Efficient Encoding/Decoding of Irreducible Words for Codes Correcting Tandem Duplications

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Abstract—Tandem duplication is the process of inserting a copy of a segment of DNA adjacent to the original position. Motivated by applications that store data in living organisms, Jain et al. (2017) proposed the study of codes that correct tandem duplications. Known code constructions are based on irreducible words.

We study efficient encoding/decoding methods for irreducible words. First, we describe an \((\ell,m)\)-finite state encoder and show that when \(m = \Theta(1/\epsilon)\) and \(\ell = \Theta(1/\epsilon)\), the encoder achieves rate that is \(\epsilon\) away from the optimal. Next, we provide ranking/unranking algorithms for irreducible words and modify the algorithms to reduce the space requirements for the finite state encoder.

I. INTRODUCTION

Advances in synthesis and sequencing technologies have made DNA macromolecules an attractive medium for digital information storage. Besides being biochemically robust, DNA strands offer ultrahigh storage densities of \(10^{15} - 10^{20}\) bytes per gram of DNA, as demonstrated in recent experiments (see [1] Table 1).

These synthetic DNA strands may be stored \textit{ex vivo} or \textit{in vivo}. When the DNA strands are stored \textit{ex vivo} or in a non-biological environment, code design takes into account the synthesising and sequencing platforms being used (see [2] for a survey of the various coding problems). In contrast, when the DNA strands are stored \textit{in vivo} or recombined with the DNA of a living organism, we design codes to correct errors due to the biological mutations.

This work looks at the latter case, and specifically, examines codes that correct errors due to tandem duplications. Tandem duplications or repeats is one of the two common repeats found in the human genome [3] and they are caused by slipped-strand mispairings [4]. They occur in DNA when a pattern of one or more nucleotides is repeated and the repetitions are directly adjacent to each other. For example, consider the string or word AGTAGTCTGC. The substring AGTAGT is a tandem repeat, and we say that AGTAGTCTGC is generated from AGTCTGC by a tandem duplication of length three.

Jain et al. [5] first proposed the study of codes that correct errors due to tandem duplications. In the same paper, Jain et al. used \textit{irreducible} words (see Section I-A for definition) to construct a family of codes that correct tandem duplications of lengths at most \(k\), where \(k \in \{2,3\}\). While these codes are optimal in size for the case \(k = 2\), these codes are not optimal for \(k = 3\), and in fact, Chee et al. [6] constructed a family of codes with strictly larger size. Recently, Jain et al. [7] looked at other error mechanisms, and studied the capacity of these tandem-duplication systems in the presence of point-mutation noise (substitution errors).

In this paper, we look at encoding/decoding methods for irreducible words. In particular, we provide polynomial-time algorithms that encodes either exactly the rates of irreducible words or close to the asymptotic rates of irreducible words. While the encoding/decoding algorithms are standard in constrained coding [8] and combinatorics literature [9], our contribution is a detailed analysis of the space and time complexities of the respective algorithms. Before we state the main results of the paper, we go through certain notations.

A. Notation and Terminology

Let \([n]\) denotes the set \(\{1,2,\ldots,n\}\). Let \(\Sigma_q = \{0,1,\ldots,q-1\}\) be an alphabet of \(q \geq 2\) symbols. For a positive integer \(n\), let \(\Sigma_n^q\) denote the set of all words of length \(n\) over \(\Sigma_q\), and let \(\Sigma_q^*\) denote the set of all words over \(\Sigma_q\) with finite length. Given two words \(x,y \in \Sigma_q^*\), we denote their concatenation by \(xy\).

We state the \textit{tandem duplication} rules. For integers \(k \leq n\) and \(i \leq n-k\), we define \(T_{i,k} : \Sigma_n^q \rightarrow \Sigma_n^{q+k}\) such that \(T_{i,k}(x) = uvvw\), where \(x = uvw\), \(|u| = i\), \(|v| = k\).

If a finite sequence of tandem duplications of length at most \(k\) is performed to obtain \(y\) from \(x\), then we say that \(y\) is a \(\leq k\)-\textit{descendant} of \(x\), or \(x\) is a \(\leq k\)-\textit{ancestor} of \(y\). Given a word \(x\), we define the \(\leq k\)-\textit{descendant cone} of \(x\) is the set of all \(\leq k\)-descendants of \(x\) and denote this cone by \(D_{\leq k}(x)\).

Example 1. Consider \(x = 012100\) over \(\Sigma_3\). We have \(T_{1,3}(x) = 01211210\) and \(T_{0,2}(01211210) = 0101211210\). So, \(0101211210 \in D_{\leq 3}(x)\).

Definition 1 (\(\leq k\)-Tandem-Duplication Codes). A subset \(\mathcal{C} \subseteq \Sigma_q^n\) is a \(\leq k\)-\textit{tandem-duplication code} if for all \(x,y \in \mathcal{C}\) and \(x \neq y\), we have that \(D_{\leq k}(x) \cap D_{\leq k}(y) = \emptyset\). We say that \(\mathcal{C}\) is an \((n,\leq k;q)\)-TD code.

The size of \(\mathcal{C}\) refers to \(|\mathcal{C}|\), while the rate of \(\mathcal{C}\) is given by \((1/n)\log_q|\mathcal{C}|\). Given an infinite family \(\{\mathcal{C}_n | \mathcal{C}_n\text{ is of length }n\}_{n=1}^\infty\), its \textit{asymptotic rate} is given by \(\lim_{n \to \infty} (1/n)\log_q|\mathcal{C}_n|\).

B. Irreducible Words

Of interest is a family of tandem-duplication codes constructed by Jain et al. [5]. Crucial to the code construction is the concept of irreducible words and roots.

Definition 2. A word is \(\leq k\)-\textit{irreducible} if it cannot be deduplicated into shorter words with deduplications of length at most \(k\). We use \(I_{r_{\leq k}}(n,q)\) to denote the set of all \(\leq k\)-irreducible words of length \(n\) over \(\Sigma_q\). The \(\leq k\)-ancestors of \(x \in \Sigma_q^*\) that are \(\leq k\)-irreducible words are called the \(\leq k\)-\textit{roots} of \(x\).
Construction 1 (Jain et al. \[5\]). For \(k \in \{2, 3\}\) and \(n \geq k\). An 
\((n, k; q; \ell)-T\)D-code \(C(n, k; q)\) is given by
\[
C(n, k; q) \triangleq \bigcup_{i=1}^{n} \{ \xi_{n-i}(x) \mid x \in \text{Irr}_{\leq k}(i, q) \}.
\]
Here, \(\xi_{\ell}(x) = xz^{\ell}\), where \(z\) is the last symbol of \(x\).

We point out certain advantages of Construction 1.

(a) Almost optimal rates. Jain et al. demonstrated that Construction 1 is optimal for \(k = 2\). However, when \(k = 3\), Chee et al. \[6\] provided constructions that achieve almost twice the size in Construction 1 (see \[6\] Table I). Unfortunately, the asymptotic rate of the latter is the same as Construction 1. Therefore, the set of irreducible words gives the best known asymptotic rates for \(k = 3\).

Furthermore, for \(q \geq 5\) and \(k = 3\), the asymptotic rates of Construction 1 differs from a theoretical upper bound (see \[6\] Proposition 4) and Table I by at most 0.01. In other words, Construction 1 is almost optimal in terms of rates.

(b) Linear-time decoding. Consider \(x \in C(n, k; q)\) and we read \(y \in D_{\leq k}(x)\). To retrieve the codeword \(x\), we simply compute the \(\leq k\)-root of \(y\) and extend the root if the root is shorter than \(n\). Jain et al. showed that there is at most one root when \(k \in \{2, 3\}\), while Chee et al. provided algorithms to compute these roots in linear time \[6\].

In view of these points, we study other practical aspects of Construction 1. Specifically, we look at efficient encoding of messages in \(\sum_{i=1}^{n}\) to codewords in \(x \in C(n, k; q)\) for some \(\ell < n\).

To this end, we look at the rates of \(C(n, k; q)\). Let \(I_{\leq k}(n, q) \triangleq \text{Irr}_{\leq k}(n, q)\). Then the size of \(C(n, k; q)\) is given by \(\sum_{i=1}^{n} I_{\leq k}(i, q)\). Let \(\text{rate}_{\leq k}(n, q)\) and \(\text{rate}_{\leq k}(q)\) denote the rate and asymptotic rate of \(C(n, k; q)\), respectively. In other words, \(\text{rate}_{\leq k}(n, q) \triangleq \frac{1}{n} \log q |C(n, k; q)|\) and \(\text{rate}_{\leq k}(q) \triangleq \lim_{n \to \infty} \text{rate}_{\leq k}(n, q)\). Jain et al. observed that \(\lim_{n \to \infty} \text{rate}_{\leq k}(n, q)\) is a regular language and hence,
\[
\text{rate}_{\leq k}(q) = \lim_{n \to \infty} \frac{\log q |I_{\leq k}(n, q)|}{n}.
\]
Furthermore, using Perron-Frobenius theory (see \[8\]), Jain et al. computed \(\text{rate}_{\leq 3}(3)\) to be approximately 0.347934. In view of (I), we look at encoding of the words in \(\text{Irr}_{\leq k}(n)\) and the extension of our encoding methods to \(C(n, k; q)\) is straightforward.

C. Our Contributions

We first develop a recursive formula for \(I_{\leq k}(n, q)\) and hence, provide a formula for the asymptotic rate for \(C(n, k; q)\). We then provide two efficient encoding methods and use combinatorial insights provided by the recursive formula to analyse the space and time complexities.

Specifically, our main contributions are as follows.

(A) We compute \(\text{rate}_{\leq k}(q)\) for all \(q\) and \(k \in \{2, 3\}\) in Section III.

(B) In Section IV we propose an \((\ell, m)\)-finite state encoder with rate \(\ell/m\). Furthermore, we show that we can choose the lengths \(\ell\) and \(m\) to be small and yet come close to the asymptotic rate. In particular, if we choose \(m = \Theta(1/\epsilon)\) and \(\ell = \Theta(1/\epsilon)\), we showed that the rate is at least \(\text{rate}_{\leq k}(q)\).

Here, the running time for the encoder is linear in codeword length \(n\) for constant \(\epsilon\).

(C) Using bijections developed Section III we provide a ranking/unranking algorithm that encodes with rate equal to \((1/n) \log q (\text{Irr}_{\leq k}(n, q))\) in Section IV. This algorithm runs in \(O(n^{2})\) time using \(O(n^{2})\) space. Furthermore, this ranking/unranking technique can be modified to reduce the space requirement to \(O(m^{2})\) in the \((\ell, m)\)-finite state encoder.

Due to space constraints, we present proofs and illustrate examples for the case \(k = 2\) and simply state the relevant results for \(k = 3\). The detailed proofs are deferred to the full paper.

II. ENUMERATING IRREDUCIBLE WORDS

In this section, we compute \(\text{rate}_{\leq k}(q)\) for all \(q\) and \(k \in \{2, 3\}\) by obtaining a recursive formula for \(I_{\leq k}(n, q)\). While the Perron-Frobenius theory (see \[8\]) is sufficient to determine the asymptotic rates, the recursive formula is useful in the analysis of the finite state encoder in Section III and the development of the ranking/unranking methods in Section IV.

To this end, we partition the set of irreducible words into two classes and provide bijections from irreducible words of shorter lengths into them. Specifically, notice that the suffix of an irreducible word is of the form either \(aba\) or \(abc\), where \(a, b, c\) are distinct symbols. Hence, we let \(\text{Irr}_{\leq 3}(2, n, q)\) and \(\text{Irr}_{\leq 3}(3, n, q)\) denote the set of irreducible words with length-three suffixes that have two and three distinct symbols, respectively.

In the case \(k = 2\), we consider the following maps for \(n \geq 4\),
\[
\phi: \text{Irr}_{\leq 2}(n-1, q) \times [q - 2] \to \text{Irr}_{\leq 2}(3, n, q),
\]
\[
\psi: \text{Irr}_{\leq 2}(n-2, q) \times [q - 2] \to \text{Irr}_{\leq 2}(2, n, q).
\]
We first define \(\phi\). If \(x = x_1x_2 \ldots x_{n-1} \in \text{Irr}_{\leq 2}(n-1, q)\) and \(i \in [q-2]\), set \(s\) to be the \(i\)th element in \(\Sigma_q \setminus \{x_{n-2}, x_{n-1}\}\). Then set \(\phi(x, i) = x_1x_2 \ldots x_{n-1}s\).

For \(\psi\), let \(x = x_1x_2 \ldots x_{n-2} \in \text{Irr}_{\leq 2}(n-2, q)\) and \(i \in [q-2]\) and set \(s\) to be the \(i\)th element in \(\Sigma_q \setminus \{x_{n-3}, x_{n-2}\}\). Then set \(\psi(x, i) = x_1x_2 \ldots x_{n-2}s\).

Proposition 1. The maps \(\phi\) and \(\psi\) are bijections.

Proof. We construct the inverse map for \(\phi\). Specifically, we set \(\phi^{-1}: \text{Irr}_{\leq 2}(3, n, q) \to \text{Irr}_{\leq 2}(n-1, q) \times [q - 2]\) such that \(\phi^{-1}(x) = (x_1 \ldots x_{n-1}, i)\), where \(i\) is the index of \(x_n\) in \(\Sigma_q \setminus \{x_{n-2}, x_{n-1}\}\). It can be verified that \(\phi \circ \phi^{-1}\) and \(\phi^{-1} \circ \phi\) are identity maps on their respective sets. Similarly, the inverse map for \(\psi\) is given by \(\psi^{-1}: \text{Irr}_{\leq 2}(2, n, q) \to \text{Irr}_{\leq 2}(n-2, q) \times [q - 2]\) such that \(\psi^{-1}(x) = (x_1 \ldots x_{n-2}, i)\), where \(i\) is the index of \(x_{n-1}\) in \(\Sigma_q \setminus \{x_{n-3}, x_{n-2}\}\).

The following corollary is then immediate.

Corollary 1. We have that \(I_{\leq 2}(2, q) = q(q - 1), I_{\leq 2}(3, q) = q(q - 1)^2\), and
\[
I_{\leq 2}(n, q) = (q - 2)I_{\leq 2}(n - 1, q) + (q - 2)I_{\leq 2}(n - 2, q)
\]
for \(n \geq 4\). Therefore, the asymptotic rate is \(\text{rate}_{\leq 2}(q) = \log_{q} \lambda_{2}\), where \(\lambda_{2} = (q - 2 + \sqrt{q^2 - 4})/2\).

In the next section, we are interested in irreducible words with certain prefixes or suffixes. Specifically, let \(p\) be a word of length
If $x = x_1 \ldots x_{n-3}x_{n-2}x_{n-1} \in \text{Irr}^{(s)}(2, n-1, q)$ where $x_{n-3} = x_{n-1}$ and $i \in \{q - 2\}$, set $\sigma$ to be the $i$th element in $\Sigma_q \setminus \{x_{n-2}, x_{n-1}\}$.

$\varphi_1(x, i) = x\sigma = x_1 \ldots x_{n-3}x_{n-2}x_{n-1}\sigma$.

Similarly, we now define $\varphi_2, \varphi_3$ as follows.

If $x = x_1 \ldots x_{n-5}x_{n-4}x_{n-3}x_{n-2} \in \text{Irr}^{(s)}(3, n-2, q)$ and $i \in \{q - 3\}$, we set

$\varphi_2(x, i) = x\sigma x_{n-3} = x_1 \ldots x_{n-5}x_{n-4}x_{n-3}x_{n-2}x_{n-1}\sigma x_{n-3}$.

We provide the recursion for $\text{Irr}^{(s)}(p, n, q)$.

**Corollary 2.** Let $p \in \Sigma_q^\ell$ for $n \geq \ell + 2$.

$$\left[\text{Irr}^{(s)}_{\leq k}(p, n, q)\right] = (q - 2)\left[\text{Irr}^{(s)}_{\leq k}(p, n-1, q)\right] + (q - 2)\left[\text{Irr}^{(s)}_{\leq k}(p, n-2, q)\right].$$

We now define $\varphi_2, \varphi_3$ as follows.

If $x = x_1 \ldots x_{n-5}x_{n-4}x_{n-3}x_{n-2} \in \text{Irr}^{(s)}(3, n-2, q)$ and $i \in \{q - 3\}$, we set

$\varphi_2(x, i) = x\sigma x_{n-3} = x_1 \ldots x_{n-5}x_{n-4}x_{n-3}x_{n-2}x_{n-1}\sigma x_{n-3}$.

We provide the recursion for $\text{Irr}^{(s)}(p, n, q)$.

**Proposition 2.** We have that $I_{\geq 3}(3, q) = q(q-1)^2, I_{\geq 3}(4, q) = q^2(q-1)(q-2), I_{\geq 3}(5, q) = q(q-1)(q-2)(q^2-1)$ and $I_{\geq 3}(n, q) = (q - 2)I_{\geq 3}(n-1, q) + (q - 3)I_{\geq 3}(n-2, q)$.

We provide the recursion for $\text{Irr}^{(s)}(p, n, q)$.

We consider the following maps for $n \geq 6$.

$\varphi_1 : \text{Irr}^{(s)}_{\leq k}(n-1, q) \times \{q-2\} \rightarrow \text{Irr}^{(s)}_{\leq k}(L_1, n, q)$,

$\varphi_2 : \text{Irr}^{(s)}_{\leq k}(n-2, q) \times \{q-2\} \rightarrow \text{Irr}^{(s)}_{\leq k}(L_2, n, q)$,

$\varphi_3 : \text{Irr}^{(s)}_{\leq k}(n-3, q) \times \{q-3\} \rightarrow \text{Irr}^{(s)}_{\leq k}(L_3, n, q)$.

We consider the following maps for $n \geq 6$.

$\varphi_1 : \text{Irr}^{(s)}_{\leq k}(n-1, q) \times \{q-2\} \rightarrow \text{Irr}^{(s)}_{\leq k}(L_1, n, q)$,

$\varphi_2 : \text{Irr}^{(s)}_{\leq k}(n-2, q) \times \{q-2\} \rightarrow \text{Irr}^{(s)}_{\leq k}(L_2, n, q)$,

$\varphi_3 : \text{Irr}^{(s)}_{\leq k}(n-3, q) \times \{q-3\} \rightarrow \text{Irr}^{(s)}_{\leq k}(L_3, n, q)$.

Recall that $\text{Irr}^{(s)}(p, n, q) = \text{Irr}^{(s)}_{\leq k}(3, n, q) \cup \text{Irr}^{(s)}_{\leq k}(2, n, q)$. We first define $\varphi_1$. If $x = x_1 \ldots x_{n-3}x_{n-2}x_{n-1} \in \text{Irr}^{(s)}_{\leq k}(3, n-1, q)$ and $i \in \{q - 2\}$, set $\sigma$ to be the $i$th element in $\Sigma_q \setminus \{x_{n-3}, x_{n-1}\}$.

Then set

$\varphi_1(x, i) = x\sigma = x_1 \ldots x_{n-3}x_{n-2}x_{n-1}\sigma$.
Corollary 3. Let \( p \in \Sigma_q^\ell \) for \( n \geq \ell + 3 \),

\[
\left| \text{Irr}_{\leq 3}^{(p)}(p, n, q) \right| = (q - 2) \left| \text{Irr}_{\leq 3}^{(p)}(p, n - 1, q) \right| + \\
(q - 3) \left| \text{Irr}_{\leq 3}^{(p)}(p, n - 2, q) \right| + (q - 2) \left| \text{Irr}_{\leq 3}^{(p)}(p, n - 3, q) \right|.
\]

We compute the values of rate\( \leq k \) for \( k \in \{2, 3\} \) in Table I. Let \( T(n, q) \) be the largest size of an \( (n, \leq 3; q) \)-TD code and define \( \tau(q) \equiv (1/n) \limsup_{n \to \infty} \log_q T(n, q) \). From [5], [6], we have that that rate\( \leq 3 \) \( \leq \tau(q) \leq \text{rate}\leq 2 \). Therefore, Table I demonstrates that \( \mathcal{C}(n, \leq 3; q) \) is almost optimal for \( q \geq 5 \).

TABLE I: The asymptotic information rates for \( k \)-irreducible words for \( k \in \{2, 3\} \)

| \( q \) | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|
| rate\( \leq 2 \) | 0.4486 | 0.7249 | 0.8281 | 0.8785 | 0.9081 | 0.9269 |
| rate\( \leq 3 \) | 0.3479 | 0.7054 | 0.8208 | 0.8753 | 0.9062 | 0.9275 |

III. Finite State Encoder

For integers \( \ell < m \), an \((\ell, m)\)-finite state encoder is triple \((S, \mathcal{E}, \mathcal{L})\), where \( S \) is a set of states, \( \mathcal{E} \subseteq S \times S \) is a set of directed edges, and \( \mathcal{L} : \mathcal{E} \to \Sigma^\ell_f \times \Sigma^n_0 \) is an edge labeling.

To encode irreducible words, we choose \( m \geq 2k - 1 \), and set

\[
S \equiv \text{Irr}_{\leq k}(m, q) \quad \text{and} \quad \mathcal{E} \equiv \{(x, x') : xx' \in \text{Irr}_{\leq k}(2m, q)\}.
\]

For \( x \in S \), we define the neighbors of \( x \) to be \( N(x) \equiv \{x' : (x, x') \in \mathcal{E}\} \). We also consider the quantity \( \Delta_{\leq k}(m, q) \equiv \min\{|N(x) : x \in S| \} \) and choose \( \ell \) such that

\[
\Delta_{\leq k}(m, q) \geq q^\ell. \quad (5)
\]

We now define the edge labelling \( \mathcal{L} \) using this choice of \( \ell \). For \( x \in S \), since \( |N(x)| \geq q^\ell \), we may use the set \( \Sigma^\ell \) to index the first \( q^\ell \) words in \( N(x) \). Hence, for \( x' \in S \), if \( x' \) is one of the first \( q^\ell \) words, we let \( y_{x'} \in \Sigma^\ell \) denote the index. Otherwise, we simply set \( y_{x'} = - \). Therefore, for \( (x, x') \in \mathcal{E} \), we set \( \mathcal{L}(x, x') = (y_{x'}, x') \). Finally, we call this triple an \((\ell, m)\)-finite state encoder for irreducible words.

Example 2. Let \( k = 2 \), \( q = 3 \), \( m = 3 \). Then \( S = \{010, 012, 020, 021, 101, 102, 120, 121, 201, 202, 210, 212\} \), and

\[
\begin{align*}
N(010) &= \{201, 210, 212\}, \\
N(012) &= \{010, 012, 021, 101, 102\}.
\end{align*}
\]

We verify that \( \Delta_{\leq 2}(3, 3) = 3 \) and so, we choose \( \ell = 1 \). So, we can set \( \mathcal{L} \) to map the edges exiting the state 010 as follow:

\[
(010, 201) \mapsto (0, 201), \quad (010, 210) \mapsto (1, 210), \quad (010, 212) \mapsto (2, 212).
\]

We represent the mapping \( \mathcal{L} \) using the following lookup table.

| \( x \) | \( N(x) \) |
|---|---|
| \( 010 \) | 201, 210, 212 |
| \( 012 \) | 010, 012, 021, 101, 102 |
| \( 020 \) | 102, 120, 121, 210, 212 |
| \( 021 \) | 012, 020, 021, 201, 202 |
| \( 101 \) | 201, 202, 210 |
| \( 102 \) | 010, 012, 101, 102, 120 |
| \( 120 \) | 102, 120, 121, 210, 212 |
| \( 121 \) | 012, 020, 021, 201, 202 |
| \( 201 \) | 201, 202, 210 |
| \( 202 \) | 101, 102, 120 |
| \( 210 \) | 120, 121, 210, 212 |
| \( 212 \) | 010, 012, 201 |

Here, to determine \( \mathcal{L}(x, x') \), we look at the row corresponding to \( x \) and look at the column corresponding to \( x' \). If the column is \( y_{x'} \), then \( \mathcal{L}(x, x') = (y_{x'}, x') \). So, \( \mathcal{L}(012, 010) = (0, 010) \).

A. Encoding

Let \( s \) be a positive integer and set \( n = sl \). Suppose the message \( y = y_1 y_2 \ldots y_s \in \Sigma^s \).

To encode \( y \) using an \((\ell, m)\)-finite state encoder for irreducible words, we do the following:

(I) Set \( x_0 \) to the first word in \( S = \text{Irr}_{\leq k}(m, q) \).

(II) For \( i \in [s] \), set \( x_i \) to be the unique word such that \( \mathcal{L}(x_{i-1}, x_i) = (y_i, x_i) \).

(III) The encoded irreducible word is \( x = x_1 x_2 \ldots x_s \).

Example 3 (Example 2 continued). Let \( s = 3 \) and consider the message \( y = 012 \). First, we set \( x_0 = 010 \). Then \( x_1 = 201 \) since \( \mathcal{L}(010, 201) = (0, 201) \). Similarly, \( x_2 = 021 \) and \( x_3 = 021 \).

Therefore, the encoded word \( x \) is 2102021.

Since the encoded word has length \( sn \), the \((\ell, m)\)-finite state encoder for irreducible words has rate \( \ell/m \). In the next subsection, we see that \( \ell \) and \( m \) can be chosen in such a way that the rate \( \ell/m \) approaches \text{rate}\leq k \( q \) quickly.

B. Approaching the Asymptotic Information Rate

Pick \( \epsilon > 0 \). We find suitable values for \( \ell \) and \( m \) so that the encoding rate satisfies

\[
\ell/m \geq \text{rate}_{\leq k}(q) - \epsilon. \quad (6)
\]

In particular, we show that \( \ell = \Theta(1/\epsilon) \) and \( m = \Theta(1/\epsilon) \) suffice to guarantee [6].

Recall that \( \ell \) and \( m \) are required to satisfy [5]. Hence, we determine \( \Delta_{\leq k}(m, q) \). Surprisingly, these values have the same recursive structure as \( I_{\leq k}(m, q) \) and therefore, have the same growth rate.

Proposition 3. We have that \( \Delta_{\leq 2}(3, q) = (q-2)/(q-1) \), \( \Delta_{\leq 2}(4, q) = (q-2)/(q-2) \), and for \( m \geq 5 \),

\[
\Delta_{\leq 2}(m, q) = (q-2)\Delta_{\leq 2}(m-1, q) + (q-2)\Delta_{\leq 2}(m-2, q). \quad (7)
\]

Proof. Observe that by symmetry, we have \( |N(x)| = |N(x')| \) for \( x, x' \in \text{Irr}_{\leq 2}(2, m, q) \). Similarly, \( |N(y)| = |N(y')| \) for \( y, y' \in \text{Irr}_{\leq 2}(3, m, q) \).
We first show that $|N(x)| \leq |N(y)|$ for $x \in \text{Irr}_{\leq 2}^{(s)}(2, m, q)$ and $y \in \text{Irr}_{\leq 2}^{(s)}(3, m, q)$. Without loss of generality, we assume $x \in \text{Irr}_{\leq 2}^{(s)}(010, m, q)$ and $y \in \text{Irr}_{\leq 2}^{(s)}(210, m, q)$. Then the neighbours of $x$ and $y$ are given by

$$N(x) = \left\{ x' : 10x' \in \bigcup_{\sigma \notin \{0,1\}} \text{Irr}_{\leq 2}^{(s)}(10\sigma, m + 2, q) \right\},$$

$$N(y) = \left\{ y' : 10y' \in \bigcup_{\sigma \neq 0} \text{Irr}_{\leq 2}^{(s)}(10\sigma, m + 2, q) \right\}.$$

Since $N(x) \subseteq N(y)$, the inequality $|N(x)| \leq |N(y)|$ follows. Hence, $\Delta_{\leq 2}(m, q) = |N(x)|$ where $x \in \text{Irr}_{\leq 2}^{(s)}(010, m, q)$.

Since $\Delta_{\leq 2}(m, q) = \sum_{\sigma \notin \{0,1\}} |\text{Irr}_{\leq 2}^{(s)}(10\sigma, m + 2, q)|$, the recursive equation (7) follows from Corollary 2.

For $k = 3$, we have the following recursive equation.

**Proposition 4.** We have that

$$\Delta_{\leq 3}(5, q) = (q - 2)(q^2 - 2q - 2),$$

$$\Delta_{\leq 3}(6, q) = (q - 1)(q^3 - 6q^2 + 9q^2 + 4q^2 - 8q - 9),$$

$$\Delta_{\leq 3}(7, q) = (q - 2)(q^6 - 6q^4 + 9q^3 + 4q^2 - 8q - 10q + 3),$$

and for $m \geq 8$,

$$\Delta_{\leq 3}(m, q) = (q - 2)\Delta_{\leq 3}(m - 1, q) + (q - 3)\Delta_{\leq 3}(m - 2, q) + (q - 2)\Delta_{\leq 3}(m - 3, q).$$

**Proof.** Let $L$ be the set of all possible suffixes of length five of an irreducible word. We can then verify that $L = \{ abcab, abacac, abacac, abca, abcbcd, abaca, abacac, ababc, ababc, abaca, abaca, abaca \mid a, b, c, d, e \text{ distinct in } \Sigma_q \}$. We first show that $\Delta_{\leq 3}(m, q) = |N(x)|$ where $x \in \text{Irr}_{\leq 2}^{(s)}(abac, m, q)$. In other words, we need to show that $|N(x)| \leq |N(y)|$ for $x \in \text{Irr}_{\leq 2}^{(s)}(abac, m, q)$ and $y \in \text{Irr}_{\leq 2}^{(s)}(p, m, q)$, where $p \in L$.

We demonstrate the inequality in the case when $p = abcd, abac$ and the remaining cases can be done similarly. Without loss of generality, we assume $x \in \text{Irr}_{\leq 2}^{(s)}(01201, m, q)$ and $y \in \text{Irr}_{\leq 2}^{(s)}(03201, m, q)$. Then the neighbours of $x$ and $y$ are given by

$$N(x) = \left\{ x' : 201x' \in \bigcup_{\sigma \notin \{1,2\}} \text{Irr}_{\leq 2}^{(s)}(201\sigma, m + 3, q) \right\},$$

$$N(y) = \left\{ y' : 201y' \in \bigcup_{\sigma \neq 1} \text{Irr}_{\leq 2}^{(s)}(201\sigma, m + 3, q) \right\}.$$

Since $N(x) \subseteq N(y)$, the inequality $|N(x)| \leq |N(y)|$ follows. Hence, $\Delta_{\leq 3}(m, q) = |N(x)|$ where $x \in \text{Irr}_{\leq 2}^{(s)}(01201, m, q)$.

Since $\Delta_{\leq 3}(m, q) = \sum_{\sigma \notin \{1,2\}} |\text{Irr}_{\leq 2}^{(s)}(201\sigma, m + 3, q)|$, the recursive equation (10) follows from Corollary 3.

Recall that $\lambda_2$ and $\lambda_3$ are roots of the equations $x^2 - (q - 2)x - (q - 2) = 0$ and $x^3 - (q - 2)x^2 - (q - 3)x - (q - 2) = 0$, respectively.

Set $\kappa_2$ such that $\Delta_{\leq 2}(m, q) \geq \kappa_2 \lambda_2^n$ for $m \in \{3, 4\}$. Similarly, set $\kappa_3$ so that $\Delta_{\leq 3}(m, q) \geq \kappa_3 \lambda_3^n$ for $m \in \{5, 6, 7\}$. Then it follows from an inductive argument and recursions (7) and (10) that

$$\Delta_{\leq k}(m, q) \geq \kappa_k \lambda_k^n \text{ for all } m.$$ (13)

We are now ready to present the main theorem of this section.

**Theorem 1.** Let $k \in \{2, 3\}$. Set $c_k = \text{rate}_{\leq k}(q) = \log_q \lambda_k$. For $\epsilon > 0$, if we choose $m$ and $\ell$ such that

$$\ell = \frac{\left( (c_k - \epsilon)(c_k - \log_q \lambda_k) \right)}{\epsilon},$$

$$m = \frac{\ell - \log_q \lambda_k}{c_k},$$

then the $(\ell, m)$-finite state encoder has rate at least $\text{rate}_{\leq k}(q) - \epsilon$.

**Proof.** We have to verify that (5) and (6) hold for the choice of $\ell$ and $m$. Now, (14) implies that $\ell \geq (c_k - \epsilon)(c_k - \log_q \lambda_k)$, and equivalently, $c_k \ell/(\ell - \log_q \lambda_k + c_k) \geq c_k - \epsilon$. Therefore,

$$\frac{\ell}{m} \geq 1 + \left( \ell - \log_q \lambda_k \right)/c_k = \frac{c_k \ell}{\ell - \log_q \lambda_k + c_k} \geq c_k - \epsilon.$$

Thus, we verify (6).

Next, from (13) and (15), we have that

$$\Delta_{\leq k}(m, q) \geq \kappa_k \lambda_k^{(\ell - \log_q \lambda_k)/\log_q \lambda_k} = q^\ell.$$

Hence, we verify (5) and complete the proof.

Therefore, to achieve encoding rates at least $\text{rate}_{\leq k}(q) - \epsilon$, we only require $\ell = \Theta(1/\epsilon)$ and $m = \Theta(1/\epsilon)$. If we naively use a lookup table to represent $(S, E, \ell)$, we require $\Omega(1/\epsilon)$ space. Furthermore, using binary search, the $(\ell, m)$-finite state encoder for irreducible words encodes in $O(n/\epsilon)$ time. In the next section, we use combinatorial insights from (2) and (4) to reduce the space requirement to $O(1/\epsilon^2)$.

**IV. RANKING/UNRANKING ALGORITHM**

A ranking function for a finite set $S$ of cardinality $N$ is a bijection rank : $S \to [N]$. Associated with the function rank is a unique unranking function unrang : $[N] \to S$, such that rank(s) = j if and only if unrang(j) = s for all s \in S and j \in [N]. In this section, we present an algorithm for ranking and unranking $\text{Irr}_{\leq k}(n, q)$. For ease of exposition, we focus on the case where $k = 2$ and present the ranking/unranking algorithm for $k = 3$ at the end of the section. The basis of our ranking and unranking algorithms is the bijections defined in Section III. As implied by the codomains of $\phi$ and $\psi$, for $n \geq 4$, we order the words in $\text{Irr}_{\leq 2}(n, q)$ such that words in $\text{Irr}_{\leq 2}(3, n, q)$ are ordered before words in $\text{Irr}_{\leq 2}(2, n, q)$. For words in $\text{Irr}_{\leq 2}(2, q)$ and $\text{Irr}_{\leq 2}(3, q)$, we simply order them lexicographically. We illustrate the idea behind the unranking algorithm through an example.

**Example 4.** Let $n = 6$ and $q = 3$. Then the values of $I_{\leq 2}(m, q)$ are as follow:

| m   | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|
| $I_{\leq 2}(m, q)$ | 6 | 12 | 18 | 30 | 48 |
Suppose we want to compute unrank(40). Proposition 1 gives
\[ \text{Irr}_{\ell}(6, 3) = \phi(\text{Irr}_{\ell}(5, 3) \times [1]) \cup \psi(\text{Irr}_{\ell}(4, 3) \times [1]). \]

Now, we are interested in the 40th word of \( \text{Irr}_{\ell}(6, 3). \) Since
\( 40 > I_{\ell}(5, 3) = 30, \) the 40th word of \( \text{Irr}_{\ell}(6, 3) \) is the image of
the 40 – 30 = 10-th word in \( \text{Irr}_{\ell}(4, 3) \) under \( \psi. \) Recursing tells us that
the 10-th word in \( \text{Irr}_{\ell}(4, 3) \) is the 10-th element in
\( \phi(\text{Irr}_{\ell}(3, 3) \times [1]). \) The 10-th element of \( \text{Irr}_{\ell}(3, 3) \) is 202.
This gives
\[ \text{unrank}(40) = \psi(\phi(202, 1), 1) = \psi(2021, 1) = 202101. \]

The formal unranking algorithm is described in Algorithm 1.

**Algorithm 1 unrank(n,q,j)**

Input: Integers \( n \geq 2, q \geq 3, 1 \leq j \leq I_{\ell}(n,q) \)

Output: \( x, \) where \( x \) is the codeword of rank \( j \) in \( \text{Irr}_{\ell}(n,q) \)

if \( n \leq 3 \) then
return \( j \)-th codeword in \( \text{Irr}_{\ell}(n,q) \)

if \( j \leq (q - 2)I_{\ell}(n - 1, q) \) then
\[ j' \leftarrow 1 + [(j - 1)/(q - 2)] \]
\[ i \leftarrow (j - 1) \mod (q - 2) + 1 \]
return \( \psi(\text{unrank}(n - 1, q, j'), i) \)

else
\[ j' \leftarrow 1 + [(j - (q - 2)I_{\ell}(n - 1, q) - 1)/(q - 2)] \]
\[ i \leftarrow (j - (q - 2)I_{\ell}(n - 1, q) - 1) \mod (q - 2) + 1 \]
return \( \psi(\text{unrank}(n - 2, q, j'), i) \)

The corresponding ranking algorithm for \( \text{Irr}_{\ell}(n,q) \) has a similar recursive structure and is described in Algorithm 2.

**Algorithm 2 rank(n,q,x)**

Input: \( n \geq 2, q \geq 3 \) and irreducible word \( x \) of length \( n \)

Output: \( j, \) where \( 1 \leq j \leq I_{\ell}(n,q) \), the rank of \( x \) in \( \text{Irr}_{\ell}(n,q) \)

if \( n \leq 3 \) then
return \( \text{rank}(x) \) in \( \text{Irr}_{\ell}(n,q) \)

if \( x_0 \neq x_{n-2} \) then
\[ x' \leftarrow x_1 x_2 \ldots x_{n-1} \]
\[ i \leftarrow \text{the index of } x_0 \text{ in } \Sigma_q \setminus \{x_{n-2}, x_{n-1}\} \]
return \( \text{rank}(n - 1, q, x') + (q - 2) + i \)

else
\[ x' \leftarrow x_1 x_2 \ldots x_{n-2} \]
\[ i \leftarrow \text{the index of } x_{n-1} \text{ in } \Sigma_q \setminus \{x_{n-2}, x_{n-1}\} \]
return \( \text{rank}(n - 2, q, x') + (q - 2) + i + (q - 2)I_{\ell}(n - 1, q) \)

The set of values of \( \{I_{\ell}(m,q) : m \leq n\} \) required in Algorithms 1 and 2 can be precomputed based on the recurrence (2).
Since the numbers \( I_{\ell}(m,q) \) grow exponentially, these \( n \) stored values require \( O(n^2) \) space.

Next, Algorithms 1 and 2 involve \( O(n) \) iterations and each iteration involves a constant number of arithmetic operations. Therefore, Algorithms 1 and 2 involve \( O(n) \) arithmetic operations and have time complexity \( O(n^3) \). Similarly, the corresponding ranking/unranking algorithm for \( \text{Irr}_{\ell}(n,q) \) have similar recursive structures and are described in Algorithm 3 and 4.

**Algorithm 3 unrank(n,q,j)**

Input: Integers \( n \geq 3, q \geq 3, 1 \leq j \leq I_{\ell}(n,q) \)

Output: \( x, \) where \( x \) is the codeword of rank \( j \) in \( \text{Irr}_{\ell}(n,q) \)

if \( n \leq 5 \) then
return \( j \)-th codeword in \( \text{Irr}_{\ell}(n,q) \)

if \( j \leq (q - 2)I_{\ell}(n - 1, q) \) then
\[ j' \leftarrow 1 + [(j - 1)/(q - 2)] \]
\[ i \leftarrow (j - 1) \mod (q - 2) + 1 \]
return \( \varphi_1(\text{unrank}(n - 1, q, j'), i) \)

else
\[ j' \leftarrow j - (q - 2)I_{\ell}(n - 1, q) \]
\[ i \leftarrow (j' - 1) \mod (q - 3) + 1 \]
return \( \varphi_3(\text{unrank}(n - 2, q, j'), i) \)

**Algorithm 4 rank(n,q,x)**

Input: \( n \geq 3, q \geq 3 \) and irreducible word \( x \) of length \( n \)

Output: \( j, \) where \( 1 \leq j \leq I_{\ell}(n,q) \), the rank of \( x \) in \( \text{Irr}_{\ell}(n,q) \)

if \( n \leq 5 \) then
return \( \text{rank}(x) \) in \( \text{Irr}_{\ell}(n,q) \)

if \( x \in \text{Irr}_{\ell}(n,q)(L_1, n,q) \) then
\[ (x', i) \leftarrow \varphi_1^{-1}(x) \]
return \( \text{rank}(n - 1, q, x') - 1) + (q - 2) + i \)

if \( x \in \text{Irr}_{\ell}(n,q)(L_2, n,q) \) then
\[ (x', i) \leftarrow \varphi_2^{-1}(x) \]
return \( \text{rank}(n - 2, q, x') - 1) + (q - 3) + i + (q - 2)I_{\ell}(n - 1, q) \)

if \( x \in \text{Irr}_{\ell}(n,q)(L_3, n,q) \) then
\[ (x', i) \leftarrow \varphi_3^{-1}(x) \]
return \( \text{rank}(n - 3, q, x') - 1) + (q - 2) + i + (q - 2)I_{\ell}(n - 1, q) + \]
\( (q - 3)I_{\ell}(n - 2, q) \)

A. Reducing the Space Requirement for the Finite State Encoder

As discussed earlier, a naive implementation of the \( (\ell, m) \)-finite state encoder in Section [III] requires \( q^{\ell m}(m) \) space (assuming \( \ell = \Theta(m) \)). Here, we modify our unranking algorithm to reduce the space requirement \( O(m) \) integers or \( O(m^2) \) bits.

Recall the notation in Section [III]. We discuss only for the case \( k = 2 \) as the case \( k = 3 \) is similar. In particular, let \( x_{i-1} \in \text{Irr}_{\ell}(m,q) \) and \( y_i \in \Sigma_q^m \). Our encoding task is to determine the irreducible word \( x_i \) in \( N(x_i) \) whose index corresponds to \( y_i \). Equivalently, if \( j \) is the rank of \( y_i \in \Sigma_q^m \), then our task is to find \( x_i \) such that its rank in \( N(x_{i-1}) \) is \( j \). Since \( x_{i-1} \) is irreducible and using symmetry, we assume that \( x_{i-1} \in \text{Irr}_{\ell}(010, m,q) \) or \( x_{i-1} \in \text{Irr}_{\ell}(210, m,q) \). Furthermore, Propositions 11 and 12 imply that \( N(x_{i-1}) \) corresponds to a union of \( \leq 2 \)-irreducible words
with prefixes of the form $10\sigma$. Therefore, it suffices to provide ranking/unranking algorithms for $\text{Irr}_{\leq 2}^{(p)}(10\sigma, m, q)$.

Since (3) implies that $\text{Irr}_{\leq 2}^{(p)}(10\sigma, m, q)$ has the same recursive structure as $\text{Irr}_{\leq 2}(m, q)$, we can modify Algorithms 1 and 2 to unrank and rank $\text{Irr}_{\leq 2}^{(p)}(10\sigma, m, q)$.

Now, to rank/unrank $\text{Irr}_{\leq 2}^{(p)}(10\sigma, m, q)$ require $O(m)$ precomputed integers. Assuming $q$ is constant, we require only $O(m)$ integers or $O(m^2)$ bits. However, the running time is increased to $O(m^2)$.

V. CONCLUSION

For $k \in \{2, 3\}$ and all $q$, we provided an explicit recursive formula for $\text{Irr}_{\leq k}(n, q)$ and hence, derived the expressions for $\text{rate}_{\leq k}(q)$.

We design efficient encoders/decoders for $\text{Irr}_{\leq k}(n, q)$.

(i) We provide an $(\ell, m)$-finite state encoder and show that for all $\epsilon > 0$, if we choose $m = \Theta(1/\epsilon)$ and $\ell = \Theta(1/\epsilon)$, the encoder achieves rate that is at least $\text{rate}_{\leq k}(q) - \epsilon$. The implementation of the finite state encoder with a lookup table runs in $O(n/\epsilon)$ time and requires $q^{\Theta(1/\epsilon)}$ space. However, if we use the ranking/unranking method in Section IV, the encoder runs in $O(n/\epsilon^2)$ time and requires $O(1/\epsilon)$ space.

(ii) We provide an unranking algorithm for irreducible words whose encoding rate is $(1/n)\log_q(|\text{Irr}_{\leq k}(n, q)|) \geq \text{rate}_{\leq k}(q)$. The encoder runs in $O(n^2)$ time and requires $O(n^2)$ space.

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