Making $K_{r+1}$-Free Graphs $r$-partite

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Abstract

The Erdős–Simonovits stability theorem states that for all $\varepsilon > 0$ there exists $\alpha > 0$ such that if $G$ is a $K_{r+1}$-free graph on $n$ vertices with $e(G) > \text{ex}(n, K_{r+1}) - \alpha n^2$, then one can remove $\varepsilon n^2$ edges from $G$ to obtain an $r$-partite graph. Füredi gave a short proof that one can choose $\alpha = \varepsilon$. We give a bound for the relationship of $\alpha$ and $\varepsilon$ which is asymptotically sharp as $\varepsilon \to 0$.

1 Introduction

Erdős asked how many edges need to be removed in a triangle-free graph on $n$ vertices in order to make it bipartite. He conjectured that the balanced blow-up of $C_5$ with class sizes $n/5$ is the worst case, and hence $1/25n^2$ edges would always be sufficient. Together with Faudree, Pach and Spencer [5], he proved that one can remove at most $1/18n^2$ edges to make a triangle-free graph bipartite. Further, Erdős, Győri and Simonovits [6] proved that for graphs with at least $n^2/5$
edges, an unbalanced $C_5$ blow-up is the worst case. For $r \in \mathbb{N}$, denote $D_r(G)$ the minimum number of edges which need to be removed to make $G$ $r$-partite.

**Theorem 1.1** (Erdős, Győri and Simonovits [6]). Let $G$ be a $K_3$-free graph on $n$ vertices with at least $n^2/5$ edges. There exists an unbalanced $C_5$ blow-up of $H$ with $e(H) \geq e(G)$ such that

$$D_2(G) \leq D_2(H).$$

This proved the Erdős conjecture for graphs with at least $n^2/5$ edges. A simple probabilistic argument (e.g. [6]) settles the conjecture for graphs with at most $2/25n^2$ edges.

A related question was studied by Sudakov; he determined the maximum number of edges in a $K_4$-free graph which need to be removed in order to make it bipartite [13]. This problem for $K_6$-free graphs was solved by Hu, Lidický, Martins, Norin and Volec [10].

We will study the question of how many edges in a $K_{r+1}$-free graph need at most to be removed to make it $r$-partite. For $n \in \mathbb{N}$ and a graph $H$, $ex(n, H)$ denote the Turán number, i.e. the maximum number of edges of an $H$-free graph. The Erdős–Simonovits theorem [7] for cliques states that for every $\varepsilon > 0$ there exists $\alpha > 0$ such that if $G$ is a $K_{r+1}$-free graph on $n$ vertices with $e(G) > ex(n, K_{r+1}) - \alpha n^2$, then $D_r(G) \leq \varepsilon n^2$.

Füredi [8] gave a nice short proof of the statement that a $K_{r+1}$-free graph $G$ on $n$ vertices with at least $ex(n, K_{r+1}) - t$ edges satisfies $D_r(G) \leq t$; thus providing a quantitative version of the Erdős–Simonovits theorem. In [10] Füredi’s result was strengthened for some values of $r$. For small $t$, we will determine asymptotically how many edges are needed. For very small $t$, it is already known [3] that $G$ has to be $r$-partite.

**Theorem 1.2** (Brouwer [3]). Let $r \geq 2$ and $n \geq 2r + 1$ be integers. Let $G$ be a $K_{r+1}$-free graph on $n$ vertices with $e(G) \geq ex(n, K_{r+1}) - \lfloor \frac{n}{r} \rfloor + 2$. Then

$$D_r(G) = 0.$$

This result was rediscovered in [11][12][13]. We will study $K_{r+1}$-free graphs on fewer edges.

**Theorem 1.3.** Let $r \geq 2$ be an integer. Then for all $n \geq 3r^2$ and for all $0 \leq \alpha \leq 10^{-7}r^{-12}$ the following holds. Let $G$ be a $K_{r+1}$-free graph on $n$ vertices with $e(G) \geq ex(n, K_{r+1}) - t$,

where $t = \alpha n^2$, then

$$D_r(G) \leq \left( \frac{2r}{3\sqrt{3}} + 30r^3\alpha^{1/6} \right) \alpha^{3/2}n^2.$$
Note that we did not try to optimize our bounds on \( n \) and \( \alpha \) in the theorem. One could hope for a slightly better error term of \( 30r^3\alpha^{5/3} \) in Theorem 1.3 but the next natural step would be to prove a structural version.

To state this structural version we introduce some definitions. The blow-up of a graph \( G \) is obtained by replacing every vertex \( v \in V(G) \) with finitely many copies so that the copies of two vertices are adjacent if and only if the originals are.

For two graphs \( G \) and \( H \), we define \( G \otimes H \) to be the graph on the vertex set \( V(G) \cup V(H) \) with \( gg' \in E(G \otimes H) \) if \( gg' \in E(G) \), \( hh' \in E(G \otimes H) \) iff \( hh' \in E(H) \) and \( gh \in E(G \otimes H) \) for all \( g \in V(G), h \in V(H) \).

**Conjecture 1.4.** Let \( r \geq 2 \) be an integer and \( n \) sufficiently large. Then there exists \( \alpha_0 > 0 \) such that for all \( 0 \leq \alpha \leq \alpha_0 \) the following holds. For every \( K_{r+1} \)-free graph \( G \) on \( n \) vertices there exists an unbalanced \( K_{r-2} \otimes C_5 \) blow-up \( H \) on \( n \) vertices with \( e(H) \geq e(G) \) such that

\[
D_r(G) \leq D_r(H).
\]

This conjecture can be seen as a generalization of Theorem 1.1. We will prove that Theorem 1.3 is asymptotically sharp by describing an unbalanced blow-up of \( K_{r-2} \otimes C_5 \) that needs at least that many edges to be removed to make it \( r \)-partite. This gives us a strong evidence that Conjecture 1.4 is true.

**Theorem 1.5.** Let \( r, n \in \mathbb{N} \) and \( 0 < \alpha < \frac{1}{4r^2} \). Then there exists a \( K_{r+1} \)-free graph on \( n \) vertices with

\[
e(G) \geq \text{ex}(n, K_{r+1}) - \alpha n^2 + \frac{4r}{3\sqrt{3}} \alpha^{3/2} n^2 - \frac{2r(r-3)}{9} \alpha^2 n^2
\]

and

\[
D_r(G) \geq \frac{2r}{3\sqrt{3}} \alpha^{3/2} n^2.
\]

In Kang-Pikhurko’s proof [11] of Theorem 1.2 the case \( e(G) = \text{ex}(n, K_{r+1}) - \lfloor n/r \rfloor + 1 \) is studied. In this case they constructed a family of \( K_{r+1} \)-free non-\( r \)-partite graphs, which includes our extremal graph, for that number of edges.

We recommend the interested reader to read the excellent survey [12] by Nikiforov. He gives a good overview on further related stability results, for example on guaranteeing large induced \( r \)-partite subgraphs of \( K_{r+1} \)-free graphs.

We organize the paper as follows. In Section 2 we prove Theorem 1.3 and in Section 3 we give the sharpness example, i.e. we prove Theorem 1.5.

**2 Proof of Theorem 1.3**

Let \( G \) be an \( n \)-vertex \( K_{r+1} \)-free graph with \( e(G) \geq \text{ex}(n, K_{r+1}) - t \), where \( t = \alpha n^2 \). We will assume that \( n \) is sufficiently large. Furthermore, by Theorem 1.2 we can assume
that

\[ \alpha \geq \frac{\lceil \frac{n}{r} \rceil - 2}{n^2} \geq \frac{1}{2rn}. \]

This also implies that \( t \geq r \) because \( n \geq 3r^2 \). During our proof we will make use of Turán’s theorem and a version of Turán’s theorem for \( r \)-partite graphs multiple time. Turán’s theorem [14] determines the maximum number of edges in a \( K_{r+1} \)-free graph.

**Theorem 2.1** (Turán [14]). Let \( r \geq 2 \) and \( n \in \mathbb{N} \). Then,

\[
\frac{n^2}{2} \left( 1 - \frac{1}{r} \right) - \frac{r}{2} \leq ex(n, K_{r+1}) \leq \frac{n^2}{2} \left( 1 - \frac{1}{r} \right).
\]

Denote \( K(n_1, \ldots, n_r) \) the complete \( r \)-partite graph whose \( r \) color classes have sizes \( n_1, \ldots, n_r \), respectively. Turán’s theorem for \( r \)-partite graphs states the following.

**Theorem 2.2** (folklore). Let \( r \geq 2 \) and \( n_1, \ldots, n_r \in \mathbb{N} \) satisfying \( n_1 \leq \cdots \leq n_r \). For a \( K_r \)-free subgraph \( H \) of \( K(n_1, \ldots, n_r) \), we have

\[ e(H) \leq e(K(n_1, \ldots, n_r)) - n_1 n_2. \]

For a proof of this folklore result see for example [2, Lemma 3.3].

We denote the maximum degree of \( G \) by \( \Delta(G) \). For two disjoint subsets \( U, W \) of \( V(G) \), write \( e(U, W) \) for the number of edges in \( G \) with one endpoint in \( U \) and the other endpoint in \( W \). We write \( e^c(U, W) \) for the number of non-edges between \( U \) and \( W \), i.e. \( e^c(U, W) = |U||W| - e(U, W) \).

Füredi [8] used Erdős’ degree majorization algorithm [4] to find a vertex partition with some useful properties. We include a proof for completeness.

**Lemma 2.3** (Füredi [8]). Let \( t, r, n \in \mathbb{N} \) and \( G \) be an \( n \)-vertex \( K_{r+1} \)-free graph with \( e(G) \geq ex(n, K_{r+1}) - t \). Then there exists a vertex partition \( V(G) = V_1 \cup \ldots \cup V_r \) such that

\[
\sum_{i=1}^{r} e(G[V_i]) \leq t, \quad \Delta(G) = \sum_{i=2}^{r} |V_i| \quad \text{and} \quad \sum_{1 \leq i < j \leq r} e^c(V_i, V_j) \leq 2t. \quad (1)
\]

**Proof.** Let \( x_1 \in V(G) \) be a vertex of maximum degree. Define \( V_1 := V(G) \setminus N(x_1) \) and \( V_1^+ = N(x_1) \). Iteratively, let \( x_i \) be a vertex of maximum degree in \( G[V_1^+] \). Let \( V_i := V_{i-1}^+ \setminus N(x_i) \) and \( V_i^+ = V_{i-1}^+ \cap N(x_i) \). Since \( G \) is \( K_{r+1} \)-free this process stops at \( i \leq r \) and thus gives a vertex partition \( V(G) = V_1 \cup \ldots \cup V_r \).

In the proof of [8, Theorem 2], it is shown that the partition obtained from this algorithm satisfies

\[
\sum_{i=1}^{r} e(G[V_i]) \leq t.
\]
By construction,
\[ \sum_{i=2}^{r} |V_i| = |V_1^+| = |N(x_1)| = \Delta(G). \]

Let \( H \) be the complete \( r \)-partite graph with vertex set \( V(G) \) and all edges between \( V_i \) and \( V_j \) for \( 1 \leq i < j \leq r \). The graph \( H \) is \( r \)-partite and thus has at most \( \text{ex}(n, K_{r+1}) \) edges. Finally, since \( G \) has at most \( t \) edges not in \( H \) and at least \( \text{ex}(n, K_{r+1}) - t \) edges total, at most \( 2t \) edges of \( H \) can be missing from \( G \), giving us
\[ \sum_{1 \leq i < j \leq r} e^c(V_i, V_j) \leq 2t \]
and proving the last inequality.

For this vertex partition we can get bounds on the class sizes.

**Lemma 2.4.** For all \( i \in [r] \), \( |V_i| \in \left\{ \frac{n}{r} - \frac{5}{2}\sqrt{\alpha n}, \frac{n}{r} + \frac{5}{2}\sqrt{\alpha n} \right\} \) and thus also
\[ \Delta(G) \leq \frac{r-1}{r}n + \frac{5}{2}\sqrt{\alpha n}. \]

**Proof.** We know that
\[ \sum_{1 \leq i < j \leq r} |V_i||V_j| \geq e(G) - \sum_{i=1}^{r} e(G[V_i]) \geq \left( 1 - \frac{1}{r} \right) \frac{n^2}{2} - r - 2t. \]

Also,
\[ \sum_{1 \leq i < j \leq r} |V_i||V_j| = \frac{1}{2} \sum_{i=1}^{r} |V_i|(n - |V_i|) = \frac{n^2}{2} - \frac{1}{2} \sum_{i=1}^{r} |V_i|^2. \]

Thus, we can conclude that
\[ \sum_{i=1}^{r} |V_i|^2 \leq \frac{n^2}{r} + r + 4t. \quad (2) \]

Now, let \( x = |V_1| - n/r \). Then,
\[ \sum_{i=1}^{r} |V_i|^2 = \left( \frac{n}{r} + x \right)^2 + \sum_{i=2}^{r} |V_i|^2 \geq \left( \frac{n}{r} + x \right)^2 + \frac{\sum_{i=2}^{r} |V_i|^2}{r-1} \]
\[ \geq \left( \frac{n}{r} + x \right)^2 + \frac{n \left( 1 - \frac{1}{r} \right) - x}{r-1} \geq \frac{n^2}{r} + x^2. \]

Combining this with (2), we get
\[ |x| \leq \sqrt{r + 4t} \leq \frac{5}{2}\sqrt{\alpha n}, \text{ and thus } \]
\[ \frac{n}{r} - \frac{5}{2}\sqrt{\alpha n} \leq |V_i| \leq \frac{n}{r} + \frac{5}{2}\sqrt{\alpha n}. \]

In a similar way we get the bounds on the sizes of the other classes. \( \square \)
Lemma 2.5. The graph $G$ contains $r$ vertices $x_1 \in V_1, \ldots, x_r \in V_r$ which form a $K_r$ and for every $i$
\[ \deg(x_i) \geq n - |V_i| - 5r\alpha n. \]

Proof. Let $V_i^c := V(G) \setminus V_i$. We call a vertex $v_i \in V_i$ small if $|N(v_i) \cap V_i^c| < |V_i^c| - 5r\alpha n$ and big otherwise. For $1 \leq i \leq r$, denote $B_i$ the set of big vertices inside class $V_i$. There are at most $4t < \frac{4 \alpha n}{5r}$ small vertices in total as otherwise (1) is violated. Thus, in each class there are at least $n/10r$ big vertices, i.e. $|B_i| \geq n/10r$. The number of missing edges between the sets $B_1, \ldots, B_r$ is at most $2t < \frac{1}{100}r^2 n^2$. Thus, using Theorem 2.2 we can find a $K_r$ with one vertex from each $B_i$. \hfill \Box

Lemma 2.6. There exists a vertex partition $V(G) = X_1 \cup \ldots \cup X_r \cup X$ such that all $X_i$s are independent sets, $|X| \leq 5r^2\alpha n$ and
\[ \frac{n}{r} - 3\sqrt{\alpha n} \leq |X_i| \leq \frac{n}{r} + 3r\sqrt{\alpha n} \]
for all $1 \leq i \leq r$.

Proof. By Lemma 2.5 we can find vertices $x_1, \ldots, x_r$ forming a $K_r$ and having $\deg(x_i) \geq n - |V_i| - 5r\alpha n$. Define $X_i$ to be the common neighborhood of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$ and $X = V(G) \setminus (X_1 \cup \ldots \cup X_r)$. Since $G$ is $K_{r+1}$-free, the $X_i$s are independent sets. Now we bound the size of $X_i$ using the bounds on the $V_i$s. Since every $x_j$ has at most $|V_j| + 5r\alpha n$ non-neighbors, we get
\[ |X_i| \geq n - \sum_{1 \leq j \leq r \atop j \neq i} (|V_j| + 5r\alpha n) \geq |V_i| - 5r^2\alpha n \geq \frac{n}{r} - 3\sqrt{\alpha n}. \]
and
\[ \sum_{i=1}^r \deg(x_i) \geq n(r-1) - 5r^2\alpha n. \quad (3) \]

A vertex $v \in V(G)$ cannot be incident to all of the vertices $x_1, \ldots, x_r$, because $G$ is $K_{r+1}$-free. Further, every vertex from $X$ is not incident to at least two of the vertices $x_1, \ldots, x_r$. Thus,
\[ \sum_{i=1}^r \deg(x_i) \leq n(r-1) - |X|. \quad (4) \]
Combining (3) with (4), we conclude that
\[ |X| \leq 5r^2\alpha n. \]
For the upper bound on the sizes of the sets $X_i$ we get

$$|X_i| \leq n - \sum_{1 \leq j \leq r, j \neq i} |X_j| \leq n - \frac{r-1}{r} n + 3r \sqrt{\alpha n} = \frac{n}{r} + 3r \sqrt{\alpha n}.$$ 

We now bound the number of non-edges between $X_1, \ldots, X_r$.

**Lemma 2.7.**

$$\sum_{1 \leq i < j \leq r} e^c(X_i, X_j) \leq t + e(X, X^c) + |X|^2 - \left(1 - \frac{1}{r}\right) n|X| + r.$$ 

**Proof.**

$$\frac{n^2}{2} \left(1 - \frac{1}{r}\right) - \frac{r}{2} - t \leq e(G) = e(X, X^c) + e(X) + \sum_{1 \leq i < j \leq r} e(X_i, X_j)$$

$$\leq e(X, X^c) + \frac{|X|^2}{2} + \left(1 - \frac{1}{r}\right) \left(\frac{n - |X|^2}{2}\right) - \sum_{1 \leq i < j \leq r} e^c(X_i, X_j).$$

This gives the statement of the lemma. 

Let

$$\bar{X} = \left\{ v \in X \mid \deg_{X_1 \cup \cdots \cup X_r}(v) \geq \frac{r - 2}{r} n + 3\alpha^{1/3} n \right\} \quad \text{and} \quad \hat{X} := X \setminus \bar{X}.$$ 

Let $d \in [0, 1]$ such that $|\bar{X}| = d|X|$. Further, let $k \in [0, 5r^2]$ such that $|X| = k\alpha n$.

Now we shall give an upper bound the number of non-edges between $X_1, \ldots, X_r$.

**Lemma 2.8.**

$$\sum_{1 \leq i < j \leq r} e^c(X_i, X_j) \leq 20r^2 \alpha^{4/3} n^2 + \left(1 - (1 - d) \frac{1}{r} k\right) \alpha n^2.$$ 

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Proof. By Lemma 2.7
\[
\sum_{1 \leq i < j \leq r} e^c(X_i, X_j) \leq t + e(X, X^c) + |X|^2 - \left(1 - \frac{1}{r}\right)n|X| + r \\
\leq t + d|X|\Delta(G) + (1 - d)|X| \left(\frac{r - 2}{r}n + 3\alpha^{1/3}n\right) + |X|^2 - \left(1 - \frac{1}{r}\right)n|X| + r \\
\leq t + d|X| \left(n - \frac{1}{r} + \frac{5}{2}\sqrt{\alpha n}\right) + (1 - d)|X| \left(\frac{r - 2}{r}n + 3\alpha^{1/3}n\right) \\
+ |X|^2 - \left(1 - \frac{1}{r}\right)n|X| + r \\
\leq \frac{5}{2}d|X|\sqrt{\alpha n} + 3(1 - d)|X|\alpha^{1/3}n + |X|^2 + t + n|X|\frac{d - 1}{r} + r \\
\leq \frac{5}{2}k\alpha^{3/2}n^2 + 3k\alpha^{4/3}n^2 + |X|^2 + \left(1 - (1 - d)\frac{1}{r}k\right)\alpha n^2 + r \\
\leq \frac{25}{2}r^2\alpha^{3/2}n^2 + 15r^2\alpha^{4/3}n^2 + 25r^2\alpha^{2}n^2 + \left(1 - (1 - d)\frac{1}{r}k\right)\alpha n^2 + r \\
\leq 20r^2\alpha^{4/3}n^2 + \left(1 - (1 - d)\frac{1}{r}k\right)\alpha n^2.
\]

Let
\[
C(\alpha) := 20r^2\alpha^{4/3} + \left(1 - (1 - d)\frac{1}{r}k\right)\alpha.
\]
For every vertex \(u \in X\) there is no \(K_r\) in \(N_{X_1}(u) \cup \cdots \cup N_{X_r}(u)\). Thus, by applying Theorem 2.2 and Lemma 2.8 we get
\[
\min_{i \neq j} |N_{X_i}(u)||N_{X_j}(u)| \leq \sum_{1 \leq i < j \leq r} e^c(X_i, X_j) \leq C(\alpha)n^2. \tag{5}
\]
Bound (5) implies in particular that every vertex \(u \in X\) has degree at most \(\sqrt{C(\alpha)n}\) to one of the sets \(X_1, \ldots, X_r\), i.e.
\[
\min_i |N_{X_i}(u)| \leq \sqrt{C(\alpha)n}. \tag{6}
\]
Therefore, we can partition \(\bar{X} = A_1 \cup \ldots \cup A_r\) such that every vertex \(u \in A_i\) has at most \(\sqrt{C(\alpha)n}\) neighbors in \(X_i\). By the following calculation, for every vertex \(u \in \bar{X}\) the second smallest neighborhood to the \(X_i\)’s has size at least \(\alpha^{1/3}n\).
\[
\min_{i \neq j} |N_{X_i}(u)| + |N_{X_j}(u)| \geq \frac{r - 2}{r}n + 3\alpha^{1/3}n - (r - 2)\left(n \frac{r}{r} + 3r\sqrt{\alpha n}\right) \geq 2\alpha^{1/3}n,
\]

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where we used the definition of $\bar{X}$ and Lemma 2.6. Combining the lower bound on the second smallest neighborhood with (5) we can conclude that for every $u \in X$

$$\min_i |N_{X_i}(u)| \leq \frac{C(\alpha)}{\alpha^{1/3}} n. \quad (7)$$

Hence, we can partition $\bar{X} = B_1 \cup \ldots \cup B_r$ such that every vertex $u \in B_i$ has at most $C(\alpha)\alpha^{-1/3}n$ neighbors in $X_i$. Consider the partition $A_1 \cup B_1 \cup X_1, A_2 \cup B_2 \cup X_2, \ldots, A_r \cup B_r \cup X_r$. By removing all edges inside the classes we end up with an $r$-partite graph. We have to remove at most

$$e(X) + d|X| \frac{C(\alpha)}{\alpha^{1/3}} n + (1 - d)|X|\sqrt{C(\alpha)} n \leq 6r^2 \alpha^{5/3} n^2 + (1 - d)k\sqrt{C(\alpha)} \alpha n^2$$

$$\leq 6r^2 \alpha^{5/3} n^2 + (1 - d)k \left(\sqrt{20r^2 \alpha^{4/3}} + \sqrt{\left(1 - (1 - d)\frac{1}{r}\alpha\right)} \alpha n^2\right)$$

$$\leq \left(\frac{2r}{3\sqrt{3}} + 30r^3 \alpha^{1/6}\right) \alpha^{3/2} n^2$$

edges. We have used (6), (7) and the fact that

$$(1 - d)k \sqrt{1 - \left(1 - d\frac{k}{r}\right)} \leq \frac{2r}{3\sqrt{3}},$$

which can be seen by setting $z = (1 - d)k$ and finding the maximum of $f(z) := z\sqrt{1 - \frac{z}{r}}$, which is obtained at $z = 2r/3$.

### 3 Sharpness Example

In this section we will prove Theorem 1.5, i.e. that the leading term from Theorem 1.3 is best possible.

**Proof of Theorem 1.5.** Let $G$ be the graph with vertex set $V(G) = A \cup X \cup B \cup C \cup D \cup X_1 \cdots \cup X_{r-2}$, where all classes $A, X, B, C, D, X_1, \ldots, X_{r-2}$ form independent sets; $A, X, B, C, D$ form a complete blow-up of a $C_5$, where the classes are named in cyclic order; and for each $1 \leq i \leq r - 2$, every vertex from $X_i$ is incident to all vertices from $V(G) \setminus X_i$.

The sizes of the classes are

$$|X| = \frac{2r}{3} \alpha n, \quad |A| = |B| = \sqrt{\frac{\alpha}{3}} n, \quad |C| = |D| = \frac{1 - \frac{2r}{3}}{\sqrt{\frac{\alpha}{3}}} n - \sqrt{\frac{\alpha}{3}} n, \quad |X_i| = \frac{1 - \frac{2r}{3}}{r} \alpha n.$$ 

The smallest class is $X$ and the second smallest are $A$ and $B$. By deleting all edges between $X$ and $A$ ($|X||A| = \frac{2r}{3\sqrt{3}} \alpha^{3/2} n^2$) we get an $r$-partite graph. Since the classes $A$
Figure 1: Graph $G$

and $X$ are the two smallest class sizes, one cannot do better as observed in [6, Theorem 7]. Hence

$$D_r(G) \geq \frac{2r}{3\sqrt{3}} \alpha^{3/2} n^2.$$  

Let us now count the number of edges of $G$. The number of edges incident to $X$ is

$$e(X, X^c) = \left(\frac{2r}{3} \alpha\right) \left(2\sqrt{\frac{\alpha}{3}}\right) n^2 + \left(\frac{2r}{3} \alpha\right) \left(\frac{1 - 2r}{r} \alpha (r - 2)\right) n^2$$

$$= \left(\frac{2}{3} (r - 2) \alpha + \frac{4r}{3\sqrt{3}} \alpha^{3/2} - \frac{4r(r - 2)}{9} \alpha^2\right) n^2.$$  

Using that $|A| + |C| = |B| + |D| = |X_1|$, we have that the number of edges inside $A \cup B \cup C \cup D \cup X_1 \cup \cdots \cup X_{r-2}$ is

$$e(X^c) = |X_1|^2 \left(\frac{r}{2}\right) - |A||B| = \left(1 - \frac{2r}{3} \alpha\right) n^2 \left(\frac{r}{2}\right) - \frac{1}{3} \alpha n^2$$

$$= \frac{1}{r^2} \left(\frac{r}{2}\right) n^2 - \frac{4r}{3} \frac{1}{r^2} \alpha \left(\frac{r}{2}\right) n^2 + \frac{4}{9} \alpha^2 \left(\frac{r}{2}\right) n^2 - \frac{1}{3} \alpha n^2$$

$$= \left(1 - \frac{1}{r}\right) \frac{n^2}{2} - \frac{2}{3} (r - 1) \alpha n^2 - \frac{1}{3} \alpha n^2 + \frac{4}{9} \alpha^2 \left(\frac{r}{2}\right) n^2.$$
Thus, the number of edges of $G$ is

$$e(G) = e(X^c) + e(X, X^c) = \left(1 - \frac{1}{r}\right) \frac{n^2}{2} - \alpha n^2 + \frac{4r}{3\sqrt{3}} \alpha^{3/2} n^2 - \frac{2r(r-3)}{9} \alpha^2 n^2$$

$$\geq \text{ex}(n, K_{r+1}) - \alpha n^2 + \frac{4r}{3\sqrt{3}} \alpha^{3/2} n^2 - \frac{2r(r-3)}{9} \alpha^2 n^2,$$

where we applied Turán’s theorem in the last step.

\[\square\]

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