Cooperative Resonance Interaction Between One- and Two-Photon Super-Fluorescences Through the Vacuum Field

Nicolae A. Enaki
Academiei str.5, Institute of Applied Physics of Academy of Sciences of Moldova,
Chisinau, MD 2028, Republic of Moldova
E-mail: enakinicolae@yahoo.com

Abstract. The theoretical approach takes into consideration the cooperative phenomena which appear between three particles in two-photon resonance in the process of single- and two-photon decay. This type of single- and two-quanta cooperative effect between three subsystems of radiators are described by master equation which takes into account three-particle cooperative resonance in the system. The resonance between the spontaneous emissions by two- and single photon transitions of three inverted radiators from the ensemble is proposed in order to accelerate the collective decay rate of the entangled photon pairs generated by the system. This effect is accompanied with the interferences between one-photon and two-quantum collective transitions of three inverted radiators from the ensemble. The three particle collective decay rate is defined in the description of three atomic correlation functions.

1. Introduction
Recently much attention is devoted to the problem of coherence which appears not only between the "individual" photons but also between groups of quanta. The generation of non-classical coherent electromagnetic field (EMF) in multi-photon emission and the interaction of coherent radiation with matter (nuclei, atoms and solids) have been the subjects of several theoretical and experimental studies in recent years [1]-[4]. Examples include the higher-order coherence in multi-photon generation of light the two-photon micro-maser emission [4], two-photon lasers the parametric down conversion, four-wave mixing and other effects in optical diapason [2], and the possibility of coherent generation of photons in x- and gamma -spectral regions [3].

This article aims to investigate the cooperative emission of the inverted system of radiators taking into account the resonance between one- and two-photon cooperative transitions of three atomic subsystems. Since the two-photon cooperative phenomenon has a small two-quantum cooperative emission rate[5], we propose to extend our attention to the new type of cooperative resonance interaction between three radiators, in which the single-photon transitions of two radiators enter the two-quantum resonance with dipole forbidden transition of third atom. This cooperative effect between three particles occurs through the vacuum fluctuations of EMF and can amplify or diminish the spontaneous emission rates of each atomic subsystem as a function of the distances between them. To obtain more powerful pulses of entangled photons, it is proposed the cooperative interaction of three radiator subsystems in which one of them are inverted relatively the dipole forbidden transition of Hydrogen-like or Helium-like atoms (ions).
The big number of classical experimental [7] and theoretical [6] descriptions of two-photon decay relatively the transitions \(2^2S_{1/2} \to 1^2S_{1/2}\) and \(2^1S_0 \to 1^1S_0\) give us the larger possibilities in the experimental realization of such resonances [8, 9]. Taking into account the elementary acts of two photon resonance between radiators, we have demonstrated the increasing of two-photon emission rate in one of radiator subsystem comparison with traditional two-photon super-fluorescence \([5, 10]\). The mutual influence of two single-photon super-fluorescence processes on the two-photon cooperative emission of the inverted atomic subsystem relatively dipole forbidden transition is examined in this paper. The elementary acts of new cooperative phenomenon consider the three-particle mutual interactions through the vacuum of EMF field, in which the product of vacuum polarization of two atoms enters in resonance with two-photon polarization of dipole forbidden transition.

In the second section of this article it is proposed the effective method of elimination of EMF operators from generalized equation of atomic subsystems in two-photon resonance. Note, that the master equation of such a system of radiators was obtained for the sample of atoms with the dimension commensurable with the radiation wavelengths. The quantum approach of elimination of bi-boson operators from master equation of atomic subsystem is proposed. The method gives us the possibility to take into account the quantum fluctuations among three radiators in two-quantum resonance through the vacuum field. The generalized equation of such three particle interaction is given in the next section.

In the third part of the paper we studied the spontaneous emission of three radiators \(j\)-th, \(l\)-th and \(m\)-th, situated at distance \(r = r_{mn} = r_{nl}\) relatively the dipole forbidden atom \(D\). As the one-photon resonance between atoms \(S\) and \(R\) is absent, the distance \(r_{jl}\) is considered as an arbitrary basis of isosceles triangle \(SRD\). In this paper it is studied the mutual influence between the spontaneous emissions of three atomic subsystems localized in the isosceles triangle vertices \(SRD\), considering that the exchange integral around the subsystem \(D\) consists from two domain.

2. Interaction Hamiltonian and Master Equation

Let us consider the interaction of three subsystems of radiators \(R\), \(S\), and \(D\) through the vacuum of EMF. The first two groups of radiators, \(R\) and \(S\), prepared in excited state \(|e_r\rangle \otimes |e_s\rangle\) can pass into Decke super-radiance \([11\), \([12]\) relatively the dipole active transitions \(e_r \to g_r\) and \(e_s \to g_s\) at frequencies \(\omega_r\) and \(\omega_s\). Preparing in the excited state relative the dipole forbidden transition \(2_d \to 1_d\), the \(D\) atomic subsystem passes in the ground state \(|1_d\rangle\) simultaneously generation two quanta under the influences of first two atomic subsystems and vacuum fluctuations of EMF. This transition takes place through the virtual levels, which we replaced by the state \(|3_d\rangle\) with opposite symmetry relative the states \(|1_d\rangle\) and \(|2_d\rangle\). Considering that the conservation energy law is established between the transitions of three atoms, \(\hbar(\omega_r + \omega_s) = 2\hbar\omega_0\), one can propose the following Hamiltonian of the system

\[
\dot{H} = \dot{H}_0 + \lambda \dot{H}_I, \tag{1}
\]

where

\[
\dot{H}_0 = \sum_k h\omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_{j=1}^{N_r} h\omega_r \hat{R}_{xz}^j + \sum_{l=1}^{N_s} h\omega_s \hat{S}_{zd}^l + 2 \sum_{m=1}^{N} h\omega_0 \hat{D}_{zm}
\]

is the free and

\[
\lambda \dot{H}_I = -\sum_{k, j=1}^{N_r} \langle d_r, g_k \rangle \{ \hat{R}_j^+ \hat{a}_k \exp[i(k, r_j)] + \hat{R}_j^- \hat{a}_k^\dagger \exp[-i(k, r_j)] \}
\]
\[
- \sum_{k} \sum_{i=1}^{N_k} (d_s, g_k) \{ \hat{S}_i^+ \hat{a}_k \exp[i(\mathbf{k}, r_i)] + \hat{S}_i^- \hat{a}_k^\dagger \exp[-i(\mathbf{k}, r_i)] \}
- \sum_{k_1, k_2} \sum_{m=1}^{N_k} (n_{31}, e_{\lambda_1})(n_{32}, e_{\lambda_2}) q(\omega_1, \omega_2) \{ \hat{D}_m^+ \hat{a}_{k_1} \hat{a}_{k_2} \exp[i(\mathbf{k}_1 + \mathbf{k}_2, r_m)] \\
+ \hat{D}_m^\dagger \hat{a}_{k_1}^\dagger \hat{a}_{k_2} \exp[-i(\mathbf{k}_1 + \mathbf{k}_2, r_i)] \}
\]

is the interaction parts of Hamiltonian (1). Here we have:
\[
q(\omega_1, \omega_2) = \frac{d_{32} d_{31} g_{k_1} g_{k_2}}{2 \hbar} \left( \frac{1}{\omega_{32} + \omega_{k_1}} + \frac{1}{\omega_{31} - \omega_{k_2}} \right), \quad g_k = \sqrt{\frac{2 \pi \hbar \omega_k}{V \epsilon_s}},
\]
\(\hat{a}_k\) and \(\hat{a}_k^\dagger\) are annihilation and creation operators of EMF photons with wave vector \(\mathbf{k}\), polarization \(\epsilon_s\), and frequency \(\omega_k\); \(d_s\) and \(d_r\) are dipole momentum transition between the ground and excited states for \(R\) and \(S\) atomic subsystems; \(d_{32i}\) and \(d_{31}\) are dipole momentum transitions in the three level system of atomic group \(D\). The operators of \(R\), \(S\), and \(D\) atomic subsystems satisfy the commutation relations for \(SU(2)\) algebra [\(J^+, J^-\) = 2\(J_z\); \([J_z, J^\pm]\) = ±\(J^\pm\), where \(J^\pm\) is equivalent with \(R^\pm\), \(S^\pm\) and \(D^\pm\). The inversion operator \(J_z\) is one of \(R_z\), \(S_z\) and \(D_z\) operators respectively. The operators of EMF satisfy the commutation relation \([\hat{a}_k, \hat{a}_{k'}^\dagger\] = \(\delta_{k, k'}\); \([\hat{a}_k^\dagger, \hat{a}_{k'}]\) = 0, where \(k = (\mathbf{k}, \lambda)\) is the wave vector and polarization of the photon.

According to the Hamiltonian (1), let us represent the solution of Haisenberg equation for EMF operators \(\hat{a}_k(t)\) and \(\hat{a}_k^\dagger(t)\) through the sources and free operator parts
\[
\hat{a}_k(t) = \hat{a}_k(0) \exp[-i\omega_k t] + \hat{a}_k^\dagger(t), \quad \hat{a}_k^\dagger(t) = [\hat{a}_k(t)]^\dagger,
\]
where \(\hat{a}_k^\dagger(t) = \hat{a}_k(0) \exp[-i\omega_k t] \) and \(\hat{a}_k(t) \) are the vacuum and source parts respectively. Here the source part is represented through the atomic polarization operators of the system
\[
\hat{a}_k^\dagger(t) = 2i \sum_{n=1}^{N} \sum_{k_1} (n_{31}, e_{\lambda_1})(n_{32}, e_{\lambda_2}) q(\omega_1, \omega_2) \frac{i(d_s, g_k)}{\hbar} \left( \int_0^t d\tau \exp[-i\omega_k \tau] \right)
\times \hat{D}_m^-(t - \tau) \hat{a}_{k_1}^\dagger(t - \tau) + \sum_{l=1}^{N} \frac{i(d_r, g_k)}{\hbar} \left( \int_0^t d\tau \exp[-i\omega_k \tau] \hat{R}_l^-(t - \tau) \right)
+ \frac{i(d_s, g_k)}{\hbar} \left( \sum_{j=1}^{N_s} \int_0^t d\tau \exp[-i\omega_k \tau] S_j^-(t - \tau) \hat{a}_k^\dagger(t - \tau) \right)
\]

Let us now consider the Heisenberg equation for the mean value of the arbitrary atomic operator \(\langle 0, \Psi_0(0) | \hat{O}(t) | 0, \Psi_0(0) \rangle\), where \(\langle 0, \Psi_0(0) \rangle = |0\rangle |\Psi_0(0)\rangle\) is the initial state of EMF and atomic subsystems. Taking into account, the action of vacuum parts of operators \(\hat{a}_k(t)\) and \(\hat{a}_k^\dagger(t)\) on the vacuum state of EMF \(\hat{a}_k(0) |0\rangle_{ph} = |0\rangle |\hat{a}_k(0)\rangle = 0\), we can partially eliminate the free parts of EMF field operators from the equation for atomic operators. Indeed considering that at initial time \(t = 0\) the EMF is in the vacuum state, we can represent the Heisenberg equation for the mean value of the arbitrary atomic operator \(\langle \hat{O}(t) \rangle\) through the normal product of EMF operators
\[
\frac{d}{dt} \langle \hat{O}(t) \rangle = -i \sum_{k} \sum_{j=1}^{N_s} \frac{(d_r, g_k)}{\hbar} \langle [\hat{R}_j^+(t), \hat{O}(t)] |\hat{a}_k(t)\rangle \exp[i(\mathbf{k}, r_j)]
\]
Lemma 1

If the Bose \(\hat{a}_k(t)\) and \(\hat{a}_k^\dagger(t)\) operators lie between two operators of atomic subsystem \(\hat{A}(t_1)\) and \(\hat{B}(t_2)\), which belong to other time moments \(t_1\) and \(t_2\), the elimination of the free part of these operators yields to the following expression for the correlation

\[
\begin{align*}
\langle \hat{A}(t_1)\hat{a}_k(t)\hat{B}(t_2) \rangle &= \langle \hat{A}(t_1)\hat{a}_k^\dagger(t)\hat{B}(t_2) \rangle - e^{-i\omega(t-t_2)} \langle \hat{A}(t_1)\hat{a}_k^\dagger(t)\hat{B}(t_2) \rangle, \\
\langle \hat{A}(t_1)\hat{a}_k(t)\hat{B}(t_2) \rangle &= \langle \hat{A}(t_1)\hat{a}_k(t)\hat{B}(t_2) \rangle - e^{i\omega(t-t_1)} \langle \hat{A}(t_1)\hat{a}_k^\dagger(t)\hat{B}(t_2) \rangle.
\end{align*}
\] (4)

Proof.
The commutations in (4) play the highest role in the two-photon spontaneous emission and such commutations bring the main contribution to the two-photon resonances between three particles. The problem is reduced to the elimination of vacuum part, which lies between the operators \(\hat{A}(t_1)\) and \(\hat{B}(t_2)\)

\[
\langle \hat{A}(t_1)\hat{a}_k(t)\hat{B}(t_2) \rangle = \langle \hat{A}(t_1)(\hat{a}_k^\dagger(t) + \hat{a}_k(t))\hat{B}(t_2) \rangle.
\]

Since \(\hat{a}_k(t) = \hat{a}_k^\dagger(t) + \hat{a}_{ks}(t)\), we will represent the vacuum part \(\hat{a}_k^\dagger(t) = \hat{a}_k(0)\exp[-i\omega_k t]\) through the vacuum operator at the time moment \(t_1\). Taking into account the identity (2), we can represent the vacuum part in the following form \(\hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(t_2)e^{-i\omega_k(t-t_2)} = \{\hat{a}_k(t_2) - \hat{a}_{ks}(t_2)\}e^{-i\omega_k(t-t_2)}. \) After the substitution of \(\hat{a}_k^\dagger(t)\) into correlation it is obtained the expression

\[
\langle \hat{A}(t_1)\hat{a}_k(t)\hat{B}(t_2) \rangle = \langle \hat{A}(t_1)\hat{a}_k^\dagger(t)\hat{B}(t_2) \rangle + e^{-i\omega_k(t-t_2)}\langle \hat{A}(t_1)\{\hat{a}_k(t_2) - \hat{a}_k^\dagger(t_2)\}\hat{B}(t_2) \rangle.
\]

We observe that operator \(\hat{a}_k(t_2)\) commutes with the operator \(\hat{B}(t_2)\). Consequently taking into account, that \(\hat{a}_k(t_2)\hat{B}(t_2)\hat{B}(t_2)\hat{a}_k(t_2)|0>=\hat{B}(t_2)\hat{a}_k(t_2)|0\), it is easily obtained that

\[
\langle \hat{A}(t_1)\{\hat{a}_k(t_2) - \hat{a}_k^\dagger(t_2)\}\hat{B}(t_2) \rangle = -\langle \hat{A}(t_1)\hat{a}_k^\dagger(t_2)\hat{B}(t_2) \rangle.
\]

This identity proofs the Lemma (4). □
This lemma (4) can be used in the last term of generalized equation (3) for correlation functions like $\langle [\hat{D}^+_j(t) , \hat{O}(t)]a_k(t)\hat{S}^-_n(t - \tau) \rangle$, $\langle [\hat{D}^+_j(t) , \hat{O}(t)]a_k(t)\hat{R}^-_n(t - \tau) \rangle$ and $\langle [\hat{R}^+_j(t) , \hat{O}(t)]\hat{D}^-_n(t - \tau)a_{k_1}^\dagger(t - \tau) \rangle$, $\langle [\hat{S}^+_n(t) , \hat{O}(t)]\hat{D}^-_n(t - \tau)a_{k_1}^\dagger(t - \tau) \rangle$. Taking into account the lemma (4), we observe that the above correlation functions can be simply represented through atomic operators

$$\langle [\hat{D}^+_j(t) , \hat{O}(t)]a_k(t)\hat{S}^-_n(t - \tau) \rangle = \langle [\hat{D}^+_j(t) , \hat{O}(t)]\hat{a}_{k_2}(t)\hat{S}^-_n(t - \tau) \rangle - e^{-i\omega_k \tau} \langle [\hat{D}^+_j(t) , \hat{O}(t)][\hat{a}_{k_2}(t) , \hat{S}^-_n(t - \tau)] \rangle,$$  

$$\langle [\hat{R}^+_j(t) , \hat{O}(t)]a_{k_1}^\dagger(t - \tau)\hat{D}^-_n(t - \tau) \rangle = \langle [\hat{R}^+_j(t) , \hat{O}(t)]a_{k_2}(t)\hat{D}^-_n(t - \tau) \rangle - e^{-i\omega_k \tau} \langle [\hat{R}^+_j(t) , \hat{O}(t)][\hat{a}_{k_2}(t) , \hat{D}^-_n(t - \tau)] \rangle.$$  

The interaction between the electromagnetic field and atomic subsystems can be found in the third order of interaction constants $(d_a, g_k)(d_s, g_k)q(\omega_1, \omega_2)$ respectively. According to this condition, the smooth correlation functions can be obtained only for the following terms of expressions (5) and (6) : $\langle R^+_1(t - \tau')[D^+_j(t), O(t)]S^-_n(t - \tau) \rangle$ and $\langle R^+_1(t - \tau')[R^+_j(t), O(t)]D^-_n(t - \tau) \rangle$. Other terms of expressions (5) and (6) give the contribution in the higher order of decomposition on the small parameter $\lambda$ (see the interaction Hamiltonian (1)).

The lemma (4) is non-applicable for correlation functions in which it is used simultaneously the creation and annihilations Boson operators belonging to different time moments : $\langle [D^+_j(t), O(t)]a_k(t)D^-_n(t - \tau)a_{k_1}^\dagger(t - \tau) \rangle$ and its hermit conjugate part $\langle a_{k_1}(t - \tau)D^-_n(t - \tau)a_k(t)\rangle$. To eliminate the vacuum part of operators $a_k(t)$ and $a_{k_1}(t')$ let us formulate the following rule.

**Lemma 2** If the arbitrary operator $\hat{A}(t_1)$ contains the creation operators of EMF and $\hat{B}(t_2)$ contains the annihilation operators of EMF, the elimination of vacuum part of annihilation $a_k(t)$ or creation $a_{k_1}^\dagger(t)$ operators situated between these operators $\hat{A}(t_1)$ and $\hat{B}(t_2)$ takes place according with Lemma 1. In opposite situation, we can represent the operator $\hat{A}(t_1)$ through the product of atomic operator $A(t_1)$ and annihilation field operators $A(t_1)=A(t_1)a_k(t_1)a_{k_2}(t_1)...a_{k_n}(t_1)$ and the operator $\hat{B}(t_2)$ through the product of creation field operators and atomic operator $B(t_2)$ so that $B(t_2)=B(t_2)a_{k_1}(t_2)a_{k_2}(t_2)...a_{k_m}(t_2)$. Here the elimination of vacuum part of the operators $a_k(t)$ and $a_{k_1}^\dagger(t)$ can be represented through the following identities

$$\langle A(t_1)a_k(t)B(t_2) \rangle = \langle A(t_1)a_k^\dagger(t)B(t_2) \rangle - \exp[-i\omega_k(t - t_2)]\{\langle A(t_1)[a_k^\dagger(t),B(t_2)] \rangle - \delta_{k,k_1}\langle A(t_1)B(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2) \rangle - \delta_{k,k_1}\langle A(t_1)B(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2) \rangle \},$$

$$\langle A(t_1)a_{k_1}^\dagger(t)B(t_2) \rangle = \langle A(t_1)a_{k_1}^\dagger(t)B(t_2) \rangle - \exp[i\omega_k(t - t_2)]\{\langle A(t_1),a_{k_1}^\dagger(t_1)B(t_2) \rangle - \delta_{k,k_1}\langle A(t_1)a_{k_2}(t_1)...a_{k_n}(t_1)B(t_2) \rangle - \delta_{k,k_1}\langle A(t_1)a_{k_2}(t_1)...a_{k_n}(t_1)B(t_2) \rangle \}. $$

**Proof.** Taking in-to account the lemma (4), we can represent the third correlation function $\langle A(t_1)a_k(t)B(t_2) \rangle$ of expression (7) in the following form

$$\langle A(t_1)a_k(t)B(t_2) \rangle = \langle A(t_1)a_k^\dagger(t)B(t_2) \rangle - \exp[-i\omega_k(t - t_2)]\{\langle A(t_1)[a_k^\dagger(t),B(t_2)] \rangle + \exp[-i\omega_k(t - t_2)]\langle A(t_1)[a_k(t)B(t_2) - B(t_2)a_k^\dagger(t_2)] \rangle \}. $$
According to explicit expression of operator, $\hat{B}(t_2) = \mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)\ldots a_{k_m}^\dagger(t_2)$, let us introduced it in the second term of right hand site of expression (9). Following the commutation rules of the boson operators of EMF, the operator $\hat{a}_k(t_2)$ can be permutated in the right hand site of the correlation function. Taking into consideration that $(\hat{a}_k^\dagger(t_2) + \hat{a}_{k_2}(t_2))|0\rangle = \hat{a}_k^\dagger(t_2)|0\rangle$, this term becomes

$$\langle A(t_1)a_k(t_2)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)\ldots a_{k_m}^\dagger(t_2)\rangle = \delta_{k,k_1}\langle A(t_1)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)\ldots a_{k_m}^\dagger(t_2)\rangle + \ldots + \delta_{k,k_m}\langle A(t_1)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)\ldots a_{k_{m-1}}^\dagger(t_2)\rangle$$

$$+ \langle A(t_1)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)\ldots a_{k_{m-1}}^\dagger(t_2)\rangle.$$

(10)

Introducing this relation in expression (10), it is not difficult to observe that the new expression for correlation $\langle \hat{A}(t_1)\hat{a}_k(t)\hat{B}(t_2) \rangle$ coincides with expression (7) from the lemma. The similar procedure of permutation of vacuum part of creation operator $a_{k}^\dagger(t)$ demonstrates the identities (8) of lemma 2.

After the substitution of source part of the EMF operators (2), the next step consists in the elimination of operator $\hat{a}_k(t - \tau')$ from the correlation $(|\hat{R}_j^+(t), \hat{O}(t)||\hat{a}_k(t - \tau')\hat{D}_{mn}^+(t - \tau'))$ of first term of equation (3). According to lemma (4) and taking into account only the resonance terms in the right hand site of equation (3), it is not difficult to observe, that in the third order of the interaction constant $\lambda$ the term proportional to $(d_s, g_k)q(\omega_{k_1}, \omega_k)$ describes the two-photon resonance between the atomic subsystems $D$ and $S,R$. Neglecting the terms proportional to the $\lambda^3$, we obtain the following resonance between $S$, $R$ and $D$ subsystems

$$\langle |\hat{R}_j^+(t), \hat{O}(t)||\hat{a}_k(t - \tau')\hat{D}_{mn}^+(t - \tau') \rangle = \frac{i(d_s, g_k)}{\hbar} \sum_{l=1}^{N_0} \int_0^t d\tau \exp[i\omega_k(\tau - \tau')]$$

$$\times \langle |\hat{S}_l^+(t - \tau)|\hat{R}_j^+(t), \hat{O}(t)|| + \theta(\tau - \tau')|\hat{R}_j^+(t), \hat{O}(t)||\hat{S}_l^+(t - \tau')\hat{D}_{mn}^+(t - \tau') \rangle.$$

(11)

Replacing in expression (11) the operator of $S$ subsystem with operators of $R$ atoms, it is obtained the similar correlation (11) from second term of equation (3).

As follows from the representation (2) and generalized equation (3), the procedure of elimination of EMF field operators must continue in the last term of expression (3) too. For correlation $\langle |\hat{D}_{mn}^+(t), \hat{O}(t)||\hat{a}_{k_1}(t)\hat{R}_j(t - \tau)\rangle$ it is obtained the same resonances in the third order of small interaction constant

$$\langle |\hat{D}_{mn}^+(t), \hat{O}(t)||\hat{a}_{k_1}(t)\hat{R}_j(t - \tau) \rangle = \frac{i(d_s, g_k)}{\hbar} \sum_{l=1}^{N_0} \int_0^t d\tau' \exp[-i\omega_k(\tau - \tau')]$$

$$\times \langle |\hat{D}_{mn}^+(t), \hat{O}(t)||\hat{S}_l^+(t - \tau')\hat{R}_j(t - \tau) - \theta(\tau' - \tau)|\hat{R}_j(t - \tau), \hat{S}_l^+(t - \tau') \rangle \rangle.$$

(12)

According to the lemma (7) from the last term of equation (3), we obtain the following contribution to two-photon cooperative decay diagrams of $D$ subsystem.

$$\langle |\hat{D}_j^+(t), \hat{O}(t)||\hat{a}_k(t)\hat{D}_{mn}^+(t - \tau)\hat{a}_{k_1}^\dagger(t - \tau) \rangle = -\{\langle |\hat{D}_j^+(t), \hat{O}(t)||\hat{a}_{k_1}(t - \tau)\hat{D}_{n_1}^+(t - \tau)\hat{a}_{k_1}^\dagger(t - \tau) \rangle - \delta_{k,k_1}\langle |\hat{D}_j^+(t), \hat{O}(t)||\hat{D}_{mn}^+(t - \tau) \rangle \} \exp[-i\omega_k\tau] + \langle |\hat{D}_j^+(t), \hat{O}(t)||\hat{a}_{k_1}(t)\hat{D}_{mn}^+(t - \tau)\hat{a}_{k_1}^\dagger(t - \tau) \rangle.$$

(13)

Neglecting the non-resonance terms in equation (3) according with expressions (11), (12) and (13), we obtain the following master equation for arbitrary operator $\hat{O}(t)$ in thread
approximation of interaction constant $\lambda$

$$\frac{d\langle O(t) \rangle}{dt} = \sum_k \sum_{i,j=1}^{N_e} \frac{(d_r, g_k)^2}{\hbar^2} \int_0^t d\tau \exp[-i\omega_k \tau + i(k, r_j - r_i)] \langle [R_j^+ (t), O(t)] R_i^- (t - \tau) \rangle$$

$$+ \sum_k \sum_{i, l=1}^{N_e} \frac{(d_s, g_k)^2}{\hbar^2} \int_0^t d\tau \exp[-i\omega_k \tau + i(k, r_j - r_i)] \langle [S_j^+ (t), O(t)] S_i^- (t - \tau) \rangle$$

$$+ \sum_{k_1, k_2, l_1, l_2}^{N} \frac{(\mathbf{n}_{31}, e_{\lambda_1})}{\hbar^2} (\mathbf{n}_{23}, e_{\lambda_2}) q(\omega_1, \omega_2) \int_0^t d\tau \langle [D_j^+ (t), O(t)] D_i^- (t - \tau) \rangle$$

$$\times \exp[-i(2\omega q_1 - \omega_k - \omega_{k_2}) \tau] \exp[i(k_1 + k_2, r_j - r_i)]$$

$$+ i \sum_{k, k', n=1}^{N} \sum_{l=1}^{l} \frac{(\mathbf{n}_{31}, e_{\lambda})}{\hbar^2} (\mathbf{n}_{23}, e_{\lambda}) q(\omega_k, \omega_{k'}) \int_0^t d\tau' \exp[i(k, r_n - r_j) + i(k', r_n - r_l)]$$

$$\times \int_0^t d\tau \int_0^t d\tau' \exp[-i\omega_k \tau - i\omega_{k'} \tau'] \langle (d_s, g_{k'}) (d_r, g_k) \rangle \langle [D_n^+ (t), O(t)] R_i^- (t - \tau) S_j^- (t - \tau') \rangle$$

$$+ (d_r, g_{k'}) (d_s, g_k) \langle [D_n^+ (t), O(t)] S_j^- (t - \tau) R_i^- (t - \tau') \rangle$$

$$- i \sum_{k, k', m=1}^{N} \sum_{l=1}^{l} \frac{(\mathbf{n}_{31}, e_{\lambda})}{\hbar^2} (\mathbf{n}_{23}, e_{\lambda}) q(\omega_k, \omega_{k'}) \exp[-i(k, r_m - r_l) + i(k', r_j - r_m)]$$

$$\times \int_0^t d\tau' \int_0^t d\tau \exp[i\omega_k \tau - i\omega_{k'} \tau' - i\omega_{k'} \tau] \langle (d_s, g_{k'}) (d_r, g_k) \rangle \langle [S_j^+ (t - \tau) [R_j^+ (t), O(t)] D_m^- (t - \tau') \rangle$$

$$\times \theta(\tau - \tau') [R_j^+ (t), O(t)] S_j^+ (t - \tau') D_m^- (t - \tau')] + (d_s, g_{k'}) (d_r, g_k)$$

$$\times \langle [R_j^+ (t - \tau) [S_j^+ (t), O(t)] + \theta(\tau - \tau') [S_j^+ (t), O(t)] R_j^+ (t - \tau') D_m^- (t - \tau')] \rangle$$

$$+ H.c. (O^+ \rightarrow O).$$

(14)

This method give as the possibility to eliminate the vacuum parts of bi-boson operators $\hat{a}_{k_1}(\tau) \hat{a}_{k}(\tau)$. The master equation (14) for three atomic subsystem in interaction through the vacuum of EMF is obtained, taking into account the normal ordering of annihilation and creation operators of EMF in correlation functions.

To avoid these divergences it is proposed the integration procedure, which takes into account the causality appearing between the radiators [13]. Following this method the right hand side of master equation (14) takes the form

$$\frac{d}{dt} \langle O(t) \rangle = \sum_{l=1}^{N_e} \sum_{j=1}^{N_e} \chi_r(j, l) \langle [R_j^+ (t), O(t)] R_i^- (t) \rangle + \sum_{l=0}^{N_e} \sum_{i=1}^{N_e} \chi_o(j, l) \langle [S_j^+ (t), O(t)] S_i^- (t) \rangle$$

$$+ \sum_{i=1}^{N_e} \sum_{l=0}^{N_e} \sum_{j=1}^{N_e} \sum_{l=0}^{N_e} \sum_{m=0}^{N_e} \sum_{l=0}^{N_e} \sum_{m=0}^{N_e} U(j, l, m) \langle [D_m^+ (t), O(t)] R_j^- (t) S_i^- (t) \rangle$$

$$- \sum_{i=1}^{N_e} \sum_{l=0}^{N_e} \sum_{j=1}^{N_e} \sum_{l=0}^{N_e} \sum_{m=0}^{N_e} \sum_{l=0}^{N_e} \sum_{m=0}^{N_e} V(j, l, m) \langle [S_j^+ (t)] R_j^- (t) O(t)] D_m^- (t) \rangle$$

$$+ \langle R_j^+ (t) [S_j^+ (t), O(t)] D_m^- (t) \rangle + H.C. (O^+ \rightarrow O).$$

(15)
Here $\chi_p(j,l) = \frac{\exp[i \omega_p r_{jl}/c] - 1}{i \omega_p r_{jl}/c}$, $p = S, R$ is the intrinsic exchange integrals between atoms of subsystems $S$ or $R$. The two-photon exchange integral between $j$-th and $l$-th radiators according to the theory of super-radiance [5] is described by $\Re \chi_d(j,l) = \sin[(2k_0 - k)r_{jl}]/[(2k_0 - k)r_{jl} + \sin[kr_{jl}] / (kr_{jl})]$ respectively, where $k$ is the wave vector of emitted photons. The spontaneous emission times, $\tau_s$ and $\tau_r$, for $S$ and $R$ dipole active atomic subsystems are defined in the literature $\tau^{-1}_\alpha = 4d^3\omega^3_\alpha/(3hc^3)$; $\alpha = r, s$.

We observe, that in the right hand part of the equation (15) the third order terms contain resonances between the single-photon radiators $S$, $R$ and two-photon radiator $D$, described by the correlation functions $(S^+_j(t) [R^+_j(t), O(t)] D^-_m(t))$, $(D^+_m(t), O(t)) [R^+_j(t) S^-_i(t)]$ and $(R^+_k(t) [S^+_i(t), O(t)] D^-_m(t))$. Indeed, representing the exchange integral between three atoms in the similar form as in the paper [5], we can find cooperative emission rate of three atoms belong to $S$, $R$ and $D$ subsystems situated at small distance $r \ll \lambda_0$

$$\frac{1}{\tau^-_0}_{srd} = \langle 1/3 \rangle^2 \frac{d^3 d^3 d^3_3 \psi_3^3(\omega_p)^3}{4 \pi \hbar^2} \left\{ \frac{1}{\omega_{32} + \omega_s} + \frac{1}{\omega_{31} - \omega_r} \right\},$$

(16)

The resonance interaction of a dipole forbidden atom $m$ and two dipole active radiators $l$ and $j$ are described by the interference function

$$U(j,l,m) = \frac{-c^2[\exp[i \omega_r r_{mj}/c] - 1][\exp[i \omega_s r_{ml}/c] - 1]}{\omega_r \omega_s r_{mj} r_{ml}}$$

$$V(j,l,m) = \exp[-i \omega_r r_{mj}/c] U(j,l,m).$$

(17)

The master equation (15) can be used for description of cooperative interaction between the dipole forbidden and dipole active subsystems of radiators.

3. Kinetic equations for correlation functions

In this section it is proposed to simplify the problem of interaction between three subsystems of radiators, taking into account the generalized equation (15). The closed system of equations for correlation functions can be found studying the chain of equations for correlations between dipole forbidden transitions of $D$ subsystem and dipole active transitions of $S$ and $R$ subsystems of radiators. Below it is studied two situations: A. the small number of atoms in which it is possible two photon resonance, B. the large number of atoms in each $S, R$ and $D$ subsystems for which the semiclassical approach is possible.

A. Considering, that the numbers of atoms in the subsystems are relatively small, the generalized equation (15) can be reduced to closed systems of equations. Let us annualize the simple situation for three particle system which contains one atom in each subsystems $S$, $R$ and $D$. According to the master equation (15), we can introduce the mean value of exited numbers of atoms, $\langle \hat{N}_a \rangle = \langle \hat{J}_{za}(t) \rangle + 0.5$, $\alpha = s, r, d$ of type $S$, $R$, and $D$. Here operator $\hat{J}_{za}$ is considered equivalent to the inversions of each atom $S_z$, $R_z$, $D_z$. Let us consider the situation, when two atoms $R$ and $S$ are situated at equal distance $r$ relatively the radiator $D$. This distance may have the same magnitude as the emission wavelength of photons from the system. When emission frequencies of the one-photon radiators coincides $\omega_s \simeq \omega_s$, this dependence of exchange integral $V(j,l,m)$ becomes real defined function $[\exp[2 \pi i \omega_r r/c] - 1][\exp[-2 \pi i \omega_s r/c] - 1 \simeq 2[1 - \cos(\omega_r r/c)]$ . In this case the mutual interference between $S$, $R$ and $D$ subsystems are described by more simple exchange integrals

$$V_{srd} = \frac{2(1 - \cos(2 \pi r/\lambda_s))}{(2 \pi r/\lambda_s)^2}, \quad U_{srd} = \exp[i \omega_s r/c] V_{srd},$$

(18)

Here $r$ is the distance between $R, S$ and $D$ subsystems, $\lambda_s \simeq \omega_s$. Introducing the correlation function between the polarizations of three atoms $\langle F \rangle = i(\langle \hat{D}^+(t) \hat{S}^-(t) \hat{R}^-(t) \rangle - \langle \hat{D}^-(t) \hat{S}^+(t) \hat{R}^+(t) \rangle)$
\( \langle \hat{S}^+(t)\hat{R}^+(t)\hat{D}^-(t) \rangle \), we obtain the closed system of equations. According to this procedure the closed system of equations is obtained from master equation (15)

\[
\frac{d}{dt} \langle \hat{N}_s(t) \rangle = -\langle \hat{N}_s \rangle \tau_s - \frac{1}{2\tau_{srd}} \langle \hat{F} \rangle , \quad \frac{d}{dt} \langle \hat{N}_d(t) \rangle = -\langle \hat{N}_d \rangle \tau_d - \frac{\cos(x)}{\tau_{srd}} \langle \hat{F} \rangle , \quad \frac{d}{dt} \langle \hat{N}_s\hat{N}_d \rangle = -C\langle \hat{N}_s\hat{N}_d \rangle , \quad \frac{d}{dt} \langle \hat{N}_s \hat{N}_d \rangle = -D\langle \hat{N}_s\hat{N}_d \rangle , \quad \frac{d}{dt} \langle \hat{F}(t) \rangle = -\frac{A}{2} \langle \hat{F}(t) \rangle + \frac{1}{\tau_{srd}} [2(\cos(x) + 1)\langle \hat{N}_s\hat{N}_d \rangle - \cos(x)\langle \hat{N}_s\hat{N}_d \rangle - \langle \hat{N}_s\hat{N}_d \rangle /2 - \langle \hat{N}_s\hat{N}_d \rangle /2] , \quad \frac{d}{dt} \langle \hat{N}_s \hat{N}_d \rangle = -A\langle \hat{N}_s\hat{N}_d \rangle .
\]

(19)

Here the following correlation functions between the atomic excitations was introduced: \( \langle \hat{N}_s\hat{N}_r\hat{N}_d \rangle , \langle \hat{N}_s\hat{N}_r \rangle , \langle \hat{N}_s\hat{N}_d \rangle \), and \( \langle \hat{N}_s\hat{N}_d \rangle \). The new three particle cooperative time between the atoms situated at relative distance \( x \) is introduced \( \tau_{srd}(x) = \tau_{0srd} / V_{srd}(x) , x = \omega_s r / c \). This system of equation is exactly solvable. The solution is

\[
\langle \hat{N}_s\hat{N}_r\hat{N}_d \rangle = \exp(-At) , \quad \langle \hat{N}_s\hat{N}_r \rangle = \exp(-Bt) , \quad \langle \hat{N}_s\hat{N}_d \rangle = \exp(-Ct) , \quad \langle \hat{N}_d\hat{N}_r \rangle = \exp(-Dt) , \quad \langle \hat{F}(t) \rangle = [4(\cos(x) + 1)1 - \exp[-At / 2] - 2\cos(x)1 - \exp[-(B - \tau_d^{-1})t / 2] - \exp[(C - \tau_s^{-1})\tau_d t / 2] / C - \tau_s^{-1}\tau_{srd}] A / B - \tau_b^{-1} , \quad \langle \hat{N}_d \rangle = \exp[-t / \tau_d] - \cos x / \tau_{srd} \int_0^t \exp[-(t - t') / \tau_d] \langle \hat{F}(t') \rangle , \quad \langle \hat{N}_s \hat{N}_d \rangle = \exp[-(1 / \tau_s + 1 / \tau_d)\tau_d] , \quad \langle \hat{N}_s \hat{N}_d \rangle = \exp[-t / \tau_d] , \quad \langle \hat{N}_s \hat{N}_d \rangle = \exp[-t / \tau_d] / 2 \tau_{srd} \int_0^t \exp[-(t - t') / \tau_d] \langle \hat{F}(t') \rangle , i, j \to s, r .
\]

where the collective rates are defined \( A = 1 / \tau_r + 1 / \tau_d + 1 / \tau_s ; B = 1 / \tau_r + 1 / \tau_s ; C = 1 / \tau_d + 1 / \tau_s ; D = 1 / \tau_d + 1 / \tau_s \). The solution of this system of equation is plotted in Figs. 1. The dependence of the atomic inversion \( \langle \hat{D}_s(t, r) \rangle \) as functions of relative time \( t / \tau_d \) and relative distance \( r / \lambda_s \) between \( S, R \) and \( D \) subsystems for the parameters of the system \( \tau_d / \tau_r = \tau_d / \tau_s = 5 , \tau_d / \tau_{srd} = 2 \) is plotted in the Figs.1. The significant influence of single-photon transition on the two-quantum transitions is observed at small distance \( r \) with the magnitude comparable with emission wavelength of the photons. This dependence follows from analytical expression (20). According to two-dimension surface \( (t, r) \) represented in Figs.1 and 1, we can observe that with decreasing of distance between the radiators the relative decay rate increases \( \tau_s / \tau_{srd} \). This process is accompanied with the substantially increasing of two-photon emission intensity of subsystem \( D \).

As follows from Fig.1 the cooperative exchanges between the radiator accelerate the two-photon decay processes so that the the emission rate \( V = -\frac{d\theta}{dt} \) achieves the maximal value. After same time this rate decreases till zero value as in the typical super-radiant process.

B. Let us now describe the time behavior of big number of radiators in subsystems. Neglecting the quantum fluctuations of inversion operators \( \langle \hat{R}_s \rangle , \langle \hat{S}_d \rangle , \langle \hat{D}_s \rangle \), we obtain the following closed system of nonlinear equations

\[
\frac{d}{dt} \langle \hat{R}_s(t) \rangle = -\frac{1}{\tau_r} [\langle \hat{N}_r(\hat{N}_r + 2) / 4 - \hat{R}_s^2 + \hat{R}_s \rangle - 1 / 2\tau_{srd} \hat{F}(t) , \quad 9\]


Figure 1. The decay process of inversion $D_z(t)$ and the two-photon decay rate $V = -dD_z(t)/dt$ stimulated by single photon processes for the parameters of the system relative decay rates of S- and R- atoms $\tau_d/\tau_r = \tau_d/\tau_s = 5$, $\tau_d/\tau_{srd} = 2$.

\[
\frac{d}{dt} S_z(t) = -\frac{1}{\tau_s} \{ N_s(N_s + 2)/4 - S_z^2 + S_z \} - \frac{1}{2\tau_{srd}} F(t),
\]

\[
\frac{d}{dt} D_z(t) = -\frac{1}{\tau_d} \{ N(2N + 1) - D_z^2 + D_z \} - \frac{\cos(x)}{\tau_{srd}} F(t),
\]

\[
\frac{d}{dt} F(t) = \{ \frac{1}{\tau_s} [S_z(t) - 1] + \frac{1}{\tau_r} [R_z(t) - 1] + \frac{1}{\tau_d} [D_z(t) - 1] \} F
\]

\[ + \frac{1}{\tau_{srd}} \{ 2\cos(x) D_z \{ N_s^2/4 - S_z^2 \} \{ N_r^2/4 - R_z^2 \} \}
\]

\[ + R_z \{ N_s^2/4 - S_z^2 \} \{ N_r^2/4 - R_z^2 \} + S_z \{ N_r^2/4 - R_z^2 \} \{ N_r^2/4 - D_z^2 \}
\]

\[ + 2(1 + \cos(x)) N_s N_r N \exp[-A \ast t] - \cos(x) N_s \ast N_r \ast \exp[-Bt]
\]

\[ - 0.5 N_s N \exp[-Ct] - 0.5 N_r N \exp[-Dt]. \tag{20} \]

the solution of which for three atomic system pass into (19). Here $R_z(t) = \sum_{j=0}^{N_s} (R_{zj}(t))$, $S_z(t) = \sum_{j=0}^{N_s} (S_{zj}(t))$ and $D_z(t) = \sum_{m=0}^{N_s} (D_{zj}(t))$ are the atomic inversions of $R$ $S$ and $D$ subsystems respectively; $F(t) = i(\{ D^+(t)S^-(t)R^-(t) \} - \{ S^+(t)R^+(t)D^-(t) \})$ describes three particle correlation function in collective interaction between subsystems, $D^+(t) = \sum_{m=0}^{N_s} D_{zj}^m(t)$, $R^+(t) = \sum_{j=0}^{N_r} R_{zj}^+(t)$ and $S^+(t) = \sum_{j=0}^{N_s} S_{zj}^+(t)$ describe the collective atomic excitation operators for subsystems $D$, $R$ and $S$ with the dimension less than emission wavelength.

In comparison with three atoms cooperative interaction, described by the closed system of quantum equations (19), the system of equations (20) is obtained in the semi-classical approximation and contains the quantum aspect of interaction between three atoms of subsystems $S$, $R$ and $D$. In other words when the number of atoms in each subsystems is equal to one, $N_s = N_r = N = 1$, the additional terms in the system of equations 20) give us the possibility to reduce this equations to the following system (19).

Taking into account the relative expressions of the decay rates described in Figs. 2, we obtain the numerical simulation of the inversion $D_z$ and its derivative. As follows from the Figs. 2 for relative parameter $\tau_d/\tau_{srd} > 0.2$ the decay rate of of two-photon emission passes from a monotonic time function to the oscillatory time dependence. The influence of single photon cooperative emission to two-quantum decay process increases with decreases of the relative distance $\tau_d/\lambda < 1$ between the subsystems. In Figs. 2 is plotted the time dependence of the inversion $D_z(t)$ of dipole forbidden radiators as function of the relative time $t/\tau_d$ and relative
Figure 2. The time dependence of inversion $D_z(t)$ Relative emission rate $V = -\frac{\tau_d}{N} \frac{dD_z(t)}{dt}$ of dipole forbidden radiator subsystem for the values of the parameters number of atoms in the subsystems $S, R$ and $D$ is $N_s = N_r = N = 50$ respectively; the relative decay times of the subsystems are $\tau_b/\tau_s = \tau_b/\tau_r = 6$ and $\tau_d/\tau_{drs}^0 = 1/6$; The distance $r$ between the subsystems $S, R$ and $D$ is changed between 0 and 2.

The behavior of decay rate of $D$-atomic subsystem is plotted in Figs. 2 for different values of cooperative rate $1/\tau_{srd}(x)$ and distance $r$, where $\tau_d/\tau_{drs}^0 = 0.2$.

The strong mutual influences between single and two-photon super-radiance processes in three particle interaction are characteristic for these numerical representations. It is observed again two regions $0 \leq r \leq \lambda_s/2$ and $\lambda_s/2 \leq r \leq \lambda_s$ in which the exchange integral $U_{srd}$ changes the sign. In the point $r = \lambda_s/2$ we observe non-significant influence of subsystems $S$ and $R$ in two-photon decay process. The oscillations in the two and single photon rates are observed in all subsystems for the parameter $\tau_d/\tau_{drs}^0 > 0.2$. This effect plays an important role in the collective decay process of the systems of radiators with the dimension smaller then wavelength.

4. Conclusions

It finds the effective interaction between three subsystems of radiators in two-photon resonance, using the method of elimination of vacuum field operators. The new cooperative interaction between one dipole-forbidden and two-dipole active subsystems of radiators was proposed. For the description of cooperative emission of three radiators (19) in two-quantum resonance interaction through the vacuum of EMF, the new master equation (15) is obtained. The exact analytical solution of three radiators in interaction with vacuum field is represented in expressions (20) and in Figs. 1. This solution takes into account quantum interference between two dipole active and one dipole forbidden radiators situated at distance $r$. The exchange integrals between such subsystems can be described by the expressions (18) and describe the acceleration of decay rates of the radiators. The magnitude of this cooperative emission depends on $1/\tau_{srd}$ as a function of the distances between the radiators. The exchange integral $U_{srd}$ pass from positive to negative sign in the regions, $0 \leq r \leq \lambda_s/2$ and $\lambda_s/2 \leq r \leq \lambda_s$ around the $D$-atom. If the atoms $S$ and $R$ are situated on the circle with radius $r = \lambda_s/2$ around the radiator $D$, the influence of single photon radiators on two-photon decay of $D$ radiator can be neglected. From numerical results Figs. 1 follows that the mutual influence between the radiators depends only on the module of interaction integral (18).

The semi-classical approach is proposed for numerical studies of cooperative resonance
between single-photon and two-quantum transitions described by correlation function 
\( \langle S^+ (t) R^+ (t) D^- (t) \rangle \). Using the master equation (15), and neglecting the quantum fluctuation of the inversion, the closed system of kinetic equation (20) for three atomic subsystems have been obtained. The big number of radiators drastically changes the decay rate as a function of time and distance \( r \). For small relative rate \( \tau_d / \tau^0_{sdr} = 0.2 \) the energy exchanges between the atoms is plotted on the Fig. 2. Studying the time behavior of the cooperative rate of the atomic subsystems we observe two regions in the decay rates as in the case of three radiator system. From the behavior of inversion of \( D \) subsystem and emission rate of bi-photons plotted in Figs. 2 respectively we observe, that the mutation process stimulated by mutual influences between the subsystems of radiators substantially increases with increasing the number of radiators. For equal numbers of radiators in all subsystems, \( N = 50 \), it is established the point \( \tau_d / \tau^0_{rsd} = 0.2 \) in which the inversion of subsystem \( D \) passes into nutation process.

References

[1] Nikolaus B, Zhang D, and Toschek P 1981 Phys. Rev. Lett. 47 171
[2] Gauthier D J, Wu Q, Morin S E and Mossberg T W 1987 Phys. Rev. Lett 68 464
[3] Rivlin L A 1997 Hyperfine Interaction 107, 57
[4] M. Bruno, J. M. Raimond, P. Goy 1987 Phys. Rev. Lett. 59 1899
[5] N.A. Enaki 1988 Zh. Eksp. Teor.Fiz. 94 135; N.A. Enaki 1990 Zh. Eksp. Teor.Fiz., 98 783
[6] Goeppert-Mayer M 1931 Ann. Phys. 9 273; Breit G and Teller E 1940 Astrophys. J. 91 215; Spitzer L and Greenstein J.L. 1940 Astrophys. J. 144 215
[7] Pearl A S 1970 Phys. Rev. Lett. 24 703; Van Dyck R S Jr, Jonson C E, Shugart H A 1971 Phys. Rev. Lett. 25, 1403; Marrus R and Schmieder R 1972 Phys. Rev. A 5 1160
[8] Alexander S G and Esazaros P M 1991 Astrophysics J. 327, 554; Chluba J. and Sunyaev R. A. 2006 Astronomy and Astrophysics, 446 39
[9] Jentschura U.D. 2008 J.Phys.A, 41 155307; U.D. Jentschura and A. Surzhykov 2008 Phys. Rev. A 77 042507
[10] Kalinkin A A , Kalachev A A , and Samartsev V.V. 2004 Optics and Spectroscopy 96 803
[11] Dicke R H 1954 Phys. Rev. 93, 99
[12] Andreev A V, Emelyanov V I, and Ilinskii Yu A 1992 Cooperative Effects in Optics IOP Publishing, Bristol
[13] Enaki N A 1996 Soviet Phys. JETP 82 607; Enaki N A and Galeamov E 2008 International Journal of Theoretical Physics 47 911