Asymptotic Self-Similarity of Minimizers and Local Bounds in a Model of Shape-Memory Alloys

Sergio Conti1 · Johannes Diermeier2 · Melanie Koser3 · Barbara Zwicknagl3

Received: 7 June 2021 / Accepted: 6 October 2021 / Published online: 21 December 2021
© The Author(s) 2021

Abstract
We prove that microstructures in shape-memory alloys have a self-similar refinement pattern close to austenite-martensite interfaces, working within the scalar Kohn-Müller model. The latter is based on nonlinear elasticity and includes a singular perturbation representing the energy of the interfaces between martensitic variants. Our results include the case of low-hysteresis materials in which one variant has a small volume fraction. Precisely, we prove asymptotic self-similarity in the sense of strong convergence of blow-ups around points at the austenite-martensite interface. Key ingredients in the proof are pointwise estimates and local energy bounds. This generalizes previous results by one of us to various boundary conditions, arbitrary rectangular domains, and arbitrary volume fractions of the martensitic variants, including the regime in which the energy scales as $\varepsilon^{2/3}$ as well as the one where the energy scales as $\varepsilon^{1/2}$.

Keywords Calculus of variations · Martensitic phase transformation · Microstructure · Asymptotic self-similarity · Local bounds

Mathematics Subject Classification (2010) 74N15 · 74G65 · 49N99

1 Introduction
Fractal microstructures, characterized by self-similar refinement of oscillations close to a Dirichlet boundary, have been studied in models of a variety of physical problems, including micromagnetism, shape-memory alloys, and compressed thin elastic sheets. They were first introduced by Landau in the 30s in the context of magnetism [44, 45], and meanwhile a rich mathematical literature has been developed. In most cases the main result is a rigorous characterization of the scaling of the energy in terms of the various parameters of the system, up to a universal multiplicative constant. The upper bound is typically obtained by an explicit
construction which is based on self-similar microstructures. This is complemented by a matching lower bound, which proves that this microstructure achieves the optimal energy up to a universal factor, and thereby supports the idea that the actual minimizers are indeed, at least approximately, self-similar. Only in very few cases it has been possible to prove detailed properties of the minimizers themselves, including in particular the fact that they exhibit, up to boundary effects, the expected self-similar structure.

Mathematical work on the subject was initiated by Kohn and Müller in the 90s [40, 41]. They proposed a scalar, two-dimensional model for microstructure in shape-memory alloys close to an austenite-martensite interface. In their model one partial derivative is constrained to take only two values, leading to the characteristic non-convexity. Their work has been meanwhile generalized to different volume fractions [19, 59], to vectorial settings in the context of linearized elasticity [3, 10, 11, 23, 36, 38, 39, 49, 52] and to geometrically nonlinear formulations [12, 50]. These vectorial generalizations have confirmed that the Kohn-Müller scalar model indeed captures the correct scaling of the energy. A number of other problems have then been addressed with similar tools, including magnetization patterns in uniaxial ferromagnets [14, 15, 37], diblock copolymers [1, 13], flux tubes in type-I superconductors [16, 20, 22, 28], wrinkling in thin elastic films [4, 6, 7, 34], dislocation microstructures [18, 19], transport network structures [8, 9], and compliance minimization [42].

We consider here a version of the Kohn-Müller model which is appropriate for modeling materials where the austenite is almost compatible with one martensitic variant, which are known to have particularly low hysteresis [24, 58]. Specifically, we work in a rectangle $R^{L_x,L_y} := (0, L_x) \times (0, L_y)$ and minimize the functional

$$I(u, R^{L_x,L_y}) := \int_{R^{L_x,L_y}} u_x^2 \, d\mathcal{L}^2 + \varepsilon |u_{yy}|(R^{L_x,L_y})$$

over the set $\mathcal{A}_0$ of admissible $u$, defined by

$$\mathcal{A}_0(R^{L_x,L_y}) := \{u \in W^{1,2}(R^{L_x,L_y}) : u(0, \cdot) \equiv 0, \ u_y \in \{-\theta, 1 - \theta\} \ \text{a.e.}, \ u_{yy} \ \text{is a finite Radon measure}\}.$$ 

Here and below, $u_x, u_y$ and $u_{yy}$ denote distributional partial derivatives of the function $u$, and $|u_{yy}|(\omega)$ denotes the total variation of the measure $u_{yy}$ on the set $\omega$. The interface with austenite is modeled by the boundary condition $u(0, \cdot) = 0$. The parameter $\varepsilon$ represents the surface energy per unit length, and $\theta \in (0, 1)$ represents the volume fraction of the minority phase, by symmetry it suffices to consider the case $\theta \leq \frac{1}{2}$. The interpretation of the parameter $\theta$ in terms of the crystallography of the phase transition, the relation of the limit $\theta \to 0$ to the so-called “$\lambda_2 = 1$”-condition, and its importance for the development of low-hysteresis materials have been discussed in [2, 24, 29, 33, 46, 53, 56–59], see the recent survey [32] for an overview. In the case of equal volume fractions, $\theta = \frac{1}{2}$, the present model reduces to the one originally studied by Kohn and Müller. A variant of $I$ in which the Dirichlet boundary condition is replaced by the elastic energy of austenite outside $R^{L_x,L_y}$ has been studied in [19], a vectorial extension in [23], resulting in a rich phase diagram. The limit $\varepsilon \sim \theta^2 \to 0$ of a similar model was addressed in [21].

1.1 Main Results

In this paper, we study the functional in (1), characterize the energy scaling (Theorem 1), prove asymptotic self-similarity of minimizers (Theorem 2), and obtain quantitative pointwise bounds on the minimizer (Theorem 3). In particular, we generalize the results from
Asymptotic Self-Similarity and Local Bounds in a Model of SMAs

151

Fig. 1 Sketch of the upper bound constructions in the proof of Theorem 1. Both have self-similar refinement close to the Dirichlet boundary on the left. If $L_y \gg \varepsilon^{1/3} \theta^{-2/3} L_x^{2/3}$ the construction is periodic in the vertical direction, and the microstructure continues over the entire domain (left image). If instead $L_y \ll \varepsilon^{1/3} \theta^{-2/3} L_x^{2/3}$ then the minority phase does not extend to the entire domain (right image).

[17] in several ways (see below). We build on the theses [25, 43] by two of the authors, in which some of the results from [17, 41] have been extended to arbitrary volume fractions.

Related results for a three-dimensional analogue of the Kohn-Müller model, which arises in the study of uniaxial ferromagnets, have been obtained in [55]. Local bounds for the energy and the minimizers have been obtained for a related nonlocal isoperimetric problem, motivated by the study of diblock copolymers, in [1] and recently for surface charges in [5], and for the three-well problem in [51, Lemma 4].

We focus here on the minimization of $I$ over the set $A_0$ (see (2)) but we point out that all our results can be extended to the case of periodic boundary conditions, see Sect. 8. Existence of miminizers was proven for $\theta = \frac{1}{2}$ in [41, Th. 2.1], the argument works for any value of $\theta$ without changes.

We first characterize the scaling of the optimal energy.

Theorem 1 (Global scaling laws) There is $c > 0$ such that for all $\varepsilon, L_x, L_y > 0$ and $\theta \in (0, \frac{1}{2}]$ one has

$$\frac{1}{c} \min\{\varepsilon^{1/2} \theta L_y^{3/2}, \varepsilon^{2/3} \theta^{2/3} L_x^{1/3} L_y\} \leq \min_{u \in A_0(R L_x, L_y)} I(u, R L_x, L_y) \leq c \min\{\varepsilon^{1/2} \theta L_y^{3/2}, \varepsilon^{2/3} \theta^{2/3} L_x^{1/3} L_y\}.$$  

This result extends [41] and is a special case of [19]. For convenience of the reader, the short proof in this context is given in Sect. 4, with the upper bounds being an easy byproduct of the constructions we present in Sect. 3. In both scaling regimes, the upper bound can be achieved by constructions that refine in a self-similar way close to the interface, see Fig. 1. This observation has been refined in [17] where asymptotic self-similarity of minimizers for (1) is proven under rather strong Dirichlet boundary conditions in the case of $\theta = \frac{1}{2}$ and very short rectangles, deep into the second scaling regime $\varepsilon^{2/3} \theta^{2/3} L_x^{1/3} L_y$ from Theorem 1. In Theorem 2 below we generalize this result to the physically important case of Neumann (and periodic, see below) boundary conditions, allowing for arbitrary volume fractions $\theta$ (including the low-hysteresis case $\theta \ll 1/2$), and to more general domains, including long thin rectangles (corresponding to the first energy scaling regime from Theorem 1). Our results in particular show that the Dirichlet boundary conditions and the shape of the domain...
chosen in [17] do not modify significantly the behavior of the minimizers. Indeed, one important tool in our proof is a method to obtain effective boundary conditions on subsets of the domain from the assumption that the scaling of the energy is the optimal one, and then to iteratively improve these bounds passing to smaller and smaller subsets of the domain, as explained in more detail in Sect. 2. Precisely, in Sect. 7, we prove the following result.

**Theorem 2** (Asymptotic self-similarity of a minimizer) Let \( \varepsilon, L_x, L_y > 0 \) and \( \theta \in (0, \frac{1}{2}] \). Let \( u \in \mathcal{A}_0(R^{L_x,L_y}) \) be a minimizer of \( I(u, R^{L_x,L_y}) \). For any \( y_0 \in (0, L_y) \) and any sequence \( v_j \to 0, v_j > 0 \) we define

\[
u_j(x,y) := v_j^{-2/3}u(v_jx, y_0 + v_j^{2/3}y)
\]
(implicitly extending \( u \) by zero to the rest of \( \mathbb{R}^2 \)). Then the sequence \( \nu_j \) has a subsequence that converges strongly in \( W_{1,2}^0((0, \infty) \times \mathbb{R}) \) towards a function \( u^\infty \in \mathcal{A}_0((0, \infty) \times \mathbb{R}) \), and \( u^\infty \) is a local minimizer.

For the precise definition of a local minimizer we refer to the notation below. The main step in the proof of asymptotic self-similarity is the proof of local bounds. Roughly speaking, on suitable subrectangles we show pointwise bounds of the form

\[|u(x,y)| \leq d_1 \varepsilon^{1/3} \theta^{1/3} x^{2/3},\]
and local energy bounds of the form

\[I(u, (0,l) \times (a,b)) \leq d_2 \varepsilon^{2/3} \theta^{2/3} l^{1/3} (b-a).\]

Optimality of these exponents follows from Remark 2 and Theorem 1, respectively. Since we do not impose boundary conditions on the top and bottom boundaries, the estimates degenerate close to the boundaries. This is made quantitative in Theorem 3 by the dependence of the constants on \( \eta \). The option of making the aspect ratio of the considered regions larger, by enlarging \( c_1 \), will be important in the proof of Theorem 2. We remark that in [17, Theorem 2.1] (see [43] for the case of arbitrary \( \theta \in (0, 1/2) \)), corresponding results are proven under rather restricted Dirichlet boundary conditions on the top and bottom of the rectangle, analogously to what was discussed above for asymptotic self-similarity. In this case, one can set \( \eta = 1 \) in (6) and \( \eta = 0 \) everywhere else.

Precisely, we prove the following result, see Sect. 6.

**Theorem 3** (Local bounds) Let \( \eta \in (0, \frac{1}{6}) \), \( k_1 > 0 \). Then there are constants \( c_1^*, c_2 > 0 \) such that for all \( c_1 \geq c_1^* \) there exist constants \( d_1, d_2 > 0 \) with the following property. Let \( \varepsilon, L_x, L_y > 0 \) and \( \theta \in (0, \frac{1}{2}] \), and let \( l_x \in (0, L_x] \) such that

\[2c_1 \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3} \leq \eta L_y\]
and let \( u \in \mathcal{A}_0(R^{L_x,L_y}) \) be a local minimizer of \( I(\cdot, R^{L_x,L_y}) \) such that

\[|u(l_x, y)| \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} l_x^{2/3} \quad \text{for all } y \in [\eta L_y, (1-\eta)L_y],\]
and

\[I(u, R^{l_x,L_y}) \leq k_1 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y.\]
Then

\[ |u(x, y)| \leq d_1 \varepsilon^{1/3} \theta^{1/3} x^{2/3} \text{ for all } x \in (0, l_x], y \in [3\eta L_y, (1 - 3\eta)L_y] \]  

(9)

and, for all \( l \in (0, l_x] \) and \((a, b) \subseteq (3\eta L_y, (1 - 3\eta)L_y)\) with \( b - a \geq 4c_1 \varepsilon^{1/3} \theta^{-2/3} l^{2/3}\),

\[ I(u, (0, l) \times (a, b)) \leq d_2 \varepsilon^{2/3} \theta^{2/3} l^{1/3} (b - a). \]  

(10)

We note that here we capture also the expected scaling in the volume fraction \( \theta \). This introduces additional technical difficulties compared to [17, 41]. In particular, in the construction of Sect. 3, the choice of the position of the interfaces needs to more accurately reproduce the local volume fraction induced from the boundary conditions. In the proof of the local bounds in Sect. 6 the relation between \( L^2 \) and \( L^\infty \) bounds changes, and requires a different treatment of the regions close to the top and bottom boundaries. While the explicit behaviour on \( \theta \) is not necessary to obtain asymptotic self-similarity of minimizers (see Theorem 2), we expect that such bounds might be helpful for proving explicit self-similarity of minimizers in the limit of low volume fractions, which could be taken along the lines of [21]. For a slightly simpler model arising in the variational study of type-I-semiconductors, the explicit self-similar minimizer is completely characterized in the limit of low-volume fractions in [28]. To the best of our knowledge, this is the only case in which self-similarity of a minimizer is known.

1.2 Outline of the Article

The rest of the article is structured as follows. After briefly introducing the notation, we give a summary of the mathematical strategy in Sect. 2 and then provide an explicit branching-type construction of a test function for given Dirichlet boundary data in Sect. 3 (see Proposition 1). In Sect. 4, we recall the global scaling law (see Theorem 1) for the minimal energy. Subsequently, in Sect. 5, we show that the energy of a minimizer restricted to subrectangles of the form \((0, l_x) \times (0, L_y)\) for \( l_x \leq L_x \) has the same scaling behaviour as the minimal energy on the smaller rectangle (see Theorem 4). In Sect. 6, we prove the analogous energy scaling result on suitable subrectangles of the form \((0, l_x) \times (a, b) \subseteq R^{l_x \times L_y}\) complemented with an \( L^\infty \)-bound on minimizers (see Theorem 3). Section 7 is devoted to the proof of asymptotic self-similarity of minimizers (see Theorem 2). Finally, in Sect. 8, the results are generalized to periodic boundary conditions on top and bottom.

1.3 Notation

For a rectangle \( R \) we denote by \( \mathcal{A}(R) \) the set of admissible functions on \( R \),

\[ \mathcal{A}(R) := \{ u \in W^{1,2}(R) : u_y \in \{-\theta, 1-\theta\} \text{ a.e.,} \quad u_{yy} \text{ is a finite Radon measure} \}, \]

so that the set \( \mathcal{A}_0 \) defined in (2) is given by those \( u \in \mathcal{A} \) such that \( u(0, y) = 0 \) for all \( y \). Elements of \( \mathcal{A}(R) \) have a Hölder-continuous representative (see Lemma 7) and in particular have a trace on \( \partial R \). The set \( \mathcal{A}((0, \infty) \times \mathbb{R}) \) is defined as the set of those \( u : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) which belong to \( \mathcal{A}((0, L) \times (-H, H)) \) for all \( L, H > 0 \), and analogously for \( \mathcal{A}_0 \).
A function \( u \in \mathcal{A}_0((0, l) \times (a, b)) \) is called local minimizer if the energy cannot be decreased by compact perturbations, in the sense that

\[
I(u, (0, l) \times (a, b)) \leq I(v, (0, l) \times (a, b)) \text{ s.t. } \exists \delta > 0 \text{ with } \{ v \neq u \} \subset (0, l - \delta) \times (a + \delta, b - \delta).
\]

(11)

A function \( u \in \mathcal{A}_0((0, \infty) \times \mathbb{R}) \) is a local minimizer if all restrictions of \( u \) to rectangles \((0, l) \times (a, b)\) are local minimizers. Moreover, a function \( u \in \mathcal{A}(R^{l_x,l_y}) \) is called minimizer with respect to its own boundary conditions if

\[
I(u, R^{l_x,l_y}) \leq I(v, R^{l_x,l_y}) \text{ for all } v \in \mathcal{A}(R^{l_x,l_y}) \text{ with } v = u \text{ on } \partial R^{l_x,l_y}.
\]

We stress that in this definition it is not required that the values of \( u_y \) and \( v_y \) on the boundary coincide.

If \( u \) is a local minimizer or a minimizer with respect to its own boundary data on \( R^{l_x,l_y} \), and \( R' = (0, l - \delta) \times (\delta, l_y - \delta) \) for some \( \delta \in (0, \min\{\frac{1}{2}l_y, l_x\}) \), then \( u \) is automatically a local minimizer on \( R' \). However, it is not clear that \( u \) is a minimizer with respect to its own boundary data in \( R' \). Indeed, in the second case for the construction of a competitor one may insert jumps in the normal derivative on the two horizontal boundaries.

For \( f : [0, l_x] \to \mathbb{R} \), we denote the linear interpolation by

\[
f^f(x) := \frac{x}{l_x} f(l_x) + \frac{l_x - x}{l_x} f(0), \quad \text{for } x \in [0, l_x].
\]

(12)

2 Outline of the Main Arguments

In this section we provide a brief summary of the general strategy and of the main mathematical ideas leading to the proof of asymptotic self-similarity. Many ideas and results are extensions of those developed in [17, 25, 41, 43]. For the sake of simplicity, we suppress the constants in the following estimates, using \( f \lesssim g \) to mean that there is \( c > 0 \) such that \( f \leq cg \) for all values of the parameters, and \( f \sim g \) to mean \( f \lesssim g \) and \( g \lesssim f \). We remark however that one important technical difficulty in the proofs is finding suitable values of the constants which permit to carry out the inductive argument sketched below (Step 3 in the proof of Theorem 3), therefore in the proofs we name the important constants explicitly. In this discussion we focus on a minimizer \( u \in \mathcal{A}_0(R^{l_x,l_y}) \) of \( I(\cdot, R^{l_x,l_y}) \), some results are more general.

The starting point is a method to bound the energy of \( u \) on a subset of its domain, say \( R' := (0, l_0) \times (a, b) \), obtained by constructing a competitor \( v \) which coincides with \( u \) outside \( R' \). Proposition 1 below presents the explicit construction of a test function which matches given boundary data on the four sides of \( R' \). The boundary data are called \( u^T \) (top side), \( u^B \) (bottom side), \( u^L \) = 0 (left side), \( u^R \) (right side).

The first part of the construction (Steps 1-4) considers only the vertical boundaries. By convexity it is immediate to see that \( v(0, \cdot) = 0 \) and \( v(l_0, \cdot) = u^R \) imply

\[
\int_{(0, l_0) \times (a, b)} v^2 \, dx \, dy \geq \int_{(0, l_0) \times (a, b)} (v')^2 \, dx \, dy = \int_a^b \frac{(u^R)^2}{l_0} \, dy,
\]

where \( v' \) is the linear interpolation of \( v \) on \( R' \), i.e. \( v'(x, y) := \frac{x}{l_0} v(l_0, y) = \frac{x}{l_0} u^R(y) \) (see (12)). Obviously, the boundary data yield \( v' = u' \). Further, due to the quadratic nature of
the energy, one can separate this “linear” contribution (in the sense of being the elastic energy of the linear interpolation between the boundary values), which only depends on the boundary data, from the “excess” energy, which arises from the oscillations around the linear interpolation,

\[ I(v, (0, l_0) \times (a, b)) = \int_0^{l_0} \int_a^b (v - v')^2 \, dx \, dy + \varepsilon |v_{yy}|((0, l_0) \times (a, b)) + \frac{1}{l_0} \int_a^b |v(l_0, y)|^2 \, dy. \]  

(13)

As the third term is fixed, we focus on the first two. The test function in Proposition 1 exhibits self-similar branching near the left and right boundaries (as illustrated in Fig. 1 for the left boundary), and its excess energy has the optimal scaling \( \varepsilon^{2/3} \theta^{2/3} l_0^{1/3} (b - a) \). The branching construction includes a careful choice of the vertical subdivision of the domain, which is important if \( \theta \) is small. This requires that the domain is (in the vertical direction) larger than the natural length scale of the microstructure, which is of order \( \varepsilon^{1/3} \theta^{-2/3} l_0^{2/3} \). This is the natural lower bound on the height of the subsets we can consider, there is no lower bound on the width \( l_0 \).

In the final step (Step 5), we modify the construction in order to fulfill also the boundary conditions on the top and bottom sides. Due to the hard constraint \( v_y \in \{1 - \theta, -\theta\} \) we cannot use a smooth cutoff function. We instead take locally the maximum or the minimum between the function obtained with branching and new functions which obey the boundary data on the horizontal boundaries (but not on the vertical ones); for example,

\[ w(x, y) := u_B(x) + (1 - \theta)(y - a) \]

is the largest function which obeys \( w_y \in \{1 - \theta, -\theta\} \) everywhere, \( w(\cdot, a) \leq u_B \) and creates at most one additional interface. Since \( \tilde{v}_x \in \{v_x, w_x\} = \{v_x, u_B^x\} \), the additional cost of elastic energetic can be controlled by

\[ \frac{\omega_B}{\theta} \int_0^{l_0} (u_B^x - (u_B^x)_l)_x^2 \, dx \]  

(14)

where \( \omega_B/\theta \) is a bound on the thickness of the interpolation layer, which in turn can be estimated in terms of \( u_B - v \). As the branching construction gives a good bound on \( |v - v'| \), \( \omega_B \) can be estimated via the distance between \( u_B^x(x) = (u_B^x)_l \) and \( u_B^x(x, y) = u(l_0, y) \). Therefore the procedure requires control of (14) and uniform control of \( u(l_0, \cdot) \). An analogous construction is carried out on the other side, and Proposition 1 follows.

The rest of the paper builds on this local bound to prove specific properties of minimizers, using several bootstrapping arguments in order to obtain appropriate control of the local boundary conditions. As explained above, Proposition 1 can be used to obtain the local energy bound (5), of the form

\[ I(u, (0, l) \times (a, b)) \lesssim \varepsilon^{2/3} \theta^{2/3} l^{1/3} (b - a) \quad \text{for} \quad \varepsilon^{1/3} \theta^{-2/3} l^{2/3} \lesssim b - a, \]  

(15)

only if we have a good bound on the quantity in (14) and on \( \omega_B \) (and the same on the top side), which in turn requires a local uniform bound of the type

\[ |u(l, y)| \lesssim \varepsilon^{1/3} \theta^{1/3} l^{2/3} \quad \text{for} \quad y \in [a, b]. \]  

(16)

The latter is essentially equivalent to (4), which for this reason is closely linked to (5).
We shall discuss below how Proposition 1 permits to prove (15) from (16), and how the constraint can be used to prove (16) from (15). The apparent circularity of this argument can be circumvented, using the fact that (15) deteriorates only moderately if \( l \) is decreased by a fixed factor. Therefore one can prove the two estimates jointly by an inductive procedure, where at step \( i \) one considers \( l \sim x \sim \phi^i L_x \), for some \( \phi \in (0, 1) \) chosen below. It is crucial to make sure that the constants implicit in the two bounds do not deteriorate in the inductive step. This is technically subtle but possible, since both estimates contain a “main” contribution from the construction, which has a universal constant, and smaller error terms from the “previous round”, whose constant deteriorates but which do not constitute the leading-order contribution, at least if the shape of the rectangles is chosen appropriately. We refer to the proof of Theorem 3 for details.

In the following we briefly outline the mathematical arguments to obtain the bounds of (15) and (16). In order to get started, we assume the two bounds to hold for some \( l = l_0 \) and some (well-chosen) \( a_0, b_0 \), with \([a_0, b_0]\) covering most of \([0, L_x]\) (for any minimizer we can find values where this holds, see Step 1 in the proof of Theorem 3). We then consider the function

\[
 f(a, b) := \int_0^{l_0} \int_a^b (u - u^l)^2 \, dx \, dy + \varepsilon |u_{yy}|((0, l_0) \times (a, b)).
\]

We know that \( f(a_0, b_0) \lesssim \varepsilon^{2/3} \theta^{2/3} l_0^{1/3} (b - a_0) \). Let us for a moment fix \( b = b_0 \) and focus on the nonincreasing function \( f(\cdot, b_0) \). If it has a large derivative, then \( f \) becomes rapidly smaller with increasing \( a \), and the desired bound follows. If instead the derivative is small, then we obtain a good bound on the quantity in (14) and, given that we globally control \( \omega^\theta \), Proposition 1 and the comparison discussed above (considering the box \((0, l) \times (a, b_0)\)) lead us to the desired upper bound for \( f(a, b_0) \). Of course, we need to make sure that the “top” boundary condition does not cause problems, this is the key criterion in choosing \( b_0 \).

Combining the two cases, and repeating the procedure with \( b \), one obtains

\[
 f(a, b) \lesssim \varepsilon^{2/3} \theta^{2/3} l_0^{1/3} (b - a) \quad \text{for} \quad \varepsilon^{1/3} \theta^{-2/3} l_0^{2/3} \lesssim b - a, \tag{17}
\]

which is (15) for \( l = l_0 \). This leads easily to an \( L^2 \) bound on \((u - u^l)(x, \cdot)\) over any admissible segment \((a, b)\), for any \( x \in (0, l_0) \),

\[
 \int_a^b |u - u^l|^2(x, y) \, dy \leq x \int_a^b \int_0^x (u - u^l)^2 \, dx \, dy \leq x f(a, b).
\]

Combining this with the definition of \( u^l \) one obtains an \( L^2 \) bound on \( u(x, \cdot) \). This can be directly turned into an \( L^\infty \) bound using the fact that \( u(x, \cdot) \) is \( 1 \)-Lipschitz, see (48) below. However, the resulting estimate does not have the optimal scaling in \( \theta \). Indeed, \( u_\varepsilon \in [1 - \theta, -\theta] \) gives a stronger condition on the negative part of the derivative. This is exploited in Lemma 4 to obtain

\[
 |u(x, y)| \leq \frac{x}{l_0} |u(l_0, y)| + \left( \frac{1}{\delta} \int_a^b |u - u^l|^2(x, y) \, dy \right)^{1/2} + \theta \delta
\]

for \( y \in (a + \delta, b - \delta) \) and \( \delta \in (0, \frac{b-a}{2}) \), we shall choose \( b - a \sim \varepsilon^{1/3} \theta^{-2/3} l_0^{2/3} \) and \( \delta := (x/l_0)^{1/3} (b - a) \). The first term can be controlled from (16), the second one from (15). The fact that the first term is linear in \( x \), whereas the desired estimate scales as \( x^{2/3} \), permits to escape the iterative deterioration of the constant. This leads to a proof of (16) for \( l_1 := \phi l_0 \).
In order to continue on \((0, l_1)\) one observes that (15) for \(l = l_0\) implies (15) for \(l = l_1 := \phi l_0\) with a constant which is \(\phi^{-1/3}\) times larger, and continues inductively.

The local bounds in (16) and (15) are one ingredient in the proof of the asymptotic self-similarity of a minimizer \(u \in \mathcal{A}_0(\mathbb{R}^{l_x, l_y})\) in the sense of blow-ups with respect to local strong convergence in \(W^{1,2}\). The main difficulty in proving the strong convergence of the sequence \((u^j)\) (introduced in (3)) is the proof of the strong convergence of the \(x\)-derivatives in \(L^2_{\text{loc}}\). The Hölder-continuity of \(u\) (see Lemma 7) and the local energy scaling law presented in Lemma 4 allow us to choose a subrectangle \((0, l) \times (0, l_y) \subseteq \mathbb{R}^{l_x, l_y}\) on which the assumption of Theorem 3 are satisfied. Due to Theorem 3 we know that (15) and (16) hold on \(R := (0, l_x) \times (3\eta l_y, (1 - 3\eta) l_y)\) for some fixed \(\eta \in (0, \frac{1}{6})\). Thus, we have the uniform bounds

\[
|u^j(l, \cdot)| \lesssim \varepsilon^{1/3} \theta^{1/3} l^{2/3}
\]

and by a change of variables,

\[
I(u^j, (0, l) \times (a, b)) \lesssim \varepsilon^{2/3} \theta^{2/3} l^{1/3} (b - a)
\]

for \(\varepsilon^{1/3} \theta^{-2/3} l^{2/3} \lesssim b - a\). Taking a diagonal sequence, we obtain the existence of a function \(u^\infty \in W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R})\) obeying

(i) \(u^j \rightharpoonup u^\infty\) weakly in \(W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R})\),

(ii) \(u^j_{y} \rightharpoonup u^\infty_{y}\) weakly as Radon measures on \((0, \infty) \times \mathbb{R}\),

(iii) \(u^j \to u^\infty\) uniformly in every compact set \(K \subseteq [0, \infty) \times \mathbb{R}\).

Using compensated compactness, (i) and (ii) imply the strong convergence of \((u^j_y)\) towards \(u^\infty_y\) in \(L^2_{\text{loc}}\). By lower-semi-continuity, (18) and (19) are also true for \(u^\infty\).

We continue by proving the existence of a subsequence \((u^j)\) such that the sequence \((u^j_y)\) strongly converges towards \(u^\infty_x\) in \(L^2(R^{l_x, l_y})\) for all \(l, h > 0\) with \(R_{l,h} := (0, l) \times (-h, h)\). For that, we show

\[
\|u^j_x - u^\infty_x\|_{L^2(R_{l,h})} \lesssim \frac{\varepsilon^{2/3} \theta^{5/6}}{H^{1/2}} \|u^j_x - u^\infty_x\|_{L^2(R_{l,h})}
\]

for \(l > 0\) and \(\varepsilon^{1/3} \theta^{-2/3} l^{2/3} \lesssim H\). Applying (20) two times and using (19) to estimate the \(L^2\)-norm of \((u^j_y)\) and \(u^\infty\) implies \(u^j_x \to u^\infty_x\) in \(L^2(R^{l_x, l_y})\) as \(j \to \infty\) for any \(l, h > 0\).

The proof of (20) contains several technical difficulties. Starting point is the identity

\[
|u^\infty_x - u^j_x|^2 + |u^\infty_y|^2 - |u^j_y|^2 = 2u^\infty_x (u^\infty_x - u^j_x),
\]

which motivates to introduce for an appropriately chosen fixed \(H \gtrsim \varepsilon^{1/3} \theta^{-2/3} l^{2/3}\) the function \(f_j : (0, H) \to \mathbb{R}\),

\[
f_j(h) := 2 \int_{R_{l,h}} u^\infty_x (u^\infty_x - u^j_x) \, dx \, dy + \varepsilon \max\{ |u^\infty_y| \|(R_{l,h}) - |u^j_y| \|(R_{l,h})\}, 0\]

\[
\geq \int_{R_{l,h}} |u^\infty_x - u^j_x|^2 \, dx \, dy + I(u^\infty, R_{l,h}) - I(u^j, R_{l,h}).
\]

By weak convergence and lower semi-continuity, \(f_j \to 0\) pointwise, and hence, by Egorov’s theorem, uniformly on a set of large measure. It remains to bound the difference \(I(u^j, R_{l,h}) - I(u^\infty, R_{l,h})\) which is again done by constructing a competitor \(w^j\) to \(u^j\).
Roughly speaking, $w^j$ is constructed to agree with $u^\infty$ well in the interior of the rectangle and equals to $u^j$ far outside. The main difficulty (compared to the construction in Proposition 1) is that we want the difference of the energies of $w^j$ and $u^\infty$ to converge towards zero and not just to be uniformly bounded. For that, we consider larger rectangles, and provide a careful treatment of the interpolation layers, with different arguments at the right (vertical) boundary and the top and bottom (horizontal) boundaries, respectively, see Sect. 7 and in particular Fig. 8 there. Let us briefly explain the main ideas of these two interpolations.

For the right interpolation, we consider a small interpolation layer of width $l_j := l \|u^j - u^\infty\|_{L^\infty(R_{2l_j}, H)}$ which for $j \to \infty$ tends to zero by (iii). Here, we take the function $u^j$ and truncate it at $u^\infty(x, y) \pm \frac{l_j}{2}$ from above and below, respectively. Note that this indeed interpolates between $u^\infty$ for $x \leq l$ and $u^j$ for $x \geq l + l_j$, and yields an admissible test function $\tilde{u}^j$. While the elastic energy of the interpolation is easily estimated by explicit computation, the surface energy requires a counting argument that is presented after (102).

The interpolation on top and bottom of the rectangle is more subtle and is worked out in Lemma 8. Let us consider only the top boundary. The interpolation takes place on (large)

Let us briefly explain the main ideas of these two interpolations.

The key observation is that a small change in the boundary values can be generated with careful treatment of the interpolation layers, with different arguments at the right (vertical) and not just to be uniformly bounded. For that, we consider larger rectangles, and provide a careful treatment of the interpolation layers, with different arguments at the right (vertical) boundary and the top and bottom (horizontal) boundaries, respectively, see Sect. 7 and in particular Fig. 8 there. Let us briefly explain the main ideas of these two interpolations.

For the right interpolation, we consider a small interpolation layer of width $l_j := l \|u^j - u^\infty\|_{L^\infty(R_{2l_j}, H)}$ which for $j \to \infty$ tends to zero by (iii). Here, we take the function $u^j$ and truncate it at $u^\infty(x, y) \pm \frac{l_j}{2}$ from above and below, respectively. Note that this indeed interpolates between $u^\infty$ for $x \leq l$ and $u^j$ for $x \geq l + l_j$, and yields an admissible test function $\tilde{u}^j$. While the elastic energy of the interpolation is easily estimated by explicit computation, the surface energy requires a counting argument that is presented after (102).

The interpolation on top and bottom of the rectangle is more subtle and is worked out in Lemma 8. Let us consider only the top boundary. The interpolation takes place on (large) layers $(0, l + l_j) \times (h_j - h, h_j)$ of height $h$, see Fig. 8. Here the $h_j$ have to be chosen carefully such that in particular $u_y$ does not jump on $\{y = h_j\}$ and the elastic energy of $u^j$ or $u^\infty$, the difference of the $x$- and the $y$-derivatives of $u^j$ and $u^\infty$ do not concentrate on $\{y = h_j\}$. The last condition in particular means that

$$\epsilon l \limsup_{j \to \infty} \eta_j^{1/2} \leq \frac{\epsilon^{2/3} l^{5/6}}{\theta^{1/3} H^{1/2}} \limsup_{j \to \infty} \|u^j_x - u^\infty_x\|_{L^2(R_{2l_j}, H)}$$

(22)

where

$$\eta_j := \frac{1}{\epsilon^{2/3} \theta^{2/3} l^{1/3}} \int_0^{l + l_j} (u^j(x, h_j) - \tilde{u}^j(x, h_j))^2 \, dx.$$

Our goal is to construct an admissible function $w^{j,T}$ on $R^T_j := (0, l + l_j) \times (h_j - h, h_j)$ which agrees (up to the derivative) with $\tilde{u}^j$ on the top boundary, with $u^j$ on the bottom boundary, and with both on the right boundary, and

$$I(w^{j,T}, R^T_j) \lesssim I(\tilde{u}^j, R^T_j) + \epsilon l (\eta_j^{1/2} + \eta_j^{3/2}).$$

(23)

The key observation is that a small change in the boundary values can be generated with a small change in energy, if one refrains from creating new interfaces but instead moves smoothly the existing ones, thereby varying the local volume fraction of the two phases on each segment $\{x\} \times (h_j - h, h_j)$, as sketched in Fig. 6 below. Indeed, this local volume fraction is in one-to-one correspondence with the difference between the value of the function on the top and bottom boundaries,

$$\mathcal{L}^1(\{y \in (h_j - h, h_j): \tilde{u}^j(x, y) = 1 - \theta\}) = \tilde{u}^j(x, h_j) + \theta h - \tilde{u}^j(x, h_j - h).$$

Therefore the required boundary values can be attained by changing this volume fraction by a factor $\alpha^j(x)$ depending on $\tilde{u}^j(x, h_j), \tilde{u}^j(x, h_j - h),$ and $u^j(x, h_j)$. This factor is close to one since $\tilde{u}^j$ and $u^j$ converge locally uniformly to the same function. We refer to Steps 1 and 2 of the proof of Lemma 8 for details. One then can verify that the energy estimate (23) is fulfilled. Putting things together, $I(u^j, R_{l,h_j}) - I(u^\infty, R_{l,h_j}) \lesssim \epsilon l \eta_j^{1/2}$. Recalling (22) one obtains (20) together with (21), (23) for the bottom and top area and $|f_j(h_j)| \to 0$ as $j \to \infty$. This concludes the proof of strong convergence.

Springer
3 Explicit Construction

In this section, we will present the construction of a test function with given Dirichlet boundary conditions for which the energy can be controlled in terms of the boundary conditions. This will be used later to modify a given function on subrectangles of the domain. The construction is taken from [25] and is a generalization to the unequal volume-fraction case of the construction from [17, Sect. 2.1], which in turn builds upon [41, Sect. 2].

**Proposition 1** (Local estimate of the energy) There is \( \tilde{c}_0 > 0 \) such that for all \( \varepsilon > 0, \theta \in (0, 1/2), l_x > 0, \) and \( l_y \in [\varepsilon^{1/3}\theta^{-2/3}l_x^{1/3}, +\infty) \) the following holds: Suppose that \( u^T, u^B : [0, l_x] \to \mathbb{R} \) and \( u^L, u^R : [0, l_y] \to \mathbb{R} \) are continuous, weakly differentiable with \( u^L_y := (u^L)' \in [\theta, 1 - \theta] \) and \( u^R_y := (u^R)' \in [\theta, 1 - \theta] \) a.e., and

\[
\begin{align*}
\int_{0}^{l_x} \int_{0}^{l_y} (u - u^L)^2 \, dL^2 + \varepsilon |u_{yy}|(R_{l_x,l_y})^2 \\ \leq \frac{\omega^T}{\theta} \int_{0}^{l_x} (u^T - u^T)^2 \, dx + \frac{\omega^B}{\theta} \int_{0}^{l_y} (u^B - u^B)^2 \, dy + \tilde{c}_0 \varepsilon^{2/3} \theta^{-4/3} l_x^{1/3} l_y.
\end{align*}
\]

**Remark 1** Explicit integration, using (25) and (i), immediately gives

\[
I(u, R_{l_x,l_y}) = \frac{1}{l_x} \int_{0}^{l_y} |u^B(y) - u^L(y)|^2 dy + \int_{R_{l_x,l_y}} (u - u^L)^2 \, dL^2 + \varepsilon |u_{yy}|(R_{l_x,l_y}).
\]
The condition (28) can be replaced with

$$-\theta l_y \leq u^T - u^B \leq \min\{K\theta, (1 - \theta)l_y\}$$

(29)

for a constant $K > 0$, the value of $\tilde{c}_0$ then depends on $K$. The choice $K = 1$ corresponds to (28).

Further, in (iii) the factors $\omega^T / \theta$ and $\omega^B / \theta$ can be replaced by $\min\{\omega^T / \theta, L_y\}$ and $\min\{\omega^B / \theta, L_y\}$, respectively.

The proof shows that we can choose $\tilde{c}_0 = 144$.

**Proof** We first construct in Step 1-4 an admissible function $\tilde{u}$ satisfying (i),

$$\int_0^{l_x} \int_0^{l_y} (\tilde{u} - u^l)^2 \, dL^2 + \varepsilon \left| \tilde{u}_{yy} \right| \left( (0, l_x) \times (0, l_y) \right)$$

(30)

$$\leq \tilde{c}_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} l_y - 2\varepsilon l_x,$$

and

$$\|\tilde{u} - u^l\|_{L^\infty(R^{l_x,l_y})} \leq \varepsilon^{1/3} \theta^{1/3} l_x^{2/3},$$

(31)

and modify it close to the upper and lower boundaries in Step 5 to obtain a function as claimed. We describe the construction only on the right half of the rectangle $(l_x/2, l_x) \times (0, l_y)$, the construction in the other part can be done similarly.

**Step 1: Geometry of the construction.**

We decompose the rectangle in smaller rectangles, as illustrated in Fig. 2. In $x$-direction we use a geometrically refining decomposition. To shorten notation, we write $\eta := 1/3$ and for $i \in \mathbb{N}$ we set

$$x_i := l_x \left( 1 - \frac{1}{2} \eta^i \right).$$

Then

$$x_{i+1} - x_i = l_x \left( \frac{1}{2} \eta^i - \frac{1}{2} \eta^{i+1} \right) = \frac{1}{2} \cdot (1 - \eta) \eta^i l_x = \frac{1}{3} l_x \left( \frac{1}{3} \right)^i.$$

The decomposition in $y$-direction is more involved. In contrast to the decomposition used for $\theta = \frac{1}{2}$ in [41, Lemma 2.3] and [17, Lemma 2.3], the separation points are not equally distributed in $(0, l_y)$. We fix $N \in \mathbb{N}$, $N \geq 1$, chosen below (see (42)) and for $i \in \mathbb{N}$ choose $2^i N + 1$ ordered points $y_{i,k} \in [0, l_y]$ such that, roughly speaking, in the intervals $\{x_i\} \times (y_{i,k}, y_{i,k+1})$, the volume fraction of the minority variant $\tilde{u}_y = 1 - \theta$ is of order $\theta$. More precisely, we define

$$f : [0, l_y] \to \mathbb{R}, \quad f(y) := \int_0^y \left( 3 + \frac{u^L_y(s) + u^R_y(s)}{\theta} \right) \, ds.$$

The function $f$ can be seen as a measure of the portion of the minority variant in $u^L$ and $u^R$ in $(0, y)$. By the assumption $u^L_y, u^R_y \in [-\theta, 1 - \theta]$ the function $f$ is Lipschitz continuous.
with $1 \leq f' \leq 1 + \frac{2}{\theta}$ almost everywhere, and in particular strictly monotonically increasing. From (28) and (24) we obtain $\max \{ |u^L(l_v) - u^L(0)|, |u^R(l_v) - u^R(0)| \} \leq \theta l_v$ and therefore

$$M := f(l_v) = 3l_v + \frac{u^L(l_v) - u^L(0) + u^R(l_v) - u^R(0)}{\theta} \in [l_v, 5l_v] \quad (32)$$

In particular, $f : [0, l_v] \to [0, M]$ is bijective.

We now select decomposition points according to this density. Precisely, for $i \in \mathbb{N}$ and $k \in \{0, \ldots, 2iN\}$ we define $y^{i,k}$ by

$$f(y^{i,k}) = \frac{Mk}{2iN}. \quad (32')$$

We observe that $y^{i,2k} = y^{i-1,k}$ for $k \in \{0, \ldots, 2^{i-1}N\}$, $f(y^{i,k+1}) - f(y^{i,k}) = \frac{M}{2iN}$ for all $i \in \mathbb{N}$ and all $k \in \{0, \ldots, 2iN - 1\}$, and, since $f' \geq 1$,

$$0 < y^{i,k+1} - y^{i,k} \leq f(y^{i,k+1}) - f(y^{i,k}) = \frac{M}{2iN} \leq \frac{5l_v}{2iN}. \quad (32')$$

**Step 2: Construction on** $\{x_i\} \times [0, l_v]$. For $i \in \mathbb{N}$, let $\tilde{u}(x_i, \cdot) : (0, l_v) \to \mathbb{R}$ be the unique continuous, piecewise affine function with the following properties (see Fig. 3):

(i) $\tilde{u}(x_i, y^{i,k}) = u^l(x_i, y^{i,k})$ for all $k \in \{0, \ldots, 2iN\}$, where $u^l$ is the linear interpolation from (25); and

(ii) for all $k \in \{0, \ldots, 2iN - 1\}$, there exists $m^{i,k} \in [y^{i,k}, y^{i,k+1}]$ such that

$$\tilde{u}_y(x_i, \cdot) = \begin{cases} 1 - \theta, & \text{in } (y^{i,k}, m^{i,k}), \\ -\theta, & \text{in } (m^{i,k}, y^{i,k+1}). \end{cases}$$

This is possible since $u^l_{y} \in [-\theta, 1 - \theta]$ (see Fig. 3). Note that

$$(1 - \theta) (m^{i,k} - y^{i,k}) - \theta (y^{i,k+1} - m^{i,k}) = u^l(x_i, y^{i,k+1}) - u^l(x_i, y^{i,k})$$
Fig. 3 Sketch of the geometry in the $y$ direction in Step 2 of the proof of Proposition 1. The lower, green line is the lower bound used to prove (34)

\[
\int_{y^{i,k}}^{y^{i,k+1}} u^l_y(x_i, s) \, ds = \int_{y^{i,k}}^{y^{i,k+1}} \left[ \frac{x_i}{l_x} u^R_y(s) + \frac{l_x - x_i}{l_x} u^L_y(s) \right] \, ds
\]

\[
\int_{y^{i,k}}^{y^{i,k+1}} \left[ \frac{x_i}{l_x} (u^R_y(s) + \theta) + \frac{l_x - x_i}{l_x} (u^L_y(s) + \theta) - \theta \right] \, ds
\]

\[
\leq \int_{y^{i,k}}^{y^{i,k+1}} \left[ u^R_y(s) + u^L_y(s) + \theta \right] \, ds.
\]

Recalling that $\theta f' = 3\theta + u^R_y + u^L_y$, and rearranging terms, we obtain

\[
(m^{i,k} - y^{i,k}) - \theta \left( y^{i,k+1} - y^{i,k} \right) \leq \int_{y^{i,k}}^{y^{i,k+1}} \left[ u^R_y(s) + u^L_y(s) + \theta \right] \, ds
\]

\[
= \theta \left( f \left( y^{i,k+1} \right) - f \left( y^{i,k} \right) \right) - 2\theta \left( y^{i,k+1} - y^{i,k} \right)
\]

which yields, recalling the definition of $y^{i,k}$ and (32),

\[
m^{i,k} - y^{i,k} \leq \theta \left( f \left( y^{i,k+1} \right) - f \left( y^{i,k} \right) \right) = \frac{\theta M}{2^i N} \leq \frac{5\theta l_y}{2^i N}.
\]

The condition $u^l_y \in [-\theta, 1 - \theta]$ leads to (see Fig. 3)

\[
\max\{u^l(x_i, y^{i,k}) - \theta(y - y^{i,k}), u^l(x_i, y^{i,k+1}) + (1 - \theta)(y - y^{i,k+1})\} \leq u^l(x_i, y)
\]

\[
\leq \min\{u^l(x_i, y^{i,k}) + (1 - \theta)(y - y^{i,k}), u^l(x_i, y^{i,k+1}) - \theta(y - y^{i,k+1})\} = \tilde{u}(x_i, y)
\]

for $y \in [y^{i,k}, y^{i,k+1}]$. The difference between the first and the third expression is bounded by $m^{i,k} - y^{i,k}$. Therefore, by (33),

\[
|\tilde{u} - u^l|(x_i, \cdot) \leq \frac{5\theta l_y}{2^i N}.
\]

Step 3: Construction in $(x_i, x_{i+1}) \times (0, l_y)$.

For $i \in \mathbb{N}$, consider the points $0 = z^{i,0} < z^{i,1} < \ldots < z^{i,l_i} = l_y$ with

\[
\{z^{i,0}, \ldots, z^{i,l_i}\} = \bigcup_{k=0}^{2^i N-1} \{m^{i,k}\} \cup \bigcup_{k=0}^{2^{i+1} N} \{y^{i+1,k}\} \cup \bigcup_{k=0}^{2^{i+1} N-1} \{m^{i+1,k}\}.
\]
We now construct a continuous, piecewise affine function $\tilde{u}$ on $(x_i, x_{i+1}) \times (0, l_y)$ in the following (iterative) way (see Fig. 4). Let $j \in \{0, \ldots, J_i - 1\}$, and assume that the construction on $(x_i, x_{i+1}) \times (0, z_i^{j})$ is done. By the construction in Steps 1 and 2 the functions $\tilde{u}(x_i, \cdot)$ and $\tilde{u}(x_{i+1}, \cdot)$ are affine on $(z_i^{j}, z_i^{j+1})$.

(i) If $\tilde{u}_y(x_i, \cdot) = \tilde{u}_y(x_{i+1}, \cdot)$ on $(z_i^{j}, z_i^{j+1})$ then we use the linear interpolation and set

$$\tilde{u}(x, y) := \frac{x - x_i}{x_{i+1} - x_i} \tilde{u}(x_i, y) + \frac{x_{i+1} - x}{x_{i+1} - x_i} \tilde{u}(x_i, y)$$

in $(x_i, x_{i+1}) \times (z_i^{j}, z_i^{j+1})$.

(ii) Assume now $\tilde{u}_y(x_i, \cdot) = -\theta \neq 1 - \theta = \tilde{u}_y(x_{i+1}, \cdot)$ on $(z_i^{j}, z_i^{j+1})$, the other case can be treated analogously. We construct a piecewise affine function such that $\tilde{u}_y$ does not jump on $(x_i, x_{i+1}) \times \{z_i^{j}\}$. Precisely, if $j = 0$ or $\lim_{y \rightarrow z_i^{j}} \tilde{u}_y(x, y) = -\theta$ for $x \in (x_i, x_{i+1})$, then we set

$$\tilde{u}(x, y) := \begin{cases} (1 - \lambda_x)\tilde{u}(x_i, z_i^{j}) + \lambda_x \tilde{u}(x_{i+1}, z_i^{j}) - \theta(y - z_i^{j}), & \text{if } y \leq \lambda_x z_i^{j} + (1 - \lambda_x) z_i^{j+1}, \\ (1 - \lambda_x)\tilde{u}(x_i, z_i^{j+1}) + \lambda_x \tilde{u}(x_{i+1}, z_i^{j+1}) + (1 - \theta)(y - z_i^{j+1}), & \text{otherwise}, \end{cases}$$

where $\lambda_x := \frac{x - x_i}{x_{i+1} - x_i}$. If instead $\lim_{y \rightarrow z_i^{j}} \tilde{u}_y(x, y) = 1 - \theta$, we set

$$\tilde{u}(x, y) := \begin{cases} (1 - \lambda_x)\tilde{u}(x_i, z_i^{j}) + \lambda_x \tilde{u}(x_{i+1}, z_i^{j}) + (1 - \theta)(y - z_i^{j}), & \text{if } y \leq (1 - \lambda_x) z_i^{j} + \lambda_x z_i^{j+1}, \\ (1 - \lambda_x)\tilde{u}(x_i, z_i^{j+1}) + \lambda_x \tilde{u}(x_{i+1}, z_i^{j+1}) - \theta(y - z_i^{j+1}), & \text{otherwise}. \end{cases}$$

For later reference, we remark that this construction satisfies

$$\tilde{u}_x \in \left\{ \frac{\tilde{u}(x_i, z_i^{j}) - \tilde{u}(x_i, z_i^{j+1})}{x_{i+1} - x_i}, \frac{\tilde{u}(x_{i+1}, z_i^{j+1}) - \tilde{u}(x_i, z_i^{j+1})}{x_{i+1} - x_i} \right\}$$

which, since $\tilde{u}_y(x_i, \cdot) - \tilde{u}_y(x_{i+1}, \cdot) = \pm 1$ in this interval, implies (with (33))

$$\left| \tilde{u}_x(x, y) - \frac{\tilde{u}(x_i, y) - \tilde{u}(x_i, y)}{x_{i+1} - x_i} \right| \leq \frac{z_i^{j+1} - z_i^{j}}{x_{i+1} - x_i} \leq \frac{1}{x_{i+1} - x_i} \frac{5\theta l_y}{2iN} \quad (35)$$

almost everywhere in $(x_i, x_{i+1}) \times (z_i^{j}, z_i^{j+1})$. 

---

**Fig. 4** Sketch of the possible constructions in $(x_i, x_{i+1}) \times (y_i^1, y_i^{k+1})$. Gray regions correspond to the minority variant $\tilde{u}_y = 1 - \theta$. There are at most three “inner” interfaces, and in each block (except for $i = 0$) we count the lower boundary.
By construction (see Fig. 4),
\[ |\tilde{u}_{yy}| ((x_i, x_{i+1}) \times (0, l_y)) \leq (4 \cdot 2^i N - 1) (x_{i+1} - x_i). \quad (36) \]

**Step 4: Energy estimate for \( \tilde{u} \).**

Summing (36) over all \( i \in \mathbb{N} \), and inserting a factor of 2 for the other half of the rectangle,
\[ |\tilde{u}_{yy}| ((0, l_x) \times (0, l_y)) \leq 2 \sum_{i=0}^{\infty} 4 \cdot 2^i N l_x \frac{1}{3^{i+1}} = 4 N l_x \sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^i = 8 N l_x. \]

It remains to estimate the elastic energy. To simplify notation, we set \( R_i := (x_i, x_{i+1}) \times (0, l_y) \) and introduce the linear interpolation in \((x_i, x_{i+1})\),
\[ \tilde{u}^{i,l}(x, y) := \frac{x_{i+1} - x}{x_{i+1} - x_i} \tilde{u}(x_i, y) + \frac{x - x_i}{x_{i+1} - x_i} \tilde{u}(x_{i+1}, y) \quad \text{for} (x, y) \in R_i. \]

Straightforward expansion and explicit integration shows, as in [17, Lemma 2.3], that
\[ \int_{R_i} (\tilde{u} - u^l)^2 \, d\mathcal{L}^2 = \int_{R_i} (\tilde{u} - \tilde{u}^{i,l})^2 \, d\mathcal{L}^2 + \int_{R_i} (\tilde{u}^{i,l} - u^l)^2 \, d\mathcal{L}^2. \quad (37) \]

We start from the first term on the right hand side, and treat each subrectangle \((x_i, x_{i+1}) \times (z_i^{1,j}, z_i^{l,j+1})\) separately. By the construction from Step 3, there are two cases:

If \( \tilde{u}_y(x_i, \cdot) = \tilde{u}_y(x_{i+1}, \cdot) \) on \((z_i^{1,j}, z_i^{l,j+1})\) then \( \tilde{u} = \tilde{u}^{i,l} \) in \((x_i, x_{i+1}) \times (z_i^{1,j}, z_i^{l,j+1})\). Otherwise, \( \tilde{u}_y(x_i, \cdot) = 1 - \theta \) or \( \tilde{u}_y(x_{i+1}, \cdot) = 1 - \theta \) in \((z_i^{1,j}, z_i^{l,j+1})\). By construction, there are at most \(2^i N\) such intervals (see Fig. 4), and by (33) for each of them \( z_i^{l,j+1} - z_i^{1,j} \leq \frac{5 \theta l_y}{2 N} \).

Furthermore, using (35), for almost every \( y \in (z_i^{1,j}, z_i^{l,j+1}) \) we have
\[ \int_{x_i}^{x_{i+1}} (\tilde{u} - \tilde{u}^{i,l})_x^2 \, dx \leq \int_{x_i}^{x_{i+1}} \frac{(z_i^{1,j+1} - z_i^{1,j})^2}{(x_{i+1} - x_i)^2} \, dx \leq \frac{1}{x_{i+1} - x_i} \left( \frac{5 \theta l_y}{2 N} \right)^2, \quad (38) \]

and hence
\[ \int_{R_i} (\tilde{u} - \tilde{u}^{i,l})_x^2 \, d\mathcal{L}^2 \leq \frac{2^i N \cdot 5 \theta l_y}{2 N (x_{i+1} - x_i)} \left( \frac{5 \theta l_y}{2 N} \right)^2 = \frac{3 \cdot 2 \cdot 5^3 \theta^3 l_y^3}{N^2 l_x} \left( \frac{3}{4} \right)^i. \quad (39) \]

For the last term in (37), we use first that \( \tilde{u}^{i,l} - u^l \) is affine in \( x \)-direction in \((x_i, x_{i+1}) \times (0, l_y)\) and then (34). Hence
\[ \int_{R_i} (\tilde{u}^{i,l} - u^l)_x^2 \, d\mathcal{L}^2 = \int_{0}^{l_y} \int_{x_i}^{x_{i+1}} (\tilde{u}^{i,l} - u^l)_x^2 \, dx \, dy \quad (40) \]
\[ = \int_{0}^{l_y} \left[ (\tilde{u}(x_{i+1}, y) - u^l(x_{i+1}, y)) - (\tilde{u}(x_i, y) - u^l(x_i, y)) \right] \, dy \]
\[ \leq \frac{l_y}{x_{i+1} - x_i} \left( \frac{2 \cdot 5 \theta l_y}{2 N} \right)^2 = \frac{4 \cdot 5^2 \theta^2 l_y^3}{4^i N^2 (x_{i+1} - x_i)} = \frac{3 \cdot 4 \cdot 5^2 \theta^2 l_y^3}{N^2 l_x} \left( \frac{3}{4} \right)^i. \]

Inserting (39) and (40) in (37) and summing over \( i \), we obtain (since \( \theta \leq 1/2 \))
\[ \int_{0}^{l_y} \int_{l_x/2}^{l_x} (\tilde{u} - u^l)_x^2 \, d\mathcal{L}^2 \leq \sum_{i=0}^{\infty} \int_{R_i} (\tilde{u} - \tilde{u}^{i,l})_x^2 \, d\mathcal{L}^2 + \int_{R_i} (\tilde{u}^{i,l} - u^l)_x^2 \, d\mathcal{L}^2 . \]
\[
\leq (3 \cdot 5^3 + 3 \cdot 4 \cdot 5^2) \frac{\theta^2 I_x^3}{N^2 I_x} \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = 4 \cdot 3 \cdot 5^2 (5 + 4) \frac{\theta^2 I_y^3}{N^2 I_x} = 2700 \frac{\theta^2 I_y^3}{N^2 I_x}.
\]

(41)

In order to balance this term and the estimate for \(|\tilde{u}_{yy}|\) we choose

\[
N := \lceil 10 \cdot \varepsilon^{-1/3} \theta^{2/3} l_x^{-2/3} l_y \rceil.
\]

(42)

Therefore, recalling that \((0, l_x/2) \times (0, l_y)\) is treated symmetrically, we obtain (30), using that \(l_x \geq \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3}\) implies that \(\varepsilon l_x \leq \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} l_y\) and \(N \leq 11 \varepsilon^{-1/3} \theta^{2/3} l_x^{-2/3} l_y\),

\[
\int_0^{l_x} \int_0^{l_y} (\tilde{u} - u^l)^2 \, d\mathcal{L}^2 + \varepsilon |\tilde{u}_{yy}| \left( (0, l_x) \times (0, l_y) \right)
\leq 8N \varepsilon l_x + 2 \cdot 2700 \frac{\theta^2 I_y^3}{N^2 I_x}
\leq (8N + 2) \varepsilon l_x + 2 \cdot 2700 \frac{\theta^2 I_y^3}{N^2 l_x} - 2 \varepsilon l_x
\leq \left( 8 \cdot 11 + 2 \cdot \frac{2 \cdot 2700}{10^2} \right) \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} l_y - 2 \varepsilon l_x = \tilde{c}_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} l_y - 2 \varepsilon l_x
\]

where in the last step we set \(\tilde{c}_0 := 144\).

From (38) and H"older’s inequality we obtain \(|\tilde{u} - \tilde{u}^l| \leq \frac{so l_y}{2N} \) in \(R_i\). Since \(\tilde{u}^l - u^l\) is affine in the \(x\) direction inside each \(R_i\), it attains its maximum either at \(x = x_i\) or at \(x = x_{i+1}\), and by (34) we obtain \(|\tilde{u}^l - u^l| \leq \frac{so l_y}{2N} \) in \(R_i\). With a triangular inequality and (42) we obtain (31),

\[
\|\tilde{u} - u^l\|_{L^\infty(R\times J^y)} \leq \sup_i \|\tilde{u} - u^l\|_{L^\infty(R_i)} \leq \frac{10 \theta l_y}{N} \leq \varepsilon^{1/3} \theta^{1/3} l_x^{2/3}.
\]

Step 5: Conclusion.

It remains to modify \(\tilde{u}\) so that the boundary conditions on the top and bottom boundaries are fulfilled. We proceed as in [17, Lemma 2.6] and set

\[
u := \min\{\eta^B, \eta^T, \max\{\varphi^T, \varphi^B, \tilde{u}\}\}
\]

where \(\varphi^B, \eta^B\) are the smallest and largest function compatible with \(u^B\) which have \(y\)-derivative in \([-\theta, 1 - \theta]\),

\[
\varphi^B(x, y) := u^B(x) - \theta y, \quad \eta^B(x, y) := u^B(x) + (1 - \theta) y
\]

and correspondingly

\[
\varphi^T(x, y) := u^T(x) - (1 - \theta)(l_y - y), \quad \eta^T(x, y) := u^T(x) + \theta(l_y - y).
\]

By (28) we have \(|u^T - u^B| \leq \theta l_y\) and therefore

\[
\varphi^B \leq \eta^T \text{ and } \varphi^T \leq \eta^B.
\]
For $y = 0$ we have $\varphi^T \leq \eta^B = \varphi^B = u^B$, which implies $u = u^B$, and correspondingly for $y = l_y$. For $x = l_x$ we have $\tilde{u}(l_x, y) = u^R(y)$ with derivative in $[-\theta, 1 - \theta]$, and recalling $u^B(l_x) = u^B(0)$ we obtain

$$\varphi^B(l_x, y) \leq \tilde{u}(l_x, y) = u^R(y) \leq \eta^B(l_x, y).$$

The corresponding estimate holds for $\varphi^T, \eta^T$, and at $x = 0$. Therefore $u$ satisfies the boundary conditions (i) and (ii).

The estimate on the energy follows by direct computation. Using that the construction of $u$ from $\tilde{u}$ creates at most two new interfaces, one obtains that the surface energy grows at most $2\varepsilon l_x$. By (31) and the definition of $\omega^B$ we have

$$|\tilde{u}(x, y) - u^B(x)| \leq |u^B|(x) + |u^l|(x, y) + \varepsilon^{1/3} \theta^{1/3} l_x^{2/3} \leq \omega^B$$

so that $\{u < \varphi^B\} \cup \{\eta^B < u\} \subseteq (0, l_x) \times (0, \omega^B/\theta)$, and the same on the other side. The increase of the elastic energy in these strips is then estimated using Lemma 1 below. This proves (iii).

In closing we recall the following result from [17, Lemma 2.5] that has been used in the final step.

**Lemma 1** Let $v, \varphi_i, \eta_j \in W^{1,2}(0, l_x)$ be given such that the inequality $\varphi_i \leq v \leq \eta_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ at $x = 0$ and $x = l_x$ holds. Then for

$$w := \min\{\eta_1, \ldots, \eta_m, \max\{\varphi_1, \ldots, \varphi_n, v\}\}$$

the following estimate holds

$$\int_0^{l_x} (w - w^l)^2 dx \leq \int_0^{l_x} (v - v^l)^2 dx + \sum_{i=1}^n \int_0^{l_x} (\varphi_i - \varphi_i^l)^2 x \chi_{\{v < \varphi_i\}} dx$$

$$+ \sum_{i=1}^m \int_0^{l_x} (\eta_i - \eta_i^l)^2 x \chi_{\{\eta_i < v\}} dx$$

$$+ \sum_{j=1}^m \int_0^{l_x} (\eta_j - \eta_j^l)^2 x \chi_{\{v < \eta_j < \max\{\varphi_1, \ldots, \varphi_n\}\}} dx.$$  

We remark that the statement in [17] misses the last term, which is irrelevant in the present situation since by construction $\max\{\varphi^B, \varphi^T\} \leq \min\{\eta^T, \eta^B\}$.

### 4 Global Scaling Laws

We discuss the global scaling laws (see Theorem 1) and then give a more precise characterization of the minimizers in the case of $L_x$ large.

**Proof of Theorem 1.** For the convenience of the reader, we give an explicit self-contained proof of the global scaling law in our setting. This is a simplified version of [19, Theorem 1.2] for $\beta = \infty$. 
Upper bound. The constructions are all based on Proposition 1.
If \( \varepsilon^{1/3} \theta^{-2/3} L_x^{2/3} \leq L_y \) then setting \( u^L := 0, u^R := 0, u^T := 0 \) and \( u^B := 0 \) in Proposition 1 we obtain a function which is zero on the boundary and has energy bounded by \( c_0 \varepsilon^{2/3} \theta^{2/3} L_x^{1/3} L_y \).

Otherwise, we define \( l_x := \varepsilon^{-1/2} \theta L_x^{3/2} < L_x \), so that \( L_y = \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3} \). We set \( u(x, y) := -\theta y \) on \( [l_x, L_x] \times (0, L_y) \), and extend it to the entire domain using Proposition 1 in \( (0, l_x) \times (0, L_y) \) with \( u^L = 0, u^B = 0, u^R = -\theta y \), and \( u^T (x) = -\theta L_y x / l_x \).

Lower bound. Fix \( u \in A_0(R^{L_x,L_y}) \). Let \( \ell \in (0, L_x) \) and \( \lambda \in (0, 1) \), chosen below. Then there is an interval \( J \subseteq (0, L_y) \) of length \( \lambda L_y \) such that
\[
I(u, (0, \ell) \times J) \leq 2 \lambda I(u, R^{\ell,L_y}).
\]
By the slicing theorem there is \( \bar{x} \in (0, \ell) \) such that \( u_y(\bar{x}, \cdot) \in (-\theta, 1-\theta) \mathcal{L}^1 \)-almost everywhere and
\[
\varepsilon |u(\bar{x}, \cdot)_{yy}|(J) \leq \frac{1}{\ell} I(u, (0, \ell) \times J).
\]
If \( u_y(\bar{x}, \cdot) \) is not constant on \( J \) then \( |u(\bar{x}, \cdot)_{yy}|(J) \geq 1 \) and
\[
I(u, R^{\ell,L_y}) \geq \frac{1}{2 \lambda} \varepsilon \ell. \tag{44}
\]
Otherwise \( u(\bar{x}, \cdot) \) is affine on \( J \). If \( u_y(\bar{x}, \cdot) = -\theta \) on \( J \), then by Jensen’s inequality and the boundary condition \( u(0, \cdot) = 0 \),
\[
I(u, (0, \ell) \times J) \geq \frac{1}{\ell} \int_J u^2(\bar{x}, y) \, dy \geq \frac{1}{\ell} \min_{a \in \mathbb{R}} \int_J (a - \theta y)^2 \, dy
\]
\[
= \frac{2}{\ell} \int_0^{2\lambda L_y} \theta^2 y^2 \, dy = \frac{\theta^2 \lambda^2 L_y^3}{12 \ell},
\]
which gives
\[
I(u, R^{\ell,L_y}) \geq \frac{\theta^2 \lambda^2 L_y^3}{24 \ell}. \tag{45}
\]
If \( u_y(\bar{x}, \cdot) = 1 - \theta \) then we obtain a larger lower bound with \( \theta \) replaced by \( 1 - \theta \). Summarizing, for all \( \ell \in (0, L_x) \) and all \( \lambda \in (0, 1) \), by (44) and (45),
\[
I(u, R^{L_x,L_y}) \geq \min \left\{ \frac{\varepsilon \ell}{2 \lambda}, \frac{\theta^2 \lambda^2 L_y^3}{24 \ell} \right\}. \tag{46}
\]
If \( \varepsilon \leq \frac{\theta^2 L_y^3}{L_x} \) then we expect branching on the whole domain. This is reflected in the choice of the parameters \( \ell = L_x \) and \( \lambda = \theta^{-2/3} \varepsilon^{1/3} L_x^{2/3} L_y^{-1} \). If on the other hand \( \varepsilon > \frac{\theta^2 L_y^3}{L_x} \) then we choose \( \ell = \varepsilon^{-1/2} \theta L_y^{3/2} \) and \( \lambda = 1 \). The result follows from (46).

Remark 2 This argument also shows that the scaling of the bound in (9) is optimal for small \( x \). Fix any \( l_x \leq \varepsilon^{-1/2} \theta L_y^{3/2} \), and set \( \lambda = c \theta^{-2/3} \varepsilon^{1/3} l_x^{2/3} L_y^{-1} \). If \( c \) is chosen sufficiently
small, then the option $I \geq \varepsilon \lambda/(2\lambda) = (2c)^{-1}e^{1/3}\theta^{2/3}L_x^{1/3}L_y$ is incompatible with the upper bound on the energy. Therefore (44) does not hold, and there is a segment of length $\lambda L_y$ on which $u$ is affine, with derivative $1 - \theta$ or $-\theta$. This implies that $\|u(\tilde{x}, \cdot)\|_{L^\infty(J)} \geq \frac{1}{2}\lambda L_y = \frac{1}{2}e^{1/3}\theta^{1/3}L_x^{1/3}L_y$.

For $L_x > \varepsilon^{-1/2}\theta L_y^{3/2} = \varepsilon l_x$ the competitor for the upper bound constructed in the proof of Theorem 1 is affine on $(l_x, L_x) \times (0, L_y)$. Hence, the energy vanishes on $(l_x, L_x) \times (0, L_y)$. We show that this is also the case for a minimizer, up to a multiplicative factor in the definition of $l_x$. The proof combines the scaling law with a result from [27, Sect. 4] that we present in this simplified setting for completeness.

**Lemma 2** Let $\varepsilon, L_x, L_y > 0$, and $\theta \in (0, \frac{1}{2})$. Let $u \in \mathcal{A}_0(R^{L_x, L_y})$ be a minimizer of $I(\cdot, R^{L_x, L_y})$ on $\mathcal{A}_0(R^{L_x, L_y})$. If $L_x > c\varepsilon^{-1/2}\theta L_y^{3/2} = \varepsilon l_x$, then $u$ is affine on $(l_x, L_x) \times (0, L_y)$, where $c > 0$ is the constant introduced in Theorem 1.

**Proof** Assume first that there is $\bar{x} \in (0, L_x)$ such that $u(\bar{x}, \cdot)$ is affine, with $u(\bar{x}, \cdot) \in \{-\theta, 1 - \theta\}$. We then consider the competitor

$$v(x, y) := \begin{cases} u(x, y), & \text{if } x \leq \bar{x}, \\ u(\bar{x}, y), & \text{if } x > \bar{x}. \end{cases}$$

It is easy to see that $v \in \mathcal{A}_0(R^{L_x, L_y})$, and that

$$I(u, R^{L_x, L_y}) \geq I(v, R^{L_x, L_y}) + \int_{(\bar{x}, L_x) \times (0, L_y)} u^2 \, d\mathcal{L}^2.$$

Since $u$ is a minimizer, we deduce $u_x = 0$ for $x \geq \bar{x}$, and in particular $u = v$ (cf. [27, Sect. 4]).

Let $v_j \to 0$, $v_j > 0$. If there exists no $x \in (0, l_x + v_j)$ such that $u(x, \cdot)$ is affine then we obtain the contradiction

$$c\varepsilon^{1/2}\theta L_y^{3/2} + \varepsilon v_j = \varepsilon l_x + \varepsilon v_j \leq I(u, R^{L_x, L_y}) \leq c\varepsilon^{1/2}\theta L_y^{3/2},$$

where we have used the definition of $l_x$ and the upper bound from Theorem 1. Hence, there exists a sequence $x_j \in (0, l_x + v_j)$, $j \in \mathbb{N}$ such that $u$ is affine on $(x_j, L_x) \times (0, L_y)$ for all $j \in \mathbb{N}$, and hence on $(l_x, L_x) \times (0, L_y)$. Since $u$ is continuous, it is affine on the closure of this set. 

---

5 Local in $x$ Energy Scaling Law

In this section we provide a local bound similar to the scaling law in Theorem 1 on rectangles $(0, l_x) \times (0, L_y)$ for $l_x \in (0, L_x)$. In particular, we prove that any minimizer $u \in \mathcal{A}_0(R^{L_x, L_y})$ restricted to a subrectangle $R^{l_x, L_y}$ obeys the same energy scaling as a minimizer on $R^{L_x, L_y}$. The case $\theta = 1/2$ was first presented by Kohn and Müller in [41, Theorem 2.6]. The results and proofs of this section for $\theta \in (0, 1/2]$ are part of the thesis [25].

**Theorem 4** (Local energy bound) Let $\varepsilon, L_x, L_y > 0$, and $\theta \in (0, \frac{1}{2}]$. Suppose that $u \in \mathcal{A}_0(R^{L_x, L_y})$ is a minimizer of $I(\cdot, R^{L_x, L_y})$. Further, let $l_x \in (0, \min\{\varepsilon^{-1/2}\theta L_y^{3/2}, L_x\})$ and
assume \(|u(l_x, L_y) - u(l_x, 0)| \leq \theta L_y\). Then

\[
\frac{1}{C} \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y \leq I(u, R^{l_x, L_y}) \leq C \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y
\]  

(47)

holds for a universal constant \(C\).

**Remark 3** In analogy to Remark 1, the assumption \(|u(l_x, L_y) - u(l_x, 0)| \leq \theta L_y\) can be replaced with the weaker condition

\[-\theta L_y \leq u(l_x, L_y) - u(l_x, 0) \leq K \theta L_y\]

for some constant \(K > 0\). Note, that we do not need the upper bound \((1 - \theta)L_y\) as in Remark 1 since we only need to construct an admissible function which is equal to \(u(l_x, \cdot)\) on \([l_x] \times (0, L_y)\) and equal to zero on \([0] \times (0, L_y)\).

The lower bound of (47) immediately follows from the scaling law, see Theorem 1. The proof of the upper bound in (47) is instead based on a result about the equipartition of energy. Precisely, we show that the horizontal distribution of the terms \(u_x^2\) and \(\varepsilon |uy_y|\) of a minimizer is the same, in the sense that there is \(\tau \in \mathbb{R}\) with

\[
\varepsilon |u(x, \cdot)_{yy}|((0, L_y)) = \tau + \int_0^{L_y} u_x^2(x, y) \, dy \text{ for almost all } x \in (0, L_x).
\]

**Lemma 3 (Equipartition of energy)** Let \(\varepsilon, L_x, L_y > 0\), and \(\theta \in (0, \frac{1}{2})\). Let \(u \in A_{0}(R^{L_x, L_y})\) be a minimizer of \(I(\cdot, R^{L_x, L_y})\) on \(A_0(R^{L_x, L_y})\). Then the following statements are true:

1. There is \(\sigma \in L^1((0, L_x))\) such that

\[
\varepsilon |uy_y|(E \times (0, L_y)) = \int_E \sigma \, d\mathcal{L}^1
\]

for all open sets \(E \subseteq (0, L_x)\).

2. There exists \(\tau \in \mathbb{R}\) such that

\[
\sigma(x) = \tau + \int_0^{L_y} u_x^2(x, y) \, dy
\]

for almost every \(x \in (0, L_x)\).

3. One has

\[
|\tau| \leq c \varepsilon^{2/3} \theta^{2/3} L_x^{-2/3} L_y,
\]

where \(c > 0\) is the constant from Theorem 1.

**Proof** Assertions (i) and (ii) can be proven by considering inner variations \(v_\rho(x) := u(x + \rho \phi(x), y)\), as discussed in [41, Lemma 2.4]. In order to prove the bound on \(\tau\) we observe that

\[
\tau L_x = \varepsilon |uy_y|(R^{L_x, L_y}) - \int_{R^{L_x, L_y}} u_x^2 \, d\mathcal{L}^2
\]
which implies
\[ |\tau| L_x \leq I(u, R^{L_x,L_y}). \]

The assertion follows then from the upper bound in Theorem 1.

The construction of an energy efficient competitor on a subrectangle has already been done in the last subsection, see the proof of Proposition 1. This construction is a crucial ingredient in the following proof.

**Proof of Theorem 4.** Let \( u \in \mathcal{A}_0(R^{L_x,L_y}) \) be a minimizer, \( \tau \) as in the equipartition result (Lemma 3), \( l_x \in (0, L_x) \). We have
\[
I(u, R^{l_x,L_y}) = \varepsilon |u_{yy}|((0, l_x) \times (0, L_y)) + \int_{(0,l_x)\times(0,L_y)} u_x^2 \, d\mathcal{L}^2
\]
\[
= \tau l_x + 2 \int_{(0,l_x)\times(0,L_y)} u_x^2 \, d\mathcal{L}^2.
\]

By Hölder’s inequality,
\[
\frac{1}{l_x} \int_0^{L_y} u^2(l_x, y) \, dy \leq \int_{R^{l_x,L_y}} u_x^2(x, y) \, dx \, dy = \frac{1}{2} I(u, R^{l_x,L_y}) - \frac{1}{2} \tau l_x.
\]

By Proposition 1, applied with \( u^T \) and \( u^B \) equal to the linear interpolation on \((0, l_x)\), there is \( v \in \mathcal{A}_0(R^{l_x,L_y}) \) such that \( v = u \) for \( x = l_x \) and
\[
I(v, R^{l_x,L_y}) \leq \frac{1}{l_x} \int_0^{L_y} u^2(l_x, y) \, dy + \tilde{c}_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y.
\]

Let \( \tilde{u} \) be equal to \( v \) on \((0, l_x)\), and equal to \( u \) on \([l_x, L_x]\). Then minimality of \( u \) leads to
\[
I(u, R^{l_x,L_y}) \leq I(v, R^{l_x,L_y}) \leq \frac{1}{l_x} \int_0^{L_y} u^2(l_x, y) \, dy + \tilde{c}_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y
\]
\[
\leq \frac{1}{2} I(u, R^{l_x,L_y}) - \frac{1}{2} \tau l_x + \tilde{c}_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y
\]

which implies
\[
\frac{1}{2} I(u, R^{l_x,L_y}) \leq \frac{1}{2} |\tau| l_x + \tilde{c}_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y.
\]

Inserting the bound on \( \tau \) from Lemma 33 and using \( l_x \leq L_x \) concludes the proof of the upper bound. The lower bound follows from Theorem 1.

**6 Local in \( x \) and \( y \) Energy Scaling Law**

This section is devoted to the proof of the local energy bounds and pointwise bounds for minimizers as given in Theorem 3. We follow the general lines of [17] with several changes to address the additional difficulties for \( \theta \ll 1 \).
Let us briefly explain where the main difficulties arise compared to the equal volume fraction case. An important ingredient in [17] is an interpolation inequality for functions \( v \in W^{1,2}(0, h) \) with \( |v'| \leq \alpha \) for some \( \alpha > 0 \). It states that
\[
\int_0^h v^2(y) \, dy \geq \min \left\{ \frac{1}{3} \alpha \left( \sup_y |v(y)| \right)^3, \alpha^2 h^3 \right\}.
\] (48)

In our setting, we would essentially need a replacement for functions with \( v_y \in [-\theta, 1-\theta] \). Having periodic laminates in mind, we would aim for an estimate with \( \alpha \) replaced by \( \theta \).

Precisely, for the function
\[
w(y) := \begin{cases} 
-\theta y, & \text{if } y \in (0, (1-\theta)h), \\
(1-\theta)y - (1-\theta)h, & \text{if } y \in ((1-\theta)h, h)
\end{cases}
\]
we have for all \( \theta \in (0, 1/2] \),
\[
\sup |w(y)| = \theta(1-\theta)h \quad \text{and} \quad \int_0^h w^2(y) \, dy \geq \frac{1}{24} \theta^2 h^3,
\]
i.e.,
\[
\int_0^h w^2(y) \, dy \geq \frac{1}{24} \min \left\{ \frac{1}{3} \theta \left( \sup_y |w(y)| \right)^3, \theta^2 h^3 \right\}.
\]

However, such an estimate does not hold for general functions \( u \) with \( u_y \in [-\theta, 1-\theta] \): Consider for example,
\[
v(y) := \begin{cases} 
0, & \text{if } y \in (0, (1-\theta)h), \\
(1-\theta)(y - (1-\theta)h), & \text{if } y \in ((1-\theta)h, h).
\end{cases}
\]

Then
\[
\sup |v(y)| \geq \frac{1}{2} \theta h \quad \text{and} \quad \int_0^h v^2(y) \, dy \leq \frac{1}{3} \theta^3 h^3.
\]

We point out that the estimate (48) with \( \alpha = 1 \) is not sufficient to obtain the expected scaling in \( \theta \), for details see [43]. The estimate is replaced by Lemma 4, which however does not cover the entire set. Therefore a different treatment of the boundary region is needed.

**Lemma 4** (Pointwise estimate of an admissible function) *Let \( I \subseteq \mathbb{R} \) be an interval, \( f \in W^{1,1}(I) \) with \( f' \geq -\theta \) almost everywhere for some \( \theta > 0 \). Let \( \delta > 0 \) and \( y \in I \). Then, choosing the continuous representative,*
\[
f(y) \leq \delta \theta + \left( \frac{1}{\delta} \int_y^{y+\delta} |f'|^2(t) \, dt \right)^{1/2} \quad \text{if } [y, y+\delta) \subseteq I \quad \text{and} \quad 
\]
\[
-f(y) \leq \delta \theta + \left( \frac{1}{-\delta} \int_{y-\delta}^y |f'|^2(t) \, dt \right)^{1/2} \quad \text{if } (y-\delta, y] \subseteq I.
\]
We start with the first bound. If \( f(y) \leq \delta \theta \), we are done. If \( f(y) > \delta \theta \), we use
\[
\frac{1}{\delta} \int_y^{y+\delta} |f|^2(t) \, dt \geq (f(y) - \delta \theta)^2
\]
which implies
\[
f(y) \leq \delta \theta + \left( \frac{1}{\delta} \int_y^{y+\delta} |f|^2(t) \, dt \right)^{1/2}.
\]
The second estimate follows by applying this one to \( y \mapsto -f(y) \).

One important strategy in the proof of Theorem 3 will be to study the local behavior of the excess elastic energy of an admissible function \( u \in A_0(R^{L_x,L_y}) \), in the sense of Proposition 1. We denote the localization of this modified functional to rectangles \((0,l_x) \times (a,b) \subseteq R^{L_x,L_y}\) by
\[
\beta_u(l_x,a,b) := \int_0^{l_x} \int_a^b (u - u^l)^2 \, dx \, dy + \varepsilon |u_{yy}|((0,l_x) \times (a,b)),
\]
where the function \( u^l \) is the linearization of \( u \) on \( R^{l_x,L_y} \),
\[
u^l(x,y) := \frac{x}{l_x} u(l_x,y), \quad \text{for} \ (x,y) \in R^{l_x,L_y}.
\]

**Lemma 5 (Estimate for the excess elastic energy)** Let \( c_0, c_1, c_2, k_0 > 0 \), with \( c_1 \geq 1 \), \( 2^{1/3} k_0 \leq c_0 \) and
\[
\tilde{c}_0 + \frac{2(1 + 2c_1 c_2 + c_0^{1/2})c_0 + 4}{c_1} \leq k_0
\]
where \( \tilde{c}_0 \) is the constant from Proposition 1.

Let \( \varepsilon, l_x > 0, \ \theta \in (0, \frac{1}{2}] \), \( A, B \in \mathbb{R} \) with \( B \geq A + c_1 \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3} \), and let \( u \in A_0((0,l_x + \delta) \times (A - \delta, B + \delta)) \) be a local minimizer, for some \( \delta > 0 \). Assume
\[
|u|(l_x, y) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} l_x^{2/3} \quad \text{for all} \ y \in [A, B]
\]
and
\[
\max \left\{ \int_0^{l_x} (u - u^l)^2(x,A) \, dx, \int_0^{l_x} (u - u^l)^2(x,B) \, dx \right\} \leq c_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3}.
\]
Then
\[
\beta_u(l_x, a, b) < k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} (b - a) \text{ if } [a, b] \subseteq [A, B] \text{ and } b - a \geq c_1 \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3}.
\]
Proof Step 0. Preliminaries.

For brevity we denote by $h_{\min} := c_1 \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3}$ the minimal height of the rectangles considered.

We shall construct competitors using Proposition 1. To estimate the coefficients $\omega^B$ and $\omega^T$ in Proposition 1 we notice that the fundamental theorem of calculus, Hölder’s inequality, $|u'|(x, y) \leq |u|(l_x, y)$ and (51) imply that for any $y_\ast \in [A, B]$ we have

$$\omega(y_\ast) := \sup_{x \in [0, l_x]} |u|(x, y_\ast) + \sup_{y \in [A, B]} |u|(l_x, y) + \varepsilon^{1/3} \theta^{1/3} l_x^{2/3} \leq \sup_{x \in [0, l_x]} |u - u'|((x, y_\ast) + 2 \sup_{y \in [A, B]} |u|(l_x, y) + \varepsilon^{1/3} \theta^{1/3} l_x^{2/3}$$

$$\leq l_x^{1/2} \left( \int_0^{l_x} (u - u')^2(x, y_\ast) \, dx \right)^{1/2} + (1 + 2c_1c_2)\varepsilon^{1/3} \theta^{1/3} l_x^{2/3}. \tag{54}$$

In particular, for $y_\ast = A$ and $y_\ast = B$, respectively, using (52) we obtain

$$\max\{\omega(A), \omega(B)\} \leq (1 + 2c_1c_2 + c_0^{1/2})\varepsilon^{1/3} \theta^{1/3} l_x^{2/3}. \tag{55}$$

We shall use a comparison with functions constructed in Proposition 1 to obtain bounds on the energy of $u$ on subsets of its domain. Specifically, if $v \in \mathcal{A}_0((0, l_x) \times (a, b))$ for some $(a, b) \subseteq (A, B)$, the fact that $u$ is local minimizer implies

$$\beta_u(l_x, a, b) \leq \beta_u(l_x, a, b) + 2\varepsilon l_x \quad \text{whenever } u = v \text{ on } \partial((0, l_x) \times (a, b)). \tag{56}$$

To see this, it suffices to consider a competitor $w$ which coincides with $u$ outside $(0, l_x) \times (a, b)$, and with $v$ inside. The possible jump of $w_y$ on the horizontal boundaries gives a contribution not larger than $2\varepsilon l_x$.

Finally, we remark for later reference that the assumption $2^{1/3}k_0 \leq c_0$ and (50) imply

$$\tilde{c}_0 + \frac{2(1 + 2c_1c_2 + (2k_0)^{1/2})k_0 + 4}{c_1} \leq k_0 \tag{57}$$

as well as

$$1 + 2c_1c_2 + k_0^{1/2} \leq 1 + 2c_1c_2 + c_0^{1/2} \leq \frac{1}{2}c_1, \tag{58}$$

and, using that (57) implies $\frac{2(1 + 2c_1c_2 + (2k_0)^{1/2})k_0}{c_1} \leq k_0$,

$$1 + 2c_1c_2 + (2k_0)^{1/2} \leq \frac{1}{2}c_1. \tag{59}$$

Step 1. We prove (53) in the case that $a = A$ or $b = B$.

We assume $a = A$ and consider the function

$$f : [A + h_{\min}, B) \to \mathbb{R}, \quad f(y) := \beta_u(l_x, A, y) - k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3}(y - A).$$

If $f(y) < 0$ for all $y$ we are done. If this is not the case, we define

$$\bar{y} := \sup\{y \in [A + h_{\min}, B) : f(y) \geq 0\} \in [A + h_{\min}, B]. \tag{60}$$
Since \( y \mapsto \beta_u(l_x, A, y) \) is nondecreasing, necessarily \( f(\tilde{y}) \geq 0 \).

We first apply Proposition 1 on the rectangle \((0, l_x) \times (A, B)\), which is admissible since \((54), (55)\) and \((58)\) imply \(|u(x, A)|, |u(x, B)| \leq \frac{1}{2} \theta h_{\min} \leq \frac{1}{2} \theta l_x\), where we write for brevity \( l_x := B - A \). We obtain a function \( v \in \mathcal{A}_0((0, l_x) \times (A, B)) \) which coincides with \( u \) on the boundary and, using \((55)\) and \((52)\), obeys

\[
\beta_v(l_x, A, B) \leq \frac{2}{\theta} (1 + 2c_1c_2 + c_0^{1/2}) e^{1/3} \theta^{1/3} l_x^{2/3} \cdot c_0 e^{2/3} \theta^{2/3} l_x^{1/3} + \bar{c}_0 e^{2/3} \theta^{2/3} l_x^{1/3} l_y.
\]

With \( l_y \geq h_{\min}, e l_x \leq e l_x \frac{l_y}{h_{\min}} = c^{-1}_1 e^{2/3} \theta^{2/3} l_x^{1/3} l_y \) and \((50)\), we obtain

\[
\beta_v(l_x, A, B) \leq \left[ \frac{2(1 + 2c_1c_2 + c_0^{1/2})c_0 + 4}{c_1} + \bar{c}_0 \right] e^{2/3} \theta^{2/3} l_x^{1/3} l_y - 4e l_x.
\]

Using \((56)\) we get \( \beta_v(l_x, A, B) < k_0 e^{2/3} \theta^{2/3} l_x^{1/3} l_y \) and therefore \( f(B) < 0 \), which implies \( \tilde{y} < B \).

Assume for a moment that there is a sequence \( y_j \in (\tilde{y}, B), y_j \to \tilde{y}, \) such that

\[
\int_0^{l_x} (u - u')^2(x, y_j) \, dx \leq k_0 e^{2/3} \theta^{2/3} l_x^{1/3} \quad \text{for all } j \tag{61}
\]

(this condition as usual includes existence of the integral). By \((54)\), this implies

\[
\omega(y_j) \leq (1 + 2c_1c_2 + k_0^{1/2}) e^{1/3} \theta^{1/3} l_x^{2/3} \tag{62}
\]

As above, with \((58)\) this leads to \(|u(x, y_j)| \leq \frac{1}{2} \theta h_{\min} \) for all \( x \), hence \((28)\) holds. We use Proposition 1 on \((0, l_x) \times (A, y_j)\) with \((55)\) and \((52)\) at \( y = A, (61)\) and \((62)\) at \( y = y_j, \) and obtain, recalling \( \tilde{y} - A \geq h_{\min}, \) that there is a competitor \( v \) with

\[
\beta_v(l_x, A, y_j) \leq \frac{\omega(A)}{\theta} \int_0^{l_x} (u - u')^2(x, A) \, dx + \frac{\omega(y_j)}{\theta} \int_0^{l_x} (u - u')^2(x, y_j) \, dx + \bar{c}_0 e^{2/3} \theta^{2/3} l_x^{1/3} (y_j - A) \leq \left[ (1 + 2c_1c_2 + c_0^{1/2})c_0 + (1 + 2c_1c_2 + k_0^{1/2})k_0 \right] \frac{1}{c_1} + \bar{c}_0 \right] e^{2/3} \theta^{2/3} l_x^{1/3} (y_j - A) \leq k_0 e^{2/3} \theta^{2/3} l_x^{1/3} (y_j - A) - 4e l_x,
\]

where in the last step we used the average of \((50)\) and \((57)\). Recalling \((56)\) and monotonicity of \( \beta_u(l_x, A, \cdot) \), this implies

\[
\beta_u(l_x, A, \tilde{y}) \leq \beta_u(l_x, A, y_j) \leq k_0 e^{2/3} \theta^{2/3} l_x^{1/3} (y_j - A) - 2e l_x
\]

for all \( j \). Taking \( j \to \infty \) leads to \( f(\tilde{y}) < 0 \), against the definition of \( \tilde{y} \) (see \((60)\)). Therefore, no sequence as in \((61)\) exists. Hence, there is \( y_\ast > \tilde{y} \) such that

\[
\int_0^{l_x} (u - u')^2(x, y) \, dx > k_0 e^{2/3} \theta^{2/3} l_x^{1/3} \quad \text{for almost every } y \in (\tilde{y}, y_\ast),
\]

\( \bowtie \) Springer
and recalling the definition of $f$ we obtain

$$f(y_s) \geq f(\tilde{y}) + \int_{\tilde{y}}^{y_s} \left[ \int_0^{l_x} (u-u')^2(x, y) \, dx - k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} \right] \, dy > f(\tilde{y}) \geq 0,$$

against the definition of $\tilde{y}$ (see (60)). Therefore $f < 0$ everywhere and the proof of Step 1 for $a = 0$ is concluded. The case $b = B$ is identical, working on intervals $[y, B]$ instead of $[A, y]$.

**Step 2. We prove (53) for $a > A$ and $b < B$.**

The argument is similar to the one of Step 1. Fix $a, b \in (A, B)$ with $b - a \geq h_{\min}$ and let $h_{\max} := \min\{a + b - 2A, 2B - (a + b)\} \geq h_{\min}$. We define $g : [h_{\min}, h_{\max}] \rightarrow \mathbb{R}$,

$$h \mapsto g(h) := \beta_u(l_x, a + b - h, a + b + h) - k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} h.$$

If $g < 0$ everywhere we are done (set $h := b - a$). Otherwise, we set

$$\tilde{h} := \sup \{h \in [h_{\min}, h_{\max}] : g(h) \geq 0\} \in [h_{\min}, h_{\max}].$$

As above, since $h \mapsto \beta_u(l_x, a + b - h, a + b + h)$ is nondecreasing we obtain $g(\tilde{h}) \geq 0$. By Step 1 we obtain $g(h_{\max}) < 0$, so that necessarily $\tilde{h} < h_{\max}$. The proof proceeds then similar to the one of Step 1. Assume first that there is a sequence $h_j \in (\tilde{h}, h_{\max})$ such that

$$\int_0^{l_x} (u-u')^2(x, a + b - h_j) \, dx + \int_0^{l_x} (u-u')^2(x, a + b + h_j) \, dx \leq 2k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3}$$

for all $j$. By (54), this implies

$$\max \left\{ \omega \left( \frac{a + b - h_j}{2} \right), \omega \left( \frac{a + b + h_j}{2} \right) \right\} \leq (1 + 2c_1 c_2 + (2k_0)^{1/2}) \varepsilon^{1/3} \theta^{1/3} l_x^{2/3}.$$

With (59) and $|u(x, \frac{a + b \pm h_j}{2})| \leq \omega(\frac{a + b \pm h_j}{2})$ we obtain $|u(x, \frac{a + b \pm h_j}{2})| \leq \frac{1}{4} \theta h_{\min}$, hence also in this case (28) holds. Using Proposition 1 on $(0, l_x) \times ((a + b - h_j)/2, (a + b + h_j)/2)$ and $h_j \geq h_{\min}$ we obtain a competitor $v$ with

$$\beta_v \left( l_x, \frac{a + b - h_j}{2}, \frac{a + b + h_j}{2} \right) \leq \left[ 2(1 + 2c_1 c_2 + (2k_0)^{1/2}) k_0 \varepsilon^{1/3} l_x^{2/3} \theta^{2/3} h_{\min} + \tilde{c}_0 \right] \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} h_j \leq k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} h_j - 4\varepsilon l_x,$$

where in the second step we used (57). As above this implies, using monotonicity of $h \mapsto \beta_u(l_x, a + b - h, a + b + h)$ and then (56),

$$\beta_u \left( l_x, \frac{a + b - \tilde{h}}{2}, \frac{a + b + \tilde{h}}{2} \right) \leq \beta_u \left( l_x, \frac{a + b - h_j}{2}, \frac{a + b + h_j}{2} \right) \leq k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} h_j - 2\varepsilon l_x.$$
Taking $j \to \infty$ leads to $g(\tilde{h}) < 0$, against the definition of $\tilde{h}$. Therefore no such sequence exists, and there is $h_\ast \in (\tilde{h}, h_{\max})$ such that

$$
\int_0^{l_x} (u - u')^2(x, a + b - h) \, dx + \int_0^{l_x} (u - u')^2(x, a + b + h) \, dx > 2k_0\epsilon^{2/3}\theta^{2/3}l_x^{1/3}
$$

for a.e. $h \in (\tilde{h}, h_\ast)$, so that

$$
g(h_\ast) > g(\tilde{h}) \geq 0,
$$

against the definition of $\tilde{h}$. Therefore $g < 0$ everywhere and the proof is concluded. \hfill \Box

**Lemma 6** Under the same assumptions as in Lemma 5, if additionally

$$k_0 \leq c_1^2,
$$

and

$$
k_0^{1/3} \frac{c_1^{2/3}c_2\phi^{1/3}}{c_1^{2/3}c_2\phi^{1/3}} \leq \frac{1}{4}, \quad \text{and} \quad \frac{c_0^{1/2}}{c_1c_2\phi^{1/6}} \leq \frac{1}{4} \quad \text{for some } \phi \in (0, \frac{1}{8}],
$$

then the pointwise estimate

$$
|u|(x, y) \leq c_1c_2\epsilon^{1/3}\theta^{1/3}x \frac{\phi^{1/3}l_x^{1/3}}{\phi^{1/3}l_x^{1/3}}
$$

holds for all $x \in [\phi l_x, l_x]$ and $y \in [A, B]$.

We remark that (64) implies in particular $|u|(\phi l_x, y) \leq c_1c_2\epsilon^{1/3}\theta^{1/3}(\phi l_x)^{2/3}$, which is a key estimate for the inductive proof below.

**Proof** We first remark for later reference that the assumptions on the constants imply

$$
\phi^{1/3} + 2 \frac{k_0^{1/3}}{c_1^{2/3}c_2\phi^{1/3}} \leq 1
$$

and

$$
\phi^{1/3} + \frac{k_0^{1/3}}{c_1^{2/3}c_2\phi^{1/3}} + \frac{c_0^{1/2}}{c_1c_2\phi^{1/6}} \leq 1.
$$

We write as above $h_{\min} := c_1\epsilon^{1/3}\theta^{-2/3}l_x^{2/3}$. Fix some $x \in [\phi l_x, l_x]$, and let $\delta := (k_0x/c_1^2l_x)^{1/3}$, both fixed for the entire proof. Note that $\delta \in (0, 1]$ by (63) and $x \leq l_x$. Pick any $y \in [A, B]$. If $y < B - \delta h_{\min}$, then by Lemma 4 applied to the map $u(x, \cdot) : [A, B] \to \mathbb{R}$ we obtain

$$
u(x, y) \leq \theta \delta h_{\min} + \left( \frac{1}{\delta h_{\min}} \int_y^{y+\delta h_{\min}} |u|^2(x, t) \, dt \right)^{1/2}.
$$

Since $u'(x, t) = \frac{1}{l_x} \int_{l_x} |u|(l_x, t)$, the triangle inequality in $L^2$ gives

$$
u(x, y) \leq \theta \delta h_{\min} + \frac{x}{l_x} \max_{t \in [A, B]} |u|(l_x, t) + \left( \frac{1}{\delta h_{\min}} \int_y^{y+\delta h_{\min}} |u - u'|^2(x, t) \, dt \right)^{1/2}.
$$

\[\square\]
We choose an interval $J \subseteq [A, B]$ of length $h_{\text{min}}$ which contains $(y, y + \delta h_{\text{min}})$ (this is possible since $\delta \leq 1$ and $h_{\text{min}} \leq B - A$). By Lemma 5,

$$
\int_{0 \times J} |u_x - u_x^1|^2(s, t) \, ds \, dt \leq k_0 \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} h_{\text{min}} = c_1 k_0 \varepsilon l_x,
$$

so that with Hölder’s inequality we obtain

$$
\int |u - u^1|^2(x, t) \, dt \leq \int_{0 \times J} (u - u^1)^2 \, ds \, dt \leq c_1 k_0 \varepsilon l_x.
$$

Inserting this estimate and assumption (51) in (67) gives

$$
u(x, y) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} + \theta \delta h_{\text{min}} + \left( \frac{1}{\delta h_{\text{min}}} \int_{J} |u - u^1|^2(x, t) \, dt \right)^{1/2}
$$

$$
\leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} + \theta \delta h_{\text{min}} + \left( c_1 k_0 \varepsilon l_x \right)^{1/2}.
$$

The choice $\delta = (k_0 x / c_1^2 l_x)^{1/3}$ balances the second and the third term when replacing $\varepsilon$ using the definition of $h_{\text{min}}$. Using $x \geq \phi l_x$ in the last term,

$$
u(x, y) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} + 2 \theta \delta h_{\text{min}} + \left( \frac{k_0^{1/3} x^{1/3}}{c_1^{2/3} l_x^{1/3}} \right)^{1/2}
$$

$$
\leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} \phi^{1/3} \left( \frac{\phi^{1/3} + 2}{\frac{k_0^{1/3}}{c_1^{2/3} c_2 \phi^{1/3}}} \right).
$$

With (65), the estimate

$$
u(x, y) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} \phi^{1/3} \phi^{-1/3} \text{ for all } y \in [A, B - \delta h_{\text{min}}]
$$

(68)

follows. Assume now $y \in [B - \delta h_{\text{min}}, B]$. Lemma 4 applied to the map $u(x, \cdot)$ gives

$$
-u(x, y) \leq \theta \delta h_{\text{min}} + \left( \frac{1}{\delta h_{\text{min}}} \int_{y - \delta h_{\text{min}}}^y |u|^2(x, t) \, dt \right)^{1/2}.
$$

The argument then proceeds as above, and leads to

$$
-u(x, y) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} \phi^{1/3} \phi^{-1/3} \text{ for all } y \in [A + \delta h_{\text{min}}, B].
$$

(69)

It remains to deal with the two boundary regions. Assume that $y \in [B - \delta h_{\text{min}}, B]$, which implies $B - y \leq k_0^{1/3} c_1^{1/3} \varepsilon^{1/3} \theta^{-2/3} x^{1/3} l_x^{1/3}$. Assumption (52) and Hölder’s inequality imply

$$
|u - u^1|(x, B) \leq x^{1/2} \left( \int_{(0, x)} (u - u^1)^2(s, B) \, ds \right)^{1/2} \leq c_0^{1/2} \varepsilon^{1/3} \theta^{1/3} x^{1/2} l_x^{1/6},
$$

and using as above $u^1(x, y) = \frac{x}{l_x} u(l_x, y)$ and assumption (51) we obtain

$$
u(x, B) \leq u^1(x, B) + |u - u^1|(x, B) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x l_x^{-1/3} + c_0^{1/2} \varepsilon^{1/3} \theta^{1/3} x^{1/2} l_x^{1/6}.
$$
Since \( u_y \geq -\theta \) almost everywhere and \( \phi I_x \leq x \),

\[
    u(x, y) \leq u(x, B) + (B - y) \theta 
\]

\[
    \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x \phi^{-1/3} \left( \phi^{1/3} + \frac{\theta^{1/2}}{c_1 c_2 \phi^{1/6}} + \frac{k_0^{1/3}}{c_1^{2/3} c_2^{1/3}} \right). 
\]

Using (66) this leads to the desired bound.

Finally, for \( y \in [A, A + \delta h_{\text{min}}] \) we analogously obtain

\[
    -u(x, y) \leq -u(x, A) + y \theta 
\]

\[
    \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} x \phi^{-1/3} \left( \phi^{1/3} + \frac{\theta^{1/2}}{c_1 c_2 \phi^{1/6}} + \frac{k_0^{1/3}}{c_1^{2/3} c_2^{1/3}} \right). 
\]

Recalling (68) and (69), the proof is concluded. \qed

We are now ready to prove the main result of this section.

**Proof of Theorem 3.** The proof consists of four steps, where in the first three we assume \( l_x < L_x \). First, we prove two local bounds on \( u \) and \( \beta u \) via induction and the two previous Lemmas. Afterwards, we show that they imply the claim of Theorem 3. In the third step, we show how the constants can be chosen. In a last step, the case \( l_x = L_x \) is treated.

For the first two steps we assume that there are constants \( c_0, c_1, c_2, k_0 > 0, \phi \in (0, \frac{1}{8}] \) such that the assumptions of Lemma 5 and 6 hold and additionally

\[
    k_1 \leq c_0 \eta \quad (70) \tag{70}
\]

and

\[
    k_0 \leq c_0 \phi^{1/3}. \quad (71) \tag{71}
\]

For \( n \in \mathbb{N} \) we set \( l_{x,n} := \phi^n l_x, h_n := \varepsilon^{1/3} \theta^{-2/3} l_{x,n}^{2/3}, r_0 := 2\eta L_y, \) and for \( n \geq 1 \)

\[
    r_n := 2\eta L_y + c_1 \sum_{i=0}^{n-1} h_i \leq 2\eta L_y + c_1 \varepsilon^{1/3} \theta^{-2/3} l_{x,n}^{2/3} \frac{1}{1 - \phi^{2/3}} \leq \frac{8}{3} \eta L_y,
\]

where in the last step we used \( \phi \leq 1/8 \) and (6). We denote by \( u_{n}^{l_{x,n}} \) the linear interpolation on \((0, l_{x,n}), \) defined as usual by \( u_{n}^{l_{x,n}}(x, y) := \frac{x}{l_{x,n}} u(l_{x,n}, y), \) as in the definition of \( \beta_{n}(l_{x,n}, \cdot, \cdot). \)

**Step 1.** We prove by induction the following two statements:

\[
    \beta_{n}(l_{x,n}, a, b) \leq k_0 \varepsilon^{2/3} \phi^{2/3} l_{x,n}^{1/3} (b - a) \text{ for all } (a, b) \subseteq (r_n, L_y - r_n) \text{ with } b - a \geq c_1 h_n \quad (72)
\]

and

\[
    |u|(x, y) \leq c_1 c_2 \varepsilon^{1/3} \theta^{1/3} l_{x,n}^{1/3} \frac{x}{l_{x,n}^{1/3}} \text{ for all } x \in [l_{x,n+1}, l_{x,n}], y \in [r_n, L_y - r_n]. \quad (73)
\]
We start with $n = 0$, and refer to Fig. 5 for a sketch of the geometry. By assumption (8) and Fubini’s theorem there is $a^* \in (\eta L_y, 2\eta L_y)$ with

$$
\int_0^{l_x} (u - u_0^t)^2(x, a_n) \, dx \leq \int_0^{l_x} u_0^t(x, a_n) \, dx \leq \frac{1}{\eta L_y} \int_{\eta L_y}^{2\eta L_y} \int_0^{l_x} u_0^t(x, y) \, dx \, dy \\
\leq \frac{1}{\eta L_y} I(u, R_{L_y}^{l_x}) \leq \frac{k_1}{\eta} e^{2/3} \theta^{2/3} l_x^{1/3}
$$

and analogously $b^* \in ((1 - 2\eta)L_y, (1 - \eta)L_y)$.

We first apply Lemma 5 with $A := a^*$, $B := b^*$, and some $0 < \delta < \min\{\eta L_y, L_x - l_x\}$. Condition (51) follows from assumption (7), condition (52) from the choice of $a^*_n$ and $b^*_n$ and (70). We obtain that for any interval $(a, b) \subseteq (a_*, b_*)$ with $b - a \geq c_1 h_0$ one has

$$
\beta_u(l_x, a, b) \leq k_0 e^{2/3} \theta^{2/3} l_x^{1/3}(b - a),
$$

which proves (72) for $n = 0$. Next we use Lemma 6 with the same choices to obtain

$$
|u|(x, y) \leq c_1 c_2 e^{1/3} \theta^{1/3} \frac{x}{(\phi l_x)^{1/3}} \text{ for } x \in [\phi l_x, l_x], y \in [a_*, b_*],
$$

which proves (73) for $n = 0$ and concludes the initial step of the induction.

Assume (72) and (73) hold for some index $n \geq 0$. Using (72) with $a = r_n, b = r_n + c_1 h_n$ and Fubini’s theorem we see that there is $a_n \in (r_n, r_n + c_1 h_n) = (r_n, r_{n+1})$ so that

$$
\int_0^{l_{x,n}} (u - u_n^t)^2(x, a_n) \, dx \leq \frac{1}{c_1 h_n} \beta_u(l_{x,n}, r_n, r_n + c_1 h_n) \leq k_0 e^{2/3} \theta^{2/3} l_{x,n}^{1/3}
$$
and analogously $b_n \in (L_y - r_{n+1}, L_y - r_n)$. By the properties of the linearization,

$$
\int_0^{l_{x,n+1}} (u - u_{n+1}^l(x, a_n))^2 \, dx \leq \int_0^{l_{x,n+1}} (u - u_n^l(x, a_n))^2 \, dx \leq k_0 e^{2/3} \theta^{2/3} l_{x,n+1}^{1/3} (b - a) \tag{74}
$$

and analogously for $b_n$. We now apply Lemma 5 with $l_x = l_{x,n+1}$, $A := a_n$, $B := b_n$ and some $0 < \delta < r_n$. Condition (51) follows from (73) with $x = l_{x,n+1}$, condition (52) from (74) and (71). We obtain that for any interval $(a, b) \subseteq (a_n, b_n)$ with $b - a \geq c_1 h_{n+1}$ one has

$$
\beta_u(l_{x,n+1}, a, b) \leq k_0 e^{2/3} \theta^{2/3} l_{x,n+1}^{1/3} (b - a).
$$

Since $(r_{n+1}, L_y - r_{n+1}) \subseteq (a_n, b_n)$, this proves (72) for $n + 1$.

Next we use Lemma 6 with the same choices to obtain

$$
|u|(x, y) \leq c_1 c_2 e^{1/3} \theta^{1/3} \frac{x}{(\phi l_{x,n+1})^{1/3}} \text{ for } x \in [\phi l_{x,n+1}, l_{x,n+1}], y \in [a_n, b_n],
$$

which proves (73) for $n + 1$.

This concludes the proof of (72) and (73).

**Step 2. We show that (72) and (73) imply the assertion of the Theorem.** We start with (9). Let $(x, y) \in (0, l_x) \times [3\eta L_y, (1 - 3\eta)L_y]$. Since $x > 0$, there is $n \in \mathbb{N}$ such that $l_{x,n+1} < x \leq l_{x,n}$. The assertion with $d_1 := \frac{C \Omega}{\phi^{1/3}}$ follows from (73) using $x \leq l_{x,n} = l_{x,n+1}/\phi$ and recalling that $r_n \leq \frac{8}{3} \eta L_y \leq 3\eta L_y$, and correspondingly $L_y - r_n \leq \frac{8}{3} \eta L_y$.

Now we turn to (10). Let $l \in (0, l_x)$, $(a, b)$ as in the statement of the theorem. Choose $n$ such that $l_{x,n+1} < l \leq l_{x,n}$. Then, recalling the definition of $\beta_u$ and of the linear interpolation,

$$
I(u, (0, l) \times (a, b)) \leq I(u, (0, l_{x,n}) \times (a, b)) = \beta_u(l_{x,n}, a, b) + \frac{1}{l_{x,n}} \int_{(a,b)} |u|^2(l_{x,n}, y) \, dy.
$$

If we can choose $\phi \geq \frac{1}{8}$, then $b - a \geq 4c_1 e^{1/3} \theta^{-2/3} l^{2/3} \geq c_1 h_n$, so that estimating the two terms with (72) and (73) leads to

$$
I(u, (0, l) \times (a, b)) \leq k_0 e^{2/3} \theta^{2/3} l_{x,n}^{1/3} (b - a) + c_1^2 c_2^2 e^{2/3} \theta^{2/3} \frac{l_{x,n}^2}{l_{x,n+1}^{1/3}} (b - a)
$$

$$
\leq (k_0 \phi^{-1/3} + c_1^2 c_2^2 \phi^{-1}) e^{2/3} \theta^{2/3} l_{x,n}^{1/3} (b - a),
$$

which concludes the proof, with $d_2 := k_0 \phi^{-1/3} + c_1^2 c_2^2 \phi^{-1}$.

**Step 3. We choose the constants.**

We are given $k_1 > 0$, $\eta \in (0, \frac{1}{6})$, and $c_0 \in [1, \infty)$ from Proposition 1. We first set $\phi := \frac{1}{8}$ and, for some $c_0 > 0$ chosen below,

$$
k_0 := k_0(c_0) := c_0 \phi^{1/3}
$$
so that (71) and \( c_0 \geq 2^{1/3}k_0 \) are satisfied. With this definition, (50) reduces to

\[
\frac{\tilde{c}_0}{\phi^{1/3}c_0} + \frac{2(1 + c_0^{1/2})}{c_1\phi^{1/3}} + \frac{4c_2}{\phi^{1/3}} + \frac{4}{c_0c_1\phi^{1/3}} \leq 1. \tag{75}
\]

We set

\[
c_2 := \frac{\phi^{1/3}}{16}
\]

and assume that

\[
c_0 \geq 4\tilde{c}_0\phi^{-1/3}
\]

so that the first and third terms of the left hand side of (75) is at most 1/4 each. This also implies \( c_0 \geq 1 \), so that the last term is not larger than the second. Therefore to fulfill (50) it suffices to ensure that

\[
c_1 \geq 8(1 + c_0^{1/2})\phi^{-1/3}.
\]

In turn, the assumptions of Lemma 6 are true if we ensure

\[
c_1 \geq \max \left\{ 1, k_0^{1/2}, 4^{3/2} \frac{c_0^{1/2}}{c_2^{3/2}\phi^{1/3}}, 4 \frac{c_0^{1/2}}{c_2\phi^{1/6}} \right\}.
\]

We finally set

\[
c_0 := c_0(k_1) := \max\left\{ \frac{k_1}{\eta}, 4\tilde{c}_0\phi^{-1/3} \right\},
\]

so that also (70) is automatically enforced, and

\[
c_1^* := c_1^*(c_0) := \max \left\{ 8(1 + c_0^{1/2})\phi^{-1/3}, 4^{3/2} \frac{c_0^{1/2}}{c_2^{3/2}\phi^{1/3}}, 4 \frac{c_0^{1/2}}{c_2\phi^{1/6}} \right\}.
\]

Here, we used that \( 8(1 + c_0^{1/2})\phi^{-1/3} \geq \max\{1, k_0^{1/2}\} \), and we remark that \( c_0 \) (and hence \( k_0 \) and \( c_1^* \)) continuously depend on \( k_1 \) and \( \eta \).

**Step 4. The case** \( l_x = L_x \).

We want to use the continuity of \( u \) and the fact that we have already proven the statement for \( l \in (0, L_x) \). Our goal is to prove the estimates of Step 2, for all constants satisfying the conditions of Step 3.

Let \( k_1 > 0 \) and \( \eta \in (0, \frac{1}{6}) \) be given and let \( c_1^*, c_2, \tilde{c}_0, c_0 \) and \( \phi \in (0, 1) \) be chosen as in Step 3. Moreover, we assume that \( c_1 \geq c_1^* \) is such that (6) holds, that for all \( y \in [\eta L_y, (1 - \eta)L_y] \)

\[
|u(L_x, y)| \leq c_1 c_2 e^{1/3} \theta^{1/3} L_x^{2/3} \quad \text{and that} \quad I(u, R^{L_x,L_y}) \leq k_1 \varepsilon^{2/3} \theta^{2/3} L_x^{1/3} L_y. \tag{76}
\]

Note, that the maps \( k_1 \mapsto c_1^*(c_0(k_1)) \) and \( k_1 \mapsto k_0(c_0(k_1)) \) are continuous. We introduce the slightly changed constants \( k_{1,j} := k_1 + \frac{1}{j} \), \( k_{0,j} := k_0(c_0(k_{1,j})) \) and \( c_{1,j} :=
max\{c_1 + \frac{1}{j}, c_1^* (c_0(k_1,j))\} for \ j \in \mathbb{N}, which satisfy the condition from Step 3. Further, for large \ j a weaker version of (6) holds, namely,

\[
\frac{5}{3} c_{1,j} e^{1/3\theta^{-2/3} l_x^{2/3}} \leq \eta L_y
\]

so that the condition \( r_n \leq \frac{8}{3} \eta L_y \) at the beginning of Step 2 is replaced by \( r_n \leq (3 - \frac{1}{2}) \eta L_y \).

By Hölder-continuity of \( u \) (see Lemma 7 below), there exists \( \delta_j \in (0, L_x) \) such that

\[
|u(l,y)| \leq c_{1,j} c_2 e^{1/3\theta^{1/3} l_x^{2/3}} \quad \text{and} \quad I(u, (0, l) \times (a, b)) \leq k_{1,j} e^{2/3\theta^{2/3} l_x^{1/3} L_y} \tag{77}
\]

holds for all \( l \in (L_x - \delta_j, L_x) \) and \( y \in [\eta L_y, (1 - \eta) L_y] \). Let \( x \in (0, L_x) \). For \( j \) sufficiently large we have \( x \leq L_x - \delta_j \), and by Step 2 we conclude

\[
|u|(x,y) \leq d_{1,j} e^{1/3\theta^{1/3} x^{2/3}} \tag{78}
\]

where \( d_{1,j} := \frac{c_{1,j} c_2}{\phi^{1/3}} \), for all \( y \in [3\eta L_y, (1 - 3\eta) L_y] \). Taking the limit \( j \to \infty \) the same holds with \( d_1 \) instead of \( d_{1,j} \), and by continuity of \( u \) we obtain (9).

Let now \( l \in (0, L_x) \) and \( (a, b) \subseteq (3\eta L_y, (1 - 3\eta) L_y) \) with \( b - a \geq 4c_{1} e^{1/3\theta^{-2/3} l_x^{2/3}} \). For \( j \) large enough we can choose \( a_j, b_j \) with \( (a, b) \subseteq (a_j, b_j) \subseteq ((3 - \frac{1}{2})\eta L_y, (1 - (3 - \frac{1}{2})\eta) L_y) \), \( b_j - a_j \geq 4c_{1,j} e^{1/3\theta^{-2/3} l_x^{2/3}} \), and \( b_j \to b, a_j \to a \). By Step 2 we conclude, for any \( j \) sufficiently large,

\[
I(u, (0, l) \times (a, b)) \leq I(u, (0, l) \times (a_j, b_j)) \leq d_{2,j} e^{2/3\theta^{2/3} l_x^{1/3} (b_j - a_j)} \tag{79}
\]

where \( d_{2,j} := k_{0,j} \phi^{-1/3} + c_{1,j} c_2^2 \phi^{-1} \to d_2 \) as \( j \to \infty \). Taking \( j \to \infty \) in (79) we obtain (10) for all \( l < L_x \), and by continuity of \( l \mapsto I(u, (0, l) \times (a, b)) \) also for \( l = L_x \). \qed

Remark 4 We recall that the parameter \( \eta \) is not necessary in the setting of [17]. Indeed, the assumptions on the top and bottom boundary conditions considered there in particular allow to choose \( a_n = 0 \) and \( b_n = L_y \) for all \( n \in \mathbb{N} \) in Step 1 of the proof, using in Lemma 5 that \( u \) is a minimizer on \( R^{k_x L_y} \), subject to full Dirichlet boundary conditions, which renders the extension of the domain in terms of \( \delta \) unnecessary.

7 Asymptotic Self-Similarity of a Minimizer

We now turn to the proof of Theorem 2. For the proof it is useful to show that the minimizer \( u \) is uniformly continuous. Indeed, the anisotropic structure of \( \mathcal{A} \) implies Hölder-continuity with exponent 1/3, see [47, Lemma 3]. For completeness we recall the self-contained argument of this well-known fact.

Lemma 7 The space \( X(L_x, L_y) := \{ u \in W^{1,2}(R^{L_x L_y}) : u_y \in L^{\infty}(R^{L_x L_y}) \} \) is continuously embedded in \( C^{1/3}(R^{L_x L_y}) \).

Proof After a linear change of variables we can assume \( R^{L_x L_y} = R := (0, 1)^2 \). It suffices to show that there is \( c > 0 \) such that for any \( u \in X(1, 1) \) for almost every \( p_0, p_1 \in R \) we have

\[
|u(p_0) - u(p_1)| \leq c (\|u_x\|_{L^{\infty}(R)} + \|u_y\|_{L^{2}(R)}) |p_0 - p_1|^{1/3}. \tag{80}
\]
To see this, let \( M := \|u_\ell\|_{L^\infty(R)} + \|u_\ell\|_{L^2(R)} \). For \( L^2 \)-almost every \( p_0 = (x_0, y_0) \) the function \( u(x_0, \cdot) \) has an \( M \)-Lipschitz representative which has a Lebesgue point at \( y_0 \), and the same for \( p_1 = (x_1, y_1) \). For \( \ell \in (0, 1] \) chosen below, with \( \ell \geq |y_0 - y_1| \), let \( I_\ell \) be an interval of length \( \ell \) such that \( y_0, y_1 \in I_\ell \subseteq [0, 1] \). Choose \( y_2 \in I_\ell \) such that \( u(x_0, \cdot) \) and \( u(x_1, \cdot) \) have a Lebesgue point at \( y_2 \) and

\[
\int_0^1 u^2_\ell(x, y_2) \, dx \leq \frac{1}{\ell} \int_0^1 \int_0^1 u^2_\ell(x, y) \, dx \, dy \leq \frac{M^2}{\ell},
\]

so that by Hölder’s inequality

\[
|u(x_0, y_2) - u(x_1, y_2)| \leq |x_0 - x_1|^{1/2} M \ell^{-1/2}.
\]

We then estimate

\[
|u(p_0) - u(p_1)| \leq |u(p_0) - u(x_0, y_2)| + |u(x_0, y_2) - u(x_1, y_2)| + |u(x_1, y_2) - u(p_1)|
\]

\[
\leq M \ell + \frac{M}{\ell^{1/2}} |p_0 - p_1|^{1/2} + M \ell.
\]

Choosing \( \ell := \min\{1, |p_0 - p_1|^{1/3}\} \) concludes the proof of (80).

The key ingredient in the proof of the strong convergence of the blow-ups is a construction that permits to continuously modify a function in \( \mathcal{A}_0 \) by shifting the interfaces. The aim is to show that a small change in the boundary values corresponds to a small change in energy. We present here the construction from [17, Lemma 3.6], adding some detail and extending it to the case of general \( \theta \). This requires, as in other parts of this paper, a different treatment of the region where \( u_y = 1 - \theta \) and the one where \( u_y = -\theta \), using somewhat different estimates.

**Lemma 8** For any \( \hat{c} > 0 \) there are \( c_y, c_\ell > 0 \) such that the following holds. Let \( \varepsilon > 0 \), \( \theta \in (0, 1/2], l_x, l_y > 0 \), \( u \in \mathcal{A}_0(R^{l_x,l_y}) \), \( v^T \in W^{1,2}((0, l_x)) \) with \( v^T(0) = 0 \), and set \( h_0 := \varepsilon^{1/3} \theta^{-2/3} l_x^{2/3} \). Assume \( l_y \geq c_y h_0 \),

\[
\max\{|v^T|(x), |u|(x, 0), |u|(x, l_y)| \leq \hat{c} \theta h_0 \text{ for all } x \in (0, l_x),
\]

\[
\int_{R^{l_x,l_y}} u^2_x \, dx \, dy \leq \hat{c} \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} l_y
\]

and

\[
\int_0^{l_x} (u^2_x(x, 0) + u^2_x(x, l_y)) \, dx < \hat{c} \varepsilon^{2/3} \theta^{2/3} l_x^{1/3}.
\]

Then there is \( v \in \mathcal{A}_0(R^{l_x,l_y}) \) such that \( v(\cdot, 0) = u(\cdot, 0) \), \( v(\cdot, l_y) = v^T \), \( v_y(\cdot, l_y) = u_y(\cdot, l_y) \), \( v_y(\cdot, 0) = u_y(\cdot, 0) \) (in the sense of traces), and

\[
I(v, R^{l_x,l_y}) \leq I(u, R^{l_x,l_y}) + c_y \eta^{1/2} + \eta^{3/2} \varepsilon l_x l_y h_0
\]

where

\[
\eta := \frac{1}{\varepsilon^{2/3} \theta^{2/3} l_x^{1/3}} \int_0^{l_x} (v^T(x) - u(x, l_y))^2 \, dx.
\]

If \( v^T(l_x) = u(l_x, l_y) \), then additionally \( v(l_x, \cdot) = u(l_x, \cdot) \).
Fig. 6 Sketch of the construction in Lemma 8 at fixed \( x \). The black curve displays the old function \( u(x, \cdot) \), the red one the new one, \( v(x, \cdot) \). They coincide at \( y = 0 \), but differ at \( y = l_y \). The intervals which compose \( \{ v_y = 1 - \theta \} \) are stretched by a factor \( \alpha(x) \) with respect to those that compose \( \{ u_y = 1 - \theta \} \), for the other set the factor is \( \beta(x) \).

Proof We write for brevity \( R := R^{l_y, l_y} \), \( u^T(x) := u(x, l_y) \) and \( u^B(x) := u(x, 0) \). We choose \( c_y := \max\{3, 6\hat{c}\} \) and notice that \( u^T(0) = v^T(0) = 0 \), Hölder’s inequality and (85) imply

\[
|u^T - v^T|(x) \leq \eta^{1/2} \epsilon^{1/3} \theta^{1/3} l_y^{2/3} = \eta^{1/2} \theta h_0. \tag{86}
\]

Step 1. Shifting the interfaces in the correct way.

One key ingredient in the proof is that (for a.e. \( x \)) the value \( u^T(x) - u^B(x) \) is in one-to-one correspondence with the measure of the set of \( y \in (0, l_y) \) with \( u_y(x, y) = 1 - \theta \). We define

\[
M(x) := \mathcal{L}^1(\{ y \in (0, l_y) : u_y(x, y) = 1 - \theta \}) = \int_0^{l_y} (u_y(x, s) + \theta) \, ds = u^T(x) + \theta l_y - u^B(x).
\]

This equation is equivalent to \( u^T(x) = u^B(x) + (1 - \theta) M(x) - \theta (l_y - M(x)) \).

The main idea is to move the interfaces in order to modify the volume of the minority phase by a factor \( \alpha(x) \), bringing it to the value required by the new boundary data,

\[
\hat{M}(x) := v^T(x) + \theta l_y - u^B(x).
\]

Correspondingly the volume of the majority phase is modified by a factor \( \beta(x) \), so that the total length of \( (0, l_y) \) is unchanged, see Fig. 6. We define

\[
\alpha(x) := \frac{\hat{M}(x)}{M(x)}, \quad \beta(x) := \frac{l_y - \hat{M}(x)}{l_y - M(x)}.
\]

From (81), \( c_y \geq 6\hat{c} \) and \( l_y \geq c_y h_0 \) we obtain \( |u^B(x) - u^T(x)| \leq \frac{1}{3} \theta l_y \) and therefore

\[
\frac{2}{3} \theta l_y \leq M(x) \leq \frac{4}{3} \theta l_y \tag{87}
\]

and \( l_y - M(x) \geq (1 - \frac{4}{3} \theta) l_y \geq \frac{1}{3} l_y \geq \frac{2}{3} \theta l_y \). Similarly, with \( \hat{M} - M = v^T - u^T \),

\[
|\alpha - 1| \leq \frac{|u^T - v^T|}{\frac{3}{2} \theta l_y} \leq \frac{1}{2} \quad \text{and} \quad |\beta - 1| \leq \frac{|u^T - v^T|}{\frac{1}{3} l_y} \leq \frac{1}{2},
\]
so that in particular $\alpha, \beta \in [\frac{1}{2}, 2]$ a.e. Using (86) and $c_y \geq 3$ we also obtain

$$|\alpha - 1| \leq \frac{1}{2} \eta^{1/2} \quad \text{and} \quad |\beta - 1| \leq \eta^{1/2} \theta. \quad (88)$$

Further, $\alpha \in W^{1,2}((0, l_x))$ with

$$\alpha_x = \frac{\hat{M}_x M - \hat{M} M_x}{M^2} = \frac{\hat{M}_x - M_x}{M} + \frac{(M - \hat{M}) M_x}{M^2}$$

so that

$$|\alpha_x| \leq 2 \frac{|v^T - u^T_x|}{\theta l_y} + 4 \frac{|v^T - u^T| |u^T_x - u^B_x|}{\theta^2 l_y^2}. \quad (87)$$

Analogously, using $l_y - M \geq \frac{1}{3} l_y$, we obtain

$$|\beta_x| \leq 3 \frac{|v^T - u^T_x|}{l_y} + 9 \frac{|v^T - u^T| |u^T_x - u^B_x|}{l_y^2}. \quad (88)$$

Using (85), (86), $h_0 \leq l_y$ and (83), we obtain

$$\int_0^{l_x} \max\{\theta^2 \alpha^2_x, \beta^2_x\} \, dx \leq \tilde{c} \eta^{2/3} \theta^{2/3} l_x^{1/3} \quad (89)$$

for some $\tilde{c} > 0$ depending only on $\hat{c}$.

**Step 2. Construction of a candidate $v$.**

Fix any $x \in (0, l_x)$ such that $u(x, \cdot)$ is Lipschitz continuous with $u_y \in \{-\theta, 1 - \theta\}$ a.e. We set

$$m(x, y) := \mathcal{L}^1\{s \in (0, y) : u_y(x, s) = 1 - \theta\} = u(x, y) + \theta y - u^B(x).$$

Then $m(x, \cdot)$ is Lipschitz continuous with $m_y \in [0, 1]$ a.e., by Lemma 7, $m \in C^{1/3}(R)$. The rescaling maps $[0, y]$ into $[0, F(x, y)]$, where

$$F(x, y) := \alpha(x)m(x, y) + \beta(x)(y - m(x, y)). \quad (90)$$

From $m(x, l_y) = M(x)$ we obtain $F(x, l_y) = l_y$, and from $m_y \in [0, 1]$ we obtain $F_y = a m_y + \beta(1 - m_y) \in [\alpha, \beta] \subset [\frac{1}{2}, 2]$ a.e., so that $F(x, \cdot)$ is a bilipschitz map from $[0, l_y]$ onto itself. Further, $u \in W^{1,2}(R)$, (83) and (89) imply $F \in W^{1,2}(R)$.

We aim to define $v(x, y)$ such that $v(x, 0) = u^B(x)$, $v_y(x, \cdot) = 1 - \theta$ on a subset of measure $am$ of $[0, F(x, y)]$, and $v_y(x, \cdot) = -\theta$ on the rest, which has measure $\beta(y - m)$. This leads to the definition

$$v(x, F(x, y)) := (1 - \theta) \alpha(x)m(x, y) - \theta \beta(x)(y - m(x, y)) + u^B(x)$$

$$= \alpha(x)m(x, y) - \theta F(x, y) + u^B(x), \quad (91)$$

which also satisfies $v(x, l_y) = v^T(x)$. Additionally, since $v^T(0) = 0 = u^T(0)$ we have $\beta(0) = \alpha(0) = 1$ and hence $v(0, F(0, y)) = u(0, y) = 0$. By the same argument, if $v^T(l_x) = u^T(l_x)$ we have $F(l_x, y) = y$ and $v(l_x, y) = u(l_x, y)$.
Step 3. Verify the desired properties of $v$.

Since $F(x, \cdot)$ is bilipschitz, for almost every $y$ we have

$$v_y(x, F(x, y))F_y(x, y) = \alpha(x)m_y(x, y) - \theta F_y(x, y).$$

Recalling that (90) implies $F_y = \alpha m_y + \beta (1 - m_y)$ and that $m_y \in [0, 1]$ almost everywhere, we obtain $v_y \in [-\theta, 1 - \theta]$. By the same reasoning we see that the number of jump points of $v_y$ is the same as for $u_y$, so that $|v_{yy}|(R) = |u_{yy}|(R)$. Moreover, $v_y(\cdot, 0) = u_y(\cdot, 0)$ and $v_y(\cdot, l_y) = u_y(\cdot, l_y)$ in the sense of traces.

It remains to estimate the derivative in $x$. We first show that $v$ is weakly differentiable. Let $\Phi(x, y) := (x, F(x, y))$. Since $F \in C^0(R)$ and $F(x, \cdot)$ is bilipschitz from $[0, l_x]$ onto itself, $\Phi$ is a continuous bijective map from $R$ onto itself. The inverse $\Psi$ obeys $\Psi_1(x, y) = x$.

Further, $\Phi \in W^{1,2}(R; \mathbb{R}^2)$, with $\det D\Phi = F_y \in [\frac{1}{2}, 2]$ a.e., which implies $\Psi \in W^{1,2}_\text{loc}(R; \mathbb{R}^2)$ (see [26, Th. 3.1]); by $\Psi_y = (0, 1/F_y) \in L^\infty$ and Lemma 7 $\Psi$ is continuous. In order to prove that $v \in W^{1,2}(R)$, we rewrite (91) as

$$v(x, y) = \alpha(x)m(\Psi(x, y)) - \theta y + u^B(x).$$

We first show that $m \circ \Psi \in W^{1,1}(R)$. This follows from general results on mappings of finite distortion, see [30, Th. 1.1], [35, Th. 1.3] or [31, Chap. 5]; for simplicity we give a self-contained argument. Let $m^\varepsilon \in C^1_\text{loc}(\mathbb{R}^2)$ be such that $m^\varepsilon \to m$ uniformly and strongly in $W^{1,2}(R)$. Obviously $m^\varepsilon \circ \Psi \to m \circ \Psi$ uniformly. Further, $D(m^\varepsilon \circ \Psi) = Dm^\varepsilon \circ \Psi D\Psi$; with $\det D\Psi \in [\frac{1}{2}, 2]$ and the change-of-variables formula we obtain $Dm^\varepsilon \circ \Psi \to Dm \circ \Psi$ in $L^2(R; \mathbb{R}^2)$. As $D\Psi \in L^2(R; \mathbb{R}^{2\times 2})$, by Hölder’s inequality $D(m^\varepsilon \circ \Psi) \to Dm \circ \Psi D\Psi$ in $L^1(R; \mathbb{R}^2)$. By continuity of the distributional derivative we obtain $D(m \circ \Psi) = Dm \circ \Psi D\Psi \in L^1(R; \mathbb{R}^2)$.

Using $m_y \in L^\infty(R)$, $m_x \in L^2(R)$ and $D\Psi_1 = e_1 \in L^\infty(R; \mathbb{R}^2)$, we obtain $Dm \circ \Psi D\Psi \in L^2(R; \mathbb{R}^2)$, so that $m \circ \Psi \in W^{1,2}(R; \mathbb{R}^2)$. With $\alpha \in W^{1,2}((0, l_x)) \cap L^\infty((0, l_x))$, recalling $m \in L^\infty(R)$ and using the product rule we obtain $v \in W^{1,2}(R)$. This argument also proves that the chain rule applies in (91).

Differentiating (91) in $x$, and dropping for brevity the arguments, we obtain

$$v_x + v_y F_x = (\alpha m)_x - \theta F_x + u^B_x$$

which, by $u(x, y) = u^B(x) + m(x, y) - \theta y$, is the same as

$$v_x + (v_y + \theta) F_x = (\alpha m)_x - m_x + u_x$$

or, using that by (90) $F_x = (\alpha m)_x + (\beta (y - m))_x$, the same as

$$v_x + (v_y + \theta - 1) F_x = (\beta (m - y))_x - m_x + u_x.$$

Recalling that $v_y \in [-\theta, 1 - \theta]$ a.e. we then obtain $v_x \in \{ (\alpha m)_x - m_x + u_x, (\beta (m - y))_x - m_x + u_x \}$ almost everywhere. Using $0 \leq m \leq M$ and (87), this leads to

$$|v_x - u_x| \leq |\alpha| m + (\alpha - 1)m_x | \leq 2|\alpha| |\theta l_y| + |\alpha - 1||u_x - u^B_x|$$

in the first case, and, using $0 \leq y - m \leq l_y$, to

$$|v_x - u_x| = |\beta_x (m - y) + (\beta - 1)m_x | \leq |\beta x| |l_y| + |\beta - 1||u_x - u^B_x|$$
in the other case, so that with (88) we obtain that almost everywhere
\[ |v_x(x, F(x, y)) - u_x(x, y)| \leq \max \{2\theta |\alpha_x|(x), |\beta_x|(x)|F_y| + \eta^{1/2} |u_x(x, y) - u_B^R(x)| \}. \] (92)

By a change of variables,
\[ \int_R v^2_x(x, y) \, dx \, dy = \int_0^{l_x} \int_0^{l_y} v^2_x(x, F(x, y)) F_y(x, y) \, dy \, dx. \]

By triangle inequality
\[ v^2_x \leq (1 + \eta^{1/2}) u^2_x + (1 + \eta^{-1/2}) (v_x - u_x)^2, \]

which, using \( F_y \leq 1 + \eta^{1/2} \) (which follows from (88) and \( F_y \in \{\alpha, \beta\} \)) and (92), leads to the pointwise bound
\[ v^2_x F_y \leq (1 + \eta^{1/2}) u^2_x(x, y) + \]
\[ + 24(\eta^{-1/2} + \eta^{1/2}) \left( \max \{\theta^2 \alpha_x^2, \beta_x^2\} l_x^{1/3} |F_y| + \eta |u_x|^2 + \eta |u_B^R|^2 \right). \]

By integration, together with (82), (83) and (89), we obtain
\[ \int_R v^2_x \, dx \, dy \leq (1 + \eta^{1/2}) \int_R u^2_x \, dx \, dy + \]
\[ + 24(\eta^{-1/2} + \eta^{1/2}) (\bar{c} \eta \epsilon^{2/3} \theta^{2/3} l_x^{1/3} |F_y| + 2\eta \hat{c} \epsilon^{2/3} \theta^{2/3} l_x^{1/3} |F_y|) \]

which, using again (82), leads to
\[ \int_R v^2_x \, dx \, dy \leq \int_R u^2_x \, dx \, dy + (24\bar{c} + 50\hat{c})(\eta^{1/2} + \eta^{3/2}) \epsilon^{2/3} \theta^{2/3} l_x^{1/3} |F_y|. \]

Recalling that \( \bar{c} \) only depends on \( \hat{c} \), this concludes the proof. \( \square \)

We then turn to asymptotic self-similarity. The first step is to find a subrectangle of \( R^{L_x, L_y} \) containing \((0, y_0)\) on which the bounds of Theorem 3 hold. This is done using Theorem 4. In a second step we prove asymptotic self-similarity using the argument from [17, Sect. 3], which was extended to arbitrary \( \theta \in (0, 1/2] \) in [43, Sect. 9].

**Proof of Theorem 2** Let \( u \in A_0(R^{L_x, L_y}) \) be a minimizer of \( I(\cdot, R^{L_x, L_y}) \) and \( y_0 \in (0, L_y) \).

**Step 1.** We choose a subrectangle on which the local bounds of Theorem 3 hold.

We replace \( u \) by its continuous representative and let \( C_H \) be its Hölder constant (Lemma 7). For any \( l_x \in (0, L_x] \) and \( y \in (0, L_y] \) we have
\[ |u(l_x, y) - u(0, y)| \leq C_H l_x^{1/3}. \]

In the following, we assume
\[ 0 < l_x \leq l_x^* := \min \left\{ L_x, \left( \frac{\theta L_y}{2C_H} \right)^3, \left( \frac{\theta L_y^{3/2}}{\varepsilon^{1/2}} \right)^3 \right\}, \]
which implies that $|u|(l_x, y) \leq \frac{1}{2} \theta L_y$, so that we can apply Theorem 4. We obtain

$$I(u, R_{l_x, L_y}) \leq C \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y. \tag{93}$$

Further, let $\eta \in (0, \frac{1}{4})$ satisfy $\gamma_0 \in (3\eta L_y, (1 - 3\eta)L_y)$. Let $c_1^*, c_2 > 0$ be the constants from Theorem 3 corresponding to our choice of $\eta$ and $k_1 := C$. We next show that we can choose $c_1 \geq c_1^*$ and $l_x \in (0, l_x^*)$ such that (6) and (7) are satisfied; Condition (8) follows immediately from (93). For some $c_1$ chosen below, we set

$$l_x := \left( \frac{\eta L_y \theta^{2/3}}{2c_1 \varepsilon^{1/3}} \right)^{3/2}, \tag{94}$$

so that (6) is automatically satisfied. For $c_1$ sufficiently large, in the sense that

$$c_1 \geq \max\{c_1^*, \frac{\eta L_y \theta^{2/3}}{2 \varepsilon^{1/3} (l_x^*)^{2/3}}\}, \tag{95}$$

both $c_1$ and $l_x$ are admissible.

It remains to choose $c_1$ so that (95) and (7) hold. By (93) and Hölder’s inequality,

$$\int_0^{L_y} |u|^2(l_x, y) \, dy \leq C \varepsilon^{2/3} \theta^{2/3} l_x^{4/3} L_y.$$

We fix some $\alpha \in (0, 1]$ chosen below, and apply Lemma 4 to $u(l_x, \cdot)$ with $\delta := \alpha \eta L_y$. We obtain

$$|u|(l_x, y) \leq \alpha \eta L_y \theta + \left( \frac{C}{\alpha \eta} \varepsilon^{2/3} \theta^{2/3} l_x^{4/3} \right)^{1/2} \quad \text{for all } y \in (\eta L_y, (1 - \eta)L_y).$$

By the definition of $l_x$ we have $c_1 c_2 \varepsilon^{1/3} \theta^{1/3} l_x^{2/3} = \frac{1}{2} c_2 \eta L_y \theta$. Therefore

$$\frac{|u|(l_x, y)}{c_1 c_2 \varepsilon^{1/3} \theta^{1/3} l_x^{2/3}} \leq \frac{2\alpha}{c_2} + \frac{C^{1/2}}{c_1 c_2 \alpha^{3/2} \eta^{1/2}}.$$

We set $\alpha := \min\{1, \frac{1}{4} c_2\}$, so that the first term is not larger than 1/2, and then choose $c_1$ sufficiently large that (95) is satisfied and the second term is also not larger than 1/2. Therefore, Theorem 3 can be applied to the rectangle $(0, l_x) \times (0, L_y)$.

**Step 2. We prove the asymptotic self similarity.**

We follow the lines of [17, Sect. 3], where one has to substitute $R_{L_x, L_y}$ with

$$R_* := (0, l_x) \times (3\eta L_y, (1 - 3\eta)L_y),$$

in which (9) and (10) hold. We define $u^j(x, y) = v_j^{-2/3} u(v_j x, y_0 + v_j^{2/3} y)$ as in (3) and observe that $u^j \in \mathcal{A}_0(R^j_*)$, where

$$R^j_* := (0, v_j^{-1} l_x) \times (v_j^{-2/3} (3\eta L_y - y_0), v_j^{-2/3} ((1 - 3\eta)L_y - y_0)).$$

Clearly $\bigcup_{j \in \mathbb{N}} R^j_* = (0, \infty) \times \mathbb{R}$. Further, the estimates of Theorem 3 remain true for $u^j$ on $R^j_*$. Indeed,

$$|u^j|(x, y) = v_j^{-2/3} |u|(v_j x, y_0 + v_j^{2/3} y) \leq v_j^{-2/3} d_4 \varepsilon^{1/3} \theta^{1/3} (v_j x)^{2/3}.$$
for \((x, y) \in R^j_x\) and, by a change of variables,

\[
I(u^j, (0, l) \times (a, b)) = \int_0^l \int_a^b (u^j_x)^2 \, dy dx + \varepsilon |u^j_{yy}|((0, l) \times (a, b))
\]

\[
= v_j^{-1} \left[ \int_0^{v_j} \int_{y_0 + v_j/3} y_0 + v_j/3 (u^j_x)^2 \, dy dx + \varepsilon |u^j_{yy}|((0, v_j l) \times (y_0 + v_j/3, y_0 + v_j/3)) \right]
\]

\[
\leq v_j^{-1} d_2 \varepsilon^{2/3} \theta^{2/3} (v_j l)^{1/3} (v_j (b - a)) = d_2 \varepsilon^{2/3} \theta^{2/3} l^{1/3} (b - a)
\]

for \(b - a \geq 4c_1 \varepsilon^{1/3} \theta^{-2/3} l^{2/3}\), provided \(j \in \mathbb{N}\) is sufficiently large that \((0, l) \times (a, b) \subseteq R^j_x\). In particular, in each such rectangle there is a subsequence which converges weakly in \(W^{1,2}\). Taking a diagonal subsequence, still denoted by \((u^j)_j\), we see that there exists \(u^\infty \in W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R})\) with

(i) \(u^j \rightarrow u^\infty\) weakly in \(W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R})\),

(ii) \(u^j_{yy} \rightarrow u^\infty_{yy}\) weakly as Radon measures on \((0, \infty) \times \mathbb{R}\),

(iii) \(u^j \rightarrow u^\infty\) uniformly in every compact set \(K \subseteq [0, \infty) \times \mathbb{R}\)

as \(j \rightarrow \infty\). The last convergence follows from Lemma 7 and the compact embedding \(C^{1,1}(K) \hookrightarrow C^0(K)\) for any compact subset \(K \subseteq [0, \infty) \times \mathbb{R}\). These conditions in particular imply that \(u^\infty(0, \cdot) = 0\) and \(u^\infty_{yy}\) is a Radon measure. From (i) and (ii) we obtain, using compensated compactness [48, 54], that \(u^j \rightarrow u^\infty\) strongly in \(L^2_{\text{loc}}((0, \infty) \times \mathbb{R})\) (see [41, Lemma 2.2] for a self-contained argument). Therefore \(u^\infty \in A_0((0, l) \times (a, b))\) for all \(l > 0\) and \(a < b\). By lower semicontinuity of the various norms involved, (96) and (97) hold also for \(u^\infty\).

We next prove the existence of a subsequence \((u^j)_j\) such that \((u^j)_j\) converges strongly in \(L^2_{\text{loc}}((0, \infty) \times \mathbb{R})\) towards \(u^\infty\), which means that it converges strongly in \(L^2(R_{l, h})\) for all \(l, h > 0\), where we write \(R_{l, h} := (0, l) \times (-h, h)\). We shall show below that there is \(\tilde{c} > 0\) such that for any \(l_x > 0\) and \(H \geq \tilde{c} \varepsilon^{1/3} \theta^{-2/3} l^{2/3}_{x}\),

\[
\limsup_{j \rightarrow \infty} \|u^j_x - u^\infty_x\|^2_{L^2(R_{l_x, H/2})} \leq \frac{c \varepsilon^{2/3} l^{5/6}_{x}}{\theta^{1/3} H^{1/2}} \limsup_{j \rightarrow \infty} \|u^j_x - u^\infty_x\|^2_{L^2(R_{2l_x, H})},
\]

(98)

Fix \(l_x > 0, h > 0\). For \(H\) sufficiently large, using twice (98) we obtain

\[
\limsup_{j \rightarrow \infty} \|u^j_x - u^\infty_x\|^2_{L^2(R_{l_x, h})} \leq \limsup_{j \rightarrow \infty} \|u^j_x - u^\infty_x\|^2_{L^2(R_{2l_x, h})}
\]

\[
\leq \frac{C}{H^{1/2}} \limsup_{j \rightarrow \infty} \|u^j_x - u^\infty_x\|^2_{L^2(R_{2l_x, h})}
\]

\[
\leq \frac{C}{H^{1/2}} \left(2 H \right)^{1/4} \limsup_{j \rightarrow \infty} \|u^j_x - u^\infty_x\|^2_{L^2(R_{4l_x, 2H})},
\]

where \(C := c \varepsilon^{2/3} l^{5/6}_{x} \theta^{-1/3}\) is independent of \(H\). By (97), we get

\[
\|u^j_x - u^\infty_x\|^2_{L^2(R_{4l_x, 2H})} \leq 8d_2 \varepsilon^{2/3} \theta^{2/3} l^{1/3}_{x} H
\]
for $H$ large. Taking $H \to \infty$ shows that $\limsup_{j \to \infty} \|u_j^I - u_\infty^I\|_{L^2(R_{l_i,h})}^2 = 0$ which, since $l_i$ and $h$ were arbitrary, concludes the proof of strong convergence.

Step 3. Proof of (98).

The idea is to relate the $L^2$ norm of the difference to the difference in energy via

$$|u_\infty^I - u_j^I|^2 + |u_\infty^I|^2 - |u_j^I|^2 = 2u_\infty^I(u_\infty^I - u_j^I),$$

where the integral of the last term converges to zero by weak convergence. We fix $H \geq \text{e}^{1/3} \theta^{-2/3} l_i^{2/3}$, recall the notation $R_{l_i,h} = (0,l) \times (-h,h)$ and define $f_j : (0,H) \to \mathbb{R}$ by

$$f_j(h) := 2 \int_{R_{l_i,h}} u_\infty^I(u_\infty^I - u_j^I) \, dx \, dy + \epsilon \max\{|u_\infty^I|(R_{l_i,h}) - |u_j^I|(R_{l_i,h}),0\} \geq \int_{R_{l_i,h}} |u_\infty^I - u_j^I|^2 \, dx \, dy + I(u_\infty^I, R_{l_i,h}) - I(u_j^I, R_{l_i,h}).$$

The first integral converges to zero for any $h \in (0,H)$. By lower semi-continuity, $|u_\infty^I|(R_{l_i,h}) \leq \liminf |u_j^I|(R_{l_i,h})$ for all $h \in (0,H)$. Therefore $f_j(h) \to 0$ pointwise. By Egorov’s theorem there is $A_f \subset (0,H)$ with

$$\mathcal{L}^1(A_f) \leq \frac{1}{12} H \quad \text{and} \quad \lim_{j \to \infty} \sup\{|f_j|(y) : y \in (0,H) \setminus A_f\} = 0. \quad (100)$$

To estimate the differences in energy we will construct a competitor $u_j^I$ that coincides with $u_\infty^I$ on some inner rectangle and is larger in energy than $u_j^I$ on a larger rectangle. We set

$$s_j := \|u_j^I - u_\infty^I\|_{L^\infty(R_{2l_i,h})}. \quad (101)$$

By (iii) we have $s_j \to 0$; for large $j$ we can assume $s_j < 1$. Let $\ell_j := s_j l_i \in (0,l_i)$. We define $\bar{u}_j : R_{2l_i,h} \to \mathbb{R}$ by

$$\bar{u}_j(x,y) := \begin{cases} \min\{u_\infty^I(x,y) + s_j \frac{x-l_i}{\ell_j}, \max\{u_\infty^I(x,y) - s_j \frac{x-l_i}{\ell_j}, u_j^I(x,y)\}\}, & \text{if } x \geq l_i, \\ u_\infty^I(x,y) & \text{otherwise.} \end{cases}$$

We verify that $\bar{u}_j = u_j^I$ for $x \geq l_i + \ell_j$, $\bar{u}_j \in \{u_j^I, u_\infty^I\} \subseteq \{-\theta, 1-\theta\}$, $|\bar{u}_j| \leq \max\{|u_j^I|, |u_\infty^I|\}$, and $\bar{u}_j \in \{u_\infty^I + s_j \ell_j, u_\infty^I - s_j \ell_j, u_\infty^I, u_j^I\}$ almost everywhere, which implies

$$\|\bar{u}_j^I\|^2 \leq 2(u_\infty^I)^2 + \frac{2s_j^2}{\ell_j^2} + (u_j^I)^2. \quad (102)$$

To estimate $|\bar{u}_j^{yy}|$ we use the following fact. If $f, g : (a, b) \to \mathbb{R}$ are continuous and piece-wise affine, with $f', g' \in (-\theta, 1-\theta)$, then $h := \min\{f, g\}$ obeys $\#Jh' \leq 1 + \#Jf' + \#Jg'$, where $Jf'$ is the set of discontinuity points of $f'$ and $\#Jf'$ its cardinality, which equals the total variation of the measure $|f"|$ over $(a,b)$. We first prove this if $\{f = g\}$ is a finite set. By definition of $h$, $Jh' \subseteq Jf' \cup Jg' \cup \{f = g\}$ (see Fig. 7). If $\#\{f = g\} \leq 1$, the claim follows. Otherwise, let $x < y$ be two consecutive points in $\{f = g\}$. Assume that $f < g$ in $(x,y)$. Then necessarily $(x,y) \cap Jg' \neq \emptyset$, and this point does not contribute to $Jh'$. Therefore $\#\{f = g\}$ is at most one plus the number of points in $Jf'$ and $Jg'$ which do not generate
a point in $Jh'$. This proves $\#Jh' \leq 1 + \#Jf' + \#Jg'$ if $\{f = g\}$ is finite. Otherwise we consider $h_\delta := \min\{f, g + \delta\}$, send $\delta \to 0$, and use lower semicontinuity of $\#Jh_\delta$.

Therefore $|\tilde{u}^j_y|(l_x, l_x + \ell_j) \times (a, b)) \leq |u^j_y|(l_x, l_x + \ell_j) \times (a, b)) + \varepsilon|l_x|$. Further, $\tilde{u}^j \in A_0(R_{2\ell_x, H})$, and for any $(a, b) \subseteq (-H, H)$ we obtain

$$I(\tilde{u}^j, (l_x, l_x + \ell_j) \times (a, b)) \leq I(u^j, (l_x, l_x + \ell_j) \times (a, b)) + \omega_j, \quad (103)$$

where, recalling (102), the choice of $\ell_j$ and $s_j \to 0$,

$$\omega_j := 2I(u^\infty, (l_x, l_x + \ell_j) \times (-H, H)) + \varepsilon\ell_j + 4\ell_j H \frac{s_j^2}{\ell_j^2} \to 0. \quad (104)$$

Next we intend to use Lemma 8 with $\tilde{c} := \max\{32d_2, 2d_1\}$. Let $c_y$ be the corresponding constant and $h := \max\{c_y, 8c_1\}h_0$. We fix $\tilde{c}$ so that $H \geq 2h$ and then select $h_j \in (\frac{1}{2}H, H)$ which satisfies several properties. First, for almost all $h_j$ we have $|u^j_y|(0, 2l_x) \times \{h_j\}) = |u^\infty_y|(0, 2l_x) \times \{h_j\}) = 0$, the same for $-h_j$, and for at least three quarters of the $h_j$

$$r^j(h_j) := \int_0^{2l_x} |u^j_y - u^\infty_y|(x, h_j) \ dx + \int_0^{2l_x} |u^j_y - u^\infty_y|(x, -h_j) \ dx \leq \frac{8}{H} \|u^j_y - u^\infty_y\|_{L^1((0, 2l_x) \times (-H, H))}. \quad (105)$$

Second, using (97) on $(0, 2l_x) \times (-H, H)$, for at least three quarters of the $h_j$ we have

$$\int_{(0, 2l_x)} [(u^j)^2(x, h_j - h) + (u^j)^2(x, h_j) + (u^j)^2(x, -h_j + h) + (u^j)^2(x, -h_j)] \ dx \leq 16d_2\varepsilon^2/3)l_x^2/3. \quad (106)$$

Third, for three quarters of the $h_j$

$$\int_{(0, 2l_x)} [(u^j - u^\infty)^2(x, h_j) + (u^j - u^\infty)^2(x, -h_j)] \ dx \leq \frac{8}{H} \int_{(0, 2l_x) \times (-H, H)} (u^j - u^\infty)^2 \ dx \ dy. \quad (107)$$

Finally, since $L^1(A_f) \leq \frac{1}{12}H$, we can choose a value of $h_j \in (\frac{1}{2}H, H) \setminus A_f$ which obeys all those properties.

We check that we can apply Lemma 8 on $(0, l_x + \ell_j) \times (h_j - h, h_j)$ with $u = \tilde{u}^j$, $v^T = u^j(\cdot, h_j)$. Indeed, from the definition of $\tilde{u}^j$ we have $|\tilde{u}^j| \leq \max\{|u^j|, |u^\infty|\}$, so that (96) and
2d₁ ≤ ˆc imply (81). In turn, (102) and (97) with (a, b) = (h_j − h, h_j) show that
\[
\int_{(0,l_x+\ell_j) \times (h_j-h, h_j)} (u_i^\partial)^2 \, dx \, dy \leq 3d_2 \epsilon^{2/3} \theta^{2/3} (2l_x)^{1/3} h + 2^{s_j} l_x
\]
so that, recalling 8d₂ ≤ ˆc, for sufficiently large j (82) holds. Finally, (106) and 32d₂ ≤ ˆc imply (83). From |u_j^\partial - u_j|^n ≤ |u_j^\partial - u_j^\infty| + s_j / \ell_j ∈ (0,l_x+\ell_j) × R and (107) we obtain
\[
\eta_j := \frac{1}{\epsilon^{2/3} \theta^{2/3} (l_x + \ell_j)^{1/3}} \int_{(0,l_x+\ell_j)} (u_j^\partial - u_j^\infty)^2(x, h_j) + (u_j^\partial - u_j^\infty)^2(x, -h_j) \, dx
\]
\[
\leq \frac{2}{\epsilon^{2/3} \theta^{2/3} l_x^{1/3}} \left[ \int_{(0,2l_x)} (u_j^\partial - u_j^\infty)^2(x, h_j) + (u_j^\partial - u_j^\infty)^2(x, -h_j) \, dx + 2s_j^2 \ell_j \right]
\]
\[
\leq \frac{16}{\epsilon^{2/3} \theta^{2/3} l_x^{1/3}} H \int_{(0,2l_x) \times (-H,H)} (u_j^\partial - u_j^\infty)^2 \, dx \, dy + \frac{4s_j}{\epsilon^{2/3} \theta^{2/3} l_x^{1/3}}.
\]
By (97), η_j ≤ 42d₂ for large j. Lemma 8 gives a function w_i^\partial with
\[
I(w_i^\partial, (0, l_x + \ell_j) \times (h_j-h, h_j)) \leq I(u_i^\partial, (0, l_x + \ell_j) \times (h_j-h, h_j)) + C_1 \eta_j^{1/2} \epsilon l_x
\]
where C_1 depends on c_1, d_2, c_y and c_1. We apply the same procedure on the other side using again u = u_i^\partial, v^B = u_i^\partial(−h_j) and obtain w_i^\partial with (0, l_x + \ell_j) \times (−h_j, −h_j + h). We set
\[
w_i^\partial := \begin{cases}
w_i^\partial \quad \text{in } (0, l_x + \ell_j) \times (h_j-h, h_j), \\
\hat{u}_i^\partial \quad \text{in } (0, l_x + \ell_j) \times (−h_j + h, h_j-h), \\
w_i^\partial \quad \text{in } (0, l_x + \ell_j) \times (−h_j, −h_j + h), \\
w_i^\partial \quad \text{in the rest of } R_{2l_x,H},
\end{cases}
\]
see Fig. 8. By the boundary values given by Lemma 8, w_i^\partial \in A_0(R_{2l_x,H}). Since u_i^\partial is a local minimizer, I(u_i^\partial, R_{2l_x,H}) ≤ I(w_i^\partial, R_{2l_x,H}). Using u_i^\partial = w_i^\partial outside (0, l_x + \ell_j) \times (−h_j, h_j), recalling (105) and w_i^\partial(−h_j, ±h_j) = u_i^\partial(−h_j, ±h_j), we have
\[
I(u_i^\partial, (0, l_x + \ell_j) \times (−h_j, h_j)) \leq I(u_i^\partial, (0, l_x + \ell_j) \times (−h_j, h_j)) + \epsilon r_j(h_j)
\]
\[
\leq I(\hat{u}_i^\partial, (0, l_x + \ell_j) \times (−h_j, h_j)) + 2C_1 \eta_j^{1/2} \epsilon l_x + \epsilon r_j(h_j)
\]
\[
\leq I(u_i^\infty, (0, l_x) \times (−h_j, h_j)) + I(u_i^\partial, (l_x + \ell_j) \times (−h_j, h_j)) + \omega_j + 2C_1 \eta_j^{1/2} \epsilon l_x + \epsilon r_j(h_j)
\]
where in the last step we used (103). Therefore
\[
I(u_i^\partial, R_{l_x,h_j}) \leq I(u_i^\infty, R_{l_x,h_j}) + \omega_j + 2C_1 \eta_j^{1/2} \epsilon l_x + \epsilon r_j(h_j).
\]
Using h_j \geq \frac{1}{2} H and (99), this implies
\[
\|u_i^\partial - u_i^\infty\|_{L^2(R_{l_x,H/2})}^2 \leq \|u_i^\partial - u_i^\infty\|_{L^2(R_{l_x,h_j})}^2 \\
\leq f_j(h_j) + \omega_j + 2C_1 \eta_j^{1/2} \epsilon l_x + \epsilon r_j(h_j).
\]
Equation (100) implies $f_j(h_j) \to 0$, and (104) implies $\omega_j \to 0$. By (105) and strong convergence of $u^j$, we have $r_j(h_j) \to 0$. In turn, (107) gives

$$\varepsilon l_x \limsup_{j \to \infty} \eta_j^{1/2} \leq 3 \frac{\varepsilon^{2/3} l_x^{5/6}}{\theta^{1/3} H^{1/2}} \limsup_{j \to \infty} \|u^j - u^\infty_x\|_{L^2(\mathbb{R}^2lx,H)}.$$ 

Combining the last two inequalities concludes the proof of (98) and of strong convergence.

**Step 4. Proof that $u^\infty$ is a local minimizer.**

Assume that $\tilde{u}^\infty \in \mathcal{A}_0((0, \infty) \times \mathbb{R})$ is such that $\{u^\infty \neq \tilde{u}^\infty\} \subset K$ for some compact set $K \subset [0, \infty) \times \mathbb{R}$. Pick $l_x$ and $H$ such that $K \subset (0, l_x) \times (-H/2, H/2)$, with $H$ sufficiently large such that the computation of Step 3 can be performed. We remark that the function $w^j$ coincides with $u^\infty$ in $(0, l_x) \times (-h_j, h_j)$. Therefore $\tilde{w}_j := w_j + (\tilde{u}^\infty - u^\infty) \in \mathcal{A}_0((0, \infty) \times \mathbb{R})$, and it coincides with $u^j$ outside $(0, l_x + \ell_j) \times (-h_j, h_j)$. By the same estimate that lead to (110),

$$I(u^j, R_{l_x,h_j}) \leq I(\tilde{u}^\infty, R_{l_x,h_j}) + \omega_j + 2C_f l_x^{1/2} \varepsilon l_x + \varepsilon r_j(h_j).$$

By (99),

$$I(u^\infty, R_{l_x,h_j}) \leq f_j(h_j) + I(u^j, R_{l_x,h_j}).$$
Recalling \( \{u^\infty \neq \hat{u}^\infty\} \subseteq (0,l_x) \times (-H/2, H/2) \) and \( h_j > H/2 \), this leads to
\[
I(u^\infty, R_{l_x,H/2}) \leq I(\hat{u}^\infty, R_{l_x,H/2}) + f_j(h_j) + \omega_j + 2CI\eta_j^{1/2}\varepsilon l_x + \varepsilon r_j(h_j)
\]
and, taking the limit \( j \to \infty \) and using strong convergence to obtain \( \eta_j \to 0 \),
\[
I(u^\infty, R_{l_x,H/2}) \leq I(\hat{u}^\infty, R_{l_x,H/2}).
\]
Therefore \( u^\infty \) is a local minimizer. \( \square \)

We next show convergence of the energy on most rectangles. This argument is similar to the one of [17, 43], where however the choice of \( a \) and \( b \) is imprecise.

Lemma 9 Under the assumptions of Theorem 2, the subsequence additionally obeys
\[
I(u^\infty, R_{l_x,a,b}) = \lim_{j \to \infty} I(u_j, R_{l_x,a,b})
\]
for almost all \( a, b \in \mathbb{R} \) with \( a < b \).

Proof The following is based on the estimates and observations of Step 3 in the proof of Theorem 2.

Fix \( l_x > 0 \). We introduce the set of admissible boundary lines
\[
P := \{ h \in \mathbb{R} : |u^\infty_{yy}|((0, l_x) \times \{h\}) = 0 \} = \bigcup_{n \in \mathbb{N}} \{ h \in (-n, n) : |u^\infty_{yy}|((0, l_x) \times \{h\}) = 0 \}.
\]

We have \( |u^\infty_{yy}|((0, l_x) \times (-n, n)) < \infty \) for any \( n \in \mathbb{N} \) due to the weak lower semi-continuity of the total variation and (97). Thus, there can only be countably many \( h \in (-n, n) \) with \( |u^\infty_{yy}|((0, l_x) \times \{h\}) > 0 \). Hence, \( \mathbb{R} \setminus P \) is a countable set.

Let \( R_{l_x,a,b} := (0,l_x) \times (a,b) \subseteq (0, \infty) \times \mathbb{R} \) with \( a, b \in P \) and \( H \geq 2 \max\{|a|, |b|\} \). Observe, that the weak lower semi-continuity of the total variation and the strong convergence proven in Theorem 2 yield
\[
I(u^\infty, R_{l_x,a,b}) \leq \lim \inf_{j \to \infty} I(u^j, R_{l_x,a,b}).
\]
(111)
Hence, we only need to prove
\[
\lim \sup_{j \to \infty} I(u^j, R_{l_x,a,b}) \leq I(u^\infty, R_{l_x,a,b}).
\]
(112)

Similar to (100) we aim at an application of Egorov’s Theorem. Therefore, we introduce the functions \( g_j : (\frac{1}{2}H, H) \to \mathbb{R} \) via the formula
\[
g_j(h) := \int_{R_{l_x,h}\setminus R_{l_x,a,b}} (u_{\infty}^j)^2 \, dx \, dy - \int_{R_{l_x,h}\setminus R_{l_x,a,b}} (u^j)^2 \, dx \, dy + \varepsilon \left( |u_{\infty}^j| R_{l_x,h} \setminus R_{l_x,a,b} - |u^j| R_{l_x,h} \setminus R_{l_x,a,b} \right)
\]
for \( h \in (\frac{1}{2}H, H) \) and \( j \in \mathbb{N} \). The weak lower-semicontinuity of the total variation and the choice \( a, b \in \mathcal{P} \) imply

\[
\limsup_{j \to \infty} \left( |u^\infty_y|(R_{lx,h} \setminus R_{lx,a,b}^l) - |u_y^j|(R_{lx,h} \setminus R_{lx,a,b}^l) \right) \\
\leq |u^\infty_y|(R_{lx,h} \setminus R_{lx,a,b}^l) - \liminf_{j \to \infty} |u_y^j|(R_{lx,h} \setminus R_{lx,a,b}^l) \\
= |u^\infty_y|((R_{lx,h} \setminus R_{lx,a,b}^l) \circ \theta) - \liminf_{j \to \infty} |u_y^j|(R_{lx,h} \setminus R_{lx,a,b}^l) \leq 0,
\]

which implies \( \limsup_{j \to \infty} g_j(h) \leq 0 \) for almost every \( h \in (\frac{1}{2}H, H) \). Hence, the sequence \( (\max\{g_j, 0\})_j \) converges pointwise almost everywhere towards zero. By Egorov’s Theorem there exists a measurable set \( A_g \subseteq (\frac{1}{2}H, H) \) with

\[
\mathcal{L}^1(A_g) \leq \frac{1}{48} H \quad \text{and} \quad \limsup_{j \to \infty} \left\{ \max\{g_j, 0\}(h) : h \in (\frac{1}{2}H, H) \setminus A_g \right\} = 0. \quad (113)
\]

Using \( A_f \) from (100) we choose \( \tilde{h}_j \in (\frac{1}{2}H, H) \setminus (A_g \cup A_f) \) such that all the properties (105), (106) and (107) are satisfied. Observe

\[
I(u^j, R_{lx,a,b}^l) - I(u^\infty, R_{lx,a,b}^l) = I(u^j, R_{lx,\tilde{h}_j}^l) - I(u^\infty, R_{lx,\tilde{h}_j}^l) + g_j(\tilde{h}_j) \\
\leq \omega_j + 2C_1 \eta_j^{1/2} \varepsilon l_x + \varepsilon r_j(h_j) + \sup \left\{ \max\{g_j, 0\}(h) : h \in (\frac{1}{2}H, H) \setminus A_g \right\}
\]

where we have used the estimate (110). Together with (111) and (113) this yields

\[
I(u^\infty, R_{lx,a,b}^l) \leq \liminf_{j \to \infty} I(u^j, R_{lx,a,b}^l) \leq \limsup_{j \to \infty} I(u^j, R_{lx,a,b}^l) \\
\leq \limsup_{j \to \infty} \left( I(u^j, R_{lx,a,b}^l) - I(u^\infty, R_{lx,a,b}^l) \right) + I(u^\infty, R_{lx,a,b}^l) \leq 0 + I(u^\infty, R_{lx,a,b}^l),
\]

which implies the desired convergence. \( \square \)

### 8 Remarks on the Periodic Problem

In this section, we briefly discuss how the results of the preceding sections can be transferred to the periodic case, i.e. to the set of admissible functions which are periodic in \( y \)-direction,

\[
A_p(R_{lx, L_y}) := \{ u \in A_0(R_{lx, L_y}) : u(\cdot, 0) = u(\cdot, L_y) \}.
\]

On \( A_p(R_{lx, L_y}) \) we consider both, the functional \( I(\cdot, R_{lx, L_y}) \) and a variant which takes into account interfaces of the \( L_y \)-periodic extension of \( u \) on \( (0, L_x) \times \{L_y\} \). Precisely, denoting by \( Eu : (0, L_x) \times \mathbb{R} \to \mathbb{R} \) the \( L_y \)-periodic extension of \( u \), and writing \( u^+_y(\cdot, 0) \) and \( u^-_y(\cdot, L_y) \) for the upper and lower traces, respectively, we set

\[
I_p(u, R_{lx, L_y}) := \int_{R_{lx, L_y}} u_x^2 \, dx \, dy + \varepsilon |Eu_y|((0, L_x) \times (0, L_y)) \\
= \int_{R_{lx, L_y}} u_x^2 \, dx \, dy + \varepsilon |u_y|(R_{lx, L_y}) + \varepsilon \int_0^{L_x} |u^+_y(x, 0) - u^-_y(x, L_y)| \, dx.
\]
Remark 5 Let \( u \in A_p(R^{L_x,L_y}) \subseteq A_0(R^{L_x,L_y}) \). Then the following assertions hold:

(i) For almost every \( x \in (0, L_x) \), we have

\[
0 = u(x, L_y) - u(x, 0) = \int_0^{L_y} u_y(x, y) \, dy
\]

\[
= -\theta \left| \{ y : u_y(x, y) = -\theta \} \right| + (1 - \theta) \left| \{ y : u_y(x, y) = 1 - \theta \} \right|
\]

which implies that the volume fraction of the minority variant \( \{ u_y(x, \cdot) = 1 - \theta \} \) on the slice \( \{ x \} \times (0, L_y) \) is equal to \( \theta \). Recalling \( 0 < \theta < 1/2 < 1 \), we obtain

\[
|u_{yy}(x, \cdot)|(0, L_y) = |E u_{yy}(x, \cdot)|(0, L_y) \geq 1,
\]

and

\[
|E u_{yy}(x, \cdot)|(0, L_y) + 1 \geq |E u_{yy}(x, \cdot)|(0, L_y) \geq 2.
\]

In particular,

\[
I(u, R^{L_x,L_y}) \leq I_p(u, R^{L_x,L_y}) \leq I(u, R^{L_x,L_y}) + \varepsilon L_x.
\]

(ii) For all \( a \in \mathbb{R} \) we have

\[
I_p(u, R^{L_x,L_y}) = I_p(E u, (0, L_x) \times (a, a + L_y)).
\]

(iii) If \( u \) is a minimizer of \( I(\cdot, R^{L_x,L_y}) \) then its restriction is a local minimizer in the sense of (11) on any subrectangle \( R^{L_x,a,b} = (0, L_x) \times (a, b) \subseteq R^{L_x,L_y} \).

(iv) If \( u \) is a minimizer of \( I_p(\cdot, R^{L_x,L_y}) \) then \( u \) is also a local minimizer of \( I(\cdot, R^{L_x,L_y}) \).

Also, for any subrectangle \( R^{L_x,a,b} \subseteq (0, \infty) \times \mathbb{R} \) with \( b - a \leq L_y \)

\[
I_p(E u, R^{L_x,a,b}) \leq I_p(v, R^{L_x,a,b}) \text{ for all } v \in A_0(R^{L_x,a,b}) \text{ such that there exists } \delta > 0 \text{ with } \{ v \neq u \} \subset (0, l_x - \delta) \times (a + \delta, b - \delta).
\]

Theorem 5 (Global scaling laws) There is \( c_p > 0 \) such that for all \( \varepsilon, L_x, L_y > 0 \) and \( \theta \in (0, 1/2] \) one has

\[
\frac{1}{c_p} \min \{ e^{2/3} \theta^{2/3} L_x^{1/3} L_y, e^{1/2} \theta L_y^{3/2} + \varepsilon L_x \} \leq \min_{u \in A_0(R^{L_x,L_y})} I(u, R^{L_x,L_y})
\]

\[
\leq \min_{u \in A_0(R^{L_x,L_y})} I_p(u, R^{L_x,L_y}) \leq c_p \min \{ e^{2/3} \theta^{2/3} L_x^{1/3} L_y, e^{1/2} \theta L_y^{3/2} + \varepsilon L_x \}.
\]

Proof This is also a special case of [19, Theorem 1.2]. Observe that (116) implies the second inequality. The first inequality (lower bound) follows from the fact that \( |u_{yy}|(R^{L_x,L_y}) \geq \varepsilon L_x \) holds for any function \( u \in A_p(R^{L_x,L_y}) \subseteq A_0(R^{L_x,L_y}) \) and the lower bound in Theorem 1.

For the third inequality (upper bound), we consider the two cases separately. If \( L_x \leq l_x := \varepsilon^{-1/2} \theta L_y^{3/2} \), then \( \varepsilon L_x \leq e^{2/3} \theta^{2/3} L_x^{1/3} L_y \) and the respective construction from the proof of Theorem 1 is an admissible function. If \( L_x \geq l_x \), we set

\[
u(x, y) := \begin{cases} -\theta y, & \text{for } y \in (0, (1 - \theta)L_y/2), \\ (1 - \theta)y - (1 - \theta)L_y/2, & \text{for } y \in ((1 - \theta)L_y/2, (1 + \theta)L_y/2), \\ -\theta(y - L_y), & \text{otherwise} \end{cases}
\]

on \([l_x, L_x]) \times (0, L_y)\), and use Proposition 1 in \((0, l_x) \times (0, L_y)\) with \( u^L = 0, u^B = u^T = 0, u^R(y) = u(l_x, y) \), see Fig. 9.
Asymptotic Self-Similarity and Local Bounds in a Model of SMAs

Fig. 9 Sketch of the upper bound constructions in the periodic case

**Theorem 6** (Local bound) Let \( u \in A_P(R^{L_x,L_y}) \) be a minimizer of \( I(\cdot, R^{L_x,L_y}) \) and \( u^P \in A_P(R^{L_x,L_y}) \) a minimizer of \( I_P(\cdot, R^{L_x,L_y}) \). Furthermore, let \( l_x \in (0, \min\{\varepsilon^{-1/3}\theta L_y^{3/2}, L_x\}) \).

There are constants \( C_P, C'_P > 0 \), independent of \( l_x, L_x, L_y, \varepsilon \) and \( \theta \), such that

\[
\frac{1}{C_P} \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y \leq I(u, R^{l_x,L_y}) \leq C_P \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y \tag{118}
\]

and

\[
\frac{1}{C'_P} \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y \leq I_P(u^P, R^{l_x,L_y}) \leq C'_P \varepsilon^{2/3} \theta^{2/3} l_x^{1/3} L_y. \tag{119}
\]

**Proof** The lower bounds follow from Theorem 5. To prove the upper bounds, we proceed along the lines of the proof of Theorem 4. As in Lemma 3 there exists a constant \( \tau_P \in \mathbb{R} \) and a density \( \sigma_P \in L^2((0,L_x)) \) such that

\[
\sigma_P(x) = \tau_P + \int_0^{L_y} (u_x)^2(x, y) \, dy \quad \text{for almost every } x \in (0, L_x), \text{ and}
\]

\[
|\tau_P| \leq c_P \varepsilon^{2/3} \theta^{2/3} L_x^{-2/3} L_y
\]

hold, where \( c_P > 0 \) is the constant introduced in Theorem 5. We then construct a competitor \( v \in A_P(R^{l_x,L_y}) \) using Proposition 1. Note that in view of Lemma 7, we may assume that \( u \) is continuous, and thus by Remark 5, \( u^T(l_x, 0) = u^B(l_x, L_y) \). Therefore, setting \( u^T := u^{T,l} \) and \( u^B := u^{B,l} \), the assumption (28) is satisfied, and we obtain a competitor in \( A_P \) which allows us to conclude the proof of (118) along the lines of the proof of Theorem 4. Finally, the upper bound in (119) follows analogously using (116).

**Remark 6** By the arguments given in [27, Sect. 4] (cf. the first part of the proof of Lemma 2), we have the following property for the continuous representatives of minimizers \( u \) (respectively \( u^P \)) of \( I \) (respectively \( I_P \)): If \( |u_{yy}(\bar{x}, \cdot)|(0, L_y) = \min\{|u_{yy}(x, \cdot)|(0, L_y) : x \in (0, L_x)\} \) then \( u(x, y) = u(\bar{x}, y) \) for all \( x \in [\bar{x}, L_x] \). Similarly, if \( |u_{yy}^P(\bar{x}, \cdot)|(0, L_y) = \min\{|u_{yy}^P(x, \cdot)|(0, L_y) : x \in (0, L_x)\} \) then \( u^P(x, y) = u^P(\bar{x}, y) \) for all \( x \in [\bar{x}, L_x] \). Proceeding as in the proof of Lemma 2, using (114) and (115) we obtain the following (quantitative) results with the constant \( c_P > 0 \) from Theorem 5:

(i) If \( L_x \geq l_x := \frac{\varepsilon}{2} \theta L_y^{3/2} \) and \( u \in A_P(R^{L_x,L_y}) \) is a minimizer of \( I(\cdot, R^{L_x,L_y}) \) then \( u(x, y) = u(l_x, y) \) and \( |u_{yy}(x, \cdot)|(0, L_y) = 1 \) hold for \( x \in [l_x, L_x] \).
The asymptotic self-similarity of a minimizer of \( I(\cdot, R^{L_x, L_y}) \) or \( I_P(\cdot, R^{L_x, L_y}) \) on \( A_P(R^{L_x, L_y}) \) can be shown in the same way as in the Neumann case (Theorem 2), using Remark 5.

**Theorem 7 (Asymptotic self-similarity of a minimizer)** Let \( \epsilon, L_x, L_y > 0 \) and \( \theta \in (0, \frac{1}{2}] \), and fix any sequence \( v_j \to 0, v_j > 0 \). Then, the following two results hold:

(i) Let \( u \in A_P(R^{L_x, L_y}) \) be a minimizer of \( I(\cdot, R^{L_x, L_y}) \) and \( y_0 \in (0, L_y) \). Then the sequence \( u^j(x, y) := \frac{v_j^{-2/3}}{\epsilon} Eu(v_j x, y_0 + v_j^{2/3} y) \) has a subsequence that converges strongly in \( W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R}) \) towards an element \( u^\infty \in A_0((0, \infty) \times \mathbb{R}) \), and \( u^\infty \) is a local minimizer.

(ii) Let \( u_P \in A_P(R^{L_x, L_y}) \) be a minimizer of \( I_P(\cdot, R^{L_x, L_y}) \) and \( y_0 \in [0, L_y] \). Then the sequence \( u_P^j(x, y) := \frac{v_j^{-2/3}}{\epsilon} Eu_P(v_j x, y_0 + v_j^{2/3} y) \) has a subsequence that converges strongly in \( W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R}) \) towards an element \( u_P^\infty \in A_0((0, \infty) \times \mathbb{R}) \), and \( u_P^\infty \) is a local minimizer.

Both \( Eu \) and \( Eu_P \) are implicitly extended by zero to the rest of \((0, \infty) \times \mathbb{R}\).

**Remark 7** We point out that we obtain the asymptotic self-similarity of minimizers of \( I_P(\cdot, R^{L_x, L_y}) \) also for the boundary points \( y_0 \in \{0, L_y\} \).

**Proof** Both assertions can be derived following the proof of Theorem 2, where the first step can be omitted due to periodicity.

To prove that assertion (ii) holds also for the boundary points \([0, L_y]\), we use Remark 5 (ii) and (iv). Precisely, for given \( y_0 \in \{0, L_y\} \), we replace \( R^{L_x, L_y} \) with \((0, L_x) \times (y_0 - L_y/2, y_0 + L_y/2)\), choose \( \eta = \frac{1}{12} \) and proceed as in the proof of Theorem 2. \( \square \)

**Acknowledgements** We are grateful to Peter Bella and Michael Goldman for interesting discussions, and in particular to Stefan Müller for his continuous support, and for, in different moments, bringing this problem to our attention. This work was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) via projects 211504053 - SFB 1060 and 195170736 - TRR 109, and under Germany’s Excellence Strategy – The Berlin Mathematics Research Center MATH+ and the Berlin Mathematical School (BMS) (EXC-2046/1, project 390685689).

**Funding Note** Open Access funding enabled and organized by Projekt DEAL.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**References**

1. Alberti, G., Choksi, R., Otto, F.: Uniform energy distribution for an isoperimetric problem with long-range interactions. J. Am. Math. Soc. 22(2), 569–605 (2009). https://doi.org/10.1090/S0894-0347-08-00622-X
2. Bechtold, C., Chluba, C., Lima de Miranda, R., Quandt, E.: High cyclic stability of the elastocaloric effect in sputtered TiNiCu shape memory films. Appl. Phys. Lett. 101(9), 091903 (2012)

3. Bella, P., Goldman, M.: Nucleation barriers at corners for cubic-to-tetragonal phase transformation. Proc. R. Soc. Edinb., Sect. A 145, 715–724 (2015)

4. Bella, P., Kohn, R.V.: Wrinkles as the result of compressive stresses in an annular thin film. Commun. Pure Appl. Math. 67(5), 693–747 (2014)

5. Bellova, K., Julia, A., Otto, F.: Uniform energy distribution in a pattern-forming system of surface charges. Rev. Mat. Iberoam. https://doi.org/10.4171/RMI/1308. (Online first 2021)

6. Ben Belgacem, H., Conti, S., DeSimone, A., Müller, S.: Energy scaling of compressed elastic films – three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

7. Bourne, D., Conti, S., Müller, S.: Energy bounds for a compressed elastic film on a substrate. J. Nonlinear Sci. 27, 453–494 (2017). https://doi.org/10.1007/s00332-016-9339-0

8. Brancolini, A., Wirth, B.: Optimal energy scaling for micropatterns in transport networks. SIAM J. Math. Anal. 49(1), 311–359 (2017). https://doi.org/10.1137/15M1050227

9. Brancolini, A., Rossmanith, C., Wirth, B.: Optimal micropatterns in 2D transport networks and their relation to image inpainting. Arch. Ration. Mech. Anal. 228, 279–308 (2018)

10. Capella, A., Otto, F.: A rigidity result for a perturbation of the geometrically linear three-well problem. Commun. Pure Appl. Math. 62(12), 1632–1669 (2009). https://doi.org/10.1002/cpa.20297

11. Capella, A., Otto, F.: A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy. Proc. R. Soc. Edinb., Sect. A 142(2), 273–327 (2012). https://doi.org/10.1017/S0308210510000478

12. Chan, A., Conti, S.: Energy scaling and branched microstructures in a model for shape-memory alloys with \(SO(2)\) invariance. Math. Models Methods Appl. Sci. 25, 1091–1124 (2015). https://doi.org/10.1142/S0218202515500281

13. Choksi, R.: Scaling laws in microphase separation of diblock copolymers. J. Nonlinear Sci. 11, 223–236 (2001)

14. Choksi, R., Kohn, R.V.: Bounds on the micromagnetic energy of a uniaxial ferromagnet. Commun. Pure Appl. Math. 51(3), 259–289 (1998). https://doi.org/10.1002/(SICI)1097-0312(199803)51:3<259::AID-CPA3>3.0.CO

15. Choksi, R., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. Commun. Math. Phys. 201(1), 61–79 (1999)

16. Choksi, R., Kohn, R.V., Otto, F.: Energy minimization and flux domain structure in the intermediate state of a type-I superconductor. J. Nonlinear Sci. 14, 119–171 (2004)

17. Conti, S.: Branched microstructures: scaling and asymptotic self-similarity. Commun. Pure Appl. Math. 53, 1448–1474 (2000)

18. Conti, S., Ortiz, M.: Dislocation microstructures and the effective behavior of single crystals. Arch. Ration. Mech. Anal. 176, 103–147 (2005)

19. Conti, S., Zwicky Nagl, B.: Low volume-fraction microstructures in martensitic and crystal plasticity. Math. Models Methods Appl. Sci. 26, 1319–1355 (2016)

20. Conti, S., Otto, F., Serfaty, S.: Branched microstructures in the Ginzburg-Landau model of type-I superconductors. SIAM J. Math. Anal. 48, 2994–3034 (2016). https://doi.org/10.1137/15M1028960

21. Conti, S., Diermeier, J., Zwicky Nagl, B.: Deformation concentration for martensitic microstructures in the three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

22. Conti, S., Goldman, M., Müller, S.: Energy scaling of compressed elastic films – three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

23. Conti, S., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. Commun. Math. Phys. 201(1), 61–79 (1999)

24. Conti, S., Kohn, R.V., Otto, F.: Energy minimization and flux domain structure in the intermediate state of a type-I superconductor. J. Nonlinear Sci. 14, 119–171 (2004)

25. Conti, S.: Branched microstructures: scaling and asymptotic self-similarity. Commun. Pure Appl. Math. 53, 1448–1474 (2000)

26. Conti, S., Ortiz, M.: Dislocation microstructures and the effective behavior of single crystals. Arch. Ration. Mech. Anal. 176, 103–147 (2005)

27. Conti, S., Zwicky Nagl, B.: Low volume-fraction microstructures in martensitic and crystal plasticity. Math. Models Methods Appl. Sci. 26, 1319–1355 (2016)

28. Conti, S., Otto, F., Serfaty, S.: Branched microstructures in the Ginzburg-Landau model of type-I superconductors. SIAM J. Math. Anal. 48, 2994–3034 (2016). https://doi.org/10.1137/15M1028960

29. Conti, S., Diermeier, J., Zwicky Nagl, B.: Deformation concentration for martensitic microstructures in the three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

30. Conti, S., Goldman, M., Müller, S.: Energy scaling of compressed elastic films – three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

31. Conti, S., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. Commun. Math. Phys. 201(1), 61–79 (1999)

32. Conti, S., Kohn, R.V., Otto, F.: Energy minimization and flux domain structure in the intermediate state of a type-I superconductor. J. Nonlinear Sci. 14, 119–171 (2004)

33. Conti, S.: Branched microstructures: scaling and asymptotic self-similarity. Commun. Pure Appl. Math. 53, 1448–1474 (2000)

34. Conti, S., Ortiz, M.: Dislocation microstructures and the effective behavior of single crystals. Arch. Ration. Mech. Anal. 176, 103–147 (2005)

35. Conti, S., Zwicky Nagl, B.: Low volume-fraction microstructures in martensitic and crystal plasticity. Math. Models Methods Appl. Sci. 26, 1319–1355 (2016)

36. Conti, S., Otto, F., Serfaty, S.: Branched microstructures in the Ginzburg-Landau model of type-I superconductors. SIAM J. Math. Anal. 48, 2994–3034 (2016). https://doi.org/10.1137/15M1028960

37. Conti, S., Diermeier, J., Zwicky Nagl, B.: Deformation concentration for martensitic microstructures in the three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

38. Conti, S., Goldman, M., Müller, S.: Energy scaling of compressed elastic films – three-dimensional elasticity and reduced theories. Arch. Ration. Mech. Anal. 164, 1–37 (2002)

39. Conti, S., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. Commun. Math. Phys. 201(1), 61–79 (1999)

40. Conti, S., Kohn, R.V., Otto, F.: Energy minimization and flux domain structure in the intermediate state of a type-I superconductor. J. Nonlinear Sci. 14, 119–171 (2004)

41. Conti, S.: Branched microstructures: scaling and asymptotic self-similarity. Commun. Pure Appl. Math. 53, 1448–1474 (2000)

42. Conti, S., Ortiz, M.: Dislocation microstructures and the effective behavior of single crystals. Arch. Ration. Mech. Anal. 176, 103–147 (2005)
30. Hencl, S., Koskela, P.: Mappings of finite distortion: composition operator. In: Ann. Acad. Sci. Fenn. Math., vol. 33, pp. 65–80. Finnish Academy of Science and Letters, Helsinki (2008)
31. Hencl, S., Koskela, P.: Lectures on Mappings of Finite Distortion (2014)
32. James, R.D.: Materials from mathematics. Bull. Am. Math. Soc. 56, 1–28 (2019)
33. James, R.D., Zhang, Z.: A way to search for multiferroic materials with “unlikely” combinations of physical properties. In: Manosa, L., Planes, A., Saxena, A. (eds.) The Interplay of Magnetism and Structure in Functional Materials, vol. 79. Springer, Berlin (2005)
34. Jin, W., Sternberg, P.: Energy estimates of the von Kármán model of thin-film blistering. J. Math. Phys. 42, 192–199 (2001)
35. Kleprlík, L.: Mappings of finite signed distortion: Sobolev spaces and composition of mappings. J. Math. Anal. Appl. 386(2), 870–881 (2012)
36. Knüpfer, H., Kohn, R.V.: Minimal energy for elastic inclusions. Proc. R. Soc., Math. Phys. Eng. Sci. 467(2127), 695–717 (2011). https://doi.org/10.1098/rspa.2010.0316
37. Knüpfer, H., Muratov, C.B.: Domain structure of bulk ferromagnetic crystals in applied fields near saturation. J. Nonlinear Sci. 21, 921–962 (2011). https://doi.org/10.1007/s00332-011-9105-2
38. Knüpfer, H., Otto, F.: Nucleation barriers for the cubic-to-tetragonal phase transition in the absence of self-accommodation. Z. Angew. Math. Mech. 99, e201800, 179 (2019)
39. Knüpfer, H., Kohn, R.V., Otto, F.: Nucleation barriers for the cubic-to-tetragonal phase transformation. Commun. Pure Appl. Math. 66(6), 867–904 (2013). https://doi.org/10.1002/cpa.21448
40. Kohn, R.V., Müller, S.: Branching of twins near an austenite-twinned martensite interface. Philos. Mag. A 66, 697–715 (1992)
41. Kohn, R.V., Müller, S.: Surface energy and microstructure in coherent phase transitions. Commun. Pure Appl. Math. XLVII, 405–435 (1994)
42. Kohn, R.V., Wirth, B.: Optimal fine-scale structures in compliance minimization for a shear load. Commun. Pure Appl. Math. 69(8), 1572–1610 (2016). https://doi.org/10.1002/cpa.21589
43. Koser, M.: On a variational model for low-volume fraction microstructures in martensites: Properties of minimizers. Master’s thesis, TU, Berlin (2020)
44. Landau, L.: The intermediate state of supraconductors. Nature 141, 688 (1938)
45. Landau, L.: On the theory of the intermediate state of superconductors. J. Phys. USSR 7, 99 (1943)
46. Louie, M., Kislitsyn, M., Bhattacharya, K., Haile, S.: Phase transformation and hysteresis behavior in Cs$_{1-x}$Rb$_x$H$_2$PO$_4$. Solid State Ion. 181, 173–179 (2010)
47. Lu, V.: On imbedding theorems for spaces of functions with partial derivatives of various degrees of summability. Vestn. Leningr. Univ. 16, 23–37 (1961)
48. Murat, F.: Compacité par compensation. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 5, 489–507 (1978)
49. Rüland, A.: A rigidity result for a reduced model of a cubic-to-orthorhombic phase transition in the geometrically linear theory of elasticity. J. Elast. 123(2), 137–177 (2016)
50. Seiner, H., Plucinsky, P., Dabade, V., Benešová, B., James, R.D.: Branching of twins in shape memory alloys revisited. J. Mech. Phys. Solids 141, 103,961 (2020)
51. Simon, T.M.: Quantitative aspects of the rigidity of branching microstructures in shape memory alloys via h-measures. SIAM J. Math. Anal. 53, 4537–4567 (2021)
52. Simon, T.M.: Rigidity of branching microstructures in shape memory alloys. Arch. Ration. Mech. Anal. 241, 1707–1783 (2021)
53. Srivastava, V., Chen, X., James, R.D.: Hysteresis and unusual magnetic properties in the singular Heusler alloy Ni$_{45}$Co$_5$Mn$_{40}$Sn$_{10}$. Appl. Phys. Lett. 97, 014,101 (2010)
54. Tartar, L.: Compensated compactness and applications to partial differential equations. In: Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. 4, Edinburgh, 1979. Res. Notes in Math., vol. 39, pp. 136–212 (1979)
55. Viehmann, T.: Uniaxial ferromagnets. Ph.D. thesis, Universität, Bonn (2009). Available from https://bib.math.uni-bonn.de/downloads/bms/BMS-396.pdf
56. Zarnetta, R., Takahashi, R., Young, M.L., Savan, A., Furuya, Y., Thienhaus, S., Maaß, B., Rahim, M., Frenzel, J., Brunken, H., Chu, Y.S., Srivastava, V., James, R.D., Takeuchi, I., Eggle, G., Ludwig, A.: Identification of quaternary shape memory alloys with near-zero thermal hysteresis and unprecedented functional stability. Adv. Funct. Mater. 20(12), 1917–1923 (2010)
57. Zhang, Z.: Special lattice parameters and the design of low hysteresis materials. Ph.D. thesis, University of Minnesota (2007)
58. Zhang, Z., James, R.D., Müller, S.: Energy barriers and hysteresis in martensitic phase transformations. Acta Mater. 57(15), 4332–4352 (2009)
59. Zwicky, T.: Microstructures in low-hysteresis shape memory alloys: scaling regimes and optimal needle shapes. Arch. Ration. Mech. Anal. 213(2), 355–421 (2014)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.