On the robustness of stabilizing feedbacks for quantum spin-$\frac{1}{2}$ systems

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Abstract—In this paper, we consider stochastic master equations describing the evolution of quantum spin-$\frac{1}{2}$ systems interacting with electromagnetic fields undergoing continuous-time measurements. We suppose that the initial states and the exact values of the physical parameters are unknown. We prove that the feedback stabilization strategy considered in [16] is robust to these imperfections. This is shown by studying the asymptotic behavior of the coupled stochastic master equations describing the evolutions of the actual state and the estimated one under appropriate assumptions on the feedback controller. We provide sufficient conditions on the feedback controller and a valid domain of estimated parameters which ensure exponential stabilization of the coupled system. Furthermore, our results allow us to answer positively to [15, Conjecture 4.4] in the case of spin-$\frac{1}{2}$ systems with unknown initial states, even in presence of imprecisely known physical parameters.

I. INTRODUCTION

Classical stochastic filtering theory [14], [24] provides tools to optimize the estimation of dynamics described by stochastic differential equations, in presence of noisy observations. A primitive theory of quantum filtering theory appeared in the work of Davies in the 1960s [9], [10]. Belavkin in the 1980s established original results in quantum filtering and feedback control of quantum systems, as a natural extension of classical filtering and control [1], [2], [3], [4]. The development of quantum probability theory and quantum stochastic calculus [13], [12], [18] provided essential mathematical tools to describe open quantum systems and quantum filtering. In the physics community, quantum filtering theory is also known as quantum trajectory theory, after it has been established in a more heuristic manner, by Carmichael in the 1990s [8]. A modern introduction to quantum filtering theory may be found in [5].

Continuous-time quantum filters describe the time evolution of the states of open quantum systems interacting with electromagnetic fields undergoing continuous-time measurements. Quantum filters are solutions of matrix-valued stochastic differential equations called stochastic master equations.

Quantum filtering theory plays a major role in the development of quantum feedback control. A measurement-based feedback is designed based on the information obtained from quantum filters. The systematic design of stabilizing feedback control laws for quantum systems is a crucial step towards engineering of quantum devices. In particular, feedback stabilization of pure states has received particular interest [21], [19], [23]. In real experiments different sorts of imperfections may be present, as for instance inefficient detectors, unknown initial states, imprecise knowledge of the detector efficiency and other physical parameters, etc. Hence, from a practical point of view, choosing feedback controls which are robust to such imperfections is an important, and challenging, problem.

Concerning quantum angular momentum systems with known initial states and parameters, in [23], based on numerical approaches, the authors designed for the first time a quantum feedback controller that globally stabilizes a quantum spin-$\frac{1}{2}$ system towards an eigenstate of $\sigma_z$ in presence of imperfect measurements. More recently, in [19], by analyzing the stochastic flow and by using stochastic Lyapunov techniques, the authors constructed a switching feedback controller which globally stabilizes the $N$-level quantum angular momentum system, in presence of imperfect measurements, to the target eigenstate. In [16], [17], by using stochastic and geometric control tools, we provided sufficient conditions on the feedback control law ensuring almost sure exponential convergence to a predetermined eigenstate of the measurement operator for spin-$\frac{1}{2}$ and spin-J systems respectively (see [7], [6] for exponential stabilization results via a different approach).

In [15], we considered controlled quantum spin-$\frac{1}{2}$ systems in the case of unawareness of initial states and in presence of measurement imperfections. We proved that the fidelity between the quantum filter and the associated estimated filter converges to one under appropriate assumptions on the feedback controller. For spin-J systems, we considered feedback controls of a particular form, and we conjectured that such control laws are capable of exponentially stabilize the system towards an eigenstate of the measurement operator.

In this paper, we study the feedback exponential stabilizability of spin-$\frac{1}{2}$ systems in presence of measurement imperfections and unawareness of the initial states and of the physical parameters (namely, the detection efficiency, the difference between the energies of the excited state and the ground state, and the strength of the interaction between the system and the probe). We find general conditions on the feedback control guaranteeing robust exponential stabilization with respect to such imperfections. The dynamics is defined by a coupled system of equations describing the evolutions of the quantum filter and the associated estimated filter, with the feedback controller being a function of the estimated quantum filter. In order to show our main result, Theorem 3, we analyze the asymptotic behavior of the cou-
plied system under appropriate assumptions on the feedback controller. In particular, we provide sufficient conditions on the feedback controller and a valid domain for the estimated parameters which ensure exponential feedback stabilization of the coupled quantum spin-$\frac{1}{2}$ system. Moreover, we give explicit forms of feedback controllers which guarantee such feedback exponential stabilization. We precise that the stabilizing feedback controllers that we proposed in [16] satisfy the assumptions of Theorem 3 and also that the results of this paper prove [15, Conjecture 4.4] for spin-$\frac{1}{2}$ systems and for a more complicated case, since in [15] we assumed unknown initial conditions but precise knowledge of the physical parameters. Numerical simulations are provided in order to illustrate our results and to support the efficiency of the proposed candidate feedback.

a) Notations: The imaginary unit is denoted by $i$. We indicate by $I$ the identity matrix. We denote the conjugate transpose of a matrix $A$ by $A^*$. The function $\text{Tr}(A)$ corresponds to the trace of a square matrix $A$. The commutator of two square matrices $A$ and $B$ is denoted by $[A, B] := AB - BA$.

We denote by $\text{int}(S)$ the interior of a subset $S$ of a topological space and by $\partial S$ its boundary.

II. System description

We consider quantum spin-$\frac{1}{2}$ systems. In the following we describe the evolutions of the actual quantum state and its associated estimated state assuming that the initial state and the physical and experimental parameters are not known. The corresponding coupled system is given by the following stochastic master equations, in Itô form

$$d\rho_t = L^u_{\omega,M}(\rho_t)dt + G_{\eta,M}(\rho_t)dW_t,$$

where

- the actual quantum state of the spin-$\frac{1}{2}$ system is denoted as $\rho_t$ and belongs to the space $S_2 := \{\rho \in C^2 \times 1 \mid \rho = \rho^*, \text{Tr}(\rho) = 1, \rho \geq 0\}$. The associated estimated state is denoted as $\hat{\rho} \in \hat{S}_2$,
- the matrices $\sigma_x, \sigma_y$ and $\sigma_z$ correspond to the Pauli matrices,
- $L^u_{\omega,M}(\rho) := -i/2[\omega \sigma_x + u \sigma_y, \rho] + M/4(\sigma_z \rho \sigma_z - \rho)$ and $G_{\eta,M}(\rho) := \sqrt{\eta M}/2(\sigma_x \rho + \rho \sigma_x - 2\text{Tr}(\sigma_x \rho))$.

$Y_t$ denotes the observation process of the actual quantum spin-$\frac{1}{2}$ system, which is a continuous semimartingale whose quadratic variation is given by $\langle Y_t, Y_t \rangle_t = t$. Its dynamics satisfies $dY_t = dW_t + \sqrt{\eta M}\text{Tr}(\sigma_x \rho_t)dt$, where $W_t$ is a one-dimensional standard Wiener process,

$u := u(\hat{\rho}_t)$ denotes the feedback controller as a function of the estimated state $\hat{\rho}_t$,

$\omega \geq 0$ is the difference between the energies of the excited state and the ground state, $\eta \in (0, 1]$ describes the efficiency of the detector, and $M > 0$ is the strength of the interaction between the system and the probe. The estimated parameters $\hat{\omega} \geq 0, \hat{\eta} \in (0, 1]$ and $\hat{M} > 0$, which may not equal to the actual ones.

By replacing $dY_t = dW_t + \sqrt{\eta M}\text{Tr}(\sigma_x \rho_t)dt$ in the equation above, we obtain the following matrix-valued stochastic differential equations describing the time evolution of the pair $(\rho_t, \hat{\rho}_t) \in S_2 \times \hat{S}_2$,

$$d\rho_t = L^u_{\omega,M}(\rho_t)dt + G_{\eta,M}(\rho_t)dW_t,$$

$$d\hat{\rho}_t = L^u_{\omega,M}(\hat{\rho}_t)dt + G_{\eta,M}(\hat{\rho}_t)dW_t$$

$$+ G_{\eta,M}(\hat{\rho}_t)(\sqrt{\eta M}\text{Tr}(\sigma_x \rho_t) - \sqrt{\eta M}\text{Tr}(\sigma_x \hat{\rho}_t))dt$$

If $u \in C^1(S_2, \mathbb{R})$, the existence and uniqueness of the solution of (1)-(2) can be proved along the same lines of [19, Proposition 3.5]. Similarly, it can be shown as in [19, Proposition 3.7] that $(\rho_t, \hat{\rho}_t)$ is a strong Markov process in $S_2 \times \hat{S}_2$.

Recall that a density operator $\rho \in S_2$ can be uniquely characterized by the Bloch sphere coordinates $(x, y, z)$ as

$$\rho = \frac{1}{2} \left[ \begin{array}{cc} 1 + z & x - iy \\ x + iy & 1 - z \end{array} \right],$$

where the vector $(x, y, z)$ belongs to the ball $\mathcal{B} := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

The stochastic differential equations (1)-(2) expressed in Bloch sphere coordinates take the following form

$$dx_t = \left( -\omega y_t - M \frac{y_t}{2} + u z_t \right) dt - \sqrt{\eta M} y_t z_t dW_t,$$

$$dy_t = \left( \omega x_t - M \frac{y_t}{2} \right) dt - \sqrt{\eta M} y_t z_t dW_t,$$

$$dz_t = -ux_t dt + \sqrt{\eta M}(1 - z_t^2)dt,$$

$$d\hat{x}_t = (\hat{\omega} \hat{y}_t - \sqrt{\eta M} \hat{x}_t) dt - \sqrt{\eta M} \hat{y}_t \hat{z}_t dW_t,$$

$$d\hat{y}_t = (\hat{\omega} \hat{x}_t - \sqrt{\eta M} \hat{y}_t) dt - \sqrt{\eta M} \hat{y}_t \hat{z}_t dW_t,$$

$$d\hat{z}_t = (u \hat{x}_t - (1 - \hat{z}_t^2) \mathcal{E}(\hat{z}_t, \hat{z}_t)) dt + \sqrt{\hat{\eta} M}(1 - \hat{z}_t^2) dW_t,$$

where $\mathcal{E}(z, \hat{z}) := \sqrt{\hat{\eta} M} \left( \sqrt{\hat{\eta} M} \hat{z} - \sqrt{\hat{\eta} M} z \right)$.

III. Basic stochastic tools

In this section, we introduce some basic definitions and classical results which are fundamental for the rest of the paper.

Given a stochastic differential equation $d\hat{q}_t = f(q_t)dt + g(q_t)dW_t$, where $q_t$ takes values in $Q \subset \mathbb{R}^p$, the infinitesimal generator is the operator $\mathcal{L}$ acting on twice continuously differentiable functions $V : Q \times \mathbb{R}_+ \to \mathbb{R}$ in the following way

$$\mathcal{L}V(q, t) := \frac{\partial V(q, t)}{\partial t} + \sum_{i=1}^{p} \frac{\partial V(q, t)}{\partial q_i} f_i(q) + \frac{1}{2} \sum_{i,j=1}^{p} \frac{\partial^2 V(q, t)}{\partial q_i \partial q_j} g_i(q) g_j(q).$$
Ito’s formula describes the variation of the function $V$ along solutions of the stochastic differential equation and is given as follows

$$dV(q, t) = \mathcal{L}V(q, t)dt + \sum_{i=1}^{p} \frac{\partial V(q, t)}{\partial q_i} g_i(q)dW_i.$$ 

From now on, the operator $\mathcal{L}$ is associated with (1)–(2).

We recall that the Bures distance between two density matrices $\rho^{(1)}, \rho^{(2)}$ in $S_2$, is given by

$$d_B(\rho^{(1)}, \rho^{(2)}) := \sqrt{2 - 2\sqrt{\mathcal{F}(\rho^{(1)}, \rho^{(2)})}},$$

where $\mathcal{F}(\rho^{(1)}, \rho^{(2)}) := \text{Tr}(\rho^{(1)}\rho^{(2)}) + 2\sqrt{\det(\rho^{(1)})\det(\rho^{(2)})}$. In particular, the Bures distance between $\rho \in S_2$ and a pure state $\rho = \psi\psi^*$ with $\psi \in \mathbb{C}^2$, is given by $d_B(\rho, \rho) = \sqrt{2 - 2|\psi|^2}$.

In view of defining the notion of stochastic exponential stability for the coupled system (1)–(2), we introduce the distance

$$d_B((\rho^{(1)}, \rho^{(1)}), (\rho^{(2)}, \rho^{(2)})) := d_B(\rho^{(1)}, \rho^{(2)}) + d_B(\rho^{(1)}, \rho^{(2)})$$

between two elements of $S_2 \times S_2$. We denote the ball of radius $r$ around $(\rho, \rho)$ as

$$B_r(\rho, \rho) := \{(\sigma, \hat{\sigma}) \in S_2 \times S_2 | d_B((\rho, \rho), (\sigma, \hat{\sigma})) < r\}.$$

**Definition 1.** An equilibrium $(\rho, \rho)$ of the coupled system (1)–(2) is said to be almost surely exponentially stable if

$$\limsup_{t \to \infty} \frac{1}{t} \log d_B((\rho_t, \rho_t), (\rho, \rho)) < 0, \ a.s.$$ 

whenever $(\rho_0, \rho_0) \in S_2 \times S_2$. The left-hand side of the above inequality is called the sample Lyapunov exponent of the solution.

Denote $\rho_g := \text{diag}(1, 0)$ and $\rho_e := \text{diag}(0, 1)$, which are the pure states corresponding to the eigenvectors of $\sigma_\zeta$. Note that a pair $(\rho, \rho)$ is an equilibrium of (1)–(2) if and only if $(\rho, \rho) \in \{\rho_e, \rho_g\}$ and $u(\rho) = 0$. In order to introduce the final result of this section, we recall that any stochastic differential equation in Ito form in $\mathbb{R}^K$

$$dx_t = \tilde{X}_0(x_t)dt + \sum_{k=1}^{n} \tilde{X}_k(x_t)dW_k, \quad x_0 = x,$$

can be written in the following Stratonovich form [20]

$$dx_t = X_0(x_t)dt + \sum_{k=1}^{n} X_k(x_t) \circ dW_k, \quad x_0 = x,$$

where $X_0(x) = \tilde{X}_0(x) - \frac{1}{2} \sum_{k=1}^{K} \sum_{l=1}^{K} \partial_{x_l} \tilde{X}_k(x)(\tilde{X}_k)(x)$, $(\tilde{X}_k)$ denoting the component $l$ of the vector $\tilde{X}_k$, and $X_k(x) = \tilde{X}_k(x)$ for $k \neq 0$.

The following classical theorem relates the solutions of a stochastic differential equation with those of an associated deterministic one.

**Theorem 1** (Support theorem [22]). Let $X_0(t, x)$ be a bounded measurable function, uniformly Lipschitz continuous in $x$ and $X_k(t, x)$ be continuously differentiable in $t$ and twice continuously differentiable in $x$, with bounded derivatives, for $k \neq 0$. Consider the Stratonovich equation

$$dx_t = X_0(t, x_t)dt + \sum_{k=1}^{n} X_k(t, x_t) \circ dW_k, \quad x_0 = x.$$ 

Let $\mathbb{P}_x$ be the probability law of the solution $x_t$ starting at $x$. Consider in addition the associated deterministic control system

$$\frac{d}{dt} \hat{v}(t) = X_0(t, \hat{v}(t)) + \sum_{k=1}^{n} X_k(t, \hat{v}(t))v^k(t), \quad \hat{v}(0) = x,$$

with $v^k \in V$, where $V$ is the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$. Now we define $\mathcal{W}_x$ as the set of all continuous paths from $\mathbb{R}_+$ to $\mathbb{R}^K$ starting at $x$, equipped with the topology of uniform convergence on compact sets, and $\mathcal{I}_x$ as the smallest closed subset of $\mathcal{W}_x$ such that $\mathbb{P}_x(x \in \mathcal{I}_x) = 1$. Then, $\mathcal{I}_x = \{x_v(\cdot) \in \mathcal{W}_x | v \in \mathcal{V}_m \} \subset \mathcal{W}_x$.

**IV. FEEDBACK EXPONENTIAL STABILIZATION OF QUANTUM SPIN-$\frac{1}{2}$ SYSTEMS**

Our aim here is to provide sufficient conditions on the feedback controller $u(\rho)$ and a valid domain of the estimated parameters $\hat{\omega}, \hat{M}$ and $\hat{\eta}$ allowing us to exponentially stabilize the coupled system (1)–(2) towards the target state $(\rho_e, \rho_e)$. By symmetry, the case in which the target state is $(\rho_g, \rho_g)$ can be treated in the same manner.

By employing arguments similar to those in [15, Lemma 3.1], we obtain the following invariance properties for the coupled system (1)–(2).

**Lemma 1.** Let $(\rho_1, \rho_1)$ be the solution of (1)–(2) starting from $(\rho_0, \rho_0)$. If $\rho_0 > 0$, then $\mathbb{P}(\rho_t > 0, \forall t \geq 0) = 1$. Similarly, if $\rho_0 > 0$, then $\mathbb{P}(\rho_t > 0, \forall t \geq 0) = 1$. In other words, the sets $\text{int}(S_2) \times S_2$ and $S_2 \times \text{int}(S_2)$ are almost surely invariant for (1)–(2).

We make the following hypothesis on the feedback controller.

**H:** $u \in C(S_2, \mathbb{R}) \cap C^1(S_2 \setminus \{\rho_e\}, \mathbb{R}), (\rho_e, 0) = 0$ and $u(\rho_g) \neq 0$.

If $\mathcal{H}$ is satisfied, then the coupled system (1)–(2) admits exactly two equilibria $(\rho_e, \rho_e)$ and $(\rho_g, \rho_g)$.

The following result, analogous to [15, Lemma 3.3], provides sufficient conditions guaranteeing that $\rho_1$ and $\rho_1$ immediately become positive definite, almost surely.

**Lemma 2.** Assume that $\eta, \hat{\eta} \in (0, 1)$ and $\mathcal{H}$ is satisfied. Then, for all initial condition $(\rho_0, \rho_0) \in \partial(S_2 \times S_2) \setminus \{(\rho_e, \rho_e) \cup (\rho_g, \rho_g)\}, (\rho_1, \rho_1) \in \text{int}(S_2) \times \text{int}(S_2)$ for all $t > 0$ almost surely.

Next, we show the instability of the equilibrium $(\rho_g, \rho_e)$.

**Lemma 3.** Suppose that $\mathcal{H}$ is satisfied and $|u(\rho)| \leq C(1 - \text{tr}(\rho))^\alpha$ for $C > 0$ and $\alpha > 0$, then there exists $\lambda > 0$ such that, for all initial condition $(\rho_0, \rho_0) \in B_{\lambda}(\rho_g, \rho_e) \setminus (\rho_g, \rho_e)$, the trajectories of the coupled system (1)–(2) exit $B_{\lambda}(\rho_g, \rho_e)$ in finite time almost surely.
Proof. We first show that for a small enough neighborhood of \((\rho_{\eta}, \rho_{e})\) there exists a constant \(\Gamma > 2\bar{\eta}M\) such that \(\mathcal{L}(1 - \bar{z}) \geq \Gamma(1 - \bar{z})\).

Indeed, by using the fact that \(|\bar{z}| \leq \sqrt{1 - z}\), we get
\[
\mathcal{L}(1 - \bar{z}) = u\bar{z} + (1 - \bar{z}^2)E(z, \bar{z}) \\
\geq \left[-C(1 - \bar{z})^{\alpha - \frac{1}{2}} + (1 + \bar{z})E(z, \bar{z})\right](1 - \bar{z}).
\]

The bracketed expression converges to \(2\bar{\eta}M + 2\sqrt{\bar{\eta}M(1 - \bar{z})}\) as \((z, \bar{z})\) converges to \((-1, 1)\), so that for any \(\Gamma \in (2\bar{\eta}M, 2\bar{\eta}M + 2\sqrt{\bar{\eta}M(1 - \bar{z})})\) there exists a small enough neighborhood \(\mathcal{U}\) of \((\rho_{\eta}, \rho_{e})\) such that \(\mathcal{L}(1 - \bar{z}) \geq \Gamma(1 - \bar{z})\) holds true. Let \(\tau\) be the first exit time from \(\mathcal{U}\). Due to Lemma\[1\] we can apply Itô’s formula to \(\log(1 - \bar{z})\), obtaining \(\mathcal{L}(1 - \bar{z}) \geq \Gamma - 2\bar{\eta}M > 0\) on \(\mathcal{U}\). By applying Dynkin formula\[20\] to \(\log(1 - \bar{z})\) we obtain the following
\[
(\Gamma - 2\bar{\eta}M)E(\tau) \leq E(\log(1 - \bar{z})) - \log(1 - \bar{z}_0) \\
\leq (2\log(1 - \bar{z}_0) < 0.
\]

Then by Markov inequality, we get
\[
\mathbb{P}(\tau = \infty) = \lim_{m \to \infty} \mathbb{P}(\tau > m) \leq \lim_{m \to \infty} E(\tau)/m = 0.
\]

The proof is complete.\[\square\]

Denote by \(\tau_r\) the first time such that the trajectories of the coupled system\[\[1\]-\[2\]\] enter inside \(B_r(\rho_c, \rho_e)\), that is
\[
\tau_r := \inf\{t > 0 \mid (\rho_t, \tilde{\rho}_t) \in B_r(\rho_c, \rho_e)\}.
\]

We have the following lemma.

Lemma 4. Consider the coupled system\[\[1\]-\[2\]\] and suppose that the feedback controller satisfies the assumptions of Lemma\[3\]. Then, for all \(r > 0\) and any given initial state \((\rho_0, \tilde{\rho}_0) \in (S_2 \times S_2)\) \(\setminus B_r(\rho_c, \rho_e)\), \(\mathbb{P}(\tau_r < \infty) = 0\).

Proof. The lemma holds trivially true for \((\rho_0, \tilde{\rho}_0) \in B_r(\rho_c, \rho_e)\), as in that case \(\tau_r = 0\). Let us suppose that \((\rho_0, \tilde{\rho}_0) \notin (S_2 \times S_2) \setminus B_r(\rho_c, \rho_e)\).

Consider the deterministic control system\[5\] associated with\[\[1\]-\[2\]\]. Following the proof of [16, Lemma 4.1], we can easily show that, for every initial condition \((\rho_0, \tilde{\rho}_0)\) and \(\epsilon > 0\), there exist \(T \in (0, \infty)\) and a piecewise constant controller \(\pi(t)\) such that the corresponding trajectory reaches \(B_r(\rho_c, \rho_e) \cup B_r(\rho_g, \rho_e)\) by time \(T\). Due to Theorem\[1\] there exists \(\zeta \in (0, 1)\) such that \(\mathbb{P}(\rho_{\tau_e} < T) < \zeta\), where \(\mu_{\tau_e} := \inf\{t > 0 \mid (\rho_t, \tilde{\rho}_t) \in B_r(\rho_c, \rho_e) \cup B_r(\rho_g, \rho_e)\}\). By the compactness of \(S_\tau := (S_2 \times S_2) \setminus (B_r(\rho_c, \rho_e) \cup B_r(\rho_g, \rho_e))\) and the Feller continuity of \((\rho_t, \tilde{\rho}_t)\), we have \(\zeta \geq 0 > 0\) for \((\rho_0, \tilde{\rho}_0) \in S_\tau\), so that \(\sup_{(\rho_0, \tilde{\rho}_0) \in S_\tau} E(\rho_{\tau_e} \geq T) \leq 1 - \zeta_0 < 1\). Then by Dynkin inequality\[11\],
\[
\sup_{(\rho_0, \tilde{\rho}_0) \in S_\tau} E(\rho_{\tau_e} \geq T) \leq \frac{T}{1 - \zeta_0} < \infty.
\]

\[1\]Recall that \(\mathbb{P}(\rho_0, \tilde{\rho}_0)\) corresponds to the joint probability law of \((\rho_t, \tilde{\rho}_t)\) starting at \((\rho_0, \tilde{\rho}_0)\); the associated expectation is denoted by \(E(\rho_{\tau_e})\).
(see [17, Theorem 6.2] for more details). The result then follows from condition (i).

Next, under an additional assumption on the physical parameters \( \eta, \hat{\eta}, M, \hat{M} \), we show the stabilizability of (1)–(2) by explicitly exhibiting a Lyapunov function satisfying the assumptions of Theorem 2.

**Theorem 3.** Consider the coupled system (1)–(2) with \((\rho_0, \hat{\rho}_0) \in (S_2 \times S_2) \setminus (\rho_g, \hat{\rho}_g)\). If \(\hat{\eta}M < 4\eta M\) and \(u\) satisfies the assumptions of Lemma 3 then \((\rho, \hat{\rho})\) is almost surely exponentially stable with sample Lyapunov exponent less than or equal to \(-\sqrt{\hat{\eta}MM} - \frac{1}{2} \min\{\eta M - \hat{\eta}M, 0\}\).

**Proof.** We set \(V(\rho, \hat{\rho}) = \sqrt{1 - z} + \sqrt{1 - \hat{z}}\) as a candidate Lyapunov function, and we show that it satisfies the assumptions of Theorem 2.

The nonnegative function \(V\) is equal to zero at the equilibrium, it is continuous on \(S_2 \times S_2\) and twice continuously differentiable on \(\text{int}(S_2) \times \text{int}(S_2)\).

Condition (ii) of Theorem 2 follows from straightforward computations.

We show that the condition (ii) holds true. The infinitesimal generator of the candidate Lyapunov function is given by \(\dot{\mathcal{L}}V(\rho, \hat{\rho}) = uU_1(\rho, \hat{\rho}) + U_2(\rho, \hat{\rho})\), where

\[
U_1(\rho, \hat{\rho}) = \frac{1}{2}(x(1 - z)^{-\frac{1}{2}} + \hat{x}(1 - \hat{z})^{-\frac{1}{2}}),
\]

\[
U_2(\rho, \hat{\rho}) = -\frac{1}{8}(\eta M(1 + z^2)\sqrt{1 - z} + \hat{\eta}M(1 + \hat{z})\sqrt{1 - \hat{z}}) + \frac{1}{2}\sqrt{\hat{\eta}M(1 + \hat{z})}\left(\sqrt{\hat{\eta}M\hat{z}} - \sqrt{\eta Mz}\right)\sqrt{1 - \hat{z}}.
\]

Using the fact that \(|x| \leq \sqrt{2(1 - z)}\) and \(|\hat{x}| \leq \sqrt{2(1 - \hat{z})}\) we get that \(|U_1(\rho, \hat{\rho})| \leq \sqrt{2}\). Since \(|u| \leq c(1 - \text{tr}(\hat{\rho}\rho))^{\alpha} = c(\frac{1}{\Pi z})^\alpha\) for some \(c > 0\) and \(\alpha > \frac{1}{2}\), we then have

\[
\lim_{(\rho, \hat{\rho}) \to (\rho_g, \hat{\rho}_g)} \frac{uU_1(\rho, \hat{\rho})}{V(\rho, \hat{\rho})} = 0.
\]

Hence

\[
\limsup_{(\rho, \hat{\rho}) \to (\rho_g, \hat{\rho}_g)} \frac{\dot{\mathcal{L}}V(\rho, \hat{\rho})}{V(\rho, \hat{\rho})} = \limsup_{(\rho, \hat{\rho}) \to (\rho_g, \hat{\rho}_g)} \frac{U_2(\rho, \hat{\rho})}{V(\rho, \hat{\rho})} = \limsup_{(z, \hat{z}) \to (1, 1)} \frac{-\eta M\sqrt{1 - z} - 2\sqrt{\hat{\eta}MM - \hat{\eta}M}\sqrt{1 - \hat{z}}}{2V(z, \hat{z})}
\]

\[
= \frac{1}{2}\max\{-\eta M - 2\sqrt{\hat{\eta}MM - \hat{\eta}M}, \hat{\eta}M\} = -\sqrt{\hat{\eta}MM} + \frac{1}{2}\hat{\eta}M,
\]

which is negative under the assumptions of the theorem. This proves the condition (ii) of Theorem 2.

Furthermore, in the notations of Theorem 2 we have

\[
K = \liminf_{(\rho, \hat{\rho}) \to (\rho_g, \hat{\rho}_g)} \varphi^2 = \min\{\eta M, \hat{\eta}M\},
\]

and the sample Lyapunov exponent is less than or equal to

\[-\sqrt{\hat{\eta}MM} - \frac{1}{2} \min\{\eta M - \hat{\eta}M, 0\}.
\]

Next, we give an example of feedback controller satisfying the assumptions of the theorem above.

**Proposition 1.** Consider the coupled system (1)–(2) with \((\rho_0, \hat{\rho}_0) \in (S_2 \times S_2) \setminus (\rho_g, \hat{\rho}_g)\) and suppose \(\hat{\eta}M < 4\eta M\). Define the feedback controller

\[
u(\rho) = \alpha(1 - \text{Tr}(\hat{\rho}\rho))^{\beta},
\]

where \(\alpha > 0\) and \(\beta \geq 1\). Then, \((\rho, \hat{\rho})\) is almost surely exponentially stable with sample Lyapunov exponent less than or equal to \(-\sqrt{\hat{\eta}MM} - \frac{1}{2} \min\{\eta M - \hat{\eta}M, 0\}\).

Note that in [15, Conjecture 4.4] we proposed candidate feedback laws in order to exponentially stabilize spin-J systems in the case of unknown initial states. Proposition 1 provides a positive answer to such a conjecture assuming, in addition to unknown initial states, unawareness of the physical parameters.

**Remark 1.** By a symmetric reasoning, the feedback controller

\[
u(\rho) = \alpha(1 - \text{Tr}(\rho\hat{\rho}))^{\beta},
\]

with \(\alpha > 0\) and \(\beta \geq 1\), almost surely exponentially stabilizes the coupled system (1)–(2) with \((\rho_0, \hat{\rho}_0) \in (S_2 \times S_2) \setminus (\rho_g, \hat{\rho}_g)\), towards \((\rho_g, \hat{\rho}_g)\) with sample Lyapunov exponent less than or equal to \(-\sqrt{\hat{\eta}MM} - \frac{1}{2} \min\{\eta M - \hat{\eta}M, 0\}\).

V. Simulation

In this section, we first illustrate the convergence of the coupled system (1)–(2) starting at \((x_0, y_0, z_0) = (1, 0, 0)\) and \((x_0, y_0, z_0) = (0, 1, 0)\) towards the target state \((\rho_g, \hat{\rho}_g)\) by applying a feedback law of the form (7). This is shown in Figure 1. Then, in Figure 2 we show the convergence of the coupled system starting at the same initial states, towards the target state \((\rho_g, \hat{\rho}_g)\) by a feedback law of the form (5).

By Equation (6), heuristically we have that the rate of convergence of the expectation of the Lyapunov function is less than or equal to \(\nu_w := -\sqrt{\hat{\eta}MM} + \hat{\eta}M/2\). This property is confirmed through simulations, see Fig. 1 and Fig. 2 (for the target state \((\rho_g, \hat{\rho}_g)\), we take \(V(\rho, \hat{\rho}) = \sqrt{1 + z} + \sqrt{1 + \hat{z}}\)). In the figures, the blue curves represent the exponential reference with the exponent \(\nu_w\), and the black curves describe the mean values of the Lyapunov functions (Bures distances) of ten samples. On the figures, in particular in the semi-log versions, we can see that the black and the blue curves have similar asymptotic behaviors. The red curves describe the exponential reference with exponent \(\nu_s := -\sqrt{\hat{\eta}MM} + \hat{\eta}M/2\). They represent the behaviors of ten sample trajectories. We observe that the red curves and the cyan curves have similar asymptotic behaviors.

VI. Conclusion

In this paper, we studied the robustness of the stabilizing feedback strategy proposed in [16] for the case of spin-1/2 systems if initial states and physical parameters are unknown. We showed such a robustness property by analyzing the asymptotic behavior of the coupled system describing the evolutions of the quantum filter and the associated estimated
state under appropriate assumptions on the feedback controller. More precisely, we showed exponential stabilization of the coupled system towards a pair \((\hat{\rho}, \hat{\rho})\), with \(\hat{\rho}\) being a chosen eigenstate of the measurement operator \(\sigma_z\). Moreover, we gave an example of feedback control law proving [15, Conjecture 4.4] for spin-\(\frac{1}{2}\) systems and supposing, in addition to unknown initial states and unlike [15], that the exact values of the physical parameters are not accessible. A future research line will concern the robustness properties of the feedback controller considered in [17] for spin-\(J\) systems.

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