APPROXIMATE OBLIQUE DUAL FRAMES

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ABSTRACT. In representations using frames, oblique duality appears in situations where the analysis and the synthesis has to be done in different subspaces. In some cases, we cannot obtain an explicit expression for the oblique duals and in others there exists only one oblique dual frame which has not the properties we need. Also, in practice the computations are not exact. To give a solution to these problems, in this work we introduce and investigate the notion of approximate oblique dual frames first in the setting of separable Hilbert spaces. We present several properties and provide different characterizations of approximate oblique dual frames. We focus then on approximate oblique dual frames in shift-invariant subspaces of $L^2(\mathbb{R})$ and give different conditions on the generators that assure their existence. The importance of approximate oblique dual frames from a numerical and computational point of view is illustrated with an example of frame sequences generated by $B$-splines, where the previous results are used to construct approximate oblique dual frames which have better attributes than the exact ones. We provide an expression for the approximation error and study its behaviour.

Key words: Frames, Oblique dual frames, Approximate dual frames, Oblique projections, Shift-invariant spaces.

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1. Introduction

A frame is a sequence of vectors in a separable Hilbert space. Frames generalize bases [2, 5, 6, 9, 16, 23]. The main difference to bases is that the reconstruction of each vector is not necessarily unique, i.e. there exists more than one collection of coefficients to represent each vector as a combination of the elements of a frame. These frame coefficients are associated to other sequences called dual frames.

In practice, the computation of duals is not exact and sometimes it is even not possible to give in theory an analytic expression of it. Another difficulty is that in applications we need to truncate the frame representations in order to work with finite sequences of numbers. Also the frame coefficients can in general only be computed approximately. Approximate dual frames

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appear as an answer to these problems \cite{1, 8, 14, 20}. They are also easier to build and can be adapted to our needs.

For frames in a subspace, the reconstruction can also be done with coefficients that depend on duals that do not necessarily belong to the same subspace. These are known as oblique dual frames \cite{12, 3, 10, 11, 13, 17, 18, 19, 24}. These dual frames are related with oblique projections and are of particular interest in signal processing, where the goal is to recover the signal itself. In this situation we again have the limitations and the computational difficulties described before. Moreover, sometimes the subspaces are given by the problem and there exists a unique oblique dual frame. This unique oblique dual could need to be improved in some aspects. In order to respond to these different problems, in this work we introduce the concept of approximate oblique dual frames and study its properties, first in the setting of separable Hilbert spaces and then in shift-invariant subspaces of $L^2(\mathbb{R})$. Since in general there exists more than one approximate oblique dual, we also gain freedom in its construction.

In Section 2 we give a brief review of existing definitions and results. In Sections 3 to 6, we introduce and investigate the notion of approximate oblique dual frames in the setting of separable Hilbert spaces. In Sections 7 and 8, we focus on approximate oblique dual frames in shift-invariant subspaces of $L^2(\mathbb{R})$.

In Section 3, we introduce the notion of $\epsilon$-approximate oblique dual frames, where $\epsilon > 0$, and investigate some of its properties. We also give its interpretation in the setting of signal processing theory and the conditions of uniqueness and consistency in order to have a good approximation of a signal.

In Section 4, we obtain characterizations of approximate oblique dual frames, in terms of series expansions as well as in terms of operators.

In Section 5, we extend to the oblique setting an important property of approximate dual frames, that is to obtain a reconstruction of the vectors as close as we desire starting from any pair of approximate oblique dual frames. We also show how to construct oblique dual frames from approximate oblique dual frames.

In Section 6, we obtain approximate oblique dual frames from the perturbation of oblique duals. We prove that if two frames are “close”, each oblique dual frame of any of them is an approximate oblique dual frame of the other, expressing its bounds in terms of the original ones.

Shift-invariant subspaces with frames of translates are widely used in the applications, and in particular in image and signal processing. Moreover, shift-invariant spaces play a key role in the construction of wavelets frames and Gabor frames. In Section 7, we consider shift-invariant subspaces of $L^2(\mathbb{R})$ and obtain conditions on their generators to provide an approximate reconstruction in one of them. The technique introduced in the proof of the main result of this section (Theorem 7.2) is the key to the obtainment of the central results of section 8 about approximate oblique dual frames.

We begin Section 8 stating sufficient conditions for approximate oblique duality, that are derived from results of section 7. We then give an expression for the Fourier transform of the
oblique projection when the spaces are shift-invariant. We use it to provide a more manageable sufficient condition, and also a necessary condition on the generators of shift-invariant subspaces for the existence of approximate oblique dual frames.

In Section 9 we describe the necessity of working in the applications with approximate oblique dual frames. The importance of approximate oblique dual frames from a numerical and computational point of view is illustrated with an example. We consider frame sequences of translates generated by $B$-splines. In this case, the generator of the smooth unique oblique dual frame has not compact support whereas the generator of each obtained approximate oblique dual frame is not only smooth but also compactly supported. We give an expression for the approximation error and analyze its behaviour.

2. Preliminaries

We consider $H, K$ separable Hilbert spaces. The space of bounded operators from $H$ to $K$ will be denoted by $L(H, K)$. For $T \in L(H, K)$ denote the image, the null space and the adjoint of $T$ by $R(T), N(T)$ and $T^*$, respectively. If $T$ has closed range we also consider the Moore-Penrose pseudo-inverse of $T$ denoted by $T^\dagger$. The inner product and the norm in $H$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The Lebesgue measure of a measurable subset $A$ of $\mathbb{R}$ will be denoted by $|A|$. We denote the characteristic function of $A$ with $\chi_A$ and the complement of $A$ with $A^c$. We consider the class $C^p_{\text{per}}$ of 1-periodic functions that restricted to $[0, 1)$ belong to $L^p(0, 1)$.

We now review briefly definitions and present properties that we use later. In the sequel $V$ and $W$ will be two closed subspaces of $H$.

2.1. Oblique projections. Let $H = W \oplus V^\perp$. The oblique projection onto $W$ along $V^\perp$, is the unique operator that satisfies

$$\pi_{WV^\perp} f = f \text{ for all } f \in W, \quad \pi_{WV^\perp} f = 0 \text{ for all } f \in V^\perp.$$

Equivalently, $R(\pi_{WV^\perp}) = W$ and $N(\pi_{WV^\perp}) = V^\perp$. Observe that $(\pi_{VW^\perp})^* = \pi_{WV^\perp}$ and $(\pi_{VW^\perp})_V = I_V$. If $V = W$ we obtain the orthogonal projection onto $V$, which we denote by $P_V$. The next result can be deduced from [15, Lemma 2.1] and the definitions of orthogonal and oblique projections:

**Lemma 2.1.** Let $H = W \oplus V^\perp$. The following holds:

(i) $(P_W)_V : V \to W$ and $(P_V)_W : W \to V$ are isomorphisms, $((P_W)_V)^{-1} = (\pi_{WV^\perp})_W$ and $((P_V)_W)^{-1} = (\pi_{WV^\perp})_V$.

(ii) $(P_W)_V^* = (P_V)_W^*$ and $((\pi_{WV^\perp})_V)^* = (\pi_{WV^\perp})_W$.

The concept of angle between two closed subspaces that we will use is the following:

**Definition 2.2.** Given $W$ and $V$ two closed subspaces of $H$, we define the angle from $V$ to $W$ as the unique real number $\theta(V, W) \in [0, \frac{\pi}{2}]$ such that

$$\cos \theta(V, W) = \inf_{v \in V, \|v\| = 1} \|P_Wv\|.$$


Theorem 2.3. [21] Let $\mathcal{V}$, $\mathcal{W}$ be closed subspaces of a separable Hilbert space $\mathcal{H}$. Then the following assertions are equivalent:

(i) $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}$.  

(ii) $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}$.  

(iii) $\cos \theta(\mathcal{V}, \mathcal{W}) > 0$ and $\cos \theta(\mathcal{W}, \mathcal{V}) > 0$.

We have $\|\pi_{\mathcal{W}^\perp}\| = \frac{1}{\cos \theta(\mathcal{V}, \mathcal{W})} \geq 1$ [1] Lemma 2.4.

2.2. Frames. Frames have been introduced by Duffin and Schaeffer in [9]. Using a frame, each element of a Hilbert space has a representation which in general is not unique. This flexibility makes them attractive for many applications involving signal expansions.

We will now recall the definition of frame for a closed subspace of $\mathcal{H}$.

Definition 2.4. Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$ and $\{f_k\}_{k=1}^\infty \subset \mathcal{W}$. Then $\{f_k\}_{k=1}^\infty$ is a frame for $\mathcal{W}$, if there exist constants $0 < \alpha \leq \beta < \infty$ such that

\begin{equation}
\alpha \|f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k \rangle|^2 \leq \beta \|f\|^2 \quad \text{for all } f \in \mathcal{W}.
\end{equation}

If the right inequality in (2.2) is satisfied, $\{f_k\}_{k=1}^\infty$ is a Bessel sequence for $\mathcal{W}$. The constants $\alpha$ and $\beta$ are the frame bounds. In case $\alpha = \beta$, we call $\{f_k\}_{k=1}^\infty$ an $\alpha$-tight frame, and if $\alpha = \beta = 1$ it is a Parseval frame for $\mathcal{W}$.

To a Bessel sequence $\mathcal{F} = \{f_k\}_{k=1}^\infty$ for $\mathcal{W}$ we associate the synthesis operator

$$T_\mathcal{F} : \ell^2 \to \mathcal{H}, T_\mathcal{F} \{c_k\}_{k=1}^\infty = \sum_{k=1}^\infty c_k f_k,$$

the analysis operator

$$T_\mathcal{F}^* : \mathcal{H} \to \ell^2, T_\mathcal{F}^* f = \{\langle f, f_k \rangle\}_{k=1}^\infty,$$

and the frame operator

$$S_\mathcal{F} = T_\mathcal{F} T_\mathcal{F}^*.$$

A Bessel sequence $\mathcal{F} = \{f_k\}_{k=1}^\infty$ for $\mathcal{W}$ is a frame for $\mathcal{W}$ if and only if $R(T_\mathcal{F}) = \mathcal{W}$, or equivalently, $S_\mathcal{F}$ is invertible when restricted to $\mathcal{W}$. Furthermore, $\mathcal{F}$ is an $\alpha$-tight frame for $\mathcal{W}$ if and only if $S_\mathcal{F} = \alpha P_\mathcal{W}$. A Riesz basis for $\mathcal{W}$ is a frame for $\mathcal{W}$ which is also a basis. If $\{f_k\}_{k=1}^\infty$ is a frame for $\text{span}\{f_k\}_{k=1}^\infty$ we say that $\{f_k\}_{k=1}^\infty$ is a frame sequence.

The optimal upper frame bound, i.e., the infimum over all upper frame bounds is $\|S_\mathcal{F}\| = \|T_\mathcal{F}\|^2$, and the optimal lower frame bound, i.e., the supremum over all lower frame bounds is $\|S_\mathcal{F}^\dagger\|^{-1} = \|T_\mathcal{F}^*\|^{-2}$.

Definition 2.5. Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$. Let $\mathcal{F} = \{f_k\}_{k=1}^\infty$ and $\mathcal{G} = \{g_k\}_{k=1}^\infty$ be frames for $\mathcal{W}$. If $T_\mathcal{G} T_\mathcal{F}^* = P_\mathcal{W}$, we say that $\mathcal{G}$ is a dual frame of $\mathcal{F}$ in $\mathcal{W}$.

The sequence $\{S_\mathcal{F}^\dagger f_k\}_{k=1}^\infty$ is the canonical dual frame of $\{f_k\}_{k=1}^\infty$ in $\mathcal{W}$.

We recall the definition of oblique dual frames [10, 8]:

Definition 2.6. Let $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $\mathcal{F} = \{f_k\}_{k=1}^\infty$ and $\mathcal{G} = \{g_k\}_{k=1}^\infty$ be frames for $\mathcal{W}$ and $\mathcal{V}$, respectively. We call $\mathcal{F}$ and $\mathcal{G}$ oblique dual frames if $T_\mathcal{F} T_\mathcal{G}^* = \pi_{\mathcal{W} V^\perp}$ or $T_\mathcal{G} T_\mathcal{F}^* = \pi_{\mathcal{V} W^\perp}$. 
we denote $\{g_k\}_{k=1}^\infty$ is an oblique dual frame of $\{f_k\}_{k=1}^\infty$ in $\mathcal{V}$ and that $\{f_k\}_{k=1}^\infty$ is an oblique dual frame of $\{g_k\}_{k=1}^\infty$ in $\mathcal{W}$. The canonical oblique dual frame of $\{f_k\}_{k=1}^\infty$ in $\mathcal{V}$ is $\mathcal{G} = \{\pi_{\mathcal{V}V^\perp}S_T^*f_k\}_{k=1}^\infty$ and $T_\mathcal{G} = \pi_{\mathcal{V}V^\perp}S_T^*T_\mathcal{F}$ [3, Theorem 3.2].

Let $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $\mathcal{F} = \{f_k\}_{k=1}^\infty$ and $\mathcal{G} = \{g_k\}_{k=1}^\infty$ be Bessel sequences for $\mathcal{W}$ and $\mathcal{V}$, respectively. We will use the next equalities that follow directly from the definitions of the operators:

$$T_\mathcal{F}^* = T_\mathcal{F}^*|_{\mathcal{W}V^\perp}$$

and

$$(T_\mathcal{F}T_\mathcal{G}^*)|_\mathcal{V} = (T_\mathcal{F}T_\mathcal{G}^*)|_\mathcal{W}(\pi_{\mathcal{V}V^\perp})|_\mathcal{V}.$$  

For more details about frames we refer the reader to [2, 5, 23].

### 2.3. Shift-invariant frame sequences

For $f \in L^1(\mathbb{R})$, its Fourier transform, that we denote by $\hat{f}$, is defined by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \gamma} dx$$

for $\gamma \in \mathbb{R}$. The Fourier transform can be extended as a unitary operator in $L^2(\mathbb{R})$. Given $k \in \mathbb{Z}$, the translation operator is

$$T_k : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_k f(x) = f(x - k),$$

and $T_k \hat{f}(\gamma) = e^{-2\pi i k \gamma} \hat{f}(\gamma)$ holds.

Let $\phi \in L^2(\mathbb{R})$ be such that $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a Bessel sequence. Let $f \in L^2(\mathbb{R})$. Then $f = \sum_{k \in \mathbb{Z}} c_k T_k \phi$ with $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ if and only if its Fourier transform is $\hat{f} = H \hat{\phi}$, where $H \in \mathcal{C}^2_{\text{per}}$ and restricted to $[0, 1)$ is equal to $\sum_{k \in \mathbb{Z}} c_k e^{-2\pi i k \cdot}$. [2, Lemma 9.2.2].

Let $\mathcal{W} := \text{span}\{T_k \phi\}_{k \in \mathbb{Z}}$. A space of this type is called shift-invariant. If $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a frame sequence, then

$$\mathcal{W} = \{\sum_{k \in \mathbb{Z}} c_k T_k \phi : \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})\}.$$  

For $\phi, \phi_1 \in L^2(\mathbb{R})$, we use the bracket notation

$$[\hat{\phi}, \hat{\phi}_1] : \mathbb{R} \to \mathbb{R}, \quad [\hat{\phi}, \hat{\phi}_1](\gamma) = \sum_{k \in \mathbb{Z}} \hat{\phi}(\gamma + k) \hat{\phi}_1(\gamma + k),$$

we denote

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(\gamma) = \sum_{k \in \mathbb{Z}} |\hat{\phi}(\gamma + k)|^2$$

and $N(\Phi) = \{\gamma : \Phi(\gamma) = 0\}$. The functions $[\hat{\phi}, \hat{\phi}_1]$ and $\Phi$ belong to $\mathcal{C}^1_{\text{per}}$.

**Theorem 2.7.** Let $\phi \in L^2(\mathbb{R})$. Then

(i) $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a Bessel sequence with bound $\beta$ if and only if $\Phi(\gamma) \leq \beta$ a.e. $\gamma \in \mathbb{R}$.

(ii) $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a frame sequence with bounds $\alpha, \beta$ if and only if $\alpha \leq \Phi(\gamma) \leq \beta$ a.e. $\gamma \in N(\Phi)$.

Let $\phi, \phi_1 \in L^2(\mathbb{R})$ be such that $\{T_k \phi\}_{k \in \mathbb{Z}}$ and $\{T_k \phi_1\}_{k \in \mathbb{Z}}$ are Bessel sequences with bounds $\beta$ and $\beta$, respectively. By Theorem 2.7(i),

$$[|\phi|, |\phi_1|](\gamma) \leq \sqrt{\beta \beta} \quad \text{a.e. } \gamma \in \mathbb{R}.$$  

(2.4)
Proposition 2.8. [3] Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \), and assume that \( \{T_k \phi\}_{k \in \mathbb{Z}} \) and \( \{T_k \phi_1\}_{k \in \mathbb{Z}} \) are frame sequences. Let \( \mathcal{W} := \text{span}\{T_k \phi\}_{k \in \mathbb{Z}} \) and \( \mathcal{V} := \text{span}\{T_k \phi_1\}_{k \in \mathbb{Z}} \). Then the following are equivalent:

(i) \( L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp \).

(ii) \( N(\Phi) = N(\Phi_1) \) and there exists a constant \( c > 0 \) such that

\[
| \langle \widehat{\phi}, \widehat{\phi}_1 \rangle(\gamma) | \geq c \ a.e. \ \gamma \in N(\Phi)^c.
\]

Let \( \mathcal{W} \) and \( \mathcal{V} \) be two closed subspaces of \( L^2(\mathbb{R}) \) such that \( L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp \). In [3, Proposition 4.8] it is shown that if \( \{T_k \phi\}_{k \in \mathbb{Z}} \) is a frame for \( \mathcal{W} \) and \( \{T_k \phi_1\}_{k \in \mathbb{Z}} \) is a frame for \( \mathcal{V} \) then

(2.5) \[ \cos \theta(\mathcal{W}, \mathcal{V}) \geq \inf_{\gamma \in N(\Phi)^c} \frac{\| \langle \widehat{\phi}, \widehat{\phi}_1 \rangle(\gamma) \|}{\sqrt{\Phi(\gamma) \Phi_1(\gamma)}}. \]

3. Definition and fundamental properties

In this section we present the concept of approximate oblique duality and explore some of its properties. First we introduce the following definition of approximate oblique dual frames:

Definition 3.1. Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} = \{f_k\}_{k=1}^\infty \) and \( \mathcal{G} = \{g_k\}_{k=1}^\infty \) be frames for \( \mathcal{W} \) and \( \mathcal{V} \), respectively. Let \( \epsilon \geq 0 \). We say that \( \mathcal{F} \) and \( \mathcal{G} \) are \( \epsilon \)-approximate oblique dual frames if

(3.1) \[ \| \pi_{\mathcal{W}^\perp} - T_T^\mathcal{G} \| \leq \epsilon \text{ or } \| \pi_{\mathcal{V}^\perp} - T_T^\mathcal{F} \| \leq \epsilon. \]

In this case we also say that \( \{g_k\}_{k=1}^\infty \) is an \( \epsilon \)-approximate oblique dual frame of \( \{f_k\}_{k=1}^\infty \) in \( \mathcal{V} \) and that \( \{f_k\}_{k=1}^\infty \) is an \( \epsilon \)-approximate oblique dual frame of \( \{g_k\}_{k=1}^\infty \) in \( \mathcal{W} \). If \( \mathcal{V} = \mathcal{W} \), then \( \mathcal{F} \) and \( \mathcal{G} \) will be called \( \epsilon \)-approximate dual frames in \( \mathcal{W} \).

Note that the conditions of the definition can be written as

(3.2) \[ \| \pi_{\mathcal{W}^\perp} f - \sum_{k=1}^\infty \langle f, g_k \rangle g_k \| \leq \epsilon \| f \| \text{ for all } f \in \mathcal{H} \]

and

(3.3) \[ \| \pi_{\mathcal{V}^\perp} f - \sum_{k=1}^\infty \langle f, f_k \rangle f_k \| \leq \epsilon \| f \| \text{ for all } f \in \mathcal{H}, \]

respectively.

Remark 3.2. (i) If \( \epsilon = 0 \), \( \mathcal{F} \) and \( \mathcal{G} \) are oblique dual frames. If \( \mathcal{V} = \mathcal{W} = \mathcal{H} \) and \( \epsilon < 1 \), then \( \mathcal{F} \) and \( \mathcal{G} \) are approximate dual frames as defined in [3].

(ii) If \( \epsilon < 1 \), by Neumann’s Theorem \( (T_T^\mathcal{G})_{|\mathcal{W}} \) is an invertible operator from \( \mathcal{W} \) to \( \mathcal{W} \). So, by (2.3) and Lemma 2.1(i), \( (T_T^\mathcal{G})_{|\mathcal{V}} \) is an invertible operator from \( \mathcal{V} \) to \( \mathcal{V} \).

(iii) If \( \epsilon < 1 \), it is sufficient that \( \{f_k\}_{k=1}^\infty \) and \( \{g_k\}_{k=1}^\infty \) are Bessel sequences, since \( \mathcal{R}(T_T) = \mathcal{W} \) by (ii). Hence \( \{f_k\}_{k=1}^\infty \) is a frame for \( \mathcal{W} \). Analogously \( \{g_k\}_{k=1}^\infty \) is a frame for \( \mathcal{V} \).

(iv) Assume that \( \epsilon < 1 \). If \( f \in \mathcal{W} \), by (ii),

\[ f = (T_T^\mathcal{G})_{|\mathcal{W}} \]^{-1} T_T^\mathcal{G} f = \sum_{k=1}^\infty \langle f, g_k \rangle (T_T^\mathcal{G})_{|\mathcal{W}}^{-1} g_k. \]

Similarly, for \( f \in \mathcal{V} \) we have

\[ f = (T_T^\mathcal{F})_{|\mathcal{V}} \]^{-1} T_T^\mathcal{F} f = \sum_{k=1}^\infty \langle f, f_k \rangle (T_T^\mathcal{F})_{|\mathcal{V}}^{-1} f_k. \]
Proposition 3.3. Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} = \{f_k\}_{k=1}^\infty \subset \mathcal{W} \) and \( \mathcal{G} = \{g_k\}_{k=1}^\infty \subset \mathcal{V} \) be \( \epsilon \)-approximate oblique dual frames within \( \mathcal{H} \) with \( \epsilon \geq 0 \). Let \( U : \mathcal{H} \to \mathcal{H} \) be a unitary operator. Then \( UF = \{Uf_k\}_{k=1}^\infty \) and \( UG = \{Ug_k\}_{k=1}^\infty \) are \( \epsilon \)-approximate oblique dual frames.

Proof. By [2, Corollary 5.3.4], the sequences \( UF \) and \( UG \) are frames for \( UW \) and \( UV \), respectively. We have \( T_{UF} = UT_{\mathcal{F}} \) and \( T_{UG} = UT_{\mathcal{G}} \). Since \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \) and \( I = UU^* \), \( I = U\pi_{\mathcal{W}^\perp}U^* + U\pi_{\mathcal{V}^\perp}U^* \) and \( \mathcal{H} = UW \oplus (UV)^\perp \). From here, we have \( \pi_{UV(U)^\perp} = U\pi_{\mathcal{W}^\perp}U^* \). Then

\[
\|\pi_{UV(U)^\perp} - T_{UF}T_{\mathcal{G}}^*\| = \|U(\pi_{\mathcal{W}^\perp} - T_{\mathcal{F}}T_{\mathcal{G}}^*)U^*\| = \|\pi_{\mathcal{W}^\perp} - T_{\mathcal{F}}T_{\mathcal{G}}^*\| \leq \epsilon,
\]

and so \( UF \) and \( UG \) are \( \epsilon \)-approximate oblique dual frames. \( \square \)

Now we briefly discuss the interpretation of the approximate oblique duality in the context of signal processing theory. Let \( \mathcal{F} \) be a frame for \( \mathcal{W} \). Assume that the samples \( T_{\mathcal{F}}f = \{(f, f_k)\}_{k=1}^\infty \) of an unknown signal \( f \in \mathcal{H} \) are given. Our goal is the reconstruction of \( f \) from these samples using a frame \( \mathcal{G} \) for \( \mathcal{V} \) in such a way that the reconstruction \( f_r \in \mathcal{V} \) is a good approximation of \( f \). This means on one hand that there don’t exist two different signals in \( \mathcal{V} \) with the same samples. On the other, that the samples of the reconstruction \( f_r \) and those of the original signal \( f \) are close. Specifically the following two conditions are required. Let \( \epsilon \geq 0 \),

(i) **Uniqueness of the reconstruction:** If \( f, g \in \mathcal{V} \) and \( T_{\mathcal{F}}f = T_{\mathcal{F}}g \), then \( f = g \).

(ii) **\( \epsilon \)-consistent reconstruction:** For every \( f \in \mathcal{H} \), \( \|T_{\mathcal{F}}f_r - T_{\mathcal{F}}f\| \leq \epsilon \|f\| \).

If (ii) holds, we say that \( f_r \) is an \( \epsilon \)-consistent reconstruction of \( f \) in \( \mathcal{V} \). Note that a 0-consistent reconstruction is a consistent reconstruction as defined in [10, 11, 13]. The condition (i) is equivalent to \( \mathcal{V} \cap \mathcal{W}^\perp = \{0\} \).

If \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \), using an oblique dual frame of \( \mathcal{F} \) in \( \mathcal{V} \), we can obtain the following bound for \( \|f_r - \pi_{\mathcal{V}^\perp}f\| \):

**Proposition 3.4.** Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} = \{f_k\}_{k=1}^\infty \) be a frame for \( \mathcal{W} \) and \( \mathcal{G} \) be an oblique dual frame of \( \mathcal{F} \) in \( \mathcal{V} \). Let \( \epsilon \geq 0 \). If \( f \in \mathcal{H} \) and \( f_r \) is an \( \epsilon \)-consistent reconstruction of \( f \) in \( \mathcal{V} \), then

\[
\|f_r - \pi_{\mathcal{V}^\perp}f\| \leq \epsilon \|T_{\mathcal{G}}\| \|f\|.
\]

Proof. Using that \( T_{\mathcal{G}}T_{\mathcal{F}} = \pi_{\mathcal{V}^\perp} \) and \( f_r \in \mathcal{V} \), we have

\[
\|f_r - \pi_{\mathcal{V}^\perp}f\| = \|T_{\mathcal{G}}T_{\mathcal{F}}f_r - T_{\mathcal{G}}T_{\mathcal{F}}f\| \leq \|T_{\mathcal{G}}\| \|T_{\mathcal{F}}f_r - T_{\mathcal{F}}f\| \leq \epsilon \|T_{\mathcal{G}}\| \|f\|.
\]

\( \square \)

The following result relates an \( \epsilon \)-consistent reconstruction \( f_r \) to the oblique projection \( \pi_{\mathcal{V}^\perp}f \) of \( f \).
Lemma 4.1. Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} \) be a frame for \( \mathcal{W} \), \( f \in \mathcal{H} \) and \( \epsilon \geq 0 \). The following holds:

(i) If \( f_r \) is an \( \epsilon \)-consistent reconstruction of \( f \) in \( \mathcal{V} \), then
\[
\| f_r - \pi_{\mathcal{V}^\perp} f \| \leq \epsilon \| \pi_{\mathcal{V}^\perp} \| T_f^\perp \| f \|.
\]
(ii) If \( \| f_r - \pi_{\mathcal{V}^\perp} f \| \leq \frac{\epsilon}{\| T_f^\perp \|} \| f \| \), then \( f_r \) is an \( \epsilon \)-consistent reconstruction of \( f \) in \( \mathcal{V} \).

Proof. If \( \mathcal{G} \) is the canonical oblique dual frame of \( \mathcal{F} \) then \( T_\mathcal{G}^\perp = \pi_{\mathcal{V}^\perp} S_f^\perp T_\mathcal{F} \). We also have, \( T_f^\perp = T_\mathcal{F}^* S_f^\perp \). So (i) follows from Proposition 3.3.

To prove (ii), assume that \( \| f_r - \pi_{\mathcal{V}^\perp} f \| \leq \frac{\epsilon}{\| T_f^\perp \|} \| f \| \). Then,
\[
\| T_f^\perp f_r - T_f^\perp f \| = \| T_f^\perp f_r - T_f^\perp \pi_{\mathcal{V}^\perp} f \| \leq \epsilon \| f \|.
\]

□

The previous theorem allows to link \( \epsilon \)-consistent reconstruction with approximate oblique duality via the next corollary.

Corollary 3.6. Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} \) be a frame for \( \mathcal{W} \), \( \mathcal{G} \) a frame for \( \mathcal{V} \) and \( \epsilon \geq 0 \). Then

(i) If \( f_r = T_\mathcal{G} T_f^\perp f \) is an \( \frac{\epsilon}{\| \pi_{\mathcal{V}^\perp} \| \| T_f^\perp \|} \)-consistent reconstruction of \( f \) in \( \mathcal{V} \), for each \( f \in \mathcal{H} \), then \( \mathcal{F} \) and \( \mathcal{G} \) are \( \epsilon \)-approximate oblique dual frames.

(ii) If \( \mathcal{F} \) and \( \mathcal{G} \) are \( \epsilon \)-approximate oblique dual frames, then \( f_r = T_\mathcal{G} T_f^\perp f \) is an \( \epsilon \| T_f^\perp \| \)-consistent reconstruction of \( f \) in \( \mathcal{V} \), for each \( f \in \mathcal{H} \).

4. Characterization of approximate oblique dual frames

The following lemma gives equivalent conditions for approximate oblique duality.

Lemma 4.1. Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} = \{f_k\}_{k=1}^\infty \), \( \mathcal{G} = \{g_k\}_{k=1}^\infty \) be Bessel sequences in \( \mathcal{H} \) such that \( \text{span}\{f_k\}_{k=1}^\infty = \mathcal{W} \) and \( \text{span}\{g_k\}_{k=1}^\infty = \mathcal{V} \). Let \( \epsilon \geq 0 \). Then the following assertions are equivalent:

(i) \( \{f_k\}_{k=1}^\infty \) and \( \{g_k\}_{k=1}^\infty \) are \( \epsilon \)-approximate oblique dual frames.

(ii) \( \| \pi_{\mathcal{V}^\perp} f - \sum_{k=1}^\infty \langle f, g_k \rangle f_k \| \leq \epsilon \| f \| \) for all \( f \in \mathcal{H} \).

(iii) \( \| \pi_{\mathcal{V}^\perp} f - \sum_{k=1}^\infty \langle f, f_k \rangle g_k \| \leq \epsilon \| f \| \) for all \( f \in \mathcal{H} \).

(iv) \( |\langle \pi_{\mathcal{V}^\perp} f, g \rangle - \sum_{k=1}^\infty \langle f, g_k \rangle \langle f_k, g \rangle| \leq \epsilon \| f \| \| g \| \) for all \( f, g \in \mathcal{H} \).

Proof. Recall that (ii) and (iii) are the equivalent conditions (3.2) and (3.3) of Definition 3.1.

Let \( f, g \in \mathcal{H} \). Assume that (ii) holds. Using that
\[
|\langle \pi_{\mathcal{V}^\perp} f, g \rangle - \sum_{k=1}^\infty \langle f, g_k \rangle \langle f_k, g \rangle| = |\langle \pi_{\mathcal{V}^\perp} f - \sum_{k=1}^\infty \langle f, g_k \rangle f_k, g \rangle|,
\]
and Cauchy-Schwarz inequality, we obtain (iv).

Assume now that (iv) holds. We have
\[
\left\| \pi_{\mathcal{V}^\perp} g - \sum_{k=1}^\infty \langle g, g_k \rangle f_k \right\| = \sup_{\| f \|=1} \left| \langle \pi_{\mathcal{V}^\perp} g - \sum_{k=1}^\infty \langle g, g_k \rangle f_k, f \rangle \right| \leq \epsilon \| g \|.
\]

So (ii) holds.

The proof of (iii) \( \iff \) (v) is similar to (ii) \( \iff \) (iv). □
Condition (ii) of Lemma 4.1 implies

\[
\|f - \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k \| \leq \epsilon \|f\| \quad \text{for } f \in \mathcal{W},
\]

whereas (iii) implies \(\|g - \sum_{k=1}^{\infty} \langle g, f_k \rangle g_k\| \leq \epsilon \|g\|\) for \(g \in \mathcal{V}\). We conclude that the elements of each subspace can be reconstructed approximately, with a uniform relative error.

Let \(\{f_k\}_{k=1}^{\infty}\) be a Bessel sequence in \(\mathcal{W}\) and \(\{g_k\}_{k=1}^{\infty}\) be a Bessel sequence in \(\mathcal{V}\). Observe that for \(f \in \mathcal{H}\),

\[
\|\pi_{\mathcal{W}^\perp} f - \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k\| = \|\pi_{\mathcal{W}^\perp} f - \sum_{k=1}^{\infty} \langle \pi_{\mathcal{W}^\perp} f, g_k \rangle f_k\|.
\]

So, we have the following result:

**Proposition 4.2.** Let \(\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp\). Let \(\{f_k\}_{k=1}^{\infty}\) be a frame for \(\mathcal{W}\) and \(\{g_k\}_{k=1}^{\infty}\) be a frame for \(\mathcal{V}\) such that (4.1) is satisfied with \(\epsilon \geq 0\). Then \(\{f_k\}_{k=1}^{\infty}\) and \(\{g_k\}_{k=1}^{\infty}\) are \(\epsilon \|\pi_{\mathcal{W}^\perp}\|\)-approximate oblique dual frames.

Similarly we have:

**Proposition 4.3.** Let \(\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp\). Let \(\{f_k\}_{k=1}^{\infty}\) be a Bessel sequence in \(\mathcal{W}\) and \(\{g_k\}_{k=1}^{\infty}\) be a Bessel sequence in \(\mathcal{H}\) such that (4.1) holds with \(\epsilon \geq 0\). If \(\epsilon \|\pi_{\mathcal{W}^\perp}\| < 1\), then \(\{f_k\}_{k=1}^{\infty}\) is a frame for \(\mathcal{W}\), \(\{\pi_{\mathcal{W}^\perp} g_k\}_{k=1}^{\infty}\) is a frame for \(\mathcal{V}\) and \(\{f_k\}_{k=1}^{\infty}\) and \(\{\pi_{\mathcal{W}^\perp} g_k\}_{k=1}^{\infty}\) are \(\epsilon \|\pi_{\mathcal{W}^\perp}\|\)-approximate oblique dual frames.

In [12, Lemma B.1] a characterization of oblique dual frames is given by oblique inverses. We have a similar characterization for approximate oblique dual frames using approximate oblique left inverses.

**Definition 4.4.** Let \(\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp\). Let \(\mathcal{F} = \{f_k\}_{k=1}^{\infty}\) be a frame for \(\mathcal{W}\) and \(\epsilon \geq 0\). We say that a linear bounded operator \(A : \ell^2(\mathbb{N}) \to \mathcal{V}\) is an \(\epsilon\)-approximate oblique left inverse of \(T_\mathcal{F}\) in \(\mathcal{V}\) along \(\mathcal{W}^\perp\) if \(\|\pi_{\mathcal{W}^\perp} - AT_\mathcal{F}\| \leq \epsilon\).

We now give the following characterization.

**Lemma 4.5.** Let \(\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp\). Let \(\mathcal{F} = \{f_k\}_{k=1}^{\infty}\) be a frame for \(\mathcal{W}\). Let \(\{\delta_k\}_{k=1}^{\infty}\) be the canonical basis of \(\ell^2(\mathbb{N})\) and \(0 \leq \epsilon < 1\). The \(\epsilon\)-approximate oblique dual frames of \(\mathcal{F}\) in \(\mathcal{V}\) are the sequences \(\{g_k\}_{k=1}^{\infty} = \{A \delta_k\}_{k=1}^{\infty}\), where \(A : \ell^2(\mathbb{N}) \to \mathcal{V}\) is an \(\epsilon\)-approximate oblique left inverse of \(T_\mathcal{F}\) in \(\mathcal{V}\) along \(\mathcal{W}^\perp\).

**Proof.** Clearly, if \(\mathcal{G} = \{g_k\}_{k=1}^{\infty}\) is an \(\epsilon\)-approximate oblique dual frame of \(\{f_k\}_{k=1}^{\infty}\) in \(\mathcal{V}\), then \(T_\mathcal{G}\) is an \(\epsilon\)-approximate oblique left inverse of \(T_\mathcal{F}\) in \(\mathcal{V}\) along \(\mathcal{W}^\perp\) and \(\{g_k\}_{k=1}^{\infty} = \{T_\mathcal{G} \delta_k\}_{k=1}^{\infty}\).

Let now \(A : \ell^2(\mathbb{N}) \to \mathcal{V}\) be an \(\epsilon\)-approximate oblique left inverse. Let \(\mathcal{G} = \{g_k\}_{k=1}^{\infty} = \{A \delta_k\}_{k=1}^{\infty}\). By [2, Theorem 3.2.3], \(\mathcal{G}\) is a Bessel sequence in \(\mathcal{V}\) with \(T_\mathcal{G} = A\). We have \(\|\pi_{\mathcal{W}^\perp} - T_\mathcal{G} T_\mathcal{F}\| \leq \epsilon\). By Remark 3.2(iii), \(\mathcal{G}\) is a frame for \(\mathcal{V}\). Thus, \(\mathcal{G}\) is an \(\epsilon\)-approximate oblique dual frame of \(\{f_k\}_{k=1}^{\infty}\) in \(\mathcal{V}\). \(\square\)
We note that $A : \ell^2(\mathbb{N}) \rightarrow \mathcal{V}$ is an $\epsilon$-approximate oblique left inverse of $T^*_F$ in $\mathcal{V}$ if and only if $A = L + R$ where $L : \ell^2(\mathbb{N}) \rightarrow \mathcal{V}$ is an oblique left inverse of $T^*_F$ in $\mathcal{V}$ and $R : \ell^2(\mathbb{N}) \rightarrow \mathcal{V}$ is a bounded operator such that $\|RT^*_F\| \leq \epsilon$. So, from Lemma 4.5 and [3] Lemma B.2, we obtain a description of the approximate oblique dual frames:

**Theorem 4.6.** Let $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $\mathcal{F} = \{f_k\}_{k=1}^\infty$ be a frame for $\mathcal{W}$ and $0 \leq \epsilon < 1$. Then any $\epsilon$-approximate oblique dual frame $\mathcal{G} = \{g_k\}_{k=1}^\infty \subset \mathcal{V}$ of $\mathcal{F}$ in $\mathcal{V}$ is of the form

$$\{g_k\}_{k=1}^\infty = \{\pi_{\mathcal{W}^\perp}(S^*_F f_k + h_k - \sum_{j=1}^\infty (S^*_F f_k, f_j) h_j + r_k)\}_{k=1}^\infty$$

where $\{h_k\}_{k=1}^\infty$ and $\{r_k\}_{k=1}^\infty$ are Bessel sequences in $\mathcal{V}$ and $\|\sum (f, f_k) r_k\| \leq \epsilon \|f\|$ for all $f \in \mathcal{H}$.

5. **IMPROVING APPROXIMATION**

Given a frame $\mathcal{F} = \{f_k\}_{k=1}^\infty$ in $\mathcal{W}$ and an approximate oblique dual frame $\mathcal{G} = \{g_k\}_{k=1}^\infty \subset \mathcal{V}$ of it, the following proposition says that it is possible to construct from $\mathcal{G}$ other approximate oblique dual frames for $\mathcal{F}$ in $\mathcal{V}$ that provide a reconstruction as "good" as needed. This is analogous to a result for approximate dual frames [3] Proposition 3.2.

**Proposition 5.1.** Let $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $\mathcal{F} = \{f_k\}_{k=1}^\infty \subset \mathcal{W}$ and $\mathcal{G} = \{g_k\}_{k=1}^\infty \subset \mathcal{V}$ be $\epsilon$-approximate oblique dual frames with $0 \leq \epsilon < 1$. For a fixed $N \in \mathbb{N}$, let

$$\tilde{g}^{(N)}_k = \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^n g_k$$

and $\tilde{G}^N = \{\tilde{g}^{(N)}_k\}_{k=1}^\infty$. Then $\tilde{G}^N$ is an $\epsilon^{N+1}$-approximate oblique dual frame of $\mathcal{F}$ in $\mathcal{V}$ and

$$\left\|\pi_{\mathcal{W}^\perp} - T^*_F \tilde{G}^N f\right\| \rightarrow 0 \text{ as } N \rightarrow \infty.$$  

**Proof.** Let $\tilde{g}^{(N)}_k = \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^n g_k$ for $N \in \mathbb{N}$ and $\tilde{G}^N = \{\tilde{g}^{(N)}_k\}_{k=1}^\infty$.

Since $\{g_k\}_{k=1}^\infty$ is a Bessel sequence and $\pi_{\mathcal{W}^\perp} - T^*_F T^*_F$ is a bounded operator, it follows that $\{\tilde{g}^{(N)}_k\}_{k=1}^\infty$ is a Bessel sequence. Hence, by [3] Theorem 3.2.3 its synthesis operator $T^*_F \tilde{G}^N$ is well defined. Let $f \in \mathcal{H}$, then

$$T^*_F \tilde{G}^N f = \sum_{k=1}^\infty (f, f_k) \tilde{g}^{(N)}_k = \sum_{k=1}^\infty (f, f_k) \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^n g_k$$

$$= \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^n T^*_F f$$

$$= \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^n (\pi_{\mathcal{W}^\perp} - (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)) \pi_{\mathcal{W}^\perp} f$$

$$= \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^n \pi_{\mathcal{W}^\perp} f - \sum_{n=0}^N (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^{n+1} \pi_{\mathcal{W}^\perp} f$$

$$= \pi_{\mathcal{W}^\perp} f - (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^{N+1} \pi_{\mathcal{W}^\perp} f$$

Now, using that $\mathcal{W}^\perp \subseteq N(\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)$,

$$\pi_{\mathcal{W}^\perp} f - T^*_F \tilde{G}^N f = (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^{N+1} \pi_{\mathcal{W}^\perp} f = (\pi_{\mathcal{W}^\perp} - T^*_F T^*_F)^{N+1} f$$
Therefore, \( \pi_{\mathcal{V}W} - T_{\tilde{g}_N}^* T_{\tilde{F}} = (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}})^{N+1} \) and
\[
\left\| \pi_{\mathcal{V}W} - T_{\tilde{g}_N}^* T_{\tilde{F}} \right\| \leq \left\| \pi_{\mathcal{V}W} - T_g T_{\tilde{F}} \right\|^{N+1} \leq \epsilon^{N+1} \to 0
\]
as \( N \to \infty \).

**Remark 5.2.** Note that in the previous proof we obtain
\[
\pi_{\mathcal{V}W} - T_{\tilde{g}_N}^* T_{\tilde{F}} = (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}})^{N+1}.
\]

Considering
\[
L_N = \sum_{n=0}^{N} (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}}) \cdot (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}})^n,
\]
the operator \( T_{\tilde{g}_N} \) can be expressed as \( T_{\tilde{g}_N} = L_N T_g \).

Setting \( \tilde{f}_k^{(N)} = \sum_{n=0}^{N} (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}})^n f_k \) and \( \tilde{F}_N = \{ \tilde{f}_k^{(N)} \}_{k=1}^{\infty} \), we have \( T_{\tilde{F}_N} = L_N^* T_{\tilde{F}}, \pi_{\mathcal{V}W} - T_g T_{\tilde{F}_N} = (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}})^{N+1} \) and
\[
\left\| \pi_{\mathcal{V}W} - T_g T_{\tilde{F}_N} \right\| \leq \left\| \pi_{\mathcal{V}W} - T_g T_{\tilde{F}} \right\|^{N+1} \leq \epsilon^{N+1} \to 0.
\]

From (5.3) and (5.5), we observe that we can improve the approximation given by the approximate oblique dual frames \( \mathcal{F} \) and \( \mathcal{G} \), modifying either of them.

The following proposition shows that the frames \( \tilde{F}_N \) and \( \tilde{G}_N \) defined previously are related with oblique dual frames of \( \mathcal{F} \) and \( \mathcal{G} \).

**Proposition 5.3.** Let \( \mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \mathcal{F} = \{ f_k \}_{k=1}^{\infty} \subset \mathcal{W} \) and \( \mathcal{G} = \{ g_k \}_{k=1}^{\infty} \subset \mathcal{V} \) be \( \epsilon \)-approximate oblique dual frames with \( 0 \leq \epsilon < 1 \). Then \( I - (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}}) \) is invertible and
\[
L := (I - (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}}))^{-1} = I + \sum_{n=1}^{\infty} (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}})^n.
\]
Let \( \tilde{G} = \{ \tilde{g}_k \}_{k=1}^{\infty} \) with \( \tilde{g}_k = LT_g \delta_k \) and \( \tilde{F} = \{ \tilde{f}_k \}_{k=1}^{\infty} \) with \( \tilde{f}_k = L^* T_{\tilde{F}} \delta_k \). Then:

(i) \( \tilde{G} \) is an oblique dual frame of \( \mathcal{F} \) in \( \mathcal{V} \).

(ii) \( \tilde{G} \) is an oblique dual frame of \( \mathcal{G} \) in \( \mathcal{W} \).

(iii) \( \tilde{G} \) is an \( \frac{\epsilon}{1-\epsilon} \)-approximate oblique dual frame of \( \tilde{F} \) in \( \mathcal{V} \).

**Proof.** The assertions about \( I - (\pi_{\mathcal{V}W} - T_g T_{\tilde{F}}) \) follow from Neumann’s Theorem.

If \( L_N \) is as in (5.3), then \( L_N \to L \) as \( N \to \infty \). So, \( T_{\tilde{g}_N} = L_NT_{\tilde{G}} \to T_{\tilde{G}} = LT_G \) as \( N \to \infty \).

Using (5.3), we have
\[
\left\| \pi_{\mathcal{V}W} - T_{\tilde{g}} T_{\mathcal{F}} \right\| = \left\| \pi_{\mathcal{V}W} - T_{\tilde{g}_N}^* T_{\tilde{F}} \right\| + \left\| T_{\tilde{g}_N}^* T_{\tilde{F}} - T_{\tilde{g}} T_{\mathcal{F}} \right\| \leq \epsilon^{N+1} + ||T_{\tilde{g}_N} - T_{\tilde{g}}|| \left\| \mathcal{F} \right\|
\]
Since the right hand side tends to 0 as \( N \to \infty \), \( T_{\tilde{g}} T_{\mathcal{F}} = \pi_{\mathcal{V}W} \). This shows that \( \tilde{G} \) is an oblique dual frame of \( \mathcal{F} \) in \( \mathcal{V} \). Similarly, it can be proved that \( \tilde{F} \) is an oblique dual frame of \( \mathcal{G} \) in \( \mathcal{W} \).

Since \( \mathcal{R}(I - L) \subseteq \mathcal{V} \), by (i),
\[
\pi_{\mathcal{V}W} - T_{\tilde{g}} T_{\mathcal{F}} = \pi_{\mathcal{V}W} - T_{\tilde{g}_N}^* T_{\tilde{F}} L = \pi_{\mathcal{V}W} (I - L) = \sum_{n=1}^{\infty} (\pi_{\mathcal{V}W} - T_{\tilde{g}} T_{\mathcal{F}})^n.
\]
Then \( \left\| \pi_{\mathcal{V}W} - T_{\tilde{g}} T_{\mathcal{F}} \right\| \leq \frac{\epsilon}{1-\epsilon}. \)

**Remark 5.4.** We note that \( \tilde{G} \) is an oblique dual frame of \( \tilde{F} \) in \( \mathcal{V} \) if and only if \( \mathcal{G} \) is an oblique dual frame of \( \mathcal{F} \) in \( \mathcal{V} \). In this case, \( \tilde{G} = \mathcal{G} \) and \( \tilde{F} = \mathcal{F} \). In effect, assume that \( \tilde{G} \) is an oblique dual frame of \( \tilde{F} \) in \( \mathcal{V} \). Then, from the equalities above \( L = I \). Conversely, if \( \mathcal{G} \) is an oblique...
dual frame of $F$ in $V$, then $\epsilon = 0$ and, from the definition of $L$, $L = I$. In both cases, $\tilde{G} = G$ and $\tilde{F} = F$.

**Remark 5.5.** It is natural to ask if we can have an approximation as close as wanted to $\pi_{VW^\perp} f$ using simultaneously the frames $F_N$ and $G_N$. The answer is no. More precisely, since

$$\|\pi_{VW^\perp} - T_{\tilde{g}}T_{F_N}^*\| \leq \|\pi_{VW^\perp} - T_{\tilde{g}}^*T_{F_N}^*\| + \|T_{\tilde{g}}(T_{F_N} - T_{F_N}^*)\|$$

and

$$\lim_{N \to \infty} \|T_{\tilde{g}}(T_{F_N}^* - T_{F_N}^*)\| = 0,$$

we obtain, using Remark 5.4,

$$\liminf_{N \to \infty} \|\pi_{VW^\perp} - T_{\tilde{g}}T_{F_N}^*\| \geq \|\pi_{VW^\perp} - T_{\tilde{g}}T_{F_N}^*\| > 0.$$  

6. Approximate oblique dual frames by perturbation

The following propositions say that if two frames are “close”, each oblique dual frame of one of them is an approximate oblique dual frame of the other. They are extensions of [4] Proposition 4.1 and Proposition 4.3, which are results about approximate dual frames.

**Proposition 6.1.** Let $H = W \oplus V^\perp$. Let $0 \leq \epsilon < 1$, $\tilde{G} = \{\tilde{g}_k\}_{k=1}^\infty$ be a frame for $V$, $G = \{g_k\}_{k=1}^\infty \subset V$ a sequence such that $\sum_{k=1}^\infty |\langle f, g_k - \tilde{g}_k \rangle|^2 \leq r \|f\|^2$ for each $f \in V$, and $F = \{f_k\}_{k=1}^\infty$ an oblique dual frame of $G$ in $W$ whose upper frame bound $\beta$ satisfies $\sqrt{\beta}r \leq \epsilon$. Then, $G$ is an $\epsilon$-approximate dual frame of $F$ in $V$.

**Proof.** We will first see that $\{g_k\}_{k=1}^\infty$ is a Bessel sequence. Let $f \in V$. Applying the Cauchy-Schwarz inequality,

$$\sum_{k=1}^\infty |\langle f, g_k \rangle|^2 \leq \sum_{k=1}^\infty (|\langle f, g_k - \tilde{g}_k \rangle| + |\langle f, \tilde{g}_k \rangle|)^2$$

$$= \sum_{k=1}^\infty |\langle f, g_k - \tilde{g}_k \rangle|^2 + 2 \sum_{k=1}^\infty |\langle f, g_k - \tilde{g}_k \rangle||\langle f, \tilde{g}_k \rangle| + \sum_{k=1}^\infty |\langle f, \tilde{g}_k \rangle|^2$$

$$\leq (r + 2\sqrt{r\beta + \tilde{\beta}}) \|f\|^2,$$

where $\tilde{\beta}$ is the upper frame bound for $\{\tilde{g}_k\}_{k=1}^\infty$. Hence, $\{g_k\}_{k=1}^\infty$ is a Bessel sequence with bound $(\sqrt{r} + \sqrt{\beta})^2$.

Note that the condition given for $\{g_k\}_{k=1}^\infty$ can be written as $\|T_{\tilde{g}}^* - T_{g}^\perp\| \leq \sqrt{r}$.

We have

$$\|\pi_{VW^\perp} - T_{\tilde{g}}T_{g}^\perp\| = \|T_{\tilde{g}}T_{g}^\perp - T_{\tilde{g}}T_{g}^\perp\| \leq \|T_{\tilde{g}}\| \|T_{\tilde{g}}^* - T_{g}^\perp\| \leq \sqrt{\beta}r \leq \epsilon.$$ 

Therefore, by Definition 3.1 and Remark 3.2(iii), $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are $\epsilon$-approximate oblique dual frames.

**Proposition 6.2.** Let $H = W \oplus V^\perp$. Let $\epsilon \geq 0$, $F = \{f_k\}_{k=1}^\infty$ be a frame for $W$ with bounds $\alpha$ and $\beta$ and $\tilde{G} = \{\tilde{g}_k\}_{k=1}^\infty \subset W$ such that $\sum_{k=1}^\infty |\langle f, g_k - \tilde{g}_k \rangle|^2 \leq r \|f\|^2$, for $f \in W$, with $r \leq \frac{\alpha^2}{(\|\pi_{VW^\perp}\| + \epsilon)^2}$. Then,
(i) $\mathcal{G}$ is a frame for $\mathcal{W}$ with bounds $(\sqrt{\alpha} - \sqrt{\beta})^2$ and $(\sqrt{\beta} + \sqrt{\beta})^2$.

(ii) If $\mathcal{G} = \{g_k\}_{k=1}^\infty \subset \mathcal{V}$ is given by $g_k = \pi_{\mathcal{W}^\perp} (\widetilde{T}_f T^*_\mathcal{G}) g_k$, then $\mathcal{G}$ is an $\epsilon$-approximate oblique dual frame of $\mathcal{F}$ in $\mathcal{V}$.

Proof. (i) Since $r < \alpha$, this part follows from [2, Corollary 22.1.5].

To see (ii), observe that, by [2, Lemma 5.1.5], the canonical dual frame $\{(\widetilde{T}_f T^*_\mathcal{G}) g_k\}_{k=1}^\infty$ in $\mathcal{W}$ has bounds $\frac{1}{(\sqrt{\beta} + \sqrt{\beta})^2}$ and $\frac{1}{(\sqrt{\beta} - \sqrt{\beta})^2}$.

By [3, Theorem 3.2], the sequence $\mathcal{G} = \{g_k\}_{k=1}^\infty \subset \mathcal{V}$ given by $g_k = \pi_{\mathcal{W}^\perp} (\widetilde{T}_f T^*_\mathcal{G}) g_k$ is an oblique dual frame of $\mathcal{G}$ in $\mathcal{V}$. Its synthesis operator satisfies $\|T_\mathcal{G}\| \leq \frac{\|\pi_{\mathcal{W}^\perp}\|}{\sqrt{\alpha} - \sqrt{\beta}}$. Hence,

$$\|\pi_{\mathcal{W}^\perp} - T_\mathcal{G} T_f^*\| = \|T_\mathcal{G} T^*_f - T_\mathcal{G} T_f^*\| \leq \|T_\mathcal{G}\| \|T^*_f - T_f^*\| \leq \frac{\|\pi_{\mathcal{W}^\perp}\|}{\sqrt{\alpha} - \sqrt{\beta}} \leq \epsilon.$$ 

So $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are $\epsilon$-approximate oblique dual frames. \qed

7. APPROXIMATE RECONSTRUCTION IN SHIFT-INVARIANT SUBSPACES

In this section we give conditions on the generators of shift-invariant subspaces $\mathcal{W}$ and $\mathcal{V}$ of $\mathcal{H} = L^2(\mathbb{R})$, in order to obtain an approximate reconstruction in one of them. We emphasize that for these results the subspaces don’t necessarily decompose $L^2(\mathbb{R})$ in direct sum, as we assumed before. As a consequence we obtain in the next section sufficient conditions on the generators for approximate oblique duality (see Corollaries 8.1 and 8.2). In order to prove the results we will use this lemma:

**Lemma 7.1.** Let $\phi, \tilde{\phi} \in L^2(\mathbb{R})$ be such that $\{T_k \phi\}_{k \in \mathbb{Z}}$ and $\{T_k \tilde{\phi}\}_{k \in \mathbb{Z}}$ are Bessel sequences. The following holds:

(i) If $f \in L^2(\mathbb{R})$, then $[\hat{f}, \hat{\phi}] \in C^2_{\text{per}}$ and

$$\sum_{k \in \mathbb{Z}} \langle \hat{f}, \hat{T_k \phi} \rangle T_k \hat{\phi} = \hat{\phi} [\hat{f}, \hat{\phi}].$$

(ii) If $f \in \text{span}\{T_k \phi\}_{k \in \mathbb{Z}}$, then

$$(7.1) \quad \sum_{k \in \mathbb{Z}} \langle \hat{f}, \hat{T_k \phi} \rangle T_k \hat{\phi} = \hat{f} [\hat{\phi}, \hat{\phi}].$$

Proof. By the Cauchy-Schwarz inequality and Theorem 2.7(i), if $f \in L^2(\mathbb{R})$ then

$$\int_0^1 |[\hat{f}, \hat{\phi}]| d\gamma \leq \beta \int_0^1 \sum_{n \in \mathbb{Z}} |\hat{f}(\gamma + n)|^2 d\gamma \leq \beta \|f\|^2,$$

where $\beta$ is an upper frame bound of $\{T_k \phi\}_{k \in \mathbb{Z}}$. So the equality in (i) – which coincides with [3, Equality (12)] – holds.

From (i), if $f = \sum_{k \in \mathbb{Z}} c_k T_k \phi$, where $\{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ has a finite number of nonzero elements,

$$\sum_{k \in \mathbb{Z}} \langle \hat{f}, \hat{T_k \phi} \rangle T_k \hat{\phi} = \hat{\phi}(\gamma) \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} c_k e^{-i2\pi k \gamma} \right) \hat{\phi}(\gamma + n) \tilde{\phi}(\gamma + n)$$

$$= \left( \sum_{k \in \mathbb{Z}} c_k e^{-i2\pi k \gamma} \right) \hat{\phi}(\gamma) \hat{\phi}(\gamma) \tilde{\phi}(\gamma) \tilde{\phi}(\gamma) = \hat{f}(\gamma) [\hat{\phi}, \hat{\phi}](\gamma).$$

It follows that (7.1) holds for $f \in \text{span}\{T_k \phi\}_{k \in \mathbb{Z}}$. Since the operator $f \mapsto \sum_{k \in \mathbb{Z}} \langle f, T_k \tilde{\phi} \rangle T_k \phi$ is continuous and by [2, (i)] $[\hat{\phi}, \hat{\phi}] \in L^\infty(\mathbb{R})$, (7.1) holds for $f \in \text{span}\{T_k \phi\}_{k \in \mathbb{Z}}$. \qed
The key result of this section is the following:

**Theorem 7.2.** Let \( \phi, \tilde{\phi} \in L^2(\mathbb{R}) \) be such that \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_k\tilde{\phi}\}_{k \in \mathbb{Z}} \) are Bessel sequences. Let \( W = \text{span}\{T_k\phi\}_{k \in \mathbb{Z}} \) and \( V = \text{span}\{T_k\tilde{\phi}\}_{k \in \mathbb{Z}} \). Then, given \( \epsilon \geq 0 \), the following statements are equivalent:

(i) \( \|f - \sum_{k \in \mathbb{Z}} \langle f, T_k\tilde{\phi} \rangle T_k\phi\| \leq \epsilon \|f\| \), for \( f \in W \).

(ii) \( \|f - \sum_{k \in \mathbb{Z}} \langle f, T_kP_w\tilde{\phi} \rangle T_k\phi\| \leq \epsilon \|f\| \), for \( f \in W \).

(iii) \( |[\hat{\phi}, \tilde{\phi}](\gamma) - 1| \leq \epsilon \) for a.e. \( \gamma \in N(\Phi)^c \).

Moreover, if \( \epsilon < 1 \), then statements (i) to (iii) are also equivalent to:

(iv) \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_kP_w\tilde{\phi}\}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate dual frames in \( W \).

**Proof.** (i) \( \Leftrightarrow \) (ii): The operators \( P_W \) and \( T_k \) commute. So, given \( f \in W \),

\[
\|f - \sum_{k \in \mathbb{Z}} \langle f, T_kP_w\tilde{\phi} \rangle T_k\phi\| = \|f - \sum_{k \in \mathbb{Z}} \langle f, T_k\tilde{\phi} \rangle T_k\phi\|.
\]

(i) \( \Rightarrow \) (iii): Assume that (iii) does not hold. Then there exists \( E \subseteq N(\Phi)^c \cap [0,1) \) such that \( |E| > 0 \) and

\[
\|f - \sum_{k \in \mathbb{Z}} \langle f, T_k\tilde{\phi} \rangle T_k\phi\| \geq \epsilon \|f\|.
\]

We will see that there exists \( f \in W \) such that

\[
\|f - \sum_{k \in \mathbb{Z}} \langle f, T_k\tilde{\phi} \rangle T_k\phi\| > \epsilon \|f\|.
\]

We can write \( E = \bigcup_{k \in \mathbb{N}} E_k \), where

\[
E_k = \left\{ \gamma \in N(\Phi)^c \cap [0,1) : |[\hat{\phi}, \tilde{\phi}](\gamma) - 1| \geq \frac{1}{k} + \epsilon \right\}
\]

for \( k \in \mathbb{N} \). If \( |E_k| = 0 \) for all \( k \), then \( |E| = 0 \), which is a contradiction. So, there exists \( \epsilon' > \epsilon \) and \( E' \subseteq E \) such that \( |E'| > 0 \) and

\[
|[\hat{\phi}, \tilde{\phi}](\gamma) - 1| \geq \epsilon' \quad \text{for} \quad \gamma \in E'.
\]

Let \( E'_p = \bigcup_{k \in \mathbb{Z}} (E' + k) \) and \( F \in C^2_{\text{per}} \) such that \( \text{supp}(F) \subseteq E'_p \) and \( |\text{supp}(F)| > 0 \). Let \( f \in W \) that verifies \( f = F\hat{\phi} \). Then \( f \neq 0 \) and by Lemma 7.1(ii) and (7.2),

\[
\|f - \sum_{k \in \mathbb{Z}} \langle f, T_k\tilde{\phi} \rangle T_k\phi\|^2 \geq (\epsilon')^2 \||\hat{f}\|^2\|^2 = (\epsilon')^2 \||\hat{f}\|^2\|^2 > 2\epsilon^2 \||f\|^2\|^2,
\]

which contradicts (i).

(iii) \( \Rightarrow \) (i): Let \( f = \sum_{k \in \mathbb{Z}} c_k T_k\phi \), where \( \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \) has a finite number of nonzero elements. Note that if \( \Phi(\gamma) = 0 \) then \( \hat{f}(\gamma) = 0 \). By Lemma 7.1(ii) and (iii), we have that

\[
\|f - \sum_{k \in \mathbb{Z}} \langle f, T_k\tilde{\phi} \rangle T_k\phi\| = \|\chi_{N(\Phi)^c} \hat{f}(1 - [\hat{\phi}, \tilde{\phi}])\| \leq \epsilon \||\hat{f}\|^2\| = \epsilon \|f\|.
\]

So, as in the proof of Lemma 7.1(ii), (i) holds for \( f \in \text{span}\{T_k\phi\}_{k \in \mathbb{Z}} \).

If \( \epsilon < 1 \), (ii) \( \Leftrightarrow \) (i) follows from Remark 7.2(iii). \( \square \)

**Remark 7.3.** If \( \epsilon < 1 \), from condition (iii), \( [\hat{\phi}, \tilde{\phi}](\gamma) \neq 0 \) for a.e. \( \gamma \in N(\Phi)^c \). So \( N(\tilde{\Phi}) \subseteq N(\Phi) \) a.e..
Let \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) be a Bessel sequence, \( W = \text{span} \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \phi_1 \in L^2(\mathbb{R}) \). The theorem below gives conditions so that there exists \( \tilde{\phi} \in \text{span} \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \) whose translates allow approximate reconstruction in \( W \). Furthermore, it yields a result about approximate dual frames in shift-invariant subspaces.

**Theorem 7.4.** Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \) are Bessel sequences. Let \( W = \text{span} \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( V = \text{span} \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \). Let \( \tilde{\phi}_1 \in W \) and \( \tilde{\phi} \in V \) such that \( \tilde{\phi}_1 = \overline{H} \tilde{\phi} \) and \( \tilde{\phi} = \overline{H} \phi_1 \) where \( H \in C^\infty_{\text{per}} \). Assume \( \epsilon \geq 0 \). Then \( \{ T_k \tilde{\phi} \}_{k \in \mathbb{Z}} \) and \( \{ T_k \tilde{\phi}_1 \}_{k \in \mathbb{Z}} \) are Bessel sequences and the following are equivalent:

1. \( \| f - \sum_{k \in \mathbb{Z}} \langle f, T_k \tilde{\phi} \rangle T_k \phi \| \leq \epsilon \| f \| \) for \( f \in W \).
2. \( \| \overline{H}(\gamma) [\tilde{\phi}, \tilde{\phi}_1](\gamma) - 1 \| \leq \epsilon \) for a.e. \( \gamma \in N(\Phi)^c \).

Moreover, if (ii) holds with \( \epsilon < 1 \) and \( N(\Phi_1) = N(\Phi) \), then \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k P_W \tilde{\phi} \}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate dual frames in \( W \), whereas \( \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \) and \( \{ T_k P_V \tilde{\phi}_1 \}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate dual frames in \( V \).

**Proof.** Observe that

\[
\tilde{\Phi}(\gamma) = \sum_{k \in \mathbb{Z}} |\overline{H}(\gamma) \tilde{\phi}_1(\gamma + k)|^2 = |\overline{H}(\gamma)|^2 \Phi_1(\gamma) \leq \| \overline{H} \|_{L^2(0,1)}^2 \beta_1,
\]

where \( \beta_1 \) is the Bessel bound of \( \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \). So, by Theorem 2.7(i), \( \{ T_k \tilde{\phi}_1 \}_{k \in \mathbb{Z}} \) is a Bessel sequence. Analogously it can be seen that \( \{ T_k \tilde{\phi}_1 \}_{k \in \mathbb{Z}} \) is a Bessel sequence.

Since \( |[\tilde{\phi}, \tilde{\phi}_1](\gamma) - 1| = |\overline{H}(\gamma)| |\tilde{\phi}, \tilde{\phi}_1|(\gamma) - 1| \), (i) \( \Leftrightarrow \) (ii) follows from Theorem 7.2.

The proof of the last part is similar to the proof of (iii) \( \Rightarrow \) (i) of Theorem 7.2.

With \( \epsilon = 0 \), from Theorem 7.2 we obtain [3, Theorem 4.1.] and from Theorem 7.2 we obtain [3, Theorem 4.3.].

8. APPROXIMATE OBLIQUE DUAL FRAMES IN SHIFT-ININVARIANT SPACES

In this section we study the concept of approximate oblique duality when \( H = L^2(\mathbb{R}) \) and \( W \) and \( V \) are shift-invariant subspaces of \( L^2(\mathbb{R}) \).

The following corollaries give conditions on the generators of two shift-invariant subspaces for approximate oblique duality. The first is a consequence of Theorem 7.2 and Proposition 4.2.

**Corollary 8.1.** Let \( L^2(\mathbb{R}) = W \oplus V^\perp \). Let \( \phi, \tilde{\phi} \in L^2(\mathbb{R}) \) be such that \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k \tilde{\phi} \}_{k \in \mathbb{Z}} \) are frames for \( W \) and \( V \) respectively. Assume \( \epsilon \geq 0 \). If

\[
|[\tilde{\phi}, \tilde{\phi}_1](\gamma) - 1| \leq \frac{\epsilon}{\| \pi_{WV^\perp} \|} \quad \text{for a.e. } \gamma \in N(\Phi)^c
\]

then \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k \tilde{\phi}_1 \}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate oblique dual frames.

Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \) are frames for \( W \) and \( V \) respectively. For what follows we recall that, by Proposition 2.8 if \( L^2(\mathbb{R}) = W \oplus V^\perp \) then \( N(\Phi) = N(\Phi_1) \).

**Corollary 8.2.** Let \( L^2(\mathbb{R}) = W \oplus V^\perp \). Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \) are frames for \( W \) and \( V \) respectively. Let \( \tilde{\phi} \in V \) be such that \( \tilde{\phi} = \overline{H} \phi_1 \) where \( H \in C^\infty_{\text{per}} \). Assume \( 0 \leq \epsilon < 1 \). If
\[ \left| \bar{H}(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) - 1 \right| \leq \frac{\epsilon}{\| \pi_{W^\perp V} \|} \text{ for a.e. } \gamma \in N(\Phi)^c \]
then \( \{T_k\phi\}_{k \in \mathbb{Z}} \) is a frame for \( \mathcal{V} \) such that \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_k\phi\}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate oblique dual frames.

**Proof.** By Proposition 2.8, there exists \( c > 0 \) such that
\[ \left| \bar{H}(\gamma) \right| \leq \frac{1}{\|\phi_{\Phi_1}(\gamma)\|} \left( 1 + \frac{\epsilon}{\|\pi_{W^\perp V}\|} \right) \leq \frac{1}{c} \left( 1 + \frac{\epsilon}{\|\pi_{W^\perp V}\|} \right) \]
for a.e. \( \gamma \in N(\Phi)^c \). So, by Theorem 2.7(i),
\[ \bar{\Phi}(\gamma) = |\bar{H}(\gamma)|^2 \Phi_1(\gamma) \leq \frac{1}{c} \left( 1 + \frac{\epsilon}{\|\pi_{W^\perp V}\|} \right)^2 \beta_1, \]
where \( \beta_1 \) is an upper frame bound of \( \{T_k\phi_1\}_{k \in \mathbb{Z}} \). Applying again Theorem 2.7(i) we conclude that \( \{T_k\phi\}_{k \in \mathbb{Z}} \) is a Bessel sequence.

Now the conclusion follows from Theorem 7.1, Proposition 1.2 and Remark 3.2(iii). \( \square \)

The following theorem gives an expression for the Fourier transform of the oblique projection when the subspaces are shift-invariant, in terms of the corresponding generators.

**Theorem 8.3.** Let \( L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_k\phi_1\}_{k \in \mathbb{Z}} \)
are frames for \( \mathcal{W} \) and \( \mathcal{V} \) respectively. Then
\[ \pi_{W^\perp V^\perp} \hat{f}(\gamma) = \begin{cases} \frac{[\hat{\phi}, \hat{\phi}_1](\gamma)}{|\phi_{\Phi_1}(\gamma)|} \hat{\phi}(\gamma) & \gamma \in N(\Phi)^c \\ 0 & \gamma \in N(\Phi) \end{cases} \]
for \( f \in L^2(\mathbb{R}) \).

**Proof.** Let \( f \in L^2(\mathbb{R}) \). Since \( \{T_k\phi\}_{k \in \mathbb{Z}} \) is a frame for \( \mathcal{W} \) there exists \( \tau \in C^2_{\text{per}} \) such that
\[ \pi_{W^\perp V^\perp} \hat{f} = \tau \hat{\phi}. \]
So, if \( \gamma \in N(\Phi) \), \( \pi_{W^\perp V^\perp} \hat{f}(\gamma) = 0 \). By [21, Lemma 2.8],
\[ [\hat{\phi}, \hat{\phi}_1](\gamma) = [\pi_{W^\perp V^\perp} \hat{f}, \hat{\phi}_1](\gamma) = \tau(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma). \]

If \( \gamma \in N(\Phi)^c \), by Proposition 2.8 \( [\hat{\phi}, \hat{\phi}_1](\gamma) \neq 0 \), hence \( \tau(\gamma) = \frac{[\hat{\phi}, \hat{\phi}_1](\gamma)}{|\phi_{\Phi_1}(\gamma)|} \) and \( \pi_{W^\perp V^\perp} \hat{f}(\gamma) = \frac{[\hat{\phi}, \hat{\phi}_1](\gamma)}{|\phi_{\Phi_1}(\gamma)|} \). \( \square \)

If \( \mathcal{W} = \mathcal{V} \) the previous theorem reduces to [21, Theorem 2.9]. Using the expression for the Fourier transform of the oblique projection given in Theorem 8.3 we obtain the following sufficient condition for approximate duality which is different to the one of Corollary 8.2. If \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_k\phi_1\}_{k \in \mathbb{Z}} \) are Riesz bases for \( \mathcal{W} \) and \( \mathcal{V} \) respectively, we have an equality in [25] [22]. So, in this case the sufficient condition of the next theorem is weaker than the one of Corollary 8.2.

**Theorem 8.4.** Let \( L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_k\phi_1\}_{k \in \mathbb{Z}} \)
are frames for \( \mathcal{W} \) and \( \mathcal{V} \) respectively. Let \( 0 \leq \epsilon < 1 \). If \( \tilde{\phi} \in \mathcal{V} \) verifies \( \hat{\phi} = \hat{H}\phi_1 \), where \( \hat{H} \in C^2_{\text{per}} \)
is such that
\[ (8.1) \left| \bar{H}(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) - 1 \right| \leq \epsilon \left| \frac{[\hat{\phi}, \hat{\phi}_1](\gamma)}{\Phi(\gamma) \Phi_1(\gamma)} \right| \]
for a.e. \( \gamma \in N(\Phi)^c \), then \( \{T_k\phi\}_{k \in \mathbb{Z}} \) and \( \{T_k\phi\}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate oblique dual frames.
Proof. Let \( f \in L^2(\mathbb{R}) \). Using Theorem 8.3 and Lemma 7.1(i), we obtain

\[
\|\pi_{\mathcal{W}^\perp} f - \sum_{k \in \mathbb{Z}} \langle f, T_k \hat{\phi} \rangle T_k \phi \| = \|\pi_{\mathcal{W}^\perp} f - \sum_{k \in \mathbb{Z}} \langle f, T_k \hat{\phi} \rangle \hat{T}_k \phi \|
\]

\[
= \int_{N(\Phi)} \left| \frac{\hat{f}(\gamma)}{\hat{\phi}_1(\gamma)} - \hat{\phi}(\gamma) \right|^2 \, d\gamma
\]

\[
= \int_{N(\Phi) \cap (0,1)} \Phi(\gamma) \left| \frac{\hat{f}(\gamma)}{\hat{\phi}_1(\gamma)} - \frac{1}{[\hat{\phi}, \hat{\phi}_1](\gamma)} - \frac{1}{\bar{H}(\gamma)} \right|^2 \, d\gamma
\]

\[
\leq \int_{N(\Phi) \cap (0,1)} \Phi(\gamma) \Phi(1) \left| \frac{\hat{f}(\gamma + n)}{[\hat{\phi}, \hat{\phi}_1](\gamma)} \right|^2 \, d\gamma
\]

\[
\leq c^2 \int_0^1 \left| \hat{f}(\gamma + n) \right|^2 \, d\gamma = \varepsilon^2 \|f\|^2.
\]

Hence, by Definition 8.1 and Remark 8.2(iii), \( \{T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{T_k \hat{\phi} \}_{k \in \mathbb{Z}} \) are \( \varepsilon \)-approximate oblique dual frames.

\[ \square \]

Remark 8.5. Assume that (8.1) does not hold. Then there exists \( E \subseteq N(\Phi) \cap [0,1) \) with \( |E| > 0 \) and that

\[
\left| \bar{H}(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) - 1 \right| > \varepsilon \frac{|\hat{\phi}, \hat{\phi}_1|(|\gamma|)}{\Phi(\gamma)\Phi_1(\gamma)}
\]

for \( \gamma \in E \). Let \( \varepsilon' > 0 \) with \( \varepsilon' < \frac{c}{\sqrt{\beta_2}} \) (where \( c > 0 \) is the constant of Proposition 7.2 and \( \beta, \beta_1 \) are upper frame bounds of \( \{T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{T_k \hat{\phi} \}_{k \in \mathbb{Z}} \), respectively). Note that \( \varepsilon' < \frac{c}{\sqrt{\beta_2}} \leq \frac{c}{\sqrt{\beta_1}} \). Let \( E_p = \bigcup_{k \in \mathbb{Z}} (E + k) \) and \( F \in C^2_{\text{per}} \) such that \( \text{supp}(F) \subseteq E_p \) and \( |\text{supp}(F)| > 0 \). Let \( f \in L^2(\mathbb{R}) \) given by \( \hat{f} = F \hat{\phi} \). Then \( f \in \mathcal{W}, f \neq 0 \) and \( \pi_{\mathcal{W}^\perp} f = f \). Taking into account Lemma 7.1(ii), we get

\[
\|\pi_{\mathcal{W}^\perp} f - \sum_{k \in \mathbb{Z}} \langle f, T_k \hat{\phi} \rangle T_k \phi \| = \int_{N(\Phi) \cap (0,1)} \left| \hat{f}(\gamma) \right|^2 \left| 1 - \bar{H}(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) \right|^2 \, d\gamma
\]

\[
\geq \varepsilon^2 \int_{N(\Phi) \cap (0,1)} \left| \hat{f}(\gamma) \right|^2 \left| \frac{[\hat{\phi}, \hat{\phi}_1](\gamma)}{\Phi(\gamma)\Phi_1(\gamma)} \right|^2 \, d\gamma \geq \left( \frac{c}{\sqrt{\beta_1}} \right)^2 \int_{N(\Phi) \cap (0,1)} \left| \hat{f}(\gamma) \right|^2 \, d\gamma > \varepsilon^2 \|f\|^2.
\]

This shows that \( \{T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{T_k \hat{\phi} \}_{k \in \mathbb{Z}} \) are not \( \varepsilon' \)-approximate oblique dual frames.

We have the following necessary condition for approximate oblique duality.

**Theorem 8.6.** Let \( L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp \). Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{T_k \phi_1 \}_{k \in \mathbb{Z}} \) are frames for \( \mathcal{W} \) and \( \mathcal{V} \) respectively. Set \( \hat{\phi} \in \mathcal{V} \) such that \( \hat{\phi} = \bar{H} \hat{\phi}_1 \), where \( \bar{H} \in C^2_{\text{per}} \). Let \( \varepsilon \geq 0 \).
If \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ \hat{T}_k \phi \}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate oblique dual frames, then
\[
(8.2) \quad \left| H(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) - 1 \right| \leq \epsilon
\]
for a.e. \( \gamma \in N(\Phi)^c \).

Proof. We proceed in a similar way to the proof of Theorem 7.2. Assume (8.2) does not hold. Then there exists \( \epsilon' > \epsilon \) and \( E' \subseteq N(\Phi)^c \cap [0, 1) \) such that \( |E'| > 0 \) and
\[
\left| H(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) - 1 \right| \geq \epsilon' \quad \text{for} \quad \gamma \in E'.
\]
Let \( E'_p = \bigcup_{k \in \mathbb{Z}} (E' + k) \) and \( F \in C^2_{\text{per}} \) such that \( \text{supp}(F) \subseteq E'_p \) and \( |\text{supp}(F)| > 0 \). Let \( f \in \mathcal{W} \) with \( \hat{f} = F \hat{\phi} \). In this case, \( \pi_{\mathcal{W} \perp} f = f \) and \( f \neq 0 \). Using Lemma 7.1(ii), we obtain
\[
\| \pi_{\mathcal{W} \perp} f - \sum_{k \in \mathbb{Z}} \langle f, T_k \phi \rangle T_k \phi \| \leq \int_{N(\Phi)^c} \left| \hat{f}(\gamma) \right|^2 \left| 1 - H(\gamma)[\hat{\phi}, \hat{\phi}_1](\gamma) \right|^2 d\gamma
\leq \epsilon'^2 \int_{N(\Phi)^c} |\hat{f}(\gamma)|^2 d\gamma = \epsilon'^2 \int |\hat{f}(\gamma)|^2 d\gamma > \epsilon'^2 \| f \|^2.
\]
This contradicts the fact that \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ \hat{T}_k \phi \}_{k \in \mathbb{Z}} \) are \( \epsilon \)-approximate oblique dual frames. \( \square \)

9. Numerical and Computational Aspects of Approximate Oblique Dual Frames

In this section we highlight the importance of approximate oblique dual frames from a numerical and computational point of view, illustrating our analysis with an example of \( B \)-splines.

Assume that \( L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V} \perp \). Let \( \phi, \phi_1 \in L^2(\mathbb{R}) \) be such that \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) and \( \{ T_k \phi_1 \}_{k \in \mathbb{Z}} \) are frames for \( \mathcal{W} \) and \( \mathcal{V} \) respectively. By [8, Theorem 4.3] and Proposition 2.8, the unique oblique dual frame of \( \{ T_k \phi \}_{k \in \mathbb{Z}} \) in \( \mathcal{V} \) is \( \{ T_k \psi \}_{k \in \mathbb{Z}} \), where \( \psi = \sum_{k \in \mathbb{Z}} c_k T_k \phi_1 \) and \( \{ c_k \}_{k \in \mathbb{Z}} \) are the Fourier coefficients of
\[
H(\gamma) = \frac{1}{\langle \hat{\phi}, \hat{\phi}_1 \rangle(\gamma)}.
\]
In practice, the generator of the oblique dual frame can only be obtained approximately since the computation of the previous series and the calculus of the Fourier coefficients are not exact. So, in fact, in the applications we always have to work with an approximate oblique dual frame. In the following example we illustrate this situation. We consider here the effect of the truncation required for the series that defines \( \psi \) when \( H \) is not a trigonometric polynomial. In this case, the used \( \epsilon(K) \)-approximate dual frame is \( \{ T_k \psi_K \}_{k \in \mathbb{Z}} \) where \( \psi_K = \sum_{|k| \leq K} c_k T_k \phi_1 \) for some \( K \in \mathbb{N} \). Taking into account (8.1) we have the error
\[
\epsilon(K) = \sup_{\gamma \in [0, 1]} \sqrt{\Phi(\gamma) \Phi_1(\gamma)} | T_K(\gamma) - H(\gamma) |
\]
where \( H_K \in C^\infty_{\text{per}} \) and restricted to \( [0, 1) \) is equal to \( \sum_{|k| \leq K} c_k e^{-2i\pi k \gamma} \).

Consider now in particular shift-invariant subspaces generated by \( B \)-splines (see [8, Example 4.5]). We recall that \( B \)-splines \( B_n \) are functions which are piecewise polynomials. They are defined inductively as
\[
B_1(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x), \quad B_{n+1}(x) = \int_{-\infty}^{\infty} B_n(x - t)B_1(t)dt = \frac{1}{n+1} B_n(x),
\]
For each \( n \geq 2 \),
\[
B_n(x) = \frac{n!}{(n-1)!} \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \left( x + \frac{j}{n} - j \right)^{n-1}, \quad x \in \mathbb{R},
\]
where \( f(x)_+ = \max\{f(x), 0\} \) if \( f: \mathbb{R} \to \mathbb{R} \). B-splines have the following properties: \( B_n \in C^{n-2}(\mathbb{R}) \) for \( n \geq 2 \), \( \text{supp}(B_n) = \left[ -\frac{n}{2}, -\frac{n}{2}\right] \) and \( B_n > 0 \) on \( \left( -\frac{n}{2}, -\frac{n}{2}\right) \). See e.g. [2, A.8] for more details.

Let \( n, m \in \mathbb{Z} \), \( \phi = B_n \) and \( \phi_1 = B_{n+2m} \). Here,

\[
H(\gamma) = \frac{1}{\sum_{z \in \mathbb{Z}} \left(\frac{\sin \pi (\gamma + z)}{\pi (\gamma + z)}\right)^{2(n+m)}}
\]

is not a trigonometric polynomial (see [2, Example 9.4.3]). So, the generator of the oblique dual frame of \( \{T_kB_n\}_{k \in \mathbb{Z}} \) in \( \mathcal{V} \) has not compact support whereas for the generator \( \psi_K \) of the approximate oblique dual frame we have \( \text{supp}(\psi_K) = \left[ -\frac{n+2m+2K}{2}, \frac{n+2m+2K}{2}\right] \). We also note that \( \psi_K \) has the same smoothness as \( B_{n+2m} \).

In the figures, we show the plots obtained with MATLAB of \( H \) (solid line), \( H_K \) (dash-dot line), \( \psi \) (solid line) and \( \psi_K \) (dash-dot line) for \( K = 0, 1, 2, 3 \). Figure 1 corresponds to the case \( n = 1 \) and \( m = 3 \) and Figure 2 to the case \( n = 2 \) and \( m = 1 \).

According to the expression of \( H \) and \( H_K \), the errors \( \varepsilon(K) \) depend on \( n + m \). The entries of the tables that appear below were computed using MATLAB. The first three show the supports of \( \psi_K \), the absolute values \( |c_{K+1}| \), the norms \( \|\psi_K - \psi\| \) and the estimated errors \( \varepsilon(K) \) for \( K = 0, 1, \ldots, 6 \). Table 1 corresponds to the case \( n = 1 \) and \( m = 3 \) and Table 2 to the case \( n = 2 \) and \( m = 1 \). We observe that \( |c_{K+1}| \to 0 \) as \( K \to 0 \). This convergence is slower when \( n + m \) increases. This is reflected in the behavior of \( \|\psi_K - \psi\| \) and \( \varepsilon(K) \) which tend to zero more slowly when \( n + m \) increases too. The error \( \varepsilon(K) \) decreases exponentially. Applying the function \text{polyfit} to \( \log_2 \varepsilon(K) \) we obtained approximate expressions of \( \varepsilon(K) \) for \( n + m = 1, \ldots, 8 \) that appear in Table 3.
Figure 2. Case $n = 2$ and $m = 1$. (a) The functions $H$ (solid line) and $H_K$ (dash-dot line); (b) The generators $\psi$ (solid line) and $\psi_K$ (dash-dot line).

| $K$ | $\text{supp}(\psi_K)$ | $|c_{K+1}|$ | $\|\psi_K - \psi\|$ | $\epsilon(K)$ |
|-----|------------------------|-------------|------------------|-------------|
| 0   | $[-3.5, 3.5]$          | 3.0910      | 4.3233           | 3.9647      |
| 1   | $[-4.5, 4.5]$          | 1.7080      | 2.4003           | 2.2172      |
| 2   | $[-5.5, 5.5]$          | 0.9208      | 1.2954           | 1.1984      |
| 3   | $[-6.5, 6.5]$          | 0.4937      | 0.6947           | 0.6428      |
| 4   | $[-7.5, 7.5]$          | 0.2644      | 0.3720           | 0.3442      |
| 5   | $[-8.5, 8.5]$          | 0.1415      | 0.1992           | 0.1842      |
| 6   | $[-9.5, 9.5]$          | 0.0758      | 0.1066           | 0.0986      |

Table 1. Case $n = 1$ and $m = 3$.

| $K$ | $\text{supp}(\psi_K)$ | $|c_{K+1}|$ | $\|\psi_K - \psi\|$ | $\epsilon(K)$ |
|-----|------------------------|-------------|------------------|-------------|
| 0   | $[-2, 2]$              | 1.3217      | 2.2396           | 1.8421      |
| 1   | $[-3, 3]$              | 0.5733      | 0.9715           | 0.8012      |
| 2   | $[-4, 4]$              | 0.2470      | 0.4186           | 0.3453      |
| 3   | $[-5, 5]$              | 0.1064      | 0.1803           | 0.1487      |
| 4   | $[-6, 6]$              | 0.0458      | 0.0776           | 0.0640      |
| 5   | $[-7, 7]$              | 0.0197      | 0.0334           | 0.0276      |
| 6   | $[-8, 8]$              | 0.0085      | 0.0144           | 0.0119      |

Table 2. Case $n = 2$ and $m = 1$.

| $n + m$ | $\epsilon(K)$ |
|---------|----------------|
| 2       | $2^{-1.9K}$ 0.73 |
| 3       | $2^{-1.21K}$ 1.84 |
| 4       | $2^{-0.89K}$ 3.96 |
| 5       | $2^{-0.69K}$ 8.24 |
| 6       | $2^{-0.56K}$ 17.1 |
| 7       | $2^{-0.46K}$ 35.93 |
| 8       | $2^{-0.38K}$ 76.63 |

Table 3. Approximated expressions of $\epsilon(K)$ for different values of $n + m$. 
DECLARATION OF INTEREST: none

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