ON THE SHARP EFFECT OF ATTACHING A THIN HANDLE ON THE SPECTRAL RATE OF CONVERGENCE

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Abstract. Consider two domains connected by a thin tube: it can be shown that the resolvent of the Dirichlet Laplacian is continuous with respect to the channel section parameter. This in particular implies the continuity of isolated simple eigenvalues and the corresponding eigenfunctions with respect to domain perturbation. Under an explicit nondegeneracy condition, we improve this information providing a sharp control of the rate of convergence of the eigenvalues and eigenfunctions in the perturbed domain to the relative eigenvalue and eigenfunction in the limit domain. As an application, we prove that, again under an explicit nondegeneracy condition, the case of resonant domains features polynomial splitting of the two eigenvalues and a clear bifurcation of eigenfunctions.

1. Introduction and statement of the main results

The aim of this paper is to investigate the behavior of Dirichlet eigenvalues in varying domains, when a shrinking cylindrical handle is attached to a smooth region, seeking not only for the rate of convergence but also for sharp asymptotics. Since we consider a tubular handle with a cross-section of radius of order $\varepsilon \to 0^+$ (see Figure 1), it is quite natural to expect the rate of convergence of the eigenvalues to rely essentially on the capacity of the junction points and hence to be of order $\varepsilon^N$, being $N$ the space dimension.

Referring to Figure 1 let $u_0$ be the $k$-th eigenfunction on the limit (disconnected) domain $D^- \cup D^+$ completely supported only in the connected component $D^+$. By the attachment of a handle $C_\varepsilon$ with cross section of radius of order $\varepsilon$, its mass will be pushed into the channel in order to spread over the new entire domain $D^- \cup C_\varepsilon \cup D^+$. Besides the tubular shape of the connecting tube, we require, as a basic assumption to start our analysis, that the handle is attached at a point $e_1$ of $\partial D^+$ where $u_0$ has a zero of order one, i.e. its normal derivative is different from zero. If moreover $u_0$ is simple and suitably normalized, the corresponding eigenvalue $\lambda_k$ can be continued into a family $\lambda_k(\varepsilon)$ of eigenvalues corresponding to normalized eigenfunctions $u_\varepsilon$ on the perturbed domain.

Figure 1. The disconnected domain $D^- \cup D^+$ becomes connected by the attachment of the handle $C_\varepsilon$.

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We prove that, in such a setting, there exists the limit
\[
\lim_{\varepsilon \to 0} \varepsilon^{-N} (\lambda_k - \lambda_{k+1}) = \left( \frac{\partial u_0}{\partial \nu}(e_1) \right)^2 \mathcal{C}(\Sigma),
\]
where \(\mathcal{C}(\Sigma)\) is a positive constant depending only on the geometry of the junction section \(\Sigma\) (see Theorems 1.1 and 1.2 below). Thinking to the eigenfunction \(u_0\) as pushed into the channel, we can imagine that a force acts over the junction between the channel and the domain where \(u_0\) is supported. The constant \(\mathcal{C}(\Sigma)\) represents indeed the compliance of the channel’s junction, under a constant force concentrated at the junction section; the compliance, which can be expressed as the \(L^1\)-norm of the trace of a suitable harmonic function over the channel section, measures the faculty of an elastic membrane to adjust or to resist to a force applied on the section, see [10] for a precise definition. Our proof consists in a sharp differentiation with respect to the parameter, which requires first a careful analysis of the transition functions which have to be attached to \(u_0\) in order to push it over the channel. In this way, we will prove that, once more,
\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-\lambda} \| u_\varepsilon - u_0 \|_{H^2(\mathbb{R}^N)}^2 = \left( \frac{\partial u_0}{\partial \nu}(e_1) \right)^2 \mathcal{C}(\Sigma),
\]
where \(u_\varepsilon\) and \(u_0\) are trivially extended to the whole \(\mathbb{R}^N\).

As an application of the sharp asymptotics (1) we are able to treat also the resonant case: if \(\lambda_k = \lambda_{k+1}\) is a double eigenvalue on the limit disconnected domain which is a simple eigenvalue both on \(D^-\) and on \(D^+\), an asymptotics for eigenvalues of type (1) still holds if the limit problem is asymmetrical, e.g. under the assumption that the normal derivatives of the limit eigenfunctions at the junctions are different from each other (see Theorem 1.3). In this case, it turns out that the splitting of the two subsequent eigenvalues \(\lambda_k, \lambda_{k+1}\) has the polynomial vanishing order \(\varepsilon^N\) (see Remark 5.3); such result complements those in [6], where it was proved that, in a symmetric dumbbell domain with a shrinking handle, the splitting of the first two eigenvalues vanishes with exponential rate. Moreover, in contrast with [6], we can localize each approximating eigenfunction on its corresponding region, up to an exponentially vanishing tail, see Theorem 1.2.

For expository reasons, the present paper discusses the effect of attaching a thin handle on the spectral rate of convergence only for dumbbell domains. However, up to minor modifications, the results obtained here hold true in quite general contexts, since they rely essentially on the attachment of a shrinking handle at a point in which the limit eigenfunction has a zero of order 1; therefore the presence/lack of a second domain beyond the channel and its shape seem to be irrelevant for the validity of the asymptotics we are going to derive. The choice of focusing on the dumbbell structure is motivated not only by the large attention devoted to this peculiar case of singularly perturbed domain in the literature, due to the many interesting related spectral phenomena (see 1.3 below), but also by the fact that some preliminary results required in our analysis have been obtained for dumbbell domains in [1,18], where the singular asymptotic behavior of eigenfunctions at the second junction of the tube is described.

1.1. Dumbbell domains. As a paradigmatic example, we consider a dumbbell domain where each “chamber” has a constant section, namely we straighten out the handle and assume its section \(\Sigma\) to be constant along its whole length, whereas we spread out the two domains \(D^+\) and \(D^-\) assuming they are two entire half-spaces, see Figure 2. We observe that such a simplification of the domain’s geometry does not imply a substantial loss of generality if a suitable weight is introduced in the eigenvalue problem under investigation: indeed, the effect of a diffeomorphism transforming a generic dumbbell in a dumbbell with two half-spaces as chambers is the transformation of the eigenvalue problem into a weighted one.

Let \(N \geq 3\). We denote
\[D^- = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 < 0\}, \quad D^+ = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 > 1\},\]
and, for all \(t > 0\),
\[B_t^+ := D^+ \cap B(e_1, t), \quad B_t^- := D^- \cap B(0, t), \quad \Gamma_t^+ = D^+ \cap \partial B_t^+, \quad \Gamma_t^- = D^- \cap \partial B_t^-\],
where $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$, $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^N$, and $B(P, t) := \{ x \in \mathbb{R}^N : |x - P| < t \}$ denotes the ball of radius $t$ centered at $P$. Let $\Sigma \subset \mathbb{R}^{N-1}$ be an open bounded set with $C^{2,\alpha}$-boundary containing $0$. For simplicity of notation, we assume that $\Sigma$ satisfies

$$(2) \quad \{ x' \in \mathbb{R}^{N-1} : |x'| \leq \frac{1}{2} \} \subset \Sigma \subset \{ x' \in \mathbb{R}^{N-1} : |x'| < 1 \}.$$ 

Let $p \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ be a weight satisfying

$$(3) \quad p \geq 0 \text{ a.e. in } \mathbb{R}^N, \quad p \in L^{N/2}(\mathbb{R}^N), \quad \nabla p \cdot x \in L^{N/2}(\mathbb{R}^N), \quad \frac{\partial p}{\partial x_1} \in L^{N/2}(\mathbb{R}^N),$$

$$(4) \quad \begin{cases} p \neq 0 \text{ in } D^-, & p \neq 0 \text{ in } D^+, \\ p(x) = 0 \text{ for all } x \in \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : \frac{1}{2} \leq x_1 \leq 1, x' \in \Sigma \} \cup B_1^+ . \end{cases}$$

Assumption (4) is required for technical reasons as in [1, 18]; it is used in (2.2) to prove some preliminary estimates of eigenfunctions on the perturbed domain.

For every open set $\Omega \subset \mathbb{R}^N$, we denote as $\sigma_p(\Omega)$ the set of the diverging eigenvalues

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_\ell(\Omega) \leq \cdots$$

(where each $\lambda_k(\Omega)$ is repeated as many times as its multiplicity) of the weighted eigenvalue problem

$$(5) \quad \begin{cases} -\Delta \varphi = \lambda p \varphi, & \text{in } \Omega , \\ \varphi = 0, & \text{on } \partial \Omega . \end{cases}$$

It is easy to verify that $\sigma_p(D^- \cup D^+) = \sigma_p(D^-) \cup \sigma_p(D^+)$. Let $\Omega^\varepsilon \subset \mathbb{R}^N$ be the domain formed by connecting the two half-spaces $D^+, D^-$ with a tube of length 1 and cross-section $\varepsilon \Sigma$, i.e.

$$(6) \quad \Omega^\varepsilon = D^- \cup C_\varepsilon \cup D^+, \quad \text{where } \varepsilon \in (0, 1) \text{ and } C_\varepsilon = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 \leq x_1 \leq 1, \frac{x_1}{\varepsilon} \in \Sigma \},$$

see Figure 2.

![Figure 2. The domain $\Omega^\varepsilon$.](image)

Here and in the sequel, for every open set $\Omega \subset \mathbb{R}^N$, $\mathcal{D}^{1,2}(\Omega)$ denotes the functional space obtained as completion of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm $\| u \|_{\mathcal{D}^{1,2}(\Omega)} = \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}$.

1.2. **Main results.** By standard minimization methods, it is easy to prove that the minimum

$$(7) \quad m(\Sigma) = \min_{w \in \mathcal{D}^{1,2}(\tilde{D})} J_{\Sigma}(w),$$

is achieved, where

$$\tilde{D} = D^+ \cup T^-_1, \quad T^-_1 = \{(x_1, x') : x' \in \Sigma, x_1 \leq 1 \},$$

and

$$(8) \quad J_{\Sigma} : \mathcal{D}^{1,2}(\tilde{D}) \to \mathbb{R}, \quad J_{\Sigma}(w) = \frac{1}{2} \int_{\tilde{D}} |\nabla w(x)|^2 \, dx - \int_{\Sigma} w(1, x') \, dx', \quad \text{for every } w \in \mathcal{D}^{1,2}(\tilde{D}).$$

It is easy to verify that

$$m(\Sigma) < 0,$$
such that

\[
\langle \mathcal{D}^{1,2}(\mathbb{R}^N) \rangle \cdot \langle \mathcal{F}, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\Sigma} w(1, x') \, dx',
\]

where in the above formula we mean the functions \( u \) associated to \( \varepsilon \). Furthermore, for every \( \lambda \),

\[
\min_{\lambda \in \mathcal{D}^{1,2}(\mathbb{R}^N)} \left( \frac{1}{2} \int_{\Sigma} |\nabla w(x)|^2 \, dx - \langle \mathcal{D}^{1,2}(\mathbb{R}^N) \rangle \cdot \langle \mathcal{F}, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \right) = -\frac{\mathcal{C}(\Sigma)}{2}
\]

(9) can be rewritten as

\[
m(\Sigma) = \min_{w \in \mathcal{D}^{1,2}(\Sigma)} \left( \frac{1}{2} \int_{\Omega} |\nabla w(x)|^2 \, dx - \langle \mathcal{D}^{1,2}(\mathbb{R}^N) \rangle \cdot \langle \mathcal{F}, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \right) = -\frac{\mathcal{C}(\Sigma)}{2}
\]

where

\[
\mathcal{C}(\Sigma) := \max_{w \in \mathcal{D}^{1,2}(\Sigma)} \left( \langle \mathcal{D}^{1,2}(\mathbb{R}^N) \rangle \cdot \langle \mathcal{F}, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \int_{\Omega} |\nabla w(x)|^2 \, dx \right)
\]

represents the compliance functional associated to the force \( \mathcal{F} \) concentrated on the section \( \Sigma \) in the flavor of [10, 11]. In general, the compliance functional measures the rigidity of a membrane subject to a given (vertical) force: the maximal rigidity is obtained by minimizing the compliance functional \( \mathcal{C}(\Sigma) \) in the class of admissible regions \( \Sigma \). With this notation and concepts in mind we state our main results.

Let us first assume that there exists \( k_0 \geq 1 \) such that

\[
\lambda_{k_0}(D^+) \text{ is simple and the corresponding eigenfunctions have in \( e_1 \) a zero of order 1,}
\]

(11)\[
\lambda_{k_0}(D^+) \notin \sigma_p(D^-).
\]

We can then fix an eigenfunction \( u_0 \in \mathcal{D}^{1,2}(D^+) \setminus \{0\} \) associated to \( \lambda_{k_0}(D^+) \), i.e. solving

\[
\begin{cases}
-\Delta u_0 = \lambda_{k_0}(D^+) p u_0, & \text{in } D^+, \\
u_0 = 0, & \text{on } \partial D^+,
\end{cases}
\]

(13) such that

\[
\frac{\partial u_0}{\partial x_1}(e_1) > 0 \quad \text{and} \quad \int_{D^+} p(x) u_0^2(x) \, dx = 1.
\]

From [15] Example 8.2, Corollary 4.7, Remark 4.3] (see also [18] Lemma 1.1]), it follows that, letting

\[
\lambda_\varepsilon = \lambda_k(\Omega^\varepsilon)
\]

where \( k = k_0 + \text{card } \{ j \in \mathbb{N} \setminus \{0\} : \lambda_j(D^-) \leq \lambda_{k_0}(D^+) \} \), so that \( \lambda_{k_0}(D^+) = \lambda_k(D^- \cup D^+) \), there holds

\[
\lambda_\varepsilon \to \lambda_{k_0}(D^+) \quad \text{as } \varepsilon \to 0^+.
\]

We will denote

\[
\lambda_0 = \lambda_{k_0}(D^+).
\]

Furthermore, for every \( \varepsilon \) sufficiently small, \( \lambda_\varepsilon \) is simple and there exists an eigenfunction \( u_\varepsilon \) associated to \( \lambda_\varepsilon \), i.e. satisfying

\[
\begin{cases}
-\Delta u_\varepsilon = \lambda_{\varepsilon p} u_\varepsilon, & \text{in } \Omega^\varepsilon, \\
u_\varepsilon = 0, & \text{on } \partial \Omega^\varepsilon,
\end{cases}
\]

(16) such that

\[
\int_{\Omega^\varepsilon} p(x) u_\varepsilon^2(x) \, dx = 1 \quad \text{and} \quad u_\varepsilon \to u_0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ as } \varepsilon \to 0^+,
\]

where in the above formula we mean the functions \( u_\varepsilon, u_0 \) to be trivially extended to the whole \( \mathbb{R}^N \). We refer to [13, §5.2] for uniform convergence of eigenfunctions.
Then the limit eigenfunctions are different from each other, the double eigenvalue
we prove that by attaching the shrinking handle at two points where the normal derivatives of
read in theorems 1.1 and 1.2, since the limits in (18) and (19) attain the
maximum (positive)
of the thin handle if compared to the case of more indented sections. This phenomenon can be
\( \tilde{\epsilon} \rightarrow 0 \). In other words, this means that among all the admissible sections \( \Sigma \), the disk attains the
section of the tube is the one which makes as slow as possible the convergence of the eigenvalues
spherical one, as we will show in proposition 3.2 by Steiner rearrangement. Hence, the disk-shaped
section, the eigenfunctions located in the right domain
\( \lambda_{\epsilon} \) as a simple eigenvalue on \( \overline{D} \), a simple eigenvalue on \( \overline{D} \),
\( \lambda_{\epsilon} = \lambda_{k}(\Omega^{\epsilon}) \) be the k-th eigenvalue of problem (16) on the domain \( \Omega^{\epsilon} \) defined in (3).

Then
\[
\lim_{\epsilon \to 0^+} \frac{\lambda_{\epsilon} - \lambda_c}{\epsilon^N} = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 C(\Sigma),
\]
with \( u_0 \) as in (13) and (14), and \( C(\Sigma) \) as in (10).

Theorem 1.2. Under assumptions (2), (3), (4), (11), and (12), let \( u_\epsilon \) and \( u_0 \) as in (13), (14), (15), (16), (17). Then
\[
\lim_{\epsilon \to 0^+} \epsilon^{-N} \| u_\epsilon - u_0 \|_{2; \Omega^{\epsilon}}^2 = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 C(\Sigma),
\]
where \( u_\epsilon \) and \( u_0 \) are trivially extended to the whole \( \mathbb{R}^N \) and \( C(\Sigma) \) is defined in (10).

We observe that, once the measure of the section \( \Sigma \) is fixed, the shape minimizing \( m(\Sigma) \) and hence maximizing both the limits \( \lim_{\epsilon \to 0^+} \epsilon^{-N} (\lambda_{\epsilon} - \lambda_c) \) and \( \lim_{\epsilon \to 0^+} \epsilon^{-N} \| u_\epsilon - u_0 \|_{2; \Omega^{\epsilon}}^2 \) is the spherical one, as we will show in proposition 3.2 by Steiner rearrangement. Hence, the disk-shaped section of the tube is the one which makes as slow as possible the convergence of the eigenvalues on the perturbed domain to the eigenvalues on the limit domain, as the handle thickness shrinks to zero. In other words, this means that among all the admissible sections \( \Sigma \), the disk attains the
minimum of the rigidity of the domain \( \overline{D} \) from the opposite point of view, in the case of a round section, the eigenfunctions located in the right domain \( D^+ \) are the most sensitive to the attachment of the thin handle if compared to the case of more indented sections. This phenomenon can be read in theorems 1.1 and 1.2 since the limits in (18) and (19) attain their maximal (positive) constant at a disk-shaped section: symmetrization of the section makes the difference \( \lambda_{\epsilon} - \lambda_c \) and \( \| u_\epsilon - u_0 \|_{2; \Omega^{\epsilon}}^2 \) drift away from being \( o(\epsilon^N) \).

The proof of theorem 1.1 which is presented in section 5 is based on the Courant-Fisher minimax characterization of eigenvalues: the estimates from above and below of the Rayleigh quotient used to prove the theorem are based on the analysis of proper test functions introduced in section 2. Theorem 1.2 is proved in section 3 using some blow-up analysis developed in section 2 and the invertibility of an operator associated to the eigenvalue problem on \( D^+ \) (see (10)).

In section 5 we drop assumption (12) and assume that \( \lambda_{k}(D^{-} \cup D^{+}) \in \sigma_{p}(D^{-}) \cap \sigma_{p}(D^{+}) \) is a simple eigenvalue on \( D^{-} \), a simple eigenvalue on \( D^{+} \), and a double eigenvalue on \( D^{-} \cup D^{+} \).
We prove that by attaching the shrinking handle at two points where the normal derivatives of the limit eigenfunctions are different from each other, the double eigenvalue \( \lambda_{k}(D^{-} \cup D^{+}) \) on the limit domain is approximated by two different branches of eigenvalues on the perturbed domain as \( \epsilon \to 0^+ \).

Theorem 1.3. Let us assume that (2), (3), (4), (12) hold and \( p(x) = 0 \) for all \( x \in B_3^\alpha \cup C_\epsilon \cup B_3^\alpha \). Let \( \lambda_{k}(D^{-} \cup D^{+}) = \lambda_{k+1}(D^{-} \cup D^{+}) \in \sigma_{p}(D^{-}) \cap \sigma_{p}(D^{+}) \) be a simple eigenvalue on \( D^{-} \) with corresponding eigenfunctions having in 0 a zero of order 1, a simple eigenvalue on \( D^{+} \) with corresponding eigenfunctions having in \( e_1 \) a zero of order 1, and a double eigenvalue on \( D^{-} \cup D^{+} \). Let \( u_0^+ \in D^{1,2}(D^{+}) \) and \( u_0^- \in D^{1,2}(D^{-}) \) be the eigenfunctions associated to \( \lambda_{k}(D^{-} \cup D^{+}) = \lambda_{k+1}(D^{-} \cup D^{+}) \) on \( D^{+} \) and \( D^{-} \) respectively satisfying
\[
\int_{D^{+}} p(x)|u_0^+(x)|^2 \, dx = \int_{D^{-}} p(x)|u_0^-(x)|^2 \, dx = 1.
\]
If
\[
\frac{\partial u_0^+}{\partial x_1} \left( e_1 \right) > \left| \frac{\partial u_0^-}{\partial x_1} \left( 0 \right) \right|,
\]
Moreover, for \( \varepsilon \to 0^+ \)

\[
\lim_{\varepsilon \to 0^+} \frac{\lambda_k(D^- \cup D^+)}{\varepsilon^N} - \lambda_k(\Omega^\varepsilon) = \mathcal{C}(\Sigma) \left( \frac{\partial u_0^+}{\partial x_1}(e_1) \right)^2,
\]

and

\[
\lim_{\varepsilon \to 0^+} \frac{\lambda_{k+1}(D^- \cup D^+)}{\varepsilon^N} - \lambda_{k+1}(\Omega^\varepsilon) = \mathcal{C}(\Sigma) \left( \frac{\partial u_0^-}{\partial x_1}(e_1) \right)^2,
\]

where \( \Omega^\varepsilon \) is defined in (5) and \( \mathcal{C}(\Sigma) \) is defined in (10).

In section [5] we also prove that, in the resonant case, under condition (20), each approximating eigenfunction is localized as \( \varepsilon \to 0^+ \) on the corresponding component of the limit domain, i.e. an asymmetrical limit configuration prevents dumbbell eigenfunctions from spreading their mass over both components and forces them to concentrate in one of the two regions.

**Theorem 1.4.** Under the same assumptions and with the same notations of Theorem [13] there exist two continuously parametrized families \( v_k^\varepsilon, v_{k+1}^\varepsilon \in D^{1,2}(\Omega^\varepsilon) \) of eigenfunctions on \( \Omega^\varepsilon \), i.e. solutions to

\[
\begin{cases}
-\Delta v^\varepsilon = \lambda \rho v^\varepsilon, & \text{in } \Omega^\varepsilon, \\
v^\varepsilon = 0, & \text{on } \partial \Omega^\varepsilon,
\end{cases}
\]

for \( \lambda = \lambda_k(\Omega^\varepsilon) \) and \( \lambda = \lambda_{k+1}(\Omega^\varepsilon) \) respectively, such that

\[
\varepsilon k \bar{\varepsilon} \to u_0^+ \text{ and } v_{k+1}^\varepsilon \to u_0^- \text{ in } D^{1,2}(\mathbb{R}^N) \text{ as } \varepsilon \to 0^+.
\]

Moreover, for \( v^\varepsilon = v_k^\varepsilon \) and \( v^\varepsilon = v_{k+1}^\varepsilon \) there holds

\[
\int_{D^\varepsilon \cup ([0,1/8] \times (\varepsilon \Sigma))} |\nabla v^\varepsilon|^2 dx = O(\varepsilon^{-(N+1)} e^{-\frac{\sqrt{\lambda_k(\Omega^\varepsilon)}}{\varepsilon}}), \quad \text{as } \varepsilon \to 0^+.
\]

where \( D^* = D^- \) and \( D^* = D^+ \) respectively, and \( \lambda_1(\Sigma) \) denotes the first eigenvalue of the Laplace operator on \( \Sigma \) under null Dirichlet boundary conditions.

For the two families of eigenfunctions \( v_k^\varepsilon, v_{k+1}^\varepsilon \) we provide a sharp asymptotics, extending the result of Theorem [12] in the resonant asymmetrical case.

**Theorem 1.5.** Under the same assumptions and with the same notations of Theorem [13] let \( v_k^\varepsilon, v_{k+1}^\varepsilon \in D^{1,2}(\Omega^\varepsilon) \) be as in Theorem [14]. Then

\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-N} \|v_k^\varepsilon - u_0^+\|_{D^{1,2}(\mathbb{R}^N)}^2 = \left( \frac{\partial u_0^+}{\partial x_1}(e_1) \right)^2 \mathcal{C}(\Sigma),
\]

\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-N} \|v_{k+1}^\varepsilon - u_0^-\|_{D^{1,2}(\mathbb{R}^N)}^2 = \left( \frac{\partial u_0^-}{\partial x_1}(e_1) \right)^2 \mathcal{C}(\Sigma),
\]

where \( v_k^\varepsilon, v_{k+1}^\varepsilon, u_0^+, \) and \( u_0^- \) are trivially extended to the whole \( \mathbb{R}^N \) and \( \mathcal{C}(\Sigma) \) is defined in (10).

1.3. Motivations and references to the literature. The continuity of eigenvalues and eigenfunctions of the Laplace operator under Dirichlet boundary conditions in varying domains including the dumbbell case has been studied in [9] [15]. We also refer to [5] for a first result about spectral continuity for less general domain’s perturbations and to [19] (and references therein) for a detailed survey.

As far as estimates of the rate of convergence are concerned, we mention [20], where, among other results, the authors prove that, in the case of a Helmholtz resonator with a cavity, the effect of adding a tubular region with a section of radius of order \( \varepsilon \) is to shift the eigenvalues by a small amount of order at most \( \varepsilon^{1/2} \). This generalizes a previous result of [4] where an \( \varepsilon^{1/2} \)-rate of convergence for resonances of a Helmholtz resonator was obtained in dimension 3. We stress that the case treated in present paper does not allow continuous spectrum for the Dirichlet Laplacian. As far as we know, no sharp estimates similar to ours can be found in the literature. Similar to our settings, we mention [24] which contains an \( \varepsilon^0 \)-bound from above for the a rate of convergence,
but not the exact asymptotics. Some other estimates on the rate of convergence of Dirichlet eigenvalues for different domain’s perturbations can be found in [16] [25].

We note that there exists an extensive literature dealing with Neumann boundary conditions, but, in the case of dumbbell domains with thin handles, the eigenvalues of the Laplacian may not be continuous, as observed in [3, 14, 21] (see also [24]).

Spectral analysis in thin branching domains arises naturally in the study of models of propagation of waves in quasi one-dimensional systems: in this framework we meet the theory of quantum graphs which provide simplified models of quantum wires, photonic crystals, carbon nano-structures, thin waveguides and many other problems, see e.g. [6, 23] for details. Similarly quantum graphs which provide simplified models of quantum wires, photonic crystals, carbon

2. Preliminaries, notation and technical lemmas

The proof of Theorem 1.1 is based on the Courant-Fisher minimax characterization of eigenvalues and some estimates from above and below on the associated Rayleigh quotient computed at suitable test functions. In this section we introduce the proper test functions on which the Rayleigh quotient will be estimated to prove upper/lower bounds, and prove some properties (i.e. point-wise estimates, blow-up analysis) of such test functions and of eigenfunctions on the domain $\Omega^\varepsilon$.

2.1. Transition functions. We start by introducing some functions describing the domain’s change of geometry at the junction, which will be used for the construction of super-solutions needed for deriving point-wise estimates on eigenfunctions and for estimating the Rayleigh quotient associated to the eigenvalue problem. More precisely, we consider

- the unique function $\Phi$ which is harmonic in the domain $\tilde{D}$, has finite energy in $T_1^-$, and behaves as $(x_1 - 1)^+$ as $|x - e_1| \to +\infty$ in $D^+$ (here $s^+ = \max\{s, 0\}$ denotes the positive part of $s$ for all $s \in \mathbb{R}$);
- for every $R > 2$, the function $z_R$ defined as the harmonic extension of $\Phi|_{\Gamma_R^+}$ in the domain $B_R^+$ vanishing on $\partial B_R^+ \cap \partial D^+$;
- for every $R > 2$, the function $v_R$ defined as the harmonic extension of $(x_1 - 1)^+|_{\Gamma_R}$ in the domain $T_1^- \cup B_R^+$ vanishing on $\partial(T_1^- \cup D^+)$.

For all $R > 1$, we denote as $\mathcal{H}_R$ the completion of $C_c^\infty \left(\left((-\infty, 1) \times \mathbb{R}^{N-1}\right) \cup B_R^+\right)$ with respect to the norm $\left(\int_{\left((-\infty, 1) \times \mathbb{R}^{N-1}\right) \cup B_R^+} |\nabla v|^2 dx\right)^{1/2}$, i.e. $\mathcal{H}_R$ is the space of functions with finite energy in $\left((-\infty, 1) \times \mathbb{R}^{N-1}\right) \cup B_R^+$ vanishing on $\{(1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : |x'| \geq R\}$.

In the sequel, we also denote as $\lambda_1(\Sigma)$ the first eigenvalue of the Laplace operator on $\Sigma$ under null Dirichlet boundary conditions, and as $\psi_1^\Sigma(x')$ the corresponding positive $L^2(\Sigma)$-normalized eigenfunction, so that

$$\begin{cases} -\Delta_{x'} \psi_1^\Sigma(x') = \lambda_1(\Sigma) \psi_1^\Sigma(x'), & \text{in } \Sigma, \\ \psi_1^\Sigma = 0, & \text{on } \partial \Sigma, \end{cases}$$

being $\Delta_{x'} = \sum_{j=2}^N \frac{\partial^2}{\partial x_j^2}$, $x' = (x_2, \ldots, x_N)$. 

ATTACHING A THIN HANDLE ON THE SPECTRAL RATE OF CONVERGENCE 7
2.1.1. The function $\Phi$. In [18, Lemma 2.4], it is proved that there exists a unique function $\Phi$ satisfying

\[
\begin{aligned}
&\int_{\Gamma_{+} \cup \partial B_{\tilde{R}}} \left( |\nabla \Phi(x)|^2 + |\Phi(x)|^2 \right) \, dx < +\infty \text{ for all } R > 1, \\
&-\Delta \Phi = 0 \text{ in a distributional sense in } \tilde{D}, \quad \Phi = 0 \text{ on } \partial \tilde{D}, \\
&\int_{\tilde{D}^+} |\nabla (\Phi - (x_1 - 1))| \, dx < +\infty.
\end{aligned}
\]

Furthermore $\Phi > (x_1 - 1)^+$ in $\tilde{D}$ (by the Strong Maximum Principle) and, by [18, Lemma 2.9], there holds

\[
\begin{aligned}
\Phi(x) &= (x_1 - 1)^+ + O(|x - e_1|^{1-N}) \quad \text{in } D^+ \text{ as } |x - e_1| \to +\infty, \\
\Phi(x) &= O(\sqrt{\lambda_1(\Sigma)}) \quad \text{as } x_1 \to -\infty \text{ uniformly with respect to } x' \in \Sigma.
\end{aligned}
\]

Let

\[
\Psi : \mathbb{S}^{N-1} \to \mathbb{R}, \quad \Psi(\theta_1, \theta_2, \ldots, \theta_N) = \frac{\theta_1}{\sqrt{N}},
\]

being $\mathbb{S}^{N-1} = \{ (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^N : \sum_{i=1}^N \theta_i^2 = 1 \}$ the unit $(N-1)$-dimensional sphere and

\[
\tilde{\Upsilon}_N = \sqrt{\frac{2}{N}} \int_{\mathbb{S}^{N-1}} \theta_1 d\sigma(\theta) = \sqrt{\frac{\omega_{N-1}}{2N}},
\]

where $\omega_{N-1}$ denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1} = \int_{\mathbb{S}^{N-2}} d\sigma(\theta)$. Here and in the sequel, the notation $d\sigma$ is used to denote the volume element on $(N-1)$-dimensional surfaces.

We notice that, letting

\[
\mathbb{S}_{+}^{N-1} := \{ \theta = (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{S}^{N-1} : \theta_1 > 0 \},
\]

\[
\Psi^+ = \Psi|_{\mathbb{S}_{+}^{N-1}} = \frac{\theta_1}{\tilde{\Upsilon}_N}
\]

is the first positive $L^2(\mathbb{S}_{+}^{N-1})$-normalized eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}_{+}^{N-1}$ under null Dirichlet boundary conditions satisfying

\[
-\Delta_{\mathbb{S}_{+}^{N-1}} \Psi^+ = (N - 1) \Psi^+ \quad \text{on } \mathbb{S}_{+}^{N-1}.
\]

**Lemma 2.1.** Let $\Phi, \Psi^+=\frac{\phi}{\sqrt{N}},$ and $\tilde{\Upsilon}_N$ be as in (26), (30), and (29) respectively, and, for every $r > 1$, let us define

\[
\varphi(r) = \int_{\mathbb{S}_{+}^{N-1}} \Phi(e_1 + r\theta)\Psi^+(\theta) \, d\sigma(\theta).
\]

Then

\[
\begin{aligned}
\varphi(r) &= \varphi(1) r^{1-N} + \tilde{\Upsilon}_N (r - r^{1-N}), \quad \text{for every } r > 1; \\
\int_{\Gamma_{+}^1} \frac{\partial \Phi}{\partial \nu} (x_1 - 1) \, d\sigma &= \mathcal{N} r^N \left( N \mathcal{N} + (1-N) \frac{\varphi(r)}{r} \right), \quad \text{for every } r > 1.
\end{aligned}
\]

**Proof.** Part (i) is proved in [11, Lemma 2.2]. To prove (ii) we observe that

\[
\int_{\Gamma_{+}^1} \frac{\partial \Phi}{\partial \nu} (x_1 - 1) \, d\sigma = \mathcal{N} r^N \varphi'(r)
\]

so that the thesis immediately follows from differentiation of (i) and simple calculations. \qed
Lemma 2.2. Let \( \Phi \) as in (26) and \( J_\Sigma : D^{1,2}(\tilde{D}) \to \mathbb{R} \) as in (8). Then
\[
(i) \quad r^{N-1} \left( \Phi(e_1 + r\theta) - \Phi(e_1) \right) \to \frac{1}{4N} \left( \int_{\Sigma} \left( \Phi(e_1 + \theta) - \Phi(e_1) \right) \, d\sigma \right) \theta_1
\]
in \( C^{1,\alpha}(\Sigma^{N-1}) \) as \( r \to +\infty \), for every \( \alpha \in (0,1) \);
\[
(ii) \quad J_\Sigma(\Phi - (x_1 - 1)^+) = \min_{w \in D^{1,2}(\tilde{D})} J_\Sigma(w);
\]
\[
(iii) \quad \int_D \nabla(\Phi - (x_1 - 1)^+) \cdot \nabla v(x) \, dx = \int_\Sigma v(1, x') \, dx', \quad \text{for every } v \in D^{1,2}(\tilde{D}).
\]

Proof. Let \( w : D^+ \to \mathbb{R} \), \( w(x) = \Phi(x) - (x_1 - 1) \). From (26) we have that \( -\Delta w = 0 \) in \( D^+ \), \( w > 0 \) in \( D^+ \), \( w = 0 \) on \( \{(1, x') : |x'| > 1\} \), and \( \int_{D^+} |\nabla w(x)|^2 \, dx < +\infty \). Then, (i) follows from [17] Theorem 1.5 applied to the function \( w \).

Statements (ii) and (iii) are contained in [18] Lemma 2.4 and [2].

As a consequence of the previous lemma, it is possible to characterize the minimum \( m(\Sigma) \) defined in (7) in terms of the function \( \Phi \).

Corollary 2.3. Let \( J_\Sigma \) as in (8), \( m(\Sigma) \) as in (7), and \( \Phi \) as in (26). Then
\[
m(\Sigma) = -\frac{1}{2} \int_\Sigma \Phi(1, x') \, dx'.
\]

Proof. From Lemma 2.2(ii), we have that
\[
m(\Sigma) = \frac{1}{2} \int_{D^+} |\nabla(\Phi - (x_1 - 1)^+)|^2 \, dx - \int_\Sigma \Phi(1, x') \, dx'
\]
and the conclusion follows taking \( v = \Phi - (x_1 - 1)^+ \) in Lemma 2.2(iii).

2.1.2. The function \( z_R \). For every \( R > 1 \), we denote as \( z_R \) the unique solution to the minimization problem
\[
\int_{B^+_R} |\nabla z_R|^2 \, dx = \min \left\{ \int_{B^+_R} |\nabla v|^2 \, dx : v \in H^1(B^+_R), \ v = 0 \text{ on } \partial D^+, \ v = \Phi \text{ on } \Gamma^+_R \right\},
\]
which then solves
\[
-\Delta z_R = 0, \quad \text{in } B^+_R,
\]
\[
z_R = \Phi, \quad \text{on } \Gamma^+_R,
\]
\[
z_R = 0, \quad \text{on } \partial D^+.
\]

Lemma 2.4. Let \( z_R, \Psi^+ = \frac{\phi_{\Sigma^+}}{\sqrt{N}} \), and \( \Upsilon_N \) be as in (32), (30) and (29) respectively, and, for every \( R > 2 \) and \( r \in (0, R] \), let us define
\[
\phi_R(r) = \int_{\Sigma^+} z_R(e_1 + r\theta) \Psi^+(\theta) \, d\sigma(\theta).
\]

Then
\[
\int_{\Gamma^+} \frac{\partial z_R}{\partial \nu}(x_1 - 1) \, d\sigma = \Upsilon_N r^N \frac{\phi_R(R)}{R}, \quad \text{for every } r \in (0, R].
\]

Proof. We first observe that
\[
\phi'(r) = \frac{1}{\Upsilon_N r^N} \int_{\Gamma^+} \frac{\partial z_R}{\partial \nu}(x_1 - 1) \, d\sigma, \quad \text{for all } r \in (0, R].
\]

Since \( z_R \) is harmonic in \( B^+_R \), there exists \( C_R \in \mathbb{R} \) such that \( \left( \frac{\phi_R(r)}{r} \right)' = \frac{C_R}{r^{N+1}} \) in \( (0, R] \), so that
\[
\phi_R(r) = r \frac{\phi_R(R)}{R} + \frac{C_R}{N} r R^{-N} = \frac{C_R}{N} r^{1-N}, \quad \text{for every } r \in (0, R].
\]
Lemma 2.5. For every \( v \in \mathbb{R} \) and then \( \phi_R(r) = r^{\phi_R(R)} \). Hence
\[
\phi_R(r) = \frac{\phi_R(R)}{R}, \quad \text{for every } r \in (0, R].
\]
The thesis follows from (33) and (35).

2.1.3. The function \( v_R \). For every \( R > 1 \), we denote as \( v_R \) the unique solution to the minimization problem
\[
\int_{T_1^- \cup B_R^+} |\nabla v_R|^2 \, dx = \min \left\{ \int_{T_1^- \cup B_R^+} |\nabla v|^2 \, dx : v \in \mathcal{H}_{0,R} \text{ and } (x_1 - 1) \right\},
\]
where \( \mathcal{H}_{0,R} \) is the completion of \( \{ v \in C^\infty_0(T_1^- \cup B_R^+ ; \phi = 0 \text{ on } \partial(T_1^- \cup D^+)) \} \) with respect to the norm \( (\int_{T_1^- \cup B_R^+} |\nabla v|^2 \, dx)^{1/2} \). This function then solves
\[
\begin{cases}
-\Delta v_R = 0, & \text{in } T_1^- \cup B_R^+; \\
v_R = (x_1 - 1), & \text{on } \Gamma_R^+; \\
v_R = 0, & \text{on } \partial(T_1^- \cup D^+). 
\end{cases}
\]

Lemma 2.5. For every \( R > 2 \), let \( v_R, \Psi^+ = \frac{\psi}{\sqrt{N}}, \text{ and } \Upsilon_N \) be as in (30), (30), and (29) respectively, and, for every \( r \in (1, R] \), let us define
\[
\chi_R(r) = \int_{S_0^N} v_R(v \theta + r \theta) \Psi^+ (\theta) \, d\sigma(\theta).
\]

Then
\[
(i) \quad v_R \to \Phi^+ \text{ as } R \to +\infty \text{ in } \mathcal{H}_t \text{ for all } t > 2;
\]
\[
(ii) \quad \int_{T_1^+} \frac{\partial v_R}{\partial \nu}(x_1 - 1) \, d\sigma = \Upsilon_N \left( \frac{\Upsilon_N(R^N + N - 1)}{1 - R^{-N}} - N \chi_R(1) \right).
\]

Proof. To prove (i), for any \( t < R \), we estimate
\[
\int_{T_1^- \cup B_R^+} |\nabla (v_R - \Phi)|^2 \, dx \leq \int_{T_1^- \cup B_R^+} |\nabla (v_R - \Phi)|^2 \, dx \\
\quad \leq \int_{T_1^- \cup B_R^+} |\nabla (\eta_R(x_1 - 1 - \Phi))|^2 \, dx
\]
via the Dirichlet Principle since \( v_R - \Phi \) is harmonic in \( T_1^- \cup B_R^+ \) and \( (v_R - \Phi)|_{T_1^+} = (x_1 - 1 - \Phi)|_{T_1^+} \), being \( \eta_R \) a smooth cut-off function such that
\[
\eta_R \equiv 0 \text{ in } T_1^- \cup B_{R/2}^+, \quad \eta \equiv 1 \text{ in } D^+ \setminus B_R^+, \quad 0 \leq \eta_R \leq 1, \quad |\nabla \eta_R| \leq \frac{4}{R} \text{ in } B_R^+ \setminus B_{R/2}^+.
\]
In view of (26) and (27)
\[
\int_{T_1^- \cup B_R^+} |\nabla (\eta_R(x_1 - 1 - \Phi))^2 \, dx \\
\quad \leq 2 \int_{B_R^+ \setminus B_{R/2}^+} |\nabla \eta_R|^2 (x_1 - 1 - \Phi)^2 \, dx + 2 \int_{D^+ \setminus B_{R/2}^+} \eta_R^2 |\nabla (x_1 - 1 - \Phi)|^2 \, dx \\
\quad \leq \text{const } R^{-2} R^{2-2N} R^N + o(1) = o(1)
\]
as \( R \to +\infty \), thus proving (i).

To prove (ii), we observe that
\[
\int_{T_1^+} \frac{\partial v_R}{\partial \nu}(x_1 - 1) \, d\sigma = R^N \Upsilon_N \chi_R(r) \quad \text{for every } r \in (1, R].
\]
From (39), \(\chi_R(r)\) solves the equation \(\left(r^{N+1}\left(\frac{x}{r}\right)\right)' = 0\) in the interval \((1, R)\), hence by integration we obtain that there exists \(C_R \in \mathbb{R}\) such that

\[
\frac{\chi_R(r)}{r} - \chi_R(1) = \frac{C_R}{N}(1 - r^{-N}), \quad \text{for every } r \in (1, R).
\]

Replacing \(r = R\) in the above identity and observing that the boundary condition in (36) implies that

\[
\frac{\chi_R(R)}{R} = \Upsilon_N,
\]

we obtain that

\[
\frac{C_R}{N} = \frac{1}{1 - R^{-N}} \left(\Upsilon_N - \chi_R(1)\right).
\]

Hence

\[
\chi_R(r) = \frac{\Upsilon_N - \chi_R(1)R^{-N}}{1 - R^{-N}} - \frac{\Upsilon_N - \chi_R(1)}{1 - R^{-N}} r^{1-N}, \quad r \in (1, R],
\]

and then

\[
\chi'_R(r) = \frac{\Upsilon_N - \chi_R(1)R^{-N}}{1 - R^{-N}} + \frac{(N - 1)(\Upsilon_N - \chi_R(1))}{1 - R^{-N}} r^{-N}, \quad r \in (1, R].
\]

The conclusion follows by plugging \(r = R\) in (30) and (40). \(\square\)

2.2. Point-wise and energy control for eigenfunctions on the varying domain. In order to prove Theorem 1.1, quite precise decaying estimates of eigenfunctions on the varying domain are needed. In this subsection we pursue this analysis.

Lemma 2.6. Let \(j \in \mathbb{N}\) and, for all \(\varepsilon \in (0, 1)\), let \(v^\varepsilon \in \mathcal{D}^{1,2}(\Omega^\varepsilon)\) solve

\[
\begin{cases}
-\Delta v^\varepsilon = \lambda_j(\Omega^\varepsilon)pv^\varepsilon, & \text{in } \Omega^\varepsilon, \\
v^\varepsilon = 0, & \text{on } \partial \Omega^\varepsilon, \\
\int_{\Omega^\varepsilon} pv^\varepsilon^2 \, dx = 1.
\end{cases}
\]

(i) For every sequence \(\varepsilon_n \to 0\) there exist a subsequence \(\varepsilon_{n_k}\) and \(v_0 \in \mathcal{D}^{1,2}(D^- \cup D^+)\) solving

\[
\begin{cases}
-\Delta v_0 = \lambda_j(D^- \cup D^+)pv_0, & \text{in } D^+ \cup D^-, \\
v_0 = 0, & \text{on } \partial(D^+ \cup D^-), \\
\int_{D^+ \cup D^-} pv_0^2 \, dx = 1,
\end{cases}
\]

such that \(v_{\varepsilon_{n_k}} \to v_0\) in \(\mathcal{D}^{1,2}(\mathbb{R}^N)\).

(ii) There exists \(\varepsilon_0 \in (0, 1)\) and \(C_1, C_2 > 0\) such that, for all \(\varepsilon \in (0, \varepsilon_0)\) and \(R, R_1, R_2 > 1\) with \(R_1 > R_2\), there holds

\[
|v_\varepsilon(x)| \leq C_1, \quad \text{for all } x \in \Omega^\varepsilon,
\]

\[
\lim_{\varepsilon \to 0^+} \sup_{B_{R_2} \setminus B_{R_1}} |v_\varepsilon| = 0,
\]

\[
\sup_{(B_{R_1} \setminus B_{R_2}) \cup (B_{R_1} \setminus B_{R_2})} |
\nabla v_\varepsilon| = O(1/\varepsilon), \quad \text{as } \varepsilon \to 0^+,
\]

\[
|v_\varepsilon(x)| \leq C_2 \left(\sup_{\partial C_\varepsilon} |v_\varepsilon|\right) \left(e^{-\frac{1}{x_1}} - e^{-\frac{1}{x_1} \frac{1}{x_1}} \frac{1}{x_1}ight), \quad \text{for all } x \in C_\varepsilon.
\]

Proof. From the spectral continuity analyzed in [15], \(\lambda_j(\Omega^\varepsilon) \to \lambda_j(D^+ \cup D^-)\); hence the proof of (i) follows easily from classical compactness argument in view of the compactness of the map \(\mathcal{D}^{1,2}(\mathbb{R}^N) \to (\mathcal{D}^{1,2}(\mathbb{R}^N))^*\), \(u \mapsto pu\). Estimate (42) follows by an iterative Brezis-Kato type argument (see e.g. [18] Lemma 2.2).

From [9] Lemma 5.2 it follows that solutions to (41) converging in \(\mathcal{D}^{1,2}(\mathbb{R}^N)\) actually converge in \(L^\infty_{loc}(\mathbb{R}^N)\); then [9] and part (i) imply that for every sequence \(\varepsilon_n \to 0^+\) there exist a subsequence \(\varepsilon_{n_k}\) such that \(v_{\varepsilon_{n_k}} \to v_0\) in \(L^\infty_{loc}(\mathbb{R}^N)\) and hence, for every \(R > 1\),

\[
\lim_{k \to +\infty} \sup_{B_{R_{n_k}} \cup B_{R_{n_k}} \cup C_{n_k}} |v_{\varepsilon_{n_k}}| = 0.
\]
Since the above limit depends neither on the sequence nor on the subsequence, we deduce the limit as $\varepsilon \to 0^+$ thus proving (13).

Estimate (12), together with classical elliptic estimates for $x \mapsto v_\varepsilon(\varepsilon x)$ over an annulus $B_{R_1}^- \setminus B_{R_2}^-$ and $x \mapsto v_\varepsilon(\varepsilon(x - \varepsilon_1))$ over $B_{R_1\varepsilon}^+ \setminus B_{R_2\varepsilon}^+$, yield (14).

To prove estimate (15), let us consider the function

$$
\Psi_\varepsilon(x_1, x') = C_\varepsilon \left( e^{-\frac{\sqrt{2\lambda(1+\varepsilon)}}{\varepsilon} x_1} + e^{-\frac{\sqrt{2\lambda(1+\varepsilon)}}{\varepsilon} (1-x_1)} \right) \psi_\varepsilon^2 \left( \frac{x'}{2\varepsilon} \right)
$$

where

$$
C_\varepsilon = \left( \min_{|y'|\leq 1/2} \psi_\varepsilon(y') \right)^{-1} \sup_{\partial C_\varepsilon} |v_\varepsilon|.
$$

We note that $C_\varepsilon = o(1)$ as $\varepsilon \to 0^+$ in view of estimate (13). For all $x' \in \varepsilon \Sigma$,

$$
\Psi_\varepsilon(0, x') \geq C_\varepsilon \psi_\varepsilon^2 \left( \frac{x'}{2\varepsilon} \right) \geq \sup_{\partial C_\varepsilon} |v_\varepsilon|, \quad \Psi_\varepsilon(1, x') \geq C_\varepsilon \psi_\varepsilon^2 \left( \frac{x'}{2\varepsilon} \right) \geq \sup_{\partial C_\varepsilon} |v_\varepsilon|,
$$

so that $\Psi_\varepsilon \geq |v_\varepsilon|$ on $\partial C_\varepsilon$. Moreover

$$
-\Delta \Psi_\varepsilon = \frac{3 \lambda_1(\Sigma)}{16 \varepsilon^2} \Psi_\varepsilon, \quad \text{in } C_\varepsilon,
$$

whereas, via Kato’s inequality (22),

$$
-\Delta |v_\varepsilon| \leq \lambda_j(\Omega')|p|v_\varepsilon|, \quad \text{in } C_\varepsilon,
$$

so that there exists a constant $c > 0$ independent of $\varepsilon$ such that, for $\varepsilon$ sufficiently small, $\Psi_\varepsilon - |v_\varepsilon|$ weakly solves

$$
-\Delta (\Psi_\varepsilon - |v_\varepsilon|) - c(\Psi_\varepsilon - |v_\varepsilon|) \geq 0, \quad \text{in } C_\varepsilon.
$$

The boundary conditions (18) imply that $(\Psi_\varepsilon - |v_\varepsilon|)^- = \max\{0, -\Psi_\varepsilon - |v_\varepsilon|\} \in H_0^1(C_\varepsilon)$; hence testing (19) with $-(\Psi_\varepsilon - |v_\varepsilon|)^-$ and using Hölder and Sobolev inequalities, we obtain that

$$
\int_{C_\varepsilon} \left| \nabla (\Psi_\varepsilon - |v_\varepsilon|)^- \right|^2 dx \leq c \int_{C_\varepsilon} \left| (\Psi_\varepsilon - |v_\varepsilon|)^- \right|^2 dx
$$

$$
\leq c \left( \int_{C_\varepsilon} \left| (\Psi_\varepsilon - |v_\varepsilon|)^- \right|^{2^*/2} dx \right)^{2^*/2} \left( \varepsilon^{N-1}|\Sigma| \right)^{2/N}
$$

$$
\leq c |\Sigma|^{2/N} \varepsilon^{2(N-1)/N} S^{-1} \int_{C_\varepsilon} \left| \nabla (\Psi_\varepsilon - |v_\varepsilon|)^- \right|^2 dx
$$

where $S$ denotes the best constant in the Sobolev inequality $S \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2$ and $|\Sigma|$ is the Lebesgue $(N - 1)$-dimensional measure of $\Sigma$. If $(\Psi_\varepsilon - |v_\varepsilon|)^- \neq 0$, the above estimate would imply that $1 \leq c |\Sigma|^{2/N} \varepsilon^{2(N-1)/N} S^{-1}$ for $\varepsilon$ small, thus giving rise to a contradiction. Then $(\Psi_\varepsilon - |v_\varepsilon|)^- \equiv 0$ and $|v_\varepsilon| \leq \Psi_\varepsilon$ in $C_\varepsilon$, which implies estimate (15).

\textbf{Corollary 2.7.} Let $v_\varepsilon$ be as in Lemma 2.6. Then for every $\delta \in (0, 1/2)$ and $R > 1$

$$
|v_\varepsilon(x_1, x')| = O(e^{-\frac{\sqrt{2\lambda(1+\varepsilon)}}{\varepsilon} x_1}), \quad \text{as } \varepsilon \to 0^+ \text{ uniformly in } (\delta, 1-\delta) \times (\varepsilon \Sigma),
$$

$$
\int_{B_{R\varepsilon}^+} pv_\varepsilon^2 dx = o(\varepsilon^N), \quad \text{as } \varepsilon \to 0^+
$$

$$
\int_{C_\varepsilon} pv_\varepsilon^2 dx = o(\varepsilon^N), \quad \text{as } \varepsilon \to 0^+.
$$
We recall the following result stated in [18]. In order to prove (52), we observe that estimates (45) and (43) imply that
\[ f \leq 0 \]
from (43). To prove (52), we invoke estimates (50) of Corollary 2.7 and Lemma 2.8.

2.3. \( \varepsilon \)

In a similar way, for every \( \eta \), let \( \bar{\varepsilon} \)

\[ \int_{\Omega} |\nabla u_\varepsilon|^2 dx = O\left(\varepsilon N \right) \] as \( \varepsilon \to 0^+ \).

For \( r \in (0, +\infty) \setminus (1, 1 + \varepsilon) \), we define

\[ \Omega_r^\varepsilon = \begin{cases} D^- \cup \{(x_1, x') \in C_\varepsilon : x_1 < r\}, & \text{if } 0 < r \leq 1, \\ D^- \cup C_\varepsilon \cup B^+_{1-\varepsilon}, & \text{if } r \geq 1 + \varepsilon \end{cases} \]

We recall the following result stated in [18].

Lemma 2.8. Let \( u_\varepsilon \) be as in (13) under the assumptions (14), (15), (11), (12), (13), (14). Then for every \( f \in L^{N/2}(\mathbb{R}^N) \) and \( M > 0 \), there exists \( \varepsilon_M, f > 0 \) such that for all \( r \in (0, 1) \) and \( \varepsilon \in (0, \varepsilon_M, f) \)

\[ \int_{\Omega_r^\varepsilon} |\nabla u_\varepsilon|^2 dx \geq M \int_{\Omega_r^\varepsilon} |f(x)| u_\varepsilon^2(x) dx. \]

Corollary 2.9. Let \( u_\varepsilon \) be as in Lemma 2.8. Then, for every \( \delta \in (0, 1/4) \)

\[ \int_{\Omega_{1-2\delta}^\varepsilon} |\nabla u_\varepsilon|^2 dx = O\left(\varepsilon N \right) \] as \( \varepsilon \to 0^+ \).

Proof. Let \( \eta \) be a smooth cut-off function such that \( \eta \equiv 1 \) in \( \Omega_{1-2\delta}^\varepsilon \), \( \eta \equiv 0 \) in \( \Omega^\varepsilon \setminus \Omega_{1-\delta}^\varepsilon \), \( 0 \leq \eta \leq 1 \), and \( |\nabla \eta| \leq 2\delta \). Let \( w_\varepsilon = \eta u_\varepsilon \). Then, taking into account (14),

\[ \int_{\Omega_{1-2\delta}^\varepsilon} (|\nabla w_\varepsilon|^2 - \lambda \varepsilon w_\varepsilon^2) dx \leq \int_{\Omega_{1-\delta}^\varepsilon} (|\nabla w_\varepsilon|^2 - \lambda w_\varepsilon^2) dx \]

\[ = -\int_{\Omega_{1-\delta}^\varepsilon \setminus \Omega_{1-2\delta}^\varepsilon} \eta (\Delta \eta) w_\varepsilon^2 dx + \frac{1}{2} \int_{\Omega_{1-\delta}^\varepsilon \setminus \Omega_{1-2\delta}^\varepsilon} w_\varepsilon^2 \Delta (\lambda \varepsilon) dx \]

from which the thesis follows invoking estimate (50) of Corollary 2.7 and Lemma 2.8.

2.3. Point-wise estimates and blow-up analysis of the test functions. For every \( \varepsilon \in (0, \varepsilon_0) \) and \( R > 1 \), let \( \bar{u}_\varepsilon,R \) be the unique solution to the minimization problem

\[ \int_{\Omega_{1+R\varepsilon}} |\nabla \bar{u}_\varepsilon,R|^2 dx = \min \left\{ \int_{\Omega_{1+R\varepsilon}} |\nabla v|^2 dx : v \in \mathcal{H}_R^\varepsilon \text{ and } v = u_0 \text{ on } \Gamma_{R\varepsilon}^+ \right\}, \]

where \( \mathcal{H}_R^\varepsilon \) is the completion of \( \left\{ v \in C^\infty_c(\overline{\Omega_{1+R\varepsilon}}) : v = 0 \text{ on } \partial \Omega^\varepsilon \right\} \) with respect to the norm \( (\int_{\Omega_{1+R\varepsilon}} |\nabla v|^2 dx)^{1/2} \); in particular \( \bar{u}_\varepsilon,R \) solves

\[ \begin{cases} -\Delta \bar{u}_\varepsilon,R = 0, & \text{in } \Omega_{1+R\varepsilon}^+, \\ \bar{u}_\varepsilon,R = u_0, & \text{on } \Gamma_{R\varepsilon}^+, \\ \bar{u}_\varepsilon,R = 0, & \text{on } \partial \Omega^\varepsilon. \end{cases} \]

In a similar way, for every \( \varepsilon \in (0, \varepsilon_0) \) and \( R > 1 \), we denote as \( \bar{v}_\varepsilon,R \) the unique solution to the minimization problem

\[ \int_{B^+_{R\varepsilon}} |\nabla \bar{v}_\varepsilon,R|^2 dx = \min \left\{ \int_{B^+_{R\varepsilon}} |\nabla v|^2 dx : v \in H^1(B^+_{R\varepsilon}), v = 0 \text{ on } \partial D^+, \text{ and } v = u_\varepsilon \text{ on } \Gamma_{R\varepsilon}^+ \right\}. \]
In particular \( \hat{u}_{\varepsilon,R} \) solves
\[
\begin{cases}
-\Delta \hat{u}_{\varepsilon,R} = 0, & \text{in } B_{R\varepsilon}^+, \\
\hat{u}_{\varepsilon,R} = u_{\varepsilon}, & \text{on } \Gamma_{R\varepsilon}^+, \\
\hat{u}_{\varepsilon,R} = 0, & \text{on } \partial D^+.
\end{cases}
\]
Hence, for every \( \varepsilon \in (0,\varepsilon_0) \) and \( R > 1 \), we define \( \hat{u}_{\varepsilon,R} \in D^{1,2}(\Omega^c) \) as
\[
\hat{u}_{\varepsilon,R} := \begin{cases}
u_0, & \text{in } D^+ \setminus B_{R\varepsilon}^+, \\
\hat{u}_{\varepsilon,R}, & \text{in } \Omega_{1+R\varepsilon}^c,
\end{cases}
\]
and \( \hat{u}_{\varepsilon,R} \in D^{1,2}(D^+) \) as
\[
\hat{u}_{\varepsilon,R} := \begin{cases}
u_{\varepsilon,R}, & \text{in } D^+ \setminus B_{R\varepsilon}^+, \\
\hat{v}_{\varepsilon,R}, & \text{in } B_{R\varepsilon}^+.
\end{cases}
\]

**Lemma 2.10.** Let \( R > 1 \) and \( \hat{u}_{\varepsilon,R} \) be as in \[(56)\]. Then, for every \( \delta \in (0,1/4) \),
\[
|\hat{u}_{\varepsilon,R}(x)| = O\left(e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{2\varepsilon}}\right)
\]
as \( \varepsilon \to 0^+ \) uniformly in \( \left\{(x_1,x') \in C_{\varepsilon} : x_1 \in (\delta,1-\delta)\right\} \). Moreover
\[
\int_{\Omega_{1+2\varepsilon}} |\nabla \hat{u}_{\varepsilon,R}|^2 dx = O\left(\varepsilon^{N-1}e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{2\varepsilon}}\right)
\]
as \( \varepsilon \to 0^+ \).

**Proof.** From \[(54)\] and the maximum principle, it follows that, for \( \varepsilon \) sufficiently small, \[(60)\] holds for all \( x \in \Omega_{1+R\varepsilon}^c \).

We argue as in the proof of estimate \[(45)\] in Lemma 2.6. Let us consider the function
\[
\tilde{\Psi}(x_1,x') = \left(\min_{|y'|\leq1/2} \psi_1(y')\right)^{-1} \|u_0\|_{L^\infty(\Omega^c)} \left(e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{2\varepsilon}x_1} + e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{2\varepsilon}(1-x_1)}\right) \psi_1 \left(\frac{x'}{2\varepsilon}\right).
\]
From \[(61)\] we obtain that, for all \( x' \in \varepsilon \Sigma \),
\[
\tilde{\Psi}(0,x') \geq \|u_0\|_{L^\infty} \geq \sup_{\partial C_{\varepsilon}} \|\hat{u}_{\varepsilon,R}\| \quad \text{and} \quad \tilde{\Psi}(1,x') \geq \|u_0\|_{L^\infty} \geq \sup_{\partial C_{\varepsilon}} \|\hat{u}_{\varepsilon,R}\|,
\]
so that \( \tilde{\Psi} \geq \hat{u}_{\varepsilon,R} \) on \( \partial C_{\varepsilon} \). Moreover,
\[
-\Delta \tilde{\Psi} = \frac{3\lambda_1(\Sigma)}{16} \tilde{\Psi} \geq 0, \quad \text{in } C_{\varepsilon},
\]
whereas \( \hat{u}_{\varepsilon,R} \) is nonnegative and harmonic in \( C_{\varepsilon} \), so that \( -\Delta(\tilde{\Psi} - \hat{u}_{\varepsilon,R}) \geq 0 \) on \( C_{\varepsilon} \) and, by the Maximum Principle, we deduce that
\[
0 \leq \hat{u}_{\varepsilon,R}(x) \leq \tilde{\Psi}(x), \quad \text{for all } x \in C_{\varepsilon},
\]
from which estimate \[(58)\] follows.

Let \( \eta \) be a smooth cut-off function such that \( \eta \equiv 1 \) in \( \Omega_{1-2\varepsilon} \), \( \eta \equiv 0 \) in \( \Omega^c \setminus \Omega_{1-\delta} \), \( 0 \leq \eta \leq 1 \), and \( |\nabla \eta| \leq 2\varepsilon \). Let \( w_\varepsilon = \eta \hat{u}_{\varepsilon,R} \). Then
\[
\int_{\Omega_{1-2\varepsilon}} |\nabla w_\varepsilon|^2 dx \leq \int_{\Omega_{1-\delta}} |\nabla w_\varepsilon|^2 dx = -\int_{\Omega_{1-\delta}} \eta(D\eta)\hat{u}_{\varepsilon,R}^2 dx + \frac{1}{2} \int_{\Omega_{1-\delta}} \hat{u}_{\varepsilon,R}^2 \Delta(\eta^2) dx
\]
from which \[(59)\] follows invoking \[(58)\]. \( \Box \)

For all \( R > 1 \) and \( \varepsilon \in (0,\varepsilon_0) \), let us define
\[
U_\varepsilon(x) := \frac{u_\varepsilon(e_1 + \varepsilon(x - e_1))}{\varepsilon}, \quad u_{0,\varepsilon}(x) := \frac{u_0(e_1 + \varepsilon(x - e_1))}{\varepsilon},
\]
\[
Z_\varepsilon(x) := \frac{\hat{u}_{\varepsilon,R}(e_1 + \varepsilon(x - e_1))}{\varepsilon}, \quad V_\varepsilon(x) := \frac{\hat{u}_{\varepsilon,R}(e_1 + \varepsilon(x - e_1))}{\varepsilon}.
\]
Lemma 2.11. The following convergences hold as $\varepsilon \to 0^+$:

\begin{align*}
(63) & \quad U_\varepsilon \to \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) \Phi \quad \text{in } \mathcal{H}_R \text{ for any } R > 2, \\
(64) & \quad u_{0,\varepsilon} \to \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) (x_1 - 1) \quad \text{in } C^2_{\text{loc}}(D^+), \\
(65) & \quad Z_\varepsilon^R \to \left( \frac{\partial u}{\partial x_1}(e_1) \right) z_R \quad \text{in } H^1(B_R^+ \setminus \overline{B_{R/2}^-}) \text{ for any } R > 2, \\
(66) & \quad V_\varepsilon^R \to \left( \frac{\partial u}{\partial x_1}(e_1) \right) v_R \quad \text{in } \mathcal{H}_R \text{ for any } R > 2
\end{align*}

with $z_R$ and $v_R$ being as in \([32]\) and \([30]\) respectively.

Proof. The convergence \([63]\) follows from \([18]\) Lemma 4.1 and Corollary 4.4] and \([11]\) Lemmas 2.1 and 2.4]. In order to prove \([64]\), we notice that

$$u_{0,\varepsilon} \to \nabla u_0(e_1) \cdot (x - e_1) = \frac{\partial u_0}{\partial x_1}(e_1)(x_1 - 1), \quad \text{for all } x \in D^+.$$ 

Furthermore, for every $t > 0$,

$$\int_{B_R^+} |\nabla u_{0,\varepsilon}|^2 \, dx = \int_{B_R^+} |\nabla u_0(e_1 + \varepsilon(x - e_1))|^2 \, dx = \varepsilon^{-N} \int_{B_R^+} |\nabla u_0|^2 \, dx \leq \text{const } t^N$$

for some constant $\geq 0$ independent of $\varepsilon$ and $t$. Then, by a diagonal process, one can easily prove that, up to subsequences, $u_{0,\varepsilon}$ weakly converges in $H^1(B_R^+)$ for all $t > 0$. By elliptic regularity theory we conclude that $u_{0,\varepsilon}$ converges to its point-wise limit in $C^2_{\text{loc}}(D^+)$ (since such a limit does not depend on the convergence, the actual convergence actually holds as $\varepsilon \to 0^+$).

In order to prove \([65]\), we notice that $Z_\varepsilon^R = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) z_R$ solves

\begin{align*}
-\Delta(Z_\varepsilon^R - \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) z_R) &= 0, \quad \text{in } B_R^+, \\
Z_\varepsilon^R - \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) z_R &= U_\varepsilon - \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) \Phi, \quad \text{on } \Gamma^R_+,
\end{align*}

and, by the Dirichlet principle and \([63]\),

\begin{align*}
\int_{B_R^+} \left| \nabla(Z_\varepsilon^R - \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) z_R) \right|^2 \, dx &\leq \int_{B_R^+} \left| \nabla \left( \eta(U_\varepsilon - \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) \Phi) \right) \right|^2 \, dx \\
&\leq 2 \left( \int_{B_R^+} |\nabla \eta|^2 \, dx + \int_{B_R^+} \eta^2 \left| \nabla \left( U_\varepsilon - \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) \Phi \right) \right|^2 \, dx \right) \\
&= o(1) \quad \text{as } \varepsilon \to 0^+,
\end{align*}

where $\eta$ is a smooth cut-off function such that $\eta \equiv 1$ in $B_{R/2}^+$, $\eta \equiv 0$ in $D^+ \setminus B_R^+$. Then

$Z_\varepsilon^R \to \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) z_R$

as $\varepsilon \to 0^+$ in $H^1(B_R^+)$ and convergence \([65]\) is proved.

In order to prove \([66]\), we first notice that, in view of \([63]\),

$$\left| V_\varepsilon^R \right|_{\mathcal{H}_R}^2 = \varepsilon^{-N} \int_{\Omega_{1+R}^+} |\nabla \tilde{u}_{\varepsilon,R}|^2 \, dx = \varepsilon^{-N} \int_{\Omega_{1+R}^+} |\nabla u_{\varepsilon,R}|^2 \, dx \leq \varepsilon^{-N} \int_{B_{1+R}^+} |\nabla u_0|^2 \, dx \leq \text{const } \varepsilon^{-N}$$

some constant $\geq 0$ independent of $\varepsilon$. Then, up to subsequences, $V_\varepsilon^R \to w$ weakly in $\mathcal{H}_R$ and strongly in $L^2(\Gamma_+^R)$ as $\varepsilon \to 0^+$ for some $w \in \mathcal{H}_0(R)$ which is harmonic in $T^-_{1+R} \cup B_R^+$. Since, by \([61]\),

$V_\varepsilon^R|_{\Gamma_+^R} = u_{0,\varepsilon} \to \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) (x_1 - 1)$ in $L^2(\Gamma_+^R)$, we conclude that $w = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) v_R$; in particular, since the weak $\mathcal{H}_R$-limit of $V_\varepsilon^R$ does not depend on the convergence, the convergence actually holds as $\varepsilon \to 0^+$.

Moreover, by standard interior elliptic estimates, it is easy to prove that the convergence is strong in $\mathcal{H}_R$ for every $r \in (1, R)$. In addition, we can prove that

$$V_\varepsilon^R \to \left( \frac{\partial u}{\partial x_1}(e_1) \right) v_R \quad \text{in } H^1(B_R^+ \setminus \overline{B_{R/2}^-}).$$

Indeed, since $V_\varepsilon^R = \left( \frac{\partial u}{\partial x_1}(e_1) \right) v_R$ is harmonic on $B_R^+ \setminus \overline{B_{R/2}^-}$, then its energy is less or equal to the energy of any other $H^1$-function with the same boundary conditions on $\partial(B_R^+ \setminus \overline{B_{R/2}^-})$. In
In particular, letting $\eta$ be a smooth cut-off function such that $\eta \equiv 0$ in $B_{R/2}^+$ and $\eta \equiv 1$ in $D^+ \setminus B_{R}^+$, and $\varphi$ be a smooth cut-off function such that $\varphi \equiv 1$ in $B_{R/2}^+$ and $\varphi \equiv 0$ in $D^+ \setminus B_{(3/4)R}^+$, we obtain that

$$\int_{B_R^+ \setminus B_{R/2}^+} |\nabla (V R - (\frac{\partial u_0}{\partial x_1}(e_1))v_R)|^2 dx$$

$$\leq \int_{B_{R}^+ \setminus B_{R/2}^+} |\nabla \left( \eta (u_0, x - (\frac{\partial u_0}{\partial x_1}(e_1))(x_1 - 1)) \right) + \varphi (V R - (\frac{\partial u_0}{\partial x_1}(e_1))v_R))|^2 dx$$

$$\leq 4 \int_{B_{R}^+ \setminus B_{R/2}^+} |\nabla \eta|^2 |u_0, x - (\frac{\partial u_0}{\partial x_1}(e_1))(x_1 - 1)|^2 dx$$

$$+ 4 \int_{B_{R}^+ \setminus B_{R/2}^+} \eta^2 |\nabla (u_0, x - (\frac{\partial u_0}{\partial x_1}(e_1))(x_1 - 1))|^2 dx$$

$$+ 4 \int_{B_{R}^+ \setminus B_{R/2}^+} |\nabla \varphi|^2 (V R - (\frac{\partial u_0}{\partial x_1}(e_1)))^2 dx$$

$$+ 4 \int_{B_{(3/4)R}^+ \setminus B_{R/2}^+} \varphi^2 |\nabla (V R - (\frac{\partial u_0}{\partial x_1}(e_1)))v_R)|^2 dx = o(1) \quad \text{as} \quad \varepsilon \to 0^+.$$

Hence $V R \to (\frac{\partial u_0}{\partial x_1}(e_1))v_R$ in $H^1(B_R^+ \setminus B_{R/2}^+)$, which, together with $H_\nu$-convergence for $r \in (1, R)$, implies (60).

**Remark 2.12.** Convergences (65) and (66) together with the normal trace embedding theorem for $H(\text{div}; \Omega)$ (see e.g. [26, Chapter 20]), imply that, for all $R > 2$,

$$\frac{\partial Z \varepsilon}{\partial \nu} \to \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) \frac{\partial Z \nu}{\partial \nu} \quad \text{in} \quad H^{-1/2}(\Gamma_R^+) \quad \text{as} \quad \varepsilon \to 0,$$

$$\frac{\partial V \varepsilon}{\partial \nu} \to \left( \frac{\partial u_0}{\partial x_1}(e_1) \right) \frac{\partial V \nu}{\partial \nu} \quad \text{in} \quad H^{-1/2}(\Gamma_R^+) \quad \text{as} \quad \varepsilon \to 0,$$

where $\nu = \nu(x) = \frac{\nu}{|\nu|}$ is the normal external unit vector to $\Gamma_R^+$.

As a straightforward corollary of the blow-up analysis performed in Lemma 2.11 we obtain the following result, which will play a crucial role in the proof of Theorem 1.2.

**Corollary 2.13.** Under assumptions (38) and (42), let $u_\varepsilon$ and $u_0$ as in (16) and (17) and (43) and (44). Then

$$\lim_{\varepsilon \to 0^+} \varepsilon N \int_{\Omega_{1+u_\varepsilon}} |\nabla (u_\varepsilon - u_0)|^2 dx = (\frac{\partial u_0}{\partial x_1}(e_1))^2 \int_{\Omega_{1+u_0} \cup B_R} |\nabla (\Phi - (x_1 - 1)^+)|^2 dx$$

for all $R > 2$.

**Proof.** The thesis follows from (63) and (64) through a change of variable. □

3. Proof of Theorem 1.1

Let us recall and fix some notation we are going to use throughout this section. We recall that $\lambda_\varepsilon = \lambda_\varepsilon(\Omega^\varepsilon)$ denotes the $k$-th eigenvalue of problem (61) on the domain $\Omega^\varepsilon$ and $\lambda_0 = \lambda_k(D^- \cup D^+)$ denotes the $k$-th eigenvalue on $D^- \cup D^+$ which is equal to the simple $k_0$-th eigenvalue on $D^+$. Let $u_\varepsilon$ be the eigenfunction on $\Omega^\varepsilon$ associated to $\lambda_\varepsilon$ satisfying (10) and (17).

For every $j = 1, 2, \ldots, k - 1$, we fix an eigenfunction $v_j^\varepsilon \in D^{1,2}(\Omega^\varepsilon)$ associated to $\lambda_j(\Omega^\varepsilon)$ on $\Omega^\varepsilon$ such that $\int_{\Omega^\varepsilon} |v_j^\varepsilon|^2 dx = 1$ and an eigenfunction $v_j^0 \in D^{1,2}(D^- \cup D^+)$ associated to the eigenvalue $\lambda_j(D^- \cup D^+)$ on $D^- \cup D^+$ such that $\int_{D^- \cup D^+} |v_j^0|^2 dx = 1$. In particular, we can choose such
eigenfunctions in such a way that
\[
\int_{D-\cup D^+} \nabla v_j^0 \cdot \nabla v_i^0 \, dx = 0, \quad \int_{D^+} \nabla v_j^0 \cdot \nabla v_i^0 \, dx = 0, \quad \text{if } i \neq j, \ 1 \leq i, j \leq \bar{k} - 1,
\]
\[
\int_{D-\cup D^+} \nabla v_j^0 \cdot \nabla u_0 \, dx = 0, \quad \nabla \cdot (\nabla v_j^0 \cdot u_{x_j}) = 0, \quad \text{for all } 1 \leq j \leq \bar{k} - 1.
\]

In the sequel we will denote \(\lambda_j(D^- \cup D^+)\) as \(\lambda_j^0\) and \(\lambda_j(\Omega^\varepsilon)\) as \(\lambda_j^\varepsilon\) (we recall that the eigenvalues are repeated as many times as their own multiplicity).

The proof of Theorem 1.1 is based on the following preliminary result.

**Theorem 3.1.** Under assumptions (2), (3), (4), (11), and (12), let \(\lambda_\varepsilon = \lambda_k(\Omega^\varepsilon)\) be the \(k\)-th eigenvalue of problem (16) on the domain \(\Omega^\varepsilon\) defined in (6) and \(\lambda_0 = \lambda_{k0}(D^+) = \lambda_k(D^- \cup D^+)\) be the \(k\)-th eigenvalue of problem (5) on \(D^- \cup D^+\) (which is equal to the simple \(k_0\)-th eigenvalue on \(D^+\)). Then
\[
\lim_{\varepsilon \to 0^+} \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^N} = \left(\frac{\partial u_0}{\partial x_1}(e_1)\right)^2 N \int_{\mathbb{R}^{N-1}} (\Phi(e_1 + \theta) - \theta_1) \, d\sigma(\theta) \in (0, +\infty),
\]
where \(\Phi\) is defined in (20).

**Proof.** We observe that a straightforward consequence of the minimax principle for eigenvalues is that \(\lambda_\varepsilon \leq \lambda_0\). We are going to prove first two estimates for the quantity \(\frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^N}\), one from below and one from above, in order to reach, for every \(R > 2\), an estimate of the type
\[
K_1(\varepsilon, R) \leq \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^N} \leq K_2(\varepsilon, R)
\]
for some constants \(K_1(\varepsilon, R), K_2(\varepsilon, R) > 0\) depending \(\varepsilon\) and \(R\); secondly, we will prove that
\[
\lim_{R \to +\infty} \lim_{\varepsilon \to 0^+} K_1(\varepsilon, R) = \lim_{R \to +\infty} \lim_{\varepsilon \to 0^+} K_2(\varepsilon, R) = \left(\frac{\partial u_0}{\partial x_1}(e_1)\right)^2 N \int_{\mathbb{R}^{N-1}} (\Phi(e_1 + \theta) - \theta_1) \, d\sigma(\theta)
\]
thus implying the stated asymptotics.

**Step 1:** estimate from below. From the Courant-Fisher minimax characterization of the Dirichlet eigenvalues, we have that

\[
\lambda_\varepsilon = \min \left\{ \max_{u \in E \setminus \{0\}} \frac{\int_{\Omega^\varepsilon} |\nabla u|^2 \, dx}{\int_{\Omega^\varepsilon} pu^2 \, dx} : E \text{ is a subspace of } \mathcal{D}^{1,2}(\Omega^\varepsilon) \text{ such that } \dim E = k \right\}.
\]

Let \(R > 2\). If we choose the space \(E = \text{span}\{v_1^0, v_2^0, \ldots, v_{k-1}^0, \tilde{u}_{\varepsilon,R}\}\) (where the functions \(v_j^0\) are trivially extended to the whole \(\Omega^\varepsilon\)), we have that \(\dim E = k\) and then

\[
\lambda_\varepsilon \leq \max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta) \in \mathbb{R}^k} \frac{\int_{\Omega^\varepsilon} \left| \nabla \left( \sum_{j=1}^{k-1} \alpha_j v_j^0 + \beta \tilde{u}_{\varepsilon,R} \right) \right|^2 \, dx}{\int_{\Omega^\varepsilon} p \left( \sum_{j=1}^{k-1} \alpha_j v_j^0 + \beta \tilde{u}_{\varepsilon,R} \right)^2 \, dx}
\]

\[
= \max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta) \in \mathbb{R}^k} \frac{\sum_{j=1}^{k-1} \alpha_j^2 \int_{\Omega^\varepsilon} |\nabla v_j^0|^2 + 2 \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} \nabla v_j^0 \cdot \nabla \tilde{u}_{\varepsilon,R} + 2 \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} \nabla v_j^0 \cdot \nabla \tilde{u}_{\varepsilon,R} \, dx}{\sum_{j=1}^{k-1} \alpha_j^2 \int_{\Omega^\varepsilon} p |v_j^0|^2 + \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} p \nabla \tilde{u}_{\varepsilon,R} \cdot \nabla \tilde{u}_{\varepsilon,R} + 2 \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} \nabla v_j^0 \cdot \nabla \tilde{u}_{\varepsilon,R} \, dx}
\]

\[
\leq \max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta) \in \mathbb{R}^k} \frac{\sum_{j=1}^{k-1} \alpha_j^2 \lambda_j^0 + \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} |\nabla \tilde{u}_{\varepsilon,R}|^2 + 2 \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} \nabla v_j^0 \cdot \nabla \tilde{u}_{\varepsilon,R} \, dx}{\sum_{j=1}^{k-1} \alpha_j^2 \lambda_j^0 + \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} p |v_j^0|^2 + \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega^\varepsilon} p \nabla \tilde{u}_{\varepsilon,R} \cdot \nabla \tilde{u}_{\varepsilon,R} + \alpha(\varepsilon^N)}
\]
in view of the estimate
\[
\left| \int_{\Omega_1^+} p u_j^0 \varpi_{\varepsilon,R} \right| \leq \| p \|_{L^{N/2}(\mathbb{R}^N)} \| v_j^0 \|_{L^{2^*}(\mathbb{R}^N)} \| \varpi_{\varepsilon,R} \|_{L^{2^*}(\Omega_{1/2}^+)} = o(\varepsilon^N)
\]
which holds by Lemma 2.10 and Sobolev inequality. Then
\[
(69) \quad \lambda_\varepsilon - \lambda_0 \leq \max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta)} \left\{ \frac{\varepsilon_N}{k} \alpha_j^2 \beta^2 + 2 \sum_{j=1}^{k-1} \alpha_j \beta \int_{\Omega_1^+} |\nabla \varpi_{\varepsilon,R}^j|^2 + 2 \sum_{j=1}^{k-1} \alpha_j^2 \beta^2 \lambda_0 \right\} + o(\varepsilon^N)
\]
\[
(70) \quad a_i := \lambda_0^i - \lambda_0 < 0.
\]
From convergences (64) and (66) established in Lemma 2.11 and Remark 2.12 it follows that
\[
(71) \quad b_{\varepsilon,R} := \int_{\Omega_1^+} |\nabla \varpi_{\varepsilon,R}^j|^2 \, dx - \int_{\Omega_1^+} |\nabla u_0|^2 \, dx = \int_{\Omega_1^+} |\nabla \varpi_{\varepsilon,R}^j|^2 \, dx - \int_{B_{1/2}^+} |\nabla u_0|^2 \, dx
\]
\[
= \int_{\Omega_1^+} u_0 \left( \frac{\partial \varpi_{\varepsilon,R}^j}{\partial \nu} - \frac{\partial u_0}{\partial \nu} \right) \, d\sigma
\]
\[
= \varepsilon^{-N} \int_{\Gamma_R^+} u_0 (e_1 + \varepsilon (x - e_1)) \left( \frac{\partial \varpi_{\varepsilon,R}^j}{\partial \nu} - \frac{\partial u_0}{\partial \nu} \right) (e_1 + \varepsilon (x - e_1)) \, d\sigma(x)
\]
\[
= \varepsilon^{-N} (b_R + o(1)), \quad \text{as } \varepsilon \to 0^+,
\]
where
\[
(72) \quad b_R = \left( \frac{\partial u_0}{\partial x_1} (e_1) \right)^2 \int_{\Gamma_R^+} (x_1 - 1) \left( \frac{\partial v_R}{\partial \nu} - \frac{\partial (x_1 - 1)}{\partial \nu} \right) (x) \, d\sigma(x)
\]
For every \( i = 1, \ldots, k - 1 \) let us denote
\[
(73) \quad c^i_{\varepsilon,R} = \int_{\Omega_1^+} \nabla v_j^0 (x) \cdot \nabla \varpi_{\varepsilon,R}^i (x) \, dx.
\]
In view of the orthogonality in \( D_+ \cup D_- \) between \( v_j^0 \) and \( u_0 \)
\[
(74) \quad c^i_{\varepsilon,R} = \int_{D_+} \nabla v_j^0 (x) \cdot \nabla \varpi_{\varepsilon,R}^i (x) \, dx
\]
\[
= \int_{D_-} \nabla v_j^0 (x) \cdot \nabla \varpi_{\varepsilon,R}^i (x) \, dx + \int_{B_{1/2}^+} \nabla v_j^0 (x) \cdot \nabla u_0 (x) \, dx
\]
\[
= \int_{D_-} \nabla v_j^0 (x) \cdot \nabla \varpi_{\varepsilon,R}^i (x) \, dx + \int_{B_{1/2}^+} \nabla v_j^0 (x) \cdot \nabla \varpi_{\varepsilon,R}^i (x) \, dx - \int_{B_{1/2}^+} \nabla v_j^0 (x) \cdot \nabla u_0 (x) \, dx
\]
\[
= O(\varepsilon^N) \quad \text{as } \varepsilon \to 0^+,
\]
taking into account Lemma 2.10 and the fact that
\[
\int_{B_{1/2}^+} |\nabla \varpi_{\varepsilon,R}^i (x)|^2 \, dx \leq \int_{\Omega_1^+} |\nabla \varpi_{\varepsilon,R}^i (x)|^2 \, dx \leq \int_{B_{1/2}^+} |\nabla u_0 (x)|^2 \, dx
\]
by Dirichlet Principle and (56).

We claim that

$$
\begin{equation}
\max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta)} \left\{ \sum_{j=1}^{k-1} \alpha_j^2 a_j + \beta^2 b_{\varepsilon, R} + 2 \sum_{j=1}^{k-1} \alpha_j \beta c_{\varepsilon, R} \right\} = \varepsilon^N (b_R + o(1)).
\end{equation}
$$

To prove (74), let \( \beta_\varepsilon \in \mathbb{R}, \alpha_{j, \varepsilon} \in \mathbb{R}, j = 1, \ldots, k - 1, \) be such that \( \sum_{j=1}^{k-1} \alpha_j^2 + \beta_\varepsilon^2 = 1 \) and

$$
\begin{equation}
\sum_{j=1}^{k-1} \alpha_j^2 a_j + \beta_\varepsilon^2 b_{\varepsilon, R} + 2 \sum_{j=1}^{k-1} \alpha_j \beta c_{\varepsilon, R} = \max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta)} \left\{ \sum_{j=1}^{k-1} \alpha_j^2 a_j + \beta^2 b_{\varepsilon, R} + 2 \sum_{j=1}^{k-1} \alpha_j \beta c_{\varepsilon, R} \right\}.
\end{equation}
$$

We first prove that

$$
\beta_\varepsilon = 1 + o(\varepsilon^N), \quad \text{as} \ \varepsilon \to 0^+.
$$

Indeed from (71), and (73), it follows that

$$
\varepsilon^N (b_R + o(1))(1 - \beta_\varepsilon^2) \leq (1 - \beta_\varepsilon^2) \max_{i=1, \ldots, k-1} a_i + O(\varepsilon^N)
$$

which implies that \( 1 - \beta_\varepsilon^2 = O(\varepsilon^N). \) Assuming by contradiction that (74) does not hold, there should exist a sequence \( \varepsilon_n \to 0^+ \) such that \( \lim_{n \to \infty} \varepsilon_n^N (1 - \beta_{\varepsilon_n}^2) = \ell \in (0, +\infty). \) Then, up to subsequences, there would exist \( L < 0 \) such that \( \lim_{n \to \infty} \varepsilon_n^N \sum_{j=1}^{k-1} \alpha_j^2 \varepsilon_n a_j = L. \) Therefore (74) and (71) would imply

$$
\varepsilon_n^N (b_R + o(1)) \leq \varepsilon_n^N (L + o(1)) + o(\varepsilon_n^N) + o(\varepsilon_n^N), \quad \text{as} \ n \to \infty,
$$

i.e. \( b_R + o(1) \leq L + b_R + o(1) \) as \( n \to \infty, \) thus contradicting \( L < 0. \) Estimate (76) is thereby proved.

From (76) we deduce that \( \alpha_{j, \varepsilon} = o(\varepsilon^{N/2}) \) as \( \varepsilon \to 0^+, \) then from (71), (76), (73), and (75) it follows that

$$
\begin{equation}
\max_{(\alpha_1, \ldots, \alpha_{k-1}, \beta)} \left\{ \sum_{j=1}^{k-1} \alpha_j^2 a_j + \beta^2 b_{\varepsilon, R} + 2 \sum_{j=1}^{k-1} \alpha_j \beta c_{\varepsilon, R} \right\} = \varepsilon^N (1 + o(\varepsilon^N))(b_R + o(1)) + o(\varepsilon^N) = \varepsilon^N (b_R + o(1)), \quad \text{as} \ \varepsilon \to 0^+,
\end{equation}
$$

thus proving claim (74). From (69) and (74), we deduce that

$$
\lambda_\varepsilon - \lambda_0 \leq \varepsilon^N (b_R + o(1)), \quad \text{as} \ \varepsilon \to 0^+,
$$

and hence, for every \( R > 2, \)

$$
\frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^N} \geq K_1(\varepsilon, R),
$$

where, for every \( R > 2, \)

$$
\lim_{\varepsilon \to 0^+} K_1(\varepsilon, R) = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \int_{\Gamma_R} (x_1 - 1) \left( \frac{\partial (x_1 - 1)}{\partial v} - \frac{\partial v}{\partial x_1} \right) (x) \, d\sigma(x).
$$

**Step 2:** estimate from above. By the Courant-Fisher *minimax characterization* of the eigenvalue \( \lambda_0 = \lambda_k(D^+ \cup D^-), \) we have that

$$
\lambda_0 = \min \left\{ \max_{u \in F \setminus \{0\}} \frac{\int_{D^+ \cup D^-} \left| \nabla u \right|^2 \, dx}{\int_{D^+ \cup D_-} \left| p u \right|^2 \, dx} : F \text{ is a subspace of } D^{1,2}(D^+ \cup D^-), \dim F = k \right\}.
$$
Let $R > 2$ and $\eta_{\varepsilon, R}$ be a smooth cut-off function such that $\eta_{\varepsilon, R} \equiv 1$ in $(D^+ \setminus B_{Rc}^+) \cup (D^- \setminus B_{Rc}^-)$, $\eta_{\varepsilon, R} \equiv 0$ in $B_{(R/2)c}^+ \cup B_{(R/2)c}^-$, $0 \leq \eta_{\varepsilon, R} \leq 1$ and $|\nabla \eta_{\varepsilon, R}| \leq 4/(\varepsilon R)$ in $D^- \cup D^+$. We choose the $\bar k$-dimensional space $F = \text{span}\{\eta_{\varepsilon, R}v^1_j, \ldots, \eta_{\varepsilon, R}v^\bar k_j, \tilde u_{\varepsilon, R}\}$ in $L^2$. Then

$$\lambda_0 \leq \max_{(\alpha_1, \ldots, \alpha_{\bar k}) \in \mathbb{R}^k} \frac{\int_{D^- \cup D^+} |\nabla \left( \sum_{j=1}^{\bar k-1} \alpha_j \eta_{\varepsilon, R}v_j^2 + \alpha_k \tilde u_{\varepsilon, R} \right)|^2 dx}{\int_{D^- \cup D^+} \left( \sum_{j=1}^{\bar k-1} \alpha_j \eta_{\varepsilon, R}v_j^2 + \alpha_k \tilde u_{\varepsilon, R} \right)^2 dx}.$$

We notice that

$$\int_{D^- \cup D^+} \left| \nabla \left( \sum_{j=1}^{\bar k-1} \alpha_j \eta_{\varepsilon, R}v_j^2 + \alpha_k \tilde u_{\varepsilon, R} \right) \right|^2 dx = \sum_{j=1}^{\bar k-1} \alpha_j^2 \int_{D^- \cup D^+} |\nabla \eta_{\varepsilon, R}v_j^2|^2 dx + \alpha_k^2 \int_{D^- \cup D^+} |\nabla \tilde u_{\varepsilon, R}|^2 dx$$

$$+ \sum_{i,j,k} \alpha_i \alpha_j \int_{D^- \cup D^+} \eta_{\varepsilon, R}v_i^2 \cdot \nabla \eta_{\varepsilon, R}v_j^2 dx + \sum_{j=1}^{\bar k-1} \alpha_j \alpha_k \int_{D^- \cup D^+} \eta_{\varepsilon, R}v_j^2 \cdot \nabla \tilde u_{\varepsilon, R} dx,$$

while, from Lemma 2.8 Corollary 2.9 assumption (4), and Corollary 2.7 it follows that

$$\int_{D^- \cup D^+} \left( \sum_{j=1}^{\bar k-1} \alpha_j \eta_{\varepsilon, R}v_j^2 + \alpha_k \tilde u_{\varepsilon, R} \right)^2 dx = \sum_{j=1}^{\bar k-1} \alpha_j^2 \int_{D^- \cup D^+} \left( \eta_{\varepsilon, R}v_j^2 \right)^2 dx + \alpha_k^2 \int_{D^- \cup D^+} \left( \tilde u_{\varepsilon, R} \right)^2 dx$$

$$+ \sum_{i,j,k} \alpha_i \alpha_j \int_{D^- \cup D^+} \eta_{\varepsilon, R}v_i^2 \cdot \eta_{\varepsilon, R}v_j^2 dx + \sum_{j=1}^{\bar k-1} \alpha_j \alpha_k \int_{D^- \cup D^+} \eta_{\varepsilon, R}v_j^2 \cdot \tilde u_{\varepsilon, R} dx$$

$$= \sum_{j=1}^{\bar k-1} \alpha_j^2 \left( 1 + \int_{B_{(R/2)c}^+} p(\eta_{\varepsilon, R}^2 - 1) |v_j^2|^2 dx \right) + \alpha_k^2 \left( 1 - \int_{B_{(R/2)c}^-} p |u_{\varepsilon, R}|^2 dx \right)$$

$$+ \sum_{i,j,k} \alpha_i \alpha_j \int_{B_{(R/2)c}^+} p(\eta_{\varepsilon, R}^2 - 1) v_i^2 v_j^2 dx - \sum_{j=1}^{\bar k-1} \alpha_j \alpha_k \int_{\Omega_{1/2}} p v_j^2 u_{\varepsilon, R} dx$$

$$= 1 + o(\varepsilon^N), \text{ as } \varepsilon \to 0^+.$$

Then

$$(79) \quad \lambda_0 - \lambda_0 \leq \max_{(\alpha_1, \ldots, \alpha_{\bar k}) \in \mathbb{R}^k} \left\{ \sum_{j=1}^{\bar k} \alpha_j^2 a^2_{j, R} + \sum_{i,j=1}^{\bar k} \alpha_i \alpha_j c^2_{i,j, R} \right\} + o(\varepsilon^N)$$

where we have set

$$a^2_{k, R} = \int_{D^+} |\nabla \tilde u_{\varepsilon, R}|^2 dx - \int_{D^+} |\nabla u_{\varepsilon, R}|^2 dx,$$

$$a^2_{k, R} = \int_{D^- \cup D^+} |\nabla \eta_{\varepsilon, R}v_j^2|^2 dx - \int_{\Omega_{1/2}} |\nabla u_{\varepsilon, R}|^2 dx, \text{ for every } j = 1, \ldots, \bar k - 1,$$

$$c^2_{i,j, R} = \int_{D^- \cup D^+} \nabla (\eta_{\varepsilon, R}v_i^2) \cdot \nabla (\eta_{\varepsilon, R}v_j^2) dx, \text{ for every } i, j = 1, \ldots, \bar k - 1, i \neq j,$$

$$c^2_{j,k, R} = c^2_{k,j, R} = \int_{D^+} \nabla (\eta_{\varepsilon, R}v_j^2) \cdot \nabla \tilde u_{\varepsilon, R} dx, \text{ for every } j = 1, \ldots, \bar k - 1.$$

Let us study each coefficient of the quadratic form above. From (82), estimates (44) of Lemma 2.6 and estimates (51) and (52) of Corollary 2.7 we deduce that
\[ a_{k,R}^\varepsilon = \varepsilon^N (a_R + o(1)), \quad \varepsilon \to 0^+, \]
where
\[ a_R = (\frac{\partial u_0}{\partial x_1}(e_1))^2 \int_{\Gamma_R} \Phi(x) \left( \frac{\partial z_R}{\partial \nu} - \frac{\partial \Phi}{\partial \nu} \right)(x) d\sigma(x). \]

For every \( j = 1, \ldots, \tilde{k} - 1 \), in view of estimates (42) and (44) of Lemma 2.6, we have that
\[ a_{j,R}^\varepsilon = \lambda_j^\varepsilon - \lambda_0 - \int_{B_{R_n}^+ \cup B_{R_0}^- \cup B_{R_0}^+} |\nabla v_j^\varepsilon|^2 dx + \int_{B_{R_n}^+ \cup B_{R_0}^-} |\nabla \eta_{R} v_j^\varepsilon|^2 dx + \int_{B_{R_0}^+} \eta_{R}^2 |\nabla v_j^\varepsilon|^2 dx \]
\[ + 2 \int_{B_{R_0}^-} |\nabla \eta_{R} v_j^\varepsilon|^2 dx + \int_{B_{R_0}^-} \eta_{R}^2 |\nabla v_j^\varepsilon|^2 dx = O(\varepsilon^{N-2}), \quad \varepsilon \to 0^+. \]
From (82), estimate (14) of Lemma 2.6 and estimates (51) and (52) of Corollary 2.7, we deduce that
\[ a_{j,R}^\varepsilon = \lambda_j^\varepsilon - \lambda_0 - \int_{B_{R_n}^+ \cup \Omega_{R_0}^+} |\nabla v_j^\varepsilon|^2 dx + \int_{B_{R_0}^+} |\nabla \eta_{R} v_j^\varepsilon|^2 dx \]
\[ = \lambda_j^\varepsilon - \lambda_0 - \lambda_j^\varepsilon \int_{B_{R_0}^+} \eta_{R}^2 |\nabla v_j^\varepsilon|^2 dx - \int_{\Gamma_{R_n}^+ \cup \Gamma_{R_0}^-} v_j^\varepsilon \frac{\partial v_j^\varepsilon}{\partial \nu} d\sigma + O(\varepsilon^{N-2}) \]
\[ = \lambda_j^\varepsilon - \lambda_0 + O(\varepsilon^{N-2}) = \lambda_j^0 - \lambda_0 + o(1), \quad \varepsilon \to 0^+. \]

For every \( i, j = 1, \ldots, \tilde{k} - 1 \) such that \( i \neq j \), from (82), estimates (43) and (44) of Lemma 2.6 and the orthogonality of \( v_i^\varepsilon \) and \( v_j^\varepsilon \) in \( D^{1,2}(\Omega^c) \), it follows that
\[ \tilde{c}_{i,j,R}^\varepsilon = - \int_{B_{R_n}^+ \cup B_{R_0}^- \cup B_{R_0}^+} \nabla v_i^\varepsilon \cdot \nabla v_j^\varepsilon dx + \int_{B_{R_n}^+ \cup B_{R_0}^-} \nabla (\eta_{R} v_i^\varepsilon) \cdot \nabla (\eta_{R} v_j^\varepsilon) dx \]
\[ = - \int_{B_{R_0}^-} \eta_{R}^2 v_i^\varepsilon v_j^\varepsilon dx - \int_{\Gamma_{R_n}^+ \cup \Gamma_{R_0}^-} v_i^\varepsilon \frac{\partial v_j^\varepsilon}{\partial \nu} d\sigma + O(\varepsilon^{N-2}) \]
\[ = O(\varepsilon^{N-2}) \quad \varepsilon \to 0. \]

From (55), (57), and Lemma 2.10 and 2.11, we have that
\[ \int_{B_{R_0}^+} |\nabla \tilde{u}_{R}^\varepsilon|^2 dx \leq \int_{B_{R_0}^+} |\nabla \tilde{u}_{R}^\varepsilon|^2 dx + \int_{B_{R_0}^+} |\nabla \eta_{R} u_{R}^\varepsilon|^2 dx \]
\[ \leq 2 \int_{B_{R_0}^+ \cup B_{R_0}^+} |\nabla \eta_{R} u_{R}^\varepsilon|^2 dx + 2 \int_{B_{R_n}^+ \cup B_{R_0}^+} \eta_{R}^2 |\nabla u_{R}^\varepsilon|^2 dx = O(\varepsilon^N), \quad \varepsilon \to 0^+. \]
From (82), (85), (12), (41), and [18] Lemma 2.10] (which in particular yields \( \sup_{\Omega_t^\varepsilon} |u_\varepsilon| = O(\varepsilon) \) as \( \varepsilon \to 0^+ \)), it follows that, for every \( j = 1, \ldots, k-1 \),

\[
\alpha_{j,\varepsilon} = \max_{\Omega_t^\varepsilon} \left\{ \sum_{i=1}^{k} a_{i,j,R} \alpha_i^2 + \sum_{i,j \neq j} c_{i,j,R} \alpha_i \alpha_j \right\} = \varepsilon^N (a_R + o(1)),
\]

as \( \varepsilon \to 0^+ \). To prove (87), let \( \alpha_{j,\varepsilon} \in \mathbb{R} \), such that \( \sum_{j=1}^{k} \alpha_{j,\varepsilon}^2 = 1 \) and

\[
\sum_{j=1}^{k} a_{j,R} \alpha_j^2 + \sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j, \quad \text{max}_{(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k} \left\{ \sum_{j=1}^{k} a_{j,R} \alpha_j^2 + \sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j \right\}.
\]

From

\[
a_{k,R}^\varepsilon \leq \sum_{j=1}^{k} a_{j,R} \alpha_j^2 + \sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j, \quad \text{it follows that}
\]

\[
\left( - \max_{j < k} a_{j,R}^\varepsilon + a_{k,R}^\varepsilon \right) (1 - \alpha_{k,\varepsilon}^2) \leq \sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j, \quad \text{and hence, in view of (80) and (83)},
\]

\[
(\max_{j < k} (\lambda_j^\varepsilon - \lambda_0) + o(1)) (1 - \alpha_{k,\varepsilon}^2) \leq \sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j.
\]

From (89), (84), and (90), it follows that

\[
1 - \alpha_{k,\varepsilon}^2 = O(\varepsilon^{N-2}), \quad \text{as } \varepsilon \to 0^+.
\]

Since \( 1 - \alpha_{k,\varepsilon}^2 = \sum_{j < k} \alpha_{j,\varepsilon}^2 \), we obtain that, for every \( j = 1, \ldots, k-1 \),

\[
\alpha_{j,\varepsilon}^2 = O(\varepsilon^{N-2}), \quad \text{as } \varepsilon \to 0^+.
\]

From (84), (91), (80) and (90), it follows that

\[
\sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j, \quad \text{as } \varepsilon \to 0^+.
\]

In the case \( N = 3 \), (89) and (92) imply that \( 1 - \alpha_{k,\varepsilon}^2 = O(\varepsilon^2) \) as \( \varepsilon \to 0^+ \), hence for all \( j \neq k \) \( \alpha_{j,\varepsilon} = O(\varepsilon) \). Therefore, in view of (84) and (89), we obtain that \( \sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j, \quad \text{as } \varepsilon \to 0^+ \), thus improving estimate (92) for \( N = 3 \). Then, for any dimension \( N \geq 3 \) we obtain

\[
\sum_{i \neq j} c_{i,j,R} \alpha_i \alpha_j = O(\varepsilon^N), \quad \text{as } \varepsilon \to 0^+.
\]

Arguing a third time in the same way, from (89) and the improved estimate (92) on the mixed terms, we can improve (90) and (91) obtaining

\[
1 - \alpha_{k,\varepsilon}^2 = O(\varepsilon^N), \quad \alpha_{j,\varepsilon}^2 = O(\varepsilon^N) \text{ for all } j < k, \quad \text{as } \varepsilon \to 0^+.
\]
Therefore, from Lemma 2.5(ii), we have that

\[
\alpha_{k,\varepsilon}^2 - \max_{j < k}(\lambda_j^0 - \lambda_0) + o(1)(1 - \alpha_{k,\varepsilon}^2) \leq \sum_{j \neq k} c_{j,j,R}^2 \alpha_{i_1,\varepsilon} \alpha_{j,\varepsilon} + \sum_{j \neq k} c_{j,\varepsilon} \rho_{j,k,R}^2 \alpha_{j,\varepsilon} \alpha_{k,\varepsilon} = O(\varepsilon^{N-2+N}) + O(\varepsilon^{N-1+(N/2)}) = o(\varepsilon^N), \text{ as } \varepsilon \to 0^+,
\]

thus implying

\[
1 - \alpha_{k,\varepsilon}^2 = o(\varepsilon^N), \quad \alpha_{j,\varepsilon} = o(\varepsilon^N) \text{ for all } j < k, \quad \text{as } \varepsilon \to 0^+.
\]

From (80), (96), (83), and (95), it follows that

\[
\sum_{j \neq k} c_{j,j,R}^2 \alpha_{i_1,\varepsilon} \alpha_{j,\varepsilon} = O(\varepsilon^{N-2+N}) + O(\varepsilon^{N-1+(N/2)}) = o(\varepsilon^N), \text{ as } \varepsilon \to 0^+,
\]

thus proving claim (87). From (79) and (87), we deduce that

\[
\lambda_0 - \lambda_\varepsilon \leq \varepsilon^N(a_R + o(1)), \quad \text{as } \varepsilon \to 0^+,
\]

and hence, for every \( R > 2 \),

\[
\frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^N} \leq K_2(\varepsilon, R),
\]

where, for every \( R > 2 \),

\[
\lim_{\varepsilon \to 0^+} K_2(\varepsilon, R) = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \int_{\Gamma_R^+} \Phi(x)(\frac{\partial z_R}{\partial \nu} - \frac{\partial \Phi}{\partial \nu})(x) d\sigma(x).
\]

**Step 3:** asymptotic behavior. Up to now we have proved the following estimate

\[
K_1(\varepsilon, R) \leq \frac{\lambda_0 - \lambda_\varepsilon}{\varepsilon^N} \leq K_2(\varepsilon, R)
\]

for any \( R \in (2, +\infty) \), where

\[
\lim_{\varepsilon \to 0^+} K_1(\varepsilon, R) = -b_R, \quad \lim_{\varepsilon \to 0^+} K_2(\varepsilon, R) = a_R,
\]

with \( b_R \) and \( a_R \) defined in (72) and (81) respectively. We now claim that

\[
\lim_{R \to +\infty} (-b_R) = \lim_{R \to +\infty} a_R = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 N \int_{\mathbb{R}^N_n} (\Phi(e_1 + \theta) - \Phi(\theta)) \frac{\partial^2 \Theta}{\partial \sigma^2} d\sigma(\theta).
\]

As far as \(-b_R\) is concerned, we first observe that

\[
\int_{\Gamma_R^+} (x_1 - 1) \frac{\partial (x_1 - 1)}{\partial \nu} d\sigma = R^{N-1} \int_{\mathbb{R}^N_n} R \theta_1^2 d\sigma = R^N \mathcal{T}_N^2.
\]

Therefore, from Lemma 2.3 ii), we have that

\[
-b_R = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \int_{\Gamma_R^+} (x_1 - 1) \left( \frac{\partial (x_1 - 1)}{\partial \nu} - \frac{\partial \Theta}{\partial \nu} \right)(x) d\sigma(x)
\]

\[
= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \left( R^N \mathcal{T}_N^2 - \mathcal{T}_N^2 \frac{R^{N + N - 1} - N \chi_R}{1 - R^{-N}} \right)
\]

\[
= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \frac{N \mathcal{T}_N}{1 - R^{-N}} \left( \chi_R(1) - \mathcal{T}_N \right)
\]

\[
= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \frac{N \mathcal{T}_N}{1 - R^{-N}} \left( \int_{\mathbb{R}^N_n} \theta_1^2 \Phi(\theta) d\sigma(\theta) - \mathcal{T}_N \right).
\]
Therefore, from Lemma \eqref{lemma2.1}, it follows that
\begin{equation}
\lim_{R \to +\infty} (-b_R) = \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \int_{S^N_{1+}} \left( \Phi(e_1 + \theta - \theta_1) \right) \, d\sigma(\theta).
\end{equation}

Let us now study the limit of $a_R$ as $R \to +\infty$. We split \eqref{51} as
\begin{equation}
\begin{aligned}
a_R &= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \int_{\Gamma_R^+} \Phi \left( \frac{\partial z_R}{\partial v} - \frac{\partial \Phi}{\partial v} \right) \, d\sigma \\
&= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \left( \int_{\Gamma_R^+} (x_1 - 1) \left( \frac{\partial z_R}{\partial v} - \frac{\partial \Phi}{\partial v} \right) \, d\sigma + \int_{\Gamma_R^+} \left( \Phi - (x_1 - 1) \right) \left( \frac{\partial z_R}{\partial v} - \frac{\partial \Phi}{\partial v} \right) \, d\sigma \right).
\end{aligned}
\end{equation}

We first prove that the second term in the sum vanishes as $R \to \infty$. Indeed,
\begin{equation}
\begin{aligned}
\int_{\Gamma_R^+} \left( \Phi - (x_1 - 1) \right) \left( \frac{\partial z_R}{\partial v} - \frac{\partial \Phi}{\partial v} \right) \, d\sigma &= \int_{\Gamma_R^+} \left( \Phi - (x_1 - 1) \right) \left( \frac{\partial (x_1 - 1)}{\partial v} - \frac{\partial \Phi}{\partial v} \right) \, d\sigma \\
&+ \int_{\Gamma_R^+} \left( \Phi - (x_1 - 1) \right) \left( \frac{\partial z_R}{\partial v} - \frac{\partial (x_1 - 1)}{\partial v} \right) \, d\sigma.
\end{aligned}
\end{equation}

Testing equation $-\Delta(\Phi(x) - (x_1 - 1)) = 0$ in $D^+ \setminus B^+_R$ with $\Phi(x) - (x_1 - 1)$, we have that
\begin{equation}
\begin{aligned}
\int_{\Gamma_R^+} \left( \Phi(x) - (x_1 - 1) \right) \left( \frac{\partial (x_1 - 1)}{\partial v} - \frac{\partial \Phi}{\partial v} \right)(x) \, d\sigma(x) &= \int_{D^+ \setminus B^+_R} |\nabla(\Phi(x) - (x_1 - 1))|^2 \, dx \\
\int_{B^+_R} \nabla(z_R - (x_1 - 1)) \cdot \nabla(\eta_R(\Phi - (x_1 - 1))) \, dx \\
&\leq \left( \int_{B^+_R} |\nabla(z_R - (x_1 - 1))|^2 \, dx \right)^{1/2} \left( \int_{B^+_R} |\nabla(\eta_R(\Phi - (x_1 - 1)))|^2 \, dx \right)^{1/2} \\
&\leq \int_{B^+_R} |\nabla(\eta_R(\Phi - (x_1 - 1)))|^2 = o(1) \quad \text{as } R \to +\infty
\end{aligned}
\end{equation}

thanks to the Dirichlet Principle and estimate \eqref{35}. Therefore, from \eqref{100}, \eqref{101}, \eqref{102}, \eqref{103}, Lemmas \eqref{lemma2.1} and \eqref{lemma2.4}, and the fact that, in view of \eqref{32}, \eqref{33}, and \eqref{34}, $\phi_R(R) = \varphi(R)$ for all $R > 2$, it follows that
\begin{equation}
\begin{aligned}
a_R &= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 \int_{\Gamma_R^+} (x_1 - 1) \left( \frac{\partial z_R}{\partial v} - \frac{\partial \Phi}{\partial v} \right)(x) \, d\sigma(x) + o(1) \\
&= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 N \mathcal{T}_N R^N \left( \frac{\varphi(R)}{R} - \gamma_N \right) + o(1) \\
&= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 N \mathcal{T}_N (\varphi(1) - \gamma_N) + o(1) \\
&= \left( \frac{\partial u_0}{\partial x_1}(e_1) \right)^2 N \int_{S^N_{1+}} (\Phi(e_1 + \theta - \theta_1) \theta_1 \, d\sigma + o(1) \quad \text{as } R \to +\infty.
\end{aligned}
\end{equation}

Combining \eqref{99} and \eqref{104}, we prove claim \eqref{98}. The conclusion follows from \eqref{97} and \eqref{98}, observing that $\int_{S^N_{1+}} (\Phi(e_1 + \theta - \theta_1) \theta_1 \, d\sigma > 0$ due to the fact that, by the Strong Maximum Principle, $\Phi > (x_1 - 1)^+ \in D^+$.
Proof of Theorem 1.1. From Lemma 2.1 it follows that
\[
\int_{S^N_{N-1}} (\Phi(e_1 + \theta) - \theta_1) \theta_1 d\sigma = \frac{1}{1 - N} \int_{\Gamma^+_1} \frac{\partial(\Phi - (x_1 - 1))}{\partial \nu}(x_1 - 1) d\sigma.
\]
On the other hand, testing first equation \(-\Delta(\Phi - (x_1 - 1)) = 0\) in \(B^+_1\) with \((x_1 - 1)\) and then equation \(-\Delta(x_1 - 1) = 0\) in \(B^+_1\) with \(\Phi - (x_1 - 1)\), we obtain that
\[
\int_{\Gamma^+_1} \frac{\partial(\Phi - (x_1 - 1))}{\partial \nu}(x_1 - 1) d\sigma = \int_{B^+_1} \nabla(\Phi - (x_1 - 1)) \cdot \nabla(x_1 - 1) dx
\]
\[= \int_{S^N_{N-1}} (\Phi(e_1 + \theta) - \theta_1) \theta_1 d\sigma - \int_{\Sigma} \Phi(1, x') dx'.
\]
Combining (105) and (106), we deduce that
\[
N \int_{S^N_{N-1}} (\Phi(e_1 + \theta) - \theta_1) \theta_1 d\sigma = \int_{\Sigma} \Phi(1, x') dx'.
\]
The conclusion follows from Theorem 3.1, Corollary 2.3 and (9). \(\square\)

Steiner rearrangement allows proving that the shape of the section \(\Sigma\) minimizing \(m(\Sigma)\) and hence maximizing \(\lim_{\varepsilon \to 0^+} \varepsilon^{-N}(\lambda_0 - \lambda_1)\) is the spherical one.

Proposition 3.2. For every \(\Sigma \subset \mathbb{R}^{N-1}\) being an open bounded domain containing 0, let \(m(\Sigma)\) be defined in (4). Then, for every \(\mu > 0\),
\[
\min \left\{ m(\Sigma) : \Sigma \subset \mathbb{R}^{N-1} \text{ is a bounded domain, } 0 \in \Sigma, |\Sigma| = \mu \right\} = m\left( B'(0, (\frac{N-1}{\omega_{N-2}})^{\frac{1}{N-1}}) \right)
\]
where \(\cdot \) denotes the Lebesgue measure of \(\mathbb{R}^{N-1}\), \(B'(0, r) := \{ x' \in \mathbb{R}^{N-1} : |x'| < r \}\) denotes the \((N-1)\)-dimensional ball of radius \(r\) centered at \(0\), and \(\omega_{N-2}\) denotes the volume of the unit \((N-2)\)-dimensional sphere.

Proof. For every \(\Sigma \subset \mathbb{R}^{N-1}\) being an open bounded domain containing 0, let us consider the domain \(D_\Sigma := ((-\infty, 1] \times \Sigma) \cup D^+\) and its Steiner symmetral \(D'_\Sigma\) in codimension \(N-1\) defined as
\[
D'_\Sigma = ((-\infty, 1] \times B'(0, r_{\Sigma})) \cup D^+ = D_{B'(0, r_{\Sigma})},
\]
where \(r_{\Sigma} = (\frac{\omega_{N-2}}{\omega_{N-1}})^{\frac{1}{N-1}}\). For every \(w \in D^{1,2}(D_\Sigma)\), \(w \geq 0\) a.e., its Steiner rearrangement in codimension \(N-1\) is the function \(w^\sigma \in D^{1,2}(D'_\Sigma)\) defined as
\[
w^\sigma(x_1, x') = \inf \left\{ t > 0 : |y \in \mathbb{R}^{N-1} : w(x_1, y) > t| \leq \frac{\omega_{N-2}}{N-1} |x'|^{N-1} \right\}.
\]
For every open bounded domain \(\Sigma \subset \mathbb{R}^{N-1}\) containing 0 and \(w \in D^{1,2}(D_\Sigma)\) such that \(w \geq 0\) a.e., the Pólya-Szegö inequality for the Steiner rearrangement (see e.g. [11] and [12]) implies that
\[
\int_{D_\Sigma} |\nabla w(x_1, x')|^2 dx_1 dx' \geq \int_{D'_\Sigma} |\nabla w^\sigma(x_1, x')|^2 dx_1 dx' = \int_{D_{w^\sigma}(0, r_{\Sigma})} |\nabla w^\sigma(x_1, x')|^2 dx_1 dx',
\]
whereas the Cavalieri principle yields
\[
\int_{\Sigma} w(1, x') dx' = \int_{B'(0, r_{\Sigma})} w^\sigma(1, x') dx'.
\]
Therefore, letting \(J_\Sigma : D^{1,2}(D_\Sigma) \to \mathbb{R}\),
\[
J_\Sigma(w) := \frac{1}{2} \int_{D_\Sigma} |\nabla w|^2 dx - \int_{\Sigma} w(1, x') dx',
\]
we have that \(J_\Sigma(w) \geq J_{B'(0, r_{\Sigma})}(w^\sigma)\) for every \(w \in D^{1,2}(D_\Sigma)\) such that \(w \geq 0\) a.e.. Since the minimum of \(J_\Sigma\) over \(D^{1,2}(D_\Sigma)\) is attained by a nonnegative function, we then conclude that
\[
m(\Sigma) \geq m(B'(0, r_{\Sigma}))
\]
for every open bounded domain \(\Sigma \subset \mathbb{R}^{N-1}\) containing 0, thus completing the proof. \(\square\)
4. Rate of convergence for eigenfunctions

In this section we prove a sharp estimate for the rate of convergence of eigenfunctions. In view of Corollary 2.13 it will be sufficient to obtain an estimate of \( \| u \|_{D^{1,2}(D^+)} \) to this aim, we consider the following operator

\[
(\lambda, u) \mapsto (\| u \|^2 - \lambda_0, -\Delta u - \lambda pu),
\]

where the symbol \( \| \cdot \| \) stands for the \( D^{1,2}(D^+) \)-norm, i.e.

\[
\| u \| := \left( \int_{D^+} |\nabla u|^2 \, dx \right)^{1/2},
\]

\( (D^{1,2}(D^+))^* \) is the dual space of \( D^{1,2}(D^+) \), and, for all \( u \in D^{1,2}(D^+) \), \( -\Delta u - \lambda pu \in (D^{1,2}(D^+))^* \) acts as

\[
(D^{1,2}(D^+))^*(\lambda, u) = \int_{D^+} \nabla u(x) \cdot \nabla v(x) \, dx - \lambda \int_{D^+} p(x)u(x)v(x) \, dx.
\]

We recall from (11) that \( \int_{D^+} pu_0^2 \, dx = 1 \) and hence \( \| u_0 \|^2 = \lambda_0 \). Therefore \( F(\lambda_0, u_0) = (0, 0) \).

**Lemma 4.1.** Under assumptions (2), (3), (4), (11), and (12), let \( \lambda_0 = \lambda_{k_0}(D^+) = \lambda_0(D^- \cup D^+) \) be the \( k \)-th eigenvalue of problem (5) on \( D^- \cup D^+ \) (which is equal to the simple \( k_0 \)-th eigenvalue on \( D^+ \)) and \( u_0 \) be as in (13) and (14). Then, the operator \( F \) defined in (108) is Fréchet-differentiable at \( (\lambda_0, u_0) \) and its Fréchet-differential \( dF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times D^{1,2}(D^+), \mathbb{R} \times (D^{1,2}(D^+))^*) \) is invertible.

**Proof.** For all \( (\lambda, u) \in \mathbb{R} \times D^{1,2}(D^+) \), there holds

\[
F(\lambda_0 + \lambda, u_0 + u) = \left( \| u_0 + u \|^2 - \lambda_0, -\Delta (u_0 + u) - \lambda pu_0 - \lambda pu \right) = \left( 2 \int_{D^+} \nabla u_0 \cdot \nabla u \, dx + \| u \|^2, -\Delta u - \lambda pu_0 - \lambda puu \right)
\]

\[
= \left( 2 \int_{D^+} \nabla u_0 \cdot \nabla u \, dx, -\Delta u - \lambda pu_0 - \lambda puu \right) + o(\| \lambda \| + \| u \|)
\]

as \( (\lambda, u) \to 0 \) in \( \mathbb{R} \times D^{1,2}(D^+) \). Therefore \( F \) is Fréchet-differentiable at \( (\lambda_0, u_0) \) and

\[
dF(\lambda_0, u_0)(\lambda, u) = \left( 2 \int_{D^+} \nabla u_0 \cdot \nabla u \, dx, -\Delta u - \lambda pu_0 - \lambda puu \right)
\]

for every \((\lambda, u) \in \mathbb{R} \times D^{1,2}(D^+) \). It remains to prove that \( dF(\lambda_0, u_0) : \mathbb{R} \times D^{1,2}(D^+) \to \mathbb{R} \times (D^{1,2}(D^+))^* \) is invertible. To this aim, by exploiting the compactness of the map \( D^{1,2}(D^+) \to (D^{1,2}(D^+))^* \), \( u \mapsto pu \), it is easy to prove that, if \( \mathcal{R} : (D^{1,2}(D^+))^* \to D^{1,2}(D^+) \) is the Riesz isomorphism and \( \text{Id}_x \) denotes the identity on \( \mathbb{R} \), then the operator \( (\text{Id}_x \times \mathbb{R}) \circ dF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times D^{1,2}(D^+)) \) is a compact perturbation of the identity. Therefore, from the Fredholm alternative, \( dF(\lambda_0, u_0) \) is invertible if and only if it is injective.

Let \( (\lambda, u) \in \mathbb{R} \times D^{1,2}(D^+) \) be such that \( dF(\lambda_0, u_0)(\lambda, u) \) vanishes, i.e.

\[
\begin{aligned}
2 \int_{D^+} \nabla u_0 \cdot \nabla u \, dx &= 0, \\
-\Delta u - \lambda pu_0 - \lambda puu &= 0,
\end{aligned}
\]

i.e.

\[
\int_{D^+} (\nabla u \cdot \nabla v - \lambda_0 puv - \lambda pu_0 v) \, dx = 0 \quad \text{for all } v \in D^{1,2}(D^+).
\]

Therefore,

\[
\lambda_0 \int_{D^+} puv_0 \, dx = \int_{D^+} \nabla u_0 \cdot \nabla u \, dx = 0
\]

and then, choosing \( v = u_0 \) in (109), we obtain that \( 0 = \lambda \int_{D^+} pu_0^2 \, dx = \lambda \). It follows that \( u \) is a weak \( D^{1,2}(D^+) \)-solution to \(-\Delta u = \lambda_0 pu \) in \( D^+ \). Since, by assumption (11), the eigenvalue \( \lambda_0 \) is simple on \( D^+ \), we conclude that \( u = \alpha u_0 \) for some \( \alpha \in \mathbb{R} \). From (110) it follows that
0 = \alpha \lambda_0 \int_{D^+} \mu_0^2 \, dx \quad \text{which implies} \quad \alpha = 0 \quad \text{and then} \quad u \equiv 0 \quad \text{in} \quad D^+. \quad \text{We conclude that} \quad dF(\lambda_0, u_0) \quad \text{is injective and then invertible.}

**Theorem 4.2.** Let \( \tilde{u}_{\epsilon,R} \) be as in definition [57]. Then, for every \( \epsilon > 0 \) and \( R > 2 \) there exist \( K(\epsilon,R), K(R) \in \mathbb{R} \) such that

\[
\epsilon^{-N/2} \| \tilde{u}_{\epsilon,R} - u_0 \|_{D^{1,2}(D^+)} \leq K(\epsilon,R),
\]

\[
\lim_{\epsilon \to 0^+} K(\epsilon,R) = K(R), \quad \text{for every} \quad R > 2, \quad \text{and} \quad \lim_{R \to +\infty} K(R) = 0.
\]

**Proof.** Let us fix \( R > 2 \) and notice that \( \tilde{u}_{\epsilon,R} \to u_0 \) in \( D^{1,2}(D^+) \) as \( \epsilon \to 0^+ \). Indeed, from [61], [62], [64], [65], and [17], we deduce that

\[
\int_{D^+} |\nabla (\tilde{u}_{\epsilon,R} - u_0)|^2 \, dx = \int_{D^+ \setminus B^+_{R_{\epsilon}}} |\nabla (u_0 - u_0)|^2 \, dx + \epsilon^N \int_{B^+_{R}} |\nabla (Z_{\epsilon} - u_0, \epsilon)|^2 \, dx = o(1)
\]

as \( \epsilon \to 0^+ \). Therefore

\[(111) \quad F(\lambda, \tilde{u}_{\epsilon,R}) = dF(\lambda_0, u_0)(\lambda - \lambda_0, \tilde{u}_{\epsilon,R} - u_0) + o(|\lambda - \lambda_0| + \|\tilde{u}_{\epsilon,R} - u_0\|), \quad \text{as} \quad \epsilon \to 0^+.
\]

In view of Lemma [4.1] the operator \( dF(\lambda_0, u_0) \) is invertible (and its inverse is continuous by the Open Mapping Theorem), then (111) implies that

\[(112) \quad |\lambda - \lambda_0| + \|\tilde{u}_{\epsilon,R} - u_0\|
\]

\[
\leq \| (dF(\lambda_0, u_0))^{-1} \|_{L^\infty([D^{1,2}(D^+) \setminus D^{1,2}(\{0\})])} \| \tilde{u}_{\epsilon,R} - u_0 \|
\]

\[
= \| (dF(\lambda_0, u_0))^{-1} \|_{L^\infty([D^{1,2}(D^+) \setminus D^{1,2}(\{0\})])} \| F(\lambda_0, \tilde{u}_{\epsilon,R}) \|_{L^\infty([D^{1,2}(D^+) \setminus D^{1,2}(\{0\})])}
\]

\[
(1 + o(1)) \quad \text{as} \quad \epsilon \to 0^+.
\]

In order to prove the theorem, we are going to estimate the norm of

\[(113) \quad F(\lambda, \tilde{u}_{\epsilon,R}) = (\mu_{\epsilon}, w_{\epsilon}) = \left( \|\tilde{u}_{\epsilon,R}\|^2 - \lambda_0, -\Delta \tilde{u}_{\epsilon,R} - \lambda_\epsilon \mu_{\epsilon,R} \right).
\]

As far as \( \mu_{\epsilon} \) is concerned, from Theorem [1.3] and [80] it follows that

\[(114) \quad \mu_{\epsilon} = \int_{D^+} |\nabla \tilde{u}_{\epsilon,R}|^2 \, dx - \lambda_0 = \int_{D^+ \setminus B_{R_{\epsilon}}} |\nabla u_{\epsilon}|^2 \, dx + \int_{B^+_{R_{\epsilon}}} |\nabla \tilde{u}_{\epsilon,R}|^2 \, dx - \lambda_0
\]

\[
\leq \int_{B^+_{R_{\epsilon}}} |\nabla \tilde{u}_{\epsilon,R}|^2 \, dx - \lambda_0 + \int_{D^+ \setminus B_{R_{\epsilon}}} |\nabla u_{\epsilon}|^2 \, dx + \lambda_0 - \lambda_0
\]

\[
= \alpha_{k,R}^2 + \lambda_0 = O(\epsilon^N), \quad \text{as} \quad \epsilon \to 0^+,
\]

where \( \alpha_{k,R}^2 \) is as in the proof of Theorem [5.1]. In particular \( \lim_{\epsilon \to 0^+} \epsilon^{-N/2} \mu_{\epsilon} = 0 \).

As far as \( w_{\epsilon} \) is concerned, we observe that, for every \( \varphi \in D^{1,2}(D^+) \),

\[
(D^{1,2}(D^+) \setminus D^{1,2}(\{0\})) \cdot (w_{\epsilon}, \varphi) \quad \text{on} \quad D^{1,2}(D^+) = \int_{D^+} \langle \nabla \tilde{u}_{\epsilon,R}, \nabla \varphi - \lambda_\epsilon \mu_{\epsilon,R} \varphi \rangle \, dx = \int_{\Gamma_{e,\epsilon}} \left( \frac{\partial \tilde{u}_{\epsilon,R}}{\partial \nu} - \frac{\partial u_{\epsilon}}{\partial \nu} \right) \varphi \, d\sigma.
\]

Thus, letting \( Z^R_{\epsilon} \) and \( U_{\epsilon} \) as in [62] and [61] respectively, we have that

\[
\epsilon^{-N/2} \| w_{\epsilon} \|_{D^{1,2}(D^+) \setminus D^{1,2}(\{0\})} = 1 \quad \epsilon^{-N/2} \sup_{\| \varphi \|=1} \varphi_{D^{1,2}(D^+) \setminus D^{1,2}(\{0\})} \int_{\Gamma_{e,\epsilon}} \left( \frac{\partial Z^R_{\epsilon}}{\partial \nu} - \frac{\partial U_{\epsilon}}{\partial \nu} \right) \varphi \, d\sigma
\]

\[
= \sup_{\| \varphi \|=1} \varphi_{D^{1,2}(D^+) \setminus D^{1,2}(\{0\})} \int_{\Gamma_{e,\epsilon}} \left( \frac{\partial Z^R_{\epsilon}}{\partial \nu} - \frac{\partial U_{\epsilon}}{\partial \nu} \right) \varphi \, d\sigma
\]

\[
= \sup_{\| \varphi \|=1} \varphi_{D^{1,2}(D^+) \setminus D^{1,2}(\{0\})} \int_{\Gamma_{e,\epsilon}} \nabla \varphi \left( Z^R_{\epsilon} - U_{\epsilon} \right) \cdot \nabla \varphi \, d\sigma
\]
where $\mathcal{T}_\varepsilon : D^{1,2}(D^+) \to D^{1,2}(D^+)$ is defined as $\mathcal{T}_\varepsilon(\varphi)(x) = \varepsilon \frac{x^+ - x^-}{\varepsilon(x + 1 - e_1)}$. Since $\mathcal{T}_\varepsilon$ is an isometry of $D^{1,2}(D^+)$, we deduce that

$$
\varepsilon^{-N/2}\|w_\varepsilon\|_{(D^{1,2}(D^+))^*} = \sup_{\varphi \in D^{1,2}(D^+)} \int_{B_R^+} \nabla(Z_{\varepsilon}^R - U_\varepsilon) \cdot \nabla \varphi \, dx.
$$

From the convergences (63) and (65) established in Lemma 2.11 it follows that

$$
\lim_{\varepsilon \to 0^+} \varepsilon^{-N/2}\|w_\varepsilon\|_{(D^{1,2}(D^+))^*} = \sup_{\varphi \in D^{1,2}(D^+)} \int_{B_R^+} \nabla(z_R - \Phi) \cdot \nabla \varphi \, dx.
$$

We observe that, for every $\varphi \in D^{1,2}(D^+)$ such that $\|\varphi\| = 1$, Lemma 2.2(iii) implies that

$$
\int_{B_R^+} \nabla(z_R - \Phi) \cdot \nabla \varphi \, dx
$$

(116)

$$
= \int_{B_R^+} \nabla((x_1 - 1)^+ - \Phi) \cdot \nabla \varphi \, dx + \int_{B_R^+} \nabla(z_R - (x_1 - 1)^+) \cdot \nabla \varphi \, dx
$$

$$
= - \int_{D^+ \setminus B_R^+} \nabla((x_1 - 1)^+ - \Phi) \cdot \nabla \varphi \, dx + \int_{B_R^+} \nabla(z_R - (x_1 - 1)^+) \cdot \nabla \varphi \, dx
$$

$$
\leq \left( \int_{D^+ \setminus B_R^+} \nabla((x_1 - 1)^+ - \Phi)^2 \, dx \right)^{1/2} + \left( \int_{B_R^+} \nabla(z_R - (x_1 - 1)^+)^2 \, dx \right)^{1/2}.
$$

Since $z_R - (x_1 - 1)^+$ is harmonic in $B_R^+$, $(z_R - (x_1 - 1)^+)|_{\Gamma^+} = (x_1 - 1)^+|_{\Gamma^+}$, and vanishes on $\partial B_R^+ \cap \partial D^+$, if $\eta_R$ is a smooth cut-off function satisfying (37), from the Dirichlet Principle, (26), and (24), we can estimate

$$
\int_{B_R^+} \nabla(z_R - (x_1 - 1)^+)^2 \, dx \leq \int_{B_R^+} \nabla(\eta_R(\Phi - (x_1 - 1)^+))^2 \, dx
$$

(117)

$$
\leq 2 \int_{B_R^+} \nabla^2 \eta_R(\Phi - (x_1 - 1)^+)^2 \, dx + 2 \int_{D^+ \setminus B_R^{N/2}} \eta_R^2 \nabla(\Phi - (x_1 - 1)^+)^2 \, dx
$$

$$
\leq \text{const } R^{-2} R^{2-2N} R^N + o(1) = o(1)
$$

as $R \to +\infty$. From (116), (117), and (23) we deduce that

$$
\lim_{R \to +\infty} \sup_{\varphi \in D^{1,2}(D^+)} \int_{B_R^+} \nabla(z_R - \Phi) \cdot \nabla \varphi \, dx = 0.
$$

The conclusion follows combining (18) with (112), (113), (114), (115), and (118). \qed

**Proof of Theorem 1.2.** Let $\delta > 0$. From Theorem 1.2 and (29), there exists $R_0 = R_0(\delta) > 2$ such that

$$
K^2(R_0) \in [0, \delta) \quad \text{and} \quad \left( \frac{\partial u_{\delta}(e_1)}{\partial x_1} \right)^2 \int_{D^+ \setminus B_R^+} \nabla(\Phi - (x_1 - 1))(x)^2 \, dx < \delta.
$$

(118)
From Theorem 4.2 it follows that
\[
\left| \frac{1}{\varepsilon} \int_{\Omega^+} |\nabla (u_\varepsilon - u_0)|^2 \, dx - \left( \frac{\partial u_\varepsilon}{\partial x_1} (e_1) \right)^2 \int_{\Omega} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx \right| \\
\leq \left( \frac{\partial u_\varepsilon}{\partial x_1} (e_1) \right)^2 \int_{\Omega^+ \setminus B_{R_0}} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx + \frac{1}{\varepsilon^N} \int_{\Omega^+ \setminus B_{R_0}} |\nabla (u_\varepsilon - u_0)|^2 \, dx \\
+ \frac{1}{\varepsilon^N} \int_{\Omega^+ \setminus B_{R_0}} |\nabla (u_\varepsilon - u_0)|^2 \, dx - \left( \frac{\partial u_\varepsilon}{\partial x_1} (e_1) \right)^2 \int_{\Omega} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx \\
\leq \delta + K^2(\varepsilon, R_0) \\
+ \frac{1}{\varepsilon^N} \int_{\Omega^+ \setminus B_{R_0}} |\nabla (u_\varepsilon - u_0)|^2 \, dx - \left( \frac{\partial u_\varepsilon}{\partial x_1} (e_1) \right)^2 \int_{\Omega} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx
\]
and hence, from Corollary 2.13 and Theorem 4.2, we deduce that there exists \( \varepsilon(\delta) > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon(\delta)) \),
\[
\left| \frac{1}{\varepsilon^N} \int_{\Omega^+} |\nabla (u_\varepsilon - u_0)|^2 \, dx - \left( \frac{\partial u_\varepsilon}{\partial x_1} (e_1) \right)^2 \int_{\Omega} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx \right| \leq 3\delta + K^2(R_0) < 4\delta,
\]
thus proving that
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\Omega^+} |\nabla (u_\varepsilon - u_0)|^2 \, dx = \left( \frac{\partial u_\varepsilon}{\partial x_1} (e_1) \right)^2 \int_{\Omega} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx.
\]
On the other hand, Lemma 2.2(iii), Corollary 2.3, and (4) imply that
\[
\int_{\Omega} |\nabla (\Phi - (x_1 - 1)^+)|^2 \, dx = \int_{\Omega} \Phi(1, x') \, dx = -2m(\Sigma) = \mathcal{C}(\Sigma).
\]
The proof is thereby complete. \( \square \)

5. The resonant case

In this section we drop assumption (12) and treat the case in which \( \lambda_0 \) is a double eigenvalue on \( D^+ \cup D^- \) and a simple eigenvalue on each of the components \( D^+ \) and \( D^- \). To this aim, we exploit twice the sharp asymptotics provided by Theorem 1.1 in two domains obtained by attaching small handles to each chamber \( D^+, D^- \).

Besides (2) and (3), we assume that \( p \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N) \) satisfies
\[
p \not\equiv 0 \text{ in } D^- \quad p \not\equiv 0 \text{ in } D^+ \quad p(x) = 0 \text{ for all } x \in B_{\varepsilon} \cup B_{\varepsilon}^+ 
\]
and that there exist \( k^+_0, k^-_0 \geq 1 \) such that
\[
(120) \quad \lambda_0 \in \sigma_p(D^+) \cap \sigma_p(D^-), \\
(121) \quad \lambda_0 = \lambda_{k^-_0}(D^-) \text{ is simple on } D^- \text{ and the corresponding eigenfunctions have in } e_1 \text{ a zero of order } 1, \\
(122) \quad \lambda_0 = \lambda_{k^+_0}(D^+) \text{ is simple on } D^+ \text{ and the corresponding eigenfunctions have in } 0 \text{ a zero of order } 1.
\]
Since \( \sigma_p(D^+ \cup D^-) = \sigma_p(D^+) \cup \sigma_p(D^-) \), (120), (121), and (122) imply that \( \lambda_0 \) is a double eigenvalue on \( D^+ \cup D^- \) and hence there exists \( k \geq 1 \) such that
\[
(123) \quad \lambda_0 = \lambda_k(D^+ \cup D^-) = \lambda_{k+1}(D^+ \cup D^-).
\]
Let \( u^+_0 \in D^{1,2}(D^+) \setminus \{0\} \) and \( u^-_0 \in D^{1,2}(D^-) \setminus \{0\} \) be the eigenfunctions associated to \( \lambda_0 \) on \( D^+ \) and \( D^- \) respectively, i.e. solving
\[
(124) \begin{cases} 
-\Delta u^+_0 = \lambda_0 p u^+_0, & \text{in } D^+, \\
u^+_0 = 0, & \text{on } \partial D^+,
\end{cases}
\quad \begin{cases} 
-\Delta u^-_0 = \lambda_0 p u^-_0, & \text{in } D^-, \\
u^-_0 = 0, & \text{on } \partial D^-,
\end{cases}
\]
such that
\[(125) \quad \frac{\partial u_0^+}{\partial x_1}(e_1) > 0, \quad \frac{\partial u_0^-}{\partial x_1}(0) < 0, \quad \int_{D^+} p(x) |u_0^+(x)|^2 \, dx = \int_{D^-} p(x) |u_0^-(x)|^2 \, dx = 1.\]

Let us introduce the following domains
\[(126) \quad D_\epsilon^+ = D^+ \cup \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : \frac{3}{4} \leq x_1 \leq 1, \frac{x'}{\epsilon} \in \Sigma \right\},\]
\[(127) \quad D_\epsilon^- = D^- \cup \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 \leq x_1 \leq \frac{1}{4}, \frac{x'}{\epsilon} \in \Sigma \right\},\]
\[(128) \quad \tilde{\Omega}^\epsilon = D_\epsilon^+ \cup D_\epsilon^- .\]

We observe that the asymptotics of eigenvalues stated in Theorem 1.1 can be proved, up to minor modifications, replacing the dumbbell perturbed domain $\tilde{\Omega}^\epsilon$ defined in (9) with either the domain $D_\epsilon^+$ or $D_\epsilon^-$; since the proof just relies on the attachment of a shrinking handle at a point in which the limit eigenfunction has a zero of order 1; therefore, arguing as in the proof of Theorem 1.1, we can prove that, under assumptions (2), (3), (119), (120), (121), and (122), there holds
\[(129) \quad \lambda_{k_0}^+(D_\epsilon^+) = \lambda_{k_0}^-(D_\epsilon^-) + \epsilon^N \mathcal{C}(\Sigma) \left( \frac{\partial u_0^-}{\partial x_1}(e_1) \right)^2 + o(\epsilon^N),\]
\[(130) \quad \lambda_{k_0}^-(D_\epsilon^-) = \lambda_{k_0}^-(D_\epsilon^-) + \epsilon^N \mathcal{C}(\Sigma) \left( \frac{\partial u_0^-}{\partial x_1}(0) \right)^2 + o(\epsilon^N),\]
as $\epsilon \to 0^+$, where $\mathcal{C}(\Sigma)$ is defined in (10).

Under the non-symmetry condition that the normal derivatives of the limit eigenfunctions at the two junctions are different, we observe that the double eigenvalue $\lambda_0$ is approximated by two different branches of eigenvalues in $\tilde{\Omega}^\epsilon$, see figure 3.

**Proposition 5.1.** Under assumptions (2), (3), (119), (120), (121), and (122), let $u_0^+$ and $u_0^-$ be as in (124) and (125), and let $D_\epsilon^+$, $D_\epsilon^-$ be as in (124), (127) respectively. If
\[(131) \quad \left\| \frac{\partial u_0^+}{\partial x_1}(e_1) \right\| > \left\| \frac{\partial u_0^-}{\partial x_1}(0) \right\|,\]
then, for $\epsilon$ sufficiently small,
\[
\lambda_{k_0}^+(D_\epsilon^+) = \lambda_k(\tilde{\Omega}^\epsilon) \quad \text{and} \quad \lambda_{k_0}^-(D_\epsilon^-) = \lambda_{k+1}(\tilde{\Omega}^\epsilon),
\]
where $\tilde{\Omega}^\epsilon$ is defined in (126) and $k$ is as in (129).

**Figure 3.** Two different branches of eigenvalues approximating the same double eigenvalue.

**Proof.** We note that $\sigma_p(\tilde{\Omega}_\epsilon) = \sigma_p(D_\epsilon^+) \cup \sigma_p(D_\epsilon^-)$. Expansions (129) and (130) together with assumptions (121), (122) yield
\[
\lambda_{k_0}^+(D_\epsilon^+) - \lambda_{k_0}^-(D_\epsilon^-) = \epsilon^N \mathcal{C}(\Sigma) \left( \left( \frac{\partial u_0^+}{\partial x_1}(0) \right)^2 - \left( \frac{\partial u_0^-}{\partial x_1}(e_1) \right)^2 \right) + o(\epsilon^N), \quad \text{as } \epsilon \to 0^+,
\]
and hence (131) implies that
\[
\lambda_{k_0}^+(D_\epsilon^+) - \lambda_{k_0}^-(D_\epsilon^-) < 0.
\]
for $\varepsilon$ sufficiently small, which gives the conclusion in view of the convergence of eigenvalues on $\tilde{\Omega}^\varepsilon$ proved by Daners in [15].

We now evaluate the difference between corresponding eigenvalues on the dumbbell domain $\tilde{\Omega}^\varepsilon$ and on the disconnected domain $\Omega^\varepsilon$.

**Lemma 5.2.** For $\ell = k, k + 1$, $\lambda_\ell(\tilde{\Omega}^\varepsilon) = \lambda_\ell(\Omega^\varepsilon) + O\left(\varepsilon^{\frac{N-k}{2}}e^{-\frac{\sqrt{\lambda_{k+1}^{\Omega^\varepsilon}}}{16\pi}}\right)$ as $\varepsilon \to 0^+$.

**Proof.** For the sake of brevity, we prove the lemma just for $\ell = k$; the proof for $\ell = k + 1$ is similar. By the Courant-Fisher *minimisation characterization* of eigenvalues we have that

$$
\lambda_k(\tilde{\Omega}^\varepsilon) = \min \left\{ \max_{u \in F \setminus \{0\}} \frac{\int_{\tilde{\Omega}^\varepsilon} |\nabla u|^2 \, dx}{\int_{\tilde{\Omega}^\varepsilon} pu^2 \, dx} : F \text{ is a subspace of } \mathcal{D}^{1,2}(\tilde{\Omega}^\varepsilon) \text{ such that dim } F = k \right\}.
$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be a smooth cut-off function such that

- $\eta \equiv 1$ in $\{(x_1, x') \in \mathbb{R}^N : x_1 \leq 1/8 \text{ or } x_1 \geq 7/8\}$,
- $\eta \equiv 0$ in $\{(x_1, x') \in \mathbb{R}^N : 1/4 \leq x_1 \leq 3/4\}$,
- $0 \leq \eta \leq 1$ in $\mathbb{R}^N$.

For every $j = 1, 2, \ldots, k$ and $\varepsilon$ small, we fix an eigenfunction $v_j^\varepsilon \in \mathcal{D}^{1,2}(\Omega^\varepsilon)$ associated to $\lambda_j(\Omega^\varepsilon)$ on $\Omega$ such that $\int_{\tilde{\Omega}^\varepsilon} p|v_j^\varepsilon|^2 \, dx = 1$ and $\int_{\tilde{\Omega}^\varepsilon} \nabla v_j^\varepsilon \cdot \nabla v_j^\varepsilon \, dx = 0$ if $i \neq j$. Choosing the $k$-dimensional space $F = \text{span}\{\eta v_1^\varepsilon, \ldots, \eta v_k^\varepsilon\}$ in (132) we obtain that

$$
\lambda_k(\tilde{\Omega}^\varepsilon) - \lambda_k(\Omega^\varepsilon) \leq \max_{\sum_{j=1}^k \alpha_j^2 = 1} \left\{ \frac{\int_{\tilde{\Omega}^\varepsilon} |\nabla (\sum_{j=1}^k \alpha_j \eta v_j^\varepsilon)|^2 \, dx}{\int_{\tilde{\Omega}^\varepsilon} p(\sum_{j=1}^k \alpha_j \eta v_j^\varepsilon)^2 \, dx} - \int_{\tilde{\Omega}^\varepsilon} |\nabla v_k^\varepsilon|^2 \, dx \right\}.
$$

From Corollary 2.7 it follows that, for every $j = 1, \ldots, k$,

$$
\int_{\tilde{\Omega}^\varepsilon} \int (\eta v_j^\varepsilon)^2 \, dx = 1 - \int_{\{(1/8,7/8) \times \{0\}\}} (1 - \eta^2)|v_j^\varepsilon|^2 \, dx = 1 + O\left(\varepsilon^{N-1}e^{-\frac{\sqrt{\lambda_{k+1}^{\Omega^\varepsilon}}}{16\pi}}\right) \text{ as } \varepsilon \to 0^+,
$$

whereas, exploiting the orthogonality of eigenfunctions, if $i \neq j$ we have that

$$
\int_{\tilde{\Omega}^\varepsilon} \int \eta v_i^\varepsilon v_j^\varepsilon \, dx = - \int_{\{(1/8,7/8) \times \{0\}\}} p(1 - \eta^2)v_i^\varepsilon v_j^\varepsilon \, dx = O\left(\varepsilon^{N-1}e^{-\frac{\sqrt{\lambda_{k+1}^{\Omega^\varepsilon}}}{16\pi}}\right) \text{ as } \varepsilon \to 0^+.
$$

From (133), (134), and (135) it follows that

$$
0 \leq \lambda_k(\tilde{\Omega}^\varepsilon) - \lambda_k(\Omega^\varepsilon)
$$

$$
\leq \max_{\sum_{j=1}^k \alpha_j^2 = 1} \left\{ \alpha_k^2 \left( \int_{\tilde{\Omega}^\varepsilon} |\nabla (\eta v_k^\varepsilon)|^2 \, dx - \int_{\tilde{\Omega}^\varepsilon} |\nabla v_k^\varepsilon|^2 \, dx \right) 
+ \sum_{j=1}^{k-1} \alpha_j^2 \left( \int_{\tilde{\Omega}^\varepsilon} |\nabla (\eta v_j^\varepsilon)|^2 \, dx - \int_{\tilde{\Omega}^\varepsilon} |\nabla v_j^\varepsilon|^2 \, dx \right) 
+ \sum_{i \neq j} \alpha_i \alpha_j \int_{\tilde{\Omega}^\varepsilon} \nabla (\eta v_i^\varepsilon) \cdot \nabla (\eta v_j^\varepsilon) \, dx \right\} + O\left(\varepsilon^{N-1}e^{-\frac{\sqrt{\lambda_{k+1}^{\Omega^\varepsilon}}}{16\pi}}\right), \text{ as } \varepsilon \to 0^+.
$$
From Corollary 2.7 it follows that
\[(137) \quad \int_{\tilde{\Omega}} |\nabla (\eta v^j_k)|^2 \, dx - \int_{\Omega^*} |\nabla v^j_k|^2 \, dx\]
\[= \int_{\tilde{\Omega}} \left( |\nabla \eta|^2 |v^j_k|^2 + \eta^2 |\nabla v^j_k|^2 + 2\eta v^j_k \nabla \eta \cdot \nabla v^j_k \right) \, dx - \int_{\Omega^*} |\nabla v^j_k|^2 \, dx\]
\[\leq O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right) + \int_{\tilde{\Omega}} (\eta^2 - 1) |\nabla v^j_k|^2 \, dx \leq O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right), \quad \text{as} \ \varepsilon \to 0^+,
\]
\[(138) \quad \int_{\tilde{\Omega}} |\nabla (\eta v^j_k)|^2 \, dx - \int_{\Omega^*} |\nabla v^j_k|^2 \, dx = \int_{\tilde{\Omega}} \eta^2 |\nabla v^j_k|^2 \, dx - \int_{\Omega^*} |\nabla v^j_k|^2 \, dx + O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right)\]
\[\leq \lambda_j(\Omega^*) - \lambda_k(\Omega^*) + O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right)\]
\[\leq O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right), \quad \text{as} \ \varepsilon \to 0^+,
\]
for all \( j < k \), and, if \( i \neq j \),
\[(139) \quad \int_{\tilde{\Omega}} \nabla (\eta v^j_k) \cdot \nabla (\eta v^i_k) \, dx\]
\[= \int_{\tilde{\Omega}} \eta^2 v^i_k \cdot \nabla v^j_k \, dx + \int_{\tilde{\Omega}} \eta \nabla \eta \cdot (v^i_k \nabla v^j_k + v^j_k \nabla v^i_k) \, dx + \int_{\tilde{\Omega}} |\nabla \eta|^2 v^i_k v^j_k \, dx\]
\[= \int_{(1/8,7/8) \times (\varepsilon \Sigma)} (\eta^2 - 1) \nabla v^i_k \cdot \nabla v^j_k \, dx + O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right)\]
\[= -2 \left| \int_{(1/8,7/8) \times (\varepsilon \Sigma)} \eta v^j_k \nabla v^i_k \cdot \nabla \eta \, dx + O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right) \right|\]
\[= O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right), \quad \text{as} \ \varepsilon \to 0^+,
\]
where in the third equality we have tested \(-\Delta v^i_k = \lambda^*_k \eta v^i_k\) with \((\eta^2 - 1) v^i_k \) for \( i \neq j \) and integrated by parts over \((1/8,7/8) \times (\varepsilon \Sigma)\), using assumption (119).

From (136), (137), (138), and (139) it follows that
\[0 \leq \lambda_k(\tilde{\Omega}^*) - \lambda_k(\Omega^*) \leq O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right), \quad \text{as} \ \varepsilon \to 0^+,
\]
thus yielding the conclusion. \( \square \)

**Remark 5.3.** We observe that, under assumption (151), expansions (129), (130), Proposition 5.1 and Lemma 5.2 imply that the splitting of the two subsequent eigenvalues \( \lambda_k(\tilde{\Omega}^*) \), \( \lambda_{k+1}(\Omega^*) \) approximating the same double eigenvalue \( \lambda_k(D^- \cup D^+) = \lambda_{k+1}(D^- \cup D^+) \) has a polynomial vanishing order, i.e.
\[\lambda_{k+1}(\Omega^*) - \lambda_k(\Omega^*) = \varepsilon^N \mathcal{C}(\Sigma) \left( \left( \frac{\partial u_0}{\partial x_1} (0,1) \right)^2 - \left( \frac{\partial u_0}{\partial x_1} (0) \right)^2 \right) + o(\varepsilon^N), \quad \text{as} \ \varepsilon \to 0^+.
\]
We emphasize that non-symmetry assumption (151) is crucial for having a polynomial splitting: indeed it was proved in [8] that in the case of a symmetric dumbbell domain the splitting of the first two eigenvalues vanishes with exponential rate.

Combining (129), (130) with Proposition 5.1 and Lemma 5.2 we derive the asymptotics of the eigenvalues \( \lambda_k(\tilde{\Omega}^*) \) and \( \lambda_{k+1}(\Omega^*) \) thus proving Theorem 1.3.

**Proof of Theorem 1.3.** From Proposition 5.1 and Lemma 5.2 we have that
\[\lambda_0 - \lambda_k(\tilde{\Omega}^*) = (\lambda_0 - \lambda_k(\tilde{\Omega}^*)) + (\lambda_k(\tilde{\Omega}^*) - \lambda_k(\Omega^*))\]
\[= (\lambda_0 - \lambda_{k+1}(D^+)) + O\left( \varepsilon^{\frac{N-1}{2}} e^{-\frac{\sqrt{\lambda_{k+1}(\Sigma)}}{2\varepsilon}} \right), \quad \text{as} \ \varepsilon \to 0^+,
\]
and
\[
\lambda_0 - \lambda_{k+1}(\Omega^e) = (\lambda_0 - \lambda_{k+1}(\Omega^e)) + (\lambda_{k+1}(\Omega^e) - \lambda_{k+1}(\Omega^e))
\]
\[
= (\lambda_0 - \lambda_{k+1}(D^e)) + O(\varepsilon^{N/2} + \varepsilon^{N}) , \quad \text{as } \varepsilon \rightarrow 0^+.
\]
Hence, by \[(129)\] and \[(130)\] we obtain
\[
\lambda_0 - \lambda_k(\Omega^e) = \varepsilon^N \mathsf{C}(\Sigma) \left( \left( \frac{\partial u^0}{\partial x_1}(e_1) \right)^2 \right) + o(\varepsilon^N) \quad \text{and} \quad \lambda_0 - \lambda_{k+1}(\Omega^e) = \varepsilon^N \mathsf{C}(\Sigma) \left( \left( \frac{\partial u_0}{\partial x_1}(0) \right)^2 \right) + o(\varepsilon^N)
\]
thus completing the proof. \(\square\)

A key ingredient for the proof of Theorem \[(14)\] is the following spectral estimate.

**Lemma 5.4.** Under the same assumptions as in Theorem \[(13)\] let
\[
L^+_\varepsilon := -\Delta - \lambda_{k+1}(\Omega^e)p : \mathcal{D}^{1,2}(D^e_\varepsilon) \rightarrow (\mathcal{D}^{1,2}(D^e_\varepsilon))^*,
\]
\[
L^-_\varepsilon := -\Delta - \lambda_k(\Omega^e)p : \mathcal{D}^{1,2}(D^-_\varepsilon) \rightarrow (\mathcal{D}^{1,2}(D^-_\varepsilon))^*,
\]
be defined as
\[
\langle (\mathcal{D}^{1,2}(D^e_\varepsilon)); (L^+_\varepsilon u, v) \rangle = \int_{D^e_\varepsilon} \nabla u \cdot \nabla v \, dx - \lambda_{k+1}(\Omega^e) \int_{D^e_\varepsilon} p uv \, dx, \quad u, v \in \mathcal{D}^{1,2}(D^e_\varepsilon),
\]
\[
\langle (\mathcal{D}^{1,2}(D^-_\varepsilon)); (L^-_\varepsilon u, v) \rangle = \int_{D^-_\varepsilon} \nabla u \cdot \nabla v \, dx - \lambda_k(\Omega^e) \int_{D^-_\varepsilon} p uv \, dx, \quad u, v \in \mathcal{D}^{1,2}(D^-_\varepsilon),
\]
where \((\mathcal{D}^{1,2}(D^e_\varepsilon))^*\) is the dual space of \(\mathcal{D}^{1,2}(D^e_\varepsilon)\) and \((\mathcal{D}^{1,2}(D^-_\varepsilon))^*\) is the dual space of \(\mathcal{D}^{1,2}(D^-_\varepsilon)\). Then, for \(\varepsilon\) sufficiently small, \(L^+_\varepsilon\) and \(L^-_\varepsilon\) are invertible and
\[
\| (L^+_\varepsilon)^{-1} \|_{\mathcal{L}(\mathcal{D}^{1,2}(D^e_\varepsilon)^*, \mathcal{D}^{1,2}(D^e_\varepsilon))} = O(\varepsilon^{-N}), \quad \text{as } \varepsilon \rightarrow 0^+,
\]
\[
\| (L^-_\varepsilon)^{-1} \|_{\mathcal{L}(\mathcal{D}^{1,2}(D^-_\varepsilon)^*, \mathcal{D}^{1,2}(D^-_\varepsilon))} = O(\varepsilon^{-N}), \quad \text{as } \varepsilon \rightarrow 0^+.
\]

**Proof.** From \[(129), (130),\] and Theorem \[(13)\] it follows that, for \(\varepsilon\) sufficiently small,
\[
\| (L^+_\varepsilon)^{-1} \|_{\mathcal{L}(\mathcal{D}^{1,2}(D^e_\varepsilon)^*, \mathcal{D}^{1,2}(D^e_\varepsilon))} \leq \sup_{\lambda \in \sigma_p(D^e_\varepsilon)} \frac{\lambda}{|\lambda - \lambda_{k+1}(\Omega^e)|} \leq 1 + \frac{\lambda}{\text{dist}(\lambda_{k+1}(\Omega^e), \sigma_p(D^e_\varepsilon))} = O(\varepsilon^{-N}), \quad \text{as } \varepsilon \rightarrow 0^+,
\]
\[
\| (L^-_\varepsilon)^{-1} \|_{\mathcal{L}(\mathcal{D}^{1,2}(D^-_\varepsilon)^*, \mathcal{D}^{1,2}(D^-_\varepsilon))} \leq \sup_{\lambda \in \sigma_p(D^-_\varepsilon)} \frac{\lambda}{|\lambda - \lambda_k(\Omega^e)|} \leq 1 + \frac{\lambda}{\text{dist}(\lambda_k(\Omega^e), \sigma_p(D^-_\varepsilon))} = O(\varepsilon^{-N}), \quad \text{as } \varepsilon \rightarrow 0^+.
\]

The proof is thereby complete. \(\square\)

**Proof of Theorem \[(14)\].** Due to simplicity of the eigenvalue \(\lambda_0 = \lambda_{k+1}^+(D^e) = \lambda_{k}^-(D^-)\) on each component \(D^-\) and \(D^+\), it is is enough to prove estimates \[(33)\] and \[(33)\] for any family
of eigenfunctions $v_k^\varepsilon \in \mathcal{D}^{1,2}(\Omega^\varepsilon)$ on $\Omega^\varepsilon$ associated to $\lambda_k(\Omega^\varepsilon)$ and any family of eigenfunctions $v_{k+1}^\varepsilon \in \mathcal{D}^{1,2}(\Omega^\varepsilon)$ on $\Omega^\varepsilon$ associated to $\lambda_{k+1}(\Omega^\varepsilon)$ such that
\[ \int_{\Omega^\varepsilon} p(x)|v_k^\varepsilon(x)|^2 \, dx = 1, \quad \int_{\Omega^\varepsilon} p(x)|v_{k+1}^\varepsilon(x)|^2 \, dx = 1. \]

We prove only (23), being the proof of (24) analogous. Let $\eta \in C^\infty(\mathbb{R}^N)$ be a smooth cut-off function such that
\[ \eta \equiv 1 \text{ in } \{(x_1, x') \in \mathbb{R}^N : x_1 \leq 1/8\}, \quad \eta \equiv 0 \text{ in } \{(x_1, x') \in \mathbb{R}^N : x_1 \geq 1/4\}, \quad 0 \leq \eta \leq 1 \text{ in } \mathbb{R}^N. \]

A direct computation shows that, letting $L_\varepsilon$ as in Lemma 5.4,
\[ L_\varepsilon(v_k^\varepsilon) = h_\varepsilon \]
where $h_\varepsilon \in (\mathcal{D}^{1,2}(D_\varepsilon^-))^*$ acts as
\[
(D^{1,2}(D_\varepsilon^-))^* (h_\varepsilon, v)_{D^{1,2}(D_\varepsilon^-)} = \int_{D_\varepsilon^-} (\Delta \eta v + 2 \nabla \eta \cdot \nabla v) v_k^\varepsilon \, dx, \quad \text{for every } v \in D^{1,2}(D_\varepsilon^-).
\]

From Corollary 2.7 it follows that
\[ \|h_\varepsilon\|_{(D^{1,2}(D_\varepsilon^-))^*} = O(e^{\frac{N-1}{2} e^{-\frac{\sqrt{\lambda}}{\varepsilon^\alpha}}}), \quad \text{as } \varepsilon \to 0^+, \]
and then Lemma 5.4 implies that
\[
\|\eta v_k^\varepsilon\|_{D^{1,2}(D_\varepsilon^-)} = \|(L_\varepsilon)v_k^\varepsilon\|_{D^{1,2}(D_\varepsilon^-)} \leq \|(L_\varepsilon)\|_{(D^{1,2}(D_\varepsilon^-))^* \to D^{1,2}(D_\varepsilon^-)} \|h_\varepsilon\|_{(D^{1,2}(D_\varepsilon^-))^*},
\]
\[ = O(e^{-N})O(e^{\frac{N-1}{2} e^{-\frac{\sqrt{\lambda}}{\varepsilon^\alpha}}}) = O(e^{\frac{N-1}{2} e^{-\frac{\sqrt{\lambda}}{\varepsilon^\alpha}}}), \quad \text{as } \varepsilon \to 0^+.
\]
Estimate (23) is thereby proved. \(\square\)

**Proof of Theorem 1.5** We prove only (24), being the proof of (25) analogous. To this aim, we first observe that, in view of the simplicity of the eigenvalue $\lambda_{k_0}^+(D^+)\) and by Theorem 1.2 (adapted to the easier case of the perturbed domain $D_\varepsilon^+$), there exists a family of eigenfunctions $v_\varepsilon^+ \in D^{1,2}(D_\varepsilon^+)$ on $D_\varepsilon^+$ associated to $\lambda_{k_0}^+(D_\varepsilon^+)$ such that
\[
\begin{cases}
-\Delta v_\varepsilon^+ = \lambda_{k_0}^+(D_\varepsilon^+)p v_\varepsilon^+, & \text{in } D_\varepsilon^+, \\
v_\varepsilon^+ = 0, & \text{on } \partial D_\varepsilon^+
\end{cases}
\]
and
\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-N} \|v_\varepsilon^+ - \eta_\varepsilon^+\|^2_{D^{1,2}(D_\varepsilon^+)} = \left(\frac{\partial u_0^+}{\partial x_1}(e_1)\right)^2 \mathcal{E}(\Sigma).
\]

In view of (122), Theorem 1.3 and Corollary 2.7, to prove (24) it is enough to show that
\[ \|\eta v_k^\varepsilon - v_k^\varepsilon\|^2_{D^{1,2}(D_\varepsilon^+)} = o(\varepsilon^N), \quad \text{as } \varepsilon \to 0^+, \]
for some $\eta \in C^\infty(\mathbb{R}^N)$ being a smooth cut-off function such that
\[ \eta \equiv 1 \text{ in } \{(x_1, x') \in \mathbb{R}^N : x_1 \geq 7/8\}, \quad \eta \equiv 0 \text{ in } \{(x_1, x') \in \mathbb{R}^N : x_1 \leq 3/4\}, \quad 0 \leq \eta \leq 1 \text{ in } \mathbb{R}^N.
\]
To prove (143), we argue as in section 4 and consider the operator
\[ F_\varepsilon : \mathbb{R} \times \mathcal{D}^{1,2}(D_\varepsilon^+) \to \mathbb{R} \times (\mathcal{D}^{1,2}(D_\varepsilon^+))^* \]
\[ (\lambda, u) \mapsto (\|u\|^2_{D^{1,2}(D_\varepsilon^+)} - \lambda \eta_\varepsilon^+(D_\varepsilon^+), -\Delta u - \lambda p u), \]
where, for all $u \in \mathcal{D}^{1,2}(D_\varepsilon^+)$,
\[ -\Delta u - \lambda p u \in (\mathcal{D}^{1,2}(D_\varepsilon^+))^* \text{ acts as } \]
\[ \langle -\Delta u - \lambda p u, v \rangle_{D^{1,2}(D_\varepsilon^+)} = \int_{D_\varepsilon^+} \nabla u(x) \cdot \nabla v(x) \, dx - \lambda \int_{D_\varepsilon^+} p(x)u(x)v(x) \, dx. \]
We observe that $F_\varepsilon(\lambda_{k_0}^+(D_\varepsilon^+), v_\varepsilon^+) = (0, 0)$ and $F_\varepsilon$ is Frechét-differentiable at $(\lambda_{k_0}^+(D_\varepsilon^+), v_\varepsilon^+)$; moreover, arguing as in the proof of Lemma 4.4, we can prove that, since $\lambda_{k_0}^+(D_\varepsilon^+)$ is simple per small $\varepsilon$, the Frechét-differential $dF_\varepsilon(\lambda_{k_0}^+(D_\varepsilon^+), v_\varepsilon^+) \in \mathcal{L}(\mathbb{R} \times \mathcal{D}^{1,2}(D_\varepsilon^+), \mathbb{R} \times (\mathcal{D}^{1,2}(D_\varepsilon^+))^*)$ is
invertible. Due to the fact that $\lambda_{k_0^+}(D^+_\varepsilon)$ converges to a simple eigenvalue on $D^+$, it is also easy to verify that
\begin{equation}
(144) \quad \|(dF_\varepsilon(\lambda_{k_0^+}(D^+_\varepsilon), v^+_\varepsilon))^{-1}\|_{L(R \times (D^{1,2}(D^+_\varepsilon))^*, R \times D^{1,2}(D^+_\varepsilon))} = O(1), \quad \text{as } \varepsilon \to 0^+.
\end{equation}
Since
\begin{equation}
F_\varepsilon(\lambda_k(\Omega^\varepsilon), \eta v^\varepsilon) = dF_\varepsilon(\lambda_{k_0^+}(D^+_\varepsilon), v^+_\varepsilon)(\lambda_k(\Omega^\varepsilon) - \lambda_{k_0^+}(D^+_\varepsilon), \eta v^\varepsilon - v^+_\varepsilon)
+ o(\|\lambda_k(\Omega^\varepsilon) - \lambda_{k_0^+}(D^+_\varepsilon)\| + \|\eta v^\varepsilon - v^+_\varepsilon\|_{D^{1,2}(D^+_\varepsilon)}), \quad \text{as } \varepsilon \to 0^+,
\end{equation}
from (144) we deduce that, for $\varepsilon$ small,
\begin{equation}
(145) \quad |\lambda_k(\Omega^\varepsilon) - \lambda_{k_0^+}(D^+_\varepsilon)| + \|\eta v^\varepsilon - v^+_\varepsilon\|_{D^{1,2}(D^+_\varepsilon)} \leq \text{const} \|F_\varepsilon(\lambda_k(\Omega^\varepsilon), \eta v^\varepsilon)\|_{R \times (D^{1,2}(D^+_\varepsilon))^*}.
\end{equation}
We have that
\begin{equation}
F_\varepsilon(\lambda_k(\Omega^\varepsilon), \eta v^\varepsilon) = (\mu_\varepsilon, w_\varepsilon) = \left(\|\eta v^\varepsilon\|_{D^{1,2}(D^+_\varepsilon)}^2 - \lambda_{k_0^+}(D^+_\varepsilon), -\Delta(\eta v^\varepsilon) - \lambda_k(\Omega^\varepsilon)p(\eta v^\varepsilon)\right).
\end{equation}
By direct calculations and using Proposition 5.1, Lemma 5.2, Corollary 2.7, and Theorem 1.4, we have that
\begin{equation}
(146) \quad \mu_\varepsilon = \|\eta v^\varepsilon\|_{D^{1,2}(D^+_\varepsilon)}^2 - \lambda_{k_0^+}(D^+_\varepsilon) = \lambda_k(\Omega^\varepsilon) - \lambda_{k_0^+}(D^+_\varepsilon) + \int_{\Omega^\varepsilon} |\nabla \eta|^2 |v^\varepsilon|^2 dx
+ 2 \int_{\Omega^\varepsilon} v^\varepsilon \eta \nabla \eta \cdot \nabla v^\varepsilon dx + \int_{D^+\setminus\{(0,7/8)\times(\varepsilon \Sigma)}} (\eta^2 - 1) |\nabla v^\varepsilon|^2 dx
= \lambda_k(\Omega^\varepsilon) - \lambda_{k_0^+}(D^+_\varepsilon) + \int_{D^+\setminus\{(3/4,7/8)\times(\varepsilon \Sigma)}} |\nabla \eta|^2 |v^\varepsilon|^2 dx
+ \lambda_k(\Omega^\varepsilon) \int_{D^+\setminus\{(0,7/8)\times(\varepsilon \Sigma)}} p(\eta^2 - 1) |v^\varepsilon|^2 dx = o(\varepsilon^N), \quad \text{as } \varepsilon \to 0^+.
\end{equation}
In order to estimate $\|w_\varepsilon\|_{(D^{1,2}(D^+_\varepsilon))^*}$, we observe that $w_\varepsilon \in (D^{1,2}(D^+_\varepsilon))^*$ acts as
\begin{equation}
(147) \quad w_\varepsilon \cdot (w_\varepsilon, v)_{D^{1,2}(D^+_\varepsilon)} = \int_{D^+} (\Delta \eta v + 2 \nabla \eta \cdot \nabla v) v^\varepsilon dx, \quad \text{for every } v \in D^{1,2}(D^+_\varepsilon).
\end{equation}
From Corollary 2.7, it follows that
\begin{equation}
(148) \quad \|w_\varepsilon\|_{(D^{1,2}(D^+_\varepsilon))^*} = o(\varepsilon^N), \quad \text{as } \varepsilon \to 0^+.
\end{equation}
Combining (145), (146), (147), and (148), we obtain (133), thus completing the proof. \hfill \Box

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