ARITHMETIC PROPERTIES OF CUBIC AND BIQUADRATIC
THETA SERIES

LUCA GHIDELLI

Abstract. A cubic (resp. biquadratic) theta series is a power series whose
n-th coefficient is equal to 1 if n is a perfect cube (resp. fourth power) and
zero otherwise. We improve on a result of Bradshaw by showing that such
series is not a cubic (resp. biquadratic) algebraic number when evaluated at
reciprocals of integers. The proof relies on a “nested gaps technique” for linear
independence and on recent results by the author on Waring’s problem for
cubes and biquadrates.

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1. Introduction

In this paper we consider numbers of the form

\[ \theta_\ell(q) = \sum_{n=0}^{\infty} \frac{1}{q^{n^\ell}}, \]

for \( \ell \in \{3, 4\} \) and \( q > 1 \). These numbers can be thought as being values (at \( z = 1/q \))
of cubic/biquadratic generalizations of the well-known theta series \( \sum_{n=0}^{\infty} z^{n^2} \). As
usual for values of transcendental series, we expect that \( \theta_\ell(q) \) is transcendental at
algebraic inputs, possibly with some well-motivated exceptions. Our main result is
the following.

Theorem 1.1. Let \( \ell \in \{3, 4\} \), let \( q \geq 2 \) be an integer and suppose that \( \theta_\ell(q) \) is
algebraic. Then \( \deg \theta_\ell(q) \geq \ell + 1 \).

The proof of Theorem 1.1 is based on a variation of Bradshaw’s technique of nested
gaps for lacunary series. It also involves some delicate considerations about the
natural numbers that can (or cannot) be represented as sums of three nonnegative
cubes or as sums of four fourth powers. In Proposition 9.2 below we will quantify
the conclusion of Theorem 1.1 by providing a measure of linear independence for the $(\ell + 1)$-tuple $(1, \theta_\ell(q),\ldots, \theta_\ell(q)^\ell)$.

1.1. Notation. We will denote the set of nonnegative integers by \( \mathbb{N} := \{0, 1, \ldots\} \) and by \( \mathbb{N}_+ := \mathbb{N} \setminus \{0\} \) the set of positive integers. The notation \( \log \) will denote the natural logarithm and \( \log_2 \) will denote the logarithm in base 2.

2. Remarks on the method and comparison with the literature

To prove that a number is not algebraic, it is a common technique to seek for good rational approximations. Since \( \theta_\ell(q) \) is defined as a series, it is natural to approximate it by its truncations. However their relatively slow rate of convergence implies only that \( \theta_\ell(q) \) is irrational at integer inputs. The method of Bradshaw [4] improves on the above strategy when the series is “lacunary.” It is based on the construction of “nested gaps” and on the following easy observation.

Remark 2.1. Let \( S = \sum_{n \geq 0} s_n \) be a series for which a tail bound of the form

\[ \sum_{n \geq N} s_n \leq f(N) \]

is given. Suppose that for some \( K, n_0 \in \mathbb{N} \) we have \( s_{n_0 + i} = 0 \) for all \( 0 \leq i < K \) we say that the series \( S \) has a gap of length \( \geq K \) at \( n_0 \). When we have such a gap, the bound for the tail at \( n_0 \) can be improved to

\[ \sum_{n \geq n_0} s_n \leq f(n_0 + K). \]

By applying this method to the (lacunary!) series representation of \( \theta_\ell(q)^{\ell-1} \) Bradshaw was able to show [4, Theorem 2.0.1], for all integer \( \ell \), that \( \theta_\ell(q) \) is not an algebraic number of degree \( \ell \). To extend the non-algebraicity of \( \theta_\ell(q) \) up to degree = \( \ell \) one faces technical difficulties related to Waring’s problem (I thank Martin Rivard-Cooke for pointing this to me). More precisely, we need the existence of arbitrarily long sequences of consecutive integers none of which is a sum of \( \ell \) nonnegative \( \ell \)-th powers. This result was recently proved by the author [9, Thm. 1.1, 1.2, 8.8] for \( \ell \in \{3, 4\} \) and is open for \( \ell \geq 5 \). The aim of this article is to check that this, together with the consideration of suitable “mild” gaps (see section 3), is enough for the proof of Theorem 1.1. As a side note, we would like to remark that our lower bound for the size of gaps between sums of fourth powers, although growing to infinity, it does so very slowly. Therefore, it came with some surprise that these estimates are in fact good enough to have arithmetic consequences on the biquadratic theta series.

In the literature variants of the above series have been considered. The irrationality and nonquadraticity of classical theta values \( \theta_2(q) \) were studied by Duverney [7, 8]. Irrationality and irrationality measures of similar numbers have been considered in many works, such as [14, 5, 1]. Bézivin [3] proved the nonquadraticity of values of the more general Tschakaloff function \( T_q(z) = \sum_{n=0}^\infty z^n q^{-n(n-1)/2} \). The results of Bézivin have been simplified by Bradshaw [4, Chapter 3] and extended by some authors [12]. Last but not least, a celebrated result of Nesterenko [13] implies that \( \theta_2(q) \) is transcendental for all nonzero algebraic \( q \) satisfying \( |q| > 1 \) [2, Theorem 4]. His proof relies on an appropriate multiplicity estimate and it exploits the differential Ramanujan identities between the quasi-modular functions \( E_2(q) \), \( E_4(q) \) and \( E_6(q) \).

3. A nested gaps principle for linear independence

In section 2 we mentioned that Bradshaw [4] took advantage of sufficiently large gaps for the series representation of \( \theta_\ell(q)^{\ell-1} \), and that he applied a certain “nested gaps” argument to prove his results. We are going to reproduce a variation of his technique by considering the series representation of \( \theta_\ell(q)^{\ell} \), for \( \ell \in \{3, 4\} \), and by considering only those “gaps” that are followed by coefficients with controlled size.
We call these gaps “mild” in Definition 3.2 below. Although we are ultimately interested in (non)-algebraicity properties of $\theta_3(q)$, a careful inspection reveals that Bradshaw’s method is more naturally seen as a lemma for linear independence of lacunary series. We think it is worthwhile to recast Bradshaw’s technique in this setting. However, we will not try to enunciate a criterion valid in maximal generality, in order not to obfuscate the underlying idea. We need a few definitions.

**Definition 3.1.** We define a $\frac{1}{2}$-function to be a powerseries $f(z) = \sum_{n \in \mathbb{N}} a_n z^n$ with integer coefficients that is absolutely convergent for all $|z| \leq 1/2$.

In particular, a $\frac{1}{2}$-function can be evaluated at reciprocals of integers $q \geq 2$.

**Definition 3.2.** Let $f(z) = \sum_{n \in \mathbb{N}} a_n z^n$ be a $\frac{1}{2}$-function, let $K \in \mathbb{N}_+$ and $E > 0$. We say that an index $n \in \mathbb{N}$ is a mild gap point for $f(z)$, with gap-length $\geq K$ and $K$-tail-norm $\leq E$, if $a_{n+k} = 0$ for all $0 \leq k < K$ and

$$\sum_{i=0}^{\infty} |a_{n+K+i}| 2^{-i} \leq E.$$

We denote by $\text{MildGap}(f(z); K, E)$ the set of such mild gap points for $f$.

The next theorem is the promised criterion, abstracted from Bradshaw’s method, for $\mathbb{Q}$-linear independence of the values $f(1/q)$, $g(1/q)$ of two lacunary $\frac{1}{2}$-functions at the reciprocal of an integer. It essentially states that the linear independence necessarily occurs when pairs of (large enough) mild gaps of $f$ can be found inside one (larger) gap of $g$. As Damien Roy pointed out to me, the proof also yields a measure of linear independence between $f(1/q)$ and $g(1/q)$. We explore this quantitative refinement in section 9.

**Theorem 3.3** (Nested Gaps Principle). Let $q \geq 2$ be an integer and let $f(z) = \sum_{n \in \mathbb{N}} a_n z^n$ and $g(z) = \sum_{n \in \mathbb{N}} b_n z^n$ be $\frac{1}{2}$-functions. Suppose that for every $H > 0$ there are positive integers $K_1 \leq K_2 < K' \in \mathbb{N}_+$, indices $n' \leq n_1 < n_2 \in \mathbb{N}$ and real numbers $E, E' > 0$ such that:

(i) $n_1 + K_1 < n_2$ and $n_2 + K_2 \leq n' + K'$;
(ii) $n_1, n_2 \in \text{MildGap}(f(z); K_1, E)$ and $n' \in \text{MildGap}(g(z); K', E')$;
(iii) $\sum_{n=n_1}^{n_2-1} a_n q^{-n} \neq 0$;
(iv) $q^{K_1} > HE$ and $q^{K_2} \geq HE'$.

Then either $g(1/q) = 0$ or $f(1/q)$ and $g(1/q)$ are linearly independent over $\mathbb{Q}$.

**Proof.** Suppose the contrary. Then there exist integers $\alpha, \beta$ such that $\alpha \neq 0$ and

$$0 = \alpha f(1/q) + \beta g(1/q) = \sum_{n \in \mathbb{N}} \frac{R(n)}{q^n},$$

where $R(n) := \alpha a_n + \beta b_n$. Let $H = \max\{|\alpha|, |\beta|\}$, then choose $K_1, K_2, K', E, E'$ and $n_1, n_2, n'$ as above. Now pick $i \in \{1, 2\}$ arbitrarily. By hypothesis (ii) and since $q \geq 2$ we have

$$\left| \sum_{n=n_i}^{\infty} \frac{R(n)}{q^n} \right| \leq |\alpha| \frac{E}{q^{n_1} + K_1} + |\beta| \frac{E'}{q^{n_2} + K_2}.$$

From the estimates (iv), eq. (3.1) and $n_1 + K_2 \leq n' + K'$, we deduce that

$$\sum_{n=0}^{n_i-1} \frac{R(n)}{q^n} < \frac{2}{q^{n_i}}.$$
However, the left-hand side of eq. (3.3) is a rational number with denominator at most $q^{n_i-1}$ and so it must be equal to zero. Having concluded this for both $n_1$ and $n_2$, we deduce that

$$0 = \sum_{n=n_1}^{n_2-1} R(n) \frac{R(n)}{q^n} = \alpha \sum_{n=n_1}^{n_2-1} a_n q^{-n},$$

against hypothesis (iii).

\section{Simple tail bounds}

In this section we present a pair of lemmas to estimate the “tail-norms” of a function when suitable bounds are known for its coefficients (see Definition 3.2).

\begin{lemma}
Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of numbers with $|a_n| \leq c(n+1)$ for all $n \in \mathbb{N}$ and some $c > 0$. Then for every $n_0 \in \mathbb{N}_+$ we have

$$\sum_{i=0}^{\infty} |a_{n_0+i}| 2^{-i} \leq 8cn_0.$$

\begin{proof}
The positive function $\psi(x) = x^2 - x$ satisfies $\psi(x) \geq \psi(2)$ for all $x \in [1,2]$ and it is monotone decreasing for $x > 1/\log 2 = 1.4427\ldots$, hence

$$\sum_{i=0}^{\infty} |a_{n_0+i}| 2^{-i} \leq 2^{n_0+1} \sum_{n \geq n_0} c(n+1) \frac{1}{2^{n+1}} \leq c2^{n_0+1} \int_{n_0}^{\infty} \frac{t}{2^t} dt.$$

By partial integration we obtain

$$\int_{n_0}^{\infty} \frac{t}{2^t} dt = \frac{2^{-n_0}}{\log 2} \left( n_0 + \frac{1}{\log 2} \right) \leq \left( \frac{1}{\log 2} + \frac{1}{(\log 2)^2} \right) n_0 \frac{2^{n_0}}{2^{n_0}} \leq 4n_0 2^{-n_0}.$$

Together with eq. (4.1), this gives the lemma.
\end{proof}

\begin{lemma}
Let $(a_n)_{n \in \mathbb{N}}$ be as in Lemma 4.1 for some $c > 0$, and let $\kappa, n_0 \in \mathbb{N}_+$ with $n_0 + \kappa \leq N$ and $\kappa \geq \log_2 N$ for some $N$. Suppose that for all $0 \leq i < \kappa$ and some $E \geq 8c$ we have $|a_{n_0+i}| \leq (3/2)^i E$. Then

$$\sum_{i=0}^{\infty} |a_{n_0+i}| 2^{-i} \leq 5E.$$

\begin{proof}
From $|a_{n_0+i}| \leq (3/2)^i E$ we get

$$\sum_{i=0}^{\kappa-1} |a_{n_0+i}| 2^{-i} \leq \sum_{i=0}^{\infty} \left( \frac{3}{4} \right)^i \cdot E = 4E.$$

On the other hand, by Lemma 4.1 and the various inequalities relating the constants, we have

$$\sum_{i=\kappa}^{\infty} |a_{n_0+i}| 2^{-i} \leq \frac{1}{2^\kappa} 8c(n_0 + \kappa) \leq \frac{1}{N} 8cN \leq E.$$

\end{proof}
\end{lemma}

\section{Linear independence of powers of $\theta_\ell$}

Fix $\ell \in \{3,4\}$. For all $s \in \{1,\ldots,\ell\}$ and $n \in \mathbb{N}$ we set

$$r_{\ell,s}(n) = \# \{ (n_1,\ldots,n_s) \in \mathbb{N}^s : n_1^\ell + \cdots + n_s^\ell = n \}$$

so that for all $q > 1$

$$\theta_\ell(q)^s = \sum_{n=0}^{\infty} \frac{r_{\ell,s}(n)}{q^n}.$$
We observe that $\theta_\ell(q)^s$ is the value at $1/q$ of the $\frac{1}{2^n}$-function

$$f_{\ell,s}(z) := \sum_{n=0}^{\infty} r_{\ell,s}(n)z^n$$

for all $\ell, s$. Therefore we may apply Theorem 3.3 to prove the following criterion.

**Proposition 5.1.** Let $q \geq 2$ be an integer. Suppose that for every $J > 0$ there are $E, N > 0$, integers $K_1 \leq K_2 \in \mathbb{N}$, and $n_1, n_2 \in \text{MildGap}(f_{\ell,\ell}; K_1, E)$ such that:

(i) $n_1 + K_1 < n_2$ and $n_2 + K_2 \leq N$;
(ii) $r_{\ell,\ell-1}(n) = 0$ for all $n \leq n_1 < n_2 + K_2$;
(iii) there exists $n_3 \in [n_1, n_2)$ with $r_{\ell,\ell}(n_3) > 0$;
(iv) $q^{K_1} > JE$ and $q^{K_2} > JN$.

Then either $\theta_\ell(q)$ is transcendental or it is algebraic with degree at least $\ell + 1$.

**Proof.** Suppose that $\theta_\ell(q)$ is algebraic of degree at most $\ell$. Then there exist integers $\alpha_0, \ldots, \alpha_\ell$ with $\alpha_\ell \neq 0$ such that

$$\alpha_0 + \alpha_1 \theta_\ell(q) + \cdots + \alpha_\ell \theta_\ell(q)^\ell = 0.$$ (5.1)

We define $f(z) := f_{\ell,\ell}(z)$ and

$$g(z) := \alpha_0 + \alpha_1 f_{\ell,1}(z) + \cdots + \alpha_{\ell-1} f_{\ell,\ell-1}(z).$$

We notice that for all $s \leq \ell$ and all $n \in \mathbb{N}$ we have the (loose) estimate

$$0 \leq r_{\ell,s}(n) \leq (\sqrt[n]{n} + 1)^s \leq 2^\ell (n + 1).$$ (5.2)

In particular for all $n \in \mathbb{N}$ the $n$-th coefficient of $g(z)$ has absolute value $\leq c(n + 1)$ where $c = \ell \cdot 2^\ell \cdot \max\{\alpha_i : i < \ell\}$. We also notice that for $n_1 \leq n < n_2 + K_2$ the condition (ii) implies that $r_{\ell,s}(n) = 0$ for all $s < \ell$, i.e. that the $n$-th coefficient of $g(z)$ vanishes. By Lemma 4.1, this means that $n_1 \in \text{MildGap}(g; K', E')$, where $K' = n_2 - n_1 + K_2$ and $E' = 8cN$. Moreover (iii) is equivalent to $\sum_{n=n_1}^{n_2} r_{\ell,\ell}(n)q^{-n} \neq 0$ because $r_{\ell,\ell}$ is nonnegative. Thus, for any $H > 0$, the hypotheses of the current proposition for $J = 8cH$ imply those of Theorem 3.3 with $n' = n_1$ and $E' = 8cN$. By eq. (5.1) the numbers $f(1/q)$, $g(1/q)$ are linearly dependent. But since $f(1/q) > 0$ and $\alpha_\ell \neq 0$ we also have $g(1/q) \neq 0$, so we arrive at a contradiction. \(\square\)

6. Sums of Powers Modulo M and Existence of Mild Gaps

By the previous proposition, Theorem 1.1 is reduced to the problem of finding suitable mild gaps of $f_{\ell,\ell}$. In this section we present a proposition that provides “many” mild gaps of a prescribed type. This result is proved via an elementary technique known as the Maier matrix method [11]. We require the following definition: for every $m \in \mathbb{Z}$ and $M \in \mathbb{N}$, let

$$r_{\ell,\ell}(m, M) := \{(x_1, \ldots, x_\ell) \in (\mathbb{Z}/M\mathbb{Z})^\ell : x_1 + \cdots + x_\ell \equiv m \mod M\}.$$ 

**Proposition 6.1.** Let $K, M, m \in \mathbb{N}$ with $m + K < M$. Now let $\epsilon_0, \ldots, \epsilon_K > 0$ such that $r_{\ell,\ell}(m + k, M) \leq \epsilon_k M^{\ell-1}$ for all $0 \leq k \leq K$ and let $E_0, \ldots, E_K \in \mathbb{N}$ such that $\alpha < 1$, where

$$\alpha := \frac{\epsilon_0}{E_0 + 1} + \cdots + \frac{\epsilon_K}{E_K + 1}.$$ 

Then for each $N > 0$ with $N \geq M^\ell$ we have

$$\# \left\{ n \in [0, N - K) \mid r_{\ell,\ell}(n + k) \leq E_k \text{ for all } 0 \leq k \leq K \right\} \geq \frac{1 - \alpha N}{2^\ell M}.$$
Therefore the 0 ≤ \alpha \leq The proposition follows because for each such K γ M for each large enough T from this we deduce that
\[ \#\{i \in [0, I) : r_{\ell,\ell}(m + k + iM) > E_k\} \leq \frac{\epsilon_k I}{E_k + 1}. \]

Therefore the 0 ≤ i < I such that r_{\ell,\ell}(m + k + iM) ≤ E_k for all 0 ≤ k ≤ K are at least
\[ (1 - \alpha)L^\ell M^{\ell - 1} = (1 - \alpha) \left( \frac{L}{L + 1} \right)^\ell \frac{(L + 1)^\ell M^\ell}{M} \geq 1 - \alpha N \frac{2^\ell}{M}. \]

The proposition follows because for each such i we have m + iM < N - K. □

7. Key results from Waring’s problem

In order to find mild gap points with gap-length K_1 using Proposition 6.1 it is crucial that we make r_{\ell,\ell}(m + k, M) as small as possible for \ell < K_1 and that we can estimate it from above for larger values of k.

Lemma 7.1. Let \ell \in \{3, 4\} and define the following auxiliary functions of T
\[ \kappa_3(T) := \frac{\sqrt{T}}{(\log T)^2} \quad \kappa_4(T) := \frac{\log \log T}{\log \log \log T} \quad \Xi_3(T) := \log \log T \quad \Xi_4(T) := 1. \]

For each large enough T there are natural numbers M, m, K_1, with max\{2m, 4K_1\} < M and M even, and positive constants C_0, C_1, C_2, C_3 such that:

(i) C_0 T ≤ log M ≤ C_1 T;
(ii) K_1 ≥ C_2 \cdot \kappa_\ell(T);
(iii) r_{\ell,\ell}(m + k, M) ≤ \frac{1}{2^\ell} \cdot M^{\ell - 1} for all 0 ≤ k < K_1;
(iv) r_{\ell,\ell}(m', M) ≤ e^{C_2 \Xi_\ell(T)} \cdot M^{\ell - 1} for all m' ∈ \mathbb{Z}.

Proof. We are going to follow the arguments of [9, Sec. 8] applied to the diagonal form F = x_1^\ell + \cdots + x_\ell^\ell. In [9, Prop. 8.1] we construct a set \mathcal{P}_\ell with positive density in the set of all primes p ≡ 1 (mod \ell). By the Prime Number Theorem the product M_T := \prod_{p \in \mathcal{P}_\ell} p of the primes p ∈ \mathcal{P}_\ell \cap [1, T] satisfies T ≪ \log M_T ≪ T. In [9, Sec. 8.4] we prove that there exist two natural numbers m, K_1 < M_T that fulfill condition (iii) provided that T ≥ \tau(\gamma_\ell, K_1), where [9, Def. 8.6]
\[ \tau_3(\gamma, K) := \gamma K^2 (\log K)^4 \quad \tau_4(\gamma, K) := \exp(\exp(\gamma K \log K)) \]

and \gamma_3, \gamma_4 > 0 are some absolute constants. If T is large enough, we may take K_1 so that C_2 \kappa_\ell(T) ≤ K_1 < \frac{4}{3} M_T for some small enough C_2. By [9, Prop. 8.2] with \mathcal{P}_1 = 0 and \mathcal{P}_2 = \mathcal{P}_\ell \cap [1, T] we have that for all m' ∈ \mathbb{Z} the inequality
r_{\ell,\ell}(m', M_T) ≤ \xi M_T^{\ell - 1} holds with \xi > 0 given by
\[ \log \xi = (\ell - 1)^\ell \sum_{p \in \mathcal{P}_\ell \cap [1, T]} p^{-(\ell - 1)/2}. \]

The sum to the right is again estimated via the Prime Number Theorem (see [9, Lemma 5.4]): if \ell = 3 this sum is \ll \log \log T; if \ell = 4 it is bounded. In both cases we get the estimate r_{\ell,\ell}(m', M) ≤ e^{C_3 \Xi_\ell(T)} \cdot M_T^{\ell - 1} for all m' ∈ \mathbb{Z} and some C_3.
Finally, we define $M := 2M_T$. All the statements in the lemma now follow because for every $m' \in \mathbb{Z}$ we have
\[ r_{\ell,\ell}(m', M) = r_{\ell,\ell}(m', 2)r_{\ell,\ell}(m', M_T) = 2^{\ell-1}r_{\ell,\ell}(m', M_T). \]

As we will see, the above lemma together with Proposition 6.1 implies that the series attached to $\theta_l(q)^\ell$ has gaps of arbitrarily large size. On the other hand, we need to produce two distinct such gaps inside a single gap attached to $\theta_l(q)^{\ell-1}$. The typical gap (in $[1, N]$) between sums of $\ell - 1$ perfect $\ell$-th powers is of size $\approx N^{1/\ell}$. Therefore we need to show that most gaps between sums of $\ell$ perfect $\ell$-th powers have size $\leq N^\gamma$ for some $\gamma < 1/\ell$. Such a result is easy to establish for $\ell = 3$ with the following greedy argument.

**Lemma 7.2.** For every $b \in \mathbb{N}$ there is $n \in (b - 25b^{8/27}, b)$ with $r_{3,3}(n) > 0$.

*Proof.* First notice that for every $B \in \mathbb{N}$ there is $x_1 \in \mathbb{N}$ such that $x_1^3 \leq B < (x_1 + 1)^3$. Such $x_1$ satisfies $B - x_1^3 \leq 6B^{2/3}$. Iterating this procedure, we find in turn $x_1, x_2, x_3 \in \mathbb{N}$ such that
\[ 0 \leq (B - x_1^3) - x_2^3 \leq 6(6B^{2/3})^{2/3} \text{ and } 0 \leq B - x_1^3 - x_2^3 - x_3^3 \leq 6^{1+2/3+4/9}B^{8/27} < 25B^{8/27}. \]

The lemma follows by choosing $B = b$ and $n = x_1^3 + x_2^3 + x_3^3$. □

As mentioned above, the crucial point is that $8/27 < 1/3$. The greedy argument above, for $\ell = 4$, only gives $x_1, x_2, x_3, x_4$ such that
\[ B - x_1^4 - x_2^4 - x_3^4 - x_4^4 = O(B^{(3/4)^4}) \]
and $(3/4)^4 = \frac{5184}{16384} > 1/4$. One way to overcome this problem is to prove the existence of suitable $x_1, \ldots, x_4$ via the so-called “circle method with diminishing ranges”, which might be thought as a (nontrivial) improved version of the greedy argument. Since the proof is technical, we perform the required computation in a separate paper [10]. In that article, we extend to sums of four powers a result of Daniel for sums of three cubes [6] and in particular we are able to show the following [10, Corollary 1.2].

**Lemma 7.3.** For almost every $a \in \mathbb{N}$ there is $n \in (a - a^{4059/16384} + \varepsilon, a]$ with $r_{4,4}(n) > 0$, where $\varepsilon > 0$ is arbitrary.

By “almost every $a$” in the above lemma we mean that for every $\varepsilon > 0$ and all $\delta \in (0,1)$ there is some $N_{\varepsilon,\delta} \in \mathbb{N}$ such that, for all $N \geq N_{\varepsilon,\delta}$, we have that the set
\[(7.1) \quad A_N := \{a \in [1, N]: r_{4,4}(n) = 0 \text{ for all } n \in (a - a^{4059/16384} + \varepsilon, a]\} \]
has cardinality $\# A_N \leq \delta N$.

8. Proof of Theorem 1.1

Fix $\ell \in \{3, 4\}$, an integer $q \geq 2$ and an arbitrary $J > 0$. Choose $\sigma_1 \in (3, \frac{47}{16})$ and $\sigma_2 \in (4, \frac{4613}{16384})$, then take $T = T(q, J, \sigma_1)$ large enough for the following arguments to be valid.

8.1. Choice of parameters. Given $T$, we choose $M, m, K_1, C_1$ as in Lemma 7.1, then we set $N = M^\sigma_1$ and $K_2 = \frac{1}{2}M > 2K_1$. We also define $\xi_3 = (\log T)^{\sigma_1}$ and $\xi_4 = \max(C_1, 32/3)$, and finally $E = 60\xi_4$. It is clear that the inequalities $q^{K_1} > JE$ and $q^{K_2} > JN$ hold if $T$ is large enough. In other words, condition (iv) of Proposition 5.1 is fulfilled.
8.2. A set of mild gap points. We apply Proposition 6.1 with $K = K_2$ and:

1. $\epsilon_k = \frac{1}{2^{k+1}}$ and $E_k = 0$ for $0 \leq k < K_1$;
2. $\epsilon_{K_1+k} = \xi_k$ and $E_{K_1+k} = 12\xi_k(3/2)^k$ for $0 \leq k \leq K_2 - K_1$.

In addition to $m + K_2 < \frac{1}{2}M + \frac{1}{2}M = M$ and $M^\ell < M^{\sigma_3} = N$, we have

$$
\alpha := \frac{\epsilon_0}{E_0 + 1} + \cdots + \frac{\epsilon_K}{E_K + 1} < K_1 + \frac{1}{2K_1} + \sum_{k=0}^{\infty} \frac{\xi_k}{12\xi_k(3/2)^k} = \frac{3}{4}
$$

So Proposition 6.1 provides a set

$$
\mathcal{B} = \{b_1 < b_2 < \ldots \} \subseteq [0, N - K_2) \cap (m + ZM)
$$

with cardinality $\#\mathcal{B} \geq N/(2^{\ell+2}M)$ such that $r_{\ell,n}(b_i + k) \leq E_k$ for all $0 \leq k \leq K_2$.

In particular, by condition (1) above we have that all elements of $\mathcal{B}$ are mild gap points for $f_{\ell,n}$ with gap-length $\geq K_1$. We recall from eq. (5.2) that $r_{\ell,n}(n) \leq 2^\ell(n+1)$ for all $n \in \mathbb{N}$.

Moreover we observe that

$$
12\xi_k \geq 8 \cdot 2^k \quad \text{and} \quad \kappa := K_2 - K_1 \geq K_1 \geq \log_2 N
$$

if $T$ is large enough. Therefore, by Lemma 4.2 and condition (2), every $b_i \in \mathcal{B}$ has $K_1$-tail-norm $\leq 5 \cdot 12\xi_k \leq E$. In other words, we have $\mathcal{B} \subseteq \text{MildGap}(f_{\ell,n}; K_1, E)$.

8.3. “Nested” pairs of mild gaps. We now seek to apply Proposition 5.1 to a pair of consecutive points $n_1 = b_1, n_2 = b_{i+1}$ from $\mathcal{B}$. We already argued that condition (iv) is satisfied by our choice of parameters. Condition (i) is fulfilled as well: $n_1 + K_1 < n_2$ because $b_i \equiv b_{i+1} \equiv m \pmod{M}$ and $K_1 < M$; while $n_2 + K_2 < N$ because $b_{i+1} \leq \max B < N - K_2$. In order to fulfill condition (ii) we need to exclude any $b_i$ from the set

$$
\mathcal{B}^{\text{bad}} := \{b_i \in \mathcal{B} : \exists n \in [b_i, b_{i+1} + K_1] \text{ with } r_{\ell,n-1}(n) \geq 1\}.
$$

Since $b_{i+1} + K_2 < b_{i+1} + M \leq b_{i+2}$ for all $i \leq \#\mathcal{B} - 2$, it is clear that

$$
\#\mathcal{B}^{\text{bad}} \leq 2 \sum_{n=0}^{N} r_{\ell,n-1}(n),
$$

which in turn is $\leq 2(\sqrt{N} + 1)^{\ell-1} \leq 2^\ell N^{1/\ell}$. On the other hand, $\#\mathcal{B} \geq 2^{-\ell-2}N^{1-1/\ell}$, so $\#\mathcal{B}^{\text{bad}} < (\#\mathcal{B})/2$ if $T$ (and so $N$) is sufficiently large. In particular, the complementary set $\mathcal{B}^{\text{good}} := \mathcal{B} \setminus \mathcal{B}^{\text{bad}}$ has cardinality at least $N/(2^{\ell+3}M)$. For every pair $(n_1, n_2) = (b_i, b_{i+1})$ with $b_i \in \mathcal{B}^{\text{good}}$, condition (ii) of Proposition 5.1 is fulfilled.

8.4. “Separated” pair of mild gaps. If $\ell = 3$ then every pair $(n_1, n_2) = (b_i, b_{i+1})$ with $b_i \in \mathcal{B}^{\text{good}}$ satisfies condition (iii) of Proposition 5.1. Indeed, recall that $n_1$ and $n_2$ are congruent (to $m$) modulo $M$, so $n_2 - n_1 \geq M$. By our choice of $\sigma_3$ we have

$$
25n_2^{8/27} \leq 25N^{8/27} < N^{1/\sigma_3} = M
$$

for every $T$ large enough, so the claim follows from Lemma 7.2. If $\ell = 4$ we define

$$
\varepsilon = \frac{1}{2} \left( \frac{4}{3} - \frac{4059}{16384} \right)
$$

and we consider the intervals of the form $I(a) := (a - a 16384^{-1/\varepsilon}, a]$, where $a$ is an element of the set $A \subseteq [1, N]$ given by

$$
A := \mathbb{N} \cap \bigcup_{b_i \in \mathcal{B}^{\text{good}}} [b_i + \frac{1}{2}M, b_i + M].
$$

We observe that $\frac{1}{2}M > N 16384^{1/\varepsilon}$ for every $T$ large enough, so each $I(a)$ with $a \in A$ is contained in an interval $(b_i, b_{i+1})$, for some $b_i \in \mathcal{B}^{\text{good}}$. Suppose that no pair $(n_1, n_2) = (b_i, b_{i+1})$ with $b_i \in \mathcal{B}^{\text{good}}$ satisfies condition (iii) of Proposition 5.1.
We present a quantitative version of the Nested Gaps Principle.

Then for every $a \in \mathcal{A}$ and every $n \in I(a)$ we have $r_{4,4}(n) = 0$: in other words, $\mathcal{A} \subseteq \mathcal{A}_N$, where $\mathcal{A}_N$ is as in eq. (7.1). However,

$$\# \mathcal{A} = \frac{1}{2} M \cdot (\#\mathcal{B}^{\text{good}}) \geq 2^{-\ell-4} N$$

and this contradicts Lemma 7.3, if $T$ is large enough.

8.5. Conclusion. For every $J > 0$ we proved the existence of $E, N, K_1, K_2$ and $n_1, n_2$ that meet all requirements of Proposition 5.1. Theorem 1.1 follows.

9. Measure of linear independence

We present a quantitative version of the Nested Gaps Principle.

**Proposition 9.1.** Let $f(z), g(z)$ and $q \geq 2$ be as in Theorem 3.3. Suppose there are positive integers $K_1 \leq K_2 < K' \in \mathbb{N}_+$, indices $n' \leq n_1 < n_2 \in \mathbb{N}$ and real numbers $E, E' > 0$ meeting all conditions (i)-(iv) of Theorem 3.3 for some $H > 0$. If $\alpha$ and $\beta$ are integers with $\alpha \neq 0$ and $|\alpha| + |\beta| \leq H$ then

$$|\alpha f(1/q) + \beta g(1/q)| \geq q^{-n_2}.$$

**Proof.** We let $R(n) := \alpha a_n + \beta b_n$ and for $i \in \{1, 2\}$ we write

$$S_i = \sum_{n=0}^{n_i-1} \frac{R(n)}{q^n}.$$

Since $\alpha \neq 0$ we have that $S_2 - S_1 \neq 0$ by conditions (ii) and (iii). Thus, there exists $i_0 \in \{1, 2\}$ such that $S_{i_0} \neq 0$. Since $S_{i_0}$ is a rational number with denominator $q^{-n_{i_0}+1}$, we have $S_{i_0} \geq q^{-n_{i_0}+1}$. On the other hand, as in the proof of Theorem 3.3 we get

$$\left| \sum_{n=n_{i_0}}^{\infty} \frac{R(n)}{q^n} \right| \leq \frac{\alpha E}{q^{n_{i_0}+K_2}} + \frac{\beta E'}{q^{n_{i_0}+K_1}} \leq q^{-n_{i_0}}.$$

Therefore

$$\alpha f(1/q) + \beta g(1/q) = S_{i_0} + \sum_{n=n_{i_0}}^{\infty} \frac{R(n)}{q^n} \geq \frac{q-1}{q^{n_{i_0}}} \geq q^{-n_2}.$$

□

From the above quantitative result we get the following measure of linear independence for the first powers of $\theta_\ell(q)$.

**Proposition 9.2.** Let $\ell \in \{3, 4\}$ and $\Theta := (1, \theta_\ell(q), \ldots, \theta_\ell(q)^{\ell}) \in \mathbb{R}^{\ell+1}$, where $q \geq 2$ is an integer. Let $P(T) = \sum_{j=0}^m \alpha_j T_j$ be a nonzero linear form with integer coefficients satisfying $|\alpha_j| \leq q^A$ for some $A > 1.1$ and $\alpha_\ell \neq 0$. If $\ell = 3$ we have

$$|P(\Theta)| \geq \exp(-\log q \exp(c_3 A^2 (\log A)^2))$$

for some $c_3 > 0$, while if $\ell = 4$ we have

$$|P(\Theta)| \geq \exp(-\log q \exp \exp(c_4 A \log A))$$

for some $c_4 > 0$.

**Proof.** We wish to apply Proposition 9.1 to the pair of $1/2$-functions

$$f(z) := f_\ell,3(z) \quad g(z) := \sum_{j=0}^{\ell-1} \alpha_j f_{\ell,j}(z).$$
We let $c = \ell 2^\ell$ and $J = 8c(q^A + 1)$. Then we set
\begin{align*}
T := c_j A^2 (\log A)^3 & \quad (\text{if } \ell = 3) \\
T := c_j \exp(\exp(c_j A \log A)) & \quad (\text{if } \ell = 4)
\end{align*}
for some $c_j > 0$ large enough and we choose $K_1, K_2, N, E$ as in section 8.1. The above formula for $T$ is chosen so that the inequality $q^{K_1} > J$ holds if $c_j > 0$ is larger than some absolute constant. Notice that if $c_j$ is large enough we also have the inequality $q^{K_2} > JN$. Moreover, all the arguments of sections 8.2 to 8.4 are valid for every $T$ larger than some $T_0$ independent of $q$ and $J$. In particular, if $c_j$ is large enough, there are some $n_1, n_2$ such that all the itemized conditions of Proposition 5.1 are fulfilled with this choice of $J, K_1, K_2, N, E$. As in the proof of Proposition 5.1 we then see that the hypotheses of Proposition 9.1 are fulfilled, with $n' = n_1, E' = 8cN$, $H = J/(8c)$ and $f(z), g(z)$ as in eq. (9.2). Since $n_2 < N$ and $\log N = O(T)$, we get from Proposition 9.1 the required estimate for $P(\Theta) = \alpha \ell f(1/q) + g(1/q)$, for some $c_\ell > 0$.

Notice that the hypothesis $\alpha \ell \neq 0$ on $P(T)$ is not restrictive. In fact, if $\alpha_{\ell+1-h} = \cdots = \alpha_{\ell} = 0$ for some $h \geq 1$, we have that $P(\Theta) = \theta_{\ell}(q)^{-h}P'(\Theta)$ where $P'(T) = \sum_{j=h}^{\ell} \alpha_{j-h} T_j$. We notice that $\theta_{\ell}(q) \leq \theta_{2}(2) \leq 2$ and so $|P'(\Theta)| \geq 2^{-\ell} |P'(\Theta)|$. Therefore the estimates of Proposition 9.2 still hold if we replace $c_\ell$ by some larger absolute constant. However, we remark that in this situation one could apply Proposition 9.1 to the pair of $\frac{1}{2}$-functions
\begin{align*}
f(z) := f_{\ell-h}(z) & \quad g(z) := \sum_{j=0}^{\ell-h-1} \alpha_j f_{\ell,j}(z).
\end{align*}
and obtain a measure of linear independence that is single-exponential in “$A$” (as opposed to Proposition 9.2, where the estimate is doubly or quadruply exponential).

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References

[1] M. Amou and M. Katsurada. Irrationality results for values of generalized Tschakaloff series. \textit{J. Number Theory}, 104(1):132–155, 2004.
[2] D. Bertrand. Theta functions and transcendence. \textit{Ramanujan J.}, 1(4):339–350, 1997.
[3] J. P. Bézivin. Sur les propriétés arithmétiques d’une fonction entière. \textit{Math. Nachr.}, 190:31–42, 1998.
[4] R. Bradshaw. Arithmetic properties of values of lacunary series. Master’s thesis, University of Ottawa, 2013.
[5] P. Bundschuh and I. Shiokawa. A measure for the linear independence of certain numbers. \textit{Results Math.}, 7(2):130–144, 1984.
[6] S. Daniel. On gaps between numbers that are sums of three cubes. \textit{Mathematika}, 44(1):1–13, 1997.
[7] D. Duverney. Propriétés arithmétiques d’une série liée aux fonctions thêta. \textit{Acta arithmetica}, 64:175–188, 1993.
[8] D. Duverney. Sommes de deux carrés et irrationalité de valeurs de fonctions têta. \textit{C. R. Acad. Sci. Paris Sér. I Math.}, 320:1041–1044, 1995.
[9] L. Ghidelli. Arbitrarily long gaps between the values of positive-definite cubic and biquadratic diagonal forms. (submitted), 2018.
[10] L. Ghidelli. On gaps between sums of four fourth powers. (submitted), 2019.
[11] A. Granville. Unexpected irregularities in the distribution of prime numbers. Proceedings of the International Congress of Mathematicians, 1:388–399, 1995.
[12] C. Krattenthaler, I. Rochev, K. Väänänen, and W. Zudilin. On the non-quadraticity of values of the $q$-exponential function and related $q$-series. Acta Arith., 136(3):243–269, 2009.
[13] Y. Nesterenko. Modular functions and transcendence problems. C. R. Acad. Sci. Paris Sér. I Math., 322(10):909–914, 1996.
[14] T. Stihl. Arithmetische Eigenschaften spezieller Heinescher Reihen. Math. Ann., 268(1):21–41, 1984.

150 Louis-Pasteur Private, Office 608, Department of Mathematics and Statistics, University of Ottawa, Ottawa ON K1N 9A7, Canada
E-mail address: luca.ghidelli@uottawa.ca