Compress and Estimate in Multiterminal Source Coding

Alon Kipnis †, Stefano Rini∗ and Andrea J. Goldsmith†

† Department of Electrical Engineering, Stanford University, Stanford, CA 94305 USA
∗ Department of Electrical Engineering, National Chiao-Tung University, Hsinchu, 30010, Taiwan

Abstract

We consider a multiterminal remote source coding problem in which a source sequence is estimated from the output of multiple source encoders, each having access only to a noisy observation of the source realization. Each remote encoder compresses its noisy observation sequence so as to minimize a local distortion measure which depends only on the distribution of its observed sequence, and is otherwise independent from the distribution of the underlying source. The latter is estimated at a central location from the output of each of the remote encoders. This source compression and estimation scenario leads to an achievable scheme for the remote multiterminal source coding problem which we term the “compress-and-estimate” (CE) scheme. For the case of a source with independently and identically distributed (i.i.d) elements observed through multiple memoryless channels, we derive a single-letter expression for the distortion in the CE scheme, which we refer to as the CE distortion-rate function (CE-DRF). We prove that the CE-DRF can be achieved by estimating the source realization from the output of any set of encoders, as long as each encoder attains its local rate-distortion function. We prove in addition a converse result saying that, for large enough blocklength, the distortion in estimating a finite sub-block of the source from the output of such encoders, averaged over all sub-blocks, does not exceed the CE-DRF. Finally, we derive closed-form expressions for the CE-DRF in the case of a Gaussian source observed through multiple AWGN channels under quadratic distortion, and for the case of a binary source observed through multiple biffip channels under Hamming distortion. By comparing the resulting expressions to known optimal source coding performance, we highlight the performance lost due to the lack of statistical knowledge of the relationship between a source and its observations at the encoders.

Index Terms

Remote source coding; Indirect source coding; Noisy source coding; Mismatched source coding; Gaussian source coding; Binary source coding; Compress-and-estimate; Oblivious rate-distortion;

I. INTRODUCTION

Consider the problem of estimating an information source according to some distortion criterion from the output of a noisy channel, where in addition this output is compressed at rate \( R \) bits per source symbol. If \( R \) is larger than the entropy of the observed sequence at the output of the channel, then the distortion in estimating the source is only due to the channel
noise. If, however, $R$ is smaller than this entropy, then the problem of minimizing the distortion is called the *remote* or *indirect* source coding problem (ISC). It is well-known that an optimal solution to this problem is obtained by compressing at rate $R$ the best estimate of the source from its noisy observations, according to the same distortion criterion [3]. The resulting minimal distortion is called the *indirect* rate-distortion function (iDRF) of the source given the channel output. The multiterminal version of the ISC problem, in which more than one noisy version of the source is compressed at multiple locations, is called the *chief executive officer* (CEO) problem [4]. In this setting, a multiple observers (agents) obtain different noisy versions of the source. Each agent then encodes or compresses its observation and aid the recovery of the original source at a central processing unit. The optimal coding strategy in the CEO setting is known only in some special cases [7], [8], and is dependent on the source’s statistics, the number of agents, and their respective quality of observations and rate of communication. Arguably, in many useful scenarios the agents are unaware of each other and cannot decide on such an optimal strategy, even if this strategy is known.

In this paper we ask what if, instead of applying the optimal ISC or CEO strategy, the $i$th agent compresses its observations at rate $R_i$ in a way that minimizes some local distortion measure with respect to these observations. Since the underlying source is eventually estimated from the collection of all encoded observations, we denote this remote compression and estimation scenario as the *compress-and-estimate* (CE) scheme. The choice of the term CE is inspired by the compress-and-forward achievability scheme for the relay channel [5], in which the relay compresses the received signal according to some locally-optimal source code without decoding for the message first.

Our study of the CE setting is motivated by various restrictions on distributed compression and estimation systems arising in practice. For both the ISC and the CEO problem, optimal codes are generally dependent on the statistics of the underlying source, as well as the fidelity criterion according to which the reconstruction distortion is measured. If any of the encoders lack information about the source statistics, the reconstruction fidelity criterion, or even the fact that the underlying source is present, it may try to minimize the distortion with respect to its local observations, resulting in the CE coding scheme. This lack of knowledge often occurs in large distributed systems, where each node communicates its own observations to a central processing unit but is unaware of the existence of an underlying remote source or the requirement to reconstruct it at the central processor. In fact, in the special case of a single remote encoder, a particular local distortion measure can be used with respect to the observations to attain the optimal source coding performance described by the iDRF [3]. If the encoder instead compresses its observations in order to minimize a different distortion measure, then the CE scheme can be seen as a form of mismatched source encoding [6]. In the case of more than one remote encoder, the minimal CEO distortion is not known in general and it is not clear whether it can be derived from a coding strategy that minimizes some local distortion measures. Moreover, in the few cases where the CEO distortion is known, the optimal coding strategy is critically dependent on the communication rates of all encoders and their quality of observations [7], [8]. In contrast, the code at each encoder in the CE scheme depends only on local parameters, and therefore does not require any offline exchange between the remote agents and/or the CEO. Since many distributed systems have limited communication between different nodes in the system, it is important to characterize the excess distortion that results from this lack of cooperation among the observers.
Related literature

The ISC problem was first studied by Dobrushin and Tsybakov in [9] where they derived a closed form solution for the indirect RDF in the Gaussian stationary case and, implicitly, showed an equivalence of the indirect problem to a direct source coding problem with an amended fidelity criterion. Berger [19, Ch. 3.5] noted the equivalence of the indirect problem to a modified direct problem with a new fidelity criterion. Witsenhausen [10] extended this equivalence to the case in which side information is available at the decoder. Wolf and Ziv [11] showed that, in the case of a quadratic distortion, the new fidelity criterion identified in [10] decomposes into the sum of two terms, only one of which depends on the coding rate. The multiterminal version of the ISC problem is referred to as the CEO problem: it was introduced by Berger, Zhang, and Viswanathan in [4] where they study the case of a discrete data sequence. For this setting they characterized the asymptotic behavior of the distortion as the number of remote encoders grows to infinity while the total communication rate between sensors and central units is kept constant. In this limit, they show that this distortion vanishes exponentially fast in the overall communication rate. Oohama [12] characterized the minimum distortion for a fixed sum-rate for the quadratic Gaussian version of the CEO problem, the model in which a Gaussian source is observed in Gaussian additive noise and is to be reconstructed under quadratic distortion. The full rate distortion region for this problem was later derived in Oohama [13] and Prabhakaran, et al. [7]. In [14] an improved outer bound for the general CEO problem is derived and is shown to be tight for the binary erasure CEO problem. Lastly, [15] considers the CEO setting with a general source observed in additive Gaussian noise and estimated under quadratic distortion with a sum-rate constraint. Here a lower bound to the optimal performance is obtained using the I-MMSE relationship. A channel coding problem related to the CE scheme was considered in [16] and was denoted oblivious processing. The oblivious processing problem addresses the limits of communication over multiple remote relays, where the codebook is selected by a random key which is known at the transmitter and the receiver but not at the relays.

Since the CE scheme employs source coding at the encoders which is optimal for the noisy source observation but not necessarily for the source itself, it can be seen as an instance of the mismatched encoding problem considered in [6] and [17]. We elaborate more on the connection between the mismatched source coding setting and the CE setting in Section IV below.

Contributions

This paper studies the CE coding scheme for the ISC and the CEO problems. We can divide the contributions of the paper into two main results: (i) we present a single-letter information theoretic distortion expression, and derive positive and negative coding theorems which associate this expression with the distortion in the CE setting (ii) we evaluate this expression in a few relevant examples and compare the CE performance with the optimal source coding performance.

For the single-letter expression of the CE performance, denoted as the CE-DRF, we derive the following results:

- **Existence** – our first result shows that there exists a sequence of codes that attains the local DRF at each encoder, and an estimator of that source from the output of this encoders that attains the CE-DRF.

- **Achievability** – we next prove a stronger achievability result, which asserts that the CE-DRF can be attained by estimating the source sequence from the encoded noisy observations when each encoder’s code satisfies the property (ii) above.
Namely, each of these encoders can employ any coding scheme that is optimal with respect to its local source coding problem, and the resulting distortion in estimating the source would still approach the CE-DRF.

- **Converse** – our last coding result is a particular form of a converse theorem with respect to the CE-DRF. This theorem states that the distortion in estimating a finite sub-block of the source sequence using the output of any sequence of distributed codes which satisfy property (2), when averaged over all sub-blocks, is not smaller than the CE-DRF. This negative coding result uses properties of empirical distributions of codes that achieves the RDF from [18], and, in many instances, cannot be strengthened further, as we show through an example.

In the second part of the paper we compare the CE-DRF to the optimal source coding performances. Our main contributions in this part are as follows:

- **Quadratic Gaussian CE** – we evaluate the single-letter expression for the special case of a Gaussian source observed through $L$ independent AWGN channels, compressed at the remote encoders under quadratic distortion and reconstructed at the central unit also under quadratic distortion. Compared to the intricate expression for the quadratic Gaussian CEO [7], the CE distortion is given by a simple analytic expression in terms of the SNR and code-rate at each remote observer. We further study this expression for a different number of remote observers. For the case of a single remote encoder, we show that there is no performance loss in using the CE scheme instead of the optimal indirect source coding scheme. This, rather surprisingly, implies that the optimal compression does not require the statistical knowledge of the relationship between the source and noisy observation. This property also holds in the quadratic Gaussian case with multiple observations and a centralized encoder operating at a small rate. At high rates, instead, the distortion is greater than the iDRF by a factor of one bit per number of observers. In the fully distributed multiterminal setting, there is a strictly positive gap between the CE performance and the optimal RMSC performance described by the optimal CEO distortion. Finally, we evaluate the CE-DRF for an asymptotically large number of observers and symmetric observations under a fixed sum-rate constraint. We show that both the CE and the optimal CEO distortion vanish at a rate inversely proportional to the available sum-rate. We then conclude that neither the optimal rate allocation nor the knowledge of the SNR at the encoders is necessary to optimally scale the decrease of the distortion in the total code-rate as the number of observers goes to infinity.

- **Bit-flip Binary CE** – we also evaluate the CE-DRF for the case in which a binary source is observed through $L$ independent binary symmetric channels, this observation is compressed according to the Hamming distortion measure and the source realization is reconstructed at the central unit also according to the Hamming distortion measure. For this setting, we show that, even for the case of a single remote observer, CE coding performs strictly worst than the optimal indirect source coding. To derive this result, we also provide the first exact evaluation for the DRF of the indirect source coding problem. In addition, we derive a closed-form expression for the CE-DRF for a symmetric Bernoulli source, with an equal rate allocation to each user in the limit of a large number of observers. The distortion in this case is shown to vanish exponentially fast in the total number of bits sent by the remote observers to the central unit. For comparison, an exponential decay of the distortion in the total number of bits for discrete sources is the main result of the paper [4] introducing the CEO problem. As in the quadratic Gaussian case, we conclude that the optimal rate of decay can be
Fig. 1: Remote multiterminal source coding (RMSC) problem.

attained also in this model without employing the optimal coding strategy or having knowledge of the source statistics at the encoders.

Paper Organization

The remainder of the paper is organized as follows: Sec. II presents the problem formulation and the definition of the CE-DRF. In Sec. III we consider relevant results in source coding theory and their connection to the CE scheme. Our main results with respect to coding theorems for the CE-DRF are given in Sec. IV. The case of a Gaussian source observed in Gaussian noise and reconstructed under quadratic distortion and the case of a binary source observed in bit-flipping noise and reconstructed under Hamming distortion are studied in Sec. V. Finally, Sec. VI concludes the paper.

II. PROBLEM DEFINITION

We consider the remote multiterminal source coding (RMSC) setting of Figure 1: the random source sequence $X^n = (X_1 \ldots X_n)$ is obtained through $n$ independent draws from the distribution $P_X$ with alphabet $\mathcal{X}$. This source sequence is observed in noise at $L$ remote encoders: the $l$th encoder obtains the sequence $Y^n_l \in \mathcal{Y}^n_l$ generated by passing the sequence $X^n$ through the memoryless channel with transition probability $P_{Y_l|X}$, i.e.

$$P_{Y^n_l|X^n}(y^n_l|x^n) = \prod_{i=1}^{n} P_{Y_l|X}(y_l,i|x_i) \overset{\Delta}{=} P_{Y_l|X}^n(y^n_l|x^n) \quad l \in [1, \ldots L],$$

and produces the index $m_l \in \{1 \ldots 2^{\lfloor nR_l \rfloor}\}$ to encode its observation. The central processing unit collects the vector of indices $m = (m_1 \ldots m_L)$ and produces the reconstruction sequence $\hat{X}^n(m) \in \hat{\mathcal{X}}^n$.

Following the setting described above, we define a distributed source code of rate-vector $R = (R_1, \ldots, R_L)$ and blocklength $n$ as the $L$-tuple of encoders $(f^n_1, \ldots, f^n_L)$ of the form

$$f^n_l : \mathcal{Y}^n_l \to \{1, \ldots, 2^{\lfloor nR_l \rfloor}\}, \quad l \in [1, \ldots L] \quad (1)$$

and a decoder $g^n$ of the form

$$g^n : \prod_{l=1}^{L} \times \{1, \ldots, 2^{\lfloor nR_l \rfloor}\} \to \hat{\mathcal{X}}^n. \quad (2)$$
The distortion between the original source sequence $X^n$ and its reconstruction $\hat{X}^n$ is measured according to a single-letter fidelity criterion $d : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}_+$ and the average distortion between $X^n$ and $\hat{X}^n$ is defined as

$$\mathbb{E}d(X^n, \hat{X}^n) \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ d(X_i, \hat{X}_i) \right],$$

where the expectation is with respect to all source and channels realization, governed by the distribution:

$$P_{X^n,Y^n} \triangleq P_{X^n,Y^n_1,\ldots,Y^n_L} = P_X^n \prod_{l=1}^{L} P_{Y_l|X}.$$  

The distortion $D$ is called achievable at the rate-vector $R$ in the RMSC problem if there exists a sequence $\{f^n_1, \ldots, f^n_L, g^n\}$ of distributed source codes of rate-vector $R$ such that (3) converges to $D$ as $n$ goes to infinity.

In the standard RMSC setting, the goal is to characterize the infimum of all achievable distortions for a given rate-vector, or, vice-versa, the characterization of all rate-vectors $R$ for which a target distortion $D$ is achievable. In contrast, in the following we consider a specific attainable distortion, as a function of $R$, which we denote as the compress-and-estimate distortion-rate function (CE-DRF). In order to define the CE-DRF, we assume the existence of $L$ reconstruction alphabets $\hat{\mathcal{Y}}_1, \ldots, \hat{\mathcal{Y}}_L$ and $L$ distortion functions $d_l : \mathcal{Y}_l \times \hat{\mathcal{Y}}_l \to \mathbb{R}_+$, $l = 1, \ldots, L$, for the observation sequences $Y^n \triangleq (Y^n_1, \ldots, Y^n_L)$.

Denote by $D_l(R_l)$ the (information) DRF of the sequence $Y^n_l$ with respect to the distortion $d_l$, namely

$$D_l(R_l) = \inf_{P_{Y_l,Y_l} \in \mathcal{D}(R_l)} \mathbb{E}d_l\left( Y_l, \hat{Y}_l \right),$$

where the infimum is over all joint probability distributions such that $I(Y_l; \hat{Y}_l) \leq R_l$, and whose marginal over the alphabet $\mathcal{Y}$ coincides with the distribution induced by $P_X$ through $P_{Y_l|X}$. The CE-DRF, which is the object of this study, is defined as follows.

**Definition 2.1:** Fix $L$ conditional probability distributions $P^n_{Y_l,Y_l}$ that each satisfy $I(Y_l; \hat{Y}_l) = R_l$ and $\mathbb{E}d_l(Y_l, \hat{Y}_l) = D_l(R_l)$. The compress-and-estimate DRF (CE-DRF) for the remote multiterminal source coding problem at a rate $L$-tuple $R = (R_1, \ldots, R_L)$ is defined as

$$D_{CE}(R) \triangleq \inf \mathbb{E}d\left( X, \hat{X}(\hat{Y}) \right),$$

where $\hat{Y} = (\hat{Y}_1, \ldots, \hat{Y}_L)$, and the infimum is over all estimators of the form

$$\hat{X} : \hat{\mathcal{Y}}_1 \times \cdots \times \hat{\mathcal{Y}}_L \to \hat{\mathcal{X}}.$$

For ease of notation, when $L = 1$, we drop the subscript one from the above expressions, namely (5) is indicated as $D_{CE}(R) \triangleq D_{CE}(R_1)$.

The CE-DRF is defined by a single-letter expression involving the minimal average distortion in estimating the random variable $X$ from $L$ random variables $\hat{Y}_1, \ldots, \hat{Y}_L$. For each $l = 1, \ldots, L$, the random variable $\hat{Y}_l$ is generated by the conditional probability distribution that minimizes the RHS of (4). The fact that there exist $L$ probability distributions for which $I(Y_l; \hat{Y}_l) = R_l$ and $\mathbb{E}d_l(Y_l, \hat{Y}_l) = D_l(R_l)$ follows from the properties of the solution to the variational optimization problem that defines Shannon’s DRF [19].
**Distributed vs. centralized encoding:** The encoding scenario in Fig. 1 is an instance of *distributed* source coding as the noisy observations are encoded separately at each remote unit. For the sake of comparison, we also consider the *centralized* source coding scenario in Fig. 2 in which the noisy observations are processed by a single encoder. We note that the centralized setting can be obtained from the distributed setting by considering the case of a single remote encoder for the observation sequence $Y^n$ in which each observation symbol $Y_i$ is the vector of $L$ components $Y_i$. 

**III. RELATED RESULTS**

In this section we review the results available in the literature for the optimal source coding in the RMSC problem in Section II.

**A. Indirect Source Coding**

Consider the centralized encoding setting of Fig. 2: the problem of finding the infimum over all achievable distortions in (3) for a given rate $R$ is denoted the *indirect* (aka remote or noisy) source coding (ISC) problem. The minimum attainable distortion at rate $R$ is called the *indirect* DRF (iDRF) of $X^n$ given the observation vector $Y^n$, denoted by $D_{X|Y}(R)$. A single-letter expression for the iDRF is given by [19, p.78-81]:

$$D_{X|Y}(R) = \inf \mathbb{E} d(X, \hat{X}),$$

where the infimum is taken over all joint probability functions of $Y$ and $\hat{X}$ such that the per letter mutual information $I(Y; \hat{X})$ does not exceed $R$.

**B. The CEO Problem**

The CEO problem is the multiterminal extension of the ISC problem. It considers the minimal distortion in estimating the source given a rate-vector $R$ in Figure 1. The CEO problem was first introduced by Berger and Zhang in [4] for the case of discrete alphabets. The authors showed that, given an asymptotically large number of observers $L$ and under a sum-rate constrain, i.e. $\sum_{l=1}^{L} R_l \leq R_S$, the DRF of the CEO problem, denoted as $D_{CEO}(R)$, vanishes exponentially fast in the overall sum-rate $R_S$. The setting of a Gaussian source observed through multiple AWGN channels and estimated under quadratic
distortion was consequently considered in [20], and will be referred to henceforth as the quadratic Gaussian CEO (QG-CEO).

The main result of [20] is that, at asymptotically large \( L \), \( D_{\text{CEO}}(R) \) vanishes inversely linear in the sum-rate \( R_{\Sigma} \), where in [12] the exact constant in this convergence was shown to be one over twice the SNR in each channel \( P_{Y|X} \). Finally, an upper bound on the minimal sum-rate \( R_{\Sigma} \) required in order to attain a prescribed distortion in the QG-CEO problem, as well as the optimal rate-allocation strategy among the encoders, was derived in [21].

The set of all \( L \)-tuples of rates \( R \) such that the distortion \( D \) is achievable in the RMSC setting is referred to as the rate region of the CEO problem corresponding to a target distortion \( D \). A full characterization of the rate-region for the QG-CEO was derived in [12], [7], and for the log-loss distortion under any source distribution in [8].

As noted in [22], the CEO problem is a special case of the distributed lossy source coding problem in which multiple correlated sources have to be reconstructed at the decoder, each to within a prescribed distortion. For this reason, the Berger-Tung inner and outer bounds [23], [24] for the lossy source coding problem can be used to derive bounds on the CEO rate-region.

In the next section we prove that the function \( D_{\text{CEO}}(R) \) can be obtained by a particular sequence of distributed source codes for the RMSC setting. Since the CEO distortion is defined as the infimum over all achievable distortions for the RMSC, we conclude that necessarily

\[
D_{\text{CEO}}(R) \leq D_{CE}(R),
\]

for any rate \( L \)-tuple \( R = (R_1, \ldots, R_L) \in \mathbb{R}_+^L \).

IV. CODING THEOREMS

In this section we present three coding theorems that link the CE-DRF to the RMSC problem. We first present an achievability result which shows that there exists a sequence of distributed encoders, oblivious of the underlying source, that attains the CE-DRF. Next, we provide a stronger achievability result which asserts that the CE-DRF is achievable using any sequence of distributed encoders that asymptotically attains optimal source coding performances with respect to their individual inputs. We close this section by providing a converse result, which says that the CE-DRF is a lower bound to the distortion, averaged over finite sub-blocks, in estimating a sequence of the underlying source from the output of such oblivious encoders.

A. Achievability of the CE-DRF

**Theorem 4.1:** Assume that the alphabets \( \mathcal{X} \) and \( \mathcal{Y}_1, \ldots, \mathcal{Y}_L \) are discrete, and in addition there exists a reference letter \( b \in \bar{X} \) such that \( \sum_{x \in X} d(x, b)P_X(x) < \infty \). Let \( R \in [0, \infty)^L \) and \( \rho > 0 \). There exists a sequence of distributed source codes \( \{(f^n_1, \ldots, f^n_L, g^n)\} \) at rate-tuple \( R + \rho \) for the RMSC setting such that for every \( \delta > 0 \) there exists \( n_0 \) for which the following holds:

(i) For all \( n > n_0 \),

\[
\mathbb{E}d(X^n, g^n(f^n_1(Y^n_1), \ldots, f^n_L(Y^n_L))) \leq D_{CE}(R) + \delta.
\]
(ii) The $l$th encoder $f^n_l$ is defined only in terms of $P_{Y_l}$ and $d_l$.

(iii) For every $l = 1, \ldots, L$ and $n > n_0$, there exists a decoder $g^n_l : \{1, \ldots, 2^{n(R_l + \rho)}\} \to \hat{Y}_l^n$ such that

$$\text{Ed}_l(Y_l^n, g^n_l(f^n_l(Y_l^n))) = D_l(R_l) + \delta.$$  

**Proof:** In order to show that $D_{CE}(R)$ is achievable as in (i), it is enough to show that $D_\phi(R) \triangleq \text{Ed}(X, \phi(Y))$ is achievable for any estimator $\phi : \hat{Y}_1 \times \ldots \times \hat{Y}_L \to \hat{X}$. We will show that given such $\phi(\cdot)$ and $\delta > 0$, there exists a sequence of distributed codes that attains average distortion less than $D_\phi(R) + \delta$ and, in addition, also satisfies conditions (ii) and (iii).

For each $l = 1, \ldots, L$, fix $P_{Y_l, \hat{Y}_l}$ such that the conditions in Definition 2.1 hold, and let $P^n_{Y_l, \hat{Y}_l}(\hat{y}_l) = \sum_{y_l \in Y_l} P^{(n)}_{Y_l, \hat{Y}_l}(\hat{y}_l, y_l)$. For $\delta > 0$ define the distortion (weakly) typical set $A^{(n)}_\delta \subseteq Y_l^n \times \hat{Y}_l^n$ as the set of sequences satisfying the following 4 conditions:

$$\delta > \left| - \frac{1}{n} \log P^n_{Y_l}(y^n_l) - H(Y_l) \right|,$$

$$\delta > \left| - \frac{1}{n} \log P^n_{Y_l}(\hat{y}^n_l) - H(\hat{Y}_l) \right|,$$

$$\delta > \left| - \frac{1}{n} \log P^n_{Y_l, \hat{Y}_l}(y^n_l, \hat{y}^n_l) - H(Y_l, \hat{Y}_l) \right|,$$

$$\delta > \left| \sum_{i=1}^{n} d_i(y_{i}, \hat{y}_{i}) - D_l(R_l) \right|.$$  

Given $\delta > 0$, randomly and independently construct $2^{n(R_l + \rho)}$ sequences $\hat{Y}^n_l$ drawn i.i.d. from the distribution $\prod_{i=1}^n P_{\hat{Y}_l}(\hat{y}_l)$ and index those sequences by $m_l \in \{1, \ldots, 2^{nR_l}\}$. The set of indexed sequences is referred to the codebook $B$. The $l$th encoder, upon receiving a sequence $Y_l^n$, transmits $m_l$ such that $(Y_l^n, \hat{Y}_l^n(m_l)) \in A^{(n)}_{\delta_l, \delta}$. If there is more than one such $m_l$, the smallest index is sent. If there is no such $m_l$ then an error is reported. Note that this encoder satisfies condition (iii) of the theorem. The decoder first forms the $L$ sequences $\hat{Y}^n_l = \hat{Y}^n_1, \ldots, \hat{Y}^n_L$, and then declares a reconstruction sequence $\hat{X}^n$ in which the $i$th coordinate is given by $\hat{X}_i = \phi \left( \hat{Y}_1(i, m_1), \ldots, \hat{Y}_L(i, m_L) \right)$. If an encoding error has been reported by one of the encoders, the decoder produces the sequence $b^n \in \hat{X}^n$ which consists of $n$ copies of the reference letter $b \in \hat{X}$.

We next calculate the expected distortion over all source and channels realizations and over the random choice of the codebook $B$ that assigns a unique index $m_l$ to $Y_l^n$. We denote this distortion by $E = E_{X^n, Y^n, B} d(X^n, \hat{X}^n)$. Let us denote by $\text{Err}$ the event where an (encoding) error has been reported and by $\text{Err}^c$ its complement.

Since the sequences $X^n, Y^n, \hat{Y}^n$ are all generated according to an i.i.d distribution, we have

$$E_{X^n, Y^n, B} \left[ d(X^n, \hat{X}^n) \mid \text{Err}^c \right] \leq \sum_{X^n, Y^n, \hat{Y}^n} \frac{1}{n} \sum_{i=1}^{n} d(x_i, \phi(\hat{y}_i)) P^n_{X^nY^n}(x^n, y^n) P_{Y^n}(y^n, \hat{Y}^n)$$

$$= \sum_{x_i, y_i, \hat{y}_i} d(x_i, \phi(\hat{y}_i)) P_{X|Y}(x|y_i) P_{Y}(y_i, \hat{Y}^n) = D_\phi(R),$$

where we used the notation $\phi(\hat{Y}^n)$ to denote the vector $(\phi(\hat{Y}_{1,1}), \ldots, \phi(\hat{Y}_{1,n}))$. In the event of an error, the per-symbol distortion is bounded by

$$d_{\text{max}} \triangleq \sum_{x \in \mathcal{X}} d(x, b) P_X(x).$$
The average distortion can therefore be bounded as

$$\overline{D} \leq D_\phi(R)(1 - p_e) + p_e d_{\text{max}} \leq D_\phi(R) + p_e d_{\text{max}},$$

where $p_e = \mathbb{P}(\text{Err})$ is the probability that $(Y^n_l, \hat{Y}^n_l) \notin A^{(n)}_{d_{l, \delta}}$ for at least one $l \in \{1, \ldots, L\}$. This last probability can be bounded by $\sum_{l=1}^L p_{e,l}$, where $p_{e,l} \triangleq \mathbb{P}\left[ (Y^n_l, \hat{Y}^n_l) \notin A^{(n)}_{d_{l, \delta}} \right]$.

It follows from the achievability proof of the standard source coding theorem with respect to the sequence $Y^n_l$ [25, Ch. 10.5] that $p_{e,l}$ can be made arbitrarily small for all $n$ larger than some $n_l$ that depends on $\epsilon$. It follows that by taking $n > \max_l\{n_l\}$ and $\epsilon$ small enough, $p_e d_{\text{max}}$ can be made smaller than $\delta$, so

$$\overline{D} \leq D_\phi(R) + \delta. \quad (8)$$

We now show that for each $l = 1, \ldots, L$, there exists a decoder for $Y^n_l$ of rate $R_l + \rho$ that attains expected distortion smaller than $D_l(R_l) + \delta$. We define such a decoder by $m_l \rightarrow \hat{Y}^n_l(m_l)$. The achievability side of the standard source coding theorem with respect to $Y^n_l$ now implies that there exists $n'_l$ large enough such that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ d\left( Y_{l,i}, \hat{Y}_{l,i} \right) \right] \leq D_l(R_l) + \delta, \quad (9)$$

for all $n \geq n'_l$. It follows that for any choice of $\delta > 0$, both (8) and (9) are satisfied for an appropriate choice of $\epsilon > 0$ and $n \geq n_0 \triangleq \max_{l=1,\ldots,L}\{n'_l, n_l\}$. We conclude that for $n > n_0$ there must exist at least one distributed code with properties (i)-(iii). \hfill \blacksquare

**Remark 4.1:** A sequence of codes $(f^n_l, g^n_l)$ with the property (iii) in Theorem 4.1 is denoted a good sequence\(^1\) of rate-distortion codes with respect to $D_l(R_l)$.

It is important to note that Theorem 4.1 only asserts the existence of a distributed code that attains $D_{CE}(R)$. That is, the code that attains $D_{CE}(R)$ can be seen as if it was tailored specifically for the task of describing $X^n$, and not as the result of ad-hoc estimating at the decoder of some underlying source sequence correlated with multiple lossy encoded sequences. In other words, Theorem 4.1 does not explicitly provide the minimum distortion when reconstructing the source $X$ using a given distributed code which was designed to compress each of the observation sequences independently.

The question that we ask next is what can be guaranteed when $X^n$ is reconstructed from arbitrary good sequences of codes with respect to $D_l(R_l), \ldots, D_L(R_L)$. As the following theorem shows, the CE-DRF can be attained even if the distributed code is a collection of arbitrary codes designed to attain the DRF with respect to each one of the observation sequences.

**Theorem 4.2:** Let $R = (R_1, \ldots, R_L)$ and assume that $d$ is bounded from above by some $d_{\text{max}}$ satisfying $d(x, \hat{x}) \leq d_{\text{max}}$

\(^1\)This is slightly different than the term good rate-distortion codes used in [26], [27], [28], [29], [18]. Indeed, the “goodness” is associated with the sequence, rather than with a particular code.
for all $x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}$. For every $l = 1, \ldots, L$ and $\rho > 0$, let $(f_l^n, g_l^n)$ be a sequence of codes

$$\gamma_l^n \xrightarrow{f_l^n} \{1, \ldots, 2^{n(R_l + \rho)}\} \xrightarrow{g_l^n} \hat{\gamma}_l^n,$$

with the property that for every $\epsilon > 0$ there exists $n_0$ such that for all $n \geq n_0$,

$$\mathbb{E} d_t(Y_l^n, g_l^n(f_l^n(Y_l^n))) \leq D_t(R_l) + \epsilon.$$

Then for any $\delta > 0$, there exists $n$ large enough and an estimation rule $g_n^* : \prod_{l=1}^L \{1, \ldots, 2^{n(R_l + \rho)}\} \rightarrow \hat{\mathcal{X}}^n,$

such that

$$\mathbb{E} d(X^n, g_n^n(f_1^n(Y_1^n), \ldots, f_L^n(Y_L^n))) \leq D_{CE}(R) + \delta.$$

**Proof:** The proof relies on the fact that with high probability, any reconstruction sequence $\hat{Y}_l^n$ and the $Y^n$ generated from $P^n_Y$ are jointly typical with respect to the distribution that attains the DRF $D_t(R_l)$. Therefore, the decoder can use the same form of memoryless decoding as in the proof of Theorem 4.1. The main difficulty is to guarantee that the sequences at the input and the output of each decoder be jointly typical with high probability. This part of the proof relies on a particular property of good rate-distortion codes proved in [29, Lem. 2.2].

Throughout this proof we use the notations $\mathcal{Y} = \prod_{l=1}^L \mathcal{Y}, \hat{\mathcal{Y}} = \prod_{l=1}^L \hat{\mathcal{Y}}$ and $\gamma^n$ and $\hat{\gamma}^n$, for their $n$th Cartesian products, respectively. Assume that the alphabet $\hat{\mathcal{X}}$ is finite (assuming this does not reduce generality since $\hat{\mathcal{Y}}$ is finite and $\hat{\mathcal{X}}$ is always a function of the latter). Let $h : \hat{\mathcal{Y}} \rightarrow \hat{\mathcal{X}}$ be an estimator such that

$$\mathbb{E} d(X, h(\hat{\gamma})) - \delta \leq D_{CE}(R),$$

where the expectation is taken with respect to the distribution $P_{\mathcal{Y},\hat{\mathcal{Y}}}^* = \prod_{l=1}^L P_{Y_l,\hat{Y}_l}^*$ that attains the $L$ DRF of $Y_1^n, \ldots, Y_L^n$ with respect to $d_1, \ldots, d_L$, respectively. Namely,

$$\mathbb{E} d(X, h(\hat{\gamma})) = \sum_{x,y,\hat{y}} d(x, h(\hat{y})) P_{X|Y}(x|y) P_{\gamma,Y,\hat{\gamma}}^*(y,\hat{\gamma}).$$

In (10) and throughout the rest of the proof, sets over which summations are performed are omitted in cases they can be understood from the context. For example, in (10) the summation is over $\mathcal{X} \times \mathcal{Y} \times \hat{\mathcal{Y}}$.

For any $y_l^n \in \gamma_l^n$ denote

$$u_l(y_l^n) = g_l^n(f_l^n(y_l^n))$$

the output of the $l$th decoder to an input $y_l^n$. We also use the notations

$$u(y^n) = (g_L^n(f_L^n(y_L^n)), \ldots, g_L^n(f_L^n(y_L^n))).$$
and $u_i(y^n)$ as the $i$th coordinate of the length $n$ sequence $u(y^n)$, which is an $L$-tuple. Finally, we define an estimation rule $g_* : \hat{Y}^n \rightarrow \hat{X}^n$ for $x^n$ by

$$g_* (f^n(y^n)) \triangleq h(u(y^n)),$$

where $h(u) \triangleq [h(u_1), \ldots, h(u_L)]$. In words, this decoder receives the output $\hat{Y}^n$ of the $L$ decoders and reconstructs the sequence $X^n$ symbol-by-symbol.

In order to analyze the average distortion attained by this decoder, we define the type $T_{u^n}$ of a sequence $u^n \in U^n$ to be a probability distribution on $U$ defined as the frequency of occurrences of the symbol $u \in U$ in $u^n$, i.e.,

$$T_{u^n}(u) = \frac{1}{n} \sum_{i=1}^{n} \delta_{u_i}, \quad u \in U.$$

In addition, denote by $T^n_\epsilon \subset Y^n$ the set of sequences that are $\epsilon$ (strongly) jointly typical with the output to the decoders, namely

$$T^n_\epsilon = \{ y^n \in Y^n : \| T(y^n, u(y^n)) - P_{Y, \hat{Y}}^* \|_{TV} < \epsilon \},$$

where the total variation distance between two probability distributions $P$ and $Q$ over the same alphabet $U$ is defined by

$$\| P - Q \|_{TV} = \sum_{u \in U} |P(u) - Q(u)|.$$

The average distortion attained by $g_*$ satisfies

$$E [d(X^n, g_* (u(Y^n)))] = E [d(X^n, g_* (u(Y^n))) | Y^n \in T^n_\delta]$$

$$+ E [d(X^n, g_* (u(Y^n))) | Y^n \notin T^n_\delta].$$

The probability of a pair of input and output sequences which are non-typical for the rate-distortion achieving distribution was studied in [29]. In particular, [29, Lem. 2.2] shows that under the assumptions on the codes in the theorem, the probability that $Y^n \notin T^n_\delta$ goes to zero as $n$ goes to infinity. As a result, we can take $n$ large enough such that

$$E [d(X^n, g_* (u(Y^n))) | Y^n \notin T^n_\delta]$$

$$\leq d_{\text{max}} P( Y^n \notin T^n_\delta) \leq \delta$$

(13)

Considering the term (11) and using the fact that the estimation rule $g_*$ operates symbol-by-symbol, we have

$$E [d(X^n, g_* (u(Y^n))) | Y^n = y^n \in T^n_\delta] = \frac{1}{n} \sum_{i=1}^{n} \sum_{x_i} d(x_i, h(u_i(y^n))) P_{Y|X}(y_i|x_i)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{x_i} \sum_{\hat{y}_i} d(x_i, h(\hat{y}_i)) P_{X|Y}(x_i|y_i) 1_{\{\hat{y}_i = u_i(y^n)\}}$$

$$= \sum_{x} \sum_{y} \sum_{\hat{y}} d(x, h(\hat{y})) P_{X|Y}(x|y) \frac{1}{n} \sum_{i=1}^{n} 1_{\{y_i = y^n\}} 1_{\{\hat{y}_i = u_i(y^n)\}},$$

that is, the expected distortion given $Y^n = y^n$ is equal to the expected distortion where $P_{Y, \hat{Y}}$ is the sample distribution of
the sequence $[y^n, u(y^n)]$. Hence

$$\mathbb{E}[d(X^n, g_r(u(Y^n)))|Y^n \in T^n] = \sum_{y^n \in T^n} \mathbb{E}[d(X^n, g_r(u(Y^n)))|Y^n = y^n] P_Y^*(y^n)$$

$$= \sum_{x} \sum_{y} \sum_{\tilde{y}} d(x, h(\tilde{y})) P_{X|Y}(x|y) \sum_{y^n \in T^n} P_Y^*(y^n) \frac{1}{n} \sum_{i=1}^{n} 1_{\{y = y_i, \tilde{y} = u(y^n)\}}.$$  \hfill (14)

Finally we have

$$|D_{CE}(R) - \mathbb{Ed}(X^n, g_r(u(Y^n)))|$$

(a) $\leq \delta + |\mathbb{Ed}(X, h(\tilde{Y})) - \mathbb{Ed}(X^n, g_r(u(Y^n)))|$

(b) $\leq 2\delta + |\mathbb{Ed}(X, h(\tilde{Y})) - \mathbb{Ed}(X^n, g_r(u(Y^n)))| |Y^n \in T^n|$

(c) $\leq 2\delta + \sum_{x, y, \tilde{y}} d(x, h(\tilde{y})) P_{X|Y}(x|y) \sum_{y^n \notin T^n} P_Y^*(y^n) \left|P_{Y,\tilde{Y}}^*(y, \tilde{y}) - \frac{1}{n} 1_{\{y = y_i, \tilde{y} = u(y^n)\}} \right|$ + $\mathbb{Ed}(X, h(\tilde{Y})) \sum_{y^n \notin T^n} P_Y^*(y^n)$

(d) $\leq 2\delta + \sum_{x, y, \tilde{y}} d(x, h(\tilde{y})) P_{X|Y}(x|y) \sum_{y^n \in T^n} P_Y(y^n) \delta + d_{\text{max}} \mathbb{P}(Y^n \notin T^n)$

(e) $\leq 2\delta + d_{\text{max}} \delta + \delta.$

where: (a) follows from (10), (b) follows from (12) and (13), (c) follows by applying the triangle inequality multiple times, (d) follows from the properties of sequences in $T^n$, and (e) follows from (13).

Remark 4.2: The assumption of $d$ bounded in Theorem 4.2 is critical in the proof (see (12)): the estimation rule $g_r$ infers information on the observations sequence $Y^n$ only by observing the outputs of the decoders $g_{I_i}^n, \ldots, g_{L_i}^n$. Therefore, without any additional assumption on the structure of the codes $(f_i^n, g_i^n)$, the decoder cannot detect non-typical realizations $Y^n$, but only non-typical decoder outputs. Although the probability of this event is vanishing by the law of large numbers, without a bounded distortion measure this might result in an unbounded overall distortion.

Remark 4.3: A reference codeword might be used to indicate a non-typical realization of the observations, as in the achievable strategy of Theorem 4.1. Hence, the assumption of $d$ bounded can be relaxed in exchange for the following additional assumption on the codes: each sequence $y_i^n$ which leads to a non-typical $(y_i^n, \tilde{y}_i^n)$ is encoded into a special symbol $b_i \in \hat{X}$ that is revealed to the decoder $g_i^n$. This way, the decoder is informed on the non-typical realizations and uses and accordingly chooses as reconstruction the reference letter $b \in \hat{X}$ to yield a bounded distortion.

Remark 4.4: With the exception of Remarks 4.2 and 4.3, the proof of Theorem 4.2 shows that the only statistical information used by the decoder $g_r^n$ that attains the CE-DRF are $P_{X,Y}$ and the joint distribution $P_{Y,\tilde{Y}}^*$ that attains the $L$ DRF $D_l(R_l), l = 1, \ldots, L$. Moreover, the latter can be computed at the decoder using only $d_l, R_l$ and the marginal $P_{Y_l}$ for
Fig. 3: The function $D_{CE}(R)$ is achievable using a distributed code in which the $l^{th}$ encoder “solves” a private source coding problem for the information source $Y_l^n$ and fidelity criterion $d_l$.

$l = 1, \ldots, L$, respectively. In particular, this decoder is independent of the particular code employed by each encoder, as long as it asymptotically attains $D_l(R_l)$.

B. Discussion

Thm. 4.1 asserts that the distortion $D_{CE}(R)$ is achievable using a sequence of distributed codes with the additional property that the $l^{th}$ encoder is defined only in terms of the distribution $P_{Y_l}$ and the distortion $d_l$. In particular, it is independent of the distortion $d_l$, and the distributions $P_X$, $P_{Y_k}$ for $k \neq l$, and $P_{Y|X}$. Moreover, property (iii) in Thm. 4.1 says the $l^{th}$ encoder is also optimal for the source coding problem with source sequence $P_{Y_l}^n$ and distortion $d_l$. These properties imply that the CE-DRF can be seen as the result of a distributed code for the RMSC setting in which each encoder applies a codebook that is optimal for compressing its local observations using its local distortion measure. The central processing unit, on the other hand, requires full knowledge of the system, that is: the codebooks used by the $L$ encoders, the distribution of the remote source sequence $X^n$, that of the noisy observations and the local fidelity criterion. Thm. 4.2 shows that the same performance can actually be achieved when the decoder does not have knowledge of the codebook employed by the encoders when these codes attain the local rate distortion function.

C. Relation to Mismatched and Indirect Encoding

Consider the centralized encoding setting of Fig. 2 and the optimal source coding distortion given by the iDRF $D_{X|Y}(R)$ : an optimal coding scheme that achieves the iDRF can be obtained by solving a standard source coding problem with respect to the observation sequence $Y^n$ and using an amended distortion measure $\hat{d} : Y^L \times \hat{X} \rightarrow \mathbb{R}_+$ defined by [19]:

$$\hat{d}(y, \hat{x}) \triangleq \mathbb{E}[d(X, \hat{x})|Y = y]. \quad (15)$$

That is, the amended distortion is defined as the averaged original distortion between the reconstruction symbol and all possible realizations of the source, given the observable symbol. An optimal source coding of $Y^n$ based on the amended distortion leads, in the limit of large blocklength $n$, to the optimal source coding performance in recovering $X^n$ given $Y^n$ with respect to the original distortion $d$ [10]. In particular, note that $\hat{d}$ in (15) depends on $d$ and the joint distribution $P_{X,Y}$.
In contrast, the CE scheme corresponds to encoding $Y^n$ in a way that is optimal with respect to the distortion $d_1$, although not necessarily with respect to $\hat{d}$. Note, moreover, that our target distortion with respect to $X^n$ can always be written as

$$E(d(X^n, \hat{X}^n)) = \frac{1}{n} \sum_{i=1}^{n} E \left[ d(X_i, \hat{X}_i) | Y_i \right]$$

$$= E \left[ \hat{d}(Y^n, \hat{X}^n) \right].$$

Therefore, the CE setting can be seen as an instance of the mismatched encoding problem considered in [6] with the sequence $Y^n$ as its source: the remote observer encodes $Y^n$ so as to minimize $d_1$, while the decoder tries to minimize $\hat{d}$. The general setting of [6], however, has a subtle but very important difference from the problem we consider: the encoding rule in [6] is restricted to be the minimal distance with respect to $d_1$, while there are no restrictions on the set of codewords. In particular, this set of codewords may depend on $d_1$ as well as $\hat{d}$, and does not necessarily form a good code. For example, the codewords in [6] can be drawn i.i.d from the marginal of the DRF-achieving distribution with respect to $\hat{d}$, but encoded using $d_1$. Indeed, as noted in [6, Thm. 1], when dependency in the decoding distortion is forbidden, the loose upper bound on the distortion due to mismatched encoding derived there reduces to (i) in Thm. 4.1. Therefore, our achievability and converse results in this paper agrees with the intuition that the constraint on using a good sequence of codes with respect to $D_1(R)$ is in general more restrictive than constraining to minimal distance encoding with respect to $d_1$.

**D. Relation to the CEO problem**

Let us define the rate-region for the CE associated with a distortion level $D$, as the closure in $\mathbb{R}^L$ of all rate $L$-tuples for which $D_{CE}(R) \leq D$. Denote this rate-region by $R_{CE}$. Since for any $R$, $D_{CE}(R)$ is achievable for the CEO problem, we necessarily have $R_{CE} \subset R_{CEO}$. In fact, the CE rate-region is contained in the inner bound of the CEO rate-region derive by Berger and Tung [23], [24], [22]. In order to see this fact, set the auxiliary random variables $U_1, \ldots, U_L$ in the Berger and Tung inner bound to be $\hat{Y}_1, \ldots, \hat{Y}_L$, as in Definition 2.1.

Recall that with the scalar case $L = 1$, the CE distortion-rate function coincides with the optimal source coding performance iDRF for a particular choice of the encoding distortion $d_1$. An interesting open question is whether this is also the case for the multiterminal setting, i.e., can the CEO rate region be exhausted by a particular choice of the distortions $d_1, \ldots, d_L$. Answering this question affirmatively would imply that optimal coding schemes for the RMSC problem can be studied using standard single-terminal source coding problems.

**E. Converse for the CE-DRF**

The CE-DRF is achievable using a sequence of codes with $L$ encoder mappings that also attains the optimal source coding performance with respect to the $L$ observation sequences. We now ask what is the minimal achievable distortion $d$ with respect to $X^n$, that can be obtained using *any* sequences of codes which attain the local DRFs $D_l(R_l)$, $l = 1, \ldots, L$ at the remote observers. If we consider a finite blocklength $k$, then the average distortion in estimating any $k$-block of the source from the lossy compressed observations converges to the CE-DRF in (5). The precise statement is given by the following theorem.

---

2In the notation of [6], $d_0 = d_1$ and $d_1 = \hat{d}$. 

---
Theorem 4.3: Let \( \{(f^n_1, \ldots, f^n_L, g^n)\} \) be a sequence of distributed source codes at rate vector \( R \) for the RMSC setting of Fig. 1. Moreover assume that for every \( \delta > 0 \) there exists \( n \) and \( L \) decoders \( g^*_i : \{1, \ldots, 2^{nR_i}\} \to \hat{Y}_i \) for \( l = 1, \ldots, L \), such that

\[
\mathbb{E}d_l(Y^n, g^*_i(f^n_l(Y^n))) < D_l(R_l) + \delta. \tag{16}
\]

Fix \( k \) and let \( I \) be uniformly distributed over \( 1, \ldots, n-k \) and independent from \( X^n \) and \( Y^n \). Then for any \( k \) we have

\[
\lim_{n \to \infty} \mathbb{E}d\left( X^{l+k}_I, \hat{X}^{l+k}_I \right) \geq D_{CE}(R),
\]

where \( \hat{X}^n = g^n(f^n(Y^n)), \ldots, f^n_L(Y^n) \).

Thm. 4.3 indicates that the average distortion in estimating \( k \) consecutive symbols of \( X^n \) from a distributed code that asymptotically attains the DRFs of the observations cannot be smaller than the CE-DRF provided the blocklength \( n \) is large enough.

**Proof:** Assume first that \( L = 1 \). Fix \( k \in \mathbb{N}, \delta > 0 \). Given the realization \( y^n = y^n_i \), denote by \( \hat{x}^n \) the sequence at the output of the decoder \( g^n(f^n(y^n)) \). Also denote by \( p^*(y, \hat{y}) \) the probability distribution that attains the DRF \( D_1(R) \) of \( Y^n \) and \( p^*(y^k, \hat{y}^k) \) the i.i.d. product distribution of \( k \) variables (in this proof we use the shortened notation \( p(u|v) \) to denote \( P_{U|V}(u|v) \) for two random variables \( U, V \)). For any \( n > k \) and decoder \( g^*_i : \{0, 1\}^{\lfloor nR_i \rfloor} \to \hat{y}^n_i \), we have

\[
\mathbb{E}d\left( X^{l+k}_I, \hat{X}^{l+k}_I \right) = \sum_{x^n, y^n} \frac{1}{n-k} \sum_{i=1}^{n-k} d\left( x^{i+k}_i, \hat{x}^{i+k}_i \right) p(x^{i+k}_i|y^{i+k}_i)p(y^{i+k}_i) \tag{17}
\]

\[
= \sum_{x^n, y^n, \hat{y}^n} \frac{1}{n-k} \sum_{i=1}^{n-k} d\left( x^{i+k}_i, \hat{x}^{i+k}_i \right) p(x^{i+k}_i|y^{i+k}_i)p(y^{i+k}_i)\left[ p(y^{i+k}_i)1(g_1(f(y^n))^{i+k}_i) - p^*(y^{i+k}_i, \hat{y}^{i+k}_i) \right] \tag{17}
\]

\[
+ \sum_{x^n, y^n, \hat{y}^n} \frac{1}{n-k} \sum_{i=1}^{n-k} d\left( x^{i+k}_i, \hat{x}^{i+k}_i \right) p(x^{i+k}_i|y^{i+k}_i)p^*(y^{i+k}_i, \hat{y}^{i+k}_i), \tag{18}
\]

where \( p^*(y, \hat{y}) \) indicates \( p(y)p^*(\hat{y}|y) \) and \( p^*(y^k, \hat{y}^k) \) the i.i.d. product distribution of \( k \) variables. We now prove that, as \( n \) goes to infinity, the term (17) converges to zero while the term (18) is bounded from below by \( D_{CE}(R) \). Following [18], for a pair of length-\( k \) blocks \( (y^k, \hat{y}^k) \) over \( Y^k \times \hat{Y}^k \), we define their \( k \)th order empirical distribution with respect to sequences \( y^n \) and \( \hat{y}^n \) as

\[
\hat{p}_k(y^k, \hat{y}^k|y^n, \hat{y}^n) \triangleq \frac{1}{n-k} \sum_{i=1}^{n-k} 1(y^{i+k} = y^{i+k}_i, \hat{y}^{i+k} = \hat{y}^{i+k}_i). \tag{19}
\]

Namely, the \( k \)th order empirical distribution is the fraction of times a pair of blocks appears with the same shift as subsequences of the two other sequences. The absolute value of (17) can be bounded from above by

\[
\sum_{x^n, y^n} \frac{1}{n-k} \sum_{i=1}^{n-k} d\left( x^{i+k}_i, \hat{x}^{i+k}_i \right) p(x^{i+k}_i|y^{i+k}_i)p(y^{i+k}_i)\left[ p(y^{i+k}_i)1(g_1(f(y^n))^{i+k}_i) - p^*(y^{i+k}_i, \hat{y}^{i+k}_i) \right] \leq \sum_{x^n, y^n} \max_{\hat{y}^n} \frac{1}{n-k} \sum_{i=1}^{n-k} d\left( x^{i+k}_i, \hat{x}^{i+k}_i \right) p(x^{i+k}_i|y^{i+k}_i)p(y^{i+k}_i)\left[ p(y^{i+k}_i)1(g_1(f(y^n))^{i+k}_i) - p^*(y^{i+k}_i, \hat{y}^{i+k}_i) \right].
\]
\begin{align*}
= d \max \sum_{y^n, \hat{y}^n} \frac{1}{n-k} \sum_{i=1}^{n-k} p(y_i^{i+k}) 1 \left( \hat{y}_i^{i+k} = g_1(f(y_i^n)) + k \right) - \frac{1}{n-k} \sum_{i=1}^{n-k} p^*(y_i^{i+k}) 1 \left( y_i^{i+k} = \hat{y}_i^{i+k} \right)
\end{align*}

The first term within the absolute value above is the \( k \)th order empirical distribution between \( Y^n \) and \( g_1^n(f^n(Y^n)) \). The second term is the i.i.d distribution \( p^*(\hat{y}^k) \): it follows from [18, Thm. 9] that for any sequence of source codes that satisfies condition (16), the \( k \) order empirical distribution \( \hat{y}_k \) converges almost surely to the unique distortion-rate achieving distribution \( p^*(\hat{y}^k) \). This implies that the term (17) converges to zero as \( n \) goes to infinity.

As for the term (18), we can assume without loss of generality that any decoder \( g^n : \{0,1\}^{nR_l} \rightarrow \hat{Y}^n \) is bijective, so that a decoder \( g^n : \{0,1\}^{nR_l} \rightarrow \mathcal{X}^n \) for \( X^n \) attains the same performance if it uses the sequences \( \hat{Y}_i^n = g^n(f^n(Y_i^n)) \), \( l = 1, \ldots, L \), as its inputs (this is true since it is always possible to augment the reconstruction alphabet \( \mathcal{Y}_l^n \) to track its pre-image under the decoder \( g^n \), while re-defining the distortion \( d_i \) ignores this augmentation. In fact, up to a negligible number of sequences, any good code to describe \( Y^n \) must use different reconstruction sequence \( \hat{Y}_l^n \) with each one of the possible \( 2^{nR_l} \) indices). As a result, the decoder \( g^n(f^n(Y^n)) = g^n(\hat{Y}^n) \) induces \( n \) estimators \( \phi_j : \hat{Y}_1 \times \cdots \times \hat{Y}_L \rightarrow \hat{X} \) which maps \( \hat{y}_{1,j}, \ldots, \hat{y}_{L,j} \) to \( \hat{x}_j \), for \( j = 1, \ldots, n \). Note that while the decoder is operating on blocks, the estimators \( h_j \) only map \( L \) single coordinates to another single coordinate. We interpret \( \phi(u^n) \) to be a length-\( n \) sequence whose \( j \)th coordinate is \( \phi_j(u_j) \). With these notations we have \( \phi(\hat{Y}^n) = \hat{X}^n \).

It follows from the definition of \( DCE(R) \) that for every \( i \), and estimator \( \phi_i \), we have

\[
DCE(R) \leq \mathbb{E}d(X, \phi_j(\hat{Y})) = \sum_{x, y, \hat{y}} d(x, \phi_i(\hat{y})) p(x|y)p^*(y, \hat{y}),
\]

and thus, for any \( n, k \) and \( i \),

\[
DCE(R) \leq \sum_{x^n, y^n, \hat{y}^n} \frac{1}{n-k} \sum_{i=1}^{n-k} d(x_i^{i+k}, \phi(\hat{y}_i^n y_i^{i+k}) p(x_i^{i+k}|y_i^{i+k}) p^*(y_i^{i+k}, \hat{y}_i^{i+k})).
\]

From the definition of \( h \), we see that (18) is in fact the RHS of the last equation, which is bounded from below by \( DCE(R) \).

In order to go from the case \( L = 1 \) to an arbitrary \( L \), we replace \( Y^n \) with \( Y^n \). The only non-trivial adaptation required is in the definition of the empirical distribution of \( Y^n \) and the output of the \( L \) encoders \( g_1^n(f_1^n(y_1^n)), \ldots, g_L^n(f_L^n(g_L^n(Y))) \). The argument that uses [18, Thm. 9] implies convergence at each coordinate, and the second term stays essentially the same. 

Compared to the proof of Theorem 4.2, typicality properties of sequences of good codes are not enough to derive a lower bound valid for all estimators of the source from their output. The reason is that unlike the particular estimator defined in the proof of Theorem 4.2, a general estimator cannot be seen as operating element-wise over the estimate \( \hat{Y}^n \) of \( Y^n \). Therefore Thm. 4.3 relied on the convergence in total variation distance of the empirical output distribution of a good sequence of distortion-rate code to the unique i.i.d distribution that minimizes Shannon’s DRF [28]. This convergence, however, only holds when the window size \( k \) is kept fixed as in the statement of the theorem. Excluding degenerate cases, the total variation distance between the DRF-achieving distribution and the output distribution induced by the code does not go to zero [30], [27]. It is only the normalized total variation that converges to zero [26], but this fact is not enough to establish a stronger converse. In
addition, it is impossible to replace the average over sliding windows of a finite length with a specific window choice, since a specific section of a code that attains the DRF of the observation may attain distortion strictly smaller than the CE-DRF. This last situation is best shown by the following example.

**Example 4.1:** Let $X^n, Z^n$ be two independent memoryless sources and let $(f^n_x, g^n_z), (f^n_x, g^n_z)$ be two sequences of rate $R$ codes that attain the DRFs of $X^n$ and $Z^n$, respectively. Denote by $D(R)$ the DRF of the source $X^n$ with respect to the distortion measure $d$ and by $D_z(R)$ the DRF of $Z^n$ with respect to distortion $d_z$. Let $Y^n$ be a sequence defined by

$$Y_i = (X_i, Z_i), \quad i = 1, \ldots, n,$$

and let $d_1$ be the distortion measure that only takes into account the second coordinate of $Y_i$, namely

$$d_1((x, z), (\hat{x}, \hat{z})) = d_z(z, \hat{z}).$$

Since the distortion of $Y^n$ with respect to $d_1$ equals the distortion of $Z^n$ with respect to $d_z$, the distribution that attains the DRF of $Y^n$ is specified only in terms of $P_Z$. As a result, estimating $X$ from this distribution is as difficult as a random guess, so we have

$$D_{CE}(R) = \min_{\hat{x} \in \hat{X}} \mathbb{E}d(X, \hat{x}). \quad (19)$$

Assume now that there exists $\delta_0 > 0$ such that $D(R) + \delta_0 < D_{CE}(R)$ (this assumption holds whenever $D(R)$ is strictly smaller than (19), which is the case for any source with non-degenerate DRF). Let $n_0$ be large enough such that

$$\mathbb{E}d(X^{n_0}, g_x^{n_0}(f_x^{n_0}(X^{n_0}))) < \delta_0/2 + D(R)$$

For $n > n_0$, we define an encoding of $Y^n$ as follows:

$$f^{n_0+n}(Y^{n_0+n}) = f_x^{n_0}(X^{n_0})||f_z^n(Z^n) \quad (20)$$

where $||$ denotes the concatenation operator between the two binary strings.

Note that the length of the codeword defined by (20) does not exceed $n_0R + nR$ binary symbols, so the rate of the code is still $R$. We now use the decoder $g_z^n$ to decode the last $n$ symbols of $Y^{n_0+n}$. This encoder attains distortion $D_z(R) = D_1(R)$ up to $\delta$, which can be made arbitrary small. On the other hand, using the decoder $g_z^{n_0}$ on the first part of the codeword in the RHS of (20) to recover $X^{n_0}$ results in distortion $d$ with respect to $X^n$ smaller than $D_{CE}(R) - \delta_0/2$.

Note that this example relies on the fact that adding a fixed-length prefix to a distortion-rate achieving code does not change its asymptotic behavior. In general, the addition (or removal) of any sub-exponential number of codeswords from a code that attains the DRF with respect to the observations does not affect the “goodness” of the code. As a result, it seems that the statement of Thm. 4.3 cannot be strengthened without other assumptions on the distributed code that attains the DRFs with respect to the observations.

Thms 4.1, 4.2 and 4.3 comprise the main theoretical results of this paper with respect to the general form of the CE-DRF in
the RMSC problem. In the following section we consider particular examples of information sources and observation models, as well as the distortion measures by which these are encoded and reconstructed. We focus in particular on comparing the CE-DRF to the optimal ISC performance, described by the iDRF in the case of a single terminal or the CEO distortion for multi-terminal.

V. EXAMPLES

As an example of the result in Sec. IV, we next consider two classic rate distortion scenarios: the case of a Gaussian source observed in Gaussian noise and reconstructed under quadratic distortion and the case of a binary source observed in bit-flipping noise and reconstructed under Hamming distortion. We use these examples to compare the CE performance with both the iDRF and the CEO problem, as well as investigate the asymptotic behavior as the number of remote encoders grows large.

A. Quadratic Gaussian

In this section we assume that the source is Gaussian and it is observed through \( L \) AWGN channels and estimated according to a quadratic distortion at both the remote encoders and the central unit. We will consider both the centralized encoding scenario in Fig. 2 and the distributed encoding scenario in Fig. 1. The performance of the centralized scenario are compared with the iDRF of the ISC problem, the distributed scenario, instead, is compared with the Gaussian CEO distortion in the RMSC problem. In both cases we consider the observation model

\[
Y^m_l = \sqrt{\gamma_l} X^n + Z^m_l, \quad l = 1, \ldots, L,
\]

where \( Z^m_l \) and \( X^n \) are a sequence of i.i.d standard normal variables, \( \gamma_1, \ldots, \gamma_L \in \mathbb{R}_+ \), and quadratic distortions, i.e.

\[
d(u, \hat{u}) = d_l(u, \hat{u}) = (u - \hat{u})^2, \quad l = 1, \ldots, L.
\]

In accordance with notations introduced in Sec. III-B for the CEO problem, the observation model (21) with the quadratic distortion is referred to as the quadratic Gaussian compress-and-estimate (QG-CE) setting.

1) Centralized Encoding: We begin by considering the centralized encoding setting of Fig. 2: we evaluate the CE-DRF for this scenario and compare its performance to that of the vector Gaussian iDRF.

The CE-DRF can be obtained by evaluating (5) in the QG-CE setting. This evaluation leads to the following result:

**Proposition 5.1:** The CE-DRF in the QG setting for the centralized encoding scenario of Fig. 2 is given by

\[
D_{CE}(R) = \frac{1}{\gamma_\Sigma + 1} + \frac{\gamma_\Sigma}{\gamma_\Sigma + 1} \theta,
\]

where \( \gamma_\Sigma \triangleq \sum_{l=1}^L \gamma_l \) and \( \theta \) is determined by

\[
\theta = \begin{cases} 
2^{-2R} & R \geq \frac{1}{2} \log(1 + \gamma_\Sigma), \\
2^{-\frac{2R}{L}} & R \leq \frac{1}{2} \log(1 + \gamma_\Sigma).
\end{cases}
\]
Proof: The evaluation of (5) for the QG-CE setting involves two main steps: first (i) obtain the optimal backward channel corresponding to the source coding problem of estimating the vector Gaussian source $Y^n$ under quadratic distortion, then (ii) estimate the remote source sequence $X^n$ from the output of this inverted channel. The details are given in App. A. 

We next wish to compare the result in Prop. 5.1 with the optimal source coding performance described by the iDRF of $X^n$ given the observation vector $Y^n$ as in (21). The expression for this iDRF in the QG setting of (21) is given in [31, Eq. 10] as follows:

$$D_{X|Y}(R) = \text{mmse}(X|Y) + (1 - \text{mmse}(X|Y)) 2^{-2R}$$

where $\text{mmse}(X|Y)$ is the minimal mean square error (MMSE) in estimating $X$ given $Y$.

By comparing (23) and (25) we observe that the CE-DRF coincides with the iDRF either when the rate $R$ is smaller than $0.5 \log(1 + \gamma \Sigma)$ or when $L = 1$. In other words, for this choice of the observation distortion, it is possible to attain the optimal source coding performance despite not having knowledge of the joint distribution between the source and the observations.

When the rate $R$ is larger than $0.5 \log(1 + \gamma \Sigma)$, the minimal distortion under CE coding decreases roughly $L$ times slower than the equivalent term in $D_{X|Y}(R)$. The distinction between these two regimes can be explained as follows: when the rate is low, an optimal coding with respect to $Y^n$ compresses the source signal along the component with the highest energy. This component happens to coincide with the subspace in which the estimation $\mathbb{E}[X|Y]$ actually resides. At higher rates, an optimal encoding with respect to $Y^n$ also incorporates the other $L - 1$ components which are orthogonal to $\mathbb{E}[X|Y]$ and thus span the noise sub-space. As a result, the part of the code that describes the noise components increases compared to that of the signal components.

Additional intuition regarding the difference between distortions (25) and (23) can be obtained by studying the amended distortion measure in (15). Under quadratic distortion, the amended distortion measure (15) that reduces the ISC problem to a regular source coding problem takes the form

$$\hat{d}(y, \tilde{x}) = \mathbb{E}[(X - \tilde{x})^2|Y = y]$$

$$= \text{mmse}(X|Y = y) + \mathbb{E}[(\mathbb{E}[X|Y] - \tilde{x})^2|Y = y].$$

Since in the QG case, the MMSE estimation of $X$ from $Y$ is of the form $\mathbb{E}[X|Y = y] = b^T y$, the amended distortion (26) can be written as

$$\hat{d}(y, \tilde{x}) = \text{mmse}(X|Y = y) + \mathbb{E}[(b^T y - \tilde{x})^2]$$

$$= \text{mmse}(X|Y = y) + \mathbb{E} \left[ \left( b^T \left( y - \frac{1}{b^T b} \tilde{x} \right) \right)^2 \right]$$

$$= \text{mmse}(X|Y = y) + \sum_{l=1}^{L} b_l^2 (y_l - \tilde{y}_l)^2$$

$$+ \sum_{l \neq p} b_l b_p (y_l - \tilde{y}_l) (y_p - \tilde{y}_p),$$

(27)
where we denoted $\tilde{y} \triangleq \frac{b^T}{b^T b} \tilde{x}$ and $b = (b_1, \ldots, b_L)$. We see that the amended distortion (27) can be seen as the sum of three terms. The first term is the MSE which is independent of the reconstruction symbol and therefore does not affect the encoding rule (when $X$ and $Y$ are jointly Gaussian as in (21), the MSE term is also independent of the source symbol $y$). The dependency of $\hat{d}$ on the joint distribution of $X^n$ and $Y^n$ is only through the minimal MSE estimator coefficients vector $b$: when $b_1 = \ldots = b_L$ (as in the case where $\gamma_1 = \ldots = \gamma_L$ in (25)), the only difference between the encoding rule induced by (26) and the quadratic distortion with respect to $Y$ used by the CE encoder is due to the cross term $b_l^2 \sum_{i \neq p} (y_i - \tilde{y}_i) (y_p - \tilde{y}_p)$. This cross term does not exist for $L = 1$, hence the equality between (25) and (23) in this case.

A comparison between the CE-DRF and the iDRF is illustrated in Fig. 4 for $L = 3$ and different values of the SNR parameter $\gamma$. Note that, as $R$ increases, both the iDRF and CE-DRF tend to $1/(\gamma_\Sigma + 1)$, although they decrease at a different rate.

2) Distributed Encoding: We next consider the QG-CE distortion in the full distributed encoding setting which corresponds to Fig. 2. Evaluating (5) in the QG setting leads to the following result:

**Proposition 5.2:** The CE-DRF with the QG setting in the distributed encoding scenario of Fig. 1 is given by

$$D_{CE} (R) = \left( 1 + \sum_{l=1}^{L} \gamma_l \frac{1 - 2^{-2R_l}}{1 + \gamma_l 2^{-2R_l}} \right)^{-1}. \quad (28)$$

**Proof:** As in Prop. 5.1, two main steps are required in order to derive (28): first (i) obtain the optimal backward channel corresponding to the source coding problem of estimating the Gaussian source $Y_l^n$ under quadratic distortion, then (ii) estimate the source sequence $X^n$ from the outputs of these inverted channels. The details are given in App. B. \hfill \Box

Note that when $L = 1$, (28) reduces to the centralized encoding scheme with a single observation of Prop. 5.1, which, as discussed in Sec. V-A1, coincides with the optimal source coding performance (25).
Fig. 5: The rate distortion region for the CE problem in (28) (green) and the CEO problem (blue) for $L = 2$, $D \in \{1/4, 1/16\}$ and $\gamma = 20$.

We next compare the CE-DRF in (28) to the minimal distortion in the QG-CEO problem. Using the characterization of the rate-region for the QG-CEO from [7], Fig. 7 shows that for $L \geq 2$ there exists a rate-vector $R$ for which $D_{CEO}(R) < D_{CE}(R)$. That is, the CE scheme is strictly sub-optimal for the RMSC setting when $L \geq 2$. Note that, as $R$ increases, both distortions converge to the MMSE of estimating $X^n$ from the noisy observations given by $1/(1 + \gamma \Sigma)$.

A comparison between the two rate-regions for $L = 2$ is presented in Fig. 5. Note that by definition the CE rate region is contained in the CEO rate region and the two regions share the same corner points, since when one rate is set to zero the two rate-regions are identical. Also note that unlike the CEO region, that time-sharing strategy implies its convexity, the CE rate-region is in general not convex.

Asymptotically large number of observations: As pointed out in [12], one is often interested in the minimal attainable distortion when the number of observations goes to infinity while the overall transmission rate between the sensors and the central unit is kept constant. This limit provides an indication of the performance gains that are achievable by adding more sensors to the system while keeping the communication rate constant. In order to study this limit, we impose the following two simplifying assumptions:

(A1) The SNR between the source to each observation sequence is identical and equals $\gamma$.

(A2) The rate allocated to each remote encoder is identical and is given by $R_l = R_S/L$, where $R_S$ is the total rate-budget. Under this assumption, we also define $R_L = (R_S/L, \ldots, R_S/L)$.

We note that (A1) leads to the same observation model as in the QG-CEO setting of [12], [20], where the CEO distortion was considered in the limit as $L$ goes to infinity.
Fig. 6: The CE rate-region boundaries for various distortion values $D_{CE}$ with $L = 2$ and $\gamma_1 = \gamma_2 = 4$. Note that the rate-region is not convex for $D_{CE} < \text{mmse}(X|Y)$.

Fig. 7: The QG-CE distortion (red) and the CEO distortion (dotted) with symmetric quality of observations $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ and rate-allocation $R_l = R_\Sigma/3$, $l = 1, 2, 3$, for two different values of $\gamma$.

Under (A1) and (A2), the QG-CE distortion takes the form

$$D_{CE}(R_L) = \left(1 + L\gamma^2 \frac{1 - 2^{-2R_\Sigma/L}}{1 + \gamma^2 2^{-2R_\Sigma/L}}\right)^{-1}$$

which, in the limit as $L \to \infty$, leads to

$$\lim_{L \to \infty} D_{CE}(R_L) = \left(1 + \frac{2\gamma^2 \ln 2}{1 + \gamma^2 R_\Sigma}\right)^{-1}.$$
which is
\[ \frac{1 + \gamma^2}{2\gamma^2 \ln 2} \] (31)
for the CE-DRF and \( 1/(2\gamma^2) \) for the CEO problem [12, Cor. 2]. The main difference between these constants is that (31) does not vanish as the SNR \( \gamma^2 \) increases, implying that the performance under the CEO setting significantly outperforms that of the performance in the CE setting at high SNR values.

It must be noted that when \( \gamma > 1 \), the symmetric rate-allocation assumed by (A2) is not necessarily the optimal rate-allocation that minimizes the CE-DRF of (28) under the sum-rate constraint \( R_\Sigma \). This can be seen, for example, from the fact that the CE rate-region in Fig. 5 is not convex. Thus, the expression (30) only provides an upper bound for the asymptotic behavior of the minimal distortion under this sum-rate constraint. In fact, for high values of \( \gamma \) and low values of \( R \), it can be seen that the strategy of allocating all rate to a single encoder strictly outperforms (30). A general rate-allocation strategy for the QG-CE is discussed in [32].

B. Binary Source in Bit-flip Noise

We now consider the binary counterpart of the previous QG example of Subsection V-A. Here the source \( X^n \) is an i.i.d. binary process with \( \mathbb{P}[X_i = 1] = p \) while the observations at the \( l \)th encoder are obtained as
\[ Y^n_l = X^n \oplus Z^n_l, \] (32)
where, for each \( l \), \( Z^n_l \) is a sequence of i.i.d binary random variables with \( \mathbb{P}[Z_{l,i} = 1] = \alpha_l \) independent of \( X^n \). We assume that \( p \) and the \( \alpha_l \)s are smaller than \( 1/2 \), since the other cases can be treated in a symmetric manner. Both the observation distortion and the source distortion are assumed to be the Hamming distortion, i.e.
\[ d(u, \hat{u}) = d_l(u, \hat{u}) = u \oplus \hat{u}, \quad l \in \{1, \ldots, L\}. \] (33)

In order to evaluate (5) for this scenario, it is required to obtain the optimal estimation of the source symbol from the observations. The optimal reconstruction symbol \( \hat{X} \in \{0, 1\} \) is the solution to a binary hypothesis testing problem with sample \( \hat{Y}_1 \ldots \hat{Y}_L \). The error in this hypothesis testing is the resulting Hamming distortion in the CE scheme, given as follows:

**Proposition 5.3:** The CE-DRF in distributed encoding with binary observation and rates \( R = R_1, \ldots, R_L \) is given by
\[
D_{CE}(R) = \bar{p} \mathbb{P} \left[ \sum_{l=1}^{L} S_l \log \left( \frac{\hat{\xi}_l}{\xi_l} \right) \geq \log \left( \frac{\bar{p}}{p} \right) \right] \\
+ p \mathbb{P} \left[ - \sum_{l=1}^{L} S_l \log \left( \frac{\hat{\xi}_l}{\xi_l} \right) < \log \left( \frac{\bar{p}}{p} \right) \right],
\] (34)
where \( \bar{x} = 1 - x \), \( \xi_l \) is defined as
\[ \xi_l \triangleq \alpha_l \star D_l \triangleq \alpha_l D_l + \bar{\alpha}_l D_l, \]
$D_l$ is the DRF of $Y_l$ evaluated at $R_l$, i.e.

$$D_l \triangleq D_l(R_l) = h^{-1}(\{h(p \star \alpha_l) - R_l\}^+),$$  \hspace{1cm} (35)$$

and $S_1 \ldots S_L$ are independent random variables with$^4$ $\mathbb{P}[S_l = 1] = \xi_l$, $\mathbb{P}[S_l = -1] = \bar{\xi}_l$.

**Proof:** Recall that the pdf that attains the DRF of a Bernoulli source can be described by a binary symmetric channel [25]. The estimator is required to estimate the most likely source symbol from the output of $L$ binary symmetric channels, where the bitflip probability in each channel is determined by the quality of observations $\alpha_l$ and the rate of compression $R_l$. The value of $D_{CE}(R)$ is the probability of an error in the source estimation. The full proof is provided in App. C. \hfill \Box

When $L = 1$ expression (34) reduces to

$$D_{CE}(R) = \mathbb{P}[S = 1] = \alpha_1 \bar{D}_0 + \bar{\alpha}_1 D_1.$$  \hspace{1cm} (36)$$

Writing (36) in terms of the rate $R$ as a function of the distortion $D$ (by using the relation (35) for $D_1$ and $R$), we obtain

$$R(D) = h(p \star \alpha_1) - h(D_1) = h(p \star \alpha_1) - h\left(\frac{D_1 - \alpha_1}{1 - 2\alpha_1}\right),$$  \hspace{1cm} (37)$$

Note that the expression in (37) for $\alpha_1 = 0$ coincides with the classical rate-distortion function of a Bernoulli random variable estimated under Hamming distortion. Moreover, when $p = 1/2$, (37) reduces to the indirect rate-distortion function of a Bernoulli random variable observed through a binary symmetric channel with crossover probability $\alpha_1$ [19, Exc. 3.8]:

$$R_{X|Y}(D) = 1 - h\left(\frac{D_1 - \alpha_1}{1 - 2\alpha_1}\right).$$  \hspace{1cm} (38)$$

The equality in (38) implies that there is no loss in performance due to CE encoding compared to the optimal source coding performance. This result is analogous to the QG case for $L = 1$ in Sec. V-A. Unlike in the QG case, though, this equivalence does not hold for all source and channel parameters: when $p_1 \neq 1/2$ we have that (37) is strictly larger then the indirect rate-distortion in (38) [2]. The difference in performance for the CE problem and the indirect rate distortion problem is illustrated in Fig. 8.

Note that both the iDRF and the CE-DRF attain the minimum distortions $D_{\min} = \alpha_1$ at rate $h(p \star \alpha_1)$: this rate is the entropy of the observation sequence $Y_1^n$, so the latter can be described almost losslessly by the decoder at this rate. Indeed, $D_{\min} = \alpha_1$ is the probability of making an error in estimating $X^n$ from the output of the channel (32). On the other hand, the iDRF attains the maximum distortion $D_{\max} = p$ at rate zero, while the CE-DRF attains the same distortion at rate $h(\alpha_1)$. This difference is rather interesting as it highlights that, for a rate less that $h(\alpha_1)$, the CE scheme does not provide any advantage over estimating the source outcome to be the all zero sequence.

$^3$Here we assume that the inverse binary entropy function returns a value in $[0, 1/2]$, for the reason it is a one-to-one mapping from $[0, 1]$ to $[0, 1/2]$.

$^4$Note that for $p, \alpha_l \leq 1/2$ we have $D_l \leq p \star \alpha_l$, and $D_l \leq \xi_l \leq p \star \alpha_l \star \alpha_l$. 


Fig. 8: The CE-DRF for a Bernoulli source with $p = 0.25$ and a single observation with bitflip probability of $\alpha_1 = 0.05$ (red). The dotted curve is the indirect DRF from [2] and the dashed curve is the DRF of the source corresponding to the case $\alpha_1 = 0$.

1) Statistically equivalent observations: In the case of a uniform source, i.e. $\alpha = 1/2$, the expression in (34) reduces to

$$D_{CE}(R) = \mathbb{P} \left[ \sum_{l=1}^{L} S_l \log \left( \frac{\xi_l}{\xi} \right) > 0 \right] + \frac{1}{2} \mathbb{P} \left[ \sum_{l=1}^{L} S_l \log \left( \frac{\xi_l}{\xi} \right) = 0 \right].$$

(39)

If we further assume that the noisy observations are statistically equivalent, that is $p \triangleq p_1 = \ldots = p_L$, and compressed at equal rate at each encoder $R_1 = \ldots = R_L = R_{\Sigma}/L$, we obtain $\xi \triangleq \xi_1 = \ldots = \xi_L$ in (39) and Prop. 5.3 leads to

$$D_{CE}(R) = \mathbb{P} \left[ \sum_{l=1}^{L} S_l > 0 \right] + \frac{1}{2} \mathbb{P} \left[ \sum_{l=1}^{L} S_l = 0 \right],$$

(40)

where the terms $S_l$s are i.i.d. with

$$\mathbb{P}[S_l = 1] = \xi = pD + \overline{p}D,$$

(41)

where

$$\frac{R_{\Sigma}}{L} = 1 - h(D).$$

(42)

2) Number of observations asymptotically large: We further explore the expression in (40) as the number of observations $L$ increases while the sum-rate $R_{\Sigma}$ is kept constant, so each encoder operates at rate $R_{\Sigma}/L$. Note that this expression holds for the case of a uniform source, statistically equivalent observations and equal rate allocation at each encoder. This rate allocation is not necessarily optimal but allows one to greatly simplify the expression in (39).

As $L$ increases, more compressed observations are available at the decoder but each compressed observation becomes noisier due to the reduction in compression rate of each encoder. These two trends suggest that the exact expression for the distortion (40) as $L$ goes to infinity requires a careful examination of the dependency of $\xi$ on $L$. The result of this examination is given by the following proposition.
Proposition 5.4: Consider the binary CE distributed setting with uniform source, i.e. $\alpha = 1/2$ with $p_1 = \ldots, p_L = p$ and symmetric rate allocation $R_S = (R_\Sigma/L, \ldots, R_\Sigma/L)$, then

$$\lim_{L \to \infty} D_{CE}(R_S) = Q \left( 2\sqrt{\ln 4R_\Sigma} \left( \frac{1}{2} - p \right) \right),$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$.

Proof: See App. D. \hfill \Box

By bounding $Q(x)$ from above as $Q(x) \leq \frac{1}{2} \exp(-x^2/2)$, the distortion in (43) Prop. 5.4 can also be upper bounded as

$$\lim_{L \to \infty} D_{CE}(R_L) \leq \frac{1}{2} 2^{-4(\frac{1}{2} - p)^2 R_\Sigma}.$$ (44)

From the expression in (44) we conclude that the CE distortion converges to zero exponentially fast in the total sum-rate. We note that an exponential convergence rate was observed in the CEO problem for discrete alphabets in [4]. Moreover, this result from [4] was obtained under the assumption of an optimal coding scheme applied by all encoders and optimal rate allocation under a sum-rate constraint. Prop. 5.4 therefore implies that an exponential convergence rate of the distortion in the RMSC setting as a function of $R_{\Sigma}$, as $L$ goes to infinity, is neither a unique property of the optimal source code nor of the optimal rate allocation strategy. Indeed, at least in the example of a binary source with bitflip noise in Prop. 5.4, neither of these assumptions were given in the setting that leads to Prop. 5.4.

Fig. 9 illustrates $D_{CE}(R_L)$ and (44) as a function of $L$ for a fixed value of $R_\Sigma$. Since in our setting increasing the number of observations under a fixed sum-rate reduces the rate of each observation, the distortion in general does not decrease monotonically in the number of observations. As can be seen in Fig. 9, the asymptotic value of the distortion (43) may not even be the minimum.
VI. CONCLUSIONS

We have considered a remote multiterminal source coding problem in which a decoder estimates a source sequence given multiple lossy compressed versions of its noisy observations. Unlike in traditional source coding settings, we do not assume that each observation is encoded in an optimal way with respect to the ultimate reconstruction problem. Instead, we considered a coding scheme called compress-and-estimate (CE), where each observer first compressed its observation according to a local distortion measure and rate-per-symbol constraint. A central processing unit receives these compressed observations and uses them to estimate the source sequence. By providing an achievability and a converse coding theorem, we have shown that the minimal source coding performance of the CE scheme in the RMSC setting can be described by a simple single-letter expression. To illustrate our results concretely, we applied them to a Gaussian source observed through multiple AWGN channels and a Bernoulli source observed through multiple binary symmetric channels. We have shown that for both cases, in the limit of a large number of observations, the distortion in CE vanishes at the same rate as a function of the sum-rate as if the optimal source code were applied.

In the CE scheme each encoder operates independently of the others and without knowledge of the statistics of the underlying source. Despite this limitation, we showed examples where this coding scheme can still achieve the optimal or near optimal source coding performance, i.e., the performance as if the statistics of the underlying source were known at each encoder. Other cases presented demonstrate that this form of encoding is in general sub-optimal compared to the optimal source coding performance. In those cases where CE is optimal, a CE separation principle exists: the compression at the observer can be designed separately from the estimator.

The work presented here leaves a number of interesting open questions with respect to the CE scheme. First, as we have seen that the CE is sub-optimal in the quadratic Gaussian case with more than one observation, perhaps a different choice of the distortion metric at each encoder might yield better performance. In particular, we ask whether the CEO rate-region can be exhausted by a particular choice of such distortion measures, as is the case in the single terminal ISC problem. In addition, it would be interesting to examine the CE performance in multiterminal settings where different distortions metrics are applied to different encoders. Finally, it seems that the CE scheme can be easily extended to cases where some encoders (but not all of them) can confer and deliver a joint message. Similarly, the encoders may only be allowed to exchange other parameters such as their respective SNR information or the total number of observers. The CE distortion in these cases is particularly relevant in understanding the benefits of wide-scale cooperation in sensor networks.

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Rewrite the observations in (21) as a jointly Gaussian i.i.d. vector $Y^n$ with covariance matrix $\Sigma_Y$. Denote by $D_Y(R)$ the quadratic DRF of the i.i.d source $Y^n$. The pdf $p(Y|\tilde{Y})$ that achieves this DRF is induced by the backward channel:

$$Y = \tilde{Y} + UV,$$

where:

- $U$ is a unitary matrix such that
  $$U^*\Sigma_Y U = \text{diag}(\lambda_1, \ldots, \lambda_L),$$
  and $\lambda_1, \ldots, \lambda_L$ are the eigenvalues of $\Sigma_Y$ [19].
- $V$ is a Gaussian vector independent of $\tilde{Y}$ whose $l^{th}$ coordinate has variance
  $$\sigma^2_{V_l} = \min\{\theta, \lambda_l\},$$
  and $\theta$ is chosen such that
  $$R = \frac{1}{2} \sum_{l=1}^{L} \log^+ \left( \frac{\lambda_l}{\theta} \right).$$

Note that from (21), the covariance matrix of the vector $Y$ can be written as

$$\Sigma_Y = \alpha\alpha^* + I,$$

where $a = (\gamma_1, \ldots, \gamma_L)$. In addition, for $\lambda \in \mathbb{R}$ we must have

$$\lambda I - \Sigma_Y = (\lambda - 1)I - a a^*,$$

so the eigenvalues of $\Sigma_Y$ equal one plus the eigenvalues of the matrix $\alpha\alpha^*$. The latter has a single non-zero eigenvalue: $a^*a$ and we thus conclude that the $\lambda_l$s are given by

$$\lambda_1 = \alpha^*\alpha + 1,$$
$$\lambda_2 = 1,$$
$$\vdots$$
$$\lambda_L = 1.$$

$$\log(\theta) = \begin{cases} 
\frac{\log(1+a^*a) - 2R}{L}, & R > \frac{1}{2} \log(1+a^*a), \\
\log(1+a^*a) - 2R, & R \leq \frac{1}{2} \log(1+a^*a). 
\end{cases}$$
In addition, note that the eigenvector $u_1$ corresponding to $\lambda_1$ equals $a\|a\|_2$, while all the remaining eigenvectors $u_2, \ldots, u_L$ are orthogonal to $a$. When considering the equivalent forward channel to (45), it is possible to define $\tilde{Y} \triangleq U^*\hat{Y}$ to obtain an orthogonal backward channel equivalent to (45),

$$u_l^*Y_l = \tilde{Y}_l + V_l, \quad l = 1, \ldots, L,$$

which leads to the following interpretation of the reverse waterfilling scheme (46): each orthogonal component $U^*Y$ suffers distortion $D_l = \sigma^2_{Z_l} = \min \{\theta, \lambda_l\}$, so that

$$D_Y(R) = \text{Tr}\mathbb{E}(Y - \hat{Y})^2 = \text{Tr}\mathbb{E}(U^*Y - \hat{Y})^2 = \sum_{l=1}^L D_l = \sum_{l=1}^L \sigma^2_{V_l}.$$ 

This leads to the following forward channel representation with respect to $\tilde{Y} = (Y_1, \ldots, Y_L)$:

$$\tilde{Y}_l = (1 - 2^{-2R_l}) u_l^*Y + \sqrt{D_l(1 - 2^{-2R_l})} G_l,$$

$$= (1 - 2^{-2R_l}) u_l^*aX + (1 - 2^{-2R_l}) u_l^*W$$

$$+ \sqrt{D_l(1 - 2^{-2R_l})} G_l,$$

(49)

where $R_l = \frac{1}{2} \log^+(\lambda_l/\theta)$, $u_l$ is the $l$th normalize eigenvector of $\Sigma_Y$ (that forms the $l$th column in $U$) and the $G_l$s are zero-mean unit-variance Gaussian independent of each other and the other variables. Write (49) in a matrix form

$$\tilde{Y} = bX + \eta,$$

where $b_l = (1 - 2^{-2R_l}) u_l^*a$ and $\eta_l = (1 - 2^{-2R_l}) u_l^*W + \sqrt{D_l(1 - 2^{-2R_l})} G_l$. The MMSE in estimating $X$ from the vector $\tilde{Y}$ is found as:

$$\text{mmse}(X|\tilde{Y}) = \text{mmse}(X|\bar{Y})$$

$$= (1 + b^*\Sigma^{-1}_\eta b)^{-1}$$

$$= \left(1 + \sum_{l=1}^L \frac{(1 - 2^{-2R_l})(a^*u_l)^2}{(1 - 2^{-2R_l}) + D_l}\right)^{-1}$$

$$= \left(1 + \sum_{l=1}^L \frac{(1 - 2^{-2R_l})(a^*u_l)^2}{(1 - 2^{-2R_l}) + \min \{\theta, \lambda_l\}}\right)^{-1}.$$ 

Since $u_1 = a/\|a\|$, $R_1 = \frac{1}{2} \log ((1 + a^*a)/\theta)$ and the rest of the $u_l$s are orthogonal to $a$, we get

$$\text{mmse}(X|\tilde{Y}) = \left(1 + a^*a \frac{1 + a^*a - \theta}{1 + a^*a(\theta + 1)}\right)^{-1} = \frac{1}{a^*a + 1} + \frac{a^*a}{(a^*a + 1)^2_\theta},$$

and (23) is obtained by setting $\theta' = \theta/(1 + a^*a)$. 


APPENDIX B

PROOF OF PROP. 5.2

For \( l = 1, \ldots, L \), the forward channel that induces the pdf that achieves

\[
D_l = 2^{-2R_l} \sigma_Y^2 = 2^{-2R_l}(\gamma_l + 1),
\]

is given by

\[
\hat{Y}_l = \left(1 - \frac{D_l}{\sigma_Y^2} \right) Y_l + \sqrt{D_l} \left(1 - \frac{D_l}{\sigma_Y^2} \right) V_l
\]

\[
= \gamma_l \left(1 - 2^{-2R_l} \right) X + \left(1 - 2^{-2R_l} \right) W_l
\]

\[
+ \sqrt{(\gamma_l + 1)2^{-2R_l}(1 - 2^{-2R_l})} V_l,
\]

(50)

where \( V_1, \ldots, V_L \) are zero-mean unit-variance i.i.d Gaussian random variables. Equation (50) can be written in the matrix form as

\[
\hat{Y} = a X + \eta,
\]

where \( \hat{Y} = (\hat{Y}_1, \ldots, \hat{Y}_L) \), and where

\[
a_l = \gamma_l \left(1 - 2^{-2R_l} \right),
\]

and

\[
\eta_l \triangleq (1 - 2^{-2R_l}) W_l + \sqrt{2^{-2R_l}(\gamma_l + 1)(1 - 2^{-2R_l})} V_l.
\]

Since the relation between \( \hat{Y} \) to \( X \) is similar to (49), the MMSE in estimating \( X \) from the vector \( \hat{Y} \), can be found as:

\[
\text{mmse}(X|\hat{Y}) = 1 - a^* (aa^* + \Sigma_\eta)^{-1} a
\]

\[
= \left(1 + a^* \Sigma_\eta^{-1} a \right)^{-1}
\]

\[
= \left(1 + \sum_{l=1}^L \gamma_l \frac{1}{(\gamma_l + 1)2^{-2R_l} + 1 - 2^{-2R_l}} \right)^{-1}
\]

\[
= \left(1 + \sum_{l=1}^L \gamma_l \frac{1 - 2^{-2R_l}}{1 + \gamma_l 2^{-2R_l}} \right)^{-1}
\]

(51)

where in the second transition we used Woodbury’s matrix identity and in the last transition we used the fact that \( \Sigma_\eta = \mathbb{E}[\eta \eta^*] \) is diagonal.
APPENDIX C

PROOF OF PROP. 5.3

Proof: Since \( Y_{l,i} = X_i \oplus Z_{l,i} \), the process \( Y_{l}^n \) is binary i.i.d with \( \mathbb{P}(Y_l = 1) = \alpha \cdot p_l = \alpha(1 - p_l) + (1 - \alpha)p_l \). The backward channel that achieves the DRF of each \( Y_l \) is given by

\[
Y_l = \hat{Y}_l \oplus W_l, \tag{52}
\]

where \( \mathbb{P}(W_l = 1) = D_l(R_l) \). Fix \( L \) joint pdfs

\[
p(x, y_1, \ldots, y_L, \hat{x}_1, \ldots, \hat{x}_L) = p(x) \left( \prod_{l=1}^{L} p(\hat{y}_l|y_l) \right) p(\hat{x}|\hat{y}_1, \ldots, \hat{y}_L),
\]

such that each of \( p(\hat{y}_l|y_l) \) satisfies (52). Denote \( \xi_l \triangleq \mathbb{P}(X = \hat{Y}_l) \). For each \( l \) we have that

\[
\xi_l = p_l D_l(R_l) + (1 - p_l)(1 - D_l(R_l)),
\]

where \( D_l(R_l) \) is the DRF of the binary process \( Y_{l}^n \) given by

\[
D_l(R_l) = h^{-1} \left( h(\alpha \cdot p_l) - R_l \right)^+.
\]

The most likely \( \hat{X} \in \{0, 1\} \) given \( \hat{Y}_1, \ldots, \hat{Y}_L \) is determined by the likelihood ratio:

\[
\hat{X} = \begin{cases} 
1, & L > 1, \\
0, & L < 1,
\end{cases} \tag{53}
\]

where

\[
L = L(\hat{y}_1, \ldots, \hat{y}_L) = \frac{\mathbb{P}(X = 1, \hat{Y}_1 = \hat{y}_1, \ldots, \hat{Y}_L = \hat{y}_L)}{\mathbb{P}(X = 0, \hat{Y}_1 = \hat{y}_1, \ldots, \hat{Y}_L = \hat{y}_L)} = \frac{\alpha}{1 - \alpha} \prod_{l=1}^{L} \frac{\mathbb{P}(1 \oplus Z_l \oplus W_l = \hat{y}_l)}{\mathbb{P}(Z_l \oplus W_l = \hat{y}_l)} = \frac{\alpha}{1 - \alpha} \prod_{l=1}^{L} \frac{\mathbb{P}(1 - U_l = \hat{y}_l)}{\mathbb{P}(U_l = \hat{y}_l)} = \frac{\alpha}{1 - \alpha} \prod_{l=1}^{L} \left( \frac{1 - \xi_l}{\xi_l} \right) \left( \frac{\xi_l}{1 - \xi_l} \right)^{\hat{y}_l \oplus 1},
\]

where we set \( U_l \triangleq W_l \oplus Z_l \) and \( \xi_l = \mathbb{P}(U_l = 1) = p_l \cdot D_l(R_l) \). Thm. 4.1 implies that \( D_{CE}(R) \) is given by the error in the estimation rule (53). This error is given by

\[
\mathbb{P}(X \neq \hat{X}) = \mathbb{P}(X = 0, L \geq 1) + \mathbb{P}(X = 1, L < 0) = (1 - \alpha) \mathbb{P} \left( \prod_{l=1}^{L} \left( \frac{1 - \xi_l}{\xi_l} \right)^{U_l} \left( \frac{\xi_l}{1 - \xi_l} \right)^{U_l \oplus 1} \geq \frac{1 - \alpha}{\alpha} \right)
\]
\[ + \alpha \mathbb{P} \left( \prod_{l=1}^{L} \left( \frac{1 - \xi_l}{\xi_l} \right)^{U_l+1} \left( \frac{\xi_l}{1 - \xi_l} \right)^{U_l} < \frac{1 - \alpha}{\alpha} \right) . \]

Taking the logarithm of both sides inside the probability and denoting \( S_l = U_l - \bar{U}_l \) leads to (34).

\[ \square \]

**APPENDIX D**

**PROOF OF PROP. 5.4**

Note that the probability that the sum \( \sum_{l=1}^{L} S_l \) equals zero is positive only when \( L \) is even, and anyway converges to zero regardless of the value of \( \xi \). For this reason in the limit of large \( L \) it is only required to consider the probability that \( \sum_{l=1}^{L} S_l \) is strictly positive. For this probability we have

\[ P \left[ \sum_{l=1}^{L} S_l > 0 \right] = P \left[ \frac{\sum_{l=1}^{L} (S_l - (2\xi - 1))}{\sqrt{4\xi(1 - \xi)L}} > \frac{\sqrt{L}(1 - 2\xi)}{\sqrt{4\xi(1 - \xi)}} \right] . \]  

(54)

Since the distortion at each encoder, \( D_l \), converges to \( 1/2 \) as the code rate at each remote encoder \( R_S/L \) goes to zero, in order to obtain the probability \( \xi = \mathbb{P}(S_l = 1) \) as a function of \( R \) and \( L \), for large \( L \), it is required to examine the behavior of the binary entropy function \( h(x) \) around \( x = 1/2 \). Expanding \( h(x) \) in a power series around \( x = 1/2 \) leads to

\[ h(x) = 1 - \frac{2}{\ln 2} \left( \frac{1}{2} - x \right)^2 + \mathcal{O} \left( \frac{1}{2} - x \right)^4 , \]

so using (42) we get

\[ D_l = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{R_S \ln 2}{L}} + \mathcal{O} \left( \frac{R_S}{L} \right)^2 . \]  

(55)

Combining (55) in (41) leads to

\[ \xi = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{R_S \ln 4}{L}} + \sqrt{\frac{R_S \ln 4}{L}} + \mathcal{O} \left( \frac{R_S}{L} \right)^2 , \]

from which we conclude that

\[ \lim_{L \to \infty} \frac{(1 - 2\xi)\sqrt{L}}{\sqrt{4\xi(1 - \xi)}} = 2 \left( \frac{1}{2} - p \right) \sqrt{R_S \ln 4} . \]  

(56)

The central limit theorem applied to the expression in the LHS of the brackets in (54) implies (43).