Shot noise in chaotic cavities with an arbitrary number of open channels

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(Received 23 December 2005; Published 13 February 2006 in: Phys. Rev. B 73, 081307(R) (2006))

Using the random matrix approach, we calculate analytically the average shot-noise power in a chaotic cavity at an arbitrary number of propagating modes (channels) in each of the two attached leads. A simple relationship between this quantity, the average conductance and the conductance variance is found. The dependence of the Fano factor on the channel number is considered in detail.

PACS numbers: 73.23.-b, 73.50.Td, 05.45.Mt, 73.63.Kv

The time dependent fluctuations in electrical currents caused by random transport of the electron charge e, which (unlike thermal fluctuations) persist down to zero temperature, are known as shot noise. In mesoscopic systems, an adequate description of this phenomenon is achieved in the scattering theory framework. In particular, for the two-terminal setup (with a small voltage difference V) it is well-known that the zero-frequency shot-noise spectral power is given by

\[ P = P_0 \sum_{p=1}^{N} T_p(1 - T_p), \quad P_0 = 2e|V|G_0, \]  

where \( T_p \) are \( n = \min(N_1, N_2) \) transmission eigenvalues of a conductor, \( G_0 \) is the conductance quantum, and \( N_{1,2} \) denotes the number of scattering channels in each of the two leads. \( T_p \) are mutually correlated random numbers between 0 and 1 whose distribution depends on the type of the conductor.

In the case of chaotic cavities considered below, universal fluctuations of \( T_p \) are believed to be provided by the random matrix theory (RMT). The latter is characterized by the symmetry index \( \beta \), distinguishing between universality classes of systems according to the absence (\( \beta = 2 \), unitary ensemble) or presence (\( \beta = 1 \), orthogonal ensemble) of time-reversal symmetry and spin-flip symmetry (\( \beta = 4 \), symplectic ensemble). Various RMT related aspects of the shot noise are under active study now, both theoretically (which is the purely classical one) and experimentally (see also the references in these papers). However, exact results for the average shot-noise power \( \langle P \rangle \) were reported in the literature only in the limiting cases of \( N_{1,2} \gg 1 \) (which is the purely classical one) or \( N_1 = N_2 = 1 \) (the experimentally relevant case of few channels being an open problem).

An alternative consideration was undertaken very recently by Braun et al., who developed the semiclassical trajectory approach to build up the \( 1/N \) expansion for \( \langle P \rangle \), extending earlier results to all orders of the inverse total number of channels, \( N = N_1 + N_2 \) (see also Ref. 23). They were able (for \( \beta = 1, 2 \)) to sum up the resulting series in a compact form, which we represent introducing \( \beta \) as follows:

\[ \langle P \rangle = \frac{N_1(N_1 - 1 + \frac{2}{\beta})N_2(N_2 - 1 + \frac{2}{\beta})}{(N - 2 + \frac{2}{\beta})(N - 1 + \frac{2}{\beta})(N - 1 + \frac{2}{\beta})}. \]  

This result surprisingly turned out to remain valid down to \( N_{1,2} = 1 \), as was checked by comparison to numerics.

Our aim here is to provide the exact RMT derivation of Eq. valid at arbitrary \( N_{1,2} \) and all \( \beta \). There are several ways to perform the calculation. First, \( T_p \) are defined as the singular values of a transmission matrix \( t \) (which is a \( N_1 \times N_2 \) off-diagonal block of a \( N \times N \) unitary scattering matrix). Finding \( \langle P \rangle = P_0 \text{tr } [tt^\dagger(1 - tt^\dagger)] \) amounts thus to an integration over the unitary group which is a quite complicated problem in general. Second, one can think of \( P_0 \) as a linear statistic on the transmission eigenvalues, so that \( \langle P \rangle = P_0 \int_0^1 dT \rho(T)T(1 - T) \) is provided by the transmission eigenvalue density \( \rho(T) \). Unfortunately, the latter is explicitly known only in the above-mentioned limiting cases.

We follow below yet another route. Contrary to the density \( \rho(T) \), the joint probability distribution function \( P_0(\{T_p\}) \) of all transmission eigenvalues is known to have the following attractively simple form at arbitrary \( N_{1,2} \):

\[ P_0(\{T_p\}) = N_\beta^{-1} |\Delta(T)|^\beta \prod_{j=1}^{n} T_j^{(\beta/2)(|N_2 - N_1| + 1) - 1}, \]  

where \( \Delta(T) = \prod_{i<j}(T_i - T_j) \) is the Vandermonde determinant. The key idea is to appreciate a relation of (3) to the integral kernel of Selberg’s integral defined as follows:

\[ I(a, b, c, n) = \int_0^1 \cdots \int_0^1 |\Delta(T)|^{2c} \prod_{j=1}^{n} T_j^{a-1}(1 - T_j)^{-1} dT_j \]

\[ = \frac{\Gamma(1 + c + j c)\Gamma(a + j c)\Gamma(b + j c)}{\Gamma(1 + c)\Gamma(a + b + (n + j - 1)c)}, \]

with \( \Gamma(x) \) being the gamma function. This result [as well as Eqs. (3) and (2) below] holds generally for complex \( a, b \) and \( c \) with positive real parts. One readily sees that (3) corresponds to the following particular values of these parameters

\[ a = (\beta/2)(|N_2 - N_1| + 1), \quad b = 1, \quad \text{and} \quad c = \beta/2. \]

It is worth noting that at these values the second line of (4) provides us with the normalization constant \( N_\beta \).

Selberg’s integral can be seen as a multidimensional generalization of Euler’s beta function. Due to the specific structure of the integral kernel in (3) very useful recursion relations may be established for certain moments (see Ref. 25). In particular, the moments of the first type of the integral kernel in (3) provide us with the normalization constant \( N_\beta \).
In the case of uncorrelated electrons, electron transfer is modified in other regimes (e.g., in the crossover between Poisson process that results in the value $F = 1$ for other types of mesoscopic conductors and how it holds for other types of mesoscopic conductors and how it is customarily described by a different method. It would be interesting to understand whether such a relation is always suppressed more strongly in the symplectic case.

This result was derived earlier by a different method. Finally, along the same lines we arrive at expression for $\langle P \rangle = P_0 \langle T_1 \rangle$.

Comparing the average shot-noise power, conductance variance, and conductance variance, one immediately finds the following relationship between them at arbitrary $N_{1,2}$:

$$\frac{2G_0 \langle P \rangle \langle G \rangle}{\beta \overline{P_0 \text{var}(G)}} = N_1 N_2.$$  

It would be interesting to understand whether such a relation holds for other types of mesoscopic conductors and how it is modified in other regimes (e.g., in the crossover between ensembles).

We proceed now with the discussion of the obtained results. In the case of uncorrelated electrons, electron transfer is a Poisson process that results in the value $P_0 = 2e\langle I \rangle = P_0 \langle G \rangle$ for the mean power. The suppression of the actual noise with respect to this Poisson value is customarily described by the Fano factor $F = \langle P \rangle / P_0$. One finds from (2) and (3) that

$$F = \frac{(N_1 - 1 + 2/\beta)(N_2 - 1 + 2/\beta)}{(N - 2 + 2/\beta)(N - 1 + 4/\beta)},$$  

(11)

at arbitrary $N_{1,2}$. In the semiclassical limit of large number of channels, $N_{1,2} \gg 1$, one readily gets from (11)

$$F \approx N_1 N_2 / N^2 - (1 - \frac{2}{\beta}) (N_1^2 - N_1 N_2 + N_2^2) / N^3 ,$$

i.e. the known classical value and the first weak-localization correction.

In the symmetric case, $N_1 = N_2 = n$, Eq. (11) reduces to

$$F = \frac{(n - 1 + 2/\beta)^2}{(2n - 2 + 2/\beta)(2n - 1 + 4/\beta)}.$$  

(12)

The Fano factor starts from the value $2/\beta + 1$ for $\beta = 1, 2, 4$, respectively at $n = 1$ and tends to the classical value $1/4$ as $n \to \infty$. For the orthogonal or unitary ensemble ($\beta = 1$ or 2) this is a monotonic decrease in $n$, whereas for symplectic ensemble ($\beta = 4$) $F$ has a minimum $\approx 0.225$ at $n \approx 2$. Figure 1 illustrates these dependencies. The shot noise is always suppressed more strongly in the symplectic case. In the general case of asymmetric cavities, it is instructive to consider the Fano factor at given fixed number $N_1$ of channels in one lead as a function of the channel number $N_2$ in the other lead. One easily finds from (11) (see Fig. 2) that $F$ starts from the value $\frac{2}{\beta + 1} N_2$ at $N_2 = 1$ and then develops a maximum at $N_2 = \sqrt{(N_1 - 1)(N_1 + 2/\beta) + 1 - 2/\beta}$, taking the following value at the maximum:

$$F^*_{\text{max}} = \frac{N_1 + 2/\beta - 1}{2\sqrt{(N_1 - 1)(N_1 + 2/\beta) + 2N_1 + 2/\beta - 1}}.$$  

(13)

As $N_2$ grows further, $F$ decreases down to zero according to $F \approx (N_1 - 1 + 2/\beta)/N_2$. This fact could be understood qualitatively: the lead with $N_2 \gg 1$ becomes almost classical with a deterministic transport through it that suppresses fluctuations of $T_p$, thus $P \to 0$. Such a suppression of the shot noise in strongly asymmetric cavities was indeed observed in the

![FIG. 1: The Fano factor as a function of the channel number in symmetric ($N_1 = N_2 = n$) chaotic cavities of different RMT ensembles.](image1)

![FIG. 2: The Fano factor at fixed number $N_1$ of channels in one lead and varied one $N_2$ in the other lead. The value at the maximum at $N_2 \approx N_1$ is close to $F^*_{\text{max}} \approx \frac{2}{\beta + 1} (1 - \frac{2}{\beta}) N_1^{-1}$ if $N_1 \gg 1$. Plotted is the result for cavities with time-reversal symmetry ($\beta = 1$).](image2)
recent experiment.\textsuperscript{15} [We note, however, that this experiment deals with asymmetric cavities when both $N_{1,2}$ are large, their ratio $\eta = N_2/N_1$ being varied. In this case the classical result $F = \eta/(1 + \eta)^2$ applies.]

In summary, we have exactly calculated the average shot noise power at an arbitrary number of open channels by relating the problem to Selberg’s integral. The proposed method is not restricted by linear statistics only and may be applied further to study, e.g., higher-order charge fluctuations as well as the whole distribution of shot-noise power. It would be highly interesting to check experimentally the predicted finite $N$ behavior of the Fano factor.

Recently, we became aware of the related study by Bulgakov et al.\textsuperscript{29} done at $N_1 = N_2$ and $\beta = 1, 2$. We thank V. Gopar for this communication.

We are grateful to P. Braun, F. Haake, S. Heusler and S. Müller for useful discussions. The financial support by the SFB/TR 12 of the DFG is acknowledged with thanks.

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