Stable multiple-charged localized optical vortices in cubic-quintic nonlinear media

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The stability of two-dimensional bright vortex solitons in a media with focusing cubic and defocusing quintic nonlinearities is investigated analytically and numerically. It is proved that above some critical beam powers not only one- and two-charged but also multiple-charged stable vortex solitons do exist. A vortex soliton occurs robust with respect to symmetry-breaking modulational instability in the self-defocusing regime provided that its radial profile becomes flattened, so that a self-trapped wave beam gets a pronounced surface. It is demonstrated that the dynamics of a slightly perturbed stable vortex soliton resembles an oscillation of a liquid stream having a surface tension. Using the idea of sustaining effective surface tension for spatial vortex soliton in a media with competing nonlinearities the explanation of a suppression of the modulational instability is proposed.

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I. INTRODUCTION

Spatial optical solitons and vortex solitons are self-trapped light beams of finite cross section that are supported by a balance between linear diffraction (dispersion) and nonlinear self-focusing of the intense wave. These structures have been predicted and experimentally demonstrated in various nonlinear optical media (see [? ] and references therein). An optical vortex soliton has embedded phase singularity and carries intrinsic angular momentum. The phase circulation around the axis of propagation is equal to $2\pi n$, where integer number $n$ is the topological charge of the vortex. We investigate the possibility of formation and stability of two-dimensional multiple-charged (having topological charge $m$ to be equal up to five) localized envelope vortices on the basis of the Nonlinear Schrödinger Equation (NSE) with competing cubic-quintic (CQ) nonlinearities.

It is well known, that ordinary nonspinning bright solitons are stable in CQ nonlinear media. Though all localized optical vortex solitons (LOVS) were believed [? ] to possess a strong azimuthal modulation instability. As a result, an unstable vortex decays into several spatial solitons with zero topological charges, which fly off tangently to the initial vortex ring conserving the total angular momentum. Recent investigations [? ? ? ? ] have shown, however, that in CQ nonlinear media one- and two-charged LOVS become stable if their numbers of quanta (beam power) exceed the critical values $N_{cr}$. Nevertheless, vortices with $m > 2$ were still regarded to be unstable because of the modulation instability [? ]. We show here that, in contrast with what was previously believed, even higher-order (with $m > 2$) vortices become stable above certain threshold value $N_{cr}$, where $N_{cr}$ increases with topological charge.

The important, but still open question is: what is the physical essence of a vortex ($m \neq 0$) symmetry-breaking instability suppression? It is remarkable, that in self-defocusing regime (when an increase of the input power leads to broadening of the light beam) vortex as well as soliton shape changes abruptly above certain critical power: the radial intensity distributions becomes practically uniform with pronounced surface. As was demonstrated previously [? ], this modification of the soliton profile corresponds to some bifurcation. It is interesting that here the light beam gets some common features with a liquid stream. Actually, the sharpness of the soliton shape modification corroborates the conception [? ] of "phase gas-liquid transition". The authors of Ref. [? ] demonstrated that a laser light gets some kind of surface tension above certain threshold beam power (in a "liquid" state). By numerical simulations of soliton collisions against planar boundaries and localized inhomogeneities they proved that two-dimensional "liquid solitons" behave like liquid streams having a surface tension.

It is reasonable to expect that such "gas-liquid transitions" do happen with vortex solitons too. In this paper we propose and validate the explanation of a vortex soliton stabilization on the basis of the sustaining surface tension conception. We show here that the dynamics of slightly perturbed stable vortex resembles the oscillation of a liquid stream having a surface tension. The remarkable analogy of light "condensation" with an effect of condensate droplet formation [? ] in the $N$-particle boson system at the large $N$ limit is revealed.

II. VORTEX SOLITONS

We consider here the intense laser light beam propagating in CQ media with focusing cubic and defocusing quintic nonlinearities. Refractive index of CQ materials can be approximated as follows: $n = n_0 + n_2 I - n_4 I^2$, here $I$ is the intensity of light beam, coefficients $n_2, n_4 > 0$ determine nonlinear response of the media. It is important to note, that for some nonlinear materials [? ] this fitting
formulas works even in strongly nonlinear self-defocusing regime, when \( I > n_2/2n_4 \), so that \( \partial n/\partial I < 0 \).

The electromagnetic field envelope \( \Psi(x, y, z) \) of linearly polarized laser beam, which propagates along \( z \) through a CQ nonlinear optical material, is described in the paraxial approximation by generalized NSE (GNSE):

\[
i\partial_2 \Psi + D \Delta_\perp \Psi + B |\Psi|^2 \Psi - K |\Psi|^4 \Psi = 0, \tag{1}\]

where \( \Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the Laplacian operator, \( D = 1/2n_3, B = \kappa n_2, K = \kappa n_4, \kappa \) is the wave number of the light beam.

For localized solution of the GNSE \( \Psi \) the following integrals of motion are assumed to be finite: (i) beam power (or number of quanta): \( N = \int |\Psi|^2 \ d^2r \), (ii) momentum: \( \vec{P} = \int \vec{p} d^2r \), where \( \vec{p} = -i \left( \Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) \), (iii) angular momentum: \( \vec{M} = \int i \vec{r} \times \vec{p} d^2r \), (iv) Hamiltonian: \( H = \int h d^2r \), where \( h = D |\Psi|^2 - \frac{i}{2} B |\Psi|^4 + \frac{1}{2} K |\Psi|^6 \). Vortex solitons are stationary solutions of the GNSE of the form

\[
\Psi(r, z) = \psi(r)e^{im\varphi+i\Lambda z}, \tag{2}\]

where \( m \) is the topological charge and \( \Lambda \) is the propagation constant. The radial profile \( \psi(r) \) of the localized vortex soliton can be found by means of numerical solution of the ordinary differential equation:

\[
-\lambda U + \Delta^{(m)} \rho U + U^3 - U^5 = 0, \tag{3}\]

where

\[
\Delta^{(m)} = \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2},
\]

and we have introduced the following rescaling transformation: \( \rho = r \sqrt{B/DB}, \lambda = \Lambda K/DB, U(\rho) = \psi(r)\sqrt{K/B} \). The typical profiles of vortices with different topological charges are presented in the Fig. (a).

Each value of the propagation constant \( \Lambda \) corresponds to vortex soliton solution having definite beam power \( N(\Lambda) \).

The main properties of the stationary solutions can be investigated by approximate variational method. The variational analysis for nonlinear spinor solitons, which takes into account soliton shape modification has been performed in Ref. \( \text{[?]} \). Here we restrict variational analysis of the LOVS using the trial function with unchanged radial profile:

\[
\Psi(r, z) = A(z)r^{a_0}\exp \left\{ -\frac{1}{2}\left[ r/a(z) \right]^2 + i\gamma(z)r^2 + im\varphi \right\},
\]

where \( \gamma(z) \) is the phase curvature, parameters \( a(z) \) and \( A(z) \) characterize beam width and amplitude respectively. The parameter \( A(z) \) can be expressed in terms of \( N \) and \( a(z) \) using definition of the number of quanta \( N \). After the procedure of Ritz optimization one can obtain for the variables \( a(z) \) and \( \beta(z) = \gamma(z)a(z) \) the set of equations in canonical form, which describes the evolution (in \( z \)-direction) of the vortex parameters:

\[
\begin{align*}
\frac{N(m + 1)}{2} \partial_2 a &= \partial H/\partial \beta, \tag{4} \\
\frac{N(m + 1)}{2} d\beta &= -\partial H/\partial a. \tag{5}
\end{align*}
\]

The Hamiltonian in the variational approximation is given as follows:

\[
H = N \left\{ D(m + 1)(a^2 + \beta^2) - \frac{1}{2} bN - \frac{1}{3} kN^2 \right\}, \tag{6}
\]

where

\[
\begin{align*}
b &= \frac{B(2m)!}{\pi(m!)^2 2^{2m+1}}, & k &= \frac{K(3m)!}{\pi^2(3m!)^3 3^{3m+1}}.
\end{align*}
\]

The vortex soliton corresponds to the stationary point of the set \( \Psi, \beta \):

\[
\frac{\partial H}{\partial a} = 0, \frac{\partial H}{\partial \beta} = 0, \tag{7}
\]

which determine parameters \( a \) and \( \beta \) of a vortex:

\[
a_0^2 = \frac{4k}{3b} \frac{N^2}{N - N_m}, \quad \beta_0 = 0, \tag{8}
\]

where \( N_m = 2D(m + 1)/b \).

From Eq. [3] it follows that (i) \( m \)-charged vortex soliton exists only above the threshold beam power: \( N > N_m \), (ii) if beam power exceeds the doubled threshold value: \( N > 2N_m \), the self-focusing regime turns into the self-defocusing one, (iii) for \( N \gg N_m \) the effective radius \( a_0 \) of the two-dimensional structure has the asymptotic behaviour of the form: \( a_0 \sim N^{1/2} \). As was found in Ref. \( \text{[?]} \), the radial profile of an ordinary soliton in self-defocusing regime changes abruptly, if beam power exceeds quadruple threshold power for soliton existence: \( N > 4N_0 \). It is remarkable, that the point of "gas-liquid" condensation of nonspining light beam, which has been revealed in Ref. \( \text{[?]} \), also corresponds to power \( N = 4N_0 \); laser light with \( N > 4N_0 \) attains some kind of effective surface tension. Above certain value of the beam power a vortex soliton gets a sharp boundary too [see Fig. (b)], likewise the gas cloud, which condenses into the liquid droplet. This state with nearly uniform density of quanta \( |\Psi|^2 \) can be considered as a condensate stream flowing along the axis \( z \).

A deeper insight into physical essence of this phenomena can be obtained using the analogy between the models, describing the photon gas and bosons interacting via \( \delta \)-like potential. In Ref. \( \text{[?]} \) the one-dimensional system of a finite number of bosons with the pairwise attraction and three-body repulsion has been considered in the self-consistent field approximation on the basis of
the one-dimensional NSE with CQ nonlinearity. As was demonstrated in Ref. [?], an addition of particles to such boson system causes the increase of the density of the particles until the three-body repulsion becomes significant. The further increase of N leads to appearance of the state having nearly homogeneous distribution of the particles and constant binding energy. This N-particle bound state in the large N limit has been considered as condensate droplet in the coordinate space [?]. The “trapping energy” per one particle (the Hamiltonian per one quantum) tends to the finite negative value, for two-dimensional LOVS under consideration, as it had been found for self-trapped bosons [?]:

$$\varepsilon = \lim_{N \to \infty} H/N = -\frac{3\hbar^2}{16|k|}.$$ 

It is clear that binding energy per unit length along z (the Hamiltonian H) for the finite size system is less than it would be for the infinite one, since not all “binding interreactions” act on “surface particles”. This leads to appearance of a surface tension and negative surface energy. As was shown in [?], the value

$$H_s = \lim_{N \to \infty} (H - N\varepsilon) = -2N_m\varepsilon,$$

can be treated as an analog of the surface power. In the next section we investigate how the surface power, which appears as an effective surface tension, influences on the vortex structure evolution in z-direction.

### III. DYNAMICS OF PERTURBED VORTEX SOLITON

As known, in contrast to nonspining solitons, which are stable with respect to rather large symmetric and asymmetric perturbations, LOVS are stable for small radially-symmetric perturbation, but unstable with respect to asymmetric ones. A small azimuthal perturbation may grow forming several filaments, and, as the result, a LOVS decays into a few ordinary solitons, which fly off in such a way, that the total angular momentum of the system conserves. A typical example of the dynamics (in z-direction) of unstable vortex is drawn in Fig. 2.

To consider an initial stage of the instability, we have performed the linear stability analysis of the LOVS with respect to small perturbations:

$$\Psi(\vec{r}, z) = \{\psi(r) + \varepsilon(\vec{r}, z)\} \exp\{im\varphi + i\Lambda z\}$$

with different azimuthal periods:

$$\varepsilon(\vec{r}, z) = a^+(r, z)e^{iL\varphi} + a^-(r, z)e^{-iL\varphi},$$

where $|\varepsilon(\vec{r}, z)| \ll |\psi(r)|$. The azimuthal number of the perturbation $L = 0, 1, 2, \ldots$ determines the number of humps on the envelope surface. Since we investigate the linear stage of instability, all unstable modes may be considered independently and treated as exponentially growing: $\varepsilon(\vec{r}, z) \sim e^{iLz}$. Therefore, the linear stability analysis after the linearization is reduced to solving the eigenvalue problem for growth rates $\Gamma_L$. If $\Gamma_L > 0$, the perturbation grows up, so that LOVS would decay into L filaments. We have solved this eigenvalue problem numerically. The growth rates of the all unstable eigenmodes for LOVS with $m = 3$ are presented in the Fig. 4.

It is seen that the widest instability region corresponds to $L = 2$ and the higher-order perturbation are suppressed even for small $\Lambda$ in the vicinity of the threshold $N_m$. As seen from Fig. 5 above some critical beam power $N_c$ the modulation instability of LOVS with $m = 3$ is completely suppressed – all growth rates are equal to zero. The same analysis has been performed for higher-order LOVS (up to $m = 5$), and it has been found that above some critical beam powers $N > N_c$ multiple-charged LOVS are stable.

The LOVS stability has been confirmed by means of direct numerical simulation of the nonstationary GNSE with the perturbed LOVS as initial condition. The dynamics of stable LOVS has been found to be quasiperiodical – effective width and amplitude of perturbed vortex oscillate with $z$.

Let us consider the dynamics of small oscillations of a LOVS surface in more details. In the framework of variational analysis the parameters of the stationary LOVS are determined by (8). Expanding the Hamiltonian around the stationary point one straightforwardly obtains that dynamics of effective width $a(z)$ of a vortex wave beam is described by the Newton-like motion equation:

$$\mu d^2\xi dz^2 = -\xi \left(\frac{\partial^2 H}{\partial a^2}\right)_{\beta=0,a=a_0},$$

where $\xi(z) = a(z) - a_0$, $\mu(N) = N(m + 1)/8D$. Thus, the frequency of small oscillations of slightly perturbed vortex soliton is:

$$\omega^2 = \frac{4H_s}{a_0^2\mu_0} \left(1 - \frac{N_m}{N}\right)^2,$$

where $\mu_0 = \mu(N_m)$, $H_s$ is the surface power. If $N \gg N_m$, the frequency tends to $\omega \to \sqrt{4H_s/a_0^2\mu_0}$. Hence, the dynamics of perturbed vortex soliton at large N limit looks like oscillations of liquid stream having the effective surface tension $\sigma \sim H_s/2\pi a_0$ and the effective density $\rho_0 \sim \mu_0/\pi a_0^2$.

In conclusion, we have found the conditions for the existence of stable multiple-charged localized optical vortex solitons in a CQ nonlinear media. They occur stable above some critical beam powers $N_c$, in the self-defocusing regime provided that their profiles become flat-topped. The increase of LOVS beam power above $N_c$ leads to formation of the vortex structure having the pronounced plateau on the radial intensity distribution and the sharp boundary. We have found out that the dynamics of the slightly perturbed stable vortex is similar to oscillations of liquid stream having a surface tension.
We have drawn a parallel between the known phenomena of condensation in the boson system and the laser light beam “condensation”. We have estimated the surface beam power and corroborated that LOVS has an effective surface tension, which causes a vortex stabilization. Actually, the sustaining surface tension suppresses small modulations of the vortex surface. It impedes the formation of the filaments, therefore LOVS becomes stable in a “liquid” state.

We believe that described mechanism can be used to explain LOVS stabilization in other materials with competing focusing-defocusing nonlinearities, e.g. in the media with defocusing cubic and focusing quadratic ones \cite{1}. Moreover, the insight into the physical reasons of LOVS stabilization would allow one to choose more suitable nonlinear material for experimental realization of the stable vortex solitons.
FIG. 1: Radial profiles of vortex solitons: (a) $m = 1, \ldots, 5$ ($\lambda = 0.1$), (b) the one-charged vortices with different $\lambda$.

FIG. 2: Gray-scale intensity distribution of vortex wave beam propagating in $z$-direction. The unstable vortex soliton with $m = 3$, $\lambda = 0.1$ decays into ordinary solitons.
FIG. 3: Maximum growth rates of all unstable azimuthal eigenmodes vs propagation constant $\lambda$ for vortex solitons with $m = 3$. Integers near the curves indicate azimuthal numbers $L$. 