RANDOM PLANAR LATTICES AND INTEGRATED SUPERBROWNIAN EXCURSION

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ABSTRACT. In this paper, a surprising connection is described between a specific brand of random lattices, namely planar quadrangulations, and Aldous’ Integrated SuperBrownian Excursion (ISE). As a consequence, the radius $r_n$ of a random quadrangulation with $n$ faces is shown to converge, up to scaling, to the width $r = R - L$ of the support of the one-dimensional ISE, or precisely:

$$n^{-1/4} r_n \xrightarrow{law} (8/9)^{1/4} r.$$

More generally the distribution of distances to a random vertex in a random quadrangulation is described in its scaled limit by the random measure ISE shifted to set the minimum of its support in zero.

The first combinatorial ingredient is an encoding of quadrangulations by trees embedded in the positive half-line, reminiscent of Cori and Vauquelin’s well labelled trees. The second step relates these trees to embedded (discrete) trees in the sense of Aldous, via the conjugation of tree principle, an analogue for trees of Vervaat’s construction of the Brownian excursion from the bridge.

From probability theory, we need a new result of independent interest: the weak convergence of the encoding of a random embedded plane tree by two contour walks $(e^{(n)}, \hat{W}^{(n)})$ to the Brownian snake description $(e, \hat{W})$ of ISE.

Our results suggest the existence of a Continuum Random Map describing in term of ISE the scaled limit of the dynamical triangulations considered in two-dimensional pure quantum gravity.

1. INTRODUCTION

From a distant perspective, this article uncovers a surprising, and hopefully deep, relation between two famous models: random planar maps, as studied in combinatorics and quantum physics, and Brownian snakes, as studied in probability theory and statistical physics. More precisely, our results connect some distance-related functionals of random quadrangulations with functionals of Aldous’ Integrated SuperBrownian Excursion (ISE) in dimension one.

Quadrangulations. On the one hand, quadrangulations are finite plane graphs with four-regular faces (see Figure 1 and Section 2 for precise definitions). Random quadrangulations, like random triangulations, random polyhedra, or the $\phi^4$-models of physics, are instances of a general family of random lattices that has received considerable attention both in combinatorics (under the name random planar maps, following Tutte’s terminology [34]) and in physics (under the name Euclidean two-dimensional discretised quantum geometry, or simply dynamical triangulations or fluid lattices [3, 10, 17]).

Many probabilistic properties of random planar maps have been studied, that are local properties like vertex or face degrees [3, 16], or 0 – 1 laws for properties expressible in first order logic [8]. Other well documented families of properties are related to connectedness and constant size separators [6], also known as branchings...
In this article we consider another fundamental aspect of the geometry of random maps, namely \textit{global properties of distances}. The \textit{profile} $(H^k_n)_{k \geq 0}$ and \textit{radius} $r_n$ of a random quadrangulation with $n$ faces are defined in analogy with the classical profile and height of trees: $H^k_n$ is the number of vertices at distance $k$ from a basepoint, while $r_n$ is the maximal distance reached. The profile was studied (with triangulations instead of quadrangulations) by physicists Watabiki, Ambjørn \textit{et al.} \cite{4,35} who gave a consistency argument proving that the only possible scaling for the profile is $k \sim n^{1/4}$, a property which reads in their terminology \textit{the internal Hausdorff dimension is 4}. Independently the conjecture that $\mathbb{E}(r_n) \sim cn^{1/4}$ was proposed by Schaeffer \cite{31}.

\textbf{Integrated SuperBrownian Excursion.} On the other hand, ISE was introduced by Aldous as a model of random distributions of masses \cite{1}. He considers random embedded discrete trees as obtained by the following two steps: first an abstract tree $t$, say a Cayley tree with $n$ nodes, is taken from the uniform distribution and each edge of $t$ is given length 1; then $t$ is embedded in the regular lattice on $\mathbb{Z}^d$, with the root at the origin, and edges of the tree randomly mapped on edges of the lattice. Assigning masses to leaves of the tree $t$ yield a random distribution of mass on $\mathbb{Z}^d$. Upon scaling the lattice to $n^{-1/4} \mathbb{Z}^d$, these random distributions of mass admit, for $n$ going to infinity, a continuum limit $\mathcal{J}$ which is a random probability measure on $\mathbb{R}^d$ called ISE.

Derbez and Slade proved that ISE describes in dimension larger than eight the continuum limit of a model of lattice trees \cite{15}, while Hara and Slade obtained the same limit for the incipient infinite cluster in percolation in dimension larger than six \cite{18}. As opposed to these works, we shall consider ISE in dimension one and our embedded discrete trees should be thought of as folded on a line. The support of ISE is then a random interval $(L, R)$ of $\mathbb{R}$ that contains the origin.

\textbf{From quadrangulations to ISE.} The purpose of this paper is to draw a relation between, on the one hand, random quadrangulations, and, on the other hand, Aldous’ ISE: upon proper scaling, the profile of a random quadrangulations is described in the limit by ISE translated to have support $(0, R - L)$. This relation implies in particular that the radius $r_n$ of random quadrangulations, again upon scaling, weakly converges to the width of the support of ISE in one dimension, that is the continuous random variable $r = R - L$. We shall indeed prove (Corollary \ref{cor:main})
that
\[ n^{-1/4} r_n \xrightarrow{\text{law}} (8/9)^{1/4} r, \]
as well as the convergence of moments. While this proves the conjecture \( \mathbb{E}(r_n) \sim cn^{1/4} \), the value of the constant \( c \) remains unknown because, as mentioned by Aldous [1], little is known on \( R \) or \( R - L \).

The path from quadrangulations to ISE consists of three main steps, the first two of combinatorial nature and the last with a more probabilistic flavor. Our first step, Theorem 1, revisits a correspondence of Cori and Vauquelin [13] between planar maps and some well labelled trees, that can be viewed as plane trees embedded in the positive half-line. Thanks to an alternative construction [31, Ch. 7], we show that under this correspondence the profile can be mapped to the mass distribution on the half-line. In particular, the radius \( r_n \) of a random quadrangulation is equal in law to the maximal label \( \mu_n \) of a random well labelled tree.

Safe for the positivity condition, well labelled trees would be constructed exactly according to Aldous’ prescription for embedded discrete trees. Well labelled trees are thus to Aldous’ embedded trees what the Brownian excursion is to the Brownian bridge, and we seek an analogue of Vervaat’s relation. At the discrete level a classical elegant explanation of such relations is based on Dvoretzky and Motzkin’s cyclic shifts and cycle lemma. Our second combinatorial step, Theorem 3, consists in the adaptation of these ideas to embedded trees. More precisely, via the conjugation of tree principle of [31, Chap. 2], we bound the discrepancy between the mass distribution of our conditioned trees on the positive half-line and a translated mass distribution of freely embedded trees. In particular we construct a coupling between well labelled trees and freely embedded trees such that the largest label \( \mu_n \), and thus the radius \( r_n \), is coupled to the width of the support \( (L_n, R_n) \) of random freely embedded trees:

\[ |r_n - (R_n - L_n)| \leq 3. \]

Since our freely embedded trees are constructed according to Aldous’ prescription, one could expect to be able to conclude directly. However two obstacles still need to be bypassed at this point.

**Contour walks and Brownian snakes.** The first obstacle is that the construction of ISE as a continuum limit of mass distributions supported by embedded discrete trees was only outlined in Aldous’ original paper. The original mathematical definition is by embedding a continuum random tree (CRT), which amounts to exchanging the embedding and the continuum limit. But Borgs et al. proved that indeed ISE is the limit of mass distributions supported by embedded Cayley trees [12] and their proof could certainly be adapted to other simple classes of trees and in particular to our embedded plane trees.

The second, more important, obstacle is that weak convergence of probability measures is not adequate to our purpose, since we are interested in particular in convergence of the width of the support, which is not a continuous functional on the space of measures. In order to circumvent this difficulty, we turn to the description of ISE in terms of superprocesses: ISE can be constructed from the Brownian snake with lifetime \( e \), the standard Brownian excursion [1, 22].

From the discrete point of view, we consider the encoding of an embedded plane tree by a pair of contour walks \((x_k, y_k)\), that encode respectively the height of the
node visited at time \( k \) and its position on the line. Our last result, Theorem 4, is the weak convergence, upon proper scaling, of this pair of walks to the Brownian snake with lifetime \( e \):

\[
\left( e^{(n)}(s), \hat{W}^{(n)}(s) \right) \xrightarrow{\text{law}} (e(s), \hat{W}_s).
\]

As \( R = \sup_s \hat{W}_s \) and \( L = \inf_s \hat{W}_s \) this convergence, together with some deviation bounds obtained in the proof allows us to conclude on the radius. (A similar weak convergence was independently proved by Marckert and Mokkadem [26] but without the deviation bounds we need here.)

More generally the joint convergence of the minimum and the mass distribution of discrete embedded trees implies that, upon scaling, the label distribution of well labelled trees converges to ISE translated to have the minimum of its support at the origin. The same then holds for the profile of random quadrangulations.

**Dynamical triangulations and a Continuum Random Map.** Although we concentrate in this article on the radius and profile of random quadrangulations, our derivation suggests a much tighter link between random quadrangulations and ISE. We conjecture that a Continuum Random Map (CRM) can be built from ISE that would describe the continuum limit of scaled random quadrangulations, in a similar way as the CRT describes the continuum limit of scaled random discrete trees. From the point of view of physics, the resulting CRM would describe in the limit the geometry of scaled dynamical triangulations as studied in discretised two-dimensional Euclidean pure quantum geometries [3, 10, 17]. We plan to discuss this connection further in future work.

**Organization of the paper.** Section 2 contains the definition the combinatorial model of random lattice. Sections 3 and 4 are devoted to the first combinatorial steps. Section 5 contains to the definition of probabilistic models, and the statement of the convergence result. Finally in Section 6 we give the proof of this convergence.

2. **The combinatorial models of random lattice**

2.1. **Planar maps and quadrangulations.** A **planar map** is a proper embedding (without edge crossings) of a connected graph in the plane. Loops and multiple edges are *a priori* allowed. A planar map is **rooted** if there is a root, *i.e.* a distinguished edge on the border of the infinite face, which is oriented counterclockwise.
The origin of the root is called the root vertex. Two rooted planar maps are considered identical if there exists an homeomorphism of the plane that sends one map onto the other (roots included).

The difference between planar graphs and planar maps is that the cyclic order of edges around vertices matters in maps, as illustrated by Figure 2. Observe that planar maps can be equivalently defined on the sphere. In particular Euler’s characteristic formula applies and provides a relation between the numbers $n$ of edges, $f$ of faces and $v$ of vertices of any planar map: $f + v = n + 2$.

The degree of a face or of a vertex of a map is its number of incidence of edges. A planar map is a quadrangulation if all faces have degree four. All (planar) quadrangulations are bipartite: their vertices can be colored in black or white so that the root is white and any edge joins two vertices with different colors. In particular a quadrangulation contains no loop but may contain multiple edges. See Figures 1 and 3 for examples of quadrangulations.

Let $Q_n$ denote the set of rooted quadrangulations with $n$ faces. A quadrangulation with $n$ faces has $2n$ edges (because of the degree constraint) and $n + 2$ vertices (applying Euler’s formula). The number of rooted quadrangulations with $n$ faces was obtained by W.T. Tutte:

$$|Q_n| = \frac{2^n}{n+2} \frac{3^n}{n+1} \binom{2n}{n}.$$  

Various alternative proofs of this result have been obtained (see e.g. 10, 13, 31). Our treatment will indirectly provide another proof, related to 13, 31.

2.2. Random planar lattices. Let $L_n$ be a random variable with uniform distribution on $Q_n$. Formally, $L_n$ is the $Q_n$-valued random variable such that for all $Q \in Q_n$

$$\Pr(L_n = Q) = \frac{1}{|Q_n|} = \frac{1}{\frac{2^n}{n+2} \frac{3^n}{n+1} \binom{2n}{n}}.$$  

The random variable $L_n$ is our random planar lattice. To explain this terminology, taken from physics, observe that locally the usual planar square lattice is a planar map whose faces and vertices all have degree four. Our random planar lattice corresponds to a relaxation of the constraint on vertices.
Classical variants of this definition are obtained by replacing quadrangulations with \( n \) faces by triangulations with \( 2n \) triangles, or by (vertex-)4-regular maps with \( n \) vertices, or by all planar maps with \( n \) edges, etc. All these random planar lattices have been considered both in combinatorics (see [6] and references therein) and in mathematical physics (see [3] and references therein; in the physics literature, definitions are usually phrased using “symmetry weights” instead of rooted objects, but this is strictly equivalent to the combinatorial definition). Although details of local topology vary between families, most probabilistic properties are believed to be “universal”, that is qualitatively analogue for all “reasonable” families. Observe also that random maps in classical families have exponentially small probability to be symmetric, so that all results hold as well as in the model of uniform unrooted maps [29].

In this article we focus on quadrangulations because of their combinatorial relation, detailed in Section 3, to well labelled trees.

2.3. The profile of a map. The distance \( d(x, y) \) between two vertices \( x \) and \( y \) of a map is the minimal number of edges on a path from \( x \) to \( y \) (in other terms all edges have abstract length 1).

The profile of a rooted map \( M \) is the sequence \((H_k)_{k \geq 1}\), where \( H_k = H_k^M \) is the number of vertices at distance \( k \) of the root vertex \( v_0 \). We shall also consider the cumulated profile \( \hat{H}_k^M = \sum_{\ell=1}^k H_k^M \). By construction the support of the profile of a rooted map is an interval, i.e. \( \{ k \mid H_k > 0 \} = [1, r] \) where \( r \) is the radius of the map (sometimes also called eccentricity). The radius \( r \) is closely related to the diameter, that is the largest distance between two vertices of a map: in particular \( r \leq d \leq 2r \). The quadrangulation of Figure 3 has radius 3.

The profile of the random planar lattice \( L_n \) is the random variable \((H_k^{(n)})_{k \geq 1}\) that is defined by taking the profile \((H_k^{[L_n]})_{k \geq 1}\) of an instance of \( L_n \), while \((\hat{H}_k^{(n)})_{k \geq 1}\) denotes the cumulated profile of \( L_n \). Similarly the radius of a random planar lattice is a positive integer valued random variable \( r_n \).

3. Encoding the profile with well labelled trees

3.1. Well labelled trees and the encoding result. A plane tree is a rooted planar map without cycle (and thus with only one face). Equivalently plane trees can be recursively defined as follows:

- the smallest tree is made of a single vertex,
- any other tree is a non-empty sequence of subtrees attached to a root.
In other terms, each vertex has a possibly empty sequence of sons, and each vertex but the root has a father. The number of plane trees with \( n \) edges is the well known Catalan number
\[
C(2n) = \frac{1}{n+1} \binom{2n}{n}.
\]

A plane tree is well labelled if all its vertices have positive integral labels, the labels of two adjacent vertices differ at most by one, and the label of the root vertex is one. Let \( W_n \) denote the set of well labelled trees with \( n \) edges.

The label distribution of a well labelled tree \( T \) is the sequence \( (\lambda_k[T])_{k \geq 1} \) where \( \lambda_k[T] \) is the number of vertices with label \( k \) in the tree \( T \). The cumulated label distribution is defined by \( \bar{\lambda}_k[T] = \sum_{\ell=1}^{k} \lambda_\ell[T] \). By construction the support of the label distribution is an interval: there exists an integer \( \mu \) such that \( \{ k \mid \lambda_k > 0 \} = [1, \mu] \). This integer \( \mu \) is the maximal label of the tree. These definitions are illustrated by Figure 4.

The following theorem will serve us to reduce the study of the profile of quadrangulations to the study of the label distribution of well labelled trees.

**Theorem 1** (Schaeffer [31]). There exists a bijection \( T \) between rooted quadrangulations with \( n \) faces and well labelled trees with \( n \) edges, such that the profile \( (H_k[Q])_{k \geq 1} \) of a quadrangulation \( Q \) is mapped onto the label distribution \( (\lambda_k[T])_{k \geq 1} \) of the tree \( T = T(Q) \).

Theorem 1 and Tutte’s formula (1) imply that the number of well labelled trees with \( n \) edges equals
\[
|W_n| = \frac{2}{n+2} \frac{3^n}{n+1} \binom{2n}{n}.
\]

This result was proved already by Cori and Vauquelin [13], who introduced well labelled trees to give an encoding of all planar maps with \( n \) edges. Because of a classical bijection between the latter maps and quadrangulations with \( n \) faces, their result is equivalent to the first part of Theorem 1. Their bijection has been extended to bipartite maps by Arquès [5] and to higher genus maps by Marcus and Vauquelin [25]. All these constructions were recursive and based on encodings of maps with permutations (also known as rotation systems).

However, our interest in well labelled trees lies in the relation between the profile and the label distribution, which does not appear in Cori and Vauquelin’s bijection. The bijection we use here is much simpler and immediately leads to the second part of Theorem 1. This approach was extended to non separable maps by Jacquard [19] and to higher genus by Marcus and Schaeffer [24].

We postpone to Section 4 the discussion of the interesting form of Formula (2) and its relation to Catalan’s numbers. Instead the rest of this part is concerned with the proof of Theorem 1, which goes in three steps. First some properties of distances in quadrangulations are indicated (Section 3.2). This allows in a second step to define the encoding, as a mapping \( T \) from quadrangulations to well labelled trees (Section 3.3). A decoding procedure allows then to prove that \( T \) is faithful (Section 3.4).
3.2. Properties of distances in a quadrangulation. Let $Q$ be a rooted quadrangulation and denote $v_0$ its root vertex. The labelling $\phi$ of the map $Q$ is defined by $\phi(x) = d(x, v_0)$ for each vertex $x$, where $d(x, y)$ denote the distance in $Q$ (cf. Figure 3). Observe that in the number of label $k$ in the labelling of the map $Q$ is precisely the number of vertices at distance $k$ of $v_0$, that is $H_k^{(Q)} = |\{x \mid \phi(x) = k\}|$. This labelling satisfies the following immediate properties:

**Property 1.** If $x$ and $y$ are joined by an edge, $|\phi(x) - \phi(y)| = 1$. Indeed the quadrangulation being bipartite, a vertex $x$ is white if and only if $\phi(x)$ is even, black if and only if $\phi(x)$ is odd.

**Property 2.** Around a face, four vertices appear: a black $x_1$, a white $y_1$, a black $x_2$ and a white $y_2$. These vertices satisfy at least one of the two equalities $\phi(x_1) = \phi(x_2)$ or $\phi(y_1) = \phi(y_2)$ (cf. Figure 3).

A face will be said simple when only one equality is satisfied and confluent otherwise (see Figure 3). It should be noted that one may have $x_1 = x_2$ or $y_1 = y_2$.

3.3. Construction of the encoding $T$. Let $Q$ be a rooted quadrangulation with its distance labelling. The map $Q'$ is obtained by dividing all confluent faces $Q$ into two triangular faces by an edge joining the two vertices with maximal label. Let us now define a subset $T(Q)$ of edges of $Q'$ by two selection rules:

- In each confluent face of $Q$, the edge that was added to form $Q'$ is selected.
- For each simple face $f$ of $Q$, an edge $e$ is selected: let $v$ be the vertex with maximal label in $f$, then $e$ is the edge leaving $v$ with $f$ on its left.

These two selection rules are illustrated by Figure 5. The first selected edge around the endpoint of the root of $Q$ is taken to be the root of $T(Q)$.

The proof of Theorem 1 is now completed in two steps. First, in the rest of this section, $T(Q)$, which is a priori only defined as a subset of edges of $Q'$ together with their incident vertices, is shown to be in fact a well labelled tree with $n$ edges. Second, in the next section the inverse mapping is described and used to prove that the mapping $T$ is faithful.

**Proposition 1.** The mapping $T$ sends a quadrangulation $Q$ with $n$ faces on a well labelled trees with $n$ edges.
Figure 6. Impossibility of cycles.

Proof. If the vertex $x$ is not the root of $Q$, then one of its neighbors in $Q$, say $y$, has a smaller label. The edge $(x, y)$ can be incident to: at least a confluent face; at least a simple face in which $x$ has maximal label; or two simple faces in which $x$ has intermediate label. In all three cases, $x$ is incident to the selected edge of at least one face. Thus all vertices of $Q$ but its root are also vertices of $\mathcal{T}(Q)$: in particular $\mathcal{T}(Q)$ has $n + 1$ vertices. Next, the number of edges of $\mathcal{T}(Q)$ is $n$, because this is the number of faces of $Q$ and two faces cannot select the same edge (as immediately follows from inspection of the selection rules). Now the planarity of $Q$ and thus of $Q'$ grants that each connected component of $\mathcal{T}(Q)$ is planar. Provided we can rule out cycles, this imply that $\mathcal{T}(Q)$ is a forest of trees with $n$ edges and $n + 1$ vertices, i.e. a single tree. This tree is then clearly well labelled.

Suppose now that there exists a cycle in $\mathcal{T}(Q)$ and let $e \geq 0$ be the value of the smallest label of a vertex of this cycle. Either all these labels are equal to $e$, or there is in the cycle an edge $(e, e + 1)$ and an edge $(e + 1, e)$. In both cases the rules of selection of edges imply that each connected component of $Q'$ defined by the cycle contains a vertex $x$ (resp. $y$) with label $e - 1$, as shown by Figure 6. According to Jordan’s theorem, either the shortest path from $x$ to the root or the shortest path from $y$ to the root has to intersect the cycle, leading to a contradiction with the definition of labels by distances. There are thus no cycles and $\mathcal{T}(Q)$ is a tree.

3.4. The inverse $Q$ of the mapping $\mathcal{T}$. Let $T$ be a well labelled tree with $n$ edges. Recall that the tree $T$ can be viewed as a planar map that has a unique face $F_0$. A corner is a sector between two consecutive edges around a vertex. A vertex of degree $k$ defines $k$ corners and the total number of corners of $T$ is $2n$. The label of a corner is by definition the label of the corresponding vertex.

The image $Q(T)$ is defined in three steps.

1. A vertex $v_0$ with label 0 is placed in the face $F_0$ and one edge is added between this vertex and each of the $\ell$ corners with label 1. The new root is taken to be the edge arriving from $v_0$ at the corner before the root of $T$.

After Step (1) a uniquely defined rooted map $T_0$ with $\ell$ faces has been obtained (see Figure 3, with $\ell = 5$). The next steps take place independently in each of those faces and are thus described for a generic face $F$ of $T_0$. Let $k$ be the degree of $F$ (by construction $k \geq 3$). Among the corners of $F$ only one belongs to $v_0$ and has label 0. Let the corners be numbered from 1 to $k$ in clockwise order along the border, starting right after $v_0$. Let moreover $e_i$ be the label of corner $i$ (so that

1The infinite face is only apparently different from the others: imagine the map on the sphere.
$e_1 = e_{k-1} = 1$ and $e_k = 0$). In Figure 7, the corners are explicitly represented with their numbering for one of the faces.

2. The function successor $s$ is defined for all corners $1, \ldots, k - 1$ by

$$s(i) = \inf \{ j > i \mid e_j = e_i - 1 \}.$$

For each corner $i \geq 2$ such that $s(i) \neq i + 1$, a cord $(i, s(i))$ is added inside the face, in such a way that the various cords do not intersect (Property 3).

Once this construction has been carried on in each face, a planar map $T'$ is obtained.

3. All edges of with labels of the form $(e, e)$ of $T'$ are deleted. The resulting map is a quadrangulation $Q(T)$ with $n$ faces (Property 4).

The following proposition ends the proof of Theorem 1.

**Proposition 2.** The mapping $Q$ is the inverse bijection of the mapping $T$.

Let us first prove the two properties that validate the preceding construction.

**Property 3.** The cords $(i, s(i))$ do not intersect.

**Proof.** Suppose that two cords $(i, s(i))$ and $(j, s(j))$ cross each other. Upon maybe exchanging $i$ and $j$ one has $i < j < s(i) < s(j)$. The first two inequalities imply, together with the definition of $s$, that $e_j > e_{s(i)}$, while the two last inequalities imply $e_{s(i)} \geq e_j$. This is a contradiction. \hfill \Box

**Property 4.** The faces of $T'$ are of one of the two types of Figure 8: either triangular with labels $e, e + 1, e + 1$, or quadrangular with labels $e, e + 1, e + 2, e + 1$. The faces of $Q(T)$ are all quadrangular.

**Proof.** Let $f$ be a face of $T'$. The face $f$ is included in a face $F$ of $T_0$ so that its corners inherit the numbering and labelling of those of $F$. Let $j$ be the corner with largest number in $f$ and $i_1 < i_2 < j$ its two neighbors in $f$ (cf. Figure 8). Let us compute the label of the corners $i_1$ and $i_2$: the edge $(i_1, j)$ is a cord by construction so that $j = s(i_1)$ and $e_{i_1} = e_j + 1$; moreover, as $i_1 < i_2 < j$, this imply $e_{i_2} \geq e_{i_1}$ (or $j$ would not be $s(i_1)$) and finally $e_{i_2} = e_{i_1} = e_j + 1$.

By construction and planarity, no cord can arrive at $i_1$ between the unique leaving cord $(i_1, j)$ and the edge $(i_1, i_1 + 1)$ of $F$. The latter edge thus borders the face $f$. There are then two cases, as illustrated by Figure 8:

- if $e_{i_1 + 1} = e_{i_1}$, then $i_2 = i_1 + 1$, and the face is triangular (left hand figure),
that of cardinalities, as granted for instance by the alternative bijection of [3], proves between well labelled trees and a subset \( T \). The selection rules for accordingly, \( T \) selects an edge of \( T \) (otherwise \( s(t_1 + 1) < t_2 \) with \( e_{s(t_1 + 1)} = e_{t_1 + 1} - 1 = e_t \) and a cord \( f(s(t_1 + 1), j) \) would exclude \( i_2 \) from the face \( f \): the face is quadrangular (right hand side in Figure 8).

Observe finally that the deletion of edges with labels of the form \((e, e)\) will join triangular faces two by two so that \( Q(T) \) has only quadrangular faces.

**Proof of Proposition 2.** Given a well labelled tree \( T \), faces of its image \( Q(T) \) are as described by Figure 8. The selection rules for \( T \) then shows that each face correctly selects an edge of \( T \), so that \( T(Q(T)) = T \). Thus \( T \) and \( Q \) are inverse bijections between well labelled trees and a subset \( \tilde{Q}_n \) of the set of quadrangulations. Equality of cardinalities, as granted for instance by the alternative bijection of [3], proves that \( \tilde{Q}_n \) is the full set of quadrangulations with \( n \) faces and concludes the proof. However, we provide below a direct proof of the equality \( Q(T(Q)) = Q \) for any quadrangulation \( Q \), for this provides better understanding of the bijection.

Let \( Q \) be a quadrangulation, \( Q' \) and \( T = T(Q) \) the map and tree as in Section 3.3 and \( T_0, T' \) as in the construction of \( Q(T) \). Consider first the selection rules applied around the root to construct \( T(Q) \). Each edge (with labels) 1-2 of \( T(Q) \) forms a directly oriented corner with an edge 1-0 in its face of creation, while each edge 1-1 forms two such corners (one on each side). Hence, in accordance with Step (1) of the reciprocal construction, an edge 1-0 arrives at each corner with label 1. Thus the submap \( T_0 \) of \( T' \) is also a submap of \( Q' \). Moreover \( T_0 \) covers all vertices of \( T' \) (resp. \( Q' \)), so that edges of \( T' \) (resp. \( Q' \)) not in \( T_0 \) are cords of faces of \( T_0 \). Accordingly, \( T' = Q' \) if, inside each face of \( T_0 \), both \( T' \) and \( Q' \) have the same cords.

The maps \( Q' \) and \( T' \) have the same vertices, and, due to Property 4, the same number of faces of degree 4 (that is, the number of edges \( i - (i + 1) \) in \( T \)), and the same number of faces of degree 3 (that is, the number of edges \( i - i \) in \( T \)). By Euler’s formula, they thus have also the same number of edges and finally, they are equal as planar maps if, inside all faces of \( T_0 \), each cord of \( T' \) is a cord of \( Q' \).

Let us now work inside a face \( F \) of \( T_0 \) (see Figure 8). By construction of \( T_0 \) the face \( F \) has only one corner with label 0 (incident to the root) and two corners with label 1 (since such a corner is incident to an edge \((0, 1)\) after Step (1)). If \( F \) has degree 3 (resp. 4), the corners around the face, in clockwise order, are labelled 0-1-1 (resp. 0-1-2-1), and there is no cord, neither in \( Q' \) nor in \( T' \). Let us thus assume that \( F \) has degree \( k \) larger than 4, and number the \( k \) corners of \( F \) in clockwise order starting after the root corner. Let \( e_i \) be the label of the \( i \)th corner, so that \( e_1 = 1, e_2 = 2, e_{k-2} = 2, e_{k-1} = 1, e_k = 0, \) and \( e_i \geq 2 \) otherwise. This numbering

**Figure 8.** Two possible sizes for \( f \): triangular or quadrangular.
corresponds to the one used to define cords of $T'$; for each corner $i$ with $e_i = 2$ but the last one, $s(i) = k - 1$ and the cord $(i, k - 1)$ appears in $T'$.

In order to check that these cords appear also in $Q'$ we consider corners with label 2 in increasing order: they are numbered $2 = i_1 < i_2 < \ldots < i_p = k - 2$. In $Q'$ let $f_1$ be the face that contains the corners numbered $k, 1$ and $k - 1$ (with labels 0, 1 and 1). The face $f_1$ contains a fourth corner, with label 2: it can be $i_p$ (if the cord $(1, i_p)$ is in $Q'$) or $i_1$ (if the cord $(i_1, k - 1)$ is in $Q'$). In the first case the face $f_1$ in $Q'$ has corners $(1, i_p, k - 1, k)$ and there is a contradiction with the fact that the edge $(i_p, k - 1)$ has not been selected in $F$ by the construction $T$. Hence the second case hold: the cord $(i_1, k - 1)$ is in $Q'$ and the edge $(1, i_1)$ was selected.

Now assume that the edges $(i_j', k - 1)$ belong to $Q'$ for $j' < j < k - 2$ and check that $(i_j, k - 1)$ belongs to $Q'$. Consider in $Q'$ the face $f_j$, included in the face $F$ and bordering the edge $(i_j - 1, i_j)$ of the cycle $F$. If $e_{i_j - 1} = 2$ then $i_j - 1 = i_{j - 1}$ and the face $f_j$ is triangular (since the selection rule of confluent faces was applied by $T$) and contains the cord $(i_{j - 1}, k - 1)$. Otherwise $e_{i_{j - 1}} = 3$ and the face $f_j$ is quadrangular and of simple type (since the selection rule of simple faces was applied by $T$). Therefore there is an edge from $i_j$ to a corner with label 1, which can only be the cord $(i_j, k - 1)$, for a cord $(i_j, 1)$ would cross $(i_1, k - 1)$.

All cords with labels 1-2 are thus identical in $T'$ and $Q'$. Let $T_1$ be the union of $T_0$ and these cords. In view of the previous discussion, the faces of $T_1$ are exactly the previous subdivisions into faces $f_j$ of all faces $F$ of $T_0$. Moreover each face $f$ of $T_1$ has only one corner with label 1 and two with label 2, all other labels being at least 3. Shifting down all labels by one inside face $f$, the situation is exactly equivalent to that of the face $F$ above (observe that the rules for the construction $T$ and $Q$ remain unaffected by the shift since only vertices with label greater or equal to two are considered). The identity between cords of $T'$ and $Q'$ of successively greater orders can thus be checked inductively.

Finally $Q'$ contains all the edges of $T'$, that is $Q' = T'$, and since the deletion of edges with labels of the form $(e, e)$ in $Q'$ (resp. $T'$) produces $Q$ (resp. $Q(T(Q))$), we obtain $Q(T(Q)) = Q$.  

4. Well labelled and embedded trees
4.1. **Unconstrained well labelled trees as embedded trees.** Formula (2) for the number of well labelled trees with $n$ edges,

$$|W_n| = \frac{2^n}{n+1} \cdot 3^n \cdot C(2n),$$

is remarkably simple and yet not immediately clear from definition. Indeed, even though $C(2n)$ is known to be the number of plane trees, the positivity of labels makes it difficult to count labellings that make a plane tree well labelled.

It is thus natural to work first without this positivity condition: define a plane tree to be an *unconstrained well labelled tree* if its vertices have integral labels, the labels of two adjacent vertices differ at most by one, and the label of the root vertex is one. Let $U_n$ denote the set of unconstrained well labelled trees with $n$ edges.

The labelling of a labelled tree can be recovered uniquely from the label of its root and the variations of labels along all edges. We shall denote $\kappa(\epsilon) \in \{-1, 0, 1\}$ the variation of labels along the edge $\epsilon$ when it is traversed away from the root.

Since there is no positivity condition on the labels of unconstrained well labelled trees, all $\kappa(\epsilon)$ can be set independently and the number of labellings of a plane tree that yield an unconstrained well labelled tree is just $3^n$. That is,

$$|U_n| = \frac{3^n}{n+1} \cdot \binom{2n}{n} = 3^n \cdot C(2n).$$

The definition of label distribution extends to unconstrained well labelled trees. For $U \in U_n$ let $(\lambda_k)_{m < k < M} \equiv (\lambda_k^U)_{k \in \mathbb{Z}}$ be the number of vertices with label $k$ in the tree $U$. The label distribution of $U$ is supported by an interval $[m, M]$ with $m \leq 1 \leq M$. The cumulated label distribution is defined with respect to the minimum label $m$ by $\tilde{\lambda}_k^U = \sum_{l=1}^k \lambda_{m+l-1}^U$. These definitions are illustrated by Figure 11.

Observe moreover that similar unconstrained labellings have been considered by D. Aldous [1] with the following interpretation (we restrict to our special one-dimensional case). The tree is folded on the lattice $\mathbb{Z}$ with the root set at position 1 and each edge mapped on an elementary vector (here $+1$, 0, or $-1$). The label of a node then describe its position on the line and, upon counting the number of nodes at position $j$, a mass distribution is obtained. More precisely, with our notations, Aldous’ discrete mass distribution associated to a tree $U \in U_n$ is just the empirical
measure of labels
\[ \mathcal{J}[U] = \frac{1}{n} \sum_{k \in \mathbb{Z}} \lambda_k[U] \delta_k, \]
where \( \delta_k \) denote the dirac mass at \( k \).

In view of this interpretation and for concision’s sake, let us rename unconstrained well labelled trees and call them instead embedded trees.

4.2. Random trees and random quadrangulations. Let \( W_n \) and \( U_n \) be random variables with uniform distribution on \( W_n \) and \( U_n \). More precisely,
\[ \Pr(W_n = W) = \frac{1}{2^n + 2} \cdot \frac{n}{2^n}, \quad \text{and} \quad \Pr(U_n = U) = \frac{1}{3^n} \cdot \frac{n}{2^n}, \]
for all \( W \in W_n \) and \( U \in U_n \).

The label distribution of the corresponding random trees are two random variables that we shall denote \((\lambda_k^{(n)})_{k \geq 1}\) for random well labelled trees, and \((\lambda_k^{(n)})_{k \in \mathbb{Z}} \equiv (\lambda_k^{[U_n]}(\mathbb{Z}))_{k \geq 1}\) for random embedded trees. For random well labelled trees we also use the notation \( \mu_n \) for the maximal label, and for random embedded trees the notations \( m_n \) and \( M_n \) for the minimal and maximal label respectively. Finally cumulated profiles \( \hat{\lambda}_k^{(n)} = \sum_{\ell=1}^k \lambda_{\ell[W_n]} \) and \( \hat{\Lambda}_k^{(n)} = \sum_{\ell=1}^k \lambda_{[U_n]_{m_n + \ell - 1}} \) are defined accordingly (the minimum \( m_n \) in \( \hat{\Lambda}_k^{(n)} \) is understood for the same realisation \( U_n \)).

At this point we are given three random variables: random quadrangulations \( L_n \), random well labelled trees \( W_n \) and random embedded trees \( U_n \). On the one hand, according to Theorem 1, random quadrangulations “are” random well labelled trees, as illustrated by the next corollary.

**Corollary 1.** The label distribution of random well labelled trees has the same distribution as the profile of quadrangulations:
\[ (\lambda_k^{(n)})_{k \geq 1} \overset{\text{law}}{=} (H_k^{(n)})_{k \geq 1}. \]
In particular \( r_n = \mu_n \).

On the other hand, random embedded trees seem to be a simple variant of well labelled trees that has the great advantage to be defined in accordance with Aldous’ prescription for discrete embedded trees. This leads us to study more precisely the relation between \( W_n \) and \( U_n \). By definition, \( W_n \subset U_n \), and according to Tutte’s formula (3),
\[ |W_n| = \frac{2}{n+2} \cdot |U_n|. \]
For combinatorists, this relation could be reminiscent of the relation between the number of Dyck walks and the number of bilatere Dyck walks (see [33, Ch. 5]).

Equivalently, from a more probabilistic point of view, the relation reads
\[ \Pr(U_n \in W_n) = \frac{2}{n+2}, \]
and random well labelled trees are random embedded trees conditioned to positivity. This is exactly similar to Kemperman’s formula for the probability that a simple symmetric walk on \( \mathbb{Z} \) starting from \( k > 0 \) and ending at 0 after \( n \) steps remains positive until the last step (see [23]).
4.3. Cyclic shifts and the cycle lemma. The idea to consider cyclic shifts originates in Dvoretzky and Motzkin’s work and was used by Raney to prove Lagrange inversion formula and by Takács to prove and extend Kemperman’s formula for random walks (we refer to [33, Ch. 5] and [28] for these historical references and many more). We shall prove a consequence of this idea to the study of “height distribution” of simple walks, that will be fundamental in the next section.

Let $n$ and $k$ be nonnegative integers and let $B_{n,k}$ denote the set of walks of length $2n+k$ with $n$ increments $+1$ and $n+k$ increments $-1$, that end with a negative increment $-1$. A walk $w \in B_{n,k}$ is described either by its sequence of increments $w = (w_1, \ldots, w_{2n+k})$, $w_i \in \{+1, -1\}$, or by the partial sums $w(p) = \sum_{i=1}^{p} w_i$, $p = 0, \ldots, 2n+k$. By construction, $w(0) = 0$, $w_{2n+k} = -1$, $w(2n+k) = -k$, and $w(2n+k-1) = -k + 1$. Finally for $k \geq 1$, consider the subset $D_{n,k}$ of $B_{n,k}$ of “positive” walks defined by the condition: $w(p) > -k$ for all $0 \leq p < 2n+k$.

Two walks $w$ and $w'$ of $B_{n,k}$ belong the same conjugacy class if they differ by a cyclic shift, that is, if there exists $s$ such that

$$w = (w_1, \ldots, w_{2n+k}) \quad \text{and} \quad w' = (w_{s}, \ldots, w_{s+2n+k}),$$

where indices are considered modulo $2n+k$.

Define a (left-to-right) record to be a step $p \geq 1$ at which a minimum is reached for the first time: $w(q) > w(p)$ for all $q < p$. Since $w(2n+k) = -k$ there are at least $k$ records. Let us denote $p_1 < \cdots < p_k$ the $k$ lowest records, so that in particular $w(p_k)$ is the minimum value reached by the walk and $w(p_i) = k - i + w(p_k)$. The steps $p_i, i = 1, \ldots, k$ are called the low records of $w$.

The following immediate properties are illustrated by Figure 11.

**Property 5.** A walk of $B_{n,k}$ belongs to $D_{n,k}$ if and only if its lowest record $p_k = 2n+k$.

**Property 6.** Cyclic shifts transport low records: Let $w = (w_1, \ldots, w_{2n+k})$ and $w' = (w_s, \ldots, w_{s+2n+k})$ and assume $\{p_1, \ldots, p_k\}$ are the low records of $w$. Then the low records of $w'$ are $\{p_1 + s, \ldots, p_k + s\}$ (modulo $2n+k$).

The classical cycle lemma follows from Properties 5 and 6.

**Lemma 1** (Cycle lemma). Let $C$ be a conjugacy class of $B_{n,k}$. Then

$$(n+k) \cdot |C \cap D_{n,k}| = k \cdot |C|.$$
Proof. Let us apply a double counting argument:

- The left hand side counts walks in \( C \) with a (low) record at the last position (\( w \in \mathcal{D}_{n,k} \)) and a down step marked (\( n + k \) choices).
- The right hand side counts walks in \( C \) with a down step at the last position (\( w \in \mathcal{B}_{n,k} \)) and a low record marked (\( k \) choices).

Now a bijection is obtained between these two sets upon sending the marked step to the last position by a cyclic shift and marking the former last step.

Given a walk \( w \) with lowest record \( p_k \), the height-to-min of a step \( p \) is \( \hat{w}(p) = w(p) - w(p_k) \), which is nonnegative by definition. In order to study \( \hat{w} \), it will be convenient to consider the height of the walk relatively to the \( k \) low records: given a walk \( w \) with low records \( p_1 < \cdots < p_k \), let us define the Dyck height at step \( p \) by

\[
\hat{w}(p) = \begin{cases} 
  w(p) - w(p_i) + 1, & \text{if } 0 \leq p < p_i, \\
  w(p) - w(p_i), & \text{if } p_i \leq p < p_{i+1}, \text{ with } 1 \leq i \leq k - 1 \\
  w(p) - w(p_k), & \text{if } p_k \leq p \leq 2n + k.
\end{cases}
\]

The Dyck height can be understood as the height inside each of the \( k \) Dyck factors separated by low records. Let \( \hat{\ell}_i(w) \) (resp. \( \hat{h}_i(w) \)) denote the number of down steps of \( w \) ending at Dyck height (resp. height-to-min) at most \( i \),

\[
\hat{\ell}_i(w) = |\{ p \mid \hat{w}(p) \leq i, w_p = -1 \}|, \quad \text{and} \quad \hat{h}_i(w) = |\{ p \mid \hat{w}(p) \leq i, w_p = -1 \}|.
\]

Then by construction, for all \( w \) and \( p \), \( \hat{w}(p) \leq \hat{\ell}_i(w) \leq \hat{\ell}_{i+k}(w) \) and, for all \( i \), \( \hat{h}_i(w) \leq \hat{\ell}_{i+k}(w) \). The following lemma immediately follows from Property 3.

**Lemma 2.** The Dyck height commutes with cyclic shift. In particular the Dyck height distribution \( \hat{\ell}_i \) is invariant under cyclic shift:

\[
(\hat{\ell}_i(w))_{i \geq 0} = (\hat{\ell}_i(w'))_{i \geq 0}, \quad \text{for all } w \text{ and } w' \text{ in the same conjugacy class.}
\]

The height-to-min distribution thus satisfies the following weaker invariance:

\[
\hat{h}_i(w) \leq \hat{h}_{i+k}(w'), \quad \text{for all } i \geq 0 \text{ and } w, w' \text{ in the same conjugacy class.}
\]

From the probabilistic point of view this result can be understood as a simplified discrete version of Vervaat’s relation between the Brownian excursion and the Brownian bridge and their local times relatively to the minimum.

4.4. How to lift the positivity condition for labelled trees. In view of Relation 3 one can expect to apply ideas of the previous section to related well labelled trees to embedded trees. As a matter of fact we shall prove the following theorem.

**Theorem 2.** There exists a partition of \( \mathcal{U}_n = \bigcup_{C \in \mathcal{C}_n} C \) into disjoint conjugacy classes each of size at most \( n + 2 \) and such that in each class \( C \in \mathcal{C}_n \)

- well labelled trees are fairly represented:

\[
2 \cdot |C| = (n + 2) \cdot |C \cap \mathcal{W}_n|,
\]

- and for any \( W \in \mathcal{W}_n \cap C \), \( U \in C \) and \( k \geq 1 \),

\[
\hat{\lambda}_{k-2}(U) \leq \hat{\lambda}_k(W) \leq \hat{\lambda}_{k+2}(U).
\]

**Corollary 2** (Cori-Vauquelin, 1981). The number of well labelled trees with \( n \) edges, (which is also the number of quadrangulations with \( n \) faces), is

\[
|\mathcal{W}_n| = \frac{2}{n + 2} \cdot |\mathcal{U}_n| = \frac{2}{n + 2} \cdot \frac{3^n}{n + 1} \left( \frac{2n}{n} \right).
\]
The proof of Theorem 2 is presented in the next section. It relies on an encoding of plane trees in terms of another family of trees, called blossom trees, and on the conjugation of trees principle which is an analogue of the cycle lemma for blossom trees. This principle was introduced in [31] in order to give a direct combinatorial proof of Corollary 2 based on the cycle lemma. However that proof did not rely on well labelled trees and does not provide the link to the profile.

Theorem 2 admits the following probabilistic restatement.

Theorem 3. There is a coupling \((W_n, U_n)\) (i.e. a distribution on \(W_n \times U_n\) such that the marginals are \(W_n\) and \(U_n\) as previously defined) such that the induced joint distribution \((\lambda^{(n)}, \Lambda^{(n)})\) satisfies for all \(k\)

\[
\hat{\lambda}_{k-2}^{(n)} \leq \hat{\lambda}_k^{(n)} \leq \hat{\lambda}_{k+2}^{(n)},
\]

and in particular

\[
|\mu_n - (M_n - m_n)| \leq 3.
\]

Proof of Theorem 3. The distribution on \(W_n \times U_n\) is immediately obtained from the partition \(U_n = \bigcup_{C \in C_n} C\) as follows: for any \((W, U)\) in \(W_n \times U_n\), let

\[
\Pr((W_n, U_n) = (W, U)) = \begin{cases} 
\frac{1}{|U_n|} & \text{if } U, W \text{ are both in } C \text{ with } |C \cap W_n| = 2, \\
\frac{1}{|U_n|} & \text{if } U, W \text{ are both in } C \text{ with } |C \cap W_n| = 1, \\
0 & \text{if } U \in C_1 \text{ and } W \in C_2 \text{ with } C_1 \neq C_2.
\end{cases}
\]

In view of the first part of Theorem 2, the marginals are uniformly distributed. The second part of Theorem 2 gives the two inequalities.

4.5. Blossom trees and the conjugation of trees. Theorem 2 is clearly analogous to Lemma 1 and 2. However we were not able to define directly conjugacy classes on embedded trees. Instead we first construct an encoding of embedded trees in terms of another family, blossom trees, and then define conjugacy classes of blossom trees and prove Theorem 2.

Following [31], let a blossom trees be a plane tree with the following properties:

- Vertices of degree one are of two types: arrows and flags. The root of the a blossom tree is a flag, which is said to be special, as opposed to the other normal flags.
- All inner nodes have degree four and each of them is adjacent to exactly one arrow.

Let \(B_n\) be the set of blossom trees with \(n\) inner nodes. By construction these trees have \(n\) vertices of degree four and thus \(2n + 2\) of degree one, that is \(n\) arrows and \(n + 2\) flags. The labelling of a blossom tree is given by the following labelling process:

- Start with current label 2 just after the root.
- Turn around the border of the tree in counterclockwise direction.
  - Each time an arrow is reached, the current label is increased by 1.
  - Each time a flag is reached, the current label is decreased by one and then written on the flag.
- Stop when the root flag is reached again (no label is written there).

This definition is illustrated by Figure 12.

Lemma 3. Embedded trees with \(n\) edges are in one-to-one correspondence with blossom trees with \(n\) inner nodes with the same label distribution.
Figure 12. An example of blossom tree and its labelling.

Figure 13. From embedded trees to blossom trees: rules.

Figure 14. From embedded trees to blossom trees: an example.
Proof. In order to prove this lemma we work on a set of decorated blossom trees: in these trees, the root flag is special and any flag with label $e$ (as given by the labelling process) can either be empty or be decorated by an embedded tree with root label $e$ (for $e \neq 1$ the immediate generalisation of embedded trees is meant). The combined label distribution of a decorated blossom tree counts labels of decorations (embedded trees on flags) and of empty flags. Examples of decorated blossom trees are given in Figure 13 (in these figures, labels along edges indicate values taken by the current label during the labelling process).

The first step of the encoding of an embedded tree consists in writing it on the normal flag of the unique blossom tree with two flags and no inner node (Figure 13 top-left example). Then the encoding is performed by recursively transforming the decorated blossom tree according to the local rules of Figure 13. Each time the leftmost rule is applied to a flag decorated by an embedded tree reduced to a vertex with label $e$, this vertex is suppressed the flag becomes empty, with label $e$ (by construction, $e$ agrees with the labelling process; the combined label distribution is left unchanged). When one of the other three rules is applied, a new inner node is created while an edge of embedded tree is suppressed. The relation between the position of the created arrow and the root labels of the embedded trees grants that the compatibility with the labelling process is preserved (observe that in the middle rule subtrees have been switched for this purpose).

As long as there are decorated flags a rule can be applied. Once there is no more decorated flag, a blossom tree is obtained. Rules are local so that rules applied in distinct subtrees commute. As a consequence the final blossom tree does not depend on the order in which rules are applied. Each rule is uniquely reversible so that the encoding is bijective.

Proof of Theorem 2. The partition $\mathcal{U}_n \equiv \mathcal{B}_n = \bigcup_{C \in C_n} C$ is the partition of blossom trees in conjugacy classes: two blossom trees $A$ and $B$ are in the same conjugacy class $C$ if $B$ is obtained from $A$ by first replacing the root flag of $A$ by a normal flag and then choosing a new special flag. This operation is called a cyclic shift of the tree. In other terms each conjugacy class $C$ is the set of blossom trees that can be obtained from a specific unrooted blossom tree (the flags of which are all normal) by selecting a special flag (root flag) in all the possible ways.

Given a blossom tree $B$ with $n$ arrows and $n+2$ flags, the evolution of the current label, while performing its labelling process, is a walk $w_B$ with $n$ increments $+1$ and $n + 2$ negative increments $-1$, that starts from 2, and whose last step, when the process reaches again the root flag, is a negative increment. Upon decreasing all labels by two, the walk $w_B$ is thus a walk of $\mathcal{B}_{n,2}$ as defined in Section 4.3.

Moreover each cyclic shift of the tree $B$ is equivalent to the corresponding cyclic shift of the walk $w_B$. Finally a blossom tree encodes a well labelled tree if and only if all its labels are positive, that is, if and only if the walk $w_B$ belongs to $\mathcal{D}_{n,2}$ (upon decreasing all label by two). The first statement of Theorem 2 is thus exactly the cycle lemma (Lemma 1).

Finally, let us consider label distributions. Given $W$ a well labelled tree and $U$ an embedded tree in the same conjugacy class of trees, the corresponding walks $w_W$ and $w_U$ belong to the same conjugacy class of walks. But the cumulated label distributions satisfies $\hat{\lambda}^{[W]}_k = \hat{h}_k(w_W)$ and $\hat{\Lambda}^{[U]}_k = \hat{h}_k(w_U)$ so that Lemma 2 gives the second statement of Theorem 2.
5. Quadrangulations, Brownian snake and ISE

5.1. Encoding embedded trees by pairs of contour walks. Let \( \tilde{U}_n \) be the set of embedded trees with root label zero instead of one. These trees, that are simply obtained from trees of \( U_n \) by shifting all labels down by one, will be more convenient for our purpose.

Let \( U \) be an embedded tree of \( \tilde{U}_n \) and consider the following traversal of \( U \), where traversing an edge takes unit time:

- At time \( t = 0 \), the traversal arrives at the root.
- If the traversal reaches at time \( t \) a vertex \( v_t \) having \( k \) sons for the \( \ell \)th time with \( \ell \leq k \), its next step is toward the \( \ell \)th son of \( v_t \).
- If the traversal reaches at time \( t \) a vertex \( v_t \) having \( k \) sons for the \((k + 1)\)th time, its next step is back toward the father of \( v_t \).

This traversal is called the contour traversal because, as exemplified by Figure 16, it turns around the tree. In particular every edge is traversed twice (first away from and then toward the root) and the complete traversal takes \( 2n \) steps. The contour pair of \( U \) is then defined by the height (i.e. distance to the root in the abstract tree), \( E[U](t) \) and label \( V[U](t) \) of vertex \( v_t \) traversed at time \( t = 0, \ldots, 2n \). (The path \( E \) is often called the Dyck path associated to the tree \( U \) [33, Ch. 5], or the contour process in [22, Ch. I.3].)

The following proposition is immediate from the definition of contour pairs.

**Proposition 3.** The contour pair construction is a one-to-one correspondence between \( \tilde{U}_n \) (or \( U_n \)) and the set \( EV_{2n} \) of pairs of walks of length \( 2n \) such that:
the walk $E$ is an excursion with increment $\pm 1$ or Dyck path, that is $E(0) = E(2n) = 0$, $|E(t) - E(t + 1)| = 1$ and $E(t) \geq 0$ for all $t = 0, \ldots, 2n - 1$;

- the walk $V$ is a bridge with increment $\{-1, 0, 1\}$ or bilatere Motzkin path, that is $V(0) = V(2n) = 0$ and $(V(t) - V(t + 1)) \in \{-1, 0, 1\}$ for all $t$;

- and the consistency condition hold:

  \( E(t) = E(t') \) and $E(s) \geq E(t)$ for all $t < s < t' \Rightarrow V(t) = V(t'). \)

The excursion $E$ alone determines a unique unlabelled rooted plane tree, while the walk $V$ describes one of the 3

\( n \)

labelling of the tree encoded by $E$. Recall that for an embedded tree $U$, \( \kappa(\epsilon) \in \{-1, 0, 1\} \) denotes the variation along edge $\epsilon$ when traversed away from the root. In particular if $\epsilon$ is traversed for the first time between time $t$ and $t + 1$ and for again between $t'$ and $t' + 1$, then

\[ \kappa(\epsilon) = V(t + 1) - V(t) = V(t') - V(t' + 1). \]

This local condition is equivalent to the consistency condition of Proposition 3.

5.2. Random trees as random contour pairs. Endow now $\tilde{U}_n$ with the uniform distribution and let $(E^{(n)}, V^{(n)}) \equiv (E[U_{n}], V[U_{n}])$ denote the contour pair of the random tree $\tilde{U}_n$. According to Proposition 3, the random contour pair $(E^{(n)}, V^{(n)})$ is uniformly distributed on $\mathcal{E}V_{2n}$ and $E_n$ is uniformly distributed on $\mathcal{E}V_{2n}$, the set of Dyck walks of length $2n$. More precisely, for all $(E, V) \in \mathcal{E}V_{2n}$,

\[
\Pr((E^{(n)}, V^{(n)}) = (E, V)) = \frac{1}{\frac{4}{n+1}\left(\frac{2n}{n}\right)}, \quad \Pr(E^{(n)} = E) = \frac{1}{\frac{4}{n+1}\left(\frac{2n}{n}\right)}.
\]

In order to state convergence results, let us now defined scaled version of these random walks: given a random tree $U_n$ and its contour pair $(E^{(n)}, V^{(n)})$, let

\[ e^{(n)} = \left(\frac{E^{(n)}(\lfloor 2ns \rfloor)}{\sqrt{2n}}\right)_{0 \leq s \leq 1} \quad \text{and} \quad \tilde{W}^{(n)} = \left(\frac{V^{(n)}(\lfloor 2ns \rfloor)}{(8n/9)^{1/4}}\right)_{0 \leq s \leq 1}. \]

The random variables $e^{(n)}$ and $\tilde{W}^{(n)}$ take their values in the Skorohod space $D([0, 1], \mathbb{R})$ of càdlàg real functions (right continuous with left limits).

As was proved by Kaigh [21], the scaled version $e^{(n)}$ of the contour process converges weakly to the normalised Brownian excursion $e$. Our aim is to state an analogous result for the random variable

\[ X^{(n)} = (e^{(n)}, \tilde{W}^{(n)}), \]

that takes its value in the Skorohod space $D([0, 1], \mathbb{R}^2)$.
5.3. A Brownian snake. Let $e$ be the normalised Brownian excursion and

$$W = (W_s(t))_{0 \leq s \leq 1, \ 0 \leq t \leq e(s)}$$

be the Brownian snake with lifetime $e$, as studied previously in [1, 12, 13, 22, 32].

More precisely, the process $W$ can be defined as follows:

- for all $0 \leq s \leq 1$, $t \to W_s(t)$ is a standard Brownian motion defined for $0 \leq t \leq e(s)$ (see Figure 17);
- the application $s \to W_s(\cdot)$ is a path-valued Markov process with transition function satisfying: for $s_1 < s_2$, and for $m = \inf_{s_1 \leq u \leq s_2} e(u)$, conditionally given $W_{s_1}(\cdot)$ (see Figure 18),
  - on the one hand we have that
    $$\left(W_{s_1}(t) \right)_{0 \leq t \leq m} = \left(W_{s_2}(t) \right)_{0 \leq t \leq m},$$
  - and on the other hand $\left(W_{s_2}(m + t) \right)_{0 \leq t \leq e(s_2) - m}$ is a standard Brownian motion starting from $W_{s_2}(m)$, independent of $W_{s_1}(\cdot)$.

The Brownian snake can be viewed as a branching Brownian motion, or as an embedded continuum random tree (see [1]). More precisely the excursion $e$ can be thought of as the contour walk obtained by contour traversal of a continuum random tree, while the snake $W_s(\cdot)$ at times $s$ describes the embedding of the branch to the root at time $s$. 

**Figure 17.** Spacial extension of the snake at time $s_1$.

**Figure 18.** Consistency of the snake between times $s_1$ and $s_2$. 

Instead of considering the full Brownian snake $W_s(t)$ we shall concentrate, as we did in the discrete case, on its description by a contour pair (or “head of the snake” description) $X = (X_s)_{0 \leq s \leq 1}$, defined by (see also Figure 19)

$$\hat{W}_s = W_s(e(s)), \quad X_s = \left(e(s), \hat{W}_s\right), \quad \text{for } 0 \leq s \leq 1.$$ 

In complete analogy with the discrete case, the full Brownian snake can be reconstructed from its contour pair description since

$$W_s(t) = \hat{W}_{\sigma(s,t)}$$

where

$$\sigma(s, t) = \sup\{s' \leq s \mid e(s') = t\}.$$ 

However we need only and shall content with results in terms of $X$ (see [26] for a complete discussion of the relation between the full snake and its contour description).

5.4. **Integrated SuperBrownian Excursion.** Let $J_n$ denote the empirical measure of labels of a random embedded tree:

$$J_n = \frac{1}{n} \sum_k \Lambda_k^{(n)}(\delta_k).$$

Following Aldous [1], for any simple family of trees like our embedded trees, $J_n$ is expected to converge upon scaling to a random mass distribution $J$ supported by a random interval $0 \in [L, R] \subset \mathbb{R}$. This random measure $J$ is called Integrated SuperBrownian Excursion (ISE) by Aldous, in view of its relation to $W$ through

$$\int g \, dJ = \int_0^1 g \left(\hat{W}_s\right) \, ds,$$

for any measurable test function $g$, see [22, Ch. IV.6]. In [12] the convergence of $J_n$ to $J$ is proved for random embedded Cayley trees. Although these trees are not exactly our random embedded plane trees, the proof could easily be adapted.

According to Corollary 3 and Theorem 3, the radius $r_n$ is given by the width of the support of $J_n$. However the weak convergence of $J_n$ to $J$, as obtained in [12] is not sufficient for our purpose since $r = R - L$, the width of the support of $J$, is not a continuous functional of the measure $J$.

5.5. **Convergence of snakes.** Instead of weak convergence of $J_n$ to $J$, we shall thus prove in Section 6 the following stronger result.

**Theorem 4.** The scaled contour pair $X^{(n)}$ converges weakly to $X$ in $D([0, 1], \mathbb{R}^2)$.

This theorem establishes weak convergence of the scaled contour (or head of the snake) description of embedded trees to the head of the snake description of the Brownian snake with lifetime $e$. We moreover obtain a deviation bound for the maximal extension of the snake $\hat{W}_s^{(n)}$.

**Proposition 4.** There exists $y_0 > 0$ such that for all $y > y_0$ and $n$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq 1} \hat{W}_s^{(n)} > \left(\frac{8}{9}\right)^{1/4} y\right) \leq e^{-y}.$$ 

Theorem 4 was independently obtained by Marckert and Mokkadem [26]. They extend the convergence result to the explicit full description $(W_s(t))_{s,t}$ but their alternative proof does not provide the exponential bound of Proposition 4.
5.6. The radius of a random quadrangulation and the width of ISE. According to Corollary 1 and to Theorem 3, the radius \( r_n \) of the quadrangulation corresponding to \( U_n \) satisfies

\[
\left| \left( \frac{8}{9} \right)^{1/4} \left( \sup_{0 \leq s \leq 1} \hat{W}_s^{(n)} - \inf_{0 \leq s \leq 1} \hat{W}_s^{(n)} \right) - n^{-1/4} r_n \right| \leq 3n^{-1/4}.
\]

Theorem 4 and Proposition 4 thus prove the conjecture \( \mathbb{E}(r_n) = \Theta(n^{1/4}) \) and lead to a much more precise characterization:

**Corollary 3.** The random variable \( n^{-1/4} r_n \) converges weakly to \( (8/9)^{1/4} r \), in which

\[
r = \sup_{0 \leq s \leq 1} \hat{W}_s - \inf_{0 \leq s \leq 1} \hat{W}_s.
\]

Furthermore, convergence of all moments holds true.

In view of Relation (4) the random variable \( r \) is also the width of ISE process \( J \).

5.7. The profile and a CRM. Actually, Theorems 1 and 3 suggest that not only the scaled radius but the full scaled profile converges (at least in distribution) to the ISE mass distribution. More precisely, define the distribution function \( F(x) \) of the translated ISE by

\[
W_{\text{min}} = \inf_{0 \leq s \leq 1} \hat{W}_s, \quad F(x) = J((-\infty, W_{\text{min}} + x]) = J([W_{\text{min}}, W_{\text{min}} + x]),
\]

and the scaled distribution function of the profile of random quadrangulations by

\[
F_n(x) = \frac{1}{n+1} \hat{\lambda}^{(n)}_{\lfloor (8n/9)^{1/4} x \rfloor} = \frac{1}{n+1} \hat{H}^{(n)}_{\lfloor (8n/9)^{1/4} x \rfloor}.
\]
where \( \hat{\lambda}^{(n)}_k \) is the cumulated distribution of labels of a random well labelled tree (as defined in Section 3) and \( \hat{H}^{(n)}_k \) is the cumulated profile of a random quadrangulation (as defined in Section 3).

Then we prove the following corollary of Theorems 1, 3, 4 and Corollary 3.

**Corollary 4.** The scaled profile \( F_n \) converges weakly to \( F \) in \( D([0, +\infty), \mathbb{R}) \).

A natural conjecture is that there is a continuum analogue to Theorem 1 that allows to define from ISE a Continuum Random Map (CRM), such that the properties of scaled distances in random quadrangulations (distances between arbitrary pairs of points, not only with respect to a basepoint) would be described by the properties of distance in the CRM. In view of the interpretation of random quadrangulations as 2d Euclidean pure quantum geometries, this CRM might be considered as a natural candidate model of continuum 2d pure quantum geometry. We plan to discuss this connection further in a subsequent paper.

6. **Random embedded trees and the Brownian snake**

In this section we prove Theorem 4. Finite dimensional density functions are first calculated (Section 6.1). Section 6.2 then provides the deviation bound for the maximum of the label walk. Finally tightness is proved using the previous bound (Section 6.3). The theorem then follows from standard results on weak convergence in the space \( D([0, 1], \mathbb{R}^2) \) [9].

6.1. **Finite dimensional density functions.** From now on in this section, \( p \) and \( \tau = (\tau_1, \ldots, \tau_p) \) are fixed with \( 0 < \tau_1 < \cdots < \tau_p < 1 \), and we prove the following finite dimensional density convergence result.

**Proposition 5.** The sequence of random variables

\[
X^{(n)}(\tau) = \left( \frac{E^{(n)}([2\tau_i n])}{(2n)^{1/2}}, \frac{V^{(n)}([2\tau_i n])}{(8n/9)^{1/4}} \right)_{1 \leq i \leq p} = \left( e^{(n)}(\tau_i), \hat{W}^{(n)}_{\tau_i} \right)_{1 \leq i \leq p},
\]

weakly converges to \( X(\tau) = \left( e(\tau_i), W_{\tau_i} \right)_{1 \leq i \leq p} \), that is,

\[
\lim_n E \left[ \Phi \left( X^{(n)}(\tau) \right) \right] = E \left[ \Phi \left( X(\tau) \right) \right]
\]

or any bounded continuous function \( \Phi \) on \( \mathbb{R}^{2p} \).

For this aim it will be convenient to prove first the weak convergence of

\[
Y^{(n)}(\tau) = \left( \left( e^{(n)}(\tau_i) \right)_{1 \leq i \leq p}, \left( \inf_{[\tau_i, \tau_{i+1}]} e^{(n)} \right)_{1 \leq i \leq p-1} \right),
\]

then that of

\[
Z^{(n)}(\tau) = \left( Y^{(n)}(\tau), \left( \hat{W}^{(n)}_{\tau_i} \right)_{1 \leq i \leq p}, \left( \mu^{(n)}_i \right)_{1 \leq i \leq p-1} \right),
\]

in which \( \mu^{(n)}_i \) is defined by

\[
\mu^{(n)}_i = \hat{W}^{(n)}_{\tau_{i}'} \quad \text{for any} \quad \tau_{i}' \in \arg\inf_{[\tau_i, \tau_{i+1}]} e^{(n)}.
\]

(The consistency condition of Proposition 3 grants that \( \mu^{(n)}_i \) is indeed independent of the exact choice of \( \tau_{i}' \) in \( \arg\inf_{[\tau_i, \tau_{i+1}]} e^{(n)} \).)
The weak convergence of \( Y^{(n)}(\tau) \) follows from [21], as a special case, but does not fill our needs. In the next section we prove a local limit theorem for finite dimensional distributions of \( e^{(n)} \) which is not a consequence of [21]. Once the local limit theorem for \( Y^{(n)}(\tau) \) is proved, we recall its interpretation in terms of trees, using the key notion of shape. This leads us to split \( \mathbb{R}^{4p-2} \) in \((p-1)!\) regions \((R_\sigma)_{\sigma \in S_{p-1}}\) and to prove weak convergence of \( Z^{(n)}(\tau) \) separately on each region. Finally we identify the limit and the weak convergence of \( X^{(n)}(\tau) \) follows from that of \( Z^{(n)}(\tau) \).

6.1.1. A local limit theorem for \( Y^{(n)}(\tau) \). The characteristics of the walk \( e^{(n)} \) we are interested in are contained in the sequence of successive heights at (resp. minima between) the \( \tau_i \). Let \((x, m) = ((x_i)_{1 \leq i \leq p}, (m_i)_{1 \leq i \leq p-1})\) be a typical value of \( Y^{(n)}(\tau) \), that is, \( x, \sqrt{2n} \) and \( m, \sqrt{2n} \) are integers, and they satisfy

\[
\inf(x_i, x_{i+1}) \geq m_i, \quad 1 \leq i \leq p - 1.
\]

For the first result, we need a more handy parametrisation, by the successive up and down relative variations: let \( \gamma_0 = x_1, \beta_p = x_p \),

\[
\gamma_i = x_{i+1} - m_i, \quad \beta_i = x_i - m_i, \quad \text{for} \quad i = 1, \ldots, p - 1,
\]

and by convention \( \beta_0 = \gamma_p = 0 \).

**Proposition 6.** Let \( K \) be a compact subset of

\[
\{(x, m) \in \mathbb{R}^p \times \mathbb{R}^{p-1} \mid 0 < m_i < \inf(x_i, x_{i+1}), 1 \leq i \leq p - 1\},
\]

that is, of the domain of definition of coherent values of the \( x_i \) and \( m_i \). Then, uniformly for \((x, m) \in K \cap (\mathbb{Z}^{(2n)^{-1}})^{p-1}\),

\[
\mathbb{P}\left(Y^{(n)}(\tau) = (x, m)\right) = (2n)^{-2\delta} \cdot \zeta(x, m) \cdot \left(1 + O_K(n^{-1/2})\right),
\]

where \( \delta = (2p - 1)/4 \), and \( \zeta(x, m) \) reads, in terms of the \( \beta_i \) and \( \gamma_i \) as given by relations (4):

\[
\zeta(x, m) = 2^{2p} (2\pi)^{-p/2} \prod_{i=0}^{p-1} \frac{(\beta_i + \gamma_i) e^{-\frac{(\beta_i+\gamma_i)^2}{4(\tau_{i+1} - \tau_i)^3/2}}}{(\tau_{i+1} - \tau_i)^{3/2}}.
\]

The notation \( O_K \) is used to stress the fact that the error term is uniform for fixed \( K \).

**Proof.** Let, for all \( n \) and \( a \) non negative integers,

\[
C(n; a) = \frac{a + 1}{n + 1} \binom{n + 1}{(n - a)/2} = \frac{\binom{2n+1}{a+1}}{\sqrt{2\pi n}} \frac{1}{n^{a/2}} e^{-\frac{a^2}{2n}} \left(1 + O\left(\frac{a}{n}\right)\right),
\]

where the error term is uniform for \( a = O(n^{1/2}) \).

The Catalan numbers \( C(2n; 0) \) are well known to give the cardinality of \( \mathcal{E}_{2n} \). More generally, the reflection principle proves that \( C(n; a) \) is the number of meanders with increments \( \pm 1 \) (aka left factors of Dyckk walks), that have length \( n \) and end at height \( a \). Given non negative numbers \( b \) and \( c \), \( C(n; b+c) \) is also the number of walks with increments \( \pm 1 \) that have length \( n \), minimum \(-b\) and final height \( c-b \), as follows from a decomposition at first and last passage at the minimum.
We thus have
\[ \mathbb{P}(Y(x,m) = (x,m)) = \prod_{i=0}^{p} C\left(\left\lfloor 2n\tau_{i+1} \right\rfloor - \left\lfloor 2n\tau_{i} \right\rfloor; (2n)^{1/2}\beta_{i} + (2n)^{1/2}\gamma_{i}\right)/C(2n;0). \]
Combined with (6), it yields Proposition 6.

The expression
\[ \zeta(x,m) = 2^{2p}(2\pi)^{-p/2} \prod_{i=0}^{p} \frac{(\beta_{i} + \gamma_{i})e^{-\frac{(\beta_{i} + \gamma_{i})^{2}}{2(\tau_{i+1} - \tau_{i})^{1/2}}}}{\tau_{i+1} - \tau_{i}} \mathbf{1}_{\beta_{i} \geq 0} \mathbf{1}_{\gamma_{i} \geq 0} \mathbf{1}_{m_{i} \geq 0} \]

where the \( \beta_{i} \) and \( \gamma_{i} \) are given by relations (5), is the expected limit density of probability for the \( x_{i} \) and \( m_{i} \); in particular it is coherent with the density of evaluations of the normalized Brownian excursion at \( p \) points, and of the \( p - 1 \) minima between them, as given in [32].

6.1.2. Shapes. Let us now define a shape to be a rooted plane tree \( T \) with \( q \) edges, that we call superedges to distinguish them from edges of embedded trees. We shall endow each superedge with a length: let \( \eta_{1}, \ldots, \eta_{q} \) denote the edges of \( T \) in prefix order (i.e. the order induced by first visits in contour traversal) and \( L(\eta_{i}) \) the length of \( \eta_{i} \).

Given the \( p \) normalised times \( 0 < \tau_{1} < \cdots < \tau_{p} < 1 \) and \( U \) an embedded tree of \( U_{n} \), let us extract a shape \( T \) and, for each superedge \( \eta \) of \( T \), a length \( L(\eta) \). In order to do this, let \( t_{i} = \lfloor 2n\tau_{i} \rfloor \) and consider \( v_{t_{i}} \) the vertex visited at time \( t_{i} \) by the contour traversal. The \( p \) fixed vertices \( v_{t_{i}} \), and the root \( r \) span a minimal subtree of \( U \) (the union of the branches from \( v_{t_{i}} \) to \( r \), see Figure 20, left hand side). Apart from the \( v_{t_{i}} \), this subtree contains vertices of two types: branchpoints that have at least two sons in the subtree (black vertices in Figure 20), and smooth vertices that have exactly one son in the subtree (grey vertices). Let us define the shape \( T \) by taking the fixed vertices, the root and the branchpoints as vertex set, and the paths connecting these vertices as superedges (Figure 20, right hand side). A superedge \( \eta \) of \( T \) is thus by definition a set of edges of \( U \) and we let \( L(\eta) = |\eta| \), which is just the length of the path \( \eta \) in the tree \( U \) (for instance, superedge \( \eta_{4} \) in Figure 20 is made of 7 edges).

**Figure 20.** A random tree \( U_{n} \), the decomposition of its normalised contour \( e^{(m)} \) at times \( (\tau_{1}, \tau_{2}, \tau_{3}) \) and at minima in between, and the extracted shape (prefix order shown on edges and vertices). In particular: \( L(\eta_{4}) = 7 = (x_{2} - m_{1})\sqrt{2n} \).
Observe now that the shape and superedge lengths extracted from a random tree $U_n$ completely determine $Y^{(n)}(\tau)$. To check this assertion, let us assume that the extraction yields a shape $T$ and superedges $\eta_i$ of normalised length $\ell(\eta_i)$ with $\mathcal{L}(\eta_i) = \ell(\eta_i)\sqrt{2n}$, $i = 1, \ldots, q$. Let $A_i$ be the set of superedges on the unique path from $v_{t_i}$ to the root, so that $\sum_{\eta \in A_i} \ell(\eta) = \ell^{(n)}(\tau_i) = x_i$. The contour traversal of the shape $T$ starts from the root $v_{t_0} = v_{t_{i+1}}$ and reaches successively each leaf $v_{t_i}$, $\ldots$, $v_{t_{p_t}}$. From $v_{t_i}$ to $v_{t_{i+1}}$, a set $B_i$ of superedges are first traversed toward the root, followed by a set $C_i$ of superedges that are traversed away from the root (so that $B_i = A_i \setminus A_{i+1}$ while $C_i = A_{i+1} \setminus A_i$). Then $\sum_{\eta \in B_i} \ell(\eta) = \beta_i$ and $\sum_{\eta \in C_i} \ell(\eta) = \gamma_i$ are the lengths of these journeys, with $\beta_i$ and $\gamma_i$ given by Relations (5). In particular the normalised lengths $\ell(\eta)$ of superedges are all of the form

$$(x_i - m_i), \quad (x_{i+1} - m_i) \quad \text{or} \quad |m_j - m_i| \quad \text{with} \quad j < i.$$ 

In the latter case, $m_j$ has to be a record, that is for $j < k < i$, $m_k > \sup(m_i, m_j)$. These relations are exemplified by Figure 21.

Conversely, as explained in [22, Ch. 3] or [32, Section 2], the $x_i$ and $m_i$ (or the $\beta_i$ and $\gamma_i$ as defined by Relations (5)) exactly determine the shape and superedge lengths of $U_n$.

From the limit density $\zeta(x,m)$ of the previous section, the probability that $m_i = m_j$ for some $i \neq j$ is seen to tend to zero as $n$ goes to infinity. This implies that, with probability tending to one as $n$ goes to infinity, the shape is a binary tree and $q = 2p - 1$ (indeed the existence of a branchpoint of larger degree corresponds to the equality of two minima $m_i$ and $m_j$). From now on we thus restrict our attention to binary shapes $T$.

Associated to any binary shape $T$, there is a matrix $M_T$ with size $2p - 1 \times 2p - 1$ and entries zero or one, that sends the $2p - 1$ normalised lengths of superedges $(\ell(\eta_i))$ on the $(x_i, m_i)$s. Moreover, provided the lengths of superedges on the one hand, and the $x_i$ and $m_i$ on the other hand, are sorted according to the prefix order, the matrix $M_T$ is lower triangular with ones on the diagonal. As a consequence, $M_T$ is Lebesgue measure preserving. Indeed the $k$th vertex in the prefix order is
reached by the $k$th edge and all other edges on the path to the root have already been visited.

Finally let us consider labels of $U_n$: the variation of label along edges extends to superedges, upon setting for any superedge $\eta$

$$\kappa(\eta) = \sum_{\epsilon \in \eta} \kappa(\epsilon).$$

Let $\tilde{M}_T$ be the double action of $M_T$ on $\mathbb{R}^{4p-2}$. Since the matrix $M_T$ also sends the normalized increments $(k_i)_{1 \leq i \leq 2p-1}$ of labels along the $2p-1$ superedges onto the normalized labels $(w_i, \mu_i)$ of the $2p-1$ vertices of the tree, $\tilde{M}_T$ describes the one-to-one correspondence between shapes equipped with superedges’ lengths and label variations on the one hand, and $Z^{(n)}(\tau)$ on the other hand. More precisely, $\tilde{M}_T$ describes the restriction of this correspondance to shape $T$.

6.1.3. A local limit theorem for $Z^{(n)}(\tau)$ in a fixed region. As already observed, the limit law of $Y^{(n)}(\tau)$ charges only the region

$$R = \{(x, m, w, \mu) \in (\mathbb{R}^p \times \mathbb{R}^{p-1})^2 \mid 0 < m_i < \inf(x_i, x_{i+1}), 1 \leq i \leq p - 1\}.$$

For a given permutation $\sigma$ on $p - 1$ symbols, define

$$R_{\sigma} = \{(x, m, w, \mu) \in R \mid 0 < m_{\sigma(1)} < m_{\sigma(2)} < \cdots < m_{\sigma(p-1)}\}.$$

Observe that the shape is constant on $R_{\sigma}$ and denote it $T_{\sigma}$. (Conversely a given shape may appear in many different regions $R_{\sigma}$, as there are only $C(p-1)$ different shapes.) Let

$$R_{\sigma, \varepsilon} = \{(x, m, w, \mu) \in R_{\sigma} \mid d((x, m, w, \mu), \partial R_{\sigma}) \geq \varepsilon\}.$$

(The distance is the usual distance in $\mathbb{R}^{4p-2}$ and $\partial R_{\sigma}$ denote the boundary of $R_{\sigma}$, which is clearly a finite union of closed $(4p-3)$-dimensional cones.)

Let $\Delta(n, K)$ denote the set of possible values of $Z^{(n)}(\tau)$ that belong to a compact subset $K$ of $\mathbb{R}^{4p-2}$, that is

$$\Delta(n, K) = K \cap \left(\left((2n)^{-1/2} \mathbb{Z}\right)^{2p-1} \times \left((8n/9)^{-1/4} \mathbb{Z}\right)^{2p-1}\right).$$

Furthermore, let $\xi$ (resp. $f$) be defined, on $\mathbb{R}^{2p-1} \times \mathbb{R}^{2p-1}$ (resp. $(\mathbb{R}^p \times \mathbb{R}^{p-1})^2$), by

$$\xi(\ell, k) = (2\pi)^{-2\delta} \prod_{i=1}^{2p-1} \ell_i^{-1/2} \exp\left(-\frac{k_i^2}{2\ell_i}\right),$$

$$\xi_T = \xi \circ \tilde{M}_T^{-1},$$

$$f(x, m, w, \mu) = \zeta(x, m) \sum_{\sigma} \xi_{T_{\sigma}}(x, m, w, \mu) \cdot 1_{R_{\sigma}}(x, m).$$

The function $f$ is exactly the density of $Z(\tau)$, that is, the density of the evaluation of $X$ at the $p$ points $\tau_i$ and at the $p-1$ minima of $\epsilon$ between them. This density was described in [2, Propositions 2 & 3]. The function $\xi_{T_{\sigma}}$ is the conditional density of the labels $(w, \mu)$ given $(x, m)$. We shall prove the following local limit law for $Z^{(n)}(\tau)$.
Lemma 4. Let $K$ be a compact subset of $R_{\sigma, \varepsilon}$, $\varepsilon > 0$. Then, uniformly for $(x, m, w, \mu) \in \Delta(n, K)$,
\[
P \left( Z^{(n)}(\tau) = (x, m, w, \mu) \right) = (2n)^{-2\delta} \cdot \left( \frac{8n}{9} \right)^{-\delta} \cdot \left( f(x, m, w, \mu) \left( 1 + O_K \left( n^{-1/2} \right) \right) \right),
\]

Corollary 5. Let $K$ denote a compact subset of $R_{\sigma, \varepsilon}$. For any uniformly continuous function $\Phi$ with support $K$,
\[
\lim_n \mathbb{E} \left[ \Phi \left( Z^{(n)}(\tau) \right) \right] = \mathbb{E} \left[ \Phi(Z(\tau)) \right].
\]

Proof of Lemma 4. Our aim is to compute for $(x, m, w, \mu) \in \Delta(n, K)$ the probability
\[
P \left( Z^{(n)}(\tau) = (x, m, w, \mu) \right) = \mathbb{P} \left( Y^{(n)}(\tau) = (x, m) \right) \cdot \mathbb{P} \left( Z^{(n)}(\tau) = (x, m, w, \mu) \mid Y^{(n)} = (x, m) \right).
\]

From Proposition 3, we already have
\[
P \left( Y^{(n)}(\tau) = (x, m) \right) = (2n)^{-2\delta} \cdot \left( 1 + O \left( n^{-1/2} \right) \right).
\]

As discussed in the previous section, the normalised lengths $\ell_i = (2n)^{-1/2}L_i$ of superedges, obtained from $(x, m)$ through $M_T$, are of the form
\[(x_i - m_i), \quad (x_{i+1} - m_i) \quad \text{or} \quad |m_j - m_i| \quad \text{with} \quad j < i.
\]

In particular the fact that $(x, m, \cdot, \cdot) \in K \subset R_{\sigma, \varepsilon}$ for some $\varepsilon > 0$ grants that these normalized lengths are uniformly bounded away from 0. In turn the variation $\kappa^{(n)}(\eta)$ of labels along any superedge $\eta$ is the sum of at least $O(\sqrt{n})$ i.i.d. uniform random variables on $\{1, 0, -1\}$. Therefore we can apply uniform bounds for the local limit theorem [27], pages 189–197: if $S_n$ denotes the sum of $n$ i.i.d. random variables uniform on $\{+1, 0, -1\}$, we have
\[
P \left( S_n = \kappa \right) = \sqrt{\frac{3}{4\pi n}} \cdot e^{-3n^2/4n} + O \left( \frac{1}{n} \right).
\]

This allows us to calculate the probability that the variations of labels along superedges $\kappa^{(n)}(\eta_i)$ are equal to $\kappa_i = k_i (8n/9)^{1/4}$, $i = 1, \ldots, 2p - 1$: uniformly for $(x, m, w, \mu) \in \Delta(n, K)$,
\[
P \left( \kappa^{(n)}(\eta_i) = \kappa_i, \quad 1 \leq i \leq 2p - 1 \mid Y^{(n)}(\tau) = (x, m) \right)
\]
\[
= \prod_{i=1}^{2p-1} \left( \sqrt{\frac{3}{4\pi L_i}} \cdot e^{-3\kappa_i^2/4L_i} + O \left( \frac{1}{\sqrt{n}} \right) \right),
\]
\[
= \left( \frac{9}{8n} \right)^{\delta} \cdot \xi(\ell, k) \cdot \left( 1 + O_K \left( n^{-1/2} \right) \right),
\]

in which the sequence of normalised variations of labels along superedges, $k = (k_i)_{1 \leq i \leq 2p-1}$ is the inverse image of $(w, \mu)$ through $M_T$. In other terms,
\[
(\ell, k) = M_T^{-1}(x, m, w, \mu),
\]
and the previous relation can be written
\[ \mathbb{P} \left( Z^{(n)}(\tau) = (x, m, w, \mu) \mid Y^{(n)} = (x, m) \right) = (8n/9)^{-\delta} \xi_{T_n} (x, m, w, \mu) \left( 1 + O_K(n^{-1/2}) \right), \]
leading to the desired result, through Proposition 6.

Next, difference of a function \( g \) if \( \int M_T(\ell, k) f \circ M_T(\ell, k) d\ell dk. \)

Therefore, the lemma follows upon proving that
\[ \lim_n \left| \mathbb{E} \left( \Phi(Z^{(n)}(\tau)) \right) - \int \Phi \circ M_T(\ell, k) f \circ M_T(\ell, k) d\ell dk \right| = 0. \]

Set \( \phi_T = \Phi \circ M_T, \quad f_T = f \circ M_T. \)
The function \( \phi_T \) has a compact support \( \tilde{K} = M_T^{-1}K \) that is included in \( M_T^{-1}K_{\sigma, \epsilon} \subset (\epsilon, \infty)^{2p-1} \times \mathbb{R}^{2p-1}. \) Since \( \tilde{K} \) is compact, there exists an \( \epsilon' > 0 \) and a compact \( K' \) such that \( K \subset K' \subset (\epsilon, \infty)^{2p-1} \times \mathbb{R}^{2p-1} \) and \( \text{d}(\partial K, \partial K') > \epsilon'. \) We shall use the fat boundary \( K' \setminus \tilde{K} \), on which \( \phi_T \) is identically zero, to deal with boundary effects.

Let finally \( \Delta(n, K) = M_T^{-1} \Delta(n, K) \) be the discretized version of \( \tilde{K} \) and similarly \( \Delta'(n, K) = M_T^{-1} \Delta(n, M_T K') \) that of \( K' \); by construction \( \Delta(n, K) \subset \Delta'(n, K) \) are finite sets with \( O_K(n^{-\delta}) \) elements, and \( \phi_T \) is identically zero over \( \Delta'(n, K) \setminus \Delta(n, K). \)

First, we have
\[ \mathbb{E} \left( \Phi(Z^{(n)}(\tau)) \right) = \sum_{(\ell, k) \in \Delta'(n, K)} \phi_T(\ell, k) \mathbb{P} \left( (\ell^{(n)}, k^{(n)}) = (\ell, k) \right) = (2n)^{-2\delta} (8n/9)^{-\delta} \sum_{(\ell, k) \in \Delta'(n, K)} \phi_T(\ell, k) f_T(\ell, k) + \| \Phi \|_{\infty} \cdot O_K(n^{-1/2}), \]
the second equality due to the local limit convergence (Lemma 4). Next, difference between the discrete summation and the integral is bounded in terms of the modulus of continuity \( \omega(\phi_T \cdot f_T, K', \cdot) \) of \( \phi_T \cdot f_T \) on \( \tilde{K} \) (recall that the modulus of continuity of a function \( g \) on a compact \( K \) is \( \omega(g, K, \epsilon) = \sup_{0 < d(x, y) < \epsilon} |g(x) - g(y)| \), and that if \( g \) is uniformly continuous on \( K \), it satisfies \( \omega(g, K, \epsilon) = O(\epsilon) \) as \( \epsilon \) tends to zero). This yields
\[ \left| (2n)^{-2\delta} (8n/9)^{-\delta} \sum_{(\ell, k) \in \Delta'(n, K)} \phi_T(\ell, k) f_T(\ell, k) - \int \phi_T(\ell, k) f_T(\ell, k) d\ell dk \right| \leq \text{measure}(K') \cdot \omega(\phi_T \cdot f_T, K', n^{-1/4}). \]
Observe that in this summation the compact \( K' \) has been approximated by a union of boxes of diameter \( O(n^{-1/4}) \) and boundary effect should be considered. However \( \phi_T \) is identically zero on a region \( K' \setminus \tilde{K} \) with \( \text{d}(\delta K, \delta K') > \epsilon' \), that contains all boxes intersecting the boundary for \( n \) large enough. The boundary effect is thus null. \( \square \)
6.1.4. Weak convergence of $Z^{(n)}(\tau)$. According to the Porte-Manteau Theorem [9, Ch. 1], we need
\[
\lim_n \mathbb{E} \left[ \Phi \left( Z^{(n)}(\tau) \right) \right] = \int \Phi \ f = \mathbb{E} [\Phi (Z(\tau))]
\]
to hold for any bounded uniformly continuous $\Phi$. Now consider
\[
K_m = \overline{B}(0, \rho_m) \cap \left( \bigcup_{\sigma} R_{\sigma, \varepsilon_m} \times \mathbb{R}^{2p-1} \right),
\]
in which $\overline{B}(0, \rho_m)$ is the closed ball, in $\mathbb{R}^{4p-2}$, with radius $\rho_m$ and let simultaneously $\rho_m$ increase to $+\infty$ and $\varepsilon_m$ decrease to 0. We obtain a increasing sequence of compacts $K_m$, each of these compacts having $(p-1)!$ connected components. As the limit of this sequence has a Lebesgue–negligible complement in $\mathbb{R}^{4p-2}$, we can choose the sequences $(\rho_m)_{m>1}$ and $(\varepsilon_m)_{m>1}$ in such a way that
\[
P (Z(\tau) \in K_m) \geq 1 - \frac{1}{m}.
\]
There exist uniformly continuous functions $\Psi_m : \mathbb{R}^{4p-2} \to [0, 1]$, such that
\[
\Psi_m \mid_{K_m} \equiv 1, \quad \Psi_m \mid_{K_m^c} \equiv 0.
\]
By construction,
\[
\left| \mathbb{E} (\Phi (Z(\tau))) - \mathbb{E} (\Psi_m \cdot \Phi (Z(\tau))) \right| \leq \frac{\|\Phi\|_{\infty}}{m},
\]
Moreover the product $\Psi_m \cdot \Phi$ is now a finite sum of functions satisfying the assumptions of Corollary 5: this yields
\[
\lim_n \mathbb{E} (\Psi_m \cdot \Phi (Z^{(n)}(\tau))) = \mathbb{E} (\Psi_m \cdot \Phi (Z(\tau))).
\]
Next observe that, by definition of $\Psi_m$,
\[
P (Z^{(n)}(\tau) \in K_m) \geq \mathbb{E} (\Psi_{m-1} (Z^{(n)}(\tau))),
\]
and
\[
\mathbb{E} (\Psi_{m-1} (Z(\tau))) \geq P (Z(\tau) \in K_{m-1}).
\]
Therefore, applying Corollary 5 to $\Psi_{m-1}$,
\[
\liminf_n P (Z^{(n)}(\tau) \in K_m) \geq P (Z(\tau) \in K_{m-1}) \geq 1 - \frac{1}{m - 1}.
\]
Moreover
\[
\mathbb{E} \left( (1 - \Psi_m) \cdot \Phi (Z^{(n)}(\tau)) \right) \leq P (\Psi_m (Z^{(n)}(\tau)) < 1) \cdot \|\Phi\|_{\infty},
\]
\[
\leq P (Z^{(n)}(\tau) \not\in K_m) \cdot \|\Phi\|_{\infty}.
\]
Thus, taking limit for $n$ going to infinity and applying the previous lower bound,
\[
\limsup_n \mathbb{E} \left( (1 - \Psi_m) \cdot \Phi (Z^{(n)}(\tau)) \right) \leq \frac{\|\Phi\|_{\infty}}{m - 1}.
\]
Finally, the decomposition
\[
\mathbb{E} (\Phi (Z^{(n)}(\tau))) = \mathbb{E} (\Psi_m \cdot \Phi (Z^{(n)}(\tau))) + \mathbb{E} \left( (1 - \Psi_m) \cdot \Phi (Z^{(n)}(\tau)) \right),
\]
yields
\[
\limsup_n \left| \mathbb{E} \left( \Phi(Z^{(n)}(\tau)) \right) - \mathbb{E}(\Phi(Z(\tau))) \right| \leq \frac{2\|\Phi\|_{\infty}}{m - 1}
\]

Letting \( m \) go to infinity gives the weak convergence of \( Z^{(n)}(\tau) \) to \( Z(\tau) \) as claimed. The convergence of \( X^{(n)}(\tau) \) is a by-product.

6.2. A deviation bound for the largest label. In this section, a rough but exponential deviation bound for the value of the largest label in a forest of \( k \) embedded trees with \( n \) edges is obtained. For \( k = 1 \), Proposition 3 is exactly obtained.

Let \( \mathcal{E}V_{k,2n} \) denote the set of \( k \)-uples of element of \( \mathcal{E}V \) of total length \( 2n \); an element \( [(E_1,V_1),\ldots,(E_k,V_k)] \) of \( \mathcal{E}V_{k,2n} \) codes for a forest of \( k \) embedded trees (each \( (E_i,V_i) \) codes for a tree, according to Proposition 3). Equivalently, one may concatenate the \( k \) pairs and view any element of \( \mathcal{E}V_{k,2n} \) as a pair \( (E,V) = (E_1\cdots E_k,V_1\cdots V_k) \in \mathcal{E}V_{2n} \) together with a set of concatenation times \( 0 = t_0 \leq t_1 \leq \cdots \leq t_k = 2n \), subject to the conditions \( E(t_i) = V(t_i) = 0 \) for all \( i = 1,\ldots,k \).

In this identification, \( E_i \) can be seen through the following weaker formulation (with \( k = 1 \)).

Proposition 7. There exists \( y_0 > 0 \) such that, for all \( n, k \) and \( y \geq y_0 \),

\[
\mathbb{P} \left( \sup_{0 \leq t \leq 2n} V^{k,n}(t) > yn^{1/4} \right) < e^{-y}.
\]

This is the key fact in the proof of tightness of the sequence \( X^{(n)} \), given in the last subsection, and it also leads to the convergence of moments in Corollary 3 through the following weaker formulation (with \( k = 1 \)): for \( y > y_0 \), and for all \( n,\)

\[
\mathbb{P} \left( \sup_{0 \leq t \leq n} W^{(n)}(t) > (8/9)^{1/4} y \right) \leq e^{-y},
\]

which is exactly Proposition 4.

The proof is based on a branch decomposition, that is discussed in the next paragraph. Then, after two preliminary results on parameters of the middle branch of a random tree (Paragraphs 6.2.2 and 6.2.3), Proposition 7 is proved by induction (Paragraphs 6.2.4, 6.2.5 and 6.2.6). At the price of more technical details in these latter paragraphs, the bound could be improved to \( e^{-c_y y^{1/4 - \epsilon}} \), for any fixed \( \epsilon > 0 \).

6.2.1. The branch decomposition at time \( t \). Let \( (E,V) \in \mathcal{E}V_{k,2n} \) with concatenation times \( 0 \leq t_1 \leq \ldots \leq t_k \leq 2n \) and let \( t \in (0,2n) \). Suppose moreover that \( t_{p-1} < t < t_p \), that is \( t \) occurs during the contour traversal of \( (E_p,V_p) \), the \( p \)-th component of the forest encoded by \( (E,V) \) and let \( U \) be the tree encoded by \( (E_p,V_p) \).

To any vertex \( v \) of \( U \) is associated the set of edges in the unique simple path from \( v \) to the root of \( U \), denoted \( \text{br}(v) \) (for the branch of \( v \)). Recall that \( v_t \) denote the vertex visited at time \( t \) of the contour traversal of \( U \). Observe that the height \( E(t) \) is \( |\text{br}(v_t)| \), the length of the branch from the root \( r \) to \( v_t \), while the label \( V(t) \) of \( v_t \) is given by

\[
V(t) = \sum_{e \in \text{br}(v_t)} \kappa(e).
\]
Let $\ell = E(t) = |\partial(v_t)|$, and call $\epsilon_i$ the edge of $\partial(v_t)$ between heights $i-1$ and $i$, for $i = 1, \ldots, \ell$. The maximal label on the branch $\partial(v_t)$ is

$$H(t) = \sup_{0 \leq j \leq \ell} \sum_{i=1}^j s(\epsilon_i)$$

where $\epsilon_i$ is the $i$th edge of $\partial(v_t)$.

The branch of $v_t$ induces a decomposition of $U$ into two forests of trees, the branch decomposition, which we now phrase in terms of $(E, V)$.

- From time $t_{p-1}$ to $t$, the edges $\epsilon_1, \ldots, \epsilon_\ell$ are successively traversed away from the root of the branch $\partial(v_t)$, at times $t'_1 < \cdots < t'_\ell < t$. Let $(E'_1, V'_1)$ be the part of the contour walks $(E, V)$ between $t'_1$ and $t'_{\ell+1}$ (with the convention that $t'_0 = t_{p-1}$ and $t'_{\ell+1} = t$).
- From time $t$ to $t_p$, the edges $\epsilon_\ell, \ldots, \epsilon_1$ are successively traversed back toward the root of the branch $\partial(v_t)$, at times $t''_{\ell+1} < \cdots < t''_1$. Let $(E''_1, V''_1)$ be the part of the contour walks $(E, V)$ between $t''_{\ell+1}$ and $t''_1$ (with the convention that $t''_{\ell+1} = t$ and $t''_0 = t_p$).

The contour walks $(E'_1, V'_1)$ (resp. $(E''_1, V''_1)$) encodes for the left (resp. right) subtree attached at the $i$th vertex of the branch $\partial(v_t)$. One can see $t'_i$ as the time of the last upcrossing of heights $(i-1, i)$ before time $t$, and $t''_i$ as the time of the first downcrossing of heights $(i, i-1)$ after time $t$.

Upon shifting the walks $E'_1$ and $E''_1$ down by $i$ so that they start from zero, and also shifting the walks $V'_1$ and $V''_1$ by $V(t'_i)$ (resp. $V(t''_i+1)$): the two forests

$$\text{left}(t) = [(E_1, V_1), \ldots, (E_{p-1}, V_{p-1}), (E'_0, V'_0), \ldots, (E'_\ell, V'_\ell)]$$

$$\text{right}(t) = [(E''_0, V''_0), \ldots, (E''_{p-1}, V''_{p-1}), (E''_p, V''_p), \ldots, (E''_{k''}, V''_{k''})]$$

belong respectively to $\mathcal{V}^{k',n'}$ and $\mathcal{V}^{k'',n''}$ with $k' = p + \ell$, $n' = t - \ell$, $k'' = k - p + 1 + \ell$, and $n'' = 2n - t - \ell$.

Let us apply this branch decomposition to a random forest $(E^{k,n}, V^{k,n})$. In view of expression (18), conditionally on $E^{(k,n)}(t) = \ell$,

$$V^{(k,n)}(t) \overset{\text{law}}{=} S_\ell.$$
where \( S_k \) denote the sum of \( k \) i.i.d. random variables uniform on \( \{-1, 0, 1\} \). Moreover, again conditionally on \( E^{(k,n)}(t) = \ell \),

\[
\text{left}^{(k,n)}(t) \overset{\text{law}}{=} (E^{k',n'}, V^{k',n'}), \quad \text{right}^{(k,n)}(t) \overset{\text{law}}{=} (E^{k'',n''}, V^{k'',n''}).
\]

6.2.2. The middle branch length in a random forest. The first step of the proof is a bound for the tail probability of \( E^{k,n}(n) \), the length of the middle branch \( (t = n) \).

**Lemma 5.** For \( 4\ell^2 > 27 \cdot n \),

\[
\Pr(E^{k,n}(n) > \ell) < A \cdot \frac{\ell}{n^{1/2}} \exp \left( -\frac{4\ell^2}{27n} \right).
\]

**Proof.** With the notations of Subsection 6.2.1, for \( 1 \leq p \leq k \), let \( t_{p}^{k,n} \) denote the \( p \)th concatenation time of \( E^{k,n} \). Then

\[
\Pr \left( E^{k,n}(n) = \ell \left| t_{p+1}^{k,n} - t_{p}^{k,n} = 2m, 0 \leq n - t_{p}^{k,n} = a \leq 2m \right. \right) = \Pr \left( E^{(m)}(a) = \ell \right),
\]

so the proof reduces to bound \( \Pr \left( E^{(m)}(a) > \ell \right) \) uniformly on pairs \((a, m)\) such that \( 0 \leq a \leq 2m \leq 2n \). We have

\[
\Pr \left( E^{(m)}(a) = \ell \right) = \frac{C(\alpha; \ell) C(2m - a; \ell)}{C(2m; 0)} = \frac{C(\alpha; \ell) C(\beta; \ell)}{C(2m; 0)},
\]

with an obvious change of variables. From [1], Ch. I.3, Th 2], for \( k > n/2 + h \) and \( 1 \leq h \leq n/6 \)

\[
\sqrt{\frac{\pi n}{2}} \binom{n}{k} < 2^n \exp \left( -\frac{2h^2}{n} + \frac{2h}{n} + \frac{4h^3}{n^2} \right).
\]

With \( n = \alpha + 1, k = (\alpha + \ell)/2 + 1, 1 \leq h = \ell/6 \leq \alpha/6 \leq n/6 \), this bound yields:

\[
C(\alpha; \ell) = \frac{\ell + 1}{\alpha + 1} \left( \frac{\alpha + 1}{(\alpha + \ell)/2 + 1} \right) < \sqrt{\frac{8}{\pi}} \cdot \frac{\ell}{n^{3/2}} 2^n \exp \left( -\frac{\ell^2}{18n} + \frac{\ell}{3n} + \frac{\ell^3}{54n^2} \right),
\]

\[
< e^{1/3} \sqrt{\frac{8}{\pi}} \cdot \frac{\ell}{\alpha^{3/2}} 2^n \exp \left( -\frac{\ell^2}{27n} \right)
\]

\[
< 2e^{10/27} \sqrt{\frac{8}{\pi}} \cdot \frac{\ell}{\alpha^{3/2}} 2^n \exp \left( -\frac{\ell^2}{27\alpha} \right),
\]

Thus,

\[
\Pr \left( E^{(m)}(a) = \ell \right) < B_1 \cdot \frac{\ell^3}{(\alpha\beta)^{3/2}} \exp \left( -\frac{\ell^2}{27\alpha} - \frac{\ell^2}{27\beta} \right),
\]

\[
< B_2 \cdot \frac{\ell^2}{m^{3/2}} \exp \left( -\frac{4\ell^2}{27m} \right),
\]

the latter inequality since the maximum of the function \((xy)^{-3/2} \exp \left( -x^{-1} - y^{-1} \right)\), subject to \( x + y = 27m/\ell^2 < 1 \), is obtained for \( x = y \). The last inequality entails
that, for $4\ell^2 > 27n$,
\[
\Pr\left(E^{(m)}(a) > \ell\right) < B_3 \cdot \frac{\ell}{m^{1/2}} \exp\left(-\frac{4\ell^2}{27m}\right)
\]
\[
< B_3 \cdot \frac{\ell}{n^{1/2}} \exp\left(-\frac{4\ell^2}{27n}\right).
\]

6.2.3. The largest label on the middle branch.

Lemma 6. Let $\bar{H}^{k,n}$ be the largest label on the branch between the root and the vertex reached at $t = n$ by the contour traversal of $(E^{k,n}, V^{k,n})$. Then there exists $c_0$ such that for all $k$, $n$ and $h$,
\[
\Pr(\bar{H}^{k,n} > h) \leq c_0 e^{-\frac{1}{3}(hn^{-1/4})^{4/3}}.
\]

Proof. As already discussed the conditional probability that the largest label is $h$ knowing that the branch has length $\ell$ is exactly the probability that a random walk with steps \{1, 0, -1\} of length $\ell$ has maximal value $h$. Using the reflection principle, Azuma’s inequality [3, Th. 2.1, p. 85] then reads
\[
\Pr(\bar{H}^{k,n} > h \mid E^{k,n}(n) = \ell) \leq 2 e^{-h^2/2\ell}.
\]

Next, as previously calculated, for $4\ell^2 \geq 27n$,
\[
\Pr(E^{k,n}(n) > \ell) < A \cdot (\ell/n^{1/2}) e^{-4\ell^2/27n}
\]
Finally, assuming $h > (27/4)^{3/4}n^{1/4}$, so that the previous inequality holds,
\[
\Pr(\bar{H}^{k,n} > h) \leq \Pr(\bar{H}^{k,n} > h \mid E^{k,n}(n) \leq h^{2/3}n^{1/3}) + \Pr(E^{k,n}(n) > h^{2/3}n^{1/3})
\]
\[
\leq 2e^{-h^{4/3}/2n^{1/3}} + A \cdot (h^{2/3}n^{-1/6}) e^{-4h^{4/3}/27n^{1/3}}.
\]
The lemma follows for $h > (27/4)^{3/4}n^{1/4}$, taking $c_0$ large enough, and also for $h \leq (27/4)^{3/4}n^{1/4}$, taking $c_0 \geq 3^{3/4}$.

6.2.4. Conditional induction in the case $E(n) > 0$.

Lemma 7. Assume the bound [3] holds true for some $y_0$ for $V^{k,m}$ with $m < n$. Then, for all $\ell > 0$ and $k \geq p \geq 0$, the probability
\[
p_{k,n}(y; h, \ell, p) = \Pr\left(\sup_{0 \leq t \leq n} V^{k,n}(t) > yn^{1/4} \mid \bar{H}^{k,n} = h, E^{k,n}(n) = \ell, t_{p-1}^{k,n} < n < t_p^{k,n}\right),
\]
satisfies
\[
p_{k,n}(y; h, \ell, p) \leq 4e^{-2/4(y-hn^{-1/4})}, \text{ provided } h \leq n^{1/4}(y - y_0/2^{1/4}).
\]

Proof. Assume $(E^{k,n}, V^{k,n}) = (E_1 \cdots E_k, V_1 \cdots V_k)$ is such that $E^{k,n}(n) = \ell > 0$ and $t_{p-1}^{k,n} < n < t_p^{k,n}$, so that $t = n$ occurs inside $(E_p, V_p)$. Apply the branch decomposition (see Section 6.2.1) at $t = n$ to $(E^{k,n}, V^{k,n})$ and let
\[
(E^{k,n}, \bar{V}^{k,n}) = \text{left}^{k,n}(n) = [(E_1, V_1), \ldots, (E_{p-1}, V_{p-1}), (E_p, V_p'), \ldots, (E_{\ell'}, V_{\ell'}')],
\]
\[
(E^{k,n}, \tilde{V}^{k,n}) = \text{right}^{k,n}(n) = [(E_p', V_p''), \ldots, (E_{p+1}, V_{p+1}''), (E_{\ell'+1}, V_{\ell'+1}''), \ldots, (E_k, V_k)].
\]
Upon taking $k' = p + \ell$, $k'' = k - p + \ell + 1$ and $n' = (n - \ell)/2$,
\[
(E^{k,n}, \bar{V}^{k,n}) \overset{\text{law}}{=} (E^{k',n'}, V^{k',n'}) \quad \text{and} \quad (E^{k,n}, \tilde{V}^{k,n}) \overset{\text{law}}{=} (E^{k'',n''}, V^{k'',n'}).
Observe now that, in the previous decomposition,
\[ \sup_{0 \leq t \leq 2n} V(t) \leq \sup \left( H + \sup_{0 \leq t \leq 2n'} \bar{V}(t), H + \sup_{0 \leq t \leq 2n'} \bar{V}(t) \right), \]
so that
\[ p_{k,n}(y; h, \ell, p) \leq \mathbb{P}\left( \sup_{0 \leq t \leq 2n'} V^{k,n}(t) > yn^{1/4} - h \ \big| \ \bar{H}^{k,n} = h, E^{k,n}(n) = \ell, t_{p-1}^{k,n} < n < t_p^{k,n} \right) + \mathbb{P}\left( \sup_{0 \leq t \leq 2n'} V^{k,n}(t) > yn^{1/4} - h \ \big| \ \bar{H}^{k,n} = h, E^{k,n}(n) = \ell, t_{p-1}^{k,n} < n < t_p^{k,n} \right). \]
Hence, in view of the preceding identities in law,
\[ p_{k,n}(y; h, \ell, p) \leq \mathbb{P}\left( \sup_{0 \leq t \leq 2n'} V^{k',n'}(t) > yn^{1/4} - h \right) + \mathbb{P}\left( \sup_{0 \leq t \leq 2n'} V^{k'',n''}(t) > yn^{1/4} - h \right). \]
Observe that \( 2n' \leq n \). Hence
\[ p_{k,n}(y; h, \ell, p) \leq \mathbb{P}\left( \sup_{0 \leq t \leq 2n'} V^{k',n'}(t) > 2^{1/4}(y - hn^{-1/4})n^{1/4} \right) + \mathbb{P}\left( \sup_{0 \leq t \leq 2n'} V^{k'',n''}(t) > 2^{1/4}(y - hn^{-1/4})n^{1/4} \right). \]
The induction hypothesis now implies, for \( 2^{1/4}(y - hn^{-1/4}) \geq y_0 \), that is, for all \( h \leq n^{1/4}(y - y_0)/2^{1/4} \),
\[ p_{k,n}(y; h, \ell, p) \leq 2e^{-2^{1/4}(y - hn^{-1/4})}, \]
which is exactly the lemma, up to a factor 2 added for later convenience.

6.2.5. Conditional induction in the case \( E(n) = 0 \).

**Lemma 8.** Assume the bound \( [\ref{bound5}] \) holds true for some \( y_0 \) for \( V^{k,m} \) with \( m < n \). Then, provided \( y \geq y_0/2^{1/4} \),
\[ \mathbb{P}\left( \sup_{0 \leq t \leq 2n} V^{k,n}(t) > yn^{1/4} \ \big| \ E^{k,n}(n) = 0 \right) \leq 4e^{-2^{1/4}y}. \]
The bound \( [\ref{bound6}] \) thus remain valid in the case \( \ell = 0 \).

**Proof.** In this case the decomposition at \( t = n \) is even simpler. Let \( p = \sup \{ i \ | \ t_i < n \} \), and \( q = \inf \{ i \ | \ t_i > n \} \), and consider a decomposition in four parts, cutting at times \( t_p \), \( n \), and \( t_q \). The two contour walks for \( 0 \leq t \leq t_p \) and \( t_q \leq t \leq 2n \) are uniform on \( \mathcal{E}V_{p,t_p} \) and \( \mathcal{E}V_{k-q,2n-t_q} \). The other two contour walks are uniform on \( \mathcal{E}V_{n-t_p} \) and \( \mathcal{E}V_{t_q-n} \).

The result then follows using the induction hypothesis on the 4 parts. \( \square \)
6.2.6. Complete induction and proof of Proposition \[21\].

Observe that the bounds of Lemmas \[9\] and \[10\] do not depend on \( \ell \) or \( p \), so that, assuming (9) holds for some \( \ell \), for all

\[
\mathbb{P} \left( \sup_{0 \leq t \leq 2n} V^{k,m}(t) > y/n^{1/4} \left| \tilde{H}^{k,n} = h \right. \right) \leq 4e^{-2^{1/4}(y-hn^{-1/4})},
\]

provided \( h \leq n^{1/4}(y-y_0/2^{1/4}) = h_0 \). Using this bound,

\[
f_{k,n}(y) = \mathbb{P} \left( \sup_{0 \leq t \leq 2n} V^{k,n}(t) > y/n^{1/4} \right) \leq 4e^{-2^{1/4}y} \sum_{h=0}^{h_0} e^{2^{1/4}hn^{-1/4}} \mathbb{P}(\tilde{H}^{k,n} = h) + \mathbb{P}(\tilde{H}^{k,n} > h_0),
\]

\[
\leq 4e^{-2^{1/4}y} \left( 1 + \sum_{h=1}^{\infty} (e^{2^{1/4}hn^{-1/4}} - e^{2^{1/4}(h-1)n^{-1/4}}) \mathbb{P}(\tilde{H}^{k,n} \geq h) \right) + \mathbb{P}(\tilde{H}^{k,n} > n^{1/4}(y-y_0/2^{1/4})).
\]

In view of Lemma \[3\], the summation is bounded by a convergent integral that evaluates to a constant \( c_1 \), Lemmas \[9\] allows also to dispose of the second term:

\[
f_{k,n}(y) \leq 4e^{-2^{1/4}y}(1 + c_1) + c_0 e^{-\frac{1}{3}(y-y_0/2^{1/4})^{4/3}}
\]

Now observe that

\[
e^{-\frac{1}{3}(y-y_0/2^{1/4})^{4/3}} = e^{-y} e^{y_0/2^{1/4}} e^{(y-y_0/2^{1/4})} e^{-(y-y_0/2^{1/4})^{4/3}}
\]

\[
\leq e^{-y} e^{y_0/2^{1/4}} e^{y-y_0/2^{1/4}} e^{-\frac{1}{3}(y-y_0/2^{1/4})^{4/3}}
\]

\[
= e^{-y} e^{y_0/2^{1/4}} e^{y-y_0/2^{1/4}} e^{-\frac{1}{3}(y-y_0/2^{1/4})^{4/3}}
\]

for \( y \geq y_0 \), as soon as \( x \rightarrow x^{-\frac{1}{3}}x^{4/3} \) is decreasing on the interval \([y_0 - y_0/2^{1/4}, +\infty]\), that is, for \( y_0 \geq 1933 \). The bound can thus be rewritten as

\[
f_{k,n}(y) \leq e^{-y} \left( 4e^{-2^{1/4}y}(1 + c_1) + c_0 e^{y_0/2^{1/4}} e^{-\frac{1}{3}(y-y_0/2^{1/4})^{4/3}} \right)
\]

which is smaller than \( e^{-y} \) for \( y_0 \) large enough, so that induction can be carried on (Recall that \( c_0 \) and \( c_1 \) do not depend on \( n \)). The case \( n = 1 \) holds true for \( y_0 \geq 1 \), and the proof of Proposition \[21\] is complete.

6.3. Tightness. Tightness for the bidimensional path follows from the tightness of the two projections. The tightness of the first projection was proved by Kaigh \[21\]. Thus we only have to prove the following proposition (cf. \[1\], Ch. 3).

**Proposition 8.** For all \( \varepsilon > 0 \) and \( \delta > 0 \) there exists \( m \) such that for \( n \) large enough

\[
\mathbb{P} \left( \sup_{0 \leq s \leq 1} \left| \tilde{W}^{(n)}_{s} - \tilde{W}^{(n)}_{[ms]/m} \right| \geq \delta \right) \leq \varepsilon.
\]

We proceed by bounding, for \( m \) large enough and all \( i \) with \( 1 \leq i \leq m, \)

\[
p_{m,i}(n) = \mathbb{P} \left( \sup_{(i-1)/m \leq s \leq i/m} \left| \tilde{W}^{(n)}_{s} - \tilde{W}^{(n)}_{(i-1)/m} \right| \geq \delta \right)
\]

\[
= \mathbb{P} \left( \sup_{t_1 \leq t \leq t_2} \left| W^{(n)}(t) - W^{(n)}(t_1) \right| \geq \delta n^{1/4} \right),
\]

where \( t_1 = [2(i-1)n/m] \), \( t_2 = [2in/m] \) and \( \delta' = (8/9)^{1/4} \delta \).
The proposition is an immediate consequence of the following lemma, upon taking \( m \) large enough and summing on \( i \).

**Lemma 9.** For all \( \varepsilon > 0 \) and \( \delta > 0 \) there exist \( m_0 \) such that for all \( m > m_0 \) and \( 1 \leq i \leq m \),

\[
\exists n_0, \forall n > n_0, \quad p_{m,i}(n) \leq \frac{\varepsilon}{2m} + C_\delta(m),
\]

where \( C_\delta(m) \) is exponentially decreasing in a positive power of \( m \).

**Proof.** For simplicity of notations let us assume \( t_2 - t_1 = 2n/m \). (The general case is identical but obscured by a collection of \( |n/m| \).)

Let us consider the shape \( T \) associated with the times \( t_1 \) and \( t_2 \), as defined in Section 6.1.2. The branches \( \text{br}(v_{t_1}) \) from \( v_{t_1} \) to the root and \( \text{br}(v_{t_2}) \) from \( v_{t_2} \) to the root meet at the unique branchpoint \( v \) of the shape \( T \). Let \( B = \text{br}(v_{t_1}) \setminus \text{br}(v_{t_2}) \) be the branch between \( v_{t_1} \) and \( v \) and \( C = \text{br}(v_{t_2}) \setminus \text{br}(v_{t_1}) \) the branch between \( v_{t_2} \) and \( v \). Between \( t_1 \) and \( t_2 \), a total of \( k \) edges \( e_i, i = 1, \ldots, k \) of \( U \) are successively traversed on the branch from \( v_{t_1} \) to \( v_{t_2} \) by the contour walk. The \( k' \) first are along the branch \( B \) and traversed toward the root, while the \( k'' \) next are along the branch \( C \) and traversed away from the root. By construction,

\[
k' = E^{(n)}(t_1) - \inf_{[t_1, t_2]} E^{(n)} = \sqrt{2n} \left[ e^{(n)} \left( \frac{i-1}{m} \right) - \inf_{\left[ \frac{i-1}{m}, \frac{i}{m} \right]} e^{(n)} \right],
\]

\[
k'' = E^{(n)}(t_2) - \inf_{[t_1, t_2]} E^{(n)} = \sqrt{2n} \left[ e^{(n)} \left( \frac{i}{m} \right) - \inf_{\left[ \frac{i}{m}, \frac{i+1}{m} \right]} e^{(n)} \right],
\]

and the total length \( \Delta_i^{(n)} = k = k' + k'' \) of the branch from \( v_{t_1} \) to \( v_{t_2} \) is

\[
\Delta_i^{(n)} = E^{(n)}(t_1) + E^{(n)}(t_2) - 2 \inf_{[t_1, t_2]} E^{(n)} = \sqrt{2n} \Delta_{m,i} e^{(n)},
\]

with the notation

\[
\Delta_{m,i} = \left[ f \left( \frac{i-1}{m} \right) + f \left( \frac{i}{m} \right) - 2 \inf_{\left[ \frac{i-1}{m}, \frac{i}{m} \right]} f \right].
\]

The walk \( (E^{(n)}(t))_{t_1 \leq t \leq t_2} \) is decomposed along the branch from \( v_{t_1} \) to \( v_{t_2} \) into a sequence

\[
E_0, \epsilon_1, E_1, \epsilon_2, \ldots, E_{k-1}, \epsilon_k, E_k,
\]

where the \( \Delta_i^{(n)} = k \) passages on the branch separate \( \Delta_i^{(n)} \) subtrees (each coded by a \( E_i \)) of total size (number of edges) \( N_i^{(n)} = (2n/m - \Delta_i^{(n)})/2 \).

In this decomposition, conditionally given \( \Delta_i^{(n)} = k \) (and \( N_i^{(n)} = n' = n/m - k/2 \)), the forest satisfies

\[
\left[ (E_0 \cdots E_k, V_0 \cdots V_k) \right] \overset{\text{law}}{=} \left( E^{k+1, n'}, V^{k+1, n'} \right),
\]

upon resetting all walks to start at zero. Under the same conditions, the variation of labels on edges of \( B \) and \( C \) are i.i.d. random variables \( \zeta(\epsilon) \), uniform on \( \{-1, 0, +1\} \): the random variable

\[
\tilde{H}_i^{(n)} = \sup_{0 \leq t \leq \Delta_i^{(n)}} \sum_{j=0}^t \zeta(\epsilon_j)
\]

has tail distribution bounded by Azuma’s inequality.
Our strategy is to bound $\Delta_i^{(n)}$ and, conditionally given $\Delta_i^{(n)}$, to bound separately the variations on the forest and on the branch:

- First fix $\alpha > 0$ and let $a_1$ denote the tail probability for $\Delta_i^{(n)}$:
  
  $$a_1 = \mathbb{P} \left( \Delta_i^{(n)} \geq m^{-\alpha} \sqrt{2n} \right) = \mathbb{P} \left( \Delta_{m,i}e^{(n)} \geq m^{-\alpha} \right).$$

- On the complementary event for $\Delta_i^{(n)}$ we consider the variation in a forest,
  
  $$a_2 = \mathbb{P} \left( \sup_{0 \leq t \leq 2N_i^{(n)}} \left| V^{\Delta_i^{(n)}+1,N_i^{(n)}}(t) \right| \geq \frac{\delta'}{4}n^{1/4} \text{ and } \Delta_{m,i}e^{(n)} \leq m^{-\alpha} \right),$$
  
  $$\leq 2\mathbb{P} \left( \sup_{0 \leq t \leq 2N_i^{(n)}} \left| V^{\Delta_i^{(n)}+1,N_i^{(n)}}(t) \right| \geq \frac{\delta'}{4}n^{1/4} \text{ and } \Delta_{m,i}e^{(n)} \leq m^{-\alpha} \right),$$

- and the variations on the branch from $t_1$ to $t_2$,
  
  $$a_3 = \mathbb{P} \left( \sup_{0 \leq t \leq \Delta_i^{(n)}} \left| \sum_{j=0}^{t} \zeta(\epsilon_j) \right| \geq \frac{\delta'}{4}n^{1/4} \text{ and } \Delta_i^{(n)} \leq m^{-\alpha}n^{1/4} \right),$$
  
  $$\leq 2\mathbb{P} \left( \hat{H}_i^{(n)} \geq \frac{\delta'}{4}n^{1/4} \text{ and } \Delta_i^{(n)} \leq m^{-\alpha}n^{1/4} \right).$$

In view of the decomposition we have

$$p_{m,i}^{(n)} \leq a_1 + a_2 + a_3.$$

We now bound separately each term $a_1$, $a_2$, and $a_3$. Let us start with $a_2$ and write

$$a_2 = \sum_{k=0}^{m^{-\alpha} \sqrt{2n}} \mathbb{P} \left( \sup_{0 \leq t \leq 2n'} V^{k+1,n'}(t) \geq \frac{\delta'}{4}n^{1/4} \right) \mathbb{P} \left( \Delta_i^{(n)} = k \right),$$

where $n' = n/m - k/2$. In order to apply the tail estimate of the previous subsection (Proposition \ref{prop:tail}), we need

$$y := \frac{\delta'}{4}(n/n')^{1/4} \geq y_0.$$ 

Since $n' \leq n/m$ this condition is satisfied as soon as

$$\frac{\delta'}{4} m^{1/4} \geq y_0, \quad \text{that is} \quad m \geq (4y_0/\delta')^4.$$ 

Then,

$$\mathbb{P} \left( \sup_{0 \leq t \leq 2n'} V^{k+1,n'}(t) \geq \frac{\delta'}{4}n^{1/4} \right) \leq \exp \left( -\delta' m^{1/4}/4 \right).$$

The latter bound being independent of $k$ we obtain

$$a_2 \leq 2\exp \left( -\delta' m^{1/4}/4 \right).$$

Let us now turn to $a_3$. Conditionally given that $\Delta_i^{(n)} = k$, the maximum on the branch $\hat{H}_i^{(n)}$ is distributed as the maximum of a random walk with $k$ steps that are
independent increments uniform on \{-1, 0, 1\}. Thus, applying Azuma’s inequality and the reflection principle, we have, for \(k \leq m^{-\alpha} \sqrt{2n}\),
\[
\mathbb{P}\left(\tilde{H}_i^{(n)} \geq \frac{\delta'}{4} n^{1/4} \mid \Delta_i^{(n)} = k\right) \leq 2 \exp\left(-\frac{\delta'^2}{32\sqrt{2}} m^\alpha\right).
\]
Again the latter bound is independent of \(k\) so that
\[
a_3 \leq \exp\left(-\frac{\delta'^2}{32\sqrt{2}} m^\alpha\right).
\]
Finally let us deal with \(a_1\). For \(n\) large enough, the convergence of \(e^{(n)}\) to the normalised excursion \(e\) entails
\[
\left|a_1 - \mathbb{P}\left(\Delta_{m,1} e \geq m^{-\alpha}\right)\right| \leq \frac{\varepsilon}{2m}.
\]
Let us thus consider, for \(1 \leq i \leq m\), the probability \(\pi_{m,i} = \mathbb{P}\left(\Delta_{m,i} e \geq m^{-\alpha}\right)\), and restrict the choice of \(\alpha\) to \(0 < \alpha < 1/2\). The finite dimensional distribution of the value of \(e\) at two points and the minimum between them was considered in Section 6.1 (take \(p = 2\) in the continuum limit of Proposition 3 or see [32]). For \(i \neq 1, m\) this entails,
\[
\pi_{m,i} = \frac{m^{3/2}}{\pi(\tau_1 \tau_2)^{3/2}} \int_{\beta > m^{-\alpha}, y > 0, x > 0, y > 0} xy\beta e^{-y^2/2\tau_1} e^{-x^2/2\tau_2} e^{-m\beta^2/2} \text{d}x \text{d}y \text{d}\beta,
\]
in which
\[
\tau_1 = \frac{i - 1}{m}, \quad \tau_2 = \frac{i}{m}, \quad \text{and} \quad \tau'_2 = 1 - \tau_2.
\]
Thus
\[
\pi_{m,i} \leq \frac{m^{3/2}}{\pi(\tau_1 \tau_2)^{3/2}} \int_{\beta > m^{-\alpha}} xy\beta e^{-y^2/2\tau_1} e^{-x^2/2\tau_2} e^{-m\beta^2/2} \text{d}x \text{d}y \text{d}\beta
\]
\[
\leq \frac{m^{1/2}}{\pi(\tau_1 \tau_2)^{1/2}} e^{-m^{1-2\alpha}/2}
\]
\[
\leq \frac{m^{1/2}}{\pi} e^{-m^{1-2\alpha}/2}.
\]
For the remaining two cases \(i = 1\) and \(i = m\), we have with \(\tau_1 = \frac{m^{-1}}{m}\),
\[
\pi_{m,1} = \pi_{m,m} = \sqrt{\frac{2m^3}{\pi \tau_1^2}} \int y^2 e^{-y^2/2\tau_1} e^{-y^2/2\tau_1} \text{d}y
\]
Now \(x \rightarrow xe^{-x^2/\alpha}\) is bounded by \(\frac{\sqrt{\pi}}{\sqrt{2e}}\), so that
\[
\pi_{m,1} \leq \sqrt{\frac{2m}{\pi \tau_1^2 e}} \int u e^{-u^2/2} \text{d}u
\]
\[
\leq \frac{m^{1/2}}{e} e^{-m^{1-2\alpha}/2}.
\]
Thus for all \(i\), \(\pi_{m,i}\) is bounded by an exponentially decreasing function of \(m\).
The proof of the lemma is then concluded by summing the contribution of \( a_1 \), \( a_2 \) and \( a_3 \): for all \( m \) there exists \( n_0 \) such that for all \( n > n_0 \),

\[
\sum_{i=1}^{m} p_{m,i}(n) \leq \frac{\varepsilon}{2} + C_3(m),
\]

in which \( C_3(m) \) is exponentially small in a power of \( m \).

Taking \( m \) and then \( n \) large enough the tightness is proved, and together with Lemma 5 this concludes the proof of Theorem 4.

6.4. Convergence of the profile. In view of the Skorohod representation theorem \( \text{\cite{30}} \), we may assume the joint existence, on some probabilistic triple \((\Omega, A, \mathbb{P})\), of a sequence of copies of \( X^{(n)} \), and of a copy of \( X \) (and we keep the same notation \( X^{(n)} \) and \( X \), as for the original), such that, for almost any \( \omega \in \Omega \), \( \left( X^{(n)}_t(\omega) \right)_{0 \leq t \leq 1} \) converges to \( (X_t(\omega))_{0 \leq t \leq 1} \) in the Skorohod topology of \( D([0, 1], \mathbb{R}^2) \). In this section we build copies of \( F_n \) and \( F \), such that, almost surely, \( F_n \) converges to \( F \).

First, the set \( \Omega_1 = \{ \omega \mid t \rightarrow X_t(\omega) \text{ is continuous} \} \) has probability 1, so that uniform convergence of \( \left( X^{(n)}_t(\omega) \right)_{0 \leq t \leq 1} \) to \( (X_t(\omega))_{0 \leq t \leq 1} \) holds almost surely. Set

\[
W^{(n)}_{\min} = \inf_{0 \leq s \leq 1} W^{(n)}_s(\omega),
\]

\[
\delta_n(\omega) = \sup_{0 \leq s \leq 1} \left| W^{(n)}_s(\omega) - \hat{W}_s(\omega) \right|,
\]

\[
\Psi_n(x, \omega) = \int_0^1 \mathbf{1}_{W^{(n)}_s(\omega) \leq x} \, ds,
\]

\[
F_n(x) = \Psi_n \left( W^{(n)}_{\min} + x, \omega \right),
\]

\[
\Psi(x, \omega) = \int_0^1 \mathbf{1}_{\hat{W}_s(\omega) \leq x} \, ds = J \left( (-\infty, x] \right).
\]

It follows from general results on superprocesses that \( \Phi \), the distribution function of the random measure ISE, is almost surely continuous \( \text{\cite{23}} \). Now from

\[
\Psi(x - \delta_n) \leq \Psi_n(x) \leq \Psi(x + \delta_n)
\]

and the almost sure continuity of \( \Psi \), it follows that the set

\[
\Omega_3 = \{ \omega \mid \forall x, \lim_{n} \Psi_n(x, \omega) = \Psi(x, \omega) \text{ and } x \rightarrow \Psi(x, \omega) \text{ is continuous} \}
\]

has probability 1. In \( \Omega_3 \), as we deal with increasing functions, uniform convergence of \( \Psi_n \) to \( \Psi \) holds true. Hence, on the set

\[
\Omega_4 = \{ \omega \in \Omega_3 \mid \lim_{n} W^{(n)}_{\min}(\omega) = W_{\min}(\omega) \},
\]

i.e. almost surely, uniform convergence of \( F_n \) to \( F \) holds true.

On the other hand, we explain below why \( F_n \) is close to some copy of \( F_n \), that we shall denote \( F_n \) too. As in section \( \text{\cite{5,2}} \) from \( X^{(n)} \) one recovers a random uniform
contour pair \((E^{(n)}, V^{(n)})\), and a random uniform embedded tree \(U_n \in \mathcal{U}_n\). Now we can choose at random a well labelled tree \(W_n\) in the conjugacy class of \(U_n\) in such a way that Theorem 3 holds for \((W_n, U_n)\). Theorem 3 entails that \(\hat{\lambda}^{(n)}\) and \(\hat{\lambda}^{(n)}\) have the same asymptotic behavior, and \(F_n\) is just \(\hat{\lambda}^{(n)}\) suitably normalised. So we have now to establish the relation between \(\hat{\lambda}^{(n)}\) and \(\hat{\Lambda}^{(n)}\). First, set

\[
\hat{f}^{(n)}_y = 2n \hat{F}_n \left( y(8n/9)^{-1/4}, \omega \right).
\]

As we have

\[
m_n = (8n/9)^{1/4} W^{(n)}_{\min},
\]

\(\hat{f}^{(n)}_y\) is the number of visits of the contour traversal of \(U_n\) to a node whose label is not larger than \(m_n + y\), but the number of visits, by the contour traversal, of a given node, is exactly 1 plus the number of children of this node, so that

\[
\hat{\Lambda}^{(n)}_y + \hat{\Lambda}^{(n)}_{y-1} - 1 \leq \hat{f}^{(n)}_y \leq \hat{\Lambda}^{(n)}_y + \hat{\Lambda}^{(n)}_{y+1}.
\]

Hence, due to Theorem 3,

\[
2\hat{\Lambda}^{(n)}_{y-3} - 1 \leq \hat{f}^{(n)}_y \leq 2\hat{\Lambda}^{(n)}_{y+3},
\]

or, equivalently,

\[
\frac{n}{n+1} \hat{F}_n \left( x - cn^{-1/4}, \omega \right) \leq F_n \left( x, \omega \right) \leq \frac{n}{n+1} \hat{F}_n \left( x + cn^{-1/4}, \omega \right) + \frac{1}{2n+2},
\]

\(c\) being a constant. That is, on \(\Omega_4\), uniform convergence of \(F_n\) to \(F\) holds true.

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