$C^{2,\alpha}$-estimate for Monge-Ampere equations with Hölder-continuous right hand side

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Abstract

We present a somewhat new proof to the $C^{2,\alpha}$-apriori estimate for the uniform elliptic Monge-Ampere equations, in both the real and complex settings. Our estimates do not need to differentiate the equation, and only depends on the $C^{\alpha'}$-norm of the right hand side of the equation, $0 < \alpha < \alpha'$.

1 Introduction

Given an uniformly elliptic Monge-Ampere equation, historically, there are various methods to obtain higher order estimates. One of the pioneering work is the celebrated third derivatives by E. Calabi [35], where he requires the solution is of class $C^5$. In 1980s, from a complete different point of view, Evans-Krylov-Safonov [22],[23],[34] gave the famous Schauder estimate. While their method requires less regularity, their estimates still rely on differentiating both sides of the equation. Thus, the estimates they give depend on $W^{1,p}$-norms (for $p$ big) or higher derivative norms of the right-hand side of the equation.

Later, Safonov [40] and Caffarelli [7] discovered the following celebrated $C^{2,\alpha}$-estimate without differentiating the Monge-Ampere equations.

**Theorem 1.1.** Suppose $B$ is a unit ball in $\mathbb{R}^n$, and $u \in C^{2,\alpha'}(\bar{B})$ is a convex function. Suppose
\[ \det u_{ij} = e^f > 0, \text{ where } f \in C^{\alpha'}(\bar{B}). \] Then, for any $\alpha \in (0, \alpha')$, we have
\[ [\nabla^2 u]_{\alpha, \frac{4}{\alpha}} \leq C, \]
where $C$ depends on the $C^0(B)$-norm of $\nabla^2 u$, the $C^{\alpha'}(\bar{B})$-norm of $f$, the dimension $n$, the $\alpha'$, and $\alpha$.

**Remark 1.2.** For $\alpha \in [0,1]$, $| \cdot |_{\alpha, \Omega}$ ( $| \cdot |_{\alpha, \Omega}$) means the $C^\alpha(\bar{\Omega})$-norm (semi-norm) in the domain $\Omega$, $| \cdot |_{k,\alpha, \Omega}$ ( $| \cdot |_{k,\alpha, \Omega}$) means the $C^k,\alpha(\bar{\Omega})$-norm (semi-norm). Most of the notations here follow the conventions in [20].
Remark 1.3. One of the main features of Safonov and Caffarelli’s results is that the $C^{2,\alpha}$-norm of the solution only depends mainly on the $C^{\alpha'}$ of the right hands side, provided the $C^2$-estimate is already obtained (or equivalently, the equation is uniformly elliptic).

Remark 1.4. The above mentioned Schauder estimates are never a complete list of existing beautiful estimates of this kind. Historically, on the Schauder estimates on nonlinear uniformly-elliptic equations, we also have the work of C, Burch [5], J, Kovats [33], Q,B, Huang [30]. More recently, X,J Wang [44] gives a nice Schauder estimate for both linear and nonlinear equations. For more work on Schauder estimates, we refer to the readers to [44] and the references therein.

The corresponding theory of Safonov and Caffarelli’s results in complex settings also has important progress in recent years. Assuming full second derivative bound, Dinew-Zhang-Zhang [20] showed a $C^{2,\alpha}$-estimate for complex Monge-Ampere equations, which only depends on the Hölder continuity of the right hand side. Very recently, a theorem to the same strength as Safonov and Caffarelli’s was proved by Yu Wang [45], which relies on a clever trick to convert the complex Monge-Ampere equation to a real equation, then apply Caffarelli’s more general estimates in [7].

In this note, following [13], [1], we present another proof of Theorem 1.1 and its complex analogue Theorem 1.5. While we believe our presentation/proof contains some new element (i.e., input from Riemannian geometry), the idea of rescaling and Liouville type theorem in general goes back to long time ago, for example, see Leon Simon’s beautiful proof of Schauder estimates for linear operators [41]. For this reason, we hope that our proof to this classical theorem on full nonlinear PDEs, is still somewhat valuable.

Theorem 1.5. Suppose $B$ is a unit ball in $\mathbb{C}^n$, and $\phi \in C^{2,\alpha'}(\overline{B})$ is a pluri-subharmonic function. Suppose

$$\det \phi_{ij} = e^f > 0, \text{ where } f \in C^{\alpha'}(\overline{B}).$$

(2)

Then, for any $\alpha \in (0, \alpha')$, we have

$$[\sqrt{-1} \partial \bar{\partial} \phi]_{\alpha, \frac{n}{4}} \leq C,$$

where $C$ depends on the $C^0(\overline{B})$-norm of $\sqrt{-1} \partial \bar{\partial} \phi$, the $C^{\alpha'}(\overline{B})$-norm of $f$, the dimension $n$, the $\alpha'$, and $\alpha$.

Remark 1.6. Theorem 1.5 is similar to but slightly different from Yu Wang’s theorem in [45]. First, Theorem 1.5 is an aprori estimate (it assumes the solution is in $C^{2,\alpha'}$), while the theorem of Yu Wang is a stronger regularity theorem. Second, the norm bound in Theorem 1.5 does not depend on $|\phi|_{L^\infty}$ i.e the lower order bound on the potential. This is essentially because equation (2) is a geometric equation for the Kähler-metric $\sqrt{-1} \partial \bar{\partial} \phi$. Given
the importance of the Calabi’s, Evan-Krylov-Safonov’s, Safonov’s, and Caffarelli’s Schauder estimates, we hope our new proof is worthwhile to present separately here.

We hope our new proof has further applications in fully nonlinear equations on singular spaces. Actually, our new proof (in section 2) is developed in the proof of Theorem 1.7 in [17] on Kähler-Ricci flows with conic singularities. The purpose of this note is to give a more simplified and direct proof than the one in [17] (from page 13 to page 19), and to show our method also works for real Monge-Ampere equations. Namely, the following theorem in essentially proved in [17] (from page 13 to page 19).

**Theorem 1.7.** Suppose $\beta \in (0, 1)$ and $0 < \alpha' < \min\{\frac{1}{\beta} - 1, 1\}$. Suppose $\bar{B}$ is the unit ball centered at the origin with respect to the model cone metric

$$\omega_{\beta} = \sqrt{-1} \frac{\beta^2}{|z|^{2-2\beta}} dz \wedge d\bar{z} + \sqrt{-1} \sum_{k=2}^{n} dv_k \wedge d\bar{v}_k,$$

which is defined over $\mathbb{C} \times \mathbb{C}^{n-1}$ with cone singularity of angle $2\beta\pi$ along the divisor $\{0\} \times \mathbb{C}^{n-1}$. Suppose $\phi$ is a pluri-subharmonic function in $C^{2,\alpha',\beta}(\bar{B})$ such that

$$\frac{1}{K} \omega_{\beta} \leq \sqrt{-1} \partial \bar{\partial} \phi \leq K \omega_{\beta} \text{ over } \bar{B} \setminus D \text{ for some } K \geq 1.$$

Denote $F_{\phi}$ as $\log(|z|^{2-2\beta} \det \phi_{ij})$, which means

$$\det \phi_{ij} = \frac{e^{F_{\phi}}}{|z|^{2-2\beta}} \text{ over } \bar{B} \setminus D.$$

Then for any $\alpha \in (0, \alpha')$, there exists a constant $C$ depending on $|F_{\phi}|_{\alpha',\beta,B}$, $K$, $n$, $\alpha$, $\alpha'$, and $\beta$, such that

$$[\sqrt{-1} \partial \bar{\partial} \phi]_{\alpha,\beta,B} \leq C.$$

The $C^{2,\alpha,\beta}$ and $C^{\alpha,\beta}$ function spaces are defined by Donaldson in [19]. For further references on definition of these function spaces, see [16], [17], [47].

Since this is a short note in the smooth case, we will not go into the ever-growing list of works in conical settings, instead we refer the readers to the following list of authors and their work related to the $C^{2,\alpha}$—estimate in conical Kähler geometry: Donaldson [19], Brendle [4], Guenancia-Paun [25], Chen-Donaldson-Sun [13], Jeffres-Mazzeo-Rubinstein [31], Calamai-Zheng [9]...

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2 A new proof of the aprori version of Yu-Wang’s Cafferelli type estimate for complex Monge-Ampere equations.

Our proof is based on Anderson’s rescaling idea in [11] and Chen-Donaldson-Sun’s trick in [13].

In Kähler geometry, a Kähler metric usually means a closed positive \((1, 1)\)-form. Give a \(\phi\) as in Theorem 1.5, \(\sqrt{-1} \partial \bar{\partial} \phi\) is then a Kähler-metric.

In general, given a Kähler-metric \(\omega\) in an open set \(\Omega\), a pluri-subharmonic function \(\phi\) is said to be a potential of \(\omega\) in \(\Omega\) if

\[
\omega = \sqrt{-1} \partial \bar{\partial} \phi, \text{ or equivalently } \omega_{k\bar{l}} = \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} \text{ over } \Omega,
\]

where \(\omega_{k\bar{l}}\) is defined as \(\omega = \sqrt{-1} \omega_{k\bar{l}} d\bar{z}_l \wedge dz_k\). Under the coordinates \(z_1, ..., z_n\), \(\omega_{k\bar{l}} (\frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l})\) is a Hermitian-matrix-valued function. In the rest of this section, we work with the metrics \(\omega\) most of time rather than the potentials \(\phi\). This is because our proof is essentially Riemannian geometry.

Our proof depends on the following 3 building blocks.

(1): The solvability of \(\sqrt{-1} \partial \bar{\partial} \phi\) equation with tame estimates.

Lemma 2.1. Suppose \(\frac{1}{\Lambda} < r < \Lambda\) for some \(\Lambda > 0\), then there exists a constant \(C_\Lambda\) depending on \(\Lambda, n, \text{ and } \alpha\) with the following properties.

Suppose \(\eta \in C^\alpha(\mathbb{B}_r)\) is a closed real \((1, 1)\)-form. Then there exists a real valued solution \(\phi \in C^{2, \alpha}(\mathbb{B}_r)\) to

\[
\sqrt{-1} \partial \bar{\partial} \phi = \eta \text{ over } \mathbb{B}_r
\]

such that \(|\phi|_{0, B_{r/4}} \leq C r^2 |\eta|_{0, B_r}\), where \(C\) is constant depending on \(n\). Consequently,

\[
|\phi|_{2, \alpha, B_{r/4}} \leq C_\Lambda |\eta|_{\alpha, B_r}.
\]

Remark 2.2. Lemma 2.1 is a simpler version of Lemma 7.1 in [18].

(2): The Liouville theorem in the complex case.

Theorem 2.3. (Riebesehl; Schulz)([39]) Suppose \(\omega\) is a Kähler-metric defined over \(\mathbb{C}^n\) which admits a \(C^{2, \alpha}\)-potential over any finite ball. Suppose there is a constant \(K\) such that

\[
det \omega_{k\bar{l}} = 1, \quad \frac{1}{K} I \leq \omega_{k\bar{l}} \leq K I \text{ over } \mathbb{C}^n.
\]

Then, for any \(1 \leq k, l \leq n\), \(\omega_{k\bar{l}}\) is a constant.

Remark 2.4. By the proof of Lemma 2.1 (see Lemma 7.1 in [18]), the \(\omega\) in Theorem 2.3 actually admits a global potential \(\phi\) over \(\mathbb{C}^n\), thus Theorem 2.3
is actually the same as Theorem 2 in [39]. However, we would like to emphasize that to prove Theorem 2.3 we don’t need to assume the metric admits a global potential. Therefore, Theorem 2.3 can be carried over exactly and directly to the proof of section 3 in the real case, without involving more issues.

(3): The Chen-Donaldson-Sun’s trick in [13]. This following version is proved simply by using Lemma 2.1 to inequality (39) in [12] (with lower order term changed from $|\phi|_{\alpha}$ to $|\phi|_{0}$).

**Proposition 2.5.** For any constant coefficient Kähler metric $\omega_{C}$, there exist a small enough positive number $\delta$ and a big enough constant $C_{\omega_{C}}$, both depending on the positive lower and upper bounds on the eigenvalues of $\omega_{C}$, the dimension $n$, and $\alpha'$, with the following properties. Suppose $\omega$ is a Kähler-metric over $B_{0}(1)$ which admits a potential in $C^{2,\alpha'}[B_{0}(1)]$. Suppose

$$\det\omega_{kl} = e^{f}, \quad \frac{\omega_{C}}{1+\delta} \leq \omega \leq (1+\delta)\omega_{C} \text{ over } B_{0}(1),$$

(5)

then the following estimate holds in $B(\frac{1}{4})$.

$$[\omega]_{\alpha',B(\frac{1}{4})} \leq C_{\omega_{C}}[e^{f}]_{\alpha',B(1)} + [\omega]_{0,B(1)}.$$

Now we are ready to prove of Theorem 1.5.

**Proof.** of Theorem 1.5 In this proof, while different “C” can be different constants, the dependence of each ”C” is the same as the last sentence of Theorem 1.5. We add more subletter to C if it depends on more things.

Notice that by the Monge-Ampere equation (2), the $C^{0}$-norms of $\sqrt{-1}\partial\bar{\partial}\phi$ and $f$ in Theorem 1.5 determines a $K \geq 1$ such that

$$\frac{I}{K} \leq \sqrt{-1}\partial\bar{\partial}\phi \leq KI.$$

(6)

Our proof can be divided into 3 steps.

Step 1: The notion of Hölder-radius and contradiction hypothesis.

Denote $\omega_{Euc}$ as the Euclidean metric in the underline coordinates, and $d_{q} = dist_{\omega_{Euc}}(q, \partial B(1))$.

**Definition 2.6.** Hölder Radius: given a closed $(1,1)$-form $\omega \in C^{\alpha}[B(1)]$, for all $q \in B(1)$, we define $h_{\omega,q}$ as the supremum of the radiiuses $h \in (0, d_{q})$ with the following properties.

$$[\omega]_{\alpha,B_{q}(h)} = \Sigma_{k,l}[\omega_{kl}]_{\alpha,B_{q}(h)} \leq \delta_{0}h^{-\alpha},$$

(7)

where $\delta_{0} > 0$ is small enough with respect to the data in the last sentence of Theorem 1.5. Notice definition (7) depends on the coordinates, thus when we rescale the coordinates, (10) and (11) hold. Since $\omega = \sqrt{-1}\partial\bar{\partial}\phi$
is assumed to be $C^{\alpha'}$, and we are considering open balls, then actually the supremum of radiuses can be attained. However, we don’t need the Hölder radius to be attainable in our proof.

By definition, we obtain the following extremely simple but extremely important property of the Hölder radius.

**Claim 2.7.** For any $0 < r < h_q$, we have $[\omega]_{\alpha,B_q(r)} \leq \delta_0 r^{-\alpha}$.

To prove Theorem 1.5, it suffices to show

$$\frac{h_{\omega,q}}{d_q} \geq c_1 > 0,$$

for some $c_1$ depending only on $K$ and $|f|_{\alpha',B(1)}$. (8)

We prove by contradiction. Were (8) not true, there exists a sequence of Kähler metrics

$$\omega_i = \sqrt{-1} \partial \bar{\partial} \phi_i,$$

functions $F_i$, and points $q_i$ such that

1. $\det \omega_i,_{k\bar{l}} = e^{F_i}$ over $B(1)$;
2. $\omega_{Euc} \leq \omega_i \leq K \omega_{Euc}$, $K \geq 1$, $|F_i|_{\alpha',B(1)} \leq c$;
3. for any fixed $i$, $\frac{h_{\omega_i,q_i}}{d_{q_i}} = \epsilon_i > 0$ (since we are doing aprori estimate);
4. $\frac{h_{\omega_i,q_i}}{d_{q_i}} = \epsilon_i \to 0$, $\frac{h_{\omega_i,q_i}}{d_{q_i}} \leq 2 \min \frac{h_{\omega_i,q_i}}{d_q}$, for any $q \in B(1)$. (9)

We shall derive a contradiction.

**Step 2: Rescaling, norm bounds, and bootstrapping.**

We consider the rescaling

1. $\hat{z}_1 = \frac{z_1 - z_1(q_i)}{h_{\omega_i,q_i}}$, ..., $\hat{z}_n = \frac{z_n - z_n(q_i)}{h_{\omega_i,q_i}}$, denote the defined inverse map from $B_0(\frac{1}{\epsilon_i}) \subset \mathbb{C}^n$ to $B(1)$ as $\Gamma_i$;
2. $\hat{\omega}_i = \frac{1}{h_{\omega_i,q_i}} \Gamma_i^* \omega_i$, $\hat{F}_i = \Gamma_i^* F_i$.

Denote the Euclidean metric with respect to the new coordinates $(\hat{z}_1, ..., \hat{z}_n)$ as $\hat{\omega}_{Euc}$. Thus, in $B_0(\frac{1}{\epsilon_i})$ with respect to the new coordinates, then following holds.

$$\det \hat{\omega}_i,_{k\bar{l}} = e^{\hat{F}_i}.$$ (10)

Moreover, the Hölder radius of $\hat{\omega}_i$ is 1 at the origin i.e

$$h_{\hat{\omega}_i,0} = 1.$$ (11)

From now on, we add $\hat{}$ to those objects in the rescaled coordinates, so the reader can figure out that everything with $\hat{}$ is after rescaling.

For any $\infty > \lambda > 0$, when $i$ is large enough, the metrics $\hat{\omega}_i$ live in $B_{\hat{0}}(\lambda)$ in the rescaled coordinates. For any $\hat{p}$ in $\mathbb{C}^n$, when $i$ is large enough such that $\hat{p} \in B_{\hat{0}}(\frac{1}{\lambda})$, consider the preimage of $\hat{p}$ under the rescaling map as

$$p_i = \Gamma_i(\hat{p}) = \hat{p} h_{\omega_i,q_i} + q_i$$

with respect to the coordinates $(z_1, ..., z_n)$. 


By the 4th item in (9), we have $h_{\omega_i, p_i} \geq \frac{d_{p_i}}{2d_{q_i}}$. Then after rescaling (with the factor $\frac{1}{h_{\omega_i, q_i}}$), we have

$$h_{\hat{\omega}, \hat{p}} \geq \frac{d_{p_i}}{2d_{q_i}}. \quad (12)$$

Notice $\frac{d_{p_i}}{d_{q_i}}$ is invariant under rescaling i.e

$$\frac{d_{p_i}}{d_{q_i}} = \text{dist}_{\hat{\omega}}(\hat{p}, \partial \tilde{B}) = \text{dist}_{\hat{\omega}}(\hat{p}, \partial \hat{B}). \quad (13)$$

where $\tilde{B}$ is the image of $B(1)$ under the rescaling map. Since

$$\text{dist}_{\hat{\omega}}(\hat{p}, \partial \hat{B}) < \infty,$$

then (9) and (13) imply

$$\lim_{i \to \infty} \frac{d_{p_i}}{d_{q_i}} = 1. \quad (14)$$

Therefore when $i$ is large, (12) and (14) imply $h_{\hat{\omega}, \hat{p}} > \frac{1}{3}$.

Hence, by Claim 2.7 we have

$$[\hat{\omega}_i]_{\alpha, B_{\hat{p}}(\frac{1}{3})} \leq 3^{\alpha} \delta_0. \quad (15)$$

Choosing $\omega_c = \hat{\omega}_i(\hat{p})$, and $\delta_0$ small enough with respect to $K$, (15) implies the small oscillation condition in Proposition 2.5 is fulfilled in $B_{\hat{p}}(\frac{1}{3})$. Then applying Proposition 2.5 (rescaled to $B_{\hat{p}}(\frac{1}{3})$), we end up with

$$[\hat{\omega}_i]_{\alpha', B_{\hat{p}}(\frac{1}{20})} \leq C. \quad (16)$$

Then, (13) and the second item in (9) imply that for any $\lambda > 0$, when $i$ is large enough such that $\frac{1}{\epsilon_i} > 1000(R+1)$, the following crucial bootstrapping estimate holds:

$$|\hat{\omega}_i|_{\alpha', B_{\hat{p}}(R)} \leq C. \quad (17)$$

Step 3: Strong convergence of the rescaled sequence, rigidity of bubble, and contradiction.

Then, by the Areila-Ascoli theorem, the sequence $\hat{\omega}_i$ subconverges to an $(\hat{\omega}_\infty, \mathbb{C}^n)$ in $C^\alpha(B(\lambda))$-topology, for any $\lambda > 0$, $\alpha < \hat{\alpha} < \alpha'$. In particular, we have

$$\lim_{i \to \infty} |\hat{\omega}_i - \hat{\omega}_\infty|_{\alpha, B_0(200)} = 0. \quad (18)$$

By the hypothesis that $|F_i|_{\alpha', B(1)} \leq c$ in (9), and the hypothesis that $\frac{1}{h_{\omega_i, q_i}} \to \infty$, the pulled back functions $\hat{F}_i$ subconverges to a constant $C_1$ in $C^\alpha[B(\lambda)]$-topology for any $\lambda > 0$. Then the following holds on $\hat{\omega}_\infty$. 

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• As a form, $\hat{\omega}_\infty \in C^\alpha(\mathbb{C}^n)$, for all $0 < \hat{\alpha} < \alpha'$;

• $\text{det} \hat{\omega}_{\infty,kl} = e^{C_1}$ over $\mathbb{C}^n$;

• $\frac{\hat{\rho}_{Euc}}{K} \leq \hat{\omega}_\infty \leq K \hat{\omega}_{Euc}$.

• $\hat{\omega}_\infty$ admits potential over any finite ball, therefore $\hat{\omega}_\infty$ is closed. To see this, for any ball $\lambda > 0$, applying (17) and Lemma 2.1 to $B_0(100\lambda + 100)$, we obtain potentials $\hat{\phi}_{i,\lambda}$ such that

$$\hat{\omega}_i = \sqrt{-1}\partial \bar{\partial} \hat{\phi}_{i,\lambda}, \quad |\hat{\phi}_{i,\lambda}|_{2,\alpha',B_0(\lambda+1)} \leq C_\lambda \text{ over } B(\lambda + 1).$$ (19)

Then, $\hat{\phi}_{i,\lambda}$ subconverges (strongly) in $C^{2,\alpha}[B_0(\lambda)]$–topology to a potential $\hat{\phi}_{\infty,\lambda}$ such that

$$\hat{\omega}_\infty = \sqrt{-1}\partial \bar{\partial} \hat{\phi}_{\infty,\lambda} \text{ over } B_0(\lambda), \quad |\hat{\phi}_{\infty,\lambda}|_{2,\alpha,B_0(\lambda)} \leq C_\lambda.$$ (20)

Thus, the above 4 items imply the conditions in Theorem 2.3 are fulfilled. According to Theorem 2.3, $\hat{\omega}_\infty$ is of constant coefficients, thus

$$[\hat{\omega}_\infty]_{\alpha,B_0(200)} = 0.$$ (21)

Hence (18) and (21) imply

$$\lim_{i \to \infty} [\hat{\omega}_i]_{\alpha,B_0(200)} = 0.$$ (22)

Then when $i$ is large enough, we deduce

$$[\hat{\omega}_i]_{\alpha,B_0(100)} \leq \frac{\delta_0}{100^\alpha}.$$  

This means

$$h_{\hat{\omega}_i,0} \geq 100,$$

which contradicts (11)!

The proof of Theorem 1.5 is completed. \(\Box\)

Remark 2.8. Actually, in the item containing (20) in Step 3 of the above proof, to prove the existence of potentials for $\hat{\omega}_\infty$ over all finite balls, it is easier to prove first by definition that $\omega$ is a closed current, and then apply Lemma 2.1. However, since we want to carry our proof in this section exactly and directly to section 3 without involving more issues, we still want to take $\hat{\phi}_{\infty,\lambda}$ as the limit of the potentials of $\hat{\omega}_i$. 

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3 Appendix: A new proof of the aprori version of Cafferelli’s estimate for real Monge-Ampere equations.

The proof of Theorem 1.1 is exactly parallel to the proof in section 2. Namely, to translate the ”complex” proof in section 2 to the real case of Theorem 1.1 we only need to

• change the $\sqrt{-1} \partial \bar{\partial}$ in section 2 to ”$\nabla^2$” (Hessian);
• change the $\omega_{kj}$ to $g_{kl}$, $\phi_{kj}$ to $u_{kl}$;
• change the complex coordinates ”$z_1...z_n$” in section 2 to real coordinates ”$x_1...x_n$”;
• change the words ”plurisubharmonic” to ”convex”;
• change the equation from (2) to (1).

By translating as above, Lemma 2.1 corresponds to Lemma 3.1, Theorem 2.3 corresponds to Theorem 3.2, Proposition 2.5 corresponds to Proposition 3.4. One thing worth mentioning is, while the proof of Lemma 2.1 requires Griffith-Harris’ trick [24] and Hormander’s results [29], Lemma 3.1 can be proved in one line.

Lemma 3.1. Suppose $\frac{1}{\Lambda} < r < \Lambda$ for some $\Lambda > 0$, then there exists a constant $C_{\Lambda}$ depending on $\Lambda$ and $n$ with the following properties.

Suppose $g$ is a matrix-valued function over $B_r$, such that $g = \nabla^2 u$ for some function $u \in C^{2,\alpha}(B_r)$. Then there exists a function $v \in C^{2,\alpha}(B_{\frac{r}{2}})$ such that $g = \nabla^2 v$ and

$$|v|_{0,B_{\frac{r}{2}}} \leq C r^2 |g|_{0,B_r},$$

where $C$ is constant depending on $n$. Consequently,

$$|v|_{2,\alpha,B_{\frac{r}{4}}} \leq C_{\Lambda} |g|_{0,B_r}.$$

Proof. of Lemma 3.1 The proof can not be easier. Just take $v$ as $u$ minus its linearization i.e

$$v = u - u(0) - x \cdot \nabla u(0),$$

then

$$\nabla^2 v = g, \; v(0) = 0, \; (\nabla v)(0) = 0.$$ 

Thus the estimate of $|v|_{L^\infty(B(\frac{r}{2}))}$ follows by applying the mean value theorem to $\nabla v$ and then to $v$. 

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Theorem 3.2. (Calabi [8]) (Pogorelov [38]) Suppose $g$ is a symmetric-matrix-valued function defined over $\mathbb{R}^n$. Suppose $g$ admits a $C^{2,\alpha}$—potential over any finite ball i.e for any ball $B \in \mathbb{R}^n$, there exists a function $u_B \in C^{2,\alpha}(B)$ such that

$$g = \nabla^2 u_B \text{ over } B.$$ 

Suppose there is a constant $K$ such that

$$\det g_{kl} = 1, \quad \frac{1}{K} I \leq g_{kl} \leq K I \text{ over } \mathbb{R}^n. \quad (24)$$

Then, for any $1 \leq k, l \leq n$, $g_{kl}$ is a constant.

Remark 3.3. Actually Calabi’s and Pogorelov’s original theorems in [8] and [38] are much stronger than Theorem 3.2, but all we need here is Theorem 3.2. In [8], $g$ is assumed to admit a global potential. Though in our new proof of Theorem 1.1 of Caffarelli, we have a global potential, we still want to state the Liouville theorem as Theorem 2.3 to emphasize that it does not need a global potential.

Proposition 3.4. For any constant coefficient Riemannian metric $g_c$, there exist a small enough positive number $\delta$ and a big enough constant $C_{g_c}$, both depending on the positive lower and upper bounds on the eigenvalues of $g_c$, the dimension $n$, and $\alpha'$, with the following properties. Suppose $u$ is a $C^{2,\alpha'}$ convex function defined over $B_0(1)$ such that

$$\det u_{ij} = e^f, \quad \frac{g_c}{1 + \delta} \leq \nabla^2 u \leq (1 + \delta)g_c \text{ over } B_0(1), \quad (25)$$

then the following estimate holds in $B(\frac{1}{4})$.

$$[\nabla^2 u]_{\alpha', B(\frac{1}{4})} \leq C_{g_c} (|e^f|_{\alpha', B(1)} + |\nabla^2 u|_{0, B(1)}) .$$

With the above discussion in section 3, the proof of Theorem 1.1 is complete.

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