D-BRANES IN AN AdS$_3$ BACKGROUND

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ABSTRACT. We study the possible D-brane configurations in an AdS$_3 \times S^3 \times T^4$ background with a NS-NS B field. We use its WZW model description and the boundary state formalism, and we analyse the bosonic and the $N=1$ supersymmetric cases separately. We also discuss the corresponding classical open string sigma model. We determine the spacetime supersymmetry preserved by the supersymmetric D-brane configurations.

1. Introduction

In the framework of the AdS/SYM correspondence conjectured by Maldacena in [1], important particles and vacuum configurations (including interaction vertices) in the gauge theory are described, on the string theory side, by D-brane configurations (wrapped or not) in the corresponding type IIB background. Thus in [2] Witten showed how various D-brane configurations in the AdS$_5 \times S^5$ and AdS$_5 \times \mathbb{C}P^5$ backgrounds are associated with the baryon vertex and the Pfaffian particle, respectively, in the four-dimensional gauge theory at the boundary of AdS$_5$. Later, in [3, 4], such D-brane configurations were obtained as classical solutions of the Born-Infeld action.

Although the general belief seems to be that the AdS/SYM connection only makes sense in the Higgs branch of the SYM theory, a proposal was made [5] to identify certain vacuum configurations of the large $N$ limit of the $N=4$ SU($N$) SYM corresponding to the Coulomb branch with particular D3-brane configurations in the bulk of the AdS$_5 \times S^5$, whereby the D3-branes have worldvolumes parallel to the AdS boundary. This suggestion was further pursued in [6] where, by using a low energy analysis, more general D3-brane configurations were analysed in the AdS bulk, and the corresponding amount of supersymmetry preserved by these D-brane configurations was determined.

In the face of these facts it appears desirable to carry out a microscopic SCFT analysis of the possible D-brane configurations in various AdS-type backgrounds. Such an analysis however would rely essentially on the explicit knowledge of the CFT underlying these string backgrounds. Unfortunately, despite the progress made in constructing type IIB string theories with RR background fields on AdS spaces...
in the framework of the Green-Schwarz formalism), we are still lacking a satisfactory worldsheet description for most of these backgrounds. The only tractable case so far remains that of type IIB string theory on AdS$_3 \times K$, where $K$ is a compact manifold, and we allow only a NS-NS $B$ field. These backgrounds have received a great deal of attention recently, in the cases where $K$ is $S^3 \times T^4$ [13, 14], or $S^3 \times S^3 \times S^1$ [15], and have been studied in detail by using perturbative methods.

Here we will initiate a study of the possible D-branes which can be formulated consistently in a superstring background defined on

$$\text{AdS}_3 \times S^3 \times T^4.$$  

We will restrict ourselves to the case with a purely NS-NS $B$ field. This will allow us to use the known superconformal theory (SCFT) underlying this background, in order to apply the boundary state formalism [16, 17, 18, 19, 20, 21, 22, 23].

The paper is organised as follows. In Section 2 we will start with a short summary describing the bosonic background, in order to set the notation and exhibit the conformal structure. In Section 3 we discuss the boundary state formalism adapted to this particular model. We consider two classes of gluing conditions: they both satisfy the basic requirement of conformal invariance, but are distinguished by the amount of the bulk symmetry they preserve. In Section 4 we analyse the geometry of the resulting D-brane configurations. We consider the boundary conditions that each of the two classes of gluing conditions give rise to, and determine the types of D-branes they describe. The configurations described by gluing conditions which preserve the infinite-dimensional symmetry underlying the WZW model generically describe conjugacy classes translated by elements of the group. This result agrees in part with the similar analysis in [23]. The other type of gluing conditions gives rise to more complicated D-brane configurations which include open submanifolds of dimension equal to the dimension of the target manifold, subgroups and cosets; although a direct comparison appears to be rather difficult, the two classes of D-brane configurations obtained here bear a certain similarity to the ones obtained in [24] by analysing the open string WZW model.

In Section 5 we briefly discuss the classical boundary conditions produced by the corresponding open string sigma model in order to investigate the possibility of having D-branes whose worldvolume fills the entire target space. We find that this depends crucially on whether the corresponding group is compact or not.

In Section 6 we extend our analysis to the supersymmetric case, and determine which D-brane configurations admit an $\mathcal{N}=1$ supersymmetric generalisation. In Section 7 we discuss the fraction of spacetime supersymmetry preserved by these configurations. We end with a brief
discussion and summary of results, comparing our results with known D-brane configurations in other AdS backgrounds. The paper also contains an appendix (written with JM Figueroa-O’Farrill) in which we investigate the conjugacy classes of SL(2, R).

2. The bosonic AdS$^3 \times S^3 \times T^4$ background

The background we are interested in appears in type IIB string theory on $\mathbb{R}^3 \times \mathbb{R}^2 \times T^4$ as the near-horizon limit of a system of $p$ fundamental strings stretched along the $\mathbb{R}^3$ factor and $k$ NS-fivebranes stretched along the same $\mathbb{R}^2$ and wrapped on the $T^4$. Its metric corresponds to an exact string background consisting of a flat space piece corresponding to $T^4$, a level $k$ SU(2) WZW theory and a level $k$ SL(2, $\mathbb{R}$) WZW theory. We therefore start with a WZW model having as the target space a direct product of group manifolds $G = G_1 \times G_2$, where $G_1 = \text{SL}(2, \mathbb{R})$, $G_2 = \text{SU}(2)$. The corresponding WZW action will therefore be a sum of two independent terms

$$I = I_{\text{SU}(2)}[g_2] + I_{\text{SL}(2, \mathbb{R})}[g_1], \quad (1)$$

where each term is of the form

$$\frac{2}{k} I[g_i] = \int_{\Sigma} \langle g_i^{-1} \partial g_i, g_i^{-1} \bar{\partial} g_i \rangle + \frac{1}{k} \int_{B} \langle g_i^{-1} dg_i, [g_i^{-1} dg_i, g_i^{-1} dg_i] \rangle.$$

The full action of the theory includes, of course, a term corresponding to $T^4$ and describing four commuting (compact) bosons. Each of the fields $g_i$ is a map from a closed orientable Riemann surface $\Sigma$ to the Lie group $G_i$, $i = 1, 2$. We denote by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ the corresponding Lie algebra, where $\mathfrak{g}_1 = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{g}_2 = \mathfrak{su}(2)$. For these algebras we choose the following bases of generators: $\{X_a\}$ for $\mathfrak{g}_1$, and $\{Y_a\}$ for $\mathfrak{g}_2$ satisfying

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2,$$

and

$$[Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2.$$

We also need to specify an invariant metric on $\mathfrak{g}$, which has a diagonal form

$$\eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix},$$

with components $(\eta_1)_{ab} \equiv \langle X_a, X_b \rangle = \text{diag}(+, +, -)$ and $(\eta_2)_{ab} \equiv \langle Y_a, Y_b \rangle = \text{diag}(+, +, +)$. 
We will use the following parametrisation\footnote{This parametrisation of AdS$_3$ is different from the one given by the Gauss decomposition,}
for the group manifold $G$:
\[ g_1 = e^{\theta_2 X_2} e^{\theta_1 X_1} e^{\theta_3 X_3}, \quad g_2 = e^{\phi_2 Y_2} e^{\phi_1 Y_1} e^{\phi_3 Y_3}, \] (2)
where $\theta_\mu$ and $\phi_\mu$, $\mu = 1, 2, 3$, play the rôle of the spacetime fields. In terms of them (1) becomes a sigma-model action, with the spacetime metric and 2-form given by
\[ ds_1^2 = d\theta_1 d\theta_1 + d\theta_2 d\theta_2 - d\theta_3 d\theta_3 + 2 \sinh \theta_1 d\theta_2 d\theta_3, \]
\[ dB_1 = -\cosh \theta_1 d\theta_1 d\theta_2, \]
on $\text{SL}(2, \mathbb{R})$, and
\[ ds_2^2 = d\phi_1 d\phi_1 + d\phi_2 d\phi_2 + d\phi_3 d\phi_3 - 2 \sin \phi_1 d\phi_2 d\phi_3, \]
\[ dB_2 = \cos \phi_1 d\phi_1 d\phi_2 d\phi_3, \]
for $\text{SU}(2)$. (The torus will have of course flat metric and no $B$ field.)

The exact conformal invariance of this model is based, as is well known, on its infinite-dimensional symmetry group $G(z) \times G(\bar{z})$ characterised by the conserved currents
\[ I_a(z) = -k \partial g_1 g_1^{-1} \] and \[ I_\bar{a}(\bar{z}) = k g_1^{-1} \partial g_1 \]
corresponding to $\mathfrak{sl}(2, \mathbb{R})$, and
\[ J_a(z) = -k \partial g_2 g_2^{-1} \] and \[ J_\bar{a}(\bar{z}) = k g_2^{-1} \partial g_2 \]
corresponding to $\mathfrak{su}(2)$. These currents generate an affine Lie algebra $\hat{g}_1 \oplus \hat{g}_2$, with $\hat{g}_1$ described by
\[ I_a(z) I_b(w) = \frac{k(\eta_{1})_{ab}}{(z-w)^2} + \frac{f_{ab} c I_c(w)}{z-w} + \text{reg}, \] (3)
where the parameter $k$ is related to the level $x$ of the affine algebra by
\[ k = x + g^*, \]
where $g^*$ is the dual Coxeter number. Similarly, for $\hat{g}_2$ we have
\[ J_a(z) J_b(w) = \frac{k(\eta_{2})_{ab}}{(z-w)^2} + \frac{f_{ab} c J_c(w)}{z-w} + \text{reg}, \] (4)
whereas the free bosons on the torus, described by the fields $\varphi_i$, $i = 1, 2, 3, 4$, satisfy the standard OPEs
\[ \partial \varphi_i(z) \partial \varphi_j(w) = \frac{\delta_{ij}}{(z-w)^2} + \text{reg}, \] (5)
with similar OPEs for the antiholomorphic sector. The corresponding CFT is then described by the energy-momentum tensor
\[ T = \Omega_{1}^{ab} (I_a I_b) + \Omega_{2}^{ab} (J_a J_b) + \sum_{i=1}^{4} (\partial \varphi_i \partial \varphi_i), \]
where $\Omega_1^{ab}$ and $\Omega_2^{ab}$ are components of the inverse of the following invariant metric

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix},$$

with the components given by $\Omega_1 = 2(k + 1)\eta_1$ and $\Omega_2 = 2(k - 1)\eta_2$. The central charge of this CFT is given by

$$c = \frac{3k}{k+1} + \frac{3k}{k-1} + 4.$$

The choice of equal levels for $\hat{g}_1$ and $\hat{g}_2$ was motivated by the fact that, in this case, the corresponding $N=1$ supersymmetric background is a critical superstring theory, as we will see in Section 6.

3. Boundary states

The boundary state formalism (see, e.g., [25, 26, 27, 28]) has become in the last years one of the main approaches to the study of D-branes in type II string backgrounds [16, 17, 18, 19, 20, 21]. Using an explicit knowledge of the CFT underlying a given string background, one describes a D-brane configuration through a set of boundary conditions, relating the left– and the right– moving conformal structures in such a way that conformal invariance is preserved. In the case of a background described by a WZW model the fields in terms of which the conformal structure of the model is realised are the affine currents. It is therefore convenient to impose the boundary conditions on these currents in order to have under control the conformal invariance of the resulting configurations.

In what follows we will consider two different classes of gluing conditions which give rise to different configurations of D-branes. Both of them will be defined in terms of a Lie algebra automorphism, $R : \mathfrak{g} \rightarrow \mathfrak{g}$, which preserves the metric $\eta$; in other words we have

$$[R(Z_a), R(Z_b)] = R([Z_a, Z_b]),$$

$$R^T \eta R = \eta,$$

where $\{Z_a\}$ is a given basis in $\mathfrak{g}$, in terms of which $R$ is given by $R(Z_a) = Z_a R^b_a$. We can now define the two classes of gluing conditions as being the following:

(i) type-$N$ gluing conditions, which can be thought of as a generalisation of the Neumann conditions, are given by

$$J_a(z) + R^b_a \bar{J}_b(\bar{z}) = 0,$$

in order to avoid confusion between the boundary conditions satisfied by the chiral currents, $J$ and $\bar{J}$, and the ones satisfied the field $g$ or its components, we will refer to the boundary conditions on the chiral currents as ‘gluing conditions’, reserving the term of boundary conditions to the fields themselves.

3We will be working here in the open string picture.
at the boundary of the worldsheet. Notice that these gluing conditions do not preserve the infinite-dimensional affine symmetry of the current algebra (correcting a statement made in [22]);

(ii) type-D gluing conditions, which can be thought of as a generalisation of Dirichlet conditions, read

\[ J_a(z) - R^b_a J_b(\bar{z}) = 0 \]

By contrast, this type of gluing conditions does preserve the current algebra of the bulk theory.

These gluing conditions have to satisfy the basic consistency requirement, which is conformal invariance. This means that the holomorphic and the antiholomorphic sectors are related by an automorphism of the corresponding CFT. In the bosonic case, since the automorphism group of the Virasoro algebra is trivial, we impose

\[ T(z) = \bar{T}(\bar{z}) \]

at the boundary. In this case, the requirement of conformal invariance translates into the condition

\[ R^T \Omega R = \Omega \]

for either of the two types of gluing conditions.

Ignoring the flat part of the target space—that is, \( T^4 \)—for which the possible D-branes are known, we take \( \mathfrak{g} \) to be the direct sum of the two Lie algebras \( \mathfrak{g}_1 = \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{g}_2 = \mathfrak{su}(2) \). Because these Lie algebras are different real forms of the same complex Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \), there is no nontrivial homomorphism between them. This implies that the matrix of gluing conditions \( R \), defined by the automorphism \( R : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2 \), must take a block-diagonal form

\[ R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \]

where \( R_1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1 \) and \( R_2 : \mathfrak{g}_2 \rightarrow \mathfrak{g}_2 \). Then from (8) and (9), we deduce that \( R_1 \) and \( R_2 \) must separately preserve the metric on \( \text{SL}(2, \mathbb{R}) \) and \( \text{SU}(2) \) respectively, and therefore \( R_1 \) defines an element of \( \text{O}(2, 1) \), whereas \( R_2 \) is an element of \( \text{O}(3) \).

On the other hand, from (8) it follows that both \( R_1 \) and \( R_2 \) are Lie algebra automorphisms, corresponding to \( \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{su}(2) \), respectively. Explicitly, each of these two automorphism conditions translates into a condition on the corresponding matrix, that is

\[ \det(R_1) = \det(R_2) = 1 \]

which makes \( R_1 \) belong to \( \text{SO}(2, 1) \), and \( R_2 \) to \( \text{SO}(3) \).

These results can be summarised as follows. We have identified two classes of gluing conditions on the group manifold \( \text{SL}(2, \mathbb{R}) \times \text{SU}(2) \) which preserve conformal invariance. Each of them is described in terms of an automorphism of the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2) \), but only
the type-D gluing conditions \( \text{[3]} \) preserve also the infinite-dimensional symmetry of the current algebra of the bulk theory. In what follows we will analyse in detail both these types of configurations in order to identify the D-branes they describe.

4. D-brane solutions and geometry

The geometric interpretation of the gluing conditions defined on the chiral currents of the WZW theory in terms of D-brane configurations is arguably the most subtle issue of this approach. The precise statement of the problem is the following: given a set of gluing conditions for the chiral currents and a fixed but otherwise arbitrary point \( g \) in the target group manifold, find the possible D-branes which pass through \( g \) and are described by these gluing conditions.

The difficulty lies with the fact that the gluing conditions imposed on the affine currents, despite being a natural nonabelian generalisation of the boundary conditions in a free theory are not, strictly speaking, boundary conditions. The flat space boundary conditions are defined in the tangent space of the target manifold and therefore the eigenvalues and eigenvectors of the corresponding matrix \( R \) identify the Neumann and Dirichlet directions. In the group manifold case the gluing conditions take values in the tangent space of \( G \) at the identity, that is \( T_gG \equiv g \), because the currents themselves are Lie algebra valued objects. Hence, in order to interpret geometrically the algebraic gluing conditions we must first of all ‘translate’ them into boundary conditions in \( T_gG \), and then determine what the Neumann and Dirichlet directions are in this case.

In case of the AdS\(_3 \times S^3\) background, the matrix \( R \) has a block diagonal form \( \text{[4]} \), which allows us to analyse the D-brane configurations on AdS\(_3\) and \( S^3 \) separately. Thus \( R_1 \), which is an element of SO(2, 1) can be thought of as a Lorentz transformation in \( \mathbb{R}^{2,1} \). In three dimensions, any Lorentz transformation leaves a vector invariant. Depending on the causal type of this vector, we can distinguish three types of Lorentz transformations:

(i) spatial rotations, typically of the form

\[
R_1 = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \alpha \in \mathbb{R}.
\] (12)

In this case \( R_1 \) has generically one \(+1\) eigenvalue, with the corresponding eigenvector being time-like;

(ii) boosts, typically of the form

\[
R_1 = \begin{pmatrix}
\cosh \alpha & 0 & \sinh \alpha \\
0 & 1 & 0 \\
\sinh \alpha & 0 & \cosh \alpha
\end{pmatrix}, \quad \alpha \in \mathbb{R}.
\] (13)
In this case $R_1$ has generically one +1 eigenvalue, but with the corresponding eigenvector being space-like;

(iii) null rotations, typically of the form

\[
R_1 = \begin{pmatrix}
1 & -a & a \\
-a & 1 - \frac{1}{2}a^2 & \frac{1}{2}a^2 \\
a & -\frac{1}{2}a^2 & 1 + \frac{1}{2}a^2
\end{pmatrix}, \quad a \in \mathbb{R}.
\] (14)

In this case $R_1$ has generically a +1 eigenvalue whose eigenvector is light-like. In contrast with the other two cases, a null rotation is not diagonalisable over the complex numbers. This makes the geometric interpretation of the corresponding configuration relatively difficult.

By contrast, a D-brane configuration in SU(2) is described by an element $R_2$ in SO(3) that is, a rotation in $\mathbb{R}^3$, typically of the form

\[
R_2 = \begin{pmatrix}
\cos \beta & \sin \beta & 0 \\
-\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \beta \in \mathbb{R}.
\] (15)

It is generically characterised by a +1 eigenvalue, whose eigenvector corresponds to the direction which is left invariant by the rotation.

**Type-N configurations.** We now arrive at the basic fact on which our geometric interpretation of the gluing conditions is based. Here we will only state this result (details will appear elsewhere) and proceed to apply it to the case of the group manifold corresponding to AdS$_3 \times S^3$.

Let us parametrise the group manifold $G$ by introducing the coordinates $X^\mu$, with $\mu = 1, ..., \dim G$; we also introduce the left- and right-invariant vielbeins, $e$ and $\bar{e}$, defined by

\[
g^{-1}dg = e^a_\mu \, dX^\mu Z_a, \quad dgg^{-1} = \bar{e}^a_\mu \, dX^\mu Z_a.
\]

The gluing conditions (8) will then give rise to the following boundary conditions at a generic point $g$ in $G$:

\[
\partial X^\mu = \tilde{R}(g)^\mu_\nu \partial X^\nu,
\]

where the matrix of boundary conditions $\tilde{R}(g)$ is given by

\[
\tilde{R}(g) = \bar{e}^{-1} R e.
\]

Notice that $\tilde{R}(g)$, which describes the boundary conditions at a given point in the target space, depends on that point through the invariant vielbeins. One can now identify the Neumann and Dirichlet directions by analogy with the flat space case. At a given point, a Dirichlet boundary condition corresponds to a $-1$ eigenvalue of the matrix $\tilde{R}(g)$, which means that the directions normal to the worldvolume of the D-brane are spanned by the corresponding eigenvectors of $\tilde{R}(g)$. All the other eigenvalues describe Neumann boundary conditions (in the
presence of a $B$ field) and the corresponding eigenvectors span the tangent space of the worldvolume of the D-brane.

We now start with $\text{SL}(2, \mathbb{R})$, and consider the simplest possible case where $R_1 = 1$. In this case the boundary conditions at a point $g_1(\theta_\mu)$ will read

$$\partial \theta^\mu = \tilde{R}_1(g_1)^\mu_\nu \partial \theta^\nu,$$

where $\tilde{R}_1(g_1) = e^{-1} e_1$. If we evaluate the matrix of boundary conditions at the identity we obtain $\tilde{R}_1(e) = 1$, which indicates that we have three Neumann directions spanning the whole $\mathfrak{sl}(2, \mathbb{R})$. In other words, the identity in $\text{SL}(2, \mathbb{R})$ belongs to a D2-brane.

If we now move away from the identity, $\tilde{R}_1$ will no longer be $1$. Instead, we obtain that $\tilde{R}_1$ always has one $+1$ eigenvalue and two complex conjugate eigenvalues. Hence, at generic points in the group manifold, $\tilde{R}_1(g_1)$ will still give rise to three Neumann directions; however there will be a submanifold of $\text{SL}(2, \mathbb{R})$ where $\tilde{R}_1(g_1)$ will have at least one $-1$ eigenvalue (two, in fact, since $\det \tilde{R}_1 = 1$). In other words, at each point on this particular submanifold we will have one Neumann and two Dirichlet directions. This submanifold can be described as the zero locus $F_1$ of a function, $F_1(g_1) \equiv \text{Tr} \tilde{R}_1(g_1) + 1$, which in our parametrisation is given by

$$F_1 = 1 + \cosh \theta_1 \cosh \theta_2 + \cosh \theta_2 \cos \theta_3 + \cos \theta_3 \cosh \theta_1$$

$$+ \sinh \theta_1 \sinh \theta_2 \sin \theta_3.$$

One can show that $F_1$ is a class function; that is, $F_1(h^{-1} gh) = F_1(g)$ for all group elements $g, h$. Indeed, if we consider the vector fields that generate the adjoint action of the group $\text{SL}(2, \mathbb{R})$

$$H_a(g_1) = ((e_1^{-1})^\mu_a - (e_1^{-1})^\mu_a) \partial_\mu, \quad a = 1, 2, 3, \quad (16)$$

we can check that $F_1$ is annihilated by them, that is

$$H_a(g_1) \cdot F_1(g_1) = 0,$$

for all $a = 1, 2, 3$. Hence we have that $\mathcal{F}_1$ consists of adjoint orbits—that is, conjugacy classes.

At every point in $\mathcal{F}_1$ we have one Neumann and two Dirichlet boundary conditions. The vector field corresponding to the Neumann direction therefore spans the worldline of a D-particle. In order for this picture to be consistent we must verify that the worldlines of these D0-branes lie within $\mathcal{F}_1$. Indeed, a straightforward calculation shows that we have

$$V_1(g_1) \cdot F_1(g_1)|_{F_1(g_1) = 0} = 0,$$

which allows us to conclude that $V_1$ is tangent to $\mathcal{F}_1$.

What happens now if we start with a matrix $R_1$ which is different from the identity? First of all we recall that $\mathfrak{sl}(2, \mathbb{R})$ is a simple Lie
algebra, and therefore $R_1$ is an inner automorphism; hence it can be identified with $\text{Ad}_{r_1}$ for some group element $r_1$. As we will see in the next paragraph, the effect of an inner automorphism at the level of the gluing conditions is a translation in the group manifold. More precisely, the submanifold on which $\tilde{R}_1 = e_1^{-1}R_1e_1^{-1}$ has $-1$ eigenvalues, which we denote by $\mathcal{F}_1^{(r_1)}$, is the zero locus of a function $F_1^{(r_1)}$ which satisfies $F^{(r_1)}(g) = F(gr_1^{-1})$. Thus $\mathcal{F}_1^{(r_1)}$ is nothing but the translation of the previous $\mathcal{F}_1$ by the group element $r_1$, that is $\mathcal{F}_1^{(r_1)} = \mathcal{F}_1r_1$. Hence through every point in $\mathcal{F}_1r_1$ passes a D0-brane whose worldline lies on $\mathcal{F}_1r_1$. Moreover, the Neumann eigenvectors tangent to the worldline of the D0-branes in $\mathcal{F}_1r_1$ can be obtained by translating accordingly the corresponding Neumann eigenvectors tangent to the D0-branes in $\mathcal{F}_1$.

A particularly interesting case is the one where $R_1$ itself has two $-1$ eigenvalues (this can be obtained by taking $R_1$ of the form $[12]$ with $\alpha = \pi$). In this case, the corresponding surface $\mathcal{F}_1r_1$ passes through the identity element in $\text{SL}(2, \mathbb{R})$, and therefore there exists a particular D0-brane whose worldline passes through the identity. Its tangent vector takes a particularly simple form at the identity, being given by $V_1 = \partial_{\theta_1}$, as expected. The worldline of this D0-brane is nothing but the subgroup of $\text{SL}(2, \mathbb{R})$ generated by $X_3$. Moreover, the translation of this particular solution gives rise to D0-brane configurations whose worldlines are cosets in $\text{SL}(2, \mathbb{R})$.

We can now analyse the $\text{SU}(2)$ case in a similar fashion. If we start with $R_2 = \mathbb{1}$, the boundary conditions at a point $g_2(\phi_i)$ will read

$$\partial \phi^\mu = \tilde{R}_2(g_2)^\mu_\nu \tilde{\partial} \phi^\nu,$$

where $\tilde{R}_2(g_2) = e_2^{-1}e_2$. At the identity we have $\tilde{R}_2(\mathbb{1}) = \mathbb{1}$, which gives three Neumann directions spanning $\text{su}(2)$. We can therefore conclude that the identity in $\text{SU}(2)$ belongs to an euclidean D2-brane. Away from the identity, $\tilde{R}_2$ will have one $+1$ and two complex conjugate eigenvalues, which generically describe three Neumann directions. Similarly to the previous case, there will be a submanifold $\mathcal{F}_2$ of $\text{SU}(2)$ where $\tilde{R}_2$ has two $-1$ eigenvalues. This submanifold can be described as the zero locus of the function $F_2(g_2) \equiv \text{Tr} \tilde{R}_2(g_2) + 1$ which reads

$$F_2 = 1 + \cos \phi_1 \cos \phi_2 + \cos \phi_2 \cos \phi_3 + \cos \phi_3 \cos \phi_1 + \sin \phi_1 \sin \phi_2 \sin \phi_3.$$

As before, one can show that $F_2$ is a class function, and hence that $\mathcal{F}_2$ consists of conjugacy classes.

One can show, along the same lines, that $\mathcal{F}_2$ is foliated by the worldlines of the euclidean D0-branes whose tangent vectors are given by the eigenvectors $V_2$ of $\tilde{R}_2$ corresponding to the $+1$ eigenvalue.

Finally, if instead of $R_2 = \mathbb{1}$ we take $R_2 = \text{Ad}_{r_2}$ (since, as before, $R_2$ is an inner automorphism) the resulting D-brane configurations can be
understood as translations in the group manifold of the ones obtained in the case $R_2 = 1$. Thus, we obtain in particular euclidean D0-branes whose worldlines lie in $F_2 r_2$.

**Type-D configurations.** This type of gluing gluing conditions (9) have been recently analysed in [23], where the resulting D-brane configurations have been identified with conjugacy classes. Here however, by using a slightly different point of view, we will arrive at partially different conclusions.

As we mentioned before, the reason for which the gluing conditions (9) cannot immediately be interpreted as boundary conditions in the target space is the fact that the currents themselves are Lie algebra valued objects. In order to obtain a boundary condition from the gluing condition (9), we must translate the currents, which take values in $g$, into objects taking values in $T_g G$. In this way one obtains

$$\partial g = R(g) \partial g,$$

with the matrix of boundary conditions given by

$$R(g) = - (\rho_g)_* \circ R \circ (\lambda_g)^{-1},$$

where $\lambda_g$ and $\rho_g$ stand for left- and right-multiplication by $g$ in $G$. The condition (17) now takes place in $T_g G$, and it is the corresponding, point-dependent matrix $R(g)$ which determines the Neumann and Dirichlet directions. Indeed, at a given point $g$ in $G$, a Dirichlet boundary condition corresponds to a $-1$ eigenvalue of $R(g)$, which means that the directions normal to the worldvolume of the D-brane are spanned by the corresponding eigenvectors of $R(g)$. All the other eigenvalues describe Neumann boundary conditions and the corresponding eigenvectors span the tangent space of the worldvolume of the D-brane.

If $R$ is taken to be the identity matrix, then $R(g) = - \text{Ad}_{g^{-1}}$, and the corresponding D-branes can be identified with the conjugacy classes of the group $G$ [23]. Indeed, in this case, and provided that the metric $G$ restricts nondegenerately to the conjugacy class $C$ of $g$, the tangent space at $g$ splits into the tangent space to the conjugacy class and its perpendicular complement, which can be identified with the tangent space to the centraliser subgroup $Z$ of $g$:

$$T_g G = T_g C \oplus T_g Z \quad \text{with} \quad T_g C \perp T_g Z.$$ 

Moreover $\text{Ad}_{g^{-1}}$ restricts to the identity on $T_g Z$, which means that the Dirichlet directions span $T_g Z$. Furthermore, the Neumann directions span $T_g C$, and hence the worldvolume of the D-brane can be identified with $C$.

Let us now consider the case of an arbitrary $R$. In our case, since both $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$ are simple Lie algebras, we can restrict ourselves to the case where $R$ is an inner automorphism, and hence can be identified with $\text{Ad}_r$, for some group element $r$. Therefore the corresponding
boundary conditions can be written in the following form

$$\partial \tilde{g} = - \text{Ad}_{\tilde{g}} \cdot \partial \tilde{g},$$

with $\tilde{g} = gr^{-1}$. This implies that the corresponding D-brane lies along the right–translate $Cr$ of the conjugacy class of $g$ by the element $r$. This result contradicts the statement made in [23] according to which inner automorphisms, being a “symmetry of the model” cannot result in D-brane configurations different from the ones already described by $R = \mathbb{I}$. Although inner automorphisms are symmetries of the background, they are not necessarily symmetries the theory containing a D-brane. This fact is not surprising, as D-branes break some of the bulk symmetries even in flat space (e.g., translational symmetry).

We can now specialise to our particular background where, as usual, we will consider the two groups separately. The conjugacy classes of $SU(2)$ are well known, and have been recently discussed in [23]. We have listed them in Table 1. They are parametrised by $S^1/\mathbb{Z}_2$, which we can understand as the interval $\theta \in [0, \pi]$. The conjugacy classes corresponding to $\theta = 0, \pi$ are points, corresponding to the elements $\pm e$ in the centre of $SU(2)$, whereas the classes corresponding to $\theta \in (0, \pi)$ are spheres. If we picture $SU(2)$, which is homeomorphic to the 3-sphere, as the one-point compactification of $\mathbb{R}^3$ where the sphere at infinity is collapsed to a point, the foliation of $SU(2)$ by its conjugacy classes coincides with the standard foliation of $\mathbb{R}^3$ by 2-spheres with two degenerate spheres at the origin and at infinity.

| Class     | Element          | Topology |
|-----------|------------------|----------|
| $C_e'$    | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | point    |
| $C_e'$    | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | point    |
| $C_\theta$ | $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ | $S^2$    |

Table 1. $SU(2)$ conjugacy classes

Let us now turn to $SL(2, \mathbb{R})$. Its conjugacy classes are computed in the appendix and can be labelled by eight types of $2 \times 2$ matrices. Those classes which are metrically nondegenerate can be interpreted as D-branes and are listed in Table 2. As in the case of $SU(2)$ we have two point-like D-branes corresponding to the two elements in the centre of $SL(2, \mathbb{R})$ as well as a family of euclidean D-strings with planar topology, but in addition there is also a family of D-strings with cylindrical topology.
| Class | Element | Topology |
|-------|---------|----------|
| $\mathcal{C}_e$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | point |
| $\mathcal{C}_{-e}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | point |
| $\mathcal{C}_\theta$ | $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ | $\mathbb{R}^2$ |
| $\mathcal{C}_\lambda$ | $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ | $\mathbb{R} \times S^1$ |

Table 2. SL(2, $\mathbb{R}$) D-branes based on conjugacy classes

Let us now summarise our findings so far. We have seen that type–N gluing conditions give rise to D5-, D3-, and D1-branes in AdS$_3 \times S^3$ whose worldvolumes are of the form $N_1 \times N_2$, with $N_1$ and $N_2$ three– or one–dimensional submanifolds of AdS$_3$ and $S^3$, respectively. Moreover, particular solutions for $N_1$ and $N_2$ include subgroups and cosets of SL(2, $\mathbb{R}$) and SU(2), respectively. By contrast, type–D gluing conditions, for a given $R = Ad_r$, describe D-branes whose “worldvolumes” are shifted conjugacy classes of the form $\mathcal{C}_r \times \mathcal{C}'_{r_2}$, which can be 0-, 2- or 4-dimensional.

5. Relation to the sigma model approach

One of the most surprising results of the boundary state analysis of the possible D-branes in AdS$_3 \times S^3$ is the absence of D-brane configurations which fill the entire group manifold. In this section we pause for a moment our analysis via the boundary state approach to consider the classical sigma model which corresponds to our WZW theory. Our main aim here is to investigate the possibility of having D-branes which fill the whole target space, that is AdS$_3 \times S^3$.

The action of a generic WZW model on a 2-space with a disc topology with an additional interaction (1-form field $A$) at the boundary reads

$$S = \int_{\Sigma} (g^{-1} \partial g, g^{-1} \bar{\partial} g) + \int_{\Sigma} g^* B + \int_{\partial \Sigma} g^* A.$$  \hspace{1cm}(18)

Here the worldsheet $\Sigma$ is a two-dimensional manifold with boundary $\partial \Sigma$, and $B$ represents a particular choice for the antisymmetric tensor field. A D-brane configuration is characterised in this setting (for more details see [24]) by a two-form $\alpha$ living on the worldvolume $D$ of the D-brane (in which the boundary of the string worldsheet $\partial \Sigma$ is included), and satisfying $d \alpha = dB|_D$. Since $d(B - \alpha)|_D = 0$, one can define locally the one-form potential $A$ such that $dA = B - \alpha$. $S$ may
be viewed as a special case of an action for an open string propagating on a group manifold and coupled to $A$ at the boundary. In the case of $\text{AdS}_3 \times S^3$ the action consists of two independent terms, $S_1 + S_2$, such that each of them is of the form (18), only with different target spaces, corresponding to the two groups, $\text{SL}(2, \mathbb{R})$ and $\text{SU}(2)$, respectively. However for the most part we will work with the generic form of the action (18), and we will consider the two components separately only at the very end.

The infinitesimal variation of $S$ contains a bulk term (yielding the same equations of motion as in the closed string case) and a boundary term which reads

$$
\int_{\partial \Sigma} d\tau (g^{-1}\delta g)^a [\eta_{ab}(g^{-1}\partial_\sigma g)^b - i\alpha_{ab}(g^{-1}\partial_\tau g)^b] \bigg|_{\sigma=\pi}^{\sigma=0} = \int_{\partial \Sigma} d\tau \delta X^\mu p_\mu \bigg|_{\sigma=0}^{\sigma=\pi},
$$

where $\eta$ is the generic metric on the group manifold and $X^\mu$ are the coordinates introduced in the previous section. We have denoted by $p_\mu$ the component of the 2-momentum normal to the boundary $\partial \Sigma$ which is given by

$$
p_\mu = G_{\mu\nu} \partial_\sigma x^\nu - i\alpha_{\mu\nu} \partial_\tau x^\nu,
$$

where $G_{\mu\nu} = e^a_{\mu} \eta_{ab} e^b_{\nu}$, $\alpha_{\mu\nu} = e^a_{\mu} \alpha_{ab} e^b_{\nu}$.

Having Neumann boundary conditions in all directions amounts to imposing $p_\mu|_{\partial \Sigma} = 0$, for all $\mu$. In order to compare these conditions with the boundary conditions (8), we must express them in terms of the same quantities—that is, in terms of the conserved currents. For this we use the expressions of the currents in terms of the spacetime fields

$$
J_a = -\eta_{ab} e^b_{\mu} \partial X^\mu, \quad \bar{J}_a = \eta_{ab} e^b_{\mu} \bar{\partial} X^\mu,
$$

in order to rewrite $p_\mu$ as follows:

$$
p_\mu = -\left[\delta^\rho_{\mu} - \alpha_{\mu\nu} G^{\nu\rho}\right] e^a_{\rho} J_a - \left[\delta^\rho_{\mu} + \alpha_{\mu\nu} G^{\nu\rho}\right] e^a_{\rho} \bar{J}_a.
$$

Then the Neumann boundary conditions take the following form

$$
J + M \bar{J} = 0,
$$

where the matrix $M$ depends on the background fields, being given by

$$
M \equiv e^{-T} \frac{1 + \alpha G^{-1}}{1 - \alpha G^{-1}} e^T.
$$

Since we imposed Neumann conditions in all directions, it is natural to expect that we obtain a D-brane whose worldvolume has the same dimension as the dimension of the target space. Notice however that from this we cannot immediately deduce that the D-brane worldvolume
literally fills the whole target space. We will return to this point at the end of this section.

In our case, since all the relevant quantities (that is, the background fields and the corresponding vielbeins) take a block-diagonal form with respect to the two group components, the matrix $M$ will do so as well,

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$ 

Therefore we can compute the two components separately.

5.1. **The $\text{SL}(2, \mathbb{R})$ component.** We can now compute the matrix of boundary conditions $M_1$ corresponding to the open WZW action $S_1$ with target space $\text{SL}(2, \mathbb{R})$. In order to do this we use the parametrisation of $\text{SL}(2, \mathbb{R})$ given by the first expression in (3). Then the invariant vielbeins $e_1$ and $\bar{e}_1$ are given by

$$e_1 = \begin{pmatrix} \cos \theta_3 & - \cosh \theta_1 \sin \theta_3 & 0 \\ \sin \theta_3 & \cosh \theta_1 \cos \theta_3 & 0 \\ 0 & - \sinh \theta_1 & 1 \end{pmatrix},$$

$$\bar{e}_1 = \begin{pmatrix} \cosh \theta_2 & 0 & - \cosh \theta_1 \sin \theta_2 \\ 0 & 1 & \sinh \theta_1 \\ - \sinh \theta_2 & 0 & \cosh \theta_1 \cosh \theta_2 \end{pmatrix}.$$ 

The corresponding background metric will be $(G_1)_{\mu\nu} = e_{\mu}^a (\eta_1)_{ab} e_{\nu}^b$. If we choose $\alpha_1 = \sinh \theta_1 d\theta_2 d\theta_3$ we find that the matrix of the boundary conditions is given by

$$M_1 = \begin{pmatrix} \cosh \theta_2 & 0 & \sinh \theta_2 \\ 0 & 1 & 0 \\ \sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta_1 & \sinh \theta_1 \\ 0 & \sinh \theta_1 & \cosh \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ - \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Alternatively, one can write $M_1$ in a more succinct form by using the adjoint action of the group, in terms of which we have $M_1 = \text{Ad}(e^{\theta_2 X_2} e^{-\theta_1 X_1} e^{\theta_3 X_3})$. One can therefore see that $M_1$ is indeed an element of $\text{SO}(2, 1)$, as we obtained in Section 3, where the parameters are given by the fields themselves.

5.2. **The $\text{SU}(2)$ component.** We now turn to $S_2$, whose matrix of boundary conditions we denote by $M_2$. We use the parametrisation of
SU(2) given by the second expression in (2), and compute the corresponding invariant vielbeins \( e_2 \) and \( \tilde{e}_2 \) which read

\[
e = \begin{pmatrix}
\cos \phi_3 & \cos \phi_1 \sin \phi_3 & 0 \\
-\sin \phi_3 & \cos \phi_1 \cos \phi_3 & 0 \\
0 & -\sin \phi_1 & 1
\end{pmatrix}
\]

\[
\tilde{e} = \begin{pmatrix}
\cos \phi_2 & 0 & \cos \phi_1 \sin \phi_2 \\
0 & 1 & -\sin \phi_1 \\
-\sin \phi_2 & 0 & \cos \phi_1 \cos \phi_2
\end{pmatrix}.
\]

From this we obtain the corresponding background metric, \( (G_2)_{\mu\nu} = e^a_\mu (\eta_2)_{ab} e^b_\nu \). If we now choose \( \alpha_2 = \sin \phi_1 d\phi_2 d\phi_3 \) we find for the matrix of the boundary conditions

\[
M_2 = \begin{pmatrix}
\cos \phi_2 & 0 & \sin \phi_2 \\
0 & 1 & 0 \\
-\sin \phi_2 & 0 & \cos \phi_2
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi_1 & \sin \phi_1 \\
0 & -\sin \phi_1 & \cos \phi_1
\end{pmatrix} \times \begin{pmatrix}
\cos \phi_3 & -\sin \phi_3 & 0 \\
\sin \phi_3 & \cos \phi_3 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Also here we can use the adjoint action of the group to write \( M_2 = \text{Ad}(e^{\phi_2 Y_2} e^{-\phi_1 Y_1} e^{\phi_3 Y_3}) \). In this case the matrix describing the boundary conditions is an element of SO(3), as obtained in Section 3, and again the parameters are given by the fields themselves.

By putting \( M_1 \) and \( M_2 \) together we obtain the matrix of classical boundary conditions for the open string sigma model on AdS_3 \( \times \) S^3. It is important to remark that the classical boundary conditions we obtained in this way are described by a field-dependent automorphism of the corresponding Lie algebra, which preserves the metric. Thus, on the one hand, these configurations do preserve conformal invariance. On the other hand, they give rise to gluing conditions which have a similar form with the type-N gluing conditions introduced in Section 3, the only difference being that, here, they are field-dependent.

We now finally come to the main point of this section, which is to investigate the possibility of having D-branes in AdS_3 \( \times \) S^3 which fill the whole target space. In order to see this, we must analyse the eigenvalues of the matrix of boundary conditions. We therefore need to rewrite the Neumann boundary conditions (20) in a slightly different form:

\[
\partial X^\mu = \tilde{M}^{\mu}_{\nu} \partial X^\nu,
\]

where the field-dependent matrix \( \tilde{M} \) is given by

\[
\tilde{M} = \frac{G + \alpha}{G - \alpha}.
\]

Clearly, in order to have a consistent configuration, \( \tilde{M} \) should not possess \(-1\) eigenvalues, which would correspond to Dirichlet directions.
An explicit calculation shows that \( \tilde{M}_1 \) possesses no \(-1\) eigenvalues, whereas \( \tilde{M}_2 \) does possess \(-1\) eigenvalues, but only at the points where our parametrisation of SU(2) is singular, that is when \( \cos \phi_1 = 0 \). In order to explain these different results, we must take into account that, in spite of our similar treatment of the two groups SL(2, \( \mathbb{R} \)) and SU(2), there is however one important distinction between them: SU(2) is a compact group, whereas SL(2, \( \mathbb{R} \)) is noncompact. Consequently the SU(2) part of our sigma model analysis involves some subtleties, for instance our parametrisation of SU(2) in (2) becomes singular for \( \cos \phi_1 = 0 \), and the two-form \( B \) cannot be not globally defined. This indicates that, even if we impose Neumann boundary conditions in all directions the corresponding D-brane will not fill the whole \( S^3 \), but rather a three-dimensional submanifold of \( S^3 \).

In fact, one can show that these singular points make up a disjoint union of two circles inside \( S^3 \), since

\[
g_2(\pm \pi/2, \phi_2, \phi_3) = e^{\pm \pi/2Y_1} e^{(\phi_3 \mp \phi_2)Y_3},
\]

and hence the corresponding D3-brane in \( S^3 \) is given by the complement of the above circles in \( S^3 \). Although a direct comparison is not easy, this agrees at least morally with [24]. Hence we must conclude that one can construct D-brane configurations which fill the whole group manifold SL(2, \( \mathbb{R} \)), but not the SU(2) manifold.

6. THE \( N=1 \) SUPERSYMMETRIC EXTENSION

Let us start by introducing the \( N=1 \) supersymmetric extension of the affine Lie algebra \( \hat{\mathfrak{g}} \), which we will denote by

\[
\hat{\mathfrak{g}}_{N=1} = \hat{\mathfrak{sl}}(2, \mathbb{R})_{N=1} \oplus \hat{\mathfrak{su}}(2)_{N=1},
\]

with generators \((I_a, \psi_a)\) for the \( \hat{\mathfrak{sl}}(2, \mathbb{R})_{N=1} \) piece satisfying

\[
I_a(z)I_b(w) = \frac{k(\eta_1)_{ab}}{(z-w)^2} + \frac{f_{ab}c I_c(w)}{z-w} + \text{reg}, \tag{22}
\]

\[
I_a(z)\psi_b(w) = \frac{f_{ab}c \psi_c(w)}{z-w} + \text{reg}, \tag{23}
\]

\[
\psi_a(z)\psi_b(w) = \frac{k(\eta_1)_{ab}}{z-w} + \text{reg}. \tag{24}
\]

and \((J_a, \chi_a)\) for the \( \hat{\mathfrak{su}}(2)_{N=1} \) piece satisfying

\[
J_a(z)J_b(w) = \frac{k(\eta_2)_{ab}}{(z-w)^2} + \frac{f_{ab}c J_c(w)}{z-w} + \text{reg}, \tag{25}
\]

\[
J_a(z)\chi_b(w) = \frac{f_{ab}c \chi_c(w)}{z-w} + \text{reg}, \tag{26}
\]

\[
\chi_a(z)\chi_b(w) = \frac{k(\eta_2)_{ab}}{z-w} + \text{reg}. \tag{27}
\]
Apart from this, we also have the contribution of the free fields \((\varphi, \lambda_i)\) on \(T^4\), with the standard OPEs
\[
\partial \varphi_i(z) \partial \varphi_j(w) = \frac{\delta_{ij}}{(z-w)^2} + \text{reg} , \tag{28}
\]
\[
\lambda_i(z) \lambda_j(w) = \frac{\delta_{ij}}{z-w} + \text{reg} . \tag{29}
\]
Then the generators of the \(N=1\) SCA will be given by
\[
T(z) = \frac{1}{2k} \eta^{ab}_1 (\tilde{I}_a \tilde{I}_b) + \frac{1}{2k} \eta^{ab}_2 (\tilde{J}_a \tilde{J}_b) + \frac{1}{2} \sum_{i=1}^4 (\partial \varphi_i \partial \varphi_i) + \frac{1}{2k} \eta^{ab}_1 (\partial \psi_a \psi_b) + \frac{1}{2k} \eta^{ab}_2 (\partial \chi_a \chi_b) + \frac{1}{2} \sum_{i=1}^4 (\partial \lambda_i \lambda_i) ,
\]
\[
G(z) = \frac{1}{k} \eta^{ab}_1 (\tilde{I}_a \psi_b) + \frac{1}{k} \eta^{ab}_2 (\tilde{J}_a \chi_b) + \sum_{i=1}^4 (\partial \varphi_i \lambda_i) - \frac{1}{6k^2} f^{abc}(\psi_a \psi_b \psi_c) - \frac{1}{6k^2} f^{abc}(\chi_a \chi_b \chi_c) ,
\]
where
\[
\tilde{I}_a \equiv I_a - \frac{1}{2k} \eta^{bd}_1 f^{abc}(\psi_c \psi_d) \quad \text{and} \quad \tilde{J}_a \equiv J_a - \frac{1}{2k} \eta^{bd}_2 f^{abc}(\chi_c \chi_d) .
\]
This SCFT has a central charge \(c = 15\). Notice that although so far we have only considered the holomorphic sector, we have a similar structure for the antiholomorphic sector as well. In other words, we have a \((1,1)\) SCFT.

As in the bosonic case, we consider two classes of gluing conditions. The gluing conditions of type-N are given by
\[
J_a(z) + R^b_a \tilde{J}_b(\tilde{z}) = 0 , \quad \psi_a(z) + S^b_a \tilde{\psi}_b(\tilde{z}) = 0 , \tag{30}
\]
whereas the gluing conditions of type-D read
\[
J_a(z) - R^b_a \tilde{J}_b(\tilde{z}) = 0 , \quad \psi_a(z) - S^b_a \tilde{\psi}_b(\tilde{z}) = 0 . \tag{31}
\]
In both cases the coefficients \(R^b_a\) and \(S^b_a\) are defined by \(R, S : g \to g\), with \(R(Z_a) = Z_b R^b_a\) and \(S(Z_a) = Z_b S^b_a\), for any \(Z_a\) in \(g\). These conditions are to be understood as supersymmetric generalisations of the gluing conditions written down in Section 3; therefore, \(R\) is taken to be an automorphism of \(g\) which preserves the metric. Moreover, since we want to obtain supersymmetric configurations, the gluing conditions satisfied by the fermions will undoubtedly be related to the ones of the bosons; however we do not impose here any specific conditions on \(S\).

These gluing conditions have to satisfy a similar consistency requirement as in the bosonic case. In this context, consistency means that the holomorphic SCFT is set equal to the antiholomorphic SCFT up
to an automorphism of the $N=1$ SCA; in other words, at the boundary we must have

$$T(z) = \bar{T}(\bar{z}) \quad \text{and} \quad G(z) = \pm \bar{G}(\bar{z}) .$$

These boundary conditions have been written down previously in [19], in the context of Kazama–Suzuki models.

The first requirement translates into a number of conditions on the matrices $R$ and $S$. Let us start with the type-D boundary conditions. From the quadratic terms in the currents we obtain that

$$R^T \eta R = \eta , \quad S^T \eta R = \pm \eta ,$$

which immediately implies that

$$S = \pm R , \quad (32)$$

as one would expect from supersymmetry. Further, from the cubic terms in the currents we have that

$$[S(Z_a), S(Z_b)] = \pm S([Z_a, Z_b]), \quad [R(Z_a), S(Z_b)] = S([Z_a, Z_b]) ,$$

which, together with (32), implies that

$$[R(Z_a), R(Z_b)] = R([Z_a, Z_b]) . \quad (33)$$

In other words, the conditions that $R$ must satisfy in order for the corresponding type-D configurations to preserve superconformal invariance match exactly the assumptions already made on $R$. Furthermore, it follows that these gluing conditions preserve the infinite-dimensional symmetry of the $N=1$ current algebra (22)-(27).

If we turn now to the type-N gluing conditions, we obtain once again that $S = \pm R$, from the quadratic terms of $T$ and $G$. However, from the cubic terms we obtain

$$[S(Z_a), S(Z_b)] = \mp S([Z_a, Z_b]), \quad [R(Z_a), S(Z_b)] = -S([Z_a, Z_b]) ,$$

which, together with (32), implies that

$$[R(Z_a), R(Z_b)] = -R([Z_a, Z_b]) . \quad (34)$$

This implies that $R$ must be an anti-automorphism, contrary to our Ansatz. From this we deduce that the type-N gluing conditions do not preserve the $N=1$ superconformal invariance of the background. For this reason, in the remaining of this paper, we will concentrate on the type-D configurations.

From the bosonic case we know that $R$ takes a block-diagonal form (11); therefore the condition (32) implies that $S$ must have a similar form

$$S = \pm \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} . \quad (35)$$
This shows that every bosonic type-D configurations that we determined previously can be made into an $N=1$ supersymmetric configuration without having to impose additional conditions.

7. Spacetime supersymmetry

One of the most important properties of D-brane configurations is that they preserve some of the spacetime supersymmetry of the background in which they live, which translates into the fact that they satisfy the BPS condition. In the context of superconformal field theories spacetime supersymmetry appears as a by-product of $N=2$ superconformal invariance, being related, via bosonisation, to the $U(1)$ current. Instead of following this standard approach, here we will analyse the spacetime symmetry preserved by the D-branes we found using a different route, which was described in [13].

We will therefore consider the spacetime supercharges to be constructed directly from the $N=1$ SCFT, by choosing five fermion bilinears and bosonising them into five scalar fields $H_I$, with $I = 1, \ldots, 5$ as follows

$$\partial H_1 = \frac{1}{k} (\psi_1 \psi_2), \quad \partial H_2 = \frac{1}{k} (\chi_1 \chi_2), \quad i \partial H_3 = \frac{1}{k} (\psi_3 \chi_3),$$

$$\partial H_4 = (\lambda_1 \lambda_2), \quad \partial H_5 = (\lambda_3 \lambda_4).$$

The corresponding spacetime supercharges will then read

$$Q = \oint dz e^{\frac{1}{2} \bar{\phi} S(z)},$$

where $\phi$ is the scalar field which appears in the bosonised superghost system of the fermionic string, and

$$S(z) = e^{\frac{1}{2} \sum_{I} \epsilon_I H_I},$$

is a linear combination of the spin fields, where the coefficients $\epsilon_I = \pm 1$ label the possible supercharges, subject to a number of requirements (for a detailed discussion see [13]). Thus, due to the requirement of mutual locality between the various supercharges, and of BRST invariance these coefficients must satisfy the following conditions:

$$\epsilon_1 \epsilon_2 \epsilon_3 = 1, \quad \epsilon_4 \epsilon_5 = 1.$$

This yields eight supercharges for each of the holomorphic and antiholomorphic sectors of the superstring background, which are displayed in Table 3.

Given a certain D-brane configuration, we can use the gluing conditions of the fermionic fields in order to derive the boundary conditions satisfied by the supercharges, and determine, in this way, the fraction of spacetime supersymmetry preserved by that particular boundary
To illustrate, let us consider the case of the configurations described by a matrix $R$, with $R_1$ of the form \((\mathbb{Z})\). The corresponding conditions satisfied by the fermions read

\begin{align*}
\psi_1 - \cos \alpha \bar{\psi}_1 - \sin \alpha \bar{\psi}_2 &= 0, \\
\psi_2 + \sin \alpha \bar{\psi}_1 - \cos \alpha \bar{\psi}_2 &= 0, \\
\psi_3 - \bar{\psi}_3 &= 0, \\
\chi_1 - \cos \beta \bar{\chi}_1 - \sin \beta \bar{\chi}_2 &= 0, \\
\chi_2 + \sin \beta \bar{\chi}_1 - \cos \beta \bar{\chi}_2 &= 0, \\
\chi_3 - \bar{\chi}_3 &= 0,
\end{align*}

where we have systematically ignored a $\pm$ sign coming from \((\mathbb{Z})\), which does not affect the fermion bilinears in the expression of $S(z)$. Therefore we deduce that $H_I = \bar{H}_I$, for $I = 1, 2, 3$. Since we are considering D-branes embedded in AdS$_3 \times S^3$, the boundary conditions corresponding to the fermions on the torus are always the same (that is, Dirichlet), and will therefore give $H_I = \bar{H}_I$, for $I = 4, 5$. From this it follows that

$$Q_\alpha = \bar{Q}_\alpha, \quad \alpha = 1, \ldots, 8 \quad (38)$$

This means that all the corresponding D-brane configurations discussed in Section 4 preserve half of the spacetime supersymmetry of the background.

In order to analyse the spacetime supersymmetry properties of the D-brane configurations described by matrices $R$ with $R_1$ of the form \((\mathbb{Z})\), we need to adopt a slightly different choice for the five fermion bilinears and, thus, for the corresponding scalar fields $H_I$:

\begin{align*}
\partial H_1 &= \frac{1}{k} (\psi_1 \psi_3), \\
\partial H_2 &= \frac{1}{k} (\chi_1 \chi_2), \\
\partial H_3 &= \frac{1}{k} (\bar{\psi}_2 \bar{\chi}_3), \\
\partial H_4 &= (\lambda_1 \bar{\lambda}_2), \\
\partial H_5 &= (\lambda_3 \bar{\lambda}_4).
\end{align*}

Similarly, we obtain that for the corresponding D-brane configurations the supercharges satisfy the same conditions \((\mathbb{Z})\). Due to the non-local nature of the dependence of the spacetime supercharges on the fermionic fields and to the particular form of the boundary conditions satisfied by the fields in the third case of the discussion in Section 4 (that is, where $R_1$ is of the form \((\mathbb{Z})\)), it is rather difficult to determine the fraction of spacetime supersymmetry preserved this type of configurations.

|   | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ | $\epsilon_4$ | $\epsilon_5$ |
|---|-------------|-------------|-------------|-------------|-------------|
| $Q_1$ | +          | +          | +          | +          | +          |
| $Q_2$ | +          | -          | -          | +          | +          |
| $Q_3$ | -          | +          | -          | +          | +          |
| $Q_4$ | -          | -          | +          | +          | +          |

|   | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ | $\epsilon_4$ |
|---|-------------|-------------|-------------|-------------|
| $Q_5$ | +          | +          | +          | -          |
| $Q_6$ | +          | -          | -          | -          |
| $Q_7$ | -          | +          | -          | -          |
| $Q_8$ | -          | -          | +          | -          |

Table 3. The spacetime supercharges of the holomorphic sector.
To summarise, we have obtained that all the D-brane configurations which preserve the superconformal invariance of the background and have a geometrical description give rise to BPS states preserving half of the spacetime supersymmetry.

Let us conclude this section with a remark. Every boundary state that we identified and which gives rise to a D$p$-brane in AdS$_3 \times S^3$ (as shown in Table 2), can also describe (with appropriate boundary conditions in the ‘flat’ directions) D$(p+2)$- and D$(p+4)$-branes wrapped on $T^2$ and $T^4$, respectively.

8. Discussion

In this paper we have studied, using the SCFT framework and the boundary state formalism, the possible D-brane configurations which can be consistently defined in an AdS$_3 \times S^3$ background characterised by a purely NS-NS B field. We have seen that at the bosonic level one can define two classes of gluing conditions which preserve conformal invariance, which we called type–N and type–D conditions. Type–D gluing conditions have the additional property that they preserve the infinite-dimensional symmetry of the bulk theory generated by the chiral currents.

In order to determine the geometry of the corresponding D-brane configurations we had to first obtain the boundary conditions encoded in the algebraic gluing conditions. Then, by analysing these boundary conditions, we were able to show that type–D gluing conditions describe D-branes whose worldvolume is given by shifted conjugacy classes in the group manifold. Furthermore, this type of configurations admits an $N=1$ supersymmetric generalisation which preserves not only the superconformal invariance of the corresponding background, but also the underlying $N=1$ affine superalgebra of the bosonic and fermionic currents. By contrast, type–N gluing conditions do not preserve the current algebra of the bulk theory. They describe D-brane configurations which are slightly more difficult to characterise geometrically, in the most general case. We have however seen that we obtain in particular D-branes whose worldvolume is an open submanifold of dimension equal to the dimension of the target manifold, and also some subgroups and cosets. This type of D-brane configurations does not seem to generalise to the supersymmetric case, in the sense that it does not yield superconformal configurations.

The two classes of bosonic D-brane configurations found here bear a certain degree of similarity with the D-brane configurations obtained in [24] using a open string WZW model analysis. There it was moreover shown that the two distinct classes of D-brane configurations are related by Poisson–Lie T–duality. It would be interesting to investigate this possible relationship also in our setting.
All D-brane configurations whose spacetime supersymmetry properties we have been able to analyse have in common the fact that the complex structure (implicitly defined through the choice of fermion bilinears) on the ten-dimensional background gives rise, when restricted to the tangent space of the worldvolume of a given D-brane, to a complex structure on the corresponding submanifold of the target. In other words, these D-brane configurations correspond to pseudocomplex cycles in the sense of [19].

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APPENDIX A. CONJUGACY CLASSES OF SL(2, R)
(with JM Figueroa-O’Farrill)

In this appendix we determine the conjugacy classes of the noncompact Lie group SL(2, R). This is probably a classic result, but we are unaware of any reference. We also analyse their causal structure relative to the natural bi-invariant metric on the group.

A.1. Jordan normal forms. We will think of SL(2, R) as the group of 2 × 2 real matrices with unit determinant:

\[ \text{SL}(2, \mathbb{R}) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid ad - bc = 1 \right\} \]

This shows that SL(2, R) is a three-dimensional Lie group, which can be represented as a hyperboloid in \( \mathbb{R}^4 \).

Let us embed SL(2, R) in GL(2, R), and in this way think of every element in SL(2, R) as the matrix of a linear transformation in \( \mathbb{R}^2 \) relative to some basis. Conjugation by GL(2, R) will then correspond to a change of basis in \( \mathbb{R}^2 \). The GL(2, R)-orbits in SL(2, R) will then be labelled by, say, normal forms of the linear transformations. One such normal form is the Jordan normal form. Although this usually is presented in a way that requires a complex change of basis—that is, conjugation in GL(2, \mathbb{C})—it is easy to restrict oneself to real changes of basis.

According to the main theorem in the theory of Jordan normal forms, any 2 × 2 complex matrix is conjugate under GL(2, \mathbb{C}) to one of the following normal forms:

\[ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \]

If we start with a matrix which actually belongs to SL(2, R), the normal form must have unit determinant and real trace, since the trace and
determinant are invariant under conjugation. That means that the normal forms of matrices in $\text{SL}(2, \mathbb{R})$ are of the form

$$
\begin{pmatrix}
\lambda & 0 \\
0 & 1/\lambda
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\pm1 & 1 \\
0 & \pm1
\end{pmatrix},
$$

where $\lambda + 1/\lambda$ is real.

If the resulting normal form is real, then it is plain to see that the matrices are actually conjugate in $\text{GL}(2, \mathbb{R}) \subset \text{GL}(2, \mathbb{C})$. To see this let $M$ and $M'$ be real $2 \times 2$ matrices which are conjugate under $\text{GL}(2, \mathbb{C})$. This means that there exists some matrix $S \in \text{GL}(2, \mathbb{C})$ such that

$$
MS = SM'.
$$

We can interpret this as a system of homogeneous linear equations for the entries of $S$ with real coefficients. Because the constraint that the determinant of $S$ be nonzero is an open condition, it means that we can choose the entries of $S$ to be real. (Of course, there will be other choices of $S$ which are complex.)

If the resulting normal form is not real, then it is necessarily of the form

$$
\begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{pmatrix}
$$

with $\theta$ real.

This matrix is itself conjugate under $\text{GL}(2, \mathbb{C})$ to the real matrix

$$
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}.
$$

Hence the original matrix in $\text{SL}(2, \mathbb{R})$ is conjugate to the above matrix under $\text{GL}(2, \mathbb{C})$ and by the previous argument, the conjugation can actually be taken to be in $\text{GL}(2, \mathbb{R})$.

In summary, the $\text{GL}(2, \mathbb{R})$ orbits of $\text{SL}(2, \mathbb{R})$ are labelled by the following matrices

$$
\begin{pmatrix}
\lambda & 0 \\
0 & 1/\lambda
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\pm1 & 1 \\
0 & \pm1
\end{pmatrix},
$$

(39)

where we can choose $\theta$ in $[0, \pi]$ and $\lambda$ real with $0 < |\lambda| \leq 1$. These choices correspond to the choice in ordering the eigenvalues of the (diagonalisable) normal forms.

In order to recover the $\text{SL}(2, \mathbb{R})$ conjugacy classes from these $\text{GL}(2, \mathbb{R})$ orbits it is necessary to decompose every $\text{GL}(2, \mathbb{R})$ orbit into $\text{SL}(2, \mathbb{R})$ orbits. Let $M$ be a matrix in $\text{SL}(2, \mathbb{R})$ and let $\mathcal{O}$ denote its $\text{GL}(2, \mathbb{R})$ orbit:

$$
\mathcal{O} := \{ gMg^{-1} \mid g \in \text{GL}(2, \mathbb{R}) \}.
$$
The Lie group $GL(2, \mathbb{R})$ has two connected components corresponding to those elements with positive and negative determinant. Correspondingly $\mathcal{O}$ breaks up into two connected components: $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_-$, where

$$\mathcal{O}_\pm := \{ gMg^{-1} \mid g \in GL(2, \mathbb{R}) \text{ and } \pm \det g > 0 \} .$$

Now every matrix $g \in GL(2, \mathbb{R})$ with $\det g > 0$ can be written as

$$g = \sqrt{\det g} s \quad \text{where} \quad s := \frac{1}{\sqrt{\det g}} g ,$$

where the matrix $s$ has unit determinant and hence belongs to $SL(2, \mathbb{R})$. Since for such a matrix $g$,

$$gMg^{-1} = sMs^{-1} ,$$

we see that $\mathcal{O}_+$ is precisely the $SL(2, \mathbb{R})$-orbit of $M$.

On the other hand, let $g \in GL(2, \mathbb{R})$ with $\det g < 0$. Then we can write it as

$$g = \sqrt{|\det g|} st ,$$

where $t$ is any matrix with determinant $-1$, for example

$$t := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and where $s$ is now given by

$$s := \frac{1}{\sqrt{|\det g|}} gt^{-1} ,$$

and has unit determinant again. Now for such a $g$, we have that

$$gMg^{-1} = stMt^{-1}s^{-1} ,$$

whence $\mathcal{O}_-$ is the $SL(2, \mathbb{R})$-orbit of the matrix $tMt^{-1}$ which belongs to $SL(2, \mathbb{R})$.

In summary we see that the $GL(2, \mathbb{R})$-orbit of a matrix $M \in SL(2, \mathbb{R})$ breaks up in at most two $SL(2, \mathbb{R})$ orbits: that of $M$ and that of $tMt^{-1}$. It might happen that $M$ and $tMt^{-1}$ are actually in the same $SL(2, \mathbb{R})$ orbit, and one has to check this case by case. At any rate, it is now a simple matter to enumerate the conjugacy classes of $SL(2, \mathbb{R})$ from the enumeration of the $GL(2, \mathbb{R})$ orbits in (39). Every matrix in $SL(2, \mathbb{R})$ is conjugate in $SL(2, \mathbb{R})$ to one of the following matrices:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} ,$$

(40)

where to avoid repetition we must now take $\lambda$ real with $0 < |\lambda| < 1$ and $\theta \in [0, 2\pi)$.

The results are summarised in Table 4. There are two conjugacy classes $\mathcal{C}_\pm e$ consisting of a point each, corresponding to the elements $\pm e$ in the centre of $SL(2, \mathbb{R})$. The remaining conjugacy classes are
two-dimensional: four one-parameter families corresponding to $C_\lambda$ for $\lambda \in (0, 1)$ and $\lambda \in (-1, 0)$ and to $C_\theta$ for $\theta \in (0, \pi)$ and $\theta \in (-\pi, 0)$, and four isolated classes $C_{\pm \pm}$.

| Class | Element | Topology | Causal type |
|-------|---------|----------|-------------|
| $C_e$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | point |  |
| $C_{-e}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | point |  |
| $C_{++}$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\mathbb{R} \times S^1$ | degenerate |
| $C_{--}$ | $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ | $\mathbb{R} \times S^1$ | degenerate |
| $C_{+-}$ | $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ | $\mathbb{R} \times S^1$ | degenerate |
| $C_{-+}$ | $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ | $\mathbb{R} \times S^1$ | degenerate |
| $C_\theta$ | $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ | $\mathbb{R}^2$ | euclidean |
| $C_\lambda$ | $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ | $\mathbb{R} \times S^1$ | minkowskian |

**Table 4.** $\text{SL}(2, \mathbb{R})$ conjugacy classes, with typical element, topology and causal type.

A.2. **The geometry of the conjugacy classes.** It is possible to understand the geometry of these conjugacy classes in $\text{SL}(2, \mathbb{R})$. To this end we let us reconsider the embedding of $\text{SL}(2, \mathbb{R})$ as a hyperboloid in $\text{Mat}(2, \mathbb{R}) \cong \mathbb{R}^4$, this time using a different coordinate system for $\mathbb{R}^4$:

$$\text{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} x + u & y + v \\ y - v & x - u \end{pmatrix} \mid x^2 + y^2 = 1 + u^2 + v^2 \right\}.$$

This embedding has the virtue of exhibiting the $\text{SO}(2, 2)$ isometry group of $\text{SL}(2, \mathbb{R})$ manifestly. The isometries are nothing but the product of left and right transformations (modulo the centre which acts trivially):

$$\text{SO}(2, 2) \cong \frac{\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})}{\mathbb{Z}_2}.$$

For our present purpose, a more interesting property of this embedding is that the coordinate $x$ equals half the trace, which is an invariant of
the conjugacy class. Therefore the conjugacy classes will be contained in the intersection of the three-dimensional hyperboloid

\[ H_3 : \quad x^2 + y^2 = 1 + u^2 + v^2 \]  

(41)

with the affine hyperplanes \( x = \text{constant} \). We can distinguish several regions of interest: \(|x| < 1\), \(|x| = 1\), and \(|x| > 1\). We will now analyse each of them in turn. The results are illustrated in Figure 1.

![Figure 1. The different types of conjugacy classes of SL(2, R).](image)

We start with the regions \(|x| > 1\). The intersection of the affine hyperplane \( x = \text{constant} \) with \( H_3 \) is given by \((x^2 - 1) + y^2 = u^2 + v^2\) which is a two-dimensional one-sheeted hyperboloid. It is connected and hence it is a conjugacy class: the class \( C_\lambda \), where \( x = \lambda + 1/\lambda \).

The intersection of \( H_3 \) with the affine hyperplane \( x = \pm 1 \) is the light-cone \( y^2 = u^2 + v^2 \). This breaks up into three conjugacy classes: the apex of the cone, which consists of the class \( C_{\pm} \), the upper light-cone \( y > 0 \) with the apex removed, which is the class \( C_{\pm +} \) and the lower light-cone \( y < 0 \) with the apex removed, which is the class \( C_{\pm -} \).

Finally, for \(|x| < 1\), the intersection of \( H_3 \) with the affine hyperplane \( x = \text{constant} \) is the two-dimensional two-sheeted hyperboloid \( y^2 = u^2 + v^2 + (1 - x^2) \). Each sheet is one conjugacy class. If we let \( x = \cos \theta \) then the upper sheet is the class \( C_\theta \) when \( \theta \in (0, \pi) \) and the lower sheet is the class \( C_{\bar{\theta}} \) when \( \theta \in (-\pi, 0) \).

As a concluding comment, let us remark that the method presented here can be employed with a little extra effort to enumerate the conjugacy classes of SL\((n, \mathbb{R})\).

A.3. The causal structure of the conjugacy classes. Due to their possible interpretation as D-branes, it is important to establish the
causal structure of the conjugacy classes which were found above: only those classes which are nondegenerate can be straightforwardly interpreted as boundary conditions for strings. The determination of the causal structure is made easy by the fact that these classes are described by the intersection of affine hyperplanes in \( \mathbb{R}^4 \) with the hyperboloid \( H_3 \) in \( \mathbb{R}^4 \) defining the embedding of \( \text{SL}(2, \mathbb{R}) \) in \( \mathbb{R}^4 \). As was mentioned above, this embedding is isometric provided we endow \( \mathbb{R}^4 \) with a split metric of signature \((2,2)\). In the coordinates \((x, y, u, v)\) chosen above, such a metric is given by

\[
d s^2 = d u^2 + d v^2 - d x^2 - d y^2 .
\] (42)

It is then a simple matter to work out the induced metric on the conjugacy classes. Let us now summarise the results. Of course, we only need concern ourselves with those conjugacy classes which are not pointlike.

A.3.1. \( \mathcal{C}_{\pm\pm} \). These conjugacy classes are the deleted halves of the light-cones at \( x = \pm 1 \). They are defined by this equation together with \( y^2 = u^2 + v^2 \). Let us parametrise the conjugacy class by \((\rho, \vartheta)\) in the following way:

\[
x = \pm 1 \quad y = \pm \rho \quad u = \rho \cos \vartheta \quad v = \rho \sin \vartheta .
\]

The induced metric is then given by

\[
d s^2 = \rho^2 d \vartheta^2 ,
\]

which is clearly degenerate. This means that the conjugacy classes \( \mathcal{C}_{\pm\pm} \) (with signs uncorrelated) cannot be interpreted as D-branes, at least straightforwardly.

A.3.2. \( \mathcal{C}_\theta \). These are two-sheeted hyperboloids obtained by intersecting the affine hyperplane defined by constant \( x \) with \( |x| < 1 \) and the hyperboloid \( H_3 \). We parametrise these classes by \((\rho, \vartheta)\) in the following way:

\[
y = \pm \sqrt{\rho^2 + (1 - x^2)} \quad u = \rho \cos \vartheta \quad v = \rho \sin \vartheta .
\]

The induced metric is then given by

\[
d s^2 = \rho^2 d \vartheta^2 + \frac{(1 - x^2)}{\rho^2 + (1 - x^2)} d \rho^2 ,
\]

which is clearly euclidean. Therefore the corresponding D-branes are euclidean D-strings.

A.3.3. \( \mathcal{C}_\lambda \). These are the one-sheeted hyperboloids obtained by intersecting the affine hyperplane defined by constant \( x \) with \( |x| > 1 \) and the hyperboloid \( H_3 \). We parametrise these classes by \((y, \vartheta)\) in the following way:

\[
u = \sqrt{y^2 + (x^2 - 1)} \cos \vartheta \quad v = \sqrt{y^2 + (x^2 - 1)} \sin \vartheta .
\]
The induced metric is then given by
\[ ds^2 = \left( y^2 + (x^2 - 1) \right) \, d\vartheta^2 - \frac{(x^2 - 1)}{y^2 + (x^2 - 1)} \, dy^2 , \]
which is clearly minkowskian. Therefore the corresponding D-branes are D-strings.

References

[1] J. Maldacena, “The large \( N \) limit of superconformal field theories and supergravity.” [hep-th/9711200]
[2] E. Witten, “Baryons and branes in Anti de Sitter space.” [hep-th/9805112]
[3] Y. Imamura, “Supersymmetries and BPS configurations on Anti-de Sitter space.” [hep-th/9807179]
[4] C. Callan, A. Guijosa, and K. Savvidy, “Baryons and string creation from the fivebrane worldvolume action.” [hep-th/9810092]
[5] M. Douglas and W. Taylor, “Branes in the bulk of Anti-de Sitter space.” [hep-th/9807225]
[6] A. Bilal and C. Chu, “D3-brane(s) in AdS\(_5 \times S^5\) and \( N=4,2,1 \) SYM.” [hep-th/9810195]
[7] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in AdS\(_5 \times S^5\) background.” [hep-th/9805028]
[8] I. Pesando, “A \( \kappa \) gauge fixed type IIB superstring action on AdS\(_5 \times S^5\).” [hep-th/9809145, 1998.]
[9] R. Kallosh and J. Rahmfeld, “The GS string action on AdS\(_5 \times S^5\),” Phys. Lett. B443 (1998) 143. [hep-th/9808038].
[10] I. Pesando, “The GS type IIB superstring action on AdS\(_3 \times S^3 \times S^1\) with ramond-ramond charge.” [hep-th/9809164, 1998.]
[11] J. Rahmfeld and A. Rajaraman, “The GS string action on AdS\(_3 \times S^3\) with \( N=4,2,1 \) SYM.” [hep-th/9812062, 1998.]
[12] A. Giveon, D. Kutasov, and N. Seiberg, “Comments on string theory on AdS\(_3\).” [hep-th/9806192]
[13] J. de Boer, H. Ooguri, H. Robins, and J. Tannenhauser, “String theory on AdS\(_3\).” [hep-th/9812046]
[14] S. Elitzur, O. Feinerman, A. Giveon, and D. Tsabar, “String theory on AdS\(_3 \times S^3 \times S^1\).” [hep-th/9811245]
[15] H. Ooguri, Y. Oz, and Z. Yin, “D-branes on Calabi–Yau spaces and their mirrors,” Nucl. Phys. B477 (1996) 407–430. [hep-th/9606112]
[16] K. Becker, M. Becker, D. R. Morrison, H. Ooguri, Y. Oz, and Z. Yin, “Supersymmetric cycles in exceptional holonomy manifolds and Calabi–Yau 4-folds,” Nucl. Phys. B480 (1996) 225. [hep-th/9608116]
[17] M. Kato and T. Okada, “D-branes on group manifolds,” Nucl. Phys. B499 (1997) 583–595. [hep-th/9612148]
[18] S. Stanciu, “D-branes in Kazama–Suzuki models,” Nucl. Phys. B526 (1998) 295–310. [hep-th/9708166]
[19] A. Recknagel and V. Schomerus, “D-branes in Gepner models.” [hep-th/9712180]
[20] J. Fuchs and C. Schweigert, “Branes: from free fields to general backgrounds.” [hep-th/9712257]
[21] S. Stanciu and A. A. Tseytlin, “D-branes in curved spacetime: the Nappi–Witten background,” JHEP 06 (1998) 010. [hep-th/9805006]
[23] A. Alekseev and V. Schomerus, “D-branes in the WZW model.”
hep-th/9812193.

[24] C. Klimčík and P. Severa, “Open strings and D-branes in WZNW models,”
Nuc. Phys. B488 (1997) 653–676. hep-th/9609112.

[25] C. Callan, C. Lovelace, C. Nappi, and S. Yost, “Loop corrections to
superstring equations of motion,” Nuc. Phys. B308 (1988) 221–284.

[26] J. Polchinski and Y. Cai, “Consistency of open superstring theories,” Nuc.
Phys. B296 (1988) 91.

[27] M. Li, “Boundary states of D-branes and dy-strings,” Nuc. Phys. B460
(1996) 351–361. hep-th/9510161.

[28] C. Callan, Jr. and I. Klebanov, “D-brane boundary state dynamics,” Nuc.
Phys. B465 (1996) 473–486. hep-th/9511173.

[29] S. Stanciu (in preparation).

[30] D. Friedan, E. Martinec, and S. Shenker, “Conformal invariance,
supersymmetry and string theory,” Nuc. Phys. B271 (1986) 93–165.

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