Circular Sound Wave Scattering Derivation for Acoustic Cloak Detection

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Abstract

In this Letter we develop analytical formulations to describe sound scattering in lossless medium due to 2D circular wave incident on an acoustic cloak. A perfect acoustic cloak is reflectionless and can completely hide the cloaked object from any sound waves. However, the realization of a perfect acoustic cloak is difficult. Compared to plane wave, our analytic calculations show that circular wave from an annular line source generates distinct scattering patterns from an imperfect cloak design. Large modification in reflection directivities can be observed if the focal point of the incident wavefront is slightly customized. Hence, our work might find applications in acoustic cloak detection, which should have significant impact on cloak design and defense.

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A cloak bends wave fields in desired directions to shield any interior object from detection [1–4]. The theory behind cloak design is a conformal map that transforms a physical region with an interior cloaked hole to a virtual domain that is mathematically simply connected. This interesting principle is firstly proposed by Pendry et al. [5] and Schurig et al. [6] for electromagnetic wave cloaking. Cummer and Schurig presented a 2D acoustic cloaking design based on the isomorphism between acoustic equations and Maxwells equations [3]. Chen and Chan elegantly extended the transformation based acoustic cloak design to 3D spherical cloak cases [2]. Cummer et al. reported the same 3D design through a different route that relies on a spherical harmonic scattering analysis [7]. Acoustic cloak designs are generally confirmed via numerical simulations with a plane, progressive, harmonic wave. Some experimental demonstrations of acoustic cloak have been conducted for linear surface waves (at 10 Hz) [8], audible sound waves (at 1 to 3 kHz) [9], and ultrasound waves (at 52 to 60 kHz) [10].

In this Letter, we develop an analytical solution that describes the interaction between an acoustic cloak and circular waves radiating from an annular line source. Our results demonstrate the existence of peculiar scattering patterns if metamaterial properties [11] of an acoustic cloak slightly differ from idealized ones. In addition, distinctively different scattering patterns can be found from circular cloak shell simply by slightly adjusting the focal point of the wavefront. In contrast to normally incident plane wave, this annular line source setup might find applications in cloak detection.

Our investigation is conducted in an analytical way that can clearly provide physical insights. Before starting the analytical derivations, we have need to recall the theoretical foundation behind acoustic cloak. The propagation of linear sound wave perturbations is governed by linear Euler equations. With an $e^{-i\omega t}$ time dependence, these equations have the following form for a cloaked region,

$$i\omega p = \kappa\kappa_0\nabla \cdot \vec{v}, \quad (1)$$

$$\nabla p = i\omega\rho\rho_0\vec{v}, \quad (2)$$

where $p$ is sound pressure; $\vec{v}$ is the associated particle velocity; $\kappa$ is anisotropic fluid bulk modulus relative to $\kappa_0$; $\rho$ is anisotropic cloak density relative to fluid density $\rho_0$; $\rho_0$ and $\kappa_0$ are normalized to unity. For simplicity, the discussion in the following is focused on circular cloaked shell in 2D polar $(r – \theta)$ coordinates (as shown in Fig. 1). The outer shell radius
is a and inner radius is b, which is nil in the transformed virtual region for a perfect 2D acoustic cloak. In other words, the transformation for an ideal cloak maps the physical region \((b < r < a)\) to the virtual region \((0 < r < a)\). In the physical region, the cloak density is a second order tenser, that is, \(\rho = \text{diag}(\rho_r, \rho_\theta)\). The relative anisotropic density and bulk modulus are

\[
\rho_r = \frac{r}{r-b}, \rho_\theta = \frac{r-b}{r}, \kappa = \left(\frac{a}{a-b}\right)^2 \frac{r-b}{r}.
\]  

(3)

Outside a cloak, \(\rho_r, \rho_\theta\) and \(\kappa\) are unity. It is easy to see that the above design is impractical as \(\rho_r\) will go infinity as \(r\) approaches \(b\). To avoid this potential singularity, the physical region \((b < r < a)\) can be mapped to a virtual region \((r_0 < r < a, r_0 > 0)\) by a linear transformation \(f\). Accordingly, Norris [12] recently developed generic material specifications,

\[
\kappa(r) = \frac{1}{f'} \left(\frac{r}{f}\right)^{d-1}, \rho_r = \left(\frac{r}{f}\right)^{d-1} f', \rho_\theta = \left(\frac{r}{f}\right)^{d-1} \frac{f^2}{r^2 f'},
\]  

(4)

where \(f' = df/dr\), and \(d = 2\) for 2D cases; \(d = 3\) for 3D cases.

From Eqs. (1)-(2) it can be seen that harmonic acoustic pressure \(p\) inside acoustic cloak is governed by the following wave equation

\[
\nabla \cdot (\rho^{-1} \nabla p) + \frac{k^2}{\kappa} p = 0,
\]  

(5)

where \(k\) is the normalized wavenumber. In 2D polar coordinates, this wave equation has the following form,

\[
\frac{\partial}{\partial r} \left( \frac{1}{\rho_r} \frac{\partial p}{\partial r} \right) + \frac{1}{r \rho_r} \frac{\partial p}{\partial r} + \frac{1}{r^2 \rho_\theta} \frac{\partial^2 p}{\partial \theta^2} - \frac{k^2}{\kappa} p = 0.
\]  

(6)

Adopting the method of separation of variables, we let \(p(r, \theta) = R(r)\Theta(\theta)\) and set \(\Theta(\theta) = e^{im\theta}\), where \(m\) is an integer. Hence, \(R(r)\) satisfies the following ordinary differential equation,

\[
r \rho_\theta \frac{\partial}{\partial r} \left( \frac{r}{\rho_r} \frac{\partial R(r)}{\partial r} \right) + \left( \frac{k^2 \rho_\theta r^2}{\kappa} - m^2 \right) R(r) = 0.
\]  

(7)

From Eq. (4) we have \(\rho_r = rf'/f\) and \(\rho_\theta = f/(rf')\) for 2D cases, Eq. (7) becomes

\[
\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{f} \frac{\partial R(r)}{\partial f} + \left( k^2 - \frac{m^2}{f^2} \right) R(r) = 0.
\]  

(8)

It is thus easy to find that the solution of Eq. (8) has the form: \(R(r) = B_m J_m(kf(r)) + C_m H_m(kf(r))\), where \(J_m\) is the \(m\)th order Bessel function of the first kind, \(H_m\) is the \(m\)th
order Bessel function of the third kind, and $B_m$ and $C_m$ are parameters to be determined. As a consequence, the sound pressure in an acoustic cloak is

$$p^{clk}(r) = \sum_{m=-\infty}^{\infty} [B_m J_m(k f(r)) + C_m H_m(k f(r))] e^{im\theta}. \quad (9)$$

Taking into account Sommerfeld radiation condition, the series forms of sound solutions in other regions are

$$p^{inc} = \sum_{m=-\infty}^{\infty} K_m J_m(k r) e^{im\theta},$$

$$p^{scat} = \sum_{m=-\infty}^{\infty} A_m H_m(k r) e^{im\theta},$$

$$p^{int} = \sum_{m=-\infty}^{\infty} D_m J_m(k r) e^{im\theta}, \quad (10)$$

where $p^{inc}$ is incident sound pressure, $p^{scat}$ is sound pressure scattered from an acoustic cloak, and $p^{int}$ is sound pressure in the interior of the cloaked hollow region.

The normal velocities can be derived using Eq. (2), which yields

$$v_r^{inc} = \frac{k}{\rho_0} \sum_{m=-\infty}^{\infty} K_m J'_m(k r) e^{im\theta},$$

$$v_r^{scat} = \frac{k}{\rho_0} \sum_{m=-\infty}^{\infty} A_m H'_m(k r) e^{im\theta},$$

$$v_r^{clk} = \frac{k f'(r)}{\rho_0 \rho_r} \sum_{m=-\infty}^{\infty} (B_m J'_m(k f(r)) + C_m H'_m(k f(r))) e^{im\theta},$$

$$v_r^{int} = \frac{k}{\rho_0} \sum_{m=-\infty}^{\infty} D_m J'_m(k r) e^{im\theta}, \quad (11)$$

where $'$ stands for $d/dr$ and $f'(r)/\rho_r = (f/r)$ for 2D cases. Sound pressure and normal velocity should be continuous at cloak interfaces, that is, $r = a$ and $r = b$ in the physical region. We can therefore have the following relations,

$$K_m J_m(k b) + A_m H_m(k b) = B_m J_m(k b) + C_m H_m(k b),$$

$$K_m J'_m(k b) + A_m H'_m(k b) = B_m J'_m(k b) + C_m H'_m(k b),$$

$$B_m J_m(k r_0) + C_m H_m(k r_0) = D_m J_m(k a),$$

$$\frac{r_0}{a} (B_m J'_m(k r_0) + C_m H'_m(k r_0)) = D_m J'_m(k a), \quad (12)$$
which yield

\[ A_m = \frac{K_m}{G}, \quad B_m = K_m, \quad C_m = \frac{K_m}{G}, \]
\[ D_m = \frac{E_1 E_6 - E_3 E_4}{E_2 E_6 - E_3 E_5} K_m, \]

where the following formulas are used,

\[ E_1 = J_m(kr_0), \quad E_2 = H_m(kr_0), \quad E_3 = J_m(ka), \]
\[ E_4 = \frac{r_0}{a} J_m'(kr_0), \quad E_2 = \frac{r_0}{a} H_m'(kr_0), \quad E_3 = J_m'(ka), \]
\[ G = \frac{E_2 E_6 - E_3 E_5}{E_1 E_5 - E_2 E_4}. \]

A 2D plane, progressive, harmonic wave can be described by \( p^{inc}(x) = e^{ikx} \), which equals \( e^{ikr\cos\theta} \), that is, \( \sum_{m=-\infty}^{\infty} i^n J_m(kr)e^{im\theta} \). Hence, the corresponding \( K_m \) in Eq. (10) is \( i^m \) for plane wave case. Figure 1 shows a plane wave sound pressure field calculated from above series solutions. The 2D plane wave is incident from the left onto an acoustic cloak with \( a = 1 \) and \( b = 0.5 \). The normalized wavenumber is 10. For a perfect cloaking design, where \( f : (b < r < a) \to (0 < r < a) \), no reflection can be observed in Fig. 1(a). Since a perfect cloak design is difficult to implement, \( f : b \mapsto r_0, r_0 > 0 \) is adopted in practical implementations. For example, \( r_0 = 0.02 \) is represented in Fig. 1(b) by the dashed circle. However, scattering due to imperfect cloak design is now clearly visible in Fig. 1(b).

It is of interest to study the scattering patterns if a circular sound wave is incident on an imperfect acoustic cloak. Figure 2 shows the setup of the problem. The circular wave incident on a cloak is generated by an annular line sound source, which presumably consists of numerous point sources obeying a uniform distribution. The radius of the annular line source is \( R \), which is set to 3 in the following demonstration. The distance between the origin of the cloaked hollow region and the expected focal point of wavefront is \( L \). The 2D annular line source gives

\[ J_0(kR) = \sum_{m=-\infty}^{\infty} J_m(kL) J_m(kr)e^{im\theta}, \]

that is, \( K_m = J_m(kL) \) for the circular wave case. Without the presence of an cloak, the origin of the line source will be the focal point of wavefronts. In the following demonstrations, we slightly move the focal point along the focal path in Fig. 2. Practical implementation can be realized by marrying the concept of beamforming (in array signal processing) \[13, 14\] with harmonic analysis introduced in this Letter.
In our demonstration, Eqs. (9)-(14) can be used to calculate sound propagation and reflection. Figure 3 shows sound reflections due to circular waves with \( L = 0.1 \) and \( L = 0.3 \), respectively. It can be seen that sound reflections are symmetric with respect to the \( x \)-axis. The resultant scattering patterns are quite different than plane wave cases. In addition, distinctive scattering patterns appear if the focal point of circular wavefront is modified. When \( L = 0.1 \), the directivity of sound reflections are approximately in the \( x \) direction. When \( L = 0.3 \), sound reflections move around the \( y \) direction. For imperfect cloak cases in Fig. 1(b), it can be seen that almost no sound energy reaches the interior region. In contrast, it appears that a small fraction of sound energy arrives the cloaked hollow region. More details of sound transmission and reflections can be found in the online supplementary materials (including animations of plane wave and circular wave cases). It is for this reason that our work might find applications in imperfect acoustic cloak detection.

In summary, the method proposed in this Letter permits a purely analytical design of acoustic cloak detection strategy. Using 2D harmonic analysis, our study demonstrates unique scattering pattern due to incident circular waves generated by an annular line source. We have also developed 3D formulations, which are omitted here for brevity. In this work we conducted 2D simulations using the proposed analytical formulations at various \( r_0 \) and \( L \) and similar conclusions can be drawn. It is worthwhile to note that the series solution works to any wavelength condition. Hence, we conclude that an annular line source might be useful in detecting an enclosed imperfect acoustic cloak, which should have a significant impact on acoustic cloak design and defense.

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FIG. 1: The real part of sound pressure field in the physical region. The plane wave is incident from the left. (a) Perfect cloak case, material properties by Eq. (3). (b) Imperfect cloak, material properties by Eq. (4), $f$ maps $b = 0.5$ to $r_0 = 0.02$.

FIG. 2: Configuration of circular wave from an annular line source incident on an acoustic cloak.
FIG. 3: Sound scattering due to circular waves incident on an imperfect acoustic cloak by Eq. (4), $f$ maps $b = 0.5$ to $r_0 = 0.02$. (a) $L = 0.1$. (b) $L = 0.3$, where the black solid dot stands for the expected focal point.