Schauder and Sobolev Estimates of Parabolic Equations

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Abstract

In this note, we use the non-homogeneous Poisson stochastic process to show how knowing Schauder and Sobolev estimates for the one-dimensional heat equation allows one to derive their multidimensional analogs. The method is probability. We generalize the result of Krylov-Priola \cite{7}.

Keywords: Poisson stochastic process; Schauder estimate; Sobolev estimate

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1 Introduction

For the classical theory of partial differential equations, the Schauder and Sobolev estimates are important issues, see the book \cite{2} \cite{5}. In \cite{7}, Krylov-Priola used the Poisson stochastic process to obtain the Schauder and Sobolev estimates of multi-dimensional heat equation from the one-dimensional case. More precisely, they first know the Schauder and Sobolev estimates for the following equation

\[
\begin{aligned}
\partial_t u(t, x) &= D_x^2 u(t, x) + f(t, x), \quad t \in (0, T), \quad x \in \mathbb{R}, \\
u(0, x) &= 0, \quad x \in \mathbb{R},
\end{aligned}
\tag{1.1}
\]

then they derive the Schauder and Sobolev estimates for multi-dimensional equation. Actually, they obtained more abundant results.

The regularity of partial differential equations has been studied by many authors, steady transport equation \cite{6}, stochastic evolution equations \cite{1} \cite{12}, stochastic partial differential equations \cite{3} \cite{4} and so on. There are a lot of work about regularity focusing on stochastic process, for example \cite{10} \cite{11}.

In the present paper, we aim to use the non-homogeneous Poisson stochastic process to find some new results. The main difference between this paper and \cite{7} is that we use the non-homogeneous Poisson stochastic process but Krylov-Priola used the homogeneous Poisson stochastic process. The method used in \cite{7} is probability and the results are interesting.

Throughout this paper, \(T\) is a fixed positive number, \(\mathbb{R}^d\) denotes Euclidean space and \(C^\alpha(\mathbb{R}^d)\), \(\alpha \in (0, 1)\) is the space of all real-valued functions \(f\) on \(\mathbb{R}^d\) with the norm

\[
\|f\|_{C^\alpha(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |f(x)| + [f]_{C^\alpha(\mathbb{R}^d)} < +\infty,
\]
Moreover, there exists only one solution \( L \) (independently identically distribution) random variables. we can not write the non-homogeneous Poisson process as the sum of a sequence which is an i.i.d and non-homogeneous Poisson process is that the latter case is not a stationary process. Thus the parameter of the process is a function of time. The significant difference between the homogeneous satisfies \( \phi \) satisfying that where \( Df \) is the gradient of \( f \) and \( D^2f \) is its Hessian.

The rest of this paper is arranged as follows. In Sections 2, we present some preliminaries and main result. Section 3 is the proof of main result.

## 2 Preliminaries and Main Results

Consider the following Cauchy problem

\[
\begin{aligned}
\partial_t u(t, x) &= a(t)D_x^2u(t, x) + f(t, x), \quad t \in (0, T), \quad x \in \mathbb{R}, \\
\quad u(0, x) &= 0,
\end{aligned}
\]

(2.1)

where \( a(t) \) is a positive bounded function. Denote \( B_c((0, T), C_0^\infty(\mathbb{R}^d)) \) as the space of functions \( \varphi \) satisfying that \( \varphi \) is Borel bounded function and \( \varphi(t, \cdot) \in C_0^\infty(\mathbb{R}^d) \) for any \( t \in (0, T) \).

It follows from [5, 8, 9] that if \( f \) belongs to \( B_c((0, T), C_0^\infty(\mathbb{R})) \), then (2.1) has a solution \( u(t, x) \) satisfying

(i) \( u \) is a continuous function in \( [0, T] \times \mathbb{R} \);
(ii) for any fixed \( t \in [0, T] \), \( u \) belongs to \( C^{2+\alpha}(\mathbb{R}) \) and has the following estimate

\[
\sup_{t \in [0, T]} \|u(t, \cdot)\|_{C^{2+\alpha}(\mathbb{R})} \leq N(T, \alpha) \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^\alpha(\mathbb{R})}.
\]

(2.2)

Moreover, there exists only one solution \( u \) satisfying the following properties

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} |u(t, x)| \leq T \sup_{(t, x) \in [0, T] \times \mathbb{R}} |f(t, x)|,
\]

(2.3)

\[
\sup_{t \in [0, T]} \|D_x^2u(t, \cdot)\|_{C^\alpha(\mathbb{R})} \leq N(\alpha) \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^\alpha(\mathbb{R})},
\]

(2.4)

\[
\|D_x^2u\|_{L^p((0, T) \times \mathbb{R})}^p \leq N_p \|f\|_{L^p((0, T) \times \mathbb{R})}^p.
\]

(2.5)

Here \( L^p \)-space is defined as usual.

Now we recall some knowledge of Poisson stochastic process. A non-homogeneous Poisson process \( \pi(t, \omega) \) (\( \pi_t \) for short) is a Poisson process with rate parameter \( \lambda(t) \) such that the rate parameter of the process is a function of time. The significant difference between the homogeneous and non-homogeneous Poisson process is that the latter case is not a stationary process. Thus we can not write the non-homogeneous Poisson process as the sum of a sequence which is an i.i.d (independently identically distribution) random variables.

As usual, \( \pi_t \) is a counting process with the following properties

(i) \( \mathbb{P}(\pi_t - \pi_s = k) = \frac{[m(t) - m(s)]^k}{k!} e^{-[m(t) - m(s)]}, \quad m(t) = \int_0^t \lambda(s)ds; \)

(ii) \( \pi_t - \pi_s \) is independent of the trajectory \( \{\pi_r, r \in [0, s]\} \).

For simplicity, in this paper, we only consider the 2-dimensional heat equation. For \( x, y \in \mathbb{R} \), we set \( z = (x, y) \in \mathbb{R}^2 \). For \( l \in \mathbb{R}^2 \), denote \( D_l^2 = \partial^2 / \partial x_i \partial x_j \) and \( D_{ij} = D_iD_j \), where \( i, j = 1, 2 \) and \( x_1 = x, x_2 = y \). We obtain the following result.
Theorem 2.1 Let \( a(t) > 0 \) be a bounded Borel measurable function. Then for any \( f \in B_c((0, T), C_0^\infty(\mathbb{R}^2)) \), there exists a unique continuous in \([0, T] \times \mathbb{R}^2\) solution \( v(t, z) \) of the equation

\[
\begin{align*}
\partial_t v(t, z) &= a(t)\Delta v(t, z) + f(t, z), \quad t > 0, \quad z \in \mathbb{R}^2, \\
v(0, z) &= 0,
\end{align*}
\]

(2.6)

Moreover, \( v(t, \cdot) \in C^{2+\alpha}(\mathbb{R}^2) \) satisfies

\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^2} |v(t, z)| \leq T \sup_{(t, z) \in [0, T] \times \mathbb{R}^2} |f(t, z)|,
\]

\[
\sup_{t \in [0, T]} [D_{ij} v(t, \cdot)]_{C^\alpha(\mathbb{R}^2)} \leq N_0(\alpha) \sup_{t \in [0, T]} [f(t, \cdot)]_{C^\alpha(\mathbb{R}^2)},
\]

\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^2} [D_t^2 v(t, z + l \cdot)]_{C^\alpha(\mathbb{R}^2)} \leq N_0(\alpha) \sup_{(t, z) \in [0, T] \times \mathbb{R}^2} [D_t^2 f(t, z + l \cdot)]_{C^\alpha(\mathbb{R}^2)},
\]

\[
|D_t^2 u|^p_{L^p([0, T] \times \mathbb{R}^2)} \leq N_p \|f\|^p_{L^p([0, T] \times \mathbb{R}^2)},
\]

where \( N_0(\alpha) \) and \( N_p \) are positive constants.

Remark 2.1 The result of this paper has a little difference from [7] in the following part. If \( a(t) = 1 \), that is, \( \lambda(t) \equiv \lambda \), then Theorem 2.1 is exactly the second part of [7]. The big difference is that we can assume \( \lambda(t) = h^2 a(t) \) and then the equation will keep the same form as the dimensional case. Of course, in [7, Section 3], Krylov-Priola used a suitable transform to consider the problem (2.1). Here we emphasize that we can use another stochastic process to deal with the problem (2.1).

One can use renew process to study the regularity of parabolic equations. The difference is that in the following Lemma [7,1]: \( \mathbb{E}[\pi_{(k+1)2^{-n}} - \pi_{k2^{-n}}] \) will be different. But for parabolic equation, the Poisson process is the best choice.

3 The Proof of Theorem [2.1]

In this section, we prove the main result. Similar to [7], we consider the following equations

\[
\begin{align*}
\partial_t u(t, x, y, \omega) &= a(t)D_x^2 u(t, x, y, \omega) + f(t, x, y - h\pi_t(\omega)), \quad t > 0, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \\
u(0, x, y) &= 0,
\end{align*}
\]

(3.1)

where \( a(t) > 0 \) is a bounded Borel measurable function and \( h \in \mathbb{R} \) is a parameter. As usual in probability theory, we do not indicate the dependence on \( \omega \) in the sequence. From the result of one-dimensional case, we get that there exists a unique solution \( u(t, x, y) \), depending on \( y \) and \( \omega \) as parameters. And thus estimates (2.2)-(2.5) hold for each \( \omega \in \Omega \) and \( y \in \mathbb{R} \) if we replace \( u(t, x) \) and \( f(t, x) \) with \( u(t, x, y) \) and \( f(t, x, y - h\pi_t) \), respectively.

The solution of (3.1) can be written as

\[
\begin{align*}
u(t, x, y + h\pi_t) &= \int_0^t [a(s)D_x^2 u(s, x, y + h\pi_s) + f(s, x, y)] ds + \int_{(0, t]} g(s, x, y) d\pi_s,
\end{align*}
\]

(3.2)

where

\[
g(s, x, y) = u(s, x, y + h + h\pi_{s-}) - u(s, x, y + h\pi_{s-})
\]

(3.3)

is the jump of the process \( u(t, x, y + h\pi_t) \) as a function of \( t \) at moment \( s \) if \( \pi_t \) has a jump at \( s \). Here \( \pi_{s-} \) is the left-continuous w.r.t. \( s \).

In order to prove the main result, we need to study the function \( g \).
Lemma 3.1 For $g$ defined as (3.1) and $t \leq T$ we have
\[
E \int_{[0,t]} g(s, x, y) d\pi_s = \int_0^t \lambda(s)[v(s, x, y + h) - v(s, x, y)] ds,
\]
where
\[
v(t, x, y) := Eu(t, x, y + h).\]

**Proof.** Assume that $t = 1$ for simplicity. Fix $x$ and $y$, and denote $g(s) = g(s, x, y)$. Note that $g$ is bounded on $\Omega \times (0, T)$, and thus if we define
\[
g_n(s) = g(k2^{-n}) = u(k2^{-n}, x, y + h + h\pi_{k2^{-n}}) - u(k2^{-n}, x, y + h\pi_{k2^{-n}})
\]
for $s \in (k2^{-n}, (k + 1)2^{-n}]$, $k = 0, 1, \ldots$, then $g_n(s) \to g(s)$ as $n \to \infty$ for any $s \in (0, t]$ and $\omega \in \Omega$, and
\[
\xi_n := \int_{[0,t]} g_n(s) d\pi_s \to \int_{[0,t]} g(s) d\pi_s =: \xi
\]
for any $\omega \in \Omega$. Dominated convergence theorem implies that $E\xi_n \to E\xi$.

Notice that
\[
E\xi_n = \sum_{k=0}^{2^n-1} Eg(k2^{-n})(\pi_{(k+1)2^{-n}} - \pi_{k2^{-n}}). \tag{3.4}
\]
Since the non-homogeneous Poisson process is an independent increment process, the expectations of he products on the right in (3.4) are equal to the products of expectations, and since $E\pi_t = m(t)$, we arrive at
\[
E\xi_n = E \sum_{k=0}^{2^n-1} g(k2^{-n})[m(k + 1)2^{-n} - m(k2^{-n})] = E \int_0^t g_n(s)\lambda(s) ds
\]
\[
\to E \int_0^t g(s)\lambda(s) ds = \int_0^t \lambda(s)Eg(s) ds.
\]
Noting that for any $s > 0$, we have $\pi_s = \pi_{s-}$ almost surely, and thus
\[
Eg(s) = v(s, x, y + h) - v(s, x, y).
\]
The proof is complete. \(\square\)

Taking expectations on both sides of (3.2), we obtain the following result.

**Lemma 3.2** Let $f \in Bc(0, T), C^\infty_0(\mathbb{R}^2)$, $h \in \mathbb{R}$ and $\lambda(t) > 0$ for all $t \in [0, T]$. Then there exists a unique continuous function $v(t, x, y)$, $t \in [0, T]$, $x, y \in \mathbb{R}$, satisfying the equation
\[
\partial_t v(t, x, y) = a(t)D^2_vv(t, x, y) + \lambda(t)[v(t, x, y + h) - v(t, x, y)] + f(t, x, y) \tag{3.5}
\]
for $t \in (0, T)$, $x, y \in \mathbb{R}$, with zero initial condition and such that $v(t, \cdot, y) \in C^{2+\alpha}(\mathbb{R})$ for any $t \in (0, T)$, $y \in \mathbb{R}$ and
\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} \|v(t, \cdot, y)\|_{C^{2+\alpha}(\mathbb{R})} \leq N(T, \alpha) \sup_{(t, y) \in [0, T] \times \mathbb{R}} \|f(t, \cdot, y)\|_{C^{\alpha}(\mathbb{R})}.
\]
Furthermore,
\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^2} |v(t, z)| \leq T \sup_{(t, z) \in [0, T] \times \mathbb{R}^2} |f(t, z)|,
\]
\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} [D^2_vv(t, \cdot, y)]_{C^{\alpha}(\mathbb{R})} \leq N({\alpha}) \sup_{(t, y) \in (0, T) \times \mathbb{R}} [f(t, \cdot, y)]_{C^{\alpha}(\mathbb{R})},
\]
\[
\|D^2_vv\|_{L^p([0, T] \times \mathbb{R}^2)}^p \leq N_p \|f\|_{L^p([0, T] \times \mathbb{R}^2)}^p.
\]
we have, for any unit vector \( l \) and using the fact that the solution \( v \) satisfies (3.6), we then have

\[
\partial_t v(t, x, y) = a(t) D_x^2 v(t, x, y) + \lambda(t) [v(t, x, y + h) - v(t, x, y)] + f(t, x, y + h \pi_t)
\]

with zero initial condition. Then it follows from the above computations, we have the function \( w(t, x, y) = Ev(t, x, y - h \pi_t) \) satisfies

\[
\partial_t w(t, x, y) = a(t) D_x^2 w(t, x, y) + \lambda(t) [w(t, x, y + h) - 2w(t, x, y) + w(t, x, y - h)] + f(t, x, y).
\]

Furthermore, \( w(t, x, y) \) has the same estimates as in Lemma 3.2.

Proof of Theorem 2.1

Taking \( \lambda(t) = h^2 a(t) \) in (3.6) and letting \( h \to 0 \), we have the solution \( w = w_h \) of (3.6) will converge to a function \( v(t, x, y) \), which satisfies the equation (2.6). Furthermore, \( v \) is continuous in \([0, T] \times \mathbb{R}^2\), and is infinitely differentiable w.r.t. \((x, y)\) for any \( t \in (0, T) \) and all the estimates in Lemma 3.2 hold true. Therefore, the following estimate holds obviously

\[
\sup_{(t, x, y) \in [0, T] \times \mathbb{R}^2} |v(t, x, y)| \leq T \sup_{(t, x, y) \in [0, T] \times \mathbb{R}^2} |f(t, x, y)|.
\]

Next we will use the rotation invariant of Laplacian operator and the estimates of Lemma 3.2 to derive the desire results. In order to do that, we define \( S \) as an orthogonal transformation of \( \mathbb{R}^2 \): \( S e_i = l_i, i = 1, 2 \), where \( e_i \) is the standard basis in \( \mathbb{R}^2 \), \( l_i \) is a unit vector in \( \mathbb{R}^2 \) and \( l_2 \) is orthogonal to \( l_1 \). Let

\[
f(t, xe_1 + ye_2) = f(t, x, y), \quad v(t, xe_1 + ye_2) = v(t, x, y), \quad S(x, y) = xl_1 + yl_2, \quad g(t, x, y) = f(t, S(x, y)), \quad w(t, x, y) = v(t, S(x, y)),
\]

then \( w \) satisfies

\[
\partial_t w(t, x, y) = a(t) \Delta w(t, x, y) + g(t, x, y),
\]

where we used the rotation invariant of Laplacian operator.

It follows from Lemma 3.2 that

\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} \sup_{x_1 \neq x_2} \frac{|D_x^2 w(t, x_1, y) - D_x^2 w(t, x_2, y)|}{|x_1 - x_2|^{\alpha}} \leq N(\alpha) \sup_{(t, y) \in [0, T] \times \mathbb{R}} \sup_{x_1 \neq x_2} \frac{|g(t, x_1, y) - g(t, x_2, y)|}{|x_1 - x_2|^{\alpha}}.
\]

Notice that

\[
D_x^2 w(t, x, y) = D_{l_1}^2 v(t, S(x, y)) = D_{l_1}^2 v(t, x l_1 + y l_2),
\]

and using the fact that the solution \( v \) of (2.6) has continuous second-order derivatives w.r.t. \((x, y)\), we have, for any unit vector \( l \in \mathbb{R}^2 \)

\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^2} \sup_{\mu \neq \nu} \frac{|D_1^2 v(t, \mu l + z) - D_1^2 v(t, \nu l + z)|}{|\mu - \nu|^{\alpha}} \leq N(\alpha) \sup_{(t, z) \in [0, T] \times \mathbb{R}^2} \sup_{\mu \neq \nu} \frac{|f(t, \mu l + z) - f(t, \nu l + z)|}{|\mu - \nu|^{\alpha}}.
\]
That is to say, we get
\[
\sup_{(t,z) \in [0,T] \times \mathbb{R}^2} [D^2_l v(t, z + l \cdot)]_{C^0(\mathbb{R}^2)} \leq N(\alpha) \sup_{(t,z) \in (0,T) \times \mathbb{R}^2} [D^2_l f(t, z + l \cdot)]_{C^0(\mathbb{R}^2)}.
\]

In particular, if we choose \( z = 0 \), we get the estimate
\[
\sup_{t \in [0,T]} [D_{ij} v(t, \cdot)]_{C^0(\mathbb{R}^2)} \leq N_0(\alpha) \sup_{t \in [0,T]} [f(t, \cdot)]_{C^0(\mathbb{R}^2)}.
\]

Since the Jacobian of \( S(x, y) \) equals to 1, then we have for any unit vector \( l \in \mathbb{R}^2 \)
\[
\int_0^T \int_{\mathbb{R}^2} |D^2_l v(t, z)|^p dz dt \leq N_p \int_0^T \int_{\mathbb{R}^2} |f(t, z)|^p dz dt.
\]

The proof is complete. \( \square \)

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