ON THE LUSTERNIK-SCHNIRELMANN CATEGORY OF PEANO CONTINUA

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Abstract. We define the LS-category \( \text{cat}_g \) by means of covers of a space by general subsets, and show that this definition coincides with the classical Lusternik-Schnirelmann category for compact metric ANR spaces. We apply this result to give short dimension theoretic proofs of the Grossman-Whitehead theorem and Dranishnikov's theorem. We compute \( \text{cat}_g \) for some fractal Peano continua such as Menger spaces and Pontryagin surfaces.

1. Introduction

We recall that the Lusternik-Schnirelmann category \( \text{cat} \) of a topological space \( X \) is the smallest integer \( k \) such that \( X = \bigcup_{i=0}^{k} A_i \), where each \( A_i \) is an open set contractible in \( X \). A set \( A \subset X \) is said to be contractible in \( X \) if its inclusion map \( A \to X \) is homotopic to the constant map. It is known that for absolute neighborhood retracts (ANR spaces) the sets \( A_i \) can be taken to be closed. In this paper we investigate what happens if the \( A_i \) are arbitrary subsets. Thus, we consider the following:

1.1. Definition. For any space \( X \), define the general LS-category \( \text{cat}_g \) of \( X \) to be the smallest integer \( k \) such that \( X = \bigcup_{i=0}^{k} A_i \), where each \( A_i \) is contractible in \( X \).

We apply this definition to Peano continua, i.e., path connected and locally path connected compact metric spaces. On these spaces, \( \text{cat}_g \) has the same upper and lower bounds as \( \text{cat} \), i.e.,

\[
\text{cup-length}(X) \leq \text{cat}_g X \leq \text{dim} X.
\]

Moreover, the Grossman-Whitehead theorem (Corollary 4.3) and Dranishnikov's theorem (Corollary 4.11) can be proven for \( \text{cat}_g \), using dimension theoretic arguments. This allows us to compute \( \text{cat}_g \).
for fractal spaces such as the Sierpinski carpet, Menger spaces and Pontryagin surfaces.

We show that \( \text{cat}_g X = \text{cat} X \) for compact metric ANRs, which yields new short proofs of the Grossman-Whitehead theorem and Dranishnikov’s theorem.

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3. LS-CATEGORY FOR GENERAL SPACES

The following result is known.

3.1. Proposition. Let \( X \) be a metric space, \( A \) a subset of \( X \) and \( V' = \{ V'_i \}_{i \in I} \) a cover of \( A \) by sets open in \( A \). Then \( V' \) can be extended to a cover \( V = \{ V_i \}_{i \in I} \) of \( A \) by sets open in \( X \) with the same nerve and such that \( V_i \cap A = V'_i \) for all \( i \in I \).

Proof. Let
\[
V_i = \bigcup_{a \in V'_i} B(a, d(a, A - V'_i)/2),
\]
and let \( V = \{ V_i \} \). Clearly \( V \) is an extension of \( V' \) and \( V_i \cap A = V'_i \). We claim that \( V_i \cap \cdots \cap V_{i_k} \neq \emptyset \) if and only if \( V'_i \cap \cdots \cap V'_{i_k} \neq \emptyset \). We prove the claim only for \( k = 2 \), since we apply it only in this case. A similar proof holds for \( k > 2 \).

Thus, we show that if \( V_i \cap V_j \neq \emptyset \), then \( V'_i \cap V'_j \neq \emptyset \). If \( x \in V_i \cap V_j \), then there exist \( a_i \in V'_i \) and \( a_j \in V'_j \) for which \( d(x, a_i) < d(a_i, A - V'_i)/2 \) and \( d(x, a_j) < d(a_j, A - V'_j)/2 \). Suppose \( d(a_i, A - V'_i) = \max\{d(a_i, A - V'_i), d(a_j, A - V'_j)\} \). Then \( d(a_i, a_j) < d(a_i, A - V'_i) \), so \( a_j \in V'_{i_j} \), which means that \( a_j \in V'_i \cap V'_j \).

We refer to [Hu] for the definitions of absolute neighborhood retracts (ANRs) and absolute neighborhood extensors (ANEs).

3.2. Theorem. [Hu] A metrizable space \( X \) is an ANR for metrizable spaces iff it is an ANE for metrizable spaces.

The following is a version of a lemma that appears in [Wa].

3.3. Theorem (Walsh Lemma). Let \( X \) be a separable metric space, \( A \) a subset of \( X \), \( K \) a metric separable ANR, and \( f : A \to K \) a map. Then, for any \( \epsilon > 0 \), there is an open set \( U \supset A \) and a map \( g : U \to K \) such that:

(1) \( g(U) \) is contained in an \( \epsilon \)-neighborhood of \( f(A) \)
is an open neighborhood  

We use the fact that any Polish space is homeomorphic to a closed subspace of a Hilbert space $H$. Since $K$ is an ANR, there is an open neighborhood $O$ of $K$ in $H$, and a retraction $r : O \to K$. For every $y \in K$, pick $\delta_y > 0$ such that:

(i) $B(y, 2\delta_y) \subset O$,

(ii) For all $y_1, y_2 \in B(y, 2\delta_y)$, we have $d(r(y_1), r(y_2)) < \epsilon$.

For each $a \in A$, pick a neighborhood $U_a$ of $a$ that is open in $A$ so that $f(U_a) \subset B(f(a), \delta_{f(a)})$. Let $\mathcal{V}' = \{V_i'\}$ be a locally finite refinement of the collection of $U_a$, and $\mathcal{V} = \{V_i\}$ the cover obtained by applying Proposition 3.1 to $\mathcal{V}'$. Let $U = \bigcup_i V_i$.

For each $i$, fix $a_i \in V_i'$. Let $\{f_i : V_i \in \mathcal{V}\}$ be a partition of unity subordinate to $\mathcal{V}$. Define $h : U \to H$ by $h(u) = \sum_i f_i(u) f(a_i)$.

Choose any $u \in U$. Then $u$ lies in precisely $k$ of the $V_i$, say in $V_{i_1}, \ldots, V_{i_k}$. Assume that $\delta_{f(a_{i_j})} = \max\{\delta_{f(a_{i_1})}, \ldots, \delta_{f(a_{i_k})}\}$. Then all the $f(a_{i_j})$ lie in $B(f(a_{i_j}), 2\delta_{f(a_{i_j})})$, so $h(u)$ lies in this ball too. This means that $h$ is a map from $U$ to $O$.

Define $g : U \to K$ by $g = r \circ h$. If $u \in U$, then we have seen that $h(u)$ lies in $B(f(a_u), 2\delta_{f(a_u)})$ for some $a_u \in A$, so $d(g(u), f(a_u)) < \epsilon$, which implies that $g(U) \subset N_{\epsilon}(f(A))$. Note here that we could have taken the $V_i$ to have as small a diameter as required, so the distance between $u$ and $a_u$ can be made as small as necessary. This will be made use of in Lemma 3.7.

For every $a \in A$, $h(a)$ and $f(a)$ lie in some convex ball, so $h|_A$ is homotopic to $f$ in $O$ via the straight line homotopy. The composition of this homotopy with $r$ is then a homotopy in $K$ between $g|_A$ and $f$.

$\square$

3.4. Theorem. Let $K$ be a compact ANE. Then there exists a constant $\epsilon(K) > 0$ such that for any metric space $X$ and maps $f, g : X \to K$, if $d(f(x), g(x)) < \epsilon(K)$ for all $x \in X$, then $f$ is homotopic to $g$.

3.5. Proposition. Let $K$ be a compact ANE, $y_0 \in K$ and $PK$ the path space $\{\phi : [0, 1] \to K | \phi(1) = y_0\}$ with sup norm $D$. If $\epsilon(K)$ is as in Theorem 3.4, then any two maps $f, g : X \to PK$ such that $D(f(x), g(x)) < \epsilon(K)$ for all $x \in X$, are homotopic to each other.

Proof. Since $PK$ is endowed with the sup norm, we have

$$d(f(x)(t), g(x)(t)) < \epsilon(K)$$

for every $x \in X, t \in [0, 1]$. Define $F : X \times I \to K$ by $F(x, t) = f(x)(t)$ and $G : X \times I \to K$ by $G(x, t) = g(x)(t)$. 

Since \( d(F(x,t), G(x,t)) < \epsilon(K) \) for all \((x,t) \in X \times I\), there exists a homotopy \( h_s : X \times I \to K \) between \( F \) and \( G \). Define \( \tilde{h}_s : X \to PK \) by \( \tilde{h}_s(x) = h_s(x,t) \). Then \( \tilde{h}_s \) is the required homotopy between \( f(x) \) and \( g(x) \).

The following is well known.

3.6. Proposition. If \( K \) is an ANE, so is \( PK \).

Proof. Suppose \( K \) is an ANE. Let \( A \) be a closed subspace of a metric space \( X \) and \( f : A \to PK \) a map. Define \( F : A \times I \to K \) by \( F(x,t) = f(x)(t) \). By hypothesis, \( F \) extends over an open neighborhood \( V \) of \( A \times I \) in \( X \times I \). For each \( a \in A \), find an open neighborhood \( U_a \) such that \( U_a \times I \subset V \). Let \( U = \bigcup_{a \in A} U_a \). The map \( \tilde{f} : U \to PK \) given by \( \tilde{f}(x)(t) = \tilde{F}(x,t) \) is the required extension.

3.7. Lemma. Let \( X \) be a compact metric space, \( A \subset X \), \( K \) a compact metric ANR, and \( f : X \to K \) a map such that the restriction of \( f \) to \( A \) is nullhomotopic. Then there exists \( U \supset A \) open in \( X \) such that the restriction of \( f \) to \( U \) is nullhomotopic.

Proof. Since \( K \) is compact, there is an \( \epsilon \) such that any two \( \epsilon \)-close maps to \( K \) are homotopic. Let \( D \) be the metric on \( K \) and \( d \) the metric on \( X \). Let \( PK \) be the path space \( \{ \phi : [0,1] \to K | \phi(1) = y_0 \} \) (for some \( y_0 \in K \)), under the sup metric \( D' \). As \( f|_A \) is nullhomotopic, there is a map \( F : A \to PK \) satisfying \( F(a)(0) = f(a) \) for all \( a \in A \). By the uniform continuity of \( f \), there is a \( \delta > 0 \) such that \( d(x,y) < \delta \Rightarrow D(f(x), f(y)) < \frac{\epsilon}{2} \).

By Proposition 3.6 and Theorem 3.2, \( PK \) is an ANR for metric spaces. By Proposition 3.5, any two \( \epsilon \)-close maps to it are homotopic. We apply Theorem 3.3 to \( F \), and construct an open neighborhood \( U \) of \( A \) and a map \( G : U \to PK \) such that for every \( u \in U \), there exists \( a_u \in A \) such that \( D'(G(u), F(a_u)) < \frac{\epsilon}{2} \). As noted in the proof of Theorem 3.3 we can construct \( U \) so that \( \text{diam}V_i < \delta \) for all \( i \), so \( d(u,a_u) < \delta \). Let \( g : U \to K \) be given by \( g(u) = G(u)(0) \). For any \( u \in U \), we have

\[
D(g(u), f(u)) \leq D(g(u), f(a_u)) + D(f(a_u), f(u)) \\
\leq D(G(u)(0), F(a_u)(0)) + \frac{\epsilon}{2} \\
\leq D'(G(u), F(a_u)) + \frac{\epsilon}{2} \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
It follows that 
\[ f\big|_U \text{ is homotopic to } g. \]

But \( h_t : U \to K \) given by \( h_t(u) = G(u)(t) \) is a homotopy between \( g \) and the constant map at \( y_0 \), so \( f\big|_U \) is nullhomotopic. \( \square \)

3.8. **Corollary.** Let \( X \) be a compact metric ANR, and let \( A \subset X \) be contractible in \( X \). Then there exists an open set \( U \supset A \) that is contractible in \( X \).

3.9. **Theorem.** For a compact metric ANR \( X \), \( \text{cat}_g X = \text{cat} X \).

**Proof.** Clearly \( \text{cat}_g X \leq \text{cat} X \), and equality holds if \( \text{cat}_g X \) is infinite.

Suppose \( \text{cat}_g X = n \). Then we can write \( X = \bigcup_{i=0}^{n} A_i \), where the \( A_i \) are contractible in \( X \). By Corollary 3.8 there exist open sets \( U_i \) containing \( A_i \) that are contractible in \( X \) for each \( i \). It follows that \( \text{cat} X \leq n \). \( \square \)

The following proposition illustrates some of the basic properties of \( \text{cat}_g \). The proofs are nearly identical to those in the case of the usual definition of category (see [CLOT]), and so are omitted.

3.10. **Proposition (Properties of \( \text{cat}_g \)).** If \( A, B, X, Y \) are spaces and \( f : X \to Y \) is a map with mapping cone \( C_f \), then the following hold:

i) \( \text{cat}_g(A \cup B) \leq \text{cat}_g A + \text{cat}_g B + 1 \)

ii) If \( f \) has a right homotopy inverse, then \( \text{cat}_g X \geq \text{cat}_g Y \)

iii) If \( f \) is a homotopy equivalence, then \( \text{cat}_g X = \text{cat}_g Y \)

iv) \( \text{cat}_g(C_f) \leq \text{cat}_g Y + 1 \).

The following proposition shows that the sets in the definition of the LS-category can be assumed to be \( G_\delta \).

3.11. **Proposition.** Let \( X \) be a complete metric space. If \( A \subset X \) is contractible in \( X \), then there exists a \( G_\delta \) set \( B \) with \( A \subset B \subset X \) such that \( B \) is contractible in \( X \).

The proof is based on the following classical theorem [E2]:

3.12. **Theorem (Lavrentieff).** If \( Y \) is a complete metric space, then any continuous map \( f : A \to Y \), where \( A \) is a dense subset of a space \( X \), can be extended to \( F : B \to Y \), where \( B \) is a \( G_\delta \) set in \( X \) containing \( A \).

**Proof of Proposition 3.11.** Consider \( A \) as a subspace of its closure \( \bar{A} \). Since \( A \) is contractible in \( X \), there is a map \( f : A \to PX \) satisfying \( f(a)(0) = a \). By Theorem 3.12 \( f \) can be extended to \( F : B \to PX \), where \( B \) is a \( G_\delta \) set in \( \bar{A} \) containing \( A \). Since \( B \) is clearly also \( G_\delta \) in \( X \), it only remains to show that it is contractible in \( X \).

Pick any \( b \in B \). Then \( b \) is the limit of some sequence \( \{a_n\} \) in \( A \), so
\[
F(b)(0) = F\left( \lim_{n \to \infty} a_n \right)(0) = \lim_{n \to \infty} F(a_n)(0) = \lim_{n \to \infty} f(a_n)(0) = \lim_{n \to \infty} a_n = b,
\]
so $B$ is contractible in $X$. 

4. Upper bounds

We need the following definitions for the Grossman-Whitehead theorem. An absolute extensor in dimension $k$, or $AE(k)$, is a space $X$ with the property that for every space $Z$ with $\dim Z \leq k$ and closed subset $Y \subset Z$, any map $f : Y \to X$ can be extended over all of $Z$. A $k$-connected or $C^k$ space is a space whose first $k$ homotopy groups are trivial. A locally $k$-connected or $LC^k$ space is a space $X$ with the property that for every $x \in X$ and neighborhood $U$ of $x$, there is a neighborhood $V$ with $x \in V \subset U$ such that every map $f : S^r \to V$ is nullhomotopic in $U$ for $r \leq k$.

4.1. **Theorem** (Kuratowski). For $k \geq 0$, a metrizable space $X$ is $AE(k + 1)$ for metrizable spaces iff it is $LC^k$ and $C^k$.

We use the notation $CX$ for the cone over $X$, 

$$CX = (X \times [0, 1])/(X \times \{1\}).$$

4.2. **Theorem.** For $k \geq 0$, let $X$ be an $LC^k$ and $C^k$ compactum. Then $\text{cat}_g X \leq \dim X/(k + 1)$.

**Proof.** Suppose $\dim X = n = p(k + 1) + r$, where $0 \leq r < k + 1$. We can write $X = \bigcup_{i=0}^{p(k+1)+r} X_i$, where $\dim X_i = 0$ for each $i$. Let $A_i = X_{(k+1)i} \cup X_{(k+1)i+1} \cup \ldots \cup X_{(k+1)(i+1)-1}$ for $i = 0, \ldots, p - 1$, and let $A_p = X_{p(k+1)} \cup \ldots \cup X_{p(k+1)+r}$. Then $\dim A_i \leq k$ for each $i$, so $\dim CA_i \leq k + 1$.

For each $i$, consider the inclusion map $A_i \hookrightarrow X$. Since $X$ is $k$-connected and locally $k$-connected, $X$ is an $AE(k + 1)$ space, and so the inclusion maps extend over $CA_i$ for each $i$. Hence we have $X = \bigcup_{i=0}^{p} A_i$, where each $A_i$ is contractible in $X$, which implies that $\text{cat}_g X \leq p \leq \dim X/(k + 1)$.

4.3. **Corollary** (Grossman-Whitehead Theorem). For a $k$-connected complex $X$, 

$$\text{cat} X \leq \dim X/(k + 1).$$

4.4. **Example.** For the $n$-dimensional Menger space $\mu^n$,

$$\text{cat}_g \mu^n = 1.$$ 

**Proof.** The $n$-dimensional Menger space is $(n - 1)$-connected and $(n - 1)$-locally connected [Be], so $\text{cat}_g \mu_n \leq 1$. Since $\mu^n$ is not contractible, $\text{cat}_g \mu_n = 1$. 

□
4.5. **Definition.** i) A family \(\{A_i\}\) of subsets of \(X\) is called an \(n\)-cover if every subfamily of \(n\) sets forms a cover: \(X = A_{i_1} \cup \cdots \cup A_{i_n}\).

ii) Given an open cover \(U\) of \(X\) and a point \(x \in X\), the order of \(U\) at \(x\), \(\text{Ord}_x U\), is the number of elements of \(U\) that contain \(x\).

We will make use of the following result that appears in [Dr2]:

4.6. **Proposition.** A family \(U\) that consists of \(m\) subsets of \(X\) is an \((n+1)\)-cover of \(X\) if and only if \(\text{Ord}_x U \geq m - n\) for all \(x \in X\).

**Proof.** See [Dr1, Dr2]. \(\square\)

4.7. **Theorem.** [Os] For every \(m > n\), every \(n\)-dimensional compactum \(X\) admits an \((n+1)\)-cover by \(m\) 0-dimensional sets.

**Proof.** This result follows from a slight modification to a proof given in [Os].

Since \(\dim X \leq n\), \(X\) can be decomposed into 0-dimensional sets as \(X = X_0 \cup \ldots \cup X_n\). We assume that the \(X_i\) are \(G_\delta\) sets [El Theorem 1.2.14], and proceed inductively. For any \(m > n + 1\), suppose an \((n+1)\)-cover \(\{X_0, ..., X_{m-1}\}\) consisting of 0-dimensional \(G_\delta\) sets has been constructed. Let

\[X_m = \{x \in X| x \text{ lies in exactly } (m - n) \text{ of the } X_i\}\]

Since the \(X_0, ..., X_{m-1}\) form an \((n+1)\)-cover, Proposition [El] implies that each \(x \in X\) lies in at least \(m - n\) of the \(X_i\). Then \(X_m\) is the complement in \(X\) of a finite union of finite intersections of \(G_\delta\) sets, and is therefore \(F_\sigma\). Similarly, for \(0 \leq i \leq m - 1\), each \(X_m \cap X_i\) is a 0-dimensional set that is \(F_\sigma\) in \(X\), and therefore in \(X_m\). As the finite union of 0-dimensional \(F_\sigma\) sets, \(X_m\) is also 0-dimensional [El Corollary 1.3.3]. It is also clear from the construction of \(X_m\) that \(\{X_0, ..., X_m\}\) is an \((n+1)\)-cover of \(X\). \(\square\)

4.8. **Corollary.** For every \(m > \lfloor n/2 \rfloor\) every \(n\)-dimensional compactum admits an \((\lfloor n/2 \rfloor + 1)\)-cover by \(m\) 1-dimensional sets.

**Proof.** Decompose \(X\) into 0-dimensional sets \(X_0, ..., X_n\) as before, and group these into pairs to get (at most) 1-dimensional \(G_\delta\) sets \(Y_0, ..., Y_{\lfloor n/2 \rfloor}\) that cover \(X\).

We proceed by induction again. For any \(m > \lfloor n/2 \rfloor\), suppose an \((\lfloor n/2 \rfloor + 1)\)-cover \(\{Y_0, ..., Y_{m-1}\}\) consisting of 1-dimensional \(G_\delta\) sets has been constructed. Let \(Y_m\) be the \(F_\sigma\) set \(\{x \in X| x \text{ lies in exactly } (m - \lfloor n/2 \rfloor) \text{ of the } Y_i\}\). Each \(Y_m \cap Y_i\) is \(F_\sigma\) of dimension \(\leq 1\), so \(\dim Y_m \leq 1\) [El Theorem 1.5.3], and \(\{Y_0, ..., Y_m\}\) is the desired \((\lfloor n/2 \rfloor + 1)\)-cover of \(X\). \(\square\)
The following lemma can be traced back to Kolmogorov (see [Os], [Dr2, Proof of Theorem 3.2]).

4.9. Lemma. Let $A_0, \ldots, A_{m+n}$ be an $(n+1)$-cover of $X$ and $B_0, \ldots, B_{m+n}$ an $(m+1)$-cover of $Y$. Then $A_0 \times B_0, \ldots, A_{m+n} \times B_{m+n}$ is a cover of $X \times Y$.

We recall that the geometric dimension $gd(\pi)$ of a group $\pi$ is defined as the minimum dimension of all Eilenberg-Maclane complexes $K(\pi, 1)$. It is known that $gd(\pi)$ coincides with the cohomological dimension of the group, $cd(\pi)$, for all groups with $gd(\pi) \neq 3$ [Br]. The Eilenberg-Ganea conjecture asserts that the equality $gd(\pi) = cd(\pi)$ holds true for all groups $\pi$.

The following theorem was proven by Dranishnikov [Dr2] for CW complexes. We present here a new short proof based on his idea to use the general LS-category.

4.10. Theorem. Let $X$ be a semi-locally simply connected Peano continuum. Then

$$\text{cat}_g X \leq gd(\pi_1(X)) + \frac{\dim X}{2}.$$ 

Proof. The conditions imply that $X$ has the universal covering space $p: \tilde{X} \to X$. Let $\pi = \pi_1(X)$ and let $q: E \to K(\pi, 1)$ be the universal cover. Note that the orbit space $\tilde{X} \times \pi E$ under the diagonal action of $\pi$ on $\tilde{X} \times E$ has the projections $p_1: \tilde{X} \times \pi E \to X$ and $p_2: \tilde{X} \times \pi E \to K(\pi, 1)$ which are locally trivial bundles. Since $E$ is contractible, $p_1$ admits a section $s$, so by Proposition 3.10, $\text{cat}_g X \leq \text{cat}_g(\tilde{X} \times \pi E)$.

The projection $p \times q: \tilde{X} \times E \to X \times K(\pi, 1)$ is the projection onto the orbit space of the action of the group $\pi \times \pi$. Therefore, it factors through the projection $\xi: \tilde{X} \times E \to \tilde{X} \times \pi E$ of the orbit action of the diagonal subgroup $\pi \subset \pi \times \pi$, $p \times q = \psi \circ \xi$ as follows:

$$\tilde{X} \times E \xrightarrow{\xi} \tilde{X} \times \pi E \xrightarrow{\psi} X \times K(\pi, 1).$$

Let $\dim X = n$ and $\dim(K(\pi, 1)) = gd(\pi) = m$. We apply Corollary 4.8 to $X$ and Theorem 4.7 to $K(\pi, 1)$ to obtain an $(\lceil n/2 \rceil + 1)$-cover $A_0, \ldots, A_r$ of $X$ by 1-dimensional sets and an $(m+1)$-cover $B_0, \ldots, B_r$ of $K(\pi, 1)$ by 0-dimensional sets, for $r = m + \lceil n/2 \rceil$. By Lemma 4.9, $A_0 \times B_0, \ldots, A_r \times B_r$ is a cover of $X \times K(\pi, 1)$ by 1-dimensional sets.

If $f: X \to Y$ is an open surjection between metric separable spaces such that the fiber $f^{-1}(y)$ is discrete for each $y \in Y$, then $\dim X = \dim Y$ [ET]. The map $\psi \circ \xi = (p, q)$ is an open map, each fiber of which is discrete. An open set in $\tilde{X} \times \pi E$ is taken to an open set in $\tilde{X} \times E$ by $\xi^{-1}$, which is taken to an open set in $X \times K(\pi, 1)$ by $\psi \circ \xi$, and so,
using the surjectivity of $\xi$, we can say that $\psi$ is open. Similarly, the
image under $\xi$ of any discrete set is still discrete, so $\psi^{-1}(y)$ is discrete
for every $y \in X \times K(\pi, 1)$. It follows that $\{\psi^{-1}(A_i \times B_i)\}_{i=0}^n$ is a cover
of $\tilde{X} \times_{\pi} E$ by 1-dimensional sets.

We show that each $\psi^{-1}(A_i \times B_i)$ is contractible in $\tilde{X} \times_{\pi} E$. This will
imply that
\[
\text{cat}_{g}(\tilde{X} \times_{\pi} E) \leq gd(\pi) + \lfloor n/2 \rfloor \leq gd(\pi) + \frac{\dim X}{2}.
\]

Note that $p_2(\psi^{-1}(A_i \times B_i)) = B_i$. Since $B_i$ is contractible to a point in
$K(\pi, 1)$, the set $\psi^{-1}(A_i \times B_i)$ can be homotoped to a fiber
$p_2^{-1}(x_0) \cong \tilde{X}$. Since $\tilde{X}$ is a simply connected and each $\psi^{-1}(A_i \times B_i)$ is 1-dimensional,
Theorem 4.1 implies that the inclusion map of each subspace can be
extended over its cone, and so each $\psi^{-1}(A_i \times B_i)$ can be contracted to
a point in $\tilde{X} \times_{\pi} E$. □

4.11. Corollary (Dranishnikov’s Theorem). For a finite CW complex
$X$,
\[
\text{cat} X \leq gd(\pi_1(X)) + \frac{\dim X}{2}.
\]

5. Lower bounds

5.1. Definition. Let $R$ be a commutative ring. The $R$-cup-length
$\text{cup-length}_R X$ of a space $X$ is the smallest integer $k$ such that all cup-
products of length $k+1$ vanish in the Čech cohomology ring $\tilde{H}^*(X; R)$.

5.2. Theorem. Let $X$ be a compactum with $\text{cat}_g X \leq m$. Then
\[
\text{cup-length}_R X \leq m
\]
for any ring $R$.

Proof. Let $\{A_i\}_{i=0}^m$ be as in Definition 4.1. Assume the contrary, i.e.,
that there exists a non-zero product $\alpha_0 \sim \alpha_1 \sim \ldots \sim \alpha_m$ in $\tilde{H}^*(X; R)$,
where $\alpha_i \in H^k(X; R)$, $k_i > 0$. For each $i$, there exists a function
$f_i : X \rightarrow K(R, k_i)$ such that $\alpha_i$ belongs to the homotopy class $[f_i]$
(in view of the isomorphism between the group of homotopy classes
$[X, K(R, n)]$ and the Čech cohomology group $\tilde{H}^n(X; R)$ [SP], and the
fact that the Čech cohomology agrees with the Alexander-Spanier co-
homology). Since $X$ is compact, for each $i$ there is a finite subcomplex
$K_i \subset K(R, k_i)$ such that $f_i(X) \subset K_i$. Since each $A_i$ is contractible
in $X$, $f_i|_{A_i} : A_i \rightarrow K_i$ is nullhomotopic. By Lemma 3.7, there ex-
sts an open neighborhood $U_i$ of $A_i$ such that $f_i|_{U_i} : U_i \rightarrow K_i$ is also
nullhomotopic.
Let \( j_i \) be the inclusion \( U_i \to X \), and \( q_i \) the map \( X \to (X, U_i) \). We consider the exact sequence of the pair \( (X, U_i) \) in the Alexander-Spanier cohomology (see [Sp], p. 308-309):

\[
\ldots \to H^k(X, U_i; R) \xrightarrow{j_i^*} H^k(X; R) \xrightarrow{q_i^*} H^k(U_i; R) \to \ldots
\]

Using, once more, the fact that the Alexander-Spanier cohomology coincides with the Čech cohomology on \( X \) and \( U_i \) and is, therefore, representable, we have, for each \( \alpha_i \), \( j_i^* (\alpha_i) = [f_i \circ j_i] = 0 \). By exactness, there is an element \( \bar{\alpha}_i \in H^k(X, U_i; R) \) satisfying \( q_i^* (\bar{\alpha}_i) = \alpha_i \).

The rest of the proof goes in the same vein as in the case of a CW complex [CLOT]. Namely, in view of the cup-product formula for the Alexander-Spanier cohomology (see [Sp], pp. 315),

\[
H^k(X, U; R) \times H^l(X, V; R) \xrightarrow{\cup} H^{k+l}(X, U \cup V; R),
\]

and the fact that \( H^n(X, X; R) = 0 \), we obtain a contradiction.

5.3. Example. Let \( \Pi_2 \) denote a Pontryagin surface constructed from the 2-sphere for the prime 2. Then \( \text{cat}_g \Pi_2 = 2 \).

Proof. We recall that \( \Pi_2 \) is the inverse limit of a sequence [Ku], [Dr4],

\[
L_1 \xleftarrow{p_2} L_2 \xleftarrow{p_3} L_3 \xleftarrow{p_4} \ldots
\]

where \( L_1 = S^2 \) with a fixed triangulation, each simplicial complex \( L_{i+1} \) is obtained from \( L_i \) by replacing every 2-simplex in the barycentric subdivision by the (triangulated) Möbius band, and the bonding map \( p_i \) sends this Möbius band back to the simplex. Thus each \( L_i, i > 1 \), is a non-orientable surface. It is well known that there is an \( \alpha \in H^1(L_2; \mathbb{Z}_2) \) with \( \alpha \sim \alpha \neq 0 \) [H]. Since for \( i > 1 \), \( (p_i)_* : H_1(L_i; \mathbb{Z}_2) \to H_1(L_{i-1}; \mathbb{Z}_2) \) is an isomorphism, this cup-product survives to the limit. Thus, \( \text{cup-length}_{\mathbb{Z}_2} \Pi_2 > 1 \). Since we also have \( \text{cat}_g \Pi_2 \leq \text{dim} \Pi_2 = 2 \), we must have \( \text{cat}_g \Pi_2 = 2 \). □

5.4. Question. Let \( D_2 \) be a Pontryagin surface constructed from the 2-disk. What is \( \text{cat}_g D_2 \) ?

5.5. Remark. The above computation works for all Pontryagin surfaces \( \Pi_p \) constructed from the 2-sphere, where \( p \) is any prime number. The cup-length estimate in the case \( p \neq 2 \) requires cohomology with twisted coefficients. Another approach to obtaining a lower bound for the category of \( \Pi_p \) is to use the category weight [CLOT]. Both approaches require substantial work.
References

[Be] M. Bestvina, Characterizing $k$-dimensional universal Menger compacta. Memoirs Amer. Math. Soc., 71, (1988), no. 330.

[B] K. Borsuk, Theory of Retracts. PWN, 1967.

[Br] K. Brown, Cohomology of groups. Springer, 1982.

[Ch] A. Chigogidze, Inverse spectra. North Holland, 1996.

[CLOT] O. Cornea, G. Lupton, J. Oprea, D. Tanré, Lusternik-Schnirelmann category. AMS, 2003.

[Dr1] A. Dranishnikov, The Lusternik-Schnirelmann category and the fundamental group. Algebr. Geom. Topol. 10 (2010), no. 2, 917-924.

[Dr2] A. Dranishnikov, On the Lusternik-Schnirelmann category of spaces with 2-dimensional fundamental group. Proc. of AMS. 137 (2009), no. 4, 1489-1497.

[Dr4] A. Dranishnikov, Homological dimension theory. Russian Math. Surveys 43 (1988), no. 4, 11 -63.

[E1] R. Engelking, Dimension Theory. North Holland Publishing Co., 1978.

[E2] R. Engelking, General Topology. Revised and completed edition Heldermann Verlag, 1989.

[Gr] D. P. Grossman, An estimation of the category of Lusternik-Schnirelman C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 1946.

[H] A. Hatcher, Algebraic Topology. Cambridge University Press 2002.

[Hu] S. T. Hu, Theory of Retracts. Wayne State Univ. Press, 1965.

[Ku] V. I. Kuzminov, Homological dimension theory. (Russian) Uspehi Mat. Nauk 23 1968 no. 5 (143), 3-49.

[Os] Ph. Ostrand, Dimension of metric spaces and Hilbert’s problem 13. Bull. Amer. Math. Soc. 71 1965, 619-622.

[Sp] E. Spanier, Algebraic Topology. McGraw-Hill Book Co., 1966.

[Wa] J. J. Walsh, Dimension, cohomological dimension, and cell-like mappings. Lecture Notes in Math., 870, Springer, 1981.

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