CONTINUOUS HAMILTONIAN DYNAMICS AND
AREA-PRESERVING HOMEOMORPHISM GROUP OF $D^2$

YONG-GEUN OH

ABSTRACT. The main purpose of this paper is to propose a scheme of a proof of
the nonsimpleness of the group $\text{Homeo}^+(D^2, \partial D^2)$ of area preserving homeo-
morphisms of the 2-disc $D^2$. We first establish the existence of Alexander
isotopy in the category of Hamiltonian homeomorphisms. Next, consider-
ing the graph of contractible topological Hamiltonian loop $\phi_F$ on $D^2 \subset S^2$
generated by the Hamiltonian $F$ with $\text{supp} F \subset \text{Int} D^2$, we prove that the
basic phase function $f_F$ associated to the graph and the normalized Hamil-
tonian $F$ is a constant function whose value coincides with the Calabi in-
variant $\text{Cal}_{\text{path}}(\phi_F) = \text{Cal}(F)$ of the topological Hamiltonian loop $\phi_F$. This
reduces the question of extendability of the well-known Calabi homomorphism
$\text{Cal} : \text{Diff}^+(D^2, \partial D^2) \to \mathbb{R}$ to a homomorphism $\overline{\text{Cal}} : \text{Homeo}(D^2, \partial D^2) \to \mathbb{R}$
to that of the vanishing of $f_F$ which is the main conjecture proposed in this
article. Here $\text{Homeo}(D^2, \partial D^2)$ is the group of Hamiltonian homeomorphisms
introduced by Müller and the author [OM] which they showed is a normal
subgroup by construction. We then provide an evidence of this conjecture
by proving the conjecture for the special class of weakly graphical topologi-
cal Hamiltonian loops on $D^2$ via a study of the associated Hamilton-Jacobi
equation.

MSC2010: 53D05, 53D35, 53D40; 28D10.

CONTENTS

1. Introduction and statements of main results 2
   1.1. Calabi invariant on $D^2$ 2
   1.2. Basic phase function and Calabi invariant 4
   1.3. Graphical Hamiltonian diffeomorphism on $D^2$ and its Calabi invariant 6

Part 1. Calabi invariant and basic phase function 6

2. Calabi homomorphism $\text{Cal}_{\text{path}}$ on the path space 7

Date: January 2015.

Key words and phrases. Area-preserving homeomorphism group, Calabi invariant, Lagrangian
submanifolds, generating function, basic phase function, topological Hamiltonian loop, Hamilton-
Jacobi equation.

The present work is supported by the IBS project # IBS-R003-D1.
Section 1. Introduction and Statements of Main Results

1.1. Calabi invariant on $D^2$. Denote by $Diff^\Omega(D^2, \partial D^2)$ the group of area-preserving diffeomorphisms supported in the interior of $D^2$ with respect to the standard area form $\Omega = dq \wedge dp$ on $D^2 \subset \mathbb{R}^2$. For any $\phi \in Diff^\Omega(D^2, \partial D^2)$

$$\phi^* \Omega = \Omega$$

by definition. Write $\Omega = d\alpha$ for some choice of $\alpha$. Then this equation leads to the statement $\phi^* \alpha - \alpha$ is closed. Furthermore since $\phi$ is supported in the interior, the one-form

$$\phi^* \alpha - \alpha$$

vanishes near $\partial D^2$ and so defines a de Rham cohomology class lying in $H^1(D^2, \partial D^2)$. Since the latter group is trivial, we can find a function $h_{\phi, \alpha}$ supported in the interior such that

$$dh_{\phi, \alpha} = \phi^* \alpha - \alpha. \quad (1.1)$$

Then the following is the well-known definition of Calabi invariant $[Ca]$.  

**Definition 1.1** (Calabi invariant). We define

$$\text{Cal}(\phi) = \frac{1}{2} \int_{D^2} h_{\phi, \alpha}. $$

One can show that this value does not depend on the choice of the one-form $\alpha$ but depends only on the diffeomorphism. We will fix one such form $\alpha$ and so suppress the dependence $\alpha$ from our notation, and just denote $h_{\phi} = h_{\phi, \alpha}$.

Another equivalent definition does not involve the choice of one-form $\alpha$ but uses the ’past history’ of the diffeomorphism in the setting of Hamiltonian dynamics $[Ba]$. More precisely, this definition implicitly relies on the following three facts:

1. $\Omega$ on two dimensional surface is a symplectic form and hence $Diff^\Omega(D^2, \partial D^2) = \text{Symp}_\omega(D^2, \partial D^2)$

where $\omega = \Omega$. 


(2) $D^2$ is simply connected, which in turn implies that any symplectic isotopy is a Hamiltonian isotopy.

(3) The group $Diff^0(D^2, \partial D^2)$ is contractible. (For this matter, finiteness of $\pi_1(Diff^0(D^2, \partial D^2), id) \cong \{0\}$ is enough.)

It is well-known (see [GG], [Oh6] for example) and easy to construct a sequence $\phi_i \in Diff^0(D^2, \partial D^2)$ such that $\phi_i \to id$ in $C^0$ topology but

$$Cal(\phi_i) = 1$$

for all $i$'s. This implies that $Cal$ cannot be continuously extended to the full group $Homeo^0(D^2, \partial D^2)$ of area-preserving homeomorphisms.

However here is the main slogan of the paper:

**Main slogan:** If a sequence $\phi_i$ of area preserving diffeomorphisms converges to $id$ in $C^0$ and is anchored by the Hamiltonian path $\phi_H$ in addition (i.e., if $\phi_i = \phi^1_H$) with convergent $H_i$ in $L^{1, \infty}$, then $\lim_{i \to \infty} Cal(\phi_i) = 0$.

As we did in [Oh5], we first define a homomorphism on the path spaces

$$Cal^{\text{path}}(\lambda) : \mathcal{P}^{\text{ham}}(\text{Symp}(D^2, \partial D^2), id) \to \mathbb{R}$$

by

$$Cal^{\text{path}}(\lambda) = \frac{1}{\text{Area}(D^2)} \int_0^1 \int_{D^2} H(t, x) \Omega \, dt.$$  

We will also denote this average by $Cal(H)$ depending on the circumstances. Based on these facts (1) and (2), we can represent $\phi = \phi^1_H$, for the time-one map $\phi^1_H$ of a time-dependent Hamiltonian $H = H(t, x)$ supported in the interior. Then based on (3) and some standard calculations in Hamiltonian geometry using the integration by parts, one proves that this integral does not depend on the choice of Hamiltonian $H \to \phi$. Therefore it descends to $Ham(D^2, \partial D^2) = Diff^0(D^2, \partial D^2)$. Then another application of Stokes' formula, one can prove that this latter definition indeed coincides with that of Definition 1.1 (See [Ba] for its proof.)

It is via this second definition how the author attempts to extend the classical Calabi homomorphism $Cal : Ham(D^2, \partial D^2) \to \mathbb{R}$ to its topological analog $Cal : Homeo(D^2, \partial D^2) \to \mathbb{R}$. In [Oh5], the definition (1.2) is extended to a homomorphism

$$Cal^{\text{path}} : \mathcal{P}^{\text{ham}}(\text{Symp}(D^2, \partial D^2), id) \to \mathbb{R}$$

on the set $\mathcal{P}^{\text{ham}}(\text{Symp}(D^2, \partial D^2), id)$ of topological Hamiltonian paths. (See section 2 for the precise definition.) Here following the notation from [OM], we denote by $\text{Symp}(D^2, \partial D^2)$ the $C^0$-closure of $\text{Symp}(D^2, \partial D^2)$. Gromov-Eliashberg’s $C^0$ symplectic rigidity theorem [El] states

$$Diff(D^2, \partial D^2) \cap \text{Symp}(D^2, \partial D^2) = \text{Symp}(D^2, \partial D^2).$$

In [Oh5], [Oh6], a proof of descent of $Cal^{\text{path}}$ to the group $Homeo(D^2, \partial D^2) := \text{ev}_1(\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id))$ of Hamiltonian homeomorphisms (or more succinctly homeomorphisms) will follow from the following extension result of Calabi homomorphism. (See [Oh5], [Oh6] for such a reduction.)

The author learned from A. Fathi in our discussion on the group $Homeo(D^2, \partial D^2)$ [El] that the following conjecture will be important in relation to the question on the nonsimplesness of the $Homeo^0(D^2, \partial D^2)$. 

Conjecture 1.2. Let $\text{Hameo}(D^2, \partial D^2) \subset \text{Hameo}^0(D^2, \partial D^2)$ be the subgroup of Hamiltonian homeomorphisms on the two-disc. Then the Calabi homomorphism $\text{Cal}: \text{Diff}^0(D^2, \partial D^2) \to \mathbb{R}$ extends continuously to $\text{Hameo}(D^2, \partial D^2)$ in Hamiltonian topology in the sense of $\text{OM}$. 

An immediate consequence would be a proof of nonsimplesness of the area-preserving homeomorphism group $\text{Hameo}^0(D^2, \partial D^2)$. (We refer to $\text{Oh5}$ for the argument needed to complete this nonsimplesness proof out of this conjecture.) 

One important ingredient in our scheme towards the proof of Conjecture 1.2, which itself has its own interest, is the existence of the Alexander isotopy in the topological Hamiltonian category. Recall that the well-known Alexander isotopy on the disc $D^2$ exists in the homeomorphism category but not in the differentiable category. We will establish that such an Alexander isotopy defines contractions of topological Hamiltonian loops to the identity constant loop in the topological Hamiltonian category.

Theorem 1.3 (Alexander isotopy; Theorem 3.3). Any topological Hamiltonian loop in $\text{Hameo}(D^2, \partial D^2)$ is contractible to the identity loop via topological Hamiltonian homotopy of loops.

1.2. Basic phase function and Calabi invariant. The scheme of the proof of Conjecture 1.2 we propose is based on the following conjectural result of the basic phase function introduced in $\text{Oh1}$. This conjecture is also a crucial ingredient needed in the proof of homotopy invariance of the spectral invariance of topological Hamiltonian paths laid out in $\text{Oh7}$. Explanation of this conjecture is now in order. 

Recall the classical action functional on $T^*N$ for an arbitrary compact manifold $N$ is defined as

$$A^0_H(\gamma) = \int_{\gamma} \gamma^* \theta - \int_{0}^{1} H(t, \gamma(t)) \, dt$$

for an arbitrary compact manifold $N$. (We refer to $\text{Oh5}$ for the argument needed to complete this nonsimplesness proof out of this conjecture.)

Any topological Hamiltonian loop in $\text{Hameo}(D^2, \partial D^2)$ is contractible to the identity loop via topological Hamiltonian homotopy of loops.

The basic phase function graph selector is canonical in that the assignment

$$H \mapsto f_H; \quad C^\infty([0, 1] \times T^*N; \mathbb{R}) \to C^0(N)$$

varies continuously in (weak) Hamiltonian topology of $C^\infty([0, 1] \times T^*N; \mathbb{R})$. The construction $f_H$ in $\text{Oh1}$ is given by considering the Lagrangian pair

$$(o_N, T_q^*N), \quad q \in N$$

and its associated Floer complex $\text{CF}(H; o_N, T_q^*N)$ generated by the Hamiltonian trajectory $z : [0, 1] \to T^*N$ satisfying

$$\dot{z} = X_H(t, z(t)), \quad z(0) \in o_N, z(1) \in T_q^*N.$$

Denote by $\text{Chord}(H; o_N, T_q^*N)$ the set of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases}
\frac{\partial u}{\partial \tau} + J (\frac{\partial u}{\partial \tau} - X_H(u)) = 0 \\
u(\tau, 0) \in o_N, \ u(\tau, 1) \in T_q^*N.
\end{cases}$$

(1.5)
The resulting spectral invariant \( \rho^{\text{lag}}(H; [q]) \) is to be defined by the mini-max value

\[
\rho^{\text{lag}}(H; [q]) = \inf_{\alpha \in [q]} \lambda_H(\alpha)
\]

where \([q]\) is a generator of the homology group \( HF(o_N, T^*_qN) \cong \mathbb{Z} \). The basic phase function \( f_H : N \to \mathbb{R} \) is then defined by \( f_H(q) = \rho^{\text{lag}}(H; [q]) \) first for generic \( q \in N \) and then extending to the rest of \( M \) by continuity. (See \[Oh1\] for the detailed construction and section 6 of the present paper for a summary.)

Next we relate the basic phase function to the Calabi invariant on the two-disc as follows. Let \( F \) be a topological Hamiltonian generating a topological Hamiltonian path \( \phi_F \) on the 2-disc \( D^2 \) with \( \text{supp} \ F \subset \text{Int} \ D^2 \). We consider an approximating sequence \( F_i \) with \( \text{supp} \ F_i \subset \text{Int} \ D^2 \). We embed \( D^2 \) into \( S^2 \) as the upper hemisphere and then extend \( F_i \) canonically to whole \( S^2 \) by zero.

We now specialize the above discussion on the basic phase function to the cases of the Lagrangianization of symplectic diffeomorphisms, i.e., consider their graphs

\[
\text{Graph} \phi = \{ (\phi(x), x) \mid x \in S^2 \} \subset S^2 \times S^2.
\]

Applying this to \( \phi_{F_i} \) and noting \( \text{supp} \phi_{F_i}^2 \subset D^2_+ \times D^2_+ \), we obtain

\[
\text{Graph} \phi_{F_i} \cap \Delta \supset \Delta D^2 \cup \Delta D^2 \setminus (1-\delta)
\]

for some \( \delta > 0 \) for all \( t \in [0, 1] \), independently of sufficiently large \( i \)'s but depending only on \( F \). (See \[OM\] or Definition 2.7 of the present paper for the precise definition of approximating sequence on open manifolds.) Then we consider the normalization \( F_i \) of \( F_i \) on \( S^2 \) and define Hamiltonian

\[
\mathbb{F}_i(t, x) := \chi(x) F_i(t, x), \quad x = (x, y)
\]

on \( T^* \Delta \) with a slight abuse of notation for \( \mathbb{F}_i \).

Two kinds of the associated generating functions, denoted by \( \mathbb{h}_{\mathbb{F}_i} \) and \( h_{\mathbb{F}_i} \), respectively, are given by

\[
\mathbb{h}_{\mathbb{F}_i}(q) = A^{\mathbb{g}_i}(z^q_{\mathbb{F}_i}), \quad h_{\mathbb{F}_i}(q) = A^{g_i}(z^x_{\mathbb{x}}), \quad (1.6)
\]

where the Hamiltonian trajectories \( z^q_{\mathbb{F}_i} \) and \( z^x_{\mathbb{x}} \) are defined by

\[
z^q_{\mathbb{F}_i}(t) = \phi_{\mathbb{F}_i}(q), \quad q \in o_{\Delta}
\]

\[
z^x_{\mathbb{x}}(t) = \phi_{\mathbb{x}}((\phi_{\mathbb{F}_i})^{-1}(x)), \quad x \in \phi_{\mathbb{F}_i}(o_{\Delta}).
\]

We note that \( z^q_{\mathbb{F}_i}(0) = q \) and \( z^x_{\mathbb{x}}(1) = x \). Then a Floer theoretic graph selector, which is called the basic phase function in \[Oh1\] \[Oh9\], is defined by

\[
f_{\mathbb{F}_i} = h_{\mathbb{F}_i} \circ \sigma_{\mathbb{F}_i}, \quad (1.7)
\]

for any given Hamiltonian \( F = F(t, x) \). Here \( \sigma_{\mathbb{F}_i} : N \to \phi_{\mathbb{F}_i}(o_{\Delta}) = L_{\mathbb{F}_i} \) is the Lagrangian selector introduced in \[Oh9\], which has the explicit formula

\[
\sigma_{\mathbb{F}_i}(q) = (q, df_{\mathbb{F}_i}(q)) \in T^* \Delta
\]

whenever \( df_{\mathbb{F}_i}(q) \) exists. This ends the review of construction of basic phase function.

The following theorem exhibits the relationship between the limit of Calabi invariants and that of the basic phase function.
Theorem 1.4 (Theorem 7.1). Let $(M, \omega)$ be an arbitrary closed symplectic manifold. Let $U = M \setminus B$ where $B$ is a closed subset of nonempty interior. Let $\lambda = \phi_F$ be any engulfed topological Hamiltonian loop in $\mathcal{P}^\text{ham}(\text{Sympeo}_U(M, \omega), \text{id})$ with $\phi_t^F \equiv \text{id}$ on $B$. Then

$$\lim_{i \to \infty} f_{F_i}(x) = \text{Cal}_U(F)$$

(1.8)

uniformly over $x \in M$, for any approximating sequence $F_i$ of $F$. In particular, the limit function $f_F$ defined by $f_F(x) := \lim_{i \to \infty} f_{F_i}(x)$ is constant.

It is crucial for the equality (1.8) to hold in the general case that we are considering topological Hamiltonian loop, not just a path. (We refer readers to the proof of Theorem 7.1 to see how the loop property is used therein. We also refer to the proof of Lemma 7.5 [Oh9] for a similar argument used for a similar purpose.)

The following is the main conjecture to beat which was previously proposed by the present author in [Oh7].

Conjecture 1.5 (Main Conjecture). Let $M = S^2$ be the 2 sphere with standard symplectic structure. Let $\Lambda = \{\phi_{H(s)}\}_{(s,t) \in [0,1]^2}$ be a hameotopy contracting a topological Hamiltonian loop $\phi_F$ with $F = H(1)$ such that $H(s) \equiv \text{id}$ on $D^2_-$ where $D^2_-$ is the lower hemisphere of $S^2$. Then $f_F = 0$.

It turns out that this conjecture itself is strong enough to directly give rise to Conjecture 1.2 in a rather straightforward manner with little usage of Floer homology argument in its outset except a few functorial properties of the basic phase function that are automatically carried by the Floer theoretic construction given in [Oh1].

We indicate validity of this conjecture by proving the conjecture for the following special class consisting of weakly graphical topological Hamiltonian loops.

1.3. Graphical Hamiltonian diffeomorphism on $D^2$ and its Calabi invariant. We start with the following definition. We refer readers to Definition 4.2 for the definition of engulfed diffeomorphisms.

Definition 1.6. Let $\Psi : U_{\Delta} \to V$ be a Darboux-Weinstein chart of the diagonal $\Delta \subset M \times M$ and denote $\pi_{\Delta} = \pi_{\Delta}^\Psi : U_{\Delta} \to \Delta$ to be the composition of $\Psi$ followed by the canonical projection $T^* \Delta \to \Delta$.

(1) We call an engulfed symplectic diffeomorphism $\phi : M \to M$ $\Psi$-graphical if the projection $\pi_{\Delta}|_{\text{Graph}\phi} \to \Delta$ is one-one, and an engulfed symplectic isotopy $\{\phi_t\}$ $\Psi$-graphical if each element $\phi_t$ is $\Psi$-graphical.

(2) We call a topological Hamiltonian loop $F$ is strongly (resp. weakly) $\Psi$-graphical, if it allows an approximating sequence $F_i$ each element of which is $\Psi$-graphical (resp. whose time-one map $\phi_{F_i}^1$ is $\Psi$-graphical).

Denote by $F^a$ the time-dependent Hamiltonian generating the path $t \mapsto \phi_{F}^t$. The statement (2) of this definition is equivalent to saying that each $F^a$ is $\Psi$-graphical for $a \in [0,1]$.
We remark that any symplectic diffeomorphisms sufficiently \( C^1 \)-close to the identity is graphical, but not every \( C^0 \)-close one. We also remark that \( \pi_A^{\text{Graph} \phi} \) is surjective and hence a diffeomorphism if \( \phi \) is a \( \Psi \)-graphical symplectic diffeomorphism isotopic to the identity via a \( \Psi \)-engulfed isotopy.

In 2 dimension, we prove the following interesting phenomenon. We doubt that similar phenomenon occurs in high dimension. This theorem will not be used in the proofs of main results of the present paper but has its own interest.

**Theorem 1.7.** Let \( M \) be a closed 2 dimensional surface. Suppose \( \phi : M \to M \) is a \( \Psi \)-graphical symplectic diffeomorphism isotopic to the identity via \( \Psi \)-graphical isotopy. and let \( \text{Graph} \phi = \text{Image} \alpha_\phi \) for a closed one-form \( \alpha_\phi \). Then for any \( 0 \leq r \leq 1 \), the projection \( \pi_2 : M \times M \to M \) restricts to a one-one map to \( \text{Image} r \alpha_\phi \subset M \times M \). In particular

\[
\text{Image} r \alpha_\phi = \text{Graph} \phi_r
\]

for some symplectic diffeomorphism \( \phi_r : M \to M \) for each \( 0 \leq r \leq 1 \).

Finally we prove Conjecture 1.5 for the weakly graphical topological Hamiltonian loop on \( S^2 \) that arises as follows.

**Theorem 1.8.** Conjecture 1.5 holds for any weakly graphical topological Hamiltonian loop on \( S^2 \) arising from one on \( D^2 \) as in subsection 1.2.

The proof of this theorem strongly relies on Theorem 1.3.

An immediate corollary of Theorem 1.4 and 1.8 is the following vanishing result of Calabi invariant.

**Corollary 1.9.** Suppose \( \lambda = \phi_F \) is a weakly graphical topological Hamiltonian loop on \( D^2 \). Then \( \text{Cal}^{\text{path}}(\lambda) = 0 \).

We will study elsewhere general engulfed topological Hamiltonian loop dropping the graphicality condition, which heavily uses the piecewise smooth Hamiltonian geometry involving the Cliff-wall surgery introduced in [Oh9].

Previously the author announced a ‘proof’ of the nonsimpleness result in [Oh6] modulo the proof of Conjecture 1.5 in which we derived nonsimplicity out of the homotopy invariance of spectral invariants whose proof also strongly relied on this vanishing result. Unlike the previously proposed scheme of the proof, the current scheme does not rely on the homotopy invariance of spectral invariants of topological Hamiltonian paths but more directly follows from the above mentioned vanishing result.

We thank M. Usher for his careful reading of the previous version of the present paper and useful discussions in relation to the proof of Theorem 1.11. We also take this opportunity to thank A. Fathi for explaining us, during his visit of KIAS in year 2005, how the question of extendability of the Calabi homomorphism on \( \text{Diff}^{\text{th}}(D^2, \partial D^2) \) to \( \text{Homeo}^{\text{th}}(D^2, \partial D^2) \) is related to the non-simplicity of the area-preserving homeomorphism group \( \text{Homeo}^{\text{th}}(D^2, \partial D^2) \).

**Part 1. Calabi invariant and basic phase function**

2. **Calabi homomorphism** \( \text{Cal}^{PATH} \) **on the path space**
2.1. Hamiltonian topology and hamiltonian homotopy. In [OM], Müller and the author introduced the notion of Hamiltonian topology on the space

\[ P^{\text{ham}}(\text{Symp}(M, \omega), id) \]

of Hamiltonian paths \( \lambda : [0, 1] \to \text{Symp}(M, \omega) \) with \( \lambda(t) = \phi^t_H \) for some time-dependent Hamiltonian \( H \). We would like to emphasize that we do not assume that \( H \) is normalized unless otherwise said explicitly. This is because we need to consider both compactly supported and mean-normalized Hamiltonians and suitably transform one to the other in the course of the proof of the main theorem of this paper.

We first recall the definition of this Hamiltonian topology.

We start with the case of closed \((M, \omega)\). For a given continuous function \( h : M \to \mathbb{R} \), we denote \( \text{osc}(h) = \max_h h - \min_h h \).

We define the \( C^0 \)-distance \( d \) on \( \text{Homeo}(M) \) by the symmetrized \( C^0 \)-distance

\[ d(\phi, \psi) = \max \{ d_{C^0}(\phi, \psi), d_{C^0}(\phi^{-1}, \psi^{-1}) \} \]

and the \( C^0 \)-distance, again denoted by \( d \), on

\[ P^{\text{ham}}(\text{Symp}(M, \omega), id) \subset P(\text{Homeo}(M), id) \]

by

\[ d(\lambda, \mu) = \max_{t \in [0, 1]} d(\lambda(t), \mu(t)). \]

The Hofer length of Hamiltonian path \( \lambda = \phi_H \) is defined by

\[ \text{leng}(\lambda) = \int_0^1 \text{osc}(H_t) \, dt = \| H \|. \]

Following the notations of [OM], we denote by \( \phi_H \) the Hamiltonian path \( \phi_H : t \mapsto \phi^t_H; [0, 1] \to \text{Ham}(M, \omega) \) and by \( \text{Dev}(\lambda) \) the associated normalized Hamiltonian

\[ \text{Dev}(\lambda) := \underline{H}, \quad \lambda = \phi_H \quad (2.1) \]

where \( H \) is defined by

\[ H(t, x) = H(t, x) - \frac{1}{\text{vol}_\omega(M)} \int_M H(t, x) \omega^n. \quad (2.2) \]

We normalize \( \omega \) so that \( \text{vol}_\omega(M) = \int_M \omega^n = 1 \) but do not remove the normalizing factor \( \frac{1}{\text{vol}_\omega(M)} \) to make the meaning of \( H \) more conspicuous.

**Definition 2.1.** Let \((M, \omega)\) be a closed symplectic manifold. Let \( \lambda, \mu \) be smooth Hamiltonian paths. The Hamiltonian topology is the metric topology induced by the metric

\[ d_{\text{ham}}(\lambda, \mu) := d(\lambda, \mu) + \text{leng}(\lambda^{-1} \mu). \quad (2.3) \]

Now we recall the notion of topological Hamiltonian flows and Hamiltonian homeomorphisms introduced in [OM].

**Definition 2.2** (\( L^{1, \infty} \) topological Hamiltonian flow). A continuous map \( \lambda : \mathbb{R} \to \text{Homeo}(M) \) is called a topological Hamiltonian flow if there exists a sequence of smooth Hamiltonians \( H_i : \mathbb{R} \times M \to \mathbb{R} \) satisfying the following:

1. \( \phi_{H_i} \to \lambda \) locally uniformly on \( \mathbb{R} \times M \).
(2) the sequence $H_t$ is Cauchy in the $L^{(1,\infty)}$-topology locally in time and so has a limit $H_\infty$ lying in $L^{(1,\infty)}$ on any compact interval $[a,b]$.

We call any such $\phi_H$ or $H_t$ an approximating sequence of $\lambda$. We call a continuous path $\lambda: [a,b] \to \text{Homeo}(M)$ a topological Hamiltonian path if it satisfies the same conditions with $\mathbb{R}$ replaced by $[a,b]$, and the limit $L^{(1,\infty)}$-function $H_\infty$ called a $L^{(1,\infty)}$ topological Hamiltonian or just a topological Hamiltonian.

Following the notations from [OM], we denote by $\text{Sympeo}(M,\omega)$ the closure of $\text{Symp}(M,\omega)$ in $\text{Homeo}(M)$ with respect to the $C^0$-metric $\bar{d}$, and by $\mathcal{H}_m([0,1] \times M,\mathbb{R})$ the set of mean-normalized topological Hamiltonians, and by

$$ev_1: \mathcal{P}^{\text{ham}}_{[0,1]}(\text{Sympeo}(M,\omega),\text{id}) \to \text{Sympeo}(M,\omega),\text{id}$$

(2.4)

the evaluation map defined by $ev_1(\lambda) = \lambda(1)$. By the uniqueness theorem of Buhovsky-Seyfaddini [BS], we can extend the map $\text{Dev}$ given in (2.1) to

$$\overline{\text{Dev}}: \mathcal{P}^{\text{ham}}_{[0,1]}(\text{Sympeo}(M,\omega),\text{id}) \to \mathcal{H}_m([0,1] \times M,\mathbb{R})$$

in an obvious way. Following the notation of [OM] Oh5, we denote the topological Hamiltonian path $\lambda = \phi_H$ when $\overline{\text{Dev}}(\lambda) = H$ in this general context.

**Definition 2.3** (Hamiltonian homeomorphism group). We define

$$\text{Hameo}(M,\omega) = ev_1 \left( \mathcal{P}^{\text{ham}}_{[0,1]}(\text{Sympeo}(M,\omega),\text{id}) \right)$$

and call any element therein a Hamiltonian homeomorphisms.

The group property and its normality in $\text{Sympeo}(M,\omega)$ are proved in [OM].

**Theorem 2.4** ([OM]). Let $(M,\omega)$ be a closed symplectic manifold. Then $\text{Hameo}(M,\omega)$ is a normal subgroup of $\text{Sympeo}(M,\omega)$.

Especially when $\dim \Sigma = 2$, we have a smoothing result

$$\text{Sympeo}(\Sigma,\omega) = \text{Homeo}^\Omega(\Sigma)$$

(2.5)

of area-preserving homeomorphisms by area-preserving diffeomorphisms (see [Oh3], [Sl] for a proof). Therefore combining this with the above theorem, we obtain the following corollary, which is the starting point of our research to apply continuous Hamiltonian dynamics to the study of the simpleness question of the area-preserving homeomorphism group of $D^2$ (or $S^2$).

**Corollary 2.5.** Let $\Sigma$ be a compact surface with or without boundary and let $\Omega$ be an area form of $\Sigma$, which we also regard as a symplectic form $\omega = \Omega$. Then $\text{Hameo}(M,\omega)$ is a normal subgroup of $\text{Homeo}^\Omega(\Sigma)$.

Both results have their counterparts even when $\partial M \neq \emptyset$. We refer the discussion below to the end of this subsection.

Next we consider the notion of homotopy in this topological Hamiltonian category. The following notion of Hamiltonian homotopy, which we abbreviate as hameotopy, of topological Hamiltonian paths is introduced in [Oh6, Oh8].

**Definition 2.6** (Hameotopy). Let $\lambda_0, \lambda_1 \in \mathcal{P}^{\text{ham}}(\text{Sympeo}(M,\omega),\text{id})$. A hameotopy $\Lambda : [0,1]^2 \to \text{Sympeo}(M,\omega)$ between $\lambda_0$ and $\lambda_1$ based at the identity is the map such that

$$\Lambda(0,t) = \lambda_0(t), \Lambda(1,t) = \lambda_1(t),$$

(2.6)

and $\Lambda(s,0) \equiv \text{id}$ for all $s \in [0,1]$, and arises as follows: there is a sequence of smooth maps $\Lambda_j : [0,1]^2 \to \text{Ham}(M,\omega)$ that satisfy
(1) $\Lambda_j(s, 0) = \text{id}$,
(2) $\Lambda_j \rightarrow \Lambda$ in $C^0$-topology,
(3) Any $s$-section $\Lambda_j, s : \{s\} \times [0, 1] \rightarrow \text{Ham}(M, \omega)$ converges in hamiltonian topology in the following sense: If we write
$$\text{Dev}(\Lambda_j, \Lambda_j^{-1}) =: H_j(s),$$
then $H_j(s)$ converges in hamiltonian topology uniformly over $s \in [0, 1]$. We call any such $\Lambda_j$ an approximating sequence of $\Lambda$.

When $\lambda_0(1) = \lambda_1(1) = \psi$, a homotopy relative to the ends is one that satisfies $\Lambda(s, 0) = \text{id}, \Lambda(s, 1) = \psi$ for all $s \in [0, 1]$ in addition.

We say that $\lambda_0, \lambda_1 \in \mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), \text{id})$ are homotopic (resp. relative to the ends), if there exists a homotopy (resp. a homotopy relative to the ends).

We emphasize that by the requirement (3),
$$H_j(0) \equiv 0 \quad (2.7)$$
in this definition.

All the above definitions can be modified to handle the case of open manifolds, either noncompact or compact with boundary, by considering $H$’s compactly supported in the interior as done in section 6 [OM]. We recall the definitions of topological Hamiltonian paths and Hamiltonian homeomorphisms supported in an open subset $U \subset M$ from [OM].

We first define $\mathcal{P}^{\text{ham}}(\text{Symple}(M, \omega), \text{id})$ to be the set of smooth Hamiltonian paths supported in $U$. The following definition is taken from Definition 6.2 [OM] to which we refer readers for more detailed discussions. First for any open subset $V \subset U$ with compact closure $\overline{V} \subset U$, we can define a completion of $\mathcal{P}^{\text{ham}}(\text{Symple}(M, \omega), \text{id})$ using the same metric given above.

**Definition 2.7.** Let $U \subset M$ be an open subset. Define $\mathcal{P}^{\text{ham}}(\text{Symple}_U(M, \omega), \text{id})$ to be the union
$$\mathcal{P}^{\text{ham}}(\text{Symple}_U(M, \omega), \text{id}) := \bigcup_{K \subset U} \mathcal{P}^{\text{ham}}(\text{Symple}_K(M, \omega), \text{id})$$
with the direct limit topology, where $K \subset U$ is a compact subset. We define $\text{Hameo}_c(U, \omega)$ to be the image
$$\text{Hameo}_c(U, \omega) := \text{ev}_1(\mathcal{P}^{\text{ham}}(\text{Symple}_U(M, \omega), \text{id})).$$

We would like to emphasize that this set is not necessarily the same as the set of $\lambda \in \mathcal{P}^{\text{ham}}(\text{Symple}(M, \omega), \text{id})$ with compact supp $\lambda \subset U$. The same definition can be applied to general open manifolds or manifolds with boundary.

### 2.2. Calabi invariants of topological Hamiltonian paths in $D^2$.

Denote by $\mathcal{P}^{\text{ham}}(\text{Symple}(D^2, \partial D^2); \text{id})$ the group of Hamiltonian paths supported in $\text{Int}(D^2)$, i.e.,
$$\bigcup_{t \in [0, 1]} \text{supp}H_t \subset \text{Int}(D^2).$$

We denote by $\mathcal{P}^{\text{ham}}(\text{Symple}(D^2, \partial D^2), \text{id})$ the $L^{1, \infty}$ hamiltonian completion of $\mathcal{P}^{\text{ham}}(\text{Symple}(D^2, \partial D^2); \text{id})$.

We recall the extended Calabi homomorphism defined in [Oh5] whose well-definedness follows from the uniqueness theorem from [BS].
Definition 2.8. Let \( \lambda \in P^{ham}(\text{Symp}(D^2, \partial D^2), \text{id}) \) and \( H \) be its Hamiltonian supported in \( \text{Int } D^2 \). We define
\[
\text{Cal}^{\text{path}}(\lambda) = \text{Cal}^{\text{path}}_{D^2}(\lambda) := \text{Cal}(H).
\]

It is immediate to check that this defines a homomorphism. The main question to be answered is whether this homomorphism descends to the group \( \text{Hameo}(D^2, \partial D^2) \). We recall that one crucial ingredient needed in the proof of well-definedness of this form of the Calabi invariant defined on \( \text{Diff}_{\Omega}(D^2, \partial D^2) \) of area-preserving diffeomorphisms is the fact that \( \text{Diff}_{\Omega}(D^2, \partial D^2) = \text{Ham}(D^2, \partial D^2) \) and it is contractible. In this regard, we would like to prove the following conjecture.

Conjecture 2.9. Let \( \lambda \) be a contractible topological hamiltonian loop based at the identity. Then
\[
\text{Cal}^{\text{path}}(\lambda) = 0.
\]

In the next section, we will establish the existence of Alexander isotopy in the topological Hamiltonian category and prove that any topological hamiltonian loop (based at the identity) on \( D^2 \) is indeed contractible and so the contractibility hypothesis in this conjecture automatically holds.

By the homomorphism property of \( \text{Cal}^{\text{path}} \), an immediate corollary of this conjecture would be the following: Suppose that Conjecture 2.9 holds. Let \( \text{Cal}^{\text{path}} : P^{ham}(\text{Sympeo}(D^2, \partial D^2), \text{id}) \to \mathbb{R} \) be the above extension of the Calabi homomorphism \( \text{Cal}^{\text{path}} \) such that \( \lambda_0(1) = \lambda_1(1) \). Then we have
\[
\text{Cal}^{\text{path}}(\lambda_0) = \text{Cal}^{\text{path}}(\lambda_1).
\]

In the next section, we will elaborate this point further.

3. Alexander isotopy of loops in \( P^{ham}(\text{Sympeo}(D^2, \partial D^2), \text{id}) \)

For the description of Alexander isotopy, we need to consider the conjugate action of rescaling maps of \( D^2 \)
\[
R_a : D^2(1) \to D^2(a) \subset D^2(1)
\]
for \( 0 < a < 1 \) on \( \text{Hameo}(D^2, \partial D^2) \), where \( D^2(a) \) is the disc of radius \( a \) with its center at the origin. We note that \( R_a \) is a conformally symplectic map and so its conjugate action maps a symplectic map to a symplectic map whenever it is defined.

Furthermore the right composition by \( R_a \) defines a map
\[
\phi \mapsto \phi \circ R_a^{-1} : \text{Hameo}(D^2(a), \partial D^2(a)) \subset \text{Hameo}(D^2, \partial D^2) \to \text{Hameo}(D^2, \partial D^2)
\]
and then the left composition by \( R_a \) followed by extension to the identity on \( D^2 \setminus D^2(a) \) defines a map
\[
\text{Hameo}(D^2, \partial D^2) \to \text{Hameo}(D^2(a), \partial D^2(a)) \subset \text{Hameo}(D^2, \partial D^2).
\]
We have the following important formula for the behavior of Calabi invariants under the Alexander isotopy.
Lemma 3.1. Let $\lambda \in \mathcal{P}^{ham}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ be a given a continuous Hamiltonian path on $D^2$. Suppose $\text{supp} \lambda \subset D^2(1 - \eta)$ for a sufficiently small $\eta > 0$. Consider the one-parameter family of maps $\lambda_a$ defined by

$$\lambda_a(t, x) = \begin{cases} a\lambda(t, \frac{x}{a}) & \text{for } |x| \leq a(1 - \eta) \\ x & \text{otherwise} \end{cases}$$

for $0 < a \leq 1$. Then $\lambda_a$ is also a topological Hamiltonian path on $D^2$ and satisfies

$$\text{Cal}^{path}(\lambda_a) = a^4 \text{Cal}^{path}(\lambda). \quad (3.1)$$

Proof. A straightforward calculation proves that $\lambda_a$ is generated by the (unique) continuous Hamiltonian, which we denote by $\text{Dev}(\lambda_a)$ following the notation of [OM, Oh5], which is defined by

$$\text{Dev}(\lambda_a)(t, x) = \begin{cases} a^2 H(t, \frac{x}{a}) & \text{for } |x| \leq a(1 - \eta) \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

where $H = \text{Dev}(\lambda)$. Obviously the right hand side function is the Hamiltonian-limit of $\text{Dev}(\lambda_{i,a})$ for a sequence $\lambda_{i,a}$ of smooth Hamiltonian approximations of $\lambda$ where $\lambda_{i,a}$ is defined by the same formula for $\lambda_i$.

From these, we derive the formula

$$\text{Cal}^{path}(\lambda_a) = \lim_{i \to \infty} \text{Cal}^{path}(\lambda_{i,a}) = \lim_{i \to \infty} a^4 \text{Cal}^{path}(\lambda_i)$$

$$= a^4 \lim_{i \to \infty} \int_0^1 \int_{D^2} H_i(t, y) \Omega \wedge dt$$

$$= a^4 \lim_{i \to \infty} \text{Cal}^{path}(\lambda_i) = a^4 \text{Cal}^{path}(\lambda).$$

This proves (3.1). $\square$

We would like to emphasize that the $s$-Hamiltonian $F_\Lambda$ of $\Lambda(s, t) = \lambda_t^s$ does not converge in $L^{(1, \infty)}$ topology and so we cannot define its Hamiltonian limit.

Explanation of this relationship is now in order in the following remark.

Remark 3.2. Let $D^{2n} \subset \mathbb{R}^{2n}$ be the unit ball. Consider a smooth Hamiltonian $H$ with $\text{supp} \phi_H \subset \text{Int} D^{2n} \subset \mathbb{R}^{2n}$ and its Alexander isotopy

$$\Lambda(s, t) = \phi^t_{H^s} = \lambda_s(t), \quad \lambda = \phi_H$$

Denote by $H_\Lambda$ and $K_\Lambda$ the $t$-Hamiltonian and the $s$-Hamiltonian respectively. Then we derive the formula

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial s}(H \circ \phi^t_{H^s}) \circ (\phi^t_{H^s})^{-1}. \quad (3.3)$$

But we compute

$$H_t \circ \phi_{H^t}^s(x) = s^2 H_t \left( \frac{\phi_{H^t}^s(x)}{s} \right) = s^2 H \left( t, \frac{\phi_{H^t}^s(x)}{s} \right).$$

Therefore we derive

$$K(s, t, x) = 2s \int_0^t H(u, \frac{x}{s}) \, du + s \int_0^t \left( dH \left( u, \frac{(\phi_{H^t}^s)^{-1}(x)}{s} \right), (\phi_{H^t}^s)^{-1}(x) \right) \, du. \quad (3.4)$$
For the second summand, we use the identity $\mathcal{H}(t, x) = -H(t, \phi_H^t(x))$. From this expression, we note that $K$ involves differentiating the Hamiltonian $H_t$ and hence goes out of the $L^{(1, \infty)}$ hamiltonian category.

Recall that the well-known Alexander isotopy on the disc $D^2$ exists in the homeomorphism category but not in the differentiable category. We will establish that such an Alexander isotopy defines contractions of topological Hamiltonian loops to the identity constant loop in the topological Hamiltonian category.

**Theorem 3.3.** Let $\lambda$ be a loop in $\mathcal{P}^{ham}(\text{Sympeo}(D^2, \partial D^2), id)$. Define $\Lambda : [0, 1]^2 \to \text{Sympeo}(D^2, \partial D^2)$ by

$$\Lambda(s, t) = \lambda_s(t).$$

Then $\Lambda$ is a homeotopy between $\lambda$ and the constant path id.

**Proof.** We have $\lambda \in \mathcal{P}^{ham}(\text{Sympeo}(D^2, \partial D^2), id)$ with $\lambda(0) = \lambda(1)$. Then $\lambda_s$ defines a loop contained in $\mathcal{P}^{ham}(\text{Sympeo}(D^2, \partial D^2), id)$ for each $0 \leq s \leq 1$. Let $H_t$ be an approximating sequence of the topological Hamiltonian loop $\lambda$.

We fix a sequence $\varepsilon_i \searrow 0$ and define a 2-parameter Hamiltonian family $\Lambda_{i, \varepsilon_i}$ defined by

$$\Lambda_{i, \varepsilon_i}(s, t) := \lambda_{i, \chi_i(s)}(t, \cdot) \circ \Lambda_{i, \varepsilon_i}^{-1}(t, \cdot)$$

where $\chi_i : [0, 1] \to [\varepsilon_i, 1]$ is a monotonically increasing surjective function with $\chi_i(t) = \varepsilon_i$ near $t = 0$, $\chi_i(1) = 1$ near $t = 1$, and $\chi_i \to \text{id}_{[0, 1]}$ in the Hamiltonian norm (see Definition 3.19 and Lemma 3.20 [OM] for this fact). It follows that the sequence $\Lambda_{i, \varepsilon_i}$ is smooth and uniformly converges in hamiltonian topology as $i \to \infty$ over $s \in [0, 1]$ and $\Lambda_{i, \varepsilon_i}^t(1) \to \lambda(t)$ since the Alexander isotopy is smooth as long as $s > 0$ and by definition $\Lambda_{i, \varepsilon_i}$ involves the Alexander isotopy for $s \geq \varepsilon_i > 0$. The convergence immediately follows from the explicit expression of $\lambda_\varepsilon$ in Lemma 3.4.2.

Finally we need to check

$$\| \text{Dev}(\Lambda_{i, \varepsilon_i}(s, \cdot)) - \text{Dev}(\Lambda_{j, \varepsilon_j}(s, \cdot)) \| \to 0$$

uniformly over $s \in [0, 1]$ as $i, j \to \infty$. For this, we apply the standard formula of Dev for the composed flow,

$$\text{Dev}(\lambda_{\mu^{-1}})(t, x) = \text{Dev}(\lambda)(t, x) - \text{Dev}(\mu)(t, \mu_t^{-1}(x))$$

to $\Lambda_{i, \varepsilon_i} := \lambda_{i, \chi_i(s)}(t, \cdot) \circ \lambda_{i, \varepsilon_i}^{-1}(t, \cdot)$, which amounts to the more familiar formula $(H \# G)_t = H_t - G_t \circ \phi_H^{-1}(\phi_H^t)^{-1}$ in the literature. Then we get

$$\text{Dev}(\Lambda_{i, \varepsilon_i}(s, \cdot))(t, x) = \text{Dev}(\lambda_{i, \chi_i(s)})(t, x) - \text{Dev}(\lambda_{i, \varepsilon_i})(t, \lambda_{i, \varepsilon_i}^{-1}(\chi_{i, \chi_i(s)}))$$

where

$$\text{Dev}(\lambda_{i, \chi_i(s)})(t, x) = \begin{cases} \chi_i(s)^2 H_t(t, \chi_{i, \chi_i(s)}) & \text{for } |x| \leq \chi_i(s)(1 - \eta) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{Dev}(\lambda_{i, \varepsilon_i})(t, x) = \begin{cases} \varepsilon_i^2 H_t(t, \varepsilon_i) & \text{for } |x| \leq \varepsilon_i(1 - \eta) \\ 0 & \text{otherwise}. \end{cases}$$

From these expressions, (3.6) immediately follows. This finishes the proof. \(\square\)

**Corollary 3.4.** If $\lambda_0, \lambda_1 \in \mathcal{P}^{ham}(\text{Sympeo}(D^2, \partial D^2), id)$ and $\lambda_0(1) = \lambda_1(1)$, then they are homeotopic relative to the end.
Proof. Theorem 3.3 implies that the Alexander isotopy is a hameotopy contracting any topological Hamiltonian loop to the identity in $\mathcal{P}_{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ with ends points fixed. This proves that the product loop $\lambda_0\lambda_1^{-1}$, which is based at the identity, is contractible via a hameotopy relative to the ends. Then this implies that $\lambda_0$ and $\lambda_1$ are hameotopic to each other relative to the ends. □

An immediate consequence of Corollary 3.4 is the following

**Proposition 3.5.** Suppose Conjecture 2.9 holds. Then we have

\[ \overline{\text{Cal}}_{\text{path}}(\lambda_0) = \overline{\text{Cal}}_{\text{path}}(\lambda_1) \]

if $\lambda_0, \lambda_1 \in \mathcal{P}_{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ and $\lambda_0(1) = \lambda_1(1)$.

This theorem implies that $\overline{\text{Cal}}_{\text{path}}$ restricted to $\mathcal{P}_{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ depends only on the final point and so gives rise to the following main theorem on the extension of Calabi homomorphism.

**Theorem 3.6.** Suppose Conjecture 2.9 holds. Define a map $\overline{\text{Cal}} : \text{Hameo}(D^2, \partial D^2) \to \mathbb{R}$ by

\[ \overline{\text{Cal}}(g) := \overline{\text{Cal}}_{\text{path}}(\lambda) \]

for a (and so any) $\lambda \in \mathcal{P}_{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ with $g(1) = \lambda(1).$ Then this is well-defined and extends the Calabi homomorphism $\overline{\text{Cal}} : \text{Diff}^0(D^2, \partial D^2) \to \mathbb{R}$ to $\overline{\text{Cal}} : \text{Hameo}(D^2, \partial D^2) \to \mathbb{R}$.

Once this theorem is established, nonsimpleness of $\text{Hameo}(D^2, \partial D^2)$ immediately follows. (See [Oh5] for the needed argument.)

### 4. Reduction to the engulfed case

In this section, we reduce the proof of Conjecture 2.9 to the engulfed topological Hamiltonian loops on $S^2$. Using the given identification of $D^2$ as the upper hemisphere denoted by $D^2_+$, we can embed

\[ \iota^+ : \mathcal{P}_{\text{ham}}(\text{Symp}(D^2, \partial D^2); \text{id}) \to \mathcal{P}_{\text{ham}}(\text{Symp}(S^2); \text{id}) \]

by extending any element $\phi_H \in \mathcal{P}_{\text{ham}}(\text{Symp}(D^2, \partial D^2); \text{id})$ to the one that is identity on the lower hemisphere $D^2_-$ by setting $H \equiv 0$ thereon.

We first recall the definition of engulfed Hamiltonians from [Oh8].

**Definition 4.1.** Let $(M, \omega)$ be a symplectic manifold. Let a Darboux-Weinstein chart

\[ \Phi : V \subset T^*\Delta \to U_\Delta \subset (M \times M, \omega \oplus -\omega) \]

be given. We call $U$ a Darboux-Weinstein neighborhood of the diagonal with respect to $\Phi$. In general we call a neighborhood $U_\Delta$ of the diagonal a *Darboux-Weinstein neighborhood* if it is the image of a Darboux-Weinstein chart.

With this preparation, we are ready to recall the following definition from [Oh8].

**Definition 4.2.**

1. We call an isotopy of Lagrangian submanifold $\{L_t\}_{0 \leq t \leq 1}$ of $L$ is called $V$-engulfed if there exists a Darboux neighborhood $V$ of $L$ such that $L_s \subset V$ for all $s$. When we do not specify $V$, we just call the isotopy engulfed.
(2) We call a (topological) Hamiltonian path $\phi_H U$-engulfed if its graph $\text{Graph} \phi_H$ is engulfed in a Darboux-Weinstein neighborhood $U$ of the diagonal $\Delta$ of $(M \times M, \omega \oplus -\omega)$.

Now let $\lambda = \phi_F$ be a contractible topological Hamiltonian loop contained in $\mathcal{P}^{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ and $\Lambda = \{\lambda(s)\}_{s \in [0,1]}$ a given homotopy contracting the loop.

Let $\lambda \in \mathcal{P}^{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ and consider its extension $\iota^+(\lambda)$ as an element in $\mathcal{P}^{\text{ham}}(\text{Sympeo}_{D^2}(S^2), \text{id})$ obtained via the embedding $\iota^+$. Denote by $D^1(T^*S^2)$ the unit cotangent bundle and by $\overline{\Delta}$ the anti-diagonal 
$$\overline{\Delta} = \{(x, \overline{x}) \in S^2 \times S^2 \mid x \in S^2\}.$$ Then it is well-known that the geodesic flow of the standard metric on $S^2$ induces a symplectic diffeomorphism 
$$\Phi: D^1(T^*S^2) \to S^2 \times S^2 \setminus \overline{\Delta}$$ (4.1) 
where $\overline{\Delta}$ is the involution along a (fixed) equator. We regard the image $U = S^2 \times S^2 \setminus \overline{\Delta}$ as a Darboux-Weinstein neighborhood of the diagonal $\Delta \subset S^2 \times S^2$.

It is then easy to see the following

**Lemma 4.3.** Let $\lambda \in \mathcal{P}^{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id})$ and denote by $\lambda^+ = \iota^+(\lambda) \in \mathcal{P}^{\text{ham}}(\text{Sympeo}_{D^2}(S^2), \text{id})$ constructed as above. Then 
$$(\lambda_t^+ \times \text{id})(\Delta) \cap \overline{\Delta} = \emptyset.$$ In particular, the path $\lambda^+$ is $U$-engulfed.

Motivated by the above discussion, we will always consider only the engulfed case in the rest of the paper, unless otherwise said.

5. **Lagrangianization of engulfed Hamiltonian flows**

Now let $F : [0,1] \times M \to \mathbb{R}$ be a mean normalized engulfed Hamiltonian on a closed symplectic manifold $(M, \omega)$. The manifold $M$ carries a natural Liouville measure induced by $\omega^n$. Consider the diagonal Lagrangian $\Delta \subset (M \times M, \omega \oplus -\omega)$ identified with the zero section $\Delta \subset T\Delta$ in a Darboux chart $(V_\Delta, -d\Theta)$ of $\Delta$ in $M \times M$. Put a density $\rho_\Delta$ on $\Delta \subset M \times M$ induced by $\omega^n$ by the diffeomorphism of the second projection $\pi_2 : \Delta \to M$.

We fix Darboux neighborhoods 
$$V_\Delta \subset \overline{\Delta} \subset U_\Delta$$ and let $\omega \oplus -\omega = -d\Theta$ on $U_\Delta$ regarded as a neighborhood of the zero section of $T\Delta$ once and for all. Then 
$$\text{Graph} \phi_F^t \subset V_\Delta \quad \text{for all } t \in [0,1].$$

We consider the Hamiltonian $\pi_1^* F$, i.e., the one defined by 
$$\pi_1^* F(t,(x,y)) = F(t,x)$$ on $T\Delta$. This itself is not supported in $U_\Delta$ but we can multiply a cut-off function $\chi$ of $U_\Delta$ so that 
$$\chi \equiv 1 \quad \text{on } V_\Delta, \quad \text{supp } \chi \subset U_\Delta$$ and consider the function $F$ defined by $F(t,(x,y)) = \chi(x,y)\pi_1^* F(t,(x,y)) = \chi(x,y)F(t,x)$ so that the associated Hamiltonian deformations of $\psi^t(o_N)$ are unchanged. We note
that $F$ is compactly supported in $T^*\Delta$. and automatically satisfies the normalization condition
\[ \int_{\Delta} F(t, \phi_t^F(q)) \rho_{\Delta} = 0 \] (5.1)
for all $t \in [0, 1]$ where $\rho_{\Delta}$ is the measure on $\Delta$ induced by the Liouville measure on $M$ under the projection $\pi_2 : \Delta \subset M \times M \rightarrow M$.

Now we denote by $f_\theta$ the basic phase function of $\text{Graph} \phi^1_F = \phi^1_F(o_{\Delta})$. In the next section, we will examine the relationship between this function and the Calabi invariant of $F$.

6. Basic phase function $f_H$ and its axioms

In this section, we first recall the definition of basic phase function constructed in [Oh1] and summarize its axiomatic properties. Following the terminology of [PPS], we first introduce the following definition.

**Definition 6.1.** Let $L \subset T^*N$ be a Hamiltonian deformation of the zero section $o_N$. We call any continuous function $f : N \rightarrow \mathbb{R}$ a graph selector such that
\[ (q, df(q)) \in L \]
where $df(q)$ exists.

Existence of such a single-valued continuous function was proved by Sikorav, Chaperon [Cha] by the generating function method and by the author [Oh1] using the Lagrangian Floer theory. Lipschitz continuity of this particular graph selector follows from the continuity result established in section 6 [Oh1] specialized to the submanifold $S$ to be a point. The detail of another proof of this Lipschitz continuity is also given in [PPS] using the generating function techniques.

We denote by Sing $f$ the set of non-differentiable points of $f$. Then by definition
\[ N_0 = \text{Reg} f := N \setminus \text{Sing} f \]
is a subset of full measure and $f$ is differentiable thereon. In fact, for a generic choice of $L = \phi^1_H(o_N)$, $N_0$ is open and dense and Sing $f$ is a stratified submanifold of $N$ of codimension at least 1. (See [Oh9] for its proof.)

By definition,
\[ |df(q)| \leq \max_{x \in L} |p(x)| \] (6.1)
for any $q \in N_0$, where $x = (q(x), p(x))$ and the norm $|p(x)|$ is measured by any given Riemannian metric on $N$.

The following is an immediate corollary of the definition. We denote by $d_H$ the Hausdorff distance.

**Corollary 6.2.** As $d_H(\phi^1_H(o_N), o_N) \rightarrow 0$, $|df(q)| \rightarrow 0$ uniformly over $q \in N_0$.

However this result itself does not tell us much about the convergence of the values of the function $f$ itself because a priori the value of $f$ might not be bounded for a sequence $H_i$ such that $d_H(\phi^1_H(o_N), o_N) \rightarrow 0$.

In [Oh1], a canonical choice of $f$ is constructed via the chain level Floer theory, provided the generating Hamiltonian $H$ of $L = \phi^1_H(o_N)$ is given. The author called the corresponding graph selector $f$ the basic phase function of $L = \phi^1_H(o_N)$ and denoted it by $f_H$. We give a quick outline of the construction referring the readers to [Oh1] for the full details of the construction.
Consider the Lagrangian pair
\[(o_N, T^*_q N), \quad q \in N\]
and its associated Floer complex \(CF(H; o_N, T^*_q N)\) generated by the Hamiltonian trajectory \(z : [0, 1] \to T^* N\) satisfying
\[
\dot{z} = X_H(t, z(t)), \quad z(0) \in o_N, \ z(1) \in T^*_q N.
\] (6.2)
Denote by \(Chord(H; o_N, T^*_q N)\) the set of solutions. The differential \(\partial_{(H, s)}\) on \(CF(H; o_N, T^*_q N)\) is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation
\[
\begin{align*}
\frac{d\rho}{dT} + J \left(\frac{d\rho}{dT} - X_H(u)\right) &= 0 \\
\rho(T, 0) &\in \alpha_N, \ \rho(T, 1) \in T^*_q N.
\end{align*}
\] (6.3)
An element \(\alpha \in CF(H; o_N, T^*_q N)\) is expressed as a finite sum
\[
\alpha = \sum_{z \in Chord(H; o_N, T^*_q N)} a_z[z], \quad a_z \in \mathbb{Z}.
\]
We denote the level of the chain \(\alpha\) by
\[
\lambda_H(\alpha) := \max_{z \in \text{supp} \alpha} \{A^t_H(z)\}.
\]
The resulting invariant \(\rho^{ag}(H; [q])\) is to be defined by the mini-max value
\[
\rho^{ag}(H; [q]) = \inf_{\alpha \in [q]} \lambda_H(\alpha)
\]
where \([q]\) is a generator of the homology group \(HF(o_N, T^*_q N) \cong \mathbb{Z}\).

A priori, \(\rho^{ag}(H; [q])\) is defined when \(H^l\) intersects \(T^*_q N\) transversely but can be extended to non-transversal \(q\)'s by continuity. By varying \(q \in N\), this defines a function \(f_H : N \to \mathbb{R}\) which is precisely the one called the basic phase function in [Oh1].

**Proposition 6.3** (Section 7 [Oh1]). There exists a solution \(z : [0, 1] \to T^* N\) of \(\dot{z} = X(t, z)\) such that \(z(0) = q, \ z(1) \in o_N\) and \(A^t_H(z) = \rho^{ag}(H; [q])\) whether or not \(H^l\) intersects \(T^*_q N\) transversely.

We summarize the main properties of \(f_H\) established in [Oh1].

**Proposition 6.4** (Theorem 9.1 [Oh1]). When the Hamiltonian \(H = H(t, x)\) such that \(L = \phi^l_H(o_N)\) is given, there is a canonical lift \(f_H\) defined by \(f_H(q) := \rho^{ag}(H; \{pt\})\) that satisfies
\[
f_H \circ \pi(x) = h_H(x) = A^t_H(z^H_x)
\] (6.4)
for some Hamiltonian chord \(z^H_x\) ending at \(x \in T^*_q N\). This \(f_H\) satisfies the following property in addition
\[
\|f_H - f'_H\|_{\infty} \leq \|H - H'\|.
\] (6.5)

An immediate corollary of this proposition is the following proved in [Oh1] [Oh2].

**Proposition 6.5.** If \(H_i\) converges in \(L^{(1, \infty)}\), then \(f_{H_i}\) converges uniformly.

**Proof.** We set \(H' = 0\) in (6.5) and get the inequality
\[
\|f_{H_i}\|_{\infty} \leq \|H_i\|.
\]
By the convergence of \(H_i \to H\) in \(L^{(1, \infty)}\)-topology, the functions \(f_{H_i}\) are bounded. Then from Corollary 6.2 this proposition follows. \(\Box\)
**Remark 6.6.** We would like to emphasize that there is no such $C^0$-control of the basic generating function $h_H$ even when $H \to 0$ in hamiltonian topology.

Based on the above proposition, we define

**Definition 6.7.** Denote by $H_a$ the Hamiltonian generating the rescaled isotopy $t \mapsto \phi_{H_a}^t$ for $a > 0$. For any given topological Hamiltonian $H = H(t, x)$, we define its timewise basic phase function by

$$f_H(t, x) := \lim_{i \to \infty} f_{H_i}(x)$$

(6.6)

for any approximation sequence $H_i$ of $H$.

We will always denote a parametric version in bold-faced letters.

We note that the basic generating function $h_H$ could behave wildly as a whole. In particular, the total wave front $W_H \subset [0, 1] \times J^1(N)$ may behave wildly. But Proposition restricted to the basic Lagrangian selector converges nicely. Note that $\pi_H = \pi|_{L_H} : L_H = \phi_H^1(o_N) \to N$ is surjective for all $H$ and so $\pi_H^{-1}(q) \subset o_N$ is a non-empty compact subset of $o_N \cong N$. Therefore we can regard the 'inverse' $\pi_H^{-1} : N \to L_H \subset T^*N$ as an everywhere defined multivalued section of $\pi : T^*N \to N$.

We introduce the following general definition

**Definition 6.8.** Let $L \subset T^*N$ be a Lagrangian submanifold projecting surjectively to $N$. We call a single-valued section $\sigma$ of $T^*N$ with values lying in $L$ a Lagrangian selector of $L_H$.

Once the graph selector $f_H$ of $L_H$ is picked out, it provides a natural Lagrangian selector defined by

$$\sigma_H(q) := \text{Choice}\{x \in L_H | \pi(x) = q, A_H^c(z_x^H) = f_H(q)\}$$

via the axiom of choice where Choice is a choice function. It satisfies

$$\sigma_H(q) = df_H(q)$$

(6.7)

whenever $df_H(q)$ is defined. We call this particular Lagrangian selector of $L_H$ the basic Lagrangian selector. The general structure theorem of the wave front (see [EL], [PPS] for example) proves that the section $\sigma_H$ is a differentiable map on a set of full measure for a generic choice of $H$ which is, however, not necessarily continuous: This is because as long as $q \in N \setminus \text{Sing} f_H$, we can choose a small open neighborhood of $U \subset N \setminus \text{Sing} f_H$ of $q$ and $V \subset L_H = \phi_H^1(o_N)$ of $x \in V$ with $\pi(x) = q$ so that the projection $\pi|_V : V \to U$ is a diffeomorphism.

### 7. Calabi homomorphism and basic phase function

We first prove the following general theorem in arbitrary dimension. We recall that $f_{\Sigma}$ converges to $f_{\Sigma}$ uniformly.

**Theorem 7.1.** Let $\lambda = \phi_F$ be any contractible topological Hamiltonian loop in $\mathcal{P}^{bam}(\text{Sympeo}_U(M, \omega), \text{id})$ and with $U = M \setminus B$ where $B$ is a closed subset of nonempty interior. Choose an approximating sequence $F_i$. Denote by

$$\overline{\text{Cal}}(F) = \frac{1}{\text{vol}_{\omega}(M)} \int_0^1 \int_M F \mu_\omega \, dt$$

for the Liouville measure associated to $\omega$. Then

$$f_{\Sigma}(x) = \overline{\text{Cal}}(F)$$

(7.1)
for all $x \in M$.

**Proof.** Let $F_i = \text{Dev}(\phi_{F_i})$ which is given by

$$F_i(t, x) = F_i(t, x) - c_i(t)$$

where

$$c_i(t) = \frac{1}{\text{vol}_\omega(M)} \int_M F_i(t, x) \mu_\omega.$$  

Then we have

$$F_i(t, x) \equiv -c_i(t)$$

and so

$$\int_0^1 F_i(t, x) \, dt = -\int_0^1 c_i(t) \, dt = -\text{Cal}_U(F_i)$$

for all $x \in B$.

Since $F_i$ is an approximating sequence of topological Hamiltonian $F$, it follows $F_i \to F$ in $L^{1, \infty}$-topology. Therefore applying (6.5) to $H = F_i$ and $H' = 0$ and using the convergence \(|F - F_i| \to 0\) as $i \to \infty$, we obtain the inequality

$$\|F\| - \frac{1}{2} \leq f_{F_i} \leq \|F\| + \frac{1}{2}$$

for all sufficiently large $i$’s.

Here now enters in a crucial way the fact that $\phi_F$ generates a topological Hamiltonian loop, not just a path. Together with the inequality (see (6.1))

$$|df_{F_i}| \leq \|\phi_{F_i}, id\| \to 0,$$

it follows that we can choose a subsequence, again denoted by $F_i$, so that $f_{F_i} \to c$ uniformly for some constant $c$.

Therefore it remains to show that this constant is indeed the value $\text{Cal}(F)$. Denote $K = \text{supp} F$ which is a compact subset of $U = M \setminus B$. We now recall the definition of Hamiltonian topology on noncompact manifolds, Definition 2.7. By definition, there exists $\delta > 0$ such that

$$\text{supp} F_i \subset \text{Int} K(1 + \delta/2) \subset K(1 + \delta) \subset U$$

where $K(1 + \delta)$ is the (closed) $\delta$-neighborhood of $K$ for all sufficiently large $i$’s. In particular,

$$B(1 + \delta/2) \subset M \setminus K(1 + \delta/2).$$

For any such $i$’s, we also have

$$F_i \equiv 0, \quad \phi^d_{F_i} \equiv \text{id}$$

on $B(1 + \frac{\delta}{2})$. In particular,

$$\text{Graph} F_i \cap o_\Delta \supset o_{B(1 + \frac{\delta}{2})}.$$  

Therefore the same properties stated above as for $F_i$ still hold for $F_i$ except the values thereof on $B$ are changed to $-c_i(t)$.

Let $q \in o_B$ be any point in its interior. By the spectrality of the values of $f_{\mathcal{Z}}(q)$ (Theorem 5.3 [Oh9]), there is a point $x \in T^*_q M \setminus \text{Graph} \phi^d_{F_i}$ such that $(\phi^d_{F_i})^{-1}(x) \in o_\Delta$ and

$$f_{\mathcal{Z}}(q) = A^d(\frac{F_i}{x}).$$

We denote $(\phi^d_{F_i})^{-1}(x) = (q', q')$.  

Because of this, \( \phi_{F_i}^t \to id \) as \( i \to \infty \) by definition of the approximating sequence \( F_i \) of \( F \). Combining this with \( q = (q, q) \in \text{Int} \, o_B, \, \pi_{\Delta}(x) = q \), we derive

\[
d((\phi_{F_i}^t)^{-1}(x), x), \, d(x, \pi_{\Delta}(x)) < \frac{\delta}{4}
\]

for all sufficiently large \( i \)'s. Then \( d((\phi_{F_i}^t)^{-1}(x), (q, q)) < \frac{\delta}{4} \). Since \( F_i(t, x) = F_i(t, x) \)

for \( x = (x, y) \), the associated Hamiltonian trajectory \( z_{x}^F \) has the form \( (\phi_{F_i}^t(q'), q') \)

where \( (\phi_{F_i}^t)^{-1}(x) = (q', q') \). But \( d(q, q') < \frac{\delta}{4} \) and hence \( q' \in M \setminus B(1+\delta) \subset K(1+\frac{\delta}{4})). \)

(We refer to the proof of Lemma 7.5 [Oh9] for a similar argument used for a similar purpose.)

Therefore \( \phi_{F_i}^t(q') \equiv q' \) for all \( t \in [0, 1] \). This proves that \( z_{x}^F \) must be the constant trajectory \( z_{x}^F(t) \equiv q \). Then we compute its action value

\[
f_{\tilde{F}_i}(q) = A^c_i((\tilde{F}_i)^{-1})
\]

\[
= - \int_0^1 F_i(t, q) \, dt = - \int_0^1 F_i(t, q) \, dt = \int_0^1 c_i(t) \, dt = \text{Cal}_U(F_i).
\]

Since \( F_i \to F \) in \( L^{1,\infty} \)-topology and \( \text{supp} \, \phi_{F_i}, \, \text{supp} \, \phi_F \subset U \), it also follows \( \text{Cal}_U(F_i) \to \text{Cal}_U(F) \) as \( i \to \infty \). This proves indeed \( f_{\tilde{F}_i} \to \text{Cal}_U(F) \).

An examination of the argument at the end of the proof leading to the identification of the constant with the \( \text{Cal}_U(F) \) shows that the reason why the convergence \( \phi_{F_i}^t \to id \) enters is because we need for the projection \( \pi_{\Delta}(x) \) to lie outside \( \text{supp} \, F_i \) to get the required identification. This needed property automatically holds for the projection \( U_\Delta \to \Delta \) with \( \delta = D^2 \) of the canonical Darboux-Weinstein neighborhood obtained through the embedding \([4.1]\) in section 4. This is because under this embedding the projection \( \pi_{\Delta}(x, y) \) is nothing but the mid-point projection of \( (x, y) \) along the geodesic connecting the points \( x, y \in S^2 \). Since the upper hemisphere \( D^2_+ \subset S^2 \) is gedeasically convex, \( \pi_{\Delta}(x, y) \) is always contained in \( \text{Int} \, D^2_+ \) whenever \( x, y \in \text{Int} \, D^2_+ \). In particular \( \pi_{\Delta}(\phi_{F_i}^t(q'), q') \in \text{Int} \, D^2_+ \) if \( q' \in \text{Int} \, D^2_+ \) and hence the point \( \pi_{\Delta}(x) = \pi_{\Delta}(\phi_{F_i}^t(q'), q') \) cannot be projected to a point \( (q, q) \) with \( q \in B = D^2 \) irrespective of the convergence \( \phi_{F_i}^t \to id \). This eliminates the above somewhat subtle argument for the case of our main interest. An implication of this consideration leads to the following stronger result for this case in that it applies to an arbitrary path not just to loops.

**Theorem 7.2.** Let \( \lambda = \phi_F \) be any topological Hamiltonian path supported in \( \text{Int} \, D^2 \).

Denote by \( F \) the associated Hamiltonian on \( D^1(T^* \Delta_{S^2}) \cong S^2 \times S^2 \setminus \Delta_{S^2} \) constructed as before (via the embedding \([4.1]\)). Then

\[
f_{\tilde{F}}(x) = \text{Cal}(F)
\]

for all \( x \in D^2_+ \).

Of course, in this case, \( f_{\tilde{F}} \) will not be constant on \( D^2_+ \) in general.

8. Extension of Calabi homomorphism

We recall from the definition of \( \mathcal{P}^{\text{hom}}(\text{Symp}(M, \omega), id) \) with \( U = M \setminus B \) that \( \phi_{\text{Ham}(s)}^t \equiv id \) and \( H \equiv 0 \) on \( B = M \setminus U \) for a nonempty open subset of \( M \), and hence
$H(s) \equiv c(s)$ on $B$ with

$$c(s) = \text{Cal}_U(H(s)) = \frac{1}{\text{vol}_U(M)} \int_0^1 \int_M H(s) \omega \, dt.$$  

Engulfedness of $H$ enables us to do computations on a Darboux-Weinstein neighborhood $V_\Delta$ of the diagonal $\Delta \subset M \times M$, which we regard either as a subset of $M \times M$ or that of $T^*\Delta$ depending on the given circumstances. At the end, we will apply the computations to the given approximating sequence of homeotopy of contractible topological Hamiltonian loop.

Now we further specialize to the case of our main interest $D^2$. We embed $D^2$ into $S^2$ as the upper hemisphere $D^2_+$ and denote $B = D^2_-$, the lower hemisphere.

The following is the main conjecture to beat which was originally proposed in [Oh7]. This is the only place where the restriction to the two-disc $D^2$ is needed, but we expect the same vanishing result hold for higher dimensional disc $D^{2n}$ or even for general pair $(M, B)$, which is a subject of future study.

**Conjecture 8.1.** Assume $M = S^2$ and $B = D^2_-$ be the lower hemisphere as above. Let $\Lambda = \{\phi^t_H(s)\}_{(s,t) \in [0,1]^2}$ be a homeotopy contracting a topological Hamiltonian loop $\phi_F$ with $F = H(1)$ such that $\phi^1_H(s) \equiv \text{id}$. Let $\underline{f}_F$ be the limit basic phase function defined by $\underline{f}_F = \lim_{i \to \infty} f_{F_i}$. Then $\underline{f}_F = 0$.

Combining Theorem 7.1 and Conjecture 8.1 we have obtained the proof of the following.

**Corollary 8.2.** Suppose Conjecture 8.1 holds and let $F_i$ be as therein. Then

$$\lim_{i \to \infty} \text{Cal}(F_i) = 0.$$  

Equivalently, we have $\overline{\text{Cal}}(F) = 0$.

Therefore we have proved

**Theorem 8.3.** Suppose Conjecture 8.1 holds. Then the homomorphism $\overline{\text{Cal}}_{\text{path}} : \mathcal{P}_{\text{ham}}(\text{Sympeo}(D^2, \partial D^2), \text{id}) \to \mathbb{R}$ descends to a homomorphism

$$\overline{\text{Cal}} : \text{Hameo}(D^2, \partial D^2) \to \mathbb{R}$$

which restricts to $\text{Cal} : \text{Ham}(D^2, \partial D^2) \to \mathbb{R}$.

So the main remaining task is to prove Conjecture 8.1 which will prove all the conjectures stated in the present paper. In the next section, we will prove the conjecture for the weakly graphical topological Hamiltonian loop on the disc.

**Part 2.** Weakly graphical topological Hamiltonian loops on $D^2$

**9.** Geometry of graphical symplectic diffeomorphisms in 2-dimension

We start with the following definition in general dimension.

**Definition 9.1.** Let $\Psi : U_\Delta \to V$ be a Darboux-Weinstein chart of the diagonal $\Delta \subset M \times M$ and $\pi_\Delta : U_\Delta \to \Delta$ the associated projection.
(1) We call an engulfed symplectic diffeomorphism \( \phi : M \to M \) \( \Psi \)-graphical if the projection \( \pi \Delta \) is one-one, and an engulfed symplectic isotopy \( \{ \phi^t \} \) \( \Psi \)-graphical if each element \( \phi^t \) \( \Psi \)-graphical. We call a Hamiltonian \( F = F(t, x) \) \( \Psi \)-graphical if its associated Hamiltonian isotopy \( \phi^t F \) is \( \Psi \)-graphical.

(2) We call a topological Hamiltonian loop \( F \) is strongly (resp. weakly) \( \Psi \)-graphical, if it allows an approximating sequence \( F_i \) each element of which is \( \Psi \)-graphical (resp. whose time-one map \( \phi^1 F_i \) is \( \Psi \)-graphical). Denote by \( F^a \) the time-dependent Hamiltonian generating the path \( t \mapsto \phi^a(t) \). The statement (2) of this definition is equivalent to saying that each \( F^a \) is \( \Psi \)-graphical for \( a \in [0, 1] \). We remark that any symplectic diffeomorphisms sufficiently \( C^1 \)-close to the identity is graphical, but not every \( C^0 \)-close one.

Recalling the geodesic flow is a Hamiltonian flow, we simplify our notation

\[
\exp_y t E(y, \phi(y)) =: (1 - t)y + t \phi(y)
\]

\[
E(y, x) := \exp_y x =: x - y.
\]  

Note that when \( d(\phi^1 F^a, id) \) is sufficiently small, we have the inclusion \( U_y \times U_y \subset V_\Delta \subset M \times M \) for a sufficiently small Darboux neighborhood \( U_y \subset M \).

In the rest of this section, we restrict ourselves to the two dimensional case. We identify \( U_y \times U_y \hookrightarrow T^*_\Delta \) by the explicit linear coordinate changes

\[
Q = q_1 - \frac{p_2}{2}, \quad q = q_1 + \frac{p_2}{2}, \quad p = q_1 + \frac{p_2}{2}, \quad \frac{1}{2} j \phi = (Q, P) \circ \pi_1 \quad \text{and} \quad (q, p) = (Q, P) \circ \pi_2 \quad \text{in this Darboux-Weinstein chart.}
\]

Then we have

\[
Q = q_1 - \frac{p_2}{2}, \quad q = q_1 + \frac{p_2}{2}, \quad p = q_1 + \frac{p_2}{2}.
\]

In short, we write

\[
x = (Q, P) = q - \frac{1}{2} j p, \quad y = (q, p) = q + \frac{1}{2} j p
\]

where \( j : \mathbb{R}^2 \times \mathbb{R}^2 \) is the linear map given by \( j(p_1, p_2) = (-p_2, p_1) \).

In dimension 2, we prove the following interesting phenomenon. Although we have not checked it, it is unlikely that similar phenomenon occurs in higher dimensions. This theorem will not be used in the proofs of main results of the present paper but of its own interest.

**Theorem 9.2.** Suppose \( \phi : M \to M \) is a \( \Psi \)-graphical symplectic diffeomorphism and let \( \text{Graph} \phi = \text{Image} \alpha_\phi \) for a closed one-form \( \alpha_\phi \) on \( \Delta \). Then for any \( 0 \leq r \leq 1 \), the projection \( \pi_2 : M \times M \to M \) restricts to a one-one map to \( \text{Image} r \alpha_\phi \subset M \times M \). In particular

\[
\text{Image} r \alpha_\phi = \text{Graph} \phi_r
\]

for some symplectic diffeomorphism \( \phi_r : M \to M \) for each \( 0 \leq r \leq 1 \).

**Proof.** We have only to prove the map

\[
q \mapsto q + \frac{r}{2} j \alpha_\phi(q)
\]

is one-one. This is because it is the composition of the maps

\[
\Delta \to \text{Image} \alpha_\phi; \quad q \mapsto (q, r \alpha_\phi(q))
\]
and the projection $\pi_2 : \text{Image } r \alpha_\phi \to M$ where the first map is a bijective map. Denote this map by $\psi_r$.

Since the map $\psi_r$ has degree 1, it will be enough to prove that is an immersion since the latter will imply that the map must be a covering projection. Therefore we need to prove that the derivative

$$d\psi(q) = I + \frac{r}{2} j \nabla \alpha_\phi(q)$$

is invertible for all $q$ and $0 \leq r \leq 1$. Here $\nabla \alpha_\phi$ is the covariant derivative of the one-form $\alpha_\phi$ with respect to the flat affine connection $\nabla$. We regard it as a section of $\text{Hom}(T\Delta, T^*\Delta)$, i.e., a bundle map

$$\nabla \alpha_\phi : T\Delta \to T^*\Delta.$$

**Lemma 9.3.** At each point $q \in \Delta$, the linear map

$$\nabla : v \mapsto \nabla_v \alpha_\phi$$

is a symmetric operator, i.e., it satisfies

$$\langle \nabla_v \alpha_\phi, w \rangle = \langle \nabla_w \alpha_\phi, v \rangle$$

(9.6)

for all $v, w \in T_q\Delta$ at any $q \in \Delta$.

**Proof.** This immediately follows from the fact that any closed one-form can be locally written as $\alpha_\alpha = df_\phi$ for some function on $\Delta$. Then $\nabla \alpha_\phi = D^2 f_\phi$ which is the Hessian of the function $f_\phi$ which is obviously symmetric. $\square$

We first prove the following general result on the set the set of $2 \times 2$ symplectic matrices.

**Lemma 9.4.** Let $A$ be a $2 \times 2$ symmetric matrix. Then

$$\det(I + r j A) > 0$$

(9.7)

for all $r \in [0, 1]$, provided it holds at $r = 1$, i.e., provided

$$\det(I + j A) > 0.$$ 

The same holds for the opposite inequality.

**Proof.** Denote $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$. Then straightforward computation shows

$$jA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} -c & -b \\ a & c \end{pmatrix}.$$ 

In particular $\text{tr}(jA) = 0$ and hence

$$\det(I + j A) = 1 + \det(j A) = 1 + (ab - c^2).$$

Therefore $\det(I + j A) > 0$ is equivalent to

$$1 + (ab - c^2) > 0.$$ 

For $r = 0$, $I + r j A = I$ and so the inequality obviously holds. On the other hand, if $r \in (0, 1]$, we derive

$$1 + r^2(ab - c^2) \geq r^2(1 + (ab - c^2)) > 0$$

which finishes the proof. $\square$
Remark 9.5. Note that if $A$ is symmetric, then $jA \in sp(2)$ the Lie algebra of the symplectic group $Sp(2)$. Then the set $\{B \in sp(2) \mid \det(I - B) = 0\}$ is given by the equation

$$1 + (ab - c^2) = 0; \quad B = \begin{pmatrix} c & b \\ -a & -c \end{pmatrix}$$

which defines a hypersurface in $sp(2)$. If we denote $sp_{\pm}(2) = \{B \in sp(2) \mid \pm \det(I - B) > 0\}$

what this lemma shows that each component thereof can be written as the union

$$sp_{\pm}(2) = \bigcup_{B \in sp_{\pm}(2)} \{r \cdot B \mid r \neq 0\}$$

respectively.

By the hypothesis, it follows that $\psi = \psi_1$ is an orientation preserving diffeomorphism and so $\det d\psi(q) > 0$. We now compute

$$\det d\psi(q) = \det \left( I + \frac{1}{2} \nabla \phi(q) \right)$$

and

$$d\psi_r(q) = I + \frac{r}{2} \nabla \phi(q).$$

By Lemma 9.4 we derive $d\psi_r(q) > 0$ and so $\psi_r$ is immersed for all $r$. This finishes the proof of Theorem 9.2.

□

10. Homotopy invariance of basic phase function

Let $\Lambda = \{\phi^t_H(s)\}$ be a smooth two-parameter family satisfying $H \equiv 0$ on a neighborhood of $B \subset M$ by definition of $\mathcal{P}^{ham}(\text{Symp}_U(M, \omega), id)$ with $U = M \setminus B$. We denote by $K = K(s, t, x)$ a $s$-Hamiltonian of the 2-parameter family $\Lambda = \{\phi^t_H(s)\}$

with $K(s, 0, \cdot) \equiv 0$: The latter choice is possible we have the $s$-Hamiltonian flow $s \mapsto \phi^t_H(s) \equiv id$ and so we can set $K(s, 0, \cdot) \equiv 0$.

We first prove a few lemmata.

The following lemma immediately follows from the same calculation done in [P, section 6.1],[Oh2]. For readers’ convenience, we give its complete proof.

Lemma 10.1. $K \equiv 0$ on a neighborhood of $B \subset M$.

Proof. We recall the identity

$$\frac{\partial K}{\partial t} = \frac{\partial H}{\partial s} - \{K, H\}.$$  \hfill (10.1)

Recall $H(s, t, x) \equiv 0$ on a neighborhood of $B$ because we assume that $H$ is compactly supported in $U = M \setminus B$ by definition. From this, it follows $\frac{\partial K}{\partial t} \equiv 0$ thereon.

Together with the initial condition $K(s, 0, \cdot) \equiv 0$, this proves $K(s, 0, x) \equiv 0$ for all $x$ in a neighborhood of $B$.

This in particular implies $\phi^1_K \in \mathcal{P}^{ham}(\text{Symp}_U(M, \omega), id)$. Next we have the following coincidence of the Calabi invariant.

Lemma 10.2. $\text{Cal}_U(K^1) = \text{Cal}_U(H(1))$
Proof. First note $\phi_{K^1}^t = \phi_{H(1)}^t$. Denote by $\Lambda(s, t) = \phi_{H(s)}^t$ the two-parameter family associated to $H$. Then

$$\Lambda(0, t) \equiv id \equiv \Lambda(s, 0)$$

by the requirement $H(0, t, x) \equiv 0$. Therefore the Hamiltonian path $t \mapsto \phi_{H(1)}^t := \Lambda(1, t)$ is smoothly homotopic to the path $s \mapsto \phi_{K^1}^s := \Lambda(s, 1)$ relative to the ends and hence we have the lemma by the smooth homotopy invariance of $\text{Cal}_U$: In fact, an explicit homotopy $\Upsilon : [0, 1]^2 \to \text{Symp}_U(M, \omega)$ between them is given by the formula

$$\Upsilon(s, t) = \begin{cases} 
\Lambda(t, 1 + 2s(t - 1)) & \text{for } 0 \leq s \leq \frac{1}{2} \\
\Lambda(2(s - 1/2) + 2t(1 - s), t) & \text{for } \frac{1}{2} \leq s \leq 1.
\end{cases}$$

The map $\Upsilon$ satisfies

$$\Upsilon(0, t) = \Lambda(t, 1) = \phi_{K^1}^t, \quad \Upsilon(1, t) = \phi_{H(1)}^t, \quad \Upsilon(s, 0) = id, \quad \Upsilon(s, 1) = \Lambda(1, 1) = \phi_{H(1)}^1 = \phi_{F^1}^1$$

and hence is the required homotopy relative to the ends. \qed

Now we prove homotopy invariance of the basic generating function and the basic phase functions.

**Proposition 10.3.** $\tilde{h}_{K^1} = \tilde{h}_{H(1)}$ and $f_{K^1} = f_{H(1)}$

**Proof.** We apply the first variation formula (1.3) to $z_{K^1}^q(s)$ and $z_{H(1)}^q(t)$ respectively, and obtain

$$\tilde{d}h_{K^1}(v) = \langle \Theta(\phi_{K^1}(q)), T\phi_{K^1}^1(v) \rangle$$
$$\tilde{d}h_{H(1)}(v) = \langle \Theta(\phi_{H(1)}(q)), T\phi_{H(1)}^1(v) \rangle$$

for any $v \in T_q\Delta$. Since $\phi_{K^1}^1 = \phi_{H(1)}^1$, we have proved $\tilde{d}h_{K^1} = \tilde{d}h_{H(1)}$. On the other hand, for any point $q \in \Delta_B$, $\tilde{h}_{H(1)} \equiv 0 \equiv \tilde{h}_{K^1}$ on a neighborhood of $q$ in $T^*\Delta$ and so both $z_{K^1}^q$ and $z_{H(1)}^q$ are constant. Therefore the values of both $\tilde{h}_{K^1}$ and $\tilde{h}_{H(1)}$ are zero at such a point $q \in \Delta_B$. This finishes the proof of the first equality.

For the proof of $f_{K^1} = f_{H(1)}$, the first equality in particular implies that the sets of critical values of the action functionals

$$\mathcal{A}_{K^1}^{cl}, \mathcal{A}_{H(1)}^{cl} : \Omega(O, T^*M) \to \mathbb{R}$$

coincide. Then standard homotopy argument used in the homotopy invariance of (in fact any type of) the spectral invariant applies to prove $\rho^\text{lag}(H, \{ q \}) = f_H(q)$ for each $q \in N$ for general $H$. This finishes the proof. \qed

Then combining Lemma 10.2 and Proposition 10.3 we also derive

$$f_{K^1} = f_{K^1} + \text{Cal}_U(K^1) = f_{H(1)} + \text{Cal}_U(H(1)) = f_{H(1)} \quad (10.2)$$

With this preparation, in the proof of Theorem 11.1 later, we will use $K^1$ instead of $H(1)$ in our proof. This is because we exploit a special nature of Alexander isotopy, Proposition 11.3, that is not shared by a general two-parameter family of Hamiltonian isotopies.
11. Vanishing of basic phase function for the graphical case on $D^2$

Now we restrict to the context of Theorem 11.1. Let $F$ be a topological Hamiltonian generating a topological Hamiltonian loop $\phi_F$ on the 2-disc $D^2$ with $\text{supp} \ F \subset \text{Int} \ D^2$. We consider an approximating sequence $F_i$ and $F_i = H_i \circ (1)$ with $\text{supp} \ F_i \subset \text{Int} \ D^2$. We embed $D^2$ into $S^2$ as the upper hemisphere and then extend $F_i$ canonically to whole $S^2$ by zero, and consider the graphs $\text{Graph} \phi_1 F_i$ in $S^2 \times S^2$. Note $\text{supp} \phi_1 F_i \subset D^2_+ \times D^2_-$ and hence

$$\text{Graph} \phi_1 F_i \cap \Delta \supset \Delta D^2_+ \cup \Delta D^2_- \setminus (1-\delta)$$

for some $\delta > 0$ for all $t \in [0, 1]$ independent of sufficiently large $i$'s depending only on $F$, provided $d(\phi_1 F_i, \text{id}) \leq \frac{\delta}{2}$. We fix the given topological Hamiltonian loop $\phi_F$ and fix such $\delta > 0$.

Then we consider the normalization $F_i$ of $F_i$ on $S^2$ and define Hamiltonian

$$F_i(t, x) := \chi(x) F_i(t, x), \quad x = (x, y)$$
on $T^* \Delta$ with a slight abuse of notation for $F_i$.

Theorem 11.1. Conjecture 1.5 holds for any weakly graphical topological Hamiltonian loop on $S^2$ arising as above.

An immediate corollary of Theorem 11.1 and 11.4 is the following vanishing result of Calabi invariant.

Corollary 11.2. Suppose $\lambda = \phi_F$ be an engulfed topological Hamiltonian loop as in Theorem 7.1. Assume $\lambda$ is weakly graphical. Then $\text{Cal} \text{path}(\lambda) = 0$.

The remaining section will be occupied by the proof of Theorem 11.1. Let $F$ be a graphical topological Hamiltonian loop and $F_i$ be an approximating sequence that is $\Psi$-graphical for a Darboux-Weinstein chart $\Psi$.

The following proposition reflects some special characteristic of Alexander isotopy relative to the general hameotopy.

Proposition 11.3. Suppose that $\phi_1 F_i$ is $\Psi$-graphical. Then $\phi_{F_i, a}$ defined as in Lemma 3.1 is also $\Psi$-graphical for all $0 \leq a \leq 1$.

Proof. The proof of this proposition is similar to that of Theorem 9.2 in its spirit but is much simpler than it. It is enough to prove the map $\kappa_a = \pi_\Delta \circ (\pi_\Delta)^{-1} : S^2 \to S^2$ is one-one since the map $(\pi_\Delta)^{-1} : S^2 \to \text{Graph} \phi_1 F_i$ is bijective. But the map $\kappa_a$ is given by

$$\kappa_a(y) = \frac{1}{2}(y + \phi_{F_i, a}(y))$$
in the affine chart. (See (9.1) for the notational convention too.) A straightforward computation shows

$$d\kappa_a(y) = \begin{cases} \frac{1}{2}(Id + d\phi_{F_i(a)}(y)) & \text{for } y \text{ with } |y| \leq a(1 - \eta) \\ Id & \text{otherwise} \end{cases}$$

Since $\kappa_1$ is an orientation-preserving diffeomorphism and $S^2$ is compact, there exists $\delta > 0$ such that

$$\det(d\kappa_1(y)) > \delta > 0$$
for all \( y \in S^2 \). From the expression of \( d\kappa_a(y) \), it follows \( d\kappa_a(y) = \frac{d\kappa_1(y)}{a} \) and hence
\[
\det(d\kappa_a(y)) = \det\left(\frac{d\kappa_1}{a}\right) > \delta > 0 \tag{11.1}
\]
for all \( a \in [0,1] \) and \( y \in S^2 \). This implies \( \kappa_a : S^2 \to S^2 \) is an immersion and so a covering map of degree 1. Therefore it must be a one-one map. \( \square \)

Now we are ready to give the proof of Theorem 11.1.

Proof of Theorem 11.1. Let \( F \) be the mean-normalized Hamiltonian associated to the topological Hamiltonian loop on \( S^2 \) arising from the compactly supported Hamiltonian \( F \) on \( D^2 \) of the given topological Hamiltonian that is weakly \( \Psi \)-graphical. We fix a sequence of weakly \( \Psi \)-graphical approximating sequence \( F_i \) and its Alexander isotopy \( \Lambda_i = \Lambda_{i,e} \) defined as in \[95\]. We then consider the corresponding mean-normalized Hamiltonian \( F_i \) and the isotopy lifted to \( S^2 \) as before. We also denote \( K_i = K_i(a,t,x) \) the associated mean-normalized \( a \)-Hamiltonian and \( G_i(a,x) = K_i(a,1,x) \). It was proved in \[Oh2\] that if \( t \)-Hamiltonian is normalized and the \( a \)-Hamiltonian is so at one point of \( a \), then \( a \)-Hamiltonian is automatically normalized for all \( a \). (Actually, this fact is an easy consequence of \[10.1\].) Therefore \( G_i \) is automatically mean-normalized and so \( G_i = G_i \) if we set \( G_i(0,\cdot) = 0 \). Recall \( \Lambda_i(0,t) = id \) for all \( t \in [0,1] \).

We denote \( H_i(a) \) the \( t \)-Hamiltonian defined by \( H_i(a,t,x) = H_i(a,t,x) \). By the \( \Psi \)-graphicality of \( H_i(a) \) from Proposition 11.3 for \( a \in [0,1] \), \( G_i^a \) is \( \Psi \)-graphical where \( G_i^a = G_i(a,s,x) = aG_i(as,x) \). We recall from Proposition 11.3 \( \phi_{H_i(a)} = \phi_{G_i}^a \), and
\[
\phi_{G_i(a)} = \phi_{G_i}^a.
\]
By the \( \Psi \)-graphicality of \( G_i^a \), the basic phase function \( f_{G_i^a} \) is defined everywhere on \( \Delta \) as a smooth (single-valued) function.

Then the function \( f_{G_i^a}^a : [0,1] \times \Delta \to \mathbb{R} \) defined by \( f_{G_i^a}(a,q) = f_{G_i^a}(q) \) satisfies the Hamilton-Jacobi equation
\[
\frac{\partial f_{G_i^a}}{\partial a}(a,q) - G_i(a,df_{G_i^a}(q)) = 0. \tag{11.2}
\]
We postpone the derivation of this equation till Appendix. We note \( d_a f_{G_i^a}(a,q) = df_{G_i^a}(q) \) by definition of \( f_{G_i^a} \).

We consider the integrals
\[
g_i(a) := \int_\Delta f_{G_i^a} \pi_2^\omega.
\]
Then
\[
g'_i(a) = \int_\Delta \frac{\partial f_{G_i^a}}{\partial a} \pi_2^\omega = \int_\Delta G_i(a,df_{G_i^a}(q)) \pi_2^\omega.
\]
and so
\[
g_i(1) = \int_0^1 \int_\Delta f_{G_i^a} \pi_2^\omega da.
\]
By the identity
\[
\phi_{G_i}(o_\Delta) = \text{Graph} \phi_{G_i}^a = \text{Image} df_{G_i^a}
\]
and the bijectivity of the projection,
\[
\pi_2 : \phi_{G_i}^a(o_\Delta) \to S^2
\]
we can write \( df_G(a)(q) = \phi^a_{G_i}(y(q)) \) for the unique \( y(q) \) satisfying
\[
\pi_2(df_G(a)(q)) = y(q)
\]
for each given \( q \). For a subset \( L \subset S^2 \times S^2 \), we denote by \( \pi^L_2 : L \to S^2 \) the restriction of \( \pi_2 \) to \( L \). Then
\[
df_G(a)(q) = \phi^a_{G_i}(y, y) = \phi^a_{G_i}((\pi_2^L)^{-1}(y))).
\]
Furthermore, we have the equality
\[
\phi^a_{G_i}(o_{\Delta}) = \pi^L_2(o_{\Delta}) \circ \pi_2^L
\]
where \( \pi^L_2(o_{\Delta}) : \phi^a_{G_i}(o_{\Delta}) \to \Delta \) is the projection of \( \phi^a_{G_i}(o_{\Delta}) \) to \( \Delta \) along \( \pi_2^L \)-direction.

Combining these, we do integration over the fiber of the projection \( \pi_2 = \pi^L_2 \)
and derive
\[
\int_{\Delta} G_i(a, df_G(a)(q)) \pi^L_2 \omega = \int_{S^2} G_i(a, \phi^a_{G_i}(y, y))) \omega
\]
\[
= \int_{S^2} G_i(a, \pi^L_1 \phi^a_{G_i}(y, y))) \omega
\]
\[
= \int_{S^2} G_i(a, \phi^a_{G_i}(y)) \omega = \int_{S^2} G_i(a, y) \omega = 0
\]
where the last vanishing occurs by the mean-normalization condition of \( G_i \) and
the next to the last by the symplectic property of \( \phi^a_{G_i} \). This proves \( g_i(1) = 0 \) in
particular.

But we have \( f_{G_i} = f_{G_i(1)} = f_{G_i} \) by Proposition \( 10.3 \) and in particular
\[
f_{G_{i,1}} = f_{G_{i,1}} \to f_{G_{i,1}}
\]
uniformly. Combining the above discussion, we have proved \( \int_{\Delta} f_{G_{i}} \pi^L_2 \omega = 0 \). This
finishes the proof. \( \square \)

**Remark 11.4.**

(1) We would like to point out that while the average of \( G_i \) vanishes and \( \phi^0_{G_i} \to id \) uniformly over \( s \in [0, 1] \), unlike the \( t \)-Hamiltonian \( F_i \) which converges in hamiltonian topology, there is no a priori control of the \( C^0 \) behavior of the \( s \)-Hamiltonian \( G_i \) itself in general according to the definition of approximation sequence \( H_i \) of the hameotopy in Definition \( 2.6 \) (See \( 3.3 \) for the explicit form of \( G_i(s, \cdot) = K_i(s, 1, \cdot) \) in the case of Alexander isotopy, which evidently involves taking the derivative the \( t \)-Hamiltonian.) In this regard, the properties \( G_i, \phi^0_{G_i} = id \) and \( G_i(0, \cdot) = 0 \)
for \( s = 0 \) play a crucial role in the above proof since they give rise to the property that \( G_i(s, 1) \) become automatically normalized.

(2) The above proof strongly relies on the graphicality of the topological Hamiltonian \( F_i \), or more precisely on graphicality of its approximation sequence \( F_i \). Without this graphicality, one has to deal with emergence of the caustics of the projection \( \pi_\Delta : \text{Graph} \phi_{F_i} \to \Delta \) or equivalently the nondifferentiability locus of the basic phase function \( f_{G_{i,1}} \). More specifically the mean-normalization of Hamiltonian \( F_i \) does not seem to give rise the convergence
of
\[ \int_0^1 \int_\Delta G_i(a, df_x(q)) \pi_2^\ast \omega da \to 0 \]
as in the graphical case, for which a bit of measure theory argument gave rise to the required convergence. Here seems to enter the piecewise smooth Hamiltonian geometry of Lagrangian chains. We will elaborate this aspect elsewhere.

Appendix A. Timewise basic phase function as a solution to Hamilton-Jacobi equation

In this section, we show that the space-time basic phase function \( f_H \) defined by
\[ f_H(t, q) = f_{H^t}(q) \]
satisfies the Hamilton-Jacobi equation. More precise description of this statement is now in order.

Let \( N \) be an arbitrary compact manifold without boundary and let \( H = H(t, x) \) be a time-dependent Hamiltonian defined on the cotangent bundle \( T^\ast N \) and \( L = \phi^1_H(o_N) \) be the associated Hamiltonian deformation of \( o_N \). In this case, there is a canonical generating function of \( L \) associated to the Hamiltonian \( H \) given as follows.

We first start with the discussion on the basic generating function. (We refer the readers to [Oh9] for more detailed exposition on this.) For any given time-dependent Hamiltonian \( H = H(t, x) \), the classical action functional on the space \( P(T^\ast N) := C^\infty([0, 1], T^\ast N) \) is defined by
\[ A_{cl}^H(\gamma) = \int \gamma^\ast \theta - \int_0^1 H(t, \gamma(t)) \, dt. \] (A.1)
We denote \( L_H = \phi^1_H(o_N) \) and by \( i_H : L_H \hookrightarrow T^\ast N \) the inclusion map. For given \( x \in L_H \), we define the Hamiltonian trajectory
\[ z_H^x(t) = \phi_H^t((\phi^1_H)^{-1}(x)) \]
which is one satisfying
\[ z_H^x(0) \in o_N, \quad z_H^x(1) = x. \]

The function \( h_H : L_H \to \mathbb{R} \), called the basic generating function in [Oh9], is defined by
\[ h_H(x) = A_H(z_H^x). \]
It satisfies \( i_H^\ast \theta = dh_H \) on \( L_H \), i.e., \( h_H \) is a canonical generating function of \( L_H \) in that it satisfies
\[ i_H^\ast \theta = dh_H. \]

Then we consider the parametric version of basic generating function [1.1] which is defined by
\[ h_H(t, x) := h_H(t, x) \] (A.2)
on \( \text{Tr}_{\phi_H(o_N)} := \bigcup_{t \in [0, 1]} \{ t \} \times \phi_H(o_N) \). A straightforward calculation leads to
Proposition A.1. Consider the map

\[ \Psi_H : [0, 1] \times N \to T^*[0, 1] \times T^*N \cong T^*[0, 1] \times N \]

defined by the formula

\[ \Psi_H(t, q) = (t, -H(t, \phi_H^t(q)), \phi_H^t(q)) \] (A.3)

where \( \alpha_q \in \alpha_N \) associated to the point \( q \in N \). Then \( \Psi_H \) is an exact Lagrangian embedding of \([0, 1] \times N\). Denote the associated exact Lagrangian submanifold by \( \tilde{L} := \text{Image} \Psi_H \) and by \( i_L : \tilde{L} \to T^*[0, 1] \times N \) the inclusion map. Let \((t, a)\) be the canonical coordinate of \( T^*[0, 1] \). Then the timewise basic generating functions \( \tilde{h}_H, h_H \) satisfy

\[ d\tilde{h}_H = \Psi_H^* (\theta + a \, dt) \]

\[ dh_H = i_L^*(\theta + a \, dt) \] (A.4)

on \([0, 1] \times N\) and on \( \tilde{L} \) respectively. In particular, \( h_H \) is a generating function of the exact Lagrangian submanifold \( \tilde{L} \subset T^*[0, 1] \times N \).

In particular, we derive from (A.3) and (A.4) the following (phase space) Hamilton-Jacobi equation

\[ \frac{\partial h_H}{\partial t} + H = 0, \quad d_x h_H = i_L^* \theta \] (A.5)

on the smooth locus of \( \text{Tr} \phi_H^t (\alpha_N) \).

As a function on \( N \), not on \( L_H \), the basic generating function \( h_H \) is a multi-valued function. But the basic phase function \( f_H \) as a timewise graph selector, it satisfies

\[ h_H(t, x) = f_H(t, \pi(x)). \] (A.6)

In particular, substituting \( x = d_x f_H(t, q) \) into (A.6) and noting \( \pi(d_x f_H(t, q)) = q \), we obtain

\[ f_H(t, q) = h_H(t, d_x f_H(t, q)) = h_H \circ \sigma_H(t, q). \]

Here \( \sigma_H \) is the timewise version of the definition (6.7), whose image is contained in \( \tilde{L} \). Therefore

\[ df_H = d(h_H \circ \sigma_H) = \sigma_H^*(dh_H) = \sigma_H^*(i_L^*(\theta + a \, dt)) = (i_L \circ \sigma_H)^*(\theta + a \, dt) \]

on the smooth locus of \( f_H \) in \([0, 1] \times N\). But

\[ i_L \circ \sigma_H(t, q) = (t, -H(t, \sigma_H(t, q)), \sigma_H(t, q)). \]

Therefore

\[ (i_L \circ \sigma_H)^*(\theta + a \, dt) = \sigma_H^* \theta - H(t, \sigma_H(t, q)) \, dt \]

and hence

\[ df_H = \sigma_H^* \theta - H(t, \sigma_H(t, q)) \, dt. \]

We also have \( d\pi d\sigma_H (\partial H / \partial t) = 0 \) since \( \pi \sigma_H(t, q) = q \) for all \( t \). This implies

\[ \sigma_H^* \theta (\partial H / \partial t) = \sigma_H(t, q) \left( d\pi d\sigma_H (\partial H / \partial t) \right) = 0. \]

In particular, we have derived

\[ \frac{\partial f_H}{\partial t} = -H(t, \sigma_H(t, q)) \]
(on the smooth locus $N \setminus \text{Sing}(df_H)$). This is equivalent to the Hamilton-Jacobi equation
\[
\frac{\partial f_H}{\partial t}(t, q) + H(t, df_H(t, q)) = 0.
\]

When we are given a two-parameter Hamiltonian $H = H(s, t, x)$, a straightforward calculation shows the following proposition, whose proof we leave to the readers or to Appendix [Oh4].

**Proposition A.2.** Assume that $\{H(s)\}$ is a 1-parameter family of $t$-Hamiltonians and denote by $G = G(s, t, x)$ its $s$-Hamiltonian. Denote by $\Lambda = \{\phi^t_{H(s)}\}$ the 2-parameter Hamiltonian diffeomorphism. Consider the map $\Psi_{\Lambda}$ defined by the formula
\[
\Psi_{\Lambda}(s, t, q) = (s, G(s, t, \Lambda(s, t)(q)), t, -H(s, t, \Lambda(s, t)(q), \Lambda(s, t)(q))).
\]

Then it defines an exact Lagrangian embedding of $T^*[0, 1]^2 \times T^*N \cong T^*([0, 1]^2 \times N)$. Furthermore the 2-parameter timewise basic generating function $h_H$ given by $h_H(s, t, x) := h_H(s, t, x)$ satisfies
\[
\frac{dh_H}{dt} = i_\ast \hat{L}(\theta, L(s, t)) = \phi^t_{H(s)}(o_N) \quad (A.8)
\]
where $(s, t, b, a)$ is the canonical coordinates of $T^*[0, 1]^2$ and $i_\ast : \hat{L} \to T^*[0, 1]^2 \times T^*N$ is the inclusion map. In other words, $h_H$ is a generating function of the Lagrangian submanifold $\hat{L} \subset T^*[0, 1]^2 \times T^*N \cong T^*([0, 1]^2 \times N)$ given by
\[
\hat{L} := \text{Image } \Psi_{\Lambda}
\]

In particular, the following (phase space) Hamilton-Jacobi equation
\[
\begin{cases}
\frac{\partial h_H}{\partial t} + H = 0, & \frac{\partial h_H}{\partial s} - G = 0 \\
d_s h_H = i^\ast_{(s, t)} \theta, & L(s, t) = \phi^t_{H(s)}(o_N)
\end{cases}
\]

on $\hat{L}$ is derived from (A.7) and (A.8). By repeating the above discussion starting with the second equation for our present context of $N = \Delta$, and $f_H = f_H(s, t, x)$, we have derived the Hamilton-Jacobi equation (11.2).

**References**

[Ba] Banyaga, A., *Sur la structure du groupe des difféomorphismes qui préser vent une forme symplectique*, Comment. Math. Helvetici 53 (1978), 174–227.

[BS] Buhovsky, L, Seyfaddini, S., *Uniqueness of generating Hamiltonians for continuous Hamiltonian flows*, J. Symplectic Geom. 11 (2013), no. 1, 37–52.

[Ca] Calabi, E., *On the group of automorphisms of a symplectic manifold*, In Problems in analysis (ed. Gunning R.), pp 1-26, Princeton University Press, 1970.

[Cha] Chaperon, M., *Lois de conservation et géométrie symplectique*, C. R. Acad. Sci. 312 (1991), 345-348.

[El] Eliashberg, Y., *A theorem on the structure of wave fronts and its applications*, (Russian) Funktsional. Anal. i Prilzhen, 21 (1987), no. 3, 65–72.

[F] Fathi, A., private communication, 2005.

[GG] Gambaudo, J.-M., Ghys, E., *Commutators and diffeomorphisms of surfaces*, Ergod. Th. & Dynam. Sys. 24 (2004), 1591-1617.

[Oh1] Oh, Y.-G., *Symplectic topology as the geometry of action functional, I*, J. Differ. Geom. 46 (1997), 499–577.

[Oh2] Oh, Y.-G., *Normalization of the Hamiltonian and the action spectrum*, J. Korean Math. Soc. 42 (2005), 85–93.

[Oh3] Oh, Y.-G., *$C^0$-coerciveness of Moser’s problem and smoothing area preserving homeomorphisms*, preprint. [arXiv:math/0601183]
[Oh4] Oh, Y.-G., *Locality of continuous Hamiltonian flows and Lagrangian intersections with the conormal of open subsets*, J. Gökova Geom. Topol. 1 (2007), 1–32.

[Oh5] Oh, Y.-G., *The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows*, pp 149-177, Contemp. Math., 512, Amer. Math. Soc., Providence, RI, 2010.

[Oh6] Oh, Y.-G., *Extension of Calabi homomorphism and nonsimpleness of the area-preserving homeomorphism group of $D^2$*, arXiv.1010.1081 (withdrawn).

[Oh7] Oh, Y.-G., *Homotopy invariance of spectral invariants of topological hamiltonian flows*, arXiv.1111.5992 (withdrawn).

[Oh8] Oh, Y.-G., *Localization of Floer homology of engulfed topological Hamiltonian loop*, Commun. Info. Systems 13 (2014), no. 4, 399-443.

[Oh9] Oh, Y.-G., *Geometry of generating functions and Lagrangian spectral invariants*, submitted, arXiv:1206.3788.

[OM] Oh, Y.-G., Müller, S., *The group of Hamiltonian homeomorphisms and $C^0$ symplectic topology*, J. Symp. Geom. 5 (2007), 167 – 219.

[PPS] Paternain, G., Polterovich, L., Siburg, K., *Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory*, Mosc. Math. J. 3 (2003), no. 2, 593–619.

[P] Polterovich, L., *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures in Mathematics ETH Zurich. Birkhauser Verlag, Basel, 2001.

[Si] Sikorav, J.C., *Approximation of a volume-preserving homeomorphism by a volume-preserving diffeomorphism*, 2007, available at [http://www.umpa.ens-lyon.fr/symplexe/publications.php](http://www.umpa.ens-lyon.fr/symplexe/publications.php)

---

Center for Geometry and Physics, Institute for Basic Sciences (IBS), Pohang, Korea & Department of Mathematics, POSTECH, Pohang, Korea

E-mail address: yongoh1@postech.ac.kr