Non-Noetherian representation categories of generalized fields

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Abstract

In this paper, we show that the categories of finitely generated projective $\mathbb{B}$-modules and $F_\infty$-modules with morphisms being (splittable) injections are not locally Noetherian. This provides another instance of the fact that these generalized fields have strange homological behavior.

1 Introduction

The framework of $FI$-modules introduced by Church, Ellenberg and Farb [3] and its generalization, the notion of quasi-Gröbner Category, introduced by Sam and Snowden [14] provided a powerful framework to prove representation-theoretic results in various fields of mathematics. These results include the representation stability of cohomology groups of configuration spaces of points on an arbitrary (connected, oriented) manifold [3, 4], the proof of the Lannes-Schwartz Artinian Conjecture [13, 14]; and polynomial growth (on the number of edges) of the dimension of homology groups of configuration spaces of points on trees and graphs [12, 11].

The Lannes-Schwartz Artinian Conjecture states (dually) that for any finite field, $F_q$, the category of finitely generated vector spaces, $\text{Vect}_{F_q}$ over $F_q$ with morphisms being splittable injections is locally Noetherian. It is natural to ask whether this statement extends to finite generalized fields. Generalized fields were introduced by Durov in his thesis [6] as a part of his new approach to Arakelov geometry [1, 2]. In this paper, we focus on two generalized fields, $\mathbb{B}$ and $F_\infty$. In the theory of Durov, the generalized field, $F_\infty$, is the residue field at infinity, while the Boolean semi-ring $\mathbb{B}$ is treated as a semi-field of characteristic one [5]. Even though basic algebro-geometric properties of the generalized fields, $\mathbb{B}$ and $F_\infty$ resemble the properties of usual finite fields [6, 10, 8], the homological properties are far from ordinary [5, 13].

In fact, in this paper, we show that the category of finitely generated projective $\mathbb{B}$-modules (or $F_\infty$-modules) with morphisms being injections is not locally Noetherian illustrating another instance of the fact that generalized fields have strange homological behavior.
2 \( \mathbb{B} \)-modules

Definition 2.1. The Boolean semi-ring \( \mathbb{B} \) is defined as the set \( \{0, 1\} \) equipped with two operations: a binary \( + \) and \( \cdot \), with the following properties:

- \((a + b) + c = a + (b + c), a + b = b + a;\)
- \(0 + a = a + 0 = a, \) hence 0 is an neutral element;
- \(a + a = a;\)
- \((a \cdot b) \cdot c = a \cdot (b \cdot c), a \cdot b = b \cdot a;\)
- \(1 \cdot a = a \cdot 1 = a.\)

Given this algebraic structure, a \( \mathbb{B} \)-module is defined as follows.

Definition 2.2. A \( \mathbb{B} \)-module is a structure \((V, 0, -, +)\) such that the following identities hold:

- \((a + b) + c = a + (b + c), a + b = b + a;\)
- \(a + a = a, a + 0 = a.\)

A notions of submodule, congruence, quotient module and module homomorphism, finitely generated module are defined as usual.

Note that a \( \mathbb{B} \)-module is join-semilattice with a minimal element (0), the join of two elements is exactly \( + \).

We say that a \( \mathbb{B} \)-module \( P \) is projective, if the usual lifting property holds: given any surjective morphism of \( \mathbb{B} \)-module \( f : M \to N \) and a \( \mathbb{B} \)-module morphism \( g : P \to N \), there exists a \( \mathbb{B} \)-module morphism \( h : P \to M \) so that \( g = f \circ h. \) It is not hard to show that a finitely generated \( \mathbb{B} \)-module \( P \) is projective if and only if there exists a splittable surjection \( F \to P \) from a free \( \mathbb{B} \)-module \( F \) to \( P \) (by splittable surjection, we mean a surjection which has a right inverse). In this way, the projective \( \mathbb{B} \)-modules can be equipped with a structure of a finite distributive lattice \([9]\).

Example 1. Consider the following finite distributive lattices of size \( 4n - 3 \) (see the figure below) denoted by \( D_n \) (here \( n > 1 \). The elements of the lattice are denoted by \( a_{ik} \) representing the position of the element in the lattice. The operations (join and meet) \( \lor \) and \( \land \) are defined as follows:

\[
a_{ik} \lor a_{lm} = a_{\max(i,l),\max(k,m)}, \quad a_{ik} \land a_{lm} = a_{\min(i,l),\min(k,m)}.\]
We remark that the $\mathcal{B}$-module structure is given by the join operation $\vee$.

It is easy to see that there exists a splittable surjection $F \to D_n$. Indeed, the surjective map $g : \mathcal{B}[A_1, \ldots, A_{2n-1}] \to D_n$ from the free $\mathcal{B}$-module generated by $A_1, A_2, \ldots, A_{2n-1}$ given by

$$g(A_1) = a_{11}, g(A_2) = a_{12}, g(A_3) = a_{21}, \ldots, g(A_{2i}) = a_{i-1,i+1}, g(A_{2i+1}) = a_{i+1,i}, \ldots$$

splits: the injection $h : D_n \to \mathcal{B}[A_1, \ldots, A_{2n-1}]$ generated by

$$h(a_{11}) = A_1, h(a_{12}) = A_1 + A_2, h(a_{21}) = A_1 + A_3, \ldots$$

$$h(a_{i-1,i+1}) = A_1 + A_2 + \ldots + A_{2i-2} + A_{2i}, h(a_{i,i+1}) = A_1 + A_2 + \ldots + A_{2i-1} + A_{2i+1},$$

provides a right inverse.

We have the following property for these distributive lattices.

**Proposition 2.3.** Let $D_n$ and $D_m$ be lattice as in Example 1 with elements $a_{11}, \ldots, a_{nn}$ and $b_{11}, \ldots, b_{mm}$ respectively. Then, any injective $\mathcal{B}$-module morphism $f : D_n \to D_m$ satisfying

$$f(a_{11}) = b_{11}, f(a_{12}) = b_{12}, f(a_{21}) = b_{21}, f(a_{22}) = b_{22}, \text{ and}$$
\( f(a_{n-1,n-1}) = b_{m-1,m-1}, f(a_{n-1,n}) = b_{m-1,m}, f(a_{n,n-1}) = b_{m,m-1}, f(a_{nn}) = b_{mm} \)
is the identity morphism (and hence \( n = m \)).

**Proof.** First, we show that \( f(a_{13}) = b_{13} \). Indeed, \( a_{13} \) is the only element (other than \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \)) in the lattice which is not bigger than \( a_{22} \).

As a consequence, since \( f \) is a \( B \)-module map, we have to have that

\[
\begin{align*}
  f(a_{23}) &= f(a_{22} + a_{13}) = b_{22} + b_{13} = b_{23}.
\end{align*}
\]

Now, we show that \( f(a_{32}) = b_{32} \). Indeed, \( a_{32} \) is the only element (other than the previously discussed elements) which is not bigger than \( a_{23} \). Hence, \( f(a_{32}) = b_{32} \). By continuing this process, we can show step by step that \( f(a_{ij}) = b_{ij} \) which concludes the proof.

**Example 2.** Similarly, we consider the finite distributive lattice of 9 elements depicted in the figure below. We will denote this lattice by \( D_0 \).

Note that this is a sub-lattice of every \( D_n \) for \( n > 3 \).

**Lemma 2.4.** The injections \( i : D_0 \to D_n \) for \( n > 3 \) defined as

\[
\begin{align*}
  i(A_{11}) &= a_{11},
  i(A_{12}) &= a_{12},
  i(A_{21}) &= a_{21},
  i(A_{22}) &= a_{22},
  i(A_{33}) &= a_{n-1,n-1},
  i(A_{34}) &= a_{n-1,n},
  i(A_{43}) &= a_{n,n-1},
  i(A_{44}) &= a_{nn},
\end{align*}
\]
are splittable injections of \( B \)-modules.
Proof. It is easy to see that the map \( j : \mathcal{D}_n \to \mathcal{D}_0 \) given by

\[
\begin{align*}
j(a_{kl}) = \begin{cases} 
A_{kl} & k, l \leq 2, \\
A_{33} & \text{if } a_{12} < a_{kl} \leq a_{n-1,n-1} \text{ and } a_{kl} \neq a_{22}, \\
A_{34} & a_{kl} = a_{n-1,n} \text{ or } a_{kl} = a_{n-2,n}, \\
A_{43} & a_{kl} = a_{n,n-1}, \\
A_{44} & a_{kl} = a_{nn}
\end{cases}
\end{align*}
\]

provides a left inverse. \( \square \)

3 \( \mathbb{F}_\infty \)-modules

In this section, we introduce our motivating example, the category of finitely generated \( \mathbb{F}_\infty \)-modules. Our main reference for this section is \([8]\).

We start with the definition of the generalized ring, \( \mathbb{F}_\infty \) (see \([6]\), 5.1.16).

**Definition 3.1.** The generalized field or field \( \mathbb{F}_\infty \) is defined as the set \( \{-1, 0, 1\} \) equipped with three operations: a unary \( \cdot \), and a binary \( \cdot \) and \( \cdot \), with the following properties:

- \( (a \cdot b) \cdot c = a \cdot (b \cdot c), a \cdot b = b \cdot a; \)
- \( 0 \cdot a = a \cdot 0 = 0, \) hence 0 is an absorbing element;
- \( a \cdot a = a; \)
- \( a \cdot (-a) = 0; \)
- \( -1) = 1, -(1) = 1, -0 = 0; \)
- \( (a \cdot b) \cdot c = a \cdot (b \cdot c), a \cdot b = b \cdot a; \)
- \( 1 \cdot a = a \cdot 1 = a; \)
- \( (-1) \cdot (-1) = 1. \)

We remark that the operator \( \cdot \) is unlike the ordinary addition: the element 0 is an absorbing element \((0 + a = 0); \) and the operation \( - \) is not the additive inverse, for instance, \((a + a) + (-a) \) is in general not \( a, \) but rather \( a + (a + (-a)) = a + 0 = 0. \)

This discussion leads us to the definition of a module over \( \mathbb{F}_\infty \) (see \([6]\), 4.3.7).

**Definition 3.2.** An \( \mathbb{F}_\infty \)-module is a structure \((V, 0, -, \cdot)\) such that:

- \( (a \cdot b) + c = a \cdot (b + c), a \cdot b = b \cdot a; \)
- \( a \cdot a = a, a \cdot (-a) = 0; \)
- \( -a + b = (-a) + (-b); -(-a) = a. \)

Again, the notions of submodule, congruence, quotient module, module homomorphism and finitely generated modules are defined as usual.
We remark that contrary to usual conventions, 0 is not a neutral element, and \(-a\) is not an additive inverse. In fact, an \(F_\infty\)-module structure on a set \(V\) gives rise to a natural partial order on \(V\) given by \(a \leq b\) if \(a + b = b\); in this partial order 0 is the maximal element.

We, again, consider projective \(F_\infty\)-modules, modules with the usual lifting property. Similarly as with usual modules, a finitely generated \(F_\infty\)-module \(P\) is projective if and only if there exists a splittable surjection \(F \to P\) from a free \(F_\infty\)-module \(F\) to \(P\).

**Example 3.** We modify Example 1 to obtain projective \(F_\infty\)-modules. Let \(D_n\) be the projective \(B\)-module as in Example 1. For each element \(a_{ij}\) we add another element \(-a_{ij}\). The addition \(\dot{+}\) is defined in an opposite fashion as before, explicitly, on the \(a_{ij}\), we have
\[
a_{ij} \dot{+} a_{kl} = a_{\min(i,k),\min(j,l)},
\]
on the \(-a_{ij}\), we have
\[
(-a_{ij}) \dot{+} (-a_{kl}) = -a_{\min(i,k),\min(j,l)},
\]
and finally
\[
a_{ij} \dot{+} (-a_{kl}) = 0.
\]
In this fashion, we obtain an \(F_\infty\)-module, which we call \(E_n\) (the figure below illustrates the module \(E_3\)).

Again, there exists a surjection \(g : F_\infty[A_1, A_2, ..., A_{2n-1}] \to E_n\) generated by the images
\[
g(A_1) = a_{nn}, g(A_2) = a_{n,n-1}, g(A_3) = a_{n-1,n}, ...
\]
\[
g(A_{2i}) = a_{n-i,n-i+2}, g(A_{2i+1}) = a_{n-i+1,n-i}.
\]
This surjection splits, the map \(h : E_n \to F_\infty[A_1, A_2, ..., A_{2n-1}]\) given by the images
\[
h(a_{nn}) = A_1, h(a_{n,n-1}) = A_1 \dot{+} A_2, h(a_{n-1,n}) = A_1 \dot{+} A_3, ...
\]
\[
h(a_{n-i,n-i+2}) = A_1 \dot{+} A_2 \dot{+} ... \dot{+} A_{2i-2} + A_{2i}, h(a_{n-i+1,n-1}) = A_1 \dot{+} ... \dot{+} A_{2i-1} \dot{+} A_{2i+1}
\]
provides a right inverse.
As before, we have the following property.

**Proposition 3.3.** Let $E_n$ and $E_m$ be $\mathbb{F}_\infty$-modules as in Example 3 with elements $a_{11}, \ldots, a_{nn}$ and $b_{11}, \ldots, b_{mm}$ respectively. Then, any injective $\mathbb{F}_\infty$-module morphism $f : E_n \to E_m$ satisfying

\[
\begin{align*}
    f(a_{11}) = b_{11}, & \quad f(a_{12}) = b_{12}, \quad f(a_{21}) = b_{21}, \quad f(a_{22}) = b_{22}, \\
    f(a_{n-1,n-1}) = b_{m-1,m-1}, & \quad f(a_{n,n}) = b_{m,n}, \quad f(a_{n,n-1}) = b_{m,n-1}, \quad f(a_{nn}) = b_{mm}
\end{align*}
\]

is the identity morphism (and hence $n = m$).

We omit the proof of the proposition above, it follows the same strategy as in Proposition 2.3, but in this case, the recursive process begins with $a_{n-2,n}$ instead of $a_{13}$.

**Example 4.** As in Example 3 we define an $\mathbb{F}_\infty$-module of 17 elements depicted below. We denote this module by $E_0$. For every $n > 3$, we have an injection $E_0 \to E_n$ defined as

\[
\begin{align*}
    k(A_{11}) = a_{11}, & \quad k(A_{12}) = a_{12}, \quad k(A_{21}) = a_{21}, \quad k(A_{22}) = a_{22}, \\
    k(A_{33}) = a_{n-1,n-1}, & \quad k(A_{34}) = a_{n-1,n}, \quad k(A_{43}) = a_{n,n-1}, \quad k(A_{44}) = a_{nn}
\end{align*}
\]
and $k(-A_{ij}) = -k(A_{ij})$. This injection is a splittable injection as in Lemma 2.4, we omit the proof of this statement, it is completely parallel (reversed).

4 Representation categories

We consider the categories, $F(R)$ (and $FI(R)$), of finite rank free $R$-modules with morphisms being injections (or splittable injections resp.) for a generalized ring. Similarly, let $P(R)$ (and $PI(R)$), denote the category of finitely generated projective $R$-modules with morphisms being injection (or splittable injection resp.) for a generalized ring. In this section, we show that the categories $F(\mathbb{B})$, $F(\mathbb{F}_\infty)$, $FI(\mathbb{B})$ and $FI(\mathbb{F}_\infty)$ are locally Noetherian, however, the categories $P(\mathbb{B})$, $PI(\mathbb{B})$, $P(\mathbb{F}_\infty)$ and $PI(\mathbb{F}_\infty)$ are not locally Noetherian. This gives another illustration of the
strange homological behavior of the generalized rings $B$ and $F_{\infty}$.

We start by recalling the basic definitions of $[14]$ and $[13]$ concerning representation categories. Let $\mathcal{C}$ be a category and $\text{Mod}_k$ the category of (left) modules over a (left) Noetherian ring $k$. The representation category, $\text{Rep}_k(\mathcal{C})$ is defined as the category whose objects are covariant functors $\mathcal{C} \to \text{Mod}_k$ and whose morphisms are natural transformations between functors. Objects of the representation category $\text{Rep}_k(\mathcal{C})$ are called $\mathcal{C}$-modules. For instance, for every $X \in \text{obj}(\mathcal{C})$ we obtain a $\mathcal{C}$-module (called the principal project module) defined as follows. For an object $Y \in \text{obj}(\mathcal{C})$, we assign the vector space generated by all maps $f : X \to Y$ in $\mathcal{C}$:

$$P_X(Y) = k[\text{Hom}_\mathcal{C}(X,Y)] = \bigoplus_{f : X \to Y} k \cdot e_f$$

and for a morphism $g : Y \to Z$ we assign the $k$-module morphism

$$k[\text{Hom}_\mathcal{C}(X,Y)] \to k[\text{Hom}_\mathcal{C}(X,Z)] : e_f \mapsto e_{gf}.$$  

The representation category, $\text{Rep}_k(\mathcal{C})$ is an Abelian category, with kernels, cokernels, images, direct sums, ... calculated point-wise. We say that a $\mathcal{C}$-module is finitely generated if it is a quotient of a finite direct sum of $P_X$. This definition is equivalent to the definition given in the Introduction (see $[14]$).

We say that the representation category $\text{Rep}_k(\mathcal{C})$ is Noetherian if every submodule of a finitely generated $\mathcal{C}$-module is also finitely generated. Similarly, we say that $\mathcal{C}$ is locally Noetherian if the representation category $\text{Rep}_k(\mathcal{C})$ is Noetherian for every (left-)Noetherian ring $k$.

### 4.1 Negative results

The following proposition shows that Property (G2) of $[14]$ is a necessary condition for a category to be locally Noetherian. For the sake of completeness, we provide the proof of this proposition.

**Proposition 4.1.** Let $\mathcal{C}$ be an essentially small category. Assume $\mathcal{C}$ contains a fixed object $X_0$ and countably infinity distinct objects $Y_i$. If for each $i$ there exists a morphism $f_i \in \text{Hom}_\mathcal{C}(X_0,Y_i)$ that does not factor as $X_0 \to Y_j \to Y_i$ for some $j < i$, then $\mathcal{C}$ is not locally Noetherian.

**Proof.** Consider the principal projective module $P_{X_0} \in \text{Rep}_k(\mathcal{C})$ and the submodule $M$ generated by $\bigcup_{i=1}^{\infty} P_{X_0}(Y_i)$. Suppose $M$ is finitely generated by $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_j \in M(X_j)$ for some object $X_j \in \mathcal{C}$. By the definition of $M$, each $\alpha_j$ is equal to a finite sum of the form

$$\sum_{i=1}^{N_j} \sum_{g_{ij} : Y_i \to X_j} M(g_{ij})(\beta_{g_{ij}}),$$

where $\beta_{g_{ij}} \in P_{X_0}(Y_i)$. Hence, we conclude $M$ is also generated by $\bigcup_{i=1}^{N} P_{X_0}(Y_i)$, where $N = \max\{N_j \mid j \in \{1,2,\ldots,\ell\}\}$. However, by assumption $e_{f_{N+1}} \in P_{X_0}(Y_{N+1})$ can not be generated by these elements. We conclude that $M$ is not finitely generated and therefore that $\mathcal{C}$ is not locally Noetherian.
The proposition above implies that \( P(\mathcal{B}) \), \( PI(\mathcal{B}) \), \( P(F_\infty) \) and \( PI(F_\infty) \) are not locally Noetherian.

**Corollary 4.2.** The categories \( P(\mathcal{B}) \), \( PI(\mathcal{B}) \), \( P(F_\infty) \) and \( PI(F_\infty) \) are not locally Noetherian.

**Proof.** The remark below implies that it is enough to prove the statement for \( P(\mathcal{B}) \) and \( P(F_\infty) \). Finally, Proposition 4.1 combined with Proposition 2.3 and 3.3 imply that \( P(\mathcal{B}) \) and \( P(F_\infty) \) are not locally Noetherian (with \( X_0 \) being the projective module \( D_0 \) or \( E_0 \) respectively).

**Remark:** If the category \( C \) is a full subcategory of \( D \), and \( C \) satisfies the conditions of Proposition 4.1, then \( D \) also satisfies the conditions of Proposition 4.1 with the same objects. Similarly, if \( C \) is a faithful subcategory of \( D \) containing the objects \( X_0 \), the \( Y_i \) and the morphisms \( f_i : X_0 \to Y_i \) so that \( D \) satisfies the conditions of Proposition 4.1 then \( C \) also satisfies the conditions of Proposition 4.1. As an easy consequence, we obtain that the category of finite posets with injections and the category of finite lattices with injections are also not locally Noetherian (see, for instance, [7]). Similarly, we obtain that the category of all finitely generated \( \mathcal{B} \)-modules (or \( F_\infty \)-modules) with injection is also not locally Noetherian.

### 4.2 Positive results

We conclude the paper by showing that the category of free \( \mathcal{B} \)-modules (or \( F_\infty \)-modules) of finite rank with (splittable) injections is locally Noetherian. We will only prove the statement for \( \mathcal{B} \)-modules, the same proof applies for \( F_\infty \)-modules with minimal changes. The proof is similar to the proofs of Theorem 8.3.1. in [14] and of Theorem C in [13], we only highlight the key steps.

**Proposition 4.3.** The category of free \( \mathcal{B} \)-modules of finite rank with (splittable) injections is locally Noetherian.

**Proof.** We first show the statement in the case of splittable injections. Let \( FS \) denote the category of finite sets with morphisms being surjections. By Theorem 8.1.2 in [14], the opposite category \( FS^{op} \) is quasi-Gröbner. Consider the essentially surjective functor \( \Phi : FS^{op} \to F(\mathcal{B}) : S \mapsto \mathcal{B}[S]^* = \text{Hom}_{\mathcal{B}}(\mathcal{B}[S], \mathcal{B}) \). Let \( \mathcal{B}^n \) be a free \( \mathcal{B} \)-module and let \( f : \mathcal{B}^n \to \mathcal{B}[S]^* \) be a splittable injection for a finite set \( S \). The dual surjection \( f^* : \mathcal{B}[S] \to (\mathcal{B}^n)^* \) factorises as \( \mathcal{B}[S] \to \mathcal{B}[T] \to (\mathcal{B}^n)^* \), where the first map is induced by a surjection of finite sets. As a consequence, \( \Phi \) satisfies Property (F) of [14] implying that \( F(\mathcal{B}) \) is locally Noetherian. Now, we turn our attention to prove that \( F(\mathcal{B}) \) is also locally Noetherian. Note that any map \( f : \mathcal{B}^n \to \mathcal{B}^m \) can be given by an \([m \times n] \) matrix, \( A \), whose entries are 0 or 1. Let \( l \) denote the number of distinct rows of this matrix. Note, that \( l \leq 2^n \) and we obtain a map \( \mathcal{B}^n \to \mathcal{B}^l \) of \( f \), given by the matrix \( B \) consisting of the distinct rows of \( A \). Moreover, we have a map \( \mathcal{B}^l \to \mathcal{B}^m \) sending the standard bases vector indexed by a row of \( B \) to the sum of standard bases vector indexed by the equal rows in \( A \). This map is a splittable injection and the two maps above yield a factorisation \( \mathcal{B}^n \to \mathcal{B}^l \to \mathcal{B}^m \).
of $f$. As a result, the forgetful functor $FI(B) \rightarrow F(B)$ satisfies Property (F). Since this functor is also essentially surjective, we obtain that $F(B)$ is also locally Noetherian.

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