^{-1} = f_{i_k}^{-1} f_{i_{k-1}}^{-1} \ldots f_{i_1}^{-1}$. Obviously, $f_{w'} = f_w f_{w'}$.

For each $w \in F_m^+$, a new metric $d_w$ on $X$ (named Bowen metric) is given by
\[
d_w(x_1, x_2) = \max_{w' \leq \overline{w}} d(f_{w'}(x_1), f_{w'}(x_2)).
\]

Clearly, if $\overline{w} \leq \overline{w'}$, then $d_{w'}(x_1, x_2) \leq d_{w''}(x_1, x_2)$ for all $x_1, x_2 \in X$.

Let $\varepsilon > 0$, a subset $E$ of $X$ is said to be a $(w, \varepsilon, f_0, \ldots, f_{m-1})$-spanning subset if for any $x \in X$, there exists $y \in E$ with $d_w(x, y) < \varepsilon$. The minimal cardinality of a $(w, \varepsilon, f_0, \ldots, f_{m-1})$-spanning subset of $X$ is denoted by $B(w, \varepsilon, f_0, \ldots, f_{m-1})$.

A subset $F$ of $X$ is said to be a $(w, \varepsilon, f_0, \ldots, f_{m-1})$-separated subset if for any $x, y \in F, x \neq y$ implies $d_w(x, y) \geq \varepsilon$. The maximal cardinality of a $(w, \varepsilon, f_0, \ldots, f_{m-1})$-separated subset of $X$ is denoted by $N(w, \varepsilon, f_0, \ldots, f_{m-1})$.

Let
\[
B(n, \varepsilon, f_0, \ldots, f_{m-1}) = \frac{1}{m^n} \sum_{|w| = n} B(w, \varepsilon, f_0, \ldots, f_{m-1}),
\]
\[
N(n, \varepsilon, f_0, \ldots, f_{m-1}) = \frac{1}{m^n} \sum_{|w| = n} N(w, \varepsilon, f_0, \ldots, f_{m-1}).
\]

In [10], Bufetov defined the topological entropy of a free semigroup action by the formula
\[
h(f_0, \ldots, f_{m-1}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log B(n, \varepsilon, f_0, \ldots, f_{m-1})
\]
\[ \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, f_0, \ldots, f_{m-1}). \]

We also denote \( h(f_0, \ldots, f_{m-1}) \) by \( h(G) \) with \( G := \{f_0, \ldots, f_{m-1}\} \).

**Remark 1.** If \( m = 1 \), this definition coincides with the Bowen entropy for the classical dynamical system \((X, f)\) [7, 36].

### 2.4. Local entropy

Let \((X, d)\) be a compact metric space, \( f : X \to X \) a continuous map and \( \mu \) a Borel probability measure on \( X \). For any \( n \geq 1, \varepsilon > 0 \) and \( x \in X \), define the Bowen ball centered at \( x \) by
\[
B_n(x, \varepsilon) = \{ y \in X : d(f^i(x), f^i(y)) < \varepsilon, \text{ for } 0 \leq i \leq n - 1 \}.
\]
Brin and Katok [9] introduced the notion of local entropy for a single map \( f \) in the following way.
\[
h_{\mu}^L(x) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon))
\]
is called the lower local entropy of \( \mu \) at point \( x \in X \) while the quantity
\[
h_{\mu}^U(x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon))
\]
is called the upper local entropy of \( \mu \) at point \( x \in X \).

### 2.5. Hausdorff dimension

Let \((X, d)\) be a metric space. Given a subset \( Z \subset X, s \geq 0 \) and \( \delta > 0 \), we define
\[
H^s_\delta(Z) = \inf \left\{ \sum_{U \in B} (\text{diam}(U))^s : B \text{ is a cover of } Z \text{ and } \text{diam}(U) < \delta \text{ for all } U \in B \right\}.
\]
As \( \delta \) decreases, \( H^s_\delta(Z) \) increases, therefore there exists the limit
\[
H^s(Z) = \lim_{\delta \to 0} H^s_\delta(Z)
\]
called the \( s \)-dimensional Hausdorff measure of \( Z \). The number
\[
HD(Z) = \inf\{s : H^s(Z) = 0\} = \sup\{s : H^s(Z) = \infty\}
\]
is called the Hausdorff dimension of \( Z \).

### 3. Topological entropy, lower and upper capacity topological entropies of a free semigroup and their properties

In this section, we introduce the definitions of topological entropy, lower and upper capacity topological entropies of a free semigroup by using C-P structure and provide some properties of them.

#### 3.1. Topological entropy and lower and upper capacity topological entropies

Let \( X \) be a compact metric space with metric \( d, f_0, f_1, \ldots, f_{m-1} \) continuous transformations from \( X \) to itself. Suppose that a free semigroup with \( m \) generators \( G = \{f_0, f_1, \ldots, f_{m-1}\} \) acts on \( X \).

Considering a finite open cover \( U \) of \( X \), let
\[
S_{n+1}(U) := \{ U = (U_0, U_1, \ldots, U_n) : U \in U^{n+1} \},
\]
where \( U^{n+1} = \prod_{i=1}^{n+1} U \) and \( n \geq 0 \). For any string \( U \in S_{n+1}(U) \), define the length of \( U \) to be \( m(U) := n + 1 \). We put \( S = S(U) = \cup_{n \geq 1} S_n(U) \). For any \( \omega = \)
(i_1, i_2, \ldots, i_n, \ldots) \in \Sigma_m^+, n \geq 1$, and a given string $U = (U_0, U_1, \ldots, U_n) \in \mathcal{S}_{n+1}(\mathcal{U})$, we associate the set
$$X_\omega(U) = \{x \in X : x \in U_0, f_{i_1} \circ \ldots \circ f_{i_n}(x) \in U_j, j = 1, 2, \ldots, n\}.$$ If $w = i_1 i_2 \ldots i_n = \omega|_{[0,n-1]} \in F_m^+$, we also denote $X_\omega(U)$ by $X_w(U)$ for the sake of convenience. Define
$$\mathcal{F} = \{X_\omega(U) : U \in \mathcal{S}(\mathcal{U}) \text{ and } \omega \in \Sigma_m^+\}.$$ and three functions $\xi, \eta, \psi : \mathcal{S} \to \mathbb{R}^+$ as follows
$$\xi(U) \equiv 1, \eta(U) = \exp(-m(U)), \psi(U) = m(U)^{-1}.$$

It is easy to verify that the set $\mathcal{S}$, collection of subsets $\mathcal{F}$, and the functions $\xi, \eta, \psi$ satisfy the Conditions (1), (2) and (3) in section 2.1 and hence they determine a C-P structure $\tau = (\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$ on $X$.

Given $w \in F_m^+, |w| = N, Z \subset X$ and $\alpha \geq 0$, we define
$$M_w(Z, \alpha, U, N) := \inf_{\mathcal{G}_w} \left\{ \sum_{U \in \mathcal{G}_w} \exp(-\alpha m(U)) \right\},$$
where the infimum is taken over all finite or countable collections of strings $\mathcal{G}_w \subset \mathcal{S}(\mathcal{U})$ such that $m(U) \geq N + 1$ for all $U \in \mathcal{G}_w$ and $\mathcal{G}_w$ covers $Z$ (i.e., for any $U \in \mathcal{G}_w$, there exists $\omega_U \in \Sigma_m^+$ such that $\omega_U|_{[0,N-1]} = w$ and $\bigcup_{U \in \mathcal{G}_w} X_{\omega_U}(U) \supset Z$). or equivalently, for any $U \in \mathcal{G}_w$, there is $w_U \in F_m^+$ such that $w \leq w_U$ and $\bigcup_{U \in \mathcal{G}_w} X_{w_U}(U) \supset Z$.

Let
$$M(Z, \alpha, U, N) = \frac{1}{m^N} \sum_{|w|=N} M_w(Z, \alpha, U, N).$$

We can easily verify that the function $M(Z, \alpha, U, N)$ is non-decreasing as $N$ increases. Therefore there exists the limit
$$m(Z, \alpha, U) = \lim_{N \to \infty} M(Z, \alpha, U, N).$$

Furthermore, given $w \in F_m^+$ and $|w| = N$, by the Condition (3') in section 2.1, we can define
$$R_w(Z, \alpha, U, N) := \inf_{\mathcal{G}_w} \left\{ \sum_{U \in \mathcal{G}_w} \exp(-\alpha(N+1)) \right\},$$
where $\Lambda_w(Z,U,N) = \inf \{|\text{card}(\mathcal{G}_w)|\}$, the infimum is taken over all finite or countable collections of strings $\mathcal{G}_w \subset \mathcal{S}(\mathcal{U})$ such that $m(U) = N + 1$ for all $U \in \mathcal{G}_w$ and $\mathcal{G}_w$ covers $Z$ (i.e., for any $U \in \mathcal{G}_w$, there is $w_U \in F_m^+$ such that $w_U = w$ and $\bigcup_{U \in \mathcal{G}_w} X_{w_U}(U) \supset Z$).

Let
$$R(Z, \alpha, U, N) = \frac{1}{m^N} \sum_{|w|=N} R_w(Z, \alpha, U, N),$$
and
$$\Lambda(Z,U,N) = \frac{1}{m^N} \sum_{|w|=N} \Lambda_w(Z,U,N).$$
It is easy to see that
\[ R(Z, \alpha, U, N) = \Lambda(Z, U, N) \exp(-\alpha(N + 1)). \]

We set
\[ \rho(Z, \alpha, U) = \liminf_{N \to \infty} R(Z, \alpha, U, N), \]
\[ \tau(Z, \alpha, U) = \limsup_{N \to \infty} R(Z, \alpha, U, N). \]

The C-P structure \( \tau \) generates the Carathéodory-Pesin dimension of \( Z \) and the lower and upper Carathéodory-Pesin capacities of \( Z \) with respect to \( G \). We denote them by \( h_Z(U, G), \mathcal{C}h_Z(U, G), \) and \( \mathcal{CH}_Z(U, G) \) respectively. We have
\[ h_Z(U, G) = \inf \{ \alpha : m(Z, \alpha, U) = 0 \} = \sup \{ \alpha : m(Z, \alpha, U) = \infty \}, \]
\[ \mathcal{C}h_Z(U, G) = \inf \{ \alpha : \tau(Z, \alpha, U) = 0 \} = \sup \{ \alpha : \tau(Z, \alpha, U) = \infty \}, \]
and
\[ \mathcal{CH}_Z(U, G) = \inf \{ \alpha : \tau(Z, \alpha, U) = 0 \} = \sup \{ \alpha : \tau(Z, \alpha, U) = \infty \}. \]

**Theorem 3.1.** For any set \( Z \subset X \), the following limits exist:
\[ h_Z(G) = \lim_{|U| \to 0} h_Z(U, G), \]
\[ \mathcal{C}h_Z(G) = \lim_{|U| \to 0} \mathcal{C}h_Z(U, G), \]
\[ \mathcal{CH}_Z(G) = \lim_{|U| \to 0} \mathcal{CH}_Z(U, G). \]

**Proof.** We use the analogous method as that of [28]. Let \( V \) be a finite open cover of \( X \) with diameter smaller than the Lebesgue number of \( U \). Then each element \( V \in \mathcal{V} \) is contained in some element \( U(V) \in U \). To any \( V = (V_0, V_1, \ldots, V_k) \in S_k(\mathcal{V}) \subset S(\mathcal{V}) \), we associate the string \( U(V) = \{U(V_0), U(V_1), \ldots, U(V_k)\} \in S_k(U) \subset S(U) \). Let \( w = i_1 i_2 \ldots i_N \in F^*_m. \) If \( \mathcal{G}_w \subset S(\mathcal{V}) \) covers \( Z \), where, for each \( V \in \mathcal{G}_w, m(V) \geq N + 1 \) and there is \( w \in F^*_m \) such that \( w \leq wV \). We denote the word that corresponds to \( U(V) \) by \( w_{U(V)} \) such that \( w_{U(V)} \) is contained in \( S(U) \). Then for every \( \alpha > 0 \) and \( N > 0 \). One can easily see that
\[ M_w(Z, \alpha, U, N) \leq M_w(Z, \alpha, V, N). \]
It follows that
\[ M(Z, \alpha, U, N) \leq M(Z, \alpha, V, N). \]

Moreover,
\[ m(Z, \alpha, U) \leq m(Z, \alpha, V). \]
This implies that
\[ h_Z(U, G) \leq h_Z(\mathcal{V}, G). \]

Since \( X \) is compact it has finite open covers of arbitrarily small diameter. Therefore,
\[ h_Z(U, G) \leq \liminf_{|V| \to 0} h_Z(V, G). \]

Let \( |U| \to 0 \), we have
\[ \limsup_{|U| \to 0} h_Z(U, G) \leq \liminf_{|V| \to 0} h_Z(V, G). \]
This implies the existence of the first limit. The existence of the other two limits can be proved in similar ways. \( \square \)
The quantities \( h_Z(G), \overline{h}_Z(G), \) and \( \overline{h}_Z(G) \) are called the topological entropy and lower and upper capacity topological entropies of \( G \) on the set \( Z \) respectively.

**Remark 2.** (1) It is easy to see that \( h_Z(G) \leq \underline{h}_Z(G) \leq \overline{h}_Z(G) \).

(2) Indeed, let \( f : X \to X \) be a continuous transformation and \( G = \{ f \} \). Then \( h_Z(G) = h_Z(f) \), \( \underline{h}_Z(G) = \underline{h}_Z(f) \), \( \overline{h}_Z(G) = \overline{h}_Z(f) \), for any set \( Z \subset X \), where \( h_Z(f) \), \( \underline{h}_Z(f) \) and \( \overline{h}_Z(f) \) are the topological entropy and lower and upper capacity topological entropies defined by Pesin \[28\]. If \( Z = X \), then \( h(G) = h(f) = \underline{h}_X(f) = \overline{h}_X(f) \), i.e., the classical topological entropy defined by Adler et al \[1\].

### 3.2. Properties of topological entropy and lower and upper capacity topological entropies.

Using the basic properties of the Carathéodory-Pesin dimension \[28\] and definitions, we get the following basic properties of topological entropy and lower and upper capacity topological entropies of a free semigroup action.

**Proposition 1.** (1) \( h_\emptyset(G) \leq 0 \).

(2) \( h_{Z_1}(G) \leq h_{Z_2}(G) \) if \( Z_1 \subset Z_2 \).

(3) \( h_Z(G) = \sup_{i \geq 1} h_{Z_i}(G) \) where \( Z = \cup_{i \geq 1} Z_i \) and \( Z_i \subset X, i = 1, 2, \ldots \).

**Proposition 2.** (1) \( \underline{h}_\emptyset(G) \leq 0, \overline{h}_\emptyset(G) \leq 0 \).

(2) \( \underline{h}_Z(G) \leq \underline{h}_{Z_2}(G) \) and \( \overline{h}_Z(G) \leq \overline{h}_{Z_2}(G) \) if \( Z_1 \subset Z_2 \).

(3) \( \underline{h}_Z(G) \geq \sup_{i \geq 1} \underline{h}_{Z_i}(G) \) and \( \overline{h}_Z(G) \geq \sup_{i \geq 1} \overline{h}_{Z_i}(G) \), where \( Z = \cup_{i \geq 1} Z_i \) and \( Z_i \subset X, i = 1, 2, \ldots \).

(4) If \( g : X \to X \) is a homeomorphism which commutes with \( G \) (i.e., \( f_i \circ g = g \circ f_i, \forall f_i \in G = \{ f_0, f_1, \ldots, f_{m-1} \} \) then

\[
\underline{h}_Z(G) = h_{g(Z)}(G), \quad \overline{h}_Z(G) = \overline{h}_{g(Z)}(G), \quad \underline{h}_Z(G) = \underline{h}_{g(Z)}(G).
\]

Obviously, the function \( \eta \) and \( \psi \) satisfy Condition (4) in section 2.1. Therefore, similar to the Theorem 2.2 in \[28\], we obtain the following lemma.

**Lemma 3.2.** For any open cover \( \mathcal{U} \) of \( X \) and any set \( Z \subset X \), there exist the limits

\[
\underline{h}_Z(\mathcal{U}, G) = \lim \inf_{N \to \infty} \frac{\log \Lambda(Z, \mathcal{U}, N)}{N},
\]

\[
\overline{h}_Z(\mathcal{U}, G) = \lim \sup_{N \to \infty} \frac{\log \Lambda(Z, \mathcal{U}, N)}{N}.
\]

**Proof.** We will prove the first equality; the second one can be proved in a similar fashion. Put

\[
\alpha = \underline{h}_Z(\mathcal{U}, G), \quad \beta = \lim \inf_{N \to \infty} \frac{\log \Lambda(Z, \mathcal{U}, N)}{N}.
\]

Given \( \gamma > 0 \), choose a sequence \( N_i \to \infty \) such that

\[
0 = \varepsilon(Z, \alpha + \gamma, \mathcal{U}) = \lim_{i \to \infty} R(Z, \alpha + \gamma, \mathcal{U}, N_i).
\]

It follows that \( R(Z, \alpha + \gamma, \mathcal{U}, N_i) \leq 1 \) for all sufficiently large \( i \). Therefore, for such numbers \( i \)

\[
\Lambda(Z, \mathcal{U}, N_i) \exp(-(\alpha + \gamma)(N_i + 1)) \leq 1.
\]

Moreover,

\[
\alpha + \gamma \geq \frac{\log \Lambda(Z, \mathcal{U}, N_i)}{N_i + 1}.
\]

Therefore,

\[
\alpha + \gamma \geq \lim \inf_{N \to \infty} \frac{\log \Lambda(Z, \mathcal{U}, N)}{N}.
\]
Hence,
\[ \alpha \geq \beta - \gamma. \]  
(1)

Let us now choose a sequence \( N'_i \) such that
\[ \beta = \lim_{i \to \infty} \log \Lambda(Z, U, N'_i). \]

We have that
\[ \lim_{i \to \infty} R(Z, \alpha - \gamma, U, N'_i) \geq r(Z, \alpha - \gamma) = \infty. \]

This implies that \( R(Z, \alpha - \gamma, U, N'_i) \geq 1 \) for all sufficiently large \( i \). Therefore, for such \( i \),
\[ \Lambda(Z, U, N'_i) \exp(-(\alpha - \gamma)(N'_i + 1)) \geq 1, \]
and hence,
\[ \alpha - \gamma \leq \frac{\log \Lambda(Z, U, N'_i)}{N'_i + 1}. \]

Taking the limit as \( i \to \infty \) we obtain that
\[ \alpha - \gamma \leq \lim_{N \to \infty} \frac{\log \Lambda(Z, U, N)}{N} = \beta, \]
and consequently,
\[ \alpha \leq \beta + \gamma. \]  
(2)

Since \( \gamma \) can be chosen arbitrarily small the inequalities (1) and (2) imply that \( \alpha = \beta \).

**Remark 3.** By the Theorem 3.1 and Lemma 3.2, we can obtain
\[
\text{Ch}_Z(G) = \lim_{|u| \to 0} \liminf_{N \to \infty} \frac{\log \Lambda(Z, U, N)}{N},
\]
\[
\text{Ch}_Z(U, G) = \lim_{|u| \to 0} \limsup_{N \to \infty} \frac{\log \Lambda(Z, U, N)}{N}.
\]

For a free semigroup with \( m \) generators acting on \( X \), denote the maps corresponding to the generators by \( G = \{ f_0, f_1, \ldots, f_{m-1} \} \), a set \( Z \subset X \) is called \( G \)-invariant if \( f_i^{-1}(Z) = Z \) for all \( f_i \in G \). For invariant sets, similar to the lower and upper capacity topological entropies of a single map [28], we have the following theorems.

**Theorem 3.3.** For any \( G \)-invariant set \( Z \subset X \), we have
\[ \text{Ch}_Z(G) = \text{Ch}_Z(U, G); \]
moreover, for any open cover \( U \) of \( X \), we have
\[ \text{Ch}_Z(U, G) = \text{Ch}_Z(U, G). \]

**Proof.** Let \( Z \subset X \) be a \( G \)-invariant set. For any \( w^{(1)}, w^{(2)} \in F^+_m \) where \( |w^{(1)}| = p \) and \( |w^{(2)}| = q \), we choose two collections of strings \( \mathcal{G}_{w^{(1)}} \subset \mathcal{S}_{p+1}(U) \) and \( \mathcal{G}_{w^{(2)}} \subset \mathcal{S}_{q+1}(U) \) which cover \( Z \). Supposing that \( U = (U_0, U_1, \ldots, U_p) \in \mathcal{G}_{w^{(1)}} \) and \( V = (V_0, V_1, \ldots, V_q) \in \mathcal{G}_{w^{(2)}} \), we define
\[ UV = (U_0, U_1, \ldots, U_p, V_0, V_1, \ldots, V_q). \]
For a fixed $i \in \{0, 1, \ldots, m-1\}$, we consider
\[ G_w := \{ UV : U \in G_w^{(1)}, V \in G_w^{(2)} \} \subset S_{p+q+2}(\mathcal{U}), \]
where $w = w^{(1)}w^{(2)}$. Then
\[
X_w(UV) = X_w^{(1)}(U) \cap (f_i \circ f_{w^{(2)}})^{-1}(X_w^{(2)}(V)),
\]
and $m(UV) = m(U) + m(V)$. Since $Z$ is a $G$-invariant set, the collection of strings $G_w$ also covers $Z$. By the definition of $\Lambda_w(Z, U, p + q + 1)$, we have
\[
\Lambda_w(Z, U, p + q + 1) = \frac{1}{m} \sum_{i=0}^{m-1} \Lambda_w(Z, U, p + q + 1) \leq \Lambda_w^{(1)}(Z, U, p) \times \Lambda_w^{(2)}(Z, U, q).
\]
It follows that
\[
\Lambda(Z, U, p + q + 1) \leq \Lambda(Z, U, p) \times \Lambda(Z, U, q).
\]
Let $a_p = \log \Lambda(Z, U, p)$. Note that $\Lambda(Z, U, p) \geq 1$. Therefore, $\inf_{p > 1} \frac{a_p}{p} > -\infty$. The desired result is now a direct consequence of the following Lemma 3.4 (The proof follows the Theorem 4.9 in [36]).

**Lemma 3.4.** Let $\{a_p\}, p = 1, 2, \ldots$ be a sequence of numbers satisfying $\inf_{p > 1} \frac{a_p}{p} > -\infty$ and $a_{p+q+1} \leq a_p + a_q$ for all $p, q > 1$. Then the limit $\lim_{p \to \infty} \frac{a_p}{p}$ exists and coincides with $\inf_{p > 1} \frac{a_p}{p}$.

Next, we discuss the relationship between the topological entropy and upper capacity topological entropy of a free semigroup action generated by $G$ on $Z$ when $Z$ is a compact $G$-invariant set. Given a compact $G$-invariant set $Z \subset X$ and an open cover $\mathcal{U}$ of $X$, we choose any $\alpha > h_Z(\mathcal{U}, G)$, then
\[
m(Z, \alpha, \mathcal{U}) = \lim_{N \to \infty} M(Z, \alpha, \mathcal{U}, N) = 0.
\]
Since $M(Z, \alpha, \mathcal{U}, N)$ is non-decreasing as $N$ increases and non-negative, it follows that $M(Z, \alpha, \mathcal{U}, N) = 0$ for any $N$. Therefore, for any $w \in F_m^{+}$ and $|w| = N$, we have $M_w(Z, \alpha, \mathcal{U}, N) = 0$. For $M_w(Z, \alpha, \mathcal{U}, 2) = 0$, there exists $A_w \subset S(\mathcal{U})$ such that $A_w$ covers $Z$ (i.e., for any $U \in A_w$, there exists $w_U \in F_m^{+}$ such that $|w_U| = m(U) - 1$, $w \leq w_U$ and $\bigcup_{U \in A_w} X_{w_U}(U) \supset Z$) and
\[
Q(Z, \alpha, A_w) = \sum_{U \in A_w} \exp(-\alpha \cdot m(U)) < p < 1, \tag{3}
\]
where $p$ is a constant. Since $Z$ is compact we can choose $A_w$ to be finite and $K \geq 3$ to be a constant and
\[
A_w \subset \bigcup_{m=3}^{K} S_m(\mathcal{U}). \tag{4}
\]
For any $w^{(1)}, w^{(2)} \in F_m^{+}, |w^{(1)}| = |w^{(2)}| = 2$ and $j \in \{0, 1, \ldots, m-1\}$, we can construct
\[
A_w^{(1),jw^{(2)}} = \{ UV : U \in A_w^{(1)} \text{ and } V \in A_w^{(2)} \},
\]
where $A_w^{(1)}, A_w^{(2)}$ satisfy (3), (4). Then
\[
X(UV) = X_{w_U}(U) \cap (f_j \circ f_{w^{(2)}})^{-1}(X_{w_{V}}(V)),
\]
where the word corresponds to $U V$ is $w_{U} j w_{V}$ and $m(UV) = m(U) + m(V) \geq 6$. Since $Z$ is $G$-invariant, then $A_{w(1)jw(2)}$ covers $Z$. It is easy to see that

$$Q(Z, \alpha, A_{w(1)jw(2)}) \leq Q(Z, \alpha, A_{w(1)}) \cdot Q(Z, \alpha, A_{w(2)}) < p^2.$$ 

By mathematical induction, for each $n \in \mathbb{N}$ and $j_1, \ldots, j_{n-1} \in \{0, 1, \ldots, m-1\}$, we can define $A_{w(1)j_1w(2)j_2\ldots w(n-1)j_{n-1}w(n)}$ which covers $Z$ and satisfies

$$Q(Z, \alpha, A_{w(1)j_1w(2)j_2\ldots w(n-1)j_{n-1}w(n)}) < p^n.$$ 

Let $\Gamma_{w(1)j_1w(2)\ldots} = A_{w(1)} \cup A_{w(1)j_1w(2)} \cup \ldots$. Since $Z$ is $G$-invariant, then $\Gamma_{w(1)j_1w(2)\ldots}$ covers $Z$ and

$$Q(Z, \alpha, \Gamma_{w(1)j_1w(2)\ldots}) \leq \sum_{n=1}^{\infty} p^n < \infty.$$ 

Therefore, for any $\omega \in \Sigma_r^+$, there exists $\Gamma_{\omega}$ covering $Z$ and $Q(Z, \alpha, \Gamma_{\omega}) < \infty$. Put

$$\mathcal{F} = \{\Gamma_{\omega} : \omega \in \Sigma_r^+\}.$$ 

**Condition 3.5.** For any $N > 0$ and any $w = i_1i_2\ldots i_N \in F_m^+$, there exists $\Gamma_{\omega} \in \mathcal{F}$ such that for any $U \in \Gamma_{\omega}$, $\overline{w} \leq \overline{wU}$ and $N + 1 \leq m(U) \leq N + K$, where $wU$ is the word corresponds to $U$ and $K$ is a constant as that in (4).

**Theorem 3.6.** Under the condition 3.5, for any compact $G$-invariant set $Z \subset X$, we have

$$h_Z(G) = \overline{h}_Z(G) = \cl h_Z(G);$$

moreover, for any open cover $\mathcal{U}$ of $X$, we have

$$h_Z(\mathcal{U}, G) = \overline{h}_Z(\mathcal{U}, G) = \cl h_Z(\mathcal{U}, G).$$

**Proof.** Under the condition 3.5. For any $N > 0$ and any $w = i_1i_2\ldots i_N \in F_m^+$, there is $\Gamma_{\omega} \in \mathcal{F}$ covering $Z$ such that for any $U \in \Gamma_{\omega}$, the word corresponds to $U$ is $wU$ and $\overline{w} \leq \overline{wU}$. Then for any $x \in Z$, there exists a string $U = (U_0, U_1, \ldots, U_N, \ldots, U_{N+P}) \in \Gamma_{\omega}$ such that $x \in X_{wU}(U)$, where $0 \leq P < K$. Let $U^* = \{U_0, U_1, \ldots, U_N\}$. Then $X_{wU}(U) \subset X_w(U^*)$. If $\Gamma^*_w$ denotes the collection of all substrings $U^*$ constructed above then

$$e^{-\alpha(N+1)} \Lambda_w(Z, U, N) \leq e^{-\alpha(N+1)} \cdot \text{card}(\Gamma^*_w) \leq \max\{1, e^{\alpha K}\} Q(Z, \alpha, \Gamma_{\omega})$$

$$\leq \max\{1, e^{\alpha K}\} \cdot \sum_{n=1}^{\infty} p^n < \infty,$$

Therefore,

$$e^{-\alpha(N+1)} \Lambda(Z, U, N) = \frac{1}{m^N} \sum_{|w|=N} e^{-\alpha(N+1)} \Lambda_w(Z, U, N) < \infty.$$ 

By Lemma 3.2 we obtain that $\alpha > \cl h_Z(\mathcal{U}, G)$, and hence the desired result follows Remark 2(1).

4. Two equivalent definitions of topological entropy in the present paper.

Now, we describe two other approaches to redefine the topological entropy and lower and upper capacity topological entropies of $G = \{f_0, \ldots, f_{m-1}\}$ on any subset of $X$. 

4.1. Definition using Bowen balls. Fix a number $\delta > 0$. Given $w \in F_m^+$ and a point $x \in X$, define the $(w, \delta)$-Bowen ball at $x$ by

$$B_w(x, \delta) = \{ y \in X : d(f_w(x), f_w(y)) \leq \delta, \text{ for } w' \leq w \}.$$ 

Put $S = X \times \mathbb{N}$. We define the collection of subsets

$$\mathcal{F} = \{ B_w(x, \delta) : x \in X, w \in F_m^+, |w| = n \text{ and } n \in \mathbb{N} \}.$$ 

and three functions $\xi, \eta, \psi : S \to \mathbb{R}$ as follows

$$\xi(x, |w|) \equiv 1, \quad \eta(x, |w|) = \exp(-n+1), \quad \psi(x, |w|) = (n+1)^{-1}, \quad |w| = n.$$ 

We can easily verify that the collection of subsets $\mathcal{F}$ and the functions $\xi, \eta, \psi$ satisfy Conditions (1), (2), (3) and (3') in section 2.1. Therefore, they determine a C-P structure $\tau = (S, \mathcal{F}, \xi, \eta, \psi)$ on $X$. Given $w \in F_m^+, |w| = N, Z \subset X$ and $\alpha \geq 0$, we define

$$M_w(Z, \alpha, \delta, N) := \inf \left\{ \sum_{B_w(x, \delta) \in G_w} \exp(-\alpha \cdot (|w'|+1)) \right\},$$

where the infimum is taken over all finite or countable subcollections $G_w \subset \mathcal{F}$ covering $Z$ (i.e., for any $B_w(x, \delta) \in G_w$, $w' \leq w$ and $\bigcup_{B_w(x, \delta) \in G_w} B_w(x, \delta) \supset Z$).

Let

$$\overline{M}(Z, \alpha, \delta, N) = \frac{1}{mN} \sum_{|w| = N} \overline{M}_w(Z, \alpha, \delta, N).$$

We can easily verify that the function $\overline{M}(Z, \alpha, \delta, N)$ is non-decreasing as $N$ increases. Therefore, there exists the limit

$$\overline{m}(Z, \alpha, \delta) = \lim_{N \to \infty} \overline{M}(Z, \alpha, \delta, N).$$

Furthermore, by the Condition (3') in section 2.1, we can define

$$\overline{R}_w(Z, \alpha, \delta, N) = \inf \left\{ \sum_{B_w(x, \delta) \in G_w} \exp(-\alpha \cdot (N+1)) \right\}$$

where $\overline{R}_w(Z, \delta, N) = \inf \{ \text{card}(G_w) \}$, the infimum is taken over all finite or countable subcollections $G_w \subset \mathcal{F}$ covering $Z$ and the words correspond to every ball in $G_w$ are all equal.

Let

$$\overline{R}(Z, \alpha, \delta, N) = \frac{1}{mN} \sum_{|w| = N} \overline{R}_w(Z, \alpha, \delta, N).$$

$$\overline{\Lambda}(Z, \delta, N) = \frac{1}{mN} \sum_{|w| = N} \overline{\Lambda}_w(Z, \delta, N).$$

We set

$$\underline{r}(Z, \alpha, \delta) = \lim \inf_{N \to \infty} \overline{R}(Z, \alpha, \delta, N),$$

$$\underline{r}(Z, \alpha, \delta) = \lim \sup_{N \to \infty} \overline{R}(Z, \alpha, \delta, N).$$
The C-P structure \( \tau \) generates the Carathéodory-Pesin dimension of \( Z \) and the lower and upper Carathéodory-Pesin capacities of \( Z \) with respect to \( G \). We denote them by \( h_Z(\delta, G), \overline{Ch}_Z(\delta, G) \), and \( \overline{Ch}_Z(\delta, G) \) respectively. We have that

\[
\begin{align*}
    h_Z(\delta, G) &= \inf \{ \alpha : m(Z, \alpha, \delta) = 0 \} = \sup \{ \alpha : m(Z, \alpha, \delta) = \infty \}, \\
    \overline{Ch}_Z(\delta, G) &= \inf \{ \alpha : \tau(Z, \alpha, \delta) = 0 \} = \sup \{ \alpha : \tau(Z, \alpha, \delta) = \infty \}, \\
    \overline{Ch}_Z(\delta, G) &= \inf \{ \alpha : \pi(Z, \alpha, \delta) = 0 \} = \sup \{ \alpha : \pi(Z, \alpha, \delta) = \infty \}.
\end{align*}
\]

**Theorem 4.1.** For any set \( Z \subset X \), the following limits exist:

\[
\begin{align*}
    h_Z(G) &= \lim_{\delta \to 0} h_Z(\delta, G), \\
    \overline{Ch}_Z(G) &= \lim_{\delta \to 0} \overline{Ch}_Z(\delta, G), \\
    \overline{Ch}_Z(G) &= \lim_{\delta \to 0} \overline{Ch}_Z(\delta, G).
\end{align*}
\]

**Proof.** Let \( U \) be a finite open cover of \( X \), and \( \delta(U) \) is the Lebesgue number of \( U \). It is easy to see that for every \( x \in X \), if \( x \in X_\omega(U) \) for some \( U \in \mathcal{S}_{k+1}(U) \) and some \( \omega \in \Sigma^+_m \) then

\[
B_{\omega|_{[0,k-1]}}(x, \frac{1}{2}\delta(U)) \subset X_\omega(U) \subset B_{\omega|_{[0,k-1]}}(x, 2\|U\|).
\]

If \( \omega|_{[0,N-1]} = w \), then we have

\[
\overline{M}_w(Z, \alpha, 2\|U\|, N) \leq M_w(Z, \alpha, U, N) \leq \overline{M}_w(Z, \alpha, \frac{1}{2}\delta(U), N).
\]

It follows that

\[
\overline{M}(Z, \alpha, 2\|U\|, N) \leq M(Z, \alpha, U, N) \leq \overline{M}(Z, \alpha, \frac{1}{2}\delta(U), N).
\]

This implies that

\[
m(Z, \alpha, 2\|U\|) \leq m(Z, \alpha, U) \leq m(Z, \alpha, \frac{1}{2}\delta(U)).
\]

Hence

\[
h_Z(2\|U\|, G) \leq h_Z(U, G) \leq h_Z(\frac{1}{2}\delta(U), G).
\]

Let \( \|U\| \to 0 \), then \( \delta(U) \to 0 \) and hence the theorem is proved. The existence of the two other limits can be proved in a similar fashion.

\( \square \)

4.2. **Definition using Bowen’s approach.** Let \((X, d)\) be a compact metric space and \( f : X \to X \) be a continuous map. Bowen [8] used a way which resembles the Hausdorff dimension to construct the topological entropy of \( f \) for non-compact sets. In the following, we give the topological entropy of free semigroup actions for non-compact sets by Bowen’s approach [8] and prove that the new definition is equivalent to the definition of topological entropy in section 3.1. Let \( U \) be a finite open cover of \( X \). For a set \( B \subset X \) we write \( B \prec U \) if \( B \) is contained in some element of \( U \). For any \( \omega = (i_1, i_2, \ldots) \in \Sigma^+_m \). Let \( n_{\omega,U}(B) \) be the largest non-negative integer \( n \) such that \( B \prec U \) and \( f_{i_1} \circ \cdots \circ f_{i_k}(B) \prec U \) for \( k = 1, \ldots, n-1 \). If \( B \notin U \) then \( n_{\omega,U}(B) = 0 \) and if \( f_{i_k} \circ \cdots \circ f_{i_1}(B) \prec U \) for all \( k \) then \( n_{\omega,U}(B) = \infty \). Now we set

\[
D_{\omega,U}(B) = \exp \left( - n_{\omega,U}(B) \right).
\]
Given $Z \subset X$, $\varepsilon \in (0,1)$ and $\alpha \geq 0$, let $N = \lfloor -\log(\varepsilon) \rfloor + 1$, where $\lfloor -\log(\varepsilon) \rfloor$ is the integer portion of $-\log(\varepsilon)$ and $w = i_1 i_2 \ldots i_N \in F_m^+$, we define

$$
\mu_w(Z, \alpha, \mathcal{U}, \varepsilon) := \inf_B \left\{ \sum_{B \in \mathcal{B}} \left( D_{\omega, \mathcal{U}}(B) \right)^{\alpha} \right\} = \inf_B \left\{ \sum_{B \in \mathcal{B}} \exp \left( -\alpha n_{\omega, \mathcal{U}}(B) \right) \right\},
$$

where the infimum is taken over all covers $\mathcal{B}$ of $Z$ such that for any $B \in \mathcal{B}$ there exists $\omega_B \in \Sigma_m^+$ satisfied $\omega_B [[0,N-1]] = w$ and $D_{\omega, \mathcal{U}}(B) < \varepsilon$.

Then we set

$$
\mu(Z, \alpha, \mathcal{U}, \varepsilon) = \frac{1}{m^N} \sum_{|w|=N} \mu_w(Z, \alpha, \mathcal{U}, \varepsilon).
$$

We can easily verify that the function $\mu(Z, \alpha, \mathcal{U}, \varepsilon)$ is non-decreasing as $\varepsilon$ increases. Therefore, there exists the limit

$$
\mu(Z, \alpha, \mathcal{U}) = \lim_{\varepsilon \to 0} \mu(Z, \alpha, \mathcal{U}, \varepsilon).
$$

$\mu(Z, \alpha, \mathcal{U})$ has similar properties as $m(Z, \alpha, \mathcal{U})$, that is, there exists $h'_Z(\mathcal{U}, G)$ such that

$$
h'_Z(\mathcal{U}, G) = \inf \{\alpha : \mu(Z, \alpha, \mathcal{U}) = 0\} = \sup \{\alpha : \mu(Z, \alpha, \mathcal{U}) = \infty\}.
$$

Finally, we set

$$
h'_Z(G) = \sup \{h'_Z(\mathcal{U}, G) : \mathcal{U} \text{ is a finite open cover of } Z\},
$$

and we have

$$
h'_Z(G) = \lim_{|\mathcal{U}| \to 0} h'_Z(\mathcal{U}, G).
$$

**Theorem 4.2.** For any set $Z \subset X$, $h'_Z(G) = h'_Z(G)$.

In order to prove this theorem, we propose the following lemma.

**Lemma 4.3.** Let $\mathcal{U}$ be an open cover of $X$, for any set $Z \subset X$, $0 < \varepsilon < 1$, $N = \lfloor -\log(\varepsilon) \rfloor + 1$, $w = i_1 i_2 \ldots i_N \in F_m^+$ and $\alpha \geq 0$, we have:

$$
\frac{M_w(Z, \alpha, \mathcal{U}, N)}{2} \leq \mu_w(Z, \alpha, \mathcal{U}, \varepsilon) \leq M_w(Z, \alpha, \mathcal{U}, N).
$$

**Proof.** According to the definition of $M_w(Z, \alpha, \mathcal{U}, N)$, we choose collection of strings $\mathcal{G}_w \subset S(\mathcal{U})$ such that $m(\mathcal{U}) \geq N + 1$ for all $\mathcal{U} \in \mathcal{G}_w$ and $\mathcal{G}_w$ covers $Z$. For any $\mathcal{U} = (U_{i_0}, U_{i_1}, \ldots, U_{i_{N-1}}) \in \mathcal{G}_w$, then there exists $\omega_\mathcal{U} \in \Sigma_m^+$ such that $\omega_\mathcal{U} [[0,N-1]] = w$ and we associate the set

$$
X_{\omega_\mathcal{U}}(\mathcal{U}) = U_{i_0} \cap f_{i_1}^{-1} U_{i_1} \cap \ldots \cap (f_{i_{N-1}} \circ \ldots \circ f_{i_1})^{-1} U_{i_{N-1}}.
$$

Moreover, we have

$$
n_{\omega_\mathcal{U}, \mathcal{U}}(X_{\omega_\mathcal{U}}(\mathcal{U})) \geq m(\mathcal{U}) \geq N + 1.
$$

It follows that

$$
\exp \left( -n_{\omega_\mathcal{U}, \mathcal{U}}(X_{\omega_\mathcal{U}}(\mathcal{U})) \right) < \varepsilon.
$$

Therefore, we get a cover $\mathcal{B} = \{ X_{\omega_\mathcal{U}}(\mathcal{U}) : \mathcal{U} \in \mathcal{G}_w \}$ of $Z$ and $D_{\omega, \mathcal{U}}(X_{\omega_\mathcal{U}}(\mathcal{U})) < \varepsilon$ for all $X_{\omega_\mathcal{U}}(\mathcal{U}) \in \mathcal{B}$. For $\alpha \geq 0$, this implies that

$$
\sum_{\mathcal{U} \in \mathcal{G}_w} \exp \left( -\alpha m(\mathcal{U}) \right) \geq \sum_{\mathcal{U} \in \mathcal{G}_w} \exp \left( -\alpha n_{\omega_\mathcal{U}, \mathcal{U}}(X_{\omega_\mathcal{U}}(\mathcal{U})) \right) \geq \inf_{\mathcal{B}} \left\{ \sum_{B \in \mathcal{B}} \exp \left( -\alpha n_{\omega, \mathcal{U}}(B) \right) \right\}.
$$
Moreover, we have
\[
\inf_{\mathcal{G}_w} \left\{ \sum_{U \in \mathcal{G}_w} \exp \left( -\alpha m(U) \right) \right\} \geq \inf_{B \in \mathcal{B}} \left\{ \sum_{B \in \mathcal{B}} \exp \left( -\alpha n_{\omega, U}(B) \right) \right\}.
\]
Hence
\[
M_w(Z, \alpha, \mathcal{U}, N) \geq \mu_w(Z, \alpha, \mathcal{U}, \varepsilon).
\]
On the other hand, for any \( \lambda > \mu_w(Z, \alpha, \mathcal{U}, \varepsilon) \), there is a cover \( \mathcal{B} \) of \( Z \) such that
\[
\sum_{B \in \mathcal{B}} (D_{\omega, U}(B))^{\alpha} = \sum_{B \in \mathcal{B}} \exp \left( -\alpha n_{\omega, U}(B) \right) < \lambda.
\]
For every \( B \in \mathcal{B} \), there exists \( \omega_B \in \Sigma^+_{\mathcal{U}} \) such that \( \omega_B|_{[0, N-1]} = w \) and \( D_{\omega, U}(B) < \varepsilon \). In the following, we discuss two cases of \( n_{\omega, U}(B) \).

**Case 1.** If \( n_{\omega, U}(B) < \infty \). There exists \( U_B = (U_{i_0}, U_{i_1}, \ldots, U_{i_{n_{\omega, U}(B)-1}}) \) such that
\[
B \subset U_{i_0} \cap f_{i_1}^{-1}U_{i_1} \cap \ldots \cap (f_{i_{n_{\omega, U}(B)-1}} \circ \ldots \circ f_{i_1})^{-1}U_{i_{n_{\omega, U}(B)-1}} = X_{\omega_B}(U_B).
\]
Hence
\[
\sum_{n_{\omega, U}(B) < \infty} \exp \left( -\alpha n_{\omega, U}(B) \right) = \sum_{n_{\omega, U}(B) < \infty} \exp \left( -\alpha m(U_B) \right) < \lambda.
\]

**Case 2.** If \( n_{\omega, U}(B) = \infty \). For such set \( B \), they are at most countable. We denote these sets by \( B_1, B_2, \ldots \). For \( B_1 \), there exists \( \omega_{B_1} \in \Sigma^+_m \) and \( U_{B_1} = (U_{i_0}^{(1)}, U_{i_1}^{(1)}, \ldots, U_{i_{m(U_{B_1})-1}}^{(1)}) \in \mathcal{S}(\mathcal{U}) \) such that \( \omega_{B_1}|_{[0, N-1]} = w \) and \( B_1 \subset X_{\omega_{B_1}}(U_{B_1}) \), \( m(U_{B_1}) \geq N + 1 \) and \( \exp \left( -\alpha m(U_{B_1}) \right) < \lambda/2 \); For \( B_2 \), there exists \( \omega_{B_2} \in \Sigma^+_m \) and \( U_{B_2} = (U_{i_0}^{(2)}, U_{i_1}^{(2)}, \ldots, U_{i_{m(U_{B_2})-1}}^{(2)}) \in \mathcal{S}(\mathcal{U}) \) such that \( \omega_{B_2}|_{[0, N-1]} = w \) and \( B_2 \subset X_{\omega_{B_2}}(U_{B_2}) \), \( m(U_{B_2}) \geq N + 1 \) and \( \exp \left( -\alpha m(U_{B_2}) \right) < \lambda/2^2 \);
\[
\vdots
\]
For \( B_n \), there exists \( \omega_{B_n} \in \Sigma^+_m \) and \( U_{B_n} = (U_{i_0}^{(n)}, U_{i_1}^{(n)}, \ldots, U_{i_{m(U_{B_n})-1}}^{(n)}) \in \mathcal{S}(\mathcal{U}) \) such that \( \omega_{B_n}|_{[0, N-1]} = w \) and \( B_n \subset X_{\omega_{B_n}}(U_{B_n}) \), \( m(U_{B_n}) \geq N + 1 \) and \( \exp \left( -\alpha m(U_{B_n}) \right) < \lambda/2^n \);
\[
\vdots
\]
Let \( \mathcal{G}_B = \{ U_B : B \in \mathcal{B} \} \). Then \( \mathcal{G}_B \) is a collection of strings that covers \( Z \) and for any \( U \in \mathcal{G}_B \), we have \( m(U) \geq N + 1 \).
\[
\sum_{U \in \mathcal{G}_B} \exp \left( -\alpha m(U) \right) = \sum_{n_{\omega, U}(B) < \infty} \exp \left( -\alpha m(U_B) \right) + \sum_{n_{\omega, U}(B) = \infty} \exp \left( -\alpha m(U_B) \right)
\]
\[
< \lambda + \sum_{i=1}^{\infty} \frac{\lambda}{2^i} = 2\lambda.
\]
Moreover, we obtain that
\[
M_w(Z, \alpha, \mathcal{U}, N) < 2\lambda,
\]
and consequently,
\[ \frac{M_w(Z, \alpha, U, N)}{2} \leq \mu_w(Z, \alpha, U, \varepsilon). \]

\( \square \)

The proof of Theorem 4.2. According to Lemma 4.3, it follows that
\[ \frac{M(Z, \alpha, U, N)}{2} \leq \mu(Z, \alpha, U, \varepsilon) \leq M(Z, \alpha, U, N), \]

Let \( \varepsilon \to 0 \), then \( N \to \infty \), we have
\[ \frac{m(Z, \alpha, U)}{2} \leq \mu(Z, \alpha, U) \leq m(Z, \alpha, U). \]

By the definitions of \( h_Z(G) \) and \( h'_Z(G) \), we have
\[ h_Z(U, G) = h'_Z(U, G). \]

Letting \( |U| \to 0 \), and the theorem is proved. \( \square \)

Remark 4. If \( G = \{ f \} \), then for any \( Z \subset X \), \( h'_Z(G) \) is equal to the topological entropy of \( f \) on \( Z \) defined by Bowen [8].

5. Relationship between the upper capacity topological entropies of a skew-product transformation and a free semigroup action. Let \( X \) be a compact metric space with metric \( d \), suppose a free semigroup with \( m \) generators acts on \( X \), the generators are continuous transformations \( G = \{ f_0, f_1, \ldots, f_{m-1} \} \) of \( X \).

For any subset \( Z \subset X \), \( w \in F^+_m \) and \( \varepsilon > 0 \), a subset \( E \subset X \) is said to be a \( (w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \)-spanning set of \( Z \), if for any \( x \in Z \), there exists \( y \in E \) such that \( d_w(x, y) < \varepsilon \). Define \( B(w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \) to be the minimum cardinality of any \( (w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \)-spanning sets of \( Z \). A subset \( F \subset Z \) is said to be a \( (w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \)-separated set of \( Z \), if \( x, y \in F \), \( x \neq y \) implies \( d_w(x, y) \geq \varepsilon \). Let \( N(w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \) denotes the maximum cardinality of any \( (w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \)-separated sets of \( Z \).

For any \( n \geq 1 \), let
\[ B(n, \varepsilon, Z, f_0, \ldots, f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} B(w, \varepsilon, Z, f_0, \ldots, f_{m-1}), \]
\[ N(n, \varepsilon, Z, f_0, \ldots, f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} N(w, \varepsilon, Z, f_0, \ldots, f_{m-1}). \]

Obviously,
\[ B(w, \frac{\varepsilon}{2}, Z, f_0, \ldots, f_{m-1}) \geq N(w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \geq B(w, \varepsilon, Z, f_0, \ldots, f_{m-1}), \]

hence,
\[ B(n, \frac{\varepsilon}{2}, Z, f_0, \ldots, f_{m-1}) \geq N(n, \varepsilon, Z, f_0, \ldots, f_{m-1}) \geq B(n, \varepsilon, Z, f_0, \ldots, f_{m-1}). \]
For any \( w \in F_m^+ \), \(|w| = N\), since \( \overline{X}_w(Z, \varepsilon, N) = B(w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \), then \( \overline{X}(Z, \varepsilon, N) = B(n, \varepsilon, Z, f_0, \ldots, f_{m-1}) \). Therefore
\[
\overline{h}_Z(G) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log B(n, \varepsilon, Z, f_0, \ldots, f_{m-1})
= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, Z, f_0, \ldots, f_{m-1}).
\]

**Remark 5.** If \( Z = X \), then \( \overline{h}_X(G) = \overline{h}_X(G) = h(G) \). Thus upper and lower capacity topological entropies of a free semigroup action on \( X \) are as same as the topological entropy of the free semigroup action on \( X \) defined by Bufetov [10].

A skew-product transformation \( F : \Sigma_m \times X \to \Sigma_m \times X \) is defined by the formula
\[
F(\omega, x) = (\sigma_m \omega, f_\omega(x)),
\]
where \( \omega = (\ldots, \omega_{1}, \omega_0, \omega_1, \ldots) \in \Sigma_m \). Here \( f_\omega \) stands for \( f_0 \) if \( \omega_0 = 0 \), and for \( f_1 \) if \( \omega_0 = 1 \), and so on. Let \( \omega = (\ldots, \omega_{1}, \omega_0, \omega_1, \ldots) \in \Sigma_m \), and the metric \( D \) on \( \Sigma_m \times X \) is defined as
\[
D((\omega, x), (\omega', x')) = \max(d(\omega, \omega'), d(x, x')).
\]

For \( F : \Sigma_m \times X \to \Sigma_m \times X \), let \( G = \{F\} \), we can get a C-P structure on \( \Sigma_m \times X \). For any set \( Z \subset X \), from Pesin [28], we call \( \overline{h}_{\Sigma_m \times X}(F) \) the upper capacity topological entropy of \( F \) on the set \( \Sigma_m \times Z \). Our purpose is to find the relationship between the upper capacity topological entropy \( \overline{h}_{\Sigma_m \times X}(F) \) of the skew-product transformation \( F \) and the upper capacity topological entropy \( \overline{h}_Z(G) \) of a free semigroup action generated by \( G = \{f_0, \ldots, f_{m-1}\} \).

**Theorem 5.1.** For any set \( Z \subset X \), then
\[
\overline{h}_{\Sigma_m \times Z}(F) = \log m + \overline{h}_Z(G).
\]

**Remark 6.** Recall that \( h(\sigma_m) = \log m \), where \( h(\sigma_m) \) denotes the topological entropy of \( \sigma_m \) in symbol space \( \Sigma_m \).

To prove this theorem, we first give the following two lemmas. The proofs of these two lemmas are similar to that of Bufetov [10]. Therefore, we omit the proof.

**Lemma 5.2.** For any subset \( Z \) of \( X \), \( n \geq 1 \) and \( 0 < \varepsilon < \frac{1}{2} \), we have
\[
N(n, \varepsilon, \Sigma_n \times Z, F) \geq \sum_{|w| = n} N(w, \varepsilon, Z, f_0, \ldots, f_{m-1}).
\]

**Lemma 5.3.** For any subset \( Z \) of \( X \), \( n \geq 1 \) and \( \varepsilon > 0 \), we have
\[
B(n, \varepsilon, \Sigma_m \times Z, F) \leq K(\varepsilon) \left( \sum_{|w| = n} B(w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \right),
\]
where \( K(\varepsilon) \) is a positive constant that depends only on \( \varepsilon \).

**Proof of Theorem 5.1.** From Lemma 5.2 we have for any subset \( Z \) of \( X \),
\[
N(n, \varepsilon, \Sigma_n \times Z, F) \geq m^n \cdot N(n, \varepsilon, Z, f_0, \ldots, f_{m-1}),
\]
and then obtain that
\[
\overline{h}_{\Sigma_m \times Z}(F) \geq \log m + \overline{h}_Z(G).
\]
From Lemma 5.3 we have
\[
B(n, \varepsilon, \Sigma_m \times Z, F) \leq K(\varepsilon)m^n \cdot N(n, \varepsilon, Z, f_0, \ldots, f_{m-1}),
\]
and hence
\[ \overline{C} h_{\Sigma_m \times Z}(F) \leq \log m + \overline{C} h_Z(G). \]
and the proof is complete. \(\square\)

**Remark 7.** If \( Z = X \), by Theorem 5.1, we have \( \overline{C} h_{\Sigma_m \times X}(F) = \log m + \overline{C} h_X(G) \). Since \( h_{\Sigma_m \times X}(F) = \overline{C} h_{\Sigma_m \times X}(F) \) and \( h(f_0, \ldots, f_{m-1}) = \overline{C} h_X(G) \), then we get \( h_{\Sigma_m \times X}(F) = \log m + h(f_0, \ldots, f_{m-1}) \). Therefore, this gives a generalization of the result of Bufetov [10].

**Problem.** For any subset \( Z \subset X \), is it true that \( h_{\Sigma_m \times X}(F) = \log m + h_Z(G) \)?

6. Some estimates of topological entropies of free semigroup actions.

Given a compact metric space \((X, d)\), a free semigroup with \( m \) generators acts on \( X \), the generators are continuous transformations \( G = \{f_0, f_1, \ldots, f_{m-1}\} \) of \( X \). In this section, we give the concepts of lower and upper local entropies. Moreover, similar as the proof in Ma and Wen [26], we get two estimates.

**Definition 6.1.** Let \( \mu \) be a Borel probability measure on \( X \). Then we call
\[ h_{\mu,G}^L(x) = \lim_{{r \to 0}} \lim_{{n \to \infty}} \frac{1}{n+1} \log \min \{ \mu(B_w(x, r)) \} \]  
(5)
the \( L^+ \) lower local entropy of \( \mu \) at point \( x \) with respect to \( G \), while the quantity
\[ h_{\mu,G}^L(x) = \lim_{{r \to 0}} \lim_{{n \to \infty}} \frac{1}{n+1} \log \max \{ \mu(B_w(x, r)) \} \]  
(6)
is called \( L^- \) lower local entropy of \( \mu \) at point \( x \) with respect to \( G \).

**Remark 8.** If \( G = \{f\} \), then \( h_{\mu,G}^L(x) = h_{\mu,G}^L(x) \), i.e., the lower local entropy for \( f \) defined by Brin and Katok [9]. One can define the \( U^+(U^-) \) upper local entropy by replacing “\( \lim \inf \)” in (5), (6) with “\( \lim \sup \)” respectively.

**Theorem 6.2.** Let \( \mu \) denote a Borel probability measure on \( X \), \( Z \) be a Borel subset of \( X \) and \( s \in (0, \infty) \). If
\[ h_{\mu,G}^L(x) \geq s \text{ for all } x \in Z \text{ and } \mu(Z) > 0 \text{ then } h_Z(G) \geq s. \]

**Proof.** Fix an \( \epsilon > 0 \). For each \( k \geq 1 \), put
\[ Z_k = \left\{ x \in Z : \lim_{{n \to \infty}} \frac{1}{n+1} \log \max \{ \mu(B_w(x, r)) \} > s - \epsilon \text{ for all } r \in (0, 1/k) \right\}. \]
Since \( h_{\mu,G}^L(x) \geq s \) for all \( x \in Z \), the sequence \( \{Z_k\}_{k=1}^\infty \) increases to \( Z \). So by the continuity of the measure, we have
\[ \lim_{{k \to \infty}} \mu(Z_k) = \mu(Z) > 0. \]
Then select an integer \( k_0 \geq 1 \) with \( \mu(Z_{k_0}) > \frac{1}{2} \mu(Z) \). For each \( N \geq 1 \), put
\[ Z_{k_0,N} = \left\{ x \in Z_{k_0} : -\frac{1}{n+1} \log \max \{ \mu(B_w(x, r)) \} > s - \epsilon \text{ for all } n \geq N \text{ and } r \in (0, 1/k_0) \right\}. \]
Since the sequence \( \{Z_{k_0,N}\}_{N=1}^\infty \) increases to \( Z_{k_0} \), we can pick an \( N^* \geq 1 \) such that \( \mu(Z_{k_0,N^*}) > \frac{1}{2} \mu(Z_{k_0}) \). Write \( Z^* = Z_{k_0,N^*} \) and \( r^* = \frac{1}{k_0} \). Then \( \mu(Z^*) > 0 \) and
\[ \max \{ \mu(B_w(x, r)) \} < (s-\epsilon)(n+1) \] for all \( x \in Z^*, 0 < r \leq r^* \) and \( n \geq N^* \),
(7)
For any $N \geq N^*$, take any $w \in F^+_m$, $|w| = N$. Set a cover of $Z^*$

$$F_w = \left\{ B_{\omega_i|_{[0,n_i-1]}}(y_i, \frac{r}{2}) : \omega_i \in \Sigma_m^+ \text{ and } \omega_i|_{[0,N-1]} = w \right\}$$

which satisfies

$$Z^* \cap B_{\omega_i|_{[0,n_i-1]}}(y_i, \frac{r}{2}) \neq \emptyset, n_i \geq N \text{ for all } i \geq 1 \text{ and } 0 < r \leq r^*.$$ 

For each $i$, there exists an $x_i \in Z^* \cap B_{\omega_i|_{[0,n_i-1]}}(y_i, \frac{r}{2})$. By the triangle inequality

$$B_{\omega_i|_{[0,n_i-1]}}(y_i, \frac{r}{2}) \subset B_{\omega_i|_{[0,n_i-1]}}(x_i, r).$$

In combination with (7), we can get

$$\sum_{i \geq 1} \exp \left( - (s - \varepsilon) \cdot (n_i + 1) \right) \geq \sum_{i \geq 1} \mu(B_{\omega_i|_{[0,n_i-1]}}(x_i, r)) \geq \mu(Z^*) > 0.$$ 

Therefore, $\overline{m}(Z^*, s - \varepsilon, r, N) \geq \mu(Z^*) > 0$ for all $N \geq N^*$, and we obtain

$$\overline{m}(Z^*, s - \varepsilon, r, N) = \frac{1}{m^N} \sum_{|w| = N} \overline{m}_w(Z^*, s - \varepsilon, r, N) \geq \mu(Z^*) > 0,$$

and consequently

$$\underline{m}(Z^*, s - \varepsilon, r) = \lim_{N \to \infty} \overline{m}(Z^*, s - \varepsilon, r, N) > 0,$$

which in turn implies that $h_{Z^*}(r, G) \geq s - \varepsilon$. Then we have $h_{Z^*}(G) \geq s - \varepsilon$ by letting $r \to 0$. It follows that $h_{Z}(G) \geq h_{Z^*}(G) \geq s - \varepsilon$ and hence $h_{Z}(G) \geq s$ since $\varepsilon > 0$ is arbitrary. The proof is completed now. \hfill \Box

**Lemma 6.3.** Let $r > 0$ and $\mathcal{B}(r) = \{ B_w(x, r) : x \in X, w \in F^+_m \}$. For any family $\mathcal{F} \subset \mathcal{B}(r)$, there exists a (not necessarily countable) subfamily $\mathcal{G} \subset \mathcal{F}$ consisting of disjoint balls such that

$$\bigcup_{B \in \mathcal{F}} \subset \bigcup_{B_w(x, r) \in \mathcal{G}} B_w(x, 3r).$$

**Proof.** The proof follows [26] and is omitted. \hfill \Box

**Theorem 6.4.** Let $\mu$ denote a Borel probability measure on $X$, $Z$ be a Borel subset of $X$ and $s \in (0, \infty)$. If

$$h_{\mu, G}^L(x) \leq s \text{ for all } x \in Z \text{ then } h_{Z}(G) \leq s.$$ 

**Proof.** Since $h_{\mu, G}^L(x) \leq s$ for all $x \in Z$, then for any $\omega \in \Sigma_m^+$ and $x \in Z$,

$$\lim_{r \to 0} \liminf_{n \to \infty} - \frac{1}{n + 1} \log \mu(B_{\omega|_{[0,n-1]}}(x, r)) \leq h_{\mu, G}^L(x) \leq s.$$ 

For any $N \geq 1$ and $w = i_1i_2\ldots i_N \in F^+_m$. Choose $\varepsilon > 0$, we have $Z = \bigcup_{k \geq 1} Z_k$, where

$$Z_k = \left\{ x \in Z : \liminf_{n \to \infty} - \frac{1}{n + 1} \log \mu(B_{\omega|_{[0,n-1]}}(x, r)) < s + \varepsilon \right\}$$

for all $r \in (0, 1/k), \omega \in \Sigma_m^+$ and $\omega|_{[0,N-1]} = w$. 

Now fix $k \geq 1$ and $0 < r < \frac{1}{3k}$. For each $x \in Z_k$, we take $\omega_x \in \Sigma^+_m$ such that $\omega_x|_{[0,N-1]} = w$, there exists a strictly increasing sequence $\{n_j(x)\}_{j=1}^\infty$ such that
\[
\mu(B_{\omega_x|_{[0,n_j(x)-1]}(x,r)}) \geq \exp \left( -(s + \varepsilon) \cdot (n_j(x) + 1) \right)
\]
for all $j \geq 1$. So, the set $Z_k$ is contained in the union of the sets in the family
\[
\mathcal{F}_w = \left\{ B_{\omega_x|_{[0,n_j(x)-1]}(x,r)} : x \in Z_k, \omega_x \in \Sigma^+_m, \omega_x|_{[0,N-1]} = w \text{ and } n_j(x) \geq N \right\}.
\]
By Lemma 6.3, there exists a subfamily $\mathcal{G}_w = \{ B_{\omega_x|_{[0,n_i(x)-1]}(x_i,r)} \}_{i \in I} \subset \mathcal{F}_w$ consisting of disjoint balls such that for all $i \in I$.
\[
Z_k \subset \bigcup_{i \in I} B_{\omega_x|_{[0,n_i(x)-1]}(x_i,3r)},
\]
and
\[
\mu(B_{\omega_x|_{[0,n_i(x)-1]}(x_i,r)}) \geq \exp \left( -(s + \varepsilon) \cdot (n_i + 1) \right).
\]
The index set $I$ is at most countable since $\mu$ is a probability measure and $\mathcal{G}_w$ is a disjointed family of sets, each of which has positive $\mu$-measure. Therefore,
\[
\mathcal{M}_w(Z_k, s + \varepsilon, 3r, N) \leq \sum_{i \in I} \exp \left( -(s + \varepsilon) \cdot (n_i + 1) \right) \leq \sum_{i \in I} \mu(B_{\omega_x|_{[0,n_i(x)-1]}(x_i,r)}) \leq 1,
\]
where the disjointness of $\{ B_{\omega_x|_{[0,n_i(x)-1]}(x_i,r)} \}_{i \in I}$ is used in the last inequality. It follows that
\[
\mathcal{M}(Z_k, s + \varepsilon, 3r, N) = \frac{1}{m^N} \sum_{|w|=N} \mathcal{M}_w(Z_k, s + \varepsilon, 3r, N) \leq 1
\]
and consequently
\[
\mathcal{M}(Z_k, s + \varepsilon, 3r) = \lim_{N \to \infty} \mathcal{M}(Z_k, s + \varepsilon, r, N) \leq 1.
\]
which in turn implies that $h_{Z_k}(3r, G) \leq s + \varepsilon$ for any $0 < r < \frac{1}{3k}$. Letting $r \to 0$ yields
\[
h_{Z_k}(G) \leq s + \varepsilon \text{ for any } k \geq 1.
\]
By Proposition 1(3),
\[
h_Z(G) = h_{\cup_{k=1}^\infty Z_k}(G) = \sup_{k \geq 1} \{h_{Z_k}(G)\} \leq s + \varepsilon.
\]
Therefore, $h_Z(G) \leq s$ since $\varepsilon > 0$ is arbitrary.

\textbf{Remark 9.} For the topological entropy of a pseudogroup introduced in [5], Biś proved a similar result to the Theorem 6.2. In [12], the authors obtained a similar result using the skew product.

\textbf{Remark 10.} If $m = 1$, then the above theorems coincide with the main results that Ma and Wen proved in [26].
7. Topological entropy and Hausdorff dimension. In dynamical systems, the relation between topological entropy and Hausdorff dimension is a very important problem which many people have studied, such as Dai et al. [16], Misiurewicz [27], Ma and Wu [24], etc.

Let $X$ be a nonempty compact metric space. Supposing that a free semigroup with $m$ generators $G = \{f_0, f_1, \ldots, f_{m-1}\}$ acts on $X$, we assume that these maps are Lipschitz self-maps with Lipschitz constant $L_i$ respectively. In this section, we show that the topological entropy of a free semigroup action generated by $G$ on any subset $Z$ of $X$ is upper bounded by Hausdorff dimension of $Z$ multiplied by the maximum logarithm of $\{L_i\}$.

**Theorem 7.1.** Given a free semigroup with $m$ generators $G := \{f_0, \ldots, f_{m-1}\}$ acting on a compact metric space $X$ and $f_i$ is a Lipschitz continuous map with the Lipschitz constant $L_i > 1$. Then for any $Z \subset X$ we get

$$h_Z(G) \leq HD(Z) \cdot \max_i \{\log L_i\}.$$ 

In particular, if $Z = X$, then

$$h_X(G) \leq HD(X) \cdot \max_i \{\log L_i\}.$$ 

**Proof.** Fix a finite open cover $\mathcal{U}$ of $X$ with a Lebesgue number $\delta$. Let $L = \max_i \{L_i\}$.

If $B \subset X$ and $\text{diam}(B) < \delta$ then $B \prec \mathcal{U}$. If $\text{diam}(B) < \delta/L^{N-1}$ then for any $w \in F_m^+$ and $|w| = N$, we have $\text{diam}(f_{w'}(B)) < \delta$ for $w' \leq w$. So for any $\omega \in \Sigma_m^+$, we have $n_{\omega, \mathcal{U}}(B) \geq N$. In other words, for every non-negative $N$ and any $\omega \in \Sigma_m^+$, if $N < \frac{\log \delta - \log \text{diam}(B)}{\log L} + 1$ then $N \leq n_{\omega, \mathcal{U}}(B)$. Hence,

$$\frac{\log \delta - \log \text{diam}(B)}{\log L} \leq n_{\omega, \mathcal{U}}(B). \quad (8)$$

The number $c = \frac{\log L}{\log L}$ is a constant, as long as $G$ and $\mathcal{U}$ are fixed. We can rewrite (8) as

$$D_{\omega, \mathcal{U}}(B) = \exp(-n_{\omega, \mathcal{U}}(B)) \leq e^{-c(\text{diam}(B))^{1/\log L}}, \quad (9)$$

where $\omega \in \Sigma_m^+$.

Therefore if $B$ is a cover of $Z$, then for any $\lambda > 0$, $w = i_1i_2\ldots i_N \in F_m^+$ and every $B \in \mathcal{B}$ with $\omega_B \in \Sigma_m^+$ such that $\omega_B|_{[0,N-1]} = w$, we have

$$\sum_{B \in \mathcal{B}} \exp(-\lambda n_{\omega_B, \mathcal{U}}(B)) \leq e^{-c\lambda} \sum_{B \in \mathcal{B}} (\text{diam}(B))^{\lambda/\log L}. \quad (10)$$

Choose $\lambda > HD(Z) \cdot \log L$. By definition of Hausdorff dimension we can write $H^{\text{dim}}(Z) = 0$, so for any $\varepsilon \in (0, 1)$ there is a cover $\mathcal{B}$ of $Z$ such that for any $B \in \mathcal{B}$

$$e^{-c(\text{diam}(B))^{1/\log L}} < \varepsilon$$

and

$$e^{-c\lambda} \sum_{B \in \mathcal{B}} (\text{diam}(B))^{\lambda/\log L} < \varepsilon.$$
For this $B$ we get $D_{\omega_B \mathcal{U}}(B) < \varepsilon$ by (9) and $\sum_{B \in \mathcal{B}} \exp(-\lambda n_{\omega_B \mathcal{U}}(B)) < \varepsilon$ by (10). So

$$\mu_w(Z, \lambda, \mathcal{U}, \varepsilon) := \inf_{B} \left\{ \sum_{B \in \mathcal{B}} \exp(-\lambda n_{\omega_B \mathcal{U}}(B)) : B \text{ covers } Z, D_{\omega_B \mathcal{U}}(B) < \varepsilon, \right.$$\[\omega_B \in \Sigma_m^+, N = \lfloor -\log(\varepsilon) \rfloor + 1 \text{ and } \omega_B|_{[0,N-1]} = w \right\} < \varepsilon.$$

This implies that

$$\mu(Z, \lambda, \mathcal{U}, \varepsilon) = \frac{1}{m^N} \sum_{|w|=N} \mu_w(Z, \lambda, \mathcal{U}, \varepsilon) < \varepsilon.$$

Taking the limit as $\varepsilon \to 0$ we get that

$$\mu(Z, \lambda, \mathcal{U}) = 0$$

and thus $h'_Z(\mathcal{U}, \mathcal{G}) \leq \lambda$. Taking supremum over all finite open covers $\mathcal{U}$ of $X$, we get

$$h_Z(\mathcal{G}) = h'_Z(\mathcal{G}) \leq HD(Z) \cdot \max_i \{\log L_i\}.$$

\[\square\]

**Remark 11.** (1) Let $f : X \to X$ be a Lipschitz continuous map with Lipschitz constant $L > 1$. If $G = \{f\}$, for any $Z \subset X$, Misiewicz [27] proved that $h'_Z(G) \leq HD(Z) \cdot \log L$.

(2) In the future we will introduce the topological pressure of free semigroup actions for noncompact sets.

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**REFERENCES**

[1] R. Adler, A. Konheim and J. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.*, 114 (1965), 309–319.

[2] L. Barreira, Ya. Pesin and J. Schmeling, On a general concept of multifractality: Multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal rigidity, *Chaos*, 7 (1997), 27–38.

[3] A. Biś, Entropies of a semigroup of maps, *Discrete Contin. Dyn. Syst.*, 11 (2004), 639–648.

[4] A. Biś, Partial variational principle for finitely generated groups of polynomial growth and some foliated spaces, *Colloq. Math.*, 110 (2008), 431–449.

[5] A. Biś, An analogue of the variational principle for group and pseudogroup actions, *Ann. Inst. Fourier.*, 63 (2013), 839–863.

[6] A. Biś and M. Urbański, Some remarks on topological entropy of a semigroup of continuous maps, *Cubo*, 8 (2006), 63–71.

[7] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.*, 153 (1971), 401–414.

[8] R. Bowen, Topological entropy for non-compact sets, *Trans. Amer. Math. Soc.*, 184 (1973), 125–136.

[9] M. Brin and A. Katok, On local entropy, *Geometric Dynamics*, Springer, Berlin, Heidelberg, 1007 (1983), 30–38.

[10] A. Bufetov, Topological entropy of free semigroup actions and skew-product transformations, *J. Dynam. Control Systems*, 5 (1999), 137–143.

[11] M. Carvalho, F. Rodrigues and P. Varandas, Semigroup actions of expanding maps, *J. Stat. Phys.*, 166 (2017), 114–136.

[12] M. Carvalho, F. Rodrigues and P. Varandas, A variational principle for free semigroup actions, *Advances in Math.*, 334 (2018), 450–487.
[13] M. Carvalho, F. Rodrigues and P. Varandas, Quantitative recurrence for free semigroup actions, Nonlinearity, 31 (2018), 864–886.
[14] E. C. Chen, T. Küpper and L. Shu, Topological entropy for divergence points, Ergodic Theory Dynam. Systems, 25 (2005), 1173–1208.
[15] V. Climenhaga, Multifractal formalism derived from thermodynamics for general dynamical systems, Electron. Res. Announc. Math. Sci., 17 (2010), 1–11.
[16] X. Dai, Z. Zhou and X. Geng, Some relations between Hausdorff-dimensions and entropies, Sci. China Ser. A, 41 (1998), 1068–1075.
[17] E. I. Dinaburg, The relation between topological entropy and metric entropy, Soviet Math. Dokl., 11 (1970), 13–16.
[18] Y. Dong and X. Tian, Multifractal analysis of the new level sets, arXiv:1510.06514 (2015).
[19] S. Friedland, Entropy of graphs, semigroups and groups, London Mathematical Society Lecture Note Series, 228. Cambridge University Press, Cambridge, 1996, 319–343.
[20] E. Ghys, R. Langevin and P. Walczak, Entropie geometrique des feuilletages, Acta Math., 160 (1988), 105–142.
[21] S. Kolyada and L. Snoha, Topological entropy of nonautonomous dynamical systems, Random Comput. Dyn., 4 (1996), 205–233.
[22] X. Lin, D. Ma and Y. Wang, On the measure-theoretic entropy and topological pressure of free semigroup actions, Ergodic Theory Dynam. Systems, 38 (2018), 686–716.
[23] D. Ma and S. Liu, Some properties of topological pressure of a semigroup of continuous maps, Dyn. Syst, 29 (2014), 1–17.
[24] D. Ma and M. Wu, On Hausdorff dimension and topological entropy, Fractals, 18 (2010), 363–370.
[25] D. Ma and M. Wu, Topological pressure and topological entropy of a semigroup of maps, Discrete Contin. Dyn. Syst., 31 (2011), 545–557.
[26] J. H. Ma and Z. Y. Wen, A Billingsley type theorem for Bowen entropy, C. R. Math. Acad. Sci., Paris, 346 (2008), 503–507.
[27] M. Misiurewicz, On Bowen definition of topological entropy, Discrete Contin. Dyn. Syst., 10 (2004), 827–833.
[28] Y. Pesin, Dimension Theory in Dynamical Systems, Chicago: The university of Chicago Press, 1997.
[29] C. Pfister and W. Sullivan, On the topological entropy of saturated sets, Ergodic Theory Dynam. Systems, 27 (2007), 929–956.
[30] F. B. Rodrigues and P. Varandas, Specification and thermodynamical properties of semigroup actions, J. Math. Phys., 57 (2016), 052704, 27 pp.
[31] F. Takens and E. Verbitski, Multifractal analysis of local entropies for expansive homeomorphisms with specification, Comm. Math. Phys., 203 (1999), 593–612.
[32] F. Takens and E. Verbitski, Multifractal analysis of dimensions and entropies, Regul. Chaotic Dyn., 5 (2000), 361–382.
[33] F. Takens and E. Verbitski, On the variational principle for the topological entropy of certain non-compact sets, Ergodic Theory Dynam. Systems, 23 (2003), 317–348.
[34] Y. Wang and D. Ma, On the topological entropy of a semigroup of continuous maps, J. Math. Anal. Appl., 427 (2015), 1084–1100.
[35] Y. Wang, D. Ma and X. Lin, On the topological entropy of free semigroup actions, J. Math. Anal. Appl., 435 (2016), 1573–1590.
[36] P. Waters, An Introduction to Ergodic Theory, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

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