ERGODICITY OF KUSUOKA MEASURES ON QUANTUM TRAJECTORIES

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ABSTRACT. In 1989 Kusuoka started the study of probability measures on the shift space that are defined with the help of products of matrices. In particular, he derived a sufficient condition for the ergodicity of such measures, which have since been referred to as Kusuoka measures. We observe that repeated measurements of a unitarily evolving quantum system generate a Kusuoka measure on the space of sequences of measurement outcomes. We show that if the measurement consists of scaled projections, then Kusuoka’s sufficient ergodicity condition can be significantly simplified. We then prove that this condition is also necessary for ergodicity if the measurement consists of uniformly scaled rank-1 projections (i.e., it is a rank-1 POVM), or of exactly two projections, one of which is rank-1. For the latter class of measurements we also show that the Kusuoka measure is reversible in the sense that every string of outcomes has the same probability of being emitted by the system as its reverse.

KEYWORDS: Kusuoka measures, ergodicity, symbolic dynamics, unitary matrices, quantum information

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1. Introduction & Preliminaries

Consider successive (isochronous) measurements on a $d$-dimensional ($d \geq 2$) quantum-mechanical system that between two subsequent measurements undergoes deterministic time evolution governed by a unitary operator $U$ (see Fig. 1). Such a procedure results in the system emitting a sequence of measurement outcomes from $I_k := \{1, \ldots, k\}$, while the joint evolution of the system can be modelled by a Partial Iterated Function System (PIFS).

![Figure 1](image.png)

**Figure 1.** Repeatedly measured quantum system that between each two consecutive measurements evolves in accordance to a unitary operator $U$. State dynamics $(\rho_0, \rho_1, \ldots)$ is Markovian, outcome dynamics $(i_1, i_2, \ldots)$ need not be Markovian.
**Definition 1.** [29, p. 59] *The triple* $(X, F_i, p_i)_{i \in I_k}$ *is called a partial iterated function system (PIFS) on* $X$ *if* $p_i : X \to [0, 1]$, $\sum_{j \in I_k} p_j = 1$, *and* $F_i : \{x \in X : p_i(x) > 0\} \to X$, *where* $i \in I_k$.

Under the action of a PIFS, a given initial state $x \in X$ is transformed into a new state $F_i(x)$ with (place-dependent) probability $p_i(x)$ and the symbol $i$ corresponding to this evolution is emitted, $i \in I_k$. The repeated action of a PIFS generates a Markov chain on $X$ and yields sequences of symbols from $I_k$, which can be modelled by a hidden Markov chain. The probability and evolution functions related to these strings are defined inductively in the following natural way. Let $n \in \mathbb{N}$, $i := (i_1, \ldots, i_n) \in I_k^n$ and $j \in I_k$. For $n = 1$ both $p_i$ and $F_i$ are given. The probability of the system outputting $i_j := (i_1, \ldots, i_n, j) \in I_k^{n+1}$ is defined as

$$p_{ij}(x) := \begin{cases} p_j(F_i(x))p_i(x) & \text{if } p_i(x) > 0 \\ 0 & \text{if } p_i(x) = 0 \end{cases}$$

and the corresponding evolution map is defined as $F_{ij}(x) := F_j(F_i(x))$ if $p_i(x) > 0$. Obviously, we have the total probability formula

$$p_i(x) = \sum_{j \in I_k} p_{ij}(x).$$

**Remark.** The notion of a PIFS generalizes, slightly but significantly, that of an Iterated Function System (IFS) with place-dependent probabilities (see, e.g., [2, 28]) by allowing the evolution map $F_i$ to remain undefined on the states that have zero probability of being subject to the action of $F_i$, $i \in I_k$. Such a generalization is necessary in considering quantum measurements, because the evolution associated with a given measurement outcome cannot be defined on the states with zero probability of producing this outcome, see (4).

From this point forward we restrict our attention to PIFSs acting on the set of quantum states $S(\mathbb{C}^d) := \{\rho \in \mathcal{L}(\mathbb{C}^d) : \rho \geq 0, \operatorname{tr} \rho = 1\}$, where $\mathcal{L}(\mathbb{C}^d)$ denotes the space of (bounded) linear maps on $\mathbb{C}^d$. The Markov chain generated by such a PIFS on $S(\mathbb{C}^d)$ corresponds to the so-called *discrete quantum trajectories*, see, e.g., [1, 7, 21, 23, 26], and the sequences of emitted symbols, interpreted as measurement outcomes, form what we can call *coarse-grained quantum trajectories*. The study of symbolic dynamics generated by quantum dynamical systems goes back to [5, 31], see also [14, 15]. In this paper we employ PIFSs to model repeated measurements performed on unitarily evolving quantum systems and focus on the probability measures that such systems induce on the shift space.

Let us recall the basic mathematical framework of quantum mechanics. A *measurement* of a $d$-dimensional quantum system with $k \in \mathbb{N}$ possible outcomes is given by a *positive operator-valued measure (POVM)*, i.e., a set of positive semi-definite (non-zero) operators $\Pi_1, \ldots, \Pi_k$ on $\mathbb{C}^d$ that sum to the identity, i.e.,

$$\sum_{j \in I_k} \Pi_j = \mathbb{I}.$$
We distinguish two special classes of measurements:

- \( \Pi \) is called a \textit{projection valued measure (PVM)} or a \textit{Lüders–von Neumann measurement} \cite{25} if \( \Pi_i \) is a projection for every \( i \in I_k \). We then have \( k \leq d \) and the projections constituting \( \Pi \) are necessarily orthogonal as self-adjoint projections on a Hilbert space. Moreover, they are mutually orthogonal, i.e., \( \Pi_i \Pi_j = 0 \) for \( i, j \in I_k, i \neq j \) [16, p. 46].

- \( \Pi \) is called a \textit{(normalised) rank-1 PVM} if \( \text{tr} \Pi_1 = \text{tr} \Pi_2 = \ldots = \text{tr} \Pi_k \) and \( \text{rank} \Pi_i = 1 \) for every \( i \in I_k \). Then, necessarily, \( \text{tr} \Pi_i = \frac{d}{k} \) for every \( i \in I_k \) and \( k \geq d \). It follows that there exist unit vectors \( \varphi_1, \ldots, \varphi_k \in \mathbb{C}^d \) associated with \( \Pi \) via \( \Pi_i = \frac{d}{k} \rho_i \), where \( \rho_i \) is an orthogonal projection on \( \text{span}\{\varphi_i\}, i \in I_k \).

If the state of the system before the measurement is \( \rho \in \mathcal{S}(\mathbb{C}^d) \), then the \textit{Born rule} dictates that the probability of obtaining the \( i \)-th outcome \( (i \in I_k) \) is given by \( \text{tr}(\Pi_i \rho) \) \cite{9}. The measurement process generically alters the state of the system, but the PVM alone is not sufficient to determine the post-measurement state. This can be done by defining a \textit{measurement instrument} (in the sense of Davies and Lewis \cite{12}) compatible with \( \Pi \), see also \cite{10, 11, Ch. 10, 17, Ch. 5}. We consider the \textit{generalised Lüders instruments}, disturbing the initial state in the minimal way, where the input-output state transformation reads

\[
\mathcal{S}(\mathbb{C}^d) \ni \rho \longmapsto \frac{\sqrt{\Pi_i \rho \Pi_i}}{\text{tr}(\Pi_i \rho)} \in \mathcal{S}(\mathbb{C}^d),
\]

provided that the measurement yielded the result \( i \in I_k \) \cite{13, p. 404}, see also \cite{3, 4}.

Fix a PVM \( \Pi = \{\Pi_1, \ldots, \Pi_k\} \) and \( U \in \mathcal{U}(\mathbb{C}^d) \), where \( \mathcal{U}(\mathbb{C}^d) \) stands for the set of unitary operators on \( \mathbb{C}^d \). In what follows we define the PIFS corresponding to a quantum system that evolves in accordance to \( U \) and is repeatedly measured with \( \Pi \). Recall that the deterministic time evolution of a quantum system is said to be governed by \( U \) if it is given by the \textit{unitary channel} acting as

\[
\mathcal{S}(\mathbb{C}^d) \ni \rho \longmapsto U \rho U^* \in \mathcal{S}(\mathbb{C}^d).
\]

Taking into account the Born rule, for an input state \( \rho \in \mathcal{S}(\mathbb{C}^d) \) we define the probability of obtaining the outcome \( i \in I_k \) as

\[ p_i(\rho) := \text{tr}(\Pi_i U \rho U^*). \]

The evolution map \( F_i \) is defined as the composition of the unitary channel (5) with the state transformation due to \( \Pi \) described in (4), i.e.,

\[ F_i(\rho) := \frac{\sqrt{\Pi_i U \rho U^*} \sqrt{\Pi_i}}{\text{tr}(\Pi_i U \rho U^*)}, \]

provided that \( p_i(\rho) > 0 \). Clearly, \( (\mathcal{S}(\mathbb{C}^d), F_i, p_i)_{i \in I_k} \) is a PIFS.

Next, for \( \rho \in \mathcal{S}(\mathbb{C}^d) \) we put \( \Lambda_i(\rho) := \sqrt{\Pi_i U \rho U^*} \sqrt{\Pi_i} \) and observe that \( p_i(\rho) = \text{tr}(\Lambda_i(\rho)) \) and \( F_i(\rho) = \Lambda_i(\rho) / \text{tr}(\Lambda_i(\rho)) \), provided that \( \text{tr}(\Lambda_i(\rho)) > 0 \). It follows that for any initial state
\[ \rho \in \mathcal{S}(\mathbb{C}^d) \] the probability of the system generating the string of measurement outcomes \((i_1, \ldots, i_n) \in I^n_k\), where \(n \in \mathbb{N}\), is given by \(\text{tr}(A_{i_n} \cdots A_{i_1}(\rho))\), i.e.,

\[ p_{(i_1, \ldots, i_n)}(\rho) = \text{tr}(\sqrt{\Pi_{i_n}U} \cdots \sqrt{\Pi_{i_1}U} \rho U^* \sqrt{\Pi_{i_1}} \cdots \sqrt{\Pi_{i_n}}). \]

In a more general setting, probability measures on the shift space that are defined on cylinder sets with the help of products of matrices corresponding to respective symbols were first investigated by Kusuoka in [22]. Under the name of Kusuoka measures they have been mostly explored in the context of fractal geometry, see, e.g., [6, 8, 18, 19, 22, 24, 32]. We stick to the definition of a Kusuoka measure given by Johansson et al. [18]:

**Definition 2.** Let \( \{A_1, \ldots, A_k\} \subset \mathcal{L}(\mathbb{C}^d) \) be such that \( \sum_{i=1}^k A_i A_i^* = \mathbb{I} \) and \( \sum_{i=1}^k A_i^* \rho A_i = \rho \) for some positive-definite operator \( \rho \in \mathcal{L}(\mathbb{C}^d) \) with \( \text{tr} \rho = 1 \). A probability measure \( \mathbb{P}_\rho \) on \( I^n_k \) with the \( \sigma \)-algebra generated by the family of all cylinder sets is called a Kusuoka measure associated to \( \{A_1, \ldots, A_k\} \) if

\[ \mathbb{P}_\rho(C_{(i_1, \ldots, i_n)}) = \text{tr}(A_{i_n}^* \cdots A_{i_1}^* \rho A_{i_1} \cdots A_{i_n}) \]

where \( C_{(i_1, \ldots, i_n)} \) stands for the cylinder set corresponding to the string \((i_1, \ldots, i_n) \in I^n_k\), i.e.,

\[ C_{(i_1, \ldots, i_n)} := \left\{ (s_i)_{i=1}^\infty \in I^n_k : s_1 = i_1, \ldots, s_n = i_n \right\}, n \in \mathbb{N}. \]

Observe that the conditions imposed on the operators \( A_1, \ldots, A_k \) assure that \( \mathbb{P}_\rho \) is consistent (well-defined) and shift-invariant, i.e., for every \( \iota \in I^n_k \), where \( n \in \mathbb{N} \), we have

\[ \mathbb{P}_\rho(C_{\iota}) = \sum_{j \in I_k} \mathbb{P}_\rho(C_{\iota j}) \quad \text{and} \quad \mathbb{P}_\rho(C_{\iota}) = \sum_{j \in I_k} \mathbb{P}_\rho(C_{j \iota}). \]

Let us get back to the quantum system modelled by the PIFS \((\mathcal{S}(\mathbb{C}^d), F, \rho_\iota)_{\iota \in I_k}\). From now on, we fix the maximally mixed state \( \rho_\iota := \mathbb{I}/d \) as the initial state of this system. For the cylinder set \( C_{\iota} \) corresponding to \( \iota = (i_1, \ldots, i_n) \in I^n_k \), \( n \in \mathbb{N} \), we put \( \mathbb{P}_\iota(C_{\iota}) \) for the probability of the system generating \( \iota \) as the string of measurement outcomes, i.e.,

\[ \mathbb{P}_\iota(C_{\iota}) := \rho_\iota(p_\iota) = \text{tr}(\sqrt{\Pi_{i_n}U} \cdots \sqrt{\Pi_{i_1}U} \rho_\iota U^* \sqrt{\Pi_{i_1}} \cdots \sqrt{\Pi_{i_n}}). \]

Note that, denoting the Hilbert-Schmidt norm by \( || \cdot ||_{\text{HS}} \), we can rewrite (8) as

\[ \mathbb{P}_\iota(C_{\iota}) = \frac{1}{d} || U^* \sqrt{\Pi_{i_1}} \cdots U^* \sqrt{\Pi_{i_n}} ||^2_{\text{HS}}. \]

By the Kolmogorov extension theorem, \( \mathbb{P}_\iota \) extends to a measure on the space of sequences of measurement outcomes \( I^n_k \) with the \( \sigma \)-algebra generated by all cylinder sets. It follows easily that \( \mathbb{P}_\iota \) is a Kusuoka measure associated to \( \{U^* \sqrt{\Pi_1}, \ldots, U^* \sqrt{\Pi_k}\} \) since the normalization condition (3) gives

\[ \sum_{j \in I_k} (U^* \sqrt{\Pi_j})(\sqrt{\Pi_j}U) = \mathbb{I} \quad \text{and} \quad \sum_{j \in I_k} (\sqrt{\Pi_j}U)\rho_\iota(U^* \sqrt{\Pi_j}) = \rho_\iota. \]
Example 3. To illustrate the notions introduced so far, let us discuss in detail the case of $\Pi$ being a rank-1 POVM. Recall that for each $i \in I_k$ we have $\Pi_i = \frac{d}{k} \rho_i$, where $\rho_i$ is an orthogonal projection on the subspace spanned by some unit vector $\varphi_i \in \mathbb{C}^d$. This implies that the evolution maps are all constant since for every $i \in I_k$ we have $F_i(\rho) = \rho_i$ for every $\rho \in S(\mathbb{C}^d)$ such that $p_i(\rho) > 0$. That is, to each measurement outcome there corresponds a single post-measurement state, so from an outcome we can recover the underlying quantum state. In consequence, the Kusuoka measure $\mathbb{P}_*$ is a Markov measure, as we now show.

Firstly, we establish the Markov chain that arises on the space of quantum states $S(\mathbb{C}^d)$. In the first measurement each outcome is equally likely:

$$p_i(\rho_*) = \text{tr}(\Pi_i U \rho_* U^*) = \frac{1}{d} \text{tr}(\Pi_i) = \frac{1}{k}$$

for every $i \in I_k$. Hence, the state space of this Markov chain is $\{\rho_1, \ldots, \rho_k\}$, its initial distribution is uniform and the transition matrix reads $[p_{ij}(\rho_i)]_{i,j \in I_k}$. Since

$$p_j(\rho_i) = \text{tr}(\Pi_j U \rho_i U^*) = \frac{d}{k} |\langle \varphi_j, U \varphi_i \rangle|^2$$

(9)

for $i, j \in I_k$, we see that the transition matrix is bistochastic, and so the uniform distribution is stationary.

The dynamics induced by this system on the space of measurement outcomes $I_k$ turns out to be Markovian as well. Actually, it mirrors the Markov chain generated on $S(\mathbb{C}^d)$, i.e., the sequence of quantum states occupied by the system at consecutive time steps can be reconstructed from the sequence of measurement outcomes. To see this, let $(i_1, \ldots, i_n) \in I_k^n$, $n \in \mathbb{N}$. We show that

$$\mathbb{P}_*(C_{(i_1, \ldots, i_n)}) = p_{(i_1, \ldots, i_n)}(\rho_*) = p_{i_1}(\rho_*) p_{i_2}(\rho_{i_1}) p_{i_3}(\rho_{i_2}) \cdots p_{i_n}(\rho_{i_{n-1}}).$$

(10)

Indeed, if $p_{(i_1, \ldots, i_n)}(\rho_*) > 0$, then (1) implies that $p_{(i_1, \ldots, i_r)}(\rho_*) > 0$ for each $r \in \{1, \ldots, n-1\}$, so $F_{(i_1, \ldots, i_r)}(\rho_*) = \rho_{i_r}$. Thus, $p_{i_{r+1}}(F_{(i_1, \ldots, i_r)}(\rho_*)) = p_{i_{r+1}}(\rho_{i_r})$, and (10) follows from (1). If $p_{(i_1, \ldots, i_n)}(\rho_*) = 0$, then there exists $r \in \{1, \ldots, n-1\}$ such that $p_{(i_1, \ldots, i_r)}(\rho_*) > 0$ and $p_{(i_1, \ldots, i_{r+1})}(\rho_*) = 0$. Using (1) again, we obtain $p_{i_{r+1}}(\rho_{i_r}) = 0$, and so (10) holds in this case as well.

Hence, the measurement outcomes form a Markov chain on $I_k$ with uniform initial distribution and with transition matrix $Q = [Q_{ij}]_{i,j \in I_k}$ such that $Q_{ij} = p_j(\rho_i)$ for $i, j \in I_k$, and the Kusuoka measure $\mathbb{P}_*$ is a Markov measure, as claimed.

Remark. Mimicking the arguments from Example 3, one can easily see that if the operators $A_1, \ldots, A_k$ that generate a Kusuoka measure are all rank-1, then this Kusuoka measure is a Markov measure. Note that every rank-1 operator can be written as the composition of a (scaled) rank-1 projection with a unitary operator, as in the case of rank-1 POVMs.

Next, we discuss the ergodicity of Kusuoka measures.
Definition 4. We say that \( \{A_1, \ldots, A_k\} \subset \mathcal{L}(\mathbb{C}^d) \) is irreducible if there does not exist a non-trivial subspace of \( \mathbb{C}^d \) invariant under \( A_i \) for every \( i \in \{1, \ldots, k\} \).

Kusuoka showed that the irreducibility of a family of operators guarantees the existence and uniqueness of \( \rho \) from Definition 2, thus also the existence and uniqueness of the probability measure associated with these operators [22, Thm. 1.2], see also [27, Prop. 15]. Moreover, Kusuoka proved that irreducibility constitutes a sufficient condition for the ergodicity of this measure ([22, Thm. 2.12], see also [18, eq. (5)]):

**Theorem 5.** If \( \{A_1, \ldots, A_k\} \) is irreducible, then the associated Kusuoka measure is ergodic.

Actually, when it comes to irreducibility, it does not matter whether one considers the operators \( A_i \)'s or their adjoints. Namely, from the simple fact that a subspace \( V \subset \mathbb{C}^d \) is invariant under \( A \in \mathcal{L}(\mathbb{C}^d) \) if and only if \( V^\perp \) is invariant under \( A^* \), we quickly deduce the following

**Observation 6.** \( \{A_1, \ldots, A_k\} \) is irreducible if and only if \( \{A_1^*, \ldots, A_k^*\} \) is irreducible.

In our context and with Observation 6 taken into account, Theorem 5 can be restated as

**Theorem 5’.** If \( \sqrt{\Pi_1}U, \ldots, \sqrt{\Pi_k}U \) is irreducible, then \( \mathbb{P}_* \) is ergodic.

The main aim of this paper is to show that Theorem 5’ can be reversed, i.e., that the irreducibility of \( \sqrt{\Pi_1}U, \ldots, \sqrt{\Pi_k}U \) is a necessary condition for the ergodicity of \( \mathbb{P}_* \), in the case of \( \Pi \) being a rank-1 POVM (Theorem 10) or a PVM consisting of exactly two projections, the ranks of which are equal to 1 and \( d - 1 \), respectively (Theorem 14). Since for rank-1 POVMs \( \mathbb{P}_* \) is a Markov measure, the characterization of ergodicity via Kusuoka’s condition provides an alternative to the well-known characterization in terms of the irreducibility of the transition matrix of the corresponding Markov chain.

A key step in reversing Theorem 5’ is the simplification of the irreducibility condition in the case of POVMs consisting of scaled orthogonal projections (Theorem 7). For rank-1 POVMs this condition has a particularly straightforward geometric description (Proposition 8). As a result, we can easily characterize when the Kusuoka measure induced by a unitarily evolving qubit (two-dimensional quantum system) undergoing repeated measurements described by a rank-1 POVM is ergodic (Corollary 11).

Additionally, for the PVMs consisting of two projections with respective ranks \( d - 1 \) and 1 we prove that the Kusuoka measure is reversible in the sense that the probability of the system emitting a given string of measurement outcomes is equal to the probability that the reverse string will be produced (Theorem 15), i.e.,

\[
\mathbb{P}_*(C_{i_1, \ldots, i_n}) = \mathbb{P}_*(C_{i_n, \ldots, i_1})
\]

for every \( (i_1, \ldots, i_n) \in I_k^n, n \in \mathbb{N} \).
2. Results

Firstly, we show that Kusuoka’s sufficient ergodicity condition, i.e., the irreducibility of \( \{\sqrt{\Pi_1}U, \ldots, \sqrt{\Pi_k}U\} \), can be significantly simplified if the POVM \( \Pi \) consists of scaled projections. Namely, instead of verifying the invariance of a subspace of \( \mathbb{C}^d \) under the composed operators \( \sqrt{\Pi_1}U, \ldots, \sqrt{\Pi_k}U \), it suffices to verify its invariance under \( U \) and under the measurement operators \( \Pi_1, \ldots, \Pi_k \).

**Theorem 7.** Let \( U \in \mathcal{U}(\mathbb{C}^d) \) and let \( \Pi = \{\Pi_1, \ldots, \Pi_k\} \) be a POVM such that \( \Pi_i = c_i P_i \) for every \( i \in I_k \), where \( c_i > 0 \) and \( P_i \in \mathcal{L}(\mathbb{C}^d) \) is an orthogonal projection (i.e., \( P_i = P_i^2 = P_i^* \)). Let \( W \) be a non-trivial subspace of \( \mathbb{C}^d \). Then for every \( i \in I_k \) we have

\[
\sqrt{\Pi_i}U(W) \subset W \iff U(W) = W \text{ and } \Pi_i(W) \subset W.
\]

**Proof.** Note that \( \Pi_i^2 = c_i \Pi_i \), and so \( \sqrt{\Pi_i} = \Pi_i/\sqrt{c_i} \), which guarantees that the images of any linear subspace of \( \mathbb{C}^d \) under \( \sqrt{\Pi_i} \) and \( \Pi_i \) coincide.

(\(\Rightarrow\)) We have \( \Pi_iU(W) \subset W \) for every \( i \in I_k \) since \( \Pi_iU(W) = \sqrt{\Pi_i}U(W) \subset W \). Let \( w \in W \). From (3) we obtain \( Uw = \sum_{i=1}^k \Pi_iUw \). It follows that \( Uw \in W \), because \( \Pi_iUw \in W \) for every \( i \in I_k \) and \( W \) is a subspace of \( \mathbb{C}^d \). Hence, \( U(w) \subset W \), thus also \( U(W) = W \) as \( U \) is an isometry. Therefore, \( \Pi_i(W) = \Pi_iU(W) \subset W \) for every \( i \in I_k \), as desired.

(\(\Leftarrow\)) It suffices to observe that \( \sqrt{\Pi_i}U(W) = \sqrt{\Pi_i}(W) = \Pi_i(W) \subset W \), where \( i \in I_k \). \( \square \)

In the case of rank-1 POVMs, which consist of uniformly scaled one-dimensional projections, Kusuoka’s sufficient ergodicity condition can be simplified further. Namely, the invariance of a subspace under \( \Pi_1, \ldots, \Pi_k \) can be expressed in terms of the vectors associated with \( \Pi \) belonging to this subspace or to its orthogonal complement. In the two following propositions we let \( \Pi = \{\Pi_1, \ldots, \Pi_k\} \) be a rank-1 POVM and denote the associated unit vectors by \( \varphi_1, \ldots, \varphi_k \), i.e., for \( i \in I_k \) we have \( \Pi_i = \frac{1}{\|\varphi_i\|^2} \varphi_i \), where \( \varphi_i \) is an orthogonal projection on \( \text{span}\{\varphi_i\} \).

**Proposition 8.** Let \( W \) be a non-trivial subspace of \( \mathbb{C}^d \). Then for every \( i \in I_k \) we have

\[
\Pi_i(W) \subset W \iff \varphi_i \in W \cup W^\perp.
\]

**Proof.** Fix \( i \in I_k \) and note that

\[
\Pi_i(W) = \langle \varphi_i, w \rangle \varphi_i: w \in W \rangle = \begin{cases} 
\{0\} & \text{if } \varphi_i \in W^\perp, \\
\text{span}\{\varphi_i\} & \text{if } \varphi_i \notin W^\perp.
\end{cases}
\]

(\(\Rightarrow\)) If \( \varphi_i \notin W^\perp \), then, by assumption, we have \( \text{span}\{\varphi_i\} = \Pi_i(W) \subset W \), which in turn implies that \( \varphi_i \in W \). We conclude that \( \varphi_i \in W \cup W^\perp \), as required.

(\(\Leftarrow\)) If \( \varphi_i \in W^\perp \), then \( \Pi_i(W) = \{0\} \), so \( \Pi_i(W) \subset W \). If \( \varphi_i \in W \), then \( \Pi_i(W) = \text{span}\{\varphi_i\} \); hence, we again obtain \( \Pi_i(W) \subset W \), which concludes the proof. \( \square \)
Recall from Example 3 that if $\Pi$ is a rank-1 POVM, then $P_*$ is a Markov measure. It is well known that the ergodicity of a Markov measure is equivalent to the irreducibility of the corresponding transition matrix [20, Thm. 6.2.6]. Hence, if $\Pi$ is a rank-1 POVM, then $P_*$ is ergodic if and only if $[p_j(\rho_i)]_{i,j\in I_k}$ is irreducible. In what follows we show that in the case of rank-1 POVMs Kusuoka’s sufficient ergodicity condition follows from the irreducibility of the transition matrix, and so from the non-ergodicity of $P_*$.  

**Proposition 9.** Let $U \in \mathcal{U}(\mathbb{C}^d)$. If there exists a non-trivial subspace $W$ of $\mathbb{C}^d$ such that $U(W) = W$ and $\varphi_i \in W \cup W^\perp$ for every $i \in I_k$, then $[p_j(\rho_i)]_{i,j\in I_k}$ is reducible.

**Proof.** Put $I_W := \{i \in I_k : \varphi_i \in W\}$. Note that $1 \leq |I_W| \leq k - 1$, because (3) implies that $\{\varphi_i\}_{i \in I_k}$ spans $\mathbb{C}^d$, and so neither $\{\varphi_i\}_{i \in I_k} \subset W$ nor $\{\varphi_i\}_{i \in I_k} \subset W^\perp$ can hold. Recall from (9) that $p_j(\rho_i) = \frac{1}{d} |\langle \varphi_j, U \varphi_i \rangle|^2$ for $i, j \in I_k$. Let $r \in I_W$ and $s \in I_k \setminus I_W$. As both $W$ and $W^\perp$ are invariant under $U$, we have $\varphi_r \in W$ and $\varphi_s \in W^\perp$, as well as $U \varphi_r \in W$ and $U \varphi_s \in W^\perp$. Hence, $p_r(\rho_s) = p_s(\rho_r) = 0$, so it follows easily that $[p_j(\rho_i)]_{i,j\in I_k}$ is reducible. □

As a result, for rank-1 POVMs Kusuoka’s sufficient ergodicity condition is also necessary, i.e., Theorem 5’ can be reversed.

**Theorem 10.** Let $U \in \mathcal{U}(\mathbb{C}^d)$ and let $\Pi = \{\Pi_1, \ldots, \Pi_k\}$ be a rank-1 POVM. The following conditions are equivalent:

(i) $P_*$ is ergodic,

(ii) $\{\sqrt{\Pi_1}U, \ldots, \sqrt{\Pi_k}U\}$ is irreducible,

(iii) there is no non-trivial subspace $W$ of $\mathbb{C}^d$ such that $U(W) = W$ and $\varphi_i \in W \cup W^\perp$ for every $i \in I_k$, where $\varphi_i$ is associated with $\Pi$ via $\text{im} \, \Pi_i = \text{span} \{\varphi_i\}, i \in I_k$,

(iv) the transition matrix $[p_j(\rho_i)]_{i,j\in I_k}$ is irreducible.

**Proof.** As explained above, (i) $\Leftrightarrow$ (iv) is a classical result [20, Thm. 6.2.6], (iv) $\Rightarrow$ (iii) is the contraposition of Proposition 9, (iii) $\Rightarrow$ (ii) follows from Theorem 7 coupled with Proposition 8, and (ii) $\Rightarrow$ (i) is Theorem 5’. □

In particular, for qubits (two-dimensional quantum systems) we obtain

**Corollary 11.** Let $U \in \mathcal{U}(\mathbb{C}^2)$ and let $\Pi$ be a rank-1 POVM. Then $P_*$ is non-ergodic if and only if $\Pi$ is the POVM corresponding to an eigenbasis of $U$.

We now move on to consider the other class of measurements, i.e., the POVMs consisting of exactly two projections, of which one has rank 1, and so the other has rank $d - 1$. If $d > 2$, then the latter measurement operator gives rise to a non-constant evolution map. In consequence, there may be infinitely many quantum states corresponding to the same measurement outcome, which, in principle, causes the symbolic dynamics to be non-Markovian. We start with a simple example of such a PVM producing a non-ergodic Kusuoka measure.
Example 12. Let $U \in \mathcal{U}(\mathbb{C}^d)$ and let $\Pi = \{\Pi_1, \Pi_2\}$ be a PVM such that $\Pi_1$ and $\Pi_2$ are projections on $\text{span}\{e_1, \ldots, e_{d-1}\}$ and $\text{span}\{e_d\}$, respectively, where $\{e_1, \ldots, e_d\}$ is an orthonormal eigenbasis of $U$.

In the first measurement both outcomes are achievable and their probabilities are proportional to the dimensions of the respective subspaces:

$$p_1(\rho) = \frac{1}{d} \text{tr}(\Pi_1) = \frac{d-1}{d} \quad \text{and} \quad p_2(\rho) = \frac{1}{d} \text{tr}(\Pi_2) = \frac{1}{d}.$$ 

Provided that the outcome ‘1’ or ‘2’ has been obtained, the post-measurement state reads

$$F_1(\rho) = \frac{1}{d-1} \Pi_1 \quad \text{or} \quad F_2(\rho) = \Pi_2,$$

respectively. Next, the probability of the system emitting $j$, provided that the first measurement yielded $i$, is equal to $p_j(F_i(\rho)) = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta and $i, j \in \{1, 2\}$. Indeed, observe that if $i \neq j$, then $\text{tr}(\Pi_j U \Pi_i U^*) = \text{tr}(\Pi_j \Pi_i) = 0$, where the first equality is due to the fact that $U$ and $\Pi_i$ share the eigenbasis $\{e_1, \ldots, e_{d}\}$, which implies that $U \Pi_i U^* = \Pi_i$, and the second follows from the mutual orthogonality of $\Pi_1$ and $\Pi_2$.

Hence, the only possible sequences of measurement outcomes are the constant sequence of 1’s, which is generated with probability $\frac{d-1}{d}$, and the constant sequence of 2’s, generated with complementary probability $\frac{1}{d}$. That is, $P_\pi = \frac{d-1}{d} P_1 + \frac{1}{d} P_2$, where $P_i$ stands for the Dirac delta probability measure on $\{1, 2\}^N$ supported on the constant sequence of $i$’s, where $i \in \{1, 2\}$. Obviously, $P_\pi$ is not ergodic.

In the above example all but one eigenvector of $U$ belong to $\text{im} \Pi_1$, where $\Pi_1$ is assumed to be the projection of rank $d - 1$. It turns out that the presence of an eigenvector of $U$ in $\text{im} \Pi_1$ is equivalent to the non-ergodicity of $P_\pi$, as we now show. Note that Theorem 14 is in fact the reverse of Theorem 5’. A crucial role in the proof is played by the following result:

Lemma 13. [30, Lemma 1] Let $U \in \mathcal{U}(\mathbb{C}^d)$ and let $z \in \mathbb{C}^d$ be a unit vector. Put $\sigma(U)$ for the set of eigenvalues of $U$ and $P$ for the orthogonal projection on $\Theta := \text{span}\{z\}^\perp$. Then

$$\lim_{m \to \infty} \text{tr}((PU)^m (PU)^*^m) = \sum_{\lambda \in \sigma(U)} \text{dim}(\Theta \cap \ker(U - \lambda I)).$$

Theorem 14. Let $U \in \mathcal{U}(\mathbb{C}^d)$ and let $\Pi = \{\Pi_1, \Pi_2\}$ be a PVM such that $\text{rank} \Pi_1 = d - 1$ and $\text{rank} \Pi_2 = 1$. Put $z$ for a unit vector that spans $\text{im} \Pi_2$ and $\Theta := \text{im} \Pi_1 = \text{span}\{z\}^\perp$. The following conditions are equivalent:

(i) $P_\pi$ is not ergodic,

(ii) there exists a non-trivial subspace of $\mathbb{C}^d$ invariant under $U$ and under $\Pi_2$ (and thus necessarily also under $\Pi_1$),

(iii) $z$ belongs to a non-trivial subspace of $\mathbb{C}^d$ invariant under $U$,

(iv) an eigenvector of $U$ belongs to $\Theta$. 
Proof.

(i) ⇒ (ii) This implication follows from Theorem 5’ and Theorem 7.

(ii) ⇒ (iii) Let $W$ be a non-trivial subspace of $\mathbb{C}^d$ invariant under $U$ and under $\Pi_2$. Clearly, $W^\perp$ is non-trivial and invariant under $U$ as well. It follows easily that $z \in W$ or $z \in W^\perp$. Indeed, if $z \notin W^\perp$, then $\text{span}\{z\} = \Pi_2(W) \subset W$, so $z \in W$, as desired.

(iii) ⇒ (iv) Let $V$ be a non-trivial subspace of $\mathbb{C}^d$ invariant under $U$ and such that $z \in V$. We can choose an orthonormal basis $\mathcal{B}_V$ of $V$ consisting of the eigenvectors of $U$. As $V^\perp$ is invariant under $U$ as well, we can extend $\mathcal{B}_V$ to an orthonormal basis $\mathcal{B}$ of $\mathbb{C}^d$ consisting of the eigenvectors of $U$. Each vector from $\mathcal{B} \setminus \mathcal{B}_V$ is an eigenvector of $U$ orthogonal to $V$, thus also to $z$, which means that it lies in $\Theta$.

(iv) ⇒ (i) Consider $\mathcal{W} := \{(s_i)_{i=1}^\infty \in I_2^\infty : s_i = 1 \text{ for almost all } i \in \mathbb{N}\}$. Obviously, $\mathcal{W}$ is invariant under the shift operator. Putting

\[ \mathcal{W}_{n,m} := \{(s_i)_{i=1}^\infty \in I_2^n : s_{n+1} = \ldots = s_{n+m} = 1\}, \]

we have $\mathcal{W} = \bigcup_{n=0}^\infty \bigcap_{m=1}^\infty \mathcal{W}_{n,m}$, so from the continuity of $P_\ast$, we obtain

\[ P_\ast(\mathcal{W}) = \lim_{n \to \infty} \lim_{m \to \infty} P_\ast(\mathcal{W}_{n,m}). \]

Fix $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. For strings consisting exclusively of 1’s we adopt the notation $1^m := (1, \ldots, 1) \in I_2^m$. Since $\mathcal{W}_{n,m} = \bigcup_{\kappa \in I_2^n} C_{\kappa 1^m}$, we obtain

\[ P_\ast(\mathcal{W}_{n,m}) = \sum_{\kappa \in I_2^n} P_\ast(C_{\kappa 1^m}) = P_\ast(C_{1^m}) = \text{tr}((\Pi_1 U)^m \rho_\ast(U^* \Pi_1)^m) = \frac{1}{d} \text{tr}((\Pi_1 U)^m (\Pi_1 U)^*), \]

where the second equality follows from (7) and the third from (8). In consequence, Lemma 13 gives

\[ P_\ast(\mathcal{W}) = \frac{1}{d} \lim_{m \to \infty} \text{tr}((\Pi_1 U)^m (\Pi_1 U)^*) = \frac{1}{d} \sum_{\lambda \in \sigma(U)} \text{dim}(\Theta \cap \ker(U - \lambda I)). \]

By assumption, there is an eigenvector of $U$ in $\Theta$. Denoting the corresponding eigenvalue by $\tilde{\lambda}$, we obtain $\text{dim}(\Theta \cap \ker(U - \tilde{\lambda} I)) \geq 1$; hence $P_\ast(\mathcal{W}) \geq \frac{1}{d} > 0$.

It remains to observe that $\Theta \setminus \Theta \cap \ker(U - \lambda I) \subset \Theta$, and so

\[ \sum_{\lambda \in \sigma(U)} \text{dim}(\Theta \cap \ker(U - \lambda I)) \leq \text{dim} \Theta = d - 1; \]

hence, $P_\ast(\mathcal{W}) \leq \frac{d-1}{d} < 1$, which concludes the proof. \(\square\)

Finally, we show that PVMs consisting of two projections with ranks equal to $d - 1$ and 1, respectively, lead to Kusuoka measures that are reversible in the sense that any given cylinder set has the same measure as the cylinder set corresponding to the reverse string. In other words, the probability of the system outputting any given string of measurement outcomes
Let \( \omega \) be a PVM such that \( \rho \) be a state. Since \( \Pi = \{ \Pi_1, \Pi_2 \} \) be a PVM such that rank \( \Pi_1 = d - 1 \) and rank \( \Pi_2 = 1 \). Then \( \Pi_1 = \rho \) and \( \Pi_2 = 1 \). Then

\[
\Pi_1 = \rho \Pi_2 = \rho \cdot \rho \Pi_2 = \rho \cdot \rho = \rho
\]

Indeed, if \( \rho \Pi_2 = 0 \), then \( \rho \Pi_2 = 0 \), and the repeated application of (1) yields the desired formula. Similarly, if \( \rho \Pi_2 = 0 \), then \( \rho \Pi_2 = 0 \).

\textbf{Fact 2.} Let \( m \in \mathbb{N} \). We have

\[
\rho_{(1, \ldots, 1, 2)}(\rho^*_m) = \frac{1}{d} \rho_{(1, \ldots, 1)}(\Pi_2).
\]

Indeed, it follows that

\[
\rho_{(2, \ldots, 1, \ldots, 1)}(\rho^*_m) = \frac{1}{d} \rho_{(1, \ldots, 1)}(\Pi_2) = \frac{1}{d} \rho_{(1, \ldots, 1)}(\Pi_2),
\]

where we first use (11) and then the fact that \( \rho_{(1, \ldots, 1)}(\rho^*_m) = \frac{1}{d} \text{tr}(\Pi_2) = \frac{1}{d} \). It remains to show that \( \rho_{(2, \ldots, 1, \ldots, 1)}(\rho^*_m) = \rho_{(1, \ldots, 2)}(\rho^*_m) \). Since \( \Pi_2 = \mathbb{I} - \Pi_1 \), from (8) we have

\[
\rho_{(2, \ldots, 1, \ldots, 1)}(\rho^*_m) = \text{tr}(\Pi U \cdots \Pi U \Pi_1 \Pi_2 U \rho U^* \Pi_2 U^* \Pi_1 \cdots U^* \Pi_1)
\]

\[
= \frac{1}{d} \text{tr}(\Pi U \cdots \Pi U \Pi_2 U^* \Pi_1 \cdots U^* \Pi_1)
\]

\[
= \frac{1}{d} \text{tr}(\Pi U \cdots \Pi U (\mathbb{I} - \Pi_1) U \rho U^* \Pi_1 \cdots U^* \Pi_1)
\]

\[
= \text{tr}(\Pi U \cdots \Pi U \rho U^* \Pi_1 \cdots U^* \Pi_1) - \text{tr}(\Pi U \cdots \Pi U (\Pi U \rho U^* \Pi_1 U \Pi_2 U^* \Pi_1) U \rho U^* \Pi_1 \cdots U^* \Pi_1)
\]

\[
= \rho_{(1, \ldots, 1)}(\rho^*_m) - \rho_{(1, \ldots, 1)}(\rho^*_m)
\]

\[
= \rho_{(1, \ldots, 1, 2)}(\rho^*_m),
\]

where the last equality follows from (2). We conclude that (12) holds.
Also, we let $\epsilon$ stand for the empty string and define $p_\epsilon(\rho) := 1$ for every $\rho \in \mathcal{S}(\mathbb{C}^d)$. We have $p_2(\rho_s) = \frac{1}{2} = \frac{1}{2}p_1(\Pi_2)$, and so (12) holds for $m = 0$ as well.

Now, we fix $\iota = (i_1, \ldots, i_n) \in I^n_2, n \in \mathbb{N}$. If $n = 1$ or $\iota$ is a string of identical symbols, then the assertion of the theorem holds trivially. We therefore assume that $n \geq 2$ and that both symbols ‘1’ and ‘2’ appear in $\iota$. Let $1 \leq j_1 < j_2 < \ldots < j_s \leq n$, where $s \in \{1, \ldots, n - 1\}$, stand for the positions in $\iota = (i_1, \ldots, i_n)$ occupied by 2, and denote by $l_r := j_{r+1} - j_r - 1$ the number of times that 1 appears between the $r$-th and $(r + 1)$-th occurrence of 2, where $r \in \{1, \ldots, s - 1\}$. Moreover, put $l_0 := j_1 - 1$ and $l_s := n - j_s$ for the number of 1’s that appear before the first and after the last appearance of 2 in $\iota$, respectively. That is, we have

$$\iota = (i_1, \ldots, i_n) = (1, \ldots, 1, 2, 1, \ldots, 1, 2, \ldots, 2, 1, \ldots, 1, 2, 1, \ldots, 1).$$

The repeated application of (11) yields the following factorization

$$\mathbb{P}_s(C(i_1, \ldots, i_n)) = p_{(i_1, \ldots, i_1)}(\rho_s) \cdot p_{(i_2, i_1)}(\Pi_2) \cdot \ldots \cdot p_{(i_{j_s - 1}, \ldots, i_{j_1})}(\Pi_2) \cdot p_{(i_{j_s - 1}, \ldots, i_1)}(\Pi_2)$$

while for the reversed string we obtain

$$\mathbb{P}_s(C(i_n, \ldots, i_1)) = p_{(i_n, \ldots, i_{j_s})}(\rho_s) \cdot p_{(i_{j_s - 1}, \ldots, i_{j_1})}(\Pi_2) \cdot \ldots \cdot p_{(i_{j_2 - 1}, \ldots, i_1)}(\Pi_2) \cdot p_{(i_{j_1 - 1}, \ldots, i_1)}(\Pi_2).$$

Clearly, if $p_{(1, \ldots, 2)}(\Pi_2) = 0$ for some $r \in \{1, \ldots, s - 1\}$, then $\mathbb{P}_s(C(i_1, \ldots, i_n)) = \mathbb{P}_s(C(i_n, \ldots, i_1)) = 0$.

Otherwise, the desired equality $\mathbb{P}_s(C(i_1, \ldots, i_n)) = \mathbb{P}_s(C(i_n, \ldots, i_1))$ is equivalent to

$$p_{(1, \ldots, 2)}(\rho_s) \cdot p_{(1, 1)}(\Pi_2) = p_{(1, \ldots, 2)}(\rho_s) \cdot p_{(1, 1)}(\Pi_2).$$

To conclude the proof, it remains to observe that (12) implies that both sides of (13) are equal to $\frac{1}{4} p_{(1, 1)}(\Pi_2) \cdot p_{(1, 1)}(\Pi_2).$ \hfill $\square$

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