Light propagation in the field of a moving axisymmetric body: theory and application to Juno

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Given the extreme accuracy of modern space science, a precise relativistic modeling of observations is required. We use the Time Transfer Functions formalism to study light propagation in the field of moving axisymmetric bodies, which extends the field of application of previous works. We first derive a space-time metric adapted to describe the geometry of an ensemble of moving bodies. Then, we show that the expression of the Time Transfer Functions in the field of a uniformly moving body can be easily derived from its well-known expression in a stationary field by using a change of variables. We also give a general expression of the Time Transfer Function in the case where the motion of the body is arbitrary. This result is given in the form of an integral easily computable numerically. We also provide the derivatives of the Time Transfer Function in this case, which are mandatory to compute Doppler and astrometric observables. We particularize our results in the case of moving axisymmetric bodies. Finally, we apply our results to study the different relativistic contributions to the range and Doppler tracking for the Juno mission in the Jovian system.

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I. INTRODUCTION

In modern times, the accuracy of spacecraft tracking requires a very detailed modeling of the light propagation in order to compute range and Doppler observables. For example, the Cassini spacecraft reaches the level of few meters accuracy for the range and $3 \times 10^{-6}$ m/s for the Doppler [1–3] while the future BepiColombo mission should reach an accuracy of 10 cm on the range and $10^{-6}$ m/s on the Doppler [4, 5]. Similar accuracies are expected for the Juno mission [6], which shall reach the Jovian system by mid-2016.

The computation of radio-science observables as well as the determination of astrometric observables (VLBI tracking [7]) requires to determine the propagation of light in a curved space-time. In this context, several approaches exist. Assuming that the metric is known, solving the null geodesic equations [8] or the eikonal equation [9] is the standard method allowing to get all the information about light propagation between two point-events. Many solutions have been proposed in the post-Newtonian (PN) and the post-Minkowskian (PM) approximations when dealing with the bending effects due to the mass multipole moments of the bodies in the Solar System [10–17]. On the other hand, the effects of the motion of monopoles on the light propagation have also been studied [10, 18–20]. A different approach is also available, initially based on the Synge World Function [21–23] and then on the Time Transfer Functions (TTF) [24, 25]. In this formalism, the computation of the coordinate light time, the frequency shift and the light deflection can be computed as integrals of functions of the components of the metric tensor over a straight line joining the emitter and the receiver of the signal [24, 25]. This method has already been successfully used to compute the propagation of light in different configurations. For example, the TTF in the field of a stationary axisymmetric body has been determined at the first post-Newtonian (1PN) approximation [21, 26]. The light propagation in the field of moving monopoles at 1.5 post-Newtonian order has also been treated [27]. Finally, the TTF in the field of a static multipole up and the second and third post-Minkowskian (3PM) approximation has also been determined [25, 28, 29].

In this paper, we use the Time Transfer Functions formalism to compute the coordinate propagation time, the frequency shift and the deflection of light in the field of moving axisymmetric bodies. In Sec. III, we briefly review how the radio-sience and astrometric observables can be determined from the TTF and its derivatives. Then, in Sec. IV, we determine the space-time metric describing the geometry in the field of a moving axisymmetric body. In Sec. V, we use this metric to develop a general expression of the TTF in the field of an arbitrary moving body. A general result is given in the form of an integral computable numerically. Moreover, an analytical result is developed in the case of a uniform motion. The derivatives of the TTF are also determined. In Sec. VI, we particularize our results in the case of a moving axisymmetric body by determining analytically the contribution of each multipole to the TTF. Finally, in Sec. VII, we apply our results to determine the different relativistic contributions to the radio-science tracking of the Juno spacecraft in the Jovian system. The contributions of the Sun and Jupiter moving monopoles and of the Jupiter moving $J_2$ are identified. Finally, we give our conclusions and general remarks in Sec. VIII.

II. NOTATION AND CONVENTIONS

In this paper $c$ is the speed of light in a vacuum and $G$ is the Newtonian gravitational constant. The Lorentzian metric of space-time $V_4$ is denoted by $g$. The signature adopted for $g$ is $(+−−−)$. We suppose that space-time is covered by some global quasi-Galilean coordinate system $(x^\mu) = (x^0, \mathbf{x})$, where $x^0 = ct$, $t$ being a time coordinate, and $\mathbf{x} = (x^i)$. We assume that the curves of equations $x^i = \text{const}$ are timelike, which means that $g_{00} > 0$ anywhere. We employ the vector notation $\mathbf{a}$ in order to denote $(a^1, a^2, a^3) = (a^\mu)$. Considering two such quantities $\mathbf{a}$ and $\mathbf{b}$ we use $\mathbf{a} \cdot \mathbf{b}$ to denote $a^\mu b_\mu$ (Einstein convention on repeated indices is used). The quantity $|\mathbf{a}|$ stands for the ordinary Euclidean norm of $\mathbf{a}$. For any quantity $f(x^\mu)$, $f,_{\alpha}$ denotes the partial derivative of $f$ with respect to $x^\alpha$.

III. TIME TRANSFER FUNCTION AND OBSERVABLES

Let $x_A = (ct_A, \mathbf{x}_A)$ and $x_B = (ct_B, \mathbf{x}_B)$ be two events of space-time supposed to be connected by a unique light ray. They denote the emission and reception point of the electromagnetic signal. The coordinate light time of a photon connecting $x_A$ and $x_B$ is given by the TTF [22, 24, 30, 31] as

$$t_B - t_A = \mathcal{T}(x_A, t_B, \mathbf{x}_B) = \frac{R_{AB}}{c} + \frac{1}{c} \Delta(x_A, t_B, \mathbf{x}_B) \quad (1)$$
where $\mathcal{T}(x_A, t_B, x_B)$ is the TTF\(^1\), $R_{AB} = |x_B - x_A|$ and $\Delta(x_A, t_B, x_B)$ is the so-called "delay function"\(^2\).

As developed in details in \[25\], the range, Doppler and astrometric observables can all be computed from the TTF. The range is directly related to the time of flight of the photon (see also \[32\]). The frequency shift is given by \[23, 32, 33\]

\[
\nu_B/\nu_A = \left[ \frac{g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j}{g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j} \right]^{1/2} \times \frac{1 - N^1_{AB}\beta^1_B - \beta^i_B \frac{\partial \Delta}{\partial x^i_A} - \frac{1}{c} \frac{\partial \Delta}{\partial t_B}}{1 - N^1_{AB}\beta^1_A + \beta^1_A \frac{\partial \Delta}{\partial x^1_A}} \tag{2}
\]

where $\beta^{i}_{A/B} = dx^i_{A/B}/c dt$ is the coordinate velocity.

The astrometric observables are directly related to the TTF through the use of \[22\]

\[
\left( \hat{k}_i \right)_B = \left( \frac{k_i}{k_0} \right)_B = -c \frac{\partial \mathcal{T}_r}{\partial x^i_B} \left[ 1 - \frac{\partial \mathcal{T}_r}{\partial t_B} \right]^{-1} = - \left( N^1_{AB} + \frac{\partial \Delta_r}{\partial x^i_B} \right) \times \left[ 1 - \frac{1}{c} \frac{\partial \Delta_r}{\partial t_B} \right]^{-1}, \tag{3}
\]

where $k_\mu$ are the covariant components of the tangent vector to the photon trajectory $(k^\mu)_B = dx^\mu/d\lambda|_B$ ($\lambda$ being an affine parameter) at $x_B$ and $N_{AB} = \frac{R_{AB}}{R_B} = \frac{x_B - x_A}{R_{AB}}$.

Finally, the angular distance between two light rays coming from two different sources can also be related to $(\hat{k}_i)_B$ \[25, 34\].

Therefore, the computation of the TTF (or equivalently the computation of the delay function) and its derivatives is crucial in order to analyze different effects on observations done using light propagation.

**IV. METRIC AT FIRST POST-MINKOWSKIAN APPROXIMATION**

Let us suppose that the gravitational field is generated by an ensemble of axisymmetric bodies. We are interested in calculating the contributions of the mass multipole and of the motion of the bodies on light propagation. The first step is to consider the metric describing such a space-time. The metric for each of the bodies at 1PM order in calculating the contributions of the mass multipoles and of the motion of the bodies on light propagation. The procedure is similar to what is developed in \[32\].

We can now perform a Poincaré transformation in order to obtain the metric in the case of a moving body. The coordinate transformation is given by \[32\]:

\[
x^\mu = b^\mu + \Lambda^\mu_0 X^\alpha \tag{5}
\]

where $x^\mu = (ct, \mathbf{x})$ are the coordinates of the global reference system and $\Lambda^\mu_0$ is given by

\[
\Lambda^0_0 = \gamma_p, \quad \Lambda^i_0 = \Lambda^i_0 = \gamma_p \beta^i_p, \quad \Lambda^0_i = \delta_{ij} + \frac{\gamma_p^2}{1 + \gamma_p^2} \beta^i_p \beta^j_p, \tag{6}
\]

where $\beta^i_p = v^i_p/c$, $v^i_p$ is the coordinate velocity of the body and $\gamma_p = 1/\sqrt{1 - \beta^2_p}$ with $\beta_p = |\beta_p|$. Note that $b^\mu$ is a constant four-vector who specifies the origin of the coordinate system: it points from the origin of the global reference system to the origin of the co-moving frame at $T = 0$ \[36\]. We have

\[
b^i = x^i_p(t_0) \quad \text{and} \quad b^0 = ct_0 \tag{7}
\]

\[1\] In this paper, we used the reception TTF. Similar results can be obtained using the emission TTF which depends on $t_A$ instead of $t_B$ \[24\].

\[2\] In this paper, we call for simplicity $\Delta(x_A, t_B, x_B)$ a "delay function" even though it has the dimension of a distance.
and the trajectory of the moving body in the global frame is given by

\[ x_p(t) = x_p(t_0) + c\beta_p(t - t_0). \] 

(8)

The inverse coordinate transformation is given by

\[ X^\alpha = \tilde{\Lambda}_\mu^\alpha (x^\mu - b^\alpha), \] 

(9)

where \( \tilde{\Lambda}_\mu^\alpha \) is the inverse of \( \Lambda^\mu_\alpha \) and is given by

\[ \tilde{\Lambda}_0^0 = \gamma_p, \quad \tilde{\Lambda}_0^i = \tilde{\Lambda}_i^0 = -\gamma_p\beta^i_p, \quad \tilde{\Lambda}_0^j = \delta_{ij} + \frac{\gamma_p^2}{1 + \gamma_p^2} \beta^i_p \beta^j_p. \] 

(10)

The metric transformation is given by

\[ g^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta G^{\alpha\beta}. \] 

(11)

Representing the metric of the global frame by \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), we have

\[ h^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta H^{\alpha\beta}. \] 

(12)

From Eq. (4), we have \( H^{\alpha\beta} = \frac{2W}{c^2}\delta^{ab} \) and using the expression of \( \Lambda^\mu_\alpha \) given in Eq. (6) into Eq. (12) leads to

\[ h^{00} = \frac{2W(X^0)}{c^2} \gamma_p^2 (1 + \beta^2_p) + \mathcal{O}(G^2), \] 

(13a)

\[ h^{0i} = \frac{4W(X^0)}{c^2} \beta^i_p \gamma_p + \mathcal{O}(G^2), \] 

(13b)

\[ h^{ij} = \frac{2W(X^0)}{c^2} (\delta_{ij} + 2\beta^i_p \beta^j_p \gamma_p^2) + \mathcal{O}(G^2). \] 

(13c)

It is worth mentioning that this metric is a generalization of the IAU metric [35] which can be recovered in the post-Newtonian limit when \( \beta_p \) are small. Let us also stress that \( W \) still depends on the local coordinates \( X^\alpha \). Therefore, we still need to use the coordinates transformation (9) to express the potential \( W \) as a function of the global coordinates \( x^\alpha \). More precisely, we get

\[ W = W(X^\alpha) = W(\tilde{\Lambda}_\mu^\alpha (x^\mu - b^\alpha)). \] 

(14)

The metric (13) describes the geometry determined by a uniformly moving body at 1PM. The metric for an ensemble of \( N \) bodies is then given by

\[ h^{00} = \sum_{p=1}^{N} \frac{2W_p}{c^2} \gamma_p^2 (1 + \beta^2_p) + \mathcal{O}(G^2), \] 

(15a)

\[ h^{0i} = \sum_{p=1}^{N} \frac{4W_p}{c^2} \beta^i_p \gamma_p + \mathcal{O}(G^2), \] 

(15b)

\[ h^{ij} = \sum_{p=1}^{N} \frac{2W_p}{c^2} (\delta_{ij} + 2\beta^i_p \beta^j_p \gamma_p^2) + \mathcal{O}(G^2). \] 

(15c)

In the case of an axisymmetric body, the Newtonian potential can be decomposed in a multipolar expansion

\[ W_p(X^i) = \frac{GM_p}{R} \left[ 1 - \sum_{n=2}^{\infty} J_{np} \left( \frac{r_{pe}}{R} \right)^n P_n \left( \frac{k_p \cdot X}{R} \right) \right], \] 

(16)

where \( k_p \) denotes the unit vector along the symmetry axis of the body \( p \), \( M_p \) is the mass of the body \( p \), \( J_{np} \) are its mass multipoles moments, \( P_n \) are the Legendre polynomials, \( r_{pe} \) is the equatorial radius of body \( p \) and \( R = |X| \).

In this paper, we assume that the symmetry axis of the body \( k_p \) is time independent, which means we neglect the precession and nutation of the body.
V. TIME TRANSFER FUNCTION AT GENERALIZED 1PM APPROXIMATION

In [23], a PM expansion of the TTF is presented. It develops the TTF in terms of integrals of functions of the metric over a straight line between the emitter and the receiver of a light signal. At 1PM order, the delay function is given by [33] as

$$\Delta(x_A, t_B, x_B) = \frac{R_{AB}}{2} \int_0^1 \left[ h^{00} - 2N^{i}_{AB} h^{0i} + N^{i}_{AB} N^{j}_{AB} \right] z^i(\lambda) \ d\lambda + \mathcal{O}(G^2).$$

where the integral is taken along a straight line defined by

$$z^0(\lambda) = ct = ct_B - \lambda R_{AB}$$
$$z(\lambda) = x_B - \lambda R_{AB} = x_B(1 - \lambda) + \lambda x_A.$$  \hfill (18a)

As it can be seen from the expression of the metric (15), $\Delta$ can be written as a sum of delay functions generated by each individual body $\Delta = \sum_{p=1}^N \Delta_p$. Replacing the expression of the metric (15) in (17) gives

$$\Delta_p(x_A, t_B, x_B) = \frac{2R_{AB}}{c^2} \int_0^1 \gamma_p^2 (1 - N_{AB} \cdot \beta_p)^2 W_p(\tilde{\Lambda}_\mu^i(z^{\mu}(\lambda) - b^{\mu})) d\lambda.$$  \hfill (19)

A. General expression in the case of uniform motions

It is useful to express the argument appearing in the expression of the potential $W_p$ in the right hand side of (19) as

$$\tilde{\Lambda}_\mu(z^{\mu}(\lambda) - b^{\mu}) = -\beta_p c (t - t_0) + z^{i}(\lambda) - x^{i}_{p0} + \frac{\gamma_p^2}{1 + \gamma_p} \beta^i_p \beta_p \cdot (z(\lambda) - x_{p0})$$
$$= x^i_B + \frac{\gamma_p^2}{1 + \gamma_p} \beta^i_p [\beta_p \cdot (x_B - x_{p0})] - x^i_{p0} - \gamma_p v^i_p(t_B - t_0)$$
$$- \lambda R_{AB} \left[ N^{i}_{AB} - \gamma_p \beta^i_p + \frac{\gamma_p^2}{1 + \gamma_p} \beta^i_p (\beta_p \cdot N_{AB}) \right],$$

by using Eq. (8), Eq. (10) and Eq. (18) and by posing $x_{p0} \equiv x_p(t_0)$, where $t_0$ is a time taken between emission and reception of the light signal. It is also possible to rewrite Eq. (20) in a more compact form as

$$\tilde{\Lambda}_\mu(z^{\mu}(\lambda) - b^{\mu}) = R^{i}_{pB} - \lambda G^{i}_{AB}$$

by setting

$$R^{i}_{pX} = x^i_X + \frac{\gamma_p^2}{1 + \gamma_p} \beta^i_p [\beta_p \cdot (x_X - x_{p0})] - x_{p0} - \gamma_p v^i_p(t_B - t_0),$$ \hfill (22a)

$$G_{AB} = R_{AB} \left[ N_{AB} - \gamma_p \beta_p + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p (\beta_p \cdot N_{AB}) \right] = R_{AB} g_{pAB}$$ \hfill (22b)

and

$$g_{pAB} = N_{AB} - \gamma_p \beta_p + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p (\beta_p \cdot N_{AB}).$$ \hfill (22c)

Let us denote by $I$ the integral in the TTF expression for the static case so that

$$I(x_{pA}, x_{pB}) = \tilde{I}(R_{AB}, x_{pB}) = \int_0^1 W_p(x_{pB} - \lambda R_{AB}) d\lambda,$$

where $x_{pA} = x_A - x_p$ and $x_{pB} = x_B - x_p$. Usually, the solution of this integral is given in terms of $x_{pA}$ and $x_{pB}$ but formally, the integral depends on $R_{AB}$ and $x_{pB}$. The transition between the two expressions of $I$ and $\tilde{I}$ in the
static case is trivial because \( x_{pA} = x_{pB} - R_{AB} \). However, this transition no longer applies in the moving case and it has to be replaced in Eq. (19) by

\[
\tilde{I}(G_{AB}, R_{pB}) = \int_0^1 W_p (R_{pB} - \lambda G_{AB}) d\lambda
\]  

(24)

with the two variables defined by Eqs. (22) and similarly to what proposed in [27].

Therefore, all the results in the moving case can be derived from the expressions used in the static case by replacing \( x_{pB} \) by \( R_{pB} \) (22a) and \( R_{AB} \) by \( G_{AB} \) (22b). We can use the conversions given below, where for each "static case" quantity on the left we give the "moving case" equivalent on the right. We get

\[
x_{pB} \rightarrow R_{pB} = x_B + \frac{\gamma_p^2 \beta_p (x_B - x_{p0})}{1 + \gamma_p \beta_p (x_B - x_{p0})},
\]

(25a)

\[
r_{pB} = |x_{pB}| \rightarrow R_{pB} = |R_{pB}|,
\]

(25b)

\[
n_{pB} \rightarrow N_{pB} = \frac{R_{pB}}{R_{pB}},
\]

(25c)

\[
R_{AB} \rightarrow G_{AB} = R_{AB} (N_{AB} - \gamma_p \beta_p + \frac{\gamma_p^2}{1 + \gamma_p \beta_p} \beta_p (\beta_p \cdot N_{AB}))
\]

(25d)

\[
R_{AB} \rightarrow R_{AB} \gamma_p (1 - \beta_p \cdot N_{AB}),
\]

(25e)

\[
N_{AB} \rightarrow g_{pAB} = \gamma_p (1 - \beta_p \cdot N_{AB})
\]

(25f)

\[
x_{pA} = x_{pB} - R_{AB} \rightarrow R_{pB} - G_{AB} = R_{pA} + \gamma_p \beta_p R_{AB},
\]

(25g)

\[
r_{pA} = |x_{pA}| \rightarrow R_{pA} = |R_{pA}|,
\]

(25h)

\[
n_{pA} \rightarrow R_{pA} + \gamma_p \beta_p R_{AB}
\]

(25i)

with

\[
g_{pAB} = |g_{pAB}| = \gamma_p (1 - \beta_p \cdot N_{AB}),
\]

(26)

and \( R_{pX} \) given by Eq. (22a). Therefore, we can rewrite Eq. (19) as

\[
\Delta_p(x_A, t_B, x_B) = \frac{2R_{AB}}{c^2} \gamma_p^2 (1 - N_{AB} \cdot \beta_p)^2 \int_0^1 W_p (R_{pB} - \lambda G_{AB}) d\lambda
\]

(27)

Then, using the definition of \( I \) from Eqs. (23)-(24) and the correspondences (25), we are able to express the exact form of the TTF in the field of moving bodies as

\[
\Delta_p(x_A, t_B, x_B) = \frac{2R_{AB}}{c^2} \gamma_p^2 (1 - N_{AB} \cdot \beta_p)^2 I(R_{pA} + \gamma_p \beta_p R_{AB}, R_{pB})
\]

(28)

with \( R_{pX} \) given by (22a). The last expression can also be written as

\[
\Delta_p(x_A, t_B, x_B) = \gamma_p^2 (1 - N_{AB} \cdot \beta_p)^2 \frac{g_{pAB}}{g_{pAB}} \tilde{\Delta}(R_{pA} + \gamma_p \beta_p R_{AB}, R_{pB})
\]

(29)

\[
= \gamma_p (1 - N_{AB} \cdot \beta_p) \tilde{\Delta}(R_{pA} + \gamma_p \beta_p R_{AB}, R_{pB}),
\]

(30)

where \( \tilde{\Delta}(x_{pA}, x_{pB}) \) is the expression of the static TTF. This particularly simple equation is very useful since it allows one to determine the TTF of a moving body from the corresponding static TTF.

The derivatives of the TTF, needed to compute the frequency shift (2) and the astrometric direction (3), can be
computed from (30) remembering Eq. (22). In the case of a uniformly moving body, their expressions are given by

\[
\frac{\partial \Delta_p(x_A, t_B, x_B)}{\partial x_A} = \gamma_p (1 - N_{AB} \cdot \beta_p) \tilde{\Delta}_{p,jA}(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) \left[ \delta_{ij} - \gamma_p \beta_p^i \left( N_{AB}^i - \frac{\gamma_p}{1 + \gamma_p \beta_p^i} \right) \right]
+ \gamma_p \beta_p^i \frac{N^i_{AB} \beta_p \cdot N_{AB}}{R_{AB}} \Delta_p(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) ,
\]

\[
\frac{\partial \Delta_p(x_A, t_B, x_B)}{\partial x_B} = \gamma_p (1 - N_{AB} \cdot \beta_p) \tilde{\Delta}_{p,jB}(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) \left[ \delta_{ij} + \frac{\gamma_p^2}{1 + \gamma_p \beta_p^i} \right]
+ \gamma_p \beta_p^i N_{AB}^i \tilde{\Delta}_{p,jA}(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) \right) - \gamma_p \beta_p^i \frac{N^i_{AB} \beta_p \cdot N_{AB}}{R_{AB}} \Delta_p(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) ,
\]

\[
\frac{\partial \Delta_p(x_A, t_B, x_B)}{\partial \beta_p} = -c_p^2 (1 - N_{AB} \cdot \beta_p) \tilde{\beta}_p^i \left[ \tilde{\Delta}_{p,iA}(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) 
+ \tilde{\Delta}_{p,iB}(R_{PA} + \gamma_p \beta_p R_{AB}, R_{PB}) \right] ,
\]

where \( \tilde{\Delta}_{p,iX}(x_A, x_B) \) is the expression of the derivative of the static TTF with respect to \( x_X \)

\[
\tilde{\Delta}_{p,iX}(x_A, x_B) = \frac{\partial \tilde{\Delta}_p(x_A, x_B)}{\partial x_X} .
\]

It is worth mentioning that in the static case, we have the relation

\[
\tilde{\Delta}_p(x_A, x_B) = \tilde{\Delta}_p(x_B, x_A)
\]

and consequently

\[
\tilde{\Delta}_{p,iB}(x_A, x_B) = \tilde{\Delta}_{p,iA}(x_B, x_A) .
\]

Therefore, the expression of the derivatives of the TTF in the "moving case" is also obtained by inserting into Eqs. (31) the static TTF and its derivatives. We present an application in the field of moving axisymmetric bodies in Section VI.

### B. Case of a non-uniform motion

The previous section gives the exact solution of the TTF in the field of uniformly moving bodies. If the bodies undergo an acceleration, it is still possible to use the previous formula, which corresponds to neglect higher order terms related to the acceleration of the body. In this case, the choice of the parameter \( t_0 \) introduced in Eq. (20) becomes critical. It has been shown [10, 12, 20] that a good choice of \( t_0 \) (i.e. which minimizes the approximation error) is given by the time of closest approach of the photon with respect to the body which is given by

\[
t_0 = \max \left( t_A, t_B - \max \left( 0, \frac{g \cdot (x_B - x_p(t_B))}{c |g|^2} \right) \right) 
\]

with \( g = N_{AB} - \beta_p(t_B) \).

Another approach consists in the numerical integration of the TTF. This approach has the convenience to be strictly valid at the 1PM order whatever the motion of the bodies. In the case where the body \( p \) has a general motion, it is still possible to use the coordinates transformation (5) but only locally. This means that at each time \( t \), we have a coordinates transformation, so that the transformation \( \Lambda^i_0 \) becomes time-dependent. Therefore, relation (19) remains valid in the general case but with \( \gamma_p \) and \( \beta_p \) functions of time and with the argument entering the expression of the potential \( W_p \) so that

\[
\tilde{\Lambda}_p^i(t) (z^\mu - b^\mu(t)) = z^i(\lambda) - \tilde{x}_p^i(t) + \frac{\gamma_p^2}{1 + \gamma_p \beta_p^i} \beta_p^i \cdot (z(\lambda) - x(t)) ,
\]
with \( z(\lambda) = x_b - \lambda R_{AB} \). Therefore, the expression of the TTF valid at 1PM in the case of an arbitrary moving body is given by

\[
\Delta_p(x_A, t_B, x_B) = \frac{2R_{AB}}{c^2} \int_0^1 \gamma_p^2 (1 - N_{AB} \cdot \beta_p)^2 W_p \left( z^i(\lambda) - x^i_p(t) + \frac{\gamma_p^2}{1 + \gamma_p^2} \beta_p^0 \beta_p \cdot (z(\lambda) - x(t)) \right) d\lambda , \tag{37}
\]

where \( \gamma_p \) and \( \beta_p \) depend on the time coordinate \( t \) that is related to \( \lambda \) through (18a). The integral in Eq. (37) can then be evaluated numerically whatever the motion of the body \( x_p(t) \).

C. Moving emitter

In the previous sections, we handle the case where the source of the gravitational field is moving. In general, the emitter and the receiver of the electromagnetic signal are also moving. In this case, the determination of the time transfer requires to solve Eq. (1), which is now implicit

\[
t_B - t_A = \mathcal{T}(x_A(t_A), t_B, x_B) = \frac{|x_B - x_A(t_A)|}{c} + \frac{1}{c} \Delta(x_A(t_A), t_B, x_B). \tag{38}
\]

In practice, the solution of this implicit equation can be determined by an iterative procedure to find \( t_A \) (for example, see Eq. (7) of [25]). Another solution consists in a post-Newtonian expansion of \( t_A \) from the TTF (for example, see Eq. (6) of [25]). Let us denote by \( \tilde{t}_A \) the coordinate time of emission solution of

\[
t_B - \tilde{t}_A = \frac{|x_B - x_A(\tilde{t}_A)|}{c}, \tag{39}
\]

then we can write at first order in \( \beta_A = v_A(\tilde{t}_A)/c \)

\[
t_B - t_A = \frac{\tilde{R}_{AB}}{c} + \Delta(x_A(\tilde{t}_A), t_B, x_B) - \frac{\tilde{R}_{AB}}{c} \beta_A \frac{\partial \Delta(x_A(\tilde{t}_A), t_B, x_B)}{\partial x_A}, \tag{39}
\]

where the "bar" denotes quantities evaluated at \( \tilde{t}_A \) like \( \tilde{R}_{AB} = |x_B - x_A(\tilde{t}_A)| \). The contribution proportional to \( \beta_A \) is also known as a Sagnac term. It has the same form as the contribution from the velocity of the source of the gravitational field at first post-Newtonian order as can be seen from Eq. (47). The order of magnitude of this contribution can reach a few meters for a Juno-Earth signal as can be seen from Fig. 4. Therefore, when solving iteratively the light-time equation, one needs to include the relativistic perturbations or to take into account the Sagnac terms to avoid the risk of significant errors.

VI. CASE OF MOVING AXISYMMETRIC BODIES

We can now use the general procedure presented in the previous section in the case of moving axisymmetric bodies whose potential is given by the multipole expansion (16). The TTF in the case of a static axisymmetric body has been computed in [26] and is given by

\[
\Delta_p(x_{pA}, x_{pB}) = \Delta_{Mp}(x_{pA}, x_{pB}) + \Delta_{p\mu}(x_{pA}, x_{pB}) \tag{40}
\]

where \( \Delta_{Mp} \) represents the mass monopole contribution and \( \Delta_{p\mu} \) represents the mass multipole contribution.

The TTF corresponding to a static monopole is well known [21] and is given by

\[
\bar{\Delta}_{Mp}(x_{pA}, x_{pB}) = \frac{2GM_p}{c^2} \ln \frac{r_{pA} + r_{pB} + R_{AB}}{r_{pA} + r_{pB} - R_{AB}}. \tag{41}
\]

By inserting (41) into (30) and using the substitutions (25), we obtain the TTF in the field of monopoles in uniform motion as

\[
\Delta_M(x_A, t_B, x_B) = 2\frac{GM_p}{c^2} \gamma_p (1 - N_{AB} \cdot \beta_p) \ln \frac{|R_{pA} + \gamma_p \beta_p R_{AB}| + R_{pB} + \gamma_p R_{AB} (1 - \beta_p \cdot N_{AB})}{|R_{pA} + \gamma_p \beta_p R_{AB}| + R_{pB} - \gamma_p R_{AB} (1 - \beta_p \cdot N_{AB})}, \tag{42}
\]
with \( R_{pA} \) given by (22a). On the other hand, the mass multipole contribution \( \Delta_{J,m} \) has been computed in [26] as

\[
\tilde{\Delta}_{J,m}(x_{pA}, x_{pB}) = K_{pm} \sum_{m=1}^{\infty} \left[ \frac{1}{(r_{pA} + r_{pB} - R_{AB})^{n-m}} - \frac{1}{(r_{pA} + r_{pB} + R_{AB})^{n-m+1}} \right] \Theta_{nm}(x_{pA}, x_{pB}),
\]

(43a)

with \( K_{pm} \equiv (1 + \gamma)GM_pJ_{pp}^{r_p}/c^2 \) and

\[
\Theta_{nm}(x_{pA}, x_{pB}) = (-1)^{n-m} \sum_{i_1, \ldots, i_m} (n-m)! \prod_{l=1}^{m}[S_l(x_{pA}, x_{pB})]^{i_l},
\]

(43b)

where the sum \( \sum_{i_1, \ldots, i_m} \) denotes the summation over the sets of nonnegative integers \( i_1, i_2, \ldots, i_m \) satisfying the pair of equations

\[
\begin{align*}
&\begin{cases}
  i_1 + 2i_2 + 3i_3 + \cdots + mi_m = n \\
i_1 + i_2 + \cdots + i_m = n - m + 1
\end{cases}
\]

(43c)

and where \( S_l(x_{pA}, x_{pB}) \) is defined by

\[
S_l(x_{pA}, x_{pB}) = \frac{1}{r_{pA}} C_l^{(-1/2)} \left( \frac{k}{r_{pA}} \right) + \frac{1}{r_{pB}} C_l^{(-1/2)} \left( \frac{k}{r_{pB}} \right)
\]

(43d)

with \( C_l^{(-1/2)}(x) \) the Gegenbauer polynomial of degree \( l \) and of parameter \(-1/2\).

Therefore, the multipole term of the TTF for the case of moving axisymmetric bodies is given by inserting (43) into the relation (30) and using the substitutions (25)

\[
\Delta_{J,m}(x_{A}, t_B, x_{B}) = \frac{2GM_pJ_{pp}^{r_p} \gamma_p}{c^2} (1 - \mathbf{N}_{AB} : \mathbf{\beta}_p)
\]

\[
\times \sum_{m=1}^{\infty} \left[ \frac{1}{(|R_{pA} + \gamma_p\mathbf{\beta}_p R_{AB}| + R_{pB} - R_{AB}\gamma_p(1 - \mathbf{\beta}_p : \mathbf{N}_{AB}))^{n-m+1}} - \frac{1}{(|R_{pA} + \gamma_p\mathbf{\beta}_p R_{AB}| + R_{pB} + R_{AB}\gamma_p(1 - \mathbf{\beta}_p : \mathbf{N}_{AB}))^{n-m+1}} \right] \Theta_{nm}(R_{pA} + \beta_p R_{AB}, R_{pB})
\]

(44)

with \( R_{pA} \) given by (22a).

In order to compute the derivatives of the TTF in the case of moving bodies from Eq. (31), one also needs the derivatives of the TTF in the static case. The derivative of the TTF in the case of a static monopole is known (see for example [37]) and it is given by

\[
\tilde{\Delta}_{M,p,iA}(x_{pA}, x_{pB}) = -4GM_p \frac{N_{AB}^i (r_{pA} + r_{pB}) + R_{AB} n_{pA}^i}{(r_{pA} + r_{pB})^2 - R_{AB}^2},
\]

(45a)

\[
\tilde{\Delta}_{M,p,iB}(x_{pA}, x_{pB}) = +4GM_p \frac{N_{AB}^i (r_{pA} + r_{pB}) - R_{AB} n_{pB}^i}{(r_{pA} + r_{pB})^2 - R_{AB}^2} = \tilde{\Delta}_{p,iA}(x_{pB}, x_{pA})
\]

(45b)

Also, the derivatives of Eq. (43) can be computed as

\[
\tilde{\Delta}_{J,m,iA}(x_{pA}, x_{pB}) = K_{pm} \sum_{m=1}^{\infty} \left\{ -\left( n - m + 1 \right) \frac{n_{pA} + N_{AB}}{(r_{pA} + r_{pB} - R_{AB})^{n-m+2}} - \frac{n_{pA} - N_{AB}}{(r_{pA} + r_{pB} + R_{AB})^{n-m+2}} \right\} \Theta(x_{pA}, x_{pB})
\]

\[
+ \left[ \frac{1}{(r_{pA} + r_{pB} - R_{AB})^{n-m+1}} - \frac{1}{(r_{pA} + r_{pB} + R_{AB})^{n-m+1}} \right] \mathbf{Y}_{A|nm}(x_{pA}, x_{pB}),
\]

(46a)

\[
\tilde{\Delta}_{J,m,iB}(x_{pA}, x_{pB}) = K_{pm} \sum_{m=1}^{\infty} \left\{ -\left( n - m + 1 \right) \frac{n_{pB} - N_{AB}}{(r_{pA} + r_{pB} - R_{AB})^{n-m+2}} - \frac{n_{pB} + N_{AB}}{(r_{pA} + r_{pB} + R_{AB})^{n-m+2}} \right\} \Theta(x_{pA}, x_{pB})
\]

\[
+ \left[ \frac{1}{(r_{pA} + r_{pB} - R_{AB})^{n-m+1}} - \frac{1}{(r_{pA} + r_{pB} + R_{AB})^{n-m+1}} \right] \mathbf{Y}_{B|nm}(x_{pA}, x_{pB})
\]

(46b)
where

\[
\Upsilon_{X|m}(x_{pA}, x_{pB}) = (-1)^{n-m} \sum_{i_1, \ldots, i_m} \frac{(n-m)!}{i_1!i_2! \cdots i_m!} \sum_{l=1}^{m} i_l [S_l(x_{pA}, x_{pB})]^{i_l-1} \prod_{q=1, q \neq l}^{m} [S_q(x_{pA}, x_{pB})]^{i_q} \left\{ P_{l-1}(k_p \cdot n_{pX})k_p - P_l(k_p \cdot n_{pX})n_{pX} \right\} r_{pX}^l.
\]

The derivatives of the TTF function in the case of a moving axisymmetric body is then given by combining Eq. (41) and Eq. (46) into Eq. (40) and by using it together with Eq. (45) and Eq. (46) into Eqs. (31) (using the correspondances (25)).

A. Particular case: post-Newtonian expansion

The section V A gives a way to compute the TTF in the field of uniformly moving bodies. The obtained expressions are exact at any order in $\beta_p$. Nevertheless, a post-Newtonian expression can sometimes be more practical to use in the case of slowly moving bodies. Therefore, we present here an expansion of the previous results in terms of the small parameter $\beta_p$. An expansion of (28) gives

\[
\Delta_p(x_A, t_B, x_B) = (1 - \beta_p \cdot NAB) \tilde{\Delta}_p(x_{pA}, x_{pB}) + (R_{AB} - c(t_B - t_0)) \beta_p \tilde{\Delta}_{p,iA}(x_{pA}, x_{pB})
\]

\[
- c(t_B - t_0) \beta_p \tilde{\Delta}_{p,iB}(x_{pA}, x_{pB}),
\]

with $x_{pX} = x_X - x_p(t_0)$. For example, the use of this of formula in the case of the moving monopoles leads to

\[
\Delta_p(x_A, t_B, x_B) = 2 \frac{GM_p}{c^2} (1 - \beta_p \cdot N_{AB}) \ln \frac{r_{pA} + r_{pB} + R_{AB}}{r_{pA} + r_{pB} - R_{AB}}
\]

\[
\quad - 4 \frac{GM_p R_{AB}}{c^2} \left( \frac{r_{pA} + r_{pB} + N_{AB} \cdot \beta_p + R_{AB} n_{pA} \cdot \beta_p}{(r_{pA} + r_{pB})^2 - R_{AB}^2} \right)
\]

\[
\quad + 4 \frac{GM_p R_{AB}}{c} (t_B - t_0) \beta_p \cdot \frac{(n_{pA} + n_{pB})}{(r_{pA} + r_{pB})^2 - R_{AB}^2} + O(c^{-4})
\]

with $n_{pX} = \frac{x_{pX}}{r_{pX}}$. This expression is equivalent of the one given by Eq. (20) of [27]. To obtain this result in such a straightforward way, illustrates the effectiveness of the TTF approach.

B. Particular case: the quadrupolar term

An explicit calculation for each of the multipole is straightforward given the above formulas. As an example, let us develop explicitly the expression for the quadrupolar term $j_2$. The only sets of integers solutions to Eqs. (43c) are $i_1 = 2$ for $m = 1$ and $\{i_1 = 0, i_2 = 1\}$ for $m = 2$. As shown in [26], we obtain

\[
\tilde{\Delta}_{j_2}(x_{pA}, x_{pB}) = \frac{GM_p}{c^2} \frac{J_p^2 r^2 pe}{r_{pA} r_{pB}} \frac{R_{AB}}{1 + n_{pA} \cdot n_{pB}} \times \left[ \frac{1 - (k_p \cdot n_{pA})^2}{r_{pA}} + \frac{1 - (k_p \cdot n_{pB})^2}{r_{pB}} \right]
\]

\[
\quad - \left( \frac{1}{r_{pA}} + \frac{1}{r_{pB}} \right) \frac{k_p \cdot (n_{pA} + n_{pB})}{1 + n_{pA} \cdot n_{pB}} \right].
\]

Therefore, inserting (49) into (30) and using the substitutions (25), we obtain

\[
\Delta_{j_2}(x_{pA}, x_{pB}) = \frac{GM_p}{c^2} \frac{J_p^2 r^2 pe}{r_{pA} r_{pB}} \frac{R_{AB}}{[R_{pA} + \gamma_p \beta_p R_{AB}] [R_{pB} + \gamma_p \beta_p R_{AB}]}
\]

\[
\times \left[ \frac{1 - (k_p \cdot \tilde{N}_A)^2}{R_{pA} + \gamma_p \beta_p R_{AB}} + \frac{1 - (k_p \cdot n_{pB})^2}{R_{pB}} - \left( \frac{1}{[R_{pA} + \gamma_p \beta_p R_{AB}]} + \frac{1}{R_{pB}} \right) \frac{k_p \cdot (\tilde{N}_A + N_{pB})}{1 + \tilde{N}_A \cdot n_{pB}} \right]^2
\]

and

\[
\tilde{N}_A = \frac{R_{pA} + \gamma_p \beta_p R_{AB}}{R_{pA} + \gamma_p \beta_p R_{AB}}.
\]
The derivative of (49) with respect to \( x_{pA} \) can be computed using Eq. (46a) and is given by

\[
\Delta_{J_{pA}}(x_{pA}, x_{pB}) = 2 \frac{GM_p}{c^2} J_{pA} \left\{ \left[ k_p \cdot (n_{pA} + n_{pB}) \right]^2 \frac{n_{pA} + N_{AB}}{(r_{pA} + r_{pB} - R_{AB})^3} - \frac{n_{pA} - N_{AB}}{(r_{pA} + r_{pB} + R_{AB})^3} \right\}
- \frac{1}{2} \left[ \frac{1}{r_{pA}} \left( 1 - (k_p \cdot n_{pA})^2 \right) + \frac{1}{r_{pB}} \left( 1 - (k_p \cdot n_{pB})^2 \right) \right] \frac{n_{pA} + N_{AB}}{(r_{pA} + r_{pB} - R_{AB})^2} - \frac{n_{pA} - N_{AB}}{(r_{pA} + r_{pB} + R_{AB})^2}
- \left( \frac{1}{r_{pA}^3} \frac{R_{AB}(r_{pA} + r_{pB})}{r_{pB}^2} k_p \cdot (n_{pA} + n_{pB}) \left[ k_p - (k_p \cdot n_{pA}) n_{pA} \right] \right)
- \frac{1}{2r_{pA}^3} \frac{R_{AB}}{r_{pB}} \frac{2(k_p \cdot n_{pA}) k_p + [1 - 3(k_p \cdot n_{pA})^2] n_{pA}}{1 + n_{pA} \cdot n_{pB}}
\]  

while the derivatives with respect to \( x_{pB} \) can be obtained by symmetry as

\[
\Delta_{J_{pB}}(x_{pA}, x_{pB}) = \Delta_{J_{pA}}(x_{pB}, x_{pA}).
\]

In order to evaluate the contribution of the moving quadrupole to the derivatives of the time transfer, it is then sufficient to combine Eq. (52) and Eq. (49) as shown in Eq. (31).

VII. APPLICATION TO JUNO

As an example, we use the equations presented in previous sections to give estimates of the relativistic corrections on the range and range-rate for the Juno mission. Juno is currently on his way to Jupiter that will be reach in 2016. The spacecraft will orbit Jupiter during one year. Some of the relativistic perturbations on Juno orbit have been studied in [38, 39]. The main goal of this section is to assess the order of magnitude produced by different effects due to the Sun and Jupiter on the time transfer. We shall use the nominal orbit of the mission around Jupiter obtained using the Naif SPICE toolkit [40] and kernels as well as the DE430 planetary ephemeris [41]. The expected accuracy for Juno is of the order of 10 cm on the range and \( 10^{-6} \) m/s on the Doppler [6].

Fig. 1 represents the lower order time transfer and range rate between Juno and Earth as well as the relativistic Shapiro correction from the Sun. These corrections are standard.

![Fig. 1](image-url)

FIG. 1. Representation of different contributions on the range (left) and range-rate (right) between Juno and Earth over one year. The contributions represented are the lower order contribution (the actual value of the observables) and the corrections produced by the Shapiro due to the monopole of the Sun.

Fig. 2 represents the contributions of the mass monopole of Jupiter on the range and on the range-rate. These contributions have been splitted into two parts: a part related to the case where Jupiter is static and a contribution proportional to Jupiter velocity \( \beta_{Jup} \). The static part is computed using (41) with the position of Jupiter taken at the critical time \( t_0 \) given in Eq. (35). The contribution relative to the velocity is computed by taking the difference...
between the relations (42) and (41). As one can see, the contributions relative to the motion of Jupiter are 2 orders of magnitude below Juno expected accuracy and can safely be neglected in the modeling of the time transfer. A similar conclusion holds for the motion of the Sun around the Solar System barycenter which is even smaller. Note that the analytical results presented in these graphs have been checked by integrating numerically the TTF (37).

![Graphs showing contributions to range and range-rate between Juno and Earth](image)

**FIG. 2.** Representation of different contributions on the range (left) and range-rate (right) between Juno and Earth over one year. The contributions represented are the corrections produced by the Shapiro due to the monopole of the Jupiter (top) and the contributions due to the velocity of Jupiter (bottom).

Fig. 3 represents the contributions of the quadrupole of Jupiter ($J_2$) on the range and on the range-rate of Juno. As above, we have split these contributions into two parts: one related to the case where Jupiter is static and one proportional to Jupiter velocity $\beta_{\text{Jup}}$. The static part is computed using (49) with the position of Jupiter taken at the critical time $t_0$ given in Eq. (35). The contribution relative to the velocity is computed by taking the difference between the relations (50) and (49). As one can see, the contributions relative to the $J_2$ of Jupiter is of the same order as the expected Juno’s accuracy. Therefore the effect of the $J_2$ should be taken into account in the reduction of the tracking data. The contribution related to the velocity of the $J_2$ is far beyond the current tracking accuracy.

Once again, the analytical results presented in these graphs have been checked by integrating numerically the TTF (37). It is important to notice that the curves depend highly on the geometry of the probe orbit. Since Juno has a polar orbit and is never in conjunction with Jupiter, the velocity effects are not detectable. Therefore, the situation can be different for another space mission like JUICE.

Fig. 4 is given for illustrative purpose and shows more effects on the range of Juno. First of all, the effects of the second order in $\beta_{\text{Jup}}$ is represented. It is computed by making the difference between the formula valid at all order in $\beta_{\text{Jup}}$ (42) and the 1PN expansion (47). This shows one can safely use the PN expansion presented in Section VI A within the Solar System. The effect of the acceleration of Jupiter on the range is also presented. This is computed by making the difference between the numerical integration of the TTF in which we are using the real Jupiter trajectory (37) and the result valid at all order in the velocity (42). The small rapid oscillations come from oscillations in Jupiter acceleration which results from the perturbations due to the Galilean satellites.

Finally, on the right of Fig. 4 is represented the Sagnac effects due to the motion of Juno. The contributions represented are due to the Shapiro of the Sun and Jupiter and it has been computed using (39). These contributions should be included in the analysis of Juno data either as a perturbations, either when solving the light-time iterations.

**VIII. CONCLUSIONS**

In this paper, we compute the TTF and its derivatives in the field of axisymmetric bodies in motion, which is useful in order to evaluate range, Doppler and astrometric observables. First, in Section IV we compute a metric adapted to describe the space-time geometry due to $N$ bodies in a global reference system by using a Poincaré transformation.

Then we present a general method to compute the TTF and its derivatives in the case where the bodies generating the gravitational field are in uniform motion. We show that the TTF in the case of uniform motion can be directly derived from the static TTF as can be seen from Eq. (30) and Eqs. (31). This result is very powerful. Moreover,
in Section V B, we have developed a general expression of the TTF in the case where the body generating the gravitational field has an arbitrary motion. The result is given as an integral over a straight line between the emitter and the receiver (37) which can be computed numerically. This general formulation has been used to numerically check our analytical derivations but is also useful to assess the effects due to the acceleration of the body on the light propagation.

Then, in Section VI we show how our method can be easily applied to the metric presented in Section IV to compute analytically the TTF and its derivatives (and thus the range, frequency-shift and astrometric direction) for a light signal propagating in the field of one or more axisymmetric bodies in motion. The results of this paper complete the work of [26, 27] and in general extend the field of applicability of the TTF formalism [24].

Finally, as an example of our method, we compute the range and Doppler for the Juno mission during its orbit around Jupiter and study in detail the different perturbations due to the Sun and Jupiter on light propagation. In particular, we have shown that in addition to the standard Shapiro contributions due to the mass monopole of Jupiter

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FIG. 3. Representation of different contributions on the range (left) and range-rate (right) between Juno and Earth over one year. The contributions represented are the corrections produced by the $J_2$ of the Jupiter (top) and the contributions produced by the fact that the $J_2$ is moving.

FIG. 4. Representation of different contributions on the range between Juno and Earth over one year. Top Left: The 2PN contribution from the monopole of Jupiter (contribution proportional to $\beta_{Jup}^2$). Bottom Left: The contribution proportional to the acceleration of Jupiter. Right: the Sagnac contributions (proportional to the Sun mass and to Jupiter mass) due to the motion of Juno.
and the Sun, the contribution of Jupiter $J_2$ is also relevant at the level of accuracy expected for Juno. The motion of the Sun and of Jupiter produces effects too small compared to Juno accuracy. Nevertheless, this conclusion depends highly on the geometry of Juno orbit and it should be assessed carefully for other space mission (JUICE for example).

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