Loss of non-Gaussianity for damped photon-subtracted thermal states

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Abstract

We investigate non-Gaussianity properties for a set of classical one-mode states obtained by subtracting photons from a thermal state. Three distance-type degrees of non-Gaussianity used for these states are shown to have monotonic behaviour with respect to their mean photon number. Decaying of their non-Gaussianity under damping is found to be consistently described by the distance-type measures considered here. We also compare the dissipative evolution of non-Gaussianity when starting from M-photon-subtracted and M-photon-added thermal states.

Keywords: classicality, photon-number distribution, non-Gaussianity, damping, photon-subtracted thermal states

(Some figures may appear in colour only in the online journal)

1. Introduction

In quantum optics, non-Gaussian states were studied in connection with some non-classical properties such as photon antibunching and quadrature or amplitude-squared squeezing. A survey on non-classicality defined as the non-existence of the Glauber–Sudarshan $P$ representation as a genuine probability density can be found in [1]. Interest in the non-Gaussian states was then renewed in quantum information processing due to their efficiency in some quantum protocols [2, 3]. In general, their usefulness was connected to certain non-classicality properties detected by negative values of the Wigner function. However, it has recently been realized that quantum states with non-negative (Gaussian or non-Gaussian) Wigner functions can be efficiently simulated on a classical computer [4, 5]. It appears that non-Gaussianity of a quantum state is a useful feature in quantum information processing, regardless of being non-classical or classical according to the concepts of quantum optics. To quantify this property as a resource, some non-Gaussianity measures were recently defined as distances between the given state $\hat{\rho}$ and its associate Gaussian state $\hat{\tau}_G$. Here $\hat{\tau}_G$ is the unique Gaussian state having the same mean displacement and covariance matrix as $\hat{\rho}$ [6–10]. In [6–8], Genoni et al used the Hilbert–Schmidt metric and the relative entropy as distances and gave a comprehensive discussion of the general properties of non-Gaussianity degrees for large sets of one-mode, two-mode and multimode states. Later, in [9, 10], a degree of non-Gaussianity based on the Bures metric was similarly introduced. We stress now that all the distance-type measures considered so far used the same state $\hat{\tau}_G$ as a reference Gaussian state in evaluating non-Gaussianity. It was only very recently that two of us succeeded in proving that the relative entropy of any N-mode state to its associate Gaussian one $\hat{\tau}_G$ is an exact distance-type measure of non-Gaussianity [11].

All these measures were already employed to evaluate non-Gaussianity of states studied in some interesting experiments. In the experiment reported in [12] the relative-entropy measure was used for single-photon-added coherent states, while in [13] the same distance-type degree was evaluated in an experiment with multiple-photon subtraction from a thermal state. The three above-mentioned degrees of non-Gaussianity were determined and compared in some recent experiments on phase-averaged coherent states [14, 15].

This work parallels some of our recent findings on non-Gaussianity and its decay in contact with a thermal reservoir for an interesting class of Fock-diagonal one-mode states: the photon-added thermal states [9, 10]. Here we intend to compare the three distance-type amounts of non-Gaussianity during the damping of two different excitations on a single-mode thermal state: an $M$-photon-added...
thermal state (PATS) and an $M$-photon-subtracted thermal state (PSTS).

The plan of our paper is as follows. In section 2, we recall some statistical properties of a PSTS. In section 3, the three usual degrees of non-Gaussianity for a Fock-diagonal one-mode state are recapitulated. Then we derive an analytic expression of the Hilbert–Schmidt measure of non-Gaussianity for a PSTS. Plots of the entropic and Bures amounts of non-Gaussianity are shown to be in agreement with the Hilbert–Schmidt measure. Section 4 examines the evolution of a PSTS due to the interaction of the field mode with a thermal reservoir, which is governed by the quantum optical master equation. We finally compare the decay of the evolution of a PSTS due to the interaction of the field mode with a thermal reservoir, which is governed by the quantum optical master equation. We finally compare the decay of

This is actually a negative binomial distribution [29] with the stopping parameter $r = M + 1$. Its generating function is

$$\sigma_{M}^{\text{sub}}(\tilde{n}, v) := \sum_{n=0}^{\infty} p_{n}^{\text{sub}} v^{n} = \left( 1 - x \right)^{M+1} \left( 1 - 1/x v \right)^{M+1} \quad (-1 \leq v \leq 1).$$

(2.4)

Accordingly, the mean number of photons in the PSTS (2.1),

$$\langle \hat{n} \rangle := \sum_{n=0}^{\infty} n p_{n}^{\text{sub}} \left[ \frac{\partial}{\partial v} \sigma_{M}^{\text{sub}}(\tilde{n}, v) \right]_{v=1},$$

(2.5)

has the expression

$$\langle \hat{n} \rangle = (M + 1) \tilde{n}.$$  

(2.6)

It is therefore proportional to the thermal mean occupancy $\tilde{n}$ and, rather counterintuitively, increases with the number $M$ of extracted photons. Note that the purity of a PSTS (2.1),

$$\text{Tr}[(\hat{\rho}_{M}^{\text{sub}}(\tilde{n}))^{2}] = \sum_{n=0}^{\infty} \left( p_{n}^{\text{sub}} \right)^{2} = (1 - x)^{2M+2} F_{1}(M + 1, M + 1; 1; x^{2}),$$

(2.7)

where $F_{1}$ is a Gauss hypergeometric function (A.1), coincides with that of the PATS $\hat{\rho}_{M}^{\text{add}}(\tilde{n})$, which was written in [9]:

$$\text{Tr}[(\hat{\rho}_{M}^{\text{sub}}(\tilde{n}))^{2}] = \text{Tr}[(\hat{\rho}_{M}^{\text{add}}(\tilde{n}))^{2}] \quad (M = 1, 2, 3, \ldots).$$

(2.8)

An equivalent form of equation (2.7) in terms of a Legendre polynomial $P_{M}$ can be obtained by using equations (A.2) and (A.3):

$$\text{Tr}[(\hat{\rho}_{M}^{\text{sub}}(\tilde{n}))^{2}] = \left( 1 - x \right)^{2M+1} P_{M}\left( 1 + \frac{2x^{2}}{1 - x^{2}} \right).$$

(2.9)

3. Non-Gaussianity of a photon-subtracted thermal state

We do not insist on the general properties of the distance-type degrees of non-Gaussianity introduced in [6, 7, 9]. We just recall the three degrees of non-Gaussianity written for a Fock-diagonal state $\hat{\rho}$ (our case in the following):

$$\hat{\rho} = \sum_{n=0}^{\infty} p_{n} |n\rangle \langle n|, \quad \text{with} \quad \sum_{n=0}^{\infty} p_{n} = 1.$$  

(3.1)

In this case, the associate Gaussian state is a thermal one with the same mean photon occupancy, $\langle \hat{n} \rangle = \sum_{n=0}^{\infty} n p_{n}$:

$$\hat{\rho}_{G} = \sum_{n=0}^{\infty} s_{n} |n\rangle \langle n| \quad \text{with} \quad s_{n} := \frac{1}{\langle \hat{n} \rangle + 1} \sigma^{n}, \quad \sigma := \langle \hat{n} \rangle / (\langle \hat{n} \rangle + 1).$$

(3.2)

The Hilbert–Schmidt and entropic amounts of non-Gaussianity were written in [6, 7, 9] as

$$\delta_{\text{HS}}[\hat{\rho}] = \frac{1}{2} \left[ 1 + \frac{\sum_{n=0}^{\infty} p_{n}^{2} (\sigma_{n}^{2} - 2 \sigma_{n} p_{n})}{\sum_{n=0}^{\infty} p_{n}^{2} \sigma_{n}^{2}} \right] \quad = \frac{1}{2} + \frac{1}{2 \text{Tr} [\hat{\rho}^{2}]} \left[ \frac{1}{2 \langle \hat{n} \rangle + 1} - \frac{2}{\langle \hat{n} \rangle + 1} G_{p}(\sigma) \right].$$

(3.3)
and, respectively,

$$\delta_{RE}[\hat{\rho}] = \sum_{n=0}^{\infty} p_n \ln p_n + (\langle \hat{n} \rangle + 1) \ln(\langle \hat{n} \rangle + 1) - \langle \hat{n} \rangle \ln(\langle \hat{n} \rangle).$$

(3.4)

Here we have used the purity of the thermal state $\bar{\gamma}_G$ arising from equation (3.2), while $G(y) := \sum_{n=0}^{\infty} p_n y^n$ is the generating function of the photon-number distribution in the given state $\hat{\rho}$.

The third measure of interest was introduced in terms of the Bures distance between the state $\hat{\rho}$ and its associate Gaussian state $\hat{\gamma}_G$ [9, 10]. In the case of a Fock-diagonal state, we notice the commutation relation $[\hat{\rho}, \hat{\gamma}_G] = 0$, which implies the simpler formula

$$\delta_{F}[\hat{\rho}] = 1 - \sum_{n=0}^{\infty} \sqrt{p_n} s_n. \quad (3.5)$$

The Hilbert–Schmidt degree of non-Gaussianity, equation (3.3), can readily be evaluated for any PSTS. First, we employ equations (2.6) and (3.2) to write the generating function (2.4) for $v = \sigma$:

$$G_{M}^{\text{sub}}(\bar{n}, \sigma) = \left[ \frac{(M + 1)\bar{n} + 1}{(M + 2)\bar{n} + 1} \right]^{M+1}. \quad (3.6)$$

Then, by replacing equations (2.9) and (3.6) into equation (3.3), one finds the formula

$$\delta_{\text{HS}}[\hat{\rho}_M^{\text{sub}}(\bar{n})] = \frac{1}{2} + \frac{(2\bar{n} + 1)^M}{2 \mathcal{P}_M} \left\{ \frac{1}{1 + \frac{M^2 \pi}{2\bar{n} + 1}} \right\} - \frac{2}{1 + \frac{M^2 \pi}{2\bar{n} + 1}} \left[ \frac{(M + 1)\bar{n} + 1}{(M + 2)\bar{n} + 1} \right]^{M}. \quad (3.7)$$

Plots of the Hilbert–Schmidt measure (3.7) versus the number $M$ of subtracted photons at some values of the thermal mean occupancies are shown in figure 1 (right). In figure 2 (right) we keep constant the value of $M$ and give the dependence of $\delta_{\text{HS}}$ on the thermal parameter $x$.

![Figure 1](image1)

**Figure 1.** Dependence of the distance-type measures of non-Gaussianity on the number of subtracted photons. All the plots start from the origin. The lowest curve is for $\tilde{n} = 0.1$. For the upper ones we have used $\tilde{n} = 1, 2, 5, 10$, respectively.

![Figure 2](image2)

**Figure 2.** Dependence of the distance-type measures of non-Gaussianity on the thermal parameter $x$ for PSTSs. All the plots start from the origin. The lowest plot is for $M = 1$. For the upper ones we have used $M = 4, 5, 8, 9$, respectively.

4. **Gaussification by damping**

In our paper [10], the evolution under the quantum optical master equation of a Fock-diagonal density matrix was conveniently written in the interaction picture

$$\rho_{jk}(t) = \delta_{jk} \left[ \frac{\bar{n}_T(t)}{\bar{n}_T(t) + 1} \right]^{j+i} \sum_{l=0}^{\infty} \rho_l(0) \left[ \frac{(\bar{n}_R + 1)(1 - e^{-\gamma t})}{\bar{n}_T(t) + 1} \right]^{l} \times \binom{j}{l-j} e^{-\gamma t} \cdot \left( \bar{n}_R + 1 \right) \left( 1 - e^{-\gamma t} \right).$$

(4.1)
In equation (4.1), \( \bar{n} \) and \( \gamma \) are constants of the thermal bath and \( \tilde{n}_T(t) := \bar{n}_R(1 - e^{-\gamma t}) \). The limit \( t \to \infty \) in equation (4.1) represents a thermal state with the Bose–Einstein mean photon occupancy \( \bar{n}_R \). We thus deal with an evolving Gaussification process which eventually destroys both the non-Gaussianity and the non-classicality properties of any input state. The corresponding time-dependent associate Gaussian state is a thermal one whose mean occupancy is equal to the average photon number of the damped field state. We find

\[
\langle \hat{n} \rangle_t = [\bar{n}(M + 1)] e^{-\gamma t} + \tilde{n}_T(t). \tag{4.2}
\]

By employing equations (A.2) and (A.4), we obtain the following expression of the photon-number distribution in a damped PSTS:

\[
p^{\text{sub}}_n(t) := \left[ \hat{\rho}^{\text{sub}}_M(\bar{n}) \right]_{\text{sub}} = \frac{[\hat{n}_T(t) + 1]^M [\bar{n} e^{-\gamma t} + \hat{n}_T(t)]^n}{[\bar{n} e^{-\gamma t} + \hat{n}_T(t) + 1]^{M+n+1}} \times \text{$_2F_1$} (-M, -n; 1; \frac{\bar{n} e^{-\gamma t}}{[\hat{n}_T(t) + 1][\bar{n} e^{-\gamma t} + \hat{n}_T(t)]}). \tag{4.3}
\]

We have used the time-dependent probability distribution (4.3) and the mean photon occupancy (4.2) to evaluate numerically two of the distance-type measures of non-Gaussianity we are interested in: \( \delta_{\text{RE}}[\hat{\rho}^{\text{sub}}_M(\bar{n})] \) and \( \delta_{\text{HS}}[\hat{\rho}^{\text{sub}}_M(\bar{n})] \), via equations (3.4) and (3.5), respectively. The last degree of non-Gaussianity we consider here is the Hilbert–Schmidt one, equation (3.3). As in the case of PATSs [10], the necessary ingredients to evaluate the Hilbert–Schmidt degree of non-Gaussianity can be obtained analytically. However, we do not write here the explicit expressions of the time-dependent purity and generating function. Our results are displayed in figure 3, where the time evolutions of the three non-Gaussianity measures are presented for the same values of the parameters. We now take advantage of our previous results on the non-Gaussianity of damped PATSs [10]. In figure 4 we present a comparison between the time evolution of non-Gaussianity for photon-added (dotted curves) and photon-subtracted (continuous curves) corresponding to the same parameters of states and reservoir.

5. Conclusions

Non-Gaussianity of pure states is often associated with their non-classicality. In this work, we have examined the class of photon-subtracted one-mode thermal states, which are always mixed and classical. Subtraction of \( M \) photons from a thermal state results in a photon-number distribution that turns out to be a negative binomial one whose stopping number is equal to \( M + 1 \). The principal feature we have looked for was the non-Gaussianity of the PSTSs, as indicated by some recently introduced distance-type measures. Also investigated was the decrease of this property during the interaction of the field mode with a thermal reservoir described by the quantum optical master equation. We have shown that decaying of non-Gaussianity was consistently pointed out by three distance–type measures. Thus, the fidelity-based degree, the Hilbert–Schmidt one and the entropic measure evolve monotonically, as expected for any good measure of non-Gaussianity [9, 10]. Moreover, we have compared this evolution to that of the PATSs, which, by contrast, are non-classical. We have found that a given PATS has a larger amount of non-Gaussianity than the corresponding PSTS. This inequality between the degrees of non-Gaussianity of PATSs and PSTSs is maintained during their dissipative evolution. We conclude by stressing the significance of the agreement between the three measures of non-Gaussianity employed here, which is equally valid for both the PATSs and the PSTSs. Because the entropic measure \( \delta_{\text{HS}}[\hat{\rho}] \) is an exact one [11], the other two, \( \delta_{\text{RE}}[\hat{\rho}] \) and \( \delta_{\text{HS}}[\hat{\rho}] \), albeit approximate, are nevertheless reliable.
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Appendix. Some useful formulae involving Gauss hypergeometric functions

A Gauss hypergeometric function is the sum of the corresponding hypergeometric series,
\[ _2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \frac{z^n}{n!} \quad (|z| < 1), \tag{A.1} \]
where \((a)_n := \Gamma(a+n)/\Gamma(a)\) is Pochhammer’s symbol standing for a rising factorial. This definition is extended by analytic continuation [30]. Recall Pfaff’s transformation formula [30],
\[ zF_1(a, b; c; z) = (1-z)^{-b} \left[ (c-a, b; c; \frac{z}{z-1}) \right], \tag{A.2} \]
as well as Murphy’s expression of the Legendre polynomial of degree \(l\) in terms of a Gauss hypergeometric function [30],
\[ \mathcal{P}_l(z) = _2F_1\left(-l, l+1; 1; \frac{1-z}{2}\right) \quad (l = 0, 1, 2, 3, \ldots). \tag{A.3} \]
The following sum [30] has been used for obtaining the photon-number distribution (4.3) in a damped PSTS:
\[ \sum_{n=0}^{\infty} \frac{(-\xi)_n}{n!} (-t)^n _2F_1(-n, b; c; z) = (1+t)^c \xi^{\frac{t_z}{1+t}} \quad (t, z > 0). \tag{A.4} \]

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