Statistical mechanics for dilatations in $\mathcal{N} = 4$ super Yang–Mills theory

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Abstract

Matrix model describing the anomalous dimensions of composite operators in $\mathcal{N} = 4$ super Yang–Mills theory up to one-loop level is considered at finite temperature. We compute the thermal effective action for this model, which we define as the log of the partition function restricted to the states of given fixed length and spin. The result is obtained in the limit of high as well as low temperature.

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1 Introduction

The gauge/string correspondence has a long history starting from the ’t Hooft paper [1], showing that the perturbative expansion of a gauge theory can be organized according to the geometrical genus expansion for the Feynman diagrams. The idea behind this is that string theory may serve as an effective description for a strongly interacting gauge system. The physical picture motivating such a description is given by condensation of the gauge field flux into tiny tubes leading to a linearly growing potential between quarks/antiquarks and string-like behavior.

The development of string theory lead to the conjecture of AdS/CFT correspondence [2] (see the classical review on the subject [3]). According to this conjecture the string theory can be described in terms of gauge fields which are nothing else that the collective coordinates of $D$-branes, the non-perturbative objects on which the fundamental strings can end. Therefore,
the gauge field description of strings makes sense in the limit of strong string interactions.

In this picture the rank $N$ of the gauge group corresponds to the number of $D$-branes. In the 't Hooft limit of large $N$ and small gauge coupling $g_{\text{YM}}$ such that $g_{\text{YM}}\sqrt{N}$ remains finite, the above duality relation is reduced to the correspondence between non-interacting IIB superstrings on $\text{AdS}_5 \times S^5$ background and $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM) on the Minkowski space which is the conformal boundary to the anti-de Sitter space.

An important feature of the AdS/CFT correspondence is that it is a true i.e. Ising-type duality which means that the strong coupling dynamics is mapped to a weak coupling one and viceversa (see [4] for a review of these type of dualities in lattice theories and spin systems). This property of AdS/CFT correspondence, beyond having a huge predictive force, also prevents from any direct proof of the correspondence itself, since for such a proof one should solve at least one of the models at strong coupling.

Last years a considerable progress was registered in the study of both string and gauge theories in the way of approaching the AdS/CFT conjecture. Among this an important point is the discovery of the integrability (see e.g. [5–8]).

It was a breakthrough to realize that there are certain limits in which both string theory and SYM are reachable in the framework of perturbation theory. On the string theory side these limits are special geometrical limits while on the gauge theory side they correspond to particular subclasses of SYM composite operators for which one can extend the applicability of the perturbation theory [9–11].

All this time the analysis was performed at infinite $N$, which in particular allows one to use the integrability [7, 8, 12]. Since $1/N$ corrections are expected to break explicitly the integrability, during this development they got much less attention.

In [13–18], however, it was shown that one can map the anomalous dimension operator on $\mathcal{N} = 4$ SYM into an interacting spin system taking into account the finite $N$ rank of the gauge group. The finiteness of $N$, results in the chain splitting and joining interaction. This process mimics the string interaction.

There is, however, still another approach to the same problem. It consists in the interpretation of the operator of anomalous dimensions as the Hamiltonian of a gauged matrix model [19–21]. Spin chains in this picture

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2To be precise, the above approach neglects the trace identities for $U(N)$ matrices. This means that only polynomial $1/N$ contribution is taken into account and not the exponential corrections. This is very similar to the Quantum Field Theory, where non-analytic corrections are systematically dropped in perturbation theory.
correspond to gauge invariant collective states. The problem with such a
description is that there are not enough developed tools for the study of such
type of matrix models independently of the spin chain results.

One possibility is to describe the model in terms of a noncommutative
gauge theory by expanding the matrix fluctuations about an appropriate
classical solution [19]. This approach however, has its own drawbacks since
it leaves a number of unanswered questions regarding the arbitrariness of
such a description as well as regarding the spin chain/string interpretation.
In particular, in the noncommutative description the matrix model of it is
not very clear what is e.g. a spin chain.

The present paper aims to fill, at least partially, the gaps in the under-
standing of the matrix model for the anomalous dimensions of
$\mathcal{N} = 4$ SYM.

Here we consider statistically large but finite values of $N$. This means that
we do not exclude the string interactions from our description. According
to BMN analysis [9], the string interaction is rated as $L^2/N$, where $L$ is the
aggregate length of all spin chains. This interaction becomes important when
one considers long enough operators.

In our approach we find it convenient to put the matrix theory in a finit-
temperature background. In the Yang–Mills theory this temperature corre-
ponds to the Euclidean compactified Liouville or scale direction. According
to AdS/CFT correspondence this is a compact time cycle in the AdS space.
As it was pointed in [22], the thermalization of AdS space occurs due to the
presence of black holes.

The $\mathcal{N} = 4$ SYM theory is a conformal invariant theory, therefore its
dilatation operator can be identified with the Hamiltonian dual to the ra-
dial time. Since the dilatation operator and the SYM Hamilton are related
through a similarity transformation, the spectrum and therefore, the ther-
modynamical properties of the matrix model should be the same as those of
the thermal Yang–Mills which were considered in [23–25]. There is a differ-
ence, however. In the contrast to the above cited works where the thermal
Yang–Mills was considered on $S^1 \times S^3$, the radial time temperature should
correspond rather to SYM compactified on $S^4$.

On the other hand, a similar partition function was considered in [29–31].
In particular, in [31], the thermal partition function of the gauged oscillator
is derived by counting the trace states using the Pólya Enumeration Theorem
(PET). The problem of such approach is that, strictly speaking, it requires
$N$ to be infinite. Because of this the analysis can not be extended beyond
the Hagedorn temperature. In contrast to this approach we compute the
path integral for the matrix model rather then just counting the states. The

\footnote{The SU(2) sector can be regarded as a particular limit of $\mathcal{N} = 4$ SYM [26–28].}
advantage of our approach is that it allows a conceptually simple (although technically evolved) extension to higher loops in the spirit of “conventional” perturbation theory. It appears that the description we obtain is in some sense complimentary to the one of the PET approach: for very long operators we get a $N^2$ scaling of the extensive thermodynamical functions, which signals for the string bit phase, while for the operators below critical length the thermodynamical potentials scale as $N^0$, which is compatible with $N$-independent description obtained in the PET approach.

The plan of the paper is as follows. In the next section we review some basic properties of the matrix model: we rewrite it in the real form in terms of Hermitian matrices and pass to the “second order formalism”. The second order formalism is important for the comparison the matrix model under study with the DLCQ-inspired BMN matrix model [9]. The comparison reveals that in our matrix model there are additional terms which seem unlikely to be canceled by the higher loop correction or to be generated from higher orders in the BMN matrix model.

Next, we describe the canonical quantization and conserved charges of the model. In the third section we analyze the spin chains as collective states of the matrix model. In particular, we analyze the regimes at which the matrix model describes the gas of free spin chains. In section 3 we compute the free energy and the entropy (which we will like to call ‘effective action’) in the high temperature limit. During the computation we observe a couple of rather remarkable features of the model. One feature is that the matrix model substantially simplifies in the mode expansion which allows one to solve the model exactly at large $N$ analytically in exponentials of the chemical potentials.

We analyze the situation in both cases of small and large chemical potentials and find out that for large enough value of the potentials (above the Hagedorn one) all terms scaling as $N^2$ and $N$ cancel out leaving us with the next leading contribution.

On the opposite side, when the chemical potentials are small the model behaves as a system of $N^2$ interacting particles. In section 4 we consider small temperature limit. In this limit the typical behavior is one of a $N$-particle system. Finally we give the string theory interpretation and discuss the results.

\[4\] An approach based on random walks allows one to find this contribution too [32].
2 Matrix model

2.1 Dilatation operator and anomalous dimensions of \( N = 4 \) SYM

According to the AdS/CFT conjecture, the string states correspond to composite operators of the gauge theory. Furthermore, identification of the respective charges in the symmetry groups on both sides puts into correspondence the string energy levels to the eigenvalues of the dilatation operator of \( N = 4 \) SYM. Classically, the value of the dilatation operator is given by the composition of the dimensions of its elementary components. In the quantum theory this is corrected by the *anomalous dimension* contribution. It also appears that the dilatation mixes different operators, which form, however, invariant classes which do not mix. A particular example which we analyze in this paper is given by the SU(2)-sector of SYM, which consists of all gauge invariant polynomial operators built from two complex scalars,

\[
\Phi_1 \equiv Z = \frac{1}{2}(\phi_5 + i\phi_6), \\
\Phi_2 \equiv \phi = \frac{1}{2}(\phi_1 + i\phi_2),
\]

where \( \phi_i, i = 1, 2, \ldots, 6 \) are the scalars of the \( N = 4 \) SYM.

The mixing matrix/dilatation operator in the context of the perturbation theory was considered in [8, 12]. In particular, the one-loop contribution to the mixing matrix for operators in SU(2) sector was found to be [33],

\[
H = -\frac{g_{YM}^2}{16\pi^2} : \text{tr}[\Phi^a, \Phi^b][\bar{\Phi}_a, \bar{\Phi}_b] : \equiv -\frac{g_{YM}^2}{8\pi^2} : \text{tr}[\phi, Z][\bar{\phi}, \bar{Z}] :,
\]

where the checked character corresponds to the differential operator,

\[
(\bar{\Phi}_a)_{mn} = \frac{\partial}{\partial \Phi^a_{mn}},
\]

and the colon "::" denotes such an ordering in which no derivative acts on the operators within the same group.

2.2 The matrix action

Consider the matrix model describing the anomalous dimensions of \( N = 4 \) super Yang–Mills model. As it was found in [19] this matrix model is
described by the action\textsuperscript{5}

\[ S(\Psi, \bar{\Psi}, A) = \int dt \left( \frac{1}{2} \text{tr} \left[ i(\bar{\Psi}_a \nabla_0 \Psi^a - \nabla_0 \bar{\Psi}_a \Psi^a) \right] - \text{tr} \bar{\Psi}_a \Psi_a + \frac{g^2_{YM}}{16\pi^2} \text{tr}[\bar{\Psi}_a, \bar{\Psi}_b][\Psi^a, \Psi^b] \right) \]  

(2.5)

where\textsuperscript{6} \( \Psi_a, a = 1, 2 \) are (non-Hermitian) \( N \times N \) matrices, and \( \nabla_0 \Psi = \partial_0 \Psi + [A_0, \Psi] \), with Hermitian matrix \( A_0 \), is the covariant time derivative.

### 2.3 Real form

To compare our model with other matrix models arising in the context of AdS/CFT correspondence it is convenient to pass from the variables \( s \Psi \) and \( \bar{s} \bar{\Psi} \) to the Hermitian ones \( X \) and \( Y \) representing the Hermitian and the anti-Hermitian parts of \( \Psi \) as follows,

\[ \Psi_a = \frac{1}{\sqrt{2}}(P_a + iX_a), \quad \bar{\Psi}_a = \frac{1}{\sqrt{2}}(P_a - iX_a). \]  

(2.6)

Substitution of (2.6) into the action (2.5) yields,

\[ S(P, X, A) = \int dt \left( \frac{1}{2} \text{tr} \left[ P_a(\nabla_0 X_a) - (\nabla_0 P_a)X_a \right] - \frac{1}{4}(P_a^2 + X_a^2) \right. \]

\[ \left. + \frac{g^2_{YM}}{64\pi^2} \text{tr} \left\{ [P_a, P_b]^2 + [X_a, X_b]^2 + [P_a, X_b]^2 \right\} \right), \]  

(2.7)

where we used the constraint \( [P_a, X_a] = 0 \) and dropped off the total time derivative term \( (i/4)\frac{d}{dt}(P_a^2 + X_a^2) \). The action (2.7) can be written in a more economical way using a common variable \( X_A \) of extended dimensionality \( A = 1, \ldots, 4 \),

\[ X_A = (P_a, X_b). \]  

(2.8)

In this case the action (2.7) can be rewritten as

\[ S(X, A) = \int dt \left( \frac{1}{2}\omega_{AB} \text{tr} X_A \nabla_0 X_B - \frac{1}{2} \text{tr} X_A^2 + \frac{g^2_{YM}}{64\pi^2} \text{tr}[X_A, X_B]^2 \right), \]  

(2.9)

\textsuperscript{5}Note, however, the difference in the notations.

\textsuperscript{6}In [19] these fields are denoted by \( X \) and \( \bar{X} \). Here, for convenience, we replace them by \( \Psi \) and \( \bar{\Psi} \), respectively. Also we introduce the term \( \text{tr} \bar{\Psi} \Psi \) which describes the classical dimension.
where the anti-symmetric matrix $\omega_{AB}$ is given in terms of $(X, Y)$ decomposition of $X_A$ by the following block structure

$$\|\omega_{AB}\| = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (2.10)$$

Let us note that the form $(2.9)$ of the action is very close to the one of the “classical” Yang–Mills type matrix models. The only difference is the mass term and the first order kinetic term.

### 2.4 “Second order” formalism

The existend matrix models related to non-perturbative string dynamics are second order acrtion i.e. their kinetic terms have the form of velocity-squared. Our model is described by first order action i.e. with at most linear velocity dependence. On the other hand the matrix variables are non-Hermiian matrices, from which we can deduce that the action $(2.5)$ the “first order form” of higher order action. Indeed, the example of ordinary harmonic oscillator with the classical action

$$S = \int dt \left( \frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2 \right), \quad (2.11)$$

can be equivalently rewritten in the canonical form

$$S = \int dt \left( p \dot{x} - \frac{1}{2} (p^2 + x^2) \right), \quad (2.12)$$

by the Legendre thransform with canonical momentum $p = \partial \mathcal{L} / \partial \dot{x}$. Further one can pass to the complex coordinate $a$ which is the classical counterpart of the oscillator annihilation operator

$$a = \frac{1}{\sqrt{2}} (p + ix), \quad \bar{a} = \frac{1}{\sqrt{2}} (p - ix). \quad (2.13)$$

In the last coordinates the action takes the form

$$S = \int dt \left( \frac{1}{2} (\dot{a} \bar{a} - \dot{\bar{a}} a) - \bar{a} a \right) + \text{boundary terms}. \quad (2.14)$$

If we replace $a$ with the matrix valued $\Psi$ and take the trace we see that the quadratic part of the action $(2.5)$ is nothing else that the harmonic matrix oscillator in the complex first order form, while $(2.7)$ gives the ordinary first order formalism. Now, let us try to reconstruct the corresponding “second order” action. Classically, one can switch back to the Lagrangian formalism
just by solving equations of motion with respect to $p$ in terms of $x$ and $\dot{x}$ and substituting $p$ in the action by this solution.

The equations of motions resulting from the variation of $P_a$ in (2.7) read

$$P_a = \dot{X}_a - \frac{g^2_{YM}}{16\pi^2} \left( [P_b, [P_b, P_a]] + [X_b, [X_b, P_a]] \right).$$  \hspace{1cm} (2.15)$$

We can not solve this equation exactly, but we do not need the exact solution either. Recall that the action (2.5) is only the order $g^2_{YM}$ approximation of the “exact” matrix model describing the anomalous dimensions to all orders in perturbation theory as well as nonperturbative effects. So, for us it suffices to solve the equation (2.15) to only the first order in $g^2_{YM}$. This solution is given by,

$$P_a = \dot{X}_a - \frac{g^2_{YM}}{16\pi^2} \left( [\dot{X}_b, [\dot{X}_b, \dot{X}_a]] + [X_b, [X_b, \dot{X}_a]] \right).$$  \hspace{1cm} (2.16)$$

Then, the resulting “second order formalism” action takes the following form (up to the order $g^2_{YM}$),

$$S = \int dt \ tr \left( \frac{1}{2}(\dot{X}^2 - X^2) + \frac{g^2_{YM}}{64\pi^2} \left( [X_a, X_b]^2 + 2[\dot{X}_a, X_b] + [\dot{X}_a, \dot{X}_b]^2 \right) \right).$$  \hspace{1cm} (2.17)$$

The Gauss law constraint at the same time takes the following form,

$$G = [\dot{X}_a, X_a] + \frac{g^2_{YM}}{16\pi^2} \left( [\dot{X}_b, [\dot{X}_b, \dot{X}_a]] + [X_b, [X_b, \dot{X}_a]] \right).$$  \hspace{1cm} (2.18)$$

Let us note the fact, that for slow varying matrices: $\dot{X} \to 0$, the matrix action (2.17) is equivalent to the matrix model of [34]. This is not surprising since the respective matrix model reproduce the BMN spectrum, while our model reproduces the leading order in $\lambda$ of this spectrum. The surprising part is that beyond the ordinary commutator term our action contains also commutator of velocities. One can see that such terms can not be canceled by higher loop contribution. It would be interesting to see whether these terms can be removed by a proper redefinition of the fields.

2.5 Quantization, gauge invariance

To quantize the model given by the action (2.5) we have to impose first the gauge fixing condition. The most natural condition in our case is the temporal gauge $A = 0$.

\footnote{In quantum theory in addition to this one should also take care on modification of the measure (or of the scalar product).}

\footnote{We use the gauge $A = 0$.}
As it can be seen, the variation of the action with the respect to the gauge field $A$ fails to produce a *bona fide* equations of motion, one has instead the Gauss law constraint,

$$G = [\Psi_a, \bar{\Psi}_a] = \frac{1}{2} \omega_{AB}[X^A, X^B] \approx 0. \quad (2.19)$$

From the simplectic structure of the classical action (2.5) one can extract the canonical Poisson bracket of the model. In components it reads:

$$\{(\bar{\Psi}_a)_i^j, (\Psi_b)_k^l\} = -i \delta_i^l \delta_k^j. \quad (2.20)$$

This can be written in the componentless form using the "quantum group" notations

$$\{\bar{\Psi}_a^{(1)}, \Psi_b^{(2)}\} = -i P_{12}, \quad (2.21)$$

where the $N^2 \times N^2$-dimensional matrices $\Psi_a^{(1,2)}$ and $P_{12}$ are defined

$$\Psi_a^{(1)} = \Psi_a \otimes I, \quad \Psi_a^{(1)} = I \otimes \Psi_a, \quad (2.22)$$

while $P_{12}$ is the permutation operator

$$P_{12} a \otimes b = b \otimes a. \quad (2.23)$$

The canonical quantization consist in the promoting of the Poisson bracket in either form (2.20) or (2.21) to the quantum commutator for which

$$[(\bar{\Psi}_a)_i^j, (\Psi_b)_k^l] = \delta_i^l \delta_k^j. \quad (2.24)$$

The interpretation of the quantum algebra (2.24) is the following. The operator $(\bar{\Psi}_a)_i^j$ creates a matrix element in the row $i$ and column $j$ having the polarization $a$ while the respective component of $(\Psi_a)_j$ destroys it. Let us note that the quantum commutator is denoted by the double braces in order to distinguish it from the ordinary matrix one which is related to different contractions of the matrix indices $i, j, \ldots$ but not related to the permutation of the operator valued matrix elements.

In the complete analogy with the Harmonic oscillator the (extended) Fock space can be constructed by the action of the rising operators $(\bar{\Psi}_a)_i^j$ on the oscillator vacuum state

$$|C\rangle = \sum_k (C^{a_1 \ldots a_k})_{j_1 \ldots j_k} (\bar{\Psi}_{a_1})_i^{j_1} \ldots (\bar{\Psi}_{a_k})_i^{j_k} |\Omega\rangle, \quad (2.25)$$

where

$$(\Psi_a)_i^j |\Omega\rangle = 0, \quad \forall a, i, j. \quad (2.26)$$
Due to the constraint (2.19) the extended Fock space is too large. One has to restrict it to the gauge invariant subspace by imposing the condition,
\[ \hat{G} |\Phi\rangle = 0, \]  
(2.27)
where \( \hat{G} =: [\Psi_a, \bar{\Psi}_a] : \) is the quantum version of the constraint (2.19). (Where the colon denotes an ordering prescription according to which all the components of \( \bar{\Psi} \) stay always to the right of those of \( \Psi \).)

As in the case of ordinary gauge fields the operator \( \hat{G} \) corresponding to the constraint is the generator of gauge transformations:
\[ [\hat{G}(u), \Psi_a] = -[u, \Psi_a], \quad [\hat{G}(u), \bar{\Psi}_a] = [u, \bar{\Psi}_a], \]
(2.28)
where \( \hat{G}(u) \) is the operator \( G \) smeared with an element of \( su(N) \) algebra:
\[ \hat{G}(u) = \text{tr} \hat{G} u = \text{tr} : [\bar{\Psi}_a, \Psi_a] : u. \]
(2.29)

As a result of application the condition (2.27) the physical space is generated by only those \( |C\rangle \) which correspond to gauge invariant polynomials of \( \Psi_a \). In terms of equation (2.25) this means that the quotients \( (C^{a_1...a_k})_{i_1...i_k}^{\gamma_1...\gamma_k} \) should be linear combinations of the products of delta symbols \( \delta^i_j \). The most generic such monomial of order \( L \) is given by a permutation of \( L \) elements \( \gamma \in S_L \),
\[ (C^{a_1...a_k})_{i_1...i_k}^{\gamma_1...\gamma_k} = C^{a_1...a_k} \gamma \delta_{i_1}^{\gamma_1} ... \delta_{i_k}^{\gamma_k}, \]
(2.30)
where \( \gamma_k = \gamma(k) \) is the permutation of the \( k \)-th element.

### 2.6 Conserved charges

In the model (2.5) there is a number of quantities which are conserved. Among these let us consider the quadratic ones
\[ L_{ab} = \text{tr} \bar{\Psi}_a \Psi_b, \quad L^\dagger_{ab} = L_{ba}, \quad \dot{L}_{ab} = 0. \]
(2.31)
They can be split into Hermitian operator \( L \) and operator valued vector \( \vec{S} \),
\[ L = L_{aa}, \quad \vec{S} = \vec{\sigma}_{ab} L_{ab}, \]
(2.32)
where \( \vec{\sigma} \) is the vector of Pauli matrices. As we will see below these operators correspond to the total length and total spin operators of the spin chains. Both \( L \) and \( \vec{S} \) generate a representation of the algebra \( U(2) \) which can be split into irreducible components described by positive integer value of \( L \) (\( U(1) \) spin) and the spin \( s \) irreducible representation of the \( SU(2) \) component.

In particular, due to the normal ordering of the operators all above charges vanish on the vacuum state \( |\Omega\rangle \). The value of the operator \( L \) corresponds to the total number of the excited oscillator modes, while e.g. \( S_3 = \frac{1}{2} \text{tr} \bar{\Psi}_1 \Psi_1 - \bar{\Psi}_2 \Psi_2 \) corresponds to the difference in the number of excited modes for the first and the second oscillator.
2.7 Matrices as spin chain gas

As we discussed earlier, the quantum Hamiltonian of the matrix model can be mapped to the Hamiltonian of a spin system with a chaining interaction [13,14] (see also [20,21]). The limit $N \to \infty$ with $\lambda = g^2 N$-fixed reduces that model to the integrable Heisenberg XXX$_{1/2}$ model. In this picture spin chains correspond to one-trace operators. As soon as $N$ is finite the subspace of such operators is not invariant with respect to dynamics and, therefore, in a gauge invariant configurations one is forced to consider a mixture of matrix states with all possible number of traces.

Naively, in $N \to \infty$ limit the non-planar interaction vanishes and one ends up with an “ideal gas” of spin chains, where each chain is conserved by the dynamics.

This simple picture, which we have for the spin chain/matrix model correspondence in the quantum theory, breaks down as soon as we consider the (semi)classical limit of either spin chains or matrix theory. As it turns out, the semiclassical regimes for two descriptions appear at different if not contradictory conditions. Indeed, the matrix model when the expectation values for the occupation numbers of most modes are larger than one. This gives us the condition,

$$L = \text{tr} \bar{\Psi} a \Psi a \sim N^2.$$  \hspace{1cm} (2.33)

On the other hand, as we know from the BMN evaluation, which should hold also in our case, the non-planar interaction rate is given by a factor proportional to $L^2/N$. Then, the regime (2.33) for the spin description means that the non-planar interactions are not only strong but overwhelmingly dominating the dynamics. In this case one can not even speak about the spin chains. In contrast, for $L \ll \sqrt{N}$, when one can neglect the non-planar interaction and, hence, the spin chains are stable the matrix model is in essentially quantum regime.

A gauge invariant matrix state generally will contain a mixture (gas, condensate, etc.) of spin chains of different lengths. The dynamics of this “spin soup” is such that the very short chain will tend to join bigger ones while too long chains will tend to split into smaller ones. Therefore, an equilibrium state should be statistically dominated by chains of a particular length.

As one can see, the statistical description of such a complicate system appears very naturally as it can catch the properties of the dynamics of both spins and matrices. Moreover, moving from one regime to another we can extract the relevant information like which sort of behavior dominates at different regimes. We do this in the next sections.
3 Spin chain gas at high temperature

Let us consider the closed sector of matrix model given by states of the length $L$, and let us further restrict ourself to the states of definite value of spin $\vec{S}$. The partition function restricted to such a subspace of the Hilbert space is given by,

$$Z(L, \vec{S}) = \text{tr} \Pi(L, \vec{S}) e^{-\beta H(\bar{\Psi}, \Psi)},$$

(3.1)

where $\Pi(L, \vec{S})$ is the projector to the $(L, \vec{S})$-subspace and $H(\bar{\Psi}, \Psi)$ is the Hamiltonian operator of (2.5). The projection to given values of $L$ and $\vec{S}$ can be realized in the following way:

$$Z(L, \vec{S}) = \int d\mu d\vec{x} e^{\mu L + \vec{x} \cdot \vec{S}} Z(\mu, \vec{x}),$$

(3.2)

where $Z(\mu, \vec{x})$ is the (grand canonical) partition function:

$$Z(\mu, \vec{x}) = \text{tr} e^{-H_{\beta, \mu, \vec{x}}(\bar{\Psi}, \Psi)},$$

(3.3)

where the chemical potentials $\mu$ and $\vec{x}$ enter into the Hamiltonian $H_{\beta, \mu, \vec{x}}(\bar{\Psi}, \Psi)$ in the following way,

$$H_{\beta, \mu, \vec{x}}(\bar{\Psi}, \Psi) =$$

$$\text{tr} \left( -i \bar{\Psi}_a [A, \Psi_a] + \mu \bar{\Psi}_a \Psi_a + \frac{1}{2} \vec{x} \bar{\Psi}_a \sigma_{ab} \Psi_b - \frac{\beta g^2 YM}{16 \pi^2} [\bar{\Psi}_a, \bar{\Psi}_b] [\Psi^a, \Psi^b] \right).$$

(3.4)

The original Hamiltonian is recovered at $\mu = \beta$ and $\vec{x} = 0$.

To find the partial partition function (3.2) let us first compute the grand canonical partition function $Z(\mu, \vec{x})$.

An important issue in this computation is one of gauge symmetry and the gauge fixing. There are several possibilities with this. One possible approach is given by imposing the diagonal gauge fixing to one of the components, say the real part of $\Psi_1$. As is claimed in [34] this leads to the description of the model in terms of a fermionic liquid. In the present case we use a different approach. To discuss the problem of gauge fixing it is useful to switch for a while to the path integral interpretation of the partition function. In this interpretation the inverse time $\beta$ plays the role of the size of compact Euclidean time. Therefore, the path integral formulation for the partition function implies periodic boundary conditions for the matrices: $X(\tau + \beta) = X(\tau)$.

Let us now turn to the gauge field. It is clear, that because of periodic time we can not impose the temporal gauge: $A = 0$. Indeed, out of the gauge
field $A(\tau)$ one can form a gauge covariant object called Polyakov loop:

$$W[A] = T \exp \oint d\tau A(\tau),$$

whose eigenvalues are gauge invariant and, therefore, can not be canceled by a gauge transformation. Hence, the admissible gauge we can impose is to fix $A$ to be at most constant and diagonal.

In the operator formalism one can still overcome the above restrictions and impose the temporal gauge $A = 0$, but one has supply it with the Gauss law constraint on the matrices. As the Gauss law constraint is a conserved quantity, it suffices to impose it once at a preferred time instant. This is equivalent to the restriction of the gauge field $A$ in the Hamiltonian (3.4) to be constant and diagonal.

The Gauss law constraint act on the quantum states as generator of gauge transformations. Hence, vanishing of the constraint is equivalent to the gauge invariance of the respective state. As we discuss in the Appendix B, in the case of compact Lie gauge group, projecting to the subspace of gauge invariant states can be implemented by the group integration. The advantage of this approach is that the compactness of the gauge group is taken into account as well.

At large temperature and fixed value of the length operator $L$ the commutator term of the Hamiltonian (3.4) is suppressed by a factor $\beta g^2_{YM}$, therefore we can treat it as a perturbation over the quadratic part of the Hamiltonian.

In what follows we will use the holomorphic representation which corresponds to the quantum oscillator described by the quadratic part of the Hamiltonian (3.4). For the convenience of the reader we give a brief summary of the holomorphic representation in the Appendices A and B.

### 3.1 Quadratic part

As we discussed above, we can consider the commutator term to be a small perturbation over the quadratic part of the Hamiltonian, such that we can apply the perturbation theory. Let us first compute the bare partition function which is the trace over gauge invariant subspace,

$$Z_0(\beta) = \text{tr}_{g.i.} e^{-H_{0,\mu,\vec{x}}},$$

where $H_{0,\mu,\vec{x}}$ is the quadratic part of the Hamiltonian (3.4), which consists of chemical potentials only:

$$H_{0,\mu,\vec{x}}(\bar{\Psi}, \Psi) = \text{tr} \left( \mu \bar{\Psi}_a \Psi_a + \frac{1}{2} \bar{\Psi}_a (\vec{x} \cdot \vec{\sigma})_{ab} \Psi_b \right).$$

---

9This equation defines $W[A]$ up to a gauge transformation related to the choice of initial point.
The Gaussian integral in (3.7) is convergent provided $\sqrt{\vec{x}^2} < \mu$. For larger values of $\vec{x}$ it diverges because the quadratic form in (3.7) becomes indefinite. Physically, however, more interesting situations are when spin is suppressed stronger than the length operator. In principle this can be done by a “Wick rotation” to imaginary spin chemical potential, $\vec{x} \mapsto i\vec{x}$.

Restriction of the trace to the gauge invariant subspace can implemented, as discussed in Appendix B, by averaging over the global gauge group,

$$Z_0(\mu, \vec{x}) \equiv \text{tr}_{g.i.} e^{-H_{0,\mu,\vec{x}}} = \int dU \int \frac{d\bar{\Psi}d\Psi}{\pi N^2} \exp \left(-\text{tr} \bar{\Psi} U^{-1} \Psi U + \bar{\Psi} e^{-\mu - \vec{x} \cdot \vec{\sigma}/2} \Psi\right), \quad (3.8)$$

where the exponent in the last term is understood as a $2 \times 2$ matrix exponent. Let us note that, in contrast to ordinary gauge fixing, averaging over the gauge group keeps the gauge invariance explicit i.e. the integral is invariant with respect to global gauge transformations:

$$U \rightarrow V^{-1} U V, \quad \Psi \rightarrow V^{-1} \Psi V, \quad \bar{\Psi} \rightarrow V^{-1} \bar{\Psi} V. \quad (3.9)$$

In the basis of eigenvectors $|\pm\rangle$ of the matrix $\vec{x} \cdot \vec{\sigma}$,

$$\vec{x} \cdot \vec{\sigma} |\pm\rangle = \pm x |\pm\rangle, \quad x = \sqrt{\vec{x}^2}, \quad (3.10)$$

the matrix $e^{-\mu - \vec{x} \cdot \vec{\sigma}/2}$ becomes diagonal with values $e^{-\mu \pm}$, where

$$\mu \pm = \mu \pm \frac{x}{2}. \quad (3.11)$$

The symmetry which is left allows us fixing the matrix $U$ to be diagonal. Also changing the variable in the $U$-integral to the diagonal values $U = \text{diag}\{e^{i \theta_n}\}$ produces square of the Vandermonde determinant,

$$|\Delta(\theta)|^2 = \prod_{m>n} 2(1 - \cos \theta_{mn}), \quad \theta_{mn} = \theta_m - \theta_n. \quad (3.12)$$

We also used the fact that the integral does not depend on the $U(1)$ factor. Hence one can fix the sum of the eigenvalues (“center of mass”) to vanish

$$\sum_n \theta_n = 0 \mod 2\pi. \quad (3.13)$$

---

10 This is in the direct relation to the fact that the gauge field can also be fixed to be constant diagonal.

11 In what follows we just systematically drop the contributions proportional to $\sum \theta_n$. 

15
As a result of above transformations the gauge invariant trace takes the following ‘normal form’,

\[ Z_0(\mu, \vec{x}) = \prod_n d\theta_n |\Delta(\theta)|^2 \times \int \frac{d\bar{\Psi} d\Psi}{\pi N^2} \exp \left( - \sum_{m,n} h^{(\pm)}_{mn}(\mu, \vec{x}, \theta) \bar{\Psi}^\pm_{mn} \Psi^\pm_{mn} \right), \quad (3.14) \]

where the normal modes \( h^{(\pm)}_{mn}(\mu, \vec{x}, \theta) \), are given by,

\[ h^{(\pm)}_{mn} = e^{i \theta_{mn}} - e^{-\mu_\pm}. \quad (3.15) \]

As one can see, the integral over the matrices \( \Psi \) and \( \bar{\Psi} \) is Gaussian and can be easily taken. The quadratic partition function then is reduced to the integral over \( \theta \)'s only,

\[ Z_0(\mu, \vec{x}) = \prod_n d\theta_n |\Delta(\theta)|^2 \prod_{m,n} [h^{(+)\prime}_{mn} h^{(-)\prime}_{mn} h^{-}_{mn} h^{+}_{mn}]^{-1} \]

\[ = \frac{2^{-\frac{1}{2}} N(N+1) e^{N^2 \mu}}{[\sinh(\mu_+ / 2) \sinh(\mu_- / 2)]^N} \int \prod_n d\theta_n \times \prod_{m>n} \frac{1 - \cos \theta_{mn}}{(\cosh \mu_+ - \cos \theta_{mn})(\cosh \mu_- - \cos \theta_{mn})}. \quad (3.16) \]

Using (3.12), one can pass to the variables \( L \) and \( \vec{S} \). As soon as a large number of degrees of freedom is concerned this integration can be done by the saddle-point approximation, i.e. by replacing \( \mu \) and \( \vec{x} \) with their solutions in the presence of the sources \( L \) and \( \vec{S} \) to the equations of motion corresponding to the effective action resulting from (3.16). The saddle point conditions coincide with the Legendre equations. Hence the Laplace transform in this limit is reduced to the Legendre transform of the free energy:

\[ L = \frac{\partial F(\mu, \vec{x})}{\partial \mu}, \quad \vec{S} = -\frac{\partial F(\mu, \vec{x})}{\partial \vec{x}}; \quad (3.17a) \]

\[ S_{\text{eff}}(L, \vec{S}) = L \mu(L, \vec{S}) + i \vec{S} \cdot \vec{x}(L, \vec{S}) - F(\mu(L, \vec{S}), \vec{x}(L, \vec{S})). \quad (3.17b) \]

where functions \( \mu(L, \vec{S}) \) and \( \vec{x}(L, \vec{S}) \) in the second line are found as solutions to the first line equations (3.17a). The free energy \( F \) is defined as the log of the partition function,

\[ F(\mu, \vec{x}) = -\ln Z_0(\mu, \vec{x}). \quad (3.18) \]
Thus, to obtain the effective action we have to integrate over the gauge field eigenvalues $\theta_n$. In some cases this can be also done by the saddle-point approximation, i.e. replace the integral by the value of the integrand at its maxima.

### 3.2 The free energy

Consider the saddle point approach to the integral (3.16). Finding the maximal value of the integrand in the partition function (3.16) is reduced to the problem for finding extrema with respect to variation of $\theta_n$ for the following action,

$$F(\theta; \mu, \vec{x}) = -N^2 \mu + N[\ln \sinh(\mu_+/2) + \ln \sinh(\mu_-/2)]$$

$$+ \frac{1}{2} \sum_{m,n \neq m} (-\ln(1 - \cos \theta_{mn}) + \ln(\cosh \mu_+ - \cos \theta_{mn}) + \ln(\cosh \mu_- - \cos \theta_{mn})),$$

(3.19)

where we dropped the constant terms, which can be absorbed by a redefinition of the measure.

The equations for the Legendre transform (3.17) then read,

$$L = -N^2 + \frac{1}{2} N[\coth(\mu_+/2) + \coth(\mu_-/2)]$$

$$+ \frac{1}{2} \sum_{m,n \neq m} \left( \frac{\sinh \mu_+}{\cosh \mu_+ - \cos \theta_{mn}} + \frac{\sinh \mu_-}{\cosh \mu_- - \cos \theta_{mn}} \right)$$

(3.20a)

$$2 \vec{S} = \frac{1}{2} N[\coth(\mu_+/2) - \coth(\mu_-/2)]$$

$$+ \frac{1}{2} \sum_{m,n \neq m} \left( \frac{\sinh \mu_+}{\cosh \mu_+ - \cos \theta_{mn}} - \frac{\sinh \mu_-}{\cosh \mu_- - \cos \theta_{mn}} \right) \vec{x},$$

(3.20b)

$$\sum_m \left( -\frac{\sin \theta_{nm}}{1 - \cos \theta_{nm}} + \frac{\sin \theta_{nm}}{\cosh \mu_+ - \cos \theta_{nm}} + \frac{\sin \theta_{nm}}{\cosh \mu_- - \cos \theta_{nm}} \right) = 0,$$

(3.20c)

where the summation in the last equation should run through $m$ different from $n$. As some $\theta_m$ approaches the $\theta_n$, the first term in (3.20c) diverges as $\sim 1/(\theta_n - \theta_m)$. In what follows we assume that this term is regularized by replacing 1 by a term $\cosh \varepsilon$, where $\varepsilon$ afterwards will be sent to zero. Having this regularization in mind we can extend the last sum to all $m$.

Let us first discuss the equations (3.20) from the qualitative point of view. The last equation, (3.20c) describes an equilibrium static configuration of
the system consisting of \( N \) particles on a circle with the following pairwise interaction potential,

\[
\varphi = -\frac{1}{2} \ln(1 - \cos \lambda) + \frac{1}{2} \ln(\cosh \mu_+ - \cos \lambda) + \frac{1}{2} \ln(\cosh \mu_- - \cos \lambda). \tag{3.21}
\]

As one can see, when \( \mu_\pm \neq 0 \) the interaction potential consists of one repulsive and two attracting terms. Normally, at small distances the repulsive term dominates while at large separations the interaction becomes attractive. Now several variants are possible. First, suppose the scale at which the attractive force starts to win over the repulsive one is larger than \( 2\pi \). In this case the repulsive dominant interaction will force the eigenvalues to arrange uniformly over the circle. This is a configuration with unbroken subgroup of the translation symmetry for \( \theta \): \( \theta_n \rightarrow \theta_n + \eta \). This happens when \( \mu_\pm \) is large enough. When \( \mu_\pm \) decrease the scale at which interaction becomes attractive decreases as well and at some point it becomes less than the circle’s length \( 2\pi \). At this stage for eigenvalues it becomes more “convenient”, from the point of view of “total energy”, to condense around some point(s) rather than be uniformly spread over the circle. In particular, when at least one of \( \mu_\pm \) vanishes the repulsion term is canceled completely and all eigenvalues tend to collapse to a single point. The last is the extremal case of breaking of the translational symmetry of the eigenvalue distribution.

### 3.2.1 Expansion in powers of \( e^{-\mu_\pm} \)

Let us pass to a quantitative analysis rather a qualitative description. The problem dramatically simplifies if one looks for solutions which are analytic in \( e^{-\mu_\pm} \). This appears possible due to the fact that the expansion in terms of powers of \( e^{-\mu_\pm} \) of the ‘equation of motion’ (3.20) is remarkably simple and takes the following form,

\[
2 \sum_{\omega=1}^{\infty} \left(-1 + e^{-\omega\mu_+} + e^{-\omega\mu_-}\right) \sum_m \sin(\omega \theta_{nm}) = 0. \tag{3.22}
\]

As one can see, this expansion organizes itself into something similar to a Fourier series.

Applied to the free energy (3.19) the expansion yields,

\[
F(\theta; \mu, \vec{x}) = N[\ln \sinh(\mu_+/2) + \ln \sinh(\mu_-/2) - \mu] \\
+ \sum_{\omega=1}^{\infty} \frac{1}{\omega} (1 - e^{-\omega\mu_+} - e^{-\omega\mu_-}) \sum_{m, n} \cos(\omega \theta_{nm}). \tag{3.23}
\]
In (3.22) and (3.23) one can observe that the terms scaling as $N^2$ were canceled away by the zeroth term in the expansion. To obtain (3.22) and (3.23) we used the following expansion,

$$
\ln (\cosh \mu - \cos \theta) = \mu - \ln 2 - \sum_{n=1}^{\infty} \frac{e^{-n\mu}}{n} \cos[n\theta],
$$

(3.24)

as well as its derivative.

Let us continue with the succession of cancelations. Observe the following properties of trigonometric sums,

$$
\begin{align*}
\sum_m \sin \omega \theta_{mn} &= \sin \omega \theta_n \sum_m \cos \omega \theta_m - \cos \omega \theta_n \sum_m \sin \omega \theta_m, \quad (3.25a) \\
\sum_{m \neq n} \cos \omega \theta_{mn} &= \left( \sum_m \cos \omega \theta_m \right)^2 + \left( \sum_m \sin \omega \theta_m \right)^2 - \sum_m (\cos^2 \omega \theta_m + \sin^2 \omega \theta_m) = \tilde{\rho}_\omega^2 + \hat{\rho}_\omega^2 - N, \quad (3.25b)
\end{align*}
$$

where,

$$
\tilde{\rho}_\omega = \sum_n \cos(\omega \theta_n), \quad \hat{\rho}_\omega = \sum_n \sin(\omega \theta_n)
$$

(3.26)

for $\omega = 1, 2, 3, \ldots$

It is natural to assume that the equilibrium $\theta$-distribution possesses at least one reflection symmetry such that one can choose a point with respect to which the distribution of $\theta$’s is even. In this case the sums over sines $\hat{\rho}_\omega$ vanish and we are left with the following equations

$$
\sum_{\omega=1}^{\infty} \left( -1 + 2e^{-\omega \mu} \cosh \left( \frac{\omega x}{2} \right) \right) \sin(\omega \theta_n) \tilde{\rho}_\omega = 0,
$$

(3.27)

as well as with

$$
F(\theta; \mu, \bar{x}) = \sum_{\omega=1}^{\infty} \frac{1}{\omega} \left( 1 - 2e^{-\omega \mu} \cosh \left( \frac{\omega x}{2} \right) \right) \tilde{\rho}_\omega^2.
$$

(3.28)

A glance at (3.27) and (3.28) reveals that the free terms scaling linearly in $N$ are canceled if we extend summation to all possible pairs of eigenvalue numbers.

It is interesting to note that equations of motion do not depend on the constant part of eigenvalue distribution $\tilde{\rho}_0$. Hence, an obvious solution to
the equation (3.27) is given by the uniform distribution of the eigenvalues:
\[ \theta_n = \frac{2\pi n}{N}, \quad n = 0, 1, 2, \ldots, N - 1, \]
or \[ \rho(\lambda) = \frac{N}{2\pi} \equiv \text{constant}. \]  

Let us find the other solutions. In the limit of large number of eigenvalues their distribution can be described by a continuous density \( \rho(\theta) \) such that,
\[ \int_{-\pi}^{\pi} d\theta \rho(\theta) = N, \quad \rho(\theta) \geq 0, \]  
and a sum over eigenvalues can be replaced by an integral as follows,
\[ \sum_n f(\theta_n) = \oint d\theta \rho(\theta) f(\theta), \]  
for some function \( f(\theta) \) which is smooth enough. In particular, \( \tilde{\rho}_\omega \) in this limit become coefficients of Fourier series,
\[ \tilde{\rho}_\omega = \oint d\lambda \rho(\lambda) \cos(\omega \lambda), \quad \rho(\lambda) = \rho_0 + \sum_{\omega=1}^{\infty} \tilde{\rho}_\omega \pi \cos \omega \lambda. \]  

Consider now the equation (3.27). The equality in (3.27) holds when each term in the sum vanishes separately i.e.,
\[ \tilde{\rho}_\omega \left( -1 + 2e^{-\omega \mu} \cosh \left( \frac{\omega x}{2} \right) \right) = 0, \quad \omega = 1, 2, 3, \ldots. \]  

Eq. (3.32) in particular means that the Fourier expansion of \( \rho(\theta) \) can have only such modes (\( \tilde{\rho}_\omega \neq 0 \)) for which the expression inside the parenthesis of (3.32) vanishes by itself,
\[ -1 + 2e^{-\omega \mu} \cosh \left( \frac{\omega x}{2} \right) = 0, \quad \omega \in \mathbb{Z}_+. \]  

So, let us analyze the possible solutions to (3.33). Basically the equation (3.33) admits at most one solution when \( \mu \geq \frac{x}{2} \). Even in this case, due to the discrete nature of \( \omega \) such a solution is possible only for particular pairs \((\mu, x)\). However, when \( \mu \) and \( x \) become small \( \omega \) can be treated as a continuous variable, therefore the set of pairs \((\mu, x)\) for which one can find an integer solution becomes dense.

In general, for each value of \( \omega \) there is a one-parameter family of \((\mu_\omega, x_\omega)\) satisfying (3.33).

Now, let us turn to the “free energy” \( F(\mu, \vec{x}) \). Due to the factor \((1 - 2e^{-\omega \mu} \cosh \frac{\omega x}{2})\) multiplied to each mode \( \tilde{\rho}_\omega \) the “on-shell” value of \( F(\theta; \mu, \vec{x}) \) vanishes. Physically this can be related to the following fact. For simplicity consider the well-studied case of the gauged one-matrix oscillator. This
system, unlike the multi-matrix oscillator, can be solved by imposing the diagonal gauge to the matrix field. The Faddeev–Popov determinant, or better to say the Jacobian, which arises when passing to the description in terms of eigenvalues is again a Vandermonde determinant. Upon quantization the determinant can be absorbed into the wave function by rescaling it by the square root of the Vandermonde determinant. Such a rescaling makes the wave function odd with respect to permutation of the eigenvalues of the matrix field. Thus, the diagonal matrix oscillators behave like fermionic ones. In particular, they have to obey the Pauli exclusion principle: no two diagonal oscillators could be in the same state. One of the effects of this is that the states with \( L < N \) are banned. In terms of the chemical potential this means that for \( \mu \) larger than a critical value \( \mu_c \) the free energy should become trivial i.e. independent of \( \mu \). Rather than disappointing this fact seems to be compatible with the PET calculation of the partition function [31], which establishes that the free energy in this regime is finite (and \( N \) independent) as \( N \) goes to infinity.

In spite that it appears that our approximation was not sensitive enough to obtain some nonzero thermodynamical quantities we still can try to extract some information from the system by observing that writing down the Legendre transformation equations (3.20) before solving equations of motions for \( \theta \)'s lead to a non-trivial result for \( L \) and \( S \). This corresponds to the interchange of integration in \( \theta \)'s with one of \( \mu \) and \( x \).

In the case of a single nontrivial mode \( \omega \), satisfying (3.33), the Legendre equations (3.20) are reduced to,

\[
L = \tilde{\rho}_\omega^2, \tag{3.34a}
\]
\[
2S = \tanh \left( \frac{\omega x}{2} \right) \tilde{\rho}_\omega^2, \tag{3.34b}
\]

where we used (3.33) to eliminate the factor \( 2e^{-\omega \mu} \cosh(\omega x/2) \) in the first equation and \( 2e^{-\omega \mu} \) in the second one. Equations (3.34) together with the mode equation (3.33) seem to be enough to eliminate \( \mu \) and \( x \) in favor of \( L \) and \( S \) for given \( \omega \):

\[
x = \frac{2}{\omega} \tanh^{-1} \left( \frac{2S}{L} \right), \tag{3.35a}
\]
\[
\mu = \frac{1}{\omega} \ln \left( \frac{2}{\sqrt{1 - \left( \frac{2S}{L} \right)^2}} \right). \tag{3.35b}
\]

As the “free energy” \( F \) vanishes on-shell the effective action \( S_{\text{eff}}(L, \vec{S}) \) is
given by,

\[ S_\omega = L\mu(L, \tilde{S}) + \tilde{S} \cdot x(L, \tilde{S}) = \frac{1}{\omega} \left( L \ln \frac{2}{\sqrt{1 - \left( \frac{2S}{L} \right)^2}} + 2S \tanh^{-1} \left( \frac{2S}{L} \right) \right). \]  

\[ (3.36) \]

Next, according to the saddle point approximation, we have to sum over the contributions of all saddle points i.e. to sum over \( \omega \),

\[ e^{S_{\text{eff}}} = \sum_{\omega=1}^{\infty} e^{S_\omega}. \]  

\[ (3.37) \]

Some comments are in order. The sum (3.37) diverges, but let us recall that the validity of our approach was given by the approximation of smooth eigenvalue distribution. This is definitely not true for the frequencies \( \omega \gtrsim N \), since they involve wavelengths shorter than the average distance between the eigenvalues \( \theta_n \). So, the applicability of the approach is restricted to still relatively large values of \( \mu \) and \( x \). Under relatively large we understand much bigger than \( \sim 1/N \). Also the equation (3.27) implies (3.32) only in the limit \( \mu N \gg 1 \). Indeed, for finite \( N \) the sines/cosines of \( \omega \gtrsim N \) can be expressed in terms of sines/cosines of smaller arguments. Therefore, (3.32) will get corrections from the higher modes: \( \omega \gtrsim N \). For \( \mu \gg 1/N \) this contribution is exponentially suppressed and we can neglect the modes higher than \( N \), otherwise they should be taken into account. As a conclusion the sum in (3.37) should be understood as regularized by restricting it to \( \omega \leq N \).

It is clear, however, that as soon as large numbers are concerned, \( S_\omega \gg 1 \) and we have,

\[ e^{S_1} \gg e^{S_\omega} = \left[ e^{S_1} \right]^{\frac{1}{\omega}}, \quad \forall \omega > 1. \]  

\[ (3.38) \]

Hence, for large \( L \) all terms except the first one can be neglected and we have \( S_{\text{eff}} = S_1 \).

As one can see from (3.36), there is no \( N \) and \( N^2 \) scaling left in the effective action. It would be also interesting to check the effective action (3.36) against the PET result.

### 3.2.2 Solution for small \( \mu \) and \( x \)

Small chemical potentials correspond to larger expectation values for respective thermodynamical quantities, since the suppression of configurations with large charges is weaker. Therefore considering small values of \( \mu \) and \( x \) will favor contributions with large \( L \) and, respectively, \( S \).
In the approach of previous subsection, however, this limit is singular since the expansion in powers of $e^{-\mu_{\pm}}$ diverges for small $\mu_{\pm}$.

Now, since $\mu_{\pm}$ are small we expect that the eigenvalues $\theta_n$ should condense around some arbitrary point breaking the translational symmetry. Let us pass to the quantitative analysis of this case as well.

For small values of $\mu_{\pm}$ the trigonometric and hyperbolic cosines can be replaced by the first terms of their Taylor expansion, $1 \pm \frac{x^2}{2} + \ldots$. In this limit the action assumes the following form,

$$S(\theta; \mu, \vec{x}) = -N^2(\mu - \ln 2) + N[\ln(\mu_+ + \mu_-) - 2 \ln 2] + \frac{1}{2} \sum_{m,n} \left( -\ln \frac{\theta_{mn}^2}{\mu_+^2 + \theta_{mn}^2} + \ln(\mu_+^2 + \theta_{mn}^2) + \ln(\mu_-^2 + \theta_{mn}^2) \right). \tag{3.39}$$

The “action” (3.39) leads to the following “equations of motion”,

$$\sum_{m \neq n} \left( -\frac{1}{\theta_{nm}} + \frac{\theta_{nm}}{\mu_+^2 + \theta_{nm}^2} + \frac{\theta_{nm}}{\mu_-^2 + \theta_{nm}^2} \right) = 0. \tag{3.40}$$

Let us note, that when at least one of $\mu_{\pm}$ vanishes, the potential becomes purely attractive and all eigenvalues collapse to a single point. In general, potential becomes attractive outside the region $\theta_{nm} \lesssim \sqrt{\mu_+ \mu_-}$, so the eigenvalues should condense to a region of this typical size.

The problem of finding the eigenvalue configuration satisfying (3.40) is equivalent to finding an equilibrium distribution of particles on a line with a Van-der-Waals-like pair interaction given by the potential

$$\varphi(\lambda) = \frac{1}{2} \left( -\ln \lambda^2 + \ln(\mu_+^2 + \lambda^2) + \ln(\mu_-^2 + \lambda^2) \right). \tag{3.41}$$

Again, in the limit of large $N$, the the eigenvalues become dense and one can replace the equation (3.40) by a continuous integral one:

$$\int_{-\infty}^{\infty} d\eta \rho(\eta) \left( -\frac{1}{\lambda - \eta} + \frac{\lambda - \eta}{\mu_+^2 + (\lambda - \eta)^2} + \frac{\lambda - \eta}{\mu_-^2 + (\lambda - \eta)^2} \right) = 0, \tag{3.42}$$

where the equation should hold for $\lambda \in \text{supp } \rho(\lambda)$.

We approach this problem in the Appendix D. The resulting ‘energy’ function (3.10) generates following Legendre equations\(^{12}\)

$$L = 2N^2 \left( \mu_+^{-1} - \mu_-^{-1} \right) \xi \frac{\partial \xi}{\partial \xi} \big|_{\xi = \sqrt{\mu_-}}, \tag{3.43a}$$

$$2S = 2N^2 \left( \mu_+^{-1} + \mu_-^{-1} \right) \xi \frac{\partial \xi}{\partial \xi} \big|_{\xi = \sqrt{\mu_-}}. \tag{3.43b}$$

\(^{12}\)Note that here and on we replaced the “original” length $L$ by a “renormalized” one $L \rightarrow L - N^2$. 

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where the function $\mathcal{E}$ is given by (see (D.10) in the Appendix D),

$$
\mathcal{E}(\xi) = \xi \arctan \xi^{-1} + \xi^{-1} \arctan \xi, \quad \xi = \sqrt{\frac{\mu_+}{\mu_-}}.
$$

(3.44)

In terms of $L_\pm = L \pm 2S$ Legendre equations (3.43a) take the form,

$$
L_\pm = \pm 4N^2 \mu_\pm^{-1} \xi \mathcal{E}'(\xi).
$$

(3.45)

Dividing the equation for “+” by the one for “−”, we obtain,

$$
\xi^2 \equiv \frac{\mu_+}{\mu_-} = -\frac{L_-}{L_+}.
$$

(3.46)

A remark is in order. Physically meaningful region is one where $0 \leq L_- \leq L \leq L_+$. This corresponds to such a regime where chemical potentials statistically suppress the spin stronger than the length $L$. On the other hand, we know that the matrix integral diverges at $\mu \leq x/2$. So, to reach the physically relevant region we are forced to make an analytic continuation to complex $x$ and $\xi^2 < 0$.

Eq. (3.45) together with (3.46) yield,

$$
\mu_\pm = \pm 4N^2 L_\pm^{-1} \xi \mathcal{E}'(\xi),
$$

(3.47)

while in terms of $\mu$ and $x$ this solution takes the form

$$
\mu = 2N^2 (L_+^{-1} - L_-^{-1}) \xi \mathcal{E}'(\xi),
$$

(3.48)

$$
x = 2N^2 (L_+^{-1} - L_-^{-1}) \xi \mathcal{E}'(\xi)
$$

(3.49)

Plugging this solution into the Legendre transform equation (3.17b), we obtain,

$$
S_{\text{eff}}(L, S) \equiv \frac{1}{2} (L_+ \mu_+ + L_- \mu_-) - 4N^2 \mathcal{E}(\xi) = -4N^2 \mathcal{E}(\xi) = 4N^2 \left( \frac{L_+}{L_-} \ln \left| \frac{1 + \sqrt{L_+}}{1 - \sqrt{L_-}} \right| + \frac{L_-}{L_+} \ln \left| \frac{1 + \sqrt{L_-}}{1 - \sqrt{L_+}} \right| \right),
$$

(3.50)

where $L_\pm$ are given by,

$$
L_\pm = L \pm 2S.
$$

(3.51)

Let us recall that $e^{S_{\text{eff}}(L, S)}$ gives the number of composite SYM operators of length $L$ and spin $S$. It is instructive to compare eq. (3.50) to the log of the number of states of given spin in a system of $L$ completely randomly oriented spins.
### 3.3 One-loop contribution

So far we considered the purely oscillator part of the matrix model. As we discussed above in the limit of high temperature and small coupling the quartic commutator term can be considered to be a small perturbation. Therefore, in this limit one can find the correction to the oscillator effective action by just evaluating the average of the quartic term in the oscillator dominated background. This amounts to neglecting among others the back reaction of the background to the presence of the perturbation. At the one loop level it is a justified approximation, since the back reaction comes at the order $g_{YM}^4$ and are of the same order as two-loop contribution from SYM.

In the previous section we considered the quadratic part (3.7) of the matrix model Hamiltonian (3.4),

$$H = H_0 + V = \text{tr} \left( \mu \sqrt{2} \Psi_a \Psi_a + \frac{1}{2} \sqrt{x} \cdot \Psi_a \sigma_{ab} \Psi_b \right) - \frac{\beta g_{YM}^2}{16\pi^2} \text{tr}[\bar{\Psi}_a, \bar{\Psi}_b][\Psi^a, \Psi^b]. \quad (3.52)$$

At high temperatures the coupling $\beta g_{YM}^2$ is small so we can expand the partition function as follows

$$Z = e^{-S_{\text{eff}}} = \text{tr} e^{-(H_0 + V)} = \text{tr} e^{-H_0} - \text{tr} e^{-H_0} V + \cdots = Z_0(1 - \langle V \rangle_0 + \ldots) \approx e^{-S_{\text{eff}}^0 - \langle V \rangle_0}, \quad (3.53)$$

where the mean value $\langle V \rangle_0$ is defined in a standard way

$$\langle V \rangle_0 = Z_0^{-1} \text{tr} e^{-H_0} V. \quad (3.54)$$

Thus, computation of non-zero coupling correction at the leading order resides in the calculation of the mean (3.54) of the following operator

$$V = \frac{\beta g_{YM}^2}{16\pi^2} \text{tr}[\bar{\Psi}_a, \bar{\Psi}_b][\Psi^a, \Psi^b]. \quad (3.55)$$

To do it let us use again the anti-holomorphic representation (see Appendix A).

Since the operator (3.55) is a normal one its anti-holomorphic kernel is given by Eq. (A.9),

$$K_V(\Phi, \bar{\Phi}) = \frac{\beta g_{YM}^2}{16\pi^2} e^{\text{tr} \Phi_a \Phi_a} \text{tr}[\bar{\Phi}_a, \bar{\Phi}_b][\Phi_a, \Phi_b]. \quad (3.56)$$

---

13 Taking into account corrections from higher orders could be similar to one-loop renormalization group improvement.
Since both the bare quadratic Hamiltonian and the perturbation are
gauge invariant it suffices to insert the projection to the gauge invariant
subspace only once. Therefore, the trace (3.54) is given by

\[ Z_0\langle V \rangle_0 = \frac{\beta g_{YM}^2}{8\pi^2} \int d\theta \Delta(\theta) \int \frac{d\Phi d\Phi}{\pi N^2} \exp \left( - \sum_{mn} \left( h_{mn}(+\Phi_{mn}^{(+)} \Phi_{mn}^{(+)} + h_{mn}(+\Phi_{mn}^{(-)} \Phi_{mn}^{(-)}) \right) \times \right. \\
left. \text{tr}[\Phi_+ e^{-\mu^+}, \Phi_- e^{-\mu^-}][\Phi_+, \Phi_-], \right) \tag{3.57} \]

where \( h_{mn}^{(\pm)} \) was defined in (3.15) of subsection 3.1.

Making the substitution: \( h_{mn}^{(\pm)} \Phi_{mn}^{(\pm)} \rightarrow \Phi_{mn}^{(\pm)} \) the integral (3.57) transforms into

\[ \langle V \rangle_0 = \frac{\beta g_{YM}^2}{8\pi^2} \int \frac{d\Phi d\Phi}{\pi N^2} e^{-\Phi \Phi} e^{-2\mu} \text{tr}[\Phi_+/h^{(\pm)}], \Phi_-/h^{(\pm)}] [\Phi_+, \Phi_-], \tag{3.58} \]

where \( (\Phi_+/h^{(\pm)}) \) is a matrix given by the elements

\[ (\Phi_+/h^{(\pm)})_{mn} = \Phi_{mn}^{(\pm)}/h_{mn}^{(\pm)}, \tag{3.59} \]

and we used \( Z_0 = 1/\prod h^{(\pm)}h^{(-)} \) to cancel \( Z_0 \) in the l.h.s.

This matrix integral can be computed taking into account that,

\[ I_{k_1,k_2,m_1,n_1,m_2,n_2}^{a_1,a_2,b_1,b_2} \equiv \int \frac{d\Phi d\Phi}{\pi N^2} (\Phi_{k_1,k_2}^{a_1} \Phi_{m_1,n_1}^{b_1} \Phi_{m_2,n_2}^{b_2}) e^{-\text{tr} \Phi \Phi} = \delta^a_b \delta \delta_{a_1b_1} \delta \delta_{a_2b_2} \delta \delta_{m_1k_1} \delta \delta_{m_2k_2} \delta \delta_{m_2k_1} \delta \delta_{m_1k_2}. \tag{3.60} \]

Now, plugging (3.60) into (3.58) we get,

\[ \langle V \rangle_0 = \frac{\beta g_{YM}^2}{4\pi^2} e^{-2\mu} \left( \sum_n \frac{1}{h_{mn}^{(+)}} - \frac{1}{2} \sum_{k,m} \left( \frac{1}{h_{mn}^{(+)}} + \frac{1}{h_{mn}^{(-)}} \right) \right) = \frac{\beta g_{YM}^2}{4\pi^2} N \left( (\mu^+ - 1)(\mu^- - 1) \right) - \frac{1}{4} \sum_{k,m} \frac{\cos \theta_{nk} e^{-\mu^+} - \cos \theta_{nk} e^{-\mu^-} + e^{-2\mu}} {\left( \cosh \mu_+ - \cos \theta_{mn} \right) \left( \cosh \mu_- - \cos \theta_{nk} \right)} \equiv \langle \tilde{V} \rangle_0 + \langle V \rangle_0'. \tag{3.61} \]

The term in the first line of the r.h.s. of (3.58) can be readily evaluated to be,

\[ \langle \tilde{V} \rangle_0(\mu, \vec{x}) = \frac{\beta g_{YM}^2 N}{4\pi^2} \frac{e^{-\mu_+ - \mu_-}}{(1 - e^{-\mu_+})(1 - e^{-\mu_-})}, \tag{3.62} \]

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or in terms of the level \( \omega \) solution to the Legendre equations (see (3.35)):

\[
\mu_{\pm} = \frac{1}{\omega} \ln \left( \frac{2L}{L \mp 2S} \right),
\]

(3.63)

the same quantity can be written as,

\[
\langle V \rangle_0(L, S) = \frac{\beta \lambda}{4\pi^2} \ln \left( \frac{2L}{L - 2S} \right)^{1/2} - 1 \frac{1}{\left( \frac{2L}{L + 2S} \right)^{1/2} - 1}.
\]

(3.64)

The \( \omega = 1 \) term is a constant,

\[
\langle V \rangle_0 = \frac{\beta \lambda}{4\pi^2}.
\]

(3.65)

It is interesting to note that for large \( \omega \) the mean value (3.64) behaves like,

\[
\langle V \rangle_0 = \frac{\beta \lambda}{4\pi^2} \left\{ \omega^2 \ln \left( \frac{2L}{L - 2S} \right) \ln \left( \frac{2L}{L + 2S} \right) - \omega \left( \frac{1}{\ln \left( \frac{2L}{L - 2S} \right)} + \frac{1}{\ln \left( \frac{2L}{L + 2S} \right)} \right) + \ldots \right\}.
\]

(3.66)

The overall negative sign in front of \( \omega^2 \) term makes the terms with large \( \omega \) to be even more strongly suppressed.

### 3.3.1 Expansion in powers of \( e^{-\mu_{\pm}} \)

Let us consider the second line of (3.58) and expand it in powers of \( e^{-\mu_{\pm}} \) the same way as we did in the previous section. Remarkably, again the expansion is not only doable, but also appears in a simple form,

\[
\langle V \rangle_0 = \langle V \rangle_0 - \langle V \rangle_0 = -\frac{\beta g_{YM}^2}{4\pi^2} \sum_{\omega, \omega'} e^{-(\omega + 1)\mu_{\pm} - (\omega' + 1)\mu_{\pm}} \sum_{m, n, k} \cos(\omega \theta_{mn} + \omega' \theta_{kn}),
\]

(3.67)

where \( \omega, \omega' = 1, 2, \ldots \) enumerate the modes of the eigenvalue density distribution.

As a warmup it is not difficult to see that for homogeneous distribution of eigenvalues \( \theta_n \) the average of each such term vanishes i.e.,

\[
\langle V \rangle_0 = \langle V \rangle_0.
\]

(3.68)
Now let us turn to the situation where the eigenvalue distribution is ‘perturbed’ by a non-constant mode $\omega$.

A straightforward way to compute the expectation value $\langle V \rangle_0$ would be to plug the eigenvalue distribution into (3.67) and get the answer. In the case of analytic solution we have the possibility of an indirect way of evaluation of (3.67) even without knowledge of the details relating to the density $\rho(\lambda)$.

Indeed, let us expand the cosine in (3.67) and use the even character of $\rho$ (see the Appendix C) to get,

$$\langle V \rangle'_{0} = -\beta g_{YM}^{2} \frac{4\pi}{2\pi} \sum_{\omega,\omega'} e^{-\omega - \omega'} \tilde{\rho}_\omega \tilde{\rho}_{\omega + \omega'} = -\beta g_{YM}^{2} \frac{4\pi}{2\pi} e^{-2\mu} \sum_{\omega} \left( e^{-\omega + \omega'} \rho_\omega^2 + \sum_{\omega,\omega' \neq 0} e^{-\omega + \omega'} \tilde{\rho}_\omega \tilde{\rho}_{\omega + \omega'} \right), \quad (3.69)$$

where we separated explicitly the zero mode $\rho_0 = N/2\pi$ of the eigenvalue distribution density. Using the property (3.33) of the modes of the density one can readily evaluate the first term of the last equality of (3.69) to be,

$$\langle V \rangle'_{0} = -\beta g_{YM}^{2} \frac{4\pi}{2\pi} e^{-2\mu} \sum_{\omega} \left( e^{-\omega + \omega'} \rho_0^2 + \sum_{\omega,\omega' \neq 0} e^{-\omega + \omega'} \tilde{\rho}_\omega \tilde{\rho}_{\omega + \omega'} \right), \quad (3.70)$$

or for $\omega = 1$ it becomes,

$$\langle V \rangle'_{0,\omega = 1} = -\beta \frac{\lambda}{(2\pi)^3} \left( L^2 - 4S^2 \right) / L. \quad (3.71)$$

By contrast, for $\omega \to \infty$, this is,

$$\langle V \rangle'_{0,\omega \to \infty} = -\beta \frac{\lambda}{(2\pi)^3} L. \quad (3.72)$$

As about the second term of (3.69), it should vanish, since there is only one non-trivial mode $\omega$. 

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3.3.2 Solution for small $\mu$ and $x$

Consider now the case of small $\mu$ and $x$. In this case the only available approach is to plug the solution for the eigenvalue density \((D.9)\) into \((3.61)\) to evaluate $\langle V \rangle_0$.

The term $\langle V \rangle_0$ in this approach is of the order $N^0$, and, therefore is small with respect to the terms $\langle V \rangle'$.

Let us turn to the evaluation of $\langle V \rangle'_0$. The leading term in $\mu_\pm$ and $\theta_{mn}$ of the second line of \((3.61)\) reads,

\[
\langle V \rangle'_0 = \frac{\beta g^2_{YM}}{4\pi^2} \sum_{mnk} \frac{-\theta_{mn} \theta_{nk} + \mu_+ \mu_-}{(\mu_+^2 + \theta_{mn}^2)(\mu_-^2 + \theta_{nk}^2)} = \\
\frac{\beta g^2_{YM}}{4\pi^2} \int d\lambda \rho(\lambda) \int d\eta \rho(\eta) \int d\nu \rho(\nu) \\
\times \frac{\mu_+ \mu_- - (\eta - \lambda)(\lambda - \nu)}{(\mu_+^2 + (\eta - \lambda)^2)((\mu_-^2 + (\nu - \lambda)^2)}.
\]

In the incompressible liquid approximation of the Appendix \([D]\) the integral in \((3.73)\) becomes,

\[
\langle V \rangle'_0 = N^2 \frac{\beta \lambda}{2(2\pi)^2 \Lambda} \int_{-\Lambda}^{\Lambda} d\lambda \times \\
\left\{ -\ln \left( \frac{\lambda^2 - (\Lambda + i\mu_+)^2}{\lambda^2 - (\Lambda - i\mu_+)^2} \right) \ln \left( \frac{\lambda^2 - (\Lambda + i\mu_-)^2}{\lambda^2 - (\Lambda - i\mu_-)^2} \right) \\
+ \frac{1}{4} \ln \left( \frac{\mu_+^2 + (\lambda + \Lambda)^2}{\mu_-^2 + (\lambda - \Lambda)^2} \right) \ln \left( \frac{\mu_+^2 + (\mu_+ + \Lambda)^2}{\mu_-^2 + (\mu_- + \Lambda)^2} \right) \right\},
\]

where we use the shortcut notation $\Lambda = \sqrt{\mu_+ \mu_-}$. Performing a substitution $\lambda \to \lambda \Lambda$ in \((3.74)\) we get for the correction,

\[
\langle V \rangle'_0 = N^2 \frac{\beta \lambda}{2(2\pi)^2} F\left( \sqrt{\frac{L_-}{L_+}} \right),
\]

where the function $F(\xi)$ is given by the integral,

\[
F(\xi) = \int_{-1}^{1} d\lambda \left\{ -\ln \left| \frac{\lambda^2 - (1 - \xi)^2}{\lambda^2 - (1 + \xi)^2} \right| \ln \left| \frac{\lambda^2 - (1 + \xi^{-1})^2}{\lambda^2 - (1 - \xi^{-1})^2} \right| \\
+ \frac{1}{4} \ln \left| \frac{-\xi^2 + (\lambda + 1)^2}{-\xi^2 + (\lambda - 1)^2} \right| \ln \left| \frac{-\xi^{-2} + (\lambda + 1)^2}{-\xi^{-2} + (\lambda - 1)^2} \right| \right\},
\]

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which can be computed explicitly and is a combination (rather long) of log and poly-log functions. The integration over the main branch of the log is assumed. $F(\xi)$ has the following properties: It is invariant with respect to inversion: $\xi \rightarrow \xi^{-1}$. Also $F(0) = F(\infty) = 0$ and it has a maximum at one.
4 Spin chain gas at low temperature

Consider the Hamiltonian (3.4) in the regime when the inverse temperature $\beta$ is large. In this regime the commutator term in the Hamiltonian becomes dominating over the quadratic part. If at the same time the Yang–Mills coupling is small enough, the one loop term becomes dominant also over the higher loop contributions. As discussed in [35] (see [36] for the background), at the large value of coupling in front of the commutator term causes configurations with non-vanishing commutator to be statistically suppressed. In the extremal case, when $\beta g_{YM}^2 \rightarrow \infty$, the field is forced to remain in the valley of the potential i.e. the allowed space will be restricted to matrices satisfying,

$$\text{tr} [\bar{\Psi}_a, \bar{\Psi}_b][\Psi_a, \Psi_b] = 0. \quad (4.1)$$

This condition is equivalent to restriction to commuting matrices,

$$[\Psi_a, \Psi_b] = 0. \quad (4.2)$$

Since $\Psi_a$ are not Hermitian, the condition (4.2) is not enough for them to be diagonalizable. The Gauss law constraint,

$$G \equiv [\bar{\Psi}_a, \Psi_a] = 0, \quad a = 1, 2 \quad (4.3)$$

implies, however, that the commuting matrices are also normal. Even more generally, they should satisfy,

$$[\bar{\Psi}_a, \Psi_b] = 0. \quad (4.4)$$

The condition (4.4) is already sufficient for simultaneous diagonalization of all $\Psi_a$ as well as $\bar{\Psi}_a$ whose eigenvalues are complex conjugate to those of $\Psi_a$.

Passing to description in terms of eigenvalues $\Psi^a_n$ of $\Psi_a$ as well as $\Psi^a_n$ of $\bar{\Psi}_a$ in quantum theory can be associated with change of variables in the path integral. The Hamiltonian takes the form,

$$H = \mu \text{tr} \bar{\Psi}_a \Psi_a + \frac{1}{2} \bar{x} \cdot \bar{\sigma}_{ab} \text{tr} \bar{\Psi}_a \Psi_b = \mu_+ \text{tr} \bar{\Psi}_+ \Psi_+ + \mu_- \text{tr} \bar{\Psi}_- \Psi_-, \quad (4.5)$$

where in the last equality we used the eigenbasis of matrix ($\bar{x} \cdot \bar{\sigma}$). The change of variables to diagonal values and angular component gives rise to a Jacobian which is the product of the Vandermonde determinants for all $\Psi$ and $\bar{\Psi}$.

$$\Delta^2(\bar{\Psi}, \Psi) = \prod_a \prod_{m > n} |\Psi^a_m - \Psi^a_n|^2. \quad (4.6)$$
A common trick in the Schrödinger equation approach is to absorb this measure (see e.g. [37]) by a redefinition of the wave function \[ \psi \mapsto \sqrt{\Delta^2(\bar{\Psi}, \Psi)}\psi. \] (4.7)

The continuity of the wave function requires that the sign of the root is chosen in such a way that after the redefinition the wave function becomes antisymmetric in diagonal modes. From the point of view of path integral approach vanishing of the measure at such points means that the configurations with coinciding eigenvalues are prohibited since they have zero probability density.

As large \( N \) or large \( L \) are concerned the number of excluded states is negligible with respect to the total number of states. Then, the partition function can be evaluated by the direct computation of the trace,

\[
Z(\mu, \bar{x}) = \text{tr} e^{-H(\mu, \bar{x})} = \sum_{\{N_n^\pm = 0, 1, 2, \ldots\}} e^{-\mu_+ \sum_n N_n^+ - \mu_- \sum_n N_n^-} = \\
\left[ \frac{1}{1 - e^{-\mu_+}} + \frac{1}{1 - e^{-\mu_-}} \right]^N = \exp \left(-N[\ln(1 - e^{-\mu_+}) - \ln(1 - e^{-\mu_-})] \right). \] (4.8)

As we discussed earlier, large \( L \) and \( \vec{S} \) correspond to small values of the chemical potentials \( \mu_\pm \). In this limit the partition function becomes,

\[ Z(\mu, \vec{x}) = \exp (-N \ln \mu_+ \mu_-). \] (4.9)

Solution to the Legendre transform equations (3.20) gives the following expression for the effective action,

\[ S_{\text{eff}}(L, \vec{S}) = N \ln L_+ L_- = N \ln (L^2 - 4\vec{S}^2). \] (4.10)

5 String theory interpretation

Let us give a qualitative string theory interpretation of above results. We considered the effective action which is the thermodynamical potential counting the number of SYM operators of a given length \( L \) and given spin \( S \). According to the AdS/CFT dictionary this corresponds to the entropy of AdS (multi)string states having \( S \) unites of angular momentum in the (12) plane and \( L - S \) unites of the angular momentum in the (56) plane of \( S^5 \), regarded

\[^{14}\text{Note, however, that in this case the redefinition is an operator action.}\]
as embedded into $\mathbb{R}^6$. (This corresponds to the choice $\phi \propto (\phi_1 + i\phi_2)$ and $Z \propto (\phi_5 + i\phi_6).$)

Basically, we found three different regimes in the behavior of the model which are distinguished by different scaling of the thermodynamical quantities at large $N$: I. $N$-independent, II. linear $N$-scaling and, finally, III. quadratic $N$-scaling. Recall that the meaning of $N$ on the string side is the number of $D3$-branes providing us with $\text{AdS}_5 \times S^5$ background in their vicinity. Consider all three cases in detail.

I. The effective number of degrees of freedom does not depend on $N$ rather on $L$ and $S$. This means that we are in a genuine stringy phase as strings with no internal structure present the effective fundamental excitations which we are counting. This phase occurs at moderate values of $L$: $L \lesssim \sqrt{N}$.

II. Linear $N$-dependence says that we have a number of degrees of freedom proportional to the number of $D3$-branes. Therefore, it is natural to assume, that it is the $D3$-branes which are the fundamental excitations of the theory in this phase. This phase corresponds to large values of $\beta g^2_{YM}$ and moderate $L$. (To have the one-loop contribution leading one has to require also that $\beta g^4$ is small.)

III. The $N^2$ scaling, generally means that the $D3$-brane is not anymore a fundamental excitation but can be considered as a condensate or bound state of smaller objects. The situation is very similar to brane condensation in noncommutative gauge theories [38–41] (for a review see also [42]). This occurs at large $L$: $L > \sqrt{N}$. Remarkably, the model in this regime is best described in terms of noncommutative gauge theory [19].

### 6 Discussion

In this paper we considered a non-Hermitian matrix model describing the anomalous dimension spectrum of $\mathcal{N} = 4$ SYM theory. In this study we did not recur to the integrability at infinite $N$ neither to the description in terms of spin system. Moreover, $N$ all the time was kept a finite statistically large number. We introduce a notion of temperature, which in the SYM theory plays the role of auxiliary parameter allowing restoration the density of the anomalous dimensions of composite SYM operators. At the same time due to the fact that the dilatation operator is the Hamiltonian corresponding to the
radial time in conformal theory this temperature corresponds to compactification of the SYM to the four-dimensional sphere $S^4$. In the framework of string/gauge duality this temperature is the genuine temperature of the dual string theory and can be attributed to the presence of a black hole [22].

For the two-matrix model we succeeded to compute the thermal partition function and the effective action depending on the total occupation number, which in the spin language is the aggregate spin chain length, and the occupation number difference, which corresponds to the total spin. We did this in the both approximation of high as well as of low temperature. The most attention and effort was given to the high temperature regime. In this regime we found a rather elegant approach based on the analytic expansion in terms of exponentials of chemical potentials. This approach allows an exact saddle point evaluation of the partition function. For sufficiently large chemical potentials $\mu_{\pm}$ the contribution to thermodynamical potentials which scale like $N^2$ and $N$ does not appear. This is compatible with the PET (Pólya Enumeration Theorem) evaluations which holds for $\mu$ above the Hagedorn critical value. As in our case the spin is also included we may conjecture that the phase separation should occur along the critical line given by\footnote{This is also supported by the random walk approach [32].}:

$$1 - e^{-\mu_+} - e^{-\mu_-} = 0,$$

at which the first non-trivial mode to the saddle point equation appear. (In the case of zero $x$ the above critical line reduces exactly to the Hagedorn criticality condition for $\mu$, note that $\mu$ for the oscillator has the same meaning as the temperature.) In this regime the fundamental excitations are stringy-like.

Using the incompressible liquid model approximation for the gauge field eigenvalue condensate we found a description of the model at small chemical potentials $\mu_{\pm}$, which can be trusted at least qualitatively. For this regime we found a $N^2$ behavior, which corresponds to string bit phase. This phase is characterized by melting of strings into point objects: string bits. As a result the model looks as a system of $N^2$ interacting particles.

At small temperatures the matrix fluctuations are bind to the diagonal. Therefore the effective number of degrees of freedom is proportional to $N$. As $N$ is the number of $D3$-branes in the string theory, the natural assumption is that they are the elementary excitations in this regime.

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A Anti-holomorphic representation

A very useful representation of wave functions of quantum oscillators is given by the anti-holomorphic representation. For the convenience of the Reader and to fix the notations we give here a brief account of this representation.

Consider the oscillator algebra generated by the ladder operators $a$ and $a^\dagger$ which are subject to the commutation relations

$$[a, a^\dagger] = 1. \quad (A.1)$$

We can represent the quantum states in the theory as anti-holomorphic function $f(z)$ of a complex variable $z$. The action of the ladder operators on such states is given by

$$a^\dagger \to \bar{z}, \quad a \to \partial / \partial \bar{z}, \quad (A.2)$$
i.e. the raising operator $a^\dagger$ acts by multiplication by $\bar{z}$, while the lowering $a$ acts as a derivative. Obviously the commutation relation (A.1) is satisfied by such definition.

Now it is not very difficult to recover all formulas used to construct the representation. The oscillator vacuum is given by the zero mode of the lowering operator

$$a |0\rangle = 0 = \partial \varphi_0 / \partial \bar{z} \Rightarrow \varphi_0 = \text{constant}. \quad (A.3)$$

We can choose $\varphi_0 = 1$. In this case the orthonormal set of the oscillator hamiltonian eigenstates is represented by

$$|n\rangle \sim \bar{z}^n / \sqrt{n!}, \quad (A.4)$$

from which we can immediately extract the scalar product rule for arbitrary (anti-holomorphic) states described by $f(\bar{z})$ and $g(\bar{z})$:

$$\langle f | g \rangle = \int d\bar{z} d\bar{z} / \pi \ e^{\bar{z} \bar{z}} f^*(z) g(\bar{z}). \quad (A.5)$$

A “well defined” operator $F$ can be represented by an anti-holomorphic kernel $F(\bar{z}, z)$. The scalar product (A.5) implies the following rule for the kernel multiplication:

$$F \cdot G \sim \int d\bar{w} dw / \pi \ e^{-\bar{w} w} F(\bar{z}, w) G(\bar{w}, z), \quad (A.6)$$
In particular, the trace of operator $F$ is given by

$$\text{tr} \ F = \int \frac{d\bar{w}dw}{\pi} e^{-\bar{w}w} F(\bar{w}, w). \quad (A.7)$$

In many cases the normal symbol of an operator is used. Therefore, it is useful to know how to pass from one description to another.

Consider an operator $F$ represented in the normal form:

$$F = F_N(a^\dagger, a) \equiv \sum_{m,n} f_{m,n}(a^\dagger)^m a^n. \quad (A.8)$$

Since the operators $a^\dagger$ and $a$ are normal ordered inside $F_N(a^\dagger, a)$ one can treat it as usual function $F_N(\bar{z}, z)$ of complex variable $z$. By analyzing the the matrix element of $F$ between two oscillator states $\langle m |$ and $| n \rangle$ one finds, that the normal symbol $F_N(a^\dagger, a)$ is related to the anti-holomorphic symbol in the following simple way:

$$F(\bar{z}, z) = e^{\bar{z}z} F_N(\bar{z}, z). \quad (A.9)$$

The last useful formula we want to give here is the anti-holomorphic symbol of the exponent of an operator with quadratic normal form:

$$U = e^{-\beta H}, \quad H = ha^\dagger a, \quad (A.10)$$

e.g. the anti-holomorphic symbol for the evolution operator of the harmonic oscillator. Again, the direct computation gives,

$$U(\bar{z}, z) = \exp \left( \bar{z} e^{-\beta h} z \right). \quad (A.11)$$

As a simple test let us compute the trace of (A.11). Modulo vacuum contribution this gives the oscillator partition function,

$$\text{tr} \ e^{-\beta H} = \int \frac{d\bar{w}dw}{\pi} \exp \left[ -\bar{w}(1 - e^{-\beta h}) w \right] = \frac{1}{1 - e^{-\beta}}. \quad (A.12)$$

The generalization to higher dimensions is straightforward.

B Anti-Holomorphic representation with gauge symmetry

Let us consider a situation when oscillator possesses a gauge symmetry,

$$z \mapsto z^g, \quad g \in G. \quad (B.1)$$
This pulls back as a $G$-transformation of the Hilbert space,

$$\psi(\bar{z}) \mapsto T(g)\psi(\bar{z}). \quad (B.2)$$

where $T(g)$ is the generator of the induced action. Infinitesimally,

$$T(g = e^u) \approx I + t(u), \quad t(u)\psi(z) \approx \psi(z^e) - \psi(\bar{z}), \quad (B.3)$$

where $t(u) = t_\alpha(\bar{z})u^\alpha$ are the generators of infinitesimal gauge transformations. Generally $t_\alpha$ have the form,

$$t_\alpha(\bar{z}) = t^i_\alpha(\bar{z})\bar{\partial}_i, \quad \bar{\partial}_i \equiv \frac{\partial}{\partial \bar{z}^i} \quad (B.4)$$

where $t^i_\alpha(\bar{z})$ are just ordinary functions,

$$t^i_\alpha(\bar{z}) = \frac{\partial^2 z^g(u)}{\partial u^\alpha \partial \bar{z}^i} \bigg|_{g=1}. \quad (B.5)$$

The gauging consists in projecting the Hilbert space to the subspace of invariant functions, which amounts to finding subspace

$$t_\alpha \cdot \psi(\bar{z}) = 0. \quad (B.6)$$

According to the canonical quantization procedure (B.6) is equivalent to imposing the on-shell constraint condition since according to the second theorem by E. Noether, operator $t_\alpha$ correspond to first-class constraint in the classical theory. When $G$ possesses an invariant Haar measure e.g. is a compact Lie group solution to (B.6) is remarkably simple

$$\psi_{\text{gauge invariant}}(\bar{z}) = \int_G d\bar{g} \psi(\bar{z}^\bar{g}), \quad (B.7)$$

where we integrate over the gauge group using the invariant Haar measure $dg$: $d(h^{-1}gh') = dg$. Hence the Haar integral (B.7) can be used as the projector to gauge invariant subspace of the Hilbert space. It is not difficult to check that for an operator with gauge invariant kernel: $K(\bar{z}, z) = K(\bar{x}, x)$ action commutes with averaging over the gauge group,

$$K \cdot \Pi \cdot \psi = \int \frac{d\bar{w} dw}{\pi} e^{-\bar{w}w} K(\bar{z}, w)\psi(\bar{w}) = \int \frac{d\bar{w} dw}{\pi} e^{-\bar{w}w} K(\bar{z}^\bar{g}, w)\psi(\bar{w}) = \Pi \cdot K \cdot \psi. \quad (B.8)$$

\(^{16}\)I am grateful to Jeong-Hyuck Park for pointing my attention to this possibility.
In particular, the gauged analog of (A.12) reads,
\[
\text{tr } \Pi \cdot e^{-\beta H} = \int \frac{d\bar{w}d\bar{w}}{\pi} \exp \left[ -\bar{w}w^g + \bar{\omega}e^{-\bar{\omega}}w \right] = \int dg \left[ \det (T(g) - e^{-\omega}) \right]^{-1}. \tag{B.9}
\]

C The trigonometric sums

We use the following trigonometric sums,
\[
\sum_{mnk} \cos (\omega \theta_{mn} + \omega' \theta_{kn}) = \sum_{mnk} \cos (\omega \theta_m) \cos (\omega' \theta_k) \cos ((\omega + \omega') \theta_n)
- \sum_{mnk} \sin (\omega \theta_m) \sin (\omega' \theta_k) \cos ((\omega + \omega') \theta_n)
+ \sum_{mnk} \sin (\omega \theta_m) \cos (\omega' \theta_k) \sin ((\omega + \omega') \theta_n)
+ \sum_{mnk} \cos (\omega \theta_m) \sin (\omega' \theta_k) \sin ((\omega + \omega') \theta_n)
= \tilde{\rho}_\omega \tilde{\rho}_{\omega'} \tilde{\rho}_{\omega+\omega'} - \tilde{\rho}_\omega \tilde{\rho}_{\omega'} \tilde{\rho}_{\omega+\omega'} + \tilde{\rho}_\omega \tilde{\rho}_{\omega'} \tilde{\rho}_{\omega-\omega'} + \tilde{\rho}_\omega \tilde{\rho}_{\omega'} \tilde{\rho}_{\omega+\omega'} + \tilde{\rho}_\omega \tilde{\rho}_{\omega'} \tilde{\rho}_{\omega-\omega'}, \tag{C.1}
\]

where,
\[
\tilde{\rho}_\omega = \sum_n \cos \omega \theta_n, \quad \tilde{\rho}_{\omega'} = \sum_n \sin \omega \theta_n. \tag{C.2}
\]

As we are considering even distributions of \(\theta_n\), sums involving the sines vanish, i.e. \(\tilde{\rho}_{\omega'} = 0\) for any \(\omega\). Hence, the sum is reduced to,
\[
\sum_{mnk} \cos (\omega \theta_{mn} + \omega' \theta_{kn}) = \tilde{\rho}_\omega \tilde{\rho}_{\omega'} \tilde{\rho}_{\omega+\omega'}. \tag{C.3}
\]

D Incompressible liquid of eigenvalues

Consider the static equilibrium problem for a system of particles with the pairwise interaction potential \(\varphi(\lambda_{mn})\) given by (3.41) where \(\lambda_{mn}\) is the distance between particles with numbers \(m\) and \(n\). The structure of the potential is such that at small distance the interaction has repulsive character while at large separation it is attractive. Thus we may expect that \(N\) particles interacting with such a potential should have a static equilibrium position.
Let us further assume that this arrangement is compact for large $N$, i.e. the eigenvalue density vanishes exactly outside some finite region. Let us assume this region to be connected of size $\Lambda$. The condition of equilibrium requires that in the presence of non-zero density the potential $\Phi(\lambda)$ defined as

$$\Phi(\lambda) = \int_{-\Lambda/2}^{\Lambda/2} d\eta \rho(\eta) \varphi(\lambda - \eta), \quad (D.1)$$

is constant $\Phi(\lambda) \equiv \Phi_0$. Due to this condition the total energy is given by,

$$E = \int d\lambda \rho(\lambda) \Phi(\lambda) = N\Phi_0, \quad (D.2)$$

where we used the normalisability condition for the density.

Let us find the size of the distribution in the approximation of incompressible condensate. For this let us consider the Taylor expansion coefficients of the potential in the vicinity of origin:

$$\Phi(\lambda) \equiv \Phi_0 = \Phi(0) + \Phi'(0)\lambda + \frac{1}{2}\Phi''(0)\lambda^2 + \ldots \quad (D.3)$$

The zeroth term should give just $\Phi_0$,

$$\Phi_0 = \rho_* \int_{-\Lambda/2}^{\Lambda/2} d\eta \varphi(\eta), \quad (D.4)$$

where $\rho_*$ is an average value of the density, in present approximation we can replace it by,

$$\rho_* = \frac{N}{\Lambda}. \quad (D.5)$$

The first term in the expansion vanishes automatically due to the symmetry of the distribution. Consider the third term:

$$\Phi''_0 = \rho_* \int_{-\Lambda/2}^{\Lambda/2} d\eta \rho(\eta) \varphi'(\eta) = 2\rho_* \varphi'(\Lambda/2). \quad (D.6)$$

The vanishing of $\Phi''_0$, which is zero pressure condition at the origin allows one to find the size of the condensate $\Lambda$,

$$\varphi'(\Lambda/2) = 0. \quad (D.7)$$

Now, plugging the the potential (3.41) into into Eq. (D.7) we get:

$$\Lambda = 2\sqrt{\mu_+\mu_-}, \quad \rho_* = \frac{N}{2\sqrt{\mu_+\mu_-}}, \quad (D.8)$$
i.e. the eigenvalue density is described by,

\[ \rho(\lambda) = \begin{cases} \frac{N}{2\sqrt{\mu_+\mu_-}}, & -\sqrt{\mu_+\mu_-} < \lambda < \sqrt{\mu_+\mu_-} \\ 0, & |\lambda| \geq \sqrt{\mu_+\mu_-}. \end{cases} \]  

(D.9)

All this leads to the following result for the “energy”,

\[ 4N^2\mathcal{E} = -4N^2 \left( \sqrt{\frac{\mu_+}{\mu_-}} \tan^{-1} \left( \sqrt{\frac{\mu_-}{\mu_+}} \right) + \sqrt{\frac{\mu_-}{\mu_+}} \tan^{-1} \left( \sqrt{\frac{\mu_+}{\mu_-}} \right) \right). \]  

(D.10)

E Entropy of random spin states

Let us compute the number of the states of spin \( s \) of a set of \( L \) spin 1/2 states. It is instructive to compare the entropy of this set of random spin states with the one of the matrix model.

The problem is related to one of computation of the multiplicity \( \nu_s^{(L)} \) of the irreducible representation of spin \( s \) in the expansion of the product of \( L \) spin-1/2 representations:

\[ \underbrace{\frac{1}{2} \times \frac{1}{2} \times \ldots \times \frac{1}{2}}_{L \text{-times}} = \sum_s \nu_s^{(L)} s, \]  

(E.1)

where we use the boldface letters \( s \) to denote the irreducible representations of spin \( s \).

To find the expansion (E.1), let us observe that the product of an irreducible representation of \( s \) with \( 1/2 \) results in

\[ \frac{1}{2} \times s = (s + \frac{1}{2}) + (s - \frac{1}{2}). \]  

(E.2)

Consequent multiplication by \( 1/2 \) in the l.h.s. of spin \( 1/2 \) representation in (E.1) can be represented by the diagram \( \square \) where each dot represent an irreducible representation of \( \text{su}(2) \). According to (E.2) multiplication by an additional factor of \( 1/2 \) gives rise to a representation of representations of spin differing by \( \pm 1/2 \) with factor one each. On the picture this is represented by arrows. Thus the the total number of factors of spin \( s \) one gets after multiplication of \( L \) \( 1/2 \) factors is given by the number of distinct paths of length \( L \) one can reach the level corresponding to the spin \( s \) from the left-most position following the arrows.

Thus, the problem is reduced to the computation of the number of distinct paths to reach the level \( s \). To do this let us note that the representations form families corresponding to straight lines starting from the bottom. Let us assign a number \( k = 0, \ldots, L \) to each line. The multiplicity of irreducible
Figure 1: Triangle of $L$ products of $1/2$ representations. The horizontal levels count (in the direction $s$) the spin of irreducible representations. The vertical dotted lines correspond to the length of the product. Thus the maximal length in this picture corresponds to the product of 11 $1/2$. The number to the left of each dot represents the number of distinct ways the dot can be reached from the “zero point” following the arrows. This number gives the degeneracy of the representation given by the level in the decomposition of the product of $1/2$ in the number of factors is given by the horizontal coordinate $L$.

representations found on a particular line satisfy simple properties. For example the multiplicities on the zeroes line are all equal to one. On the line number one the multiplicities are given by the sum of multiplicities from the zeroes line up to the next number of the sequence term. In general, the multiplicity on the $k$th line is connected to ones on the $(k - 1)$th one by the following recurrence relation

$$
\nu_k^n = \sum_{l=2}^{n+1} \nu_{k-1}^l, \quad \nu_k^1 = \nu_{k-1}^2.
$$

(E.3)

The recurrence relation (E.3) can be solved which leads to the following
expression for the multiplicities,

\[
\nu_k^n = n \frac{(n + 2k - 1)!}{k!(n + k)!}.
\]  

(E.4)

Now taking into account that \(k\) and \(n\) can be expressed in terms of the number of spins \(L\) and and the spin \(s\) as follows\(^{17}\)

\[
L = n + 2k - 1, \quad s = L/2 - k = (n - 1)/2,
\]

(E.5)

one can rewrite the multiplicities in the following form

\[
\nu_s^{(L)} = (2s + 1) \frac{L!}{(L/2 - s)!(L/2 + s + 1)!}.
\]

(E.6)

The check that the sum of the dimensions all irreducible representations taking into account the multiplicities is indeed

\[
\sum_s (2s + 1)\nu_s^{(L)} = 2^L,
\]

(E.7)

is left to the reader.

In the limit of large \(L\) one can use the Stirling’s approximation to the factorials. This gives the log number of \(L\) random spin states of spin \(s\) to be,

\[
S_{\text{random spin}} = L \ln L - \frac{1}{2} \{ (L - 2s) \ln(L - 2s) + (L + 2s) \ln(L + 2s) \}
\]

(E.8)

It is useful to compare this quantity to the spin entropy of the matrix oscillator in the main body of the paper.

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