Deriving the time-dependent Schrödinger $m$- and $p$-equations from the Klein-Gordon equation.

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I present an alternative and rather direct way to derive the well known Schrödinger equation for a quantum wavefunction, by starting with the Klein Gordon equation and applying a directional factorization scheme. And since if you have a directionally factorizing hammer, everything looks like a factorizable nail, I also derive an alternative wavefunction propagation equation in the momentum-dominated limit. This new Schrödinger $p$-equation therefore provides a potentially useful complement to the traditional Schrödinger $m$-equation’s mass-dominated limit.

I. INTRODUCTION

There have been many and varied (re)derivations of the Schrödinger equation [1], based on a variety of principles – e.g. Feynman path integrals [2], stochastics (e.g. [3, 4]), utilizing axioms [5], or by applying various ad hoc approximations to variants of the Klein-Gordon equation (e.g. [6]). Here I present another method, inspired by the success of both the Salpeter Hamiltonian and a gravitational potential.

Klein & Gordon started with the relativistic equation for the energy of a massive particle,

$$ E^2 = m^2 c^4 + p^2 c^2, \quad (1) $$

and, by replacing $E$ and $p$ with operators using the correspondence principle [4]. From a mathematical perspective, the correspondence principle is just the process of switching between one domain and its Fourier transformed counterpart. Here, the correspondence is

$$ E \leftrightarrow i\hbar \partial_t, \quad p \leftrightarrow -i\hbar \nabla, \quad (2) $$

which allows us to directly convert eqn. (1) into the Klein-Gordon (KG) equation for the wavefunction of a single massive particle, i.e.

$$ \left[ \hbar^2 \partial_t^2 + m^2 c^4 - \hbar^2 c^2 \nabla^2 \right] \Phi(r,t) = 0, \quad (4) $$

or

$$ \left[ \frac{1}{c^2} \partial_t^2 + \frac{m^2 c^2}{\hbar^2} - \nabla^2 \right] \Phi(r,t) = 0. \quad (5) $$

This Klein-Gordon second order wave equation can, if desired, be factorized using spinors to give the first order Dirac equation. However, this does not allow for anything that might alter the wavefunction behaviour away from that in a simple vacuum, so to address this lack I consider modifications inspired by both the Salpeter Hamiltonian and a gravitational potential.

The Salpeter Hamiltonian: It is useful – especially when deriving the Schrödinger equations – to be able to include the effect of a static potential within which the particle is moving. We might therefore start with the Salpeter Hamiltonian [11]

$$ H\Phi = \left[ \sqrt{m^2 c^4 + p^2 c^2} + V(r) \right] \Phi, \quad (6) $$

where the Hamiltonian can be applied twice to the wavefunction $\Phi$. Then, by identifying $H$ with the energy $E$, we get the squared form

$$ (E - V)^2 \Phi(r,t) = \left[ m^2 c^4 + p^2 c^2 \right] \Phi(r,t), \quad (7) $$

which matches up to the Klein-Gordon starting point under the condition that $V = 0$. As would be expected, the same as the Klein-Gordon equation in a Coulomb potential if $V = -e^2/r$.

In the following, I will call the potential $V$ the “Salpeter potential” to specify its conceptual origin.

Gravitational potential: Although it might seem unlikely that gravitational potentials have sufficient variation in either space or time to produce effects that apply to quantum phenomena, it is nevertheless interesting to see how gravity might appear in the Schrödinger equation. In general relativity, the Newtonian limit for a gravitational potential $\Xi(r,t)$ gives an expression for $E^2$ which is

$$ E^2 = m^2 c^4 \left[ 1 + 2\Xi(r,t) \right] + p^2 c^2. \quad (8) $$

In an operator form, applied to some wavefunction $\Phi$, this would then be

$$ E^2 \Phi(r,t) = \left[ m^2 c^4 \left[ 1 + 2\Xi(r,t) \right] + p^2 c^2 \right] \Phi(r,t). \quad (9) $$
Since there is no elegant way to handle the time dependent $\Xi$ term elegantly as part of the energy (i.e. on the LHS), it is best left as a perturbation and treated in the same way as the momentum – both are small in the non-relativistic limits. Note that when properly scaled, we can also use $\Xi(r,t)$ as a proxy for any other space and time dependent potential that might affect our system.

Combined potentials: So we only have to perform the following calculation once, I will combine the Salpeter energy expression eqn. (8) with that allowing for a gravitational potential eqn. (9). For an operator-like form, applied to a wavefunction $\Phi$, we have

$$[E - V(r)]^2 \Phi(r,t) = [m^2 c^4 \left(1 + 2\Xi(r,t)\right) + p^2 c^2] \Phi(r,t)$$

$$= [m^2 c^4 + 2m^2 c^4 \Xi(r,t) + p^2 c^2] \Phi(r,t).$$

(10)

Here the potential $V(r)$ has no $t$ dependence, because allowing that would complicate the transformation of the LHS into a time derivative. However, any time dependent part of a more general $V$ could easily be merged into $\Xi(r,t)$. In a Klein-Gordon wave equation form, this is

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{m^2 c^2}{h^2} \left[1 + 2\Xi(r,t)\right] - \nabla^2 \right\} \Phi(r,t) = 0. \quad (12)$$

In frequency space ($\omega, \mathbf{r}$-space), this becomes

$$\left\{ -\frac{\omega^2}{c^2} + \frac{m^2 c^2}{h^2} \left[1 + 2\Xi(r,\omega)\right] - \nabla^2 \right\} \Phi(r,\omega) = 0, \quad (13)$$

where the brie (here $\hat{\Xi}$) tells us to convolve $\Xi$ with $\Phi$ over $\omega$. Alternatively, in wavevector space ($t, \mathbf{k}$-space), this becomes

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{m^2 c^2}{h^2} \left[1 + 2\hat{\Xi}(k,\omega)\right] + k^2 \right\} \Phi(k,\omega) = 0, \quad (14)$$

where the hat (here $\hat{\Xi}$) tells us to convolve $\Xi$ with $\Phi$ over $k$. Both types of convolution play no interesting role in the following calculations, and are merely an intermediate stage which disappears when the equations being used are converted back into their primary $t, \mathbf{r}$ domain.

Method: In what follows, I use eqn. (12) which contains two different types of potential, to derive approximate equations which have only first order derivatives in the propagation variable; i.e. $t$ for the usual temporally propagated Schrödinger equation. To complement the Schrödinger equation derivation, I also derive a spatially-propagated version, which is applicable in a different limit. For a more systematic look at the differences between temporal propagation and spatial propagation, the reader is referred to Ref. [9]. Further, although here we factorize in Cartesian coordinates, this is not the only possible choice [8]. Finally, note that my original source for the factorization method used was by Ferrando et al. [14].

II. MASS DOMINANT: THE SCHRÖDINGER EQUATION

We can see from the correspondence principle described above that the energy $E$ is related to evolution in time $t$, while also noting that in non-relativistic scenarios the bulk of a massive particle’s energy is frozen in its rest mass. Thus to reduce the second-order-in-time KG equations down to the first-order-in-time Schrödinger equation we need to manipulate the starting equations while focussing on the energy $E$, and the rest mass $m$.

To proceed I will follow the directional factorization method recently popularized in optics [1], albeit with an alternative physical focus on temporal propagation (see e.g. [8]). This is the most physically motivated factorization, and we decompose the system behaviour (waves) into directional components that then evolve either forward or backward in space, as shown in Fig. 1. To analyse temporal propagation, we need a useful reference parameter to characterise it, and it should preferably be one that remains constant. In this case, a frequency domain analysis is called for: we might therefore use either an energy or a frequency $\omega$. This means that the parts of the physics we wish to ascribe to the role of “reference propagation” must be time independent.

Start by defining $E' = E - V(r) = h\omega$ to work in a scaled frequency ($\omega$) space, so that

$$h^2 \omega^2 \Phi(r,\omega) = \left[ m^2 c^4 + 2m^2 c^4 \Xi(r,\omega) - h^2 \omega^2 \nabla^2 \right] \Phi(r,\omega)$$

(15)

$$\left[ h^2 \omega^2 - m^2 c^2 \right] \Phi = \left[ 2m^2 c^4 \Xi - h^2 \omega^2 \nabla^2 \right] \Phi \quad (16)$$

$$(h\omega - mc^2) \left( h\omega + mc^2 \right) \Phi = \left[ 2m^2 c^4 \Xi - h^2 \omega^2 \nabla^2 \right] \Phi \quad (17)$$

$$\Phi = \frac{\Phi}{(h\omega - mc^2) (h\omega + mc^2)} \quad (18)$$

$$\Phi = \left[ \frac{1/2mc^2}{h\omega - mc^2} - \frac{1/2mc^2}{h\omega + mc^2} \right] \Phi \quad (19)$$

where $\tilde{Q}(r,\omega) = 2m^2 c^4 \Xi(r,\omega) - h^2 \omega^2 \nabla^2 \quad (20)$
We can see from the term in square brackets on the RHS of eqn. (19) that \( \Phi \) evolves according to two complementary parts of differing sign. The term proportional to \((\hbar \omega - mc)^{-1}\) generates a forward-like evolution, and that proportional to \((\hbar \omega + mc)^{-1}\) generates a backward-like evolution \([19]\). As a result we can likewise split the wavefunction into corresponding pieces, with \( \Phi = \Phi_+ + \Phi_- \). When we transform back into the time domain, these will (must!) propagate forward in time \( t \), all the while holding information about the wavefunction as a function of \( r \). To avoid notational clutter, we use this fact as an excuse to omit the time argument, and only the \( r \) argument of \( \Phi_\pm \) will be given.

Further, since the \( \Phi_+ \) forward evolving component is by definition propagating to later times, its excitations therefore must (also) be understood to be evolving forward in space \( (r \to \infty) \). In contrast, the \( \Phi_- \) backward evolving component (also propagating to later times), therefore has excitations evolving backward in space \( (r \to -\infty) \).

Continuing the separation of \( \Phi_+ \) and \( \Phi_- \), we see that

\[
\Phi_+ (r) + \Phi_- (r) = \left[ \frac{1/2mc^2}{\hbar \omega - mc^2} \right] \Phi_\omega (r) \tag{21}
\]

\[
\Phi_\pm (r) = \pm \left\{ \frac{2m^2c^2 \Xi - \hbar^2 \nabla^2}{(2mc^2)(\hbar \omega \pm mc^2)} \right\} [\Phi_+ (r) + \Phi_- (r)] \tag{22}
\]

\[
\Phi_\pm (r) = \pm \left\{ \frac{m^2c^2 \Xi - \hbar^2 \nabla^2}{\hbar \omega \mp mc^2} \right\} [\Phi_+ (r) + \Phi_- (r)] \tag{23}
\]

which enables us to write

\[
(\hbar \omega \mp mc^2) \Phi_\pm (r) = \pm \left\{ mc^2 \Xi - \frac{\hbar^2 \nabla^2}{2m} \right\} [\Phi_+ (r) + \Phi_- (r)] \tag{24}
\]

\[
(E - V \mp mc^2) \Phi_\pm (r) = \pm \left\{ mc^2 \Xi - \frac{\hbar^2 \nabla^2}{2m} \right\} [\Phi_+ (r) + \Phi_- (r)] \tag{25}
\]

and finally

\[
E \Phi_\pm (r) = \pm mc^2 \Phi_\pm (r) + V \Phi_\pm (r) \tag{26}
\]

\[
\pm \left\{ mc^2 \Xi - \frac{\hbar^2 \nabla^2}{2m} \right\} [\Phi_+ (r) + \Phi_- (r)].
\]

If \( \Phi_- \) is set to zero, and only the \( \Phi_+ \) is considered, we see that in terms of momentum \( p = \hbar k \), and for a time-independent \( \Xi \), this will have the dispersion relation \( E = E(p) = mc^2 + V + mc^2 \Xi(k) + p^2/2m \). By using the correspondence principle \([10]\) to replace \( E \leftrightarrow i\hbar \partial_\tau \), we see that back in \( t, r \) space the convolution vanishes, and we get a pair of coupled differential equations,

\[
\begin{align*}
\hbar \partial_\tau \Phi_\pm (r) &= \pm mc^2 \Phi_\pm (r) + V \Phi_\pm (r) \\
&\pm \left\{ mc^2 \Xi - \frac{\hbar^2 \nabla^2}{2m} \right\} [\Phi_+ (r) + \Phi_- (r)].
\end{align*}
\]

(28)

This is a pair of first order wave equations coupled only by the gravitational potential \( \Xi(r, t) \) and the momentum squared term (i.e. that \( \propto \nabla^2 \)); the potential \( V \) does not couple the two because it was chosen to be time independent. Those couplings, along with the rest mass, the wavefunction(s), and their spatial derivatives, then tell us how \( \Phi_\pm (r) \) will change on propagating forward in time.

If both \( \Xi \) and the momentum are small compared to the (dominant) mass term, as is true in Newtonian and non-relativistic scenarios, then any finite \( \Phi_+ \) will only weakly drive \( \Phi_- \), and any finite \( \Phi_- \) will only weakly drive \( \Phi_+ \). Further, the two components evolve very differently, one “forwards” in space at \( \omega \sim mc^2/\hbar \) and the other “backwards” at \( \omega \sim -mc^2/\hbar \). Thus any finite cross-coupling that does occur will be very poorly phase matched, and will almost certainly average out to zero \(^1\). This smallness criteria, viz.

\[
\left\lfloor \left( E - \frac{\hbar^2 \nabla^2}{2m^2 c^2} \right) \right\rfloor \Phi_\pm \ll \left\lfloor 1 \pm \frac{V}{mc^2} \right\rfloor \Phi_\pm,
\]

(29)

is therefore the minimum criteria which must hold for the Schrödinger equation to be valid; although we should also be sure that any periodicities in \( \Xi(r, t) \) or \( V(r) \) do not phase match the cross-coupling terms and allow them to accumulate to a significant level.

Assuming for now that this is true, as is indeed likely for non-relativistic low-momentum situations, we get

\[
\begin{align*}
\hbar \partial_\tau \Phi_\pm (r) &= \pm mc^2 \Phi_\pm (r) + V \Phi_\pm (r) \\
&\pm mc^2 \Xi \Phi_\pm (r) \mp \frac{\hbar^2 \nabla^2}{2m} \Phi_\pm (r).
\end{align*}
\]

(30)

Next we can choose to – but are not compelled to – factor out the fixed rest-mass part, which gives rise to fast oscillations induced by the energy of the particle’s rest mass \( m \). This is done by introducing

\[
\Phi_\pm (r) = \psi_\pm (r) e^{\pm mc^2i/\hbar},
\]

(31)

so that

\[
\hbar \partial_\tau \psi_\pm (r) = \pm V \psi_\pm (r) \pm mc^2 \Xi \psi_\pm (r) \mp \frac{\hbar^2 \nabla^2}{2m} \psi_\pm (r).
\]

(32)

Then we can choose our preferred direction – forwards in time – as indicated by a choice of upper signs, so that

\[
\hbar \partial_\tau \psi_+ (r) = [V(r) + mc^2 \Xi(r, t)] \psi_+ (r) \mp \frac{\hbar^2 \nabla^2}{2m} \psi_+ (r),
\]

(33)

\(^1\) See appendix B of \([19]\), and also e.g. \([5]\).
which is the usual expression for the Schrödinger equation; and we see that the effect of both Salpeter and gravitational potentials ends up essentially the same in this limit. Since this derivation of the Schrödinger equation is for cases where the rest mass $m$ is dominant, we might denote it the Schrödinger “$m$-equation”.

If we were to consider propagating the wavefunction $\psi_+$ forward in time, we might divide both sides by $\hbar$ to get

$$\partial_t \psi_+ (r) = -\frac{i}{\hbar} \left[ V(r) + mc^2 \Xi(r,t) \right] \psi_+ (r) + \frac{i\hbar V^2}{2m} \psi_+ (r).$$

(34)

It is worth noting that the last term in eqn. (33) (or indeed eqn. (34)) is a diffusion term, and causes wavefunctions to spread outwards. While this is the usually expected behaviour, it is worth noting that being a diffusion does generate a causal problem – if starting from a strictly bounded wavefunction, the diffusion term immediately generates some non-zero wavefunction values at arbitrarily large distances. Thus parts of the wavefunction have propagated faster than light-speed! Of course, this simply an artifact introduced by our mass-dominated non-relativistic approximation; it is not a feature of the initial Klein-Gordon wave equation, which remains properly causal [24]. Having made such an approximation, we should certainly not expect it to give useful (or even sensible) results for any effects propagating at or near lightspeed. The artifacts are outside the scope allowed by the approximations used, and however annoying, they do not represent inherent physical failings. If those artifacts are problematic in a particular case, then the conclusion should be that the Schrödinger equation is too approximate to use.

Lastly, we can extract dispersion relations rather directly from eqns. (33) or (34), by returing to the wavevector $\mathbf{k}$ domain and using $\mathbf{p} = \hbar \mathbf{k}$. It is

$$E - mc^2 = V(\mathbf{k}) + mc^2 \Xi(\mathbf{k},t) + p^2 / 2m,$$

(35)
as expected in the mass-dominated limit considered.

### III. MOMENTUM DOMINANT: THE $p$-EQUATION

In contrast to the intent of the Schrödinger derivation, here we focus on momentum-dominated systems, which naturally propagate with a strong spatial orientation. This means we must aim to reduce the second-order-in-space KG equations down to the first-order-in-space “$p$-equation” by manipulating and approximating the starting equations treating momentum as the quantity of primary importance. Such a treatment typically makes most sense with very light or massless particles, and indeed a spatially propagated description is very widely used in optics (see e.g. [7] and references therein). This factorization assumes a propagation forward in space, whilst decomposing the system behaviour (waves) into components that evolve either forward or backward in time, as shown in fig. 2.

To analyse spatial propagation, we need a useful reference parameter to characterise it, and it should preferably be one that remains constant. In this case, a spatial frequency domain analysis is called for: we might therefore use either linear momentum $\mathbf{p}$ or a wavevector $\hat{\mathbf{k}}$. This means that the parts of the physics we wish to ascribe to the role of “reference propagation” must be independent of the primary propagation direction.

For the procedure to work, we need to assume a direction along which the waves will primarily propagate. Without loss of generality, we will assume this to be the $z$-axis, with the $x$ and $y$-axes to account for any transverse properties. Thus we will focus on the $p_z$ momentum component, and relegate $p_x$ and $p_y$ to the status of corrections. After defining $\bar{E}^2 = E^2 - m^2 c^4 = \hbar^2 \omega^2 - m^2 c^4$, and $p^2_z = p^2_z + p^2_x$, we work in a spatial frequency (wavevector) space $\mathbf{k}$. Remembering that $\mathbf{p} = \hbar \mathbf{k}$, we proceed in the following way

$$[E - \bar{V}(\mathbf{k})]^2 \Phi(\mathbf{k},\omega)$$

$$= \left[ m^2 c^4 + 2m^2 c^2 \bar{\Xi}(\mathbf{k},\omega) + \hbar^2 c^2 k^2_z \right] \Phi(\mathbf{k},\omega)$$

(36)

$$\left[ \hbar^2 c^2 k^2_z - (E^2 - m^2 c^4) \right] \Phi$$

$$= \left[ -\hbar^2 c^2 k^2_z + \bar{V}^2 - (E \bar{V} + \bar{V} E) - 2m^2 c^4 \bar{\Xi} \right] \Phi$$

(37)

$$\left[ \hbar^2 c^2 k^2_z - \bar{E}^2 \right] \Phi = \bar{\mathbf{W}} \Phi$$

(38)

$$\left( \hbar c k_z - \bar{E} \right) \left( \hbar c k_z + \bar{E} \right) \Phi = \bar{\mathbf{W}} \Phi$$

(39)

$$\Phi = \frac{1}{\left( \hbar c k_z - \bar{E} \right) \left( \hbar c k_z + \bar{E} \right)} \bar{\mathbf{W}} \Phi$$

(40)

$$\Phi = \left[ \frac{1}{2\bar{E}} \frac{1}{\hbar c k_z - \bar{E}} - \frac{1}{2\bar{E}} \frac{1}{\hbar c k_z + \bar{E}} \right] \bar{\mathbf{W}} \Phi$$

(41)
where for convenience I have defined
\[
\hat{W}(k, \omega) = -c^2 p_x^2 + \hat{V}^2 - E \hat{V} - \hat{V} E - 2m^2 \hat{c}^2 \hat{z},
\]
retaining the ordering of \( E \) and \( V \) as is needed when \( E \) is (re)turned to operator form (i.e. as a time derivative).

We can see from the term in square brackets on the RHS of eqn. \([41]\) that \( \Phi \) evolves according to two complementary parts of differing sign. The term proportional to \((\hbar c k_z - E)^{-1}\) generates a forward-like evolution, and that proportional to \((\hbar c k_z + E)^{-1}\) generates a backward-like evolution \([44]\). As a result we can likewise split the wavefunction into matching pieces, with \( \Phi \equiv \Phi^+ + \Phi^- \). When we transform back into the spatial domain, these will (must!) propagate forward in space \( z \), all the while holding information about the wavefunction as a function of \( x, y, t \). To avoid notational clutter, we use this as an excuse to omit the spatial argument \( z \), and only the \( x, y, t \) arguments of \( \Phi^\pm \) will be given.

Further, since the \( \Phi^+ \) forward evolving component is by definition propagating to larger \( z \), it therefore must (also) be understood to have excitations that evolve forward in time \( t \). In contrast, the \( \Phi^- \) backward evolving component (also propagating to larger \( z \)), will contain excitations that evolve backward in time. While the notion of treating waves that evolve backward in time would (or perhaps should) typically be viewed with suspicion, it can nevertheless be defended as a useful approximation in many circumstances – notably, this picture allows a remarkably powerful way of treating dispersion \([5]\).

Continuing the separation of \( \Phi^+ \) and \( \Phi^- \), we see that
\[
\Phi^+(x, y, \omega) + \Phi^-(x, y, \omega) = \left[ \frac{1/2E}{\hbar c k_z - E} - \frac{1/2E}{\hbar c k_z + E} \right] \hat{W} \Phi(x, y, \omega),
\]
and this enables us to write
\[
(\hbar c k_z \pm E) \Phi^\pm(x, y, \omega) = \pm \frac{\hat{W}}{2E} \left[ \Phi^+(x, y, \omega) + \Phi^-(x, y, \omega) \right]
\]
\[
h c k_z \Phi^\pm(x, y, \omega) = \pm \frac{\hbar}{2E} \Phi^\pm(x, y, \omega)
\]
By again using the correspondence principle to convert back from an \( \omega, k \) based description, into a \( t, r \) form, and with \( \nabla_T = (\partial_x, \partial_y, 0) \), we get a pair of coupled differential equations,
\[
-\imath \hbar c \partial_T \Phi^\pm(x, y, t) = \mp \imath \hbar c \partial_T \Phi^\mp(x, y, t)
\]
\[
\pm \frac{1}{2E} \left[ \Phi^+(x, y, t) + \Phi^-(x, y, t) \right]
\]
\[
\pm \frac{1}{2E} \left[ V^2 - \imath \hbar \partial_t - \imath \hbar \partial_t V - 2m^2 \hat{c}^4 \hat{z} \right]
\]
\[
\times \left[ \Phi^+(x, y, t) + \Phi^-(x, y, t) \right]
\]
\[
\partial_T \Phi^+ = \pm c^{-1} \partial_t \Phi^+ \pm \frac{\imath \hbar c \nabla_T^2}{2E} (\Phi^+ + \Phi^-)
\]
\[
\pm \frac{\imath}{2\hbar c \hat{E}} \left[ V^2 - \imath \hbar \partial_t V - 2m^2 \hat{c}^4 \hat{z} \right] (\Phi^+ + \Phi^-).
\]

On combining the time derivative terms, this becomes
\[
\partial_T \Phi^+ = \pm \frac{1}{c} \left[ 1 + \frac{V}{E} \right] \partial_t \Phi^+ \pm \frac{V}{cE} \partial_T \Phi^\mp \pm \frac{\imath \hbar c \nabla_T^2}{2E} (\Phi^+ + \Phi^-)
\]
\[
\pm \frac{\imath}{2\hbar c E} \left[ V^2 - \imath \hbar (\partial_t V) - 2m^2 \hat{c}^4 \hat{z} \right] (\Phi^+ + \Phi^-).
\]

Whichever of eqns. \([43]\) or \([49]\) you might prefer, either consists of a pair of first order wave equations coupled only by the potentials \( V \) and \( \Xi \), as scaled by the mass compensated energy component (\( \hat{E} \)). Those couplings, along with the wavefunction(s), then tell us how \( \Phi^\pm \) will change on propagating forward in space \( \z \). Note that unlike in the Schrödinger (\( m \)) equation case, there is consequently no explicit mass dependent oscillation; the effect of the mass appears solely as a correction to the effect of the potential; although the rest-mass oscillation remains a legitimate contribution to \( \Phi^\pm \).

If this potential-based coupling is small, as is perhaps likely for light or massless particles, then any finite \( \Phi^+ \) will only weakly drive \( \Phi^- \), and any finite \( \Phi^- \) will only weakly drive \( \Phi^+ \). Further, the two components evolve very differently, one “forwards” in time at \( p \sim E/c \) and the other “backwards” at \( p \sim -E/c \). Thus any finite cross-coupling that does occur will be very poorly phase matched, and will almost certainly average out to zero. This smallness criteria, viz.
\[
\left\{ \frac{V}{E} + \frac{\imath \hbar c \nabla_T^2}{2E} + \frac{\imath}{2\hbar E} \left[ V^2 - \imath \hbar (\partial_t V) - 2m^2 \hat{c}^4 \hat{z} \right] \right\} \Phi^+ \ll \left| 1 + \frac{V}{E} \right| \Phi^\mp
\]
is therefore the minimum criteria which must hold for this particular equation to be valid; although we should also be sure that any periodicities in \( \Xi(r, t) \) or \( V(r) \) do not phase match the cross-coupling terms and allow them to accumulate significantly.

Assuming for now that this is true, as is indeed it might be for energetic but low-mass objects, we get
\[
\partial_T \Phi^+ = \pm c^{-1} \partial_t \Phi^+ \pm \frac{\imath \hbar c \nabla_T^2}{2E} \Phi^+
\]
\[
\pm \frac{\imath}{2\hbar c E} \left[ V^2 - \imath \hbar \partial_t V - 2m^2 \hat{c}^4 \hat{z} \right] \Phi^\mp.
\]
Or, with combined time derivatives,

\[
\partial_z \Phi^\pm = \pm \frac{1}{c} \left[ 1 + \frac{V}{E} \right] \partial_t \Phi^\pm \pm \frac{i \hbar c^2}{2E} \Phi^\pm
\]

\[
\pm \frac{i}{2 \hbar E} \left[ V^2 - i \hbar (\partial_t V) - 2m^2 c^4 \Xi \right] \Phi^\pm.
\]

(52)

In this last form, we see that the typical (or "reference") wavevector \( K \) for a wavefunction component evolving with frequency \( \omega \) is

\[
K(\omega) = \frac{\omega}{c} \left[ 1 + V(\omega/c) / \bar{E} \right].
\]

(53)

Again we have a diffusion-like term in our first order wave equation (52), here dependent on \( \nabla^2 T \). Now, however, because we are propagating a wavefunction known as a function of time forward in space, the extremes of the diffusion behaviour (which in this case actually a diffraction) correspond to very slow processes, and therefore are not acausal artifacts².

IV. SUMMARY

I have shown how to derive the Schrödinger equation for a particle of mass \( m \), starting from the Klein-Gordon equation, while taking into account the possible effects of both static and/or dynamic potential landscapes influencing the evolution of the wavefunction. This "\( m \)-equation" is found using an approximation which assumes that the object’s energy is dominated by its rest mass – i.e. that it is moving as non-relativistic speed. The method is an adaption [9] of a factorization scheme recently applied in optics [7, 8, 15], but not originating from there.

Further, I also derive an alternative to the Schrödinger equation in a different and complementary limit, i.e. that of large momentum \( p \). This equation does not propagate the wave equation forward in time, as the usual Schrödinger equation does, but forward in space. This alternate "\( p \)-equation" is presented here primarily as an exercise in technique, and discussions of its possible utility are left for later work.

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