On Stochastic Quantisation of Supersymmetric Theories

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Abstract

We explain how stochastic TQFT supersymmetry can be made compatible with space supersymmetry. Taking the case of $N = 2$ supersymmetric quantum mechanics, (the proof would be the same for the Wess–Zumino model), we determine the kernels that ensure the convergence of the stochastic process toward the standard path integral, under the condition that they are covariant under supersymmetry. They depend on a massive parameter $M$ that can be chosen at will and modifies the course of the stochastic evolution, but the infinite stochastic time limit of the correlation functions is in fact independent on the choice of $M$. 

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1 introduction

Giving the existence of stochastic quantisation, an interesting problem is to understand the way stochastic evolution can preserve space supersymmetry, or, more mathematically, how physical space supersymmetry and stochastic TQFT supersymmetry can be made compatible.

Stochastic quantisation computes the Euclidean quantum correlation functions of a given field theory in a $d$-space $\{x^\mu\}$ by interpreting this space as the $\tau = \infty$ boundary of a $d+1$-space, $\{x^\mu\} \to \{x^\mu, \tau\}$. The stochastic time $\tau$ is in fact a bulk coordinate and stochastic quantisation gives a microscopic understanding of the Euclidean path integral measure $\exp -S/\hbar$ as an equilibrium distribution at $\tau = \infty$, analogously as Langevin equations determine the thermal equilibrium Boltzmann formula [1][2]. It is also a fact that a BRST-TQFT localisation procedure can enforce Langevin equations in a supersymmetric way so that the whole apparatus of stochastic quantisation can be expressed as a TQFT in $d+1$ dimensions [3]. This gives a good handle on the arguments in [4].

The drift force of the stochastic process is basically proportional to the Euclidean equations of motion of the theory on the boundary, modulo some other forces that can possibly improve the convergence of the process. It is often the case that kernels that realise factors of multiplication of the equations of motion must be carefully chosen to improve, and, sometimes, to define, the convergence of the stochastic process [1]. For maintaining space supersymmetry, if any, appropriate kernels are needed. The goal of this note is to explain how they can be determined.

It is natural to ask how symmetries, which are realised on the boundary at $\tau = \infty$, are represented in the bulk, that is, at finite values of the stochastic time. For gravity and gauge symmetries, and even string theory, one solves the question rather elegantly in the framework of equivariant cohomology [3][5]. It leads one to define a stochastic equivariant topological BRST invariance. One then obtains a universal frame for the stochastic quantisation of any given theory with a gauge invariance, including a method to fix the gauge invariance in the bulk with a BRST symmetry construction.

The case of theories with space supersymmetry is puzzling. The equations of motion have a different supersymmetry covariance than the fields. Thus, they cannot be used as drift forces without introducing appropriate kernels to modify consistently their covariance. In this note, we study in details the case of theories with global supersymmetry. We take the case of the simple $N = 2$ supersymmetric mechanics and explain how the supersymmetry of the theory on the boundary can be enforced in the bulk. The case of the Wess and Zumino model can be handled in exactly the same way, modulo some notational complications, and we let the reader make the correspondence. The superfield formalism is handy to get Langevin equations that are compatible with supersymmetry for all values of the stochastic time. The method we follow to compute the appropriate kernels and enforce supersymmetry in the stochastic bulk can be actually applied to more delicate cases, including when supersymmetry mixes with gauge symmetries, as will be shown in a separate publication.

2 A reminder of kernels in stochastic quantisation

Consider a quantum field theory of fields $\varphi_a(x)$ with a local action $S[\varphi] = \int dx L(\varphi_a(x))$, where $a$ is some index that labels the fields, and the $x$’s denote Euclidean coordinates in $d$ dimensions. Stochastic quantisation introduces the stochastic time coordinate $\tau$, with $\varphi_a(x) \to \varphi_a(x, \tau)$. The $\tau$ evolution of the field $\varphi_a(x, \tau)$ is defined by a Langevin process, such that the path integral weight $\exp -S[\varphi]$ is its equilibrium distribution, when it exists. Given a correlation function $\langle f(\varphi(x)) \rangle$ in the space $\{x^\mu\} = \{x^\mu, \tau = \infty\}$, which can be computed by the usual Euclidean path integral on the space $\{x^\mu\}$,
The fundamental property of stochastic quantisation is that

$$< f(\varphi) > \equiv \int [d\varphi]_x f(\varphi) \exp -S[\varphi] = \lim_{\tau \to \infty} << f(\varphi^0(x, \tau) >>^{K,S}$$  

(1)

The $(x, \tau)$ dependent correlators $<< f(\varphi^0(x, \tau)) >>^{K,S}$ are computed as an average over the noises $\eta_a(x, \tau)$. Here, $\varphi^0$ is computed in function of the noises $\eta_a$ by solving the following Langevin differential equation with some any given initial condition $\varphi^0(x, \tau = \tau_0) = \varphi_0(x)$,

$$\frac{d\varphi_a}{d\tau} = -K_{ab} \frac{\delta S}{\delta \varphi_b} + \eta_a$$  

(2)

The dependence on $\hbar$ is through a rescaling of the noise $\eta$. The correlations of the noise are in fact defined by the following formula, true for any given sufficiently regular functional $f$

$$<< f(\varphi^0) >>^K \equiv \int [d\eta_a]_x f(\varphi^0) \exp -\frac{1}{2} \int dxdt \eta_a K_{ab}^{-1} \eta_b$$  

(3)

It is often the case that $K_{ab}$ must be different from the trivial identity to ensure a well-behaved drift force in the r.h.s of the Langevin equation [4], and a proper behaviour of the stochastic time evolution at $\tau \to \infty$. One must indeed have a uniformly attractive force in the Langevin process for all components of fields. Thus $\eta$ is not necessarily a white noise, and this happens for instance for the quantisation of a spinor [4].

The dependence on $K_{ab}$ evaporates when one reaches the limit $\tau = \infty$. Indeed, given different kernels $K$ that ensure a convergence of the stochastic process, the value of $\lim_{\tau \to \infty} << f(\varphi(x, \tau)) >>^{K,S}$ is independent on the choice of $K$, and is given by the standard part integral formula in the r.h.s. of Eq. (1).

This property can be demonstrated from the Fokker–Planck equation that is implied by the $K$ dependent Langevin equation. Indeed, the Langevin equation (2) implies the existence of a (Fokker–Planck) kernel $P^{S,K}(\varphi(x), \tau)$, which permits one to computes the time evolution of equal time stochastic correlators as follows

$$<< \varphi(x_1, \tau), .., \varphi(x_n, \tau) >> = \int [d\varphi]_x \varphi(x_1), .., \varphi(x_n) P^{S,K}(\varphi, \tau)$$  

(4)

Here, $P^{S,K}$ is the solution of the Fokker–Planck equation

$$\frac{\partial P^{S,K}(\varphi, \tau)}{\partial \tau} = \int dx \frac{\delta}{\delta \varphi_a(x)} K_{ab} \left( \frac{\delta}{\delta \varphi_b(x)} + \frac{\delta S}{\delta \varphi_b(x)} \right) P^{S,K}(\varphi, \tau)$$  

(5)

This equation needs an initial condition, eg: $P^{S,K}(\varphi(x), \tau = \tau_0) = \delta(\varphi(x) - \varphi_0(x))$. For sufficiently regular theories, one can check an exponential damping $\sim O(exp - \tau)$ in the dependence on the initial condition $\varphi_0(x)$ when $\tau \to \infty$. The details of the evolution depend on the choice of $K$, but not the limit. If a stationary distribution $\lim_{\tau \to \infty} P^{S,K}(\varphi(x, \tau) exists, Eq. (5) implies

$$P^{S,K}(\varphi, \tau = \infty) = \exp -\int S[\varphi]$$  

(6)

For a free Dirac action $S = \int dx \bar{\chi}(\gamma \cdot \partial - m) \lambda$ one can use the following Langevin equation $\dot{\lambda} = (\gamma \cdot \partial + m)(\gamma \cdot \partial - m) \lambda + \eta_\lambda$, with fermionic noise correlation functions $<< \eta_\lambda(x, \tau) \eta_\lambda(x, \tau) >> = \delta(x-x')\delta(\tau - \tau')$. It gives the correct result $P^{K=\gamma \cdot \partial + m}(\lambda, \tau = \infty) = \exp -\int dx \bar{\chi}(\gamma \cdot \partial - m) \lambda$. If, instead, we take $\dot{\lambda} = (\gamma \cdot \partial - m) \lambda + \eta_\lambda$ the Langevin process is ill-defined. In the former case the eigenvalues of the drift force $(\partial^2 - m) \lambda$ are positive, and the Langevin process converges. In the later case, the drift force is $(\gamma \cdot \partial - m) \lambda$, with positive and negative eigenvalues, so that the limit $\tau = \infty$ cannot be reached.
independently of the choice of $K$. At finite $\tau$, the $K$-dependance of $P^{S,K}$ is, explicitly:

$$P^{S,K}(\varphi(x),\tau) = \left( \exp \tau \int d\tau \frac{\delta}{\delta \varphi_a(x)} K_{ab}(\varphi(x)) \right) P^{S,K}(V,\tau = \tau_0)$$

This standard result of statistical mechanics can be simplified by the use of stochastic TQFT supersymmetry [3][4]. There is indeed a supersymmetric path integral that expresses the stochastic process in the bulk $\{ x, \tau \}$, where $\varphi_a(x) \rightarrow \varphi(a)(x, \tau)$ [1][4]. It enlarges the phase space of the fields as follows:

$$\varphi_a(x) \rightarrow \left( \varphi_a(x, \tau), \Psi_a(x, \tau), \overline{\Psi}_a(x, \tau), b_a(x, \tau) \right)$$

The stochastic supersymmetry acts on this enlarged space of fields and it is defined by a nilpotent graded differential operator $s_{stoc}$ [3]. $s_{stoc}$ is alike a BRST topological symmetry operation, which justifies the name stochastic BRST supersymmetry, where $\varphi_a, \Psi_a, \overline{\Psi}_a, b_a$ are identified as two trivial BRST multiplets under $s_{stoc}$ and

$$s_{stoc} \varphi = \Psi, \quad s_{stoc} \Psi = 0, \quad s_{stoc} \overline{\Psi} = b, \quad s_{stoc} b = 0 \tag{9}$$

If there a gauge symmetry acting on $\varphi_a(x)$, some refinements are necessary and one must build an equivariant supersymmetry [5]. The physical conclusions remain the same for the relaxation to a stable equilibrium and they justify the gauge theory path integral formula on the boundary.

We now indicate a universal formula that will be useful in the following. It leaves aside the unessential indices $x^\mu$ that can be discretised. Given a pair of boson fields $(m(\tau), n(\tau))$ and a pair of fermion fields $(p(\tau), q(\tau))$, or the reversed case where on interchanges bosons and fermions, determinant formula imply that one has indeed the following identity

$$\int [dm]_\tau [dn]_\tau [dp]_\tau [dq]_\tau \exp \int d\tau \left[ n \left( \mathcal{M}(m) + \alpha \right) + p \frac{\delta \mathcal{M}(m)}{\delta m} q \right] = 1 \tag{10}$$

where $\mathcal{M}(m)$ is any given local functional of $m$. $\alpha$ is independent on $m, n, p, q$ and the formula holds true whatever $\alpha$ is. Defining $s_{stoc} m = q, s_{stoc} q = 0, s_{stoc} p = n, s_{stoc} n = 0$ and $s_{stoc}$, one has

$$s_{stoc} \left( p(\mathcal{M}(m) + \alpha) \right) = n(\mathcal{M}(m) + \alpha) + p \frac{\delta \mathcal{M}}{\delta m} q \tag{11}$$

$s_{stoc}$ acts as a graded differential operator, so $s_{stoc}^2 = 0$ on any functional of $m, n, p, q$.

The identity (10) is important, as it explains the chain of identities in the next equation (12). Indeed, replacing $\alpha \rightarrow \eta$, $m \rightarrow \varphi, n \rightarrow b, p \rightarrow \overline{\Psi}, q \rightarrow \Psi$ and $\mathcal{M}(\varphi) \rightarrow \frac{d\varphi}{dt} + K \frac{S}{\delta \varphi}$ in the functional identity (10), one can write the following succession of equations:

$$<< f(\varphi) >>^K = \int [d\eta]_\tau f(\varphi) \exp - \frac{1}{2} \int dt \eta_a \frac{\delta \mathcal{L}}{\delta \varphi_b} = \int [d\varphi_a][d\Psi_a][d\overline{\Psi}_a] f(\varphi) \delta \left( \frac{d\varphi_a}{dt} + K_{ab} \frac{\delta \mathcal{L}}{\delta \varphi_b} - \eta_a \right) \exp - \int d\tau \left[ \frac{1}{2} \eta_a \frac{\delta \mathcal{L}}{\delta \varphi_b} - \overline{\Psi}_a \left( \frac{d}{dt} + K_{ab} \frac{\delta \mathcal{L}}{\delta \varphi_b} \right) \psi_c \right]$$

These equalities hold modulo normalising factors for each line. The first line enforces the definition of $<< f(\varphi) >>^K$ as an average over noises, using the Langevin equation $\frac{d\varphi_a}{dt} = -K_{ab} \frac{\delta \mathcal{L}}{\delta \varphi_b} + \eta_a$ and an arbitrarily given initial condition $\varphi(\tau_0)$ at some reference time $\tau_0$ that one must introduce to solve $\varphi$.
in function of $\eta$. The last line teaches us that the correlator $<< f(\varphi) >>^K$ is computed by the path integral of a topological quantum field theory, with a localising gauge function related to the Langevin equation. In other words, the path integral of the $s_{stoc}$-exact Lagrangian in the last line defines the same correlation functions as those computed either from the Langevin equations, or from their corresponding Fokker-Planck equations (if one restricts oneself to equal stochastic time correlators). This supposes that the possible zero modes of the operators that stand between the ghosts $\Psi$ and $\overline{\Psi}$ are handled properly with periodic boundary conditions for $\Psi$.

The Hamiltonian that defines the stochastic time evolution is the $\tau$ Legendre transform of the supersymmetric Lagrangian expressed in the third or fourth lines of Eq. (12), as generically explained in [4].

Eq. (12) is valid whether $\varphi_a$ is bosonic or fermionic, so it can applied when $\varphi_a$ stands for any one of the field components $x, \lambda, \lambda, A$ of the superfield $X(t)$ that we will shortly introduce.

In the stochastic BRST-TQFT supersymmetric formulation, the proof of the $K$ independence of the $\tau = \infty$ limit of correlators, if they exist, follows from the properties of determinants, for the bosonic case as well as for the fermionic case for $\varphi$.

## 3 Supersymmetric kernels

Let us now consider a supersymmetric theory with a set of auxiliary fields that gives a closed system of transformations. We will prove that supersymmetry is compatible with the stochastic time evolution.

For the notational simplicity of the proof we consider the $N = 2$ supersymmetric quantum mechanics. One has $\{x^\mu\} = \{t\}$, two supersymmetry generators $\delta_i = (\delta, \hat{\delta})$ and one multiplet $(1, 2, 1)$, that is, one propagating boson $q(t)$, two fermions $\lambda(t)$ and $\hat{\lambda}(t)$ and one auxiliary field $A(t)$. The construction we will detail can be generalised to all other multiplets. The proof can be generalised to the Wess and Zumino model with no difficulty but notational complications. We first consider that $t$ is the real time and we will shortly shift to the Euclidean time, $t \rightarrow \imath t$. Call $X(t, \theta, \hat{\theta})$ the superfield

$$X(t, \theta, \hat{\theta}) = q(t) + \theta \lambda(t) + \hat{\theta} \hat{\lambda}(t) + \theta A(t)$$

where $\theta$ and $\hat{\theta}$ are Grassman coordinates, with $\int \hat{d}\theta = \int \hat{d}\hat{\theta} = 0$ and $\int \hat{d} \theta = \int \hat{d} \hat{\theta} = 1$. Define also

$$D_\theta \equiv \partial_\theta + \theta \partial_t \quad D_{\hat{\theta}} \equiv \partial_{\hat{\theta}} + \hat{\theta} \partial_t$$

$$\{D_\theta, D_{\hat{\theta}}\} = \{D_{\hat{\theta}}, D_\theta\} = 0 \quad \frac{1}{2} \{D_\theta, D_{\hat{\theta}}\} = \partial_t \quad (13)$$

The action of supersymmetry transformations on the superfield $X$ is given by $Q = \partial_\theta - \theta \partial_t$ and $\hat{Q} = \partial_{\hat{\theta}} - \hat{\theta} \partial_t$. By expansion in components, one finds the following action of both supersymmetries.

$$\delta q = \lambda \quad \delta \lambda = \partial_\theta q \quad \delta \hat{\lambda} = -A \quad \delta A = -\partial_t \hat{\lambda}$$

$$\hat{\delta} q = \hat{\lambda} \quad \hat{\delta} \lambda = \partial_{\hat{\theta}} q \quad \hat{\delta} \hat{\lambda} = A \quad \hat{\delta} A = \partial_t \lambda \quad (14)$$

They satisfy $\frac{1}{2} \{\delta_i, \delta_j\} = \delta_{ij} \partial_t$ and $\delta_i d + d \delta_i = 0$. The massless supersymmetric free action is

$$\Sigma_0 \equiv \int dt d\theta d\hat{\theta} d\tau (\frac{1}{2} XD_\theta D_{\hat{\theta}} X) = \int dt (\frac{1}{2} A^2 - \frac{1}{2} q \partial_t^2 q - \frac{1}{2} \lambda \partial_t \lambda - \frac{1}{2} \hat{\lambda} \partial_t \hat{\lambda}) \quad (15)$$

A general supersymmetric interaction, including a mass term, is determined by the prepotential $W(X)$

$$\Sigma_W \equiv \int dt d\theta d\hat{\theta} d\tau (W(X)) = \int dt (\partial_t A W_q(q) - \lambda W_{qq}(q) \hat{\lambda}) \quad (16)$$
(W_q, W_{qq} mean \partial_q W, \partial^2_q W). So, the generic standard interacting supersymmetric action that we consider is

$$\mathcal{I}_W \equiv \Sigma_0 + \Sigma_W = \int dt (\frac{1}{2}(A + W_q)^2 - \frac{1}{2}W^2_q + \frac{1}{2}(\partial_q q)^2 - \frac{1}{2}\lambda\partial_t \lambda - \frac{1}{2}\hat{\lambda}\partial_t \hat{\lambda} - \lambda W_{qq}\hat{\lambda})$$

With \( t \to it \), one gets the following equivalent (modulo boundary terms) expressions of the Euclidean action \( E \)

$$\mathcal{I}_W^{\text{twisted}} = \int dt \left( \frac{1}{2}A^2 + AW_q - \frac{1}{2}(\partial_q q)^2 - \frac{i}{2}\lambda\partial_t \lambda - \frac{i}{2}\hat{\lambda}\partial_t \hat{\lambda} - \lambda W_{qq}\hat{\lambda} \right)$$

$$= \int dt \left( \frac{1}{2}(A + W_q)^2 - \frac{1}{2}W^2_q - \frac{1}{2}(\partial_q q)^2 - \frac{i}{2}\lambda\partial_t \lambda - \frac{i}{2}\hat{\lambda}\partial_t \hat{\lambda} - \lambda W_{qq}\hat{\lambda} \right)$$

$$= \int dt \left( \frac{1}{2}(A + W_q)^2 - \frac{1}{2}(\partial_q q + W_q)^2 - \frac{i}{2}\lambda\partial_t \lambda - \frac{i}{2}\hat{\lambda}\partial_t \hat{\lambda} - \lambda W_{qq}\hat{\lambda} \right)$$

(17)

From now on we will denote \( I_w \equiv -\mathcal{I}_W^{\text{twisted}} \), keeping in mind that we compute Euclidean correlation functions in the \( \tau = \infty \) limit. In order to achieve the stochastic quantisation, one uses the Euclidean action. It is unclear if the Wick rotation on \( t \) can be done at finite values of the stochastic time. The possibility of a Wick rotation can be however checked in the limit \( \tau = \infty \).

For polynomial interactions, it is convenient to separate the mass term \( \frac{1}{2}m^2 q^2 \) from the rest of the interactions in \( W(q) \), which are of degree higher than 2, with \( W \equiv \frac{1}{2}mq^2 + V(q) \) and \( m = W_{qq}(0) \). This decomposition will be used shortly, to check perturbatively the possible stochastic equilibrium of the supersymmetric model.

We must define the supersymmetry covariance of all fields of the stochastic process. Consider the noise fields \( \eta_a \). Given a component field \( a \) of the multiplet \( X \), its noise \( \eta_a \) is a random fluctuations of \( a \), modulo terms proportional to some equations of motion. Thus, if stochastic quantisation is compatible with supersymmetry, the noises of \( q, A, \lambda, \hat{\lambda} \) build a superfield

$$\eta = \eta_q + \theta \eta_\lambda + \hat{\theta} \eta_{\hat{\lambda}} + \theta \hat{\theta} \eta_A$$

(18)

and transform accordingly. As we will demonstrate, non-trivial kernels \( K_{ab} \) are necessary in order to obtain supersymmetry covariant Langevin equations as well as a proper convergence of the Langevin process. The Langevin equation that respects supersymmetry must be written as

$$\dot{q} = - \sum_{a=q,\lambda,\hat{\lambda},b} K_{q,a} \frac{\delta I_w}{\delta a} + \eta_q$$

$$\dot{A} = - \sum_{a=q,\lambda,\hat{\lambda},b} K_{A,a} \frac{\delta I_w}{\delta a} + \eta_A$$

$$\dot{\lambda} = - \sum_{a=q,\lambda,\hat{\lambda},b} K_{\lambda,a} \frac{\delta I_w}{\delta a} + \eta_{\lambda}$$

$$\dot{\hat{\lambda}} = - \sum_{a=q,\lambda,\hat{\lambda},b} K_{\hat{\lambda},a} \frac{\delta I_w}{\delta a} + \eta_{\hat{\lambda}}$$

(19)

We will use superfield arguments to compute the appropriate kernels \( K_{ab} \).

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\(^\dagger\)As all supersymmetric theories, this model has a twisted formulation, where one can rewrite the supersymmetry algebra as a nilpotent one. In this case, one redefines \( \psi = \frac{A_1}{\sqrt{2\tau}} \) and \( \lambda = \frac{A_2}{\sqrt{2\tau}} \). By doing this change of variables, the Euclidean action becomes \( I_{\text{twisted}} = \int dt \left( \frac{1}{2}A^2 + W_q A - \frac{1}{2}(\partial_q q)^2 + \bar{\psi}\partial_t \Psi - \bar{\Psi}\partial_t \bar{\psi} - \bar{\psi}W_{qq}\psi \right) \) and the twisted nilpotent supersymmetry is \( \delta_{\text{twisted}} q = \psi, \delta_{\text{twisted}} \psi = 0, \delta_{\text{twisted}} b = 0, \) with \( I_{\text{twisted}} = \delta_{\text{twisted}} \int dt \bar{\psi}(\frac{1}{2}b + \partial q + W_q) \). In this expression, the \( N = 2 \) supersymmetry appears twisted, with some interesting properties. As for its generalisation in the stochastic formulation, one can proceed as in the untwisted case, and, once one has computed the stochastic TQFT supersymmetry, one gets an interesting 4-simplex, because both supersymmetries of the stochastically quantised action can be described as nilpotent ones.
4 Determination of the kernels to enforce supersymmetry at all values of stochastic time

We will use the following remark: $X$ being a superfield, the supersymmetric invariance of the action $I_w(X)$ implies that its equations of motion build the following "dual" superfield

$$X^*_w = \frac{\delta I_w}{\delta A} - \theta \frac{\delta I_w}{\delta \lambda} + \dot{\theta} \frac{\delta I_w}{\delta A} + \theta \dot{\theta} \frac{\delta I_w}{\delta q},$$

(20)

independently of the details of the supersymmetric action and thus of $W$. The proof is simple: if one varies $X \rightarrow X + \delta X$, the variation of $I_w$ is the supersymmetric term $\int dx \left( \frac{\delta I_w}{\delta q} \delta q + \frac{\delta I_w}{\delta \lambda} \delta \lambda + \frac{\delta I_w}{\delta A} \delta A \right)$. This variation is nothing but the integral over $d\theta d\dot{\theta}$ of the superfield $\delta X$ times $X^*_w$, which picks out the coefficients term of $\theta \dot{\theta}$ from the product $X^*_w \delta X$. Thus $X^*_w$ is a superfield, and one has:

$$X^*_w = -A - W_q + \theta(-i\partial_\lambda + W_{qq} \lambda) - \dot{\theta}(-i\partial_\lambda - W_{qq} \dot{\lambda}) - \theta \dot{\theta} \left( \partial_t^2 q + AW_{qq} - \lambda W_{qq} \dot{\lambda} \right)$$

(21)

The stochastic time derivative of $X$ is also a superfield,

$$\dot{X}(t, \theta, \dot{\theta}, \tau) = \dot{q} + \theta \dot{\lambda} + \dot{\theta} \lambda + \theta \dot{\theta} A$$

(22)

We will need more independent superfields to define the stochastic BRST supersymmetry framework.

Let $a, b, \ldots$ denote the components $x, \lambda, \dot{\lambda}, A$ of the superfield $X$. The goal is to write a supersymmetric action of the form $s_{stoc} \Psi_a \left[ K_{ab} b_b + \frac{dX_a}{dt} + K_{ab} \frac{dX_b}{dt} \right]$, where $s_{stoc}$ is the stochastic BRST symmetry operation we defined for Eq. (12). In the case we discuss, one has $s_{stoc} X_a = \Psi_a$, $s_{stoc} \Psi_a = 0$, $s_{stoc} \Psi_a = b_a$, $s_{stoc} b_a = 0$. We will express the $s_{stoc}$-exact action depending on $X$ and its BRST topological partners as the integral of a super-Lagrangian.

Then, we will integrate this super-Lagrangian over superspace $(\theta, \dot{\theta})$ and get the supersymmetric $s_{stoc}$-exact action in components. Using Eq. (12), we will identify in a reversed way the Langevin equation for each component in $X$. Indeed, they are encoded in Eq. (12) as the coefficient of the terms linear in the $b$’s. Such Langevin equations will be covariant under supersymmetry, by construction.

To impose the supersymmetry transformation of the $b_a$’s, we simply declare that

$$B \equiv b_A + \theta b_\lambda + \dot{\theta} b_\lambda + \theta \dot{\theta} b_q$$

(23)

is a superfield. This determines the supersymmetry transformations of all components $b_a$.

Thus, the program is to first write a super-action depending on $B$, which has the form of the $b$-dependant terms in Eq. (12) after integration over superspace.

To get the stochastic ghost depending part of Eq. (12), and enforce the compatibility of the supersymmetry of our model with the stochastic quantisation, we also introduce the stochastic antighost superfield $\Psi_X$, with $s_{stoc} \Psi_X = B$, $s_{stoc} B = 0$,

$$\Psi_X = \Psi_A + \theta \Psi_\lambda + \dot{\theta} \Psi_\lambda + \theta \dot{\theta} \Psi_q$$

(24)

and the stochastic ghost superfield $\Psi_X$ of $X$, with $s_{stoc} X = \Psi_X$, $s_{stoc} \Psi_X = 0$ and the decomposition

$$\Psi_X = \Psi_q + \theta \Psi_\lambda + \dot{\theta} \Psi_\lambda + \theta \dot{\theta} \Psi_A$$

(25)

For the sake of the definition of the kernels, we now introduce a mass parameter $M$. This parameter may have to be fine-tuned, to ensure the eventual convergence of the stochastic process, using the kernels that we shall shortly determine. The values of the correlators at $\tau = \infty$, if the limit exists, will not depend on the choice of $M$ according to the general proof of the $K$ independence of the limit.
One then defines the following supersymmetric quadratic functional of $B$:

$$I_M = \int dt d\theta d\hat{\theta} dt (\frac{1}{2} BD_{\hat{\theta}} D_{\theta} B - \frac{1}{2} MB^2)$$

$$= \int dt \left( \frac{1}{2} b_q^2 - Mb_q b_A - \frac{1}{2} (\partial_i b_A)^2 - i b_\lambda \partial_i b_\lambda - b_\lambda \partial_i b_\lambda + Mb_\lambda b_\lambda \right)$$

(26)

Whatever the value of $M$ is in this supersymmetric auxiliary free action, the equations of motion of $B$ determine the "dual" superfield:

$$B_{I_{stoc}}^* \equiv b_q - Mb_A + \theta (i \partial_i b_\lambda + Mb_\lambda) + \hat{\theta} (-i \partial_i b_\lambda + Mb_\lambda) + \theta \hat{\theta} (\partial_i^2 b_A - Mb_q)$$

(27)

Since $\dot{X}, X_W, B, B_{I_{stoc}}^*$ are all superfields, the following action is supersymmetric:

$$I_{stoc,B}^* = \int dt d\theta d\hat{\theta} dt \left[ \frac{1}{2} BB_{I_{stoc}}^* + B\dot{X} + B_{I_{stoc}}^* X_W^* \right]$$

(28)

It can be expressed in components as

$$I_{stoc,B}^* = \int dt d\theta d\hat{\theta} dt \left[ \frac{1}{2} b_q^2 \lambda + \frac{1}{2} b_A \delta^2 b_A - Mb_q b_A - i \frac{1}{2} (b_\lambda \partial_i b_\lambda + b_\lambda \partial_\lambda b_\lambda) - Mb_\lambda b_\lambda \right]$$

$$+ b_q (\dot{q} + \delta I_w \lambda - \frac{M \delta I_w \lambda}{\delta A}) + b_A (\dot{A} + \delta I_w \lambda - \frac{M \delta I_w \lambda}{\delta A})$$

$$+ b_\lambda (\dot{\lambda} + i \partial_i \delta I_w \lambda - \frac{M \delta I_w \lambda}{\delta \lambda}) - b_\lambda (\dot{\lambda} + i \partial_i \delta I_w \lambda + \frac{M \delta I_w \lambda}{\delta \lambda})$$

(29)

where $b_q^* = b_q - Mb_A$. This supersymmetric action has a quadratic dependance in the fields $b_a = (b_q, b_\lambda, b_{\lambda}, b_A)$, which are the auxiliary fields of the topological BRST symmetry of stochastic quantisation, as in Eq. (12). The coefficients of the linear terms in the $b$’s provide the supersymmetry covariant Langevin equations

$$\dot{q} = -\frac{\delta I_w}{\delta q} + M \frac{\delta I_w}{\delta A} + \eta_q$$

$$\dot{\lambda} = -i \partial_i \frac{\delta I_w}{\delta \lambda} + M \frac{\delta I_w}{\delta \lambda} + \eta_\lambda$$

$$\dot{\lambda} = -i \partial_i \frac{\delta I_w}{\delta \lambda} - M \frac{\delta I_w}{\delta \lambda} + \eta_\lambda$$

$$\dot{A} = - (\partial_i^2 - M^2) \frac{\delta I_w}{\delta A} - M (\dot{\eta_q} + \frac{\delta I_w}{\delta \eta_q}) + \eta_A = - (\partial_i^2 - M^2) \frac{\delta I_w}{\delta A} + \eta_A - M \eta_q$$

(30)

The quadratic terms in $b$ define the Gaussian noise distribution by

$$\langle \langle f(\eta) \rangle \rangle = \int [d\eta_q] [d\eta_\lambda] [d\eta_A] f(\eta)$$

$$\exp \left( -\frac{1}{2} \int dt d\theta d\hat{\theta} dt \left( \eta_q^2 + \eta_A \frac{\partial^2}{\partial \eta_q^2} - M^2 \eta_q + \frac{1}{2} \eta_\lambda \frac{\partial^2}{\partial \eta_\lambda^2} - M^2 \eta_\lambda + \frac{1}{2} \eta_\lambda \frac{\partial^2}{\partial \eta_\lambda^2} - M^2 \eta_\lambda + \eta_\lambda \frac{M}{\partial^2 - M^2} \eta_\lambda \right) \right)$$

(31)

The general theorem applies and the limit of the field correlators in the limit $\tau \to \infty$ is independent on the chosen value of $M$.  

8
To see perturbatively the $\tau \to \infty$ convergence of the solutions of Langevin equations, one expands $W_{qq}(q) = m + V_{qq}(q)$ to verify that the drift forces are negative for all fields for $W = \frac{m}{2}q^2$.\[\begin{align*}
\dot{q} &= (\partial_t^2 - m M)q + (m - M)A + \eta_q \\
\dot{\lambda} &= (\partial_t^2 - m M)\lambda - i(m - M)\partial_t\dot{\lambda} + \eta_\lambda \\
\dot{A} &= (\partial_t^2 - M^2)(A + mq) + \eta_A - M\eta_q
\end{align*}\]Both operators $\partial_t^2 - m M$ and $\partial_t^2 - M^2$ have negative eigenvalues in the Fourier transform space. (This requires the Euclidian formulation for the stochastic quantisation). One must verify the negativity of the eigenvalues of the differential operators that define the $\tau$ evolution of $A, q$ and $\lambda, \dot{\lambda}$. They are

\[
\begin{pmatrix}
\partial_t^2 - m M \\
(m(\partial_t^2 - M^2)) \quad \partial_t^2 - M^2
\end{pmatrix}
\begin{pmatrix}
m - M \\
i(m - M)\partial_t - i(m - M)\partial_t
\end{pmatrix}.
\]

The eigen values are non-degenerate for $M \neq m$ and degenerate for the simplest and easiest choice $M = m$. The determinant of the first matrix is $(\partial_t^2 - M^2)(\partial_t^2 - m^2) > 0$ and for the second matrix it is $(\partial_t^2 - m M)^2 - (m - M)^2\partial_t^2 > 0$. Moreover the traces of both matrices are negative. Thus, whatever the value of $M$ is, we have the non positivity requirement and a normalisable vacuum for the supersymmetric Fokker–Planck process, and the stochastic process is converging at $\tau \to \infty$ for the fields $q, A, \lambda, \dot{\lambda}$. The values of correlators at $\tau \to \infty$ is independent on the choice of $M$, although the details of the evolution are $M$ dependent. The solutions have a different dependence on $\tau$ for $M \neq m$ and $M = m$, but the limit $M \to m$ is continuous.

We have thus obtained a set of Langevin equations with a well defined convergence at $\tau = \infty$ that are by construction covariant under the $N = 2$ supersymmetry. For $M = m$, the stochastic process is simplest because no mixing occurs between the drift forces of all fields at the free level, and the Fokker–Planck Hamiltonian only involves the eigenvalues of the single operator $-\partial_t^2 + M^2$ for all fields.

We can now write the complete $s_{\text{stoc}}$-exact action associated to these supersymmetric Langevin equations, that is, a stochastic BRST-TQFT supersymmetric action in the 2d-space $\{t, \tau\}$, which is the integral of a superfield, and thus a $N = 2$ supersymmetric 2d action.

We use the already anticipated stochastic topological superghosts $\Psi_X$ and $\overline{\Psi}_X$, upon which the graded differential operator $s_{\text{stoc}}$ acts as $s_{\text{stoc}}X = \Psi_X$, $s_{\text{stoc}}\Psi_X = 0$, $s_{\text{stoc}}\overline{\Psi}_X = B$, $s_{\text{stoc}}B = 0$. Defining,

\[
\overline{\Psi}_{1_M} = \overline{\Psi}_q - M\overline{\Psi}_A + \theta(\partial_t\overline{\Psi}_\lambda + M\overline{\Psi}_\lambda) + \theta(-i\partial_t\overline{\Psi}_\lambda + M\overline{\Psi}_\lambda) + \theta(\partial_t^2\overline{\Psi}_A - M\overline{\Psi}_q)
\]

one has $s_{\text{stoc}}\overline{\Psi}_{1_M} = B_{1_M}^*$ and $s_{\text{stoc}}B_{1_M}^* = 0$. The complete $N=2$ supersymmetric action that is invariant under the stochastic BRST supersymmetry and expresses the Langevin equations \[\text{(32)}\] is thus

\[
I_{\text{stoc}} = \int dt d\theta d\dot{\theta} ds_{\text{stoc}} \left[ \overline{\Psi}_{1_M} \left( \frac{1}{2}B + X^*_W + \overline{\Psi}_X \dot{X} \right) \right]
\]

It can be simply expanded in components. Then, all the details of the stochastic evolution of the correlators of all components of the superfield can be studied while maintaining the supersymmetry.

5 Conclusion

Using a superfield construction, we explained the way the stochastic time evolution of the stochastically quantised $N = 2$ supersymmetric quantum mechanics correlation functions is compatible with the

\[\text{The stochastic equation for } A \text{ is consistent if one eliminates } A \text{ by its equation of motion, } A = -mq, \text{ so there is no evolution for } A \text{ provided } \eta_A = M\eta_q. \text{ But the the various closure relations hold modulo equations of motion.}\]
$N = 2$ supersymmetry. The stochastic TQFT supersymmetry and the space supersymmetry satisfy

$$\left[s_{\text{stoc}}, \frac{d}{d\tau}\right] = 0, \quad \{s_{\text{stoc}}, Q\} = 0, \quad \{s_{\text{stoc}}, \hat{Q}\} = 0$$

(35)

for every finite value of the stochastic time, which is the required property. We introduced kernels that ensure the convergence of the stochastic process toward the standard path integral of $N = 2$ supersymmetric quantum mechanics. These kernels are covariant under supersymmetry and depend on a massive parameter that can be chosen at will. Its value modifies the course of the stochastic quantisation, but the limit of the correlation functions when the stochastic time runs to $\tau = \infty$ is independent on its choice.

The same demonstration can be mutatis mutandi repeated for the Wess and Zumino model. It is also possible to repeat this construction to more sophisticated supersymmetric theories, including those with a gauge invariance, provided one uses the method of equivariant cohomology.

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**References**

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