ON PROPERTIES OF CERTAIN ANALYTIC MULTIPLIER TRANSFORM OF COMPLEX ORDER

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ABSTRACT. The focus of this paper is to investigate the subclasses \(S^*C(\gamma, \mu, \alpha, \lambda; b)\), \(TS^*C(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*C(\gamma, \mu, \alpha, \lambda; b)\) and obtain the coefficient bounds as well as establishing its relationship with certain existing results in the literature.

1. Introduction

Let \(A\) be the class of normalized analytic functions \(f\) in the open unit disc \(U = \{z \in \mathbb{C} : |z| < 1\}\) with \(f(0) = f'(0) = 0\) and of the form

\[f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in C,\]

and \(S\) the class of all functions in \(A\) that are univalent in \(U\). Also, the subclass of functions in \(A\) that are of the form

\[f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,\]

is denoted by \(T\) and the subclasses \(S^*(\alpha), \ C(\gamma)\) are given respectively by

\[S^*(\alpha) = \left\{ f \in S : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma \ z \in U \right\} \]

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Meanwhile, the author in [4] defined a linear transformation $D$. An interesting unification of the classes $S$ and $C$ defined by $D$ is the subclass of functions $f \in T$ such that $f$ is starlike of order $\gamma$ and respectively, $T \cap S$ is the class of function $f \in T$ such that $f$ is convex of order $\gamma$.

An interesting unification of the classes $S^*(\alpha)$ and $C(\gamma)$ defined by $S^*(\alpha)$ and $C(\gamma)$ which satisfies the condition

$$\text{Re} \left\{ \frac{zf'(z) + \beta z^2 f'(z)}{\beta z f'(z) + (1 - \beta) f(z)} \right\} > \gamma, \quad 0 \geq \gamma < 1, z \in U. \quad \text{(1.5)}$$

has been extensively studied by different researchers, for example, see [6] and [1,2,3]. The special cases for $\beta = 0, 1$ are given by $S^*(\gamma)$ and $C(\gamma)$ respectively.

Furthermore, the class $T S^*(\gamma, \beta)$ which is the subclass of function $f \in T$ such that $f$ belongs the class $S^*(\gamma, \beta)$, was studied by Altintas et al. and other researchers. For details see [3, 5, 6].

Using the unification in (5), Nizami Mustafa [6] introduced and investigated the class $S^*(\gamma, \beta; \tau)$ and $T S^*(\gamma, \beta; \tau)$, $0 \leq \alpha < 1; \beta \in [0, 1]; \tau \in C$ which he defined as follows

A function $f \in S$ given by (1.1) is said to belong to the class $S^*(\gamma, \beta; \tau)$ if the following condition is satisfied

$$\text{Re} \left\{ 1 + \frac{1}{\tau} \left[ \frac{zf'(z) + \beta z^2 f'(z)}{\beta z f'(z) + (1 - \beta) f(z)} - 1 \right] \right\} > \gamma, \quad 0 \geq \gamma < 1; \beta \in [0, 1]; \tau \in C \setminus \{0\}, z \in U. \quad \text{(1.6)}$$

Meanwhile, the author in [4] defined a linear transformation $D_{\alpha, \lambda}^m f$ by

$$D_{\alpha, \lambda}^m f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad 0 \leq \lambda \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0 \quad \text{(1.7)}$$

Motivated by the work of Mustafa in [6], we study the effect of the application of the linear operator $D_{\alpha, \lambda}^m f$ on the unification of the classes of the functions $S^*(\gamma, \beta; \tau)$.

Now, we define the class $S^*(\gamma, \alpha, \lambda; b)$ to be class of functions $f \in S$ which satisfies the condition

$$\text{Re} \left\{ 1 + \frac{1}{b} \left[ \frac{z(D_{\alpha, \lambda}^m f)'(z) + \mu z^2(D_{\alpha, \lambda}^m f)''(z)}{\mu z(D_{\alpha, \lambda}^m f)''(z) + (1 - \mu)(D_{\alpha, \lambda}^m f)'(z)} - 1 \right] \right\} > \gamma, 0 \geq \gamma < 1, z \in U; 0 \leq \lambda, \mu \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0 \quad \text{(1.8)}$$

Also, we denote by $D_T$ the subclass of the class of functions in (7) which is of the form

$$D_{\alpha, \lambda}^m f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad 0 \leq \lambda, \mu \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0 \quad \text{(1.9)}$$

and denote by $T S^*(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*(\gamma, \mu, \alpha, \lambda; b)$ which is the class of functions $f$ in (1.9) such that $f$ belong to the class $S^*(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*(\gamma, \mu, \alpha, \lambda; b)$.

In this paper, we investigate the subclasses $S^*(\gamma, \mu, \alpha, \lambda; b)$ and $T S^*(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*(\gamma, \mu, \alpha, \lambda; b)$.
2. Coefficient bounds for the classes $S^*C^\lambda_\alpha(\gamma, \mu; b)$ and $TS^*C^\lambda_\alpha(\gamma, \mu; b)$

**Theorem 2.1.** Let $f$ be as defined in (1.1). Then the function $D^m_{\alpha, \lambda} f$ belongs to the class $S^*C(\gamma, \mu, \alpha, \lambda; b)$, $0 \leq \gamma < 1, z \in U, 0 \leq \mu, \lambda \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0$

if

$$\sum_{n=2}^{\infty} \left[ \alpha \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{\gamma}{2}} \left| 1 + \mu(n - 1) \right| \left| n + |b|(1 - \gamma) - 1 \right| \right] a_n \leq |b|(1 - \gamma)$$

The result is sharp for the function

$$D^m_{\alpha, \lambda} f(z) = z + \frac{|b|(1 - \gamma)(1 + \lambda(\alpha - 1))^m}{\alpha[1 + \mu(n - 1)] |n + |b|(1 - \gamma)| (1 + \lambda(n + \alpha - 2))^{m} z^n} \quad n \geq 2$$

**Proof.** By (1.8), $f$ belong to the class $S^*C(\gamma, \mu, \alpha, \lambda; b)$ if

$$\Re \left\{ 1 + \frac{1}{\beta} \left[ \frac{z(D^m_{\alpha, \lambda} f)'(z) + \mu z^2(D^m_{\alpha, \lambda} f)''(z)}{\mu z(D^m_{\alpha, \lambda} f)'(z) + (1 - \mu)(D^m_{\alpha, \lambda} f)(z)} - 1 \right] \right\} > \gamma$$

It suffices to show that:

$$\left| \frac{1}{\beta} \left[ \frac{z(D^m_{\alpha, \lambda} f)'(z) + \mu z^2(D^m_{\alpha, \lambda} f)''(z)}{\mu z(D^m_{\alpha, \lambda} f)'(z) + (1 - \mu)(D^m_{\alpha, \lambda} f)(z)} - 1 \right] \right| < 1 - \gamma \quad (2.1)$$

Simple computation in (2.1), using (1.7), we have:

$$\left| \frac{1}{\beta} \left[ \frac{z(D^m_{\alpha, \lambda} f)'(z) + \mu z^2(D^m_{\alpha, \lambda} f)''(z)}{\mu z(D^m_{\alpha, \lambda} f)'(z) + (1 - \mu)(D^m_{\alpha, \lambda} f)(z)} - 1 \right] \right| = \left| \frac{1}{\beta} \left[ \frac{z + \sum_{n=2}^{\infty} n \alpha \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n + \mu \sum_{n=2}^{\infty} n(n - 1) \alpha \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n}{\mu z + \sum_{n=2}^{\infty} \frac{\mu n \alpha}{1 - \frac{\mu}{1 + \lambda(\alpha - 1)}} \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n + (1 - \mu) \left( z + \sum_{n=2}^{\infty} \frac{\mu \alpha}{1 - \frac{\mu}{1 + \lambda(\alpha - 1)}} \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n \right)} - 1 \right| \right|$$

$$= \frac{1}{\beta} \left| \frac{z + \sum_{n=2}^{\infty} n \alpha \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n + \mu \sum_{n=2}^{\infty} n(n - 1) \alpha \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n}{\mu z + \sum_{n=2}^{\infty} \frac{\mu n \alpha}{1 - \frac{\mu}{1 + \lambda(\alpha - 1)}} \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n + (1 - \mu) \left( z + \sum_{n=2}^{\infty} \frac{\mu \alpha}{1 - \frac{\mu}{1 + \lambda(\alpha - 1)}} \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n z^n \right)} - 1 \right| \right|$$

$$\leq \frac{1}{\beta} \left| \frac{\sum_{n=2}^{\infty} \alpha(n - 1) \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n}{1 - \sum_{n=2}^{\infty} \alpha(n - 1) \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n} \right| |a_n|$$

which is bounded by $1 - \gamma$ if

$$\sum_{n=2}^{\infty} \alpha(n - 1) \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n \leq |b|(1 - \gamma) 1 - \sum_{n=2}^{\infty} \alpha(n - 1) \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n$$

which is equivalent to

$$\sum_{n=2}^{\infty} \alpha(n - 1) \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n \leq |b|(1 - \gamma)$$

$$\alpha(n - 1) \left[ 1 + \mu(n - 1) \right] \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} a_n \leq |b|(1 - \gamma)$$

Which implies that

$$\sum_{n=2}^{\infty} \alpha \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^{\frac{m}{2}} \left[ 1 + \mu(n - 1) \right] |n + |b|(1 - \gamma) - 1| |a_n| \leq |b|(1 - \gamma) \quad (2.2)$$
Thus, (2.1) is satisfied if (2.2) is satisfied.

Corollary 2.1. Let $f$ be as defined in (1) and the function $D^{m}_{\alpha,\lambda}f$ belongs to the class $S^{*}C(\gamma, \mu, \alpha, \lambda; b)$, $0 \geq \gamma < 1$, $z \in U; 0 \leq \lambda, \mu \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0$. Then

$$|a_n| \leq \frac{|b|(1-\gamma)(1+\lambda(\alpha-1))^m}{\alpha[1+\mu(n-1)][n+|b|(1-\gamma)-1][1+\lambda(n+\alpha-2)]^m}$$

Corollary 2.2. Let $f$ be as defined in (1.1). Then the function $D^{m}_{\alpha,\lambda}f$ belongs to the class $S^{*}C(\gamma, \mu, 1, \lambda; b)$, $0 \geq \gamma < 1$, $z \in U; 0 \leq \lambda, \mu \leq 1; m \in \mathbb{N} \cup 0$ if

$$\sum_{n=2}^{\infty} [(1+\lambda(n-1))^m [1+\mu(n-1)]|n+|b|(1-\gamma)-1]|a_n| \leq |b|(1-\gamma)$$

(2.3)

The result is sharp for the function

$$D^{m}_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{[1+\mu(n-1)][n+|b|(1-\gamma)-1][1+\lambda(n(n-1))]^m}z^n, \quad n \geq 2$$

Corollary 2.3. Let $f$ be as defined in (1.1). Then the function $D^{m}_{\alpha,\lambda}f$ belongs to the class $S^{*}C(\gamma, \mu, 1, \lambda; b)$, $0 \geq \gamma < 1$, $z \in U; 0 \leq \lambda, \mu \leq 1; m \in \mathbb{N} \cup 0$ if

$$\sum_{n=2}^{\infty} [(1+\lambda(n-1))^m [1+\mu(n-1)]|n+|b|(1-\gamma)-1]|a_n| \leq |b|(1-\gamma)$$

(2.4)

The result is sharp for the function

$$D^{m}_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{[1+\mu(n-1)][n+|b|(1-\gamma)-1][1+\lambda(n(n-1))]^m}z^n, \quad n \geq 2$$

Corollary 2.4. Let $f$ be as defined in (1.1). Then the function $D^{m}_{\alpha,\lambda}f$ belongs to the class $S^{*}C(\gamma, \mu, 1, 1; b)$, $0 \geq \gamma < 1$, $z \in U; 0 \leq \lambda, \mu \leq 1; m \in \mathbb{N} \cup 0$ if

$$\sum_{n=2}^{\infty} [n+|b|(1-\gamma)-1]|a_n| \leq |b|(1-\gamma)$$

(2.5)

The result is sharp for the function

$$D^{m}_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{n[1+\mu(n-1)][n+|b|(1-\gamma)-1]}z^n, \quad n \geq 2$$

Corollary 2.5. Let $f$ be as defined in (1.1). Then the function $D^{m}_{\alpha,\lambda}f$ belongs to the class $S^{*}C(\gamma, \mu, 1, 0; b)$, $0 \geq \gamma < 1$, $z \in U; 0 \leq \lambda, \mu \leq 1; m \in \mathbb{N} \cup 0$ if

$$\sum_{n=2}^{\infty} [n+|b|(1-\gamma)-1]|a_n| \leq |b|(1-\gamma)$$

(2.6)

The result is sharp for the function

$$D^{m}_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{[1+\mu(n-1)][n+|b|(1-\gamma)-1]}z^n, \quad n \geq 2$$

This result agrees with the Theorem 2.1 in [6].
Corollary 2.6. Let $f$ be as defined in (1.1). Then the function $D_{\alpha,\lambda}^m f$ belongs to the class $S^*C(\gamma, 0, 1, \lambda, 0; 1)$, $0 \leq \gamma < 1$, $z \in \mathbb{U}$; $0 \leq \lambda, \mu \leq 1$; $m \in \mathbb{N} \cup \{0\}$ if

$$\sum_{n=2}^{\infty} [(1 + \mu(n - 1))[n - \gamma] |a_n| \leq 1 - \gamma \quad (2.7)$$

The result is sharp for the function

$$D_{\alpha,\lambda}^m f(z) = z + \frac{1 - \gamma}{1 + \mu(n - 1)[n - \gamma]} z^n, \; n \geq 2$$

This result agrees with the Corollary 2.2 in [6].

Corollary 2.7. Let $f$ be as defined in (1.1). Then the function $D_{\alpha,\lambda}^m f$ belongs to the class $S^*C(\gamma, \mu, 1, \lambda, 0; 1)$, $0 \leq \gamma < 1$, $z \in \mathbb{U}$; $0 \leq \lambda, \mu \leq 1$; $m \in \mathbb{N} \cup \{0\}$ if

$$\sum_{n=2}^{\infty} (n - \gamma) |a_n| \leq 1 - \gamma \quad (2.8)$$

The result is sharp for the function

$$D_{\alpha,\lambda}^m f(z) = z + \frac{1 - \gamma}{n - \gamma} z^n, \; n \geq 2$$

This result agrees with the Corollary 2.2 in [6].

Theorem 2.2. Let $f \in D_T$. Then the function $D_{\alpha,\lambda}^m f$ belongs to the class $D_T S^*C(\gamma, \mu, \alpha, \lambda; b)$, $0 \leq \gamma < 1$, $z \in \mathbb{U}$; $0 \leq \lambda, \mu \leq 1$; $\alpha \geq 1$; $m \in \mathbb{N} \cup \{0\}$ if and only if

$$\sum_{n=2}^{\infty} \alpha(n - 1)[1 + \mu(n - 1)][n + b(1 - \gamma)] \left(\frac{x}{y}\right)^m |a_n| \leq |b|(1 - \gamma)$$

Proof. We shall prove only the necessity part of the Theorem as the sufficiency proof is similar to the proof of Theorem 1.

Let $f$ be as defined in (1.1) and $D_{\alpha,\lambda}^m f$ belongs to the class $TS^*C(\gamma, \mu, \alpha, \lambda; b)$, $0 \leq \gamma < 1$, $z \in \mathbb{U}$; $0 \leq \lambda, \mu \leq 1$; $\alpha \geq 1$; $m \in \mathbb{N} \cup \{0\}$; $b \in \mathbb{R} \setminus \{0\}$, we have

$$Re \left\{ 1 + \frac{1}{b} \left[ \frac{z(D_{\alpha,\lambda}^m f)'(z) + \mu z^2(D_{\alpha,\lambda}^m f)''(z)}{\mu z(D_{\alpha,\lambda}^m f)'(z) + (1 - \mu)(D_{\alpha,\lambda}^m f)(z) - 1} \right] \right\} > \gamma \quad (2.9)$$

Using (1.7) in (2.9) and by algebraic simplification, we have

$$Re \left\{ -\frac{\sum_{n=2}^{\infty} \alpha(n - 1)[1 + \mu(n - 1)] \left(\frac{1 + \lambda(n + n - 2)}{1 + \lambda(n - 1)}\right)^m a_n z^n}{b \left(1 - \sum_{n=2}^{\infty} \alpha(1 + \mu(n - 1)) \left(\frac{1 + \lambda(n + n - 2)}{1 + \lambda(n - 1)}\right)^m a_n z^n \right)} \right\} \geq \gamma - 1$$

Choosing $z$ to be real and $z \to 1$, we have

$$-\frac{\sum_{n=2}^{\infty} \alpha(n - 1)[1 + \mu(n - 1)] \left(\frac{1 + \lambda(n + n - 2)}{1 + \lambda(n - 1)}\right)^m a_n}{b \left(1 - \sum_{n=2}^{\infty} \alpha(1 + \mu(n - 1)) \left(\frac{1 + \lambda(n + n - 2)}{1 + \lambda(n - 1)}\right)^m a_n \right)} \geq \gamma - 1 \quad (2.10)$$
Let $b > 0$ in (19), we have

$$\sum_{n=2}^{\infty} \alpha(n-1)[1 + \mu(n-1)] \left( \frac{x}{y} \right)^m |a_n| \geq (\gamma - 1)b \left\{ 1 - \sum_{n=2}^{\infty} \alpha(1 + \mu(n-1)) \left( \frac{x}{y} \right)^m |a_n| \right\}$$

(2.11)

where $x = 1 + \lambda(n + \alpha - 2)$ and $y = 1 + \lambda(\alpha - 1)$ From (20), we have

$$\sum_{n=2}^{\infty} \alpha(n-1)[1 + \mu(n-1)][n + b(1 - \gamma)] \left( \frac{x}{y} \right)^m |a_n| \leq b(1 - \gamma)$$

(2.12)

Now suppose $b < 0$, which implies that $b = -|b|$ and substituting $b = -|b|$ in (19), we have

$$\sum_{n=2}^{\infty} \alpha(n-1)[1 + \mu(n-1)] \left( \frac{\bar{z}}{y} \right)^m |a_n| \geq \frac{\sum_{n=2}^{\infty} \alpha(n-1)[1 + \mu(n-1)] \left( \frac{\bar{z}}{y} \right)^m |a_n|}{|b| \left\{ 1 - \sum_{n=2}^{\infty} \alpha(1 + \mu(n-1)) \left( \frac{\bar{z}}{y} \right)^m |a_n| \right\}}$$

(2.13)

$$\sum_{n=2}^{\infty} \alpha(n-1)[1 + \mu(n-1)] \left( \frac{\bar{z}}{y} \right)^m |a_n| \geq (\gamma - 1)|b| \left\{ 1 - \sum_{n=2}^{\infty} \alpha(1 + \mu(n-1)) \left( \frac{\bar{z}}{y} \right)^m |a_n| \right\}$$

which implies

$$\sum_{n=2}^{\infty} \alpha(n-1)[1 + \mu(n-1)][n + b(1 - \gamma)] \left( \frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^m |a_n| \geq -b(1 - \gamma)$$

(2.14)

From (21) and (23), the proof of the necessity is completed. □

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