Approximate Performance Measures for a Two-Stage Reneging Queue

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Abstract

We study a two-stage reneging queue with Poisson arrivals, exponential services, and two levels of exponential reneging behaviors, extending the popular Erlang A model that assumes a constant reneging rate. We derive approximate analytical formulas representing performance measures for the two-stage queue following the Markov chain decomposition approach. Our formulas not only give accurate results spanning the heavy-traffic to the light-traffic regimes, but also provide insight into capacity decisions.

Key words: two-stage queue, exponential reneging, Markov chain decomposition

1. Introduction

Models representing service systems with impatient customers have been studied extensively due to their practical importance. When analyzing such systems, it is often assumed that all customers are equally impatient and randomly renge (i.e., leave a queue after entering but before reaching service) at the same rate. This constant patience assumption may be reasonable when customers have no information about their position in queue (e.g., in some call centers). However, customers often change their level of patience, for example

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Preprint submitted to Elsevier May 8, 2020
based on their queue position if they experience a different level of comfort, or
when they know they are close to, or far from, the front of the queue [1]. For
example, many restaurants provide some seats (capacity) for customers waiting
inside (1st stage queue), but not everybody can sit there; if all seats inside are
filled, any remaining waiting customers must wait outside where it may be cold
or raining (2nd stage queue). Our model aims to provide simple, yet accurate
formulas for practitioners who make capacity (in this case waiting room seat-
ing capacity) decisions when their customers show stage-dependent reneging
behaviors. Specifically, we consider a two-stage reneging queue with Poisson
arrivals, exponential services, and stage-dependent exponential reneging rates.
We analyze three performance measures: probability of queueing, probability of
customer abandonment (via reneging or blocking), and average queue length.
Our two-stage reneging model can cope with both finite and infinite queues.

When the two reneging rates match, our queue reduces to the Erlang A
model, an M/M/n+M queueing model with reneging. This Erlang A model and
its variations have been utilized for the analysis of various real-world problems
such as public housing [2], kidney transplantation [3], on-street parking [4], and
call centers [see, e.g., [5, 6, 7, 8, 9]. Most of the studies are based on either heavy
traffic approximation or asymptotic analysis [10], which are very effective for
analyzing a single stage reneging queue. Since our model has two stages, we
take a different, Markov chain decomposition approach [11], and evaluate each
decomposed sub-system separately using a Poisson-Normal approximation [12]
with a continuity correction term [13, 14, 15]. This continuity correction term
makes our formulas accurate and robust over a wide range of parameters.

2. Two-Stage Reneging Model

In this section, we present our model and derive several basic performance
measures following the Markov chain decomposition approach [11]. The for-
mulas we derive in this section are exact; approximations are discussed in the
following section.
2.1. Markov Chain Structure and Its Decomposed Subchains

Our two-stage model is a simple birth-death Markov chain (MC) (Fig. 1). Customers arrive according to Poisson process with rate $\lambda$, and are served by one of the $s$ servers with exponential rate $\mu$ following a first-in first-out (FIFO) rule. The queue has two stages ($i = 1, 2$); stage $i$ has $n_i$ spaces, which can be either finite or infinite. Each customer in the $i^{th}$ stage reneges after an exponentially distributed time with rate $\theta_i$ ($> 0$). When both stages of the queue are fully occupied in state $s + n_1 + n_2$, a new customer is unable to enter the system.
The arrival and departure rates at state $k$ are
\[ \lambda_k = \begin{cases} 
\lambda & 0 \leq k < s + n_1 + n_2 \\
0 & s + n_1 + n_2 \leq k 
\end{cases} \]
\[ \mu_k = \begin{cases} 
k\mu & 1 \leq k \leq s \\
s\mu + (k-s)\theta_1 & s < k \leq s + n_1 \\
s\mu + n_1\theta_1 + (k-s-n_1)\theta_2 & s + n_1 < k \leq s + n_1 + n_2 \\
0 & s + n_1 + n_2 < k. 
\end{cases} \]

To analyze our system we decompose the full MC (Fig. 1) into subchains: an $M/M/s/s$ subchain (Subchain 0 in Fig. 2a MC comprised of states 1 to $s + n_1 + n_2$). Subchain $q$ is further decomposed into the 1st stage queue with reneging rate $\theta_1$ (Subchain 1 in Fig. 2c: MC comprised of states $s$ to $s + n_1$) and the 2nd stage queue with reneging rate $\theta_2$ (Subchain 2 in Fig. 2d: MC comprised of states $s + n_1$ to $s + n_1 + n_2$). Let $\Omega$ and $A_j$ represent a set of states for the full MC and subchain $j (= 0, 1, 2, q)$, respectively. Thus: $\Omega = A_0 \cup A_q$, $A_q = A_1 \cup A_2$, $A_0 \cap A_q = A_0 \cap A_1 = \{s\}$, and $A_1 \cap A_2 = \{s + n_1\}$.

### 2.2. Representation of Performance Measures

We denote the steady-state probability of state $k$ in the full MC and subchain $j$ as $\pi_k$ and $\pi_k^j$, respectively. Note that for any subchain $j$, $\pi_k \propto \pi_k^j$ for $k \in A_j$. Define $\mathbb{E}[f(X)] = \sum_{k \in \Omega} f(k)\pi_k$, $\mathbb{E}_j[f(X)] = \sum_{k \in A_j} f(k)\pi_k^j$, and \( r_1 = \frac{\pi^1_{s+n_1}}{\pi^1_s}. \) The following proposition holds as a special case of the Markov chain decomposition method based on the total expectation theorem \[11\]. All proofs are given in appendices.

**Proposition 1.** For any function of states $f(X)$ for the two-stage reneging queue, the following equations hold:
\[ \frac{\mathbb{E}[f(X)]}{\pi_s} = \frac{\mathbb{E}_0[f(X)]}{\pi_0^s} + \frac{\mathbb{E}_q[f(X)]}{\pi_q^s} - f(s), \]  
\[ \frac{\mathbb{E}_q[f(X)]}{\pi_q^s} = \frac{\mathbb{E}_1[f(X)]}{\pi_1^s} + r_1 \left( \frac{\mathbb{E}_2[f(X)]}{\pi_2^{s+n_1}} - f(s + n_1) \right). \]

Combining above, we obtain
\[ \frac{\mathbb{E}[f(X)]}{\pi_s} = \frac{\mathbb{E}_0[f(X)]}{\pi_0^s} + \left( \frac{\mathbb{E}_1[f(X)]}{\pi_1^s} - f(s) \right) + r_1 \left( \frac{\mathbb{E}_2[f(X)]}{\pi_2^{s+n_1}} - f(s + n_1) \right). \]
Denote the queueing probability as $P_Q$, the abandonment probability (due to reneging and blocking) as $P_A$, and the expected number of customers in queue as $L$. Using Proposition 1, we can obtain $\frac{1}{\pi_s}$ by setting $f(X) = 1$; $P_Q$ by setting $f(X) = 1_{A_q}$ (an indicator function); $P_A$ by setting $f(X) = N_A$ (a random variable representing the steady-state number of customer abandonments per unit time); and $L$ by setting $f(X) = N$ (a random variable representing the steady-state number of customers in queue).

**Proposition 2.** Performance measures of the two-stage reneging queue are

\[
\frac{1}{\pi_s} = \frac{1}{\pi_1^0} + \frac{1}{\pi_2^0} - 1 = \frac{1}{\pi_1^0} + \left(\frac{1}{\pi_2^0} - 1\right) + r_1 \left(\frac{1}{\pi_{s+n_1}^0} - 1\right),
\]

\[
P_Q = \frac{1}{\pi_s^0} + r_1 \left(\frac{1}{\pi_{s+n_1}^0} - 1\right),
\]

\[
P_A = p \left(\frac{1}{\pi_1^0} - 1\right) + 1 - p \left[\left(\frac{1}{\pi_2^0} - 1\right) + r_1 \left(\frac{1}{\pi_{s+n_1}^0} - 1\right)\right] + 1,
\]

\[
L = \frac{1}{\pi_s} \left[p \left(\frac{1}{\pi_1^0} - 1\right) + 1 - r_1\right] + \frac{\mu}{\lambda} \left[p + \frac{n_1(\theta_2 - \theta_1)}{\lambda}\right] \left(\frac{1}{\pi_{s+n_1}^0} - 1\right) + 1 - r_2,
\]

where

\[
r_1 = \frac{\pi_{s+n_1}^1}{\pi_1^1}, \quad r_2 = \frac{\pi_{s+n_1+n_2}^2}{\pi_{s+n_1}^2}, \quad \text{and} \quad p = 1 - \frac{s\mu}{\lambda}.
\]

Proposition 2 implies the following exact relationships.

**Corollary 1.** Abandonment probability $P_A$ can be represented as follows:

\[
P_A = p (P_Q - \pi_s) + \pi_s,
\]

\[
= \frac{p - \pi_0^0}{1 - \pi_0^0} P_Q + \frac{1 - p}{1 - \pi_0^0} \pi_s.
\]

3. Approximation Procedure

Proposition 2 expresses the performance measures of interest as simple functions of steady-state probabilities. To obtain approximate performance measures, we derive an approximate analytical representation of these steady-state probabilities.
3.1. Steady-State Probabilities in Poisson and Normal Representations

We first show the Poisson-Normal conversion formulas, which involve a Poisson random variable (RV) with mean \( R \) and the standard Normal RV, whose cumulative distribution functions (CDFs) are \( F_P(\cdot; R) \) and \( \Phi(\cdot) \); their corresponding probability mass function (PMF) and probability density function (PDF) are \( f_P(\cdot; R) \) and \( \phi(\cdot) \), respectively. Define the standard Normal hazard function as

\[
h(x) = \frac{\phi(x)}{1 - \Phi(x)} = \frac{\phi(-x)}{\Phi(-x)}
\]

and introduce parameters \( c = (s - R)/\sqrt{R} \) and \( \Delta = 0.5/\sqrt{R} \). The following result is based on [15].

**Lemma 1.** The Poisson CDF/PMF are approximated by the standard Normal CDF/PDF as:

\[
F_P(s; R) \approx \Phi(c + \Delta), \tag{10}
\]

\[
f_P(s; R) \approx \frac{\phi(c + \Delta)}{\sqrt{R}}, \tag{11}
\]

\[
\frac{f_P(s; R)}{1 - F_P(s; R)} \approx \frac{h(c + \Delta)}{\sqrt{R}}, \tag{12}
\]

\[
\frac{f_P(s; R)}{F_P(s; R)} \approx \frac{h(-c - \Delta)}{\sqrt{R}}. \tag{13}
\]

Following conventions in capacity planning, we call the argument \( s \) of the Poisson PMF/CDF the *staffing level*, the mean \( R \) of the Poisson RV the *resource requirement*, and the argument \( c \) of the standard Normal PDF/CDF the *square-root coefficient*. The term \( \Delta \) is the *continuity correction term*, which is introduced due to the conversion from a discrete function (Poisson) to continuous (Normal). The term \( \Delta \) is non-negligible when \( R \) is small (around 10 or less), but diminishes to zero in the asymptotic limit of large \( R \).

Corresponding to the three subchains (SC) 0 (M/M/s/s queue), 1 (1st stage queue), and 2 (2nd stage queue), we define parameters necessary for Poisson and standard Normal representations in Table 1. Note that the staffing levels \( s, s_1, \) and \( s_2 \) must be non-negative integers when they appear in a Poisson representation; if they are non-integer, we round them to their nearest integer values. (In other words, a Poisson representation is exact when staffing levels are integers and otherwise is an approximation.) This integer condition can be dropped when we convert to the Normal; a major benefit of using the Normal
Table 1: Parameters for Poisson and Standard Normal Representations

| SC | Staffing Level | Sq-root Coef. | Resource Req. | Cont. Correction |
|----|----------------|---------------|---------------|-----------------|
| 0  | s              | $c = \frac{s-R}{\sqrt{R}}$ | $R = \frac{1}{\mu}$ | $\Delta = \frac{0.5}{\sqrt{R}}$ |
| 1  | $s_1 = \frac{\mu_1}{\theta_1}$ | $c_1 = \frac{\mu_1-R}{\sqrt{R_1}}$ | $R_1 = \frac{1}{\mu_1}$ | $\Delta_1 = \frac{0.5}{\sqrt{R_1}}$ |
|    | $s_{1+} = s_1 + n_1$ | $c_{1+} = \frac{s_1+R_1}{\sqrt{R_1}}$ | ($= \frac{\mu_1}{\mu_1}R_1$) | ($= \sqrt{\frac{2}{\mu_1}}\Delta$) |
| 2  | $s_2 = \frac{\mu_1+n_1}{\theta_2}$ | $c_2 = \frac{\mu_2-R_2}{\sqrt{R_2}}$ | $R_2 = \frac{1}{\mu_2}$ | $\Delta_2 = \frac{0.5}{\sqrt{R_2}}$ |
|    | $s_{2+} = s_2 + n_2$ | $c_{2+} = \frac{s_2+R_2}{\sqrt{R_2}}$ | ($= \frac{\mu_2}{\mu_2}R_2$) | ($= \sqrt{\frac{2}{\mu_2}}\Delta$) |

representation. Using Lemma 1 and parameters defined in Table 1 we obtain Proposition 3. The continuity correction terms ($\Delta$, $\Delta_1$, $\Delta_2$) in this proposition can be dropped at the limit of large resource requirements ($R, R_1, R_2 \to \infty$, respectively).

**Proposition 3.** The steady-state probabilities of subchains are approximately expressed in Poisson and Normal representations as follows:

1. $M/M/s/s$ (subchain 0):

   \[
   \frac{1}{\bar{\pi}_0} = \frac{F_P(s; R)}{f_P(s; R)} \approx \frac{\sqrt{R}}{h(-c - \Delta)} =: \bar{h}.
   \]  

2. 1st stage queue (subchain 1):

   \[
   r_1 = \frac{s_{1+}}{\bar{\pi}_1} = \frac{f_P(s_{1+}; R_1)}{f_P(s_1; R_1)} \approx \frac{\phi(c_{1+} + \Delta_1)}{\phi(c_1 + \Delta_1)} =: \bar{r}_1,
   \]

   \[
   \frac{1}{\bar{\pi}_1} - 1 = \frac{1 - F_P(s_1; R_1)}{f_P(s_1; R_1)} = \sqrt{R_1} \left( \frac{1}{h(c_1 + \Delta_1)} - \frac{\bar{r}_1}{h(c_{1+} + \Delta_1)} \right) =: \bar{h}_1.
   \]  

3. 2nd stage queue (subchain 2):

   \[
   r_2 = \frac{s_{2+}}{\bar{\pi}_2} = \frac{f_P(s_{2+}; R_2)}{f_P(s_2; R_2)} \approx \frac{\phi(c_{2+} + \Delta_2)}{\phi(c_2 + \Delta_2)} =: \bar{r}_2,
   \]

   \[
   \frac{1}{\bar{\pi}_2} - 1 = \frac{1 - F_P(s_2; R_2)}{f_P(s_2; R_2)} = \sqrt{R_2} \left( \frac{1}{h(c_2 + \Delta_2)} - \frac{\bar{r}_2}{h(c_{2+} + \Delta_2)} \right) =: \bar{h}_2.
   \]
Figure 3: Comparison between our approximation and exact values for different 1st stage reneging rates $\theta_1$.

(a) $P_Q$  
(b) $P_A$  
(c) $L$

Notes: $\lambda = 50, \mu = 1, n_1 = 10, n_2 = 20, \theta_2 = 2$. $s$ and $\theta_1$ are variables. The open circles in the figures correspond to exact values.

3.2. Performance Measures in Normal Representation

Combining Propositions 2 and 3 and using parameters defined in Table 1, we derive the approximate representations of our performance measures.

Proposition 4. Performance measures of the two-stage reneging queue are approximately expressed in the standard Normal representation as follows:

$$\frac{1}{\pi_s} = \tilde{h} + \tilde{h}_1 + \tilde{r}_1 \tilde{h}_2, \quad P_Q = \frac{1}{\pi_s} = 1 + \tilde{h}_1 + \tilde{r}_1 \tilde{h}_2, \quad P_A = p \left( \tilde{h}_1 + \tilde{r}_1 \tilde{h}_2 \right) + 1,$$

$$\frac{L}{\pi_s} = R_1 \left( p \tilde{h}_1 + 1 - \tilde{r}_1 \right) + \tilde{r}_1 R_2 \left[ (p + \frac{n_2}{n_1} - \frac{n_2}{n_1} \tilde{h}_2 + 1 - \tilde{r}_2 \right].$$

Proposition 4 can represent both finite and infinite queues as well as the Erlang A model by choosing parameters appropriately. Notice that for $i \in \{1, 2\}$, $n_i \to \infty \iff \tilde{r}_i = 0$; and $n_i = 0 \iff \tilde{r}_i = 1, \tilde{h}_i = 0$. The above formulas reduce to the Erlang A formulas either by setting $\theta_1 = \theta_2$ and $\tilde{r}_2 = 0$ or by setting $\tilde{r}_1 = 1, \tilde{h}_1 = 0$, and $\tilde{r}_2 = 0$. Proposition 4 can be extended to more stages if desired (see Appendix G).

4. Numerical Experiments

We first demonstrate the accuracy of our approximate formulas and then examine the impact of selected parameters of the two-stage reneging model.
Figure 4: Comparison between our approximation and exact values for different 2nd stage reneging rates $\theta_2$.

(a) $P_Q$  
(b) $P_A$  
(c) $L$

Notes: $\lambda = 50, \mu = 1, n_1 = 5, n_2 = 20, \theta_1 = 2$. $s$ and $\theta_2$ are variables; the lines for $P_A$ are hardly distinguishable due to the insensitivity of $P_A$ to $\theta_2$ in this example. The open circles in the figures correspond to exact values.

4.1. Comparison with Exact Values

Fig. 3 compares the exact values with our approximation for fixed $\lambda, \mu, n_1, n_2$, and $\theta_2$. (We tested many parameter settings; results were similar.) Fig. 3 shows staffing level $s$ on the horizontal axis and different lines for different 1st stage reneging rates $\theta_1$. Similarly, Fig. 4 compares the exact values with our approximation when staffing level $s$ changes for different 2nd stage reneging rates $\theta_2$. The results shown in Figs. 3 and 4 demonstrate that our approximation is accurate and robust to a wide range of parameters.

Finally, we notice from Figs. 3b and 4b that $P_A$ is largely insensitive to reneging rates. This is because customer abandonment is independent from reneging rates in the two extremes: When servers are always busy, all customers not served by busy servers abandon, and when servers are not busy, almost nobody abandons. Figs. 3b and 4b confirm this view: $P_A$ approaches $p = (\lambda - s\mu)/\lambda$, a linear function of $s$, when $s$ is small (in the heavy-traffic limit; $s < 40$ in our example), while $P_A$ is close to zero when $s$ is large (in the light-traffic limit; $s > 60$ in our example). A reneging rate has some impact on $P_A$ only when $s \approx R$ ($40 \lesssim s \lesssim 60$ in our example).
4.2. Performance Measures’ Dependence on Parameters

It is intuitive that if customers rarely exceed the 1st stage capacity—for example, due to large $n_1$ and/or large $\theta_1$—then the system will become insensitive to $n_2$ and $\theta_2$. Numerical experiments verify this: Figs. 5 and 6 show how $n_1$ and $\theta_1$ (horizontal axis) impact performance measures for different $n_2$ and $\theta_2$, respectively; we observe all lines converge as $n_1$ and $\theta_1$ increase. To illustrate how the 1st stage queue affects the system’s dependence on the 2nd stage queue, we conduct the following experiment: Fig. 7 shows the impact of $\theta_2$ (horizontal axis) on performance measures for different $n_1$; a steeper line corresponds to higher impact of $\theta_2$. We observe that, in this example, at least about $n_1 = 8$ is necessary to make the system insensitive to the change of $\theta_2$. A real-world implication of this result is that, by conducting a similar analysis on a spreadsheet, a restaurant owner could decide how many seats to provide for customers waiting inside (1st stage queue) to make the quality of service robust to the weather, which directly affects the level of patience for customers waiting outside (2nd stage queue).

Before we conclude this section, we provide further insight into this insensitivity. First, notice that the 1st stage queue is (approximately) represented by the standard Normal distribution ranging from $c_1(= \frac{s-\mu}{\sqrt{\lambda \theta_1}})$ to $c_1+(= \frac{s+\mu-n\theta_1-\lambda}{\sqrt{\lambda \theta_2}})$. To contain most waiting customers within the 1st stage queue, its distribution should cover a wide range of the standard Normal distribution in-
Figure 6: Impact of reneging rates $\theta_1$ and $\theta_2$ on performance measures.

Note: $\lambda = 50$, $\mu = 1$, $s = 30$, $n_1 = 6$, $n_2 = 20$. $\theta_1$ and $\theta_2$ are variables.

Figure 7: Impact of 2nd stage reneging rate $\theta_2$ and 1st stage queue capacity $n_1$ on performance measures.

Note: $\lambda = 50$, $\mu = 1$, $s = 30$, $\theta_1 = 4$, $n_2 = \infty$. $\theta_2$ and $n_1$ are variables.

Including the center of the distribution. Since $c_1 < 0$ (i.e., $\lambda > s\mu$) in the case of interest (otherwise, there are only a small number of waiting customers), we require $c_1 + z$, where $z$ is a decision variable representing a certain positive threshold on the standard Normal distribution. For example, if we use $z = 1$, we obtain $n_1 \geq 15$ in Fig. 5; $\theta_1 \geq 6.3$ in Fig. 6; and $n_1 \geq 8.5$ (or 9 since $n_1$ should be an integer) in Fig. 7; these estimates turn out to be sufficiently good. Although this rule of thumb cannot replace accurate numerical calculations based on Proposition 4, it allows practitioners to make ballpark parameter estimates for containing most waiting customers in the 1st stage queue.
5. Conclusion and future work

In this paper we study a two-stage reneging model with finite or infinite queue capacity. Our two-stage model has wide applicability in the real world; for example, practitioners can utilize our formulas to make capacity decisions efficiently without solving Markov chains or conducting simulations.

There are several possible extensions of our results. First, our two-stage model can be straightforwardly extended to three or more stages if further accuracy is needed. Second, our paper assumes a constant arrival rate with no balking, but extension to a stage-dependent arrival rate (i.e., stage-dependent balking) is straightforward. Third, in addition to the three performance measures we analyze in this paper, we can also analyze other performance measures such as waiting time distribution. Our two-stage reneging model is simple, but more versatile than the Erlang A model that has inspired researchers for several decades; we hope our two-stage model opens a new avenue of research on service systems with impatient customers.

Appendix A Proof of Proposition 1

Since the MC we consider is a birth-death type, all decomposed subchains maintain a stationary distribution proportional to the full MC: for any subchain \( j \), \( \pi_k^j \propto \pi_k, \forall k \in A_j \), or equivalently, \( \pi_k/\pi_k^j = \pi_k/\pi_k^j, \forall k, k' \in A_j \). Also, note that \( A_0 \cap A_q = \{s\} \). Define \( \mathbb{E}[f(X)] = \sum_{k \in \Omega} f(k)\pi_k \) and \( \mathbb{E}_j[f(X)] = \sum_{k \in A_j} f(k)\pi_k^j \). Applying the total expectation theorem to the full MC, we derive

\[
\mathbb{E}[f(X)] = \sum_{k \in A_0} f(X)\pi_k + \sum_{k \in A_q} f(X)\pi_k - f(s)\pi_s
\]

\[
= \sum_{k \in A_0} f(X)\pi_k \frac{\pi_s}{\pi_s^2} + \sum_{k \in A_q} f(X)\pi_k^q \frac{\pi_s}{\pi_s^2} - f(s)\pi_s
\]

\[
= \mathbb{E}_0[f(X)]\frac{\pi_s}{\pi_s^2} + \mathbb{E}_q[f(X)]\frac{\pi_s}{\pi_s^2} - f(s)\pi_s,
\]

from which we obtain (1). The derivation of (2) is almost identical. Note that \( A_1 \cap A_2 = \{s + n_1\} \). By decomposing subchain \( q \) into subchains 1 and 2, we
Appendix B Proof of Proposition 2

For (4), we set \( f(X) = 1 \) in Proposition 1 and obtain \( \frac{1}{\pi_s} = \frac{1}{\pi_1} + \frac{1}{\pi_2} - 1 \) and \( \frac{1}{\pi_2} = \frac{1}{\pi_1} + r_1 \left( \frac{1}{\pi_{s+n_1}} - 1 \right) \), from which we obtain the result. For (5), we set \( P[A] = E[1_A] \), an indicator function. Note that \( P_Q = E[1_{A_q}] \), \( E_0[1_{A_0}] = \pi_0^s \), \( E_1[1_{A_1}] = 1 \), and \( E_s[1_{A_s}] = 1 \). Thus, we obtain \( \frac{P_A}{\pi_s} = E[1_A] = \frac{1}{\pi_s} = \frac{1}{\pi_1} + r_1 \left( \frac{1}{\pi_{s+n_1}} - 1 \right) \). For (6), we set \( f(X) = N_A \), a random variable representing the steady-state number of customer abandonments (via reneging and blocking) per unit time. Note that \( P_A = \frac{E[N_A]}{\lambda} \), \( E_0[N_A] = \lambda \pi_0^s \), \( E_s[N_A] = \lambda \), and \( E_1[N_A] = \lambda - s \mu (1 - \pi_0^s) \) (the flow conservation law). Thus, we obtain \( \frac{P_A}{\pi_s} = \frac{E[N_A]}{\lambda \pi_s} = p \left( \frac{1}{\pi_1} - 1 \right) + 1 \), where \( p = 1 - \frac{s \mu}{\lambda} \). Finally, as for (7), we set \( f(X) = N \), a random variable representing the steady-state number of customers in queue. Let \( L = E[N] \), \( L_1 = E_1[N] \), and \( L_2 = E_2[N] \). We obtain \( \frac{L}{\pi_s} = \frac{E[N]}{\pi_s} = \frac{L}{\pi_1} + r_1 \left( \frac{L_2}{\pi_{s+n_1}} - n_1 \right) \). The remaining task is to find \( L_1 \) and \( L_2 \). We use the flow conservation law for subchains 1 and 2: 
\[
\lambda - \pi_{s+n_1}^1 = s \mu (1 - \pi_1^s) + \theta_1 L^1 \\
\lambda - \pi_{s+n_1}^2 = (s \mu + n_1 \theta_1)(1 - \pi_{s+n_1}^2) + \theta_2 (L^2 - n_1) ,
\]
from which we obtain the expressions for \( L_1 \) and \( L_2 \):
\[
\frac{L^1}{\pi_1^2} = \theta_1 \left[ p \left( \frac{1}{\pi_1^1} - 1 \right) + 1 - r_1 \right] ,
\]
\[
\frac{L^2}{\pi_{s+n_1}} - n_1 = \theta_2 \left[ \left( p - \frac{n_1 (\theta_2 - \theta_1)}{\lambda} \right) \left( \frac{1}{\pi_{s+n_1}^2} - 1 \right) + 1 - r_2 \right] ,
\]
where \( r_1 = \frac{\pi_{s+n_1}^1}{\pi_1^2} \) and \( r_2 = \frac{\pi_{s+n_1}^2}{\pi_{s+n_1}} \).

Appendix C Proof of Corollary 1

The results directly follow from Proposition 2. We obtain (8) by eliminating \( \pi_s^2 \) in (3) and (5). We obtain (9) by eliminating \( \pi_s \) in (3) using (4).
Appendix D  Proof of Lemma 1

The Poisson-to-Normal conversion is based on the Central Limit Theorem and is described in many textbooks [e.g., 13]. The approximation achieves high accuracy when the continuity correction term \( \Delta \) is small \((R \text{ is large})\); at the limit of small \( \Delta \) \((\text{large } R)\), the error of the approximation approaches zero. For (10), we make a discrete-to-continuous conversion from Poisson to Normal with a continuity correction \( +0.5 \) following [15]. We then convert Normal to standard Normal using
\[
c = s - R \sqrt{R} \quad \text{and} \quad \Delta = 0.5 \sqrt{R}.
\]

For (11), we use the result above and the assumption that \( \Delta \) is sufficiently small, we obtain
\[
\begin{align*}
    f_P(s; R) &= F_P(s; R) - F_P(s - 1; R) \\
    &\approx \Phi(c + \Delta) - \Phi(c - \Delta) \\
    &\approx \phi(c + \Delta) \sqrt{R}. 
\end{align*}
\]

For (12) and (13), we use the above results and the definition of the hazard function for the standard Normal distribution. We obtain
\[
\begin{align*}
    f_P(s; R) &= 1 - F_P(s; R) \\
    &\approx \phi(c + \Delta) \sqrt{R(1 - \Phi(c + \Delta))} \\
    &= h(c + \Delta). 
\end{align*}
\]

Appendix E  Proof of Proposition 3

We discuss each subchain separately. We use parameters defined in Table 1. We assume that all staffing levels \( s, s', \text{ and } s'' \) are non-negative integers in a Poisson representation; if they are non-integer, we round them to their nearest integer values. This integer condition is dropped when we convert to the Normal.

1. M/M/s/s (subchain 0): This result is familiar [see, e.g., 16]. Let \( R = \frac{\lambda}{\mu} \) and consider a random variable \( X \sim \text{Pois}(R) \); thus \( \Pr\{X = i\} = e^{-R} R^i / i! \), \( \forall i \in \mathbb{Z}_{\geq 0} \). This subchain is a birth-death MC with the total departure rate \( k\mu \) at state \( k \). Hence, for all \( k = 1, 2, \ldots, s \),
\[
\pi_k = \pi_{k-1} \frac{\lambda}{k\mu} = \pi_{k-1} R k = \ldots = \pi_0 R^k / k! \propto \Pr\{X = k\}.
\]

Note that \( \pi_k \propto \Pr\{X = k\} \) means the distributions of \( \pi_k \) and \( \Pr\{X = k\} \) for \( k \in A_0 \) are the same except for the normalization constant. By summing up the terms with respect to \( k \) and applying the normalization condition, we obtain
\[
\frac{1}{\pi_0} = \frac{\Pr(0 \leq X \leq s)}{\Pr(X = s)} = \frac{F_P(s; R)}{f_P(s; R)}.
\]

Let \( c = \frac{s - R \sqrt{R}}{\sqrt{R}} \) and \( \Delta = \frac{0.5 \sqrt{R}}{\sqrt{R}} \). Using (13), we obtain (14).
2. 1st stage queue (subchain 1): Let \( R_1 = \frac{\lambda}{\theta_1} \), \( s_1 = \frac{\mu}{\theta_1} \), and \( s_{1+} = s_1 + n_1 \). Consider a random variable \( X_1 \sim \text{Pois}(R_1) \); thus \( \Pr\{X_1 = i\} = e^{-R_1} \frac{(R_1)^i}{i!}, \forall i \in \mathbb{Z}_{\geq 0} \). This subchain is a birth-death MC with the total departure rate \( s\mu + k\theta_1 = \frac{s\mu + k}{s_1 + n_1} \theta_1 = (s_1 + k)\theta_1 \) at state \( s + k \). For all \( k = 1, 2, \ldots, n_1 \),
\[
\pi_{s+k}^1 = \pi_{s+k-1}^1 \frac{\lambda}{(s_1 + k)\theta_1} = \pi_{s+k-1}^1 \frac{R_1}{s_1 + k} = \cdots = \pi_{s}^1 (R_1)^{s+k} \frac{s!}{(s_1 + k)! (R_1)^{s_1}} \propto \Pr\{X_1 = s + k\}.
\]

By summing up the terms with respect to \( k \) and applying the normalization condition, we obtain
\[
\frac{1}{\pi_{s}^1} = \frac{\Pr\{s_1 \leq X_1 \leq s_1 + n_1 \}}{\Pr\{X_1 = s_1\}} = \frac{\Pr\{X_1 = s_1\} - \Pr\{X_1 \leq s_1\} + \Pr\{X_1 \leq s_1 + n_1\}}{\Pr\{X_1 = s_1\}}
= 1 + \frac{1 - F_P(s_1; R_1)}{f_P(s_1; R_1)} - \frac{1 - F_P(s_{1+}; R_1)}{f_P(s_1; R_1)}
= 1 + \frac{1 - F_P(s_1; R_1)}{f_P(s_1; R_1)} \frac{f_P(s_{1+}; R_1)}{f_P(s_1; R_1)} \frac{1 - F_P(s_{1+}; R_1)}{f_P(s_{1+}; R_1)}.
\]

Let \( c_1 = \frac{s_1 - R_1}{\sqrt{R_1}} \), \( c_1^+ = \frac{s_1 - R_1}{\sqrt{R_1}} \), and \( \Delta_1 = \frac{0.5}{\sqrt{R_1}} \). Using \([11]\) and \([12]\), we obtain \([15]\).

3. 2nd stage queue (subchain 2): The argument is almost identical to the case of the 1st stage queue. Let \( R_2 = \frac{\lambda}{\theta_2} \), \( s_2 = \frac{s_1 + n_1 \theta_1}{\theta_2} \), and \( s_{2+} = s_2 + n_2 \). Consider a random variable \( X_2 \sim \text{Pois}(R_2) \); thus \( \Pr\{X_2 = i\} = e^{-R_2} \frac{(R_2)^i}{i!}, \forall i \in \mathbb{Z}_{\geq 0} \). This subchain is a birth-death MC with the total departure rate \( s\mu + n_1 \theta_1 + k\theta_2 = (s_2 + k)\theta_2 \) at state \( s + n_1 + k \). For all \( k = 1, 2, \ldots, n_2 \),
\[
\pi_{s+n_1+k}^2 = \pi_{s+n_1+k-1}^2 \frac{\lambda}{(s_2 + k)\theta_2} = \pi_{s+n_1+k-1}^2 \frac{R_2}{s_2 + k} = \cdots = \pi_{s+n_1}^2 (R_2)^{s+k} \frac{s!}{(s_2 + k)! (R_2)^{s_2}} \propto \Pr\{X_2 = s_2 + k\},
\]
from which we obtain
\[
\frac{1}{\pi_{s+n_1}^2} = \frac{\Pr\{s_2 \leq X_2 \leq s_2 + n_2 \}}{\Pr\{X_2 = s_2\}} = 1 + \frac{1 - F_P(s_2; R_2)}{f_P(s_2; R_2)} - \frac{1 - F_P(s_{2+}; R_2)}{f_P(s_2; R_2)} \frac{1 - F_P(s_{2+}; R_2)}{f_P(s_{2+}; R_2)}.
\]

Let \( c_2 = \frac{s_2 - R_2}{\sqrt{R_2}} \), \( c_2^+ = \frac{s_2 - R_2}{\sqrt{R_2}} \), and \( \Delta_2 = \frac{0.5}{\sqrt{R_2}} \). Using \([11]\) and \([12]\), we obtain \([16]\).

\[\text{Appendix F} \quad \text{Proof of Proposition 4}\]

The results directly follow from Propositions 2 and 3.
Appendix G  Extension to a Three-Stage Reneging Queue

To extend the two-stage reneging model to three-stage, we introduce the 3rd subchain with capacity $n_3$ and reneging rate $\theta_3$. Define parameters accordingly:

\[ R_3 = \frac{\lambda}{\theta_3}, \quad s_3 = \frac{s_1 + n_1 \theta_1 + n_2 \theta_2}{\theta_3}, \quad s_{3+} = s_3 + n_3, \quad c_3 = \frac{s_3 - R_3}{\sqrt{R_3}}, \quad c_{3+} = \frac{s_{3+} - R_3}{\sqrt{R_3}}, \]

\[ \Delta_3 = \frac{0.5}{\sqrt{R_3}}. \]

Also, we introduce

\[ r_3 = \frac{\pi_3 \left( \frac{1}{\pi_3} - \frac{s_3 - R_3}{\phi(c_3 + \Delta_3)} \right)}{\phi(c_3 + \Delta_3)} =: \tilde{r}_3 \]

and

\[ 1 - \frac{1}{\pi_{s+1} + n_2} \approx \sqrt{R_3} \left( \frac{1}{h(c_3 + \Delta_3)} - \frac{s_3 - R_3}{h(c_3 + \Delta_3)} \right) =: \tilde{h}_3. \]

It is straightforward to obtain the expressions for the three-stage reneging queue.

\[ L_{\pi_3} = \frac{R_1 (p \tilde{h}_1 + 1 - \tilde{r}_1) + \tilde{r}_1 R_2 \left( p + \frac{n_1}{R_2} - \frac{n_1}{R_1} \right) \tilde{h}_2 + 1 - \tilde{r}_2 + \tilde{r}_1 \tilde{r}_2 \tilde{h}_3}{R_1 (p \tilde{h}_1 + 1 - \tilde{r}_1) + \tilde{r}_1 R_2 \left( p + \frac{n_1 + n_2}{R_2} - \frac{n_1}{R_1} - \frac{n_2}{R_2} \right) \tilde{h}_3 + 1 - \tilde{r}_3}. \]

Observe that the formulas we obtain here are simply the formulas in Proposition 4 with extra terms added to account for the 3rd subchain. If it is necessary, an extension to four or more stages is also straightforward.

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