WHEN DOES NIP TRANSFER FROM FIELDS TO HENSELIAN EXPANSIONS?

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Abstract. Let $K$ be an NIP field and let $v$ be a henselian valuation on $K$. We ask whether $(K, v)$ is NIP as a valued field. By a result of Shelah, we know that if $v$ is externally definable, then $(K, v)$ is NIP. Using the definability of the canonical $p$-henselian valuation, we show that whenever the residue field of $v$ is not separably closed, then $v$ is externally definable. In the case of separably closed residue field, we show that $(K, v)$ is NIP as a pure valued field.

1. Introduction and Motivation

There are many open questions connecting NIP and henselianity, most prominently

Question 1.1. (1) Is any valued NIP field $(K, v)$ henselian?

(2) Let $K$ be an NIP field, neither separably closed nor real closed. Does $K$ admit a definable non-trivial henselian valuation?

Both of these questions have been recently answered positively in the special case where ‘NIP’ is replaced with ‘dp-minimal’ (cf. Johnson’s results in [Joh15]). Moreover, Johnson also showed that question (1) can be answered affirmatively when the characteristic of $K$ is positive and showed a positive answer to question (2) when ‘NIP’ is replaced by ‘dp-finite of positive characteristic’ ([Joh19]).

The question discussed here is the following:

Question 1.2. Let $K$ be an NIP field (in an expansion of the language of rings) and $v$ a henselian valuation on $K$. Is $(K, v)$ NIP?

Note that this question neither implies nor is implied by any of the above questions, it does however follow along the same lines aiming to find out how close the bond between NIP and henselianity really is.

The first aim of this article is to show that the answer to Question 1.2 is ‘yes’ if $Kv$ is not separably closed:

Theorem A. Let $(K, v)$ be henselian and such that $Kv$ is not separably closed. Then $v$ is definable in the Shelah expansion $K^{\text{Sh}}$.

See section 2.1 for the definition of $K^{\text{Sh}}$. The theorem follows immediately from combining Propositions 2.4 and 2.5. If $v$ is definable in $K^{\text{Sh}}$, then one can add a symbol for the valuation ring $\mathcal{O}$ to any language $\mathcal{L}$ extending $\mathcal{L}_{\text{ring}}$ and obtain that if $K$ is NIP as an $\mathcal{L}$-structure, then $(K, v)$ is NIP as an $\mathcal{L} \cup \{\mathcal{O}\}$-structure. Theorem A is proven using the definability of the canonical $p$-henselian valuation. We make a case distinction between when $Kv$ is neither separably closed nor real closed (Proposition 2.4) and when $Kv$ is real closed (Proposition 2.5).

On the other hand, if $Kv$ is separably closed, then we cannot hope for a result in the same generality: it is well-known that any algebraically closed valued field is NIP in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$, however, any algebraically closed field with two independent valuations has IP ([Joh13, Theorem 6.1], see also Example 3.2). In this case, we can still consider the question in the language of rings: given an NIP field $K$ and a henselian valuation $v$ on $K$, is $(K, v)$ NIP in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$? The answer to this is again positive:

Theorem B. Let $K$ be NIP, $v$ henselian on $K$. Then $(K, v)$ is NIP as a pure valued field.

Theorem B is proven as Theorem 3.11 in section 3. The proof of the theorem uses two NIP transfer theorems recently proven in [JS19] and [AJ19a]. A transfer theorem gives criteria under which dependence of the residue field implies dependence of the (pure) valued field. Delon proved a transfer theorem for henselian valued fields of equicharacteristic 0 (see [Del81] or [Sim15, Theorem A.15]), and Béclair proved a
version for equicharacteristic Kaplansky fields which are algebraically maximal (see [Bél99]). The transfer theorems proven in [JS19] Theorem 3.3 and [AJ19a] Proposition 4.1 generalizes these results to separably algebraically maximal Kaplansky fields, in particular, they also works in mixed characteristic. See section 3 for definitions and more details. Combining these transfer theorems with an idea of Scanlon and some standard trickery concerning definable valuations yields that under the assumptions of Theorem B we can find a decomposition of \( v = \bar{v} \circ w \) into two NIP valuations. The question whether the composition of two henselian NIP valuations is again NIP seems to be open. For the case when the residue field of the coarser valuation is stably embedded, this follows from [JS19] Proposition 2.5. Using that the residue field in separably algebraically maximal Kaplansky fields and unramified henselian fields is stably embedded, this allows us to prove the second part of Theorem B.

The paper is organized as follows: In section 2 we first recall the necessary background concerning the Shelah expansion. We then discuss the definition and definability of the canonical \( p \)-henselian valuation. In the final part, we use these two ingredients to prove Theorem A. In particular, we conclude that for any henselian NIP field the residue field is NIP as a pure field. We also obtain as a consequence that if a field admits a non-trivial henselian valuation and is NIP in some \( \mathcal{L} \supseteq \mathcal{L}_{\text{ring}} \), then there is some non-trivial henselian valuation \( v \) on \( K \) such that \((K,v)\) is NIP in \( \mathcal{L} \cup \{O\}\) (Corollary 2.8).

In the third section, we treat the case of separably closed residue fields. We first recall an example which shows that we have to restrict Question 1.2 to the language of pure valued fields. We then briefly review different ingredients, starting with the transfer theorem for separably algebraically maximal Kaplansky fields. After quoting a result by Delon that separably closed valued fields are NIP, we state and prove a Proposition by Scanlon (Proposition 3.6) which implies that on an NIP field, any valuation with non-perfect residue field is \( \mathcal{L}_{\text{ring}} \)-definable. We then recall some facts about stable embeddedness and the composition of NIP valuations. In the final subsection, we prove Theorem B.

Finally, in section 4 we treat the simpler case of convex valuation rings on an ordered field \((K,v)\). As any convex valuation ring is definable in \((K,<)^{\text{Sh}}, we conclude that if \((K,<)\) is an ordered NIP field in some language \( \mathcal{L} \supseteq \mathcal{L}_{\text{ring}} \cup \{<\} \) and \( v \) is a convex valuation on \( K \), then \((K,v)\) is NIP in \( \mathcal{L} \cup \{O\} \) (Corollary 4.3).

Throughout the paper, we use the following notation: for a valued field \((K,v)\), we write \( vK \) for the residue field and \( \mathcal{O}_v \) for the valuation ring of \( v \).

2. Non-separably closed residue fields

2.1. Externally definable sets. Throughout the subsection, let \( M \) be a structure in some language \( \mathcal{L} \).

**Definition.** Let \( N \succ M \) be an \(|M|^+\)-saturated elementary extension. A subset \( A \subseteq M \) is called externally definable if it is of the form

\[
\{a \in M^{\lceil \bar{z} \rceil} \mid N \models \varphi(a, b)\}
\]

for some \( \mathcal{L} \)-formula \( \varphi(x, \bar{y}) \) and some \( b \in N^{\lceil \bar{y} \rceil} \).

The notion of externally definable sets does not depend on the choice of \( N \). See [Sim15] Chapter 3 for more details on externally definable sets.

**Definition.** The Shelah expansion \( M^{\text{Sh}} \) is the expansion of \( M \) by predicates for all externally definable sets.

Note that the Shelah expansion behaves well when it comes to NIP:

**Proposition 2.1** (Shelah, [Sim15] Corollary 3.14]). If \( M \) is NIP then so is \( M^{\text{Sh}} \).

The way the Shelah expansion is used in this paper is to show that any coarsening of a definable valuation on an NIP field is an NIP valuation. Thus, the following example is crucial:

**Example 2.2.** Let \((K,w)\) be a valued field and \( v \) be a coarsening of \( w \), i.e., a valuation on \( K \) with \( \mathcal{O}_v \supseteq \mathcal{O}_w \). Then, there is a convex subgroup \( \Delta \leq wK \) such that we have \( vK \cong wK/\Delta \). As \( \Delta \) is externally definable in the ordered abelian group \( wK \), the valuation ring \( \mathcal{O}_v \) is definable in \((K,w)^{\text{Sh}}\).

2.2. \( p \)-henselian valuations. Throughout this subsection, let \( K \) be a field and \( p \) a prime. We recall the main properties of the canonical \( p \)-henselian valuation on \( K \). We define \( K(p) \) to be the compositum of all Galois extensions of \( K \) of \( p \)-power degree (in a fixed algebraic closure). Note that we have

- \( K \neq K(p) \) iff \( K \) admits a Galois extension of degree \( p \) and
• if $[K(p) : K] < \infty$ then $K = K(p)$ or $p = 2$ and $K(2) = K(\sqrt{-1})$ (see [EP05 Theorem 4.3.5]).

A field $K$ which admits exactly one Galois extension of 2-power degree is called Euclidean. Any Euclidean field is uniquely ordered, the positive elements being exactly the squares (see [EP05 Proposition 4.3.4 and Theorem 4.3.5]). In particular, the ordering on a Euclidean field is $\mathcal{L}_{\text{ring}}$-definable.

**Definition.** A valuation $v$ on a field $K$ is called $p$-henselian if $v$ extends uniquely to $K(p)$. We call $K$ $p$-henselian if $K$ admits a non-trivial $p$-henselian valuation.

Every henselian valuation is $p$-henselian for all primes $p$. Assume $K \neq K(p)$. Then, there is a canonical $p$-henselian valuation on $K$: We divide the class of $p$-henselian valuations on $K$ into two subclasses,

$$H^2_1(K) = \{ v \text{ $p$-henselian on } K \mid K v \neq K v(p) \}$$

and

$$H^2_2(K) = \{ v \text{ $p$-henselian on } K \mid K v = K v(p) \}.$$  

One can show that any valuation $v_2 \in H^2_2(K)$ is finer than any $v_1 \in H^2_1(K)$, i.e. $O_{v_2} \subset O_{v_1}$, and that any two valuations in $H^2_1(K)$ are comparable. Furthermore, if $H^2_2(K)$ is non-empty, then there exists a unique coarsest valuation $v^0_K$ in $H^2_2(K)$; otherwise there exists a unique finest valuation $v^0_K \in H^2_1(K)$. In either case, $v^0_K$ is called the canonical $p$-henselian valuation (see [Koe95] for more details).

The following properties of the canonical $p$-henselian valuation follow immediately from the definition:

- If $K$ is $p$-henselian then $v^0_K$ is non-trivial.
- Any $p$-henselian valuation on $K$ is comparable to $v^0_K$.
- If $v$ is a $p$-henselian valuation on $K$ with $K v \neq K v(p)$, then $v$ coarsens $v^0_K$.
- If $p = 2$ and $K v^2_K$ is Euclidean, then there is a (unique) 2-henselian valuation $v^2_K$ such that $v^2_K$ is the coarsest 2-henselian valuation with Euclidean residue field.

**Theorem 2.3 ([JK15b Corollary 3.3]).** Let $p$ be a prime and consider the (elementary) class of fields $K = \{ K \mid K$ $p$-henselian, with $\zeta_p \in K$ in case $\text{char}(K) \neq p \}$

There is a parameter-free $\mathcal{L}_{\text{ring}}$-formula $\psi_p(x)$ such that

1. if $p \neq 2$ or $K v^2_K$ is not Euclidean, then $\psi_p(x)$ defines the valuation ring of the canonical $p$-henselian valuation $v^0_K$, and
2. if $p = 2$ and $K v^2_K$ is Euclidean, then $\psi_p(x)$ defines the valuation ring of the coarsest 2-henselian valuation $v^2_K$ such that $K v^2_K$ is Euclidean.

### 2.3. External definability of henselian valuations.

In this subsection, we apply the results from the previous two subsections to prove Theorem A from the introduction.

**Proposition 2.4.** Let $(K,v)$ be henselian such that $Kv$ is neither separably closed nor real closed. Then $v$ is definable in $K^{\text{Sh}}$.

**Proof.** Assume $Kv$ is neither separably closed nor real closed. For any finite separable extension $F$ of $K$, we use $u$ to denote the (by henselianity unique) extension of $v$ to $F$. Choose any prime $p$ such that $Kv$ has a finite Galois extension $k$ of degree divisible by $p^2$. Consider a finite Galois extension $N \supseteq K$ such that $Nu = k$. Note that such an $N$ exists by [EP05 Corollary 4.1.6]. Now, let $P$ be a $p$-Sylow of $\text{Gal}(Nu/Kv)$.

Recall that the canonical restriction map

$$\text{res} : \text{Gal}(N/K) \to \text{Gal}(Nu/Kv)$$

is a surjective homomorphism ([EP05 Lemma 5.2.6]). Let $G \leq \text{Gal}(N/K)$ be the preimage of $P$ under this map, and let $L := \text{Fix}(G)$ be the intermediate field fixed by $G$. In particular, $L$ is a finite separable extension of $K$. By construction, the extension $Lu \subseteq Nu$ is a Galois extension of degree $p^n$ for some $n \geq 2$, in particular, we have $Lu \neq Lu(p)$.

Hence, we have constructed some finite separable extension $L$ of $K$ with $Lu \neq Lu(p)$. Moreover, we may assume that $L$ contains a primitive $p$th root of unity in case $p \neq 2$ and $\text{char}(K) \neq p$: The field $L' := L(\zeta_p)$ is again a finite separable extension of $K$ and its residue field is a finite extension of $Lu$. Thus, by [EP05 Theorem 4.3.5], we get $L'u \neq L'u(p)$. Similarly, in case $p = 2$ and $\text{char}(K) = 0$, we may assume that $L$ contains a square root of $\sqrt{-1}$: By construction, $Lu$ has a Galois extension of degree $p^n$ for some $n \geq 2$.  

Consider $L' := L(\sqrt{-1})$, then $L'u$ is not 2-closed and not orderable. In this case, no 2-henselian valuation on $L'$ has Euclidean residue field (see [EP03] Lemma 4.3.6).

Finally, $v_L^p$ is definable on $L$ by a parameter-free $\mathcal{L}$-formula $\varphi_p(x)$. It follows from the defining properties of $v_L^p$ that $\mathcal{O}_{v_L^p} \subseteq \mathcal{O}_u$ holds. As $L/K$ is finite, $L$ is interpretable in $K$. Hence, $\mathcal{O}_w := \mathcal{O}_{v_L^p} \cap K$ is an $L_{\text{ring}}$-definable valuation ring of $K$ with $\mathcal{O}_w \subseteq \mathcal{O}_v$. By Example 2.2, $v$ is definable in $K_{\text{Sh}}$. \hfill \Box

**Proposition 2.5.** Let $(K,v)$ be henselian such that $Kv$ is real closed. Then $v$ is definable in $K_{\text{Sh}}$.

**Proof.** Assume that $(K,v)$ is henselian and $Kv$ is real closed. Then $K$ is orderable. We first reduce to the case that $K$ is Euclidean: Note that $v$ is a 2-henselian valuation with Euclidean residue field. Let $v_K^2$ be the coarsest 2-henselian valuation on $K$ with Euclidean residue field, which is $\emptyset$-definable on $K$ in $L_{\text{ring}}$ by Theorem 2.3. Now, if the induced valuation $\bar{\pi}$ on $K_{v_K^2}$ is definable in $(K_{v_K^2})_{\text{Sh}}$, then the valuation ring of $v$, which is the composition of $v_K^2$ and $\bar{\pi}$, is also definable in $K_{\text{Sh}}$.

Thus, we may assume that $K$ is Euclidean. In this case, $K$ is uniquely ordered and the ordering on $K$ is $L_{\text{ring}}$-definable. Let $\mathcal{O}_w \subseteq K$ be the convex hull of $\mathbb{Z}$ in $K$. Then, $\mathcal{O}_w$ is definable in $K_{\text{Sh}}$. By [EP03] Theorem 4.3.7, $(K,w)$ is a 2-henselian valuation ring on $K$ with Euclidean residue field. As $w$ has no proper refinements, $w$ is the canonical 2-henselian valuation on $K$. Thus, we get $\mathcal{O}_w \subseteq \mathcal{O}_v$ and hence $\mathcal{O}_v$ is also definable in $K_{\text{Sh}}$ by Example 2.2. \hfill \Box

Note that combining Propositions 2.4 and 2.5 immediately yields Theorem A from the introduction. Applying Proposition 2.1, we conclude:

**Corollary 2.6.** Let $K$ be a field and $v$ a henselian valuation on $K$. Assume that $\text{Th}(K)$ is NIP in some language $\mathcal{L} \supseteq L_{\text{ring}}$. If $Kv$ is not separably closed, then $(K,v)$ is NIP in the language $\mathcal{L} \cup \{\mathcal{O}\}$.

As separably closed fields are always NIP in $L_{\text{ring}}$, we note that the residue field of a henselian valuation on an NIP field is NIP as a pure field.

**Corollary 2.7.** Let $K$ be a field and $v$ henselian on $K$. Assume that $\text{Th}(K)$ is NIP in some language $\mathcal{L} \supseteq L_{\text{ring}}$. Then $Kv$ is NIP as a pure field.

Recall that a field $K$ is called henselian if it admits some non-trivial henselian valuation.

**Corollary 2.8.** Let $K$ be a henselian field such that $\text{Th}(K)$ is NIP in some language $\mathcal{L} \supseteq L_{\text{ring}}$. Assume that $K$ is neither separably closed nor real closed. Then $K$ admits some non-trivial externally definable henselian valuation $v$. In particular, $(K,v)$ is NIP in the language $\mathcal{L} \cup \{\mathcal{O}_v\}$.

**Proof.** If $K$ admits some non-trivial henselian valuation $v$ such that $Kv$ is not separably closed, the result follows immediately by Propositions 2.4 and 2.5. Otherwise, $K$ admits a non-trivial $L_{\text{ring}}$-definable henselian valuation by [JK15] Theorem 3.8. \hfill \Box

The question of what happens in case $Kv$ is separably closed is addressed in the next section.

3. Separably closed residue fields

In this section, we give a partial answer to Question 1.2 in case the residue field is separably closed. Recall that when $(K,v)$ is henselian and the residue field is not separably closed, we may add a symbol for $\mathcal{O}_u$ to any NIP field structure on $K$ and obtain an NIP structure. First, we note that we cannot expect the same when it comes to separably closed (residue) fields:

**Example 3.1** ([HHJ19] Example 5.5). Let $K$ be a separably closed field and assume that $v_1$ and $v_2$ are two independent valuations on $K$. Then $(K,v_1,v_2)$ has IP in $L_{\text{ring}} \cup \{\mathcal{O}_1\} \cup \{\mathcal{O}_2\}$.

There are of course many examples of separably closed fields with independent valuations:

**Example 3.2.** Let $Q_{\text{alg}}$ be an algebraic closure of $Q$ and let $p \neq l$ be prime. Consider a prolongation $v_p$ (respectively $v_l$) of the $p$-adic (respectively $l$-adic) valuation on $Q$ to $Q_{\text{alg}}$. Then $v_p$ and $v_l$ are independent, thus the bi-valued field $(Q_{\text{alg}}, v_p, v_l)$ has IP.

As any separably closed valued field has NIP in $L_{\text{ring}} \cup \{\mathcal{O}\}$ and any valuation is henselian on a separably closed field, we cannot expect an analogue of Corollary 2.6 to hold for separably closed residue fields. We will instead focus on the following version of Question 1.2.
Question 3.3. Let $K$ be NIP as a pure field and $v$ a henselian valuation on $K$ with $Kv$ separably closed. Is $(K, v)$ NIP in $\mathcal{L}_{\text{ring}} \cup \{O\}$?

3.1. Ingredients for the proof of Theorem [B]. We will split the proof of Theorem [B] into an equicharacteristic case and a mixed characteristic case. In both cases, separably algebraically maximal Kaplansky fields play an important role.

Definition. Let $(K, v)$ be a valued field and $p = \text{char}(Kv)$.

1. We say that $(K, v)$ is (separably) algebraically maximal if $(K, v)$ has no immediate (separable) algebraic extensions.
2. We say that $(K, v)$ is Kaplansky if $p = 0$ or if $p > 0$ and the value group $vK$ is $p$-divisible and the residue field $Kv$ is perfect and admits no Galois extensions of degree divisible by $p$.

Note that separable algebraic maximality always implies henselianity. See [Kuh10] for more details on (separably) algebraically maximal Kaplansky fields. As mentioned in the introduction, there is a transfer theorem which works for separably algebraically maximal Kaplansky fields:

Theorem 3.4 ([JS19 Theorem 3.3] and [AJ19a Proposition 4.1]). Any complete theory of separably algebraically maximal Kaplansky fields is NIP if and only if corresponding theories of the residue field and value group are NIP.

The fact that the theory SCVF of separably closed valued fields is NIP has been proven (independently) by Delon and Hong; however, Delon’s proof is unpublished and Hong’s proof only works for finite degree of imperfection. It is also an immediate consequence of Theorem [3.4].

Corollary 3.5 ([AJ19a Corollary 4.2]). Let $K$ be separably closed and let $v$ be a valuation on $K$. Then $(K, v)$ has NIP as a pure valued field.

Using an argument by Scanlon, we reduce Question [1.2] to the case of algebraically closed residue fields.

Proposition 3.6 (Scanlon). Let $(K, v)$ be a henselian valued field with $\text{char}(Kv) = p$, such that $Kv$ is not perfect and has no separable extensions of degree divisible by $p$. Then $O_v$ is definable in $\mathcal{L}_{\text{ring}}$.

Proof. Choose $t \in O_v$ such that we have $\bar{t} \in Kv \setminus Kv^p$. Consider the $\mathcal{L}_{\text{ring}}$-definable subset of $K$ given by $S := \{a \in K \mid \exists L \supseteq K \text{ with } [L : K] < p \text{ and } \exists y \in L : y^p - ay = t\}$.

We claim that $S = \{a \in K \mid v(a) \leq 0\}$ holds. We first show the inclusion $S \subseteq \{a \in K \mid v(a) \leq 0\}$. Assume for a contradiction that there is some $a \in S$ with $v(a) > 0$. Take $L \supseteq K$ and $y \in L$ witnessing $a \in S$, i.e., we have $[L : K] < p$ and $y^p - ay = t$. Let $w$ denote the unique prolongation of $v$ to $L$. Note that, as $w(t) \geq 0$ and $w(a) > 0$, we have $w(y) \geq 0$. Hence, we get $\bar{y}^p = \bar{t} \in Lw$. However, as $[Lw : Kv] \leq [L : K] < p$, this gives the desired contradiction.

For the other inclusion, suppose that we have $v(a) \leq 0$. Choose any $b \in K^{\text{alg}}$ with $b^{p-1} = a$ and set $L := K(b)$. In particular, we have $[L : K] \leq p - 1 < p$. Let $w$ denote the unique extension of $v$ to $L$. Consider the equation $baZ^p - Zba - t = (bZ)^p - a(bZ) - t = 0$ over $L$. As we have $w(ba) \leq 0$, this equation has a solution in $L$ if and only if the equation $Z^p - Z - \frac{t}{ba} = 0$ over $O_w$ has a solution in $O_w$. As $(L, w)$ is henselian and $Lw$, a separable extension of $Kv$, also has no separable extensions of degree divisible by $p$, there is some $z \in O_w$ with $z^p - z = \frac{t}{ba}$. For $y = zb$, we conclude $y^p - ay = t$ as desired.

It now follows immediately from the claim that $O_v$ is also definable.

□
3.2. Compositions of NIP valuations. In the proof of Theorem 3.11, we decompose the valuation $v$ on $K$ into several pieces: a coarsening $u$ of $v$ and a valuation $\bar{v}$ on $Ku$ such that $v$ is the composition of $\bar{v}$ and $u$. However, in general, it is not clear whether showing that each of these is NIP is sufficient to show that $v$ is NIP. The situation is simpler if the residue field $Ku$ of $u$ is stably embedded.

**Definition.** Let $M$ be a structure in some language $\mathcal{L}$ and $\mathcal{N} \succ M$ sufficiently saturated. A definable set $D$ is said to be stably embedded if for every formula $\phi(x; y)$, $y$ a finite tuple of variables from the same sort as $D$, there is a formula $d\phi(z; y)$ such that for any $a \in \mathcal{N}^{\mathcal{L}|x|}$, there is a tuple $b \in D^{\mathcal{L}|z|}$, such that $\phi(a; D) = d\phi(b; D)$.

See [Sim15, Chapter 3] for more on stable embeddedness. Note that [JS19, Proposition 2.5] proves that we can add NIP structure on a stably embedded set and stay NIP. This always works in separably algebraically maximal Kaplansky fields:

**Proposition 3.7** ([JS19, Lemma 3.1] and [AJ19a, Proof of Proposition 4.1]). Let $(K, v)$ be a separably algebraically maximal Kaplansky field. Then, the residue field $Kv$ is stably embedded.

There are more natural examples of henselian fields with stably embedded residue fields.

**Definition.** Let $(K, v)$ be a valued field of characteristic $(\text{char}(K), \text{char}(Kv)) = (0, p)$ for some prime $p > 0$. We say that $(K, v)$ is

1. unramified if $v(p)$ is the smallest positive element of $vK$ and
2. finitely ramified if the interval $[0, v(p)] \subseteq vK$ is finite.

The residue field of any unramified henselian valued field is purely stable embedded as an $\mathcal{L}_{\text{ring}}$-structure ([AJ19b, Corollary 13.7]). However, the residue field of a finitely ramified henselian valued field need not be stably embedded as a pure field (cf. [AJ19b, Example 12.8]), and it is not known whether it is stably embedded in case the residue field is not perfect. Nonetheless, using that every finitely ramified henselian valued field is up to elementary equivalence a finite extension of an unramified field with the same residue field, one can nonetheless prove an NIP transfer:

**Proposition 3.8** ([AJ19a, Corollary 4.7]). Let $(K, v)$ be a henselian valued field of mixed characteristic and $u$ a coarsening of $v$ such that $(K, u)$ is finitely ramified and $(Ku, \bar{v})$ is NIP. Then $(K, v)$ is NIP.

We finish the section with some open problems.

**Questions 3.9.**

1. Is the composition of (henselian) NIP valuations NIP?
2. Is the residue field of every henselian valuation on an NIP field stably embedded?

By [JS19, Proposition 2.5], a positive answer to the second question would imply a positive answer to the henselian case of the first question. Moreover, there seems to be no known example of a henselian field in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}_v\}$ such that the residue field is not stably embedded.

3.3. The case of separably closed residue fields. In this subsection, we prove our second main result which was mentioned as Theorem 3 in the introduction. We start with the equicharacteristic case:

**Proposition 3.10.** Let $K$ be NIP, $v$ henselian on $K$ with $\text{char}(K) = \text{char}(Kv)$. Then, $(K, v)$ is NIP as a pure valued field.

**Proof.** We may assume that $v$ is non-trivial as otherwise the statement is clear. In case $Kv$ is non-separably closed, the statement follows from Corollary 2.7. Now assume that $Kv$ is separably closed, in particular, $Kv$ is NIP as a pure field. Moreover, we assume that $K$ is not separably closed since otherwise the conclusion follows from Corollary 3.5. If $\text{char}(Kv) = 0$, the statement follows immediately from Delon’s classical result ([Sim15, Theorem A.15]) - or by the fact that any equicharacteristic 0 henselian valued field is separably algebraically maximal Kaplansky. On the other hand, if $\text{char}(K) = p > 0$, then $K$ admits no Galois extensions of degree divisible by $p$ by [KSW11, Corollary 4.4]. Thus, $vK$ is $p$-divisible and $Kv$ is perfect (for an argument for the latter, see the proof of [JS19, Proposition 4.1]). As $Kv$ is separably closed, we conclude that $(K, v)$ is Kaplansky. Moreover, any immediate separable extension of $K$ has degree divisible by $p$ by the lemma of Ostrowski ([Kuh11, see (3) on p. 280 for the statement and p. 300 for the proof]). Thus, $(K, v)$ is separably algebraically maximal with algebraically closed residue field. By Theorem 3.11, $(K, v)$ is NIP. □
We now come to the general case:

**Theorem 3.11.** Let $K$ be NIP, $v$ henselian on $K$. Then $(K, v)$ is NIP as a pure valued field.

**Proof.** If $Kv$ is not separably closed, the statement follows from [2.7]. In the case when $Kv$ is separably closed and non-perfect, the theorem holds by Proposition [3.6]. Thus, we may assume that $Kv$ is algebraically closed. Moreover, by Corollary [3.3], we may assume that $K$ is not separably closed. The equicharacteristic case follows from Proposition [3.10]. Thus, we now assume char$(K) = 0$ and char$(Kv) = p > 0$. Furthermore, we assume that $(K, v)$ is $K_1$-saturated.

We consider the following decomposition of the place $\phi_+ : K \to Kv$ corresponding to $v$:

$$K = K_0 \xrightarrow{\Gamma/\Delta_0} K_1 \xrightarrow{\Delta_0/\Delta_p} K_2 \xrightarrow{\Delta_p} K_3 = Kv$$

where every arrow is labelled with the corresponding value group. Note that char$(K) = char(K_1) = 0$ and char$(K_2) = char(Kv) = p$. Let $v_i$ denote the valuation on $K_i$ corresponding to the place $K_i \to K_{i+1}$.

By [AK16 Theorem 1.13], the value group $v_1 K_1 = \Delta_0/\Delta_p$ of $(K_1, v_1)$ is either isomorphic to $\mathbb{Z}$ or $\mathbb{R}$. Moreover, by saturation (and since $\Delta/\Delta_0$ has rank 1), $(K_1, v_1)$ is spherically complete and thus algebraically maximal (compare also the proof of [16h15 Lemma 6.8]). We now consider two cases:

In case $v_1 K_1$ is isomorphic to $\mathbb{Z}$, the composition $u$ of $v_1$ and $v_2$ is finitely ramified. Since $u$ is henselian, Corollary [2.7] implies that $Ku = K_2$ is an NIP field, thus $(K_2, v_2)$ is NIP (Proposition [3.10]). Applying Proposition [3.8], we conclude that $(K, v)$ is NIP.

On the other hand, in case $(K_1, v_1)$ has divisible value group, we first show that $K_2$ is perfect. Assume $K_2$ is not perfect. Recall that it is NIP by Corollary [2.7] and hence admits no Galois extensions of degree divisible by $p$. Hence, by Proposition [3.6], the composition $u$ of $v_1$ and $v_2$ is again definable. But this contradicts $K_1$-saturation: Recall that $\Delta_p$ is the biggest convex subgroup not containing $v(p)$, and that $\Delta_0$ is the smallest convex subgroup containing $v(p)$. If $\Delta_p$ is definable in $(K_1, v_1)$, then there is always a minimum positive element in $\Delta_0/\Delta_p$, since by saturation, $\Delta_0/\Delta_p$ must otherwise contain a convex subgroup. Thus, if $\Delta_0/\Delta_p$ is isomorphic to $\mathbb{R}$, $K_2$ is perfect.

We now argue that $(K_1, v_1)$ is an algebraically maximal Kaplansky field. By what we have just shown, its residue field $K_2$ is perfect and NIP (using Corollary [2.7] again), and by assumption we are in the case when $(K_1, v_1)$ is divisible. Thus, $(K_1, v_1)$ is Kaplansky. Since we have already argued that $(K_1, v_1)$ is algebraically maximal, Proposition [3.4] now implies that $(K_1, v_1)$ is NIP. By [3.10] also $(K, v_0)$ and $(K_2, v_2)$ are NIP. Moreover, by Proposition [2.7], $K_2 = K_1 v_1$ is stably embedded as a pure field in $(K_1, v_1)$ and of course, being an equicharacteristic 0 henselian valued field, $K_1 = Kv_0$ is stably embedded as a pure field in $(K, v_0)$. Thus, applying [JS19 Proposition 2.5] twice, we finally conclude that $(K, v)$ is NIP.

4. Ordered fields

In this section, we use the same technique as in the proof of Proposition [2.5] to study convex valuation rings on an ordered field. We show that any convex valuation ring $O_v$ on $K$ is definable in $(K, <)^{sh}$. The idea to consider convex valuation rings on ordered fields was suggested by Sahma Kuhlmann.

**Definition.** Let $(K, <)$ be an ordered field and $R \subseteq K$ a subring.

1. The $<\text{-convex hull}$ of $R$ in $K$ is defined as

$$O_R(%) := \{x \in K : x, -x < a \text{ for some } a \in R\}.$$ 

2. We say that $R$ is $<\text{-convex}$ if $O_R(%) = R$.

The following facts about convex valuation rings are well-known.

**Fact 4.1.** [EP05, p.36]. Let $(K, <)$ be an ordered valued field.

1. Any convex subring of $K$ containing 1 is a valuation ring.
2. A subring $R \subseteq K$ is $<\text{-convex}$ if and only if $R$ is a convex subgroup of the additive group of $K$. Thus, any two valuations $v, w$ on $K$ which are convex with respect to $<$ are comparable.
3. There is a (unique) finest valuation $v_0$ on $K$ which is convex with respect to $<$. It is called the natural valuation of $(K, <)$. The valuation ring $O_{v_0}$ is the convex hull of the integers in $(K, <)$. 

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It is now an easy consequence of the properties of the natural valuation that convex valuation rings are definable in the Shelah expansion:

**Proposition 4.2.** Let \((K,\langle\rangle)\) be an ordered field and \(O_v\) a convex valuation ring on \(K\). Then \(O_v\) on \(K\) is definable in \((K,\langle\rangle)_{\text{Sh}}\).

**Proof.** As the valuation ring of the natural valuation \(v_0\) is exactly the convex closure of \(\mathbb{Z}\) in \(K\), it is definable in \((K,\langle\rangle)_{\text{Sh}}\). As any convex valuation \(v\) on \(K\) is a coarsening of \(v_0\), the valuation ring of \(v\) is also definable in \((K,\langle\rangle)_{\text{Sh}}\). \(\Box\)

Applying Proposition 2.1, this yields the following

**Corollary 4.3.** Let \(K\) be an ordered field such that \(\text{Th}(K)\) is NIP in some language \(\mathcal{L} \supseteq \mathcal{L}_\text{cf}\) and let \(v\) be a convex valuation on \(K\). Then, \((K,v)\) is NIP in \(\mathcal{L} \cup \{O_v\}\).

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