LARGE DEVIATION THEOREMS FOR DIRICHLET DETERMINANTS OF ANALYTIC QUASI-PERIODIC JACOBI OPERATORS WITH BRJUNO-RÜSSMANN FREQUENCY

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(Communicated by Enrico Valdinoci)

Abstract. In this paper, we first study the strong Birkhoff Ergodic Theorem for subharmonic functions with the Brjuno-Rüssmann shift on the Torus. Then, we apply it to prove the large deviation theorems for the finite scale Dirichlet determinants of quasi-periodic analytic Jacobi operators with this frequency. It shows that the Brjuno-Rüssmann function, which reflects the irrationality of the frequency, plays the key role in these theorems via the smallest deviation. At last, as an application, we obtain a distribution of the eigenvalues of the Jacobi operators with Dirichlet boundary conditions, which also depends on the smallest deviation, essentially on the irrationality of the frequency.

1. Introduction. We study the following quasi-periodic analytic Jacobi operators $H(x,\omega)$ on $l^2(\mathbb{Z})$:

$$[H(x,\omega)\phi](n) = -a(x+(n+1)\omega)\phi(n+1) - \overline{a(x+n\omega)}\phi(n-1) + v(x+n\omega)\phi(n), \quad n \in \mathbb{Z},$$

(1.1)

where $v : \mathbb{T} \to \mathbb{R}$ is a real analytic function called potential, $a : \mathbb{T} \to \mathbb{C}$ is a complex analytic function and not identically zero. The characteristic equations $H(x,\omega)\phi = E\phi$ can be expressed as

$$\begin{pmatrix} \phi(n+1) \\ \phi(n) \end{pmatrix} = \frac{1}{a(x+(n+1)\omega)} \begin{pmatrix} v(x+n\omega) - E & -\overline{a(x+n\omega)} \\ a(x+(n+1)\omega) & 0 \end{pmatrix} \begin{pmatrix} \phi(n) \\ \phi(n-1) \end{pmatrix}.$$  

(1.2)

Define

$$M(x,E,\omega) := \frac{1}{a(x+\omega)} \begin{pmatrix} v(x) - E & -\overline{a(x)} \\ a(x+\omega) & 0 \end{pmatrix}.$$  

(1.3)

2020 Mathematics Subject Classification. 37C55, 37F10, 37C40.

Key words and phrases. Large deviation theorems; Jacobi operators; finite scale Dirichlet determinants; Brjuno-Rüssmann frequency; strong Birkhoff ergodic theorem.

The second author was supported by the Fundamental Research Funds for the Central Universities(Grant B200202004) and China Postdoctoral Science Foundation (Grant 2019M650094).

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and call a map 
\[(\omega, M) : (x, \vec{v}) \mapsto (x + \omega, M(x)\vec{v})\]
a Jacobi cocycle. Due to the fact that an analytic function only has finite zeros, 
\[M(x, E, \omega)\] and the n-step transfer matrix 
\[M_n(x, E, \omega) := \prod_{k=n}^1 M(x + k\omega, E)\]
make sense almost everywhere. By the Kingman’s subadditive ergodic theorem, the 
Lyapunov exponent 
\[L(E, \omega) = \lim_{n \to \infty} L_n(E, \omega) = \inf_{n \to \infty} L_n(E, \omega) \geq 0 \quad (1.4)\]
always exists, where 
\[L_n(E, \omega) = \frac{1}{n} \int_T \log \|M_n(x, E, \omega)\|dx.\]

Let \(H_{[m,n]}(x, \omega)\) be the Jacobi operator defined by (1.1) on a finite interval \([m,n]\) with Dirichlet boundary conditions, 
\[\phi(m-1) = 0 \text{ and } \phi(n+1) = 0.\] Let \(f_{[m,n]}^a(x, E, \omega) = \det(H_{[m,n]}(x, \omega) - E)\) be its characteristic polynomial. One has 
\[f_{[m,n]}^a(x, E, \omega) = f_{n-m+1}^a(x + (m-1)\omega, E, \omega), \quad (1.5)\]
where

\[
\begin{vmatrix}
  v(x + \omega) - E & -a(x + 2\omega) & 0 & \cdots & 0 \\
  -a(x + 2\omega) & v(x + 2\omega) - E & -a(x + 3\omega) & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & -a(x + n\omega) & v(x + n\omega) - E
\end{vmatrix}
\]

In this paper, the aim is to study the properties of \(f_{[m,n]}^a(x, E, \omega)\). To state our 
conclusions, we first make some introductions to the background of our topic.

The operator (1.1) has the following important special case, which is called the 
Schrödinger operator and has been studied extensively:
\[\left[ H^s(x, \omega) \phi \right](n) = \phi(n+1) + \phi(n-1) + v(x + n\omega)\phi(n), \quad n \in \mathbb{Z}. \quad (1.6)\]
Then, \(M_s^a(x, E, \omega), L_s^a(E, \omega), L_s^a(E, \omega)\) and \(f_s^a(x, E, \omega)\) have the similar definitions. In [6], Bourgain and Goldstein proved that if 
\(L_s^a(E, \omega) > 0\), then for almost all \(\omega\), the operator \(H^s(0, \omega)\) has Anderson Localization, which means that it has pure-point spectrum with exponentially decaying eigenfunction. In [11], Goldstein and Schlag obtained the Hölder continuity of \(L_s^a(E, \omega)\) in \(E\) with the strong Diophantine \(\omega\), i.e. for some \(\alpha > 1\) and any integer \(n\),
\[\|n\omega\| > \frac{C_\omega}{|n| (\log |n| + 1)^\alpha}, \quad (1.7)\]
where
\[\|x\| = \min_{n \in \mathbb{Z}} |x + n|.\]
It is well known that for a fixed $\alpha > 1$, almost every irrational $\omega$ satisfies (1.7). Obviously, if we define the Diophantine number as

$$\|n\omega\| > \frac{C_\omega}{|n|^\alpha}, \quad (1.8)$$

then it also has a full measure. In these two references, the key lemmas are the following so-called large deviation theorems (LDTs for short) for matrix $M_n^s(x, E, \omega)$ with these two types of frequencies: for the Diophantine $\omega$, it was proved in [6] that there exists $0 < \sigma < 1$ such that

$$\n \begin{align*}
\text{mes} \left\{ x : \frac{1}{n} \log \|M_n^s(x, E, \omega)\| - L_n^s(E, \omega) > n^{-\sigma} \right\} < \exp (-n^\sigma); \quad (1.9)
\end{align*}$$

for the strong Diophantine $\omega$, it was proved in [11] that there exists $\delta_0(n)$ such that for any $\delta > \delta_0(n)$

$$\n \begin{align*}
\text{mes} \left\{ x : \frac{1}{n} \log \|M_n^s(x, E, \omega)\| - L_n^s(E, \omega) > \delta \right\} < \exp (-c\delta^2 n). \quad (1.10)
\end{align*}$$

Here $\delta_0(n)$ is called the smallest deviation in the LDT and very important in our paper.

Compared with the Schrödinger cocycle, one of the distinguishing features of the Jacobi cocycle is that it is not $SL(2, \mathbb{C})$. Then Jitomirskaya, Koslover and Schulteis [17], and Jitomirskaya and Marx [18] proved that the LDT (1.9) for $M_n(x, E, \omega)$ and the weak Hölder continuity of the Lyapunov exponent of the analytic $GL(2, \mathbb{C})$ cocycles hold with the Diophantine frequency. In [20], we showed that (1.10) can hold for $M_n(x, E, \omega)$ with the strong Diophantine $\omega$ and the continuity of the Lyapunov exponent of the Jacobi cocycles $L(E, \omega)$ can be Hölder in $E$.

For any irrational $\omega$, there exist its continued fraction approximates $\{p_s/q_s\}_{s=1}^\infty$, satisfying

$$\frac{1}{q_s(q_{s+1} + q_s)} < |\omega - \frac{p_s}{q_s}| < \frac{1}{q_sq_{s+1}}. \quad (1.11)$$

Define $\beta$ as the exponential growth exponent of $\{p_s/q_s\}_{s=1}^\infty$ as follows:

$$\beta(\omega) := \lim \sup _s \frac{\log q_{s+1}}{q_s} \in [0, \infty].$$

Obviously, both the sets of the strong Diophantine frequency and the Diophantine one are the subsets of $\{\omega : \beta(\omega) = 0\}$. We say $\omega$ is the Liouville number, if $\beta(\omega) \geq 0$.

Recently, more and more attentions are paid to the question that what will happen to these operators with more generic $\omega$, such as the one satisfying $\beta(\omega) = 0$, the finite Liouville one satisfying $0 < \beta(\omega) < \infty$ and the irrational one. So far, the most striking answers are mainly for the almost Mathieu operators (AMO for short), which is also a special case of the Jacobi ones

$$[H^m(x, \omega, \lambda)\phi](n) = \phi(n+1) + \phi(n-1) + 2\lambda \cos (2\pi(x + n\omega)) \phi(n), \quad n \in \mathbb{Z}. \quad (1.12)$$

The most famous one, the Ten Martini Problem, which was dubbed by Barry Simon and conjectures that for any irrational $\omega$, the spectrum of AMO is a Cantor set, was completely solved by Avila and Jitomirskaya [1]. In that reference, they also proved that $H^m(x, \omega, \lambda)$ has Anderson Localization for almost every $x \in \mathbb{T}$ with $\lambda > e^{4\beta}$. In [4], Avila, You and Zhou improved it to $\lambda > e^\beta$.

While, the answers for the Schrödinger or Jacobi operators in the positive Lyapunov exponent regimes are mainly in the study of the continuity of the Lyapunov
exists a monotone increasing and continuous function $\Delta(t)$ which is a famous extension of the strong Diophantine number. It says that there is a finite-volume determinant $f_n(x, E, \omega)$ with a general analytic $v_0$ and $C$ is a positive constant also depending only on $v_0$. Our second author also proved the corresponding conclusion for the Jacobi operators in [21]. These two results are optimal, because Avila, Last, Shamis and Zhou [3] showed that the continuity of the Lyapunov exponent of the almost Mathieu operators can’t be Hölder if $\beta > 0$ and $e^{-\beta} < \lambda < e^\beta$.

Until now, we do not know much about the spectrum of the Schrödinger or Jacobi operators in the positive Lyapunov exponent regimes when the frequency is not strong Diophantine. The main reason is that we do not know much about the finite-volume determinant $f_n(x, E, \omega)$. While, for the almost Mathieu operators, it can be handled explicitly via the Lagrange interpolation for the trigonometric polynomial. This method can be applied for the following extend Harper’s operators, which also have the cosine potential, to obtain many spectral conclusions with the generic frequency, such as [2] and [14]:

$$a(x) = \lambda_3 \exp[-2\pi i (x + \frac{\omega}{2})] + \lambda_2 + \lambda_1 \exp[2\pi i (x + \frac{\omega}{2})], \quad 0 \leq \lambda_2, 0 \leq \lambda_1 + \lambda_3,$$

$$v(x) = 2 \cos(2\pi x).$$

However, the Lagrange interpolation can not work for the more general Schrödinger operators, since their potentials both are generic analytic functions. Therefore, in [12], Goldstein and Schlag applied the LDT (1.10) and the relationship that

$$M_n^* (x, E, \omega) = \begin{pmatrix} f_n(x, E, \omega) & f_n(x + \omega, E, \omega) \\ f_n(x, E, \omega) & f_n(x + \omega, E, \omega) \end{pmatrix}$$

(1.13)

to estimate the BMO norm of $f_n^*(x, E, \omega)$. Then they obtained the following LDT for $f_n^*(x, E, \omega)$ with the strong Diophantine $\omega$ by the John-Nirenberg inequality:

$$\text{mes} \{ x \in \mathbb{T} : |\log |f_n^*(x)| - \langle \log |f_n^*| \rangle > n\delta \} \leq C \exp \left( -c\delta n \left( \frac{h_n}{\delta} \right)^{-1} \right). \quad (1.14)$$

This LDT was applied to get the Hölder exponent of the Hölder continuity of $L^*(E, \omega)$ in $E$ and the upper bound on the number of eigenvalues of $H_n^*(x, \omega)$ contained in an interval of size $n^{-C}$. What’s more, with its help, the estimation on the separation of the eigenvalues of $H_n^*(x, \omega)$ and the property that the spectrum of $H^*(x, \omega)$, denoted by $\mathcal{S}_\omega$, is a Cantor set were obtained in [13], and the homogeneity of $\mathcal{S}_\omega$ was proved in [10]. In [8] and [9], Binder and Voda applied this method to our analytic Jacobi operators (1.1). It must be noted that the above conclusions all hold only for the strong Diophantine $\omega$ and the LDTs for $f_n^*(x, E, \omega)$ and $f_n^*(x, E, \omega)$ are the key lemma in the method.

Now, we can declare that the concrete content of our main aim is to obtain the LDT for the finite-volume determinant $f_n^*(x, E, \omega)$ with more generic $\omega$. It is important in our field and is the key preparation for the study of the spectrum problem for discrete quasiperiodic operators of second order in the future.

In this paper, we assume that the frequency $\omega$ is the Brjuno-Rüssmann number, which is a famous extension of the strong Diophantine number. It says that there exists a monotone increasing and continuous function $\Delta(t) : [1, \infty) \to [1, \infty)$ such
that $\Delta(1) = 1$ and for any integer $k > 0$,
\[
\|k\omega\| > \frac{C_\omega}{\Delta(k)},
\]
and
\[
\int_1^\infty \frac{\log \Delta(t)}{t^2} < +\infty.
\]

For example, this Brjuno-Rüssmann function $\Delta(t)$ can be $t(\log t + 1)^\alpha$, $\exp(t^{1/\alpha})$, $t^\alpha$, $\exp((\log t)^\alpha)$ and $\exp(\frac{t}{(\log t)})$ with $\alpha > 1$. Define $\Gamma_\omega(n) = \|n\omega\|^{-1}$. Due to (1.15), we have that
\[
q_{s+1} < \Gamma_\omega(q_s) < q_s + q_{s+1}, \text{ and } \Gamma_\omega(n) < \Gamma_\omega(q_s), \forall n \in (q_s, q_{s+1}).
\]
Therefore, there exists another definition of the Brjuno-Rüssmann number as follow: There exists a function $\Psi_\omega(t) = \max(\|k\omega\|^{-1}, \forall 0 < k, k \in \mathbb{Z})$ satisfying (1.16). Note that the denominator series ${\|k\omega\|}_s^{\infty}$ and the function $\Psi_\omega$ depend on $\omega$. Thus, to make almost every irrational number satisfy (1.15), we assume that
\[
\Delta(t) > t(\log t + 1).
\]
It implies that $\log \Delta(t) > \log t$ but it is false that $\frac{\Delta(t)}{\Delta(t)} \neq \frac{\log \Delta(t)}{\log t}$.

However, it holds for all examples which we have given. So we make the following hypothesis:

Hypothesis H.1. $\Delta(t) > t(\log t + 1)$ and $t\Delta'(t) \geq \Delta(t)$ for any $t \geq 1$.

Then, our first LDT for $f_n^a(x, E, \omega)$ is

**Theorem 1.1.** Let $\omega$ be the Brjuno-Rüssmann number satisfying Hypothesis H.1 and $L(E, \omega) > 0$. There exist constants $c = c(a, v, E, \omega)$ and $C = C(a, v, \omega)$, and absolute constant $C$ such that for any integer $n \geq 1$ and $\delta > \delta_{H.1}(n) := \frac{C\log \Delta(n)}{\Delta^{-1}(C_\omega n)}$, where $\Delta^{-1}(\cdot)$ means the inverse function of $\Delta(\cdot)$, then
\[
\text{mes } \{x \in T : |\log |f_n^a(x)| - \log |f_n^a| > n\delta\} \leq C \exp \left(-c\delta_{H.1}(n)^{-1}\right).
\]

**Remark 1.** In this paper, the notation $A^{1-}\epsilon$ means $A^{1-\epsilon}$ for any small absolute $\epsilon > 0$. And $A^{1+}$ has the similar definition.

**Remark 2.** If we assume the potential $v$ is of the form $\lambda v_0$ with a general analytic $v_0$, then the second author [21] proved that there exists $\lambda_0 = \lambda_0(v_0, a)$ such that the Lyapunov exponent $L(E, \omega)$ is always positive for any $E$ and any irrational $\omega$ under the condition $\lambda < \lambda_0$.

If $\Delta(t) = t(\log t + 1)^\alpha$, then $\delta_{H.1}(n) = \frac{\hat{C}\log n}{n^{1+\alpha}}$ which is very close to the smallest deviation for the strong Diophantine number $\frac{\log n}{n^{1+\alpha}}$. But if $\Delta(t) = \exp(t^{1/\alpha})$, then $\delta_{H.1}(n) = \hat{C}n^{1/\alpha}$ which is too large for us to apply Theorem 1.1 to the research of the spectrum of the analytic quasi-periodic operators (1.1) in our future work. Thus, we need to make some hypothesis to improve this smallest deviation when $\Delta(t)$ grows fast:

Hypothesis H.2. $\omega$ satisfies Hypothesis H.1 and for any $t \geq 1$,
\[
\Delta(t) < \exp\left(\frac{t}{\log t}\right).
\]

From (1.16), $\Delta(t)$ has an upper bound of $\exp(\frac{t}{\log t})$ generally. But it is possible that it grows very fast and exceeds this upper bound in some intervals, and in
the rest it grows very slowly and makes the integral converge. Therefore, the aim of this hypothesis is to avoid this extreme possibility. Then, our second LDT for \( f_n^a(x, E, \omega) \) is

**Theorem 1.2.** Let \( \omega \) be the Brjuno-Rüssmann number satisfying Hypothesis H.2 and \( L(E, \omega) > 0 \). There exist constants \( c = c(a, v, E, \omega) \) and \( \bar{C} = \bar{C}(a, v, \omega) \), and absolute constant \( C \) such that for any integer \( n \geq 1 \) and \( \delta > \delta_{H,2}(n) := \frac{C}{[\log(\Delta^{-1}(C, \omega))]^{-1}} \),

\[
\text{mes } \{ x \in T : |\log |f_n^a(x)| - \langle \log |f_n^a| \rangle | > n\delta \} \leq C \exp \left( -c\delta(\delta_{H,2}(n))^{-1} \right).
\]

**Remark 3.** With this hypothesis, no matter \( \Delta(t) \) equals to \( \exp(t^{\frac{1}{n}}) \) or \( \exp(\frac{t}{(\log t)^{c_0}}) \), the smallest deviation \( \delta_{H,2}(n) = \frac{C}{[\log(\Delta^{-1}(C, \omega))]^{-1}} \ll 1 \) which satisfies what we need for the study of the spectrum, such as Theorem 1.5.

As mentioned above, what we want to avoid is the case that \( \Delta(t) \) grows faster than \( \exp(t) \) in some intervals. But Hypothesis H.2 only requires that \( \Delta(t) \) has an upper bound, but has no restriction on its derivative. Thus, we make the following hypothesis, which gives the mutual restriction between \( \Delta(t) \) and \( \Delta'(t) \) and looks also very reasonable:

**Hypothesis H.3.** \( \omega \) satisfies Hypothesis H.1 and \( \frac{\log(\Delta(t))}{t} \) is non-increasing for any \( t \geq 1 \). Easy computation shows that this hypothesis is equivalent to the inequality

\[
\Delta'(t) \leq \frac{\Delta(t) \log(\Delta(t))}{t}.
\]

Combined it with Hypothesis H.1, it shows that the bound of \( t\Delta'(t) \) is determined by \( \Delta(t) \). Obviously, all examples of functions mentioned above satisfy this hypothesis. With its help, we improve Theorem 1.1 and 1.2 as follows:

**Theorem 1.3.** Let \( \omega \) be the Brjuno-Rüssmann number satisfying Hypothesis H.3 and \( L(E, \omega) > 0 \). There exist constants \( c = c(a, v, E, \omega) \) and \( \bar{C} = \bar{C}(a, v, \omega) \), and absolute constant \( C \) such that for any integer \( n \geq 1 \) and \( \delta > \delta_{H,3}(n) := \frac{C \log(C, \omega)}{[\Delta^{-1}(C, \omega)]^{-1}} \),

\[
\text{mes } \{ x \in T : |\log |f_n^a(x)| - \langle \log |f_n^a| \rangle | > n\delta \} \leq C \exp \left( -c\delta(\delta_{H,3}(n))^{-1} \right).
\]

**Remark 4.** Since \( t(\log t + 1) < \Delta(t) \), it is obvious that

\[
\delta_{H,1}(n) = \frac{\log(\Delta(n))}{(\Delta^{-1}(C, \omega))]^{-1} \gg \delta_{H,3}(n) = \frac{\log(C, \omega)}{[\Delta^{-1}(C, \omega)]^{-1}}.
\]

On the other hand, due to the fact that \( 0 < \Delta^{-1}(t) < \Delta(t) \), \( n \gg \Delta(\log^2 \Delta^{-1}(n)) \), we have that

\[
\delta_{H,2}(n) = \frac{\log(C, \omega)}{[\Delta^{-1}(C, \omega)]^{-1} \gg \delta_{H,3}(n) = \frac{\log(C, \omega)}{[\Delta^{-1}(C, \omega)]^{-1}}.
\]

The key to prove these three LDTs for \( f_n^a(x, E, \omega) \) is an ergodic theorem for the subharmonic function shifting on \( T \). Specifically, we know that if \( T : X \rightarrow X \) is an ergodic transformation on a measurable space \( (X, \Sigma, m) \) and \( f \) is an \( m \)-integrable function, then the Birkhoff Ergodic Theorem tells that the time average functions

\[
f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)
\]

 converge to the space average \( \langle f \rangle = \frac{1}{m(X)} \int_X f dm \) for almost every \( x \in X \). But it doesn’t tell us how fast do they converge. So, we call a theorem the strong Birkhoff Ergodic Theorem, if it gives the convergence rate. The
following strong Birkhoff Ergodic Theorem for the subharmonic function shifting on $\mathbb{T}$ is the key which we just mentioned above:

**Theorem 1.4.** Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$ and $\omega$ be the Brjuno-Rüssman number satisfying Hypothesis H.1, or H.2, or H.3. Suppose that $\partial \omega$ and $T$ consists of finitely many piece-wise $C^1$ curves and $\mathbb{T}$ is contained in $\Omega'$ if $\omega$, $\Omega'$ is a compactly contained subregion of $\Omega$. There exist constants $c = c(\omega, u)$ and $\tilde{C} = \tilde{C}(\Omega, u)$ such that for any positive $n$ and $\delta > \delta^{(n)}_0$,

$$\mes \left( \left\{ x \in \mathbb{T} : \left| \sum_{k=1}^{n} u(x + k\omega) - n(u) \right| > \delta n \right\} \right) \leq \exp(-c\delta n), \quad (1.20)$$

where

$$\delta^{(n)}_0 = \begin{cases} \delta_{H.1}(n) := \frac{C \log \Delta(n)}{(\Delta^{-1}(C\omega))}, & \text{if } \omega \text{ satisfies H.1}, \\ \delta_{H.2}(n) := \frac{C \log \Delta(n)}{[\log \Delta^{-1}(C\omega)]}, & \text{if } \omega \text{ satisfies H.2}, \\ \delta_{H.3}(n) := \frac{\tilde{C} \log (C\omega)}{[\Delta^{-1}(C\omega)]}, & \text{if } \omega \text{ satisfies H.3}. \end{cases} \quad (1.21)$$

A very interesting thing we find is that no matter the irrational frequency is, the convergence rate of the exceptional measure is always $\exp(-c\delta n)$. The only difference is the smallest deviation $\delta^{(n)}_0$. If $\beta(\omega) > 0$, then our second author obtained in [21] that $\delta^{(n)}_0 = c\beta$ which is proved to be optimal in [3]; if $\beta(\omega) = 0$, we obtain (1.21) which includes the result for the strong Diophantine number by Goldstein and Schlag. Correspondingly, the three LDTs we obtain in this paper can be unified into the following form:

$$\mes \{ x \in \mathbb{T} : |\log |f^n_a(x)| - \langle \log |f^n_a| \rangle| > n\delta \} \leq C \exp \left(-c\delta \left(\delta^{(n)}_0\right)^{-1}\right), \forall \delta > \delta^{(n)}_0. \quad (1.22)$$

While, the exceptional measure in (1.22) will not converge when $\delta^{(n)}_0 = c\beta$! The method created by Goldstein and Schlag and applied in this paper should be improved for the Liouville frequency. We think it is a good question for our further research in the future.

Here we need to emphasize that our paper is not weaker version of [21]. That shows that the smallest deviation is $\delta^{(n)}_0 = c\beta$, and then the strong Birkhoff ergodic theorem and the LDTs for matrices hold when the deviation is larger than $\delta^{(n)}_0$. Letting the positive Lyapunov exponent be this deviation, our second author obtained the H"older continuity of the Lyapunov exponent. However, if we applied these results in our condition that $\beta = 0$, then the smallest deviation is 0! It is absurd! So, compared to [21], the main aim of our second section is to find the smallest deviation when $\beta = 0$. What’s more, we will find that in technology the key is to estimate $\sum_{j=1}^{2m-1} \frac{q_{s-j+1}}{q_{s-j}} \log q_{s-j+1}$. It is easy when $\beta > 0$:

$$\sum_{j=1}^{2m-1} \frac{q_{s-j+1}}{q_{s-j}} \log q_{s-j+1} \leq 2\beta \sum_{j=1}^{2m-1} q_{s-j+1} \leq 8\beta n.$$  

While, when $\beta = 0$, the fact that $\left\{ \frac{\log q_{s+1}}{q_s} \right\}_{s=1}^{\infty}$ has different speeds, which depend on $\Delta(t)$, to converge to 0 makes this estimation much harder. On the other hand, the aims of our Section 3 and 4 are to obtain the LDT for $f^{n}_a$ and its applications, which are nonexistent in [21]. In summary, the focus point of our paper is to show the importance of the smallest deviation of the strong Birkhoff ergodic theorem and
calculate it when $\beta = 0$. Of course, when we need the LDTs for matrices and the H"{o}lder continuity of the Lyapunov exponent, such as Lemma 3.1 and 4.1, we can use the results from [21] directly.

At last, we have an application of our LDTs, which estimates the upper bound on the number of eigenvalues of $H_n(x, \omega)$ contained in an interval of size $(\delta_0^{(n)}/\delta)^{1/2}$, where $h$ is the H"{o}lder exponent of the H"{o}lder continuity of $L(E, \omega)$, see Lemma 4.1. The distribution of the eigenvalues is very important in the further study of the spectrum problem for discrete quasiperiodic operators of second order. With fixed $x$ and $\omega$, the matrix $H_n(x, \omega)$ has $n$ eigenvalues. So we have an intuition that these eigenvalues have a more uniform distribution when the frequency $\omega$ is "more irrational". For the Brjuno-R"{u}ssmann number, it means that $\Delta(t)$ grows more slowly and then $\delta_0^{(n)}$ is smaller. The following theorem verifies our intuition:

**Theorem 1.5.** Let $\omega$ be the Brjuno-R"{u}ssmann number satisfying Hypothesis H.1, or H.2, or H.3 and $L(E, \omega) > 0$. Then, for any $x_0 \in T$ and $E_0 \in \mathbb{R}$,

$$\# \left\{ E \in \mathbb{R} : f_n^a(x_0, E, \omega) = 0, \ |E - E_0| < \left( \delta_0^{(n)} \right)^{1/\delta} \right\} \leq 13n\delta_0^{(n)}.$$  

We organize this paper as follows. In Section 2, we prove Theorem 1.4, the strong Birkhoff Ergodic theorem for the subharmonic function shifting on $T$ with our Brjuno-R"{u}ssmann frequency. We apply it to the analytic quasi-periodic Jacobi operator and obtain Theorem 1.1, Theorem 1.2 and Theorem 1.3 in Section 3, which are all the LDTs for $f_n^a(x, E, \omega)$ with different hypotheticals. Then, we prove Theorem 1.5, an application of them, in the last section.

2. **Strong birkhoff ergodic theorem for subharmonic functions with the Brjuno-R"{u}ssmann shift.** Let \{x\} = $x - \lfloor x \rfloor$. For any positive integer $q$, complex number $\zeta = \xi + i\eta$ and $0 \leq x < 1$, define

$$F_{q,\zeta}(x) = \sum_{0 \leq k < q} \log |\{x + k\omega\} - \zeta| \text{ and } I(\zeta) = \int_0^1 \log |y - \zeta| dy.$$  

Let $|\{x + k\omega\} - \zeta| = \min_{k=1}^{q-1} |\{x + k\omega\} - \zeta|$, where $q_s$ is the denominator of the continued fraction approximants. In [11], Goldstein and Schlag proved Lemma 3.1 that for any irrational $\omega$, there exists an absolute constant $C$ such that

$$|F_{q_s,\zeta}(x) - q_s I(\zeta)| \leq C \log q_s + |\log |\{x + k\omega\} - \zeta||. \quad (2.1)$$  

Then

**Lemma 2.1.** For any irrational $\omega$ and integer $l < \frac{q_s + 1}{q_s}$,

$$|F_{l, q_s,\zeta}(x) - l_s q_s I(\zeta)| < C l_s \log q_s + |\log D(x - \xi, \omega, l_s q_s)| + 2l_s \log q_{s+1},$$

where $D(x, \omega, n) := \min_{k=0}^{n-1} \{x + k\omega\}$.

**Proof.** Define $x_h = x + hq_s\omega$ and $|\{x_h + k_h\omega\} - \zeta| = \min_{k=0}^{q_s-1} |\{x_h + k\omega\} - \zeta|$. Due to (2.1), we have

$$|F_{l, q_s,\zeta}(x) - l_s q_s I(\zeta)| \leq \sum_{h=0}^{l_s-1} |F_{l, q_s,\zeta}(x_h) - l_s q_s I(\zeta)|$$

$$\leq \sum_{h=0}^{l_s-1} |\log |\{x_h + k_h\omega\} - \zeta|| + C l_s \log q_s. \quad (2.2)$$
We declare that if there exists \( 0 \leq j < q_s \) such that \(|\{x+j\omega\} - \xi| \leq \frac{1}{2q_s} - \frac{1}{q_{s+1}}\), then \( j = k_0 \). Actually, if \(|\{x+j\omega\} - \xi| \leq \frac{1}{2q_s} - \frac{1}{q_{s+1}}\) and \( j \neq k_0 \), then \(|\{x+k_0\omega\} - \xi| \leq \frac{1}{2q_s} - \frac{1}{q_{s+1}}\), which implies

\[
|\{x + k_0\omega\} - \{x + j\omega\}| \leq \frac{1}{q_s} - \frac{2}{q_{s+1}}.
\]

Due to (1.11), it has

\[
\frac{k}{q_s(q_s+1)q_s} < |k\omega - kp_s| < \frac{k}{qsq_{s+1}} \leq \frac{1}{q_{s+1}}, \quad 0 < k < q_s.
\]

Then,

\[
|\{x + j\omega\} - \{x + k_0\omega\}| < \frac{1}{q_s},
\]

It is a contraction. Thus, there is at most one integer \( 0 \leq k_0 < q_s \) such that \(|\{x+k_0\omega\} - \xi| < \frac{1}{2q_s} - \frac{1}{q_{s+1}}\) and

\[
|\{x + k\omega\} - \xi| > \frac{1}{2q_s} - \frac{1}{q_{s+1}} > \frac{1}{4q_s}, \quad k \neq k_0.
\]

Due to (1.11) again, it has

\[
\frac{1}{2q_{s+1}} < \frac{1}{q_s+q_{s+1}} < |q_s\omega - p_s| < \frac{1}{q_{s+1}}.
\]

Define \( Q = \lfloor \frac{q_{s+1}}{q_s} \rfloor \) and let \( j \) be the number such that \(|\{x_j + kj\omega\} - \xi| < \frac{1}{4q_{s+1}}\).

Then by (2.5) and the above declaration, we have for any \( j - 2Q + 1 \leq h < j \) and \( j < h \leq j + 2Q - 1 \),

\[
|\{x_h + kh\omega\} - \xi| > \frac{1}{4q_{s+1}}.
\]

Thus there are at most one point which is small than \( \frac{1}{4q_{s+1}} \). Combining it with (2.4), we have

\[
|F_{l_s,q_s,\xi}(x) - l_s q_s I(\xi)| \leq |\log D(x - \xi, \omega, l_s q_s)| + Cl_s \log q_s + l_s \left| \log \frac{1}{4q_{s+1}} \right|
\]

\[
\leq |\log D(x - \xi, \omega, l_s q_s)| + Cl_s \log q_s + 2l_s \log q_{s+1}.
\]

\[\square\]

**Lemma 2.2.** For any \( 0 < \sigma < 1 \), irrational \( \omega \) and integer \( l_s < \frac{q_{s+1}}{q_s} \),

\[
\int_{T} \exp \left( \sigma |F_{l_s,q_s,\xi}(x) - l_s q_s I(\xi)| \right) dx < \exp \left( 5\sigma l_s \log q_{s+1} \right).
\]

**Proof.** We first apply Lemma 3.2 in [11]. It says that if \( \Omega \subset T \) is an arbitrary finite set, then for any \( 0 < \sigma < 1 \),

\[
\int_{T} \exp \left( \sigma |\log \text{dist}(x, \Omega)| \right) dx \leq \frac{2^\sigma}{1 - \sigma} (2\Omega)^n.
\]

Set \( \Omega = \{m\omega : 0 \leq m < l_s q_s\} \). Then \( 2\Omega = l_s q_s \) and \( \text{dist}(x - \xi, \Omega) = D(x - \xi, \omega, l_s q_s) \).

Thus, by (2.6),

\[
\int_{T} \exp \left( \sigma |\log D(x - \xi, \omega, l_s q_s)| \right) dx = \int_{T} \exp \left( \sigma |\log \text{dist}(x, \Omega)| \right) dx \leq \frac{2^\sigma}{1 - \sigma} (l_s q_s)^n.
\]

By Lemma 2.1, we have

\[
\int_{T} \exp \left( \sigma |F_{l_s,q_s,\xi}(x) - l_s q_s I(\xi)| \right) dx
\]
Then we have

\[ \leq \exp(2C\sigma \log(l_s q_s) + C\sigma l_s \log q_s + 2\sigma l_s \log q_{s+1}) < \exp(5\sigma l_s \log q_{s+1}). \]

Now for any \( n \), there exist \( q_s \) and \( q_{s+1} \) such that \( q_s \leq n < q_{s+1} \). Let \( l_s = \left[ \frac{n}{q_s} \right] \) and \( l_i = \left[ \frac{2^{i+1}}{q_i} \right] \) for \( i < s \). Then there exists \( r_{s-2m+1} \) satisfying \( 0 \leq r_{s-2m+1} < q_{s-2m+1} \) such that

\[ n = l_s q_s + l_{s-1} q_{s-1} + \cdots + l_{s-2m+1} q_{s-2m+1} + r_{s-2m+1}. \]

Define

\[ \bar{n} = l_s q_s + l_{s-1} q_{s-1} + \cdots + l_{s-2m+1} q_{s-2m+1}. \] (2.7)

Then we have

**Lemma 2.3.** For any compact \( \Omega \subset \mathbb{C} \), there exist constants \( \bar{c} = \bar{c}(\omega) \) such that for any \( 0 < \sigma \leq \bar{c} \), we have

\[ \int_0^1 \exp(\sigma |F_{\bar{n}, \omega}(x) - \bar{n}I(\omega)|)dx \leq \exp \left( \bar{c} \sigma n \delta_0(n) \right), \] (2.8)

where

\[ \delta_0(n) = \begin{cases} \delta_{H,1}(n) := \frac{1}{[\Delta^{-1}(C_{\omega}n)]^1}, & \text{if } \omega \text{ satisfies } H.1, \\
\delta_{H,2}(n) := \left[ \frac{\log(\Delta^{-1}(C_{\omega}n))}{\log(C_{\omega}n)} \right]^1, & \text{if } \omega \text{ satisfies } H.2, \\
\delta_{H,3}(n) := \left[ \frac{\log(\Delta^{-1}(C_{\omega}n))}{\log(C_{\omega}n)} \right]^1, & \text{if } \omega \text{ satisfies } H.3. \end{cases} \] (2.9)

**Proof.** Let \( r_s = \bar{n} - l_s q_s \). Due to the Hölder inequality and Lemma 2.2,

\[ \int_0^1 \exp(\sigma |F_{\bar{n}, \omega}(x) - \bar{n}I(\omega)|)dx \]

\[ \leq \left[ \int_0^1 \exp(2\sigma |F_{l_s q_s, \omega}(x) - l_s q_s I(\omega)|)dx \right]^\frac{1}{2} \times \left[ \int_0^1 \exp(2\sigma |F_{r_s, \omega}(x) - r_s I(\omega)|)dx \right]^\frac{1}{2} \]

\[ \leq \exp(5\sigma l_s \log q_{s+1}) \left[ \int_0^1 \exp(2\sigma |F_{r_s, \omega}(x) - r_s I(\omega)|)dx \right]^\frac{1}{2}. \]

Let \( r_{s-i+1} = l_{s-i} q_{s-i} + r_{s-i} \), where \( l_{s-i} = \left[ \frac{r_{s-i+1}}{q_{s-i}} \right], 0 \leq r_{s-i} = r_{s-i+1} - l_{s-i} q_{s-i} < q_{s-i} \). Then

\[ \left[ \int_0^1 \exp(2\sigma |F_{r_{s-i+1}, \omega}(x) - r_{s-i+1} I(\omega)|)dx \right]^\frac{1}{2} \]

\[ \leq \left[ \int_0^1 \exp(2\sigma |F_{r_{s-i} q_{s-i}, \omega}(x) - l_{s-i} q_{s-i} I(\omega)|)dx \right]^\frac{1}{2} \]

\[ \times \left[ \int_0^1 \exp(2\sigma |F_{r_{s-i}, \omega}(x) - r_{s-i} I(\omega)|)dx \right]^\frac{1}{2} \]

\[ \leq \exp(5\sigma l_{s-i} \log q_{s-i+1}) \times \left[ \int_0^1 \exp(2\sigma |F_{r_{s-i+1}, \omega}(x) - r_{s-i} I(\omega)|)dx \right]^\frac{1}{2}. \]

Therefore,

\[ \int_0^1 \exp(\sigma |F_{\bar{n}, \omega}(x) - \bar{n}I(\omega)|)dx \]

\[ \leq \exp [5\sigma (l_s \log q_{s+1} + l_{s-1} \log q_s + \cdots + l_{s-2m+1} \log q_{s-2m+2})] \]
Therefore, we have

\[ \frac{1}{(\Delta^{-1}(C_\omega t))'} = \Delta' \left( \Delta^{-1}(C_\omega t) \right), \]

we have

\[ \Delta^{-1}(C_\omega t) (\log t + 1) > C_\omega t \log t \left( \Delta^{-1} \right)'(C_\omega t). \]  

(2.11)

Now we finish the proof of the assertion as (2.11) shows that the numerator of the derivative of \( \Delta^{-1}(C_\omega t) \) is positive.

Due to (1.15) and (1.11),

\[ C_\omega q_i < \Delta(q_i - 1) \text{ and } q_i - 1 > \Delta^{-1}(C_\omega q_i). \]  

(2.12)

Therefore,

\[ \frac{q_i}{q_{i-1}} \log q_i < \frac{q_i}{\Delta^{-1}(C_\omega q_i)} \log q_i. \]

We apply the assertion and obtain

\[ \sum_{j=1}^{2m-1} \frac{q_{s-j+1}}{q_{s-j}} \log q_{s-j+1} < (2m - 1) \frac{n}{\Delta^{-1}(C_\omega n)} \log n. \]  

(2.13)

Recall that \( \Delta(t) \) is monotone increasing and continuous. And so is \( \Delta^{-1}(t) \). Combining it with (2.12), we have

\[ q_s \Delta^{-1}(C_\omega q_{s+1}) > \Delta^{-1}(C_\omega n), \]  

(2.14)

and for any \( n > n_0(\omega) \),

\[ l_s \log q_{s+1} \leq \frac{n}{q_s} \log \left( \frac{\Delta(q_s)}{C_\omega} \right) \leq \frac{2n \log \Delta(q_s)}{q_s} \leq \frac{2n \log \Delta(q_s)}{\Delta^{-1}(C_\omega n)} \leq \frac{2n \log \Delta(n)}{\Delta^{-1}(C_\omega n)}. \]  

(2.15)

Choosing

\[ m = m_{H,1} = \log_2 \Delta^{-1}(C_\omega n), \]  

(2.16)

and due to (2.10), (2.13), (2.15), we have

\[ \int_0^1 \exp(\sigma|F_n(\zeta)|)dx \]

\[ \leq \exp \left\{ 5\sigma \left[ \frac{2n \log \Delta(n)}{\Delta^{-1}(C_\omega n)} + 2 \log_2 \Delta^{-1}(C_\omega n) \frac{n}{\Delta^{-1}(C_\omega n)} \log n \right] \right\} \]

\[ \leq \exp \left\{ 20\sigma n \log \Delta(n) \frac{\log C \Delta^{-1}(C_\omega n)}{\Delta^{-1}(C_\omega n)} \right\} \]

\[ \leq \exp \left\{ 20\sigma n \left( \frac{\log \Delta(n)}{(\Delta^{-1}(C_\omega n))^T} \right) \right\}. \]  

(2.17)
Assume that $\omega$ satisfies Hypothesis H.2 which implies that $\Delta(t) < \exp\left(\frac{t}{\log t}\right)$ holds. Then
\[
\frac{\log \Delta(q_s)}{q_s} < \frac{1}{\log q_s} \quad \text{and} \quad \frac{\log n}{\Delta^{-1}(C_\omega n)} < \frac{1}{\log \Delta^{-1}(C_\omega n)}.
\] (2.18)
Combining them with (2.13) and (2.14),
\[
l_s \log q_{s+1} < \frac{2n \log \Delta(q_s)}{q_s} < \frac{2n}{\log q_s} < \frac{2n}{\log(\Delta^{-1}(C_\omega n))},
\]
and
\[
\sum_{j=1}^{2m-1} \frac{q_{s-j+1}}{q_{s-j}} \log q_{s-j+1} < (2m - 1) \frac{n}{\Delta^{-1}(C_\omega q_s) \log q_s}
\]
\[
< (2m - 1) \frac{n}{\Delta^{-1}(C_\omega n) \log n}
\]
\[
< (2m - 1) \frac{n}{\log(\Delta^{-1}(C_\omega n))}.
\]
Letting
\[
m = m_{H.2} = \log_2 \log \Delta^{-1}(C_\omega n),
\] (2.19)
we obtain that
\[
\int_0^1 \exp(\sigma|F_{n,\xi}(x) - \bar{n}I(\xi)||)dx \leq \exp \left\{ 20\sigma \log_2 \log \Delta^{-1}(C_\omega n) \right\}
\]
\[
\leq \exp \left\{ 20\sigma \frac{n}{[\log(\Delta^{-1}(C_\omega n))]} \right\}. \quad (2.20)
\]
Assume the $\omega$ satisfies Hypothesis H.3 which implies that $\frac{\log \Delta(t)}{t}$ is non-increasing. Due to (2.14) and (2.15),
\[
l_s \log q_{s+1} \leq \frac{2n \log \Delta(q_s)}{q_s} \leq \frac{n \log \Delta^{-1}(C_\omega n)}{\Delta^{-1}(C_\omega n)} = \frac{n \log(C_\omega n)}{\Delta^{-1}(C_\omega n)}.
\] (2.21)
By (2.13),
\[
\sum_{j=1}^{2m-1} \frac{q_{s-j+1}}{q_{s-j}} \log q_{s-j+1} < (2m - 1) \frac{n}{\Delta^{-1}(C_\omega q_s) \log q_s}
\]
\[
\leq (2m - 1) \frac{n}{\Delta^{-1}(C_\omega n) \log n}.
\] (2.22)
Choosing
\[
m = m_{H.3} = \log_2(\Delta^{-1}(C_\omega n)),
\] (2.23)
we have
\[
\int_0^1 \exp(\sigma|F_{n,\xi}(x) - \bar{n}I(\xi)||)dx \leq \exp \left( 20\sigma \frac{n \log(C_\omega n)}{[\Delta^{-1}(C_\omega n)]^{1-}} \right). \quad \Box
\] (2.24)
To finish the proof of Theorem 1.4, we need the following Riesz’s theorem proved in [12]:
Lemma 2.4. Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial \Omega$ consists of finitely many piece-wise $C^1$ curves. There exists a positive measure $\mu$ on $\Omega$ such that for any $\Omega_1 \Subset \Omega$ (i.e., $\Omega_1$ is a compactly contained subregion of $\Omega$),

$$u(z) = \int_{\Omega_1} \log |z - \zeta| d\mu(\zeta) + h(z), \quad (2.25)$$

where $h$ is harmonic on $\Omega_1$ and $\mu$ is unique with this property. Moreover, $\mu$ and $h$ satisfy the bounds

$$\mu(\Omega_1) \leq C(\Omega, \Omega_1) \sup_{\Omega} u - \sup_{\Omega_1} u, \quad (2.26)$$

and

$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \leq C(\Omega, \Omega_1, \Omega_2) \sup_{\Omega} u - \sup_{\Omega_1} u \quad (2.27)$$

for any $\Omega_2 \Subset \Omega_1$.

What’s more, we can obtain the following strong Birkhoff Ergodic Theorem for this harmonic function $h$ with the Brjuno-Rüssmann frequency easily:

Lemma 2.5. Let $h$ be a $1 -$ periodic harmonic function defined on a neighborhood of the real axis. There exists a constant $C$ depending only on $h$ such that for any integer $n$ and Brjuno-Rüssmann $\omega$,

$$\left| \sum_{k=1}^{n} h(x + k\omega) - n \int_0^1 h(y)dy \right| < \frac{C}{C_\omega},$$

where $C_\omega$ comes from (1.15).

Proof. Easy computation shows that

$$\left| \sum_{k=1}^{n} h(x + k\omega) - n \int_0^1 h(y)dy \right| = \left| \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} \hat{h}(j) e^{2\pi i j(x + k\omega)} - n\hat{h}(0) \right| \leq \sum_{j \neq 0} \left( \left| \hat{h}(j) \right| \left| \sum_{k=1}^{n} e^{2\pi i jk\omega} \right| \right),$$

where $\hat{h}(j)$ is the $j -$th Fourier coefficients of $h$. Due to its harmonicity, there exists two constants $\hat{C}_h$ and $\rho$ depending only on $h$ such that

$$\left| \hat{h}(j) \right| \leq \hat{C}_h \exp(-\rho j). \quad (2.28)$$

Therefore, we obtain this lemma by (2.28) and

$$\sum_{j=1}^{\infty} \frac{\log \Delta(j)}{j} < \infty \text{ from (1.16).}$$

The proof of Theorem 1.4. Notice that the ergodic measure for the shift on the Torus is the Lebesgue measure and $m(\mathbb{T}) = 1$. Then, $\langle u \rangle = \int_{\mathbb{T}} u(x) dx$, and

$$\sum_{k=1}^{n} u(x + k\omega) - n \langle u \rangle = \sum_{k=1}^{n} \int_{\Omega_1} \log \{|x + k\omega\} - \zeta| d\mu(\zeta) - n \int_{\Omega_1} I(\zeta) d\mu(\zeta) + \sum_{k=1}^{n} h(x + k\omega) - n \int_0^1 h(y) dy.$$
Recall that
\[ \sum_{k=1}^{n} \int_{\Omega_1} \log |x + k\omega| - \zeta |d\mu(\zeta)| = \int_{\Omega_1} F_{n,\zeta}(x)d\mu(\zeta). \]

Thus, due to Lemma 2.5, it yields that for any \( n \) and \( \delta \gg \frac{1}{n} \),
\[
\left\{ x \in \mathbb{T} : \sum_{k=1}^{n} u(x + k\omega) - n(u) > \delta n \right\} \subseteq \left\{ x \in \mathbb{T} : \int_{\Omega_1} |F_{n,\zeta}(x) - nI(\zeta)|d\mu(\zeta) > \frac{\delta n}{2} \right\}. \tag{2.29}
\]

To estimate the measure of the upper set, we use the Markov’s inequality: For any measurable extended real-valued function \( f(x) \) and \( \epsilon > 0 \), we have
\[
\text{mes } \{ x \in \mathbb{X} : |f(x)| \geq \epsilon \} \leq \frac{1}{\epsilon} \int_{\mathbb{X}} |f|dx.
\]

Let \( f(x) = \exp \left( \sigma \int_{\Omega_1} |F_{n,\zeta}(x) - nI(\zeta)|d\mu(\zeta) \right) \) and \( \epsilon = \exp(\sigma\delta n/2) \), then
\[
\leq \exp \left( -\frac{\sigma\delta n}{2} \right) \int_{0}^{1} \exp \left( \sigma \int_{\Omega_1} |F_{n,\zeta}(x) - nI(\zeta)|d\mu(\zeta) \right) \left( \int_{\mathbb{X}} |f|dx \right) \tag{2.30}
\]

Due to the Hölder inequality,
\[
\int_{0}^{1} \exp \left( \sigma \int_{\Omega_1} |F_{n,\zeta}(x) - nI(\zeta)|d\mu(\zeta) \right) dx \leq \left[ \int_{0}^{1} \exp \left( 2\sigma \int_{\Omega_1} |F_{n,\zeta}(x) - (n - \tilde{n})I(\zeta)|d\mu(\zeta) \right) dx \right]^{\frac{1}{2}} \times \left[ \int_{0}^{1} \exp \left( 2\sigma \int_{\Omega_1} |F_{n,\zeta}(x) - \tilde{n}I(\zeta)|d\mu(\zeta) \right) dx \right]^{\frac{1}{2}}, \tag{2.31}
\]
where \( \tilde{n} \) comes from (2.7). Since \( \exp(\cdot) \) is a convex function, the Jensen’s inequality and Lemma 2.3 imply that
\[
\int_{0}^{1} \exp \left( \sigma \int_{\Omega_1} |F_{n,\zeta}(x) - \tilde{n}I(\zeta)|d\mu(\zeta) \right) dx \leq \int_{0}^{1} \int_{\Omega_1} \exp(\sigma\mu(\Omega_1)|F_{n,\zeta}(x) - \tilde{n}I(\zeta)|) \frac{d\mu(\zeta)}{\mu(\Omega_1)} dx \\
= \int_{\Omega_1} \int_{0}^{1} \exp(\sigma\mu(\Omega_1)|F_{n,\zeta}(x) - \tilde{n}I(\zeta)|) \frac{d\mu(\zeta)}{\mu(\Omega_1)} dx \\
\leq \int \exp \left( 20\sigma\mu(\Omega_1)n\tilde{\omega}(n) \right) dx \leq \exp \left( 20\sigma\mu(\Omega_1)n\tilde{\omega}(n) \right). \tag{2.32}
\]

On the other hand, due to the facts that it is integrable for \( \log |z| \) on the disc \( |z| < r \) and \( |I(\zeta)| \leq \|\text{Im}(\zeta)\| \) if \( |\text{Im}\zeta| \) is close to 0, it is obvious that there exists a constant \( C = C(\Omega_1) \) such that
\[
\left| \int_{\Omega_1} \log |x - \zeta| - I(\zeta) d\mu(\zeta) \right| \leq \mu(\Omega_1) \left| \int_{\Omega_1} \log |x - \zeta| - I(\zeta) \frac{d\mu(\zeta)}{\mu(\Omega_1)} \right| \leq \mu(\Omega_1) C.
\]
Thus,
\[
\left[ \int_0^1 \exp \left( 2\sigma \left| \int_{\Omega_1} (F_{n-n,z}(x) - (n-n)I(\zeta))d\mu(\zeta) \right| \right) dx \right]^\frac{1}{2} 
\leq \exp \left( \sigma \mu(\Omega_1) C(n-n) \right) \leq \exp \left( \sigma \mu(\Omega_1) C r_{s-2m+1} \right). 
\tag{2.33}
\]
Note that for any irrational \( \omega \), the denominators of its continued fraction approximates satisfy
\[ q_{t+1} = a_{t+1}q_t + q_{t-1} > 2q_{t-1}. \]
Thus
\[ q_t > 2^m q_{t-2m} \quad \text{and} \quad r_{s-2m+1} < 2^{-m} n. \]
Recall that in the proof of Lemma 2.3, we choose \( m = m_{H,1} = \log_2 \Delta^{-1}(C_\omega n) \) when \( \omega \) satisfy the hypothesis H.1. Then
\[ C r_{s-2m+1} < \frac{n}{\Delta^{-1}(C_\omega n)} \ll n \delta_{H,1}(n) = n \delta_0^{(n)}. \]
Similarly, \( C r_{s-2m+1} \) also has the same upper bound in the other two hypotheses. Combined it with (2.30), (2.31), (2.32) and (2.33), we have that for any \( 0 < \sigma \leq \frac{c}{\mu(\Omega_1)} \),
\[
\mu \left( \left\{ x \in \mathbb{T} : \left| \int_{\Omega_1} (F_{n,z}(x) - nI(\zeta))d\mu(\zeta) \right| > \frac{\delta_n}{2} \right\} \right) 
\leq \exp \left( -\sigma \delta_n^2 + 21\sigma \mu(\Omega_1)n \delta_0^{(n)} \right). 
\]
Thus, we finish this proof by (2.29) and setting \( \tilde{C} = 100\mu(\Omega_1) \) with the fact that \( \delta_0^{(n)} \gg \frac{1}{n} \).
\[ \square \]

3. Large deviation theorems for \( f_n^a(x, E, \omega) \). To apply Theorem 1.4, we first need to define some subharmonic functions. Let
\[
M_n^a(x, E, \omega) := \prod_{j=1}^n a(x + j\omega) \quad M_n(x, E, \omega) 
= \prod_{j=1}^n \begin{vmatrix} v(x + j\omega) - E & a(x + j\omega) \\ a(x + (j+1)\omega) & 0 \end{vmatrix}. 
\tag{3.1}
\]
Note that a real function \( f(x) \) on \( \mathbb{T} \) has its complex analytic extension \( f(z) \) on the complex strip \( T_\rho = \{ z : |\text{Im}z| < \rho \} \) and the complex analytic extension of \( \tilde{a}(x) \) should be defined on \( T_\rho \) by
\[ \tilde{a}(z) := \overline{\tilde{a} \left( \frac{1}{z} \right)}. \]
Then, the extension of \( M_n^a(x, E, \omega) \) is
\[ M_n^a(z, E, \omega) = \prod_{j=1}^n \begin{vmatrix} v(z + j\omega) - E & \tilde{a}(z + j\omega) \\ a(z + (j+1)\omega) & 0 \end{vmatrix}, \tag{3.2} \]
where \( z + \omega \) means \( z \exp(2\pi i \omega) \). Moreover, simple computations yield that
\[
M_n^a(z, E, \omega) = \begin{pmatrix} f_n^a(z, E, \omega) & \tilde{a}(z)f_n^a(z + \omega, E, \omega) \\ a(z + n\omega)f_{n-1}^a(z, E, \omega) & -\tilde{a}(z)a(z + n\omega)f_{n-1}^a(z + \omega, E, \omega) \end{pmatrix}. \tag{3.3} \]
where
\[ f_n^a(z, E, \omega) = \det(H_n(z, \omega) - E) \]
\[
\begin{vmatrix}
 v(z + \omega) - E & -a(z + 2\omega) & 0 & \cdots & 0 \\
 -a(z + 2\omega) & v(z + 2\omega) - E & -a(z + 3\omega) & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \cdots & 0 & \cdots & v(z + n\omega) - E
\end{vmatrix}.
\]

Note that if \( \text{Im} z = 0 \), then \( H_n(z, \omega) = H_n(x, \omega) \) is Hermitian. Now with fixed \( E \) and \( \omega \), the function \( \frac{1}{n} \log \| M_N(z, E, \omega) \| \) is subharmonic. In this paper, we only need to consider \( E \in \mathcal{E} \), where
\[
\mathcal{E} := [-2]|a(x)|_{L^\infty(T)} - \|v(x)\|_{L^\infty(T)} \cdot 2\|a(x)\|_{L^\infty(T)} + \|v(x)\|_{L^\infty(T)},
\]
as the spectrum \( \mathcal{S}_\omega \subset \mathcal{E} \). Thus, for any irrational \( \omega \) and \( 1 \leq n \in \mathbb{N} \),
\[
\sup_{E \in \mathcal{E}, x \in T} \frac{1}{n} \log \| M_N^a(z, E, \omega) \| \leq M_0,
\]
where
\[
M_0 := \log \left( 3\|a\|_{L^\infty(T)} + 2\|v\|_{L^\infty(T)} \right).
\]
We also need to define the unimodular matrix
\[
M_n^a(x, E, \omega) := \frac{M_n(x, E, \omega)}{|\det M_n(x, E, \omega)|^{\frac{1}{2}}},
\]
which makes sense a.e. \( x \in T \) and has the relationship
\[
\int_T \frac{1}{n} \log \| M_n^a(x, E, \omega) \| dx = L_n(E, \omega).
\]

Then, we have the LDTs for the matrices as follows:

**Lemma 3.1.** Let \( \omega \) be the Brjuno-Rüssmann number satisfying Hypothesis H.1, or H.2, or H.3 and \( L(E, \omega) > 0 \). There exist \( \hat{c} = \hat{c}(v, a, \omega) \) and \( \hat{c} = \hat{c}(v, a, \omega) \) such that for any \( n \geq 0 \) and \( \delta > \delta_0(n) \),
\[
\text{mes} \left\{ \{u\} : |u(x, E, \omega) - \langle u \rangle| > \delta \right\} < \exp \left(-\hat{c}\delta n \right) + \exp \left(-\hat{c}\delta^2 n \right),
\]
where \( u(x, E, \omega) \) can be \( \frac{1}{n} \log \| M_n^a(x, E, \omega) \|, \frac{1}{n} \log \| M_n(x, E, \omega) \|, \frac{1}{n} \log \| M_n^u(x, E, \omega) \| \).

What’s more, there exists \( \hat{c} = \hat{c}(v, a, \omega) \) such that if \( \delta = \kappa L(E, \omega) \) with \( \kappa < \frac{1}{10} \), then the exception measure in \( (3.6) \) will be less than \( \exp \left(-\hat{c}\kappa^2 L(E, \omega) n \right) \).

**Proof.** When \( u = \frac{1}{n} \log \| M_n^a(x, E, \omega) \| \), the LDT \( (3.6) \) is about the analytic matrix; when \( u = \frac{1}{n} \log \| M_n(x, E, \omega) \| \), the LDT \( (3.6) \) is about the Jacobi cocycles; when \( u = \frac{1}{n} \log \| M_n^u(x, E, \omega) \| \), the LDT \( (3.6) \) is about the unimodular matrix to satisfy the hypotheses of Lemma 3.8, Lemma 3.9 and the Avalanche Principle(Proposition 1). In [21], our second author obtained these LDTs with finite Liouville frequency, which means that \( \beta(\omega) < \infty \). The proofs in that paper are also available here for \( \beta = 0 \) and \( \delta > \delta_0(n) \).

What’s more, the following lemma shows that \( L_n(E, \omega) \) and \( L_n^a(E, \omega) \), which is defined by
\[
L_n^a(E, \omega) = \int_T \frac{1}{n} \log \| M_n^a(x, E, \omega) \| dx,
\]
in the above LDTs can be exchanged by \( L(E, \omega) \) and \( L^n(E, \omega) \), respectively. Here,
\[
L^n(E, \omega) = \lim_{n \to \infty} L_n^a(E, \omega) = L(E, \omega) + D, \tag{3.7}
\]
and
\[
D := \int_T \log |a(x)| \, dx = \int_T \log |\bar{a}(x)| \, dx. \tag{3.8}
\]

**Lemma 3.2.** Let \( L(E, \omega) > 0 \). For any integer \( n > 1 \), we have
\[
0 \leq L_n - L = L_n^a - L^n = L_n^a - L^n < C_0 \frac{(\log n)^2}{n}
\]
where \( C_0 = C_0(a, \nu, \omega, E) \).

**Proof.** It is the same as Lemma 3.9 in [8], which was for the strong Diophantine \( \omega \). They applied the same LDTs, whose \( \delta_0(n) = \frac{(\log n)^4}{n} \) with that frequency, to obtain its proof. It is available here, since it only need the fact, which our LDTs also satisfy, that \( \delta_0(n) \) is much less than the positive Lyapunov exponent. \( \square \)

Although the details of the proof of Lemma 3.1 can be found in [21], we still give a brief introduction here, to make the readers understand the methods we apply in this section to obtain Theorem 1.1-1.3. Easy computations show that the differences between \( \frac{1}{n} \log \|M_N^a(x, E, \omega)\| \) and \( \frac{1}{n} \log \|M_N(x, E, \omega)\| \) and between \( \frac{1}{n} \log \|M_N^a(x, E, \omega)\| \) and \( \frac{1}{n} \log \|M_N^a(x, E, \omega)\| \) are constructed by the combination of \( \frac{1}{n} \sum_{j=1}^N \log |a(x + j\omega)| \) and \( \frac{1}{n} \sum_{j=1}^N \log |\bar{a}(x + j\omega)| \), whose complex extensions can be estimated by our Theorem 1.4 easily. Therefore, we only need to prove the LDT for \( \frac{1}{n} \log \|M_N^a(x, E, \omega)\| \), which also has a subharmonic extension. Due to this subharmonicity,
\[
\operatorname{mes} \left\{ x : \left| \frac{1}{n} \sum_{j=1}^{n} \log \|M_{N}^a(x + j\omega, E, \omega)\| - L_{n}^a(E, \omega) \right| > \delta \right\} < \exp(-c\delta n). \tag{3.9}
\]

On the other hand, for any \( k \in \mathbb{Z} \),
\[
- \frac{2M_0 k}{n} + \sum_{j=0}^k \frac{k-j}{nk} d(x + j\omega)
\leq \frac{1}{n} \log \|M_{N}^a(x, E, \omega)\| - \frac{1}{nk} \sum_{j=1}^{k} \log \|M_{N}^a(x + j\omega, E, \omega)\|
\leq \frac{2M_0 k}{n} - \sum_{j=0}^{k-1} \frac{k-j}{nk} d(x + (n+j-1)\omega),
\]
where \( d(x) = \log |a(x + \omega)\bar{a}(x)| \). Obviously, it also can be solved by our Theorem 1.4. Now, we can explain why we apply the BMO norm and the John-Nirenberg inequality, not the method for (3.6), to obtain the LDTs for \( f_n^a(x, E, \omega) \). The reason is that Theorem 1.4 holds for \( f_n^a(x, E, \omega) \), but we can not handle the difference between \( \frac{1}{n} \log |f_n^a(x, E, \omega)| \) and \( \frac{1}{kn} \sum_{j=1}^{k} \log |f_n^a(x + j\omega, E, \omega)| \).

Next, we will apply the analyticity of \( f_n^a(x, E, \omega) \) and the subharmonicity of \( \frac{1}{n} \log |f_n^a(x, E, \omega)| \) via the following lemmas in this paper.
**Definition 3.3.** Let \( H > 1 \). For any arbitrary subset \( B \subset D(z_0, 1) \subset \mathbb{C} \) we say \( B \in \text{Car}_1(H, K) \) if \( B \subset \bigcup_{j=1}^{j_0} D(z_j, r_j) \) with \( j_0 < K \), and
\[
\sum_j r_j < e^{-H}.
\]
(3.10)

Here \( D(z, r) \) means the complex platform center at \( z \) with radius \( r \). If \( d \) is a positive integer greater than one and \( B \subset \prod_{j=1}^{d} \subset \mathbb{C}^d \) then we define inductively that \( B \in \text{Car}_d(H, K) \) for any \( z \in \mathbb{C} \setminus B_j \), here \( B^{(j)}_j = \{(z_1, \ldots, z_d) \in B : z_j = z\} \).

**Lemma 3.4** (Cartan estimate, Lemma 2.4 in [13]). Let \( \phi(z_1, \ldots, z_d) \) be an analytic function defined in a polydisk \( P = \prod_{j=1}^{d} D(z_j, 0, 1) \), \( z_j, 0 \in \mathbb{C} \). Let \( M \geq \sup_{z \in P} \log |\phi(z)| \), \( m \leq \log |\phi(z_0)| \), \( z_0 = (z_{1,0}, \ldots, z_{d,0}) \). Given \( H > 1 \) there exists a set \( B \subset P, B \in \text{Car}_d(H^2, K), K = C_d H(M - m) \), such that
\[
\log |\phi(z)| > M - C_d H(M - m)
\]
for any \( z \in \prod_{j=1}^{d} D(z_j, 0, \frac{1}{H}) \setminus B \).

**Lemma 3.5** (Lemma 2.4 in [12]). Let \( u \) be a subharmonic function defined on \( A_{\rho} \) such that \( \sup_{A_{\rho}} u \leq M \). There exist constants \( C_1 = C_1(\rho) \) and \( C_2 \) such that, if for some \( 0 < \delta < 1 \) and some \( L \) we have
\[
\text{mes} \{ x \in \mathbb{T} : u(x) < -L \} \geq \delta,
\]
then
\[
\sup_{\mathbb{T}} u \leq C_1 M - \frac{L}{C_1 \log (C_2/\delta)}.
\]

Recalling the definitions of \( M_n(x, E, \omega), M_n^a(x, E, \omega), M_n^u(x, E, \omega) \) and the expression (3.3), we have
\[
M_n(z, E, \omega) = \left( \begin{array}{c}
\frac{f_n(z, E, \omega)}{a(z)} - \frac{\bar{a}(z)}{a(z + \omega)} f_{n-1}(z + \omega, E, \omega) \\
\frac{f_{n-1}(z, E, \omega)}{a(z + \omega)} - \frac{\bar{a}(z)}{a(z + \omega)} f_{n-2}(z + \omega, E, \omega)
\end{array} \right),
\]
(3.11)

and
\[
M_n^u(z, E, \omega) = \left( \begin{array}{c}
\frac{f_n^u(z, E, \omega)}{a(z + \omega)} - \frac{\bar{a}(z)}{a(z + \omega)} \frac{a(z + \omega)}{\bar{a}(z)} f_{n-1}^u(z + \omega, E, \omega) \\
\frac{f_{n-1}^u(z, E, \omega)}{a(z + \omega)} - \frac{\bar{a}(z)}{a(z + \omega)} \frac{a(z + \omega)}{\bar{a}(z)} a(z + (n-1)\omega) f_{n-2}^u(z + \omega, E, \omega)
\end{array} \right),
\]
where
\[
f_n(z, E, \omega) = \frac{1}{\prod_{j=1}^{n} a(z + j\omega)} f_n^a(z, E, \omega),
\]
(3.12)
and
\[
f_n^u(z, E, \omega) = \frac{1}{\prod_{j=0}^{n-1} a(z + (j+1)\omega) \bar{a}(z + j\omega)} \frac{1}{2} f_n^u(z, E, \omega)
\]
\[
= \left( \prod_{j=0}^{n-1} \frac{|a(z + (j+1)\omega)|^2 \bar{a}(z + j\omega)}{a(z + (j+1)\omega)} \right) f_n(z, E, \omega).
\]
(3.13)
Assume $L(E, \omega) = \gamma > 0$. Then, we can obtain a particular deviation theorem as follow:

**Lemma 3.6.** There exists $l_0 = l_0(a, v, \gamma)$ such that

$$\text{mes}\{x \in T : |f_l(x)| \leq \exp(-l^3)\} \leq \exp(-l)$$

for all $l \geq l_0$.

**Proof.** It is the same as Lemma 4.2 in [8], which was for the strong Diophantine $\omega$. We have the same reason which we just stated in the proof of Lemma 3.2 to omit this proof. \qed

Note that in order to simplify the notation, we suppressed the dependence on $E$ and $\omega$. We will be doing this throughout this paper if there is no confusion.

According to Lemma 3.5 and 3.6, we can have more choices of the deviation and the exceptional measure.

**Lemma 3.7.** Let $\sigma > 0$ and $g(n) > 0$. There exist constants $l_0 = l_0(a, v, \gamma)$ and $n_0 = n_0(a, v, \gamma)$ such that

$$\text{mes}\{x \in T : |f_{l_n}(x)| \leq \exp(-g(n))\} \leq \exp\left(-\frac{g(n)l^{-3}}{2}\right)$$

for any $n \geq n_0$ and for any $l_0 \leq l \lesssim g(n)$. The same result, but with possibly different $l_0$ and $n_0$, holds for $f_{l_n}$.

**Proof.** Assume

$$\text{mes}\{x \in T : |f_l(x)| \leq \exp(-g(n))\} > \exp\left(-\frac{g(n)l^{-3}}{2}\right).$$

We have that

$$|f_{l_n}(x)| = |f_l(x)| \prod_{j=1}^l |a(x + j\omega)| \leq \exp(-g(n)) C^{l-1} \leq \exp\left(-\frac{1}{2}g(n)\right)$$

on a set of measure greater than $\exp\left(-\frac{g(n)l^{-3}}{2}\right)$. By Lemma 3.5, it implies that for any $x \in T$,

$$|f_{l_n}(x)| \leq \exp\left(C_l l - \frac{g(n)}{2C_1 \log\left(C_2 \exp\left(g(n)l^{-3}\right)\right)}\right) \leq \exp\left(-Cl^3\right).$$

Due to Theorem 1.4,

$$\text{mes}\left\{x \in T : \sum_{k=1}^l \log|a(x + k\omega) - lD| > l\right\} \leq \exp(-cl).$$

Therefore, recalling (3.12), we have

$$|f_l(x)| \leq \exp\left(l(1 - D) - Cl^3\right) \leq \exp\left(-Cl^3\right)$$

for all $x$ except for a set of measure less than $\exp(-cl)$. It contradicts with the previous lemma. At last, by (3.13), we can prove the result for $f_{l_n}$ by similar methods. \qed

Now we need some facts about stability of contracting and expanding directions of unimodular matrices. It follows from the polar decomposition that if $A \in SL(2, \mathbb{C})$ then there exist unit vectors $u_+^A \perp u_-^A$ and $v_+^A \perp v_-^A$ such that $Au_+^A = \|A\| v_+^A$ and $Au_-^A = \|A\|^{-1} v_-^A$. 

Lemma 3.8 (Lemma 2.5 in [12]). For any $A, B \in SL(2, \mathbb{C})$ we have
\[
|Bu_{AB} \wedge u_A^\perp| \leq ||A||^{-2} ||B||, \quad |u_{BA} \wedge u_A^\perp| \leq ||A||^{-2} ||B||^2
\]
\[
|v_{AB}^+ \wedge v_A^\perp| \leq ||A||^{-2} ||B||^2, \quad |v_{BA}^+ \wedge v_A^\perp| \leq ||A||^{-2} ||B||.
\]

Lemma 3.9 (Lemma 4.5 in [8]). If $A \in SL(2, \mathbb{C})$ and $w_1, w_2, \text{ and } w_3$ are unit vectors in the plane then
\[
|w_1 \wedge Aw_2| \leq |w_1 \wedge Aw_3| + \sqrt{2} ||A||^{-1} |w_2 \wedge w_3|
\]
and
\[
|w_1 \wedge Aw_2| \leq |w_3 \wedge Aw_2| + \sqrt{2} ||A|| |w_1 \wedge w_3|
\]

Now, we can improve the Lemma 3.7, but the LDT is about three determinants.

Lemma 3.10. There exist constants $0 < \kappa = \kappa(\omega) < 1$, $0 < \tau = \tau(\omega) < 1$, $l_0 = l_0(a, v, \gamma)$ and $n_0 = n_0(a, v, \gamma)$ such that
\[
\operatorname{mes}\left\{ x \in \mathbb{T}: |f_n^a(x)| + |f_n^a(x + j_1 \omega)| + |f_n^a(x + j_2 \omega)| \leq \exp\left(nL_n - 100n\delta_0^{(n)}\right)\right\}
\]
\[
< \exp\left(-n^{-\kappa}\right) \quad (3.14)
\]
for any $l_0 \leq j_1 \leq j_1 + l_0 \leq j_2 \leq n^\tau$ and $n \geq n_0$.

Proof. Here we assume $\delta_0^{(n)} \geq n^{-\frac{1}{4}}$, since the proof of Lemma 4.6 in [8] can be applied without any change when $\delta \leq n^{-\frac{1}{4}}$.

For any $1 \leq j \leq n$, due to Lemma 3.1 and 3.2, choose the deviation $\delta = \frac{n\delta_0^{(n)}}{j} > \delta^{(j)}$ and then
\[
\operatorname{mes}\left\{ x : |\log ||M_n^a(x)|| - jL| > n\delta_0^{(n)}\right\}
\]
\[
< \exp\left(-\delta_0^{(n)}\right) + \exp\left(-\delta_0^{(n)}2j\right) \leq 2 \exp\left(-\delta_0^{(n)}, 2j\right)\right\}
\]
\[
< \exp\left(-n^{-\kappa}\right). \quad (3.15)
\]
Let $G_n$ be the set of points $x \in \mathbb{T}$ such that for any $1 \leq j \leq n$ and $|l| \leq 2n$,
\[
|\log ||M_n^a(x + l\omega)|| - jL| \leq n\delta_0^{(n)}
\]
and
\[
|\log |a(x + j\omega)| - D| \leq n\delta_0^{(n)}
\]
Due to (3.15) and Theorem 1.4 for $|\log |a(x)||$, we have that
\[
\operatorname{mes}(\mathbb{T} \setminus G_n) \leq 4n^2 \exp\left(-\delta_0^{(n)}, 2j\right)\right\}
\]
\[
< \exp\left(-c\delta_0^{(n)}, 2j\right)\right\}
\]
Note that $\det M_n^a(x, E, \omega) \equiv 1$. Therefore, for any $x$, $E$ and $\omega$,
\[
||M_n^a(x, E, \omega)|| = ||\left(M_n^a\right)^{-1}(x, E, \omega)||
\]
Let \{e_1, e_2\} be the standard basis of $\mathbb{R}^2$ and for any integer $j$, $u_j^+, u_j^-$, $v_j^+$ and $v_j^-$ be the unit vectors satisfying $u_j^+ \perp u_j^-$, $v_j^+ \perp v_j^-$, $M_n^a u_j^+ = \|M_n^a\| v_j^+$ and $M_n^a u_j^- = \|M_n^a\|^{-1} v_j^-$. Then
\[
f_n^a(x) = M_n^a(x) e_1 \wedge e_2 = (M_n^a(x) \left(\left(u_n^+(x) \cdot e_1\right) u_n^+(x) + \left(u_n^-(x) \cdot e_1\right) u_n^-(x)\right)) \wedge e_2
\]
\[
= (u_n^+(x) \cdot e_1) ||M_n^a(x)|| v_n^+(x) \wedge e_2 + (u_n^-(x) \cdot e_1) ||M_n^a(x)||^{-1} v_n^-(x) \wedge e_2.
If $|f_n^u(x)| \leq \exp \left( nL_n - 100n\delta_0^{(n)} \right)$, then

$$\|M_n^u(x)\| \|u_n^+(x) \cdot e_1\| |v_n^+(x) \wedge e_2| - \|M_n^u(x)\|^{-1} \|u_n^-(x) \cdot e_1\| |v_n^-(x) \wedge e_2| \leq \exp \left( nL_n - 100n\delta_0^{(n)} \right).$$

Due to Lemma 3.2, for any $x \in G_n$,

$$|u_n^-(x) \wedge e_1| |v_n^+(x) \wedge e_2| \leq \|M_n^u(x)\|^{-1} \exp \left( nL_n - 100n\delta_0^{(n)} \right) + \|M_n^u(x)\|^{-2} \leq \exp \left( n(L_n - L) - 99n\delta_0^{(n)} \right) + \exp \left( 2n\delta_0^{(n)} - 2nL \right) \leq \exp \left( -90n\delta_0^{(n)} \right).$$

Hence,

$$|u_n^-(x) \wedge e_1| \leq \exp \left( -40n\delta_0^{(n)} \right) \text{ or } |v_n^+(x) \wedge e_2| \leq \exp \left( -40n\delta_0^{(n)} \right). \quad (3.16)$$

Suppose (3.14) fails. Let $\sigma < \kappa < 1/2$. Recall $n\delta_0^{(n)} \geq n^{1-2\sigma}$ and set

$$\tilde{G}_n = \left\{ x \in G_n : |f_n^u(x)| + |f_n^u(x + j_1 \omega)| + |f_n^u(x + j_2 \omega)| \leq \exp \left( nL_n - 100n\delta_0^{(n)} \right) \right\}.$$

We have

$$\text{mes } \tilde{G}_n > \exp \left( -n^{1-\kappa} \right) - \exp \left( -n\delta_0^{(n)} \right) > \frac{1}{2} \exp \left( -n^{1-\kappa} \right).$$

If $x \in \tilde{G}_n$, then either

$$|u_n^-(x + j_1 \omega) \wedge e_1| \leq \exp \left( -40n\delta_0^{(n)} \right)$$

or

$$|v_n^+(x + j_2 \omega) \wedge e_2| \leq \exp \left( -40n\delta_0^{(n)} \right)$$

has to hold for two of the points $x, x + j_1 \omega, x + j_2 \omega$.

We first assume that

$$|u_n^-(x + j_1 \omega) \wedge e_1|, |u_n^-(x + j_2 \omega) \wedge e_1| \leq \exp \left( -40n\delta_0^{(n)} \right). \quad (3.17)$$

From Lemma 3.9 and Lemma 3.8, we have that if $x \in G_n$, then

$$|u_n^u(x + j_2 \omega) \wedge M_{j_2-j_1}^u(x + j_1 \omega)u_n^u(x + j_1 \omega)| \leq |u_n^u(x + j_2 \omega) \wedge M_{j_2-j_1}^u(x + j_1 \omega)u_n^u(x + j_1 \omega)| + C \|M_{j_2-j_1}^u(x + j_1 \omega)\| |u_n^u(x + j_1 \omega) \wedge u_n^u(x + j_1 \omega)|$$

$$= |u_n^u(x + j_2 \omega) \wedge M_{j_2-j_1}^u(x + j_1 \omega)u_n^u(x + j_1 \omega)| + C \|M_{j_2-j_1}^u(x + j_1 \omega)\| |u_n^u(x + j_1 \omega) \wedge u_n^u(x + j_1 \omega)|$$

$$\leq \|M_n^u(x + j_2 \omega)\|^2 \|M_n^u(x + j_1 \omega)\| \left( \left| -2n + j_2 - j_1 \right| L + 3n\delta_0^{(n)} \right)$$

$$+ C \exp \left( -2n + 3(j_2 - j_1) \right)L + 5n\delta_0^{(n)} \leq \exp \left( -nL \right).$$

Combined it with Lemma 3.9 and (3.17), we obtain

$$|e_1 \wedge M_{j_2-j_1}^u(x + j_1 \omega) e_1|$$
Due to the fact that
\[ |e_1 \wedge M_{j_2-j_1}^u (x + j_1 \omega) e_1| = \left| \frac{a(x + j_2 \omega)}{a(x + (j_2 - 1) \omega)} \right|^{1/2} |f_{j_2-j_1-1}^u (x + j_1 \omega)|, \]
and the setting of \( G_n \), we have
\[ |f_{j_2-j_1-1}^u (x + j_1 \omega)| \leq C \exp \left( \frac{1}{2} \left( n\delta_0^{(n)} - D \right) - 30n\delta_0^{(n)} \right) \leq \exp \left( -20n\delta_0^{(n)} \right). \]
Similarly, we can obtain
\[ |f_{j_2-j_1-1}^u (x + (n + j_1 + 1) \omega)| \leq \exp \left( -20n\delta_0^{(n)} \right), \]
if we assume that
\[ |v_n^+ (x + j_1 \omega) \wedge e_2|, \ |v_n^+ (x + j_2 \omega) \wedge e_2| \leq \exp \left( -40n\delta_0^{(n)} \right). \quad (3.18) \]
What’s more, the same type of estimates are obtained if we replace \((j_1, j_2)\) in (3.17) and (3.18) with \((0, j_1)\) or \((0, j_2)\).

In conclusion
\[ \text{mes} \left\{ x \in \mathbb{T} : |f_l^u (x)| \leq \exp \left( -20n\delta_0^{(n)} \right) \right\} > \frac{1}{2} \exp \left( -n^{1-\kappa} \right) \]
for some choice of \( l \) from \( j_1 - 1, j_2 - 1, j_2 - j_1 - 1 \). However, choosing \( g(n) = 20n\delta_0^{(n)} \) in Lemma 3.7 and \( \tau = \frac{n^{\alpha-1}}{4} \) in the hypothesis of this lemma, we have
\[ \text{mes} \left\{ x \in \mathbb{T} : |f_l^u (x)| \leq \exp \left( -20n\delta_0^{(n)} \right) \right\} \leq \exp \left( -20n\delta_0^{(n)} l^{-3} \right) \ll \exp \left( -cn^{1-\kappa} \right). \]
Thus, we complete the proof by this contradiction. \( \square \)

One of our methods to obtain a large deviation estimate for a single determinant is the \( BMO(\mathbb{T}) \) norm. \( BMO(\mathbb{T}) \) is the space of functions of bounded mean oscillation on \( \mathbb{T} \). Identifying functions that differ only by an additive constant, then norm on \( BMO(\mathbb{T}) \) is given by
\[ \|f\|_{BMO(\mathbb{T})} := \sup_{l \in \mathbb{T}} \frac{1}{|l|} \int_l |f - \langle f \rangle_l| |dx|, \quad (3.19) \]
where \( \langle f \rangle_l := \int_l f(x) |dx| \). Applying the previous lemma, we obtain the following lower bound of the mean value of \( \frac{1}{n} |f_n^u (x)| \), which will help us estimate the BMO norm.
Lemma 3.11. There exist constants $0 < c_0 = c_0(\omega) \leq 1$ and $n_0 = n_0(a, v, \gamma)$ such that for $n \geq n_0$ we have
\[
\int_{\mathbb{R}} \frac{1}{n} |f_n^u(x)| \, dx > L_n - \left( \delta_n(0) \right)^{c_0}.
\]

Proof. Set
\[
\Omega_n := \left\{ x \in \mathcal{G}_n : \min \left\{ |f_n^u(x + j_1\omega)| + |f_n^u(x + j_2\omega)| + |f_n^u(x + j_3\omega)| : 0 < j_1 < j_1 + l_0 \leq j_2 < j_2 + l_0 \leq j_3 \leq n \right\} > \exp \left( nL_n - 100n\delta_0(n) \right) \right\}.
\]
Then, \( \text{mes} \left( T \setminus \Omega_n \right) \leq n^{\gamma} \exp \left( -n^{1-\gamma} \right) < \exp \left( -\frac{n^{1-\gamma}}{2} \right) \).

Define $\nu_n^u(x) = \log |f_n^u(x)|/n$ and set $M = \left\lfloor \frac{n^{\gamma}}{t_0} \right\rfloor \geq n^{\tilde{\gamma}}$ for large $n$. For any $x \in \Omega_n$ we have that $\nu_n^u(x) + kl_0\omega > L_n - 100\delta_0(n) - \frac{\log 3}{n}$ for all but at most two $k$’s, $1 \leq k \leq M$. We have
\[
\langle \nu_n^u \rangle = \int_T \nu_n^u(x) \, dx = \frac{1}{M} \sum_{k=1}^M \int_T u(x + kl_0\omega) \, dx
\]
\[
\geq \int_{\Omega_n} \left( \frac{M - 2}{M} \right) \left( L_n - 100\delta_0(n) - \frac{\log 3}{n} \right) + \frac{2}{M} \inf_{1 \leq k \leq M} \nu_n^u(x + kl_0\omega) \, dx
\]
\[
+ \frac{1}{M} \sum_{k=1}^M \int_{T \setminus \Omega_n} \nu_n^u(x + kl_0\omega) \, dx. \tag{3.20}
\]

Define $\nu_n^a(x) = \log |f_n^a(x)|/n$. Note that $\nu_n^a(x)$ can be extended to the complex trip $T_p$ where $\nu_n^a(z)$ is subharmonic. Due to (3.3), we have that
\[
S := \sup_{z \in A_{n_0}} \nu_n^a(z) \leq \sup_{z \in A_{n_0}} \frac{1}{n} \log \| M_n^a(z) \| < M_0.
\]

Applying Cartan’s estimate, Lemma 3.4, to $f_n^a(z)$ with $M = Sn$, $m = < \nu_n^a > n$ and $H = n^{\tilde{\gamma}}$, we have
\[
\inf_{1 \leq k \leq M} \nu_n^a(x + kl_0\omega) \geq S - C(S - \langle \nu \rangle) n^{\tilde{\gamma}} > -C(2|S| - \langle \nu_n^a \rangle) n^{\tilde{\gamma}} \tag{3.21}
\]
up to a set not exceeding $CM \exp \left( -n^{\tilde{\gamma}} \right)$ in measure. Combining it with the relationship that
\[
\nu_n^a(x) = \nu_n^a(x) - \frac{1}{2n} \left( \sum_{j=1}^n + \sum_{j=0}^{n-1} \right) \log |a(x + j\omega)| \tag{3.22}
\]
and applying (3.21) and Theorem 1.4 for $\frac{1}{2n}(\sum_{j=1}^n + \sum_{j=0}^{n-1}) \log |a(x + j\omega)|$ with deviation $|D|$, we have
\[
\inf_{1 \leq k \leq M} \nu_n^a(x + kl_0\omega) > -C(2|S| - \langle \nu_n^a \rangle) n^{\tilde{\gamma}} - 2|D| > -C' n^{\tilde{\gamma}}
\]
up to a set $B_n$ not exceeding $CM \exp \left( -n^{\tilde{\gamma}} \right) + \exp(-c|D|n) < \exp \left( -\frac{n^{1-\gamma}}{2} \right)$ in measure. Therefore,
\[
\langle \nu_n^u \rangle \geq \left( 1 - \frac{2}{M} \right) \left( L_n - 100\delta_0(n) - \frac{\log 3}{n} \right) - \frac{C' n^{\tilde{\gamma}}}{M} - \frac{2}{M} \sum_{k=1}^M \int_{\Omega_n \cup B_n} |\nu_n^u(x + kl_0\omega)|.
\]
Let \( g(n) = n^3 \) in Lemma 3.7. Then simple calculations shows that \( \| u_n^u \|_{L^2(T)} \leq Cn^3 \). Thus,

\[
\int_{\Omega_n^\epsilon \cup B_n} |\mu_n^u(x + kl_0\omega)| \, dx \leq (\text{mes} \{ \Omega_n^\epsilon \cup B_n \})^{1/2} \| u \|_{L^2(T)}
\]

\[
\leq Cn^3 \exp \left( -\frac{1}{4} n^\frac{2}{3} \right) \leq C \exp \left( -\frac{1}{8} n^\frac{2}{3} \right).
\]

Above all,

\[
\langle \nu_n^u \rangle \geq L_n - 100\delta_0^{(n)} - \frac{2}{M} L_n - C' n^{-\frac{2}{3}} - C \exp \left( \frac{1}{8} n^\frac{2}{3} \right) \geq L_n - \left( \delta_0^{(n)} \right)^c_0. \tag{3.23}
\]

**Remark 5.** Due to the setting of \( \tau \) and (3.23), easy computations shows that

\[
c_0 = \begin{cases} 
A, & \text{if } \Delta(t) > t^5; \\
\frac{1}{5}, & \text{if } \Delta(t) \sim t^4, \ 1 < A < 5.
\end{cases}
\]

We will show that the supermum of the subharmonic function \( u_n^a(z, E, \omega) \) on \( T \) is closed to its mean value. Here, we will apply the property that a subharmonic function at a point is small than the its integration on the platform center at that point. From the proof of Theorem 1.4, it is easily seen that the sharp LDT for \( u_n^a(x) \) can be extended to the complex region \( T_{\rho} \):

\[
\text{mes} \{ x : |u_n^a(re(x), E, \omega) - L_n^a(r, E, \omega)| > \delta \}
\]

\[
< \exp(-\delta^2) + \exp(-\delta^2 n), \ \forall \delta > \delta_0^{(n)}, \tag{3.24}
\]

where

\[
L_n^a(r, E, \omega) = \int_T u_n^a(re(x), E, \omega) \, dx.
\]

Lemma 4.1 in [12] proved that there exists \( C_0 = C_0(M_0, \rho) \) such that for any \( r_1, r_2 \in (1 - \rho, 1 + \rho) \) we have

\[
|L_n^a(r_1) - L_n^a(r_2)| \leq C_0 |r_1 - r_2|. \tag{3.25}
\]

**Lemma 3.12.** For any integer \( n > 1 \) we have that

\[
\sup_{x \in T} \log \| M_n^a(x) \| \leq nL_n^a + 2n\delta_0^{(n)}.
\]

**Proof.** Due to (3.24) with \( \delta = \delta_0^{(n)} \), we have

\[
\log \| M_n^a(re(x)) \| - nL_n^a(r) \leq n\delta_0^{(n)}
\]

except for a set of measure less than \( \exp(-\delta n\delta_0^{(n)}) + \exp(-\delta_0^{(n)^2} n) \). By the subharmonicity of \( \log \| M_n^a \| \) we have

\[
\log \| M_n^a(x) \| - nL_n^a \leq \frac{1}{\pi n^{-2}} \int_{D(x, n^{-1})} (\log \| M_n^a \| - nL_n^a) \, dA(z)
\]

\[
\leq \frac{1}{\pi n^{-2}} \int_{1-n^{-1}}^{1+n^{-1}} \int_{x-2n^{-1}}^{x+2n^{-1}} |\log \| M_n^a \| - L_n^a| \, dy \, dr. \tag{3.26}
\]

For \( r \in (1 - n^{-1}, 1 + n^{-1}) \) we have

\[
\int_{x-2n^{-1}}^{x+2n^{-1}} |\log \| M_n^a \| - L_n^a| \, dy
\]
\[
\begin{align*}
\leq \int_{x-2n^{-1}}^{x+2n^{-1}} \left| \log |M_n\phi (ry)| - L_n \phi \right| dy + |L_n \phi - L_n \phi| \\
\leq n\delta_0^{(n)} + C_n \left[ \exp \left( -\frac{c}{2} n\delta_0^{(n)} \right) + \exp \left( -\frac{c}{2} \left( \delta_0^{(n)} \right)^2 n \right) \right] + C_3 n^{-1} < 2n\delta_0^{(n)}. \quad \Box
\end{align*}
\]

Then, we will use the following lemma proved by Bourgain, Goldstein and Schlag in [7], not the definition, to calculate the BMO norm of subharmonic functions.

**Lemma 3.13 (Lemma 2.3 in [7]).** Suppose \( u \) is subharmonic on \( \mathbb{T}_\rho \), with \( \mu(\mathbb{T}_\rho) + \sup_{z \in \mathbb{T}_\rho} h(z) \leq n \) where \( \mu(\mathbb{T}_\rho) \) and \( h(z) \) comes from Lemma 2.4. Furthermore, assume that \( u = u_0 + u_1 \), where

\[
\|u_0 - \langle u_0 \rangle\|_{L^\infty(\mathbb{T})} \leq \epsilon_0 \quad \text{and} \quad \|u_1\|_{L^1(\mathbb{T})} \leq \epsilon_1. \quad (3.27)
\]

Then for some constant \( C_\rho \) depending only on \( \rho \),

\[
\|u\|_{BMO(\mathbb{T})} \leq C_\rho \left( \epsilon_0 \log \left( \frac{n}{\epsilon_1} \right) + \sqrt{n\epsilon_1} \right).
\]

**Lemma 3.14.** There exist constant \( c_1 = c_1(a,v,E,\rho,\gamma) \) and absolute constant \( C \) such that for every integer \( n \) and any \( \delta > 0 \) we have

\[
\text{mes} \{ x \in \mathbb{T} : |\log |f_n^a(x)| - \langle |f_n^a| \rangle | > n\delta \} \leq C \exp \left( -c_1 \delta (\delta_0^{(n)})^{-c_0} \right).
\]

where \( c_0 \) comes from Remark 5. The same estimate with possibly different \( c_1 \) holds for \( f_n^a \).

**Proof.** It is enough to establish the estimate for \( n \) large enough. By Lemma 3.11 and Lemma 3.12,

\[
\left\{ \begin{array}{l}
\langle \nu_n^a \rangle \geq L_n^\rho - \left( \delta_0^{(n)} \right)^{c_0} \\
\sup_{T} \nu_n^a \leq L_n^\rho + 2\delta_0^{(n)}.
\end{array} \right.
\]

This implies that

\[
\|\nu_n^a - \langle \nu_n^a \rangle\|_{L^1(\mathbb{T})} \leq 3 \left( \delta_0^{(n)} \right)^{c_0}.
\]

Due to Lemma 3.13 with setting \( \epsilon_0 = 0 \), we have

\[
\|\nu_n^a\|_{BMO(\mathbb{T})} = \|\nu_n^a - \langle \nu_n^a \rangle\|_{BMO(\mathbb{T})} \leq C_\rho \|\nu_n^a - \langle \nu_n^a \rangle\|_{L^1(\mathbb{T})}^{1/2} \leq 3C_\rho \left( \delta_0^{(n)} \right)^{c_0}.
\]

Then, the well-known John-Nirenberg inequality tells us how to apply this MBO norm to obtain the large deviation theorem: Let \( f \) be a function of bounded mean oscillation on \( \mathbb{T} \). Then there exist the absolute constants \( C \) and \( c \) such that for any \( \gamma > 0 \)

\[
\text{mes} \{ x \in \mathbb{T} : |f(x) - \langle f \rangle| > \gamma \} \leq C \exp \left( -\frac{c\gamma}{\|f\|_{BMO} \} \right). \quad (3.28)
\]

Thus,

\[
\text{mes} \{ x \in \mathbb{T} : |\nu_n^a(x) - \langle \nu_n^a \rangle| > \delta \} \leq C \exp \left( -c_1 \delta (\delta_0^{(n)})^{-c_0} \right). \quad \Box
\]

Now, due to the above proof and Remark 5, to prove Theorem 1.1-1.3, the only thing we need to do is obtain \( \|\frac{1}{2} \log |f_n^a|\|_{BMO} = O(\delta_0^{(n)}) \), when \( \Delta(t) \sim t^A \) and \( 1 < A < 5 \). In the following proof, we will use the Avalanche Principle to refine the previous estimation:
**Proposition 1** (Avalanche Principle). Let $A_1, \ldots, A_n$ be a sequence of $2 \times 2$-matrices whose determinants satisfy

$$\max_{1 \leq j \leq n} |\det A_j| \leq 1. \quad (3.29)$$

Suppose that

$$\min_{1 \leq j \leq n} \|A_j\| \geq H > n \quad \text{and} \quad \max_{1 \leq j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|] < \frac{1}{2} \log H. \quad (3.30, 3.31)$$

Then

$$\left|\log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\|\right| < C \frac{n}{H} \quad (3.32)$$

with some absolute constant $C$.

**The Proof of Theorem 1.1 to 1.3.** Define

$$\left[\begin{array}{cc} f^n_i(x) & 0 \\ 0 & 0 \end{array}\right] M^n_i(x) \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] =: M^n_i(x).$$

and $M^n$ analogously. Obviously, $|f^n(x)| = ||M^n_i(x)||$. Let $c'$ be a small constant constant, $l \sim n^{c''}$ be an integer and $n = l + (m - 2)l + l'$ with $2l \leq l' \leq 3l$. Set

$$A^n_j(x) = M^n_j(x + (j - 1)l\omega), j = 2, \ldots, m - 1,$$

$$A^n_1(x) = M^n_1(x) \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{cc} f^n_1(x) & 0 \\ 0 & 0 \end{array}\right],$$

and

$$A^n_m(x) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] M^n_1(x + (m - 1)l\omega) = \left[\begin{array}{cc} f^n_1(x + (m - 1)l\omega) & * \\ 0 & 0 \end{array}\right].$$

The matrices $A^n_j$ have similar definitions. By Lemma 3.1, for any $j = 2, \ldots, m - 1,$

$$\text{mes} \left\{ x : \frac{1}{l} \log \|A_j(x)\| - L_l > \frac{1}{20} L_l \right\} < \exp (-cL_l).$$

Due to the fact that

$$\log |f^n_1(x)| \leq \log \|A^n_1(x)\| \leq \log \|M^n_1(x)\|,$$

Lemma 3.14, 3.11 and 3.1, we have

$$\text{mes} \left\{ x : \frac{1}{l} \log \|A_1(x)\| - L_l > \frac{1}{10} L_l \right\} < \exp \left(-cL_l \left(\delta_0 \right)^{-c_0} \right),$$

and an analogous estimate for $\log ||A^n_n||$. Now the hypothesis of Avalanche Principle are satisfied and hence

$$\log ||M^n_n(x)|| + \sum_{j=2}^{m-1} \log ||A^n_j(x)|| - \sum_{j=1}^{m-1} \log ||A^n_{j+1}(x) A^n_j(x)|| = O \left(\frac{1}{l} \right) \quad (3.33)$$

up to a set of measure less than $3m \exp(-cL_l \left(\delta_0 \right)^{-c_0})$. By the definitions of $M^n$ and $M^n_n$, easy computations show that

$$\log ||M^n_n(x)|| + \sum_{j=2}^{m-1} \log ||A^n_j(x)|| - \sum_{j=1}^{m-1} \log ||A^n_{j+1}(x) A^n_j(x)||.$$
\[
= \log \| \mathcal{M}_n^a (x) \| + \sum_{j=2}^{m-1} \log \| A_j^a (x) \| - \sum_{j=1}^{m-1} \log \| A_j^a + 1 (x) A_j^a (x) \|.
\]

Thus, (3.33) also holds for \( \mathcal{M}_n^a \). If we set
\[
 u_0 (x) = \log \| A_m^a (x) A_{m-1}^a (x) \| + \log \| A_1^a (x) A_1^a (x) \|,
\]
then the previous relation can be rewritten as
\[
\log \| \mathcal{M}_n^a (x) \| + \sum_{j=2}^{m-1} \log \| M_j^a (x + (j-1) \omega) \|
- \sum_{j=2}^{m-2} \log \| M_{2j}^a (x + (j-1) \omega) \| - u_0 (x) = O \left( \frac{1}{l} \right).
\]

Similarly, for any \( 0 \leq k < \ell - 1 \),
\[
\log \| \mathcal{M}_n^a (x) \| + \sum_{j=2}^{m-1} \log \| M_j^a (x + \omega j + (j-1) \omega) \|
- \sum_{j=2}^{m-2} \log \| M_{2j}^a (x + \omega j + (j-1) \omega) \| - u_k (x) = O \left( \frac{1}{l} \right),
\]
where
\[
u_k (x) = \log \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} M_{\ell - k}^a (x + \omega (m-1) \omega) \cdot A_{m-1}^a (x + \omega) \right\|
+ \log \left\| A_2^a (x + \omega) \cdot M_{1-k}^a (x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|
\]
which means that we decrease the length of \( A_m^a \) by \( k \) and increase the length of \( A_1^a \) by \( k \). Adding these equations and dividing by \( l \) yields
\[
\log \| \mathcal{M}_n^a (x) \| + \sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \| M_j^a (x + j \omega) \|
- \sum_{j=l}^{(m-2)l-1} \frac{1}{l} \log \| M_{2j}^a (x + j \omega) \| - \sum_{k=0}^{l-1} \frac{1}{l} u_k (x) = O \left( \frac{1}{l} \right)
\]
up to a set of measure less than \( 3n \exp (c L ((\delta_{0}^l)^{c_{0}})) \). Note that Theorem 1.4 can be applied for \( \sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \| M_j^a (x + j \omega) \| \) and \( \sum_{j=l}^{(m-2)l-1} \frac{1}{l} \log \| M_{2j}^a (x + j \omega) \| \)
and \( ml \sim n \). So, the deviation \( \delta \) is the smallest deviation \( \delta_{0}^{(m)} \) we can choose here. Then,
\[
\sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \| M_j^a (x + j \omega) \| - \sum_{j=l}^{(m-2)l-1} \frac{1}{l} \log \| M_{2j}^a (x + j \omega) \|
= (m-2) l L_l^a - (m-3) l L_{2l}^a + O \left( n_0^{(m)} \right)
\]
up to a set of measure less than \( \exp (c n_0^{(m)}) \). Note that \( u_k , k = 0 , \ldots , l-1 \) have the subharmonic extensions. Therefore, for any \( \frac{n}{k} \), Theorem 1.4 can be applied.
with \( n = 1 \) and \( \delta = \frac{n\delta_0(n)}{l} \), and obtain that
\[
\sum_{k=0}^{l-1} \frac{1}{l} u_k(x) - \sum_{k=0}^{l-1} \frac{1}{l} \langle u_k \rangle = O\left(\frac{n\delta_0(n)}{l}\right)
\]
up to a set of measure less than \( l \exp(-cn^{1-c}\delta_0(n)) \). Thus, combining these equations, we have that
\[
\log |f_n^a(x)| + (m-2) LL_1^a - (m-3) LL_2^a - \sum_{k=0}^{l-1} \frac{1}{l} \langle u_k \rangle = O\left(\frac{n\delta_0(n)}{l}\right)
\]
up to a set \( B \), satisfying that
\[
\text{mes}(B) \leq 3n \exp\left(-cL_1 \left(\frac{n\delta_0}{cn}\right)^{l-\alpha_0}\right) + l \exp\left(-cn^{1-c}\delta_0(n)\right) + \exp\left(-cn\delta_0(n)\right).
\]
Recalling that \( \delta_0(n) = C_n n^{-\frac{d}{2}+} \), \( \alpha_0 = \frac{1}{2} \) and \( l \sim n^{c'} \), we have \( \text{mes}(B) \leq \exp\left(-c''n^{-c'}\right) \), where \( c'' \) is a small constant depending on \( a, v, \omega \) and \( E \). Integrating (3.34) and using the fact that \( \|\log |f_n^a|\|_{L^\infty(\mathbb{T})} \leq Cn \), yields
\[
\langle \log |f_n^a(x)| \rangle + (m-2) LL_1^a - (m-3) LL_2^a - \sum_{k=0}^{l-1} \frac{1}{l} \langle u_k \rangle = O\left(\frac{n\delta_0(n)}{l}\right) + Cn \exp\left(-c''n^{c'}\right).
\]
Combining it with (3.34), we have
\[
\left| \log |f_n^a(x)| - \langle \log |f_n^a| \rangle \right| = O\left(\frac{n\delta_0(n)}{l}\right)
\]
up to \( B \). Define
\[
\frac{1}{n} \log |f_n^a| - \left\langle \frac{1}{n} \log |f_n^a| \right\rangle = u_0 + u_1
\]
where \( u_0 = 0 \) on \( B \) and \( u_1 = 0 \) on \( \mathbb{T} \setminus B \). Obviously, \( \|u_0 - \langle u_0 \rangle\|_{L^\infty(\mathbb{T})} = O(\delta_0(n)) \) and
\[
\|u_1\|_{L^2(\mathbb{T})} \leq C\sqrt{\text{mes}(B)} \leq C \exp\left(-c''n^{c'}\right).
\]
Due to Lemma 3.13,
\[
\|\log |f_n^a|\|_{BMO(\mathbb{T})} = O\left(\frac{n\delta_0(n)}{l}\right).
\]

Similar to Lemma 3.2, we also can prove that \( \left\langle \frac{1}{n} \log |f_n^a| \right\rangle \) in Theorems 1.1-1.3 can be exchanged by \( L^a \).

**Lemma 3.15.** There exists a constant \( C_0 = C_0(a, v, E, \omega, \gamma) \) such that
\[
\left| \langle \log |f_n^a| \rangle - nL_n^a \right| \leq C_0
\]
for all integers.

**Proof.** Recall that
\[
\log ||\mathcal{M}_n^a(x)|| + \sum_{j=2}^{m-1} \log ||A_j^a(x)|| - \sum_{j=1}^{m-1} \log ||A_j^a(x) A_{j+1}^a(x)|| = O\left(\frac{1}{l}\right)
\]
up to a set of measure less than $3m \exp(-cL_l(\delta_0^{(n)}))$. Similarly,

$$\log \|M_n^a(x)\| + \sum_{j=2}^{m-1} \log \|A_j^a(x)\| - \sum_{j=1}^{m-1} \log \|A_{j+1}^a(x) A_j^a(x)\|$$

$$- \log \|M_l^a(x+(m-1)\omega)M_l^a(x+(m-2)\omega)\| = O\left(\frac{1}{l}\right)$$

up to a set of measure less than $3m \exp(-cL_l(E))$. Subtracting these two expressions and then integrating, yields

$$|\langle \log |f_n^a| \rangle - nL_n^a| \leq CR(l) + O\left(\frac{1}{l}\right)$$

where

$$R(n) = \sup_{n/2 \leq m \leq n} |\langle \log |f_m^a| \rangle - mL_m^a|, \text{ and } \log n \ll l \ll n.$$ 

Then, our conclusion is obtained by iterating this estimate. \hfill \Box

4. The proof of Theorem 1.5. We used the LDTs and the Avalanche Principle together in the above two proofs. As we have mentioned in the introduction, this method was first created in [11] to prove the Hölder continuity of Lyapunov exponent $L^*(E,\omega)$ in $E$ with the strong Diophantine $\omega$. Recently, our second author also applied it to obtain the same continuity of $L(E,\omega)$ with any irrational $\omega$ in [21]. Our proof of Theorem 1.5 needs this result. Therefore, we list it as a lemma:

**Lemma 4.1.** Assume $\beta(\omega) = 0$ and $L(E_0,\omega) > 0$. There exists $r_E = r_E(a,v,E_0,\omega)$ such that for any $|E - E_0| \leq r_E$,

$$\frac{3}{4} L(E_0,\omega) < L(E,\omega) < \frac{5}{4} L(E_0,\omega).$$

Furthermore, there exists a constant $h = h(a,v)$ called Hölder exponent such that for any $E_1, E_2 \in [E_0 - r_E, E_0 + r_E]$,

$$|L(E_1,\omega) - L(E_2,\omega)| < |E_1 - E_2|^h. \quad (4.1)$$

**The proof of Theorem 1.5.** From Theorem 1.1-1.3 and Lemma 3.15, we have that for any $\delta > \delta_0^{(n)}$ and $(x,E) \in \mathbb{T} \times \mathcal{E}$ except for a set of measure $C \exp(-c\delta(\delta_0^{(n)})^{-1})$,

$$|\log |f_n^a(x,E,\omega)| - nL_n^a(E,\omega)| \leq n\delta. \quad (4.2)$$

Then, due to Fubini’s Theorem and Chebyshev’s inequality, there exists a set $\mathcal{B}_{n,\delta} \subset \mathbb{T}$ with $\text{mes} \mathcal{B}_{n,\delta} < C \exp(-c\delta(\delta_0^{(n)})^{-1})$, such that for each $x \in \mathbb{T} \setminus \mathcal{B}_{n,\delta}$ there exists $\mathcal{E}_{n,\delta,x} \subset \mathcal{D}$, with $\text{mes} \mathcal{E}_{n,\delta,x} < C \exp(-c\delta(\delta_0^{(n)})^{-1})$ such that (4.2) holds for any $E \in \mathcal{E} \setminus \mathcal{E}_{n,\delta,x}$. Therefore, there exist $x_1, E_1$ satisfying

$$|x_1 - x_0| \leq C \exp\left(-c \left(\frac{\delta_0^{(n)}}{c_0}\right)^{-\frac{1}{2}}\right),$$

and

$$|E_1 - E_0| \leq C \exp\left(-c \left(\frac{\delta_0^{(n)}}{c_0}\right)^{-\frac{1}{2}}\right),$$

such that

$$\log |f_n^a(x_1,E_1)| \geq nL_n(E_1) - n \left(\frac{\delta_0^{(n)}}{c_0}\right)^{\frac{1}{2}}. \quad (4.3)$$

Define

$$R := \left(\frac{\delta_0^{(n)}}{c_0}\right)^{\frac{1}{2}} \gg C \exp\left(-c_0 \left(\frac{\delta_0^{(n)}}{c_0}\right)^{-\frac{1}{2}}\right),$$
and
\[ N_{x,E}(r) = \# \{ E : f_n(x, E') = 0, |E' - E| \leq r \}. \]

The Jensen formula states that for any function \( f \) analytic on a neighborhood of \( D(z_0, R) \), see [19],
\[ \int_0^1 \log |f(z_0 + Re(\theta))| d\theta - \log |f(z_0)| = \sum_{\zeta : f(\zeta) = 0} \log \frac{R}{|\zeta - z_0|} \]
provided \( f(z_0) \neq 0 \). Thus, we have that
\[ N_{x,E_i}(3R) \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f_n(x_1, E_1 + 4Re^{i\theta})| d\theta - \log |f_n(x_1, E_1)|. \]

By Lemma 3.12, it yields
\[ N_{x_1,E_1}(3R) \leq \left( \sup_{|E - E_1| = 4R} (n (L_n^a(E) - L_n^a(E_1))) \right) + 3n\delta_0^{(n)}. \]

Due to Lemma 4.1, if \( |E_1 - E_2| < 4R \), then
\[ |L(E_1) - L(E_2)| < |E_1 - E_2|^b < 4\delta_0^{(n)}. \]

Combining it with Lemma 3.2 and the fact that \( \delta_0^{(n)} \gg \frac{(\log n)^2}{n} \), we have
\[ \left( \sup_{|E - E_1| = 4R} (n (L_n^a(E) - L_n^a(E_1))) \right) < 10n\delta_0^{(n)}. \]

Thus,
\[ N_{x_1,E_1}(3R) \leq 13n\delta_0^{(n)}. \]

Recalling that \( |E_0 - E_1| \ll R \), we have
\[ N_{x_1,E_0}(2R) \leq N_{x_1,E_1}(3R) \leq 13n\delta_0^{(n)}. \]

Note that \( H_n(x, \omega) \) is Hermitian. Thus, by the Mean Value Theorem,
\[ \|H_n(x_0, \omega) - H_n(x_1, \omega)\| \leq C |x_0 - x_1| \leq C \exp \left( -c_0 \left( \frac{1}{\delta_0^{(n)}} \right)^{-\frac{1}{2}} \right). \]

Let \( E_j^{(n)}(x, \omega), j = 1, \ldots, n \) be the eigenvalues of \( H_n(x, \omega) \) ordered increasingly. Then,
\[ \left| E_j^{(n)}(x_0) - E_j^{(n)}(x_1) \right| \leq C \exp \left( -c_0 \left( \frac{1}{\delta_0^{(n)}} \right)^{-\frac{1}{2}} \right). \]

This implies that
\[ N_{x_0,E_0}(R) \leq N_{x_1,E_0}(2R) < 13n\delta_0^{(n)}. \]

**Remark 6.** Similarly,
\[ \# \left\{ z \in \mathbb{C} : f_n^a(z, E_0, \omega) = 0, |z - x_0| < \left( \frac{1}{\delta_0^{(n)}} \right)^{\frac{1}{2}} \right\} \leq 13n\delta_0^{(n)}. \]

**Acknowledgments.** We would like to thank the referees very much for their valuable comments and suggestions.
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Received October 2019; revised July 2020.

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