Extended-Lorentz Quantum-Cosmology Symmetry Group

$U(1) \times SD(2,c)_L \times SL(2,c)_R$

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Summary

Unitarily representable by transformations of Milne quantum-universe (MQU) Hilbert-space vectors is a 9-parameter ‘extended-Lorentz’ Lie group whose algebra comprises 9 conserved MQU-constituent (‘quc’) attributes: electric charge, energy, spirality, 3-vector momentum and 3-vector angular momentum. Commutation with the full symmetry algebra by the 3-element Lorentz-extending sub-algebra identifies any quc by its (permanent) trio of charge, spirality and energy integers.

Milne’s redshift-specifying ‘universe age’ is invariant under the MQU symmetry group. Also invariant is the (elsewhere specified) universe hamiltonian—a self-adjoint age-dependent Hilbert-space operator (not a symmetry-algebra member) that generates universe evolution with increasing age through a ‘Schrödinger’ (first-order) differential equation.

Remark: Ontological (not mathematical) language recognizes the ‘objective reality’ of a ‘particle’—e.g., a photon. An MQU particle is an approximately-stable ‘relationship’ between different qucs—a ‘marriage’ whose persistence allows ‘recognition’ by intermediate-scale quc aggregates endowed with ‘consciousness’. ‘Intermediate scale’ means particle-scale-huge while Hubble-scale-tiny.
Introduction

Dirac quantum theory--based on self-adjoint Hilbert-space operators--has for physics been impeded by the absence of unitary finite-dimensional Lorentz-group representations. The Gelfand-Naimark (GN) unitary Hilbert-space infinite-dimensional representation, (1) although unusable by a physics founded on positive-energy objective reality, applies cosmologically to a Milne (redshifting, ‘big-bang’) (2) quantum universe (MQU) where constituent energies may be positive or negative and where total energy vanishes. (3)

Milne’s cosmological application of the Lorentz group--to a universe spacetime whose hyperbolic 3-space is invariantly metricized, is profoundly different from the physics application of the Lorentz group--to a spacetime where 3-space is Euclidean with non-invariant metric. Widely-misunderstood Milne cosmology associates ‘redshift’ so directly to universe age as to render redshift ‘trivial’.

Four-dimensional Milne spacetime occupies the interior of a forward lightcone, where the positive (arrowed) ‘age’, τ, of any location is its ‘Minkowski distance’ from the lightcone vertex. MQU is governed by the symmetries of a here-specified 9-parameter Lie group that we designate as U(1) × SD(2,c)l × SL(2,c)r—a product of four individually-semisimple subgroups, three of which, each 1-parameter, commute with the entire group. Universe age (not a Hilbert-space operator) and universe hamiltonian (a self-adjoint operator, although not a member of the group algebra) are both invariant under the 9-parameter MQU symmetry group.

A 2×2 unimodular-matrix meaning for the subscripts L and R in the foregoing notation will here be explained, as well as use of the symbol D in a notational location familiarly occupied by symbols such as U or L. Each of the nine members of the MQU algebra is (Stone) representable by a self-adjoint Hilbert-space operator. The (conserved) algebra corresponds to electric charge, ‘spirality’, energy, and a 6-vector (nonabelian) combination of momentum and angular momentum.

Particle physics is a local positive-energy restricted-scale approximation to MQU that ignores (redshifting) universe expansion. Not yet established but plausible is an approximate relation between charge, spirality and baryon number that accompanies approximate CP and CPT particle-physics symmetries. Reference (3) addresses both the meaning of spirality and the physics ‘spatial-flattening’ of MQU symmetry to the 10-parameter group that Wigner associated to the name of Poincaré.

The present paper and Reference (3) both employ the (pronounceable) acronym ‘quc’ to denote an ‘MQU constituent’. Each quc separately represents the symmetry group. The total number of qucs, although huge, is finite and constant--age independent because the hamiltonian lacks quc annihilation or creation operators. Each of the nine group generators represents a ‘nameable’ conserved quc attribute. The integer-valued spirality attribute distinguishes odd-integer ‘fermionic’ qucs from even-integer ‘bosonic’.
Objective reality—positive-energy temporarily-stable spatially-localized ‘measurement-accessible’ current density—involves at least two qucs. A single quc cannot represent ‘matter’—the definition of which requires some stable self-sustaining relationship between different qucs.

Fiber-Bundle Factorization of a Quc’s 6-Dimensional Unimodular 2×2 Matrix Coordinate

The 6-dimensional complex-unimodular 2×2 matrix coordinate of a quc (uncovered by GN in a purely-mathematical non-cosmological context\(^{(1)}\)) admits factorization into a product of individually-unimodular 3-dimensional unitary and positive-hermitian 2×2 coordinate matrices. The unitary factor represents the unmetricized (‘directional’) fiber of a bundle whose metricized (geometric) base space is represented by the positive-hermitian factor. Either order of the two factors is possible with the same unitary factor, but when the hermitian factor stands on the right it is different from, although unitarily equivalent to, the hermitian factor when standing on the left.

Three ‘Euler’ angles,

\[ 0 \leq \phi' < 4\pi, \ 0 \leq \theta < \pi, \ 0 \leq \varphi < 2\pi, \]

specify, collectively, the unitary 2×2 matrix,

\[ u = \exp{(i\sigma_3\phi')/2} \exp{(i\sigma_1\theta/2)} \exp{(i\sigma_3\varphi/2)}. \]

The matrix (2) is isomorphic to a unit-radius 3-sphere.

In Formula (2) the symbols \( \sigma_1 \) and \( \sigma_3 \) denote (Pauli) hermitian traceless self-inverse real 2×2 matrices, each with determinant -1, \( \sigma_3 \) being diagonal and \( \sigma_1 \) off-diagonal. (Any boldface symbol in the present paper denotes a 2×2 matrix.) If the full 6-dimensional unimodular quc-coordinate matrix is denoted by the symbol \( a \), then

\[ a = u \ h_R = h_l \ u, \text{ with } h_l = u \ h_R \ u^{-1}. \]

A positive-trace hermitian 2×2 matrix may represent either a right or left Lorentz positive 4-vector. A right 4-vector transforms under the group \( \text{SL}(2,\mathbb{C})_R \) by multiplication from the right by the unimodular matrix representing this group and multiplication from the left by the hermitian conjugate of this same matrix. A left 4-vector transforms under the group \( \text{SL}(2,\mathbb{C})_L \) by multiplication from the left by the unimodular matrix representing the latter and multiplication from the right by this matrix’s hermitian conjugate.
Hyperbolic Base-Space Metric

Milne’s Lorentz-invariantly-metricized hyperbolic 3-space is isomorphic to the ‘base space’ of a (classical) fiber bundle that requires reference neither to Hilbert space nor to ‘quc’. Reference (3), on the other hand, represents the invariant MQU hamiltonian’s quc kinetic-energy as a ‘hyperbolic laplacian’—a self-adjoint Dirac operator acting on complex normed functions of quc base-space.

The MQU kinetic-energy operator, a GN-uncovered positive function of extended-Lorentz-group Casimirs, (1) maintains such fiber-ignoring base-space meaning when fiber space has dimensionality 2 rather than 3. Reduction of fiber dimensionality will below be related to the meaning of ‘spirality’. The discrete meaning of both electric charge and spirality involves Hilbert space.

The combination of quc Hilbert space with quc classical fiber-bundle we call ‘quc fiber package’. This paper will identify a package with 2-dimensional fiber. The present section, however, addresses two alternative coordinations of a 6-dimensional ‘classical’ bundle where the fiber occupies a 3-sphere. Here, despite our use of Pauli matrices and complex numbers, we are ignoring Hilbert space.

Unitary equivalence of right and left hermitian base-space coordinate is conveniently representable in Pauli-matrix notation through 3-vector inner products. The imaginary Pauli matrix,

\[ \sigma_2 = -i\sigma_3\sigma_1, \]  
(4)

also hermitian (\( \sigma_3 \) and \( \sigma_1 \) anticommute), self-inverse, traceless and with determinant -1, combines with \( \sigma_1 \) and \( \sigma_3 \) to define a ‘handed matrix 3-vector’. Defining a unit-magnitude ‘direction 3-vector’ \( n \) by the ordered set of components

\[ n_1 = \sin \theta \cos \phi, \quad n_2 = \sin \theta \sin \phi, \quad n_3 = \cos \theta, \]  
(5)

with

\[ 0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi, \]  
(6)

the symbol \( \sigma \cdot n \) represents the 3-vector inner product

\[ \sigma \cdot n \equiv n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3, \quad (n \cdot n = 1). \]  
(7)

The positive-hermitian (base-space) factor of the 6-dimensional quc-coordinate unimodular matrix, when standing to the right (left) of the unitary (fiber) factor \( u \), will be denoted by the symbol \( h_R \) (\( h_L \)). Any positive-hermitian unimodular 2×2 matrix may be written in the form \( \exp(-\frac{i\theta}{2} \sigma \cdot n) \), with \( \theta \geq 0 \).
We shall denote the left (right) fiber-bundle base-space (positive-hermitian) factor by the symbol: \( \exp(-\frac{1}{2}\beta \sigma \cdot n_L R) \). It follows from (3) that \( \sigma \cdot n_L = u \sigma \cdot n_R u^{-1} \)—the left and right directional (unit) 3-vectors being related by a rotation that leaves \( \beta \) unchanged. The symbols \( \beta_{R(L)} \) will sometimes be employed to denote the 3-vectors \( \beta n_{R(L)} \).

Right-left invariance of (non-negative) \( \beta \) reflects this symbol’s interpretability as ‘shortest distance’ between two different quc locations in the 3-dimensional hyperbolic base space, one of the two locations being at this space’s origin.

The hyperbolic 3-dimensional fiber-bundle base space, occupied by a (fixed) finite although huge set of qucs at some fixed value of age, is invariantly metricized by

\[
(d\beta)^2 + \sinh^2 \beta (dn_L \cdot dn_L = dn_R \cdot dn_R),
\]

with

\[
(d\theta_{R(L)})^2 + \sin^2 \theta_{R(L)} (d\phi_{R(L)})^2.
\]

This paper will later denote simply by \( n \) a unit 3-vector equal to \( n_R \), but any coordination of 3-dimensional base-space maintains Milne’s (redshift-stipulating) hyperbolic \( \text{not elliptic} \) curvature.

Any ‘Lorentz boost’, either from right or left, when applied to all quc coordinates, merely shifts base-space origin to a new location, without altering the relationship between different qucs (such as the spatial distance between members of a pair). More generally, any element of \( \text{SL}(2,c)_L \times \text{SL}(2,c)_R \) , applied to all quc coordinates of MQU, leaves the universe unchanged.

Although the present paper considers only right 4-vectors, it attends to a 2-parameter abelian left-acting group \( \text{SD}(2,c)_L \) , a subgroup of \( \text{SL}(2,c)_L \) that transforms a quc’s coordinate matrix \( a \) by multiplication from the left by a diagonal 2×2 complex unimodular matrix. The reader may understand the meaning of the symbol \( D \) either as ‘diagonal matrix’ or as ‘displacement of a complex number’. (In the following section the displaced complex quc coordinate will be denoted by the symbol \( s \).)

The nonabelian 6-parameter group \( \text{SL}(2,c)_R \) transforms the coordinate \( a \) through right multiplication by a unimodular complex 2×2 matrix. The groups \( \text{SD}(2,c)_L \) and \( \text{SL}(2,c)_R \) commute with each other. The factorization \( a = u h_R \) of the 6-dimensional quc coordinate matches our later Hilbert-space representation of a 9-parameter group that is isomorphic to \( \text{U}(1) \times \text{SD}(2,c)_L \times \text{SL}(2,c)_R \) , rather than to this group’s L↔R equivalent. (In their representation of \( \text{SL}(2,c) \), GN \( ^{(1)} \) made the arbitrary choice between L and R that we employ here. The reader, if a physicist, is warned that in representing SU(2) Wigner made the opposite choice.)
Alternative Factorizations of the Quc-Coordinate-Matrix; Spirality

An alternative to Formula (3) is a factorization of the (6-dimensional) unimodular 2×2 complex quc-coordinate matrix $a$ into a product of three unimodular 2×2 matrices, each coordinating the manifold of a 2-parameter abelian $\text{SL}(2,c)$ subgroup (acting from either right or left):

$$a(s, y, z) = \exp(-\sigma_3 s) \times \exp(\sigma_+ y) \times \exp(\sigma_- z),$$

where the real-matrix pair $\sigma_{\pm}$ is defined as $\frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and each of the symbols $s, y, z$ denotes a complex variable.

The leftmost factor in (10) is a diagonal 2×2 matrix which may itself be written as the product, $\exp(-i\sigma_3 \text{Im } s) \times \exp(-\sigma_3 \text{Re } s)$, of a commuting pair of unitary and positive-hermitian unimodular diagonal 2×2 matrices. It is useful to define a 5-parameter quc-coordinate matrix

$$b \equiv \exp(i\sigma_3 \text{Im } s) \times a$$

$$= \exp(\sigma_3 \text{Re } s) \times \exp(\sigma_+ y) \times \exp(\sigma_- z),$$

that depends on $\text{Re } s$, $y$, $z$ but not on $\text{Im } s$.

One dimension of the 3-sphere unitary factor in (3) thereby becomes recognized as enjoying status distinct from that of a remaining dimension pair. [The 3-sphere of Formula (1) is the product of an ‘ordinary’ 2-sphere and a $4\pi$-circumference circle.] The distinguished fiber coordinate, $\text{Im } s$, is Dirac conjugate to the self-adjoint operator representing the symmetry-group generator we call ‘spirality’. (3) The (ordinary) 2-sphere ‘remainder’ of quc-fiber 3-space we call ‘quc velocity-direction space’.

Positive Right 4-Vectors that Specify Quc Location in Base and Velocity-Direction Spaces

Two right 4-vectors, one positive-timelike and one positive lightlike, are equivalent to a quintet of real quc coordinates (the $y, z$ complex-coordinate pair plus $\text{Re } s$) that specify the 2×2 unimodular matrix $b$. In the corresponding ‘quc package’ neither a quc’s location in 3-dimensional metricized package base space nor its location in a 2-dimensional quc velocity-direction space depends on $\text{Im } s$. Locations in base-space and velocity-direction space are collectively equivalent to $b$.

Formulas (3) and (11) together expose the positive-hermitian unimodular 2×2 matrix,

$$B \equiv b^\dagger b$$

$$= e^\beta \sigma \cdot \eta,$$
as a dimensionless positive-timelike right 4-vector. The dimensionful positive factor $\tau$ (age) then allows the symbol $x \equiv \tau B$ to denote a 4-vector which locates a quc within Milne spacetime—the interior of a forward lightcone—by prescribing the quc’s displacement from the lightcone vertex. In a more familiar notation the 4 components of $x$ are $\tau \cosh \beta$, $\tau \sinh \beta$.

Complementing dimensionless $B$, which coordinates a fiber-bundle’s metricized base space, is a second dimensionless positive right 4-vector—this one lightlike—to be denoted by the symbol $v$ and coordinating a 2-dimensional unmetricized fiber. The 4-vector pair $B, v$ is equivalent to $b$—an equivalence related below to invariant 4-vector inner products. (The inner product of any two positive 4-vectors is non-negative.)

The invariant inner product of two right 4-vectors will be denoted by the symbol $\cdot$. The inner product of any two right 4-vectors may be shown equal to the inner product of the unitarily-equivalent left 4-vector pair, so either product is invariant under $\text{SL}(2,c) \times \text{SL}(2,c)_R$ and, thereby, under the extended Lorentz group.

The quc velocity-direction right 4-vector is defined to be the dimensionless zero-determinant positive-hermitian matrix

$$v \equiv b^\dagger (\sigma_0 - \sigma_3)b,$$

(14)

where the symbol $\sigma_0$ denotes the unit $2 \times 2$ matrix. Equivalence of the 5-dimensional coordinate matrix $b$ to the positive 4-vector pair $B, v$ accompanies the trio of inner products, $B \cdot v = 1$, $B \cdot B = 1$ and $v \cdot v = 0$, deducible by right-transforming to a special frame where $b = \sigma_0$. Explicit evaluation of Formula (14) via (10) and (11) reveals $v$ independence of $y$—the 4-vector $v$ being determined entirely by $z$ and $\text{Re} \ s$.

**Fiber-Package Unitary Representation of Extended-Lorentz Group**

An MQU ray is a sum of (‘tensor’) products, each with a below-discussed age-independent number of factors, of single-quc Hilbert-space vectors that each unitarily represents the 9-parameter extended-Lorentz group. A regular-basis electric-charge-$Q$ Hilbert-space single-quc vector for Age $\tau$ is a complex differentiable function $\psi_Q^{\tau}(a)$ with invariant (finite) norm,

$$\int d\alpha \left| \psi_Q^{\tau}(a) \right|^2.$$

(15)

The invariant 6-dimensional volume element (Haar measure) $d\alpha$ we now express through the trio (10) of complex-variable coordinates equivalent to $a$.

In the interest of notational simplicity we shall henceforth omit the age superscript $\tau$. (Already omitted is a label to distinguish the quc in question from others with the same electric charge.) Further to be ignored except in Eq. (21) is the integer-$Q$ quc-charge subscript; $U(1)$ transformation (Kaluza-Klein) will be seen merely to shift wave-function phase by an increment proportional to $Q$.

The 6-dimensional Haar measure,

$$d\alpha = ds dy dz,$$

(16)
is invariant under \( a \rightarrow a' = a \Gamma^{-1} \), with \( \Gamma \) a 2×2 unimodular matrix representing a right Lorentz transformation of the coordinate \( a \). The measure (16) is also invariant under analogous left transformation. The ‘volume-element’ symbol \( d \xi \) in (16), with \( \xi \) complex, means \( d \text{Re} \, \xi \times d \text{Im} \, \xi \).

The Hilbert-vector norm-defining integration (15) is, wrt \( \text{Im} \, s \), over any continuous 2π interval of \( \text{Im} \, s \). Below we shrink the Hilbert space so that \( \text{Re} \, s \) and \( \text{Im} \, s \) enjoy similar status in vector-norm definition. The norm (16) is then not invariant under the full left nonabelian group of Lorentz transformations but only under the 2-parameter abelian diagonal-matrix \( (D) \) left subgroup. Invariance under \( \text{SL}(2, \mathbb{C}) \) \( \mathbb{R} \) is unaffected.

A symmetry transformation specified by the 2×2 complex unimodular right-acting matrix \( \Gamma \) is unitarily Hilbert-space represented by

\[
\Psi(a) \rightarrow \Psi(a \Gamma^{-1}).
\]

Calculation shows \( a \Gamma^{-1} \) to be equivalent to

\[
\begin{aligned}
z' &= (\Gamma_{22}z - \Gamma_{21}) / (\Gamma_{11} - \Gamma_{12}z), \\
y' &= (\Gamma_{11} - \Gamma_{12}z)(\Gamma_{11} - \Gamma_{12}z)y - \Gamma_{12}, \\
s' &= s + \ln (\Gamma_{11} - \Gamma_{12}z).
\end{aligned}
\]

We now make explicit the single-quc Hilbert-space representation of \( \text{U}(1) \times \text{SD}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})_\mathbb{R} \), an element of which is specified by a \( \text{U}(1) \)-representing angle \( \omega \), \( 0 \leq \omega < 2\pi \), an \( \text{SD}(2, \mathbb{C})_\mathbb{C} \)-representing complex displacement \( \Delta \) and an \( \text{SL}(2, \mathbb{C})_\mathbb{R} \)-representing 2×2 complex unimodular matrix \( \Gamma \). Under an \( \omega, \Delta, \Gamma \)-specified extended Lorentz-group element, a regular-basis single-quc Hilbert-space vector transforms to

\[
\Psi_Q^{\omega, \Delta, \Gamma} (s, y, z) = e^{iQ\omega} \Psi_Q(s' + \Delta, y', z').
\]

Under the 9-parameter group the 2-dimensional volume element \( ds \) within the Haar measure (16) is invariant, as also is the 4-dimensional volume element \( dy \, dz \). Such Haar-measure factorizability dovetails with the Formula (11) factorization of quc coordinate space.

**Periodicity in Quc Hilbert-Vector Dependence on Re s**

Displacement in the coordinate \( \text{Re} \, s \), at fixed \( \text{Im} \, s, y, z \) and \( \tau \), displaces what we loosely call the ‘time’ of an individual quc at fixed values of ‘everything else’. Quc energy, as a self-adjoint Hilbert-space operator representing a well-defined member of the left symmetry-subgroup algebra, is canonically-conjugate in Dirac sense to what we choose (with a dimensionality-endowing factor of \( \tau \)) to call the ‘time’ of this quc. In the QMU hamiltonian \(^3\) each quc’s gravitational potential energy is proportional to the quc’s energy—paralleling proportionality of quc electromagnetic potential energy to quc electric charge.
Although lightlikeness of the \textit{quc} velocity-direction 4-vector \( \mathbf{v} \) allows confusion (especially with \( c = 1 \)) between ‘temporal’ and ‘spatial’ \textit{quc} displacement, the group algebra unambiguously distinguishes right-invariant \textit{quc} energy from a \textit{quc}-momentum (3-vector) component of a right 6-vector—a component that generates (fixed-age) infinitesimal \textit{quc} displacement in some direction through curved metricized base-space.

Because the infinitesimal-displacement direction must be specified in some \textit{fixed} right-Lorentz frame, whereas a geodesic follows a curved path, the Reference (3) invariant self-adjoint \textit{quc} hyperbolic laplacian—a positive Casimir function of ‘geodesic-following’ second derivatives to which a \textit{quc’s kinetic} energy is proportional \(^{(1)}\)—is \textit{not} (as in Schrödinger’s flat-space Hamiltonian) proportional to the inner product with itself of a \textit{quc’s} 3-vector momentum.

Already noted has been the explicit confirmation by Formula (20) that (fixed-\( \tau, y, z \)) displacements in \( s \) are right invariant (both real and imaginary parts). They further are invariant under the 3-parameter symmetry subgroup (with energy, spirality and electric charge as generators) that defines \textit{quc} type, despite failure to be invariant under the full left-Lorentz subgroup. A Hilbert-space \textit{shrinkage} specifying ray \textit{periodicity} in dependence on \( Re \ s \) (periodicity for the hand of a ‘\textit{quc} timepiece’) maintains \textit{quc} capacity to represent D-left-extended right-Lorentz transformations.

We therefore diminish each \textit{quc’s} Hilbert space by the periodicity constraint,

\[
\Psi(s, y, z) = \Psi(s + 2\pi, y, z),
\]

with a matching redefinition of Hilbert-vector norm as integration in (15) over any (single) \( 2\pi \) interval of both \( Im \ s \) \textit{and} \( Re \ s \). The constraint (22) specifies \textit{integer} eigenvalues for the self-adjoint operator \( M \) that is Dirac-conjugate to \( Re \ s \).

Each of the three members of the extension subalgebra that complements the 6-member \( SL(2,\mathbb{C})_R \) subalgebra is then Hilbert-space represented by a self-adjoint operator with (integer-specified) \textit{discrete} spectra. The \textit{quc} energy integer \( M \) is joined by the charge integer \( Q \) and the spirality integer \( N \). (The spacing between neighboring \textit{quc} energies is \( \hbar/2\tau \)—minuscule in the present universe on \textit{any} of the scales of particle physics.)

**Conclusion**

Any \textit{quc} is distinguished from all others by its conserved-integer trio, \( Q, N, M \)—the first integer, \( Q \), specifying \textit{quc} electric charge, the second, \( N \), specifying \textit{quc} spirality (distinguishing fermionic \textit{qucs} from bosonic) while the third, \( M \), specifies \textit{quc} energy (in units of \( \hbar/2\tau \)). The finite total number of different \textit{qucs}—the population of the set of allowed integer trios—is addressed in Reference (3). Because, within this set, any positive integer is accompanied by the corresponding negative integer, the total value \textit{vanishes} of universe electric charge, spirality and energy.

Also vanishing is total-universe angular momentum—QMU’s version of Mach’s principle. Because \textit{quc} momentum is continuous, Hilbert-space meaning for vanishing total-universe-momentum is more subtle; the meaning is addressed in an appendix to Ref. (3).

A ‘type-basis’ Hilbert-space vector \( \Phi_{Q,N,M}(y, z) \), for the \textit{quc} identified by the integer trio \( Q, N, M \), is a differentiable normed complex function of two complex variables. In terms of the complex displacement (linear in \( \Delta \) although \( z \)-dependent via \( \mathfrak{I} \)) that is specified by the formula
\[ \delta (\Delta, \Gamma, z) \equiv \Delta + \ln (\Gamma_{11} - \Gamma_{12} z), \]  

(23)

any quc-type-basis Hilbert-space vector represents the group element specified by \( \omega, \Delta, \Gamma \) through the transformation

\[ \Phi_{Q,N,M} (y, z) \rightarrow e^{[Q \omega + N \text{Im} \delta + M \text{Re} \delta] \Phi_{Q,N,M} (y', z')} \]  

(24)

The role of the quc coordinate \( \text{Re} s \), absent from (24), deserves attention in our conclusion. This coordinate is resurrectable by defining the Hilbert vector

\[ \psi_{Q,N,M} (b) \equiv e^{-iM \text{Re} s} \Phi_{Q,N,M} (y, z), \]  

(25)

that recalls equivalence of the matrix coordinate \( b \) to the quintet \( \text{Re} s, y, z \) which coordinates the fiber bundle of 3-dimensional metricized hyperbolic base space and 2-dimensional velocity-direction fiber.

Established has been (classical) \( b \) equivalence to the right positive 4-vector pair \( B \) (base-space location) and \( v \) (velocity direction). Although both these 4-vectors depend on \( \text{Re} s \) and \( z \), because \( v \) fails to depend on \( y \), the latter may described classically as a ‘2-dimensional base-space quc-location coordinate’. Contrastingly, because \( \text{Re} s \) and \( z \) collaborate in ‘fiber-package’ roles that involve Hilbert space, this coordinate trio fails to admit a classical name.

Ontologically-justifiable names for 8 of 9 members of the extended-Lorentz algebra are unproblematic. The name ‘spirality’, assigned here to one algebra member, may or may not survive the test of usefulness.

Acknowledgements

Decades of discussions with Henry Stapp and Jerry Finkelstein have led to the quantum cosmology here proposed. Also contributing have been remarks by Korkut Bardakci, David Finkelstein, Eyvind Wichmann, Bruno Zumino, and Nicolai Reshetikhin.

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