TRIGONAL TODA LATTICE EQUATION

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Abstract. In this article, we give the trigonal Toda lattice equation

$$\frac{1}{2} \frac{d^3 u}{du^3} q_\ell(u) = e^{q_{\ell+1}(u)} + e^{q_{\ell+1}(u)} + e^{q_{\ell+12}(u)} - 3e^{q_{\ell}(u)},$$

for a lattice points $\ell \in \mathbb{Z} \zeta_3$ of a 3-regular graph, and its elliptic solution for the curve

$$y(y - s) = x^3, \ (s \neq 0) \text{ and } \zeta_3 = e^{2\pi i / 3}.$$

1. Introduction

The elliptic functions have high symmetries and generate many interesting relations, including the Toda lattice equation [To]. Recently J. C. Eilbeck, S. Matsutani and Y. Ônishi introduced some new addition formulae for the Weierstrass $\wp$ functions in genus one, especially for the equiharmonic elliptic curve which has the trigonal cyclic symmetry as a Galois group.

In this article, we use the new addition formulae of the equiharmonic elliptic curve $E$ given by $y(y - s) = x^3$ and derive the the trigonal Toda lattice equation following the derivation of Toda equation in [KMP] as in Proposition 2.2. This third order differential-difference equation reflects the trigonal cyclic symmetry of the curve.

The third order differential equations reminds us of the Chazy equations [CO]. However the trigonal Toda lattice equation cannot have non-trivial continuum limit because it becomes the three rational curves if we put $s = 0$. In other words, the trigonal Toda equation is out of the framework of Chazy equations. Since the algebraic curves sometimes have automorphism with higher symmetries, this equation might be applied to such curves. For example, it may be extended to the trigonal curves of higher genus, e.g., $y^3 = x(x - b_1)(x - b_2)(x - b_3)$ [EEMOP, Appendix].

Acknowledgments: I would like to thank Yuji Kodama for helpful comments and pointing out its relation to the Chazy equation, and Yoshihiro Ônishi for variable discussions.

1.1. Addition formula of $\sigma$ of the equiharmonic elliptic curve. J. C. Eilbeck, S. Matsutani and Y. Ônishi showed the relations of the elliptic sigma function [EMO]

$$\frac{\sigma(u - v)\sigma(u - \zeta_3 v)\sigma(u - \zeta_3^2 v)}{\sigma(u)^3\sigma(v)^3} = (y(u) - y(v)).$$

Date: S. Matsutani June 8, 2019.
and

\[ (1.2) \quad -\frac{\sigma(u + v)\sigma(u + \zeta_3 v)\sigma(u + \zeta_3^2 v)}{\sigma(u)^3\sigma(v)^3} = (y(u) + y(v)) \]

for the curve

\[ y^2 + \mu_3 y = x^3 + \mu_6, \]

where \( \zeta_3 = e^{2\pi \sqrt{-1}/3} \) since the curve has the symmetry of the trigonal cyclic action. It corresponds to the Weierstrass standard form,

\[ (\wp')^2 = 4\wp^3 - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \]

where \( g_3 = -4(\mu_3^2 + \mu_6) \) and \( \wp(u)' = 2y + \mu_3 \). Then we have

\[ \sigma(\zeta_3 u) = \zeta_3 \sigma(u), \quad \wp(\zeta_3 u) = \zeta_3 \wp(u), \quad \wp'(\zeta_3 u) = \wp'(u) \]

for \( u \in \mathbb{C} \).

In this article, we consider the curve \( E \) given by

\[ y(y - s) = x^3, \quad \left( y - \frac{s}{2} \right)^2 = \left( x + \sqrt[3]{\frac{s^2}{4}} \right) \left( x + \zeta_3 \sqrt[3]{\frac{s^2}{4}} \right) \left( x + \zeta_3^2 \sqrt[3]{\frac{s^2}{4}} \right). \]

It means \( \mu_3 = -s, \mu_6 = 0, e_j = -\zeta_3^{1-j} \sqrt[3]{\frac{s^2}{4}} \) (\( j = 1, 2, 3 \)) and \( g_3 = -4s^2 \).

### 1.2. The integrals of \( \omega_0 \) and \( \omega_s \).

Let us the integral to \((x, y) = (0, s)\) from the infinity point \( \infty \) denoted by \( \omega_s \), and similarly that to \((0, 0)\) by \( \omega_0 \),

\[ \omega_s = \int_{\infty}^{(0,s)} du, \quad \omega_0 = \int_{\infty}^{(0,0)} du. \]

The complete elliptic integrals of the first and the second kinds are given by

\[ \omega' = \omega_1 = \int_{\infty}^{e_1} du, \quad \omega_2 = \zeta_3^2 \omega', \quad \omega'' = \omega_3 = \zeta_3 \omega'. \]

\[ \eta_i = \int_{\infty}^{e_i} xdu, \quad \eta_1 = \eta_1, \quad \eta_2 = \eta_3 = \zeta_3^2 \eta', \]

where

\[ \omega_1 + \omega_2 + \omega_3 = 0, \quad \eta_1 + \eta_2 + \eta_3 = 0, \quad \eta' \omega'' - \eta'' \omega' = \frac{\pi \sqrt{-1}}{2}. \]

Thus we have the relation

\[ \eta' \omega' = \frac{\pi \sqrt{-1}}{2(\zeta_3^2 - \zeta_3^3)} = \frac{\pi}{2\sqrt{3}} \in \mathbb{R}. \]

From [EMO], we have

\[ \wp(u) = x(u), \quad \wp'(u) = 2y(u) - s, \quad y(u) = \frac{1}{2}(\wp'(u) + s). \]
Lemma 1.1.

\[
\begin{align*}
    x(\omega_0) = x(\omega_s) &= 0, \quad y(\omega_0) = 0, \quad y(\omega_s) = s, \\
    \omega_0 &= \frac{1 - \zeta_3}{3} 2\omega', \quad \omega_s = \frac{1 - \zeta_2^3}{3} 2\omega'.
\end{align*}
\]

Proof. The first three equations are obvious from [EMO]. We use the covering \( \pi_2 : E \to \mathbb{P} \ ((x, y) \mapsto y) \). We note that the integral \( \omega_s \) is a contour integral from \( \infty \) to \((0, s)\) and the point \((0, s)\) is a branch point of \( \pi_2 \). Thus when we consider the contour integral another sheet of \( \pi_2^{-1} \) as the return path from \((0, s)\) to \( \infty \). It must be a period and thus \((1 - \zeta_3)\omega_s \) is a point of the lattice \( \mathbb{Z} 2\omega' + \mathbb{Z} 2\omega'' \). There exist \( n \) and \( m \) satisfying

\[
(1 - \zeta_3)\omega_s = 2\omega'n + 2\omega''m = 2(n + \zeta_3m)\omega'
\]

Thus

\[
\omega_s = \frac{1}{3} (1 - \zeta_3^2) 2(n + \zeta_3m)\omega'
\]

We fix \( \omega_s \) modulo lattice and then, there are two possibilities,

\[
\omega_s = \pm \frac{1}{3} (1 - \zeta_3^2) 2\omega' \text{ modulo lattice}.
\]

We find it \( \omega_s = \frac{1}{3} (1 - \zeta_3^2) 2\omega' \) numerically using Maple. \( \square \)

The image of the incomplete elliptic integrals is acted by \( \text{SL}(2, \mathbb{Z}) \) and the cyclic group \( \zeta_3 \). For \( v \in \mathbb{C} \), we define

\[
Z_v := \mathbb{Z}[\zeta_3^2]v = \{ \ell_1 v + \ell_2 \zeta_3^2 v \mid \ell_1, \ell_2 \in \mathbb{Z} \}
\]

Since the lattice in the above proof is given by \( \mathbb{Z} 2\omega' \) and the Jacobian \( J_E \) of the curve \( E \) is given by

\[
J_E = \mathbb{C}/\mathbb{Z} 2\omega'.
\]

These points of the integrals for the branch points of the curve \( E \) are illustrated in Figure 1.

2. The trigonal Toda equation

Proposition 2.1. The quantities,

\[
q(u, v) = \log(y(u) - y(v)), \quad y(u) - y(v) = e^{q(u,v)},
\]

and

\[
\bar{q}(u, v) = \log(y(u) + y(v)), \quad y(u) + y(v) = e^{\bar{q}(u,v)},
\]

satisfy the relations,

\[
-\frac{1}{2} \frac{d^3}{du^3} q(u, v) = e^{q(u-v,v)} + e^{q(u-\zeta_3 v,v)} + e^{q(u-\zeta_3^2 v,v)} - 3e^{q(u,v)}
\]

\[
-\frac{1}{2} \frac{d^3}{du^3} \bar{q}(u, v) = e^{\bar{q}(u-v,v)} + e^{\bar{q}(u-\zeta_3 v,v)} + e^{\bar{q}(u-\zeta_3^2 v,v)} - 3e^{\bar{q}(u,v)}
\]

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Figure 1. The lattice points of $\mathbb{Z}_{2\omega'}$: The lattice points of $\mathbb{Z}_{2\omega'}$ are denoted by the black dots, $\omega_0$ and $\omega_s$ with $\mathbb{Z}_{2\omega'}$ translations are denoted by gray dots and white dots respectively.

Proof. We consider the logarithm of both sides of (1.1) and differentiate both side third times with respect to $u$. Then we have

$$-\frac{d^3}{du^3} \log(y(u) - y(v)) = 2(y(u - v) + y(u - \zeta_3 v) + y(u - \zeta_3^2 v)) - 6(y(u)).$$

Similarly we have the second relation from (1.2). □

We regard $\mathbb{Z}_v$ as the set of nodes of an infinite 3-regular graph $G_v$. The right hand sides in the equations in Proposition 2.1 can be regarded as actions of the graph Laplacian of a 3-regular graph. For an Eisenstein integer $\ell \in \mathbb{Z}[\zeta_3]$ or $\ell v \in \mathbb{Z}_v$, we introduce the quantities,

$$q_\ell(u, v) = q(u - \ell v, v) = \log(y(u - \ell v) - y(v)).$$

and

$$\tilde{q}_\ell(u, v) = \tilde{q}(u - \ell v, v) = \log(y(u - \ell v) + y(v)).$$

by substituting $u - \ell v$ into $u$ in $q$ and $\tilde{q}$.

It is obvious that they satisfy the relations over the lattice $\mathbb{Z}_v$,

$$-\frac{1}{2} \frac{d^3}{du^3} q_\ell(u, v) = e^{q_{\ell+1}(u,v)} + e^{q_{\ell+\zeta_3}(u,v)} + e^{q_{\ell+\zeta_3^2}(u,v)} - 3e^{q_\ell(u,v)},$$

$$-\frac{1}{2} \frac{d^3}{du^3} \tilde{q}_\ell(u, v) = e^{\tilde{q}_{\ell-1}(u,v)} + e^{\tilde{q}_{\ell-\zeta_3}(u,v)} + e^{\tilde{q}_{\ell-\zeta_3^2}(u,v)} - 3e^{\tilde{q}_\ell(u,v)}$$

noting $\zeta_3^2 = -1 - \zeta_3 \in \mathbb{Z}[\zeta_3]$. The $q_\ell(u, v)$ and $\tilde{q}_\ell(u, v)$ are functions over the $J_E$.

Noting $y(\omega_0) = 0$ from Lemma 1.1, we have the higher symmetrical point $\omega_0$. It implies that $q_\ell(u, \omega_0) = \tilde{q}_\ell(u, \omega_0)$. From the results in [EMO], we have the relation:
Proposition 2.2. For $\ell \omega_0 \in \mathbb{Z}_{\omega_0}$, the quantity

$$q_{\ell}(u) = q(u - \ell \omega_0) = \log(y(u - \ell \omega_0))$$

satisfies the relations over the lattice $\mathbb{Z}_v$,

$$-\frac{1}{2} \frac{d^2}{du^2} q_{\ell}(u) = e^{q_{\ell+1}(u)} + e^{q_{\ell+\zeta_3}(u)} + e^{q_{\ell+\zeta_2^3}(u)} - 3e^{q_{\ell}(u)}$$

$$= e^{q_{\ell-1}(u)} + e^{q_{\ell-\zeta_3}(u)} + e^{q_{\ell-\zeta_2^3}(u)} - 3e^{q_{\ell}(u)}$$

We note that $\mathbb{Z}_{\omega_0}$ means the lattice whose point corresponds all dots in Figure 1 without distinguishing white, gray and black color since $3\omega_0 \in \mathbb{Z}_{2\omega}$. Let us consider a certain $\ell \omega_0$ in $\mathbb{Z}_{\omega_0}$. As an example, we consider one of the black points. Then $(\ell + 1, \ell + \zeta_3, \ell + \zeta_2^3)$ means the adjacent gray dots of the black $\ell$ dot, whereas $(\ell - 1, \ell - \zeta_3, \ell - \zeta_2^3)$ means the adjacent white dots of the black $\ell$ dot. It means that we consider the dual 3-regular subgraphs $G_{\omega_0}$ and $G_{\omega_0}^*$ whose set of the nodes is $\mathbb{Z}_{\omega_0}$. The right hand side of the first equation in Proposition 2.2 corresponds to the action by graph Laplacian of $G_{\omega_0}$ whereas that of the second equation does to $G_{\omega_0}^*$. Proposition 2.2 means that both are consistent. This equation is a dynamical equation for the lattice $\mathbb{Z}_{\omega_0}$ as the ordinary Toda lattice is for the one-dimensional lattice. Thus we call it trigonal Toda equation.

The elliptic function solutions of this equation is illustrated in Figure 2 and Figure 3.

Figure 2. The elliptic solution of the trigonal Toda lattice equation $q_0(t\sqrt{-1}), s = 0.5, t \in \mathbb{R}$: (a) shows its real part whereas (b) corresponds to its imaginary part.
Figure 3. The elliptic solution of the trigonal Toda lattice equation $q_0(t), s = 0.5, t \in \mathbb{R}$: (a) shows its real part whereas (b) corresponds to its imaginary part.

3. Discussion

We derived the trigonal Toda lattice equation based on the symmetry of the curve $E$ associated with the automorphism of the curve. Since the algebraic curve has higher symmetries, this approach could be generalized to more general curves, e.g., the genus three curve [EEMOP].

It should be noted that since this curve becomes trivial three rational curves if we take the limit $s \to 0$, it cannot be related to the Chazy equations directly. However if we find a similar equation of the curve with higher genus, we could consider its relation to the Chazy equation.

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