Finite-size scaling of free energy in the dimer model on a hexagonal domain

A.A. Nazarov

1Department of Physics, St. Petersburg State University,
Ulyanovskaya 1, 198504 St. Petersburg, Russia
email: antonnaz@gmail.com

Abstract

We consider dimer model on a hexagonal lattice. This model can be seen as a “pile of cubes in the corner”. The energy of configuration is given by the volume of the pile and the partition function is computed by the classical MacMahon formula or, more formally, by the determinant of Kasteleyn matrix. We use the expression for the partition function to derive the scaling behavior of free energy in the limit of lattice mesh tending to zero and temperature tending to infinity. We discuss the universality and physical meaning of expansion coefficients.

Introduction

The dimer model appeared in attempt to extend a statistical theory of perfect solutions in chemistry to the case of liquid mixtures with molecules of two very distinct sizes [1]. The molecules were represented by the rigid tiles on a lattice and the number of tilings was approximately estimated. But an exact computation was not accessible at the time.

In 1961, Kasteleyn [2], Temperley and Fisher [3] represented the partition function of the dimer model as a Pfaffian of the signed adjacency matrix (“Kasteleyn matrix”) thus allowing the computation of the number of tilings and of free energy scaling limit. This result was very elegantly used by Fisher to solve Ising model [4] and by Fan and Wu [5] to compute free energy for a certain case of eight-vertex model.

Further studies of dimer models revealed the connection to the theory of alternating matrices [6, 7]. Later, the well-known limit shape phenomenon [8] was discovered for dimer models. First, the “Arctic circle” theorem was proven for domino tilings of the domain in the form of “Aztec diamond” [9]. Then similar result was obtained for a hexagonal domain on the hexagonal lattice [10]. Soon the connection of these results with the theory of random matrices was established [11]. The papers [12, 13] present a detailed exposition of the limit shape phenomenon in the dimer models.
Dimer models are the integrable lattice models of statistical physics that are now under an active theoretical [14, 15] and numerical investigation [16]. Computation of correlation functions is a common problem for all vertex models [17] as well as for dimer models. Another problem of great interest is the study of limit shapes in various cases [18, 19].

Configurations of dimer model on a hexagonal lattice are in one-to-one correspondence with the configurations of five-vertex model that appears for certain choice of parameters in the well-known six-vertex model [20, 21].

Study of dimer models on various lattices and domains led to interesting connections with the geometry of curved manifolds and with spectra of discrete and continuous Dirac and Laplace operators [22, 23]. Scaling limit of dimer model is proven to be described by a Gaussian free-field theory [24], but finite-size corrections were not considered previously. These corrections are important to close the gap between numerical simulations and theoretical results.

We consider a particular case of the dimer model on a hexagonal domain of hexagonal lattice, that can be seen as a “pile of cubes in the corner”. The energy of configuration is the total number of cubes. For this particular case we use an exact combinatorial formula for the partition function to derive the expressions for scaling limit of free energy and first three terms of the finite-size corrections. We show that the first term is identically zero. The second term has a universal part that is given by the Euler constant and a geometry-dependent part that is written explicitly in terms of elementary functions. The third term which encapsulates logarithmic dependence on the mesh size is connected with the central charge of the effective field theory.

This result is supported by numeric simulations presented in our previous publication [25].

1 Model definition

The configurations of the dimer model are perfect matchings (sets of non-touching edges, covering all the vertices) on some graph $G$ with some choice of weights $\omega(\epsilon)$ on the edges. The model is solvable on the bipartite graphs, i.e. the partition function can be computed if the weights are introduced in such a way that for each face bounded by 0 mod 4 edges there is an odd number of negative edge weights and each face with 2 mod 4 edges has an even number of negative edge weights. Then the signs of the edge weights form a so-called “Kasteleyn orientation” on graph, the weighting is called “Kasteleyn weighting” [21, 13].

For a bipartite graph $G$, color the vertices black and white in such a way that all the vertices adjacent to the black one are white. Denote by $B, W$ the sets of black and white vertices and by $b, w$ the elements of these sets.

The weights can be encoded as the “Kasteleyn matrix” – weighted, signed adjacency matrix $K$ with the matrix elements $K(w, b)$ equal to the weight of the edge $w \rightarrow b$: $K(w, b) = \omega(w \rightarrow b)$.

Then the partition function is equal to the absolute value of the determinant
of the Kasteleyn matrix\cite{2, 3}:

\[ Z = \sum_{\text{conf}} \prod_{e \in \text{conf}} w(e) = |\det K| \]  

(1)

Kasteleyn matrix defines a discrete Dirac operator $D$, the action of $D$ on a function $f$ defined on vertices is given by:

\[ (Df)(v) = \sum_u K(v, u)f(u) \]  

(2)

Kenyon \cite{22, 23} and others \cite{26} considered asymptotics of the determinants of the discrete Dirac and Laplace operators, the problem that, as can be seen from the above, is very close to the scaling of the free energy. But the finite size corrections to the free energy scaling were not computed.

We consider coverings of the hexagonal domain on the hexagonal lattice consisting of the subsets of lattice edges such that every vertex is the endpoint of exactly one edge.

We can draw a rhombus on a dual lattice around each edge in the configuration. The picture of “cubes in the corner” presented in the Fig. 1 is obtained. Let us write on the top of each uppermost cube the height of its column of cubes. Looking at this picture from the top, we obtain a height function defined on the rectangular domain of the square lattice.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A configuration of dimers on the hexagonal lattice and a corresponding picture of “cubes in the corner”.
}
\end{figure}

Let us define the sizes $M$, $N$, and $K$ of the sides of the hexagon. The above description can be formalized by setting the non-negative numbers up to $K$ in the boxes of the rectangular $M \times N$ table so that a value in each box is not greater than values in the adjacent upper and left boxes

\[ h_{ij} \leq h_{i-1,j}, \quad h_{ij} \leq h_{i,j-1}. \]  

(3)
The weight of a particular configuration is given by the exponent of the volume of all cubes or by a sum of the height function values:

\[ E[\text{conf}] = \sum_{i,j} h_{ij} = \text{Volume} \]

We set Boltzmann constant equal to 1 and choose system of units in such a way that there is no coupling constant in the expression for energy. Then the partition function is

\[ Z = \sum_{\text{conf}} e^{-E[\text{conf}]/T} = \sum_{\text{conf}} q^{\text{Volume}[\text{conf}]}, \]

where \( q = \exp(-1/T) \).

For this particular case, the partition function is given by the classical Macmahon combinatorial formula \[27\]

\[ Z[M, N, K, q] = \prod_{i=1}^{M} \prod_{j=1}^{N} \prod_{k=1}^{K} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \] (4)

MacMahon formula is obtained for the following definition of the Kasteleyn matrix. Embed the hexagonal lattice in the complex plane \( \mathbb{C} \) in such a way that some edges are parallel to the real line with and corresponding vertices have coordinates with integer real and imaginary parts. Then take

\[ K(w, b) = q^{\Re w + \Im w} \text{ if } \Im w = \Im b \] (5)

\[ K(w, b) = 1 \text{ if } \Im w \neq \Im b \] (6)

The free energy per site is defined as\(^1\)

\[ f = -\frac{1}{V} \ln Z(M, N, K, q) \]

Here \( V \) is the number of vertices, it is twice the number of dimers and twice the number of cube faces:

\[ V = 2(MN + NK + MK) \]

We are interested in the scaling limit, combined with the thermodynamic limit, when \( T \to \infty \), and \( M, N, K \to \infty \), such that ratios \( \frac{M}{T} = a, \frac{N}{T} = b, \frac{K}{T} = c \) remain fixed. In what follows we use \( \varepsilon = \frac{1}{T} \), which can be seen as the scale of the model, e.g. mesh size due to our choice of the system of units.

In the next section we compute the asymptotic expansion of the free energy \( f \) in \( \varepsilon \) and derive \( \varepsilon \)-independent closed expressions for the first several coefficients in this expansion.

\(^1\)For convenience we omit the factor \( \frac{1}{T} \) in the usual definition of the free energy.
2 The computation of the free energy asymptotic expansion

First we substitute MacMahon formula (4) into the free energy definition (1) and obtain

\[
f = - \frac{1}{V} \ln Z = - \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{K} \frac{1}{V} \ln \left( \frac{1 - e^{-\varepsilon(i+j+k)}e^\varepsilon}{1 - e^{-\varepsilon(i+j+k)}e^{2\varepsilon}} \right)
\] (7)

Expanding the exponents \(e^\varepsilon, e^{2\varepsilon}\) and logarithms in powers of \(\varepsilon\) we obtain

\[
f = - \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{K} \frac{1}{V} \left\{ g(i\varepsilon, j\varepsilon, k\varepsilon) + \frac{3}{2} \varepsilon \left[ g(i\varepsilon, j\varepsilon, k\varepsilon) + g(i\varepsilon, j\varepsilon, k\varepsilon)^2 \right] 
+ \frac{7}{6} \varepsilon^2 \left[ g(i\varepsilon, j\varepsilon, k\varepsilon) + 3g(i\varepsilon, j\varepsilon, k\varepsilon)^2 + 2g(i\varepsilon, j\varepsilon, k\varepsilon)^3 \right] \right\} + O(\varepsilon^3)
\] (8)

Here and below we will use the notation:

\[
g(x, y, z) = \frac{1}{e^{x+y+z} - 1}
\] (9)

This expression looks like a combination of Riemann sums for some integrals. Note that these sums are finite, but some of the corresponding integrals are divergent.

Let \(G\) be an analytic function in the cube \([0, a] \times [0, b] \times [0, c]\), then

\[
\int_0^a \int_0^b \int_0^c G(x, y, z)dx \ dy \ dz \approx \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{K} \varepsilon^3 G(i\varepsilon, j\varepsilon, k\varepsilon) - \\
- \left[ \frac{\varepsilon^4}{2} \left( (\partial_x + \partial_y + \partial_z)G \right) (i\varepsilon, j\varepsilon, k\varepsilon) \right] + \\
\varepsilon^5 \left( \frac{1}{6} (\partial_x^2 + \partial_y^2 + \partial_z^2) + \frac{1}{4} (\partial_x \partial_y + \partial_x \partial_z + \partial_y \partial_z) \right) G(i\varepsilon, j\varepsilon, k\varepsilon) \} + O(\varepsilon^6)
\] (10)

This formula can be easily derived by dividing the volume of integration into cubes with side \(\varepsilon\) and substituting Taylor series for \(G\) into integrals.

We can use this formula to approximate sums by integrals. Similar formula can be written for a function of two variables and double sums. We will not need higher order terms, but will use the approximation:

\[
\sum_{j=1}^{N} \sum_{k=1}^{K} \varepsilon^2 G(i\varepsilon, j\varepsilon, k\varepsilon) \approx \int_0^b \int_0^c G(i\varepsilon, y, z)dy \ dz + O(\varepsilon^3))
\] (11)

Note that

\[
\partial_x g(x, y, z) = -g(x, y, z) - g(x, y, z)^2 \\
\partial_x^2 g(x, y, z) = g(x, y, z) + 3g(x, y, z)^2 + 2g(x, y, z)^3
\] (12)
Taking this into account and using the relation (10) to express the triple sum
\[
\sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{K} g(i\varepsilon, j\varepsilon, k\varepsilon)
\] in (8) as the integral plus corrections, we obtain
\[
f = -\frac{1}{2(ab + bc + ca)} \left\{ \int_{0}^{a} \int_{0}^{b} \int_{0}^{c} \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{e^{x+y+z} - 1} \right\} + \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{K} \frac{\varepsilon^5}{12} \left[ g(i\varepsilon, j\varepsilon, k\varepsilon) + 3g(i\varepsilon, j\varepsilon, k\varepsilon)^2 + 2g(i\varepsilon, j\varepsilon, k\varepsilon)^3 \right]
\] (13)

Due to the equations (12), the corrections are of the same form as the higher order terms in \(\varepsilon\). The linear in \(\varepsilon\) term in (8) is cancelled by three partial derivatives in (10). The coefficient of the quadratic term is changed by the addition of second derivatives from (10) to \(\frac{1}{12}\).

There is no singularity in the sum, but \(\int_{0}^{a} \int_{0}^{b} \int_{0}^{c} \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{\exp(x+y+z) - 1}\) diverges logarithmically. So we cannot just apply the integral representation (10) directly.

We apply the formula (11) to the internal sums in the triple sum, since there is no singularity in \(\frac{1}{(\exp(x+y+z) - 1)}\) for \(i \neq 0\):

\[
\sum_{i=1}^{M} \left( \sum_{j=1}^{N} \sum_{k=1}^{K} \varepsilon^5 \frac{e^{2(i\varepsilon+j\varepsilon+k\varepsilon)} + e^{i\varepsilon+j\varepsilon+k\varepsilon}}{(e^{i\varepsilon+j\varepsilon+k\varepsilon} - 1)^3} \right) \approx \frac{\varepsilon^3}{12} \sum_{i=1}^{M} \int_{0}^{b} \int_{0}^{c} \left( \frac{e^{2(i\varepsilon+y+z)} + e^{i\varepsilon+y+z}}{(e^{i\varepsilon+y+z} - 1)^3} \right) \mathrm{d}y \, \mathrm{d}z + \mathcal{O}(\varepsilon^3)
\] (14)

The double integral can be taken explicitly, so we obtain
\[
\frac{\varepsilon^3}{12} \sum_{i=1}^{M} \left( \frac{1}{e^{b+c+i\varepsilon} - 1} + \frac{1}{1 - e^{b+i\varepsilon}} + \frac{1}{1 - e^{c+i\varepsilon}} + \frac{1}{e^{i\varepsilon} - 1} \right)
\] (15)

The function \(\frac{1}{e^{x} - 1}\) has a divergence, so we can not apply integral approximation directly. We first need to subtract and add \(\frac{1}{i\varepsilon}\). The sum over \(i\) of this term can be approximated using Euler formula:
\[
\sum_{i=1}^{M} \frac{1}{i} = \frac{1}{M} \sum_{i=1}^{M} \frac{1}{i/M} = \int_{\frac{1}{M}}^{1} \frac{\mathrm{d}x}{x} + \gamma + \mathcal{O}(\varepsilon) = -\ln \varepsilon + \ln a + \gamma + \mathcal{O}(\varepsilon)
\] (16)

Thus we get the integral
\[
\frac{\varepsilon^3}{12} \left[ \sum_{i=1}^{M} \left( \frac{1}{e^{b+c+i\varepsilon} - 1} + \frac{1}{1 - e^{b+i\varepsilon}} + \frac{1}{1 - e^{c+i\varepsilon}} + \frac{1}{e^{i\varepsilon} - 1} \right) - \ln \varepsilon + \ln a + \gamma + \mathcal{O}(\varepsilon^3) \right]
\] \approx \frac{\varepsilon^2}{12} \left[ \int_{0}^{a} \left( \frac{1}{e^{b+c+x} - 1} + \frac{1}{1 - e^{b+x}} + \frac{1}{1 - e^{c+x}} + \frac{1}{e^{x} - 1} \right) \mathrm{d}x - \ln \varepsilon + \ln a + \gamma \right] + \mathcal{O}(\varepsilon^3)
\] (17)
The integral is now taken explicitly, it is equal to
\[
\frac{\varepsilon^2}{12} \left[ \int_0^a \left( \frac{1}{e^{b+c+x} - 1} + \frac{1}{1 - e^{b+x}} + \frac{1}{1 - e^{c+x}} + \frac{1}{e^x - 1} - \frac{1}{x} \right) \, dx - \ln \varepsilon + \ln a + \gamma \right] =
\]
\[
\frac{\varepsilon^2}{12} \left[ \ln \left( \frac{(e^a - 1)(e^b - 1)(e^c - 1)(e^{a+b+c} - 1)}{a(e^{a+b} - 1)(e^{b+c} - 1)(e^{a+c} - 1)} \right) - \ln \varepsilon + \ln a + \gamma \right]
\]
(18)

The term \(\ln a\) is cancelled by the and symmetry between \(a, b, c\) is restored. Substituting this approximation into the expression (13) we obtain the final expression:
\[
f = \frac{1}{2(ab + bc + ca)} \left\{ \int_0^a \int_0^b \int_0^c \frac{dx \, dy \, dz}{e^{x+y+z} - 1} \right\} + \frac{\varepsilon^2}{12} \left[ \ln \left( \frac{(e^a - 1)(e^b - 1)(e^c - 1)(e^{a+b+c} - 1)}{a(e^{a+b} - 1)(e^{b+c} - 1)(e^{a+c} - 1)} \right) - \gamma \right] + \frac{\varepsilon^2}{12} \ln \varepsilon \right\} + \mathcal{O}(\varepsilon^3)
\]
(19)

3 Physical meaning of the expansion coefficients

The expansion (19) can be written as
\[
f = f_0 + f_1 \varepsilon + f_2 \varepsilon^2 \ln \varepsilon + f_3 \varepsilon^2,
\]
(20)
where the coefficients are
\[
f_0 = -\frac{1}{2(ab + bc + ca)} \left\{ \int_0^a \int_0^b \int_0^c \frac{dx \, dy \, dz}{e^{x+y+z} - 1} \right\},
\]
(21)
\[
f_1 = 0,
\]
(22)
\[
f_2 = -\frac{1}{2(ab + bc + ca)} \frac{1}{12},
\]
(23)
and
\[
f_3 = -\frac{1}{2(ab + bc + ca)} \frac{1}{12} \left[ \ln \left( \frac{(e^a - 1)(e^b - 1)(e^c - 1)(e^{a+b+c} - 1)}{a(e^{a+b} - 1)(e^{b+c} - 1)(e^{a+c} - 1)} \right) - \gamma \right].
\]
(24)

In the paper [25] we have presented results of Monte Carlo simulations using Metropolis and Wang-Landau algorithms that support the form of expansion (20). In particular we got \(f_1 = 0.04 \pm 0.04\) which is consistent with \(f_1 = 0\).

In the scaling limit \(\varepsilon \to 0\) so called “limit shape phenomenon” [9] [10] appears in the dimer model. The areas around the corners of the domain are “frozen” with height function values being fixed. An analytical “Arctic curve” delimits frozen regions from the region where the behavior is described by the effective free field theory [13] [28] [12].
The behavior of the logarithm of the partition function is generic in two-dimensional models.

First two terms $f_0$ and $f_1$ are interpreted as a bulk and boundary free energies in the corresponding field theory. Since we have $f_1 = 0$, we can conclude that boundary tension is zero. The value of the first term $f_0$ can be rewritten using polylogarithm functions as

$$f_0 = \frac{1}{2(ab + bc + ca)} \left[ abc + \text{Li}_3(e^a) + \text{Li}_3(e^b) + \text{Li}_3(e^c) - \text{Li}_3(e^{a+b}) - \text{Li}_3(e^{b+c}) - \text{Li}_3(e^{a+c}) + \text{Li}_3(e^{a+b+c}) - \zeta(3) \right]$$ (25)

Here Riemann zeta function appears as a particular value of polylogarithm $\text{Li}_3(1) = \zeta(3)$.

The term proportional to the logarithm of the scale $\varepsilon$ is also universal and should appear in all two-dimensional theories with boundary. In the paper it was argued that on a manifold of a characteristic length $L$ with a smooth boundary such a term must have the following form:

$$\delta F = -\frac{1}{6} c \chi \ln L,$$ (26)

where $c$ is central charge of the effective field theory and $\chi$ is the Euler characteristic of the manifold

$$\chi = 2 - 2h - b,$$ (27)

where $h$ is the number of handles and $b$ is the number of boundaries.

Since the non-frozen domain is delimited by a smooth boundary, we have $\chi = 1$ and we interpret $2(ab + bc + ca)$ as a volume of the domain, thus

$$\delta F = (2(ab + bc + ca)) f_2 \ln \varepsilon,$$ (28)

which together with formula (25) suggests the value of central charge $c = \frac{1}{2}$. This value is in agreement with the identification of the dimer model with free fermions in the paper.

The last term $f_3$ depends only on the shape of the domain through $a, b, c$ with a universal contribution that is equal to the Euler constant $\gamma$. We conjecture that Euler constant will appear even for a different choice of Kasteleyn weighting, due to the treatment of logarithmic divergency. In the future work we will consider coordinate dependent values of $q$ and more complex geometries, as was done in the paper, to check this suggestion.

**Conclusion and outlook**

In the present paper we computed the asymptotic expansion of the free energy in the dimer model on a hexagonal domain of the hexagonal lattice. We’ve discussed the physical meaning of the expansion coefficients and argued that
our results support the identification of the scaling behavior of the dimer model with the free-fermion field theory.

In further work we will show the connection of the expansion coefficients with the spectral properties of Dirac operator on the non-frozen domain and study the universality of the presented expressions by considering the model on more generic domain geometries with the non-uniform Kasteleyn weighting with \( q \) depending on the coordinate.

Acknowledgments

I am grateful to professor Nikolai Reshetikhin for his guidance in this work. I thank Pavel Belov for useful discussions and general support.

I thank the organizers and participants of the conference MQFT-2018 for the opportunity to present our results and useful discussions.

This research is supported by RFBR grant No. 18-01-00916.

References

[1] G. S. Fowler, R. H.; Rushbrooke, “An attempt to extend the statistical theory of perfect solutions,” Transactions of the Faraday Society 33 (1937).

[2] P. Kasteleyn, “The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice,” Physica 27 (1961).

[3] H. N. V. Temperley and M. E. Fisher, “Dimer problem in statistical mechanics—an exact result,” The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics 6 (1961) no. 68, 1061–1063.

[4] M. E. Fisher, “On the dimer solution of planar Ising models,” Journal of Mathematical Physics 7 (1966) no. 10, 1776–1781.

[5] F. Y. Fan, Chungpeng; Wu, “General Lattice Model of Phase Transitions,” Physical Review B 2 (8, 1970).

[6] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, “Alternating-sign matrices and domino tilings (Part I),” Journal of Algebraic Combinatorics 1 (1992) no. 2, 111–132.

[7] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, “Alternating-sign matrices and domino tilings (Part II),” Journal of Algebraic Combinatorics 1 (1992) no. 3, 219–234.
[8] A. M. Vershik and S. V. Kerov, “Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux,” in Soviet Math. Dokl, vol. 18, pp. 527–531. 1977.

[9] W. Jockusch, J. Propp, and P. Shor, “Random Domino Tilings and the Arctic Circle Theorem,” arXiv Mathematics e-prints (Jan., 1998) math/9801068, arXiv:math/9801068 [math.CO]

[10] H. Cohn, M. Larsen, and J. Propp, “The shape of a typical boxed plane partition,” New York J. Math 4 (1998) no. 137, 165.

[11] K. Johansson, “Non-intersecting paths, random tilings and random matrices,” Probability theory and related fields 123 (2002) no. 2, 225–280.

[12] R. Kenyon, A. Okounkov, and S. Sheffield, “Dimers and amoebae,” Annals of mathematics (2006) 1019–1056.

[13] R. Kenyon, “Lectures on dimers,” arXiv preprint arXiv:0910.3129 (2009).

[14] P. Zinn-Justin, “Six-vertex model with domain wall boundary conditions and one-matrix model,” Phys. Rev. E 62 (Sep, 2000) 3411–3418.

[15] P. L. Ferrari and H. Spohn, “Domino tilings and the six-vertex model at its free-fermion point,” Journal of Physics A: Mathematical and General 39 (2006) no. 33, 10297.

[16] D. Keating and A. Sridhar, “Random Tilings with the GPU,” ArXiv e-prints (Apr., 2018), arXiv:1804.07250 [cs.OH]

[17] F. Colomo and A. G. Pronko, “An approach for calculating correlation functions in the six-vertex model with domain wall boundary conditions,” Theoretical and Mathematical Physics 171 (2012) no. 2, 641–654.

[18] A. Borodin, V. Gorin, and E. M. Rains, “q-Distributions on boxed plane partitions,” Selecta Mathematica 16 (2010) no. 4, 731–789.

[19] P. Di Francesco and E. Guitter, “A tangent method derivation of the arctic curve for q-weighted paths with arbitrary starting points,” arXiv preprint arXiv:1810.07936 (2018).

[20] V. Kapitonov and A. G. Pronko, “Weighted enumerations of boxed plane partitions and inhomogeneous five-vertex model,” Zapiski Nauchnykh Seminarov POMI 398 (2012) 125–144.

[21] V. Kapitonov and A. G. Pronko, “The five-vertex model and boxed plane partitions,” Zapiski Nauchnykh Seminarov POMI 360 (2008) 162–179.

[22] R. Kenyon, “The Laplacian and Dirac operators on critical planar graphs,” Inventiones mathematicae 150 (2002) no. 2, 409–439.
[23] R. Kenyon, “The asymptotic determinant of the discrete Laplacian,” Acta Mathematica 185 (2000) no. 2, 239–286.

[24] R. Kenyon, “Dominoes and the Gaussian free field,” Annals of probability (2001) 1128–1137.

[25] P. A. Belov, A. I. Enin, and A. A. Nazarov, “Finite size scaling in the dimer and six-vertex model,” Journal of Physics: Conference Series 1135 (2018) no. 1, 012024, arXiv:1809.05599.

[26] A. Sridhar, “Asymptotic Determinant of Discrete Laplace-Beltrami Operators,” arXiv preprint arXiv:1501.02057 (2015).

[27] M. Vuletić, “A generalization of MacMahon’s formula,” Transactions of the American Mathematical Society 361 (2009) no. 5, 2789–2804.

[28] R. Kenyon, “Height fluctuations in the honeycomb dimer model,” Communications in Mathematical Physics 281 (2008) no. 3, 675.

[29] J. L. Cardy and I. Peschel, “Finite-size dependence of the free energy in two-dimensional critical systems,” Nuclear Physics B 300 (1988) 377–392.

[30] R. Dijkgraaf, D. Orlando, S. Reffert, et al., “Dimer models, free fermions and super quantum mechanics,” Advances in Theoretical and Mathematical Physics 13 (2009) no. 5, 1255–1315.

[31] A. Okounkov and N. Reshetikhin, “Random skew plane partitions and the Pearcey process,” Communications in mathematical physics 269 (2007) no. 3, 571–609.