Recent progress in calculating weak matrix elements using staggered fermions

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We present a chronological review of the progress in calculating weak matrix elements using staggered fermions. We review the perturbative calculation of one-loop matching formula including both current-current diagrams and penguin diagrams using improved staggered fermions. We also present preliminary results of weak matrix elements relevant to CP violation calculated using the improved (HYP (II)) staggered fermions. Since the complete set of matching coefficients at the one-loop level became available recently, we have constructed lattice operators with all the $g^2$ corrections included. The main results include both $\Delta I = 3/2$ and $\Delta I = 1/2$ contributions.

1. INTRODUCTION

Staggered fermions preserve enough chiral symmetry to prevent those operators of our physical interest from mixing with operators of wrong chirality and to protect quark mass from additive renormalization. This chiral symmetry is essential to calculate $\epsilon'/\epsilon$. Staggered fermions have the advantages over DWF of requiring less computing time and dynamical simulations are already possible with relatively light quark masses.

By construction, staggered fermions possess four degenerate flavors (also called “tastes”). They allow large taste changing quark-gluon interactions, which make the perturbative correction large even at the one loop level. In addition, four-fermion operator mixing matrix is huge ($65536 \times 65536$), which makes it impractical to determine the matching coefficients in a completely non-perturbative way. Another disadvantage is that the unimproved staggered fermion action and operators receive large perturbative corrections [12] and have large scaling violations of order $a^2$ [3]. Both of these disadvantages can be alleviated by improving staggered fermions using fat links. It has been shown that taste symmetry breaking in the pion spectrum is significantly reduced with fat links [15].

The first goal of this staggered $\epsilon'/\epsilon$ project is to check the results ($\epsilon'/\epsilon < 0$) obtained using quenched domain wall fermions (DWF) by CP-PACS [6] and RBC collaboration [7]. The second goal is to extend the calculation to dynamical simulation. The main goal is to find a window for new physics or to confirm the standard model, when our numerical results are compared with the experimental results.

This project has progressed through the following small steps.

2. PERTURBATION

Since the penguin diagrams had already been known for the gauge invariant operators from [8], we calculated the remaining diagrams of the current-current type [11]. This provided a complete set of matching formula for $\epsilon'/\epsilon$, combined with the existing results for penguin diagrams [8]. We also observed that the perturbative correction for some of the operators such as $O_7$, $O_8$, and $O_6$ are so large that the uncertainty from the truncated contribution is of order 100% for using unimproved staggered fermions.

3. NUMERICAL STUDY

Using unimproved staggered fermions, we performed a numerical study on $\epsilon'/\epsilon$. Using the
matching formula given in [18], we constructed fully one loop matched gauge invariant operators. As expected, we found large perturbative corrections for $B_7$ and $B_6$, so that we can not quote quantitative results for this although the statistical uncertainties are under control. We also observed that different quenching transcriptions of the continuum operators on the lattice (proposed by Golterman and Pallante [9] lead to noticeably different values for $B_6$ [10]. This indicates a large quenching uncertainty, which deserves further investigation.

4. IMPROVEMENT SCHEME

Hence, it was essential to reduce the large perturbative correction by improving the staggered fermions. Hence, the main goal was to find an improvement scheme which can reduce the perturbative correction down to 10% or less. In order to find the best scheme, we calculated, explicitly, one loop matching coefficients for various improved staggered actions and operators [11]: 1) Fat7, 2) Fat7+Lepage, 3) HYP and 4) AsqTad-like (Fat7+Lepage+Naik) actions. We observe that all the above improvement schemes significantly reduce the size of matching coefficients. After the second level of mean-field improvement, the HYP and Fat7 links lead to the smallest one-loop corrections [11]. Since the HYP fat link reduces the non-perturbative flavor symmetry breaking more efficiently in the pion spectrum than the Fat7 link, we adopted the HYP scheme in our numerical study.

5. HYP/Fat7

We studied further on the HYP link [12]. The HYP links possess some universal properties, which are summarized in the following 5 theorems.

**Theorem 1 (SU(3) Projection)**

Any fat link can be expanded in powers of gauge fields ($A_\mu$).

\[
B_\mu = B^{(1)}_\mu + B^{(2)}_\mu + B^{(3)}_\mu + \cdots \\
B^{(n)}_\mu = O(A^n)
\]

1. The linear term, $B^{(1)}_\mu$ is invariant under SU(3) projection.

2. The quadratic term, $B^{(2)}_\mu$ is antisymmetric in gauge fields.

**Theorem 2 (Triviality of renormalization)**

1. At one loop level, only the $B^{(1)}_\mu$ term contributes to the renormalization of the gauge-invariant staggered fermion operators.

2. At one loop level, the contribution from $B^{(n)}_\mu$ for any $n \geq 2$ vanishes.

3. At one loop level, the renormalization of the gauge-invariant staggered operators can be done by simply replacing the propagator of the $A_\mu$ field by that of the $B^{(1)}_\mu$ field.

This theorem is true, regardless of details of the smearing transformation.

**Theorem 3 (Multiple SU(3) projections)**

1. The linear gauge field term $B^{(1)}_\mu$ in the perturbative expansion is universal.

2. In general, the quadratic terms may be different from one another. But all of them are antisymmetric in gauge fields.

3. This theorem is true, regardless of the details of smearing.

**Theorem 4 (Uniqueness)**

If we impose the perturbative improvement condition of removing the flavor changing interactions on the HYP action, the HYP link satisfies the following:

1. The linear term $B^{(1)}_\mu$ in perturbative expansion is identical to that of the SU(3) projected Fat7 links.

2. The quadratic term $B^{(2)}_\mu$ is antisymmetric in gauge fields.

**Theorem 5 (Equivalence at one loop)**

If we impose the perturbative improvement condition to remove the flavor changing interactions, at one loop level,
Operators \( [P \times P][P \times P]_{II} \)

| Operators | \( [P \times P][P \times P]_{II} \) |
|-----------|----------------------------------|
| NAIVE     | \( 2 \times (111.3 - 2C_N) \) |
| HYP/Fat7  | \( 2 \times (6.925 - 2C_H) \) |

Table 1
One-loop correction to \( (O_3)_{II} \).

1. the renormalization of the gauge invariant staggered operators is identical between the HYP staggered action and those improved staggered actions made of the SU(3) projected Fat7 links (often called “Fat7”),

2. the contribution to the one-loop renormalization can be obtained by simply replacing the propagator of \( A_\mu \) by that of \( B_\mu^{(1)} \).

The first two theorems were used in [2] and [13], although they did not present their derivation. For derivations of all five theorems and further details, refer to [12]. As a result of these theorems, we can prove that for each Feynman diagram,

\[ \| C_{fat} \| < \| C_{thin} \|. \]

Here, \( C_{fat} \) (\( C_{thin} \)) represents perturbative corrections to gauge-invariant staggered operators constructed using SU(3) projected fat links (thin links). This inequality is not valid for those fat links without SU(3) projection. Hence, this lead to a conclusion that we may view the SU(3) projection of fat links as a tool of tadpole improvement for the staggered fermion doublers [12]. We also present alternative choices of constructing fat links to improve staggered fermions in [12]. The above five theorems make the perturbative calculation simpler for the HYP scheme, because one can perform the calculation merely by replacing the thin link propagator with that of the HYP links. This simplicity is extensively used in calculating the renormalization constants of the four-fermion operators in the next stage.

6. PERTURBATION FOR HYP/Fat7 (1)

We calculated the current-current diagrams to obtain perturbative matching coefficients for the staggered four-fermion operators constructed using the HYP/Fat7 links [14]. In particular, we are interested in the \( (O_3)_{II} \) operator (we use the same notation as in [1]), because this receives large perturbative corrections (\( \approx 1 \)) in the case of unimproved staggered fermions.

\[ (O_3)_{II} = 2([P \times P][P \times P] - [S \times P][S \times P])_{II} \]

In Tables 1 and 2 we present values of one-loop corrections to \( (O_3)_{II} \) constructed using both the HYP/Fat7 link and unimproved thin link (denoted as NAIVE). Here, \( C_N = 13.159 \) and \( C_H = 1.4051 \), which correspond to tadpole improvement contributions. The results shows that, by choosing the HYP/Fat7 scheme, the perturbative corrections are reduced to \( \approx 10\% \) level. In the case of the HYP/Fat7 scheme, note that the size of one-loop correction is already under control even without tadpole improvement. For further details, refer to [14].

7. PERTURBATION FOR HYP/Fat7 (2)

In order to obtain a complete set of matching formula to convert the lattice results into continuum values, we need to calculate the penguin diagrams for the gauge-invariant staggered operators constructed using various fat links [15]. One of the main results regarding the diagonal mixing from penguin diagrams can be summarized into the following theorem:

**Theorem 1 (Equivalence)**

At the one loop level, the diagonal mixing coefficients of penguin diagrams are identical between (a) the unimproved (naive) staggered operators constructed using the thin links and (b) the improved staggered operators constructed using the fat links such as HYP (I), HYP (II), Fat7, Fat7+Lepage, and Fat7.\(^2\)

\(^2\)Note that AsqTad is NOT included on the list. In this case, by construction the operators are made of the fat links which are not the same as those used in the action due to the Naik term. In addition, the choice of the fat links are open and not unique.
The details on the proof of this theorem will be given in [15,16].

By construction, gluons carrying a momentum close to \( k \sim \pi/a \) are physical in staggered fermions and lead to taste changing interactions, which is a pure lattice artifact. In the case of unimproved staggered fermions, it is allowed to mix with wrong taste \( (\neq 1) \) and the mixing coefficient is substantial. In contrast, in the case of improved staggered fermions using fat links of our interest such as Fat7, Fat7 and HYP (II), the off-diagonal mixing with wrong taste vanishes and is absent. In the case of the improvement using HYP (I) and Fat7 + Lepage, the off-diagonal mixing with wrong taste is significantly suppressed. The details of this off-diagonal mixing will be given in [16].

8. NUMERICAL STUDY FOR HYP/Fat7

Recently we have performed a numerical study using Columbia QCDSP supercomputer with the HYP (II) staggered fermion. Since we know the complete set of matching formula for HYP/Fat7 staggered operators from [14,15], we constructed fully one-loop matched gauge-invariant operators to study \( \epsilon'/\epsilon \). Here, we present preliminary estimates of \( B_K, B_{7}^{5/2}, B_{9}^{3/2} \) and \( B_{6}^{1/2} \) calculated using the HYP (II) staggered fermions at \( \beta = 6.0 \) on a \( 16^3 \times 64 \) lattice with 218 configurations.

8.1. \( B_K \)

Fig. 1 shows \( B_K \) as a function of \( M_K^2 \), where the mesons are made of degenerate quarks. We fit \( B_K \) to the form suggested by quenched chiral perturbation theory [17]:

\[
B_K = c_0 + c_1(M_K^2)^2 + c_2(M_K^2)\log(M_K^2).
\]

The cross symbol in Fig. 1 corresponds to the value interpolated to the physical kaon mass. Our preliminary result is \( B_K = 0.581(18) \), which is consistent with the continuum extrapolated value calculated using unimproved staggered fermions [3]. However, our value is substantially different from the value \( (B_K = 0.6790(16)) \) calculated using unimproved staggered fermions at finite lattice spacing \( (1/a = 2.01 \text{ GeV}) \). Therefore, this leads to the preliminary conclusion that, using the HYP staggered fermions, the scaling behavior is so improved that our results at \( \beta = 6.0 \) \( (1/a = 1.95 \text{ GeV}) \) is already in agreement with the continuum extrapolated values of the unimproved staggered results. Of course, this claim needs further investigation at weaker couplings. In the chiral limit, we obtain \( c_0 = 0.13(15) \), which is also consistent with those...
results obtained using the NLO, large $N_c$ calculation \[18,19\]. The value for $c_2/c_0$ is consistent with the predictions of quenched chiral perturbation theory \[17\] within large errors.

8.2. $B_7^{(3/2)}$ and $B_8^{(3/2)}$

A major contribution to the $\Delta I = 3/2$ amplitudes comes from $B_7^{(3/2)}$ and $B_8^{(3/2)}$. We fit $B_7^{(3/2)}$ and $B_8^{(3/2)}$ to the form of our trial function: $f_1(M_K^2) = c_0 + c_1(M_K^2) + c_2(M_K^2) \log(M_K^2)$. The results of $B_7^{(3/2)}$ and $B_8^{(3/2)}$ are presented in Fig. 3 and Fig. 4 respectively. Our preliminary values at the physical kaon mass are $B_7^{(3/2)} = 0.919(13)$ and $B_8^{(3/2)} = 1.047(15)$. We also fit $B_7^{(3/2)}$ and $B_8^{(3/2)}$ to the form suggested by the quenched chiral perturbation \[3\]: $f_2(M_K^2) = c_0 + c_1(M_K^2)$. These fitting results of $B_7^{(3/2)}$ and $B_8^{(3/2)}$ are shown in Fig. 5 and Fig. 6 respectively. Our preliminary values at the physical kaon mass from these fits are $B_7^{(3/2)} = 0.917(13)$ and $B_8^{(3/2)} = 1.044(15)$. Hence, we observe not only that the interpolated values at the physical kaon mass are insensitive to the fitting functions, but also that the extrapolated values in the chiral limit are extremely sensitive to the fitting functions. We can think of two possibilities for the chiral extrapolation. One possibility is that the data point at the lightest quark mass might be shifted due to a finite volume effect, which certainly need further investigation in near future. Another possibility is that the truncated higher order corrections from the quenched chiral perturbation theory are not negligible so that we might have to include higher order terms to do better chiral extrapolation. For these reasons, we do not quote our values for $B_7^{(3/2)}$ and $B_8^{(3/2)}$ in the chiral limit but we only quote the values interpolated to the physical kaon mass as above.

Note that we calculated $B_7^{(3/2)}$ at the scale $\mu = 1/a$ using the HYP staggered fermions, which would not have been meaningful for unimproved staggered fermions due to large perturbative corrections. Compared with previous calculation done using Landau-gauge operators \[20\], the systematics of the HYP staggered operators are significantly reduced and the results are as much more reliable.

8.3. $B_6^{(1/2)}$

A major contribution to $\Delta I = 1/2$ amplitudes comes from $B_6^{(1/2)}$. There are two independent methods to calculate $B_6^{(1/2)}$ in (partially) quenched QCD: the standard (STD) method
First, we fit $B(1/2)^{6}$ to the form suggested by the quenched chiral perturbation theory \cite{9}:

$$f_1(M_K^2) = c_0 + c_1(M_K^2) + c_2(M_K^2) \log(M_K^2).$$

The results are presented in Fig. 6. Our preliminary values at the physical kaon mass are

$B(1/2)^{6}_{\text{STD}}(\mu = 1/a) = 0.714(91)$

$B(1/2)^{6}_{\text{GP}}(\mu = 1/a) = 0.974(69)$.

where $1/a = 1.95$ GeV, set by the $\rho$ meson mass.

We also fit $B(1/2)^{6}$ to another form suggested by the quenched chiral perturbation \cite{9}:

$$f_2(M_K^2) = c_0 + c_1(M_K^2).$$

The results are presented in Fig. 6. Our preliminary values at the physical kaon mass from this fit are

$B(1/2)^{6}_{\text{STD}}(\mu = 1/a) = 0.701(93)$

$B(1/2)^{6}_{\text{GP}}(\mu = 1/a) = 0.973(70)$.

As in the case of $B(3/2)^{7}$ and $B(3/2)^{8}$, we observe that the extrapolated values in the chiral limit are extremely sensitive to the fitting function choice. At present, we can not exclude the possibility that the data point at the lightest quark mass could be shifted due to a finite volume effect (in the $\epsilon$ region of the chiral perturbation theory). It is also possible that truncated higher order corrections from the quenched chiral perturbation theory are so significant that we might have to include these neglected terms for chiral extrapolation. Therefore, we cannot quote our values of $B(1/2)^{6}$ in the chiral limit.

Note that unlike the unimproved staggered fermion calculations where the perturbative corrections to the STD calculation are $\approx 50\%$, the perturbative corrections in this calculation are modest for both the STD and GP methods. We also observe that the gap between the STD and GP methods is reduced at the physical kaon mass using the HYP staggered fermions compared that of the unimproved staggered fermions.

8.4. Preliminary $\epsilon'/\epsilon$

We use the formula provided in \cite{21} to convert $(m_s + m_d)_{\mu = m_c}$, $B(1/2)^{6}(\mu = m_c)$ and $B(3/2)^{8}(\mu = m_c)$ at $m_c = 1.3$ GeV into $\epsilon'/\epsilon$. As mentioned before, we have two different methods to calculate $B(1/2)^{6}$ in quenched QCD: the STD and GP methods. When we use the STD method for $B(1/2)^{6}(\mu = m_c)$, we obtain $\epsilon'/\epsilon(\text{STD}) = 0.00046(23)$. For the GP method for $B(1/2)^{6}(\mu = m_c)$, we obtain $\epsilon'/\epsilon(\text{GP}) = 0.00115(17)$. These values are very preliminary and we have not in-
Figure 7. $B_6^{\Delta I=1/2}(\mu = 1/a)$

cluded an analysis of the systematic errors. In addition, we did not use lattice values for any $B_i^{(1/2)}$ except for $B_6^{(1/2)}$ since we have not yet extracted them.

9. FUTURE PERSPECTIVES

We plan to calculate all the $B_i^{(1/2)}$ and incorporate all of them into the calculation of $\epsilon'/\epsilon$. We also plan to obtain the optimal matching scale, $q^*$ [22]. We plan to extend our calculation to dynamical simulation. Current calculation of $\epsilon'/\epsilon$ uses the leading order chiral perturbation results to convert $K \to \pi$ and $K \to 0$ amplitudes into $K \to \pi\pi$ amplitudes. In principle, it is possible to calculate $K \to \pi\pi$ amplitudes directly on the lattice. A number of attempts in this direction have been tried [23][24]. There has been a numerical study on Fat7 as a new fat link computationally cheaper than the HYP link, while preserving all the nice features of the HYP link [25].

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