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Solving the noncommutative Batalin-Vilkovisky equation.

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Abstract

I show that summation over ribbon graphs give the construction of the solutions to the noncommutative Batalin-Vilkovisky equation, including the equivariant version, introduced in my previous papers. This generalizes the known construction of $A_\infty$-algebra via summation over ribbon trees. In particular, the quadratic Maurer-Cartan equation is replaced by the linear equation $\Delta(\Psi) = 0$. Such solutions are naturally the matrix action functionals giving the equivariantly closed differential forms on the matrix spaces, described in my previous paper.

I explain that the summation over ribbon graphs with legs produces solutions to the noncommutative Batalin-Vilkovisky equation, introduced in [B06a]. This generalizes the known construction of $A_\infty$-structure via summation over trees, see [M].

Notations. I work in the tensor category of super vector spaces, over an algebraically closed field $k$, $\text{char}(k) = 0$. Let $V = V_0 \oplus V_1$ be a super vector space. I denote by $\bar{\alpha}$ the parity of an element $\alpha$ and by $\Pi V$ the super vector space with inversed parity. Element $(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ of $A^\otimes n$ is denoted by $(a_1, a_2, \ldots, a_n)$. I denote by $V^*$ the dual vector space $\text{Hom}(V, k)$.

1 Algebra with odd differentiation.

I consider a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $A$, $\dim_k A < \infty$, with multiplication denoted by $m_2 : A^\otimes 2 \to A$ and an odd differentiation $I : A \to \Pi A$

$$Im_2(a, b) = m_2(Ia, b) + (-1)^{\bar{a}} m_2(a, Ib),$$

in particular, if $I^2 = 0$ then this is a d$(\mathbb{Z}/2\mathbb{Z})g$-algebra. I assume, at the beginning, that the algebra is cyclic with odd scalar product

$$\beta : A^\otimes 2 \to \Pi A,$$
so that the three tensor

\[ m \in ((\Pi A)^{\otimes 3})^\vee \]

\[ m(\pi a, \pi b, \pi c) = (-1)^\beta m_2(a, b, c) \]

is cyclically invariant

\[ m(\pi a, \pi b, \pi c) = (-1)^{\pi + \beta} m(\pi c, \pi a, \pi b) \]

and that \( \beta \) is preserved by \( I \):

\[ \beta(Ia, b) + (-1)^\beta \beta(a, Ib) = 0. \]

The modification for the variant with an even scalar product are described below.

Below I consider also the variant for general \( d(\mathbb{Z}/2\mathbb{Z}) \) algebra without scalar product. It is reduced to the case with even/odd scalar product by putting \( \bar{A} = A \oplus A^\vee \), or \( \bar{A} = A \oplus \Pi A^\vee \) with their natural scalar products.

I have

\[ m(Ia, b, c) + (-1)^\beta m(a, Ib, c) + (-1)^{\pi + \beta} m(a, b, Ic) = 0 \quad (1) \]

which reflect the Leibnitz rule for the differentiation \( I \). Denote by \( \beta^\pi \in (\Pi A)^{\otimes 2} \) the tensor of the scalar product on the dual vector space, then for any \( a, b, c, d \in A \),

\[ \langle m(\pi a, \pi b, \cdot m(\cdot, \pi c, \pi d), \beta^\pi) = (-1)^\pi \langle m(\pi d, \pi a, \cdot m(\cdot, \pi b, \pi c), \beta^\pi) \quad (2) \]

which is the associativity of the multiplication \( m \).

Let \( H \) be an odd selfadjoint operator

\[ H : A \to \Pi A, \quad H^\vee = H \]

such that

\[ Id - [I, H] = P \quad (3) \]

is an idempotent operator \( P : A \to A \),

\[ P^2 = P. \]

I assume also that \( H \) commutes with \( I^2 \), this is automatic if \( I^2 = 0 \). I denote by \( B \) the subspace which is the image of the idempotent \( P \).

Let \( \Gamma \) be a tri-valent ribbon graph, i.e. the trivalent graph with fixed cyclic orders on the sets of the three flags attached to every vertex. Let \( \Sigma_\Gamma \) be the corresponding two-dimensional surface. Then I put:
• the three-tensors
\[ m^v \in \left( (\Pi A)^{\otimes \text{Flag}(v)} \right)^\vee \]
on every vertex \( v \) and define
\[ \alpha_\Gamma = \bigotimes_{v \in \text{Vert}(\Gamma)} m^v \]
• the two tensors
\[ \beta^\nu,e_H \in (\Pi A)^{\otimes \{f,f'\}}, \]
\[ \beta^\nu,e_H = \beta^\nu(H^\nu u_f, v_{f'}) = (-1)^{\nu f'} \beta^\nu(H^\nu v_{f'}, u_f) \]
for any interior edge \( e = (f,f') \)
• element \( a_l \in \Pi B \), for any exterior leg \( l \), this gives a partition of the set of elements \( \{a_l\}_{l \in \text{Leg}(\Gamma)} \) to the subsets corresponding to the components of the boundary \( \partial \Sigma_\Gamma \) and cyclic orders on these subsets.

Notice that both \( m^v \) and \( \beta^\nu,e_H \) are even elements, so that the products \( \bigotimes_{v \in \text{Vert}(\Gamma)} m^v \) and \( \otimes_{e \in \text{Edge}(\Gamma)} \beta^\nu,e_H \) are canonically defined.

**Definition 1** I define the tensor \( W_\Gamma \) as the contraction
\[ W_\Gamma \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left( \bigotimes_{v \in \text{Vert}(\Gamma)} m^v ; \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^\nu,e_H \right) \right) \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) \tag{4} \]

Notice that \( W_\Gamma \) is cyclically invariant on every subset of \( \{a_l\}_{l \in \text{Leg}(\Gamma)} \) corresponding to a component of the boundary of \( \Sigma_\Gamma \). Moreover the cyclic orders on flags at vertices induce the orientation on the ribbon graph \( \Gamma \), whose more detailed analysis, see e.g. [B09b], shows that \( W_\Gamma \) lies in
\[ W_\Gamma \in \text{Symm}(\bigoplus_{j=1}^\infty (\Pi B^{\otimes j})^{Z/2Z})^\vee \]
Let \( \chi(\Sigma_\Gamma) \) denotes the genus of \( \Sigma_\Gamma \),
\[ \chi(\Sigma_\Gamma) = 2 - 2g(\Sigma_\Gamma) - i(\Sigma_\Gamma), \]
where \( g(\Sigma_\Gamma), i(\Sigma_\Gamma) \) are the genus and the number of boundary components of \( \Sigma_\Gamma \). I put
\[ S = \sum_{\{\Gamma\}} h^{-\chi(\Sigma_\Gamma)} W_\Gamma \tag{5} \]
where the sum is over isomorphism classes of connected trivalent graphs with nonempty subsets of legs on every boundary component of \( \Sigma_\Gamma \). One can include the graphs with empty subsets of legs on boundary components by adding the constant extra term to the Batalin-Vilkovisky operator \( \Delta \) below, I leave the details to the interested reader.

I’ve described in [B06a],[B06b] the Batalin-Vilkovisky formalism associated with a \( Z/2Z \)-graded vector space with odd scalar product and in particular I’ve defined the second order Batalin-Vilkovisky operator \( \Delta \) and the odd Poisson bracket \( \{,\} \) on \( \text{Symm}(\bigoplus_{j=1}^\infty (\Pi B^{\otimes j})^{Z/2Z})^\vee \).
Theorem 2 The sum over graphs \( S \) defined in (5) satisfy the equivariant non-commutative Batalin-Vilkovisky equation:

\[
\hbar \Delta S + \frac{1}{2} \{S, S\} + I^\vee S = 0, 
\]

in particular if \( I |_B \) is zero then \( S \) is the solution of the non-commutative Batalin-Vilkovisky equation from [B06a],[B06b]

\[
\hbar \Delta S + \frac{1}{2} \{S, S\} = 0
\]

Remark 3 If \( I |_B \neq 0 \), but \( I^2 |_B = 0 \), then \( S + S_{0,2} \) is also a solution to the non-commutative Batalin-Vilkovisky equation, where \( S_{0,2} = \beta(I, \cdot) |_B \) is the quadratic term corresponding to the differential \( I |_B \).

Proof. The proof is straightforward, see e.g. [B09b]. For a trivalent graph \( \Gamma \) and an internal edge \( e \in \text{Edge}(\Gamma) \) consider the three tensors

\[
W_{\Gamma,e}^{[I,H]}, W_{\Gamma,e}^{Id}, W_{\Gamma,e}^{P} \in \text{Symm}((\oplus_{j=1}^{\infty}((\Pi B)^{\otimes j})^{\otimes_j})^{\vee})
\]

which are defined by the same contraction as \( W_{\Gamma} \) except that at the edge \( e \in \text{Edge}(\Gamma) \) I put the tensors

\[
\beta^\vee([I^\vee, H^\vee] u_f, v_f), \beta^\vee(u_f, v_f), \beta^\vee(P^\vee u_f, v_f)
\]

correspondingly instead of \( \beta^\vee_{\Gamma,e} \). Then, from (3)

\[
W_{\Gamma,e}^{P} = W_{\Gamma,e}^{Id} - W_{\Gamma,e}^{[I,H]}
\]

By summing over \( v \in \text{Vert}(\Gamma) \) of the Leibniz rule (1) and noticing that

\[
\beta^\vee(H^\vee I^\vee u_f, v_f) + \beta^\vee(u_f, H^\vee I^\vee v_f) = -\beta^\vee([I^\vee, H^\vee] u_f, v_f)
\]

I get

\[
I^\vee W_{\Gamma} - \sum_e W_{\Gamma,e}^{[I,H]} = 0.
\]

Next I use (2) to substitute in \( W_{\Gamma,e}^{Id} \) the contraction

\[
\langle m(\pi a, \pi b, \cdot) m(\cdot, \pi c, \pi d), \beta^\vee \rangle
\]

corresponding to the internal edge \( e \in \text{Edge}(\Gamma) \) by

\[
(-1)^\vee \langle m(\pi d, \pi a, \cdot) m(\cdot, \pi b, \pi c), \beta^\vee \rangle.
\]

This corresponds to passing from the trivalent ribbon graph \( \Gamma \) to the trivalent ribbon graph \( \Gamma' \) obtained by the standard transformation on the edge \( e \), preserving the overall cyclic order of the flags corresponding to \( \pi a, \pi b, \pi c, \pi d \). This transformation preserves the surface \( \Sigma_\Gamma \) and the distribution of elements of \( \text{Leg}(\Gamma) \) over the boundary components of \( \Sigma_\Gamma \). Therefore the sum of \( W_{\Gamma,e}^{Id} \)
over all internal edges and over the set of trivalent graphs, having the same \( \Sigma_\Gamma \) with same distribution of \( \text{Leg}(\Gamma) \) over the boundary components, is zero:

\[
\sum_{(\Gamma), \Sigma_\Gamma} = \sum_{e \in \text{Edge}(\Gamma)} W_{\Gamma,e}^{1d} = 0.
\]

Notice that \( P^2 = P \) implies that

\[
\beta^\vee (P^\vee u_f, v_f) = \beta^\vee (P^\vee u_f, P^\vee v_f).
\]

Then, from the definition of the Batalin-Vilkovisky operator and the odd Poisson bracket on \( B \) it follows that

\[
\sum_{\Gamma} h^{-\chi(\Sigma_\Gamma)} \Delta W_\Gamma + \frac{1}{2} \left\{ \sum_{\Gamma} h^{-\chi(\Sigma_\Gamma)} W_\Gamma, \sum_{\Gamma'} h^{-\chi(\Sigma_{\Gamma'})} W_{\Gamma'} \right\} = \sum_{(\Gamma), e \in \text{Edge}(\Gamma)} h^{-\chi(\Sigma_e)} W_{\Gamma,e}^{P}
\]

where each term on left hand side corresponds precisely to the right hand side term \( h^{-\chi(\Sigma_e)} W_{\Gamma,e}^{P} \), where \( \Gamma \) is obtained by gluing two legs to form the edge \( e \) from either the single surface or the two surfaces . Notice that the condition, that \( \Delta \) does not get contributions from the neighboring points on the same circle, corresponds precisely to the fact that the resulting surface \( \Sigma_\Gamma \) has always nonempty subsets of \( \text{Legs}(\Gamma) \) on the boundary components.}

**Remark 4** In general, for infinite dimensional cyclic \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra \( A \) the tensor \( \beta^\vee (H^\vee, \cdot) \) belongs to some completion of \( A \otimes_k A \), and extra work is needed in the infinite dimensional situation in order to verify that the tensors \( W_\Gamma \) are well-defined. One can either apply the technique of analysis and construct the propagators as forms with singularity on the diagonal and then verify that the resulting integrals defining \( W_\Gamma \) converge, or the technique of algebraic geometry by working in the \( \mathbb{Z} \)-graded category and with dg- algebras whose resolution of the diagonal is constructed from modules with finite-dimensional graded pieces, for example, for smooth dg-algebras, and this implies that the tensors \( W_\Gamma \) are well-defined, see ??.

**Corollary 5** Given a \( \mathbb{Z}/2\mathbb{Z} \)-graded cyclic \( A_\infty \)-algebra \( B \), if \( B \) has a cyclic d\( (\mathbb{Z}/2\mathbb{Z})g \) associative model \( A \), such that for \( A \) the contractions (4) over trivalent ribbon graphs are well-defined, then the summation over such graphs gives an extension of the cyclic \( A_\infty \)-algebra on \( B \) to the quantum \( A_\infty \)-algebra, i.e. to the solution of the non-commutative Batalin-Vilkovisky equation.

1.1

1.2 Even scalar product.

Assume now that the scalar product on \( A \) is even:

\[
\beta : A^{\otimes 2} \to A.
\]
Then given an odd differentiation $I : A \rightarrow \Pi A$ and an odd self-adjoint operator $H : A \rightarrow \Pi A$, satisfying (3), I construct the tensors $W_{\Gamma}$ for any ribbon trivalent graph by the same contraction (4). The only difference is that in this case, both the three-tensors $m^v$ attached to the vertices and the two-tensors $\beta_H^{\vee,e}$ are odd and an analysis of the corresponding orientation on $\Gamma$, similar to [B09b], shows that $W_{\Gamma}$ belongs to the exterieur power of the space of cyclic tensors

$$W_{\Gamma} \in \text{Symm}(\oplus_{j=1}^{\infty} \Pi(\Pi B^{\otimes j})^{Z/2Z})^\vee$$

For the case of even scalar product on $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V$ I’ve also described a variant of noncommutative Batalin-Vilkovisky formalism in [B06a].

**Theorem 6** The sum over trivalent ribbon graphs $S$ satisfy the equivariant noncommutative Batalin-Vilkovisky equation:

$$\hbar \Delta S + \frac{1}{2} \{S, S\} + I^\vee S = 0,$$

in particular if $|I|_B = 0$ then $S$ is the solution of the non-commutative Batalin-Vilkovisky equation from [B06a], [B06b]

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0$$

### 1.3 General algebras.

Let now $A$ be an arbitrary $\mathbb{Z}/2\mathbb{Z}$-graded algebra, $\dim_k A < \infty$, with an odd differentiation $I : A \rightarrow \Pi A$. This case is reduced to the two previous cases by putting $A = A \oplus (\Pi A)^\vee$ with odd scalar product $\beta$ given by natural odd pairing between $A$ and $(\Pi A)^\vee$, or by putting $A = A \oplus A^\vee$ with even scalar product $\beta$ given by natural even pairing between $A$ and $A^\vee$. Then $\tilde{A}$ is naturally an associative algebra with odd, respectively even, scalar product and the odd differentiation $I$, whose action extends naturally to $\tilde{A}$.

Consider the case of odd scalar product. Suppose that $H$ is an odd operator

$$H : A \rightarrow \Pi A,$$

such that

$$\text{Id} - [I, H] = P$$

(6)

is an idempotent operator $P : A \rightarrow A$,

$$P^2 = P.$$

Then both $H$ and $P$ act naturally on $\tilde{A}$ as self-adjoint operators and I apply to this situation the construction of tensors $W_{\Gamma}$ for ribbon graphs described above. The tensors

$$W^B_{\Gamma} \in \text{Symm}(\oplus_{j=1}^{\infty} ((\Pi B \oplus B^\vee)^{\otimes j})^{Z/2Z})^\vee$$
defined by the contraction (4) are given by the sum over markings

$$\text{Flag}(\Gamma) \to \{\Pi A, A^\vee\}$$

such that for any edge, its two flags are marked differently, and for any vertex there is exactly one flag which is marked by $\Pi A$, with no other extra restrictions. In particular such marking gives an orientation for every edge, from $A^\vee$ to $\Pi A$, and there must be exactly one edge exiting every vertex. There are no other restrictions, in particular the edges can in principle form cycles. The legs of $\Gamma$, which correspond to the points sitting on the boundary of the surface $\Sigma_{\Gamma}$, are also marked as either entries ($B^\vee$) or exits ($B_\vee$). And I define $S^B$ by the summation as above

$$S^B = \sum_{(\Gamma)} \hat{h}_{\chi(\Sigma_{\Gamma})} W^B$$

where the sum is over isomorphism classes of connected trivalent graphs with such markings and with nonempty subsets of legs on every boundary component of $\Sigma_{\Gamma}$.

Similarly I define the tensors $W^B$ in the case of $\tilde{A} = \Pi A^\vee$ with its even pairing. These tensors are from the space of exterior powers of linear functionals on cyclic words consisting of elements from $B^\vee$ and $B_\vee$, respectively from $B_\vee$ and $B^\vee$:

$$W^B \in \text{Sym}(\oplus_{j=1}^\infty \Pi ((\Pi B \oplus \Pi B^\vee)^{\otimes j})^{Z/2Z})^\vee$$

and I define $S^{\Pi B}$ as their sum over ribbon graphs as above.

**Theorem 7** Let $A$ an arbitrary $\mathbb{Z}/2\mathbb{Z}$-graded algebra, $\dim_k A < \infty$, with an odd differentiation $I : A \to \Pi A$ and a homotopy $H$, such that the operator (6) is idempotent. The sums over ribbon graphs $S^B$ and $S^{\Pi B}$ give the solutions to the two variants of the equivariant noncommutative Batalin-Vilkovisky equation in the spaces of symmetric, respectfully exterior powers, of cyclic words, consisting of elements from $\Pi B^\vee$ and $B$, respectfully from $\Pi B^\vee$ and $\Pi B$:

$$h \Delta S^B + \frac{1}{2} \{S^B, S^B\} + I^\vee S^B = 0$$

$$h \Delta S^{\Pi B} + \frac{1}{2} \{S^{\Pi B}, S^{\Pi B}\} + I^\vee S^{\Pi B} = 0.$$

2 Graphs with the insertion of $A_\infty$-tensors.

Assume now that $A$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $A_\infty$-algebra, $\dim_k A < \infty$, as above I relax the condition $d^2 = 0$, and assume that it is simply an odd operator $I : A \to \Pi A$, which together with other structure maps $m_n \in ((\Pi A)^{\otimes n})^\vee \otimes A$, $n \geq 2$, satisfy the standard $A_\infty$-constrains, except the very first, so that $I^2$ is in general nonzero: for any $n \geq 2$

$$Im_n(v_1, \ldots, v_n) - \sum_l (-1)^l m_{n-l}(v_1, \ldots, Iv_l, \ldots v_n) =$$

$$= \sum_{i+j=n+1} (-1)^j m_i(v_1, \ldots, m_j(\ldots), \ldots v_n)$$
I assume first that $A$ has also an invariant odd scalar product $\beta$ so that all tensors 

$$m_n \in ((\Pi A)^{\otimes n+1})^\vee, \beta( m_n(v_1, \ldots, v_n), v_{n+1})$$

are cyclic invariant, the variant without scalar product is reduced as above to this case by taking $A = A \oplus (\Pi A)^\vee$.

Let as above $H$ be an odd selfadjoint operator 

$$H : A \to \Pi A, \quad H^\vee = H$$

such that 

$$Id - [I, H] = P$$

is an idempotent operator $P : A \to A$, whose image $I$ denote by $B$.

Now I define the tensors $W_\Gamma$, by inserting the cyclic tensors $m_n(v) \in ((\Pi A)^{\otimes \text{Flag}(v)})^\vee$ at vertices, as above, where $\Gamma$ is now a ribbon graph, with valency $n(v)$ for any vertex at least three:

$$W_\Gamma(\bigotimes_{l \in \text{Leg}(\Gamma)} a_i) = \left( \bigotimes_{v \in \text{Vert}(\Gamma)} m_n(v) \cdot \left( \bigotimes_{e \in \text{Edge}(\Gamma)} B^\vee_{H, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_i \right)$$

and

$$W_\Gamma \in \text{Symm}(\bigoplus_{j=1}^{\infty} (\Pi B^{\otimes j})_{\Sigma j/2})^\vee$$

Next however, looking carefully at the proof of the equation for $S$ above, one sees that one immediately runs into a problem because of tadpoles, unless the important condition

$$\Delta m_n = 0$$

is imposed, which I assume from now on.

I define now, similarly to above,

$$S = \sum_\{\Gamma\} \chi(\Sigma_\Gamma) W_\Gamma$$

where the sum is over isomorphism classes of connected ribbon graphs with vertices of valency $n(v) \geq 3$, and with nonempty subsets of legs on every boundary component of $\Sigma_\Gamma$.

**Theorem 8** Let the odd operator $I$ and the cyclically invariant tensors $m_n \in ((\Pi A)^{\otimes n+1})^\vee$, $n \geq 2$, satisfy

$$I^\vee m + \{m, m\} = 0$$

$$\Delta m = 0$$

Then the sum over ribbon graphs $S$ satisfy the equivariant noncommutative Batalin-Vilkovisky equation associated with $(B, \beta|_B)$:

$$\hbar \Delta S + \frac{1}{2} \{S, S\} + I^\vee S = 0,$$

in particular if $I|_B = 0$ then $S$ is the solution of the non-commutative Batalin-Vilkovisky equation from [B06a],[B06b]

$$\hbar \Delta S + \frac{1}{2} \{S, S\} = 0$$

8
Proof. The proof is parallel to the above. ■

The same result holds starting from the arbitrary solution to the non-commutative Batalin-Vilkovisky equation and the tensors $W_T$ constructed similarly for stable ribbon graphs. Details will appear elsewhere.

Analogous result holds for arbitrary modular operad.

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