HYPERSURFACE COMPLEMENTS, MILNOR FIBERS and MINIMALITY of ARRANGEMENTS

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1. The main results

There is a gradient map associated to any reduced homogeneous polynomial $h \in \mathbb{C}[x_0, \ldots, x_n]$ of degree $d > 0$, namely

$$\text{grad}(h) : D(h) \rightarrow \mathbb{P}^n, \quad (x_0 : \ldots : x_n) \mapsto (h_0(x) : \ldots : h_n(x))$$

where $D(h) = \{x \in \mathbb{P}^n; h(x) \neq 0\}$ is the principal open set associated to $h$ and $h_i = \frac{\partial h}{\partial x_i}$. Our first result is the following topological description of the degree of the gradient map $\text{grad}(h)$.

**Theorem 1.** For any reduced homogeneous polynomial $h \in \mathbb{C}[x_0, \ldots, x_n]$, the complement $D(h)$ is homotopy equivalent to a CW complex obtained from $D(h) \cap H$ by attaching $\deg(\text{grad}(h))$ cells of dimension $n$, where $H$ is a generic hyperplane in $\mathbb{P}^n$. In particular, one has

$$\deg(\text{grad}(h)) = (-1)^n \chi(D(h) \setminus H).$$

Note that the meaning of 'generic' here is quite explicit: the hyperplane $H$ has to be transversal to a stratification of the projective hypersurface $V(h)$ defined by $h = 0$ in $\mathbb{P}^n$.

**Corollary 2.** Let $h^{(i)}$ denote the homogeneous polynomial obtained by restricting $h$ to a generic $i$-codimensional linear subspace in $\mathbb{C}^{n+1}$. Then

$$\chi(D(h)) = \sum_{i=0,n} (-1)^{n-i} \deg(\text{grad}(h^{(i)}))$$

where $\deg(\text{grad}(h^{(n)}) = 1$ by convention.
Using this result and the additivity of the Euler characteristic with respect to constructible partitions, one obtains formulas for the Euler characteristic of any constructible set in terms of an alternating sum of degrees. This result should be compared with results by Szafraniec [Sz], where degrees of real polynomials play a similar role.

Let \( f \in \mathbb{C}[x_0, \ldots, x_n] \) be a homogeneous polynomial of degree \( e > 0 \) with global Milnor fiber \( F = \{ x \in \mathbb{C}^{n+1} | f(x) = 1 \} \), see for instance [D1] for more on such varieties. Let \( g : F \setminus N \to \mathbb{R} \) be the function \( g(x) = h(x)\overline{h}(x) \), where \( N = \{ x \in \mathbb{C}^{n+1} | h(x) = 0 \} \). Then we have the following.

\[ \textbf{Theorem 3.} \quad \text{For any reduced homogeneous polynomial } h \in \mathbb{C}[x_0, \ldots, x_n] \text{ and for any generic polynomial } f \text{ in the space of homogeneous polynomials of degree } e > 0 \text{ one has the following.} \]

(i) the function \( g \) is a Morse function.

(ii) the Milnor fiber \( F \) is homotopy equivalent to a CW complex obtained from \( F \cap N \) by attaching \( |C(g)| \) cells of dimension \( n \), where \( C(g) \) is the critical set of the Morse function \( g \).

We point out that both Theorem 1. and Theorem 3. follow from the results by Hamm in [H]. In the case of Theorem 1. the homotopy type claim is a direct consequence from [H], Theorem 5. and also from Goresky and MacPherson [GM], Theorem 4.1, the new part being the relation between the number of \( n \)-cells and the degree of the gradient map \( \text{grad}(h) \). We establish this equality by using polar curves, see section 2.

On the other hand, in Theorem 3. the main claim is that concerning the homotopy type and this follows from a very general result, see [H], Proposition 3. by a geometric argument described in section 3.

Our results above have interesting implications for the topology of hyperplane arrangements and these implications were our initial motivation in this study. Let \( \mathcal{A} \) be a hyperplane arrangement in the complex projective space \( \mathbb{P}^n \), with \( n > 0 \). Let \( d > 0 \) be the number of hyperplanes in this arrangement and choose a linear equation \( H_i : \ell_i(x) = 0 \) for each hyperplane \( H_i \) in \( \mathcal{A} \), for \( i = 1, \ldots, d \).

Consider the homogeneous polynomial \( Q(x) = \prod_{i=1,d} \ell_i(x) \in \mathbb{C}[x_0, \ldots, x_n] \) and the corresponding principal open set \( M = D(Q) = \mathbb{P}^n \setminus \cup_{i=1,d} H_i \). The topology of the hyperplane arrangement complement \( M \) is a central object of study in the theory of hyperplane arrangements, see Orlik-Terao [OT1]. As a consequence of Theorem 1. we prove the following.
Corollary 4. For any projective arrangement $\mathcal{A}$ as above one has

$$b_n(D(Q)) = \deg(\text{grad}(Q)).$$

In particular, the following are equivalent.

(i) the morphism $\text{grad}(Q)$ is dominant;
(ii) $b_n(D(Q)) > 0$;
(iii) the projective arrangement $\mathcal{A}$ is essential, i.e. the intersection $\cap_{i=1}^d H_i$ is empty.

To obtain Corollary 4. from Theorem 1. all we need is the following.

Lemma 5. For any arrangement $\mathcal{A}$ as above one has $(-1)^n \chi(D(f) \setminus H) = b_n(D(f))$.

This easy lemma has another very interesting consequence. We say that a topological space $Z$ is minimal if $Z$ has the homotopy type of a CW-complex $K$ whose number of $k$-cells equals $b_k(K)$ for all $k \in \mathbb{N}$.

The importance of this notion for the topology of hyperplane arrangements was recently discovered by S. Papadima and A. Suciu, see [PS] for various applications. The following result was independantly obtained by Randell, see [R], using similar techniques.

Corollary 6. The complement $M$ is a minimal space.

It is easy to see that for $n > 1$, the open set $D(f)$ is not minimal for $f$ generic of degree $d > 1$ (just use $H_1(D(f), \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$), but the Milnor fiber $F$ defined by $f$ is clearly minimal. Note that conversely, in spite of Corollary 6., the Milnor fiber $\{Q = 1\}$ associated to an arrangement is not minimal in general.

From Theorem 3. we get a substantial strengthening of some of the main results by Orlik and Terao in [OT2]. Let $\mathcal{A}'$ be the affine hyperplane arrangement in $\mathbb{C}^{n+1}$ associated to the projective arrangement $\mathcal{A}$. Note that $Q(x) = 0$ is a reduced equation for the union $N$ of all the hyperplanes in $\mathcal{A}'$. Let $f \in \mathbb{C}[x_0, ..., x_n]$ be a homogeneous polynomial of degree $e > 0$ with global Milnor fiber $F = \{x \in \mathbb{C}^{n+1} | f(x) = 1\}$ and let $g : F \setminus N \to \mathbb{R}$ be the function $g(x) = Q(x)Q(x)$ associated to the arrangement. The polynomial $f$ is called $\mathcal{A}'$-generic if

(\text{GEN1}) the restriction of $f$ to any intersection $L$ of hyperplanes in $\mathcal{A}'$ is non-degenerate, in the sense that the associated projective hypersurface in $\mathbb{P}(L)$ is smooth, and
(\text{GEN2}) the function $g$ is a Morse function.

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Orlik and Terao have shown in [OT2] that for an essential arrangement $\mathcal{A}'$, the set of $\mathcal{A}'$-generic functions $f$ is dense in the set of homogeneous polynomials of degree $e$, and, as soon as we have an $\mathcal{A}'$-generic function $f$, the following basic properties hold for any arrangement.

(P1) $b_q(F, F \cap N) = 0$ for $q \neq n$ and

(P2) $b_n(F, F \cap N) \leq |C(g)|$, where $C(g)$ is the critical set of the Morse function $g$.

Moreover, for a special class of arrangements called pure arrangements it is shown in [OT2] that (P2) is actually an equality. In fact, the proof of (P2) in [OT2] uses some Morse theory, but we are unable to see the details behind the Corollary (3.5).

With this notation the following is a direct consequence of Theorem 3.

**Corollary 7.**

For any arrangement $\mathcal{A}'$ the following hold.

(i) the set of $\mathcal{A}'$-generic functions $f$ is dense in the set of homogeneous polynomials of degree $e > 0$;

(ii) the Milnor fiber $F$ is homotopy equivalent to a CW complex obtain from $F \cap N$ by attaching $|C(g)|$ cells of dimension $n$, where $C(g)$ is the critical set of the Morse function $g$. In particular $b_n(F, F \cap N) = |C(g)|$.

This paper represents a strengthening of the results in [D2] (in which the homological version of Theorem 1. and 3. above was proven).

The author thanks Stefan Papadima for raising the question answered by Corollary 4 above and for lots of helpful comments. In particular he informed me that Corollary 4 was proved by Paltin Ionescu in the case $n = 2$ by completely different methods. I also thank Pierrette Cassou-Noguès for drawing my attention on Richard Randell’s preprint [R].

2. Polar curves, affine Lefschetz theory and degree of gradient maps

The use of the local polar varieties in the study of singular spaces is already a classical subject, see Lê [Lê], Lê -Teissier [LT] and the references therein. Global polar curves in the study of the topology of polynomials (or, equivalently, the affine Lefschetz theory; for more on this equivalence see the beginning of the proof of Theorem 3.) is a topic under intense investigations, see for instance Cassou-Noguès and Dimca [CD], Hamm [H], Némethi [N1-2], Siersma and Tibăr [ST], [T]. For all the proofs in this paper, the classical (local) theory is sufficient: indeed, all the objects being homogeneous, one can localize at the origin of
in the standard way, see [D1]. However, for the sake of geometric intuition, it seems to us easier to work with global (algebraic) objects, and hence we adopt this viewpoint in the sequel.

We recall briefly the notation and the results from [CD] and [N1-2]. Let \( h \in \mathbb{C}[x_0, ..., x_n] \) be a polynomial (even non-homogeneous to start with) and assume that the fiber \( F_t = h^{-1}(t) \) is smooth and connected, for some fixed \( t \in \mathbb{C} \).

For any hyperplane in \( \mathbb{P}^n \), \( H : \ell = 0 \) where \( \ell(x) = h_0x_0 + h_1x_1 + ... + h_nx_n \), we define the corresponding polar variety \( \Gamma_H \) to be the union of the irreducible components of the variety

\[
\{ x \in \mathbb{C}^{n+1} \mid \text{rank}(dh(x), d\ell(x)) = 1 \}
\]

which are not contained in the critical set \( S(h) = \{ x \in \mathbb{C}^{n+1} \mid dh(x) = 0 \} \) of \( h \).

**Lemma 8.** (see [CD], [ST])

For a generic hyperplane \( H \) we have the following properties.

(i) The polar variety \( \Gamma_H \) is either empty or a curve, i.e. each irreducible component of \( \Gamma_H \) has dimension 1.

(ii) \( \dim(F_t \cap \Gamma_H) \leq 0 \) and the intersection multiplicity \( (F_t, \Gamma_H) \) is independent of \( H \).

(iii) The multiplicity \( (F_t, \Gamma_H) \) is equal to the number of tangent hyperplanes to \( F_t \) parallel to the hyperplane \( H \). For each such tangent hyperplane \( H_\alpha \), the intersection \( F_t \cap H_\alpha \) has precisely one singularity, which is an ordinary double point.

The non-negative integer \( (F_t, \Gamma_H) \) is called the polar invariant of the hypersurface \( F_t \) and is denoted by \( P(F_t) \). Note that \( P(F_t) \) corresponds exactly to the classical notion of class of a projective hypersurface, see [L].

We think of a projective hyperplane \( H \) as above as the direction of an affine hyperplane \( H' = \{ x \in \mathbb{C}^{n+1} \mid \ell(x) = s \} \) for \( s \in \mathbb{C} \). All the hyperplanes with the same direction form a pencil, and it is precisely the pencils of this type that are used in the affine Lefschetz theory, see [N1-2]. One of the main results in [CD] is the following, see also [ST] or [T] for similar results.

**Proposition 9.**

For a generic hyperplane \( H' \) in the pencil of all hyperplanes in \( \mathbb{C}^{n+1} \) with a fixed generic direction \( H \), the fiber \( F_t \) is homotopy equivalent to a CW-complex obtained from the section \( F_t \cap H' \) by attaching \( P(F_t) \) cells of dimension \( n \). In particular

\[
P(F_t) = (-1)^n(\chi(F_t) - \chi(F_t \cap H')) = (-1)^n\chi(F_t \setminus H').
\]
Moreover in this statement 'generic' means that the hyperplane $H'$ has to verify the following two conditions.

(g1) its direction, which is the hyperplane in $\mathbb{P}^n$ given by the homogeneous part of degree one in an equation for $H'$ has to be generic, and

(g2) the intersection $F_t \cap H'$ has to be smooth.

These two conditions are not stated in [CD], but the reader should have no problem in checking them by using Theorem 3′ in [CD] and the fact proved by Némethi in [N1-2] that the only bad sections in a good (i.e. the analog of a Lefschetz pencil in the projective Lefschetz theory, see [L]) pencil are the singular sections. Completely similar results hold for generic pencils with respect to a closed smooth subvariety $Y$ in some affine space $\mathbb{C}^N$, see [N1-2], but note that the polar curves are not mentioned there.

**Proof of Theorem 1.**

In view of Hamm’s affine Lefschetz theory, see [H], Theorem 5. and also from Goresky and MacPherson [GM], Theorem 4.1, the only thing to prove is the equality between the number $k_n$ of $n$-cells attached and the degree of the gradient.

Assume from now on that the polynomial $h$ is homogeneous of degree $d$ and that $t = 1$. It follows from (g1) and (g2) above that we may choose the generic hyperplane $H'$ passing through the origin.

Moreover, in this case, the polar curve $\Gamma_H$, being defined by homogeneous equations, is a union of lines $L_j$ passing through the origin. For each such line we choose a parametrization $t \mapsto a_j t$ for some $a_j \in \mathbb{C}^{n+1}, a_j \neq 0$. It is easy to see that the intersection $F_1 \cap L_j$ is either empty (if $h(a_j) = 0$) or consists of exactly $d$ distinct points with multiplicity one (if $h(a_j) \neq 0$). The lines of the second type are in bijection with the points in $\text{grad}(h)^{-1}(D_{H'})$, where $D_{H'} \in \mathbb{P}^n$ is the point corresponding to the direction of the hyperplane $H'$. It follows that

$$d \cdot \text{deg(\text{grad}(h))} = P(F_1).$$

The $d$-sheeted unramified coverings $F_1 \to D(h)$ and $F_1 \cap H' \to D(h) \cap H$ give the result, where $H$ is the projective hyperplane corresponding to the affine hyperplane (passing through the origin) $H'$. Indeed, they imply the equalities: $\chi(F_1) = d \cdot \chi(D(h))$ and $\chi(F_1 \cap H') = d \cdot \chi(D(h) \cap H)$. Hence we have $\text{deg(\text{grad}(h))} = \chi(F_1, F_1 \cap H')/d = \chi(D(h), D(h) \cap H) = k_n$.

**Remark 10.** The gradient map $\text{grad}(h)$ has a natural extension to the larger open set $D'(h)$ where at least one of the partial derivatives of $h$ does not vanish. It is obvious (by a dimension argument) that this extension has the same degree as the map $\text{grad}(h)$.
3. Non-proper Morse Theory

For the convenience of the reader we recall, in the special case we need, a basic result of Hamm, see [H], Proposition 3, with our addition concerning the condition (c0) in [DP], see Lemma 3. and Example 2. The final claim on the number of cells to be attached is also standard, see for instance [ST] and [T].

Proposition 11.

Let \( A \) be a smooth algebraic subvariety in \( \mathbb{C}^p \) with \( \dim A = m \). Let \( f_1, \ldots, f_p \) be polynomials in \( \mathbb{C}[x_1, \ldots, x_p] \). For \( 1 \leq j \leq p \), denote by \( \Sigma_j \) the set of critical points of the mapping \( (f_1, \ldots, f_j) : A \setminus \{ z \in A; f_1(z) = 0 \} \rightarrow \mathbb{C}^j \) and let \( \Sigma_j' \) denote the closure of \( \Sigma_j \) in \( A \). Assume that the following conditions hold.

(c0) The set \( \{ z \in A; |f_1(z)| \leq a_1, \ldots, |f_p(z)| \leq a_p \} \) is compact for any positive numbers \( a_j, j = 1, \ldots, p \).

(c1) The critical set \( \Sigma_1 \) is finite.

(cj) (for \( j = 2, \ldots, p \)) The map \( (f_1, \ldots, f_{j-1}) : \Sigma_j' \rightarrow \mathbb{C}^{j-1} \) is proper.

Then \( A \) has the homotopy type of a space obtained from \( A_1 = \{ x \in A; f_1(x) = 0 \} \) by attaching \( m \)-cells and the number of these cells is the sum of the Milnor numbers \( \mu(f_1, x) \) for \( x \in \Sigma_1 \).

Proof of Theorem 3.

We set \( X = h^{-1}(1) \). Let \( v : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^N \) be the Veronese mapping of degree \( e \) sending \( x \) to all the monomials of degree \( e \) in \( x \) and set \( Y = v(X) \). Then \( Y \) is a smooth closed subvariety in \( \mathbb{C}^N \) and \( v : X \rightarrow Y \) is an unramified (even Galois) covering of degree \( c \), where \( c = \gcd(d, e) \). To see this, use the fact that \( v \) is a closed immersion on \( \mathbb{C}^N \setminus \{0\} \) and \( v(x) = v(x') \) iff \( x' = u \cdot x \) with \( u^e = 1 \).

Let \( H \) be a generic hyperplane direction in \( \mathbb{C}^N \) with respect to the subvariety \( Y \) and let \( C(H) \) be the finite set of all the points \( p \in Y \) such that there is an affine hyperplane \( H'_p \) in the pencil determined by \( H \) that is tangent to \( Y \) at the point \( p \) and the intersection \( Y \cap H'_p \) has a complex Morse (alias non-degenerated, alias \( A_1 \)) singularity. Under the Veronese mapping \( v \), the generic hyperplane direction \( H \) corresponds to a homogeneous polynomial of degree \( e \) which we call from now on \( f \).

To prove the first claim (i) we proceed as follows. It is known that doing affine Lefschetz theory for a pencil of hypersurfaces \( \{ h = t \} \) is equivalent to doing (non-proper) Morse theory for the function \( |h| \) or, what amounts to the same, for the function \( |h|^2 \). More explicitly, in view of the last statement at the end of the proof of Lemma (2.5) in
In this way we obtain an induced Whitney stratification $S_C(V)$. Assume to be a smooth point on this fiber. Whitney stratification $S_{\phi}$ dimension $q$ a point $S_{\phi}$. Note that the requirement of $f$ proper in that Corollary is not necessary in our case, as any algebraic map can be compactified). Here and in the sequel, for a map $\phi : \mathcal{X} \to \mathcal{Y}$ and a point $q \in \mathcal{X}$ we denote by $T_q(\phi)$ the tangent space to the fiber $\phi^{-1}(\phi(q))$ at the point $q$, assumed to be a smooth point on this fiber.

Since in our case $h$ is a homogeneous polynomial, we can find a stratification $S$ as above such that all of its strata are $\mathbb{C}^*$-invariant, with respect to the natural $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$. In this way we obtain an induced Whitney stratification $S'$ on the projective hypersurface $V(h)$. We choose our polynomial $f$ such that the corresponding projective hypersurface $V(f)$ is smooth and transversal to the stratification $S'$. In this way we get an induced Whitney stratification $S'_1$ on the projective complete intersection $V_1 = V(h) \cap V(f)$.

We use Proposition 11. above with $A = F$ and $f_1 = h$. All we have to show is the existence of polynomials $f_2, ..., f_{n+1}$ satisfying the conditions listed in Proposition 11.

We will choose these polynomials inductively to be generic linear forms as follows. We choose $f_2$ such that the corresponding hyperplane $H_2$ is transversal to the stratification $S'_1$. Let $S'_2$ denote the induced stratification on $V_2 = V_1 \cap H_2$. Assume that we have constructed $f_2, ..., f_{j-1}$, $S'_1, ..., S'_{j-1}$ and $V_1, ..., V_{j-1}$. We choose $f_j$ such that the corresponding hyperplane $H_j$ is transversal to the stratification $S'_{j-1}$. Let $S'_j$ denote the induced stratification on $V_j = V_{j-1} \cap H_j$. Do this for $j = 3, ..., n$ and choose for $f_{n+1}$ any linear form.

With this choice it is clear that for $1 \leq j \leq n$, $V_j$ is a complete intersection of dimension $n - 1 - j$. In particular, $V_n = \emptyset$, i.e.

$$(c0') \quad \{x \in \mathbb{C}^{n+1}; f(x) = h(x) = f_2(x) = ... = f_n(x) = 0\} = \{0\}.$$  

Then the map $(f, h, f_2, ..., f_n) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is proper, which clearly implies the condition $(c0)$.

The condition $(c1)$ is fulfilled by our construction of $f$. Assume that we have already
checked that the conditions \((ck)\) are fulfilled for \(k = 1,\ldots, j - 1\). We explain now why the
next condition \((cj)\) is fulfilled.

Assume that the condition \((cj)\) fails. This is equivalent to the existence of a sequence
\(p_m\) of points in \(\Sigma'_j\) such that
\[
(*) \quad |p_m| \to \infty \text{ and } f_k(p_m) \to b_k \text{ (finite limits) for } 1 \leq k \leq j - 1.
\]
Since \(\Sigma_j\) is dense in \(\Sigma'_j\), we can even assume that \(p_m \in \Sigma_j\).

Note that \(\Sigma_{j-1} \subset \Sigma_j\) and the condition \(c(j-1)\) is fulfilled. This implies that we may
choose our sequence \(p_m\) in the difference \(\Sigma_j \setminus \Sigma_{j-1}\). In this case we get
\[
(**) \quad f_j \in \text{Span}(df(p_m), dh(p_m), f_2, \ldots, f_{j-1})
\]
the latter being a \(j\)-dimensional vector space.

Let \(q_m = \frac{p_m}{|p_m|} \in S^{2n+1}\). Since the sphere \(S^{2n+1}\) is compact we can assume that the
sequence \(q_m\) converges to a limit point \(q\). By passing to the limit in \((*)\) we get \(q \in V_{j-1}\).
Moreover, we can assume (by passing to a subsequence) that the sequence of \((n-j+1)\)-planes
\(T_{q_m}(h, f, f_2, \ldots, f_{j-1})\) has a limit \(T\). Since \(p_m \notin \Sigma_{j-1}\), we have
\[
T_{q_m}(h, f, f_2, \ldots, f_{j-1}) = T_{q_m}(h) \cap T_{q_m}(f) \cap H_2 \cap \ldots \cap H_{j-1}
\]
As above, we can assume that the sequence \(T_{q_m}(h)\) has a limit \(T_1\) and, using the \(a_h\)-
condition for the stratification \(S\) we get \(T_q S_i \subset T_1\) if \(q \in S_i\). Note that we have \(T_{q_m}(f) \to T_q(f)\) and hence \(T = T_1 \cap T_q(f) \cap H_2 \cap \ldots \cap H_{j-1}\). It follows that
\[
T_q S_{i,j-1} = T_q S_i \cap T_q(f) \cap H_2 \cap \ldots \cap H_{j-1} \subset T
\]
where \(S_{i,j-1} = S_i \cap V(f) \cap H_2 \cap \ldots \cap H_{j-1}\) is the stratum corresponding to the stratum \(S_i\)
in the stratification \(S'_{j-1}\). On the other hand, the condition \((**)\) implies that \(T_q S_{i,j-1} \subset T \subset H_j\), a contradiction to the fact that \(H_j\) is transversal to \(S'_{j-1}\).

4. Complements of hyperplane arrangements

**Proof of Lemma 5.**
Here we just give the main idea, since the details are standard. One has to use the
method of deletion and restriction, see [OT1], p. 17, the obvious additivity of the Euler
characteristics and, more subtly, the additivity of the top Betti numbers coming from the
exact sequence (8) in [OT1], p. 20 or (3.8) in [DL].
Proof of Corollary 4.

To complete this proof we only have to explain why the claims (ii) and (iii) are equivalent. If the projective arrangement is not essential, then using a projection onto $\mathbb{P}^{n-1}$ with center a point in all the hyperplanes $H_i$ we get a fiber bundle $D(Q) \to U$ with fiber $\mathbb{C}$ and base $U$, an affine variety of dimension $n - 1$. This implies $b_n(D(Q)) = 0$.

If the arrangement is essential, then $d \geq n + 1$ and we may assume that $\ell_i(x) = x_{i-1}$ for $i = 1, ..., n + 1$. In the case $d = n + 1$, we are done, since in this case $D(Q) = (\mathbb{C}^*)^n$ and hence $b_n(D(Q)) = 1$. In the remaining case $d > n + 1$, one should use the additivity of the top Betti numbers alluded above in the proof of Lemma 5.

Proof of Corollary 6.

Using the Affine Lefschetz Theorem of Hamm, see Theorem 5 in [H], we know that for a generic projective hyperplane $H$, the space $M$ has the homotopy type of a space obtained from $M \cap H$ by attaching $n$-cells. The number of these cells is given by

$$(-1)^n \chi(M, M \cap H) = (-1)^n \chi(M \setminus H) = b_n(M)$$

see Lemma 4. above.

To finish the proof of the minimality of $M$ we proceed by induction using the equalities

$$b_k(M) = b_k(M \cap H)$$

for $0 \leq k < n$. Indeed, for $0 \leq k < n - 1$, this is obvious since we attach only $n$-cells. The equality for $k = n - 1$ follows from these previous equalities and a new application of Lemma 5.

Remark 12.

Let $\mu_e$ be the cyclic group of the $e$-roots of unity. Then there is a natural algebraic action of $\mu_e$ on the space $F \setminus N$ occurring in Theorem 1'. The corresponding weight equivariant Euler polynomial (see [DL] for a definition) gives information on the relation between the induced $\mu_e$-action on the cohomology $H^*(F \setminus N)$ and the functorial Deligne mixed Hodge structure present on cohomology.

When $N$ is a hyperplane arrangement $\mathcal{A}'$ and $f$ is an $\mathcal{A}'$-generic function, this weight equivariant Euler polynomial can be combinatorically computed from the lattice associated to the arrangement (see Corollary (2.3) and Remark (2.7) in [DL]) using the fact that the weight equivariant Euler polynomial of the $\mu_e$-variety $F$ is known, see for instance [D1].
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