TOPOLOGICAL STRUCTURE OF SPACES OF STABILITY CONDITIONS AND TOPOLOGICAL FUKAYA TYPE CATEGORIES

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ABSTRACT. This is a survey on two closely related subjects. First, we review the study of topological structure of ‘finite type’ components of spaces of Bridgeland’s stability conditions on triangulated categories \([W, KQ, Q1, QW, BPP]\). The key is to understand Happel-Reiten-Smalø tilting as tiling of cells. Second, we review topological realizations of various Fukaya type categories \([BZ, QZ1, Q2, Q3, QZ2, HKK]\), namely cluster/Calabi-Yau and derived categories from surfaces. The corresponding spaces of stability conditions are of ‘tame’ nature and can be realized as moduli spaces of quadratic differentials due to Bridgeland-Smith and Haiden-Katzarkov-Kontsevich \([BS, HKK, I, KQ2]\).

1. INTRODUCTION

The notion of a stability condition on a triangulated category \(\mathcal{D}\) was introduced by Bridgeland \([B]\). The idea was inspired from physics by studying D-branes in string theory. Bridgeland showed a key result that the space \(\text{Stab}(\mathcal{D})\) of stability conditions on \(\mathcal{D}\) is a finite dimensional complex manifold, provided that the rank of the Grothendick group \(\mathcal{K}(\mathcal{D})\) is finite. Moreover, these spaces carry interesting geometric/topological structures which shed light on the properties of the original triangulated categories.

While the spaces of stability conditions in geometric setting are hard to studied in general, the algebraic (quivery) case is much easier. Namely, when the heart \(\mathcal{H}\) of a stability conditions is finite (i.e. a length category with finitely many simples), the stability conditions supported on \(\mathcal{H}\) is a complex half open half closed cell, cf. \((3.1)\). This allows us to study the global topological structure of \(\text{Stab}(\mathcal{D})\). The skeleton of a finite type component \(\text{Stab}_0\) of a space of stability conditions is a (connected component of) exchange graph \(\text{EG}_0\). The vertices and edges of such a graph are (finite) hearts and simple HRS-tilting. In fact, \(\text{EG}_0\) is the dual of some of the stratification of \(\text{Stab}_0\) (see \([Q1, \text{Theorem 3.4}], [KQ2, \S\ 4.7]\) and Figure 1). Applying this method, we show that such a component is always contractible (Theorem 3.4).

Next class of components to be studied is the ‘tame type’. Our philosophy/definition is that in a tame type component, any stability condition, up to a \(C\)-action, admits a finite heart. The corresponding categories, which have been studied, are from surfaces. The key here is to understand (directly or via other theory, i.e. cluster theory) objects/morphisms in this type of categories via arcs/intersections on surfaces. In some sense, they are of topological Fukaya type (cf. Remark 4.4).

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Moreover, these stability spaces are related to Kontsevich’s homological mirror symmetry for a pair of mirror varieties \((X, X^\vee)\). More precisely, the Kähler moduli space \(\mathcal{M}(X^\vee)\) of \(X^\vee\) can be (conjecturally) embedded into (a quotient of) the space of stability conditions on the categories

\[ D\text{Fuk}(X) \cong D^b(\text{Coh} X^\vee), \]

where \(D\text{Fuk}(X)\) is the derived Fukaya category of \(X\) and \(D^b(\text{Coh} X^\vee)\) the bounded derived category of coherent sheaves on \(X^\vee\).

Topological Fukaya type categories are the test ground for this idea/conjecture. Two pioneer works have been carried out by Bridgeland-Smith and Haiden-Katzarkov-Kontsevich in [BS, HKK]. Our upcoming works [IQ1, IQ2] aim to understand the relations between their results.

**Context.** In §2 we recall couple of structures on abelian/triangulated categories. In §3 we review a series of works [W, KQ, QW], which use HRS-tilting as tiling (and stratification), to understand the (global) topological structure of finite type components of spaces of stability conditions. In §4 we review a series of works [BZ, QZ1, Q2, Q3, QZ2, HKK], which use (graded) string model, to understand cluster/derived categories via arcs/intersections on surfaces. In §5 we collect two key results [BS, HKK] in the direction of the study of stability conditions via quadratic differentials.

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2. Structures on triangulated categories

2.1. Torsion pairs and stability functions on abelian categories.

**Definition 2.1.** A torsion pair in an abelian category $\mathcal{H}$ is an ordered pair of full subcategories $(\mathcal{T}, \mathcal{F})$, such that $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and any object $E \in \mathcal{H}$ sits in a short exact sequence $0 \to E^T \to E \to E^F \to 0$, for some $E^T \in \mathcal{T}$ and $E^F \in \mathcal{F}$. We will call $\mathcal{T}$ the torsion part and $\mathcal{F}$ the torsion free part. We will write $\mathcal{H} = (\mathcal{T}, \mathcal{F})$ for a torsion pair of $\mathcal{H}$.

Here is a well-known lemma (cf. [KQ, Lemma 3.6]).

**Lemma 2.2.** Let $S$ be a rigid simple object in a Hom-finite abelian category $\mathcal{H}$. Then $\mathcal{H}$ admits a torsion pair $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{F} = \langle S \rangle$. Similarly, there is also a torsion pair with the torsion part $\mathcal{T} = \langle S \rangle$.

Next we have the notion of stability function on abelian categories, which is closely related to the corresponding notion on triangulated categories. Denote by $K(\mathcal{H})$ the Grothendick group of $\mathcal{H}$.

**Definition 2.3.** A stability function on an abelian category $\mathcal{H}$ is a group homomorphism $Z : K(\mathcal{H}) \to \mathbb{C}$ such that for any non-zero object $M$, we have $Z(M) = m(M) \exp(\phi(Z(M))i\pi)$ for some $m(M) \in \mathbb{R}_{>0}$ and $\phi(Z(M)) \in (0, 1]$, i.e. $Z(M)$ lies in the upper half-plane

$$\mathbb{H} = \{re^{\theta i} \mid r \in \mathbb{R}_{>0}, 0 < \theta \leq 1\} \subset \mathbb{C}. \quad (2.1)$$

Call $\phi_Z(M)$ the phase of $M$. We say an object $0 \neq M \in \mathcal{C}$ is semistable (with respect to $Z$) if every subobject $0 \neq L$ of $M$ satisfies $\phi_Z(L) \leq \phi_Z(M)$. Further, we say a stability function $Z$ on $\mathcal{C}$ satisfies HN-property, if for an object $0 \neq M \in \mathcal{C}$, there is a collection of short exact sequences

$$0 = M_0 \xrightarrow{\cdot} M_1 \xrightarrow{\cdot} \cdots \xrightarrow{\cdot} M_{k-1} \xrightarrow{\cdot} M_k = M$$

in $\mathcal{C}$ such that $L_1, \ldots, L_k$ are semistable objects (with respect to $Z$) and their phases are in decreasing order, i.e. $\phi(L_1) > \cdots > \phi(L_k)$.

The stability functions on abelian categories can be used to define (motivic) Donaldson-Thomas invariants via quantum dilogarithm identities (due to Reineke and Kontsevich-Soibelman, cf. the survey [K2]). One can also use Keller’s green mutation to define such invariants combinatorially. [Q1, §8] provides an alternative proof of quantum dilogarithm identities in the Dynkin case via topological structure of spaces of stability conditions that will be discussed in the following.

2.2. Stability structures on triangulated categories. A torsion pair in a triangulated category $\mathcal{D}$ is an ordered pair of full subcategories $(\mathcal{T}, \mathcal{F})$, such that $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and any object $E \in \mathcal{D}$ sits in a triangle

$$E^T \to E \to E^F \to E^T[1],$$
for some $E^T \in T$ and $E^F \in F$. As triangulated categories admit distinguish auto-equivalence, the shift $[1]$, we have the following upgraded version of torsion pairs.

**Definition 2.4.** A t-structure $\mathcal{P}$ is the torsion part of a torsion pair satisfying $\mathcal{P}[1] \subset \mathcal{P}$.

The heart of a t-structure $\mathcal{P}$ is the full subcategory

$$\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P},$$

where $\mathcal{P}^\perp$ is the corresponding torsion free part.

A t-structure $\mathcal{P}$ is **bounded** if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j].$$

Equivalently, one can define the heart as follows.

**Definition 2.5.** A heart $\mathcal{H}$ of a triangulated category $\mathcal{D}$ is an abelian subcategory satisfying the following conditions:

1. if $k_1 > k_2 (\in \mathbb{Z})$ and $A_i \in \mathcal{H}[k_i]$ ($i = 1, 2$), then $\text{Hom}(A_1, A_2) = 0$,
2. for $0 \neq E \in \mathcal{D}$, there is a finite sequence of integers $k_1 > k_2 > \cdots > k_m$ and a collection of exact triangles (known as HN-filtration)

$$0 = E_0 \to E_1 \to \cdots \to E_{m-1} \to E_m = E \quad (2.2)$$

with $A_i \in \mathcal{H}[k_i]$ for all $i$.

It is well-known that

- there is a bijection between the set of bounded t-structures and the set of hearts on a triangulated category, sending a t-structure to its heart.

Note that (2.2) means that a heart $\mathcal{H}$ provides a homology (or a $\mathbb{Z}$-structure) on $\mathcal{D}$, that one can define the $k$-th homology of $E$, with respect to $\mathcal{H}$, to be

$$\mathcal{H}_k(E) = \begin{cases} A_i & \text{if } k = k_i \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

Then $\mathcal{P}$ consists of those objects with no (nonzero) negative homology, $\mathcal{P}^\perp$ those with only negative homology and $\mathcal{H}$ those with homology only in degree 0.

One can refine the (bounded) t-structure ($\mathbb{Z}$-structure) on triangulated categories by further dividing, i.e. to get the $\mathbb{R}$-structure as follows.

**Definition 2.6.** A slicing on $\mathcal{D}$ is a collection of additive subcategories

$$\{ \mathcal{P}(\phi) \subset \mathcal{D} \mid \phi \in \mathbb{R} \}$$

such that, if one defines (by triangulated extension)

$$\mathcal{P}(I) = \langle \mathcal{P}(\phi) \mid \phi \in I \rangle$$

for an interval $I \subset \mathbb{R}$, then
• $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1], \forall \phi \in \mathbb{R}$.
• $\mathcal{P}(\phi, +\infty)$ and $\mathcal{P}(\phi, +\infty)$ are t-structures for any $\phi \in \mathbb{R}$.

Here is Bridgeland’s $C$-upgrading of slicing on triangulated categories.

**Definition 2.7.** [B] A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism

$$Z : K(\mathcal{D}) \to \mathbb{C}$$

called the central charge and a slicing $\{\mathcal{P}(\phi) \mid \phi \in \mathbb{R}\}$ satisfying the following compatible condition:

• if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)e^{\phi \pi i}$ for some $m(E) \in \mathbb{R}_{>0}$.

The (non-zero) objects in the slicing are said to be semistable and the simple objects in (the abelian category) $\mathcal{P}(\phi)$ are said to be stable.

Note that a slicing does not necessarily admit a compatible central charge to form a stability conditions. We will always assume the support property on stability conditions [KoSo], i.e. for some norm $\| \cdot \|$ on $K(\mathcal{D}) \otimes \mathbb{R}$, there is a constant $C > 0$ such that

$$\| \alpha \| < C|Z(\alpha)|$$

for all classes $\alpha \in K(\mathcal{D})$ represented by $\sigma$-semistable objects in $\mathcal{D}$.

**Theorem 2.8.** [B, Theorem 1.2] All stability conditions on a triangulated category $\mathcal{D}$ form a complex manifold, denoted by $\text{Stab}(\mathcal{D})$; each connected component of $\text{Stab}(\mathcal{D})$ is locally homeomorphic to a linear sub-manifold of $\text{Hom}_Z(K(\mathcal{D}), \mathbb{C})$, sending a stability condition $(Z, \mathcal{P})$ to its central change $Z$.

There is a natural $\mathbb{C}$-action on the set $\text{Stab}(\mathcal{D})$ of all stability conditions on $\mathcal{D}$, namely:

$$s \cdot (Z, \mathcal{P}) = (Z \cdot e^{-i\pi s}, \mathcal{P}_{\text{Re}(s)}),$$

where $\mathcal{P}_x(\phi) = \mathcal{P}(\phi + x)$. There is also a natural action on $\text{Stab}(\mathcal{D})$ induced by the auto-equivalence group $\text{Aut} \mathcal{D}$, namely:

$$\Phi(Z, \mathcal{P}) = (Z \circ \Phi, \Phi(\mathcal{P})).$$

The heart $\mathcal{H}(\sigma)$ of a stability condition $\sigma = (Z, \mathcal{P})$ is $\mathcal{P}(0, 1]$. And we will say $\sigma$ is supported on $\mathcal{H}$ if $\mathcal{H}(\sigma) = \mathcal{H}$.

### 3. Tilting as tiling

Following [W, KQ, Q1, QW], we review the idea of (simple) HRS-tilting as tiling to study finite type components of spaces of stability conditions. Let $\mathcal{D}$ be a triangulated category as above with $K(\mathcal{D}) \cong \mathbb{Z}^n$.

#### 3.1. HRS-tilting

The canonical partial order on the set of hearts in $\mathcal{D}$ are defined by

$$\mathcal{H}_1 \leq \mathcal{H}_2 \iff \mathcal{P}_1 \supset \mathcal{P}_2$$

where $\mathcal{P}_i$ are the corresponding t-structures.
**Theorem 3.1.** [HRS] For any torsion pair of a heart \( \mathcal{H} = (\mathcal{T}, \mathcal{F}) \) (in \( \mathcal{D} \)) there exists the following two hearts with torsion pairs
\[
\mathcal{H}^\flat = (\mathcal{F}[1], \mathcal{F}), \quad \mathcal{H}^\flat = (\mathcal{F}, \mathcal{F}[-1]).
\]
We call \( \mathcal{H}^\flat \) the forward tilt of \( \mathcal{H} \) with respect to the torsion pair \( (\mathcal{F}, \mathcal{F}) \), and \( \mathcal{H}^\flat \) the backward tilt of \( \mathcal{H} \).

Clearly \( \mathcal{T} = \mathcal{H} \cap \mathcal{H}^\flat \) and \( \mathcal{F} = \mathcal{H} \cap \mathcal{H}^\flat[-1] \). Also we have \( \mathcal{H}^\flat = \mathcal{H}^\flat[-1] \) and
\[
\mathcal{H}[-1] \leq \mathcal{H} \leq \mathcal{H}^\flat \leq \mathcal{H}[1].
\]

It is well-known that there are canonical bijections between
- the set of torsion pairs in \( \mathcal{H} \),
- the set of hearts in \( \mathcal{D} \) between \( \mathcal{H}[-1] \) and \( \mathcal{H} \),
- the set of hearts in \( \mathcal{D} \) between \( \mathcal{H} \) and \( \mathcal{H}[1] \),

which send a torsion pair to its backward and forward tilts respectively.

**Definition 3.2.** [KQ] A forward tilt of a heart \( \mathcal{H} \) is simple, if, in the corresponding torsion pair \( (\mathcal{T}, \mathcal{F}) \), the part \( \mathcal{F} \) is generated by a single rigid simple \( S \) (and thus \( \mathcal{T} = \perp S \)).

We denote the new heart by \( \mathcal{H}^\flat S \). Similarly, a backward tilt of \( \mathcal{H} \) is simple if \( \mathcal{T} \) is generated by a rigid simple \( S \) (and thus \( \mathcal{F} = S \perp \)). The new heart is denoted by \( \mathcal{H}^\flat S \).

The total exchange graph \( \operatorname{EG}(\mathcal{D}) \) of a triangulated category \( \mathcal{D} \) is the oriented graph whose vertices are all hearts in \( \mathcal{D} \) and whose edges correspond to simple forward tiltings between them.

For instance, Lemma 2.2 says when a simple is rigid, then the corresponding forward/backward simple tilts exist. One of the most important feature of simple tilting is that one can iterated tilt a (rigid) simple (up to shifts). Let \( \mathcal{H} \) be a heart in a triangulated category \( \mathcal{D} \). Denote by \( \operatorname{Sim} \mathcal{H} \) the set of simples in an abelian category \( \mathcal{H} \). For \( S \in \operatorname{Sim} \mathcal{H} \), we set \( \mathcal{H}^\flat S = \mathcal{H} \) and inductively define ([KQ, Def. 5.11])
\[
\mathcal{H}^\flat_S^{m+1} = \left( \mathcal{H}^\flat_S^{m+1} \right)^\flat_S^{m+1} \mathcal{H}^\flat_S^{m+1},
\]
for \( m \geq 1 \), and similarly \( \mathcal{H}^\flat_S^{m-1} \), for \( m \geq 1 \). For \( m < 0 \), we also set \( \mathcal{H}^\flat_S^{m-1} = \mathcal{H}^{-m} \).

The formulae to calculate new simples in a forward/backward simple tilt is given in [KQ, Prop. 5.4].

**3.2. Finite type contractible connected components.** A finite/algebraic heart \( \mathcal{H} \) is a length category with finitely many simples. In this case, the stability conditions that supported on \( \mathcal{H} \), denoted by \( \operatorname{U}(\mathcal{H}) \) is
\[
\operatorname{U}(\mathcal{H}) \cong \mathbb{H}^n, \quad (3.1)
\]
where the coordinates are given by the central charge \( \{ Z(S) \mid S \in \operatorname{Sim} \mathcal{H} \} \) on the simples of \( \mathcal{H} \).

**Definition 3.3.** A finite component \( \operatorname{Stab}_0 \) in a space \( \operatorname{Stab}(\mathcal{D}) \) of stability conditions is the union
\[
\operatorname{Stab}_0 = \bigcup_{\mathcal{H} \in \operatorname{EG}_0} \operatorname{U}(\mathcal{H})
\]
for some connected component $E G_0$ of $E G D$ consisting of hearts that are all finite.

We have the following result.

**Theorem 3.4.** [QW] A finite component $\text{Stab}_0$ is a contractible connected component of $\text{Stab}(D)$.

There are the following three classes of finite components. For simplicity, let $k$ be a fixed algebraically-closed field. Recall that a triangulated category $\mathcal{D}$ is called Calabi-Yau-$N$ (CY-$N$) if, for any objects $L, M$ in $\mathcal{D}$ we have a natural isomorphism

$$
\mathcal{S} : \text{Hom}_{\mathcal{D}}^n(L, M) \cong \text{Hom}_{\mathcal{D}}^n(M, L)^{\vee}[N].
$$

(3.2)

**Example 3.5.** Let $Q$ be a Dynkin quiver (in general $Q$ can be an acyclic quiver). Denote by $\Gamma_N^Q$ the Calabi-Yau-$N$ Ginzburg (dg) algebra associated to $Q$, which is constructed as follows ([K1, § 7.2], [G]):

- Let $Q^N$ be the graded quiver whose vertex set is $Q_0$ and whose arrows are: the arrows in $Q$ with degree 0; an arrow $a^*: j \to i$ with degree $2 - N$ for each arrow $a: i \to j$ in $Q$; a loop $e^*: i \to i$ with degree $1 - N$ for each vertex $e$ in $Q$.
- The underlying graded algebra of $\Gamma_N^Q$ is the completion of the graded path algebra $kQ^N$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $Q^N$.
- The differential of $\Gamma_N^Q$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values on the arrows of $Q^N$:

$$
d a^* = 0, \quad d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*].
$$

Our primary example is when $\mathcal{D} = D_{fd}(\Gamma_N^Q)$, the finite dimensional derived category of $\Gamma_N^Q$, and $\text{Stab}_0 = \text{Stab}(D)$.

**Example 3.6.** Another two classes of finite type components are (cf. [QW, BPP])

- $\mathcal{D}$ is a discrete derived category of finite global dimension and $\text{Stab}_0 = \text{Stab}(D)$, which includes the cases when $\mathcal{D} = D^b(Q)$ for a Dynkin quiver $Q$.
- $\mathcal{D}$ is a locally-finite triangulated category with finite rank Grothendieck group and $\text{Stab}_0$ is any connected component of $\text{Stab}(D)$.

### 3.3. Tiling and stratification.

In this section, we explain the idea of the proof of Theorem 3.4. We will fix a finite component $\text{Stab}_0$ that corresponds to $E G_0$.

First, each finite heart corresponds to a cell in $\text{Stab}(D)$.

**Proposition 3.7.** [B] To give stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a heart with a stability function satisfying the HN-property. Further, to give a stability condition on $\mathcal{D}$ with a finite heart $\mathcal{H}$ is equivalent to give a function $\text{Sim} \mathcal{H} \to \mathbb{H}$, where $\mathbb{H}$ is the upper half plane as in (2.1).

Moreover, simple HRS-tilting corresponds to tiling (cf. [B, W, Q1]) in the sense as below.

**Lemma 3.8.** Let $\mathcal{H}_1, \mathcal{H}_2$ be two finite hearts. Then $U(\mathcal{H}_1) \cap U(\mathcal{H}_2)$ is a codimension one wall of the cells $U(\mathcal{H}_i)$ if and only if $\mathcal{H}_1$ is a simple tilt of $\mathcal{H}_2$. 

This implies the following. Here $\overline{U}$ is the closure of $U$.

**Proposition 3.9.** [Q1, Thm. 3.4] There is a canonical embedding $\iota : EG_0 \to \text{Stab}_0$, unique up to homotopy, such that

- $\iota(\mathcal{H}) \in (U(\mathcal{H}))^\circ$ for any heart $\mathcal{H} \in EG_0$.
- $\iota(s)$ is contained in $(U(\mathcal{H}) \cup U(\mathcal{H}^t))^{\circ}$ and transversally intersects $(U(\mathcal{H}) \cap U(\mathcal{H}^t))^{\circ}$ at exactly one point, for any edge $s : \mathcal{H} \to \mathcal{H}^t$.

Further, there is a surjection $\pi_1 EG_0 \twoheadrightarrow \pi_1 \text{Stab}_0$.

Pushing this idea to the limit, we can construct a stratification of $\text{Stab}_0$ in [QW].

**Construction 3.10.** For $\mathcal{H} \in EG_0$ and $I \subset \text{Sim} \mathcal{H}$, define the stratum

$$U_{\mathcal{H}, I} = \{ \sigma = (Z, \mathcal{P}) \in U(\mathcal{H}) \mid \mathcal{P}(1) = \langle I \rangle \} \subset U(\mathcal{H}).$$

Then $U_{\mathcal{H}, I} \cong \mathbb{H}^{n-i} \times \mathbb{R}_{<0}$ for $i = |I|$. Of course we have

$$\text{Stab}_0 = \bigcup_{\mathcal{H} \in EG_0} U(\mathcal{H}) = \bigcup_{\mathcal{H} \in EG_0} \left( \bigcup_{I \subset \text{Sim} \mathcal{H}} U_{\mathcal{H}, I} \right)$$

Moreover,

$$\overline{U_{\mathcal{H}, I}} = \bigcup_{I \subset K \subset \text{Sim} \mathcal{H}} \partial_K U_{\mathcal{H}, I}$$

for

$$\partial_K U_{\mathcal{H}, I} = \{ \sigma = (Z, \mathcal{P}) \in \overline{U_{\mathcal{H}, I}} \mid Z(S) \in \mathbb{R} \setminus \{0\} \iff S \in K \}.$$ 

The important observation is the following, which encodes stratification structure of $\text{Stab}_0$.

**Proposition 3.11.** [QW, Cor. 3.13] Let $\mathcal{H}, \mathcal{G} \in EG_0$ and $I \subset \text{Sim} \mathcal{H}, J \subset \text{Sim} \mathcal{G}$.

$$U_{\mathcal{H}, I} \subset \overline{U_{\mathcal{G}, J}} \iff \mathcal{H}^I \leq \mathcal{G}^J \leq \mathcal{G} \leq \mathcal{H} \quad (3.3)$$

where $\mathcal{H}^I$ is the backward tilt of $\mathcal{H}$ w.r.t. the torsion pair whose torsion part is generated by $I$ and similarly for $\mathcal{G}^J$.

This implies that the stratum of $\text{Stab}_0$ makes it a (regular and totally-normal) CW-cellular stratified space, see [QW] for details. Then one can prove $\text{Stab}_0$ is contractible as in Theorem 3.4.

**Remark 3.12.** Proposition 3.9 also holds in some general setting, c.f. [BS] and [KQ2, § 4.3] in the marked surface case.
3.4. **On spherical twists: Dynkin case.** One of most crucial classes of auto-equivalences is spherical twists \([ST]\), which provides braid like actions on \(\text{Stab}(D)\).

**Definition 3.13.** An object \(S\) in a CY-\(N\) category \(D\) is \((N-)spherical\) if \(\text{Hom}^*(S, S) = k \oplus k[-N]\) and induces a twist functor \(\phi_S \in \text{Aut} D\), such that
\[
\Psi_S(X) = \text{Cone}(S \otimes \text{Hom}^*(S, X) \to X)
\]
with inverse
\[
\Psi_S^{-1}(X) = \text{Cone}(X \to S \otimes \text{Hom}^*(X, S)^\vee)[-1].
\]

In the case when \(D = D_{fd}(\Gamma_NQ)\), any simple in the canonical heart
\[
\mathcal{H}_N^Q : = \text{mod } \Gamma_NQ.
\]
is spherical and the **spherical twist group** is
\[
\text{ST}(\Gamma_NQ) = \langle \Psi_S \mid S \in \text{Sim } \mathcal{H}_N^Q \rangle \subset \text{Aut } D.
\]
We have the following result. See the survey \([Q4]\) for the full credit on this topic (generalization of braid group, in particular, spherical twist group).

**Theorem 3.14.** \([QW]\) If \(Q\) is a Dynkin quiver and \(N \geq 2\), then \(\text{ST}(\Gamma_NQ)\) is isomorphic to the braid group \(\text{Br}(Q)\), sending the standard generators to the corresponding ones, and acts faithfully on \(\text{Stab}(D_{fd}(\Gamma_NQ))\).

4. **Topological models for categories**

4.1. **Quivers with potential.** Let \((Q, W)\) be a quiver with potential, that is, a directed graph \(Q\) and a linear combination of cycles in \(Q\).

**Definition 4.1.** \([G, K1]\) The Ginzburg dg algebra (of degree 3) \(\Gamma(Q, W)\) associated to \((Q, W)\) is constructed as follows:
- \(Q^3\) is constructed in Example 3.5 that provides underlying graded algebra structure of \(\Gamma(Q, W)\).
- The differential of \(\Gamma(Q, W)\) is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values
\[
d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*], \quad d \sum_{a \in Q_1} a^* = \partial W,
\]
where \(\partial W\) is the full cyclic derivative of \(W\) with respect to the arrows.

There are three triangulated categories associated to \(\Gamma = \Gamma(Q, W)\).
- the **finite dimensional derived category** \(D_{fd}(\Gamma)\) as mentioned above, which is CY-3;
- the **perfect derived category** per \(\Gamma\);
- the **cluster category** \(\mathcal{C}(\Gamma)\) (which is CY-2) that fits into Amiot’s short exact sequence of triangulated categories \(([K1])\)
\[
0 \to D_{fd}(\Gamma) \to \text{per } \Gamma \xrightarrow{\pi} \mathcal{C}(\Gamma) \to 0.
\]
Denote the quotient map by \(\pi\).
4.2. **Marked surfaces.** Following [FST], let $S$ be a *marked surface* with non-empty boundary with a finite set $M$ of marked points on its boundary $\partial S$ and a finite set $P$ of punctures in its interior $S \setminus \partial S$ such that the following conditions hold

- each connected component of $\partial S$ contains at least one marked point,
- the rank

$$n = 6g + 3p + 3b + m - 6$$

of the surface is positive, where $g$ is the genus of $S$, $b$ the number of boundary components, $m = |M|$ the number of marked points and $p = |P|$ the number of punctures.

For the most part of the paper, we will assume $P = \emptyset$ for simplicity. The exceptional case (i.e. results are proved in general setting) will be pointed out.

**Definition 4.2.** An open arc in $S$ is a curve (up to isotopy) in its interior, whose endpoints are marked points in $M$, and which is neither homotopic to a boundary segment nor to a point. Denote by $\text{OA}(S)$ the set of open arcs on $S$.

A triangulation $T$ of $S$ is a maximal collection of open arcs in $S$ which do not intersect each other in the interior of $S$. It is well-known that any triangulation $T$ of $S$ consists of $n$ open arcs and divides $S$ into

$$\aleph = \frac{2n + |M|}{3}$$

triangles.

There are the following correspondences between topological data of $S$ and the associated cluster algebras, which work even when $P \neq \emptyset$ with certain modification (i.e. tagging). See [FST] for details.

**Table 1. Correspondences I**

| Topology                          | Cluster Algebra |
|----------------------------------|-----------------|
| (Tagged) Open arcs in $S$        | $1 \to 1$       | Cluster variables |
| (Tagged) Triangulations of $S$    | $1 \to 1$       | Clusters          |
| (Tagged) Flip                    | $1 \to 1$       | Mutation          |

Let $T$ be a triangulation of $S$. Then there is an associated quiver $Q_T$ with a potential $W_T$, constructed as follows (see, e.g. [FST] and [L] for the precise definition):

- the vertices of $Q_T$ are (indexed by) the arcs in $T$;
- for each triangle in $T$, there are three arrows between the corresponding vertices as shown in Figure 2;
- these three arrows form a 3-cycle in $Q_T$ and $W_T$ is the sum of all such 3-cycles.
4.3. **Cluster categories from marked surfaces.** We have the following result for the cluster categories $\mathcal{C}(\Gamma_T)$. For the unpuncture case, it is due to [BZ] and the general case ($P \neq \emptyset$) is due to [QZ1]. Note that different triangulations will give the same category up to equivalence. By abuse of notation, we will write $\mathcal{C}(S)$ for $\mathcal{C}(\Gamma_T)$ sometimes.

**Theorem 4.3.** [BZ, ZZZ, QZ1] There is a bijection

$$M : \text{OA}(S) \to \mathcal{C}^0(S), \quad \gamma \mapsto M_\gamma,$$

where $\mathcal{C}^0(S)$ is the set of string objects in the cluster category $\mathcal{C}(S)$, with the formula

$$\text{Int}(\gamma_1, \gamma_2) = \dim \text{Hom}^1(M_{\gamma_1}, M_{\gamma_2}).$$

Note that (4.5) is compatible with the fact that $\mathcal{C}(S)$ is CY-2 as interchanging $\gamma_i$ does not effect the value on both hand sides. Moreover, the Auslander-Reiten translation $\tau$ on $\mathcal{C}(S)$, which is the shift functor [1], is realized as the universal (tagged) rotation (cf. [BQ, Def. 3.4]) of OA$(S)$. Then one can apply (4.5) to calculate Hom of any degree.

**Remark 4.4.** We consider a category that admits this type of topological realization is of topological Fukaya type. More precisely, the topological realization means

- An (important) class of objects is bijective to curves/arcs on a surface.
- The basis of homomorphism spaces is bijective to intersections between the corresponding curves/arcs.

In the following, we collect couple of results of this type.

4.4. **Derived categories from decorated marked surfaces.** Recall that any triangulation divides $S$ into $\aleph$ triangles, cf. (4.3).

**Definition 4.5.** [Q2, Def. 3.1] The **decorated marked surface** $S_\triangle$ is a marked surface $S$ with a set $\triangle$ of $\aleph$ decorating points in the interior of $S$. Moreover, we have the following notions.

- A **closed arc** in $S_\triangle$ is a curve (up to isotopy) in $S_\triangle \setminus \triangle$, whose endpoints are different points in $\triangle$. Denote by CA$(S_\triangle)$ the set of simple closed curve in $S_\triangle$.
- An **open arc** in $S_\triangle$ is a curve (up to isotopy) in $S_\triangle \setminus \triangle$, whose endpoints are marked points in $M$, and which is neither homotopic to a boundary segment nor to a point. Denote by OA$(S_\triangle)$ the set of open arcs on $S_\triangle$.
- A **triangulation** $T$ of $S_\triangle$ is a maximal collection of open arcs in $S_\triangle$ that divides $S_\triangle$ into once-decorated triangles. More precisely, $S_\triangle$ will be divided into $\aleph$ triangles, each of which contains exactly one decorating point in $\triangle$. 

There is an obvious forgetful map $F: S_{\Delta} \to S$, that induces a map $F: OA(S_{\Delta}) \to OA(S)$. We choose an initial triangulation $T$ of $S_{\Delta}$ and let $T = F(T)$ be the induced triangulation of $S$. The quiver with potential $(Q_T, W_T)$ and the associated Ginzburg dga $\Gamma_T$ of $T$ are defined to be the ones of $T$.

As $S$ provides a topological model for the cluster category $C(\Gamma_T)$, $S_{\Delta}$ provides a topological model for the finite dimensional/perfect derived categories $D_{fd}(\Gamma_T) \subset \text{per} \Gamma_T$. Similar to the case in § 3.4, any simple in the canonical heart $H_T: = \text{mod} \Gamma_T$ is spherical. Let

$$\text{ST}(\Gamma_T): = \langle \Psi_S \mid S \in \text{Sim} H_T \rangle$$

be the spherical twist group and

$$\text{Sph}(\Gamma_T): = \text{ST}(\Gamma_T) \cdot \text{Sim} H_T$$

be the set of reachable spherical objects.

**Theorem 4.6.** [Q2, QZ2] There is a bijection

$$X : CA(S_{\Delta}) \to \text{Sph}(\Gamma_T)/[1],$$

\[\eta \mapsto X_\eta \mathbb{Z},\]  

with the formula

$$2 \text{Int}(\eta_1, \eta_2) = \text{dim} \text{Hom}^* (X_{\eta_1}, X_{\eta_2}).$$

Note that (4.6) is compatible with the fact that $D_{fd}(\Gamma_T)$ is CY-3 as interchanging $\eta_i$ does not effect the value on both hand sides.

Moreover, we have the following.

**Theorem 4.7.** [Q3, QZ2] There is a bijection

$$\tilde{M} : \text{OA}^\circ(S_{\Delta}) \to \text{Ind}^\circ \text{per} \Gamma_T,$$

\[\tilde{\gamma} \mapsto \tilde{M}_\tilde{\gamma},\]  

where $\text{OA}^\circ(S_{\Delta})$ is the set of open arcs that appear in triangulation obtained from $T$ by repeated flipping and $\text{Ind}^\circ \text{per} \Gamma_T$ is a class of rigid indecomposables in $\text{per} \Gamma_T$, with the formula

$$\text{Int}(\tilde{\gamma}, \eta) = \text{dim} \text{Hom}^* (\tilde{M}_\tilde{\gamma}, X_\eta).$$

for $\tilde{\gamma} \in \text{OA}^\circ(S_{\Delta})$ and $\eta \in \text{CA}(S_{\Delta})$.

**4.5. On the spherical twists: surface case.** The main application of Theorem 4.6 is the following, which identifies the spherical twist group with a subgroup, the braid twist group $\text{BT}(T)$, of the mapping class group $\text{MCG}(S_{\Delta})$ of $S_{\Delta}$. More precisely,

$$\text{BT}(T): = \langle B_\eta \mid \eta \in T^* \rangle$$

is generated by the braid twists $B_\eta$ along the closed arcs $\eta$ in the dual graph $T^*$ of the triangulation $T$, cf. Figure 3.

**Theorem 4.8.** [Q2, Theorem 1] The bijection $X$ in (4.6) induces an isomorphism

$$\iota_T : \text{BT}(T) \cong \text{ST}(\Gamma_T)$$

$$B_\eta \mapsto \Psi_{X_\eta},$$
Figure 3. The dual graph of a triangulation and the braid twist along a closed arc

sending the generators to the corresponding generators, which both can be indexed by vertices of $Q_T$ (i.e. open arcs in $T$).

In summary, we have Table 2 of correspondences.

**Table 2. Correspondences II**

| Topology                          | Category                          |
|-----------------------------------|-----------------------------------|
| Braid twist group $BT(T)$         | Spherical twist group $ST(\Gamma_T)$ |
| Simple closed arcs in $S_\Delta$  | $1\to1$ up to $|1|$ Spherical obj. in $D_{fd}(\Gamma_T)$ |
| Dual Tri. with Whitehead move     | Hearts with simple tilting        |
| graph dual                        | sim.-proj. dual                   |
| Reachable open arcs in $S_\Delta$ | $1\to1$ Reachable ind. in per $\Gamma_T$ |
| Triangulations with flip          | Silting objects with mutation     |
| forgetful map $S_\Delta$          | forgetful map $S$                 |
|                                  | quotient map $\mathbb{C}(\Gamma_T)$ |
| Open arcs in $S$                  | $1\to1$ Rigid ind. in $\mathbb{C}(\Gamma_T)$ |
| Triangulations with flip          | Cluster tilting objects with mutation |

4.6. **Topological Fukaya categories.** There is a closely related result in [HKK], which is in the same spirit of Theorem 4.3 and Theorem 4.6. Let $S$ be a graded flat surface, $GA(S)$ the set of graded arcs on $S$, $TFuk(S)$ the associated topological Fukaya category (see [HKK, § 2, § 3] for details). They prove the following (we only give a simplified version).
Theorem 4.9. [HKK] There is a bijection
\[ \text{GA}(S) \rightarrow \text{IndTFuk}(S). \] (4.10)

The corresponding \( \text{Int} = \dim \text{Hom} \) formula will be proved in [IQZ].

Example 4.10. The prototype for TFuk is when \( S \) is a disk and \( \text{TFuk}(S) \cong D^b(A_n) \) for an \( A_n \) quiver.

5. Stability conditions via quadratic differentials

In this section, we review two important results realizing spaces of stability conditions as moduli spaces of quadratic differentials. In both cases, the key idea is the following formula
\[ Z(S) = \int \sqrt{\phi}, \] (5.1)
that the central charge \( Z \) of a semistable object \( S \) should be given by the the period (integration) of some differential (i.e. \( \sqrt{\phi} \)) along some curve \( \tilde{\eta} \) — the double cover of a closed arc \( \eta \) corresponds to \( S \). The correspondence
\[ \eta \leftrightarrow S \]
is in fact the ones in Theorem 4.6 and Theorem 4.9.

5.1. Calabi-Yau-3 case. Let \( X \) be a compact Riemann surface and \( \omega_X \) its holomorphic cotangent bundle. A meromorphic quadratic differential \( \phi \) on \( X \) is a meromorphic section of the line bundle \( \omega_X^2 \). We will only consider signed GMN differentials \( \Phi = (\phi, \zeta) \) on \( X \), consists of a meromorphic quadratic differential \( \phi \) whose zeros are all simple (order is one) and a function \( \zeta : \text{Pol}_2(\phi) \rightarrow \{\pm 1\} \) on the set of order 2 poles of \( \phi \). Denote by \( X^\Phi \) the real blow-up of \( X \) at the poles with order at least 3. Denote by \( \text{Quad}(S) \) the moduli space of signed quadratic differentials on \( S \), consisting of a signed GMN differentials \( \Phi \) on some Riemann surface \( X \) and a diffeomorphism \( S \rightarrow X^\Phi \) (sending \( P \) to the set of poles with order one or two).

In [BS], they prove the following.

Theorem 5.1. [BS, Thm. 11.2] As complex manifolds, there is an isomorphism
\[ \text{Stab}^\delta(D_{fd}(\Gamma_T))/\text{Aut}^\delta D_{fd}(\Gamma_T) \cong \text{Quad}^\times(S). \] (5.2)

When \( P = \emptyset \), there is no sign involved (so \( \text{Quad}^\times(S) = \text{Quad}(S) \)). Moreover, there are couple of variations (framed version) of (5.2) given in [KQ2]. In particular, we have the following.

Theorem 5.2. [KQ2, Thm. 4.14 and Thm. 4.16] Let \( S_\Delta \) be a decorated marked surface with a triangulation \( T \). Then
\[ \text{Stab}^\delta(\Gamma_T) \cong F\text{Quad}^T(S_\Delta) \]
is simply connected, where \( F\text{Quad} \) is the \( S_\Delta \)-framed version of \( \text{Quad}(S) \).

Also, [I] generalizes Theorem 5.1 to CY-\( N \) case when \( S \) is a polygon (and the quiver with potential is of type \( A_n \)).
5.2. **Non-Calabi-Yau case.** Building on Theorem 4.9, there is a similar result of Theorem 5.1 as follows.

In [HKK], they prove the following.

**Theorem 5.3.** [HKK, Thm. 5.3] *As complex manifolds, there is an isomorphism*

\[
\text{Quad}(S) \cong \text{Stab}^\bullet(\text{TFuk}(S)),
\]

*where Quad(S) is the S-framed quadratic differential with exponential type singularities and Stab^\bullet(\text{TFuk}(S)) is the union of certain connected components of Stab(\text{TFuk}(S)).*

Note that when S is a disk with three marked points/boundary components,

\[
\text{TFuk}(S) \cong \mathcal{D}^b(A_2)
\]

and the result above is shown in [BQS].

5.3. **Further studies.** Clearly, the results in the two cases, namely,

- Theorem 4.6 for category and Theorem 5.1 for geometry in CY-3 case;
- Theorem 4.9 for category and Theorem 5.3 for geometry in non-CY case,

are of the same sort. In the upcoming series of works [IQ1, IQ2, IQZ], we will introduce new categories and q-deformation of stability conditions to relate them in precise statements, both categorically and geometrically.

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