Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions

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Abstract

Recently, folk questions on the smoothability of Cauchy hypersurfaces and time functions of a globally hyperbolic spacetime $M$, have been solved. Here we give further results, applicable to several problems:

(1) Any compact spacelike acausal submanifold $H$ with boundary can be extended to a spacelike Cauchy hypersurface $S$. If $H$ were only achronal, counterexamples to the smooth extension exist, but a continuous extension (in fact, valid for any compact achronal subset $K$) is still possible.

(2) Given any spacelike Cauchy hypersurface $S$, a Cauchy temporal function $T$ (i.e., a smooth function with past-directed timelike gradient everywhere, and Cauchy hypersurfaces as levels) with $S = T^{-1}(0)$ is constructed -thus, the spacetime splits orthogonally as $\mathbb{R} \times S$ in a canonical way.

Even more, accurate versions of this last result are obtained if the Cauchy hypersurface $S$ were non-spacelike (including non-smooth, or achronal but non-achronal). Concretely, we construct a smooth function $\tau : M \to \mathbb{R}$ such that the levels $S_t = \tau^{-1}(t), t \in \mathbb{R}$ satisfy: (i) $S = S_0$, (ii) each $S_t$ is a (smooth) spacelike Cauchy hypersurface for any other $t \in \mathbb{R}\{0\}$. If $S$ is also acausal then function $\tau$ becomes a time function, i.e., it is strictly increasing on any future-directed causal curve.

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1 Introduction

The question whether the Cauchy hypersurfaces and Cauchy time functions of a globally hyperbolic spacetime $(M, g)$ can be taken smooth, have remained as an open folk question in Lorentzian Geometry until the recent full solution [2, 3]. Essentially, $(M, g)$ not only contains a (smooth) spacelike Cauchy hypersurface [2], but also a Cauchy temporal function $T$ and, thus, a global smooth splitting $M = \mathbb{R} \times S_0$, such that each level at constant $T$ is a Cauchy hypersurface, orthogonal to the vector field $\partial/\partial T$ naturally induced from the $\mathbb{R}$ part [3] (see also [4, Section 2] for a brief history of the problem, and [15] for related questions, including the smoothability of time functions in stably causal spacetimes). Even though such results are enough from the conceptual viewpoint as well as for typical applications in General Relativity ([16], [1], [6], etc.), the importance of spacelike Cauchy hypersurfaces for both, classical GR (Cauchy problem for Einstein’s equation) and its quantization suggests related problems.

In the present note, two related problems on smoothability are solved. The first one concerns the problem on extendability of a compact spacelike submanifold to a full smooth Cauchy hypersurface $S$. Concretely, we will prove:

**Theorem 1.1** Let $(M, g)$ be a globally hyperbolic spacetime, and let $H \subset M$ be a spacelike and acausal compact $m$–submanifold with boundary. Then, there exists a spacelike Cauchy hypersurface $S$ such that $H \subset S$.

We emphasize that all the hypotheses are necessary and, in fact, if $H$ were only achronal instead of acausal, then no smooth Cauchy hypersurface extending $H$ will exist in general (Example 3.4) –even though a non-smooth one does exist, Theorem 4.8. Moreover, in the definitions below we will assume that submanifolds are connected (an usual simplifying convention) but the proof of Theorem 1.1 is extended to the non-connected case easily, Remark 4.14.

For the second problem, recall first that the Cauchy temporal function $T$ in [3] is constructed by means of a quite long process, which uses the previously constructed Geroch’s Cauchy time function [9]. So, even though the level hypersurfaces of $T$ are proven to be Cauchy, they become rather uncontrolled. Therefore, it is natural to wonder:

Given a spacelike Cauchy hypersurface $S$, is there a Cauchy temporal function $T$ such that $S$ is one of its levels?

Even more, if $S$ were only a topological Cauchy hypersurface, we can wonder if it can be approximated by spacelike ones $S_t$ in the strongest possible sense, that is, by finding a Cauchy time function $\tau$ (everywhere smooth and with timelike gradient except at most on $S$) such that $S_t = \tau^{-1}(t)$, for all $t \in \mathbb{R} \setminus \{0\}$, and $S = \tau^{-1}(0)$.

Our purpose is to answer affirmatively these questions. First, we study the last one. A continuous function $\tau$ fulfilling all the other required properties is shown to exist easily, Proposition 5.17. By using techniques in [2, 3], we show that $\tau$ can be chosen smooth everywhere (with gradient vanishing only at $S$, etc.):
which maybe non-smooth), Theorem 5.15. In this theorem, if $S$ is acausal then $\tau$ will be automatically a smooth time function. Nevertheless, it is not evident to prove that, if $S$ is spacelike, then $\tau$ can be chosen with timelike gradient at $S$. This is carried out by using again techniques in \[3\], proving finally:

**Theorem 1.2** Let $(M, g)$ be globally hyperbolic, and $S$ a spacelike Cauchy hypersurface. Then, there exists a Cauchy temporal function $\mathcal{T} : M \rightarrow \mathbb{R}$ such that $S = \mathcal{T}^{-1}(0)$.

These problems become natural as purely geometric problems in Lorentzian Geometry, applicable, for example, to the initial value problem in General Relativity. In fact, the typical initial 3-Riemannian manifold for Einstein equation would be (a posteriori) one of the levels of a Cauchy temporal function. Thus, from the viewpoint of foundations, one should expect that any spacelike Cauchy hypersurface can be chosen as one such level, whatever the stress-energy tensor in Einstein’s equation maybe –as ensured by Theorem 1.2. Even more, some gluing methods to solve Einstein equations make natural to extend a compact spacelike hypersurface with boundary $H$ to a spacelike Cauchy hypersurface $S$ under some restrictions \[7, 11\]. Theorem 1.2 is then a fundamental claim, valid in general. On the other hand, the interest of such a $S$ when $H$ is only a 2-surface (composed by the fixed points of a 1-parameter group of isometries) is stressed in the report by Kay and Wald \[10\].

Nevertheless, in spite of these “classical” applications, we are in debt with researchers in quantization who suggested us both problems. Concretely, the first problem applies in locally covariant quantum field theories \[5\], and allows to simplify Ruzzi’s proof of the punctured Haag duality \[14\]. The second one is useful to prove the uniqueness of solutions to hyperbolic equations in the context of quantization.

This paper is organized as follows. In Section 2, conventions and basic facts are recalled. The extendability of subsets to Cauchy hypersurfaces is studied in Section 3 from the topological viewpoint; then, smooth extensions are found in Section 4. In Section 5 any Cauchy hypersurface is proven to be the level of a Cauchy function $\tau$ which is not only continuous but also smooth. The final improvement, i.e., $\tau$ can be chosen temporal if $S$ is spacelike, is carried out in Section 6.

### 2 Set-up

Throughout this paper we will use background results, notation and conventions as in \[2, 3\]. In particular, $(M, g)$ will denote a spacetime, i.e., a connected time-oriented $C^k$ Lorentzian $n_0$-manifold, $n_0 \geq 2, k = 1, 2, \ldots, \infty$; smooth will mean $C^k$-differentiable. The signature will be chosen $(-, +, \ldots, +)$ and, thus, for a timelike (resp. causal, lightlike, spacelike) tangent vector $v \neq 0$, one has $g(v, v) < 0$ (resp. $\leq 0, = 0, > 0$); following \[12\], vector 0 will be regarded as spacelike. The topological closure of a subset $G$ is denoted as $\overline{G}$; if $G$ is a submanifold with boundary, the boundary is denoted $\partial G$. 


A convex neighborhood $C_p$ of $p \in M$ is an open subset which is a normal neighborhood of all its points $q \in C_p$; in particular, each two points $q, q' \in C_p$ can be joined by a unique geodesic entirely contained in $C_p$. We will also consider that a complete Riemannian metric $g_R$ is fixed and the $g_R$–diameter of any $C_p$ is assumed to be $< 1$ (further conditions on convex neighbourhoods are frequent; see for example simple neighbourhoods in [13]).

Hypersurfaces and submanifolds will be always embedded, and they will be regarded as connected and without boundary, except if otherwise specified. In general, they are only topological, but the spacelike ones will be regarded as smooth.

Recall that a Cauchy hypersurface is a subset $S \subset M$ which is crossed exactly once by any inextendible timelike curve (and, then, at least once by any inextendible causal curve); this implies that $S$ is a topological hypersurface (see, for example, [12, Lemma 14.29]). Cauchy hypersurfaces are always achronal, but not always acausal; nevertheless, spacelike Cauchy hypersurfaces are necessarily acausal.

A time function is a continuous function $t$ which increases strictly on any future-directed causal curve. A temporal function is a smooth function with past-directed timelike gradient everywhere. Temporal functions are always time functions, but even smooth time functions may be non-temporal (as will be evident below). Time and temporal functions will be called Cauchy if their levels are Cauchy hypersurfaces. Notation as $S_{t_i} = t^{-1}(t_i)$ or, simply, $S_i$ will be used for such levels.

In what follows, $(M, g)$ will be assumed globally hyperbolic. By Geroch’s theorem [9], it admits a Cauchy time function $t : M \to \mathbb{R}$ and, from the results in [3], even a Cauchy temporal function $T$ (these functions will be assumed onto without loss of generality). Given such a $T$, the metric admits a (globally defined) pointwise orthogonal splitting

$$g = -\beta dT^2 + g_T,$$

where $\beta > 0$ is a smooth function on $M$, and $g_T$ is a Riemannian metric on each level $S_T$.

Given a Cauchy hypersurface $S$, it is well known that $M$ can be regarded topologically as a product $\mathbb{R} \times S$, where $(s, x) \ll (s', x)$ if $s < s'$ (if $S$ is smooth then $M$ is also smoothly diffeomorphic to $\mathbb{R} \times S$). This result is obtained by moving $S$ with the flow of any complete timelike vector field, but then the slices at constant $s$ maybe non-Cauchy (even non-achronal). Nevertheless, the aim of Theorem 1.2 is to prove that (when $S$ is spacelike) one can take $s = T$ for some temporal function and, then, the prescribed $S$ is one of the levels of $T$ in (1).

3 Topological versions of Theorem 1.1

For any subset $K$, we write $J(K) := J^+(K) \cup J^-(K)$. By using global hyperbolicity, it is straightforward to check that, if $K$ is compact then $J^\pm(K)$ and $J(K)$ are closed.
First, we will prove a topological version of Theorem 1.1. For this version, the following simple property becomes relevant.

**Lemma 3.3** Assume that \( K \subset M \) is compact and acausal. Then \( K \) separates \( J(K) \), i.e., any curve \( \rho \) from \( J^-(K) \setminus K \) to \( J^+(K) \setminus K \) enterly contained in \( J(K) \) must cross \( K \).

**Proof.** Given such \( \rho \), there is a point \( p = \rho(s_0) \) which belongs to \( J^+(K) \cap J^-(K) \), because \( J^+(K) \) and \( J^-(K) \) are closed. As \( K \) is acausal, necessarily \( p \in K \).

If \( K \) is assumed only achronal instead of acausal, then the conclusion does not hold, as the following counterexample shows.

**Example 3.4** Consider the canonical Lorentzian cylinder \((\mathbb{R} \times S^1, g = -dt^2 + d\theta^2)\) and put \( K = \{(\theta/2, (\cos \theta, \sin \theta)) : \theta \in [0, 4\pi/3]\} \). Then, \( K \) is spacelike and achronal, but not acausal because \((0, (0, 0)) < (2\pi/3, (-1/2, -\sqrt{3}/2))\). Even more, \( J(K) \) is the whole cylinder, but \( K \) does not separates it.

Nevertheless, given such an achronal \( K \), a new achronal set \( K^C \supset K \) can be constructed which satisfies the conclusion in Lemma 3.3. Concretely, recall that if \( K \) is achronal but not acausal then there are points \( p, q \in K \), \( p < q \) which will be connectable by a lightlike geodesic segment \( \gamma_{pq} \) (without conjugate points except at most the extremes). Define the causal hull of \( K \) as the set \( K^C \subset M \) containing \( K \) and all the lightlike segments which connect points of \( K \). Obviously, \( J(K) = J(K^C) \) and, if \( K \) is compact and achronal, then so is \( K^C \). A straightforward modification of Lemma 3.3 yields:

**Lemma 3.5** Assume that \( K \subset M \) is compact and achronal. Then the causal hull \( K^C \) separates \( J(K) \).

The topological version of Theorem 1.1 is the following one (recall that \( K \) does not need to be a submanifold here):

**Proposition 3.6** Let \((M, g)\) be a globally hyperbolic spacetime, and let \( K \subset M \) be an acausal (resp. achronal) compact subset.

Then, there exists an acausal Cauchy (resp. a Cauchy) hypersurface \( S'_K \) such that \( K \subset S'_K \).

**Proof.** We will reason the acausal case; the achronal one is analogous, replacing along the proof \( K \) by \( K^C \) and using Lemma 3.5 instead of Lemma 3.3.

Notice first that, as \( J(K) \) is closed, its complement \( M' = M \setminus J(K) \), if non-empty, is an open subset of \( M \) which, regarded as a (possibly non-connected) spacetime, becomes globally hyperbolic too. In fact, \( M' \) is obviously strongly causal and, given any \( p, q \in M' \), the compact diamond \( J^+(p) \cap J^-(q) \subset M \) is included in \( M' \).

Now, consider any (possibly non-connected) acausal Cauchy hypersurface \( S' \) of \( M' \) (if \( M' = \emptyset \), put \( S' = \emptyset \)) and let us check that the choice \( S'_K = S' \cup K \)
is the required hypersurface. Recall first that $S'_K$ is an acausal subset of $M$, because if a (inextendible, future-directed) causal curve $\gamma$ crosses $K$ then it is completely contained in $J(K)$ and cannot cross $S'$.

On the other hand, if $\gamma$ is not completely contained in $J(K)$, it crosses obviously $S'$. Otherwise, it will cross $K$ because if, say, $\gamma(s_0) \in J^+(K)$, then $Z = J^-(\gamma(s_0)) \cap J^+(K)$ is compact and, as $\gamma$ cannot remain imprisoned towards the past in $Z$, it will reach $J^-(K)$, and Lemma \ref{lem:compact} can be claimed.

**Remark 3.7** The compactness of $K$ becomes essential in Proposition \ref{prop:compact} as one can check by taking $K$ as the upper component of hyperbolic space (or closed non-compact subsets of it) in Lorentz-Minkowski spacetime.

The following technical strengthening of Proposition \ref{prop:compact} shows (also at a topological level) that the Cauchy hypersurface can be controlled further, inside the levels of a Cauchy time function.

**Theorem 3.8** Fix a Cauchy time function $t : M \to \mathbb{R}$ in $(M,g)$. If $K$ is a compact acausal (resp. achronal) subset then there exists $t_1 < t_2$ and an acausal Cauchy (resp. a Cauchy) hypersurface $S_K \supset K$ such that:

$$S_K \subset J^+(S_1) \cap J^-(S_2).$$ (2)

**Proof.** Again, we will consider only the acausal case. From Proposition \ref{prop:compact} we can assume that $K$ is a subset of an acausal Cauchy hypersurface $S'_K$.

Let $t_1$ (resp. $t_2$) be the minimum (resp. maximum) of $t(K)$, choose any $t_0 \in \mathbb{R}$, and regard $S_1, S_2, S'_K$ as graphs on $S_0$, i.e.:

$$S'_K = \{(t'(x), x) : x \in S_0\}$$

for some continuous $t'_K : S_0 \to \mathbb{R}$, and analogously for $S_i$ (each function $t_i(x)$ is constantly equal to $t_i$, for $i = 1, 2$). Define $t_K : S_0 \to \mathbb{R}$ as:

$$t_K(x) = \begin{cases} 
  t'_K(x) & \text{if } t_1 \leq t'_K(x) \leq t_2 \\
  t_1 & \text{if } t'_K(x) < t_1 \\
  t_2 & \text{if } t_2 < t'_K(x) 
\end{cases}$$ (3)

Clearly, $t_K$ is continuous ($t_K = \text{Max}(t_1, \text{Min}(t_2, t'_K))$), and the corresponding graph $S_K$ is a closed topological hypersurface which includes $K$. To check that $S_K$ is Cauchy, recall first that it is crossed by any inextendible timelike curve $\gamma$ (as $\gamma$ must cross $S_1$ and $S_2$). It is also achronal because, if $p, q \in S_K$ were connectable by means of a future-directed causal curve $\gamma$, then at least one of them is not included in $S'_K$, say, $p = (t_1, x_1) \in S_1$. Recall that, as $t'_K(x_1) < t_1$, then $S'_K \ni (t'_K(x_1), x_1) \ll p < q$, and the acausality of $S'_K$ forces $q \notin S'_K$. Thus $q = (t_2, x_2) \in S_2$ but then $(t'_K(x_1), x_1) \ll p < q \ll (t'_K(x_2), x_2) \in S'_K$, in contradiction again with the acausality of $S'_K$. □

Recall from Example 3.4 that, in general, one cannot hope to extend a spacelike achronal compact hypersurface to a smooth Cauchy hypersurface. Thus,
Theorem 3.8 is our best result for this case\(^1\). Nevertheless, in the next section, we will see that the acausal ones can be smoothly extended. The following simple result for the acausal case, also suggest the main obstruction for the achronal one (again, the conclusion would not hold in Example 3.4).

**Lemma 3.9** In the hypotheses of Th. 1.1, there exist two hypersurfaces \(G_1, G_2\) such that the closures \(\bar{G}_1, \bar{G}_2\) are acausal, compact, spacelike hypersurfaces with boundary, and \(H \subset G_1, \bar{G}_1 \subset G_2\).

**Proof.** To prove the existence of \(G_1\) is enough. It is straightforward to prove the extendability of the compact \(k\)-submanifold with boundary \(H\) to an hypersurface \(G_1\) which can be chosen spacelike by continuity (and with \(\bar{G}_1\) compact). Let us prove that \(\bar{G}_1\) can be also chosen acausal.

Otherwise, taken any sequence of such hypersurfaces with boundary \(\{\bar{G}_n\}_n\) with \(G_{n+1} \subset \bar{G}_n\), and \(H = \cap_n G_n\), then no hypersurface \(G_n\) would be acausal. Thus, we can construct two sequences \(p_n < q_n, p_n, q_n \in \bar{G}_n\) (and the corresponding sequence of connecting causal curves \(\gamma_n\)) which, up to subsequences, converge to limits \(p, q\) in \(H\). If \(p \neq q\) this would contradict the acausality of \(H\) (the sequence \(\{\gamma_n\}_n\) would have a causal limit curve); otherwise, the strong causality of \((M, g)\) would be violated at \(p\). \(\blacksquare\)

### 4 The smoothing procedure for Theorem 1.1

In what follows, the notation and ambient hypotheses of Theorem 1.1 and Lemma 3.9 are assumed, and \(S_0 = S_{t_0}\) for a choice \(t_0 < t_1\). \(S_K\) will denote the Cauchy hypersurface in Theorem 3.8 obtained for \(K = \bar{G}_2\). Our aim is to smooth \(S_K\) outside \(H\).

**Remark 4.10** Notice that, by using the smoothness results \([2, 3]\), the Cauchy hypersurface \(S_K'\) in Proposition 3.6 is the graph of a function \(t_K'\), which can be assumed smooth everywhere but the points corresponding to \(\partial \bar{G}_2\). This suggests the possibility to obtain the required smooth hypersurface by smoothing \(t_K'\). Nevertheless, the limits of the derivatives of \(t_K'\) as one approaches \(\partial \bar{G}_2\) from outside are widely uncontrolled (say, for a sequence \(\{p_n\}_n \subset M\backslash \bar{G}_2\) converging to a point \(p \in \partial \bar{G}_2\), the sequence of –spacelike– tangent spaces \(T_{p_n} S_K'\) may converge to a degenerate hyperplane of \(T_p M\)). Thus, we prefer to follow systematically the approach in \([2]\).

We will need first the following two technical results. The first one was proved in \([2]\):

**Lemma 4.11** Let \(G_0\) be an open neighbourhood of \(H\) in \(G_1\), fix \(p \in S_K \backslash G_0\), and any convex neighborhood of \(p, C_p \subset I^+(S_0)\) which does not intersect \(H\).

\(^1\) The proofs of Theorem 3.8 and Proposition 3.9 also show that, if \(K\) is assumed only achronal, the points in \(S_K\) where the acausality is violated belongs to the convex hull \(K^C\).
Then there exists a smooth function
\[ h_p : M \to [0, \infty) \]
which satisfies:
(i) \( h_p(p) = 2 \).
(ii) The support of \( h_p \) is compact and included in \( \mathcal{C}_p \).
(iii) If \( q \in J^-(S_K) \) and \( h_p(q) \neq 0 \) then \( \nabla h_p(q) \) is timelike and past-directed.

In fact, we can take \( h_p(q) = a_p e^{-1/d(q,p')^2} \) where \( p' \in I^-(p) \) is chosen close to \( p \) and \( a_p \) is a constant of normalization (see [2, Lemma 4.12] for details). The second one can be seen as a refinement of previous lemma, in order to deal with the submanifold \( H \) instead of a point \( p \).

**Lemma 4.12** There exists a function
\[ h_G : M \to [0, \infty) \]
which is smooth on (a neighborhood of) \( J^-(S_K) \) and satisfies:
(i) \( h_G(q) = 1 \) for all \( q \) in some neighbourhood \( G_0 \) of \( H \) in \( G_1 \); \( h_G(S_K \setminus G_2) \equiv 0 \).
(ii) The support of \( h_G \) (i.e., the closure of \( h_G^{-1}(0, \infty) \)) is included in \( I^+(S_0) \) for some \( t_0 \in \mathbb{R} \).
(iii) If \( q \in J^-(S_K) \) and \( h_G(q) \neq 0 \) then \( \nabla h_G(q) \) is timelike and past-directed.

*Proof.* Essentially, \( h_G \) will be the time-separation function (up to a normalization and a suitable exponentiation, to make it smooth at 0) to a hypersurface \( \tilde{G} (= G^\epsilon \text{ for small } \epsilon) \) which behaves as in the figure.

**Figure 1:** \( \tilde{G}^0 \) is a spacelike graph on \( \tilde{G} (H \subset G_1; \tilde{G}_1 \subset G) \). Moreover, \( \tilde{G}_1 \subset G^0 \) and, thus, \( \phi_\epsilon(G_1) \subset \phi_\epsilon(G^0) \). For small \( \epsilon \), \( J^-(G_0) \cap G^\epsilon \subset \phi_\epsilon(G_1) \) and \( d(G^\epsilon, p) = \epsilon \) for all \( p \in G_0 \). \( \tilde{G} \) is taken equal to such \( G^\epsilon \).

Rigourously, let \( N \) be the unitary future-directed normal vector field to \( G_2 \), and let \( d_R \) be the distance associated to the induced Riemannian metric on \( G_2 \).
Denote as $d_R(\cdot, \tilde{G}_1)$ the $d_R$-distance function to $\tilde{G}_1$ on $\tilde{G}_2$, and let

$$\rho : \tilde{G}_2 \to [0, \infty), \quad \rho(p) = \exp(-d_R(p, \tilde{G}_1)^{-2}).$$

Note that this function is 0 on $\tilde{G}_1$, and smooth on some open neighbourhood $G$ of $\tilde{G}_1$. Even more, taking if necessary a smaller $G$, we can assume that the graph

$$\tilde{G}^0 = \{\exp(\rho(p)N_p) : p \in \tilde{G}\}$$

is a spacelike compact hypersurface with boundary.

Let $\epsilon_0 > 0$ be the minimum of $\rho$ on the boundary $\partial G$, and let

$$\Phi : [0, \delta_0) \times \tilde{G} \to M, \quad \Phi(s, p) = \exp(-sN_p),$$

where $\delta_0 < \epsilon_0$ is small enough to make $\Phi$ a diffeomorphism onto its image (see, for example, [13, Section 7] or [12, Ch. 10] for background details). Now, for any $\epsilon \in (0, \delta_0)$ the hypersurface

$$\tilde{G}^\epsilon = \{\Phi(\epsilon, p) = \exp(-\epsilon N_p) : p \in \tilde{G}^0\}$$

satisfies:

(a) $\partial G^\epsilon \subset I^+(S_K)$
(b) there are no focal points of $G^\epsilon$ in $I^+(G^\epsilon) \cap J^-(S_K)$.

Therefore, the time-separation function (Lorentzian distance) to $\tilde{G}^\epsilon$, $d(\cdot, \tilde{G}^\epsilon)$, is smooth on a neighbourhood of $I^+(G^\epsilon) \cap J^-(S_K)$, with past-directed timelike gradient. Even more, choosing $\epsilon$ small enough we have, for some neighbourhood $G_0$ of $H$:

(c) $J^-(G_0) \cap G^\epsilon \subset \Phi(\epsilon, G_1)$ and, thus, $d(p, G^\epsilon) = \epsilon$, for all $p \in G_0$.

Now, the required hypersurface $\tilde{G}$ is any such $G^\epsilon$ satisfying (a), (b), (c), and $h_G$ is then:

$$h_G(p) = \exp(\epsilon^{-2} - d(p, \tilde{G})^{-2}).$$

Proof: Consider for any $p \in S_K \setminus G_0$ the convex neighborhood $C_p$ in Lemma 4.11 and take the corresponding function $h_p$. Let $W_p = h_p^{-1}((1, \infty)) (W_p \subset C_p)$, and $W_G = h_G^{-1}(0, \infty)$. Obviously,

$$\mathcal{W} = \{W_p, p \in S_K \setminus G_0\} \cup \{W_G\}$$
covers the closed hypersurface $S_K$, and admits a locally finite subcovering $\mathcal{W'} = \{W_p, \ i \in \mathbb{N}\} \cup \{W_G\}$. Notice that $W_G$ must be included necessarily in the subcovering, because $H \cap W_p = \emptyset$ for all $p$. Then, it is easy to check that

$$h = h_G + \sum_i h_i. \quad (4)$$

fulfills all the requirements. ■

Proof of Theorem 1.1

Notice that, by continuity, property (iii) of Proposition 4.13 also holds in some neighbourhood $V$ of $S_K$. Thus, $h$ admits 1 as a regular value in $V \cup J^-(S_K)$ and the connected component $S$ of $h^{-1}(1)$ which contains $H$ is included in the closed subset $J^-(S_K)$. Thus, $S$ is a closed spacelike hypersurface of $M$ which lies between two Cauchy hypersurfaces $S \subset J^+(S_0) \cap J^-(S_K)$ and, then, it is itself a Cauchy hypersurface [2, Corollary 3.11]. ■

Remark 4.14 If $H$ where not connected (but still acausal), then the proof would remain essentially equal, just considering in (4) the sum of functions $h^j_G, j = 1, \ldots, l$ for each connected component $H^j$ of $H$ (instead of only one $h_G$). Notice that $h^{-1}(1) \cap J^-(S_K)$ cannot have more than one connected component and, thus, a hypersurface containing all the $H^j$'s is obtained. In fact, the connectedness of $h^{-1}(1) \cap J^-(S_K)$ follows because otherwise each connected component would be a Cauchy hypersurface. But if there were more than one component, a timelike curve $\gamma$ in $J^-(S_K)$ would cross two such Cauchy hypersurfaces, in contradiction with property (iii) ($h$ increases on $\gamma$ but if it is constant on $h^{-1}(1) \cap J^-(S_K)$).

5 Continuous and smooth versions of Theorem 1.2

Along this section $S$ will be one of the topological Cauchy hypersurfaces of $(M, g)$ (non-necessarily smooth nor acausal). If $U \ni p$ is a neighbourhood, $J^+(p, U)$ denotes the causal future of $p$ computed in $U$, regarded $U$ as a spacetime.

Our aim will be to prove:

Theorem 5.15 Let $(M, g)$ be globally hyperbolic, and $S$ a Cauchy hypersurface. There exists a smooth onto function $\tau : M \to \mathbb{R}$ such that:

(i) $S = \tau^{-1}(0)$.

(ii) The gradient $\nabla \tau$ is past-directed timelike on $M \setminus S$.

(iii) Each level $S_t = \tau^{-1}(t), t \in \mathbb{R} \setminus \{0\}$ is a spacelike Cauchy hypersurface.

Even more, if $S$ is acausal then $\tau$ is also a smooth time function.

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1 The assumption on the bound by 1 of the $g_R$-diameter of $C_p$ (Section 2) is used here; see [2] Lemma 4.13 for related details.
Lemma 5.16 $I^+(S)$ and $I^-(S)$, regarded as spacetimes, are globally hyperbolic.

Proof: Straightforward by taking into account (for $I^+(S)$):
\[ J^+(p; I^+(S)) \cap J^-(q; I^+(S)) = J^+(p) \cap J^-(q), \quad \forall p, q \in I^+(S). \]

Our construction of the function with the required properties (along this section and the next one), starts at the following result, which will be subsequently refined:

Proposition 5.17 There exists a continuous onto function $\tilde{\tau} : M \to \mathbb{R}$ such that:

(i) $S = \tilde{\tau}^{-1}(0)$.

(ii) $\tilde{\tau}$ is smooth with (past-directed) timelike gradient on $M \setminus S$.

(iii) Each $S_t, t \in \mathbb{R}\setminus\{0\}$ (inverse image of the regular value $t \neq 0$) is a spacelike Cauchy hypersurface.

Even more, if $S$ is acausal then $\tilde{\tau}$ is also a time function.

Proof: Let $T_{S^+}$ (resp. $T_{S^-}$) be a (onto) Cauchy temporal function on $I^+(S)$ (resp. $I^-(S)$), as constructed in [3]. Define
\[
\tilde{\tau}(p) = \begin{cases} 
\exp(T_{S^+}(p)), & \forall p \in I^+(S) \\
0, & \forall p \in S \\
-\exp(-T_{S^-}(p)), & \forall p \in I^-(S).
\end{cases}
\]

In order to check the continuity of $\tilde{\tau}$ on $S$, take any sequence $\{x_k\} \to x_0 \in S$. Assume $\{x_k\} \subset I^+(S)$ (the case $I^-(S)$ is analogous), and recall that all the points $x_k$ must lie in a compact neighbourhood $W$ of $x_0$. Moreover, each $x_k$ belongs to a level $S_{t_k} = \tilde{\tau}^{-1}(t_k)$ of $\tilde{\tau}$ (and $T_{S^+}$). If $\{t_k\}$ did not converge to 0 then, up to a subsequence, $\{t_k\} \to t_0 > 0$ and the limit $x_0$ of $\{x_k\}$ would lie in $W \cap S_{t_0} \subset I^+(S)$, a contradiction.

To check that $\tilde{\tau}$ satisfies the remainder of required properties becomes straightforward.

Proposition 5.18 Let $S_-$ and $S_+$ be two disjoint Cauchy hypersurfaces of $M$ with $S_- \subset I^-(S_+)$. Then any connected, closed (as a topological subspace of $M$), spacelike hypersurface contained in $I^+(S_-) \cap I^-(S_+)$ is a Cauchy hypersurface.

Proof of Theorem 5.15
Notice that the function $\tilde{\tau}$ constructed in Proposition 5.17 satisfies all the conditions but the smoothability at $S$. In order to obtain this, consider a sequence of smooth functions $\varphi^+_k, \varphi^-_k : \mathbb{R} \to \mathbb{R}$ such that:

$$\varphi^+_k(t) = \begin{cases} 
0, & \forall t \leq 1/k \\
t, & \forall t \geq 2 
\end{cases} \quad \varphi^-_k(t) = \begin{cases} 
t, & \forall t \leq -2 \\
0, & \forall t \geq -1/k 
\end{cases}$$

$$\frac{d\varphi^+_k}{dt}(t_0) > 0, \forall t_0 > 1/k, \quad \frac{d\varphi^-_k}{dt}(t_0) > 0, \forall t_0 < -1/k.$$ 

Choose constants $C_k \geq 1$, such that:

$$\left| \frac{d^m\varphi^+_k}{dt^m}(t_0) \right| < C_k, \quad \left| \frac{d^m\varphi^-_k}{dt^m}(t_0) \right| < C_k, \quad \forall m \in \{1, \ldots, k\}. \quad (5)$$

Now, put:

$$\tau^+_k = \varphi^+_k \circ \tilde{\tau}, \quad \tau^-_k = \varphi^-_k \circ \tilde{\tau}.$$ 

Recall that $\nabla \tau^+_k$, (resp. $\nabla \tau^-_k$) is timelike past-directed in $I^+(S_1/k)$ (resp. $I^-(S_{-1/k})$) and 0 otherwise. The required function is then:

$$\tau = \Lambda \sum_{k=1}^{\infty} \frac{1}{2^k C_k} (\tau^+_k + \tau^-_k)$$

with $\Lambda = \sum_k 2^k C_k$. In fact, the smoothness of $\tau$ follows from the definition of $\tau^\pm_k$ and the bounds in (5) (see Theorem 3.11] for related computations).

Property (i) is trivial, and (ii) follows from the convexity of the timecones (recall [3, Lemma 3.10]). For (iii), recall that $\tau^{-1}(t) = \tilde{\tau}^{-1}(t)$ for $|t| \geq 2$; for the case $|t| < 2$, use Proposition 5.18 with $S_- = S_{-2}, S_+ = S_2$.

6 From smooth time functions to temporal functions

Along this section the Cauchy hypersurface $S$ will be spacelike, and we will fix a smooth time function $\tau$ as in Theorem 5.15. Our purpose is to complete the proof of Theorem 1.2.

Lemma 6.19 Let $W \subset \tau^{-1}(-1, 1)$ be an open neighborhood of $S$. Then, there exist a function $h_+$ (resp $h_-$) on $M$ such that:

(i) $h_+ \geq 0$ (resp $h_- \leq 0$) on $M$, and $h_+ \equiv 0$ on $I^-(S) \setminus W$ (resp. $h_- \equiv 0$ on $I^+(S) \setminus W$).

(ii) $h_+ \equiv 1$ on $J^+(S)$ (resp. $h_- \equiv -1$ on $J^-(S)$).

(iii) If $\nabla h_+(p)$ (resp. $\nabla h_-(p)$) does not vanish at $p (\in W)$ then $\nabla h_+(p)$ (resp. $\nabla h_-(p)$) is timelike past-directed.
Proof: For $h_-$, take the function $h^-$ in Lemma 3.8] with $U = W \cup J^-(S)$ (function $h_+$ can be constructed analogously). Taking into account the notion of “time step function” in [3, Proposition 3.6] such that $S$ is a level of $\tau_0$. More precisely:

**Proposition 6.20** Given the Cauchy hypersurfaces $S_i = \tau^{-1}(i), i = -1, 0, 1$ ($S \equiv S_0$) there exists a function $\tau_0$ such that:

1. $\nabla \tau_0$ is timelike and past-directed where it does not vanish, that is, in the interior of its support $V := \text{Int}(\text{Supp}(\nabla \tau_0))$.
2. $-1 \leq \tau_0 \leq 1$.
3. $\tau_0(J^+(S_1)) \equiv 1$, $\tau_0(J^-(S_{-1})) \equiv -1$. In particular, $V \subset \tau^{-1}(-1, 1)$.
4. $S = \tau_0^{-1}(0) \subset V$.

Proof: Consider the function $p \rightarrow d(p, S)$, where $d(\cdot, S)$ is the signed distance to $S$, i.e., $d(p, S)$ is the maximum length of the causal curves from $p$ to $S$, $d(p, S) \geq 0$ if $p \in J^+(S)$, and $d(p, S) < 0$ otherwise. The function $h_S(p) = d(p, S) + 1$ satisfies $h_S(S) \equiv 1$ obviously. Moreover, $h_S$ is smooth, positive and with past-directed timelike gradient in the closure of some open neighbourhood $W$ of $S$. Without loss of generality, we will assume $W \subset \tau^{-1}(-1, 1)$.

Now, consider the function $h^+$ provided in Lemma 6.19. Clearly, $h^+ = h_S h^+$ is smooth, non-negative and well defined in $W \cup J^-(S)$ (putting $h^+ \equiv 0$ on $J^-(S) \backslash W$). Even more, $\nabla h^+$ is timelike and past-directed where it does not vanish (simply, use Leibniz’s rule and the conditions imposed on $h_+, h_S$; notice that the role of $h^+$ is similar to the function also labelled $h^+$ in [3, Proposition 3.6]). Taking $h_-$ from Lemma 5.16, the required function will be

$$\tau_0 = 2 \frac{h^+}{h^+ - h_-} - 1$$

($\tau_0$ is extended as 1 on $J^+(S) \backslash W$). In fact, notice that

$$\nabla \tau_0 = 2 \frac{h^+ \nabla h_- - h_- \nabla h^+}{(h^+ - h_-)^2}$$

is either timelike or 0 everywhere, and also the other required properties hold.

**Proof of Theorem 1.2**

The required function is

$$\tau = \tau + \tau_0,$$

where $\tau$ is taken from Theorem 5.15 and $\tau_0$ from previous proposition. In fact, $\tau$ is obviously a temporal function with $S = \tau^{-1}(0)$. Even more, the levels of $\tau$ are Cauchy hypersurfaces: $\tau^{-1}(t) = \tau^{-1}(t)$ for $|t| > 1$ and, for the case $|t| < 1$, use Proposition 5.16 with $S_- = S_{-1}, S_+ = S_1$.
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