Resetting Infinite Time Blum-Shub-Smale-Machines

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Abstract. In this paper, we study strengthenings of Infinite Times Blum-Shub-Smale-Machines (ITBMs) that were proposed by Seyfferth in [16] and Welch in [17] obtained by modifying the behaviour of the machines at limit stages. In particular, we study Strong Infinite Times Blum-Shub-Smale-Machines (SITBMs), a variation of ITBMs where lim is substituted by lim inf in computing the content of registers at limit steps. We will provide upper and lower bounds to the computational strength of such machines. Finally, we will study the computational strength of restrictions of SITBMs to rational numbers strengthening a result in [17] and partially answering a question posed by Welch in [17].

1 Introduction

In [2] Blum, Shub and Smale introduced register machines which compute over real numbers called Blum-Shub-Smale-Machines (BSSMs). A BSSM is a register machine whose registers contain real numbers and that at each step of the computation can either check the content of the registers and perform a jump based on the result, or apply a rational function to the registers. While being quite powerful, BSSMs are still bound to work for a finite amount of time. This limitation is in contrast with the fact that every real number is usually thought as to encode an infinite amount of information. It is therefore natural to ask if a transfinite version of these machines is possible. An answer to this question was first given by Koepke and Seyfferth who in [16] and Welch in [17] introduced the notion of Infinite Time Blum, Shub and Smale machine (ITBM). These machines execute classical BSSM-program¹ but are capable of running for a transfinite amount of time. More precisely, the behaviour of ITBMs at successor stages is completely analogous to that of a normal BSSM. At limit stage an ITBM computes the content of each register using the limit operation on ℜ, and updates the program counter using inferior limits. Infinite Time Blum-Shub-Smale-Machines were further studied in [10].

As mentioned in [8], the approach taken in extending BSSMs to ITBMs is analogous to that used by Hamkins and Lewis in [9] and by Koepke and Miller

¹ With the limitation that every rational polynomial appearing in the program has rational coefficients.
in \[10\] to introduce Infinite Time Turing machines (ITTMs) and Infinite Time Register machines (ITRMs), respectively. A different approach to the generalisation of BSSMs analogous to that used by Koepke in \[11\] to introduce Ordinal Turing Machines (OTM) was taken by the second author in \[8, 7\] where he introduced the notion of Surreal Blum, Shub and Smale machine (SBSSM). A complete account of all these models of computation can be found in \[3\].

As shown in \[10\], because of the way in which they compute the content of registers at limit stages, ITBMs have a limited computational power, which is way below that of other notions of transfinite computability such as ITTMs, OTMs, SBSSMs, ITRMs. In this paper we consider strengthenings of the limit rule for ITBMs. We will first consider four natural modifications of the behaviour of ITBMs at limit stages. For each such modification we show that it can either be simulated by ITBMs or by machines that we will call Strong Infinite Time Register machines. The rest of the paper is devoted to the study of SITBMs. First we provide upper and lower bounds to their computational power. Then, we restrict our attention to programs whose SITBM-computation is of low complexity. In particular, we will strengthen a result mentioned without proof by Welch in \[17\] and we will give a partial negative answer to the question asked by Welch of whether \(\Pi_3\)-reflection is an optimal upper bound to the halting times of BSSM-programs whose SITBM-computation uses only rational numbers.

Many of the arguments in this paper are inspired by those in \[4\] and \[5\].

### 2 ITBM Limit Rules

As we mentioned in the introduction, the value of registers of an ITBM at limit stages is computed using Cauchy limits, i.e., the content of each register at limit steps is the limit of the sequence obtained by considering the content of the register at previous stages of the computation. The machine is assumed to diverge if for at least one of the registers this limit does not exist. In this section we will consider modified versions of the limit behaviour of ITBMs.

Let \(f : \alpha \to \mathbb{R}\) be an \(\alpha\)-sequence on \(\mathbb{R}\). We call \(\ell \in \mathbb{R}\) a finite limit point of \(f\) if for all \(\beta < \alpha\) and for all positive \(\varepsilon \in \mathbb{R}\) there is \(\beta < \gamma < \alpha\) such that \(|f(\gamma) - \ell| < \varepsilon\). If for all \(x \in \mathbb{R}\) and for all \(\beta < \alpha\) there is \(\beta < \gamma < \alpha\) such that \(x < f(\gamma)\) then we will say that \(+\infty\) is an infinite limit point of \(f\). Similarly for \(-\infty\). If for all \(x \in \mathbb{R}\) there is \(\beta < \alpha\) such that for all \(\beta < \gamma < \alpha\) we have \(x < f(\gamma)\) then we will say that \(+\infty\) is an infinite strong limit point of \(f\). Similarly, if for all \(x \in \mathbb{R}\) there is \(\beta < \alpha\) such that for all \(\beta < \gamma < \alpha\) we have \(x > f(\gamma)\) then we will say that \(-\infty\) is an infinite strong limit point of \(f\).

We will denote by \(\text{LimP}(f) \subset \mathbb{R} \cup \{-\infty, +\infty\}\) the set of finite and infinite limit points of \(f\), and by \(\text{sLimP}(f) \subset \mathbb{R} \cup \{-\infty, +\infty\}\) the set of finite limit points and infinite strong limit points of \(f\). Note that, if \(f\) has an infinite strong limit point, then \(\text{LimP}(f)\) is a singleton containing either \(-\infty\) or \(+\infty\).

**Remark 1.** If \(f\) is Cauchy with limit \(\ell\), then \(\text{sLimP}(f) = \text{LimP}(f) = \{\ell\}\). The converse is not true. Indeed, fix \(\rho : \omega \times \omega \to \omega\) to be any computable bijection.
Let \( f_n(m) = \ell + \frac{n}{m} \) for \( n > 0 \). Each \( f_n \) is a countable sequence which converges to \( \ell \). Then, setting \( s(\rho(n, m)) = f_n(m) \) we see that the sequence \( s \) is such that \( \text{sLimP}(s) = \{ \ell \} \) but is not Cauchy.

Remark 2. Note that if \( \alpha \) is a limit ordinal, \( f : \alpha \to \mathbb{R} \) is an \( \alpha \)-sequence, \( \gamma < \alpha \) is an ordinal, and \( g \) is the sequence \( g(\beta) = f(\gamma + \beta) \), then \( \text{LimP}(f) = \text{LimP}(g) \), and \( \text{sLimP}(f) = \text{sLimP}(g) \).

Let \( \lambda \) be a limit ordinal and \( R_i \) be a register. Assume that for each \( \alpha < \lambda \) the register \( R_i \) had content \( R_i(\alpha) \) in the \( \alpha \)th step of the computation. We consider ITBMs whose limit rule is modified as follows:

1. **Weak ITBM (WITBM):**

   \[
   R_i(\lambda) = \begin{cases} 
   \ell & \text{if } R_i^\lambda \text{ is Cauchy with limit } \ell; \\
   0 & \text{if } \min(\text{sLimP}(R_i^\lambda)) \text{ is an infinite strong limit.}
   \end{cases}
   \]

2. **Strong ITBM (SITBM):**

   \[
   R_i(\lambda) = \begin{cases} 
   \min(\text{LimP}(R_i^\lambda)) & \text{if } \min(\text{sLimP}(R_i^\lambda)) \in \mathbb{R}; \\
   0 & \text{if } \min(\text{sLimP}(R_i^\lambda)) \text{ is an infinite strong limit.}
   \end{cases}
   \]

3. **Bounded-strong ITBM (BSITBM):**

   \[
   R_i(\lambda) = \min(\text{sLimP}(R_i^\lambda)) \text{ if } \min(\text{sLimP}(R_i^\lambda)) \in \mathbb{R}.
   \]

4. **Super-strong ITBM (SSITBM):**

   \[
   R_i(\lambda) = \begin{cases} 
   \min(\text{LimP}(R_i^\lambda)) & \text{if } \text{LimP}(R_i^\lambda) \in \mathbb{R}; \\
   0 & \text{if } \text{LimP}(R_i^\lambda) \text{ is infinite.}
   \end{cases}
   \]

If \( \text{LimP}(R_i) = \emptyset \), \( \text{sLimP}(R_i) = \emptyset \), or in general if \( R_i(\lambda) \) is not defined, we will assume that the machine crashes, i.e., the computation is considered divergent. In each case the rest of the machine is left unchanged. As for ITBMs all the machines we consider run BSSM-programs and work on real numbers. Therefore, given \( \Gamma \in \{ \text{SITBM, BSITBM, SSITBM} \} \), the definitions of \( \Gamma \)-computation, \( \Gamma \)-computable function, \( \Gamma \)-computable set, and \( \Gamma \)-clockable ordinal are exactly the same as the correspondent notions for ITBMs, see, e.g., [14, Definition 1, Definition 2, Definition 3]. As noted in [14, Algorithm 4] ITBMs are capable of computing the binary representation of a real number and perform local changes to this infinite binary sequence. The same algorithms work for SITBM, and therefore for BSITBM, and SSITBM. For this reason, in the rest of the paper we will sometimes treat the content of the each register as an infinite binary sequence. Since not all the binary sequences can be represented in this way, whenever we need to treat a register as an infinite binary sequence, we will do so by representing the sequence \( t : \omega \to 2 \) by the real \( r \) whose binary representation is such...
that for every \( n \in \mathbb{N} \) the \((2n + 1)\)st bit is \( t(n) \) and all the bits in even position are 0. In this case we will call \( t \) the binary sequence represented by \( r \).

Given a set \( A \subseteq \omega \) we say that it is \( \Gamma \)-writable if there is a BSSM-program whose \( \Gamma \)-computation with no input outputs a real \( r \) such that for every \( n \), the \( n \)th bit in the binary sequence represented by \( r \) is 1 if and only if \( n \in A \). Similarly, we say that a countable ordinal \( \alpha \) is \( \Gamma \)-writable if there is a \( \Gamma \)-writable set \( A \) such that \((\alpha, <) \cong (\mathbb{N}, \{(n, m) : \rho(n, m) \in A\})\).

Let \( P \) be a \( n \)-register \( 2 \) BSSM-program, \( r_1, \ldots, r_n \in \mathbb{R} \) be a real numbers, and \((C_\alpha)_{\alpha<\Theta} = (R_1(\alpha) \ldots R_n(\alpha), I(\alpha))_{\alpha<\Theta} \) be the SITBM-computation of \( P \) on input \( r_1, \ldots, r_n \). For every \( \alpha < \Theta \) we will call \( C_\alpha \) the snapshot of the execution at time \( \alpha \). Given an SITBM-program \( P \) and the SITBM-computation \(((R_1(\alpha), \ldots, R_n(\alpha), I(\alpha)))_{\alpha<\Theta} \) of \( P \) on input \( R_1(0), \ldots, R_n(0) \), for every \( i \in \{1, \ldots, n\} \) and \( \alpha < \Theta \) we will denote by \( R_\alpha^i : \alpha \to \mathbb{R} \) the \( \alpha \)-sequence such that \( R_\alpha^i(\beta) = R_i(\beta) \) for all \( \beta < \alpha \). In the rest of the paper we will omit the superscript \( \alpha \) when it is clear from the context.

As Lemma 6, Lemma 7, and Lemma 8 show, SSITBM-s, BSITBM-s, and WITBM-s are inessential modifications of ITBM-s and SITBM-s. For this reason in the rest of the paper we focus on the study of the computational strength of SITBM-s. It is worth noticing that a version of SITBM-s was briefly considered by Welch in [17], see, §4.

**Lemma 3.** Every ITBM-computable function is SITBM-computable.

**Proof.** The claim follows by Remark 1.

**Remark 4.** Since every ITRM program is essentially a BSSM-program, every ITRM-computable function is SITBM-computable. Moreover, the simulation can be performed in exactly the same number of steps.

By Remark 4 the result of Lemma 3 cannot be reversed.

**Lemma 5.** There is a SITBM-computable function which is not ITBM-computable.

**Proof.** It is enough to note that \( \omega^\omega \) is ITRM-computable [6, Lemma 1 & Theorem 7] and therefore SITBM-writable but not ITBM-writable.

**Lemma 6.** A real function \( f \) is ITBM-computable iff it is WITBM-computable.

**Proof.** Trivially every ITBM-computable function is WITBM-computable. Now, let \( f \) be any ITBM computable field isomorphism between \( \mathbb{R} \) and \((0, 1)\), e.g., \( G : x \mapsto \frac{x}{2^{\lfloor \log_2 x \rfloor}} \). Let \( f \) be a WITBM computable function and \( P \) be a program computing it. One can define a program \( P' \) in which every constant \( c \) is replaced by \( G(c) \) and all the fields operations are replaced with the correspondent operations in \((0, 1)\). Moreover, at each step of the computation the program checks if one of the registers is 0 or 1, if so the program sets the register to \( G(0) \). Finally the program should compute \( G^{-1} \) of the output. It is not hard to see that the program computes \( f \).

\(^2\) Since the number of registers is not of importance in our results, and because of Lemma 9 in the rest of the paper we will just refer to \( n \)-registers BSSM-programs as BSSM-programs.
Lemma 7. A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is SITBM-computable iff it is BSITBM-computable.

Proof. The proof is very similar to that of Lemma 6. Every BSITBM-computable function is SITBM-computable. Now, let \( f \) be any ITBM-computable field isomorphism between \( \mathbb{R} \) and \( (0, 1) \), e.g., \( G : x \mapsto e^{2x} + 1 \). Let \( f \) be an SITBM-computable function and \( P \) be a program computing it. One can define a program \( P' \) in which every constant \( c \) is replaced by \( G(c) \) and all the fields operations are replaced with the correspondent operations in \( (0, 1) \). For each register \( R_i \) the machine should have two auxiliary registers \( C_i \) and \( D_i \). Each time \( R_i \) is modified the machine does the following:

If \( D_i = 0 \) and the new value of \( R_i \) is smaller than the old value then set \( D_i = 1 \) and \( C_i = 0 \), if \( D_i = 0 \) and the new value of \( R_i \) is bigger or equal to the old value then set \( C_i = 1 \), if \( D_i = 1 \) and the new value of \( R_i \) is smaller than or equal to the old value then set \( C_i = 1 \), if \( D_i = 1 \) and the new value of \( R_i \) is bigger than the old value then set \( D_i = 0 \) and \( C_i = 0 \). At each step of the computation the machine should check that all the registers of the original program are not 0 or 1. If a register \( R_i \) is 0 or 1 and \( C_i \neq 0 \) then the machine sets \( R_i \) to \( G(0) \) and continues the computation otherwise the program enters an infinite loop.

Finally, the program should compute \( G^{-1} \) of the output. It is not hard to see that the program computes \( f \).

Finally, we note that, once more by shrinking the computation to \( (0, 1) \) as we did in the proof of Lemma 7 one can show that SSITBMs can be simulated by SITBMs, and therefore the two models compute exactly the same functions.

Lemma 8. A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is SITBM-computable iff it is SSITBM-computable.

We end this section by noticing that, coding finitely many real numbers in one register one can easily build a universal SITBM machine.

Lemma 9. There is a BSSM-program \( U \) which given a real number \( r \) coding a BSSM-program \( P \) and the values \( r \) of the input registers of \( P \), executes \( P \) on \( r \).

3 Bounding the Computational Power of SITBMs

In this section we will provide upper and lower bounds to the computational strength of SITBMs.

First we show that SITBMs are stronger than ITRMs. Given \( X \subseteq \omega \) we will denote by \( O^X \) the hyperjump of \( X \), see, e.g., [16].

In order to show that SITBMs are stronger than ITRMs we will prove that SITBMs can compute the \( \omega \) iteration of the hyperjump function and much

\[3 \text{ This algorithm checks if the content of } R_i \text{ is due to a proper limit to infinity.} \]
more. First, we note that as in [10, Proposition 2.8], if a function $f$ is SITBM-computable, then iterations of $f$ along a SITBM-clockable ordinal are also computable. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $\Theta$ be an infinite countable ordinal, and $h : \omega \to \Theta$ be a bijection, we define:

$$
\begin{align*}
 f^0_h &= 0, \\
 f^{\alpha+1}_h &= f(f^\alpha_h) \text{ for } \alpha + 1 \leq \Theta, \\
 f^\lambda_h &= \bigoplus_{\alpha \in \lambda} f^\alpha_h \text{ for } \lambda \leq \Theta \text{ limit},
\end{align*}
$$

where $\bigoplus_{\alpha \in \lambda} f^\alpha_h$ is the real $r$ such that if $n = \rho(i, h(\alpha))$ then the $n$th bit of the binary sequence represented by $r$ is the same as the $i$th bit of the binary representation of $f^\alpha_h$ for all $i \in \mathbb{N}$ and $\alpha < \lambda$, and 0 for $\lambda < \alpha$.

Similarly to [14, Proposition 2.7], one can easily see that SITBMs can compute iterations of SITBM-computable function over an SITBM-writable ordinal.

**Lemma 10 (Iteration Lemma).** Let $f : \mathbb{R} \to \mathbb{R}$ be a SITBM-computable function, and $\Theta$ be SITBM-writable. Then there is a bijection $h : \omega \to \Theta$ such that $f^\Theta_h$ is SITBM-computable.

**Proof (Sketch).** We will only sketch the proof and we leave the details to the reader.

Assume that $\Theta$ is SITBM-writable. Let $h$ be such that $h : (\mathbb{N}, \{(n, m) : \text{ the } \rho(n, m) \text{th bit of the binary sequence represented by } r \text{ is 1}\}) \cong (\Theta, <)$ where $r$ is the SITBM-writable real coding $\Theta$. There is a machine that computes $f^\Theta_h$.

Note that, since $\Theta$ is SITBM-writable, essentially by using the classical ITTM algorithm, given a set of natural numbers we can always compute their least upper bound (if it exists) according to the order induced by $h$. Therefore, we can generate the sequence $h^{-1}(0)h^{-1}(2)\ldots h^{-1}(\omega)\ldots$.

As noted in [10] Proposition 2.8 the main problem in proving this lemma is to show that we can iterate the same program infinitely many times ensuring that no register in the program will diverge because of the iteration. To solve this problem we note that by Lemma 8 it is enough to show that the function is SSITBM-computable. Note that SSITBM-computations are obviously closed under transfinite iterations since SSITBMs never brake because of a diverging register.

The SSITBM algorithm to compute the function is the following: the machine first writes $\Theta$ in one of the registers say $R_1$. Then, our machine starts computing $f(0)$ and saves the result in the register $R_2$ by copying the $i$th bit of $f(0)$ in position $\rho(i, h^{-1}(0))$ of the binary sequence represented by $R_2$. In general, the machine should compute $f(f^\alpha_h)$ and save the result in $R_2$ according to $h^{-1}$ as we did for $f(0)$. Checking the correctness of the sketched algorithm is left to the reader.

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4 Note that, while the example mentioned in [10, p.42] is not problematic for SITBMs, our machines could still diverge if for example in the infinite iteration one of the registers of the program assumes values $0, 1, -1, 2, -2, 3, -3, \ldots$. 

Now, we can prove that SITBMs are strictly stronger than ITRMs. First note that since ITRMs can compute the hyperjump of a set, by Remark \[4\] so can SITBMs.

**Lemma 11.** Let $X \subseteq \omega$. The hyperjump $O^X$ in the oracle $X$ is SITBM-computable.

By Lemma \[10\] we have the following result:

**Lemma 12.** Let $\alpha$ be SITBM-clockable. Then the $\alpha$th iteration of the hyperjump is SITBM-computable.

But since by \[12, Proposition 12\] ITRMs cannot compute the $\omega$-iteration of the hyperjump we have that SITBMs are strictly stronger than ITRMs.

**Corollary 13.** There is a subset of natural numbers $A \subseteq \mathbb{N}$ which is SITBM-computable but not ITRM-computable, and thus $A \notin \mathbb{L}_{\omega,\text{CK}}$.

We will now show that the first $\Sigma_2$-admissible ordinal is an upper bound of the set of SITBM-clockable ordinal. We refer the reader to \[1\] for an introduction to admissibility. We will start by proving a sufficient condition for SITBM-computations to diverge. We prove a looping criterion for the divergence of BSSMs, which is analogous to those for ITTM \[9, Theorem 1.1\], ITRMs \[6, Theorem 5\] and ITBMs \[14, Theorem 1\].

**Lemma 14 (Strong Loop Lemma).** Let $P$ be a BSSM-program, $(C_\alpha)_{\alpha<\Theta} = ((R_1(\alpha), \ldots, R_n(\alpha), I(\alpha)))_{\alpha<\Theta}$ be a SITBM-computation of $P$, and let $\gamma < \beta < \Theta$ be such that:

1. $(R_1(\gamma), \ldots, R_n(\gamma), I(\gamma)) = (R_1(\beta), \ldots, R_n(\beta), I(\beta))$;
2. for all $\gamma \leq \alpha \leq \beta$ we have $I(\beta) \leq I(\alpha)$ and for all $i \in \{1 \ldots n\}$ we have $R_i(\beta) \leq R_i(\alpha)$.

Then $P$ diverges.

**Proof.** Without loss of generality we will assume that $n = 1$, a similar proof works for an arbitrary number of registers. Note that since $(R_1(\gamma), I(\gamma)) = (R_1(\beta), I(\beta))$ the machine is in a loop. Let us call $L = ((R_1(\alpha), I(\alpha)))_{\gamma \leq \alpha \leq \beta}$ the looping block of the computation. Let $\delta$ be the smallest ordinal such that $\gamma + \delta = \beta$, we will call $\delta$ the length of $L$.

**Claim 1** If $\alpha = \gamma + \delta \times \nu$ for some $\nu > 0$ is such that $(R_1(\gamma), I(\gamma)) = (R_1(\alpha), I(\alpha))$ then for all $\mu < \delta$ we have $(R_1(\gamma + \mu), I(\gamma + \mu)) = (R_1(\alpha + \mu), I(\alpha + \mu))$, i.e., if $\alpha = \gamma + \delta \times \nu$ is such that $(R_1(\gamma), I(\gamma)) = (R_1(\alpha), I(\alpha))$ then the computation from $\alpha$ to $\alpha + \delta$ is the loop $L$.

**Proof.** We prove the claim by induction on $\mu$. If $\mu = 0$ the claim follows trivially from the hypothesis. If $\mu = \eta + 1$ then by inductive hypothesis $(R_1(\gamma + \eta), I(\gamma + \eta)) = (R_1(\alpha + \eta), I(\alpha + \eta))$. But then $(R_1(\gamma + \eta + 1), I(\gamma + \eta + 1)) = (R_1(\alpha + \eta + 1), I(\alpha + \eta + 1))$ follows from the fact that our machines are deterministic. Finally for $\mu$ limit the claim follows from the inductive hypothesis and from Remark \[2\]
Lemma 14 A strong loop is a cofinal subsequence of \( R \{ \be \max(\cdots) \} \). Finally, note that Claim 2 implies that the computation of \( P \) of \( i \in \{ \sigma \} \). 

Proof. We prove the claim by induction on \( \nu \). If \( \nu = 1 \) the claim follows from the assumptions. For \( \nu = \eta + 1 \) then the claim follows from Claim 1. Assume that \( \nu \) is a limit ordinal. By inductive hypothesis and by Claim 1 the computation from step \( \gamma \) to step \( \gamma + \delta \times \nu \) consists of \( \nu \)-many repetitions of the loop \( L \). In particular this means that the snapshot \( (R_1(\gamma), I(\gamma)) \) appears cofinally often in the computation up to \( \nu \). Therefore \( R_1(\gamma) \in s\text{Lim}(R_1^{\gamma+\delta \times \nu}) \). Finally, by 2 we have that \( I(\gamma) = \lim_{\alpha<\gamma+\delta \times \nu} I(\gamma) = I(\gamma+\delta \times \nu) \) and \( R_1(\gamma) = \min(s\text{Lim}(R_1^{\gamma+\delta \times \nu})) \). Therefore \( R_1(\gamma) \) as desired.

Finally, note that Claim 2 implies that the computation of \( P \) diverges as desired.

We call a computation \( L = ((R_1(\alpha), \ldots, R_n(\alpha), I(\alpha)))_{\alpha<\Theta} \) as the one in Lemma 14. A strong loop.

We are now able to show that the first \( \Sigma_2 \)-admissible ordinal is an upper bound of \( \text{SITBM} \)-clockable ordinals.

Lemma 15. Let \( \Theta \) be a \( \Sigma_2 \)-admissible ordinal. Let \( P \) be a BSSM-program and let \( ((R_1(\alpha), \ldots, R_n(\alpha), I(\alpha)))_{\alpha<\Theta} \) be the initial segment of the \( \text{SITBM} \)-computation of \( P \) of length \( \Theta + 1 \). Then for every \( i \in \{ 1, \ldots, n \} \) we have that \( R_i : \Theta \to \mathbb{R} \) is not Cauchy and does not have infinite strong limit.

Proof. Assume that \( R_i \) is Cauchy, a similar proof works with the infinite strong limit case. Then for all positive \( n \in \mathbb{N} \) there is \( \gamma < \Theta \) such that \( |R_i(\gamma) - R_i(\Theta)| < \frac{1}{n} \). For each positive \( n \in \mathbb{N} \) let \( \gamma_n \) be the first such ordinal. The function \( n \mapsto \gamma_n \) would be \( \Sigma_2 \)-definable on \( \mathbb{L}_\Theta \) and cofinal in \( \Theta \). But this contradicts the fact that \( \Theta \) is \( \Sigma_2 \)-admissible. So \( R_i \) cannot be Cauchy.

Theorem 16. Let \( \Theta \) be a \( \Sigma_2 \)-admissible ordinal. Let \( P \) be a BSSM-program and let \( ((R_1(\alpha), \ldots, R_n(\alpha), I(\alpha)))_{\alpha<\Theta} \) be the initial segment of the \( \text{SITBM} \)-computation of \( P \) of length \( \Theta + 1 \). Then the computation diverges. In particular every \( \text{SITBM} \)-computation must halt before the first \( \Sigma_2 \)-admissible or diverge.

Proof. Let \( P, \Theta, \) and \( C := ((R_1(\alpha), \ldots, R_n(\alpha), I(\alpha)))_{\alpha<\Theta} \) be as in the statement. We will prove that there is a strong loop below \( \Theta \). Note that \( I(\Theta) \) must appear cofinally often in \( C \) and that there is \( \sigma_\ell < \Theta \) such that for all \( \beta > \sigma_\ell \) we have \( I(\beta) \geq I(\Theta) \).

Claim 3 For each \( i \in \{ 1, \ldots, n \} \) and for every \( r \in \mathbb{R} \) with \( r < R_i(\Theta) \) there is \( \sigma < \Theta \) such that for all \( \beta \geq \sigma \) we have \( R_i(\beta) \geq r \).

Proof. Let \( r \) be such that for every \( \sigma \) there is \( \beta \geq \sigma \) with \( R_i(\beta) \leq r \). Then there is a cofinal subsequence of \( R_i \) which is bounded by \( r \). By Bolzano-Weierstraß theorem this sequence has a convergent subsequence with limit \( \leq r < R_i(\Theta) \). Contradiction!

Note that by Claim 3, since \( \Theta \) is \( \Sigma_2 \)-admissible, we have that for every \( i \in \{ 0, \ldots, n \} \) there is \( \sigma_i \) such that for all \( \beta \geq \sigma_i \) we have \( R_i(\beta) \leq R_i(\Theta) \). Let \( \sigma \) be \( \max(\{\sigma_i; 0 < i \leq n\} \cup \{\sigma_\ell\}) \).
Claim 4 For every $i \in \{1, \ldots, n\}$ and for every $\gamma < \Theta$ there is $\beta \geq \gamma$ such that the $R_i(\Theta) = R_i(\beta)$.

Proof. Note that it follows from Lemma 16 that the case in which $R_i(\Theta) = 0$ due to an infinite strong limit is impossible. Therefore, if $R_i(\Theta)$ can be $0$ only because of a finite limit. So, we only need to show the claim for the finite limit case.

If there is $\gamma < \Theta$ such that for all $\beta \geq \gamma$ we have $R_i(\beta) \neq R_i(\Theta)$, then consider the map $f : n \mapsto \beta_{\frac{1}{n+1}}$ where $\beta_{\frac{1}{n+1}}$ is the first ordinal larger than $\gamma$ with $R_i(\beta_{\frac{1}{n+1}}) - R_i(\Theta) < \frac{1}{n+1}$. Then $f$ is cofinal in $\Theta$ and $\Sigma_2$-definable over $L_\Theta$. Contradiction!

Now, let $\tau_0$ be the smallest ordinal $\geq \sigma$ such that for every $i \in \{1, \ldots, n\}$ there is $\sigma \leq \beta_i \leq \tau_0$ with $R_i(\beta_i) = R_i(\Theta)$ and there is $\sigma \leq \beta_i \leq \tau_0$ with $I(\beta_i) = I(\Theta)$. In general, let $\tau_{j+1}$ be the smallest ordinal bigger than $\tau_j$ such that for every $i \in \{1, \ldots, n\}$ there is $\tau_j \leq \beta_i \leq \tau_{j+1}$ with $R_i(\beta_i) = R_i(\Theta)$, and there is $\tau_j \leq \beta_i \leq \tau_{j+1}$ with $I(\beta_i) = I(\Theta)$. Let $\tau$ be sup$_{j \in \omega} \tau_j$. Note that by $\Sigma_2$-admissibility $\tau < \Theta$ and $(R_1(\tau), \ldots, R_n(\tau), I(\tau)) = (R_1(\Theta), \ldots, R_n(\Theta), I(\Theta))$. Similarly, let $\tau'_0$ be the smallest ordinal that for every $i \in \{1, \ldots, n\}$ there is $\tau \leq \beta_i \leq \tau'_0$ with $R_i(\beta_i) = R_i(\Theta)$ and there is $\tau \leq \beta_i \leq \tau'_0$ with $I(\beta_i) = I(\Theta)$. In general, let $\tau'_{j+1}$ be the smallest ordinal bigger than $\tau'_j$ such that for every $i \in \{1, \ldots, n\}$ there is $\tau'_j \leq \beta_i \leq \tau'_{j+1}$ with $R_i(\beta_i) = R_i(\Theta)$, and there is $\tau'_j \leq \beta_i \leq \tau'_{j+1}$ with $I(\beta_i) = I(\Theta)$. Let $\tau'$ be sup$_{j \in \omega} \tau'_j$. Once more $\tau' < \Theta$ and $(R_1(\tau'), \ldots, R_n(\tau'), I(\tau')) = (R_1(\Theta), \ldots, R_n(\Theta), I(\Theta))$.

But this implies that for every $i \in \{1, \ldots, n\}$, $R_i(\tau)$, $R_i(\tau')$, $I(\tau)$, $I(\tau')$ is a strong loop, therefore the claim follows from Lemma 14.

Given Theorem 16 it is natural to ask if the first $\Sigma_2$-admissible ordinal is an optimal bound for $\Sigma_2$-admissible ordinals. The following lemma shows that this is not the case.

Lemma 17. The supremum of the $\Sigma_2$-clockable ordinals is not $\Sigma_2$-admissible.

Proof. Assume that the supremum $\Theta$ of the $\Sigma_2$-clockable ordinals is $\Sigma_2$-admissible. Let $f$ be the function sending every program to its halting time if it exists and to $0$ otherwise. Note that $f$ is $\Sigma_2$-definable over $L_\Theta$. Moreover, by assumption $f$ would be cofinal in $\Theta$, but this contradicts the $\Sigma_2$-admissibility of $\Theta$.

We end this section by noticing that while in this paper we focused on $\Sigma_2$-clockable ordinals, the notions of eventual (ew) and accidental writability (aw) known from the context of ITTMs make sense for $\Sigma_2$-clockable ordinals. Denoting by $\tilde{\Sigma}_2$ the supremum of the ordinals coded by aw real number, it turns out that, in sharp contrast to ITTMs, there are aw real numbers not contained in $L_{\tilde{\Sigma}_2}$. The reason is roughly that, if there were not, then Welch’s argument for ITTMs would show that the supremum of the $\Sigma_2$-clockable ordinals would be bigger than or equal to the supremum $\lambda$ of the ITTM-writable ordinals. Since $\lambda$ is much bigger than the first $\Sigma_2$-admissible this would contradict Theorem 16.
4 Low Complexity Machines

In the rest of the paper we investigate the ordinals which are clockable by SITBMs whose computations are of low complexity. In particular, we strengthen Lemma 18 which was mentioned without proof in [17].

We will call a BSSM-program $P$ whose SITBM-computations are such that every snapshot only contains rational numbers possibly except for the input registers a rational BSSM-program.

**Theorem 18 (Welch).** Let $\beta$ be the first $\Pi_3$-reflecting ordinal. Then, for every rational BSSM-program $P$ we have that the SITBM-computation of $P$ with input 0 either diverges or halts before $\beta$.

**Lemma 19.** Let $\beta$ be $\Pi_3$-reflecting, $\Theta > \beta$, $r \in \mathbb{R} \cap L_\beta$ be a real number, and $(C_\alpha)_{\alpha \in \Theta}$ be an SITBM-computation with $R_i(\beta) = r$. Then $R_i$ has value $r$ cofinally often below $\beta$, i.e., for all $\gamma < \beta$ there is $\gamma < \alpha < \beta$ such that $R_i(\alpha) = r$.

**Proof.** Note that for every $\tau < \beta$ the following sentences $\forall r' \prec r \exists \alpha > \tau \forall \gamma > \alpha R_i(\gamma) > r'$, and $\forall r' \succ r \forall \alpha > \tau \exists \gamma > \alpha R_i(\gamma) < r'$ are $\Pi_3$ in $L_\beta$. Moreover, they are both true in $L_\beta$ since $R_i(\beta) = r$. Therefore, for every $\gamma < \beta$ there is $\tau < \alpha < \beta$ such that $R_i(\alpha) = r$.

**Theorem 20.** Let $\beta$ be the first $\Pi_3$-reflecting ordinal. Then, for every SITBM-computation $(C_\alpha)_{\alpha \in \Theta}$ such that every snapshot is in $L_\beta$ we have that either $\Theta = \text{On}$ or $\Theta < \beta$.

**Proof.** Assume that $\Theta > \beta$. By Lemma 19 we have that the snapshot $C_\beta$ appears cofinally often before $\beta$. Now, we want to show that this means that the program is in a strong loop. Let $\delta < \beta$ be such that $C_\delta = C_\beta$ and $\delta$. If for some $\gamma < \beta$ some register $R_i$ is such that $R_i(\delta + \gamma) < R_i(\delta)$, then $R_i(\delta + \gamma)$ would have value $R_i(\delta + \gamma)$ cofinally often below $\beta$ but this contradicts the fact that $R_i(\delta + \gamma) < R_i(\delta)$ for all $\gamma < \beta$ but then the computation is in a strong loop. The claim follows by Lemma 14.

In [17, p.31] Welch asked if the bound of Lemma 18 was optimal. The following lemma shows that this is not the case.

**Lemma 21.** The supremum of the SITBM-clockable ordinals is not a $\Pi_3$-reflecting ordinal.

**Proof.** It is enough to note that the sentence expressing the fact that every SITBM-computation either diverges or stops is $\Pi_3$, and can therefore be reflected below any $\Pi_3$-reflecting ordinal.

\(^5\) In [17, p.31] Welch mentions that the first $\Pi_3$-reflecting ordinal is an upper bound to the computational strength of SITBMs. In a private communication he clarified to the authors that the machines he was referring to are actually machines which only work on rational numbers. The question of whether the first $\Pi_3$-reflecting ordinal is an upper bound to the computational strength of SITBMs is still open.
Corollary 22. The supremum of the SITBM-clockable ordinals clockable by a rational BSSM-program is strictly smaller than the first $\Pi_3$-reflecting ordinal.

Let $P$ be a BSSM-program whose SITBM-computation $(C_\beta)_{\beta \in \text{On}}$ diverges. We will call the strong looping time of $P$ the least ordinal $\alpha$ such that there is $\beta < \alpha$ and $(C_\gamma)_{\beta \leq \gamma \leq \alpha}$ is a strong loop. Note that by Theorem 16 every divergent program has a strong limit time.

Lemma 23. The supremum of the strong looping times is not $\Pi_2$-reflecting thus in particular not admissible.

Proof. It is enough to note that the sentence expressing the fact that a program is in a strong loop is $\Pi_2$, and can therefore be reflected below the first $\Pi_2$-reflecting ordinal.

We end this section by showing that the model of computation obtained by restricting SITBMs to rational BSSM-programs is strictly weaker than the unrestricted one.

Lemma 24. There is a BSSM-program $P$ that decides the halting problem restricted to rational BSSM-programs, i.e., given the code $c$ of a rational BSSM-program, $P$ halts with output 0 if the program coded by $c$ with input 0 halts, and $P$ halts with output 1 if the program coded by $c$ with input 0 diverges.

Proof. Let $U$ be the universal SITBM machine of Lemma 9. We will describe the program $P$ and leave the precise implementation to the reader.

Given a code $c$ for a rational BSSM-program $P'$ as input $P$ starts executing $U$ on $c$. The program $P$ will use the register $R$ to keep track of the snapshots that appeared in the computation. At each step of the computation $P$ first checks if the simulation halted, in which case $P$ halts with output 0. If the current snapshot of the simulation of $P'$ is not an halting snapshot, $P$ runs a new copy of $U$ looking for the code $n$ of a program computing the current snapshot of the simulation of $P'$.

Once $P$ finds the code for the snapshot it will check if the $n$th bit of binary sequence represented by the real in register $R$ is one, in which case the $P$ halts with output 1. If not then for every $i \in \mathbb{N}$ the program $P$ checks if the $i$th bit of the binary sequence represented by the register register $R$ is 1 in this case computes the snapshot coded by $i$ and checks that every element of the snapshot is smaller than or equal to the corresponding element in the snapshot coded by $n$. If this is the case then $P$ sets the $(2n+1)$st bit of $R$ to 1 and continues. If not then machine erases $R$, set the $(2n+1)$st bit of $R$ to 1 and continues.

One can check that the $P$ does compute the halting problem restricted to rational BSSM-programs.

As usual a subset $A$ of real numbers is called SITBM-semi decidable if there is a BSSM-program $P$ such that for every $r \in A$ the SITBM-computation of $P$ with input $r$ halts, and for every $r \notin A$ the SITBM-computation of $P$ with input $r$ diverges. Moreover, if $P$ can be chosen to be a rational BSSM-program then we will say that $A$ is rationally SITBM-semi decidable.
Corollary 25. Every rationally SITBM-semi decidable set is SITBM-computable.

Corollary 26. There is a SITBM-semi decidable set which is not rationally SITBM-semi decidable.

References
[1] Barwise, J.: Admissible Sets and Structures. Perspectives in Logic, Cambridge University Press (2017)
[2] Blum, L., Shub, M., Smale, S.: On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines. Bulletin of the American Mathematical Society 21, 1–46 (1989)
[3] Carl, M.: Ordinal Computability. An Introduction to Infinitary Machines. De Gruyter (2019)
[4] Carl, M.: Space and time complexity for Infinite Time Turing Machines (2019), arXiv:1905.06832
[5] Carl, M.: Taming Koepke’s zoo II: Register machines (2019), arXiv:1907.09513
[6] Carl, M., Fischbach, T., Koepke, P., Miller, R., Nasf, M., Weckbecker, G.: The basic theory of infinite time register machines. Archive for Mathematical Logic 49(2), 249 – 273 (2010)
[7] Galeotti, L.: The theory of generalised real numbers and other topics in logic. Ph.D. thesis, Universität Hamburg (2019)
[8] Galeotti, L.: Surreal blum-shub-smale machines. In: Manea, F., Martin, B., Paulusma, D., Primiero, G. (eds.) Computing with Foresight and Industry. pp. 13–24. Springer International Publishing, Cham (2019)
[9] Hamkins, J.D., Lewis, A.: Infinite time Turing machines. Journal of Symbolic Logic 65, 567–604 (2000). https://doi.org/10.2307/2586556
[10] Koepke, P., Morozov, A.S.: The computational power of infinite time Blum-Shub-Smale machines. Algebra and Logic 56(1), 37–62 (2017)
[11] Koepke, P., Seyfferth, B.: Ordinal machines and admissible recursion theory. Annals of Pure and Applied Logic 160(3), 310 – 318 (2009)
[12] Koepke, P.: Ordinal computability. In: Ambos-Spies, K., Löwe, B., Merkle, W. (eds.) Mathematical Theory and Computational Practice. pp. 280–289. Springer Berlin Heidelberg, Berlin, Heidelberg (2009)
[13] Koepke, P., Miller, R.: An enhanced theory of infinite time register machines. In: Beckmann, A., Dimitracopoulos, C., Löwe, B. (eds.) Logic and Theory of Algorithms. pp. 306–315. Springer (2008)
[14] Koepke, P., Seyfferth, B.: Towards a theory of infinite time Blum-Shub-Smale Machines. In: Cooper, S.B., Dawar, A., Löwe, B. (eds.) How the World Computes: Turing Centenary Conference and 8th Conference on Computability in Europe, CiE 2012, Cambridge, UK, June 18-23, 2012. Proceedings. vol. 7318, pp. 405–415. Springer (2012)
[15] Sacks, G.E.: Higher Recursion Theory. Perspectives in Logic, Cambridge University Press (2017). https://doi.org/10.1017/9781316717301
[16] Seyfferth, B.: Three Models of Ordinal Computability. Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universität Bonn (2013)
[17] Welch, P.D.: Turing’s legacy: developments from Turing’s ideas in logic. Lecture Notes in Logic (42), 493–529 (2014)
[18] Welch, P.: Eventually infinite time turing machine degrees: infinite time decidable reals. Journal of Symbolic Logic 65(3), 11931203 (2000). https://doi.org/10.2307/2586695