A NOTE ON THE PAPER “SUFFICIENT OPTIMALITY CONDITIONS USING CONVEXIFACTORS FOR OPTIMISTIC BILEVEL PROGRAMMING PROBLEM”

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Abstract. In this work, some reasoning’s mistakes in the paper by Kohli (doi:10.3934/jimo.2020114) are highlighted. Furthermore, we correct the flaws, propose a correct formulation of the main result (Theorem 5.1) and give alternative proofs.

1. Introduction. Bilevel problems have been investigated by many authors [3, 4, 5, 7, 8, 9]. In the paper [9], Kohli investigated the following bilevel optimization problem

\[(BLPP): \min_{x,y} F(x,y) \text{ s.t. } G_j(x,y) \leq 0, \ j \in J, \ y \in \psi(x),\]

where, for each \( x \in \mathbb{R}^{n_1} \), \( \psi(x) \) is the set of optimal solutions of the following parametric optimization problem

\[\min_y f(x,y) \text{ s.t. } g_i(x,y) \leq 0, \ i \in I,\]

where \( F, f, g_i, G_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}, \ i \in I = \{1, \cdots, m_1\}, \ j \in J = \{1, \cdots, m_2\}, \) are given functions; \( n_1 \geq 1, \ n_2 \geq 1, \ m_1 \geq 1, \ m_2 \geq 1 \) are integers.

Under asymptotic pseudoconvexity and asymptotic quasiconvexity given in terms of convexifactors, using an upper estimate of Clarke subdifferential of value function, the author gave sufficient optimality conditions for optimistic bilevel programming problems with convex lower-level problems. The main theorem, where the author gave sufficient optimality conditions, is Theorem 5.1 [9].

Looking closely, we realized that the proof of Theorem 5.1 [9] is erroneous. In this note, several reasoning’s mistakes in Kohli’s argument are highlighted (see Remark 2 and Remark 3) and in support of our comments, some counterexamples are given (see Example 3.1 and Example 3.2). Some anomalies in [9, Example 5.1], the example provided by the author to illustrate [9, Theorem 5.1], are also pointed out (see Remark 4). For the convenience of the reader, we propose a correct formulation of Theorem 5.1 and give an alternative proof (see Theorem 4.3). When the lower level functions are continuously differentiable around the considered point, we give sufficient optimality conditions for the bilevel programming problem (BLPP) with

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the help of a subdifferential formula for the optimal value function established by Tanino and Ogawa [12] (see Theorem 4.4).

The rest of the paper is organized in this way: Section 2 contains basic definitions and preliminary material. Counterexamples and comments are given in Section 3. Section 4 addresses our main results (corrected optimality conditions).

2.

Preliminaries. Throughout this paper, \( \mathbb{R}^n \) is the usual \( n \)-dimensional Euclidean space with a norm \( \| \cdot \| \). We denote by \( \langle \cdot, \cdot \rangle \) and \( \mathbb{R}^+_n \) the inner product and the non-negative orthant of \( \mathbb{R}^n \) defined by
\[
\mathbb{R}^+_n = \{ (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0 \}.
\]
For a subset \( S \) of \( \mathbb{R}^n \), the sets \( cl S, conv S \) and \( cone S \) stand for the closure of \( S \), the convex hull of \( S \) and the convex cone generated by \( S \), respectively.

Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a given function and let \( x \in \mathbb{R}^n \) where \( f(x) \) is finite. The expressions
\[
f^-_d(x,v) = \liminf_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}
\]
and
\[
f^+_d(x,v) = \limsup_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}
\]
signify, respectively, the lower and upper Dini directional derivatives of \( f \) at \( x \) in the direction \( v \).

**Definition 2.1.** [6] The function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to have an upper convexifactor (UCF) \( \partial^* f(x) \) at \( x \) if \( \partial^* f(x) \subseteq \mathbb{R}^n \) is closed and, for each \( v \in \mathbb{R}^n \),
\[
f^-_d(x,v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle.
\]
The function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to have an upper semiregular convexifactor (USRCF) \( \partial^* f(x) \) at \( x \) if \( \partial^* f(x) \) is an upper convexifactor at \( x \) and, for each \( v \in \mathbb{R}^n \),
\[
f^+_d(x,v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle.
\]

**Remark 1.** [6] The Clarke, Michel-Penot and Mordukhovich subdifferentials are upper semiregular convexifactors of \( f \) when \( f \) is a locally Lipschitz function. However, the convex hull of an upper semiregular convexifactor of a locally Lipschitz function may be strictly contained in both the Clarke and the Michel-Penot subdifferentials.

3.

Comments and counterexamples. In order to reformulate the bilevel problem \( (BLPP) \) into a single-level programming problem, Kohli used the value function of the lower level problem defined by
\[
V(x) := \inf_y \left\{ f(x,y) : g_i(x,y) \leq 0, \ i \in I \right\}.
\]
According to [3], \( (BLPP) \) is globally equivalent to
\[
(ROBLPP) : \quad \begin{aligned}
\min_{x,y} & \quad F(x,y) \\
\text{s.t.} & \quad G_j(x,y) \leq 0, \ j \in J, \\
& \quad g_i(x,y) \leq 0, \ i \in I, \\
& \quad f(x,y) - V(x) \leq 0, \\
& \quad (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.
\end{aligned}
\]
The feasible set of \( (ROBLPP) \) is given by
\[
X := \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : f(x,y) - V(x) \leq 0, \ g_i(x,y) \leq 0, \ i \in I, \ G_j(x,y) \leq 0, \ j \in J \}.
\]
Remark 2. In [9], substituting the upper estimate of $\partial V(x)$ in (7) (see page 8, line -1), the author overlooked the fact that the inclusion

$$\partial V(x) \subseteq \bigcup_{y \in \psi(x)} \left\{ \bigcup_{(\lambda_1, \ldots, \lambda_{m_1}) \in \Lambda(x,y)} \left\{ \partial_x f(x,y) + \sum_{i \in I} \lambda_i \partial_x g_i(x,y) \right\} \right\}$$

(1)

where

$$\Lambda(x,y) = \left\{ (\lambda_1, \ldots, \lambda_{m_1}) \in \mathbb{R}^{m_1} : \begin{array}{c} 0 \in \partial_y f(x,y) + \sum_{i \in I} \lambda_i \partial_y g_i(x,y), \lambda_i \geq 0, \lambda_i g_i(x,y) = 0, \ i \in I \end{array} \right\}$$

is not necessarily an equality (see Example 3.1 where inclusion (1) is strict). This error has seriously impacted the remaining of the proof of [9, Theorem 4.4]. As the use of

$$\bigcup_{y \in \psi(x)} \left\{ \bigcup_{(\lambda_1, \ldots, \lambda_{m_1}) \in \Lambda(x,y)} \left\{ \partial_x f(x,y) + \sum_{i \in I} \lambda_i \partial_x g_i(x,y) \right\} \right\}$$

instead of $\partial V(x)$ is an integral part of the proof of [9, Theorem 5.1], the result obtained by the author as well as its proof are false.

The next example shows that the inclusion (1) can be strict.

Example 3.1. Let

$$f(x,y) := |x + y| \text{ and } g_1(x,y) := y, \quad \forall (x,y) \in \mathbb{R}^2.$$  

Then,

$$\psi(x) = \begin{cases} \{-x\} & \text{if } x \geq 0 \\ \{0\} & \text{if } x < 0 \end{cases} \text{ and } V(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Notice that $f$ and $g_1$ are convex continuous functions and $\psi$ is inner semicompact at $x = 0$ where

$$\psi(x) = \{y\}, \text{ with } y = 0, \text{ and } \partial V(x) = [-1,0].$$

• Since

$$\Lambda(x,y) = \{ \lambda \in \mathbb{R}^+ \text{ such that } 0 \in [-1,1] + \lambda \}$$

we get

$$\Lambda(x,y) = [0,1].$$

• Let

$$\Theta(x) := \bigcup_{y \in \psi(x)} \left\{ \bigcup_{(\lambda_1, \ldots, \lambda_{m_1}) \in \Lambda(x,y)} \left\{ \partial_x f(x,y) + \sum_{i \in I} \lambda_i \partial_x g_i(x,y) \right\} \right\}.$$

Then,

$$\Theta(x) = \bigcup_{\lambda \in [0,1]} \left\{ \partial_x f(x,y) + \lambda \partial_x g_1(x,y) \right\}.$$

Consequently,

$$\Theta(x) = \bigcup_{\lambda \in [0,1]} [-1,1].$$
Thus,

\[ \Theta (x) = [-1, 1]. \]

Notice that

\[ \partial V (x) \not\subset \Theta (x). \]

**Remark 3.** Contrary to what is stated on page 7 (line -5), Inequality (4) [9] is not correct. The author made a reasoning mistake when, from [9, Inequality (1)]

\[ \langle \xi, (x_n - \bar{x}, y_n - \bar{y}) \rangle < 0 \text{ for all } \xi \in cl\text{conv} \partial^* F (x, y) \]

he deduced

\[ \left\{ \lim_{n \to \infty} \langle \xi'_n, (x_n - \bar{x}, y_n - \bar{y}) \rangle < 0 \text{ for some } \xi'_n \in conv \partial^* F (x, y) \right\}
\]

\[ \lim_{n \to \infty} \xi'_n = \xi. \]

(see Example 3.2 where (2) does not imply (3)).

**Example 3.2.** Setting

\[ F (x, y) = x + y, \quad x = 0, \quad y = 0, \quad x_n = -\frac{1}{n} \text{ and } y_n = -\frac{1}{n} \]

we have

\[ \partial^* F (x, y) = \{ \xi \}, \text{ with } \xi = (1, 1). \]

Notice that \( F \) is \( \partial^* \)-asymptotic pseudoconvex at \( (x, y) \).

- (2) is satisfied. Indeed, since

\[ conv \partial^* F (x, y) = \{ \xi \} \]

we have

\[ \langle \xi, (x_n - \bar{x}, y_n - \bar{y}) \rangle = \left\langle (1, 1), \left( -\frac{1}{n}, -\frac{1}{n} \right) \right\rangle = -\frac{2}{n} < 0. \]

- (3) is not satisfied. By contrary, suppose that there exists \( \xi'_n \in conv \partial^* F (x, y) \) such that \( \lim_{n \to \infty} \xi'_n = \xi \) and

\[ \lim_{n \to \infty} \langle \xi'_n, (x_n - \bar{x}, y_n - \bar{y}) \rangle < 0. \]

Since

\[ conv \partial^* F (x, y) = \{ \xi \} \]

we have \( \xi'_n = \xi = (1, 1) \). Consequently,

\[ \lim_{n \to \infty} \langle \xi'_n, (x_n - \bar{x}, y_n - \bar{y}) \rangle = \lim_{n \to \infty} \left\langle (1, 1), \left( -\frac{1}{n}, -\frac{1}{n} \right) \right\rangle = \lim_{n \to \infty} -\frac{2}{n} = 0, \]

which contradicts (4).

**Remark 4.** Contrary to what is stated by the author, [9, Example 5.1] is not suitable to illustrate [9, Theorem 5.1]. Notice that the convexity of the function \( f \) is required in [9, Theorem 5.1].

- In [9, Example 5.1], the function

\[ f (x, y) = \begin{cases} x^2 + y^2 & \text{if } x \geq 0, \ y \in \mathbb{R} \\ x^2 + y & \text{if } x < 0, \ y \in \mathbb{R} \end{cases} \]
is not convex. Indeed, for \( \alpha := \frac{1}{2} \in [0, 1] \), \( u_1 := \left( \frac{1}{4}, \frac{1}{2} \right) \) and \( u_2 := \left( -\frac{1}{4}, -\frac{1}{2} \right) \), we have

\[
f(\alpha u_1 + (1 - \alpha) u_2) = f \left( \frac{1}{2} \times \left( \frac{1}{4}, \frac{1}{2} \right) + \frac{1}{2} \times \left( -\frac{1}{4}, -\frac{1}{2} \right) \right) = f(0, 0) = 0
\]

and

\[
\alpha f(u_1) + (1 - \alpha) f(u_2) = \frac{1}{2} f \left( \frac{1}{4}, \frac{1}{2} \right) + \frac{1}{2} f \left( -\frac{1}{4}, -\frac{1}{2} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{16} + \frac{1}{4} \right) + \frac{1}{2} \left( \frac{1}{16} - \frac{1}{2} \right) = -\frac{1}{16} < 0.
\]

Consequently,

\[
f(\alpha u_1 + (1 - \alpha) u_2) > \alpha f(u_1) + (1 - \alpha) f(u_2).
\]

• This inappropriate choice of \( f \) has impacted the optimal value function

\[
V(x) = \begin{cases} 
x^2 & \text{if } x \geq 0 \\
x^2 - 1 & \text{if } x < 0
\end{cases}
\]

which is accordingly nonconvex too. Notice that \( V \) is not continuous at zero.

• Let

\[
S_a(f - V) := \{(x, y) \in \mathbb{R}^2 \setminus f(x, y) - V(x) \leq a\}
\]

be the sublevel set of \( f - V \) for \( a \in \mathbb{R} \). Contrary to what is stated by the author, the function \( f - V \) is not quasiconvex at \((0, 0)\). Indeed, since

\[
f(x, y) - V(x) = \begin{cases} 
y^2 & \text{if } x \geq 0, y \in \mathbb{R} \\
y + 1 & \text{if } x < 0, y \in \mathbb{R}
\end{cases}
\]

we have

\[
f(x, y) - V(x) \leq \frac{1}{4} \iff \left( \left[ x \geq 0 \text{ and } -\frac{1}{2} \leq y \leq \frac{1}{2} \right] \text{ or } \left[ x < 0 \text{ and } y \leq -\frac{3}{4} \right] \right)
\]

Then,

\[
S_{\frac{1}{4}}(f - V) = \left( \mathbb{R}^+ \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) \cup \left( -\infty, 0 \right] \times \left( -\infty, -\frac{3}{4} \right]
\]

Since \( S_{\frac{1}{4}}(f - V) \) is not convex, we deduce that \( f - V \) is not quasiconvex.

• Since \( f \) and \( V \) are not convex, the use of the convex analysis subdifferentials is no longer appropriate. As a conclusion, we can say that [9, Example 5.1] is wrong.

4. Optimality conditions. For each \( x \in \mathbb{R}^{n_1} \), let \( K(x) \) be the feasible set of the lower level problem be given by

\[
K(x) := \{ y \in \mathbb{R}^{n_2} \setminus g_i(x, y) \leq 0, \ i \in I \}.
\]

We shall need the following notions.

**Definition 4.1.** [3] We say that the bilevel program \((OBLPP)\) is lower-level regular if for each \( x \in \mathbb{R}^{n_1} \), with \( K(x) \neq \emptyset \), there is \( y_x \in \mathbb{R}^{n_2} \) such that

\[
g_i(x, y_x) < 0 \text{ whenever } i \in I.
\]

(5)

**Remark 5.** [3] For smooth convex problems, the regularity condition (5) implies that in [9, Definition 2.3].
Definition 4.2. [10] The function \( h : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \cup \{+\infty\} \) is said to be \( \partial^* \)-asymptotic pseudocoxconvex at \( \bar{\pi} \) with respect to \( X \) iff for every \( u \in X \),

\[
\text{for some } u_n^* \in \text{conv} \partial^* h (\bar{\pi}) , \quad \lim_{n \to \infty} \langle u_n^*, u - \bar{\pi} \rangle \geq 0 \implies h (u) \geq h (\bar{\pi}) .
\]

The function \( h : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \cup \{+\infty\} \) is said to be \( \partial^* \)-asymptotic quasiconvex at \( \bar{\pi} \in X \) with respect to \( X \) iff for every \( u \in X \),

\[
h (u) \leq h (\bar{\pi}) \implies \text{for any } u_n^* \in \text{conv} \partial^* h (\bar{\pi}) , \quad \lim_{n \to \infty} \langle u_n^*, u - \bar{\pi} \rangle \leq 0.
\]

The following result is a corrected version of [9, Theorem 5.1].

**Theorem 4.3.** Let \((\pi, \overline{y})\) be a feasible solution of \((OBLPP)\). Assume that the following assertions hold.

- \( F, g_i, i \in I, \) and \( G_j, j \in J, \) admit respectively upper convexifiers \( \partial^* F (\pi, \overline{y}), \partial^* g_i (\pi, \overline{y}), i \in I, \) and \( \partial^* G_j (\pi, \overline{y}), j \in J. \)
- \( f \) and \( g_i, i \in I, \) are continuous and fully convex functions (i.e. convex jointly with respect to all their variables).
- For each vector \( y \in \psi (\pi), (\pi, y) \) is lower-level regular in the sense of [9, Definition 2.3].
- The value function \( V \) is finite around \( \pi, f - V \) is quasiconvex with respect to \( X, F \) is \( \partial^* \)-asymptotic pseudocoxconvex at \((\pi, \overline{y})\) with respect to \( X, \) and \( G_j, j \in J, \) is \( \partial^* \)-asymptotic quasiconvex at \((\pi, \overline{y})\) with respect to \( X. \)
- There exist scalars \( \lambda \geq 0, \mu_k \geq 0, k \in I (\pi, \overline{y}), \) and \( \tau_j \geq 0, j \in J (\pi, \overline{y}), \) such that

\[
0 \in \text{cl} \left[ \text{conv} \partial^* F (\pi, \overline{y}) + \sum_{j \in J (\pi, \overline{y})} \tau_j \text{conv} \partial^* G_j (\pi, \overline{y}) + \sum_{k \in I (\pi, \overline{y})} \mu_k \partial g_k (\pi, \overline{y}) + \lambda (\partial f (\pi, \overline{y}) - \partial V (\pi) \times \{0\}) \right] .
\]

Then, \((\pi, \overline{y})\) is a global optimal solution of \((OBLPP)\).

**Proof.** Let \((\pi, \overline{y})\) be a feasible solution of \((OBLPP)\). By (6), we can find \( \xi_n \in \text{conv} \partial^* F (\pi, \overline{y}), a_{jn} \in \text{conv} \partial^* G_j (\pi, \overline{y}), j \in J (\pi, \overline{y}), b_{kn} \in \partial g_k (\pi, \overline{y}), k \in I (\pi, \overline{y}), c_n \in \partial f (\pi, \overline{y}) \) and \( d_n \in \partial V (\pi) \times \{0\} \) such that

\[
0 = \lim_{n \to \infty} \left[ \xi_n + \sum_{j \in J (\pi, \overline{y})} \tau_j a_{jn} + \sum_{k \in I (\pi, \overline{y})} \mu_k b_{kn} + \lambda (c_n - d_n) \right] .
\]

Then, for all \((x, y) \in X, \) we have

\[
0 = \lim_{n \to \infty} \left( \xi_n , (x, y) - (\pi, \overline{y}) \right) + \sum_{j \in J (\pi, \overline{y})} \tau_j \lim_{n \to \infty} \left( a_{jn} , (x, y) - (\pi, \overline{y}) \right) + \sum_{k \in I (\pi, \overline{y})} \mu_k \lim_{n \to \infty} \left( b_{kn} , (x, y) - (\pi, \overline{y}) \right) + \lambda \lim_{n \to \infty} \left( c_n - d_n , (x, y) - (\pi, \overline{y}) \right) .
\]

- Observing that

\[
f (\pi, \overline{y}) - V (\pi) = 0, \quad g_k (\pi, \overline{y}) = 0, \quad G_j (\pi, \overline{y}) = 0, \quad \forall k \in I (\pi, \overline{y}), \quad j \in J (\pi, \overline{y}),
\]

we have

\[
g_k (x, y) \leq g_k (\pi, \overline{y}), \quad G_j (x, y) \leq G_j (\pi, \overline{y}), \quad \forall k \in I (\pi, \overline{y}), \quad j \in J (\pi, \overline{y}),
\]

and

\[
f (x, y) - V (x) \leq f (\pi, \overline{y}) - V (\pi) .
\]
SUFFICIENT OPTIMALITY RESULTS

- Taking into account the convexity of $g_k$, $k \in I$, and the asymptotic quasiconvexity of $G_j$, $j \in J$, at $(\bar{x}, \bar{y})$, we have
  \[
  \lim_{n \to \infty} \langle a_{jn}, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0 \quad \text{and} \quad \lim_{n \to \infty} \langle b_{kn}, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0.
  \]
  Consequently,
  \[
  \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \lim_{n \to \infty} \langle a_{jn}, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0 \tag{8}
  \]
  and
  \[
  \sum_{k \in L(\bar{x}, \bar{y})} \mu_k \lim_{n \to \infty} \langle b_{kn}, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0. \tag{9}
  \]

- Since the value function $V$ is convex for the fully convex lower-level problem under consideration, it is locally Lipschitzian around $(\bar{x}, \bar{y}) \in intdom(V)$. Since $f - V$ is quasiconvex and regular at $(\bar{x}, \bar{y})$, using [1, Theorem 5.4.2], we deduce that $f - V$ is $\partial^\ell$-quasiconvex at $(\bar{x}, \bar{y})$. Consequently,
  \[
  \langle \sigma, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0, \quad \forall \sigma \in \partial^\ell (f - V) (\bar{x}, \bar{y}),
  \]
  where $\partial^\ell (f - V) (\bar{x}, \bar{y})$ is the Clarke subdifferential of $f - V$ at $(\bar{x}, \bar{y})$. By [2, Theorem 2.4.1], we get
  \[
  \langle \sigma, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0, \quad \forall \sigma \in (\partial f (\bar{x}, \bar{y}) - \partial V (\bar{x}) \times \{0\}).
  \]

  Since
  \[
  c_n - d_n \in \partial f (\bar{x}, \bar{y}) - \partial V (\bar{x}) \times \{0\}
  \]
  we have
  \[
  \langle c_n - d_n, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0.
  \]
  Then,
  \[
  \lim_{n \to \infty} \langle c_n - d_n, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0.
  \]
  Thus,
  \[
  \lambda \lim_{n \to \infty} \langle c_n - d_n, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0. \tag{10}
  \]

- Summing (8), (9) and (10), we obtain
  \[
  \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \lim_{n \to \infty} \langle a_{jn}, (x, y) - (\bar{x}, \bar{y}) \rangle + \sum_{k \in L(\bar{x}, \bar{y})} \mu_k \lim_{n \to \infty} \langle b_{kn}, (x, y) - (\bar{x}, \bar{y}) \rangle + \lambda \lim_{n \to \infty} \langle c_n - d_n, (x, y) - (\bar{x}, \bar{y}) \rangle \leq 0. \tag{11}
  \]

- Combining (7) and (11), we get
  \[
  \lim_{n \to \infty} \langle \xi_n, (x, y) - (\bar{x}, \bar{y}) \rangle \geq 0, \quad \forall (x, y) \in X.
  \]

  Now, using the asymptotic pseudoconvexity of $F$ at $(\bar{x}, \bar{y})$, we obtain
  \[
  F(x, y) \geq F(\bar{x}, \bar{y}), \quad \forall (x, y) \in X.
  \]

  We conclude that $(\bar{x}, \bar{y})$ is a global optimal solution of (OBLPP).

\[\square\]

If in addition, $f$ and $g_i$, $i \in I$, are continuously differentiable around $(\bar{x}, \bar{y})$, we have the following result.

**Theorem 4.4.** Let $(\bar{x}, \bar{y})$ be a feasible solution of (OBLPP). Assume that the following assertions hold.

- $F, g_i, i \in I, and G_j, j \in J, admit respectively upper convexificators $\partial^* F(\bar{x}, \bar{y}), \partial^* g_i(\bar{x}, \bar{y}), i \in I, and \partial^* G_j(\bar{x}, \bar{y}), j \in J$. 

$f$ and $g_i$, $i \in I$, are continuously differentiable around $(\bar{x}, \bar{y})$ and fully convex.

- The value function $V$ is finite around $\bar{x}$, $f - V$ is quasiconvex with respect to $X$, $F$ is $\partial^*$-asymptotic pseudoconvex at $(\bar{x}, \bar{y})$ with respect to $X$, and $G_j$, $j \in J$, is $\partial^*$-asymptotic quasiconvex at $(\bar{x}, \bar{y})$ with respect to $X$.

- There exist scalars $\lambda \geq 0$, $\lambda_i \geq 0$, $i \in I$, $\mu_k \geq 0$, $k \in I(\bar{x}, \bar{y})$, and $\tau_j \geq 0$, $j \in J(\bar{x}, \bar{y})$, such that

\[
\begin{align*}
0 &\in \text{cl} \left\{ \nabla f(\bar{x}, \bar{y}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}, \bar{y}) \times \{0\} - \sum_{k \in I(\bar{x}, \bar{y})} \mu_k \nabla g_k(\bar{x}, \bar{y}) \right. \\
&\quad + \lambda \left( \nabla f(\bar{x}, \bar{y}) - \nabla_x f(\bar{x}, \bar{y}) \times \{0\} \right) \times \left( \sum_{i \in I} \lambda_i \nabla_x g_i(\bar{x}, \bar{y}) \times \{0\} \right) \left. \right\}, \\
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{where the union is taken over the set} \\
&\quad \Lambda(\bar{x}, \bar{y}) := \left\{ (\lambda_1, \ldots, \lambda_{m_1}) \in \mathbb{R}^{m_1} : 0 \in \nabla_y f(\bar{x}, \bar{y}) + \sum_{i \in I} \lambda_i \nabla_y g_i(\bar{x}, \bar{y}) + \lambda_i g_i(\bar{x}, \bar{y}) = 0, \lambda_i g_i(\bar{x}, \bar{y}) = 0, \forall i \in I \right\} \quad (12)
\end{align*}
\]

- $G_j$, $j \in J$, is $\partial^*$-asymptotic quasiconvex at $(\bar{x}, \bar{y})$ with respect to $X$.

Then, $(\bar{x}, \bar{y})$ is a global optimal solution of (OBLPP).

Proof. Under the assumptions made in this theorem, we obtain the following subdifferential formula for the optimal value function

\[
\partial V(\bar{x}) = \bigcup_{(\lambda_1, \ldots, \lambda_{m_1}) \in \Lambda(\bar{x}, \bar{y})} \left\{ \nabla f(\bar{x}, \bar{y}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}, \bar{y}) \right\} 
\]

where the union is taken over the set

\[
\Lambda(\bar{x}, \bar{y}) := \left\{ (\lambda_1, \ldots, \lambda_{m_1}) \in \mathbb{R}^{m_1} : 0 \in \nabla_y f(\bar{x}, \bar{y}) + \sum_{i \in I} \lambda_i \nabla_y g_i(\bar{x}, \bar{y}) + \lambda_i g_i(\bar{x}, \bar{y}) = 0, \lambda_i g_i(\bar{x}, \bar{y}) = 0, \forall i \in I \right\}
\]

established by Tanino and Ogawa [12] (see also [11, Theorem 6.6.7] and [3, Theorem 4.2]). As specified in [3], the uniform boundedness requirement on $\psi$ around $\bar{x}$ is imposed in [11, 12] while the proof therein works under the inner semicompactness of the argminimum map. Substituting (12) into (6), we complete the proof of the theorem.

Remark 6. In Theorem 4.4, we derive sufficient optimality conditions without any inner semicontinuity assumption on the argminimum map $\psi$.

Remark 7. Unlike [9, Theorem 5.1], Theorem 4.3 and Theorem 4.4 provide sufficient global optimality conditions for (OBLPP).

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