THE IVP FOR THE KURAMOTO-SIVASHINSKY EQUATION IN LOW REGULARITY SOBOLEV SPACES

ALYSSON CUNHA AND EDUARDO ALARCON

Abstract. In this work, we study the initial-value problem associated with the Kuramoto-Sivashinsky equation. We show that the associated initial value problem is locally and globally well-posed in Sobolev spaces $H^s(\mathbb{R})$, where $s > 1/2$. We also show that our result is sharp, in the sense that the flow-map data-solution is not $C^2$ at origin, for $s < 1/2$. Furthermore, we study the behavior of the solutions when $\mu \downarrow 0$.

1. Introduction

This paper is concerned with the initial-value problem (IVP) for the Kuramoto-Sivashinsky equation (KS)

$$
\begin{cases}
    u_t - \partial_x^2 u - \mu (1 - \partial_x^2)^{-1/2} u - \frac{1}{2}(\partial_x u)^2 = 0, & x \in \mathbb{R}, \ t \geq 0, \\
    u(x, 0) = \phi(x),
\end{cases}
$$

where $\mu > 0$ is a constant and $u$ is a real-valued function.

First we present a derivation of the equation (1.1). Indeed, an initial value problem equivalent to (1.1)

$$
\begin{cases}
    \partial_t H - D_C \partial_x^2 H - \frac{1}{2}(\partial_x H)^2 - \frac{\delta G}{8\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ik(x-x')} \frac{H(x', t)}{\sqrt{\frac{1}{4} + k^2}} \, dx' \, dk = 0, \\
    u(x, 0) = \phi(x),
\end{cases}
$$

where $D_C$ is dimensionless catalyst diffusivity, $\delta$ is relative density and $G$ is dimensionless acceleration of gravity, was derived by G. I. Sivashinsky, et al (19), to describe vertical propagation of chemical waves fronts in the presence instability due to density gradients (possibly thermally induced). Assuming an interaction region thin enough to be described as a surface ($z = H(x, y, t)$), where $H$ is the vertical position of the front, they use thermo-hydrodynamic equations in the regions of reacted fluid ($z < H(x, y, t)$) and unreacted fluid ($z > H(x, y, t)$) together with conservation of energy, matter and momentum to derive jump conditions on discontinuities at the interface. The equations governing these autocatalytic systems involving propagating reaction-diffusion fronts have been derived in (19), where they consider the reaction front to be very thin chemically, other assumption in use involves how the densities of the fluids change with temperature. Since the density changes due to thermal expansion of the fluids are small, write the density of the fluids to first order as $\rho(T) = \rho_1[1 - \alpha(T - T_1)]$, where $\rho(T)$ is the density at temperature $T$, $\rho_1$ is density at the reference temperature $T_1$ and $\alpha$ is the classical thermal expansion coefficient at constant pressure. The relative difference between the densities of these two fluids at the front is one of the key parameters in this study, and is defined by $\delta = \frac{\rho_a - \rho_b}{\rho_b}$, where $\rho_a$ and $\rho_b$ are the densities of the
fluid above the front (unreacted fluid) and of that below the front (reacted fluid), respectively. This is due to the fact that $\rho$ is dependent on the thermal diffusivity of the fluids, in (30) the diffusivity is assumed to be infinite.

As in (19), they obtain the following system of equations

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla P_r + \nu \nabla^2 V,$$

$$\nabla \cdot V = 0,$$

$$c = \hat{n} \cdot \frac{\partial H}{\partial t} - \hat{n} \cdot V|_{z=H},$$

where $V$ is fluid velocity, $z = H(x, y, t)$ is the vertical position of the front as a function of $x$, $y$ and $t$, $\hat{n}$ unit vector pointing normal to the front into the unreacted fluid, $c$ the normal front velocity with respect to the unreacted fluid and $\nu$ is kinematic viscosity.

Together with jump conditions across the interface between the reacted and unreacted fluids given by:

1. $[\hat{n} \cdot V]^\pm = 0,$
2. $[\hat{n} \times V]^\pm = 0,$
3. $[P_r]^\pm - [n_i n_j T_{ij}^V]^\pm = -\delta \rho g H,$
4. $[\epsilon_{ijk} n_j n_l T_{kl}^V]^\pm = 0,$

and viscous stress tensor $T_{ij}^V = -\nu \rho \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right)$, $P_r$ is the reduced pressure given by $P_r = P + \rho g z$ and $\epsilon_{ijk}$ is the totally antisymmetric tensor.

By making asymptotic expansions in the delta parameter of the variables $V$, $P_r$ and $H$, we obtain equation (1.3) (see 30).

As the usual, we are assuming the well-posedness in the Kato’s sense, that is, includes, existence, uniqueness, persistence property and smoothness of the map data-solution, see 14, 15, 22, 23 and 24. In 2 and 13 the authors, using the Banach fixed point theorem obtained the local and global well-posedness for the IVP (1.1). More precisely they proved the following theorems.

Theorem A (Local well-posedness) Let $s \geq 1$. Then for any $\phi \in H^s(\mathbb{R})$, there exists a positive $T = T_{\mu}(\|\phi\|_{H^s})$ and a unique solution $u \in C([0, T]; H^s(\mathbb{R}))$ of the IVP (1.1). Furthermore, the flow-map $\phi \mapsto u(t)$ is continuous in the $H^s$-norm.

Theorem B (Global well-posedness) The problem (1.1) is globally well posed in $H^s(\mathbb{R})$, for $s \geq 1$.

In this paper, we are mainly interested in improving the last two theorems. For this we look at the dissipative effect of the IVP (1.1) and we will use the same methods of Dix (16) (see also [7]). In general terms, the Dix method consists of an application of the fixed point theorem in a suitable time-weighted function space. Recently, many authors have used this technique, see, for example, Carvajal and Panthee [11, 12], Esfahani [17], Fonseca, Pastrán and Rodríguez-Blanco [8] and Pilot [27]. See also [10]. We observe that, in these works the highest order dissipative term of the equations, often is greater than or equal to three (unless in [12], which is greater than 5/2). In our work, the degree of the highest dissipation term is two, this show, in particular, that this technique is also useful when we have a low order of dissipation.
Our results are sharp in the sense that the flow-map data-solution, for the IVP (1.1), is not $C^2$ at origin, for $s < 1/2$. As it is well known, a consequence of this fact is that the Cauchy problem (1.1), for $s < 1/2$, cannot solve by a contraction argument on the integral equation (see [9], [25], [26], [29] and references therein).

Now we state the main results of this paper.

**Theorem 1.1.** (Local well-posedness). Let $\mu > 0$ and $s > 1/2$, then for all $\phi \in H^s(\mathbb{R})$, there exists $T = T(||\phi||_{H^s})$, a space $
abla^p \to C([0,T];H^s(\mathbb{R}))$

and a unique solution $u$ of (1.1) in $\nabla^p$. In addition, the flow map data-solution $S : H^s(\mathbb{R}) \to \nabla^p \cap C([0,T];H^s), \phi \mapsto u$

is smooth and

$u \in C((0,T];H^\infty(\mathbb{R}))$.

Moreover, if $s' > s$ then the solution with initial data $\phi \in H^{s'}(\mathbb{R})$ is defined in the same interval $[0,T]$, with $T = T(||\phi||_{H^{s'}})$.

**Theorem 1.2.** (Global well-posedness). Let $\mu > 0$ and $s > 1/2$, then the initial value problem (1.1) is globally well-posed in $H^s(\mathbb{R})$.

**Theorem 1.3.** (Ill-posedness). Let $s < 1/2$, if there exists some $T > 0$, such that the problem (1.1) is locally well-posed in $H^s(\mathbb{R})$, then the flow-map data solution $S : H^s(\mathbb{R}) \to C([0,T];H^s(\mathbb{R})), \phi \mapsto u$,

is not $C^2$ at zero.

An open problem about the IVP (1.1) is to investigate the existence of a global attractor (see [4], [5] and [6]). In view of the ideas in [1], [4], [6], [20] and [28] we believe that it’s possible to show the existence of the global attractor in $H^s(\mathbb{R})$, where $s > 1/2$. For a general theory about the global attractor, see [11] and [28].

Another interesting question would be to explore the well-posedness for the IVP (1.1) with bore-like data. That is, we deal with the problem (1.1), with $g$ instead of $\phi$, where $g$ satisfies

i) $g(x) \to C_\pm$ with $x \to \pm \infty$;

ii) $g' \in H^s$, for some $s > 1/2$;

iii) $(g - C_\pm) \in L^2([0,\infty))$ and $(g - C_\pm) \in L^2(-\infty,0])$;

iv) $\partial_x^{-1}g$ and $\partial_x^{-1}g' \in H^{3/2}(\mathbb{R})$.

and $\hat{\partial_x^{-1}g}(\xi) = \hat{g}(\xi) / i\xi$.

In particular, these results on bore-like data, would improve those obtained in [3]. More information on bore-like data, can be found in [22].

This paper is organized as follows. In the next section, we derive some preliminary estimates. The well-posedness for the IVP (1.1), for $s > 1/2$, is established in section 3. In section 4 we deal with the limit when $\mu \downarrow 0$. Finally, in section 5 we state the results about Ill-posedness for the IVP (1.1).
1.1. Notation. In this article, we use the following notation. We say $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$. By $a \sim b$ we mean that $a \lesssim b$ and $b \lesssim a$. We write $a \lesssim b$ when the constant depends on only parameter $l$. The Fourier transform of $f$, is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx.
\]
If $s \in \mathbb{R}$, $H^s := H^s(\mathbb{R})$ represents the nonhomogeneous Sobolev space defined as
\[
H^s(\mathbb{R}) = \{ f \in \mathcal{S}(\mathbb{R}) : \| f \|_{H^s} < \infty \},
\]
where
\[
\| f \|_{H^s} = \| \langle \xi \rangle^s \hat{f} \|_{L^2_\xi},
\]
and $\langle \xi \rangle = (1 + \xi^2)^{1/2}$.

In addition, we define the Bessel potential $J^s$ by
\[
(J^s f)^\wedge(\xi) = \langle \xi \rangle^s \hat{f}(\xi), \quad \text{for all } f \in \mathcal{S}(\mathbb{R}),
\]
hence $\| J^s f \|_{L^2_\xi} = \| f \|_{H^s}$.

In the rest of the paper, we will denote the $L^2$-norm in the $x$ variable by $\| \cdot \|_{L^2_x} := \| \cdot \|$.

2. Preliminary estimates

By defining
\[
\Phi(\xi) = -\xi^2 + \frac{\mu}{(1 + \xi^2)^{1/2}},
\]
the semigroup associated with the linear part of (1.1) is defined via Fourier transform by
\[
E_\mu(t) \psi = \left( e^{t \Phi(\xi)} \hat{\phi}(\xi) \right)^\vee,
\]
and the integral equation associated to (1.1)
\[
F_\mu(u)(t) := u(t) = E_\mu(t)\phi + \frac{1}{2} \int_0^t E_\mu(t - \tau)(\partial_x u)^2(\tau) d\tau.
\]

The following result is useful in establishing of estimates for the semigroup $E_\mu$.

Lemma 2.1. Let $\mu > 0$, $\lambda \geq 0$, $\nu \in \mathbb{R}$ and $t \in [0, T]$. Then
\[
\| \xi^{2\lambda} e^{t(-\xi^2 + \mu/(1 + \xi^2)^{1/2})} \|_{L^\infty_\xi} \leq e^{\mu T} \left( \frac{\lambda}{e} \right) ^{\lambda/\mu} t^{-\lambda}
\]
and
\[
\| \| \xi^\nu e^{-t\xi^2} \|_{L^2_\xi} = c_\nu t^{-\frac{\nu}{2} - \frac{1}{2}}.
\]

Proof. First we will establish (2.6). For this, note that for all $\xi \in \mathbb{R}$ and $t \in [0, T]$
\[
\xi^{2\lambda} e^{t \Phi(\xi)} \leq e^{\mu t} \xi^{2\lambda} e^{-t \xi^2}.
\]
Therefore, looking at the maximum value of function $(\cdot)^{2\lambda} e^{-t(\cdot)^2}$, we obtain the result.

About identity (2.7), by using the change of variables $\xi = t^{-1/2} w$
\[
\| \| \xi^\nu e^{-t\xi^2} \|_{L^2_\xi}^2 = t^{\nu - 1/2} \int w^{2\nu} e^{-2w^2} dw = c_\nu^2 t^{\nu - 1/2}.
\]
This finish the proof.
In the following, we deal with the well-posedness for the IVP (1.1) where \( s > 1/2 \).

First, we need a technical lemma, which will be useful in our linear estimates. This is a new version of Lemma 2.1 of [12].

**Lemma 2.2.** There exists \( M > 0 \) such that if \( |\xi| \geq M \), then
\[
\Phi(\xi) = -\xi^2 + \mu(\xi)^{-1} < -1
\]  
and
\[
|\Phi(\xi)| \geq \frac{\xi^2}{2}.
\]  

**Proof.** The inequalities (2.8) and (2.9) follows from
\[
\lim_{|\xi| \to \infty} \frac{\Phi(\xi)}{\xi^2} = -1
\]
and
\[
\lim_{|\xi| \to \infty} \frac{\mu(\xi)^{-1}}{\xi^2} = 0.
\]

The next lemma is a simple result about calculus.

**Lemma 2.3.** Let \( f(t) = t^\alpha e^{t^\beta}, \alpha > 0 \) and \( \beta < 0 \). Then, for all \( t \geq 0 \)
\[
f(t) \leq \left( \frac{\alpha}{|\beta|} \right)^\alpha e^{-\alpha}.
\]

**Proof.** See Lemma 2.3 of [12].

Next, we present the function spaces appropriated for to show the existence of a solution.

Let \( 0 < s < 1 \) and \( 0 < T \leq 1 \), then we define
\[
X^s_T = \{ u \in C([0, T]; H^s) : \| u \|_{X^s_T} < \infty \},
\]  
where
\[
\| u \|_{X^s_T} := \sup_{[0, T]} \left( \| u(t) \|_{H^s} + t^{\frac{1-s}{2}} \| \partial_x u(t) \|_{L^2} \right).
\]

In the following, we present the linear estimates in the spaces \( X^s_T \). The proof follows the same ideas contained in Lemma 2.6 of [12].

**Lemma 2.4.** Let \( \mu > 0 \), \( 0 < T \leq 1 \), \( s < 1 \), \( t \in [0, T] \) and \( \phi \in H^s(\mathbb{R}) \). Then
\[
\| E_\mu(t) \phi \|_{X^s_T} \leq C \| \phi \|_{H^s},
\]  
where \( C \) depends on \( s, \mu, T \) and \( M \), with \( M \) as in Lemma 2.2.

**Proof.** By the definition of semigroup \( E_\mu \), the first term in (2.11) can be estimated as follows
\[
\| E_\mu(t) \phi \|_{H^s} \leq \| e^{t\Phi(\xi)}(\xi)^s \| \\
\leq \| e^{t\Phi(\xi)} \|_{L^\infty(\xi)} \| \phi \|_{H^s} \\
\lesssim e^{tT} \| \phi \|_{H^s}.
\]  

\[\square\]
For estimate the second term of the $X^s_T$-norm, putting $\alpha = \frac{1-s}{2}$ and using the Plancherel identity, we have

\begin{align}
\int_0^t \| \partial_x E_\mu(t) \phi \|_{L^2} &= \int_0^t \| \xi e^{t\Phi(\xi)} \hat{\phi} \|_{L^2} \\
&= \int_0^t \| \xi \langle \xi \rangle^{-\alpha} e^{t\Phi(\xi)} \langle \xi \rangle\hat{\phi} \|_{L^2} \\
&\leq \int_0^t \| \xi \langle \xi \rangle^{-\alpha} e^{t\Phi(\xi)} \|_{L^\infty} \| \phi \|_{H^s}.
\end{align}

We can write

\begin{align}
\| \xi \langle \xi \rangle^{-\alpha} e^{t\Phi(\xi)} \|_{L^\infty} \leq \| \langle \xi \rangle^{1-s} e^{t\Phi(\xi)} \chi_{|\xi| \leq M} \|_{L^\infty} + \| \langle \xi \rangle^{1-s} e^{t\Phi(\xi)} \chi_{|\xi| \geq M} \|_{L^\infty} \\
:= C_M + I,
\end{align}

then

\begin{align}
\int_0^t \| \partial_x E_\mu(t) \phi \|_{L^2} \leq C_M.
\end{align}

By using Lemmas 2.3 and 2.16, with $\alpha = \frac{1-s}{2}$ and $\beta = \Phi(\xi)$, follows that

\begin{align}
t^\alpha I &= t^\alpha \| \langle \xi \rangle^{1-s} e^{t\Phi(\xi)} \chi_{|\xi| \geq M} \|_{L^\infty} \\
&\leq \| \langle \xi \rangle^{1-s} \left( \frac{\alpha}{\Phi(\xi)} \right)^\alpha e^{-\alpha \chi_{|\xi| \geq M}} \|_{L^\infty} \\
&\leq (2\alpha)^\alpha \| \langle \xi \rangle^{1-s} \chi_{|\xi| \geq M} \|_{L^\infty} \\
&\leq (2\alpha)^\alpha \| \langle \xi \rangle^{-\alpha} \chi_{|\xi| \geq M} \|_{L^\infty} \\
&\leq (2\alpha)^\alpha (M^{-2\alpha} + 1),
\end{align}

where above, we used

\begin{align}
\langle \xi \rangle^{1-s} \lesssim 1 + |\xi|^{1-s}.
\end{align}

Therefore, by (2.13)–(2.17), we conclude the proof. \hfill \Box

Next, we deduce some bilinear estimates useful to prove the Theorem 1.1.

**Proposition 2.5.** Let $\mu > 0$, $0 < T \leq 1$ and $1/2 < s < 1$. Then

\begin{align}
\left\| \int_0^t E_\mu(t-\tau)(\partial_x u \partial_x v)(\tau) d\tau \right\|_{X^s_T^\mu} \leq e^{\mu T} T^{\frac{s}{2} + \frac{s}{2}} \left\| u \right\|_{X^s_T} \left\| v \right\|_{X^s_T},
\end{align}

for all $u, v \in X^s_T$. 

**Proof.** Let $0 < t \leq T$, therefore by the definition of norms $X^s_T$, we obtain

\begin{align}
\tau^{\frac{1}{2} - s} \left\| \partial_x u \right\|_{L^2} \leq \left\| u \right\|_{X^s_T} \quad \text{and} \quad \tau^{\frac{1}{2} - s} \left\| \partial_x v \right\|_{L^2} \leq \left\| v \right\|_{X^s_T}.
\end{align}

Since $s > 0$, we see that $(1 + \xi^2)^{s/2} \lesssim 1 + |\xi|^s$, then

\begin{align}
\left\| \int_0^t E_\mu(t-\tau)(\partial_x u \partial_x v)(\tau) d\tau \right\|_{H^s} \leq \left\| \int_0^t E_\mu(t-\tau)(\partial_x u \partial_x v)(\tau) d\tau \right\|_{L^2} + \left\| \int_0^t E_\mu(t-\tau)(\partial_x u \partial_x v)(\tau) d\tau \right\|_{H^s}
\end{align}

:= A + B.
Therefore, by Young’s inequality for convolution and identity \(2.7\)

\[
B \leq \int_0^t \| \xi^s e^{-\xi(s-t)} (\partial_x u \partial_x v (\tau)) \|_{L_x^2} d\tau
\]

\[
\leq e^{\mu t} \int_0^t \| \xi^s e^{-\xi(s-t)} \|_{L_x^2} \| \tilde{\partial}_x u \ast \tilde{\partial}_x v \|_{L_x^2} d\tau
\]

(2.20)

\[
\lesssim s e^{\mu t} t^{-\frac{s}{2}} \int_0^1 (1 - \sigma)^{-\frac{s}{2}} \sigma^{s-1} d\sigma \| u \|_{X_s^1} \| v \|_{X_s^1},
\]

and

\[
A \leq e^{\mu t} \int_0^t \| e^{-\xi(s-t)} \|_{L_x^2} \| \tilde{\partial}_x u \ast \tilde{\partial}_x v \|_{L_x^2} d\tau
\]

(2.21)

\[
\lesssim e^{\mu t} t^{\frac{s}{2}} \int_0^1 (1 - \sigma)^{-\frac{s}{2}} \sigma^{s-1} d\sigma \| u \|_{X_s^1} \| v \|_{X_s^1}.
\]

In the above, again we used the change of variables \(\sigma = \frac{t}{\tau}\).

With respect to second norm in \(2.11\)

\[
\int_0^t \left\| \int_0^\tau E_\mu (t - \tau) \partial_x (uv) (\tau) d\tau \right\| \leq e^{\mu t} \int_0^t \int_0^\tau e^{\mu (t - \tau)} \| \partial_x u \partial_x v (\xi, \tau) \| d\tau d\tau
\]

\[
\lesssim e^{\mu t} \int_0^t \int_0^\tau e^{\mu (t - \tau)} \| \tilde{\partial}_x u \ast \tilde{\partial}_x v \| d\tau d\tau
\]

\[
\lesssim e^{\mu t} t^{-\frac{s}{2}} \int_0^t (t - \tau)^{-\frac{s}{2}} \| \partial_x u \| \| \partial_x v \| d\tau d\tau
\]

\[
\lesssim e^{\mu t} t^{-\frac{s}{2}} \int_0^t (t - \tau)^{-\frac{s}{2}} \tau^{s-1} d\tau \| u \|_{X_s^1} \| v \|_{X_s^1}
\]

(2.22)

\[
\lesssim e^{\mu t} t^{\frac{s}{2} + \frac{1}{2}} \int_0^1 (1 - \sigma)^{-\frac{s}{2}} \sigma^{s-1} d\sigma \| u \|_{X_s^1} \| v \|_{X_s^1}.
\]

Therefore, by \(2.19\)–\(2.22\) we obtain the proof.

The next result will be useful to obtain regularity of the solutions, in Theorem \(1.1\)

**Proposition 2.6.** Let \(\mu > 0\), \(0 < T \leq 1\), \(1/2 < s < 1\) and \(\lambda \geq 0\). If \(s + \lambda < 3/2\) and \(\lambda < s - 1/2\), then the application

\[
W_\mu : t \in [0, T] \mapsto \int_0^t E_\mu (t - \tau) (\partial_x u)^2 (\tau) d\tau \in H^{s+\lambda} (\mathbb{R}),
\]

(2.23)

is continuous.

**Proof.** Fixed \(0 \leq t_0 < t \leq T\), then

\[
W_\mu (t) - W_\mu (t_0) = \int_{t_0}^t (E_\mu (t - \tau) - E_\mu (t_0 - \tau)) (\partial_x u)^2 (\tau) d\tau +
\]

\[
+ \int_{t_0}^t E_\mu (t - \tau) (\partial_x u)^2 (\tau) d\tau
\]

(2.24)

\[
:= \psi_1 (t, t_0) + \psi_2 (t, t_0).
\]
Case a): \( s + \lambda \geq 0 \). In view of \((1 + \xi^2)^{\frac{s+\lambda}{2}} \lesssim 1 + |\xi|^{s+\lambda}\), we obtain

\[
\|\psi_2(t, t_0)\|_{H^{s+\lambda}} \leq \int_{t_0}^{t} \|E_\mu(t - \tau)(\partial_x u)^2(\tau)\|d\tau + \int_{t_0}^{t} \|E_\mu(t - \tau)(\partial_x u)^2(\tau)\|_{H^{s+\lambda}}d\tau
\]

\[= A(t, t_0) + B(t, t_0). \quad (2.25)\]

Then

\[
A(t, t_0) \leq e^{\mu T} \int_{t_0}^{t} \|e^{-(t-\tau)^2} \|_{L^2_x} \|\bar{\partial}_x u * \bar{\partial}_x u\|d\tau
\]

\[\lesssim e^{\mu T} \int_{t_0}^{t} (t - \tau)^{-1/4} \tau^{-1/2} d\tau \|u\|_{X_T^s}\]

\[\lesssim (t - t_0)^{s+1/4} e^{\mu T} \int_{t_0}^{1} (1 - \sigma)^{-1/4} \sigma^{s+1} d\sigma \|u\|_{X_T^s}^2 \rightarrow 0,
\]

with \( t \rightarrow t_0 \).

As for the other integral

\[
B(t, t_0) \lesssim e^{\mu T} \int_{t_0}^{t} \|\xi|^{s+\lambda} e^{-(t-\tau)^2} \|_{L^2_x} \|\bar{\partial}_x u * \bar{\partial}_x u\|d\tau
\]

\[\lesssim e^{\mu T} \int_{t_0}^{t} (t - \tau)^{-\frac{s+\lambda}{2} - \frac{1}{4}} \tau^{-1} d\tau \|u\|_{X_T^s}^2
\]

\[\lesssim e^{\mu T} (t - t_0)^{-\frac{s+\lambda}{2} - \frac{1}{4}} \int_{t_0}^{1} (1 - \tau)^{-\frac{s+\lambda}{2} - \frac{1}{4}} \tau^{-1} d\tau \|u\|_{X_T^s}^2 \rightarrow 0,
\]

with \( t \rightarrow t_0 \). Where, in the above arguments we used the change of variables \( \tau = t_0 + \sigma(t - t_0) \) and the inequality \( \tau^s \leq \sigma^s(t - t_0)^s \).

With respect to the first integral in (2.24)

\[
\|\psi_1(t, t_0)\|_{H^{s+\lambda}} \leq \|\psi_1(t, t_0)\| + \int_{t_0}^{t} \|\xi|^{s+\lambda} (E_\mu(t - \tau) - E_\mu(t_0 - \tau))(\partial_x u)^2(\tau)\|d\tau
\]

\[= E(t, t_0) + F(t, t_0). \quad (2.28)\]

Then from way analogous to the above case \( E(t, t_0) \rightarrow 0 \), with \( t \rightarrow t_0 \).

With respect to second integral

\[
F(t, t_0) \lesssim_{\mu, T} \left( \int_{t_0}^{t} \|h(t, t_0, \tau, \xi)\| \tau^{-s-1} d\tau \right) \|u\|^2_{X_T^{s}},
\]

where

\[h(t, t_0, \tau, \xi) := \|\xi|^{s+\lambda} \left( e^{-(t-\tau)^2} - e^{-(t_0-\tau)^2} \right)^2\]

Note that the function \( g \) above converges to zero, with \( t \) goes to \( t_0 \), for all \( \xi \in \mathbb{R} \). We also have by the inequality \( t - \tau > t_0 - \tau \) that \( |h(t, t_0, \tau, \xi)| \leq 2e^{-\sigma(t_0-\tau)^2} |\xi|^{s+\lambda} \in L^2_\xi \).

Then by the Lebesgue dominated convergence theorem

\[
\|h\|_{L^2_\xi} \rightarrow 0, \text{ with } t \rightarrow t_0.
\]
Moreover
\[ \|h(t, t_0, \tau, \xi)\|_{\dot{L}_2^s} \leq 2\|\xi\|_{\dot{H}^{s-\lambda}} e^{-(t_0-\tau)^{2/3}} \|\xi\|_{\dot{H}^{s-1}} \]
(2.29)

A new application of the Lebesgue dominated convergence theorem gives us
\[ F(t, t_0) \to 0, \text{ with } t \to t_0. \]

This concludes the proof of case a).

Case b): \( s + \lambda < 0 \). Since \( \dot{H}^{s+\lambda} \hookrightarrow H^{s+\lambda} \), we obtain

\[ \|\psi_2(t, t_0)\|_{H^{s+\lambda}} \lesssim e^{\mu T} \int_{t_0}^t \|\xi\|_{\dot{H}^{s+\lambda}} e^{-(t_0-\tau)^{2/3}} \|\xi\|_{L^2} \|\partial_x u \|_{L^2} d\tau \]
\[ \lesssim e^{\mu T} \int_{t_0}^t \|\xi\|_{\dot{H}^{s+\lambda}} e^{-(t_0-\tau)^{2/3}} \|u\|_{X_T^s}^2 d\tau \]
\[ \lesssim e^{\mu T} \int_{t_0}^t (t_0 - \tau)^{-\frac{2}{3}} \frac{d\tau}{\tau} \|u\|_{X_T^s}^2 \]
(2.30)

with \( t \to t_0 \).

Finally, we see that \( \psi_2(t, t_0) \) converges to zero in \( H^{s+\lambda} \)-norm, by way analogous to the term \( F(t, t_0) \), in (2.28).

Therefore, the proof of proposition is finalized. \( \square \)

**Remark 2.7.** Let \( s' > s > 1/2 \), then modifying the space \( X_T^{s'} \) by
\[ \dot{X}_T^{s'} = \{ u \in X_T^{s'} : \|u\|_{\dot{X}_T^{s'}} < \infty \} \]
with
\[ \|u\|_{\dot{X}_T^{s'}} = \|u\|_{X_T^{s'}} + \sup_{t \in [0, T]} \left( t^{s/2} \|J^{s-s} u(t)\|_{L^2} \right) \]
and using the fact that
\[ (1 + \xi^2)^{s'/2} \lesssim (1 + \xi^2)^{s'/2} (1 + \xi^2)^{(s'-s)/2} + (1 + \xi^2)^{s/2} (1 + (\xi_1)^2)^{(s'-s)/2}, \]
we obtain, from way similar to the Proposition 2.5,
\[ \left\| \int_0^t E_\mu(t - \tau) (\partial_x u \partial_x v) \tau d\tau \right\|_{\dot{X}_T^{s'}} \lesssim e^{\mu T} T^{s}(\|u\|_{\dot{X}_T^{s'}} \|v\|_{X_T^s} + \|u\|_{X_T^s} \|v\|_{\dot{X}_T^{s'}}). \]

**3. PROOF OF THEOREMS 1.1 AND 1.2**

**Proof of Theorem 1.1.** Let \( \mu > 0 \) and \( 1/2 < s < 1 \). Our strategy is to show that the operator \( F_\mu \) given by (2.25) is a contraction in some closed ball in \( X_T^s \). In fact, by (2.25)
\[ \|F_\mu(t)\|_{X_T^s} \lesssim \|E\|_{X_T^s} \|\phi\|_{X_T^s} + \left\| \int_0^t E_\mu(t - \tau) \partial_x u \partial_x v \tau d\tau \right\|_{X_T^s} \]
(3.31)

and
\[ \lesssim C_s, \|\phi\|_{H^s} + T^s \|u\|_{X_T^s}^2, \]
\[ \| F_\mu(u) - F_\mu(v) \|_{X^s_t} \leq \| \int_0^t E_\mu(t - \tau)((\partial_x u)^2 - (\partial_x v)^2)(\tau)d\tau \|_{X^s_t} \]
\[ = \| \int_0^t E_\mu(t - \tau)(\partial_x(u-v)\partial_x(u+v))(\tau)d\tau \|_{X^s_t} \leq C_s,\mu T^{\delta(s)}\| u - v \|_{X^s_t}\| u + v \|_{X^s_t}, \]

for all \( u, v \in X^s_t \) and \( 0 < T \leq 1 \).

Therefore, given \( R = 2C_s,\mu \| \phi \|_{H^s} \) we define
\[ X^s_t(R) = \{ u \in X^s_t; \| u \|_{X^s_t} \leq R \}. \] (3.33)

Then taking
\[ 0 < T < \min \left\{ \left( 4C_s^2,\mu \| \phi \|_{H^s} \right)^{-1/\delta(s)}, 1 \right\}, \] (3.34)

the estimates (3.31) and (3.32) implies that \( F_\mu \) is a contraction on the \( X^s_t(R) \). Then by the Banach fixed point theorem, there exists a unique solution \( u \) of the integral equation (3.33) in \( X^s_t(R) \). By the Proposition 2.6 follows that \( u \in C([0, T]; H^s(\mathbb{R})) \). The uniqueness in whole space \( X^s_t \) and the smoothness of the flow-map solution follows by known arguments, see for example [2], [8] and [21].

Let \( s' > s \), then a similar contraction argument using the norm \( \tilde{X}^s_t \), defined in Remark 2.7, shows that the solution with initial data \( \phi \in H^{s'} \) is defined on \([0, T] \) with \( T = T(\| \phi \|_{H^s}) \).

With respect to regularity, we note that \( t \in (0, T] \mapsto E_\mu(t)\phi \in H^\infty(\mathbb{R}) \) is continuous with respect to the topology of \( H^\infty \), see [2] and [13]. From the Proposition 2.6 there exists \( \lambda > 0 \) such that \( V_\mu \in C([0, T]; H^{s + \lambda}(\mathbb{R})) \), thus \( u \in C((0, T]; H^{s + \lambda}(\mathbb{R})) \).

Therefore, by a well known bootstrapping argument, using the uniqueness result and the fact that \( T \) only depends on the \( H^s \)-norm of the initial data, we obtain \( u \in C((0, T]; H^\infty(\mathbb{R})) \).

\[ \square \]

Proof of Theorem 1.2. Let \( u \) be the local solution given by Theorem 1.1. In view of \( u \in C((0, T]; H^\infty) \), we only need an a priori estimate in \( H^1 \). For this, putting \( w = \partial_x u \), we get the following
\[ \begin{cases} 
\partial_t w - \partial_x^2 w - w\partial_x w - \mu(1 - \partial_x^2)^{-1/2} w = 0 \\
 w(x, 0) = \phi'(x). \end{cases} \] (3.35)

Multiplying (3.35) by \( w \) and integrating over the real line we obtain
\[ \frac{d}{dt} \| w \|^2 = -2\| \partial_x w \|^2 + 2\mu\| w \|_{H^{1/2}}^2 \leq 2\mu\| w \|^2. \]

Then by the Gronwall’s Lemma
\[ \| \partial_x u \|^2 \leq \| \phi' \|^2 \exp \left[ 2\mu \int_0^t dt' \right] \leq e^{2\mu T}\| \phi' \|^2. \] (3.36)
In the following, multiplying (1.1) by \( u \) and integrating over \( \mathbb{R} \) we get

\[
\frac{d}{dt} \| u \|^2 = 2 \langle u, \partial_x^2 u \rangle + \langle u, (\partial_x u)^2 \rangle + 2 \mu (u, (1 - \partial_x^2)^{-1/2} u) \\
\leq -2 \| \partial_x u \|^2 + \| u \|_{L^\infty} \| \partial_x u \|^2 + 2 \mu \| u \|^2 \\
\lesssim \| u \|^2 \| \partial_x u \|^2 + 2 \mu \| u \|^2 \\
\lesssim_{\mu,T} (\| u \|^2 + \| u \|_{H^s}^2),
\]

where above, we use respectively, the Gagliardo-Nirenberg’s inequality, and the Young’s inequality. Therefore, an application of Gronwall’s Lemma in (3.37) give us the desired result. This finish the proof.

\[\square\]

4. CONVERGENCE OF SOLUTIONS WHEN \( \mu \downarrow 0 \)

In this section we study the behavior of the solutions of IVP (1.1), when \( \mu \) goes to zero. In the following, we define by \( u_\mu \), the solution of the IVP (1.1) constructed in Theorem 1.1 on parameter \( \mu > 0 \) and defined in the interval \([0, T]\). Recall that by the proof of Theorem 1.1, \( T \) is independent of \( \mu \). Here we are using arguments similar to [21] (see also [27]).

**Theorem 4.1.** Let \( 1/2 < s < 1, \phi \in H^s(\mathbb{R}) \) and \( \mu > 0 \). If \( u_\mu \) is the solution defined as above, for \( u_\mu(0) = \phi \), then

\[
\lim_{\mu \downarrow 0} \sup_{t \in [0, T]} \| u_\mu - u \|_{H^s} = 0,
\]

where \( u \) is the solution of (1.1), on parameter \( \mu = 0 \), with \( u(0) = u_0(0) = \phi \).

**Proof.** Putting \( E := E_0 \), after straightforward computations, follows that \( w := u_\mu - u \) satisfies the integral equation

\[
w = (E_\mu(t) - E(t))\phi + \frac{1}{2} \int_0^t E_\mu(t - \tau) \partial_x u_\mu \partial_x w + \partial_x u_\mu \partial_x w \, d\tau \\
+ \frac{1}{2} \int_0^t (E_\mu(t - \tau) - E(t - \tau)) (\partial_x u)^2 \, d\tau \\
:= \alpha_1 + \alpha_2 + \alpha_3.
\]

Let \( u_\mu \) solutions constructed in Theorem 1.1 on interval \([0, T_0]\), such that

\[
\| u_\mu \|_{X^s_{T_0}} \leq R, \text{ where } 0 < T_0 \leq \min\{1, T\}.
\]

In view of inequality

\[
\sup_{t \in [0, T_0]} \| u_\mu - u \|_{H^s} \leq \| w \|_{X^s_{T_0}},
\]

is enough to examine the convergence on spaces \( X^s_{T_0} \). Then, by using Proposition 2.5

\[
\| \alpha_2 \|_{X^s_{T_0}} \lesssim e^{\mu T_0 T^{\delta(s)}} (\| u_\mu \|_{X^s_{T_0}} \| w \|_{X^s_{T_0}} + \| u \|_{X^s_{T_0}} \| w \|_{X^s_{T_0}}) \\
\lesssim 2 e^{\mu T_0 T^{\delta(s)}} R \| w \|_{X^s_{T_0}}.
\]

(4.41)
Taking $T_0$ such that $2e^{\mu T_0^2(s)} R < 1/2$, follows by (4.39)
\[
\|w\|_{X_T^s} \leq \|\alpha_1\|_{X_T^s} + \|\alpha_2\|_{X_T^s} + \|\alpha_3\|_{X_T^s}
\leq \|\alpha_1\|_{X_T^s} + \frac{1}{2}\|w\|_{X_T^s} + \|\alpha_3\|_{X_T^s}.
\]
(4.42)
Then
\[
\|w\|_{X_T^s} \leq 2(\|\alpha_1\|_{X_T^s} + \|\alpha_3\|_{X_T^s}).
\]
(4.43)
By the last inequality its enough study the limit on $\varphi_1$ and $\varphi_3$. For this, we observe
that by the definition of the $X_T^s$-norms and the Lebesgue dominated convergence theorem, follows that
\[
\|\alpha_1\|_{X_T^s} \to 0, \text{ with } \mu \downarrow 0.
\]
(4.44)
About $\alpha_3$, again by the Lebesgue dominated convergence theorem and using the
same ideas as in the Propositions 2.5 and 2.6
\[
\|\alpha_3\|_{X_T^s} \to 0, \text{ with } \mu \downarrow 0.
\]
(4.45)
Therefore, by (4.40)–(4.45) we conclude that
\[
\sup_{t \in [0,T]} \|u_\mu - u\|_{H^s} \to 0, \text{ with } \mu \downarrow 0.
\]
(4.46)
To conclude, we can use an interactive process to extend the solution for all interval
$[0,T]$. This finish the proof. $\Box$

**Remark 4.2.** By a modification of the space $X_T^s$ we can show the existence of
solutions to the IVP (1.1), when $s > 0$. In this case, the uniqueness of IVP (1.1)
fail, once that’s in [16], the author obtained the non-uniqueness in the initial value
problem for the Burgers’ equation, where $s < -1/2$.

5. **ILL-POSEDNESS**

In this section we use analogous arguments contained in [27] (see also [8], [12],
[17] and [18]).

**Theorem 5.1.** Let $s < 1/2$ and $T > 0$. Then there no exists a space $X_T^s$ continu-
ously embedded in $C([0,T]; H^s(\mathbb{R}))$ such that
\[
\|E_\mu(t)\phi\|_{X_T^s} \lesssim \|\phi\|_{H^s}, \forall \phi \in H^s(\mathbb{R})
\]
and
\[
\left\| \int_0^t E_\mu(t-\tau)b(u,v)(\tau)\,d\tau \right\|_{X_T^s} \lesssim \|u\|_{X_T^s} \|v\|_{X_T^s}, \forall u,v \in X_T^s,
\]
where
\[
b(u,v) = \frac{1}{2}\partial_x u \partial_x v.
\]
(5.49)

**Proof.** The proof follows by a contradiction argument. Suppose that there exists
a space $X_T^s$ as in theorem 5.56. Let $u(t) = E_\mu(t)\phi$ and $v(t) = E_\mu(t)\psi$ where
$\phi, \psi \in H^s(\mathbb{R})$ and $0 < t < T$ is fixed. Using (5.47) and (5.48)
\[
\left\| \int_0^t E_\mu(t-\tau)b(u,v)(\tau)\,d\tau \right\|_{X_T^s} \lesssim \|\phi\|_{H^s} \|\psi\|_{H^s}
\]
(5.50)
Now, we will construct functions $\phi$ and $\psi$ such that (5.50) fails. Let $\phi$ and $\psi$
defined by
\[
\phi = r^{-1/2} N^{-(s-1/2)}(\chi_{I_1})^\vee \quad \text{and} \quad \psi = r^{-1/2} N^{-(s-1/2)}(\chi_{I_2})^\vee,
\]
(5.51)
where

\[ I_1 = [-N, -N + r], I_2 = [N + r, N + 2r], \quad N >> 1 \text{ and } r \sim 1. \]

We observe that

\[ \|\phi\|_{L^2_x} \lesssim 1 \text{ and } \|\psi\|_{L^2_x} \lesssim 1. \quad (5.52) \]

Recalling that

\[ \Phi(\xi) = -\xi^2 + \frac{\mu}{(1 + \xi^2)^{1/2}}, \]

by taking the Fourier transform and using Fubini’s Theorem

\[
f(\xi, t) := \left( \int_0^t E_\mu(t - \tau)b(u(\tau), v(\tau))d\tau \right)^\wedge(\xi)
= \int_0^t e^{(t-\tau)\Phi(\xi)}b(u(\tau), v(\tau))^\wedge(\xi)d\tau
= \frac{1}{2} \int_0^t e^{(t-\tau)\Phi(\xi)} i\xi e^{\tau\Phi(\xi)} \hat{\phi} + i\xi e^{\tau\Phi(\xi)} \hat{\psi} d\tau,
\]

where, by the change of variables \( z = \xi - \eta \) we obtain

\[
g(\xi, \tau) := -\int (\xi - \eta)e^{\tau\Phi(\xi - \eta)} \hat{\phi}(\xi - \eta)\eta e^{\tau\Phi(\eta)} \hat{\psi}(\eta) d\eta
= -\int z e^{\tau\Phi(z)} \hat{\phi}(z)\eta e^{\tau\Phi(\eta)} \hat{\psi}(\eta) d\eta. \quad (5.54)
\]

Then the integral above can be written

\[
f(\xi, t) = -\frac{1}{2} \int_0^t e^{(t-\tau)\Phi(\xi)} \int z\eta e^{\tau\Phi(z)} \hat{\phi}(z) e^{\tau\Phi(\eta)} \hat{\psi}(\eta) d\eta d\tau
= -\frac{e^{\Phi(\xi)}}{2} \int_0^t \int z\eta e^{-\tau\Phi(\xi)} e^{-\tau\Phi(z)} \hat{\phi}(z) e^{\tau\Phi(\eta)} \hat{\psi}(\eta) d\eta d\tau
= -\frac{e^{\Phi(\xi)}}{2rN^{2(s-1/2)}} \int_{K_\xi} z\eta \chi_{I_1}(z) \chi_{I_2}(\eta) \int_0^t e^{\tau(-\Phi(\xi) + \Phi(z) + \Phi(\eta))} d\tau d\eta
= -\frac{e^{\Phi(\xi)}}{2rN^{2(s-1/2)}} \int_{K_\xi} z\eta \left( \frac{e^{t(-\Phi(\xi) + \Phi(z) + \Phi(\eta))} - 1}{-\Phi(\xi) + \Phi(z) + \Phi(\eta)} \right) d\eta,
\]

where \( K_\xi = \{\eta \in \mathbb{R} : z \in I_1 \text{ and } \eta \in I_2\} \).

We observe that if \( \eta \in I_1 \) and \( z \in I_2 \), then \( |\eta| \sim |z| \sim N, \ r \leq \xi \leq 3r \) and \( |\eta z| \sim N^2 \). Therefore, we obtain \( |f(\xi, t)|^2 \gtrsim N^{-4(s-1/2)} \), so that

\[
(1 + \xi^2)^s |f(\xi, t)|^2 \gtrsim (1 + \xi^2)^s N^{-4(s-1/2)}.
\]

Thus

\[
\left\| \int_0^t E_\mu(t - \tau)b(u(\tau), v(\tau))d\tau \right\|_{H^s}^2 \gtrsim N^{-4(s-1/2)}. \quad (5.56)
\]

Then from (5.48), (5.52) and (5.56) follows that

\[ N^{-2(s-1/2)} \lesssim 1, \quad \forall N >> 1, \]

which is a contradiction, taking account our hypothesis on \( s \).

□
Proof of Theorem 1.3. If the flow-map data solution would be $C^2$ at origin, by a computation of the Fréchet derivative we would obtain

$$
\left\| \int_0^t E_\mu(t-\tau)b(u(\tau),v(\tau))d\tau \right\|_{X_t^s} \lesssim \|\phi\|_{H^s} \|\psi\|_{H^s}, \forall \phi, \psi \in H^s(\mathbb{R}).
$$

But as we have seen in (5.50) the above inequality fails. This finish the proof. □

References

[1] F. Abergel, Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains, *J. Differential Equations* 83, 85–108, 1990.

[2] E. A. Alarcon and P. J. Otterson, On the Initial Value Problem for a nonlinear wave front evolution equation describing waves with convection, *Anais do 45 Seminário Brasileiro de Análise*, I, 263–282, 1997.

[3] E. A. Alarcon, On the Initial Value Problem for a nonlinear wave front evolution equation with Bore-Like data, *58 Seminário Brasileiro de Análise*, IMECC-UNICAMP, 326–332, 2003.

[4] E. A. Alarcon, Existence of the Global Attractor for the n-dimensional Kuramoto-Sivashinsky Equation, *52 Seminário Brasileiro de Análise*, São José dos Campos-SP, 291–298, 2000.

[5] E. A. Alarcon and R. J. Iorio, The existence of global attractors for a class of nonlinear dissipative evolution equations, *Proc. Roy. Soc. Edinburgh Sect. A*, 135, No. 5, 887–913, 2005.

[6] E. A. Alarcon, Existence and finite dimensionality of the global attractor for a class of nonlinear dissipative equations, *Proc. Roy. Soc. Edinburgh Sect. A* 123, No. 5, 893–916, 1993.

[7] D. Bekiranov, The initial value problem for the generalized Burgers’ equation, *Diff. Int. Equations* 9, 1253–1265, 1996.

[8] G. Fonseca, R. Pastrán and G. Rodriguez, The IVP for a nonlocal perturbation of the Benjamin-Ono equation in classical and weighted Sobolev spaces, *Arxiv*, 2018.

[9] J. Bourgain, Periodic Korteweg de Vries equation with measures as initial data, *Selecta Math. (N.S.)* 3, No. 2, 115–159, 1997.

[10] X. Carvajal and M. Panthe, Well-Posedness for some Perturbations of Kdv equation with regularity data, *Electronic Journal of Differential Equations*, Vol. 2008, No. 02, 1–18, 2008.

[11] X. Carvajal and M. Panthe, Sharp local well-posedness of Kdv type equations with Dissipative perturbations, *Quarterly of Applied Mathematics*, Vol. LXXIV, No. 03, 571–594, 2016.

[12] X. Carvajal and M. Panthe, A note on local well-posedness of generalized KdV type equations with dissipative perturbations, *Springer Proc. Math. Stat.*, v. 209, 85–100, 2017.

[13] A. Cunha, O Problema de Cauchy para uma Equação de Evolução Não Linear, Master Thesis, UFG, 2003.

[14] A. Cunha and A. Pastor, The IVP for the Benjamin-Ono-Zakharov-Kuznetsov equation in weighted Sobolev spaces, *J. Math. Anal. Appl.*, v. 417, 660–693, 2014.

[15] A. Cunha and A. Pastor, The IVP for the Benjamin-Ono-Zakharov-Kuznetsov equation in low regularity Sobolev spaces, *J. Differential Equations*, v. 261, 2041–206, 2016.

[16] D. B. Dix, Nonuniqueness and uniqueness in the initial value problem for Burgers’ equation, *SIAM J. Math. Anal. 27*, 708–724, 1996.

[17] A. Esfahani, Sharp well-posedness of the Ostrovsky, Stepanyans and Tsimring equation, *Math. Commun.*, 18 (2013), 323–335, 2013.

[18] A. Esfahani and A. Pastor, Ill-posedness results for the (generalized) Benjamin-Ono-Zakharov-Kuznetsov equation, *Proc. Amer. Math. Soc.*, Vol. 139, 943–956, 2011.

[19] B. F. Edwards, J. W. Wilder, K. Showalter, Finite thermal diffusivity at onset of convection in autocatalytic systems: Continuous fluid density, *Phys. Rev. A*, 45, 2320–2327 (1992).

[20] O. Goubet, L. Molinet, Global attractor for weakly damped nonlinear Schrödinger equations in $L^2(\mathbb{R})$, *Nonlinear Anal.* 71, No. 1-2, 317–320, 2009.

[21] H. Biagioni, J. L. Bona, R. Iorio and M. Scialom, On the Korteweg-de Vries-Kuramoto-Sivashinsky equation, *Adv. Diff. Eq. 1*, 1–20, 1996.

[22] R. J. Iório & V. M. Iório, *Fourier Analysis and Partial Differential Equations*, Cambridge Studies in Advanced Mathematics 70, Cambridge University Press, 2001.
[23] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de-Vries, *Studies in Applied Mathematics, Advances in Mathematics Supplementary Studies*, Academic Press, Vol.8, 93–128, 1983.

[24] T. Kato and G. Ponce, Commutator estimates and Euler and Navier-Stokes equation, *Comm. Pure Appl. Math*, Vol.41, 891–907, 1988.

[25] L. Molinet, F. Ribaud and A. Youssfi, Ill-posedness issues for a class of parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A* 132, 1407–1416, 2002.

[26] L. Molinet, F. Ribaud, On the low regularity of the Korteweg-de Vries-Burgers equation, *Int. Math. Res. Not.*, No.37, 1979–2005, 2002.

[27] D. Pilot, Sharp well-posedness results for the Kuramoto-Velarde equation, *Communications on Pure and Applied Analysis*, 7(4), 867–881, 2008.

[28] R. Teman, Infinite Dimensional Systems in Mechanics and Physics, Springer-Verlag, Berlin/New York, 1988.

[29] N. Tzvetkov, Remark on the local ill-posedness for Kdv equation, *C. R. Acad. Sci. Paris Sér. I Math.*, Vol.39, No. 12, 1043–1047, 1999.

[30] J. W. Wilder, B. F. Edwards, D. A. Vasques, G. I. Sivashinsky, Derivation of a nonlinear front evolution equation for chemical waves involving convection, *Physica D*, 73, 217–226 (1994).

Universidade Federal de Goiás, Instituto de Matemática e Estatística. Universidade Federal de Goiás - UFG - Campus II, Goiânia, Brazil.
E-mail address: alysson@ufg.br

Universidade Federal de Goiás, Instituto de Matemática e Estatística. Universidade Federal de Goiás - UFG - Campus II, Goiânia, Brazil.
E-mail address: arbietoalarcon@ufg.br