HYPERBOLIC GROUPS THAT ARE NOT ABSTRACTLY COHOPFIAN

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Abstract. Sela proved every torsion-free one-ended hyperbolic group is coHopfian. We prove there exist torsion-free one-ended hyperbolic groups that are not abstractly coHopfian. In particular, we show that the fundamental group of every simple surface amalgam is not abstractly coHopfian.

1. Introduction

A group is coHopfian if it is not isomorphic to any of its proper subgroups. Sela [Sel97], building on work of Rips–Sela [RS94], proved a torsion-free hyperbolic group is coHopfian if and only if it is not freely indecomposable. Thus, by Stallings’ Theorem [Sta68], a torsion-free one-ended hyperbolic group is coHopfian.

A group \( \Gamma \) is abstractly coHopfian if no finite-index subgroup of \( \Gamma \) is isomorphic to an infinite-index subgroup of \( \Gamma \). Strebel [Str77] proved that the infinite-index subgroups of a Poincaré duality group have strictly smaller cohomological dimension than the group. Thus, Poincaré duality groups are abstractly coHopfian. In particular, fundamental groups of closed hyperbolic manifolds are abstractly coHopfian.

In this paper, we exhibit one-ended hyperbolic groups that are not abstractly coHopfian, answering a question of Whyte on Bestvina’s Problem list [Bes00, (Whyte, Q. 1.12)] and also asked by Kapovich [Kap12, Section 5].

Theorem 1.1. There exist one-ended hyperbolic groups that are not abstractly coHopfian.

Our proof of Theorem 1.1 is topological. We exhibit in Section 2 a simple surface amalgam \( X \) and two finite covers \( X_1 \to X \) and \( X_2 \to X \) so that the space \( X_1 \) properly includes in the space \( X_2 \). See Figure 1. The construction given in Section 2 does not immediately extend to simple surface amalgams in which the subsurfaces have different Euler characteristics. Nonetheless, we prove the following in Section 3.

Theorem 1.2. The fundamental group of a simple surface amalgam is not abstractly coHopfian.

Bowditch [Bow98] proved that if \( G \) is a one-ended hyperbolic group that is not Fuchsian, then there is a canonical graph of groups decomposition of \( G \), called the JSJ decomposition of \( G \), with edge groups that are two-ended and vertex groups of three
types: two-ended; maximally hanging Fuchsian; and quasi-convex rigid vertex groups not of the first two types. For background, see [SW79, Ser80, GL17]. We conjecture that for a one-ended hyperbolic group the abstractly coHopfian property is related to the existence of maximal hanging Fuchsian vertex groups in the JSJ decomposition of the group over two-ended subgroups.

**Conjecture 1.3.** Let \( \Gamma \) be a one-ended hyperbolic group that is not Fuchsian. If \( \Gamma \) is not abstractly coHopfian, then its JSJ decomposition contains a maximal hanging Fuchsian vertex group. Moreover, if the JSJ decomposition of \( \Gamma \) only contains maximal hanging Fuchsian vertex groups and 2-ended vertex groups, then \( \Gamma \) is not abstractly coHopfian.

In this paper, the embeddings constructed are quasi-isometric embeddings and are surely not representative. Indeed, highly distorted subgroups may be counterexamples to Conjecture 1.3, so a quasi-convexity assumption may be required.

In Section 4 we present two examples of one-ended hyperbolic groups whose JSJ decomposition contains both maximal hanging Fuchsian and rigid vertex groups and so that one group is abstractly coHopfian and the other is not. We summarize our examples and open problems in Section 6.

**Motivation of the terminology.** Kapovich [Kap12] uses the term weakly coHopfian instead of abstractly coHopfian. We abandon this terminology, since the property is not weaker than the coHopfian property: every one-ended hyperbolic group is coHopfian by the theorem of Sela, but not every one-ended hyperbolic group is abstractly coHopfian as shown here. However, in general, the abstractly coHopfian property defined in this paper is not a stronger condition than the coHopfian property. For example, the integers \( \mathbb{Z} \) are abstractly coHopfian, but not coHopfian. The adjective abstractly is justified by the fact that being abstractly coHopfian is an abstract commensurability invariant, which follows from the lemma below. The coHopfian property, on the other hand, is not an abstract commensurability invariant by work of Cornulier [Cor16, Appendix A].

**Lemma 1.4.** If \( H \leq G \) is a finite-index subgroup, then \( H \) is abstractly coHopfian if and only if \( G \) is abstractly coHopfian.

**Proof.** If \( G \) is abstractly coHopfian, then \( H \) is abstractly coHopfian. Indeed, otherwise, there exists a finite-index subgroup \( H' \leq H \leq G \) with an infinite-index embedding \( \varphi : H' \to H \leq G \), contradicting the abstract coHopficity of \( G \).

Conversely, suppose \( G \) is not abstractly coHopfian. Then, there exists a finite-index subgroup \( G' \leq G \) and an embedding \( \varphi : G' \to G \) so that \( \varphi(G') \) is an infinite-index subgroup of \( G \). The intersection \( \varphi^{-1}(H) \cap H \) is a finite-index subgroup of \( H \), since the preimage \( \varphi^{-1}(H) \) is a finite-index subgroup of \( G \). Therefore \( \varphi \) restricts to an infinite-index embedding \( \varphi^{-1}(H) \cap H \to H \). Thus, \( H \) is not abstractly coHopfian.

This abstract commensurability invariant provokes the following question.

**Question 1.5.** Is being abstractly coHopfian a quasi-isometry invariant for finitely generated/presented groups?
We also note that the following question of Bestvina remains open.

**Question 1.6.** Does there exist a one-ended hyperbolic group that contains isomorphic finite-index subgroups of different index?

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### 2. The main example

A *simple surface amalgam* $X$ is the union of a finite set of surfaces $\Sigma_1 \ldots \Sigma_k$ with negative Euler characteristic such that $\partial \Sigma_i \cong S^1$ and all boundary components are identified to a single copy of the circle $S^1$ by a homeomorphism. Let $\mathcal{C}_k$ denote the set of groups $\pi_1(X)$ such that $X$ is the simple surface amalgam obtained from $k$ surfaces with boundary. Simple surface amalgams have been studied in [Mal10, Sta17, DST18, SW18].

The following lemma determines the finite covers of a surface with boundary.

**Lemma 2.1.** [Neu01, Lemma 3.2] Let $\Sigma$ be an oriented surface with positive genus. Fix a positive integer $d$. For each boundary component of $\Sigma$ take a collection of degrees summing to $d$. Then a $d$-sheeted covering $\Sigma' \to \Sigma$ exists with the prescribed degree coverings in the preimage of each boundary component of $\Sigma$ if and only if the total number of boundary components of $\Sigma'$ has the same parity as $d\chi(\Sigma)$.

**Proof of Theorem 1.1.** Let $X$ be a simple surface amalgam with subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$, where $\Sigma_i$ is a surface of genus one with a single boundary component. Demonstrating that $\pi_1(X)$ is not abstractly coHopfian follows from considering Figure 1.

We first construct a degree-3 cover $f_1 : X_1 \to X$. By Lemma 2.1, there exists a degree-3 cover $\Sigma'_i \to \Sigma_i$ so that $\Sigma'_i$ has a single boundary component for $i \in \{1, 2, 3\}$.
By an elementary Euler characteristic computation, the surface \( \Sigma_i \) has genus two. The boundary components of each \( \Sigma_i \) for \( i \in \{1, 2, 3\} \) can be identified to each other by a homeomorphism to construct a 3-sheeted cover \( f_1 : X_1 \to X \).

We now build a degree-4 cover \( f_2 : X_2 \to X \). By Lemma 2.1, there exists a degree-2 cover \( \Sigma''_i \to \Sigma_i \) so that \( \Sigma''_i \) has two boundary components for \( i \in \{1, 2, 3\} \). Again, by an elementary Euler characteristic computation, the surface \( \Sigma''_i \) has genus one. By identifying a single boundary component from each \( \Sigma''_i \) and attaching copies of \( \Sigma_j, \Sigma_k \) to the other boundary component of \( \Sigma''_i \) we obtain the 4-sheeted covering \( f_2 : X_2 \to X \).

There is a \( \pi_1 \)-injective proper embedding \( \phi : X_1 \to X_2 \) as shown in Figure 1 that yields an embedding of \( \pi_1(X_1) \) in \( \pi_1(X_2) \) as an infinite-index subgroup. Therefore, \( \pi_1(X) \) contains a finite-index subgroup that is isomorphic to an infinite-index subgroup. \( \square \)

3. Simple surface amalgams are not abstractly coHopfian

**Theorem 3.1.** If \( G \) is the fundamental group of a simple surface amalgam, then \( G \) is not abstractly coHopfian.

**Proof.** Let \( G \) be the fundamental group of a simple surface amalgam \( X \) with \( k \) sub-surfaces \( \Sigma_1, \ldots, \Sigma_k \). Let \( \chi_i = \chi(\Sigma_i) \). We construct a degree-2 cover \( \hat{X} \to X \) by an application of Lemma 2.1. (This step allows us to resolve any parity issues in the future application of Lemma 2.1.) Let \( \hat{X} \) be the union of \( k \) surfaces \( \Sigma_1, \ldots, \Sigma_k \), where \( \chi(\Sigma_i) = 2\Sigma_i \), the surface \( \Sigma_i \) has two boundary components \( \gamma_i \) and \( \gamma'_i \), and \( \hat{X} \) is obtained by identifying the curves \( \{\gamma_i | 1 \leq i \leq k\} \) to a single curve \( \gamma \) and the curves \( \{\gamma'_i | 1 \leq i \leq k\} \) to a single curve \( \gamma' \).

As in Section 2, we will construct two finite covers \( X' \) and \( X'' \) of the space \( \hat{X} \) so that the space \( X' \) properly embeds in the space \( X'' \). See Figure 2. This construction relies on the next claim.

**Claim 3.2.** Given negative integers \( \chi_1, \ldots, \chi_k \), there exists a set of positive integers \( \{D, d_i | 1 \leq i \leq k\} \) so that

\[
(6D + 2d_1)2\chi_1 + (2d_1)2\chi_k = 2D\chi_1 \\
(6D + 2d_2)2\chi_2 + (2d_2)2\chi_1 = 2D\chi_2 \\
\vdots \\
(6D + 2d_k)2\chi_k + (2d_k)2\chi_{k-1} = 2D\chi_k.
\]

**Proof of Claim.** Rewrite the \( i \)-th equation in the following form mod \( k \):

\[
(4d_i)\chi_i + \chi_{i-1} = -4D\chi_i.
\]

Choosing \( D \) to be divisible by \( \text{lcm}\{\chi_i + \chi_{i-1} | 1 \leq i \leq k\} \), there exists \( d_i \) such that

\[
d_i = \frac{-D\chi_i}{\chi_i + \chi_{i-1}} \geq 1.
\]

\( \square \)

There exists a degree-\( D \) cover \( X' \to \hat{X} \) as follows, where \( D \) is the positive integer in Claim 3.2. Let \( X' \) be the union of \( k \) surfaces \( \Sigma'_1, \ldots, \Sigma'_k \), so that \( \chi(\Sigma'_i) = D\chi(\Sigma_i) = 2D\chi_i \),
the surface $\Sigma_i'$ has two boundary components $\rho_i$ and $\rho'_i$, and $\hat{X}$ is obtained by identifying the curves $\{\rho_i \mid 1 \leq i \leq k\}$ to a single curve $\rho$ and the curves $\{\rho'_i \mid 1 \leq i \leq k\}$ to a single curve $\rho'$. There exists a degree-$D$ cover $X' \to \hat{X}$ by Lemma 2.1.

We now construct the finite cover $X'' \to \hat{X}$ so that the space $X'$ properly embeds in $X''$. To build $X''$, for $i \in \{1,\ldots,k\}$ we will partition the surface $\Sigma_i' \subset X'$ into three subsurfaces, $\Sigma_{i1}', \Sigma_{i2}', \Sigma_{i3}'$ and attach additional subsurfaces to the boundary curves of $\Sigma_{i2}'$ as follows. The construction is illustrated in Figure 2. Let $\Sigma_{i1}'$ be the subsurface with Euler characteristic $(6D + i)d_i \chi_i$ and four boundary components, two of which are the curves $\rho_i$ and $\rho_i'$; call the other boundary curves $\rho_{i1}$ and $\rho'_{i1}$. Let $\Sigma_{i2}'$ be the subsurface with Euler characteristic $(2d_i)2\chi_{i-1}$ (subscript mod $k$) and four boundary components, two of which are $\rho_{i1}$ and $\rho'_{i1}$; call the other boundary curves $\rho_{i2}$ and $\rho'_{i2}$. Finally, let $\Sigma_{i3}'$ be the subsurface with Euler characteristic $2d_i\chi_i$ and two boundary curves, $\rho_{i2}$ and $\rho'_{i2}$.

Claim 3.2 implies that $\Sigma_i' \cong \Sigma_{i1}' \cup \Sigma_{i2}' \cup \Sigma_{i3}'$.

For $i \in \{1,\ldots,k\}$ attach $k - 2$ surfaces $\{\Sigma_j' \mid j \in \{1,\ldots,k\}, j \neq i, i - 1\}$ with two boundary components and Euler characteristics $\chi(\Sigma_j') = 2d_i\chi_j$ to the pair of curves $\{\rho_{i1}, \rho'_{i1}\}$. Similarly, attach $k - 2$ surfaces $\{\Sigma_j'^{i'} \mid j \in \{1,\ldots,k\}, j \neq i, i - 1\}$ with two boundary components and Euler characteristics $\chi(\Sigma_j'^{i'}) = 2d_i\chi_j$ to the pair of curves $\{\rho_{i2}, \rho'_{i2}\}$.

We prove there exists a degree $6D + \sum_{i=1}^{k} 2d_i$ cover $X'' \to X$. We describe the cover on the branching curves of $X''$, and then we use Lemma 2.1 to show the cover extends to all of $X''$. Suppose the curves $\{\rho_i\}_{i=1}^{k}$ and $\{\rho'_i\}_{i=1}^{k}$ are glued together to form the curves $\rho$ and $\rho'$, respectively. Then, $\rho$ and $\rho'$ cover the curves $\gamma$ and $\gamma'$ by degree $6D$. For all $i \in \{1,\ldots,k\}$, the curves $\rho_{i1}$ and $\rho_{i2}$ cover the $\gamma$ by degree $d_i$, and the curves $\rho'_{i1}$ and $\rho'_{i2}$ cover the curve $\gamma'$ by degree $d_i$. Then, by Lemma 2.1, there exists a degree $6D + d_i$ cover $\Sigma_{i1} \to \Sigma_i'$, a degree $2d_i$ cover $\Sigma_{i2} \to \Sigma_{i-1}'$, and a degree $d_i$ cover $\Sigma_{i3} \to \Sigma_i'$.
Figure 3. A degree-3 cover of a surface $\Sigma$ of genus one with one boundary component by a surface of genus two and one boundary component. The red curve on $\Sigma$ has two pre-images that lie in subsurfaces separated by the blue curve.

By Lemma 2.1, there are degree $d_i$ covers $\Sigma_j^i \to \Sigma_j'$ and $\Sigma_j'^i \to \Sigma_j'$. Since these covering maps agree on their intersection, there exists a finite cover $X'' \to X$. \hfill $\square$

4. Examples with mixed JSJ decomposition

Example 4.1. (Not abstractly coHopfian.) We adapt the proof in Section 2 to exhibit a one-ended hyperbolic group $G$ whose JSJ decomposition contains both maximal hanging Fuchsian vertex groups and rigid vertex groups and so that $G$ is not abstractly coHopfian. An illustration of this example appears in Figure 3 and Figure 4. Let $X_0$ be a simple surface amalgam with subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$, where $\Sigma_i$ is a surface of genus one with a single boundary component. Let $a_i$ be an essential simple closed curve on $\Sigma_i$ that is not homotopic to the boundary. There exists a homeomorphism $\phi_{ij} : \Sigma_i \to \Sigma_j$ so that $\phi_{ij}(a_i) = a_j$ for all $i, j \in \{1, 2, 3\}$.

Let $H$ be a torsion-free one-ended hyperbolic group that does not split over a virtually cyclic subgroup, and let $X_H$ be a finite cell complex with $\pi_1(X_H) \cong H$. Suppose there exists an infinite-order element $h \in H$ represented by a closed curve $a_h$ on $X_H$ so that there exists a degree-2 cover $X'_H \to X_H$ in which $a_h$ lifts to a single closed curve on $X'_H$. (For a concrete example, let $H \cong \pi_1(S) \rtimes \langle t \rangle$ be the fundamental group of a closed fibered hyperbolic 3-manifold with fiber a closed surface $S$, and let $h = t$.) For $i \in \{1, 2, 3\}$, let $\phi_i : X_H \to X_{H_i}$ be a homeomorphism, and let $a_{hi} = \phi(a_h)$. For $i \in \{1, 2, 3\}$ let $A_i$ be an annulus. Glue one boundary component of $A_i$ to the curve $a_i$ and the other boundary component of the annulus to the curve $a_{hi}$ by homeomorphisms. Let $X$ be the resulting complex, and let $G$ be the fundamental group of $X$. The JSJ decomposition of $G$ over
2-ended vertex groups contains three maximal hanging Fuchsian vertex groups, $\pi_1(\Sigma_i)$, and three rigid vertex groups, $\pi_1(X_{H_i})$.

**Claim 4.2.** The group $G$ is not abstractly coHopfian.

**Proof.** We first construct a degree-3 cover $X' \to X$. As shown in Figure 3, for $i \in \{1, 2, 3\}$ there exists a degree-3 cover $\Sigma'_i \to \Sigma_i$ so that $\Sigma'_i$ has one boundary component and genus two and so that the preimage of the curve $a_i$ has two components $a'_i$ and $a''_i$, where $a'_i$ covers $a_i$ by degree one and $a''_i$ covers $a_i$ by degree two. Moreover, there exists a closed curve $\gamma_i$ (shown in blue in Figure 3) that separates $\Sigma'_i$ into two subsurfaces; one subsurface has boundary $\gamma_i$ and contains the curve $a'_i$, and the other subsurface has two boundary components and contains the curve $a''_i$. Thus, as in Section 2, the boundary components of $\Sigma'_i$ can be glued together to form a degree three cover of the simple surface amalgam $X'_0 \to X_0$. By assumption on the group $H$, the degree-3 cover of the simple surface amalgam extends to a degree-3 cover of $X$ obtained by taking copies of $X_H$ and copies of the degree-two cover $X'_H$ and attaching them along annuli to lifts of the curves $a_{hi}$ on $X'_0$. See Figure 4.

The degree-4 cover $X'' \to X$ is constructed in analogy to the construction in Section 2. The space $X''$ contains the space $X'$ as a proper subspace, and for $i \in \{1, 2, 3\}$ to each of the curves $\gamma_i \subset X'$ defined in the paragraph above a copy of $\Sigma_i \cup X_{H_i}$ is glued along the boundary component of $\Sigma_i$. As above, the space $X''$ forms a degree-4 cover of $X$. Since $X'$ properly embeds in $X''$, the group $G = \pi_1(X)$ contains a finite-index subgroup isomorphic to $\pi_1(X')$ and an infinite-index subgroup isomorphic to $\pi_1(X')$. Thus, $G$ is not abstractly coHopfian. \qed

Examples 4.3 and 5.1 make use of the notion of an *acylindrical submanifold*. Let $M$ be a Riemannian manifold and $N \subseteq M$ a locally convex submanifold. Let $A$ denote
the annulus. The submanifold $\mathcal{N}$ is said to be \textit{acylindrical} if any $\pi_1$-injective map $(A, \partial A) \to (M, N)$ is relatively homotopic to a map $(A, \partial A) \to (\mathcal{N}, \mathcal{N})$. Equivalently, the subgroup $\pi_1(\mathcal{N}) \leq \pi_1(M)$ is \textit{malnormal} in the sense that $\pi_1(\mathcal{N}) \cap \pi_1(M)^g = \{1\}$ for all $g \in \pi_1(M) - \pi_1(\mathcal{N})$. In particular, if $M$ is a closed hyperbolic manifold and $\mathcal{N}$ is a simple closed geodesic, then $\mathcal{N}$ is acylindrical in $M$.

\textbf{Example 4.3.} (Abstractly coHopfian.) Let $M$ be a closed hyperbolic 3-manifold, and let $\gamma$ be a embedded locally geodesic closed curve in $M$. Let $\Sigma$ be a compact surface with positive genus and boundary $\partial \Sigma$ homeomorphic to $S^1$. Identify $\gamma$ with $\partial \Sigma$ via a homeomorphism to obtain a quotient space $X$. The fundamental group $G = \pi_1(X)$ is a one-ended hyperbolic group given by the amalgamation of the 3-manifold group and the free group $\pi_1(\Sigma)$ along the cyclic groups corresponding to $\gamma$ and $\partial \Sigma$. This amalgamated free product corresponds to the canonical JSJ decomposition for $\pi_1 X$.

\textbf{Claim 4.4.} The group $G$ is abstractly coHopfian.

\textit{Proof.} Let $G' \leq G$ be a finite-index subgroup and $\varphi : G' \to G$ an injective homomorphism. Without loss of generality, assume that $G'$ is a normal subgroup of $G$ and let $\pi : X' \to X$ denote the corresponding finite regular cover. Take the $\pi$-preimages of $M$ and $\Sigma$ to decompose $X'$ as a collection of homeomorphic 3-manifolds $M'_1, \ldots, M'_n$ and a collection of homeomorphic surfaces with boundary $\Sigma'_1, \ldots, \Sigma'_m$ such that $M'_i \to M$ and $\Sigma'_j \to \Sigma$ are regular covers.

We first argue that the homomorphism $\varphi : G' \to G$ is induced by a map $\Phi : X' \to X$ such that the restriction of $\Phi$ to each 3-manifold $M'_i$ is a covering map $\Phi_i : M'_i \to M$. Let $T$ denote the Bass-Serre tree of the JSJ-splitting of $G$. The subgroup $\varphi(G') \leq G$ acts on $T$. Since the group $\pi_1(M'_i)$ does not split over a virtually cyclic subgroup, the subgroup $\varphi(\pi_1(M'_i))$ stabilizes a vertex in $T$. Thus, there exists $g_i \in G$, such that $\varphi(\pi_1(M'_i))^{g_i} \leq \pi_1(M)$. As $\pi_1(M)$ is abstractly coHopfian, $\varphi(\pi_1(M'_i))^{g_i}$ is a finite-index subgroup of $\pi_1(M)$. By Mostow rigidity [Mos68], the covering space corresponding the subgroup $\varphi(\pi_1(M'_i))^{g_i}$ is isometric to $M'_i$, so the homomorphism $\varphi^{g_i} : \pi_1(M'_i) \to \pi_1(M)$ is induced by a covering map $\Phi_i : M'_i \to M$. Since $X$ and $X'$ have contractible universal covers, there exists a continuous map $\Psi : X' \to X$ such that $\Psi_{\ast} = \varphi$. As $\varphi^{g_i}$ induces the same map on the fundamental group as $\Phi_i$, by an application of Whitehead’s theorem (see [Hat02, Thm 4.5]), we can homotope $\Psi$ to a map $\Phi$ so that it restricts to $\Phi_i$ on each $M'_i \subseteq X'$. Thus, the resulting map $\Phi$ is as specified.

Suppose towards a contradiction that $\Phi : X' \to X$ is not homotopic to a covering map. Let $C_j \subset \Phi^{-1}(\gamma)$ be the set of curves in the full preimage of the amalgamating curve $\gamma \subset X$ that lie in the surface $\Sigma'_j$. After homotopy, we may assume that $C_j$ is a set of disjoint curves containing the boundary curves $\partial \Sigma'_j$. Moreover, since the curves $\gamma \subset M$ and $\partial \Sigma \subset \Sigma$ are acylindrical subspaces, applying a suitable homotopy removes parallel curves in the set $C_j$. Since $\Phi$ is not homotopic to a covering map, without loss of generality, the set $C_1$ contains a curve that is not a component in $\partial \Sigma'_1$.

Let $\sigma_1, \ldots, \sigma_\ell$ denote the closures of the components of $\Sigma'_1 - C_1$. Each $\sigma_i$ is mapped by $\Phi$ into either $\Sigma$ or $M$ and we refer to the subsurfaces as either $\Sigma$-\textit{type} or $M$-\textit{type}, accordingly. If $\sigma_i$ and $\sigma_j$ intersect in $\Sigma'_1$, then they must be of different types. If $\sigma_i$ contains a boundary curve in $\partial \Sigma'_1$ then $\sigma_1$ must be $\Sigma$-type. Therefore we can deduce that there is some proper subset of $\sigma_1, \ldots, \sigma_\ell$ that are $\Sigma$-type, and they contain $\partial \Sigma'_1$. 

Moreover, for each Σ-type σ_i, since Φ induces a π_1-injective map (σ_i, ∂σ_i) → (Σ, ∂Σ) we deduce that it is boundary homotopic to a covering map.

Under the regular covering map π : X' → X corresponding to the finite-index subgroup G' ≤ G, each boundary component in ∂Σ_j' covers ∂Σ with degree d for some d ∈ N. Thus, the Euler characteristic satisfies

\[ \chi(\Sigma_j') = d|\partial \Sigma_j'| \cdot \chi(\Sigma). \]

As Φ_i maps M'_i to M via a covering map, the boundary components ∂Σ'_i are mapped to γ via Φ by a degree d map. Thus, we can deduce that

\[ \chi(\Sigma'_i) = \sum \chi(\sigma_i) > \sum_{\sigma_i \text{ is Σ-type}} |\chi(\sigma_i)| \]

\[ = \sum_{\sigma_i \text{ is Σ-type}} \deg(\Phi : \partial \sigma_i \rightarrow \gamma)|\chi(\Sigma)| > d|\partial \Sigma_j'| \cdot \chi(\Sigma). \]

(Note that we let deg(Φ : ∂σ_i → γ) denote the sum of the degrees of the map restricted to each component in ∂σ_i.) The first inequality follows from discarding the M-type surfaces σ_i, and the second inequality follows from only counting the degrees of the curves in ∂Σ'_j. This contradicts the previous equality, and thus, Φ is homotopic to a covering map. Therefore, ϕ(G') is a finite-index subgroup of G. \(\square\)

5. Abstractly coHopfian groups

In this section we provide an example of a one-ended hyperbolic group with non-trivial JSJ decomposition and only rigid and two-ended vertex groups. The key point is that we choose the rigid vertex groups to be abstractly coHopfian.

Example 5.1. Let M and N be closed hyperbolic 3-manifolds. For simplicity we will assume that π_1(M) and π_1(N) are incommensurable. Let γ ⊆ M and σ ⊆ N be simple closed geodesics, and let A be an annulus. Let X be the space obtained from M ∪ A ∪ N by gluing one boundary component of the annulus to γ and the other to σ.

Claim 5.2. G = π_1X is abstractly coHopfian.

Proof. The proof follows a similar strategy to Claim 4.4. Let G' ≤ G be a finite-index subgroup and ϕ : G' → G is an injective homomorphism. Assuming G' is a normal subgroup, let X' → X be the finite-sheeted, regular cover corresponding to G'. Considering the ϕ-preimages of M, N, and A, decompose X' as a graph of spaces with vertex spaces M_{u_1}, ..., M_{u_m} and N_{v_1}, ..., N_{v_n} and edge spaces A_{e_1}, ..., A_{e_a}.

As π_1(M) and π_1(N) do not split over a virtually cyclic group, are abstractly co-Hopfian, and are incommensurable with each other, there exists g_i, h_i ∈ G such that ϕ(π_1(M_{u_i}))^{g_i} is a finite-index subgroup of π_1(M) and ϕ(π_1(N_{v_i}))^{h_i} is a finite-index subgroup of π_1(N). By Mostow rigidity, there exist covering maps Φ_{u_i} : M_{u_i} → M and Φ_{v_i} : N_{v_i} → N that correspond to the embeddings ϕ^{g_i} : π_1(M_{u_i}) → π_1(M) and ϕ^{h_i} : π_1(N_{v_i}) → π_1(N).

The spaces X' and X have contractible universal covers, so there exists a continuous map Φ : X' → X such that Φ_γ = ϕ. After homotopy, Φ restricts to Φ_{u_i} on M_{u_i} and Φ_{v_i} on N_{v_i}. The map Φ may be homotoped to a covering map since any annulus mapping
(A, ∂) to either (M, γ) or (N, σ) can be homotoped into γ or σ, since γ and σ are acylindrical subspaces of M and N. Thus, ϕ(G') is a finite-index subgroup of G. □

6. Summary

The table below summarizes the results in this paper and related open problems.

| G | JSJ decomposition has only 2-ended and maximal hanging Fuchsian vertex groups | JSJ decomposition has 2-ended, maximal hanging Fuchsian, and rigid vertex groups | JSJ decomposition has only 2-ended and rigid vertex groups | G does not split over Z |
|---|---|---|---|---|
| Abstractly coHopfian | Open Problem | Example 4.3 | Example 5.1 | Poincaré duality groups [Str77] |
| Not abstractly coHopfian | Section 2; Theorem 3.1 | Example 4.1 | Open Problem | Open Problem |

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