Upper continuity bounds on the relative $q$-entropy for $q > 1$

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Generalized entropies and relative entropies are the subject of active research. Similar to the standard relative entropy, the relative $q$-entropy is generally unbounded for $q > 1$. Upper bounds on the quantum relative $q$-entropy in terms of norm distances between its arguments are obtained in finite-dimensional context. These bounds characterize a continuity property in the sense of Fannes.

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I. INTRODUCTION

The relative entropy has been adopted as very important measure of statistical distinguishability. In the classical regime, the relative entropy of probability distribution $\{a_i\}$ to $\{b_i\}$ is defined by \[ D(a||b) := -\sum_i a_i \ln(b_i/a_i) \] and also known as Kullback-Leibler divergence \[2\]. The usual convention is that $-\infty \ln 0 = 0$. The quantum relative entropy for (normalized) density operators $\rho$ and $\sigma$ is defined as

\[ D(\rho||\sigma) := \text{Tr}(\rho \ln \rho - \rho \ln \sigma) . \] (1.1)

This expression is well-defined whenever the kernel of $\sigma$ does not intersect with the support of $\rho$, i.e. $\text{ker}(\sigma) \subset \text{ker}(\rho)$ \[1, 3\]. Otherwise, the relative entropy is defined to be $+\infty$. Many fundamental results of quantum information theory are closely related to properties of the relative entropy \[1, 4\].

There exist various generalizations of the standard entropic functionals. The Tsallis entropy has been found to be very useful in numerous problems of physics and other sciences \[5\]. In the context of statistical physics, Tsallis defined the $q$-entropy for $q \neq 1$ by \[6\]

\[ S_q(a_i) := (1-q)^{-1} \left( \sum_i a_i^q - 1 \right) \equiv - \sum_i a_i^q \ln_q a_i \] (1.2)

where the $q$-logarithm $\ln_q x = (x^{1/q} - 1) / (1-q)$. This functional has also been derived within an axiomatic approach \[6\]. Entropic uncertainty principle has been expressed in terms of the Tsallis entropies \[8, 9\]. A generalization of the relative entropy for classical probability distributions is naturally introduced as \[10\]

\[ D_q(a||b) := -\sum_i a_i \ln_q(b_i/a_i) \equiv (1-q)^{-1} \left( 1 - \sum_i a_i^q b_i^{-1/q} \right) . \] (1.3)

Note that the term \[13\] can be recast as the sum $\sum_i a_i^q (\ln_q a_i - \ln_q b_i)$, similar to the standard analytical form. In the quantum case, the sums in Eq. \[13\] are replaced by the traces of corresponding fractional powers of density operators. For $q \in (0; 1)$, when no singularities occur, basic properties of the quantum relative $q$-entropy are examined in Refs. \[11, 12\]. Up to a factor, the relative $q$-entropy is a particular case of Petz’s quasi-entropies \[13, 14\]. In the case $q > 1$, singularities may occur in the expression for the relative $q$-entropy. Like the standard relative entropy, the relative $q$-entropy is defined to be $+\infty$ in the singular case.

Relations between distinguishability measures are of interest. For certain applications, some of them may be more appropriate than others. For instance, cloning processes are often studied with respect to fidelity-based measures \[15\], but the relative entropy has found use as well \[16\]. Relations of such a kind are frequently expressed in the form of inequalities. Various upper bounds on the relative entropy \[11, 14\] were obtained in Ref. \[17\]. As given in terms of difference distances, these bounds characterize a continuity property in the sense of Fannes \[17\]. Recall that Fannes’ inequality bounds from above a potential change of the von Neumann entropy in terms of trace norm distance \[18\]. Fannes’ inequality has been extended to the Tsallis $q$-entropy \[13, 20\] and its partial sums \[21\]. In the classical regime, continuity properties of wide classes of entropies and relative entropies are considered in Ref. \[22\]. In the present paper, we are interested in upper continuity bounds on the relative $q$-entropy for $q > 1$.

II. DEFINITIONS AND BACKGROUND

In this section, the definition of the relative $q$-entropy for $q > 1$ and related questions are discussed. Required mathematical tools are briefly outlined. By ker($X$) we denote the kernel of operator $X$. The support supp($X$) is the subspace orthogonal to ker($X$).
For $q > 1$ and any pair of normalized density operators $\rho$ and $\sigma$, the quantum relative $q$-entropy is defined by

$$D_q(\rho \| \sigma) := \begin{cases} \frac{1}{1-q} \left( 1 - \text{Tr}(\rho^q \sigma^{1-q}) \right), & \text{ker} (\sigma) \subset \text{ker} (\rho), \\ +\infty, & \text{otherwise}. \end{cases} \tag{2.1}$$

Let $\{a\} = \text{spec}(\rho)$ and $\{b\} = \text{spec}(\sigma)$ denote the spectra of $\rho$ and $\sigma$, and let $\{|a\}\}$ and $\{|b\}\}$ denote the related orthonormal bases. Let $f(x)$ and $g(x)$ be functions of scalar variable $x$. Recall that the formula

$$\text{Tr} (f(\rho) g(\sigma)) = \sum_a \sum_b |\langle a | b \rangle|^2 f(a) g(b) \tag{2.2}$$

is regarded as the definition of $\text{Tr} (f(\rho) g(\sigma))$ when $f(\rho)$ or $g(\sigma)$ is unbounded. The trace in (2.1) is written as

$$\text{Tr}(\rho^q \sigma^{1-q}) = \sum_a \sum_b |\langle a | b \rangle|^2 a^q b^{1-q}, \tag{2.3}$$

which is generally unbounded for $q > 1$ and singular $\sigma$. If $\text{ker} (\sigma) \subset \text{ker} (\rho)$, i.e. $\text{supp}(\rho) \subset \text{supp}(\sigma)$, then we correctly define this trace as the trace taken just over $\text{supp}(\sigma)$, i.e. the sign $\sum_b$ is interpreted as $\sum_{b \neq 0}$. Effectively, the sum with respect to $a$ is also restricted to the nonzero $a$’s.

The quantum relative $q$-entropy enjoys some properties similar to the standard relative entropy (1.1). In particular, the measure (2.1) is positive and vanishes only for $\rho = \sigma$. It is also pseudoadditive for all $q > 1$. These points can be established by a relevant modification of the reasons given for $q \in (0;1)$ in Ref. [12]. One of most important properties of the standard relative entropy (1.1) is its monotonicity under the action of stochastic maps. The relative $q$-entropy is jointly convex and monotone for $0 \leq q \leq 2$. These properties follow from the results of Refs. [23, 24]. More details and conditions for equality can be found in the recent review [25].

Let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $d$-dimensional Hilbert space $\mathcal{H}$. By $\mathcal{L}_+(\mathcal{H})$ and $\mathcal{L}_{++}(\mathcal{H})$ we denote the sets of positive and strictly positive operators respectively. For any $X \in \mathcal{L}(\mathcal{H})$, we put $|X| = \sqrt{X^*X} \in \mathcal{L}_+(\mathcal{H})$. The eigenvalues of $|X|$ counted with their multiplicities are the singular values $s_j(X)$ of $X$ [26]. For $p \geq 1$, the Schatten $p$-norm of operator $X$ is given by [26, 27]

$$\|X\|_p = \left( \sum_{j=1}^d s_j(X)^p \right)^{1/p}. \tag{2.4}$$

This family includes the trace norm $\|X\|_1$ for $p = 1$ and the spectral norm $\|X\|_\infty = \max \{s_j(X) : 1 \leq j \leq d\}$ for $p = \infty$. These norms and relations between them have found use in various questions of quantum information [28–30]. For each $p \in [1;\infty]$ and $X, Y, Z \in \mathcal{L}(\mathcal{H})$, there holds (see, e.g., section 2.4 in [27])

$$\|XYZ\|_p \leq \|X\|_\infty \|Y\|_p \|Z\|_\infty. \tag{2.5}$$

Since the Schatten $p$-norms are non-increasing in $p$, these norms satisfy submultiplicativity $\|XY\|_p \leq \|X\|_p \|Y\|_p$. Combining Eq. (2.5) for $p = 1$ with $|\text{Tr}(X)| \leq \text{Tr}|X|$, we obtain

$$|\text{Tr}(XYZ)| \leq \|X\|_\infty \|Z\|_\infty \text{Tr}|Y|. \tag{2.6}$$

We will extensively use the integral representations of matrix fractional power based on the formulas

$$a^r = \frac{\sin r \pi}{\pi} \int_0^\infty x^{r-1} dx \frac{a}{a+x} = \frac{\sin r \pi}{\pi} \int_0^\infty y^{-r} dy \frac{1}{y + a^{-1}}, \quad (0 < r < 1). \tag{2.7}$$

The second integral follows from the first one by substituting $x = 1/y$.

### III. UPPER BOUNDS FOR $1 < q \leq 2$

Let us briefly mention lower continuity bounds. Lower bounds on the relative $q$-entropy for $1 < q \leq 2$ can be derived from the inequalities

$$D_p(\rho \| \sigma) \leq D_1(\rho \| \sigma) \leq D_q(\rho \| \sigma), \tag{3.1}$$
where \(0 < p < 1\). The relations \(3.1\) are actually proved in Ref. \(31\). (Note that the definition of relative entropy in Ref. \(31\) differs from \(1.1\) in the sign, and the formula \(3.1\) is obtained as the formula (9) of that paper with the reversed sign.) The quantum relative entropy obeys (see, e.g., theorem 1.15 in Ref. \(32\))

\[
\frac{1}{2} \| r - \sigma \|_1^2 \leq D_1(r\|\sigma).
\]

(3.2)

This is a quantum analog of the Pinsker inequality from classical information theory. Other lower bounds on the relative entropy \((1.1)\) are presented in Ref. \(17\). By Eq. \(3.1\), these lower bounds are all valid for \(D_q(\rho\|\sigma)\) with \(1 < q \leq 2\). Due to this fact and unboundedness of relative \(q\)-entropy for \(q > 1\), we focus on nontrivial upper bounds. The first upper bound extends the derivation given for the relative entropy \((1.1)\) in Ref. \(33\) (see Example 6.2.31).

**Theorem III.1.** If both the density operators \(\rho\) and \(\sigma\) are strictly positive then for \(1 < q \leq 2\),

\[
D_q(\rho\|\sigma) \leq \frac{1}{q - 1} \rho^{q-1} \| \rho - \sigma \|_{\infty} \leq \frac{1}{2(q - 1)} \rho^{q-1} \| \rho - \sigma \|_1,
\]

(3.3)

\[
D_q(\rho\|\sigma) \leq \frac{1}{q - 1} \rho^{q-1} \| \rho - \sigma \|_1.
\]

(3.4)

where \(a_1 := \max\{a : a \in \text{spec}(\rho)\}, \lambda_0 := \min\{\lambda : \lambda \in \text{spec}(\rho) \cup \text{spec}(\sigma)\}\).

**Proof.** Using the second integral from Eq. \(2.7\) with \(q - 1 = r \in (0; 1)\) and the linearity of the trace, one gives

\[
rD_q(\rho\|\sigma) = \text{Tr}\left\{ \rho^{q - r}(\sigma^r - \rho^r) \right\} = \rho^{q - r} \frac{\sin r\pi}{\pi} \int_0^\infty y^{-r} dy \left\{ (y^2 + \sigma^{1-r} - (y\mathbb{1} + \rho)^{1-r}) \right\}
\]

\[
= \frac{\sin r\pi}{\pi} \int_0^\infty y^{-r} dy \text{Tr}\left( \rho^{q - r}(y^2 + \sigma^{1-r} - (y\mathbb{1} + \rho)^{1-r}) \right).
\]

(3.5)

Here we used \(A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}\) and put \(\Delta = \rho - \sigma\). Due to Eq. \(2.6\), the relation \(3.5\) leads to

\[
rD_q(\rho\|\sigma) \leq \text{Tr}\left| \Delta \rho^q \frac{\sin r\pi}{\pi} \int_0^\infty y^{-r} dy \cdot \| (y\mathbb{1} + \sigma^{1-r}) - (y\mathbb{1} + \rho)^{1-r} \|_{\infty} \right| \| (y\mathbb{1} + \rho)^{-1} \|_{\infty} \leq \text{Tr}\left| \Delta \rho^q \frac{1}{\lambda_0^q} \right|.
\]

(3.6)

Indeed, the eigenvalues of \((y\mathbb{1} + \sigma)^{-1}\) are equal to \((y + b)^{-1}, b \in \text{spec}(\sigma)\), and we find \(\| (y\mathbb{1} + \sigma)^{-1} \|_{\infty} = (y + b_0)^{-1}\) in terms of \(b_0 = \min\{b : b \in \text{spec}(\sigma)\}\). Putting \(\lambda_0 = \min\{a_0, b_0\}\), we further obtain

\[
\frac{\sin r\pi}{\pi} \int_0^\infty y^{-r} dy \frac{1}{(y + b_0)(y + a_0)} = \frac{b_0^{-r} - a_0^{-r}}{a_0 - b_0} \leq \frac{1}{\lambda_0^q}.
\]

(3.7)

From Eq. \(2.5\), we get \(\|XY\|_1 \leq \|X\|_\infty \|Y\|_1\). By \(\text{Tr}\left| \Delta \rho^q \right| \leq \|\Delta\|_\infty \text{Tr}(\rho^q) \leq a_1^{q-1} \|\Delta\|_\infty\), we have the first inequality in Eq. \(3.3\). The second inequality in Eq. \(3.3\) holds due to the fact that the operator \(\Delta\) is traceless and, hence, \(\|\Delta\|_\infty \leq (1/2) \|\Delta\|_1\) (see lemma 4 in Ref. \(17\)). In view of \(\text{Tr}\left| \Delta \rho^q \right| \leq \|\rho^q\|_\infty \|\Delta\|_1 = a_1^q \|\Delta\|_1\), the relation \(3.6\) gives Eq. \(3.3\) as well. The above arguments go through for all \(q \in (1; 2]\). Since the term \(2.3\) is a continuous function of \(q\), the claim remains valid for \(q = 2\). 

The bounds \(3.3\) and \(3.4\) are linear in the corresponding distance between \(\rho\) and \(\sigma\) and characterize a continuity of the relative \(q\)-entropy in the Fannes sense. The former is more appropriate, when the maximal eigenvalue \(a_1\) of \(\rho\) is unknown and replaced with one. The latter is stronger for sufficiently small values of \(a_1\). They both show that the \(D_q(\rho\|\sigma)\) increases no faster than \(\lambda_0^{-q}\) as \(\lambda_0\) goes to zero. However, we would be interested in a bound which involves only the minimal eigenvalue of \(\sigma\). This can be obtained by a \(q\)-parametric extension of the quadratic upper bound from Ref. \(17\) (see theorem 2 therein). Here one auxiliary statement is required. 

**Lemma III.2.** For any two \(A, B \in \mathcal{L}_{++}(\mathcal{H})\) and \(0 < r < 1\), there holds

\[
A^{-r} - B^{-r} \leq \frac{\sin r\pi}{\pi} \int_0^\infty y^{-r} dy (y\mathbb{1} + A)^{-1}(B - A)(y\mathbb{1} + A)^{-1}.
\]

(3.8)

**Proof.** The left-hand side of Eq. \(3.5\) can be recast as \(g(B) - g(A)\), where the function \(g(t) = -t^{-r}\) is operator concave for all \(r \in (0; 1)\). The concavity leads to

\[
(1 - \theta)g(A) + \theta g(B) \leq g((1 - \theta)A + \theta B) = g(A + \theta D),
\]

(3.9)
where $D = B - A$ and $0 < \theta \leq 1$. Due to the integral \(2.7\), we rewrite \(3.9\) as

$$g(B) - g(A) \leq \frac{g(A + \theta D) - g(A)}{\theta} = \frac{\sin \frac{\pi}{\theta} \int_0^\infty y^{-r}dy (y1 + A)^{-1}D (y1 + A + \theta D)^{-1}}{\theta}. \quad (3.10)$$

In the limit $\theta \to +0$, the right-hand side of Eq. \(3.10\) tends to the right-hand side of Eq. \(3.8\), which is herewith the Fréchet differential of $g(\cdot)$ at $A$ in the direction $D$. ■

**Theorem III.3.** For $1 < q \leq 2$, if $\ker(\sigma) \subset \ker(\rho)$ then

$$D_q(\rho||\sigma) \leq \frac{\ln_2(b_1/b_0)}{1 - b_0/b_1} \frac{a_1^{-1}}{b_0^2} ||\rho - \sigma||_1 + \frac{a_1^{-1}}{b_0^2} ||\rho - \sigma||_1, \quad (3.11)$$

where $a_1 := \max\{a : a \in \text{spec}(\rho)\}$, $b_1 := \max\{b : b \in \text{spec}(\sigma)\}$, $b_0 := \min\{b \neq 0 : b \in \text{spec}(\sigma)\}$.

**Proof.** Since $X \leq Y$ gives $\text{Tr}(AX) \leq \text{Tr}(AY)$ for all $A \in \mathcal{L}_+(\mathcal{H})$, the formula \(3.8\) and the trace linearity lead to

$$\text{Tr}\left\{\rho^q(\sigma^{r - \rho})^{r - \rho} \right\} \leq \frac{\sin \frac{\pi}{\rho} \int_0^\infty y^{-r}dy \text{Tr}\left((\rho^{q-1}(\sigma + \Delta)(y1 + \sigma)^{-1}(\sigma + \Delta)(y1 + \sigma)^{-1}\right)}{\pi} \cdot (3.12)$$

The right-hand side of Eq. \(3.12\) can be rewritten as the sum of two terms. Using Eq. \(2.6\) and the submultiplicativity of the spectral norm, the first term is estimated from above by

$$\frac{\sin \frac{\pi}{\rho} \int_0^\infty y^{-r}dy \text{Tr}\left((\rho^{q-1}(\sigma + \Delta)(y1 + \sigma)^{-1}\Delta (y1 + \sigma)^{-1}\right)}{\pi} \leq ||\rho^{q-1}||_\infty \text{Tr}[\Delta] \frac{\sin \frac{\pi}{\rho} \int_0^\infty y^{-r}dy \sigma(y1 + \sigma)^{-1}\sigma(y1 + \sigma)^{-1}||_{\infty}||\sigma(y1 + \sigma)^{-1}||_{\infty} \cdot (3.13)$$

Here we have inserted $||\sigma(y1 + \sigma)^{-1}||_{\infty} = b_1(y1 + b_1)^{-1}$, since the eigenvalues of $\sigma(y1 + \sigma)^{-1}$ are equal to $b(y1 + b)^{-1}$, and the latter is an increasing function of $b$. Dividing the right-hand side of Eq. \(3.13\) by $r = q - 1$, we obtain the first term of the right-hand side of Eq. \(3.11\). Using Eq. \(2.6\) and the submultiplicativity again, the second term of the right-hand side of Eq. \(3.12\) is estimated by

$$\frac{\sin \frac{\pi}{\rho} \int_0^\infty y^{-r}dy \text{Tr}\left((\rho^{q-1}(\sigma + \Delta)(y1 + \sigma)^{-1}\Delta (y1 + \sigma)^{-1}\right)}{\pi} \leq ||\rho^{q-1}||_\infty ||\sigma(y1 + \sigma)^{-1}||_{\infty} ||\sigma(y1 + \sigma)^{-1}||_{\infty} \frac{\text{Tr}[\Delta]}{\pi} \frac{b_1}{b_1 - 1} \frac{1}{b_1 - 1} = a_1^{-1} ||\Delta||_1 \frac{b_1 - 1}{b_1 - 1} \cdot (3.14)$$

Dividing this by $r = q - 1$, we obtain the second term of the right-hand side of Eq. \(3.11\). ■

The bound \(3.11\) is not purely quadratic, but its second term dominant for small $b_0$ is just quadratic in a distance between $\rho$ and $\sigma$. For traceless $\Delta = \rho - \sigma$, we have $||\Delta||_\infty \leq (1/2) ||\Delta||_1$ (see lemma 4 in Ref. \[17\]), whence

$$D_q(\rho||\sigma) \leq \frac{\ln_2(b_1/b_0)}{1 - b_0/b_1} \frac{a_1^{-1}}{b_0^2} ||\rho - \sigma||_1 + \frac{a_1^{-1}}{b_0^2} ||\rho - \sigma||_1^2. \quad (3.15)$$

The upper continuity bounds \(3.11\) and \(3.15\) involve the minimal eigenvalue only of $\sigma$ and suit for singular $\rho$. These points are advantages of these bounds. At the same time, we would like to find an upper bound with the dependence $b_0^{1-q}$ in view of the term $\sigma^{1-q}$ in the definition of $D_q(\rho||\sigma)$. Indeed, for the standard relative entropy \[17\] we have the bound which is logarithmic in the minimal eigenvalue of $\sigma$ \[17\].

**IV. UPPER BOUNDS FOR ARBITRARY $q > 1$**

In this section, we present upper continuity bounds that cover all the values $q > 1$. Moreover, these bounds has the dominant term with a dependence $b_0^{1-q}$ in the minimal eigenvalue of $\sigma$. We first consider the case of integer powers.
Lemma IV.1. For integer $n \geq 1$ and $X, Y \in \mathcal{L}(\mathcal{H})$, there holds
\[ ||X^n - Y^n||_p \leq n\lambda_1^{n-1} ||X - Y||_p ,\]
where $\lambda_1 := \max\{||X||_\infty, ||Y||_\infty\}$.

Proof. We shall proceed by induction. For $n = 1$, the claim is obvious. For $n > 1$, we write
\[ X^{n+1} - Y^{n+1} = X^{n+1} - X^nY + X^nY - Y^{n+1} = X^n(X - Y) + (X^n - Y^n)Y. \]
By means of the triangle inequality, Eq. (4.3) and $||X^n||_\infty \leq \lambda_1^n$, we then obtain
\[ ||X^{n+1} - Y^{n+1}||_p \leq ||X^n||_p ||X - Y||_p + ||(X^n - Y^n)Y||_p \leq \lambda_1^n ||X - Y||_p + \lambda_1 ||X^n - Y^n||_p. \]
Assuming Eq. (4.1), we derive $||X^{n+1} - Y^{n+1}||_p \leq (n+1)\lambda_1^n ||X - Y||_p$ too. \( \blacksquare \)

Remark IV.2. Let $||.||$ be a submultiplicative norm and $X, Y \in \mathcal{L}(\mathcal{H})$. For integer $n \geq 1$, there holds
\[ ||X^n - Y^n|| \leq n \theta^{n-1} ||X - Y||. \]
where $\theta := \max\{||X||, ||Y||\}$. The claim can be proved similarly to the statement of Lemma IV.1.

Using Lemma IV.1 we now obtain the upper bound of a kind $b_0^{1-q}$ for any $q > 1$, including non-integer $q$. By $[q]$ and $\lceil q \rceil$ we respectively denote the floor and ceiling of real $q$. The main result of this section is stated as follows.

Theorem IV.3. For arbitrary $q > 1$, if $\ker(\sigma) \subset \ker(\rho)$ then
\[ D_q(\rho||\sigma) \leq \frac{[q] - 1}{q - 1} \frac{\lambda_1^{q-1}}{b_0^q} ||\rho - \sigma||_1, \]
where $\lambda_1 := \max\{\lambda : \lambda \in \sigma(\rho) \cup \sigma(\sigma)\}$, $b_0 := \min\{b \neq 0 : b \in \sigma(\sigma)\}$.

Proof. For non-integer $q > 1$, we write $n = [q]$, $s = q - n$, and the identity
\[ \rho^{n+s} - \sigma^{n+s} = (\rho^n - \sigma^n)\rho^s + \sigma^n(\rho^s - \sigma^s). \]
By the linearity of the trace and $r = q - 1$, we then have
\[ rD_q(\rho||\sigma) = \text{Tr} \{\sigma^{-r}(\rho^s - \sigma^s)\} = \text{Tr} \{\sigma^{-r}(\rho^n - \sigma^n)\rho^s\} + \text{Tr} \{\sigma^{1-s}(\rho^s - \sigma^s)\}. \]
Due to Eqs. (2.10) and (4.11), the first trace in the right-hand side of Eq. (4.7) is bounded from above by the quantity
\[ ||\sigma^{-r}||_\infty ||\rho^n - \sigma^n||_1 ||\rho^s||_\infty \leq na_1^s \lambda_1^{q-1} b_0^{-r} ||\rho - \sigma||_1 \leq [q] \lambda_1^{q-1} b_0^{-r} ||\rho - \sigma||_1. \]
The second trace is merely $\text{Tr}(\sigma^{1-s}\rho^s) - 1 = -(1-s)\text{D}_q(\rho||\sigma) \leq 0$ in view of $s \in (0; 1)$ and the positivity of the relative $s$-entropy. Combining this fact with Eqs. (4.7) and (4.8) then gives the claim for non-integer $q > 1$. For $q \in (n; n+1)$, the right-hand side of Eq. (4.8) is a continuous function of the parameter $q$. The expression for $D_q(\rho||\sigma)$ is continuous for all $q \neq 1$. So the bound (4.5) remains valid when $q$ tends to an integer from below. \( \blacksquare \)

With respect to small $b_0$, the right-hand side of Eq. (4.5) has a dependence $b_0^{1-q}$. So the upper continuity bound (4.9) is stronger than the bounds of Theorems III.1 and III.3. The upper bound (4.5) also involves $\lambda_1^{q-1}$ and the trace norm distance between density operators. By a structure, the right-hand side of Eq. (4.5) is in good agreement with the expression for the relative $q$-entropy. Another advantage of the bound (4.5) is that it covers the range $q > 2$ as well. At the same time, the factor $(\lceil q \rceil - 1)/(q - 1)$ is not continuous in $q$. For very interesting values, $1 < q < 2$, the relation (4.8) leads to the inequality
\[ D_q(\rho||\sigma) \leq \frac{1}{q-1} \frac{a_1^{q-1}}{b_0^{q-1}} ||\rho - \sigma||_1, \]
with $a_1$ instead of $\lambda_1$. In general, the upper continuity bound (4.9) seems to be more appropriate among the ones presented above for $1 < q \leq 2$. For sufficiently small $b_0$, this bound is clearly tighter than (3.11) and (3.15). On the other hand, the upper bounds (3.11) and (3.15) may be more sharpening in the sense of closeness of the states $\rho$ and $\sigma$, when $b_0$ is not very small. Due to $a_1 \leq \lambda_1 \leq 1$, all the above bounds can be rewritten with one instead of the maximal eigenvalue. Finally, we note that our methods could be used for estimating the modulus of the second trace in the right-hand side of Eq. (4.7). We present one continuity bound of such a kind, though it was not required for the proof of Theorem IV.3.
Lemma IV.4. Let $A \in \mathcal{L}_+(\mathcal{H})$, $B \in \mathcal{L}_+(\mathcal{H})$ and $\text{Tr}(A) = \text{Tr}(B) = \tau$. For $0 < s < 1$, there holds

$$|\text{Tr}(B^{1-s}A^s) - \tau| \leq \frac{a_1}{b_0} \|A - B\|_1,$$

(4.10)

where $a_1 := \max\{a : a \in \text{spec}(A)\}$, $b_0 := \min\{b : b \in \text{spec}(B)\}$.

**Proof.** Using the first integral representation from Eq. (2.22) and the properties of the trace, we have

$$\text{Tr}(B^{-s}A^s) = \frac{\sin s\pi}{\pi} \int_0^\infty x^{s-1} dx \text{Tr}\left(B^{-s}A(A + x\mathbb{1})^{-1}\right),$$

(4.11)

$$\tau = \text{Tr}(B^{-s}AB^s) = \frac{\sin s\pi}{\pi} \int_0^\infty x^{s-1} dx \text{Tr}\left(B^{-s}A(B + x\mathbb{1})^{-1}\right).$$

(4.12)

Combining Eqs. (4.11) and (4.12) with the identity

$$A(A + x\mathbb{1})^{-1}B - A(B + x\mathbb{1})^{-1}B = A(A + x\mathbb{1})^{-1}\{(B + x\mathbb{1}) - (A + x\mathbb{1})\}(B + x\mathbb{1})^{-1}B$$

(4.13)

and putting $D = B - A$, we obtain

$$\text{Tr}(B^{1-s}A^s) - \tau = \frac{\sin s\pi}{\pi} \int_0^\infty x^{s-1} dx \text{Tr}\left(B^{-s}A(A + x\mathbb{1})^{-1}D(B + x\mathbb{1})^{-1}B\right).$$

(4.14)

Using the cyclic property of the trace and Eq. (2.9), one gets

$$|\text{Tr}(B^{1-s}A^s) - \tau| \leq \text{Tr}|D| \frac{\sin s\pi}{\pi} \int_0^\infty x^{s-1} dx \|A(A + x\mathbb{1})^{-1}\|_\infty \|(B + x\mathbb{1})^{-1}B^{1-s}\|_\infty.$$

(4.15)

The eigenvalues of $A(A + x\mathbb{1})^{-1}$ are equal to $a(a + x)^{-1} \leq a_1(a_1 + x)^{-1}$, the eigenvalues of $(B + x\mathbb{1})^{-1}B^{1-s}$ are equal to $(b + x)^{-1}b^{1-s} \leq b^{-s} \leq b_0^{-s}$, whence

$$\|A(A + x\mathbb{1})^{-1}\|_\infty = a_1(a_1 + x)^{-1}, \quad \|(B + x\mathbb{1})^{-1}B^{1-s}\|_\infty \leq b_0^{-s}.$$ (4.16)

Due to these relations and $\text{Tr}|D| = \|A - B\|_1$, the inequality (4.15) leads to

$$|\text{Tr}(B^{1-s}A^s) - \tau| \leq \frac{1}{b_0} \|A - B\|_1 \frac{\sin s\pi}{\pi} \int_0^\infty x^{s-1} dx \frac{a_1}{a_1 + x}.$$

(4.17)

By the first integral of Eq. (2.27), this is equivalent to Eq. (4.10). \(\blacksquare\)

V. SUMMARY

Similar to the standard relative entropy, the Tsallis relative $q$-entropy is generally unbounded for $q > 1$. Hence upper bounds on this functional are of interest. In this paper, we have obtained several upper bounds on the relative $q$-entropy for $q > 1$. These bounds are expressed in terms of norm distances between its two arguments and the minimal eigenvalue of the second argument. The presented inequalities characterize the property of continuity of the relative $q$-entropy for states which are close in the trace norm sense. They also estimate from above the rate of divergence of the relative $q$-entropy when the minimal eigenvalue of its second argument goes to zero. The considered upper bounds can be regarded as some $q$-parametric extensions of those bounds that have been obtained in the literature for the standard relative entropy. To derive the results, we have extensively used general properties of the trace and spectral norms as well as the known integral representations of matrix fractional powers.

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