On relation between Nekrasov functions and BS periods in pure $SU(N)$ case

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Abstract

We investigate the duality between the Nekrasov function and the quantized Seiberg-Witten prepotential, first guessed in [1] and further elaborated in [2] and [3]. We concentrate on providing more thorough checks than the ones presented in [3] and do not discuss the motivation and historical context of this duality. The check of the conjecture up to $o(h^6, \ln(\Lambda))$ is done by hands for arbitrary $N$ (explicit formulas are presented). Moreover, details of the calculation that are essential for the computerization of the check are worked out. This allows us to test the conjecture up to $h^6$ and up to higher powers of $\Lambda$ for $N = 2, 3, 4$. Only the case of pure $SU(N)$ gauge theory is considered.

1 Introduction

It is often realized during late decades that the quantities, calculated in two theories with different underlying mathematical machinery and/or physical origin, coincide. When this takes place one says that there is a duality between these two theories, and it often means that there is some sort of underlying structure or a unifying concept, which makes the duality evident.

Discovering and investigating dualities is very important, since once an underlying concept is discovered it provides the right point of view for the both theories. Moreover, since some theories in modern mathematical physics still lack experimental evidence, the fact that they are dual to some other theories may serve as an indication that they do describe the real world. Moreover, even if the underlying concept is missing, the duality can be used to solve longstanding problems in one theory via using techniques from the other theory.

Recently, the duality that connects a huge number of different theories in modern mathematical physics had been conjectured. It connects Seiberg-Witten theory [11], [15]-[17], the Nekrasov functions from the quiver theories [18]-[22], conformal field theory [23]-[29] (the relation to CFT is provided by the celebrated AGT conjecture [30]-[41]), and matrix models in the Dijkgraaf-Vafa phase [42]-[45]. Importance of this unification, its traces being present in a number of new [2], [3], [30]-[41], [59] and less recent [1], [46]-[58] papers, cannot be overestimated. However, although at conjectural level the whole picture is relatively clear [59], checks and proofs still need to be done to be certain that this unification really takes place.

In this paper we are concentrating on a small part of the unification, that is, on the statement that the Nekrasov function with $\epsilon_2 = 0$ is equal to the quantized SW prepotential provided $\epsilon_2 = h$. Furthermore, we restrict ourselves to the case of pure gauge theory, without matter hypermultiplets. The idea of this relation first appeared in [11] and then was investigated in [2], [3]. However, for $SU(N)$ with $N > 2$ only simple checks were made (up to $o(h^2, \Lambda^{2N})$). In this paper we present more thorough check, partly by presenting explicit formulas (up to $o(h^6, \ln(\Lambda))$ for arbitrary $N$) and partly by performing computer experiment (for $N = 2, 3, 4$ where we check that the duality holds with higher precision but formulas are too lengthy to be manifestly written down). A general proof of the conjecture is still missing.

2 Outline

The conjecture itself and the way to check it are described in detail in [3]. In this section we recall the construction very briefly.

The statement that is made is that the Nekrasov function with one of the regularization parameters set to zero is equal to the quantized SW prepotential.

$$F_{Nek}(\epsilon_1, 0, \Lambda)\bigg|_{\epsilon_1 = h} = F_{SW}(h, \Lambda),$$  (1)
where the prepotential is defined by equations

\[ a_i = \Pi^h_{A_i}(\lambda) \]
\[ -\frac{1}{4} \frac{\partial F_{SW}}{\partial a_i} = \Pi^h_{B_i}(\lambda) \]

and \( \Pi \)'s are the Bohr-Sommerfeld periods of the quantum SW differential, and \( \lambda \)'s are the roots of the polynomial that enters the Fourier transform of the Baxter equation (see below).

The proof includes evaluation of \( \Pi^h_{A_i} \) and \( \Pi^h_{B_i} \), which can be expressed through \( \Pi^0_{A_i} \) and \( \Pi^0_{B_i} \) - the periods for the classical SW differential. While \( \Pi^0_{0A_i} \) are easy to obtain by direct integration [4], direct calculation of \( \Pi^0_{0B_i} \) is not so easy, and one, for example, can use the well-known fact, that for \( \hbar = 0 \) the conjecture is valid

\[ F_{Nek}(0, 0, \Lambda) = F_{SW}(0, \Lambda) \]  

From which one deduces

\[ \Pi^0_{0B_i} = -\frac{1}{4} \frac{\partial F_{SW}}{\partial a_i}(0, 0, \Lambda) \]  

Then one constructs \( \Pi^h_{B_i} \) and looks if the result coincides with \( -\frac{1}{4} \frac{\partial F_{Nek}}{\partial a_i}(\hbar, 0, \Lambda) \). The subtlety here is, that \( \Pi^h_{B_i} \) and \( -\frac{1}{4} \frac{\partial F_{Nek}}{\partial a_i}(\hbar, 0, \Lambda) \) are expressed in terms of different variables (\( \lambda \)'s and \( a \)'s), so one needs to express \( a \)'s in terms of \( \lambda \)'s.

Mnemonically, one can write the conjecture in the following form

\[ \Pi^h_{B_i}(\hat{\Omega}\Pi^0_{A_i}) = \hat{\Omega}(\Pi^0_{B_i}(\Pi^0_{A_i})) \]

which contains the way to check it.

The rest of the paper is organized as follows. In sections 3 and 4 the SW and Nekrasov sides of the conjecture are described in more detail. Attention is paid to the questions of automatization of calculations. Section 5 presents checks themselves.

3 Seiberg-Witten side

3.1 Outlook

The classical Seiberg-Witten prepotential \( F_{SW} \) is determined by equations

\[ a_i = \int_{A_i} \lambda_{SW} = \Pi^0_{A_i} \]
\[ -\frac{1}{4} \frac{\partial F_{SW}}{\partial a_i} = \int_{B_i} \lambda_{SW} = \Pi^0_{B_i}, \]

where \( \lambda_{SW} \) is the Seiberg-Witten differential \( \lambda_{SW} = pdx \) defined on the spectral curve given by equation

\[ K(p) + \gamma \cos(x) = 0, \]

where \( K = \sum_j u_j p^j = u_N \prod_j (p - \lambda_j) \) is a polynomial. \( A_i \) and \( B_i \) form the symplectic basis of 1-cycles on this curve.

The original idea of [3] is to substitute “classical” SW-differential \( pdx \) with “quantum” differential \( Pdx \), where \( P \) solves quantum version of (6) that is the Fourier transform of the Baxter equation

\[ \left( K' (\frac{\hbar}{i}) \partial + \gamma \cos(x) \right) e^x P dx = 0, \]

where \( P = p + O(\hbar) \).

Periods of this quantized differential then define \( F_{SW}(\hbar) \) in the similar way

\[ a_i = \Pi^h_{A_i}(\lambda), \quad -\frac{1}{4} \frac{\partial F_{SW}(\hbar)}{\partial a_i} = \Pi^h_{B_i}(\lambda), \]

where the implicit dependency \( \lambda(a) \) needs to be resolved from the first set of equations (for A-cycles) and substituted into the second (for B-cycles).
From the above it is clear that in order to define the prepotential we need to know only periods of \( \lambda_{SW} \), not \( \lambda_{SW} \) itself, which means that we can add arbitrary exact terms to \( \lambda_{SW} \) to simplify the calculations. Further, the idea of \([3]\) is to represent the quantized SW differential as some differential operator \( \hat{O} \) acting on the classical one \( Pdx = \hat{O}(pdx) \). Then for the periods one also gets

\[
\Pi^0_{A_i} = \hat{O}\Pi^0_{A_i}, \quad \Pi^0_{B_i} = \hat{O}\Pi^0_{B_i},
\]

where \( \hat{O} = 1 + O(\hbar) \).

Thus, to find the quantum SW periods one needs to do the following:

- solve the Baxter equation for \( P \) perturbatively,
- simplify \( P \) by adding full derivatives,
- rewrite the resulting \( P \) in the form \( \hat{O}(p) \).

The rest of this section is devoted to the details of these steps.

### 3.2 Solving Baxter equation

The first step is to evaluate \( P \) perturbatively. When one considers conjugation of \( \partial^n \) with \( e^{\hbar\int f^x pdx} \), one finds

\[
e^{-\hbar\int f^x pdx} \partial^n e^{\hbar\int f^x pdx} = P^n + \frac{\hbar}{2} n(n-1) \dot{P} + \left( \frac{\hbar}{6} \right)^2 \left( \frac{n(n-1)(n-2)}{6} P^n + \frac{1}{24} \cdot 3n(n-1)(n-2)(n-3)P^{n-4} \dot{P}^2 + \ldots \right), \tag{9}
\]

where summation at level \( k \) goes over partitions (Young diagrams) of weight \( k \). Namely, the contribution of the partition \( \overline{k} = (k_1, \ldots, k_m) \) equals ((\( \tilde{k}_1 \ldots \tilde{k}_n \)) denotes the conjugated partition)

\[
\left( \frac{\hbar}{i} \right)^{\alpha} \frac{1}{\alpha!} \frac{\beta}{\gamma} \prod \frac{1}{(n-\alpha_m)!} \cdot \frac{1}{(n-\alpha_m)!} P^{(k_1)} \cdots P^{(k_m)} = \left( \frac{\hbar}{i} \right)^{\alpha} n! \prod (n-\alpha_m)! \prod \frac{1}{(n-\alpha_m)!} P^{(k_1)} \cdots P^{(k_m)} C(\overline{k}),
\]

where the equivalence is the definition of the coefficients \( C(\overline{k}) \) and

\[
\alpha_j = \prod_{i=1}^j (k_i + 1), \quad \beta = \prod_{i=1}^m \frac{1}{C(\alpha_i)^{k_i+1}}, \quad \gamma = \prod_{i=1}^n \frac{1}{(\tilde{k}_i - k_{i-1})!}.
\]

Hence, the Baxter equation becomes

\[
\sum\limits_{\overline{k}} \left( \frac{\hbar}{i} \right)^{\alpha} K^{(\alpha_m)}(P)C(\overline{k}) + \gamma \cos(x) = 0, \tag{10}
\]

where \( K^{(i)} \) stands for the \( i \)-th derivative of \( K \). Having this, one can calculate \( P = p + \hbar p_1 + \ldots p_n \), up to any desired order.

### 3.3 Simplification of \( P \)

Since we are interested in \( A \) and \( B \) periods of \( Pdx \), not just in \( P \) itself, we can add to it terms, which are exact, in order to simplify it.

A typical term in \( P \) looks like

\[
\frac{K^{(i_1)}(p) \cdots K^{(i_j)}(p)}{(K^{(p)})^{k}} V^{(j_1)} \cdots V^{(j_l)}.
\]

The simplified form is such that the degree of \( P \) in \( \frac{1}{K} \) is the smallest. The motivation for this definition is that we want to rewrite our \( P \) as some differential operator \( \hat{O} \) acting on \( p \), and the order of this operator is roughly speaking half of the degree of \( P \) in \( \frac{1}{K} \).

We suggest the following ansatz for exact terms: they are themselves the derivatives of something of the form \([11]\).

To get the idea of how to simplify \( P \) one should notice that among the terms of our ansatz the one which comes from differentiation of \( 1/K \) gives the biggest contribution to the degree of \( P \). Indeed, compare these two lines

\[
\frac{\partial}{\partial x} \frac{1}{K^k} = \frac{K'' V'}{(K')^{k+2}}. \tag{12}
\]
It is clear that the simplification procedure looks as follows:

- one looks for a term in P that contains \( \frac{K''}{(K')^n} V' \) with the biggest \( n \),
- subtracts from \( P \) the corresponding full derivative
- repeats the first two steps until there are terms in \( P \) of that form.

### 3.4 Perturbative answer → differential operator

Let’s recall, that in order to find the periods of the quantum SW differential one needs to find the differential operator \( \hat{O} \) such that \( P = \hat{O} p \). It turns out, that up to \( \hbar^2 \) the simplified form of \( P \) contains only even derivatives of \( V \), which for \( V = \cos(x) \) are proportional to \( V \) itself. These are good news, since the differential operator \( \hat{O} \) can be composed from the following elementary differential operators.

\[
D_{\gamma} = \gamma \frac{\partial}{\partial \gamma} \tag{13}
\]

\[
D_i = \sum_{j \geq i} \frac{j!}{(j - i)!} u_j \frac{\partial}{\partial u_{j-i}} = i! \sum_{j \geq i} C_i^j u_j \frac{\partial}{\partial u_{j-i}} \]

The following identities are helpful

\[
D_{\gamma}(\gamma) = -\frac{\gamma \cos(x)}{K'} = -\frac{V}{K'} \quad D_i(p) = -\frac{K^{(i)}}{K'} \tag{14}
\]

\[
D_{\gamma}(K^{(j)}) = -\frac{K^{(j+1)} V}{K'} \quad D_i(K^{(j)}) = K^{(j+i)} - \frac{K^{(j+1)} K^{(i)}}{K'} \tag{15}
\]

One can see, that the term of the highest degree in \( \frac{1}{K'} \) in the result of the action of differential operator \( D_{i_1} \cdots D_{i_n}(D_{\gamma})^m \) on \( p \) is proportional to

\[
\left. D_{i_1} \cdots D_{i_n}(D_{\gamma})^m(p) \right|_{\text{highest degree}} \sim \frac{V^m (K'')^{m-1} (K'')^n K^{(i_1)} \cdots K^{(i_n)}}{(K')^{2(n-1)}} \tag{16}
\]

So, the procedure of finding the differential operator looks as follows.

- one finds a term of the form [16] in \( P \),
- deduces the differential operator \( D \), that generates such term,
- subtracts the result of the action of \( D \) on \( p \) from \( P \),
- repeats until \( P \) is equal to zero.

Summing up all \( D \)’s found during this procedure, one obtains \( \hat{O} \).

After this procedure is applied, one ends with the following expression for the differential operator

\[
\hat{O} = 1 + \frac{\hbar^2}{2^3 \cdot 3} D_2 D_{\gamma} + \frac{7\hbar^4}{2^7 \cdot 3^2 \cdot 5} \left( D_2 D_2 D_2 D_{\gamma} D_{\gamma} - \frac{2}{7} D_4 D_{\gamma} D_{\gamma} - \frac{2}{7} D_2 D_2 D_2 D_{\gamma} \right) + \hat{O}^{(6)} + o(\hbar^6), \tag{17}
\]

and the formula for \( \hat{O}^{(6)} \) will be written below, since its derivation includes few additional tricks.

The important thing to stress is that the resulting operator is not just an arbitrary operator in \( u_i \) and \( \gamma \), how it could in principle have happened, but is composed of \( D_i \) and \( D_{\gamma} \), and hence lies in the really strict class of differential operators. One may hope that the same simplification occurs in the case of gauge theory with matter (the XXX chain).
the 6th order in $\hbar$  At the sixth order in $\hbar$ a new subtlety appears: simplified $P$ contains $(V^{(3)})^2 \sim \sin^2(x)$, so at first sight the set $(\mathcal{D}_\gamma, \mathcal{D}_i)$ is not sufficient to express $\tilde{O}^{(6)}$. However, let’s examine the situation in more detail.

Consider the more general form of the classical Baxter equation

$$K(p) + \frac{\gamma}{2} e^{ix} + \frac{\beta}{2} e^{-ix} = 0,$$

so at first sight the set $(\mathcal{D}_\gamma, \mathcal{D}_i)$ is not sufficient to express $\tilde{O}^{(6)}$. However, let’s examine the situation in more detail.

Consider the more general form of the classical Baxter equation

$$K(p) + \frac{\gamma}{2} e^{ix} + \frac{\beta}{2} e^{-ix} = 0,$$

so the previous form corresponds to $\beta = \gamma$.

It is convenient to introduce the following operators

$$\mathcal{D}_{\gamma^+} = \gamma \frac{\partial}{\partial \gamma} + \beta \frac{\partial}{\partial \beta},$$
$$\mathcal{D}_{\gamma^-} = \gamma \frac{\partial}{\partial \gamma} - \beta \frac{\partial}{\partial \beta}.$$  

It is obvious that

$$(\mathcal{D}_{\gamma^+})^n(p) \bigg|_{\beta=\gamma} = \frac{\cos^n(x)(K'')^n}{(K')^{2n-1}} + \cdots$$

and in order to express $\tilde{O}^{(6)}$ we need only

$$(\mathcal{D}_{\gamma^-})^2(p) \bigg|_{\beta=\gamma} = -\frac{\cos(x)}{K'} - \frac{\sin^2(x)K''}{(K')^3}$$

The less obvious thing is that

$$\left( \frac{\gamma}{\partial \gamma} - \beta \frac{\partial}{\partial \beta} \right) \Pi^0 = 0$$

Indeed, $\oint p \, dx = -\oint xdp$, and if one performs the shift $x \rightarrow x + i \ln(\gamma)$, (18) becomes

$$K(p) + \frac{1}{2} e^{ix} + \frac{\beta\gamma}{2} e^{-ix} = 0,$$

and calculating $x(p)$ perturbatively one always gets a function of $\gamma\beta$, which is mapped to zero by $\mathcal{D}_{\gamma^-}$. Further, since

$$[\mathcal{D}_{\gamma^-}, \mathcal{D}_{\gamma^+}] = \mathcal{D}_{\gamma^-},$$

only those terms in $\tilde{O}$ which are free of $\mathcal{D}_{\gamma^-}$ give non-zero contribution, when acting on $\Pi^0$. And this means, that after finding $\tilde{O}$ in terms of $\mathcal{D}_{\gamma^+}$ and $\mathcal{D}_{\gamma^-}$, terms with $\mathcal{D}_{\gamma^-}$ can be dropped out and $\mathcal{D}_{\gamma^+}$ can be substituted by $\gamma \frac{\partial}{\partial \gamma}$.

After all this is performed, one gets the following answer for $\tilde{O}^{(6)}$

$$\tilde{O}^{(6)} = \hbar^6 \left[ \frac{31}{8} (D_2)^3 - \frac{15}{4} D_2 D_4 + \frac{1}{3} (D_3)^2 + D_6 \right] (D_\gamma)^3 +$$
$$\left[ -\frac{15}{4} (D_2)^3 + 2D_2 D_4 - (D_3)^2 - D_6 \right] (D_\gamma)^2 +$$
$$\left[ (D_2)^3 - D_2 D_4 + \frac{2}{3} (D_3)^2 \right] D_\gamma.$$  

$D_i$ in terms of roots  Since $A$ and $B$ periods are conveniently written in terms of roots of $K$, not its coefficients, it is necessary to express $D_i$, and hence $\tilde{O}$ in terms of roots. It can be verified by direct check, that the following expression for $D_i$ holds

$$D_i = - \sum_{m} \frac{K^{(i)}(\lambda_m)}{K'(\lambda_m)} \frac{\partial}{\partial \lambda_m},$$

which is rather simple.
3.5 Classical A-periods in terms of roots

In [4] the general expression for the classical A-periods was obtained

$$\Pi^0_{A_i} = \lambda_i + \sum_{n=1}^{\infty} \frac{1}{n!(2n)!} \left( \frac{\partial}{\partial \lambda_i} \right)^{2n-1} \prod_{k \neq i} (\lambda_k - \lambda_i)^{-2n} $$

(30)

Up to $o(\Lambda^{4N})$ one gets

$$\Pi^0_{A_i} = \lambda_i - \frac{\Lambda^{2N}}{2} \frac{1}{\Lambda^2} \sum \frac{1}{\lambda} - \Lambda^{4N} \frac{1}{\Pi \Lambda^4} \left( \left( \sum \frac{1}{\lambda} \right)^3 - \frac{3}{4} \left( \sum \frac{1}{\lambda} \right)^2 \right) + \frac{1}{8} \sum \frac{1}{\lambda^3}$$

(31)

where $\Pi \lambda^i$ is a shorthand for $\prod_{k \neq i} \lambda_i^k$ and $\sum \lambda^k$ is a shorthand for $\sum_{k \neq i} \lambda_i^k$.

4 Nekrasov side

The definition of the Nekrasov function can be found in numerous papers, for example [21] and [3]. Here we give the definition for $\epsilon_2 = 0$. The Nekrasov function $F_{Nek}$ is equal to the sum of perturbative and instantonic contributions.

$$F_{Nek} = F_{pert} + F_{inst}$$

(32)

For $F_{pert}$ no nice expression is available, but there is one for $\frac{\partial F_{pert}}{\partial a_i}$, which is sufficient for our needs.

$$- \frac{\partial F_{pert}}{\partial a_i} = \sum_{j \neq i} 4a_{ij} \left[ (\ln \frac{a_{ij}}{\Lambda} - 1) + \sum_{m=1} B_{2m} 2m(2m-1) \left( \frac{\epsilon_1}{a_{ij}} \right)^{2m} \right]$$

(33)

For such definition of $F_{pert}$, $F_{inst}$ should be defined as follows (this deviates from [21] slightly, but we believe it is the matter of convention)

$$F_{inst} = -2\epsilon_1 \epsilon_2 \ln Z_{inst}$$

(34)

and $Z_{inst}$ is the sum over n-tuples of partitions

$$Z_{inst} = \sum_n \left( \Lambda^{2Nn} \sum X_{|k|n} Z_{inst}^k \right)$$

(35)

$$F_{inst} = \prod_{nl} \prod_{ij} \frac{a_{nl} + \epsilon_1(i-1) + \epsilon_2(-j)}{a_{nl} + \epsilon_1(i-1) - \epsilon_2(k_{ij}) + \epsilon_2(k_{nj}) - j}$$

(36)

where

$$\Lambda^N = \frac{1}{2N} \Lambda^N$$

(37)

and $\Lambda$ is precisely the $\Lambda$ from the Seiberg-Witten side of the duality.

In two-instanton approximation, $Z_{inst}$ equals (indices in square brackets label the types of n-tuples of Young diagrams).

$$Z_{inst} = 1 + \left( \frac{\Lambda}{2} \right)^2 Z_{[1]} + \left( \frac{\Lambda}{2} \right)^4 (Z_{[2]} + Z_{[1,1]} + Z_{[1],[1]})$$

(38)

$$Z_{[1]} = -\frac{1}{\epsilon_1 \epsilon_2} \sum_{i=1}^N R_i(a_i)$$

(39)

$$Z_{[2]} = \frac{1}{2 \epsilon_1^2 \epsilon_2} \sum_{i=1}^N R_i(a_i) R_i(a_i + \epsilon_2)$$

(40)

$$Z_{[1,1]} = -\frac{1}{2 \epsilon_1 \epsilon_2} \sum_{i=1}^N R_i(a_i) R_i(a_i + \epsilon_1)$$

(41)

$$Z_{[1],[1]} = \frac{1}{2 \epsilon_1 \epsilon_2} \sum_{i \neq j} R_i(a_i) R_j(a_j) \left( \frac{a_{ij}^2 (a_{ij}^2 - \epsilon^2)}{(a_{ij}^2 - \epsilon_1^2)(a_{ij}^2 - \epsilon_2^2)} \right)$$

(42)

$$R_i(x) = \frac{1}{\prod_{j \neq i} (x - a_j)(x - a_j + \epsilon)}$$

(43)
From $Z_{\text{inst}}$, $F_{\text{inst}}(\epsilon_1, 0)$ can be readily obtained (now in $R_i(x)$ $\epsilon_2$ is also put to zero).

$$F_{\text{inst}} = \frac{1}{2} \lambda^2 \sum_i R_i(a_i) + \left( \frac{\lambda}{2} \right)^4 \left[ \frac{1}{\epsilon_1} \sum_i \left( R_i^2(a_i) \sum_{j \neq i} \left( \frac{1}{a_{ij}} + \frac{1}{a_{ij} + \epsilon_1} \right) \right) \right]$$

$$- \frac{1}{\epsilon_1} \sum_i R_i^2(a_i) + \frac{1}{\epsilon_1} \sum_i R_i(a_i) R_i(a_i + \epsilon_1) + \sum_{i \neq j} R_i(a_i) R_j(a_j) \frac{1}{a_{ij} - \epsilon_1}$$

(44)

5 Check of the conjecture

As was pointed out already in [3], the calculation of both the SW and Nekrasov sides of the duality can be easily computerized, and hence the duality can be checked up to any desired order in $\hbar$ and $\Lambda$ for any given $N$. However, if one tries to attack generic $N$, calculations should be performed by hands, and for high orders in $\Lambda$ presentation of the results is not an easy task (mainly because formulas are very lengthy and hence non-illustrative). So, for the sake of simplicity, here we present explicit calculations for arbitrary $N$ only up to $o(\hbar^6, \ln(\Lambda))$.

Explicit computer checks that we have performed for a few small $N$ allow us to state that the duality holds at least with following precision

- $N = 2$: up to $o(\hbar^6, \Lambda^{6N})$ at least,
- $N = 3$: up to $o(\hbar^6, \Lambda^{4N})$ at least,
- $N = 4$: up to $o(\hbar^6, \Lambda^{2N})$ at least.

5.1 The zeroeth order in $\Lambda^N$

Here we concentrate on checking the conjecture up to $o(\hbar^6, \ln(\Lambda))$ for arbitrary $N$. Formulas in this case are quite simple and the ideas of the check are easy to illustrate.

First of all, both classical and quantum $\Lambda$-periods on the SW side are equal to the corresponding roots of $K$.

$$\Pi^h_{A_1} = \Pi^0_{A_i} = a_i = \lambda_i,$$

(46)

since $\hat{O}$ acts nontrivially only on $\Lambda$-dependent terms, which in this case are missing.

Hence, in what follows we write all formulas in terms of $a$’s instead of $\lambda$’s and hope this will not cause any confusion.

Recall that the identity we want to check is

$$\Pi^h_{B_i} = \hat{O} \Pi^0_{B_i},$$

or, in other words

$$\frac{\partial F_{\text{pert}}}{\partial a_i}(\hbar, \Lambda) = \hat{O} \frac{\partial F_{\text{pert}}}{\partial a_i}(0, \Lambda)$$

(47)

Up to $o(\hbar^6)$ $\Pi^h_{B_i}$ is equal to

$$\Pi^h_{B_i} = - \frac{1}{4} \frac{\partial F_{\text{pert}}}{\partial a_i}(\hbar, \Lambda) = \sum_{j \neq i} \left[ a_{ij} \left( \ln \frac{a_{ij}}{\Lambda} - 1 \right) + \frac{\hbar^2}{12} \frac{1}{a_{ij}^2} - \frac{\hbar^4}{360} \frac{1}{a_{ij}} + \frac{\hbar^6}{1260} \frac{1}{a_{ij}} \right]$$

(48)

$$\Pi^0_{B_i} = - \frac{1}{4} \frac{\partial F_{\text{pert}}}{\partial a_i}(0, \Lambda) = \sum_{j \neq i} a_{ij} \left( \ln \frac{a_{ij}}{\Lambda} - 1 \right)$$

Since $\hat{O}$ acts nontrivially only on $\Lambda$-dependent part of $\Pi^0_{B_i}$ and $D_\gamma \ln \gamma = 1$, the terms in $\hat{O}$ with higher than one power of $D_\gamma$ do not contribute, and we are in fact checking the following identity

$$\sum_{j \neq i} \frac{\hbar^2}{12} \frac{1}{a_{ij}} - \frac{\hbar^4}{360} \frac{1}{a_{ij}^3} + \frac{\hbar^6}{1260} \frac{1}{a_{ij}} =$$

$$\left[ \frac{\hbar^2}{2^4 \cdot 3} D_2 + - \frac{\hbar^4}{2^6 \cdot 3^2 \cdot 5} D_2 D_2 + \frac{\hbar^6}{2^7 \cdot 3^3 \cdot 5 \cdot 7} \left( (D_2)^3 - D_2 D_4 + 2 \frac{2}{3}(D_3)^2 \right) \right] \left( - \frac{1}{N} \sum_{j \neq i} a_{ij} \right),$$

(49)
since $\mathcal{D}_i \left( \ln \frac{1}{x} \right) = -\frac{1}{x}$.

Naturally, the check splits into 3 checks for $h^2, h^4$ and $h^6$, respectively. In what follows for each order in $h$ first it is shown that the structure of both sides of the equality (49) is the same and then that the coefficients do coincide.

$h^2$:

$$D_2\Pi^0 = -\sum_{j \neq i} \left( \sum_{k \neq i} \frac{2}{a_{ik}} - \sum_{k \neq j} \frac{2}{a_{jk}} \right) = -\sum_{k \neq i} 2(N-1) + 1 \frac{1}{a_{ik}}$$

and

$$\frac{2}{2^3 \cdot 3} = \frac{1}{12}$$

$h^4$:

$$D_2D_2\Pi^0 = -4N \sum_{j \neq i} \frac{1}{a_{ij}} \left( \sum_{k \neq i} \frac{1}{a_{ik}} - \sum_{k \neq j} \frac{1}{a_{jk}} \right) = -\sum_{j \neq i} 8N a_{ij} a_{ik} a_{jk}$$

since

$$\sum_{k \neq i} \frac{1}{a_{ik}} - \sum_{k \neq j} \frac{1}{a_{jk}} = \frac{2}{a_{ij}} + \sum_{k \neq i,j} \frac{a_{ji}}{a_{ik}a_{jk}}$$

and

$$\sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{a_{ij}a_{jk}a_{ki}} = 2 \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{a_{ij}a_{kj}a_{ik}} = \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{a_{ij}a_{kj}a_{ik}} = 0$$

The corresponding coefficient

$$-\frac{8}{2^6 3^2 5} = -\frac{1}{360},$$

as it should be.

$h^6$: Again, the strategy is to rewrite the r.h.s. in terms of ‘nested sums’, that is the sums in which all summation indices are mutually different, namely

$$(D_2^3)\frac{1}{N} \sum_{j \neq i} a_{ij} = -96 \sum_{j \neq i} \frac{1}{a_{ij}} + 48 \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{a_{ij}a_{ik}a_{jk}}$$

and

$$\frac{2}{3}(D_3)\frac{1}{N} \sum_{j \neq i} a_{ij} = -24 \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{a_{ij}} \left( \frac{1}{a_{ik}} + \frac{1}{a_{jk}} \right)$$

$$-24 \sum_{j \neq i} \sum_{k \neq i} \frac{1}{a_{ij}} \sum_{l \neq i,j,k} \frac{1}{a_{jk}a_{il}} \left( \frac{1}{a_{ij}} + \frac{1}{a_{ik}} \right) \frac{1}{a_{ij}a_{il}a_{jk}}$$

$$-12 \sum_{j \neq i} \sum_{k \neq i} \frac{1}{a_{ij}a_{lk}} \sum_{n \neq i,j,k} \sum_{l \neq i,j,k,n} \frac{1}{a_{lk}a_{ll}} \left( \frac{1}{a_{ij}a_{il}a_{jk}} - \frac{1}{a_{ijn}a_{il}} \right)$$

$$D_2D_4 \frac{1}{N} \sum_{j \neq i} a_{ij} = -16 \sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \frac{1}{a_{ij}} \left( \frac{1}{a_{ik}a_{il}a_{jk}} - \frac{1}{a_{ijkl}a_{jk}} \right)$$

$$-48 \sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \frac{1}{a_{ij}a_{il}} \left( \frac{1}{a_{ijkl}a_{jk}} + \frac{1}{a_{ijk}a_{jp}} \right)$$

First of all, let’s note, that

$$\frac{96}{2^7 \cdot 3^3 \cdot 5 \cdot 7} = \frac{1}{1260}$$

so the coefficient is correct.

To complete the check, one must see, that double, triple and quadruple sums are zero. Here we do not write all of them, just a few examples to illustrate the idea, since the only trick one needs to use is relabeling of indices in sums. More precisely, one should represent given expression as a sum of ‘cyclic sums’, that is sums of the form

$$\sum_{i_1 \neq i} \cdots \sum_{i_{n-1} \neq i_n} \frac{1}{a_{i_1i_2} a_{i_2i_3} \cdots a_{i_ni_1}}$$

and these cyclic sums can be shown to be equal to zero by noticing that such sum does not change under permutation of indices $i_1 \ldots i_n$ and then summing over all permutations.
Examples. For the double sum we have
\[
\sum_{j \neq i, k \neq i, j} \frac{1}{a_{ij}^2 a_{ik}} \left( \frac{1}{a_{ik}} - \frac{1}{a_{jk}} \right) = \sum_{j \neq i, k \neq i, j} \frac{1}{a_{ij}^2 a_{ik}^2 a_{jk}} = 0,
\]
due to the antisymmetry of the summand under the exchange \( j \leftrightarrow k \).

For one of the quadruple sums we have (we omit the sum symbols to simplify formulas)
\[
\frac{1}{a_{ij}^2 a_{ik}} \left( \frac{1}{a_{in} a_{il}} - \frac{1}{a_{jn} a_{jl}} \right) \sim \frac{1}{a_{ij} a_{ik} a_{in} a_{il} a_{jn} a_{jl}} + \frac{1}{a_{ij} a_{ik} a_{in} a_{jl} a_{jn} a_{il}},
\]
where \( ' \sim ' \) here and in what follows means that sometimes \( \frac{1}{2} \) appears, which is inessential since we are showing that these sums are equal to zero.

The second sum is zero because it contains the cycle \( a_{ij} a_{jl} a_{li} \). In the first term we can exchange \( l \leftrightarrow j \) and add this to the result of exchange \( l \leftrightarrow n \)
\[
\frac{1}{a_{ik} a_{ij} a_{il} a_{in}} \sim \frac{1}{a_{ik} a_{ij} a_{jl} a_{in}} \sim \frac{1}{a_{ik} a_{ij} a_{jl} a_{in} a_{nl}} \sim \frac{1}{a_{ik} a_{ij} a_{jn} a_{ni} a_{il}} = 0.
\]

6 Conclusion

In this paper we made the next step in proving the conjecture that the quantized SW prepotential is equal to the Nekrasov function \( F_{Neck} \) with vanishing \( \epsilon_2 \). We considered the case of pure gauge \( SU(N) \) theory. Explicit formulas for the check up to \( o(\hbar^6, \ln \Lambda) \) for arbitrary \( N \) were presented, and some intermediate steps of the construction of the objects that appear at both sides of the duality, which are needed to computerize the check, were clarified. The quantization operator \( \hat{O} \), which plays the central role at the SW side, was evaluated up to \( o(\hbar^6) \). All these considerations allowed us to check the duality for non-zero instantonic numbers for a few small \( N \) via computer.

Still, our understanding of this duality is far from being clear. Many interesting questions remain, such as what is the structure of the coefficients of \( \hat{O} \), what are the right terms to formulate the conjecture so that formulas become compact and whether the duality survives if one includes matter hypermultiplets. The work is in progress in these directions.

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