Differential calculus on the $h$-superplane

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Abstract

A non-commutative differential calculus on the $h$-superplane is presented via a contraction of the $q$-superplane. An R-matrix which satisfies both ungraded and graded Yang-Baxter equations is obtained and a new deformation of the $(1 + 1)$ dimensional classical phase space (the super-Heisenberg algebra) is introduced.
I. INTRODUCTION

A possible approach to quantum groups is obtained by deforming the coordinates of a linear space to be non-commuting objects. In this scheme the quantum group structure appears if one considers linear transformations which preserve the algebraic properties of the algebra of coordinates. A natural and physically interesting question is whether one can define differentials and derivatives corresponding to these non-commuting variables. A general answer to this question was given by Connes. Who considered the differential algebra for non-commutative algebras. Differential geometry of Lie groups and supergroups plays an important role in the mathematical modeling of physical theories. Since a (graded) Hopf algebra or quantum (super) group can be regarded as a generalization of the notion of a (super) group, it is tempting to also generalize the corresponding notions of differential geometry.

Recently Wess and Zumino have shown that a consistent quantum deformation of the differential calculus is satisfied by an R-matrix which can be any solution of the quantum Yang-Baxter equation. The non-commutative plane has been studied in the course of the last few years in that the geometry underlying the above mentioned plane has a deep connection with the Yang-Baxter equation which is important in two dimensional exactly soluble statistical models. The quantum (super) plane was generalized to the supersymmetric quantum (super) plane and it was shown that the quantum superplane is related to the graded Yang-Baxter equation.

The $h$-deformation of the supergroup GL(1|1) is given by Dabrowski and Parashar, via a contraction of GL$_q$(1|1). They have also introduced a differential calculus on the $h$-superplane which is related to GL$_h$(1|1) via the Wess-Zumino formulae.

This paper considers an alternative approach where instead of adopting the Wess-Zumino calculus from the start, the R-matrix is obtained using the consistency conditions. This leads to a consistent exterior derivative. We start by introducing a non-commutative differential calculus on the $h$-superplane via a contraction of the $q$-superplane. We define derivatives and differentials on the $h$-superplane of non-commuting coordinates and give their commutation rules. We note the role of the graded Yang-Baxter equation and the connection to the quantum supergroup. We give a new deformation of the $(1+1)$ dimensional classical phase space. Finally we show that the $q$-deformed super-oscillator algebra satisfies the undeformed (classical) super-oscillator algebra when objects are transformed into new objects such that they are singular for certain values of the deformation parameter.
II. A DIFFERENTIAL CALCULUS ON $h$-SUPERPLANE

In this work we denote $q$-deformed objects by primed quantities. Unprimed quantities represent transformed coordinates. As usual, we assume that even (bosonic) objects commute with everything and odd (grassmann) objects anti-commute among themselves.

A. Quantum $h$-superplane

We begin by considering the quantum superplane which is defined by Manin.\(^9\) The commutation relation between the even coordinate $x'$ and the odd (grassmann) coordinate $\theta'$ of the quantum superplane is in the form

$$x'\theta' - q\theta'x' = 0,$$

where $q$ is a complex deformation parameter.

We now introduce new coordinates $x$ and $\theta$, in terms of $x'$ and $\theta'$ as

$$x = x', \quad \theta = \theta' - \frac{h}{q - 1} x',$$

as in Ref. 10. This transformation is singular in the $q \rightarrow 1$ limit. Using relation (1), it is easy to verify that

$$x\theta = q\theta x + hx^2,$$

where the new deformation parameter $h$ commutes with the coordinate $x$. Also, since the grassmann coordinate $\theta'$ satisfies

$$\theta'^2 = 0$$

one obtains

$$\theta^2 = -h\theta x,$$

where $h$ anti-commutes with $\theta$ and

$$h^2 = 0,$$

that is, the new deformation parameter $h$ is a grassmann number.\(^8\) Taking the $q \rightarrow 1$ limit we obtain the following relations which define the $h$-superplane

$$x\theta = \theta x + hx^2, \quad \theta^2 = -h\theta x.$$

B. Relations of Coordinates and Differentials
To establish a non-commutative differential calculus on the quantum $h$-superplane, we assume that the commutation relations between the coordinates and their differentials are in the following form

\begin{align}
x' dx' &= A dx' x', \\
x' d\theta' &= F_{11} dx'\theta' + F_{12} dx'\theta', \\
\theta' dx' &= F_{21} dx'\theta' + F_{22} dx'\theta', \\
\theta' d\theta' &= B d\theta' \theta'.
\end{align}

We demand an exterior differential $d$ obeying the condition:

\[ d^2 = 0, \]  

and the graded Leibniz rule

\[ d(fg) = (df)g + (-1)^{\hat{f}}f(dg), \]

where $\hat{f}$ is the grassmann degree of $f$ (recall that $d$ should be odd), that is, $\hat{f} = 0$ for even variables and $\hat{f} = 1$ for odd variables. Considering the differential of a function and differentiating (2) we have

\[ dx = dx', \quad d\theta = d\theta' + \frac{h}{q-1} dx'. \]

Substituting (2) and (10) into (7) one has

\begin{align}
x dx &= A dx x, \\
x d\theta &= F_{11} dx \theta + \frac{h}{q-1} (A - F_{11} - F_{12}) dx x, \\
\theta dx &= F_{21} dx \theta + F_{22} dx \theta - \frac{h}{q-1} (A + F_{21} + F_{22}) dx x, \\
\theta d\theta &= B d\theta \theta - \frac{h}{q-1} [(B + F_{12} + F_{21}) dx \theta - (B - F_{11} - F_{22}) d\theta x].
\end{align}

These relations are slightly different from the results of Dabrowski and Parashar. The reason for this difference is that in ref. 8, the commutation relations among the matrix elements of a matrix belonging to $GL_h(1|1)$ were obtained via the use of commutation relations of the dual exterior superplane. On the other hand we use the commutation relations of the dual superplane as in ref. 3. The commutation relations among the matrix elements are the same in both approaches. However the commutation relations involving the differentials turn out not to be the same.
We know that the quantum supermatrices of the quantum \( GL_q(1|1) \) supergroup can be defined as linear transformations of the variables \( x' \) and \( \theta' \) which preserve the commutation relation (1) and their duals, that is, the quantum supergroup \( GL_q(1|1) \) acts as a linear transformation on the quantum superplane which preserves (1) and the relations

\[
\varphi'^2 = 0, \quad \varphi'y' - q^{-1}y'\varphi' = 0. \tag{12}
\]

In extending this property of covariance under the coaction of \( GL_q(1|1) \), from the superplane to its calculus, it will be assumed that the deformed group structure implies and is implied by invariance of the intermediary relations (7) under linear transformations of the quantum superplane. In the present work, this will be applied to the \( h \)-deformed superplane.

The coefficients \( A, B \) and \( F_{ij} \) given in (11) can be related to \( q \) by the consistency of calculus. Thus we apply the exterior derivative \( d \) to the relation (3). From the consistency condition

\[
d(x\theta - q\theta x - hx^2) = 0 \tag{13}
\]

we find

\[
F_{11} = q(1 - F_{22}), \quad F_{12} = -(1 + qF_{21}). \tag{14a}
\]

Applying the exterior derivative \( d \) on the second and third relations of (11) and using the definitions (8), (9) one finds

\[
F_{12} = qF_{11} - 1, \quad F_{21} = q(F_{22} - 1). \tag{14b}
\]

Similarly from the last relation of (11) we get

\[
F_{12} + F_{21} = q(F_{11} + F_{22}) - (1 + q)B. \tag{14c}
\]

On the other hand, using the relation (4) we obtain

\[
\theta d\theta = d\theta \theta + hF_{21}dx \theta - h(1 - F_{22})d\theta x = d\theta \theta - \frac{h}{q} [F_{11}d\theta x + (1 + F_{12})dx \theta]. \tag{15}
\]

Thus we have

\[
B = 1, \quad F_{12} + F_{21} + 1 = (1 - q)F_{21}, \quad 1 - F_{11} - F_{22} = (1 - q)(1 - F_{22}). \tag{14d}
\]

To find the commutation relation between differentials, say \( dx \) and \( d\theta \), we apply the exterior derivative \( d \) on the first two relations of (11) and use the nilpotency of \( d \) [eq. (8)]. Then it is easy to see that

\[
dx d\theta = \frac{1}{q}d\theta dx, \quad (dx)^2 = 0. \tag{16}
\]
Solving the system (14) we now get

\begin{align*}
A & \text{ undetermined, } F_{11} = q, \\
F_{12} = q^2 - 1, & \quad F_{21} = -q, \quad F_{22} = 0.
\end{align*}

(17)

We choose $A$ equal to $q^2$, since this leads to the standard R-matrix [eq. (38)] in the $h \to 0$ limit. We are thus led to the following deformed relations containing $q$ and $h$

\begin{align*}
xdx &= q^2dx, \\
x \ d\theta &= qd\theta + hdx + (q^2 - 1)dx, \\
\theta \ dx &= -qdx - qhdx, \\
\theta \ d\theta &= d\theta + h(qdx + d\theta) .
\end{align*}

(18)

Note that although in the $q \to 1$ limit the transformations (2) and (10) are ill behaved, the resulting commutation relations are well defined. We shall not use the limit process, yet.

C. Relations of Derivatives and Variables

First, we wish to construct the curl of any one-form $w(x', \theta')$. To this end let us denote the partial derivatives with respect to $x'$ and $\theta'$ as

\[ \partial_{x'} = \frac{\partial}{\partial x'}, \quad \partial_{\theta'} = \frac{\partial}{\partial \theta'} , \]

(19)

respectively, and introduce the super-gradiend operator in vector notation

\[ \nabla = (\partial_{x'}, \partial_{\theta'}) . \]

(20)

We define the vectors $X' = (x', \theta')$ and $dX' = (dx', d\theta')$. Then we can write the differential $d$ as

\[ d = dx'\partial_{x'} + d\theta'\partial_{\theta'} = dX'.\nabla \]

(21)

where the dot denotes the inner product. If we write $w(x', \theta')$ in the basis $dX'$ as

\[ w(x', \theta') = dx'w_1(x', \theta') + d\theta'w_2(x', \theta') \]

(22)

where $w_1$ and $w_2$ are smooth functions of the variables then we get

\[ dw(x', \theta') = dx'd\theta' [q\partial_{x'}w_2(x', \theta') - \partial_{\theta'}w_1(x', \theta')] . \]

(23)

Thus the curl of the one-form $w(x', \theta') = (w_1(x', \theta'), w_2(x', \theta'))$ is given by

\[ \nabla \times w(x', \theta') = q\partial_{x'}w_2(x', \theta') - \partial_{\theta'}w_1(x', \theta') . \]

(24)
Now we can find the commutation rules of the derivatives, once we obtain the derivatives $\partial_x$ and $\partial_\theta$. For this if we demand the chain rule on the expressions (2) we find

$$\partial_x = \partial_{x'} + \frac{h}{q-1}\partial_{y'}, \quad \partial_\theta = \partial_{\theta'}. \quad (25)$$

It is easy to see that in the case of (25) the differential $d$ given in (21) preserves its form

$$d = dx\partial_x + d\theta\partial_\theta. \quad (26)$$

If we now put $w(x, \theta) = df(x, \theta)$ then we conclude that $\partial_x$ and $\partial_\theta$ generate a non-commutative algebra with the commutation relations

$$\partial_\theta \partial_x = q\partial_x \partial_\theta, \quad \partial_\theta^2 = 0. \quad (27)$$

From (26)

$$df(x, \theta) = dx\partial_x f + d\theta\partial_\theta f, \quad (28)$$

so that replacing $f$ with $xf$ and $\theta f$ we arrive at the following commutation relations between derivatives and variables

$$\partial_x x = 1 + A x \partial_x + F_{12} \theta \partial_\theta - \frac{h}{q-1}(A - F_{11} - F_{12})x\partial_\theta,$$

$$\partial_x \theta = -F_{21} \theta \partial_x - \frac{h}{q-1}[(A + F_{21} + F_{22})x\partial_x + (1 + F_{12} + F_{21})\theta \partial_\theta],$$

$$\partial_\theta x = F_{11} x \partial_\theta,$$

$$\partial_\theta \theta = 1 - \theta \partial_\theta - F_{22} x \partial_x - \frac{h}{q-1}(1 - F_{11} - F_{22})x\partial_\theta. \quad (29)$$

These commutation relations are well defined in the limit $q \rightarrow 1$. This can be checked using (17).

The covariant differential structure which is obtained so far is a concrete example of non-commutative differential geometry. The complete framework of the differential calculus requires commutation relations of the exterior differentials with derivatives.

**D. Relations of Differentials with Derivatives**

Finally we shall find the commutation relations between differentials and derivatives. We assume that they have the following form in terms of primed quantities

$$\partial_{x'}dx' = A_{11}dx'\partial_{x'} + A_{12}d\theta'\partial_{y'},$$

$$\partial_{x'}d\theta' = A_{21}d\theta'\partial_{x'} + A_{22}dx'\partial_{y'}, \quad (30)$$

$$\partial_{y'}dx' = B_{11}dx'\partial_{y'} + B_{12}d\theta'\partial_{x'}. $$


Using (25) and (10), these commutation rules can be written as

\[ \partial_x dx = A_{11} dx \partial_x + \frac{h}{q-1}(A_{11} - A_{12} + B_{11})dx \partial_\theta + A_{12} d\theta \partial_x + \frac{h}{q-1}B_{12} d\theta \partial_x, \]

\[ \partial_x d\theta = A_{21} d\theta \partial_x + \frac{h}{q-1}(A_{11} - A_{21} + B_{22})dx \partial_x + \frac{h}{q-1}(A_{12} - A_{21} + B_{12})d\theta \partial_\theta, \]

\[ \partial_\theta dx = B_{11} dx \partial_\theta + \frac{h}{q-1}(B_{22} - B_{21} - B_{11})dx \partial_\theta + B_{22} dx \partial_x - \frac{h}{q-1}B_{12} d\theta \partial_x, \]

\[ \partial_\theta d\theta = B_{21} d\theta \partial_x + \frac{h}{q-1}(B_{22} - B_{21} + B_{11})dx \partial_\theta + \frac{h}{q-1}B_{12} d\theta \partial_x, \]

\[ \partial_\theta dx = B_{11} dx \partial_\theta + B_{12} d\theta \partial_x - \frac{h}{q-1}B_{12} (d\theta \partial_\theta + dx \partial_x). \] (31)

Finding the coefficients \( A_{ij} \) and \( B_{ij} \) from these equations we get

\[ \partial_x dx = q^2 dx \partial_x - h dx \partial_\theta + (q^2 - 1) d\theta \partial_\theta, \]

\[ \partial_x d\theta = q d\theta \partial_x + qh (dx \partial_x + d\theta \partial_\theta) + \]

\[ \partial_\theta dx = -q dx \partial_\theta, \]

\[ \partial_\theta d\theta = d\theta \partial_\theta + h dx \partial_\theta. \] (32)

Here, in order to obtain these relations we used that the exterior differential \( d \) (anti-) commutes with the differentials, that is,

\[ d(dx) = -(dx) d, \quad d(d\theta) = (d\theta) d \] (33)

and the relation

\[ \partial_i (X^j dx^k) = \delta^j_i \delta^k_l dx^l \] (34)

where \( \partial_1 = \partial_x, \ \partial_2 = \partial_\theta, \ X^1 = x, \) and \( X^2 = \theta. \) Again, the relations (32) are well defined in the limit \( q \to 1. \)

### E. The R-Matrix Formalism

We now shall obtain the R-matrix satisfying the graded Yang-Baxter equations

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \] (35)

\[ \hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}. \] (36)

For this we define the commutation relations between variables and their differentials [see eq. (18)] in the following form

\[ X^i dx^j = q(-1)^{i(j+1)} K^{ji}_{kl} dx^k X^l \] (37)
where $K \in \text{End}(\mathcal{C} \otimes \mathcal{C})$. Comparing (37) with (18) we have
\[
K_{h,q} = \begin{pmatrix}
q & 0 & 0 & 0 \\
\frac{h}{1} & 0 & 0 \\
\frac{-q^{-1}h}{q - q^{-1}} & 1 & 0 \\
0 & -h & \frac{-q^{-1}h}{q - q^{-1}}
\end{pmatrix} = (K^{ij}_{kl}). \tag{38}
\]

If we define
\[
K_h = \lim_{q \to 1} K_{h,q}, \tag{39}
\]
\[
\hat{K}_h = \lim_{q \to 1} (K_{h,q} P), \tag{40}
\]
where $P$ is the super permutation matrix, that is,
\[
P^{ij}_{kl} = (-1)^{\hat{i}j} \delta_i^l \delta^j_k,
\]
we have
\[
K_h = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{h}{1} & 0 & 0 \\
\frac{-h}{1} & 1 & 0 \\
0 & -h & -h & 1
\end{pmatrix}, \quad \hat{K}_h = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{h}{1} & 0 & 1 \\
\frac{-h}{1} & 1 & 0 \\
0 & -h & -h & -1
\end{pmatrix}. \tag{41}
\]
Here the matrix $\hat{K}_h$ coincides with the $\hat{R}_h$ matrix of ref. 8. We know from ref. 8 that the matrix $\hat{K}_h$ satisfies equation (36) with the grading
\[
(\hat{K}_{12})^{abc}_{def} = \hat{K}^{ab}_{de} \delta^c_f,
\]
\[
(\hat{K}_{13})^{abc}_{def} = (-1)^{b(c+f)} \hat{K}^{ac}_{df} \delta^b_e,
\]
\[
(\hat{K}_{23})^{abc}_{def} = (-1)^{a(b+c+e+f)} \hat{K}^{bc}_{ef} \delta^a_d. \tag{42}
\]
Also, the R-matrix
\[
R_h = P \hat{K}_h \tag{43}
\]
obeyes both the ungraded and the graded Yang-Baxter equations with the grading again given by (42). This is due to the odd character of $h$. As a consequence,
\[
K_h = R_h^{-1} \tag{44}
\]
so that $K_h$ has the same properties as $R_h$.

We note that the equation
\[
\hat{K}_h T_1 T_2 = T_1 T_2 \hat{K}_h \tag{45}
\]
is satisfied for the $h$-deformed supergroup $GL(1|1)$. Here $T$ is a supermatrix in $GL_h(1|1)$ and $T_1 = T \otimes I$, $T_2 = I \otimes T$. It is assumed that the tensor product is graded, that is,
\[
(T_1)^{ij}_{kl} = T^i_{k} \delta^j_l, \quad (T_2)^{ij}_{kl} = (-1)^{\hat{i}(\hat{j}+l)} T^j_{i} \delta^k_l.
\]
It is well-known that a supermatrix has in the form
\[
T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}
\]
with two even (latin letters) and two grassmann (greek letters) matrix elements. Equation (45) explicitly reads
\[
a\beta = \beta a, \quad a\gamma = \gamma a + h(a^2 + \gamma\beta - ad), \\
d\beta = \beta d, \quad d\gamma = \gamma d - h(d^2 - \gamma\beta - da), \\
\beta^2 = 0, \quad \gamma^2 = h\gamma(d - a), \\
\beta\gamma = -\gamma\beta + h\beta(d - a), \\
ad = da + h\beta(a - d).
\] (46)

Using the \(K_h\) matrix, we now formulate to the differential calculus on the \(h\)-superplane. The commutation relations between variables and their differentials are
\[
X^i dX^j = (-1)^{\hat{i}(\hat{j} + 1)} K_{ijkl} dX^k X^l.
\] (47)
The commutation relations between variables and derivatives are
\[
\partial_j X^i = \delta^i_j + (-1)^{\hat{i}\hat{j}} K_{ijkl} X^l \partial_k,
\] (48)
and the relations between differentials and derivatives are
\[
\partial_j dX^i = (-1)^{\hat{i}(\hat{j} + 1)} (K^{-1})_{ijkl} dX^l \partial_k.
\] (49)
Note that the commutation relations between variables can be expressed using the \(\hat{K}\) matrix as
\[
X^i X^j = \hat{K}_{ijkl} X^k X^l,
\] (50)
and the relations between derivatives as
\[
\partial_i \partial_j = \hat{K}_{ijkl} \partial_l \partial_k.
\] (51)

F. The Commutation Relations

We would like to discuss the meaning of covariance in a graded version of non-commutative differential calculus of Wess-Zumino. Before proceeding, we define the dual quantum \(h\)-superplane. For this we interpret the differentials \(dx\) and \(d\theta\), as the coordinates of the dual superplane as follows [see eq. (12)]
\[
dx = \varphi, \quad d\theta = y.
\] (52)
We now formulate the differential calculus on the $h$-superplane as follows:

The commutation relations of variables and their differentials are

$$x\theta = \theta x + hx^2, \quad \theta^2 = -h\theta x,$$
$$\varphi y = y\varphi, \quad \varphi^2 = 0.$$  \hspace{1cm} (53a)

Note that if we assume that the relations (53a) have to be covariant under the coaction

$$\delta(x) = a \otimes x + \beta \otimes \theta, \quad \delta(\theta) = \gamma \otimes x + d \otimes \theta,$$
and that $\beta, \gamma$ anticommute with $\theta, \varphi$ and $h$ we get anew the relations (46).

The commutation relations among the derivatives are

$$\partial_x \partial_\theta = \partial_\theta \partial_x, \quad \partial_\theta^2 = 0,$$  \hspace{1cm} (53b)

and those between variables and derivatives are

$$\partial_x x = 1 + x\partial_x + hx\partial_\theta, \quad \partial_\theta x = x\partial_\theta,$$
$$\partial_x \theta = \theta\partial_x - h(x\partial_x + \theta\partial_\theta), \quad \partial_\theta \theta = 1 - \theta\partial_\theta + hx\partial_\theta.$$  \hspace{1cm} (53c)

The commutation relations of variables with their differentials are

$$x \ d x = d x \ x, \quad \theta \ d \theta = d \theta \ x - h(d x \ x + d \theta \ x),$$
$$x \ d \theta = d \theta \ x - h d x \ x, \quad \theta \ d x = -d x \ x - h d x \ x.$$  \hspace{1cm} (54)

The commutation relations between derivatives and differentials are

$$\partial_\theta d x = d x \partial_x - h d x \partial_\theta, \quad \partial_\theta d \theta = d \theta \partial_\theta + h d x \partial_\theta,$$
$$\partial_x d \theta = d \theta \partial_x + h(d x \partial_x + d \theta \partial_\theta), \quad \partial_\theta d x = -d x \partial_\theta.$$  \hspace{1cm} (55)

Note that the simple relations involving the differentials $\varphi$ and $y$ in (53) are not obtained using the R-matrix $\hat{K}_h$. Instead the $q \to 1$ limit of (16) has been used.

The exterior differential

$$d = \varphi \partial_x + y \partial_\theta$$  \hspace{1cm} (56)

satisfies the usual properties such that

$$d x - x d = \varphi, \quad d \theta + \theta d = y,$$  \hspace{1cm} (57a)
as expected. The relations of the exterior differential \( d \) with \( \partial_x \) and \( \partial_\theta \) are

\[
d\partial_x = \partial_x d, \quad d\partial_\theta = -\partial_\theta d \tag{57b}
\]

and those with differentials \( \varphi \) and \( y \)

\[
d\varphi = -\varphi d, \quad dy = yd. \tag{57c}
\]

Thus the basic requirement for the exterior derivative is quite consistent:

\[
d^2 = d(\varphi \partial_x + y \partial_\theta)
= -\varphi d\partial_x + y d\partial_\theta
= -(\varphi \partial_x + y \partial_\theta)d = -d^2
\]

so that \( d^2 \) must vanish.

As we understand from the language of Wess-Zumino, covariance here means that all the relations between coordinates \( x, \theta \), differentials \( dx, d\theta \) and derivatives \( \partial_x, \partial_\theta \), etc. must preserve their form when one changes the coordinates by

\[
x \rightarrow ax + \beta \theta, \quad \theta \rightarrow \gamma x + d\theta, \tag{58a}
\]

where the matrix \( T = (T^j_\ell) \) is an element of the quantum supergroup acting on the quantum superspace. Therefore we change the differentials by

\[
dx \rightarrow adx - \beta d\theta, \\
d\theta \rightarrow -\gamma dx + d\theta. \tag{58b}
\]

This is consistent since the exterior differential \( d \) anticommutes with the Grassmann variables as mentioned before. Covariance can be maintained if one defines the transformation law of the partial derivatives as follows

\[
\partial_x \rightarrow (a^{-1} - a^{-1} \gamma d^{-1} \beta a^{-1})\partial_x - a^{-1} \gamma d^{-1} \partial_\theta, \\
\partial_\theta \rightarrow (d^{-1} - d^{-1} \beta a^{-1} \gamma d^{-1})\partial_\theta + d^{-1} \beta a^{-1} \partial_x. \tag{58c}
\]

Hence the differentials transform under the action of supertranspose of supertranspose of \( T \), \((T^{st})^{st}\), whereas the derivatives transform under the inverse of the supertranspose of \( T \), \((T^{st})^{-1}\).

**III. A NEW DEFORMATION OF CLASSICAL PHASE SPACE**

We now shall obtain a new deformation of the \((1 + 1)\)-dimensional super-Heisenberg algebra (the classical phase space). We denote the algebra (53)
generated by coordinates $x$, $\theta$ and the derivatives $\partial_x$ and $\partial_\theta$ by $B_h$. It is interesting to note that simply identifying $\partial_x$ and $\partial_\theta$ with $ip_x$ and $p_\theta$ is not compatible with the hermiticity of coordinates and momenta. In other words, the algebra $B_h$ cannot be interpreted as a deformation of the $(1 + 1)$-dimensional super-Heisenberg algebra. In order to identify $\partial_x$ and $\partial_\theta$ with the momenta $ip_x$ and $p_\theta$, one must take care of hermiticity of coordinates and momenta. To this end, let us define the hermitean conjugation of the coordinates $x$ and $\theta$, respectively, as

$$x^+ = x, \quad \theta^+ = \theta + 2h_x.$$  
(59)

It is then easy to see that the deformation parameter of the algebra (53) becomes a pure imaginary parameter:

$$\bar{h} = -h$$  
(60)

where the bar denotes complex conjugation. In this case, the hermitean conjugation of the derivatives $\partial_x$ and $\partial_\theta$ are

$$\partial_x^+ = -\partial_x + 2h \partial_\theta, \quad \partial_\theta^+ = \partial_\theta.$$  
(61)

Note that the definitions (59) and (61) is for the classical case are obtained in the $h \rightarrow 0$ limit.

The relations (53) are now invariant under the definitions (59)-(61). The above involution allows us to define the hermitean operators

$$\hat{x} = x, \quad \hat{\theta} = \theta + h x,$$  
(62)

and

$$\hat{p}_x = i(\partial_x - h \partial_\theta), \quad \hat{p}_\theta = \partial_\theta.$$  
(63)

The final form of the $h$-deformed super-Heisenberg algebra is

$$\hat{x} \hat{\theta} = \hat{\theta} \hat{x} + h \hat{x}^2, \quad \hat{\theta}^2 = -h \hat{\theta} \hat{x},$$
$$\hat{p}_x \hat{p}_\theta = \hat{p}_\theta \hat{p}_x, \quad \hat{p}_\theta^2 = 0,$$
$$\hat{p}_x \hat{x} = \hat{x} \hat{p}_x + i(1 + h \hat{x} \hat{p}_\theta), \quad \hat{p}_\theta \hat{x} = \hat{x} \hat{p}_\theta,$$  
(64)
$$\hat{p}_x \hat{\theta} = \hat{\theta} \hat{p}_x - h(\hat{x} \hat{p}_x + i \hat{\theta} \hat{p}_\theta), \quad \hat{p}_\theta \hat{\theta} = 1 - \hat{\theta} \hat{p}_\theta + h \hat{x} \hat{p}_\theta.$$

IV. A COMMENT ON SUPER-OSCILLATORS

We know that introducing one 'bosonic' and one 'fermionic' oscillator, $A$ and $B$, respectively, and making the usual identification

$$x' \leftrightarrow A^+, \quad \theta' \leftrightarrow B^+,$$
$$\partial_{x'} \leftrightarrow A, \quad \partial_{\theta'} \leftrightarrow B,$$  
(65)
one constructs the quantum super-oscillator algebra which is covariant under the quantum supergroup $GL_q(1|1)$. Under identification (25) and (2) give

$$x \leftrightarrow A^+, \quad \partial_x \leftrightarrow A + \frac{\hbar}{q-1}B,$$

$$\partial_\theta \leftrightarrow B, \quad \theta \leftrightarrow B^+ - \frac{\hbar}{q-1}A^+$$

(66)

where $q$ is a real number. Substituting (66) into (29) and using (17), surprisingly all $\hbar$-dependence cancels and one obtains the usual $q$-deformed super-oscillator algebra

$$AA^+ = 1 + q^2 A^+ A + (q^2 - 1)B^+ B,$$

$$BB^+ = 1 - B^+ B, \quad B^2 = 0 = B^+ B,$$

$$AB^+ = qB^+ A, \quad AB = q^{-1}BA.$$  

(67)

In the $q \rightarrow 1$ limit, we get undeformed super-oscillator algebra.

An interesting problem is the construction of a differential calculus on the $(\hbar_1, \hbar_2)$-superplane using the methods of this paper. A differential calculus on this superplane using the Wess-Zumino formulae has been given in Ref. 12.

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1 V. G. Drinfeld, *Quantum groups*, in Proc. IMS, Berkeley, (1986);
Yu I. Manin, *Quantum groups and non-commutative geometry*, CRM, Montreal University, (1988).
2 A. Connes, *Non-commutative differential geometry*, Publ. Math. I.H.E.S. 62 (1985).
3 B. Schmidke, S. Vokos and B. Zumino, Z. Phys. C 48, 249 (1990);
E. Corrigan, B. Fairlie, P. Fletcher and R. Sasaki, J. Math. Phys. 31, 776 (1990).
4 Wess, J. and Zumino, B., Nucl. Phys. B (Proc. Suppl.) 18 B, 302 (1990).
5 F. Mueller-Hoissen, J. Phys. A 25, 1703 (1992).
6 S. Soni, J. Phys. A 24, L459 (1991); ibid 619.
7 W.S. Chung, J. Math. Phys. 35, 2484 (1994).
8 L. Dabrowski and P. Parashar, Lett. Math. Phys. 38, (1996).
9 Yu. I. Manin, Commun. Math. Phys. 123, 163 (1989).
10 A. Aghamohammadi, M. Khorrami and A. Shariati, J. Phys. A 28, L225.
V. Karimipour, *Lett. Math. Phys.* **30**, 87 (1994).

11 M. Chaichian, P. Kulish and J. Lukierski, *Phys. Lett. B* **262**, 43 (1991).

12 Salih Çelik, *Lett. Math. Phys.* **42**, 299 (1997).