Nonfactorizable soft contributions in the $B \rightarrow \eta_c K, \chi_{c0} K$ decays with the light-cone sum rules approach

Zhi-Gang Wang$^{1,2,3,*}$, Lin Li$^2$ and Tao Huang$^{1,2}$

$^1$ CCAST (World Laboratory), P.O.Box 8730, Beijing 100080, P. R. China

$^2$ Institute of High Energy Physics, P.O.Box 918 ,Beijing 100039,P. R. China

$^3$ Department of Physics, North China Electric Power University, Baoding 071003, P. R. China

† Mailing address

In this article, we calculate the nonfactorizable soft contributions in the $B \rightarrow \eta_c K, \chi_{c0} K$ decays with the light-cone QCD sum rules approach. Our results show that the nonfactorizable corrections from the soft gluon exchanges in the decay $B \rightarrow \eta_c K$ are of $(15 - 30)\%$ and should be taken into account. As for the decay $B \rightarrow \chi_{c0} K$, the factorizable contributions are zero and the nonfactorizable contributions from the soft hadronic matrix elements are too small to accommodate the experimental data.

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I. INTRODUCTION

The nonleptonic decays of the $B$ meson have attracted much attention in studying the nonperturbative dynamics of QCD, final state interactions and CP violation. The exclusive $B$ to charmonia decays are important since those decays $B \rightarrow J/\psi K, \eta_c K, \chi_{cJ} K$ are regarded as the golden channels for the study of CP violation. The quantitatively understanding of those decays depends on our knowledge about the nonperturbative hadronic matrix elements of the operators entering the effective weak Hamiltonian. In Ref. [1], the authors propose an original approach called QCD-improved factorization to deal with the two-body nonleptonic decays of the $B$ meson. In this approach, the decay amplitudes are expressed in terms of the semileptonic form factors, hadronic light-cone distribution amplitudes and hard-scattering amplitudes. The semileptonic form factors, the light-cone distribution amplitudes are taken as input parameters and the hard-scattering amplitudes including nonfactorizable corrections due to the exchanges of hard gluons are calculated by perturbative QCD. For the exclusive $B$ to charmonia decays, the QCD-improved factorization approach is broken down due to the divergence arising from the soft-gluon exchanges, moreover, the theoretical branching fractions are too small to accommodate the experimental data [1–5]. In Refs. [2,3], the authors observe
that for the exclusive $B \to \eta_c K$ decay, the nonfactorizable corrections to the naive factorization are infrared safe at leading-twist order, the spectator interactions arising from the kaon twist-3 effects are formally power-suppressed but chirally and logarithmically enhanced; for the $B \to \chi_{c0} K$ decay, there are infrared divergences arising from the nonfactorizable vertex corrections as well as logarithmic divergences due to the spectator interactions even at leading-twist order.

The effects of soft gluons which break down factorization are supposed of order $O(\Lambda_{QCD}/m_b)$ and neglected in the QCD-improved factorization studies, however, no theoretical work has ever proved that they are small quantities. For the color-suppressed $B$ to charmonia decays, there may be significant impacts of the nonfactorizable soft contributions.

On the other hand, the QCD light-cone sum rules (LCSR) approach provides a powerful tool for calculating the exclusive soft hadronic amplitudes [6–8]. The LCSR approach carries out the operator product expansion near the light-cone $x^2 \approx 0$ instead of the short distance $x \approx 0$ while the nonperturbative matrix elements are parameterized by the light-cone distribution amplitudes which classified according to their twists instead of the vacuum condensates. For detailed discussions of this approach, one can consult Ref. [9]. The LCSR approach has been applied to study the nonfactorizable hadronic matrix elements due to the soft gluons exchanges and gave satisfactory results [10–12].

Experimentally, the $B \to \eta_c K$ decay was observed by CLEO, BaBar, Belle Collaborations and the $B \to \chi_{c0} K$ decay by Belle Collaboration with relatively large branching fractions [13,14]. The large discrepancies between the theoretical and experimental values for those decays call for considerations of new ingredients and mechanisms. It is interesting to study the nonfactorizable soft contributions in the pseudoscalar and scalar charmonia decays $B \to \eta_c K, \chi_{c0} K$ with the LCSR approach.

The article is organized as follows: the factorizable contributions from the effective weak Hamiltonian are derived in Sec.II; the soft hadronic matrix elements $\langle \eta_c K | \bar{O} | B \rangle$ and $\langle \chi_{c0} K | \bar{O} | B \rangle$ are calculated with the light-cone sum rules approach in Sec.III; numerical results are presented in Sec.IV; the section V is reserved for conclusion.

II. EFFECTIVE WEAK HAMILTONIAN AND FACTORIZABLE CONTRIBUTIONS

The effective weak Hamiltonian for the $b \to sc \bar{c}$ decay modes can be written as (for detailed discussion of the effective weak Hamiltonian, one can consult Ref. [15])

$$H_w = \frac{G_F}{\sqrt{2}} \left\{ V_{cb} V_{cs}^* \left[ C_1(\mu) O_1 + C_2(\mu) O_2 \right] - V_{tb} V_{ts}^* \sum_{i=3}^{10} C_i O_i \right\} ,$$

(1)
where $V_{ij}$'s are the CKM matrix elements, $C_i$'s are the Wilson coefficients calculated at the renormalization scale $\mu \sim O(m_b)$ and the relevant operators $O_i$ are given by

\[
O_1 = (\bar{\sigma}_\alpha b_\beta)_{V-A}(\bar{\tau}_\beta c_\alpha)_{V-A}, \quad O_2 = (\bar{\sigma}_\alpha b_\beta)_{V-A}(\bar{\tau}_\beta c_\beta)_{V-A}, \\
O_{3(5)} = (\bar{\sigma}_\alpha b_\beta)_{V-A}\sum_q (\bar{\tau}_\beta q_\beta)_{V-A(V+A)}, \quad O_{4(6)} = (\bar{\sigma}_\alpha b_\beta)_{V-A}\sum_q (\bar{\tau}_\beta q_\alpha)_{V-A(V+A)}, \\
O_{7(9)} = \frac{3}{2}(\bar{\sigma}_\alpha b_\beta)_{V-A}\sum_q e_q(\bar{\tau}_\beta q_\alpha)_{V-A(V+A)}, \quad O_{8(10)} = \frac{3}{2}(\bar{\sigma}_\alpha b_\beta)_{V-A}\sum_q e_q(\bar{\tau}_\beta q_\alpha)_{V+A(V-A)}. \\
\] (2)

We can reorganize the color-mismatched quark fields into color singlet states by Fierz transformation, (for example, $O_1 = \frac{1}{N_c}O_2 + 2\tilde{O}_2$, $N_c$ is the color number and taken as 3.) and express the effective weak Hamiltonian $H_w$ in the following form,

\[
H_w = \frac{G_F}{\sqrt{2}} \left\{ V_{cb}V_{cs}^* \left[ \left( C_2(\mu) + \frac{C_1(\mu)}{3} \right) O_2 + 2C_1(\mu)\tilde{O}_2 \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_3(\mu) + \frac{C_4(\mu)}{3} \right) O_3 + 2C_4(\mu)\tilde{O}_3 \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_5(\mu) + \frac{C_6(\mu)}{3} \right) O_5 + 2C_6(\mu)\tilde{O}_5 \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_7(\mu) + \frac{C_8(\mu)}{3} \right) O_7 + 2C_8(\mu)\tilde{O}_7 \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_9(\mu) + \frac{C_{10}(\mu)}{3} \right) O_9 + 2C_{10}(\mu)\tilde{O}_9 \right] \right\}, \\
\] (3)

where

\[
O_{2(3,9)} = (\bar{\tau}_\gamma(1 - \gamma_5)c)(\bar{\sigma}\gamma^\mu(1 - \gamma_5)b), \quad \tilde{O}_{2(3,9)} = (\bar{\tau}_\gamma(1 - \gamma_5)c)(\bar{\sigma}\gamma^\mu(1 - \gamma_5)b), \\
O_{5(7)} = (\bar{\tau}_\gamma(1 + \gamma_5)c)(\bar{\sigma}\gamma^\mu(1 - \gamma_5)b), \quad \tilde{O}_{5(7)} = (\bar{\tau}_\gamma(1 + \gamma_5)c)(\bar{\sigma}\gamma^\mu(1 - \gamma_5)b), \\
\] (4)

here $\lambda^x$'s are $SU(3)$ Gell-Mann matrices.

The factorizable matrix elements of the operator $O_i$ for the decay $B \to \eta_c K$ can be parameterized as

\[
\langle \eta_c(p)K(q)|H_w|B(p + q) \rangle = \frac{G_F}{\sqrt{2}} \left\{ V_{cb}V_{cs}^* \left[ \left( C_2(\mu) + \frac{C_1(\mu)}{3} \right) \langle \eta_c(p)|C_2(\mu)\rangle + \frac{C_4(\mu) + C_{10}(\mu)}{3} \langle \eta_c(p)|C_4(\mu)\rangle + \frac{C_6(\mu) + C_8(\mu)}{3} \langle \eta_c(p)|C_6(\mu)\rangle \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_3(\mu) + C_9(\mu) + \frac{C_4(\mu) + C_{10}(\mu)}{3} \right) \langle \eta_c(p)|C_3(\mu)\rangle + \frac{C_6(\mu) + C_8(\mu)}{3} \langle \eta_c(p)|C_6(\mu)\rangle \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_5(\mu) + C_7(\mu) + \frac{C_4(\mu) + C_{10}(\mu)}{3} \right) \langle \eta_c(p)|C_5(\mu)\rangle + \frac{C_6(\mu) + C_8(\mu)}{3} \langle \eta_c(p)|C_6(\mu)\rangle \right] \\
- V_{tb}V_{ts}^* \left[ \left( C_9(\mu) + \frac{C_{10}(\mu)}{3} \right) \langle \eta_c(p)|C_9(\mu)\rangle + \frac{C_6(\mu) + C_8(\mu)}{3} \langle \eta_c(p)|C_6(\mu)\rangle \right] \right\}, \\
\] (5)
where the meson momenta are explicitly specified and chosen as \( p^2 = m^2_{\eta_c} \). The \( \eta_c \) meson decay constant is defined by the relation,

\[
\langle \eta_c(p)\bar{\tau}(0)\gamma_\mu\gamma_5c(0)|0 \rangle = -if_{\eta_c}p_\mu. \tag{6}
\]

The decay constant \( f_{\eta_c} \) can be estimated from the QCD sum rules approach with the current \( J_\alpha = \bar{c}\gamma_\alpha\gamma_5c \) [16] or phenomenological potential models, in fact, the values obtained from those approaches do not differ from each other much; new estimation based on the nonperturbative approach of coupled Schwinger-Dyson equation and Bethe-Salpeter equation is in preparation. We use the value obtained from the potential model in this article [17].

The \( B-K \) form factor can be parameterized as

\[
\langle K(q)|\bar{\tau}\gamma_\mu b|B(p+q) \rangle = (2q+p)_\mu F_0^p(p^2) - \frac{m^2_B - m^2_K}{p^2}p_\mu(F_0^p(p^2) - F_0^B(p^2)), \tag{7}
\]

the above form factors \( F_0^p(p^2), F_0^B(p^2) \) can be estimated from the light-cone sum rules approach [18–20], here we take the value

\[
F_0^B(m^2_{\eta_c}) = 0.42 \pm 0.06. \tag{8}
\]

The concise expression for the factorizable matrix elements in the decay \( B \rightarrow \eta_cK \) can be written as

\[
\langle \eta_c(p)K(q)|H_w|B(p+q) \rangle = \frac{G_F}{\sqrt{2}} \left\{ V_{cb}V_{cs}^* \left( C_2(\mu) + \frac{C_1(\mu)}{3} \right) - V_{tb}V_{ts}^* \left( C_3(\mu) - C_5(\mu) - C_7(\mu) + C_9(\mu) + \frac{C_4(\mu) - C_6(\mu) - C_8(\mu) + C_10(\mu)}{3} \right) \right\} i f_{\eta_c}m^2_B F_0^p(m^2_{\eta_c}). \tag{9}
\]

There are no factorizable contributions in the decay \( B \rightarrow \chi_{c0}K \) as the vector current \( \bar{c}\gamma_\mu c \) is conserved and has a vanishing matrix element with the \( \chi_{c0} \) meson,

\[
\langle \chi_{c0}(p)|\bar{\tau}c|0 \rangle = f_{\chi_{c0}}m_{\chi_{c0}} ,
\]

\[
\langle \chi_{c0}(p)K(q)|H_w|B(p+q) \rangle = 0 , \tag{10}
\]

here \( f_{\chi_{c0}} \) and \( m_{\chi_{c0}} \) are the decay constant and mass of the \( \chi_{c0} \) meson respectively.

**III. LIGHT-CONE SUM RULES FOR \( \langle \eta_cK|\bar{\tau}c|B \rangle \) AND \( \langle \chi_{c0}K|\bar{\tau}c|B \rangle \)**

In the following, we apply the approach developed in Ref. [10] for the \( B \rightarrow \pi\pi \) channel to estimate the contributions from the soft-gluon exchanges in the \( B \rightarrow \eta_cK, \chi_{c0}K \) decays. Firstly, let us write down the correlation functions,
\[ F^\eta_\rho(p, q, k) = i^2 \int d^4x e^{-i(p-q) x} \int d^4y e^{i(p-k) y} \langle 0| T\{ j^\eta_\rho(y) \bar{O}(0) j^B_\rho(x)\}|K(q)\rangle, \quad (11) \]
\[ F_{\chi_0}(p, q, k) = i^2 \int d^4x e^{-i(p-q) x} \int d^4y e^{i(p-k) y} \langle 0| T\{ j^{\chi_0}(y) \bar{O}(0) j^B_\rho(x)\}|K(q)\rangle, \quad (12) \]

where \( j^\eta_\rho = \bar{c} \gamma_\rho \gamma_5 c, j^{\chi_0} = \bar{c} c \) and \( j^B_\rho = m_0 \bar{u} \gamma_5 u \) are currents interpolating the \( \eta_c, \chi_{c0} \) and \( B \) meson fields, respectively. We take the same notation \( \tilde{O} \) for the operators \( \tilde{O}_i \) as our final result indicates that they have the same analytical expressions due to their special Dirac structures for the decays \( B \rightarrow \eta_c K, \chi_{c0} K \).

The correlation functions can be calculated by the operator product expansion method near the light-cone \( x^2 \sim y^2 \sim (x - y)^2 \sim 0 \) in QCD. They are functions of three independent momenta chosen to be \( q, p - k \) and \( k \) by convenience. Here we introduce the unphysical momentum \( k \) in order to avoid that the \( B \) meson has the same four-momentum before \( (p - q) \) and after the decay \( (P) \). In such a way, we can avoid a continuum of light contributions in the dispersion relation in the \( B \)-channel. The independent kinematical invariants can be taken as \( (p - q)^2, (p - k)^2, q^2, P^2 = (p - k - q)^2 \) and \( p^2 \). We set \( k^2 = 0 \) and take \( q^2 = m_K^2 = 0 \), neglecting the small corrections of the order \( O(m_K^2/m_B^2) \). The momentum \( p^2 \) is kept undefined for the moment in order to make the derivation of the sum rules without restriction. Its value is going to be set later in this section, and chosen \( p^2 = m_{\eta_c}^2, m_{\chi_{c0}}^2 \). The values of \( (p - k)^2, (p - q)^2 \) and \( P^2 \) should be spacelike and large in order to stay far away from the hadronic thresholds in the \( B, \eta_c \) and \( \chi_{c0} \) channels. All together, we have

\[ q^2 = k^2 = 0, \quad p^2 (\text{undefined}), \quad |(p - k)|^2 \gg \Lambda_{QCD}, \quad |(p - q)|^2 \gg \Lambda_{QCD}, \quad |P|^2 \gg \Lambda_{QCD}. \]

The decomposition of the correlation function in Eq.(11) with the independent momenta are straightforward and it can be divided into the following Lorentz invariant amplitudes,

\[ F^\eta_\rho(p, q, k) = (p - k)_\rho F^\eta_\rho(p, q, k) + q_\rho \tilde{F}^\eta_1(p, q, k) + k_\rho \tilde{F}^\eta_2(p, q, k) + \epsilon_{\rho\beta\lambda\xi} q^\beta p^\lambda k^\xi \tilde{F}^\eta_3(p, q, k). \quad (13) \]

According to the basic assumption of current-hadron duality in the QCD sum rules approach [16], we insert a complete series of intermediate states satisfying the unitarity principle with the same quantum numbers as the current operators \( j^\eta_\rho, j^{\chi_0} \) and \( j^B_\rho \) into the correlation functions in Eq.(11) and Eq.(12) to obtain the hadronic representation. After isolating the pole terms of the lowest pseudoscalar \( \eta_c \) and \( \chi_{c0} \) mesons in the charmonium channels, we get the following result,

\[ ^1F^\eta_\rho \quad \text{and} \quad F_{\chi_0} \quad \text{are functions of} \quad q, p - k \quad \text{and} \quad k; \quad \text{their momenta dependence will be written explicitly when necessary, for example,} \quad F^\eta_\rho(p - k) \quad \text{and} \quad F_{\chi_0}(p - k), \quad \text{with emphasis on the dependence on the momentum} \quad p - k. \]
\[
F_{\rho}^{(0)}(p, q, k) = \frac{\langle 0 | j_{\rho}^{(0)}(0) | \eta_c(p-k) \rangle}{m_{\eta_c}^2 - (p-k)^2 - i\varepsilon} \Pi_{\eta_c}(p-q)^2, P^2, p^2) + \int_{s_0^{\rho}}^\infty ds \frac{\rho_{\rho}(s, (p-q)^2, P^2, p^2)}{s - (p-k)^2}; \quad (14)
\]
\[
F_{\chi}(p, q, k) = \frac{\langle 0 | j_{\rho}^{(0)}(0) | \chi(p-k) \rangle}{m_{\chi}^2 - (p-k)^2 - i\varepsilon} \Pi_{\chi}(p-q)^2, P^2, p^2) + \int_{s_0^{\chi}}^\infty ds \frac{\rho_{\chi}(s, (p-q)^2, P^2, p^2)}{s - (p-k)^2}, \quad (15)
\]

where

\[
\Pi_{\eta_c(\chi)}((p-q)^2, P^2, p^2) = i \int d^4 x e^{-i(p-q)x} (\eta_c(\chi))((p-k)K(-q)| T[O(0)j_{\rho}^{B}(x)] | 0). \quad (16)
\]

In above equations, \( \rho_{\rho}^0 \) and \( s_0^{\rho} \) are the spectral density and threshold parameter of the lowest excited resonances and continuum states in the \( \eta_c \) channel, respectively; while \( \rho_{\chi}^{\chi} \) and \( s_0^{\chi} \) are the corresponding ones in the \( \chi \) channel.

In the limit of large spacelike momentum \((p-k)^2 \ll m_{\eta_c}^2, m_{\chi}^2\), the correlation functions in Eq.(11) and Eq.(12) can be calculated in QCD at the level of quark-gluon degrees of freedom and rewritten in the following forms by applying dispersion relation,

\[
F_{\rho}(F_{\chi}) = \frac{1}{\pi} \int_{4m_{\rho}^2}^\infty d\omega \frac{\text{Im} F_{\rho}(F_{\chi})(s, (p-q)^2, P^2, p^2)}{s - (p-k)^2}. \quad (17)
\]

We can approximate the hadronic spectral densities \( \rho_{\rho}^\eta \) and \( \rho_{\chi}^{\chi} \) above the thresholds of the lowest excited resonances and continuum states by the corresponding ones from QCD calculations with the assumption of quark-hadron duality,

\[
\rho_{\rho}^\eta(s, (p-q)^2, P^2, p^2)\Theta(s - s_0^{B}(s_{\eta_c}^{\chi})) = \frac{1}{\pi} \text{Im} F_{\rho}(F_{\chi})(s, (p-q)^2, P^2, p^2)\Theta(s - s_0^{B}(s_{\eta_c}^{\chi})). \quad (18)
\]

Now we explore the analytical properties of the amplitudes \( \Pi_{\eta_c}(p-q)^2, P^2, p^2 \), \( \Pi_{\chi}(p-q)^2, P^2, p^2 \) in the \( B \)-channel and insert a complete series of hadronic states with the same quantum numbers as the \( B \) meson into the correlation functions in Eq.(16). After isolating the lowest pole terms of the \( B \) meson contributions, we obtain the following results,

\[
\Pi_{\eta_c(\chi)}((p-q)^2, P^2, p^2) = \frac{m_{B}^2 f_B}{m_B^2 - (p-q)^2 - i\varepsilon} (\eta_c(\chi))((p-k)K(-q)| O(0)| B(p+q)) + \int_{s_0^{B}}^\infty ds \rho_{\eta_c(\chi)}^{B}(s', P^2, p^2). \quad (19)
\]

The hadronic spectral densities \( \rho_{\rho}^{\eta} \) and \( \rho_{\chi}^{\eta} \) above the threshold of the continuum \( s_0^{B} \) can be approximated by the corresponding ones at the level of quark-gluon degrees of freedom.

Finally, we obtain the correlation functions in hadronic representation,

\[
F_{\rho}^{\eta}(p, q, k) = \frac{\langle 0 | j_{\rho}^{(0)}(0) | \eta_c(p-k) \rangle}{m_{\eta_c}^2 - (p-k)^2 - i\varepsilon} (\eta_c(p-k)| \bar{O}(0)| B(p-q))| K(q) \rangle \frac{\langle B(p-q)| j_{\rho}^{B}(0)| 0 \rangle}{m_B^2 - (p-q)^2 - i\varepsilon} + \cdots,
\]

\[
= \frac{i f_{\eta_c(p-k)}^{\rho}}{m_{\eta_c}^2 - (p-k)^2 - i\varepsilon} \frac{f_B m_B^2}{m_B^2 - (p-q)^2 - i\varepsilon} (\eta_c(p-k)K(-q)| \bar{O}(0)| B(p-q)) + \cdots,
\]

6
\[ F_{\chi\alpha}(p, q, k) = \frac{\langle 0 | j_{\chi\alpha}(0) | \chi\alpha(p - k) \rangle}{m_{\chi\alpha}^2 - (p - k)^2 - i\varepsilon} \langle \chi\alpha(p - k) | \bar{\Omega}(0) | B(p - q) \rangle | K(q) \rangle \frac{B(p - q | j_{\rho}^B(0) | 0)}{m_B^2 - (p - q)^2 - i\varepsilon} + \cdots, \]

\[ \frac{f_{\chi\alpha} m_{\chi\alpha}}{m_{\chi\alpha}^2 - (p - k)^2 - i\varepsilon m_B^2 - (p - q)^2 - i\varepsilon} (\eta_c(p - k) K(-q) | \bar{\Omega}(0) | B(p - q)) + \cdots. \]  

(21)

Here we do not show the contributions from the higher resonances and continuum states above the thresholds explicitly, they can be written in terms of dispersion integrals and the spectral densities can be approximated by the quark-hadron duality ansatz. In Eq.(20), we select the relevant terms with tensor structure \((p - k)_\rho\), which corresponding to the contributions from the pseudoscalar mesons, for example, the \(\eta_c\) and B mesons.

In order to suppress the contributions from the excited and continuum states in the charmonium channels, we can perform n-th derivative with respect to the momentum \((p - k)^2\) in Eqs.(14-15) to obtain n-th moment sum rules for the correlation functions \(\Pi_{\eta_c}((p - q)^2, P^2, p^2)\) and \(\Pi_{\chi\alpha}((p - q)^2, P^2, p^2)\) in hadronic representation,

\[ i(p - k)_\rho \Pi_{\eta_c}((p - q)^2, P^2, p^2) = \frac{1}{\pi \int_{4m^2_\rho}^{s_{\eta_c}^0}} ds \frac{(m_{\eta_c}^2 + Q_0^2)^{n+1}}{(s + Q_0^2)^{n+1}} \text{Im}_s F_{\rho}^{\eta_c} (s, (p - q)^2, P^2, p^2), \]

\[ = \frac{1}{\pi^2 f_{\eta_c}} \int_{4m_\eta_c^2}^{s_{\eta_c}^0} ds \frac{(m_{\eta_c}^2 + Q_0^2)^{n+1}}{(s + Q_0^2)^{n+1}} \text{Im}_s F_{\eta_c} (s, (p - q)^2, P^2, p^2), \]

\[ \Pi_{\chi\alpha}((p - q)^2, P^2, p^2) = \frac{1}{\pi m_{\chi\alpha}^2 f_{\chi\alpha}} \int_{4m_{\chi\alpha}^2}^{s_{\chi\alpha}^0} ds \frac{(m_{\chi\alpha}^2 + Q_0^2)^{n+1}}{(s + Q_0^2)^{n+1}} \text{Im}_s F_{\chi\alpha} (s, (p - q)^2, P^2, p^2), \]

\[ = \frac{1}{\pi^2 m_{\chi\alpha}^2 f_{\chi\alpha}} \int_{4m_{\chi\alpha}^2}^{s_{\chi\alpha}^0} ds \frac{(m_{\chi\alpha}^2 + Q_0^2)^{n+1}}{(s + Q_0^2)^{n+1}} \int_{m_B^2}^{\infty} ds' \frac{(p - q)^2}{s' - (p - q)^2} \text{Im}_s \text{Im}_{s'} F_{\chi\alpha} (s, s', P^2, p^2), \]

(22-23)

where the imaginary parts with respect to \(s\) and \(s'\) are given by

\[
\text{Im}_s \text{Im}_{s'} F_{\rho}^{\eta_c} (F_{\chi\alpha}) (s, s', P^2, p^2) = i(p - k)_\rho \int_{4m^2_\rho}^{s_{\eta_c}^0} ds \frac{(m_{\eta_c}^2 + Q_0^2)^{n+1}}{(s + Q_0^2)^{n+1}} \text{Im}_s F_{\rho}^{\eta_c} (s, (p - q)^2, P^2, p^2)
\]

\[
\delta(s' - m_B^2) \delta(s - m_{\eta_c}^2), \]

(24)

Here \(Q_0\) is the parameter for QCD sum rules in the charmonium channels, the spectral densities above the thresholds can be approximated by the corresponding ones from QCD calculation and not shown explicitly.

The Borel transformations with respect to \((p - q)^2\) in the B channels in Eqs.(22-23) are straightforward. Comparing with Eqs.(14-16), we can obtain the following results\(^8\),

\[
B_{\text{trans}} F_{\eta_c}((p - k)) = i f_{\eta_c} (f_{\chi\alpha} m_{\chi\alpha}) f_B m_B^2 (\eta_c(\chi\alpha))(p - k) K(-q) | \bar{\Omega}(0) | B(p - q) \]

\[
\frac{1}{(m_{\eta_c}(\chi\alpha) + Q_0^2)^{n+1}} e^{-\frac{m_B^2}{M^2}} + \cdots, \]

(25)

\(^8\)Here we prefer the notations \(F_{\eta_c}(p - k), F_{\chi\alpha}(p - k)\) to \(F_{\eta_c}(p, q, k), F_{\chi\alpha}(p, q, k)\), with emphasis on the dependence on \(p - k\).
where $B_{trans}$ denotes both the n-th derivative and Borel transformation, $M^2$ is the Borel parameter in the B channel. The contributions from the excited and continuum states are not shown explicitly for simplicity.

Then we can analytically continue $P^2$ from the space-like region $P^2 \ll 0$ to the time-like region $P^2 \geq 0$, and choose $P^2 = m_B^2$. Now we carry out the operator product expansion near the light-cone to obtain the representation at the level of quark-gluon degrees of freedom for the amplitudes $F_\nu$ and $F_\chi$. Firstly, let us write down the propagator of a massive quark in the external gluon field in the Fock-Schwinger gauge [21,22],

$$
\langle 0| T\{q_i(x_1) \bar{q}_j(x_2)\}|0\rangle = i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \left\{ \frac{k + m}{k^2 - m^2} \delta_{ij} - \int_0^1 dv \, g_s G_{\alpha}^{\mu \nu} (vx_1 + (1-v)x_2) \left( \frac{\lambda^\alpha}{2} \right)_{ij} \right\},
$$

(26)

where $G_{\alpha}^{\mu \nu}$ is the gluonic field strength, $g_s$ denotes the strong coupling constant.

Substituting the above b and c quark propagators into the correlation functions in Eqs.(11-12), we can obtain the hadronic spectral densities at the level of quark-gluon degrees of freedom. The following three particle kaon distribution amplitudes are useful in our calculation,

- twist-3 distribution amplitude

$$
\langle 0| \bar{\sigma}_0^\mu \gamma_5 G_{\alpha \beta}(vy) u(x)|K^+(q)\rangle = i f_3 K \left[ (q_\alpha q_\mu g_{\beta \nu} - q_\beta q_\mu g_{\alpha \nu}) - (q_\alpha q_\mu g_{\beta \nu} - q_\beta q_\mu g_{\alpha \nu}) \right] \int D\alpha_i \phi_3 K (\alpha_i, \mu) e^{-iq(x_\alpha + y\nu\alpha)};
$$

(27)

- twist-4 distribution amplitudes

$$
\langle 0| \bar{\sigma}_0^\mu i \gamma_\mu \tilde{G}_{\alpha \beta}(vy) u(x)|K^+(q)\rangle = q_\mu \int D\alpha_i \tilde{\phi}_\parallel (\alpha_i, \mu) e^{-iq(x_\alpha + y\nu\alpha)} \\
+ (g_{\mu \alpha} q_\beta - g_{\mu \beta} q_\alpha) \int D\alpha_i \tilde{\phi}_\perp (\alpha_i, \mu) e^{-iq(x_\alpha + y\nu\alpha)};
$$

(28)

$$
\langle 0| \bar{\sigma}_0^\mu \gamma_5 \tilde{G}_{\alpha \beta}(vy) u(x)|K^+(q)\rangle = q_\mu \int D\alpha_i \tilde{\phi}_\parallel (\alpha_i, \mu) e^{-iq(x_\alpha + y\nu\alpha)} \\
+ (g_{\mu \alpha} q_\beta - g_{\mu \beta} q_\alpha) \int D\alpha_i \tilde{\phi}_\perp (\alpha_i, \mu) e^{-iq(x_\alpha + y\nu\alpha)},
$$

(29)

where

$$
\tilde{G}_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \rho \sigma} G^{\rho \sigma}, \quad G^{\rho \sigma} = g_s \frac{\lambda^\alpha G_{\rho \sigma}^\alpha}{2};
$$

$$
D\alpha_i = d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3), \quad g_{\alpha \beta} = g_{\alpha \beta} - \frac{x_\alpha q_\beta + x_\beta q_\alpha}{q x}.
$$

(30)
The twist-3 and twist-4 light-cone distribution amplitudes can be parameterized as

\[
\phi_{3K}(\alpha_i, \mu) = 360\alpha_1\alpha_2\alpha_3^2 \left(1 + a(\mu)\frac{1}{2}(7\alpha_3 - 3) + b(\mu)(2 - 4\alpha_1\alpha_2 - 8\alpha_3(1 - \alpha_3)) + c(\mu)(3\alpha_1\alpha_2 - 2\alpha_3 + 3\alpha_3^2) \right),
\]

(31)

\[
\phi_{\perp}(\alpha_i, \mu) = 30\delta^2(\mu)(\alpha_1 - \alpha_2)\alpha_3^2 \left[\frac{1}{3} + 2\epsilon(\mu)(1 - 2\alpha_3) \right],
\]

(32)

\[
\phi_{\parallel}(\alpha_i, \mu) = 120\delta^2(\mu)\epsilon(\mu)(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3,
\]

(33)

\[
\tilde{\phi}_{\perp}(\alpha_i, \mu) = 30\delta^2(\mu)\alpha_3^2(1 - \alpha_3) \left[\frac{1}{3} + 2\epsilon(\mu)(1 - 2\alpha_3) \right],
\]

(34)

\[
\tilde{\phi}_{\parallel}(\alpha_i, \mu) = -120\delta^2(\mu)\alpha_1\alpha_2\alpha_3 \left[\frac{1}{3} + \epsilon(\mu)(1 - 3\alpha_3) \right].
\]

(35)

Those parameters in the light-cone distribution amplitudes can be estimated with the QCD sum rules approach [9,23,24]. In practical manipulation, we can neglect the \(a(\mu), b(\mu), c(\mu)\) and \(\epsilon(\mu)\) dependence and the asymptotic forms will be taken.

After carrying out the operator product expansion near the light-cone, we obtain the following expressions for the \(F_{\eta_c}\) and \(F_{\chi_c}\),

\[
F_{\eta_c}(p - k) = \frac{m_0 f_{3K}}{4\pi^2} \int_0^1 \frac{d\alpha_1}{m_0^2 - (p - q(1 - \alpha_1))^2} \int_0^1 \frac{dx}{m_2^2 - (p - k - \alpha_3q)^2x(1 - x)} \frac{2x^2(1 - x)}{q \cdot (p - k) [(2 - v)q \cdot k + 2(1 - v)q \cdot (p - k)]} \\
- \frac{m_0^2 f_K}{4\pi^2} \int_0^1 \frac{d\alpha_1}{m_0^2 - (p - q(1 - \alpha_1))^2} \int_0^1 \frac{dx}{m_2^2 - (p - k - \alpha_3q)^2x(1 - x)} \frac{2x^2(1 - x)q \cdot (p - k)}{2x^2(1 - x)(2v - 3)x \cdot (p - k)} ;
\]

(36)

\[
F_{\chi_c}(p - k) = \frac{m_0 m_c f_{3K}}{4\pi^2} \int_0^1 \frac{d\alpha_1}{m_0^2 - (p - q(1 - \alpha_1))^2} \int_0^1 \frac{dx}{m_2^2 - (p - k - \alpha_3q)^2x(1 - x)} \frac{(2x^2 - 2x(1 - x)(1 - 2v))q \cdot (p - k)q \cdot p}{m_2^2 - (p - k - \alpha_3q)^2x(1 - x)} \\
- \frac{m_0^2 m_c f_K}{4\pi^2} \int_0^1 \frac{d\alpha_1}{m_0^2 - (p - q(1 - \alpha_1))^2} \int_0^1 \frac{dx}{m_2^2 - (p - k - \alpha_3q)^2x(1 - x)} \frac{(2x^2 - 2x(1 - x)(1 - 2v))q \cdot (p - k)}{m_2^2 - (p - k - \alpha_3q)^2x(1 - x)}.
\]

(37)

Here we will take a short digression to discuss the technical details. In calculation, we will encounter \(x\)-integral of the form

\[
\int d^4x \frac{x_\rho}{q \cdot x} f(p, q, k, x)
\]

in the coordinates representation, which can be formally written as

\[
\int d^4x \frac{x_\rho}{q \cdot x} f(p, q, k, x) = A(p, q, k)(p - k)_\rho + B(p, q, k)q_\rho + C(p, q, k)k_\rho + \epsilon_\rho\beta\alpha q^\delta p^\lambda k^\kappa D(p, q, k).
\]

(38)
Multiplying both sides of above equation by \( q_s \) and taking the Chiral limit \( q^2 = m_K^2 = 0 \), the expression of the relevant quantity \( A(p, q, k) \) can be obtained.

It is easy to perform the Feynman parameter \( x \) integral in Eqs.(36-37), we prefer this form in order to facilitate the Borel transformation and n-th derivative. For the case of massless quark loops, one can integrate out the variable \( x \) directly.

In the following, we write down the dispersion relations for the correlation functions at the level of quark-gluon degrees of freedom,

\[
F_{\eta_c}(p - k) = \frac{m_b}{4\pi^2} \int_{0}^{1} \frac{dv}{v} \int_{m_b^2}^{s_B} d\alpha_i \int_{m_b^2}^{s^{B_B}} d\alpha \int_{m_b^2}^{s^{B_B}} d\alpha \int_{m_b^2}^{s^{B_B}} d\alpha \int_{0}^{1} \frac{dx}{x(1-x)} \frac{x^2(1-x)}{s_2 - (p - k)^2} \frac{s_2 - P^2}{s_2 - (p - k)^2} \frac{d\beta}{d\gamma} \left[ f_{3K\bar{K}}(\alpha_i, \mu) \left( \frac{2 - v}{2} (P^2 + m_{\eta_c}^2 - s_1 - s_2) + (1 - v)(s_2 - P^2) \right) - m_b f_{K\bar{K}}(\alpha_i, \mu) + \tilde{f}_{\perp}(\alpha_i, \mu) \right] \\
\delta(m_{\eta_c}^2 - m_{\eta_c}^2 \alpha_1 - s_1(1 - \alpha_1)) \delta(m_{\eta_c}^2 - x(1-x)v_{\alpha_3} P^2 - x(1-x)(1 - \alpha_3) s_2) + \cdots ; \tag{39}
\]

\[
F_{\chi_{c0}}(p - k) = \frac{m_b m_c}{8\pi^2} \int_{0}^{1} \frac{dv}{v} \int_{m_b^2}^{s_B} d\alpha_i \int_{m_b^2}^{s^{B_B}} d\alpha \int_{m_b^2}^{s^{B_B}} d\alpha \int_{m_b^2}^{s^{B_B}} d\alpha \int_{0}^{1} \frac{dx}{x(1-x)} \frac{x^2 - x(1-x)(1 - 2v)}{s_1 - (p - q)^2} \frac{s_2 - P^2}{s_2 - (p - k)^2} \frac{d\beta}{d\gamma} \left[ f_{3K\bar{K}}(\alpha_i, \mu) (m_{\chi_{c0}}^2 - s_1) - m_b f_{K\bar{K}}(\alpha_i, \mu) - 2 \tilde{f}_{\perp}(\alpha_i, \mu) \right] \\
\delta(m_{\chi_{c0}}^2 - m_{\chi_{c0}}^2 \alpha_1 - s_1(1 - \alpha_1)) \delta(m_{\chi_{c0}}^2 - x(1-x)v_{\alpha_3} P^2 - x(1-x)(1 - \alpha_3) s_2) + \cdots . \tag{40}
\]

Again, the higher resonances and continuum states contributions are not shown explicitly for simplicity, as they are Borel transformation or n-th derivative suppressed. Here we can introduce some notations to simplify the cumbersome expressions in Eq.(39) and Eq.(40) respectively,

\[
x_i = \frac{1}{2} \left( 1 - \sqrt{1 - 4m_{\eta_c}^2} \right), \quad x_f = \frac{1}{2} \left( 1 + \sqrt{1 - 4m_{\eta_c}^2} \right), \\
\alpha_c = \frac{x(1-x) s - m_{\eta_c}^2}{x(1-x)(s - P^2)}, \quad \alpha_0 = \frac{m_b^2 - m_{\eta_c}^2 (m_{\chi_{c0}}^2)}{s_B - m_{\eta_c}^2 (m_{\chi_{c0}}^2)}. \tag{41}
\]

In performing the \( \delta \) functions integrals in Eq.(39) and Eq.(40), we note that in the space-like region \( P^2 \ll 0 \), the condition \( \alpha_0 > \alpha_c \) can be warranted. Performing Borel transformation in the \( B \) channel and n-th derivative in the \( \eta_c, \chi_{c0} \) channels, then matching with Eq.(25), finally we obtain the sum rules for the nonfactorizable soft matrix elements,

\[
\langle \eta_c(p - k) K(-q) | \bar{O}(0) | B(p - q) \rangle = \frac{im_b}{4\pi^2 f_B m_B} \int_{m_b^2}^{s_B} ds \int_{x_i}^{x_f} dx \int_{\alpha_0}^{\alpha_c} d\alpha \int_{\beta}^{\beta'} d\beta' \left[ f_{3K\bar{K}}(1 - \alpha, \alpha - \beta, \beta) \right. \\
\left. \left( (s - P^2)(1 - \frac{\alpha_0}{\beta}) - \left( 1 - \frac{\alpha_0}{2\beta} \right) \frac{m_b^2 - m_{\eta_c}^2 (1 - \alpha) + \alpha(s - P^2 - m_{\eta_c}^2)}{\alpha} \right) \right] 
\]

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\[-m_b f_K (\tilde{\phi}_\parallel (1 - \alpha, \alpha - \beta, \beta) + \tilde{\phi}_\perp (1 - \alpha, \alpha - \beta, \beta)) \]
\[-m_b f_K (\tilde{\phi}_\perp (1 - \alpha, \alpha - \beta, \beta)(\frac{2\alpha_c}{\beta} - 3) \]
\[\exp\left(\frac{m_B^2 \alpha + m_h^2 (1 - \alpha) - m_b^2}{M^2 \alpha}\right) \left(\frac{m_h^2 + Q_0^2}{s + Q_0^2}\right)^{n+1} \frac{1}{\alpha \beta} ; \quad (42)\]

\[\langle \chi_{00}(p-k)K(-q)|\tilde{O}(0)|B(p-q) \rangle = \frac{m_b m_c}{8\pi^2 f_B f_{\chi_{00}} m_B^2 m_{\chi_{00}}} \int_{x_{c0}}^{s_{c0}} ds \int_{x_i}^{x_f} dx \int_{\alpha_0}^{1} d\alpha \int_{\alpha_c}^{\alpha} d\beta [f_{3K}\phi_{3K}(1 - \alpha, \alpha - \beta, \beta)\]
\[\frac{m_{\chi_{00}}^2 - m_b^2}{\alpha} - m_b f_K (\phi_\parallel (1 - \alpha, \alpha - \beta, \beta) - 2\phi_\perp (1 - \alpha, \alpha - \beta, \beta)) \]
\[\left\{ x - (1 - x) \left(1 - 2\frac{\alpha_c}{\beta}\right) \right\} \]
\[\exp\left(\frac{m_B^2 \alpha + m_{\chi_{00}}^2 (1 - \alpha) - m_b^2}{M^2 \alpha}\right) \left(\frac{m_{\chi_{00}}^2 + Q_0^2}{s + Q_0^2}\right)^{n+1} \frac{1}{\alpha \beta (1 - x)} . \quad (43)\]

In above expressions, $P^2$ is chosen to be large space-like squared momentum ($|P^2| \sim m_b^2$) in order to stay far away from the hadronic thresholds in the channels of the B and charmonia currents, the values of $\alpha_c$ are small positive quantities but not always small enough to be safely neglected, we can perform the following approximation for the $\beta$ integral,

\[\int_{\alpha_c}^{\alpha} d\beta G(s, x, \alpha, \beta) = \left\{ \int_{0}^{\alpha} - \int_{0}^{\alpha_c} \right\} d\beta G(s, x, \alpha, \beta), \quad (44)\]

here $G$ is an abbreviation for the integral functions and can be written as

\[G(s, x, \alpha, \beta) = A(s, x, \alpha, \beta)\phi_{3K}(1 - \alpha, \alpha - \beta, \beta) + B(s, x, \alpha, \beta)\tilde{\phi}_\parallel (1 - \alpha, \alpha - \beta, \beta) + C(s, x, \alpha, \beta)\tilde{\phi}_\perp (1 - \alpha, \alpha - \beta, \beta)\]

or

\[G(s, x, \alpha, \beta) = D(s, x, \alpha, \beta)\phi_{3K}(1 - \alpha, \alpha - \beta, \beta) + E(s, x, \alpha, \beta)\phi_\parallel (1 - \alpha, \alpha - \beta, \beta) + F(s, x, \alpha, \beta)\phi_\perp (1 - \alpha, \alpha - \beta, \beta),\]

$A, B, C, D, E, F$ are formal notations. We can expand the light-cone distribution amplitudes $\phi_{3K}, \tilde{\phi}_\parallel, \tilde{\phi}_\perp, \phi_\parallel$ and $\phi_\perp$ in terms of Taylor series of $\beta$ **, for example,

\[\phi_{3K}(1 - \alpha, \alpha - \beta, \beta) = \phi_{3K}(1 - \alpha, \alpha - \beta, \beta)|_{\beta=0} + \frac{\partial}{\partial \beta}\phi_{3K}(1 - \alpha, \alpha - \beta, \beta)|_{\beta=0}\beta \]
\[+ \frac{1}{2} \frac{\partial^2}{\partial \beta^2}\phi_{3K}(1 - \alpha, \alpha - \beta, \beta)|_{\beta=0}\beta^2 + \cdots , \quad (45)\]

**For very small $\alpha_c$, we can approximate the integral $\int_{0}^{\alpha_c} d\beta G(s, x, \alpha, \beta)$ by $\alpha_c G(s, x, \alpha, \beta)|_{\beta\rightarrow 0}$, then analytically continue $P^2$ into the timelike region, $P^2 = m_B^2$.\]

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and analytically continue \( P^2 \) into the timelike region, \( P^2 = m_K^2 \), then complete the integral \( \int_0^{\alpha_c} d\beta G(s, x, \alpha, \beta) \). The explicit expressions for the physical matrix elements \( \langle \eta_c(p)K(-q)|\bar{O}(0)|B(p - q) \rangle \) and \( \langle \chi_{c0}(p)K(-q)|\bar{O}(0)|B(p - q) \rangle \) are lengthy due to the re-summation of all the Taylor series of \( \beta \), here we show only the leading terms explicitly,

\[
\langle \eta_c(p)K(-q)|\bar{O}(0)|B(p - q) \rangle = \frac{m_k^4}{4\pi^2 f_B f_{\eta_c} m_B^2} \int_{x_0}^{x_f} dx \int_{\alpha_0}^{\alpha} d\alpha \left\{ f_{3K} \left( \int_0^\alpha d\beta \phi_{3K}(1 - \alpha, \alpha - \beta, \beta) - \int_0^{\alpha_c} d\beta \phi_{3K}(1 - \alpha, \alpha - \beta, \beta)|_{\beta=0} \right) \right\} \\
\left\{ (s - m_B^2)(1 - \frac{\alpha_c}{\beta}) - (1 - \frac{\alpha_c}{\beta}) \frac{m_k^2 - m_{\eta_c}^2 (1 - \alpha) + \alpha(s - m_B^2 - m_{\eta_c}^2)}{\alpha} \right\} \\
- m_k f_{3K} \left\{ \int_0^\alpha d\beta \hat{\phi}_\parallel (1 - \alpha, \alpha - \beta, \beta) + \hat{\phi}_\perp (1 - \alpha, \alpha - \beta, \beta)|_{\beta=0} \right\} \\
- \int_0^{\alpha_c} d\beta \hat{\phi}_\perp (1 - \alpha, \alpha - \beta, \beta) - \int_0^{\alpha_c} d\beta \hat{\phi}_\perp (1 - \alpha, \alpha - \beta, \beta)|_{\beta=0} \right\} \\
\left( 2\frac{\alpha_c}{\beta} - 3 \right) \text{Exp} \left( \frac{m_{\eta_c}^2 + m_{\eta_c}^2 (1 - \alpha) - m_k^2}{M^2 \alpha} \right) \left( \frac{m_{\eta_c}^2 + Q_0^2}{s + Q_0^2} \right)^{n+1} \frac{1}{\alpha \beta} + \cdots, \tag{46} \right.

\[
\langle \chi_{c0}(p)K(-q)|\bar{O}(0)|B(p - q) \rangle = \frac{m_k m_{\chi_{c0}}}{8\pi^2 f_B f_{\chi_{c0}} m_B^2 m_{\chi_{c0}}} \int_{x_0}^{x_f} dx \int_{\alpha_0}^{\alpha} d\alpha \left\{ f_{3K} \frac{m_{\chi_{c0}}^2 - m_k^2}{\alpha} \right\} \\
\left\{ \int_0^\alpha d\beta \phi_{3K}(1 - \alpha, \alpha - \beta, \beta) - \int_0^{\alpha_c} d\beta \phi_{3K}(1 - \alpha, \alpha - \beta, \beta)|_{\beta=0} \right\} \\
- m_k f_{3K} \left\{ \int_0^\alpha d\beta (\phi_\parallel (1 - \alpha, \alpha - \beta, \beta) - 2\phi_\perp (1 - \alpha, \alpha - \beta, \beta)) \right\} \\
- \int_0^{\alpha_c} d\beta (\phi_\parallel (1 - \alpha, \alpha - \beta, \beta)|_{\beta=0} - 2\phi_\perp (1 - \alpha, \alpha - \beta, \beta)|_{\beta=0}) \right\} \\
\left\{ x - (1 - x) \left( 1 - 2\frac{\alpha_c}{\beta} \right) \right\} \\
\text{Exp} \left( \frac{m_B^2 \alpha + m_{\chi_{c0}}^2 (1 - \alpha) - m_k^2}{M^2 \alpha} \right) \left( \frac{m_{\chi_{c0}}^2 + Q_0^2}{s + Q_0^2} \right)^{n+1} \frac{1}{\alpha \beta (1 - x)} + \cdots. \tag{47} \right.

In performing the \( \beta \) integral \( \int_0^{\alpha_c} \), we need only the values of the light-cone distribution amplitudes \( \phi_{3K}, \hat{\phi}_\parallel, \hat{\phi}_\perp, \phi_\parallel, \phi_\perp \) and their derivations at zero momentum fraction i.e. \( \beta = 0 \), there are no problems with negative partons (quarks and gluons) momentum fractions. The analytical continuation of \( P^2 \) to its positive value ends up with an unavoidable theoretical uncertainty, if only a few terms of the Taylor series are taken, smaller \( |\alpha_c| \) (In the Chiral limit \( m_c = 0, |s/(s - m_B^2)| \)) with greater precision, however, with the re-summation to all orders of \( \beta \) in Eq.(45), the assumption of quark-hadron duality is still applicable in the case of heavy meson final states. This procedure ensures the disappearance of the unphysical momentum \( k \) from the ground state contribution and enables the extraction of the physical matrix elements \( \langle \eta_c(p)K(-q)|\bar{O}(0)|B(p - q) \rangle \) and \( \langle \chi_{c0}(p)K(-q)|\bar{O}(0)|B(p - q) \rangle \) due to the simultaneous

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conditions, $P^2 = m_B^2$ and $(p-q)^2 = m_B^2$. The light-cone distribution amplitudes $\phi_{3K}, \tilde{\phi}_\parallel, \tilde{\phi}_\perp, \phi_\parallel$ and $\phi_\perp$ are analytical functions, as well as only numerical values are concerned, we can analytically continue $P^2$ into the timelike region, $P^2 = m_B^2$, if the unphysical negative momentum fractions of the partons (quarks and gluons) are taken for granted, we can then take the integral $\int d\beta G(s, x, \alpha, \beta)$ directly, the numerical results will not make difference. In the channel $B \to \pi\pi$, the duality region is about $s_\pi = 0.7\text{GeV}^2 \ll |P^2| \sim m_B^2$, the quantity $s/(s - P^2) = -s/(P^2(1 - s/P^2))$ is small, and the expansion in terms of Taylor series of $s/P^2$ converges quickly, the results will not depend significantly on the end point values of the light-cone distribution amplitudes. As for the channels $B \to \eta_cK, \chi_{c0}K$, the duality regions are about $s_{\eta_c} = 11\text{GeV}^2$ and $s_{\chi_{c0}} = 13\text{GeV}^2$, the values of $s/|P^2|$ are about 44% and 52%, respectively. The expansion in terms of Taylor series of $\beta$ for the light-cone distribution amplitudes converges more slowly and more terms should be taken into account, for example, $|\alpha_c| \leq 36\%$ in the $B \to \eta_cK$ channel and $|\alpha_c| \leq 55\%$ in the $B \to \chi_{c0}K$ channel; the main drawback of this approach is the significant dependence on the end point values of those light-cone distribution amplitudes.

**IV. NUMERICAL RESULTS**

Next, we choose the input parameters entering the light-cone sum rules before giving numerical predictions on the nonfactorizable soft contributions.

The parameters enter the decays $B \to \eta_cK, \chi_{c0}K$ are taken as $m_B = 5.28 \text{ GeV}$, $f_B = 180 \pm 30 \text{ MeV}$, $m_b = 4.7 \pm 0.1 \text{ GeV}$, $s_B = 35 \pm 2 \text{GeV}^2$, $m_{\eta_c} = 3.0 \text{ GeV}$, $f_{\eta_c} = 0.35 \text{ GeV}$, $m_c = 1.25 \pm 0.05$, $s_{\eta_c} = 11 \pm 1 \text{GeV}^2$, $m_{\chi_{c0}} = 3.41 \text{ GeV}$, $f_{\chi_{c0}} = 0.36 \text{ GeV}$, $s_{\chi_{c0}} = 13 \pm 1 \text{GeV}^2$, and $f_K = 0.16 \text{ GeV}$ [25]. The value of decay constant $f_{\chi_{c0}}$ is taken from Ref. [26] and new estimation based on the nonperturbative approach of coupled Schwinger-Dyson equation and Bethe-Salpeter equation is in preparation. For the coefficients of the twist-3 and twist-4 kaon meson light-cone distribution amplitudes, we can make an approximation $f_{3\pi} \simeq f_{3K}$, $\delta_K^2 \simeq \delta_\pi^2$ and take $f_{3K} = 0.0026 \text{ GeV}$, $\delta^2(\mu_b) = 0.17 \text{ GeV}$, where $\mu_b = \sqrt{m_B^2 - m_b^2} \sim 2.4 \text{ GeV}$ [9,23,24].

Here we will take a short digression to discuss the duality regions. In the $B \to \eta_cK$ channel, as the axial-vector current $J_\mu = \bar{c}\gamma_\mu \gamma_5 c$ in stead of pseudoscalair current $J_5 = i\bar{c}\gamma_5 c$ is chosen to interpolate the $\eta_c$ meson, we must be careful in choosing the duality region to avoid possible pollutions from the $\eta_c(2s)$ and $\chi_{c1}$ mesons with the same quantum numbers as the interpolating current. The masses of those two mesons are about $m_{\eta_c(2s)} = 3.6\text{GeV}$, $m_{\chi_{c1}} = 3.5\text{GeV}$, and the widths of those mesons are narrow, we can choose the duality region to be $s_{\eta_c} = 11 \pm 1 \text{GeV}^2$. In the
$B \to \chi_{c0}K$ channel, due to narrow width of the $\chi_{c0}$ meson, we can choose the duality region to be $s_{\chi_{c0}} = 13 \pm 1 GeV^2$ to avoid possible pollutions from the excited and continuum states. Furthermore, larger $s$ means larger $|\alpha_c|$, more Taylor series of $\beta$ have to be re-summed, heavy dependence on the end point values of the light-cone distribution amplitudes. In those two channels, the variation of the duality thresholds $s_{\eta_c}$ and $s_{\chi_{c0}}$ can lead to large uncertainties.

The parameters $n$ and $M^2$ must be carefully chosen to guarantee the excited and continuum states to be suppressed and to obtain a reliable perturbative QCD calculation. The stable region for the Borel parameter $M^2$ is found in the interval $M^2 = 10 \pm 2 GeV^2$ which is known from the $B$ channel QCD sum rules [9]. In the charmonium channels, we usually perform $n$-th derivative and take $n$-th moment sum rules to satisfy the stability criteria [25].

For the decay $B \to \eta_cK$, the calculation is rather stable on the range $n = 3 - 7$. $Q^2_0$ is parameterized by $Q^2_0 = 4m^2_c\xi$, where $\xi$ is usually allowed to take values from 0 to 1 while the best interval is $\xi = 0.3 - 1$. In the following, we write down the numerical value for the nonfactorizable soft matrix element,

$$\langle \eta_c(p)K(q)|\bar{O}(0)|B(p + q)\rangle = 0.035 \pm 0.010 GeV^3,$$

the largest uncertainties come from the variations of the mass of the $c$ quark.

Taking into account the next-to-leading order Wilson coefficients calculated in the naive dimensional regularization scheme [15] for $\mu = m_b(m_b) = 4.40 GeV$ and $\Lambda^{(5)}_{\overline{MS}} = 225 MeV$,

$$C_1(m_b(m_b)) = 1.082, \quad C_2(m_b(m_b)) = -0.185, \quad C_3(m_b(m_b)) = 0.014,$$

$$C_4(m_b(m_b)) = -0.035, \quad C_5(m_b(m_b)) = 0.009, \quad C_6(m_b(m_b)) = -0.041,$$

here we have neglected the Wilson coefficients $C_7, C_8, C_9, C_{10}$ in numerical calculation due to their small values, finally we obtain the numerical relation between the contributions from the factorizable and nonfactorizable matrix elements,

$$\frac{\{2V_{cb}V_{cs}^*C_1(\mu) - 2V_{tb}V_{ts}^*[C_4(\mu) + C_6(\mu)]\} \langle \eta_c(p)K(q)|\bar{O}(0)|B(p + q)\rangle}{\left\{V_{cb}V_{cs}^* \left[ C_2(\mu) + \frac{C_4(\mu)}{3} \right] - V_{tb}V_{ts}^* \left[ C_3(\mu) - C_5(\mu) + \frac{C_4(\mu) - C_6(\mu)}{3} \right] \right\} f_{\eta_c}m^2_BF_0(m^2_{\eta_c})} = 0.11 \pm 0.04.$$

From above expressions, we can see that the nonfactorizable soft contributions are considerable and they must be included in analyzing the branching fraction. A rough estimation shows that the theoretical branching fraction will increase to about 1.15–1.30 times as the naive factorization result. Although there are still large mismatches between the theoretical and experimental values, we can say that the theoretical prediction is considerably improved. The consistent and complete QCD LCSR analysis should include all the contributions from $O(\alpha_s)$ corrections, however, the calculation is cumbersome and we prefer another article.
For the decay $B \to \chi_{c0}K$, the calculation is rather stable on the range $n = 5 - 10$, the parameter $\xi$ is usually allowed to take values larger than 1 while the best interval is $\xi = 1.2 - 2.0$. The nonfactorizable soft hadronic matrix element in the $B \to \chi_{c0}K$ decay is

$$
\langle \chi_{c0}(p)K(q)|\bar{\mathcal{O}}(0)|B(p+q)\rangle = 0.22 \pm 0.08 GeV^3,
$$

and the corresponding branching fraction is $(1.0 \pm 0.6) \times 10^{-4}$ which is smaller than the experimental data $(6.0 \pm 2.1) \times 10^{-4}$ [14]. From Eq.(48) and Eq.(51), we can see that value of the nonfactorizable soft hadronic matrix element $\langle \eta_c(p)K(q)|\bar{\mathcal{O}}(0)|B(p+q)\rangle$ is about 15% of the $\langle \chi_{c0}(p)K(q)|\bar{\mathcal{O}}(0)|B(p+q)\rangle$, and contributes to the branching fractions with magnitude about $10^{-5}$. Take into account for the contributions from the factorizable hadronic matrix elements, the theoretical predicted branching fraction is about 10% of the corresponding experimental data for the decay $B \to \eta_cK$. In this article, we take into account only the nonfactorizable soft contributions of twist-3 and twist-4, while higher twist contributions are neglected. Although there are large uncertainties due to discarding the higher twist contributions and varying the duality thresholds, we can estimate the order of magnitude of the nonfactorizable soft effects at least.

V. CONCLUSION

In this article, we have analyzed the contributions from the nonfactorizable soft hadronic matrix elements

$$
\langle \eta_c(p)K(q)|\bar{\mathcal{O}}(0)|B(p+q)\rangle \quad \text{and} \quad \langle \chi_{c0}(p)K(q)|\bar{\mathcal{O}}(0)|B(p+q)\rangle
$$

to the decays $B \to \eta_cK, \chi_{c0}K$ with the effective weak Hamiltonian. As the QCD-improved factorization approach breaks down in the B to charmonia decays, the contributions from the soft gluon exchanges will signalize themselves. Our numerical results show that their contributions are considerable and should not be neglected for the decay $B \to \eta_cK$. Although there are still large mismatches between the theoretical and experimental values, the theoretical predicted branching fraction is considerably improved (about $(15 - 30)\%$). As for the decay $B \to \chi_{c0}K$, there are no factorizable contributions, the nonperturbative contributions from the nonfactorizable soft matrix elements are about five times smaller than the experimental data. The consistent and complete QCD LCSR analysis should include all the contributions from $O(\alpha_s)$ corrections and higher twist contributions.
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