Variable Selection in Latent Regression IRT Models via Knockoffs: An Application to International Large-scale Assessment in Education

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Abstract

International large-scale assessments (ILSAs) play an important role in educational research and policy making. They collect valuable data on education quality and performance development across many education systems, giving countries the opportunity to share techniques, organisational structures and policies that have proven efficient and successful. To gain insights from ILSA data, we identify non-cognitive variables associated with students’ academic performance. This problem has three analytical challenges: 1) students’ academic performance is measured by cognitive items under a matrix sampling design; 2) there are often many missing values in the non-cognitive variables; and 3) multiple comparisons due to a large number of non-cognitive variables. We consider an application to data from the Programme for International Student Assessment (PISA), aiming to identify non-cognitive variables associated with students’ performance in science. We formulate it as a variable selection problem under a latent variable model and further propose a knockoff method that conducts variable selection with a controlled error rate for false selections.

Keywords: Model-X knockoffs, item response theory, missing data, variable selection, international large-scale assessment

1 Introduction

International large-scale assessments (ILSAs), including the Programme for International Student Assessment (PISA), Programme for the International Assessment of Adult Competencies (PIAAC), Progress in International Reading Literacy Study (PIRLS), and Trends in International Mathematics and Science Study (TIMSS), play an important role in educational research and policy making. They collect valuable data on education quality and performance development across many education systems in the world, giving countries the opportunity to share techniques, organisational structures and policies that have proven efficient and successful [Singer et al. 2018, von Davier et al. 2012].

PISA is a worldwide study by the Organisation for Economic Co-operation and Development (OECD) in member and non-member nations intended to evaluate educational systems by measuring 15-year-old school students’ scholastic performance on the subjects of mathematics, science, and reading, as well as a large number of non-cognitive variables, such as students’ socioeconomic status, family background, and learning experiences. Students’ scholastic performance is measured by response data from cognitive items that measure ability/proficiency in each of the three subjects, and non-cognitive variables are collected through non-cognitive questionnaires for students, school principals, teachers and parents. In this study, we focus on the knowledge domain of science in PISA 2015, where science was the assessment focus in this survey. Given the importance of science education [National Research Council 2012, Gov.UK 2015], it is of particular interest for educators and policy makers to understand what non-cognitive variables (e.g.,
socioeconomic status, family background, learning experiences) are significantly associated with students’ knowledge of science. Naturally, one would consider a regression model with students’ performance in science as the response variable and the non-cognitive variables as predictors, and identify the predictors with non-zero regression coefficients. Seemingly straightforward, constructing such a regression model and then selecting the non-null variables are nontrivial, due to three challenges brought by the complexity of the current problem. First, students’ performance in science is not directly observed, but instead, measured by a set of test items. The measurement is further complicated by a matrix sampling design adopted by PISA (Gonzalez and Rutkowski 2010). That is, each student is administered a small subset of available cognitive items, in order to cover an extensive content domain, while not overburdening students and schools in terms of their time and administration cost. Consequently, one cannot simply calculate a total score as a surrogate for student science performance. We note that OECD provides plausible values, which are obtained using a multiple imputation procedure (von Davier et al. 2009), as a summary of each student’s overall performance in each subject domain. However, it is not suitable to use a plausible value as the response variable when performing the current variable selection task. This is because, the multiple imputation procedure for producing the plausible values involves the predictors through a principal component analysis step (chapter 9, OECD 2016b), due to which all the predictors are associated with the plausible values and thus, performing variable selection is not sensible. Second, students’ non-cognitive variables are collected via survey questions, which contain many missing values. In fact, in the US sample considered in the current study, around 6% of the entries are missing and the proportion of sample points that are fully observed is less than 26%. Consequently, it is virtually impossible to conduct the regression analysis without a proper treatment of the missing values. Finally, PISA collects a large number of non-cognitive variables. In the current study of PISA 2015 data, we have 62 predictors, even though careful pre-processing is performed that substantially reduces the number of variables. Due to the multiple comparison issue, it is a challenge to control for a reasonable error when conducting variable selection.

We tackle these challenges through several methodological contributions. We introduce a latent construct for science knowledge, and use an Item Response Theory (IRT) model to measure this latent construct based on students’ responses to science items. The relationship between the latent construct and non-cognitive variables is further modelled through a structural model that regresses the latent construct onto the non-cognitive variables. This structural equation model is often known as the latent regression IRT model, or simply the latent regression model (Mislevy 1984; von Davier and Sinharay 2010). When there are many missing values in the non-cognitive variables, estimating the latent regression model is a challenge. To tackle this problem, we propose to model the predictors using a Gaussian copula model (Fan et al. 2017; Han and Pan 2012), which allows the predictors to be of mixed types (e.g., continuous, binary, ordinal). Thanks to the Gaussian copula model, we can estimate the latent regression model by a likelihood-based estimator. In dealing with multiple comparison, we consider the knockoff framework for controlled variable selection (Barber and Candès 2015; Candès et al. 2018). More specifically, we adapt the derandomised knockoffs method (Ren et al. 2021) to the current latent regression model with missing values. This approach allows us to control the Per Family Error Rate (PFER), i.e., the expected number of false positives among the detections. We choose the derandomised knockoff method instead of the Model-X knockoff method, because the latter is a randomised procedure that may suffer from a high Monte Carlo error. The derandomised knockoffs method leverages the Model-X knockoff method by aggregating the results from multiple knockoff realisations. To our best knowledge, this is the first time that missing data is considered in a knockoff approach with theoretical guarantees.
In real-world applications, especially in social sciences, missing data are commonly encountered in predictors. Although the current study focuses on data from an education survey, the proposed knockoff method can be easily generalised to other applications of variable selection that involve missing data. In particular, our method can be adapted to linear and generalised linear regression models, where the response variable is directly observable. Furthermore, although the current study considers a specific measurement model, the proposed method can be adapted to variable selection in other structural equation models (e.g., Jacobucci et al.; 2019; Serang et al.; 2017).

The remainder of the paper is structured as follows. Section 2 provides the background on the central substantive question – how students’ knowledge of science is associated with their non-cognitive variables – and a description of the PISA 2015 data. In Section 3 we introduce the latent regression IRT model for studying the relationship between a latent construct of science knowledge and non-cognitive variables, and a Gaussian copula model for handling missing predictors which are of mixed types. Section 4 proposes knockoff methods for controlled variable selection under the latent regression IRT model with missing data. The proposed method is evaluated via a simulation study in Section 5 and then applied to data from PISA 2015 in Section 6. Finally we discuss the implications of our results and possible directions for future research in Section 7. Proof of theoretical results, details of computation, and further information about the PISA data are given in the supplementary material.

2 Background and Overview of PISA 2015 Data

2.1 Academic Achievement and Non-cognitive Predictors

The term “non-cognitive” typically refers to a broad range of personal attributes, skills and characteristics representing one’s attitudinal, behavioral, emotional, motivational and other psychosocial dispositions. It is often used as a catch-all phrase encompassing variables that are potentially important for academic achievement but not measured by typical achievement or cognitive tests (Farkas; 2003). Social science researchers have devoted considerable research effort towards identifying non-cognitive predictors of students' academic achievement (e.g., Duckworth and Yeager; 2015; Richardson et al.; 2012; Lee and Stankov; 2018).

Science has changed our lives and is vital to the future prosperity of the society. Thus, science education plays an important role in the modern education system (National Research Council; 2012; Gov.UK; 2015). Identifying the predictors of science education helps educators, policy makers, and other stakeholders understand the psychosocial factors behind science education, which may lead to better policies and practices of science education. PISA, which collects both students’ science achievement and non-cognitive variables, provides a great opportunity for identifying the key non-cognitive predictors of science achievement.

2.2 PISA 2015 Data

PISA is conducted in a three-year cycle, with each cycle focusing on one of the three subjects, i.e., mathematics, science, and reading. PISA 2015 is the most recent cycle that focused on science. It collected data from 72 participating countries and economics. Computer-based tests were used, with assessments lasting a total of two hours for each student. Following a matrix sampling design, different students took different combinations of test items on science, reading, mathematics and collaborative problem solving. Test items involved a mixture of multiple-choice and constructive-response questions. See OECD (2016b) for the summary of the design and results of PISA 2015.
This study considers a subset of the PISA 2015 dataset. Specifically, to avoid modelling country heterogeneity, we considered data from a single country, the United States (US). After some data pre-processing which excluded observations with poor-quality data, the sample size is 5,685. PISA 2015 contained 184 items in the science domain that were dichotomously or polytomously scored. Due to the matrix sampling design of PISA, on average each student was only assigned 16.25% of the items.

In addition, we consider non-cognitive variables collected by the student survey, which provides information about the students themselves, their homes, and their school and learning experiences. We constructed 62 variables as candidates in variable selection. These variables include 11 raw responses to questionnaire items (e.g., GENDER (gender), LANGAH (language at home)), 34 indices that OECD constructed (e.g., CULTPO (cultural possession), HEDRES (home educational resources)), and 17 composites that we constructed based on students’ responses to questionnaire items (e.g., OUT.GAM (play games out of school), OUT.REA (reading out of school)). We decided to include these constructed variables rather than the corresponding raw responses, for better substantive interpretations. For some ordinal raw responses, certain adjacent categories were merged due to sample size considerations. Details of these 62 candidate variables are given in Section 6 and the supplementary material. Unlike the cognitive items, students were supposed to answer all the items in the student survey. However, there are still many missing responses in the student survey data. Among the candidate variables, 20 variables have more than 5% of their data missing, and the variable DUECEC (duration in early childhood education and care) has the largest missing rate, 37.17%.

3 Model Framework

In this section, we describe a latent regression model that allows for missing values in predictors. This model consists of three components: (1) a measurement model which describes the conditional distribution of item responses given the latent construct, science proficiency, (2) a structural model which regresses the latent construct on the predictors, and (3) a Gaussian copula model for the predictors under a missing at random setting. We first introduce some notation.

3.1 Notation and Problem Setup

Consider data collected from $N$ students, where data from different students are independent. For each student $i$, the data can be divided into two parts – (1) responses to cognitive items and (2) non-cognitive predictors. We use a random vector $\mathbf{Y}_i$ to denote student $i$’s cognitive responses. Due to the matrix sampling design for cognitive items in ILSAs, the length of $\mathbf{Y}_i$ can vary across students. More precisely, we use $\mathcal{B}_i$ to denote the set of cognitive items that student $i$ is assigned. Then $\mathbf{Y}_i = \{Y_{ij} : j \in \mathcal{B}_i\}$. PISA contains both dichotomous and polytomous cognitive items. For a dichotomous item, $Y_{ij} \in \{0, 1\}$, and for a polytomous item, $Y_{ij} \in \{0, 1, ..., K_j\}$, where the categories are ordered and $K_j + 1$ is the total number of categories. In addition, consider $p$ predictors collected via non-cognitive survey questions. Let $\mathbf{Z}_i = (Z_{i1}, \ldots, Z_{ip})^T$ denote the complete predictor vector for student $i$. Often, there are missing values in $\mathbf{Z}_i$. Let $\mathcal{A}_i$ denote the set of observed predictors for student $i$, and let $\mathbf{Z}_i^{obs} = \{Z_{ij} : j \in \mathcal{A}_i\}$ and $\mathbf{Z}_i^{mis} = \{Z_{ij} : j \notin \mathcal{A}_i\}$. The predictors are of mixed types. In the current study, binary, ordinal, and continuous predictors are considered.
3.2 Measurement Model

We introduce a latent variable $\theta_i$ as the latent construct which is measured by the cognitive items. In the current application, $\theta_i$ can be interpreted as student $i$’s proficiency in science. The measurement model is an IRT model that specifies the conditional distribution of $Y_{ij}$ given $\theta_i$. The path diagram of the proposed model is given in Figure 1, where the measurement model is reflected by the directed edges from $\theta_i$ to $Y_{ij}$s.

More specifically, this model assumes local independence, an assumption that is commonly adopted in IRT models (Embretson and Reise 2000). That is, $Y_{ij}$, $j \in B_i$, are conditionally independent given $\theta_i$. For a dichotomous item $j$, the conditional distribution of $Y_{ij}$ given $\theta_i$ is assumed to follow a two-parameter logistic model (2PL, Birnbaum 1969). That is,

$$P(Y_{ij} = 1|\theta_i) = \frac{\exp(a_j\theta_i + b_j)}{1 + \exp(a_j\theta_i + b_j)},$$ (1)

where $a_j$ and $b_j$ are two item-specific parameters. For a polytomous item $j$ with $K_j + 1$ categories, $Y_{ij}$ given $\theta_i$ is assumed to follow a generalized partial credit model (GPCM, Muraki 1992), for which

$$P(Y_{ij} = k|\theta_i) = \frac{\exp \left[ \sum_{r=1}^{k} (a_j\theta_i + b_{jr}) \right]}{1 + \sum_{k=1}^{K_j} \exp \left[ \sum_{r=1}^{K_j} (a_j\theta_i + b_{jr}) \right]}, \quad k = 1, \ldots, K_j,$$ (2)

where $a_j, b_{j1}, b_{j2}, \ldots, b_{jK_j}$ are item-specific parameters. In OECD’s analysis of PISA data, the item-specific parameters are first calibrated based on item response data from all the countries and then treated as known when inferring the proficiency level of students or the proficiency distributions of countries (chapter 9, OECD 2016b). We follow this routine. Specifically, the item-specific parameters are fixed to the values used by OECD for scaling PISA 2015 data.\footnote{The item parameters can be found from: https://www.oecd.org/pisa/data/2015-technical-report/PISA2015_TechRep_Final-AnnexA.pdf} We denote $P(Y_{ij} = y|\theta_i) = h_j(y|\theta_i)$, where $h_j$ is a known function given by equation (1) or (2).

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**Figure 1**: Path diagram for the proposed model. The measurement model is represented by the directed edges from $\theta_i$ to $Y_{ij}$, $j \in B_i$, the structural model is represented by the directed edges from $Z_{ij}$s to $\theta_i$, and the predictor model is represented by the undirected edges between $Z_{ij}$s.
3.3 Structural Model

The structural model regresses the latent construct $\theta_i$ onto the complete-data predictors $Z_{i1}, ..., Z_{ip}$. The structural model is reflected by the directed edges from $Z_{ij}$ to $\theta_i$ in the path diagram in Figure 1. A linear regression model is assumed for $\theta_i$ given $Z_{i1}, ..., Z_{ip}$. More specifically, for each variable $j$, we introduce a transformation $g_j(Z_j)$. When $Z_j$ is an ordinal variable with categories $\{0, ..., K_j\}$, the transformation function $g_j$ creates $K_j$ dummy variables, i.e., $g_j(Z_j) = (1(Z_j > 0), ..., 1(Z_j > K_j))^T$. For continuous and binary variables, $g_j$ is an identity link, i.e., $g_j(Z_j) = Z_j$. We assume $\theta_i|Z_i \sim N(\beta_0 + \beta_1^T g_1(Z_{i1}) + \cdots + \beta_p^T g_p(Z_{ip}), \sigma^2)$, where $\beta_0$, $\beta_1, ..., \beta_p$ are the slope parameters, and $\sigma^2$ is the residual variance. Note that $\beta_j$ is a scalar when predictor $j$ is continuous or binary, and is vector when the predictor is ordinal. Here, $\beta_0$, $\beta_1, ..., \beta_p$, and $\sigma^2$ are unknown, and will be estimated from the model. The main goal of our analysis is to find predictors for which $\|\beta_j\| \neq 0$.

3.4 Predictor Model

To handle missing values in $Z_{ij}$s, we impose a joint model for the predictors, which is reflected by the undirected edges between $Z_{ij}$s in Figure 1. Here, we consider a Gaussian copula model, which has been widely used for missing data analysis (Hoff, 2007; Murray et al., 2013; Robbins et al., 2013; Hollenbach et al., 2021). This model introduces underlying random variables $Z_i^* = (Z_{i1}^*, ..., Z_{ip}^*)^T$, for which $Z_{i1}^*, ..., Z_{ip}^*$ are independent and identically distributed, following a $p$-variate normal distribution $N(0, \Sigma)$. We assume that the normal distribution is non-degenerate, i.e., rank($\Sigma$) = $p$. Each underlying variable $Z_{ij}^*$ is assumed to marginally follow a standard normal distribution, i.e., the diagonal entries of $\Sigma$ are 1. Each predictor $Z_{ij}$ is assumed to be a transformation of its underlying variable $Z_{ij}^*$, denoted by $Z_{ij} = F_j(Z_{ij}^*)$. For a continuous predictor $j$, let $F_j(Z_{ij}^*) = c_j + d_j Z_{ij}^*$, where $c_j$ and $d_j$ are unknown parameters. For a binary or ordinal predictor $j$, let $F_j(Z_{ij}^*) = k$ if $Z_{ij}^* \in (c_{jk}, c_{jk+1}]$, $k = 0, ..., K_j$, where $c_{j1}, ..., c_{jK_j}$ are unknown parameters, and $c_{j0} = -\infty$ and $c_{jK_j + 1} = \infty$. Note that $K_j = 1$ for a binary variable, and $K_j > 1$ for an ordinal variable.

We note that the above model specifies a joint distribution for $Z_{i1}, ..., Z_{ip}$. More specifically, let $\mathcal{D} \subseteq \{1, ..., p\}$ be the set of dichotomous and polytomous predictors. We use $\Xi$ as generic notation for the unknown parameters in the Gaussian copula model, including $\Sigma$ and the parameters in the transformations between $Z_{ij}^*$ and $Z_{ij}$. We further use $\phi(\cdot|\Sigma)$ to denote the density function of the multivariate normal distribution $N(0, \Sigma)$. Then the density function of $Z_i$ takes the form

$$f(z_i|\Xi) = \int \cdots \int \left(\prod_{j \in \mathcal{D}} dz_j^*\right) \left[\phi(z_i^*|\Sigma) \times \left(\prod_{j \notin \mathcal{D}} d_j^{-1}\right) \times \left(\prod_{j \notin \mathcal{D}} \mathbb{I}(z_j^* \in (c_{jz_j-1}, c_{jz_j}])\right)\right]_{z_j^* = \frac{z_j - z_{jz_j}}{d_j}}.$$

3.5 Data Missingness and Statistical Inference

The previous model specification leads to a joint model for cognitive item response data and complete predictors. As the PISA data involve many missing values in the predictors, certain assumptions remain to be imposed on the data missingness. Here, we assume that missing data in predictors are Missing At Random (MAR), which is a quite strong assumption but commonly adopted in missing data analysis (Little and Rubin, 2019; Van Buuren, 2018).

More specifically, let $z_{i1}^{obs}$ be the realisation of $Z_{i1}^{obs}$, $i = 1, ..., N$, and recall that $\Xi$ denotes the unknown parameters of the Gaussian copula model. Under the MAR assumption, the log-likelihood function for $\Xi$ takes the form $l_1(\Xi) = \sum_{i=1}^N \log f_i(z_{i1}^{obs}|\Xi)$, where $f_i(z_{i1}^{obs}|\Xi) = \int \cdots \int \left(\prod_{j \notin \mathcal{A}_i} dz_{ij}\right) f(z_i|\Xi)$. Note that the
integral in $f_i(z_i^{obs}|\Xi)$ is with respect to the missing variables. The maximum likelihood estimator for $\Xi$ is given by

$$\arg\max_{\Xi} l_1(\Xi)$$

subject to $\Sigma_{jj} = 1$, $j = 1, \ldots, p$,

$$d_j > 0, j \notin D, c_{j1} < c_{j2} < \ldots < c_{jK_j}, j \in D.$$  \(3\)

We note that this optimisation problem involves high-dimensional integrals and constraints. We adopt a stochastic proximal gradient algorithm proposed in Zhang and Chen (2022). In this algorithm, the integrals are handled by Monte Carlo sampling of the missing values, and the unknown parameters are updated by stochastic proximal gradient descent, in which constraints are handled. The details of this algorithm can be found in the supplementary material.

Finally, given the estimated predictor model, one can estimate the unknown parameters in the structural model, including $\beta = (\beta_0, \beta_1^T, \ldots, \beta_p^T)^T$ and $\sigma^2$. Recall that the item-specific parameters in the measurement model are assumed to be known. Let $y_i$ be the realisation of $Y_i$, $i = 1, \ldots, N$. The log-likelihood for $\beta$ and $\sigma^2$ takes the form

$$l_2(\beta, \sigma^2) = \sum_{i=1}^{N} \log \left( \int \cdots \int \left( \prod_{j \notin A_i} dz_j \right) f(z_i|\hat{\Xi}) f(y_i|z_i; \beta, \sigma^2) \right),$$

where $f(y_i|z_i; \beta, \sigma^2)$ is the density function of the conditional distribution of $Y_i$ given $Z_i = z_i$

$$f(y_i|z_i; \beta, \sigma^2) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \int d\theta_i \left( \prod_{l \in B_i} h_l(y_{il}|\theta_i) \right) \exp \left( -\frac{(\theta_i - (\beta_0 + \beta_1^T g_1(z_{i1}) + \cdots + \beta_p^T g_p(z_{ip}))^2}{2\sigma^2} \right) \right).$$

We estimate $\beta$ and $\sigma^2$ by maximising $l_2(\beta, \sigma^2)$. Note that when writing down the likelihood $l_2$, it is implicitly assumed that $Y_i$ and $A_i$ are conditionally independent given $Z_i$. Similar to the optimisation \(3\), the maximisation of $l_2(\beta, \sigma^2)$ also involves high-dimensional integrals. We carry out this optimisation using a stochastic Expectation-Maximisation (EM) algorithm \(2\) (Nielsen 2000; Zhang et al. 2020). The details are given in the supplementary material.

4 Variable Selection via Knockoffs

4.1 Problem Setup and Knockoffs

As mentioned previously, our goal is to solve a model selection problem, i.e., to find the non-null predictors for which $\|\beta_j\| \neq 0$. We hope to control the statistical error in the model selection to assure that most of the discoveries are indeed true and replicable. This is typically achieved by controlling for a certain risk functions for variable selection, such as the false discovery rate, the $k$-familywise error rate, and the per familywise error (PFER); see Janson and Su (2016) and Candès et al. (2018). Let $\hat{S}$ and $S^* \subset \{1, \ldots, p\}$ be the selected and true non-null predictors, respectively. The current study concerns the control of PFER, defined as $\mathbb{E}[|\hat{S} \setminus S^*|]$, where $|\cdot|$ denotes the number of elements in a set.

The knockoff method is a general framework for controlled variable selection. The key to a knockoff method is the construction of knockoff variables, where the knockoff variables mimic the dependence structure

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2 The stochastic proximal gradient algorithm used for the optimisation problem \(3\) can also be used to solve the current optimisation problem. The stochastic EM algorithm is chosen as it tends to empirically converge faster for the current problem.
within the original variables but are null variables (i.e., not associated with the response variable). They serve as negative controls in the variable selection procedure that help identify the truly important predictors, while controlling for a certain risk function, such as the PFER. Many knockoff methods have been developed (Barber and Candès 2015; Candès et al. 2018; Fan et al. 2019; 2020; Janson and Su 2016; Sesia et al. 2019; Romano et al. 2020). Many knockoff methods are based on the model-X knockoff framework (Candès et al. 2018) which is very flexible and can be extended to the current setting involving missing data and mixed-type predictors. However, one drawback of the model-X knockoffs is that it only takes one draw of the knockoff variables through Monte Carlo sampling. As a result, this procedure often suffers from high uncertainty brought by the Monte Carlo error, even though the risk function is controlled. To alleviate this uncertainty, which has important implications on the interpretability of the variable selection results, we adopt the derandomised knockoff method (Ren et al. 2021). This method can substantially reduce the Monte Carlo error by aggregating the selection results across multiple runs of a knockoff algorithm. In what follows, we first introduce the way constructing knockoff variables under the joint model described in the above section, and then introduce a derandomised knockoff procedure for controlling PFER.

### 4.2 Constructing Knockoffs with Missing Data

We extend the concept of knockoffs to the missing data setting. To control the variable selection error with the knockoff procedure introduced below, a Stronger MAR condition is needed. It is called the SMAR condition as introduced in Definition 1 below.

**Definition 1** (SMAR condition). Consider the conditional distribution of $A_i$ given $Z_i$. Let $q(\alpha|z)$ denote the conditional probability mass function of $A_i$ given $Z_i$. We say the SMAR condition holds with respect to the non-null variables $S^*$, if $q(\alpha|z) = q(\alpha|z')$ holds, for any $\alpha$, $z = (z_1, \ldots, z_p)^\top$ and $z' = (z'_1, \ldots, z'_p)^\top$ satisfying $\{z_i : i \in \alpha \cap S^*\} = \{z'_i : i \in \alpha \cap S^*\}$.

This SMAR condition says that the probability of being missing is the same within groups defined by the observed non-null variables. It is stronger than MAR, because MAR only requires $q(\alpha|z) = q(\alpha|z')$ to hold, for any $\alpha$, $z = (z_1, \ldots, z_p)^\top$ and $z' = (z'_1, \ldots, z'_p)^\top$ satisfying $\{z_i : i \in \alpha\} = \{z'_i : i \in \alpha\}$, i.e., the probability of being missing is the same within groups defined by the observed variables, regardless whether they are in $S^*$ or not. On the other hand, the SMAR condition is weaker than Completely Missing at Random (MCAR), as MCAR implies that $q(\alpha|z) = q(\alpha|z')$ for all $\alpha$, $z$ and $z'$. Throughout the rest of the paper, the SMAR condition is assumed to hold.

**Definition 2** (Knockoffs). Suppose that the SMAR condition in Definition 1 holds for $Z_i$ and $A_i$. Under the setting in Section 3, we say that $\tilde{Z}_i^{\text{obs}}$ is a knockoff copy of $Z_i^{\text{obs}}$, if there exist complete data $Z_i = (Z_{i1}, \ldots, Z_{ip})^\top$ and $\tilde{Z}_i = (\tilde{Z}_{i1}, \ldots, \tilde{Z}_{ip})^\top$ such that

1. $Z_i^{\text{obs}} = \{Z_{ij} : j \in A_i\}$ and $\tilde{Z}_i^{\text{obs}} = \{\tilde{Z}_{ij} : j \in A_i\}$;
2. $\tilde{Z}_i$, $Y_i$, and $A_i$ are conditionally independent given $Z_i$;
3. $Z_i$ follows the Gaussian copula model in Section 3.4 and $Y_i$ given $Z_i$ follows the latent regression model in Sections 3.3 and 3.4;
4. For any subset $S \subset \{1, \ldots, p\}$, $(Z_i, \tilde{Z}_i)_{\text{swap}(S)}$ and $(Z_i, \tilde{Z}_i)$ are identically distributed.
Above, the vector \((\mathbf{Z}_i, \tilde{\mathbf{Z}}_i, \mathbf{S})\) obtained from \((\mathbf{Z}_i, \tilde{\mathbf{Z}}_i, \mathbf{S})\) by swapping the entries \(Z_{ij}\) and \(\tilde{Z}_{ij}\) for each \(j \in \mathcal{S}\); for example, with \(p = 3\) and \(\mathcal{S} = \{1, 3\}\), \((\mathbf{Z}_{i1}, \mathbf{Z}_{i3}, \tilde{\mathbf{Z}}_{i1}, \tilde{\mathbf{Z}}_{i3}, \tilde{\mathbf{Z}}_{i3})_{\text{swap}((1,3))} = (\tilde{\mathbf{Z}}_{i1}, \mathbf{Z}_{i3}, \tilde{\mathbf{Z}}_{i3}, \mathbf{Z}_{i1}, \mathbf{Z}_{i3})\).

We compare the current definition of knockoffs under a missing data setting with the standard definition for model-X knockoffs in Candès et al. (2018). The model-X knockoff framework assumes no missing data in predictors \(\mathbf{Z}_i\), and \(\mathbf{Z}_1, ..., \mathbf{Z}_N\) are independent and identically distributed. Therefore, the definition of model-X knockoff omits the subscript \(i\). On the other hand, the current analysis depends on \(\mathcal{A}_i\) which differs across observations. Consequently, knockoffs are defined for each \(\mathbf{Z}_i^{\text{obs}}\). When there is no missing data, i.e., \(\mathcal{A}_i = \{1, ..., p\}\), \(i = 1, ..., N\), the current definition coincides with the definition in Candès et al. (2018). Note that stronger conditions are needed for the construction of knockoffs when there exist missing data. These conditions (e.g., SMAR) are needed to ensure that the joint distribution of \(\mathbf{Y}_i, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}, \) and \(\mathcal{A}_i\) remains identical when swapping the null variables, which is essential for establishing the exchangeability property (Candès et al., 2018) for controlling variable selection error. Specifically, under Definition 2, \(\tilde{\mathbf{Z}}_i^{\text{obs}}\) and \(\mathbf{Y}_i\) are likely not conditionally independent given \(\mathbf{Z}_i^{\text{obs}}\), even though \(\tilde{\mathbf{Z}}_i\) and \(\mathbf{Y}_i\) are conditionally independent given \(\mathbf{Z}_i\). Consequently, when constructing the knockoff variables \(\tilde{\mathbf{Z}}_i^{\text{obs}}\), one needs information not only from \(\mathbf{Z}_i^{\text{obs}}\) but also \(\mathbf{Y}_i\), to compensate for the missing information. In other words, the joint distribution of \((\mathbf{Y}_i, \mathbf{Z}_i)\) is needed to construct \(\tilde{\mathbf{Z}}_i^{\text{obs}}\).

In what follows, we present an algorithm for constructing knockoffs \(\tilde{\mathbf{Z}}_i^{\text{obs}}\) under Definition 2. To ensure the exact satisfaction of Definition 2, we assume that the true model parameters are known. In practice, we plug an estimate of the parameters into the algorithm; see Section 4.4 for theoretical justifications and further discussions.

**Algorithm 1** (Constructing knockoff copies).

**Input:** Observed data \(\mathbf{Y}_i\) and \(\mathbf{Z}_i^{\text{obs}}, i = 1, ..., N\), the true model parameters \(\Xi\) of the Gaussian copula model, and the true parameters \(\beta, \beta_0, \sigma^2\) of latent regression model.

**Step 1:** Sample underlying variables \(\mathbf{Z}_i^*\) from their conditional distribution given \(\mathbf{Z}_i^{\text{obs}}\) and \(\mathbf{Y}_i\).

**Step 2:** Sample \(\tilde{\mathbf{Z}}_i^*\) given \(\mathbf{Z}_i^*\), where \((\mathbf{Z}_i^*, \tilde{\mathbf{Z}}_i^*)\) jointly follows a multivariate normal distribution with mean zero and covariance matrix

\[
G = \begin{pmatrix}
\Sigma & \Sigma - SS \\
\Sigma - SS & \Sigma
\end{pmatrix},
\]

where \(\Sigma\) is the correlation matrix in the Gaussian copula model, and \(SS\) is a diagonal matrix specified in such a way that the joint covariance matrix \(G\) is positive semidefinite. The construction of \(SS\) is based on the Minimize the Reconstructability (MVR) procedure (Spector and Janson, 2022).

**Step 3:** Obtain \(\tilde{\mathbf{Z}}_i\) from \(\tilde{\mathbf{Z}}_i^*\), where \(\tilde{Z}_{ij} = F_j(\tilde{Z}_{ij}^*)\) for each \(j = 1, ..., p\).

**Output:** Knockoff copy \(\tilde{\mathbf{Z}}_i^{\text{obs}} = \{\tilde{Z}_{ij}^{\text{obs}} : j \in \mathcal{A}_i\}\).

**Proposition 1.** The output \(\tilde{\mathbf{Z}}_i^{\text{obs}}\) from Algorithm 1 satisfies Definition 2.

The proof of this proposition is given in the supplementary material. Figure 2 below gives the path diagram for the generation of knockoff copies. We provide several remarks on the algorithm. This algorithm allows for mixed types of predictors under a Gaussian copula model, which extends the multivariate Gaussian model for knockoff construction considered in Barber and Candès (2015) and Candès et al. (2018). When all
the predictors are continuous, the Gaussian copula model degenerates to the multivariate Gaussian model. In that case and if there is no missing data, then Algorithm 1 coincides with the knockoff construction method in Candès et al. (2018), except that Candès et al. (2018) uses the Mean Absolute Correlation (MAC) procedure to construct the $SS$ matrix.

Figure 2: Path diagram for constructing $\tilde{Z}_{ij}$.

When $Z_{ij}$ contains binary or ordinal variables, directing sampling $Z_{ij}$ is not straightforward. We obtain approximate samples via Gibbs sampling. Thanks to the underlying multivariate normality assumption, each step of the Gibbs sampler only involves sampling from univariate normal or truncated normal distributions. Details of the Gibbs sampler are given in the supplementary material. We compute the diagonal matrix $SS$ in Step 2 of the algorithm using the MVR procedure (Spector and Janson 2022), which tends to be more powerful than the MAC procedure adopted in Barber and Candès (2015) and Candès et al. (2018).

4.3 Variable Selection via Derandomised Knockoffs

We now describe a knockoff procedure for variable selection with a controlled PFER. Suppose that knockoff copies $\tilde{Z}_{ij}$, $i = 1, \ldots, N$, have been obtained using Algorithm 1. For the ease of exposition, we denote $Z_{ij} = (Z_{i1}^{obs}, \ldots, Z_{iN}^{obs})$, $\tilde{Z}_{ij} = (\tilde{Z}_{i1}^{obs}, \ldots, \tilde{Z}_{iN}^{obs})$, and $Y = (Y_1, \ldots, Y_N)$. We define a knockoff statistic that measures the importance of each predictor.

Definition 3 (Knockoff statistic). Consider a statistic $W_j$ taking the form $W_j = w_j([Z^{obs}, \tilde{Z}^{obs}], Y)$ for some function $w_j$, where $Z^{obs}$ are knockoffs satisfying Definition 3. This statistic is called a knockoff statistic for the $j$th predictor if it satisfies the flip-sign property: that is for any subset $S \subseteq \{1, \ldots, p\}$,

$$w_j ([Z^{obs}, \tilde{Z}^{obs}]_{\text{swap}(S)}, Y) = \begin{cases} w_j ([Z^{obs}, \tilde{Z}^{obs}], Y), & j \notin S, \\
-w_j ([Z^{obs}, \tilde{Z}^{obs}], Y), & j \in S, \end{cases}$$

where $[Z^{obs}, \tilde{Z}^{obs}]_{\text{swap}(S)}$ is the matrix obtained by swapping the entries $Z_{ij}$ and $\tilde{Z}_{ij}$ for each $j \in S \cap A_i$, $i = 1, \ldots, N$.

The flip-sign property in Definition 3 is key to guaranteeing valid statistical inference from finite samples. However, to achieve a good power, $W_j$ should also provide evidence regarding whether $|\beta_j| = 0$. See Section 3 of Candès et al. (2018) for a generic method of constructing $W_j$ and specific examples. In this study, we will focus on a knockoff statistics constructed based on the likelihood function. More specifically, we incorporate the knockoff variables into the latent regression model defined in Section 3. That is, the measurement model remains the same, while the structural model becomes

$$\theta_i | Z_i, \tilde{Z}_i \sim N(\beta_0 + \beta_1^T g_1(Z_{i1}) + \cdots + \beta_p^T g_p(Z_{ip}) + \gamma_1^T g_1(\tilde{Z}_{i1}) + \cdots + \gamma_p^T g_p(\tilde{Z}_{ip}), \sigma^2),$$

where $Z_i$ and $\tilde{Z}_i$ are defined in Definition 2. Since $\tilde{Z}_i$ and $Y_i$ are conditionally independent given $Z_i$, the true value of $\gamma_j$ is $0$, $j = 1, \ldots, p$, though these parameters will be estimated when constructing the knockoff statistics. This latent regression model is illustrated by a path diagram in Figure 3.
Then a knockoff statistic can be constructed as

$$f_{\text{estimator based on}}$$

properties of the knockoff statistics to be described below. In addition, $$\tilde{\theta}$$ is constructed based on $$\tilde{\beta}, \gamma, \sigma^2$$). Specifically, consider the maximum likelihood estimator based on $$\tilde{\beta}, \gamma, \sigma^2$$)

$$\tilde{\beta}, \gamma, \sigma^2$$)

$$W_j = \text{sign} (\|\tilde{\beta}_j\| - \|\gamma_j\|) \max \left\{ \|\tilde{\beta}_j\|/\sqrt{p_j}, \|\gamma_j\|/\sqrt{p_j} \right\},$$

where $$p_j$$ is the dimension of $$\beta_j$$ (or equivalently that of $$\gamma_j$$), and $$\tilde{\beta}_j = \text{Cov}(g_j(Z_{ij}))^{1/2} \beta_j$$ and $$\tilde{\gamma}_j = \text{Cov}(g_j(Z_{ij}))^{1/2} \gamma_j$$ are standarised coefficients.

**Proposition 2.** $$W_j$$ given by (7) satisfies Definition 3.

The proof of this proposition is given in the supplementary material. Similar to the estimation of the latent regression model without knockoffs, the optimisation problem (6) can be solved using a stochastic
EM algorithm. We remark that the statistic (7) is a special case of the Lasso coefficient-difference statistic given in Candès et al. (2018), when the Lasso penalty is set to zero. Since the sample size \(N\) is often much larger than \(p\) in ILSA applications, this likelihood-based knockoff statistic performs well in our simulation study and real data analysis. For higher-dimensional settings, a Lasso coefficient-difference statistic may be preferred; see Candès et al. (2018).

We now adapt the derandomised knockoff method (Ren et al.; 2021) to the current problem. This method achieves PFER control by aggregating the results from multiple runs of a baseline algorithm proposed in Janson and Su (2016). This baseline algorithm is summarised in Algorithm 2 below.

**Algorithm 2** (Baseline algorithm for PFER control (Janson and Su; 2016)).

Input: Knockoff statistics \(W_1, \ldots, W_p\) satisfying Definition 3, and PFER level \(\nu \in \mathbb{Z}_+\).

Step 1: Compute the threshold \(\tau = \inf \{t > 0 : 1 + \sum_{j \in \mathbb{Z}} |\{W_j < -t\}| = \nu\}\). We let \(\tau = -\infty\) if the set on the right hand side is an empty set.

Output: \(S = \{j : W_j > \tau\}\).

**Proposition 3.** \(\hat{S}\) given by Algorithm 2 satisfies \(E|\hat{S}\setminus S^*| \leq \nu\), i.e., the PFER can be controlled at level \(\nu\).

**Algorithm 3** (Derandomised knockoffs (Ren et al.; 2021)).

Input: Observed data \(Z_{\text{obs}}\) and \(Y\), the number of runs \(M\) of the baseline algorithm, a selection threshold \(\eta\), and PFER level \(\nu \in \mathbb{Z}_+\).

Step 1: For each \(m = 1, \ldots, M\), generate knockoffs \(Z_{\text{obs}}^{(m)}\) independently using Algorithm 1, and then compute the set of knockoff statistics \(W_j^{(m)}\) using (7), \(j = 1, \ldots, p, m = 1, \ldots, M\).

Step 2: For each \(m = 1, \ldots, M\), run Algorithm 2 with \(W_j^{(m)}, j = 1, \ldots, p\) and PFER threshold \(\nu\), and obtain the selection set \(S^{(m)}\).

Step 3: For each \(j = 1, \ldots, p\), compute the selection frequency \(\Pi_j = \frac{1}{M} \sum_{m=1}^{M} \mathbb{I}\left(j \in S^{(m)}\right)\).

Output: \(\hat{S} = \{j \in \{1, \ldots, p\} : \Pi_j \geq \eta\}\).

Following the theoretical result in Ren et al. (2021), when the threshold \(\eta\) is chosen properly, Algorithm 3 guarantees to control PFER at level \(\nu\). We provide a simplified version of this result in Proposition 4 below.

**Proposition 4.** If for any \(\eta \in (0, 1)\), the condition \(P(\Pi_j \geq \eta) \leq E[\Pi_j]\) holds for every \(j \notin S^*\), then \(\hat{S}\) given by Algorithm 3 satisfies \(E|\hat{S}\setminus S^*| \leq \nu\), i.e., the PFER can be controlled at level \(\nu\). In particular, assuming that the probability mass function of \(\Pi_j\) is monotonically non-increasing for each \(j \notin S^*\), \(P(\Pi_j \geq \eta) \leq E[\Pi_j]\) holds for \(M = 31\) and \(\eta = 1/2\).

While noting that other choices are possible, we set \(M = 31\) and \(\eta = 1/2\), which is also the default choice in Ren et al. (2021). We also note that the statistics \(\Pi_j, j = 1, \ldots, p\), rank the importance of the predictors. The predictors with \(\Pi_j \geq \eta\) are selected as the non-null variables.
4.4 A Three-step Procedure When Model Parameters are Unknown and Its Robustness

The knockoff procedure described previously requires the true joint model for \( Y_i \) and \( Z_i \), which is infeasible in practice. When the true model is known, the variable selection problem becomes trivial since the null and non-null variables can be directly told from the true model. In practice, we first estimate the model parameters, and then conduct variable selection based on the estimated model. This procedure involves three steps. First, estimate the parameters \( \Xi \) in the Gaussian copula model. This is done by the maximum likelihood estimator (3). Second, estimate the parameters \( \beta \) and \( \sigma^2 \) based on the log-likelihood (4), where the estimated Gaussian copula model \( \hat{\Xi} \) is plugged in. Third, select variables by plugging the estimated parameters \( \hat{\beta}, \hat{\gamma}, \hat{\sigma}^2 \) into Algorithm 2 or 3.

Empirically, simulation results in Section 5 show that PFER is well controlled when we apply the above three-step procedure. Theoretically, by plugging into the estimated model rather than the true model, the PFER can no longer be exactly controlled as described in Propositions 3 and 4. Following a similar proof strategy as in Barber et al. (2020), we show that this procedure is robust, in the sense that the resulting PFER is controlled near \( \nu \), if the plug-in model is sufficiently accurate. Note that Barber et al. (2020) only consider the robustness of model-X knockoffs for controlling false discovery rate and does not cover PFER.

More precisely, we use \( \mathbb{P} \) and \( \mathbb{Q} \) to denote the true and plug-in models, respectively. Consider a pair of \( i \) and \( j \), satisfying \( j \in \mathcal{A}_i \) and \( j \notin S^* \). Let \( Z_{i,j}^{obs} = \{Z_{ik} \mid k \in \mathcal{A}_i \setminus \{j\}\} \). Let \( \mathbb{P}_{ij}(z_{ij}|Z_{i,j}^{obs}, y_i) \) denote the conditional density function of \( Z_{ij} \) given \( Z_{i,j}^{obs} = z_{i,j}^{obs} \) and \( Y_i = y_i \) under the true model \( \mathbb{P} \). Let \( \mathbb{Q}_{ij}(Z_{i,j}^{obs}, \tilde{Z}_{ij}|Z_{i,j}^{obs}, z_{ij}, y_i) \) denote the conditional density function of \( (Z_{i,j}^{obs}, \tilde{Z}_{ij}) \) given \( Z_{i,j}^{obs} = z_{i,j}^{obs}, Z_{ij} = Z_{ij} \) and \( Y_i = y_i \) under the plug-in model \( \mathbb{Q} \). Note that by the SMAR assumption, we can omit \( \mathcal{A}_i \) in these conditional densities. We define

\[
\hat{\text{KL}}_{ij} = \sum_{i \in \mathcal{A}_i} \log \left( \frac{\mathbb{P}_{ij}(Z_{ij}|Z_{i,j}^{obs}, Y_i) \cdot \mathbb{Q}_{ij}(Z_{i,j}^{obs}, \tilde{Z}_{ij}|Z_{i,j}^{obs}, Z_{ij}, Y_i)}{\mathbb{P}_{ij}(Z_{ij}|Z_{i,j}^{obs}, Y_i) \cdot \mathbb{Q}_{ij}(Z_{i,j}^{obs}, \tilde{Z}_{ij}|Z_{i,j}^{obs}, \tilde{Z}_{ij}, Y_i)} \right).
\]

Note that the numerator inside of the logarithm corresponds to the true data generation mechanism for \( (Z_{ij}, Z_{ij}^{obs}) \), and the denominator corresponds to that when switching the roles of \( Z_{ij} \) and \( \tilde{Z}_{ij} \). \( \hat{\text{KL}}_{ij} \) can be viewed as an observed Kullback–Leibler (KL) divergence that measures the discrepancy between the true model \( \mathbb{P} \) and its approximation \( \mathbb{Q} \), with \( \hat{\text{KL}}_{ij} = 0 \) when \( \mathbb{Q} = \mathbb{P} \). We remark that this definition of \( \hat{\text{KL}}_j \) is consistent with that in Barber et al. (2020). However, the \( \hat{\text{KL}}_j \) in Barber et al. (2020) can be further simplified with a pairwise exchangeable property of their procedure under a model-X knockoff setting without missing data, while this pairwise exchangeable property does not always hold for the current procedure due to the involvement of \( Y_i \) and thus, the current \( \hat{\text{KL}}_j \) cannot be further simplified.

**Theorem 1.** Under the definitions above, for any \( \epsilon \geq 0 \), consider the null variables for which \( \hat{\text{KL}}_{ij} \leq \epsilon \). If we use a modified Algorithm 3 that generates knockoffs under the plug-in model \( \mathbb{Q} \), then the expected number of rejections that correspond to such nulls obeys \( \mathbb{E}[\mathcal{S} \cap S^* \text{ and } \hat{\text{KL}}_{ij} \leq \epsilon] \leq \nu e^\epsilon \). In particular, \( \hat{\text{KL}}_{ij} = 0 \), when \( \mathbb{Q} = \mathbb{P} \).

When \( \mathbb{P} = \mathbb{Q} \), we can set \( \epsilon = 0 \), and thus, Theorem 1 implies Proposition 3. This property of robustness carries over to the derandomised procedure. We define \( \Pi_j = \left( \sum_{m=1}^{M} \mathbb{I}(j \in \hat{S}(m) \text{ and } \hat{\text{KL}}_{ij}^{(m)} \leq \epsilon) \right)/M \), where \( \hat{S}(m) \) is the selection in the \( m \)th run of modified Algorithm 3 that generates knockoffs under the plug-in model \( \mathbb{Q} \), and \( \hat{\text{KL}}_{ij}^{(m)} \) is the corresponding observed KL divergence based on the knockoffs from the
Theorem 2. Under the definitions above, for any \( \epsilon \geq 0 \), consider the null variables for which \( \hat{K}L^{(m)}_{j} \leq \epsilon \) for all \( m = 1, ..., M \). We use a modified Algorithm 3 where knockoffs are generated under the plug-in model \( Q \), and obtain selections \( \hat{S} \). If the condition \( P(\Pi_{j}^{*} \geq \eta) \leq E[\Pi_{j}^{*}] \) holds for every \( j \notin S^{*} \), then \( E\{j : j \in \hat{S} \setminus S^{*} \text{ and } \hat{K}L^{(m)}_{j} \leq \epsilon, m = 1, ..., M\} \leq \nu \epsilon \).

If the probability mass function of \( \Pi_{j}^{*} \) is monotonically non-increasing for each \( j \notin S^{*} \), \( P(\Pi_{j}^{*} \geq \eta) \leq E[\Pi_{j}^{*}] \) holds for \( M = 31 \) and \( \eta = 1/2 \).

5 Simulation Study

In this section, we conduct a simulation study to evaluate the performance of the proposed knockoff method. We check if the PFER can be controlled at the targeted level when the three-step procedure described in Section 4.4 is applied. The power of variable selection will also be assessed.

We set \( p = 100, J = 60 \), and consider \( N \in \{1000, 2000, 4000\} \) for comparing power under different sample sizes. It leads to three settings. For each setting, we generate 100 independent replications. The data are generated as follows. We divide the predictors into five blocks, each contains contain 10 continuous variables and 10 binary variables. Ordinal variables are not included in this study for simplicity. We consider the following design for the correlation matrix \( \Sigma \) of the underlying variables \( Z_{i}^{\ast} \), which is similar to the one used in Grund et al. (2021) that concerns analysing missing data in ILSAs. This correlation matrix mimics the correlation structure in ILSA data. (a) Within block 1, the correlation between every pair of variables is 0.6. (b) Within block 2, the correlation between every pair of variables is 0.6. For the 10-by-10 submatrix recording the correlations between variables in blocks 1 and 2, the diagonal entries are set to be 0.3 and the off-diagonal entries are set to be 0.15. (c) Within block 3, the correlation between every pair of variables is 0.6. The variables in block 3 has a correlation 0.15 with each variable in blocks 1 and 2. (d) Within block 4, the correlation between every pair of variables is 0.3. For the 10-by-30 submatrix recording the correlations between variables in block 4 and those in blocks 1 to 3, all the entries take value 0.15, except that the diagonal entries of the 10-by-10 submatrix corresponding to blocks 4 and 1 are set to 0.3. (e) Within block 5, the correlation between every pair of variables is 0.3. For the 10-by-40 submatrix recording the correlations between variables in block 5 and those in blocks 1 to 4, the entries are generated independently from a uniform distribution over the interval \([0.1, 0.2]\). The same correlation matrix is used in all the 100 replications. The heat map of this correlation matrix is given in Figure 4. This correlation matrix has a maximal eigenvalue as 22.73 and a minimal eigenvalue as 0.11.

![Figure 4: Heatmap of the designed correlation matrix in simulation study.](image)
The rest of the Gaussian copula model is set as follows. For continuous variables, we set \( c_j = 0 \) and \( d_j = 1 \). For the binary variables, we set their threshold parameters \( c_{j1} \) to take one of the values in \((-1.2, -0.3, 0, 0.3, 1.2)\) iteratively (i.e., \( c_{11,1} = -1.2, c_{12,1} = -0.3 \) and so on). Regarding the parameters in the structural model, we set the intercept \( \beta_0 = 0 \), \( \beta_j = 0.5 \) for \( j = 1, 22, 43, 64, 85, -0.5 \) for \( j = 11, 32, 53, 74, 95 \), and 0 for the rest of the variables. Under this setting, the non-zero coefficients are distributed uniformly in five blocks and two variable types, and for continuous and binary variables. We further set \( \sigma^2 = 1 \) for the residual variance.

Data missingness is generated following the SMAR condition. For each observation \( i \), we generate a random variable \( R_i \) from a categorical distribution with support \( \{1, 2, ..., 5\} \), satisfying \( P(R_i = k) = 0.2 \), for all \( k = 1, ..., 5 \). The data missingness is determined by \( R_i \) and the non-null variables. Let \( S_k^* \) denote the set of non-null variables in the \( k \)th block. For observation \( i \), when \( R_i = k \), we let all the variables in \( S_k^* \) be observed. For each of the rest of the variables \( j \), its probability of being missing is given by \( (1 + \exp(1 - (\sum_{j' \in S_k^*} Z_{ij'})))/2 \)\(^{-1} \). Under this setting, around 32% of the entries of the data matrix for predictors are missing.

Finally, we generate the parameters in the measurement model with only dichotomous items. We sample \( a_j \)'s from a uniform distribution \( U[0.5, 1.5] \), and \( b_j \)'s from uniform distribution \( U[-2, 0] \), where the range of these distributions is chosen to guarantee that \( a_j\theta_i + b_j \) to be in a suitable range. When generating the responses, a matrix sampling design is adopted. Here, all the items are divided into three equal-size blocks. Each observation is randomly assigned one of the three blocks, and the responses to the rest of the two blocks are missing completely at random.

We apply the three-step procedure described in Section 4.3 including both the baseline procedure based on Algorithm 2 and the derandomised procedure based on Algorithm 3. Different target levels are considered, including \( \nu \in \{1, 2, ..., 5\} \). Our results are given in Table 1. Two performance metrics are reported, including (1) the average PFER, which is calculated by averaging \(|\hat{S}\setminus S^*|\) over 100 replications, and (2) the average True Positive Rate (TPR), which is calculated by averaging \(|\hat{S} \cap S^*|/|S^*|\). As we can see, the baseline algorithm controls the PFER slightly below the nominal level, while the derandomised knockoff method tends to be more conservative that gives an average PFER much smaller than the nominal level. On the other hand, the derandomised method tends to be more powerful than the baseline algorithm, in the sense that it typically achieves a higher average TPR. This phenomenon is consistent with the findings in Ren et al. (2021) under linear and logistic regression settings.

6 Application to PISA 2015

We now apply the proposed method to the PISA 2015 dataset described in Section 2. Our results are given in Table 2. In this table, the predictors are ranked according to the value of \( \Pi_j \) when \( \nu = 1 \), from the largest to the smallest. For each predictor, we give the variable name, the variable type (continuous, binary, or ordinal), and a brief explanation of the variable. Further details about these variables are given in the supplementary material. In addition, we present the estimated coefficients of these variables under the full model (i.e., the model with all the predictors), and their standard errors based on a non-parametric bootstrap procedure with 200 replications. For each continuous variable, the standardised estimated coefficient is given, which is the estimated coefficient multiplied by the standard deviation of the corresponding variable. Variable selection results with nominal PFER levels \( \nu = 1, 2, 3 \) are given in Table 2, for which 38, 41, and 46 predictors are selected, respectively. Note that by the construction of the derandomised knockoff method,
Table 1: Simulation results. Here, “Baseline” refers to the baseline algorithm, Algorithm 2, and “DRM” refers to derandomised knockoffs, Algorithm 3. \( \nu \) refers to the nominal PFER level.

| \( \nu = 1 \) | \( \nu = 2 \) | \( \nu = 3 \) | \( \nu = 4 \) | \( \nu = 5 \) |
|---|---|---|---|---|
| PFER | Baseline | 0.96 | 1.80 | 2.93 | 4.01 | 4.92 |
| | DRM | 0 | 0.01 | 0.05 | 0.23 | 0.37 |
| TPR | Baseline | 29.3% | 38.7% | 44.1% | 48.5% | 52.1% |
| | DRM | 34.5% | 45.6% | 48.4% | 51.0% | 53.5% |
| N = 1000 | PFER | Baseline | 1.04 | 1.95 | 2.80 | 3.82 | 4.71 |
| | DRM | 0.05 | 0.22 | 0.44 | 0.78 | 1.22 |
| TPR | Baseline | 65.1% | 73.9% | 78.6% | 82.2% | 85.6% |
| | DRM | 65.8% | 76.3% | 80.0% | 82.1% | 84.4% |
| N = 2000 | PFER | Baseline | 1.02 | 1.97 | 3.06 | 3.99 | 4.93 |
| | DRM | 0.15 | 0.59 | 1.10 | 1.69 | 2.33 |
| TPR | Baseline | 87.4% | 92.6% | 95.2% | 97.2% | 98.0% |
| | DRM | 88.5% | 94.9% | 97.9% | 99.1% | 99.6% |
| N = 4000 |

These selection results are nested, in the sense that the variables selected with \( \nu = t \) are also selected with \( \nu = t + 1, t = 1, 2, \ldots \). We also point out that for the first 25 variables (ANXTES to UNFAIR), \( \Pi_j = 1 \), i.e., the variables are always selected by the baseline algorithm, and for the last 12 variables (DUECEC to WEALTH), \( \Pi_j = 0 \), for any \( \nu = 1, 2, 3 \), i.e., they are never selected by the baseline algorithm.

Table 2: Results from applying Algorithm 3 to PISA data. The variables are ordered according to the value of \( \Pi_j \) when \( \nu = 1 \), from the largest to the smallest. For variables with the same \( \Pi_j \) values, they are ordered alphabetically. Continuous, binary, and ordinal variables are indicated by C, B, and O, respectively. For an ordinal variable \( Z_j \), a coefficient corresponds to a dummy variable \( I(Z_k \geq k) \), for each non-baseline category \( k = 1, \ldots, K_j \).

| Name    | Type | Description                                                                 | Estimate | SE   |
|---------|------|-----------------------------------------------------------------------------|----------|------|
| ANXTES  | C    | Personality: test anxiety.                                                  | -0.0546  | 0.0099 |
| BELONG  | C    | Subjective well-being: sense of belonging to school.                       | -0.0461  | 0.0109 |
| COOPER  | C    | Collaboration and teamwork dispositions: enjoy cooperation.                 | 0.0546   | 0.0113 |
| CPSVAL  | C    | Collaboration and teamwork dispositions: value cooperation.                 | -0.0865  | 0.0102 |
| DISCLI  | C    | Disciplinary climate in science classes.                                    | 0.0627   | 0.0093 |
| EBSCTT  | C    | Enquiry-based science teaching and learning practices.                      | -0.0532  | 0.0099 |
| EISCED  | O    | ISCED (International Standard Classification of Education) level student expects to complete. (0/1/2 = [level 2 or 3A]/[level 4 or 5B]/[level 5A or 6]) | 0.1854   | 0.0343 |
|         |      |                                                                             | 0.0696   | 0.0282 |
| Variable | Measurement | Description                                                                                     | Value 1 | Value 2 |
|----------|-------------|-------------------------------------------------------------------------------------------------|---------|---------|
| EISEIO C | C           | Student’s expected International Socio-economic Index of occupational status.                    | 0.0394  | 0.0080  |
| ENVAWA C | C           | Environmental awareness.                                                                         | 0.0620  | 0.0101  |
| ENVOPT C | C           | Environmental optimism.                                                                           | -0.0875| 0.0088  |
| EPIST C  | C           | Epistemological beliefs.                                                                           | 0.0864  | 0.0097  |
| FISEIO C | C           | ISEI (International Socio-economic Index) of occupational status of father.                       | 0.0425  | 0.0088  |
| GENDER B | B           | Student’s gender. (0 = female/male)                                                                | 0.1933  | 0.0212  |
| JOYSCI C | C           | Enjoyment of science.                                                                             | 0.0865  | 0.0109  |
| MISEIO C | C           | ISEI (International Socio-economic Index) of occupational status of mother.                        | 0.0360  | 0.0082  |
| OUT.STU B| B           | Whether study for school or homework outside the school. (0/1 = no/yes)                           | 0.1150  | 0.0194  |
| OUT.PAR B| B           | Whether talk to parents outside the school. (0/1 = no/yes)                                        | -0.1265| 0.0236  |
| OUT.JOB B| B           | Whether work for pay outside the school. (0/1 = no/yes)                                           | 0.2023  | 0.0261  |
| OUT.SPO B| B           | Whether exercise or do a sport outside the school. (0/1 = no/yes)                                 | 0.1870  | 0.0187  |
| PERFEE C | C           | Perceived feedback.                                                                               | -0.1252| 0.0111  |
| REPEAT B | B           | Whether the student has ever repeated a grade. (0/1 = no/yes)                                     | -0.2250| 0.0303  |
| SCI.CHE B| B           | Whether attended chemistry courses in this or last school year. (0/1 = no/yes)                  | 0.1064  | 0.0188  |
| TDSCIT C | C           | Teacher-directed science instruction.                                                             | 0.0450  | 0.0103  |
| TMINS C  | C           | Learning time in class per week (minutes).                                                        | 0.0735  | 0.0073  |
| UNFAIR C | C           | Teacher unfairness.                                                                               | -0.0551| 0.0093  |
| LANGAH B | B           | Whether language at home different from the test language. (0/1 = no/yes)                         | -0.1054| 0.0266  |
| SCI.PHY B| B           | Whether attended physics courses in this or last school year. (0/1 = no/yes)                     | -0.0753| 0.0193  |
| CULTPO C | C           | Cultural possessions at home.                                                                    | 0.0401  | 0.0109  |
| INSTSC C | C           | Instrumental motivation.                                                                           | -0.0378| 0.0096  |
| SCIEEF C | C           | Science self-efficacy.                                                                             | 0.0397  | 0.0105  |
| OUTHOU C | C           | Out-of-school study time per week (hours).                                                       | -0.0340| 0.0089  |
| SKIDAY O | O           | The frequency student skipped a whole school day in the last two full weeks of school. (0/1/2 = none/[one or two times]/[three or more times]) | -0.0423| 0.0194  |
| DAYPEC O | O           | Averaged days that student attends physical education classes each week. (0/1/2/3 = [0]/[1 or 2]/[3 or 4]/[5 or more]) | -0.0750| 0.0318  |
| ARRLAT O | O           | The frequency of arriving late for school in the last two full weeks of school. (0/1/2 = none/[one or two times]/[three or more times]) | -0.0775| 0.0199  |
| GRADE   | O | Student’s grade. (0/1/2 = lower than modal grade/not lower than modal grade/higher than modal grade.) | 0.1412 | 0.0323 |
|---------|---|------------------------------------------------------------------|---------|---------|
| CHODIF  | O | Whether can choose the level of difficulty for school science course(s). (0/1/2 = no, not at all/yes, to a certain degree/yes, can choose freely) | 0.0623 | 0.0188 |
| CHONUM  | O | Whether can choose the number of school science course(s) they study. (0/1/2 = no, not at all/yes, to a certain degree/yes, can choose freely) | 0.0836 | 0.0198 |
| SCLEAR  | B | Whether attended earth and space courses in this or last school year. (0/1 = no/yes) | -0.0608 | 0.0198 |

\( \nu = 2 \)

| INTBRS  | C | Interest in broad science topics. | 0.0225 | 0.0101 |
|---------|---|----------------------------------|---------|---------|
| OUT.NET | B | Whether use Internet outside the school. (0/1 = no/yes) | 0.0579 | 0.0210 |
| ADINST  | C | Adaption of instruction. | 0.0227 | 0.0118 |

\( \nu = 3 \)

| HEDRES  | C | Home educational resources. | -0.0190 | 0.0108 |
|---------|---|-----------------------------|---------|---------|
| OUT.VED | B | Whether watch TV/DVD/Video outside the school. (0/1 = no/yes) | 0.0388 | 0.0201 |
| EMOSUP  | C | Parents’ emotional support. | -0.0180 | 0.0101 |
| CHOCOU  | O | Whether can choose the school science course(s) they study. (0/1/2 = no, not at all/yes, to a certain degree/yes, can choose freely) | 0.0617 | 0.0212 |
| OUT.MEA | B | Whether have meals before school or after school. (0/1 = no/yes) | 0.0382 | 0.0219 |

\( \nu > 3 \)

| OUT.GAM | B | Whether play video-games outside the school. (0/1 = no/yes) | 0.0311 | 0.0230 |
|---------|---|-----------------------------|---------|---------|
| SCI.BIO | B | Whether attended biology courses in this or last school year. (0/1 = no/yes) | -0.0268 | 0.0271 |
| SCI.GEN | B | Whether attended general, integrated, or comprehensive science courses in this or last school year. (0/1 = no/yes) | 0.0296 | 0.0186 |
| DAYMPA  | O | Number of days with moderate physical activities for a total of at least 60 minutes per each week. (0/1/2/3/4/5/6/7 = 0/1/2/3/4/5/6/7) | 0.0113 | 0.0397 |
| DUECEC  | O | Duration in early childhood education and care of student. (0/1/2/3 = [less than two years]/[at least two but less than three years]/[at least three but less than four years]/[at least four years]) | 0.0347 | 0.0176 |
| FISCED  | O | Father’s education in ISCED level. (0/1/2/3/4 = [none or ISCED 1]/[ISCED 2]/[ISCED 3B or 3C]/[ISCED 3A or 4]/[ISCED 5B]/[ISCED 5A or ISCED 6]) | 0.0148 | 0.0423 |

|       |       |       |       |       |       |       |

|       |       |       |       |       |       |       |
We comment on some of the variable selection results. Several variables in the data concern the socioeconomic status of students’ family, including the parents’ occupational statuses (FISEIO, MISEIO), cultural possessions at home (CULTPO; e.g., books), parents’ education levels (FISCED, MISCED), home educational resources (HEDRES), and family wealth (WEALTH), where FISEIO, MISEIO, FISCED, and MISCED are ordinal variables, and HEDRES, CULTPO and WEALTH are continuous variables. These variables are positively correlated with each other (correlations/polyserial correlations between 0.22 and 0.69). It is interesting that parents’ occupational statuses and cultural possessions seem to be important in explaining students’ performance in science (statistically significant and selected when $\nu = 1$), where the higher occupational status of father/mother or the more cultural possessions is associated with better science performance, given the rest of the variables. HEDRES seems to be a weak predictor, as it is only selected when $\nu = 3$ and is not statistically significant. On the other hand, parents’ education levels and family wealth seem to be less important (statistically insignificant and not selected even when $\nu = 3$). These results may be interpreted by a hypothetical mediation model as shown in Figure 5, which remains to be validated using additional data and statistical path analysis. That is, WEALTH naturally has direct effects on CULTPO and HEDRES which may have direct effects on students’ science achievement. Moreover, FISCED and MISCED natually have a direct effect on FISEIO and MISEIO, respectively, and also possibly have direct effects on CULTPO, HEDRES and WEALTH. However, there may not be direct paths from WEALTH, FISCED or MISCED to students’ science achievement. Students’ science achievement may be largely influenced by the genetic factors (e.g., intelligence) and environmental factors (e.g., education resources inside and outside home). It is possible that FISEIO, MISEIO, CULTPO, HEDRES and the other variables in the current analysis have provided good proxies to these genetic and environmental factors. Given these variables, FISCED, MISCED and WEALTH tend to be conditionally independent of students’ science achievement.
Several variables consider students’ behaviours attending school, including whether the student has ever repeated a grade (REPEAT), the frequency of a student skipping a whole school day in the last two full weeks of school (SKIDAY), the frequency of the student arriving late for school in the last two full weeks of school (ARRLAT), and the frequency of the student skipping some classes in the last two full weeks of school (SKICLA), where REPEAT is a binary variable, and the other three are ordinal variables. These variables are positively correlated with each other (tetrachoric/polychoric correlations between 0.07 and 0.53). The signs of the estimated coefficients are all consistent with our intuition. For instance, a student tended to perform worse in the test if they have ever repeated a grade, or if they often arrived in school late. Among these variables, REPEAT, ARRLAT, and SKIPDAY seem to be important variables, in the sense that they are all selected with $\nu = 1$. On the other hand, given these variables as well as the rest of the variables, the variable SKIPCAL seems to be irrelevant (not selected even with $\nu = 3$ and the coefficients are not significant).

A few variables are related to teachers and their teaching style, including enquiry-based teaching and learning (EBSCIT), teacher-directed science instruction (TDSCIT), perceived feedback (PERFEE), teacher unfairness (UNFAIR), adaptive instruction (ADINST), and teacher support in science classes of students’ choice (TEASUP), all of which are continuous variables. Among these variables, EBSCIT, TDSCIT, PERFEE, and UNFAIR are selected by our procedure with $\nu = 1$, ADINST is selected with $\nu = 2$, while TEASUP is not selected. Variable UNFAIR has a negative coefficient, suggesting that teacher unfairness is associated with poor student performance after controlling for the other variables. ADINST has a positive coefficient, suggesting that teachers’ flexibility with their lessons – tailoring the lessons to the students in their classes – tends to improve students’ science performance. In addition, it is interesting to see that TDSCIT has a positive coefficient while EBSCIT has a negative coefficient, which suggests that enquiry-based teaching and learning seem to have a negative effect on students’ science achievement while teacher-directed instruction has a positive effect. It is possible that enquiry-based teaching and learning can broaden students’ interests and increase their enjoyment of science (correlation between EBSCIT and JOYSCI is 0.16 and that between EBSCIT and INTBRS is 0.13), but may be less efficient in developing students’ science knowledge than teacher-directed instruction. Thus, a blended instruction model that combines the two teaching modes may be preferred. Finally, PERFEE has a negative coefficient, which may seem counter-intuitive at the first glance as providing informative and encouraging feedback is essential for improving student outcomes. This result may be due to the confounding of school types, which are not included in the current analysis. That is, students in disadvantaged schools may be more likely to report that their teachers provide them with feedback (Chapter 2, [OECD] 2016a). These students also tended to perform worse in the test, which results in the negative coefficient estimate.

Several variables concern students’ attending of science courses in this or last school year, including chem-
istry (SCI.CHE), physics (SCI.PHY), earth and space (SCI.EAR), biology (SCI.BIO), general, integrated, or comprehensive science (SCI.GEN), and applied sciences and technology (SCI.APP). All these variables are binary. Among these variables, SCI.CHE, SCI.PHY and SCI.EAR are selected with \( \nu = 1 \), and the rest are not selected even with \( \nu = 3 \). For the selected variables, SCI.CHE has a positive coefficient, while SCI.PHY and SCI.EAR have negative coefficients. We suspect that these results may be due to the different curriculum settings at different types of schools, which is not included in the current model. Besides, there are also variables that measure students’ opportunity to learn science at school. In particular, data are available on whether students can choose the number (CHONUM) and level of difficulty (CHODIF) of science courses, and whether they can choose specific science courses (CHOCOU) at school. It turns out that CHONUM and CHODIF are selected with \( \nu = 1 \), while CHOCOU is selected with \( \nu = 3 \). More specifically, the estimated coefficients for CHOCOU, CHONUM and CHODIF suggest that students with some freedom to choose the subject, number and level of difficulty of science courses tended to perform better in the test.

Students’ science achievement may also be related to their activities and received supports outside of school. The current analysis includes variables on whether a student studies for school or homework (OUT.STU), talks to parents outside the school (OUT.PAR), works for pay (OUT.JOB), exercises or does sports (OUT.SPO), uses internet (OUT.NET), watches TV/DVD/Video (OUT.VED), plays video games (OUT.GAM), has meals (OUT.MEA), meets or talks to friends (OUT.FRI), works in the household (OUT.HOL), and reads a book/newspaper/magazine (OUT.REA) outside the school, and whether they receive emotional support from their parents (EMOSUP). All these variables are binary. Among these variables, OUT.STU, OUT.PAR, OUT.JOB, and OUT.SPO are selected with \( \nu = 1 \), OUT.NET is selected with \( \nu = 2 \), OUT.VED and OUT.MEA are selected with \( \nu = 3 \), and the rest are not selected. Among the selected variables, variables OUT.STU, OUT.JOB, OUT.SPO, OUT.NET, OUT.VED and OUT.MEA have positive coefficients, suggesting that students with these outside-of-school activities also tended to perform better in the test after controlling for the rest of the variables. On the other hand, it is counter-intuitive that OUT.PAR and EMOSUP have negative coefficients, though the coefficient for EMOSUP is statistically insignificant. This phenomenon is worth future investigation.

Furthermore, the data contain variables that concern students’ perception or attitude towards science and related topics. They include the level of enjoying cooperation (COOPER), the level of valuing cooperation (CPSVAL), environmental awareness (ENVAWA), environmental optimism (ENVOPT), epistemological beliefs about science (EPIST), enjoyment of science (JOYSCI), instrumental motivation (INSTSC), science self-efficacy (SCIEEF), and interest in broad science topics (INTBRS). All these variables are continuous. They are all selected. Specifically, INTBRS is selected with \( \nu = 2 \), and the rest are selected with \( \nu = 1 \). The correlation between CPSVAL and COOPER is 0.45. It is interesting that CPSVAL has a negative coefficient, suggesting that controlling for the other variables students who more appreciate the value of cooperation and teamwork tended to perform worse in the test. In contrast, COOPER has a positive coefficient, implying that controlling for the other variables students who more enjoy cooperation and teamwork tended to perform better. Variables ENVAWA and ENVOPT have a correlation −0.14. Interestingly, ENVAWA has a positive coefficient, and ENVOPT has a negative coefficient. Moreover, INSTSC, which measures students’ perception that studying science in school is useful to their future lives and careers, has a negative coefficient. It seems slightly counter-intuitive. However, such a result is possible, given that variables like JOYSCI and INTBRS have been included in the regression model (correlation between INSTSC and JOYSCI is 0.34 and correlation between INSTSC and INTBRS is 0.24). It may be explained by a mediation model in Figure 6, where INSTSC has positive direct effects on JOYSCI and INTBRS, both of which further have positive
effects on students’ science achievement. However, given JOYSCI and INTBRS, the direct effect of INSTSC to science achievement is negative, possibly due to that INSTSC also brings pressure and stress to students when they learn science.

Finally, science achievement may also be related to other psychological factors. Specifically, the current analysis considers students’ test anxiety level (ANXTES), sense of belonging to school (BELONG), expected education level (EISCED), expected occupational status (EISEIO), and motivation to achieve (MOTIVA), all of which are continuous except that EISCED is ordinal. All these variables are selected with \( \nu = 1 \), except for MOTIVA which is not selected even when \( \nu = 3 \). For most of these variables, the signs of the estimated coefficients are consistent with our intuition. Specifically, ANXTES has a negative coefficient, suggesting a higher level of test anxiety is associated with poorer performance, controlling for the rest of the variables. EISCED and EISEIO have positive coefficients, which suggests that a higher anticipation of the future is associated with high science achievement. However, it is less intuitive that BELONG has a negative coefficient, which may be due to not accounting for school effect.

7 Discussions

In this paper, we considered identifying non-cognitive predictors of students’ academic performance based on complex data from ILSAs that involve many missing values, mixed data types, and measurement errors. This problem can naturally be formulated as a variable selection problem. However, existing statistical methods are not applicable, due to the complex data structure. For instance, variable selection methods for linear regression do not solve the current problem, due to (1) the response variable – students’ academic achievement – is not directly observable but measurement by cognitive items, and (2) there are many missing values in the predictors. We addressed these challenges by proposing a new model which combines a latent regression model and a Gaussian copula model. Furthermore, we proposed a derandomised knockoff method under the proposed method variable selection. This method tackles the multiple comparison issue of variable selection, by controlling the PFER, a familywise error rate for variable selection. Theoretical properties of the proposed method were established. We focused on an application to PISA 2015 data, with the response variable being students’ proficiency in the science domain. This analysis involved 5,685 students, 184 science items, and 62 non-cognitive variables that are of mixed types and contain many missing values. To our best knowledge, this is the first variable selection study of ILSAs that involve a dataset as large as the current one. With PFER level set to be \( \nu = 1, 2, 3 \), the proposed procedure selected 38, 41, and 46 variables, respectively. The model selection results are sensible, and signs of the parameter estimates for most of the selected variables are consistent with our intuition. The variable selection and parameter estimation results were examined from the perspectives of family socioeconomic status, school attending behaviours, teacher-related factors, science course resources and choices at school, out-of-school activities, perception and attitude towards science and related topics, and other psychological factors. These results provided
insights into non-cognitive factors that are likely associated with students' science achievement, which can be useful to educators, policy makers, and other stakeholders.

The current analysis has several limitations that will be addressed in future research. First, the current application only considers the US sample and the science domain in PISA. It is of interest to investigate how the result of model selection varies across countries and knowledge domains. In particular, we expect the selection results to be substantially different across different countries, due to the cultural and socio-economic differences. In addition, the non-null predictors for different knowledge domains may also differ, which can suggest tailored education strategies for different domains. Second, it is also of interest to extend the current analysis to other ILSAs, such as the TIMSS and PIRLS, to see how the results change with slightly different test designs and different student age groups.

The proposed method may be very useful for the scaling and reporting of ILSAs. First, the variable selection results establish a pathway between students' achievement in each subject domain and its possible influencing factors/causes. These results provide evidence that assists educators, policy makers, and related stakeholders to make informed education decisions. Second, it may improve the scaling methodology of ILSAs. Currently, a latent regression model is used in most ILSAs to estimate the performance distributions of populations (e.g., countries). This model, which is similar to the latent regression model in the current study, borrows information from non-cognitive background variables to compensate for the shortage of cognitive information. However, unlike the current model, the latent regression model adopted in ILSAs does not directly regress on the background variables. Instead, it first conducts a PCA step to reduce the background variables' dimensionality, and then incorporates the derived PCA scores as predictors in latent regression. This approach is often criticised for lacking interpretability, as the principal components often lack substantive meanings. Instead of performing PCA, we recommend to reduce the dimensionality of the background variables by variable selection, and then fit the latent regression model with the selected predictors. With the theoretical guarantee on our variable selection method and by reporting the selected variables, the estimation and reporting of performance distributions become more transparent and interpretable.

While we focus on ILSAs, the proposed method also receives many other applications. For example, the method can also be used to identify neural determinants of visual short-term memory and to identify demographic correlates of psycho-pathological traits (Jacobucci et al., 2019). Moreover, the proposed Gaussian copula model can be used with other regression models, such as linear and generalised linear regression models, for solving estimation and variable selection problems involving massive missing data and mixed types of variables. It is thus widely applicable to real-world problems involving missing data, which are commonly encountered in the social sciences, such as social surveys, marketing, and public health.

From the methodological perspective, there are several directions worth future development and investigation. First, the current model fails to account for possible multilevel structures in the data; for example, students are nested within schools. From the analysis of PISA data, the signs of some estimated coefficients are not consistent with our intuition, which is likely due to not accounting for school effect. Therefore, we believe that it is important to extend the current model by introducing random effects to model multilevel structures. New computation methods need to be developed accordingly. Second, the current analysis requires a relatively strong condition on data missingness, which is weaker than MCAR but stronger than MAR. In social science data, data may often be missing not at random. In that case, one may simultaneously model the complete data distribution and the missing data mechanism (e.g., Kuha et al., 2018). Such a joint model can be incorporated into the current analysis framework for generating knockoffs and further controlling variable selection error. Third, the knockoff method may be coupled with the multiple imputation
method for missing data analysis, as the knockoff variables can naturally be viewed as missing data. Thus, one may extend the state-of-the-arts multiple imputation methods \cite{Liu2014, VanBuuren2018} to simultaneously impute missing data and knockoff copies, and then use the imputed data for solving the variable selection problem. Finally, this paper focuses on the PFER as the performance metric for variable selection. Other performance metrics may be explored, such as false discovery rate and \( k \) family-wise error rate, which may be more sensible in other applications. Making use of recent developments on knockoff methods \cite{Ren2021, RenBarber2022}, we believe that it is not difficult to extend the current method to these error metrics.

Appendix

A Proof of Theorems

\textit{Proof of Theorem 1.} First of all, we claim that \( S^* \) is exactly the index set of non-null variables under the full rank assumption on \( \Sigma \), i.e., \( j \notin S^* \) if and only if \( Z_j \) is conditionally independent of \( Y \) given \( \{ Z_k \}_{k \neq j} \).

\textbf{Lemma 1.} Suppose the underlying correlation matrix \( \Sigma \) has full rank, then \( j \notin S^* \) if and only if \( Z_j \) is conditionally independent of \( Y \) given \( \{ Z_k \}_{k \neq j} \).

The proof Lemma 1 is given in Appendix C.

Next we consider any index \( j \in S^* \). Similar to Barber et al. \cite{Barber2020}, we have the following result.

\textbf{Lemma 2.} For any \( j \notin S^* \),

\[ \mathbb{P} \left( W_j > 0, \bar{K}L_j \leq \epsilon \|W_j\|, W_{-j} \right) \leq \epsilon \cdot \mathbb{P} \left( W_j < 0 \|W_j\|, W_{-j} \right). \]  

(A.1)

where \( W_{-j} = \{ W_k \}_{k \neq j} \).

The proof of this Lemma is also given in Appendix C.

We now denote the threshold \( \tau \) as a function of \( W \)

\[ \tau = \inf \{ t > 0 : \{ j : W_j < -t \} = v \} := T(W), \]

with \( \inf \emptyset = -\infty \). For \( j = 1, \ldots, p \), we also define

\[ T_j = T(W_1, \ldots, W_{j-1}, |W_j|, W_{j+1}, \ldots, W_p) > 0. \]

Then the threshold \( T(W) \) has a property that for any \( j, k \),

\[ \text{If } \max \{W_j, W_k\} < -\min \{T_j, T_k\}, \text{ then } T_j = T_k. \]  

(A.2)
Consider the number of false discoveries \(|\{j : j \in \hat{S} \cap S^* \text{ and } \hat{K}L_j \leq \epsilon\}|\). We have

\[
|\{j : j \in \hat{S} \cap S^* \text{ and } \hat{K}L_j \leq \epsilon\}| = \sum_{j \notin S^*} \mathbb{I}(W_j > \tau, \hat{K}L_j \leq \epsilon)
\]

\[
= \left[ 1 + \sum_{j \notin S^*} \mathbb{I}(W_j < -\tau) \right] \cdot \frac{\sum_{j \notin S^*} \mathbb{I}(W_j > \tau, \hat{K}L_j \leq \epsilon)}{1 + \sum_{j \notin S^*} \mathbb{I}(W_j < -\tau)}
\]

\[
\leq \left[ 1 + \sum_{j=1}^{p} \mathbb{I}(W_j < -\tau) \right] \cdot \frac{\sum_{j \notin S^*} \mathbb{I}(W_j > \tau, \hat{K}L_j \leq \epsilon)}{1 + \sum_{j \notin S^*} \mathbb{I}(W_j < -\tau)}
\]

\[
\leq \nu \cdot R_{\epsilon}(\tau),
\]

where we denote

\[
R_{\epsilon}(\tau) = \frac{\sum_{j \notin S^*} \mathbb{I}(W_j > \tau, \hat{K}L_j \leq \epsilon)}{1 + \sum_{j \notin S^*} \mathbb{I}(W_j < -\tau)}.
\]

It suffices to prove that

\[
\mathbb{E}[R_{\epsilon}(\tau)] \leq e^\epsilon.
\]

Since \(\tau = T_j\) if \(W_j > \tau\), we have

\[
\mathbb{E}[R_{\epsilon}(\tau)] = \mathbb{E} \left[ \frac{\sum_{j \notin S^*} \mathbb{I}(W_j > \tau, \hat{K}L_j \leq \epsilon)}{1 + \sum_{j \notin S^*} \mathbb{I}(W_j < -\tau)} \right]
\]

\[
= \sum_{j \notin S^*} \mathbb{E} \left[ \frac{\mathbb{I}(W_j > T_j, \hat{K}L_j \leq \epsilon)}{1 + \sum_{k \notin S^*, k \neq j} \mathbb{I}(W_k < -T_j)} \right]
\]

\[
= \sum_{j \notin S^*} \mathbb{E} \left[ \frac{\mathbb{P}(W_j > 0, \hat{K}L_j \leq \epsilon \mid W_j, \mathbf{W}_{-j}) \cdot \mathbb{I}(|W_j| > T_j)}{1 + \sum_{k \notin S^*, k \neq j} \mathbb{I}(W_k < -T_j)} \right]
\]

where the last equation holds since \(T_j\) is a function of \(|W_j|\) and \(\mathbf{W}_{-j}\). By equation (A.1),

\[
\sum_{j \notin S^*} \mathbb{E} \left[ \frac{\mathbb{P}(W_j > 0, \hat{K}L_j \leq \epsilon \mid W_j, \mathbf{W}_{-j}) \cdot \mathbb{I}(|W_j| > T_j)}{1 + \sum_{k \notin S^*, k \neq j} \mathbb{I}(W_k < -T_j)} \right]
\]

\[
\leq \sum_{j \notin S^*} \mathbb{E} \left[ e^\epsilon \frac{\mathbb{P}(W_j < 0 \mid W_j, \mathbf{W}_{-j}) \cdot \mathbb{I}(|W_j| > T_j)}{1 + \sum_{k \notin S^*, k \neq j} \mathbb{I}(W_k < -T_j)} \right]
\]

\[
= e^\epsilon \cdot \sum_{j \notin S^*} \mathbb{E} \left[ \frac{\mathbb{I}(W_j < -T_j)}{1 + \sum_{k \notin S^*, k \neq j} \mathbb{I}(W_k < -T_j)} \right].
\]
Finally by \(A.2\), we have
\[
\sum_{j \not\in S^*} \mathbb{E} \left[ \frac{\mathbb{I}(W_j < -T_j)}{1 + \sum_{k \not\in S^*, k \not= j} \mathbb{I}(W_k < -T_k)} \right] = \sum_{j \not\in S^*} \mathbb{E} \left[ \frac{\mathbb{I}(W_j < -T_j)}{1 + \sum_{k \not\in S^*, k \not= j} \mathbb{I}(W_k < -T_k)} \right] = 1.
\]

This finishes the proof of \(\mathbb{E}[R_c(\tau)] \leq \epsilon^c\).

For the case when \(P = Q\), we need to show that \(P_{ij}(z_{ij}|z_{i,-j}^{obs}, y_i) \cdot P_{ij}(z_{i,-j}^{obs}, z_{ij}, y_i)\) remains the same after switching the roles of \(z_{ij}\) and \(z_{ij}^{obs}\). It is equivalent to show that
\[
(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i) \overset{d}{=} (\hat{Z}_{ij}, Z_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i)
\]
under the true model. By Proposition 1, we have,
\[
p(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i) = \int p(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i) \, dZ_{i}^{mis} \, d\hat{Z}_{i}^{mis}
\]
\[
= \int p(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}) \cdot p(Y_i|Z_{ij}, z_{i,-j}^{obs}) \, dZ_{i}^{mis} \, d\hat{Z}_{i}^{mis}
\]
\[
= \int p(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}) \cdot p(Y_i|Z_{i,-j}^{obs}) \, dZ_{i}^{mis} \, d\hat{Z}_{i}^{mis},
\]
where the second equation holds since \(\hat{Z}_i\) and \(Y_i\) are conditionally independent given \(Z_i\), and the last equation holds since \(j \not\in S^*\). Again by Proposition 1, \((Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i) \overset{d}{=} (\hat{Z}_{ij}, Z_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i)\), thus
\[
p(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i) = \int p(Z_{ij}, \hat{Z}_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}) \cdot p(Y_i|Z_{i,-j}^{obs}) \, dZ_{i}^{mis} \, d\hat{Z}_{i}^{mis}
\]
\[
= \int p(\hat{Z}_{ij}, Z_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}) \cdot p(Y_i|Z_{i,-j}^{obs}) \, dZ_{i}^{mis} \, d\hat{Z}_{i}^{mis}
\]
\[
= p(\hat{Z}_{ij}, Z_{ij}, z_{i,-j}^{obs}, \hat{Z}_{i,-j}, Y_i),
\]
which finishes the proof of Theorem 1. \(\square\)

**Proof of Theorem 2** By definition, we have
\[
\mathbb{E} \{j : j \in \mathcal{S} \setminus S^* \text{ and } \hat{\mathbb{K}}_j^{(m)}(\epsilon, m = 1, \ldots, M)\}
\]
\[
= \mathbb{E} \left[ \sum_{j \in \mathcal{S} \setminus S^*} \mathbb{I} \left( \Pi_j \geq \eta \text{ and } \max_{1 \leq m \leq M} \hat{\mathbb{K}}_j^{(m)} \leq \epsilon \right) \right]
\]
\[
= \sum_{j \in \mathcal{S} \setminus S^*} \mathbb{P} \left( \Pi_j \geq \eta \text{ and } \max_{1 \leq m \leq M} \hat{\mathbb{K}}_j^{(m)} \leq \epsilon \right)
\]
\[
= \sum_{j \in \mathcal{S} \setminus S^*} \mathbb{P} \left( \frac{1}{M} \sum_{m=1}^{M} \mathbb{I} \left( j \in \mathcal{S}^{(m)} \right) \geq \eta \text{ and } \max_{1 \leq m \leq M} \hat{\mathbb{K}}_j^{(m)} \leq \epsilon \right).
\]
Note that within the event \( \{ \max_{1 \leq m \leq M} \hat{KL}_j^{(m)} \leq \epsilon \} \),
\[
\mathbb{I} \left( j \in \mathcal{S}^{(m)} \right) = \mathbb{I} \left( j \in \mathcal{S}^{(m)} \text{ and } \hat{KL}_j^{(m)} \leq \epsilon \right).
\]

Thus
\[
E \{ j : j \in \hat{S} \setminus \mathcal{S}^* \text{ and } \hat{KL}_j^{(m)} \leq \epsilon, m = 1, \ldots, M \} \\
= \sum_{j \in \hat{S} \setminus \mathcal{S}^*} \mathbb{P} \left( \frac{1}{M} \sum_{m=1}^{M} \mathbb{I} \left( j \in \mathcal{S}^{(m)} \text{ and } \hat{KL}_j^{(m)} \leq \epsilon \right) \geq \eta \text{ and } \max_{1 \leq m \leq M} \hat{KL}_j^{(m)} \leq \epsilon \right) \\
\leq \sum_{j \in \hat{S} \setminus \mathcal{S}^*} \mathbb{P} \left( \frac{1}{M} \sum_{m=1}^{M} \mathbb{I} \left( j \in \mathcal{S}^{(m)} \text{ and } \hat{KL}_j^{(m)} \leq \epsilon \right) \geq \eta \right) \\
= \sum_{j \in \hat{S} \setminus \mathcal{S}^*} \mathbb{P} \left( \Pi_j^1 \geq \eta \right)
\]

By the assumption \( \mathbb{P} \left( \Pi_j^1 \geq \eta \right) \leq E \left[ \Pi_j^1 \right] \) and the result of Theorem 1,
\[
\sum_{j \in \hat{S} \setminus \mathcal{S}^*} \mathbb{P} \left( \Pi_j^1 \geq \eta \right) \leq \sum_{j \in \hat{S} \setminus \mathcal{S}^*} E \left[ \Pi_j^1 \right] = E \left[ \sum_{j \in \hat{S} \setminus \mathcal{S}^*} \mathbb{I} \left( j \in \mathcal{S}^{(1)} \text{ and } \hat{KL}_j^{(1)} \leq \epsilon \right) \right] \leq n e^\epsilon,
\]
which proved the first part of Theorem 2. The proof of the second part of Theorem 2 is totally the same as that of Proposition 2 in [Ren et al. 2021], which we omit here.

\[\square\]

### B Proof of Propositions

**Proof of Proposition 1** Let \( Z^*_i \) be obtained from Algorithm 1 and \( Z_i \) be the complete data such that \( Z_{ij} = F_j(Z^*_i), \ j = 1, \ldots, p, \) with \( Z_{\text{min}} = \{ Z_{ij}, j \in A_i \}. \) Since \( Z^*_i \) is sampled from its conditional distribution given \( Z_{i}^{\text{obs}} \) and \( Y_i \) with true model parameters \( \Xi, \beta, \beta_0, \sigma^2, \) we can know that \( Z_i \) follows the true Gaussian copula model and \( Y_i | Z_i \) follows the true latent regression model.

From the data generating process, it is easy to see that \( \bar{Z}_i \perp (Y_i, A_i) | Z_{i} \). The conditional probability distribution function of \( (Y_i, A_i) \) given \( Z_i \) can be written as
\[
p(Y_i, A_i | Z_i) = p(A_i | Z_i, Y_i) \cdot p(Y_i | Z_i).
\]
By the MAR assumption, \( p(A_i | Z_i, Y_i) = p(A_i | Z_i). \) We then have
\[
p(Y_i, A_i | Z_i) = p(A_i | Z_i) \cdot p(Y_i | Z_i),
\]
which shows that \( Y_i \) and \( A_i \) are conditionally independent given \( Z_i \).

Finally, it is easy to see from the property of multivariate Gaussian distribution that \( (Z^*_i, \bar{Z}_i)_{\text{swap}(\mathcal{S})} \) has the same distribution with \( (Z^*_i, \bar{Z}_i) \). Hence \( (Z_i, \bar{Z}_i)_{\text{swap}(\mathcal{S})} \) and \( (Z_i, \bar{Z}_i) \) are identically distributed.

\[\square\]
Hence predicted by the other variables. Without loss of generality we let $p$ to prove that any variable in vector $\mathbf{Z}$.

Denote the underlying correlation matrix $\Sigma$, it is easy to show that when any variable in vector $\mathbf{Z}$.

For simplicity, we omit the subscript $i$.

Proof of Lemma 1.

Proof of Lemmas

Proof of Proposition 4.

This proposition is implied by Theorem 1 in the case of $P = Q$.

Proof of Proposition 2.

This proposition is implied by Theorem 2 in the case of $P = Q$.

C Proof of Lemmas

Proof of Lemma 2

Let $(\hat{\beta}, \hat{\gamma}, \hat{\rho})$ be the maximum likelihood estimator. For any subset $S \subseteq \{1, \ldots, p\}$,

$$w_j \left( \left[ \mathbf{Z}_{obs}^{obs}, \tilde{\mathbf{Z}}_{obs}^{obs} \right]_{\text{swap}(S)}, \mathbf{Y} \right) = \begin{cases} \text{sign}(\|\hat{\beta}_j\| - |\hat{\gamma}_j|) \max \left\{ \|\hat{\beta}_j\|/\sqrt{\mathbf{P}^j}, \|\hat{\gamma}_j\|/\sqrt{\mathbf{P}^j} \right\} = W_j, & j \notin S, \\ \text{sign}(\|\hat{\beta}_j\| - |\hat{\gamma}_j|) \max \left\{ \|\hat{\beta}_j\|/\sqrt{\mathbf{P}^j}, \|\hat{\gamma}_j\|/\sqrt{\mathbf{P}^j} \right\} = -W_j, & j \in S, \end{cases}$$

Hence $W_j$ given by (7) satisfies Definition 3.

Proof of Proposition 3

This proposition is implied by Theorem 1 in the case of $P = Q$.

Proof of Proposition 4

This proposition is implied by Theorem 2 in the case of $P = Q$.

C Proof of Lemmas

Proof of Lemma 2

For simplicity, we omit the subscript $i$ in this proof. Due to the linear structure of the latent regression model (5), it is easy to show that when any variable in vector $(g_1(Z_1), \ldots, g_p(Z_p))^T$ cannot be perfectly predicted by the others, $S^*$ is the index set of non-null variables. Thus we need only to prove that any variable in vector $(g_1(Z_1), \ldots, g_p(Z_p))^T$ can not be perfectly predicted by the others if the underlying correlation matrix $\Sigma$ has full rank. Suppose conversely that some variable in $g_j(Z_j)$ can be predicted by the other variables. Without loss of generality we let $j = 1$.

- If $Z_1$ is continuous, note that $Z_1^* = (Z_1 - c_1)/d_1 = (g_1(Z_1) - c_1)/d_1$ and $g_2(Z_2), \ldots, g_p(Z_p)$ are functions of $Z_2^*, \ldots, Z_p^*$, which implies that $Z_1^*$ can be predicted by $Z_2^*, \ldots, Z_p^*$. This is impossible due to the full rank assumption.
- If $Z_1$ is binary, then similarly we know $I(Z_1^* > c_1)$ can be predicted by $Z_2^*, \ldots, Z_p^*$, which also contradicts the full rank assumption since $Z_1^*$ given $Z_2^*, \ldots, Z_p^*$ follows a non-degenerated normal distribution.
- Finally, if $Z_1$ is ordinal with $K_1 + 1$ categories, then there exit $k \in \{1, \ldots, K_2\}$ such that $I(Z_1^* > c_1 k^*)$ can be predicted by $I(I(Z_1^* > c_1 k') : k' \neq k)$ along with $Z_2^*, \ldots, Z_p^*$. Consider the case when $I(Z_1^* > c_1 k') = 0$ for all $k' > k$ and $I(Z_1^* > c_1 k') = 1$ for all $k' < k$. Then it must hold that $I(Z_1^* > c_1 k)$ can be predicted by $I(c_1 k - 1 < Z_1^* < c_1 k + 1)$ (remind that $c_1 0 = -\infty, c_1 K_1 + 1 = \infty$) and $Z_2^*, \ldots, Z_p^*$. This will not hold under the full rank assumption as well.

In conclusion, we have proved our claim by argument of contradiction.

Proof of Lemma 3

Denote $\mathbf{Z}_{i,-j}^{obs} = \{Z_{ik} : k \in \mathcal{A}_i \setminus \{j\}\}$, $\mathbf{Z}_{i}^{obs} = \{Z_{ij} : j \in \mathcal{A}_i\}$ and $\mathbf{Z}_{-j}^{obs} = \{Z_{i,-j}^{obs} : j \in \mathcal{A}_i\}$.

Similarly, $\tilde{\mathbf{Z}}_{i,-j}^{obs} = \{\tilde{Z}_{ik} : k \in \mathcal{A}_i \setminus \{j\}\}$, $\tilde{\mathbf{Z}}_{i}^{obs} = \{\tilde{Z}_{ij} : j \in \mathcal{A}_i\}$ and $\tilde{\mathbf{Z}}_{-j}^{obs} = \{\tilde{Z}_{i,-j}^{obs} : j \in \mathcal{A}_i\}$. Label the unordered pair of vectors $\{\mathbf{Z}^{obs}_{j}, \tilde{\mathbf{Z}}^{obs}_{j}\}$ as $\{\mathbf{Z}^{(0)}_{j}, \mathbf{Z}^{(1)}_{j}\}$, such that

$$\begin{align*}
\text{If } \mathbf{Z}^{obs}_{j} = \mathbf{Z}^{(0)}_{j} \text{ and } \tilde{\mathbf{Z}}^{obs}_{j} = \mathbf{Z}^{(1)}_{j}, \text{ then } W_j \geq 0; \\
\text{If } \mathbf{Z}^{obs}_{j} = \mathbf{Z}^{(1)}_{j} \text{ and } \tilde{\mathbf{Z}}^{obs}_{j} = \mathbf{Z}^{(0)}_{j}, \text{ then } W_j \leq 0;
\end{align*}$$

Let $\mathbf{Y} = \{Y_i\}_{i=1}^N$. It follows from the flip-sign property that the statistics $|W_j|$ and $\mathbf{W}_{-j}$ are functions of
\(Z_j^{(0)}, Z_j^{(1)}, \tilde{Z}_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}\) and \(Y\). Ignoring the trivial case \(|W_j| = 0\), we have

\[
P(W_j > 0\mid Z_j^{(0)}, Z_j^{(1)}, \tilde{Z}_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y) = \frac{P((Z_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}) = (Z_j^{(0)}, Z_j^{(1)}))|Z_j^{(0)}, Z_j^{(1)}, \tilde{Z}_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y)}{P((Z_j^{\text{obs}}, \tilde{Z}_{j}^{\text{obs}}) = (Z_j^{(0)}, Z_j^{(1)}))|Z_j^{(0)}, Z_j^{(1)}, \tilde{Z}_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y)} = \prod_{i:j \in A_i} P((Z_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}) = (Z_i^{(0)}, Z_i^{(1)}))|Z_i^{(0)}, Z_i^{(1)}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, Y_i).
\]

(C.1)

Note that the conditional probability function of \(Z_{i,j}, \tilde{Z}_{i,j}, \tilde{Z}_{i,j}^{\text{obs}}\) given \(Z_{i,j}^{\text{obs}}, Y_i\) can be decomposed as

\[
p(Z_{i,j}, \tilde{Z}_{i,j}, \tilde{Z}_{i,j}^{\text{obs}}|Z_{i,j}^{\text{obs}}, Y_i) = P_{ij}(Z_{i,j}|Z_{i,j}^{\text{obs}}, Y_i) \cdot Q_{ij}(\tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}|Z_{i,j}^{\text{obs}}, Z_{i,j}, Y_i).
\]

Suppose \(z_{i,j}^{(0)}, z_{i,j}^{(1)}, \tilde{z}_{i,j}^{\text{obs}}, \tilde{z}_{i,j}^{\text{obs}}, y_i\) to be realizations of \(Z_{i,j}^{(0)}, Z_{i,j}^{(1)}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, Y_i\) for \(i\) such that \(j \in A_i\). Then

\[
\prod_{i:j \in A_i} P((Z_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}) = (z_{i,j}^{(0)}, z_{i,j}^{(1)}))|Z_{i,j}^{(0)}, Z_{i,j}^{(1)}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, Y_i) = \prod_{i:j \in A_i} P_{ij}(Z_{i,j}^{(0)}, Z_{i,j}^{(1)}|Z_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, Y_i).\]

Hence by the definition of \(\hat{K}_L_{ij}\), if we let

\[
\prod_{i:j \in A_i} P((Z_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}) = (z_{i,j}^{(0)}, z_{i,j}^{(1)}))|Z_{i,j}^{(0)}, Z_{i,j}^{(1)}, \tilde{Z}_{i,j}^{\text{obs}}, \tilde{Z}_{i,j}^{\text{obs}}, Y_i) = e^{\rho_j},
\]

we have \(\hat{K}_L_{ij} = \rho_j\) if \(W_j > 0\) and \(\hat{K}_L_{ij} = -\rho_j\) if \(W_j < 0\). Since \(|W_j|\) and \(W_{-j}\) are functions of \(Z_j^{(0)}, Z_j^{(1)}, \tilde{Z}_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y\), we have

\[
P \left(W_j > 0, \hat{K}_L_{ij} \leq \epsilon \mid |W_j|, W_{-j} \right)
= E \left[P \left(W_j > 0, \hat{K}_L_{ij} \leq \epsilon \mid Z_j^{(0)}, Z_j^{(1)}, Z_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y \right) \mid |W_j|, W_{-j} \right]
= E \left[P \left(W_j > 0 \mid Z_j^{(0)}, Z_j^{(1)}, Z_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y \right) I(\rho_j \leq \epsilon) \mid |W_j|, W_{-j} \right]
\leq E \left[P \left(W_j > 0 \mid Z_j^{(0)}, Z_j^{(1)}, Z_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y \right) e^{\rho_j} I(\rho_j \leq \epsilon) \mid |W_j|, W_{-j} \right],
\]

where the last equation is from \((C.1)\). Note that \(e^{\rho_j} I(\rho_j \leq \epsilon) \leq e^\epsilon, I(\rho_j \leq \epsilon)\) then holds from

\[
E \left[P \left(W_j > 0 \mid Z_j^{(0)}, Z_j^{(1)}, Z_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y \right) e^{\rho_j} I(\rho_j \leq \epsilon) \mid |W_j|, W_{-j} \right]
\leq e^\epsilon E \left[P \left(W_j > 0 \mid Z_j^{(0)}, Z_j^{(1)}, Z_{ij}^{\text{obs}}, \tilde{Z}_{ij}^{\text{obs}}, Y \right) \mid |W_j|, W_{-j} \right]
= e^\epsilon E \left[P \left(W_j > 0 \mid |W_j|, W_{-j} \right) \right].
\]
D Computation

D.1 Algorithm for estimation of Gaussian copula model

D.1.1 Algorithm

Recall the constrained optimization problem

\[
\arg\max_{\Xi} \ l_1(\Xi)
\]

subject to

\[
\begin{align*}
\Sigma_{jj} &= 1, \ j = 1, \ldots, p, \\
d_j &> 0, \text{ if } j \notin D, \\
c_{j1} &< c_{j2} < \ldots < c_{jK_j}, \text{ if } j \in D.
\end{align*}
\]

To better deal with the constraints in the optimization problem, we first reparametrize part of the parameters in the model. For the correlation matrix \(\Sigma\), we reparameterize it as \(\Sigma = LL^T\), where \(L = (l_{ab})_{p \times p}\) is the Cholesky decomposition of \(\Sigma\). Since \(L\) is a lower triangular matrix, we have \(l_{ab} = 0, \ a = j + 1, \ldots, p\). By constraining \(l_{aa} \neq 0\) and \(\sum_{a=1}^{b} l_{ab}^2 = 1\), we can also guarantee that \(LL^T\) is positive definite and has unit diagonals. For the threshold parameters of discrete variable \(Z_j\), we introduce \(u_j = (u_{j1}, u_{j2}, \ldots, u_{jK_j})^T(K_j \geq 1)\), such that

\[
u_{jk} = \begin{cases} 
c_{j1}, & \text{if } k = 1, \\
\log(c_{jk} - c_{j,k-1}), & \text{if } 2 \leq k \leq K_j.
\end{cases}
\]

Note that there are no constraints on \(u_j\) anymore.

Now we denote \(\tilde{\Xi}\) as the set of unknown parameters after reparametrization. The marginal likelihood function then becomes

\[
l_1(\tilde{\Xi}) = \sum_{i=1}^{N} \log f_i(z_{i}^{obs}|\tilde{\Xi}) = \sum_{i=1}^{N} \log \int \cdots \int \left( \prod_{j \notin A_i} dz_j \right) f(z|\tilde{\Xi}),
\]

where the density function \(f(z^{obs}|\tilde{\Xi})\) takes the form

\[
f(z|\tilde{\Xi}) = \int \cdots \int \left( \prod_{j \in D} dz_j^* \right) \phi(z^*|LL^T) \times \left( \prod_{j \notin D} d_{j_i}^{-1} \right) \times \left( \prod_{j \in D} \mathbb{I}(z_j^* \in (c_{j,z_j} - 1, c_{j,z_j})) \right)_{z_j^* = \frac{z_j - c_{j,z_j}}{\hat{d}_j}, j \notin D}.
\]

Here we slightly abuse the notations \(c_{jk}(k = 0, 1, \ldots, K_j, K_j + 1)\) for \(j \in D\) to denote functions of \(u_j\) such that

\[
c_{jk}(u_j) = \begin{cases} 
-\infty, & \text{if } k = 0, \\
u_{j1}, & \text{if } k = 1, \\
u_{j1} + \sum_{k=2}^{K_j} \exp(u_{j1}) & \text{if } 2 \leq k \leq K_j, \\
\infty, & \text{if } k = K_j + 1.
\end{cases}
\]
The constrained optimization problem (6) now becomes

$$\arg \max_{\tilde{\Xi}} \quad l_1(\tilde{\Xi})$$

(D.1)

subject to

$$l_{jj} \neq 0, \quad \sum_{a=1}^{b} t_{ab}^2 = 1, \quad l_{ab} = 0, \quad a = 1, \ldots, p, \quad b = a + 1, \ldots, p.$$ 

$$d_j > 0, \quad \text{if} \ j \notin D.$$

For convenience we denote

$$\Omega = \{ \tilde{\Xi} : l_{jj} \neq 0, \quad \sum_{a=1}^{b} t_{ab}^2 = 1, \quad l_{kj} = 0, \quad j = 1, \ldots, p, \quad k = j + 1, \ldots, p; \quad d_j > 0, \quad \text{if} \ j \notin D \}$$

as the feasible set.

We propose to solve (D.1) by gradient-based method. However, the gradient of $l_1(\tilde{\Xi})$ involves high-dimensional integrals, thus is difficult to compute exactly. Fortunately, it can be approximated by method of sampling.

For convenience, we introduce $C \subset \{1, \ldots, p\}$ as the set of continuous covariates. If $j \in C \cap A_i$, it can be shown by integral transformation that

$$\frac{\partial}{\partial \epsilon_j} \log f_i(z_i^{obs}; \tilde{\Xi}) = E_{\epsilon_i^* \sim N(0, \mathbf{L}^\top \mathbf{L})} \left[ \frac{\partial}{\partial \epsilon_j} \psi_i(\tilde{\Xi} | \epsilon_i^*) \bigg| _{\epsilon_i^* = \frac{z_{ij} - c_j}{d_j}, j \in C \cap A_i; \quad \epsilon_i^* \in (c_j, z_{ij} - 1, c_j, z_{ij}), j \in D \cap A_i} \right],$$

where

$$\psi_i(\tilde{\Xi} | \epsilon_i^*) = \log \left( \phi(\epsilon_i^* | \mathbf{L}^\top \mathbf{L}) \times \prod_{j \notin C \cap A_i} d_j^{-1} \right) \left| _{\epsilon_i^* = \frac{z_{ij} - c_j}{d_j}, j \notin C \cap A_i} \right|.$$ 

The expectation $E_{\epsilon_i^* \sim N(0, \mathbf{L}^\top \mathbf{L})} \left[ | \epsilon_i^* = \frac{z_{ij} - c_j}{d_j}, j \in C \cap A_i; \quad \epsilon_i^* \in (c_j, z_{ij} - 1, c_j, z_{ij}), j \in D \cap A_i \right]$ means that the expectation is taken with respect to $\epsilon_i^*$, whose distribution is the multivariate normal distribution $N(0, \mathbf{L}^\top \mathbf{L})$ with constraints $\epsilon_i^* = \frac{z_{ij} - c_j}{d_j}$ when $j \in C \cap A_i$ and $\epsilon_i^* \in (c_j, z_{ij} - 1, c_j, z_{ij})$ when $j \in D \cap A_i$. In the following, we use $N(0, \mathbf{L}^\top \mathbf{L})[z_i^{obs}, \mathbf{c}, \mathbf{d}]$ to denote such a distribution.

Similarly, if $j \in C \cap A_i$ we have

$$\frac{\partial}{\partial d_j} \log f_i(z_i^{obs}; \tilde{\Xi}) = E_{\epsilon_i^* \sim N(0, \mathbf{L}^\top \mathbf{L})} \left[ \frac{\partial}{\partial d_j} \psi_i(\tilde{\Xi} | \epsilon_i^*) \bigg| _{\epsilon_i^* = \frac{z_{ij} - c_j}{d_j}, j \in C \cap A_i; \quad \epsilon_i^* \in (c_j, z_{ij} - 1, c_j, z_{ij}), j \in D \cap A_i} \right],$$

and for $\mathbf{L}$ we have

$$\frac{\partial}{\partial \mathbf{L}} \log f_i(z_i^{obs}; \tilde{\Xi}) = E_{\epsilon_i^* \sim N(0, \mathbf{L}^\top \mathbf{L})} \left[ \frac{\partial}{\partial \mathbf{L}} \psi_i(\tilde{\Xi} | \epsilon_i^*) \bigg| _{\epsilon_i^* = \frac{z_{ij} - c_j}{d_j}, j \in C \cap A_i; \quad \epsilon_i^* \in (c_j, z_{ij} - 1, c_j, z_{ij}), j \in D \cap A_i} \right].$$

For parameters $\mathbf{u}_{ijk}$ where $j \in D \cap A_i$, the derivatives can be equivalently written as

$$\frac{\partial}{\partial u_{ijk}} \log f_i(z_i^{obs}; \tilde{\Xi}) = E_{\epsilon_i^* \sim N(0, \mathbf{L}^\top \mathbf{L})} \left[ \frac{\partial}{\partial u_{ijk}} \xi_{ij}(\tilde{\Xi} | \epsilon_i^*) \bigg| _{\epsilon_i^* = \frac{z_{ij} - c_j}{d_j}, j \in C \cap A_i; \quad \epsilon_i^* \in (c_j, z_{ij} - 1, c_j, z_{ij}), j \in D \cap A_i} \right].$$
where
\[ \xi_{ij}(\tilde{\xi}|\mathbf{Z}^*_i) = \log \int_{c_{ij},s_{ij}}^{c_{ij+1},s_{ij+1}} \phi(\mathbf{Z}^*_i|\mathbf{LL}^\top) \, d\mathbf{Z}^*_i. \]

More concretely, we now suppose \( \mathbf{z}_i^*(\tilde{\xi}) = (z_1^*(\tilde{\xi}), \ldots, z_p^*(\tilde{\xi}))^\top \) to be a realization of the underlying random vector that is sampled (exactly or approximately) from \( N(0, \mathbf{LL}^\top|\mathbf{z}_i^\text{obs}, \mathbf{c}, \mathbf{d}) \), which satisfies
\[ z_{ij}^*(\tilde{\xi}) = \begin{cases} z_{ij}^* - c_j, & \text{if } j \in \mathcal{C} \cap \mathcal{A}_i; \\ [c_{ij}, z_{ij}, c_{ij+1}], & \text{if } j \in \mathcal{D} \cap \mathcal{A}_i. \end{cases} \]

The derivatives of \( \tilde{\xi} \) with respect to \( \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi}) \) can be approximated by the following formulas:

- If \( Z_{ij} \) is a continuous variable and observed,
  \[ \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial c_{ij}} \approx \frac{\partial}{\partial c_{ij}} \psi_i(\tilde{\xi}|\mathbf{z}_i^*, \tilde{\xi}) = \frac{1}{d_j} (\mathbf{LL}^\top)^{-1} z_{ij}^*(\tilde{\xi}), \]
  \[ \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial d_{ij}} \approx \frac{\partial}{\partial d_{ij}} \psi_i(\tilde{\xi}|\mathbf{z}_i^*, \tilde{\xi}) = \frac{z_{ij} - 1}{d_j} (\mathbf{LL}^\top)^{-1} z_{ij}^*(\tilde{\xi}). \]  
  \( \text{(D.2)} \)

- If \( Z_{ij} \) is a continuous variable and missing,
  \[ \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial c_{ij}} = \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial d_{ij}} = 0. \]

- If \( Z_{ij} \) is a binary variable and observed,
  \[ \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial u_{ij}} \approx \frac{\partial}{\partial u_{ij}} \xi_{ij}(\tilde{\xi}|\mathbf{z}_i^*, \tilde{\xi}) = \begin{cases} -\phi(\mathbf{z}_i^*, \tilde{\xi})|\mathbf{LL}^\top|_{z_{ij}^*(\tilde{\xi}) = u_{ij}} & \text{if } z_{ij} = 1, \\ \phi(\mathbf{z}_i^*, \tilde{\xi})|\mathbf{LL}^\top|_{z_{ij}^*(\tilde{\xi}) = u_{ij}} & \text{if } z_{ij} = 0. \end{cases} \]  
  \( \text{(D.3)} \)

- If \( Z_{ij} \) is a binary variable and missing,
  \[ \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial u_{ij}} = 0. \]

- If \( Z_{ij} \) is an ordinal variable and observed,
  \[ \frac{\partial \log f_i(\mathbf{z}_i^\text{obs}|\tilde{\xi})}{\partial u_{jk}} \approx \sum_{l=1}^{K_j} \frac{\partial}{\partial c_{jl}} \xi_{ij}(\tilde{\xi}|\mathbf{z}_i^*, \tilde{\xi}) \frac{\partial c_{jl}}{\partial u_{jk}}. \]  
  \( \text{(D.4)} \)
where

\[
\frac{\partial}{\partial c_{jl}} \xi_{ij}(\tilde{\xi}|z_i^*(\tilde{\xi})) = \begin{cases} 
\phi(z_{ij}^*(\tilde{\xi})|LL^\top)|_{z_{ij}^*(\tilde{\xi}) = c_{jl}}, & \text{if } z_{ij} < K_j \text{ and } l = z_{ij} + 1, \\
\int_{c_{jl}^{t-1}}^{c_{jl}^t} \phi(z_{ij}^*(\tilde{\xi})|LL^\top)|_{z_{ij}^*(\tilde{\xi}) = c_{jl}}, & \text{if } z_{ij} > 0 \text{ and } l = z_{ij}, \\
0, & \text{others},
\end{cases}
\]

and

\[
\frac{\partial c_{jl}}{\partial u_{jk}} = \begin{cases} 
1, & \text{if } k = 1, \\
\exp(u_{jk}), & \text{if } 2 \leq k \leq l, \\
0, & \text{others}.
\end{cases}
\]

If \(Z_{ij}\) is an ordinal variable and missing,

\[
\frac{\partial \log f_i(z_i^{\text{obs}}|\tilde{\xi})}{\partial u_{jk}} = 0.
\]

- For the lower triangular matrix \(L\),

\[
\frac{\partial \log f_i(z_i^{\text{obs}}|\tilde{\xi})}{\partial L} \approx \frac{\partial}{\partial L} g_i(\tilde{\xi}|z_i^*(\tilde{\xi})) \\
= (LL^\top)^{-1} z_{ij}^*(\tilde{\xi}) z_{ij}^*(\tilde{\xi})^\top (L^\top)^{-1} - \text{diag} \left( \left( \frac{1}{l_{11}}, \ldots, \frac{1}{l_{pp}} \right)^\top \right), \tag{D.5}
\]

where \(\text{diag}(v)\) denotes the diagonal matrix whose diagonal is the given vector \(v\).

We summarize the algorithm to solve optimization problem [D.1] in Algorithm 4 following Zhang and Chen [2022].

Algorithm 4 (Stochastic proximal ascent algorithm).

**Input:** Observed data \(z_i^{\text{obs}}, i = 1, \ldots, N\), initial value of parameters \(\tilde{\xi}^{(0)}\), initial value of underlying variables \(z_i^{*(0)}, i = 1, \ldots, N\), burn-in size \(M\), number of follow-up iterations \(T\), sequence of step sizes \(\{\gamma_t\}_{t=1}^{M+T}\).

**Update:** At \(t\)-th iteration where \(1 \leq t \leq M + T\), perform the following two steps:

1. **Sampling step:** For \(i = 1, 2, \ldots, N\), sample \(z_i^{*(t)}\) from \(\mathcal{N}(0, L^{(t-1)}(L^{(t-1)})^\top | z_i^{\text{obs}}, c^{(t-1)}, d^{(t-1)}\) by a Gibbs sampler.

2. **Proximal gradient ascent step:** Based on \(z_i^{*(t)}\), compute the approximated gradient \(\tilde{\nabla}_{\Xi} \log f_i(z_i^{\text{obs}}|\tilde{\xi}^{(t-1)})\) through equations [D.2], [D.3], [D.4] and [D.5]. Let \(G^{(t)} = \sum_{i=1}^N \tilde{\nabla}_{\Xi} \log f_i(z_i^{\text{obs}}|\tilde{\xi}^{(t-1)})\). Compute a positive definite diagonal matrix \(H^{(t)}\), and update

\[
\tilde{\xi}^{(t)} = \mathcal{P}_{\Omega} \left(\tilde{\xi}^{(t-1)} + \gamma_t \left(H^{(t)}\right)^{-1} G^{(t)}\right).
\]

Here \(\mathcal{P}_{\Omega}\) is the projection operator onto the feasible set \(\Omega\).
Output: $\hat{\Xi}_T = \frac{1}{T} \sum_{t=B+1}^{B+T} \Xi^{(t)}$

**Remark 1.** In practice, we run only one scan of Gibbs sampling in the sampling step, which is enough for the algorithm to converge. If $j \not\in A_i$, the conditional distribution is a simple univariate normal distribution. If $j \in D \cap A_i$, the conditional distribution is a truncated normal distribution, which can be sampled from by standard method. Also we can employ a random scan Gibbs sampler for a better mixing rate. See Givens and Hoeting (2012) for more details.

**Remark 2.** The stepsize can be chosen as $\gamma_t = t^{-0.51}$, as is recommended in Zhang and Chen (2022).

**Remark 3.** In practice, initial value of $c$, $d$ and $u$ can be obtained by marginal distribution of the corresponding variables. Initial value of $\Sigma$ can be obtained by the mixed correlation matrix composed of the Pearson/ polychoric/ tetrachoric/ biserial/ polychoric correlation of each pair of variables, where the type of correlation depends on the combination of variables types of the corresponding pair. This mixed correlation matrix can be computed by the mixedCor function in R package ‘psych’. If the raw mixed correlation matrix is not positive definite, we can project it to its nearest positive semidefinite matrix.

**Remark 4.** As long as the matrix $H^{(t)}$ converges to a positive definite diagonal matrix, theoretical results in Zhang and Chen (2022) guarantee that the output of the algorithm converges as $T \to \infty$. In our study, during the burn-in period we let $H^{(t)}$ be the diagonal matrix with diagonal elements $-\sum_{i:j \in A_i} \hat{\rho}^2_{ij} \psi_i(\hat{\Xi}_i^{*^{(t)}})$ and $-\sum_{i:j \in D_k} \hat{\rho}^2_{ij} \psi_j(\hat{\Xi}_j^{*^{(t)}})$ for $j \in C$, $-\sum_{i:j \in A_i} \hat{\rho}^2_{ij} \psi_i(\hat{\Xi}_i^{*^{(t)}})$ and $-\sum_{i=1}^N \hat{\rho}^2_{ij} \psi_i(\hat{\Xi}_i^{*^{(t)}})$ for $a, b = 1, \ldots, p$. After the burn-in period, we fix $H^{(t)} = H^{(B)}$.

### D.2 Algorithm for estimation of latent regression model and extended latent regression model

#### D.2.1 Algorithm

In this section, we will give a detailed description of the stochastic EM algorithm to estimate the latent regression model and extended latent regression model. Since the latent regression model and its extended version are essentially the same, we only describe the algorithm to estimate the latter one here.

Recall the maximum marginal likelihood problem (6)

$$(\hat{\beta}, \hat{\gamma}, \hat{\sigma}^2) = \arg \max_{\beta, \gamma, \sigma^2} \tilde{l}_2(\beta, \gamma, \sigma^2).$$

Here the likelihood function $\tilde{l}_2(\beta, \gamma, \sigma^2)$ takes the form

$$\tilde{l}_2(\beta, \gamma, \sigma^2) = \sum_{i=1}^{N} \log \left( \prod_{j \not\in A_i} \int d\bar{z}_j \left( \prod_{j \not\in A_i} d\bar{z}_j \right) f(\bar{z}_i, \bar{z}_j| \Xi) f(y_i| \bar{z}_i, \bar{z}_j; \beta, \gamma, \sigma^2) \right),$$

where

$$f(y_i| \bar{z}_i; \beta, \sigma^2) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right) \int d\theta_i \left( \prod_{l \in B_i} h_l(y_i| \theta_l) \right) \exp \left( -\frac{(\theta_i - (\beta_0 + \beta_1 y_1 + \cdots + \beta_p y_p))^2}{2\sigma^2} \right).$$
and
\[
f(z_i, \tilde{z}_i|\Xi) = \int \cdots \int \left( \prod_{j \in \mathcal{D}} dz_j^* \right) \left( \prod_{j \in \mathcal{D}} d\tilde{z}_j^* \right) \left( \prod_{j \notin \mathcal{D}} d\tilde{z}_j^* \right) \times
\left\{ \phi([z^*, \tilde{z}^*]|G) \times \left( \prod_{j \in \mathcal{D} \cap \mathcal{A}_i} \mathbb{I}(z_j^* \in (c_{ij}, z_{ij} - 1, c_{ij}, z_{ij}], \tilde{z}_j^* \in (c_{ij}, \tilde{z}_{ij} - 1, c_{ij}, \tilde{z}_{ij}]) \right) \right\}_{z_j^* = \frac{z_{ij} - c_{ij}}{d_j}, \tilde{z}_j^* = \frac{\tilde{z}_{ij} - c_{ij}}{d_j}, j \notin \mathcal{D}}.
\]

We first summarize the algorithm to solve this problem in Algorithm 5.

**Algorithm 5** (Stochastic EM algorithm).

**Input:** Item responses \( \{y_i\}_{i=1}^N \), observed covariate data \( \{z_i^{\text{obs}}\}_{i=1}^N \), knockoff copy of observed covariate data \( \{\tilde{z}_i^{\text{obs}}\}_{i=1}^N \), initial value of underlying variables \( \{z_i^{(0)}\}_{i=1}^N \) and \( \{\tilde{z}_i^{(0)}\}_{i=1}^N \), initial value of parameters \( \Theta^{(0)} = \{\beta^{(0)}, \gamma^{(0)}, (\sigma^2)^{(0)}\} \), parameters of the Gaussian copula model \( \Xi \), extended correlation matrix \( G \), burn-in size \( M \), number of following-up iterations \( T \).

**Transformation:** For \( i = 1, 2, \ldots, N \), transform \( \{z_i^{(0)}, \tilde{z}_i^{(0)}\} \) to \( \{z_i^{(0)}, \tilde{z}_i^{(0)}\} \) according to the Gaussian copula model.

**Update:** At \( t \)-th iteration where \( 1 \leq t \leq M + T \), perform the following two steps:

1. **Sampling step:**
   1. For \( i = 1, 2, \ldots, N \), sample \( \theta_i^{(t)} \) from \( p_{\theta_i|z_i, \tilde{z}_i, Y_i}(z_i^{(t-1)}, \tilde{z}_i^{(t-1)}, y_i) \) by adaptive rejection sampling.
   2. For \( i = 1, 2, \ldots, N \), sample \( \{z_i^{(t)}, \tilde{z}_i^{(t)}\} \) from \( N(0, G|z_i^{\text{obs}}, \tilde{z}_i^{\text{obs}}, c, d) \) by a Gibbs sampler, where \( N(0, G|z_i^{\text{obs}}, \tilde{z}_i^{\text{obs}}, c, d) \) denotes the distribution of a multivariate Gaussian vector \( [z_i^*, \tilde{z}_i^*] \sim N(0, G) \) with constraints \( z_i^* = \frac{z_{ij} - c_{ij}}{d_j} \) and \( \tilde{z}_i^* = \frac{\tilde{z}_{ij} - c_{ij}}{d_j} \) when \( j \in C \cap \mathcal{A}_i \), and \( (z_{ij}, \tilde{z}_{ij}) \in (c_{ij}, z_{ij} - 1, c_{ij}, z_{ij}] \times (c_{ij}, \tilde{z}_{ij} - 1, c_{ij}, \tilde{z}_{ij}] \) when \( j \notin \mathcal{D} \cap \mathcal{A}_i \).

2. **Transformation step:** For \( i = 1, 2, \ldots, N \), transform \( \{z_i^{(t)}, \tilde{z}_i^{(t)}\} \) to \( \{z_i^{(t)}, \tilde{z}_i^{(t)}\} \) according to the Gaussian copula model.

3. **Maximization step:** Let \( (\beta^{(t)}, \gamma^{(t)}, (\sigma^2)^{(t)}) \) be the maximizer to the objective function
   \[
   L^{(t)}(\beta, \gamma, \sigma^2) = -\frac{N}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^N \left( \theta_i^{(t)} - \beta_0 - \sum_{j=1}^p \beta_j g_j(z_{ij}^{(t)}) - \sum_{j=1}^p \gamma_j g_j(\tilde{z}_{ij}^{(t)}) \right)^2.
   \]

**Output:** \( \beta_T = \frac{1}{T} \sum_{t=1}^{T} \beta^{(t)}, \gamma_T = \frac{1}{T} \sum_{t=1}^{T} \gamma^{(t)}, \sigma^2_T = \frac{1}{T} \sum_{t=1}^{T} (\sigma^2)^{(t)} \).

**Remark 5.** As in Algorithm 4, we only run one scan of Gibbs sampling in the sampling step. More details on the sampling step will be given in the next section.

### D.2.2 Sampling details

The sampling step in stochastic EM algorithm actually requires to sample \( (\theta_i, z_i^*, \tilde{z}_i^*|z_i^{\text{obs}}, \tilde{z}_i^{\text{obs}}, Y_i; \beta, \gamma, \sigma^2, \Xi, G) \).
However, exactly sampling from this posterior distributions is a difficult task. Instead, we use a Gibbs sampler and run only one scan. For this purpose, we need only to specify the conditional distribution for each variable given the others. Note that sampling of $\tilde{Z}_{ij}^*$ is almost same with $Z_{ij}^*$, therefore, we omit the details for sampling $Z_{ij}^*$ here.

Throughout this section, we denote $Z_{i,-j} = \{Z_{i1}, \ldots, Z_{ip}\}\{Z_{ij}\}$. $Z_{i,-j}^*$ is defined analogously.

**Sampling of $\theta_i$** For $\theta_i$, we have $\theta_i|\mathbf{Z}_i^*, \tilde{\mathbf{Z}}_i, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}, \mathbf{Y}_i \overset{d}{=} \theta_i|\mathbf{Z}_i, \tilde{\mathbf{Z}}_i, \mathbf{Y}_i$, where $\mathbf{Z}_i$ and $\tilde{\mathbf{Z}}_i$ are obtained from $\mathbf{Z}_i^*$ and $\tilde{\mathbf{Z}}_i^*$ by applying the link functions in the Gaussian copula model. The density of the latter distribution can be written as

$$p(\theta_i|\mathbf{Z}_i, \tilde{\mathbf{Z}}_i, \mathbf{Y}_i; \beta, \gamma, \sigma^2, \Xi, \mathbf{G}) \propto \prod_{t \in B_i} h_t(Y_{it}|\theta_i) \exp \left( -\frac{1}{2\sigma^2} \left( \theta_i - \beta_0 - \sum_{j=1}^p \beta_j^\top g_j(Z_{ij}) - \sum_{j=1}^p \gamma_j^\top g_j(\tilde{Z}_{ij}) \right)^2 \right).$$

It is easy to verify that this density is log-concave. Thus we can use the adaptive rejection sampling method (Gilks and Wild 1992) to sample $\theta_i$ from its conditional distribution.

**Sampling of $Z_{ij}^*$** For $Z_{ij}^*$, the conditional distribution is

$$p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}, \theta_i; \beta, \gamma, \sigma^2, \Xi, \mathbf{G}).$$

It suffices to consider $j \in (\mathcal{C}\setminus \mathcal{A}_i) \cup \mathcal{D}$, since $Z_{ij}^*$ is fixed when $j \in \mathcal{C}\cap \mathcal{A}_i$. From the data generation process, it is easy to see that $\mathbf{Z}_i^* \perp \mathbf{Y}_i|\theta_i$, hence the conditional distribution becomes

$$p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}, \theta_i; \beta, \gamma, \sigma^2, \Xi, \mathbf{G}). \quad (D.6)$$

1. If $j \in \mathcal{C}\setminus \mathcal{A}_i$, $Z_{ij}$ is continuous and missing. We have $Z_{ij}^* \perp (\mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}})|(Z_{i,-j}^*, \tilde{Z}_i^*)$. Since $g_j(Z_{ij}) = Z_{ij} = d_j Z_{ij}^* + c_j$, $p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}, \theta_i; \beta, \gamma, \sigma^2, \Xi, \mathbf{G}) \propto p(\theta_i|Z_{ij}^*; \tilde{Z}_i^*, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}, \theta_i; \beta, \gamma, \sigma^2) \times p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*, \mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}}; \beta, \gamma, \sigma^2, \Xi, \mathbf{G})$ $\propto \exp \left( -\frac{1}{2\sigma^2} \left[ \beta_j(d_j Z_{ij}^* + c_j) - \left( \theta_i - \beta_0 - \sum_{k \neq j} \beta_k^\top g_k(Z_{ik}) - \sum_{j=1}^p \gamma_j^\top g_j(\tilde{Z}_{ij}) \right) \right]^2 \right)$ $\times p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*; \mathbf{G})$. Note that $p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*; \mathbf{G})$ is a univariate density. It is then easy to see that $p(Z_{ij}^*|Z_{i,-j}^*, \tilde{Z}_i^*)$ is also a normal density in this case.

2. If $j \in \mathcal{D}\setminus \mathcal{A}_i$, $Z_{ij}$ is discrete and missing. Similarly, $Z_{ij}^* \perp (\mathbf{Z}_i^{\text{obs}}, \tilde{\mathbf{Z}}_i^{\text{obs}})|(Z_{i,-j}^*, \tilde{Z}_i^*)$. For convenience, we
We log-transform the variable OUT_HOU.

3. If $j \in \mathcal{D} \cap \mathcal{A}_i$, $Z_{ij}$ is discrete and observed. In this case,

$$
p(Z_{ij}^* | Z_{i,-j}^*, \tilde{Z}_{i,-j}, Z_{i}^{\text{obs}}, \tilde{Z}_{i}^{\text{obs}}, \theta; \beta, \gamma, \sigma^2, \Xi, G)
\propto p(\theta_i | Z_{ij}^*, Z_{i,-j}^*, \tilde{Z}_{i,-j}, Z_{i}^{\text{obs}}, \tilde{Z}_{i}^{\text{obs}}, \beta, \gamma, \sigma^2) \times p(Z_{ij}^* | Z_{i,-j}^*, \tilde{Z}_{i,-j}^*, G)
\propto \exp\left(\frac{1}{2\sigma^2} \left( \theta_i - \theta_0 - \sum_{k \neq j} \beta_k g_k(Z_{ik}) - \sum_{k=1}^p \gamma_k g_k(\tilde{Z}_{ik}) \right)^2 \right)
\times p(Z_{ij}^* | Z_{i,-j}^*, \tilde{Z}_{i,-j}^*, G) \times \mathbb{1}(Z_{ij}^* \in (c_j, z_{ij}, c_{j,z_{ij}}+1))
$$

This is a univariate truncated normal distribution, which can be sampled from using standard methods.

3. If $j \in \mathcal{D} \cap \mathcal{A}_i$, $Z_{ij}$ is discrete and observed. In this case,

$$
p(Z_{ij}^* | Z_{i,-j}^*, \tilde{Z}_{i,-j}, Z_{i}^{\text{obs}}, \tilde{Z}_{i}^{\text{obs}}, \theta; \beta, \gamma, \sigma^2, \Xi, G)
\propto p(\theta_i | Z_{ij}^*, Z_{i,-j}^*, \tilde{Z}_{i,-j}, Z_{i}^{\text{obs}}, \tilde{Z}_{i}^{\text{obs}}, \beta, \gamma, \sigma^2) \times p(Z_{ij}^* | Z_{i,-j}^*, \tilde{Z}_{i,-j}^*, G)
\propto \exp\left(\frac{1}{2\sigma^2} \left( \theta_i - \theta_0 - \sum_{k \neq j} \beta_k g_k(Z_{ik}) - \sum_{k=1}^p \gamma_k g_k(\tilde{Z}_{ik}) \right)^2 \right)
\times p(Z_{ij}^* | Z_{i,-j}^*, \tilde{Z}_{i,-j}^*, G) \times \mathbb{1}(Z_{ij}^* \in (c_j, z_{ij}, c_{j,z_{ij}}+1))
$$

This is a univariate truncated normal distribution, which can be sampled from using standard methods.

E Real Data

E.1 Data Preprocessing

We perform the following pre-processing steps:

1. We discard samples and variables that include too much missing values.

2. For ordinal variables, we merge the categories with too few samples (below 5% of observed samples) into their adjacent categories.

3. For highly correlated variables (correlations higher than 0.7), we discard some of them, or combine them into less correlated variables. This step is necessary since technically we don’t want $[X, \tilde{X}]$ to suffer from issue of multicollinearity.

4. We log-transform the variable OUT_HOU.

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### E.2 Candidate Variables

Table E.1: Full description of candidate variables used in real data analysis.

| Name   | Type | Description                                                                                                                                 |
|--------|------|--------------------------------------------------------------------------------------------------------------------------------------------|
| OUT.FRI| B    | Whether the student meet friends or talk to friends on the phone outside the school. This variable is obtained by combining questionnaire items ST076Q07NA (whether meet or talk to friends before school) and ST078Q07NA (whether meet or talk to friends after school). Coding: 1 = yes, 0 = no. |
| OUT.GAM| B    | Whether the student play video-games outside the school. This variable is obtained by combining questionnaire items ST076Q06NA (whether play video-games before school) and ST078Q06NA (whether play video-games after school). Coding: 1 = yes, 0 = no. |
| OUT.HOL| B    | Whether the student work in the household outside the school. This variable is obtained by combining questionnaire items ST076Q09NA (whether work in the household before school) and ST078Q09NA (whether work in the household after school). Coding: 1 = yes, 0 = no. |
| OUT.NET| B    | Whether the student use Internet outside the school. This variable is obtained by combining questionnaire items ST076Q05NA (whether use Internet before school) and ST078Q05NA (whether use Internet after school). Coding: 1 = yes, 0 = no. |
| OUT.JOB| B    | Whether the student work for pay outside the school. This variable is obtained by combining questionnaire items ST076Q10NA (whether work for pay before school) and ST078Q10NA (whether work for pay after school). Coding: 1 = yes, 0 = no. |
| OUT.MEA| B    | Whether the student have meals before school or after school. This variable is obtained by combining questionnaire items ST076Q01NA (whether have breakfast before school) and ST078Q01NA (whether have dinner after school). Coding: 1 = yes, 0 = no. |
| OUT.PAR| B    | Whether the student talk to parents outside the school. This variable is obtained by combining questionnaire items ST076Q08NA (whether talk to parents before school) and ST078Q08NA (whether talk to parents after school). Coding: 1 = yes, 0 = no. |
| OUT.REA| B    | Whether the student read a book/newspaper/magazine outside the school. This variable is obtained by combining questionnaire items ST076Q04NA (whether read a book/newspaper/magazine before school) and ST078Q04NA (whether read a book/newspaper/magazine after school). Coding: 1 = yes, 0 = no. |
| Variable   | Description                                                                 | Coding          |
|------------|-----------------------------------------------------------------------------|-----------------|
| OUT.SPO B  | Whether the student exercise or do a sport outside the school. The variable is obtained by combining questionnaire items ST076Q11NA (whether exercise or do a sport before school) and ST078Q11NA (whether exercise or do a sport after school). Coding: 1 = yes, 0 = no. |                |
| OUT.STU B  | Whether the student study for school or homework outside the school. This variable is obtained by combining questionnaire items ST076Q02NA (whether study before school) and ST078Q02NA (whether study after school). Coding: 1 = yes, 0 = no. |                |
| OUT.VED B  | Whether the student watch TV/DVD/Video outside the school. This variable is obtained by combining questionnaire items ST076Q03NA (whether watch TV/DVD/Video before school) and ST078Q03NA (whether watch TV/DVD/Video after school). Coding: 1 = yes, 0 = no. |                |
| REPEAT B   | Has the student ever repeated a grade. Coding: 1 = yes, 0 = no.               |                |
| SCI.APP B  | Whether the student attend applied sciences and technology courses in this school year or last school year. This variable is obtained by combining questionnaire items ST063Q05NA (whether attend this year) and ST063Q05NB (whether attend last year). Coding: 1 = yes, 0 = no. |                |
| SCI.BIO B  | Whether the student attend biology courses in this school year or last school year. This variable is obtained by combining questionnaire items ST063Q03NA (whether attend this year) and ST063Q03NB (whether attend last year). Coding: 1 = yes, 0 = no. |                |
| SCI.CHE B  | Whether the student attend chemistry courses in this school year or last school year. This variable is obtained by combining questionnaire items ST063Q02NA (whether attend this year) and ST063Q02NB (whether attend last year). Coding: 1 = yes, 0 = no. |                |
| SICLEAR B  | Whether the student attend earth and space courses in this school year or last school year. This variable is obtained by combining questionnaire items ST063Q04NA (whether attend this year) and ST063Q04NB (whether attend last year). Coding: 1 = yes, 0 = no. |                |
| SICGEN B   | Whether the student attend general, integrated, or comprehensive science courses in this school year or last school year. This variable is obtained by combining questionnaire items ST063Q06NA (whether attend this year) and ST063Q06NB (whether attend last year). Coding: 1 = yes, 0 = no. |                |
| SICPHY B   | Whether the student attend physics courses in this school year or last school year. This variable is obtained by combining questionnaire items ST063Q01NA (whether attend this year) and ST063Q01NB (whether attend last year). Coding: 1 = yes, 0 = no. |                |
| GENDER B   | Student’s gender. The original item is named ‘ST004D01T’. Coding: 1 = male, 0 = female. |                |
| Variable | Type | Description |
|----------|------|-------------|
| LANGAH | B | Is student’s language at home different from the test language. The original item is named ‘ST022Q01TA’. Coding: 1 = yes, 0 = no. |
| DUECEC | O | Duration in ECEC (Early Childhood Education and Care) of student. The original variable is named ‘DURECEC’ in the original dataset, which has 9 categories. We merged them into 4 categories. Coding: 0 = attended ECEC for less than two years, 1 = attended ECEC for at least two but less than three years, 2 = attended ECEC for at least three but less than four years, 3 = attended ECEC for at least four years. |
| FISCED | O | Father’s education in ISCED (International Standard Classification of Education) level. The original variable has 6 categories, we merged them into 5 categories. Coding: 0 = none or ISCED 1 (primary education), 1 = ISCED 2 (lower secondary), 2 = ISCED 3B or 3C (vocational/pre-vocational upper secondary), 3 = ISCED 3A (general upper secondary) or 4 (non-tertiary post-secondary), 4 = ISCED 5B (vocational tertiary), 5 = ISCED 5A (theoretically oriented tertiary) or 6 (post-graduate). |
| GRADE | O | Student’s grade. 0 = lower than modal grade, 1 = not lower than modal grade, 2 = higher than modal grade. Here the modal grade is the grade level that most 15-year-old students in the country attend. |
| MISCED | O | Mother’s education in ISCED (International Standard Classification of Education) level. The original variable has 6 categories, we merged them into 5 categories. Coding: 0 = none or ISCED 1 (primary education), 1 = ISCED 2 (lower secondary), 2 = ISCED 3B or 3C (vocational/pre-vocational upper secondary), 3 = ISCED 3A (general upper secondary) or 4 (non-tertiary post-secondary), 4 = ISCED 5B (vocational tertiary), 5 = ISCED 5A or ISCED 6 (theoretically oriented tertiary and post-graduate). |
| DAYPEC | O | Averaged days that student attends physical education classes each week. The original item is named ‘ST031Q01NA’, which has 8 categories. We merged them into 4 categories. Coding: 0 = 0 days, 1 = 1 or 2 days, 2 = 3 or 4 days, 3 = 5 days or more. |
| DAYMPA | O | Number of days with moderate physical activities for a total of at least 60 minutes per each week. The original item is named ‘ST032Q01NA’, which has 8 categories. Coding: 0 = 0 days, 1 = 1 day, 2 = 2 days, 3 = 3 days, 4 = 4 days, 5 =5 days, 6 = 6 days, 7 = 7 days, 8 = 8 days. |
| SKIDAY | O | How often did student skipped a whole school day in the last two full weeks of school. The original item is named ‘ST062Q01TA’, which has 4 categories. We merged them into 3 categories. Coding: 0 = none, 1 = one or two times, 2 = three or more times. |
| Variable | Type | Description |
|----------|------|-------------|
| SKICAL   | O    | How often did student skipped some classes in the last two full weeks of school. The original item is named ‘ST062Q02TA’, which has 4 categories. We merged them into 3 categories. Coding: 0 = none, 1 = one or two times, 2 = three or more times. |
| ARRLAT   | O    | How often did student arrived late for school in the last two full weeks of school. The original item is named ‘ST062Q03TA’, which has 4 categories. We merged them into 3 categories. Coding: 0 = none, 1 = one or two times, 2 = three or more times. |
| CHOCOU   | O    | Can student choose the school science course(s) he or she study. The original item is named ‘ST064Q01NA’, which has 3 categories. Coding: 0 = no, not at all; 1 = yes, to a certain degree; 2 = yes, can choose freely. |
| CHODIF   | O    | Can student choose the level of difficulty for school science course(s) he or she study. The original item is named ‘ST064Q02NA’, which has 3 categories. Coding: 0 = no, not at all; 1 = yes, to a certain degree; 2 = yes, can choose freely. |
| CHONUM   | O    | Can student choose the number of school science course(s) he or she study. The original item is named ‘ST064Q03NA’, which has 3 categories. Coding: 0 = no, not at all; 1 = yes, to a certain degree; 2 = yes, can choose freely. |
| EISCED   | O    | The ISCED level that student expects to complete. The original item is named ‘ST111Q01TA’, which has 6 categories. We merged them into 3 categories. Coding: 0 = level 2 or 3A, 1 = level 4 or 5B, 2 = level 5A or 6. |
| ADINST   | C    | Adaption of instruction. This variable is derived based on IRT scaling. The observed values range from -1.97 to 2.04. |
| ANXTES   | C    | Personality: test anxiety. This variable is named ‘ANXTEST’ in the original dataset, which is derived based on IRT scaling. The observed values range from -2.51 to 2.55. |
| BELONG   | C    | Subjective well-being: sense of belonging to school. This variable is derived based on IRT scaling. The observed values range from -3.13 to 2.61. |
| FISEIO   | C    | ISEI (International Socio-economic Index) of occupational status of father. The observed values range from 12 to 89. |
| MISEIO   | C    | ISEI (International Socio-economic Index) of occupational status of mother. The observed values range from 12 to 89. |
| EISEIO   | C    | Student’s expected ISEI of occupational status. The observed values range from 16 to 89. |
| COOPER   | C    | Collaboration and teamwork dispositions: enjoy cooperation. This variable is named ‘COOPERATE’ in the original dataset, which is derived based on IRT scaling. The observed values range from -3.33 to 2.29. |
| Variable | Type | Description |
|----------|------|-------------|
| CPSVAL   | C    | Collaboration and teamwork dispositions: value cooperation. This variable is named ‘CPSVALUE’ in the original dataset, which is based on IRT scaling. The observed values range from -2.83 to 2.14. |
| CULTPO   | C    | Cultural possessions at home. This variable is named ‘CULTPOSS’ in the original dataset, which is derived based on IRT scaling. The observed values range from -1.71 to 2.63. |
| DISCLI   | C    | Disciplinary climate in science classes. This variable is named ‘DISCLISCI’ in the original dataset, which is derived based on IRT scaling. The observed values range from -2.41 to 1.88. |
| EMOSUP   | C    | Parents emotional support. This variable is named ‘EMOSUPS’ in the original dataset, which is derived based on IRT scaling. The observed values range from -3.08 to 1.10. |
| ENVAWA   | C    | Environmental awareness. This variable is named ‘ENVAWARE’ in the original dataset, which is derived based on IRT scaling. The observed values range from -3.38 to 3.29. |
| ENVOPT   | C    | Environmental optimism. This variable is derived based on IRT scaling. The observed values range from -1.80 to 3.01. |
| EPIST    | C    | Epistemological beliefs. This variable is derived based on IRT scaling. The observed values range from -2.79 to 2.16. |
| HEDRES   | C    | Home educational resources. This variable is derived based on IRT scaling. The observed values range from -4.41 to 1.78. |
| EBSCIT   | C    | Enquiry-based science teaching and learning practices. This variable is named ‘IBTEACH’ in the original dataset, which is derived based on IRT scaling. The observed values range from -3.34 to 3.18. |
| INSTSC   | C    | Instrumental motivation. This variable is named ‘INSTSCIE’ in the original dataset, which is derived based on IRT scaling. The observed values range from -1.93 to 1.74. |
| INTBRS   | C    | Interest in broad science topics. This variable is named ‘INTBRSCT’ in the original dataset, which is derived based on IRT scaling. The observed values range from -2.55 to 2.73. |
| JOYSCI   | C    | Enjoyment of science. This variable is named ‘JOYSCIE’ in the original dataset, which is derived based on IRT scaling. The observed values range from -2.12 to 2.16. |
| MOTIVA   | C    | Achievement motivation. This variable is named ‘MOTIVAT’ in the original dataset, which is derived based on IRT scaling. The observed values range from -3.09 to 1.85. |
| OUTHOU   | C    | Out-of-school study time per week (hours). This variable is obtained from the original variable named ‘OUTHOURS’ by setting 0 as missing and performing a log-transformation. The observed values range from 0 to 4.25. |
| Variable  | Type | Description                                                                                                                                                                                                 |
|-----------|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| PERFEE    | C    | Perceived feedback. This variable is named ‘PERFEED’ in the original dataset, which is derived based on IRT scaling. The observed values range from -1.53 to 2.50.                                                                 |
| SCIACT    | C    | Index science activities. This variable is named ‘SCIACT’ in the original dataset, which is derived based on IRT scaling. The observed values range from -1.76 to 3.36.                                                                 |
| SCIEEF    | C    | Science self-efficacy. This variable is named ‘SCIEEF’ in the original dataset, which is derived based on IRT scaling. The observed values range from -3.76 to 3.28.                                                                 |
| TDSCIT    | C    | Teacher-directed science instruction. This variable is named ‘TDSCIT’ in the original dataset, which is derived based on IRT scaling. The observed values range from -2.45 to 2.08.                                                                 |
| TEASUP    | C    | Teacher support in a science classes of students choice. This variable is named ‘TEASUP’ in the original dataset, which is derived based on IRT scaling. The observed values range from -2.72 to 1.45.                                                                 |
| TMINS     | C    | Learning time in class per week (minutes). The observed values range from 0 to 3000.                                                                                                                        |
| UNFAIR    | C    | Teacher unfairness. This variable is named ‘UNFAIR’ in the original dataset, which is derived by taking the sum of the responses to questionnaire items ST039Q01NA to ST039Q06NA (questions that ask students about how often in the past 12 months they had experienced unfair treatment by teachers). The observed values range from 1 to 24. |
| WEALTH    | C    | Family wealth. This variable is derived based on IRT scaling. The observed values range from -7.01 to 4.27.                                                                                                          |

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