A SHORT NOTE ON ADDITIVE FUNCTIONS ON RIEMANNIAN CO-COMPACT COVERINGS

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Abstract. The main purpose of this note is to provide a topological approach to defining additive functions on Riemannian co-compact normal coverings.

1. Introduction

Let $X$ be a connected smooth Riemannian manifold of dimension $n$ equipped with an isometric, properly discontinuous, free, and co-compact action of a discrete deck group $G$. Notice that the deck group $G$ is finitely generated due to the Švarc-Milnor lemma and hence, $\text{Hom}(G, \mathbb{R})$ is finite dimensional. Furthermore, the orbit space $M := X/G$ is a compact Riemannian manifold when equipped with the metric pushed down from $X$. The action of an element $g \in G$ on $x \in X$ is denoted by $g \cdot x$. Let $\pi$ be the covering map from $X$ onto $M$. Thus, $\pi(g \cdot x) = \pi(x)$ for any $(g, x) \in G \times X$.

Following [6], we define the class of additive functions on the covering $X$ as follows:

Definition 1.1.

- A real smooth function $u$ on $X$ is said to be additive if there is a homomorphism $\alpha : G \to \mathbb{R}$ such that
  $$u(g \cdot x) = u(x) + \alpha(g), \quad \text{for all } (g, x) \in G \times X.$$  
  We denote by $\mathcal{A}(X)$ the space of all additive functions on $X$.
- A map $h$ from $X$ to $\mathbb{R}^m$ ($m \in \mathbb{N}$) is called a vector-valued additive function on $X$ if every component of $h$ belongs to $\mathcal{A}(X)$.

We remark that additive functions on co-compact covers appeared in various results such as studying the structure of positive $G$-multiplicative type solutions [1,6], describing the off-diagonal long time asymptotics of the heat kernel [3] and the Green’s function asymptotics of periodic elliptic operators [2] on a noncompact abelian cover of a compact Riemannian manifold.

A direct construction of additive functions on co-compact covers can be found in either [4, Section 3] or [6, Remark 2.6]. However, this construction depends on the choice of a fundamental domain for the base $M$ in $X$. A more invariant approach to defining additive functions on covers was mentioned briefly in [1,4]. Our goal in this note is to present the full details of this approach for any co-compact covering.

2. Additive functions on co-compact normal coverings

We begin with the following notion (see [1,4]):
Lemma 2.2. $\Omega^1(M, G) \cong \text{Hom}(G, \mathbb{R})$.

Proof. By Hurewicz’s theorem (see e.g., [5]), the homologies $H_1(M)$ and $H_1(X)$ are isomorphic to the abelianizations of the fundamental groups $\pi_1(M)$ and $\pi_1(X)$, respectively. Therefore, we can identify De Rham cohomologies $H^1_{DR}(M)$ and $H^1_{DR}(X)$ with $\text{Hom}(\pi_1(M), \mathbb{R})$ and $\text{Hom}(\pi_1(X), \mathbb{R})$, correspondingly. Since $X$ is a normal covering of $M$, $\pi_1(X)$ is a normal subgroup of $\pi_1(M)$ and moreover, the sequence

$$0 \to \pi_1(X) \to \pi_1(M) \to G \to 0$$

is exact. Because $\text{Hom}(\cdot, \mathbb{R})$ is a contravariant exact functor, we deduce the exactness of the following sequence of vector spaces:

$$0 \to \text{Hom}(G, \mathbb{R}) \to H^1_{DR}(M) \to H^1_{DR}(X) \to 0.$$

Hence, $\Omega^1(M, G)$ is isomorphic to $\text{Hom}(G, \mathbb{R})$. \hfill \Box

Lemma 2.3. For such 1-form $\omega$, we have:

i) Fix any $g \in G$, then $f_\omega(g \cdot x) - f_\omega(x)$ is independent of $x \in X$.

ii) If $\pi^*\omega = 0$ then $\omega = 0$.

Proof.

i) For each $g \in G$, let $L_g$ be the diffeomorphism of $X$ that maps $x$ to $g \cdot x$. Since $\pi \circ L_g = \pi$, we get $df_\omega = \pi^*\omega = L_g^*\pi^*\omega = L_g^*df_\omega = dL_g^*f_\omega = d(f_\omega \circ L_g)$. Thus, $d(f_\omega - f_\omega \circ L_g) = 0$ and so, $f_\omega \circ L_g - f_\omega$ is constant since $X$ is connected.

ii) Fix any point $p \in M$. We pick an evenly covered open subset $U$ of $M$ such that it contains $p$. Then there is a smooth local section $\sigma : U \to X$, i.e., $\pi \circ \sigma = id|_U$ (see e.g., [5] Proposition 4.36]). Hence, $\omega(p) = \sigma^*\pi^*\omega(p) = 0$. \hfill \Box

On $\mathcal{A}(X)$, we introduce an equivalent relation $\sim$ as follows: $f_1 \sim f_2$ in $\mathcal{A}(X)$ if and only if $f_1 - f_2 = f \circ \pi$ for some function $f \in C^\infty(M, \mathbb{R})$. 

\textbf{Definition 2.1.} Let $H^1_{DR}(M), H^1_{DR}(X)$ be De Rham cohomologies of $M$ and $X$, respectively. We denote by $\Omega^1(M, G)$ the image in $H^1_{DR}(M)$ of the set of all closed differential 1-forms $\omega$ on $M$ (modulo the exact ones) such that their lifts $\omega$ to $X$ are exact. In other words, $\Omega^1(M, G)$ is the kernel of the homomorphism

$$\pi^* : H^1_{DR}(M) \to H^1_{DR}(X),$$

where $\pi^*$ is the induced homomorphism of the covering map $\pi : X \to M$. 

By De Rham’s theorem, $\Omega^1(M, G)$ is a finite dimensional vector space. Indeed, more is true:

Lemma 2.2. $\Omega^1(M, G) \cong \text{Hom}(G, \mathbb{R})$. 

Proof. By Hurewicz’s theorem (see e.g., [5]), the homologies $H_1(M)$ and $H_1(X)$ are isomorphic to the abelianizations of the fundamental groups $\pi_1(M)$ and $\pi_1(X)$, respectively. Therefore, we can identify De Rham cohomologies $H^1_{DR}(M)$ and $H^1_{DR}(X)$ with $\text{Hom}(\pi_1(M), \mathbb{R})$ and $\text{Hom}(\pi_1(X), \mathbb{R})$, correspondingly. Since $X$ is a normal covering of $M$, $\pi_1(X)$ is a normal subgroup of $\pi_1(M)$ and moreover, the sequence

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Hence, $\Omega^1(M, G)$ is isomorphic to $\text{Hom}(G, \mathbb{R})$. \hfill \Box

Fixing a base point $x_0 \in X$. For any closed 1-form $\omega$ on $M$ such that its lift to $X$ is exact, there exists a unique function $f_\omega \in C^\infty(X, \mathbb{R})$ such that $\pi^*\omega = df_\omega$ and $f_\omega(x_0) = 0$. Equivalently,

$$f_\omega(x) = \int_{x_0}^x \pi^*\omega, \quad \forall x \in X.$$ 

Lemma 2.3. For such 1-form $\omega$, we have:

i) Fix any $g \in G$, then $f_\omega(g \cdot x) - f_\omega(x)$ is independent of $x \in X$.

ii) If $\pi^*\omega = 0$ then $\omega = 0$.

Proof.

i) For each $g \in G$, let $L_g$ be the diffeomorphism of $X$ that maps $x$ to $g \cdot x$. Since $\pi \circ L_g = \pi$, we get $df_\omega = \pi^*\omega = L_g^*\pi^*\omega = L_g^*df_\omega = dL_g^*f_\omega = d(f_\omega \circ L_g)$. Thus, $d(f_\omega - f_\omega \circ L_g) = 0$ and so, $f_\omega \circ L_g - f_\omega$ is constant since $X$ is connected.

ii) Fix any point $p \in M$. We pick an evenly covered open subset $U$ of $M$ such that it contains $p$. Then there is a smooth local section $\sigma : U \to X$, i.e., $\pi \circ \sigma = id|_U$ (see e.g., [5] Proposition 4.36]). Hence, $\omega(p) = \sigma^*\pi^*\omega(p) = 0$. \hfill \Box

On $\mathcal{A}(X)$, we introduce an equivalent relation $\sim$ as follows: $f_1 \sim f_2$ in $\mathcal{A}(X)$ if and only if $f_1 - f_2 = f \circ \pi$ for some function $f \in C^\infty(M, \mathbb{R})$. 

\textbf{Definition 2.1.} Let $H^1_{DR}(M), H^1_{DR}(X)$ be De Rham cohomologies of $M$ and $X$, respectively. We denote by $\Omega^1(M, G)$ the image in $H^1_{DR}(M)$ of the set of all closed differential 1-forms $\omega$ on $M$ (modulo the exact ones) such that their lifts $\omega$ to $X$ are exact. In other words, $\Omega^1(M, G)$ is the kernel of the homomorphism

$$\pi^* : H^1_{DR}(M) \to H^1_{DR}(X),$$

where $\pi^*$ is the induced homomorphism of the covering map $\pi : X \to M$.
By Lemma 2.3 (i), the map \( \omega \mapsto \tilde{f}_\omega \) induces the following linear map

\[
\Lambda : \Omega^1(M,G) \to \mathcal{A}(X)/\sim
\]

\[
[\omega] \mapsto [\tilde{f}_\omega],
\]

where \([\omega], [\tilde{f}_\omega]\) are the equivalent classes of \(\omega, \tilde{f}_\omega\) in \(\Omega^1(M,G)\) and \(\mathcal{A}(X)/\sim\), correspondingly. We now claim that \( [\omega] = 0 \) if and only if \( [\tilde{f}_\omega] = 0 \), and hence \( \Lambda \) is an injective linear map. Indeed, due to Lemma 2.3 (ii), the condition that \( \tilde{f}_\omega \) is harmonic is equivalent to \( \pi^*\omega = df \) for some \( f \in C^\infty(M,\mathbb{R}) \). But this is the same as \( df_\omega = df \circ \pi \), or \( [\tilde{f}_\omega] = 0 \).

Consider an additive function \( f \) on \( X \). According to the definition, there exists a unique group homomorphism \( \ell_f : G \to \mathbb{R} \) such that \( f(g \cdot x) = f(x) + \ell_f(g) \) for any \( g \in G, x \in X \). Then the map \( f \mapsto \ell_f \) induces the linear map

\[
\Upsilon : \mathcal{A}(X)/\sim \to \text{Hom}(G,\mathbb{R})
\]

\[
[f] \mapsto \ell_f,
\]

which is injective.

Then the composition \( \Upsilon \circ \Lambda \) is also injective. By Lemma 2.2, \( \dim \Omega^1(M,G) = \dim \text{Hom}(G,\mathbb{R}) < \infty \). These facts together imply that the linear maps \( \Upsilon \) and \( \Lambda \) are isomorphism. We conclude:

**Theorem 2.4.** The three vector spaces \( \Omega^1(M,G), \mathcal{A}(X)/\sim \) and \( \text{Hom}(G,\mathbb{R}) \) are isomorphic to each other.

In particular, we obtain:

**Corollary 2.5.** Assume that \( G = \mathbb{Z}^d \). Then there is a smooth \( \mathbb{R}^d \)-valued function \( h \) on \( X \) such that for any \( (g,x) \in \mathbb{Z}^d \times X \),

\[
h(g \cdot x) = h(x) + g.
\]

The following standard proposition says that given any additive function \( u \) on \( X \), then one can pick a harmonic additive function \( f \) such that \( f - u \) is \( G \)-periodic.

**Proposition 2.6.** For any \( \ell \in \text{Hom}(G,\mathbb{R}) \), there exists a unique (modulo a real constant) harmonic function \( f \) on \( X \) such that for any \( (g,x) \in G \times X \), we have

\[
f(g \cdot x) = f(x) + \ell(g).
\]

**Proof.** First, we show the existence part. Due to Theorem 2.4, let \( \tilde{f} \) be a function on \( X \) satisfying \( \tilde{f}(g \cdot x) = \tilde{f}(x) + \ell(g) \) for any \( (g,x) \in G \times X \). We recall the isomorphism \( \Lambda \) defined in (1). We put \( \alpha := \Lambda^{-1}([\tilde{f}]) \in \Omega^1(M,G) \). By the Hodge theorem, there exists a unique harmonic 1-form \( \omega \) on \( M \) such that \( [\omega] = \alpha \in H^1_{\text{DR}}(M) \). Let \( f \) be a smooth function such that \( f \in \mathcal{A}(X) \) and \( \pi^*\omega = df \). Then \( f \) satisfies (1) since \( [f] = [\tilde{f}] \in \mathcal{A}(X)/\sim \). Thus, it is sufficient to show that \( f \) is harmonic on \( X \). We denote by \( \delta_X, \Delta_X \) and \( \delta_M, \Delta_M \) the codifferential and Laplace-Beltrami operators on \( X \) and \( M \), respectively. Since the covering map \( \pi \) is a local isometry between \( X \) and \( M \), its pullback \( \pi^* \) intertwines the codifferential operators, i.e.,

\[
\delta_X \pi^* = \pi^* \delta_M.
\]
Since $\Delta_M \omega = 0$, it follows that $\delta_M \omega = 0$. Thus,
\[ \Delta_X f = \delta_X df = \delta_X \pi^* \omega = \pi^* \delta_M \omega = 0. \]

For the uniqueness part, let $f_1$ and $f_2$ be any two harmonic functions on $X$ such that (4) holds for each of these functions. Since $f_1 - f_2$ is $G$-periodic, it can be pushed down to a real function $f$ on $M$. Moreover, $\pi^* \Delta_M f = \Delta_X \pi^* f = \Delta_X (f_1 - f_2) = 0$. Therefore, $f$ must be constant since it is a harmonic function on a compact, connected Riemannian manifold $M$. Thus, $f_1 - f_2$ is constant.

Corollary 2.7. Fixing a base point $x_0$ in $X$. Then to each $\alpha \in \text{Hom}(G, \mathbb{R})$, there exists a unique harmonic function $f_\alpha$ defined on $X$ such that $f_\alpha(x_0) = 0$ and $\Upsilon([f_\alpha]) = \alpha$, where $\Upsilon$ is introduced in (2). Consequently,
\[ A(X) = \bigcup_{\alpha \in \text{Hom}(G, \mathbb{R})} \{ f_\alpha + \varphi \mid \varphi \text{ is periodic} \}. \]

Remark 2.8. When $G = \mathbb{Z}^d$, the Albanese pseudo-metric $d_G$ introduced in [3, Section 2] is actually the pseudo-distance arising from any harmonic vector-valued additive function $h$ satisfying (3), i.e., $d_G(x, y) = |h(x) - h(y)|$ for any $x, y \in X$.

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