On the natural modes of helical structures

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Abstract—Natural modes of helical structures are treated by using the periodic dyadic Green’s functions in cylindrical coordinates. The formulation leads to an infinite system of one-dimensional integral equations in reciprocal (Fourier) space. Due to the twisted structure of the waveguide together with a quasi-static assumption the set of non-zero coefficients in reciprocal space is sparse and the formulation can therefore be used in a numerical method based on a truncation of the set of coupled integral equations. The periodic dyadic Green’s functions are furthermore useful in a simple direct calculation of the quasi-static fields generated by thin helical wires.

Index Terms—High-voltage power cables, helical waveguides, dispersion relations, open waveguides, volume integral equations.

I. INTRODUCTION

The purpose of this paper is to formulate a volume integral equation for the determination of the natural modes of helical structures. Low-frequency applications are of particular importance where it can be anticipated the existence of twisted modes of a particularly simple structure. The presented problem formulation is largely motivated by the need of being able to accurately model the field distribution and losses inside twisting three-phase high-voltage power cables at 50 Hz, see e.g., [8, 9]. The approach could also potentially be useful for analyzing the wave propagation characteristics of the so called litz wires.

Helical waveguide structures have been treated previously such as, e.g., with helical sheaths [5, 15], and approximations for wire helices [16, 26]. Presently, there are also very promising numerical techniques being developed that are based on Finite Element Modeling [8, 9] and the Method of Moments [22]. However, to our knowledge there has not been any general presentation regarding analytical modeling of the natural modes of helical structures. It is the aim of this paper to fill in this gap. On the other hand, there is a large body of literature on the general dispersion properties of open waveguides, see e.g., [12, 13, 18, 19, 24], as well as on the general properties of the electromagnetic volume integral equations, see e.g., [2–4, 6, 25, 27]. In particular, it is well known that the volume integral operators in electromagnetics are strongly singular and that many questions regarding their spectral theory remain largely open [6]. Nevertheless, it can be shown in very general settings that modes of open waveguides exist and can be interpreted in terms of poles of a meromorphic Fourier transform and that these poles depend continuously on the model data (except for points where poles coalesce or at the boundary of the domain of meromorphicity, i.e., at infinity and at the branch-point corresponding to the wavenumber of the exterior domain) [19]. When there are sources present, the modes (the discrete set of eigenfunctions) can be obtained as the residues of the poles and the non-discrete set is manifested as an integration along the branch-cut [5]. In practical circumstances, the branch-cut contribution can often be neglected [17].

In this paper, a general helical waveguide structure is treated by using classical analytic function theory and Fourier techniques (in particular the convolution theorem) in connection with the electromagnetic volume integral formulation and a cylindrical vector wave expansion of the related dyadic Green’s functions. The Floquet modes are defined as the poles of the corresponding integral operators, and the analytic periodic Green’s functions are derived by employing the classical Poisson summation formula. Two independent approaches (with and without explicit sources) are used to derive the resulting infinite system of one-dimensional integral equations. These equations can then be discretized by truncation and by using a standard collocation method.

As a useful byproduct, the periodic dyadic Green’s functions provide a simple direct calculation of the quasi-static fields generated by thin helical wires.

II. NATURAL MODES WITH FINITE SOURCES

A. Preliminaries

Consider a straight helical waveguide structure of radius a consisting of a twisting inhomogeneous and anisotropic material. The twisting means that the cross-section of the guide is rotating along the longitudinal direction of the structure. The waveguide may consist of several layers with different twist, but it is assumed that the material has a smallest common period p in the longitudinal direction. The waveguide is assumed to be lossy, and it constitutes an open structure placed in a surrounding homogeneous and isotropic free space.

Let μ0, ϵ0, η0 and c0 denote the permeability, the permittivity, the wave impedance and the speed of light in vacuum,
electric sources

where the wavenumber of vacuum is given by

\[ k_0 = \frac{\omega}{\sqrt{\mu_0 \varepsilon_0}} \]

where \( \omega = 2\pi f \) is the angular frequency and \( f \) the frequency. The relative permeability and permittivity of the surrounding free space are denoted by \( \mu \) and \( \varepsilon \), respectively, and the corresponding wavenumber is given by \( k = k_0 \sqrt{\mu \varepsilon} \). The cylindrical coordinates are denoted by \((\rho, \phi, z)\), the corresponding unit vectors \( (\hat{\rho}, \hat{\phi}, \hat{z}) \), the transverse coordinate vector \( \rho = \hat{\rho} \rho \) and the radius vector \( r = \rho + z \hat{z} \).

Let \( E(r) \) and \( H(r) \) denote the electric and magnetic fields, respectively, where the time-harmonic factor \( e^{-i\omega t} \) has been suppressed. Further, let \( J_s(r) \) and \( M_s(r) \) denote the imposed electric and magnetic sources, respectively, and which are assumed to be constrained to a finite region \( V_s \) inside the waveguide structure. Maxwell’s equations \[ \{ \begin{align*}
\nabla \times E(r) &= \imath \omega \mu_0 \mu(r) \cdot H(r) - M_s(r), \\
\nabla \times H(r) &= -\imath \omega \varepsilon_0 \varepsilon(r) \cdot E(r) + J_s(r),
\end{align*} \]

where \( \mu(r) \) and \( \varepsilon(r) \) are the complex valued relative permittivity and permeability dyadics of the material, respectively. The Maxwell’s equations (1) are also supplemented with a radiation condition providing a unique solution with fields vanishing at infinity [4]. The equations (1) can be reformulated in terms of the surrounding free space as

\[ \{ \begin{align*}
\nabla \times E(r) &= \imath \omega \mu_0 \mu H(r) - M(r) - M_s(r), \\
\nabla \times H(r) &= -\imath \omega \varepsilon_0 \varepsilon E(r) + J(r) + J_s(r),
\end{align*} \]

where \( M(r) \) and \( J(r) \) are the equivalent magnetic and electric sources

\[ \{ \begin{align*}
M(r) &= -\imath \omega \mu_0 \chi_m(r) \cdot H(r), \\
J(r) &= -\imath \omega \varepsilon_0 \chi_e(r) \cdot E(r),
\end{align*} \]

where \( \chi_m(r) \) and \( \chi_e(r) \) are the magnetic and electric susceptibility (or contrast) dyadics, respectively, defined by

\[ \{ \begin{align*}
\chi_m(r) &= \frac{1}{\imath} \mu(r) - I, \\
\chi_e(r) &= \frac{1}{\imath} \varepsilon(r) - I,
\end{align*} \]

and where \( I \) is the identity dyadic.

The solution to (1) can now be expressed in terms of the following integral equation of the second kind

\[ \{ \begin{align*}
E(r) - k^2 \int_{V_m} G(r', r', k) \cdot \chi_e(r') \cdot E(r') \, dr' \\
- \imath \omega \mu_0 \mu \int_{V_m} G(r', r', k) \cdot \chi_m(r') \cdot H(r') \, dr' \\
= \imath \omega \mu_0 \mu \int_{V_s} G_s(r', r', k) \cdot J_s(r') \, dr' \\
- \imath \omega \varepsilon_0 \chi_e \int_{V_s} G_s(r', r', k) \cdot M_s(r') \, dr',
\end{align*} \]

where the integrals extend over the support of the material \( V_m = \{ r | (\rho, \phi, z) \in [0, a] \times [0, 2\pi] \times [-\infty, +\infty] \} \), and the finite source region \( V_s \), respectively, and where \( r \in V_m \), cf. [2–6, 25]. Here, \( G_e(r, r', k) \) and \( G_m(r, r', k) \) are the electric and magnetic dyadic Green’s functions for the surrounding free space, respectively, defined by

\[ \{ \begin{align*}
G_e(r, r', k) &= \{ I + \frac{k^2}{\imath \omega \mu_0 \mu} \nabla \nabla \} G(r, r', k), \\
G_m(r, r', k) &= \nabla G(r, r', k) \times I,
\end{align*} \]

and where \( G(r, r', k) = \frac{e^{i k |r-r'|}}{4\pi |r-r'|} \) is the corresponding scalar Green’s function, see e.g., [4, 5, 11].

In cylindrical coordinates, the electric dyadic Green’s function \( G_e(r, r', k) \) can be expressed as

\[ G_e(r, r', k) = G_e^0(r, r', k) - \frac{1}{k^2} \hat{\rho} \hat{\rho} \delta(r - r'), \]

where \( G_e^0(r, r', k) \) is the part of \( G_e(r, r', k) \) which can be expanded in transverse (solenoidal) cylindrical vector waves for \( r \neq r' \), \( \delta(r - r') \) the three-dimensional delta distribution and \( -\frac{1}{k^2} \hat{\rho} \hat{\rho} \) the corresponding source-point dyadics [4,5]. Here, the expansion in cylindrical vector waves is given by

\[ G^0_e(r, r', k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_m(\rho, \rho', k, \alpha) e^{i m (\phi - \phi')} e^{i \alpha z - z'} \, d\alpha, \]

where the dyadic \( a_m(\rho, \rho', k, \alpha) \) is defined in (60) in the appendix A. Similarly, the magnetic dyadic Green’s function \( G_m(r, r', k) \) is expanded in cylindrical vector waves based on the dyadic \( b_m(\rho, \rho', k, \alpha) \) defined in (63). Note that \( G_m(r, r', k) \) does not contain any source-point dyadics at \( r = r' \) [29].

To simplify the description below, a non-magnetic material is assumed where \( \chi_m(r) = 0 \) and \( \chi_e(r) = \chi(r) \), and only the first equation is needed in (5). A generalization to the full system (5) will be straightforward. Extracting the contribution from the source-point as defined in (7), the electric field integral equation in (5) can now be expressed as

\[ [I + \hat{\rho} \hat{\rho} \cdot \chi(r)] \cdot E(r) \]

\[ - k^2 \int_{V_m} G_e^0(r, r', k) \cdot \chi(r') \cdot E(r') \, dr' = F(r), \]

where \( r \in V_m \), and the source vector function \( F(r) \) is given by

\[ F(r) = \imath \omega \mu_0 \mu \int_{V_m} G_s(r, r', k) \cdot J_s(r') \, dr' \\
- \imath \omega \varepsilon_0 \chi_e \int_{V_s} G_s(r, r', k) \cdot M_s(r') \, dr'. \]

At each radial distance \( \rho \) the material is periodic in the \( \phi \)-coordinate with period \( 2\pi \) and in the \( z \)-coordinate with period \( \pi \). The material dyadic \( \chi(r) \) can hence be represented by the following two-dimensional Fourier series

\[ \chi(r) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \chi_{mn}(\rho) e^{i m \phi} e^{i n \pi z}, \]

where \( \chi_{mn}(\rho) \) are the corresponding dyadic Fourier series coefficients.
B. Fourier analysis

Consider the following Fourier representation of the electric field
\[
E(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E_m(\rho, \alpha)e^{im\phi}e^{in\alpha z} d\alpha, \tag{12}
\]
and similarly for the sources \(J_s(r)\) and \(M_s(r)\) and the source vector function \(F(r)\). The material dyadic is furthermore represented as
\[
\chi(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \chi_m(\rho, \alpha)e^{im\phi}e^{in\alpha z} d\alpha, \tag{13}
\]
where
\[
\chi_m(\rho, \alpha) = 2\pi \sum_{n=-\infty}^{\infty} \chi_{mn}(\rho)\delta(\alpha - n\frac{2\pi}{p}), \tag{14}
\]
and where the coefficients \(\chi_{mn}(\rho)\) have been defined in (11).

Based on the convolution theorem the Fourier transformation of (9) yields the following integral equation in Fourier space
\[
E_m(\rho, \alpha) + \hat{\rho} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \chi_{m-m'}(\rho, \alpha - \alpha') \cdot E_{m'}(\rho, \alpha') d\alpha' \\
- k^2 2\pi \int_{0}^{a} a_m(\rho, \rho', k, \alpha) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \chi_{m-m'}(\rho', \alpha-\alpha') \\
\times E_{m'}(\rho', \alpha') d\alpha' d\rho' = F_m(\rho, \alpha), \tag{15}
\]
where \(\rho \in [0, a]\), \(m \in \mathbb{Z}\) and \(\alpha \in \mathbb{R}\). The Fourier transformation of the source vector function is given by
\[
F_m(\rho, \alpha) = \frac{\iota \omega \mu_0 \mu_2\pi}{k^2} \int_{0}^{a} \sum_{m'=\infty}^{\infty} \sum_{n'=\infty}^{\infty} \chi_{m-m',n-n'}(\rho, \alpha) \cdot E_{m'}(\rho, \alpha') \rho' d\rho' \\
- \frac{\omega \mu_0}{k} \int_{0}^{a} b_m(\rho, \rho', k, \alpha) \cdot J_{m'}(\rho, \alpha') \rho' d\rho'. \tag{16}
\]
By further exploiting the distributional property given in (14), the integral equation (15) becomes
\[
E_m(\rho, \alpha) + \hat{\rho} \cdot \sum_{m'=\infty}^{\infty} \sum_{n'=\infty}^{\infty} \chi_{m-m',l}(\rho) \cdot E_{m'}(\rho, \alpha - l\frac{2\pi}{p}) \\
- k^2 2\pi \int_{0}^{a} a_m(\rho, \rho', k, \alpha) \cdot \sum_{m'=\infty}^{\infty} \sum_{n'=\infty}^{\infty} \chi_{m-m',l}(\rho') \\
\times E_{m'}(\rho', \alpha - l\frac{2\pi}{p}) \rho' d\rho' = F_m(\rho, \alpha). \tag{17}
\]
C. The Floquet theorem

It is assumed that the sources \(J_s(r)\) and \(M_s(r)\) are supported only for \(z < 0\), and hence that the source vector function \(F_m(\rho, \alpha)\) defined in (16) is an analytic function in the upper half-plane \(\text{Im} \alpha > 0\). According to the Floquet theorem [5], there will be Floquet modes in the source-free region \(z > 0\), and where the electric field can be expressed as
\[
E(r) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E_{mn}(\rho)e^{im\phi}e^{in\frac{2\pi}{p}z}e^{\beta z}, \tag{18}
\]
where \(\beta\) is the complex valued propagation constant of the Floquet mode and where \(\text{Im} \beta > 0\). The Fourier transform \(E_m(\rho, \alpha)\) can hence be written as
\[
E_m(\rho, \alpha) = \sum_{n=-\infty}^{\infty} E_{mn}(\rho) \frac{1}{\iota(\alpha - (\beta + n\frac{2\pi}{p})} + E_m(\rho, \alpha), \tag{19}
\]
with poles at \(\beta + n\frac{2\pi}{p}\). Here, \(E_{mn}(\rho, \alpha)\) is the Fourier transform of the left-sided part of \(E(r)\) which is supported in \(z < 0\), and hence \(E_m(\rho, \alpha)\) is analytic in the upper half-plane \(\text{Im} \alpha > 0\).

The residue theorem can be stated here as
\[
\frac{1}{2\pi} \oint_{C_n} \frac{f(\alpha) d\alpha}{\iota(\alpha - (\beta + q\frac{2\pi}{p})} = f(\beta + n\frac{2\pi}{p})\delta_{nq}, \tag{20}
\]
where \(C_n\) is a small closed contour enclosing the point \(\beta + n\frac{2\pi}{p}\), \(f(\alpha)\) is a function which is analytic inside \(C_n\) and \(\delta_{nq}\) denotes the Kronecker delta. By applying the residue theorem (20) to (17), the following integral equation is obtained for \(E_{mn}(\rho)\)
\[
E_{mn}(\rho) + \hat{\rho} \cdot \sum_{m'=\infty}^{\infty} \sum_{n'=\infty}^{\infty} \chi_{m-m',n-n'}(\rho, \alpha) \cdot E_{m'n'}(\rho) \\
- k^2 2\pi \int_{0}^{a} a_m(\rho, \rho', k, \alpha) \cdot J_{m'}(\rho, \alpha') \rho' d\rho' \\
\times \sum_{m'=\infty}^{\infty} \sum_{n'=\infty}^{\infty} \chi_{m-m',n-n'}(\rho') \cdot E_{m'n'}(\rho') \rho' d\rho' = 0, \tag{21}
\]
where \(\rho \in [0, a]\) and \((m, n)\) are integers.

III. NATURAL MODES WITH PERIODIC EXCITATION

A. Preliminaries

Maxwell’s equations for the free space with a periodically modulated plane wave excitation can be written
\[
\begin{align*}
\nabla \times E(r) &= \iota \omega \mu_0 \mu_2 H(r) - M(r), \\
\nabla \times H(r) &= -\iota \omega \epsilon_0 \epsilon E(r) + J(r),
\end{align*} \tag{22}
\]
where the magnetic source is given by \(M(r) = M(r)e^{i\beta z}\) with a periodic part \(M(r) = M(r + z)p\), and similarly for the electric source \(J(r) = J(r)e^{i\beta z}\), and where \(\beta\) is the Floquet wavenumber with \(\text{Im} \beta \geq 0\). In the transverse plane the sources are assumed to be constrained to the cross-sectional area \(S\) with radius \(a\). The Maxwell’s equations (22) are also supplemented with a radiation condition in the transverse plane.
Following the standard derivation based on vector potentials as in e.g., [11], the solution to (22) can be written

\[
\begin{align*}
E(r) &= i \omega \mu_0 \mu_r \int_S J_s \delta \rho e_{p}(r', \beta) \cdot \mathbf{J}(r') \, dS' \, dz' \\
&- \int_S \rho_0 \mathbf{G}_{\text{mp}}(r', \beta) \cdot \mathbf{M}(r') \, dS' \, dz', \\
H(r) &= \int_S \rho \mathbf{G}_{\text{mp}}(r', \beta) \cdot \mathbf{J}(r') \, dS' \, dz' \\
&+ i \omega e_0 \int_S J_s \rho_0 \mathbf{G}_{\text{ep}}(r', \beta) \cdot \mathbf{M}(r') \, dS' \, dz',
\end{align*}
\]

where the integration is over one unit cell \( V_c = \{ r | (\rho, z) \in S \times [0, p] \} \) and \( r \in V_c \), and where \( \mathbf{G}_{\text{ep}}(r', \beta) \) and \( \mathbf{G}_{\text{mp}}(r', \beta) \) are the periodic electric and magnetic Green’s dyadics, respectively, defined by

\[
\begin{align*}
\mathbf{G}_{\text{ep}}(r', \beta) &= \{ \mathbf{I} + \frac{i}{k^2} \nabla \nabla \} \mathbf{G}_p(r', \beta), \\
\mathbf{G}_{\text{mp}}(r', \beta) &= \nabla \mathbf{G}_p(r', \beta) \times \mathbf{I},
\end{align*}
\]

and where \( \mathbf{G}_p(r', \beta) \) is the scalar periodic Green’s function defined in (69) in Appendix B. The periodic electric Green’s dyadic can furthermore be factorized as \( \mathbf{G}_{\text{ep}}(r', \beta) = \mathbf{G}_e(r', \beta) e^{i\beta(z-z')} \), where \( \mathbf{G}_e(r', \beta) \) is periodic in \( z-z' \) with period \( p \), and similarly for the periodic magnetic Green’s dyadic \( \mathbf{G}_{\text{mp}}(r', \beta) \).

B. Periodic Green’s dyadic

The spectral representation of the electric Green’s dyadic is given by

\[
\begin{align*}
\mathbf{G}_e(r', \beta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{G}_e(\rho, \beta) e^{i\alpha(z-z')} \, d\alpha,
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{G}_e(\rho, \beta) &= \mathbf{G}_e^0(\rho, \beta) - \frac{1}{k^2} \hat{\rho} \hat{\rho} e^{i\beta(\rho-\rho')},
\end{align*}
\]

and which are based on (57) and (58) in Appendix A. From the definition (6) it follows that \( \mathbf{G}_e(r', \beta) \) can also be represented as

\[
\begin{align*}
\mathbf{G}_e(r', \beta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ \mathbf{I} + \frac{1}{k^2} \nabla \nabla \} \mathbf{G}(\rho, \beta) e^{i\alpha(z-z')} \, d\alpha,
\end{align*}
\]

where \( \mathbf{G}(\rho, \beta) \) is the scalar two-dimensional Green’s function defined in (65) in Appendix B. A comparison of (25) and (27) yields immediately that

\[
\begin{align*}
\mathbf{G}_e(\rho, \beta) e^{i\alpha(z-z')} &= \{ \mathbf{I} + \frac{1}{k^2} \nabla \nabla \} \mathbf{G}(\rho, \beta) e^{i\alpha(z-z')},
\end{align*}
\]

and where both sides can be extended analytically into the complex \( \alpha \)-plane. Based on the Poisson summation formula for the scalar periodic Green’s function given in (74), an analytic expression for the periodic electric Green’s dyadic can now be derived as follows

\[
\mathbf{G}_{\text{ep}}(r', \beta) = \{ \mathbf{I} + \frac{1}{k^2} \nabla \nabla \} \mathbf{G}_p(r', \beta) e^{i\beta(z-z')} = \frac{1}{p} \sum_{n=-\infty}^{\infty} \mathbf{G}_e(\rho, \beta) e^{i\beta n \pi p} e^{i\alpha(z-z')}.
\]

Finally, based on (26) the periodic electric Green’s dyadic is given by

\[
\begin{align*}
\mathbf{G}_{\text{ep}}(r', k, \beta) &= \mathbf{G}_{\text{ep}}^0(r', k, \beta) \\
&- \frac{1}{k^2} \hat{\rho} \hat{\rho} e^{i\beta(\rho-\rho')} \sum_{n=-\infty}^{\infty} \delta(z-z' + np) e^{i\beta(z-z')}.
\end{align*}
\]

C. Integral equation for natural modes

Consider now the source-free Maxwell’s equations as in (1) and (2) with \( \mathbf{M}_s = \mathbf{J}_s = \mathbf{0} \) and with equivalent sources \( \mathbf{M}(r) \) and \( \mathbf{J}(r) \) defined as in (3), and where the periodic material dyadics are defined as in (11). For simplicity, a non-magnetic material is assumed here with \( \chi_m(r) = 0 \) and \( \chi_e(r) = \chi(r) \). The fields are assumed to have a Floquet wavenumber \( \beta \) with \( \text{Im} \beta \geq 0 \) and can be expressed as \( \mathbf{E}(r) = \mathbf{E}(r)e^{i\beta^2} \) with a \( p \)-periodic part given by

\[
\mathbf{E}(r) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{E}_{mn}(\rho) e^{im\phi} e^{i\frac{2\pi}{p}z}.
\]

The general solution (23) then leads to the following integral equation for the electric field

\[
\begin{align*}
[ \mathbf{I} + \hat{\rho} \cdot \chi(r) ] \cdot \mathbf{E}(r) \\
- k^2 \int_{S} \rho \mathbf{G}_{\text{ep}}(r', k, \beta) \cdot \chi(r') \cdot \mathbf{E}(r') \, dS' \, dz' = 0,
\end{align*}
\]

where the integration is over one unit cell \( V_c \) and \( r \in V_c \), and where the contribution from the source-point has been extracted by evaluating the periodic distribution in (30).
writing \( \mathbf{G}^0_{ep}(r, r', k, \beta) = \tilde{\mathbf{G}}^0_{e}(r, r', k, \beta)e^{i\beta(z-z')}, \) the integral equation (36) can also be written

\[
[I + \hat{\rho}\hat{\rho} \cdot \chi(r)] \cdot \tilde{E}(r) - k^2 \int_0^a \int_0^{2\pi} \int_0^b \mathbf{G}^0_{e}(r, r', k, \beta) \cdot \chi(r') \cdot \tilde{E}(r') \rho' \, d\rho' \, d\phi' \, dz' = 0, \tag{37}
\]

where \( r \in V_c. \) Here, \( \tilde{\mathbf{G}}^0_{e}(r, r', k, \beta) \) is defined by (31) and (59) and can hence be written as

\[
\tilde{\mathbf{G}}^0_{e}(r, r', k, \beta) = \frac{1}{p} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_m(\rho, \rho', k, \beta+n\frac{2\pi}{p})e^{im(\phi-\phi')}e^{in\frac{\pi}{2}(z-z')}. \tag{38}
\]

By employing the convolution theorem for two-dimensional Fourier series, the integral equation (37) can finally be written in Fourier space exactly as in (21).

IV. APPLICATIONS

A. Twist-modes for multi-conductor power cables

Assume that the cross-section of the waveguide rotates along the longitudinal direction and that there are \( R \) radial regions \( R_r \) with distinct rotation (twist) for \( r = 1, \ldots, R. \) In each region, the period in the azimuthal direction is given by \( 2\pi/m_r \) where \( m_r \in \{1, 2, \ldots\}. \) It is furthermore assumed that there is a smallest common period \( p \) in the longitudinal direction such that the longitudinal period \( p_r \) in each radial region is given by \( p_r = p/m_r \) where \( n_r \in \{\pm 1, \pm 2, \ldots\}, \) and where the positive (negative) sign indicates a left (right) handed twist. The twist in each radial region \( R_r \) is thus characterized by the twist direction \( (m_r, n_r), \) and the non-zero Fourier coefficients \( \chi_{mn}(\rho) \) defined in (11) can only be found at the following points in the reciprocal (Fourier) space

\[
\begin{cases}
m = km_r, \\
n = km_r,
\end{cases}
\tag{39}
\]

where \( k = 0, \pm 1, \pm 2, \ldots \) and \( r = 1, \ldots, R. \) An example with three distinct (one left and two right handed) twist regions is illustrated in Fig. 1.

The primary interest here is with the twisted modes of three-phase power cables at quasi-static (50 Hz) conditions. Here, the excitation (16) is governed by a single symmetric component [28] with positive phase progression, i.e., an azimuthal Floquet-mode with factor \( e^{i\beta}. \) The corresponding field twist-mode \( E_{mn}(\rho) \) is defined by the excitation point \( (m, n) = (1, 0) \) together with the conditions of (21), and which yields the following set of feasible Fourier indices

\[
\begin{cases}
m = km_r + 1, \\
n = km_r,
\end{cases}
\tag{40}
\]

where \( k = 0, \pm 1, \pm 2, \ldots \) and \( r = 1, \ldots, R. \) Note that this is simply a one-step shift to the right in comparison to the illustration in Fig. 1.

B. Thin helical wires under quasi-static conditions

A useful application of the periodic Green’s dyadics expressed in (23) is with the calculation of the electromagnetic fields produced by a helical current distribution. As an example, consider the periodic magnetic field produced by a thin helical wire carrying the stationary current \( I \) under quasi-static conditions. The radius of the helix is \( a \) and the period is \( p. \) Here,

\[
H(r) = I \int_L G_{mp}(r, r', k, 0) \cdot dr', \tag{41}
\]

where \( \beta = 0 \) and \( L \) is a curve constituting one period of the helix defined in cartesian coordinates \( (x', y', z') \) as

\[
\begin{align*}
x' &= a \cos\left(\frac{2\pi}{p} z' + \varphi\right), \\
y' &= a \sin\left(\frac{2\pi}{p} z' + \varphi\right),
\end{align*}
\tag{42}
\]

where \( z' \in [0, p] \) and \( \varphi \) is an offset parameter of the helix. In cylindrical coordinates it is seen that \( \rho' = a, \phi' = \frac{2\pi}{p} z' + \varphi \) and \( dr' = a \frac{2\pi}{p} dz' + \hat{\rho}. \) By using (34), (62) and (63) for \( \rho > a, \) the expression (41) yields the result

\[
H(r) = \frac{I k}{4} \sum_{m=-\infty}^{\infty} \sum_{\tau=1}^{2} u_{\tau m}(\rho, -m\frac{2\pi}{p})e_{\tau m}(\rho', -m\frac{2\pi}{p})
\cdot \left(\hat{\rho} a \frac{2\pi}{p} + \hat{z}\right)e^{im(\phi-\varphi-\frac{2\pi}{p} z')}, \tag{43}
\]

where the integration over one period has been performed as

\[
\int_0^p e^{-i(m+n)\frac{2\pi}{p} z'} \, dz' = p \delta_{m,n}. \tag{44}
\]

V. NUMERICAL METHOD

A. Discretization by the collocation method

A discretization based on the collocation method [14] is devised as follows. It is assumed that the material region of the helical structure is given by the radial domain \( \rho_1 \leq \rho \leq a \) where \( \rho_1 > 0. \) An \( N \)-point discretization is defined where \( \rho_1 < \rho_2 < \ldots < \rho_N = a \) and \( L_i(\rho) \) denotes the corresponding Lagrange basis consisting of linear splines with the interpolation property \( L_i(\rho) = \delta_{ij} \) for \( i, j = 1, \ldots, N, \)

Note that the associated twist angle \( \varphi_*(\rho) = \arctan\left(\frac{2\pi}{p} \frac{\rho m_r}{m_r} \right) = \arctan\left(\frac{2\pi}{p} \frac{-\rho n_r}{m_r} \right) \) increases with the radius \( \rho. \)
see e.g., [14]. The Fourier components of the electric field defined in (18) or (35) is now expanded as
\[ E_{mn}(\rho) = \sum_{j=1}^{N} E_{mnj}(\rho)L_j(\rho), \]  
(45)
where \( E_{mnj}(\rho) \) is a vector valued coefficient with constant cylindrical components. Let \( \rho_i \) denote the radial vector \( \rho \) evaluated at the point \( \rho_i \) for \( i = 1, \ldots, N \). Due to the interpolation property of the Lagrange basis it is seen that \( E_{mn}(\rho_i) = E_{mn}(\rho) \). The integral equation (21) evaluated at the interpolation points \( \rho_i \) now yields the discrete system
\[ E_{mn}(\rho) + \sum_{m'} \sum_{n'} \rho_i \cdot \chi_{m-m',n-n'}(\rho_i) \cdot E_{m'n'i}(\rho) \]
\[ - k^2 2\pi \sum_{m'} \sum_{n'} \sum_{j=1}^{N} \int_{\Omega_j} a_m(\rho, \rho', \beta + n \frac{2\pi}{p}) \cdot \chi_{m-m',n-n'}(\rho')L_j(\rho') \cdot E_{m'n'j}(\rho') d\rho' = 0, \]  
(46)
where \( \Omega_j \) denotes the support region of \( L_j(\rho) \).

The Fourier series (35) is truncated using \( M \) terms based on the assumed twist-modes that are at hand, cf., section IV-A. A multi-index notation is introduced where \( k \leftrightarrow (m, n), l \leftrightarrow (m', n') \) and where \( k, l = 1, \ldots, M \). The following definitions are made
\[ E_{ki} = E_{mn}(\rho), \quad E_{ij} = E_{m'n'j}(\rho), \]
\[ \chi_{kil} = \chi_{m-m',n-n'}(\rho_i), \]
(47)
and
\[ a_{kilj}(\beta) = \int_{\Omega_j} a_m(\rho, \rho', \beta + n \frac{2\pi}{p}) \cdot \chi_{m-m',n-n'}(\rho')L_j(\rho') d\rho', \]  
(49)
where \( k, l = 1, \ldots, M \) and \( i, j = 1, \ldots, N \). The integration in (49) is highly regular with at most some points of discontinuity in either of the terms \( a_m(\rho, \rho', \beta + n \frac{2\pi}{p}) \) and \( \chi_{m-m',n-n'}(\rho') \). It is therefore convenient to place any possible points of discontinuity of the material function \( \chi_{mn}(\rho') \) at the grid points \( \rho_i \) of the linear interpolation, and to employ an efficient quadrature rule based on interior points such as the Gauss-Legendre quadrature [14, 20] to evaluate the integral in (49). The system (46) can now be written in the more convenient form
\[ E_{ki} + \sum_{l=1}^{M} \rho_i \cdot \chi_{kil} \cdot E_{li} - k^2 2\pi \sum_{l=1}^{M} a_{kilj}(\beta) \cdot E_{lj} = 0, \]  
(50)
which can be readily interpreted in terms of \( 3 \times 3 \) block matrices and with the row and column multi-indices \( (k, i) \) and \( (l, j) \), respectively.

Let \( A(\beta) \) denote the finite matrix corresponding to the linear system defined in (50). The propagation constant can then be computed from a numerical residue calculus based on
\[ \beta_1 = \frac{\int_{C} \beta_{1}(A(\beta)) d\beta}{\int_{C} 1 \lambda_{1}(A(\beta)) d\beta}, \]  
(51)
where \( \lambda_{1}(A(\beta)) \) is the minimum (modulus) eigenvalue of the matrix \( A(\beta) \), and where the closed loop \( C \) is circumscribing the true zero \( \beta_1 \) in such a way that there are no other zeros or branch-points of \( \lambda_{1}(A(\beta)) \) inside the loop.

B. Material smoothing

Appropriate windowing techniques can be applied directly in the Fourier domain to reduce the Gibbs phenomena in the truncation of the (material) Fourier coefficients \( \chi_{mn}(\rho) \). As an example is illustrated in Fig. 2 a comparison of the truncated Fourier series expansions of a square pulse with and without a one-dimensional coefficient smoothing based on the Kaiser window [21].

\[ \text{Truncated Fourier series} \]

\[ \text{Ordinary truncation} \]

\[ \text{Truncation using the Kaiser window} \]

Fig. 2. Truncation of the Fourier series of a square pulse, with and without spectral smoothing and with azimuthal indices \( m = -20, \ldots, 20 \).

\[ \text{MODE EXPANSIONS OF THE DYADIC GREEN’S FUNCTIONS} \]

Consider a homogeneous and isotropic cylindrical region with relative permittivity \( \varepsilon \), relative permeability \( \mu \) and wavenumber \( k = k_0 \sqrt{\varepsilon \mu} \). The solenoidal (source-free) cylindrical vector waves are defined here by
\[ w_{1m}(r, \alpha) = \frac{1}{k} \nabla \times (\tilde{\psi}_m(\kappa \rho) e^{im\phi} e^{i\alpha z}), \]
\[ w_{2m}(r, \alpha) = \frac{1}{k} \nabla \times w_{1m}(r), \]  
(52)
where \( \psi_m(\kappa \rho) \) is a Bessel function or a Hankel function of the first kind, each of order \( m \), cf., [1, 5]. Here, \( \alpha \) is the longitudinal wavenumber and \( \kappa = \sqrt{k^2 - \alpha^2} \) the transversal wavenumber where the square root is chosen such that \( 0 < \arg \kappa \leq \pi \) and hence \( \Im \kappa \geq 0 \). It can be shown by direct calculation that \( \nabla \times w_{2m}(r, \alpha) = kw_{1m}(r, \alpha) \). The following curl properties are thus obtained
\[ \nabla \times w_{\tau \tau}(r, \alpha) = kw_{\tau m}(r, \alpha), \]  
(53)
for \( \tau = 1, 2 \), and where \( \tilde{\tau} \) denotes the complement of \( \tau (1 = 2 \) and \( 2 = 1) \).

The following notation will be used
\[ w_{\tau m}(r, \alpha) = w_{\tau m}(r, \alpha)e^{i\alpha \phi} e^{i\alpha z}, \]  
(54)
where \( \tau = 1, 2 \), and the vectors \( w_{\tau m}(r, \alpha) \) are given explicitly in cylindrical coordinates as
\[ w_{1m}(\rho, \alpha) = \hat{\rho} \frac{\tilde{\psi}_m(\kappa \rho)}{\kappa \rho} - \hat{\phi} \tilde{\psi}_m(\kappa \rho), \]
\[ w_{2m}(\rho, \alpha) = \hat{\theta} \tilde{\psi}_m(\kappa \rho) + \frac{\kappa}{\kappa \rho} \tilde{\phi} \tilde{\psi}_m(\kappa \rho) + \hat{\rho} \frac{\kappa}{\kappa \rho} \tilde{\psi}_m(\kappa \rho), \]  
(55)
and where the ′ denotes differentiation with respect to the argument. Let the regular and the outgoing (radiating) cylindrical vector waves \(v_{\tau m}(r, \alpha)\) and \(u_{\tau m}(r, \alpha)\) be defined as in (52) by using the regular Bessel functions and the Hankel functions of the first kind, \(J_n(\kappa \rho)\) and \(H^{(1)}_n(\kappa \rho)\), respectively.

The electric dyadic Green’s function defined in (6) can be expanded in cylindrical vector waves as

\[
G_e(r, r', k) = \frac{i}{8\pi} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\tau=1}^{2} v_{\tau m}(r, \alpha) u_{\tau m}^\dagger(r', \alpha) \, d\alpha - \frac{1}{k^2} \hat{\mathbf{\rho}} \delta(r - r'),
\]

(56)

where \(r_+\) and \(r_-\) denote the vector in \(\{r, r'\}\) having the smallest and largest radial coordinate, respectively, i.e., \(\rho_- = \min\{\rho, \rho'\}\) and \(\rho_+ = \max\{\rho, \rho'\}\), cf. [1, 4, 5]. The dagger \(\dagger\) refers to a sign-shift in the exponentials in the definition (52), and which can be placed on any of the two vector waves \(v_{\tau m}(r, \alpha)\) and \(u_{\tau m}(r, \alpha)\) in the expression (56), cf. [1]. Note that the expression (56) also contains a delta distribution taking the source point into account, cf. [5].

By employing the notation defined in (54), the electric Green’s dyadic can now be expressed as

\[
G_e(r, r', k) = G^0_e(r, r', k) - \frac{1}{k^2} \hat{\mathbf{\rho}} \delta(r - r'),
\]

(57)

where

\[
G^0_e(r, r', k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^0_e(\rho, \rho', k, \alpha) e^{i\alpha(z - z')} \, d\alpha,
\]

(58)

\[
G^0_e(\rho, \rho', k, \alpha) = \sum_{m=-\infty}^{\infty} a_m(\rho, \rho', k, \alpha) e^{im(\phi - \phi')},
\]

(59)

and where the dyadic \(a_m(\rho, \rho', k, \alpha)\) is given by

\[
a_m(\rho, \rho', k, \alpha) = \left\{ \begin{array}{ll}
\frac{i}{4} \sum_{\tau=1}^{2} u_{\tau m}(\rho, \alpha) v_{\tau m}^\dagger(\rho', \alpha) & \rho' < \rho, \\
\frac{i}{4} \sum_{\tau=1}^{2} v_{\tau m}(\rho, \alpha) u_{\tau m}^\dagger(\rho', \alpha) & \rho' > \rho,
\end{array} \right.
\]

(60)

where the dagger \(\dagger\) has been placed on the vector with primed coordinate, and where \(u_{\tau m}^\dagger(\rho', \alpha) = u_{\tau m}(\rho', \alpha) e^{-im\phi'} e^{-i\alpha z'}\).

The magnetic dyadic Green’s function can be obtained as \(G_m(r, r', k) = \nabla \times G_e(r, r', k)\) [11], and it follows from (53) and (56) that it can be expanded in cylindrical vector waves as

\[
G_m(r, r', k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_m(\rho, \rho', k, \alpha) e^{i\alpha(z - z')} \, d\alpha,
\]

(61)

where

\[
G_m(\rho, \rho', k, \alpha) = \sum_{m=-\infty}^{\infty} b_m(\rho, \rho', k, \alpha) e^{im(\phi - \phi')},
\]

(62)

and where the dyadic \(b_m(\rho, \rho', k, \alpha)\) is given by

\[
b_m(\rho, \rho', k, \alpha) = \left\{ \begin{array}{ll}
\frac{i}{4} \sum_{\tau=1}^{2} u_{\tau m}(\rho, \alpha) v_{\tau m}^\dagger(\rho', \alpha) & \rho' < \rho, \\
\frac{i}{4} \sum_{\tau=1}^{2} v_{\tau m}(\rho, \alpha) u_{\tau m}^\dagger(\rho', \alpha) & \rho' > \rho.
\end{array} \right.
\]

(63)

### Appendix B

**Scalar periodic Green’s function**

The scalar Green’s function \(G(r, r', k) = e^{ik|r - r'|}/(4\pi|r - r'|)\) satisfies the inhomogeneous Helmholtz equation

\[
\{ \nabla^2 + k^2 \} G(r, r', k) = -\delta(r - r'),
\]

(64)

and can be represented by the Fourier integral

\[
G(r, r', k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\rho, \rho', k, \alpha) e^{i\alpha(z - z')} \, d\alpha,
\]

(65)

where the two-dimensional Green’s function \(G(\rho, \rho', k, \alpha)\) satisfies

\[
\{ \nabla^2_{\perp} + \kappa^2 \} G(\rho, \rho', k, \alpha) = -\delta(\rho - \rho'),
\]

(66)

and where \(\nabla^2_{\perp}\) is the two-dimensional transverse Laplace operator, \(\kappa = \sqrt{k^2 - \alpha^2}\) (\(\text{Im}\alpha \geq 0\)) and \(\delta(\rho - \rho')\) the transverse delta distribution. The function \(G(\rho, \rho', k, \alpha)\) is given by

\[
G(\rho, \rho', k, \alpha) = \frac{1}{4} H^{(1)}_0(\kappa|\rho - \rho'|),
\]

(67)

and which can be expanded as

\[
G(\rho, \rho', k, \alpha) = \frac{1}{4} \sum_{m=-\infty}^{\infty} J_m(\kappa \rho_{\perp}) H^{(1)}_m(\kappa \rho_{\perp}) e^{im(\phi' - \phi)},
\]

(68)

see e.g., [1, 11, 23] and the Graf’s and Gegenbauer’s addition theorem [20].

The periodic Green’s function \(G_p(r, r', k, \beta)\) for plane wave excitation \(e^{i\beta z}\) satisfies

\[
\{ \nabla^2 + k^2 \} G_p(r, r', k, \beta) = -\delta(\rho - \rho') \sum_{n=-\infty}^{\infty} \delta(z - z' + np) e^{i\beta(z - z')},
\]

(69)

together with a radiation condition in the transverse plane, and where \(p\) is the period and \(\text{Im}\beta \geq 0\). The periodic Green’s function can furthermore be factorized as

\[
G_p(r, r', k, \beta) = \widetilde{G}(r, r', k, \beta) e^{i\beta(z - z')},
\]

(70)

where \(\widetilde{G}(r, r', k, \beta)\) is periodic in \(z - z'\) with period \(p\).

To derive an analytic expression for the periodic Green’s function [23], the following Poisson summation formula [7] can be employed

\[
\sum_{n=-\infty}^{\infty} f(np) e^{i\beta pn} = \frac{1}{p} \sum_{n=-\infty}^{\infty} \hat{F}(\beta + n\frac{2\pi}{p}),
\]

(71)
where $\beta$ is real valued and where the Fourier transform is defined by
\[
\hat{F}(\alpha) = \int_{-\infty}^{\infty} f(z)e^{i\alpha z} dz,
\]
and it follows from the Poisson summation formula (71) that
\[
\beta
\]
and it converges exponentially for all $\text{Im} \beta \neq 0$, whereas the sum on the right-hand side of (74) converges exponentially for all $\beta$ with $\text{Im} \beta \geq 0$. By using the property (66) which is valid with complex valued wavenumbers $\kappa$, it is readily seen that the expression in the right-hand side of (74) satisfies (69) when $\beta$ is complex valued.

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