On the partial trace over collective spin-degrees of freedom

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Abstract

We derive analytical properties for the degeneracy $\nu(N,j)$ occurring in the decomposition $\bigoplus_j^{\infty} \nu(N,j) \mathbb{C}^{2j+1}$ of the state space $\mathbb{C}^{2\otimes N}$. We also investigate the dynamics of two qubits coupled via Ising interactions to separate spin baths, and we study the thermodynamic limit.
In recent years there has been an increasing interest in the description of the dynamics of small quantum systems interacting with their surrounding [1]. This was motivated by the necessity of understanding the phenomenon of decoherence in quantum systems [2, 3, 4, 5], and the attempt to build quantum devices that enable the implementation of quantum algorithms [6]. However, the main difficulty one faces in such a task consists in dealing with the large number of environmental degrees of freedom, which makes most of the proposed theoretical models impossible to be solved analytically even for finite sizes of the surrounding.

Among the promising candidates to quantum information processing and quantum computing, spin systems seem to be the most suitable for the construction of quantum gates [7, 8]. Recently, it has been shown that exact analytical solutions can be obtained for the dynamics of few central qubits coupled to spin baths of finite and infinite sizes [9, 10, 11]. There, the interaction Hamiltonians together with the baths Hamiltonians are functions of the collective spin operators of the environments. In order to derive the reduced density matrix of the central qubits, the partial trace over the environmental spin degrees of freedom was carried out within the subspaces corresponding to the different values of the total angular momentum of the surrounding.

Recall that the state space of single spin-\(\frac{1}{2}\) particle is given by \(\mathbb{C}^2\), where \(\mathbb{C}\) denotes the field of complex numbers. The corresponding basis is formed by the eigenvectors \(\{|-\rangle, |+\rangle\}\) associated with the eigenvalues ±\(\frac{1}{2}\) of the operator \(S_z = \frac{1}{2}\sigma_z\), where \(\sigma_z\) designates the \(z\)-component of the Pauli operator \(\vec{\sigma}\). In general, the state space of a system of \(N\) qubits is given by the \(N\)-fold tensor product of the state spaces of the individual particles, namely, \(\mathbb{C}^{2^N}\). One possible basis of the latter space consists of the state vectors \(\bigotimes_i^N |\epsilon_i\rangle\), with \(\epsilon_i = \pm\). These are eigenvectors of the collective spin operator \(J_z\), where \(\vec{J} = \frac{1}{2} \sum_{i=1}^N \vec{\sigma}_i\). Alternatively, one can construct new basis composed of the common eigenvectors of the operators \(J^2\) and \(J_z\); we shall denote them by \(|j, m\rangle\) such that \(\kappa \leq j \leq N/2\) and \(-j \leq m \leq j\), as imposed by the laws of addition of angular momentum in quantum mechanics [12]. In the above, \(\kappa = 0\) for \(N\) even, and \(\kappa = 1/2\) for \(N\) odd. Note that the scalar product of state vectors corresponding to different values of \(j\) vanishes. This means that the total space \(\mathbb{C}^{2^N}\) can be decomposed as the direct sum of subspaces \(\mathbb{C}^{2j+1}\), that is

\[
\mathbb{C}^{2^N} = \bigoplus_{j=\kappa}^{N} \nu(N, j)\mathbb{C}^{2j+1}.
\]  

The quantity \(\nu(N, j)\) is the multiplicity corresponding to the value \(j\) of the total angular
momentum; its exact form reads

$$\nu(N, j) = \frac{N}{N/2 - j} - \frac{N}{N/2 - j - 1} = \frac{2j + 1}{N/2 + j + 1} \frac{N!}{(\frac{N}{2} - j)! (\frac{N}{2} + j)!}. \quad (2)$$

Hence, given any operator $\hat{G}(\vec{J})$ on $\mathbb{C}^{2^N}$, its trace can be written as

$$\text{tr} \hat{G} = \sum_{j=\kappa}^{N/2} \nu(N, j) \sum_{m=-j}^{j} \langle j, m | \hat{G} | j, m \rangle. \quad (3)$$

Following the general ideas of the theory of open quantum systems, the problem of finding a relation between the multiplicities of the subspaces $\mathbb{C}^{2 \otimes N_i}$ and that of $\mathbb{C}^{2 \otimes N}$, where $\sum_i N_i = N$, naturally arises. In this work we illustrate how this problem can be solved, in the case $N = N_1 + N_2$, using the invariance of the trace. The latter property will also be used to describe the dynamics of two qubits in separate spin baths.

A decomposition law for the degeneracy $\nu(N, J)$. Let us denote by $| j_i, m_i \rangle$ the basis state vectors in the space $\mathbb{C}^{2 \otimes N_i}$ ($i = 1, 2$). Hence the trace of $\hat{G}(\vec{J})$ can also be expressed as

$$\text{tr} \hat{G} = \sum_{j_1, m_1=-j_1}^{N_1/2} \sum_{j_2, m_2=-j_2}^{N_2/2} \nu(N_1, j_1)\nu(N_2, j_2) \langle j_1, j_2, m_1, m_2 | \hat{G} | j_1, j_2, m_1, m_2 \rangle. \quad (4)$$

On the other hand we have

$$| j_1, j_2, m_1, m_2 \rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} (-1)^{j_1-j_2+M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} | J, M \rangle,$$  

where the quantity in matrix form denotes Wigner $3j$-symbol; obviously, the condition $m_1 + m_2 = M$ along with the triangle rule $|j_1 - j_2| \leq J \leq j_1 + j_2$ must be satisfied. By equations (4) and (5), we can write:

$$\text{tr} \hat{G} = \sum_{j_1, m_1, j_2, m_2} \nu(N_1, j_1)\nu(N_2, j_2) \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} \sum_{M'=-J'}^{J'} (-1)^{2(j_1-j_2)+M+M'} \sqrt{(2J+1)(2J'+1)} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J' \\ m_1 & m_2 & -M' \end{pmatrix} | J', M' \rangle | J, M \rangle. \quad (6)$$
where we have used the fact that $3j$-symbols are real. The operator $\hat{G}$ is arbitrary; it can be chosen such that it satisfies $\langle J', M'|\hat{G}|J, M \rangle = \langle J, M|\hat{G}|J, M \rangle \delta_{J,J'} \delta_{M,M'}$. In this case equation (6) reduces to

$$\text{tr} \hat{G} = \sum_{j_1,m_1,j_2,m_2} \nu(N_1,j_1)\nu(N_2,j_2) \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} (-1)^{2(j_1-j_2)+2M} (2J+1) \left\{ \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \right\}^2 \langle J, M|\hat{G}|J, M \rangle. \quad (7)$$

The lower and upper limits of the sum over $J$ in the above equation are, respectively, $|j_1-j_2|$ and $j_1+j_2$. For $J < |j_1-j_2|$, or $J > j_1+j_2$, the triple $(j_1, j_2, J)$ does not satisfy the triangle rule and hence the corresponding Wigner $3j$-symbol vanishes. Consequently, the right-hand side of equation (7) will not be affected if we take $\frac{N_1+N_2}{2}$ as an upper limit, and $\kappa$ as a lower limit for the sum over $J$ such that $\kappa = 0$ for $N_1 + N_2$ even and $\kappa = 1/2$ for $N_1 + N_2$ odd. This effectively allows us to exchange the order of the sums in the above equation. Then by comparing the resulting equation with (3), we obtain

$$\nu(N_1 + N_2, J) = \sum_{j_1,m_1,j_2,m_2} \nu(N_1,j_1)\nu(N_2,j_2)(-1)^{2(j_1+j_2-J)}(2J+1) \times \left\{ \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -J \end{pmatrix} \right\}^2. \quad (8)$$

Herein, we have replaced $M$ by its maximum value $J$ (or equivalently by $-J$ because of the symmetry) since the sum does not depend on this quantum number; once again the condition $m_1 + m_2 = J$ is implied.

Equation (8) can be regarded as a decomposition law for the degeneracy; many useful relations satisfied by the latter can be easily obtained from it. Let us first begin by noting that

$$\sum_{J=\kappa}^{\frac{N}{2}} \nu(N, J) = \binom{N}{\frac{N}{2} - \kappa}, \quad (9)$$

$$\sum_{J=\kappa}^{\frac{N}{2}} (2J+1)\nu(N, J) = 2^N. \quad (10)$$

The first equation can be readily proved by expanding the sum over $J$. The second one simply expresses the fact that the sum of the dimensions of the subspaces $\mathbb{C}^{2j+1}$ is equal to
the dimension of the total state space, $\mathbb{C}^{2^N}$. Furthermore, if we let $J$ to take the value $\frac{N_1 + N_2}{2}$ in equation (8), we obtain

$$(-1)^{N_1+N_2}(N_1 + N_2 + 1) \sum_{j_1,m_1} \sum_{j_2,m_2} \nu(N_1, j_1) \nu(N_2, j_2) (-1)^{2(j_1-j_2)}$$

$$\times \sum_{j_1,m_1} \sum_{j_2,m_2} \nu(N_1, j_1) \nu(N_2, j_2) \left( -1 \right)^{2(j_1-j_2)} \left( j_1 - j_2 \right) \delta_{j_1,j_2} \delta_{-m_1,m_2}.$$  

$$\sum_{j_1,m_1} \sum_{j_2,m_2} \nu(N_1, j_1) \nu(N_2, j_2) \left( -1 \right)^{2(j_1-j_2)} \left( j_1 - j_2 \right) \delta_{j_1,j_2} \delta_{-m_1,m_2}. \quad (11)$$

Now let us suppose that $J = 0$, which is possible only when $N_1$ and $N_2$ are either both even or both odd positive integers. Here it should be noted that the denominator of the corresponding Wigner 3j-symbol contains the product $(j_1 - j_2)! (j_2 - j_1)!$; but since $x! = \infty$ for $x < 0$, we conclude that when $J = 0$, the quantity under the sum sign in the right-hand side of equation (8) is nonzero only when $j_1 = j_2$. In fact one should have

$$\nu(N_1, j_1) \nu(N_2, j_2) \left( -1 \right)^{2(j_1-j_2)} \left( j_1 - j_2 \right) \delta_{j_1,j_2} \delta_{-m_1,m_2}. \quad (12)$$

By inserting the latter expression of Wigner 3j-symbol into equation (8), and performing the sum over $j_2$ and $m_2$, we obtain

$$\nu(N_1 + N_2, 0) = \sum_{j} \nu(N_1, j) \nu(N_2, j) \left( -1 \right)^{2(j-m)} \frac{1}{2j+1} \delta_{j_1,j_2} \delta_{-m_1,m_2}.$$

$$\nu(N_1, j) \nu(N_2, j), \quad (13)$$

where we have used the fact that $\sum_{m=-j}^{j} (-1)^{2m} = (-1)^{2j} (2j + 1)$. It immediately follows that

$$\sum_{j} \nu(N, j)^2 = \frac{(2N)!}{(N+1)(N!)^2}. \quad (14)$$

The above procedure can be easily generalized to further decompositions of the total number of spins.

**Dynamics of two qubits in separate spin baths.** As a second application, let us investigate the dynamics of two qubits coupled via ising interactions to separate spin environments of the same size, $N$. The total angular momentum operators of the latter are denoted by $\vec{J}$ and $\vec{J}$. The full Hamiltonian of the composite system is given by

$$H = \lambda (\sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2) + \delta \sigma_z^1 \sigma_z^2 + \frac{\gamma}{\sqrt{N}} (\sigma_z^1 J_z + \sigma_z^2 J_z) + \mu (\sigma_1^1 + \sigma_2^2) + H_{B_1} + H_{B_2}. \quad (15)$$
Here, \( \lambda \) and \( \delta \) are the strengths of interaction of the central qubits with each other, \( \gamma \) is the coupling constant to the baths, and \( \mu \) is the strength of an applied magnetic field. The operators \( H_{B_i} \), with \( i = 1, 2 \), denote the Hamiltonians of the spin baths. One can show that the interaction Hamiltonian describing the coupling of the central qubits to the environments is diagonal in the standard basis of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), namely,

\[
H_I = \frac{\gamma}{\sqrt{N}} \text{diag}(-\Sigma_z, -\Delta_z, \Delta_z, \Sigma_z),
\]

where we have introduced the operators \( \vec{\Sigma} = \vec{J} + \vec{J} \) and \( \vec{\Delta} = \vec{J} - \vec{J} \). Then it can be shown that the model Hamiltonian is given by the direct sum of the Hamiltonian operators \( H_1 \) and \( H_2 \), where

\[
H_1 = \sigma_z(2\mu + \frac{\gamma}{\sqrt{N}}\Sigma_z) + \mathbb{I}_2(H_B + \delta),
\]

\[
H_2 = 2\lambda \sigma_x + \frac{\gamma}{\sqrt{N}} \sigma_z \Delta_z + \mathbb{I}_2(H_B - \delta),
\]

with \( H_B = H_{B_1} + H_{B_2} \) and \( \mathbb{I}_2 \) is the \( 2 \times 2 \) unit matrix. Note that the basis vectors of the subspace corresponding to \( H_1 \) are given by

\[
| \downarrow \rangle \equiv | - - \rangle, \tag{18}
\]

\[
| \uparrow \rangle \equiv | + + \rangle; \tag{19}
\]

those associated with \( H_2 \) are given by

\[
| 0 \rangle \equiv | - + \rangle, \tag{20}
\]

\[
| 1 \rangle \equiv | + - \rangle. \tag{21}
\]

Thus the system under consideration can be mapped onto two pseudo two-level systems \( S_1 \) and \( S_2 \) whose dynamics is governed by the operators \( H_1 \) and \( H_2 \), respectively. Each one is coupled to a spin environment consisted of \( 2N \) spin-\( \frac{1}{2} \) particles with the only exception that \( S_1 \) and \( S_2 \) see different compositions of the total angular momentum, namely \( \Sigma_z \) and \( \Delta_z \), respectively. Notice that the above pseudo systems become completely independent from each other if the initial density matrix of the qubits takes the form

\[
\rho(0) = \begin{pmatrix}
\rho_{11}^0 & 0 & 0 & \rho_{14}^0 \\
0 & \rho_{22}^0 & \rho_{23}^0 & 0 \\
0 & \rho_{32}^0 & \rho_{33}^0 & 0 \\
\rho_{41}^0 & 0 & 0 & \rho_{44}^0
\end{pmatrix}.
\]
In such a case, it is sufficient to investigate the coupling of each pseudo system separately. For a reason that will become apparent below, we set \( H_B = H_{B_1} + H_{B_2} = h(J_z - J'_z) \), where \( h \) is the strength of an applied magnetic field. Moreover, we assume that the baths are initially in thermal equilibrium at temperatures \( T_1 = T_2 = T \) (we set \( k_B = 1 \)); the corresponding total initial density matrix is given by

\[
\rho_B(0) = \exp(-h\beta \Delta_z) / \left[ 2 \cosh \left( \frac{h\beta}{2} \right) \right]^{2N},
\]

(23)

where \( \beta = 1/T \) is the inverse temperature and \( Z = \left[ 2 \cosh \left( \frac{h\beta}{2} \right) \right]^{2N} \) is the partition function. Under the above assumptions, the contributions of the coupling constant \( \delta \) can be neglected.

The dynamics of \( S_2 \) is quite trivial since the corresponding time evolution operator is diagonal. Indeed, it is easy to show that \( \rho_{11}(t) = \rho_{11}^0 \) and \( \rho_{44}(t) = \rho_{44}^0 \). Moreover,

\[
\rho_{14}(t) = Z^{-1}\rho_{14}^0 \sum_{j_1,m_1} \sum_{j_2,m_2} \nu(N,j_1)\nu(N,j_2)
\times \exp\{2i[2\mu + \gamma(m_1 + m_2)/\sqrt{N}]t - h\beta(m_1 - m_2)\}.
\]

(24)

In the special case when \( h = 0 \) or \( T \to \infty \), we can write

\[
\rho_{14}(t) = 2^{-2N}\rho_{14}^0 e^{4it\mu} \sum_{J,M} \nu(2N,J)e^{2\sqrt{2}\mu\gamma M/N} \cos \left( \frac{\gamma t}{\sqrt{N}} \right)^{2N}.
\]

(25)

For arbitrary values of \( h \) and \( T \), the right-hand side of equation (24) can be evaluated within the computational basis; this yields

\[
\rho_{14}(t)/\rho_{14}^0 = e^{4it\mu} \left[ 1 + \cos^2 \left( \frac{\gamma t}{\sqrt{N}} \right) \right]^N.
\]

(26)

Then, by expanding the cosine function in Taylor series and taking the limit \( N \to \infty \), we obtain the Gaussian decay law:

\[
\left| \frac{\rho_{14}(t)}{\rho_{14}^0} \right| = \exp\left\{-\frac{\gamma^2 t^2}{\cosh^2(h\beta/2)} \right\}.
\]

(27)

This means that the decoherence time scale is given by \( \tau_D = \cosh(h\beta/2)/|\gamma| \). Obviously \( \tau_D \to \infty \) as \( T \to 0 \) or \( h \to \infty \).

As a measure of entanglement, we use the concurrence defined by [15]

\[
C(\rho) = \max\{0, 2 \max_i \sqrt{\lambda_i} - \sum_{i=1}^4 \sqrt{\lambda_i}\},
\]

(28)
where the quantities $\lambda_i$ are the eigenvalues of the operator $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. In our case, when applied to $\rho(t)$, the above definition of the concurrence leads to the evaluation of the eigenvalues of the operator $\rho(t)\sigma_x \rho(t)^*\sigma_x$ where $\rho(t)$ is now restricted to the subspace of $H_1$. A straightforward calculation yields

$$C(t) = 2|\rho_{14}(t)|.$$  \hspace{1cm} (29)

An example of the evolution in time of the real value of $\rho_{14}(t)$ along with the concurrence $C(t)$ corresponding to the initial state $(|\rangle - |\rangle + |\rangle + |\rangle)/\sqrt{2}$ is shown in figure 1. We notice the revival of the concurrence in the case of finite number of spins. At short times, the curves corresponding to $N \to \infty$ coincide with those of finite $N$.

![Figure 1: (Color online) Evolution in time of the real part of $\rho_{14}(t)/\rho_{14}^0$ (oscillating curve) and the concurrence (enveloping curve) corresponding to the initial state $(|\rangle - |\rangle + |\rangle + |\rangle)/\sqrt{2}$. Here, $N = 100$, $\gamma = 2$, $h\beta = 1$, and $\mu = 4$. For $t < 10$, the curves coincide with those of the limit $N \to \infty$.](image)

It should be stressed that when the Hamiltonian of the composite spin bath is given by $H_B = h(J_z + J_z^z) = h\Sigma_z$, then

$$\rho_{14}(t) = \rho_{14}^0 e^{4i\mu t} \left[ \cos(\gamma t/\sqrt{N}) - i \sin(\gamma t/\sqrt{N}) \tanh(h\beta/2) \right]^{2N}.$$  \hspace{1cm} (30)

The existence of the sine function makes it not possible to find a relation similar to (27) when $N \to \infty$. However if we rescale the coupling constant $\gamma$ by $N$ instead of $\sqrt{N}$, that
exact analytical expression can be derived for the case of an infinite number of spins, namely,

\[
\rho_{14}(t) = \rho_{14}^0 \exp\left\{-it[4\mu + \gamma \tanh(h\beta/2)]\right\},
\]

Consequently the central qubits preserve their coherence, since the decoherence time scale in this case is infinite, as indicated by formula (32). With the new scaling of \(\gamma\), the larger the number of spins to which the qubits are coupled, the less appreciable is the decoherence.

The Hamiltonian operator \(H_2\) can be diagonalized by dealing with the operator \(\Delta_z\) as a scalar. This yields the following matrix elements in \(\mathbb{C}^2\):

\[
U_{22}(t) = \cos\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right) + i\frac{\gamma}{\sqrt{N}}\Delta_z \frac{\sin\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right)}{\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}},
\]

\[
U_{23}(t) = U_{32}(t) = -\frac{2i\lambda}{\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}} \sin\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right),
\]

\[
U_{33}(t) = \cos\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right) - i\frac{\gamma}{\sqrt{N}}\Delta_z \frac{\sin\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right)}{\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}},
\]

Here we have omitted the contribution of \(H_B = h\Delta_z\) since it simply introduces a global unitary term to the dynamics.

Let us consider the case when the qubits are initially prepared in the maximally entangled state \(|\psi\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)\).

Clearly, the density matrix \(\rho(0) = |\psi\rangle\langle\psi|\) belongs to the subspace corresponding to the Hamiltonian \(H_2\). Using the fact that \(|U_{22}(t)|^2 + |U_{23}(t)|^2 = I_B\), and \(U_{22}(t)U_{23}(t)^\dagger + U_{23}(t)U_{33}(t)^\dagger = 0\), it can be shown that the elements of the above density matrix evolve in time according to

\[
\rho_{22}(t) = \frac{1}{2}[1 - g(t)], \quad \rho_{23} = \frac{1}{2}[1 - f(t)],
\]

where

\[
g(t) = \frac{4\lambda\gamma}{[2\cosh(h\beta/2)]^2N} \text{tr}\left\{\frac{\Delta_z e^{-h\beta\Delta_z}}{\sqrt{N}} \sin^2\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right) \right\},
\]

and

\[
f(t) = \frac{1}{[2\cosh(h\beta/2)]^2N} \text{tr}\left\{2\gamma^2\Delta_z^2 e^{-h\beta\Delta_z} \sin^2\left(t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right) \right\} + \frac{\gamma e^{-h\beta\Delta_z}}{\sqrt{N}} \Delta_z \sin\left(2t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right)
\]

\[
- \frac{i\gamma e^{-h\beta\Delta_z}}{\sqrt{N}} \Delta_z \sin\left(2t\sqrt{4\lambda^2 + \gamma^2\Delta_z^2/N}\right) \right\}.
\]
Figures 2 and 3 display the behavior of the concurrence as a function of time for some particular values of the model parameters. We can see that for $h\beta = 1$ (i.e., at relatively high temperature) the concurrence shows damped oscillations and converges to a certain asymptotic limit which can be analytically derived, as we shall see below, only for $h = 0$ and/or $\beta = 0$. As $h\beta$ increases, the oscillations disappear and the concurrence converges to lower asymptotic values as shown in figure 2.

![Figure 2: (Color online) Concurrence as a function of time in the case of the initial state $(|−+⟩+|+−⟩)/\sqrt{2}$ for $N = 100$, $\gamma = 4$, $h\beta = 4$, and $\lambda = 2$.](image)

In what follows we focus our attention on the infinite temperature limit, i.e., $\beta \to 0$. In this case the reduced density matrix takes the form

$$\rho(t) = \frac{1}{2} \begin{pmatrix} 1 & 1 - f(t) \\ 1 - f(t) & 1 \end{pmatrix}, \quad (38)$$

whereas the function $f(t)$ simplifies to

$$f(t) = 2^{-2N} \text{tr} \left\{ \frac{2\gamma^2 \Delta_z^2 \sin^2 \left( t\sqrt{4\lambda^2 + \gamma^2 \Delta_z^2/N} \right)}{N \left( 4\lambda^2 + \gamma^2 \Delta_z^2/N \right)} \right\}. \quad (39)$$

Notice that $0 \leq f(t) \leq 2$, in accordance with the general properties of density matrices in $\mathbb{C}^2$. This enables us to derive the following explicit expression for the concurrence:

$$C(t) = \frac{1}{2} \left[ \sqrt{f(t)^2 - 4f(t) + 4} - f(t) \right]$$

$$= 1 - f(t). \quad (40)$$
In the thermodynamic limit, \( N \to \infty \), the function \( f(t) \) can be expressed as

\[
f(t) = 4\gamma^2 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{x^2 e^{-2x^2}}{4\lambda^2 + 2\gamma^2 x^2} \sin^2 \left( t \sqrt{4\lambda^2 + 2\gamma^2 x^2} \right) dx.
\]  

(41)

Some comments are in order here: We have shown in [9] that the operator \( J_z/\sqrt{N} \) converges to a real normal random variable \( \alpha \) with the probability density function \( F(\alpha) = \sqrt{2/\pi} \exp\{-2\alpha^2\} \); this is also the case for the operator \( J_z/\sqrt{N} \). Thus we are led to the task of finding the probability distribution function \( L(\alpha) \) of the sum of two independent random variables \( \alpha_1 \) and \( \alpha_2 \) characterized by \( F(\alpha_1) \) and \( F(\alpha_2) \), respectively. (note that the probability distribution function of \( a\alpha \), where \( a \) is nonzero real number, is equal to \( (1/|a|)F(\alpha/a) \).) The function \( L(\alpha) \) is simply given by the convolution of \( F(\alpha) \) with itself, which yields \( L(\alpha) = (1/\sqrt{\pi}) \exp\{-\alpha^2\} \). This becomes apparent from the change of variable \( \alpha \to \sqrt{2}\alpha \) carried out in equation (41). An other way to see that is to simply notice that \( \Delta_z/(\sqrt{2N}) \) converges to the random variable \( \alpha \to F(\alpha) \). From equation (41) it follows that

\[
\lim_{t \to \infty} f(t) = 1 - 2\sqrt{\pi} \frac{\gamma}{\lambda} e^{4\lambda^2} \text{erfc} \left( \frac{2\lambda}{\gamma} \right),
\]  

(42)

where \( \text{erfc}(x) \) denotes the complementary error function. By virtue of equation (40), we obtain

\[
C(\infty) = \lim_{t \to \infty} C(t) = 2\sqrt{\pi} \frac{\lambda}{\gamma} e^{4\lambda^2} \text{erfc} \left( \frac{2\lambda}{\gamma} \right).
\]  

(43)
In figure 4, we have plotted the concurrence as a function of time in the limit $N \to \infty$ along with the asymptotic value given by formula (43). The behavior of $C(\infty)$ as a function of $\lambda$ and $\gamma$ is shown in figures 5 and 6. As one may expect, $\lim_{\lambda \to \infty} C(\infty) = 1$, and $\lim_{\gamma \to \infty} C(\infty) = 0$. 
Figure 6: (Color online) $C(\infty)$ as a function of $\gamma$ for $\lambda = 2$.

This confirms the results of [10] where it is shown that strong coupling between the central qubits reduces the effect of the environment on their dynamics. Finally it is worth mentioning that due to the $XY$ interaction between the central spins, entanglement will be generated between them when the initial state is $|\pm\pm\rangle$. However, the corresponding off-diagonal elements of the reduced density matrix vanish at long times, making the asymptotic state of the qubits unentangled.

In summary we have used the invariance of the trace to derive analytical properties of the degeneracy $\nu(N, j)$, and to describe the dynamics of two qubits embedded in separate spin baths. We have shown that when the baths have the same size, the form of the model Hamiltonian enables us to map the full dynamics onto the evolution in time of two pseudo two-level systems coupled to a spin bath whose size is twice larger than the physical ones. This allowed us to derive the limit of an infinite number of spins within the environments and to analytically calculate the asymptotic state. The results of this work provide more evidences regarding the role played by the mutual interactions between the central qubits in diminishing the effects of their coupling to the surrounding spin environments.

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