ON ABELIAN COVERS OF THE PROJECTIVE LINE WITH FIXED GONALITY AND MANY RATIONAL POINTS

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Abstract. A smooth geometrically connected curve over the finite field $\mathbb{F}_q$ with gonality $\gamma$ has at most $\gamma(q+1)$ rational points. The first author and Grantham conjectured that there exist curves of every sufficiently large genus with gonality $\gamma$ that achieve this bound. In this paper, we show that this bound can be achieved for an infinite sequence of genera using abelian covers of the projective line. We also argue that abelian covers will not suffice to prove the full conjecture.

1. Introduction

Throughout, we use the unqualified term “curve” to mean a smooth proper geometrically connected scheme of dimension 1 over a field. Given a curve $C$ over a finite field $\mathbb{F}_q$, the existence of a morphism $f : C \to \mathbb{P}^1$ provides a bound on the number of points of $C$. Indeed, every rational point of $C$ maps to one of the rational points of $\mathbb{P}^1$, and each fiber contains at most $\deg(f)$ points:

$$\#C(\mathbb{F}_q) \leq \deg(f)(q+1).$$

The minimum degree of a morphism to $\mathbb{P}^1$ is known as the gonality of $C$.

Write $N_q(g, \gamma)$ for the maximum number of points on a curve $C/\mathbb{F}_q$ with genus $g$ and gonality $\gamma$; we set this quantity to $-\infty$ if no such curve exists. From (1.1), we know that $N_q(g, \gamma) \leq \gamma(q+1)$. In [2, 1], the first author and Grantham gave many examples of curves where this bound is achieved, which led to the following conjecture:

Conjecture 1.1. Fix a prime power $q$ and an integer $\gamma \geq 2$. For $g$ sufficiently large, we have

$$N_q(g, \gamma) = \gamma(q+1).$$

The implicit challenge in the conjecture is to construct interesting families of curves. The first author and Grantham wrote down explicit hyperelliptic equations to handle the case $\gamma = 2$ [1]. The second author established the conjecture for $q$ odd by constructing singular curves on toric surfaces [10]. The proof uses a squarefree-discriminant trick to insure that the singular curves are smooth above non-rational points of $\mathbb{P}^1$; in characteristic 2, discriminants are never squarefree, so one must find a different approach to control singularities. The goal of this note is to investigate how much can be proved using only abelian covers of the projective line, especially in the remaining open case where $q$ is even. Recall that an abelian cover of curves $X \to Y$ is a nonconstant morphism for which the corresponding extension of function fields $\kappa(X)/\kappa(Y)$ is Galois with abelian group of automorphisms. See [7, §5.12] and [9, §6.3] for summaries of geometric class field theory; see [5, Ch. V] and [6] for more robust treatments.

Theorem 1.2. Fix a prime power $q$ and an integer $\gamma \geq 2$. There exists an infinite sequence of distinct positive integers $(g_i)$ and abelian covers $C_i \to \mathbb{P}^1$ of degree $\gamma$ such that $C_i$ has genus $g_i$ and $\gamma(q+1)$ rational points. In particular,

$$\limsup_{g \to \infty} N_q(g, \gamma) = \gamma(q+1).$$
Unfortunately, abelian covers are not sufficient to replace the limsup with a proper limit. For example, consider the case of covers of $\mathbb{P}^1$ of prime degree $\gamma \equiv 1 \pmod{4}$, where $\gamma$ does not divide $q$. If $C \to \mathbb{P}^1$ is abelian of degree $\gamma$, then all of the ramification indices are equal to 1 or $\gamma$. Let $m$ be the number of ramified points of $C$. The Hurwitz formula shows the genus of $C$ is $\frac{\gamma - 1}{2}(m - 2)$. This expression is always even, so abelian covers of degree $\gamma$ will never have anything to say about curves of odd genus.

Roughly speaking, there are three main approaches in the literature to constructing curves of large genus with many rational points:

- Modular curves, especially in towers;
- Class field theory constructions; and
- Ad hoc methods for special families.

Modular methods typically produce curves whose genus and gonality grow together, which is not helpful for Conjecture 1.1. As indicated in the preceding paragraph, methods of class field theory do not produce enough curves to address the full scope of the conjecture. This leaves ad hoc constructions. One particularly powerful construction is that of curves on toric surfaces; this was first introduced for the study of rational points in [4] and later used to prove Conjecture 1.1 for odd $q$ in [10]. See [7, p.vii] and [9, p.1] for more philosophical exposition on constructing curves over finite fields.

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2. Proof of Theorem 1.2

In our discussion below, we will identify $\mathbb{P}^1$ with $\text{Spec} \mathbb{F}_q[t] \cup \{\infty\}$. Recall that the branch locus of a finite morphism $f : X \to Y$ is the minimal closed subset $B \subset Y$ such that the induced morphism $X \setminus f^{-1}(B) \to Y \setminus B$ is étale. Equivalently, the branch locus is the support of the discriminant of the induced extension of function fields $\kappa(X)/\kappa(Y)$.

The construction of our sequence of curves involves three steps:

- Choose a morphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ that maps every point of $\mathbb{P}^1(\mathbb{F}_q)$ to $\infty$.
- Construct a sequence of morphisms $f_i : X_i \to \mathbb{P}^1$ of degree $\gamma$ such that $\infty$ splits completely in each $X_i$ and the genera of the $X_i$ tend to infinity.
- Take the fiber product of $f_i$ and $h$ to obtain a new covering of curves $C_i := X_i \times_{\mathbb{P}^1} \mathbb{P}^1 \to \mathbb{P}^1$ of degree $\gamma$ with the correct number of rational points.

We must take care in choosing the branch loci of the morphisms $f_i$ in order to have control over the genus of the curve $C_i$.

The construction of $h$ is explicit. Let $a \in \mathbb{F}_q[t]$ be an irreducible polynomial of degree $q + 1$ and define

$$h(t) = \frac{a(t)}{t^q - t}.$$ 

Then $h$ has degree $q + 1$, and each point of $\mathbb{P}^1(\mathbb{F}_q)$ maps to $\infty$.

Lemma 2.1. Fix $\gamma \geq 1$ and a proper closed subset $S \subset \mathbb{P}^1$. There exists an abelian cover $f : X \to \mathbb{P}^1$ of degree $\gamma$ such that the point $\infty$ splits completely in $X$, and such that the branch locus of $f$ is supported on a closed point outside $S$. 

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Proof. Define a modulus \( m = e[P] \), where \( P \) is a yet-to-be-determined closed point of \( \mathbb{P}^1 \) and \( e \geq 1 \) is an integer. By [5, Thm. V.1.7], the order of the ray class group \( \text{Cl}_m^0(\mathbb{P}^1) \) is given by
\[
\# \text{Cl}_m^0(\mathbb{P}^1) = \frac{d^{\text{deg}(P)} - 1}{q - 1} q^{\text{deg}(P)(e-1)}.
\]
This quantity is divisible by \( \gamma \) for infinitely many closed points \( P \) and choices of \( e \geq 1 \). In particular, we may choose \( P \) and \( e \) so that \( \gamma \) divides the order of \( \# \text{Cl}_m^0(\mathbb{P}^1) \) and so that \( P \notin S \cup \{ \infty \} \).

Let \( G \) be a quotient of \( \text{Cl}_m^0(\mathbb{P}^1) \) of order \( \gamma \). Then the map \( D \mapsto D - \text{deg}(D)[\infty] \) followed by projection to \( G \) gives a surjective homomorphism \( \alpha : \text{Cl}_m(\mathbb{P}^1) \to G \). We now invoke the existence theorem of geometric class field theory [7, p.124] to obtain a curve \( X_{/\mathbb{F}_q} \) and a Galois covering \( X \to \mathbb{P}^1 \) with group \( G \) whose branch locus is supported on \( P \), and whose Frobenius element for \( Q \neq P \) is given by \( \alpha([Q]) \). By construction, we have \( \alpha([\infty]) = 1 \), so that \( \infty \) splits completely in \( X \). \( \square \)

For a curve \( C \), write \( g(C) \) for its genus.

**Lemma 2.2.** Let \( f : Y \to X \) and \( h : Z \to X \) be morphisms of curves over a field \( k \) whose branch loci in \( X \) are disjoint. Then the fiber product \( Y \times_X Z \) is a smooth \( k \)-variety of dimension 1. If it is geometrically irreducible, then it has genus
\[
d_f g(Y) + d_f g(Z) - d_f d_h g(X) + (d_f - 1)(d_h - 1),
\]
where \( d_f = \text{deg}(f) \) and \( d_h = \text{deg}(h) \).

**Proof.** Write \( W = Y \times_X Z \) for ease of notation. The following diagram describes our situation:

\[
\begin{array}{c}
W \xrightarrow{\tilde{f}} Z \\
\downarrow h \quad \downarrow h \\
Y \xrightarrow{f} X
\end{array}
\]

Let \( B_f \) and \( B_h \) be the branch loci of \( f \) and \( h \), respectively. The base extension of a smooth morphism is smooth, so \( \tilde{f} \) is smooth above \( Z \setminus h^{-1}(B_f) \). Similarly, \( \tilde{h} \) is smooth above \( Y \setminus f^{-1}(B_h) \). Since the branch loci of \( f \) and \( h \) are disjoint, we conclude that \( W \) is regular.

For the genus formula, write \( R_Y, R_Z, \) and \( R_W \) for the ramification divisors of the morphisms \( f, h, \) and \( f \tilde{h} = \tilde{f} \tilde{h} \), respectively. Since \( f \) and \( h \) have disjoint branch loci, we learn that
\[
R_W = \tilde{f}^* R_Z + \tilde{h}^* R_Y.
\]
Taking degrees gives
\[
\text{deg}(R_W) = \text{deg}(f) \text{deg}(R_Z) + \text{deg}(h) \text{deg}(R_Y).
\]
The Hurwitz formulae for the morphisms \( f : Y \to X, h : Z \to X, \) and \( f \tilde{h} : W \to X \) show that
\[
\begin{align*}
\text{deg}(R_Y) &= 2g(Y) - 2 - \text{deg}(f) (2g(X) - 2) \\
\text{deg}(R_Z) &= 2g(Z) - 2 - \text{deg}(h) (2g(X) - 2) \\
\text{deg}(R_W) &= 2g(W) - 2 - \text{deg}(f) \text{deg}(h) (2g(X) - 2).
\end{align*}
\]
Solving for \( g(W) \) in this system of four equations gives the desired result. \( \square \)

Now we complete the proof of the Theorem 1.2. Let \( B_0 \) be the branch locus of \( h : \mathbb{P}^1 \to \mathbb{P}^1 \). Apply Lemma 2.1 to construct an abelian cover \( f_0 : X_0 \to \mathbb{P}^1 \) of degree \( \gamma \) such that \( \infty \) splits completely in \( X_0 \), and such that the branch locus of \( f_0 \) is disjoint from \( B_0 \). Next we iteratively apply Lemma 2.1 to construct a sequence of abelian covers \( f_i : X_i \to \mathbb{P}^1 \) of degree \( \gamma \) and an increasing sequence of closed subsets \( B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq \mathbb{P}^1 \) such that
• \( \infty \) splits completely in \( X_i \);
• the branch locus of \( f_i \) is a closed point disjoint from \( B_i \); and
• the set \( B_{i+1} \) is the union of \( B_i \) and the branch locus of \( f_i \).

Since the branch locus of \( f_i \) is disjoint from the branch locus of \( h \), we may apply Lemma 2.2 to see that \( C_i = X_i \times_{\mathbb{P}^1} \mathbb{P}^1 \) is a smooth 1-dimensional variety over \( \mathbb{P}_q \).

We argue next that \( C_i \) is geometrically connected. Note that \( C_i \) is a closed subscheme of \( X_i \times \mathbb{P}^1 \). The intersection theory of ruled surfaces [3, V.2.3] shows that the group of numerical equivalence classes of \( X_i \times \mathbb{P}^1 \) is generated by \( e_1 = X_i \times \{ \text{pt} \} \) and \( e_2 = \{ \text{pt} \} \times \mathbb{P}^1 \), and these two curves satisfy \( e_1^2 = e_2^2 = 0 \) and \( e_1.e_2 = 1 \). Since \( C_i \) is a fiber product, each of its geometric components dominates \( X_i \) and \( \mathbb{P}^1 \) for the respective projection maps. Consequently, any geometric component is numerically equivalent to \( ae_1 + be_2 \) for some \( a, b > 0 \). If \( C_i \) has two distinct geometric components, then those components have nonzero intersection. As any intersection point between components is non-smooth, we have a contradiction. It follows that \( C_i \) is geometrically connected.

Now we argue that the genus of \( C_i \) tends to infinity with \( i \). Lemma 2.2 shows that \( C_i \) has genus
\[
g(C_i) = (q + 1)g(X_i) + q(\gamma - 1).
\]
The branch locus of \( f_i : X_i \to \mathbb{P}^1 \) is supported on a closed point that is disjoint from the branch loci of all \( f_j \) with \( j < i \). As there are only finitely many closed points of a given degree, the degree of this branch locus must tend to infinity with \( i \). By the Hurwitz formula, \( g(X_i) \to \infty \) with \( i \), and hence the genus of \( C_i \) must also tend to infinity with \( i \).

Finally, we show that \( C_i \) has \( \gamma(q+1) \) rational points and gonality \( \gamma \). Consider the following fiber product diagram:

\[
\begin{array}{ccc}
C_i & \xrightarrow{f_i} & \mathbb{P}^1 \\
\downarrow{h} & & \downarrow{h} \\
X_i & \xrightarrow{f_i} & \mathbb{P}^1
\end{array}
\]

Note first that \( \hat{f}_i \) has the same degree as \( f_i \), namely \( \gamma \); thus \( C_i \) has gonality at most \( \gamma \). Since \( \infty \) splits completely for the map \( f_i : X_i \to \mathbb{P}^1 \), every pre-image of \( \infty \) under \( h \) splits completely in \( C_i \) under the map \( \hat{f}_i [8, \text{Prop. 3.9.6b}] \). In particular, since \( h \) maps all rational points of \( \mathbb{P}^1 \) to \( \infty \), we see that \( \hat{f}^{-1}_i(x) \) consists of \( \gamma \) rational points for each \( x \in \mathbb{P}^1(\mathbb{F}_q) \). Thus \( \# C_i(\mathbb{F}_q) \geq \gamma(q+1) \). By (1.1), \( C_i \) must have gonality at least \( \gamma \). We conclude that \( C_i \) has gonality \( \gamma \) and exactly \( \gamma(q+1) \) rational points, and the proof of Theorem 1.2 is complete.

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