Fidelity Decay Saturation Level for Initial Eigenstates

Yaakov S. Weinstein\textsuperscript{1}, Joseph V. Emerson\textsuperscript{1}, Seth, Lloyd,\textsuperscript{*} David G. Cory\textsuperscript{12}

\textsuperscript{1}Massachusetts Institute of Technology, Department of Nuclear Engineering, Cambridge, MA 02139
\textsuperscript{*}d’Arbeloff Laboratory for Information Systems and Technology, Massachusetts Institute of Technology, Department of Mechanical Engineering, Cambridge, MA 02139
\textsuperscript{2}Author to whom correspondence should be addressed

We show that the fidelity decay between an initial eigenstate state evolved under a unitary chaotic operator and the same eigenstate evolved under a perturbed operator saturates well before the $1/N$ limit, where $N$ is the size of the Hilbert space, expected for a generic initial state. We provide a theoretical argument and numerical evidence that, for intermediate perturbation strengths, the saturation level depends quadratically on the perturbation strength.

PACS numbers

Over the past twenty years different phenomenon found in quantum systems that have chaotic classical analogs have been suggested as appropriate signatures of quantum chaos \cite{1} \cite{2} \cite{3} \cite{4} \cite{5} \cite{6}. Peres \cite{4} conjectured that the initial rate and behavior of a system’s fidelity decay due to a small perturbation in the Hamiltonian may provide an appropriate signature of quantum chaos. This signature provides an analog to the sensitivity to initial conditions which characterizes classical chaos but, as a consequence of strictly unitary evolution, cannot emerge in quantum systems. Recent insights \cite{7} \cite{8} \cite{9} have lead to a more detailed understanding of this signature.

For a unitary map, $U$, the fidelity compares the evolution of an initial state under unperturbed and perturbed dynamics. The fidelity is given by

$$F(n) = |\langle \psi_i | (U^d)^n (U_p U)^n | \psi_i \rangle|^2$$

(1)

where $U_p = \exp(-i\delta V)$ is the perturbation operator of strength $\delta$, and $\psi_i$ is the initial state of the system. The fidelity decay behavior depends not only on whether the map is chaotic but also on the initial state of the system and the strength of the perturbation. For chaotic systems, the fidelity eventually approaches an asymptotic level. Here, we focus on the characteristics of this asymptotic level by studying

$$F_\infty = \lim_{n \to \infty} \frac{1}{n} \sum_{1}^{n} F(n)dn.$$  

(2)

For initial random states the fidelity saturates at $1/N$ \cite{7}, as we show below. However, for eigenstates $F_\infty$, the saturation level, in much larger and depends sensitively on the perturbation strength, $\delta$. The study of initial eigenstate fidelity decay is of particular interest since it is equivalent to the survival probability of a system eigenstate under the influence of a perturbation.

Below we provide theoretical arguments showing a region where $F_\infty$ depends quadratically on the perturbation strength. We also test this prediction numerically on quantum chaotic maps.

For chaotic systems and weak perturbation strengths the initial fidelity decay is Gaussian, as expected from perturbation theory and random matrix theory \cite{10}. For stronger perturbations the fidelity decay is exponential. The regime of exponential decay, known as the Fermi Golden Rule (FGR) regime \cite{8}, is reached when $\sigma$, a typical off-diagonal matrix element of perturbation Hamiltonian expressed in the ordered eigenbasis of the system Hamiltonian, is greater then the average system level spacing, $\Delta$. It has been shown that for some perturbations the rate of the the exponential decay increases as $\delta^2$, until saturating at a rate given by the corresponding classical system’s Lyapunov exponent \cite{11} \cite{8} \cite{12}, or the bandwidth of the system Hamiltonian \cite{8}.

Jacquod and coworkers \cite{8} showed that the fidelity decay in the FGR regime is related to the local density of states (LDOS) for initial eigenstates. We define eigenvectors and eigenangles for the unperturbed operator $U|v_j\rangle = \exp(-i\phi_j)|v_j\rangle$, and the perturbed operator $U_p U|v'_k\rangle = \exp(-i\phi'_k)|v'_k\rangle$. The LDOS is the spectral density of the original system under transition rules given by the perturbation. Hence, it is as a measure of the overlap between perturbed and unperturbed states separated by an angle $(\phi_j - \phi'_k)$

$$\eta(\phi_j - \phi'_k) = |\langle v_j|v'_k \rangle|^2.$$  

(3)

For an initial eigenstate of $U$, the fidelity decay is the Fourier transform of the LDOS

$$F(n) = \sum_m \eta(\phi_j - \phi'_k) \exp(-i(\phi_j - \phi'_k)n).$$

(4)

Previous studies suggest that the LDOS of a complex system in the regime of strong perturbation is Lorentzian \cite{13} \cite{14} \cite{15}

$$\eta(\phi_j - \phi'_k) = \frac{\Gamma}{(\phi_j - \phi'_k)^2 + (\Gamma/2)^2}$$

(5)

with a width of $\Gamma = 2\pi \sigma^2/\Delta$ where $\sigma$ is a typical off diagonal element of the perturbation operator. Thus, using the Fourier transform relation, the initial fidelity decay is exponential with a rate of $\Gamma$

$$F(n) = \exp(-\Gamma n).$$

(6)
Γ can be rewritten in terms of perturbation strength as follows: \( \sigma = \sqrt{\langle \delta V_{nm}^2 \rangle} \) where \( \delta V_{mn} \) is the second moment of the matrix elements \( V_{mn} \). \( \delta V_{mn} \) may be estimated by noting that for chaotic systems the eigenvectors are random, and, therefore, \( \delta V_{mn}^2 = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^2 \) is the variance of the eigenvalues of \( V \). The average level spacing, \( \Delta \), is equal to \( 2\pi/N \). The rate of the exponential decay, \( \Gamma \) can now be evaluated as \( \Gamma = \delta^2 \lambda^2 \).

We now turn to the study of the saturation level of the fidelity decay, \( F_{\infty} \). After a certain amount of time we would expect the initial state to become evenly spread out over a complete set of states. This implies that \( F_{\infty} \) should be of order \( 1/N \). \( F_{\infty} \) should also be independent of the perturbation strength. A weaker perturbation simply leads to a longer period of time until the saturation level is reached, but the saturation level should remain unchanged. For initial states that are eigenstates of the unperturbed system, however, \( F_{\infty} \) depends on the perturbation strength. Weaker perturbations, even in the FGR regime, lead to saturation levels significantly higher than \( 1/N \).

Prosen [7] has noted that for initial eigenstates \( F_{\infty} \to 1 \) in the limit of weak perturbation and \( F_{\infty} \to (4 - \beta)/N \) for strong perturbation where \( \beta = 1 \) for maps with circular orthogonal ensemble (COE) properties and \( \beta = 2 \) for maps with circular unitary ensemble (CUE) properties. Here, we provide a theoretical argument and numerical evidence for a quadratic behavior for \( F_{\infty} \) of initial eigenstates versus perturbation strength for perturbation strengths between these two extremes.

To evaluate the dependence of the saturation level on the perturbation strength let us start by expressing the fidelity for an initial eigenstate, \(|v_m\rangle\), as

\[
F(n) = \langle v_m \left| \sum_l a_{lm} e^{-i(n\phi_l - \phi_m)} |v_m\rangle \rangle^2
\]

where \( a_{lm} = \langle v'_l | v_m \rangle \). The above equation can be separated into a time independent term plus a time dependent term

\[
F(n) = \sum_l |a_{lm}|^4 + \sum_{lk} |a_{lm}|^2 |a_{km}|^2 \cos[(\phi_l' - \phi_k) n].
\]

The time average of the second term goes to zero while the first term determines \( F_{\infty} \) as an inverse participation ratio of the overlap between perturbed and unperturbed eigenvectors [7]. In other words, the fidelity saturation level is simply the sum of the squared elements of the LDOS.

Using equation (8), we recover the \( \simeq 1/N \) saturation level of Prosen in the limit of strong perturbation. An extremely strong perturbation could cause the initial state (though an eigenstate of the system dynamics) to become evenly spread over all eigenstates of the system, such that \( \langle |v'_l v_m| \rangle \simeq 1/\sqrt{N} \) for all \( v'_l \). \( F_{\infty} \) would then be \( \sum_l |a_{lm}|^4 \simeq 1/N \). For weaker perturbations the saturation level will depend on the number of contributing eigenvectors, \( |v'_l| \) and the coefficients \( a_{lm} \). This is equivalent to the width of the LDOS under the particular perturbation.

Hence, to estimate \( F_{\infty} \) for intermediate strengths we must have an idea of the number of contributing perturbed operator eigenvectors \( |v'_l\rangle \) to the initial eigenstate, \( |v_m\rangle \). This can be estimated by the width of the LDOS, \( |a_{lm}|^2 \). We assume all eigenvectors within the width \( \Gamma \) of the approximate Lorentzian shaped LDOS to have equal weight. With this approximation,

\[
F_{\infty} \propto 1/(\Gamma N) = 1/(\delta^2 \lambda^2 N). \quad (9)
\]

Thus, we expect a quadratic dependence of \( F_{\infty} \) on the perturbation strength in the FGR regime until the saturation level reaches \( O(1/N) \).

A similar analysis for an initial random state, \( |\psi_i\rangle = \sum_m c_m |v_m\rangle \), shows that \( F_{\infty} \simeq 1/N \) for all perturbation strengths. For random states the fidelity can be written as

\[
F(n) = \sum_{mij} c_m^* c_j a_{lj} a_{km}^* \rho_{m} e^{-i(\phi'_l - \phi'_m)} |n\rangle. \quad (10)
\]

Once again, the right hand side can be divided to a time dependent term

\[
\frac{1}{N^2} \sum_{mij} \sum_m c_m^* c_j a_{lj} a_{km}^* \cos[(\phi'_l - \phi'_m + \phi_m - \phi_m) n], \quad (11)
\]

which vanishes under time average, and time independent term

\[
F_{\infty} = \frac{1}{N^2} \sum_{mij} |a_{lj}|^2 |a_{km}|^2 \quad (12)
\]

where, in the above equations, \( |c_j|^2 \simeq 1/N \) for a random state. For any non-zero perturbation strength, the time average of time dependent term will go to zero. The time independent term is easily seen to be approximately \( 1/N \) in the limits of weak and strong perturbation. For intermediate perturbation strengths we can estimate the contribution of the time independent term by analyzing the LDOS. Again, we approximate the Lorentzian LDOS with a rectangle of width \( \Gamma \) and height \( 1/\Gamma \). Contributions to the sum will be non-zero only if the \( j \)th and \( m \)th eigenvectors are a distance of less than \( \Gamma/2 \) from the \( l \)th perturbed eigenvector. Hence, for each of the \( N \) values of \( l \) there will be \( \Gamma \) terms \( |a_{ij}|^2 \) and \( \Gamma \) terms \( |a_{lm}|^2 \) each of magnitude \( 1/\Gamma \). The value of the time independent term is thus \( 1/N \).

The above predictions were first tested on random circular unitary ensemble (CUE) maps. Random matrix theory predicts the behavior of the fidelity decay in both
the Gaussian [10] and FGR [16] perturbation strength regimes. The use of a random matrix as the evolution operator to study dynamical aspects of quantum chaos has been done in [9].

We assume that our system is composed of a collection of two-level subsystems or qubits. The perturbation used is a z-rotation of all of these qubits through an angle $\delta$

$$U_p = \prod_{j=1}^{n_q} e^{-i\delta \sigma_j^z / 2}$$

(13)

where $n_q = \log_2 N$ is the number of qubits in the system. In the context of quantum information processing, this perturbation corresponds to an error in the phase of all the quantum bits in a quantum information processor. We note that this perturbation also arises in quantum control studies as a model of coherent far-field errors [17]. For this perturbation, CUE maps exhibit exponential fidelity decay and a Lorentzian shaped LDOS [9] as shown in the insets of figure 1.

Figure 1 shows $F_{\infty}$ versus perturbation strength for CUE maps, using initial eigenstates of the CUE matrix. We see that below the FGR regime there is very little decay while in the limit of strong perturbation $F_{\infty} = 2/N$ as expected for CUE maps. Between these we see a power law decrease of $F_{\infty}$ with increased perturbation strength. Since the LDOS is Lorentzian the discrepancy seen in figure 1 must be due to the approximation made by replacing the Lorentzian LDOS with a rectangle of width $\Gamma$. The actual slope of the data is between 1.8 and 1.9. The data is compared to $F_{\infty} = C_{\text{CUE}}/(\delta^2 N^2)$, where the proportionality constant, $C_{\text{CUE}} = 3.6$ is chosen to best fit the data.

FIG. 1. Saturation level versus perturbation strength for initial eigenstates of a random CUE map of dimensions 256 (circles), 512 (stars), and 1024 (x). For weak perturbations below the FGR regime the fidelity barely decays. In the limit of strong perturbation $F_{\infty}$ saturates at $2/N$ (solid line). For intermediate values of $\delta$, $F_{\infty}$ is well approximated by the estimate of equation (9) with the proportionality constant $C_{\text{CUE}} = 3.6$. $F_{\infty}$ is obtained by averaging over 2000 map iterations starting at iteration $n = 2000$, well after the initial exponential decay. This is averaged over all $N$ initial eigenstates. The lower inset shows the initial exponential fidelity decay of the CUE map with $N = 1024$ averaged over all 1024 system eigenstates. The fidelity decay is plotted versus $1/N$ so that the exponential decay rates overlap and the saturation level is easily seen. The perturbation strengths used are $0.1, 0.2, 0.3, 0.4$ (top to bottom). The upper inset shows a semi-log plot of the local density of states for a CUE map perturbed by a collective bit z-rotation, $\delta = 0.1, 0.2, 0.3$ and 0.4 (bottom to top). The solid line is a Lorentzian of width $\Gamma = \sqrt{2}/\delta_*/\Delta$ with $\sqrt{2}/\delta_*$ determined numerically from the CUE map.

A similar analysis was carried out for random circular orthogonal ensemble (COE) maps. Random COE matrices can be created from CUE matrices, $\text{COE} = \text{CUE} \ast \text{transpose}(\text{CUE})$ [3]. Like the CUE maps, the COE maps have no classical analog and we introduce them here as models for the behavior of quantum chaotic maps with COE eigenvector statistics and energy level spacings. Figure 3 shows $F_{\infty}$ versus perturbation strength for COE maps. Again, an approximate quadratic relationship emerges but with a different proportionality coefficient, $C_{\text{COE}} = 5.4$.

The difference in proportionality constants is in line with the work of Prosen [7] who, using a random matrix theory argument, predicts a ratio of $3/2$ for $F_{\infty}^{\text{COE}} / F_{\infty}^{\text{CUE}}$ in the limit of strong perturbation. We observe that this ratio holds for all perturbation strengths in the FGR regime. The calculated numerical average of $F_{\infty}^{\text{COE}} / F_{\infty}^{\text{CUE}}$ for the three Hilbert space dimensions explored with perturbations in the FGR regime is $1.48 \approx 3/2$.

We next study $F_{\infty}$ for a quantum system with a well defined classical analog, the quantum kicked top (QKT) [18] [3]. The QKT is an exemplary model of quantum chaos and has been used in previous studies of fidelity decay [4] [7] [8]. The QKT is a unitary map $U_{\text{QKT}} = \exp(-i\pi J_y / 2) \exp(-i k J_z^2 / j)$ acting on a Hilbert space of dimension $N = 2j + 1$. $J$ is the angular momentum operator in the irreducible representation and $k$ is the kick strength. A kick strength of $k = 12$ is used which is well in the chaotic region of the QKT. Since the QKT shows anti-unitary symmetry, it is part of the COE class. The QKT has COE-like nearest neighbor level spacings [3] and eigenvector statistics [19]. The same perturbation, the collective z-rotation, is used.
FIG. 2. $F_\infty$ versus perturbation strength for initial eigenstates of a random COE map (x) and the QKT with $k = 12$ (circles) of dimensions 256, 512, and 1024 (from top to bottom). For weak perturbations below the FGR regime the fidelity barely decays. In the limit of strong perturbation $F_\infty$ saturates at $3/N$ (solid line). For intermediate values of $\delta$, $F_\infty$ is well approximated by the estimate of equation (9) with the proportionality constant $C_{COE} = 5.4$. The numerical value of $F_\infty$ is determined in the same manner as for the CUE maps.

It should be noted that the data for the QKT and COE maps are very similar. This is expected in that, as has been conjectured and demonstrated in a number of works, quantum chaotic systems have statistical [2] [19] and dynamic features [9] similar to those of the canonical random matrix theory ensembles.

The QKT is a system with a classical analog and has symmetries not found in random matrices. It is interesting to see what effect these symmetries, or invariant subspaces have on $F_\infty$. To do this, $F_\infty$ is calculated for the oe subspace (odd under $180^\circ$ rotations around the y-axis [4]) of the QKT which has dimension $N = j$. The results are shown in figure 3 and again we see that $F_\infty$ approximately follows a quadratic decrease with increased perturbation strength. However, while the saturation level at the limit of strong perturbation does reach the expected $3/N$ at the same perturbation strength as for the full QKT, the intermediate perturbation strengths lead to a saturation level that is higher then for the full QKT. The coefficient $C_{oe}$ is significantly higher than that of the CUE or COE maps.

In conclusion, we have given a theoretical argument estimating the saturation level of fidelity decay, $F_\infty$, for initial states that are eigenstates of the system for intermediate perturbation strengths. Numerical simulations for systems with and without classical analogs agree with the theoretical predictions. However, the presence of invariant subspaces appears to influence the saturation level of the fidelity decay.

This work was supported by DARPA/MTO through ARO grant DAAG55-97-1-0342 and by the Cambridge-MIT Institute.

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