\( \mathcal{N} = 1 \) Supersymmetry, Deconstruction and Bosonic Gauge Theories

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Abstract

We show how the full holomorphic geometry of local Calabi-Yau threefold compactifications with \( \mathcal{N} = 1 \) supersymmetry can be obtained from matrix models. In particular for the conifold geometry we relate F-terms to the general amplitudes of \( c = 1 \) non-critical bosonic string theory, and express them in a quiver or, equivalently, super matrix model. Moreover we relate, by deconstruction, the uncompactified \( c = 1 \) theory to the six-dimensional conformal (2,0) theory. Furthermore, we show how we can use the idea of deconstruction to connect \( 4 + k \) dimensional supersymmetric gauge theories to a \( k \)-dimensional internal bosonic gauge theory, generalizing the relation between 4d theories and matrix models. Examples of such bosonic systems include unitary matrix models and gauged matrix quantum mechanics, which deconstruct 5-dimensional supersymmetric gauge theories, and Chern-Simons gauge theories, which deconstruct gauge theories living on branes wrapped over cycles in Calabi-Yau threefolds.

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1. Introduction

In a series of papers [1,2,3] we have advanced a connections between a large class of \( \mathcal{N} = 1 \) gauge theories in four dimensions and matrix models. This connection was motivated from insights coming from string theory [4,5,6,7,8]. Moreover, inspired by the string theory derivation of this result a direct field theory proof has been obtained from surprisingly simple computations [9], where one can see how diagram by diagram the field theory computation reduces to the combinatorics of planar diagrams of the corresponding matrix model. An alternative derivation has been given in [10] based on a generalization of the Konishi anomaly.

However, our original interest, that led to the works [1,2,3], was to find a direct connection between matrix models and non-critical bosonic strings on the one hand, and topological strings and \( \mathcal{N} = 1 \) supersymmetric gauge theories on the other. Since this motivation may not be familiar to many readers we will briefly review it now.

Non-critical bosonic strings were heavily investigated more than ten years ago with some very striking results. ‘Non-critical’ refers here to the fact that the dimension of the target space is not 26, but less than that. It was found that for dimension (central charge) less than or equal to 1, one can compute string amplitudes exactly using double scaling limits of matrix models. The limiting case of \( c = 1 \) was precisely the limit where the usual bosonic ‘tachyon field’ was still non-tachyonic — in fact massless. Many papers were devoted to studying non-critical bosonic strings with \( c = 1 \), the target space represented by a circle of radius \( R \) (including the decompactification limit \( R \to \infty \)). For a review see [11].

An interesting connection between non-critical strings and topological strings [12] in the context of twisted \( \mathcal{N} = 2 \) SCFT’s was found in [13] where using the string BRST complex, the \( c = 1 \) theories were mapped to \( \mathcal{N} = 2 \) theories with \( \hat{c} = 3 \), i.e. theories with the same central charge as supersymmetric sigma models on Calabi-Yau 3-folds. Further evidence for this correspondence was found in [4] where it was seen that the scaling properties of topological string amplitudes near a deformed conifold singularity are exactly the same as that for \( c = 1 \) non-critical strings.

Properties of non-critical bosonic string on a circle strongly depends on the radius of the circle. In particular it was shown in [14,15] that the string theory enjoys an infinite enhanced symmetry algebra in the target space at the self-dual radius. Moreover, it was shown that a real three-dimensional geometry captures the ring of observables of this theory [14,16]. This three-dimensional geometry is a real version of the deformed conifold.
The connection between $c = 1$ non-critical bosonic strings and $\mathcal{N} = 2$ superconformal field theories was made more concrete in [17] where it was shown that $c = 1$ on a circle of self-dual radius is given by an $\mathcal{N} = 2$ Kazama-Suzuki GKO coset model $SL(2)/U(1)$ at level 3. Later, it was shown in [18] that this corresponds to topological B-model on the Calabi-Yau threefold given by the deformed conifold. The chiral ring of the non-critical string in this context gets mapped to the ring of holomorphic functions on the manifold, which is the chiral ring of topological B-model. Note that it is crucial that observables of the B-model involve only holomorphic functions and so this can be mapped to the ring of observables of bosonic strings at self-dual radius.

On the other hand, in trying to embed the large $N$ topological string duality of [5] in superstrings one ends up with the result that the topological B-model of the conifold describes IR properties of a pure $\mathcal{N} = 1$ supersymmetric Yang-Mills [6]. Thus it was conjectured in [6] that the totality of F-terms of the $\mathcal{N} = 1$ theory of pure Yang-Mills is captured by non-critical bosonic strings with $c = 1$ at self-dual radius.

On the other hand, following a completely different path it was also found in [19] that the deformed conifold is equivalent in some limit to the IR properties of pure $\mathcal{N} = 1$ Yang-Mills. This was obtained by considering a non-conformal deformation of the $\mathcal{N} = 1$ AdS/CFT duality found in [20], initiated in [21,22]. However the pure $\mathcal{N} = 1$ emerged only at the end of a cascade of Seiberg-like dualities. In particular if one goes to higher energies, one finds a different gauge theory description with more degrees of freedom giving rise to an affine $\hat{A}_1$ quiver theory. The duality cascade was interpreted in [23] as affine Weyl reflections of $\hat{A}_1$ (corresponding to generalized flops).

One aim of this note is to realize the suggestion of [6], relating $\mathcal{N} = 1$ SYM and $c = 1$ strings, in terms of the $\mathcal{N} = 1$ affine $\hat{A}_1$ quiver theory. In other words, we argue that the F-terms of the affine $\hat{A}_1$ theory are equivalent to correlation functions of the non-critical bosonic string at the self-dual radius. Moreover, using the recent results [1,2,3] we can then show that this in turn is captured by a (quiver) matrix model. This leads to a new realization of $c = 1$ theory at self-dual radius in terms of a matrix model (not in the double scaling sense, but à la ’t Hooft). We also show that this model can be interpreted as a supermatrix model that naturally appears from a topological brane/antibrane system. Moreover we extend this dictionary by showing that the non-critical $c = 1$ bosonic string at $k$ times the self-dual radius captures the F-terms of an $\hat{A}_{2k-1}$ affine quiver theory. In particular $c = 1$ bosonic string on a non-compact line is related to affine $\hat{A}_\infty$ quiver theory.
This is indeed interesting as this latter theory in turn is related to the (2, 0) little strings by deconstruction in a certain range of parameters [24].

Another aim of this paper is to reconstruct the full Calabi-Yau threefold geometry from gauge theory. In particular with the affine quiver theories all the holomorphic correlations of the $\mathcal{N} = 1$ theory can be computed from matrix models. Given the equivalence of these with string theories, we can thus directly compute all the relevant holomorphic quantities of the string theory directly, and easily, using the matrix model. Moreover, one may naturally expect that this captures (at the conformal point) even the structure of the D-term and so it implicitly characterizes the full string theory. This is somewhat analogous to the two-dimensional program that F-term data in $\mathcal{N} = 2$ theories characterizes the SCFT completely (with a unique compatible D-term). We thus come up with a potential dramatical reformulation of the full string theory in terms of simple matrix models, in an implicit way.

We also investigate the generalization of matrix models to higher dimensional gauge theories and their meaning in the context of $\mathcal{N} = 1$ supersymmetric gauge theories in 4 dimensions. This is done by viewing the internal theory as an infinite collections of 4d multiplets coupled in a complicated, but computable, way. More generally, we show that any bosonic gauge theory in $k$ dimensions gives rise to some $\mathcal{N} = 1$ dynamics in 4d, perhaps with infinitely many massive multiplets, interacting in a complicated way. This turns out to be a powerful way to deconstruct higher-dimensional theories.

In particular we map matrix quantum mechanics to the deconstruction of $\mathcal{N} = 1$ supersymmetric Yang-Mills in 5 dimensions interacting with matter hypermultiplets. Furthermore we map Chern-Simons gauge theory in the real or holomorphic version, to the dynamics of gauge theory on D6 branes wrapped over 3-cycles or D9 brane wrapped over Calabi-Yau 3-fold respectively, directly from a 4-dimensional point of view. However the aim of using deconstruction in our context is somewhat different from that used in [25]. In particular our aim in relating the higher-dimensional theory to a 4-dimensional theory is not to provide a potential UV completion of the higher-dimensional theory, but rather reduce its F-term content to 4 dimensions where F-term computations are simple as exemplified by the connection between matrix models and supersymmetric gauge theories.

The organization of this paper is as follows: In section 2 we review some basic aspects of $c = 1$ non-critical strings. In section 3 we explain our proposal of how this is related to
a matrix model. In section 4 we show how to identify the relevant $\mathcal{N} = 1$ gauge theory. In section 5 we discuss generalizations of this construction; this includes generalizations to $c = 1$ non-critical string at multiples of self-dual radius and the decompactified limit. We show how the decompactified $c = 1$ bosonic string theory can be viewed as computing $F$-terms for certain deformation of $(2,0)$ superconformal theory in 6 dimensions. In section 6 we discuss application of deconstruction to arbitrary $k$-dimensional bosonic gauge theory and its interpretation in 4-dimensional terms, including deconstruction of pure $\mathcal{N} = 1$ Yang-Mills in 5 dimensions. In section 7 we discuss how these ideas may lead to the full reconstruction of string theory in certain backgrounds from matrix models.

2. $c = 1$ Non-Critical String Theory

In the continuum formulation the world-sheet theory of the $c = 1$ string consists of a single bosonic coordinate $X$ that can be compactified on a circle of radius $R$

$$X \sim X + 2\pi R.$$ 

Because this string is non-critical ($c \neq 26$) the conformal mode of the world-sheet metric does not decouple and becomes a second dynamical space-time coordinate $\varphi$, the Liouville field, making the target space effectively two-dimensional. The local world-sheet Lagrangian reads

$$\int d^2z \left( \frac{1}{2} (\partial X)^2 + (\partial \varphi)^2 + \mu e^{\sqrt{2} \varphi} + \sqrt{2} \varphi R^{(2)} \right)$$

together with a pair of $(b, c)$ diffeomorphism ghosts. Here the background charge for $\varphi$ gives total central charge $c = 26$ for the matter fields. The coupling $\mu$ plays the role of a world-sheet cosmological constant. Its presence makes the CFT strongly interacting and difficult to analyze, although for specific amplitudes the Liouville potential can be treated perturbatively in $\mu$.

2.1. Physical States and The Ground Ring

At arbitrary compactification radius $R$ the physical states of this string theory are mainly of two types. (Note that all operators are dressed by an appropriate Liouville vertex operator $e^{s\varphi}$ to give total scaling dimensions $(1, 1)$.) First, there are the tachyon momentum operators

$$T_k = e^{ikX/R}$$
that create the modes of the two-dimensional massless “tachyon” field $T(X, \varphi)$. Furthermore there are the winding modes

$$\tilde{T}_m = e^{im\tilde{X}R}$$

with $\tilde{X}$ is the dual scalar field, obtained by a T-duality from $X$, that is compactified on a circle of radius $1/R$. There are no mixed momentum/winding operators, since physical vertex operators should have equal left and right conformal dimensions $h = \overline{h}$. Apart from these tachyon modes there are also so-called discrete states $\tilde{T}_m$, that are most relevant at the self-dual radius.

At the self-dual radius $R = 1$ the $U(1) \times U(1)$ symmetry of the $c = 1$ conformal field theory is extended to a $SU(2) \times SU(2)$ affine symmetry. Under this extended symmetry the vertex operators $T_k$ become part of a spin $(k/2, k/2)$ multiplet of primary fields

$$O_{n_L, n_R}^{(k)}, \quad -k \leq n_L, n_R \leq k.$$  

The highest/lowest weight operators are related to the tachyon vertex operators as

$$O_{\pm k, \pm k}^{(k)} = T_{\pm k}, \quad \tilde{O}_{\pm k, \mp k}^{(k)} = \tilde{T}_{\pm k}.$$  

These physical states satisfy interesting algebraic relations. In any string theory physical vertex operators can be represented in two pictures: either as BRST-closed 0-forms on the world-sheet, or as (1,1)-forms that can be consistently integrated over the the world-sheet. In the former representation there is a natural ring structure, given by the operator product (modulo BRST exact terms) — the so-called ground ring. In the correspondence with twisted $\mathcal{N} = 2$ superconformal field theories this ground ring can be identified with the chiral ring.

As observed by Witten [14] in this case this ring has a simple geometric structure. Introduce two doublets $(a_1, a_2)$ and $(b_1, b_2)$ of the $SU(2) \times SU(2)$ symmetry group. Then the representations $O^{(k)}$ of spin $(k/2, k/2)$ is given by expressions $P(a)Q(b)$ where $P(a)$ and $Q(b)$ are polynomials of degree $k$. Stated otherwise, we can write the basis of string observables as

$$O_{n_L, n_R}^{(k)} = a_1^{n_L}a_2^{k-n_L}b_1^{n_R}b_2^{k-n_R}. \quad (2.1)$$

The ground ring is captured by introducing the four generators

$$x_{ij} = a_i b_j.$$
In the representation as $(1,1)$ forms they can be identified as gravitationally dressed versions of the minimal momentum/winding operators

$$
x_{11} = T_{+1}, \quad x_{12} = \tilde{T}_{+1},
$$

$$
x_{21} = \tilde{T}_{-1}, \quad x_{22} = T_{-1}.
$$

The generators $x_{ij}$ satisfy a single relation that at $\mu = 0$ reads

$$
det x_{ij} = x_{11}x_{22} - x_{12}x_{21} = 0.
$$

When considered with complex variables this affine quadric defines the conifold — a singular Calabi-Yau threefold. As has been argued in [14], in the $c = 1$ string with $\mu \neq 0$ this relation is generalized to

$$
x_{11}x_{22} - x_{12}x_{21} = \mu,
$$

(2.2)

which is the real version of the deformed conifold. Viewing $x_{ij}$ as complex variables would lead to a geometry diffeomorphic (as a real 6-manifold) to $T^* S^3$. This is not an accident. In fact as shown in [15] the topological B-model on the deformed conifold is equivalent to $c = 1$ non-critical strings at the self-dual radius. Turning on the general deformations $O_k$ will deform this geometry to a general affine hypersurface

$$
x_{11}x_{22} - x_{12}x_{21} + f(x_{11}, x_{22}, x_{12}, x_{21}) = \mu.
$$

(2.3)

This geometry can be seen as a general local CY geometry in the neighbourhood of a (deformed) conifold singularity. In that sense the $c = 1$ string captures the data of the general topological B-model on a local CY three-fold. Note that by the Morse lemma we can always put a local geometry of the form (2.3) locally in the canonical form (2.2). But of course such a transformation changes dramatically the behaviour at infinity, and therefore changes the physics.

3. Affine Quiver Gauge Theories and Supermatrix Models

The $N = 1$ gauge theory that we will be discussing in this section has been investigated in great detail, because it has a holographic dual given by IIB string theory on $AdS_5 \times T^{1,1}$ with fluxes [20]. Let us briefly recall the construction of the gauge theory and how it is obtained from D-branes in a conifold geometry.
3.1. The Supersymmetric Gauge Theory

One starts with a compactification of the IIB string theory on the singular conifold

\[ x_{11}x_{22} - x_{12}x_{21} = 0. \]

One then puts \( N \) D3 branes at the singularity. Their world-volumes completely fill the space-time \( \mathbb{R}^4 \). This brane configuration gives in the near-horizon or decoupling limit a superconformal quiver gauge theory with gauge group \( U(N) \times U(N) \) \cite{20}.

Before recalling the field content of this gauge theory, it is natural to consider a more general class of models. One can break the conformal symmetry by placing \( K \) additional D5 branes that wrap the \( \mathbb{P}^1 \) obtained by a small resolution of the conifold. In that case we obtain a \( \hat{A}_1 \) quiver gauge theory \cite{21,22} with gauge group

\[ U(N_+) \times U(N_-), \]

with the ranks of the two gauge groups given as

\[ N_+ = N + K, \quad N_- = N. \]

Furthermore this gauge theory contains the following chiral matter fields. There are two fields \( \Phi_+ \) and \( \Phi_- \) in the adjoint representation of \( U(N_+) \) and \( U(N_-) \) respectively. These are supplemented by two sets of bi-fundamental fields: \( A_1, A_2 \) transforming in the representation \((N_+, N_-)\), and \( B_1, B_2 \) transforming in the representation \((N_+, N_-)\). There is an obvious \( SU(2) \times SU(2) \) global symmetry acting on these bi-fundamentals, with the \( A_i \) transforming as one doublet and the \( B_i \) as another.

The tree-level superpotential for these fields is given by \cite{20}

\[ W = \frac{1}{2} \left( \text{Tr} \Phi_+^2 - \text{Tr} \Phi_-^2 \right) - \text{Tr} A_i \Phi_+ B_i + \text{Tr} B_i \Phi_- A_i. \]  \hspace{1cm} (3.1)

Integrating out the adjoint scalars produces the quartic superpotential

\[ W = \frac{1}{2} \text{Tr} (A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1). \]  \hspace{1cm} (3.2)

At the critical point of \( W \) we find the relations

\[ \Phi_+ = B_i A_i, \quad \Phi_- = A_i B_j. \]
and

\[ A_i \Phi_+ = \Phi_- A_i, \quad \Phi_+ B_i = B_i \Phi_- . \]

These relations allow us to express the adjoints \( \Phi_\pm \) in terms of the bifundamentals \( A_i, B_i \). Furthermore, the latter satisfy relations that can be written as

\[ A_1 B_j A_2 = A_2 B_j A_1, \quad B_1 A_j B_2 = B_2 A_j B_1. \]  

(3.3)

A basis of chiral operators of this gauge theory is given by expressions of the form

\[ \mathcal{O}_{i_1 \ldots i_k, j_1 \ldots j_k}^{(k)} = \text{Tr}(A_{i_1} B_{j_1} \cdots A_{i_k} B_{j_k}). \]  

(3.4)

Because of the relations (3.3) the tensor \( \mathcal{O}^{(k)} \) is completely symmetric in the \( i \) and the \( j \) indices. In terms of the \( SU(2) \times SU(2) \) global symmetry these operators therefore transform as a spin \((k/2, k/2)\) representation. In the superconformal point these fields have scaling dimension \( 3k/2 \) and R-charge \( k \). Besides these fields there are the corresponding combinations including the \( U(N_+) \times U(N_-) \) glueball superfield \( \mathcal{W}_\alpha \)

\[ \text{Tr} \left[ \mathcal{W}_\alpha (AB)^k \right], \quad \text{Tr} \left[ \mathcal{W}_\alpha \mathcal{W}_\alpha^\alpha (AB)^k \right]. \]

All these chiral fields have been identified under the holographic duality to IIB string theory on \( AdS_5 \times T^{1,1} \) [26].

The similarity of this set of gauge theory observables (3.4) to the observables (2.1) of \( c = 1 \) at self-dual radius [14] was noticed in this context by [27]. As we will explain below this is not accidental.

3.2. The Cascade

As explained in [19] the dynamics of this gauge theory is extremely rich. Under the RG flow alternatively the gauge coupling of one of the two gauge groups will go to strong coupling, and the theory will undergo a Seiberg-duality [28]. There is also another version of description of this duality, discussed in [23] which one keeps the adjoint fields and the cascade is described as an affine Weyl reflection. This description of the duality can be immediately implemented in the matrix model. We will discuss this later in the context of matrix model. Here we will review the Seiberg duality as applied to this case in [19].
Suppose that $N_+ > N_-$ and that the gauge group $U(N_+)$ becomes strongly coupled as we go towards the IR. So we will perform a Seiberg duality with $N_+$ colors and $2N_-$ flavors. The duality will then replace the strongly coupled gauge group as 

$$U(N_+) \rightarrow U(2N_- - N_+)$$

and replace the bifundamentals $A_i, B_j$ by new bifundamentals $a_i, b_j$. Apriori there is no relation between these fields. Furthermore there are new meson fields

$$M_{ij} = A_i B_j$$

where we have contracted the $U(N_+)$ indices. So the mesons $M_{ij}$ are neutral with respect to $U(N_+)$ but transform as adjoints under the ‘flavor gauge group’ $U(N_-)$. There is further a tree-level superpotential 

$$\text{Tr}(M^{ij} a_i b_j).$$

(Here $SU(2)$ indices are raised and lowered using $\epsilon_{ij}$). Plugging in these fields in the original Klebanov-Witten superpotential (3.2) gives 

$$W = \text{Tr} (M_{11} M_{22} - M_{12} M_{21} + M^{ij} a_i b_j)$$

It is simple to integrate out the meson fields, that only appear quadratically. After a shift

$$M_{ij} \rightarrow M_{ij} + a_i b_j$$

we obtain the dual superpotential

$$W = \text{Tr} (a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1).$$

This is the same quartic superpotential now written in terms of the new bifundamentals for the $U(2N_- - N_+) \times U(N_-)$ gauge theory.

We now want to see what happens to this argument when we have deformations of $W$ by arbitrary monomials in the $A_i, B_j$

$$W = \text{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) + \sum t_{i_1 \cdots i_k, j_1 \cdots j_k} \text{Tr}(A_{i_1} B_{j_1} \cdots A_{i_k} B_{j_k}).$$

Let us keep the variation of $W$ infinitesimal. That is, we will be interested in computing correlation function of chiral operators in the undeformed theory with the couplings $t = 0$. 

In this case the cascade goes through and we get the desired result by replacing the monomials in terms of the meson fields and then eliminating meson fields.

So, for example, in this way the operator $\text{Tr}(A_1B_1)^k$ gets replaced by $\text{Tr}M_{11}^k$. One then has to eliminate the meson fields using the above superpotential. To leading order the equations for the elimination of meson fields do not get modified, and we can simply replace

$$M_{ij} \to a_ib_j.$$ 

This means that we have the same set of chiral fields, but now written in terms of the new bifundamentals $a_i, b_j$. The same chiral operator now takes the form $\text{Tr}(a_1b_1)^k$. But beyond the leading order in the couplings $t_k$, it does change the basis of chiral fields. This implies that in the process of the Seiberg duality we will have operator mixing among the chiral fields.

A particular interesting case appears if we start with a gauge theory with rank

$$N_+ = 2N, \quad N_- = N.$$ 

If we apply the duality to this case, we are left with a pure $U(N)$ gauge theory. The other gauge factor has disappeared. So there are no longer new bifundamental fields $a_i, b_j$. The only dynamical fields are the mesons $M_{ij}$ that we recall transform as adjoints under the remaining gauge group $U(N)$. This critical case with $N_c = N_f$ is more subtle, since there are now baryon degrees of freedom, and a well-known quantum correction to the moduli space. Ignoring these effects, the superpotential for this model, including arbitrary deformations will be of the form

$$W = \text{Tr}(M_{11}M_{22} - M_{12}M_{21} + f(M_{ij})) \tag{3.5}$$

with $f$ an arbitrary function of the four meson fields $M_{ij}$. We will see in a moment how, using the relation of F-term computations in gauge theory to matrix models, this suggests a four-matrix model for this $U(N)$ gauge theory with four adjoint chiral fields.

Clearly, in the connection with the $c = 1$ string we want to identify the mesons fields $M_{ij}$ with the space-time coordinates $x_{ij}$ of the conifold, and relate the superpotential deformation (3.3) to the deformed conifold (2.3). Note also that the cascade at the level of gauge theory has been interpreted as flops in $[23]$ and thus at the topological level (i.e. the theory on topological branes) the cascade continues to hold for the matrix model.
3.3. The Quiver Matrix Model

According to the results of [1, 2, 3] the effective superpotential of the quiver gauge theory can be computed in terms of an associated quiver matrix model. More precisely, the effective superpotential, considered as a function of the glueball superfields $S_+$ and $S_-$ associated to the $U(N_+)$ and $U(N_-)$ gauge groups, takes the form

$$W_{\text{eff}}(S) = N_+ \frac{\partial F_0}{\partial S_+} + N_- \frac{\partial F_0}{\partial S_-} + 2\pi i (\tau_+ S_+ + \tau_- S_-),$$

where the function $F_0(S_+, S_-)$ is the planar free energy of the associated matrix model, and $\tau_\pm$ are the bare gauge couplings of the two gauge factors.

The corresponding random matrix model consists of the same field content as the gauge theory, but the ranks $M_+, M_-$ of the matrices $\Phi_+, \Phi_-$ are unrelated to the ranks $N_+, N_-$ of the gauge groups. In the 't Hooft limit $g_s \to 0$, $M_\pm \to \infty$, we instead identify

$$S_\pm = g_s M_\pm.$$ 

The matrix integral can be written as

$$Z = \frac{1}{V} \int d\Phi_+ d\Phi_- dA_i dB_i \exp \left[-\frac{1}{g_s} W_{\text{tree}}(\Phi_+, \Phi_-, A_i, B_j)\right]$$

with normalization factor

$$V = \text{vol} \left(U(M_+) \times U(M_-)\right)$$

We now want to claim that the matrix model only depends on the combination

$$S = S_+ - S_-.$$

In particular we have a simple relation between the matrix model free energy $F_0(S)$ and the gauge theory effective superpotential

$$W_{\text{eff}}(S) = (N_+ - N_-) \frac{\partial F_0}{\partial S} + 2\pi i (\tau_+ - \tau_-) S.$$ (3.6)

We will show that the free energy $F_0(S)$ is just that of the gaussian one-matrix model.

To analyze the model we can work with a more general superpotential

$$\text{Tr} \left(W(\Phi_+) - W(\Phi_-) - A_i \Phi_+ B_i + B_i \Phi_- A_i\right).$$ (3.7)
As a first step, one can integrate out the bifundamentals $A_i, B_j$ and then go to an eigenvalue basis for the remaining adjoint fields

$$\Phi_\pm \sim \text{diag} \left( \lambda_1^\pm, \ldots, \lambda_{M_\pm}^\pm \right).$$

This reduces the matrix model to the following integral over the eigenvalues

$$Z = \int \prod_{I,K} d\lambda^+_I d\lambda^-_K \frac{\prod_{I<J} (\lambda^+_I - \lambda^+_J)^2 \prod_{K<L} (\lambda^-_K - \lambda^-_L)^2}{\prod_{I,K} (\lambda^+_I - \lambda^-_K)^2} \times \exp \left[ -\frac{1}{2g_s} \sum_I W(\lambda^+_I) + \sum_K W(\lambda^-_K) \right]$$

(3.8)

The integral (3.8) represents a gas of both positively and negatively charged eigenvalues with a Coulomb interaction in a background potential $W(x)$. To find the large $M$ saddle-point, it helps to consider it as a system of a total of $M_+ + M_-$ eigenvalues $\lambda_I$ that each can have a charge $q_I = \pm 1$. There will then be $M_+$ eigenvalues of positive charge and $M_-$ eigenvalues of negative charge. If one wishes, the negatively charged eigenvalues can be considered as “holes” or “anti-eigenvalues.”

With this notation we can write the integral as

$$Z = \int \prod_I d\lambda_I \prod_{I<J} (\lambda_I - \lambda_J)^{2q_I q_J} \exp \left[ -\frac{1}{g_s} \sum_I q_I W(\lambda_I) \right]$$

The equation of motion now reads

$$W'(\lambda_I) = 2g_s \sum_{J\neq I} \frac{q_J}{\lambda_I - \lambda_J}.$$

The saddle-point evaluation of this integral in the ’t Hooft limit proceeds now very much like the bosonic one-matrix model with action $\text{Tr} W(\Phi)$. One introduces the resolvent

$$\omega(x) = \frac{1}{M} \sum_I \frac{q_I}{x - \lambda_I},$$

with $M$ the net charge in the system

$$M = \sum_I q_I = M_+ - M-. $$
We similarly denote \( S = g_s M = S_+ - S_- \). One then straightforwardly derives the planar loop equation

\[
y^2 - W'(x)^2 + f(x) = 0 \tag{3.9}
\]

written for the variable

\[
y = W'(x) - 2S\omega(x) = W'(x) - 2g_s \sum_i \frac{q_i}{x - \lambda_i}. \tag{3.10}
\]

Here the quantum deformation \( f(x) \) is given as the weighted average

\[
f(x) = 4g_s \sum_i q_i \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i}.
\]

For a general potential \( W(x) \) of degree \( n + 1 \) this is a polynomial of degree \( n \), that encodes the dependence on the moduli \( S_i = g_s M_i \). Here the relative filling fractions \( M_i \) now indicate the total charge \( \sum q_i \) of the eigenvalues that occupy the \( i \)-th critical point of \( W \).

We conclude that the spectral curve (3.9) of the \( \hat{A}_1 \) quiver matrix model is identical to that of the bosonic matrix model [1], with the remark that the filling fractions \( M_i \) and therefore also the moduli \( S_i \) now also can be naturally negative. In fact, if we take a real potential \( W(x) \) and hermitian matrices \( \Phi_{\pm} \), then the eigenvalues of positive charge will typically sit at the (stable) minima, while those of negative charge will sit at the (unstable) maxima.

Note that the \( B \)-cycle period

\[
\frac{\partial F_0}{\partial S_i} = \int_a^\infty ydx
\]

have an interpretation as either removing a positively charged eigenvalue from the system, or adding a negatively charged one. The system is therefore insensitive to adding an eigenvalue/anti-eigenvalue pair. So the planar free energy \( F_0 \) of quiver matrix model, as a function of the variables \( S_i \), is exactly the same as that of the one-matrix model. In fact, since the full loop equations for \( \omega(x) \) are identical to those of the one-matrix model, this identity holds also for the higher genus contributions \( F_g \).

In the special case of the gaussian potential \( W(x) = \frac{1}{2}x^2 \) the spectral curve of the affine quiver is

\[
y^2 + x^2 = S
\]
and the planar free energy is given by

$$F_0(S) = \frac{1}{2} S^2 \log(S/\Lambda^3).$$

Plugging this into (3.6) we get the effective superpotential for the quiver system

$$W_{\text{eff}}(S) = (N_+ - N_-)S \log(S/\Lambda^3) + 2\pi i (\tau_+ - \tau_-)S.$$

### 3.4. Supermatrix Models

As an aside we like to point out that integrals like the quiver matrix model (3.7) have been considered before in the context of supermatrix models (here defined as integrals over super Lie algebras, see e.g. [29]). In that case we work with a $M_+|M_-$ supermatrix $\Phi$ with a decomposition

$$\Phi = \begin{pmatrix} \Phi_+ & \psi \\ \chi & \Phi_- \end{pmatrix},$$

with $\Phi_+, \Phi_-$ bosonic and the off-block diagonal components $\psi, \chi$ fermionic. The action is now written in terms of a supertrace as

$$\text{Str} \ W(\Phi), \quad \text{with} \quad \text{Str} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a - d.$$ 

This action is invariant under the supergroup $U(M_+|M_-)$. It has been observed that this supermatrix model is equivalent to the $U(M)$ bosonic matrix model with $M = M_+ - M_-$. This result is essentially equivalent to invariance under the duality cascade. Following [30] we expect this system to appear naturally from the topological field theory description a system of $M_+$ D5 branes and $M_-$ anti-D5 branes together with a set of D3 branes represented as non-trivial flux.

As in [31] we can go to a basis in which the odd components are zero

$$\Phi = \begin{pmatrix} \Phi_+ & 0 \\ 0 & \Phi_- \end{pmatrix},$$

and we break to the bosonic subgroup

$$U(M_+|M_-) \to U(M_+) \times U(M_-).$$

This introduces ghosts $b, c$ with a decomposition

$$b = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix}.$$ 

Because the ghosts have odd statistics the off-diagonal fields are even; we will decompose them as

$$A, A^* = A_1 \pm iA_2, \quad B, B^* = B_1 \pm iB_2.$$ 

The matrix model action in the gauge fixed version is given by a supertrace

$$W = \text{Str} \ (W(\Phi) + b[\Phi, c]). \quad (3.11)$$

which written in components reduces exactly to (3.7).
3.5. Stability of The Glueball Superpotential and K-Theory

The above ideas are relevant for the resolution of the following puzzle: If we fix a gauge group of finite rank $N$, the powers of the glueball superfield $S$, defined as a fermion bilinear, terminate at finite order, if we consider $S$ as a classical field. How can one then justify the perturbative computations of the glueball superpotential, as was done successfully in [9], not incorporating this fact? One resolution of this is to use the “replica trick” and embed the $U(N)$ theory in the $U(NK)$ theory, and do the computation in that context in the limit where $K$ is large. But under this transformation the superpotential changes by a factor of $K$, and so in particular one is dealing with a different effective theory. Moreover, one will have to argue why going back to the $U(N)$ theory is as simple as dividing the superpotential by a factor of $K$.

We feel a better resolution of this puzzle is what we have observed here: The F-term of the supersymmetric gauge theory is equivalent to that of a bigger gauge theory by an inverse cascade effect. This, as we have explained above, is related to adding $M$ extra brane/anti-brane pairs to an $\mathcal{N} = 1$ theory with an equivalent F-term content. In the context of this bigger gauge theory, the computation of the glueball superpotential makes sense for higher powers of $S_+$ and $S_-$, as long as we take $M$ large enough. In fact we can take $M \to \infty$ without changing the F-terms of the theory, and consider arbitrary powers of $S_\pm$’s since the ranks become infinite. This in particular justifies the perturbative computation done in [9], without needing to change the F-term content of the theory.

This is very similar to how K-theory captures D-brane charges, where one considers adding an arbitrary number of brane/anti-brane pairs [32]. We thus see a notion of “stability” of the perturbative glueball superpotential computation, where one stabilizes the gauge theory by adding sufficient numbers of branes and anti-branes so that one can effectively ignore the condition that the glueball field $S$ is nilpotent classically.

3.6. Seiberg-like Dualities From Matrix Models

Quiver theories with gauge group $\prod U(N_i)$, consisting of adjoint chiral field $\Phi_i$ on each node with some superpotential $W_i(\Phi_i)$ and certain bifundamental fields $Q_{ij}$, admit a duality discovered in [23], considered in the context of affine A-D-E quiver theories. We replace the rank of the gauge group at node $i$ by the sum of the adjacent ranks minus $N_i$ (this is the analog of $N_f - N_c$ for Seiberg duality), and we replace

$$W_i(\Phi_i) \to -W_i(\Phi_i)$$
and

\[ W_j(\Phi_j) \rightarrow W_j(\Phi_j) + e_j \cdot e_i W_i(\Phi_j), \]

where \( e_i \) denote the basis of positive roots associated to the nodes of the affine quiver theory. This duality was interpreted in [23] as a Weyl reflection on the node \( i \). Exactly the same interpretation can be done in the context of matrix model, which gives a derivation of this duality; this point was already noted in [3] and we will elaborate on it here.

The matrix model which describes the F-terms of this theory is the quiver matrix model already studied in [33]. One can integrate out the bifundamental fields (as we did above for the case of the \( \tilde{A}_1 \)) to obtain the integral in term of the eigenvalues of the \( \Phi_i \) on each node:

\[ Z = \int \prod_{i,I} d\lambda^i_I \prod_{(i,I) \neq (j,J)} (\lambda^i_I - \lambda^j_J)^{e_i \cdot e_j} \exp \sum_{i,I} W_i(\lambda^i_I) \]

This system is clearly invariant under the Weyl reflection. To make this more clear, consider the planar limit, where we associate a density eigenvalue \( \rho^i(\lambda) \) to each node \( i \). Then the effective action can be written as

\[ S = \int d\lambda \rho^i(\lambda) W_i(\lambda) + \int d\lambda d\lambda' (e_i \cdot e_j) \rho^i(\lambda) \rho^j(\lambda') \log(\lambda - \lambda') \]

Viewing \( \rho = \rho^i e_i \) and \( W = W_i e^i \) as vectors and covectors in the affine root lattice this system enjoys the natural Weyl reflection action on each node, which leads to the duality mentioned above. As was noted in [23] this leads to a matrix model derivation of the gauge theory duality.

4. \( c = 1 \) Non-Critical Bosonic String and \( \mathcal{N} = 1 \) Gauge Theory

So far we have seen that a particular gauge theory, namely the affine quiver theory based on \( \tilde{A}_1 \) with quadratic superpotential, has the same set of chiral fields as that of the \( c = 1 \) string at self-dual radius. This is not accidental, as we will explain in this section. Furthermore we give the detailed link between computations of \( c = 1 \) correlation functions and F-terms of \( \mathcal{N} = 1 \) supersymmetric gauge theory. This will generalize the correspondence of the free energies to arbitrary correlators.

---

1 In fact one can refine the statement of the duality, as was done in [23], to explain how the critical points of the matrix model (which correspond to arbitrary roots of the affine Dynkin diagram) get exchanged under the Weyl reflection.
It was argued in [18] (using the result of [17] describing $c = 1$ at self-dual radius as a $SL(2)/U(1)$ Kazama-Suzuki model at level 3) that B-model topological string theory on the deformed conifold geometry

$$x_{11}x_{22} - x_{12}x_{21} = \mu$$

is equivalent to non-critical bosonic strings with $c = 1$ at self-dual radius, where one identifies the geometry as the (complexified) ground ring of the bosonic string theory. This identification was checked to a few loop orders and extended to all loops using the results [34]. This identification was further studied in [35].

However, this topological model is equivalent (as a mirror of the duality between Chern-Simons and topological strings [5]) to the resolved conifold geometry with branes wrapping $\mathbf{P}^1$. In the connection between the topological string and the type IIB string, these branes get promoted to $D5$ branes, and in addition $D3$ branes, which are points in the internal geometry. On the other hand the gauge theory of this system involves the $\hat{A}_1$ quiver theory [20] already discussed. Since the B-model topological string on the side including branes computes $F$-terms of the corresponding $\mathcal{N} = 1$ gauge theory, and that is equivalent to the matrix integral, we come to the conclusion that the $c = 1$ non-critical bosonic string at self-dual radius is equivalent to the $\hat{A}_1$ quiver matrix model. We explain this identification now in more detail.

As discussed before for each observable of the $c = 1$ theory at self-dual radius, there is a chiral field in the matrix model and the gauge theory, namely

$$O^{(k)}_{i_1\ldots i_k,j_1\ldots j_k} \leftrightarrow \text{Tr}(A_{i_1}B_{j_1}\ldots A_{i_k}B_{j_k})$$

Consider now the partition function of $c = 1$ theory deformed by arbitrary physical operators

$$Z(\mu, t) = \left\langle \exp \sum t_{i_1\ldots i_k,j_1\ldots j_k} O^{(k)}_{i_1\ldots i_k,j_1\ldots j_k} \right\rangle$$

with genus expansion

$$Z(\mu, t) = \exp \sum_{g \geq 0} g_s^{2g-2} F_g(\mu, t).$$

Then we reach the conclusion that this is equivalent to the deformed $\hat{A}_1$ matrix model theory

$$Z(\mu, t) = \frac{1}{V} \int dA_i dB_j \exp \frac{1}{g_s} \left[ \text{Tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1) + \sum t_{i_1\ldots i_k,j_1\ldots j_k} \text{Tr}(A_{i_1}B_{j_1}\ldots A_{i_k}B_{j_k}) \right]$$

(4.1)
where $\mu = g_s(M_+ - M_-)$ ($S$ in the gauge theory) and $M_+$ and $M_-$ are the ranks of the two matrices of $\hat{A}_1$ theory. As discussed before the identification of the parameters $t_{i_1...,i_k,j_i...,j_k}$ is unambiguous to first order but beyond that there could be an operator mixing.

Moreover if $F_0(\mu, t)$ denote the contribution of planar diagrams (or equivalently the genus 0 amplitudes of $c = 1$ at self-dual radius) then in the associated $\mathcal{N} = 1$ supersymmetric gauge theory the deformed superpotential is given by

$$W_{\text{eff}}(S, t) = (N_+ - N_-) \frac{\partial F_0(S, t)}{\partial S} + (\tau_+ - \tau_-)S.$$ 

Using the cascade the associated gauge theory can also be viewed as a single $U(N)$ gauge theory with 4 adjoint fields $M_{ij}$. Ignoring the subtleties of baryons and the quantum moduli space, the naive superpotential for the associated 4-matrix model is

$$W = \text{Tr} (\text{det}_{ij} M + f(t, M)).$$

In particular if we just deform the momentum modes of the $c = 1$ self-dual string, i.e. if we want to compute the scattering amplitudes of the tachyon modes

$$Z(\mu, t) = \left\langle \exp \sum_k t_k T_k \right\rangle,$$

two adjoint fields can be integrated out. So the correlations get mapped to an $\mathcal{N} = 1$ supersymmetric gauge theory with the remaining two adjoints

$$X_+ = M_{11}, \quad X_- = M_{22},$$

and with superpotential

$$W = \text{Tr} \left( X_+ X_- + \sum_{n>0} t_n X_+^n + t_{-n} X_-^n \right). \quad (4.2)$$

This is exactly a well-known two-matrix model representation of the $c = 1$ string [38]. Using the Harish Chandra-Itzykson-Zuber integral, one can show that this matrix model is equivalent to another model where one assumes that the matrices $X_+, X_-$ commute. Since we can also choose the reality condition $X_+^\dagger = X_-$, this is sometimes also called the normal matrix model [39].

A generalization of this model that includes couplings $\tilde{t}_k$ to the winding modes or vortices $\tilde{T}_k$ has been proposed in [37]. In this three-matrix model, one also introduces a unitary matrix $U$ that captures the vortices. The full action is

$$\text{Tr} \left( X_+ X_- - UX_+ U^{-1} X_- + \sum_{n>0} (t_{+n} X_+^n + t_{-n} \text{Tr} X_-^n) + \sum_{k \in \mathbb{Z}} \tilde{t}_k \text{Tr} U^k \right). \quad (4.3)$$

It would be very interesting to relate this models directly to our quiver model. (See in this respect also our comments in section 6.)

---

1 A normal matrix $X$ satisfies $[X, X^\dagger] = 0$.  

2 A normal matrix $X$ satisfies $[X, X^\dagger] = 0$.  

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5. Generalizations

The above example can be generalized in two basic directions. On the one hand we can generalize this to the case where the radius of the \( c = 1 \) circle is \( k \) times the self-dual radius. On the other hand, as we discussed before, we can relax the condition of the adjoint superpotential to be quadratic. We will consider these two cases in turn.

5.1. Multiples of The Self-Dual Radius

The fastest way to obtain the matrix model and the gauge theory in this case is to consider a \( \mathbb{Z}_k \) orbifold of the model with self-dual radius — on the side of both the bosonic string, the matrix model, the gauge theory and the geometry. Let us consider these in turn.

Recall that in the case of the non-critical \( c = 1 \) bosonic string at self-dual radius the Calabi-Yau 3-fold geometry can be identified with the ground ring. In particular, the ring at zero cosmological constant

\[
x_{11}x_{22} - x_{12}x_{21} = 0
\]

is identified with the singular conifold. Recall also that the monomials are identified with the momentum and winding modes of \( c = 1 \) as discussed in section 2.1. Thus under the \( \mathbb{Z}_k \) orbifold they transform according to

\[
\begin{align*}
  x_{11} &\rightarrow \omega x_{11}, \\
x_{22} &\rightarrow \omega^{-1} x_{22}, \\
x_{12} &\rightarrow x_{12}, \\
x_{21} &\rightarrow x_{21},
\end{align*}
\]

where \( \omega \) is a primitive \( k \)-th root of unity

\[
\omega^k = 1.
\]

The invariant ring in this case is generated by

\[
u = (x_{11})^k, \quad v = (x_{22})^k,
\]

together with \( x_{12}, x_{21} \). This leads to the ring relation

\[
u v = (x_{12}x_{21})^k
\]

(5.1)
This is indeed the chiral ring for \( k \) times the self-dual radius \([38]\).

One can also carry this orbifolding at the level of the quiver theory. This has been done in \([39]\). One ends up with an \( \hat{A}_{2k-1} \) quiver theory with \( N = 2 \) matter content and bifundamental fields between the nodes. Moreover there is a superpotential \( +\text{Tr} \Phi_i^2 \) for the adjoint field for the even nodes and \( -\text{Tr} \Phi_i^2 \) for the odd nodes. To obtain this orbifold one simply uses the method of obtaining quiver theories on D-brane orbifolds \([40]\) where the \( \mathbb{Z}_k \) action on the fields is given, in addition to the cyclic action on the \( kM_+ \) and \( kM_- \) branes, by

\[
\begin{align*}
A_1 &\rightarrow \omega A_1, \\
B_1 &\rightarrow B_1, \\
A_2 &\rightarrow A_2, \\
B_2 &\rightarrow \omega^{-1} B_2, \\
\Phi_+ &\rightarrow \Phi_+, \\
\Phi_- &\rightarrow \Phi_-.
\end{align*}
\]

We can also obtain an alternative derivation of the quiver theory starting from the ring relations (5.1) and using the results of \([41]\) and \([42]\). Let us define \( x_{12} = y + x \) and \( x_{21} = y - x \). Then the relation (5.1) becomes

\[
u v - (y^2 - x^2)^k = 0,
\]

which can be equivalently written as

\[
u v - (y - x)(y + x)(y - x) \cdots (y + x) = 0.
\]

Here there are \( k \) monomials of \( y + x \) and \( k \) monomials of \( y - x \). According to \([41]\) if one has a geometry of the form

\[
u v - (y - e_1(x))(y - e_2(x)) \cdots (y - e_{2k}(x)) = 0,
\]

with branes wrapped over blown up 2-cycles, one obtains the \( \hat{A}_{2k-1} \) quiver theory with \( \mathcal{N} = 2 \) content where the gradient of the superpotential as a function of the \( i \)-th adjoint field is

\[
W_i'(x) = e_i(x) - e_{i+1}(x),
\]

with \( x \rightarrow \Phi_i \). Moreover the sum of the \( W_i \)'s are zero.
Applying this to the case at hand we see that we have an adjoint field with quadratic superpotential at each node, with alternating signs for nearest neighbors. Thus we have a reformulation of the $c = 1$ bosonic string at $k$ times the self-dual radius in terms of a matrix quiver theory. Note that there are more gauge groups and we can vary the rank of all $2k$ gauge groups. In particular we can choose all the ranks $M_i$ (with $i = 1, \ldots, 2k$) independently. This corresponds in the $c = 1$ theory to turning on twisted states, which are the lightest momentum modes in the theory with radius being $k$ times the self-dual radius.

5.2. More General Superpotentials

In principle we can have an independent superpotential for each of the nodes of the affine quiver theory as long as they sum up to zero [41]. For example for the $A_1$ quiver theory, as we already explained, we could put superpotentials $W(\Phi_+) - W(\Phi_-)$ for a general polynomial $W$. In this case the matrix model computations reduce to that of the single matrix model with potential $W(\Phi)$. Note that this can be viewed as a deformation of the quadratic potential by some higher powers of $\Phi$ which can be written in terms of monomials of $A_i$ and $B_j$ using

$$\text{Tr} \, \Phi^n_+ = \text{Tr}(A_i B_i)^n.$$ 

Thus the correlations of the gaussian matrix model observables $\text{Tr} \, \Phi^n$ can be viewed as computing some specific subset of correlation functions of $c = 1$ at self-dual radius.

This relation can also be seen from the geometry side. Including higher powers of $\Phi$ deforms the conifold to

$$uv + y^2 + W'(x)^2 + f(x) = 0$$

This is a special case of the universal deformation (2.3).

Similarly we can consider the $Z_k$ orbifold of this and obtain alternating $W$'s at each node.

5.3. Deconstruction and $A_\infty$ Quiver Matrix Model

A particular limit of the model we have been studying is related to the deconstruction of the six-dimensional $(2, 0)$ superconformal theory [24]. This is also an interesting limit from the point of view of $c = 1$ theory. If we consider the uncompactified limit $k \to \infty$ which leads to $c = 1$ string on an infinite real line, then we obtain the $A_\infty$ quiver theory.
The infinite array of nodes of the quiver is naturally identified as a deconstruction, or discretization, of the $c = 1$ line.

Thus, if we consider the uncompactified $c = 1$ theory with all the momentum fields (including the cosmological constant operator) turned off, \textit{i.e.} $M_i = M$ for all nodes $i$, and then turn off the superpotential $W(\Phi_i) \to 0$, we obtain the quiver theory relevant for the deconstruction of the $(2,0)$ SCFT in $d = 6$ \cite{24}. The choice of setting $W \to 0$ can be viewed as a regularization of the $(2,0)$ theory (freezing to particular points on the scalar moduli space, similarly to what was done in \cite{13} in getting $\mathcal{N} = 2$ information from the $\mathcal{N} = 1$ deformation). The most natural superpotential for the $c = 1$ at infinite radius is the quadratic potential $W(\Phi) = \Phi^2$ which freezes $\Phi$ to zero. But we can also deform away from this point by considering a more general $W(\Phi)$, alternating in sign from one node to another, to freeze to an arbitrary point in the $\Phi$ moduli space. We can also deform the $(2,0)$ theory by having varying ranks on each node of the $A_\infty$ quiver. This gets mapped to the correlations of the momentum states of $c = 1$ theory on an infinite line. Given the vast literature on correlations of the decompactified $c = 1$ string theory, this should lead to new insights into $(2,0)$ little strings, which is worth further study.

Let us explain further why obtaining the deconstruction theory of $(2,0)$ superconformal theory is not unexpected. As was shown in \cite{20} the theory of D3 branes near a conifold singularity is the same as the theory of D3 brane near an $A_1$ singularity where the $A_1$ geometry is fibered over the plane (giving rise to the superpotential $\text{Tr} \Phi^2$). Orbifolding this by $\mathbb{Z}_k$ can be viewed as orbifolding the $A_1$ singularity by $\mathbb{Z}_k$ which is the same as modding $\mathbb{C}^2$ by $\mathbb{Z}_{2k}$ producing an $A_{2k-1}$ singularity. Thus the D3 branes in this geometry are the same as what is used in the deconstruction of $(2,0)$ SCFT. Moreover, after deforming by the superpotential term, is the same as the theory we are studying which is equivalent to $c = 1$ at $k$ times the self-dual radius.

Note also that the deconstruction of the $(2,0)$ theory has effectively allowed us to describe F-terms of this theory in terms of a matrix model which is dual to the $c = 1$ (which in turn is related to 2d black hole geometry). This throat region description is reminiscent of holography in this context studied in \cite{14}. Here we are encountering this directly from the gauge theory, which is equivalent in this context to a matrix model.

6. Higher-Dimensional Field Theories

In this section we want to point out an obvious generalization to higher-dimensional field theories of the connection between four-dimensional $\mathcal{N} = 1$ gauge theories and matrix models.
6.1. 10D Super-Yang-Mills and Holomorphic Chern-Simons

Let us start with a characteristic example. Consider ten-dimensional super Yang-Mills theory on a space-time of the form $\mathbb{R}^4 \times Y$, with $Y$ a Calabi-Yau three-fold. We want to keep all the Kaluza-Klein modes on the internal manifold $Y$, also the massive ones, and thus consider this as a four-dimensional supersymmetric gauge theory with an infinite number of fields. This field theory can be regarded as an effective field theory, with some unspecified UV completion, or we can discretize the model by deconstructing the extra dimensions. As such the action can be written in terms of a $d = 4 \, \mathcal{N} = 1$ superspace notation, using $4 + 6$ bosonic coordinates $(x^\mu, y^a)$ and the usual 4d spinor coordinates $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

Written in this way the action will contain D-terms, that are integrals over $d^4\theta$, and F-term, that are integrals over $d^2\theta$. More precisely, for the $d = 10$ SYM theory the action takes the following form \[45,46\]. Choose complex coordinates $(y^1, y^2, y^3)$ for the internal space $Y$. Similarly write the internal components of the Yang-Mills gauge fields in terms of a holomorphic connection $A_i(x; y)$ ($i = 1, 2, 3$) and a conjugated anti-holomorphic connection $\bar{A}_{\dot{i}}(x; y)$. Let $W_\alpha(x; y)$ be the spinor field strength of the four-dimensional gauge connection $V(x; y)$. The fields $W_\alpha$ and $A_i$ should be considered as infinite sets of four-dimensional chiral superfields, parametrized by the internal coordinate $y \in Y$.

The F-term of the action then takes the form \[45,46\]

$$
\int d^4x d^2\theta \ W_{\text{tree}}(W_\alpha, A_i),
$$

where the tree-level superpotential is given by

$$
W_{\text{tree}} = \int_Y d^6y \text{Tr} \left[ W_\alpha^2 + \epsilon^{ijk} \left( A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right) \right]. \quad (6.1)
$$

The second term is just the holomorphic Chern-Simons action on the manifold $Y$. Note that the variation of the Chern-Simons term is the (holomorphic) curvature $F_{ij}$, so that after integrating out the auxiliary fields the action contains the term $|W'|^2 = |F_{ij}|^2$ which produces exactly the internal part of the Yang-Mills term.

There is of course also a D-term, that is in this case given by \[45,46\]

$$
\int d^4x d^4\theta \int_Y d^6y \text{Tr} \left( \bar{D}_i e^{-V} D_i e^V - \bar{\partial}_i e^{-V} \partial_i e^V \right).
$$

Here we use the notation

$$
D_i = \partial_i + A_i, \quad \partial_i = \partial / \partial y^i.
$$
We can now apply the philosophy of [3] to this model, considered as a 4d $\mathcal{N} = 1$ theory, and compute the effective superpotential in terms of an auxiliary internal field theory given by the action (6.1). So the matrix model of [3] is now replaced by the six-dimensional holomorphic Chern-Simons gauge theory. This is of course just a direct field theory derivation of the result of [4] that the four-dimensional effective action of the type IIB open string is computed in terms of the B-model topological open string on the internal manifold. In this case the open string theory is given by a collection of $N$ D9 branes wrapped over the Calabi-Yau $Y$.

Perhaps we should spell out this connection a bit more precisely in this case. Let $M$ be the rank of the holomorphic Chern-Simons gauge theory. (Again, just as in the case in 4 dimensions, we should be careful not to confuse the rank of the gauge theory $N$ and the rank of the auxiliary topological theory $M$.) Suppose that the manifold $Y$ is such that there are $n$ isolated critical points of the superpotential, i.e. $n$ inequivalent holomorphic connections without moduli. Let $M_1, \ldots, M_n$ be the rank of these connections. Then we have a symmetry breaking pattern

$$U(M) \to U(M_1) \times \cdots \times U(M_n)$$

and can consider the 't Hooft limit $M_i \to \infty$. This will then lead in the usual way to $n$ gaugino condensates $S_i = g_s M_i$. The 4d effective superpotential is now again given by

$$W_{\text{eff}} = \sum_i \left( N_i \frac{\partial F_0}{\partial S_i} + \tau_i S_i \right)$$

with $F_0$ the planar partition function of the holomorphic CS field theory. Note that if we applied this to the case of ordinary CS theory on $S^3$ we obtain the embedding of the duality of CS with topological string [5,6] into a direct 4d field theory language (for another description of this field theory see [17]).

6.2. General Philosophy

The example of 10d super Yang-Mills illustrates nicely the general philosophy. Any $4 + k$-dimensional supersymmetric gauge theory can be written in terms of 4d $\mathcal{N} = 1$ superfields $\Phi_i(x, \theta; y)$ parametrized by coordinates $y^a$ on an internal $k$-dimensional manifold $Y$. Such a theory will have a tree-level superpotential that is given by some $k$-dimensional bosonic action

$$W_{\text{tree}} = \mathcal{S}[\Phi_i] = \int_Y d^k y \mathcal{L}(\Phi_i)$$
containing in general gauge fields and matter fields on $Y$. The 4d effective *superpotential* is now expressed through (6.2) in terms of the effective *action* $F_0$ of the internal $k$-dimensional bosonic gauge theory.

$$\lim_{M \to \infty} \int D\Phi \ e^{-S[\Phi]/g_s} = e^{-F_0/g_s^2}$$

For classical groups and (bi)fundamental matter this effective action is computed in terms of planar Feynman graphs of the internal theory.

Many interesting cases can be studied in this way. In the next subsection we explain how 5-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories can be deconstructed in this way, which leads us directly to matrix quantum mechanics.

### 6.3. Deconstruction of 5D Gauge Theories and Matrix Quantum Mechanics

Let us first consider deconstruction of a pure $\mathcal{N} = 1$ Yang-Mills in 5 dimensions compactified on a circle of radius $\beta$. This case has been recently studied in [48]. Viewed from the perspective of $\mathcal{N} = 1$ in 4 dimensions this has in addition a chiral scalar field $\Phi$ given by the component of the gauge field along the 5th direction. (The extra (real) scalar field, that is part of the 5-dimensional vector multiplet, complexifies $\Phi$.) This scalar gives rise to a holonomy

$$U = \exp(\beta \Phi) \in U(N)$$

around the 5th circle of length $\beta$. Here $\Phi$ (whose eigenvalues are periodic with period $2\pi i/\beta$) represents a flat direction.

In terms of the general discussion of the previous section the internal theory is here a one-dimensional gauge theory. Such a theory is entirely described in terms of the holonomy $U$ modulo conjugation by $U(N)$. So we immediately see that according to the above philosophy 5d super Yang-Mills will be related to a *unitary* matrix model. If we break to $\mathcal{N} = 1$ in 4 dimensions by choosing some tree-level superpotential $W(U)$, the F-terms of this deformed theory are computed by the large $M$ limit of the holomorphic gauged unitary matrix model

$$Z = \frac{1}{\text{vol} U(M)} \int dU \ \exp \left[ -\frac{1}{g_s} \text{Tr} W(U) \right],$$

with $dU$ the Haar measure on $U(M)$.

To gain insight into freezing the moduli at a particular point, we consider deforming the theory by a superpotential of the same degree as the rank of the $U(N)$ group and choose
the $N$ eigenvalues of $\Phi$ to fill the $N$ distinct values of the critical points. We can recover the information about the original theory by letting the strength of the superpotential to go to zero, as in [43]. In particular we consider a superpotential $W$ of the form

$$W(U) = \sum_{i=0}^{N} g_i U^i,$$

which has $N$ distinct critical points (viewing $\Phi$ as the fundamental field). We have

$$\frac{1}{\beta} \frac{dW}{d\Phi} = P(U) = UW'(U) = \prod_{i=1}^{N} (U - a_i) = 0.$$

We can now study this theory in the planar limit to extract exact information about this theory in 4 dimensions. The planar limit of unitary matrix models such as this one have been analyzed before [49,50] and reanalyzed recently in [2]. In particular following the analysis of [2] we find that the spectral curve is given by

$$y^2 + yP(u) + f(u) = 0,$$

where $f(u)$ is a quantum correction depending on how the eigenvalues are distributed. It is given by the matrix model expectation value

$$f(u) = \left\langle \text{Tr} \left[ (P(U) - P(u)) \frac{U + u}{U - u} \right] \right\rangle.$$

Using this formula one sees immediately, as in [2], that $f(u)$ is a polynomial in $u$, with degree at most $N$, i.e.

$$f(u) = \sum_{i=0}^{N} b_i u^i$$

for some $b_i$. If we distribute the eigenvalues equally among the vacua and then extremize the superpotential (with respect to the $S_i$’s), we expect, as in [43] that the quantum correction $f(u)$ will simplify. More precisely, we expect that all $b_i = 0$ except for $b_0$, so that

$$f(u) = b_0.$$

This remains to be shown, which should be possible using the techniques of [43].

This in particular would lead to the curve of the compactified 5d theory

$$y^2 + yP(u) + b_0 = y^2 + y \prod_{i=1}^{N} (u - a_i) + b_0 = 0.$$  

(6.4)
The period matrix of this Riemann surface gives the gauge coupling constant of the \( U(1)^N \) in the 4d theory. This is indeed the correct curve [51,52] for a particular value of the level \( k \) of the 5d Chern-Simons term, namely \( k = N \). This is compatible with recent results of [48] where they argue why the simplest deconstruction procedure (which is equivalent to our setup) gives rise to this particular value of Chern-Simons term in 5d. However, as is known from [53] the allowed values of \( k \) leading to a decoupled 5d theory are \( |k| \leq N \). Moreover, from [51,52] we know that this gives rise to curve (6.3) with the correction term

\[
f(u) = bu^{N-|k|}.
\]

This is indeed compatible with allowed quantum corrections of the matrix model, which suggests that, as we change the CS level, we should be minimizing a different superpotential. This is not unreasonable, as the CS term involves from the 4d point of view a term \( \Phi F \wedge F \). So changing the coefficient of this term will change the superpotential to be extremized. It would be interesting to incorporate this into the superpotential and see why extremization now changes \( f \) to a different monomial, as is expected.

We can extend these considerations to include \( N = 1 \) gauge theories coupled to hypermultiplets in 5 dimensions. In that case we obtain a one-dimensional gauge theory with matter fields. Now we can no longer reduce to zero-modes and we get an honest quantum mechanical system. For example, a five-dimensional gauge theory coupled to a set of hypermultiplets, that we write in terms of 4d chiral multiplets as \( (Q^i, P_i) \), including some superpotential \( H(Q, P) \), leads to a matrix quantum mechanics model with action (we write the internal 5th coordinate as \( t \))

\[
\int_0^\beta dt \ Tr \left( P_i \frac{DQ^i}{dt} + H(P, Q) + W(U) \right)
\]

Here the one-dimensional gauge field \( \Phi(t) \) appears in the covariant derivative, although by a gauge transformation this connection can be eliminated in favor of a boundary condition for the matter fields twisted by the holonomy \( U \). We have also included the superpotential \( W(U) \) as we did before, for the scalar field \( \Phi \). (Compare the above action with the three-matrix model [1.3].) Note that if we take \( H(P, Q) = P^2 + V(Q) \) and integrate out \( P \) we get the standard gauge quantum matrix model. The particular case where \( H(P, Q) = P^2 - Q^2 \) is related in a suitable double scaling limit to the \( c = 1 \) theory [11]. It should be compared to [4.2] with \( X_\pm = P \pm Q \).
6.4. Other Examples

This approach opens up many further interesting possibilities. For example, we already mentioned that compactifying D6 branes on a three-manifold \( M \) gives rise to an internal Chern-Simons gauge theory. If we compactify first on a circle, and thus take \( M = S^1 \times \Sigma \), the resulting 2d action on \( \Sigma \) takes a “BF” form

\[
S = \int_{\Sigma} \text{Tr}(\Phi F)
\]

This is also known as 2d topological Yang-Mills \([14]\). We can break the supersymmetry further down to \( \mathcal{N} = 1 \) by deforming this to

\[
S = \int_{\Sigma} \text{Tr}(\Phi F + W(\Phi))
\]

Taking a quadratic term gives 2d Yang-Mills. Its large \( N \) limit has been solved in \([53]\). It would be interesting to further explore this connection.

In fact, the most general statement is the following. Take one’s favorite \( k \)-dimensional \( U(N) \) (bosonic) gauge theory and promote it to an internal superpotential of a \( 4 + k \)-dimensional theory by interpreting all fields as chiral superfields and coupling them to four-dimensional gauge fields. Choose some appropriate D-terms to complete the theory. Then the planar diagrams of the chosen gauge theory will compute all F-terms in the four-dimensional effective action. In these more general internal bosonic gauge theories one does not get a theory which can be viewed as a KK reduction of a standard higher-dimensional gauge theory (for example for 4d bosonic YM as internal theory one gets \((D_i F_{ij})^2\) as the relevant piece of the action for the extra 4 dimensions). Nevertheless they give rise to a vast collection of potentially interesting \( \mathcal{N} = 1 \) supersymmetric gauge theories in 4 dimensions.

7. Recovering The Full String Theory

Although significant progress has been made in understanding the structure of F-terms in \( \mathcal{N} = 1 \) supersymmetric gauge theories, one can ask what can be said about the non-holomorphic information (D-terms). In the most general case there is little hope that these terms are under control. But a special role should be played by superconformal fixed points. In these cases one can expect that F-terms are enough to completely specify the theory. If that is true, this would be the strongest possible sense in which \( \mathcal{N} = 1 \) systems are solvable.
This is a well-established fact for $\mathcal{N} = 2$ superconformal field theories in two dimensions. In that case the only marginal deformations are given by variations of F-terms. The proof is straightforward — D-term variations are of the form $Q^2\overline{Q}^2\Phi$, with $\Phi$ an operator of conformal dimension zero and thus equal to the identity. The only possible deformations are therefore F-terms. It would be interesting to find a proof for the corresponding statement in the four-dimensional case. To the best of our knowledge there is no counterexample to this claim.

In the present case it is clear that the holomorphic data uniquely specify the corresponding string theory. Given the complex structure and fluxes the (generalized) Calabi-Yau geometry is fixed by the world-sheet beta-functions. Furthermore this data is entirely captured by the matrix integral. In that sense we can say that the matrix model is a very efficient, though implicit, way to encode the full string theory.

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