Small Littlewood-Richardson coefficients

Christian Ikenmeyer*

May 2, 2014

Abstract
We develop structural insights into the Littlewood-Richardson graph, whose number of vertices equals the Littlewood-Richardson coefficient \( c^\nu_{\lambda, \mu} \) for given partitions \( \lambda, \mu \) and \( \nu \). This graph was first introduced in [BI12], where its connectedness was proved.

Our insights are useful for the design of algorithms for computing the Littlewood-Richardson coefficient: We design an algorithm for the exact computation of \( c^\nu_{\lambda, \mu} \) with running time \( O((c^\nu_{\lambda, \mu})^2 \cdot \text{poly}(n)) \), where \( \lambda, \mu, \) and \( \nu \) are partitions of length at most \( n \). Moreover, we introduce an algorithm for deciding whether \( c^\nu_{\lambda, \mu} \geq t \) whose running time is \( O(t^2 \cdot \text{poly}(n)) \). Even the existence of a polynomial-time algorithm for deciding whether \( c^\nu_{\lambda, \mu} \geq 2 \) is a nontrivial new result on its own.

Our insights also lead to the proof of a conjecture by King, Tollu, and Toumazet posed in [KTT04], stating that \( c^\nu_{\lambda, \mu} = 2 \) implies \( c^{M\nu}_{M\lambda, M\mu} = M + 1 \) for all \( M \in \mathbb{N} \). Here, the stretching of partitions is defined componentwise.

2010 MSC: 05E10, 22E46, 90C27

Keywords: Littlewood-Richardson coefficient, hive model, efficient algorithms, flows in networks

Contents

1 Introduction 2
2 Flow description of LR coefficients 3
  2.1 Flows on digraphs 3
  2.2 Flows on the honeycomb graph \( G \) 6
  2.3 Hive flows 6
  2.4 The LR graph and secure cycles 7
3 Enumerating hive flows 7
4 The residual digraph \( R_f \) 9
5 The Neighbourhood Generator 10
  5.1 A first approach 11
  5.2 Bypassing the secure extension problem 12
6 The Shortest Turncycle Theorem 15
7 Proof of the King-Tollu-Toumazet Conjecture 16
  7.1 Proof of Proposition 7.3 18
  7.2 Proof of Key Lemma 7.4 21
  7.3 Proof of Lemma 7.11 22

*University of Paderborn, Germany, Email address: ciken@math.upb.de
1 Introduction

Let $\lambda, \mu, \nu \in \mathbb{Z}^n$ be nonincreasing $n$-tuples of integers. The Littlewood-Richardson coefficient $c_{\lambda,\mu}^\nu$ of $\lambda$, $\mu$ and $\nu$ is defined as the multiplicity of the irreducible $\text{GL}_n(\mathbb{C})$-representation $V_\nu$ with dominant weight $\nu$ in the tensor product $V_\lambda \otimes V_\mu$. These coefficients appear not only in representation theory and algebraic geomterics, but also in topology and enumerative geometry.

Different combinatorial characterizations of the Littlewood-Richardson coefficients are known. The classic Littlewood-Richardson rule (cf. [Ful97]) counts certain skew tableaux, while in [BZ92] the number of integer points of certain polytopes are counted. A beautiful characterization was given by Knutson and Tao [KT99], who characterized Littlewood-Richardson coefficients either as the number of honeycombs or hives with prescribed boundary conditions.

The focus of this paper is on the complexity of computing the Littlewood-Richardson coefficient $c_{\lambda,\mu}^\nu$ on input $\lambda, \mu, \nu$. Without loss of generality we assume that the components of $\lambda, \mu, \nu$ are nonnegative integers and put $|\lambda| := \sum \lambda_i$. Moreover we write $\ell(\lambda)$ for the number of nonzero components of $\lambda$. Then $|\nu| = |\lambda| + |\mu|$ and $\nu_i \geq \max\{\lambda_i, \mu_i\}$ are necessary conditions for $c_{\lambda,\mu}^\nu > 0$. All known algorithms for computing Littlewood-Richardson coefficients take exponential time in the worst case. Narayanan [Nar06] proved that this is unavoidable: the computation of $c_{\lambda,\mu}^\nu$ is a $\#P$-complete problem. Hence there does not exist a polynomial time algorithm for computing $c_{\lambda,\mu}^\nu$ under the widely believed hypothesis $P \neq NP$.

Main results. This is a follow-up paper to [BI12]. We use the characterization of $c_{\lambda,\mu}^\nu$ as the number of hive flows with prescribed border throughput on the honeycomb graph $G$, cf. Figures 1–2. Besides capacity constraints given by $\lambda, \mu, \nu$, these flows have to satisfy rhombus inequalities corresponding to the ones considered in [KT99, Buc00].

The integral hive flows form the integral points of the hive flow polytope $P(\lambda, \mu, \nu)$. The vertices of this polytope, together with edges given by cycles on the honeycomb graph $G$, form an undirected graph, whose connectedness was proved in [BI12], see Section 2.3. To compute $c_{\lambda,\mu}^\nu$, we design a variant of breadth-first-search that lists all points in $P(\lambda, \mu, \nu)$ with only polynomial delay between the single outputs. This enables us to decide $c_{\lambda,\mu}^\nu \geq t$ in time $O(t^2 \cdot \text{poly}(n))$, see Theorem 5.2 where $\text{poly}(n)$ stands for a function that is polynomially bounded in $n$. Even the polynomial time algorithm for deciding $c_{\lambda,\mu}^\nu \geq 2$ is a new and nontrivial result. Also we get an algorithm for computing $c_{\lambda,\mu}^\nu$ which runs in time $O((c_{\lambda,\mu}^\nu)^2 \cdot \text{poly}(n))$, see Theorem 5.3. All algorithms in this paper only use addition, multiplication, and comparison and the running time is defined to be the number of these operations.

With only minor modifications our algorithms can be used to enumerate efficiently all hive flows corresponding to a given tensor product $V_\lambda \otimes V_\mu$, as asked in [KT04, p. 186].

Our algorithms are implemented and available. We encourage the reader to try out our Java applet at http://www-math.upb.de/agpb/flowapplet/flowapplet.html

Moreover, our insights into the structure of the hive flow polytope lead to the proof of the following conjecture of King, Tollu, and Toumazet posed in [KT04].

1.1 Theorem. Given partitions $\lambda$, $\mu$ and $\nu$ such that $|\nu| = |\lambda| + |\mu|$. Then $c_{\lambda,\mu}^\nu = 2$ implies $c_{M\lambda,\mu}^{M\nu} = M + 1$ for all $M \in \mathbb{N}$.

We remark that [KT04, Conj. 3.3] also contains the conjecture that $c_{\lambda,\mu}^\nu = 3$ implies either $c_{M\lambda,\mu}^{M\nu} = 2M + 1$ or $c_{M\lambda,\mu}^{M\nu} = (M + 1)(M + 2)/2$. We think that a careful refinement of the methods in this paper can be used to prove this and similar conjectures as well.
Several sections of this paper have a large overlap with [BI12]. Most of the original content appears in Sections 3, 5, and 7. The reader accustomed to the arguments in [BI12] will find it easier to understand this follow-up paper. The Sections 2, 5 use results from [BI12], but when such results are used, then they are explicitly stated, so that these sections can be studied without reading [BI12]. Section 6 however needs a profound understanding of the proof of Theorem 4.8 in [BI12]. Section 7 is nearly self-contained, except for some references to Section 6.

Acknowledgements I benefitted tremendously from the long, intense, and invaluable discussions with my PhD thesis advisor Peter Bürgisser. Furthermore, I thank the Deutsche Forschungsgemeinschaft for their financial support (DFG-grants BU 1371/3-1 and BU 1371/3-2).

2 Flow description of LR coefficients

2.1 Flows on digraphs

We fix some terminology regarding flows on directed graphs, compare [AMO93]. Let $D$ be a digraph with vertex set $V(D)$ and edge set $E(D)$. Let $e_{\text{start}}$ denote the vertex where the edge $e$ starts and $e_{\text{end}}$ the vertex where $e$ ends. The inflow and outflow of a map $f: E(D) \to \mathbb{R}$ at a vertex $v \in V(D)$ are defined as

$$\text{inflow}(v, f) := \sum_{e_{\text{end}} = v} f(e), \quad \text{outflow}(v, f) := \sum_{e_{\text{start}} = v} f(e),$$

respectively. A flow on $D$ is defined as a map $f: E(D) \to \mathbb{R}$ that satisfies Kirchhoff’s conservation laws: $	ext{inflow}(v, f) = \text{outflow}(v, f)$ for all $v \in V(D)$.

The set of flows on $D$ is a vector space that we denote by $\mathcal{F}(D)$. A flow is called integral if it takes only integer values and we denote by $\mathcal{F}(D)_{\mathbb{Z}}$ the group of integral flows on $D$.

A path $p$ in $D$ is defined as a sequence $x_0, \ldots, x_\ell$ of pairwise distinct vertices of $D$ such that $(x_{i-1}, x_i) \in E$ for all $1 \leq i \leq \ell$. A sequence $x_0, \ldots, x_\ell$ of vertices of $D$ is called a cycle $c$ if $x_0, \ldots, x_{\ell-1}$ are pairwise distinct, $x_\ell = x_0$, and $(x_{i-1}, x_i) \in E$ for all $1 \leq i \leq \ell$. It will be sometimes useful to identify a path or a cycle with the subgraph consisting of its vertices $x_0, x_1, \ldots, x_\ell$ and edges $(x_0, x_1), \ldots, (x_{\ell-1}, x_\ell)$. Since the starting vertex $x_0$ of a cycle is not relevant, this does not harm.

A cycle $c$ in $D$ defines a flow $f$ on $D$ by setting $f(e) := 1$ if $e \in c$ and $f(e) := 0$ otherwise. It will be convenient to denote this flow with $c$ as well. A flow is called nonnegative if $f(e) \geq 0$ for all edges $e \in E$.

We will study flows in two rather different situations. The residual digraph $R$ introduced in Section 4 has the property that it never contains an edge $(u, v)$ and its reverse edge $(v, u)$. Only nonnegative flows on $R$ will be of interest.

On the other hand, we also need to look at flows on digraphs resulting from an undirected graph $G$ by replacing each of its undirected edges $\{u, v\}$ by the directed edge $e := (u, v)$ and its reverse $-e := (v, u)$. We shall denote the resulting digraph also by $G$. To a flow $f$ on $G$ we assign its reduced representative $\bar{f}$ defined by $\bar{f}(e) := f(e) - f(-e) \geq 0$ and $\bar{f}(-e) := 0$ if $f(e) \geq f(-e)$, and setting $\bar{f}(e) := 0$ and $\bar{f}(-e) := f(-e) - f(e)$ if $f(e) < f(-e)$. It will be convenient to interpret $f$ and $\bar{f}$ as manifestations of the same flow: Formally, we consider the linear subspace $N(G) := \{ f \in \mathbb{R}^{E(G)} \mid \forall e \in E(G) : f(e) = f(-e) \}$ of “null flows” and the factor space

$$\mathcal{F}(G) := \mathcal{F}(G)/N(G).$$
We call the elements of \( \mathcal{F}(G) \) flow classes on \( G \) (or simply flows) and denote them by the same symbols as for flows. No confusion should arise from this abuse of notation in the context at hand. We usually identify flow classes with their reduced representative. A flow class is called integral if its reduced representative is integral and we denote by \( \mathcal{F}(G)_Z \) the group of integral flow classes on \( G \).

### 2.2 Flows on the honeycomb graph \( G \)

We start with a triangular array of vertices, \( n+1 \) on each side, as seen in Figure 1(a). The resulting planar graph \( \Delta \) shall be called the triangular graph with parameter \( n \), we denote its vertex set with \( V(\Delta) \) and its edge set with \( E(\Delta) \). A triangle consisting of three edges in \( \Delta \) is called a hive triangle. Note that there are two types of hive triangles: upright (‘\( \triangle \)’) and downright oriented ones (‘\( \nabla \)’). A rhombus is defined to be the union of an upright and a downright hive triangle which share a common side. In contrast to the usual geometric definition of the term rhombus we use this term here in this very restricted sense only. Note that the angles at the corners of a rhombus are either acute of \( 60^\circ \) or obtuse of \( 120^\circ \). Two distinct rhombi are called overlapping if they share a hive triangle.

To realize the dual graph of \( \Delta \), as in [Buc00], we introduce a black vertex in the middle of each hive triangle and a white vertex on each hive triangle side, see Figure 1(b). Moreover, in each hive triangle \( T \), we introduce edges connecting the three white vertices of \( T \) with the black vertex. Moreover, we add a single black vertex to the outer face and connect it with each white vertex at the border of \( \Delta \). Clearly, the resulting (undirected) graph \( G \) is bipartite and planar. We shall call \( G \) the honeycomb graph with parameter \( n \). The cycles on \( G \) that do not pass the outer vertex are called proper cycles.

We study now the vector space \( \mathcal{F}(G) \) of flow classes on \( G \) introduced in Section 2.1. Recall that for this, we have to replace each edge of \( G \) by the corresponding two directed edges. Correspondingly, we will consider \( G \) as a directed graph. Since any cycle \( c \) in the digraph \( G \) defines a flow, it also defines a flow class on \( G \) that we denote by \( c \) as well.

### 2.1 Definition (Throughput)

For an edge \( k \in E(\Delta) \) let \( e \in E(G) \) denote the edge pointing from the white vertex on \( k \) into the upright triangle. The throughput \( \delta(k, f) \) w.r.t. a flow \( f \in \mathcal{F}(G) \) is defined as \( f(e) - f(-e) \).

In this paper, it will be extremely helpful to have some graphical way of describing rhombi and throughputs. We shall denote a rhombus \( \rho \) by the pictogram
\(\diamond\), even though \(g\) may lie in any of the three positions “\(\diamond\)”, “\(\square\)” or “\(\triangle\)” obtained by rotating with a multiple of 60°. Let \(\hat{k}\) denote the edge \(k\) of \(\Delta\) given by the diagonal of \(g\) connecting its two obtuse angles. Then we denote by \(\hat{\delta}(f) := \delta(k, f)\) the throughput of \(f\) through \(k\) (going into the upright hive triangle). Similarly, we define the throughput \(\hat{-\delta}(f) := -\delta(k, f)\). The advantage of this notation is that if the throughput is positive, then the flow goes in the direction of the arrow. For instance, using the symbolic notation, we note the following consequence of the flow conservation laws:

\[
\hat{\psi}(f) + \hat{-\psi}(f) = \hat{\xi}(f) + \hat{-\xi}(f).
\]

### 2.3 Hive flows

All definitions up to this point were fairly standard. We describe now the relation to the Littlewood-Richardson coefficients.

#### 2.2 Definition.

The slack of the rhombus \(\diamond\) with respect to \(f \in F(G)\) is defined as

\[
\sigma(\diamond, f) := \hat{\psi}(f) + \hat{-\psi}(f).
\]

The rhombus \(\diamond\) is called \(f\)-flat if \(\sigma(\diamond, f) = 0\). ■

It is clear that \(F(G) \to \mathbb{R}, f \mapsto \sigma(g, f)\) is a linear form. Note also that by flow conservation, the slack can be written in various different ways:

\[
\sigma(\diamond, f) = \hat{\psi}(f) + \hat{-\psi}(f) = \hat{\xi}(f) - \hat{-\xi}(f) = \hat{\psi}(f) - \hat{-\xi}(f) = \hat{-\psi}(f) + \hat{\xi}(f).
\]

For calculating the slack values of rhombi, the **hexagon equality** described below will be useful. The straightforward proof is omitted.

#### 2.3 Claim (Hexagon equality).

The union of two overlapping rhombi \(\diamond_1\) and \(\diamond_2\) forms a trapezoid. Glueing together two such trapezoids \((\diamond_1, \diamond_2)\) and \((\diamond_1', \diamond_2')\) at their longer side, we get a hexagon. We have \(\sigma(\diamond_1, f) + \sigma(\diamond_2, f) = \sigma(\diamond_1', f) + \sigma(\diamond_2', f)\) for each flow \(f \in F(G)\). In pictorial notation, the hexagon equality can be succinctly expressed as

\[
\sigma(\diamond_1, f) + \sigma(\diamond_2, f) = \sigma(\diamond_1', f) + \sigma(\diamond_2', f).
\]

#### 2.4 Definition.

A flow \(f \in F(G)\) is called a **hive flow** iff for all rhombi \(g\) we have \(\sigma(g, f) \geq 0\). ■

Note that the set of hive flows is a cone in \(F(G)\). Figure 2 provides an example of a hive flow.

**Figure 2:** A hive flow for \(n = 3, \lambda = (4, 2, 0), \mu = (5, 2, 0), \nu = (6, 4, 3)\). The numbers give the throughputs through edges of \(\Delta\) in the directions of the arrows. The diagonals of rhombi with positive slack are drawn thicker. In this example, the slack of the rhombi ranges from 0 to 2.
We now fix the throughputs at the border of $\Delta$, depending on a chosen triple $\lambda, \mu, \nu \in \mathbb{N}^n$ of partitions satisfying $|\nu| = |\lambda| + |\mu|$. To the $i$th border edge $k$ of $\Delta$, we assign the fixed throughput $\delta(k) := \lambda_i$, see Figure 1(c). Further, we set $\delta(k)$ for the $i$th edge $k$ on the bottom border of $\Delta$, counted from right to left. Finally, we set $\delta(k') := -\nu_i$ for the $i$th edge $k'$ on the left border of $\Delta$, counted from top to bottom. Recall that $\delta(k, f)$ denotes the throughput of a flow $f$ into $\Delta$, while $-\delta(k', f)$ denotes the throughput of $f$ out of $\Delta$.

2.5 Definition. Let $\lambda, \mu, \nu \in \mathbb{N}^n$ be a triple of partitions satisfying $|\nu| = |\lambda| + |\mu|$. The hive flow polytope $P := P(\lambda, \mu, \nu) \subseteq \overline{P}(G)$ is defined to be the set of hive flows $f \in \overline{P}(G)$ satisfying $\delta(k, f) = \delta(k)$ for all border edges $k$ of $\Delta$. We also set $P_\mathbb{Z} := P \cap \overline{P}(G)_\mathbb{Z}$. 

The following description of the Littlewood-Richardson coefficient is heavily based on [KT99], see [BI12].

2.6 Proposition ([BI12, Prop. 2.7]). The Littlewood-Richardson coefficient $c_{\lambda,\mu}^\nu$ equals the number of integral hive flows in $P(\lambda, \mu, \nu)$, i.e., $c_{\lambda,\mu}^\nu = |P(\lambda, \mu, \nu)_\mathbb{Z}|$.

2.4 The LR graph and secure cycles

We now focus on cycles in $\Delta$ and their induced flow, starting out with the following fairly simple claim.

2.7 Claim. For each cycle $c$ and each rhombus $\rho$ we have $\sigma(\rho, c) \in \{-2, -1, 0, 1, 2\}$.

Proof. Fix a rhombus $\rho$. The claim follows from $\tilde{\rho}(c) \in \{-1, 0, 1\}$ and $\hat{\rho}(c) \in \{-1, 0, 1\}$.

For the exact slack calculation we introduce the following notation, in which we write paths in a pictorial notation. As always, $\rho$ can appear in any situation rotated by a multiple of $60^\circ$.

2.8 Definition. The sets of paths, interpreted as subsets of $E(G)$,

$$\Psi_+(\rho) := \{\hat{\rho}, \hat{\rho}, \tilde{\rho}, \tilde{\rho}\}, \quad \Psi_-(\rho) := \{\hat{\rho}, \hat{\rho}, \tilde{\rho}, \tilde{\rho}\}, \quad \text{and} \quad \Psi_0(\rho) := \{\tilde{\rho}, \tilde{\rho}, \tilde{\rho}, \tilde{\rho}\}$$

are called the sets of of positive, negative, and neutral slack contributions of the rhombus $\rho$, respectively.

The following Proposition 2.9 gives a method to determine the slack.

2.9 Proposition ([BI12 Obs. 3.3]). Let $c$ be a cycle in $G$, let $\rho$ be a rhombus, and let $E_0$ denote the set of edges of $G$ contained in a rhombus $\rho$. Then $c \cap E_0$ is either empty, or it is a union of one or two slack contributions $q$. The slack $\sigma(\rho, c)$ is obtained by adding 1, 0, or $-1$ over the contributions $q$ contained in $c$, according to whether $q$ is positive, negative, or neutral.

2.10 Definition. Fix a hive flow $f \in P$. A flow $d \in \overline{P}(G)$ is called $f$-hive preserving if there exists $\epsilon > 0$ such that $f + \epsilon d \in P$. We call a cycle $f$-hive preserving if its induced flow is $f$-hive preserving.

We can easily check whether a cycle is $f$-hive preserving using the following lemma.

2.11 Lemma. A cycle $c$ is $f$-hive preserving, iff $c$ does not use any negative contribution in $f$-flat rhombi.
Proof. According to Proposition 2.9, if $c$ does not use any negative contribution in $f$-flat rhombi, then $f + \epsilon c \in P$ for $\epsilon$ small enough. Conversely, assume that $c$ uses a negative contribution in an $f$-flat rhombus $\diamondsuit$. If $c$ uses $\heartsuit$ or $\spadesuit$, then $c$ uses no other contribution in $\diamondsuit$ and hence $\sigma(\diamondsuit, f + \epsilon c) = -\epsilon < 0$. If $c$ uses $\clubsuit$, then by a topological argument, the only other contribution that $c$ can use is $\heartsuit$, which is also negative. As before we conclude $\sigma(\heartsuit, f + \epsilon c) < 0$. The argument for $c$ using $\spadesuit$ is analogous. □

2.12 Definition. We say that $f, g \in P_Z$ are neighbors iff $g - f$ is induced by a cycle in $G$. The resulting graph with the set of vertices $P_Z$ is also denoted by $P_Z$ and it is called the Littlewood-Richardson graph or LR graph for short. The neighborhood of $f$ is denoted with $\Gamma(f)$.

Clearly, the neighborhood relation is symmetric. In the next Definition 2.13 and in Proposition 2.14 we focus on which cycles serve as edges in the LR graph $P_Z$.

2.13 Definition. A rhombus $g$ is called nearly $f$-flat, if $\sigma(g, f) = 1$. We call a cycle $c f$-secure, if $c$ is $f$-hive preserving and if additionally $c$ does not use both negative contributions $\heartsuit$ and $\diamondsuit$ at the acute angles of any nearly $f$-flat rhombus $\spadesuit$.

2.14 Proposition ([BI12 Prop. 3.8]). Assume $f \in P_Z$. If $g \in P_Z$ is a neighbor of $f$, then $g - f$ is an $f$-secure cycle. Conversely, if $c$ is a proper $f$-secure cycle, then $f + c \in P_Z$ is a neighbor of $f$.

One main result in [BI12] is the following Theorem 2.15.

2.15 Theorem (Connectedness Theorem, [BI12 Thm. 3.12]). The LR graph $P_Z$ is connected.

3 Enumerating hive flows

In this section we give a nontechnical overview of the enumeration algorithms.

The following theorem states that (a superset of) the neighborhood $\Gamma(f)$ can be efficiently enumerated. We postpone the proof of Theorem 3.1 to Section 5.

3.1 Theorem (Neighbourhood generator). There exists an algorithm NeighGen which on input $f \in P_Z$ outputs the elements of a set $\tilde{\Gamma}(f) \subseteq P_Z$ one by one such that $\Gamma(f) \subseteq \tilde{\Gamma}(f)$. The computation of the first $k$ elements takes time $O(k \cdot \text{poly}(n))$.

We define the directed graph $\tilde{P}_Z$ to be the graph with vertex set $V(P_Z)$ and (possibly asymmetric) neighborhood function $\tilde{\Gamma}$ as given by Theorem 3.1. Since $P_Z$ is connected by Theorem 2.15, $\tilde{P}_Z$ is strongly connected. Therefore, breadth-first-search on $\tilde{P}_Z$ started at any hive flow in $P_Z$ visits all flows in $P_Z$. A variant of this breadth-first-search is realized in the following Algorithm 1, which gets an additional threshold parameter $t$ such that Algorithm 1 visits at most $t$ flows.
The first two lines of Algorithm 1 deal with computing a hive flow \( f \in P_2 \) if there exists one. This can be done in time strongly polynomial in \( n \) using Tardos’ algorithm \cite{Tar86} \cite{GLS93} \cite{KT99} as stated in \cite{MS05} and \cite{DLM06}. We can also use the combinatorial algorithm presented in \cite{BL12} for this purpose, which is especially designed for this problem, but note that although it has a much smaller exponent in the running time, its running time depends on the bitsize of the input partitions. Here, the running time is defined to be the number of additions, multiplications and comparisons. In practice, the algorithm in \cite{BL12} may be the better choice, but in this paper we focus on algorithms whose running time does not depend on the input bitsize. If Tardos’ algorithm is used as a subalgorithm in Algorithm 1, then this is the case and hence we choose this option.

### 3.2 Theorem

Given partitions \( \lambda, \mu, \nu \) with \( |\lambda| + |\mu| = |\nu| \) and a natural number \( t \geq 1 \), then Algorithm 1 decides \( c_{\lambda, \mu}^t \geq t \) in time \( O(t^9 \cdot \text{poly}(n)) \).

**Proof.** Recall that according to Proposition 2.6 we have \( |P_2| = c_{\lambda, \mu}^t \). Now observe that, starting after line 3, Algorithm 1 preserves the three invariants \( T \subseteq S \subseteq P_2 \), \( |S| \leq t \) and \( \forall f \in T : \Gamma(f) \subseteq S \).

If the algorithm returns \( \text{TRUE} \), then \( |S| \geq t \). As \( S \subseteq P_2 \) and \( |P_2| = c_{\lambda, \mu}^t \), we have \( c_{\lambda, \mu}^t \geq t \).

If the algorithm returns \( \text{FALSE} \), then \( |S| < t \) and \( S = T \). Moreover, \( \Gamma(f) \subseteq S \) for all \( f \in S \). Since the digraph \( P_2 \) is strongly connected, it follows that \( P_2 = S \). Therefore we have \( c_{\lambda, \mu}^t = |P_2| = |S| < t \).

We have shown that the algorithm works correctly.

Now we analyze its running time. Recall that the first two lines of Algorithm 1 run in time \( \text{poly}(n) \), because of Tardos’ algorithm. The outer loop runs at most \( t \) times, because in each iteration, \( |T| \) increases and \( |T| \leq |S| \leq t \). If in the inner loop we have \( \Gamma(f) < t \), then the inner loop runs for at most \( t - 1 \) iterations and hence \( \Gamma(f) \) can be generated in time \( O(t \cdot \text{poly}(n)) \) via Theorem 3.1. If in the inner loop we have \( \Gamma(f) \geq t \), then after \( t \) iterations we have \( |S| \geq t \) and the algorithm returns immediately. The first \( t \) elements of \( \Gamma(f) \) can be generated via Theorem 3.1 in time \( O(t \cdot \text{poly}(n)) \). Therefore we get an overall running time of \( O(t^9 \cdot \text{poly}(n)) \).

### 3.3 Theorem

Given partitions \( |\lambda| + |\mu| = |\nu| \). Then \( c_{\lambda, \mu}^t \) can be computed in time \( O((c_{\lambda, \mu}^t)^2 \cdot \text{poly}(n)) \) by a variant of Algorithm 1.

**Proof.** Use Algorithm 1 with the input \( t = \infty \) as a formal symbol, but instead of returning \( \text{FALSE} \) in line 1 return 0 and instead of returning \( \text{FALSE} \) in line 12 return false.
return \(|S|\). Note that the algorithm never returns \(\text{TRUE}\), because “\(|S| > \infty\)” in line 8 is always false. If the algorithm terminates, then \(P_\mathcal{S} = S\) and thus the algorithm works correctly. Note that if started with \(t = \infty\) the algorithm behaves exactly as if started with \(t = e'_\lambda + 1\). Thus it runs in time \(\mathcal{O}(\lambda,n)\).

4 The residual digraph \(R_f\)

We would like to have a direct algorithm that prints out the elements of the neighborhood \(\Gamma(f)\). A naïve approach would list all cycles \(c\) and reject those with \(f + c \notin P\). But we cannot control this algorithm’s running time when there are many rejections. Note that there are exponentially many cycles! The solution is to a priori generate only those cycles \(c\) with \(f + c \in P\). According to Lemma 2.11 these \(c\) use no negative contributions in \(f\)-flat rhombi. So we now introduce the digraph \(R_f\) in which the possibility of using negative contributions in \(f\)-flat rhombi is eliminated. The digraph \(R_f\) will arise as a subgraph of the following digraph \(R\).

4.1 Definition (Digraph \(R\)). A turn is defined to be a path in \(G\) consisting of two edges, starting at a white vertex, using a black vertex of a hive triangle, and ending at a different white vertex. The digraph \(R\) has as vertices the turns, henceforth called turnvertices. The edges of \(R\) are ordered pairs of turns that can be concatenated to a path in \(G\), henceforth called turnedges.

Note that there are six turnvertices in each hive triangle: \(\AA, \AA, \AA, \AA, \AA, \AA\), and \(\AA\). Pictorially, we can write a turnedge like \(\circlearrowlefteq\) := \((\circlearrowlefteq, \circlearrowleftright\)). Note that a turnedge corresponds to a path in \(G\) of length 4. Each rhombus \(\circlearrowlefteq\) contains exactly the eight turnedges \(\circlearrowlefteq, \circlearrowlefteq, \circlearrowlefteq, \circlearrowlefteq, \circlearrowlefteq, \circlearrowlefteq, \circlearrowlefteq, \circlearrowlefteq\). Paths on \(R\) are called turnpaths and cycles on \(R\) are called turncycles. Note that a turnpath can for example use both the turnedge \(\circlearrowlefteq\) and the turnedge \(\circlearrowlefteq\) in a rhombus \(\circlearrowlefteq\), because both turnedges have no common turnvertex. We denote by \(\text{start}(p)\) the first turnvertex of a turnpath \(p\) and by \(\text{end}(p)\) its last turnvertex.

By using turnvertices and turnedges to define \(R_f\) and \(D_f\) by focusing on the more complicated graph structure of \(R\) instead of the easy structure of \(G\), we now have the possibility to delete turnedges, which is done in the next definition.

4.2 Definition (Digraph \(R_f\)). Let \(f \in P_\mathcal{S}\). We define the digraph \(R_f\) by deleting from \(R\) the turnvertices \(\circlearrowlefteq\) and \(\circlearrowlefteq\) and the turnedges \(\circlearrowlefteq\) and \(\circlearrowlefteq\) for each \(f\)-flat rhombus \(\circlearrowlefteq\).

Note that the deleted parts correspond exactly to negative slack-contributions in \(f\)-flat rhombi (cp. Definition 2.8).

4.3 Definition (Turncycles). We denote with \(C(R)\) the set of turncycles on \(R\), and with \(C(R_f)\) the set of turncycles on \(R_f\).

4.4 Definition (Throughput). Let \(e \in C(R)\) and let \(e \in E(\Delta)\). For a turnvertex \(v \in V(R)\) we write \(c(v) = 1\) if \(v \in e\) and \(c(v) = 0\) otherwise. If \(e\) is a diagonal of a rhombus, then we can write \(e = \circlearrowlefteq\) and we define the throughput as

\[
\bar{c}(e) := c(\circlearrowlefteq) + c(\circlearrowlefteq) - c(\circlearrowlefteq) - c(\circlearrowlefteq) \in \{-2, -1, 0, 1, 2\}.
\]

If \(e\) is a border edge of \(\Delta\), we define the throughput analogously.

4.5 Remark. Note that the throughput of cycles \(c' \in C(G)\) can only range from \(-1\) to \(1\), whereas for turncycles \(c \in C(R)\) the throughput can range from \(-2\) to \(2\). The larger throughput range makes things complicated and in fact it is the price we pay for being able to delete single turnedges in Definition 1.2.
To a turncycle we can associate a flow on $G$ as follows.

4.6 Definition. The map $\pi : C(R) \to \mathcal{F}(G)$ is the unique linear map preserving the throughput: For $c \in C(R)$ we define $\pi(c)$ such that $\phi_\tau(c) = \phi(\pi(c))$ for all edges $\tau \in E(\Delta)$.

Note that $\pi$ is well-defined, since $\phi_\tau(c) + \phi_\tau(c) + \phi_\tau(c) = 0$, see [BI12, Sec. 2.2].

We define the slack w.r.t. turncycles via $\pi$ as follows: $\sigma(\varnothing, c) = \phi_\tau(\pi(c))$ for all edges $\tau \in E(\Delta)$.

4.7 Claim ([BI12, Lemma 4.4]). Let $c$ be a turncycle and $q$ be a rhombus. The slack $\sigma(q, c)$ is obtained by adding 1, 0, or $-1$ over the turnedges $q$ used by $c$ in $q$, according to whether $q$ is positive, negative, or neutral, and by further adding 1 or $-1$ over the remaining turnvertices $q'$ used by $c$ in $q$, according to whether $q'$ is positive ($\varnothing$ or $\hat{\varnothing}$) or negative ($\varnothing$ or $\hat{\varnothing}$).

4.8 Lemma. Let $f \in \mathcal{P}$. Each turncycle $c$ in $R_f$ is $f$-hive preserving.

Proof. By construction of $R_f$ and Claim 1.7 see the discussion in [BI12] after Lemma 4.7 there.

There is a canonical injective map from the set of proper cycles $c$ on $G$ to the set of turncycles $c'$ on $R$: consecutive turns in $c$ correspond to turnedges in $c'$. A turncycle $c'$ in the image of this map is ordinary, which means the following: $c'$ uses only a single turnvertex in each hive triangle. This is the desired behaviour of turncycles when we want to simulate cycles on $G$.

4.9 Definition. Turnpaths and turncycles are called ordinary, if they use at most one turnvertex in each hive triangle.

Since $f$-secure turncycles are ordinary, there is a bijection between the set of $f$-secure proper cycles on $G$ and the set of ordinary turncycles on $R$.

The following definition of secure turnpaths is related to the Definition 2.13 of secure cycles.

4.10 Definition (Secure turnpaths). A turnpath $p$ on $R_f$ is called $f$-secure, if $p$ is ordinary and if additionally $p$ does not use both counterclockwise turnvertices $\varnothing$ and $\hat{\varnothing}$ at the acute angles of any nearly $f$-flat rhombus $\varnothing$. We define $f$-secure turncycles analogously.

Since $f$-secure turncycles are ordinary, there is a bijection between $f$-secure proper cycles and $f$-secure turncycles. To prove Theorem 3.1 we want to list all $f$-secure proper cycles (cf. Prop. 2.14). We have seen that we may as well list the $f$-secure turncycles.

5 The Neighbourhood Generator

This section is devoted to the proof of Theorem 3.1 by describing and analyzing the algorithm NeighGen. This algorithm is inspired by the binary partitioning method used in [FM94].

Given $f \in \mathcal{P}_Z$, NeighGen prints out the elements of a set $\hat{\Gamma}(f)$ with $\Gamma(f) \subseteq \hat{\Gamma}(f) \subseteq \mathcal{P}_Z$. Note that we would like to have a direct algorithm that prints the elements of $\Gamma(f)$, but we do not know how to do this efficiently.
Although we can treat $f$-hive preserving turncycles algorithmically, there are problems when it comes to $f$-secure turncycles. In fact we do not know how to solve the following crucial Secure Extension Problem 5.1.

5.1 Problem (Secure extension problem). Given $f \in P_Z$ and an $f$-secure turnpath $p$, decide in time $\text{poly}(n)$ whether there exists an $f$-secure turncycle $c$ containing $p$ or not.

If in Problem 5.1 an extension $c$ exists for a given $p$, then we call $p$ $f$-securely extendable.

The usefulness of having a solution to Problem 5.1 will be made clear in the next subsection, where we introduce an algorithm $\text{NeighGen'}$ that proves Theorem 3.1 under the assumption that Problem 5.1 is has a positive solution.

5.1 A first approach

Assume that $\mathcal{A}$ is an algorithm that on input $(f, p)$ with $f \in P_Z$ and $p$ an $f$-secure turnpath in $R_f$ returns whether $p$ is $f$-securely extendable or not. Notationally,

$$\mathcal{A}(f, p) = \begin{cases} \text{TRUE}, & \text{if } p \text{ is } f\text{-securely extendable} \\ \text{FALSE}, & \text{otherwise} \end{cases}$$

We denote by $T(\mathcal{A}, n)$ the worst case running time of $\mathcal{A}(f, p)$ over all partitions $\lambda, \mu, \nu$ into $n$ parts, all $f \in P(\lambda, \mu, \nu)_Z$ and all $f$-secure turnpaths $p$ in $R_f$. If $\mathcal{A}$ solves Problem 5.1 then $T(\mathcal{A}, n)$ is polynomially bounded in $n$ — but remember that we do not know of such $\mathcal{A}$.

The algorithms presented in this subsection use $\mathcal{A}$ as a subroutine and hence they are only polynomial time algorithms if $T(\mathcal{A}, n)$ is polynomially bounded. In fact this subsection is only meant to prepare the reader for the more complicated approach used in the Subsection 5.2, where $\mathcal{A}$ is modified in a way such that polynomial running time is achieved.

The main subalgorithm of this subsection is Algorithm 2 below. Note that the statement for both in line 4 means for all, as there are at most two turnedges $e$ in $R_f$ starting at $\text{end}(p)$.

5.2 Lemma. Algorithm 2 works according to its output specification.

| Algorithm 2 FINDNEIGHWITHBLACKBOX |
|-----------------------------------|
| **Input:** $f \in P_Z$; $p$ an $f$-securely extendable turnpath on $R_f$; $\mathcal{A}$ as in (†) |
| **Output:** Prints all integral flows $f + c \in P_Z$, where $c \in C(G)$ is a cycle that contains $p$. Prints at least one element. |
| 1: if $p$ is not just a turnpath, but a turncycle then |
| 2: print $(f + \pi(p))$ and return. |
| 3: end if |
| 4: for both turnedges $e := (\text{end}(p), z) \in E(R_f)$ do |
| 5: Concatenate $p' \leftarrow pe$. |
| 6: If $p'$ is not $f$-secure, continue with the next $e$. |
| 7: if $\mathcal{A}(f, p')$ then |
| 8: Recursively call FINDNEIGHWITHBLACKBOX($f, p', \mathcal{A}$). |
| 9: end if |
| 10: end for |
Proof. Since the input \( p \) is \( f \)-securely extendable, there exists at least one turnedge \( e = (\text{end}(p), z) \) such that \( p' = pe \) is \( f \)-secure in line 6 and \( f \)-securely extendable in line 7. Hence Algorithm 2 prints at least one element or calls itself recursively. The lemma follows now easily by induction on the number of turns in \( G \) not used by \( p \).

We will see that it is crucial for the running time that for each call of Algorithm 2 we can ensure that an element is printed.

5.3Lemma. Let \( f \in P_Z \). On input \( f \in P_Z \), Algorithm 2 prints out distinct elements. The first \( k \) elements are printed in time \( O(kn^2T(A, n)) \).

Proof. Algorithm 2 traverses a binary recursion tree of depth at most \( |E(R_f)| = O(n^2) \) with depth-first-search. The time needed at each recursion tree node is \( O(T(A)) \), if the implementation is done in a reasonable manner: Checking if a turnpath is a turncycle can be done in time \( O(1) \) and testing \( p' \) for \( f \)-security under the assumption that \( p \) was \( f \)-secure can also be in time \( O(1) \). Thus the time the algorithm spends between two leafs is at most \( O(n^2T(A)) \). The lemma is proved by the fact that at each leaf a distinct element is printed.

We can define an algorithm NeighGen’ as required for Theorem 3.1 (besides polynomial running time) as follows:

1: Let \( f \in P_Z \) be an input.
2: for all turnedges \( p \in E(R_f) \) do
3: Compute \( A(f, p) \).
4: if \( A(f, p) = \text{TRUE} \) then
5: Call Algorithm 2 on \( (f, p, A) \).
6: end if
7: end for

Lemma 5.2 ensures that \( \Gamma(f) \) is printed by NeighGen’. We now analyze the running time of NeighGen’.

Since each element in \( \Gamma(f) \) is printed at most once during each of the \( O(n^2) \) calls of Algorithm 2 and each call of Algorithm 2 prints pairwise distinct elements, it follows that each element is printed by NeighGen’ at most \( O(n^2) \) times. Hence, according to Lemma 5.3 the first \( k \) elements are printed in time \( O(k \cdot n^2 \cdot T(A, n)) \).

Since the existence of \( A \) was hypothetical, we have to bypass Problem 5.1. This is achieved in the next subsection.

5.2 Bypassing the secure extension problem

We need a polynomial time algorithm that solves a problem similar to Problem 5.1. A first approach for this is the following (which will fail for several reasons explained below): Instead of extending \( p \) to an \( f \)-secure cycle, we compute a trivial extension \( q \) of \( p \), which is a shortest turnpath \( q \) in \( R_f \) starting at \text{end}(p) \) and ending at \text{start}(p). A turncycle \( c \) containing \( p \) can then be obtained as the concatenation \( c = pq \). But \( c \) might not be secure and might not even be ordinary and in the worst case we could have \( f + \pi(pq) \notin P \). It will be crucial in the following to find \( q \) such that \( f + \pi(pq) \in P \), so in the upcoming examples we have a look at the difficulties that may arise for a trivial extension \( q \) and how we can fix them.

Example (a): Figure 3(a) shows a secure turnpath \( p \) and a trivial extension \( q \) that uses no turnvertices of \( p \). Nevertheless, \( f + \pi(pq) \notin P \). In the light of Example (a) we want to consider only those \( q \) where \( q \) uses no turnvertices in those hive triangles where \( p \) uses turnvertices.
Example (b): Figure 3(b) shows a secure turnpath $p$ and a trivial extension $q$ that uses no turnvertices of $p$ and uses no turnvertices in hive triangles where $p$ uses turnvertices. But still we have $f + \pi(pq) \notin P$. Example (b) gives the idea to consider only those $q$ that use no turnvertices in nearly $f$-flat rhombi where $p$ uses a negative slack contribution.

Example (c): Figure 3(c) shows a secure turnpath $p$ and a trivial extension $q$ that uses no turnvertices of $p$ and uses no turnvertices in hive triangles where $p$ uses turnvertices. Moreover, $q$ uses no turnvertices in nearly $f$-rhombi in which $p$ uses a negative slack contribution. Nevertheless, $f + \pi(pq) \notin P$. A situation as in Example (c) is possible, because $p$ consists of only 2 turns. We will see that 3 turns are enough to avoid these problems.

Considering more and more special subclasses of turnpaths $q$ in $R_f$ leads to the forthcoming definition of the digraph $R_f^p$. The turnpaths $q$ will be shortest turnpaths on $R_f^p$.

5.4 **Definition** (the digraph $R_f^p$). Let $f \in P_Z$ and $p$ be an $f$-secure turnpath on $R_f$. We define the digraph $R_f^p$ by further deleting from $R_f$ a subset of its turnvertices: Delete from $R_f$ each turnvertex lying in a hive triangle in which $p$ uses turnvertices. Moreover, for all nearly $f$-flat rhombi $q$ in which $p$ uses $q$ or $q$, delete all turnvertices of $q$. If we deleted $\text{start}(p)$ or $\text{end}(p)$, add them back.

We denote with $C'(R_f^p)$ the set of turnpaths in $R_f^p$ from $\text{end}(p)$ to $\text{start}(p)$. Hence for each $q \in C'(R_f^p)$ we have that $pq \in C(R_f)$. If for an $f$-secure turnpath $p$ in $R_f$ we have that $C'(R_f^p) \neq \emptyset$, then we call $p$ $f$-extendable. Note the following crucial fact: All $f$-securely extendable turnpaths are also $f$-extendable. Unlike $f$-secure extendability, $f$-extendability can be handled efficiently: We can check in time $O(n^2)$ whether an $f$-secure turnpath $p$ is $f$-extendable or not by breadth-first-search on $R_f^p$.

Definition 5.4 is precisely what we want, which can be seen in the following result, which is a variant of Theorem 4.8 in [BI12].

5.5 **Theorem** (Shortest Turncycle Theorem). Let $p$ be an $f$-secure turnpath in $R_f$, consisting of at least 3 turns. Let $q$ be a shortest turnpath in $C'(R_f^p)$. Then $f + \pi(pq) \in P$.

The proof is very similar to the one of Theorem 4.8 in [BI12] and in Section 6 we outline the main changes that need to be made when adapting it.

Consider now Algorithm 5 which is a refinement of Algorithm 2.

We will see in Remark 5.7 in which situations the condition of line 12 is satisfied.

5.6 **Lemma**. Algorithm 5 works according to its output specification.
Algorithm 3 FindNeigh

Input: $f \in P_Z$; an $f$-extendable turnpath $p$ in $R_f$ consisting of at least 3 turns
Output: (1) Prints at least all integral flows $f + c \in P_Z$, where $c \in C(G)$ is a cycle that contains $p$, but may print other elements of $P_Z$ as well. (2) Prints at least one element.

1: if start($p$) = end($p$) then
2: print($f + \pi(p)$) and return.
3: end if
4: local foundpath ← FALSE.
5: for both turnedges $e := (end(p), z) \in E(R_f^p)$ do
6: Concatenate $p' ← pe$.
7: if $p'$ is $f$-extendable then
8: Recursively call FindNeigh($f, p'$).
9: foundpath ← TRUE.
10: end if
11: end for
12: if not foundpath then
13: print($f + \pi(pq)$) with a shortest $q \in C'(R_f^p)$ and return.
14: end if

Proof. Recall that by definition, all $f$-extendable turnpaths are $f$-secure. Hence in line 2 only elements of $P_Z$ are printed. The flows printed in line 13 are elements of $P_Z$ because of the Shortest Turncycle Theorem 5.5.

If $p$ is $f$-securely extendable, then the binary recursion tree of Algorithm 2 is a subtree of the binary recursion tree of Algorithm 3 because $f$-secure extendability implies $f$-extendability. Hence in this case, Algorithm 3 meets the output specification requirement (1).

If $p$ is not $f$-securely extendable, then there exists no $f$-secure turncycle $c$ containing $p$ such that $f + \pi(c) \in P_Z$, see Proposition 2.14. Thus in this case, Algorithm 3 trivially meets the output specification requirement (1).

The output specification requirement (2) is satisfied because exactly one of the following three cases occurs: (a) Algorithm 3 prints an element in line 2 and returns or (b) Algorithm 3 calls itself recursively or (c) Algorithm 3 prints an element in line 13 and returns.

5.7 Remark. If during a run of Algorithm 3 we have foundpath=FALSE in line 12 then we have an $f$-extendable turnpath $p$ such that its concatenation $p' ← pe$ with any further turnedge $e$ results in $p'$ being not $f$-extendable. For example, this can happen if $p$ ends with $\overrightarrow{a}$ and $q \in C'(R_f^p)$ continues with $\overrightarrow{b}$, but $q$ also uses $\overrightarrow{c}$. Then $q$ uses two turnvertices in $\overrightarrow{a}$. If $q$ is the only element in $C'(R_f^p)$, then adding a turnedge to $p$ destroys $f$-extendability: A turnpath $q' \in C'(R_f^p)$ would mean that there exists a concatenated turncycle $p'q'$ that uses only a single turnvertex in the hive triangle $\overrightarrow{a}$, a contradiction to the uniqueness of $q$. In this situation, $f + \pi(pq)$ is printed in line 13 and since $pq$ is not ordinary, this means that an element of $P_Z$ is printed that does lie in $\Gamma(f)$.

5.8 Lemma. Let $f \in P_Z$. On input $f \in P_Z$, Algorithm 3 prints out distinct elements. The first $k$ elements are printed in time $O(k \cdot n^4)$.

Proof. We use the fact that $f$-extendability can be decided in time $O(n^2)$. The rest of the proof is analogous to the proof of Lemma 5.3. Note that it is crucial in this proof that for each recursive algorithm call we can ensure that at least one element of $P_Z$ is printed. This is guaranteed by line 13.
5.9 Remark. If we delete line 13 from Algorithm 3 then its execution gives a recursion tree where at some leafs no elements are printed. Then it is not clear how much time is spent visiting elementless leafs and we cannot prove Lemma 5.8.

Analogously to the definition of $\text{NeighGen}'$, we can define the algorithm $\text{NeighGen}$ as required for Theorem 3.1 as follows: Call Algorithm 3 several times with fixed $f \in P_z$, but each time with a different secure turnpath $p \in E(R_f)$ consisting of 3 turns such that $p$ is $f$-extendable. We now prove the correctness and running time of $\text{NeighGen}$.

Let $\tilde{\Gamma}(f)$ be the set of flows printed by $\text{NeighGen}$. Since $f$-secure extendability implies $f$-extendability, each flow printed by $\text{NeighGen}'$ is also printed by $\text{NeighGen}$, so $\tilde{\Gamma}(f) \subseteq \Gamma(f)$. Lemma 5.6 implies $\tilde{\Gamma}(f) \subseteq P_z$.

Since we have $O(n^2)$ calls of Algorithm 3 we get a total running time of $O(kn^6)$. This proves Theorem 3.1 with one hole remaining: The proof of the Shortest Turncycle Theorem 5.5.

6 The Shortest Turncycle Theorem

In this section we discuss how to prove the generalization of [BI12, Theorem 4.8], namely the Shortest Turncycle Theorem 5.5. At the same time, we use this section for proving Propositions 6.2 and 6.3 which will be needed in Section 7 in a little bit more general form than they are found in [BI12]. To achieve this, we slightly generalize the term “turnpath”.

6.1 Definition. Let $k \in E(\Delta)$ and let $\eta$ be one of the two directions in which $k$ can be crossed by turnedges. Then $p := (k, \eta)$ is called a turnpath of length 0. For turnpaths of length 0, the symbol $C'(R_p)$ is defined as the set of all turnpaths in $R_f$ that cross $k$ in direction $\eta$. By a generalized turnpath we understand a turnpath or a turnpath of length 0. Turnpaths of length 0 are defined to be secure.

In the following, we state some facts about shortest turncycles, which are variants from propositions proved in [BI12, Sec. 7.1].

Each turnvertex in $R$ has a reverse turnvertex in $R$ that points in the other direction, e.g., the reverse turnvertex of $\hat{\diamond}$ is $\check{\diamond}$.

6.2 Proposition. Let $f \in P_z$ and let $p$ be an $f$-secure generalized turnpath on $R_f$. A shortest turnpath in $C'(R_p)$ cannot use a turnvertex and its reverse.

Proof. Similar to the proof of [BI12 Prop. 7.1]. A little care must be taken, because $R^p$ has only a subset of the turnvertices of $R_f$.

In the following we fix $f \in P_z$, a secure generalized turnpath $p$, and a shortest turnpath $q \in C'(R_p)$. Proposition 6.2 directly implies that the diagonal of each rhombus is crossed at most twice by $q$. A rhombus $\varrho$ is called special if the turnpath $q \in C'(R_p)$ crosses its diagonal twice. If the crossing is in the same direction, then $\varrho$ is called confluent, otherwise, if the crossing is in opposite directions, $\varrho$ is called contrafluent.

6.3 Proposition. 1. In a confluent rhombus, $q$ uses exactly the turnedges $\hat{\diamond}$ and $\check{\diamond}$ and no other turnvertex. In a contrafluent rhombus, $q$ uses exactly the turnedges $\hat{\diamond}$ and $\check{\diamond}$ and no other turnvertex.

2. Special rhombi do not overlap.

Proof. Analogous to the proofs in [BI12, Sec. 7.1].
Let $c := pq \in C(R_f)$. We set
\[\varepsilon := \max\{ t \in \mathbb{R} \mid f + t \varepsilon(c) \in P \}, \quad g := f + \varepsilon \pi(c).\]

Then we have $g \in P$ and by Lemma 6.3 we have $\varepsilon > 0$. For the proof of the Shortest Turncycle Theorem 5.3 it suffices to show that $\varepsilon \geq 1$, since then $f + \pi(c) \in P_2$.

If all rhombi are $f$-flat, then there are no turncycles in $R_f$ (see [12, Prop. 4.12 and Fig. 6]), in contradiction to the existence of $q \in C(R_f^*)$. So in the following we suppose that not all rhombi are $f$-flat. We shall argue indirectly and assume that $\varepsilon < 1$.

6.4 Definition. A rhombus is called critical if it is not $f$-flat, but $g$-flat.

The following claim is easy to see.

6.5 Claim. For critical rhombi $g$ we have $\sigma(g, c) \leq -2$.

6.6 Claim. If $p$ consists of at least 3 turns, then $p$ does not use turnvertices in critical rhombi.

Proof. Assume that $p$ uses turnvertices in a critical rhombus $\diamond$. W.l.o.g. let $p$ use a turnvertex in the hive triangle $\hat{\diamond}$. If $p$ uses $\hat{\diamond}$, then $\hat{\sigma}(c) = 1$. Since $\hat{\sigma}(c) \in \{-2, -1, 0, 1, 2\}$ we have $\hat{\sigma}(\diamond, c) \geq -1$. This implies that $\hat{\sigma}$ is not critical according to Claim 6.5. Analogous arguments show that $p$ does not use $\hat{\diamond}$ or $\hat{\sigma}$.

If $p$ uses $\hat{\diamond}$ and $\sigma(\diamond, f) = 1$, then by construction of $R_f^*$, $c$ uses no further turnvertex in $\hat{\diamond}$ and hence $\hat{\sigma}$ is not critical. If $p$ uses $\hat{\diamond}$ and $\sigma(\diamond, f) > 1$, then $\sigma(\diamond, c) \leq -3$. This implies $\hat{\sigma}(c) = 2$ and $\hat{\sigma}(c) = 2$, which results in overlapping special rhombi $\hat{\diamond}$ and $\hat{\sigma}$: A contradiction to Proposition 5.3.

Assume that $p$ uses $\hat{\diamond}$. If $\hat{\sigma}(c) = 2$, then $q$ uses $\hat{\sigma}$ and $\hat{\sigma}$, in contradiction to $p$ using $\hat{\diamond}$. Hence $\hat{\sigma}(c) \leq 1$. Since $\hat{\diamond}$ is critical, it follows $\hat{\sigma}(c) = 1$. But since $p$ uses $\hat{\diamond}$, $q$ uses both $\hat{\sigma}$ and $\hat{\sigma}$. Therefore, $\hat{\sigma}$ is $f$-flat. The hexagon equality (Claim 6.6) implies that either we have $\sigma(\hat{\sigma}, f) = 0$ and $\sigma(\hat{\sigma}, f) = 1$ or we have $\sigma(\hat{\sigma}, f) = 1$ and $\sigma(\hat{\sigma}, f) = 0$. In the former case, $p$ continues with $\hat{\sigma}$. This implies that all turnvertices in the hive triangle $\hat{\sigma}$ are deleted in $R_f^*$, in contradiction to $q$ using $\hat{\diamond}$. Therefore $\sigma(\hat{\sigma}, f) = 1$ and $\sigma(\hat{\sigma}, f) = 0$. But by construction of $R_f$, $c$ uses $\hat{\sigma}$. The turnpath $p$ can continue as $\hat{\sigma}$ or as $\hat{\sigma}$, but both cases will yield a contradiction.

Assume $p$ continues as $\hat{\sigma}$. Then the hive triangle $\hat{\sigma}$ has no turnvertices in $R_f^*$, in contradiction to $c$ using $\hat{\sigma}$. So now assume that $p$ continues as $\hat{\sigma}$. Since $\hat{\sigma}$ is $f$-flat, $p$ continues as $\hat{\sigma}$; also in contradiction to $c$ using $\hat{\sigma}$.

The proof that $p$ does not use $\hat{\sigma}$ is analogous to above argument.

With Claim 6.6 in mind, the rest of the proof of the Shortest Turncycle Theorem 5.3 is analogous to [12, Sec. 7.2]. The requirement that $p$ must have at least 3 turns is needed when proving the analogues of [12, Claim 7.13] and [12, Claim 7.16] when deducing the contradiction to the choice of the first critical rhombus.

7 Proof of the King-Tollu-Toumazet Conjecture

In this section we prove Theorem 1.1 more precisely, we prove the following equivalent geometric formulation.

7.1 Theorem. Let $c'_{\lambda, \mu} = 2$ and let $f_1$ and $f_2$ be the two integral points of $P(\lambda, \mu, \nu)$. Then $P(\lambda, \mu, \nu)$ is exactly the line segment between $f_1$ and $f_2$.

7.2 Claim. Theorem 1.1 is equivalent to Theorem 1.1.
Proof. Let $c_{\lambda,\mu}^\nu = 2$ and let $P = P(\lambda, \mu, \nu)$. For every natural number $M$, the stretched polytope $MP$ contains at least the $M + 1$ integral points

$$\kappa_M := \{ Mf_1, (M - 1)f_1 + f_2, (M - 2)f_1 + 2f_2, \ldots, f_1 + (M - 1)f_2, Mf_2 \}$$

and hence $c_{\kappa_M,\mu,\mu}^M \geq M + 1$.

If $P$ contains a point besides the line segment between $f_1$ and $f_2$, then $P$ contains a rational point $x$ besides this line segment. But then there exists $M$ such that $Mx$ is integral and hence $c_{M\kappa_M,\mu,\mu}^M > M + 1$.

On the other hand, if $P$ contains no other point besides the line segment, then it is easy to see that these $M + 1$ points are the only integral points in $MP$ and hence $c_{M\kappa_M,\mu,\mu}^M = M + 1$.

From now on, fix partitions $\lambda$, $\mu$ and $\nu$ such that $c_{\lambda,\mu}^\nu = 2$. Let $f_1$, $f_2$ be the two flows in $P(\lambda, \mu, \nu)_2$. Let $c_1 := f_2 - f_1$ denote the $f_1$-secure cycle that connects the integral points in $P$, and analogously define $c_2 := c_1 = f_1 - f_2$ the $f_2$-secure cycle running in the other direction. W.l.o.g. $c_1$ runs in counterclockwise direction, otherwise we switch $f_1$ and $f_2$.

For $M \in \mathbb{N}$, the stretched polytope $MP$ contains at least the set of integral points

$$\kappa_M := \{ Mf_1 + mc_1 \mid m \in \mathbb{N}, 0 \leq m \leq M \}.$$

To prove Theorem 7.1 it remains to show that for all $M \in \mathbb{N}_{\geq 2}$ these are the only integral points in $MP$. According to the Connectedness Theorem 2.15, this is equivalent to the statement that for each integral flow $\xi \in \kappa_M$ the neighborhood $\Gamma(\xi)$ is contained in $\kappa_M$. To prove this, we fix some $M \in \mathbb{N}_{\geq 2}$. Now we choose an arbitrary $\xi \in \kappa_M$, which means choosing a natural number $0 \leq m \leq M$ such that $\xi = Mf_1 + mc_1$. If $\xi \in \{ Mf_1, Mf_2 \}$, we say that $\xi$ is extremal, otherwise $\xi$ is called inner. We are interested in the neighborhood $\Gamma(\xi)$.

If $\xi \neq f_2$, then $\xi + c_1 = Mf_1 + (m + 1)c_1 \in MP$, which implies that the cycle $c_1$ is $\xi$-secure by Proposition 2.15. Analogously, $c_2$ is $\xi$-secure for $\xi \neq f_2$. It follows that both $c_1$ and $c_2$ are $\xi$-secure for an inner $\xi$. It remains to show that for an inner $\xi$ the only $\xi$-secure proper cycles are $c_1$ and $c_2$, and for both $\xi \in \{ 1, 2 \}$ the only $f_i$-secure proper cycle is $c_i$. In fact, we are going to show the following slightly stronger statement.

For an inner $\xi$, the proper cycles $c_1$ and $c_2$ are the only $\xi$-hive preserving proper cycles. For both $i \in \{ 1, 2 \}$ the only $f_i$-hive preserving proper cycle (**) is $c_i$.

Note that a proper cycle $c$ is $\xi$-hive preserving iff $c$ is $\frac{\xi}{M}$-hive preserving. Hence for proving (**) we can assume w.l.o.g. that $\xi = xf_1 + (1 - x)f_2$ with $0 \leq x \leq 1$ is a convex combination of $f_1$ and $f_2$. To classify all $\xi$-hive preserving proper cycles we use the following important proposition which we prove in the next subsection.

7.3 Proposition. Let $c_{\lambda,\mu}^\nu = 2$ and let $f_1$ and $f_2$ be the two integral hive flows in $P(\lambda, \mu, \nu)$. For each convex combination $\xi = xf_1 + (1 - x)f_2$, $0 \leq x \leq 1$ we have that each $\xi$-hive preserving proper cycle is either $f_1$-hive preserving or $f_2$-hive preserving.

According to Proposition 7.3 for proving (**) it suffices to show that $c_1$ is the only $f_1$-hive preserving proper cycle and $c_2$ is the only $f_2$-hive preserving proper cycle. But this can be seen with the following key lemma, whose proof we postpone to Subsection 7.4.

7.4 Key Lemma. Let $c_{\lambda,\mu}^\nu = 2$ and let $f_1$ and $f_2$ be the two integral points of $P(\lambda, \mu, \nu)$. For each $i \in \{ 1, 2 \}$ each ordinary turncycle in $R_{f_i}$ is $f_i$-secure.
We apply Key Lemma 7.4 as follows. Let $\iota \in \{1, 2\}$ and let $c' \neq c_\iota$ be an $f_\iota$-hive preserving proper cycle. Then Key Lemma 7.4 implies that $c'$ is $f_\iota$-secure. The fact that $c_{\lambda, \mu} = 2$ implies $c' = c_\iota$. Therefore we have shown that $c_\iota$ is the only $f_\iota$-hive preserving proper cycle and are done proving Theorem 7.1.

It remains to prove Key Lemma 7.4 and Proposition 7.3.

### 7.1 Proof of Proposition 7.3

In this subsection we already use Key Lemma 7.4, which is proved later in Subsection 7.2.

We first prove Proposition 7.3 for extremal $\xi \in \{f_1, f_2\}$. In this case it remains to show that there is no proper cycle that is both $f_1$-hive preserving and $f_2$-hive preserving. Indeed, for the sake of contradiction, assume there is such a cycle $c$. Key Lemma 7.4 implies that $c$ is both $f_1$-secure and $f_2$-secure, in contradiction to $c_{\lambda, \mu} = 2$.

From now on, we assume that $\xi$ is inner. Recall that in this case both $c_1$ and $c_2$ are $\xi$-hive preserving.

First of all, we note that if a rhombus $\circ$ is $f_1$-flat and $f_2$-flat, then $\circ$ is also $\xi$-flat, because $\xi$ lies between $f_1$ and $f_2$. The converse is also true: if a rhombus $\circ$ is $\xi$-flat, then $\circ$ is both $f_1$-flat and $f_2$-flat, because $\sigma(\circ, \xi) = 0$ and $\sigma(\circ, f_1) > 0$ would imply $\sigma(\circ, f_2) < 0$, in contradiction to $f_2 \in P$.

From this consideration, it follows that a cycle $c$ is $\xi$-hive preserving if $c$ is $f_1$-hive preserving or $f_2$-hive preserving.

In the following Claim 7.6 we begin to rule out the existence of $\xi$-hive preserving cycles whose curves have no intersection with the curve of $c_1$. We first introduce some terminology (see Figure 4): The set of hive triangles from which $c_1$ uses turns is called the pipe. Note that this coincides with the set of hive triangles from which $c_2$ uses turns. A hive triangle of the pipe is called a pipe triangle. The pipe partitions the plane into several connected components: The pipe itself, the outer region and the inner regions enclosed by the pipe. The pipe border is defined to be the set of edges between the regions and can be divided into the inner pipe border and the outer pipe border.

#### 7.5 Claim. Rhombi whose diagonal lies on the outer pipe border are not $f_1$-flat. Likewise, rhombi whose diagonal lies on the inner pipe border are not $f_2$-flat.

**Proof.** This follows directly from $c_1$ traversing the pipe in counterclockwise direction.

#### 7.6 Claim. Each $\xi$-hive preserving cycle uses a pipe triangle.
Proof. First we show that each ξ-hive preserving cycle that runs only in the outer region is also f₁-hive preserving and each ξ-hive preserving cycle that runs only in an inner region is also f₂-hive preserving: Recall that the flows ξ, f₁, and f₂ only differ by multiples of c₁. So for a rhombus in which both hive triangles are not pipe triangles, ξ-flatness, f₁-flatness, and f₂-flatness coincide. The first claim follows with Claim 7.3.

For the sake of contradiction, assume now the existence of a ξ-hive preserving cycle c that uses no pipe triangle. Then c runs only in the outer region and is thus f₁-hive preserving, or c runs only in the inner region and is thus f₂-hive preserving. Key Lemma 7.4 ensures that c is f₁-secure or f₂-secure, which is a contradiction to cλ,µ = 2.

We show now that there are severe restrictions on the possible shape of the pipe, forcing it to have at most one inner region. The upcoming Claim 7.8 shows that the following two fundamentally different types, introduced in the next definition, are the only types of pipes that can appear.

7.7 Definition. We call the pipe thin if it has the following property, see Figure 4(a). Two pipe triangles share a side iff they are direct predecessors or direct successors when traversing c₁. Additionally, we require that the pipe encloses a single inner region, see Figure 4(c) and Figure 4(d) for counterexamples.

We call the pipe thick if it has the following property, see Figure 4(b). There exists a path ζ in Δ, called the center curve, such that ζ has only obtuse angles of 120° and the path c₁ runs around ζ as indicated in Figure 4(b). Additionally, we require that two pipe triangles Δ₁ and Δ₂ share a side k ∈ E(Δ) iff either k is an edge of ζ or Δ₁ and Δ₂ are direct predecessors or direct successors when traversing c₁. The center curve may consist of a single vertex only.

7.8 Claim. The pipe is either thin or thick.

Proof. First of all, assume that c₁ runs as depicted in ξ. Then c₁ can be rerouted in R₁, to c via ζ. However, c is f₁-hive preserving and Key Lemma 7.4 implies that c is f₁-secure, in contradiction to cλ,µ = 2. This excludes the cases in Figure 4(d).

Now suppose that the pipe is not thin, i.e., two adjacent pipe triangles are not direct successors when traversing c₁. It is a simple topological fact that then there is a rhombus ∆ where c₁ or c₂ runs like ξ. We choose ζ ∈ {1, 2} such that c₁ runs like ξ. So ζ is not f₁-flat. It suffices to construct the center curve ζ, which contains the diagonal c. To achieve this, we show that “c₁ cannot diverge”, which precisely means the following: If c₁ uses ζ, then c₁ uses ζ and ζ continues with ζ; if c₁ uses ζ, then c₁ uses ζ, and ζ continues with ζ; and also both situations rotated by 180°.

We treat only one case, the others being similar. So let c₁ use ζ. If ζ is not f₁-flat, then c₁ can be rerouted via ζ to give an f₁-hive preserving cycle c. Key Lemma 7.4 ensures that c is f₁-secure. This is a contradiction to cλ,µ = 2. Therefore, ζ is f₁-flat. If c₁ uses ζ, then c₁ can be rerouted via ζ again in contradiction to cλ,µ = 2. Therefore c₁ uses ζ. Since c₁ uses ζ, it follows that ζ is not f₁-flat. The hexagon equality (Claim 2.3) implies that ζ is not f₁-flat. If ζ were not f₁-flat, then c₁ could be rerouted via ζ, which would again result in a contradiction to cλ,µ = 2. Therefore ζ is f₁-flat. But this implies that c₁ uses ζ. □

If the pipe is thin, then we have exactly one inner region by definition. If the pipe is thick, then we have no inner region. The center curve of the thick pipe is not considered part of the pipe border. In fact, the thick pipe is defined to have an empty inner pipe border.
Flatspace sides and the Rerouting Theorem  We will need a special version of the fundamental Rerouting Theorem, Thm. 4.19 in [BI12]. For this reason we introduce some terminology.

Fix a flow $f \in P$. An $f$-flatspace side is defined to be a side of an $f$-flatspace, defined in [BI12]. But we can equivalently define $f$-flatspace sides as follows.

An $f$-flatspace side $a$ is a maximal line segment consisting of edges in $\Delta$ such that

1. each edge $\xi$ of $a$ is the diagonal of a non-$f$-flat rhombus $\iota$.
2. For two successive edges $\iota$ of $a$ all four rhombi contained in the two incident

The pipe has sides, which are defined to be maximal line segments of the pipe border. Inner pipe sides are contained in the inner pipe border, while outer pipe sides are contained in the outer pipe border.

7.9 Claim. All outer pipe sides can be partitioned into $f_1$-flatspace sides and all inner pipe sides can be partitioned into $f_2$-flatspace sides.

Proof. Claim 7.5 implies that all edges of outer pipe sides belong to $f_1$-flatspace sides. We now prove that the $f_1$-flatspace sides do not exceed the pipe sides. This is easy to see by looking at the following example, which represents the general case. Consider $c_1$ and the pipe side $a: \xi \ni \iota$. Then the following edges are diagonals of non-$f_1$-flat rhombi: $\iota \ni \iota$. This proves that $a$ can be partitioned into $f_1$-flatspace sides.

The following result is a direct corollary of [BI12, Thm. 4.19].

7.10 Proposition. If an $f$-hive preserving flow $f$ with zero throughput on the border of $\Delta$ has nonzero throughput through an edge of an $f$-flatspace side $a$, then there exists a turncycle $\hat{c}$ in $R_f$ that crosses $a$.

Proof of Proposition 7.3  Case 1: We first analyze $\xi$-hive preserving cycles that use pipe triangles only. If the pipe is thin, we note that a cycle that uses only pipe triangles necessarily equals $c_1$ or $c_2$. Since $c_1$ is $f_1$-hive preserving, the assertion follows.

Now assume that the pipe is thick and assume by way of contradiction that there is a $\xi$-hive preserving cycle $c$ that uses only pipe triangles but is neither $f_1$-hive preserving nor $f_2$-hive preserving. Hence there is a rhombus $g_1$ such that $\sigma(g_1, f_1) = 0$ and $\sigma(g_1, c) < 0$. If we had $\sigma(g_1, f_2) = 0$, then $\sigma(g_1, \xi) = 0$, which contradicts $\sigma(g_1, c) < 0$ as $c$ is $\xi$-hive preserving. Therefore $\sigma(g_1, f_2) > 0$. Similarly, by our assumption, there is also a rhombus $g_2$ with $\sigma(g_2, f_2) = 0$ and $\sigma(g_2, c) < 0$. We know that $c$ must use a negative contribution in $g_1$ and in $g_2$. We can now analyze all positions in which $g_1$ and $g_2$ can lie in the thick pipe and after a detailed but rather straightforward case distinction, which we omit here, we end up with a contradiction to $c_{\lambda, \mu} = 2$.

Case 2: For the sake of contradiction, we now assume the existence of a $\xi$-hive preserving cycle $c$ that does not use pipe triangles only. Claim 7.6 implies that $c$ uses at least one pipe triangle. Since $c$ does not use pipe triangles only, it follows that $c$ crosses the pipe border. Use Claim 7.9 and choose $i \in \{1, 2\}$ such that $c$ crosses the pipe border through an $f_i$-flatspace side $a$. Choose $\hat{x} \in \{x, 1 - x\}$ such that $\xi = f_i + \hat{x}c_i$. We have $f_i + \hat{x}c_i + \epsilon c \in P$ for a small $\epsilon > 0$ and hence $d := \hat{x}c_i + \epsilon c$ is $f_i$-hive preserving. Proposition 7.10 applied to $f_i$ and $d$ ensures the existence of a turncycle in $R_{f_i}$ that crosses the side $a$. According to Claim 7.9, $a$ is contained in the pipe border. Let $\hat{c}$ be a shortest turncycle in $R_{f_i}$ that crosses the pipe border. According to Proposition 6.2 (for turnpaths of length 0) and Proposition 6.3, $\hat{c}$ uses no reverse turnvertices and all self-intersections of the curve of $\hat{c}$ can only happen in $(\hat{c}, f_i)$-special rhombi, defined as follows:
For a turncycle \( c \) and a flow \( f \) we call a rhombus \( (c, f) \) special, if (1) \( \diamond \) is \( f \)-flat and (2) the diagonal of \( \diamond \) is crossed by \( c \) via \( \lozenge \), \( \heartsuit \), \( \spadesuit \), or \( \clubsuit \) and (3) \( c \) does not use any additional turnvertex in \( \diamond \).

First assume that \( \overline{c} \) is ordinary. But in this case, Key Lemma 7.4 implies that \( \overline{c} \) is \( f_\iota \)-secure, in contradiction to \( c_\nu \lambda, \mu = 2 \).

Now assume that \( \overline{c} \) is not ordinary, i.e., that the curve of \( \overline{c} \) has self-intersections. We will refine Key Lemma 7.4 to suit our needs (see Lemma 7.11 below). To achieve this, we now precisely analyze the situation at the self-intersections. We choose a \((\overline{c}, f_\iota)\)-special rhombus and reroute \( \overline{c} \) in \( c \) to obtain a shorter turncycle \( c' \) in \( R_{f_\iota} \) as follows:

\[
\begin{align*}
\lozenge & \rightsquigarrow \diamond \\
\heartsuit & \rightsquigarrow \lozenge \\
\spadesuit & \rightsquigarrow \spadesuit \\
\clubsuit & \rightsquigarrow \clubsuit
\end{align*}
\]

(‡)

Note that this rerouting is the unique way to reroute in an \( f_\iota \)-flat rhombus \( \diamond \) such that the resulting turncycle uses no negative slack contribution in \( \diamond \). Because of the minimal length of \( \overline{c} \), the turncycle \( c' \) does not cross the pipe border. According to Claim 7.6, \( c' \) uses pipe triangles only. We now show that \( c' \) is ordinary.

If the pipe is thin, this is obvious. Now assume that the pipe is thick. For the sake of contradiction, assume that \( c' \) is not ordinary. Since \( c' \) was obtained by rerouting from \( \overline{c} \), self-intersections of the curve of \( c' \) can only appear in \((c', f_\iota)\)-special rhombi. As \( c' \) uses pipe triangles only, the diagonals of \((c', f_\iota)\)-special rhombi are contained in the center curve, see Figure 5. But rerouting iteratively at these rhombi finally results in an ordinary \( f_\iota \)-hive preserving turncycle, which is shorter than \( c_\iota \). According to Key Lemma 7.4 this turncycle is \( f_\iota \)-secure, in contradiction to \( c_\nu \lambda, \mu = 2 \).

Hence \( c' \) is ordinary in both pipe cases. Since \( c' \) is \( f_\iota \)-hive preserving, it is \( f_\iota \)-secure (Key Lemma 7.4). The fact \( c_\nu \lambda, \mu = 2 \) implies that \( c' \) coincides with \( c_\iota \). Thus \( \overline{c} \) reroutes to \( c_\iota \), no matter in which \((\overline{c}, f_\iota)\)-special rhombus we reroute.

We can see that \( \overline{c} \) is \( f_\iota \)-secure, in contradiction to \( c_\nu \lambda, \mu = 2 \), by using the following Lemma 7.11.

**7.11 Lemma.** Let \( \iota \in \{1, 2\} \). If an \( f_\iota \)-hive preserving turncycle \( c \) has no reverse turnvertices and all self-intersections of the curve of \( c \) occur in \((c, f_\iota)\)-special rhombi in which \( c \) reroutes to \( c_\iota \) when applying (‡), then \( c \) is \( f_\iota \)-secure.

This finishes the proof of Proposition 7.3.

Lemma 7.11 is a refined version of Key Lemma 7.4. We postpone its proof to Subsection 7.3 because it is based on ideas of the following subsection.

**7.2 Proof of Key Lemma 7.4**

**Proof.** We will show that

for each ordinary turncycle in \( R_{f_\iota} \) that is not \( f_\iota \)-secure there exist two distinct shorter ordinary turncycles in \( R_{f_\iota} \).

If there exists an ordinary turncycle in \( R_{f_\iota} \) that is not \( f_\iota \)-secure, then take one of minimal length. It follows from (‡) that there exist two distinct ordinary \( f_\iota \)-secure turncycles. This is a contradiction to \( c_\nu \lambda, \mu = 2 \).
It remains to prove (**). Recall that ordinary turncycles in \( R_i \) are in bijection to proper cycles in \( G \). Let \( c \) be an ordinary turncycle on \( R_i \) that is not \( f_i \)-secure. By Definition 2.10 and Proposition 2.14 we have \( f_i + c \notin P \) and hence there exists \( 0 < \varepsilon < 1 \) such that \( f_i + \varepsilon c \in P \). According to Claim 2.7 we have \( \sigma(g, c) \in \{-2, -1, 0, 1, 2\} \) for all rhombi \( g \). Hence there exists a rhombus \( g \) with \( \sigma(g, f_i) = 1 \) and \( \sigma(g, c) = -2 \). We call such rhombi bad. Let \( \diamond \) be a bad rhombus. Then \( c \) uses \( \diamond \) by Proposition 2.9.

We begin by analyzing a very special case: If all four rhombi \( \nabla, \triangledown, \triangleright, \triangleright \), and \( \triangleright \) are not \( f_i \)-flat, then \( c \) can be rerouted twice: Once via \( \triangledown \) and once via \( \triangleright \), which results in two ordinary turncycles in \( R_i \). This proves (**)) in this special case. In the more general case, we prove the following:

In each bad rhombus \( \diamond \) the ordinary turncycle \( c \) in \( R_i \) can be rerouted via \( \triangledown \) or one of the three rhombi \( \triangledown, \triangleright, \triangleright \), or \( \triangleright \), respectively. Additionally, in each bad rhombus \( \diamond \) the ordinary turncycle \( c \) in \( R_i \) can be rerouted via \( \triangledown \) or one of the three rhombi \( \nabla, \triangledown, \triangleright, \triangleright \), or \( \triangledown \), respectively.

We first show that (**)) implies (**). According to (**)), each bad rhombus that cannot be rerouted at the left has another bad rhombus located at its left, and analogously for its right side. We continue finding bad rhombi in this manner and obtain a set of adjacent bad rhombi, which we call the chain. The chain has two endings at which \( c \) can be rerouted to shorter ordinary turncycles. Hence (**)) follows.

We now show that (**)) holds. First, we precisely characterize the situations in which \( c \) can be rerouted in \( R_i \) via \( \triangledown \): This is exactly the case when

\[
\text{both} \quad \left( \nabla \text{ is not } f_i \text{-flat or } c \text{ uses } \downarrow \right) \quad \text{and} \quad \left( \triangleright \text{ is not } f_i \text{-flat or } c \text{ uses } \uparrow \right).
\]

Now assume that \( c \) cannot be rerouted via \( \triangledown \), i.e.,

\[
\left( \nabla \text{ is } f_i \text{-flat and } c \text{ uses } \downarrow \right) \quad \text{or} \quad \left( \triangleright \text{ is } f_i \text{-flat and } c \text{ uses } \uparrow \right).
\]

We demonstrate how to prove (**)) in the following exemplary case, all others being similar: Let \( \nabla \) be \( f_i \)-flat with \( c \) using \( \downarrow \) and let \( \triangleright \) be \( f_i \)-flat with \( c \) using \( \uparrow \). The hexagon equality (Claim 2.5) applied twice implies that we have either \( \sigma(\nabla, f_i) = 0, \sigma(\triangleright, f_i) = 1, \sigma(\nabla, f_i) = 1, \sigma(\triangleright, f_i) = 1, \sigma(\nabla, f_i) = 0, \sigma(\triangleright, f_i) = 0 \). The latter is impossible, because \( c \) uses \( \triangledown \). The fact that \( c \) uses no negative contributions in \( f_i \)-flat rhombi and that \( c \) is ordinary leads to \( c \) running as desired: \( \triangledown \) with \( \nabla \) being bad. All other cases are similar.

### 7.3 Proof of Lemma 7.11

We now complete the proof of Proposition 7.3 by proving Lemma 7.11.

**Proof.** The proof is completely analogous to the proof of Key Lemma 7.10. We only highlight the technical differences here. Since, in contrast to Key Lemma 7.4, we are not dealing with ordinary turncycles only, we make the following definition.

**7.12 Definition.** Let \( i \in \{1, 2\} \). If an \( f_i \)-hive preserving turncycle \( c \) has no reverse turnvertices and all self-intersections of the curve of \( c \) occur in \( (c, f_i) \)-special rhombi in which \( c \) reroutes to \( c \), when applying \( (\dagger) \), then \( c \) is called almost ordinary.

Note that this notion depends on \( i \), which we think of being fixed in the following. In analogy to Key Lemma 7.4, Lemma 7.11 now reads as follows:

Every almost ordinary turncycle is \( f_i \)-secure.
We will show the following statement:

For each almost ordinary turncycle in $R_{f}$ that is not $f$-secure there exist two distinct shorter almost ordinary turncycles in $R_{f}$. (∗)

As in the proof of Key Lemma 7.4, in order to prove Lemma 7.11 it suffices to show (∗). So fix an almost ordinary turncycle $c$. First of all, since $c$ is not necessarily ordinary, we have $\sigma(\langle \rangle, c) \in \{-4, -3, \ldots, 3, 4\}$ for all rhombi $\langle \rangle$. But we show now that $\sigma(\langle \rangle, c) \geq -2$ for all rhombi $\langle \rangle$.

We begin by showing that there is no rhombus $\langle \rangle$ with $\sigma(\langle \rangle, c) = -4$. Recall that $\sigma(\langle \rangle, c) = \gamma(c) + \delta(c)$. If $\sigma(\langle \rangle, c) = -4$, then both $\gamma$ and $\delta$ are $(c, f_{i})$-special. But since $\sigma(\langle \rangle, c) = \gamma(c) + \delta(c)$, it follows that $\gamma$ and $\delta$ are $(c, f_{i})$-special as well. This is a contradiction to the fact that $(c, f_{i})$-special rhombi do not overlap (true by definition, cp. Proposition 6.3(2)). Hence $\sigma(\langle \rangle, c) \geq -3$ for all rhombi $\langle \rangle$.

We show next that $\sigma(\langle \rangle, c) \neq -3$. Assume the contrary. W.l.o.g. let $\gamma(c) = -2$ and $\delta(c) = -1$, the other case being the same, just rotated by 180°. Then $c$ is bound to use $\delta$. But, according to our assumption that in $(c, f_{i})$-special rhombi $c$ reroutes to $c_{i}$, this means that $c_{i}$ uses $\delta$, which is a contradiction to Claim 7.8.

So it follows that $\sigma(\langle \rangle, c) \geq -2$ for all rhombi. This is exactly the same situation as in Key Lemma 7.4. As in the proof of Key Lemma 7.4 we call rhombi $\langle \rangle$ bad if $\sigma(g, c) = -2$ and $\sigma(g, f_{i}) = 1$. There exists a bad rhombus $\langle \rangle$, since $c$ is assumed to be not $f_{i}$-secure. But here is a technical difference: Unlike in Key Lemma 7.4, $c$ does not necessarily use exactly the turnvertices $\langle \rangle$ in $\langle \rangle$, but $c$ uses exactly one of the following sets of turnvertices in $\langle \rangle$: $\langle \rangle$, $\langle \rangle$, $\langle \rangle$, or $\langle \rangle$. We now show that only the first 5 cases can appear, because the last 4 cases are in contradiction to the fact that $c$ reroutes to $c_{i}$ in $(c, f_{i})$-special rhombi: In the case where $c$ uses $\langle \rangle$, then the cycle $c_{i}$ on $G$ uses $\delta$, which is impossible. The other three cases are treated similarly.

We want to prove that there exists a chain of adjacent bad rhombi as we did in Key Lemma 7.4. Analogously to Key Lemma 7.4 we can prove the following statement, which is more technical than (∗∗):

Case $\langle \rangle$: For each bad rhombus $\langle \rangle$ where the turncycle $c$ in $R_{f}$ uses only the turnvertices $\delta$, one of the following holds: (1) $c$ can be rerouted via $\gamma$, $\delta$, $\gamma$, or $\delta$, or (2) one of the three rhombi $\langle \rangle$, $\langle \rangle$, or $\langle \rangle$ is bad such that $c$ uses $\delta$, $\delta$, or $\delta$, respectively. Additionally, as in (∗∗), this holds for the situation rotated by 180°.

Case $\langle \rangle$: For each bad rhombus $\langle \rangle$ where $c$ uses exactly the turnvertices $\langle \rangle$ in $\langle \rangle$, the turncycle $c$ can be rerouted via $\gamma$ and $\delta$ or $\delta$ is bad. Remaining cases: Results that are analogous to the second case hold for $c$ using $\langle \rangle$, $\langle \rangle$, or $\langle \rangle$, respectively.

We remark that the strange reroutings $\gamma$, $\delta$, and $\delta$ occur in the cases where $c$ uses $\delta$, $\delta$, or $\delta$, respectively. As in the proof of Key Lemma 7.4 (∗∗) can be seen to imply (∗) by constructing a chain of bad rhombi.

Theorem 1.1 is completely proved.

References

[AMO93] Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin. Network flows: theory, algorithms, and applications. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1993.
[BI12] Peter Bürgisser and Christian Ikenmeyer. Deciding Positivity of Littlewood-Richardson coefficients. arXiv:1204.2484v1 [math.RT], 2012.

[Buc00] Anders Skovsted Buch. The saturation conjecture (after A. Knutson and T. Tao) with an appendix by William Fulton. *Enseign. Math.*, 2(46):43–60, 2000.

[BZ92] A. D. Berenstein and A. V. Zelevinsky. Triple multiplicities for $\mathfrak{sl}(r + 1)$ and the spectrum of the exterior algebra of the adjoint representation. *J. Algebraic Comb.*, 1(1):7–22, 1992.

[DLM06] Jesús A. De Loera and Tyrrell B. McAllister. On the computation of Clebsch-Gordan coefficients and the dilation effect. *Experiment. Math.*, 15(1):7–19, 2006.

[FM94] K. Fukuda and T. Matsui. Finding all the perfect matchings in bipartite graphs. *Appl. Math. Lett.*, 7(1):15–18, 1994.

[Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[GLS93] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, second edition, 1993.

[KT99] Allen Knutson and Terence Tao. The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[KT01] Allen Knutson and Terence Tao. Honeycombs and sums of Hermitian matrices. *Notices Amer. Math. Soc.*, 48(2):175–186, 2001.

[KTT04] R. C. King, C. Tollu, and F. Toumazet. Stretched Littlewood-Richardson and Kostka coefficients. In *Symmetry in physics*, volume 34 of *CRM Proc. Lecture Notes*, pages 99–112. Amer. Math. Soc., Providence, RI, 2004.

[MS05] Ketan D. Mulmuley and Milind Sohoni. Geometric complexity theory III: On deciding positivity of Littlewood-Richardson coefficients. cs.ArXive preprint cs.CC/0501076, 2005.

[Nar06] Hariharan Narayanan. On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients. *J. Algebraic Combin.*, 24(3):347–354, 2006.

[Tar86] Éva Tardos. A strongly polynomial algorithm to solve combinatorial linear programs. *Oper. Res.*, 34(2):250–256, 1986.