3-stage and 4-stage tests
with deterministic stage sizes
and non-iid data

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Abstract: Given a fixed-sample-size test that controls the error probabilities under two specific, but arbitrary, distributions, a 3-stage and two 4-stage tests are proposed and analyzed. For each of them, a novel, concrete, non-asymptotic, non-conservative design is specified, which guarantees the same error control as the given fixed-sample-size test. Moreover, first-order asymptotic approximation are established on their expected sample sizes under the two prescribed distributions as the error probabilities go to zero. As a corollary, it is shown that the proposed multistage tests can achieve, in this asymptotic sense, the optimal expected sample size under these two distributions in the class of all sequential tests with the same error control. Furthermore, they are shown to be much more robust than Wald’s SPRT when applied to one-sided testing problems and the error probabilities under control are small enough. These general results are applied to testing problems in the iid setup and beyond, such as testing the correlation coefficient of a first-order autoregression, or the transition matrix of a finite-state Markov chain, and are illustrated in various numerical studies.

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1. Introduction

A typical motivation for employing a sequential test, i.e., a testing procedure whose sample size depends on the collected observations, is that its average sample size can be much smaller than that of the corresponding fixed-sample-size test. The first test of this kind in the literature was the double sampling procedure of Dodge and Romig [13], a precursor to Wald’s Sequential Probability Ratio Test (SPRT) [32] and the field of “sequential analysis”. However, the implementation of the SPRT, as well as of many sequential tests in the literature (see, e.g., [30]), requires continuous monitoring of the data collection process, which is often inconvenient, or even infeasible, in application areas such as sampling inspection and clinical trials [19, 6]. As a result, the emphasis in such applications has been on group-sequential tests, like the one in [13], i.e., sequential tests whose implementation requires the collection of only a small number of groups of samples. An equivalent terminology, which we use in this work, is multistage tests, in which the groups of samples are referred to as stages.

Most works about multistage tests, e.g., [1, 28, 25, 29, 33, 16, 15, 26, 2], (i) focus on testing the mean of iid Gaussian observations with known variance, (ii) are designed to control prescribed type-I and type-II error probabilities under two specific distributions, and (iii) require equal stage sizes. Free parameters, if any, as in [33], are selected to optimize the expected sample size under a certain distribution, such as the one under which the type-II error probability is controlled. This optimization is performed via dynamic programming in [15, 2].

Multistage tests with unequal and random stage sizes are considered in [22, 20, 18], as well as in [21]. In the latter, more general testing problems, regarding the parameters of an exponential family, are also studied.

In all the above works the stage sizes are treated as user-specified inputs. Lorden in [23] showed that 3-stage tests, with properly selected stage sizes, achieve asymptotically the optimal expected sample size, under both hypotheses, among all sequential tests with the same or smaller error probabilities as the latter go to 0. In the case of simple hypotheses for iid data, this was shown for tests with deterministic stage sizes [23, Section 2]. On the other hand, in the case of composite hypotheses for the one-sided testing problem in a one-parameter exponential family, this was shown for tests whose stage sizes are adaptive, i.e., they can depend on the data from the previous stages [23, Section 3]. Such multistage tests were also considered in [4, 5], where they were designed to be less conservative than in [23, Section 3]. All these asymptotic optimality results require certain assumptions on the decay rates of the prescribed error probabilities, which are not allowed to go to 0 very asymmetrically.

In the present work we focus on the design and analysis of multistage tests with deterministic stage sizes, and we strengthen, extend and generalize the results in [23, Section 2]. First of all, unlike all the above mentioned works, we do not require that the observations be either independent or identically distributed. Instead, we only assume that a fixed-sample-size test is given, which can control the type-I and type-II error probabilities under two specific distributions below arbitrary levels. Given such a test, we introduce and analyze
a 3-stage test, that generalizes the one in [23, Section 2], as well as two novel 4-stage tests. For each of them, we propose a novel, concrete, non-asymptotic, non-conservative specification, which guarantees the same error control as the fixed-sample-size test. This specification only requires knowledge of the number of observations and the threshold the fixed-sample-size test requires for its error control. While there are not, in general, explicit formulas for these quantities, they can be estimated via simulation. In the case of very small error probabilities, in which plain Monte-Carlo is not efficient or even feasible (see, e.g., [10]), we propose a simulation approach via importance sampling.

In order to obtain theoretical insights regarding the proposed multistage tests, we impose some structure on the above general setup. Essentially, we assume that there are thresholds for which the error probabilities of the given fixed-sample-size test, under the two prescribed distributions, decay exponentially fast in the sample size. Using the Gärtner-Ellis theorem from large deviation theory (see, e.g., [12]), we show that the required conditions are satisfied in various testing problems beyond the iid setup. Two specific examples, which we work out in detail, are testing the correlation coefficient of a first-order autoregression, and testing the transition matrix of an irreducible and recurrent finite-state Markov chain.

Assuming that the above conditions hold, we establish first-order asymptotic approximations to the expected sample sizes of the proposed multistage tests under the distributions with respect to which we control the error probabilities, as the latter go to 0. For the 3-stage test, the relative decay of the error probabilities is allowed to be much more asymmetric than the one required in [23, Section 2]. Even more asymmetric rates are allowed for each of the two 4-stage tests. As a corollary, we extend the asymptotic optimality of the 3-stage test in [23, Section 2], beyond the iid setup and for more asymmetric error probabilities. Moreover, we show that the two proposed 4-stage tests are asymptotically optimal, in the same setup as the 3-stage test, with even more asymmetric error probabilities. These results are also illustrated in a numerical study, where these multistage tests are compared with the SPRT with respect to their average sample sizes under the two prescribed distributions.

In order to obtain a more complete understanding of how the proposed multistage tests perform, especially in comparison to the SPRT, it is important to assess their behavior when the true distribution is different from those under which we control the error probabilities. Indeed, when the SPRT is applied to the one-sided testing problem for the mean of iid Gaussian observations with known variance, as suggested in [31, Chapter 7.5], its expected sample size can be much larger even than that of the corresponding optimal fixed-sample-size test when the true mean is between the values used for the design of the SPRT (see, e.g., [7]). Motivated by this phenomenon, we establish a distribution-free asymptotic upper bound on the expected sample sizes of the proposed multistage tests, as at least one of the two prescribed error probabilities goes to 0. This reveals that when the prescribed error probabilities are small enough, the proposed multistage tests are much more robust than the SPRT, thus, they may be preferable not only because of their practical advantages, but also based on
statistical considerations.

The remainder of this paper is organized as follows. In Section 2 we formulate
the testing setup and in Section 3 we introduce and analyze the proposed
multistage tests. In Section 4 we state our asymptotic results, and in Section 5 we
state sufficient conditions for this asymptotic analysis. In Section 6 we propose
an importance sampling approach for the implementation of the proposed tests
when the error probabilities are small. In Section 7 we illustrate the general
theory in three specific testing problems. In Section 8 we present the results of
our numerical studies. In Section 9 we conclude and discuss potential extensions.
The proofs of most results are presented in Appendices A, B, C.

Finally, we introduce some notations that we use throughout the paper. We
denote by $\mathbb{N}$ the set of positive integers, i.e., $\mathbb{N} \equiv \{1, 2, \ldots\}$, and by $\mathbb{R}$ the set of
real numbers. For a set $A$ we denote by $1\{A\}$ its indicator function and by $A^o$
its interior. For a function $f : \mathbb{R} \to (-\infty, \infty]$, we call $\{x \in \mathbb{R} : f(x) < \infty\}$ the
effective domain of $f$, and denote by $f(x^+)$ the right limit and by $f(x^-)$ the
left limit of $f$ at $x \in \mathbb{R}$, when they exist. For $x, y \in \mathbb{R}$ we set $x \wedge y \equiv \min\{x, y\}$
and $x \vee y \equiv \max\{x, y\}$. For positive sequences $(x_n), (y_n)$, we write $x_n \sim y_n$ for
$\lim_{n} (x_n/y_n) = 1$, $x_n \lesssim y_n$ for $\liminf_{n} (x_n/y_n) \leq 1$, $x_n \preceq y_n$ for $\lim_n (x_n/y_n) \leq 1$,
$x_n \ll y_n$ for $x_n/y_n \to 0$, and $x_n \gg y_n$ for $x_n/y_n \to \infty$.

2. Problem formulation

We consider a sequence of $S$-valued random elements, $X \equiv \{X_n, n \in \mathbb{N}\}$, where
$(S, \mathcal{S})$ is an arbitrary measurable space. For any $n \in \mathbb{N}$, we denote by $\mathcal{F}_n$ the $\sigma$-
algebra generated by the first $n$ terms of this sequence, i.e., $\mathcal{F}_n \equiv \sigma(X_1, \ldots, X_n)$.
Moreover, we denote by $\mathcal{P}$ the distribution of $X$, assume that it belongs to some
family, $\mathcal{P}$, and consider the following hypotheses for it,

$$H_0 : \mathcal{P} \in \mathcal{P}_0 \quad \text{versus} \quad H_1 : \mathcal{P} \in \mathcal{P}_1,$$

where $\mathcal{P}_0$ and $\mathcal{P}_1$ are disjoint subsets of $\mathcal{P}$.

2.1. Tests

We allow the data to be collected sequentially so that, after each observation,
the decision whether to stop sampling or not and, in the former case, whether
to select the null or the alternative hypothesis, can depend on all the already
collected data. Thus, we say that $\chi \equiv (\tau, d)$ is a test for (2.1) if the random time,
$\tau$, that represents the utilized sample size, is a stopping time with respect to the
filtration $\{\mathcal{F}_n, n \in \mathbb{N}\}$, and the Bernoulli random variable, $d$, that represents
the decision ($H_i$ being selected when $d = i$, where $i \in \{0, 1\}$) is $\mathcal{F}_\tau$-measurable,
i.e.,

$$\{\tau = n\}, \{\tau = n, d = i\} \in \mathcal{F}_n, \quad \forall \; n \in \mathbb{N}, \; i \in \{0, 1\}.$$

We denote by $\mathcal{C}$ the family of all tests, and we further introduce a subfamily
of tests that control the two error probabilities under two specific, but arbitrary,
distributions. To be specific, we fix $P_0 \in \mathcal{P}_0$, $P_1 \in \mathcal{P}_1$ and, for any $\alpha, \beta \in (0, 1)$, we denote by $\mathcal{C}(\alpha, \beta)$ the family of tests whose type-I error probability under $P_0$ does not exceed $\alpha$ and whose type-II probability under $P_1$ does not exceed $\beta$, i.e.,

$$\mathcal{C}(\alpha, \beta) \equiv \{(\tau, d) \in \mathcal{C} : P_0(d = 1) \leq \alpha \quad \text{and} \quad P_1(d = 0) \leq \beta\}. \quad (2.2)$$

### 2.2. The fixed-sample-size test

Our only standing assumption throughout the paper is that there is a sequence of test statistics, $T \equiv \{T_n, n \in \mathbb{N}\}$, such that $T_n$ is $\mathcal{F}_n$-measurable for every $n \in \mathbb{N}$ and, for any $\alpha, \beta \in (0, 1)$, there exist $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$ so that the fixed-sample-size test that rejects $H_0$ if and only if $T_n > \kappa$ belongs to $\mathcal{C}(\alpha, \beta)$. Supposing the dependence on $T$, we denote by $n^*(\alpha, \beta)$ the smallest such sample size, i.e.,

$$n^*(\alpha, \beta) \equiv \inf\{n \in \mathbb{N} : \exists \kappa \in \mathbb{R} : P_0(T_n > \kappa) \leq \alpha \quad \text{and} \quad P_1(T_n \leq \kappa) \leq \beta\}, \quad (2.3)$$

and by $\kappa^*(\alpha, \beta)$ any of the corresponding thresholds. In Section 6 we discuss the estimation of these quantities via Monte-Carlo simulation when they do not admit closed-form expressions.

### 2.3. Goals

The main goal of this work is to design multistage tests with deterministic stage sizes that

(i) belong to $\mathcal{C}(\alpha, \beta)$, for any choice of $\alpha, \beta \in (0, 1)$,

(ii) are robust, in the sense that their expected sample sizes under any plausible distribution are not much larger than $n^*(\alpha, \beta)$, when $\alpha, \beta$ are small enough,

and, if additionally the test statistic $T$ is selected appropriately,

(iii) achieve asymptotically, as $\alpha, \beta \to 0$, the optimal expected sample size in $\mathcal{C}(\alpha, \beta)$ under both $P_0$ and $P_1$, $\mathcal{L}_0(\alpha, \beta)$ and $\mathcal{L}_1(\alpha, \beta)$, where

$$\mathcal{L}_i(\alpha, \beta) \equiv \inf \{E_i[\tau] : (\tau, d) \in \mathcal{C}(\alpha, \beta)\}, \quad i \in \{0, 1\}, \quad (2.4)$$

and $E$ and $E_i$ represent expectation under $P$ and $P_i$, $i \in \{0, 1\}$.

The error control in (i) and the asymptotic optimality property in (iii) are common goals in many sequential testing formulation, including [23, Section 2]. In order to explain the necessity and importance of the robustness property in (ii), it is useful to consider the special case of the generic one-sided testing problem.
2.4. The one-sided testing problem

Consider the case where the family of plausible distributions, $\mathcal{P}$, is parametrized by a scalar parameter, $\mu$, taking values in an open interval $M \subseteq \mathbb{R}$. That is, if $\mathbb{P}_\mu$ denotes the distribution, and $\mathbb{E}_\mu$ the expectation, of $X$ when the true parameter is $\mu$, then

$$\mathcal{P} = \{ \mathbb{P}_\mu : \mu \in M \}.$$ 

Moreover, suppose that the testing problem of interest is whether the true parameter $\mu$ is smaller or larger than some user-specified value, $\mu^*_\ast \in M$, i.e.,

$$H_0 : \mu < \mu^*_\ast \text{ versus } H_0 : \mu > \mu^*_\ast,$$  

or equivalently

$$\mathcal{P}_0 = \{ \mathbb{P}_\mu : \mu < \mu^*_\ast \}, \quad \mathcal{P}_1 = \{ \mathbb{P}_\mu : \mu > \mu^*_\ast \}.$$  

(2.5)  

(2.6)

If also, it is required that the type-I error probability be controlled below $\alpha$ when $\mu = \mu_0$ and the type-II error probability below $\beta$ when $\mu = \mu_1$, where $\alpha, \beta \in (0, 1)$ and

$$\mu_0, \mu_1 \in M, \quad \mu_0 < \mu_1, \quad \mu_0 \leq \mu^*_\ast \leq \mu_1,$$

then this is a special case of the framework of this section, with

$$\mathcal{P}_i = \mathbb{P}_{\mu_i}, \quad i \in \{0, 1\}.$$  

(2.7)

In this context, the asymptotic optimality property in (iii) guarantees that the expected sample size when the true parameter is in $\{\mu_0, \mu_1\}$ will be relatively close to the optimal in $\mathcal{C}(\alpha, \beta)$, at least when $\alpha, \beta$ are small enough. However, it is well known (see e.g., [7]) that the expected sample size of such an asymptotically optimal test may be unacceptably large when the true parameter is between $\mu_0$ and $\mu_1$ (see also Subsection 4.4.1 below). This phenomenon motivates the design of sequential tests that are asymptotically optimal even when the true parameter is not in $\{\mu_0, \mu_1\}$, (see, e.g., [11, Chapter 16]). Such an asymptotic optimality property has been established for fully sequential tests (see, e.g., [30, Chapter 5]) and for multistage tests with adaptive stage sizes (see, e.g., [23], [3], [4]). However, it is not, in general, achievable by multistage tests with deterministic stage sizes, which cannot easily adapt to the true value of the parameter. Thus, the robustness property in (ii) guarantees that, even if it is asymptotically suboptimal, the average sample size of such a multistage test does not exceed, at least by much, that of the corresponding fixed-sample-size test, no matter what the true distribution is. As a result, it is a necessary complement to the asymptotic optimality property in (iii), making sure that the latter does not come at the price of an inflated expected sample size when the true parameter is between $\mu_0$ and $\mu_1$. 
Remark: In the context of the above one-sided testing problem, it is desirable that a test $\chi \equiv (\tau, d)$ in $C(\alpha, \beta)$ controls the type-I error probability below $\alpha$ for every $\mu \leq \mu_0$ and the type-II error probability below $\beta$ for every $\mu \geq \mu_1$, i.e.,

$$
P_{\mu}(d = 1) \leq \alpha \quad \forall \mu \leq \mu_0 \quad \text{and} \quad P_{\mu}(d = 0) \leq \beta \quad \forall \mu \geq \mu_1,
$$

where $\alpha, \beta \in (0, 1)$. This is obviously the case for the given fixed-sample-size test that rejects $H_0$ if and only if $T_n > \kappa$ when

$$
P_{\mu_0}(T_n > \kappa) = \sup_{\mu \leq \mu_0} P_{\mu}(T_n > \kappa),
$$

$$
P_{\mu_1}(T_n \leq \kappa) = \sup_{\mu \geq \mu_1} P_{\mu}(T_n \leq \kappa).
$$

If the monotonicity property in (2.9) holds for every $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$, then the uniform error control in (2.8) will also hold, for every $\alpha, \beta \in (0, 1)$, for the proposed multistage tests in this work.

3. The multistage tests

In this section we introduce and analyze the multistage tests that we consider in this work.

3.1. The 3-stage test

We next introduce and analyze a test that offers two opportunities to accept the null hypothesis and two to reject it. Its implementation requires the specification of three positive integers, $n_0, n_1, N$, and three real thresholds, $\kappa_0, \kappa_1, K$, so that

$$n_0 \vee n_1 < N \quad \text{and} \quad \kappa_0 \leq \kappa_1.
$$

Specifically, $n_0$ (resp. $n_1$) is the number of observations that need to be collected by the first opportunity to accept (resp. reject) $H_0$, and $N$ the maximum number of observations that can be collected. Indeed, given these parameters, the test proceeds as follows:

(i) $n_0 \wedge n_1$ observations are initially collected.
- If $n_0 \leq n_1$ and $T_{n_0} \leq \kappa_0$, then $H_0$ is accepted.
- If $n_1 \leq n_0$ and $T_{n_1} > \kappa_1$, then $H_0$ is rejected.

(ii) If the decision has not been reached yet, $(n_0 \vee n_1) - (n_0 \wedge n_1)$ additional observations are collected.
- If $n_0 \leq n_1$ and $T_{n_1} > \kappa_1$, then $H_0$ is rejected.
- If $n_1 \leq n_0$ and $T_{n_0} \leq \kappa_0$, then $H_0$ is accepted.

(iii) If the decision has not been reached yet, $N - (n_0 \vee n_1)$ additional observations are collected and $H_0$ is rejected if and only if $T_N > K$.

This testing procedure can be implemented by collecting at most three samples of deterministic sizes. Thus, in what follows we refer to it as the 3-stage test and denote it by $\tilde{\chi} \equiv (\tilde{\tau}, \tilde{d})$. 
3.1.1. Error control

By the definition of the 3-stage test it follows that, for any selection of its parameters and any \( P \in P \),

\[
P(\tilde{d} = 1) \leq P(T_{n_1} > \kappa_1) + P(T_N > K), \tag{3.1}
\]

\[
P(\tilde{d} = 0) \leq P(T_{n_0} \leq \kappa_0) + P(T_N \leq K). \tag{3.2}
\]

Consequently, by (3.1) with \( P = P_0 \) and by (3.2) with \( P = P_1 \) we can see that if the sample size and the threshold are

\[
n_0 = n^*(\gamma, \beta) \quad \text{and} \quad \kappa_0 = \kappa^*(\gamma, \beta) \quad \text{for some} \quad \gamma \in (\alpha, 1) \tag{3.3}
\]

in the first opportunity to accept \( H_0 \),

\[
n_1 = n^*(\alpha, \delta) \quad \text{and} \quad \kappa_1 = \kappa^*(\alpha, \delta) \quad \text{for some} \quad \delta \in (\beta, 1) \tag{3.4}
\]

in the first opportunity to reject \( H_0 \), and

\[
N = n^*(\alpha, \beta) \quad \text{and} \quad K = \kappa^*(\alpha, \beta) \tag{3.5}
\]

in the final stage, then

\[
P_0(\tilde{d} = 1) \leq 2\alpha \quad \text{and} \quad P_1(\tilde{d} = 0) \leq 2\beta.
\]

Thus, we have shown the following theorem.

**Theorem 3.1.** Let \( \alpha, \beta \in (0, 1) \). If (3.3)-(3.5) hold with \( \alpha \) and \( \beta \) replaced by \( \alpha/2 \) and \( \beta/2 \) respectively, then \( \tilde{\chi} \in C(\alpha, \beta) \).

Theorem 3.1 specifies a design for \( \tilde{\chi} \in C(\alpha, \beta) \) up to two free parameters, \( \gamma \in (\alpha/2, 1) \) and \( \delta \in (\beta/2, 1) \). Increasing the value of \( \gamma \) (resp. \( \delta \)) reduces the number of observations until the first opportunity to accept (resp. reject) \( H_0 \), but increases the probability of continuing to the final stage. To solve this trade-off, we propose in Subsection 3.1.3 that \( \gamma \) (resp. \( \delta \)) be selected to minimize an upper bound on \( E_0[\tilde{\tau}] \) (resp. \( E_1[\tilde{\tau}] \)) that is independent of \( \delta \) (resp. \( \gamma \)).

3.1.2. The average sample size

By the definition of the 3-stage test it follows that, for any \( P \in P \),

- if \( n_0 \leq n_1 < N \), then
  \[
  E[\tilde{\tau}] = n_0 + (n_1 - n_0) \cdot P(T_{n_0} > \kappa_0) + (N - n_1) \cdot P\left(\frac{T_{n_0} > \kappa_0}{T_{n_1} \leq \kappa_1}\right), \tag{3.6}
  \]

- if \( n_1 \leq n_0 < N \), then
  \[
  E[\tilde{\tau}] = n_1 + (n_0 - n_1) \cdot P(T_{n_1} \leq \kappa_1) + (N - n_0) \cdot P\left(\frac{T_{n_1} \leq \kappa_1}{T_{n_0} > \kappa_0}\right). \tag{3.7}
  \]
Applying to these identities the basic inequalities:
\[
\max\{0, P(A) - P(B^c)\} \leq P(A \cap B) \leq P(A),
\]
we obtain, for any selection of the test parameters, the following bounds:
\[
\begin{align*}
E[\hat{\tau}] &\geq n_0 \cdot P(T_{n_1} \leq \kappa_1) + (N - n_0) \cdot (P(T_{n_0} > \kappa_0) - P(T_{n_0} > \kappa_1))^+ \\
E[\hat{\tau}] &\leq n_0 + (N - n_0) \cdot P(T_{n_0} > \kappa_0)
\end{align*}
\]
(3.8)

and
\[
\begin{align*}
E[\tilde{\tau}] &\geq n_1 \cdot P(T_{n_0} > \kappa_0) + (N - n_1) \cdot (P(T_{n_1} \leq \kappa_1) - P(T_{n_0} \leq \kappa_0))^+ \\
E[\tilde{\tau}] &\leq n_1 + (N - n_1) \cdot P(T_{n_1} \leq \kappa_1).
\end{align*}
\]
(3.9)

When, in particular, the test parameters are selected as in Theorem 3.1, by (3.8) with \( P = P_0 \) we obtain
\[
n_0 \cdot (1 - \alpha/2) + (N - n_0) \cdot (\gamma - \alpha/2) \leq E_0[\hat{\tau}] \leq n_0 + (N - n_0) \cdot \gamma,
\]
where \( \gamma \in (\alpha/2, 1) \), \( n_0 = n^*(\gamma, \beta/2) \), \( N = n^*(\alpha/2, \beta/2) \),
(3.10)

and by (3.9) with \( P = P_1 \) we obtain
\[
n_1 \cdot (1 - \beta/2) + (N - n_1) \cdot (\delta - \beta/2) \leq E_1[\tilde{\tau}] \leq n_1 + (N - n_1) \cdot \delta,
\]
where \( \delta \in (\beta/2, 1) \), \( n_1 = n^*(\alpha/2, \delta) \), \( N = n^*(\alpha/2, \beta/2) \).
(3.11)

3.1.3. Specification of the free parameters

For any selection of \( \gamma \) (resp. \( \delta \)) we can see that, at least when \( \alpha \) (resp. \( \beta \)) is small, the upper bound in (3.10) (resp. (3.11)) is approximately equal to the lower bound and, as a result, it provides an accurate approximation to \( E_0[\hat{\tau}] \) (resp. \( E_1[\tilde{\tau}] \)). Thus, for any \( \alpha, \beta \in (0, 1) \), we suggest selecting \( \gamma \) and \( \delta \) as
\[
\gamma = \tilde{\gamma} \quad \text{and} \quad \delta = \tilde{\delta},
\]
(3.12)

where \( \tilde{\gamma} \) is a minimizer of the upper bound in (3.10) and \( \tilde{\delta} \) a minimizer of the upper bound in (3.11).

This selection of \( \gamma \) (resp. \( \delta \)) essentially minimizes \( E_0[\hat{\tau}] \) (resp. \( E_1[\tilde{\tau}] \)), at least when \( \alpha \) (resp. \( \beta \)) is small, and it is practically convenient, as it requires the minimization with respect to a single variable. Moreover, it requires knowledge of only the function \( n^* \), defined in (2.3), which is also needed for the specification of the other test parameters according to Theorem 3.1.

**Remark:** The test of this section was proposed in [23] when \( X \) is an iid sequence and the test statistic, \( T \), is the corresponding average log-likelihood ratio. Our setup here is essentially universal, as the only assumption throughout this section about \( X \) and \( T \) is that the corresponding fixed-sample-size test can control the error probabilities below arbitrary, user-specified levels, i.e., that \( n^*(\alpha, \beta) \) be finite for any \( \alpha, \beta \in (0, 1) \). At the same time, we propose a concrete, non-asymptotic specification of the test parameters, which is novel and practically useful even in the setup of [23, Section 2].
3.2. The 4-stage tests

Finally, we introduce and analyze two novel tests, $\hat{\chi} \equiv (\hat{\tau}, \hat{d})$ and $\tilde{\chi} \equiv (\tilde{\tau}, \tilde{d})$, which differ from that of the previous subsection only in that the first (resp. second) one allows for stopping and accepting (resp. rejecting) the null hypothesis if the value of the test statistic, $T$, after collecting $N_0$ (resp. $N_1$) observations is smaller (resp. larger) than $K_0$ (resp. $K_1$), where

$$n_0 < N_0 < N \quad \text{and} \quad K_0 \leq \kappa_1,$$
$$n_1 < N_1 < N \quad \text{and} \quad \kappa_0 \leq K_1.$$

Both these tests can be implemented by collecting at most 4 samples of deterministic sizes, and for this reason we refer to them as 4-stage tests. To avoid repetition, we present a detailed analysis for $\hat{\chi}$, and only state the corresponding results for $\tilde{\chi}$. Thus, given the above parameters, $\hat{\chi}$ proceeds as follows:

(i) $n_0 \land n_1$ observations are initially collected.
   - If $n_0 \leq n_1$ and $T_{n_0} \leq \kappa_0$, then $H_0$ is accepted.
   - If $n_1 \leq n_0$ and $T_{n_1} > \kappa_1$, then $H_0$ is rejected.

(ii) If the decision has not been reached yet, $((n_0 \lor n_1) \land N_0) - (n_0 \land n_1)$ additional observations are collected.
   - If $n_0 \leq n_1 \leq N_0$ and $T_{n_1} > \kappa_1$, then $H_0$ is rejected.
   - If $n_0 \leq N_0 \leq n_1$ and $T_{N_0} \leq K_0$, then $H_0$ is accepted.
   - If $n_1 \leq n_0 \leq N_0$ and $T_{n_0} \leq \kappa_0$, then $H_0$ is accepted.

(iii) If the decision has not been reached yet, $(n_1 \lor N_0) - ((n_0 \lor n_1) \land N_0)$ additional observations are collected.
   - If $n_1 \leq N_0$ and $T_{N_0} \leq K_0$, then $H_0$ is accepted.
   - If $N_0 \leq n_1$ and $T_{n_1} > \kappa_1$, then $H_0$ is rejected.

(iv) If the decision has not been reached yet, $N - (n_1 \lor N_0)$ additional observations are collected and $H_0$ is rejected if and only if $T_N > K$.

3.2.1. Error control

By the definition of $\hat{\chi}$ it follows that, for any selection of its parameters and any $P \in \mathcal{P}$,

$$P(\hat{d} = 1) \leq P(T_{n_1} > \kappa_1) + P(T_N > K), \quad (3.13)$$
$$P(\hat{d} = 0) \leq P(T_{n_0} \leq \kappa_0) + P(T_{N_0} \leq K_0) + P(T_N \leq K). \quad (3.14)$$

Therefore, if $n_0, n_1, N, \kappa_0, \kappa_1, K$ are selected as in (3.3)–(3.5) and we also set

$$N_0 = n^*(\gamma', \beta) \quad \text{and} \quad K_0 = c^*(\gamma', \beta) \quad \text{for some } \gamma' \in (\alpha, \gamma), \quad (3.15)$$
by (3.13) with $P = P_0$ and by (3.14) with $P = P_1$ we obtain
\[ P_0(\hat{d} = 1) \leq 2\alpha \quad \text{and} \quad P_1(\hat{d} = 0) \leq 3\beta. \]

With a similar analysis we obtain
\[ P_0(\hat{d} = 1) \leq 2\alpha \quad \text{and} \quad P_1(\hat{d} = 0) \leq 3\beta, \]
when $n_0, n_1, N, \kappa_0, \kappa_1, K$ are selected as in (3.3)–(3.5) and also
\[ N_1 = n^*(\alpha, \delta') \quad \text{and} \quad K_1 = \kappa^*(\alpha, \delta') \quad \text{for some} \ \delta' \in (\beta, \delta). \quad (3.16) \]

Thus, we have shown the following theorem.

**Theorem 3.2.** Let $\alpha, \beta \in (0, 1)$.

(i) If (3.3)–(3.5) and (3.15) hold with $\alpha$ and $\beta$ replaced by $\alpha/3$ and $\beta/2$ respectively, then $\tilde{\chi} \in C(\alpha, \beta)$.

(ii) If (3.3)–(3.5) and (3.16) hold with $\alpha$ and $\beta$ replaced by $\alpha/3$ and $\beta/2$ respectively, then $\tilde{\chi} \in C(\alpha, \beta)$.

Theorem 3.2 specifies designs for $\hat{\chi}$ and $\tilde{\chi}$, which guarantee the desired error control, up to three free parameters, $\gamma, \gamma', \delta$ and $\gamma, \delta, \delta'$ respectively. We next propose a specific selection for these parameters, similar to the one for the free parameters of the 3-stage test in Subsection 3.1.3.

**3.2.2. The average sample size**

By the definition of $\hat{\chi}$ it follows that, for any $P \in \mathcal{P}$,

- if $n_0 \leq n_1 \leq N_0 \leq N$, then
  \[
  \mathbb{E}[\hat{\tau}] = n_0 + (n_1 - n_0) \cdot P(T_{n_0} > \kappa_0) + (N_0 - n_1) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{n_1} \leq \kappa_1 \\
  N_0 > K_0
  \end{array} \right),
  \]
  \[
  \quad + (N - N_0) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{n_1} \leq \kappa_1 \\
  N_0 > K_0
  \end{array} \right),
  \]
  \[
  \quad + (N - n_1) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{N_0} > K_0
  \end{array} \right),
  \]
  \[
  + (N - n_0) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{N_0} > K_0
  \end{array} \right),
  \]
  \[
  \quad \text{for some} \ \delta' \in (\beta, \delta). \quad (3.17)
  \]

- if $n_0 \leq N_0 \leq n_1 \leq N$, then
  \[
  \mathbb{E}[\hat{\tau}] = n_0 + (N_0 - n_0) \cdot P(T_{n_0} > \kappa_0) + (n_1 - N_0) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{n_1} \leq \kappa_1 \\
  N_0 > K_0
  \end{array} \right),
  \]
  \[
  \quad + (N - N_0) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{n_1} \leq \kappa_1 \\
  N_0 > K_0
  \end{array} \right),
  \]
  \[
  \quad + (N - n_0) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{N_0} > K_0
  \end{array} \right),
  \]
  \[
  + (N - n_1) \cdot P\left( \begin{array}{c}
  T_{n_0} > \kappa_0 \\
  T_{N_0} > K_0
  \end{array} \right),
  \]
  \[
  \quad \text{for some} \ \delta' \in (\beta, \delta). \quad (3.18)
  \]
Applying to the above identities the following basic inequalities:

\[ \mathbb{E}[\hat{\gamma}] = n_1 + (n_0 - n_1) \cdot \mathbb{P}(T_{n_1} \leq \kappa_1) + (N_0 - n_0) \cdot \mathbb{P}\left( \begin{array}{c} T_{n_1} \leq \kappa_1 \\ T_{n_0} > \kappa_0 \end{array} \right) \]

\[ + (N - N_0) \cdot \mathbb{P}\left( \begin{array}{c} T_{n_0} > \kappa_0 \\ T_{N_0} > K_0 \end{array} \right). \]  

(3.19)

Applying to the above identities the following basic inequalities:

\[ \max\{\mathbb{P}(A) - \mathbb{P}(B^c) - \mathbb{P}(C^c), 0\} \leq \mathbb{P}(A \cap B \cap C) \leq \mathbb{P}(A), \]

we obtain, for any selection of the test parameters, the following bounds:

\[ \mathbb{E}[\hat{\gamma}] \geq n_0 \cdot \mathbb{P}(T_{n_1} \leq \kappa_1) + (N_0 - n_0) \cdot \mathbb{P}(T_{n_0} > \kappa_0) - \mathbb{P}(T_{n_1} > \kappa_1) \]

\[ + (N - N_0) \cdot \mathbb{P}(T_{N_0} > K_0) - \mathbb{P}(T_{n_0} \leq \kappa_0) - \mathbb{P}(T_{n_1} > \kappa_1))^+, \]  

(3.20)

\[ \mathbb{E}[\hat{\gamma}] \leq n_0 + (N_0 - n_0) \cdot \mathbb{P}(T_{n_0} > \kappa_0) + (N - N_0) \cdot \mathbb{P}(T_{N_0} > K_0) \]

and

\[ \mathbb{E}[\hat{\gamma}] \geq n_1 \cdot \mathbb{P}(T_{n_0} > \kappa_0) - \mathbb{P}(T_{N_0} \leq K_0) \]

\[ + (N - n_1) \cdot \mathbb{P}(T_{n_1} \leq \kappa_1) - \mathbb{P}(T_{n_0} \leq \kappa_0) - \mathbb{P}(T_{N_0} \leq K_0)) \]  

(3.21)

\[ \mathbb{E}[\hat{\gamma}] \leq n_1 + (N - n_1) \cdot \mathbb{P}(T_{n_1} \leq \kappa_1). \]

When, in particular, the parameters of \( \hat{\chi} \) are selected as in Theorem 3.2.(i), by (3.20) with \( P = P_0 \) we obtain

\[ \mathbb{E}_0[\hat{\gamma}] \leq n_0 + (N_0 - n_0) \cdot \gamma + (N - N_0) \cdot \gamma', \]

\[ \mathbb{E}_0[\hat{\gamma}] \geq n_0 \cdot (1 - \alpha/2) + (N_0 - n_0) \cdot (\gamma - \alpha/2) \]

\[ + (N - N_0) \cdot ((1 - \alpha/2) - (1 - \gamma) - (1 - \gamma'))^+, \]  

(3.22)

where \( \alpha/2 < \gamma' < \gamma < 1 \), \( n_0 = n^*(\gamma, \beta/3) \), \( N_0 = n^*(\gamma', \beta/3) \), \( N = n^*(\alpha/2, \beta/3) \),

and by (3.21) with \( P = P_1 \) we obtain

\[ n_1 \cdot (1 - 2\beta/3) + (N - n_1) \cdot (\delta - 2\beta/3) \leq \mathbb{E}_1[\hat{\gamma}] \leq n_1 + (N - n_1) \cdot \delta \]

where \( \delta \in (\beta/3, 1) \), \( n_1 = n^*(\alpha/2, \delta) \), \( N = n^*(\alpha/2, \beta/3) \).  

(3.23)

With a similar analysis it follows that when the parameters of \( \hat{\chi} \) are selected according to Theorem 3.2.(ii), then

\[ n_0 \cdot (1 - 2\alpha/3) + (N - n_0) \cdot (\gamma - 2\alpha/3) \leq \mathbb{E}_0[\hat{\gamma}] \leq n_0 + (N - n_0) \cdot \gamma, \]

where \( \gamma \in (\alpha/3, 1) \), \( n_0 = n^*(\gamma, \beta/2) \), \( N = n^*(\alpha/3, \beta/2) \).  

(3.24)
and
\[
E_1[\hat{\tau}] \leq n_1 + (N_1 - n_1) \cdot \delta + (N - N_1) \cdot \delta',
\]
\[
E_1[\hat{\tau}] \geq n_1 (1 - \beta/2) + (N_1 - n_1) (\delta - \beta/2)
+ (N - N_1) \cdot ((1 - \beta/2) - (1 - \delta) - (1 - \delta'))^+,
\]
where \(\beta/2 < \delta' < \delta < 1\), \(n_1 = n^*(\alpha/3, \delta)\),
\(N_1 = n^*(\alpha/3, \delta')\), \(N = n^*(\alpha/3, \beta/2)\).

(3.25)

3.2.3. Specification of the free parameters

For any \(\alpha, \beta \in (0, 1)\), we propose selecting the free parameters of \(\hat{\chi}\) as
\[
\delta = \hat{\delta}, \quad \gamma = \hat{\gamma}, \quad \gamma' = \hat{\gamma}',
\]
where \((\hat{\gamma}, \hat{\gamma}')\) is a minimizer of the upper bound in (3.22) and \(\hat{\delta}\) a minimizer of the upper bound in (3.23), and the free parameters of \(\tilde{\chi}\) as
\[
\gamma = \tilde{\gamma}, \quad \delta = \tilde{\delta}, \quad \delta' = \tilde{\delta'},
\]
where \(\tilde{\gamma}\) is a minimizer of the upper bound in (3.24) and \((\tilde{\delta}, \tilde{\delta}')\) a minimizer of the upper bound in (3.25).

Remark: Comparing with the corresponding results for the 3-stage test, we can see that, at least when \(\alpha\) (resp. \(\beta\)) is small, \(\hat{\delta}\) (resp. \(\hat{\gamma}\)) is close to \(\tilde{\delta}\) (resp. \(\tilde{\gamma}\)), and the expected sample size of \(\hat{\chi}\) (resp. \(\tilde{\chi}\)) close to that of \(\tilde{\chi}\) under \(P_1\) (resp. \(P_0\)). Indeed, the additional stage in \(\tilde{\chi}\) (resp. \(\tilde{\chi}\)) is useful mainly for reducing the expected sample size under \(P_0\) (resp. \(P_1\)). This reduction is illustrated numerically in Figures 2 and 3.

4. Asymptotic analysis.

In this section we obtain asymptotic bounds and approximations, as \(\alpha, \beta \to 0\), to the expected sample sizes of the multistage tests of the previous sections. For this analysis, we need to impose some structure on the almost universal setup we have considered so far.

4.1. Assumptions on the testing problem

Throughout this section, we assume that for every \(n \in \mathbb{N}\), \(P_0\) and \(P_1\) are mutually absolutely continuous when restricted to \(\mathcal{F}_n\), and denote by \(\Lambda \equiv \{\Lambda_n, n \in \mathbb{N}\}\) and \(\bar{\Lambda} \equiv \{\bar{\Lambda}_n, n \in \mathbb{N}\}\) the corresponding log-likelihood ratio and average log-likelihood ratio statistics, i.e.,
\[
\Lambda_n \equiv \log \frac{dP_1}{dP_0}(\mathcal{F}_n), \quad \bar{\Lambda}_n \equiv \Lambda_n/n, \quad n \in \mathbb{N}.
\]

(4.1)
We assume that there are numbers $I_0, I_1 > 0$ such that
\begin{equation}
P_0(\Lambda_n \to -I_0) = P_1(\Lambda_n \to I_1) = 1,
\end{equation}
\begin{equation}
\forall \epsilon > 0, \quad \sum_{n=1}^{\infty} P_0(\Lambda_n > -I_0 + \epsilon) + \sum_{n=1}^{\infty} P_1(\Lambda_n \leq I_1 - \epsilon) < \infty.
\end{equation}
These assumptions imply (see, e.g., [30, Lemma 3.4.1, Theorem 3.4.2]) an asymptotic approximation, as $\alpha, \beta \to 0$, to $L_i(\alpha, \beta)$, $i \in \{0, 1\}$, defined in (2.4). Specifically,
\begin{equation}
E_0[\tau'] \sim \mathcal{L}_0(\alpha, \beta) \sim |\log \beta| I_0 \quad \text{and} \quad E_1[\tau'] \sim \mathcal{L}_1(\alpha, \beta) \sim |\log \alpha| I_1,
\end{equation}
where $\chi' \equiv (\tau', d')$ is Wald’s SPRT, i.e.,
\begin{equation}
\tau' \equiv \inf\{n \in \mathbb{N} : \Lambda_n \notin (-A, B)\} \quad \text{and} \quad d' \equiv 1\{\Lambda_{\tau'} \geq B\},
\end{equation}
with $A$ and $B$ selected, for example, as $A = |\log \beta|$ and $B = |\log \alpha|$.

### 4.1.1. The iid setup

When $X$ is an iid sequence with common density $f_i$ under $P_i$ with respect to some dominating measure $\nu$, $i \in \{0, 1\}$, and the Kullback-Leibler divergences are positive and finite, i.e.,
\begin{equation}
D(f_0 || f_1) \equiv \int \log(f_0/f_1) f_0 \, d\nu, \quad D(f_1 || f_0) \equiv \int \log(f_1/f_0) f_1 \, d\nu \in (0, \infty),
\end{equation}
then the log-likelihood ratio statistic in (4.1) becomes
\begin{equation}
\Lambda_n = \sum_{i=1}^{n} \frac{f_1(X_i)}{f_0(X_i)}, \quad n \in \mathbb{N},
\end{equation}
and (4.2)-(4.3) hold with $I_0 = D(f_0 || f_1)$ and $I_1 = D(f_1 || f_0)$ (for more details, see Subsection 5.3.1).

### 4.2. Assumptions on the test statistic

With respect to the test statistic, $T$, throughout this section we assume that there are real numbers $J_0, J_1$, with $J_0 < J_1$, so that
\begin{equation}
P_0(T_n \to J_0) = P_1(T_n \to J_1) = 1,
\end{equation}
and, for every $\kappa \in (J_0, J_1)$, the error probabilities of the fixed-sample-size test that rejects $H_0$ if and only if $T_n > \kappa$ go to zero exponentially fast in $n$. Specifically, we assume that there are non-negative, convex, lower-semicontinuous functions
\[ \psi_0 : \mathbb{R} \to [0, \infty] \quad \text{and} \quad \psi_1 : \mathbb{R} \to [0, \infty], \]
so that
- $[J_0, J_1]$ is a subset of the effective domains of both $\psi_0$ and $\psi_1$,
- $\psi_0(J_0) = 0$ and $\psi_0$ is strictly increasing in $[J_0, J_1]$,
- $\psi_1(J_1) = 0$ and $\psi_1$ is strictly decreasing in $[J_0, J_1]$,
- for every $\kappa \in (J_0, J_1)$,
\[
\lim_{n} \frac{1}{n} \log P_0(T_n > \kappa) = -\psi_0(\kappa), \tag{4.9}
\]
\[
\lim_{n} \frac{1}{n} \log P_1(T_n \leq \kappa) = -\psi_1(\kappa). \tag{4.10}
\]

**Remarks:**
1) When $T_n = \overline{\Lambda}_n$, (4.8) is the same as (4.2) and (4.9)-(4.10) imply (4.3), with $J_0 = -I_0$ and $J_1 = I_1$.

2) In Section 5 we state sufficient conditions for the existence of functions $\psi_0$ and $\psi_1$ that satisfy (4.9)-(4.10), which we also specify. In Section 7 we show that these sufficient conditions are satisfied in various testing problems and for different statistics. The graphs of $\psi_0$ and $\psi_1$ in each of these examples are plotted in Figures 1a, 1c, 1e.

3) In the iid setup of Subsection 4.1.1, the above assumptions hold when $T = \overline{\Lambda}$ as long as (4.6) holds (see Subsection 5.3.1).

4) By assumption, the function
\[ g(\kappa) \equiv \frac{\psi_0(\kappa)}{\psi_1(\kappa)}, \quad \kappa \in (J_0, J_1) \tag{4.11} \]
is continuous and strictly increasing with $g(J_0^+) = 0$ and $g(J_1^-) = \infty$. As a result, its inverse, $g^{-1}$, is well-defined in $(0, \infty)$ and satisfies
\[ g^{-1}(0, \infty) = (J_0, J_1). \tag{4.12} \]

5) The above assumptions will suffice for obtaining first-order asymptotic upper bounds on the expected sample sizes of the proposed multistage tests under $P_0$ and $P_1$ as $\alpha, \beta \to 0$. When $T = \Lambda$, they will also suffice for obtaining the matching lower bounds. However, in order to obtain such lower bounds when $T \neq \Lambda$, we will need to additionally assume that
\[ \exists \text{ a neighborhood of } J_1 \text{ in which } \psi_0 \text{ is finite and } (4.9) \text{ holds} \]
\[ \exists \text{ a neighborhood of } J_0 \text{ in which } \psi_1 \text{ is finite and } (4.10) \text{ holds}. \tag{4.13} \]
In Section 5 we also state sufficient conditions for (4.13), which hold for all test statistics, different from $\Lambda$, that we consider in Section 7.
4.3. Asymptotic analysis for the fixed-sample-size test

The asymptotic analysis for the proposed multistage tests is based on asymptotic bounds and approximations for \( n^*(\alpha, \beta) \) as at least one of \( \alpha \) and \( \beta \) goes to 0, while the other one either goes to 0 as well or remains fixed. When any of these asymptotic regimes holds, we simply write \( \alpha \wedge \beta \to 0 \).

4.3.1. Asymptotic bounds

**Theorem 4.1.** As \( \alpha \wedge \beta \to 0 \),

\[
\min \left\{ \frac{\log \beta}{\psi_1(\kappa)}, \frac{\log \alpha}{\psi_0(\kappa)} \right\} \lesssim n^*(\alpha, \beta) \lesssim \max \left\{ \frac{\log \beta}{\psi_1(\kappa)}, \frac{\log \alpha}{\psi_0(\kappa)} \right\}
\]

(4.14)

for every \( \kappa \in (J_0, J_1) \), and consequently

\[
n^*(\alpha, \beta) \lesssim \frac{\log(\alpha \wedge \beta)}{C}, \quad \text{where} \quad C \equiv \sup_{\kappa \in (J_0, J_1)} \left\{ \psi_1(\kappa) \wedge \psi_0(\kappa) \right\}.
\]

(4.15)

**Proof.** Appendix A. \( \square \)

**Remark:** In the iid setup of Subsection 4.1.1, \( C \) is the well-known Chernoff information (see, e.g., [12, Corollary 3.4.6]).

We present the following asymptotic lower bounds separately when \( T = \bar{\Lambda} \) and when \( T \neq \bar{\Lambda} \), as in the latter case we also need assumption (4.13).

**Theorem 4.2.** (i) If \( T = \bar{\Lambda} \), then

\[
n^*(\alpha, \beta) \gtrsim \max \left\{ \frac{\log \beta}{I_0}, \frac{\log \alpha}{I_1} \right\} \quad \text{as} \quad \alpha \wedge \beta \to 0.
\]

(4.16)

(ii) If \( T \neq \bar{\Lambda} \) and (4.13) holds, then

\[
n^*(\alpha, \beta) \gtrsim \max \left\{ \frac{\log \beta}{\psi_1(J_0)}, \frac{\log \alpha}{\psi_0(J_1)} \right\} \quad \text{as} \quad \alpha \wedge \beta \to 0.
\]

(4.17)

**Proof.** Appendix A. \( \square \)

4.3.2. Asymptotic approximations

Unlike the preceding bounds, asymptotic approximations to \( n^*(\alpha, \beta) \) depend on the relative decay rate of \( \alpha \) and \( \beta \). We start with the asymptotic regime where \( \alpha, \beta \to 0 \) so that

\[
|\log \alpha| \sim r |\log \beta| \quad \text{for some} \quad r \in (0, \infty),
\]

(4.18)
in which case the approximation is expressed in terms of the function \( g \), defined in (4.11).
Corollary 4.2.1. As $\alpha, \beta \to 0$ so that (4.18) holds,
\[ n^*(\alpha, \beta) \sim \frac{|\log \alpha|}{\psi_0(g^{-1}(r))} \sim \frac{|\log \beta|}{\psi_1(g^{-1}(r))}. \]  
(4.19)
When in particular, $r = 1$,
\[ n^*(\alpha, \beta) \sim \frac{|\log \alpha|}{C} \sim \frac{|\log \beta|}{C}, \]  
(4.20)
where $C$ is defined in (4.15).

Proof. Appendix A.

Remark: From the previous corollary and the optimal asymptotic performance in (4.4) we obtain the asymptotic relative efficiency of the fixed-sample-size test as $\alpha, \beta \to 0$ so that (4.18) holds. Specifically,
\[ n^*(\alpha, \beta) \sim I_1 \psi_1(J_0) \]  
and when in particular $r = 1$,
\[ n^*(\alpha, \beta) \sim I_0 \psi_0(J_1). \]  
(4.22)
When $\alpha \land \beta \to 0$ so that $|\log \alpha|/|\log \beta|$ either goes to zero or diverges, the asymptotic lower bounds in Theorem 4.2 turn out to be sharp.

Corollary 4.2.2. Let $T = \tilde{\Lambda}$.

(i) If $\alpha \land \beta \to 0$ so that $|\log \alpha| << |\log \beta|$, then $n^*(\alpha, \beta) \sim |\log \beta|/I_0$.

(ii) If $\alpha \land \beta \to 0$ so that $|\log \alpha| >> |\log \beta|$, then $n^*(\alpha, \beta) \sim |\log \alpha|/I_1$.

Proof. Appendix A.

Corollary 4.2.3. Let $T \neq \tilde{\Lambda}$ and assume that (4.13) holds.

(i) If $\alpha \land \beta \to 0$ so that $|\log \alpha| < |\log \beta|$, then $n^*(\alpha, \beta) \sim |\log \beta|/\psi_1(J_0)$.

(ii) If $\alpha \land \beta \to 0$ so that $|\log \alpha| >> |\log \beta|$, then $n^*(\alpha, \beta) \sim |\log \alpha|/\psi_0(J_1)$.

Proof. Appendix A.

Remark: When $T = \tilde{\Lambda}$ and one of $\alpha$ and $\beta$ is fixed, Corollary 4.2.2 is known as Stein’s lemma (see, e.g., [12, Lemma 3.4.7]). We stress, however, that both $\alpha$ and $\beta$ may go to 0 in the previous corollaries.

When both $\alpha$ and $\beta$ go 0, Corollary 4.2.2, in conjunction with (4.4), implies that the fixed-sample-size test is asymptotically optimal under one of the two hypotheses, while being of larger order of magnitude compared to the optimal under the other hypothesis. This is formalized in the following corollary.
Corollary 4.2.4. Let $T = \bar{\Lambda}$.

(i) If $\alpha, \beta \to 0$ so that $|\log \alpha| << |\log \beta|$, then

$$\mathcal{L}_1(\alpha, \beta) << n^*(\alpha, \beta) \sim \mathcal{L}_0(\alpha, \beta).$$

(ii) If $\alpha, \beta \to 0$ so that $|\log \alpha| >> |\log \beta|$, then

$$\mathcal{L}_0(\alpha, \beta) << n^*(\alpha, \beta) \sim \mathcal{L}_1(\alpha, \beta).$$

We end this subsection with the corresponding result when $T \neq \bar{\Lambda}$.

Corollary 4.2.5. Let $T \neq \bar{\Lambda}$ for which (4.13) holds.

(i) If $\alpha, \beta \to 0$ so that $|\log \alpha| << |\log \beta|$, then

$$\mathcal{L}_1(\alpha, \beta) << n^*(\alpha, \beta) \sim \frac{I_0}{\psi_1(J_0)} \mathcal{L}_0(\alpha, \beta).$$

(ii) If $\alpha, \beta \to 0$ so that $|\log \alpha| >> |\log \beta|$, then

$$\mathcal{L}_0(\alpha, \beta) << n^*(\alpha, \beta) \sim \frac{I_1}{\psi_0(J_1)} \mathcal{L}_1(\alpha, \beta).$$

4.4. Asymptotic analysis for multistage tests

We now focus on the multistage tests we introduced in Section 3 and establish the main theoretical results of this work. We assume that the test parameters are selected according to Theorems 3.1 and 3.2. However, unless otherwise specified, we do not require that the free parameters are selected as in Section 3.1.3 and 3.2.3.

4.4.1. An upper bound on the maximum sample size

By the definitions of the multistage tests and the selection of their parameters according to Theorems 3.1 and 3.2 it follows that, for any $\alpha, \beta \in (0, 1)$ and any choice of the free parameters,

$$\tilde{\tau}, \hat{\tau}, \bar{\tau} \leq n^*(\alpha/3, \beta/3),$$

and consequently, in view of Theorem 4.1,

$$\tilde{\tau}, \hat{\tau}, \bar{\tau} \lesssim \frac{|\log(\alpha \wedge \beta)|}{C} \quad \text{as} \quad \alpha \wedge \beta \to 0. \quad (4.23)$$

On the other hand, it is well known (see, e.g., [7]) that, even when $X$ is an iid sequence, the SPRT, defined in (4.5), not only does not have bounded sample size, but even its expected sample size can be much larger than $n^*(\alpha, \beta)$.

To be specific, consider a $P \in \mathcal{P}$, different from $P_0$ and $P_1$, under which $\Lambda$ is a random walk whose increments have zero mean and finite variance $\sigma^2$. The
expected sample size of the SPRT, with \( A = |\log \beta| \) and \( B = |\log \alpha| \), under such a \( P \) is
\[
E[\tau] \approx |\log \alpha||\log \beta|/\sigma^2, \tag{4.24}
\]
where \( \approx \) is an equality when there is no overshoot over the boundaries (see, e.g., [30, Chapter 3.1.1.2]). Comparing with the upper bound in (4.23) suggests that all proposed multistage tests will perform much better than the SPRT under such a \( P \) when \( \alpha \) and \( \beta \) are small enough. This robustness of the proposed multistage tests is illustrated in Figure 3.

4.4.2. Asymptotic analysis under \( P_0 \) and \( P_1 \)

By the optimal asymptotic performance in (4.4) it follows that, as \( \alpha, \beta \to 0 \),
\[
E_1[\hat{\tau}], E_1[\tilde{\tau}], E_1[\check{\tau}] \gtrsim \frac{|\log \alpha|}{\psi_0(J_1)} \quad \text{and} \quad E_0[\hat{\tau}], E_0[\tilde{\tau}], E_0[\check{\tau}] \gtrsim \frac{|\log \beta|}{\psi_1(J_0)},
\]
for any selection of the free parameters and any choice of the test-statistic, \( T \). In the next lemma we obtain a sharper asymptotic lower bound when \( T \) is not \( \bar{\Lambda} \), but satisfies condition (4.13).

**Lemma 4.1.** Suppose that \( T \neq \bar{\Lambda} \) and (4.13) holds. Then, for any selection of the free parameters, as \( \alpha, \beta \to 0 \),
\[
E_1[\hat{\tau}], E_1[\tilde{\tau}], E_1[\check{\tau}] \gtrsim \frac{|\log \alpha|}{\psi_0(J_1)} \quad \text{and} \quad E_0[\hat{\tau}], E_0[\tilde{\tau}], E_0[\check{\tau}] \gtrsim \frac{|\log \beta|}{\psi_1(J_0)}.
\]

**Proof.** Appendix B.

We next state the main results of this section, according to which the previous asymptotic lower bounds are attained with an appropriate selection of the free parameters. To avoid repetition, we state these results only when \( |\log \alpha| \gtrsim |\log \beta| \), as analogous results hold when \( |\log \alpha| \lesssim |\log \beta| \).

**Theorem 4.3.** Suppose that \( T = \bar{\Lambda} \) and let the free parameters be selected according to (3.12), (3.26), (3.27).

\( (i) \) If \( \alpha, \beta \to 0 \) so that \( |\log \alpha| \gtrsim |\log \beta| \), then
\[
E_1[\hat{\tau}] \sim E_1[\tilde{\tau}] \sim E_1[\check{\tau}] \sim \frac{|\log \alpha|}{I_1} \sim \mathcal{L}_1(\alpha, \beta).
\]

\( (ii) \) If also \( |\log \alpha| \lesssim |\log \beta|/\beta^r \) for some \( r > 0 \), then
\[
E_0[\hat{\tau}] \sim \frac{|\log \beta|}{I_0} \sim \mathcal{L}_0(\alpha, \beta).
\]

\( (iii) \) If also \( |\log \alpha| \lesssim |\log \beta|^r \) for some \( r \geq 1 \), then
\[
E_0[\hat{\tau}] \sim E_0[\tilde{\tau}] \sim \frac{|\log \beta|}{I_0} \sim \mathcal{L}_0(\alpha, \beta).
\]
Proof. Appendix B.

**Theorem 4.4.** Suppose that $T \neq \bar{\Lambda}$, (4.13) holds, and let the free parameters be selected according to (3.12), (3.26), (3.27).

(i) If $\alpha, \beta \to 0$ so that $|\log \alpha| \gtrsim |\log \beta|$, then

$$E_1[\tilde{\tau}] \sim E_1[\hat{\tau}] \sim E_1[\check{\tau}] \sim \frac{|\log \alpha|}{\psi_0(J_1)} \sim \frac{I_1}{\psi_0(J_1)} \mathcal{L}_1(\alpha, \beta).$$

(ii) If also $|\log \alpha| \lesssim |\log \beta|/\beta^r$ for some $r > 0$, then

$$E_0[\tilde{\tau}] \sim \frac{|\log \beta|}{\psi_1(J_0)} \sim \frac{I_0}{\psi_1(J_0)} \mathcal{L}_0(\alpha, \beta).$$

(iii) If also $|\log \alpha| \lesssim |\log \beta|^r$ for some $r \geq 1$, then

$$E_0[\tilde{\tau}] \sim E_0[\hat{\tau}] \sim \frac{|\log \beta|}{\psi_1(J_0)} \sim \frac{I_0}{\psi_1(J_0)} \mathcal{L}_0(\alpha, \beta).$$

Proof. Appendix B.

**Remarks:** 1) As can be seen in the proof of Theorem 4.4, condition (4.13) is used only in Lemma 4.1, i.e., it is only needed for establishing the asymptotic lower bounds but not for obtaining the matching upper bounds.

2) As can be seen from their proofs, the above theorems hold even if the free parameters of the multistage tests are not selected as suggested in Subsections 3.1.3 and 3.2.3. Indeed, part (i) of each theorem holds as long as $\delta \to 0$ and $|\log \delta| << |\log \alpha|$ as $\alpha \to 0$. Similarly, part (ii) (resp. (iii)) of each theorem holds as long as the specification of $\gamma, \gamma'$ (resp. $\gamma$) is such that (B.4) (resp. (B.5)) is satisfied.

3) Part (i) in Theorems 4.3 and 4.4 states that, under the alternative hypothesis, all multistage tests in this work achieve the optimal performance to a first-order asymptotic approximation when $T = \bar{\Lambda}$, and have the same asymptotic relative efficiency when $T \neq \bar{\Lambda}$ and (4.13) holds, as $\alpha, \beta \to 0$ so that $|\log \alpha| \gtrsim |\log \beta|$. On the other hand, parts (ii) and (iii) imply that the corresponding results under the null hypothesis hold as long as $\beta$ does not go to 0 much faster than $\alpha$. This suggests that $\hat{\chi}$ will perform much better than $\tilde{\chi}$ and $\check{\chi}$ under the null hypothesis when $\alpha$ is much smaller than $\beta$. This insight is supported by Figures 2 and 3.

3) Analogous results hold when $\alpha, \beta \to 0$ so that $|\log \alpha| \lesssim |\log \beta|$. Indeed, under this asymptotic regime, all three multistage tests are asymptotically optimal when $T = \bar{\Lambda}$, and admit the same asymptotic relative efficiency when $T \neq \bar{\Lambda}$ and (4.13) holds, under the null hypothesis. The corresponding results under the alternative hypothesis hold as long as $\beta$ does not go to 0 much faster than
\(\alpha\), with this requirement being much stricter for \(\hat{\chi}\) and \(\tilde{\chi}\) than for \(\hat{\chi}\).

4) The asymptotic optimality under both hypotheses of the 3-stage test with 
\(T = \bar{\Lambda}\) was established in [23, Section 2], in the iid setup of Subsection 4.1.1, as 
\(\alpha, \beta \to 0\) so that

\[|\log \beta|/r \lesssim |\log \alpha| \lesssim r |\log \beta|\]

for some \(r \geq 1\).

Therefore, apart from extending it to a more general distributional setup, here 
we generalize this result even in the iid case. Indeed, from parts (i) and (iii) of 
Theorem 4.3 and the remark 3) we can conclude that the asymptotic optimality 
of the 3-stage test under both hypotheses holds as \(\alpha, \beta \to 0\) so that

\[|\log \beta|^{1/r} \lesssim |\log \alpha| \lesssim |\log \beta|^r\]

for some \(r \geq 1\).

At the same time, we show how adding a stage can further relax this asymptotic 
regime. Specifically, from Theorem 4.3 and remark 3) we can conclude that the 
4-stage test \(\hat{\chi}\) is asymptotically optimal under both hypotheses as \(\alpha, \beta \to 0\) so that

\[|\log \beta|^{1/r} \lesssim |\log \alpha| \lesssim |\log \beta|/\beta^k,\]

for some \(r \geq 1\) and \(k > 0\),

while the 4-stage test \(\hat{\chi}\) is asymptotically optimal under both hypotheses as 
\(\alpha, \beta \to 0\) so that

\[|\log \alpha|^{1/r} \lesssim |\log \beta| \lesssim |\log \alpha|/\alpha^k,\]

for some \(r \geq 1\) and \(k > 0\).

5) In view of Theorem 4.4, in what follows we use the following notation for the 
asymptotic relative efficiencies under \(P_0\) and \(P_1\), as \(\alpha, \beta \to 0\), of the proposed 
multistage tests when \(T \neq \bar{\Lambda}\) and (4.13) holds,

\[\text{ARE}_0 \equiv \frac{\psi_1(J_0)}{I_0}\] and 
\[\text{ARE}_1 \equiv \frac{\psi_0(J_1)}{I_1},\] (4.25)

without further reference to the relative decay rates of \(\alpha\) and \(\beta\).

5. Sufficient conditions

In this section we state sufficient conditions for the existence of functions \(\psi_0, \psi_1\) 
that satisfy (4.9)–(4.10), which we also specify. To this end, we rely on the 
\(G\aa\rnter-Ellis theorem\) from large deviation theory. We start by stating a version 
of this theorem that focuses on events of form \((\kappa, \infty)\) or \((-\infty, \kappa)\), where \(\kappa \in \mathbb{R}\), 
and requires somewhat weaker conditions compared to standard formulations 
in the literature, such as [12, Theorem 2.3.6] or [10, Theorem 3.2.1].
5.1. The Gärtner-Ellis theorem

In this subsection we consider an arbitrary $\mathbb{P} \in \mathcal{P}$ and for every $\theta \in \mathbb{R}$ we set

$$\phi_n(\theta) \equiv \frac{1}{n} \log \mathbb{E} \{ \exp \{ n \theta T_n \} \}, \quad n \in \mathbb{N},$$

and assume that

$$\phi(\theta) \equiv \lim_{n} \phi_n(\theta) \text{ exists in } (-\infty, \infty].$$

We denote by $\Theta$ the effective domain of $\phi$, i.e., $\Theta \equiv \{ \theta \in \mathbb{R} : \phi(\theta) < \infty \}$, and by $\phi^*$ its Legendre-Fenchel transform:

$$\phi^*(\kappa) \equiv \sup_{\theta \in \mathbb{R}} \{ \theta \kappa - \phi(\theta) \}, \quad \kappa \in \mathbb{R}. \quad (5.1)$$

We further assume that $\Theta^o \neq \emptyset$, and that

$\phi$ is strictly convex and continuous in $\Theta$ and differentiable in $\Theta^o$.

This assumption implies that $\phi'$ is strictly increasing in $\Theta^o$, that $\phi'(\Theta^o)$ is a non-trivial open interval, and as a result that

$$\phi^*(\kappa) = \vartheta(\kappa) \kappa - \phi(\vartheta(\kappa)), \quad \forall \kappa \in \phi'(\Theta^o),$$

where $\vartheta$ is the inverse of $\phi'$ in $\Theta^o$.

Finally, we assume that for every $\theta \in \Theta$ there exists a (unique) distribution of $X$, $Q_\theta$, such that

$$dQ_\theta(F_n) = \exp \{ n \theta T_n - \phi_n(\theta) \}, \quad \forall n \in \mathbb{N}. \quad (5.2)$$

This is known as an exponential tilting of $\mathbb{P}$, and for its existence it suffices, for example, that $\mathbb{S}$ be Polish (see, e.g., [27, p. 144, Theorem 5.1]).

**Theorem 5.1.** Suppose that the above assumptions hold.

(i) If $\Theta^o \cap (0, \infty) \neq \emptyset$, then $\phi^*(\phi'(0^+)) = 0$, $\phi^*$ is strictly increasing in $\phi'(\Theta^o \cap (0, \infty))$ and, for every $\kappa \in \phi'(\Theta^o \cap (0, \infty))$,

$$\lim_n \frac{1}{n} \log \mathbb{P} (T_n > \kappa) = -\phi^*(\kappa). \quad (5.3)$$

(ii) If $\Theta^o \cap (-\infty, 0) \neq \emptyset$, then $\phi^*(\phi'(0^-)) = 0$, $\phi^*$ is strictly decreasing in $\phi'(\Theta^o \cap (-\infty, 0))$ and, for every $\kappa \in \phi'(\Theta^o \cap (-\infty, 0))$,

$$\lim_n \frac{1}{n} \log \mathbb{P} (T_n \leq \kappa) = -\phi^*(\kappa). \quad (5.4)$$

(iii) For every $\theta \in \Theta^o$, $Q_\theta (T_n \to \phi'(\theta)) = 1$. 

and, for each \( \theta \), \( P \) thus exponentially fast in \( n \).

Remark: 1) Theorem 5.1 implies that, for any \( \epsilon > 0 \), \( P (T_n - \phi'(0+) > \epsilon) \) decays exponentially fast in \( n \) if \( \Theta^0 \) intersects \((0, \infty)\), and \( P(T_n - \phi'(0-) \leq -\epsilon) \) decays exponentially fast in \( n \) if \( \Theta^0 \) intersects \((-\infty, 0)\).

2) In standard formulations of the Gärtner-Ellis theorem, such as [12, Theorem 2.3.6] or [10, Theorem 3.2.1], it is additionally assumed that \( 0 \in \Theta^o \), in which case the conditions in both (i) and (ii) of Theorem 5.1 hold, \( \phi'(0) \) exists, and thus \( P(|T_n - \phi'(0)| > \epsilon) \) decays exponentially fast in \( n \) for any \( \epsilon > 0 \), and \( P(T_n \rightarrow \phi'(0)) = 1 \). It is also assumed that \( \phi \) is steep, i.e., \( \phi'(\Theta^0) = \mathbb{R} \), (see, e.g., [12, Definition 2.3.5]), in which case

\[
\phi'(\Theta^0 \cap (0, \infty)) = (\phi'(0), \infty) \quad \text{and} \quad \phi'(\Theta^0 \cap (-\infty, 0)) = (-\infty, \phi'(0)).
\]

5.2. Sufficient conditions for the asymptotic theory of Section 4

We next apply Theorem 5.1 to establish sufficient conditions for the asymptotic theory of Section 4. To this end, when the assumptions of Subsection 5.1 hold for \( P = P_i \), where \( i \in \{0, 1\} \), we write \( \phi_{i,n}, \phi_i, \Theta_i, \phi_i^*, \vartheta_i \) instead of \( \phi_n, \phi, \Theta, \phi^*, \vartheta \) and, for each \( \theta \in \Theta^0_i \), we denote by \( Q_{i,\theta} \) the exponential tilting of \( P_i \), i.e.,

\[
\frac{dQ_{i,\theta}}{dP_i}(\mathcal{F}) = \exp \left\{ n (\theta T_n - \phi_{i,n}(\theta)) \right\}, \quad \forall n \in \mathbb{N}. \tag{5.5}
\]

Corollary 5.1.1. Suppose (4.8) holds for some \( J_0, J_1 \in \mathbb{R} \), with \( J_0 < J_1 \).

(i) If the assumptions of Subsection 5.1 hold for \( P = P_0 \) and

\[
\Theta_0^o \cap (0, \infty) \neq \emptyset, \quad \phi_0'(0+) = J_0, \quad \text{and} \quad \exists \theta_0 \in \Theta_0 \cap (0, \infty) : \phi_0'(\theta_0-) = J_1, \tag{5.6}
\]

then (4.9) holds, with \( \psi_0 = \phi_0^* \), for every \( \kappa \in (J_0, J_1) \). If also \( \theta_0 \in \Theta_0^o \), then (4.9) holds, with \( \psi_0 = \phi_0^* \), in a neighborhood of \( J_1 \).

(ii) If the assumptions of Subsection 5.1 hold for \( P = P_1 \) and

\[
\Theta_1^o \cap (-\infty, 0) \neq \emptyset, \quad \phi_1'(0-) = J_1, \quad \text{and} \quad \exists \theta_1 \in \Theta_1 \cap (-\infty, 0) : \phi_1'(-\theta_1) = J_0, \tag{5.7}
\]

then (4.10) holds, with \( \psi_1 = \phi_1^* \), for every \( \kappa \in (J_0, J_1) \). If also \( \theta_1 \in \Theta_1^o \), then (4.10) holds, with \( \psi_1 = \phi_1^* \), in a neighborhood of \( J_0 \).

(iii) For \( i \in \{0, 1\} \), if the assumptions of Subsection 5.1 hold for \( P = P_i \), then

\[
Q_{i,\theta} (T_n \rightarrow \phi'_i(\theta)) = 1 \quad \forall \theta \in \Theta^0_i.
\]
Proof. We only prove (i), as the proof of (ii) is similar, whereas that of (iii) follows directly from Theorem 5.1.(iii). Since \( \phi'_0(\Theta_0^0) \) is, by assumption, an open interval, (5.6) implies that

\[
(J_0, J_1) \subseteq \phi'_0(\Theta_0^0 \cap (0, \infty)),
\]

and the first claim in (i) follows by an application of Theorem 5.1.(i).

If also \( \theta_0 \in \Theta_0^0 \), (5.6) implies that

\[
(J_0, J_1] \subseteq \phi'_0(\Theta_0^0 \cap (0, \infty)),
\]

and the second claim in (i) follows again by an application of Theorem 5.1.(i).

**Corollary 5.1.2.** Suppose that the assumptions of Subsection 5.1 hold for both \( P = P_0 \) and \( P = P_1 \).

(i) If \( 0 \in \Theta_0^0 \cap \Theta_1^0 \) and \( \phi'_0(0) < \phi'_1(0) \), then (4.8) holds with \( J_i = \phi'_i(0) \), \( i = 0, 1 \).

(ii) If also both \( \phi_0 \) and \( \phi_1 \) are steep, then (4.9) holds, with \( \psi_0 = \phi^*_0 \), for every \( \kappa > J_0 \), and (4.10) holds, with \( \psi_1 = \phi^*_1 \), for every \( \kappa < J_1 \).

Proof. This is a direct consequence of the remark following Theorem 5.1. □

**Remark:** In Section 7 we show that the assumptions of Corollary 5.1.2 are satisfied in various examples. However, Corollary 5.1.1 implies that, when \( T = \bar{\Lambda} \), for the asymptotic theory of Section 4 to apply, it suffices that (5.6)-(5.7) hold, and the latter can be true even if 0 is not in the interior of either \( \Theta_0 \) or \( \Theta_1 \). We explore this point in more detail next.

### 5.3. The likelihood ratio case

In what follows, we focus on the case where \( T = \bar{\Lambda} \) and the assumptions of Subsection 5.1 hold for \( P = P_0 \). Then, in view of the fact that

\[
E_1[\exp\{\theta \Lambda_n\}] = E_0[\exp\{(\theta + 1) \Lambda_n\}], \quad \forall \ n \in \mathbb{N}, \quad \theta \in \mathbb{R},
\]

the assumptions of Subsection 5.1 also hold for \( P = P_1 \), with

\[
\phi_1(\theta) = \phi_0(\theta + 1), \quad \theta \in \mathbb{R}, \tag{5.10}
\]

\[
\Theta_1 = \Theta_0 - 1, \tag{5.11}
\]

\[
\phi'_1(\kappa) = \phi'_0(\kappa) - \kappa, \quad \kappa \in \mathbb{R}. \tag{5.12}
\]

From (5.10) it follows that 1 is the non-zero root of \( \phi_0 \), and as a result that \( [0, 1] \subseteq \Theta_0 \), since \( \Theta_0 \) is an interval. Since also \( \phi_0 \) is strictly convex and continuous in \([0, 1]\), and differentiable in \((0, 1)\), we conclude that

\[-\infty < \phi'_0(0+) < 0 < \phi'_0(1-) < \infty.\]
From (5.10) and (5.11) it similarly follows that $-1$ is the non-zero root of $\phi_1$, $[-1, 0] \subseteq \Theta_1$, and

$$-\infty < \phi'_1(-1+) < 0 < \phi'_1(0-) < \infty.$$  

Based on these observations, we can see that the conditions of Corollary 5.1.1 simplify considerably.

**Corollary 5.1.3.** If $T = \bar{\Lambda}$, the assumptions of Subsection 5.1 hold for $P = P_0$, and (4.2) holds with $I_0 = -\phi'_0(0+)$ and $I_1 = \phi'_0(1-)$, then (4.9) and (4.10) hold for every $\kappa \in (-I_0, I_1)$ with $\psi_0 = \phi'_0$ and $\psi_1 = \phi'_1$, respectively. Moreover,

$$C = \phi'_0(0) = \phi'_1(0) = -\inf_{\theta \in \mathbb{R}} \phi_0(\theta), \quad (5.13)$$

where $C$ is defined in (4.15).

**Proof.** From the discussion prior to statement of Corollary 5.1.3 it follows that the conditions of Corollary 5.1.1 are satisfied. To show (5.13), we note that the supremum in the definition of $C$ in (4.15) is attained when $\psi_0 = \psi_1$, or equivalently when $\phi'_0 = \phi'_1$. Comparing with (5.12) completes the proof.

**Remark:** Suppose that $T = \bar{\Lambda}$ and that the assumptions of Subsection 5.1 hold for $P = P_0$. Then, from Corollary 5.1.2 it follows that a sufficient condition for (4.2) to hold, with $I_0 = -\phi'_0(0)$ and $I_1 = \phi'_0(1)$, is that $\{0, 1\} \subseteq \Theta'_0$. However, as we mentioned earlier, the assumptions of Corollary 5.1.3 may hold even when $\Theta_0 = [0, 1]$, in which case $\Theta_1 = [-1, 0]$ and $(-I_0, I_1) = \phi'_i(\Theta'_i)$, $i \in \{0, 1\}$. The importance of this observation becomes clear in the iid setup, on which we focus next.

### 5.3.1. The iid setup

We end this section by showing that the conditions of Corollary 5.1.3 are satisfied in the iid setup of Subsection 4.1.1 as long as the Kullback-Leibler divergences defined in (4.6) are positive and finite, or equivalently, the expectation of $\Lambda_1 = \log(f_1(X_1)/f_0(X_1))$ is non-zero and finite under both $P_0$ and $P_1$.

Indeed, in this case, (4.2) holds with $I_0 = D(f_0\|f_1)$ and $I_1 = D(f_1\|f_0)$ by Kolmogorov’s Strong Law of Large Numbers, and clearly

$$\phi_0(\theta) = \log E_0[\exp\{\theta \Lambda_1\}], \quad \theta \in \mathbb{R}.$$ 

Since $\phi_0$ is the cumulant generating function of a non-degenerate distribution and $[0, 1] \subseteq \Theta_0$, $\phi_0$ is strictly convex in $\Theta_0$, differentiable in $\Theta'_0$, continuous at 0 and 1, and satisfies

$$\phi'_0(0+) = E_0[\Lambda_1] = -I_0 \quad \text{and} \quad \phi'(1-) = \frac{E_0[\Lambda_1 \exp\{\Lambda_1\}]}{E_0[\exp\{\Lambda_1\}]} = I_1$$

(see, e.g. [12, Exercise 2.2.24]).
6. Implementation via importance sampling

The proposed designs for the multistage tests in Section 2 require knowledge of the functions $n^*$ and $\kappa^*$, defined in (2.3). These do not admit, in general, closed-form expressions and need to be approximated. For any given $\alpha$ and $\beta$ in $(0, 1)$, $n^*(\alpha, \beta)$ and $\kappa^*(\alpha, \beta)$ can be approximated by estimating $P_0(T_n > \kappa)$ and $P_1(T_n \leq \kappa)$ for different $n$ and $\kappa$, and finding the minimum $n$ for which there exists a $\kappa$ so that the first probability does not exceed $\alpha$ and the second does not exceed $\beta$.

If it is convenient to simulate the sequence $X$ under $P_0$ and $P_1$, a simple method for the estimation of $P_0(T_n > \kappa)$ and $P_1(T_n \leq \kappa)$ is plain Monte-Carlo simulation. However, when these probabilities are very small, this approach may not be efficient, or even feasible. Indeed, if the probability of interest is $10^{-a}$ for some $a > 0$, the minimum number of simulation runs needed for the relative error of the Monte-Carlo estimator to be at most $1\%$ is $10^{a+4}$. Therefore, when the probability of interest is very small, a different method may need to be applied for its estimation, such as importance sampling [10].

To illustrate this method, we focus on the estimation of $P_0(T_n > \kappa)$, as a completely analogous discussion applies to the estimation of $P_1(T_n \leq \kappa)$. We observe that if $Q$ is a distribution of $X$ that is mutually absolutely continuous with $P_0$ on $\mathcal{F}_n$ for every $n \in \mathbb{N}$, then

$$P_0(T_n > \kappa) = \mathbb{E}_Q[Z_{n,\kappa}(Q)],$$

where

$$Z_{n,\kappa}(Q) = \frac{dP_0}{dQ}(\mathcal{F}_n) \cdot 1\{T_n > \kappa\}$$

and $\mathbb{E}_Q$ denotes expectation under $Q$. Thus, if it is possible to simulate $X$ under $Q$, $P_0(T_n > \kappa)$ can be estimated by averaging $Z_{n,\kappa}(Q)$ over a large number of independent realizations of $X$ in which it is distributed according to $Q$.

The question then is how to select the importance sampling distribution $Q$, so that the relative error of the induced estimator is small even when $P_0(T_n > \kappa)$ is small. To answer it, we assume that the assumptions of Corollary 5.1.1.(i) (resp. Corollary 5.1.3) hold when $T \neq \Lambda$ (resp. $T = \Lambda$) and fix $\kappa$ in $(J_0, J_1)$ (resp. $(-I_0, I_1)$), in which case $P_0(T_n > \kappa)$ decays exponentially fast in $n$. Then, squaring both sides in (6.1), applying the Cauchy-Schwarz inequality, taking logarithms on both sides, dividing by $n$, letting $n \to \infty$, and applying (4.9), we obtain

$$\lim_{n \to \infty} \frac{n}{\log n} \mathbb{E}_Q[Z_{n,\kappa}^2(Q)] \geq -2\psi_0(\kappa).$$

The latter is essentially a universal asymptotic lower bound on the variance of any importance sampling estimator. As it is common in the relevant literature (see, e.g., [10, Chapter 5]), we refer to $Q$ as logarithmically efficient for the estimation of $P_0(T_n > \kappa)$ if it attains this asymptotic lower bound, i.e., if

$$\lim_{n \to \infty} \frac{n}{\log n} \mathbb{E}_Q[Z_{n,\kappa}^2(Q)] \leq -2\psi_0(\kappa).$$
Recalling the definition of the exponential tilting $Q_{0,\theta}$ in (5.2), for every $n \in \mathbb{N}$ and $\theta \in \Theta_0^0$ we have
\[
E_{Q_{0,\theta}}[Z_{n,\kappa}^2(Q_{0,\theta})] = E_{Q_{0,\theta}}[\exp\{-2n(\theta T_n - \phi_{0,n}(\theta))\}; T_n > \kappa] \\
\leq \exp\{-2n(\theta \kappa - \phi_{0,n}(\theta))\}.
\]
Taking logarithms, dividing by $n$ and letting $n \to \infty$ we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \log E_{Q_{0,\theta}}[Z_{n,\kappa}^2(Q_{0,\theta})] \leq -2(\theta \kappa - \phi_0(\theta)).
\]
Therefore, when $\theta = \vartheta_0(\kappa)$, where $\vartheta_0$ is the inverse function of $\phi_0'$, the right-hand-side is equal to $-2\psi_0(\kappa)$, which proves that $Q_{0,\vartheta_0(\kappa)}$ is logarithmically efficient for the estimation of $P_0(T_n > \kappa)$.

Working similarly, we can see that if the assumptions of Corollary 5.1.1.(ii) (resp. Corollary 5.1.3) hold when $T \neq \Lambda$ (resp. $T = \Lambda$), a logarithmically efficient importance sampling distribution for the estimation of $P_1(T_n \leq \kappa)$ when $n$ is large is $Q_{1,\vartheta_1(\kappa)}$, where $\vartheta_1$ is the inverse function of $\phi_1'$. In Subsection 7.1 we present an example where $Q_{0,\vartheta_0(\kappa)}$ and $Q_{1,\vartheta_1(\kappa)}$ coincide.

Finally, we observe that by Corollary 5.1.1.(iii) it follows that
\[
Q_{i,\vartheta_i(\kappa)}(T_n \to \kappa) = 1, \quad i \in \{0,1\}.
\]
This suggests that if it is not convenient to simulate $X$ under the logarithmically efficient importance sampling distributions, a potential strategy for estimating $P_0(T_n > \kappa)$ and $P_1(T_n \leq \kappa)$, simultaneously, is to apply importance sampling using a distribution under which it is convenient to simulate $X$ and $T_n$ converges almost surely to $\kappa$ as $n \to \infty$. We apply this strategy successfully in two non–iid testing problems in Section 8.

7. Examples

In this section we focus on three concrete testing problems, with which we illustrate the general results of the previous sections. Specifically, for each of these testing problems we show that the conditions of Subsection 4.1 hold, and also that the conditions of Subsection 4.2 hold for $T = \Lambda$, as well as for an alternative test statistic. For the latter, we also compute the induced asymptotic relative efficiency, defined in (4.25).

7.1. Testing in a one-parameter exponential family

In the first example of this section we let $h$ be a density with respect to a $\sigma$-finite measure $\nu$ on $\mathbb{S}$ such that $M \neq \emptyset$, where
\[
M \equiv \{\mu \in \mathbb{R} : \varphi(\mu) < \infty\}^\circ, \quad \varphi(\mu) \equiv \log \int_{\mathbb{S}} e^{\mu x} h(x) \nu(dx)
\] (7.1)
and, for each $\mu \in M$, we set

$$h_{\mu}(x) \equiv h(x) e^{\mu x - \varphi(\mu)}, \quad x \in \mathbb{S},$$

noting that $h_{\mu}$ is also a density with respect to $\nu$, with the same support as $h$. We denote by $P_\mu$ the distribution of $X$, and by $E_\mu$ the corresponding expectation, when $X$ is a sequence of independent random elements with common density $h_{\mu}$, and consider the testing setup of Subsection 2.4. In this context, the log-likelihood ratio statistic in (4.7) becomes

$$\Lambda_n = (\mu_1 - \mu_0) \sum_{i=1}^{n} X_i - n (\varphi(\mu_1) - \varphi(\mu_0)), \quad n \in \mathbb{N}, \quad (7.2)$$

and, for each $\mu \in M$, it is a random walk under $P_\mu$ with drift

$$E_\mu[\Lambda_1] = (\mu_1 - \mu_0) \varphi'(\mu) - (\varphi(\mu_1) - \varphi(\mu_0)). \quad (7.3)$$

Thus, setting $\mu$ equal to $\mu_0$ and $\mu_1$, we obtain the following expressions for the Kullback-Leibler divergences in (4.6):

$$D(f_0 || f_1) = -((\mu_1 - \mu_0) \varphi'(\mu_0) - (\varphi(\mu_1) - \varphi(\mu_0)))$$
$$D(f_1 || f_0) = (\mu_1 - \mu_0) \varphi'(\mu_1) - (\varphi(\mu_1) - \varphi(\mu_0)).$$

Since these are positive and finite, by the discussion in Subsection 5.3.1 it follows that all assumptions in Subsections 4.1-4.2 hold with

$$I_0 = D(f_0 || f_1), \quad I_1 = D(f_1 || f_0), \quad C = \psi_0(0),$$
$$\psi_0(\kappa) = \vartheta_0(\kappa) - \varphi_0(\vartheta_0(\kappa)), \quad \forall \kappa \in (-I_0, I_1)$$
$$\psi_1(\kappa) = \vartheta_1(\kappa) - \varphi_1(\vartheta_1(\kappa)), \quad \forall \kappa \in (-I_0, I_1), \quad (7.4)$$

where $\vartheta_i$ is the inverse of $\varphi_i'$, $i \in \{0, 1\}$, and

$$\varphi_0(\theta) = \varphi(\mu_0 + \theta(\mu_1 - \mu_0)) - (\varphi(\mu_0) + \theta(\varphi(\mu_1) - \varphi(\mu_0))), \quad \theta \in [0, 1]$$
$$\varphi_1(\theta) = \varphi(\mu_1 + \theta(\mu_1 - \mu_0)) - (\varphi(\mu_1) + \theta(\varphi(\mu_1) - \varphi(\mu_0))), \quad \theta \in [-1, 0]. \quad (7.5)$$

As a result, in this context, the asymptotic optimality of the proposed multistage tests holds when $T = \bar{\Lambda}$. In fact, it also holds when

$$T = \bar{X} \equiv \{ \bar{X}_n, n \in \mathbb{N} \}, \quad \text{where} \quad \bar{X}_n \equiv \frac{1}{n} \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}. \quad (7.6)$$

Indeed, from (7.2) it follows that when $T = \bar{X}$, then for any $\mu_0, \mu_1 \in M$ we have

$$\bar{\Lambda}_n = (\mu_1 - \mu_0) T_n - (\varphi(\mu_1) - \varphi(\mu_0)), \quad n \in \mathbb{N}, \quad (7.7)$$

which means that the values of $n^*(\alpha, \beta)$ and $\kappa^*(\alpha, \beta)$, which in general depend on the choice of the test statistic $T$, coincide when $T = \bar{X}$ and $T = \bar{\Lambda}$. 
7.1.1. Importance sampling distributions

In this setup, it is convenient to obtain an explicit form for the logarithmically efficient importance sampling distributions for the estimation of $P_0(\Lambda_n > \kappa)$ and $P_1(\Lambda_n \leq \kappa)$ when $T = \Lambda$ for any $\kappa \in (-I_0, I_1)$. Indeed, for any $\kappa \in (-I_0, I_1)$ we have:

$$Q_{0,\phi_0(\kappa)} = Q_{1,\phi_1(\kappa)} = P_\mu,$$

where $\mu \in (\mu_0, \mu_1)$ is such that $E_\mu[\Lambda_1] = \kappa$. To prove this statement, we first note that for any $n \in \mathbb{N}$ and $\theta \in (0, 1)$, by (7.2) we have

$$\Lambda_n (P_{\mu_0 + \theta(\mu_1 - \mu_0)}, P_0) = \Lambda_n (P_{\mu_0 + \theta(\mu_1 - \mu_0)}, P_{\mu_0})$$

$$= \theta(\mu_1 - \mu_0) \sum_{i=1}^{n} X_i - n (\varphi(\mu_0 + \theta(\mu_1 - \mu_0)) - \varphi(\mu_0))$$

$$= n (\theta \Lambda_n - \phi_0(\theta)),$$

and similarly, for any $n \in \mathbb{N}$ and $\theta \in (-1, 0)$,

$$\Lambda_n (P_{\mu_1 + \theta(\mu_1 - \mu_0)}, P_1) = n (\theta \Lambda_n - \phi_1(\theta)).$$

Therefore, the exponential tiltings of $P_0$ and $P_1$, defined in (5.5), are given by

$$Q_{0,\theta} = P_{\mu_0 + \theta(\mu_1 - \mu_0)}, \quad \theta \in (0, 1),$$

$$Q_{1,\theta} = P_{\mu_1 + \theta(\mu_1 - \mu_0)}, \quad \theta \in (-1, 0).$$

Differentiating the identities in (7.5) and comparing with (7.3) we obtain

$$\mathbb{E}_{\mu_0 + \theta(\mu_1 - \mu_0)}[\Lambda_1] = \phi_0(\theta), \quad \theta \in (0, 1),$$

$$\mathbb{E}_{\mu_1 + \theta(\mu_1 - \mu_0)}[\Lambda_1] = \phi_1(\theta), \quad \theta \in (-1, 0).$$

(7.8)

The statement now follows by the definition of $\tilde{\vartheta}_i$ as the inverse of $\phi'_i$, where $i \in \{0, 1\}$.

7.1.2. A binary statistic

An approach to the testing problem of this subsection, which can be motivated by practical constraints or robustness considerations, is to binarize the data, recording only whether each observation is larger, or not, than some user-specified value in the interior of the support of $h$, say $x_*$. Then, the test statistic can be written as

$$T = \bar{Z} \equiv \{\bar{Z}_n, n \in \mathbb{N}\}, \quad \bar{Z}_n \equiv \frac{1}{n} \sum_{i=1}^{n} Z_i, \quad Z_i \equiv 1\{X_i > x_*\}, \quad n \in \mathbb{N}, \quad (7.9)$$

and all assumptions in Subsection 4.2, including (4.13), are satisfied with

$$J_i = P_i(\{X_1 > x_*\}), \quad C = -\log \sqrt{4J_0J_1},$$

$$\phi_i(\theta) = \log (J_i e^\theta + (1 - J_i)), \quad \theta \in \mathbb{R},$$

$$\psi_i(\kappa) = \text{Ber}(\kappa|J_i), \quad \kappa \in (0, 1),$$

where $\text{Ber}(\kappa|J_i)$ denotes the Bernoulli distribution with parameter $\kappa$ conditioned on the event $J_i$.
where \( i \in \{0, 1\} \), and \( \text{Ber}(x||y) \) is the Kullback-Leibler divergence between two Bernoulli distributions with success probabilities \( x \) and \( y \) respectively, i.e.,

\[
\text{Ber}(x||y) \equiv x \log(x/y) + (1 - x) \log((1 - x)/(1 - y)), \quad x, y \in (0, 1).
\]

\[ (7.10) \]

### 7.1.3. Testing the Gaussian mean

We next specialize the above results to the special case of testing the mean of a Gaussian distribution with unit variance, i.e., when \( M = \mathbb{R} \) and \( \varphi(\mu) = \mu^2/2 \) for every \( \mu \in \mathbb{R} \) in (7.1). For simplicity, we assume that the two parameter values under which we control the two error probabilities, \( \mu_0 \) and \( \mu_1 \), are opposite, i.e.,

\[
\mu_1 = -\mu_0 = \eta \text{ for some } \eta > 0.
\]

In this case, \( n^*(\alpha, \beta) \) and \( \kappa^*(\alpha, \beta) \) in (2.3) can be computed explicitly when \( T = \bar{\Lambda} \) or \( T = \bar{\Sigma} \), for any \( \alpha, \beta \in (0, 1) \), and do not need to be estimated via simulation. Specifically, by the formulas in the general case of this subsection we obtain

\[
I_0 = I_1 = 2\eta^2 \equiv I, \quad C = 4I
\]

\[
\phi_0(\theta) = \theta(\theta - 1) I, \quad \phi_1(\theta) = \theta(\theta + 1) I, \quad \theta \in \mathbb{R}
\]

\[
\psi_0(\kappa) = (I + \kappa)^2 / (4I), \quad \psi_1(\kappa) = (I - \kappa)^2 / (4I), \quad \kappa \in \mathbb{R},
\]

and, for any \( \alpha, \beta \in (0, 1) \),

\[
n^*(\alpha, \beta) = \frac{(z_\alpha + z_\beta)^2}{2I} \quad \text{and} \quad \kappa^*(\alpha, \beta) = I \frac{z_\alpha - z_\beta}{z_\alpha + z_\beta}, \quad (7.11)
\]

where \( z_p \) is the upper \( p \)-quantile of the standard Gaussian distribution. In Figure 1a we plot the functions \( \psi_0, \psi_1 \), for \( T = \bar{\Lambda} \) and \( T = \bar{\Sigma} \), when \( \eta = 0.5 \).

Finally, we note that in this case the asymptotic relative efficiencies in (4.25) coincide when \( T = \bar{\Sigma} \), since

\[
\text{ARE}_0 = \frac{\text{Ber}(\Phi(-\eta)||\Phi(\eta))}{2\eta^2} = \frac{\text{Ber}(\Phi(\eta)||\Phi(-\eta))}{2\eta^2} = \text{ARE}_1, \quad (7.12)
\]

where \( \Phi \) denotes the cumulative distribution function of the standard Gaussian distribution and the function \( \text{Ber}(x||y) \) is defined in (7.10). We note also that this quantity converges to 0.25 as \( \eta \to \infty \) and to \( 2/\pi \) as \( \eta \to 0 \). In Figure 1b we plot the asymptotic relative efficiency in (7.12) as a function of \( \eta \) in \( \in (0, 5) \).

### 7.2. Testing the coefficient of a first-order autoregressive model

In the second example of this section we assume that \( X \) follows a Gaussian first-order autoregressive model, i.e.,

\[
X_n = \mu X_{n-1} + \epsilon_n, \quad n \in \mathbb{N},
\]
Table 1

In the left column we plot the functions images of $\psi_0$ and $\psi_1$ when $T = \bar{\Lambda}$ and when $T$ is the alternative test statistic considered in each of these examples of Section 7. To distinguish, we write $\zeta_i$ instead of $\psi_i$ when $T = \bar{\Lambda}$, $i \in \{0, 1\}$ and write $C_{\zeta_i}$ and $C_{\psi_i}$ for the quantity $C$ defined in (4.15). In the right column we plot the corresponding asymptotic relative efficiencies, $\text{ARE}_0$ and $\text{ARE}_1$, defined in (4.25).
Moreover, from (5.13) it follows, by minimizing the testing problem of Subsection 2.4.

In this setup, the log-likelihood ratio statistic in (4.1) becomes

$$\Lambda_n = (\mu_1 - \mu_0) \left( \frac{1}{n} \sum_{i=1}^{n} X_{i-1}X_i - \frac{\mu_1 + \mu_0}{2} \frac{1}{n} \sum_{i=1}^{n} X_i^2 \right), \quad n \in \mathbb{N}. \quad (7.13)$$

For any $\mu \in M$, from [9, Chapter 3] it follows that, for every $\theta$

$$\frac{1}{n} \sum_{i=1}^{n} X_{i-1}^2 \rightarrow \frac{1}{1 - \mu^2} \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} X_{i-1}X_i \rightarrow \frac{\mu}{1 - \mu^2} \quad \mathbb{P}_\mu \text{-a.s.}, \quad (7.14)$$

and consequently

$$\Lambda_n \rightarrow \frac{\mu_1 - \mu_0}{1 - \mu^2} \left( \mu - \frac{\mu_1 + \mu_0}{2} \right) \quad \mathbb{P}_\mu \text{-a.s.} \quad (7.15)$$

Moreover, from [8] it follows that, for every $\mu \in M$,

$$\frac{1}{n} \log \mathbb{E}_\mu \left[ e^{\theta \Lambda_n} \right] \rightarrow -\frac{1}{2} \log \left( \frac{1}{2} p_\mu(\theta) + \frac{1}{2} \sqrt{p_\mu^2(\theta) - 4q_\mu^2(\theta)} \right) \equiv \phi(\theta; \mu), \quad (7.16)$$

where $\theta \in D_\mu \equiv D_{\mu,1} \cup D_{\mu,2} \cup D_{\mu,3}$,

$$D_{\mu,1} \equiv \left\{ \theta \in \mathbb{R} : \mu^2 < p_\mu(\theta) \leq 2\mu^2, \quad q_\mu^2(\theta) \leq \mu^2(p_\mu(\theta) - \mu^2) \right\},$$

$$D_{\mu,2} \equiv \left\{ \theta \in \mathbb{R} : 2\mu^2 < p_\mu(\theta) < 2, \quad p_\mu(\theta) > 2|q_\mu(\theta)| \right\},$$

$$D_{\mu,3} \equiv \left\{ \theta \in \mathbb{R} : p_\mu(\theta) \geq 2, \quad q_\mu^2(\theta) \leq p_\mu(\theta) - 1 \right\},$$

and $\theta = 1 + \mu^2 + (\mu_1 - \mu_0)(\mu_1 + \mu_0)\theta$, $q_\mu(\theta) = -\mu - (\mu_1 - \mu_0)/\theta$.

the function $\phi(\cdot; \mu)$ in (7.16) is differentiable in $D_\mu^a$, and

$$0 \in D_{\mu,0,2}^a, \quad 1 \in (D_{\mu,0,1}^a \cup D_{\mu,0,2}^a)^a.$$

Thus, setting $\mu$ equal to $\mu_0$ and $\mu_1$ in (7.15)-(7.16), we conclude that all assumptions in Corollary 5.1.3 are satisfied with

$$I_0 = \frac{(\mu_1 - \mu_0)^2}{2(1 - \mu_0^2)}, \quad I_1 = \frac{\mu_1^2 - \mu_0^2}{2(1 - \mu_1^2)}, \quad \phi_i = \phi(\cdot; \mu_i), \quad i \in \{0, 1\}.$$

Moreover, from (5.13) it follows, by minimizing $\phi(\cdot; \mu_0)$, that

$$C = \log \sqrt{\frac{1 - \mu_0 \mu_1}{1 - (\mu_0 + \mu_1)^2/4}}. \quad (7.17)$$

The functions $\psi_0$ and $\psi_1$ in this context are computed numerically and are plotted in Figure 1c when $\mu_1 = -\mu_0 = 0.5$. We note that, in this case, they are symmetric about the y-axis, a property that does not hold, in general, when $\mu_1 \neq -\mu_0$. 
7.2.1. The Yule-Walker estimator

An alternative test statistic for this testing problem is the Yule-Walker estimator, i.e., \( T = \hat{\mu} = \{ \hat{\mu}_n, n \in \mathbb{N} \} \), where

\[
\hat{\mu}_n = \frac{\sum_{i=1}^n X_{i-1}X_i}{\sum_{i=1}^n X_i^2}, \quad n \in \mathbb{N}.
\] (7.18)

From (7.14) it follows that \( \hat{\mu}_n \) is a strongly consistent estimator of \( \mu \), i.e., for every \( \mu \in M \),

\[
P_\mu(\hat{\mu}_n \to \mu) = 1.
\] (7.19)

Moreover, from [8] it follows that, for any \( \mu \in M \),

\[
\frac{1}{n} \log P_\mu(\hat{\mu}_n > \kappa) \to \psi(\kappa; \mu), \quad \forall \kappa \in (\mu, 1)
\]

\[
\frac{1}{n} \log P_\mu(\hat{\mu}_n \leq \kappa) \to \psi(\kappa; \mu), \quad \forall \kappa \in (-1, \mu),
\] (7.20)

where the function

\[
\psi(\kappa; \mu) \equiv \log \sqrt{\frac{1 + \mu^2 - 2\mu\kappa}{1 - \kappa^2}}, \quad \kappa \in (-1, 1)
\]

is strictly convex, has a unique root at \( \mu \), goes to \( \infty \) as \( \kappa \) goes to \( -1 \) or 1. Therefore, setting \( \mu \) equal to \( \mu_0 \) and \( \mu_1 \) in (7.19) and (7.20), we conclude that assumptions (4.8) (4.9), (4.10), (4.13) hold with

\[
J_i = \mu_i, \quad \psi_i = \psi(\cdot; \mu_i), \quad i \in \{0, 1\}.
\]

Interestingly, equating \( \psi_0 \) and \( \psi_1 \) we obtain the same value for \( C \) as in (7.17). In view of (4.20), this implies that using \( \hat{\mu} \), instead of \( \hat{\Lambda} \), as the test statistic, does not reduce the asymptotic relative efficiency of the fixed-sample-size test as \( \alpha, \beta \to 0 \) so that \( |\log \alpha| \sim |\log \beta| \). This is not the case for the proposed multistage tests, as can be seen in Figure 1d, where we plot \( \text{ARE}_0 \) and \( \text{ARE}_1 \) when \( \mu_0 = -\mu_1 \), in which case they coincide, for different values of \( \mu_1 \) in (0, 1).

7.3. Testing the transition matrix of a Markov chain

In the third example of this section we assume that \( X \) is an irreducible and recurrent Markov chain with state space \( I = \{0, 1, \ldots, I\} \), where \( I \in \mathbb{N} \), initial value \( X_0 = 0 \), transition matrix \( \Pi \), and stationary distribution \( \pi \). Moreover, we note that (see, e.g., [14, Theorem 5.5.9])

\[
Y = \{Y_n = (X_{n-1}, X_n), n \in \mathbb{N}\}
\]

is also an irreducible and recurrent Markov chain, with state space \( \mathbb{I}^2 \), transition matrix \( \Pi^Y \) whose \((i_1, i_2), (i_3, i_4)\)-th element is

\[
\Pi(i_3, i_4) 1\{i_2 = i_3\}, \quad (i_1, i_2), (i_3, i_4) \in \mathbb{I}^2,
\]
and stationary distribution
\[ \pi^\Pi(i, j) = \pi(i) \Pi(i, j), \quad i, j \in [I]. \]

For simplicity, we identify the family of all possible distributions of \( X, \mathcal{P} \), with the class of all irreducible and recurrent transition matrices of dimension \( I + 1 \). For each \( \Pi \in \mathcal{P} \), we denote by \( \mathbb{P}_\Pi \) the distribution of \( X \), and by \( \mathbb{E}_\Pi \) the corresponding expectation, when the transition matrix of \( X \) is \( \Pi \). We consider the general testing setup of Section 2, where \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) are two arbitrary subclasses of \( \mathcal{P} \), and
\[ P_i \equiv \mathbb{P}_{\Pi_i}, \quad i \in \{0, 1\} \]
for some arbitrary \( \Pi_i \in \mathcal{P}_i, i \in \{0, 1\} \). In this setup, the log-likelihood ratio statistic in (4.1) takes the form:
\[ \Lambda_n = \sum_{(i,j) \in [I]^2} r(i, j) N_n(i, j) = \sum_{m=1}^n \mathcal{U}(Y_m), \quad n \in \mathbb{N}, \]
where, for each \( (i, j), y \in [I]^2 \),
\[ r(i, j) \equiv \log \left( \frac{\Pi_1(i, j)}{\Pi_0(i, j)} \right), \quad N_n(i, j) \equiv \sum_{m=1}^n 1\{Y_m = (i, j)\}, \]
\[ \mathcal{U}(y) = \sum_{(i, j) \in [I]^2} r(i, j) \cdot 1\{y = (i, j)\}. \]

For any \( \Pi \in \mathcal{P} \), from [14, Example 6.2.4] it follows that, for every \( (i, j) \in [I]^2 \),
\[ \frac{1}{n} N_n(i, j) \to \pi^\Pi(i, j) \quad \mathbb{P}_\Pi - \text{a.s.} \]
and, as a result,
\[ \bar{\Lambda}_n \to \sum_{(i,j) \in [I]^2} r(i,j) \pi^\Pi(i,j) \quad \mathbb{P}_\Pi - \text{a.s.} \quad (7.21) \]
Moreover, by [12, Theorem 3.1.1 & 3.1.2], it follows that, for any \( \Pi \in \mathcal{P} \),
\[ \frac{1}{n} \log \mathbb{E}_\Pi[\exp\{\theta \Lambda_n\}] \to \log \xi \left( \Pi^\theta_{\Pi, \Pi} \right) \equiv \phi(\theta; \Pi), \quad \text{for every} \quad \theta \in \mathbb{R}, \quad (7.22) \]
where \( \xi \) is the functional that maps a matrix to its greatest eigenvalue, \( \Pi^\theta_{\Pi, \Pi} \) is a matrix of the same dimension as \( \Pi^\Pi \) whose \(((i_1, i_2), (i_3, i_4))\)-th element is
\[ \Pi^\theta((i_1, i_2), (i_3, i_4)) \exp\{\theta \mathcal{U}((i_3, i_4))\}, \quad (i_1, i_2), (i_3, i_4) \in [I]^2, \]
and the limit in (7.22) is a finite and differentiable function of \( \theta \). Therefore, setting \( \Pi \) equal to \( \Pi_0 \) and \( \Pi_1 \) in (7.21)-(7.22) we conclude that all assumptions in Corollary 5.1.3 are satisfied, and \( I_i, \phi_i, i \in \{0, 1\} \) can be computed accordingly.
7.3.1. The two-state case

We next specialize the previous setup to the case that \( I = 1 \), where the transition matrix and stationary distribution of \( X \) are of the form

\[
\Pi = \begin{pmatrix} p & 1-p \\ 1-\mu & \mu \end{pmatrix}, \quad \pi = \begin{pmatrix} \frac{1-\mu}{2-p-\mu} & \frac{1-p}{2-p-\mu} \end{pmatrix}, \quad \text{where} \quad p, \mu \in (0, 1).
\]

We fix \( p \in (0, 1) \), so that the only unknown parameter is \( \mu \), which takes values in \( M = (0, 1) \). Thus, we now denote by \( P_\mu \) the distribution, and by \( E_\mu \) the corresponding expectation, of \( X \) when the unknown parameter is \( \mu \), and consider the testing setup of Subsection 2.4. In this case, (7.21) reduces to

\[
\bar{\Lambda}_n \to \frac{1-p}{2-p-\mu} \left( \text{Ber}(\mu||\mu_0) - \text{Ber}(\mu||\mu_1) \right) \quad P_\mu - \text{a.s.}, \quad (7.23)
\]

where \( \text{Ber}(x||y) \) is defined in (7.10), whereas \( I_0 \) and \( I_1 \) become:

\[
I_0 = \frac{1-p}{2-p-\mu_0} \text{Ber}(\mu_0||\mu_1), \quad I_1 = \frac{1-p}{2-p-\mu_1} \text{Ber}(\mu_1||\mu_0).
\]

An alternative test statistic in this setup is the sample average in (7.6), or equivalently,

\[
T_n = \bar{X}_n \equiv \frac{1}{n} \sum_{m=1}^{n} V(x_m), \quad \text{where} \quad V(x) = x.
\]

Unlike the first example of this section, however, this test statistic does not lead to asymptotic optimality, as it does not admit a bijection with the log-likelihood ratio, as in (7.7). To compute the resulting asymptotic relative efficiency, (4.25), we note that, by [14, Example 6.2.4], for any \( \mu \in M \),

\[
\bar{X}_n \to \sum_{i \in [I]} i \pi(i) = \frac{1-p}{2-p-\mu} \quad P_\mu - \text{a.s.} \quad (7.24)
\]

Moreover, by [12, Theorem 3.1.1 & 3.1.2] it follows that, for any \( \mu \in M \),

\[
\frac{1}{n} \log E_\mu(\exp\{\theta n \bar{X}_n\}) \to \log \xi(\Pi_{\theta, V}) \equiv \phi(\theta; \Pi), \quad \forall \theta \in \mathbb{R}, \quad (7.25)
\]

where \( \Pi_{\theta, V} \) is a matrix of the same dimension as \( \Pi \), whose \((i, j)\)-th element is

\[
\Pi(i, j) e^{\theta V(j)}, \quad i, j \in [I],
\]

and the limit is finite, differentiable and steep in \( \mathbb{R} \) as a function of \( \theta \). Therefore, setting \( \mu \) equal to \( \mu_0 \) and \( \mu_1 \) in (7.24)-(7.25) we conclude that all assumptions in Corollary 5.1.2 are satisfied with

\[
J_i = \frac{1-p}{2-p-\mu_i} \quad \text{and} \quad \phi_i(\theta) = \phi(\theta; \Pi_i), \quad \text{for every} \quad \theta \in \mathbb{R}, \quad i \in \{0, 1\}.
\]

In Figure 1e we plot the functions \( \psi_0, \psi_1 \) for \( T = \bar{\Lambda} \) and \( T = \bar{X} \) when \( \mu_0 = 1 - \mu_1 = 0.25 \). In Figure 1f we plot the asymptotic relative efficiencies in (4.25) against \( \mu_0 = 1 - \mu_1 \) for different values of \( \mu_0 \) in \( (0, 0.5) \).
8. Numerical studies

In this section we present the results of two numerical studies in which we compare the 3-stage test, $\tilde{\chi}$, the 4-stage test, $\hat{\chi}$, both with $T = \bar{\Lambda}$, against the SPRT, $\chi'$, when

- testing the mean of an iid Gaussian sequence with unit variance (Subsection 7.1.3), with $\mu_1 = -\mu_0 = 0.5$,
- testing the coefficient of an first-order autoregression (Subsection 7.2), when $\mu_1 = -\mu_0 = 0.5$,
- testing an entry in the transition matrix of a two-state Markov chain (Subsection 7.3.1), with $p = 0.5$ and $\mu_0 = 1 - \mu_1 = 0.25$.

Before we describe the two studies and present the main findings, we discuss how the tests are designed and how their average sample sizes are computed.

8.1. Design of tests

In all cases, the SPRT in (4.5) is designed with $B = |\log \alpha|$ and $A = |\log \beta|$, whereas the multistage tests are designed according to Theorems 3.1 and 3.2, with the free parameters selected according to (3.12) and (3.26). The functions $n^*$ and $\kappa^*$, defined in (2.3), are evaluated using the closed-form expressions in (7.11) in the first testing problem and the importance sampling method of Section 6 in the other two. Specifically, the importance sampling distribution employed in the second (resp. third) testing problem is the distribution $\mathbb{P}_{\mu}$ for which the limit in (7.15) (resp. (7.23)) is equal to $\kappa$. Moreover, grid search is used for the determination of the free parameters of the multistage tests.

8.2. Computation of the expected sample sizes

The expected sample sizes of the multistage tests are computed using the formulas (3.6)-(3.7) and (3.17)-(3.19) in the first testing problem, as it is possible to compute the multivariate Gaussian probabilities in these expressions, and plain Monte Carlo in the other two. The expected sample size of the SPRT is estimated with plain Monte Carlo in all cases. In each Monte Carlo application, $10^4$ replications are utilized, leading in all cases to relative errors below 5%.

8.3. The first study

In the first study we compare the expected sample sizes of $\tilde{\chi}$, $\hat{\chi}$ and $\chi'$ under $\mathbb{P}_0$, with the understanding that analogous results can be obtained when comparing $\tilde{\chi}$, $\hat{\chi}$ and $\chi'$ under $\mathbb{P}_1$. Specifically, we evaluate $E_0[\tilde{\tau}]/E_0[\tau']$ and $E_0[\hat{\tau}]/E_0[\tau']$, i.e., the ratio of the expected sample sizes under $H_0$ of $\tilde{\chi}$ and $\hat{\chi}$ over that of $\chi'$, in the context of the first testing problem, for different values of $\beta$, when $\alpha$ is given by one of the following relationships:

$$\alpha = \beta, \quad \alpha = \beta^4, \quad |\log \alpha| = |\log \beta|^{1.5}, \quad |\log \alpha| = |\log \beta|/\beta^{0.08}. \quad (8.1)$$
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(a) $\alpha = \beta$

(b) $\alpha = \beta$

(c) $|\log \alpha| = 4 |\log \beta|$

(d) $|\log \alpha| = 4 |\log \beta|$

(e) $|\log \alpha| = |\log \beta|^{1.5}$

(f) $|\log \alpha| = |\log \beta|^{1.5}$

(g) $|\log \alpha| = |\log \beta|/\beta^{0.08}$

(h) $|\log \alpha| = |\log \beta|/\beta^{0.08}$

Table 2

In the left column we plot $E_\tau h(\tau)/E_\tau h(\tau')$ and $E_\tau h(\tau)/E_\tau h(\tau')$, along with the corresponding bounds from Section 2, against $|\log \beta|$, when $\alpha$ follow the given pattern, in testing the mean of iid Gaussian sequence with unit variance. In the right column we plot the corresponding $|\log \gamma|$ in $\hat{\chi}$ and $|\log \gamma'|$ in $\hat{\chi}$ against $|\log \beta|$. 
In the left column of Figure 2 we present these ratios, together with the non-asymptotic bounds implied by (3.10)-(3.11) and (3.22)-(3.23). In these graphs we observe a slow, downward trend, as $\alpha$ and $\beta$ decrease, in all ratios but the one that corresponds to $\tilde{\chi}$ in the last asymptotic regime. This is consistent with Theorem 4.3, in which $\hat{\chi}$ is shown to achieve asymptotic optimality under $P_0$ in all asymptotic regimes in (8.1), whereas $\tilde{\chi}$ only in the first three.

From these graphs we also see that, under $P_0$, the average sample of the 4-stage test, $\hat{\chi}$, is substantially smaller than that of the 3-stage test, $\tilde{\chi}$, in all cases, and does not exceed that of the SPRT by more than 50%.

Finally, we see that the upper bounds are very accurate approximations of the expected sample sizes in all cases, even for large values of $\alpha$ and $\beta$. On the other hand, the lower bounds are similarly accurate for $\tilde{\chi}$, but relatively conservative for $\hat{\chi}$. To illustrate the selection of the free parameters of the two multistage tests, in the right column of Figure 2 we plot $\tilde{\gamma}$ in $\tilde{\chi}$ and $\hat{\gamma}, \hat{\gamma}'$ in $\hat{\chi}$, against $\beta$, all of them in the $|\log(\cdot)|$ scale.

8.4. The second study

In the second study we compare the expected sample sizes of the various tests when the true distribution is not necessarily $P_0$ or $P_1$. Specifically, we compute $E_\mu[\tau]$ for different values of $\mu$, in each of the three testing problems, when $\alpha = \beta = 10^{-4}$ and when $\alpha = 10^{-8}, \beta = 10^{-2}$. The results are presented in Figure 3. Consistently with our discussion in Subsection 4.4.1, we can see that when the true parameter is close to the middle of $\mu_0$ and $\mu_1$, the expected sample size of the SPRT is much larger than those of the multistage tests. On the other hand, the expected sample sizes of the multistage tests are not much larger than that of the SPRT when the true parameter is smaller than $\mu_0$ or larger than $\mu_1$.

9. Conclusion

Given a fixed-sample-size test that controls the error probabilities at two specific distributions, in this paper we design and analyze a 3-stage and two 4-stage tests, with deterministic stage sizes, which guarantee the same error control. Under some additional assumptions, which hold for many testing problems beyond the iid setup, we also conduct an asymptotic analysis for these tests. Specifically, we obtain asymptotic approximations for their expected sample sizes under the two distributions with respect to which we control the error probabilities, as the latter go to 0. In particular, when the test statistic is the average log-likelihood ratio between these two distributions, their expected sample sizes under these two distributions are asymptotically the optimal among all sequential tests with the same error control. Moreover, we obtain a universal asymptotic upper bound, which reveals robustness in comparison to the corresponding SPRT.

The above asymptotic optimality properties require certain constraints on how asymmetrically the two error probabilities go to 0. These constraints are
We plot the expected sample sizes of the fixed-sample-size test, the two multistage tests and the SPRT against the true value of the parameter. The values of the parameter at which we control the two types of error probabilities are highlighted on the x-axis. Each row corresponds to one of the testing problems considered in Section 7. The first column corresponds to $\alpha = \beta = 10^{-4}$, and the second $\alpha = 10^{-8}, \beta = 10^{-2}$.

(a) IID Gaussian, $\alpha = \beta = 10^{-4}$

(b) $\alpha = 10^{-8}, \beta = 10^{-2}$

(c) AR(1), $\alpha = \beta = 10^{-4}$

(d) $\alpha = 10^{-8}, \beta = 10^{-2}$

(e) Two-state Markov, $\alpha = \beta = 10^{-4}$

(f) $\alpha = 10^{-8}, \beta = 10^{-2}$
removed in [34], in an iid setup, using multistage tests in which the number of stages is fixed, but increases, without a bound, with the asymmetry between the two error probabilities. An interesting direction is the extension of these results beyond the iid setup, using similar ideas as in the present paper.

In order to have multistage tests that achieve asymptotic optimality under every distribution of the null and the alternative hypotheses, at least some stage sizes need to be random, as in [23, Section 3], [17, 4]. In these works, such a uniform asymptotic optimality property was established in the case of iid data that belong to an exponential family and under the assumption of symmetric error probabilities. Ideas from the present work can be useful for extending these results to more general distributional setups and more asymmetric error probabilities.

Finally, another direction of interest is the application of multistage tests, as the ones we consider in this work, in a multiple testing setup, similarly to [24].

Appendix A

In this Appendix we prove the results in Subsection 4.3. To this end, we start with a preliminary lemma, which holds under only some of the assumptions of Section 4.

Lemma A.1. (i) If, for every \( n \in \mathbb{N} \), \( P_1 \) and \( P_0 \) are mutually absolutely continuous when restricted to \( F_n \), then
\[
n^*(\alpha, \beta) \to \infty \quad \text{as} \quad \alpha \land \beta \to 0.
\]

(ii) If also (4.8) holds, then
\[
J_0 \leq \lim \kappa^*(\alpha, \beta) \quad \text{and} \quad \lim \kappa^*(\alpha, \beta) \leq J_1 \quad \text{as} \quad \alpha \land \beta \to 0.
\]

Proof. (i) Since \( n^* \) is decreasing in both its arguments, it suffices to show \( n^*(\alpha, \beta) \) goes to infinity when only one of \( \alpha \) and \( \beta \) goes to 0, while the other one is fixed. Without loss of generality, we assume that \( \alpha \) is fixed and \( \beta \to 0 \). We argue by contradiction and suppose that \( n^*(\alpha, \beta) \not\to \infty \) as \( \beta \to 0 \). From this assumption and the fact that \( n^* \) is decreasing in both its arguments we conclude that there exists an \( m \in \mathbb{N} \) and a sequence \( (\beta_n) \) with \( \beta_n \to 0 \) such that \( n^*(\alpha, \beta_n) = m, \forall n \in \mathbb{N} \). Then, for every \( n \in \mathbb{N} \) we have \( \kappa^*(\alpha, \beta_n) \geq z_\alpha \), where
\[
z_\alpha \equiv \inf\{z \in \mathbb{R} : P_0(T_m > z) \leq \alpha\} > -\infty,
\]
and subsequently
\[
\beta_n \geq P_1(T_{n^*(\alpha, \beta_n)} \leq \kappa^*(\alpha, \beta_n)) = P_1(T_m \leq \kappa^*(\alpha, \beta_n)) \geq P_1(T_m \leq z_\alpha).
\]
Letting \( n \to \infty \) we obtain \( P_1(T_m \leq z_\alpha) = 0 \). By the definition of \( z_\alpha \) we also have \( P_0(T_m \leq z_\alpha) \geq 1 - \alpha > 0 \). This violates the assumption that \( P_0 \) is absolutely continuous to \( P_1 \) when restricted to \( F_m \), thus, we have reached a contradiction.
We only prove the first inequality, as the proof of the second is similar. Without loss of generality, we assume that $\beta \to 0$, while $\alpha$ is either fixed or goes to 0. We argue by contradiction and suppose that $\lim \kappa^*(\alpha, \beta) < J_0$. Then, there exists an $\epsilon > 0$ so that $\lim \kappa^*(\alpha, \beta) \leq J_0 - 2\epsilon$ and we can find a sequence $(\alpha_n, \beta_n)$, such that $\beta_n \to 0$, $(\alpha_n)$ is either constant or also goes to 0, and $\kappa^*(\alpha_n, \beta_n) \leq J_0 - \epsilon$ for every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N},$

$$\alpha_n \geq P_0(T_n^*(\alpha_n, \beta_n) > \kappa^*(\alpha_n, \beta_n)) \geq P_0(T_n^*(\alpha_n, \beta_n) > J_0 - \epsilon).$$

In view of (i) and assumption (4.8), the lower bound goes to 1 as $n \to \infty$, which contradicts the fact that the sequence $(\alpha_n)$ is bounded away from 1. \hfill \Box

**Proof of Theorem 4.1.** The upper bound in (4.14) implies that

$$n^*(\alpha, \beta) \lesssim \frac{|\log \alpha| \vee |\log \beta|}{\psi_1(\kappa) \wedge \psi_0(\kappa)} = \frac{|\log(\alpha \wedge \beta)|}{\psi_1(\kappa) \wedge \psi_0(\kappa)}$$

for every $\kappa \in (J_0, J_1)$, and optimizing with respect to $\kappa$ we obtain (4.15). Therefore, it suffices to show (4.14). To lighten the notation, we set $n^* \equiv n^*(\alpha, \beta)$ and $\kappa^* \equiv \kappa^*(\alpha, \beta)$. By the definitions of these quantities we have

$$P_0(T_n^* > \kappa^*) \leq \alpha \quad \text{and} \quad P_1(T_n^* \leq \kappa^*) \leq \beta = \alpha \frac{|\log \beta|}{|\log \alpha|},$$

and as a result

$$\max \left\{ P_0(T_n^* > \kappa^*), \ P_1(T_n^* \leq \kappa^*) \right\} \lesssim \frac{|\log \alpha|}{|\log \beta|} \leq \alpha.$$ 

Since for any $n \in \mathbb{N}$ and $\kappa_1, \kappa_2 \in \mathbb{R}$ we have

either $P_0(T_n > \kappa_1) \geq P_0(T_n > \kappa_2)$ or $P_1(T_n \leq \kappa_1) \geq P_1(T_n \leq \kappa_2),$

for any $\kappa \in (J_0, J_1)$ we obtain

$$\min \left\{ P_0(T_n^* > \kappa), \ P_1(T_n^* \leq \kappa) \right\} \leq \alpha,$$

and consequently

$$\min \left\{ \frac{1}{n^2} \log P_0(T_n^* > \kappa), \ \frac{1}{n^2} \log P_1(T_n^* \leq \kappa) \right\} \leq \alpha \frac{|\log \alpha|}{|\log \beta|},$$

which proves the asymptotic upper bound in (4.14). On the other hand, the definition of $n^*$ and $\kappa^*$ implies that, for any $\alpha, \beta \in (0, 1),$

either $\alpha < P_0(T_{n^*-1} > \kappa^*)$ or $\beta < P_1(T_{n^*-1} \leq \kappa^*),$

and consequently

$$\frac{1}{n^2} \log P_0(T_n^* > \kappa), \ \frac{1}{n^2} \log P_1(T_n^* \leq \kappa) \cdot \frac{|\log \alpha|}{|\log \beta|} \geq 1.$$
and consequently
\[ \alpha < \max \left\{ P_0(T_{n^*} - 1 > \kappa^*), \ P_1(T_{n^*} - 1 \leq \kappa^*) \right\}. \]

Working as before we conclude that, for any \( \kappa \in (J_0, J_1) \),
\[ \max \left\{ \frac{1}{n^* - 1} \log P_0(T_{n^*} - 1 > \kappa), \ \frac{1}{n^* - 1} \log P_1(T_{n^*} - 1 \leq \kappa) \cdot \frac{\log \alpha}{\log \beta} \right\} < 1 \]
for every \( \alpha, \beta \in (0, 1) \), and letting \( \alpha \land \beta \to 0 \) we obtain
\[ \lim_{n^*} \frac{n^*}{\log \alpha} \max \left\{ -\psi_0(\kappa), \ -\psi_1(\kappa) \cdot \frac{\log \alpha}{\log \beta} \right\} \leq 1. \]
Thus, we have established the asymptotic lower bound in (4.14), and the proof is complete.

**Proof of Theorem 4.2.** (i) When both \( \alpha \) and \( \beta \) go to 0, this follows from the universal asymptotic lower bound in (4.4). Therefore, it suffices to consider the case that only one of them goes to 0, while the other one is fixed. Without loss of generality, we assume that \( \beta \to 0 \), while \( \alpha \) is fixed, in which case it suffices to show that, for every \( \epsilon > 0 \),
\[ \lim_{n^*} \frac{\log \beta}{n^* (\alpha, \beta)} \geq -I_0 - \epsilon. \]
To this end, we fix \( \epsilon > 0 \) and observe that, by Lemma A.1.(ii), for \( \beta \) small enough we have \( \kappa^*(\alpha, \beta) > -I_0 - \epsilon \) and consequently
\[ \beta \geq P_1(\tilde{\Lambda}_{n^* (\alpha, \beta)} \leq \kappa^*(\alpha, \beta)) \]
\[ \geq P_1(-I_0 - \epsilon < \tilde{\Lambda}_{n^* (\alpha, \beta)} \leq \kappa^*(\alpha, \beta)) \]
\[ = \mathbb{E}_0 \left[ \exp\{\Lambda_{n^* (\alpha, \beta)}\}; \ -I_0 - \epsilon < \Lambda_{n^* (\alpha, \beta)} \leq \kappa^*(\alpha, \beta) \right] \]
\[ \geq \exp\{-n^*(\alpha, \beta)(I_0 + \epsilon)\} \ P_0(-I_0 - \epsilon < \Lambda_{n^* (\alpha, \beta)} \leq \kappa^*(\alpha, \beta)). \] (A.1)
Moreover, for any \( \alpha, \beta \in (0, 1) \) we have
\[ P_0(-I_0 - \epsilon < \Lambda_{n^* (\alpha, \beta)} \leq \kappa^*(\alpha, \beta)) \]
\[ = 1 - P_0(\Lambda_{n^* (\alpha, \beta)} \leq -I_0 - \epsilon) - P_0(\Lambda_{n^* (\alpha, \beta)} > \kappa^*(\alpha, \beta)) \]
\[ \geq 1 - P_0(\Lambda_{n^* (\alpha, \beta)} \leq -I_0 - \epsilon) - \alpha, \] (A.2)
and the probability in the lower bound of (A.2) goes to zero as \( \beta \to 0 \), because of Lemma A.1.(i) and assumption (4.2). Therefore, taking logarithms on both sides of (A.1), dividing by \( n^*(\alpha, \beta) \) and letting \( \beta \to 0 \) completes the proof.

(ii) We only prove that, as \( \alpha \land \beta \to 0 \),
\[ n^*(\alpha, \beta) \gtrsim \frac{\log \beta}{\psi_1(J_0)}. \]
as the proof that \( n^*(\alpha, \beta) \gtrsim |\log \alpha|/\psi_0(J_1) \) is similar. By assumption (4.13), there is an \( \epsilon > 0 \) so that \( \psi_1 \) is finite and (4.10) holds in \( (J_0 - 2\epsilon, J_1) \). From Lemma A.1.(ii) it follows that, when at least one of \( \alpha \) and \( \beta \) is small enough, \( \kappa^*(\alpha, \beta) > J_0 - \epsilon \) and consequently

\[ \beta \geq P_1(T_{n^*(\alpha, \beta)} \leq \kappa^*(\alpha, \beta)) \geq P_1(T_{n^*(\alpha, \beta)} \leq J_0 - \epsilon). \]

Thus, taking logarithms, dividing by \( n^*(\alpha, \beta) \) and letting \( \alpha \wedge \beta \to 0 \) we obtain

\[ \lim \frac{\log \beta}{n^*(\alpha, \beta)} \geq \lim \frac{1}{n^*(\alpha, \beta)} \log P_1(T_{n^*(\alpha, \beta)} \leq J_0 - \epsilon) = -\psi_1(J_0 - \epsilon), \]

where the equality follows from Lemma A.1.(i) and assumption (4.10). Since \( \psi_1 \) is convex, it is continuous on the interior of its effective domain. By assumption, \( \psi_1 \) is finite in a neighborhood of \( J_0 \), thus, letting \( \epsilon \downarrow 0 \) completes the proof.

\[ \square \]

**Proof of Corollary 4.2.1.** The first asymptotic approximation in (4.19) follows by setting \( \kappa = g^{-1}(r) \) in (4.14), whereas the second by (4.12), which implies

\[ \psi_0(g^{-1}(r)) = r \psi_1(g^{-1}(r)) \quad \forall \ r \in (0, \infty). \]

To prove (4.20) it suffices to show that

\[ C = \psi_0(g^{-1}(1)) = \psi_1(g^{-1}(1)). \]

Indeed, the strict monotonicity of \( \psi_0 \) and \( \psi_1 \) in \( (J_0, J_1) \) implies that the supremum in (4.15) is attained when \( \psi_0 = \psi_1 \), or equivalently when \( g = 1 \).

**Proofs of Corollaries 4.2.2 and 4.2.3.** In view of the asymptotic lower bounds in Theorem 4.2, it satisfies to establish only the corresponding upper bounds. We only prove part (i) of each Corollary, as the proof of (ii) is similar.

We show first that, for any test statistic \( T \), even if (4.13) does not hold,

\[ n^*(\alpha, \beta) \lesssim |\log \beta|/\psi_1(J_0) \quad \text{or equivalently} \quad \psi_1(J_0) \lesssim |\log \beta|/n^*(\alpha, \beta), \]

as \( \alpha \wedge \beta \to 0 \) so that \( |\log \alpha| << |\log \beta| \).

By assumption, \( \psi_1 \) is convex and lower-semicontinuous, thus, it is continuous in its effective domain, and as a result in \( [J_0, J_1] \). Therefore, to prove the above claim it suffices to show that, as \( \alpha \wedge \beta \to 0 \) so that \( |\log \alpha| << |\log \beta| \),

\[ \psi_1(\kappa) \lesssim |\log \beta|/n^*(\alpha, \beta) \quad \forall \ \kappa \in (J_0, J_1), \]

which follows directly by Theorem 4.1. When \( T \neq \hat{\Lambda} \), the proof is complete. When \( T = \hat{\Lambda} \), it remains to show that \( \psi_1(-J_0) \geq I_0 \). Since \( \psi_1 \) is continuous in \( [-I_0, I_1] \), it suffices to show that \( \psi_1(\kappa) \geq -\kappa \) for every \( \kappa \in (-I_0, 0) \). Indeed, for any \( \kappa < 0 \), by Markov’s inequality we have

\[ P_1(\hat{\Lambda}_n \leq \kappa) \leq e^{n\kappa} E_0[\exp\{A_n\}] = e^{n\kappa} \quad \text{for all} \ \ n \in \mathbb{N}. \]

Taking logarithms, dividing by \( n \), letting \( n \to \infty \), and applying (4.10) for \( \kappa \) in \( (-I_0, 0) \) completes the proof. \[ \square \]
Appendix B

In this Section we prove Lemma 4.1 and Theorem 4.4. The proof of Theorem 4.3 is omitted, as it is almost identical to that of Theorem 4.4.

Proof of Lemma 4.1. We only prove the asymptotic lower bounds under $P_0$, as the proofs of the corresponding lower bounds under $P_1$ are similar. We first prove the result for $\tilde{\chi}$, in which case it suffices to show that, for all $\epsilon \in (0,1)$,

$$\inf_{\gamma \in (\alpha/2, 1)} E_0[\tilde{\tau}] \gtrsim (1 - \epsilon) \frac{|\log \beta|}{\psi_1(J_0)}$$

as $\alpha, \beta \to 0$. (B.1)

Fix $\epsilon \in (0,1)$. By the non-asymptotic lower bound in (3.10) it follows that, for any $\alpha, \beta \in (0,1)$ and $\gamma \in (\alpha/2, 1)$,

$$E_0[\tilde{\tau}] \geq \max \{ n^*(\gamma, \beta/2) \cdot (1 - \alpha/2), n^*(\alpha/2, \beta/2) \cdot (\gamma - \alpha/2) \}.$$ 

When, in particular, $\gamma \leq 1 - \epsilon$,

$$E_0[\tilde{\tau}] \geq n^*(\gamma, \beta/2) \cdot (1 - \alpha/2) \geq n^*(1 - \epsilon, \beta/2) \cdot (1 - \alpha/2).$$

and when $\gamma > 1 - \epsilon$,

$$E_0[\tilde{\tau}] \geq n^*(\alpha/2, \beta/2) \cdot (\gamma - \alpha/2) \geq n^*(\alpha/2, \beta/2) \cdot (1 - \epsilon - \alpha/2).$$

By Theorem 4.2.(ii) it then follows that, as $\alpha, \beta \to 0$,

$$\inf_{\gamma \in (\alpha/2, 1 - \epsilon)} E_0[\tilde{\tau}] \gtrsim n^*(1 - \epsilon, \beta/2) \gtrsim \frac{|\log \beta|}{\psi_1(J_0)},$$

$$\inf_{\gamma \in (1 - \epsilon, 1)} E_0[\tilde{\tau}] \gtrsim (1 - \epsilon) \cdot n^*(\alpha/2, \beta/2) \gtrsim (1 - \epsilon) \frac{|\log \beta|}{\psi_1(J_0)},$$

and this implies (B.1). The proof for $\hat{\chi}$ is similar and omitted. To prove the result for $\hat{\chi}$, it suffices to show that, for every $\epsilon \in (0,1)$,

$$\inf_{\alpha/2 < \gamma' < \gamma < 1} E_0[\hat{\tau}] \gtrsim (1 - 2\epsilon) \frac{|\log \beta|}{\psi_1(J_0)}$$

as $\alpha, \beta \to 0$. (B.2)

Fix $\epsilon \in (0,1)$. By the non-asymptotic lower bound in (3.22) it follows that, for any $\alpha, \beta \in (0,1)$ and $\alpha/2 < \gamma' < \gamma < 1$,

$$E_0[\hat{\tau}] \geq \max \{ n^*(\gamma, \beta/3) \cdot (1 - \alpha/2), n^*(\gamma', \beta/3) \cdot (\gamma - \alpha/2),$$

$$n^*(\alpha/2, \beta/3) \cdot ((1 - \alpha/2) - (1 - \gamma) - (1 - \gamma')) \}.$$ 

When, in particular, $\gamma < 1 - \epsilon$,

$$E_0[\hat{\tau}] \geq n^*(\gamma, \beta/3) \cdot (1 - \alpha/2)$$

$$\geq n^*(1 - \epsilon, \beta/3) \cdot (1 - \alpha/2),$$
when $\gamma' < 1 - \epsilon < \gamma$,
\[
E_0[\hat{\tau}] \geq n^*(\gamma', \beta/3) \cdot (\gamma - \alpha/2) \\
\geq n^*(1 - \epsilon, \beta/3) \cdot (1 - \epsilon - \alpha/2),
\]
and when $\gamma' > 1 - \epsilon$,
\[
E_0[\hat{\tau}] \geq n^*(\alpha/2, \beta/3) \cdot ((1 - \alpha/2) - (1 - \gamma) - (1 - \gamma')) \\
\geq n^*(\alpha/2, \beta/3) \cdot (1 - 2\epsilon - \alpha/2).
\]

By Theorem 4.2.(ii) it then follows that, as $\alpha, \beta \to 0$,
\[
\inf_{\gamma' \in (\alpha/2, 1 - \epsilon)} E_0[\hat{\tau}] \gtrsim n^*(1 - \epsilon, \beta/3) \gtrsim \frac{|\log \beta|}{\psi_1(J_0)}, \\
\inf_{\gamma' \in (1 - \epsilon, 1)} E_0[\hat{\tau}] \gtrsim n^*(1 - \epsilon, \beta/3) \cdot (1 - \epsilon) \gtrsim (1 - \epsilon) \frac{|\log \beta|}{\psi_1(J_0)}, \\
\inf_{\gamma' \in (1 - \epsilon, 1)} E_0[\hat{\tau}] \gtrsim n^*(\alpha/2, \beta/3) \cdot (1 - 2\epsilon) \gtrsim (1 - 2\epsilon) \frac{|\log \beta|}{\psi_1(J_0)},
\]
and this implies (B.2).

\begin{proof}[Proof of Theorem 4.4]
In view of Lemma 4.1, it remains to prove in each case the corresponding asymptotic upper bounds.

(i) Let $\delta$ be a function of $\alpha, \beta$ such that $\delta \in (\beta/2, 1)$ for every $\beta \in (0, 1)$, and
\[
\delta \to 0 \quad \text{and} \quad |\log \delta| < < |\log \alpha| \quad \text{as} \quad \alpha \to 0,
\]
e.g., $\delta = |\log \alpha|^{-\epsilon} \vee \beta$ for some $\epsilon \in (0, 1)$. By the non-asymptotic upper bound in (3.11) and the selection of the free parameters according to (3.12) it follows that, for any $\alpha, \beta \in (0, 1)$,
\[
E_1[\hat{\tau}] \leq n^*(\alpha/2, \delta) + (n^*(\alpha/2, \beta/2) - n^*(\alpha/2, \delta)) \cdot \delta \\
\leq n^*(\alpha/2, \delta) + n^*(\alpha/2, \beta/2) \cdot \delta.
\]

Then, by Corollary 4.2.2.(ii), Theorem 4.1 and (B.3) we conclude that
\[
E_1[\hat{\tau}] \lesssim \frac{|\log \alpha|}{\psi_0(J_1)} \cdot \frac{|\log(\alpha \land \beta)|}{C} \cdot \delta \sim \frac{|\log \alpha|}{\psi_0(J_1)} \cdot \delta
\]
as $\alpha, \beta \to 0$ so that $|\log \alpha| \gtrsim |\log \beta|$, and this completes the proof for $\hat{\chi}$. The proof for $\hat{\chi}$ is similar and omitted. To prove the result for $\hat{\chi}$, we observe that by the non-asymptotic upper bound in (3.25) and the selection of the free parameters according to (3.27) it follows that
\[
E_1[\hat{\tau}] \leq n^*(\alpha/3, \delta) + (n^*(\alpha/3, \delta') - n^*(\alpha/3, \delta)) \cdot \delta \\
+ (n^*(\alpha/3, \beta/2) - n^*(\alpha/3, \delta') \cdot \delta' \\
\leq n^*(\alpha/3, \delta) + n^*(\alpha/3, \beta/2) \cdot \delta
\]
for any $\alpha, \beta \in (0, 1)$ and $\delta', \delta$ such that $\beta'/2 < \delta' < \delta < 1$. The proof then continues in exactly the same way as for $\tilde{\chi}$, i.e., by selecting $\delta$ to satisfy (B.3).

(ii) Let $\gamma, \gamma'$ be functions of $\alpha$ and $\beta$ such that $\alpha < \gamma' < \gamma < 1$ and

$$
\log \gamma' \leq \log \gamma < \alpha < \beta < 0 \quad \text{so that} \quad |\log \gamma| < |\log \beta|, \quad |\log (\gamma' \wedge \gamma)| < |\log \beta|, \quad |\log \alpha| < |\log \beta|,
$$

as $\alpha, \beta \to 0$ so that $|\log \beta| \lesssim |\log \alpha| \lesssim |\log \beta|/\beta^r$ for some $r \geq 1$. (B.4)

(e.g., $\gamma = |\log \beta|^{-\epsilon} \wedge \alpha$ and $\gamma' = (\beta^r \epsilon' \wedge \gamma) \wedge \alpha$ for some $\epsilon \in (0, 1)$ and $\epsilon' > 0$. By the non-asymptotic upper bound in (3.22) and the selection of the free parameters according to (3.26) it follows that, for any $\alpha, \beta \in (0, 1),

$$
E_0[\tilde{\tau}] \leq n^*(\gamma, \beta/3) + (n^*(\gamma', \beta/3) - n^*(\gamma, \beta/3)) \cdot \gamma \\
+ (n^*(\alpha/2, \beta/3) - n^*(\gamma', \beta/3)) \cdot \gamma' \\
\leq n^*(\gamma, \beta/3) + n^*(\gamma', \beta/3) \cdot \gamma + n^*(\alpha/2, \beta/3) \cdot \gamma'.
$$

Then, by Corollary 4.2.2.(i), Theorem 4.1 and (B.4) we conclude that

$$
E_0[\tilde{\tau}] \lesssim \frac{|\log \beta|}{\psi_1(J_0)} + \frac{|\log (\gamma' \wedge \gamma)|}{C} + \frac{|\log (\gamma' \wedge \alpha)|}{C} \cdot \gamma' \sim \frac{|\log \beta|}{\psi_1(J_0)}
$$

as $\alpha, \beta \to 0$ so that $|\log \beta| \lesssim |\log \alpha| \lesssim |\log \beta|/\beta^r$ for some $r \geq 1$.

(iii) To prove the result for $E_0[\tilde{\tau}]$, we let $\gamma$ be a function of $\alpha$ and $\beta$ such that $\gamma \in (\alpha, 1)$ and

$$
|\log \gamma| < |\log \beta| \quad \text{and} \quad |\log \beta|^{-r} \gamma \to 0
$$

as $\alpha, \beta \to 0$ so that $|\log \beta| \lesssim |\log \alpha| \lesssim |\log \beta|/\beta^r$ for some $r \geq 1$. (B.5)

e.g., $\gamma = |\log \beta|^{-\epsilon} \wedge \alpha$ for some $\epsilon \in (0, 1)$. By the non-asymptotic upper bound in (3.10) and the selection of the free parameters according to (3.12) it follows that, for any $\alpha, \beta \in (0, 1),

$$
E_0[\tilde{\tau}] \leq n^*(\gamma, \beta/2) + (n^*(\alpha/2, \beta/2) - n^*(\gamma, \beta/2)) \cdot \gamma \\
\leq n^*(\gamma, \beta/2) + n^*(\alpha/2, \beta/2) \cdot \gamma.
$$

Then, by Corollary 4.2.2.(i), Theorem 4.1 and (B.5) we conclude that

$$
E_0[\tilde{\tau}] \lesssim \frac{|\log \beta|}{\psi_1(J_0)} + \frac{|\log (\gamma' \wedge \alpha)|}{C} \gamma \sim \frac{|\log \beta|}{\psi_1(J_0)}
$$

as $\alpha, \beta \to 0$ so that $|\log \beta| \lesssim |\log \alpha| \lesssim |\log \beta|/\beta^r$ for some $r \geq 1$.

\[\square\]

**Appendix C**

In this Appendix, we prove Theorem 5.1, a version of the Gärtner-Ellis Theorem. The proof is essentially the same as in [12, Theorem 2.3.6] or [10, Theorem 3.2.1], and is presented only for completeness. Specifically, we establish first the asymptotic upper bounds in (i) and (ii). Using these, we establish (iii). Finally, using (iii), we establish the asymptotic lower bounds in (i) and (ii).
Proof of Theorem 5.1. We establish the asymptotic upper bound only for (i), as the corresponding proof for (ii) is similar. Thus, we assume that $\Theta^o \cap (0, \infty) \neq \emptyset$. For any $\kappa_1, \kappa_2 \in \phi'(\Theta^o \cap (0, \infty))$ such that $\kappa_1 < \kappa_2$ and $\vartheta(\kappa_1), \vartheta(\kappa_2) > 0$,

$$\phi^*(\kappa_1) = \vartheta(\kappa_1)\kappa_1 - \phi(\vartheta(\kappa_1)) < \vartheta(\kappa_1)\kappa_2 - \phi(\vartheta(\kappa_1)) \leq \phi^*(\kappa_2),$$

which proves that $\phi^*$ is strictly increasing in $\phi'(\Theta^o \cap (0, \infty))$. From [12, Lemma 2.2.5]) it follows that $\phi^*$ is non-negative and lower-semicontinuous, and these properties imply that

$$0 \leq \phi^*(\phi'(0+)) \leq \lim_{\theta \downarrow 0} \phi^*(\phi'(\theta)) = \lim_{\theta \downarrow 0} \{\theta \phi'(\theta) - \phi(\theta)\} = 0.$$

Since $\phi'(\Theta^o \cap (0, \infty))$ is an open interval, whose right endpoint may be infinity, to show that (5.3) holds for every $\kappa \in \phi'(\Theta^o \cap (0, \infty))$ it suffices to show that it holds for every $\kappa \in \phi'((0, \theta_*))$, where $\theta_* \in \Theta^o \cap (0, \infty)$. Thus, we fix $\theta_* \in \Theta^o \cap (0, \infty)$ and denote $\phi'((0, \theta_*)) \equiv (a, b)$, where $a \equiv \phi'(0+)$ and $b \equiv \phi'(\theta_*)$.

For any $\kappa \in (a, b)$ and $\theta \in (0, \theta_*)$, we have

$$P(T_n > \kappa) \leq \exp\{ -n \theta \kappa \} \mathbb{E}[\exp\{n \theta T_n\}] = \exp\{ -n(\theta \kappa - \phi_n(\theta))\},$$

which, after taking logarithm, dividing by $n$ and letting $n \to \infty$, gives

$$\lim_{n \to \infty} \frac{1}{n} \log P(T_n > \kappa) \leq - (\theta \kappa - \phi(\theta)).$$

Optimizing the right-hand-side with respect to $\theta \in (0, \theta_*)$, we obtain $-\phi^*(\kappa)$.

Note that this asymptotic upper bound is non-trivial for every $\kappa \in (a, \infty)$, since $\phi^*(a) = 0$ and $\phi^*$ is strictly increasing in $(a, b)$. Therefore, it implies that $P(T_n - \phi'(0+) \geq \epsilon)$ is an exponentially decaying sequence for every $\epsilon > 0$. Similarly it follows that if $\Theta^o \cap (-\infty, 0) \neq \emptyset$, then $P(T_n - \phi'(0-) \leq -\epsilon)$ is an exponentially decaying sequence for every $\epsilon > 0$. From these observations we conclude that if $0 \in \Theta^o$, then $P(|T_n - \phi'(0)| \geq \epsilon)$ is exponentially decaying for every $\epsilon > 0$, and as a result $P(T_n \to \phi'(0)) = 1$. Therefore, (iii) follows using exactly the same argument as long as the sequence of functions

$$\lambda \in \mathbb{R} \to \frac{1}{n} \log \mathbb{E}_{Q_\theta} [\exp\{n \lambda T_n\}], \quad n \in \mathbb{N}$$

satisfies the assumptions of the theorem, 0 belongs to the interior of the effective domain of its limit, and the derivative of its limit at 0 is $\phi'(\theta)$. To show this, we fix $\theta \in (0, \theta_*)$. Then, for any $\lambda \in \mathbb{R}$,

$$\mathbb{E}_{Q_\theta} [\exp\{n \lambda T_n\}] = \mathbb{E} [\exp\{n((\lambda + \theta)T_n - \phi_n(\theta))\}] = \exp\{n(\phi_n(\lambda + \theta) - \phi_n(\theta))\},$$

and consequently

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{Q_\theta} [\exp\{n \lambda T_n\}] = \phi(\lambda + \theta) - \phi(\theta).$$
The limit is finite for \( \lambda \in (-\theta, \theta_s - \theta) \), which contains 0 in its interior, inherits all the smoothness properties of \( \phi \), and its derivative at \( \lambda = 0 \) is \( \phi'(-\theta) \). This completes the proof of (iii).

It remains to prove the asymptotic lower bounds in (i) and (ii). Again, we only do so for (i), as the proof for (ii) is similar. Fix \( \kappa \in (a, b) \). For any \( n \in \mathbb{N} \), \( \theta \in (0, \theta_s) \) and \( \epsilon \in (0, b - \kappa) \),

\[
P(T_n > \kappa) = E_Q_{\theta} [\exp\{-n(\theta T_n - \phi_n(\theta))\}; T_n > \kappa] \\
\geq E_Q_{\theta} [\exp\{-n(\theta T_n - \phi_n(\theta))\}; \kappa < T_n \leq \kappa + \epsilon] \\
\geq \exp\{-n(\theta(\kappa + \epsilon) - \phi_n(\theta))\} Q_{\theta}(\kappa < T_n \leq \kappa + \epsilon).
\]

If we now set \( \theta = \vartheta(\kappa + \epsilon/2) \), take logarithms, divide by \( n \) and let \( n \to \infty \), by (iii) we obtain

\[
\lim_{n} \frac{1}{n} \log P(T_n > \kappa) \geq -\vartheta(\kappa + \epsilon/2)(\kappa + \epsilon) + \phi(\vartheta(\kappa + \epsilon/2)).
\]

To complete the proof, we let \( \epsilon \downarrow 0 \) and observe that the right-hand-side converges to \(- (\vartheta(\kappa) - \phi(\vartheta(\kappa))) = -\phi^*(\kappa) \), since \( \vartheta \) and \( \phi \) are both continuous in the corresponding neighborhoods.

Acknowledgments

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