ON THE UNILATERAL SHIFT AS A HILBERT MODULE OVER
THE DISC ALGEBRA

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Abstract. We study the unilateral shift (of arbitrary countable multiplicity) as a Hilbert module over the disc algebra and the associated extension groups. In relation with the problem of determining whether this module is projective, we consider a special class of extensions, which we call polynomial. We show that the subgroup of polynomial extensions of a contractive module by the adjoint of the unilateral shift is trivial. The main tool is a function theoretic decomposition of the Sz.-Nagy–Foias model space for completely non-unitary contractions.

1. Introduction

In their pioneering work [9], Douglas and Paulsen reformulated several interesting operator theoretic questions in the language of module theory, and in doing so introduced the notion of Hilbert modules over function algebras. This suggested the use of cohomological methods to further the study of problems such as commutant lifting. Naturally, the question of identifying those Hilbert modules which are projective arose and attracted a lot of interest. The first result in that direction was obtained by Carlson and Clark in [2], where it was shown that a contractive projective Hilbert module over the disc algebra \( A(\mathbb{D}) \) must be similar to an isometric one. Soon thereafter, the same authors along with Foias and Williams proved in [4] that isometric modules over \( A(\mathbb{D}) \) are projective in the category of contractive Hilbert modules. This turns out to be an equivalence, as was later shown by Ferguson in [10]. In addition, the authors of [4] show that unitary modules over \( A(\mathbb{D}) \) are projective in the larger category of (non-necessarily contractive) Hilbert modules.

Projective Hilbert modules over \( A(\mathbb{D}) \) are to this day still quite mysterious. In fact, as things stand currently, unitary modules are the only known instances of such objects. On the other hand, by the results mentioned above a contractive projective module must be similar to an isometric module. In view of the classical Wold-von Neumann decomposition of an isometry, we see that the quest to identify the contractive projective Hilbert modules over the disc algebra is reduced to the following question: are unilateral shifts projective? A consequence of Pisier’s famous counterexample to the Halmos conjecture (see [16]) is that the answer is negative in the case of infinite multiplicity. Whether or not things are different for finite multiplicities is still an open problem.

We study extension groups associated to unilateral shifts viewed as Hilbert modules over the disc algebra. With the notation established in Section 2, our main result (Theorem 5.4) establishes the triviality of the subgroup of elements...
\([X] \in \text{Ext}^1_{A(D)}(T, S^*)\) such that \(S^*NXT^N = 0\) for some integer \(N \geq 0\) whenever \(T\) is similar to a contraction (here \(S^*\) is the adjoint of the unilateral shift of arbitrary countable multiplicity). In some sense, this supports the idea that the unilateral shift is projective. However, the reader should keep in mind that our result holds regardless of multiplicity and thus does not capture the fact that the shift of infinite multiplicity is not projective. The crucial ingredient for the proof of Theorem 5.4 is a decomposition of the Sz.-Nagy–Foias model space \(H(\Theta)\) which we think is of independent interest (see Theorem 5.3).

There has been further work on the question of projective Hilbert modules following the appearance of [2], [4] and [10]. Generalizing the fact that unitary modules are projective over \(A(D)\), it was shown in [5] that whenever the algebra \(A\) is a so-called unit modulus algebra and the module action can be extended to an action of \(C(\partial A)\) (here \(\partial A\) denotes the Shilov boundary of \(A\)), then the module is projective. An earlier paper of Guo (see [12]) establishes using essentially the same idea that the result holds for the ball algebra \(A(\mathbb{B}^N)\) under an additional continuity assumption on the module action. This assumption was later removed by Didas and Eschmeier in [8], where domains more general than the ball are considered. The case of the polydisc algebra \(A(\mathbb{D}^N)\) was first considered in [3], where results exhibiting a sharp contrast with the one dimensional case were obtained. From the point of view of reproducing kernel Hilbert spaces, Clancy and McCullough showed in [6] that the Hilbert space \(H^2(k)\) associated to a Nevanlinna-Pick kernel \(k\) considered as a Hilbert module over its multiplier algebra is projective in an appropriate category. The existence of a projective Hilbert module over very general function algebras was established in [11]. Note finally that the notion of Hilbert modules and the question of projectivity have also been studied over general operator algebras, see [13].

The paper is organized as follows. Section 2 introduces the necessary preliminaries about Hilbert modules. In Section 3 we develop some technical tools which are used in Section 5 to obtain the main result. In the meantime, we examine in Section 4 some simple examples and offer some explicit calculations of the objects introduced in Section 3. Finally, in Section 6 we briefly address the issue of non-contractive modules by considering operators of the type constructed by Pisier in [16].

2. Preliminaries

Let \(\mathcal{H}\) be a Hilbert space and let \(T : \mathcal{H} \to \mathcal{H}\) be a bounded linear operator, which we indicate by \(T \in B(\mathcal{H})\). Recall that the operator \(T\) is said to be \textit{polynomially bounded} if there exists a constant \(C > 0\) such that for every polynomial \(\varphi\), we have

\[
\|\varphi(T)\| \leq C \|\varphi\|_{\infty}
\]

where

\[
\|\varphi\|_{\infty} = \sup_{|z| < 1} |\varphi(z)|.
\]

This inequality allows one to extend continuously the polynomial functional calculus \(\varphi \mapsto \varphi(T)\) to all functions \(\varphi\) in the disc algebra \(A(\mathbb{D})\), which consists of the holomorphic functions on \(\mathbb{D}\) that are continuous on \(\overline{\mathbb{D}}\) (throughout the paper \(\mathbb{D}\) denotes the open unit disc and \(T\) denotes the unit circle).
If $T \in B(H)$ is a polynomially bounded operator, the map

$$A(D) \times H \to H$$

$$(\varphi, h) \mapsto \varphi(T)h$$

gives rise to a structure of an $A(D)$-module on $H$, and we say that $(H, T)$ is a Hilbert module (see [9] for more details). We only deal with $A(D)$-modules in this paper, so no confusion may arise regarding the underlying function algebra and we usually do not mention it explicitly. Moreover, when the underlying Hilbert space is understood, we slightly abuse terminology and say that $T$ is a Hilbert module.

Given two Hilbert modules $(H_1, T_1)$ and $(H_2, T_2)$, we can consider the extension group $\text{Ext}^1_{A(D)}(T_2, T_1)$. This group consists of equivalence classes of exact sequences

$$0 \to H_1 \to K \to H_2 \to 0$$

where $K$ is another Hilbert module and each map is a module morphism. Rather than formally defining the equivalence relation and the group operation, we simply use the following characterization from [2].

**Theorem 2.1.** Let $(H_1, T_1)$ and $(H_2, T_2)$ be Hilbert modules. Then, the group

$$\text{Ext}^1_{A(D)}(T_2, T_1)$$

is isomorphic to $A/J$, where $A$ is the space of operators $X : H_2 \to H_1$ for which the operator

$$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is polynomially bounded, and $J$ is the space of operators of the form $T_1 L - L T_2$ for some bounded operator $L : H_2 \to H_1$.

Extension groups are invariant under similarity: if $(H'_1, T'_1)$ and $(H'_2, T'_2)$ are Hilbert modules which are similar to $(H_1, T_1)$ and $(H_2, T_2)$ respectively, then the groups $\text{Ext}^1_{A(D)}(T_2, T_1)$ and $\text{Ext}^1_{A(D)}(T'_2, T'_1)$ are isomorphic.

In view of Theorem 2.1, the next lemma is useful. Before stating it, we recall a well-known estimate. Let

$$D : A(D) \to A(D)$$

be defined as

$$(D f)(z) = \frac{1}{z} (f(z) - f(0))$$

for every $z \in D$ and $f \in A(D)$. It is a classical fact that there exists a constant $M > 0$ such that

$$\|D^n\| \leq M (1 + \log n)$$

for every $n \geq 1$.

**Lemma 2.2.** Let $(H_1, T_1)$ and $(H_2, T_2)$ be Hilbert modules. Let $X : H_2 \to H_1$ be a bounded operator such that $T_1^N X T_2^N = 0$ for some integer $N \geq 0$. Then, the operator $R : H_1 \oplus H_2 \to H_1 \oplus H_2$ defined as

$$R = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is polynomially bounded.
Proof. Choose a $\varphi(z) = \sum_{k=0}^d a_k z^k$. A quick calculation shows that

$$\varphi(R) = \begin{pmatrix} \varphi(T_1) & \delta_X(\varphi) \\ 0 & \varphi(T_2) \end{pmatrix}$$

where

$$\delta_X(\varphi) = \sum_{k=1}^d a_k \sum_{j=0}^{k-1} T_1^j \sum_{j=0}^{k-1} T_2^{k-1-j}.$$ 

Since both $T_1$ and $T_2$ are polynomially bounded, there exist constants $C_1, C_2 > 0$ independent of $\varphi$ such that

$$\|\varphi(T_1)\| \leq C_1 \|\varphi\|_\infty$$

and

$$\|\varphi(T_2)\| \leq C_2 \|\varphi\|_\infty.$$ 

Therefore, we simply need to verify that there exists a constant $C > 0$ independent of $\varphi$ such that

$$\|\delta_X(\varphi)\| \leq C \|\varphi\|_\infty.$$ 

We have

$$\|\delta_X(\varphi)\| \leq \sum_{k=1}^d |a_k| \sum_{j=0}^{k-1} \|T_1^j X T_2^{k-1-j}\|$$

$$\leq C_1 C_2 \|X\| \sum_{k=1}^d k |a_k|$$

$$= C_1 C_2 \|X\| \sum_{k=1}^d k \left| \frac{(k)(0)}{k!} \right|$$

and in light of the classical Cauchy estimates, we find

(1) $\|\delta_X(\varphi)\| \leq C_1 C_2 \frac{d(d + 1)}{2} \|X\| \|\varphi\|_\infty.$

In particular, if we set

$$C = C_1 C_2 N (2N - 1) \|X\|,$$

which depends only on $X, T_1, T_2$ and $N$, then

$$\|\delta_X(\varphi)\| \leq C \|\varphi\|_\infty$$

whenever $\varphi$ has degree at most $2N - 1$. We focus therefore on the case where $d \geq 2N$. We have

(2) $\delta_X(\varphi) = \sum_{k=1}^{2N-1} a_k \sum_{j=0}^{k-1} T_1^j X T_2^{k-1-j} + \sum_{k=2N}^d a_k \sum_{j=0}^{k-1} T_1^j X T_2^{k-1-j}.$

Since

$$\left| \sum_{k=1}^{2N-1} a_k \sum_{j=0}^{k-1} T_1^j X T_2^{k-1-j} \right| \leq C \|\varphi\|_\infty$$

we have

$$\|\delta_X(\varphi)\| \leq C \|\varphi\|_\infty.$$

Therefore, we have verified the desired result.
we are left with estimating the second sum in (2), where \( k \geq 2N \). By assumption, we know that \( T_j^j X T_2^{k-1-j} \neq 0 \) only when \( j \leq N - 1 \) or \( k - 1 - j \leq N - 1 \). This allows us to write
\[
\sum_{k=2N}^{d} a_k \sum_{j=0}^{k-1} T_j^j X T_2^{k-1-j} = \sum_{k=2N}^{d} a_k \sum_{j=0}^{N} T_j^j X T_2^{k-1-j} + \sum_{k=2N}^{d} a_k \sum_{j=k-N}^{k-1} T_j^j X T_2^{k-1-j}
\]
\[
= \sum_{k=2N}^{d} a_k \sum_{j=0}^{N} T_j^j X T_2^{k-1-j} + \sum_{k=2N}^{d} a_k \sum_{j=0}^{N-1} T_j^j X T_2^{k-1-j}
\]
\[
= \sum_{j=0}^{N-1} (T_j^j X \Phi_j(T_2) + \Phi_j(T_1) XT_2^j)
\]
where
\[
\Phi_j(z) = \sum_{k=2N}^{d} a_k z^{k-1-j}
\]
whenever \( 0 \leq j \leq N - 1 \). Notice now that
\[
\Phi_j(z) + \sum_{k=j+1}^{2N-1} a_k z^{k-1-j} = (D^{j+1} \varphi)(z)
\]
whence
\[
\|\Phi_j\|_\infty \leq \|D^{j+1} \varphi\|_\infty + \sum_{k=0}^{2N-1} |a_k|
\]
\[
= \|D^{j+1} \varphi\|_\infty + \sum_{k=0}^{2N-1} \frac{|\varphi^{(k)}(0)|}{k!}
\]
for every \( 0 \leq j \leq N - 1 \). Another use of the Cauchy estimates along with the remark preceding the statement of the lemma implies the existence of a constant \( C' > 0 \) depending only on \( N \) such that
\[
\|\Phi_j\|_\infty \leq C' \|\varphi\|_\infty
\]
for every \( 0 \leq j \leq N - 1 \). Thus,
\[
\left\| \sum_{k=2N}^{d} a_k \sum_{j=0}^{k-1} T_j^j X T_2^{k-1-j} \right\| \leq 2NC' C_1 \|X\| \|\varphi\|_\infty
\]
and
\[
\|\delta_X(\varphi)\| \leq \left\| \sum_{k=0}^{2N-1} a_k \sum_{j=0}^{k-1} T_j^j X T_2^{k-1-j} \right\| + \left\| \sum_{k=2N}^{d} a_k \sum_{j=0}^{k-1} T_j^j X T_2^{k-1-j} \right\|
\]
\[
\leq C'' \|\varphi\|_\infty
\]
where \( C'' > 0 \) depends only on \( N, X, T_1, T_2 \). The proof is complete. \( \square \)

An important question in the study of extension groups is that of determining which Hilbert modules \((H_2, T_2)\) have the property that
\[
\text{Ext}^1_{A(D)}(T_2, T_1) = 0
\]
for every Hilbert module \((\mathcal{H}_1, T_1)\). Such Hilbert modules are said to be projective. It is easy to verify using Theorem 2.1 that the map \([X] \mapsto [X^*]\) establishes an isomorphism between the groups \(\text{Ext}^1_{A(D)}(T_2, T_1)\) and \(\text{Ext}^1_{A(D)}(T_1^*, T_2^*)\), so \(T_2\) is projective if and only if

\[
\text{Ext}^1_{A(D)}(T_1, T_2^*) = 0
\]

for every Hilbert module \((\mathcal{H}_1, T_1)\). A characterization of projective Hilbert modules has long been sought. This result from [4] was mentioned in the introduction.

**Theorem 2.3.** If \(T \in B(\mathcal{H})\) is similar to a unitary operator, then the Hilbert module \((\mathcal{H}, T)\) is projective.

If \(\mathcal{E}\) is a separable Hilbert space, we denote by \(L^2(\mathcal{E})\) the Hilbert space of weakly measurable square integrable functions \(f : \mathbb{T} \to \mathcal{E}\). The Hardy space \(H^2(\mathcal{E})\) is the closed subspace of \(L^2(\mathcal{E})\) consisting of functions with vanishing negative Fourier coefficients. Elements of \(H^2(\mathcal{E})\) can also be viewed as \(\mathcal{E}\)-valued functions holomorphic on \(\mathbb{D}\) with square summable Taylor coefficients. We embed \(\mathcal{E}\) in \(H^2(\mathcal{E})\) as the subspace consisting of constant functions, and we denote by \(P_\mathcal{E}\) the orthogonal projection of \(H^2(\mathcal{E})\) onto \(\mathcal{E}\). When \(\mathcal{E} = \mathbb{C}\), we simply write \(H^2(\mathcal{E}) = H^2\) and \(L^2(\mathcal{E}) = L^2\). The unilateral shift operator

\[
S_\mathcal{E} : H^2(\mathcal{E}) \to H^2(\mathcal{E})
\]

is defined as

\[
(S_\mathcal{E} f)(z) = zf(z)
\]

for every \(f \in H^2(\mathcal{E})\). Recall that the multiplicity of \(S_\mathcal{E}\) is the dimension of \(\mathcal{E}\). Note also that since \(S_\mathcal{E}\) is isometric, it gives rise to a Hilbert module structure on \(H^2(\mathcal{E})\).

We now give a rather precise description of the group \(\text{Ext}^1_{A(D)}(T, S)\) where \((\mathcal{H}, T)\) is any Hilbert module. This result was originally proved in [2] (Proposition 3.1.1 and Theorem 3.2.1) for the shift of multiplicity one. However, a quick glance at the proof of Proposition 3.1.1 shows that it can be adapted to any multiplicity, while the more general version of Theorem 3.2.1 can be found in Lemma 2.1 of [3].

**Theorem 2.4.** Let \((\mathcal{H}, T)\) be a Hilbert module. Then, an operator \(X : \mathcal{H} \to \mathcal{E}\) gives rise to an element \([X] \in \text{Ext}^1_{A(D)}(T, S_\mathcal{E})\) if and only if there exists a constant \(c > 0\) such that

\[
\sum_{n=0}^{\infty} \|XT^n h\|^2 \leq c\|h\|^2
\]

for every \(h \in \mathcal{H}\). Moreover, for every \([X] \in \text{Ext}^1_{A(D)}(T, S_\mathcal{E})\) there exists an operator \(Y : \mathcal{H} \to \mathcal{E}\) with the property that \([X] = [Y]\).

We bring the reader’s attention to the fact that the group \(\text{Ext}^1_{A(D)}(T, S_\mathcal{E})\) is really of a “scalar” nature: it consists of elements \([X]\) where the operator \(X : \mathcal{H} \to H^2(\mathcal{E})\) has range contained in the constant functions \(\mathcal{E}\). We use Theorem 2.4 throughout as a basis for comparison with our own results about \(\text{Ext}^1_{A(D)}(T, S_\mathcal{E}^*)\).

Finally, we end this section with a theorem that identifies the projective modules in the smaller category of contractive Hilbert modules (see [10]).

**Theorem 2.5.** Let \(T \in B(\mathcal{H})\) be similar to a contraction. The following statements are equivalent:

(i) \(\text{Ext}^1_{A(D)}(T, S_\mathcal{E}) = 0\) for some separable Hilbert space \(\mathcal{E}\)
(ii) the Hilbert module \((\mathcal{H}, T)\) is projective in the category of Hilbert modules similar to a contractive one.

(iii) the operator \(T\) is similar to an isometry.

3. A criterion for the projectivity of isometric Hilbert modules

Throughout the paper we will assume that \(\mathcal{E}\) is a separable Hilbert space. The first result of this section is elementary. We record it here for convenience.

**Lemma 3.1.** Let \(X, \Lambda : \mathcal{H} \to H^2(\mathcal{E})\) be bounded operators defined as

\[
Xh = \sum_{n=0}^{\infty} z^n X_n h
\]

and

\[
\Lambda h = \sum_{n=0}^{\infty} z^n L_n h
\]

for every \(h \in \mathcal{H}\), where \(L_n, X_n \in B(\mathcal{H}, \mathcal{E})\) for every \(n \geq 0\). Then, \(X = S_\varepsilon^* \Lambda - \Lambda T\) if and only if \(X_n = L_{n+1} - L_n T\) for every \(n \geq 0\).

The following observation lies at the base of our investigations.

**Lemma 3.2.** Let \((\mathcal{H}, T)\) be a Hilbert module. Let \(X : \mathcal{H} \to H^2(\mathcal{E})\) be a bounded operator defined as

\[
Xh = \sum_{n=0}^{\infty} z^n X_n h
\]

for every \(h \in \mathcal{H}\), where \(X_n \in B(\mathcal{H}, \mathcal{E})\) for every \(n \geq 0\). Let \(c > 0\) and \(L \in B(\mathcal{H}, \mathcal{E})\). Then, there exists a bounded operator \(\Lambda : \mathcal{H} \to H^2(\mathcal{E})\) such that

\[
X = S_\varepsilon^* \Lambda - \Lambda T,
\]

\[
P_\varepsilon \Lambda = -L
\]

and

\[
\|\Lambda h\|^2 \leq \|Lh\|^2 + c\|h\|^2
\]

for every \(h \in \mathcal{H}\) if and only if

\[
\sum_{n=1}^{\infty} \left\| \sum_{j=0}^{n-1} X_{n-1-j} T^j - LT^n \right\|^2 \leq c\|h\|^2
\]

for every \(h \in \mathcal{H}\).

**Proof.** Assume first that

\[
\sum_{n=1}^{\infty} \left\| \sum_{j=0}^{n-1} X_{n-1-j} T^j - LT^n \right\|^2 \leq c\|h\|^2
\]

for every \(h \in \mathcal{H}\). Set \(L_0 = -L\) and

\[
L_n = \sum_{j=0}^{n-1} X_{n-1-j} T^j - LT^n
\]

for \(n \geq 1\). Notice now that we have

\[
L_0 T = -LT = L_1 - X_0
\]
and

\[ L_nT = \sum_{j=0}^{n-1} X_{n-1-j}T^{j+1} - LT^{n+1} \]
\[ = \sum_{j=1}^{n} X_{n-j}T^j - LT^{n+1} \]
\[ = \sum_{j=0}^{n} X_{n-j}T^j - X_n - LT^{n+1} \]
\[ = L_{n+1} - X_n \]

for \( n \geq 1 \), which shows that for every \( n \geq 0 \) we have

\[ X_n = L_{n+1} - L_nT. \]

Define

\[ \Lambda h = \sum_{n=0}^{\infty} z^n L_n h \]

for every \( h \in \mathcal{H} \). By Lemma 3.1, we see that

\[ S_\xi^* \Lambda - \Lambda T = X. \]

Moreover, by assumption we have for every \( h \in \mathcal{H} \) that

\[ \|\Lambda h\|^2 = \sum_{n=0}^{\infty} \|L_n h\|^2 \]
\[ = \|Lh\|^2 + \sum_{n=1}^{\infty} \left\| \left( \sum_{j=0}^{n-1} X_{n-1-j}T^j - LT^n \right) h \right\|^2 \]
\[ \leq \|Lh\|^2 + c\|h\|^2. \]

Conversely, assume that there exists a bounded linear operator \( \Lambda : \mathcal{H} \to H^2(\mathcal{E}) \) defined as

\[ \Lambda h = \sum_{n=0}^{\infty} z^n L_n h \]

for every \( h \in \mathcal{H} \) with the property that

\[ X = S_\xi^* \Lambda - \Lambda T, \]
\[ L_0 = -L \]

and

\[ \|\Lambda h\|^2 \leq \|Lh\|^2 + c\|h\|^2 \]

for every \( h \in \mathcal{H} \). Then, by Lemma 3.1 we have that

\[ X_n = L_{n+1} - L_nT \]
for every $n \geq 0$, so that we find
\[
\sum_{j=0}^{n} X_{n-j}T^j = \sum_{j=0}^{n} (L_{n-j+1} - L_{n-j}T)T^j = \sum_{j=0}^{n} L_{n-j+1}T^j - \sum_{j=1}^{n+1} L_{n-j+1}T^j = L_{n+1} - L_0 T^{n+1} = L_{n+1} + LT^{n+1}
\]
for every $n \geq 0$. Consequently,
\[
\sum_{n=1}^{\infty} \left\| \left( LT^n - \sum_{j=0}^{n-1} X_{n-1-j}T^j \right) h \right\|^2 = \sum_{n=1}^{\infty} \| L_n h \|^2 = \| \Lambda h \|^2 - \| L h \|^2 \leq c \| h \|^2
\]
and the proof is complete. \(\square\)

As suggested by this result, we make the following definition.

**Definition 3.3.** Let \((\mathcal{H}, T)\) be a Hilbert module and let \(\mathcal{E}\) be a separable Hilbert space. We denote by \(Z_\mathcal{E}(T)\) the subspace of \(B(\mathcal{H}, \mathcal{E})\) consisting of the operators \(X \in B(\mathcal{H}, \mathcal{E})\) with the property that there exists a constant \(c_X > 0\) such that
\[
\sum_{n=0}^{\infty} \| X T^n h \|^2 \leq c_X \| h \|^2
\]
for every \(h \in \mathcal{H}\).

By Theorem 2.4, we see that the set \(Z_\mathcal{E}(T)\) consists exactly of those operators \(X : \mathcal{H} \to \mathcal{E}\) which give rise to an element \([X] \in Ext^1_{A(\mathcal{D})}(T, S_\mathcal{E})\).

We now give a criterion for an element \([X] \in Ext^1_{A(\mathcal{D})}(T, S_\mathcal{E}^*)\) to be trivial when \(X\) is particularly simple, namely of the type considered in Lemma 2.2.

**Theorem 3.4.** Let \((\mathcal{H}, T)\) be a Hilbert module. Let \(X : \mathcal{H} \to H^2(\mathcal{E})\) be defined as
\[
Xh = \sum_{n=0}^{\infty} z^n X_n h
\]
for every \(h \in \mathcal{H}\), where \(X_n \in B(\mathcal{H}, \mathcal{E})\) for every \(n \geq 0\). Assume that \(S_\mathcal{E}^* X T^N = 0\) for some integer \(N \geq 0\). Then, the element \([X] \in Ext^1_{A(\mathcal{D})}(T, S_\mathcal{E}^*)\) is trivial if and only if
\[
\sum_{j=0}^{N-1} X_j T^{N-1-j} \in B(\mathcal{H}, \mathcal{E}) T^N + Z_\mathcal{E}(T).
\]

**Proof.** By Lemma 3.2, we find that \([X] = 0\) if and only if for some \(L \in B(\mathcal{H}, \mathcal{E})\) we have
\[
\sum_{n=1}^{\infty} \left\| \left( LT^n + \sum_{j=0}^{n-1} X_{n-1-j}T^j \right) h \right\|^2 \leq c \| h \|^2
\]
for some constant $c > 0$ and every $h \in H$. Now, the condition $S_\varepsilon^N X T^N = 0$ implies that $X_n T^m = 0$ if $n \geq N$ and $m \geq N$. Thus, $X_{n-1-j} T^j \neq 0$ only if $j \leq N - 1$ or $j \geq n - N$. Therefore, for $n \geq 2N$, we can write

$$\sum_{j=0}^{n-1} X_{n-1-j} T^j = \sum_{j=0}^{N-1} X_{n-1-j} T^j + \sum_{j=n-N}^{n-1} X_{n-1-j} T^j = \sum_{j=0}^{N-1} (X_{n-1-j} T^j + X_j T^{n-1-j}).$$

Notice now that

$$\sum_{n=2N}^{\infty} \left\| \sum_{j=0}^{N-1} X_{n-1-j} T^j \right\|^2 h^2 \leq N \sum_{n=2N}^{\infty} \sum_{j=0}^{N-1} \left\| X_{n-1-j} T^j h \right\|^2 \leq N \sum_{j=0}^{N-1} \|X T^j h\|^2 \leq N^2 \|X\|^2 C_T^2 \|h\|^2$$

where as usual $C_T > 0$ is a constant satisfying

$$\|\varphi(T)\| \leq C_T \|\varphi\|_\infty$$

for every $\varphi \in A(\mathbb{D})$. Thus, $[X] = 0$ if and only if

$$\sum_{n=2N}^{\infty} \left\| \left( LT^n + \sum_{j=0}^{N-1} X_j T^{n-1-j} \right) h \right\|^2 \leq c \|h\|^2$$

which is in turn equivalent to

$$\sum_{n=2N}^{\infty} \left\| \left( LT^n + \sum_{j=0}^{N-1} X_j T^{n-1-j} \right) T^{n-N} h \right\|^2 \leq c \|h\|^2$$

and thus to

$$LT^n + \sum_{j=0}^{N-1} X_j T^{n-1-j} \in Z_\varepsilon(T).$$

\[\square\]

**Definition 3.5.** Let $E$ be a separable Hilbert space. Given two Hilbert modules $(H^2(\mathcal{E}), T_1)$ and $(\mathcal{H}, T_2)$, we define the *polynomial subgroup* $\text{Ext}^1_{\text{poly}}(T_2, T_1)$ of $\text{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$ to be the subgroup of elements $[X]$ such that $S_\varepsilon^N X T^N = 0$ for some integer $N \geq 0$.

We are primarily interested in the case of $T_1 = S_\varepsilon$ or $T_1 = S_\varepsilon^2$. In particular, we obtain the following consequence of Theorem 3.4.

**Corollary 3.6.** Let $S_\varepsilon : H^2(\mathcal{E}) \to H^2(\mathcal{E})$ be the unilateral shift and let $(\mathcal{H}, T)$ be a Hilbert module. Then

$$B(\mathcal{H}, \mathcal{E}) T + Z_\varepsilon(T) = B(\mathcal{H}, \mathcal{E})$$

if and only if

$$\text{Ext}^1_{\text{poly}}(T, S_\varepsilon) = 0.$$
Proof. Note that \( Z_\mathcal{E}(T)T \subset Z_\mathcal{E}(T) \). Thus, if
\[
B(\mathcal{H}, \mathcal{E})T + Z_\mathcal{E}(T) = B(\mathcal{H}, \mathcal{E})
\]
then by using an iterative argument we find
\[
B(\mathcal{H}, \mathcal{E})T^N + Z_\mathcal{E}(T) = B(\mathcal{H}, \mathcal{E})
\]
for every \( N \geq 0 \), and Theorem 3.4 immediately implies that
\[
\text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) = 0.
\]
Conversely, assume the polynomial subgroup vanishes and fix \( X \in B(\mathcal{H}, \mathcal{E}) \). In light of the equality \( S_\mathcal{E}^*X = 0 \), Lemma 2.2 implies that the operator \( X : \mathcal{H} \to H^2(\mathcal{E}) \) gives rise to an element \([X] \) in \( \text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) \) and by Theorem 3.4 we find
\[
X \in B(\mathcal{H}, \mathcal{E})T + Z_\mathcal{E}(T).
\]
Since \( X \in B(\mathcal{H}, \mathcal{E}) \) was arbitrary, we see that
\[
B(\mathcal{H}, \mathcal{E})T + Z_\mathcal{E}(T) = B(\mathcal{H}, \mathcal{E})
\]
\( \square \)

Corollary 3.7. Let \( S_\mathcal{E} : H^2(\mathcal{E}) \to H^2(\mathcal{E}) \) be the unilateral shift and let \( (\mathcal{H}, T) \) be a Hilbert module. If
\[
\text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}) = \text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) = 0,
\]
then \( T \) is bounded below. Conversely, if \( T \) is bounded below, then
\[
\text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) = 0.
\]
Proof. Assume first that \( T \) is bounded below. Then, \( T \) is left-invertible so that
\[
B(\mathcal{H}, \mathcal{E})T = B(\mathcal{H}, \mathcal{E})
\]
and we obtain
\[
\text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) = 0
\]
by Corollary 3.6.

Conversely, assume
\[
\text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}) = \text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) = 0.
\]
By Theorem 2.4 we know that \( X \in Z_\mathcal{E}(T) \) if and only if \([X] \in \text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}) \). Since this group is assumed to be trivial, we see \( X \in Z_\mathcal{E}(T) \) implies \( X = S_\mathcal{E}L - LT \). Now, the range of \( X \) lies in \( \mathcal{E} \), so we obtain \( X = -P_\mathcal{E}LT \), whence \( Z_\mathcal{E}(T) \subset B(\mathcal{H}, \mathcal{E})T \). Using \( \text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}) = 0 \), Corollary 3.6 implies
\[
B(\mathcal{H}, \mathcal{E}) = B(\mathcal{H}, \mathcal{E})T + Z_\mathcal{E}(T) \subset B(\mathcal{H}, \mathcal{E})T
\]
and thus \( T \) is bounded below. \( \square \)

It is known that the fact that \( T \) is bounded below isn’t sufficient for the group \( \text{Ext}^1_{A(\mathbb{D})}(T, S_\mathcal{E}) \) to vanish, so that the preceding corollary cannot be improved to an equivalence. In fact, in the case where \( T \) is a contraction, the vanishing of this extension group is equivalent to the operator \( T \) being similar to an isometry by Theorem 2.5.

We obtain another consequence of Corollary 3.6, which applies in particular to self-adjoint contractions with closed range.
Corollary 3.8. Let $S_\mathcal{E} : H^2(\mathcal{E}) \to H^2(\mathcal{E})$ be the unilateral shift and let $(\mathcal{H}, T)$ be a Hilbert module. If

$$T \mathcal{H} \subset (\ker T)^\perp \subset T^* \mathcal{H}$$

then

$$\text{Ext}^1_{\text{poly}}(T, S_\mathcal{E}^*) = 0.$$
whenever $\Lambda \mathcal{H} \subset zH^2$. Of course, this does not imply that $[X]$ is a non-trivial element of $\text{Ext}_{A(D)}^1(T, S_0^\mathcal{E})$, but it shows that the equation
\[ X = S_0^\mathcal{E} \Lambda - \Lambda T \]
has no solution if we impose the extra condition that the range of $\Lambda$ should lie entirely in the codimension one subspace $zH^2$.

4. Explicit calculations of the subspace $Z$

The goal of this section is to identify the spaces $Z_C(S_F^*)$ and $Z_C(S_F)$ for separable Hilbert spaces $\mathcal{E}, \mathcal{F}$ (recall Definition 3.3). First we set up some notation. Given a bounded operator $X : H^2(\mathcal{F}) \to \mathcal{E}$, we can write
\[ X^*e = \sum_{n=0}^{\infty} z^n X^*_n e \]
for every $e \in \mathcal{E}$, where $X^*_n : \mathcal{E} \to \mathcal{F}$ for each $n$. In particular, $X_n : \mathcal{F} \to \mathcal{E}$ and
\[ Xh = \sum_{n=0}^{\infty} X_n \hat{h}(n) \]
where $\hat{h}(n) \in \mathcal{F}$ denotes the $n$-th Fourier coefficient of the function $h \in H^2(\mathcal{F})$.

Associated to $X$, there is the Toeplitz operator
\[ T_X : H^2(\mathcal{F}) \to H^2(\mathcal{E}) \]
defined as
\[ T_X h = \sum_{n=0}^{\infty} z^n \left( \sum_{m=n}^{\infty} X_{m-n} \hat{h}(m) \right) \]
and the Hankel operator
\[ H_X : H^2(\mathcal{F}) \to H^2(\mathcal{E}) \]
defined as
\[ H_X h = \sum_{n=0}^{\infty} z^n \left( \sum_{m=0}^{\infty} X_{m+n} \hat{h}(m) \right) . \]
Typically, $T_X$ and $H_X$ are unbounded operators, but they are always defined on the dense subset of polynomials.

**Proposition 4.1.** If $S_F : H^2(\mathcal{F}) \to H^2(\mathcal{F})$ is the unilateral shift, then $Z_C(S_F^*)$ consists of the operators $X \in B(H^2(\mathcal{F}), \mathcal{E})$ with the property that $T_X$ is bounded, while $Z_C(S_F)$ consists of the operators $X \in B(H^2(\mathcal{F}), \mathcal{E})$ with the property that $H_X$ is bounded.

**Proof.** We first observe that
\[
\sum_{n=0}^{\infty} \|X S_F^* h\|^2 = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} X_m \hat{h}(m+n) \right)^2 = \sum_{n=0}^{\infty} \left( \sum_{m=n}^{\infty} X_{m-n} \hat{h}(m) \right)^2 = \|T_X h\|^2.
\]
and
\[
\sum_{n=0}^{\infty} \| X S F_n h \|^2 = \sum_{n=0}^{\infty} \left\| \sum_{m=n}^{\infty} X_m \hat{h}(m-n) \right\|^2 = \sum_{n=0}^{\infty} \left\| \sum_{m=0}^{\infty} X_{m+n} \hat{h}(m) \right\|^2 = \| H_X h \|^2.
\]

The result now follows directly from the definition of the spaces \( Z_E(S_F^*) \) and \( Z_E(S_F) \). \( \square \)

It is well-known (see Chapter 5 of [1]) that \( T_X \) is bounded if and only if the function
\[
\Phi_X(z) = \sum_{n=0}^{\infty} z^n X_n
\]
belongs to \( H^\infty(B(F, E)) \), the space of weakly holomorphic bounded functions on \( \mathbb{D} \) with values in \( B(F, E) \). Furthermore, \( H_X \) is bounded if and only if we can find for every integer \( n < 0 \) an operator \( X_n : F \to E \) with the property that the function
\[
\sum_{n=-\infty}^{\infty} z^n X_n
\]
belongs to \( L^\infty(B(F, E)) \), the space of essentially bounded weakly measurable functions from \( \mathbb{T} \) into \( B(F, E) \) (this is usually referred to as the Nehari-Page theorem, see [14]).

In light of these remarks, let us examine what Proposition 4.1 says when \( E = \mathbb{C} \). In this case, any operator \( X \in B(H^2(F), E) \) acts as \( X h = (\hat{h}, \xi) \) for some fixed \( \xi \in H^2(F) \) and thus
\[
X_n \hat{h}(n) = (\hat{h}(n), \hat{\xi}(n)).
\]
We find
\[
T_X h = \sum_{n=0}^{\infty} \left( \sum_{m=n}^{\infty} (\hat{h}(m), \hat{\xi}(m-n)) \right) z^n
\]
and
\[
H_X h = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} (\hat{h}(m), \hat{\xi}(m+n)) \right) z^n
\]
for every \( h \in H^2(F) \). This last equality shows that Corollary 3.1.6 in [2] follows from Proposition 4.1 upon taking \( E = \mathbb{C} \). Of course, this is to be expected since \( X \in Z_C(S_F) \) is equivalent to the fact that \( X \) gives rise to an element \([X]\) of \( \text{Ext}^1_{A(D)}(S_F, S_C) \), by Theorem 2.4. In addition, we see that \( X \in Z_C(S_F^*) \) if and only if \( \xi \in H^\infty(F) \), while \( X \in Z_C(S_F) \) if and only if there exists another holomorphic function \( \eta \) with the property that \( \xi + \eta \in L^\infty(F) \).
5. Vanishing of the polynomial subgroup in the case of contractions

The goal of this section is to show that \( \text{Ext}^{1}\text{poly}(T, S_{E}^{*}) = 0 \) whenever \( T \) is a contraction. To achieve it, we make use of the functional model of a completely non-unitary contraction which we briefly recall (see [18] or [1] for greater detail).

Let \( \mathcal{E}, \mathcal{E}_{*} \) be separable Hilbert spaces and let \( \Theta \in H^{\infty}(B(\mathcal{E}, \mathcal{E}_{*})) \) be a contractive (weakly) holomorphic function. Define \( \Delta \in L^{\infty}(B(\mathcal{E})) \) as follows
\[
\Delta(e^{it}) = \sqrt{I - \Theta(e^{it})^{*}\Theta(e^{it})}.
\]
If we set
\[
M(\Theta) = \{ \Theta u \oplus \Delta u : u \in H^{2}(\mathcal{E}) \},
\]
then the space \( H(\Theta) \) is defined as
\[
H(\Theta) = (H^{2}(\mathcal{E}_{*}) \oplus \Delta L^{2}(\mathcal{E})) \ominus M(\Theta)
\]
and we have
\[
S(\Theta) = P_{H(\Theta)}(S \oplus U)|H(\Theta)
\]
where \( S = S_{E_{*}} \) is the unilateral shift on \( H^{2}(\mathcal{E}_{*}) \) and \( U \) is the unitary operator of multiplication by the variable \( e^{it} \) on \( L^{2}(\mathcal{E}) \).

In order to proceed, we require two technical lemmas which are most likely well-known. We provide the calculation for the reader’s convenience.

**Lemma 5.1.** Let \( e_{*} \in \mathcal{E}_{*} \). Then,
\[
P_{M(\Theta)}(e_{*} \oplus 0) = \Theta(0)^{*}e_{*} \oplus \Delta \Theta(0)^{*}e_{*}
\]
and
\[
P_{H(\Theta)}(e_{*} \oplus 0) = (I - \Theta(0)^{*}e_{*}) \oplus (-\Delta \Theta(0)^{*}e_{*}).
\]

**Proof.** For any \( u \in H^{2}(\mathcal{E}) \) we have
\[
(e_{*} \oplus 0, \Theta u \oplus \Delta u)_{H^{2}(\mathcal{E}_{*}) \oplus L^{2}(\mathcal{E})} = (\Theta(0)^{*}e_{*}, u(0))_{\mathcal{E}}
\]
and
\[
(\Theta(0)^{*}e_{*} \oplus \Delta \Theta(0)^{*}e_{*}, \Theta u \oplus \Delta u)_{H^{2}(\mathcal{E}_{*}) \oplus L^{2}(\mathcal{E})} = (\Theta(0)^{*}e_{*}, u)_{H^{2}(\mathcal{E})} + (\Delta^{2} \Theta(0)^{*}e_{*}, u)_{L^{2}(\mathcal{E})}
\]
\[
= (\Theta(0)^{*}e_{*}, u)_{L^{2}(\mathcal{E})} + (I - \Theta^{*} \Theta(0)^{*}e_{*}, u)_{L^{2}(\mathcal{E})}
\]
\[
= (\Theta(0)^{*}e_{*}, u)_{L^{2}(\mathcal{E})}
\]
which shows the first equality, and the second follows immediately.

**Lemma 5.2.** The range of the operator \( S(\Theta) \) is
\[
\{ f_{1} \oplus f_{2} \in H(\Theta) : f_{1}(0) \in \Theta(0) \mathcal{E} \}.
\]

**Proof.** Assume that
\[
f_{1} \oplus f_{2} = S(\Theta)(v_{1} \oplus v_{2})
\]
for \( v_{1} \oplus v_{2} \in H(\Theta) \). Then, we can write
\[
f_{1} \oplus f_{2} = zv_{1} \oplus e^{it}v_{2} + \Theta u \oplus \Delta u
\]
for some \( u \in H^{2}(\mathcal{E}) \), and therefore
\[
f_{1}(0) = \Theta(0)u(0)
\]
lies in the range of \( \Theta(0) \).
Conversely, pick \( f = f_1 \oplus f_2 \in H(\Theta) \) such that \( f_1(0) = \Theta(0)e \) for some \( e \in \mathcal{E} \). Then, the function
\[
 f_1 - \Theta e \in H^2(\mathcal{E}_*)
\]
vanishes at \( z = 0 \), so we can find another function \( v_1 \in H^2(\mathcal{E}_*) \) with the property that
\[
 f_1 - \Theta e = zv_1.
\]
Since \( U\Delta = \Delta U \), we find that the function
\[
 v_2 = U^*(f_2 - \Delta e)
\]
lies in \( \overline{\Delta L^2(\mathcal{E})} \) and satisfies
\[
 Uv_2 = f_2 - \Delta e.
\]
We see that
\[
 P_{H(\Theta)}(Sv_1 \oplus Uv_2) = P_{H(\Theta)}((f_1 - \Theta e) \oplus (f_2 - \Delta e))
\]
\[
 = P_{H(\Theta)}(f_1 \oplus f_2)
\]
\[
 = f_1 \oplus f_2
\]
and therefore
\[
 f_1 \oplus f_2 = S(\Theta)P_{H(\Theta)}(v_1 \oplus v_2)
\]
lies in the range of \( S(\Theta) \).

The following is the crucial technical step in the proof of the main result.

**Theorem 5.3.** Let \( \mathcal{F}, \mathcal{F}_*, \mathcal{E} \) be separable Hilbert spaces. Let \( \Theta \in H^\infty(B(\mathcal{F}, \mathcal{F}_*)) \) be a contractive holomorphic function. Then,
\[
 B(H(\Theta), \mathcal{E}) = B(H(\Theta), \mathcal{E})S(\Theta)^* + Z_\mathcal{E}(S(\Theta)^*).
\]

**Proof.** Let \( X \in B(H(\Theta), \mathcal{E}) \). Define \( X_1 : H(\Theta) \to \mathcal{E} \) as
\[
 X_1 h = XP_{H(\Theta)}P_{\mathcal{F} \oplus \{0\}} \hat{h}(0)
\]
where for a function \( h \in H(\Theta) \) we define \( \hat{h}(n) \) to be its \( n \)-th Fourier coefficient, which lies in \( \mathcal{F}_* \oplus \mathcal{F} \). Given \( e \in \mathcal{E} \) and \( h = h_1 \oplus h_2 \in H(\Theta) \), we have
\[
 (X_1 h, e)_{\mathcal{E}} = \langle XP_{H(\Theta)}P_{\mathcal{F} \oplus \{0\}} \hat{h}(0), e \rangle_{\mathcal{E}}
\]
\[
 = \langle h_1(0) \oplus 0, X^* e \rangle_{H^2(\mathcal{F}_*) \oplus L^2(\mathcal{F})}
\]
\[
 = \langle \hat{h}(0), P_{\mathcal{F}_* \oplus \{0\}} \overline{X^* e(0)} \rangle_{\mathcal{F}_* \oplus \mathcal{F}}
\]
\[
 = \langle h, P_{H(\Theta)}P_{\mathcal{F} \oplus \{0\}} \overline{X^* e(0)} \rangle_{H(\Theta)}
\]
whence
\[
 X_1^* e = P_{H(\Theta)}P_{\mathcal{F}_* \oplus \{0\}} \overline{X^* e(0)}.
\]
Set \( X_2 = X - X_1 \) and \( \overline{X^* e(0)} = f_\ast \oplus f \in \mathcal{F}_* \oplus \mathcal{F} \). Using Lemma 5.1, we find
\[
 X_1^* e = (I - \Theta(0)^*)f_\ast \oplus (-\Delta(0)^* f_\ast).
\]
A straightforward verification using Lemma 5.2 establishes that the range of \( X_2^* \) is contained in the range of \( S(\Theta) \). By Douglas’s Lemma, this implies in turn that
\[
 X_2 \in B(H(\Theta), \mathcal{E})S(\Theta)^*.
\]
Since \( X = X_1 + X_2 \), it remains only to check that \( X_1 \in Z_\mathcal{E}(S(\Theta)^*) \). First, we note that for \( \hat{h} = h_1 \oplus h_2 \in H(\Theta) \) we have

\[
S(\Theta)^n h = (P_{H^2(\mathcal{E})} \mathcal{F}^n h_1) \oplus e^{-int} h_2
\]

and the Fourier coefficient of order zero of \( S(\Theta)^n h \) is therefore equal to \( \hat{h}(n) \). Consequently,

\[
X_1 S(\Theta)^n h = XP_{H(\Theta)} P_{\mathcal{F}^n \oplus \{0\}} \hat{h}(n)
\]

and

\[
\sum_{n=0}^{\infty} \|X_1 S(\Theta)^n h\|^2 \leq \|X\|^2 \sum_{n=0}^{\infty} \|\hat{h}(n)\|^2 \leq \|X\|^2 \|h\|^2
\]

so that \( X_1 \in Z_\mathcal{E}(S(\Theta)^*) \).

We now come to the main result of the paper (recall Definition 3.5).

**Theorem 5.4.** Let \( \mathcal{E} \) be a separable Hilbert space and let \( S_\mathcal{E} : H^2(\mathcal{E}) \to H^2(\mathcal{E}) \) be the unilateral shift. Then, \( \text{Ext}_{1}^{\text{poly}}(T, S_\mathcal{E}^*) = 0 \) for every operator \( T \) which is similar to a contraction.

**Proof.** Since extension groups are invariant under similarity, we may assume that \( T \in B(H) \) is a contraction. Then, it is well-known that there exists a reducing subspace \( M \subset H \) with the property that \( T|M \) is completely non-unitary and \( T|M^\perp \) is unitary. According to this decomposition, it is easy to verify that any bounded operator \( X : H \to H^2(\mathcal{E}) \) giving rise to an element \([X] \in \text{Ext}_{1}^{\text{poly}}(T, S_\mathcal{E}^*) \) can be written as \( X = (X_1, X_2) \), where \([X_1] \in \text{Ext}_{1}^{\text{poly}}(T|M, S_\mathcal{E}) \) and \([X_2] \in \text{Ext}_{1}^{\text{poly}}(T|M^\perp, S_\mathcal{E}) \).

Using Theorem 2.3 we see that \([X] = 0 \) if and only if \([X_1] = 0 \). Therefore, we may assume that \( T \) (and hence \( T^* \)) is completely non-unitary. By Theorem VI.2.3 of \cite{18}, we know that \( T^* \) is unitarily equivalent to \( S(\Theta) \) for some contractive operator-valued holomorphic function \( \Theta \), so for our purposes we may as well take \( T^* \) to be equal to \( S(\Theta) \). In light of Theorem 5.3, we find

\[
B(H(\Theta), \mathcal{E}) = B(H(\Theta), \mathcal{E}) S(\Theta)^* + Z_\mathcal{E}(S(\Theta)^*)
\]

and thus an application of Corollary 3.6 completes the proof. \( \square \)

Theorem 2.5 and Theorem 5.4 illustrate a clear difference between \( S_\mathcal{E} \) and \( S_\mathcal{E}^* \) on the level of extension groups: \( \text{Ext}_{1}^{\text{poly}}(T, S_\mathcal{E}^*) = 0 \) for every contraction \( T \) while \( \text{Ext}_{1}^{\text{poly}}(T, S_\mathcal{E}) = 0 \) only when the contraction \( T \) is similar to an isometry. The reader will object immediately to the fact that we are considering the polynomial subgroup in one case and the full group in the other. In some sense however, there is no discrepancy between the two settings. Indeed, by Theorem 2.4, every element in \( \text{Ext}_{1}^{\text{poly}}(T, S_\mathcal{E}) \) can be represented by an operator \( X : H \to H^2(\mathcal{E}) \) with range contained in \( \mathcal{E} \). In particular, we see that \( S_\mathcal{E} X = 0 \), and thus \( X \) is a polynomial operator. Therefore, the group \( \text{Ext}_{1}^{\text{poly}}(\mathcal{E}, S_\mathcal{E}) \) coincides with \( \text{Ext}_{1}^{\text{poly}}(\mathcal{E}, S_\mathcal{E}^*) \). This is not the case if \( S_\mathcal{E} \) is replaced by \( S_\mathcal{E}^* \), as is shown in Section 6.

6. **The case of non-contractive modules: Pisier’s counterexample**

Much of the vanishing results for extension groups obtained in the previous sections focus on extensions of the unilateral shift (and its adjoint) by contractive modules. It is natural to wonder what happens for extensions by polynomially bounded operators which are not similar to a contraction. Unfortunately, few examples of such operators are known. In fact, only the family of counterexamples
introduced by Pisier in [16] is available. Let us recall the details of his construction here.

Let \( S_F : H^2(F) \to H^2(F) \) be the unilateral shift with infinite multiplicity, where

\[ F = \bigoplus_{n=1}^{\infty} (\mathbb{C}^2)^\otimes n. \]

Define

\[ V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

and

\[ D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

For \( 0 \leq k \leq n - 1 \), define \( C_{k,n} : (\mathbb{C}^2)^\otimes n \to (\mathbb{C}^2)^\otimes n \) as

\[ C_{k,n} = V^{\otimes (k+1)} \otimes D \otimes I^{\otimes (n-k-2)} \]

and for any \( k \geq 0 \) set

\[ W_k = \bigoplus_{n=k+1}^{\infty} C_{k,n} \]

which acts on \( F \). It is well-known (see [7] or [15]) that the sequence of operators \( \{W_k\}_k \subset B(F) \) satisfies the so-called canonical anticommutation relations. Given a sequence \( \alpha = \{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{C} \), we define a Hankel operator \( X_\alpha \) acting on \( H^2(F) \) by

\[ X_\alpha = (\alpha_{i+j}W_{i+j})_{i,j=0}^{\infty} \]

and we set

\[ R(X_\alpha) = \begin{pmatrix} S_F & X_\alpha \\ 0 & S_F \end{pmatrix}. \]

The following result can be found in [7] and [17].

**Theorem 6.1.** The operator \( R(X_\alpha) \) is polynomially bounded if and only if

\[ \sup_{k \geq 0} (k+1)^2 \sum_{i=k}^{\infty} |\alpha_i|^2 \]

is finite, and it is similar to a contraction if and only if

\[ \sum_{k=0}^{\infty} (k+1)^2 |\alpha_k|^2 \]

is finite.

We noted at the end of Section 5 that

\[ \text{Ext}^1_{A(\mathbb{D})}(\cdot, S_\xi) = \text{Ext}^1_{\text{poly}}(\cdot, S_\xi). \]

However, things are different for \( \text{Ext}^1_{A(\mathbb{D})}(\cdot, S_F^*) \). Indeed, if the sequence \( \alpha = \{\alpha_n\}_n \) is chosen such that \( R(X_\alpha) \) is polynomially bounded but not similar to a contraction, then \( [X_\alpha] \) is a non-trivial element of \( \text{Ext}^1_{A(\mathbb{D})}(S_F, S_F^*) \). In particular, Theorem 5.4 implies that

\[ \text{Ext}^1_{\text{poly}}(S_F, S_F^*) \neq \text{Ext}^1_{A(\mathbb{D})}(S_F, S_F^*). \]

We do not know whether equality holds if we require that the shift be of finite multiplicity.
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The remainder of this section is dedicated to the study of the group
\[ \text{Ext}_{\text{poly}}^1(R(X_\alpha), S_C) = \text{Ext}_{A(D)}^1(R(X_\alpha), S_C). \]

Of particular interest is the case where \( R(X_\alpha) \) is not similar to a contraction, which lies outside the reach of Theorem 2.5 where little is known.

We start by giving an alternative formulation of Corollary 3.6 adapted to the unilateral shift of multiplicity one. For a Hilbert module \( (H, T) \), define \( \mathcal{Z}(T) \subset H \) to be the set consisting of those vectors \( x \in H \) with the property that there exists a constant \( c_x > 0 \) such that
\[ \sum_{n=0}^{\infty} |\langle h, T^n x \rangle| \leq c_x \|h\|^2 \]
for every \( h \in H \).

**Lemma 6.2.** Let \( (H, T) \) be a Hilbert module and let \( S_C : H^2 \to H^2 \) be the unilateral shift with multiplicity one. Then
\[ T^* H + \mathcal{Z}(T) = H \]
if and only if
\[ \text{Ext}_{\text{poly}}^1(T, S_C^*) = 0. \]

**Proof.** Note that any operator \( X : H \to \mathbb{C} \) is given by \( Xh = \langle h, \xi \rangle \) for some \( \xi \in H \). It is a routine verification to establish that under this identification, the equality
\[ B(H, \mathbb{C}) T + \mathcal{Z}(T) = B(H, \mathbb{C}) \]
corresponds to
\[ T^* H + \mathcal{Z}(T) = H \]
so the result follows from Corollary 3.6. \( \square \)

This corollary offers the advantage over the more complicated general version that the equality we are interested in takes places inside the Hilbert space \( H \) instead of inside the Banach space \( B(H, E) \). Note also that the discussion at the end of Section 4 shows that \( \mathcal{Z}(S_F^*) = H^{\infty}(\mathcal{F}) \). We now state a simple result.

**Lemma 6.3.** Let \( (H, T) \) be a Hilbert module. Any operator \( X \in Z_{\mathcal{F}}(T) \) for which \([X] = 0 \) in \( \text{Ext}_{A(D)}^1(T, S_{\mathcal{F}}) \) belongs to \( B(H, \mathcal{F}) \).

**Proof.** If \([X] = 0\), then \( X = S_{\mathcal{F}} L - LT \) and arguing as in the proof of Corollary 3.7 we find that \( X = L'T \). \( \square \)

Let us now apply this lemma to the study of \( \text{Ext}_{\text{poly}}^1(R(X_\alpha), S_C) \). Using the fact that \( S_F^* X = X S_F \), it is readily verified that
\[ R(X_\alpha)^n = \begin{pmatrix} S_F^n & 0 \\ nX^* S_F^{n-1} & S_F^n \end{pmatrix} \]
for every integer \( n \geq 1 \). Thus, for \( h \in H^2(\mathcal{F}) \) we have that \( h \oplus 0 \in Z(R(X_\alpha)) \) if and only if
\[ \sum_{n=1}^{\infty} \left| \left( \begin{pmatrix} z^n h \\ nX^* z^{-n} h \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right) \right|^2 \leq c \|g\|^2 \]
for some constant $c > 0$ and every $g = g_1 \oplus g_2 \in H^2(\mathcal{F}) \oplus H^2(\mathcal{F})$. Consequently, $h \oplus 0 \in Z(R(X_\alpha))$ is equivalent to $h \in Z(S_\mathcal{F}^2)$ and
\[
\sum_{n=1}^{\infty} |\langle nX^*z^{n-1}h, g \rangle|^2 \leq c\|g\|^2
\]
for every $g \in H^2(\mathcal{F})$. Notice at this point that for $\omega \in \mathcal{F}$ we have
\[
X^*z^n\omega = \sum_{m=0}^{\infty} \zeta_{m+n}^mW^*(m+n)\omega.
\]
Let $\omega = e_1 \oplus 0 \oplus 0 \oplus \ldots \in \mathcal{F}$ where $e_1 = (1, 0) \in \mathbb{C}^2$. Then, $W_k^*\omega = 0$ for $k \geq 1$ so that
\[
nX^*z^{n-1}\omega = 0
\]
for $n \geq 2$ and thus
\[
\sum_{n=1}^{\infty} |\langle nX^*z^{n-1}\omega, g \rangle|^2 \leq \|X^*\omega\|^2\|g\|^2
\]
for every $g \in H^2(\mathcal{F})$. In addition, it is clear that $\omega \in H^\infty(\mathcal{F}) = Z(S_\mathcal{F}^2)$ so in fact $\omega \in Z(R(X_\alpha))$. Define now $\Omega : H^2(\mathcal{F}) \oplus H^2(\mathcal{F}) \to \mathbb{C}$ by
\[
\Omega(f_1 \oplus f_2) = \langle f_1(0), \omega \rangle_{\mathcal{F}}.
\]
Since $\omega \in Z(R(X_\alpha))$, we have that $\Omega \in ZC(R(X_\alpha))$, whence
\[
[\Omega] \in \text{Ext}^1_{A(\mathcal{D})}(R(X_\alpha), S\mathcal{C})
\]
by Theorem 2.4. Moreover, $\Omega(\omega \oplus 0) = 1$ and $R(X_\alpha)(\omega \oplus 0) = 0$ so that
\[
\Omega \ker R(X_\alpha) \neq 0
\]
and thus $[\Omega] \neq 0$ in $\text{Ext}^1_{A(\mathcal{D})}(R(X_\alpha), S\mathcal{C})$ by Lemma 6.3. In other words,
\[
\text{Ext}^1_{A(\mathcal{D})}(R(X_\alpha), S\mathcal{C}) \neq 0.
\]
It is easy to see that this argument can be adapted to show that
\[
\text{Ext}^1_{A(\mathcal{D})}(R(X_\alpha), S\mathcal{E}) \neq 0
\]
for every separable Hilbert space $\mathcal{E}$. Note finally that $[\Omega] = 0$ in $\text{Ext}^1_{A(\mathcal{D})}(R(X_\alpha), S\mathcal{E}^\alpha)$ by Theorem 3.4.

In conclusion, let us mention that the question of whether or not
\[
\text{Ext}^1_{\text{poly}}(R(X_\alpha), S\mathcal{E}^\alpha)
\]
vanishes (in the case where $R(X_\alpha)$ is not similar to a contraction, of course) remains open. Given its direct relation to the projectivity of the unilateral shift of multiplicity one, this problem is obviously meaningful. We hope that Lemma 6.2 may help settle it in the future.
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