Gravitational Waves in the Nonsymmetric Gravitational Theory

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Abstract

We prove that the flux of gravitational radiation from an isolated source in the Nonsymmetric Gravitational Theory is identical to that found in Einstein’s General Theory of Relativity.

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1 Introduction

It has recently been claimed by Damour, Deser and McCarthy [1] that the Nonsymmetric Gravitational Theory (NGT) [2] is unphysical due to radiative instability. They claim that NGT predicts an infinite flux of negative energy from gravitational waves. We prove this claim to be false by demonstrating that the flux of gravitational radiation from an isolated source in NGT is identical to that found in General Relativity (GR).

Our proof is based on the following analysis. Firstly, we solve the NGT field equations for an isolated axi-symmetric source using the expansion technique employed in [1]. (The exact version of this work will be published elsewhere [3]). We find that the skew sector of the theory does not contribute to the flux of gravitational radiation due to the short-range nature of the skew fields. Secondly, since the NGT wave equations allow linear superposition in the wave zone, we see that any compact source has a wave field that can be expressed as the superposition of a suitable collection of differently aligned axi-symmetric solutions. This completes our proof.

2 NGT Vacuum Field Equations

The NGT Lagrangian without sources takes the form:

\[ \mathcal{L} = \sqrt{-g} g^\mu{}\nu R_{\mu\nu}(W), \]  

with \( g \) the determinant of \( g_{\mu\nu} \). The NGT Ricci tensor is defined as:

\[ R_{\mu\nu}(W) = W_{\mu,\beta}{}^\beta - \frac{1}{2} \left( W_{\mu\beta,\nu} + W_{\nu\beta,\mu} \right) - W_{\alpha\nu} W_{\alpha\beta} W_{\mu\nu} + W_{\mu\beta} W_{\alpha\nu}, \]  

where \( W^{\lambda}_{\mu\nu} \) is an unconstrained nonsymmetric connection:

\[ W^{\lambda}_{\mu\nu} = W^{\lambda}_{(\mu\nu)} + W^{\lambda}_{[\mu\nu]}, \]  

(Throughout this paper parentheses and square brackets enclosing indices stand for symmetrization and antisymmetrization, respectively.) The contravariant nonsymmetric tensor \( g^{\mu\nu} \) is defined in terms of the equation:

\[ g^{\mu\nu} g_{\sigma\nu} = g^{\nu\mu} g_{\nu\sigma} = \delta^\mu_\sigma. \]

If we define the torsion vector as:

\[ W^\mu = W^\mu_{[\mu\nu]} = \frac{1}{2} \left( W^\nu_{\mu\nu} - W^\nu_{\nu\mu} \right), \]
then the connection $\Gamma^\lambda_{\mu\nu}$, where

$$\Gamma^\lambda_{\mu\nu} = W^\lambda_{\mu\nu} + \frac{2}{3} \delta^\lambda_\mu W_\nu,$$

is torsion free:

$$\Gamma_\mu \equiv \Gamma^\sigma_{[\mu\alpha]} = 0.$$

Defining now:

$$R_{\mu\nu}(\Gamma) = \Gamma^\beta_{\mu\nu,\beta} - \frac{1}{2} (\Gamma^\beta_{(\mu\beta),\nu} + \Gamma^\beta_{(\nu\beta),\mu}) - \Gamma^\beta_{\alpha\nu} \Gamma^\alpha_{\mu\beta} + \Gamma^\beta_{(\alpha\beta)} \Gamma^\alpha_{\mu\nu},$$

we can write:

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \frac{2}{3} W_{[\mu,\nu]},$$

where $W_{[\mu,\nu]} = \frac{1}{2} (W_{\mu,\nu} - W_{\nu,\mu})$. Finally, the NGT vacuum field equations can be expressed as:

$$g_{\mu\nu,\sigma} - g_{\rho\nu} \Gamma^\rho_{\mu\sigma} - g_{\mu\rho} \Gamma^\rho_{\sigma\nu} = 0,$$

(10a)

$$\left(\sqrt{-g} g_{[\mu\nu]}\right)_{,\nu} = 0,$$

(10b)

$$R_{\mu\nu}(\Gamma) = \frac{2}{3} W_{[\mu,\nu]}.$$

(10c)

It is convenient to decompose $R_{\mu\nu}$ into standard symmetric and antisymmetric parts: $R_{(\mu\nu)}$, $R_{[\mu\nu]}$, and then rewrite the field equation (10d) in the following form:

$$R_{(\mu\nu)}(\Gamma) = 0,$$

(11a)

$$R_{[\mu\nu],\rho}(\Gamma) = R_{[\rho\nu],\mu}(\Gamma) + R_{[\rho\mu],\nu}(\Gamma) = 0.$$

(11b)

We shall base our analysis on the expansion used in [1], where the field equations are expanded in powers of the antisymmetric part, $h_{[\mu\nu]}$, of the metric but where the symmetric part is taken to be an exact GR background. The field equations become to lowest order:

$$^{GR}R_{\mu\nu} = 0,$$

(12a)

$$D^\alpha F_{\mu\nu\alpha} + 4 \, ^{GR}R^\alpha_{\mu\nu} \beta h_{[\alpha\beta]} = \frac{4}{3} W_{[\nu,\mu]},$$

(12b)

$$D^\mu h_{[\mu\nu]} = 0,$$

(12c)

where $F_{\mu\nu\alpha}$ is the cyclic curl of $h_{[\mu\nu]}$ and $D^\mu$ is the GR background covariant derivative.

For our current purposes, we shall be using the radiative, axi-symmetric GR background found by Bondi, van der Burg and Metzner [4]. Since their result is given as an expansion in inverse powers of the radial coordinate, we shall also have to expand the above equations in the same way. The two expansions are perfectly compatible, and both expansions are well suited to studying the equations in the wave zone.
3 The Metric

The physical situation that we shall be studying is that of an isolated spherical body which has been deformed by a perturbing compression along a single axis. We seek to study the flux of gravitational radiation emitted as the body oscillates through the cycle oblate–spherical–prolate–spherical–oblate.

Due to the physical picture sketched above, and to the fact that we are interested in the asymptotic behaviour of the field at spatial infinity (in an arbitrary direction from our isolated source), polar coordinates $x^0 = u, x = (r, \theta, \phi)$ are the natural choice. The “retarded time” $u = r - t$ has the property that the hypersurfaces with $u = \text{constant}$ are light–like. Detailed discussion of the coordinate systems permissible for investigation of outgoing gravitational waves from isolated systems can be found in [4, 5].

The covariant GR metric tensor corresponding to the situation described above:

$$ GR_{g_{\mu\nu}} = \begin{pmatrix} Vr^{-1}e^{2\beta} - U^2r^2 e^{2\gamma} & e^{2\beta} & U^2e^{2\gamma} & 0 \\ e^{2\beta} & 0 & 0 & 0 \\ U^2r^2 e^{2\gamma} & 0 & -r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}, $$

with $U, V, \beta, \gamma$ being functions of $u, r$ and $\theta$ was first given in [6]. We shall be using the forms found for the GR metric functions, and all derived quantities, as given in [4].

In order to find the NGT generalization of the metric tensor (13), we require that the symmetric part of the NGT metric tensor is formally the same as the GR metric tensor. We then impose the spacetime symmetries of the symmetric metric on the antisymmetric tensor $g_{[\mu\nu]}$. This is achieved by demanding that $\mathcal{L}_{\xi_{(i)}} g_{[\mu\nu]} = 0$, where the Killing vector field $\bar{\xi}_{(i)}$ is obtained from $\mathcal{L}_{\bar{\xi}_{(i)}} g_{(\mu\nu)} = 0$. The solution to this equation for the metric (13) yields the single Killing vector field, $\bar{\xi}_{(1)} = \xi_{(1)}^\phi \partial_\phi = \sin^2\theta \partial_\phi$. Imposing $\mathcal{L}_{\bar{\xi}_{(1)}} g_{[\mu\nu]} = 0$ yields:

$$ \xi_{(1)}^\phi \partial_\phi g_{[\mu\nu]} + g_{[\mu\lambda]} \partial_\nu \xi_{(1)}^\lambda + g_{[\nu\lambda]} \partial_\mu \xi_{(1)}^\lambda = 0. $$

This equation gives $\partial_\phi g_{[\mu\nu]} = 0$, but does not exclude any components of $g_{[\mu\nu]}$. This is markedly different from the static spherically symmetric case, where the above procedure excludes four of the six components of $g_{[\mu\nu]}$. To make the problem tractable, we need to determine which $g_{[\mu\nu]}$ should be excluded. The criteria to be applied are that our solution must reduce to the usual NGT static spherically sym-
metric solution when passing through the equilibrium position, and that for each $g_{\mu\nu}$ set to zero, a corresponding field equation must also vanish. To accomplish this, we note that the imposition of axi-symmetry splits the antisymmetric field equations \[10b,11b\] into two sets of three independent equations. (This can be seen directly from the block-diagonal form of the GR metric). The first set explicitly involves the three functions $g^{[01]}$, $g^{[02]}$, $g^{[12]}$:

$$
\left(\sqrt{-g}g^{[\mu\nu]}\right)_{,\nu} = 0 \quad (\mu = 0, 1, 2), \\
R_{[01,2]} = 0.
$$

(15)

These four equations are not independent owing to the existence of the one identity:

$$
\left(\sqrt{-g}g^{[\mu\nu]}\right)_{,\nu,\mu} = 0 \quad (\mu, \nu = 0, 1, 2).
$$

(16)

The second set of four equations explicitly involves the three functions $g^{[30]}$, $g^{[31]}$, $g^{[32]}$:

$$
\left(\sqrt{-g}g^{[3\nu]}\right)_{,\nu} = 0, \\
R_{[3\mu,\nu]} = 0.
$$

(17)

These four equations are also not independent due to the one identity:

$$
e^{3\mu\nu\rho}R_{[3\mu,\nu,\rho]} = 0.
$$

(18)

We note that eliminating one complete set of three functions simultaneously eliminates the three corresponding equations. However, eliminating one or two functions from either set fails to reduce the number of equations and leads to an over constrained system. Moreover, a Killing vector analysis performed at the instant when the system passes through its equilibrium position, shows that the set \{$g^{[01]}, g^{[02]}, g^{[12]}$\} must reduce to $g^{[01]}$, while the set \{$g^{[30]}, g^{[31]}, g^{[32]}$\} must reduce to $g^{[32]}$. Since we know from the exact spherically symmetric solution that asymptotic flatness requires $g^{[32]} = 0$, we must eliminate the entire second set of three functions \{$g^{[30]}, g^{[31]}, g^{[32]}$\}.

In view of the above, the NGT generalization of the metric tensor \[\text{13}\] is:

$$
g_{\mu\nu} = 
\begin{pmatrix}
V r^{-1} e^{2\beta} - U^2 r^2 e^{2\gamma} & e^{2\beta} + \omega & U r^2 e^{2\gamma} + \lambda & 0 \\
e^{2\beta} - \omega & 0 & \sigma & 0 \\
U r^2 e^{2\gamma} - \lambda & -\sigma & -r^2 e^{2\gamma} & 0 \\
0 & 0 & 0 & -r^2 e^{-2\gamma} \sin^2 \theta
\end{pmatrix},
$$

(19)
where $\omega, \lambda$ and $\sigma$ are functions of $u, r$ and $\theta$. The contravariant metric tensor is given to order $h^2$ (where $h = \omega, \lambda$ or $\sigma$) by:

$$
g^{\mu\nu} = \begin{pmatrix}
0 & e^{-2\beta} + g^{[01]} & g^{[02]} & 0 \\
e^{-2\beta} + g^{[10]} & e^{-2\beta}Vr^{-1} & e^{-2\beta}U + g^{[12]} & 0 \\
g^{[20]} & e^{-2\beta}U + g^{[21]} & -e^{-2\gamma}r^{-2} & 0 \\
0 & 0 & 0 & -r^{-2}e^{2\gamma} \sin^{-2}\theta
\end{pmatrix},
$$

where

$$
g^{[01]} = -(\omega-\sigma U)e^{-4\beta},$$

$$
g^{[02]} = -\sigma e^{-2\beta-2\gamma}r^{-2},$$

$$
g^{[12]} = -(Ur^2(\sigma U-\omega)e^{-4\beta}+e^{-2\beta-2\gamma}(\lambda-\sigma Vr^{-1}))r^{-2},$$

$$
g = \det(g_{\mu\nu}) = -r^4 \sin^2 \theta e^{4\beta}.
$$

4. Solving the Field Equations

We shall solve the field equations (12) using the GR background given by [4]:

$$\beta = -\frac{1}{4}c^2r^{-2} + ...$$

$$\gamma = cr^{-1} + \left(C - \frac{1}{6}c^3\right)r^{-3} + ...,$$

$$U = -(c_{,\theta} + 2c \cot \theta)r^{-2} + (2N + 3cc_{,\theta} + 4c^2 \cot \theta)r^{-3} + ...,$$

$$V = r - 2M - \left[N_{,\theta} + N \cot \theta - c_{,\theta}^2 - 4cc_{,\theta} \cot \theta - \frac{1}{2}c^2(1 + 8 \cot^2 \theta)\right]r^{-1} + ...,$$

where $c(u, \theta), N(u, \theta), M(u, \theta)$ are functions of integration and $C(u, \theta)$ satisfies:

$$4C_{,u} = 2c^2c_{,u} + 2cM + N \cot \theta - N_{,\theta}.$$

We begin by solving the three field equations ($\sqrt{-g}g^{[\mu\nu]}_{\nu} = 0, \ (\mu = 0, 1, 2)$ with the skew functions expanded in inverse powers of $r$. To this end, it is convenient to work with the following linear combinations of $\omega, \lambda$ and $\sigma$:

$$\eta = (\omega-\sigma U)e^{-2\beta},$$

$$\kappa = (\lambda-\sigma Vr^{-1})e^{-2\gamma},$$

$$\delta = \sigma e^{-2\gamma}.$$
The field equations then become:

\[
\begin{align*}
(\eta r^2)_r \sin \theta + (\delta \sin \theta)_\theta &= 0 , \\
\delta_{,u} + \kappa_r - (U r^2 \eta)_r &= 0 , \\
r^2 \eta_{,u} \sin \theta + r^2 (U \eta \sin \theta)_\theta - (\kappa \sin \theta)_\theta &= 0 .
\end{align*}
\]

Since \( g_{\mu\nu} \) must transform under the Poincaré group when \( r \to \infty \), we require that \( \eta, \kappa \) and \( \delta \) have the following expansions in \( 1/r \).

\[
\begin{align*}
\eta &= n_1/r + n_2/r^2 + \ldots , \\
\delta &= d_0 + d_1/r + d_2/r^2 + \ldots , \\
\kappa &= k_0 + k_1/r + k_2/r^2 + \ldots .
\end{align*}
\]

We begin by solving for the coefficients of \( h \) at lowest order in \( 1/r \). It is only when we find these coefficients that we can be sure that the next terms in the expansion of \( h \) are smaller than the \( h^2 \) terms that we have neglected.

Equation (23b) yields \( d_{0,u} = 0 \), and since \( d_0 = 0 \) when the system passes through its equilibrium position, \( d_0 = 0 \) always. Equation (23a) then gives \( n_1 = 0 \). Continuing to next order, we find from equation (23a) that \( (d_1 \sin \theta)_\theta = 0 \). This equation for \( d_1 \) demands \( d_1 = 0 \) in order to get a solution that is regular on the polar axis. The next equation in the set, (23b), yields \( k_1 = d_{2,u} \) while the equation (23c) yields \( n_{2,u} = \sin^{-1} \theta (k_0 \sin \theta)_\theta \).

To gain some understanding of the expansions thus far, it is instructive to calculate the NGT charge, \( L^2 \), of the system

\[
L^2 = \frac{1}{4\pi} \int \left( \sqrt{-g} g^{[0\nu]} \right)_{,\nu} d^3x = \frac{1}{2} \int_0^\pi n_2 \sin \theta \, d\theta = \langle n_2 \rangle ,
\]

where the brackets \( <> \) denote the angular average. To maintain a notation consistent with other work in NGT we define \( n_2 = l^2 Q(u, \theta) \) where \( l^2 \) is a constant (with respect to \( r, \phi, u \) and \( \theta \)) with the dimensions of a \([\text{length}]^2\) and is identified as the usual NGT charge, and \( < Q > = 1 \) for times when the system passes through its equilibrium position. Notice that \( L_{2,u}^2 = 0 \) by dint of the condition \( n_{2,u} = \sin^{-1} \theta (k_0 \sin \theta)_\theta \). This expresses the important fact that the NGT charge of a body cannot be radiated away, and is analogous to the situation in electromagnetism where the electric charge of an antenna does not change.
We can now use the results of the lowest order expansion to find out how far we can safely expand $h$ in inverse powers of $r$ before needing to incorporate $h^2$ terms. This can easily be accomplished by studying the exact form of $g^{\mu\nu}$ and $g$. We find that the linear order can accurately determine $\eta, \kappa$ and $\delta$ as follows:

$$\eta = Q_l^2 r^{-2} + R_l^3 r^{-3} + S_l^4 r^{-4}, \quad (26)$$

$$\kappa = II + J_l^2 r^{-1} + K_l^3 r^{-2}, \quad (27)$$

$$\delta = Al^2 r^{-2} + Bl^4 r^{-3}, \quad (28)$$

(Only even powers of $l$ will appear in physical quantities such as the curvature tensor). The functions of integration are dimensionless functions of both $u$ and $\theta$. They satisfy:

$$Q_{,u} = \frac{(I \sin \theta)_{,\theta}}{\sin \theta}, \quad (29a)$$

$$J = lA_{,u}, \quad (29b)$$

$$R = \frac{(A \sin \theta)_{,\theta}}{\sin \theta}, \quad (29c)$$

$$R_{,u} = \frac{(J \sin \theta)_{,\theta}}{l \sin \theta}, \quad (29d)$$

$$K = \frac{1B_{,u}}{2} - \frac{(c_{,\theta} + 2c \cot \theta)}{l}, \quad (29e)$$

$$S = \frac{(B \sin \theta)_{,\theta}}{2 \sin \theta}, \quad (29f)$$

$$S_{,u} = \frac{1}{\sin \theta} \left( \frac{\sin \theta (c_{,\theta} + 2c \cot \theta + lK)}{l^2} \right)_{,\theta}. \quad (29g)$$

Notice that (29c), (29d) can be combined to give (29b) and that (29f), (29g) can be combined to give (29e). This is a consequence of the identity (16).

Substituting the first terms in the above expansions into (12b) gives:

$$W_{[2,0]} = -\frac{3l}{4r^2}(2lJ + 2cI + lQ_{,\theta})_{,u} + O(r^{-3}), \quad (30a)$$

$$W_{[0,1]} = O(r^{-4}), \quad (30b)$$

$$W_{[1,2]} = \frac{3l}{2r^3}(2lJ + 2cI + lQ_{,\theta}) + O(r^{-4}). \quad (30c)$$

The identity $W_{\{\mu,\nu\},\rho} = 0$ then gives

$$(2l^2J + 2cI + l^2Q_{,\theta})_{,u} = 0$$

$$\Rightarrow 2l^2J + 2cI + l^2Q_{,\theta} = l^2 f(\theta). \quad (31)$$

7
The field strength $F_{\mu\nu\rho}$ is

$$F_{012} = \frac{l^2 f(\theta)}{r^2} + O(r^{-3}), \quad (32)$$

and since $F_{012} = 0$ when the system passes through its equilibrium position, $f(\theta) = 0$.

Our results can be summarised as follows:

$$h^{[\mu\nu]} = O(r^{-2}) + \ldots, \quad (33a)$$

$$W_{[\mu,\nu]} = O(r^{-3}) + \ldots, \quad (33b)$$

$$F_{[\mu\nu\rho]} = O(r^{-3}) + \ldots. \quad (33c)$$

5 Calculating the Radiation Flux

The rate of energy loss for an isolated body in NGT is given by

$$\frac{dE}{dt} = -R^2 \oint t^0 \hat{n}_i d\Omega, \quad (34)$$

where the integration is over a sphere of radius $R$ in the wave zone, $\hat{n}_i$ is an outward pointing unit vector, and $t^{\mu\nu}$ is the energy-momentum pseudo-tensor. The skew contribution to $t^{\mu\nu}$ is given by [1]:

$$t^{(\mu\nu)}_{skew} = \left( \frac{1}{2} F^{\mu\alpha\beta} F_{\nu\alpha\beta} - \frac{1}{12} GR g^{\mu\nu} F^2 \right) + \frac{2}{3} \left( 2h^{[\mu\alpha]} GR g^{\nu\beta} W_{[\beta,\alpha]} + 2h^{[\mu\alpha]} GR g^{\nu\beta} W_{[\beta,\alpha]} - GR g^{\mu\nu} h^{[\alpha\beta]} W_{[\alpha,\beta]} \right). \quad (35)$$

Inserting our expressions (33c), (33a) and (33b) for $F_{\mu\nu\alpha}$, $h^{[\mu\nu]}$ and $W_{[\mu,\nu]}$, we find that $t^{(\mu\nu)}_{skew} = O(r^{-5})$ so there is no skew contribution to the radiation flux in NGT. The only non-vanishing gravitational radiation flux in NGT comes from the leading order terms of the symmetric sector, which reproduce the usual GR radiation formula.

As explained in the introduction, the linearized wave equations allow us to obtain the radiation pattern of any generic source by linearly superimposing a suitable combination of differently aligned axi-symmetric solutions. Since we know that the radiation flux is identical in NGT and GR for the axi-symmetric case, we know that the flux will be identical in general. This line of reasoning can be rigorously supported by generalizing the work of Sachs [5], who found that the axi-symmetric wave solution of Bondi, van der Burg and Metzner [4] contained all the essential features needed to describe gravitational radiation from a generic source of no particular symmetry.
6  Conclusions

We have proved that the flux of gravitational radiation from an isolated source in NGT is identical to that found in GR, a result that clearly refutes the claims made by Damour, Deser and McCarthy.

In many respects our result is not unexpected. The two conserved sources for an isolated system in NGT are $L^2$, the NGT charge, and $T^{\mu\nu}$, the energy momentum tensor. Since $L^2$ has the dimensions of a [length]$^2$, while the constant $M$ in the GR sector has the dimensions of a length, we immediately expect the skew sector to be shorter ranged than the symmetric sector. Moreover, from the viewpoint of the phenomenological model where $L^2$ is taken to be proportional to the fermion number, we expect that $L$ will be a constant since the fermion number of an isolated body is a constant. This was borne out in our solution. Thus, there will be no dimensionless quantity $L^2_{\mu\nu}$ available in the skew sector unlike the symmetric sector where the dimensionless quantity $M_{\mu\nu}$ is available. Using dimensional analysis, we can then predict the form of the Riemann tensor, which must have the dimensions of [length]$^{-2}$ and contains at most two time derivatives. The symmetric sector can contribute super-radiative $1/r$ terms and radiative $1/r^2$ terms, while the skew sector can only contribute non-radiative $1/r^4$ (and lower) terms.

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