Chance-constrained set covering with Wasserstein ambiguity

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Received: 5 August 2020 / Accepted: 27 January 2022 / Published online: 21 March 2022
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Abstract
We study a generalized distributionally robust chance-constrained set covering problem (DRC) with a Wasserstein ambiguity set, where both decisions and uncertainty are binary-valued. We establish the NP-hardness of DRC and recast it as a two-stage stochastic program, which facilitates decomposition algorithms. Furthermore, we derive two families of valid inequalities. The first family targets the hypograph of a “shifted” submodular function, which is associated with each scenario of the two-stage reformulation. We show that the valid inequalities give a complete description of the convex hull of the hypograph. The second family mixes inequalities across multiple scenarios and gains further strength via lifting. Our numerical experiments demonstrate the out-of-sample performance of the DRC model and the effectiveness of our proposed reformulation and valid inequalities.

Keywords Chance constraints · Wasserstein ambiguity · Valid inequalities · Convex hull

Mathematics Subject Classification 90C15 · 90C47 · 90C11

1 Introduction
We consider a set covering model, in which one selects a subset of $n$ elements to cover $I$ targets with a minimum cost. If we employ $x \in \{0, 1\}^n$ to denote a binary decision vector such that, for all $j \in [n] := \{1, \ldots, n\}$, $x_j = 1$ if element $j$ is selected and...
where $\xi_i := [\xi_{i1}, \ldots, \xi_{in}]^T$ denotes a vector of binary parameters such that $\xi_{ij} = 1$ if element $j$ is able to cover target $i$ and $\xi_{ij} = 0$ otherwise. Set covering models find a variety of real-world applications, including scheduling [45], production planning [50], facility location [20], and vehicle routing [8], etc. Consider, for example, we wish to build medical facilities among $n$ locations to cover $I$ target residential regions. In this context, $\xi_i$ describes the connection between target $i$ and all candidate locations and it may depend on their distances, e.g., $\xi_{ij} = 1$ if target $i$ is near location $j$. Accordingly, constraints (1) ensure that every target is within the neighborhood of some open facilities. This model can be generalized to incorporate backup coverage (see [60]). For example, crucial targets with higher priority (e.g., nursing homes) may be covered by multiple open facilities. This generalizes constraints (1) to $x^T \xi_i \geq v_i$, $\forall i \in [I]$, where $v_i \geq 1$ denotes an integer constant that represents the coverage level of target $i$.

In emergency (e.g., natural disasters), the connection between target $i$ and open facilities may be randomly disrupted. In that case, it is convenient to model $\xi_i$ as Bernoulli random variables, denoted by $\tilde{\xi}_i$, $i \in [I]$, and formulate the generalized set covering constraints probabilistically:

$$
\mathbb{P}_{\text{true}} \left\{ x^T \tilde{\xi}_i \geq v_i, \forall i \in [I] \right\} \geq 1 - \epsilon,
$$

(2)

where $\mathbb{P}_{\text{true}}$ denotes the joint probability distribution of $\tilde{\xi}$ and $\epsilon$, often chosen to be small such as 0.1 and 0.05, denotes a pre-specified risk level. Intuitively, a selection $x$ that satisfies (2) can maintain the desired coverage level of all targets with high probability. Chance constraint (2) is called single if $I = 1$ and joint if $I \geq 2$.

In reality, our knowledge on $\mathbb{P}_{\text{true}}$ is usually ambiguous. For example, the historical data of $\tilde{\xi}_i$ may be limited because of the infrequency of natural disasters. Due to such ambiguity on $\mathbb{P}_{\text{true}}$, we adopt a distributionally robust perspective. Specifically, we assume the access to a set of $N$ independent and identically distributed (i.i.d.) samples drawn from $\mathbb{P}_{\text{true}}$, denoted as $\{\tilde{\xi}^j\}_{j \in [N]}$. This gives rise to an empirical distribution $\mathbb{P}_{\tilde{\xi}} := (1/N) \sum_{j=1}^N \delta_{\tilde{\xi}^j}$ and provides an approximation to $\mathbb{P}_{\text{true}}$, where $\delta_{\tilde{\xi}^j}$ denotes the Dirac measure on the singleton $\{\tilde{\xi}^j\}$. Unfortunately, as we will demonstrate in Sect. 5.1, simply replacing $\mathbb{P}_{\text{true}}$ with $\mathbb{P}_{\tilde{\xi}}$ in chance constraint (2) may not produce a feasible solution. That is, a solution to such empirical approximation has low confidence of satisfying the actual chance constraint (2) under $\mathbb{P}_{\text{true}}$. As an alternative, distributionally robust approaches consider all distributions lying in a neighborhood of $\mathbb{P}_{\tilde{\xi}}$. We denote this neighborhood as an ambiguity set $\mathcal{P}$, which can be defined based on various discrepancy measures between probability distributions, e.g., Hellinger distance, Prokhorov metric, Wasserstein distance, etc. In this paper, we adopt the Wasserstein distance (see, e.g., [36]), which measures the discrepancy

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between distributions $\mathbb{P}_1$ and $\mathbb{P}_2$ by

$$d_W(\mathbb{P}_1, \mathbb{P}_2) := \inf_{Q \sim (\mathbb{P}_1, \mathbb{P}_2)} \mathbb{E}_Q \left[ \|\tilde{X}_1 - \tilde{X}_2\|_p \right],$$

where $\tilde{X}_1$, $\tilde{X}_2$ are random variables following distributions $\mathbb{P}_1$, $\mathbb{P}_2$ respectively, $Q \sim (\mathbb{P}_1, \mathbb{P}_2)$ denotes that $Q$ is a joint distribution of $\tilde{X}_1$ and $\tilde{X}_2$ with marginals $\mathbb{P}_1$ and $\mathbb{P}_2$, and $\|\cdot\|_p$ denotes the $p$-norm. Intuitively, $Q$ is a plan of transporting the probability masses of $\mathbb{P}_1$ to make it coincide with $\mathbb{P}_2$, and $d_W(\mathbb{P}_1, \mathbb{P}_2)$ equals the minimum transportation cost evaluated under the $p$-norm. Accordingly, we consider the following Wasserstein ambiguity set:

$$\mathcal{P} := \left\{ \mathbb{P} \in \mathcal{P}_0((0, 1)^I \times n) : d_W(\mathbb{P}, \hat{\mathbb{P}}_{\bar{\xi}}) \leq \delta \right\},$$

where $\mathcal{P}_0((0, 1)^I \times n)$ denotes the set of all distributions supported on $(0, 1)^I \times n$ and $\delta > 0$ denotes the radius of the ambiguity set. This leads to the following generalized distributionally robust chance-constrained set covering problem:

$$(\text{DRC}) \begin{aligned} \min_x & \quad c^T x, \\
\text{s.t.} & \quad \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\left\{ x^T \bar{\xi}_i \geq v_i, \forall i \in [I] \right\} \geq 1 - \epsilon, \\
& \quad x \in \{0, 1\}^n, \end{aligned}$$

where $c \in \mathbb{R}^n$ denotes a deterministic cost vector and constraint (4) requires to satisfy the chance constraint under all distributions within $\mathcal{P}$. If $\delta$ is sufficiently large that $\mathbb{P}_{\text{true}} \in \mathcal{P}$ then any feasible solution $x$ to (DRC) satisfies the “true” chance constraint (2). In a data-driven context, it has been shown that as the data size $N$ increases, the confidence of $\mathbb{P}_{\text{true}} \in \mathcal{P}$ increases to one exponentially fast (see, e.g., [17, 18]).

We remark that the Wasserstein distance is equivalent to various other discrepancy measures, including Hellinger distance and Prokhorov metric. For example, one can bound the Hellinger distance between $\mathbb{P}$ and $\hat{\mathbb{P}}_{\bar{\xi}}$, from both above and below, by $d_W(\mathbb{P}, \hat{\mathbb{P}}_{\bar{\xi}})$ (see, e.g., [18]). This implies that we can employ constraint (4) to approximate distributionally robust chance constraints defined under other discrepancy measures.

1.1 Literature review

Chance-constrained programming (CCP) arises in a wide range of applications including power systems [62], transportation [12], facility location [35], and wireless communication [26]. Dating back to [11, 34, 38], CCPs are considered very challenging to solve because

1. checking the feasibility of a given solution $x$ demands multivariate integral, which is difficult to calculate, and
2. CCP produces a nonconvex and even disconnected feasible region in general.
To mitigate this challenge, prior work has studied convex conservative approximation [4, 37] and sample average approximation (SAA) [10, 33], both of which can efficiently search for feasible solutions with guarantee. Nevertheless, the former may remain challenging to solve if decision variables are discrete (like in this paper), and the latter often demands the capability of drawing as many samples from $P_{true}$ as needed. In addition, distributionally robust chance-constrained models (DR-CCP) have received increasing attention in recent years [9, 13–15, 28, 30, 37, 49, 58, 59, 63]. There are many successful developments on the tractability of single and joint chance constraints with moment ambiguity sets, which characterize $P$ based on moment information of $P_{true}$ [9, 22, 23, 30, 57, 61, 63]. Nevertheless, moment ambiguity sets become more conservative than their counterparts based on discrepancy measures (e.g., a Wasserstein ambiguity set) when more data samples are available. On the other hand, DR-CCP with a Wasserstein ambiguity set is not polynomially solvable in general [56]. Prior work has developed exact reformulations and valid inequalities [13, 24, 25, 27, 55] for solving this model. For example, [13, 27, 55] derived mixed-integer linear or conic reformulations. When decision variables are purely binary, [55] exploited the submodularity of the reformulation to produce extended polymatroid inequalities. In addition, [27] derived precedence valid inequalities among the binary variables they employed to indicate constraint satisfaction in each scenario. [24] and [25] focused on DR-CCP with right-hand-side (RHS) and left-hand-side (LHS) Wasserstein ambiguity, respectively. By exploring the connection between DR-CCP and the SAA formulation of CCP, they employed the mixing scheme to produce valid inequalities (see, e.g., [21, 29, 32]). Nevertheless, prior work has paid less attention to problems with binary decision variables and discrete uncertainty. In this paper, we study joint DR-CCP with LHS Wasserstein ambiguity in generalized set covering, where both decision and random variables are purely binary.

Chance-constrained integer programs [5, 32, 46, 47, 52, 54] are stochastic variants of combinatorial optimization problems when uncertainty arises. In [32], the authors developed a general decomposition framework for solving chance-constrained programs and derived strong mixed valid inequalities by combining “base” inequalities from each scenario. In addition, [47] proposed an efficient coefficient strengthening method and lifted probabilistic cover inequalities for chance-constrained bin packing problems. Recently, [52] studied a chance-constrained assignment problem and its DR-CCP variant, for which they derived strong lifted cover inequalities with efficient separation heuristics. In this work, we exploit the special structures of set covering and Wasserstein ambiguity to derive two families of valid inequalities. The first family produces the convex hull of the hypograph of a “shifted” submodular function. To the best of our knowledge, such a convex hull was not discovered in prior work. The second family mixes inequalities across multiple scenarios, bearing a resemblance to the derivation in [32]. Nevertheless, we take advantage of the binary nature of our decision and random variables and strengthen the mixed inequalities further via lifting.

Chance-constrained set covering models fall into two main categories. In the first category, uncertainty arises on the RHS [6, 42]. In [6], the authors developed a specialized branch-and-bound algorithm based on the enumeration of $p$-efficient points, which were initially introduced by [39]. Later, [42] simplified the enumeration approach and derived polarity cuts to improve the computational performance.
The second category models LHS uncertainty \[2, 16, 54\], as in this paper. For example, \[16\] studied single chance constraints, where all components of the Bernoulli random vector \(\tilde{\xi}_i\) are independent, and developed efficient cutting plane approaches. In addition, \[54\] proposed an exact approach for solving CCPs when there exists an oracle to retrieve the probability of any events under \(P_{\text{true}}\). They demonstrated this approach on chance-constrained partial set covering problems with either independence or linear threshold assumptions. To the best of our knowledge, \[2\] is the only prior work on DR-CCP for set covering, but they studied single chance constraints under a moment ambiguity set. They derived a compact equivalent reformulation and exploited its supermodularity to derive strong valid inequalities. Different from \[2\], we study joint DR-CCP under Wasserstein ambiguity. In addition, the supermodularity of their reformulation stems from the correlation among \(\tilde{\xi}_{ij}\)'s, while our “shifted” submodular function is a result of the generalized set covering and the Wasserstein ambiguity.

1.2 Contributions

We derive exact reformulations and two families of valid inequalities for (DRC). Our main contributions include

1. We establish the NP-hardness of (DRC) and show that (DRC) admits a deterministic two-stage reformulation, which can be solved efficiently by decomposition algorithms.
2. We derive a complete description of the convex hull of a basic mixed-integer set, which stems from the hypograph of a “shifted” submodular function and arises in each scenario of the two-stage reformulation.
3. We derive a family of cross-scenario inequalities by lifting the mixed valid inequalities obtained from multiple scenarios.
4. We conduct extensive numerical experiments to demonstrate (1) the out-of-sample performance of (DRC) and (2) the effectiveness of our two-stage reformulation and valid inequalities.

The rest of this paper is organized as follows. Section 2 shows the NP-hardness and develops an exact two-stage reformulation for (DRC). Section 3 derives the single- and cross-scenario valid inequalities. Section 4 extends the reformulation and valid inequalities to a distributionally robust knapsack chance constraint. Finally, Sect. 5 demonstrates the effectiveness of the proposed model and solution approaches.

Notation: \(\mathbb{Z}_+\) denotes the set of nonnegative integers. For integers \(m\) and \(n\), \([n] := \{1, \ldots, n\}\), \(1_{m \times n}\) denotes an \(m \times n\) matrix of all ones, \(1\) denotes a vector of all ones with suitable dimension, and \(e_m\) denotes the \(m\)th standard basis vector with suitable dimension. For \(x \in \mathbb{R}\), \((x)^+ := \max\{x, 0\}\). For set \(E\), \(|E|\) denotes its cardinality, \(\text{conv}(E)\) denotes its convex hull, and the indicator function \(\mathbb{1}\{x \in E\} := 1\) if \(x \in E\) and \(\mathbb{1}\{x \in E\} := 0\) otherwise.
2 Two-stage reformulation

We derive a reformulation of (DRC) based on the conditional value-at-risk (CVaR) in Sect. 2.1 and show its NP-hardness in Sect. 2.2. Then, in Sect. 2.3, we further recast it as a two-stage stochastic program.

2.1 Conditional value-at-risk reformulation

We recall the definitions of value-at-risk (VaR) and CVaR [41]. For a random variable \( \tilde{X} \) following its induced probability distribution \( P_{\tilde{X}} \), the \((1 - \epsilon)\)-VaR of \( \tilde{X} \) is defined as

\[
\text{VaR}_{1 - \epsilon}(\tilde{X}) := \inf \left\{ x : P_{\tilde{X}}\{\tilde{X} \leq x\} \geq 1 - \epsilon \right\},
\]

and its \((1 - \epsilon)\)-CVaR is defined as:

\[
\text{CVaR}_{1 - \epsilon}(\tilde{X}) = \min_{\gamma} \left\{ \gamma + \frac{1}{\epsilon} \mathbb{E} (\tilde{X} - \gamma)^+ \right\}.
\] (5)

Previously, [55] derived a CVaR reformulation for DR-CCP when \( \tilde{\xi} \) is supported in a normed vector space. Since our \( \tilde{\xi} \) is binary-valued, we adapt the framework of [55] to obtain a slightly different reformulation, which also admits a CVaR interpretation.

**Proposition 1** (Adapted from Theorem 1 in [55]) Let \( Z \) represent the feasible region produced by constraint (4) in (DRC). Then, it holds that

\[
Z = \left\{ x \in \{0, 1\}^n : \exists \gamma \in \mathbb{R}_+, z \in \mathbb{R}_+^N : \begin{align*}
\delta - \gamma \epsilon &\leq \frac{1}{N} \sum_{j \in [N]} z_j, \\
z_j + \gamma &\leq \min_{i \in [I]} \left( \left( x^\top \tilde{\xi}_i - v_i + 1 \right)^+ \right)^{1/p}, \forall j \in [N] \end{align*} \right\}. \tag{6}
\]

In addition, \( Z \) admits the following CVaR interpretation:

\[
Z = \left\{ x \in \{0, 1\}^n : \frac{\delta}{\epsilon} + \text{CVaR}_{1 - \epsilon} \left[ -g(x, \tilde{\xi}) \right] \leq 0 \right\},
\]

where

\[
g(x, \xi) = \min_{i \in [I]} \left( \left( x^\top \tilde{\xi}_i - v_i + 1 \right)^+ \right)^{1/p}.
\] (7)

**Proof (see Theorem 1 in [55] and Theorem 3 in [7])** We first rewrite constraint (4) using complement:

\[
\sup_{P \in \mathcal{P}} \mathbb{P} \left\{ \exists i \in [I] : x^\top \tilde{\xi}_i \leq v_i - 1 \right\} \leq \epsilon.
\]
By Theorem 3 in [7], we expand the above supremum to obtain

\[
\sup_{\mathcal{P}} \mathbb{P}\left\{ \exists i \in [I]: x^\top \tilde{\xi}_i \leq v_i - 1 \right\} = \min_{\lambda \geq 0} \left\{ \lambda \delta + \frac{1}{N} \sum_{j \in [N]} \left( 1 - \lambda \cdot g(x, \hat{\xi}^j) \right)^+ \right\},
\]

where

\[
g(x, \hat{\xi}^j) := \inf_{\xi \in \Xi} \left\{ \|\xi - \hat{\xi}^j\|_p : \exists i \in [I] \text{ s.t. } x^\top \xi_i \leq v_i - 1 \right\} = \min_{i \in [I]} \inf_{\xi \in \Xi} \left\{ \|\xi - \hat{\xi}^j\|_p : x^\top \xi_i \leq v_i - 1 \right\}.
\]

Then, the CVaR interpretation follows from Corollary 1 in [55]. In what follows, we recast \( g(x, \hat{\xi}^j) \). To this end, let \( T \) be the index set of \( x \), i.e., \( T := \{ k \in [n] : x_k = 1 \} \). For any \( i \in [I] \), we have

\[
\min_{x^\top \xi_i \leq v_i - 1} \|\xi - \hat{\xi}^j\|_p = \min_{x^\top \xi_i \leq v_i - 1} \left( \sum_{k=1}^n \left| \xi_{ik} - \hat{\xi}_{ik}^j \right| \right)^{1/p}
\]

\[
= \min_{x^\top \xi_i \leq v_i - 1} \left( \sum_{k \in T} \left| \xi_{ik} - \hat{\xi}_{ik}^j \right| + \sum_{k \notin T} \left| \xi_{ik} - \hat{\xi}_{ik}^j \right| \right)^{1/p}
\]

\[
= \min_{x^\top \xi_i \leq v_i - 1} \left( \sum_{k \in T} \left| \xi_{ik} - \hat{\xi}_{ik}^j \right| \right)^{1/p}
\]

\[
= \left( (x^\top \hat{\xi}_i^j - (v_i - 1))^+ \right)^{1/p}.
\]

It follows that \( g(x, \hat{\xi}^j) = \min_{i \in [I]} \left( (x^\top \hat{\xi}_i^j - v_i + 1)^+ \right)^{1/p} \). This completes the proof. \( \square \)

Although (DRC) admits the same CVaR interpretation as in [55], the reformulation \( Z \) of (DRC) is inherently different from that of [55]. This is because the uncertain parameters \( \tilde{\xi} \) are binary-valued in (DRC) while those in [55] are supported in a vector space. As a result, projecting a vector onto a half-space, which is a key step of reformulating Wasserstein chance constraints, differs in discrete and vector spaces. Specifically, in the proof of Proposition 1,

\[
\min_{x^\top \xi_i \leq v_i - 1} \|\xi - \hat{\xi}^j\|_p
\]

\[
= \begin{cases} 
(x^\top \hat{\xi}_i^j - v_i + 1)^{1/p} & \text{if } \Xi = \{0, 1\}^n \text{ (as in the proof of Proposition 1),} \\
\frac{(x^\top \hat{\xi}_i^j - v_i + 1)^+}{\|x\|_p/(p-1)} & \text{if } \Xi = \mathbb{R}^n \text{ (as in [55]).}
\end{cases}
\]
Remark 1 The reformulation of $Z$ in (6) is of independent interest because it remains valid beyond the current setting of set covering. For example, (DRC) admits the same reformulation when $x \in \{0, 1\}^n$, $\Xi = \mathbb{Z}_+^{I \times n}$, and $p = 1$. In addition, the reformulation remains valid when some target cannot be covered by certain elements. Formally, suppose that $\xi$ is supported on the set

$$\left\{ \xi \in \{0, 1\}^{I \times n} : \xi_{ik} = 0 \quad \forall (i, k) \in S \right\}$$

for a subset $S \subseteq [I] \times [n]$, and accordingly $\hat{\xi}_{ik} = 0$ for all $(i, k) \in S$. Then, the reformulation of $Z$ in (6) remains valid.

2.2 NP-hardness

We establish the NP-hardness of solving (DRC) in the following proposition.

Proposition 2 (DRC) is NP-hard to solve for any given fixed risk level $\epsilon \in (0, 1)$.

Proof We show that (DRC) is equivalent to the following problem when $N = 1$ and $v_i = 1$ for all $i \in [I]$:

$$\min_x c^T x,$$

s.t. $\frac{\delta}{\epsilon} \leq \min_{i \in [I]} \left( x^T \hat{\xi}_i \right)^{1/p}$, $x \in \{0, 1\}^n$.

Let $(x_1, \gamma_1, z_1) \in Z$ be a feasible solution to (DRC), then $x_1$ is feasible to (DRC') because

$$\frac{\delta}{\epsilon} \leq \frac{z_1}{\epsilon} + \gamma_1 \leq z_1 + \gamma_1 \leq \min_{i \in [I]} \left( x^T \hat{\xi}_i \right)^{1/p}.$$

On the other hand, if $x_2$ is feasible to (DRC'), then together with $\gamma_2 := \delta/\epsilon$, $z_2 := 0$ they are feasible to (DRC) because

$$\delta - \gamma_2 \epsilon = 0 \leq z_2,$$

$$z_2 + \gamma_2 = \delta/\epsilon \leq \min_{i \in [I]} \left( x^T \hat{\xi}_i \right)^{1/p}.$$

Therefore, (DRC) and (DRC') are equivalent. Since restricting $\delta$ to be $\epsilon$ in (DRC') recovers the set cover problem, we conclude that (DRC) is NP-hard to solve. \hfill \Box

One may be tempted to obtain the convex hull of the following mixed-integer set $Q$ arising from the reformulation (6),

$$Q := \left\{ (\theta, x) \in \mathbb{R} \times \{0, 1\}^n : \theta \leq \min_{i \in [I]} \left( x^T \hat{\xi}_i \right)^{1/p} \right\}.$$
for given $\xi_i \in \{0, 1\}^n$, $i \in [I]$. Nonetheless, we show that optimizing a linear function over $Q$ is NP-hard as well. The proof of this proposition is given in Appendix 1.

**Proposition 3** The following problem is NP-hard to solve for any given fixed $p \in \mathbb{R}, p \geq 1$:

$$\min_{x, \theta} c^\top x - \theta^{1/p}$$

s.t. $\theta \leq x^\top \xi_i, \forall i \in [I],$

$\theta \in \mathbb{R}, x \in \{0, 1\}^n.$

(8)

2.3 Two-stage reformulation

The reformulation (6) of $Z$ is potentially challenging to optimize over, particularly because function $g(x, \xi)$ defined in Proposition 1 is non-concave in $x$. Although one can linearize $g$ with the help of auxiliary binary variables and big-M coefficients, the formulation thus obtained is computationally ineffective. Another linearized formulation without big-M coefficients can be obtained by following [55] if $g(x, \xi)$ is supermodular in $x$. Unfortunately, this is not the case even when $I = 1$ (see Sect. 3.1).

As an alternative, we exploit the (hidden) submodularity of $g$ to obtain a two-stage reformulation without additional binary variables or big-M coefficients. We review key concepts of submodularity in Sect. 2.3.1 and present the reformulation in Sect. 2.3.2.

2.3.1 Polyhedral results for submodular functions

**Definition 1** A function $\phi: 2^{[n]} \to \mathbb{R}$ is submodular if for any $\mathcal{R}, \mathcal{T} \subseteq [n]$, we have

$$\phi(\mathcal{R}) + \phi(\mathcal{T}) \geq \phi(\mathcal{R} \cup \mathcal{T}) + \phi(\mathcal{R} \cap \mathcal{T}).$$

In addition, $\phi$ is supermodular if $-\phi$ is submodular. $\Box$

For ease of exposition, we use $\phi(\mathcal{T})$ and $\phi(x_\mathcal{T})$ interchangeably, where $x_\mathcal{T} \in \{0, 1\}^n$ is the indicating vector of $\mathcal{T}$ such that $x_k = 1$ if and only if $k \in \mathcal{T}$. An example of submodular function follows.

**Lemma 1** ([48, 55]) For fixed $\alpha \in \mathbb{R}_+^n, \alpha_o \in \mathbb{R}$ function $\phi(x) := \min \{-\alpha^\top x + \alpha_o, 0\}$ is submodular.

Next, consider the epigraph $\text{epi}(\phi)$ of a submodular function $\phi$:

$$\text{epi}(\phi) := \{(\theta, x) \in \mathbb{R} \times \{0, 1\}^n : \phi(x) \leq \theta\}.$$
Then, the convex hull of \( \text{epi}(\phi) \) is fully described by the extended polymatroid inequalities (EPI) [43], i.e.,

\[
\text{conv} (\text{epi}(\phi)) := \{(\theta, x) \in \mathbb{R} \times [0, 1]^n : \phi(\emptyset) + \sum_{k=1}^{n} [\phi(T_k) - \phi(T_{k-1})] x_{\sigma_k} \leq \theta, \forall \sigma \in \Sigma \},
\]

where \( \Sigma \) denotes all permutations of \([n]\) and \( T_k := \{\sigma_1, \ldots, \sigma_k\}, T_0 := \emptyset \). The separation of \( \text{conv} (\text{epi}(\phi)) \) is very efficient even though it involves \( n! \) number of constraints.

**Theorem 1** (Proposition 1 in [3], Theorem 44.3 in [43], Section 3 in [31]) If \((\hat{\theta}, \hat{x}) \notin \text{conv} (\text{epi}(\phi))\), then it violates constraint

\[
\phi(\emptyset) + \sum_{k=1}^{n} [\phi(T_k) - \phi(T_{k-1})] x_{\sigma_k} \leq \theta,
\]

where \( \sigma \in \Sigma \) is the permutation such that \( \hat{x}_{\sigma_1} \geq \hat{x}_{\sigma_2} \geq \cdots \geq \hat{x}_{\sigma_n} \).

### 2.3.2 Reformulation

We first exchange the order of applying \((\cdot)^+\) and \((\cdot)^{1/p}\) in the definition of \( g \). Since \( x^\top \hat{\xi}_j - v_i + 1 \) can be negative, we extend the domain of function \((\cdot)^{1/p}\) in the following.

**Proposition 4** For \( z \in \mathbb{Z} \), define \( \bar{f}(z) := z^{1/p} \cdot 1 \{z \geq 0\} + z \cdot 1 \{z \leq -1\} \). Then,

\[
g(x, \xi) = \min_{i \in [I]} \left( \left( x^\top \widehat{\xi}_i^j - v_i + 1 \right)^{1/p} \right) = \left( \bar{f} \left( \min_{i \in [I]} \left\{ x^\top \widehat{\xi}_i^j - v_i + 1 \right\} \right) \right)^{+}.
\]

**Proof** For \( x \in \{0, 1\}^n \), we have

\[
\min_{i \in [I]} \left( \left( x^\top \widehat{\xi}_i^j - v_i + 1 \right)^{+} \right)^{1/p} = \left( \min_{i \in [I]} \left( x^\top \widehat{\xi}_i^j - v_i + 1 \right)^{+} \right)^{1/p} \]

\[
= \left( \left( \min_{i \in [I]} \left\{ x^\top \widehat{\xi}_i^j - v_i + 1 \right\} \right)^{+} \right)^{1/p} \]

\[
= \left( \bar{f} \left( \min_{i \in [I]} \left\{ x^\top \widehat{\xi}_i^j - v_i + 1 \right\} \right) \right)^{+},
\]

where the first equality is because \((\cdot)^{1/p}\) is monotone, the second equality is because \((\cdot)^{+}\) is monotone, and the third equality is because \( \bar{f}(z) \geq 0 \) if and only if \( z \geq 0 \). □
Although \( \bar{f} \) is defined on \( \mathbb{Z} \), the conclusion of Proposition 4 holds even when \( x^\top \xi^j_i \) takes a fractional value. Next, we linearize the nonlinear function \( \bar{f} \). In what follows, Propositions 5–7 utilize the fact that \( x^\top \xi^j_i \) is an integer.

**Proposition 5** For any \( x \in \{0, 1\}^n \), it holds that

\[
\bar{f} \left( \min_{i \in [I]} \left\{ x^\top \xi^j_i - v_i + 1 \right\} \right) = \max_y c^\top_p y + \bar{f}(1 - v_m) \\
\text{s.t. } 1^\top y \leq \min_{i \in [I]} \left\{ x^\top \xi^j_i - v_i + v_m \right\}, \\
y \in \{0, 1\}^n,
\]

where \( v_m := \max \left\{ v_i : i \in [I] \right\} \) and \( c_p := [\bar{f}(2 - v_m) - \bar{f}(1 - v_m), \bar{f}(3 - v_m) - \bar{f}(2 - v_m), \ldots, \bar{f}(n + 1 - v_m) - \bar{f}(n - v_m)]^\top \).

**Proof** For fixed \( x \in \{0, 1\}^n \), define \( \bar{v} := \min \left\{ x^\top \xi^j_i - v_i + 1 : i \in [I] \right\} \). Since the above linear program is a continuous knapsack problem with an integer knapsack capacity and \( c_p \geq 0 \), there exists an optimal solution \( y^* \in \{0, 1\}^n \) such that \( 1^\top y^* = \bar{v} + v_m - 1 \). Also note that vector \( c_p \) has non-increasing entries because \( \bar{f} \) is a concave increasing function by construction. Hence, without loss of optimality, the first \( \bar{v} + v_m - 1 \) entries of \( y^* \) equal one and the remaining entries equal zero. This yields an optimal objective value

\[
c_p^\top y^* = \bar{f}(2 - v_m) - \bar{f}(1 - v_m) + \bar{f}(3 - v_m) - \bar{f}(2 - v_m) + \cdots + \bar{f}(\bar{v}) - \bar{f}(\bar{v} - 1) + \bar{f}(1 - v_m) = \bar{f}(\bar{v}) = \bar{f} \left( \min_{i \in [I]} \left\{ x^\top \xi^j_i - v_i + 1 \right\} \right).
\]

\( \square \)

We are now ready to present the main result of this section.

**Proposition 6** (DRC) admits the following reformulation:

\[
\text{(MP)} \quad \min_{x,y,z} c^\top x \\
\text{s.t. } \delta - y \epsilon \leq \frac{1}{N} \sum_{j \in [N]} z_j, \\
- \bar{z}_j - y \geq \min \left\{ Q_j(\mu, x), 0 \right\}, \quad \forall \mu \in \mathcal{M}, \forall j \in [N], \\
x \in \{0, 1\}^n, \gamma \in \mathbb{R}_+, z \in \mathbb{R}_+^N, \\
\text{where } Q_j(\mu, x) := \sum_{i \in [I]} \mu_{ii} \left( x^\top \xi^j_i - v_i + v_m \right) + 1^\top \mu_2 - \bar{f}(1 - v_m) \text{ and } \mathcal{M} := \{(\mu_1, \mu_2) \in \mathbb{R}_+^I \times \mathbb{R}_+^n : \mu_1^\top \mathbf{1}_{I \times n} + \mu_2^\top \leq -c_p^\top \}.
\]
Proof By Propositions 1 and 5, (DRC) is equivalent to:

\[
\min_{x, \gamma, z} \mathbf{c}^T x \\
\text{s.t. } \delta - \gamma \epsilon \leq \frac{1}{N} \sum_{j \in [N]} z_j, \\
- z_j - \gamma \geq \min \left\{ Q'_j(x), 0 \right\}, \quad \forall j \in [N], \\
x \in \{0, 1\}^n, \gamma \in \mathbb{R}_+, z \in \mathbb{R}_N^N,
\]

where \(Q'_j(x)\) is defined as:

\[
(SP_j) \quad Q'_j(x) := \min_{y_j} - \mathbf{c}_p^T y \ - \bar{f}(1-v_m) \\
\text{s.t. } 1^T y_j \leq x^T \hat{\xi}_i^j - v_i + v_m, \quad \forall i \in [I], \\
y_j \in [0, 1]^n.
\]

Taking dual of the above linear program yields:

\[
(DSP_j) Q'_j(x) = \max_{\mu} Q_j(\mu, x) \\
\equiv \sum_{i \in [I]} \mu_{1i} \left( x^T \hat{\xi}_i^j - v_i + v_m \right) + 1^T \mu_2 - \bar{f}(1-v_m) \\
\text{s.t. } \mu \in \mathcal{M},
\]

where dual variables \(\mu_1, \mu_2\) are associated with constraints (10a) and (10b), respectively. Then, the reformulation follows by noting that

\[
\min \left\{ \max_{\mu \in \mathcal{M}} Q_j(\mu, x), 0 \right\} = \max_{\mu \in \mathcal{M}} \min \left\{ Q_j(\mu, x), 0 \right\}. \tag{11}
\]

Remark 2 By Lemma 1, the RHS of constraints (9) is submodular in \(x\). As a result, we can replace (9) with the EPI with respect to this submodular function. This implies that (DRC) admits a mixed-integer linear reformulation.

Proposition 6 suggests a Benders decomposition (BD) algorithm of solving (DRC) in an iterative manner. In each iteration, BD solves (MP) with constraints (9) replaced by a relaxation, and sends an optimal solution (\(\hat{x}, \hat{\gamma}, \hat{z}\)) to every (DSP\(_j\)) to check feasibility. If feasible, then (\(\hat{x}, \hat{\gamma}, \hat{z}\)) is optimal to (DRC); otherwise, we identify a \(\hat{\mu} \in \mathcal{M}\) such that \(- \hat{z}_j - \hat{\gamma} < \min\{Q'_j(\hat{\mu}, \hat{x}), 0\}\). In the latter case, we obtain a violated EPI with respect to \(\min\{Q'_j(\hat{\mu}, \hat{x}), 0\}\) and add it as a Benders feasibility cut back to (MP). Naturally, the efficacy of this BD algorithm depends on that of solving (DSP\(_j\)). The next proposition gives a closed-form solution to (DSP\(_j\)).
Proposition 7  For fixed $x$, an optimal solution $(\hat{\mu}_1, \hat{\mu}_2)$ to (DSP$_j$) satisfies

$$\hat{\mu}_1 = -e_i^*(c_p)_i^*, \quad (\hat{\mu}_2)_k = \min \left\{ - (c_p)_k - 1^T \hat{\mu}_1, 0 \right\}, \quad \forall k \in [n].$$

(12)

where $v_m = \max \{v_i : i \in [I]\}$, $i^* \in \arg\min \{x^T \tilde{\xi}_j^* - v_i + v_m : i \in [I]\}$, $\tilde{v} := \min \{x^T \tilde{\xi}_i^* - v_i + v_m : i \in [I]\}$, and $v^n \in \left\{ [\tilde{v}], [\tilde{v}] \right\}$.

Proof  To maximize $Q_j(\mu, x)$, every component in $\mu_2$ should attain its upper bound at optimality; and hence (12) follows. Since (i) $\mu_1 \leq 0$, (ii) $x^T \tilde{\xi}_j^* - v_i + v_m \geq 0$ for all $i \in [I]$, and (iii) $\sum_{k \in [n]} ((c_p)_k + 1^T \mu_1)^+$ relies solely on $1^T \mu_1$, the support of $\mu_1$ is a singleton at optimum, i.e., $\mu_1 = s \cdot e_i^*$ for a nonpositive scalar $s$. Accordingly, (DSP$_j$) reduces to a one-dimensional problem

$$\max_{\mu \in \mathcal{M}} Q_j(\mu, x) = \max_{s \in \mathbb{R}_-} \left\{ s \cdot \tilde{v} - \sum_{k \in [n]} ((c_p)_k + s)^+ \right\} - \tilde{f}(1 - v_m),$$

and $1^T \mu_1 = s$. Since the entries of $c_p$ are descending, a maximizer to the above problem equals either $-(c_p)_{[\tilde{v}]}$ or $-(c_p)_{[\tilde{v}]}$. □

Therefore, by Theorem 1 the Benders feasibility cut takes the following form:

$$- z_j - y \geq \phi^i(\emptyset) + \sum_{k=1}^n \left[ \phi^i(T_k) - \phi^i(T_{k-1}) \right] x_{s_k},$$

where $\phi^i(x) := \min \left\{ 1^T \hat{\mu}_1 \left( x^T \tilde{\xi}_i^* - v_i + v_m \right) + 1^T \hat{\mu}_2 - \tilde{f}(1 - v_m), 0 \right\}$

and $\phi^i(T_k) - \phi^i(T_{k-1}) = \begin{cases} 1^T \hat{\mu}_1 & \text{if } \phi^i(T_{k-1}) < 0 \\ \phi^i(T_k) & \text{if } \phi^i(T_{k-1}) = 0 \text{ and } \phi^i(T_k) < 0 \\ 0 & \text{if } \phi^i(T_k) = 0. \end{cases}$ (13)

We compare our two-stage reformulation of (DRC) with that of [55] when decision variables are binary. As pointed out after Proposition 1, our setting is inherently different from that of [55], and as a result, the technique Xie [55] used to exploit the supermodularity does not apply to our case. To be concrete, suppose that $\tilde{\xi}$ is supported in a vector space as in [55]. Then, applying Theorem 1 and Proposition 1 of [55] to our set covering model yields the following reformulation of (DRC):

$$Z_0 = \left\{ x \in \{0, 1\}^n : \exists y \in \mathbb{R}_+, v \in \mathbb{R}_+, z \in \mathbb{R}_N^N : \delta v - y \varepsilon \leq \frac{1}{N} \sum_{j \in [N]} z_j, \\ \|x\|_{p/(p-1)} \leq v, \\ z_j + y \leq \left( x^T \tilde{\xi}_i^* - v_i \right)^+, \forall i \in [I], j \in [N] \right\}.$$
In the last constraints of $Z_0$, the supermodularity of $(x^\top \hat{\xi}_j^i - v_i)^+$ in $x$ appears naturally. In contrast, in our case the term $((x^\top \hat{\xi}_j^i - v_i + 1)^+)^{1/p}$ in (6) is neither supermodular in $x$ nor in its complements (i.e., replacing some entries $x_k$ with $1 - x_k$) because the root function $(\cdot)^{1/p}$ is strictly concave for all $p > 1$. To exploit supermodularity, we need to recast the root function as a linear program (i.e., Proposition 5) and then take dual.

3 Valid inequalities

We derive two families of valid inequalities for the reformulation (6) of $Z$. Section 3.1 provides inequalities based on a single scenario $j \in [N]$, Sect. 3.2 consists of detailed proofs for establishing a convex-hull result of these inequalities, and Sect. 3.3 provides cross-scenario inequalities.

3.1 Valid inequalities from a single scenario

We focus on a nonlinear inequality $z_j + \gamma \leq ((x^\top \hat{\xi}_j^i - v_i + 1)^+)^{1/p}$ in the reformulation (6), for a single scenario $j \in [N]$ and a single covering $i \in [I]$. If $v_i = 1$ then the RHS of this inequality becomes a submodular function in $x$ because $p \geq 1$ and $\hat{\xi}_j^i \geq 0$. For general $v_i$, the RHS can be viewed as the submodular function being “shifted” by $(v_i - 1)$. However, after such a shift the RHS becomes neither submodular nor supermodular, preventing us from using existing valid inequalities (e.g., [1, 43]). In this section, we generalize this inequality and consider the following hypograph of a shifted submodular function:

$$X := \{ (\theta, x) \in \mathbb{R} \times \{0, 1\}^n : \theta \leq f((x^\top \xi - \beta)^+) \};$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is a strictly concave increasing function with $f(0) = 0$, $\xi \in \{0, 1\}^n$ is a given binary vector, and $\beta \in \mathbb{Z}_+$ is a given integer. Let $\mathcal{Z}$ be the set of indices where $\xi$ takes 1. If $\beta \geq |\mathcal{Z}| - 1$ then $f((x^\top \xi - \beta)^+) \equiv f(1)(x^\top \xi - \beta)^+$ is supermodular in $x$. In this case, $\text{conv}(X)$ can be fully described by the EPI. Hence, we assume $1 \leq \beta \leq |\mathcal{Z}| - 2$ without loss of generality (w.l.o.g.). Additionally, since $\xi$ has at most $|\mathcal{Z}|$ numbers of ones, all possible values for $(x^\top \xi - \beta)^+$ lie in the set

$$L := [|\mathcal{Z}| - \beta].$$

We replace $f$ with its inner piecewise linear approximation with integer breakpoints in $L$ in the definition of $X$. The following lemma formalizes this.

**Lemma 2** For $\ell \in L$, let $\varphi_\ell$ be a linear function defined as $\varphi_\ell(z) := a_\ell z + b_\ell$ where $a_\ell := f(\ell) - f(\ell - 1)$ and $b_\ell := f(\ell) - a_\ell \cdot \ell$. To avoid clutter, we also define $a_{|\mathcal{Z}| - \beta + 1} := 0$ and $b_{|\mathcal{Z}| - \beta + 1} := f(|\mathcal{Z}| - \beta)$. Then, we have $f(z) = \min_{\ell \in L} \{ \varphi_\ell(z) \}$ for any $z \in L$. In particular, $f(\ell) = \varphi_\ell(\ell) = \varphi_{\ell+1}(\ell)$ for all $\ell \in L$. 

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**Proof** It is straightforward that \( f(\ell) = \varphi_\ell(\ell) = \varphi_{\ell+1}(\ell) \) for all \( \ell \in L \). To show that \( f(z) = \min_{\ell \in L} \left\{ \varphi_\ell(z) \right\} \) for any \( z \in L \), it is sufficient to argue that \( \varphi_i(\ell) \geq \varphi_\ell(\ell) \) for every \( i \in L \). When \( i < \ell \), \( \varphi_i(\ell) \geq \varphi_\ell(\ell) \) is equivalent to \( (f(i) - f(i - 1)) \geq (f(\ell) - f(i))/(\ell - i) \), which holds due to the concavity of \( f \). When \( i > \ell \), \( \varphi_i(\ell) \geq \varphi_\ell(\ell) \) is equivalent to \( (f(i) - f(\ell))/(i - \ell) \geq f(i) - f(i - 1) \), which holds due to the concavity of \( f \).

We mention properties about \( a_\ell \) and \( b_\ell \) in the following remark.

**Remark 3** For \( a_\ell \) and \( b_\ell \), it holds that

(i) \( a_\ell \) are strictly positive and decreasing in \( \ell \). This is because \( f \) is strictly concave and increasing.

(ii) \( b_\ell \geq 0 \) for all \( \ell \in L \). This is because \( b_\ell = \sum_{k=1}^\ell (f(k) - f(k - 1)) - \ell \cdot (f(\ell) - f(\ell - 1)) = \sum_{k=1}^\ell (a_k - a_\ell) \geq 0 \), where the last inequality is because \( a_\ell \) are decreasing in \( \ell \).

(iii) \( b_\ell/a_\ell \) are strictly increasing in \( \ell \). This is because \( a_\ell \) are strictly decreasing and \( b_\ell \) are increasing since

\[
b_{\ell+1} - b_\ell = \sum_{k=1}^{\ell+1} (a_k - a_\ell) - \sum_{k=1}^{\ell} (a_k - a_\ell) \geq (\ell(a_\ell - a_{\ell+1}) > 0.
\]

In addition, for ease of presenting the valid inequalities we define

\[
L_1 := \bigcup_{\ell \in L} \left\{ (\ell, \rho) : \rho = \frac{b_\ell}{a_\ell} \right\},
\]

\[
L_2 := \bigcup_{\ell \in L} \left\{ (\ell, \rho) : \rho \in [\beta] \cap \left( \frac{b_\ell}{a_\ell}, \frac{b_{\ell+1}}{a_{\ell+1}} \right) \right\},
\]

and \( h_{(\ell,\rho)} : \mathbb{R}_+ \rightarrow \mathbb{R} \),

\[
h_{(\ell,\rho)}(z) := \left( \frac{f(\ell)}{(\ell + \rho)(z - (\beta - \rho))} \right)^+, \quad \forall (\ell, \rho) \in L_1 \cup L_2.
\]

**Remark 4** For \( (\ell, \rho) \in L_1 \), we can simplify \( h_{(\ell,\rho)} \) as

\[
h_{(\ell,\rho)}(z) = (a_\ell (z - \beta) + b_\ell)^+ = (\varphi_\ell (z - \beta))^+,
\]
from which \( h_{(\ell,\rho)}(z) \) coincides with \( f((z - \beta)^+) \) for all \( (\ell, \rho) \in L_1 \) when \( z \in (\ell + \beta - 1, \ell + \beta) \). For ease of exposition, we use \( (\varphi_\ell)^+ \) and \( h_{(\ell,\rho)} \) interchangeably if \( (\ell, \rho) \in L_1 \). For \( (\ell, \rho) \in L_2 \), \( h_{(\ell,\rho)}(z) = 0 \) if \( z \leq \beta - \rho \) and \( h_{(\ell,\rho)}(z) > 0 \) if \( z > \beta - \rho \). In particular, \( h_{(\ell,\rho)}(z) = f(\ell) \) when \( z = \ell + \beta \).
Example 1 Suppose that $|Z| = 4$, $\beta = 2$, and function $f(z) = \sqrt{z}$. Then, $L = [2]$, $L_1 = \{(1, 0), (2, \sqrt{2})\}$, $L_2 = \{(1, 1), (2, 2)\}$, and the shifted function $f((z - \beta)^+) = \sqrt{(z - 2)^+}$. Accordingly, for $(\ell, \rho) \in L_1$, the function $h_{(\ell, \rho)}$ takes the following two forms:

$$h_{(1, 0)}(z) = (z - 2)^+, \quad h_{(2, \sqrt{2})}(z) = (\sqrt{2} - 1) \left(z - (2 - \sqrt{2})\right)^+;$$

and for $(\ell, \rho) \in L_2$, it takes the following two forms:

$$h_{(1, 1)}(z) = \frac{1}{2} (z - 1)^+, \quad h_{(2, 2)}(z) = \frac{\sqrt{2}}{4} (z)^+.$$

We depict the shifted function $f((z - \beta)^+)$ and $h_{(\ell, \rho)}(z)$ in Fig. 1. In particular, from Fig. 1a, we observe that, as expected, each $h_{(\ell, \rho)}(z)$, $(\ell, \rho) \in L_1$ coincides with $f((z - \beta)^+)$ when $z \in \{\ell + \beta - 1, \ell + \beta\}$. From Fig. 1b, we observe that each $h_{(\ell, \rho)}(z)$,
$(\ell, \rho) \in L_2$ has an integer $z$-intercept and it coincides with $f((z - \beta)^+)$ only at the point $(\ell + \beta, f(\ell))$.

In what follows, we study the hypograph of $h_{(\ell, \rho)}$ and its relationship with $X$. For $(\ell, \rho) \in L_1 \cup L_2$, define

$$X_{(\ell, \rho)} := \left\{ (\theta, x) \in \mathbb{R} \times \{0, 1\}^n : \theta \leq h_{(\ell, \rho)}(x^T \xi) \right\}.$$ 

First, we observe that $\{X_{(\ell, \rho)} : (\ell, \rho) \in L_1\}$ provides a reformulation for $X$, while $\{X_{(\ell, \rho)} : (\ell, \rho) \in L_2\}$ provides a set of valid inequalities.

**Lemma 3** For a given $x \in \{0, 1\}^n$, we have

$$f \left( (x^T \xi - \beta)^+ \right) = \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(x^T \xi) \leq \min_{(\ell, \rho) \in L_2} h_{(\ell, \rho)}(x^T \xi). \quad (14)$$

**Proof** Fix $\hat{x} \in \{0, 1\}^n$. We first prove the equality. If $\hat{x}^T \xi - \beta \leq 0$, then

$$f \left( (\hat{x}^T \xi - \beta)^+ \right) = 0 = \varphi_1 \left( (\hat{x}^T \xi - \beta)^+ \right) = h_{(1, 0)}(\hat{x}^T \xi)$$

$$\geq \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(\hat{x}^T \xi),$$

where the first equality is because $f(0) = 0$, the second equality is because $\varphi_1(z) = f(1)z$, and the third equality is because $\varphi_1((x^T \xi - \beta)^+) = (\varphi_1(x^T \xi - \beta))^+$. If $\hat{\ell} := \hat{x}^T \xi - \beta \geq 1$, then

$$f \left( (\hat{x}^T \xi - \beta)^+ \right) = f(\hat{\ell}) = a_{\hat{\ell}} \cdot \hat{\ell} + b_{\hat{\ell}} = \left( \varphi_{\hat{\ell}}(\hat{x}^T \xi - \beta) \right)^+$$

$$\geq \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(\hat{x}^T \xi),$$

where the last equality is because $a_{\hat{x}} b_{\hat{\ell}} \geq 0$. Next we show that $f((x^T \xi - \beta)^+) \leq \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(x^T \xi)$. Suppose that $(\ell, \rho)$ is a minimizer for $\min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(\hat{x}^T \xi)$, then

$$\min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(\hat{x}^T \xi) = h_{(\ell, \rho)}(\hat{x}^T \xi) = \left( \varphi_{\ell}(\hat{x}^T \xi - \beta) \right)^+$$

$$= \varphi_{\ell} \left( (\hat{x}^T \xi - \beta)^+ \right) 1\{\hat{x}^T \xi - \beta \geq 0\}$$

$$+ \left( \varphi_{\ell}(\hat{x}^T \xi - \beta) \right)^+ 1\{\hat{x}^T \xi - \beta < 0\}$$

$$\geq f \left( (\hat{x}^T \xi - \beta)^+ \right).$$
where the last equality is because $a_{\widehat{\ell}}, b_{\widehat{\ell}} \geq 0$ and the inequality is due to the piecewise linear representation of $f$ and the fact that $f(0) = 0$. Therefore, $f \left( (x^T \xi - \beta)^+ \right) = \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(x^T \xi)$.

It remains to show $\min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(x^T \xi) \leq h_{(\hat{\ell}, \hat{\rho})}(x^T \xi)$ for all $(\hat{\ell}, \hat{\rho}) \in L_2$. We note that

$$a_{\hat{\ell}+1}(\hat{\rho} + \hat{\ell}) \leq a_{\hat{\ell}+1} \cdot \hat{\ell} + b_{\hat{\ell}+1} = f(\hat{\ell}) = a_{\hat{\ell}} \cdot \hat{\ell} + b_{\hat{\ell}} \leq a_{\hat{\ell}}(\hat{\rho} + \hat{\ell}),$$

implying

$$a_{\hat{\ell}+1} \leq \frac{f(\hat{\ell})}{\hat{\ell} + \hat{\rho}} \leq a_{\hat{\ell}}.$$

Therefore, we have

$$h_{(\hat{\ell}, \hat{\rho})}(x^T \xi) = \left( \frac{f(\hat{\ell})}{\hat{\ell} + \hat{\rho}} (x^T \xi - \beta + \hat{\rho}) \right)^+ = \left( f(\hat{\ell}) + \frac{f(\hat{\ell})}{\hat{\ell} + \hat{\rho}} (x^T \xi - \beta - \hat{\ell}) \right)^+$$

$$= \left( f(\hat{\ell}) + \frac{f(\hat{\ell})}{\hat{\ell} + \hat{\rho}} (x^T \xi - \beta - \hat{\ell}) \right)^+ \mathbb{1}\{x^T \xi - \beta \leq \hat{\ell}\} +$$

$$\left( f(\hat{\ell}) + a_{\hat{\ell}} (x^T \xi - \beta - \hat{\ell}) \right)^+ \mathbb{1}\{x^T \xi - \beta \geq \hat{\ell} + 1\}$$

$$\geq \left( f(\hat{\ell}) + a_{\hat{\ell}} (x^T \xi - \beta - \hat{\ell}) \right)^+ \mathbb{1}\{x^T \xi - \beta \leq \hat{\ell}\} +$$

$$\left( f(\hat{\ell}) + a_{\hat{\ell}+1} (x^T \xi - \beta - \hat{\ell}) \right)^+ \mathbb{1}\{x^T \xi - \beta \geq \hat{\ell} + 1\}$$

$$= \left( \varphi_{\hat{\ell}}(x^T \xi - \beta) \right)^+ \mathbb{1}\{x^T \xi - \beta \leq \hat{\ell}\}$$

$$+ \left( \varphi_{\hat{\ell}+1}(x^T \xi - \beta) \right)^+ \mathbb{1}\{x^T \xi - \beta \geq \hat{\ell} + 1\}$$

$$\geq \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(x^T \xi).$$

\[ \square \]

The previous Lemma implies

$$f \left( (x^T \xi - \beta)^+ \right) = \min_{(\ell, \rho) \in L_1} h_{(\ell, \rho)}(x^T \xi) = \min_{(\ell, \rho) \in L_1 \cup L_2} h_{(\ell, \rho)}(x^T \xi).$$

Second, we notice that the EPIs arising from $\{X_{(\ell, \rho)}: (\ell, \rho) \in L_2\}$ and $\{X_{(\ell, \rho)}: (\ell, \rho) \in L_1\}$ are sufficient to describe $\text{conv}(\mathcal{X})$.  

\[ \square \] Springer
Theorem 2  It holds that

\[
\text{conv}(X) = \bigcap_{(\ell, \rho) \in L_1 \cup L_2} \text{conv}(X_{(\ell, \rho)}).
\]

Before presenting the proof, we discuss the implication of Theorem 2 on the separation of \(\text{conv}(X)\). To this end, by Theorem 2, we only need to separate from \(\text{conv}(X_{(\ell, \rho)})\) for all \((\ell, \rho) \in L_1 \cup L_2\), where \(|L_1| = |\mathcal{Z}| - \beta, |L_2| \leq \beta\) and so \(|L_1| + |L_2| \leq |\mathcal{Z}| \leq n\). In addition, for each \((\ell, \rho)\), \(h_{(\ell, \rho)}(x^\top \xi)\) is supermodular in \(x\) by Lemma 1. Hence, by Theorem 1, we can separate from each \(\text{conv}(X_{(\ell, \rho)})\) by running a sorting algorithm among the \(n\) entries of a given solution \(\hat{x}\). In addition, the resulting order is valid for all \((\ell, \rho)\). Therefore, we can separate from \(\text{conv}(X)\) by running a single sorting and checking the violation of at most \(n\) inequalities.

**Proof of Theorem 2**  The proof is established on Propositions 8, 9, and 10. We present their statements and relegate the proofs to Sect. 3.2. We denote the intersection on the RHS of the claim as \(X'\). By Lemma 3, any \((\hat{\theta}, \hat{x}) \in X\) satisfies

\[
\hat{\theta} \leq f \left((\hat{x}^\top \xi - \beta)^+) = \min_{(\ell, \rho) \in L_1 \cup L_2} h_{(\ell, \rho)}(\hat{x}^\top \xi).
\]

Then, \((\hat{\theta}, \hat{x}) \in X_{(\ell, \rho)}\) for all \((\ell, \rho) \in L_1 \cup L_2\), implying \((\hat{\theta}, \hat{x}) \in X'\) and \(\text{conv}(X) \subseteq X'\). It remains to prove \(X' \subseteq \text{conv}(X)\). Equivalently, we prove that \((\hat{\theta}, \hat{x}) \notin \text{conv}(X)\) implies \((\hat{\theta}, \hat{x}) \notin \text{conv}(X_{(\ell, \rho)})\) for some \((\ell, \rho) \in L_1 \cup L_2\).

First, by the hyperplane separation theorem, there exists nonzero \((\pi_0, \pi) \in \mathbb{R}^{1+n}\) satisfying

\[
\pi_0 \hat{\theta} + \pi^\top \hat{x} > \pi_0 \theta + \pi^\top x, \quad \forall (\theta, x) \in X.
\]

If \(\hat{x} \notin [0, 1]^n\) then \((\hat{\theta}, \hat{x}) \notin \text{conv}(X_{(\ell, \rho)})\) because \(\text{conv}(X_{(\ell, \rho)}) \subseteq \mathbb{R} \times [0, 1]^n\). Hence, we can assume \(\hat{x} \in [0, 1]^n\) w.l.o.g. Since \(\theta\) is unbounded below in \(X\), \(\pi_0\) should be nonnegative. If \(\pi_0 = 0\) then \(\pi^\top \hat{x} > \pi^\top x\) for all \(x \in [0, 1]^n\), a contradiction, because this nonzero \(\pi\) separates \(\hat{x}\) from \([0, 1]^n\). Hence, \(\pi_0 > 0\). We can further assume \(\pi_0 = 1\) w.l.o.g. and arrive at

\[
\exists \pi \in \mathbb{R}^n : \hat{\theta} + \pi^\top \hat{x} > \theta + \pi^\top x, \quad \forall (\theta, x) \in X.
\]

In other words, \(\hat{\theta}\) exceeds the optimal value of the following formulation:

\[
\min_{\pi \in \mathbb{R}^n} \left\{ \max_{(\theta, x) \in X} \left\{ \theta + \pi^\top x : (\theta, x) \in X \right\} - \pi^\top \hat{x} \right\}. \tag{15}
\]

The rest of the proof solves problem (15). The main idea can be summarized as follows:
\[ = \min_{\pi \in \Pi^*} \left\{ \max_{(\ell, \rho) \in L_1 \cup L_2} \left[ \max_{(\theta, x) \in \mathcal{X}(\ell, \rho)} \left\{ \theta + \pi^T x \right\} - \pi^T \hat{x} \right] \right\} \quad \text{by Propositions 8, 9} \]

\[ = \min_{\pi \in \Pi^*} \left\{ \min_{(\ell, \rho) \in L_1 \cup L_2} \left[ \max_{(\theta, x) \in \mathcal{X}(\ell, \rho)} \left\{ \theta + \pi^T x \right\} - \pi^T \hat{x} \right] \right\} \quad \text{by Proposition 10,} \]

where \( \Pi^* \) is a subset of \( \mathbb{R}^n \) given by the optimality conditions derived in Propositions 8 and 9. Proposition 10 relates \( \mathcal{X} \) with \( \{ \mathcal{X}(\ell, \rho) \}_{(\ell, \rho) \in L_1 \cup L_2} \). Hence, if \( \hat{\theta} \) exceeds the optimal value of (15), there exist \( \pi' \in \Pi^* \) and \( (\ell', \rho') \in L_1 \cup L_2 \) that separates \( (\hat{\theta}, \hat{x}) \) from \( \mathcal{X}(\ell', \rho') \).

Let \( \left\{ \sigma_i \right\}_{i=1}^{\left| \mathcal{Z} \right|} \) be a permutation of \( \mathcal{Z} \equiv \{ i \in [n] : \xi_i = 1 \} \) such that \( \hat{x}_{\sigma_1} \geq \hat{x}_{\sigma_2} \geq \cdots \geq \hat{x}_{\sigma_{\left| \mathcal{Z} \right|}} \). For the outer minimization problem in formulation (15), we derive the following optimality conditions.

There exists an optimal solution \( \pi \) to formulation (15) satisfying the following:

1. \( \pi_{\sigma_1} \geq \pi_{\sigma_2} \geq \cdots \geq \pi_{\sigma_{\left| \mathcal{Z} \right|}} \);
2. \( \pi \leq 0, \pi_{\sigma_{\beta+1}} \geq -a_1, \) and \( \pi_i = 0 \) for all \( i \notin \mathcal{Z} \).

\[ \square \]

Prop. Hence, it suffices to consider those \( \pi \in \mathbb{R}^n \) with the same ordering as \( \hat{x} \) on \( \mathcal{Z} \). It follows that

\[
\max_{(\theta, x) \in \mathcal{X}} \left\{ \theta + \pi^T x \right\} = \max_{T \in \{0\} \cup [\left| \mathcal{Z} \right|]} \left[ \max_{x^\top \xi = T} \left\{ f(\xi^\top \beta) + \sum_{i \in \mathcal{Z}} \pi_i x_i \right\} \right] = \max_{T \in \{0\} \cup [\left| \mathcal{Z} \right|]} \left\{ f(T - \beta) + \sum_{i = 1}^{\left| \mathcal{Z} \right|} \pi_{\sigma_i} \right\} = \max_{T \in \{0\} \cup [\left| \mathcal{Z} \right|]} \left\{ \min_{(\ell, \rho) \in L_1 \cup L_2} \left\{ \sum_{i = 1}^{\left| \mathcal{Z} \right|} \pi_{\sigma_i} \right\} \right\},
\]

where the first equality uses the definition of \( \mathcal{X} \) and the optimality condition \( \pi_i = 0 \) for all \( i \notin \mathcal{Z} \), the second equality uses the ordering of \( \pi \) on \( \mathcal{Z} \), and the last equality follows from Lemma 3.

Second, we show that there exists a set \( \Pi^* \subseteq \mathbb{R}^n \) of candidate optimal solutions such that a \( \pi \in \Pi^* \) solves problem (15). In addition, we show that the following minimax-type claim holds for all \( \pi \in \Pi^* \):

\[
\max_{T \in \{0\} \cup [\left| \mathcal{Z} \right|]} \left\{ \min_{(\ell, \rho) \in L_1 \cup L_2} \left\{ h_{(\ell, \rho)}(T) + \sum_{i = 1}^{\left| \mathcal{Z} \right|} \pi_{\sigma_i} \right\} \right\} = \min_{(\ell, \rho) \in L_1 \cup L_2} \left\{ \max_{T \in \{0\} \cup [\left| \mathcal{Z} \right|]} \left\{ h_{(\ell, \rho)}(T) + \sum_{i = 1}^{\left| \mathcal{Z} \right|} \pi_{\sigma_i} \right\} \right\},
\]

To this end, we consider the one-dimensional maximization problem (16) and observe that its objective function is nonincreasing on \( \{0\} \cup [\beta] \) since \( f((T - \beta)^+) \) is always
In addition, if \( T \in \{ \beta + 1, \ldots, |\mathcal{Z}| \} \), it has decreasing increments \( a_i - \beta + \pi_{\sigma_i} \) for any \( \beta + 1 \leq i \leq |\mathcal{Z}| \) because \( f \) is concave. Since \( a_1 + \pi_{\sigma_{\beta+1}} \geq 0 \) by Proposition 8, we define \( t \) to be the last occurrence of nonnegative increment, i.e.,

\[
\begin{align*}
    t : = \max \left\{ i \in \{ \beta + 1, \ldots, |\mathcal{Z}| - 1 \} : a_i - \beta + \pi_{\sigma_i} \geq 0, a_{i+1} - \beta + \pi_{\sigma_{i+1}} \leq 0 \right\}
    \quad \text{if } a_{|\mathcal{Z}|} - \beta + \pi_{|\mathcal{Z}|} \leq 0 \\
    \quad \text{if } a_{|\mathcal{Z}|} - \beta + \pi_{|\mathcal{Z}|} > 0.
\end{align*}
\]

Then, with any fixed \( \pi \) satisfying the optimality conditions in Proposition 8, a maximizer \( T^* \) of problem (16) is either 0 or \( t \), yielding the optimal value

\[
\left( \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_i - \beta + \pi_{\sigma_i}) \right)^+.
\]

This allows us to represent problem (15) as follows by partitioning the feasible region of \( \pi \):

\[
(15) = \min_{t \in \mathbb{Z}, \beta + 1 \leq t \leq |\mathcal{Z}|} \min_{\pi \in \Pi_t} \left\{ \left( \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_i - \beta + \pi_{\sigma_i}) \right)^+ - \sum_{i=1}^{\beta} \pi_{\sigma_i} \gamma_{\sigma_i} \right\},
\]

where \( \Pi_t : = \left\{ \pi \in \mathbb{R}^n : 0 \geq \pi_{\sigma_1} \geq \pi_{\sigma_2} \geq \cdots \geq \pi_{|\mathcal{Z}|} \right\}, \pi_i = 0, \forall i \notin \mathcal{Z} \).

\( \forall \beta + 1 \leq t \leq |\mathcal{Z}|. \)

For each \( t \), we further identify the structure of an optimal \( \pi^* \in \Pi_t \). For all \( t \in \{ \beta + 1, \ldots, |\mathcal{Z}| \} \), there exists an optimal solution \( \pi^* \) to the inner minimization problem

\[
\min_{\pi \in \Pi_t} \left\{ \left( \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_i - \beta + \pi_{\sigma_i}) \right)^+ - \sum_{i=1}^{\beta} \pi_{\sigma_i} \gamma_{\sigma_i} \right\}
\]

that has the following structure:

1. If \( b_{t-\beta} - \beta a_{t-\beta} \geq 0 \), then \( \pi_{\sigma_1}^* = \pi_{\sigma_2}^* = \cdots = \pi_{|\mathcal{Z}|}^* = -a_{t-\beta} \).
2. If \( b_{t-\beta} - \beta a_{t-\beta} < 0 \), then \( \pi^* \) takes one of the following two forms:

   (A) Let \( \tilde{r} := -\beta + \frac{b_{t-\beta}}{a_{t-\beta}} \). Then,

   \[
   0 = \pi_{\sigma_1}^* = \cdots = \pi_{\sigma_{\tilde{r}}^*} \geq \pi_{\sigma_{\tilde{r}+1}^*} = -a_{t-\beta} (\tilde{r} - \tilde{r}) > \pi_{\sigma_{\tilde{r}+1}^*} = \cdots = -a_{t-\beta}.
   \]

   (B) If there exists an integer \( t' \in (\beta - \frac{b_{t-\beta}+1}{a_{t-\beta}+1}, \beta + \frac{b_{t-\beta}}{a_{t-\beta}}) \cap \mathbb{Z}_+, \) then

   \[
   0 = \pi_{\sigma_1}^* = \cdots = \pi_{\sigma_{t'}^*} > \pi_{\sigma_{t'+1}^*} = \cdots = \pi_{\sigma_{|\mathcal{Z}|}^*} = -\frac{f(t - \beta)}{(t - t')}. \]

\( \Box \) prop

Accordingly, for all \( t \in \{ \beta + 1, \ldots, |\mathcal{Z}| \} \), if \( b_{t-\beta} - \beta a_{t-\beta} \geq 0 \), then we define

\[
\Pi_t^* : = \left\{ \pi \in \mathbb{R}^n : \pi_{\sigma_1} = \pi_{\sigma_2} = \cdots = \pi_{|\mathcal{Z}|} = -a_{t-\beta}, \pi_i = 0, \forall i \notin \mathcal{Z} \right\}.
\]
and if $b_t - \beta a_t - \beta < 0$, then we define

$$\Pi^*_t := \left\{ \pi \in \mathbb{R}^n : \begin{array}{l}
\pi_{\sigma_1} = \pi_{\sigma_2} = \cdots = \pi_{\sigma_{\tilde{t}}} = 0, \\
\pi_{\sigma_{\tilde{t} + 1}} = -a_{t-\beta}(\tilde{t} - \tilde{t}), \\
\pi_{\sigma_{\tilde{t} + 1} + 1} = \cdots = \pi_{\sigma_{|\tilde{Z}|}} = -a_{t-\beta}, \\
\pi_i = 0, \quad \forall i \notin \tilde{Z}
\end{array} \right\} \cup \left\{ \begin{array}{l}
\pi_1 = \cdots = \pi_{\alpha'} = 0, \\
\pi_{\sigma' + 1} = \cdots = \pi_{\sigma|\tilde{Z}|} = -\frac{f(\beta - \beta)}{(t - t')}, \\
\pi_1 = 0, \quad \forall i \notin \tilde{Z} \\
t' \in \left( \beta - \frac{b_t - b_{t-1} + 1}{a_t - a_{t-1}}, \beta - \frac{b_t - b_{t-1}}{a_t - a_{t-1}} \right) \cap \mathbb{Z}^+
\end{array} \right\}. $$

Note that if the interval $(\beta - b_t - b_{t-1} + 1/a_{t-1}, \beta - b_t/a_{t-1})$ does not contain an integer, then the second component of the above union equals $\emptyset$. Let $\Pi^*_t := \bigcup_t \Pi^*_t$, then Proposition 9 shows that $\Pi_t$ can be replaced by $\Pi^*_t$ without loss of optimality. In other words,

$$\begin{array}{c}
(15) = \min_{\beta + 1 \leq t \leq |\tilde{Z}|} \min_{\pi \in \Pi^*_t} \left\{ \left( \sum_{i=1}^t \pi_{\sigma_i} + \sum_{i=\beta+1}^t (a_{i-\beta} + \pi_{\sigma_i}) \right) + \sum_{i=1}^{|\tilde{Z}|} \pi_{\sigma_i} \tilde{x}_{\sigma_i} \right\} \\
= \min_{\pi \in \Pi^*_t} \max_{T \in \{0\} \cup \{\tilde{Z}\}} \left\{ \left( T - \beta \right)^+ + \sum_{i=1}^T \pi_{\sigma_i} \right\} - \sum_{i=1}^{|\tilde{Z}|} \pi_{\sigma_i} \tilde{x}_{\sigma_i}
\end{array}$$

Furthermore, Proposition 10 shows that, for all $\pi \in \Pi^*_t$,

$$\max_{T \in \{0\} \cup \{\tilde{Z}\}} \left\{ \left( T - \beta \right)^+ + \sum_{i=1}^T \pi_{\sigma_i} \right\} = \min_{(\ell, \rho) \in L_1 \cup L_2} \max_{T \in \{0\} \cup \{\tilde{Z}\}} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^T \pi_{\sigma_i} \right\}.$$ 

It follows that, for all $\pi \in \Pi^*_t$,

$$\max_{T \in \{0\} \cup \{\tilde{Z}\}} \left\{ \left( T - \beta \right)^+ + \sum_{i=1}^T \pi_{\sigma_i} \right\} = \min_{(\ell, \rho) \in L_1 \cup L_2} \max_{T \in \{0\} \cup \{\tilde{Z}\}} \left\{ h(\ell, \rho)(x^T \xi) + \pi^T x \right\} - \pi^T \hat{x}$$

$$= \min_{(\ell, \rho) \in L_1 \cup L_2} \max_{(\theta, x) \in X(\ell, \rho)} \left\{ \theta + \pi^T x \right\} - \pi^T \hat{x},$$

where the first equality is because $\hat{x}$ satisfies the optimality conditions in Proposition 8. Thus, the optimal value of problem (15) satisfies

$$\hat{\theta} > \min_{\pi \in \Pi^*_t} \min_{(\ell, \rho) \in L_1 \cup L_2} \max_{(\theta, x) \in X(\ell, \rho)} \left\{ \theta + \pi^T x \right\} - \pi^T \hat{x}. $$
That is, there exist \((\ell, \rho)\) such that
\[
\widehat{\theta} + \widehat{x}^\top \pi' > \theta + x^\top \pi', \quad \forall (\theta, x) \in \mathcal{X}(\ell, \rho).
\]
Therefore, \((1, \pi')\) separates \((\widehat{\theta}, \widehat{x})\) from the set \(\mathcal{X}(\ell, \rho)\), implying \((\widehat{\theta}, \widehat{x}) \notin \text{conv} (\mathcal{X}(\ell, \rho))\). This finishes the proof. \(\square\)

When applying the BD algorithm to solve \((\text{DRC})\), we add a Benders feasibility cut \((13)\) if the incumbent solution \((\widehat{x}, \widehat{\gamma}, \widehat{\pi})\) is infeasible to \((\text{DSP}_j)\). In the meantime, we add more inequalities by separating \((\widehat{x}, \widehat{\gamma}, \widehat{\pi})\) from \(\text{conv}(\mathcal{X}_{i^*}^j)\) with
\[
\mathcal{X}_{i^*}^j := \left\{(x, \gamma, z_j) \in \{0,1\}^n \times \mathbb{R}^2 : z_j + \gamma \leq \left(\left(x^\top \hat{\xi}_{i^*}^j - v_{i^*} + 1\right)^{1/p}\right)^{1/p}\right\},
\]
where \(i^*\) is identified in Proposition 7. Since \((\cdot)^{1/p}\) is strictly concave and increasing, \(\mathcal{X}_{i^*}^j\) is isomorphic to \(\mathcal{X}\) with \(z_j + \gamma\) and \(v_{i^*} - 1\) playing the roles of \(\theta\) and \(\beta\), respectively. As a consequence, separating from \(\text{conv}(\mathcal{X}_{i^*}^j)\) is efficient. In addition, with regard to any \((\ell, \rho)\), a valid inequality takes the following form:
\[
-z_j - \gamma \geq F^j(\emptyset) + \sum_{k=1}^n \left[ F^j(T_k) - F^j(T_{k-1}) \right] x_{\sigma_k}, \quad (17)
\]
where
\[
F^j(x) := \frac{\ell^{1/p}}{\ell + \rho} \min \left\{-x^\top \hat{\xi}_{i^*}^j - v_{i^*} - \rho + 1, 0\right\},
\]
and \(\sigma\) is a permutation of \([n]\) such that \(\widehat{x}_{\sigma_1} \geq \widehat{x}_{\sigma_2} \geq \cdots \geq \widehat{x}_{\sigma_n}\). In our implementation, we perform this separation via the lazy callback.

### 3.2 Proofs of Propositions 8, 9, and 10

**Proposition 8** There exists an optimal solution \(\pi\) to formulation \((15)\) satisfying the following:

1. \(\pi_{\sigma_1} \geq \pi_{\sigma_2} \geq \cdots \geq \pi_{\sigma_{|Z|}}\);
2. \(\pi \leq 0, \pi_{\sigma_{|Z|} + 1} \geq -a_1, \text{ and } \pi_i = 0 \text{ for all } i \notin Z\).

**Proof** First, for any \(\pi \in \mathbb{R}^n\) let \(\{\sigma_i : i \in Z\}\), possibly different from \(\{\sigma_i : i \in Z\}\), be a permutation of \(Z \equiv \{i \in [n] : \xi_i = 1\}\) such that \(\pi_{\sigma_1} \geq \cdots \geq \pi_{\sigma_{|Z|}}\). Then, we recast formulation \((15)\) as
\[
\min_{\pi \in \mathbb{R}^n} \left\{ \max \left\{ \theta + \pi^\top x : (\theta, x) \in \mathcal{X} \right\} - \pi^\top \hat{x} \right\}.
\]
\[= \min_{\pi \in \mathbb{R}^n} \left\{ \max_{x \in [0, 1]^n} \left\{ f \left( \left( x^\top \xi - \beta \right)^+ \right) + \sum_{i \in Z} \pi_{\sigma_i}^t x_{\sigma_i} + \sum_{i \notin Z} \pi_i x_i \right\} - \pi^\top \hat{x} \right\} \]

\[= \min_{\pi \in \mathbb{R}^n} \left\{ \max_{T \in \{0\} \cup [\mathcal{Z}] | x^\top \xi = T} \left\{ f \left( \left( x^\top \xi - \beta \right)^+ \right) + \sum_{i \in Z} \pi_{\sigma_i}^t x_{\sigma_i} + \sum_{i \notin Z} \pi_i x_i \right\} - \pi^\top \hat{x} \right\} \]

\[= \min_{\pi \in \mathbb{R}^n} \left\{ \max_{T \in \{0\} \cup [\mathcal{Z}]} \left\{ f \left( (T - \beta^+) \right) + \sum_{i=1}^T \pi_{\sigma_i}^t + \sum_{i \notin Z} (\pi_i t^+) - \pi^\top \hat{x} \right\} \right\}. \quad (18) \]

We prove the following optimality conditions:

**Condition 1.** Suppose that \( \pi \in \mathbb{R}^n \) is such that \( \pi_{\sigma_j} < \pi_{\sigma_k} \) for some \( j, k \in [\mathcal{Z}] \) and \( j < k \). We show swapping \( \pi_{\sigma_j} \) and \( \pi_{\sigma_k} \) yields a lower objective value of \((18)\). Specifically, define \( \pi' \) as

\[
\pi'_{\sigma_i} := \begin{cases} 
\pi_{\sigma_k} & \text{if } i = j, \\
\pi_{\sigma_j} & \text{if } i = k, \\
\pi_{\sigma_i} & \text{o.w.}
\end{cases} \quad \forall i \in \mathcal{Z},
\]

\[
\pi'_i := \pi_i \quad \forall i \notin \mathcal{Z}.
\]

Since \( \pi, \pi' \) differ only in ordering, the optimal value of the inner maximization problem of \((18)\) remains the same after swapping. Then, the difference between their objective values equals

\[-\hat{x}^\top \pi - \left( -\hat{x}^\top \pi' \right) = -(\pi_{\sigma_j} \hat{x}_{\sigma_j} + \pi_{\sigma_k} \hat{x}_{\sigma_k}) + (\pi'_{\sigma_j} \hat{x}_{\sigma_j} + \pi'_{\sigma_k} \hat{x}_{\sigma_k})
\]

\[= (\hat{x}_{\sigma_j} - \hat{x}_{\sigma_k})(\pi_{\sigma_k} - \pi_{\sigma_j}) \geq 0. \]

Therefore, there exists an optimal \( \pi \) that shares the same ordering with \( \hat{x} \) on \( \mathcal{Z} \).

**Condition 2.** To prove that the entries of an optimal \( \pi \) on \([n] \setminus \mathcal{Z} \) are all zero, we show that replacing any nonzero entry \( \pi_j, j \notin \mathcal{Z} \), with 0 lowers the objective value. Specifically, define \( \pi' \) to be the \( \pi \) with \( \pi_j \) replaced by zero. This yields a difference in objective value

\[
\left( (\pi_j^+) - \pi_j \hat{x}_j \right) - 0 = \pi_j (1 - \hat{x}_j) \mathbb{I} \left\{ \pi_j > 0 \right\} - \pi_j \hat{x}_j \mathbb{I} \left\{ \pi_j \leq 0 \right\} \geq 0.
\]

To prove \( \pi \leq 0 \) without loss of optimality, it suffices to show that \( \pi \) is nonpositive on \( \mathcal{Z} \). Suppose that \( \pi \) has strictly positive entries on \( \mathcal{Z} \) and let \( \pi_{\sigma_j} \) be the last such entry, i.e., either \( \pi_{\sigma_j} > 0, \pi_{\sigma_{j+1}} \leq 0 \) or \( \pi_{\sigma_{|\mathcal{Z}|}} > 0 \) with \( j = |\mathcal{Z}| \). We show that replacing \( \pi_{\sigma_j} \) with 0 lowers the objective value. Specifically, let \( \pi' \) be the \( \pi \) with \( \pi_{\sigma_j} \) replaced by 0. Then, \( \pi \) and \( \pi' \) share the same ordering. In addition, both \( \pi_{\sigma_i} \) and \( \pi'_{\sigma_i} \) are nonnegative for all \( i \in [j] \). As a result, since \( f \) is increasing, a maximizer \( T^* \) of the inner maximization problem of \((18)\) must be at least \( j \) with respect to
both \( \pi \) and \( \pi' \). It follows that \( \pi \) and \( \pi' \) yield the same maximizer \( T^* \). Therefore, the difference in objective value equals

\[
\pi_{\sigma_j}(1 - \hat{x}_{\sigma_j}) - 0 \geq 0,
\]

implying that the objective value evaluated at \( \pi' \) is smaller. To prove \( \pi_{\sigma_{\beta+1}} \geq -a_1 \), we note that if \( \pi_{\sigma_{\beta+1}} + a_1 < 0 \) then \( \pi_{\sigma_{\beta+1}} + a_1 \leq 0 \) for all \( i \in \{|Z| - \beta| \}, \) because \( \pi_{\sigma_{\beta+i}} \) and \( a_i \) are decreasing in \( i \). Since \( \pi \leq 0 \), an optimal \( T^* \) to the inner maximization problem of (18) is 0. It follows that increasing all \( \pi_{\sigma_i}, 1 \leq i \leq \beta + 1 \) to \( -a_1 \) does not increase the objective value of (18). Hence, we have \( \pi_{\sigma_{\beta+1}} + a_1 \geq 0 \) without loss of optimality.

\[ \square \]

**Proposition 9**  
For all \( t \in \{ \beta + 1, \ldots, |Z| \} \), there exists an optimal solution \( \pi^* \) to the inner minimization problem

\[
\min_{\pi \in \Pi_t} \left\{ \left( \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_i - \beta + \pi_{\sigma_i}) \right) + \sum_{i=1}^{|Z|} \pi_{\sigma_i} \hat{\pi}_{\sigma_i} \right\}
\]

that has the following structure:

1. If \( b_{t-\beta} - \beta a_{t-\beta} \geq 0 \), then \( \pi^* = \pi_{\sigma_1} = \cdots = \pi_{|Z|} = -a_t - \beta \).
2. If \( b_{t-\beta} - \beta a_{t-\beta} < 0 \), then \( \pi^* \) takes one of the following two forms:

   (A) Let \( \tilde{t} := \beta - \frac{b_{t-\beta}}{a_{t-\beta}} \). Then,

   \[
   0 = \pi^*_{\sigma_1} = \cdots = \pi^*_{\sigma_{\tilde{t}}} \geq \pi^*_{\sigma_{\tilde{t}+1}} = -a_t - \beta (|Z| - \tilde{t}) > \pi^*_{\sigma_{|Z|+1}} \\
   = \cdots = \pi^*_{\sigma_{|Z|}} = -a_t - \beta.
   \]

   (B) If there exists an integer \( t' \in \left( \beta - \frac{b_{t-\beta+1}}{a_{t-\beta+1}}, \beta - \frac{b_{t-\beta}}{a_{t-\beta}} \right) \cap \mathbb{Z}_+, \) then

   \[
   0 = \pi^*_{\sigma_1} = \cdots = \pi^*_{\sigma_{t'}} = \pi^*_{\sigma_{t'+1}} = \cdots = \pi^*_{\sigma_{|Z|}} = -\frac{f(t - \beta)}{t - t'}.
   \]

**Proof**  
Fix \( t \in \{ \beta + 1, \ldots, |Z| \} \) and we discuss the following two cases.

First, if \( b_{t-\beta} - \beta a_{t-\beta} \geq 0 \) then

\[
\sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_i - \beta + \pi_{\sigma_i}) \geq t \pi_{\sigma_t} + \sum_{i=\beta+1}^{t} a_i - \beta \\
\geq -ta_{t-\beta} + a_{t-\beta}(t - \beta) + b_{t-\beta} \\
= b_{t-\beta} - \beta a_{t-\beta} \geq 0,
\]

where the first inequality is because \( \pi_{\sigma_i} \geq \pi_{\sigma_t} \) for all \( i \in [t] \) and the second inequality is because \( \pi_{\sigma_t} \geq -a_{t-\beta} \) by the definition of \( t \) and \( \sum_{i=\beta+1}^{t} a_i - \beta = \sum_{i=\beta+1}^{t} (f(i -
\( \beta - f(i - \beta - 1) = f(t - \beta) = a_{t - \beta}(t - \beta) + b_{t - \beta}. \) Then, we recast (19) as
\[
(19) = \min_{\pi \in \Pi_t} \left\{ \sum_{i=\beta+1}^{t} a_{i - \beta} + \sum_{i=1}^{t} \pi_{\sigma_i} (1 - \hat{x}_{\sigma_i}) - \sum_{i=t+1}^{\mid Z \mid} \pi_{\sigma_i} \hat{x}_{\sigma_i} \right\}.
\]
Since \( \hat{x} \in [0, 1]^n \), decreasing \( \{\pi_{\sigma_i} : i \in [t]\} \) and increasing \( \{\pi_{\sigma_i} : t + 1 \leq i \leq \mid Z \mid\} \) improve the objective value. It follows that an optimal solution \( \pi^* \) takes the following form:
\[
\pi_{\sigma_1}^* = \cdots = \pi_{\sigma_t}^* = -a_{t - \beta} = \pi_{\sigma_{t+1}}^* = \cdots = \pi_{\sigma_{\mid Z \mid}}^*.
\]
Second, suppose that \( b_{t - \beta} - \beta a_{t - \beta} < 0 \). For \( \pi \in \Pi_t \) we define
\[
S(\pi) := \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_{i - \beta} + \pi_{\sigma_i}).
\]
We show \( S(\pi) = 0 \) at optimality of (19). To see that, we discuss the following two cases.

1. If \( S(\pi) > 0 \), then there exists an \( i \in Z, \beta + 1 \leq i \leq \mid Z \mid \) such that \( a_{i - \beta} + \pi_{\sigma_i} > 0 \). Let \( t \) be its last occurrence:
\[
t := \max \left\{ \beta + 1 \leq i \leq \mid Z \mid : a_{i - \beta} + \pi_{\sigma_i} > 0 \right\}.
\]
Note that \( \underline{t} \leq t \) by definition and \( a_{i - \beta} + \pi_{\sigma_i} = 0 \) for all \( \underline{t} < i \leq t \). Then,
\[
\max \left\{ S(\pi), 0 \right\} - \sum_{i=1}^{\mid Z \mid} \pi_{\sigma_i} \hat{x}_{\sigma_i} = \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_{i - \beta} + \pi_{\sigma_i}) - \sum_{i=t+1}^{\mid Z \mid} \pi_{\sigma_i} \hat{x}_{\sigma_i}
\]
\[
= \sum_{i=\beta+1}^{t} a_{i - \beta} + \sum_{i=1}^{t-1} \pi_{\sigma_i} (1 - \hat{x}_{\sigma_i}) + \pi_{\sigma_t} (1 - \hat{x}_{\sigma_t})
\]
\[
- \sum_{i=t+1}^{\mid Z \mid} \pi_{\sigma_i} \hat{x}_{\sigma_i}.
\]
Therefore, as long as \( S(\pi) > 0 \), decreasing \( \pi_{\sigma_i} \) lowers the objective value of (19).

2. If \( S(\pi) < 0 \), then there exists an \( i \in [\beta] \) such that \( \pi_{\sigma_i} < 0 \). Let \( \tilde{t} \) be its first occurrence:
\[
\tilde{t} := \min \left\{ i \in [\beta] : \pi_{\sigma_i} < 0 \right\}.
\]
Then, by noting that $\pi_{\sigma_i} = 0$ for all $i \in \{\bar{t} - 1\}$, we have

$$\max \left\{ S(\pi), 0 \right\} - \sum_{i=1}^{\lvert Z \rvert} \pi_{\sigma_i} \hat{x}_{\sigma_i} = -\pi_{\sigma_1} \hat{x}_{\sigma_1} - \sum_{i=\bar{t}+1}^{\lvert Z \rvert} \pi_{\sigma_i} \hat{x}_{\sigma_i}.$$  

Therefore, as long as $S(\pi) < 0$, increasing $\pi_{\sigma_\bar{t}}$ lowers the objective value of (19).

In view of the optimality condition $S(\pi) = 0$, we reformulate (19) as the following linear program:

\begin{equation}
(19) = \min_{\pi} - \sum_{i=1}^{\lvert Z \rvert} \pi_{\sigma_i} \hat{x}_{\sigma_i},
\end{equation}

\begin{align}
\text{s.t.} \quad & 0 \geq \pi_{\sigma_1}, \\
& \pi_{\sigma_i} \geq \pi_{\sigma_{i+1}}, \forall i \in [\lvert Z \rvert - 1], \\
& \pi_{\sigma_i} + a_{t-\beta} \geq 0, \\
& \pi_{\sigma_{i+1}} + a_{t-\beta+1} \leq 0, \\
& \sum_{i=1}^{t} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} a_i = 0,
\end{align}

where constraint (21) states $S(\pi) = 0$ and the remaining constraints come from the definition of $\Pi_t$. Note that constraint (20) does not exist in the case $t = \lvert Z \rvert$. Although the remaining proof focuses on $t < \lvert Z \rvert$, it can be directly adapted to the case $t = \lvert Z \rvert$.

In what follows, we analyze the basic feasible solutions of the above linear program to identify the structure of an optimal solution $\pi^*$. There are $\lvert Z \rvert$ decision variables, 1 equality constraint, and $\lvert Z \rvert + 2$ inequality constraints. By definition of $t$, the constraint $\pi_{\sigma_{t+1}} + a_{t-\beta+1} \leq 0$ must hold strictly. In addition, variables $\pi_{\sigma_{t+2}}, \ldots, \pi_{\sigma_{\lvert Z \rvert}}$ are only constrained by their ordering and that $\pi_{\sigma_{t+2}} \leq \pi_{\sigma_{t+1}}$, and their coefficients in the objective function are all negative. As a consequence, $\pi_{\sigma_{t+1}} = \pi_{\sigma_{t+2}} = \cdots = \pi_{\sigma_{\lvert Z \rvert}}$ at optimality. Hence, we only need to consider those basic feasible solutions, where either (A) $\pi_{\sigma_t} + a_{t-\beta} = 0$ and two among the constraints $0 \geq \pi_{\sigma_1} \geq \cdots \geq \pi_{\sigma_{t+1}}$ are satisfied strictly (or weakly in case of degeneracy), or (B) $\pi_{\sigma_t} + a_{t-\beta} > 0$ and one of the constraints $0 \geq \pi_{\sigma_1} \geq \cdots \geq \pi_{\sigma_{t+1}}$ is satisfied strictly (or weakly in case of degeneracy).

We discuss cases (A) and (B) to finish the proof.

(A) Since $\pi_{\sigma_t} + a_{t-\beta} = 0$, there exist $0 \leq t_1 < t_2 \leq t$ such that

$$0 = \pi_{\sigma_1} = \cdots = \pi_{\sigma_{t_1}} \geq \pi_{\sigma_{t_1+1}} = \cdots = \pi_{\sigma_{t_2}} \geq \pi_{\sigma_{t_2+1}} = \cdots = \pi_{\sigma_{\lvert Z \rvert}} = -a_{t-\beta}.$$  

By constraint (21), we have:

$$(t_2 - t_1)\pi_{\sigma_{t_2}} = - \sum_{i=t_2+1}^{t} (a_{t-\beta}) - \sum_{i=\beta+1}^{t} a_i - \beta a_{t-\beta} + b_{t-\beta},$$
where the last equality is because \( \sum_{i=\beta+1}^{t} a_{i-\beta} = \sum_{i=\beta+1}^{t} (f(i - \beta) - f(i - \beta - 1)) = f(t - \beta) = a_{t-\beta}(t - \beta) + b_{t-\beta} \). Let \( \tilde{r} := \beta - \frac{b_{t-\beta}}{a_{t-\beta}} \). Then, a valid choice of \( \pi_{\sigma_2} \) should satisfy \(-a_{t-\beta} \leq \pi_{\sigma_2} \leq 0\), which, together with the above equality, implies 

\[
-(t_2 - t_1)a_{t-\beta} \leq (\tilde{r} - t_2)a_{t-\beta} \leq 0 \iff t_1 \leq \tilde{r} \leq t_2.
\]

We assume that \( \tilde{r} \) is fractional and will return to the case of integer-valued \( \tilde{r} \) in the end. We show \( t_1 = [\tilde{r}] \) and \( t_2 = [\tilde{r}] + 1 \) at optimality. Indeed, if \( t_1 \leq [\tilde{r}] - 1 \) then increasing \( t_1 \) by 1 improves the objective value. Specifically, let \( \pi' \) be the \( \pi \) with \( t_1 \) replaced by \( t_1 + 1 \). Then, \( \pi'_{\sigma_2} \leq \pi_{\sigma_2} \) because \( \sum_{i=1}^{t} \pi_{\sigma_i} \) equals a constant due to constraint (21). In addition,

\[
(t_2 - t_1)\pi_{\sigma_2} = \sum_{i=1}^{t} \pi_{\sigma_i} - \sum_{i=t_2+1}^{t} \pi_{\sigma_i} \quad \text{by construction of } \pi
\]

\[
= - \sum_{i=\beta+1}^{t} a_{i-\beta} - \sum_{i=t_2+1}^{t} \pi_{\sigma_i} \quad \text{by constraint (21)}
\]

\[
= \sum_{i=1}^{t} \pi'_{\sigma_i} - \sum_{i=t_2+1}^{t} \pi'_{\sigma_i} \quad \text{by construction of } \pi'
\]

\[
= (t_2 - t_1 - 1)\pi'_{\sigma_2}.
\]

Hence, the change in objective value equals

\[
-\pi_{\sigma_2} \sum_{i=t_1+1}^{t_2} \hat{x}_{\sigma_i} + \pi'_{\sigma_2} \sum_{i=t_1+2}^{t_2} \hat{x}_{\sigma_i} = -\pi_{\sigma_2} \hat{x}_{\sigma_1} + (\pi_{\sigma_2} - \pi'_{\sigma_2}) \sum_{i=t_1+2}^{t_2} \hat{x}_{\sigma_i}
\]

\[
\geq -\pi_{\sigma_2} \hat{x}_{\sigma_1} + (\pi_{\sigma_2} - \pi'_{\sigma_2}) \sum_{i=t_1+1}^{t_2} \hat{x}_{\sigma_i}
\]

\[
= -\hat{x}_{\sigma_1} \sum_{i=1}^{t_2} \pi_{\sigma_2} + (\pi_{\sigma_2} - \pi'_{\sigma_2}) (t_2 - t_1 - 1)
\]

where the inequality is because \( \hat{x}_{\sigma_i} \) decreases in \( i \). Therefore, \( t_1 = [\tilde{r}] \) at optimality. Likewise, if \( t_2 \geq [\tilde{r}] + 1 \) then decreasing \( t_2 \) by 1 improves the objective value. Specifically, we let \( \pi'' \) be the \( \pi \) with \( t_2 \) replaced by \( t_2 - 1 \). Then, \( \pi''_{\sigma_2} \geq \pi_{\sigma_2} \) because \( \sum_{i=1}^{t} \pi_{\sigma_i} \) equals a constant due to constraint (21). In addition,

\[
(t_2 - t_1)\pi_{\sigma_2} = \sum_{i=1}^{t} \pi_{\sigma_i} - \sum_{i=t_2+1}^{t} \pi_{\sigma_i} \quad \text{by construction of } \pi
\]

\[
= - \sum_{i=\beta+1}^{t} a_{i-\beta} - \sum_{i=t_2+1}^{t} \pi_{\sigma_i} \quad \text{by constraint (21)}
\]
\[
\begin{align*}
&= \sum_{i=1}^{t} \pi''_{\sigma_i} - \sum_{i=t+1}^{t} \pi''_{\sigma_i} \\
&= (t_2 - t_1 - 1)\pi''_{\sigma_{t_2}} + (-a_{t-\beta}).
\end{align*}
\]

Hence, the change in objective value equals

\[
\begin{align*}
- \pi_{\sigma_{t_2}} \sum_{i=t_1+1}^{t_2} \hat{x}_{\sigma_i} + a_{t-\beta} \sum_{i=t_2+1}^{|Z|} \hat{x}_{\sigma_i} + \pi''_{\sigma_{t_2}} \sum_{i=t_1+1}^{t_2-1} \hat{x}_{\sigma_i} - a_{t-\beta} \sum_{i=t_2}^{|Z|} \hat{x}_{\sigma_i} \\
= -(\pi_{\sigma_{t_2}} - \pi''_{\sigma_{t_2}}) \sum_{i=t_1+1}^{t_2-1} \hat{x}_{\sigma_i} - (\pi_{\sigma_{t_2}} + a_{t-\beta})\hat{x}_{\sigma_{t_2}} \\
\geq - \left[ (\pi_{\sigma_{t_2}} - \pi''_{\sigma_{t_2}})(t_2 - t_1 - 1) + \pi_{\sigma_{t_2}} + a_{t-\beta} \right] \hat{x}_{\sigma_{t_2}} = 0.
\end{align*}
\]

Therefore, \( t_2 = \lceil \tilde{t} \rceil \) at optimality. It follows from constraint (21) that at optimality

\[
\pi_{\sigma_{t_2}}^* = -a_{t-\beta}(\lceil \tilde{t} \rceil - \tilde{t}).
\]

Finally, suppose that \( \tilde{t} \) is an integer. Then, following a similar analysis as above, we obtain that either (i) \( t_1 = \tilde{t} - 1, t_2 = \tilde{t}, \pi_{\sigma_{t}}^* = 0 \) or (ii) \( t_1 = \tilde{t}, t_2 = \tilde{t} + 1, \pi_{\sigma_{t_2}}^* = -a_{t-\beta} \). Both cases coincide with the claim of this proposition.

(B) Since \( \pi_{\sigma_{t}} + a_{t-\beta} > 0 \), there exists a \( 0 \leq t' \leq t \) such that

\[
0 = \pi_{\sigma_{t}} = \cdots = \pi_{\sigma_{t'}} \geq \pi_{\sigma_{t'+1}} = \cdots = \pi_{\sigma_{|Z|}}.
\]

If \( t' = t \) then \( \sum_{i=1}^{t} \pi_{\sigma_{i}} = 0 \), which violates constraint (21). Hence, \( t' \leq t - 1 \) and a valid choice of \( \pi_{\sigma_{i}} \) satisfies

\[
-a_{t-\beta+1} > \pi_{\sigma_{t}} = -\frac{1}{(t-t')} \sum_{i=\beta+1}^{t} a_{i-\beta} > -a_{t-\beta}.
\]

Since \( \sum_{i=\beta+1}^{t} a_{i-\beta} = f(t-\beta) = (t-\beta)a_{t-\beta} + b_{t-\beta} \) and \( \sum_{i=\beta+1}^{t} a_{i-\beta} = f(t-\beta) = (t-\beta)a_{t-\beta+1} + b_{t-\beta+1} \), these inequalities are equivalent to

\[
\beta - \frac{b_{t-\beta+1}}{a_{t-\beta+1}} < t' < \beta - \frac{b_{t-\beta}}{a_{t-\beta}}.
\]

Accordingly, at optimality \( \pi_{\sigma_{t'+1}}^* \) equals

\[
\frac{\sum_{i=\beta+1}^{t} a_{i-\beta}}{(t-t')} \equiv -\frac{f(t-\beta)}{(t-t')}.
\]
Proposition 10 For all $\pi \in \Pi^*$, it holds that

$$\max_{T \in [0] \cup [\|Z\|]} \left\{ f\left(\left(T - \beta\right)^+\right) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} = \min_{(\ell, \rho) \in L_1 \cup L_2} \max_{T \in [0] \cup [\|Z\|]} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\}. \quad (22)$$

Proof Since $f\left(\left(T - \beta\right)^+\right) = \min_{(\ell, \rho) \in L_1 \cup L_2} h(\ell, \rho)(T)$ by Lemma 3, we switch the order of maximization and minimization to obtain

$$\max_{T \in [0] \cup [\|Z\|]} \left\{ f\left(\left(T - \beta\right)^+\right) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} = \max_{T \in [0] \cup [\|Z\|]} \left\{ \min_{(\ell, \rho) \in L_1 \cup L_2} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \right\} \leq \min_{(\ell, \rho) \in L_1 \cup L_2} \left\{ \max_{T \in [0] \cup [\|Z\|]} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \right\}. \quad (22)$$

It remains to prove the other direction of (22). $\pi \in \Pi^*$ implies that $\pi \in \Pi^*_t$ for a $t \in \mathbb{Z}$, $\beta + 1 \leq t \leq \|Z\|$. We discuss the following cases to finish the proof.

1. If $b_t - \beta a_t - \beta \geq 0$, then by definition of $\Pi^*_t$

$$\pi_{\sigma_1} = \pi_{\sigma_2} = \cdots = \pi_{\sigma_{\|Z\|}} = -a_t - \beta.$$ 

It follows that

$$\min_{(\ell, \rho) \in L_1 \cup L_2} \max_{T \in [0] \cup [\|Z\|]} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \leq \max_{T \in [0] \cup [\|Z\|]} \left\{ (\psi_{t - \beta} (T - \beta))^+ + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \leq \max_{T \in [0] \cup [\|Z\|]} \left\{ f\left(\left(T - \beta\right)^+\right) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\},$$

where the inequality is because we select $(\ell, \rho) \in L_1$ with $\ell = t - \beta$ and $\rho = b_t - \beta / a_t - \beta$, the first equality is because $b_t - \beta a_t - \beta \geq 0$, the second equality is because $b_t - \beta a_t - \beta = b_{t - \beta} + (t - \beta) a_{t - \beta} - \beta a_t - \beta = \sum_{i=\beta+1}^{t} a_{t - \beta} - \beta a_t - \beta$, and the last equality is because $\pi \in \Pi^*_t \subseteq \Pi_t$.

2. If $b_t - \beta a_t - \beta < 0$ then we discuss the following two cases.

(A) If $0 = \pi_{\sigma_1} = \cdots = \pi_{\sigma_{\tilde{t}}} \geq \pi_{\sigma_{\tilde{t}+1}} = -a_{t - \beta} (\tilde{t} - \beta) > \pi_{\sigma_{\tilde{t}+1}} = \cdots = \pi_{\sigma_{\|Z\|}} = -a_t - \beta$ for $\tilde{t} \equiv \beta - \frac{b_t - \beta}{a_t - \beta}$, then

$$\min_{(\ell, \rho) \in L_1 \cup L_2} \max_{T \in [0] \cup [\|Z\|]} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\}$$
\[
\begin{align*}
&\leq \max_{T \in \{0\} \cup [|Z|]} \left\{ \left( a_{t-\beta} (T - \beta) + b_{t-\beta} \right)^+ + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \\
&= 0 \\
&= \max \left\{ \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_{i-\beta} + \pi_{\sigma_i}), 0 \right\} \\
&= \max_{T \in \{0\} \cup [|Z|]} \left\{ f \left( (T - \beta)^+ \right) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\},
\end{align*}
\]

where the inequality is because we select \((\ell, \rho) \in L_1\) with \(\ell = t - \beta\) and \(\rho = b_{t-\beta}/a_{t-\beta}\), the first equality is because

\[
\left( a_{t-\beta} (T - \beta) + b_{t-\beta} \right)^+ = \left( T a_{t-\beta} + b_{t-\beta} - \beta a_{t-\beta} \right)^+ \\
= (T - \tilde{\tau})^+ a_{t-\beta} = -\sum_{i=1}^{T} \pi_{\sigma_i}
\]

for all \(T \in \{0\} \cup [|Z|]\), the second equality is because \(\sum_{i=\beta+1}^{t} a_{i-\beta} = (t - \tilde{\tau}) a_{t-\beta}\), and the last equality is because \(\pi \in \Pi_t' \subseteq \Pi_t\).

(B) If \(0 = \pi_{\sigma_1} = \cdots = \pi_{\sigma_{t'}} > \pi_{\sigma_{t'+1}} = \cdots \pi_{\sigma_{|Z|}} = -\frac{f(t - \beta)}{t - t'}\) for a \(t' \in \left( \beta - \frac{b_{t-\beta+1}}{a_{t-\beta+1}}, \beta - \frac{b_{t-\beta}}{a_{t-\beta}} \right) \cap \mathbb{Z}_+\), then

\[
\begin{align*}
&\min_{(\ell, \rho) \in L_1 \cup L_2} \max_{T \in \{0\} \cup [|Z|]} \left\{ h(\ell, \rho)(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \\
&\leq \max_{T \in \{0\} \cup [|Z|]} \left\{ h_{(t, \beta-\beta-t')}(T) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \\
&= \max_{T \in \{0\} \cup [|Z|]} \left\{ \left( \frac{f(t - \beta)}{t - t'} (T - t') \right)^+ + \sum_{i=1}^{T} \pi_{\sigma_i} \right\} \\
&= 0
\end{align*}
\]
\[
= \max \left\{ \sum_{i=1}^{\beta} \pi_{\sigma_i} + \sum_{i=\beta+1}^{t} (a_i - \beta + \pi_{\sigma_i}), 0 \right\}
\]
\[
= \max_{T \in \{0\} \cup [|Z|]} \left\{ f \left( (T - \beta)^+ \right) + \sum_{i=1}^{T} \pi_{\sigma_i} \right\},
\]
where the inequality is because we select \((\ell, \rho) \in L_2\) with \(\ell = t - \beta\) and \(\rho = \beta - t'\), the second equality is due to the definition of \(\pi\), the third equality is because \(\sum_{i=\beta+1}^{t} a_i - \beta = f(t - \beta)\), and the last equality is because \(\pi \in \Pi_1^* \subseteq \Pi_I\).

\[\square\]

### 3.3 Valid inequalities from cross scenarios

Using the fact that \(x^T \xi\) is an integer, we study valid inequalities arising from multiple scenarios \(j \in [N]\) and multiple coverings \(i \in [I]\). To this end, we focus on a set \(S\) consisting of \(J\) “base” inequalities and other domain restrictions:

\[
S := \left\{ (x, \gamma, z) \in \{0, 1\}^n \times \mathbb{R}^{1+J} : \begin{array}{l}
- z_j - \gamma \geq d^j \left( x^T \bar{\xi}^j \right) + d_0^j, \quad \forall j \in [J] \\
- z_j \leq 0, \quad \forall j \in [J]
\end{array} \right\},
\]

(23)

where \(d_0^j \in \mathbb{R}\) and \(d^j < 0\) represent constant parameters. The base inequalities can come from two sources. First, the Benders feasibility cut (13) implies that

\[
- z_j - \gamma \geq \phi^j(\emptyset) + \sum_{k=1}^{n} \left[ \phi^j(T_k) - \phi^j(T_{k-1}) \right] x_{\sigma_k}
\]

\[
= \phi^j(\emptyset) + \sum_{k: \phi^j(T_{k-1}) < 0} \Delta \mu_k x_{\sigma_k} + \sum_{k: \phi^j(T_k) = 0} \phi^j(T_k) x_{\sigma_k} + \sum_{k: \phi^j(T_{k-1}) = 0, \phi^j(T_k) < 0} 0 \cdot x_{\sigma_k}
\]

\[
\geq \phi^j(\emptyset) + \left(1^T \hat{\mu}_1 \right) x^T \bar{\xi}^j,
\]

where \(\bar{\xi}^j\) denotes a binary vector, in which \(\bar{\xi}^j_{\sigma_k} = 1\) if and only if \(\phi^j(T_k) < 0\), and the last inequality is because \(\phi^j(T_k) \geq 1^T \hat{\mu}_1\) if \(\phi^j(T_{k-1}) = 0\) and \(\phi^j(T_k) < 0\). As a result, (13) generates a base inequality with \(d_0^j = \phi^j(\emptyset)\) and \(d^j = 1^T \hat{\mu}_1 < 0\).

Second, the single-scenario valid inequality (17) implies
\[-z_j - \gamma \geq \Phi^j(\emptyset) + \sum_{k=1}^{n} \left[ \Phi^j(T_k) - \Phi^j(T_{k-1}) \right] x_{\sigma_k} \]
\[= \Phi^j(\emptyset) + \sum_{k: \Phi^j(T_{k-1}) < 0} \left( -\frac{\ell^{1/p}}{\ell + \rho} \right) x_{\sigma_k} + \sum_{k: \Phi^j(T_k) = 0, \Phi^j(T_k) < 0} \Phi^j(T_k) x_{\sigma_k} \]
\[+ \sum_{k: \Phi^j(T_k) = 0} 0 \cdot x_{\sigma_k} \geq \Phi^j(\emptyset) + \left( -\frac{\ell^{1/p}}{\ell + \rho} \right) x^\top \bar{\xi}^j, \]

where $\bar{\xi}^j$ denotes a binary vector, in which $\bar{\xi}_{\sigma_k}^j = 1$ if and only if $\Phi^j(T_k) < 0$. As a result, (17) generates a base inequality with $d^j_0 = \Phi^j(\emptyset)$ and $d^j = -\ell^{1/p} / (\ell + \rho) < 0$. We remark that the number of base inequalities, $J$, need not to be the same as the number of scenarios, $N$. This is because a single scenario can generate multiple base inequalities with regard to different coverings.

We employ the mixing scheme [21] to derive valid inequalities for $S$. This scheme produces a new inequality by combining existing ones. Specifically, consider the following mixing set $\mathcal{MX}$ consisting of $J$ base inequalities:

\[\mathcal{MX} := \left\{ x \in \{0, 1\}^n : f^j(x) + B g^j(x) \geq u^j, \quad \forall j \in [J] \right\}, \]

where $B \in \mathbb{R}_+$, $u^j \in \mathbb{R}$, and $g^j(x) \in \mathbb{Z}$. Then, [21] derived the following mixed inequality.

**Theorem 3** (Theorem 2 in [21]) The following mixed inequality is valid for $\mathcal{MX}$:

\[\max_{j \in [J]} \{ f^j(x) \} \geq \sum_{j=1}^{J} (v^j - v^{j-1})(\tau^j - g^j(x)), \]

where $v^0 := 0$, $\tau^j := \lceil u^j / B \rceil$, $v^j := u^j - B(\tau^j - 1)$ for all $j \in [J]$, and $v^j$’s are sorted so that $v^j \geq v^{j-1}$ for all $j \in [J]$. \(\square\)

To apply the mixing scheme, we divide both sides of the base inequalities in (23) by $-d^j$ to obtain

\[\frac{z_j}{d^j} + x^\top \bar{\xi}^j \geq -\frac{d^j_0 + \gamma}{d^j}, \quad \forall j \in [J].\]

This would obey the structure of $\mathcal{MX}$ if $-(d^j_0 + \gamma)/d^j$ (counterpart of $u^j$) were a constant, or equivalently, if variable $\gamma$ were a constant. Although this is not true in general, we observe that, without loss of optimality, $\gamma$ can take only a finite set of values. It is worth mentioning that a similar result appeared in [24, Lemma 1].
Proposition 11 There exists an optimal solution \((x^*, \gamma^*, z^*)\) to \((DRC)\), using the reformulation \(Z\) defined in (6), such that

\[
\gamma^* = -\text{VaR}_{1-\epsilon} \left[ -g(x^*, \hat{\xi}) \right]
\]

and

\[
z_j^* = \min \left\{ 0, \ g(x^*, \hat{\xi}^j) - \gamma^* \right\}, \ \forall j \in [N],
\]

where \(g(x, \xi) = \min_{i \in [I]} \left( (x^\top \xi_i - v_i + 1)^{+} \right)^{1/p} \).

**Proof** By Proposition 1 and the definition (5) of \(\text{CVaR}\), we recast the set \(Z\) as

\[
Z = \left\{ x \in \{0, 1\}^n : \frac{\delta}{\epsilon} + \min_{\gamma'} \left\{ \gamma' + \frac{1}{N\epsilon} \sum_{j \in [N]} \left( -g(x, \hat{\xi}^j) - \gamma' \right)^{+} \right\} \leq 0 \right\}.
\]

Then, by the definition (5) of \(\text{CVaR}\), \(\gamma'\) equals \(\text{VaR}_{1-\epsilon} \left[ -g(x^*, \hat{\xi}) \right]\) at optimality. The conclusion follows. \(\Box\)

By Proposition 11, we can restrict the value of \(\gamma\) to be in a discrete set \(\{ r \in \mathbb{R} : r^p \in \{0\} \cup \{n\} \}\) because \((x^*)^\top \hat{\xi}_i\) and \(v_i\) are integers for all \(i \in [I]\). In addition, since \(g(x, \hat{\xi}) \leq g(1, \hat{\xi})\) almost surely for all \(x \in \{0, 1\}^n\) and \((DRC)\) requires that \(\text{CVaR}_{1-\epsilon} \left[ -g(x, \hat{\xi}) \right] \leq -\delta/\epsilon\), we can further restrict the value of \(\gamma\) to be that

\[
\gamma \in \left\{ r : \left[ \frac{\delta}{\epsilon} \right] \left( \frac{1}{p} \right)^{1/p} \leq r \leq -\text{VaR}_{1-\epsilon}(-g(1, \hat{\xi})), \ r^p \in \{n\} \right\}.
\]

For ease of exposition, we denote this set as \(\{ r_k : k \in \Gamma \}\), where \(\Gamma\) is the set of indices, and let \(r_1\) be the smallest element. It follows that \(\gamma \geq r_1\) at optimality and the base inequalities in (23) imply

\[
\frac{z_j}{d_j} + x^\top \hat{\xi}^j \geq -\frac{d_0^j}{d_j} + r_1, \ \forall j \in [J].
\]

Then, applying the mixing scheme in Theorem 3 produces a mixed inequality for \(S\):

\[
\max_{j \in [J]} \left\{ \frac{z_j}{d_j} \right\} \geq \sum_{j=1}^{J} (v_1^j - v_1^{j-1}) (\tau_1^j - x^\top \hat{\xi}^j),
\]

where \(\tau_1^j : = \left\lceil \frac{r_1 + d_0^j}{d_j} \right\rceil, v_1^{0} : = 0,\) and \(v_1^j : = -\frac{r_1 + d_0^j}{d_j} - (\tau_1^j - 1)\) such that \(v_1^j \geq v_1^{j-1}\) for all \(j \in [J]\). Nevertheless, the mixed inequality (24) is potentially weak because it simply relaxes \(\gamma\) to be \(r_1\) and ignores the possibility of \(\gamma\) taking other values.
in \( \{r_k : k \in \Gamma\} \). To strengthen it, we notice that \( \gamma \), which equals one and only one \( r_k \) for \( k \in \Gamma \), admits a binary encoding:

\[
\gamma = \sum_{k \in \Gamma} r_k \gamma_k, \quad \sum_{k \in \Gamma} \gamma_k = 1,
\]

where \( \{\gamma_k \in \{0, 1\} : k \in \Gamma\} \) are (auxiliary) binary variables that determine the value of \( \gamma \). Accordingly, we recast set \( S \) as

\[
S_B := \left\{ (x, \gamma, z) \in \{0, 1\}^{n+|\Gamma|} \times \mathbb{R}_+^J : \begin{array}{l}
- z_j - \sum_{k \in \Gamma} r_k \gamma_k \geq d_j^1 (x^\top \hat{\xi}_j^1) + d_0^j, \quad \forall j \in [J] \\
\sum_{k \in \Gamma} \gamma_k = 1
\end{array} \right\}.
\]

As a consequence, the mixed inequality (24) is valid for a restriction of \( S_B \) when fixing \( \gamma_1 = 1 \), i.e., \( S_B \cap \{(x, \gamma, z) : \gamma_1 = 1\} \). Hence, we can view (24) as a seed inequality and strengthen it by lifting variables \( \gamma_2, \ldots, \gamma_{|\Gamma|} \) back (see, e.g., [19, 53] for more on the lifting scheme).

**Theorem 4** The following lifted inequality is valid for \( S_B \):

\[
\max_{j \in [J]} \left\{ \frac{z_j}{d_1^j} \right\} \geq \sum_{j=1}^{J} (v_1^j - v_1^{j-1}) (\tau_1^j - x^\top \hat{\xi}_j^1) + \sum_{k \in \Gamma} \alpha_k \gamma_k, \quad \text{(25)}
\]

where \( \alpha_k := \left( \min_{j \in [J]} \min \left\{ v_k^j - v_1^j, 0 \right\} + \sum_{j=1}^{J} (v_1^j - v_1^{j-1}) (\tau_k^j - \tau_1^j) \right) \)

\[
\tau_k^j := \left[ \begin{array}{c}
\tau_1^j - \frac{r_k + d_0^j}{d_1^j} \nu_k^j, \quad \nu_k^j = 0, \text{ and } v_k^j := - \frac{r_k + d_0^j}{d_1^j} (\tau_k^j - 1) \quad \text{such that } v_1^j \geq v_1^{j-1}
\end{array} \right]
\]

for all \( j \in [J] \).

**Proof** For all \( k \in \Gamma \), let \( S_k := S_B \cap \{(x, \gamma, z) : \gamma = r_k\} \). We simultaneously lift variables \( \gamma_2, \ldots, \gamma_{|\Gamma|} \) back into inequality (24). To this end, we search for lifting coefficients \( \alpha_2, \ldots, \alpha_{|\Gamma|} \) such that

\[
\max_{j \in [J]} \left\{ \frac{z_j}{d_1^j} \right\} \geq \sum_{j=1}^{J} (v_1^j - v_1^{j-1}) (\tau_1^j - x^\top \hat{\xi}_j^1) + \sum_{k \in \Gamma} \alpha_k \gamma_k, \quad \forall (x, \gamma, z) \in S_B,
\]

where \( \alpha_1 := 0 \). Since \( \sum_{k \in \Gamma} \gamma_k = 1 \), fixing \( \gamma_k = 1 \) automatically forces all other \( \gamma_k \)'s to be zero. Hence, the lifted inequality is valid if and only if

\[
\max_{j \in [J]} \left\{ \frac{z_j}{d_1^j} \right\} \geq \sum_{j=1}^{J} (v_1^j - v_1^{j-1}) (\tau_1^j - x^\top \hat{\xi}_j^1) + \alpha_k \gamma_k, \quad \forall (x, \gamma, z) \in S_k, \forall k \in \Gamma.
\]
If \( k = 1 \), this validity requirement is valid because \( \alpha_1 = 0 \) and the inequality reduces to (24). If \( k \in \Gamma \setminus \{1\} \), the validity requirement is valid if and only if

\[
\alpha_k \leq \min_{x,z} \max_{j \in [J]} \left\{ \frac{z_j}{d_j} \right\} - \sum_{j=1}^{J} (v^j_1 - v^{j-1}_1)(\tau^j_1 - x^\top \bar{\xi}^j), \quad \text{(Lifting)}
\]

\[
\text{s.t.} \quad \frac{z_j}{d_j} \geq -\frac{r_k + d^j_0}{d_j} - x^\top \bar{\xi}^j, \quad \forall j \in [J],
\]

\[
x \in \{0, 1\}^n, \quad z \in \mathbb{R}_+^J.
\]

For fixed \( x \in \{0, 1\}^n, z_j \leq 0 \) and \( d_j < 0 \) imply

\[
\frac{z_j}{d_j} \geq \left( -\frac{r_k + d^j_0}{d_j} - x^\top \bar{\xi}^j \right)^+ = \left( v^j_k + \tau^j_k - x^\top \bar{\xi}^j - 1 \right)^+ = v^j_k \left( \tau^j_k - x^\top \bar{\xi}^j \right)^+ + (1 - v^j_k) \left( \tau^j_k - x^\top \bar{\xi}^j - 1 \right)^+
\]

for all \( j \in [J] \). Let \( \Upsilon_k := \max \left\{ \tau^j_k - x^\top \bar{\xi}^j : j \in [J] \right\} \), \( \mathcal{J}_k \) be the corresponding set of maximizers, and \( i \in \mathcal{J}_k \) be a maximizer with the largest \( v^j_k \), i.e.,

\[
\mathcal{J}_k := \left\{ j \in [J]: \tau^j_k - x^\top \bar{\xi}^j = \Upsilon_k \right\}
\]

and \( i \in \text{argmax} \left\{ v^j_k : j \in \mathcal{J}_k \right\} \).

Then, it is easy to verify that

\[
\max_{j \in [J]} \left\{ \frac{z_j}{d_j} \right\} = v^i_k \left( \tau^i_k - x^\top \bar{\xi}^i \right)^+ + (1 - v^i_k) \left( \tau^i_k - x^\top \bar{\xi}^i - 1 \right)^+
\]

at optimality of the (Lifting) problem. Specifically, if \( \Upsilon_k \leq 0 \), then \( \tau^j_k - x^\top \bar{\xi}^j \leq 0 \) for all \( j \in [J] \) and so \( \max_{j \in [J]} \left\{ \frac{z_j}{d_j} \right\} = 0 \). This implies a lower bound for the objective value of (Lifting):

\[
\max_{j \in [J]} \left\{ \frac{z_j}{d_j} \right\} - \sum_{j=1}^{J} (v^j_1 - v^{j-1}_1)(\tau^j_1 - x^\top \bar{\xi}^j) = \sum_{j=1}^{J} (v^j_1 - v^{j-1}_1)(x^\top \bar{\xi}^j - \tau^j_1)
\]

\[
\geq \sum_{j=1}^{J} (v^j_1 - v^{j-1}_1)(\tau^j_k - \tau^j_1).
\]
If $\Upsilon_k \geq 1$ then $\max_{j \in [J]} \left\{ \frac{z_j}{d^j} \right\} = (\tau^i_k - x^\top \tilde{\xi}^i - 1) + v^j_k$, and the lower bound for the objective value of (Lifting) becomes

$$\max_{j \in [J]} \left\{ \frac{z_j}{d^j} \right\} - \sum_{j = 1}^J (v^j_k - v^{j-1}_k)(\tau^j_k - x^\top \tilde{\xi}^j) = (\tau^i_k - x^\top \tilde{\xi}^i - 1) + v^j_k - \sum_{j = 1}^J (v^j_k - v^{j-1}_k)(\tau^j_k - x^\top \tilde{\xi}^j) - \sum_{j \in J_k} (v^j_k - v^{j-1}_k)(\tau^j_k - x^\top \tilde{\xi}^j - 1) - \sum_{j = 1}^J (v^j_k - v^{j-1}_k)(\tau^j_k - \tau^j_k)$$

$$= (\tau^i_k - x^\top \tilde{\xi}^i - 1) + v^j_k - \sum_{j \in J_k} (v^j_k - v^{j-1}_k) + \sum_{j = 1}^J (v^j_k - v^{j-1}_k)(\tau^j_k - \tau^j_k)$$

$$\geq v^j_k - v^j_k + \sum_{j = 1}^J (v^j_k - v^{j-1}_k)(\tau^j_k - \tau^j_k),$$

where the first inequality is because $\tau^j_k - x^\top \tilde{\xi}^j \leq \tau^j_k - x^\top \tilde{\xi}^j - 1$ if $j \notin J_k$ and the second inequality is because $\tau^j_k - x^\top \tilde{\xi}^j = \Upsilon_k \geq 1$, $v^j_k \leq 1$, and $\sum_{j \in J_k} (v^j_k - v^{j-1}_k) \leq \sum_{j = 1}^J (v^j_k - v^{j-1}_k) = v^j_k$. The two cases of the value of $\Upsilon_k$ together suggests the following valid choice of $\alpha_k$:

$$\alpha_k = \min_{j \in [J]} \min \left\{ v^j_k - v^j_k, 0 \right\} + \sum_{j = 1}^J (v^j_k - v^{j-1}_k)(\tau^j_k - \tau^j_k).$$

But $\alpha_k = 0$ is also a valid choice because, in that case, the lifted inequality reduces to (24), which is valid for $S_B \supseteq S_k$. Therefore, we can take the maximum of these two valid choices. The conclusion follows. $\square$

We remark that, thanks to constraint $\sum_{k \in \Gamma} \gamma_k = 1$, the lifted inequality in Theorem 4 is sequence-independent (cf. [19]). In our implementation, we generate the base inequalities in (23) from the single-scenario valid inequality (17). We add the lifted inequalities via the lazy callback.

### 4 Extension to knapsack chance constraint

In this section, we extend the reformulation and valid inequalities for (DRC) to knapsack chance constraints with random, binary-valued coefficients. Specifically, consider the following ambiguous knapsack chance constraint

$$\inf_{P \in \mathcal{P}} \mathbb{P} \left\{ x^\top \tilde{\xi}^i \leq v_i, \forall i \in [I] \right\} \geq 1 - \epsilon,$$  

(KCC)
where the Wasserstein ambiguity set \( \mathcal{P} \) is defined in (3), \( \tilde{\xi} \in \Xi = \{0, 1\}^{I \times n} \), and \( v_i \in [n - 1] \) for all \( i \in [I] \). In this case, we can follow the proof of Proposition 1 to show that

\[
\min_{\xi: x^T \tilde{\xi} \geq v_i + 1} \|\tilde{\xi} - \tilde{\xi}^i\|_p = \left( \left(v_i + 1 - x^T \tilde{\xi}^i\right)^+ \right)^{1/p} + \chi \left(x^T 1 \geq v_i + 1\right),
\]

where \( \chi(\cdot) \) denotes a characteristic function, which takes value 0 if \( x^T 1 \geq v_i + 1 \) and takes value \(+\infty\) otherwise. Hence, if we assume \( v_1 \leq v_2 \leq \cdots \leq v_I \) w.l.o.g. then (KCC) is equivalent to \( \bigcup_{k=0}^{I} Z_k \), where \( Z_0 := \{x \in \{0, 1\}^n: x^T 1 \leq v_1\} \) and, for all \( k \in [I] \), \( Z_k \) resembles the reformulation \( Z \) for (DRC) in Proposition 1. That is,

\[ Z_k := \{x \in \{0, 1\}^n: v_k + 1 \leq x^T 1 \leq v_{k+1}\} \cap \left\{ x: \begin{array}{l}
\exists \gamma \in \mathbb{R}_+, z \in \mathbb{R}_N^n : \\
\delta - \gamma \epsilon \leq \frac{1}{N} \sum_{j \in [N]} z_j, \\
z_j + \gamma \leq \min_{i \in [k]} \left( (v_i + 1 - x^T \tilde{\xi}^j)^+ \right)^{1/p}, \forall \epsilon \in [N] \end{array} \right\}, \]

where \( v_{I+1} := n \). As a consequence, we can readily adapt the valid inequalities in Sect. 3 to solve optimization problems involving (KCC). For example, under the complement \( y := 1 - x \), the final constraints in the definition of \( Z_k \) can be rewritten as

\[
z_j + \gamma \leq \left( (y^T \tilde{\xi}^j - 1^T \tilde{\xi}^j + v_i + 1)^+ \right)^{1/p},
\]

which is precisely the setting studied in Sect. 3 if \( (1^T \tilde{\xi}^j - v_i - 1) \geq 1 \). Hence, the valid inequalities (17) and (25) can be adapted to compute (KCC) more effectively. On the other hand, when \( \beta' := (1^T \tilde{\xi}^j - v_i - 1) \leq 0 \), it can be shown that the convex hull of the mixed-integer set

\[
\mathcal{X}' := \{(\theta, x) \in \mathbb{R} \times \{0, 1\}^n: \theta \leq f(x^T \xi - \beta')\}
\]

coincides with the continuous relaxation of its piecewise linear reformulation, i.e.,

\[
\text{conv}(\mathcal{X}') = \{(\theta, x) \in \mathbb{R} \times [0, 1]^n: \theta \leq a_{\ell}(x^T \xi - \beta') + b_{\ell}, \forall \ell \in \{1 - \beta', \ldots, n - \beta'\}\}.
\]

Finally, we notice that a special case of (KCC) is an ambiguous set packing chance constraint, in which \( v_i = 1 \) for all \( i \in [I] \). Hence, our reformulation and valid inequalities also apply to set packing chance constraints with random set membership.
5 Numerical experiments

We demonstrate the out-of-sample performance of (DRC) and the effectiveness of our two-stage reformulation and valid inequalities in the following four numerical experiments:

1. Section 5.1 reports the out-of-sample performance of (DRC) with various radii $\delta$ and data sizes $N$ when the entries of $\tilde{\xi}$ are i.i.d.;
2. Section 5.2 examines correlated $\tilde{\xi}$ entries, using a context of medical facility location under natural disasters;
3. Section 5.3 evaluates the value of incorporating the support information $\Xi = \{0, 1\}^I \times n$ into (DRC) as opposed to relaxing it to be $\Xi = \mathbb{R}^I \times n$;
4. Section 5.4 demonstrates the strength of the valid inequalities derived in Sect. 3.

All experiments in this section are conducted using the Python API of Gurobi 9.0.2 on a single core of an Intel Xeon Gold 6154 Processor provided by the UM Great Lakes cluster.

5.1 Independent uncertain parameters

In this experiment, we generate testing data of $\tilde{\xi}$ from the Bernoulli distribution with i.i.d. entries, and the probability of success $q_i := \mathbb{P}_{\text{true}}\{\tilde{\xi}_{ij} = 1\}$ is uniformly sampled from the interval $[0.4, 0.8]$. In the out-of-sample evaluation, this specific choice of $\mathbb{P}_{\text{true}}$ facilitates the feasibility checking of a given solution $x \in \{0, 1\}^n$ because

$$\mathbb{P}_{\text{true}}\left\{x^T\tilde{\xi}_i \geq v_i, \forall i \in [I]\right\} = \prod_{i=1}^{I} \mathbb{P}_{\text{true}}\left\{x^T\tilde{\xi}_i \geq v_i\right\} = \prod_{i=1}^{I} \sum_{s=v_i}^{n} \mathbb{P}_{\text{true}}\left\{\sum_{k:x_k=1} \tilde{\xi}_{ik} = s\right\} = \prod_{i=1}^{I} \sum_{s=v_i}^{n} \left(1^T x\right)_s q_i s (1 - q_i) (1^T x)_s$$

admits a closed-form expression. In addition, we generate the training historical data $\left\{\hat{\xi}_j\right\}_{j=1}^{N}$ by adding Gaussian noise to $\tilde{\xi}$:

$$\hat{\xi}_{ik} := \text{round}(\tilde{\xi}_{ik} + 0.25 \cdot \tilde{\varepsilon}), \quad \forall i \in [I], k \in [n],$$

where $\tilde{\varepsilon}$ is a standard Gaussian random variable and round$(\cdot)$ rounds real numbers to $\{0, 1\}$. For a given solution $x \in \{0, 1\}^n$, checking whether it is feasible for chance constraint (2) simplifies to computing its out-of-sample performance using (26) and comparing with the risk level $1 - \epsilon$. We compare the out-of-sample performance of
Fig. 2 Out-of-sample performance of (DRC) and (SAA) as a function of $\delta$

(DRC) with the sample average approximation formulation:

$$\begin{align*}
\min & \quad c^T x, \\
\text{s.t.} & \quad \frac{1}{N} \sum_{j=1}^{N} z_j \geq 1 - \epsilon, \\
& \quad x^T \tilde{\xi}_i^j - v_i \geq v_m(z_j - 1), \quad \forall i \in [I], \ j \in [N], \\
& \quad x \in \{0, 1\}^n, \ z \in \{0, 1\}^N.
\end{align*}$$

Random test instances with $n = 30, I = 10$ are generated in the aforementioned way. For each instance, $N \in \{100, 200, 300, 400, 500\}$ training data are generated, the RHS $v_i, i \in [I]$ are all 1, coefficients $c$ in the objective function are uniformly sampled from integers between 1 and 100, the radius $\delta$ of the Wasserstein ball enumerates $\{0.05, 0.07, 0.09, 0.11, 0.13, 0.15, 0.17, 0.19, 0.21, 0.23, 0.25, 0.27\}$, and the risk level $\epsilon = 0.1$. For each parameter setting, we compute the average of out-of-sample performance, as well as its 90% confidence interval, for the returned solution over 5 different random instances and study how the radius $\delta$ and the sample size $N$ affect the out-of-sample performance of (DRC) and (SAA).

5.1.1 Impact of the radius $\delta$ on out-of-sample performance

Figure 2 displays the impact of $\delta$ on out-of-sample performance of the (DRC) and (SAA) optimal solutions. We notice that the out-of-sample performance of (DRC) increases as $\delta$ becomes larger. This makes sense because the larger the radius $\delta$ is, the more distributions the Wasserstein ball $\mathcal{P}$ includes, and accordingly the more likely (DRC) generates a solution feasible with respect to $\mathbb{P}_{\text{true}}$. The (SAA) model, however, yields a low out-of-sample performance. Specifically, Fig. 2a shows that, with a proper choice of $\delta$, (DRC) is able to provide a reliable solution satisfying the chance constraint even when the training data is very limited. In contrast, (SAA) does not return a feasible solution even if the size of training data has a tenfold increase.
5.1.2 Impact of the data size $N$ on out-of-sample performance

Figure 3 displays how $N$ affects out-of-sample performance of the (DRC) and (SAA) optimal solutions. We observe that the out-of-sample performance of both models shows an increasing trend as the data size becomes larger. Specifically, (DRC) can return a high-quality solution with a proper choice of radius $\delta$ depending on much less data. In contrast, (SAA) does not produce a feasible solution even when 500 training data are provided. In addition, (DRC) is significantly more stable than (SAA), as (DRC) generates a much narrower 90% confidence interval (the shaded area in Fig. 3) than (SAA). This demonstrates that (DRC) is capable of generating reliable and stable solutions even with limited training data.

5.2 Dependent uncertain parameters

In this experiment, we test (DRC) with dependent uncertain parameters based on a medical facility location problem with natural disasters. Specifically, we consider an undirected network with $I$ nodes and a set $\mathcal{E} \subseteq [I] \times [I]$ of roads connecting these nodes. As a special case of (DRC) with $n = I$, this problem establishes medical facilities among these $I$ nodes in order to cover the medical needs from the same set of nodes under coverage uncertainty. The binary random variables $\tilde{\xi}_{ij}, i, j \in [I]$ indicate whether a facility in node $j$ can cover the need from node $i$. To generate correlated $\tilde{\xi}_{ij}$, we consider natural disasters and let $\tilde{H}_j$ denote a binary random variable such that $\tilde{H}_j = 1$ indicates a disaster in node $j$. In addition, we partition the nodes into subsets $\{S_k\}_{k=1}^K$ according to geographic proximity and designate that if a node in a subset is affected by the disaster then all nodes in the same subset are affected, i.e., $\tilde{H}_j = 1$ for all $j \in S_k$ whenever $\tilde{H}_i = 1$ for an $i \in S_k$. Furthermore, we consider a random travel time $\tilde{T}_e$ on each road $e \in \mathcal{E}$ and assume that $\tilde{T}_e$ depends on the two nodes road $e$ connects: if neither of the nodes is affected by the disaster then $\tilde{T}_e$ takes a constant value $T_e^0$; and if either node is affected then $\tilde{T}_e$ follows a Gaussian distribution
Fig. 4 Network of a medical facility location case study

\[ N(4 \cdot T_0^0, 1) \]. Accordingly, we define \( \tilde{\xi}_{ij} \) as

\[
\tilde{\xi}_{ij} := (1 - \tilde{H}_j) \times \prod_{e \in \text{Path}(j, i)} \left( \sum_{e \in \text{Path}(j, i)} \tilde{T}_e \leq T_{\text{max}} \right),
\]

where \( \text{Path}(j, i) \) represents the shortest path from node \( j \) to node \( i \) after the disaster and \( T_{\text{max}} \) denotes the longest time allowed to provide medical service to a disaster node. That is, \( \tilde{\xi}_{ij} = 1 \) only when node \( j \) is not affected by the disaster and the shortest path from \( j \) to \( i \) is shorter than \( T_{\text{max}} \).

Our case study is based on a network of southern US with \( n = I = 30 \) (cf. [40, 44, 51]; see Fig. 4). The partition \( \{ S_k \}_{k=1}^K \) of these nodes includes subsets \( \{ 11, 12, 13 \} \), \( \{ 14, 15 \} \), \( \{ 19, 20 \} \), \( \{ 22, 23 \} \), \( \{ 24, 25, 26 \} \), \( \{ 27, 28 \} \), and all the remaining subsets are singletons.

First, to generate \( \tilde{H}_j \), we assign all nodes into three categories: \( C_1 := \{ 2, 5, 11, 13, 14, 15, 29, 30, 21, 22 \} \), \( C_2 := \{ 1, 3, 10, 12, 25, 26, 27, 28, 23 \} \), and \( C_3 := [I] \setminus (C_1 \cup C_2) \). For each node \( j \), we independently draw a Gaussian random number centered around the category index of \( j \) with a standard deviation of 1. For example, we sample from \( N(2, 1) \) for node 1 and from \( N(1, 1) \) for node 2. Then, we set \( \tilde{H}_j = 1 \) if node \( j \) generates the smallest Gaussian random number, and we broadcast \( \tilde{H}_j = 1 \) in the subset \( j \) belongs with. For example, if \( j = 1 \) then \( \tilde{H}_1 = 1 \) and \( \tilde{H}_i = 0 \) for all \( i \in [I] \setminus \{1\} \); and if \( j = 11 \) then \( \tilde{H}_{11} = \tilde{H}_{12} = \tilde{H}_{13} = 1 \) and \( \tilde{H}_i = 0 \) for all \( i \in [I] \setminus \{11, 12, 13\} \).
Second, we generate the random travel times $\tilde{T}_e$ for all $e \in \mathcal{E}$ based on $\{\tilde{H}_j\}_{j=1}^I$ as described above and the random coverage $\tilde{\xi}$ using (27) and $T_{\text{max}} := 12$. For each node $i \in [I]$, the coverage level $v_i$ is set to be 4 minus the category index of $i$. For example, $v_1 = 2$ and $v_2 = 3$.

Third, to evaluate the out-of-sample performance of (DRC), we generate $N \in \{20, 50, 100, 200, 300, 400, 500\}$ samples as training data, the radius $\delta$ of the Wasserstein ball are taken from $\{0.01, 0.03, 0.05, 0.075, 0.1, 0.125, 0.15, 0.175, 0.2\}$, and the risk level $\epsilon = 0.1$. For each parameter setting, we generate 5 different random instances, evaluate the out-of-sample performance using 100,000 random coverage samples generated in the same way as the training data, and report the average performance along with the 90% confidence interval in Figs. 5 and 6. Figure 5 depicts the out-of-sample performance as a function of the data size $N$. From this figure, we observe that when $\delta$ is small, the performance of (DRC) and (SAA) are similar. As $\delta$ increases, e.g., when $\delta = 0.15$, we observe a large improvement in the out-of-sample performance of (DRC) as it requires feasible solutions to satisfy chance constraints with respect to more distributions. In Fig. 6a, even with a small training data size, e.g., $N = 20$, (DRC) can return a reliable solution when $\delta$ is properly chosen, say $\delta = 0.15$. In contrast, (SAA) did not produce a feasible solution in the sense of out-of-sample performance, even when $N = 500$ as displayed in Fig. 6b.
5.3 Value of incorporating support information

A key difference between this paper and Xie [55] lies in the support $\Xi$ of uncertainty parameters. We derive a reformulation $Z$ of (DRC) in (6) by assuming a binary-valued support $\Xi = \{0, 1\}^{I \times n}$. If we relax it to be $\Xi = \mathbb{R}^{I \times n}$, a different reformulation of the feasible region produced by chance constraint (4) follows from Theorem 1 and Proposition 1 of [55]:

$$Z_0 = \left\{ x \in \{0, 1\}^n : \exists \gamma \in \mathbb{R}_+, \nu \in \mathbb{R}_+, z \in \mathbb{R}_+^N : \delta v - \gamma \epsilon \leq \frac{1}{N} \sum_{j \in [N]} z_j, \right.$$  
$$\left. \|x\|_{\frac{p}{(p-1)}} \leq \nu, \right.$$  
$$\left. z_j + \gamma \leq \left( x^\top \hat{\xi}_j^i - v_i \right)^+, \forall i \in [I], j \in [N] \right\}. \quad (28)$$

Intuitively, as $Z$ relies on a more restricted support, it leads to a larger feasible region and accordingly (DRC) becomes less conservative. The following proposition formalizes this intuition.

**Proposition 12** For fixed $\delta$ and $\epsilon$, it holds that $Z_0 \subseteq Z$.

**Proof** Since $v_i \geq 1$ for all $i \in [I], x = 0$ is not feasible to (DRC). Then, we recast $Z_0$ in (28) as follows by projecting out variable $\nu$:

$$Z_0 = \left\{ x \in \{0, 1\}^n : \exists \gamma \in \mathbb{R}_+, z \in \mathbb{R}_+^N : \delta - \gamma \epsilon \leq \frac{1}{N} \sum_{j \in [N]} z_j, \right.$$  
$$\left. \|x\|_{\frac{p}{(p-1)}} \leq \nu, \right.$$  
$$\left. z_j + \gamma \leq \left( x^\top \hat{\xi}_j^i - v_i \right)^+, \forall i \in [I], j \in [N] \right\},$$

where $\|\cdot\|_* := \|\cdot\|_{\frac{p}{(p-1)}}$ represents the dual norm of $\|\cdot\|_p$. Hence, it suffices to show

$$\frac{\left( x^\top \hat{\xi} - v \right)^+}{\|x\|_*} \leq \left( \left( x^\top \hat{\xi} - v + 1 \right)^+ \right)^{1/p}$$

for any $0 \neq x \in \{0, 1\}^n, \xi \in \{0, 1\}^n$, and $v \in [n]$. To this end, we discuss the following two cases.

1. When $x^\top \hat{\xi} \leq v$, we have

$$\frac{\left( x^\top \hat{\xi} - v \right)^+}{\|x\|_*} = 0 \leq \left( \left( x^\top \hat{\xi} - v + 1 \right)^+ \right)^{1/p}.$$
2. When $x^\top \xi \geq v + 1$, let $T$ be the index set of $x \wedge \xi$, i.e., $T := \{ i \in [n] : \xi_i = x_i = 1 \}$, then $|T| \geq v + 1$. Let $T'$ be $T$ with arbitrary $v$ elements removed and $\xi'$ be the indicating vector of $T'$. That is, $\xi' \in \{0, 1\}^n$ and, for all $i \in [n]$, $\xi'_i = 1$ if and only if $i \in T'$. Then, $x^\top \xi' = 1^\top \xi' = x^\top \xi - v$ and $\|\xi'\|_p = (1^\top \xi')^{1/p}$. We finish the proof by noticing that:

\[
(x^\top \xi - v) = x^\top \xi' \\
\leq \|\xi'\|_p \|x\|_* \\
= (1^\top \xi')^{1/p} \|x\|_* \\
= (x^\top \xi - v)^{1/p} \|x\|_* \leq (x^\top \xi - v + 1)^{1/p} \|x\|_*,
\]

where the first inequality is by Cauchy–Schwarz inequality and the last inequality is because $x^\top \xi - v \geq 0$.

\[\square\]

Next, we demonstrate Proposition 12 numerically using a set of random test instances, generated in the same way as in Sect. 5.1, with various $n, I, N, \delta$ and the same risk level $\epsilon = 0.1$. For each problem size, we solve our two-stage reformulation in Proposition 6 (denoted as $\text{bin}$) and Xie [55]'s reformulation $\min_{x \in Z_0} \{ c^\top x \}$ (denoted as $\text{cont}$). We solve five random instances for each problem size and report the average optimal value, out-of-sample constraint satisfaction probability (OOS), and CPU time in Table 1. An “INF” is placed in the column “Optval” if all five random instances are infeasible, and “N/A” represents “not available”. From Table 1, we observe that for the same problem size, Xie [55]'s reformulation always returns a higher optimal value than ours. For example, for $n = 30, I = 25, N = 100, \delta = 0.01$, $\text{bin}$ returns an optimal value of 173.2, while $\text{cont}$'s optimal value 305.8 is almost twice as large. Additionally, for $n = 20, I = 25, N = 100, \delta = 0.1$, $\text{cont}$ is infeasible in all five instances while $\text{bin}$ always remains feasible and its average optimal value 304.8 is even lower than $\text{cont}$'s with a smaller radius $\delta = 0.01$. Besides, from Table 1 we notice that the out-of-sample constraint satisfaction probability of $\text{cont}$ is nearly 1.00 across all instances, while that of $\text{bin}$ is closer to the target probability (i.e., 0.90). These computational results align with the fact $Z_0 \subseteq Z$ and demonstrate that incorporating support information can make (DRC) significantly less conservative.

### 5.4 Strength of the two-stage reformulation and valid inequalities

We demonstrate the strength of our two-stage reformulation, and the single- and cross-scenario valid inequalities. Random test instances with $n \in \{60, 80\}, \; I \in \{70, 90\}$ are generated, each of which is paired with $N \in \{50, 100, 200\}, \; \delta \in \{0.05, 0.1, 0.2, 0.3\}$, and $\epsilon \in \{0.05, 0.1\}$. The numerical results are reported in Tables 2–4, where we use $2$-Stg, +single, and +cross to denote using the two-stage reformulation in
Table 1 Comparison between optimizing over $Z_0$ and $Z$

| $n$ | $I$ | $N$ | $\delta$ | \textbf{bin} & \textbf{cont} |
|----|----|----|---|------|------|
|    |    |    |    | Optval | Time | OOS | Optval | Time | OOS |
| 20 | 10 | 50 | 0.01 | 136.0 | 1.31 | 0.82 | 270.4 | 8.13 | 0.98 |
|    |    |    | 0.05 | 159.4 | 0.94 | 0.88 | 557.4 | 7.02 | 0.98 |
|    |    |    | 0.1  | 203.4 | 0.62 | 0.90 | INF   | 0.55 | N/A  |
| 100| 0.01| 115.4| 2.38 | 0.88 | 239.8 | 15.58 | 0.98 |
|    | 0.05| 128.8| 1.39 | 0.91 | 552.4 | 18.00 | 0.99 |
|    | 0.1 | 210.6| 1.16 | 0.94 | 1020.0| 1.66  | 1.00 |
| 25 | 50 | 0.01| 200.2| 2.32 | 0.84 | 365.2 | 15.99 | 0.98 |
|    | 0.05| 216.2| 1.50 | 0.87 | 615.7 | 9.96  | 0.99 |
|    | 0.1 | 258.0| 0.99 | 0.89 | INF   | 0.67  | N/A  |
| 100| 0.01| 213.8| 5.50 | 0.85 | 409.2 | 49.20 | 0.98 |
|    | 0.05| 233.8| 2.98 | 0.90 | 841.7 | 8.17  | 0.99 |
|    | 0.1 | 304.8| 1.83 | 0.95 | INF   | 1.16  | N/A  |
| 30 | 10 | 50 | 0.01| 150.2| 11.24 | 0.83 | 236.6 | 109.86| 0.96 |
|    | 0.05| 161.6| 5.83 | 0.87 | 404.0 | 109.56| 0.99 |
|    | 0.1 | 204.2| 1.63 | 0.93 | 752.6 | 44.02 | 0.99 |
| 100| 0.01| 147.0| 13.62 | 0.90 | 255.2 | 191.69| 0.97 |
|    | 0.05| 162.6| 5.58 | 0.91 | 459.2 | 108.07| 0.99 |
|    | 0.1 | 223.6| 2.32 | 0.95 | 959.2 | 33.22 | 1.00 |
| 25 | 50 | 0.01| 205.6| 19.16 | 0.85 | 322.6 | 177.63| 0.96 |
|    | 0.05| 226.2| 8.46 | 0.88 | 544.8 | 96.73 | 0.99 |
|    | 0.1 | 252.8| 3.33 | 0.91 | 1253.6| 7.09  | 0.99 |
| 100| 0.01| 173.2| 23.38 | 0.88 | 305.8 | 588.23| 0.98 |
|    | 0.05| 199.4| 10.16 | 0.91 | 579.8 | 282.40| 0.99 |
|    | 0.1 | 261.4| 3.68 | 0.96 | 1201.3| 24.23 | 1.00 |
| Average | & 197.84 | 5.47 | 0.89 | 578.41 | 79.53 | 0.99 |

Proposition 6 only, two-stage reformulation with single-scenario inequalities, and two-stage reformulation with both single- and cross-scenario inequalities, respectively. In addition, we report average CPU time (in seconds) and average optimality gap over 5 random test instances, after they are terminated due to either a proof of optimality/infeasibility (INF) or a timelimit of 3, 600 seconds, whichever occurs first. The number in the parenthesis following the gap, if displayed, denotes the number of instances we fail to prove optimality/INF within the timelimit. Finally, an “INF” is placed in the column “Gap (%)” if all five random instances are infeasible.

From Tables 2–4, we observe that $2$-Stg generally took shorter time to solve the easier instances than $+single$, while the strength of single-scenario inequalities began to reveal in the harder instances. For instance, when $n = 60, I = 90, \delta = 0.05, \epsilon = 0.05, N = 50, 2$-Stg only solved 2 out of 5 random instances and took 2333.7 seconds on average, whereas $+single$ solved all 5 instances and only took 70.1 seconds on average. In Table 2, single-scenario inequalities on average shortened
the CPU time by approximately 20% and shrunk the optimality gap significantly (around 48%). Specifically, they helped a lot on problems where the ratios of \( \delta / \epsilon \) is larger, e.g., when \( (\delta, \epsilon) = (0.05, 0.05) \) or \( (\delta, \epsilon) = (0.1, 0.1) \). This demonstrates the effectiveness of the single-scenario inequalities. In Table 4, when the training data size \( N \) became larger (\( N = 200 \)), the speedup brought by single-scenario inequalities remained considerable, even though its average CPU time and final gap were on par with \( 2 \)-\text{Stg}. For instance, when \( n = 60, I = 70, \delta = 0.1, \epsilon = 0.1, N = 200, 2 \)-\text{Stg} solved 4 out of 5 instances and spent 1731.0 seconds on average, while +\text{single} solved all the instances and took only 45% of the time on average. By comparing the performance of +\text{cross} and +\text{single}, we observe that cross-scenario inequalities can further improve the computational performance. For example, in Table 2, where \( n = 80, I = 90, \delta = 0.05, \epsilon = 0.05, N = 50, +\text{cross} \) solved 4 instances to optimality within the timelimit, while +\text{single} solved only 2, let alone \( 2 \)-\text{Stg}, which failed to solve any of the 5 instances. As for the most challenging instances, in which all three approaches failed to prove optimality within the timelimit, +\text{cross} proved significantly smaller optimality gaps than \( 2 \)-\text{Stg} and +\text{single}. This demonstrates the strength of the cross-scenario inequalities on capturing the intersections of the feasible regions arising from multiple scenarios or coverings.

Appendix A: Proof of Proposition 3

**Proof** Consider a graph \( G := (V, E) \) with vertex set \( V \) and edge set \( E \), on which the classic NP-hard vertex cover problem has the following binary linear formulation:

\[
\begin{align*}
\min & \sum_{u \in V} x_u, \\
\text{s.t.} & \quad x^\top \xi_{u,v} \geq 1, \forall (u, v) \in E, \\
& \quad x_u \in \{0, 1\}, \forall u \in V,
\end{align*}
\]

(VC)

where binary variables \( x_u \) indicate whether node \( u \in V \) is part of the vertex cover and \( \xi_{u,v} \) is a binary vector with two nonzero entries: \( \xi_{u,v} = e_u + e_v \) and \( x^\top \xi_{u,v} = x_u + x_v \) for all \( x \in \{0, 1\}^{|V|} \). In particular, a vertex cover \( x \) can cover every edge twice if and only if all nodes are in the cover, i.e., \( x = 1 \). We provide a polynomial reduction from (VC) to the following instance of (8) to finish the proof:

\[
\min_{x \in \{0, 1\}^{|V'|}} L(x) := \frac{1}{|V'|} \sum_{u \in V} x_u + \frac{1}{|V'|} x_{w'} + x_w - \min_{(u,v) \in E'} \left(x^\top \xi_{u,v}\right)^{1/p},
\]

(29)

where we add two new nodes \( w \) and \( w' \) and augment the graph \( G \) to obtain \( G' := (V', E') \), \( V' := V \cup \{w, w'\} \), and \( E' := E \cup \{(w, w')\} \). Since the optimal value of (VC) is bounded above by \( |V| \), we are only interested in whether there is a vertex cover of size less than or equal to \( K \in \mathbb{Z}_+, 1 \leq K \leq |V| - 1 \). On the one hand, suppose that there exists a vertex cover \( x \) with size less than or equal to \( K \), then together
Table 2 Benchmark between different models on synthetic data ($N = 50$)

| $n$ | $I$ | $\delta$ | $\epsilon$ | 2-Stg | +single | +cross |
|-----|-----|----------|------------|-------|---------|--------|
|     |     |          |            | Time  | Gap (%) | Time  | Gap (%) | Time  | Gap (%) |
| 60  | 70  | 0.05     | 0.05       | 1498.0| 2 (1)   | 63.5  | 0       | 38.0  | 0       |
|     |     | 0.10     | 5.3        | 0     | 11.8    | 8.7   | 0       |
|     |     | 0.10     | 836.3      | 2.9 (1)| 722.8   | 188.8 | 0       |
|     |     | 0.20     | 1.6        | INF   | 9.1     | INF   | 6.4     | INF   |
|     |     | 0.30     | 1.1        | INF   | 7.4     | INF   | 0.0     | INF   |
| 90  | 0.05 | 0.05     | 2333.7     | 8.2 (3)| 70.1    | 31.0  | 0       |
|     | 0.10 | 3600.3   | 44 (5)     | 3600.0| 30.3 (5)| 1829.0| 0       |
|     | 0.10 | 3328.5   | 9 (4)      | 905.8 | 0       | 184.5 | 0       |
|     | 0.20 | 0.05     | 1.8        | INF   | 9.0     | INF   | 6.6     | INF   |
|     | 0.30 | 0.05     | 1.2        | INF   | 7.2     | INF   | 0.0     | INF   |
| 80  | 0.05 | 0.05     | 3600.0     | 42.7 (5)| 3088.9 | 10.8 (4)| 2333.8| 6.6 (2)|
|     | 0.10 | 3600.0   | 65.2 (5)   | 3600.1| 51.2 (5)| 3600.0| 23.4 (5)|
|     | 0.10 | 3600.2   | 35 (5)     | 3431.0| 10.9 (4)| 2622.2| 7.1 (3)|
|     | 0.20 | 0.05     | 7.2        | 0     | 17.8    | 0     | 12.7    | 0     |
|     | 0.30 | 0.05     | 1.7        | INF   | 14.1    | INF   | 0.0     | INF   |
| 90  | 0.05 | 0.05     | 3600.1     | 39.6 (5)| 3126.5 | 5.2 (3)| 2203.9| 1.1 (1)|
|     | 0.10 | 3600.0   | 64.1 (5)   | 3600.1| 52.1 (5)| 3600.0| 22.5 (5)|
|     | 0.10 | 3600.1   | 35.5 (5)   | 3600.0| 16.1 (5)| 3255.6| 8.2 (4)|
|     | 0.20 | 0.05     | 6.5        | 0     | 17.0    | 0     | 12.6    | 0     |
|     | 0.30 | 0.05     | 2.0        | INF   | 14.9    | INF   | 0.0     | INF   |
|     | 0.10 | 9.2      | 0          | 19.5  | 0       | 14.7  | 0       |

Average* 1140.2 14.6 911.1 7.5 673.0 2.7

*INF instances are excluded when calculating average gap
| n  | I   | δ   | ε   | 2-Stg Time | Gap (%) | +single Time | Gap (%) | +cross Time | Gap (%) |
|----|-----|-----|-----|-----------|---------|-------------|---------|-------------|---------|
| 60 | 70  | 0.05| 0.05| 968.7     | 2 (1)   | 671.5       | 0       | 247.8       | 0       |
|    |     | 0.10| 3600.0| 43.6 (5) | 3600.0 | 29.9 (5)   | 2103.2 | 6.5 (2)     |         |
|    |     | 0.10| 10.8   | 0       | 25.6    | 0           | 18.7   | 0           |         |
|    |     | 0.10| 1903.4 | 2.9 (2) | 990.7   | 0.7 (1)     | 264.7  | 0           |         |
|    |     | 0.20| 0.05   | 2.5     | INF     | 18.4        | INF    | 13.9        | INF     |
|    |     | 0.10| 13.5   | 0       | 32.2    | 0           | 18.5   | 0           |         |
|    |     | 0.30| 0.05   | 2.0     | INF     | 14.8        | INF    | 0           | INF     |
|    |     | 0.10| 8.0    | 0       | 26.6    | 0           | 17.4   | 0           |         |
| 90 | 0.05| 2696.9 | 8 (3) | 1159.9 | 2.1 (1) | 307.0 | 0           |         |
|    | 0.10| 3600.0 | 48.1 (5) | 3600.1 | 35 (5) | 2594.6 | 8.5 (3) |         |
|    | 0.10| 180.0  | 0      | 34.0   | 0       | 22.3   | 0           |         |
|    | 0.10| 2440.3 | 4.8 (3) | 1717.0 | 1.7 (1) | 544.9  | 1.7 (1) | INF       |
|    | 0.20| 0.05   | 2.7     | INF    | 18.1    | INF    | 13.9 | INF         |
|    | 0.10| 24.0   | 0      | 53.4   | 0       | 27.4   | 0           |         |
|    | 0.30| 0.05   | 2.1     | INF    | 15.0    | INF    | 0    | INF         |
|    | 0.10| 11.6   | 0      | 33.1   | 0       | 20.9   | 0           |         |
| 80 | 0.05| 3600.0 | 30.6 (5) | 3305.9 | 9.8 (4) | 2528.1 | 6.1 (3) |         |
|    | 0.10| 3600.0 | 65 (5) | 3600.2 | 51.1 (5) | 3600.0 | 26.1 (5) |         |
|    | 0.10| 225.6  | 0      | 102.0  | 0       | 53.8   | 0           |         |
|    | 0.10| 3600.1 | 29.8 (5) | 3600.0 | 16.6 (5) | 2941.8 | 9.2 (3) |         |
|    | 0.20| 0.05   | 7.4     | 0      | 33.4    | 0       | 24.6 | 0           |
|    | 0.10| 415.2  | 0      | 234.7  | 0       | 60.2   | 0           |         |
|    | 0.30| 0.05   | 2.9     | INF    | 29.3    | INF    | 0    | INF         |
|    | 0.10| 23.8   | 0      | 60.8   | 0       | 35.5   | 0           |         |
| 90 | 0.05| 3600.0 | 30.8 (5) | 3579.4 | 10.4 (4) | 3197.5 | 7.4 (4) |         |
|    | 0.10| 3600.3 | 65.5 (5) | 3601.0 | 52 (5) | 3600.3 | 27.1 (5) |         |
|    | 0.10| 46.6   | 0      | 50.8   | 0       | 35.1   | 0           |         |
|    | 0.10| 3600.0 | 27.1 (5) | 3600.0 | 15.2 (5) | 3241.2 | 7.5 (3) |         |
|    | 0.20| 0.05   | 7.2     | 0      | 31.9    | 0       | 23.9 | 0           |
|    | 0.10| 95.5   | 0      | 70.5   | 0       | 38.1   | 0           |         |
|    | 0.30| 0.05   | 3.1     | INF    | 28.7    | INF    | 0    | INF         |
|    | 0.10| 24.4   | 0      | 59.2   | 0       | 34.1   | 0           |         |

Average* 1179.9 13.8 1062.4 8.6 800.1 3.8

*INF instances are excluded when calculating average gap
Table 4  Benchmark between different models on synthetic data ($N = 200$)

| $n$ | $I$ | $\delta$ | $\epsilon$ | 2-Stg | +single | +cross |
|-----|-----|--------|--------|------|--------|--------|
|     |     |        |        | Time | Gap (%) | Time | Gap (%) | Time | Gap (%) |
| 60  | 70  | 0.05   | 0.05   | 432.4 | 0      | 326.5 | 0      | 81.6 | 0      |
|     |     | 0.10   | 0.05   | 3600.0| 44.7 (5)| 3600.0| 32.8 (5)| 3428.5| 7.7 (4) |
|     |     | 0.10   | 0.05   | 14.2  | 0      | 47.8  | 0      | 34.2 | 0      |
|     |     | 0.10   | 0.05   | 1731.0| 0.6 (1)| 745.4 | 0      | 152.7| 0      |
|     |     | 0.20   | 0.05   | 4.3   | INF    | 35.5  | INF    | 31.5 | INF    |
|     |     | 0.10   | 0.05   | 25.2  | 0      | 67.3  | 0      | 38.1 | 0      |
|     |     | 0.30   | 0.05   | 3.4   | INF    | 30.0  | INF    | 0.0  | INF    |
|     |     | 0.10   | 0.05   | 13.4  | 0      | 50.1  | 0      | 39.5 | 0      |
| 90  | 0.05 | 0.05   | 2044.0 | 2.1 (2)| 809.0 | 0     | 105.8 | 0    | 0      |
|     | 0.10 | 0.05   | 3600.0 | 48.8 (5)| 3600.9| 36.4 (5)| 3566.4| 13 (4) |
|     | 0.10 | 0.05   | 22.9   | 0      | 63.3  | 0     | 41.1  | 0    | 0      |
|     | 0.10 | 0.05   | 2930.7 | 6.3 (4)| 2457.6| 2.8 (3)| 461.4 | 0    | 0      |
|     | 0.20 | 0.05   | 5.0    | INF   | 36.6  | INF   | 26.9  | INF  | 0      |
|     | 0.10 | 0.05   | 48.5   | 0     | 95.8  | 0     | 50.0  | 0    | 0      |
|     | 0.30 | 0.05   | 3.7    | INF   | 29.1  | INF   | 0.0   | INF  | 0      |
|     | 0.10 | 0.05   | 23.3   | 0     | 74.4  | 0     | 46.3  | 0    | 0      |
| 80  | 0.05 | 0.05   | 3600.0 | 31.3 (5)| 3600.0| 21.2 (5)| 3600.0| 12.9 (5)|
|     | 0.10 | 0.05   | 3600.1 | 75.2 (5)| 3600.4| 54.3 (5)| 3600.3| 35.4 (5)|
|     | 0.10 | 0.05   | 193.8  | 0     | 165.2 | 0     | 78.3  | 0    | 0      |
|     | 0.10 | 0.05   | 3600.0 | 31.3 (5)| 3600.2| 22.1 (5)| 3600.0| 12.9 (5)|
|     | 0.20 | 0.05   | 15.2   | 0     | 71.2  | 0     | 56.2  | 0    | 0      |
|     | 0.10 | 0.05   | 180.7  | 0     | 200.7 | 0     | 97.5  | 0    | 0      |
|     | 0.30 | 0.05   | 5.8    | INF   | 58.6  | INF   | 0.0   | INF  | 0      |
|     | 0.10 | 0.05   | 46.0   | 0     | 119.7 | 0     | 83.9  | 0    | 0      |
| 90  | 0.05 | 0.05   | 3600.0 | 23.3 (5)| 3600.0| 14.5 (5)| 2141.8| 3.2 (2) |
|     | 0.10 | 0.05   | 3600.2 | 70.2 (5)| 3601.2| 51.7 (5)| 3600.0| 30.1 (5)|
|     | 0.10 | 0.05   | 110.8  | 0     | 163.7 | 0     | 98.2  | 0    | 0      |
|     | 0.10 | 0.05   | 3600.0 | 30.3 (5)| 3600.0| 17.6 (5)| 3600.0| 6.3 (5) |
|     | 0.20 | 0.05   | 14.3   | 0     | 69.9  | 0     | 47.9  | 0    | 0      |
|     | 0.10 | 0.05   | 856.0  | 0.1   | 787.7 | 0     | 136.4| 0    | 0      |
|     | 0.30 | 0.05   | 6.0    | INF   | 56.6  | INF   | 0.0   | INF  | 0      |
|     | 0.10 | 0.05   | 123.2  | 0     | 278.2 | 0     | 110.7 | 0    | 0      |

Average*  
1176.7  9.1  1113.8  9.7  904.9  4.7

* INF instances are excluded when calculating average gap
with \( x_{w'} := 1 \) and \( x_w := 0 \), \( x' := (x, x_{w'}, x_w) \) form a feasible solution to (29) with objective value less than or equal to \((K + 1)/|V'| - 1\) because

\[
L(x') = \frac{1}{|V'|} \sum_{u \in V} x_u + \frac{1}{|V'|} x_{w'} + x_w - \min_{(u, v) \in E'} \left( x^T \xi_{u, v} \right)^{1/p} \leq \frac{1}{|V'|} (K + 1) - 1.
\]

On the other hand, suppose that there is a \( y := (y_0, y_{w'}, y_w) \in \{0, 1\}^{|V'|} \) such that \( L(y) \leq (K + 1)/|V'| - 1 \). We discuss the following three cases, with regard to the coverage number \( C(y) := \min_{(u, v) \in E'} \left( y^T \xi_{u, v} \right)^{1/p} \), to show that there exists a vertex cover with size at most \( K \).

1. \( C(y) = 0 \) is impossible because

\[
C(y) = \min_{(u, v) \in E'} \left( y^T \xi_{u, v} \right)^{1/p} \geq \frac{1}{|V'|} \sum_{u \in V} y_u + \frac{1}{|V'|} y_{w'} + y_w - \frac{1}{|V'|} (K + 1) + 1
\]

\[
\geq -\frac{1}{|V'|} (K + 1) + 1 \geq 1 - \frac{|V'| + 1}{|V'| + 2} > 0.
\]

2. \( C(y) = 1 \): suppose that \( y_w = 1 \) and \( y_{w'} \) equals zero or one. Then an alternative solution \( y' := (y_0, y_{w'}, y_w') \) with \( y_{w'} = 1 \) and \( y_w = 0 \) to formulation (29) satisfies:

\[
C(y') = \min \left\{ \min_{(u, v) \in E'} \left( y_0^T \xi_{u, v} \right)^{1/p}, (y_{w'} + y_w')^{1/p} \right\} = 1 \quad \text{and}
\]

\[
L(y') \leq L(y) \leq \frac{1}{|V'|} (K + 1) - 1.
\]

It follows that \( y_0 \) is a vertex cover of \( G \), and we may assume that \( y_{w'} = 1, y_w = 0 \) in formulation (29) without loss of optimality. Hence,

\[
L(y) = \frac{1}{|V'|} \sum_{u \in V} (y_0)_u + \frac{1}{|V'|} y_{w'} + y_w - C(y) = \frac{1}{|V'|} \left( \sum_{u \in V} (y_0)_u + 1 \right) - 1
\]

\[
\leq \frac{1}{|V'|} (K + 1) - 1,
\]

which implies that \( y_0 \) is a vertex cover of size at most \( K \).

3. \( C(y) = 2^{1/p} \) is also impossible. Indeed, since \( C(y) = 2^{1/p} \), every edge of \( G' \) is covered twice, implying that \( y_{w'} = w_w = (y_0)_u = 1 \) for all \( u \in V \). Then,

\[
L(y) = \frac{1}{|V'|} (|V'| + 1) + 1 - 2^{1/p} \leq \frac{1}{|V'|} (K + 1) - 1 \leq \frac{1}{|V'|} |V| - 1.
\]

Simplifying the two ends of the above inequalities gives us \( 2 + |V'|^{-1} \leq 2^{1/p} \), which is impossible for \( p \geq 1 \). Therefore, \( C(y) \) cannot be \( 2^{1/p} \).

To sum up, if there is a feasible solution \( y \) to formulation (29) with \( L(y) \leq (K + 1)/|V'| - 1 \), then there is a vertex cover of \( G \) with size at most \( K \).

\( \square \)
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