A MORE GENERAL METHOD TO CLASSIFY UP TO EQUIVARIANT KK-EQUIVALENCE II: COMPUTING OBSTRUCTION CLASSES

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Abstract. We describe Universal Coefficient Theorems for the equivariant Kasparov theory for C*-algebras with an action of the group of integers or over a unique path space, using KK-valued invariants. We compare the resulting classification up to equivariant KK-equivalence with the recent classification theorem involving a K-theoretic invariant together with an obstruction class in a certain Ext^2-group and with the classification by filtrated K-theory. This is based on a general theorem that computes these obstruction classes.

1. Introduction

Objects in a triangulated category, such as equivariant KK-categories, may be classified up to isomorphism using a primary homological invariant and a secondary “obstruction class” provided they have a projective resolution of length 2 in a suitable sense. This method was applied in [4] to objects in the circle-equivariant KK-category KK^T, the equivariant KK-category KK^X for C*-algebras over a finite, unique path space X, and graph C*-algebras with finite ideal lattice. Here we compute the obstruction classes that occur in this classification. This makes the classification for objects of KK^T and KK^X more explicit.

The result suggests, in fact, a different invariant for objects in these categories that is fine enough to admit a Universal Coefficient Theorem. The price to pay is that the invariant uses bivariant K-theory instead of ordinary K-theory. We explain this for the equivariant bootstrap class in the category KK^Z; the latter is equivalent to KK^T by Baaj–Skandalis duality.

An object of KK^Z is a C*-algebra A with a Z-action, which is generated by a single automorphism α. The most obvious homological invariant on KK^Z maps this to the K-theory K_*(A) with the module structure over the group ring Z[x, x^{-1}] of Z that is induced by α. Let A be the category of countable, Z/2-graded modules over Z[x, x^{-1}]. This is a stable Abelian category. The K-theory described above defines a stable homological functor F^Z: KK^Z → A. The category A has cohomological dimension 2, that is, any object has a projective resolution of length 2. Let Aδ be the additive category of pairs (A, δ) with A ∈ A and δ ∈ Ext^2_A(ΣA, A); morphisms from (A, δ) to (A', δ') in Aδ are morphisms f from A to A' in A with δ'f = fδ. It is shown in [4] that isomorphism classes of objects in the bootstrap class in KK^Z are in bijection with isomorphism classes of objects in Aδ. In particular, any object A of A lifts to an object in the bootstrap class in KK^Z. The lifting is, however, not unique: different liftings of A are in bijection with Ext^3_A(ΣA, A); here two liftings B_1, B_2 are identified if there is an isomorphism B_1 ≅ B_2 that induces the identity map on A = F^Z(B_1) = F^Z(B_2).

Here we compute the obstruction class of an object of KK^Z explicitly. This question remained open in [4]. Let (A, α) be a C*-algebra with an automorphism α

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as above. The exact sequence in the Universal Coefficient Theorem splits, so that
\[ \text{KK}_0(A, A) \cong \text{Hom}(K_*(A), K_*(A)) \oplus \text{Ext}^1(K_{*+1}(A), K_*(A)). \]

While the splitting above is not natural, it is canonical enough to associate well-defined elements \( a_0 \in \text{Hom}(K_*(A), K_*(A)) \) and \( a_1 \in \text{Ext}^1(K_{*+1}(A), K_*(A)) \) to \( \alpha \).
Here \( a_0 \) is the action on \( K_*(A) \) induced by \( \alpha \), which is part of the \( \mathbb{Z}[x, x^{-1}] \)-module structure on \( F^Z_K(A) \). We show how to compute the obstruction class from \( a_1 \).
Namely, it is \( \mu^*(\alpha_1) \) for a canonical map
\[ \mu^*: \text{Ext}^1(K_{*+1}(A), K_*(A)) \to \text{Ext}^2_{\mathbb{Z}[x, x^{-1}]}(K_{*+1}(A), K_*(A)). \]

Thus an object \((A, \alpha)\) of \( \mathcal{R}_\mathcal{R}^Z \) is determined uniquely up to isomorphism by \( A \) as an object of \( \mathcal{R}_\mathcal{R}^\mathcal{X} \) together with the class \([\alpha]\) of \( \alpha \) in \( \text{KK}_0(A, A) \). We treat the pair \((A, [\alpha])\) as an object of a certain exact category \( \mathcal{R}_\mathcal{R}[\mathbb{Z}] \) with a suspension automorphism. The forgetful functor \( F^Z: \mathcal{R}_\mathcal{R}^Z \to \mathcal{R}_\mathcal{R}[\mathbb{Z}] \) is a stable homological functor. The fact that it gives a complete invariant for objects of \( \mathcal{R}_\mathcal{R}^Z \) follows from a Universal Coefficient Theorem computing \( \text{KK}^Z(A, B) \). This Universal Coefficient Theorem is already proven in \( [11] \), and it is shown there to be equivalent to a Pimsner–Voiculescu like exact sequence for \( \text{KK}^Z(A, B) \).

The Universal Coefficient Theorem for the invariant \( F^Z: \mathcal{R}_\mathcal{R}^Z \to \mathcal{R}_\mathcal{R}[\mathbb{Z}] \) is based on projective resolutions of length 1 for the stable homological ideal \( \ker F^Z \). We use these resolutions to compute the obstruction class related to the homological invariant \( F^Z_K(A) := K_*(A) \). Namely, let \( 0 \to B_1 \to B_0 \to A \to 0 \) be a projective resolution of length 1 for the ideal \( \ker F^Z \) in \( \mathcal{R}_\mathcal{R}^Z \). This remains exact with respect to the larger stable homological ideal \( \ker F^Z_K \). The objects \( B_1 \) and \( B_0 \) are no longer projective for \( \ker F^Z \), but they have projective resolutions of length 1. Thus \( \text{KK}^Z(B_1, B_0) \) is computed by a Universal Coefficient Theorem, which splits non-naturally. So the arrow \( B_1 \to B_0 \) gives a class in \( \text{Ext}^1_{\mathbb{Z}[x, x^{-1}]}(K_{*+1}(B_1), K_*(B_0)) \). It must determine the obstruction class of \( A \) because \( A \) is \( \text{KK}^Z \)-equivalent to the cone of the map \( B_1 \to B_0 \). The result is the formula for the obstruction class of \( A \) asserted above. This computation of obstruction classes is carried out in Section 3 in the same generality as in \( [3] \), using a triangulated category with a stable homological ideal with enough projectives.

The Universal Coefficient Theorem for \( \mathcal{R}_\mathcal{R}^Z \) is developed in Section 2.1. This is followed by an analogous treatment for the category of \( C^* \)-algebras over a unique path space \( X \) in Section 2.2. In Section 4 we compare the classification theorems that our two invariants give for objects in the bootstrap classes in \( \mathcal{R}_\mathcal{R}^Z \) and \( \mathcal{R}_\mathcal{R}^X \). In Section 5 we study \( \mathcal{R}_\mathcal{R}^\mathcal{X} \) in the special case where \( X \) is totally ordered. Then filtrated K-theory is a third complete invariant for the same objects (see \( [12] \)), and we compare this invariant with the new ones. This is a rather long homological computation. In Section 5.2 the result of Section 5 is formulated more nicely if \( X \) has only two points, so that \( C^* \)-algebras over \( X \) are extensions of \( C^* \)-algebras.

2. Universal Coefficient Theorems with KK-valued functors

This section develops Universal Coefficient Theorems for the equivariant Kasparov categories \( \mathcal{R}_\mathcal{R}^Z \) and \( \mathcal{R}_\mathcal{R}^X \) for a unique path space \( X \), which are based on forgetful functors to \( \mathcal{R} \). We shall use the general framework developed in \( [11] \) for doing relative homological algebra in triangulated categories. The starting point for this is a stable homological ideal \( \mathcal{I} \) in a triangulated category \( \mathcal{T} \). Being stable and homological means that there is a stable homological functor \( F: \mathcal{T} \to \mathcal{R} \) to some stable Abelian category \( \mathcal{R} \) with
\[ \mathcal{I}(A, B) = \ker F(A, B) := \{ g \in \mathcal{T}(A, B) : F(g) = 0 \}. \]
Here stability of $\mathfrak{A}$ and $F$ means that $\mathfrak{A}$ is equipped with a suspension automorphism and that $F \circ \Sigma_T = \Sigma_{\mathfrak{A}} \circ F$.

A stable homological ideal $\mathfrak{I}$ allows to carry over various notions from homological algebra to $\mathfrak{T}$. In particular, there are $\mathfrak{I}$-exact chain complexes, $\mathfrak{I}$-projective objects and $\mathfrak{I}$-projective resolutions in $\mathfrak{T}$, which then allow to define $\mathfrak{I}$-derived functors. We shall only be interested in cases where there are enough $\mathfrak{I}$-projective objects. The thick subcategory $(\mathfrak{P}_{\mathfrak{I}})$ generated by the $\mathfrak{I}$-projective objects is an analogue of the “bootstrap class” in Kasparov theory. If $A \in (\mathfrak{P}_{\mathfrak{I}})$ has an $\mathfrak{I}$-projective resolution of length 1, then the graded group $\Sigma_{\mathfrak{I}}(A, B)$ for $B \in \mathfrak{T}$ may be computed by a Universal Coefficient Theorem (UCT). The Hom and Ext groups in this UCT are those in a certain Abelian category, namely, the target category of the universal $\mathfrak{I}$-exact stable homological functor $F^u$. The functor $F^u$ strongly classifies objects of $(\mathfrak{P}_{\mathfrak{I}})$ with an $\mathfrak{I}$-projective resolution of length 1, that is, any isomorphism $F^u(A) \cong F^u(B)$ between such objects is induced by an isomorphism $A \cong B$ in $\mathfrak{T}$. In many cases of interest, the universal functor $F^u$ is quite explicit. The following theorem allows to recognise it:

**Theorem 2.1 (\cite{11} Theorem 57).** Let $\mathfrak{T}$ be a triangulated category, let $\mathfrak{I} \subseteq \mathfrak{T}$ be a stable homological ideal, and let $F: \mathfrak{T} \to \mathfrak{A}$ be an $\mathfrak{I}$-exact stable homological functor into a stable Abelian category $\mathfrak{A}$. Let $\mathfrak{P}_{\mathfrak{A}}$ be the class of projective objects in $\mathfrak{A}$. Suppose that idempotent morphisms in $\mathfrak{T}$ split. The functor $F$ is the universal $\mathfrak{I}$-exact stable homological functor to a stable Abelian category and there are enough $\mathfrak{I}$-projective objects in $\mathfrak{T}$ if and only if

1. $\mathfrak{A}$ has enough projective objects;
2. the adjoint functor $F^\circ$ of $F$ is defined on $\mathfrak{P}_{\mathfrak{A}}$;
3. $F \circ F^\circ(A) \cong A$ for all $A \in \mathfrak{P}_{\mathfrak{A}}$.

The ideals to be treated in this section are defined as the kernel on morphisms of a triangulated functor $F: \mathfrak{T} \to \mathfrak{S}$ to another triangulated category $\mathfrak{S}$. Ideals of this form are already treated in \cite{11}, using a construction of Freyd to embed $\mathfrak{S}$ into an Abelian category. Here we unify these two cases by allowing the functor $F$ to take values in an exact category (see \cite{57}). An Abelian category $\mathfrak{A}$ is exact where all extensions are admissible. And a triangulated category $\mathfrak{S}$ is exact where only split extensions are admissible. So exact categories contain the two cases treated in \cite{11}. In many examples of classification results using a UCT, the range of the universal $\mathfrak{I}$-exact functor $F^u_{\mathfrak{A}_{\mathfrak{S}_{\mathfrak{I}}}}$ is a certain exact subcategory of a module category, and only objects in this exact subcategory have projective resolutions of length 1. This holds for the UCTs in \cite{2,5,8,12} (see Remark 5.6 for one of these cases). The phenomenon is understood better in \cite{6}.

Quillen’s Embedding Theorem allows to embed any exact category into an Abelian one in a fully faithful, fully exact way. So all results in the general theory carry over to the case where $F$ takes values in an exact category. In particular, we may modify the definition of the universal $\mathfrak{I}$-exact stable homological functor by allowing exact categories as target. Let $F^u_{\mathfrak{A}_{\mathfrak{S}}}: \mathfrak{T} \to \mathfrak{A}$ be the universal $\mathfrak{I}$-exact functor into an Abelian category as in Theorem 2.1. Let $\mathfrak{C} \subseteq \mathfrak{A}$ be the exact subcategory of $\mathfrak{A}$ generated by the range of $F^u_{\mathfrak{A}_{\mathfrak{S}}}$, and $F^u_{\mathfrak{C}_{\mathfrak{S}}}$ viewed as a functor to $\mathfrak{C}$. This is the universal $\mathfrak{I}$-exact functor in the new sense. Its universal property follows from the one of $F^u_{\mathfrak{A}_{\mathfrak{S}}}$ and Quillen’s Embedding Theorem. If $\mathfrak{I}$ is defined by a forgetful functor $F: \mathfrak{T} \to \mathfrak{S}$ to another triangulated category, then the universal homological functor to an Abelian category typically uses Freyd’s embedding of $\mathfrak{S}$ into an Abelian category. In contrast, we shall see that in many examples the universal $\ker F$-exact functor to an exact category gives an exact category that is very closely related to $\mathfrak{S}$. 

Remark 2.2. Assume that all objects in $\mathcal{Z}$ have an $\mathcal{I}$-projective resolution of finite length. Then the image of $F^u_{\text{old}}$ is contained in the subcategory of objects in $\mathfrak{A}$ with a finite-length projective resolution. This subcategory is exact. And it is already generated as an exact category by the projective objects in $\mathfrak{A}$; this is proved by induction on the length of a resolution. Hence the image of $F^u_{\text{new}}$ must be equal to the subcategory of objects with a finite-length projective resolution. So $F^u_{\text{new}} = F^u_{\text{old}}$ if and only if all objects of $\mathfrak{A}$ have a finite-length projective resolution.

2.1. Actions of the group of integers. We now treat a concrete example, namely, the case $\mathcal{Z} := \mathcal{R}\mathcal{R}^Z$. This is a triangulated category. Its objects are pairs $(A, \alpha)$ with a separable C*-algebra $A$ and $\alpha \in \text{Aut}(A)$; the latter generates an action of $\mathbb{Z}$ on $A$ by automorphisms. The arrows in $\mathcal{R}\mathcal{R}^Z$ are the Kasparov groups $\text{KK}_0(A, B)$, and the composition is the Kasparov product; we have dropped the automorphisms from our notation as usual to avoid clutter. The triangulated category structure on $\mathcal{R}\mathcal{R}^Z$ is described in [9] Appendix]. The relative homological algebra in $\mathcal{R}\mathcal{R}^Z$ is already studied in [11]. The main result is the following variant of the Pimsner–Voiculescu sequence for $\text{KK}_2(A, B)$. Let $A$ and $B$ be C*-algebras with automorphisms $\alpha$ and $\beta$, respectively. Then there is an exact sequence

$$\begin{align*}
\text{KK}_1(A, B) &\xrightarrow{\alpha \ast \beta_{*}^{-1}} \text{KK}_0^Z(A, B) \xrightarrow{\text{forget}} \text{KK}_0(A, B) \\
\text{KK}_1(A, B) &\leftarrow \text{KK}_0^Z(A, B) \leftarrow \text{KK}_0(A, B)
\end{align*}$$

(see [11] Section 5.1); here $(\alpha \ast \beta_{*}^{-1} - 1)(x) = \beta_{*}^{-1} \circ x \circ \alpha - x$ for all $x \in \text{KK}_1(A, B)$. We may rewrite this as a pair of short exact sequences

$$\text{coker}(\alpha \ast \beta_{*}^{-1} - 1) : \text{KK}_{1+i}(A, B) \xrightarrow{\text{forget}} \text{KK}_{1+i}(A, B)$$

$$\text{ker}(\alpha \ast \beta_{*}^{-1} - 1) : \text{KK}_{i}(A, B) \xrightarrow{\text{forget}} \text{KK}_{i}(A, B)$$

for $i = 0, 1$. We shall explain that the long exact sequence (2.3) is an instance of a Universal Coefficient Theorem as in [11] Theorem 66. In particular, the cokernel and kernel in it are the Ext and Hom groups in a certain exact category. This is already proven in [11], but the consequences for classification are not explored there. We shall also treat other examples by the same method later.

The forgetful functor

$$\mathcal{R}^Z : \mathcal{R}\mathcal{R}^Z \to \mathcal{R}\mathcal{R}, \quad (A, \alpha) \mapsto A,$n
is triangulated. So its kernel on morphisms

$$\mathcal{F}^Z(A, B) := \{ f \in \text{KK}_0^Z(A, B) : \mathcal{R}^Z(f) = 0 \text{ in } \text{KK}_0(A, B) \}$$

is a stable homological ideal in $\mathcal{R}\mathcal{R}^Z$. We now describe the universal $\mathcal{F}^Z$-exact functor. The main point here is to describe its target category. Let $\mathcal{R}\mathcal{R}[\mathbb{Z}]$ be the additive category of functors $\mathbb{Z} \to \mathcal{R}\mathcal{R}$ with natural transformations as arrows. Equivalently, an object of $\mathcal{R}\mathcal{R}[\mathbb{Z}]$ is an object $A \in \mathcal{R}\mathcal{R}$ with a group homomorphism $a : \mathbb{Z} \to \text{KK}_0(A, A)^{\times}$, $n \mapsto a_n$, where $\text{KK}_0(A, A)^{\times}$ denotes the multiplicative group of invertible elements in the ring $\text{KK}_0(A, A)$. And an arrow $(A, a) \to (B, b)$ in $\mathcal{R}\mathcal{R}[\mathbb{Z}]$ is an arrow $f \in \text{KK}_0(A, B)$ that satisfies $b_n \circ f = f \circ a_n$ for all $n \in \mathbb{Z}$. The homomorphism $a$ is determined by its value at $1 \in \mathbb{Z}$, which may be any element $a_1 \in \text{KK}_0(A, A)^{\times}$. So we also denote objects of $\mathcal{R}\mathcal{R}[\mathbb{Z}]$ as $(A, a_1)$. An element $f \in \text{KK}_0(A, B)$ is an arrow $(A, a) \to (B, b)$ in $\mathcal{R}\mathcal{R}[\mathbb{Z}]$ if and only if $b_1 \circ f = f \circ a_1$ or, equivalently, $b_1^{-1} \circ f \circ a_1 = f = 0$. Thus

$$\text{ker}(\alpha \ast \beta_{*}^{-1} - 1) : \text{KK}_0(A, B) \to \text{KK}_0(A, B) \cong \text{Hom}_{\mathcal{R}\mathcal{R}[\mathbb{Z}]}((A, [\alpha]), (B, [\beta])).$$
Here we have implicitly used the functor 
\[ F^\mathbb{Z} : \mathcal{R}[\mathbb{Z}] \to \mathcal{R}[\mathbb{Z}], \quad (A, \alpha) \mapsto (A, [\alpha]), \]
where \([\alpha] \in KK_0(A, A)^{\times}\) is the \(KK\)-class of the automorphism \(\alpha\). It has the same kernel on morphisms \(\mathbb{Z}\) as the forgetful functor \(R^\mathbb{Z}\).

We call a kernel-cokernel pair \(K \to E \to Q\) in \(\mathcal{R}[\mathbb{Z}]\) admissible if it splits in \(\mathcal{R}\), that is, \(E \cong K \oplus Q\) as objects of \(\mathcal{R}\). This turns \(\mathcal{R}[\mathbb{Z}]\) into an exact category (see [5, Exercise 5.3]). We use this exact structure to define projective objects and projective resolutions in \(\mathcal{R}[\mathbb{Z}]\).

Given \(A \in \mathcal{R}\) and a free Abelian group \(G = \mathbb{Z}[I]\) on a countable set \(I\), we define \(A \otimes G \in \mathcal{R}\) by
\[ (A \otimes G) := \bigoplus_{i \in I} A. \]
In particular, \(A \otimes \mathbb{Z} := A\). This construction is an additive functor in \(G\). Namely, let \(G\) and \(H\) be free Abelian groups and let \(f : G \to H\) be a group homomorphism. The Universal Coefficient Theorem implies \(KK_0(C \otimes G, C \otimes H) \cong \text{Hom}(G, H)\). So we get a functorial \(f_C \in KK_0(C \otimes G, C \otimes H)\). Identifying \(A \otimes G = A \otimes (C \otimes G)\) (with the minimal tensor product of \(C^\ast\)-algebras), we get a functorial \(f_A := id_A \otimes f_C \in KK_0(A \otimes G, A \otimes H)\).

Let \(A \in \mathcal{R}\). Then \(A \otimes \mathbb{Z}[x, x^{-1}]\) with the invertible element \(x_A = id_A \otimes x \in KK_0(A \otimes \mathbb{Z}[x, x^{-1}], A \otimes \mathbb{Z}[x, x^{-1}])\) induced by the invertible element \(x \in \mathbb{Z}[x, x^{-1}]\) is an object of \(\mathcal{R}[\mathbb{Z}]\). It behaves like a free module over \(A\) because
\[ \text{Hom}_{\mathcal{R}[\mathbb{Z}]}((A \otimes \mathbb{Z}[x, x^{-1}], x_A), (B, b)) \cong KK_0(A, B). \]
In particular, \((A \otimes \mathbb{Z}[x, x^{-1}], x_A)\) is projective in the exact category \(\mathcal{R}[\mathbb{Z}]\). The invertible element \(x_A\) in the KK-endomorphism ring of \((A \otimes \mathbb{Z}[x, x^{-1}], x_A)\) lifts to the shift automorphism
\[ \tau \in \text{Aut}(C_0(\mathbb{Z}, A)), \quad (\tau f)(n) := f(n - 1). \]
That is, \(F^\mathbb{Z}(C_0(\mathbb{Z}, A), \tau) \cong (A \otimes \mathbb{Z}[x, x^{-1}], x_A).\) And
\[ KK^\mathbb{Z}(C_0(\mathbb{Z}, A), B) \cong KK_0(A, B) \cong \text{Hom}_{\mathcal{R}[\mathbb{Z}]}((A \otimes \mathbb{Z}[x, x^{-1}], x_A), B). \]
The first isomorphism here is [9] Equation (20). It applies the forgetful functor
\[ KK^\mathbb{Z}((C_0(\mathbb{Z}, A), \tau), (B, \beta)) \to KK(C_0(\mathbb{Z}, A), B) \]
and then composes with the inclusion ∗-homomorphism \(A \to C_0(\mathbb{Z}, A)\) that maps \(A\) identically onto the summand at \(0 \in \mathbb{Z}\). Equation (25) says that the partial adjoint of the functor \(F^\mathbb{Z}\) is defined on \((A \otimes \mathbb{Z}[x, x^{-1}], x_A)\) and maps it to \((C_0(\mathbb{Z}, A), \tau)\).

**Lemma 2.6.** Any object of \(\mathcal{R}[\mathbb{Z}]\) has a free resolution of length 1.

**Proof.** The trivial representation of the group \(\mathbb{Z}\) on \(\mathbb{Z}\) corresponds to \(\mathbb{Z}\) made a module over \(\mathbb{Z}[x, x^{-1}]\) by letting \(x \in \mathbb{Z}[x, x^{-1}]\) act by 1. This module has the following free resolution of length 1:
\[ 0 \to \mathbb{Z}[x, x^{-1}] \xrightarrow{\text{mult}(x^n)} \mathbb{Z}[x, x^{-1}] \xrightarrow{ev_1} \mathbb{Z} \to 0, \]
where \(ev_1 : \mathbb{Z}[x, x^{-1}] \to \mathbb{Z}\) is the homomorphism of evaluation at 1, so \(ev_1(x^n) = 1\) for all \(n \in \mathbb{Z}\). The projective resolution in (2.7) is contractible as a chain complex of Abelian groups because \(\mathbb{Z}\) and \(\mathbb{Z}[x, x^{-1}]\) are free as Abelian groups.

The resolution (2.7) induces a chain complex
\[ 0 \to (A \otimes \mathbb{Z}[x, x^{-1}], [\alpha] \otimes x) \to (A \otimes \mathbb{Z}[x, x^{-1}], [\alpha] \otimes x) \to (A, [\alpha]) \to 0 \]
in the additive category \( \mathcal{R}[\mathbb{Z}] \). That is, the maps are arrows in \( \mathcal{R}[\mathbb{Z}] \). The resolution \((2.7)\) splits by a group homomorphism, and the tensor product construction is additive. Hence \((2.8)\) is exact, that is, it splits in \( \mathcal{R} \). The arrow
\[
A \otimes \mathbb{Z}[x, x^{-1}] = \bigoplus_{n \in \mathbb{Z}} A \xrightarrow{\bigoplus_{n \in \mathbb{Z}} [a^n]} \bigoplus_{n \in \mathbb{Z}} A = A \otimes \mathbb{Z}[x, x^{-1}]
\]
in \( \mathcal{R} \) is an isomorphism
\[
(A \otimes \mathbb{Z}[x, x^{-1}], \text{id}_A \otimes x) \cong (A \otimes \mathbb{Z}[x, x^{-1}], [a] \otimes x).
\]
Thus \((2.9)\)
\[
0 \to (A \otimes \mathbb{Z}[x, x^{-1}], 1 \otimes x) \to (A \otimes \mathbb{Z}[x, x^{-1}], 1 \otimes x) \to (A, [a]) \to 0,
\]
in \( \mathcal{R}[\mathbb{Z}] \), where the boundary map on \((A \otimes \mathbb{Z}[x, x^{-1}], 1 \otimes x)\) has changed to \([a]^{-1} \otimes \text{mult}(x) - \text{id}_{A \otimes \mathbb{Z}[x, x^{-1}]} \). □

Lemma \(2.6\) implies that the exact category \( \mathcal{R}[\mathbb{Z}] \) has enough projective objects and that an object of \( \mathcal{R}[\mathbb{Z}] \) is projective if and only if it is a direct summand of a free object. Since the partial adjoint of the functor \( F^\mathbb{Z} \) is defined on all free objects, it is defined also on all projective objects of \( \mathcal{R}[\mathbb{Z}] \), and it is inverse to \( F^\mathbb{Z} \) on this subcategory. Idempotents in the category \( \mathcal{R}[\mathbb{Z}] \) split because it is triangulated and has countable direct sums.

**Proposition 2.10.** The functor \( F^\mathbb{Z} : \mathcal{R}^\mathbb{Z} \to \mathcal{R}[\mathbb{Z}] \) is the universal \( \mathbb{Z} \)-exact stable homological functor from \( \mathcal{R}^\mathbb{Z} \) to an exact category.

**Proof.** To prove this, we first embed \( \mathcal{R}[\mathbb{Z}] \) into an Abelian category by Quillen’s Embedding Theorem. One way to do this is to embed \( \mathcal{R} \) into an Abelian category \( \mathfrak{A} \) by Freyd’s Embedding Theorem and then form \( \mathfrak{A}[\mathbb{Z}] \). The Abelian category \( \mathfrak{A}[\mathbb{Z}] \) has enough projective objects, and all its projective objects already belong to \( \mathcal{R}[\mathbb{Z}] \). So Theorem \(2.1\) applies and shows that the functor to \( \mathfrak{A}[\mathbb{Z}] \) is the universal \( \mathbb{Z} \)-exact functor to an Abelian category. The subcategory \( \mathcal{R}[\mathbb{Z}] \subseteq \mathfrak{A}[\mathbb{Z}] \) is exact and contains the range of the functor. In fact, we shall show soon that the functor \( \mathcal{R}^\mathbb{Z} \to \mathcal{R}[\mathbb{Z}] \) is surjective on objects (see Theorem \(2.11\)). Taking this for granted, Remark \(2.2\) shows that the functor \( \mathcal{R}^\mathbb{Z} \to \mathcal{R}[\mathbb{Z}] \) is the universal \( \mathbb{Z} \)-exact functor to an exact category. □

Equip \( C_0(\mathbb{Z}, A) \) with the \( \mathbb{Z} \)-action generated by the shift automorphism \( \tau \). We use \((2.7)\) to lift the resolution \((2.9)\) in \( \mathcal{R}[\mathbb{Z}] \) to the following \( \mathbb{Z} \)-projective resolution of length 1 in \( \mathcal{R}^\mathbb{Z} \):
\[
(2.11) \quad 0 \to C_0(\mathbb{Z}) \otimes A \xrightarrow{\varphi} C_0(\mathbb{Z}) \otimes A \xrightarrow{p} (A, [a]) \to 0;
\]
here \( \varphi = [\tau] \otimes [a]^{-1} - 1 \), and \( p \) is the counit of the adjunction \((2.5)\): that is, the first isomorphism in \((2.5)\) maps \( p \) to the identity element in \( 
K K_0(A, A) \). In other words, when we forget the \( \mathbb{Z} \)-actions, then \( p \) restricts to the identity map at the 0th summand in \( C_0(\mathbb{Z}) \otimes A = \bigoplus_{n \in \mathbb{Z}} A \). When we use the resolution \((2.11)\) to compute derived functors, we get
\[
\text{Ext}_{\mathcal{R}[\mathbb{Z}], \mathbb{Z}}^1((A, [a]), (B, [\beta])) \cong \text{coker}((\alpha^*)^{-1} \beta_* - 1 : \text{KK}(A, B) \to \text{KK}(A, B)).
\]
The map \((\alpha^*)^{-1} \beta_* - 1\) has the same cokernel as \( \alpha^* \beta_*^{-1} - 1 \).

A rather deep result says that the \( \mathbb{Z} \)-projective objects generate \( \mathcal{R}^\mathbb{Z} \). Equivalently, if \( H^\mathbb{Z}(A) \cong 0 \) in \( \mathcal{R} \), then already \( A \cong 0 \) in \( \mathcal{R}^\mathbb{Z} \). This is related to the proof of the Baum–Connes conjecture for the group \( \mathbb{Z} \). It follows immediately from the Pimsner–Voiculescu sequence \((2.3)\).
Theorem 2.12. Let \((A, \alpha)\) and \((B, \beta)\) be objects of \(\mathcal{R}\). Any isomorphism \(F^\mathcal{Z}(A) \cong F^\mathcal{Z}(B)\) in \(\mathcal{R}(\mathbb{Z})\) lifts to an isomorphism \(A \cong B\) in \(\mathcal{R}\). And any object of \(\mathcal{R}(\mathbb{Z})\) is isomorphic to \(F^\mathcal{Z}(A)\) for some \(A \in \mathcal{R}\), which is unique up to isomorphism.

If \(t \in \mathcal{K}K_0(A, \beta)\) is a KK-equivalence with \(t \circ [\alpha] = [\beta] \circ t\), then there is an isomorphism in \(\mathcal{K}K^\mathcal{Z}(A, B)\) that is mapped to \(t\) by the forgetful functor \(\mathcal{K}K^\mathcal{Z} \to \mathcal{K}K\).

Proof. The Universal Coefficient Theorem for \(\mathcal{K}K^\mathcal{Z}(A, B)\) and the ideal \(\mathcal{I}^\mathcal{Z}\) allows to lift any map \(t: (A, [\alpha]) \to (B, [\beta])\) in \(\mathcal{R}(\mathbb{Z})\) to an element \(t' \in \mathcal{K}K^\mathcal{Z}((A, \alpha), (B, \beta))\). The naturality of the Universal Coefficient Theorem implies that the composition vanishes for two elements of the \(\mathcal{E}\)-part of \(\mathcal{K}K^\mathcal{Z}(A, B)\). Hence \(t\) is invertible if \(t'\) is invertible. Any object of \(\mathcal{R}(\mathbb{Z})\) has a projective resolution of length 1. This allows to lift it to an object of \(\mathcal{K}K^\mathcal{Z}\) (see [4, Proposition 2.3]). This proves both assertions in the first paragraph. The second paragraph only describes isomorphisms in \(\mathcal{R}(\mathbb{Z})\) more concretely.

We make the isomorphism criterion in Theorem 2.12 more explicit under the assumption that the \(C^\ast\)-algebras \(A\) and \(B\) belong to the bootstrap class. Let \(A^\pm\) be \(C^\ast\)-algebras in the bootstrap class with

\[
\mathcal{K}K_0(A^+)=\mathcal{K}K_0(A), \quad \mathcal{K}K_1(A^+)=0, \quad \mathcal{K}K_0(A^-)=0, \quad \mathcal{K}K_1(A^-)=\mathcal{K}K_1(A).
\]

Then \(\mathcal{K}K_0(A^+ \oplus A^-) \cong \mathcal{K}K_0(A)\). So \(A \cong A^+ \oplus A^-\) by the Universal Coefficient Theorem for \(\mathcal{K}K\). We use such a KK-equivalence to map \(\alpha\) to an element of \(\mathcal{K}K_0(A^+ \oplus A^-, A^+ \oplus A^-)\). We rewrite this as a \(2 \times 2\)-matrix

\[
\begin{pmatrix}
\alpha^{++} & \alpha^{+-} \\
\alpha^{-+} & \alpha^{- -}
\end{pmatrix},
\]

\[\alpha^{++} \in \mathcal{K}K_0(A^+, A^+) \cong \text{Hom}(\mathcal{K}K_0(A), \mathcal{K}K_0(A)),\]

\[\alpha^{+-} \in \mathcal{K}K_0(A^-, A^+) \cong \text{Ext}(\mathcal{K}K_1(A), \mathcal{K}K_0(A)),\]

\[\alpha^{-+} \in \mathcal{K}K_0(A^+, A^-) \cong \text{Ext}(\mathcal{K}K_0(A), \mathcal{K}K_1(A)),\]

\[\alpha^{- -} \in \mathcal{K}K_0(A^-, A^-) \cong \text{Hom}(\mathcal{K}K_1(A), \mathcal{K}K_1(A)).\]

Here we have used the Universal Coefficient Theorem for the \(C^\ast\)-algebras \(A^\pm\). A similar decomposition \(B \cong B^+ \oplus B^-\) allows us to map \(\beta\) to an element of \(\mathcal{K}K_0(B^+ \oplus B^-, B^+ \oplus B^-)\), which we then rewrite as a \(2 \times 2\)-matrix

\[
\begin{pmatrix}
\beta^{++} & \beta^{+-} \\
\beta^{-+} & \beta^{- -}
\end{pmatrix},
\]

\[\beta^{++} \in \mathcal{K}K_0(B^+, B^+) \cong \text{Hom}(\mathcal{K}K_0(B), \mathcal{K}K_0(B)),\]

\[\beta^{+-} \in \mathcal{K}K_0(B^-, B^+) \cong \text{Ext}(\mathcal{K}K_1(B), \mathcal{K}K_0(B)),\]

\[\beta^{-+} \in \mathcal{K}K_0(B^+, B^-) \cong \text{Ext}(\mathcal{K}K_0(B), \mathcal{K}K_1(B)),\]

\[\beta^{- -} \in \mathcal{K}K_0(B^-, B^-) \cong \text{Hom}(\mathcal{K}K_1(B), \mathcal{K}K_1(B)).\]

And we may also transfer an element \(t \in \mathcal{K}K_0(A, B)\) to such a \(2 \times 2\)-matrix

\[
\begin{pmatrix}
t^{++} & t^{+-} \\
t^{-+} & t^{- -}
\end{pmatrix},
\]

\[t^{++} \in \mathcal{K}K_0(A^+, B^+) \cong \text{Hom}(\mathcal{K}K_0(A), \mathcal{K}K_0(B)),\]

\[t^{+-} \in \mathcal{K}K_0(A^-, B^+) \cong \text{Ext}(\mathcal{K}K_1(A), \mathcal{K}K_0(B)),\]

\[t^{-+} \in \mathcal{K}K_0(A^+, B^-) \cong \text{Ext}(\mathcal{K}K_0(A), \mathcal{K}K_1(B)),\]

\[t^{- -} \in \mathcal{K}K_0(A^-, B^-) \cong \text{Hom}(\mathcal{K}K_1(A), \mathcal{K}K_1(B)).\]

The naturality of the exact sequence in the Universal Coefficient Theorem implies that the Kasparov product of an element of \(\text{Ext}(\mathcal{K}K_1^*, A, B)\) with an element of \(\mathcal{K}K_0(B, C)\) depends only on the image of the latter in \(\text{Hom}(\mathcal{K}K_1(B), \mathcal{K}K_1(C))\). Thus the product of two Ext-terms always vanishes. Hence the condition \(t[\alpha] = [\beta]t\) in Theorem 2.12 is equivalent to four equations

\[
t^{++} + t^{+-} = \beta^{++} + \beta^{+-}, \quad t^{+-} + t^{-+} = \beta^{-+} + \beta^{- -} + \beta^{++} + \beta^{+-},
\]

\[
t^{-+} = \beta^{-+} - t^{- -}, \quad t^{-+} + t^{- -} = \beta^{- -} + \beta^{++} + \beta^{+-} - t^{-+}.
\]
Remark 2.14

We shall later identify this cokernel with the group

\[ \text{Ext}(K_1(A), K_0(B)) \]

where we abbreviate

\[ \alpha_\ast = (\alpha^{++}, \alpha^{--}) \quad \text{and} \quad \alpha^1 = (\alpha^{+-}, \alpha^{-+}), \]

and similarly for \( \beta \).

The choice of \( t^1 \) has no effect on the invertibility of \( t \). So the criterion in Theorem 2.12

is whether the image of \( \beta^1 t_\ast - t_\ast \alpha^1 \) vanishes in the cokernel of

\[
\text{Ext}(K_{1,++}(A), K_0(B)) \to \text{Ext}(K_{1,++}(A), K_0(B)), \quad t^1 \mapsto t^1 \alpha_\ast - \beta_\ast t^1.
\]

We shall later identify this cokernel with the group \( \text{Ext}_{\mathbb{Z}/2, x^{-1}}(K_{1,++}(A), K_0(B)) \)

and show that the image of \( \beta^1 t_\ast - t_\ast \alpha^1 \) in this cokernel is the relative obstruction class for \((A, \alpha)\) and \((B, \beta)\) and a \( \mathbb{Z}[x, x^{-1}] \)-module isomorphisms \( t_\ast : K_0(A) \cong K_0(B) \).

This gives the rule to translate between the classifying invariants in Theorem 2.12

and in [4].

Remark 2.14. If \( A = B \) is a Kirchberg algebra (separable, nuclear, unital, purely

infinite and simple), then a much finer classification theorem for automorphisms is

proved by Nakamura [13].

2.2. C*-Algebras over unique path spaces. Now we prove a Universal Coefficient

Theorem for \( \mathcal{R} \mathcal{R}^{++} \) for a unique path space \( X \). Let \( X \) be a countable set and

let \( \to \) be a relation on \( X \), which says for which points \( x, y \in X \) there is an edge

\( x \to y \). Equip \( X \) with the partial order generated by \( \to \), that is, \( x \preceq y \) if and only

if there is a chain of edges \( x = x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_t = y \) with some \( t \geq 0 \) and

\( x_0, x_1, \ldots, x_{t-1} \in X \). Equip \( X \) with the Alexandrov topology generated by this partial

order. We assume \((X, \to)\) to be a unique path space, that is, there is at most one

chain of edges \( x = x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_t = y \) between any two points \( x, y \in X \).

For \( x \in X \), the subset \( U_x := \{ y \in X : x \preceq y \} \) is the minimal open subset

containing \( x \). A C*-algebra over \( X \) is equivalent to a C*-algebra \( A \) with fixed

ideals \( A(U_x) \triangleleft A \) for all \( x \in X \), such that \( A(U_x) \subseteq A(U_y) \) for all \( x, y \in X \) with

\( x \to y \) or, equivalently, with \( y \preceq x \). The equivariant Kasparov category \( \mathcal{R} \mathcal{R}^{X} \) for

C*-algebras over \( X \) has separable C*-algebras over \( X \) as objects and the KK-groups

\( KK_0(X; A, B) \) as arrows (see also [10], where this category is denoted \( \mathcal{R} \mathcal{R}(X) \)). The

forgetful functor

\[
R^X : \mathcal{R} \mathcal{R}^{X} \to \prod_{x \in X} \mathcal{R} \mathcal{R}, \quad A \mapsto (A(U_x))_{x \in X},
\]

is a triangulated functor between triangulated categories. Its kernel on morphisms

\[
J^X(A, B) := \{ f \in KK_0^X(A, B) : f(U_x) = 0 \text{ in } KK_0(A(U_x), B(U_x)) \text{ for all } x \in X \}
\]

is a stable homological ideal. We now describe the universal \( J^X \)-exact stable

homological functor as in Section 2.1.

Let \( \mathcal{R} \mathcal{R}(X) \) be the category of functors \((X, \preceq) \to \mathcal{R} \mathcal{R} \) with natural transformations

as arrows. Since the category associated to the partially ordered set \((X, \preceq)\) is the path category of the directed graph \((X, \to)\), an object of \( \mathcal{R} \mathcal{R}(X) \) is given by

\( A_x \in \mathcal{R} \mathcal{R} \) for \( x \in X \) and \( \alpha_{y,x} \in KK_0(A_x, A_y) \) for \( x, y \in X \) with \( x \to y \), without

any relations on the \( \alpha_{y,x} \). This uniquely determines KK-classes \( \alpha_{y,x} \in KK_0(A_x, A_y) \)

for \( x, y \in X \) with \( x \preceq y \) such that \( \alpha_{y,x} = id_{A_x} \) and \( \alpha_{z,y} \circ \alpha_{y,x} = \alpha_{z,x} \) for all
$x, y, z \in X$ with $x \geq y \geq z$. An arrow $(A_x, \alpha_{y,x}) \to (B_x, \beta_{y,x})$ is a family of arrows $f_x \in \text{KK}_0(A_x, B_x)$ for $x \in X$ with $f_y \alpha_{y,x} = \beta_{y,x} f_x$ in $\text{KK}_0(A_x, B_y)$ for all $x, y \in X$ with $x \to y$; then $f_y \alpha_{y,x} = \beta_{y,x} f_x$ holds for all $x, y \in X$ with $x \geq y$. Define

$$F^X : \mathfrak{K}^X \to \mathfrak{K}[X]$$

by mapping a $C^*$-algebra $A$ over $X$ to the object of $\mathfrak{K}[X]$ where $A_x := A(U_x)$ and where $\alpha_{y,x} \in \text{KK}_0(A_x, A_y)$ for $x, y \in X$ with $x \to y$ is the KK-class of the inclusion map $A(U_x) \hookrightarrow A(U_y)$. Then $\alpha_{y,x} \in \text{KK}_0(A_x, A_y)$ for $x, y \in X$ with $x \geq y$ is the KK-class of the inclusion map as well. A kernel–cokernel pair $K \to E \to Q$ in $\mathfrak{K}[X]$ is called admissible if it splits pointwise, that is, $K_x \to E_x \to Q_x$ is a split extension in $\mathfrak{K}$ for all $x \in X$; so $E_x \cong K_x \oplus Q_x$ in $\mathfrak{K}$ for all $x \in X$, but the sections $Q_x \to E_x$ are not compatible with the structure maps $E_x \to E_y$ and $Q_x \to Q_y$ for $x \to y$. This turns $\mathfrak{K}[X]$ into an exact category (see [5, Exercise 5.3]).

Let $z \in X$ and $A \in \mathfrak{K}$. As in (10), let $i_z(A) \in \mathfrak{K}^X$ be the $C^*$-algebra $A$ with

$$i_z(A)(U_x) := \begin{cases} A & \text{if } z \in U_x, \text{ that is, } x \preceq z, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\text{KK}^X(i_z(A), B) \cong \text{KK}_0(A, B(U_z)) \quad \text{(2.15)}$$

for all $B \in \mathfrak{K}^X$ by [10] Proposition 3.13. This isomorphism applies the restriction map $\text{KK}^X(i_z(A), B) \to \text{KK}(i_z(A)(U_z), B(U_z))$ and then identifies $i_z(A)(U_z) = A$. The object

$$j_z(A) := F^X(i_z(A)) \in \mathfrak{K}[X]$$

has $j_z(A)_x = A$ for $x \preceq z$ and $j_z(A)_x = 0$ otherwise, and the map $j_z(A)_x \to j_z(A)_y$ for $x \preceq y$ is the identity map in $\text{KK}_0(A, A)$ if $z \preceq x$ and the zero map in $\text{KK}_0(0, j_z(A)_y)$ otherwise. We compute

$$\text{Hom}_{\mathfrak{K}[X]}(j_z(A), (B_x, \beta_{y,x}) \cong \text{KK}_0(A, B_z). \quad \text{(2.16)}$$

Equations (2.15) and (2.16) imply

$$\text{KK}^X(i_z(A), B) \cong \text{KK}_0(A, B(U_z)) \cong \text{Hom}_{\mathfrak{K}[X]}(j_z(A), F^X(B)) \quad \text{(2.17)}$$

for all $B \in \mathfrak{K}^X$.

Theorem 2.18. Let $(X, \to)$ be a countable unique path space. The objects of the exact category $\mathfrak{K}[X]$ of the form $\bigoplus_{z \in X} j_z(A_z)$ for $A_z \in \mathfrak{K}$ for $z \in X$ are projective. For any object $A = (A_x, \alpha_{y,x})$ of $\mathfrak{K}[X]$, there is an admissible extension

$$\bigoplus_{x \to y} j_y(A_x) \twoheadrightarrow \bigoplus_{x \in X} j_x(A_x) \rightarrow A, \quad \text{(2.19)}$$

which is a projective resolution of length 1. The exact category $\mathfrak{K}[X]$ has enough projective objects, and the partial adjoint of $F^X$ is defined on all projective objects of $\mathfrak{K}[X]$ and is a section for $F^X$ there. The functor $F^X : \mathfrak{K}^X \to \mathfrak{K}[X]$ is the universal $\mathfrak{K}[X]$-exact stable homological functor to an exact category.

Proof. The objects $j_z(A_z)$ are projective by (2.16). This is inherited by the direct sum $\bigoplus_{z \in X} j_z(A_z)$. The identity map on $A_z$ has an adjunct $a_z : j_z(A_z) \to A$ by (2.16). These maps induce the map $\pi = (a_x)_{x \in X} : \bigoplus_{x \in X} j_x(A_x) \to A$. For each edge $x \to y$ in the directed graph $(X, \to)$, the map $(\text{id}_{A_x}, -\alpha_{y,x}) : A_x \to A_x \oplus A_y$ is adjunct to a map

$$a_{x \to y} : j_y(A_x) \to j_x(A_x) \oplus j_y(A_y) \subseteq \bigoplus_{x \in X} j_x(A_x)$$
It satisfies $\pi \circ a_{x \to y} = 0$. The maps $a_{x \to y}$ combine to a map

$$\iota : \bigoplus_{x \to y} j_y(A_x) \to \bigoplus_x j_x(A_x)$$

with $\pi \circ \iota = 0$.

Now we prove that the maps $\pi$ and $\iota$ mapped to $\prod_{x \in X} \mathcal{R}$ by the forgetful functor form a split exact sequence. We consider the entries at a fixed $z \in X$. The entry of $A$ at $z$ is simply $A_z$. The entry of $\bigoplus_{x \in X} j_x(A_x)$ at $z \in X$ is the direct sum of $A_x$ over all $x \in X$ with $x \geq z$. The entry of $\bigoplus_{x \to y} j_y(A_x)$ at $z \in X$ is the direct sum of $A_x$ for all edges $x \to y$ in $X$ with $y \geq z$. The entry of $\pi$ at $z$ is

$$(\alpha_{z,x})_{x \geq z} : \bigoplus_{x \geq z} A_x \to A_z.$$

This is split surjective with the canonical section that maps $A_z$ identically onto the summand $A_x$ for $x = z$. The entry of $\iota$ at $z$ maps the summand $A_y$ for $x \to y \geq z$ to $A_x \oplus A_y \subseteq \bigoplus_{i \geq z} A_t$ using $(\text{id}_{A_x}, -\alpha_{y,z})$. We are going to define a map

$$s_1 : \bigoplus_{t \geq z} A_t \to \bigoplus_{x \to y \geq z} A_x$$

which together with $s_0$ forms a contracting homotopy for the short chain complex formed by $\iota_{t\beta}$ and $\pi_{t\beta}$. By assumption, if $t \geq z$ then there is a unique chain $t = x_0 \to x_1 \to \ldots \to x\ell = z$. We let $(s_1)_{t\beta}|_{A_t}$ for $t \geq z$ map the summand $A_t$ in $\bigoplus_{t \geq z} A_t$ to the direct sum of $A_z$ for the edges $x_i \to x_{i+1}$ for $i = 0, \ldots, \ell - 1$, where we use $\alpha_{x_i,t}$ to map $A_t$ to $A_{x_i}$. If $x \to y \geq z$, then $y = x_{\ell}$. Therefore, $s_1 \circ \iota_{|_{A_t}}$ is a map to $\bigoplus_{x \to y \geq z} A_{x_\ell}$, where the entry at $A_{x_\ell}$ is $\alpha_{x_\ell,t} - \alpha_{x_\ell,y}$ for $y = x_{\ell}$, $\pi_{x_\ell,t}$ and the identity map for $j = 0$. So $s_1 \circ \iota$ is the identity map. Finally, we claim that $s_0 \circ \pi + \iota \circ s_1$ is the identity map on $\bigoplus_{t \geq z} A_t$. This is checked on each summand $A_t$ separately. Let $t = x_0 \to x_1 \to \ldots \to x\ell = z$ be the unique chain as above. Then $s_1 \circ \iota_{|_{A_t}}$ is a telescoping sum of $\pm \alpha_{x_j,t}$ for $j = 0, \ldots, \ell$, where $\alpha_{x_j,t}$ occurs only with sign $+$ and $\alpha_{x_\ell,t}$ only with sign $-$. And $s_0 \circ \pi$ is the map $\alpha_{x_\ell,t}$. Thus $s_0 \circ \pi + \iota \circ s_1$ is the identity map on $A_t$. This finishes the proof that (2.19) is $\mathcal{X}$-exact. Its entries are $\mathcal{X}$-projective, so it is an $\mathcal{X}$-projective resolution.

The projective resolution (2.19) implies that $\mathcal{R}[X]$ has enough projective objects and that an object is projective if and only if it is a direct summand of an object of the form $\bigoplus_{x \in X} j_x(A_z)$ for some separable $C^*$-algebras $B_z$ for $x \in X$. Equation (2.17) says that the adjoint functor to $F^x : \mathcal{R}^X \to \mathcal{R}[X]$ is defined on $j_x(A)$ and maps it to $i_x(A)$. Since the partial adjoint commutes with direct sums, it is also defined on $\bigoplus_{x \in X} j_x(A_x)$ for any $A_x \in \mathcal{R}$ and maps it to $\bigoplus_{x \in X} i_x(A_x)$. Idempotents in the category $\mathcal{R}^X$ split because it is triangulated and has countable direct sums. Therefore, the partial adjoint of $F^X$ is defined on all projective objects of $\mathcal{R}[X]$ and is a section for $F^X$ there. An argument as in the proof of Proposition 2.10 shows that $F^X$ is the universal $\mathcal{X}$-exact functor to an exact category. The target category is not smaller because the functor $\mathcal{R}^X \to \mathcal{R}[X]$ is surjective on objects by Corollary 2.20.

Unlike in Section 2.1, the $\mathcal{X}$-projective objects in $\mathcal{R}^X$ do not generate $\mathcal{R}^X$. The localising subcategory $\langle \mathcal{P}_X \rangle$ generated by them is equal to the localising subcategory generated by objects of the form $i_x(A)$ for $x \in X$, $A \in \mathcal{R}$. If $X$ is finite, then it is described in several equivalent ways in [10] Definition 4.7. This is the subcategory on which Theorem 2.18 implies a Universal Coefficient Theorem, using Hom and Ext groups in the category $\mathcal{R}[X]$. This implies the following classification theorem:
Corollary 2.20. Let \((X, \to)\) be a countable unique path space and let \(A\) and \(B\) be \(C^*\)-algebras over \(X\) that belong to the localising subcategory \(\langle P_3 \rangle\) generated by objects of the form \(t_x(B)\) for \(x \in X\), \(B \in \mathcal{X}\). Any isomorphism between \(F^X(A)\) and \(F^X(B)\) in \(\mathcal{X}\) lifts to an isomorphism in \(\mathcal{X}\). And any object of \(\mathcal{X}\) lifts to an object of \(\langle P_3 \rangle\). An isomorphism \(F^X(A) \cong F^X(B)\) is a family of invertible elements \(t_x \in \text{KK}_0(A(U_x), B(U_y))\) for \(x \in X\) for which the diagrams

\[
\begin{align*}
A(U_x) & \longrightarrow A(U_y) \\
\downarrow t_x & \quad \downarrow t_y \\
B(U_x) & \longrightarrow B(U_y)
\end{align*}
\]

commute in \(\mathcal{X}\) for all \(x, y \in X\) with \(x \to y\).

The criterion above may be made more explicit if, in addition, \(A(U_x)\) and \(B(U_x)\) belong to the bootstrap class for all \(x \in X\). As in Section 2.1 we identify \(A(U_x) \cong A(U_x)^+ \oplus A(U_x)^-\) and rewrite the classes of the inclusion maps \(\alpha_{y, x} \in \text{KK}_0(A(U_x), A(U_y))\) and \(\beta_{y, x} \in \text{KK}_0(B(U_x), B(U_y))\) for \(x \to y\) as \(2 \times 2\)-matrices, and similarly for the arrows \(t_x \in \text{KK}_0(A(U_x), B(U_y))\). As in the case of \(\mathfrak{R}^2\), the equality \(\beta_{y, x} t_x = t_y \alpha_{y, x}\) for \(x \to y\) in Corollary 2.20 may be rewritten as four equalities of matrix coefficients. The equality of the diagonal terms says that the diagrams \((K_*(A(U_x)), K_0(\alpha_{y, x}))\) and \((K_*(B(U_x)), K_0(\beta_{y, x}))\) of countable \(\mathbb{Z}/2\)-graded Abelian groups are isomorphic. That is, for all \(x, y \in X\) with \(x \to y\), the following diagram commutes:

\[
\begin{array}{ccc}
K_*(A(U_x)) & \xrightarrow{K_*(\alpha_{y, x})} & K_*(A(U_y)) \\
\downarrow K_*(t_x) & & \downarrow K_*(t_y) \\
K_*(B(U_x)) & \xrightarrow{K_*(\beta_{y, x})} & K_*(B(U_y))
\end{array}
\]

The equality of the off-diagonal terms will be studied in Section 4.2.

3. Computation of the obstruction class

We recall the setup of [3]. Let \(\mathcal{I}\) be a triangulated category with countable direct sums. Let \(\mathcal{J}\) be a stable homological ideal in \(\mathcal{I}\) with enough projective objects. Let \(F : \mathcal{I} \to \mathfrak{A}\) be the universal \(\mathcal{J}\)-exact stable homological functor (in this article, we allow \(\mathfrak{A}\) to be exact). We assume that \(\mathfrak{A}\) is paired as in [4, Definition 2.14], that is, \(\mathfrak{A} = \mathfrak{A}_+ \times \mathfrak{A}_-\) with \(\Sigma \mathfrak{A}_+ = \mathfrak{A}_-\) and \(\Sigma \mathfrak{A}_- = \mathfrak{A}_+\). For instance, \(\mathfrak{A}\) could be the category of \(\mathbb{Z}\)-graded or \(\mathbb{Z}/2\)-graded modules over some ring, with the suspension automorphism shifting the grading, and \(\mathfrak{A}_+\) and \(\mathfrak{A}_-\) may be taken to be the graded modules concentrated in even or odd degrees, respectively. We want to compute the obstruction class of an object \(A \in \mathfrak{I}\). This is only meaningful if \(A\) is constructed from simpler ingredients.

We assume that there is an exact, \(\mathcal{J}\)-exact triangle

\[
(3.1) \quad B_1 \xrightarrow{\varphi} B_0 \xrightarrow{p} A \xrightarrow{i} \Sigma B_1,
\]

where \(B_0\) and \(B_1\) are objects of \(\langle P_3 \rangle\) with projective resolutions of length 1. Then \(A \in \langle P_3 \rangle\) as well. The \(\mathcal{J}\)-exactness assumption says that \(i \in \mathcal{J}\). Equivalently, \(F(i) = 0\), \(F(\varphi)\) is monic, and \(F(p)\) is epic. The objects \(B_0\) and \(B_1\) are uniquely determined up to isomorphism by \(F(B_0)\) and \(F(B_1)\) because they have projective resolutions of length 1 (compare [4, Proposition 2.3]). We may split \(B_1 \cong B_1^+ \oplus B_1^-\) with \(F(B_1^\pm) \in \mathfrak{A}_\pm\) for \(i = 0, 1\). Then \(\varphi\) becomes a \(2 \times 2\)-matrix

\[
\varphi = \\
\begin{pmatrix} \varphi^{++} & \varphi^{+-} \\ \varphi^{-+} & \varphi^{--} \end{pmatrix}
\]
with $\varphi_{++}: B_1^+ \to B_0^+$, and so on. The two diagonal entries give an element of
\[
\Sigma(B_1^+, B_0^+) \oplus \Sigma(B_1^-, B_0^-) \cong \text{Hom}_A\left(F(B_1), F(B_0)\right),
\]
and the two off-diagonal entries give an element of
\[
\Sigma(B_1^+, B_0^-) \oplus \Sigma(B_1^-, B_0^+) \cong \text{Ext}_A^3\left(\Sigma F(B_1), F(B_0)\right);
\]
here we have used the Universal Coefficient Theorem to compute $\Sigma(B_1^\pm, B_0^\pm)$. The results for these four groups only have a single Hom or a single Ext group because of the parity assumptions. Thus the splitting of $\varphi$ into $\varphi^0 := \varphi_{++} + \varphi_{--}$ and $\varphi^1 := \varphi_{+-} + \varphi_{-+}$ splits the exact sequence
\[
\text{Ext}_A^3\left(\Sigma F(B_1), F(B_0)\right) \to \Sigma(B_1, B_0) \to \text{Hom}_A\left(F(B_1), F(B_0)\right).
\]
We shall compute the obstruction class of $A$ in terms of $\varphi^1$.

By assumption, there is a short exact sequence
\[
F(B_1) \xrightarrow{F(\varphi)} F(B_0) \xrightarrow{F(p)} F(A).
\]
This induces a long exact sequence
\[
0 \leftarrow \text{Ext}_A^2\left(\Sigma F(A), F(A)\right) \xleftarrow{\partial} \text{Ext}_A^1\left(\Sigma F(B_1), F(A)\right) \xrightarrow{F(\varphi)^\ast}
\]
\[
\text{Ext}_A^1\left(\Sigma F(B_0), F(A)\right) \xleftarrow{F(p)^\ast} \text{Ext}_A^1\left(\Sigma F(A), F(A)\right) \leftarrow \cdots
\]
because $\text{Ext}_A^k\left(\Sigma F(B_i), F(A)\right) = 0$ for $i = 0, 1, k \geq 2$.

**Theorem 3.2.** The obstruction class of $A$ is
\[
\partial(p, \varphi^1) = \partial(F(p) \circ \varphi^1) \in \text{Ext}_A^2\left(\Sigma F(A), F(A)\right),
\]
where $F(p) \in \text{Hom}_A\left(\Sigma F(B_0), \Sigma F(A)\right)$ and $\varphi^1 \in \text{Ext}_A^3\left(\Sigma F(B_1), F(B_0)\right)$.

**Proof.** We shall recall the construction of obstruction classes in [4] along the way. It starts with an $\mathcal{I}$-projective resolution of $A$ of length 2. So first we have to construct this. We use the projective resolutions of $F(B_i)$ of length 1, which exist by assumption. They lift canonically to $\mathcal{I}$-projective resolutions
\[
0 \to P_{1i} \xrightarrow{d_{1i}} P_{0i} \xrightarrow{d_{0i}} B_i
\]
in $\mathfrak{A}$ for $i = 0, 1$ (see [11, Theorem 59]). Since (3.3) is a resolution, $d_{1i}$ is $\mathcal{I}$-monic and $d_{0i}$ is $\mathcal{I}$-epic. The arrow $\varphi \in \Sigma(B_1, B_0)$ lifts to a chain map
\[
\xymatrix{ P_{1i} \ar^-{d_{1i}}[r] & P_{10} \ar^-{d_{0i}}[r] & B_1 \ar^-{\varphi}[r] \\
& P_{0i} \ar^-{d_{0i}}[r] & P_{00} \ar^-{d_{00}}[r] & B_0 \ar^-{\varphi}[r] }
\]
(3.4)
between the $\mathcal{I}$-projective resolutions (3.3) (see [11, Proposition 44]). We write $\twoheadrightarrow$ for $\mathcal{I}$-monic and $\rightarrow$ for $\mathcal{I}$-epic maps. We claim that
\[
0 \to P_{11} \xrightarrow{(-d_{11}, \Phi_1)} P_{10} \oplus P_{01} \xrightarrow{(\Phi_0, d_{01})} P_{00} \xrightarrow{p_{00}} A \to 0
\]
is an $\mathcal{I}$-projective resolution of $A$ of length 2. The entries are $\mathcal{I}$-projective by construction. Next we prove that (3.5) is a resolution, that is, it becomes an exact chain complex when we apply $F$ to it.

When we apply $F$ to the diagram (3.4), the two rows become short exact sequences in $\mathfrak{A}$, and the vertical maps become a chain map between them. The mapping cone of this chain map is again an exact chain complex in $\mathfrak{A}$. It has the form
\[
0 \to F(P_{11}) \to F(P_{10}) \oplus F(P_{01}) \to F(P_{00}) \oplus F(B_1) \to F(B_0) \to 0.
\]
The map $F(\varphi) : F(B_1) \to F(B_0)$ is monic with cokernel $F(A)$. So the direct
dsummand $F(B_1)$ and its image in $F(B_0)$ together form a contractible subcomplex.
The quotient by it is again an exact chain complex in $\mathfrak{A}$. This is what we get by
applying $F$ to (3.5). So this is a resolution as asserted.

An axiom for triangulated categories provides an exact triangle

$$P_{11} \xrightarrow{(-d_{11}, \Phi_1)} P_{10} \oplus P_{01} \xrightarrow{} D \xrightarrow{} \Sigma P_{11}$$

containing $(-d_{11}, \Phi_1)$. Similarly, the map $p \circ d_{00} : P_{00} \to A$ in (3.5) is part of an
exact triangle

$$D' \xrightarrow{\gamma} P_{00} \xrightarrow{p_{00}} A \xrightarrow{} \Sigma D'.$$

The long exact sequences for $F$ applied to these two exact triangles show that $F(D)$
is the cokernel of the monomorphism $F(-d_{11}, \Phi_1)$, and that $F(D')$ is the kernel of
the epimorphism $F(p \circ d_{00})$. The exactness of (3.5) implies $F(D) \cong F(D')$. Since
$B_0$ and $B_1$ belong to $\langle \mathfrak{P}_3 \rangle$, so do $D$ and $D'$. And $F(D)$ has a projective resolution
of length 1 by construction. Hence the Universal Coefficient Theorem applies to $D$
and $D'$. Thus the isomorphism $F(D) \cong F(D')$ lifts to an isomorphism $D \cong D'$. We
shall identify $D = D'$.

The Universal Coefficient Theorem for $D$ gives a short exact sequence

$$\text{Ext}_\mathfrak{A}^1(\Sigma F(D), F(P_{00})) \xrightarrow{} \mathcal{T}(D, P_{00}) \xrightarrow{F} \text{Hom}_\mathfrak{A}(F(D), F(P_{00})).$$

We split $D = D^+ \oplus D^-$ and $P_{00} = P_{00}^+ \oplus P_{00}^-$ into objects of even and odd parity
as in the construction of $\varphi^1$ above the theorem. Then $\mathcal{T}(D, P_{00})$ splits accordingly as a
$2 \times 2$-matrix. The sum of the diagonal terms $\mathcal{T}(D^+, P_{00}^+) \oplus \mathcal{T}(D^-, P_{00}^-)$ is isomorphic
to $\text{Hom}_\mathfrak{A}(F(D), F(P_{00}))$, whereas the sum of the off-diagonal terms $\mathcal{T}(D^-, P_{00}^+) \oplus
\mathcal{T}(D^+, P_{00}^-)$ is isomorphic to $\text{Ext}_\mathfrak{A}^1(\Sigma F(D), F(P_{00}))$. This is the (unnatural) splitting
of the Universal Coefficient Theorem exact sequence (3.7) that follows because $\mathfrak{A}$
is paired. In particular, we decompose $\gamma = \gamma^0 + \gamma^1$ into its parity-preserving
and parity-reversing parts.

Let $A'$ be another object of $\langle \mathfrak{P}_3 \rangle$ with an isomorphism $F(A') \cong F(A)$. The
argument above shows that both $A$ and $A'$ are cones of some $\gamma, \gamma' \in \mathcal{T}(D, P_{00})$ as
in (3.6), which lift the inclusion map $F(D) \hookrightarrow F(P_{00})$ that we get from the
resolution (3.5). The \textit{relative obstruction class} is defined as follows: compose

$$\gamma - \gamma' \in \text{Ext}_\mathfrak{A}^1(\Sigma F(D), F(P_{00})) \subseteq \mathcal{T}(D, P_{00})$$

with the map $F(p \circ d_{00}) : F(P_{00}) \to F(A)$ and apply the boundary map for the
extension $F(D) \hookrightarrow F(P_{00}) \to F(A)$. That is, plug $\gamma - \gamma'$ into the map

$$\text{Ext}_\mathfrak{A}^1(\Sigma F(D), F(P_{00})) \xrightarrow{F(p \circ d_{00})} \text{Ext}_\mathfrak{A}^1(\Sigma F(D), F(A)) \xrightarrow{\partial_D p_{00}} \text{Ext}_\mathfrak{A}^2(\Sigma F(A), F(A)).$$

Let $\gamma^0 : D \to P_{00}$ be the unique parity-preserving arrow that lifts the inclusion map
$F(D) \hookrightarrow F(P_{00})$ and let $A^0$ be its cone. The \textit{obstruction class} of $A$ is the relative
obstruction class for $A$ and $A^0$. That is, we plug $\gamma^1 := \gamma - \gamma^0$ into (3.8).

Finally, we relate the obstruction class defined above to $\varphi^1$. The solid square in
the following diagram commutes:

$$\begin{array}{ccc}
B_1 & \xrightarrow{\varphi} & B_0 \\
\uparrow & \uparrow \mathbb{0} & \uparrow \\
D & \xrightarrow{\gamma} & P_{00} \xrightarrow{p_{00}} A \xrightarrow{} \Sigma D
\end{array}$$
By the third axiom of triangulated categories, there is an arrow $\varepsilon$ making all three squares commute. We shall only use the left square. Since $D, B_1, B_0$ and $P_{00}$ have projective resolutions of length 1 and $\mathfrak{A}$ is paired, the Universal Coefficient Theorem allows us to split them into even and odd parts. Hence each of the arrows $\gamma, \varphi, \varepsilon$ and $d_{00}$ splits into a parity-preserving and a parity-reversing part. We have already used the splittings $\varphi = \varphi^0 + \varphi^1$ and $\gamma = \gamma^0 + \gamma^1$, and now we also split $\varepsilon = \varepsilon^0 + \varepsilon^1$. The arrow $d_{00}$ is parity-preserving because $\Sigma(P_{00}, B_0) \cong \text{Hom}_\mathfrak{A}(F(P_{00}), F(B_0))$ has no parity-reversing part. The left commuting square in (3.9) implies

$$d_{00} \circ \gamma^1 = \varphi^0 \circ \varepsilon^1 + \varphi^1 \circ \varepsilon^0 = \varphi \circ \varepsilon + \varphi^1 \circ \varepsilon. \tag{3.10}$$

The second step uses that the composite of two odd terms always vanishes, that is, the Ext$^1$-term in the Universal Coefficient Theorem is nilpotent.

Composing $\gamma^1 \in \text{Ext}^1_\mathfrak{A}(\Sigma F(D), F(A))$ with $F(p \circ d_{00})$ as in (3.8) has the same effect as composing with $p \circ d_{00}$ because the exact sequence in the Universal Coefficient Theorem is natural. Thus the obstruction class of $A$ is the image of $p \circ d_{00} \circ \gamma^1 \in \text{Ext}^1_\mathfrak{A}(\Sigma F(D), F(A))$ under the boundary map

$$\partial_{DP_{00}A}: \text{Ext}^1_\mathfrak{A}(\Sigma F(D), F(A)) \to \text{Ext}^2_\mathfrak{A}(\Sigma F(A), F(A)).$$

Equation (3.10) and $p \circ \varphi = 0$ imply

$$p \circ d_{00} \circ \gamma^1 = p \circ \varphi \circ \varepsilon^1 + p \circ \varphi^1 \circ \varepsilon = p \circ \varphi^1 \circ \varepsilon.$$

The composite $p \circ \varphi^1 = p_\ast((\varphi^1) \in \text{Ext}^1_\mathfrak{A}(\Sigma F(B_1), F(A))$ also appears in the statement of the theorem. Composing with $\varepsilon$ in $\mathcal{T}$ has the same effect as composing with $F(\varepsilon)$ in the graded category $\text{Ext}^*_{\mathfrak{A}}$. When we apply $F$ to the morphism of exact triangles (3.7), we get the following morphism of extensions in $\mathfrak{A}$:

$$\xymatrix{ F(B_1) \ar[d]_{F(\varepsilon)} & F(B_0) \ar[l]_{F(\varphi)} \ar[r]^{F(p)} & F(A) \ar[d] \cr F(D) & F(\gamma) \ar[l] \ar[r] & F(P_{00}) \ar[r]^{F(p) \circ F(d_{00})} & F(A) & F(p) \circ F(d_{00}) },$$

Since boundary maps in Ext-theory are natural, the boundary map

$$\partial: \text{Ext}^1_\mathfrak{A}(\Sigma F(B_1), F(A)) \to \text{Ext}^2_\mathfrak{A}(\Sigma F(A), F(A))$$

for the top row is equal to the composite of $F(\varepsilon)$ and the boundary map $\partial_{DP_{00}A}$. Thus the obstruction class of $A$ is $\partial(F(p) \circ \varphi^1)$ as asserted. \hfill $\square$

### 4. Comparison of classification theorems

In this section, we apply Theorem 3.2 in several cases to relate the classification theorem involving the obstruction class to other classification theorems. We first compare the classification for $\mathbb{Z}$-actions in Theorem 2.12 with the one obtained in [4]. Then we compare the classification for $\mathbb{C}^*$-algebras over a unique path space $(X, \to)$ in Corollary 2.20 with the one in [4]. In both cases, the invariants can be translated into each other rather directly.

#### 4.1. Actions of the integers

The Universal Coefficient Theorem for $\mathbb{Z}$-actions in Section 2.1 is based on the stable homological ideal $\mathcal{T}^\mathbb{Z}$ defined by the forgetful functor $\mathfrak{R}\mathcal{R}^\mathbb{Z} \to \mathfrak{R}\mathcal{R}$. Now we use another ideal $\mathcal{T}^\mathbb{Z}_K$. Let $\mathfrak{A}^\mathbb{Z}$ be the category of countable $\mathbb{Z}$/2-graded $\mathbb{Z}[x, x^{-1}]$-modules. Let

$$F^\mathbb{Z}_K: \mathfrak{R}\mathcal{R}^\mathbb{Z} \to \mathfrak{A}^\mathbb{Z}, \quad (A, \alpha) \mapsto (K_\ast(A), K_\ast(\alpha)),
$$

that is, we map $(A, \alpha) \in \mathfrak{R}\mathcal{R}^\mathbb{Z}$ to the $\mathbb{Z}$/2-graded Abelian group $K_\ast(A)$ with the $\mathbb{Z}[x, x^{-1}]$-module structure given by the automorphism $K_\ast(\alpha)$ of $K_\ast(A)$. Let

$$\mathcal{T}^\mathbb{Z}_K(A, B) := \{ \varphi \in KK^\mathbb{Z}(A, B): F^\mathbb{Z}_K(\varphi) = 0 \}$$
be the kernel of $F^Z_K$ on morphisms. This example is treated in [4], but there $\mathcal{R}^Z$ is disguised as $\mathcal{R}$ for the circle group $\mathbb{T}$. These two categories are equivalent by Baaj–Skandalis duality (see [1]). The functor $F^Z_K$ corresponds to the functor on $\mathcal{R}$ that maps a $C^*$-algebra with a continuous $\mathbb{T}$-action to $K^Z_1(A)$ with the canonical module structure over the representation ring $\mathbb{Z}[x,x^{-1}]$ of $\mathbb{T}$, which is used in [4].

If $B \in KK^Z$, then (3.1) implies

$$KK^Z_1(C_0(\mathbb{Z}), B) \cong KK_0(C, B) \cong K_*(B) \cong \text{Hom}_\mathbb{Z}(\mathbb{Z}[x,x^{-1}], F^Z_K(B)).$$

Thus the partial adjoint of $F^Z_K$ is defined on the rank-1 free module $\mathbb{Z}[x,x^{-1}]$ and maps it to $C_0(\mathbb{Z})$. Since idempotents in $\mathcal{R}^Z$ split, the partial adjoint $(F^Z_K)^+$ of $F^Z_K$ is defined on all projective objects of $\mathcal{R}^Z$. Since $F^Z_K(C_0(\mathbb{Z}))$ is the rank-1 free module again, we get $(F^Z_K)^+ (P) \cong P$ for all projective objects of $\mathcal{R}^Z$. Any object in $\mathcal{R}^Z$ has a projective resolution of length 2. Hence it belongs to the image of $F^Z_K$ by [4] Lemma 2.4. Remark 2.2 and Theorem 2.1 show that $F^Z_K$ is the universal $\mathcal{J}$-exact stable homological functor both to an exact and to an Abelian category.

The category $\mathcal{R}^Z$ is paired in an obvious way, using the subcategories $\mathcal{R}^Z_+$ of countable $\mathbb{Z}[x,x^{-1}]$-modules concentrated in degree 0 and 1, respectively. So the obstruction class is defined for any object of $\mathcal{R}^Z$. Assume that $(\alpha, \alpha) \in \mathcal{R}^Z$ is such that $A$ belongs to the bootstrap class in $\mathcal{R}$. Equivalently, $(\alpha, \alpha)$ belongs to the localising subcategory of $\mathcal{R}^Z$ generated by the $\mathcal{J}$-projective object $C_0(\mathbb{Z})$. Then the main result of [4] shows that $A$ is determined uniquely up to isomorphism by $(K_*(A), K_*(\alpha)) \in \mathcal{R}^Z$ and the obstruction class.

A chain complex in $\mathcal{R}^Z$ that is $\mathcal{J}$-exact is also $\mathcal{J}^Z_K$-exact because $\mathcal{J}^Z_K \subseteq \mathcal{J}_K^Z$. So the $\mathcal{J}^Z_K$-projective resolution in (2.11) is $\mathcal{J}^Z_K$-exact. Its entries $C_0(\mathbb{Z}, A)$ are no longer $\mathcal{J}^Z_K$-projective. We claim, however, that they have $\mathcal{J}^Z_K$-projective resolutions of length 1. Let $P_1 \rightarrow P_0 \rightarrow K_*(A)$ be a resolution of the $\mathbb{Z}/2$-graded Abelian group $K_*(A)$ by countable free Abelian groups. Then

$$Z[x,x^{-1}] \otimes P_1 \rightarrow Z[x,x^{-1}] \otimes P_0 \rightarrow Z[x,x^{-1}] \otimes K_*(A)$$

is a projective resolution of

$$F^Z_K(C_0(\mathbb{Z}, A)) \cong Z[x,x^{-1}] \otimes K_*(A).$$

Here the tensor products with $Z[x,x^{-1}]$ carry the module structure defined by multiplication with $x$ in the tensor factor $Z[x,x^{-1}]$. So we are in the situation of (3.1). Theorem 3.2 implies the following formula for the obstruction class of $A$:

**Theorem 4.1.** Let $A$ be a $C^*$-algebra in the bootstrap class in $\mathcal{R}$ and $\alpha \in \text{Aut}(A)$. Split $[\alpha] \in KK_0(\mathbb{Z}, \mathbb{A})$ into parity-preserving and parity-reversing parts $\alpha^0 \in \text{Hom}(K_*(\mathbb{A}), K_*(\mathbb{A}))$ and $\alpha^- \in \text{Ext}(K_{1+}(\mathbb{A}), K_*(\mathbb{A}))$. Define

$$\gamma: \text{Ext}_{1}(K_{1+}(\mathbb{A}), K_*(\mathbb{A})) \rightarrow \text{Ext}_{1}(K_{1+}(\mathbb{A}), K_*(\mathbb{A})), \quad x \mapsto \alpha \circ x - x \circ \alpha.$$

Then $\text{Ext}_{2Z,[x,x^{-1}]}(\Sigma F^Z_K(A), F^Z_K(A)) \cong \text{coker} \, \gamma$ and the obstruction class of $A$ is the image of $-\alpha^1(\alpha^{-1})^{-1}$ in this cokernel.

**Proof.** In our case, the map $\varphi$ in (3.1) is the map $C_0(\mathbb{Z}) \otimes A \rightarrow C_0(\mathbb{Z}) \otimes A$ in (2.11). Split $A = A^+ \oplus A^-$ into its even and odd parts as before. Then

$$C_0(\mathbb{Z}) \otimes A \cong C_0(\mathbb{Z}) \otimes A^+ \oplus C_0(\mathbb{Z}) \otimes A^-$$

is the parity decomposition of $C_0(\mathbb{Z}) \otimes A$. The translation $\tau$ and the identity are parity-preserving. Decompose $[\alpha]$ and $[\alpha^{-1}]$ into their even and odd parts. The parity-reversing part of the map on $C_0(\mathbb{Z}) \otimes A$ is $\varphi^\tau = [\tau] \otimes [\alpha^{-1}]^1$. The map $p \in KK^Z_0(\mathbb{Z}, \mathbb{A})$ satisfies $p(\tau \otimes \text{id}_A) = [\alpha]p$ because it is $Z$-equivariant, and it restricts to the identity map on the 0th summand of $C_0(\mathbb{Z}) \otimes A$. The isomorphism
As above, a grading-preserving
the functor $F$ by the partial order generated by $\mathfrak{t}$. So
Section 2.2. So the latter is equivalent to an invertible element in $KK$
isomorphism between $A$ in $KK$ isomorphism to identify $K$
and the obstruction class from the classification by $F$
by the automorphism $\text{Ext}^\mathfrak{t}$. So the images of $\alpha$
formula for the obstruction class depends, of course, on the isomorphism
Theorem 3.2 now says that the obstruction class for $A$
in (2.5) forgets the $\mathbb{Z}$-action and then evaluates at 0. Therefore, the image of $\varphi$ in $KK_0(A, A)$ is the restriction of $p \circ \varphi$ to the 0th summand. And this is $[\alpha][\alpha^{-1}]^1 \in KK_0(A, A)$. Since $\alpha \alpha^{-1} = \text{id}_A$ and $\alpha \alpha^{-1} = [\text{id}_A]^1 = 0$. Theorem 3.2 now says that the obstruction class for $A$ is the image of $[\alpha][\alpha^{-1}]^1 = -[\alpha]^1[\alpha^{-1}] = -\alpha^1(\alpha^0)^{-1} \in \text{Ext}_\mathfrak{t}(K_{1+}(A), K_*(A)))$
under the boundary map to $\text{Ext}^\mathfrak{t}_2(Z, x, x^{-1})(\Sigma F_{K}(A), F_{K}(A))$. The description of the latter $\text{Ext}$ group in the theorem follows when we compute it with the projective resolution in (3.5) defined by the length-1 resolutions of $C_0(\mathbb{Z}) \otimes A$ above. This computation has already been done in [4, Section 3.2]. When the $\text{Ext}$ groups are described in this way, the boundary map in Theorem 3.2 becomes a trivial map, mapping an element of $\text{Ext}_\mathfrak{t}(K_{1+}(A), K_*(A))$ to its image in $\text{coker} \gamma$.

The formula for the obstruction class depends, of course, on the isomorphism $\text{Ext}^\mathfrak{t}_2(Z, x, x^{-1})(\Sigma F_{K}(A), F_{K}(A)) \cong \text{coker} \gamma$, and this depends on the $\mathfrak{t}$-projective resolution from which it is obtained. Our theorem uses the most obvious resolution. One may apply the automorphism of $\text{Ext}^\mathfrak{t}_2(Z, x, x^{-1})(\Sigma F_{K}(A), F_{K}(A))$ that composes with the automorphism $-\Sigma[\alpha]^0$ on $\Sigma F_{K}(A)$. This replaces the obstruction class $-\alpha^1(\alpha^0)^{-1}$ by $\alpha^1$. So the images of $-\alpha^1(\alpha^0)^{-1}$ and $\alpha^1$ in $\text{Ext}^\mathfrak{t}_2(Z, x, x^{-1})(\Sigma F_{K}(A), F_{K}(A))$ contain the same information.

Theorem 4.1 and the computations after Theorem 2.12 allow to deduce the classification by $F_K$ and the obstruction class from the classification by the invariant $F_{\mathfrak{t}}$ in Section 2.1. Let $(A, \alpha)$ and $(B, \beta)$ belong to the bootstrap class in $\mathfrak{RR}^{\mathfrak{t}}$. Assume that there is an isomorphism $t_0: F_{\mathfrak{t}}(A) \cong F_{\mathfrak{t}}(B)$ in $\mathfrak{A}^{\mathfrak{t}}$, that is, a grading-preserving $\mathbb{Z}[x, x^{-1}]-module isomorphism $K_*(A) \cong K_*(B)$. We use this isomorphism to identify $K_*(A) = K_*(B)$. By Theorem 4.1 the relative obstruction class vanishes if and only if $\alpha^1 - \beta^1 \in \text{Ext}(K_{1+}(A), K_*(A))$ vanishes in $\text{coker}(\gamma)$. Equivalently, there is $t^1 \in \text{Ext}_1(K_{1+}(A), K_*(B))$ for which $t^1 = t_0^1 + t^1 \in \text{KK}_0(A, B)$ satisfies $[\beta] \circ t = t \circ [\alpha]$. Then $t$ is an isomorphism $F_{\mathfrak{t}}(A) \cong F_{\mathfrak{t}}(B)$ in $\mathfrak{RR}(\mathbb{Z})$, and Theorem 2.12 shows that such an isomorphism lifts to an isomorphism in $KK^{\mathfrak{t}}_0(A, B)$.

Recall that Baaj–Skandalis duality is an equivalence of triangulated categories $\mathfrak{RR}^{\mathfrak{t}} \cong \mathfrak{RR}^{\mathfrak{t}}$. Hence everything said above about $\mathbb{Z}$-actions carries over to $\mathbb{T}$-actions. The functor $A \mapsto K_*(A)$ becomes $B \mapsto K_*(B)$ on $\mathfrak{RR}^{\mathfrak{t}}$, equipped with the natural module structure over the representation ring $\mathbb{Z}[x, x^{-1}]$ of $\mathbb{T}$. The automorphism $\alpha$ is replaced by an automorphism of $A \rtimes \mathbb{T}$, namely, the generator $\beta$ of the dual action of $Z$. Since $K_*(A \rtimes \mathbb{T}) \cong K_*(A)$, the Universal Coefficient Theorem splits $\text{KK}_0(A \rtimes \mathbb{T}, A \rtimes \mathbb{T})$ into the parity-preserving part $\text{Hom}(K_1(A), K_1(A))$ and the parity-reversing part $\text{Ext}(K_1(A), K_1(A))$. The obstruction class of $A \rtimes \mathbb{T}$ is the class of $-\beta^1(\beta^0)^{-1} \in \text{Ext}(K_{1+}(A), K_*(A))$ in the cokernel of the map

$$\gamma: \text{Ext}(K_{1+}(A), K_*(A)) \to \text{Ext}(K_{1+}(A), K_*(A)), \quad x \mapsto \beta^0 x - x \beta^0.$$

As above, a grading-preserving $\mathbb{Z}[x, x^{-1}]-module isomorphism $t: K_1(A) \to K_1(B)$ is compatible with the obstruction classes if and only if it lifts to an isomorphism between $A \rtimes \mathbb{T}$ and $B \rtimes \mathbb{T}$ in $\mathfrak{RR}(\mathbb{Z})$, and such an isomorphism lifts further to an isomorphism between $A \rtimes \mathbb{T}$ and $B \rtimes \mathbb{T}$ in $\mathfrak{RR}^{\mathfrak{t}}$. By Baaj–Skandalis duality, the latter is equivalent to an invertible element in $KK^{\mathfrak{t}}(A, B)$.

4.2. $C^*$-Algebras over unique path spaces. Now we return to the setup of Section 2.2. So $(X, \to)$ is a directed graph with the unique path property. Let $\succeq$ be the partial order generated by $\to$ and equip $X$ with the Alexandrov topology defined by $\succeq$. Let $\mathfrak{RR}^X$ be the category of $C^*$-algebras over $X$. Objects in the appropriate
bootstrap class in $\mathfrak{A}^X$ are classified in [4] under the extra assumption that $X$ be finite. Actually, the arguments in [4] work in the same way if $X$ is infinite. Here we treat this more general case right away.

Let $\mathfrak{A}^X$ be the category of all functors from $X$ to the category of countable $\mathbb{Z}/2$-graded Abelian groups. Equivalently, an object of $\mathfrak{A}^X$ is a family of countable $\mathbb{Z}/2$-graded Abelian groups $G_x$ for $x \in X$ with grading-preserving group homomorphisms $\gamma_{y,x} : G_x \to G_y$ for all $x,y \in X$ with $x \to y$. Morphisms $(G_x, \gamma_{y,x}) \to (H_x, \eta_{y,x})$ in $\mathfrak{A}^X$ are families of grading-preserving group homomorphisms $t_x : G_x \to H_x$ that satisfy $t_y \circ \gamma_{y,x} = \eta_{y,x} \circ t_x$ for all $x,y \in X$ with $x \to y$.

We define the functor

$$F^K_\mathfrak{A} : \mathfrak{A}^X \to \mathfrak{A}^X$$

by mapping a $C^*$-algebra over $X$ to the diagram of $\mathbb{Z}/2$-graded Abelian groups $K_x(A(U_x))$ for $x \in X$ with the maps induced by the inclusion maps $A(U_x) \hookrightarrow A(U_y)$ for $x \to y$. The target category $\mathfrak{A}^X$ is a paired, stable Abelian category, where $\mathfrak{A}^X_\pm \subseteq \mathfrak{A}^X$ are the subcategories of $\mathbb{Z}/2$-graded groups where the odd or even part vanishes, respectively. And $F^K_\mathfrak{A}$ is a stable homological functor. Let $\mathfrak{A}_X^\mathfrak{A}$ be its kernel on morphisms. Equation (2.17) implies

$$K\mathfrak{A}^\mathfrak{A}_X(j_z(C), B) \cong K\mathfrak{A}_0(C(B(U_z))) \cong \operatorname{Hom}(\mathfrak{A}^\mathfrak{A}_X(j_z(C)), F^K_\mathfrak{A}(B)).$$

Hence the partial adjoint $(F^K_\mathfrak{A})^-$ of $F^K_\mathfrak{A}$ is defined on $j_z(C)$ for all $z \in X$. As in Section 2.2 any object of $\mathfrak{A}^X$ is a quotient of a direct sum of objects of the form $j_z(C)$ for $z \in X$. Hence $(F^K_\mathfrak{A})^-$ is defined on all projective objects of $\mathfrak{A}_X^\mathfrak{A}$ and $F^K_\mathfrak{A} \circ (F^K_\mathfrak{A})^-(P) = P$ for all projective objects $P$ of $\mathfrak{A}^X$; this is proved like the corresponding statement about $F^K_\mathfrak{A}$ in Section 4.1. Therefore, the functor $F^K_\mathfrak{A} : \mathfrak{A}^\mathfrak{A} \to \mathfrak{A}^X$ is the universal $\mathfrak{A}^\mathfrak{A}$-exact stable homological functor into an exact category or into an Abelian category by Theorem 2.1 and Remark 2.2. And $F^K_\mathfrak{A}$ restricts to an equivalence of categories between the $\mathfrak{A}^\mathfrak{A}$-projective objects in $\mathfrak{A}^\mathfrak{A}$ and the projective objects in $\mathfrak{A}^X$. Let $\mathfrak{A}^X \subseteq \mathfrak{A}^\mathfrak{A}$ be the localising subcategory generated by the $\mathfrak{A}^\mathfrak{A}$-projective objects. This is the analogue of the bootstrap class in $\mathfrak{A}^\mathfrak{A}$.

**Lemma 4.2.** Let $G = (G_x, \gamma_{y,x})$ and $H = (H_x, \eta_{y,x})$ be objects of $\mathfrak{A}^X$.

1. There is a projective resolution for $G$ of length 2 in $\mathfrak{A}^X$.
2. There is $A \in \mathfrak{A}^X$ with $F^K_\mathfrak{A}(A) \cong G$.
3. Write Ext for the Ext$^1$ of Abelian groups. The group Ext$^2_{\mathfrak{A}^X}(G, H)$ is naturally isomorphic to the cokernel of the map

$$\prod_{x \in X} \operatorname{Ext}(G_x, H_x) \to \prod_{x \to y} \operatorname{Ext}(G_x, H_y),$$

$$(t_x)_{x \in X} \mapsto (\eta_{y,x} \circ t_x - t_y \circ \gamma_{y,x})_{x \to y}.$$  

**Proof.** Since $\mathfrak{A}^X = \mathfrak{A}^\mathfrak{A}$, any $\mathfrak{A}^\mathfrak{A}$-exact chain complex in $\mathfrak{A}^\mathfrak{A}$ is also $\mathfrak{A}^\mathfrak{A}$-exact. In particular, the $\mathfrak{A}^\mathfrak{A}$-projective resolutions in (2.19) are $\mathfrak{A}^\mathfrak{A}$-exact. We claim that its entries have $\mathfrak{A}^\mathfrak{A}$-projective resolutions of length 1. Hence there is a projective resolution of $F^K_\mathfrak{A}(A)$ of length 2 as in (3.5). To prove the same for all objects $G$ of $\mathfrak{A}^X$, we carry over (2.19). Recall that $j_z(B) \in \mathfrak{A}^X$ for a countable $\mathbb{Z}/2$-graded Abelian group $B$ denotes the diagram with $j_z(B)_x = B$ if $z \geq x$ and $j_z(B)_x = 0$ otherwise, with the identity map $j_z(B)_x \to j_z(B)_y$ for $z \geq x \to y$ and the zero map otherwise. Then

$$\operatorname{Hom}_{\mathfrak{A}^\mathfrak{A}}(j_z(B), H) \cong \operatorname{Hom}(B, H_z),$$

where the isomorphism simply restricts a morphism in $\mathfrak{A}^\mathfrak{A}$ to the object $z \in X$. In particular, if $H = j_y(B)$ for some $y \in X$ with $y \geq z$, then $j_y(B)_z = B$ and so
there is a canonical map \( j_y \circ z(B) : j_z(B) \to j_y(B) \) in \( \mathfrak{X} \). Explicitly, this map is the identity map on \( j_z(B) \) if \( z \geq x \) and the zero map on \( j_z(B) \) otherwise. The proof that \((2.19)\) is exact also proves the exactness of

\[
(4.5) \quad 0 \to \bigoplus_{x \to y} j_y(G_x) \overset{\psi_x}{\to} \bigoplus_{x \in X} j_x(G_x) \overset{q_y}{\to} G \to 0;
\]

here \( \psi \) restricted to the summand \( j_y(G_x) \) for \( x \to y \) is the map \((j_y \circ y, j_y(\gamma_{y,x}))\) to \( j_x(G_x) \oplus j_y(G_y) \), and \( q \) maps \( j_x(G_x) \) to \( y \) by the adjunct of the identity map on \( G_x \) under the adjunction in \((4.4)\). Explicitly, the entry of \( \bigoplus_{x \in X} j_x(G_x) \) at \( z \in X \) is \( \bigoplus_{x \geq z} G_x \) which is mapped to \( G_y \) by \((\gamma_{z,x})_{x \in X} \). The proof that \((4.5)\) is exact shows that the chain complexes of \( \mathbb{Z}/2 \)-graded Abelian groups

\[
0 \to \bigoplus_{x \to y} j_y(G_x) \overset{\psi_x}{\to} \bigoplus_{x \in X} j_x(G_x) \overset{q_y}{\to} G \to 0
\]

are contractible for all \( z \in X \).

For each \( x \in X \), there is a resolution

\[
(4.6) \quad P_{x,1} \xrightarrow{d_{x,1}} P_{x,0} \xrightarrow{d_{x,0}} G_x
\]

of \( G_x \) of length 1 by contractible free \( \mathbb{Z}/2 \)-graded Abelian groups. The homomorphism \( \gamma_{y,x} : G_x \to G_y \) for \( x \to y \) lifts to a morphism of extensions

\[
(4.7) \quad \begin{array}{cc}
P_{x,1} & \xrightarrow{d_{x,1}} & P_{x,0} & \xrightarrow{d_{x,0}} & G_x \\
\gamma_{y,x} & \downarrow & \gamma_{y,x} & \downarrow & \\
P_{y,1} & \xrightarrow{d_{y,1}} & P_{y,0} & \xrightarrow{d_{y,0}} & G_y
\end{array}
\]

The construction \( j_z \) above is an exact functor, and it maps free Abelian groups to projective objects of \( \mathfrak{X} \) by the adjunction \((4.4)\). Hence

\[
\begin{array}{cc}
\bigoplus_{x \to y} j_y(P_{x,1}) & \xrightarrow{\bigoplus_{x \to y} \psi_x} & \bigoplus_{x \in X} j_x(P_{x,1}) \\
\bigoplus_{x \in X} j_x(P_{x,1}) & \xrightarrow{\bigoplus_{x \in X} q_x} & \bigoplus_{x \in X} j_x(G_x)
\end{array}
\]

are projective resolutions of length 1 in \( \mathfrak{X} \). The resolution in \((4.5)\) gives a projective resolution of \( G \) of length 2 as in \((3.5)\). This proves the first assertion.

Now \([4]\) Lemma 2.4 shows that there is \( A \in \mathfrak{B}^X \) with \( G = \mathbb{F}_{\mathbb{K}}^X(A) \); the property that \( \mathbb{K}^X(A, B) = 0 \) for all \( \mathfrak{A}^X \)-contractible \( B \in \mathfrak{B}^X \) is equivalent to \( A \in \mathfrak{B}^X \).

The projective resolution of \( G \) built above has the form

\[
0 \to \bigoplus_{x \to y} j_y(P_{x,1}) \to \bigoplus_{x \to y} j_y(P_{x,0}) \oplus \bigoplus_{x \in X} j_x(P_{x,1}) \to \bigoplus_{x \in X} j_x(P_{x,0}) \to G,
\]

where the maps \( \bigoplus_{x \to y} j_y(P_{x,1}) \to \bigoplus_{x \in X} j_x(P_{x,1}) \) for \( i = 0, 1 \) use \( \gamma_{y,x}^i : P_{x,1} \to P_{y,1} \) in \((4.7)\). We use this resolution to compute the group \( \text{Ext}_{\mathbb{K}^X}^3(G, H) \). An element of \( \text{Ext}_{\mathbb{K}^X}^3(G, H) \) is represented by a map \( \bigoplus_{x \to y} j_y(P_{x,1}) \to H \). By the adjunction \((4.4)\), this corresponds to a family of maps \( f_{x \to y} : P_{x,1} \to H_y \). This family represents 0 in \( \text{Ext}_{\mathbb{K}^X}^3(G, H) \) if and only if the corresponding map \( \bigoplus_{x \to y} j_y(P_{x,1}) \to H \) factors through \( \bigoplus_{x \to y} j_y(P_{x,0}) \oplus \bigoplus_{x \in X} j_x(P_{x,1}) \). Using the adjunction \((4.4)\) again, a map on this direct sum corresponds to families of maps \( g_{x \to y} : P_{x,0} \to H_y \) and \( h : P_{x,1} \to H_x \). The resulting map \( \bigoplus_{x \to y} j_y(P_{x,1}) \to H \) corresponds to the family of maps

\[
-g_{x \to y} \circ d_{x,1} + \eta_{y,x} h_x - h_y \gamma_{y,x}^1 : P_{x,1} \to H_y
\]

The resolutions \((4.6)\) compute \( \text{Ext}(G_x, D) \) for any \( \mathbb{Z}/2 \)-graded Abelian group \( D \). So each \( f_{x \to y} : P_{x,1} \to H_y \) represents an element of \( \text{Ext}(G_x, H_y) \), and it represents the
zero element if and only if it is of the form \( g_{x \to y} \circ d_{x,1} \) for some \( g_{x \to y} : P_{x,0} \to H_y \). The elements \( h_x \) above represent elements of \( \text{Ext}(G_x, H_x) \). If they represent the zero element of \( \text{Ext}(G_x, H_x) \), then the term \( \eta_{y,x} h_x - h_y \gamma_{y,x} \) above may be rewritten in the form \( -g_{x \to y} \circ d_{x,1} \). Now we get the formula for \( \text{Ext}_{\mathfrak{A}^X}(G, H) \) in the third statement in the lemma.

**Theorem 4.8.** Let \( A \) and \( B \) belong to \( \mathfrak{A}^X \). An isomorphism \( t : F_{K}^{X}(A) \xrightarrow{\sim} F_{K}^{X}(B) \) in \( \mathfrak{A}^X \) lifts to an invertible element in \( KK^X(A, B) \) if and only if \( t \delta_A = \delta_B t \) holds in \( \text{Ext}_{\mathfrak{A}^X}^{2}(\Sigma F_{K}^{X}(A), F_{K}^{X}(B)) \), where \( \delta_A \) and \( \delta_B \) are the obstruction classes of \( A \) and \( B \). 

**Proof.** The lemma verifies all the conditions to apply the classification method of [4].

The proof of the lemma also gives all the ingredients needed in Section 3. So we may now compute obstruction classes:

**Theorem 4.9.** Let \( B \) be an object of \( \mathfrak{A}^X \). Let \( B_x := B(U_x) \) for \( x \in X \). Let \( \beta_{y,x} \in KK(B_y, B_x) \) be the KK-class of the inclusion map \( B_x \hookrightarrow B_y \). Split \( \beta_{y,x} = \beta_{y,x}^0 + \beta_{y,x}^1 \) with a parity-preserving part \( \beta_{y,x}^0 \in \text{Hom}(K_x(B_x), K_x(B_y)) \) and a parity-reversing part \( \beta_{y,x}^1 \in \text{Ext}(K_{1+}(B_x), K_x(B_y)) \). The obstruction class of \( B \) is the class in the cokernel of the map in (113) that is represented by

\[
(\beta_{y,x}^{1})_{x \to y} \in \prod_{x \to y} \text{Ext}(K_{1+}(B_x), K_x(B_y)).
\]

**Proof.** The projective resolution (2.19) in \( \mathfrak{A}[X] \) lifts to an exact triangle

\[
\bigoplus_{x \to y} i_y(B_x) \xrightarrow{\delta} \bigoplus_{x \in X} i_x(B_x) \xrightarrow{\beta} B \to \Sigma \bigoplus_{x \to y} i_y(B_x)
\]

in \( \mathfrak{A}^X \) by [4, Proposition 2.3]. This is how the Universal Coefficient Theorem in [11, Theorem 66] is proved. Here the map \( \varphi \) restricted to the summand \( i_y(B_x) \) is the difference of two maps: the map \( i_y(\beta_{y,x}^0) \) to \( i_y(B_y) \) and the canonical map \( i_x(\beta_{y,x}^1) : i_y(B_x) \to i_x(B_x) \) that is the adjunct of the identity map \( B_x \to i_x(B_x) = B_y \) under the adjunction (2.15). And the map \( p \) restricted to \( i_x(B_x) \) is the adjunct of the identity map \( B_x \to B_x \) under the adjunction (2.15).

Both \( \bigoplus_{x \to y} i_y(B_x) \) and \( \bigoplus_{x \in X} i_x(B_x) \) belong to \( \mathfrak{A}^X \) and have \( \mathfrak{A}^X \)-projective resolutions of length 1 (see (4.6)). So we are in the situation of (3.1). We split each \( B_x := B(U_x) \) into its even and odd parity part \( B_x = B_x^+ \oplus B_x^- \). Then \( i_y(B_x^+) \) and \( i_y(B_x^-) \) are of even or odd parity, respectively. Split \( \beta_{y,x} = \beta_{y,x}^0 + \beta_{y,x}^1 \) into a parity-preserving and a parity-reversing part. So \( \beta_{y,x}^0 \in \text{Hom}(K_x(B_x), K_x(B_y)) \) and \( \beta_{y,x}^1 \in \text{Ext}(K_{1+}(B_x), K_x(B_y)) \) by the Universal Coefficient Theorem for KK, see the discussion after Theorem 2.12. The induced maps \( i_y(\beta_{y,x}^0) \) and \( i_y(\beta_{y,x}^1) \) are parity-preserving and parity-reversing, respectively. And the map \( i_x(\beta_{y,x}^1) \) is parity-preserving. So the parity-reversing part \( \varphi^1 \) of \( \varphi \) is the map that restricts to \( i_y(\beta_{y,x}^1) : i_y(B_x) \to i_x(B_y) \) on the summand for \( x \to y \). The composite \( p \circ \varphi^1 \) is the map \( \bigoplus_{x \to y} i_y(B_x) \to B \) whose restriction to the summand \( i_y(B_x) \) is adjoint to \( \beta_{y,x}^1 : B_x \to B_y \). These maps define an element of \( \prod_{x \to y} \text{Ext}(K_{1+}(B_x), K_x(B_y)) \). The obstruction class of \( B \) is its image under the boundary map to \( \text{Ext}_{\mathfrak{A}^X}^{3}(\Sigma F_{K}^{X}(B), F_{K}^{X}(B)) \) by Theorem 3.2. We have described \( \text{Ext}_{\mathfrak{A}^X}(G, H) \) in Lemma 4.2 in such a way that this boundary map becomes tautological: it simply maps an element of \( \prod_{x \to y} \text{Ext}(G_x, H_y) \) to its class in the cokernel of the map in (113). In particular, the obstruction class of \( B \) is represented by \( (\beta_{y,x}^{1})_{x \to y} \in \prod_{x \to y} \text{Ext}(K_{1+}(B_x), K_x(B_y)) \).
Theorem 4.9 allows us to compare the classification theorems for $C^*$-algebras in $\mathcal{B}^X$ that use the invariant $F^X(B)$ or $F^K_X(B)$ with the obstruction class. Let $A, B \in \mathcal{B}^X$. An isomorphism

$$t^0 : F^K_X(A) \xrightarrow{\sim} F^K_X(B)$$

is equivalent to a family of isomorphisms

$$t^0_x : K_\ast(A(U_x)) \xrightarrow{\sim} K_\ast(B(U_x))$$

that make the diagrams

$$K_\ast(A(U_x)) \xrightarrow{t^0_x} K_\ast(B(U_x))$$

commute for all edges $x \rightarrow y$. Here the maps $K_\ast(\alpha_{y,x})$ and $K_\ast(\beta_{y,x})$ are induced by the inclusion maps of our $C^*$-algebras over $X$. The obstruction class for the isomorphism $(t^0_x)_{x \in X}$ vanishes if and only if there are $t^1_x \in \text{Ext}(K_{1+\ast}(A(U_x)), K_\ast(B(U_x)))$ for $x \in X$ such that

$$(4.10) \quad t^1_y \circ K_\ast(\alpha_{y,x}) - K_\ast(\beta_{y,x}) \circ t^1_x = \beta_{y,x}^1 \circ t^0_x - t^0_y \circ \alpha_{y,x}^1$$

for all edges $x \rightarrow y$. Here

$$\alpha_{y,x}^1 \in \text{Ext}(K_{1+\ast}(A(U_x)), K_\ast(A(U_y))), \quad \beta_{y,x}^1 \in \text{Ext}(K_{1+\ast}(B(U_y)), K_\ast(B(U_x)))$$

are the parity-reversing parts of the KK-classes of the inclusion maps. The elements $t^0_y$ and $t^1_x$ together form $t_x \in \text{KK}(A(U_x), B(U_y))$. And (4.10) is equivalent to $t_y \circ \alpha_{y,x} = \beta_{y,x} \circ t_x$ in $\text{KK}(A(U_x), B(U_y))$.

Corollary 2.20 says that any family of KK-equivalences $t_x \in \text{KK}(A(U_x), B(U_y))$ with $t_y \circ \alpha_{y,x} = \beta_{y,x} \circ t_x$ for all edges $x \rightarrow y$ lifts to an invertible element in $\text{KK}^X(A, B)$. The classification using the obstruction class says that a family of isomorphisms $t^0_x : K_\ast(A(U_x)) \xrightarrow{\sim} K_\ast(B(U_x))$ lifts to an invertible element in $\text{KK}^X(A, B)$ if and only if there exist $t^1_x$ satisfying (4.10). Corollary 2.20 makes a slightly stronger assertion because it says that any choice of elements $t^1_x$ satisfying (4.10) may be realised by an invertible element in $\text{KK}^X(A, B)$.

The result in Corollary 2.20 about the existence of liftings is also slightly stronger than the corresponding result using the invariant $F^K_X$ and the obstruction class because it does not require the $C^*$-algebras $A(U_x)$ to belong to the bootstrap class.

5. Filtrated K-theory for totally ordered spaces

Now we consider the special unique path space

$$X = \{1 \leftarrow 2 \leftarrow \cdots \leftarrow n\}$$

for some $n \in \mathbb{N}_{\geq 2}$. We are going to compare the classification for $C^*$-algebras over $X$ that follows from the Universal Coefficient Theorem in [12] to the classifications in Section 4.2. The invariant used in [12] is filtrated K-theory. This is the diagram of K-theory groups formed by $K_\ast(A(S))$ for all locally closed subsets $S \subseteq X$. Here a subset is locally closed if and only if it is of the form

$$[a, b] := \{a, a + 1, \ldots, b - 1, b\}, \quad 1 \leq a \leq b \leq n.$$ 

The maps in the filtrated K-theory diagram are those that come from natural transformations. We are going to describe these below.

Let $\mathcal{H}^X_{\text{fil}}$ be the kernel on morphisms of the filtrated K-theory functor, that is, $f \in \text{KK}^X(A, B)$ belongs to $\mathcal{H}^X_{\text{fil}}(A, B)$ if and only if it induces the zero map
We write 

\[ Yoneda Lemma, \]

where the maps

\[ \text{These groups are computed in } [12, \text{Equation (3.1)}]. \]

Theorem 4.9, we still have to compare the resolution used there with the one coming from Theorem 3.2 computes the obstruction class from the parity-reversing part of the map \( \varphi \) in (5.1). This computation is, however, quite non-trivial. We must first recall how the natural transformations in the filtrated K-theory diagram look like. Going beyond the results of [12], we then build an explicit \( J_{B,0}^{X} \)-projective resolution. Next, we observe which parts of the map \( \varphi \) are parity-reversing. This gives a class in \( \text{Ext}_{A}^{2}X \). To translate it into the setting of Theorem 4.9 we still have to compare the resolution used there with the one coming from (5.1). This requires most of the work.

We first recall the description of the \( \mathbb{Z}/2 \)-graded Abelian groups \( \mathcal{N}\mathcal{T}^{\ast}([a, b], [c, d]) \) of natural transformations \( K_{\ast}(A([a, b])) \rightarrow K_{\ast}(A([c, d])) \) in [12]. Let \( 1 \leq a \leq b \leq n \). The functor \( \mathfrak{R}X \rightarrow \mathfrak{A}b^{2}/2, A \mapsto K_{\ast}(A([a, b])) \), is represented by an object \( R_{[a, b]} \) of \( \mathfrak{A}X \), which is described in [12]. Let \( 1 \leq a \leq b \leq n \) and \( 1 \leq c \leq d \leq n \). By the Yoneda Lemma,

\[ \mathcal{N}\mathcal{T}^{\ast}([a, b], [c, d]) \cong KK_{\ast}^{\mathbb{X}}(R_{[c, d]}, R_{[a, b]}) \cong K_{\ast}(R_{[a, b]}([c, d])). \]

These groups are computed in [12, Equation (3.1)]:

\[ \mathcal{N}\mathcal{T}^{\ast}([a, b], [c, d]) \cong \begin{cases} \mathbb{Z}_{+} & \text{if } c \leq a \leq d \leq b, \\ \mathbb{Z}_{-} & \text{if } a + 1 \leq c \leq b + 1 \leq d, \\ 0 & \text{otherwise}. \end{cases} \]

We write \([a, b] \rightarrow [c, d]\) in the first two cases, that is, when there is a non-zero natural transformation in \( \mathcal{N}\mathcal{T}^{\ast}([a, b], [c, d]) \). The groups \( \mathcal{N}\mathcal{T}^{\ast}([a, b], [c, d]) \) form a \( \mathbb{Z}/2 \)-graded ring \( \mathcal{N}\mathcal{T} \). The filtrated K-theory of a separable \( C^{\ast} \)-algebra over \( X \) is a \( \mathbb{Z}/2 \)-graded, countable module over it, which we denote by \( FK(A) \).

The computations in [12] interpret the elements of \( \mathcal{N}\mathcal{T}^{\ast}([a, b], [c, d]) \) as follows. Let \( A \) be a separable \( C^{\ast} \)-algebra over \( X \) and let \( M := FK(A) \). If \( a < b \leq c \), then \([b, c]\) is relatively open in \([a, c]\) with complement \([a, b - 1]\). This induces the following natural six-term exact sequence in K-theory:

\[ \cdots \rightarrow M[b, c] \xrightarrow{i} M[a, c] \xrightarrow{r} M[a, b - 1] \xrightarrow{\delta_{\text{odd}}} M[b, c] \rightarrow \cdots \]

where the maps \( i, r \) preserve the \( \mathbb{Z}/2 \)-grading and \( \delta \) reverses it. Any natural transformation \( M[a, b] \rightarrow M[c, d] \) is an integer multiple of a product of the maps
Remark 5.6. The last column and the first row in the diagram are the same. So the diagram
This is clear in the even case. In the odd case, we use the naturality of boundary maps
vanishes if and only if it factors through one of the objects 0 on the boundary of the
extended diagram.

\begin{equation}
\begin{array}{ccc}
M[a, b] & \xrightarrow{i} & M[c, b] \\
\downarrow{r} & & \downarrow{r} \\
M[a, d] & \xrightarrow{r} & M[c, d],
\end{array}
\end{equation}

and its diagonal map \( \tau_{\{a, b\}}^{c, d} \) generates \( \mathcal{N} T_0([a, b], [c, d]) \cong \mathbb{Z} \). And if \( a + 1 \leq c \leq b + 1 \leq d \), then there is a commuting square

\begin{equation}
\begin{array}{ccc}
M[a, b] & \xrightarrow{\delta} & M[b + 1, d] \\
\downarrow{r} & & \downarrow{r} \\
M[a, c - 1] & \xrightarrow{\delta} & M[c, d],
\end{array}
\end{equation}

and its diagonal map \( \tau_{\{a, b\}}^{c, d} \) generates \( \mathcal{N} T_1([a, b], [c, d]) \cong \mathbb{Z} \). We have defined a generator \( \tau_{\{a, b\}}^{c, d} \) for \( \mathcal{N} T_*([a, b], [c, d]) \) whenever \([a, b] \rightarrow [c, d]\), that is, whenever \( \mathcal{N} T_*([a, b], [c, d]) \neq \emptyset \) by (5.2). It is convenient to define \( \tau_{\{a, b\}}^{c, d} = 0 \) if \( \mathcal{N} T_*([a, b], [c, d]) = 0 \).

By the Yoneda Lemma, the natural transformations \( \tau_{\{a, b\}}^{c, d} \) correspond to arrows

\( (\tau_{\{a, b\}}^{c, d})^* : \mathcal{R}_{\{c, d\}} \rightarrow \mathcal{R}_{\{a, b\}}. \)

Remark 5.6. An \( \mathcal{N} T \)-module is called exact if the sequences (5.3) are exact for all \( a < b \leq c \). The exact \( \mathcal{N} T \)-modules form a stable exact subcategory of the stable Abelian category of all \( \mathcal{N} T \)-modules, and the filtrated K-theory of any separable \( C^* \)-algebra over \( X \) is exact as an \( \mathcal{N} T \)-module. The results in [12] imply that any exact \( \mathcal{N} T \)-module has a projective resolution of length 1. Hence it lifts to an object of the bootstrap class \( \mathcal{B}^X \). So the image of the filtrated K-theory functor is equal to the class of exact, countable \( \mathcal{N} T \)-modules. And the filtrated K-theory functor, viewed as a functor to the subcategory of exact, countable \( \mathcal{N} T \)-modules, is the universal \( \mathcal{F}^X \)-exact functor to an exact category. So \( \mathcal{F}^X \) has the property that its universal exact functors to an Abelian and to an exact category are different.

Now we study the multiplication in \( \mathcal{N} T \). We begin with decomposing the generators in \( \mathcal{N} T \) further. We may rewrite the natural transformations \( \tau_{\{a, b\}}^{c, d} \) defined above as products of the special natural transformations

\( i = \tau_{\{a, b\}}^{0, b} : [a, b] \rightarrow [a, b], \quad a + 1 \leq b \leq n, \)

\( r = \tau_{\{a, b\}}^{a, b+1} : [a, b + 1] \rightarrow [a, b], \quad a \leq b \leq n - 1, \)

\( \delta = \tau_{\{a, b\}}^{1, a} : [1, a - 1] \rightarrow [a, n], \quad 2 \leq a \leq n. \)

This is clear in the even case. In the odd case, we use the naturality of boundary maps to rewrite the boundary map \( \delta : [a, b] \rightarrow [b + 1, d] \) as the product of \( i : [a, b] \rightarrow [1, b] \), the boundary map \( \delta : [1, b] \rightarrow [b + 1, n] \), and \( r : [b + 1, n] \rightarrow [b + 1, d] \). The generating natural transformations defined above form a commuting diagram as in Figure [1]. The last column and the first row in the diagram are the same. So the diagram repeats when we put a reflected copy of it next to it. Figure [2] shows the full diagram for \( n = 3 \). We claim that all relations among the generating natural transformations are given by this extended commuting diagram. In particular, a composite of \( i, r, \delta \) vanishes if and only if it factors through one of the objects 0 on the boundary of the extended diagram.
Let $1 \leq a \leq b \leq n$ and $1 \leq c \leq d \leq n$. A product of the generators of type $i, r$ from $[a, b]$ to $[c, d]$ exists if and only if $c \leq a$ and $d \leq b$. Figure 2 shows that all such products are equal. Equation (5.2) shows that this product is 0 unless $a \leq d$, so that $c \leq a \leq d \leq b$. Then it is equal to $\tau_{[a, b]}^{[c, d]}$ by the definition in (5.4). As a consequence, $\tau_{[c, d]}^{[c, g]} \cdot \tau_{[a, b]}^{[a, d]} = \tau_{[a, b]}^{[c, d]}$ if $c \leq e \leq d \leq g$ and $c \leq a \leq g \leq b$; this is non-zero if and only if also $a \leq d$ or, equivalently, $c \leq e \leq a \leq d \leq g \leq b$.

Now consider a product of $i, r, \delta$ going from $[a, b]$ to $[c, d]$ and containing exactly one factor of $\delta$. Using the diagram in Figure 1 we may rearrange this product in such a way that we first go right and then go down in the extended diagram as in Figure 2. If this goes through the zeros outside the drawn region, the product is 0.
If not, we may combine consecutive \( i \) and consecutive \( r \) to bring the product into the following form:

\[
[a, b] \xrightarrow{i} [1, b] \xrightarrow{\delta} [b + 1, n] \xrightarrow{r} [b + 1, d] \xrightarrow{r} [c, d].
\]

The combination \( r \circ \delta \circ i \) in the beginning is the boundary map \( \delta : [a, b] \to [b + 1, d] \). So we get \( \tau_{[c,d]}^{[e,g]} \) if \( c + 1 \leq a \leq d + 1 \leq b \) and 0 otherwise by (5.5) and (5.2). Since we may rewrite all even \( \tau_{[c,d]}^{[e,g]} \) in terms of \( i, r \), we can now compute \( \tau_{[c,d]}^{[e,g]} \cdot \tau_{[a,b]}^{[e,g]} \) if one of the transformations \( \tau_{[c,d]}^{[e,g]} \) and \( \tau_{[a,b]}^{[e,g]} \) is even and the other one is odd. Namely, the product is \( \tau_{[a,b]}^{[c,d]} \) if \( a + 1 \leq c \leq b + 1 \leq d \), and 0 otherwise. In more detail, the assumption that exactly one of the transformations \( \tau_{[c,d]}^{[e,g]} \) and \( \tau_{[a,b]}^{[e,g]} \) is even means that \( e + 1 \leq c \leq g + 1 \leq d \) and \( e \leq a \leq g \leq b \) or \( c \leq e \leq d \leq g \) and \( a + 1 \leq e \leq b + 1 \leq g \). The assumption \( a + 1 \leq c \leq b + 1 \leq d \) becomes \( e \leq a \leq c \leq g + 1 \leq b + 1 \leq d \) or \( a < c \leq e \leq b + 1 \leq d \leq g \) in these two cases, respectively.

Finally, any product with more than two factors \( \delta \) vanishes because it may be deformed in the extended diagram in Figure 2 so as to factor through one of the zeros on the boundary. We sum up our results about the multiplication in \( NT \):

\[
(5.7) \quad \tau_{[c,d]}^{[e,g]} \cdot \tau_{[a,b]}^{[e,g]} = \tau_{[a,b]}^{[c,d]} \neq 0 \iff \begin{cases} 
  e \leq e \leq a \leq d \leq g \leq b, \\
  e \leq a \leq c - 1 \leq g \leq b < d, \\
  a < c \leq e \leq b + 1 \leq d \leq g,
\end{cases}
\]
and \( \tau_{[c,d]} \cdot \tau_{[e,g]} = 0 \) otherwise. We write

\[
[a, b] \rightarrow [c, g] \rightarrow [c, d] \quad \iff \quad \tau_{[c,d]} \cdot \tau_{[e,g]} = \tau_{[c,d]} \cdot \tau_{[a,b]} = 0.
\]

It can happen that \([a, b] \rightarrow [c, g], [e, g] \rightarrow [c, d]\) and \([a, b] \rightarrow [e, g] \rightarrow [c, d] \); that is, \(\tau_{[c,d]} \cdot \tau_{[e,g]} = 0\) although \(\tau_{[a,b]} \cdot \tau_{[c,d]} \cdot \tau_{[e,g]} \neq 0\).

Given \(1 \leq a \leq b \leq n\), there are one or two proper natural transformations to \(M[a,b]\) that are shortest in the sense that all others factor through them. If \(a < b < n\), these are \(i: M[a + 1, b] \rightarrow M[a, b]\) and \(r: M[a, b + 1] \rightarrow M[a, b]\). If \(1 < a < b = n\), then \(r\) above is replaced by \(\delta: M[1, a - 1] \rightarrow M[a, n]\). One of these maps is missing if \(a = b\) or if \((a, b) = (1, n)\), that is, on the two outer diagonals in the diagram in Figure 2 (now for general \(n\)). We define \(M[a + 1, a] := 0\) for \(0 \leq a \leq n\) to make this a special case of the generic case.

Now we build an \(\mathcal{F}^n\)-projective resolution of \(A\) of length 1. This has not yet been done in [12], where only the existence of such a resolution is proven. For \(1 \leq a \leq b \leq n\), let \(M[a, b]_{\text{ss}}\) be the quotient of \(M[a, b]\) by the images of all proper natural transformations to \(M[a, b]\) or, equivalently, by the images of the two shortest natural transformations:

\[
M[a, b]_{\text{ss}} := \left\{ \begin{array}{ll}
M[a, b] / (i(M[a + 1, b]) + r(M[a, b + 1])) & \text{if } b < n, \\
M[a, b] / (i(M[a + 1, b]) + \delta(M[1, a - 1])) & \text{if } b = n 
\end{array} \right.
\]

(compare [12] Definition 3.7 and Lemma 3.8]). Choose a resolution

\[
Q_i[a, b] \xrightarrow{d_i} Q_0[a, b] \xrightarrow{d_0} M[a, b]_{\text{ss}}
\]

of \(M[a, b]_{\text{ss}}\) by countable \(\mathbb{Z}/2\)-graded free Abelian groups. For \(i = 0, 1\), let

\[
Q_i[a, b] := \mathcal{R}_{[a, b]} \otimes_{\mathbb{Z}} \mathcal{Q}_i[a, b],
\]

where the tensor product is defined as in (2.4). Since \(\mathcal{R}_{[a, b]} \otimes_{\mathbb{Z}} \mathcal{Q}_i[a, b]\) is a direct sum of copies of suspensions of \(\mathcal{R}_{[a, b]}\), the definition of \(\mathcal{R}_{[a, b]}\) as a representing object implies

\[
\text{KK}^\mathbb{Z}_0(\mathcal{Q}_i[a, b], B) \cong \text{Hom}(\mathcal{Q}_i[a, b], \mathcal{K}_* (B[a, b]))
\]

for all \(C^*\)-algebras \(B\) over \(X\); here \(\text{Hom}\) means grading-preserving group homomorphisms. This property characterises \(\mathcal{Q}_i[a, b]\) uniquely up to isomorphism in \(\text{KK}^\mathbb{Z}_0\).

Equation (5.9) implies a similar description of the \(\mathbb{Z}/2\)-graded Abelian group \(\text{KK}^\mathbb{Z}_0(\mathcal{Q}_i[a, b], B)\), replacing \(\text{Hom}\) by group homomorphisms that need not respect the grading. Given a group homomorphism \(g: \mathcal{Q}_i[a, b] \rightarrow \mathcal{K}_* (B[a, b])\), let \(g^\#\) denote the corresponding element of \(\text{KK}^\mathbb{Z}_0(\mathcal{Q}_i[a, b], B)\).

Since the \(\mathbb{Z}/2\)-graded Abelian group \(Q_0[a, b]\) is free, the homomorphism \(d_0\) in (5.8) lifts to a grading-preserving homomorphism

\[
f[a, b]: Q_0[a, b] \rightarrow M[a, b] = \mathcal{K}_* (A[a, b]).
\]

Let \(f[a, b]^\# \in \text{KK}^\mathbb{Z}_0(Q_0[a, b], A)\) correspond to \(f[a, b]\) by (5.9). Let

\[
P_i := \sum_{1 \leq a \leq b \leq n} \mathcal{Q}_i[a, b]
\]

for \(i = 0, 1\). The objects \(\mathcal{Q}_i[a, b]\) and \(P_i\) for \(i = 0, 1\) are \(\mathcal{F}^n\)-projective because of (5.9). There is a unique element \(f \in \text{KK}^\mathbb{Z}_0(P_0, A)\) that restricts to \(f[a, b]^\#\) on the summand \(\mathcal{Q}_i[a, b]\).

Lemma 5.11. The map \(FK(f): FK(P_0) \rightarrow M := FK(A)\) is surjective. Its kernel is isomorphic to \(FK(P_1)\) as an \(\mathcal{N}T\)-module.
Proof. Let \( f_\ast := FK(f) \). The functor \( M \mapsto M_{ss} \) is a tensor product functor with a certain right \( NT \)-module \( NT_{ss} \). So it is right exact and (coker \( f_\ast \))[\( a, b \)]_ss = coker(\( f_\ast[a, b]_ss \)) for all \( 1 \leq a \leq b \leq n \). The projective \( NT \)-module \( FK(P_b) \) has \( FK(P_b)[a, b]_ss \cong Q_0[a, b] \). Hence \( (f_\ast)_ss \) is surjective by construction of \( f \), and the \( NT \)-module coker \( f_\ast \) satisfies (coker \( f_\ast \))_ss = 0. This implies coker \( f_\ast = 0 \) by [12, Proposition 3.10]. That is, \( f_\ast \) is surjective. Let \( N := \ker FK(f) \). So there is an extension \( N \mapsto FK(P_b) \mapsto M \) of \( NT \)-modules. Since \( M \) and \( FK(P_b) \) are exact \( NT \)-modules, they satisfy Tor_1^{NT}(NT_{ss}, M) = 0 and Tor_1^{NT}(NT_{ss}, FK(P_b)) = 0 by [12, Lemma 3.13]. Hence

\[
N[a, b]_{ss} \rightarrow FK(P_b)[a, b]_{ss} \rightarrow M[a, b]_{ss}
\]

is a short exact sequence for all \( 1 \leq a \leq b \leq n \) and Tor_1^{NT}(NT_{ss}, N) = 0. Since \( FK(P_b)[a, b]_{ss} \cong Q_0[a, b] \), this implies \( N[a, b]_{ss} \cong Q_1[a, b] \). Now [12, Theorem 3.12] shows that \( N \) is a projective \( NT \)-module. In fact, the proof of this theorem shows that \( N \cong FK(P_1) \). More precisely, the quotient maps \( N[a, b] \rightarrow N[a, b]_{ss} \) split because \( N[a, b]_{ss} \cong Q_1[a, b] \) is free. Let

\[
(5.12) \quad \varphi[a, b]: Q_1[a, b] \cong N[a, b]_{ss} \rightarrow N[a, b] \subseteq FK(P_0)[a, b]
\]

be sections. They induce an \( NT \)-module homomorphism \( FK(P_1) \rightarrow N \) by the universal property of the “free” \( NT \)-module \( FK(P_1) \). And the proof of [12, Theorem 3.12] shows that it is an isomorphism. \( \square \)

Disregarding the \( \mathbb{Z}/2 \)-grading, we may write

\[
FK(P_b)[a, b] = \bigoplus_{[c, d] \rightarrow [a, b]} Q_0[c, d],
\]

that is, the sum runs over all \( 1 \leq c \leq d \leq n \) with \( a \leq c \leq b \leq d \) or \( c + 1 \leq a \leq d + 1 \leq b \). So the map \( \varphi[a, b] \) in (5.12) has components

\[
\varphi_{[c, d]}[a, b]: Q_1[a, b] \rightarrow Q_0[c, d]
\]

for \( [c, d] \rightarrow [a, b] \). Since \( \varphi[a, b] \) is even, the map \( \varphi_{[c, d]}[a, b] \) has the same parity as \( \tau_{[c, d]_{ss}}[a, b] \), that is, it is grading-preserving if \( a \leq c \leq b \leq d \) and grading-reversing if \( c + 1 \leq a \leq d + 1 \leq b \). The image of \( \varphi[a, b] \) for \( 1 \leq a \leq b \leq n \) is contained in \( N = \ker FK(f) \), that is, \( FK(f) \circ \varphi[a, b] = 0 \) as a map \( Q_1[a, b] \rightarrow K_* (A[a, b]) \).

Unravelling the definition of \( FK(f) \), this becomes

\[
(5.13) \quad \sum_{[c, d] \rightarrow [a, b]} \tau_{[c, d]_{ss}}[a, b] \circ f[c, d] \circ \varphi_{[c, d]}[a, b] = 0: Q_1[a, b] \rightarrow K_* (A[a, b]).
\]

The objects \( P_b \) and \( P_1 \) are \( \mathcal{F}_{ss}^X \)-projective, and \( FK \) is fully faithful on \( \mathcal{F}_{ss}^X \)-projective objects. So the arrow \( FK(P_1) \rightarrow FK(P_b) \) with components \( \varphi_{[c, d]}[a, b] \) for \( [c, d] \rightarrow [a, b] \) lifts uniquely to an arrow \( \varphi \in \mathcal{K}_{ss}^X(P_1, P_b) \). More precisely, the map \( \varphi \) is given by a matrix of maps \( \mathcal{R}_{[c, d]_0} \otimes Q_1[c, d] \rightarrow \mathcal{R}_{[a, b]} \otimes Q_0[a, b] \) for all \( 1 \leq c \leq d \leq n \) and \( 1 \leq a \leq b \leq n \). The entries of this matrix are \( \left( (\tau_{[c, d]_{ss}}^*)^* \otimes \varphi_{[c, d]}[a, b] \right)_{[c, d]_0 \rightarrow [a, b]} \), that is,

\[
(5.14) \quad \varphi = \left( \left( \tau_{[c, d]_{ss}}^* \otimes \varphi_{[c, d]}[a, b] \right)_{[c, d]_0 \rightarrow [a, b]} \right) \bigoplus_{1 \leq c \leq d \leq n} \left( \mathcal{R}_{[c, d]} \otimes Q_1[c, d] \rightarrow \bigoplus_{1 \leq a \leq b \leq n} \mathcal{R}_{[a, b]} \otimes Q_0[a, b] \right).
\]

Here we use the convention that \( \tau_{[c, d]}[a, b] = 0 \) if not \( [a, b] \rightarrow [c, d] \).

Lemma 5.11 says that (5.1) with \( f \) and \( \varphi \) as above is \( \mathcal{F}_{ss}^X \)-exact, that is, \( FK \) applied to (5.1) is an exact sequence. The \( \mathcal{F}_{ss}^X \)-exactness of (5.1) says that the
functor $B \mapsto K_\ast(B[a,b])$ maps it to an exact sequence for each $1 \leq a \leq b \leq n$. In fact, this gives projective resolutions. We write them down explicitly:

**Lemma 5.15.** Let $1 \leq a \leq b \leq n$. Then

\[
\bigoplus_{[c,d] \rightarrow [a,b]} Q_1[c,d] \xrightarrow{\left(\varphi_{[c,d]}^{|c,n]} \circ f[c,d]\right)} \bigoplus_{[e,g] \rightarrow [a,b]} Q_0[e,g] \rightarrow K_\ast(A[a,b])
\]

is a free resolution. Here $(\varphi_{[c,d]}^{|c,n]} : [c,d] \rightarrow [a,b]$ means that the matrix entry is $\varphi_{[c,d]}^{|c,n]}$ if $[c,d] \rightarrow [e,g] \rightarrow [a,b]$ as described in (5.7), and 0 otherwise.

The boundary maps in (5.16) are inhomogeneous, that is, the matrix entries of the maps may have even or odd degree.

**Proof.** Equation (5.2) computes the group $K_\ast(R_{[c,d]}([a,b])) \cong \mathcal{N}^\ast([c,d], [a,b])$: it is $\mathbb{Z}$ in even or odd degree if $[c,d] \rightarrow [a,b]$ and 0 otherwise. Therefore, $P_i[a,b] \cong \bigoplus_{[c,d] \rightarrow [a,b]} Q_i[c,d]$ for $i = 0, 1$, disregarding the grading. The map $(\varphi_{[c,d]}^{|c,n]} \circ f[c,d])$ between the summands $R_{[c,d]} \otimes Q_1[c,d]$ in $P_1$ and $R_{[c,d]} \otimes Q_0[e,g]$ in $P_0$ induces the map $\varphi_{[c,d]}^{|c,n]} : Q_1[c,d] \rightarrow Q_0[e,g]$ if $[c,d] \rightarrow [e,g] \rightarrow [a,b]$, and 0 otherwise, compare (5.7). The map $K_\ast(P_0[a,b]) \rightarrow K_\ast(A[a,b])$ corresponds to a family of maps

\[
K_\ast(R_{[c,d]}[a,b]) \otimes Q_0[c,d] \cong K_\ast(R_{[c,d]}[a,b] \otimes Q_0[c,d] \rightarrow K_\ast(A[a,b])
\]

for $1 \leq c \leq d \leq n$. As above, $K_\ast(R_{[c,d]}[a,b]) \neq 0$ only if $[c,d] \rightarrow [a,b]$, and then the map $Q_0[c,d] \rightarrow K_\ast(A[a,b])$ induced by $f : P_0 \rightarrow A$ is $\tau_{[c,d]}^{|a,b]} \circ f[c,d]$. \hfill \square

We have reached the first milestone in the computation of the obstruction class: the $\mathfrak{J}_0^X$-projective resolution (5.1). It is explicit enough to express the obstruction class that comes from filtrated $K$-theory in the terms of Theorem 4.9, namely, as being represented by a family of elements in $\text{Ext}^1(K_\ast(A[e+1,n]), K_\ast(A[e,n]))$ for $e = 1, \ldots, n - 1$. We compute these Ext-groups with the resolutions in (5.16). So the obstruction class corresponds to a sequence of maps

$$\delta_e : \bigoplus_{[a,b] \rightarrow [e+1,n]} Q_1[a,b] \rightarrow K_\ast(A[e,n]), \quad e = 1, \ldots, n - 1.$$ 

In turn, each $\delta_e$ is given by maps $\delta_{a,b}^{e,n]} : Q_1[a,b] \rightarrow K_\ast(A[e,n])$ for all $1 \leq a \leq b \leq n$ with $[a,b] \rightarrow [e+1,n]$. The following theorem computes these maps $\delta_{a,b}^{e,n]}$. It is the main result of this section. Section 5.1 is dedicated to its proof.

**Theorem 5.17.** Let

$$\delta_{a,b}^{e,n]} := \begin{cases} \sum_{[c,d] \rightarrow [a,b]} \tau_{[c,d]}^{|e,n]} \circ f[c,d] \circ \varphi_{[c,d]}^{|a,b]} & \text{if } a = c \text{ and } b < n, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting family of maps $(\delta_e)_{1 \leq e \leq n}$ lifts the obstruction class of $A$ to an element of $\prod_{e=1}^{n-1} \text{Ext}^1(K_\ast(A[e+1,n]), K_\ast(A[e,n])).$

Any $\mathfrak{J}_0^X$-epic map of $A$ into $A$ is isomorphic to one of the form above because any $\mathfrak{J}_0^X$-epic map from an $\mathfrak{J}_0^X$-projective object to $A$ is isomorphic to a map $f \in KK_0^\mathfrak{J}_0^X(P_0, A)$ as above. Since $\mathfrak{J}_0^X$-projective resolutions of $A$ are equivalent to projective resolutions of $FK(A)$, the maps $\delta_{a,b}^{e,n]}$ may, in principle, be computed from $FK(A)$ by choosing a projective resolution. This gives the maps $f[c,d]$ and $\varphi_{[a,b]}$. Such a computation may, of course, be difficult in practice.
5.1. Proof of the obstruction class formula. First we examine the smaller invariant \( F^X_K: \mathcal{R} \mathcal{K}^X \rightarrow \mathcal{A}^X \). This takes the part of filtrated K-theory consisting of \( K_*(A[a,n]) \) for \( 1 \leq a \leq n \) with the maps \( i \) between them because the minimal open subset containing \( a \) is \( U_a = [a,n] \). So the diagram \( F^X_K(A) \) is simply the first row in the diagram in Figure 1. We have \( i_a C \cong \mathcal{R}_{[a,n]} \) for \( 1 \leq a \leq n \) because both objects represent the same functor \( A \mapsto K_*(A[a,n]) \). So \( \mathcal{R}_{[a,n]} \) for \( 1 \leq a \leq n \) is \( \mathcal{J}_K \)-projective, and \( F^X_K(\mathcal{R}_{[a,n]}) \) is the diagram

\[
\mathcal{P}_{[a,n]} := F^X_K(\mathcal{R}_{[a,n]}) = \left( \begin{array}{c}
[0, \ldots, 0] \\
\text{a times}
\end{array} \right) \left( \begin{array}{c}
[0, \ldots, 0] \\
\text{n-a times}
\end{array} \right).
\]

in \( \mathcal{A}^X \). If \( 1 \leq a \leq b \leq n-1 \), then \( F^X_K(\mathcal{R}_{[a,b]}) \) is the diagram

\[
\mathcal{P}_{[a,b]} := F^X_K(\mathcal{R}_{[a,b]}) = \left( \begin{array}{c}
[0, \ldots, 0] \\
\text{a times}
\end{array} \right) \left( \begin{array}{c}
[0, \ldots, 0] \\
\text{b+1-a times}
\end{array} \right) \left( \begin{array}{c}
[0, \ldots, 0] \\
\text{b times}
\end{array} \right) \left( \begin{array}{c}
[0, \ldots, 0] \\
\text{n-b times}
\end{array} \right)
\]

because of the formula for \( K_* (\mathcal{R}_{[a,b]}([c,n])) \) in (5.2). Thus the objects \( \mathcal{R}_{[a,b]} \) for \( 1 \leq a \leq b \leq n \) are even if \( b = n \) and odd if \( b < n \).

The formula for the obstruction class in Theorem 5.2 uses the parity-reversing part of \( \varphi \in \mathcal{K}_K^X(P_1, P_0) \). This is described by the following lemma:

**Lemma 5.20.** The component \( (\tau_{[a,b]}^* ) \otimes \varphi_{[a,b]}^d \) of \( \varphi \) is parity-reversing if and only if \( [c,d] \rightarrow [a,n] \rightarrow [a,b] \) and \( b < n \).

**Proof.** If \( \tau_{[a,b]} \neq 0 \), then either \( a \leq c \leq b \leq d \) or \( c+1 \leq a \leq d+1 \leq b \). The map \( (\tau_{[a,b]}^* ) \otimes \varphi_{[a,b]}^d \) always belongs to \( \mathcal{K}_K^X (\mathcal{Q}_1[a,b], \mathcal{Q}_0[c,d]) \). So \( \varphi_{[a,b]}^d : \mathcal{Q}_1[a,b] \rightarrow \mathcal{Q}_0[c,d] \) is parity-preserving in the first case and parity-reversing in the second case. The object \( \mathcal{R}_{[a,b]} \) is even if \( b = n \) and odd if \( b < n \).

First let \( a \leq c \leq b \leq d \). If \( b = n \), then \( d = n \) as well, so that \( \mathcal{R}_{[a,b]} \) and \( \mathcal{R}_{[c,d]} \) have the same parity, and \( \varphi_{[a,b]}^d \) preserves parity. So we get a parity-preserving component of \( \varphi \). For the same reasons, we get a parity-preserving component if \( a \leq c \leq b \leq d < n \), and a parity-reversing component if \( a \leq c \leq b < d < n \).

Now let \( c+1 \leq a \leq d+1 \leq b \), so that \( \varphi_{[a,b]}^d \) reverses parity. If \( b = n \), then \( d < n \), so that \( \mathcal{R}_{[a,b]} \) and \( \mathcal{R}_{[c,d]} \) have opposite parity. Hence \( (\tau_{[a,b]}^* ) \otimes \varphi_{[a,b]}^d \) is parity-reversing altogether. If \( b < n \), then \( d < n \) and so \( \mathcal{R}_{[a,b]} \) and \( \mathcal{R}_{[c,d]} \) have the same parity. Thus \( (\tau_{[a,b]}^* ) \otimes \varphi_{[a,b]}^d \) is parity-reversing. Inspection shows that the parity-reversing components are exactly those for which \( [c,d] \rightarrow [a,n] \rightarrow [a,b] \) as in (5.7) and \( b < n \).

Let \( 1 \leq a \leq b < n \). The exact triangle

\[
\Sigma \mathcal{R}_{[a,n]} \xrightarrow{i} \Sigma \mathcal{R}_{[b+1,n]} \xrightarrow{\delta} \mathcal{R}_{[a,b]} \xrightarrow{r} \mathcal{R}_{[a,n]}
\]

in \( \mathcal{R} \mathcal{K}^X \) is \( \mathcal{J}_K \)-exact, that is, \( F^X_K(r^* ) = 0 \). Since \( \Sigma \mathcal{R}_{[a,n]} \) and \( \Sigma \mathcal{R}_{[b+1,n]} \) are \( \mathcal{J}_K \)-projective, it is an \( \mathcal{J}_K \)-projective resolution of \( \mathcal{R}_{[a,b]} \) of length 1. Since we allow both odd and even arrows in diagrams, we may drop the suspensions in (5.21). For \( 1 \leq a \leq b \leq n \), let

\[
M(a,b) := \begin{cases} 
a & \text{if } b = n, \\b+1 & \text{if } b < n.
\end{cases}
\]

Then \( \left( \tau_{[a,b]}^* \right) : \mathcal{R}_{[M(a,b),n]} \rightarrow \mathcal{R}_{[a,b]} \) is an \( \mathcal{J}_K \)-epimorphism both for \( b = n \) and \( b < n \). Its cone is \( \mathcal{R}_{[a,n]} \) if \( b < n \) and 0 if \( b = n \). Now we write down \( \mathcal{J}_K \)-projective
resolutions of \( P_i \) for \( i = 0, 1 \). Let

\[
P_{10} := \bigoplus_{1 \leq a \leq b \leq n} \mathcal{R}_{[M(a,b),n]} \otimes Q_i[a,b],
\]

\[
P_{11} := \bigoplus_{1 \leq a \leq b < n} \mathcal{R}_{[a,n]} \otimes Q_i[a,b].
\]

Then the following is an \( \mathcal{K}^X \)-projective resolution:

\[
0 \to P_{11} \xrightarrow{\tau \otimes \text{id}} P_{10} \xrightarrow{\tau \otimes \text{id}} P_1 \to 0.
\]

Here \( \bigoplus \tau \otimes \text{id} \) means the direct sum of the maps \( (\tau_{[x,y]})^* \otimes \text{id}_{Q_i[a,b]} \) between the summands for fixed \( a, b \), with the appropriate \( x, y, z, w \). So (5.24) is the direct sum of the resolutions (5.21), tensored with \( Q_i[a,b] \), over all \( 1 \leq a \leq b < n \), and the trivial resolutions \( 0 \to \mathcal{R}_{[a,n]} \to \mathcal{R}_{[a,n]} \), tensored with \( Q_i[a,b] \), over all \( 1 \leq a \leq b = n \). The maps in (5.24) are inhomogeneous, that is, some components are in \( \mathcal{K}^X \) and others in \( \mathcal{K}^A \).

The resolution (5.24) allows us to compute \( \mathcal{K}^X(P_1, P_0) \) with the Universal Coefficient Theorem for the invariant \( F^X_0 \). First, the long exact sequence for the direct sum of the exact triangles (5.21) implies a natural extension of Abelian groups

\[
\text{coker}(\mathcal{K}^X(\Sigma P_{10}, P_0) \to \mathcal{K}^X(\Sigma P_{11}, P_0)) \to \mathcal{K}^X(P_1, P_0) \to \text{ker}(\mathcal{K}^X(P_{10}, P_0) \to \mathcal{K}^X(P_{11}, P_0)).
\]

We may rewrite the kernel and cokernel here as \( \text{Hom} \) and \( \text{Ext} \) in \( \mathcal{A}^X \), using that (5.24) is an \( \mathcal{K}^X \)-projective resolution. The extension above splits unnaturally, giving the decomposition of \( \varphi \) into its parity-preserving and -reversing parts \( \varphi^+ \) and \( \varphi^- \), respectively.

**Lemma 5.25.** The image of the parity-reversing part \( \varphi^- \) of \( \varphi \) in \( \text{Ext}^1_{\mathcal{K}^X}(P_1, P_0) \) is the map

\[
P_{11} = \bigoplus_{1 \leq a \leq b \leq n} \mathcal{R}_{[a,n]} \otimes Q_1[a,b] \to \bigoplus_{1 \leq c \leq d \leq n} \mathcal{R}_{[c,d]} \otimes Q_0[c,d] = P_0
\]

with matrix coefficients \( (\tau_{[a,n]})^* \otimes \varphi_{[c,d]} \) if \( [c,d] \to [a,n] \to [a,b] \) and \( b < n \), and 0 otherwise.

**Proof.** In terms of the matrix description of \( \varphi \), each matrix entry \( (\tau_{[a,b]})^* \otimes \varphi_{[c,d]} \) has even or odd parity and thus belongs to either \( \text{Hom} \) or \( \text{Ext} \), respectively. By Lemma 5.20, the entry belongs to \( \varphi^- \) if and only if \( \tau_{[a,b]} \) factors through \( \tau^+ : \mathcal{R}_{[a,b]} \to \mathcal{R}_{[a,n]} \). In this case, it factors as \( (\tau_{[a,n]})^* \otimes \varphi_{[a,b]} \circ (\tau^+ \otimes \text{id}) \). Since \( \tau^+ \) is the boundary map in (5.21), this exhibits a map \( P_{11} \to P_0 \). The map from \( \text{Ext}^1_{\mathcal{A}^X} \) to \( \mathcal{K}^X \) in the UCT is defined by composing with the boundary map in the exact triangle that contains the given resolution. So the map \( P_{11} \to P_0 \) found above is the relevant component of \( \varphi \). The formula in the lemma follows. \( \square \)

According to the recipe in Theorem 3.2, the obstruction class in \( \text{Ext}^2_{\mathcal{K}^X}(\Sigma A, A) \) is the composite of the parity-reversing part of \( \varphi \), viewed as an element of \( \text{Ext}^1_{\mathcal{K}^X}(P_1, P_0) \), with \( f \in \mathcal{K}^X(P_0, A) \) and with the class of the extension (5.1) in \( \text{Ext}^1_{\mathcal{K}^X}(A, P_1) \). Composing the two extensions gives a length-2 resolution

\[
P_{11} \to P_{10} \to P_0 \to A.
\]
As usual, the composite maps through a resolution in (5.26) by one that is equivalent to the isomorphism (5.9). Here the sum runs only over those \([c, d]\) with \([c, d] \to [a, n] \to [a, b]\) as in Lemma 5.25. In contrast, the sum over all \([c, d]\) with \([c, d] \to [a, b]\) is 0 by (5.13).

In a sense, we have now computed the obstruction class. The length-2 resolution in (5.26) is, however, different from the one that is implicitly used in Theorem 4.9 to compute the relevant Ext^2-group and the obstruction class in it. To translate the formula for the obstruction class that we get from filtrated K-theory into the formula for the obstruction class that we get from filtrated K-theory into the formula for the obstruction class that we get from filtrated K-theory, we first compare the underlying length-2 resolutions. First, we replace the resolution in (5.26) by one that is 3\(k\)-projective.

The entries \(P_{10}\) and \(P_{11}\) are already 3\(k\)-projective, and (5.24) is an 3\(k\)-projective resolution of \(P_0\). The objects \(P_i\) and \(P_{i+1}\) are all sums over \(1 \leq a \leq b \leq n\), with summands of the form \(R_{[c, d]} \otimes Q[a, b]\) for suitable \(x, y\) depending on \(a, b\); the summands in \(P_{i+1}\) are 0 for \(b = n\). Let \((\tau^* \otimes \varphi)\) denote the map between these sums for \(i = 1\) to those for \(i = 0\) with matrix entries

\[
(\tau^* \otimes \varphi)_{[x, y]}^{[z, w]} : R_{[x, y]} \otimes Q[a, b] \to R_{[z, w]} \otimes Q_0[c, d].
\]

As usual, \(\tau^* \otimes \varphi = 0\) if \(\mathcal{N}T_*([z, w], [x, y]) = 0\).

**Lemma 5.28.** There is a commuting diagram

\[
\begin{array}{ccc}
P_{11} & \xrightarrow{\oplus \tau^* \otimes \text{id}} & P_{10} \\
\downarrow{\tau^* \otimes \varphi} & & \downarrow{\tau^* \otimes \varphi}
P_{01} & \xrightarrow{\oplus {\tau^* \otimes \text{id}}} & P_{00} \\
\end{array}
\]

**Proof.** We compare maps between direct sums by comparing their matrix coefficients. For the two composite maps \(P_{11} \to P_{00}\), these are maps \(R_{[a, n]} \otimes Q[a, b] \to R_{[c, d, n]} \otimes Q_0[c, d]\) for \(1 \leq a \leq b \leq n\) and \(1 \leq c \leq d \leq n\). The composite map through \(P_{01}\) is \(\left(\tau^* \otimes \varphi\right)_{[a, b]}^{[c, d]}\) if \([a, n] \leftarrow [M(a, b, n)] \leftarrow [M(c, d, n)]\), and 0 otherwise; and the composite map through \(P_{01}\) is \(\left(\tau^* \otimes \varphi\right)_{[a, b]}^{[c, d]}\) if \([a, n] \leftarrow [M(a, c, d, n)] \leftarrow [M(c, d, n)]\) and \(d < n\), and 0 otherwise; the condition \(d < n\) comes because \(P_{01}\) contains only summands \(R_{[c, n]} \otimes Q_0[c, d]\) with \(1 \leq c \leq d < n\). In both cases, the map vanishes unless \([c, d] \to [a, b]\) because of the factor \(\varphi_{[a, b]}^{[c, d]}\). We claim that if \([c, d] \to [a, b]\) and \(b < n\), then \([a, n] \leftarrow [M(a, b, n)] \leftarrow [M(c, d, n)]\) if and only if \([a, n] \leftarrow [c, n] \leftarrow [M(c, d, n)]\) and \(d < n\); here \(M(a, b, n) = b + 1\) because \(b < n\). Indeed, if \(d = n\), then \(M(c, d) = c\), and \([a, n] \leftarrow [b + 1, n]\) means \(a \leq b + 1 \leq c\), which contradicts \([c, n] \to [a, b]\). If \(d < n\), then \(M(c, d) = d + 1\). Then \([a, n] \leftarrow [c, n] \leftarrow [d + 1, n]\) and \([a, n] \leftarrow [M(a, b, n)] = [M(c, d, n)]\) are equivalent to \(a \leq c \leq d + 1\) and \(a \leq b + 1 \leq d + 1\), respectively. If \([c, d] \to [a, b]\), both conditions say that we are in the case \(a \leq c \leq b \leq d\). The computations above show that the two maps \(P_{11} \to P_{00}\) are equal.

Now consider the two maps \(P_{10} \to P_0\). Its matrix coefficients are maps

\[
R_{[M(a, b, n)]} \otimes Q[a, b] \to R_{[c, d]} \otimes Q_0[c, d], \quad 1 \leq a \leq b \leq n, \quad 1 \leq c \leq d \leq n.
\]

As above, the composite maps through \(P_{00}\) and \(P_1\) are \(\left(\tau^* \otimes \varphi\right)_{[a, b]}^{[c, d]}\) or 0. For the maps through \(P_1\) and \(P_{00}\), the former case occurs if \([M(a, b, n)] \leftarrow [a, b] \leftarrow [c, d]\) or \([M(a, b, n)] \leftarrow [M(c, d, n)] \leftarrow [c, d]\), respectively. We may assume \([a, b] \leftarrow [c, d]\).
[c,d] and \([M(a,b),n] \leftarrow [c,d]\) because otherwise \(\tau_{[c,d]} = 0\) or \(\tau_{[M(a,b),n]} = 0\).

Under these assumptions, \([M(a,b),n] \leftarrow [a,b] \leftarrow [c,d]\) always holds by (5.7). And \([c,d] \rightarrow [M(a,b),n]\) implies \([M(a,b),n] \leftarrow [M(c,d),n] \leftarrow [c,d]\) because any natural transformation \(K_\ast(A[c,d]) \rightarrow K_\ast(A[e,n])\) for some \(1 \leq e \leq n\) factors through \(\tau_{[M(c,d),n]}\). So the two maps \(P_{10} \rightarrow P_0\) are equal as well.

□

Using also the resolution (5.1) of \(A\), we get the following \(\mathcal{E}_K^X\)-projective resolution of \(A\):

\[
P_{11} = \bigoplus_{b=1}^{n-1} R_{[b,n]} \otimes A[b+1,n] \rightarrow \bigoplus_{b=1}^n R_{[b,n]} \otimes A[b,n] \twoheadrightarrow A.
\]

The computation of the obstruction class in Theorem 4.8 starts with the following \(\mathcal{E}_K^X\)-projective resolution of length 1 in \(\mathfrak{R}^X\):

\[
(5.29) \quad \bigoplus_{b=1}^{n-1} R_{[b,n]} \otimes A[b+1,n] \rightarrow \bigoplus_{b=1}^n R_{[b,n]} \otimes A[b,n] \rightarrow A.
\]

Since \(R_{[b,n]} = i_b(C)\) in the notation of Section 2.2, we have \(R_{[b,n]} \otimes A[b,n] \cong i_b(A[b,n])\). The restriction of the second map in (5.29) to this direct summand is the one that corresponds to the identity map on \(A[b,n]\) under the isomorphism in (2.15). The first map in (5.29) restricted to the summand \(R_{[b,n]} \otimes A[b+1,n]\), is the difference of the two maps

\[
\begin{align*}
\tau^{[b,n]}_{[b+1,n]}: R_{[b,n]} \otimes A[b+1,n] & \rightarrow R_{[b+1,n]} \otimes A[b+1,n], \\
\text{id} \otimes \tau^{[b,n]}_{[b+1,n]} : R_{[b,n]} \otimes A[b+1,n] & \rightarrow R_{[b,n]} \otimes A[b,n],
\end{align*}
\]

where \(\tau^{[b,n]}_{[b+1,n]}\) denotes the inclusion of \(A[b+1,n]\) into \(A[b,n]\); we could have written \(i\) for \(\tau^{[b,n]}_{[b+1,n]}\) as in Figure 1. It is shown in Section 2.2 that this sequence is \(\mathcal{E}_K^X\)-exact. And it is easy to prove this directly.

The projective resolutions of Abelian groups in (5.16) imply that there is an \(\mathcal{E}_K^X\)-projective resolution

\[
\bigoplus_{[c,d] \rightarrow [b+1,n]} R_{[b,n]} \otimes Q[c,d] \twoheadrightarrow \bigoplus_{[c,d] \rightarrow [b+1,n]} R_{[b,n]} \otimes Q_0[c,d] \rightarrow \bigoplus_{b=1}^{n-1} R_{[b,n]} \otimes A[b+1,n].
\]

Splicing it with the resolution in (5.29) gives an \(\mathcal{E}_K^X\)-exact chain complex

\[
W_1 \rightarrow W_0 \rightarrow \bigoplus_{b=1}^n R_{[b,n]} \otimes A[b,n] \rightarrow A
\]

with

\[
W_i := \bigoplus_{b=1}^{n-1} \bigoplus_{[c,d] \rightarrow [b+1,n]} R_{[b,n]} \otimes Q_i[c,d], \quad i = 0, 1.
\]

Next we are going to compare the two \(\mathcal{E}_K^X\)-exact chain complexes built above. We are going to build maps \(\gamma_{ij}\) and \(\delta\) for \(0 \leq i, j \leq 1\) that make the following diagram
As usual, this is 

Then $\gamma$ for $1 \leq \tau \leq a$

Using this isomorphism implicitly, we let $P_{10} := \bigoplus_{1 \leq a \leq b < n} \mathcal{R}_0^{[M(a,b),n]} \otimes Q_0[a,b]$, $P_{11} := \bigoplus_{1 \leq a \leq b < n} \mathcal{R}_1^{[a,n]} \otimes Q_1[a,b]$ for $i = 0, 1$. The matrix coefficients of $\gamma_{00}$ are maps

$$\gamma_{00}^{e,[a,b]} : \mathcal{R}_0^{[M(a,b),n]} \otimes Q_0[a,b] \to \mathcal{R}_1^{[e,n]} \otimes A[e,n]$$

for $1 \leq a \leq b \leq n$ and $1 \leq e \leq n$. We let $\gamma_{00}^{e,[a,b]} = 0$ if $e \neq M(a,b)$. Let $e = M(a,b)$. Then $\gamma_{00}^{e,[a,b]}$ corresponds to a map

$$(\gamma_{00}^{e,[a,b]})^\flat : Q_0[a,b] \to K_* (\mathcal{R}_1^{[e,n]} \otimes A[e,n])$$

under the isomorphism (5.9). We have already used above that $\mathcal{R}_1^{[e,n]} \otimes A[e,n] \cong i_e(A[e,n])$; so

$$(5.32) \quad \mathcal{R}_1^{[e,n]} \otimes A[e,n] \cong A[e,n].$$

Using this isomorphism implicitly, we let

$$(\gamma_{00}^{e,[a,b]})^\flat := \left\{ \begin{array}{ll} \gamma_{01}^{e,[c,d],[a,b]} \circ f[a,b] : Q_0[a,b] \to K_* (A[a,b]) & \text{if } [a,b] = [c,d], \\ 0 & \text{otherwise.} \end{array} \right.$$ 

As usual, this is 0 unless $[a,b] \to [e,n]$. The map $\gamma_{01} : P_{01} \to W_0$ is given by a matrix of maps

$$\gamma_{01}^{e,[c,d],[a,b]} : \mathcal{R}_1^{[a,n]} \otimes Q_1[a,b] \to \mathcal{R}_1^{[e,n]} \otimes Q_0[c,d]$$

for $1 \leq a \leq b < n$, $1 \leq c < n$, and $1 \leq e \leq d \leq n$ with $[c,d] \to [e+1,n]$. We let

$$\gamma_{01}^{e,[c,d],[a,b]} := \left\{ \begin{array}{ll} (\tau_{[e,n]}^{[a,b]})^* \otimes id_{Q_0[a,b]} & \text{if } [a,b] = [c,d], \\ 0 & \text{otherwise}. \end{array} \right.$$ 

If $a = c$ and $b = d < n$, then $[c,d] \to [e+1,n]$ if and only if $a \leq e \leq b$, so that $\tau_{[c,n]}^{[a,b]} \neq 0$ in this formula.

The map $\gamma_{10} : P_{10} \to W_0$ is given by a matrix of maps

$$\gamma_{10}^{e,[c,d],[a,b]} : \mathcal{R}_1^{[M(a,b),n]} \otimes Q_1[a,b] \to \mathcal{R}_1^{[e,n]} \otimes Q_0[c,d]$$

for $1 \leq a \leq b \leq n$, $1 \leq c < n$, and $1 \leq c \leq d \leq n$ with $[c,d] \to [e+1,n]$. We let

$$\gamma_{10}^{e,[c,d],[a,b]} := \left\{ \begin{array}{ll} (\tau_{[e,n]}^{[M(a,b),n]})^* \otimes \varphi_{[a,b]}^{[c,d]} & \text{if } M(a,b) \leq c < M(c,d), \\ 0 & \text{otherwise}. \end{array} \right.$$ 

The map $\gamma_{11} : P_{11} \to W_1$ is given by a matrix of maps

$$\gamma_{11}^{e,[c,d],[a,b]} : \mathcal{R}_1^{[a,n]} \otimes Q_1[a,b] \to \mathcal{R}_1^{[e,n]} \otimes Q_1[c,d]$$

for $1 \leq a \leq b < n$, $1 \leq c < n$, and $1 \leq c \leq d \leq n$ with $[c,d] \to [e+1,n]$. We let

$$\gamma_{11}^{e,[c,d],[a,b]} := \left\{ \begin{array}{ll} (\tau_{[e,n]}^{[a,n]})^* \otimes id_{Q_1[a,b]} & \text{if } a = c, \ b = d, \\ 0 & \text{otherwise}. \end{array} \right.$$
The map $\delta$ is given by a family of maps $\delta_{b,c,d}: R_{[b,n]} \otimes Q_1[e,d] \to A$ for $1 \leq b < n$ and $[c,d] \to [b+1,n]$. These correspond to maps $\delta_{b,c,d}[1,c,d]: Q_1[e,d] \to K_*(A[b,n])$ by the isomorphism (5.9). We define $\delta$ so that the maps $\delta_{b,c,d}$ are the maps denoted by $\delta_{b,c,d}$ in Theorem 5.17.

Now we must prove that the squares in the diagram commute. We begin on the right, comparing the two maps $P_{00} \to A$. Its restrictions $R_{[M(a,b),n]} \otimes Q_0[a,b] \to A$ correspond to maps $Q_0[a,b] \to K_*(A[M(a,b),n])$ under the isomorphism (5.9). This map is $\tau_{[a,b]}^{[M(a,b),n]} \circ f[a,b]$ both for the direct boundary map $P_{00} \to A$ and for the map through $\bigoplus_{n=1}^{n} R_{[e,n]} \otimes A[e,n]$. So this square commutes.

Next we compare the two maps from $P_{10} \oplus P_{01}$ to $\bigoplus_{b=1}^{n} R_{[b,n]} \otimes A[b,n]$. We first consider the restriction to $P_{01}$, then to $P_{10}$. The matrix coefficients of the map on $P_{01}$ are maps $R_{[a,n]} \otimes Q_0[a,b] \to R_{[e,n]} \otimes A[e,n]$ for $1 \leq a < b < n$ and $1 \leq e < n$. Such maps correspond to group homomorphisms $Q_0[a,b] \to K_*(R_{[e,n]}[a,n] \otimes A[e,n])$. Recall that $R_{[e,n]}[a,n] = \mathbb{C}$ if $a < e$ and 0 otherwise. So we may assume without loss of generality that $a \leq e$, and then we get corresponding maps $Q_0[a,b] \to K_*(A[e,n])$.

The map $\gamma_0$ picks out the summand with $e = M(a,b) = b + 1$, and $(\tau_{[a,b]}^{[M(a,b),n]})^*$ induces the identity map $\mathbb{Z} \cong K_*(R_{[e,n]}([b+1,n])) \to K_*(R_{[b+1,n]}([b+1,n])) \cong \mathbb{Z}$. Therefore, the map in the square through $\gamma_0$ contributes the map

$\delta_{e,b+1,[a,n]} \circ f[a,b]: Q_0[a,b] \to K_*(A[e,n])$.

When we map through $\gamma_0$ instead, then we first map $R_{[a,n]} \otimes Q_0[a,b]$ to the direct sum of $R_{[a,n]} \otimes Q_0[a,b]$ over all $g \in [a,b]$ using $\tau^* \otimes \text{id}$ and then apply the boundary map on $W_0$. This gives a contribution in $K_*(A[e,n])$ if $g = e$ or $g = e - 1$, and these two contributions cancel each other for $a < e \leq b$. For $e = b + 1$, we get the same term as for the map that goes through $P_{00}$. And we get 0 for $e = a$ because $\tau_{[a+1,n]}^{[a,n]} \circ f_{[a,b]} = 0$. So the two maps are equal on $P_{01}$.

The matrix coefficients of the two maps on $P_{10}$ are maps $R_{[M(a,b),n]} \otimes Q_1[a,b] \to R_{[e,n]} \otimes A[e,n]$ for $1 \leq a \leq b \leq n$ and $1 \leq e \leq n$. As above, we may assume $M(a,b) \leq e$ because otherwise any such map is zero. And then maps $R_{[M(a,b),n]} \otimes Q_1[a,b] \to R_{[e,n]} \otimes A[e,n]$ correspond to maps $Q_1[a,b] \to K_*(A[e,n])$. We shall examine the difference of the map through $P_{00}$ and the map through $W_0$. We first consider the map through $P_{00}$. It first applies the matrix $\tau^* \otimes \varphi$, going to the direct sum of $R_{[M(e,d),n]} \otimes Q_0[c,d]$ for $1 \leq c \leq d \leq n$. The map $Q_1[a,b] \to K_*(A[e,n])$ for the composite map through $R_{[M(e,d),n]} \otimes Q_0[c,d] = \delta_{M(e,d),e} \tau_{[c,d]}^{[M(e,d),n]} \circ f[c,d] \circ \varphi_{[a,b]}$.

So we get the sum of these terms over all $1 \leq c \leq d \leq n$. When we apply $\gamma_0: P_{10} \to W_0$, then we apply the maps $(\tau_{[g,n,b]}^{[M(a,b),n]})^* \otimes \varphi_{[a,b]}$ to the direct summands $R_{[g,n]} \otimes Q_0[c,d]$ of $W_0$, where $1 \leq c \leq d \leq n$ and $1 \leq g < n$. Suppose $[c,d] \to [g+1,n]$ and $M(a,b) \leq g < M(c,d)$. The condition $[c,d] \to [g+1,n]$ is equivalent to $c > g$ if $d = n$ and $c \leq g < d + 1$ if $d < n$. So the set of $g$ that are allowed is an interval $[x,y]$ or empty. The upper bound is always $y := M(c,d) - 1$. The lower bound $x$ is $M(a,b)$ if $d = n$ or the maximum of $c$ and $M(a,b)$ if $d < n$. By convention, we redefine $x := M(c,d)$ if the lower bound is bigger than $M(c,d)$. So $g$ runs through the interval $[x,y]$ if $x \leq y$, and otherwise $x = y + 1 = M(c,d)$ and the set of possible $g$ is empty.

We must compose $\gamma_0$ with the boundary map on $W_0$. As above, this only contributes to the map $R_{[M(a,b),n]} \otimes Q_1[a,b] \to R_{[e,n]} \otimes A[e,n]$ if $g = e$ or $g = e - 1$. And the contribution to the corresponding map $Q_1[a,b] \to K_*(A[e,n])$ is $-\tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}$ if $g = e$ and $+\tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}$ if $g = e - 1$. The contributions for $g = e$ and $g = e - 1$ cancel if both occur. Therefore, when we sum over all $g$ in the interval $[x,y]$ above, we get $-\tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}$ if $e = x$, $\tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}$.
if $c = y + 1$, and 0 otherwise. So we get the map

$$
\delta_{x,y+1} - \delta_{x,x} \cdot \tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}^{[c,d]} : Q_1[a,b] \to K_n(A[e,n]).
$$

This formula remains correct if no $g$ is allowed because then $x = y + 1$. Since $y + 1 = M(c,d)$, the map involving $\delta_{x,y+1}$ is equal to the one that we get from the map through $P_0$. So when we take the difference of the two maps in the square, this term is cancelled. We remain with

$$
\sum_{[c,d] \to [a,b]} \delta_{x,x} \cdot \tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}^{[c,d]},
$$

where $x$ depends on $a, b, c, d$ as above. Recall that we only need the case $c \leq M(a,b)$. We are going to prove that the sum in (5.33) vanishes under this assumption. First assume first that we have to study the lower bound

$$
\tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}^{[c,d]}(5.13).
$$

We may assume $a = b$. If $a \leq c \leq b < d < e < n$, then $M(c,d) = c$ and so $x = e$. If $c + 1 \leq a \leq b + 1 \leq d = n$, then $d < n$. So $M(c,d) = d + 1$ and $x = a$ as well. Therefore, $x = a$ and $e,n = [a,b]$ whenever $b = n$. In this case, the sum in (5.33) vanishes because of (5.13).

Now assume $b < n$, so $M(a,b) = b + 1$. If $a \leq c \leq b < d = n$, then $M(c,d) = c < b + 1$. So $x = M(c,d)$ and the summand in (5.33) vanishes because $e = x < M(a,b)$. And $\tau_{[a,b]}^{[b+1,n]} \cdot \tau_{[c,d]}^{[b+1,n]} = 0$ as well. If $a \leq c \leq b \leq d < n$, then $M(c,d) = d + 1$ and $x = b + 1$. In this case, we have $\tau_{[c,d]}^{[b+1,n]} = \tau_{[a,b]}^{[b+1,n]} \cdot \tau_{[c,d]}^{[b+1,n]} \neq 0$. So we may rewrite the sum in (5.33) as

$$
\sum_{[c,d] \to [a,b]} \delta_{x,x} \cdot \tau_{[c,d]}^{[e,n]} \circ f[c,d] \circ \varphi_{[a,b]}^{[c,d]} = \sum_{[c,d] \to [a,b]} \tau_{[a,b]}^{[b+1,n]} \cdot \tau_{[c,d]}^{[b+1,n]} \circ f[c,d] \circ \varphi_{[a,b]}^{[c,d]},
$$

and this vanishes by (5.13). This proves the vanishing of (5.33) in all cases and finishes the proof that the square of maps $P_{01} \oplus P_{10} \to \bigoplus \mathcal{R}_{[c,d]} \otimes A[e,n]$ commutes.

Next, we consider the maps $P_{11} \to W_0$. We look at the matrix coefficient $\mathcal{R}_{[c,d]} \otimes Q_1[a,b] \to \mathcal{R}_{[c,d]} \otimes Q_0[c,d]$ of the maps $P_{11} \to W_0$ through $W_1, P_{10}$ and $P_{01}$ for fixed $1 \leq a \leq b < n, 1 \leq e < n, [c,d] \to [e + 1,n]$. In $W_1$, we have summands $\mathcal{R}_{[c,d]} \otimes Q_1[a,b]$ for those $e$ with $[a,b] \to [e + 1,n]$, which is equivalent to $a \leq e < b$ because $b < n$. The map $\gamma_{11}$ maps the summands $\mathcal{R}_{[c,d]} \otimes Q_1[a,b]$ in $P_{11}$ to each of these summands through $\tau_{[a,b]}^{[e,n]} \otimes \text{id}$. The boundary map $W_1 \to W_0$ is obtained by tensoring the boundary map in (5.16) with $\text{id} \mathcal{R}_{[c,d]}$. So we get the contribution $\tau_{[a,b]}^{[e,n]} \otimes \varphi_{[a,b]}^{[c,d]}$ to our matrix coefficient if $[c,d] \to [a,b] \to [e + 1,n]$, and 0 otherwise. The boundary map to $P_{10}$ maps the summands $\mathcal{R}_{[c,d]} \otimes Q_1[a,b]$ in $P_{11}$ to the summand $\mathcal{R}_{[b+1,n]} \otimes Q_1[a,b]$ through $\tau_{[a,b]}^{[b+1,n]} \otimes \text{id}$. When we continue with $\gamma_{10}$, we get the contribution $\tau_{[a,b]}^{[e,n]} \otimes \varphi_{[a,b]}^{[c,d]}$ to our matrix coefficient if and only if $M(a,b) \leq c < M(c,d)$. The map through $P_{01}$ gives the contribution $-\tau_{[a,b]}^{[e,n]} \otimes \varphi_{[a,b]}^{[c,d]}$ to our matrix coefficient if $d < n$ and $a \leq c \leq e$ (recall that the summands $\mathcal{R}_{[c,d]} \otimes Q_0[c,d]$ of $P_{01}$ only run over $1 \leq d < n$).

We must show that the sum of these terms is 0. Again we look at different cases regarding the order among $a, b, c, d, e$. We may assume $b < n$ and $[c,d] \to [e + 1,n]$ because our matrix coefficient is only defined in this case. And we may assume $[e,n] \to [a,n]$ and $[c,d] \to [a,b]$ because otherwise $\tau_{[a,b]}^{[e,n]} \otimes \varphi_{[a,b]}^{[c,d]} = 0$. Besides $b < n$, these assumptions mean, first, that $e + 1 \leq c \leq n = d$ or $c + 1 \leq e + 1 \leq d + 1 \leq n$ holds; secondly, $c \leq e$; and, thirdly, $a \leq c \leq b < d$ or $c + 1 \leq e \leq d + 1 \leq b$. Assume first that $d = n$. Then our assumptions imply $a \leq e < c \leq b < d = n$.
In this case, none of the three maps $P_{11} \to W_0$ give a non-zero contribution because $[c, d] \to [a, b] \to [a, n]$ is impossible, $M(a, b) = b + 1 > c = M(c, d)$ and $d = n$. So we may assume $d < n$ from now on. If $c + 1 \leq a$, then it follows that $c + 1 \leq a \leq b < d + 1 \leq b < n$. Again, none of the three maps $P_{11} \to W_0$ give a non-zero contribution in this case. So we may assume $a \leq c$. Then either $a \leq c \leq e \leq b < d < n$ or $a \leq c \leq e \leq d < n < b$. In the first case, the maps through $W_1$ and $P_{01}$ give contributions that cancel each other, and the map through $P_{10}$ gives no contribution because $e < M(a, b) = b + 1$. In the second case, the maps through $P_{10}$ and $P_{01}$ give contributions that cancel each other, and the map through $W_1$ vanishes because $b < e$. Hence we get 0 in all cases, as needed. This finishes the proof that the square of maps $P_{11} \to W_0$ commutes.

Finally, we compute the composite map $\delta \circ \gamma_{11} : P_{11} \to A$ in our commuting diagram. Consider the restriction to $\mathcal{R}_{[a, n]} \otimes Q_1[a, b]$ for some $1 \leq a \leq b < n$. This map corresponds to a map $Q_1[a, b] \to K_n(A[a, n])$ by (5.9). The map $\delta^p_{c, [c, d]} = \delta^p_{c, [a, b]}$ vanishes unless $e = c$, and the matrix coefficient $\gamma_{00}^{c, [c, d], [a, b]}$ vanishes for $[a, b] \neq [c, d]$ and is the identity map if $[a, b] = [c, d]$. So the composite map corresponds simply to the map

$$
\sum_{[c, d] \to [a, n] \to [a, b]} \gamma_{[a, n]}^{[c, d]} \circ f_{c, d} \circ \varphi_{[c, d]} : Q_1[a, b] \to K_n(A[a, n]).
$$

This is exactly the formula for the obstruction class in (5.27). This finishes the proof of Theorem 5.17.

5.2. The case of extensions. We now specialise to the case $n = 2$. Then an object of $\mathfrak{R}X$ is equivalent to a $C^*$-algebra extension

$$
I \hookrightarrow A \twoheadrightarrow A/I,
$$

where $I = A[2]$, $A = A[1, 2]$ and $A/I = A[1]$ and the maps are those in (5.3). Here we abbreviate $[1] = \{1, 1\}$ and $[2] = \{2, 2\}$. The filtrated K-theory is the six-periodic exact chain complex

$$
\begin{align*}
\textbf{K}_0(I) & \xrightarrow{i^*} \textbf{K}_0(A) \xrightarrow{r^*} \textbf{K}_0(A/I) \\
\textbf{K}_1(A/I) & \xleftarrow{r_*} \textbf{K}_1(A) \xleftarrow{i_*} \textbf{K}_1(I)
\end{align*}
$$

(5.34)

The morphisms between the filtrated K-theory invariants are grading-preserving chain maps (morphisms of six-term exact sequences).

The invariant in Theorem 2.12 is the KK-class $[i] \in KK_0(I, A)$. The invariant in Theorem 4.8 is the induced map $i_* : K_*(I) \to K_*(A)$, together with the obstruction class. To compute the latter, let $i^- \in \text{Ext}(K_{1+}(I), K_n(A))$ be the parity-reversing part of $[i]$ in the Universal Coefficient Theorem for $KK_0(I, A)$. The obstruction class is the image of $i^-$ in the cokernel of the map

$$
\begin{align*}
\text{Ext}(K_{1+}(I), K_n(I)) & \oplus \text{Ext}(K_{1+}(A), K_n(A)) \to \text{Ext}(K_{1+}(I), K_n(A)), \\
(l_1, t_A) & \mapsto i^* \circ l_1 + t_A \circ i.
\end{align*}
$$

(5.35)

It follows from our theory that $i_*$ and the image of $i^-$ in the cokernel of (5.35) determine an object of $\mathfrak{R}X$ uniquely up to $KK^X$-equivalence. The cokernel comes in because there are isomorphisms of $C^*$-algebra extensions that act identically on $K_*(I)$ and $K_*(A)$, but have non-trivial components in $\text{Ext}(K_{1+}(I), K_n(I))$ or $\text{Ext}(K_{1+}(A), K_n(A))$. So the isomorphism class of an object in $\mathfrak{R}X$ does not determine $i^-$ uniquely. Only its image in the cokernel of (5.35) is unique. And our
theory shows that isomorphism classes of pairs consisting of \(i_\ast \in \text{Hom}(K_\ast(I), K_\ast(A))\) and an element in the cokernel of (5.35) are in bijection with isomorphism classes of objects in the bootstrap class \(\mathcal{B}^\infty \subseteq \mathcal{A}^\infty\).

We are going to compare this classification result with the filtrated K-theory classification by the long exact sequences in (5.34). The long exact sequence in (5.34) contains \(i_\ast\) and the extension

\[
(5.36) \quad \text{coker}(i:\ K_\ast(I) \to K_\ast(A)) \to K_\ast(A/I) \to \ker(i:\ K_{\ast+1}(I) \to K_{\ast+1}(A)),
\]

and we may reconstruct the long exact sequence from these two pieces. Two extensions as in (5.36) have the same class in Ext if and only if the long exact sequences associated to them (for the same \(i_\ast\)) are isomorphic with an isomorphism that is the identity on \(K_\ast(I)\) and \(K_\ast(A)\). Therefore, the filtrated K-theory invariant is equivalent to the pair consisting of \(i_\ast: K_\ast(I) \to K_\ast(A)\) and a class in \(\text{Ext}(\ker(i_\ast), \text{coker}(i_\ast))\). Now the following proposition clarifies the relationship between our different invariants:

**Proposition 5.37.** The cokernel in (5.35) is naturally isomorphic to the group

\[
\text{Ext}^2_{\mathcal{B}^\infty}(\Sigma A, A) \cong \text{Ext}(\ker(K_{\ast+1}(I) \xrightarrow{i_\ast} K_{\ast+1}(A)), \text{coker}(K_\ast(I) \xrightarrow{i_\ast} K_\ast(A))).
\]

And the obstruction class is the class that corresponds to minus the extension in (5.36).

**Proof.** Let \(X\) be an Abelian group. By the long exact sequence for Hom and Ext, the restriction map \(\text{Ext}(K_{\ast+1}(A), X) \to \text{Ext}(i_\ast(K_{\ast+1}(I)), X)\) is surjective and

\[
\cdots \to \text{Ext}(i_{\ast}(K_{\ast+1}(I)), X) \to \text{Ext}(K_{\ast+1}(I), X) \to \text{Ext}(\ker(i_\ast: K_{\ast+1}(I) \to K_{\ast+1}(A)), X) \to 0
\]

is exact. Hence the cokernel of the map \(i^\ast: \text{Ext}(K_{\ast+1}(A), X) \to \text{Ext}(K_{\ast+1}(I), X)\) is \(\text{Ext}(\ker(i_\ast), X)\). If we let \(X := K_\ast(A)\), then we may identify the cokernel of the map in (5.35) with the cokernel of the map

\[
\text{Ext}(K_{\ast+1}(I), K_\ast(A)) \xrightarrow{\text{restrict}} \text{Ext}(\ker(i_{\ast}), K_\ast(I)) \xrightarrow{i_\ast} \text{Ext}(\ker(i_\ast), K_\ast(A)).
\]

Since the first map is surjective, this is equal to the cokernel of

\[
i_\ast: \text{Ext}(\ker(i_{\ast}), K_\ast(I)) \to \text{Ext}(\ker(i_{\ast}), K_\ast(A)).
\]

A variant of the proof above for the second variable identifies this cokernel with the group \(\text{Ext}(\ker(i_{\ast}), \text{coker}(\delta_{\ast}))\) as claimed. Given a class \(\delta \in \text{Ext}(K_{\ast+1}(I), K_\ast(A))\), the map to \(\text{Ext}(\ker(i_{\ast}), \text{coker}(i_{\ast}))\) simply applies the bifunctoriality of Ext for the quotient map

\[
K_\ast(A) \to \text{coker}(i_{\ast}: K_\ast(I) \to K_\ast(A))
\]

and the inclusion map

\[
\ker(i_{\ast}: K_{\ast+1}(I) \to K_{\ast+1}(A)) \to K_{\ast+1}(I).
\]

It remains to compute the image of the obstruction class in Theorem [5.17] in \(\text{Ext}(\ker(i_{\ast}), \text{coker}(i_{\ast}))\). We recall the data used in Theorem [5.17] in our special case. The semi-simple part of the filtrated K-theory consists of

\[
K_\ast(A[2])_{ss} \cong \frac{K_\ast(I)}{\delta(K_{\ast-1}(A/I))} \cong i_{\ast}(K_\ast(I)) \subseteq K_\ast(A),
\]

\[
K_\ast(A[1, 2])_{ss} \cong \frac{K_\ast(A)}{i_{\ast}(K_\ast(A))} = \text{coker}(i_{\ast}),
\]

\[
K_\ast(A[1])_{ss} \cong \frac{K_\ast(A/I)}{r_{\ast}(K_\ast(A))} \cong \delta(i_{\ast}(A/I)) = \ker(i_{\ast}) \subseteq K_{\ast-1}(I).
\]

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Our construction is based on free resolutions of these $\mathbb{Z}/2$-graded Abelian groups. In particular, we use a free resolution $Q_1[1] \twoheadrightarrow Q_0[1] \twoheadrightarrow K_s(A[2])$, $Q_1[1, 2] \twoheadrightarrow Q_0[1, 2] \twoheadrightarrow K_s(A[1, 2])$, $Q_1[1] \oplus Q_1[2] \twoheadrightarrow Q_0[1] \oplus Q_0[2] \twoheadrightarrow K_s(A[1])$, which contain the maps $(\tau f[a, b])^\#$ and $\varphi_{[a, b]}^{\varphi_{[a, b]}}$. We shall need the maps in the third extension and put them into a larger diagram, which commutes because of the construction of the maps $(\tau f[a, b])^\#$ and $\varphi_{[a, b]}^{\varphi_{[a, b]}}$:

$$
\begin{array}{ccc}
Q_1[1] \oplus Q_1[2] & \xrightarrow{\varphi_{[1]}^{[1]} \oplus \varphi_{[1]}^{[2]}} & Q_0[1] \oplus Q_0[2] \\
pr_1 & & pr_2 \\
Q_1[1] & \xrightarrow{f_1^{\varphi_{[1]}^{[1]}}} & Q_0[1] \\
r_* (K_s(A)) & \xrightarrow{\text{incl.}} & K_s(A/I) \\
& & \delta_* \\
& & \ker(i_*)
\end{array}
$$

The matrix coefficient $\delta([n,b])$ in Theorem 5.17 is defined only if $e = a \leq b < n$. In our case $n = 2$, there is only one such matrix coefficient, namely, $\delta([1]) : Q_1[1] \rightarrow K_s(A[1, 2]) = K_s(A)$. The sum defining it has only one summand, which is indexed by $[1, 2] \rightarrow [1, 2] \rightarrow [1]$. So

$$
\delta([1]) = f[1, 2] \circ \varphi_{[1]}^{[1,2]} : Q_1[1] \rightarrow K_s(A).
$$

We compose this map with the quotient map $K_s(A) \rightarrow \ker(i_*)$. The exact sequence (5.34) shows that $r_*$ induces an isomorphism from $\ker(i_*)$ onto $r_*(K_s(A)) \subseteq K_s(A/I)$. So we may as well compose with the map $r_* : K_s(A) \rightarrow r_*(K_s(A))$. The composite of the two maps in the top row in (5.38) is 0. Thus $r^* \circ \delta([1]) = - f[1] \circ \varphi_{[1]}^{[1]}$ (compare (5.13)). The commuting diagram (5.38) shows that the group extension of $\ker(i_*)$ in (5.36) belongs to the map $f[1] \circ \varphi_{[1]}^{[1]}$. The obstruction class belongs to the negative of this map.

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