On the misinterpretation of conditionally-solvable quantum-mechanical problems

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Abstract

We apply the Frobenius (power-series) method to some simple exactly-solvable and conditionally-solvable quantum-mechanical models with supposed physical interest. We show that the supposedly exact solutions to radial eigenvalue equations derived in recent papers are not correct because they do not satisfy some well-known theorems. We also discuss the origin of the mistake by means of the approach indicated above.

1 Introduction

Conditionally-solvable quantum-mechanical problems have been of great interest during the last decades (see, for example, Turbiner’s remarkable review [1] and the references therein). However, the solutions to the eigenvalue equations stemming from such models have been misinterpreted in a wide variety of physical applications [2–4].

In a recent paper, Mustafa [5] derived apparently exact solutions to an eigenvalue equation that is known to be conditionally solvable [2–4]. For this reason, we deem it necessary to discuss Mustafa’s results in some detail. In section [2] we

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outline Mustafa’s models. In section 3 we solve one of them, which is actually exactly solvable, by means of the Frobenius (power-series) method. In section 4 we apply the Frobenius method to a conditionally-solvable radial eigenvalue equation that contains Mustafa’s ones as particular cases. Finally, in section 5 we summarize the main results and draw conclusions.

2 Physical models

In this section we outline the eigenvalue equations for three models discussed by Mustafa [5]. The physical meaning of the parameters is not relevant for present discussion and the interested reader is referred to that paper.

For the “KG-oscillator in cosmic string spacetime within KKT” Mustafa [5] derived the radial equation

$$U''(r) + \left[ \frac{\lambda + \frac{1}{4} - \tilde{\gamma}^2}{r^2} - \tilde{\omega}^2 r^2 \right] U(r) = 0,$$

and obtained the eigenvalues

$$\tilde{\lambda}^M = 2\tilde{\omega} \left( 2n_r + |\tilde{\gamma}| + 1 \right),$$

where $n_r = 0, 1, \ldots$ is the radial quantum number.

By means of the change of variables $\rho = \tilde{\omega}^{1/2} r$ we obtain

$$F''(\rho) + \left[ \frac{\lambda}{\tilde{\omega}} + \frac{1/4 - \tilde{\gamma}^2}{\rho^2} - \rho^2 \right] F(\rho) = 0.$$

For the “pseudo-confined PDM KG-oscillator in cosmic string spacetime within KKT” Mustafa [5] derived the radial equation

$$U''(r) + \left[ \mathcal{E} + \frac{1/4 - \tilde{\beta}^2}{r^2} - \tilde{\omega}^2 r^2 - \eta\tilde{a} r - \tilde{b} \right] U(r) = 0,$$

and obtained the eigenvalues

$$\mathcal{E}^M = 2\tilde{\omega} \left( 2n_r + |\tilde{\beta}| + 1 \right) - \frac{\tilde{a}^2 \eta^2}{4\tilde{\omega}^2}.$$

Curiously, these eigenvalues do not depend on $\tilde{b}$ in spite of the fact that the eigenvalue equation already depends on this model parameter. Well-known general
theorems are useful for testing results; here, we can resort to the celebrated Hellmann-Feynman theorem (HFT) \cite{[6],[7]} that in the present case states that
\begin{equation}
\frac{\partial E}{\partial \tilde{b}} = \langle \frac{1}{r} \rangle > 0, \quad \frac{\partial E}{\partial \eta} = \tilde{a} \langle r \rangle.
\end{equation}

Clearly, Mustafa's result cannot be correct because
\begin{equation}
\frac{\partial E}{\partial \tilde{b}} = 0, \quad \frac{\partial E}{\partial \eta} = -\frac{\tilde{a}^2 \eta}{2\tilde{\omega}^2}.
\end{equation}

Another unmistakable indication that equation (5) is not correct is that $E_M$ does not yield the eigenvalues of the Coulomb problem when $\tilde{\omega} = 0$ and $\tilde{a} = 0$. The same change of variables indicated above yields
\begin{equation}
F''(\rho) + \left[ \tilde{\lambda}^1 + \frac{1/4 - \tilde{\gamma}^2}{\rho^2} - \rho^2 - \frac{\eta \tilde{a}}{\tilde{\omega}^{3/2}}\rho - \frac{\tilde{b}}{\tilde{\omega}^{1/2}}\rho \right] F(\rho) = 0.
\end{equation}

For the "confined PDM KG-oscillator-III in cosmic string spacetime within KKT" Mustafa \cite{[5]} derived the radial eigenvalue equation
\begin{equation}
U''(r) + \left[ \tilde{\lambda}^1 + \frac{1/4 - \tilde{\gamma}^2}{r^2} - \tilde{\omega}^2 r^2 - 2mA^r - \frac{2B}{r} \right] U(r) = 0,
\end{equation}
and obtained the eigenvalues
\begin{equation}
\tilde{\lambda}^M_1 = 2\tilde{\omega}_1 (2n_r + |\tilde{\gamma}_1| + 1) - \frac{m^2 A^2}{\tilde{\omega}_1^2}.
\end{equation}

In this case, the HFT states that
\begin{equation}
\frac{\partial \tilde{\lambda}^M_1}{\partial B} = 2 \langle \frac{1}{r} \rangle > 0, \quad \frac{\partial \tilde{\lambda}^M_1}{\partial A} = 2m \langle r \rangle,
\end{equation}
which are not satisfied by Mustafa's $\tilde{\lambda}^M_1$. Besides, $\tilde{\lambda}^M_1$ does not yield the eigenvalues of the Coulomb problem when $\tilde{\omega}_1 = 0$ and $A = 0$.

The change of variables $\rho = \tilde{\omega}_1^{1/2} r$ yields
\begin{equation}
F''(\rho) + \left[ \tilde{\lambda}^1_1 + \frac{1/4 - \tilde{\gamma}^2}{\rho^2} - \rho^2 - \frac{2mA^r}{\tilde{\omega}_1^{3/2}}\rho - \frac{2B}{\tilde{\omega}_1^{1/2}}\rho \right] F(\rho) = 0.
\end{equation}

It is clear from the analysis above that Mustafa's analytical expressions for the energy in his equations (33) and (38) cannot be correct. Consequently, all the physical conclusions derived from such equations, and shown in Mustafa’s
figures 3 and 4, are based on wrong analytical expressions for the eigenvalues $E^M$ and $\tilde{\lambda}_1^M$.

The three radial eigenvalue equations outlined above are particular cases of

$$F''(\rho) + \left[ W + \frac{1/4 - \gamma^2}{\rho^2} - \rho^2 - \rho \frac{a}{\rho} - b \rho \right] F(\rho) = 0,$$

(13)

where $a$ and $b$ are arbitrary real parameters. We are interested in square-integrable solutions $F(\rho)$ that vanish at origin. Such solutions only take place for particular values of the eigenvalue $W$ that we may label as $W_{\nu,\gamma}(a, b)$, $\nu = 0, 1, \ldots$, in such a way that $W_{\nu,\gamma} < W_{\nu+1,\gamma}$.

From the HFT we conclude that

$$\frac{\partial W}{\partial a} = \left\langle \frac{1}{\rho} \right\rangle > 0, \quad \frac{\partial W}{\partial b} = \langle \rho \rangle > 0.$$  

(14)

3 The exactly-solvable case

The eigenvalue equation (13) is exactly solvable only when $a = b = 0$. In order to obtain exact solutions for this particularly simple case we resort to the well known Frobenius (power-series) method. If we insert the ansatz

$$F(\rho) = \rho^s \exp \left( -\frac{\rho^2}{2} \right) \sum_{j=0}^{\infty} c_j \rho^{2j}, \quad s = |\gamma| + \frac{1}{2},$$

(15)

into the eigenvalue equation we find that the expansion coefficients $c_j$ satisfy the two-term recurrence relation

$$c_{j+1} = \frac{4j + 2s - W + 1}{2(j + 1)(2j + 2s + 1)} c_j, \quad j = 0, 1, \ldots.$$  

(16)

For arbitrary values of $W$ the solution (15) is not square integrable [8]; however if we choose

$$W = W_{\nu,\gamma} = 2(2\nu + |\gamma| + 1), \quad \nu = 0, 1, \ldots,$$

(17)

the series reduces to a polynomial and the resulting eigenfunctions are square integrable. Note that Mustafa’s result (2) is correct because $W = \tilde{\lambda}_1^M / \tilde{\omega}$ when $\nu = n_r$. The two-term recurrence relation thus takes the following simpler form

$$c_{j+1,\nu,\gamma} = \frac{2(j - \nu)}{(j + 1)(2j + 2s + 1)} c_{j,\nu,\gamma},$$

(18)
and the eigenfunctions become

\[ F_{\nu,\gamma}(\rho) = \rho^s \exp \left( -\frac{\rho^2}{2} \right) \sum_{j=0}^{\nu} c_{j,\nu,\gamma} \rho^j. \]  

(19)

4 Conditionally-solvable models

When one of the model parameters \( a \) or \( b \) is nonzero the eigenvalue equation (13) is not exactly solvable. It is an example of conditionally-solvable quantum-mechanical problems [1]. Mustafa [5] obtained the results outlined above from a dubious analysis of the biconfluent Heun function. In this section we resort to the Frobenius method because it is not only simpler and clearer but leaves no room for doubts.

In the general case we resort to the ansatz

\[ F(\rho) = \rho^s \exp \left( -\frac{b}{2} \rho^2 - \frac{\rho^2}{2} \right) \sum_{j=0}^{\infty} c_j \rho^j, \quad s = |\gamma| + \frac{1}{2}, \]  

(20)

that leads to the three-term recurrence relation

\[ c_{j+2} = A_j c_{j+1} + B_j c_j = 0, \quad j = -1, 0, 1, \ldots, c_{-1} = 0, \quad c_0 = 1, \]

\[ A_j = \frac{a + b (j + s + 1)}{(j + 2) (j + 2s + 1)}, \quad B_j = \frac{4 (2j + 2s - W + 1) - b^2}{4 (j + 2) (j + 2s + 1)}. \]  

(21)

In order to have a polynomial of degree \( n \) we require that \( c_n \neq 0 \), \( c_{n+1} = 0 \) and \( c_{n+2} = 0 \), \( n = 0, 1, \ldots \), that clearly leads to \( c_j = 0 \) for all \( j > n \). It follows from this condition that \( B_n = 0 \) that yields

\[ W = W_{\gamma}^{(n)} = 2n + 2s + 1 - \frac{b^2}{4} = 2 (n + |\gamma| + 1) - \frac{b^2}{4}. \]  

(22)

When \( W = W_{\gamma}^{(n)} \) \( B_j \) takes the simpler form

\[ B_j = \frac{2 (j - n)}{(j + 2) (j + 2s + 1)}. \]  

(23)

Present expression for \( W \) is not consistent with Mustafa’s results (5) and (10) unless \( n = 2n_r \). However, this discrepancy is irrelevant because neither \( \mathcal{E}^M, \tilde{z}_1^M \) or \( W_{\gamma}^{(n)} \) are the eigenvalues of the corresponding radial equations [2–4]. Note that \( W_{\gamma}^{(n)} \) also fails to satisfy the HFT [14].
The most important point is that, in order to obtain such particular polynomial solutions, we need a second condition $c_{n+1}(a, b) = 0$ already omitted by Mustafa [5]. Since $c_j(a, b)$ is a polynomial function of degree $j$ in each model parameter we conclude that we have $n + 1$ roots $a^{(n,i)}(b)$, $i = 1, 2, \ldots, n + 1$, for a given value of $b$, or $b^{(n,i)}(a)$ for each value of $a$. It can be proved that all the roots are real [2, 9]. The exact polynomial solutions for a given value of $b$ are

$$F^{(n,i)}_\gamma(\rho) = \rho^s \exp \left( -\frac{b}{2} \rho - \frac{\rho^2}{2} \right) \sum_{j=0}^{n} c_j^{(n,i)} \rho^j.$$  

(24)

The meaning of this kind of solutions has been discussed extensively in recent papers [2–4]. However, in order to make present one sufficiently self contained and convince the reader that our results are correct we show some examples in what follows.

When $n = 0$ we obtain

$$c_1(a, b) = \frac{a + bs}{2s} = 0,$$  

(25)

and the exact solution

$$F^{(0)}_\gamma(\rho) = \rho^s \exp \left( -\frac{b}{2} \rho - \frac{\rho^2}{2} \right),$$  

(26)

of equation [13] with $W = W^{(0)}_\gamma$ and $a = a^{(0)}_\gamma(b) = -bs$. Note that $F^{(0)}_\gamma(\rho)$ is the ground state of a model given by $a = a^{(0)}_\gamma(b)$.

When $n = 1$ the second condition reads

$$c_2(a, b) = \frac{a^2 + ab(2s + 1) + b^2s(s + 1) - 4s}{4s(2s + 1)} = 0,$$  

(27)

with roots

$$a_\gamma^{(1,1)}(b) = \frac{\sqrt{b^2 + 16s} - b(2s + 1)}{2}, \quad a_\gamma^{(1,2)}(b) = -\frac{\sqrt{b^2 + 16s} + b(2s + 1)}{2},$$  

(28)

from which we obtain

$$F^{(1,1)}_\gamma(\rho) = \rho^s \exp \left( -\frac{b}{2} \rho - \frac{\rho^2}{2} \right) \left( 1 + \frac{\sqrt{b^2 + 16s} - b}{4s} \rho \right),$$  

$$F^{(1,2)}_\gamma(\rho) = \rho^s \exp \left( -\frac{b}{2} \rho - \frac{\rho^2}{2} \right) \left( 1 - \frac{\sqrt{b^2 + 16s} + b}{4s} \rho \right).$$  

(29)
Note that $F_\gamma^{(1,1)}(\rho)$ is the ground state of a model with $a = a_\gamma^{(1,1)}(b)$ while $F_\gamma^{(1,2)}(\rho)$ is the first excited state for a model given by $a = a_\gamma^{(1,2)}(b)$, both for $W = W^{(1)}$. Anybody can easily verify that the functions in equations (26) and (29) are solutions of equation (13) under the conditions just indicated.

As already stated above, this kind of solutions has been extensively discussed in recent papers [2–4]. We just want to point out that the occurrence of the exact polynomial solutions (24) requires a second condition (overlooked by Mustafa [5]) that restricts the values of the model parameters and that such solutions are not the only ones [2–4]. Therefore, any physical conclusion derived from eigenvalues like (5) and (10) are meaningless. In particular, Mustafa’s expressions for the energy in his equations (33) and (38) cannot be correct. Consequently, all the physical conclusions derived from such equations, and shown in Mustafa’s figures 3 and 4, are based on wrong analytical expressions for the eigenvalues $E^M$ and $\tilde{\lambda}_1^M$.

5 Conclusions

We have already shown that the exact results obtained by Mustafa [5] for the eigenvalue equations (4) and (9) are not correct because they are not exactly solvable. Those equations are conditionally solvable and one obtains some particular polynomial solutions for particular values of the model parameters determined by a second condition overlooked by Mustafa. This conclusion also applies to another paper by the same author where he apparently derived exact results for a similar conditionally-solvable problem [10]. Most physical conclusions commonly derived from the polynomial solutions to conditionally-solvable eigenvalue equations are meaningless [2–4].

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