Maurer-Cartan Equations
and Black Hole Superpotentials
in $\mathcal{N} = 8$ Supergravity

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Abstract
We retrieve the non-BPS extremal black hole superpotential of $\mathcal{N} = 8$, $d = 4$ supergravity by using the Maurer-Cartan equations of the symmetric space $\frac{E_7(7)}{SU(8)}$. This superpotential was recently obtained with different 3- and 4-dimensional techniques. The present derivation is independent on the reduction to $d = 3$. 
1 Introduction

Recently, much progress has been obtained in the description of BPS and non-BPS extremal black hole (BH) flows in $\mathcal{N} \geq 2$ supergravities in $d = 4$ space-time dimensions\[^1\] (see also Sect. 2 of \[8\]). In particular, for all theories whose non-linear scalar sigma model is a symmetric space $\mathbb{R}^1$, ”superpotentials” $W$’s exist for all BPS and non-BPS branches, thus yielding that the corresponding radial flow equations are of first order. Namely, the warp factor $U$ of the extremal BH metric and the scalar field trajectories respectively read: \[\dot{U} = -e^U W;\] \[\dot{\phi}^i = -2e^U g^{ij} \partial_j W,\] where $W$ is related to the effective BH potential \[V_{BH} = \frac{1}{2} Z_{AB} Z^{AB} + Z_I Z^I\] through \[V_{BH} = W^2 + 2g^{ij} \partial_i W \partial_j W = W^2 + 2g^{ij} \nabla_i W \nabla_j W.\] Here $Z^I$ denote the matter charges (absent e. g. in $\mathcal{N} = 8$ supergravity), and $Z_{AB} = -Z_{BA}$ is the central charge matrix, entering the supersymmetry algebra as follows: \[\{ Q^\alpha_A, Q^\beta_B \} = \epsilon^{\alpha\beta} Z_{AB} (\phi_\infty, Q).\] Moreover, Eq. (1.2) implies that attractor points \[\dot{\phi}^i = 0\] correspond to critical points of $W$ itself: \[\partial_i W = 0.\] For BPS BHs \[W (\phi, Q) = \left| z_I \right|_{\text{max}} (\phi, Q),\] where $Q$ is the symplectic charge vector, and $\left| z_I \right|_{\text{max}}$ is the highest absolute value of the skew-eigenvalues $z_I$’s of $Z_{AB}$. Furthermore, the ADM mass $M_{ADM}$ \[^9\] is related to $W$ through ($r$ denotes the radial coordinate throughout) \[M^2_{ADM} = \lim_{r \to \infty} W^2.\]

\[^1\]Note that this is always the case for $\mathcal{N} \geq 3$, $d = 4$ theories.
The Bekenstein-Hawking entropy-area formula \[10\] exploits as follows:

$$
\frac{S_{BH}(Q)}{\pi} = \frac{A_H}{4\pi} = \lim_{r \to r_H} W^2 = W^2|_{\partial W=0} = W^2(\phi_H(Q), Q), \quad (1.10)
$$

where \(r_H\) and \(A_H\) respectively stand for the radius and the area of the event horizon of the considered extremal BH, and \(\phi_H(Q)\) denotes the set of scalar fields at the horizon, stabilized in terms of the charges \(Q\).

Explicit ways of constructing \(W\) has been given in \([5, 6, 7]\) by using different methods, e.g. based on the \(\mathcal{N} = 2\) suit model \([5, 7]\) or on three-dimensional techniques \([6]\). All these exploit the fact, as generally proven in \([4]\), that

$$
W = W(i_n(\phi, Q)), \quad (1.11)
$$

where \(i_n(\phi, Q)\)'s \((n = 1, ..., 5)\) are duality invariant combinations of the scalars \(\phi^i\)'s and of charges \(Q\) \([13, 5]\). A polynomial in \(i_n\)'s gives the (unique) scalar-independent duality invariant \(\mathcal{I}(Q)\) \([11, 12, 13]\). In the \(\mathcal{N} = 2\) case, it reads \([13, 5]\):

$$
\mathcal{I} = (i_1 - i_2)^2 + 4i_4 - i_5. \quad (1.12)
$$

It is worth remarking that in the considered framework the symplectic vector of charges \(Q\) must belong to a non-degenerate \((i.e. with \mathcal{I} \neq 0)\) orbit of the \(U\)-duality group \([14, 15, 16]\).

In particular, \(\mathcal{I}\) is quartic\(^2\) in charges \(Q\) for all rank-three \(\mathcal{N} = 2\) symmetric spaces \([17]\), as well as for \(\mathcal{N} = 8\) supergravity (see Eqs. \((2.8)-(2.12)\) below). Moreover, since \(\mathcal{N} \geq 3, d = 4\) supergravities all have symmetric scalar manifolds, they all admit \(W\)'s for their various scalar flows, \(i.e.\) for each different orbit of the charge vector \([14, 15, 16]\).

For \(\mathcal{N} = 8\) supergravity, it follows that

$$
W = W(\rho_0, \rho_1, \rho_2, \rho_3, \varphi), \quad (1.13)
$$

where \(\rho_I\)'s \((I = 0, 1, 2, 3\) throughout\) are the absolute values of the skew-eigenvalues of \(Z_{AB}\), whose \(SU(8)\)-invariant phase is \(\varphi\) (see Eq. \((2.1)\) below). In \([20]\) the explicit expressions of \(\rho_I\)'s and \(\varphi\) were computed in terms of the four roots of a quartic algebraic equation, involving the quantities \((Tr(ZZ^\dagger))^{m+1}\) \((m = 0, 1, 2, 3)\), as well as the quartic invariant \(\mathcal{I}_4\) (see \(e.g.\) Eqs. \((2.10)\) and \((2.11)\) below, and also the treatment in \([13]\)).

As shown in \([21]\), two different branches of attractor scalar flows exist, namely the \(\frac{1}{8}\)-BPS and the non-BPS branches. Note that \(W\) exhibits the same flat directions of \(V_{BH}\) at its critical points; such flat directions span the moduli spaces \(E_{6(2)}\) \((\mathcal{I}_4 > 0,\ see\ Eq.\ \((2.21)\) below)\) and \(E_{6(6)}\ \(USp(8)\) \((\mathcal{I}_4 < 0,\ see\ Eq.\ \((2.26)\) below)\) \([23]\).

\(^2\)The quartic invariant \(\mathcal{I}_4\) of \(\mathcal{N} = 4\) theories was derived in \([18, 19]\).
This paper is devoted to the derivation of the $W$’s for both these branches. This is done by exploiting the $(d = 4)$ Maurer-Cartan equations of the exceptional coset $\frac{E_7(7)}{SU(8)}$ (see e.g. [24] and Refs. therein). We will show that, while $W_{\text{BPS}}$ is given by the highest absolute value of the skew-eigenvalues of $Z_{AB}$ (consistent with Eq. (1.8)), $W_{n\text{BPS}}$ is (proportional to) the $USp(8)$-singlet of the 28 of $SU(8)$. These results extend to the whole attractor scalar flow the expression of $W$ which was known for both BPS and non-BPS attractors after [21] (see also e.g. [25]). Our investigation and derivation is complementary to [6], where the expression of $W_{n\text{BPS}}$ was obtained by making use of the nilpotent orbits of the $d = 3$ geodesic flow obtained through a timelike reduction (see e.g. [26, 27, 28, 29, 30, 31, 32, 33], and Refs. therein).

The paper is organized as follows.

In Sect. 2 we recall the $SU(6) \times SU(2)$-covariant normal frame of $\mathcal{N} = 8$ supergravity, which we dub “special” normal frame, and we show that Maurer-Cartan Eqs. yield a partial differential equation (PDE) for $W$, whose simplest solution is the BPS superpotential $W_{\text{BPS}}$.

Sect. 3 is devoted to the analysis of the $USp(8)$-covariant normal frame of $\mathcal{N} = 8$ supergravity (see e.g. the analysis of [31, 35], and Refs. therein), which we dub “symplectic” normal frame. We show that in such a normal frame the Maurer-Cartan Eqs. yield a PDE for $W$, whose simplest solution is the non-BPS superpotential $W_{n\text{BPS}}$. $W$’s are nothing but the singlets in the decomposition of the 28 of $SU(8)$ into the maximal compact subgroup of the stabilizer of the corresponding supporting charge orbit, i.e. respectively into $SU(6) \times SU(2)$ (BPS) and $USp(8)$ (non-BPS).

Derivations of some relevant formulæ are given in the Appendix, which concludes the paper.

## 2 Special Normal Frame

Following [30, 37, 38], through a suitable $SU(8)$ transformation the complex skew-symmetric central charge matrix $Z_{AB}$ ($A, B = 1, \ldots, \mathcal{N} = 8$ in the 8 of $R$-symmetry $SU(8)$) can be skew-diagonalised, and thus recast in normal form (see e.g. Eq. (87) of [2], adopting a different convention on the $2 \times 2$ symplectic metric $\epsilon$; $a = 1, 2, 3$ throughout; unwritten matrix components do vanish throughout):

$$Z_{AB}^{SU(8)} \rightarrow \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \otimes \epsilon = e^{i \frac{\varphi}{4}} \begin{pmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_2 \\ \rho_2 & \rho_3 \end{pmatrix} \otimes \epsilon,$$

$\rho_0, \rho_a \in \mathbb{R}^+$, $\varphi \in [0, 8\pi)$, \hspace{1cm} (2.1)
where
\[ \epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (2.2)

Notice that the second line of Eq. (2.1) can be obtained from the first one by performing a suitable \((U(1))^3\) transformation.

The general definition (1.3) of effective BH potential \(V_{BH}\) thus yields
\[ V_{BH} = \rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2. \] (2.3)

Therefore, in the normal frame defined by (2.1) the non-vanishing components of \(Z_{AB}\) reads as follows:
\[ z_0 \equiv Z_{12} = \rho_0 e^{i\varphi}; \] (2.4)
\[ z_1 \equiv Z_{34} = \rho_1 e^{i\varphi}; \] (2.5)
\[ z_2 \equiv Z_{56} = \rho_2 e^{i\varphi}; \] (2.6)
\[ z_3 \equiv Z_{78} = \rho_3 e^{i\varphi}. \] (2.7)

Within this parametrization, the (unique) quartic invariant \(I_4\) of the \(56\) of the \(U\)-duality group \(E_{7(7)}\) (see e.g. [12, 13], and Refs. therein)
\[ I_4 = Tr \left( ZZ^\dagger ZZ^\dagger \right) - \frac{1}{2^2} Tr^2 \left( ZZ^\dagger \right) + 2^3 Re \left[ \text{Pfaff} (Z) \right]; \] (2.8)
\[ \text{Pfaff} (Z) = \frac{1}{2^4 4!} \epsilon^{ABCD\ldots GH} Z_{AB} Z_{CD} Z_{EF} Z_{GH}, \] (2.9)
reads as follows (see e.g. [39]):
\[ I_4 = \sum_1^4 \rho_i^4 - 2 \sum_{i<j} \rho_i^2 \rho_j^2 + 8 \rho_0 \rho_1 \rho_2 \rho_3 \cos \varphi = \] (2.10)
\[ = (\rho_0 + \rho_1 + \rho_2 + \rho_3) (\rho_0 + \rho_1 - \rho_2 - \rho_3) \cdot \] (2.11)
\[ \cdot (\rho_0 - \rho_1 + \rho_2 - \rho_3) (\rho_0 - \rho_1 - \rho_2 + \rho_3) + \] \[ + 8 \rho_0 \rho_1 \rho_2 \rho_3 (\cos \varphi - 1). \]

The Pfaffian of \(Z_{AB}\), defined by Eq. (2.4), simply reads
\[ \text{Pfaff} (Z) = Z_{12} Z_{34} Z_{56} Z_{78} = e^{i\varphi} \prod_1^4 \rho_i. \] (2.12)

It is worth remarking that the skew-diagonal form of \(Z_{AB}\) given by Eq. (2.4) is “democratic”, in the sense that it fixes the phases of the four skew-eigenvalues
\[ z_I \equiv \rho_I e^{i\varphi_I} \] (2.13)
of \(Z_{AB}\) to be all equal:
\[ \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 \equiv \frac{\varphi}{4}. \] (2.14)
Actually, this implies some loss of generality, because $SU(8)$ only constrains the phases of $z_I$’s as follows:

$$\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 \equiv \varphi.$$  \hfill (2.15)

Up to renamings, without loss of generality, the $|z_I|$’s can be ordered as follows:

$$\rho_0 \geq \rho_1 \geq \rho_2 \geq \rho_3.$$  \hfill (2.16)

Notice that $\rho_I$’s are $U(8)$ invariant, whereas the overall phase $\varphi$ is invariant under $SU(8)$, but not under $U(8)$.

It turns out that the special skew-diagonalization \text{(2.1)} is particularly suitable for the treatment of the $\frac{1}{8}$-BPS ("large") attractor flow, as shown in the following Subsection.

### 2.1 Attractor Solutions

In the special normal frame (2.1), the two attractor solutions of $\mathcal{N} = 8$, $d = 4$ supergravity read as follows (see e.g. [21], [34], and Refs. therein; see also the analysis of [35] for further detail):

- $\frac{1}{8}$-BPS:

  $$\begin{align*}
  \rho_0 &\equiv \rho_{BPS} \in \mathbb{R}_0^+; \\
  \rho_1 &= \rho_2 = \rho_3 = 0; \\
  \varphi &\text{ undetermined,}
  \end{align*}$$

  thus yielding:

  $$\begin{align*}
  Z_{AB,\frac{1}{8}\text{-BPS}} &= e^{i \frac{1}{4} \rho_{BPS}} \begin{pmatrix}
  1 \\
  0 \\
  0 \\
  0
  \end{pmatrix} \otimes \epsilon; \\
  \mathcal{I}_4 (Q_{BPS}) &= \rho_{BPS}^4 (Q_{BPS}) > 0,
  \end{align*}$$

  where 

  $$Q_{BPS} \in \mathcal{O}_{\frac{1}{8}\text{-BPS,non-deg}} = \frac{E_7(7)}{E_6(2)},$$

  with maximal compact symmetry $SU(6) \times SU(2)$.

- Non-BPS:

  $$\begin{align*}
  \rho_0 &= \rho_1 = \rho_2 = \rho_3 \equiv \rho_{nBPS} \in \mathbb{R}_0^+; \\
  \varphi &= \pi,
  \end{align*}$$

  $\rho_{nBPS}$:
thus yielding:

\[ Z_{AB,nBPS} = e^{i \frac{\pi}{4} \rho_{nBPS} \Omega_{AB}}; \quad (2.25) \]

\[ \mathcal{I}_4 (Q_{nBPS}) = -2^4 \rho_4^4 (Q_{nBPS}) < 0, \quad (2.26) \]

where

\[ \Omega_{AB} \equiv \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \otimes \epsilon \quad (2.27) \]

is the 8 \times 8 metric of \( USp(8) \), and \[22\]

\[ Q_{nBPS} \in O_{nBPS} = \frac{E_7(7)}{E_6(6)}, \quad (2.28) \]

with maximal compact symmetry \( USp(8) \).

### 2.2 Maurer-Cartan Equations and PDE for \( W \)

Let us now consider the Maurer-Cartan Eqs. of \( N = 8, d = 4 \) supergravity (see e.g. \[24\] and Refs. therein):

\[ \nabla_i Z_{AB} = \frac{1}{2} P_{ABCD,i} Z^{CD}, \quad (2.29) \]

where the Vielbein 1-form \( P_{ABCD} = P_{ABCD,j} \phi^j \) \((i = 1, \ldots, 70)\) of the real homogeneous symmetric scalar manifold

\[ M_{N=8,d=4} = \frac{E_7(7)}{SU(8)} \quad (2.30) \]

sits in the 4-fold antisymmetric 70 of \( SU(8) \), and it satisfies the self-dual reality condition (see e.g. \[12\])

\[ P_{ABCD} = P_{[ABCD]} = \frac{1}{4!} \epsilon_{ABCDEFGH} P^{ABCDEFGH}. \quad (2.31) \]

In order to simplify forthcoming calculations, it is convenient to group \( SU(8) \)-indices as follows:

\[ 12 \rightarrow 0; 34 \rightarrow 1; 56 \rightarrow 2; 78 \rightarrow 3. \quad (2.32) \]

Thus, for a generic skew-diagonal \( Z_{AB} \), Maurer-Cartan Eqs. \[2.29\] read

\[ \nabla_i Z_0 = P_{01,i} Z^1 + P_{02,i} Z^2 + P_{03,i} Z^3; \quad (2.33) \]

\[ \nabla_i Z_1 = P_{01,i} Z^0 + P_{12,i} Z^2 + P_{13,i} Z^3; \quad (2.34) \]

\[ \nabla_i Z_2 = P_{02,i} Z^0 + P_{12,i} Z^1 + P_{23,i} Z^3; \quad (2.35) \]

\[ \nabla_i Z_3 = P_{03,i} Z^0 + P_{13,i} Z^1 + P_{23,i} Z^2. \quad (2.36) \]
By disregarding the reality condition (2.31) of the Vielbein $P_{ABCD}$, within the considered special normal frame (2.1) one can determine the PDE for $W$ in an easy way. Indeed, Eqs. (2.29) yield

$$\nabla_i \rho_I = \frac{1}{2} \left( e^{i\varphi/4} \nabla_i \mathcal{Z}^I + e^{-i\varphi/4} \nabla_i \mathcal{Z}_J \right);$$  \hspace{1cm} (2.37)

$$\nabla_i \varphi = -2i \nabla_i \left( \ln \mathcal{Z}_I - \ln \mathcal{Z}_J \right) = \frac{2}{\rho_I} \left( i e^{i\varphi/4} \nabla_i \mathcal{Z}^I - i e^{-i\varphi/4} \nabla_i \mathcal{Z}_J \right).$$ \hspace{1cm} (2.38)

Consequently, the total covariant differential of $W$ generally reads (the sum is expanded in Eq. (A.7))

$$\nabla_i W = \frac{1}{2} \sum_{I<J} \left\{ e^{i\varphi/2} (W_I \rho_J + W_J \rho_I) + e^{-i\varphi/2} \tilde{e}^{IJKL} (\mathcal{W}_K \rho_L + \mathcal{W}_L \rho_K) \right\} P_{IJ},$$ \hspace{1cm} (2.39)

where the quantity

$$W_I \equiv \frac{\partial W}{\partial \rho_I} + i \frac{\partial W}{\partial \varphi}$$ \hspace{1cm} (2.40)

was introduced.

By performing various steps (detailed in App. A, see Eqs. (A.1)-(A.6) therein) and recalling Eq. (1.4), the final PDE for the fake superpotential $W$ reads:

$$W^2 + \sum_{I,J \neq I} \left\{ \left| (W_I \rho_J + W_J \rho_I) \right|^2 + \frac{1}{2} \left[ e^{i\varphi} \tilde{e}^{IJKL} (W_I \rho_J + W_J \rho_I) (W_K \rho_L + W_L \rho_K) + e^{-i\varphi} \tilde{e}^{IJKL} (\mathcal{W}_I \rho_J + \mathcal{W}_J \rho_I) (\mathcal{W}_K \rho_L + \mathcal{W}_L \rho_K) \right] \right\} =$$

$$= \rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2,$$ \hspace{1cm} (2.41)

where all terms of the sum can be found in Eq. (A.9).

As a consequence of $\mathcal{N} = 8$ supersymmetry, Eq. (2.41) is fully symmetric in $\{\rho_0, \rho_1, \rho_2, \rho_3\}$, and it is straightforward to check that any the $\rho_I$’s is a solution. Following [2], a natural Ansatz for $\mathcal{N} = 8$ solutions is a linear combination of the skew eigenvalues (with constant coefficients):

$$W = \sum_{I=0}^{3} \alpha_I \rho_I.$$ \hspace{1cm} (2.42)

In this way, the $\frac{1}{8}$-BPS solution reads ($a = 1, 2, 3$)

$$\alpha_0 = 1, \alpha_a = 0.$$ \hspace{1cm} (2.43)
Due to the asymptotical meaning of $W$ itself as ADM mass (see Eq. (1.9)), this corresponds to choosing

$$W_{\frac{1}{8}}^{\text{-BPS}} = \rho_0,$$

(2.44)

namely the highest of the absolute values of the skew-eigenvalues of $Z_{AB}$ as given by ordering (2.16), as the solution with the physical meaning of superpotential.

For non-BPS attractor flow, Ansatz (2.42) produces [2]

$$W_{n\text{BPS}} = \frac{1}{2} (\rho_0 + \rho_1 + \rho_2 + \rho_3),$$

(2.45)

which however is not the most general one. Indeed, $W_{n\text{BPS}}$ given by Eq. (2.45) is a solution iff the phase is fixed as

$$\varphi = \pi,$$

(2.46)

thus not describing the most general flow with all five parameters, but rather a particular case with double-extremal phase (see Sect. 3).

Let us notice that the result (2.44) is an extension to the whole attractor flow (i.e. for all range of the radial coordinate $\tau \in (-\infty, 0]$) of the well-known fact that the solution of the $\frac{1}{8}$-BPS solution to the $\mathcal{N} = 8$ Attractor Eqs. is obtained by retaining the singlet in the decomposition of $SU(8)$ with respect to the stabilizer of the $\frac{1}{8}$-BPS non-degenerate charge orbit, namely [12, 21, 40, 25]:

$$E_{7(7)} \rightarrow SU(8) \rightarrow SU(6) \times SU(2) \times U(1);$$

$$56 \rightarrow 28 + 2\overline{28} \rightarrow (15, 1)_{+1} + (6, 2)_{-1} + (1, 1)_{-3} + (15, 1)_{-1} + (\overline{6}, 2)_{+1} + (1, 1)_{+3},$$

(2.47)

where the subscripts denote the charge with respect to $U(1)$. The corresponding extension to the whole $\frac{1}{8}$-BPS attractor flow amounts to stating that the superpotential governing the evolution is given by the “singlet sector” $(1, 1)_{+3} + (1, 1)_{+3}$ in the decomposition (2.47). In the normal frame (2.1), by recalling Eqs. (2.4) - (2.7) and splitting the index of the $8$ of $SU(8)$ as $A = \hat{a}, \tilde{a}$, with $\hat{a} = 1, 2$ and $\tilde{a} = 3, ..., 8$ (consistently with (2.47)), it then follows that

$$W_{\frac{1}{8}}^{\text{-BPS}} = |Z_{12}| = \rho_0.$$ 

(2.48)

### 3 Symplectic Normal Frame:

**Maurer-Cartan Equations and PDE for $W$**

This Section is devoted to the derivation of the non-BPS “fake” superpotential uniquely from Maurer-Cartan Eqs., with suitable boundary horizon conditions.
We will obtain $W_{n\text{BPS}}$ as a solution of the Maurer-Cartan Eqs. in a suitably defined manifestly $USp(8)$-covariant normal frame \cite{6}, in which maximal compact symmetry $USp(8)$ of the non-BPS charge orbit in $E_7(7)$ \cite{22} is fully manifest (see e.g. also the treatment of \cite{25,34,35}). As it will be evident from subsequent treatment, such a normal frame is generally and intrinsically not “democratic” (in the meaning specified at the start of Sect. \cite{2}).

In order to derive the non-BPS “fake” superpotential from the geometric structure encoded in the Maurer-Cartan Eqs., we extend to the whole attractor flow the well-known fact that the non-BPS solution of the $\mathcal{N} = 8$ Attractor Eqs. is obtained by retaining the singlet in the decomposition of $SU(8)$ with respect to the stabilizer of the non-BPS charge orbit, namely \cite{21,40,25}:

$$E_{7(7)} \rightarrow SU(8) \rightarrow USp(8);$$

$$56 \rightarrow 28 + 2\bar{8} \rightarrow 27 + 1 + 27' + 1', \quad (3.1)$$

where the priming distinguishes the various real irreps. of $USp(8)$, namely the rank-2 antisymmetric skew-traceless $27^{(l)}$ and the related skew-trace $1^{(l)}$. The corresponding extension to the non-BPS attractor flow amounts to stating that the superpotential governing the evolution is given by the “singlet sector” $1 + 1'$ in the decomposition (3.1).

The branching (3.1) corresponds to decomposing the skew-diagonal complex matrix $Z_{AB}$ (within the generic normal frame given by the first line of Eq. (2.1)) into its skew-trace and its traceless part. This amounts to introducing the following quantities:

$$\begin{align*}
    z_0 & \equiv b + c_1 + c_2 + c_3; \\
    z_a & \equiv b - c_a;
\end{align*} \quad \iff \begin{align*}
    b &= \frac{1}{4} (z_0 + \sum_a z_a); \\
    c_a &= \frac{1}{4} (z_0 + \sum_a z_a - 4z_a),
\end{align*} \quad (3.2)$$

thus yielding

$$Z_{AB} = b \, \Omega_{AB} + T_{0,AB}, \quad (3.3)$$

with $b$ and $T_0$ respectively being (half of) the skew-trace and the skew-traceless part of the skew-diagonal complex matrix $Z_{AB}$ (within the generic
normal frame given by the first line of Eq. (2.1):
\[ b \equiv \frac{1}{8} Z_{AB} \Omega^{AB}; \]  
(3.4)

\[ T_{0,AB} \equiv Z_{AB} - \frac{1}{8} Z_{CD} \Omega^{CD} \Omega_{AB} = \begin{pmatrix} c_1 + c_2 + c_3 & -c_1 \\ -c_2 & -c_3 \end{pmatrix} \otimes \epsilon, \]  
(3.5)

where \( \Omega_{AB} \) is the \( 8 \times 8 \) metric of \( USp(8) \) defined in (2.27).

Following the same steps as in Sect. 2, with details explained in App. A (see Eqs. (A.10)-(A.14) therein), after some straightforward algebra, one
achieves the following result (recall \( a = 1, 2, 3 \) throughout):

\[ \nabla W \nabla W = \]
\[ = \frac{1}{8} \left\{ 4 \text{Re} \left[ \left( b \frac{\partial W}{\partial b} - \sum_a c_a \frac{\partial W}{\partial c_a} \right) + (c_2 + c_3) \left( -\frac{\partial W}{\partial c_1} + \frac{\partial W}{\partial c_2} + \frac{\partial W}{\partial c_3} \right) \right] + \right. \]
\[-2i \text{Im} \left[ (c_2 + c_3) \left( \frac{\partial W}{\partial b} - \sum_a c_a \frac{\partial W}{\partial c_a} \right) + 2b \left( -\frac{\partial W}{\partial c_1} + \frac{\partial W}{\partial c_2} + \frac{\partial W}{\partial c_3} \right) \right] + \]
\[+ 2 \left( -c_1 \frac{\partial W}{\partial c_1} + c_2 \frac{\partial W}{\partial c_2} + c_3 \frac{\partial W}{\partial c_3} \right) \left| ^2 \right. \]
\[+ \left. \left. \left| \left| 4 \text{Re} \left[ \left( b \frac{\partial W}{\partial b} - \sum_a c_a \frac{\partial W}{\partial c_a} \right) + (c_1 + c_3) \left( \frac{\partial W}{\partial c_1} - \frac{\partial W}{\partial c_2} + \frac{\partial W}{\partial c_3} \right) \right] + \right. \right. \right. \]
\[-2i \text{Im} \left[ (c_1 + c_3) \left( \frac{\partial W}{\partial b} - \sum_a c_a \frac{\partial W}{\partial c_a} \right) + 2b \left( \frac{\partial W}{\partial c_1} - \frac{\partial W}{\partial c_2} + \frac{\partial W}{\partial c_3} \right) \right] + \]
\[+ 2 \left( c_1 \frac{\partial W}{\partial c_1} - c_2 \frac{\partial W}{\partial c_2} + c_3 \frac{\partial W}{\partial c_3} \right) \left| ^2 \right. \]
\[+ \left. \left. \left. \left. \left| \left| 4 \text{Re} \left[ \left( b \frac{\partial W}{\partial b} - \sum_a c_a \frac{\partial W}{\partial c_a} \right) + (c_1 + c_2) \left( \frac{\partial W}{\partial c_1} + \frac{\partial W}{\partial c_2} - \frac{\partial W}{\partial c_3} \right) \right] + \right. \right. \right. \right. \right. \]
\[-2i \text{Im} \left[ (c_1 + c_2) \left( \frac{\partial W}{\partial b} - \sum_a c_a \frac{\partial W}{\partial c_a} \right) + 2b \left( \frac{\partial W}{\partial c_1} + \frac{\partial W}{\partial c_2} - \frac{\partial W}{\partial c_3} \right) \right] + \]
\[+ 2 \left( c_1 \frac{\partial W}{\partial c_1} + c_2 \frac{\partial W}{\partial c_2} - c_3 \frac{\partial W}{\partial c_3} \right) \right\} \right\} \right\} \right\} \right\} \right\}. \]
(3.6)

In order to proceed further, group theoretical arguments based on the reality of the \( 27 \) and \( 27' \) of \( USp(8) \) (see Eq. (3.1)) allow for the following
change of parametrization of the traceless part $T_{0,AB}$ (see Eq. (3.3))
\[
c_a \equiv \varrho_{27,a} \exp(-i\beta) \Rightarrow \left(\begin{array}{c}
\frac{\partial}{\partial c_a} \\
\frac{\partial}{\partial \bar{c}_a}
\end{array}\right) = \left(\begin{array}{cc}
e^{i\beta} & \frac{i}{\xi_a} e^{i\beta} \\
e^{-i\beta} & -\frac{i}{\xi_a} e^{-i\beta}
\end{array}\right) \left(\begin{array}{c}
\frac{\partial}{\partial \varrho_{27,a}} \\
\frac{\partial}{\partial \beta}
\end{array}\right),
\]
where, with a slight abuse of language, $\varrho_{27}$'s generally denote the degrees of freedom pertaining to the traceless part $T_{0,AB}$ of $Z_{AB}$ (see Eq. (3.3), and reasoning made above). Moreover we split the skew-trace into its real and imaginary part
\[
b \equiv x + iy, \quad x, y \in \mathbb{R}.
\]
The reasoning made at the start of the present Section (see Eqs. (3.1) and (3.3)) implies the non-BPS “fake” superpotential $W_{nBPS}$ to be related to the skew-trace $b$.

We now proceed by formulating the Ansatz that $b$ is independent on all $\varrho_{27}$'s introduced in Eq. (3.7). As we will see below, this corresponds to a natural decoupling Ansatz for the PDE (3.10) satisfied by $W$, which will yield to (the) simple(st) solution. This yields the vanishing of all the derivatives of $W$ with respect to $c_a$'s. Thus, Eq. (3.6) reduces to
\[
\nabla W \nabla W = \frac{1}{8} \left\{ 12 \left( x \frac{\partial W}{\partial x} - y \frac{\partial W}{\partial y} \right) + 
\right. 
\left. + \left[ (\varrho_{27,1} + \varrho_{27,2})^2 + (\varrho_{27,1} + \varrho_{27,3})^2 + (\varrho_{27,2} + \varrho_{27,3})^2 \right] \cdot 
\right. 
\left. \cdot \left( \cos \beta \frac{\partial W}{\partial y} - \sin \beta \frac{\partial W}{\partial x} \right)^2 \right\},
\]
so that the whole PDE for the $W$ reads
\[
W^2 + \frac{1}{4} \left\{ 12 \left( x \frac{\partial W}{\partial x} - y \frac{\partial W}{\partial y} \right)^2 + \Delta_{27} \left( \cos \beta \frac{\partial W}{\partial y} - \sin \beta \frac{\partial W}{\partial x} \right)^2 \right\} = 4 (x^2 + y^2) + \Delta_{27},
\]
where the quantity (symmetric in $\{\varrho_{27,1}, \varrho_{27,2}, \varrho_{27,3}\}$)
\[
\Delta_{27} \equiv (\varrho_{27,1} + \varrho_{27,2})^2 + (\varrho_{27,1} + \varrho_{27,3})^2 + (\varrho_{27,2} + \varrho_{27,3})^2
\]
was introduced.
Eq. (3.10) is a non-linear PDE in the real functional variables $x$ and $y$. The previous statement that $b$ is independent on all $\varrho_{27}$'s yields that $x \neq x(\Delta_{27})$; $y \neq y(\Delta_{27})$.
\[
^3\text{We should also note that this Ansatz holds for the particular solution (2.45), with } \beta = -\frac{\pi}{4} + 2k\pi \ (k \in \mathbb{Z}) \text{ but } \partial W \neq 0.
\]
Under position (3.12), PDE (3.10) naturally decouples in the following system of PDEs:

\[
W^2 + 3 \left( x \frac{\partial W}{\partial x} - y \frac{\partial W}{\partial y} \right)^2 = 4 \left( x^2 + y^2 \right); \quad (3.13)
\]

\[
\left( \cos \beta \frac{\partial W}{\partial y} - \sin \beta \frac{\partial W}{\partial x} \right)^2 = 4. \quad (3.14)
\]
PDE (3.13) admits the solution (symmetric in \(x\) and \(y\))

\[
W(x, y) = \left( x^{\frac{2}{3}} + y^{\frac{2}{3}} \right)^{\frac{2}{3}}, \quad (3.15)
\]

which plugged into PDE (3.14) yields the following algebraic equation for \(x\) and \(y\) in terms of \(\beta\):

\[
\left( x^{2/3} + y^{2/3} \right) \left( x^{1/3} \cos \beta - y^{1/3} \sin \beta \right)^2 = x^{2/3} y^{2/3}. \quad (3.16)
\]

Eq. (3.16) is in turn solved by (factor 2 introduced for later convenience)

\[
x = -2\varrho \sin^3 \beta, \quad y = 2\varrho \cos^3 \beta, \quad (3.17)
\]

where \(\varrho\) is a real strictly positive number:

\[
\varrho \in \mathbb{R}^+. \quad (3.18)
\]

In solution (3.17) \(\varrho\) is an arbitrary parameter, whose introduction is possible as a consequence of the homogeneity of degree 0 of algebraic Eq. (3.10) in \(x\) and \(y\). In other words, \(\varrho\) can be understood as an integration constant whose meaning has to be clarified by imposing proper boundary conditions. This is the case for the requirement of positivity of \(\varrho\) which is an asymptotical boundary condition due to the physical meaning of \(W\) that defines the ADM mass \(M_{ADM}\) at radial infinity (see Eqs. (1.9) and (3.19)). Thus, Eqs. (3.15) and (3.17) yield that the final solution for \(W\) reads as follows:

\[
W(x, y) = 2\varrho. \quad (3.19)
\]

By recalling Eqs. (3.3)-(3.5) and (3.7), in the resulting manifestly \(USp(8)\)-covariant normal frame the central charge matrix \(Z_{AB}\) can thus be written as:

\[
Z_{AB} = 2 \left( \cos^3 \beta + i \sin^3 \beta \right) i \varrho \Omega_{AB} +
\]

\[
+ \exp \left( -i\beta \right) \left( \begin{array}{ccc}
\varrho_0 & \varrho_1 & \varrho_2 \\
-\varrho_0 & \varrho_2 & \varrho_1 \\
-\varrho_1 & -\varrho_0 & \varrho_2 \\
\end{array} \right) \otimes \epsilon. \quad (3.20)
\]
Eq. (3.20) determines a parametrization of the symplectic normal frame (3.3)-(3.5) which is "minimal", because it contains only five parameters (see e.g. [39, 21], and Refs. therein), namely \( \{ \beta, \varrho, \varrho_{27,1}, \varrho_{27,2}, \varrho_{27,3} \} \).

In order to consistently characterize solution (3.19) as the non-BPS "fake" superpotential, one can use the boundary condition at the horizon of non-BPS BH. To this end we notice that (see reasoning at the start of the present Sect.) at non-BPS critical points of \( V_{BH,N=8} \) we have

\[ \varrho_{27,1} = \varrho_{27,2} = \varrho_{27,3} = 0 \] (3.21)

so that the parametrization (3.20) reduces to

\[ Z_{AB,nBPS} = 2 \left( \cos^3 \beta_{nBPS} + i \sin^3 \beta_{nBPS} \right) i \varrho_{nBPS} \Omega_{AB} \] (3.22)

This last equation has to be compared with Eq. (2.25), to get:

\[ 2 \left( \cos^3 \beta_{nBPS} + i \sin^3 \beta_{nBPS} \right) i \varrho_{nBPS} = e^{i \pi/4} \rho_{nBPS} \] (3.23)

whose splitting in real and imaginary parts respectively yields:

\[ \sqrt{2} \left( \sin^3 \beta_{nBPS} - \cos^3 \beta_{nBPS} \right) \varrho_{nBPS} = \rho_{nBPS}; \]

\[ \cos^3 \beta_{nBPS} + \sin^3 \beta_{nBPS} = 0. \] (3.25)

The unique solution of the system (3.21)-(3.25) (consistent with Eq. (3.18)) is found to be

\[ \beta_{nBPS} = -\frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}. \] (3.26)

\[ \varrho_{nBPS} = \rho_{nBPS}. \] (3.27)

In particular Eq. (3.26) hints the following redefinition of the angular parameter:

\[ \beta \equiv \alpha - \frac{\pi}{4}, \] (3.28)

which yields the parametrization considered in [6] (up to an overall factor \( \frac{1}{2} \), and with obvious renamings).

The non-BPS nature of the solution (3.19) implies the \( I_4 \) of the 56 of \( E_{7(7)} \) (given by Eqs. (2.10) and (2.11) in the special normal frame (2.1)) to be negative. To show this, we rewrite \( I_4 \) in the manifestly \( USp(8) \)-covariant parametrization (3.20), obtaining [6]

\[ I_4 = -2^4 \sin^2 2\beta \left( \varrho \sin 2\beta - \varrho_{27,1} - \varrho_{27,2} - \varrho_{27,3} \right) \prod_a \left( \varrho \sin 2\beta + \varrho_{27,a} \right) \] (3.29)

which evaluated at the horizon of non-BPS BH reads:

\[ I_{4,nBPS} = -2^4 \varrho_{nBPS}^4 \sin^6 (2\beta_{nBPS}). \] (3.30)
Using Eqs. (3.26) and (3.27), Eq. (3.30) implies
\[
I_{4,nBPS} = -\rho^4 \rho_{nBPS} = - W_{nBPS}^{4} |_{nBPS} < 0, \quad (3.31)
\]
which confirms the function $W$ given by Eq. (3.19) to be the non-BPS “fake” superpotential of $\mathcal{N} = 8, d = 4$ supergravity:
\[
W_{nBPS} = 2\rho. \quad (3.32)
\]

Thus, $W_{nBPS}$ given by Eq. (3.32) has been proved to be the simplest solution of the PDE (3.10), determining the non-BPS “fake” superpotential of $\mathcal{N} = 8, d = 4$ supergravity. The proof given in the treatment performed above relies completely on the geometric data encoded into Maurer-Cartan Eqs. (with suitable consistent boundary horizon conditions), and it is alternative with respect to the treatment given in [6].

As the special normal frame (2.1) has been proved in Sect. 2 to be more suitable to derive (and describe) $1/8$-BPS attractor flow, so the symplectic normal frame (3.20) has been proved in this Sect. to be more suitable to derive (and describe) non-BPS attractor flow.

The expression of $\rho$ in terms of the five parameters $\{\rho_0, \rho_1, \rho_2, \rho_3, \varphi\}$ of the special normal frame (2.1) is not trivial, and it is thoroughly treated in App. B of [6]. In general, $\rho^2$ turns out to satisfy an algebraic equation of order six with coefficients depending on $\{\rho_0, \rho_1, \rho_2, \rho_3, \varphi\}$ and their (scalar-independent) combination $I_4$, as given by Eq. (B.14) of [6] (see also the discussion in [7]).

Thus, in general $\rho^2$ seems not to enjoy an analytical expression. However, (at least one of the) solutions of Eq. (B.14) of [6] is a solution of PDE (A.9), yielding $W_{nBPS}$ in the special normal frame (2.1). Analogously, $W_{1/8-BPS}$ given by Eq. (2.44), suitably translated in the notation of the symplectic normal frame (3.20) (see treatment of App. B of [6]), is a solution of PDE (3.10), yielding $W_{1/8-BPS}$ in the symplectic normal frame (3.20). Furthermore, it is here worth mentioning that, through a suitable re-writing in $\mathcal{N} = 2$ language, the results of [5, 6, 7] are solutions of PDEs (A.9) and/or (3.10) (eventually through additional reductions to $st^2$ or $t^3$ models [5, 6, 7]).

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A Computational Details

In this Appendix we collect details of the computations determining the various formulæ of the present paper.

- Concerning Sect. 2, the details are listed below. Within the index reduction (2.32), the basic multiplication rules for the vielbein

\[
P_{ABCD}P^{EFGH} = \delta^{EFGH}_{ABCD}; \quad (A.1)
\]

\[
P_{ABCD}P_{EFGH} = \epsilon_{ABCD}^{EFGH} \quad (A.2)
\]

recast as

\[
P_{I J \bar{P}^K L} = \delta^K_I \delta^L_J; \quad (A.3)
\]

\[
P_{I J \bar{P} K L} = \tilde{\epsilon}_{IJKL} \equiv |\epsilon_{IJKL}|. \quad (A.4)
\]

Furthermore such rules and Eqs. (2.29) yield

\[
\nabla Z_I \nabla Z_J = \tilde{\epsilon}_{IJKL} Z^K Z^L; \quad (A.5)
\]

\[
\nabla Z_I \nabla Z_J = \delta^I_J |Z_I| = \delta^I_J \rho_I. \quad (A.6)
\]

Using (A.4) and the fully explicited form of Eq. (2.39) which reads

\[
\nabla_i W = \frac{1}{2} \left\{ e^{i\phi/2} (W_0 \rho_1 + W_1 \rho_0) + e^{-i\phi/2} (W_2 \rho_3 + W_3 \rho_2) \right\} P_{23} + e^{i\phi/2} (W_0 \rho_2 + W_2 \rho_0) + e^{-i\phi/2} (W_1 \rho_3 + W_3 \rho_1) P_{13} + e^{i\phi/2} (W_0 \rho_3 + W_3 \rho_0) + e^{-i\phi/2} (W_1 \rho_2 + W_2 \rho_1) P_{12} + e^{i\phi/2} (W_1 \rho_2 + W_2 \rho_1) + e^{-i\phi/2} (W_0 \rho_3 + W_3 \rho_0) P_{03} + e^{i\phi/2} (W_1 \rho_3 + W_3 \rho_1) + e^{-i\phi/2} (W_0 \rho_2 + W_2 \rho_0) P_{02} + e^{i\phi/2} (W_0 \rho_2 + W_2 \rho_0) + e^{-i\phi/2} (W_0 \rho_1 + W_1 \rho_0) P_{01} \right\}, \quad (A.7)
\]
it can be computed that
\[
g^{ij} \nabla_i W \nabla_j W = \frac{1}{2} \left\{ |(W_0 \rho_1 + W_1 \rho_0)|^2 + |(W_0 \rho_2 + W_2 \rho_0)|^2 + |(W_0 \rho_3 + W_3 \rho_0)|^2 + 
+ |(W_1 \rho_2 + W_2 \rho_1)|^2 + |(W_1 \rho_3 + W_3 \rho_1)|^2 + |(W_2 \rho_3 + W_3 \rho_2)|^2 + 
+ [e^{i\varphi} (W_0 \rho_1 + W_1 \rho_0) (W_2 \rho_3 + W_3 \rho_2) + e^{-i\varphi} (\overline{W_0 \rho_1 + W_1 \rho_0}) (\overline{W_2 \rho_3 + W_3 \rho_2})] + 
+ [e^{i\varphi} (W_0 \rho_2 + W_2 \rho_0) (W_1 \rho_3 + W_3 \rho_1) + e^{-i\varphi} (\overline{W_0 \rho_2 + W_2 \rho_0}) (\overline{W_1 \rho_3 + W_3 \rho_1})] + 
+ [e^{i\varphi} (W_0 \rho_3 + W_3 \rho_0) (W_1 \rho_2 + W_2 \rho_1) + e^{-i\varphi} (\overline{W_0 \rho_3 + W_3 \rho_0}) (\overline{W_1 \rho_2 + W_2 \rho_1})] \right\},
\]

(A.8)

that, in turns, gives the following expanded form of PDE (2.41)
\[
W^2 + 
+ \left\{ |(W_0 \rho_1 + W_1 \rho_0)|^2 + |(W_0 \rho_2 + W_2 \rho_0)|^2 + |(W_0 \rho_3 + W_3 \rho_0)|^2 + 
+ |(W_1 \rho_2 + W_2 \rho_1)|^2 + |(W_1 \rho_3 + W_3 \rho_1)|^2 + |(W_2 \rho_3 + W_3 \rho_2)|^2 + 
+ [e^{i\varphi} (W_0 \rho_1 + W_1 \rho_0) (W_2 \rho_3 + W_3 \rho_2) + e^{-i\varphi} (\overline{W_0 \rho_1 + W_1 \rho_0}) (\overline{W_2 \rho_3 + W_3 \rho_2})] + 
+ [e^{i\varphi} (W_0 \rho_2 + W_2 \rho_0) (W_1 \rho_3 + W_3 \rho_1) + e^{-i\varphi} (\overline{W_0 \rho_2 + W_2 \rho_0}) (\overline{W_1 \rho_3 + W_3 \rho_1})] + 
+ [e^{i\varphi} (W_0 \rho_3 + W_3 \rho_0) (W_1 \rho_2 + W_2 \rho_1) + e^{-i\varphi} (\overline{W_0 \rho_3 + W_3 \rho_0}) (\overline{W_1 \rho_2 + W_2 \rho_1})] \right\} = 
\]
\[
\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2. 
\]

(A.9)

- Concerning Sect. 3 the details are listed below.

Within parametrization (3.2)–(3.5), the Maurer-Cartan Eqs. (2.38)–
(2.36) read as follows:

\[ \nabla b = \frac{1}{4} \begin{bmatrix} P_{01} (2\vec{b} + \vec{v}_2 + \vec{v}_3) + P_{02} (2\vec{b} + \vec{v}_1 + \vec{v}_3) + \\ + P_{03} (2\vec{b} + \vec{v}_1 + \vec{v}_2) + P_{12} (2\vec{b} - \vec{v}_1 - \vec{v}_2) + \\ + P_{13} (2\vec{b} - \vec{v}_1 - \vec{v}_3) + P_{23} (2\vec{b} - \vec{v}_2 - \vec{v}_3) \end{bmatrix} \]  

(A.10)

\[ \nabla c_1 = \frac{1}{4} \begin{bmatrix} P_{01} (-2\vec{b} - 4\vec{v}_1 - 3\vec{v}_2 - 3\vec{v}_3) + P_{02} (2\vec{b} + \vec{v}_1 + \vec{v}_3) + \\ + P_{03} (2\vec{b} + \vec{v}_1 + \vec{v}_2) + P_{12} (-2\vec{b} - \vec{v}_1 + 3\vec{v}_2) + \\ + P_{13} (-2\vec{b} - \vec{v}_1 + 3\vec{v}_3) + P_{23} (2\vec{b} - \vec{v}_2 - \vec{v}_3) \end{bmatrix} \]  

(A.11)

\[ \nabla c_2 = \frac{1}{4} \begin{bmatrix} P_{01} (2\vec{b} + \vec{v}_2 + \vec{v}_3) + P_{02} (-2\vec{b} - 3\vec{v}_1 - 4\vec{v}_2 - 3\vec{v}_3) + \\ + P_{03} (2\vec{b} + \vec{v}_1 + \vec{v}_2) + P_{12} (-2\vec{b} + 3\vec{v}_1 - \vec{v}_2) + \\ + P_{13} (2\vec{b} - \vec{v}_1 - \vec{v}_3) + P_{23} (-2\vec{b} - \vec{v}_2 + 3\vec{v}_3) \end{bmatrix} \]  

(A.12)

\[ \nabla c_3 = \frac{1}{4} \begin{bmatrix} P_{01} (2\vec{b} + \vec{v}_2 + \vec{v}_3) + P_{02} (2\vec{b} + \vec{v}_1 + \vec{v}_3) + \\ + P_{03} (-2\vec{b} - 3\vec{v}_1 - 4\vec{v}_2 - 4\vec{v}_3) + P_{12} (2\vec{b} - \vec{v}_1 - \vec{v}_2) + \\ + P_{13} (-2\vec{b} + 3\vec{v}_1 - \vec{v}_3) + P_{23} (-2\vec{b} + 3\vec{v}_2 - \vec{v}_3) \end{bmatrix} \]  

(A.13)

Then, by following the same steps as in Sect. 2 after some algebra,
one achieves the following result (recall $a = 1, 2, 3$ throughout):

$$\nabla W =$$

$$= \frac{1}{4} \left\{ P_{01} \right. + P_{02} + P_{03} + P_{12} + P_{13} + P_{23} \left\} ,$$

$$+ \left( \sum_{a} \frac{\partial W}{\partial c_{a}} \right) \left[ \begin{array}{c}
(2b + \tau_{2} + \tau_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
+ (2b - c_{2} - c_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
-4 \left( b - c_{2} \right) \frac{\partial W}{\partial c_{3}} + \left( b - c_{3} \right) \frac{\partial W}{\partial c_{2}} + \\
-4 \left( b + \tau_{1} + \tau_{2} + \tau_{3} \right) \frac{\partial W}{\partial c_{1}} \\
\end{array} \right) +$$

$$+ \left( \sum_{a} \frac{\partial W}{\partial c_{a}} \right) \left[ \begin{array}{c}
(2b + \tau_{1} + \tau_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
+ (2b - c_{1} - c_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
-4 \left( b - c_{1} \right) \frac{\partial W}{\partial c_{2}} + \left( b - c_{3} \right) \frac{\partial W}{\partial c_{1}} + \\
-4 \left( b + \tau_{1} + \tau_{2} + \tau_{3} \right) \frac{\partial W}{\partial c_{2}} \\
\end{array} \right] +$$

$$+ \left( \sum_{a} \frac{\partial W}{\partial c_{a}} \right) \left[ \begin{array}{c}
(2b - \tau_{1} - \tau_{2}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
+ (2b + c_{1} - c_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
-4 \left( b - \tau_{1} \right) \frac{\partial W}{\partial c_{2}} + \left( b - \tau_{2} \right) \frac{\partial W}{\partial c_{1}} + \\
-4 \left( b + c_{1} + c_{2} + c_{3} \right) \frac{\partial W}{\partial c_{3}} \\
\end{array} \right] +$$

$$+ \left( \sum_{a} \frac{\partial W}{\partial c_{a}} \right) \left[ \begin{array}{c}
(2b - \tau_{1} - \tau_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
+ (2b + c_{1} - c_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
-4 \left( b - \tau_{1} \right) \frac{\partial W}{\partial c_{2}} + \left( b - \tau_{3} \right) \frac{\partial W}{\partial c_{1}} + \\
-4 \left( b + c_{1} + c_{2} + c_{3} \right) \frac{\partial W}{\partial c_{2}} \\
\end{array} \right] +$$

$$+ \left( \sum_{a} \frac{\partial W}{\partial c_{a}} \right) \left[ \begin{array}{c}
(2b - \tau_{2} - \tau_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
+ (2b + c_{2} + c_{3}) \left( \frac{\partial W}{\partial b} + \sum_{a} \frac{\partial W}{\partial c_{a}} \right) + \\
-4 \left( b - \tau_{2} \right) \frac{\partial W}{\partial c_{2}} + \left( b - \tau_{3} \right) \frac{\partial W}{\partial c_{2}} + \\
-4 \left( b + c_{1} + c_{2} + c_{3} \right) \frac{\partial W}{\partial c_{1}} \\
\end{array} \right] .$$

(A.14)

It is worth noticing that the coefficient of the vielbein $P_{1J}$ (recall
\( I = 0, 1, 2, 3 \) throughout) is the complex conjugate of the coefficient of \( P_{KL} \), with \( K, L \neq I, J \). In other words, in order to compute the term \( \nabla W \nabla W \) one has just to sum up the squares of the real and imaginary part of each coefficient, thus obtaining Eq. (3.6).

References

[1] A. Ceresole and G. Dall’Agata, *Flow Equations for Non-BPS Extremal Black Holes*, JHEP **0703**, 110 (2007), [hep-th/0702088](http://arxiv.org/abs/hep-th/0702088).

[2] L. Andrianopoli, R. D’Auria, E. Orazi and M. Trigiante, *First order description of black holes in moduli space*, JHEP **0711**, 032 (2007), [arXiv:0706.0712](http://arxiv.org/abs/0706.0712) [hep-th].

[3] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *stu Black Holes Unveiled*, Entropy Vol. 10(4), 507 (2008), [arXiv:0807.3503](http://arxiv.org/abs/0807.3503) [hep-th].

[4] L. Andrianopoli, R. D’Auria, E. Orazi and M. Trigiante, *First Order Description of D = 4 static Black Holes and the Hamilton-Jacobi equation*, [arXiv:0905.3938](http://arxiv.org/abs/0905.3938) [hep-th].

[5] A. Ceresole, G. Dall’Agata, S. Ferrara and A. Yeranyan, *First order flows for N = 2 extremal black holes and duality invariants*, [arXiv:0908.1110](http://arxiv.org/abs/0908.1110) [hep-th].

[6] G. Bossard, Y. Michel and B. Pioline, *Extremal black holes, nilpotent orbits and the true fake superpotential*, [arXiv:0908.1742](http://arxiv.org/abs/0908.1742) [hep-th].

[7] A. Ceresole, G. Dall’Agata, S. Ferrara and A. Yeranyan, *Universality of the superpotential for d = 4 extremal black holes*, [arXiv:0910.2697](http://arxiv.org/abs/0910.2697) [hep-th].

[8] S. Ferrara, A. Gneccchi and A. Marrani, *d = 4 Attractors, Effective Horizon Radius and Fake Supergravity*, Phys. Rev. **D78**, 065003 (2008), [arXiv:0806.3196](http://arxiv.org/abs/0806.3196).

[9] R. Arnowitt, S. Deser and C. W. Misner: *Canonical Variables for General Relativity*, Phys. Rev. **117**, 1595 (1960).

[10] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973). S. W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971); in C. DeWitt, B. S. DeWitt, *Black Holes (Les Houches 1972)* (Gordon and Breach, New York, 1973). S. W. Hawking, Nature **248**, 30 (1974). S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).

[11] R. Kallosh and B. Kol, *E_{7(7)} Symmetric Area of the Black Hole Horizon*, Phys. Rev. **D53**, 5344 (1996), [hep-th/9602014](http://arxiv.org/abs/hep-th/9602014).
L. Andrianopoli, R. D’Auria and S. Ferrara, *U invariants, black hole entropy and fixed scalars*, Phys. Lett. **B403**, 12 (1997), hep-th/9703156.

B. L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, *Duality, Entropy and ADM Mass in Supergravity*, Phys. Rev. **D79**, 125010 (2009), arXiv:0902.3973 [hep-th].

S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge Orbits of Symmetric Special Geometries and Attractors*, Int. J. Mod. Phys. **A21**, 5043 (2006), hep-th/0606209.

S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, Lect. Notes Phys. **755**, 115 (2008), arXiv:0711.4547 [hep-th].

S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *SAM Lectures on Extremal Black Holes in d = 4 Extended Supergravity*, arXiv:0905.3739 [hep-th].

M. Günaydin, G. Sierra and P. K. Townsend, *The Geometry of N = 2 Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. **B242**, 244 (1984).

M. Cvetic and D. Youm, *Dyonic BPS Saturated Black Holes of Heterotic String on a Six-Torus*, Phys. Rev. **D53**, 584 (1996), hep-th/9507090.

M. Cvetic and A. A. Tseytlin, *Solitonic Strings and BPS Saturated Dyonic Black Holes*, Phys. Rev. **D53**, 5619 (1996); Erratum - ibid. **D55**, 3907 (1997), hep-th/9512031.

R. D’Auria, S. Ferrara and M. A. Lledó, *On Central Charges and Hamiltonians for Ω-brane dynamics*, Phys. Rev. **D60**, 084007 (1999), hep-th/9903089.

S. Ferrara and R. Kallosh, *On N = 8 attractors*, Phys. Rev. **D73**, 125005 (2006), hep-th/0603247.

S. Ferrara and M. Günaydin, *Orbits of Exceptional Groups, Duality and BPS States in String Theory*, Int. J. Mod. Phys. **A13**, 2075 (1998), hep-th/9708025.

S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for N=2 Symmetric Manifolds*, Phys. Lett. **B652**, 111 (2007), arXiv:0706.1667 [hep-th].

L. Andrianopoli, R. D’Auria and S. Ferrara, *U-Duality and Central Charges in Various Dimensions Revisited*, Int. J. Mod. Phys. **A13** 431 (1998), hep-th/9612105.
[25] S. Ferrara and A. Marrani, $\mathcal{N}=8$ non-BPS Attractors, Fixed Scalars and Magic Supergravities, Nucl. Phys. B788, 63 (2008), arXiv:0705.3866 [hep-th].

[26] M. Güneydin, A. Neitzke, B. Pioline and A. Waldron, BPS black holes, quantum attractor flows and automorphic forms, Phys. Rev. D73, 084019 (2006), hep-th/0512296.

[27] M. Güneydin, A. Neitzke, B. Pioline and A. Waldron, Quantum Attractor Flows, JHEP 0709, 056 (2007), arXiv:0707.0267 [hep-th].

[28] D. Gaiotto, W. Li and M. Padi, Non-Supersymmetric Attractor Flow in Symmetric Spaces, JHEP 0712, 093 (2007), arXiv:0710.1638 [hep-th].

[29] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet, Generating Geodesic Flows and Supergravity Solutions, Nucl. Phys. B812, 343 (2009), arXiv:0806.2310 [hep-th].

[30] G. Bossard, H. Nicolai and K. S. Stelle, Universal BPS structure of stationary supergravity solutions, JHEP 0907, 003 (2009), arXiv:0902.4438 [hep-th].

[31] W. Chemissany, J. Rosseel, M. Trigiante and T. Van Riet, The full integration of black hole solutions to symmetric supergravity theories, arXiv:0903.2777 [hep-th].

[32] P. Fré and A. S. Sorin, The Integration Algorithm for Nilpotent Orbits of $G/H^*$ Lax systems: for Extremal Black Holes, arXiv:0903.3771 [hep-th].

[33] W. Chemissany, P. Fré and A. S. Sorin, The Integration Algorithm of Lax equation for both Generic Lax matrices and Generic Initial Conditions, arXiv:0904.0801 [hep-th].

[34] A. Ceresole, S. Ferrara, A. Gnechi and A. Marrani, More on $\mathcal{N}=8$ Attractors, Phys. Rev. D80, 045020 (2009), arXiv:0904.4506 [hep-th].

[35] A. Ceresole, S. Ferrara and A. Gnechi, 5d/4d U-dualities and $\mathcal{N}=8$ black holes, arXiv:0908.1069 [hep-th].

[36] C. Bloch and A. Messiah, Nucl. Phys. 39, 95 (1962).

[37] B. Zumino, J. Math. Phys. 3, 1055 (1962).

[38] S. Ferrara, C. A. Savoy and B. Zumino, General Massive Multiplets in Extended Supersymmetry, Phys. Lett. B100, 393 (1981).
[39] S. Ferrara, J. M. Maldacena, *Branes, central charges and U-duality invariant BPS conditions*, Class. Quant. Grav. 15, 749 (1998), hep-th/9706097.

[40] L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, Lect. Notes Phys. 737, 661 (2008), hep-th/0611345.