ENUMERATING THE DERANGEMENTS OF AN \( n \)-CUBE VIA MÖBIUS INVERSION

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ABSTRACT. In \( \mathcal{L} \), the semilattice of faces of an \( n \)-cube, we count the number of automorphisms of \( \mathcal{L} \) that fix a given subalgebra – either pointwise or as a subalgebra. By using Möbius inversion we get a formula for the number of derangements on the \( n \)-cube in terms of the Möbius function on the lattice of MR-subalgebras. We compute this Möbius function.

1. Introduction

We are interested in derangements of the \( n \)-cube, ie the automorphisms that fix only the codimension zero face of the cube. Our approach is to consider the face-semilattice of the \( n \)-cube – \( \mathcal{L}_n \) – and use the fact that this semilattice is a Metropolis-Rota implication algebra (MR-algebra), and the automorphism group of this algebra has a well-known structure. Since any automorphism that fixes a set \( X \) of edges of the \( n \)-cube must also fix the entire MR-subalgebra of \( \mathcal{L}_n \) generated by \( X \), we restrict our study to MR-subalgebras of \( \mathcal{L} \).

We begin by fixing a subalgebra \( A \) and consider the ways that \( A \) might be fixed.

\textbf{Definition 1.1.} Let \( A \) be a sub-MR-algebra of \( \mathcal{L} \). Let \( \phi \) be an automorphism of \( \mathcal{L} \).
\begin{itemize}
  \item[(a)] \( \phi \) freezes \( A \) iff \( \phi \upharpoonright A = \text{id} \upharpoonright A \).
  \item[(b)] \( \phi \) fixes \( A \) iff \( \phi[A] = A \).
\end{itemize}

This leads us to the following definition of two subgroups associated with \( A \).

\textbf{Definition 1.2.} Let \( A \) be a sub-MR-algebra of \( \mathcal{L} \).
\begin{itemize}
  \item[(a)] Fr(\( A \)) = \{\( \phi \mid \phi \) freezes \( A \)\};
  \item[(b)] Stab(\( A \)) = \{\( \phi \mid \phi \) fixes \( A \)\}.
\end{itemize}

It is easy to see that both Fr(\( A \)) and Stab(\( A \)) are subgroups of Aut(\( \mathcal{L} \)). We want to count the size of each of these groups. We also want to determine the number of automorphisms that freeze \( A \) only. This we will do by Möbius inversion, as if

\[ f(A) = |\text{Fr}(A)| \]
\[ s(A) = |\{\phi \in \text{Aut}(\mathcal{L}) \mid \phi \upharpoonright A = \text{id}_A \text{ and } \forall B > A \phi \upharpoonright B \neq \text{id}_B\}| \]
\[ g(A) = |\text{Stab}(A)| \]

then we see that

\[ f(A) = \sum_{A \subseteq B} s(B) \]
so that, by Möbius inversion, we have

\[ s(A) = \sum_{A \subseteq B} \mu(A, B) f(B) \]

where \( \mu \) is the Möbius function on the partial order of MR-subalgebras of \( \mathcal{L} \). As a special case we get \( s(\{1\}) \) is the number of derangements on \( \mathcal{L} \).

So we will compute the functions \( f \) and \( g \) and the Möbius function on the partial order of MR-subalgebras of \( \mathcal{L} \). To evaluate \( f \) and \( g \) we compute orbits.

There is a study of derangements of \( n \)-cubes by Chen & Stanley in [7]. That study concentrates on properties of signed permutations (which are the automorphisms of \( \mathcal{L} \)) and obtains an alternative counting of the derangements.

We begin by giving some basic background on cubic and MR implication algebras. The reader is referred to [1] or [8] for a more thorough introduction to cubic and MR-algebras.

1.1. Background Material.

1.1.1. Cubic and MR-algebras. We will give a brief introduction to the basic properties of cubic and MR-algebras. A cubic algebra is an upper semilattice \( \mathcal{L} \) with a binary operator \( \Delta \) on \( \mathcal{L} \) – \( \Delta(x, y) \) corresponding to reflection of \( y \) through the centre of \( x \) – satisfying the following axioms:

a. if \( x \leq y \) then \( \Delta(y, x) \lor x = y \);

b. if \( x \leq y \leq z \) then \( \Delta(z, \Delta(y, x)) = \Delta(\Delta(z, y), \Delta(z, x)) \);

c. if \( x \leq y \) then \( \Delta(y, \Delta(y, x)) = x \);

d. if \( x \leq y \leq z \) then \( \Delta(z, x) \leq \Delta(z, y) \);

Let \( xy = \Delta(1, \Delta(x \lor y, y)) \lor y \) for any \( x, y \) in \( \mathcal{L} \). Then:

e. \( (xy)y = x \lor y \);

f. \( x(yz) = y(xz) \);

In fact the face poset of an \( n \)-cube, \( \mathcal{L}_n \), is also an MR-algebra. We recall the pertinent details.

**Definition 1.3.** An MR-algebra is a cubic algebra satisfying the MR-axiom:

if \( a, b < x \) then

\[ \Delta(x, a) \lor b < x \text{ iff } a \land b \text{ does not exist.} \]

**Definition 1.4.** Let \( \mathcal{L} \) be a cubic algebra. Then for any \( x, y \in \mathcal{L} \) we define the (partial) operation \(^\wedge\) (caret) by:

\[ x ^\wedge y = x \land \Delta(x \lor y, y) \]

whenever this meet exists.

**Lemma 1.5.** If \( \mathcal{L} \) is a cubic algebra then \( \mathcal{L} \) is an MR-algebra iff the caret operation is total.

**Proof.** See [2] lemma 10 and theorem 12. \( \square \)

**Definition 1.6.** Let \( \mathcal{L} \) be a cubic algebra and \( a, b \in \mathcal{L} \). Then

\[ a \preceq b \text{ iff } \Delta(a \lor b, a) \leq b \]

\[ a \simeq b \text{ iff } \Delta(a \lor b, a) = b. \]

**Lemma 1.7.** Let \( \mathcal{L}, a, b \) be as in the definition. Then

\[ a \preceq b \text{ iff } b = (b \lor a) \land (b \lor \Delta(1, a)). \]

**Proof.** See [1] lemmas 2.7 and 2.12. \( \square \)
Lemma 1.8. Let \( \mathcal{L} \) be a cubic algebra and \( a \in \mathcal{L} \). If \( b, c \geq a \) then
\[
b \leq c \iff b \leq c.
\]

Proof. If \( b \leq \Delta(b \lor c) \) then we have \( a \leq c \) and \( a \leq b \leq \Delta(b \lor c) \) and so \( b \lor c = a \lor \Delta(b \lor c, a) \leq c \lor c = c \).

There are a number of representations of MR-algebras. For finite ones the principal three are as the face lattice of an \( n \)-cube; as the poset by signed subsets of \( \{1, \ldots, n\} \); and as the poset of closed intervals of \( \rho([1, \ldots, n]) \). We will consider the latter two briefly.

1.2. Signed Sets.

Definition 1.9. Let \( X \) be a set.

(a) A signed subset of \( X \) is a pair \( (A_1, A_2) \) where \( A_1 \subseteq X \) and \( A_1 \cap A_2 = \emptyset \).

(b) \( S(X) \) is the collection of all signed subsets of \( X \) ordered by reverse pointwise inclusion.

\( S(X) \) is an MR-algebra and if \( \mathcal{L} \) is any finite MR-algebra with \( \text{CoAt}(\mathcal{L}) \) its set of coatoms and \( C \subset \text{CoAt}(\mathcal{L}) \) is such that \( C \cup \{\Delta(1, c) \mid c \in C\} = \text{CoAt}(\mathcal{L}) \) and \( C \cup \{\Delta(1, c) \mid c \in C\} = \emptyset \) then there is a canonical isomorphism of \( \mathcal{L} \) with \( S(C) \) – see [8] for more details.

As one application of this construction we have a simple homomorphism extension result for finite MR-algebras.

Proposition 1.10. Let \( \mathcal{L} \) be a finite MR-algebra. Let \( \phi: \text{CoAt}(\mathcal{L}) \to \text{CoAt}(\mathcal{L}) \) be a \( \Delta(1, \bullet) \)-preserving bijection. Then there is a canonical extension of \( \phi \) to an automorphism of \( \mathcal{L} \).

Proof. Let \( a \) be an atom of \( \mathcal{L} \) and let \( C \) be the coatoms over \( a \). Then \( C \cup \Delta(1, C) = \text{CoAt}(\mathcal{L}) \) and \( C \cup \Delta(1, C) = \emptyset \). Likewise \( \phi[C] \) has the same properties since \( \phi \) preserves \( \Delta(1, \bullet) \). From [6] this implies \( \land \phi[C] \) exists and is an atom \( a' \) of \( \mathcal{L} \). Now \( [a, 1] \) and \( [a', 1] \) are isomorphic as Boolean algebras by an extension of \( \phi \upharpoonright C \) and so we get an extension of this mapping to an automorphism of \( \mathcal{L} \). This extends \( \phi \) also as it has to preserve \( \Delta(1, \bullet) \).

An alternative way to view this proof is via the isomorphism sequence
\[
\mathcal{L} \to S(C) \xrightarrow{\phi} S(\phi[C]) \to \mathcal{L}
\]
where \( S(\phi)((A, B)) = (\phi[A], \phi[B]) \).

1.3. Implication Algebras. Let \( I \) be an implication algebra (ie an upwards closed subset of a Boolean algebra). We define
\[
S(I) = \{ \langle a, b \rangle \mid a, b \in I, a \lor b = 1 \text{ and } a \land b \text{ exists in } I \}
\]
ordered by
\[
\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } a \leq c \text{ and } b \leq d.
\]
This is a partial order that is an upper semi-lattice with join defined by
\[
\langle a, b \rangle \lor \langle c, d \rangle = \langle a \lor c, b \lor d \rangle
\]
and a maximum element \( 1 = (1, 1) \).

We can also define a \( \Delta \) function by
\[
\text{if } \langle c, d \rangle \leq \langle a, b \rangle \text{ then } \Delta((a, b), (c, d)) = \langle a \land (b \rightarrow d), b \land (a \rightarrow c) \rangle.
\]

More properties of this construction are described in [2].
1.4. The Problem. In [2, 4, 9] the automorphism group of $\mathcal{L}$ was investigated. In this paper we wish to consider automorphisms that fix an MR-subalgebra of $\mathcal{L}$.

As described in the introduction we have the two groups Fr($A$) and Fix($A$) (= stab($A$), the stabilizer of $A$) and to find the size of each of these groups we consider the orbit of $A$.

2. Orbits

We will consider the natural group action of Aut($\mathcal{L}$) on subalgebras. First we want to examine an invariant (the type) of an MR-subalgebra.

Definition 2.1. Let $A$ be an MR-subalgebra of $\mathcal{L}$.

CoAt$_n$ is the set of coatoms of $\mathcal{L}$, CoAt($A$) is the set of coatoms of $A$.

For each $a \in$ CoAt($A$) we let $\mathcal{C}_a = \{c \in$ CoAt$_n \mid a \leq c\}$ and $\Gamma_A = \{\mathcal{C}_a \mid a \in$ CoAt($A$)$\}$.

We notice that $a \in$ CoAt($A$) implies $\Delta a \in$ CoAt($A$) and $\mathcal{C}_{\Delta a} = \Delta \mathcal{C}_a$. Thus for each $1 \leq i \leq n$ there are an even number of $\mathcal{C}_a$’s of size $i$. Let $t_i$ be such that $2t_i = ||a \in$ CoAt($A$)$||\mathcal{C}_a|| = i||$.

Definition 2.2. The type of an MR-subalgebra $A$ of $\mathcal{L}$ is the sequence $tp(A) = (t_i \mid 1 \leq i \leq n)$.

The action we are considering is the evaluation action of Aut($\mathcal{L}$) on the partial order of MR-subalgebras of $\mathcal{L}$. Thus we know that

$$|\text{Stab}(A)| = |\text{Aut}(\mathcal{L})| / |\text{Orb}(A)| = 2^n!/|\text{Orb}(A)|$$

and we therefore want to compute the size of the orbits.

Lemma 2.3. If $A$ and $B$ are two MR-subalgebras on $\mathcal{L}$ in the same orbit, then $tp(A) = tp(B)$.

Proof. This is clear.  \qed

Lemma 2.4. Let $c_1$ and $c_2$ be two coatoms of $\mathcal{L}$ and $\phi$: $\{\leftarrow, c_1\} \rightarrow \{\leftarrow, c_2\}$ be an isomorphism. Then $\phi$ extends to an automorphism of $\mathcal{L}$.

Proof. The coatoms of $\{\leftarrow, c_1\}$ are of the form $c \land c_1$ where $c \neq c_1$ and $c \neq \Delta(1, c_1)$ – see [8]. Furthermore $c \neq d$ implies $c \land c_1 \neq d \land c_1$. So we define the extension by defining what happens on the coatoms:

$$\phi'(c) = c' \text{ where } \phi(c \land c_1) = c' \land c_2$$

$$\phi'(c_1) = c_2$$

$$\phi'\Delta(1, c_1)) = \Delta(1, c_2).$$

From [6] we know that $\Delta(c_1, c_1 \land c) = c_1 \land \Delta(1, c)$ and so $\phi'(\Delta(1, c)) = \Delta(1, \phi'(c))$.

From this we extend $\phi'$ to an automorphism of $\mathcal{L}$ as usual.  \qed

Lemma 2.5. Let $v_1$ and $v_2$ be two elements of $\mathcal{L}$ of the same co-rank. Then there is an automorphism of $\mathcal{L}$ taking $v_1$ to $v_2$.

Proof. We proceed by induction on co-rank. Suppose that $v_1$ and $v_2$ are coatoms. There are two cases.

Case 1: $v_1 = \Delta(1, v_2)$ – in this case just take $\Delta(1, \bullet)$ as the automorphism.

Case 2: $v_1 \neq \Delta(1, v_2)$ – then $v_1 \land v_2$ exists and so we have a Boolean algebra $[v, 1]$, where $v$ is any vertex below $v_1 \land v_2$. As $v_1$ and $v_2$ are coatoms of $[v, 1]$ there is a Boolean automorphism of $[v, 1]$ taking $v_1$ to $v_2$. As usual this extends to an automorphism of $\mathcal{L}$.
Now let $c_1 \geq v_1$ be two coatoms and let $\phi$ be an automorphism of $\mathcal{L}$ that takes $c_1$ to $c_2$. In $\mathcal{L}$ the co-rank of $v_2$ equals the co-rank of $\phi(v_1)$ and is one less than the co-rank of $v_1$ in $\mathcal{L}$. By induction, there is an automorphism $\psi$ of $\mathcal{L}$ that takes $\phi(v_1)$ to $v_2$. By lemma 2.4 this extends to an automorphism $\psi'$ of $\mathcal{L}$. Then we have $\psi'(\phi(v_1)) = v_2$ as desired.

**Theorem 2.6.** Suppose that $A$ and $B$ are two MR-subalgebras on $\mathcal{L}$ with $tp(A) = tp(B)$. Then there is an automorphism of $\mathcal{L}$ that takes $A$ to $B$.

**Proof.** Let $tp(A) = \langle t_i \mid 1 \leq i \leq n \rangle$. Let $r = \sum_{i=1}^{n} i t_i$. This must be the corank of a vertex in $A$ or $B$. Let $k = \sum_{i=1}^{n} t_i$ – this is the dimension of $A$ (ie $A \cong \mathcal{L}_k$).

First we find $v_1 \in A$ and $v_2 \in B$ having co-rank $r$, and find an automorphism of $\mathcal{L}$ that takes $v_1$ to $v_2$.

Now the $A$-atoms over $v_1$ go to an antichain in $[v_2, 1]$ that induces a partition of the $\mathcal{L}$-covers of $v_2$. Likewise the $B$-atoms of $[v_2, 1]$ induce a partition of the $\mathcal{L}$-covers of $v_2$. Since $tp(A) = tp(B)$ these partitions are similar and so there is a permutation of the $\mathcal{L}$-covers of $v_2$ taking the first antichain to the second. This induces a Boolean automorphism $\psi$ of $[v_2, 1]$.

Now, let $a_1 \leq v_1$ be an $\mathcal{L}$-vertex and $a_2 = \phi(a_1) \leq v_2$. Then the automorphism $\psi$ can be extended to a Boolean automorphism $\psi'$ of $[a_2, 1]$ and to an automorphism $\tilde{\psi}'$ of $\mathcal{L}$. Then we have $\tilde{\psi}'(\phi(A)) = B$.

From the lemma we need only count the number of ways we get the same type. Suppose that $tp(B) = tp(A)$ and $d$ is a vertex of $B$. We can think of a set of atoms of a subalgebra of $[d, 1]$ as a partition of the $\mathcal{L}$-covers of $d$. If co-rk($d$) = $r$ then rk($d, 1$) = $r$ so these correspond to partitions of $r$. At($A$) gives one such partition $\Pi$ and we need to count the number of similar partitions.

We also need to recall that each MR-subalgebra of rank $k$ has $2^k$ vertices and so the size of the orbit is

$$\frac{|\{d \in \mathcal{L} \mid \text{co-rk}(d) = r\}|}{2^k} \times \text{the number of partitions of } r \text{ similar to } \Pi.$$

The number $|\{d \in \mathcal{L} \mid \text{co-rk}(d) = r\}|$ is known to be

$$|\{d \in \mathcal{L} \mid \text{co-rk}(d) = r\}| = 2^r \binom{n}{r}$$

– see [5] for example. The number of partitions of $r$ similar to $\Pi$ is well known to be

$$\prod_{i=1}^{n} (i!)^{y_i} t_i !.$$

Thus the size of the orbit is

$$2^r \binom{n}{r} \prod_{i=1}^{n} (i!)^{y_i} t_i !.$$

From this we infer that

$$|\text{Stab}(A)| = \frac{2^r n! \prod_{i=1}^{n} (i!)^{y_i} t_i !}{2^{-k} \frac{n^r}{(n-r)!}} = 2^{n+k-r}(n-r)! \prod_{i=1}^{n} (i!)^{y_i} t_i !.$$

There are two ways to compute the size of Fr($A$) – one directly and another by noticing that we have a group homomorphism

$$\rho : \varphi \mapsto \varphi \uparrow A$$

from Stab($A$) to Aut($A$) with kernel Fr($A$), and so we can compute the image of this homomorphism.
We will do both as each gives a viewpoint on automorphisms that we find interesting.

We can partition CoAt(A) into the sets $\Gamma_i = \{a \mid \text{co-rk}(a) = i\}$. We notice that $|\Gamma_i| = 2t_i$ and $a \in \Gamma_i$ implies $\Delta a \notin \Gamma_i$.

**Lemma 2.7.** Let $\psi \in \text{Aut}(A)$. Then $\psi \in \text{Im}(\rho)$ iff $\psi[\Gamma_i] = \Gamma_i$ for all $i$.

**Proof.** Let $\psi = \rho(\Psi)$ for some $\Psi \in \text{Stab}(A)$. The $\Psi$ preserves co-rank and takes $A$ to $A$, so it must take $\Gamma_i$ into itself. As it is one-one on $\Gamma_i$ it is also onto.

For the converse, suppose that $\psi \in \text{Aut}(A)$ is such that $\psi[\Gamma_i] = \Gamma_i$ for all $i$.

Let $N_i \subseteq \Gamma_i$ be maximal $\Delta$-independent.

We claim that $\mathcal{C}_{N_i}$ (ie the set of coatoms above some element of $N_i$) is also $\Delta$-independent in $\mathcal{C}_{\Gamma_i}$. Indeed, if $c \in \mathcal{C}_{N_i}$, there is some $a \in N_i$ and $\Delta c \geq b \in N_i$ then $c \in \mathcal{C}_a \cap \mathcal{C}_{ab}$. As $\Delta b \in N_i$ and $a \neq \Delta b$ implies $\mathcal{C}_a \cap \mathcal{C}_{ab} = \emptyset$ we must have $a = \Delta b$ — contradicting the assumption that $N_i$ is $\Delta$-independent.

Let $N_i' = \psi[N_i] \subseteq \Gamma_i$. This is also $\Delta$-independent and so is $\mathcal{C}_{N_i'}$.

$\mathcal{C}_{N_i}$ is the disjoint union of the set $\{\mathcal{C}_a \mid a \in N_i\}$ and we know that $|\mathcal{C}_a| = |\psi[\mathcal{C}_a]| = i$ so we can find a bijection from $\mathcal{C}_{N_i}$ to $\mathcal{C}_{N_i'}$ that takes $\mathcal{C}_a$ to $\mathcal{C}_{\psi(a)}$.

Finally we can patch these bijections to get a bijection between the disjoint union of the set $\{\mathcal{C}_{N_i} \mid 1 \leq i \leq n\}$ and the disjoint union of the set $\{\mathcal{C}_{N_i'} \mid 1 \leq i \leq n\}$. This is now a bijection between two maximal $\Delta$-independent subsets of CoAt($\mathcal{L}$) and so extends to an automorphism of $\mathcal{L}$.

We note that it takes $a \in \Gamma_i$ to $\psi(a)$ and so it restricts to $\psi$. 

**Lemma 2.8.** Let $P$ be a partition of CoAt$_\mathcal{L}$ such that for all $X \in P$ $\Delta[X] = X$. Let $\text{Aut}_P = \{\varphi \in \text{Aut}(\mathcal{L},_0) \mid \forall X \in P \varphi[X] = X\}$. Then

$$\text{Aut}_P \cong \prod_{X \in P} \text{Aut}(\mathcal{L},_{X/2}).$$

**Proof.** We first observe that if $\Delta[X] = X$ for some set of coatoms $X$ of $\mathcal{L}$, then $A_X = \{x \in \mathcal{L} \mid \text{the coatoms above } x \text{ are all in } X\}$ is an MR-subalgebra isomorphic to $\mathcal{L},_{X/2}$ — by taking $X' \subseteq X$ so that $X' \cap \Delta[X'] = \emptyset$ and $X' \cup \Delta[X'] = X$ and the mapping $x \mapsto (\{z \in X' \mid x \leq z\}, \{z \in \Delta[X'] \mid x \leq z\})$ is a cubic isomorphism to the algebra of signed subsets of $X'$, ie $\mathcal{L},_{X/2}$.

Thus, if $\varphi \in \text{Aut}_P$ then $\varphi' | A_X$ is a cubic automorphism of $A_X$.

This mapping is onto — as if $\langle \varphi_X | X \in P \rangle \in \prod_{X \in P} \text{Aut}(A_X)$, then we consider the mapping defined on CoAt$_\mathcal{L}$ by

$$x \mapsto \varphi_X(x) \text{ if } x \in X.$$  

This is a $\Delta$-preserving mapping of the coatoms and so lifts to an automorphism of $\mathcal{L}$, which restricts to $\varphi_X$ on each $A_X$.

From these two lemmas we see that if $P = \{\Gamma_i \mid 1 \leq i \leq n\}$ is the partition of CoAt(A) described above, then

$$|\text{Im}(\rho)| = |\text{Aut}_P| = \prod_i |\text{Aut}(\mathcal{L},_i)| = \prod_i 2^i t_i ! = 2^k \prod_i t_i !.$$ 

Thus we have

$$|\text{Fr}(A)| = \frac{2^{n+k-r}(n-r)! \prod_{i=1}^n (i)! t_i !}{2^k \prod_i t_i !} = 2^{n-r}(n-r)! \prod_{i=1}^n (i)! t_i !.$$ 

Here is another way to see this result — consider the sets $\mathcal{C}_a$ as $a$ varies over CoAt(A), and $D = \text{CoAt}_\mathcal{L} \setminus \bigcup_a \mathcal{C}_a$. Notice that $\Delta[D] = D$, and $\Delta[\mathcal{C}_a] = \mathcal{C}_{\Delta a}$ for all $a$. 

Then \( \varphi \) freezes \( A \) iff \( \varphi[C_a] = C_a \) for all \( a \in \text{CoAt}(A) \). This of course, implies \( \varphi \uparrow D \) is a \( \Delta \)-preserving mapping from \( D \) to itself.

Also we must have \( \varphi \uparrow C_{\text{co}} = \Delta(\varphi \uparrow C_a) \Delta \) – as \( \varphi \) preserves \( \Delta \).

Hence, if \( M \) is a maximal \( \Delta \)-independent set of coatoms of \( A \) then \( \varphi \) is completely determined by its action on \( C_a \) for \( a \in M \) and its action on \( D \). \( \varphi \) can be any permutation of \( C_a \) (for \( a \in M \)) and (as above) any \( \Delta \)-preserving bijection of \( D \) – of which there are \( 2^{n-r}(n-r)! \) such mappings. Hence there are

\[
2^{n-r}(n-r)! \prod_i (i!)^{t_i}
\]

such \( \varphi \) – as computed above.

3. The Möbius function on Implication lattices

As the first step in computing the Möbius function on the poset of MR-subalgebras we will look at implication sublattices of a Boolean algebra. The MR-subalgebra computation will then be reduced to this case.

Let \( B \) be a finite Boolean algebra. We will assume that \( B \cong \mathcal{P}(n) \).

**Definition 3.1.** An implication subalgebra of \( B \) is a subset closed under \( \rightarrow \).

An implication sublattice of \( B \) is a subset closed under \( \rightarrow \) and \( \land \).

In [3] we found that this function is given by the formula

\[
\mu(A, B) = (-1)^{n-k}(n-k)! \prod_{i=1}^{n} (-1)^{i-1}(i-1)!^{t_i}
\]

(1)

\[
= (-1)^{n-d}(n-k)! \prod_{i=1}^{n} [(i-1)!]^t_i
\]

as \( \sum_{i=1}^{n} (i-1)t_i = k - d \), and

(3) \( \mu(\{1\}, B) = n! \).

4. The Möbius function on MR-subalgebras

The way we will compute the Möbius function on the poset of MR-subalgebras is similar to that of the last section.

We begin by showing how \( \mu(M, N) \) can always be determined by knowing \( \mu(\{1\}, L_k) \) for all \( k \).

Then we represent the poset in terms of implication sublattices of \([0, 1]\) together with some extra information. Then we define a closure operator on this new representation and finally reduce the problem to the poset of implication sublattices of \([0, 1]\).

Let \( A \) and \( B \) be two MR-subalgebras of \( L \). As \( B \) is already an MR-algebra and \( B \) so is the face lattice of a (possibly) smaller cube, we may assume that \( B = L \). We consider a reduction showing that we may also assume that \( A = \{1\} \).

Fix an atom \( 0 \) of \( L \) below some \( A \)-atom \( a_A \). Let \( A \subseteq C \subseteq L \) be any intermediate subalgebra. Then \( C \) is determined by knowing \( C \cap [a_A, 1] \) and \( C \cap \leftarrow, a_A \). This shows us that

\[
[A, L] \cong [B \upharpoonright A \cap [a_A, 1] \subseteq B \subseteq [a_A, 1] \text{ is a Boolean algebra}] \\
\times [C \mid C \text{ is an MR-subalgebra of } \leftarrow, a_A].
\]
4.1. **Locator Pairs.** We consider MR-subalgebras in \([1, \mathcal{L}]\) using implication sublattices of \([0, 1]\). In this section we develop a way of describing subalgebras that leads to a clearer picture of the partial order.

**Definition 4.1.** A locator-pair is a pair of \((c, B)\) where \(B\) is an implication sublattice of \([0, 1]\) and \(c \geq \text{min} B\).

Locator pairs will be used to facilitate counting.

**Lemma 4.2.** Locator pairs correspond to subalgebras of \(\mathcal{L}\).

**Proof.** Let \(A\) be any subalgebra, let \(a_4\) be a vertex of \(A\). Let \(A_4 = \beta_0[A]\) and \(c = 0 \lor a_4\). Then \((c, A_4)\) is a locator-pair. We can recover \(A\) from this locator-pair by noting that if \(a_4 = \text{min} A\), then \(a = \Delta(c, a_4)\) is a vertex of \(A\) and we can move \(A_4\) to a \(g\)-filter of \(A\) using the mapping

\[
x \mapsto (x \lor a) \land (\Delta(1, x) \lor a)
\]

(as usual), and as this is a \(g\)-filter for \(A\) we can recover \(A\). \(\square\)

The implication sublattice \(A_4\) is uniquely determined by \(A\), but the other element is not – as we can choose many \(A\)-vertices.

**Definition 4.3.** Let \(A\) be an MR-subalgebra of \(\mathcal{L}\). A locator-pair \((c, B)\) that determines \(A\) is said to locate \(A\).

By an abuse of notation we will often write this as \((c, B) = A\).

**Definition 4.4.** Let \((c_1, B_1)\) and \((c_2, B_2)\) be two locator-pairs. Let \(A_i\) be the corresponding MR-subalgebra of \(\mathcal{L}\). Then

\[
(c_1, B_1) \leq (c_2, B_2) \text{ iff } A_1 \subseteq A_2 \\
(c_1, B_1) \cong (c_2, B_2) \text{ iff } A_1 = A_2.
\]

It is easy to see that \(\cong\) is an equivalence relation. We want to characterize \(\leq\) on locator-pairs more carefully.

**Definition 4.5.** Let \(c, d\) be in \(\mathcal{L}\). Then \(c +_a d\) is the Boolean sum of \(c \lor a\) and \(d \lor a\) in \([a, 1]\).

Note that in a Boolean algebra we have \(c +_a d = (c + d) \lor a\). Hence in a cubic algebra or implication algebra, if \(a_1 \leq a_2\) then \(c +_{a_1} d = (c +_{a_1} d) \lor a_2\).

We need the following technical lemma

**Lemma 4.6.** Let \(c_1\) and \(c_2\) be two elements of \(\mathcal{L}\) such that \(c_1 \land c_2\) exists. Let \(a \leq c_1 \land c_2\). Then

(a) if \(a \leq b \leq c_1\) then

\[
\Delta(c_1, b) \lor \Delta(c_2, a) = \Delta(c_1, b) \lor \Delta(c_2 \lor b, b);
\]

(b) if \(a \leq x\) then

\[
\Delta(x \lor c_1 \lor c_2, \Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x)) = c_1 +_a c_2.
\]

**Proof.** Without loss of generality we may work in an interval algebra and take \(a = [0, 0]\). Then we have \(c_i = [0, c_i], b = [0, b]\) and \(x = [0, x]\).
(a) 
\[
\Delta(c_1, b) \lor \Delta(c_2, a) = \Delta([0, c_1], [0, b]) \lor \Delta([0, c_2], [0, 0]) \\
= [c_1 \land \overline{b}, c_1] \lor [c_2, c_2] \\
= [c_1 \land c_2 \land \overline{b}, c_1 \lor c_2].
\]
\[
\Delta(c_1, b) \lor \Delta(c_2 \lor b, b) = \Delta([0, c_1], [0, b]) \lor \Delta([0, c_2 \lor b], [0, b]) \\
= [c_1 \land \overline{b}, c_1] \lor [c_2 \land \overline{b}, c_2 \lor b] \\
= [c_1 \land c_2 \land \overline{b}, c_1 \lor c_2 \lor b] \\
as \ c_1 \geq b.
\]

(b) 
\[
\Delta(x \lor c_1, x) = \Delta([0, x \lor c_1], [0, x]) \\
= [c_1 \land \overline{x}, c_1 \lor x] \\
\Delta(x \lor c_1 \lor c_2, x \lor c_1, x \lor c_2) \\
\Delta(x \lor c_2, x) = \Delta([0, x \lor c_1 \lor c_2], [c_1 \land \overline{x}, c_1 \lor x] \lor [c_2 \land \overline{x}, c_2 \lor x]) \\
= \Delta([0, x \lor c_1 \lor c_2], [c_1 \land c_2 \land \overline{x}, c_1 \lor c_2 \lor x]) \\
\Delta([0, x \lor c_1 \lor c_2], (x \lor c_1) \land (x \lor c_2)) \\
= 0, x \lor (c_1 + c_2) \\
\Delta([0, x \lor c_1 \lor c_2], (x \lor c_1) \lor (x \lor c_2)) \\
\Delta([0, x \lor c_1 \lor c_2], (x \lor c_1) \lor (x \lor c_2)) \\
= [0, c_1 + c_2].
\]

\[\Box\]

**Theorem 4.7.** Let \(\langle c_1, B_1 \rangle\) locate \(A_1\) and \(\langle c_2, B_2 \rangle\) locate \(A_2\). Then 
\(A_1 \subseteq A_2\) iff \(B_1 \subseteq B_2\) and \(c_1 \lor c_2 \in B_2\)

where \(a_i = \min B_i\).

**Proof.** Suppose that \(A_1 \subseteq A_2\). Then \(B_i\) is obtained from \(A_i\) as the image of the mapping \(x \mapsto \Delta(x \lor 0, x)\) and so clearly \(B_1 \subseteq B_2\).

The locator \(c_i\) has the property that \(\Delta(c_i, a_i)\) is an atom of \(A_i\). As \(A_1 \subseteq A_2\) this implies \(\Delta(c_1, a_1) \lor \Delta(c_2, a_2) \in A_2\). Hence \(\Delta(\Delta(c_1, a_1) \lor \Delta(c_2, a_2) \lor a_2, \Delta(c_1, a_1) \lor \Delta(c_2, a_2)) \in B_2\). Now 
\[
\Delta(c_1, a_1) \lor \Delta(c_2, a_2) \lor a_2 = \Delta(c_1, a_1) \lor c_2 \\
as \Delta(c_2, a_2) \lor a_2 = c_2 \\
\geq \Delta(c_1, a_2) \lor c_2 \\
= c_1 \lor c_2
\]

As 
\[
\Delta(c_1, a_1) \lor c_2 \leq c_1 \lor c_2
\]
we have
\[ \Delta(c_1, a_1) \lor \Delta(c_2, a_2) \lor a_2 = c_1 \lor c_2. \]
\[ \Delta(c_1, a_1) \lor \Delta(c_2, a_2) = \Delta(c_1, a_1) \lor \Delta(c_2 \lor a_1, a_1) \]
by lemma \[4.6\] (a).

Therefore
\[ \Delta(\Delta(c_1, a_1) \lor \Delta(c_2, a_2) \lor a_2), \]
\[ \Delta(\Delta(c_1, a_1) \lor \Delta(c_2, a_2)) = \Delta(c_1 \lor c_2 \lor a_1, \Delta(c_1, a_1) \lor \Delta(c_2 \lor a_1, a_1)) \]
\[ = c_1 \lor c_2 \]
by lemma \[4.6\] (b).

Now let us suppose that \(B_1 \subseteq B_2\) and \(c_1 +_{a_1} c_2 \in B_2\). It suffices to show that if \(x \in B_1\) then \(\Delta(x \lor \Delta(c_1, a_1), x) \in A_2\) – as the set of such elements forms a set that generates \(A_1\).

First we note that \(x \geq a_1\) so that \(x \lor \Delta(c_1, a_1) = x \lor c_1\). Also we know \(x \in B_2\) so that
\[ \Delta(x \lor c_2, x) = \Delta(x \lor \Delta(c_2, a_2), x) \in A_2\]
and furthermore \(\Delta(\Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x), \Delta(x \lor c_2, x)) = \Delta(x \lor c_1, x)\). Thus it suffices to show that \(\Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x) \in A_2\).

For this it is sufficient to show that the preimage over \(a_2\) is in \(B_2\) – i.e. \(\Delta(\Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x) \lor a_2, \Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x)) \in B_2\).

\[ \Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x) \lor a_2 = x \lor c_1 \lor c_2 \]
so that
\[ \Delta(\Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x) \lor a_2), \]
\[ \Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x)) = \Delta(x \lor c_1 \lor c_2, \Delta(x \lor c_1, x) \lor \Delta(x \lor c_2, x)) \]
\[ = c_1 \lor c_2 \]
by lemma \[4.6\]
\[ = (c_1 +_{a_1} c_2) \lor x \]
\[ \in B_2 \]
as \(c_1 +_{a_1} c_2 \in B_2\) and \(x \in B_1 \subseteq B_2\).

\[ \square \]

4.2. A closure operator. We will finally compute the Möbius function we want through an appeal to the following theorem about closure operators – see [10] Proposition 2.1.19.

**Theorem 4.8.** Let \(X\) be a locally finite partial order and \(x \mapsto \overline{x}\) be a closure operator on \(X\). Let \(\overline{X}\) be the suborder of all closed elements of \(X\) and \(y, z\) be in \(X\). Then
\[ \sum_{\overline{y} = \overline{z}} \mu(y, x) = \begin{cases} \mu_{\overline{X}}(y, z) & \text{if } y \in \overline{X} \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** See [10].

There is several closure operators of interest that naturally apply to locator-pairs by modifying the second component. We will consider only one of them.

If \(\langle c, B \rangle\) is a locator-pair we define \(B^*\) to be the subalgebra of \([\min B, 1]\) generated by \(\{c\} \cup B\). Then we have \(\langle c, B^* \rangle = \langle c, B \rangle\).

**Lemma 4.9.** \(\langle c, B \rangle \mapsto \langle c, B^* \rangle\) is a closure operator on locator-pairs.
Proof. Since $B^* = B$ we trivially have $\langle c, B \rangle = \langle c, B \rangle$.
\[ \langle c, B \rangle \leq \langle c, \overline{B} \rangle \text{ as } B \subseteq B^* \text{ and } c +_B c = b \in B^*. \]
If $\langle \alpha, B_1 \rangle \leq \langle \alpha, B_2 \rangle$ then $B_1 \subseteq B_2$ and so $B_1 \leq B_2$. Also $c_1, c_2 \in B_2^*$ and therefore $c_1 = (c_1 +_B c_2) +_B c_2$ is in $B_2^*$. Hence $B_1^* \subseteq B_2^*$. (The $b = \min B_i$.
\[ \square \]

Lemma 4.10. $\langle c, B \rangle$ is closed iff $\langle c, B \rangle \approx \langle \min B, B \rangle$.

Proof. Let $b = \min B$.

It is clear that $\langle c, B \rangle$ is closed iff $c \in B$. Also, if $\langle c, B \rangle \approx \langle c', B \rangle$ and $c' \in B$ then $c \in B$ -- since $c = (c +_B c') +_B c'$. Therefore $\langle c, B \rangle \approx \langle b, B \rangle$ implies $c \in B$ and so $\langle c, B \rangle$ is closed.

Conversely, if $c \in B$ then $c +_B b = c \in B$ and so $\langle c, B \rangle \approx \langle b, B \rangle$.

This lemma tells us that the poset of closed pairs is the same as the poset of implication sublattices of $[0, 1]$ -- that we have considered elsewhere [7BO: ImpMob].

We also note that $\langle 1, \{1\} \rangle$ is closed and locates $\{1\}$, and $\langle 0, \{0, 1\} \rangle$ is closed and locates $\mathcal{L}_n$.

5. Getting the Möbius Function

We need to count the subalgebras whose closure is $\mathcal{L}$ rather carefully.

We see that $\langle c, B \rangle \approx \mathcal{L}$ iff $\min B = 0$ and $B^* = [0, 1]$. Thus $B$ is actually a Boolean subalgebra of $[0, 1]$. Let $a_1, \ldots, a_m$ be the atoms of $B$. The atoms of $B^*$ are the non-zero elements of
\[ \{a_i \land c \mid 1 \leq i \leq m\} \cup \{a_i \land \neg c \mid 1 \leq i \leq m\}. \]
As these must be the atoms of $[0, 1]$ we see that every atom of $B$ must be either a $[0, 1]$-atom or the join of two such atoms.

Let $k$ be the number of $B$-atoms that are also $[0, 1]$-atoms and $\ell = m - k$. Then we have $k + 2\ell = n$ and the pair $(c, B)$ is determined by the arrangement of $[0, 1]$-atoms $(S, P)$ where
\[ \begin{align*}
S &= \{a \mid a \text{ is a } B\text{-atom}\} \\
P &= \{(a, b) \mid a \neq b \text{ and } a \lor b \text{ is a } B\text{-atom}\}. 
\end{align*} \]

The number $k + \ell$ is the dimension of $B$ and the next lemma shows that this naturally determines a partition of the locators we are interested in. We also note that $k + \ell = m$ and $k + 2\ell = n$ if $k = 2m - n$ and $\ell = n - m$, so the pair is determined by the dimension of $B$ -- as $n$ is fixed. As we need $k, \ell \geq 0$ this also implies $n \geq m \geq \lceil \frac{k}{2} \rceil$.

Lemma 5.1. The intervals $\mathbb{I}_1 = \langle 1, \{1\} \rangle, \langle c_1, B_1 \rangle$ and $\mathbb{I}_2 = \langle 1, \{1\} \rangle, \langle c_2, B_2 \rangle$ are order-isomorphic iff $B_1$ and $B_2$ are isomorphic.

Proof. Suppose that $\mathbb{I}_1 \approx \mathbb{I}_2$. Let $A_i$ be the MR-subalgebra located by $\langle c_i, B_i \rangle$. Let $a_i$ be an $A_i$-vertex and let $j > s_1 > \cdots > s_k = a_i$ be a maximal chain in $A_i \cap [a_i, 1]$. Then we have a maximal chain of subalgebras of $A_i$ induced by the intervals $[s_j, 1] \cap A_i$ -- this is a maximal chain as the $A_i$-rank goes up by one as $j$ increases by one.

From this we see that the rank of $A_1$ is equal to that of $A_2$. As the rank of $B_i$ equals that of $A_i$ we have $B_1 \approx B_2$.

Conversely, if $B_1 \approx B_2$ then we have $A_1 \approx \mathcal{I}(B_1) \approx \mathcal{I}(B_2) \approx A_2$ from which the result is clear. \[ \square \]

Now we need to count the number of MR-algebras with locator of a particular dimension.

Lemma 5.2. There are $2^n - m S(n, m)$ MR-subalgebras of $\mathcal{L}$ with dimension $m$ and locator a subalgebra of $[0, 1]$. 
Remark 5.1. $S(n, m)$ is a Stirling number of the second kind, counting the number of partitions of a set of size $n$ into $m$ pieces.

Proof. There are $S(n, m)$ Boolean subalgebras of $[0, 1]$ of dimension $m$ – since each subalgebra corresponds to a partition of the atoms of $[0, 1]$ into $m$ pieces.

Given such a subalgebra, we see that $(c, B) \sim (c', B)$ iff $c + c' \in B$ iff $c$ and $c'$ are in the same coset in $[0, 1]$ relative to $B$. Thus the number of cosets of $B$ equals the number of MR-subalgebras located by $B$, i.e., $2^{n-m}$.

Hence there are $2^{n-m}S(n, m)$ such MR-subalgebras. □

Now we are able to compute the Möbius function. First a small lemma.

Lemma 5.3. Let $M$ be the partial order of MR-subalgebras of $\mathcal{L}$. Then $\overline{M}$ is isomorphic to the partial order of implication sublattices of $[0, 1]$.

Proof. We know that $(c, B) = (c, B)$ iff $(c, B) \sim (\text{min } B, B)$. Thus the mapping that takes a closed element to the second component of a locator pair is an order isomorphism. □

Corollary 5.4.

$$\mu_M(\{1\}, \mathcal{L}) = n!.$$ 

Proof. Noting that $\{1\}$ is closed we can apply the lemma and equation (3). □

Using this result and theorem 4.8 we see that

$$\sum_{(c, B) = \mathcal{L}} \mu(\{1\}, (c, B)) = n!$$

and so

$$\mu(\{1\}, \mathcal{L}_n) = n! - \sum_{(c, B) = \mathcal{L}_{n=1}} \mu(\{1\}, (c, B))$$

$$= n! - \sum_{m=\lceil \frac{n}{2} \rceil}^{n-1} \sum_{\dim B = m} \mu(\{1\}, (c, B))$$

$$= n! - \sum_{m=\lceil \frac{n}{2} \rceil}^{n-1} 2^{n-m}S(n, m)\mu(\{1\}, \mathcal{L}_m).$$

Now let $a_n = \mu(\{1\}, \mathcal{L}_n)$ so we can rewrite this as

$$\frac{a_n}{2^n n!} = \frac{1}{2^n} - \sum_{m=\lceil \frac{n}{2} \rceil}^{n-1} (m!S(n, m)) \frac{a_m}{2^m m!}$$

or as

$$\sum_{m=\lceil \frac{n}{2} \rceil}^{n} S(n, m) \frac{a_m}{2^m} = n!.$$
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