Quantum fluctuations, gauge freedom and mesoscopic/macroscopic stability

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Abstract. We study how the mesoscopic/macroscopic stability of coherent extended domains is generated out of the phase locking between gauge field and matter field. The rôle of the radiative gauge field in sustaining the coherent regime is discussed.

1. Introduction
Quantum field theory (QFT) has become in recent years a powerful tool for understanding the formation of extended domains where some selected physical variables assume specified nonvanishing values. Within such domains the dynamics of the elementary components becomes quite different than in the usual empty space; the above domains play actually the rôle of new vacua, unitarily inequivalent to the usual one. With respect to the dynamics occurring in the empty space, the dynamics in these new vacua appears to depend on codes provided by the constraints on the behavior of the components induced by the structure of the new vacua. This feature appears promising in the understanding of phase transitions, where each phase is the consequence of the dynamics of the peculiar structure of the relevant vacuum. The set of the vacua available to an ensemble of elementary components (e.g., the vacua corresponding to the gaseous, liquid and solid phases relative to a definite ensemble of molecules) should be self-produced by the interaction of the components, which can be regarded as the quanta of the matter field, with the gauge field, whose presence is prescribed by the local phase invariance requirement. In the present contribution we study, in the framework of a model system made up of electric dipoles, the formation of domains where the interaction between the radiative electromagnetic (e.m.) field and the dipoles breaks spontaneously the original rotational invariance of the system. A permanent nonvanishing background e.m. field appears that becomes a new environment where the dynamics of the elementary dipoles assumes the features of a new macroscopic phase of the system.

Our study reveals how coherence is the dynamical response of the system aimed to preserve the theory local gauge invariance in the presence of spontaneous breakdown of global gauge invariance. In establishing such a coherence a key rôle is played by the phase locking among the

\textsuperscript{1} This paper is dedicated to Professor E.C.G. Sudarshan in the occasion of his 75th birthday.
matter field and the radiative gauge field in an extended space-time region. Of course, preserving the theory local gauge invariance is not a formal request void of physical meaning. It amounts to guarantee the stability of the system at mesoscopic and macroscopic space-time scales against the dominant feature of the quantum components at the microscopic scale, namely quantum fluctuations. Local phase invariance, indeed, is the QFT solution to the problem of building a stable system out of fluctuating components. Such a solution is achieved by prescribing that the Lagrangian of the system built upon quantum fields should be invariant under the local phase transformation of the quantum component field \( \psi(\mathbf{x}, t) \rightarrow \psi'(\mathbf{x}, t) = \exp(i\theta(\mathbf{x}, t))\psi(\mathbf{x}, t) \). The requirement of local phase invariance then demands the introduction of gauge fields, e.g. the electromagnetic (e.m.) field \( A_\mu(\mathbf{x}, t) \), such that the Lagrangian be also invariant under the local gauge transformation, \( A_\mu(\mathbf{x}, t) \rightarrow A'_\mu(\mathbf{x}, t) - \partial_\mu \theta(\mathbf{x}, t) \), devised to compensate terms proportional to \( \partial_\mu \theta(\mathbf{x}, t) \) arising from the kinetic term for the matter field \( \psi(\mathbf{x}, t) \). In this respect, the gauge field may be described as a compensating environment, or "reservoir", against variations in the many accessible microscopic configurations.

The plan of the paper is the following. In Section 2, as a preliminary to our discussion, we consider some algebraic features of the system of \( N \) non-interacting two-level atoms under the influence of an electric field. Our discussion shows the collective nature of the excitations controlling the system at a mesoscopic level and the rôle played by the large number \( N \) of the component atoms in the enhancement of their interaction with the electric field. In Section 3 we consider the model of two-level atoms with the e.m. quantum radiative modes arising in the transitions between the atomic levels. We disregard the static dipole-dipole interaction. The field equations are presented in Section 4. The system ground states and the phase locking among the e.m. field and the matter field are considered in Section 5, where we show how coherence is the system response aimed to preserve the local gauge invariance and therefore its macroscopic stability against quantum fluctuations of the matter field phases. In Section 6 we comment on the rôle played by the gauge vector potential \( A_\mu \) as a pervading background sustaining the emergence of coherence and show how the scenario depicted by our results fits with the well known Anderson-Higgs-Kibble (AHK) mechanism in spontaneously broken symmetry theory.

2. The collective modes and the large \( N \) limit

We consider a system on \( N \) two-level atoms under the action of an external driving field able to excite them. Such a system is physically interesting for its relevance to a number of problems also in quantum optics [1]. The atoms are assumed to be non-interacting among themselves and they may be described as (fermion-like) non-interacting electrical dipole doublets under the influence of an electric field. Such a system of two-level atoms can be also described as a system of \( \frac{1}{2} \) spins, according to the known formal equivalence among the two descriptions [2]. We closely follow Refs. [3, 4] in our discussion below.

The ground state and the excited state of each of the \( N \) two-level atoms are denoted by \( |0_i\rangle \) and \( |1_i\rangle \), \( i = 1, ..., N \), respectively, associated to the eigenvalues \( \pm \frac{1}{2} \) of the operator \( \sigma_3 \) of the form \( \frac{1}{2}(|1_i\rangle\langle 1_i| - |0_i\rangle\langle 0_i|) \), no-summation on \( i \). The operators \( \sigma_+ = |1_i\rangle\langle 0_i| \) and \( \sigma_- = (\sigma_+)^\dagger \) generate the transitions between the two levels induced by the action of the electric field. The \( N \)-atom system is thus described by \( \sigma_\pm = \sum_{i=1}^{N} \sigma_i^\pm \), \( \sigma_3 = \sum_{i=1}^{N} \sigma_i^3 \), with the fermion-like su(2) algebra

\[
[\sigma_3, \sigma_\pm] = \pm \sigma_\pm, \quad [\sigma_-, \sigma_+] = -2\sigma_3.
\]

The interaction \( \mathcal{H} \) of the atoms with the electrical field \( \mathbf{E} \), \( \mathcal{H} = -\mathbf{d} \cdot \mathbf{E} \), where \( \mathbf{d} \) is the atom electric dipole moment, can be written as

\[
\mathcal{H} = \hbar \gamma (\mathbf{b}^\dagger \sigma^- + b\sigma^+) \, ,
\]

which is a Jaynes-Cummings-like Hamiltonian [5]. Here \( \gamma \) is the coupling constant which is proportional to the atomic dipole moment matrix element and to the inverse of the volume.
square root $V^{-1/2}$, $b$ is the electric field quantum operator, $\sigma^\pm$ are the atomic polarization operators.

Suppose that the electric field induces the transition $|0\rangle_i \rightarrow |1\rangle_i$ for a certain number $l$ of atoms, originally assumed for simplicity in the ground state (in a realistic system one has, of course, quantum fluctuations and not all the atoms initially are in their respective ground state; our conclusions, however, do not depend of these more realistic initial conditions). The system state may be then represented as the normalized superposition $|l\rangle$ given by

$$
|l\rangle \equiv (|0\rangle_1|0\rangle_2...|0\rangle_{N-l}|1\rangle_{N-l+1}|1\rangle_{N-l+2}...|1\rangle_N + ... + |1\rangle_1|1\rangle_2...|1\rangle_l|0\rangle_{l+1}|0\rangle_{l+2}...|0\rangle_N)/\sqrt{\binom{N}{l}}.
$$

The difference between the number of atoms in the excited state and the ones in the ground state is measured by $\sigma_3$:

$$
\langle l|\sigma_3|l\rangle = l - \frac{1}{2}N
$$

and the non-zero value of this quantity (proportional to the system polarization in the case of dipoles) signals that the rotational $SU(2)$ symmetry is broken. Operating with $\sigma^\pm$ on $|l\rangle$ gives:

$$
\sigma^+|l\rangle = \sqrt{l+1}\sqrt{N-l}|l+1\rangle , \\
\sigma^-|l\rangle = \sqrt{N-(l-1)}\sqrt{l}|l-1\rangle .
$$

Eqs. (4) and (5) show that $\sigma_3$ and $\sigma^\pm$ are represented on $|l\rangle$ by

$$
\sigma_3 = S^+S^- - \frac{1}{2}N , \\
\sqrt{\frac{1}{N}} \sigma^\pm = S^+\sqrt{1 - \frac{S^+S^-}{N}} , \\
\sqrt{\frac{1}{N}} \sigma^- = \sqrt{1 - \frac{S^+S^-}{N}}S^- ,
$$

where $S^- = (S^+)^\dagger$, $[S^-, S^+] = 1$, $S^+|l\rangle = \sqrt{l+1}|l+1\rangle$ and $S^-|l\rangle = \sqrt{l}|l-1\rangle$, for any $l$. Eqs. (6) are the Holstein-Primakoff non-linear boson realization of $SU(2)$ [6, 7].

The $\sigma$'s in Eqs. (6) still satisfy the su(2) algebra (1). However, Eqs. (5) give for $N \gg l$

$$
\frac{\sigma^\pm}{\sqrt{N}}|l\rangle = S^\pm|l\rangle
$$

By defining $S_3 \equiv \sigma_3$, the su(2) algebra (1) therefore contracts to the (projective) e(2) algebra (or Weyl-Heisenberg algebra) in the large $N$ limit [3, 8, 9]

$$
[S_3, S^\pm] = \pm S^\pm , \\
[S^-, S^+] = 1 ,
$$

the contraction parameter being $\frac{1}{\sqrt{N}}$. Eq. (8) shows that, for large $N$, $S^\pm$ act as the creation and annihilation boson operators. They behave as the Nambu-Goldstone (NG) boson modes and are thus associated to the quanta of collective dipole waves excited by the electric field polarization action. The interaction (2) can be now written in terms of $S^\pm$ as

$$
\mathcal{H} = \hbar\sqrt{N}\gamma(b^\dagger S^- + bS^+) .
$$

In summary, we have reached two main conclusions:

i) in the large $N$ limit the collection of single two-level fermion-like atoms manifests itself as a collective boson system (which is the meaning of the group contraction mechanism $SU(2) \rightarrow E(2)$);
remarkably, the original coupling of the individual atoms to the electric field gets enhanced by the factor $\sqrt{N}$ and manifests itself as the coupling of the collective modes $S^\pm$ to the electric field. The meaning of this is that for large $N$ the system of atoms behaves as a collective whole.

The breakdown of symmetry, with the consequent phenomenon of group contraction, thus provides a change of scale [9], from the microscopic quantum dynamics to the macroscopic quantum system behavior. Moreover, the enhancement of the coupling by the factor $\sqrt{N}$ implies that for large $N$ the collective interaction time scale is much shorter (indeed by the factor $\frac{1}{\sqrt{N}}$) than the short range interactions among the atoms. Hence the mesoscopic/macroscopic stability of the system vs the quantum fluctuations of the microscopic components [4].

Notice that, by assuming finite density, the large $N$ limit implies the large volume limit (the thermodynamic limit). We remark that the system finite size prevents from having a persistent polarization surviving the switching off of the electric field in the $H \rightarrow 0$ limit [10, 11]. In this case, since the system dynamics does not allow an asymmetric ground state (with persistent non-zero polarization order parameter), we have explicit breakdown of symmetry induced by the $\mathcal{H} = -\mathbf{d} \cdot \mathbf{E}$ interaction term. In the limit of switching off the electric field, $H \rightarrow 0$, the dipole rotational symmetry is thus restored. The reader is referred to [6] and to [7, 9] for further analysis of the relation between the group contraction mechanism here discussed and the Holstein-Primakoff nonlinear realization of the $su(2)$ algebra. Here we observe that in the large (infinite) volume limit quantum field theoretical explicit computations show that, as an effect of the breakdown of the $SU(2)$ symmetry induced by a weak external trigger, two boson modes (corresponding to $S^\pm$) are dynamically generated as bound states of elementary fermion components. These boson modes are indeed the NG bosons. The dynamical rearrangement of the symmetry $SU(2) \rightarrow E(2)$ is then shown to be not the linear approximation of the Holstein-Primakoff nonlinear realization of the $su(2)$ algebra, but the exact realization of the symmetry at the observational level under the condition of the breakdown of the $SU(2)$ symmetry [7, 9].

The emergence of the coherence among the fermion-like atoms appears then as the result of the system internal dynamical consistency. The investigation of such an internal dynamical consistency in connection with the theory local gauge invariance is the object of our discussion in the following Sections.

3. The two-level atom model and the initial conditions

We assume there are $N$ atoms per unit volume which are collectively described by the complex dipole wave field $\phi(x,t)$. The system is assumed to be spatially homogeneous and in a thermal bath kept at a non-vanishing temperature $T$. Under such conditions the system is invariant under dipole rotations and since the atom density is assumed to be spatially uniform, the only relevant variables are the angular ones. In our discussion we use natural units $\hbar = 1 = c$ and we closely follow the presentation of ref. [4]. We denote with $d\Omega = \sin \theta d\theta d\phi$ the element of solid angle and with $(r, \theta, \phi)$ the polar coordinates of $r$. The dipole wave field $\phi(x,t)$ integrated over the sphere of unit radius $r$ gives:

$$\int d\Omega |\phi(x,t)|^2 = N . \tag{10}$$

In terms of the rescaled field $\chi(x,t) = \frac{1}{\sqrt{N}} \phi(x,t)$, this reads as

$$\int d\Omega |\chi(x,t)|^2 = 1 . \tag{11}$$

In full generality, under the assumed conditions the field $\chi(x,t)$ may be expanded in the unit sphere in terms of spherical harmonics

$$\chi(x,t) = \sum_{l,m} \alpha_{l,m}(t) Y_l^m(\theta, \phi) , \tag{12}$$
which, by setting $\alpha_{l,m}(t) = 0$ for $l \neq 0$, $1$, reduces to the expansion in the four levels $(l,m) = (0,0)$ and $(1,m), m = 0, \pm 1$. The populations of these levels are given by $N|\alpha_{l,m}(t)|^2$ and at thermal equilibrium, in the absence of interaction, they follow the Boltzmann distribution.

Thermal equilibrium and the dipole rotational invariance imply that there is no preferred direction in the dipole orientation, which means that $|\alpha_{1,m}(t)|$ for any $m$ have in the average the same value, independent of $m$. We write

\[
\begin{align*}
\alpha_{0,0}(t) & \equiv \alpha_0(t) = A_0(t) e^{i\delta_0(t)}, \\
\alpha_{1,m}(t) & \equiv A_1(t) e^{i\delta_{1,m}(t)} e^{-i\omega t} \equiv a_{1,m}(t) e^{-i\omega t},
\end{align*}
\]

where we have used $a_{1,m}(t) \equiv A_1(t) e^{i\delta_{1,m}(t)}$. The amplitudes $A_0(t)$ and $A_1(t)$ and the phases $\delta_0(t)$ and $\delta_{1,m}(t)$ are real quantities; $\omega_0 \equiv \frac{1}{3}$, where $I$ denotes the moment of inertia of the atom, which gives a relevant scale for the system, $\omega_0 \equiv k = \frac{2\pi}{\Delta}$. By denoting with $L^2$ the squared angular momentum operator, the eigenvalue of $L^2$ on the state $(1,m)$ is $\frac{(l+1)}{2} = \frac{1}{7} = \omega_0$.

The fact that the three levels $(1,m), m = 0, \pm 1$ are in the average equally populated under normal conditions is confirmed by the absence of permanent polarization in the system. In the assumed conditions, the time average of the polarization $P_n$ along any direction $n$ is indeed zero. To see this we set the $z$ axis parallel to $n$ and putting $\omega t \equiv \delta_{1,0}(t) - \delta_{0}(t)$, we find

\[
P_n = \int d\Omega \chi^*(\mathbf{x},t)(\mathbf{x} \cdot \mathbf{n}) \chi(\mathbf{x},t) = \frac{2}{\sqrt{3}} A_0(t) A_1(t) \cos(\omega - \omega_0) t,
\]

whose time average is zero, as it should be. Therefore, we can safely write $\sum_m |\alpha_{1,m}(t)|^2 = 3 |a_1(t)|^2$. The normalization condition (11) gives

\[
Q \equiv |\alpha_{0,0}(t)|^2 + \sum_m |\alpha_{1,m}(t)|^2 = |a_0(t)|^2 + 3 |a_1(t)|^2 = 1, \quad \forall t
\]

and therefore $\frac{d}{dt}Q = 0$, i.e.

\[
\frac{\partial}{\partial t}|a_1(t)|^2 = -\frac{1}{3} \frac{\partial}{\partial t}|a_0(t)|^2.
\]

The conservation law $\frac{d}{dt}Q = 0$ expresses the conservation of the total number $N$ of atoms; Eq. (16) means that, due to the rotational invariance, the rate of change of the population in each of the levels $(1,m), m = 0, \pm 1$, equally contributes, in the average, to the rate of change in the population of the level $(0,0)$, at each time $t$.

Consistently with Eq. (15) we can set, in full generality, the initial conditions at $t = 0$ as

\[
|a_0(0)|^2 = \cos^2 \theta_0, \quad |a_1(0)|^2 = \frac{1}{3} \sin^2 \theta_0, \quad 0 < \theta_0 < \frac{\pi}{2}.
\]

The $\theta_0$ values zero and $\frac{\pi}{2}$ are excluded since it is physically unrealistic for the state $(0,0)$ to be completely filled or completely empty, respectively. One can properly tune the parameter $\theta_0$ in its range of definition. For example, $\theta_0 = \frac{\pi}{4}$ describes the equipartition of the field modes of energy $E(k)$ among the four levels $(0,0)$ and $(1,m), |a_0(0)|^2 \approx |a_{1,m}(0)|^2, m = 0, \pm 1$, as given by the Boltzmann distribution when the temperature $T$ is high enough, $k_B T \gg E(k)$. We will find that the lower bound for the parameter $\theta_0$ is imposed by the dynamics in a self-consistent way.
4. The field equations

Let $u_r(k,t) = \frac{1}{\sqrt{N}} c_r(k,t)$, and $c_r(k,t)$ denote the radiative e.m. field operator with polarization $r$; $d$ the magnitude of the electric dipole moment, $\rho \equiv \frac{N}{2}$ and $\epsilon_r$ the polarization vector of the e.m. mode (for which we assume the transversality condition $\mathbf{k} \cdot \epsilon_r = 0$). The field equations for our system are [5, 12]:

$$i \frac{\partial \chi(x,t)}{\partial t} = \frac{L^2}{2\ell} \chi(x,t) - i \sum_{k,r} d \sqrt{\rho} \sqrt{\frac{k}{2}} (\epsilon_r \cdot \mathbf{x}) [u_r(k,t) e^{-ikt}$$

$$- u_r^\dagger(k,t) e^{ikt}] \chi(x,t) ,$$

$$i \frac{\partial u_r(k,t)}{\partial t} = i d \sqrt{\rho} \sqrt{\frac{k}{2}} e^{ikt} \int d\Omega(\epsilon_r \cdot \mathbf{x}) |\chi(x,t)|^2 .$$

Again, notice the enhancement by the factor $\sqrt{N}$ appearing in the coupling $d \sqrt{\rho}$ in Eqs. (18) due to the the rescaling of the fields (cf. with point ii) in Section 2). We have restricted ourselves to the resonant radiative e.m. modes, i.e. those for which $k = \frac{2\pi a_0}{\omega_0} = \omega_0$, and we have used the dipole approximation, i.e. $\exp(i\mathbf{k} \cdot \mathbf{x}) \approx 1$, since we are interested in the macroscopic behavior of the system. This means that the wavelengths of the e.m. modes we consider, of the order of $\frac{2\pi}{\omega_0}$, are larger than (or comparable to) the system linear size.

From Eqs. (18), by using $a_{1,m}(t) = \alpha_{1,m}(t) e^{i\omega_0 t}$, we obtain the set of coupled equations

$$\dot{a}_0(t) = \Omega \sum_m u^*_m(t) a_{1,m}(t)$$

$$\dot{a}_{1,m}(t) = -\Omega u_m(t) a_0(t)$$

$$u_m(t) = 2\Omega a_0^*(t) a_{1,m}(t) ,$$

where $\Omega \equiv \frac{2d}{\sqrt{3} \sqrt{\frac{\rho}{2\omega_0}}} \omega_0 \equiv G \omega_0$ and $u_m$ is the amplitude of the e.m. mode coupled to the transition $(1,m) \leftrightarrow (0,0)$. We define

$$u_m(t) = U(t) e^{i\varphi_m(t)} ,$$

where $U(t)$ and $\varphi_m(t)$ are real quantities. We remark that Eqs. (19)-(21), as well as Eqs. (18), are not invariant under time-dependent phase transformations of the field amplitudes. Our task is to investigate how the local (in time) gauge symmetry can be recovered.

Eqs. (19)-(21) are of course consistent with the conservation law $\frac{\partial}{\partial t} Q = 0$ and we have

$$\frac{\partial}{\partial t} |u_m(t)|^2 = -2 \frac{\partial}{\partial t} |a_{1,m}(t)|^2 ,$$

from which we see that $|u_m(t)|$ does not depend on $m$ since $|\alpha_{1,m}(t)| = |a_{1,m}(t)|$ does not depend on $m$. Also, we have another conservation law, i.e.

$$|u(t)|^2 + 2 |a_1(t)|^2 = \frac{2}{3} \sin^2 \theta_0 , \ \forall \ t$$

where $|u(t)| \equiv |u_m(t)|$, $|a_1(t)| \equiv |a_{1,m}(t)|$, the initial condition (17) has been used and we have set

$$|u(0)|^2 = 0 .$$
By using Eqs. (13) and (22), Eqs. (19)-(21) give
\[
\begin{align*}
\dot{A}_0(t) &= \Omega U(t) A_1(t) \cos \alpha_m(t) , \\
\dot{A}_1(t) &= -\Omega U(t) A_0(t) \cos \alpha_m(t) , \\
\dot{U}(t) &= 2\Omega A_0(t) A_1(t) \cos \alpha_m(t) , \\
\dot{\varphi}_m(t) &= 2\Omega \frac{A_0(t) A_1(t)}{U(t)} \sin \alpha_m(t) ,
\end{align*}
\]  
where the dot over the symbol as usual denotes time derivative and
\[\alpha_m \equiv \delta_{1,m}(t) - \delta_0(t) - \varphi_m(t) .\]  

Equations for \(\delta_{1,m}\) and \(\delta_0\) can be derived in a similar way. Eqs. (26)–(28) show that phases turn out to be independent of \(m\). Indeed, the right hand sides of these equations have to be independent of \(m\) since their left hand sides are independent of \(m\), so either \(\cos \alpha_m(t) = 0\) for any \(m\) at any \(t\), or \(\alpha_m\) is independent of \(m\) at any \(t\). In both cases, Eq. (29) shows that \(\varphi_m\) is then independent of \(m\), which in turn implies, together with Eq. (30), that \(\delta_{1,m}(t)\) is independent of \(m\). We therefore put \(\varphi \equiv \varphi_m\), \(\delta_1(t) \equiv \delta_{1,m}(t), \alpha \equiv \alpha_m, u(t) \equiv u_m(t)\) and \(a_1(t) \equiv a_{1,m}(t)\). In general, one can always change the phases by arbitrary constants. However, if they are equal in one frame they are unequal in a rotated frame and gauge invariance is lost. The independence of \(m\) of the phases is here of dynamical origin and the phase locking which we will find (see Eq. (35)) among \(\delta_0(t), \delta_1(t)\) and \(\varphi(t)\) has indeed the meaning of recovering the gauge symmetry.

The set of Eqs. (19)-(21) now becomes [4, 13]:
\[
\begin{align*}
\dot{a}_0(t) &= 3 \Omega u^*(t) a_1(t) , \\
\dot{a}_1(t) &= -\Omega u(t) a_0(t) , \\
\dot{u}(t) &= 2 \Omega a_0^*(t) a_1(t) .
\end{align*}
\]  

The correct selection rules in radiative and absorption processes [14, 15, 16] are satisfied by these equations. Eq. (31) describes the fact that each of the levels \((1, m)\) may find in the e.m. field the proper mode to couple with, in full respect of the selection rules. The field concept, as a full collection of e.m. modes with all possible polarizations, is crucial here.

5. The system ground states and the phase locking

The study of the system ground states for each of the modes \(a_0(t), a_1(t)\) and \(u(t)\) shows that spontaneous breakdown of the global \(SO(2)\) symmetry (the \(global\) phase symmetry) in the plane \((a_{0,R}(t), a_{0,I}(t))\) occurs [4] (here and in the following the indexes \(R\) and \(I\) denote the real and the imaginary component, respectively, of the field). In the semiclassical approximation [17], we find [4] that for the mode \(a_0(t)\) there is the quasi-periodic mode with pulsatıon \(m_0 = 2\Omega \sqrt{1 + \cos^2 \theta_0}\) (the ‘massive’ mode with real mass \(2\Omega \sqrt{1 + \cos^2 \theta_0}\)) and a zero-frequency mode \(\delta_0(t)\) corresponding to a massless mode playing the rôle of the NG field. It is remarkable that the value \(\alpha_0 = 0\) consistently appears to be the relative maximum for the potential, and therefore an instability point out of which the system (spontaneously) runs away.

We also find [4] that \(a_1(t)\) is a massive field with (real) mass (pulsation) \(\sigma^2 = 2 \Omega^2 (1 + \sin^2 \theta_0)\).

For the \(u(t)\) field, we derive [4] that the \(global\) \(SO(2)\) cylindrical symmetry around an axis orthogonal to the plane \((u_R(t), u_I(t))\) can be spontaneously broken or not, according to the negative or positive value of the squared mass \(\mu^2 = 2\Omega^2 \cos \theta_0\) of the field, respectively, as usual in the semiclassical approximation. In the case, \(\mu^2 < 0\), i.e. \(\theta_0 > \pi/4\), the potential has a relative maximum at \(u_0 = 0\) and a (continuum) set of minima given by
\[
|u(t)|^2 = -\frac{1}{3} \cos \theta_0 = -\frac{\mu^2}{6\Omega^2} = v^2(\theta_0) , \quad \theta_0 > \frac{\pi}{4} .
\]
They represent (infinitely many) possible vacua for the system and they transform into each other under shifts of the field $\varphi$: $\varphi \to \varphi + \alpha$. The global phase symmetry is broken, the order parameter is given by $v(\theta_0) \neq 0$ and one specific ground state is singled out by fixing the value of the $\varphi$ field. We have a ‘massive’ mode, as indeed expected in the AHK mechanism [17], with real mass $\sqrt{2/\mu^2} = 2\Omega \sqrt{\cos 2\theta_0}$ (a quasi-periodic mode) and the zero-frequency mode $\varphi(t)$ (the massless NG collective field, also called the "phason" field [18]). The fact that in such a case $u_0 = 0$ is a maximum for the potential means that the system dynamically evolves away from it, consistently with the similar situation noticed for the $a_0$ mode. We thus find that dynamical consistency requires $\theta_0 > \frac{\pi}{4}$.

Provided $\theta_0 > \frac{\pi}{4}$, a time-independent amplitude $U(t) \equiv U$ is compatible with the system dynamics (e.g. the ground state value of $A_0 \neq 0$ implies $U = \text{const}$). Eqs. (28) and (29) indeed show that such a time-independent amplitude $U = \text{const}$ exists, $U(t) = 0$, if and only if the *phase locking* relation

$$\alpha = \delta_1(t) - \delta_0(t) - \varphi(t) = \frac{\pi}{2} \quad (35)$$

holds. Then we have

$$\dot{\varphi}(t) = \delta_1(t) - \delta_0(t) = \omega \quad (36)$$

and this shows that any change in time of the difference between the phases of the amplitudes $a_1(t)$ and $a_0(t)$ is compensated by the change of the phase of the e.m. field. When Eq. (35) holds we also have $A_0 = 0 = A_1$ (cf. Eqs. (26), (27)). The phase relation (35) shows that, provided $\theta_0 > \frac{\pi}{4}, \dot{\alpha} = 0$. It expresses nothing but the local (in time) gauge invariance of the theory.

Since $\delta_0$ and $\varphi$ are the NG modes, Eqs. (35) and (36) exhibit the coherent feature of the collective dynamical regime. The system of $N$ dipoles and of the e.m. field is characterized by the "in phase" dynamics expressed by Eq. (35) (phase locking): the local gauge invariance of the theory is preserved by the dynamical emergence of the coherence between the matter field and the e.m. field.

Finally, suppose that in the phase locking regime the atom system is under the influence of an electric field $\mathbf{E}$ due, e.g., to an impurity, or to any other external agent. Assume $\mathbf{E}$ to be parallel to the $z$ axis. Then the term $\mathcal{H} = -d \cdot \mathbf{E}$, where $d$ is the electric dipole moment of the atom, will be added to the system energy and will break the dipole rotational symmetry. The polarization $P_n$ is now given by [4]

$$P_n = \frac{1}{\sqrt{3}} (A_0^2 - A_1^2) \sin 2\tau \quad (37)$$

$$\quad + \frac{2}{\sqrt{3}} A_0(t) A_1(t) \cos 2\tau \cos[(\omega - \sqrt{\omega_0^2 + 4H^2})t],$$

to be compared with Eq. (14) and whose time average is non-zero: $\overline{P_n} = \frac{1}{\sqrt{3}} (A_0^2 - A_1^2) \sin 2\tau$.

Here $\tau$ is given [4] by $\tan \tau = \frac{\omega - \sqrt{\omega_0^2 + 4H^2}}{2H}$. The non-zero difference in the level populations $(A_0^2 - A_1^2)$, as it is indeed found in the phase locking regime (see [4]), is therefore crucial in obtaining the non-zero polarization. As shown by Eq. (37), the polarization persists as far as the field $\mathbf{E}$ is active (i.e. $\mathcal{H} \neq 0$). As observed in Section 2, the system finite size prevents from having a persistent polarization surviving the $\mathcal{H} \to 0$ limit [10, 11].

6. Coherence and the vector potential $A_n$

Summarizing, the system may be prepared with initial conditions dictated by the conservation of the particle number and given by Eqs. (17) and (25), where the value of the parameter $\theta_0$ is in principle arbitrary within reasonable physical conditions. Starting at $t = 0$ from the
The gauge arbitrariness of the field $A_\mu$ is meant to compensate exactly the arbitrariness of the phase of the matter field in the covariant derivative $D_\mu = \partial_\mu - igA_\mu$. Should one of the two arbitrariness be removed by the dynamics, the invariance of the theory requires the other arbitrariness, too, must be simultaneously removed. This is the physical meaning of the phase locking. The link between the phase of the matter field and the gauge of $A_\mu$ is stated by the equation $A_\mu = \partial_\mu \varphi$ ($A_\mu$ is a pure gauge field). When $\varphi(x, t)$ is a regular (continuous differentiable) function then it can be easily shown that $E = 0 = B$, namely the potentials and not the fields are present in the coherent region. The existence of non-vanishing fields $E \neq 0$ and $B \neq 0$ is then connected to the topological singularities of the gauge function $\varphi(x, t)$ [10], as it happens, e.g., in the presence of the vortex or other topologically non trivial solution.

It is well known that in the process of spontaneous symmetry breakdown in the presence of a gauge field a crucial rôle is played by the AHK mechanism [20, 21]: the gauge field is expelled out of the ordered domains and confined into ”normal” regions having a vanishing order parameter, i.e. where the long range correlation modes (the NG modes) responsible for the ordering are not present. In agreement with the AHK mechanism we find that in the ordered domains the fields $E$ and $B$ are vanishing; however, the gauge potentials are there nonvanishing, as said above. In the AHK mechanism the gauge field removes the order in the regions where it penetrates (e.g. in the vortex core), here our discussion has revealed the rôle of radiative gauge field in sustaining the phase locking in the coherent regime.

We note again that the collective dynamical features presented above protect the mesoscopic/macroscopic stability of the system vs the quantum fluctuations in the short range dynamics of the microscopic components. The coupling enhancement by the factor $\sqrt{N}$ implies indeed that for large $N$ the collective interaction time scale is much shorter than the short range interaction time-scale among the atoms. For the same reason, for sufficiently large $N$ the collective interaction is protected against thermal fluctuations. Of course, much larger than $k_B T$ is the energy gap, more robust is the protection against thermal fluctuations.

In the discussion above we have not considered energy losses from the system volume. The collective dynamical features are not substantially affected by these losses. They are related with the different lifetimes of our different modes, according to the different time scales associated to the pulsations $m_0$, $\sigma$ and $\mu$. An analysis of energy losses when the system is enclosed in a cavity has been presented in [3] in connection with the problem of efficient cooling of an ensemble of $N$ atoms. Also, we have not considered the problem related to how much time the system demands to set up the collective regime. This problem is a central one in the domain formation in the Kibble-Zurek scenario [22, 23, 24]. In this connection we only remark that in the discussion presented above, since the correlation among the elementary constituents is kept by a pure gauge field, the communication among them travels at the phase velocity of the gauge field [4].
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