Primary Track Number Selection Method Based on Order Statistics

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Abstract. According to the requirements of robustness, this paper proposes an optimal method for the number of initial orbits based on order statistics. These methods are simple in calculation, strong in anti-interference ability, and have practical value.

1. Introduction

In the initial orbit calculation, always look for a kind of orbital number to make the best match with the actual observation data. [1] Due to the influence of data errors, it is possible to seriously affect the accuracy of orbit determination, especially the semi-major axis and the eccentricity of the orbit. Since the semi-major axis is the most important number, we mainly discuss the selection of the semi-major axis. Since the non-parametric statistical method has a small restriction on the model, it is naturally robust. [2-4] some applications of order statistics are non-parametric, so we study the selection of semi-major axes of orbits based on order statistics.

2. Primary track number selection method based on order statistics

Let the overall distribution of the semi-major axis have a density function , and is the symmetric center of the distribution. Then we can estimate the parameter by a variety of more robust methods, that is, estimate the true value of the semi-major axis of the orbit.

2.1. Median

Since the center of symmetry of the overall distribution is both the population mean and the population median, both the sample mean and the sample median can be used to estimate. [5-6] in theory, the benefit of the median is that it always exists, and some mathematical expectations of distribution do not exist; from the application, the median is a more reasonable measure of the “average” value of the overall indicator than the mean. Moreover, the expected value is greatly affected by individual extra large or extra small values, but the median is not, it is hardly affected by a small number of "outliers". For the above reasons, we use the median method to select the initial track.

The calculated values of a series of semi-major axes of the orbit are arranged in the order of small to large. To:

\[ a_{(1)} \leq a_{(2)} \leq \ldots \leq a_{(n)} \]  \hspace{1cm} (1)

The median of the sample is defined as:

\[ \tilde{a}_{y/2} = \begin{cases} a_{(\frac{n+1}{2})}, & \text{When } n \text{ is odd} \\ \frac{a_{(\frac{n}{2})} + a_{(\frac{n}{2}+1)}}{2}, & \text{When } n \text{ is even} \end{cases} \]  \hspace{1cm} (2)
It is known from statistics that if the population variance is \( \sigma^2 \), the variance of the sample mean is \( \frac{\sigma^2}{n} \). When \( n \) is large, the asymptotic variance of the median of the sample is \( \frac{4\sigma^2}{n} \). When the overall distribution is normally distributed, \( \frac{n\sigma^2}{f} \), and the ratio of the asymptotic variance of the two is \( \frac{4\sigma^2}{n1} \). When the overall distribution is Cauchy distribution \( \frac{1}{\pi} \), then the overall variance of the Cauchy distribution \( \Sigma2 \) is infinite, so the asymptotic variance ratio is infinite, that is, \( \tilde{a}_{1/2} \) is better than the sample mean. We can further point out that when the population distribution is Cauchy distribution, the maximal likelihood estimation of \( \theta \) will have a finite variance, which is asymptotic to \( 2/n \). The asymptotic variance of the median of the sample is \( \frac{4\sigma^2}{n1} \approx 2.47/n \). However, this maximum likelihood estimate is very complex and is determined by numerical calculations, but the sample median is very simple and reliable.

2.2. Estimating the semi-major axis from the sample P quantile
The symmetry center of the overall distribution (ie, the true value \( a \) of the semi-major axis of the orbit) can also be estimated using the sample P quantile. For \( 0<P<1 \), the sample P quantile is defined as:

\[
Pa \sim = \frac{1}{\pi} \left( nPa + (n+1)(P-1) \right),
\]

For \( 0<P<1 \), \( Pa \sim \) is actually an appropriate weighted average of both \( nPa \) and \( (n+1)(P-1) \). When \( nP<1 \), \( nPa \) can be understood as 0, and when \( n \) is large, \( (n+1)(P-1) \) can be used instead of \( Pa \sim \).

The variance of the sample mean is \( \sigma^2/n \) (\( \sigma^2 \) is the variance of the population sample). The asymptotic variance ratio of the two is \( 2\sigma^2 f^2(\sigma_p)/p \). For the general distribution to be normal \( N(0, \sigma^2) \), when \( p=1/5 \) and \( p=1/4 \), the asymptotic variance ratios are 0.7836 and 0.8078, respectively. It can be seen that the true value \( a \) of the semi-major axis is estimated by \( a Pa \). When \( p=1/5 \) and \( p=1/4 \), the efficiency of this method is about 80%.

In practice, for the sake of simplicity, after \( a_i \) is sorted from small to large, the average of the two data located at the front \( PN \) and \( (1-P)N \) (ie, the latter \( PN \)) of the series can be directly taken as the whole. Estimation of the distribution symmetry center. The above P-quantile method can also be generalized as:

\[
\hat{a} = \frac{1}{3} \left( \frac{\tilde{a}_{1/5} + \tilde{a}_{1/2} + \tilde{a}_{4/5}}{P} \right)
\]

Or

\[
\hat{a} = \frac{1}{6} \left( \frac{\tilde{a}_{1/6} + \tilde{a}_{4/6} + 2\tilde{a}_{2/6} + \tilde{a}_{5/6} + \tilde{a}_{3/6}}{P} \right)
\]

The latter two estimates can be used even if the overall distribution deviates from a normal state.
2.3. Trimmed mean
From the point of view of the order statistics, the sample mean and the sample median are two extremes: the former uses the overall order statistic, and the latter uses only one or two. Using the sample mean and the sample median to estimate the center of symmetry of the overall distribution, whichever is better or worse, depends on the overall distribution. For example, in the overall normal state, the sample mean is better than the sample median, and in the case of the overall Cauchy distribution, the sample median is better than the sample mean. It is not difficult to imagine that for some population distributions, a practice between the sample mean and the sample median may be more favorable: in
$$\left(\begin{array}{c} a_k \\ \vdots \\ a_n \end{array}\right) \leq \left(\begin{array}{c} a_{(1)} \\ \vdots \\ a_{(n)} \end{array}\right) \leq \left(\begin{array}{c} a_{(k)} \\ \vdots \\ a_{(n-k+1)} \end{array}\right)$$

is not used, and the remaining arithmetic average is:

$$T_{nk} = \frac{1}{n-2k} \sum_{i=k+1}^{n-k} a_{(i)}$$  \hspace{1cm} (6)

It is assumed here: $k < n/2$. $T_{nk}$ is called the k-th order truncation mean. More specifically, $T_{nk}$ is the k-order symmetric tail-final mean. The reason why the symmetrical tail is used is because when the symmetry center is estimated, if the population mean exists, $T_{nk}$ is an unbiased estimate of the center of symmetry. Obviously, $\bar{a}$ and $\tilde{a}$ are special cases of symmetric truncated mean: the former corresponds to $k=0$, the latter corresponds to $k = \frac{n-1}{2}$ (when $n$ is odd), or $k = \frac{n-2}{2}$ (when $n$ is even).

If the overall distribution has a very irregular shape or a thicker tail than a normal distribution, it is more useful to use the truncated mean to estimate the center of the population. In practice, the use of truncated mean values also eliminates the effects of certain “outliers” because these “outliers” are discarded in $T_{nk}$. It should be acknowledged that in the case just discussed, the truncated mean is better than $\bar{a}$; however, when the population distribution is a normal distribution or an asymptotic normal distribution, $\bar{a}$ is better than the truncated mean. Since we don’t know which kind of situation, it is important to know how the mean square error of the truncated mean is greater than the mean square error of $\bar{a}$ when the actual distribution is normal. In other words, we want to know how much damage is used when the actual distribution is normal.

**Table 1.** For a normal distribution, the mean square error is equal to the table column value multiplied by $\sigma^2/n$.

| Sample                     | n=10  | n=20  |
|----------------------------|-------|-------|
| Sample mean $\bar{a}$      | 1.00  | 1.00  |
| First-order censored mean $T_{n1}$ | 1.05  | 1.02  |
| Second-order censored mean $T_{n2}$ | 1.12  | 1.06  |
| Third-order censored mean $T_{n3}$ | 1.21  | 1.10  |
| Fourth-order censored mean $T_{n4}$ | 1.37  | 1.14  |
| Median sample $\tilde{a}_{1/2}$ | 1.37  | 1.50  |
When $a_1, \cdots, a_n$ forms a random sample from the normal population $N(0, \sigma^2)$, the probability distributions of $\bar{a}$ and $T_{nk}$ are all symmetric about $\theta$. Therefore, the mean of $\bar{a}$ and $T_{nk}$ is $\theta$, and regardless of the true value of $\theta$, their mean square error has a certain constant value. When the sample size is $n=10$ or $20$, several mean square errors of the normal distribution are shown in Table 1.

It can be seen from Table 1 that when the overall distribution is actually a normal distribution, the mean square error of the truncated mean is not much larger than the mean square error of $\bar{a}$. In fact, when $n=20$, the mean square error of the second-order censored mean ($k=2$) is only 1.06 times the $\bar{a}$ mean square error. Even for the median of the sample, the mean square error is only 1.5 times the $\bar{a}$ mean square error. These values indicate that the truncated mean can be used as a robust estimate of $\theta$.

Now, we consider the improvement in mean square error that can be obtained with the truncated mean when the actual distribution is not normal. If $a_1, \cdots, a_n$ is a random sample of size $n$ from the Cauchy distribution, the mean square error of $\bar{a}$ is infinite. For the Cauchy distribution, the mean square error of the truncated mean is given in Table 2 when the sample size is 10 or 20.

| Sample mean $\bar{a}$                          | n=10 | n=20 |
|------------------------------------------------|------|------|
| First-order censored mean $T_{n_1}$            | 27.22| 23.98|
| Second-order censored mean $T_{n_2}$           | 8.57 | 7.32 |
| Third-order censored mean $T_{n_3}$            | 3.86 | 4.57 |
| Fourth-order censored mean $T_{n_4}$           | 3.66 | 3.58 |
| Median sample $\bar{a}_m$                      | 3.66 | 2.88 |

It can be seen from Table 2 that the mean square error of the truncated mean is significantly smaller than the mean square error of $\bar{a}$. In order to use the truncated mean as an estimate of $\theta$, it is clear that the $k$ value must be chosen. If there is reason to believe that the population distribution is asymptotically normal, then 10 to 15% of the observations can be discarded at each end of the sequence sample, and then $\theta$ can be estimated using the truncated mean. If the overall distribution is far from normal, or if there are several "outliers" in the observations, then $\theta$ can be estimated using the median of the sample.

3. An instance
In a certain task, real-time orbit determination obtains 10 sets of orbital root numbers, and their semi-major axes are: (unit: km) 26647.768, 26645.128, 26648.381, 26645.613, 26649.247, 26648.942, 26647.337, 26647.970, 26647.315, 26647.954, now select a set of optimal orbital root numbers.

We assume that the true value of the semi-major axis is $\theta$, which is the symmetry center of the semi-major axis distribution. The "observations" of the above semi-major axes are now arranged from small to large: 26645.128, 26645.613, 26647.315, 26647.337, 26647.768, 26647.954, 26647.970, 26648.381, 26648.942, and 26649.247.

Using the method described above, an estimate of the true value of the semi-major axis can be obtained. It is also known that the semi-major axis of the fine track is 6,674,454 meters. The estimated values of the various methods and the corresponding root mean square error and external coincidence error $\Delta a$ (difference from the fine track) are listed in Table 3.
Table 3. Estimated semi-major axis and corresponding root mean square error

| Estimation method                        | Estimated value $\hat{a}$ (m) | $\sigma_{\hat{a}}$ (m) | $\Delta a$ (m) |
|------------------------------------------|-------------------------------|-------------------------|----------------|
| Sample mean $\bar{a}$                    | 6647566                       | 439                     | 112            |
| Median sample $\tilde{a}_{\frac{n}{2}}$  | 6647861                       | 514                     | 407            |
| $\hat{a}_{\frac{n}{2}}$                  | 6647570                       | 496                     | 116            |
| First-order censored mean $T_{n_1}$      | 6647660                       | 465                     | 116            |
| Second-order censored mean $T_{n_2}$     | 6647788                       | 465                     | 334            |

It can be seen from Table 3 that the results estimated by the various methods are very close due to the absence of abnormal data in the $a_i$. However, the sample median method is the easiest to calculate. Therefore, we can select the two sets of roots whose half-axis value is closest to $\tilde{a}_{\frac{n}{2}}$ (that is, the two sets of roots corresponding to $a_{(\frac{n}{2})}$ or $a_{(\frac{n}{2})}$) as the optimal number of initial orbits.

4. Summary

The results of the various methods studied in this paper are very close. However, the calculation method based on the order statistics of the initial track number is simpler and has stronger anti-interference ability, which is more practical.

References

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