Lens reconstruction and source redshift distribution

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Abstract. The currently used linear and nonlinear lens inversion techniques are based on distortion estimators whose complicated source redshift dependence makes the influence of the redshift distribution of the sources difficult to take into account and to analyze. However, the lens equations can be explicitly averaged over the source redshift distribution by a suitable choice of the estimators. Lens reconstruction procedures are outlined, which make use of all the information and in which all the unknown quantities of the problem can be recovered simultaneously in both the linear and nonlinear regime, for either noncritical or critical lenses. These procedures require no prior knowledge of the redshift distribution of the sources, but are possibly dependent on the Cosmology in some instances. Possible methods of recovery of the source redshift distribution itself are briefly discussed at the same time.

Key words: Cosmology – Dark Matter – Gravitational lensing

1. Introduction

A decade after the discovery of giant gravitational arcs in distant clusters of galaxies (Soucail et al. 1987, Lynds and Petrosian 1986), the strong and weak lensing distortions induced by the distribution of foreground dark matter on the position and shape of background sources have become one of the major tools in modern observational Cosmology. Early work (e.g., Kochanek 1990, or Miralda-Escudé 1991, 1993) has focused on parametric reconstruction of the lens properties; a significant progress along this line has been performed by the identification and use of multiple images of a same source (e.g., Mellier et al. 1993, Kneib et al. 1993, 1995). Two major breakthroughs have been made in the recent years: on the theoretical side, powerful linear and nonlinear nonparametric lens reconstruction methods have been proposed (e.g., Kaiser and Squires, 1993; Kaiser 1995; Seitz and Schneider 1995), while on the observational side, new and impressive techniques have been developed to measure the distortion field in the weak lensing regime from the distortion of individual sources (Bonnet et al. 1994) and from the autocorrelation function of whole images (Van Waerbeke et al. 1997). Finally the end of the long cosmological quest for the magic numbers of the Universe (Ω and Λ; see, e.g., Fort et al. 1997) and even the determination of the power spectrum of large scale structures (Bernardeau et al. 1997) seem within reach of the lensing approach.

However, a remaining vexing difficulty plagues these attempts, namely the complicated dependence of the distortion estimators on the background source redshifts, which makes the influence of the source redshift distribution on the lensing analyses difficult to quantify (see, e.g., Luppino and Kaiser, 1996). The purpose of this Letter is to show how this difficulty is resolved by an appropriate choice of the distortion estimators (this was already partially pointed out by Kochanek 1990; however, the implications for the elaboration of reconstruction procedures with a distribution of source redshifts seem to have gone unnoticed; for an alternative approach, see Seitz and Schneider 1997). Sect. 2 is devoted to an exposition of the lensing and deformation equations used in the paper, both to specify the adopted notation and introduce the distortion estimators. Sect. 3 begins with a presentation of the relevant probability distributions of images and sources, and moves on to the statistical properties of the chosen distortion estimators. Sect. 4 discusses these results and concludes this Letter by outlining possible methods of recovery of the lens projected mass distribution and of the source redshift distribution for noncritical as well as critical lenses.

2. Lensing equations and notations:

In what follows, we make use of quantities which can be defined in similar ways for the sources, the images and the lens. Such quantities are accordingly indexed with a subscript or a superscript “I”, “S” and “L”. Latin subscripts refer to the angular directions in the plane of the
sky, which are labeled $x_1$ and $x_2$. With these definitions, the lens equation reads
\[ X^S = X^I - \nabla \phi(X^I), \] (1)
where the bidimensional lensing potential $\phi$ is related to the lens surface density $\Sigma$ by the bidimensional Poisson equation
\[ \Delta \phi = 2 \frac{\Sigma}{\Sigma_c}. \] (2)
The critical surface density $\Sigma_c$ just introduced is defined by
\[ \Sigma_c = \frac{c^2}{2\pi G} \frac{D_S}{D_L D_S}, \] (3)
where $D_L$, $D_S$, and $D_{LS}$ are the angular-diameter distance of the observer to the lens, the observer to the source, and the lens to the source, respectively.

Extended objects are characterized by their surface brightness $I_S(x_S) = I_I(x^I)$, the position of their centers in the source plane and in the image plane
\[ X^{S,I}_{ij} = \int \frac{I_{I,S}(x^{I,S}) x^{I,S}_i x^{I,S}_j d^2 x^{I,S}}{\int I_{I,S}(x^{I,S}) d^2 x^{I,S}}, \] (4)
and their shape matrix
\[ M_{ij}^{S,I} = \int \frac{I_{I,S}(x^{I,S}) (x^{I,S}_i - X^{I,S}_i) (x^{I,S}_j - X^{I,S}_j) d^2 x^{I,S}}{\int I_{I,S}(x^{I,S}) d^2 x^{I,S}}. \] (5)

Following Kneib et al. (1994), it is convenient to express the normalised source and image shape matrices $M / (\det M)^{1/2}$ in terms of the orientation of the principal axes, $\theta$ (defined such that $\tan(2\theta) = 2M_{12}/(M_{11} - M_{22})$), and the major axis $a_0$ and minor axis $b_0$ of the equivalent ellipse as
\[ \frac{M}{(\det M)^{1/2}} = \begin{pmatrix} \delta + |\tau| \cos(2\theta) & |\tau| \sin(2\theta) \\ |\tau| \sin(2\theta) & \delta - |\tau| \cos(2\theta) \end{pmatrix}. \] (6)
where the modulus of the complex quantity $\tau \equiv |\tau| \exp(2i\theta)$ is given by
\[ |\tau| = \frac{a_0^2 - b_0^2}{2a_0b_0}, \] (7)
and
\[ \delta = \frac{a_0^2 + b_0^2}{2a_0b_0} (1 + |\tau|^2)^{1/2}. \] (8)

Finally, one can define a last observable, $\sigma$, which measures the amplitude of the deformation matrix:
\[ \sigma = 8(\det M)^{1/2} = 2ab. \] (9)

When the distortions are moderate (so that the amplification matrix can be assumed to be nearly constant across the image), the source and image shape matrices $M$ are related by
\[ M^S = a^{-1} M^I a^{-1}, \] (10)
where the inverse of the amplification matrix $a^{-1}$ reads
\[ a^{-1} = \begin{pmatrix} 1 - \phi_{11} & -\phi_{12} \\ -\phi_{12} & 1 - \phi_{22} \end{pmatrix}. \] (11)
The coma designates partial derivatives with respect to coordinates represented by their indices.

It is worth pointing out here that Van Waerbeke et al. (1997) have introduced a different distortion estimator, the second moments of the autocorrelation function in the image plane. This new matrix estimator possesses at least two fundamental advantages over the more conventional individual image shape matrix: first, it makes maximal use of all the information contained in CCD images, including the information contained in the noise, so that this autocorrelation matrix can be extracted from the data everywhere in the field with unprecedented accuracy, which makes it the most promising data analysis tool to this date; furthermore, this estimator transforms like the shape distortion matrix introduced above, so that the analysis developed below can be literally transposed to it.

Introducing the convergence
\[ \kappa = \frac{\Delta \phi}{2} = \frac{\Sigma}{\Sigma_c}, \] (12)
and the complex shear
\[ \gamma \equiv |\gamma| \exp(i2\theta_L) = \frac{1}{2}(\phi_{22} - \phi_{11}) - i\phi_{12}, \] (13)
the inverse amplification matrix reads
\[ a^{-1} = \begin{pmatrix} 1 - \kappa + |\gamma| \cos(2\theta_L) & |\gamma| \sin(2\theta_L) \\ |\gamma| \sin(2\theta_L) & 1 - \kappa - |\gamma| \cos(2\theta_L) \end{pmatrix}. \] (14)

Besides $\gamma$, two other quantities related to the lens potential, $\tau_L$ and $\delta_L$, are needed in the following analysis, and write
\[ \tau_L = \frac{-2(1 - \kappa)|\gamma|}{|A^{-1}|} = \frac{-2(1 - \kappa)|\gamma|}{|(1 - \kappa)^2 - |\gamma|^2|^2}, \] (15)
\[ \delta_L = \frac{(1 - \kappa)^2 + |\gamma|^2}{|A^{-1}|} = \frac{(1 - \kappa)^2 + |\gamma|^2}{|(1 - \kappa)^2 - |\gamma|^2|^2}, \] (16)
where $A^{-1} = \det(a^{-1})$. Note that $\delta_L^2 = 1 + \tau_L^2$, as the equivalent source and image quantities.

With these definitions, the shape matrix transformation equation [Eq. (10)] yields
\[ \sigma_I = \frac{\sigma_S}{|A^{-1}|}, \] (17)
\[ |	au_S| \sin(2\theta_S - 2\theta_L) = |	au_I| \sin(2\theta_I - 2\theta_L), \] (18)
\[ |	au_S| \cos(2\theta_S - 2\theta_L) = \delta_L |	au_I| \cos(2\theta_I - 2\theta_L) - \tau_L \delta_I. \] (19)
The last equation can be inverted, which gives

$$|\tau_1| \cos(2\theta_1 - 2\theta_L) = \delta_L |\tau_S| \cos(2\theta_S - 2\theta_L) + \tau_L \delta_S,$$  \hspace{1cm} (20)$$

In the rest of the paper, the complex distortion parameter \(\tau\) and the distortion amplitude \(\sigma\) are used instead of the shape matrix.

These distortion equations present several advantages over the ones which are presently used in inversion procedures. First, the linear relation between source and image distortion estimators simplifies greatly the averaging over the source orientation. Second, the \(A^{-1}\) denominator makes an explicit redshift averaging of the equations possible, as already pointed out by Kochanek (1990), and shown in Sect. 3. On the other hand, \(\tau\) is sensitive to the intrinsic ellipticity distribution of the sources, so that a single highly elongated source can in principle bias the estimation of the local averages used in the next sections; however, this seems likely to happen only rarely in practice for reasonable source redshift distributions.

3. Probability distributions and distortion statistics:

The lens and the distortion equations involve both deterministic parameters (the lens redshift and mass distribution, the cosmological parameters) and random variables (the position \(\mathbf{x^3}\), the complex distortion \(\tau_S\), the distortion amplitude \(\sigma_S\) and the redshift \(z_S\) for the sources, the position \(\mathbf{x'}\), the complex distortion \(\tau_I\) and the distortion amplitude \(\sigma_I\) for the images; the number of sources and images per unit area is also a random variable, but it is not used here). It is commonly assumed that the positions, distortion estimators, and redshifts of the sources (i.e., in the absence of the lens) are all independent random variables (however, the following argument is unaffected by a possible correlation between \(\sigma_S\) and \(|\tau_S|\)). Furthermore, the positions and orientations \(\theta_S\) of the sources are assumed to be uniformly distributed random variables.

With the assumption just recalled, the joint probability density distribution of source parameters \(p(\mathbf{x^3}, \tau_S, \sigma_S, z_S) = (1/4\pi)p(\tau_S)p(\sigma_S)p(z_S)\). The statistical quantities of interest are the averages of the distortion estimators over the source random variables, at a given position in the lens plane, i.e., the probability distribution needed is the conditional probability \(p(\tau_S, \sigma_S, z_S/\mathbf{x'}) \equiv p(\mathbf{x'}, \tau_S, \sigma_S, z_S)/p(\mathbf{x'}) = p(\mathbf{x'}, z_S)p(\tau_S)p(\sigma_S)/p(\mathbf{x'})\). Furthermore,

$$p(\mathbf{x'}, z_S) = \frac{1}{4\pi} p(z_S)|A^{-1}|,$$  \hspace{1cm} (21)

from which one obtains

$$p(\mathbf{x'}) = \frac{1}{4\pi} \langle|A^{-1}|\rangle_{z_S},$$  \hspace{1cm} (22)

where \(\langle|A^{-1}|\rangle_{z_S}\) stands for the redshift average of \(|A^{-1}|\) at given image position \(\mathbf{x'}\). Finally, the required probability distribution reads

$$p(\tau_S, \sigma_S, z_S/\mathbf{x'}) = p(\tau_S)p(\sigma_S)p(z_S) \frac{|A^{-1}|}{\langle|A^{-1}|\rangle_{z_S}}.$$  \hspace{1cm} (23)

For critical lenses, \(\int p(\mathbf{x'}) \, d\mathbf{x'} > 1\), because multiple images are absolutely correlated events and should not be coadded as independent ones. If multiplied by the number of sources in the source plane, Eq. (22) yields the expectation value of the number density in the image plane.

Averaging the distortion equations for \(\delta_I\), the complex distortion \(\tau_I\) and the distortion amplitude \(\sigma_I\) over the sources at a given position in the image plane now yields

$$\langle \sigma_I \rangle = \frac{\langle \sigma_S \rangle}{\langle |A^{-1}| \rangle_{z_S}},$$  \hspace{1cm} (24)

$$\langle \delta_I \rangle = \frac{\langle \delta_{LS} |A^{-1}| \rangle_{z_S}}{\langle |A^{-1}| \rangle_{z_S}},$$  \hspace{1cm} (25)

$$\langle \tau_I \rangle = \frac{\langle \delta_{LS} |A^{-1}| \rangle_{z_S} \gamma}{\langle |A^{-1}| \rangle_{z_S} |\gamma|},$$  \hspace{1cm} (26)

where \(\tau_I = |\tau_I| \exp(2i\theta_I)\), and where the ratio \(\gamma/|\gamma|\) does not depend on the redshifts of the sources. The left-hand sides of Eqs. (24) through (26) are provided by the observations.

It is necessary to express the redshift averages appearing in the preceding equations in more detail. To this purpose, let us introduce the lens critical radius \(b \equiv D_{LS}/D_S\), and define a redshift independent convergence \(K\) and shear \(\Gamma\) by

$$\kappa \equiv b K,$$  \hspace{1cm} (27)

$$\gamma \equiv b \Gamma,$$  \hspace{1cm} (28)

so that

$$\langle \tau_{LS} |A^{-1}| \rangle_{z_S} = -2\langle b \rangle |\Gamma| + 2\langle b^2 \rangle K |\Gamma|,$$  \hspace{1cm} (29)

$$\langle \delta_{LS} |A^{-1}| \rangle_{z_S} = 1 - 2\langle b \rangle K + \langle b^2 \rangle (K^2 + |\Gamma|^2).$$  \hspace{1cm} (30)

The sign of \(A^{-1}\) depends on the position in the image plane and on the source redshift. If the sources are all at the same redshift, the sign change occurs on the critical lines, i.e. a zero-width region. If the sources have some dispersion in redshift, the critical lines become critical zones. Two cases naturally emerge, depending on the size of these critical zones.

If the critical zones are too small to be of practical importance, one has

$$\langle |A^{-1}| \rangle_{z_S} = \text{sgn}(A^{-1}) \left[ 1 - 2\langle b \rangle K + \langle b^2 \rangle (K^2 - |\Gamma|^2) \right].$$  \hspace{1cm} (31)
where $\text{sign}(A^{-1})$ is well-defined outside the (neglected) critical zones. Both $\langle b \rangle$ and $\langle b^2 \rangle$ depend only on the distribution of source redshifts and on the Cosmology. Also, $\langle \delta \rangle^2 / \langle \delta S \rangle^2 = 1 + \langle \tau_i \rangle^2 / \langle \delta S \rangle^2$.

Otherwise, at every position $\mathbf{x}'$ one can define a redshift domain $Z^+$ for which $A^{-1} > 0$, and a complementary redshift domain $Z^-$ where $A^{-1} < 0$; these domains are known if the lens surface density is known, for a given Cosmology. Let us define $B^+ = \int_{Z^+} b^0 p(z) dS$; one has $B^++B^- = \langle b^0 \rangle (= 1$ for $n = 0)$, where $\langle b \rangle$ and $\langle b^2 \rangle$ have the same (position independent) value as in regions of constant sign of $A^{-1}$ (e.g., outside the asymptotic external critical line). Then

$$\langle |A^{-1}| \rangle_{z_0} = (\Delta B_0) - 2(\Delta B_1) K + (\Delta B_2)(K^2 - |\Gamma|^2),$$

where $\Delta B_n = B^+_n - B^-_n$. Note that the three quantities $\Delta B_n$ are all related to each other through the source redshift distribution and the Cosmology, so that the spatial dependence of $\langle |A^{-1}| \rangle_{z_0}$ provides us with an integral equation for $p(zS)$, for a given Cosmology and for a given lens surface density distribution. As $Z^\pm$ cover the whole redshift range of the distribution when one crosses the critical zones, this integral can in principle be solved for $p(zS)$, possibly in a parametric way.

4. Reconstruction procedure and discussion:

Following a lead by Kaiser and Squires (1993), Seitz and Schneider (1995) have shown that the convergence and the complex shear are related through an invertible convolution equation which yields

$$\Gamma(\mathbf{x}') = \frac{1}{\pi} \int d^2 y' G(\mathbf{x}' - \mathbf{y}') K(\mathbf{y}'),$$

$$K(\mathbf{x}') = \frac{1}{\pi} \int d^2 y' \text{Re} \left(G(\mathbf{x}' - \mathbf{y}') \Gamma^*(\mathbf{y}')\right),$$

where the kernel $G$ is given by

$$G(\mathbf{x}) = \frac{x_1^2 - x_2^2 + 2ix_1x_2}{|\mathbf{x}|^4}$$

(35)

Two different cases must be distinguished, depending on whether one can ignore the critical zones defined above. Let us first assume that we can, which is probably correct in a majority of cases. Then Equations (24) through (35) [especially Eqs. (24), (26) and (33)] provide us with enough independent constraints for the five quantities ($K$, $\text{Re}(\Gamma)$, $\text{Im}(\Gamma)$, $\langle b \rangle$ and $\langle b^2 \rangle$), which can therefore in principle be solved for as functions of $\mathbf{x}'$ (with appropriate noise filtering; Kaiser and Squires, 1993, and Seitz and Schneider, 1995). Note that $\langle |A^{-1}| \rangle_{z_0}$ is obtained from the distortion amplitude $\sigma$ but any other estimator of $\text{sign}(A^{-1})$ would do as well (e.g., it is possible that the magnification might be extracted with more accuracy from the image autocorrelation itself than from its second order moments). Also, $\sigma$ and $\delta$ are expected to be observationally less well constrained than $\tau$. However, the expected constancy of $\langle b \rangle$ and $\langle b^2 \rangle$ in the image plane provides us with an additional constraint on the inverse procedure, as well the identification of multiple images, if any, so that, although this inverse procedure would probably be difficult to implement from the observed distortions of individually identified objects alone, it seems within reach of the autocorrelation method proposed by Van Waerbeke et al. (1997). Note however that the strong amplification near the critical zones biases the observed distribution of objects towards fainter or farther sources, which can possibly invalidate the assumed constancy of $\langle b \rangle$ and $\langle b^2 \rangle$.

Note also that the local degeneracy pointed out by Schneider and Seitz (1995) is broken as long as $\langle b^2 \rangle$ is significantly different from $\langle b \rangle^2$. As expected, the global degeneracy of the distortion equations cannot be raised without the use of external information. In the same vein, one also needs an external constraint to determine the magnitude of the potential, as Eqs. (24) through (35) are invariant under the scaling $K, \Gamma \rightarrow \alpha K, \alpha \Gamma$, and $\langle b \rangle, \langle b^2 \rangle \rightarrow \langle b \rangle/\alpha, \langle b^2 \rangle/\alpha^2$ (within the limits of variation of $b$). The simplest such constraint is obtained from the knowledge of the redshift of a giant arc (if one is present in the field, which fortunately is not an exceptional situation). As for the source redshift distribution, $\langle b \rangle$ and $\langle b^2 \rangle$ depend on both the redshift of the lens and on the Cosmology besides the shape of the source redshift distribution, so that it seems likely that the combination of the application of the reconstruction procedure just outlined to a number of lenses at different (known) redshifts with other lensing methods depending on the characteristics of the redshift distribution and constraining the Cosmology (see, e.g., Fort et al. 1997, andBernardeau et al. 1997) will finally yield both the source redshift distribution and the cosmological parameters $\Omega$ and $\Lambda$.

Although it seems less likely to occur in practice, let us now also consider the other possibility, namely that the critical zones are extended enough in the image plane so that they cannot be ignored in the reconstruction procedure. Comments made above and applying to this case (with appropriate transpositions) are not repeated here. It is likely that the inversion is still possible in principle, although the reconstructed surface density becomes Cosmology dependent. For example, one can assume an a priori Cosmology and redshift distribution, which fixes the $\Delta B_n$ factors in Eq. (32). Then Eq. (32) depends only on the lens mass distribution, which can be recovered in much the same way as described above along with $\langle b \rangle$ and $\langle b^2 \rangle$ (if needed) which can be considered as free parameters for the mass distribution reconstruction. Then Eq. (32) can be solved for the redshift distribution for a given Cosmology and the previously found surface density and possibly the values of $\langle b \rangle$ and $\langle b^2 \rangle$, and the procedure iterated until convergence is obtained. The weak points of this approach are that the limited size of the concerned regions in the image plane, the possible breakdown of the approximation
underlying Eq. (10) for some sources not individually identified, the nonvanishing size of the background sources, the limited background sources number density and other observational sources of noise and bias are likely to make such a reconstruction procedure much more difficult to implement than the preceding one (not mentioning possible problems in the reconstruction of $p(z_S)$ from the integral equation). The strong point is that a single well-chosen lens, combined with the methods mentioned above to constrain the Cosmology could in principle yield at the same time the cosmological parameters and the source redshift distribution, a rather tantalizing possibility.

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