Information exclusion, uncertainty relations, and incompatibility

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Complementary information provides partial descriptions of quantum systems from different perspectives, and is inaccessible simultaneously. We introduce the notion of information operator associated with a measurement set, which, if informationally complete, allows one to analytically reconstruct arbitrary unknown quantum states directly from measurement outcomes. Further, we derive from it a universal information exclusion relation depicting the competitive balance between information extracted from generalized measurements, based on which innovative entropic uncertainty relations are also formulated. Moreover, the unbiasedness of orthonormal bases is naturally captured by the Hilbert-Schmidt norm of the corresponding information operator, it’s relationship with the white noise robustness are studied both analytically and numerically.

I. INTRODUCTION

One of the most surprising features of quantum mechanics that distinguishes it from classical theories is that one’s simultaneous accessible information of non-commuting (incompatible) observables is fundamentally limited, and can’t be extracted under a single experimental setup. The knowledge of one observable is more or less accompanied by unpredictability of measuring observables that is incompatible to it. This tradeoff between complementary information lying behind incompatible observables is to some extent captured by uncertainty relations, which has long been a hot topic since Heisenberg’s statement of uncertainty principle [1].

In quantum information theory, uncertainty is frequently quantified by the sum of entropies for different measurements [2–5]. Entropic uncertainty relations (EURs) are widely applied in quantum information theory, such as metrology [6], entanglement witness (EURs) are widely applied in quantum information measurements [2–5]. Entropic uncertainty relations are also formulated. Moreover, the unbiasedness of orthonormal bases is naturally quantified by the sum of entropies for different measurements [1].

In this work, we first develop the concept of information operator step by step, from projective measurements to POVMs. In this process its basic properties are made clear. Next, we show the operator norm of the information operator is of direct physical meaning as a quantification of how close the measurements are from being predictable with certainty simultaneously, based on which we are able to derive a general information exclusion relation as well as new EURs. Further, we examine the Hilbert-Schmidt norm of the information operator and its implications in quantifying unbiasedness of orthonormal bases.

This is part of the reason why MUBs are usually deemed as the most incompatible. But unsurprisingly, Eq. (1) is not tight for more general observables, since simply the maximal overlap is too inaccurate a description of measurements. Generalizations to multiple bases has been made in [12], and further in [13]. But it still remain an open question to find an elegant measurement-based only uncertainty relation that counts in all the overlaps.

When considering generalized measurements, i.e., positive-operator-valued measures (POVMs), a more frequently used but strictly weaker form of compatibility than simultaneous measurability is joint measurability [14,15]. Joint measurements can be implemented under a single experimental setup by measuring a parent POVM, followed by proper post-processing of the measured data. In this sense, measurements are incompatible if they are not jointly measurable. A common quantification of incompatibility is the noise robustness of a measurement set, that is, the minimal amount of white noise required for the measurements to be jointly measurable. During the past few years, many important and interesting works [16–23] have manifested the intimate connection between incompatibility and steering. Counterintuitively, there exists evidence showing that MUBs are not necessarily the most incompatible [17,24], but more detailed discussion on unbiasedness and incompatibility has not been made.

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II. INFORMATION OPERATOR

The Choi-Jamiolkowski isomorphism [25] allow one to represent an operator $A$ on $\mathcal{H}_d \otimes \mathcal{H}_d$ in the following way:

$$A \rightarrow [A] = \sqrt{d} \cdot A \otimes I_d \cdot |\phi_0\rangle = \sum_{x,x'=0} A_{xx'} |x\rangle \otimes |x'\rangle, \quad |A\rangle \rightarrow A = \sqrt{d} \cdot \text{tr}_2 ([A] \langle \phi_0 |), \quad (3)$$

where $|\phi_0\rangle = \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} (|x\rangle \otimes |x\rangle)^\ast$ is the maximally entangled state of $\mathcal{H}_d \otimes \mathcal{H}_d$ and $\text{tr}_2(\cdot)$ denotes the partial trace over the second space. A useful property of this isomorphism is $\langle A_1 | A_2 \rangle = \text{tr}(A_1^\ast A_2)$, the Hilbert-Schmidt inner product between operators $A_1$ and $A_2$.

With Eq. (3) we can represent any measurement as a set of vectors, but for brevity, let’s start with projective measurements only.

Now consider a finite set $\mathcal{O}$ of observables on $\mathcal{H}_d$, with $\{\{i_m\}_m\} (i = 0, \cdots, d - 1)$ denoting the basis for measuring the $m$th observable, we have

$$\sqrt{d} (|i_m\rangle \langle i_m \otimes I_d |\phi_0\rangle = |i_m\rangle \otimes |i_m\rangle^\ast, \quad (4)$$

which form an orthonormal basis of a $d$-dimensional subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$ for each $m = 1, \cdots, |\mathcal{O}|$, and so does their Fourier transforms

$$|\phi_{m,k}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{ik} |i_m\rangle \otimes |i_m\rangle^\ast, \quad (5)$$

here $\omega = e^{2\pi i/d}, \quad k = 0, 1, \cdots, d - 1$. Observe that $|\phi_{1,0}\rangle = \cdots = |\phi_{|\mathcal{O}|,0}\rangle = |\phi_0\rangle$ is basis independent, so we have from (4) at most $|\mathcal{O}|(d - 1) + 1$ linearly independent vectors. Especially, if $\mathcal{O}$ are complementary observables, with $\{\{i_m\}_m\}_m$ it can be easily checked that $|\phi_{m,k}\rangle$ and $|\phi_{m',k'}\rangle$ are orthogonal unless $k = k' = 0$.

Let $\mathcal{H}^0_m$ be the $d^2 - 1$ dimensional subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$ that is orthogonal to $|\phi_0\rangle$, we define the information operator of $\mathcal{O}$ as follows

$$\hat{G}(\mathcal{O}) = \sum_{m=1}^{|\mathcal{O}|} \sum_{k=1}^{d-1} |\phi_{m,k}\rangle \langle \phi_{m,k} | = \sum_{m=1}^{|\mathcal{O}|} \Phi_m \quad (6)$$

where $\Phi_m = \sum_{k=1}^{d-1} |\phi_{m,k}\rangle \langle \phi_{m,k} |$ are projections onto $d - 1$ dimensional subspaces $\{V_m\}$ of $\mathcal{H}^0_m$.

Theorem 1. Suppose $\mathcal{O}$ is a finite set of observables on $\mathcal{H}_d$ and $\hat{G}$ is the associated information operator, denote by $p_{i_m} = \langle i_m | \hat{G} | i_m \rangle$ the probability of obtaining the $i_m$th outcome when the $m$th observable is measured on a quantum state $\rho$, then we have

$$\|1 \cdot \hat{G} | \rho\rangle = \sum_{m=1}^{|\mathcal{O}|} \sum_{i=1}^d p_{i_m} |i_m\rangle \otimes |i_m\rangle^\ast = |\psi_{p,\mathcal{O}}\rangle, \quad (7)$$

where $1$ denotes the identity operator on $\mathcal{H}_d^0$, and if $\hat{G}$ is invertible (positive definite) on $\mathcal{H}_d^0$, $\rho = \sqrt{d} \cdot \text{tr}_2 (\hat{G}^{-1} |\psi_{p,\mathcal{O}}\rangle \langle \phi_0 |) + 1_d/d. \quad (8)$

We can see from Eqs. (7,8) that the complete information of $\rho$ can be extracted through measuring $\mathcal{O}$ when and only when $\hat{G}$ is invertible on $\mathcal{H}_d^0$, while if $\hat{G}$ is singular, those information of $|\rho\rangle$ stored in the null eigenspace of $\hat{G}$ would remain unknown. Another way of understanding this is the subspace projections $\{\Phi_m\}$ in Eq. (6) may fail to cover the whole space $\mathcal{H}_d^0$, implying loss of information.

Now let’s proceed to consider the information operator associated with POVMs. A POVM on $\mathcal{H}_d$ is a set of positive semi-definite operators $\{E_i\}$ (called effects) such that $\sum_i E_i = I_d$, and the probability of obtaining the $i$th outcome when measured on a quantum system $\rho$ is given by $p_i = \text{tr}(E_i \rho)$. For a finite set $\mathcal{O}$ of POVMs with weight parameters $\{w_m\} \{w_m > 0 \text{ and } \sum_m w_m = 1\}$, let $A_{i_m}$ denote the $i_m$th effect of the $m$th POVM, we define the corresponding weighted information operator as below

$$\hat{g}(\mathcal{O}) = \sum_{m,i} w_m |a_{i_m}\rangle \langle a_{i_m} |, \quad (9)$$

where $|a_{i_m}\rangle = |A_{i_m}\rangle - |\phi_0\rangle \langle \phi_0 |A_{i_m}\rangle$ and $\{w_m\}$ can be understood as probabilities of implementing different measurements. Note here $\hat{g} |\phi_0\rangle \equiv 0$, and when $\mathcal{O}$ are equal-weighted projective measurements $\hat{g}(\mathcal{O}) = G(\mathcal{O})/|\mathcal{O}|$.

We note a similar result for state reconstruction and a criterion for information completeness are obtained earlier in Ref. [29] based on frame theory [30].

III. INDEX OF UNCERTAINTY

For clarity, throughout the rest of this paper we say observables are compatible in subspace if they have one or more common eigenvectors. Denote by $\|\|_\|$ the operator norm, i.e., the largest eigenvalue of an operator, we define the index of uncertainty (IU) and its average to be the following

$$u(\mathcal{O}) = |\mathcal{O}|(1 - \|\hat{g}(\mathcal{O})\|), \quad \bar{u}(\mathcal{O}) = 1 - \|\hat{g}(\mathcal{O})\|. \quad (10)$$

The IU measures how close the measurements are from being predictable with certainty simultaneously. This is best exhibited by projective measurements.

Theorem 2. For any finite set $\mathcal{O}$ of observables on $\mathcal{H}_d$, if there exists a subset of $n_1 (n_1 \geq 1)$ observables compatible in subspace and a subset of $n_2 (n_2 \geq 1)$ complementary observables, then $n_1 \leq \|\hat{G}(\mathcal{O})\| \leq |\mathcal{O}| - n_2 + 1$.

As a consequence, $n_1 \leq \|\hat{G}(\mathcal{O})\| \leq |\mathcal{O}| - 1$. To illustrate this, observe that an information operator $\hat{G}$ in the form of Eq. (9) consists of several subspace projections $\{\Phi_m\}$, each corresponds to one observable of $\mathcal{O}$. When all the observables are complementary to each other, $\{V_m\}$ will be orthogonal subspaces, in which case $\hat{G}(\mathcal{O})$ is simply an identity operator on the space $V_1 \cup \cdots \cup V_{|\mathcal{O}|}$.
Meanwhile, if all the observables of \( O \) are compatible in subspace, say \(|0_1 \otimes 0_1\rangle = \cdots = |0_{|O|} \otimes 0_{|O|}\rangle\), then \( V_1 \cap \cdots \cap V_{|O|} \neq \emptyset \), and

\[
|v\rangle = \sqrt{d} |0_m\rangle \otimes |0_m\rangle - |\phi_0\rangle = \sum_{k=1}^{d-1} |\phi_{m,k}\rangle \tag{11}
\]

lies simultaneously in all \( V_m \). Immediately we know \(|v\rangle\) of (11) is an eigenvector of \( \hat{G}(O) \) with eigenvalue being \( |\mathcal{O}| \), correspondingly, \( u(O) = 0 \). Thus the existence of state-independent uncertainty can be equivalently rephrased as \( u(O) > 0 \). One can see in Appendixes a detailed proof of Theorem 2 as well as a perspective on IU as quantification of how much the basis vectors of different bases are overlapped.

There are some basic and interesting properties of IU (10). (i) \( u(O) \) is invariant under reordering and simultaneous unitary similarity transformation of observables \( O \). (ii) \( u(O) > 0 \) implies the existence of state-independent uncertainty, and for fixed number of observables it is maximal for complementary observables. (iii) \( u(O) \) is non-decreasing when including more observables, i.e., \( u(U \cup A) \geq u(O) \) for any observable \( A \). (iv) \( \bar{u}(O \cup A) > \bar{u}(O) \) if \( A \) is complementary to all observables of \( O \), while \( \bar{u}(U \cup A) \leq \bar{u}(O) \) if the eigenvector of \( \hat{G}(O) \) corresponding to \( \|\hat{G}(O)\| \) is invariant under \( \Phi_A \), in which case \( \|\hat{G}(O \cup A)\| = \|\hat{G}(O)\| + 1 \).

IV. INFORMATION EXCLUSION

We don’t expect the index of uncertainty (IU) defined in Eq. (11) to be with direct operational meaning, instead, we examine its implication in information tradeoff and uncertainty relations. In Ref. [34] the authors introduced the following measure of information for a probability distribution \( \hat{P} = (p_0, p_1, \cdots) \) over \( l \) measurement outcomes

\[
I(P) = \sum_i (p_i - 1/l)^2, \tag{12}
\]

The physical meaning behind (12) is clear, if a measurement outcome occurs with probability \( p \) known prior to an experimenter, then \( p(1 - p) \) quantifies the experimenter’s lack of information (uncertainty) of this outcome [35]. When \( p = 1 \) (0) of course the experimenter knows for sure that outcome will (not) occur, implying no uncertainty. In this perspective, \( I(P) \) represents the experimenter’s whole knowledge about a measurement, as a complement to the total lack of information \( \sum p_i (1 - p_i) \).

Though may have not been understood in the above way, there exists exclusion relations between information extracted from several kinds of measurements with specific symmetry, including MUBs [36,38], mutually unbiased measurements [39,40], symmetric informationally-complete POVM [11], POVMs from quantum 2-designs [12,43], and recently POVMs from equiangular tight frames [11]. Next we’ll show this is actually a very general phenomena, but before that we’d like to point out that the information measure (12) is improper for POVMs, and a modification is necessary.

The point is, when one is completely ignorant of the quantum state to be measured, his expectation of this state is an average over all possible pure states, that is, completely mixed. Then the probability distribution corresponds to the least information of the measured state should be given as \( p_i = tr(E_i)/d \), which is a uniform distribution only for equal-trace-effects POVMs (ETE-POVMs), that is, \( tr(E_0) = \cdots = tr(E_{d-1}) = d/l \). Therefore, the uncertainty of occurrence of measurement outcomes may not faithfully reflect the ignorance of quantum states. As an example, the POVM \( \{0.612, 0.431\} \) cannot be used to reveal information of any qubit, however, \( I(P) = 0.6^2 + 0.4^2 - 0.5 > 0 \). Our modification is

\[
I_o(P) = \sum_i \left(p_i - tr(E_i) / d\right)^2. \tag{13}
\]

Theorem 3. Suppose \( O \) is a finite set of weighted POVMs on \( \mathcal{H}_d \) and \( \hat{g} \) is the associated information operator, for any quantum state \( \rho \) on \( \mathcal{H}_d \) denote by \( P_m \) the probability distribution induced by performing the \( m \)-th POVM on \( \rho \), there is

\[
\sum_{m=1}^{\mid O \mid} w_m I_o(P_m) \leq \|\hat{g}\| \left(tr(\rho^2) - \frac{1}{d}\right), \tag{14}
\]

Proof. We give the proof only for ETE-POVMs, and general POVMs just take a little more calculation. In this case

\[
\hat{g} = \sum_m w_m \left(\sum_i |A_{im}\rangle \langle A_{im}| - \frac{d}{m} |\phi_0\rangle \langle \phi_0|\right). \tag{15}
\]

From Eq. (3) we have \( tr(|\phi_0\rangle \langle \phi_0| \rho \langle \rho|) = 1/d \) and \( tr(|\rho\rangle \langle \rho| \rho \langle \rho|) = tr(\rho^2) \), then

\[
tr(\hat{g} |\rho\rangle \langle \rho|) = \sum_{i,m} w_m |tr(\rho A_{im})|^2 - \sum_m W_m \frac{W_m}{l_m} \leq \|\hat{g}\| \cdot tr(1 |\rho\rangle \langle \rho|) = \|\hat{g}\| \cdot (tr(\rho^2) - 1/d),
\]

here \( 1 \) again is the identity operator on \( \mathcal{H}_d \). Q.E.D.

So we have now a universal upper bound on one’s simultaneous information regarding different measurements of \( O \), which is never trivial and decreases monotonically with the average index of uncertainty, a quantity that relies only on measurements. It is worth mentioning that when \( \hat{g} \) is proportional to \( 1 \), e.g. a complete set of MUBs, an equality of (14) can be reached by arbitrary quantum states, implying conservation of information. More over, this never happens for informationally incomplete measurements since 0 must be an eigenvalue of \( \hat{g} \). For those aforementioned measurements with special symmetry, Eq. (14) would reduce to the results obtained earlier in Ref. [36,43]. But our result is general and applies to more versatile scenarios when measurements are not necessarily implemented at random, as we derive it without any prior assumption about measurements.

Example 1. For two equal-weighted observables on \( \mathcal{H}_2 \), \( \|\hat{g}\| = c_{max} = \max_{i,j} |\langle i| j\rangle|^2 \), Eq. (14) becomes

\[
\frac{1}{2} \sum_{m=1.2} \sum_{i=0,1} p^2_{im} - \frac{1}{2} \leq c_{max} (tr(\rho^2) - 1/2). \tag{16}
\]
This simple example recovers the result of Ref. [45].

V. ENTROPIC UNCERTAINTY RELATIONS

An interesting and meaningful question here is how to derive lower bound on entropic uncertainty relations (EURs) from an upper bound on information. The Rényi entropy of a probability distribution \( P = (p_0, p_1, \cdots) \) induced by measuring \( A \) on quantum state \( \rho \) is

\[
H_\alpha(A|\rho) = \frac{1}{1 - \alpha} \log \sum_i p_i^\alpha \quad (\alpha > 0, \alpha \neq 1).
\]

(17)

In the limiting case \( \alpha \to 1 \), Eq. (17) would approach the Shannon entropy.

The index of coincidence (IC) of a probability distribution \( P = (p_0, p_1, \cdots) \) is defined as \( IC(P) = \sum p_i \). For ETE-POVMs, (13) is equivalent to (12) and Eq. (14) becomes an upper bound on the average IC, from which EURs can be formulated.

Theorem 4. Suppose \( \mathcal{O} = \{A_m\} \) is a finite set of ETE-POVMs on \( \mathcal{H}_d \), each consists of \( \ell \) effects, and \( \rho \) is a quantum state on \( \mathcal{H}_d \), for Rényi \( \alpha \)-entropy with \( \alpha \geq 2 \) we have

\[
\sum_{m=1}^{\left|\mathcal{O}\right|} w_m H_\alpha(A_m|\rho) \geq \frac{\alpha}{1 - \alpha} \log p_a + \frac{\log \ell}{(1 - \alpha) \log[1 + (l - 1)^2\frac{p_a^2}{p_b^2}]} \log \left[ 1 + (l - 1)^2\frac{p_a^2}{p_b^2} \right],
\]

where \( p_a = \frac{1 + \sqrt{(l - 1)(l - 1)}}{\ell}, \quad p_b = \frac{1 - \sqrt{(l - 1)(l - 1)}}{\ell}, \) and

\[
q = 1/l + \|\hat{g}\| \cdot (tr(\rho^2) - 1/d)
\]

is the upper bound on average IC given by Eq. (14).

Example 2. Suppose \( \mathcal{O} = \{A_m\} \) is a finite set of ETE-POVMs and \( q \) is the corresponding average IC, then

\[
\sum_{m=1}^{\left|\mathcal{O}\right|} w_m H_2(A_m|\rho) \geq -\log q.
\]

(20)

As for the Shannon entropy, the best lower bound that can be obtained based on (19) only is given in Ref. [47]. For two observables on \( \mathcal{H}_2 \), according to [47] the strongest Shannon entropic lower bound that can be obtained from (16) is \( h(b) \), with \( h(x) = -\frac{1}{a-x} \log \frac{1-x}{a-x} - \frac{1+x}{a+x} \log \frac{1+x}{a+x} \) and \( b = 2^{2\max(1, 2k)} - 1 \), which is exactly the bound proposed in Ref. [45], a tighter one than (1). But with (19) more fruitful results regarding Shannon EURs can be constructed in a similar way.

Example 3. For arbitrary three observables \( \{A_m\} \) on \( \mathcal{H}_2 \), there is

\[
\sum_{m=1}^{3} H(A_m|\rho) \geq h(\sqrt{2Q - 3 - k}) + 2 - k,
\]

with \( Q = ||\hat{G}||\cdot(\text{tr}(\rho^2) - 1/2) + 3/2 \) being an upper bound on sum of IC, \( k = 2q - 3 \).

Eqs. (20) are compared with the numerical optimal bound in Fig. 1 for three randomly generated observables on \( \mathcal{H}_2 \). We can see (20) approaches the tight entropic bound for observables that are complementary or are compatible in subspace, and it can also be tight for observables with intermediate index of uncertainty. As for (21), its state-independent form \( (\text{tr}(\rho^2) = 1) \) is generally not tight, but it is already stronger than the generalization of Eq. (1) to multiple bases presented in Refs. [12, 13]. Interestingly, though expressed in quite a different form, for MUBs the R.H.S. of (21) equals to \( 2 - \text{tr}(\rho \log \rho) \), a bound obtained earlier in [48].

VI. UNBIASEDNESS AND INCOMPATIBILITY

In Ref. [49] the authors proposed a geometric measure of unbiasedness for two orthonormal bases. Interestingly, it can be equivalently expressed in terms of the Hilbert-Schmidt norm of the corresponding information operator \( \hat{G} \) as \( D = 2(d - 1) - \frac{1}{2} ||\hat{G}||_2^2 \). We have a direct generalization to multiple bases

\[
D(\mathcal{O}) = |\mathcal{O}|(d - 1) - \frac{||\hat{G}(\mathcal{O})||_2^2}{|\mathcal{O}|}.
\]

(22)

with \( 0 \leq D(\mathcal{O}) \leq (|\mathcal{O}|-1)(d - 1) \). In the above sense eigenbases of commuting observables are not unbiased at all, and that of the complementary observables are the most unbiased.

The most commonly used quantification of incompatibility is the critical amount of white noise that render the measurements jointly measurable. Mathematically, the critical value \( \eta^* \) of \( \eta (0 \leq \eta \leq 1) \) below which \( \eta A_m + (1 - \eta)1_{\mathcal{H}/d} \) would be jointly measurable. Especially, white noise is jointly measurable. In Ref. [50] the authors proposed an upper bound on the noise robustness of arbitrary measurements, denote it

\[
\sum_{m=1}^{\left|\mathcal{O}\right|} w_m H_2(A_m|\rho) \geq -\log q.
\]
by \( \eta_0 \). Notably, for \( k \) MUBs of \( \mathcal{H}_d \) they obtain the bound
\[
\eta_{(d,k)} = \left( \frac{\sqrt{d}/k + 1}{\sqrt{d} + 1} \right),
\]
which is shown to be tight when \( k = 2, d, \) and \( d + 1 \), and its tightness for general \( k \) is left as a conjecture.

Following \[50\], we find that the noise-robustness of arbitrary two bases is upper bounded by a linear function of unbiasedness, and we have

**Theorem 5.** The noise robustness of arbitrary two orthonormal bases of \( \mathcal{H}_d \) is upper bounded by the following
\[
\eta^* \leq 1 + \frac{D}{d-1} (\eta(d,2) - 1) = \eta^1,
\]
here \( D \) is the unbiasedness \[22\] of the bases under consideration.

The proof is presented in Appendixes, where we also propose a conjectured generalization to multiple bases and some numerical results in low dimensions.

**VII. CONCLUSION**

Motivated by a better quantification of unpredictable with certainty simultaneously as a replacement for the maximal overlap between basis vectors, we introduced an operator named information operator to each set of measurements, which turns out to be of wide applications ranging from state tomography to uncertainty relations and unbiasedness of measurements.

Our investigations provide new sights for the complementarity and incompatibility of quantum measurements, and the method we use is enlightening for related future investigations.

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**Appendix A: Index of uncertainty**

1. **Proof of theorem 2**

We only need to prove \( \|\hat{G}\| \leq 1 + |\mathcal{O}| - n_2 \). Let \( S \) be the label set for the \( n_2 \) complementary observables, then
\[
\|\hat{G}\| = \left\| \sum_{m=1}^{\mathcal{O}} \Phi_m \right\| \leq \left\| \sum_{m \in S} \Phi_m \right\| + \left\| \sum_{m \notin S} \Phi_m \right\|
= 1 + \sum_{m \notin S} \Phi_m \leq 1 + |\mathcal{O}| - n_2.
\]

The second equality follows from the fact that \( \sum_{m \in S} \Phi_m \) is simply an identity operator on the space \( V_I \cup \cdots \cup V_{|\mathcal{O}|} \). The last inequality can be saturated only when one of the complementary observables and all the remaining \( |\mathcal{O}| - n_2 \) observables are compatible in some subspace.

2. **Another perspective**

To provide a different perspective on IU, observe \( |\mathcal{O}|d \times |\mathcal{O}|d \) Gram matrix \( \hat{G} \) for vectors \( \{|i_m\rangle \otimes |i_m\rangle^*\}_{i,m} \):
\[
G_{i_m,j_{m'}} = |\langle i_m | j_{m'} \rangle|^2,
\]
which is Hermitian and positive semi-definite \[52\], and fully describes the overlaps between different measurement bases. Since \( |\phi_0\rangle \) is basis independent, we know the Gram matrix \( \hat{G} \) for the Fourier transformed vectors \( \{|\phi_{m,k}\rangle\} \) must contain a \( |\mathcal{O}| \times |\mathcal{O}| \) principal minor, say \( \hat{G}^{(0)} \), with all of its elements being 1, and \( \hat{G} \) can be written in the block diagonal form
\[
\hat{G} = \begin{bmatrix}
\hat{G}^{(0)} & 0 \\
0 & \hat{G}^{(1)}
\end{bmatrix}.
\]

Note here \( \hat{G} \) is unitarily similar to \( G \) of Eq. \[A1\], and the eigenvalues of \( \hat{G}^{(0)} \) are \( \{|\mathcal{O}|, 0, \ldots, 0\} \). Meanwhile, \( \hat{G}^{(1)} \) is a \( |\mathcal{O}|(d-1) \times |\mathcal{O}|(d-1) \) Gram matrix for vectors \( \{|\phi_{m,k}\rangle\} \) \( k = 1, 2, \ldots, d-1; m = 1, 2, \ldots, |\mathcal{O}| \), whose nonzero eigenvalues must coincides with that of the information operator \( \hat{G} \), as when represented in an
what we’ll consider here is the following dual formulation

\[ \langle a|G|a' \rangle = \sum_{m,k} (a|\phi_{m,k}) (\phi_{m,k}|a') = (RR^+)_{a,a'} \]

\[ \bar{G}^{(1)}_{m,k,m',k'} = \langle \phi_{m,k}|\phi_{m',k'} \rangle = \sum_a (\phi_{m,k}|a) (a|\phi_{m',k'}) \]

\[ = (R^+R)_{m,k,m',k'} \quad \text{(A3)} \]

\[ \|G\| = \|RR^+\| = \|R^+ R\| = \|\bar{G}^{(1)}\| . \]

Appendix B: Unbiasedness

1. Proof of Theorem 5

The noise robustness of a set of measurements can be determined via a semidefinite program (SPD) [23], what we’ll consider here is the following dual formulation [17]

\[ \eta^* = \min_{X} \quad 1 + tr \sum_{i,m} X_{im} A_{im} \quad \text{(B1)} \]

s.t. \[ \quad 1 + tr \sum_{i,m} X_{im} A_{im} \geq \frac{1}{d} \sum_{i,m} tr X_{im} tr A_{im} , \]

\[ \sum_{i,m} \delta_{i,m} X_{im} \geq 0 . \quad \text{(B3)} \]

Note here the ordered Hermitian operators \( \bar{X} = (X_{01}, \cdots, X_{im}, \cdots) \) satisfying (B2) and (B3) form a convex set, and \( \eta^* \) is the minimum value of a linear function of \( \bar{X} \) optimized over it, which must be achieved at the boundary. More over, the boundary given by (B2) is a polytope with faces depending linearly on \( \{A_{im}\} \).

Consider two projective measurements, we can always fix one measurement in the computational basis \( B_0 \), and the other in the basis \( B_1 = U(\bar{\theta})B_0 \), the corresponding noise robustness is formally expressed as

\[ \eta^*(\bar{\theta}) = 1 + \sum_{i,m} tr (X_{im} A_{im}) \quad \text{(B4)} \]

For any \( \varepsilon > 0 \), we can multiply the above \( \bar{X} \) in (B4) by a proper parameter \( r \in [0,1] \) to obtain another vector \( \bar{X}^{(r)} \) such that (B2) is a strict inequality and

\[ \eta^*(\bar{\theta}) \leq \eta^{(r)}(\bar{\theta}) = 1 + \sum_{i,m} tr (X_{im}^{(r)} A_{im}) \]

\[ \leq \eta^*(\bar{\theta}) + \varepsilon , \quad \text{(B5)} \]

Note here the case \( \eta^{(0)}(\bar{\theta}) \equiv 1 \) is trivial, and in the limiting \( r \to 1 \), \( \eta^{(r)}(\bar{\theta}) = \eta^*(\bar{\theta}) \). Then with \( X_{im} + \delta X_{im} \) being the optimal solution to (B1) for measurements in the bases \( \{B_0, U(\bar{\theta} + \delta \bar{\theta})B_0 \} \), we have in a neighborhood of \( \bar{\theta} \)

\[ \eta^{(r)}(\bar{\theta} + s \delta \bar{\theta}) \leq \eta^{(r)}(\bar{\theta}) + (1 - s) \eta^{(r)}(\bar{\theta}) , \quad \text{(B6)} \]

So we know there exists a convex function of unbiasedness that is an upper bound on the noise robustness.

Figure 3. A comparison between \( \eta_0 \) (gray dots), \( \eta_2 \) (dashed lines), and the numerical optimal noise robustness \( \eta_{opt} \) (orange dots) for three random bases of \( H_2 \).

2. Conjectured generalization

For arbitrary \( N \) orthonormal bases, we conjecture that the noise robustness is bounded by below

\[ \eta^* \leq \eta_d(k - 1) + \frac{\eta_d(k) - \eta_d(k - 1)}{\Delta_d(k) - \Delta_d(k - 1)} = \eta_2 , \quad \text{(B7)} \]

with

\[ \Delta_d(N,k) = |\mathcal{O}|(d - 1) - \frac{a[N/k]^2 + b([N/k] + 1)^2}{|\mathcal{O}|} , \]

where \( a = \lfloor N/k \rfloor k + k - N \) and \( b = N - \lfloor N/k \rfloor k \).

For three bases of \( H_2 \), the numerical optimal bound is compared with our conjectured bound in Fig. 3.

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