Stochastic leverage effect in high-frequency data: a Fourier based analysis.

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Abstract

We study the finite sample properties of the Fourier estimator of the integrated leverage effect in the presence of microstructure noise contamination. Our estimation strategy is related to a measure of the contemporaneous correlation between financial returns and their volatility increments. We do not prior assume that the aforementioned correlation is constant, as mainly done in the literature. We instead consider it as a stochastic process. In this framework, we show that the Fourier estimator is asymptotically unbiased but its mean squared error diverges when noisy high-frequency data are in use. This drawback of the estimator is further analyzed in a simulation study where a feasible estimation strategy is developed to tackle this problem. The paper concludes with an empirical study on the leverage effect patterns estimated using high-frequency data for the S&P 500 futures between January 2007 and December 2008.

JEL Classification: C13, C14, C51, C58

Keywords: Fourier analysis, leverage effect, high-frequency data, microstructure noise

1 Introduction

The leverage effect is one of the most striking empirical regularity observed in financial data and has been observed across different time-scales and data sets. The empirical regularity in question refers to the apparent asymmetry observed in the dynamics of the returns and their respective volatilities. Given a data set, the leverage effect can be identified by computing different type of correlations. In fact we can compute the correlation between the volatility and current and past returns and the correlation between volatility and future returns. Both are typically expected to be negative. The presence of these negative correlations is based, respectively, on the following

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two hypotheses: the financial leverage effect and the volatility feedback effect. The first follows from the seminal papers of Black [9] and Christie [15] and refers to the raise in the volatility concurrently with a decline in the asset price: in this instance the companies are said to be more leveraged since the relative value of their debt rises relative to that of their equity causing the assets to become more volatile. The second hypothesis, see for example [20] and [12], attempts to explain as an anticipated increase in volatility generates a stock price decline: if the volatility is priced, an increase in volatility raises the required rate of return, in turn necessitating an immediate stock-price decline to allow for higher future returns. The papers [7, 11, 19, 21, 28, 32] are a not exhaustive list of the works present in the literature which use daily to monthly data in the attempt to assess the presence and the persistence of the correlations implied by the financial leverage and volatility feedback hypotheses. They obtain often inconclusive or conflicting results which depend on the data sets (the asymmetry is generally larger for aggregate market index returns than that for individual stocks, see e.g. [30]), the proxies of the volatility and the different methodologies used in the papers. Investigating further the fundamental causes behind the leverage effect is not the focus of this paper. It is however important to highlight that there is broad agreement that the effect should be present.

We instead move to examine the leverage effect when a high-frequency data set is employed, i.e. intra-daily data, which provides the opportunity to explore more closely the relation between returns and volatility. In Bollerslev et al. [10], the authors use high-frequency five-minute S&P 500 futures and the squared high-frequency returns as a simple volatility proxy, determining that a prolonged negative correlation between the volatility and the current and lagged returns lasts for several days. These results therefore support the notion of a highly significant prolonged leverage effect at the intra-daily level. At the same time, they also observe that the contemporaneous correlation between the high-frequency returns and their absolute value is most significant when the time lag is zero.

In this paper we study the contemporaneous leverage effect observed in tick-by-tick data which are price records with an average frequency of 5 minutes or higher. In the aforementioned literature the leverage effect is described as a constant correlation parameter. This empirical regularity is then incorporated in models for returns and volatilities that are stationary (or at least weak-stationary), see for more detail on this issue Section 2. However, the assumption of stationarity for high-frequency data is difficult to test. Very often common used tests, e.g. the KPSS test [8], lose their power at high-frequency and/or they have to be properly set. Moreover, there is also the problem of handling the non-equidistant grid where tick-by-tick data are typically recorded. If we assume to not have information on the stationarity of the data generating process we have to look at the data with a different perspective. In Section 2 we observe evidence of stochastic correlation between financial asset returns and their respective volatility increments by analyzing a series of the S&P 500 futures recorded in the 2008.

In the literature, there have been so far very few attempts to study the leverage effect in this modeling framework. In [13], the authors provide empirical evidences of stochastic skew in the
currency option data sets and mention the possibility to incorporate this feature by randomizing the correlation parameter between the currency return and its corresponding volatility without however investigating further their assertions. On the other hand, contrary to the prevalent literature, several works are present which show empirical evidence that the leverage effect is a time varying function, [3, 19, 33]. In Veraart and Veraart [31], for the first time, parametric stochastic volatility models with a stochastic correlation parameter have been investigated. The authors introduce a linear transformation of a Jacobi process to model the correlation between the logarithmic price and its volatility process. This leads to analyze two new models: a generalization of the Heston model [23] and of the Barndorff-Nielsen Shepard model [4, 5].

In this paper, we model the logarithmic asset price \( p \) and the volatility \( \sigma^2 \) as two continuous semi-martingale processes correlated by means of a stochastic process \( \rho \) with values in \([-1, 1]\). We do not assume any specific functional form of the volatility, of the variance of the volatility (also called volatility of volatility) and of the correlation process. The choice of a continuous stochastic volatility model for tick-by-tick data is supported by the empirical work in [14]. In this paper, data at millisecond precision are analyzed and it is highlighted that jumps account for about 1% of total price variability. We do not investigate the presence of jumps for lower frequency data being the existing tests based on a different modeling set-up than ours, see [14].

It is also important to highlight that, to the best of our knowledge, there are no statistical tests that can help us to assess the presence of stochastic leverage effect in conjunction with stochastic volatility of volatility in the data. This is however an interesting open problem that we hope to address in future investigations.

In our set-up, we define the leverage process as the covariance (to be precise a covariation) between the returns and the increments of the volatility process. A spot estimator of the leverage process can be found in [17]. In this paper, however, we focus on estimating an integrated measure of the leverage effect, i.e. the covariation between the logarithmic price and its corresponding volatility, by using the Fourier based estimation presented in [17, 18].

Several authors have proposed alternative non-parametric procedures for estimating the integrated leverage effect in an Itô semimartingale framework [1, 2, 6, 16, 27]. The common feature of these estimators is the use of a pre-estimate of the spot volatility in the definition of the integrated covariation by means of different techniques - Fourier transform method [6, 16] or local averages of integrated volatility estimators as in [1, 2, 27]. Due to the different modeling set-ups assumed by the authors in [1, 2, 6, 16, 27], comparing estimators of the integrated leverage effect is difficult. In [2], for example, the leverage effect is still considered as a constant correlation parameter. The estimators in [1, 27] are the most similar to the one defined in [17, 18].

However, they also do not allow to consider a general specification of the stochastic correlation between the logarithmic price and the volatility process, as the model set-up presented in [17, 18] does, see Remark [3, 1] for more detail on this issue. The estimation presented in [17, 18] is made possible by the Fourier transform methodology introduced by Malliavin and Mancino in [24, 25] and allows to define an estimation strategy without resorting to a pre-estimate of the volatility.
path. In fact, $N$ Fourier coefficients of the volatility process are estimated by following the procedure addressed in [24]. This is a step that requires the preliminary computation of $M$ Fourier coefficients of the returns. In the following, the parameters $M$ and $N$ are called cutting frequencies. In [17, 18], the asymptotic properties of the Fourier estimator of the integrated leverage have been studied in the absence of microstructure noise contamination. We study in this paper how to use the Fourier estimator in a real data framework, i.e. in the presence of microstructure noise contamination. In fact, when sampled at sufficiently high-frequency, asset prices tend to incorporate the mechanism of the trading process such as bid/ask bounces, the different price impact of different types of trades, limited liquidity or other types of frictions.

In the finite sample, there are typically three possible sources of bias that might be identified when estimating the leverage effect using noisy high-frequency data, see [2, 27]: bias due to discretization, latency of the volatility and, obviously, microstructure noise. While using a Fourier estimation strategy similar sources of bias can be identified by analyzing the role of the parameters $n$, $M$ and $N$, respectively, the number of observations, the Fourier coefficients of the returns and of the volatility process. It is important to highlight that the properties of the Fourier estimator do not depend on the grid where the data are recorded. In fact, the Fourier estimator can be indifferently applied to equidistant or non-equidistant data sets. Moreover, despite presenting a bias in the finite sample, it can be shown that the estimator is asymptotically unbiased in the presence of microstructure noise under the assumptions that $N^2/M \rightarrow 0$ and $MN/n \rightarrow 0$ as $n, M, N \rightarrow \infty$. However, under the same assumptions, we observe that the mean squared error of the estimator diverges. The aforementioned asymptotic ratios between the parameters $n, M, M$ differ from the one used in [17, 18] where consistency and asymptotic normality of the estimator in absence of noise are shown. We then prove that these results still hold under the new set-up.

In a simulation study, where the data are drawn by the Heston [23] and the Generalized Heston model [31], we analyze the mean squared error of the estimator and conclude that the finite sample variance is responsible for its divergence. We then define a variance corrected estimator and propose a feasible estimation strategy in a real-data framework by selecting the optimal parameters $M$ and $N$ prior to the computation of the Fourier estimator. Using Monte-Carlo data, it is shown as selecting the parameters $M$ and $N$ by direct minimization of the true mean squared error is equivalent to selecting the parameters by minimizing the sample variance of the estimator. The optimal selected parameters $M$ and $N$ always determine a bias in the obtained estimation. However, due to the intrinsic unbiasedness of the estimator, an estimation with at least a significant digit is always obtained. To conclude, using as a benchmark the integrated leverage estimator of Mikland and Wang [27] on data generated by the Heston model, we show a comparison with the performance of the Fourier estimator.

The paper is organized as follows. Section 2 documents the presence of stochastic leverage effect in the data. The data generating process and model setting can be found in Section 3. In Section 4, the Fourier estimation methodology and the consistency of the integrated estimator
are proven. Section 5 analyzes the finite sample properties of the estimator, namely, asymptotic unbiasedness and its mean squared error in the presence of microstructure noise contamination. In Section 6, the definition of a variance corrected estimator is given and a feasible selection strategy for the cutting frequency parameters is developed. Section 7 presents an empirical study based on high-frequency S&P 500 futures between January 2007 and December 2008. Section 8 concludes. The Appendix contains the proofs of all statements presented in the paper.

2 The stochastic leverage effect at high-frequency

![Figure 1: Log-price paths of the S&P 500 futures tick-by-tick data set described in Section 7 on four days during 2008.](image)

In Bollerslev et al. [10], 5 minutes intra-daily data are analyzed and the contemporaneous leverage effect is determined by calculating the correlation between returns and volatility proxies, i.e. squared or absolute returns. They obtain that the contemporaneous correlation in their data is negative and persistent. The authors then choose models for returns and volatility which are correlated by means of a negative constant parameter. As the contemporaneous correlation coefficient is defined as the covariance between two random variables divided by the product of
Figure 2: Scatter plot of the increments of the volatility, computed by using the Fourier spot volatility estimator, and a series of high-frequency returns with an average frequency of approximately 5 minutes, for the S&P 500 futures. Correlation coefficients from May 27th to Nov. 20th are, respectively, $3.80 \times 10^{-2}$, $-8.07 \times 10^{-2}$, $-2.45 \times 10^{-2}$ and $2.04 \times 10^{-1}$.

the standard deviation of the two variables globally, when considering the correlation parameter between returns and volatility constant there is an underlying assumption that these variables should be modeled as stationary (or at least weak stationary). These kind of underlying assumptions are also found in [2] where noisy high-frequency data are used to infer the leverage effect at longer horizons, e.g. a day, by using high-frequency (integrated) volatility estimates.

In our paper, we do not conduct our analysis prior assuming that returns or volatilities are generated by a stationary process. In fact, we want to study the leverage effect by freeing its analysis from restrictive model set-up assumptions as those considered in the majority of the literature.

To motivate the theoretical analysis that follows, we start by a simple empirical experiment. We study scatter plots of estimated changes of volatilities and returns which provide a way to examine graphically the relationship between the returns and the increments of their respective volatilities. We look at a series of tick-by-tick data of the S&P 500 futures, see Section [7] for more
details. We use as volatility proxy the Fourier spot volatility estimator defined in [24] which is robust under microstructure noise contamination. The series of the returns is then constructed as an aggregate with respect to the grid where the volatility process is estimated. We obtain on average, that the returns have approximately 5 minutes frequency. No significant first or higher order autocorrelation is found in the series of the returns analyzed in the following.

We select the following days of the 2008: May 27th, October 9th, October 29th and November 20th. The motivation behind the choice of these days is the following. The first day corresponds to a normal trading day where no news generating turmoil in the market have been released. October 9th corresponds to a day during the turmoil generated by the subprime mortgage crisis. The S&P 500 lost 21.6% of its values in a nine-day period from October 1st and 9th. October 29th and November 20th are two day where strong price variations have been observed in the market. See Figure [1] In Figure [2] the increments of the volatility respect to the returns show unexpected asymmetries. For extreme negative or positive returns the volatility can respond with positive or negative increments whereas based on the causal study on the leverage effect, namely, the financial leverage and volatility feedback effect, we would expect that for extreme negative and positive returns the volatility responds with positive and negative increments, respectively. In our data we observe that this is not always the case. Moreover, the correlation parameter between returns and increments of the volatility, as it can be observed by the slope of the red regression lines in Figure [2] is positive in May 27th and November 20th and negative in October 9th and October 29th. In conclusion, we observe that patterns of the increments of volatility and returns may variate through time and that often their correlation parameter switches signs and is of different magnitude.

We then have ground for claiming that a possible approach to the study of the leverage effect in a high-frequency framework is assuming stochastic correlation between the logarithmic asset price and its volatility. We define in the next section a general continuous time data generating process where this empirical regularity is implemented.

3 Data Generating Process

We assume that the log-price and the volatility processes are defined as solutions of the system of equations

\[
\begin{aligned}
    dp(t) &= a(t) \, dt + \sigma(t) \, dW(t) \\
    d\sigma^2(t) &= b(t) \, dt + \gamma(t) \, dZ(t)
\end{aligned}
\]  

(1)

where \( W(t), t \geq 0 \) and \( Z(t), t \geq 0 \) are two correlated standard Brownian motions defined on the complete probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). \( \mathcal{F} = (\mathcal{F}_t) \) is the natural filtration generated by \( W \) and \( Z \). \( \sigma^2(t) \) is the process we call volatility throughout the paper. The correlation process is defined as \( \rho(t) \) with values in \([-1, 1]\) such that \( (dW(t), dZ(t)) = \rho(t) dt \).

We perform our analysis in a time window \([0, T]\) for \( T > 0 \) and the processes appearing in
satisfy the following assumption:

- (H1) $a(t), b(t), \sigma(t), \gamma(t)$ and $\rho(t)$ are $\mathbb{R}$-valued processes, almost surely continuous on $[0, T]$ and adapted to the filtration $\mathcal{F}$ such that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |a(t)|^8 \right] < \infty, \quad \mathbb{E} \left[ \sup_{t \in [0, T]} |b(t)|^8 \right] < \infty, \\
\mathbb{E} \left[ \sup_{t \in [0, T]} |\sigma(t)|^8 \right] < \infty, \quad \mathbb{E} \left[ \sup_{t \in [0, T]} |\gamma(t)|^8 \right] < \infty, \\
\mathbb{E} \left[ \sup_{t \in [0, T]} |\rho(t)|^8 \right] < \infty.
$$

- (H2) Let $\mathbb{D}^{1,p}$ be the space of $\mathbb{R}$-valued measurable and adapted processes admitting a first order Malliavin derivative $\mathcal{D}$ that is $p$-integrable. We define $\mathbb{D}^{1,\infty} = \bigcap_{p \geq 1} \mathbb{D}^{1,p}$. Then, the processes $a(t), b(t), \sigma(t), \gamma(t) \in \mathbb{D}^{1,\infty}$ and $\forall p \geq 1$

$$
\mathbb{E} \left[ \sup_{s,t \in [0,2\pi]} |\mathcal{D}_s a(t)|^p \right] < \infty, \quad \mathbb{E} \left[ \sup_{s,t \in [0,2\pi]} |\mathcal{D}_s b(t)|^p \right] < \infty, \\
\mathbb{E} \left[ \sup_{s,t \in [0,2\pi]} |\mathcal{D}_s \sigma(t)|^p \right] < \infty, \quad \mathbb{E} \left[ \sup_{s,t \in [0,2\pi]} |\mathcal{D}_s \gamma(t)|^p \right] < \infty.
$$

We refer the reader to [29, Section 1.5] for further details regarding the construction of the space $\mathbb{D}^{1,\infty}$ and to [29] for the basic theory of Malliavin calculus.

Model (1) describes the dynamics of an underlying efficient price process in the absence of market frictions. The parametric models, e.g. Heston, CEV, and the generalized Heston model defined in [31] satisfy our assumptions, see [17, Remark 1].

**Remark 3.1.** In [1], the authors work on an underlying model that admits jumps in the logarithmic price and the volatility dynamics, see [1, Assumption (H)]. In the continuous case, the estimator in [1] can still be used but at the cost of more restrictive assumptions on the volatility process than in our Assumptions (H1). A more careful comparison can be made with the results in [27]. Here, $a(t), b(t)$ and $\gamma(t)$ are assumed to be locally bounded in absolute value and $\sigma(t)$, in particular, locally bounded away from zero. However, a stochastic correlation process $\rho(t)$ like the one in model (1) cannot be defined in the model set-up described in [27], see [27, Appendix A] therein for more details. In fact, our filtration is generated by two Brownian motions correlated by means of the process $\rho(t)$ whereas in [27] the processes are adapted to a filtration generated by two independent Brownian motions. The latter assumption implies, for example, that the data generating process in [27] does not encompass the generalized Heston model presented in [31].

The leverage process $\eta(t)$ is defined by means of the covariation between the returns and
the increments of the volatility process as

\[ \langle dp(t), d\sigma^2(t) \rangle = \eta(t)dt. \] (2)

We are interested in estimating the integrated covariation between the logarithmic price \( p \) and the volatility process \( \sigma^2 \)

\[ \hat{\eta} = \int_0^T \eta(t)dt. \] (3)

4 The Fourier estimator of the integrated leverage effect

We start by introducing some notations. We denote with \( D_N(t) \) the normalized Dirichlet kernel defined by

\[ D_N(t) = \frac{1}{2N+1} \sum_{|l|\leq N} e^{i \frac{2\pi}{T} lt} \] (4)

and with \( D'_N(t) \) its first derivative

\[ D'_N(t) = \frac{1}{2N+1} \sum_{|l|\leq N} il \frac{2\pi}{T} e^{i \frac{2\pi}{T} lt}. \] (5)

The following properties hold. The proof of the results below is straightforward and we omit it.

Proposition 4.1. Let \( D_N(t) \) be the normalized Dirichlet kernel defined in (4), then the following properties are satisfied.

1. \( \int_0^T |D_N(u)|^2 du = \frac{T}{2N+1} \),
2. \( \forall p > 1 \), there exists a constant \( C_p \) such that \( \int_0^T |D_N(u)|^p du = \frac{C_p}{2N+1} \).

The methodology then starts by defining the Fourier coefficients of the leverage process. Following [24], we define the Fourier coefficients of the returns and of the increments of the volatility process as

\[ c(l; dp) = \frac{1}{T} \int_0^T e^{-i \frac{2\pi}{T} lt} dp(t), \] (6)

and

\[ c(l; d\sigma^2) = \frac{1}{T} \int_0^T e^{-i \frac{2\pi}{T} lt} d\sigma^2(t), \] (7)

for each \( l \in \mathbb{Z} \).

Given two functions \( \Phi \) and \( \Psi \) on the integers \( \mathbb{Z} \), we say that their Bohr convolution product exists if the following limit exists for all integers \( h \)

\[ (\Phi \ast \Psi)(h) := \lim_{N \to \infty} \frac{1}{2N+1} \sum_{|l|\leq N} \Phi(l)\Psi(h-l). \]
Under Assumptions (H1), let \((p(t), \sigma^2(t))\) be a solution of system \([1]\). For a fixed \(h\), defining \(\Phi(l) := c(l; d\sigma^2)\) and \(\Psi(h - l) := c(h - l, dp)\), the limit in probability of the Bohr convolution product exists and converges to the \(h\)-th Fourier coefficient of the leverage process. This result is shown in \([25, \text{Theorem 2.1}]\) in the case of the covariance process. The \(h\)-th Fourier coefficient of \(\eta(t)\) is then

\[
c(h; \eta) = \lim_{N \to \infty} \frac{T}{2N + 1} \sum_{|l| \leq N} c(l; d\sigma^2) c(h - l; dp) = \frac{1}{T} \int_0^T e^{-i\frac{2\pi}{T}ht} \eta(t) dt. \tag{8}
\]

The identity above has the obvious drawback to be feasible only when continuous observations of the logarithmic price and the volatility process are available.

Let us assume, first, that we can observe continuously the logarithmic price and that the volatility process is latent. For all \(l \neq 0\), by means of the use of the integration by parts formula, we have that

\[
c(l; d\sigma^2) = il \frac{2\pi}{T} c(l; \sigma^2) + \frac{1}{T}(\sigma^2(T) - \sigma^2(0)), \tag{9}
\]

where

\[
c(l; \sigma^2) = \frac{1}{T} \int_0^T e^{-i\frac{2\pi}{T}lt} \sigma^2(t) dt.
\]

Therefore, the limit \((8)\) becomes

\[
c(h; \eta) = \lim_{N \to \infty} \frac{T}{2N + 1} \sum_{|l| \leq N} (il \frac{2\pi}{T} c(l; \sigma^2) + \frac{1}{T}(\sigma^2(T) - \sigma^2(0))) c(h - l; dp) \tag{10}
\]

\[
= \lim_{N \to \infty} \frac{T}{2N + 1} \sum_{|l| \leq N} il \frac{2\pi}{T} c(l; \sigma^2) c(h - l; dp) + \frac{T}{2N + 1} \sum_{|l| \leq N} \frac{1}{T}(\sigma^2(T) - \sigma^2(0)) c(h - l; dp)
\]

\[
= \lim_{N \to \infty} \frac{1}{T} \int_0^T \int_0^T e^{-i\frac{2\pi}{T}ht} D_N(t - s) \sigma^2(s) ds dp(t) + \frac{1}{T} \int_0^T (\sigma^2(T) - \sigma^2(0)) e^{-i\frac{2\pi}{T}ht} D_N(t) dp(t).
\]

The second summand converges to 0 in probability as \(N\) tends to infinity. In fact, by applying the Itô isometry and the Cauchy-Schwartz inequality

\[
\mathbb{E}\left[\left| \frac{1}{T} \int_0^T (\sigma^2(T) - \sigma^2(0)) e^{-i\frac{2\pi}{T}ht} D_N(t) dp(t) \right|^2 \right] \leq C \frac{T}{2N + 1}
\]

because of Proposition \([4.1]\) and Assumption (H1), where \(C\) is a constant independent of \(N\).

Thus, when the volatility is a latent process

\[
c(h; \eta) = \lim_{N \to \infty} \frac{T}{2N + 1} \sum_{|l| \leq N} il \frac{2\pi}{T} c(l; \sigma^2) c(h - l; dp) \tag{11}
\]

In order to construct a feasible estimation procedure for the \(h\)-th Fourier coefficient of the
leverage process, we consider the truncation of the limit in (11). Thus,
\[
c_N(h; \eta) = \frac{T}{2N + 1} \sum_{|l| \leq N} i \frac{2\pi}{T} l c(l; \sigma^2) c(h - l; dp)
\] (12)
in which only the Fourier coefficients of the return and volatility process appear. Therefore, the
error due to the estimation of a spot volatility path can be overcome defining an estimation strategy only in the frequency domain.

We now assume to observe \(p(t)\) on a discrete non-equidistant time grid.

\[S_n := \{0 = t_0 \leq t_1 \leq \ldots \leq t_n = T\} \text{, for all } i = 0, \ldots, n.\]

We define \(\tau(n) := \max_{i=0,\ldots,n-1} |t_{i+1} - t_i|\) and the discrete observed return as \(\delta_i = p(t_{i+1}) - p(t_i)\) for all \(i = 0, \ldots, n - 1.\)

Following [17, 18], an estimator of the \(h\)-th Fourier coefficient of the leverage process can be then defined as
\[
c_{n,M,N}(h; \eta) = \frac{T}{2N + 1} \sum_{|l| \leq N} i \frac{2\pi}{T} c_n(l; \sigma^2) c(h - l; dp),
\] (13)
for any integer \(h\) such that \(|h| \leq N\), where \(c_n(s; dp)\) for \(|s| \leq N + M\) are the discrete Fourier coefficients of the return process
\[
c_n(s; dp) = \frac{1}{T} \sum_{i=0}^{n-1} e^{-i s 2\pi T i} \delta_i(p)
\] (14)
and \(c_{n,M}(l; \sigma^2)\) are the Fourier coefficients of the volatility introduced in [24] for \(|l| \leq N\)
\[
c_{n,M}(l; \sigma^2) = \frac{T}{2N + 1} \sum_{|s| \leq M} c_n(s; dp) c_n(l - s; dp).
\] (15)

The above-mentioned estimators are written as functions of the parameters \(n, M\) and \(N\), which stand for the number of observations available, the number of the discrete Fourier coefficients of the returns and of the Fourier coefficients of the volatility process, respectively, as defined in [15].

An estimator of \(\hat{\eta}\) can then be simply obtained by means of definition [13] for \(h = 0\)
\[
\eta_{n,M,N} = T c_{n,M,N}(0; \eta) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} D_M(t_i - t_j) D'_N(t_k - t_j) \delta_i \delta_j \delta_k,
\] (16)
where \(D'_N\) and \(D_M\) are defined in [5] and [4].

In [18], it is shown that the estimator [16], in the absence of microstructure noise, is
consistent if \( N^2/M \to 0 \) and \( M\tau(n) \to a \) with \( a > 0 \). The cutting frequency \( M \), i.e. the number of Fourier coefficients of the return process to be used in (13), have theoretically just an upper bound given by the Nyquist frequency corresponding to \( \lceil n/2 \rceil \). As discussed in [20, Chapter 5], the Fourier estimator is designed to filter the noise components by requiring a smaller number of Fourier coefficients of the returns with respect to the Nyquist frequency. Thus, with the aim to apply the Fourier methodology to data affected by microstructure noise contamination, we have to show that the asymptotic properties of the estimator hold under the assumptions that \( M\tau(n) \) converges to zero. We then prove anew the convergence in probability of the estimator.

**Proposition 4.2.** We assume that the assumptions \((H1), (H2)\) and

\[
\frac{N^2}{M} \to 0 \quad \text{and} \quad MN\tau(n) \to 0
\]  

hold true as \( n, M, N \to \infty \) and \( \tau(n) \to 0 \). Then

\[
\eta_{n,M,N} \overset{P}{\to} \hat{\eta}.
\]

**Remark 4.3.** In the proof of Proposition 4.2, see the Appendix, it is shown that the drift components of the logarithmic price and the volatility process are negligible in probability with respect to the diffusive components. For simplicity, we will then assume throughout the paper that the drift terms \( a(t) \) and \( b(t) \) are equal to zero.

**Remark 4.4.** In [17], a central limit theorem for the estimator (16) is obtained under the assumptions \((H1), (H2)\) and the asymptotic ratios \( N^3/M \to 0 \) and \( M\tau(n) \to a \) where \( a > 0 \). By assuming \( N^3/M \to 0 \) and \( MN^{\frac{3}{2}}\tau(n) \to 0 \), by reading [17, Section 4.2-4.3], it is easy to check that the same asymptotic result as in [17, Theorem 3] applies.

## 5 Finite sample properties

In order to define an estimation strategy for high-frequency data, we add microstructure noise contamination to the efficient log-price in equilibrium, \( p(t) \), defined in [1]. Thus, the logarithm of the observed price is

\[
\tilde{p}(t_i) = p(t_i) + \zeta(t_i)
\]

where \( \zeta(t) \) is the microstructure noise. We assume

- **(H3)** The random shocks \( \zeta \) are considered i.i.d. Gaussian and independent of \( p \). We further assume that \( \mathbb{E}[\zeta^6] < \infty \).

This is typical of the bid-ask bounce effects in the case of exchange rates and, to a lesser extent, in the case of equities.
Let us define $\epsilon_i = \zeta(t_{i+1}) - \zeta(t_i)$ and $\delta_i = \bar{p}(t_{i+1}) - \bar{p}(t_i)$. Then, the Fourier estimator of the integrated leverage effect (16) in the presence of noise becomes

$$
\tilde{\eta}_{n,M,N} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} D_M(t_i - t_j)D'_N(t_k - t_j)\delta_i \delta_j \delta_k.
$$

We can disentangle the estimator (20) in the following components

$$
\sum_{i \neq j \neq k} D_M(t_i - t_j)D'_N(t_k - t_j)\delta_i \delta_j \delta_k
$$

$$
+ \sum_{i,j \neq j} D_M(t_i - t_j)D'_N(t_i - t_j)\delta_i^2 \delta_j + \sum_{i,j} D'_N(t_i - t_j)\delta_i \delta_j^2
$$

$$
= \sum_{i \neq j \neq k} D_M(t_i - t_j)D'_N(t_k - t_j)\delta_i \delta_j \delta_k
$$

$$
+ \sum_{i,j \neq j} D_M(t_i - t_j)D'_N(t_i - t_j)\delta_i^2 \delta_j + \sum_{i,j} D'_N(t_i - t_j)\delta_i \delta_j^2
$$

$$
+ \eta_{n,M,N}^\epsilon,
$$

where the sum of the components (22) and (23) correspond to the estimator in the absence of noise (16) and

$$
\eta_{n,M,N}^\epsilon = \sum_{i,j,k \neq j \neq k} D_M(t_i - t_j)D'_N(t_k - t_j)\delta_i \delta_j \delta_k
$$

$$
+ \sum_{i,j \neq j} D_M(t_i - t_j)D'_N(t_i - t_j)\delta_i^2 \delta_j + \sum_{i,j} D'_N(t_i - t_j)\delta_i \delta_j^2
$$

$$
+ \sum_{i,j} D'_N(t_i - t_j)\delta_i \delta_j.
$$

We now compute the bias of the estimator. We remind the reader that the integrated leverage $\hat{\eta}$ under the modelling assumption in (1) can be positive or negative. Therefore, we analyse the bias of the estimator in absolute value. The definition of the $\eta_{n,M,N}$ as given in Section 4 does not require equidistant data. Anyway for simplicity of computation, we will suppose that the observations are equidistant in time and $\frac{T}{n}$ is the distance between two observations, where $[0,T]$ is the time window where we observe the data. Moreover, we consider throughout the section, that the drift components of the underlying model are zero, following Remark 4.3.

**Theorem 5.1.** We assume that the assumptions (H1), (H3) and

$$
\frac{N^2}{M} \to 0 \hspace{1em} \text{and} \hspace{1em} \frac{MN}{n} \to 0
$$

hold true as $N, M, n \to \infty$ and $n \to \infty$, then the estimator $\tilde{\eta}_{n,M,N}$ is asymptotically unbiased.
More precisely,

\[ |E[\tilde{\eta}_{n,M,N} - \tilde{\eta}]| \leq |E[\eta_{n,M,N} - \int_0^T \eta(t) dt]| + |E[\eta_{n,M,N}]| \]

\[ \leq \Gamma(n, M, N) + \Lambda(n, N) + \Psi(N) + o(1) + \left| 2(n - 1) \left( D_M \left( \frac{T}{n} \right) - 1 \right) D'_N \left( \frac{T}{n} \right) \mathbb{E}[\zeta^3] \right|, \]

where

\[ \Gamma(n, M, N) \leq \frac{N(M + N)}{n} 8\pi^2 T^{\frac{3}{2}} \mathbb{E} \left[ \sup_{[0, T]} \sigma^2(t) \right]^{\frac{3}{2}} + \frac{N}{\sqrt{2M + 1}} 2\pi T^{\frac{3}{2}} \mathbb{E} \left[ \sup_{[0, T]} \sigma^2(t) \right]^{\frac{3}{2}}, \]

\[ \Lambda(n, N) \leq \frac{N}{\sqrt{n}} 4\pi T^{\frac{3}{2}} \mathbb{E} \left[ \sup_{[0, T]} \sigma^2(t) \right]^{\frac{3}{2}} + \frac{N^2}{n} 4\pi^2 (1 + T^{\frac{1}{2}}) \mathbb{E} \left[ \sup_{[0, T]} \sigma^4(t) \right]^{\frac{1}{2}} \mathbb{E} \left[ \sup_{[0, T]} \sigma^2(t) \right]^{\frac{1}{2}}, \]

and,

\[ \Psi(N) \leq \frac{1}{\sqrt{2N + 1}} T \mathbb{E} \left[ \sup_{[0, T]} \eta^2(t) \right]^{\frac{1}{2}}. \]

The following corollary is of great importance in Section 6 where a feasible estimation strategy for the integrated leverage effect is given. A detailed explanation of the role of the term (21) can then be found in Section 6.

**Corollary 5.2.** We assume that Assumptions of Theorem 5.1 hold. Then, the term defined in (21) has expected value equal to zero.

**Proof.** The term (21) can be decomposed as

\[ \sum_{i \neq j \neq k} D_M(t_i - t_j)D'_N(t_k - t_j)\delta_i\delta_j\delta_k \]

\[ + \sum_{i,j,k: i \neq j \neq k} D_M(t_i - t_j)D'_N(t_k - t_j)(\delta_i\delta_j\epsilon_k + \delta_j\delta_k\epsilon_i + \delta_k\delta_i\epsilon_j + \delta_i\epsilon_j\epsilon_k + \delta_j\epsilon_i\epsilon_k + \delta_k\epsilon_i\epsilon_j + \epsilon_i\epsilon_j\epsilon_k). \]

Because of the results shown in the proof of Theorem 5.1 it follows directly that the expected value of (21) is equal to zero.

We now focus on the study of the mean squared error of the estimator. We have that

\[ \mathbb{E}[(\tilde{\eta}_{n,M,N} - \tilde{\eta})^2] = \text{Var}(\tilde{\eta}_{n,M,N}) + \mathbb{E}[(\tilde{\eta}_{n,M,N} - \tilde{\eta})^2] + \text{Var}(\tilde{\eta}) - 2\text{Cov}(\tilde{\eta}_{n,M,N}, \tilde{\eta}). \] (25)

This mean squared error decomposition differs from the classical one given in the literature because the quantity we aim to estimate \( \tilde{\eta} \) is a random variable and not a constant. In the result below, it is shown that the mean squared error of the Fourier estimate diverges to infinity.

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**Theorem 5.3.** We assume that (H1), (H2), (H3) and
\[
\frac{N^2}{M} \to 0 \quad \text{and} \quad \frac{MN}{n} \to 0 \quad (26)
\]
hold true as \(N, M, n \to \infty\) and \(n \to \infty\), then
\[
\mathbb{E}[(\tilde{\eta}_{n,M,N} - \hat{\eta})^2] \to \infty
\]

The theorem above highlights the presence of a divergent element in (25). In the next section, we try to identify more precisely the order of magnitude of the summands in (25) by means of a numerical experiment.

### 6 Numerical simulations

We start with testing the efficiency of the estimator \(\tilde{\eta}_{n,M,N}\) in the finite sample. We assume that the underlying efficient dynamic of the price process is described by two different models: the classical Heston model [23] and the Generalized Heston model proposed in [31]. Therefore, we will work on two data-sets.

We simulate second-by-second return and variance paths over a daily trading period of \(T = 6\) hours, for a total of 100 trading days and \(n = 21600\) observations per day.

The first data-set is simulated by the model

\[
H : \begin{cases} 
  dp(t) &= \sigma(t)dW_1(t) \\
  d\sigma^2(t) &= \alpha(\beta - \sigma^2(t))dt + \nu \sigma(t)dW_2(t),
\end{cases} \quad (27)
\]

where \(W_1\) and \(W_2\) are correlated Brownian motions. The parameter values used in the simulations are \(\alpha = 0.01, \beta = 0.2, \nu = 0.05\) and the correlation parameter is chosen as \(\rho = -0.2\).

The second data-set is simulated by the model

\[
GH : \begin{cases} 
  dp(t) &= \sigma(t)dX(t) \\
  dX(t) &= \rho(t)dW_1(t) + \sqrt{1 - \rho^2(t)}dW_2(t) \\
  d\sigma^2(t) &= \alpha(\beta - \sigma^2(t))dt + \nu \sigma(t)dW_1(t),
\end{cases}
\]

and the infinitesimal variation of \(\rho(t)\) is given by

\[
d\rho(t) = ((2\xi - \eta) - \eta \rho(t))dt + \theta \sqrt{(1 + \rho(t))(1 - \rho(t))}dW_0,
\]

where \(\eta, \xi, \theta\) are positive constants and \(W_0\) is a Brownian motion. The processes \(W_0(t), W_1(t)\) and \(W_2(t)\) are assumed to be independent. The parameter values used in the simulation are \(\alpha = 0.01, \beta = 0.2, \nu = 0.05\) and \(\xi = 0.02, \eta = 0.5, \theta = 0.5\), where the last three parameters are chosen in the range prescribed in [31] such that \(\rho \in [-1, 1]\). We set the initial values as
\( \sigma^2(0) = \beta, \ p(0) = \log(100) \) and \( \rho(0) = -0.04 \). The noise-to-signal ratio \( \text{std}(\zeta)/\text{std}(r) \) is equal to 0.8, where \( r \) is the 1-second returns.

When processing simulated data, the natural approach in optimizing estimators with respect to bandwidths or other parameters is to choose those values that minimize the finite sample mean squared error (MSE). Therefore, one possible choice is to select the cutting frequencies \( M \) and \( N \), which identify the numbers of the Fourier coefficients of the return process and of the volatility process appearing in the Fourier estimator, by following this methodology.

Figure 3 shows the MSE and sample variance (VAR) of the leverage estimate \( \tilde{\eta}_{n,M,N} \) as a function of \( M \) and \( N \). It is evident from the figure that the MSE and the sample variance are almost indistinguishable. Indeed, the Fourier estimator of the integrated leverage is affected by a large asymptotic and finite sample variance so that the MSE of the estimator is almost equal to the sample variance.

This is evident also from Figure 4 where the relative difference (MSE-VAR)/MSE as a function of \( M \) and \( N \) is shown. This ratio is negligible except for the lowest values of \( M \) and

![Figure 3: MSE and sample variance of the leverage estimate as a function of \( M \) and \( N \) under microstructure effects. Left panels: H model. Right panels: GH model.](image)
Figure 4: Relative difference between MSE and variance of the leverage estimate as a function of $M$ and $N$ under microstructure effects. Left panels: H model. Right panels: GH model.

never exceeds 0.1 so that the difference MSE-VAR never exceeds 10% of the MSE. Therefore, we can state that the largest part of the MSE consists of the variance of the estimator. Going back to equation (25), we numerically find that the remaining terms are at least one order of magnitude smaller and hence they are negligible with respect to the variance. Therefore, the minimization of the MSE in the finite sample is equivalent to minimizing the estimator variance. This also means that the cutting frequencies are not corresponding to a minimum value of the bias of the estimator. The optimized estimator is then affected by a non negligible bias which is nevertheless very small. Based on these considerations, we can define a feasible procedure to select optimal cut-off frequencies $\hat{M}$ and $\hat{N}$ based on the minimization of the sample variance. Alternatively, we try to pursue a different approach in order to reduce the variance of the Fourier estimator of integrated leverage. The term

$$\Upsilon_{n,M,N} = \sum_{i,j,k: i \neq j \neq k} D_M(t_i - t_j)D'_N(t_k - t_j) \tilde{\delta}_i \tilde{\delta}_j \tilde{\delta}_k,$$

containing all the cross products of the noisy returns $\tilde{\delta}_i \tilde{\delta}_j \tilde{\delta}_k$ with $i \neq j \neq k$, has expected value equal to zero as shown in Corollary 5.2 and it is correlated to $\tilde{\eta}_{n,M,N}$.

We then define the estimator

$$\eta^*_{n,M,N} = \tilde{\eta}_{n,M,N} - b \Upsilon_{n,M,N}$$

which is an asymptotically unbiased estimator of the integrated leverage effect because of Theorem 5.1 and Corollary 5.2 and has

$$Var(\eta^*_{n,M,N}) = Var(\tilde{\eta}_{n,M,N}) - 2b Cov(\tilde{\eta}_{n,M,N}, \Upsilon_{n,M,N}) + b^2 Var(\Upsilon_{n,M,N})$$

(29)
Hence the estimator \( \eta^*_{n,M,N} \) has smaller variance than the estimator \( \tilde{\eta}_{n,M,N} \) if
\[
b^2 \text{Var}(\Upsilon_{n,M,N}) < 2b \text{Cov}(\tilde{\eta}_{n,M,N}, \Upsilon_{n,M,N}).
\]
The optimal coefficient \( b^* \) minimizing the variance of the estimator \( \eta^*_{n,M,N} \) is given by
\[
b^*_{M,N} = \frac{\text{Cov}(\tilde{\eta}_{n,M,N}, \Upsilon_{n,M,N})}{\text{Var}(\Upsilon_{n,M,N})}.
\] (30)
Substituting this value in (29) and simplifying, we find that
\[
\frac{\text{Var}(\eta^*_{n,M,N})}{\text{Var}(\tilde{\eta}_{n,M,N})} = (1 - \text{Corr}(\tilde{\eta}_{n,M,N}, \Upsilon_{n,M,N})^2),
\]
which gives us the variance reduction obtained by using the estimator (28).

**Remark 6.1.** The same variance reduction appears when the classical control variate estimator is used to reduce the variance of the sample mean estimator, see ([22, Section 4.1]).

Operatively, the variance corrected estimator (28) can be implemented by the following procedure:

**Step 1:** Given a sample of \( n \) observed returns and for all \( M \in \{\text{range}\} \) and \( N \in \{\text{range}\} \), let \( \tilde{\eta}^1_{n,M,N}, \tilde{\eta}^2_{n,M,N}, \ldots, \tilde{\eta}^d_{n,M,N} \) be \( d \) replications of the Fourier estimate of the integrated leverage in a Monte Carlo experiment. Along with \( \tilde{\eta}_{n,M,N} \), on each replication we also calculate \( \Upsilon^1_{n,M,N} \);

**Step 2:** let \( M^*, N^* := \text{argmin} \text{VAR}(\Upsilon_{n,M,N}) \) and let \( \Upsilon^* := \Upsilon_{n,M^*,N^*} \);

**Step 3:** plug the selected correction \( \Upsilon^* \) into equation (28)
\[
\eta^*_{n,M,N} = \tilde{\eta}_{n,M,N} - b^*_{M,N} \Upsilon^*,
\]
where
\[
b^*_{M,N} = \frac{\text{COV}(\tilde{\eta}_{n,M,N}, \Upsilon^*)}{\text{VAR}(\Upsilon^*)}
\]
and for each \( M \) and \( N \), compute \( d \) replications \( \eta^i_{n,M,N} \) (\( i = 1, \ldots, d \)) of the estimator (28).

**Step 4:** choose the cutting frequencies \( \hat{M} \) and \( \hat{N} \) which minimize the finite sample MSE of the corrected estimates \( \eta^i_{n,M,N} \) for \( i = 1, \ldots, d \).

The magnitude of the variance correction given by the estimator (28) is tuned by formula (30), where \( \text{Var}(\Upsilon_{n,M,N}) \) appears in the denominator. In our procedure, we first set the parameter (30) such to have the minimum possible denominator and to increase the effectiveness of 1

---

1Note that COV and VAR denote the sample covariance and the sample variance, respectively.
the correction. Afterwards, in Step 4, we choose the optimal MSE-based cutting frequencies \( \hat{M} \) and \( \hat{N} \). With these procedure, better results are empirically observed respect to simultaneously optimizing the parameters \( M \) and \( N \) appearing in \( (28) \).

Table 1 shows the MSE reduction obtained by using the estimator \( (28) \) versus the Fourier estimator \( \tilde{\eta}_{n,M,N} \). The optimal parameter values \( \hat{M} \) and \( \hat{N} \) are obtained by following the MSE-based procedures described above. Since both estimators are only asymptotically unbiased and

| \( H - \text{model} \) | \( GH - \text{model} \) |
|-------------------|-------------------|
| \( \bar{\eta} \)  | \( -1.013673e-04 \) | \( -4.603226e-05 \) |
| MSE    | BIAS    | \( \hat{M} \) | \( \hat{N} \) | MSE    | BIAS    | \( \hat{M} \) | \( \hat{N} \) |
| Fourier estimator | 2.40e-07 | 2.79e-05 | 887 | 1 | 1.70e-07 | 4.67e-06 | 2404 | 2 |
| Estimator \((28)\) | 1.43e-07 | 4.63e-05 | 889 | 1 | 1.49e-07 | 4.76e-06 | 2638 | 1 |

Table 1: Efficiency of the Fourier estimator \( \eta_{n,M,N} \) and the estimator \((28)\) under microstructure effects. \( \bar{\eta} \) represents the average real integrated leverage for each data set.

in the case of the Heston model the selected cutting frequencies \( \hat{M} \) and \( \hat{N} \) are rather small, the variance correction entails a slight increase of the bias, while for the Generalized Heston model the bias remains almost the same. We notice that in both cases the optimal MSE-based \( \hat{M} \) turns out to be much smaller than the Nyquist frequency (i.e \( \hat{M} < < n/2 \)), whereas \( \hat{N} \) is very small, as prescribed by the asymptotic growth conditions. Again, we notice from the table that in all simulations the largest part of the MSE consists of the variance of the estimator, while the remaining part is negligible. Therefore, we can define an alternative feasible procedure to select optimal cut-off frequencies \( \hat{M} \) and \( \hat{N} \) for the variance corrected Fourier estimator \((28)\) as well, which is based on the minimization of the sample variance. With respect to the MSE-based methodology explained above, we then modify the Step 4 accordingly to the minimization of the sample variance. The results are displayed in Table 2. We notice that the cut-off frequencies selected by the feasible procedure are the same as the ones selected by MSE minimization. The symbol \( \lambda \) in the table denotes the variance reduction ratio \( \text{Var}(\eta_{n,M,N}^*)/\text{Var}(\tilde{\eta}_{n,M,N}) \).

As a benchmark for our results, we consider the estimator proposed by [27] based on preaveraging and blocking that allows to deal with microstructure effects present in the data. Here

| \( H - \text{model} \) | \( GH - \text{model} \) |
|-------------------|-------------------|
| \( \bar{\eta} \)  | \( -1.013673e-04 \) | \( -4.603226e-05 \) |
| VAR    | \( \lambda \) | \( \hat{M} \) | \( \hat{N} \) | VAR    | \( \lambda \) | \( \hat{M} \) | \( \hat{N} \) |
| Fourier estimator | 2.43e-07 | 887 | 1 | 1.75e-07 | 2404 | 2 |
| Estimator \((28)\) | 1.44e-07 | 0.59 | 889 | 1 | 1.51e-07 | 0.86 | 2638 | 1 |

Table 2: Efficiency of the Fourier estimator \( \eta_{n,M,N} \) and of the estimator \((28)\). Feasible optimization based on the minimization of the sample variance under microstructure effects. The symbol \( \lambda \) denotes the variance reduction ratio \( \text{Var}(\eta_{n,M,N}^*)/\text{Var}(\tilde{\eta}_{n,M,N}) \). \( \bar{\eta} \) represents the average real integrated leverage for each data set.
two nested level of blocks are required: the first one, of size $M$, defines the range of preaveraging and the second one, of size $L$ is used for computing the realized covariance between returns and volatility increments. The assumptions at the basis of the central limit theorem for this estimator make it applicable only in the Heston model and not in the Generalized Heston one, see Remark 3.1. Our choice for the blocking parameters $M$ and $L$ is the following: we let $M$ vary from 2 seconds to 300 seconds (i.e. 5 minute-block size). Then, up to rounding, for each value of $M$ we define $n' = n/M$ and let $L = \lceil \sqrt{n'} \rceil$. The optimal MSE-based estimator is then chosen by direct minimization of the MSE over the range of $M$’s. In their paper, the authors provide a rule to choose the optimal values of $M$ and $L$ that minimize the asymptotic variance. However, the implementation of this rule requires a preliminary estimate of the integrated volatility, of the integrated quarticity and the integrated sixth power of volatility, besides the estimation of the diffusion coefficients in the volatility dynamics. In order to reduce possible sources of estimation errors, we computed the analytical values of these quantities from the model (27) by Riemann integration rule. The results are resumed in Table 3. We notice that the first procedure, which is completely unfeasible, provides a worse estimate than the Fourier methodology both in terms of bias and variance. On the other hand, the second procedure provides a very good estimate in terms of bias while the variance and MSE are nevertheless slightly larger than those obtainable by Fourier approach. This does not come as a surprise, since the estimator proposed by Wang and Mykland contains a bias correction factor while in the Fourier case the unbiasedness is achieved only asymptotically. As a further evidence, in Figure 5 we can see that the estimate proposed by Wang and Mykland is largely dependent on the choice of the block size $M$ and its MSE is increasing with this parameter, while the bias remains rather stable around zero.

### 7 Empirical Analysis

We consider now a case study based on high-frequency transaction data of the S&P500 futures recorded at the Chicago Mercantile Exchange (CME) for the period from 03 January 2007 to 31 December 2008 (502 days). During this period, United States experienced the subprime mortgage crisis, a nationwide financial crisis that contributed to the U.S. recession of December 2007 till June 2009. It was triggered by a large decline in home prices after the collapse of a housing bubble during 2006. This induced a large banking crisis in 2007 and the financial crisis in 2008. In a nine-day period from October 1 to 9, 2008 the S&P500 lost 21.6% of its value. Table 4 describes the main features of our data set.
Figure 5: MSE-based integrated leverage estimate by Wang and Mykland [27] together with its MSE and BIAS as a function of the block size $M$.

| Year | N. trades | Variable          | Mean     | Std. Dev. | Min   | Max    |
|------|-----------|-------------------|----------|-----------|-------|--------|
| 2007 | 566409    | S&P 500 index     | 1484.84  | 44.30     | 1375.00 | 1586.50 |
|      |           | log-return        | 5.00e-6  | 1.81e-2   | 1.16   | 2.33   |
| 2008 | 557982    | S&P 500 index     | 1226.55  | 186.89    | 739.00 | 1480.20 |
|      |           | log-return        | -9.03e-5 | 4.75e-2   | -8.66  | 6.12   |

Table 4: Summary statistics for the sample of the traded CME S&P500 futures for the period from 03 January 2007 to 31 December 2008 (502 days). “Std. Dev.” denotes the sample standard deviation of the variable.

Figure 6 shows the time plot of the log-prices, the log-returns and the absolute daily returns (as a proxy of the daily volatility) for the row transaction data. Large volatility periods accompanied by a strong decline of the price reveal the presence of the leverage effect. High-frequency returns are contaminated by transaction costs, bid-and-ask bounce effects, etc., leading to biases in the variance measures. Figure 7 shows the autocorrelation function for the log-returns. Row data exhibit a strongly significant positive first order autocorrelation and higher order autocorrelations remain significant up to lag 8 in 2007 and up to lag 15 in 2008.

Fig. 8 shows the daily integrated leverage estimated by the Fourier method (16) and by the estimator (28) in 2007 and 2008. The values on the vertical axes in the two plots are different from one year to another due to the different magnitudes of the estimated leverage effects. The year 2008 displays the largest values (both negative and positive) of the leverage effects, coherently with the occurrence of the financial crisis. During 2007 the integrated leverage is rather small and mostly negative. Our finding highlights the presence of persistent positive and negative integrated leverage effect, especially in periods of financial turmoil.

From a visual inspection, we notice that both the Fourier estimator (16) and the corrected estimator (28) catch the same positive and negative spikes of the integrated leverage but the
Figure 6: Time plot of the tick-by-tick log-prices, log-returns and the absolute daily returns for S&P500 futures in the years 2007 (left panels) and 2008 (right panels).

Figure 7: Autocorrelation function for S&P500 futures in the years 2007 (left panel) and 2008 (right panel).
Figure 8: Integrated leverage estimated by the Fourier estimator (16) (blue) and by the estimator (28) (red) in the years 2007 (left panel) and 2008 (right panel).

|                        | 2007          | 2008          |
|------------------------|---------------|---------------|
| **S&P500 futures**     | Estimate      | VAR           | λ   | M   | N   | Estimate | VAR           | λ   | M   | N   |
| Fourier estimator (16) | -8.66e-07     | 1.53e-11      | 363 | 1   |     | -7.25e-06 | 7.55e-09      | 281 | 3   |     |
| Estimator (28)         | -3.06e-07     | 1.31e-11      | 0.85| 313 | 1   | -1.97e-06 | 4.11e-09      | 0.54| 297 | 3   |

Table 5: Fourier estimate \( \tilde{\eta}_{n,M,N} \), estimate (28) and their sample variance. The symbol \( \lambda \) denotes the variance reduction ratio \( Var(\eta^*_n,M,N)/Var(\eta_{n,M,N}) \). The leverage effect measures are averages of daily estimates over the whole year. The optimal cutting frequencies are obtained by feasible optimization based on the minimization of the sample variance. Averaged over the all the year the leverage effect appears to be negative.

variability of the estimator (28) is mitigated. Both estimators seem to capture the leverage effect at an aggregate daily level. In particular, they both exhibit the largest negative spike on October 9 (day 194 in our sample), while the common largest positive spike is on November 20 (day 224), in agreement with the evidence in Figure 2.

When estimating the leverage effect, a larger variability in the estimates can be observed if compared to other quantities such as volatility or quarticity. According to the analysis of Section 6, for both estimators the cutting parameters \( M \) and \( N \) are chosen such to minimize the sample variance over the whole one year sample. Their optimal values are listed in Table 5 along with the sample variance achieved by the Fourier method (16) and by the estimator (28). Due to the presence of microstructure effects, the optimal cut-off frequency \( M \) turns out to be much smaller than the Nyquist frequency (i.e. \( M \ll n/2 = 2460 \)).

We remark that the Fourier estimator makes use of all the \( n \) observed prices, because it reconstructs the signal in the frequency domain and it filters out microstructure effects by a suitable choice of \( M \) and \( N \) instead of reducing the sampling frequency.
8 Conclusions

The Fourier estimator of the integrated leverage effect gives a measure of the asymmetric dynamics between returns and volatilities at an aggregate daily level. It is a measure obtainable without assuming at any stage of the estimation that financial returns or volatilities are stationary processes. The Fourier estimator is asymptotically unbiased but has a diverging variance in the presence of microstructure noise contamination. This behavior of the estimator is analyzed in a simulation study where a variance reduction correction is provided and a feasible strategy for selecting optimal cutting frequencies parameters \( M \) and \( N \) is developed. The paper concludes with an empirical study on the leverage effect patterns estimated using high-frequency data for the S&P 500 futures between January 2007 and December 2008. The latter shows how negative and positive spikes of the integrated leverage effect are especially observed in periods of financial turmoil.

Appendix: Proofs

Proof of Proposition 3.2: Throughout the proof we indicate with \( \phi_n(s) := \sup_{k=0,\ldots,n} \{ t_k : t_k \leq s \} \) and use the following integral notations to express the Fourier coefficients. As example, the Fourier coefficients of the return process become

\[
c_n(s; dp) = \frac{1}{T} \int_0^T e^{-\frac{i}{2\pi} t \phi_n(u)} dp(u).
\]

Along the proof, \( C \) will denote a positive constant, not necessarily the same at different occurrences.

Using the notations above and the product rule to the term \( c_n(s; dp) c_n(l-s; dp) \) appearing in \((15)\), we obtain that the error decomposition

\[
\eta_{n,M,N} - \hat{\eta} = \eta_{n,N}^1 - \hat{\eta} + \eta_{n,M,N}^2
\]

\[
= \frac{T^2}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \frac{1}{T} \int_0^T e^{-\frac{i}{2\pi} l t \phi_n(t)} \sigma^2(t) dt \int_0^T e^{\frac{i}{2\pi} l t \phi_n(u)} dp(u) - \int_0^T \eta(t) dt
\]

\[
+ \frac{T^2}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \left( \frac{1}{T} \int_0^T \int_0^t e^{-\frac{i}{2\pi} l t \phi_n(u)} D_M(\phi_n(t) - \phi_n(u)) dp(u) dp(t) \right)
\]

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where the variable $t \in [0, T)$. By using the Cauchy-Schwartz inequality we have that

$$
\mathbb{E}[\eta_{n,M}^2] \leq \frac{T^2}{2N + 1} \sum_{|l| \leq N} |l|^{2\pi T} \mathbb{E}\left[ \frac{1}{T} \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} D_M(\phi_n(t) - \phi_n(u)) \, du \, dp(t) \right] + \frac{1}{T} \int_0^T e^{-i\frac{2\pi}{T} \phi_n(t)} \int_0^t D_M(\phi_n(t) - \phi_n(u)) \, du \, dp(t) \right|^2 \right] \leq \left[ \mathbb{E}\left[ \frac{1}{T} \int_0^T e^{i\frac{2\pi}{T} \phi_n(u)} \, dp(u) \right] \right]^2.
$$

For each $|l| \leq N$, the $L^2$-norm of the Fourier coefficients is bounded under Assumptions (H1), in fact

$$
\mathbb{E}\left[ \int_0^T e^{i\frac{2\pi}{T} \phi_n(u)} \, dp(u) \right] \leq C.
$$

On the other hand,

$$
\mathbb{E}\left[ \frac{1}{T} \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} D_M(\phi_n(t) - \phi_n(u)) \, du \, dp(t) \right] + \frac{1}{T} \int_0^T e^{-i\frac{2\pi}{T} \phi_n(t)} \int_0^t D_M(\phi_n(t) - \phi_n(u)) \, du \, dp(t) \right|^2 \right] \leq C\mathbb{E}\left[ \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} D_M(\phi_n(t) - \phi_n(u)) \, du \, dp(t) \right]^2 + C\mathbb{E}\left[ \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} \, dp(u) \right] + C\mathbb{E}\left[ \frac{1}{T} \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} D_M(\phi_n(t) - \phi_n(u)) \, du \, dp(t) \right]^2.
$$

The addends (34) and (35) have the same order of magnitude in $L^2$-norm. Thus, we just show just the estimation of the term (34). The latter is less than or equal to

$$
C\mathbb{E}\left[ \frac{1}{T} \int_0^T \int_0^t \left( e^{-i\frac{2\pi}{T} \phi_n(u)} - e^{-i\frac{2\pi}{T} \phi_n(t)} \right) \frac{1}{2M + 1} \sum_{|s| \leq M} e^{-i\frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} \, du \, dp(t) \right]^2
$$

$$
+ C\mathbb{E}\left[ \frac{1}{T} \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} \frac{1}{2M + 1} \sum_{|s| \leq M} \left( e^{-i\frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} - e^{-i\frac{2\pi}{T} s(t - u)} \right) \, dp(u) \, dp(t) \right]^2
$$

$$
+ C\mathbb{E}\left[ \frac{1}{T} \int_0^T \int_0^t e^{-i\frac{2\pi}{T} \phi_n(u)} D_M(t - u) \, dp(u) \, dp(t) \right]^2.
$$
The term \((T_1)\) is less than or equal to
\[
C\mathbb{E}\left[ \int_0^T \left| \int_0^t \left( e^{-\frac{2\pi}{T} \phi_n(u) - \frac{2\pi}{T} t u} \right) \frac{1}{2M+1} \sum_{|s| \leq M} e^{-\frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} dp(u) \right|^2 \sigma^2(t) dt \right]
\]
\[
+ C\mathbb{E}\left[ \int_0^T \left( \int_0^t \left( e^{-\frac{2\pi}{T} \phi_n(u) - \frac{2\pi}{T} t u} \right) \frac{1}{2M+1} \sum_{|s| \leq M} e^{-\frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} dp(u) \right) \left( \int_0^z \left( e^{i\frac{2\pi}{T} \phi_n(v)} - e^{i\frac{2\pi}{T} t u} \right) \frac{1}{2M+1} \sum_{|s| \leq M} e^{i\frac{2\pi}{T} s(\phi_n(z) - \phi_n(v))} dp(v) \right) a(z) a(t) dz dt \right],
\]
after applying the Itô isometry,
\[
\leq C\mathbb{E}\left[ \int_0^T \left( \| \frac{2\pi}{T} \phi_n(t) - t \| + t^2 \frac{4\pi^2}{T^2} \| \phi_n(t) - t \|^2 \right)^2 du dt \right]
\]
\[
+ C\mathbb{E}\left[ \int_0^T \left( \frac{2\pi}{T} \phi_n(t) - t \right) \| \phi_n(t) - t \|^2 \right) \left( \frac{2\pi}{T} \phi_n(s) - s \right) + t^2 \frac{4\pi^2}{T^2} \| \phi_n(s) - s \|^2 \right) dv ds dt \right]
\]
\[
+ C(N^2 \tau(n) + o(1))\mathbb{E}\left[ \int_0^T \left( \int_0^t \frac{1}{2M+1} \sum_{|s| \leq M} e^{-\frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} dp(u) \right) \left( \int_0^z \frac{1}{2M+1} \sum_{|s| \leq M} e^{i\frac{2\pi}{T} s(\phi_n(z) - \phi_n(v))} dp(v) \right) a(z) a(t) dz dt \right]
\]
by applying Taylor’s formula, Assumption (H1) and the Hölder and Cauchy-Schwarz inequalities. Note that the first two addends of the decomposition below correspond to the estimation of the term \((T_{11})\). The estimation of the term \((T_{12})\) can be found in the third addend. The expectation appearing in this addend is finite by using again Assumption (H1), the Cauchy- Schwarz and the Hölder inequality. We can then conclude
\[
\mathbb{E}[(T_1)] \leq CN^2 \tau^2(n) + o(1).
\]
The estimation of the order of magnitude of the term \((T_2)\) in \(L_2\)-norm follows the same strategy and leads to
\[
\mathbb{E}[(T_2)] \leq CM^2 \tau^2(n) + o(1).
\]
It remains to analyze the term \((T_3)\). We use in this instance Proposition 4.1. Hence, by using Itô isometry
\[
\mathbb{E} \left[ \left| \frac{1}{T} \int_0^T \int_0^t e^{-\frac{2\pi}{T} t u} D_M(t - u) dp(u) dp(t) \right|^2 \right]
\]
\[
\leq C\mathbb{E} \left[ \int_0^T \left| \int_0^t e^{-\frac{2\pi}{T} t u} D_M(t - u) dp(u) \right|^2 \sigma^2(t) dt \right]
\]
\[ + \left[ \int_{0}^{T} \left( \int_{0}^{T} e^{-ixt} D_{M}(t-u) \, dp(u) \int_{0}^{z} e^{ix t} \, \sigma^{2}(z) \, dv \right) \, a(t) \, \sigma(z) \, dt \, dz \right]. \]

Using the Hölder inequality, the Cauchy-Schwarz inequality and for \( p' \in (1, 2) \) the term above is less than or equal to

\[ \leq C \mathbb{E} \left[ \int_{0}^{T} \left( \int_{0}^{T} D_{M}^{2}(t-u) \sigma^{2}(u) \, du \, dt \right) \right] + C \mathbb{E} \left[ \int_{0}^{T} \left( \int_{0}^{T} D_{M}(t-u) \, du \right)^{\frac{2}{p'}} \, dt \right] \]

\[ + C \mathbb{E} \left[ \int_{0}^{T} \left( \int_{0}^{T} D_{M}^{2}(t-u) \, du \right) \, dt \, dz \right] \]

By using the properties of the Dirichlet kernel, it holds

\[ \mathbb{E}[(T_{3})] \leq \frac{C}{2M+1} + \frac{C}{(2M+1)^{\frac{2}{p}}} + \frac{C}{(2M+1)^{\frac{2+2\beta}{2}}} \]

Thus

\[ E[|\eta_{n,M,N}|] \leq C \frac{N}{\sqrt{M+1}} + CN^{2}\tau(n) + CNM\tau(n) + o(1). \]

We now show that the term \( [32] \) converges to zero in probability

\[ \mathbb{E}[|\eta_{n,N} - \bar{\eta}|] \]

\[ = \mathbb{E} \left[ \frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{\frac{2\pi}{T}(t_{i}-t_{j})} \int_{t_{i}}^{t_{i+1}} \sigma^{2}(t) \, dt \int_{t_{j}}^{t_{j+1}} \, dp(u) - \int_{0}^{T} \eta(t) \, dt \right] \]

\[ \leq \mathbb{E} \left[ \frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \int_{0}^{T} \int_{0}^{T} (e^{\frac{2\pi}{T}(\phi_{n}(t)-\phi_{n}(u))) - e^{\frac{2\pi}{T}(t-u)}) \sigma^{2}(u) \, du \, a(t) \, dt \right] \]

\[ + \mathbb{E} \left[ \frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \int_{0}^{T} \int_{0}^{T} (e^{\frac{2\pi}{T}(\phi_{n}(t)-\phi_{n}(u))) - e^{\frac{2\pi}{T}(t-u)}) \sigma^{2}(u) \, du \, \sigma(t) \, dW(t) \right] \]

\[ + \mathbb{E} \left[ \frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \int_{0}^{T} e^{-ixt} \sigma^{2}(u) \, du \int_{0}^{T} e^{\frac{2\pi}{T}t} \, dp(t) - \int_{0}^{T} \eta(t) \, dt \right]. \]
By using Taylor’s formula, the term (36) is less than or equal to

\[ C \frac{1}{2N+1} \sum_{|l| \leq N} \left| l \right| \frac{2\pi}{T} \mathbb{E} \left[ \int_0^T \int_0^T e^{-\frac{2\pi}{T} (t-u)} \left( \frac{2\pi}{T} \right) \left| l \right| \phi_n(t) - \phi_n(u) + u \right] \]

\[ + l^2 \frac{4\pi^2}{T^2} o(|\phi_n(t) - \phi_n(u) + u|^2) dudt \]

\[ \leq CN^2\tau(n) + o(1). \]

The term (37) is also less than or equal to \( CN^2\tau(n) \) by proceeding similarly. Let us analyse the term (38). By using formula (9)

\[ \mathbb{E} \left[ \left| \frac{1}{2N+1} \sum_{|l| \leq N} il^2 \int_0^T e^{-i\frac{2\pi}{T} tu} \sigma^2(u) \, du \int_0^T e^{i\frac{2\pi}{T} t^2} dp(t) - \int_0^T \eta(t) \, dt \right| \right] \]

\[ = \mathbb{E} \left[ \left| \frac{T^2}{2N+1} \sum_{|l| \leq N} il \int_0^T e^{-i\frac{2\pi}{T} l \sigma^2(u) \, du \int_0^T e^{i\frac{2\pi}{T} t^2} dp(t) - \int_0^T \eta(t) \, dt \right| \right] \]

\[ = \mathbb{E} \left[ \left| \frac{1}{2N+1} \sum_{|l| \leq N} \left( c(l; \sigma^2) - \frac{1}{T} \int_0^T \sigma^2(u) \right) c(-l; dp) - \int_0^T \eta(t) \, dt \right| \right]. \]

We use now the product rule and obtain

\[ \mathbb{E} \left[ \left| \int_0^t \int_0^0 D_N(t-u) dp(u) d\sigma^2(t) + \int_0^t \int_0^t D_N(t-u) d\sigma^2(u) dp(t) \right| \right] \]

\[ - \int_0^t \int_0^t D_N(u) dp(u) d\sigma^2(t) \right] - \left[ \int_0^t \int_0^t D_N(t) d\sigma^2(u) dp(t) - \int_0^t D_N(u) \eta(u) du \right]. \]

Let us analyze the first double integral \( M_{1,N}(T) \)

\[ \mathbb{E} \left[ \left| \int_0^t \int_0^0 D_N(t-u) dp(u) d\sigma^2(s) \right| \right] = \mathbb{E} \left[ \left| \int_0^t \int_0^0 D_N(t-u) \sigma(u) dW(u) \gamma(s) dZ(s) \right| \right] \]

\[ + \int_0^t \int_0^t D_N(t-u) \sigma(u) dW(u) b(s) ds + \int_0^t \int_0^t D_N(t-u) a(u) dW(u) \gamma(s) dZ(s) \]

\[ + \int_0^t \int_0^t D_N(t-u) a(u) dW(s) ds \]

The first two summands of the decomposition above have a \( L_1 \)-norm respectively of order \( O(N^{-\frac{1}{2}}) \) and \( O(N^{-\frac{2+\beta}{2\beta}}) \) and the third and the fourth ones are of order \( O(N^{-\frac{3}{2}}) \), where \( p \in (1, 2) \).

These estimations are performed by means of the use of Proposition 4.1 the Hölder inequality.
and the duality property for the stochastic integrals, see \cite{29} Formula 1.42, see \cite{17} Section 4.3 for more details regarding the calculations. The latter can be applied because Assumption (H2) holds. The $L_1$-norm of the summands $M_{1,N}(T)$, $M_{2,N}(T)$, $M_{3,N}(T)$, $M_{4,N}(T)$ has evidently the same order of magnitude.

By means of Proposition 4.1 we have

$$
\mathbb{E}[|M_{5,N}(2\pi)|] \leq C \mathbb{E}\left[\sup_{t \in [0,T]} |\eta(t)| \right] \sqrt{N} \left( \int_0^T |D_N(u)|^p \, du \right)^{\frac{1}{p}} \leq \frac{C}{N^{\frac{2-p}{2p}}}.
$$

Choosing $p \in (1, 2)$ we obtain that the term $M_{5,N}(2\pi)$ converges to zero in $L_1$-norm as $N \to \infty$. Thus,

$$
\mathbb{E}\left[ \left| \sqrt{N} \frac{4\pi^2}{2N+1} \sum_{|l| \leq N} ic(l; \nu)c(-l; dp) - \int_0^{2\pi} \eta(t) \, dt \right| \right] \leq \frac{C}{\sqrt{N}} + \frac{C}{N^{\frac{2-p}{2p}}} + \frac{C}{N^{\frac{2-p}{2p}}}
$$

Therefore, under the asymptotic ratios (17), the estimator $\eta_{n,M,N}$ is consistent. \hfill \blackblacksquare

**Proof of Theorem 5.1.** We first analyze the Bias due to the noise components.

Because of Assumption (H3), it holds

$$
\mathbb{E}[\epsilon_i \epsilon_j \epsilon_j] = 0 \text{ if } i \neq j \neq k
$$

and,

$$
\mathbb{E}[\eta_{n,M,N}] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} D_N'(t_i - t_j) \mathbb{E}[\epsilon_i^2 \epsilon_j] + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} D_M(t_i - t_j) D_N'(t_i - t_j) \mathbb{E}[\epsilon_i^2 \epsilon_j]
$$

$$
= \sum_{i=0}^{n-2} D_N'(t_{i+1} - t_i) \mathbb{E}[\epsilon_i^2 \epsilon_{i+1}] + \sum_{i=1}^{n-1} D_N'(t_{i-1} - t_i) \mathbb{E}[\epsilon_i^2 \epsilon_{i-1}]
$$

$$
+ \sum_{i=0}^{n-2} D_M(t_{i+1} - t_i) D_N'(t_{i+1} - t_i) \mathbb{E}[\epsilon_i^2 \epsilon_{i+1}] + \sum_{i=1}^{n-1} D_M(t_{i-1} - t_i) D_N'(t_{i-1} - t_i) \mathbb{E}[\epsilon_i^2 \epsilon_{i-1}]
$$

$$
= (n-1) D_N'\left(\frac{T}{n}\right) \mathbb{E}[\epsilon_i^2 \epsilon_{i+1}] + (n-1) D_N'\left(-\frac{T}{n}\right) \mathbb{E}[\epsilon_i^2 \epsilon_{i-1}]
$$

$$
+ (n-1) D_M\left(\frac{T}{n}\right) D_N'\left(\frac{T}{n}\right) \mathbb{E}[\epsilon_i^2 \epsilon_{i+1}] + D_M\left(\frac{T}{n}\right) D_N'\left(-\frac{T}{n}\right) \mathbb{E}[\epsilon_i^2 \epsilon_{i-1}]
$$

\[ = 2(n - 1)D'_{N}(\frac{T}{n})(D_{M}(\frac{T}{n}) - 1)\mathbb{E}[\zeta^3]. \]

By using Taylor formula, \( D'_{N}(\frac{T}{n}) \sim O\left( \frac{N^2}{n^2} \right) \) and \( D_{M}(\frac{T}{n}) \sim 1 - O\left( \frac{M^2}{n^2} \right) \), then, under the asymptotic ratio [24], \( \mathbb{E}[\eta_{n,M,N}] \) converges to zero as \( N, M, n \to \infty \).

The expected value of the term [22]

\[
\sum_{i,j,k: i \neq j \neq k} D_{M}(t_i - t_j) D'_{N}(t_k - t_j) \mathbb{E}[\delta_i \delta_j \delta_k] = 0
\]

because \( \mathbb{E}[\delta_i \delta_j \delta_k] = 0 \).

The expected value of the term involving the components [23]

\[
\mathbb{E}\left[ \frac{1}{2N + 1} \sum_{|t| \leq N} il \frac{2\pi}{T} \frac{1}{2M + 1} \sum_{|s| \leq M} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{T} ((s-t_j)) \delta_i \delta_j} \int_{t_i}^{t_{i+1}} \sigma^2(u) du \right]
\]

\[
(A_1)
\]

\[
+ \mathbb{E}\left[ \frac{1}{2N + 1} \sum_{|t| \leq N} il \frac{2\pi}{T} \frac{1}{2M + 1} \sum_{|s| \leq M} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{T} ((s-t_j)) \delta_i \delta_j} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} dp(u) dp(t) \right]
\]

\[
(A_2)
\]

\[
+ \mathbb{E}\left[ \frac{1}{2N + 1} \sum_{|t| \leq N} il \frac{2\pi}{T} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{T} ((t_i-t_j)) \delta_i \delta_j} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} dp(u) dp(t) \right]
\]

\[
(A_3)
\]

\[
+ \mathbb{E}\left[ \frac{1}{2N + 1} \sum_{|t| \leq N} il \frac{2\pi}{T} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{T} ((t_i-t_j)) \delta_i \delta_j} \int_{t_j}^{t_{j+1}} \sigma^2(u) du \int_{t_i}^{t_{i+1}} dp(t) - \int_{0}^{T} \eta(t) dt \right].
\]

The term \( (A_1) \) can be further decomposed in

\[
\mathbb{E}\left[ \frac{1}{2N + 1} \sum_{|t| \leq N} il \frac{2\pi}{T} \frac{1}{2M + 1} \sum_{|s| \leq M} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{T} ((s-t_j)) \delta_i \delta_j} \int_{t_i}^{t_{i+1}} \sigma^2(u) du \int_{t_j}^{t_{j+1}} dp(t) \right]
\]

\[
(A_7)
\]
+\mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} i l \frac{2\pi}{T} \sum_{|s| \leq M} \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} e^{i \frac{2\pi}{T} (l-s)(t_i-t_j)} \int_{t_i}^{t_{i+1}} \sigma^2(u) \, du \int_{t_j}^{t_{j+1}} dp(t) \right].

The second summand is zero by using the tower property with respect to the sigma-algebra \( \mathcal{F}_{i+1} \) and the martingale property of the Itô integrals. Therefore the term \( \left| (A_1^*) \right| = \Gamma(n, M, N) \).

\[
\Gamma(n, M, N) = \mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} i l \frac{2\pi}{T} \int_0^T \int_0^t e^{i \frac{2\pi}{T} l(\phi_n(t) - \phi_n(u))} \frac{1}{2M+1} \right.
\sum_{|s| \leq M} e^{-i \frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} \, dp(u) \sigma^2(t) \, dt]
\]

\[
= \mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} i l \frac{2\pi}{T} \int_0^T \int_0^t e^{i \frac{2\pi}{T} l(\phi_n(t) - \phi_n(u))} \frac{1}{2M+1} \right.
\sum_{|s| \leq M} \left( e^{-i \frac{2\pi}{T} s(\phi_n(t) - \phi_n(u))} - e^{-i \frac{2\pi}{T} s(t-u)} \right) \, dp(u) \sigma^2(t) \, dt]
\]

\[
+ \mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} i l \frac{2\pi}{T} \int_0^T \int_0^t \left( e^{-i \frac{2\pi}{T} l(\phi_n(t) - \phi_n(u))} - e^{-i \frac{2\pi}{T} l(t-u)} \right) \frac{1}{2M+1} \sum_{|s| \leq M} e^{-i \frac{2\pi}{T} s(t-u)} \, dp(u) \sigma^2(t) \, dt]
\]

\[
+ \mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} i l \frac{2\pi}{T} \int_0^T \int_0^t e^{-i \frac{2\pi}{T} l(t-u)} D_M(t-u) \, dp(u) \sigma^2(t) \, dt \right].
\]

The third summand is less than or equal to

\[
\mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} |l| \frac{2\pi}{T} \int_0^T \int_0^t e^{-i \frac{2\pi}{T} l(t-u)} D_M(t-u) \, dp(u) \sigma^2(t) \, dt \right]
\]

\[
\leq \frac{1}{2N+1} \sum_{|l| \leq N} |l| \frac{2\pi}{T} \mathbb{E}\left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}} T \mathbb{E}\left[ \int_0^T D_M(u) \, du \right]^{\frac{3}{2}} \leq 2\pi N \mathbb{E}\left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}} \left( \frac{T}{2M+1} \right)^{\frac{1}{2}}
\]

by using the Cauchy Schwartz and Hölder inequality, the Itô isometry and the properties of the rescaled Dirichlet kernel.

By means of the Taylor’s formula, we obtain estimations for the first and second summand of \( \Gamma(n, M, N) \) as follows

\[
\left| \mathbb{E}\left[\frac{1}{2N+1} \sum_{|l| \leq N} i l \frac{2\pi}{T} \int_0^T \int_0^t e^{i \frac{2\pi}{T} l(\phi_n(t) - \phi_n(u))} \frac{1}{2M+1} \right. \right. \]

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\[
\begin{align*}
\sum_{|s|\leq M} & \left( e^{-i\frac{2\pi}{T}s} (s\phi_{n(t)} - s\phi_{n(u)}) - e^{-i\frac{2\pi}{T}s} (s(t-u)) \right) dp(u) \sigma^2(t) dt \\
& \leq \mathbb{E} \left[ \left| \frac{1}{2N+1} \sum_{|l|\leq N} \frac{il}{T} \int_0^T \int_0^t e^{i\frac{2\pi}{T}(l) (s\phi_{n(t)} - s\phi_{n(u)})} \frac{1}{2M+1} \right| \\
& \sum_{|s|\leq M} e^{-i\frac{2\pi}{T}s} (s(t-u)) \left( \frac{2\pi}{T} (t-u - \phi_{n(t)} + \phi_{n(u)}) + o(1) \right) dp(u) \sigma^2(t) dt \right] \\
& \leq \frac{1}{2N+1} \sum_{|l|\leq N} \left| \frac{2\pi}{T} \frac{1}{2M+1} \sum_{|s|\leq M} \mathbb{E} \left[ \sup_{[0,T]} \sigma^2(t) \right] \right| \\
\int_0^T & \mathbb{E} \left[ \left| \int_0^t e^{i\frac{2\pi}{T}(l) (s\phi_{n(t)} - s\phi_{n(u)})} e^{-i\frac{2\pi}{T}s} (s(t-u)) \left( \frac{2\pi}{T} (t-u - \phi_{n(t)} + \phi_{n(u)}) + o(1) \right) dp(u) \right| dt \\
& \leq \frac{1}{2N+1} \sum_{|l|\leq N} \left| \frac{2\pi}{T} \frac{1}{2M+1} \sum_{|s|\leq M} \mathbb{E} \left[ \sup_{[0,T]} \sigma^2(t) \right] \right| \int_0^t \left( \int_0^t s \frac{4\pi^2}{n^2} + o(1) du \right)^{\frac{1}{2}} dt \\
& \leq \frac{MN}{n} 8\pi^2 T^{\frac{3}{2}} \mathbb{E} \left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}} + o(1),
\end{align*}
\]
and,
\[
\begin{align*}
\left| \mathbb{E} \left[ \frac{1}{2N+1} \sum_{|l|\leq N} \frac{il}{T} \int_0^T \int_0^t (e^{-i\frac{2\pi}{T}s} (s\phi_{n(t)} - s\phi_{n(u)}) - e^{-i\frac{2\pi}{T}s} (s(t-u)) \right) \\
& \frac{1}{2M+1} \sum_{|s|\leq M} e^{-i\frac{2\pi}{T}s} (s(t-u)) dp(u) \sigma^2(t) dt \right] \right| \\
& \leq \frac{1}{2N+1} \sum_{|l|\leq N} \left| \frac{2\pi}{T} \frac{1}{2M+1} \sum_{|s|\leq M} \mathbb{E} \left[ \sup_{[0,T]} \sigma^2(t) \right] \right| \int_0^t \left( \int_0^t \left( \frac{4\pi^2}{n^2} (\phi_{n(t)} - \phi_{n(u)} - t+u)^2 + o(1) \right) du \right)^{\frac{1}{2}} dt \\
& \leq \frac{N^2}{n} 8\pi^2 T^{\frac{3}{2}} \mathbb{E} \left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}} + o(1).
\end{align*}
\]

Let us now further decompose the term \((A_4)\) as
\[(A_4) = \mathbb{E} \left[ \frac{1}{2N+1} \sum_{|l|\leq N} \frac{2\pi}{T} \int_0^T e^{-it\frac{2\pi}{T}\phi_{n(u)}} \sigma^2(u) du \right] \left( \int_0^T e^{it\frac{2\pi}{T}\phi_{n(t)}} - e^{it\frac{2\pi}{T}\phi_{n(t)}} dp(t) \right)\]
\[(A_{4,1})\]
\[+ E\left(\int_0^T e^{it_2T} \phi_n(t) \, dp(t) \left( \int_0^T e^{-it_2T} \phi_n(t) - e^{-it_2T} \sigma^2(u) \, du \right) \right)\] 

(4.2)

\[+ E\left( \int_0^T e^{it_2T} \phi_n(t) \, dp(t) \int_0^T e^{-it_2T} \sigma^2(u) \, du \right) \]

(4.3)

We call \( \Lambda(n, N) = |(A_2)| + |(A_3)| + |(A_{4.1})| + |(A_{4.2})| \).

Let us first discuss the terms \((A_2)\) and \((A_3)\). For \(i \neq j\), the terms

\[\frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \sum_{|s| \leq M} \sum_{i \neq j} e^{i2\pi s(t_2-t_i)} E\left[ \int_{t_i}^{t_{i+1}} dp(u) \int_{t_i}^{t_{i+1}} dp(t) \int_{t_j}^{t_{j+1}} dp(t) \right]\]

are zero because the Itô integrals appearing in the sums are defined on non overlapping intervals.

For \(i = j\), let us evaluate the terms \(|(A_2)|\) and \(|(A_3)|\). In this instance, \((A_2)\) and \((A_3)\) are both equal and

\[\left| E\left( \int_0^T e^{it_2T} \phi_n(t) \, dp(t) \left( \int_0^T e^{-it_2T} \phi_n(t) - e^{-it_2T} \sigma^2(u) \, du \right) \right) \right|\]

\[\leq \frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \sum_{i=0}^{n-1} E\left[ \int_{t_i}^{t_{i+1}} dp(u) \sigma^2(t) \, dt \right]\]

\[\leq \frac{1}{2N+1} \sum_{|l| \leq N} \frac{2\pi}{T} \sum_{i=0}^{n-1} E\left[ \left| \int_{t_i}^{t_{i+1}} dp(u) \sigma^2(t) \, dt \right| \right]\]

\[\leq N \frac{2\pi}{T} \sum_{i=0}^{n-1} E\left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}} (t_{i+1} - t_i)^{\frac{3}{2}}\]

\[\leq N \frac{2\pi}{T} \left( \frac{T}{n} \right)^{\frac{3}{2}} n E\left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}} = \frac{N}{\sqrt{n}} 2\pi T^{\frac{1}{2}} E\left[ \sup_{[0,T]} \sigma^2(t) \right]^{\frac{3}{2}},\]

by using the Itô isometry and the Hölder inequality. Moreover, because of the Cauchy-Schwartz inequality, \(|(A_{4.1})|\) is less than or equal to

\[\frac{1}{2N+1} \sum_{|l| \leq N} |l|^2 \pi T E\left[ \left| \int_0^T e^{-it_2T} \phi_n(t) \sigma^2(u) \, du \right| \right] \frac{1}{2} E\left[ \left| \int_0^T e^{it_2T} \phi_n(t) - e^{-it_2T} \phi_n(t) \, dp(t) \right| \right] \]
and, using Taylor’s Formula and the Cauchy-Schwartz inequality,

\[
\leq \frac{1}{2N + 1} \sum_{|l| \leq N} |l| \frac{2\pi}{T} E \left[ \int_0^T \sigma^4(u) \, du \right]^{\frac{1}{2}} \frac{1}{T} \sum_{|l| \leq N} |l| \frac{2\pi}{T} E \left[ \int_0^T \left( \frac{2\pi T}{T} \frac{o(1)}{n} \right)^2 \sigma^2(t) \, dt \right]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2N + 1} \sum_{|l| \leq N} |l| \frac{2\pi}{T} E \left[ \sup_{t \in [0,T]} \sigma^4(t) \right]^{\frac{1}{2}} \frac{1}{T} \sum_{|l| \leq N} |l| \frac{2\pi}{T} E \left[ \sup_{t \in [0,T]} \sigma^2(t) \right]^{\frac{1}{2}} T \frac{N}{n} 2\pi + o(1)
\]

\[
\leq \frac{N^2}{n} 4\pi^2 \frac{1}{T} \left[ \sup_{t \in [0,T]} \sigma^4(t) \right]^{\frac{1}{2}} \left[ \sup_{t \in [0,T]} \sigma^2(t) \right]^{\frac{1}{2}} + o(1).
\]

With the same strategy it can be shown that \(|(A_{4.2})|\) is less than or equal to

\[
\frac{N^2}{n} 4\pi^2 \frac{1}{T} \left[ \sup_{t \in [0,T]} \sigma^4(t) \right]^{\frac{1}{2}} \left[ \sup_{t \in [0,T]} \sigma^2(t) \right]^{\frac{1}{2}} + o(1).
\]

It remains to evaluate the term \(|(A_{4.3})|\) that we call \(\Psi(N)\) By using formula (9) and the Itô formula, \(|(A_{4.3})|\) can be expressed as

\[
\left| E \left[ \int_0^{T} \int_0^t D_N(t-u)dp(u)d\sigma^2(t) + \int_0^T \int_0^t D_N(t-u) d\sigma^2(u)dp(t) \right. \right.
\]

\[
\left. - \int_0^T \int_0^t D_N(u)dp(u)d\sigma^2(t) - \int_0^T \int_0^t D_N(t)d\sigma^2(u)dp(t) - \int_0^T D_N(u)\eta(u)du \right].
\]

Because the Itô integrals have mean zero, the term above is equal to

\[
\left| E \left[ - \int_0^T D_N(u)\eta(u)du \right] \right| \leq E \left[ \int_0^T D_N^2(u)du \right] \frac{T}{\sqrt{2N + 1}} E \left[ \sup_{t \in [0,T]} \eta(t)^2 \right]^{\frac{1}{2}}.
\]

after applying the Cauchy-Schwartz inequality. Then,

\[
\Psi(N) \leq \frac{T}{\sqrt{2N + 1}} E \left[ \sup_{t \in [0,T]} \eta(t)^2 \right]^{\frac{1}{2}}.
\]

Hence, the terms \(\Gamma(n, M, N)\), \(\Lambda(n, N)\) and \(\Omega(N)\) converge to zero under the assumptions \((24)\) and therefore it follows that our estimator is asymptotically unbiased. □

**Proof of Theorem 5.3.** Following the decomposition of the estimator \((16)\) in the presence of microstructure noise contamination, we obtain

\[
E[(\tilde{\eta}_{n,M,N} - \hat{\eta})^2]
\]
\[
\begin{align*}
&= \mathbb{E}\left[ \left( \sum_{i,j,k:\{i,j\neq k\}} D_M(t_i - t_j)D_N'(t_k - t_j)\delta_i\delta_j\delta_k + \sum_{i,j:i\neq j} D_M(t_i - t_j)D_N'(t_i - t_j)\tilde{\delta}_i\tilde{\delta}_j \right. \\
&\left. + \sum_{i,j} D_N'(t_i - t_j)\delta_i^2 \hat{\delta}_j^2 - \tilde{\eta}^2 \right) \right] \\
&= \mathbb{E}\left[ \left( \eta_{n,M,N} - \hat{\eta} \right. \\
&\left. + \left( \sum_{i,j,k:\{i,j\neq k\}} D_M(t_i - t_j)D_N'(t_k - t_j)\left( \delta_i\delta_j\delta_k + \delta_i\delta_k\delta_j + \delta_k\delta_i\delta_j + \delta_k\delta_j\delta_i + \delta_j\delta_k\delta_i + \delta_j\delta_i\delta_k + \delta_i\delta_j\delta_k + \delta_i\delta_k\delta_j + \delta_k\delta_i\delta_j \right) \right) \\
&\left. + \left( \sum_{i,j:i\neq j} D_M(t_i - t_j)D_N'(t_i - t_j)\left( \delta_i\epsilon_j + \epsilon_i\delta_j + \epsilon_j\delta_i \right) \right) + \left( \sum_{i,j} D_N'(t_i - t_j)\left( \delta_i^2\delta_j + \epsilon_j^2\delta_i + \epsilon_i^2\delta_j + 2\delta_i\delta_j\delta_i + 2\delta_i\delta_j\delta_j + 2\delta_j\delta_i\delta_j \right) \right)^2 \right] \\
\end{align*}
\]

Note that, in the above-mentioned mean squared error decomposition, the summand \( \mathbb{E}[\left( \eta_{n,M,N} - \hat{\eta} \right)^2] \) converges to zero as \( n, M, N \) tend to infinity. To show this result, we focus on the term

\[
\mathbb{E}[\left( \eta_{n,M,N} - \hat{\eta} \right)^2]. \tag{40}
\]

In fact, the term \( \left( \eta_{n,M,N} - \hat{\eta} \right) \) has the biggest order of magnitude in \( L_2 \)-norm. Therefore, studying \( \eta_{n,M,N} - \hat{\eta} \) is enough to analyze the behavior of the mean squared error of the estimator \( \eta_{n,M,N} \). Under zero drift assumption, see Remark 4.3, we have that (40) is equal to

\[
\begin{align*}
&= \mathbb{E}\left[ \left( \frac{1}{2N + 1} \sum_{\mid l \mid \leq N} il\frac{2\pi}{T} \int_0^T \int_0^T \left( e^{i\frac{2\pi}{T} l \phi_n(t)} - e^{i\frac{2\pi}{T} l \phi_n(u)} \right) \sigma^2(u) du \sigma(t) dW(t) \\
&+ \int_0^T \int_0^T D_N(t - u) dp(u) d\sigma^2(t) + \int_0^T \int_0^T D_N(t - u) d\sigma^2(t) dp(u) \\
&- \int_0^T \int_0^T D_N(u) dp(u) d\sigma^2(t) - \int_0^T \int_0^T D_N(u) d\sigma^2(t) dp(u) - \int_0^T D_N(u) \eta(u) du \right)^2 \right] \\
\end{align*}
\]

\[
\leq 2\mathbb{E}\left[ \left( \frac{1}{2N + 1} \sum_{\mid l \mid \leq N} il\frac{2\pi}{T} \int_0^T \int_0^T \left( e^{i\frac{2\pi}{T} l \phi_n(t)} - e^{i\frac{2\pi}{T} l \phi_n(u)} \right) \sigma^2(u) du \sigma(t) dW(t) \right)^2 \right] \tag{41}
\]

\[
+ 8\mathbb{E}\left[ \left( \int_0^T \int_0^T D_N(t - u) dp(u) d\sigma^2(t) \right)^2 \right] + 8\mathbb{E}\left[ \left( \int_0^T \int_0^T D_N(t - u) dp(u) d\sigma^2(t) \right)^2 \right] \tag{42}
\]

\[
+ 8\mathbb{E}\left[ \left( \int_0^T D_N(u) \eta(u) du \right)^2 \right] + 8\mathbb{E}\left[ \left( \int_0^T D_N(u) \eta(u) du \right)^2 \right] \tag{43}
\]

35
Under Assumption (H3), we have that (47) is equal to
\[
\leq \frac{128\pi^2 N^4}{T^2 n^2} \mathbb{E}[\sup_{t \in [0,T]} \sigma^2(t)] \mathbb{E}[\sup_{t \in [0,T]} \sigma^4(t)]
\] (44)
\[
+ 16 \frac{T^2}{2N+1} \mathbb{E}[\sup_{t \in [0,T]} \sigma^2(t)] \mathbb{E}[\sup_{t \in [0,T]} \gamma^2(t)]
\] (45)
\[
+ 16 \frac{T}{2N+1} \mathbb{E}[\sup_{t \in [0,T]} \eta(t)^2],
\] (46)
where (44), (45), (46) correspond respectively to the estimation of the summands (41), (42), (43). Thus, (40) converges to zero as \(n, N \to \infty\) and so does the mean squared error of the estimator (16). However, whenever a noise component appears in the decomposition of (39), the related terms diverge to infinity as \(n, M, N\) goes to infinity. As exemplary calculation, we will show that
\[
\mathbb{E}\left[\left(\sum_{i,j} D_N'(t_i - t_j) (\delta_j \epsilon_i + \epsilon_j^2 \delta_i + \epsilon_j^2 \epsilon_i + 2\delta_j \delta_i \epsilon_j + 2\delta_j \epsilon_i \epsilon_j)\right)^2\right]
\] (47)
diverges as \(n, N \to \infty\) and is greater than \(O(n^2N)\). In order to handle the other terms in (39), the strategies of computation below addressed, Taylor’s formula and Proposition 4.1 are used and lead to show that the remaining terms in (39) are greater than \(O(NM^2 + n^2N)\).

We have that
\[
\mathbb{E}\left[\left(\sum_{i,j} D_N'(t_i - t_j) (\delta_j \epsilon_i + \epsilon_j^2 \delta_i + \epsilon_j^2 \epsilon_i + 2\delta_j \delta_i \epsilon_j + 2\delta_j \epsilon_i \epsilon_j)\right)^2\right]
\] (48)
\[
= \sum_{i,j} (D_N'(t_i - t_j))^2 \mathbb{E}[\delta_j \epsilon_i + \epsilon_j^2 \delta_i + \epsilon_j^2 \epsilon_i + 2\delta_j \delta_i \epsilon_j + 2\delta_j \epsilon_i \epsilon_j]^2
\] (49)
Under Assumption (H3), we have that (47) is equal to
\[
\sum_{i,j} (D_N'(t_i - t_j))^2 (\mathbb{E}[(\delta_j^2)\mathbb{E}[(\epsilon_i^2)] + \mathbb{E}[(\epsilon_j^4)\mathbb{E}[(\delta_i^2)] + \mathbb{E}[(\epsilon_j^4)\mathbb{E}[(\epsilon_i^2)] + 4\mathbb{E}[(\delta_j^2)\mathbb{E}[(\delta_i^2)] + 4\mathbb{E}[(\delta_j^2)\mathbb{E}[(\epsilon_i^2)]])
\] (50)
\[
+ \sum_{i,j,j' ; i \neq i', j \neq j'} D_N'(t_i - t_j) D_N'(t_{i'} - t_{j'}) \mathbb{E}[(\epsilon_i^2 \epsilon_{i'} \epsilon_{j'} \epsilon_{j'})].
\] (51)
It holds that
\[ \mathbb{E}[\zeta_1^2] = 2\mathbb{E}[\zeta^2] \]
\[ \mathbb{E}[\epsilon_1^2] = 2\mathbb{E}[\zeta^4] + 6\mathbb{E}[\zeta^2]^2 \]
\[ \mathbb{E}[\epsilon_1^4] = \begin{cases} 
4\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] + 12\mathbb{E}[\zeta^2]^3 & \text{if } |i - j| \neq 1, \\
9\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] + \mathbb{E}[\zeta^6] + 6\mathbb{E}[\zeta^2]^3 - 4\mathbb{E}[\zeta^3]^2 & \text{if } i = j - 1, \\
13\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] + \mathbb{E}[\zeta^6] + 2\mathbb{E}[\zeta^2]^3 - 4\mathbb{E}[\zeta^3]^2 & \text{if } i = j + 1. 
\end{cases} \]
\[ \mathbb{E}[\epsilon_1^2\epsilon_i^2] = \begin{cases} 
4\mathbb{E}[\zeta^2]^2 & \text{if } |i - j| > 1, \\
3\mathbb{E}[\zeta^2]^2 + \mathbb{E}[\zeta^4] & \text{if } i = j - 1, \\
3\mathbb{E}[\zeta^2]^2 + \mathbb{E}[\zeta^4] & \text{if } i = j + 1. 
\end{cases} \]
\[ \mathbb{E}[\epsilon_1^4] = 0 \]
\[ \mathbb{E}[\epsilon_1^2\epsilon_i^2\epsilon_j^2] = \begin{cases} 
0 & \text{if } i \neq i', j \neq j', i \neq j', j \neq i', \\
0 & \text{if } i \neq i', j \neq j', i = j', j = i' \text{ and } |i - j| \neq 1, \\
\mathbb{E}[\zeta^3]^2 - \mathbb{E}[\zeta^6] - 6\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] - 9\mathbb{E}[\zeta^2]^3 & \text{if } i \neq i', j \neq j', i = j', j = i' \text{ and } i = j + 1, \\
\mathbb{E}[\zeta^3]^2 - \mathbb{E}[\zeta^6] - 6\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] - 9\mathbb{E}[\zeta^2]^3 & \text{if } i \neq i', j \neq j', i = j', j = i' \text{ and } i = j - 1. 
\end{cases} \]

Therefore [47] is equal to

\[
\sum_{i,j} (D'_{\mathcal{N}}(t_i - t_j))^2 (2\mathbb{E}[\delta_j^2]\mathbb{E}[\zeta^2] + 2\mathbb{E}[\delta_j^2]\mathbb{E}[\zeta^4] + 6\mathbb{E}[\delta_j^2]\mathbb{E}[\zeta^2]^2 + 8\mathbb{E}[\delta_j^2]\mathbb{E}[\zeta^2]^2][\mathbb{E}[\zeta^2]]) \tag{52} 
\]
\[
+ \sum_{i,j;|i-j|=1} (D'_{\mathcal{N}}(t_i - t_j))^2 16\mathbb{E}[\delta_j^2]\mathbb{E}[\zeta^2]^2 + \sum_{i,j;|i-j|=1} (D'_{\mathcal{N}}(t_i - t_j))^2 \mathbb{E}[\delta_j^2](12\mathbb{E}[\zeta^2]^2 + 4\mathbb{E}[\zeta^4]) \tag{53} 
\]
\[
+ \sum_{i,j;|i-j|\neq1} (D'_{\mathcal{N}}(t_i - t_j))^2 (4\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] + 12\mathbb{E}[\zeta^2]^3) \tag{54} 
\]
\[
+ \sum_{i,j;i=j-1} (D'_{\mathcal{N}}(t_j - t_i - t_j))^2 (9\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] + \mathbb{E}[\zeta^6] + 6\mathbb{E}[\zeta^2]^3 - 4\mathbb{E}[\zeta^3]^2) \tag{55} 
\]
\[
+ \sum_{i,j;i=j+1} (D'_{\mathcal{N}}(t_j - t_i - t_j))^2 (13\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] + \mathbb{E}[\zeta^6] + 2\mathbb{E}[\zeta^2]^3 - 4\mathbb{E}[\zeta^3]^2) \tag{56} 
\]
\[
+ \sum_{i,j;i=j-1} D'_{\mathcal{N}}(t_j - t_i - t_j)D'_{\mathcal{N}}(t_j - t_i - t_j - 1)(\mathbb{E}[\zeta^3]^2 - \mathbb{E}[\zeta^6] - 6\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] - 9\mathbb{E}[\zeta^2]^3) \tag{57} 
\]
\[
+ \sum_{i,j;i=j+1} D'_{\mathcal{N}}(t_j - t_i - t_j)D'_{\mathcal{N}}(t_j - t_i - t_j + 1)(\mathbb{E}[\zeta^3]^2 - \mathbb{E}[\zeta^6] - 6\mathbb{E}[\zeta^4]\mathbb{E}[\zeta^2] - 9\mathbb{E}[\zeta^2]^3) \tag{58} 
\]

Computing the summands from (55) to (58), we obtain
Table 6: Summing rule: the terms has to be first summed according to the indices present in each row, then the resulting addends has to be summed with respect to the indices grouped by color.

\[
(n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} + \frac{1}{(2N+1)^2} \sum_{l\neq l'} l'^2 \frac{4\pi^2}{T^2} e^{-i\frac{2\pi}{n}(l-l')}
\times (9E[\zeta^4][E[\zeta^2] + E[\zeta^6] + 6E[\zeta^2]^3 - 4E[\zeta^3]^2])
\] (59)

\[
+ (n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} e^{-i\frac{2\pi}{n}l} + \frac{1}{(2N+1)^2} \sum_{l\neq l'} l'^2 \frac{4\pi^2}{T^2} e^{-i\frac{2\pi}{n}(l-l')}
\times (13E[\zeta^4]E[\zeta^2] + E[\zeta^6] + 2E[\zeta^2]^3 - 4E[\zeta^3]^2)
\] (60)

\[
+ (n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} e^{-i\frac{4\pi}{n}l} + \frac{1}{(2N+1)^2} \sum_{l\neq l'} l'^2 \frac{4\pi^2}{T^2} e^{-i\frac{4\pi}{n}(l-l')}
\times (E[\zeta^3]^2 - E[\zeta^6] - 6E[\zeta^4]E[\zeta^2] - 9E[\zeta^2]^3)
\] (61)

\[
+ (n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} e^{-i\frac{4\pi}{n}l} + \frac{1}{(2N+1)^2} \sum_{l\neq l'} l'^2 \frac{4\pi^2}{T^2} e^{-i\frac{4\pi}{n}(l-l')}
\times (E[\zeta^3]^2 - E[\zeta^6] - 6E[\zeta^4]E[\zeta^2] - 9E[\zeta^2]^3)
\] (62)

The terms where the indices \(l\) and \(l'\) appear in (59) and (61) has to be summed up following the rule highlighted in Table 6. The same applies for the terms where the indices \(l\) and \(l'\) appear in (60) and (62).

The summands from (55) to (58) are then equal to

\[
(n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} (22E[\zeta^4][E[\zeta^2] + 2E[\zeta^6] + 8E[\zeta^2]^3 - 8E[\zeta^3]^2])
\] (63)

\[
+ (n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} 2\cos(\frac{4\pi}{n}l)(E[\zeta^3]^2 - E[\zeta^6] - 6E[\zeta^4]E[\zeta^2] - 9E[\zeta^2]^3)
\] (64)

\[
+ (n-1)\left(\frac{1}{2N+1}\right)^2 \sum_{l,l'\geq 0} l' \frac{4\pi^2}{T^2} \sin(\frac{2\pi}{n}l) \sin(\frac{2\pi}{n}l')(34E[\zeta^4]E[\zeta^2] + 4E[\zeta^6] + 26E[\zeta^2]^3 - 5E[\zeta^3]^2).
\] (65)

If \(N = n^{\frac{1}{3}}\) with \(\beta > \frac{\log(n)}{\log(n) - \log(8)}\) then \(0 \leq \cos(\frac{4\pi}{n}l) \leq 1\) for all \(|l| \leq N\). If \(\beta > \frac{\log(n)}{\log(n) - \log(2)}\) then
0 ≤ \sin(\frac{2π}{n}l) ≤ 1. We have that the choice of \( \beta > \frac{\log(n)}{\log(n) - \log(8)} \) is implied by the ratio in (26). In conclusion, the term (65) is greater than or equal to zero and the sum between (63) and (64) is greater than or equal to

\[
(n - 1) \frac{1}{(2N + 1)^2} \sum_{|l| < N} l^2 4\pi^2 \frac{T^2}{T^2} (2E[\zeta^4]E[\zeta^2] + 2E[\zeta^6] + 8E[\zeta^2]^3 - 8E[\zeta^3]^2),
\]

(66)

if (E[\zeta^2] - E[\zeta^6] - 6E[\zeta^4]E[\zeta^2] - 9E[\zeta^3]^2) > 0 and greater than or equal to

\[
(n - 1) \frac{1}{(2N + 1)^2} \sum_{|l| < N} l^2 4\pi^2 \frac{T^2}{T^2} (10E[\zeta^4]E[\zeta^2] - 10E[\zeta^2]^3 - 6E[\zeta^3]^2),
\]

(67)

if (E[\zeta^3] - E[\zeta^6] - 6E[\zeta^4]E[\zeta^2] - 9E[\zeta^3]^2) < 0.

Let us now analyze the summands from (52) to (54). They are equal to

\[
\sum_{i,j} \left( \frac{1}{(2N + 1)^2} \sum_{|l| < N} l^2 4\pi^2 \frac{T^2}{T^2} + \frac{1}{(2N + 1)^2} \sum_{l \neq l'} l^4 4\pi^2 \frac{T^2}{T^2} e^{-i\frac{2\pi}{T^2}(l'-l)(t_i-t_j)} \right) \\
\times (2E[\delta_i^2]E[\zeta^2] + 2E[\delta_i^2]E[\zeta^4] + 6E[\delta_i^2]E[\zeta^2]^2 + 8E[\delta_i^2]E[\zeta^3]^2) + \sum_{i,j:|i-j|=1} \left( \frac{1}{(2N + 1)^2} \sum_{|l| < N} l^2 4\pi^2 \frac{T^2}{T^2} + \frac{1}{(2N + 1)^2} \sum_{l \neq l'} l^4 4\pi^2 \frac{T^2}{T^2} e^{-i\frac{2\pi}{T^2}(l'-l)(t_i-t_j)} \right) \cdot 16E[\delta_i^2]^{2}E[\zeta^2]^2
\]

(68)

\[
+ \sum_{i,j:|i-j|=1} \left( \frac{1}{(2N + 1)^2} \sum_{|l| < N} l^2 4\pi^2 \frac{T^2}{T^2} + \frac{1}{(2N + 1)^2} \sum_{l \neq l'} l^4 4\pi^2 \frac{T^2}{T^2} e^{-i\frac{2\pi}{T^2}(l'-l)(t_i-t_j)} \right) \cdot (12E[\zeta^2]^{2} + 4E[\zeta^4])
\]

(69)

\[
+ \sum_{i,j:|i-j|\neq 1} \left( \frac{1}{(2N + 1)^2} \sum_{|l| < N} l^2 4\pi^2 \frac{T^2}{T^2} + \frac{1}{(2N + 1)^2} \sum_{l \neq l'} l^4 4\pi^2 \frac{T^2}{T^2} e^{-i\frac{2\pi}{T^2}(l'-l)(t_i-t_j)} \right) \cdot (4E[\zeta^4]E[\zeta^2] + 12E[\zeta^2]^3)
\]

(70)

We first take care of the sum with respect to the indices \( i, j, l \) and \( l' \) appearing in the terms (68) to (71). To simply explain the calculations below, let us consider from now on that the indices \( l \) and \( l' \) are positive and that there exists an \( s = 1, \ldots, n - 1 \) such that if \( t_i > t_j \), \( t_i - t_j = s\frac{2\pi}{n} \). We do not consider \( s = 0 \) in the calculations below because \( D^l_N(t_i - t_i) = 0 \).
In Table 7, we find, for fixed values of \(s, l, l', s\), the summands (73), (75), (77) and (79) are greater than or equal to zero. We respectively obtained for \(s\) for \(l, l'\), the same number of times. We then obtain Table 8. Summing up the blue and red terms respectively obtained for \(s\) and \(-s\) we have

\[
\sum_{i,j} \frac{1}{2(N+1)^2} \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} (2E[\delta_j^2]E[\delta^2] + 2E[\delta_j^2]E[\delta^4] + 6E[\delta_j^2]E[\delta^2]^2 + 8E[\delta_j^2\delta_j^4]E[\delta^2])
\]

(72)

\[
\frac{1}{2(N+1)^2} \sum_{|l|<N} l^4 \frac{4\pi^2}{T^2} \sin\left(\frac{2\pi}{n} l s\right) \sin\left(\frac{2\pi}{n} l' s\right)
\]

(73)

\[
+ \sum_{i,j|i-j|=1} \frac{1}{2(N+1)^2} \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} E[\delta_j^2](12E[\delta^2]^2 + 4E[\delta^4])
\]

(76)

\[
+ (n-1) \frac{1}{2(N+1)^2} \sum_{|l'|>0} l^2 \frac{4\pi^2}{T^2} \sin\left(\frac{2\pi}{n} l' s\right) E[\delta_j^2](12E[\delta^2]^2 + 4E[\delta^4])
\]

(77)

\[
+ \sum_{i,j|i-j|\neq 1} \frac{1}{2(N+1)^2} \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} (4E[\delta^4]E[\delta^2] + 12E[\delta^2]^3)
\]

(78)

\[
+ \sum_{i,j|i-j|\neq 1} \frac{1}{2(N+1)^2} \sum_{|l|<N} l^2 \frac{4\pi^2}{T^2} (4E[\delta^4]E[\delta^2] + 12E[\delta^2]^3)
\]

(79)

If \(N = \frac{1}{3}\) such that \(\beta > \log(n)\log(2n^2)\) then \(0 \leq \sin\left(\frac{2\pi}{n} l s\right) \leq 1\) for \(s = 1, \ldots, n-1\) and \(l > 0\). The latter is straightforwardly implied by (66), being \(\frac{\log(n)\log(2n^2)}{\log(n)\log(2n^2)}\) negative. Therefore, the summands (73), (75), (77) and (79) are greater than or equal to zero.
In conclusion, \((47)\) is possibly greater than or equal to two sums. The first one is

\[
\sum_{i,j} \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} \left(2E[\delta_j^2]E[\zeta^2] + 2E[\delta_j^2]E[\zeta^4] + 6E[\delta_j^2]E[\zeta^2]^2 + 8E[\delta_j^2 \delta_j^2]E[\zeta^2]\right)
\]

\[+
\sum_{i,j:|i-j|\neq 1} \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} 16E[\delta_j^2]E[\zeta^2]^2 \]

\[+
\sum_{i,j:|i-j|=1} \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} E[\delta_j^2](12E[\zeta^2]^2 + 4E[\zeta^4]) \tag{80}
\]

\[+
(n - 1) \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} (22E[\zeta^4]E[\zeta^2] + 2E[\zeta^6] + 8E[\zeta^2]^3 - 8E[\zeta^3]^2),
\]

and the second one is

\[
\sum_{i,j} \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} \left(2E[\delta_j^2]E[\zeta^2] + 2E[\delta_j^2]E[\zeta^4] + 6E[\delta_j^2]E[\zeta^2]^2 + 8E[\delta_j^2 \delta_j^2]E[\zeta^2]\right)
\]

\[+
\sum_{i,j:|i-j|\neq 1} \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} 16E[\delta_j^2]E[\zeta^2]^2 \]

\[+
\sum_{i,j:|i-j|=1} \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} E[\delta_j^2](12E[\zeta^2]^2 + 4E[\zeta^4]) \tag{81}
\]

\[+
(n - 1) \frac{1}{(2N + 1)^2} \sum_{|l| < N} t^2 \frac{4\pi^2}{T^2} (10E[\zeta^4]E[\zeta^2] - 10E[\zeta^2]^3 - 6E[\zeta^3]^2).
\]

Note that because of Assumption (H1), the terms \(E[\delta_j^2]\) and \(E[\delta_j^2 \delta_j^2]\) are positive and finite constants. The behavior of the sums \((80)\) and \((81)\) is ruled by their first summands. In fact,

\[
\sum_{i,j} \frac{1}{(2N + 1)^2} \sum_{|l| \leq N} t^2 \frac{4\pi^2}{T^2} = \frac{n^2}{(2N + 1)^2} \frac{4\pi^2}{T^2} \left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right),
\]

which diverges as \(n, N \to \infty\).
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