Torsion points and concurrent exceptional curves on del Pezzo surfaces of degree one

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Abstract

The blow-up of the anticanonical base point on a del Pezzo surface $X$ of degree 1 gives rise to a rational elliptic surface $\mathcal{E}$ with only irreducible fibers. The sections of minimal height of $\mathcal{E}$ are in correspondence with the 240 exceptional curves on $X$. A natural question arises when studying the configuration of these curves: if a point on $X$ is contained in ‘many’ exceptional curves, is it torsion on its fiber on $\mathcal{E}$? In 2005, Kuwata proved for the analogous question on del Pezzo surfaces of degree 2, where there are 56 exceptional curves, that if ‘many’ equals 4 or more, the answer is yes. In this paper, we prove that for del Pezzo surfaces of degree 1, the answer is yes if ‘many’ equals 9 or more. Moreover, we give counterexamples where a non-torsion point lies in the intersection of 7 exceptional curves. We give partial results for the still open case of 8 intersecting exceptional curves.

1 Introduction

A del Pezzo surface over a field $k$ is a smooth, projective, geometrically integral surface over $k$ with ample anticanonical divisor. The degree of a del Pezzo surface is the self-intersection number of the canonical divisor, which is an integer between 1 and 9. Over an algebraically closed field, a del Pezzo surface of degree $d$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ (for $d = 8$), or to $\mathbb{P}^2$ blown up at $9 - d$ points in general position. Over non-algebraically closed fields, this is in general not true, and the arithmetic of these surfaces has been widely studied. Del Pezzo surfaces of degree at least 2 over a field $k$ with a $k$-rational point are known to be $k$-unirational under the extra condition for degree 2 that the $k$-rational point lies outside a closed subset $\{\text{Seg43, Seg51, Man74, Kol02, Pie12, STV A14}\}$. Del Pezzo surfaces of degree 1 over a field with characteristic not 2 are known to be unirational if they admit a conic bundle structure $\{\text{KM17}\}$, but outside this case, there is no example of a minimal del Pezzo surface of degree 1 that is known to be $k$-unirational, nor of one that is known not to be $k$-unirational. Unirationality for del Pezzo surfaces of degree 1 in general is considered an extremely difficult problem. Over infinite fields there are several partial results on Zariski density of the set of rational points on these surfaces $\{\text{VAL1, Uia07, Uia08, Jab12, SvL14, Bul18, DW22}\}$, which is a weaker notion in the sense that it is implied by unirationality. However, there are still many families of del Pezzo surfaces of degree 1 for which even Zariski density of rational points is not proven.

Over an algebraically closed field, a del Pezzo surface contains a finite number of exceptional curves, depending on the degree of the surface; we often call these lines. For several of the earlier mentioned results, one requires the existence of a rational point on the surface which is not contained in too many lines. For example, in the paper $\{\text{STV A14}\}$ the authors show that a del Pezzo surface of degree 2 is unirational if it contains a point that is not contained in 4 lines, and lies outside a specific subset of the surface. As another example, one of the conditions for the set of rational points on a del Pezzo surface of degree 1 to be dense in $\{\text{SvL14}\}$ is the existence of a point not contained in 6 of these lines. At the same time, several of the results on density of rational points on a del Pezzo surface of degree 1 require the existence of a point which is non-torsion on its fiber of the elliptic surface obtained by blowing up the base point of the anticanonical linear system; see for instance $\{\text{SvL14, DW22}\}$. The paper $\{\text{SvL14}\}$ contains several examples of a point on a del Pezzo surface of degree 1 for which their method fails, and in all cases, the point is contained in the intersection of at least 6 lines, and it is torsion on its fiber. A natural question is therefore whether there is a relation between these two phenomena. More precisely, for a del Pezzo surface $X$ of degree 1 and the corresponding elliptic surface $\mathcal{E}$ obtained by blowing up the base point of the anticanonical linear system on $X$, we ask the following.

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Question 1.1. If a point on $X$ is contained in ‘many’ lines, is the corresponding point on $E$ then torsion on its fiber?

Of course, ‘many’ needs to be specified. In this paper we give a positive answer to this question for ‘many’ equal to 9:

**Theorem 1.2.** If at least 9 exceptional curves on $X$ are concurrent in a point, then the corresponding point on $E$ is torsion on its fiber.

We also show that if we take ‘many’ to be 7, the answer to this question is negative, at least in most characteristics, by providing two counterexamples (Examples 3.1 and 3.3). These are the first known examples of a non-torsion point contained in more than 6 lines; see also Section 5.1.

We call a set of exceptional curves on a del Pezzo surface concurrent in a point if that point is contained in all of them.

**Remark 1.3.** For del Pezzo surfaces of degree 2, the situation is simpler, and a result similar to Theorem 1.2 is known. A del Pezzo surface of degree 2 is a double cover of $\mathbb{P}^2$ ramified along a quartic curve. On such a surface, a point is contained in at most 4 exceptional curves, and this happens exactly when its projection to $\mathbb{P}^2$ is in the intersection of 4 bitangents of the quartic curve. In [Kuw05], Kuwata gives a construction for an elliptic surface by blowing up twice on the del Pezzo surface, and he shows that for a point contained in 4 exceptional curves, the corresponding point on the elliptic surface is torsion on its fiber [Kuw05, Proposition 7.1].

The situation for del Pezzo surfaces of degree 1 is more complex. First of all, outside characteristics 2 and 3 the maximal number of concurrent lines on a del Pezzo surface of degree 1 is 10 [vLW23, Theorems 1.1 and 1.2], but as Theorem 1.2 shows, a point contained in 9 lines is already torsion on its fiber. Moreover, 4 intersecting lines on a del Pezzo surface of degree 2 intersect pairwise with multiplicity 1, but there are a priori many different ways in which 9 or more concurrent lines on a del Pezzo surface of degree 1 can intersect; we explain this in Section 4.

The question of whether one can find an example with 8 lines on $X$ that intersect in a point which is non-torsion on its fiber of $E$ stays unsolved. We show that the lines in such an example, if it exists, intersect each other according to one of 15 prescribed configurations in general, and 13 in characteristic 0.

**Theorem 1.4.** If 8 exceptional curves on a del Pezzo surface $X$ of degree 1 are concurrent in a point, then the corresponding point on the elliptic surface $E$ obtained by blowing up the base point of the anticanonical linear system is torsion on its fiber, except possibly in the following case. The 8 exceptional curves intersect pairwise with multiplicities 1 or 2, and the graph where each vertex corresponds to an exceptional curve and two vertices are connected with an edge if and only if the corresponding exceptional curves intersect with multiplicity 2, equals one of the graphs with numbers 1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 in Figure 1. In characteristic 0, numbers 7 and 15 can be excluded from this list.

The paper is organized as follows. In Section 2 we present the necessary background on the exceptional curves and the root system $E_8$, the elliptic surface $E$ obtained from $X$, and the strict transforms on $E$ of the exceptional curves on $X$. In Section 3 we show that 7 concurrent lines do not always intersect in a torsion point, by giving two counterexamples (Examples 3.1 and 3.3). In Section 4 we prove Theorem 1.2. In Section 5 we study the case of 8 lines and prove Theorem 1.4. We close this section with an example of exactly 6 lines on a del Pezzo surface of degree 1 that meet at a non-torsion point (Example 5.5); we discuss how this example and Examples 3.1 and 3.3 are produced in Remark 5.6.

Part of this paper appeared in the PhD thesis of the second author. Specifically, Sections 2 and 4 are slight modifications of [Win21, Chapter 5 and Sections 1.4.2, 1.4.3]. The innovative material compared to what is found in the thesis is Sections 3 and 5 which includes Theorem 1.4.

Computations were done in magma [BCP97]; the code is available online [Cod].

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2 Background

In this Section we give the necessary background for the rest of this paper.

2.1 Exceptional curves and the $E_8$ root system

Definition 2.1. Let $r \leq 8$ be an integer, and let $P_1, \ldots, P_r$ be points in $\mathbb{P}^2$. We say that $P_1, \ldots, P_r$ are in general position if there is no line containing three of the points, no conic containing six of the points, and no cubic containing eight of the points with a singularity at one of them.

As mentioned in the introduction, a del Pezzo surface over an algebraically closed field is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ (for $d = 8$), or to $\mathbb{P}^2$ blown up at $9 - d$ points in general position [Man74, Theorem 24.4].

Definition 2.2. An exceptional curve on a del Pezzo surface $X$ with canonical divisor $K_X$ is an irreducible projective curve $C \subset X$ such that

$$C^2 = C \cdot K_X = -1.$$ 

We often call exceptional curves lines, since for a del Pezzo surface of degree $d \geq 3$, the images of these curves under the anticanonical embedding in $\mathbb{P}^d$ are lines. Over an algebraically closed field, we know exactly how to describe the lines on a del Pezzo surface in terms of curves in $\mathbb{P}^2$.

Theorem 2.3 ([Man74, Theorem 26.2]). For an integer $d \in \{1, \ldots, 8\}$, let $P_1, \ldots, P_{9 - d}$ be $9 - d$ points in general position in $\mathbb{P}^2$. The exceptional curves on the del Pezzo surface of degree $9 - d$ obtained by the blow-up of $P_1, \ldots, P_{9 - d}$ are

- the exceptional curves $E_i$ above the points $P_i$ for $i \in \{1, \ldots, 9 - d\}$, and the strict transforms of the following curves in $\mathbb{P}^2$.
- The line $L_{i,j}$ passing through the points $P_i$ and $P_j$ for $i \neq j$.
- The conics passing through five of the points.
- The cubic $C_{i,j}$ not passing through $P_j$, passing twice through $P_i$ and passing once through the six remaining points for $i \neq j$.
- The quartic $Q_{i,j,k}$ passing through the eight points with a double point in $P_i$, $P_j$ and $P_k$ for $i,j,k$ distinct.
- The quintics passing through the eight points with double points at 6 of them, and
- The sextics passing through the eight points with double points at 7 of them, and a triple point at one of them.

Notation 2.4. Throughout the paper, for $r$ points $P_1, \ldots, P_r$ in general position in $\mathbb{P}^2$, and for $i \in \{1, \ldots, r\}$, we use the notation $E_i$ for the exceptional curve above $P_i$ on the del Pezzo surface obtained by blowing up $P_1, \ldots, P_r$. Similarly, for $i,j,k \in \{1, \ldots, r\}$ we write $L_{i,j}$, $C_{i,j}$, $Q_{i,j,k}$ for the lines, cubics, and quartics in $\mathbb{P}^2$, as defined in Theorem 2.3.

From now on, we focus on del Pezzo surfaces of degree 1. Let $X$ be such a surface over an algebraically closed field. From Theorem 2.3 it follows that $X$ contains 240 exceptional curves. These are in one-to-one correspondence with the root system $E_8$, as we will now describe.

Let $\langle \cdot, \cdot \rangle$ be the negative of the intersection pairing on Pic $X$, and let $K_X$ be the canonical divisor of $X$. Then $\langle \cdot, \cdot \rangle$ on $\mathbb{R} \otimes_{\mathbb{Z}} \text{Pic} X$ induces the structure of a Euclidean space on the orthogonal complement $K_X^J$ of the class of the canonical divisor, and with this structure, the set

$$R = \{ D \in \text{Pic} X \mid \langle D, D \rangle = 2; \ D \cdot K_X = 0 \}$$
is a root system of type $E_8$ in $K_X^2$ ([Man74], Theorem 23.9). Let $I$ be the set of the 240 exceptional curves in $	ext{Pic } X$. For $e \in I$ we have $e + K_X \in K_X^2$ and $(e + K_X, e + K_X) = 2$, and this gives a bijection
\[ I \to R, \quad e \mapsto e + K_X. \] (1)

For $e_1, e_2 \in I$ we have $(e_1 + K_X, e_2 + K_X) = 1 - e_1 \cdot e_2$. As a consequence of this bijection, the group of permutations of $I$ that preserve the intersection pairing is isomorphic to the Weyl group $W_8$, which is the group of permutations of $E_8$ generated by the reflections in the hyperplanes orthogonal to the roots ([Man74], Theorem 23.9). Another way of stating the bijection (1) is to note that the weighted graphs on $I$ and $E_8$ and their automorphism groups are isomorphic (Remark 2.7).

**Definition 2.5.** By a graph we mean a pair $(V, D)$, where $V$ is a set of elements called vertices, and $D$ a subset of the power set of $V$ such that every element in $D$ has cardinality 2; elements in $D$ are called edges, and the size of the graph is the cardinality of $V$. A graph $(V, D)$ is complete if for every two distinct vertices $v_1, v_2 \in V$, the pair ${v_1, v_2}$ is in $D$.

By a weighted graph we mean a graph $(V, D)$ with a map $\psi : D \to A$, where $A$ is any set, whose elements we call weights; for any element $d$ in $D$ we call $\psi(d)$ its weight. If $(V, D)$ is a weighted graph with weight function $\psi$, then we define a weighted subgraph of $(V, D)$ to be a graph $(V', D')$ with map $\psi'$, where $V'$ is a subset of $V$, while $D'$ is a subset of the intersection of $D$ with the power set of $V'$, and $\psi'$ is the restriction of $\psi$ to $D'$. A clique of a weighted graph is a complete weighted subgraph.

An isomorphism between weighted graphs $(V, D)$ and $(V', D')$ with weight functions $\psi : D \to A$ and $\psi' : D' \to A'$, respectively, consists of a bijection $f$ between the sets $V$ and $V'$ and a bijection $g$ between the sets $A$ and $A'$, such that for any two vertices $v_1, v_2 \in V$, we have $\{v_1, v_2\} \in D$ with weight $w$ if and only if $\{f(v_1), f(v_2)\} \in D'$ with weight $g(w)$. We call the map $f$ an automorphism of $(V, D)$ if $(V, D) = (V', D')$, $\psi = \psi'$, and $g$ is the identity on $A$.

**Definition 2.6.** By $\Gamma$ we denote the complete weighted graph whose vertex set is the set of roots in $E_8$, and where the weight function is induced by the dot product. Similarly, by $G$ we denote the complete weighted graph whose vertex set is $I$, and where the weight function is the intersection pairing in $	ext{Pic } X$.

**Remark 2.7.** There is an isomorphism of weighted graphs between $G$ and $\Gamma$, that sends a vertex $e$ in $G$ to the corresponding vertex $e + K_X$ in $\Gamma$, and an edge $d = \{e_1, e_2\}$ in $G$ with weight $w$ to the edge $\delta = \{e_1 + K_X, e_2 + K_X\}$ in $\Gamma$ with weight $1 - w$. The different weights that occur in $G$ are $0, 1, 2, 2, 2$, and they correspond to weights $1, 0, -1, 1$, and $-2$, respectively, in $\Gamma$. As a consequence, the weighted graphs $G$ and $\Gamma$ have isomorphic automorphism groups, given by the Weyl group $W_8$.

### 2.2 The elliptic surface

Let $X$ be a del Pezzo surface of degree 1 over a field $k$ with canonical divisor $K_X$. The surface $X$ can be embedded in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates $(x : y : z : w)$ as the set of solutions to the equation
\[ y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0, \] (2)

where $a_i \in k[z, w]$ is homogeneous of degree $i$ for each $i$ in $\{1, \ldots, 6\}$. The linear system $|-K_X|$ induces a rational map $X \dasharrow \mathbb{P}^1$, which is defined everywhere except in the base point of $|-K_X|$, given by $\mathcal{O} = (1 : 1 : 0 : 0)$. The blow-up of $X$ in $\mathcal{O}$ gives a rational elliptic surface $\mathcal{E}$ with only irreducible fibres, and we denote this blow-up by $\pi : \mathcal{E} \to X$. We denote the induced elliptic fibration on $\mathcal{E}$ by $\nu : \mathcal{E} \to \mathbb{P}^1$. The generic fiber of $\mathcal{E}$ is an elliptic curve $E$ over the function field $k(t)$ of $\mathbb{P}^1$. We call the Mordell–Weil group of $E$ the **Mordell–Weil group of $\mathcal{E}$**. Points in $E(k(t))$ correspond to sections of $\nu$ that are defined over $k$ ([Sil83], Proposition 3.10). For $(z_0 : w_0) \in \mathbb{P}^1_k$, the fiber $\nu^{-1}((z_0 : w_0))$ is isomorphic to the cubic curve in $\mathbb{P}^2_k$ with affine Weierstrass equation
\[ Y^2 + a_1(z_0, w_0)XY + a_3(z_0, w_0)Y = X^3 + a_2(z_0, w_0)X^2 + a_4(z_0, w_0)X + a_6(z_0, w_0). \] (3)

The point at infinity on such a fiber is the intersection on $\mathcal{E}$ with the exceptional divisor $\mathcal{O}$ above $\mathcal{O}$. For a point $P \in X \setminus \{\mathcal{O}\}$ we denote by $P_{\mathcal{E}}$ the corresponding point on $\mathcal{E}$.

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1Reciprocally, the contraction of the zero-section on a rational elliptic surface $\mathcal{E}$ produces a del Pezzo surface of degree 1 whenever $\mathcal{E}$ has only irreducible fibers.
Remark 2.8. The linear system $|−2K_X|$ of the bi-anticanonical divisor of $X$ induces a morphism $\varphi$, which is the composition of the projection of $X$ to $\mathbb{P}(2, 1, 1)$ on the $x, z, w$-coordinates, and the 2-uple embedding of $\mathbb{P}(2, 1, 1)$ in $\mathbb{P}^3$. This morphism realizes $X$ as a double cover of a cone in $\mathbb{P}^3$ ramified over a sextic curve. Using the notation in \(\square\), the morphism $\varphi$ is ramified at the points $(x_0 : y_0 : z_0 : w_0) \in X$ for which we have $2y_0 + a_1x_0 + a_3 = 0$, and from \(\bigcirc\) it follows that these are exactly the points that are 2-torsion on their fiber on $\mathcal{E}$.

Remark 2.9. Since exceptional curves on $X$ are defined over a separable closure of $k$ \([VA09, \text{Theorem 2.1.1}]\), from \([VA08, \text{Theorem 1.2}]\) it follows that the exceptional curves on $X \subset \mathbb{P}(2, 3, 1, 1)$ are exactly the points that are 2-torsion on their fiber on $\mathcal{E}$. This gives a negative answer to Question 1.1 for ‘many’ equal to 7 or less. As explained in Remark 2.8, our examples hold in all but 10 characteristics.

In this section we give two examples that show that for a point $P$ on the del Pezzo surface of degree 1, the point $P_{\mathcal{E}}$ is not guaranteed to be torsion on its fiber. This gives a negative answer to Question 1.1 for ‘many’ equal to 7 or less. As explained in Remark 2.8, our examples hold in all but 10 characteristics.

Example 3.1. Let $X$ be the blow-up of $\mathbb{P}^2_Q$ in the eight points:

\[
\begin{align*}
P_1 &= (0 : 1 : 1); \\
P_2 &= (0 : 14 : 13); \\
P_3 &= (1 : 0 : 1); \\
P_4 &= (21 : 0 : 13); \\
P_5 &= (1 : 1 : 1); \\
P_6 &= (6 : 6 : -1); \\
P_7 &= (-2 : 2 : 1); \\
P_8 &= (-3 : 3 : -1).
\end{align*}
\]

It is easy to check that these points are in general position, thus $X$ is a del Pezzo surface of degree 1. Consider the following curves in $\mathbb{P}^2$ (see Notation \(\square\)): the line $L_{1,2}$ given by $x = 0$, the line $L_{3,4}$ given by $y = 0$, the line $L_{5,6}$ given by $x − y = 0$, the line $L_7,8$ given by $x + y = 0$, the cubic $C_{1,2}$ given by

\[
26x^3 + 42x^2y − 68x^2z − 33xy^2 − 9xyz + 42xz^2 − 36y^3 + 72y^2z − 36yz^2 = 0,
\]

the cubic $C_{3,4}$ given by

\[
36x^3 + 46x^2y − 72x^2z − 42xy^2 − 4xyz + 36x^2z − 39y^3 + 81y^2z − 42yz^2,
\]

and the quartic $Q_{2,6,7}$ given by

\[
1144x^4 + 1288x^3y − 808x^3z − 2910x^2y^2 + 4748x^2yz − 3864x^2z^2 − 1092xy^3 + 318xy^2z − 2352xyz^2 + 3528xz^3 + 1521y^4 − 4797y^3z + 5040y^2z^2 − 1764yz^3.
\]
These 7 curves all go through \( Q = (0 : 0 : 1) \), and each of them gives rise to an exceptional curve on \( X \) by Theorem 2.3. The base point \( O \) of the anticanonical linear system of \( X \) is strict transform of the base point in \( \mathbb{P}^2 \) of the pencil of cubics through \( P_1, \ldots, P_8 \), which is given by

\[
B = (27 : 68 : 109).
\]

The fiber of \( Q \) of the elliptic surface obtained by blowing up \( X \) in \( O \) is given by the strict transform of the cubic curve through \( P_1, \ldots, P_8, Q \) with origin given by \( B \). With \texttt{magma} we check that the point \( Q \) is non-torsion on this elliptic curve.

**Remark 3.2.** The previous example also holds over any other field \( k \), as long as the characteristic of \( k \) is \( p \) for all but a finite number of primes \( p \). In fact, the only characteristics for which this does not hold are the ones for which \( P_1, \ldots, P_8 \) are not in general position, and for which the fiber of \( Q \) is not an elliptic curve, i.e., for which it is singular. We compute this with \texttt{magma} and find the primes \( \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 3319\} \). It is not hard to generate similar examples that hold in some of the missing characteristics; for example, the eight points in \( \mathbb{P}^2 \) given by Example 3.3 are in general position with a non-singular fiber in all but 29 characteristics, and this gives, together with Example 3.1 examples of 7 exceptional curves that are concurrent in a point \( P \) such that \( P_\mathcal{E} \) is not torsion on its fiber for each characteristic except for \( p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 \).

**Example 3.3.** Let \( X \) be the blow-up of \( \mathbb{P}^3_\mathbb{Q} \) in the eight points:

- \( P_1 = (0 : 1 : 1) \);
- \( P_2 = (0 : 3861 : 1957) \);
- \( P_3 = (1 : 0 : 1) \);
- \( P_4 = (1188 : 0 : 19) \);
- \( P_5 = (1 : 1 : 1) \);
- \( P_6 = (780 : 780 : 1883) \);
- \( P_7 = (-52 : 52 : 51) \);
- \( P_8 = (-9 : 9 : -17) \).

It is an easy check that \( P_1, \ldots, P_8 \) are in general position, so \( X \) is a del Pezzo surface of degree 1. Again the line \( L_{1,2} \) is given by \( x = 0 \), the line \( L_{3,4} \) by \( y = 0 \), the line \( L_{5,6} \) by \( x - y = 0 \), and the line \( L_{7,8} \) by \( x + y = 0 \). As explained in Remark 5.6 choosing these equations for these for lines is in fact the first step in the construction of all examples in this paper. The cubic \( C_{1,2} \) is now given by

\[
247x^3 - 15444x^2y + 15197x^2z - 56500xy^2 + 71944xyz - 15444xz^2 - 24336y^2z - 24336yz^2 = 0,
\]

the cubic \( C_{3,4} \) is given by

\[
24336x^3 + 48425x^2y - 48672x^2z + 15444xy^2 - 63869xyz + 24336xz^2 + 7828y^2z - 23272y^2z + 15444yz^2 = 0,
\]

and the quartic \( Q_{2,6,7} \) is given by

\[
2705155115x^4 - 160214640456x^3y + 165198460765x^3z - 340717645684x^2y^2 + 583405507724x^2yz - 245421685080x^2z^2 - 86417174668x^3y + 301295315984xy^2z - 29751362880xyz^2 + 77518069200xz^3 - 6127758400y^4 + 30306884800y^2z^2 - 48030840000y^2z^2 + 23851713600y^3z^3 = 0.
\]

These 7 curves again all go through \( Q = (0 : 0 : 1) \), and completely analogous to the previous example we check with \texttt{magma} that the point \( Q \) is non-torsion on its fiber.

## 4 The case of 9 or more lines: proof of Theorem 1.2

In this Section we prove Theorem 1.2. Let \( X \) be a del Pezzo surface of degree 1 over a field \( k \), let \( \mathcal{E} \) be the corresponding elliptic surface, and \( E \) the generic fiber of \( \mathcal{E} \). We start by describing a pairing on the Mordell–Weil group of \( \mathcal{E} \).

Let \( \varphi \) be the morphism induced by the bi-anticanonical linear system on \( X \) as in Remark 2.8. Let \( L_1, \ldots, L_n \) be at least 9 exceptional curves on \( X \) that are concurrent in a point \( Q \) that lies outside the ramification curve of \( \varphi \). Let \( L_1, \ldots, L_n \) be the corresponding sections on \( \mathcal{E} \). Let \( \langle \cdot, \cdot \rangle_h \) be the symmetric and bilinear pairing on the Mordell–Weil group of \( \mathcal{E} \) as defined in \texttt{Shi90} Theorem 8.4; that is, for \( C_1, C_2 \) in \( E(k(t)) \), we have \( (C_1, C_2)_h = -(\varphi_h(C_1) \cdot \varphi_h(C_2)) \), where \( \varphi_h : E(k(t)) \to \text{Pic} \mathcal{E} \) is the map given in \texttt{Shi90} Lemmas 8.1 and 8.2, and \( \cdot \) is the intersection pairing in the Picard group of \( \mathcal{E} \). We call \( \langle \cdot, \cdot \rangle_h \) the height pairing on \( E(k(t)) \).
Lemma 4.1. For two exceptional curves in Pic $X$, the height pairing of the corresponding sections in the Mordell–Weil group of $\mathcal{E}$ is the same as the dot product of the roots in the root system $E_8$ associated to these exceptional curves under the bijection (1).

Proof. Let $C_1, C_2$ be two sections of $\mathcal{E}$ that are strict transforms of exceptional curves $c_1, c_2$ in $X$. Since $\mathcal{E}$ has no reducible fibers, by [Shi90] Lemma 8.1 we have
$$\varphi_h(C_1) \cdot \varphi_h(C_2) = ([C_1] - [\mathcal{O}] - F) \cdot ([C_2] - [\mathcal{O}] - F),$$
where $[C_1], [C_2], [\mathcal{O}]$ are the classes of $C_1, C_2$, and the zero section, respectively, and $F$ is the class of a fiber. This gives
$$\varphi_h(C_1) \cdot \varphi_h(C_2) = [C_1] \cdot [C_2] - 1,$$
where we use that the zero section is an exceptional curve, and it is disjoint from $C_1$ and $C_2$ (Remark 2.8). We conclude that we have $(C_1, C_2)_h = 1 - [C_1] \cdot [C_2]$. Since $C_1, C_2$ are disjoint from $\mathcal{O}$, the intersection pairing of $C_1$ and $C_2$ in Pic $\mathcal{E}$ is the same as the intersection pairing of $c_1$ and $c_2$ in Pic $X$. The statement now follows from the bijection (1).

Let $M$ be the height pairing matrix of $L_1, \ldots, L_n$, that is, $M$ is the $n \times n$ matrix with entries $M_{ij} = \langle \tilde{L}_i, \tilde{L}_j \rangle_h$ for $i, j \in \{1, \ldots, n\}$.

Lemma 4.2. The kernel of the matrix $M$ contains a vector $(a_1, \ldots, a_n)$ in $\mathbb{Z}^n$ with $a_1 + \cdots + a_n \neq 0$.

Proof. Recall the complete weighted graphs $G$ and $\Gamma$ as defined in Definition 2.6. Since $Q$ lies outside the ramification curve of $\varphi$, where $\varphi$ is the morphism in Remark 2.8 the exceptional curves $L_1, \ldots, L_n$ correspond to a clique of size $n$ in $G$ that is contained in a maximal clique in $G$ with only edges of weights 1 and 2 [LV23, Remark 2.1], which corresponds to a maximal clique $C$ in $\Gamma$ with only edges of weights -1 and 0 by the bijection (1). Since we have $n \geq 9$, the clique $C$ has size at least 9. The table in [Wu21, Appendix A] contains all isomorphism types of maximal cliques in $\Gamma$ with only edges of weights -1 and 0 and of size at least 9 [Wu21, Proposition 21]; there are 11 maximal cliques of size 9, which we call $a_1, \ldots, a_{11}$ in the order that they appear in the table, there are 6 maximal cliques of size 10, which we call $b_1, \ldots, b_6$ in the order that they appear in the table, and there is 1 maximal clique of size 12, which we call $c$. For each of these 18 cliques, whose elements correspond to roots in $E_8$, we compute its Gram matrix, which is the matrix where the entry $(i, j)$ is the dot product of the roots corresponding to the $i$-th and $j$-th vertex in the clique after choosing an ordering on the vertices. With magma we find the generators for the kernels of these matrices. The results are in Table 1. Let $r$ be the number of vertices of $C$, and let $N$ be the Gram matrix of $C$; then the kernel of $N$ is equal to one of the 18 kernels in the table, after rearranging the order of the vertices in $C$ if necessary. Since we have $n \geq 9$, we see from Table 1 that for any subset of $n$ vertices in $C$, there is a vector $(a_1, \ldots, a_n)$ in the kernel of $N$ which is 0 outside the entries corresponding to the $n$ vertices, and such that $a_1 + \cdots + a_n \neq 0$. By Lemma 4.1, this gives a vector in the kernel of $M$ as claimed.

Proof of Theorem 1.2. Let $P$ be a point on $X$. If $P$ is contained in the ramification curve of the morphism induced by the linear system of the bi-anticanonical divisor, then $P_\mathcal{E}$ is torsion (Remark 2.8), and we are done. Now assume that $P$ is not contained in this ramification curve, and that there is a set of at least 9 exceptional curves that are concurrent in $P$. Let $K_1, \ldots, K_n$ be the corresponding sections of $\mathcal{E}$, and let $N$ be the height pairing matrix of these sections. Let $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector in the kernel of $N$ such that $a_1 + \cdots + a_n \neq 0$, which exists by Lemma 4.2. Then for all $i \in \{1, \ldots, n\}$, we have that
$$a_1(K_i, K_1)_h + \cdots + a_n(K_i, K_n)_h = 0,$$
and since the height pairing is bilinear this implies
$$\langle K_i, a_1K_1 + a_2K_2 + \cdots + a_nK_n \rangle_h = 0 \text{ for all } i \in \{1, \ldots, n\},$$
which implies
$$\langle a_1K_1 + a_2K_2 + \cdots + a_nK_n, a_1K_1 + a_2K_2 + \cdots + a_nK_n \rangle_h = 0.$$
From the latter we conclude that $a_1K_1 + a_2K_2 + \cdots + a_nK_n$ is torsion in the Mordell–Weil group of $\mathcal{E}$ [Shi90, Theorem 8.4], and since the torsion subgroup is trivial [Shi90, Theorem 10.4], we conclude that
$$a_1K_1 + a_2K_2 + \cdots + a_nK_n = 0.$$
Since for all $i$ in $\{1, \ldots, n\}$, the section $K_i$ contains the point $P_\varepsilon$, we have, on the fiber of $P_\varepsilon$, the equality $(a_1 + \cdots + a_n)P_\varepsilon = 0$. Since $a_1 + \cdots + a_n \neq 0$, this implies that $P_\varepsilon$ is torsion on its fiber. □

### 5 The case of 8 lines

Given that the rank of the Mordell-Weil group of the elliptic surface arising from a del Pezzo surface of degree 1 is 8 (see also Remark 2.10), we conjecture that points in the intersection of 8 lines are not always torsion, that is, we expect to find an example of a non-torsion point contained in the intersection of 8 lines. However, we have not yet found such an example, nor do we have a proof that every point contained in the intersection of 8 lines is torsion on its fiber. In this section we prove Theorem 1.4 we show that 8 lines that are concurrent in a non-torsion point can only intersect each other according to one of 15 specific configurations, which reduces to 13 in characteristic 0. We also give strategies for searching for examples with non-torsion points within these cliques, or eliminating more cliques.

Let $X$ be a del Pezzo surface of degree 1 over an algebraically closed field, and let $\varepsilon$ be the associated elliptic surface. Let $G$ be the weighted graph on the 240 exceptional curves as in Definition 2.6. Note that its automorphism group, the Weyl group $W_8$, acts on the set of cliques of size 8 with only edges of weights 1 and 2.

**Proposition 5.1.** There are 47 orbits under the action of $W_8$ of cliques of size 8 with only edges of weights 1 and 2 in $G$. Their isomorphism types are represented in Figure 7.

**Proof.** Every clique in $G$ of size 8 with only edges of weights 1 and 2 contains at least one edge of weight 1, since the maximal size of cliques in $G$ with only edges of weight 2 is 3, which follows from [Wcl.21] Lemma 7], using the bijection [11]. Using this same bijection, it follows from [Wcl.21] Proposition 6] that $W_8$ acts transitively on the set of pairs of exceptional curves that intersect with multiplicity 1, so we fix two such curves; let $e_1$ be the strict transform on $X$ of the curve $L_{1,3}$ in $\mathbb{P}^2$, and $e_2$ the strict transform of $L_{3,4}$, where we use Notation 2.3. It follows that every clique of size 8 with only edges of weights 1 and 2 in $G$ is conjugate under the action of $W_8$ to a clique containing $e_1$ and $e_2$. With magma we compute that there are 136 exceptional curves that intersect both $e_1$ and $e_2$ with multiplicity 1 or 2. We define the graph $H$ on these 136 exceptional curves in magma, with an edge between two vertices if they correspond to exceptional curves intersecting with multiplicity

| Clique | Basis for the kernel |
|--------|---------------------|
| $\alpha_1$ | $\{(1,1,0,0,0,1,0,1),(0,0,1,1,1,0,2,0)\}$ |
| $\alpha_2$ | $\{(1,0,1,0,0,1,0,1),(0,0,0,1,1,0,0,0)\}$ |
| $\alpha_3$ | $\{(1,1,0,0,1,0,1,0),(0,0,0,1,1,0,1,0)\}$ |
| $\alpha_4$ | $\{(1,1,0,1,0,0,1,0),(0,0,0,0,1,1,1,0)\}$ |
| $\alpha_5$ | $\{(2,1,1,0,2,0,0,1),(0,0,0,1,0,1,0,0)\}$ |
| $\alpha_6$ | $\{(1,1,1,1,1,1,1,1)\}$ |
| $\alpha_7$ | $\{(1,1,0,1,1,1,1,1)\}$ |
| $\alpha_8$ | $\{(0,1,1,2,2,1,1,0)\}$ |
| $\alpha_9$ | $\{(2,1,1,1,2,2,2,2)\}$ |
| $\alpha_{10}$ | $\{(2,2,0,3,1,4,2,3)\}$ |
| $\alpha_{11}$ | $\{(6,3,1,4,4,2,2,5,3)\}$ |
| $\beta_1$ | $\{(0,1,0,0,2,1,0,1),(0,1,0,1,2,0,1,0)\}$ |
| $\beta_2$ | $\{(1,1,0,0,0,0,1,1),(0,0,0,0,1,1,1,0)\}$ |
| $\beta_3$ | $\{(1,1,0,1,0,0,1,0),(0,0,1,0,1,1,1,0)\}$ |
| $\beta_4$ | $\{(1,1,0,1,0,1,0,1),(0,0,0,0,1,0,1,0)\}$ |
| $\beta_5$ | $\{(1,1,0,0,0,0,0,1),(0,0,1,1,1,2,2,0)\}$ |
| $\beta_6$ | $\{(2,1,3,0,2,0,2,1,1),(0,0,0,1,0,1,0,1,0)\}$ |
| $\gamma$ | $\{(1,1,0,0,0,0,0,0,0,0,0,0,0,0,1),(0,0,1,0,0,1,0,0,0,0,0,1,0,0)\}$ |

Table 1: Bases
1 or 2, and no edge otherwise. The function $\text{AllCliques}(H,6,\text{false})$ gives all (not necessarily maximal) cliques of size 6 in this graph; there are 8963624 of them. We conclude that there are 8963624 cliques in $G$ of size 8 with only edges of weights 1 and 2, that contain $e_1$ and $e_2$. This set $A$ of cliques contains a representative for each $W_8$-orbit of cliques of size 8 with only edges of weights 1 and 2 in $G$, and we want to find a set of such representatives. To reduce computing time we first sort all cliques in $A$ according to the size of their stabilizer in $W_8$. This gives 20 different sets $A_1,\ldots,A_{20}$, where each set contains only cliques in $A$ with the same stabilizer size. Finally, for each of these sets $A_i$, we check whether the cliques inside are conjugate under the action of $W_8$, and end up with one representative for each $W_8$-orbit of the cliques in $A_i$. Doing this for all $A_i$ takes a very long time; we let magma run for two weeks straight on the compute servers of the Max Planck Institute for Mathematics in the Sciences, Leipzig. The output gave 47 cliques of size 8 with only edges of weight 1 and 2, where each clique is a representative for a different $W_8$-orbit. The isomorphism types follow from the pairwise intersection multiplicity of the exceptional curves.

Figure 1 contains the isomorphism types of 47 cliques in $G$, one representative for each of the 47 orbits in Proposition 5.1; there are 45 different isomorphism types. All graphs are fully connected subgraphs of $G$ with edges of weights 2 (the ones that are drawn) and 1 (all other edges).

![Figure 1: Isomorphism types of the cliques in 47 orbits of the set of cliques of size 8 with only edges of weights 1 and 2](image)

**Definition 5.2.** We say that a clique in $G$ contains an $n$-gon if it contains a set of $n$ vertices $e_1,\ldots,e_n$, corresponding to $n$ exceptional curves $c_1,\ldots,c_n$ that intersect with multiplicity 2 for all
\{i, j\} \in \{(a, a + 1) : a \in \{1, \ldots, n - 1\}\} \cup \{(1, n)\}, and with multiplicity 1 otherwise.

**Proposition 5.3.** If a clique in \(G\) contains an \(n\)-gon corresponding to \(n\) exceptional curves that are concurrent in a point \(P \in X\), then the point \(P_E\) is torsion on its fiber, of order dividing \(n\).

**Proof.** Let \(e_1, \ldots, e_n\) be the \(n\) exceptional curves corresponding to an \(n\)-gon, and let \(L_1, \ldots, L_n\) be the corresponding sections of \(E\). From Lemma [11] and the bijection in [11] it follows that, in the Mordell–Weil group of \(E\), we have \((L_i, L_i)_h = 2\) for all \(i \in \{1, \ldots, n\}\), and \((L_i, L_j)_h = -1\) for \(\{i, j\} \in \{(a, a + 1) : a \in \{1, \ldots, n - 1\}\} \cup \{(1, n)\}\), and \((L_i, L_j)_h = 0\) otherwise. We find

\[
(L_1 + \cdots + L_n, L_1 + \cdots + L_n)_h = \sum_{i=1}^{n} (L_i, L_i)_h + \sum_{i=1}^{n} (L_i, L_i + \cdots + L_{i-1} + L_{i+1} + \cdots + L_n)_h
\]

\[
= 2n + n(-2) = 0.
\]

Therefore we have that \(L_1 + \cdots + L_n\) is torsion in the Mordell–Weil group of \(E\) [Shi90, Theorem 8.4], and since the torsion subgroup is trivial [Shi90 Theorem 10.4], we conclude that

\[
L_1 + \cdots + L_n = 0.
\]

Specializing the section \(L_1 + \cdots + L_n\) to the fiber of \(P_E\), gives \(nP_E = 0\).

The following proposition shows that, in characteristic 0, cliques whose isomorphism type equals number 7 or 15 in Figure [11] do not correspond to lines that are concurrent in a point on a del Pezzo surface.

**Proposition 5.4.** Let \(k\) be a field of characteristic 0. Let \(P_1, \ldots, P_8\) be points in general position in \(\mathbb{P}^2_k\), such that the lines \(L_{1,2}, L_{3,4}, L_{5,6}\) and \(L_{7,8}\) are concurrent in a point \(P\). Then the curves in \(\mathbb{P}^2_k\) given by \(C_{1,2}, C_{3,4},\) and \(C_{5,6}\) are not concurrent in \(P\).

**Proof.** Notice that none of the points \(P_1, \ldots, P_8\) equals \(P\); otherwise, there exists a subset of three of the \(P_i\) such that they are aligned. Moreover, \(P\) is not collinear with any of the three points \(P_1, P_3, P_5\), since this together with \(P \in L_{1,2} \cap L_{3,4} \cap L_{5,6}\) would also contradict the general position of \(P_1, \ldots, P_8\). Thus \(P_1, P_3, P_5\) and \(P\) are in general position, hence, after applying an automorphism of \(\mathbb{P}^2\) if necessary, we may assume that we have \(P = (0 : 0 : 1)\), and

\[
P_1 = (0 : 1 : 1); \quad P_3 = (1 : 0 : 1); \quad P_5 = (1 : 1 : 1).
\]

It follows that \(L_{1,2}\) is the line given by \(x = 0\), \(L_{3,4}\) is the line given by \(y = 0\), and \(L_{5,6}\) is the line given by \(x = y\). Since \(L_{7,8}\) is unequal to \(L_{1,2}, L_{3,4}, L_{5,6}\) and contains \(P\), there exists an \(m \in k \setminus \{0, 1\}\) such that \(L_{7,8}\) is the line given by \(my = x\). Therefore there are \(a, b, c, d, e \in k \setminus \{0, 1\}\) such that we can write

\[
P_1 = (0 : 1 : 1); \quad P_2 = (0 : 1 : a); \\
P_3 = (1 : 0 : 1); \quad P_4 = (1 : 0 : b); \\
P_5 = (1 : 1 : 1); \quad P_6 = (1 : 1 : c); \\
P_7 = (m : 1 : d); \quad P_8 = (m : 1 : e); \\
P = (0 : 0 : 1).
\]

We assume by contradiction that \(C_{1,2}, C_{3,4},\) and \(C_{5,6}\) contain \(P\). The curve \(C_{1,2}\) is given by the general cubic equation:

\[
C_{1,2} : t_1x^3 + t_2y^3 + t_3z^3 + t_4xz^2 + t_5x^2z + t_6x^2y + t_7xy^2 + t_8y^2z + t_9yz^2 + t_{10}xyz = 0,
\]

where the coefficients \(t_1, \ldots, t_{10}\) are elements in \(k\). Since we assumed \(C_{1,2}\) passes through \(P\), we have \(t_3 = 0\). Moreover, since \(C_{1,2}\) contains the points \(P_1, P_3, P_4, P_5, P_6, P_7, P_8,\) and is singular in \(P_1\), we have the following.

- Evaluating (6) in \(P_1, P_3, P_4, P_5, P_6, P_7, P_8\) gives

\[
t_2 + t_8 + t_9 = 0, \quad t_1 + t_4 + t_5 = 0, \quad t_1 + b^2t_4 + bt_5 = 0,
\]

\[
(7) \quad (8) \quad (9)
\]
Contradiction. Therefore we find \( g \neq 0 \) so we can rewrite the system as the single condition:

\[
3t_2 + 2t_8 + t_9 = 0,
\]

Using conditions (17), (18), (14) and (16), we rewrite \( t_1, t_2, t_7 \) and \( t_8 \) in terms of the other variables. By replacing these in the other equations, the system reduces to:

\[
(b^2 - 1)t_4 + (b - 1)t_5 = 0,
\]

\[-t_4 + t_6 = 0,
\]

\[
(c^2 - 2)t_4 + (c - 1)t_5 + t_6 + (c - 1)t_9 = 0,
\]

\[(d^2 - m^3 - m)t_4 + (d^2 - m^3)t_5 + m^2 t_6 + (d^2 - 2d + 1)t_9 + (dm - m)t_{10} = 0,
\]

\[(e^2 - m^3 - m)t_4 + (e^2 - m^3)t_5 + m^2 t_6 + (e^2 - 2e + 1)t_9 + (em - m)t_{10} = 0.
\]

Since \( P_1, \ldots, P_k \) are in general position, we have \( c \neq 1, d \neq 1, \) and \( m(c - 1) - d + 1 \neq 0 \) (note that \( m(c - 1) - d + 1 = 0 \) would imply that \( P_1, P_6, P_7 \) are all on the line \( (c - 1)x + y - z = 0 \). Therefore we can rewrite the system as the single condition:

\[
(bcdm^4 - bcdm^3 - bcem^4 + bcem^3 + bd^2 em^2 - bd^2 em - bde^2 m^2 + bde^2 m - bdm^4 + 3bdm^3 - bdm^2 - bdm + be^2 m - be^2 m + bem^4 - 3bem^3 + bem^2 + bem - cd^2 em^2 + cd^2 em + cd^2 m^2 - cde^2 m - cdm^2 + cdm - ce^2 m^2 + cem^2 - cem - 2d^2 em^2 - 2d^2 em - 2d^2 m^2 + 2d^2 m - 2de^2 m^2 + 2dt^2 m - 2dm^2 - 2dm + 2em^2 - 2e^2 m^2 - 2e^2 m - 2em^2 + 2em^2) / ((d - 1)(m(c - 1) - d + 1)) t_9 = 0
\]

Observe that \( t_9 = 0 \) forces all the other coefficients \( t_i \) of \( P \) to be 0. This cannot be the case since \( C_{1,2} \) is a cubic. So we require the other factor to be zero. Factorizing its numerator, we obtain the following possibilities:

- \( m = 0, \)
- \( m = 1, \)
- \( d = e, \)
- \( g_1 b + g_2 = 0, \)

where \( g_1 = cm^2 + de - dm - d - em - e - m^2 + 2m + 1 \) and \( g_2 = -cde + cd + ce - c + 2de - 2d - 2e + 2. \) Note that the first three possibilities imply that \( P_1, \ldots, P_k \) are not in general position, giving a contradiction. Therefore we find \( g_1 b + g_2 = 0. \) Now assume that \( g_1 = 0, \) then it follows that \( g_2 = 0. \) But then we have \( 0 = (c - 1)g_1 + g_2 = (cm - e - m + 1)(em - d - m - 1), \) and the later is a product of two equations that each say that three of the points are collinear (\( P_1, P_6, P_8 \) and \( P_1, P_6, P_7 \), respectively). It follows that we have \( g_1 \neq 0 \) and we can write

\[
b = \frac{g_2}{g_1} = \frac{-cde + cd + ce - c + 2de - 2d - 2e + 2}{(cm^2 + de - dm - d - em - e - m^2 + 2m + 1)}.
\]

We will now repeat this process twice and summarize what we find; for the detailed computations see [C3.4]. First we assume that \( C_{3,4} \) passes through \( P \) and find \( h_1 a + h_2 = 0 \) with \( h_1 = e + de - dm - d - em - e + m^2 + 2m + 1 \) and \( h_2 = -cde + cdem + cem^2 - 2de - 2dm - 2em + 2m^2. \)
From \((c - 1)h_1 + h_2 = (c - d + m - 1)(c - e + m - 1)\) and the fact that the latter factors imply that \(P_5, P_6, P_7\) and \(P_3, P_6, P_8\) are collinear, respectively, we conclude that \(h_1 \neq 0\) and we obtain:

\[
a = -\frac{h_2}{h_1} = \frac{(-cde + cm - d + 2dm - 2c + 2m^2)}{(c + d - e - m + m^2 + 2m - 1)}.
\]  

(20)

Setting the configuration with \(a\) and \(b\) as in (19) and (20), we look at the curve \(C_{5,6}\), and in particular at what happens if it passes through \(P\). This time, when solving the system of equations, we obtain \(h_1 = m^2 - mv - mc + 2m + vc - v - c + u - 1 = 0\). But we already showed that this implied that the points are not in general position, giving a contradiction. We conclude that the cubics \(C_{1,2}, C_{3,4}, C_{5,6}\) do not all go through \(P\).

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4** Let \(e_1, \ldots, e_8\) be 8 exceptional curves that intersect in a point \(P\) on \(X\). If there are \(i, j \in \{1, \ldots, 8\}\) such that \(e_i \cdot e_j = 3\), then \(P\) lies on the ramification curve of the morphism \(\varphi\) by [LAV23, Remark 2.11], hence \(P_E\) is torsion on its fiber by Remark 2.8. Assume that all \(e_1, \ldots, e_8\) pairwise intersect with multiplicities 1 and 2. Therefore, from Proposition 5.1, there is only one \(\varphi\)-equivalent relation \(P\). Since \(P\) would be torsion on its fiber, the corresponding sections specialize to a relation \(aP_E = 0\) for some \(a \neq 0\). This proves the first part of the theorem.

For the second part, let \(Y\) be a del Pezzo surface of degree 1 over an field \(k\) of characteristic 0. Without loss of generality we can assume that \(k\) is algebraically closed, and that \(Y\) is the blow-up of points \(P_1, \ldots, P_8 \in \mathbb{P}^2\) in general position. Let \(G'\) be the weighted intersection graph of the 240 exceptional curves on \(Y\). Let \(K_1 = \{f_1, \ldots, f_8\}\) be the clique in \(G'\) corresponding to the exceptional curves on \(Y\) given by the strict transforms of \(L_{1,2}, L_{3,4}, L_{5,6}, L_{7,8}, C_{1,2}, C_{3,4}, C_{5,6}, C_{7,8} \subset \mathbb{P}^2\), and \(K_2\) the clique corresponding to the strict transforms of \(L_{1,2}, L_{3,4}, C_{5,6}, L_{7,8}, C_{1,2}, C_{3,4}, C_{5,6}, Q_{2,4,7} \subset \mathbb{P}^2\). All exceptional curves in \(K_1\) intersect pairwise with multiplicity 1, so the intersection graph of \(K_1\) is equal to number 7 in Figure 1. Similarly, all exceptional curves in \(K_2\) intersect pairwise with multiplicity 1, except the pairs \(\{L_{5,6}, Q_{2,4,7}\}, \{C_{1,2}, Q_{2,4,7}\}, \{C_{3,4}, Q_{2,4,7}\}\), which all intersect with multiplicity 2. Therefore, the intersection graph of \(K_2\) equals number 15 in Figure 1.

Let \(e_1, \ldots, e_8\) be exceptional curves on \(Y\) that pairwise intersect with multiplicities 1 and 2, and let \(C\) be their weighted intersection graph. Assume that \(C\) equals number 7 in Figure 1. By Proposition 5.1, there is only one \(W_S\)-orbit of cliques of size 8 in \(G\) with intersection graph equal to number 7. Therefore, after permuting the indices if necessary, there is an element \(w \in W_S\) such that \(e_i = w(f_i)\) for \(i \in \{1, \ldots, 8\}\). Write \(E'_i = w(E_i)\) for \(i \in \{1, \ldots, 8\}\), where \(E_i\) is as in Notation 2.8. Then, since the \(E'_i\) are pairwise disjoint, \(Y\) is isomorphic to the blow-up of \(\mathbb{P}^2\) in points \(Q_1, \ldots, Q_8\) in \(\mathbb{P}^2\) such that \(E'_i\) is the exceptional curve above \(Q_i\) for all \(i \in \{1, \ldots, 8\}\). It follows that, under this blow-up, the exceptional curves \(e_1, \ldots, e_8\) are the strict transforms of the curves \(L_{1,2}, L_{3,4}, L_{5,6}, L_{7,8}, C_{1,2}, C_{3,4}, C_{5,6}, C_{7,8} \subset \mathbb{P}^2\), but now defined with respect to the points \(Q_1, \ldots, Q_8\). Assume that the lines \(L_{1,2}, L_{3,4}, L_{5,6}, L_{7,8}\) with respect to \(Q_1, \ldots, Q_8\) are concurrent in a point \(P\). Then by Proposition 5.3, the curves \(C_{1,2}, C_{3,4}, C_{5,6}\) with respect to \(Q_1, \ldots, Q_8\) do not go through \(P\). We conclude that the exceptional curves in a clique with isomorphism type 7 are not concurrent. The same holds completely analogously for clique number 15. This finishes the proof.

### 5.1 An example with 6 lines

In the following example, a point \(P\) on \(X\) is contained in 6 lines, and the corresponding point \(P_E\) is non-torsion. This is not the first example of 6 lines contained in a non-torsion point, see [Win21, Example 5.1.5]. However, the lines in Example 5.1.5 have a different intersection graph than those in [Win21, Example 5.1.5], showing that there are several families of such examples. We explain our construction in Remark 5.6.
Example 5.5. Let $k$ be a field of characteristic 0, and consider the following points in $\mathbb{P}^2_k$:

\begin{align*}
P_1 &= (0 : 1 : 1); & P_2 &= (0 : 319 : -920); \\
P_3 &= (1 : 0 : 1); & P_4 &= (799 : 0 : 610); \\
P_5 &= (1 : 1 : 1); & P_6 &= (1 : 1 : -1); \\
P_7 &= (31 : 1 : 123); & P_8 &= (31 : 1 : 11).
\end{align*}

It is an easy check that $P_1, \ldots, P_8$ are in general position. Let $X$ be del Pezzo surface of degree 1 given by the blow-up of $\mathbb{P}^2_k$ in the eight points. Consider the curves $L_{1,2}, L_{3,4}, L_{5,6}, L_{7,8}, C_{1,2}, C_{3,4}$ in $\mathbb{P}^2$. These curves all go through $Q = (0 : 0 : 1)$, and each of them gives rise to an exceptional curve on $X$. As in Examples 3.1 and 3.3, we check with Magma that the point on $X$ corresponding to $Q$ is non-torsion on its fiber.

Remark 5.6. Let $k$ be a field of characteristic 0, and let $a, b, c, u, v, m \in k \setminus \{0\}$ be parameters describing the following set of 8 points in $\mathbb{P}^2$:

\begin{align*}
P_1 &= (0 : 1 : 1); & P_3 &= (1 : 0 : 1); & P_5 &= (1 : 1 : 1). & P_7 &= (m : 1 : v); \\
P_2 &= (0 : 1 : a); & P_4 &= (1 : 0 : b); & P_6 &= (1 : 1 : a); & P_8 &= (m : 1 : c).
\end{align*}

From the proof of Proposition 5.4 we can see how to find examples of a set of exceptional curves containing $L_{1,2}, L_{3,4}, L_{5,6}, L_{7,8}, C_{1,2}, C_{3,4}$. 

By choosing values for $c, u, v, m$, and fixing $a, b$ as in equations (19) and (20), such that $P_1, \ldots, P_8$ are in general position, we obtain a configuration of 8 points in $\mathbb{P}^2$ whose blow-up gives rise to a del Pezzo surface of degree 1, with 6 concurrent exceptional curves; the curves (21) then all go through $Q = (0 : 0 : 1)$. Examples 5.5 and 4.3 are both found by choosing such adequate values of $c, u, v, m \in \mathbb{Q}$. In Example 3.1 we require further conditions on $m$ and $u$ so that an additional quartic passes through $Q$. While random values for the free parameters lead without trouble to more examples like 5.5, the further conditions required by the construction of the quartic lead to a much sparser set of possibilities to obtain more example like 3.1 and 3.3.

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