The weak energy condition and the expansion history of the Universe

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Abstract

We examine flat models containing a dark matter component and an arbitrary dark energy component, subject only to the constraint that the dark energy satisfies the weak energy condition. We determine the constraints that these conditions place on the evolution of the Hubble parameter with redshift, \( H(z) \), and on the scaling of the coordinate distance with redshift, \( r(z) \). Observational constraints on \( H(z) \) are used to derive an upper bound on the current matter density. We demonstrate how the weak energy condition constrains fitting functions for \( r(z) \).

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1. Introduction

Observational evidence [1,2] indicates that roughly 70% of the energy density in the universe is in the form of an exotic, negative-pressure component, dubbed dark energy. (See Ref. [3] for a recent review.) If \( \rho_{DE} \) and \( p_{DE} \) are the density and pressure, respectively, of the dark energy, then the dark energy can be characterized by the equation of state parameter \( w \), defined by

\[
w = \frac{p_{DE}}{\rho_{DE}}. \quad (1)
\]

Although the simplest possibility for the dark energy is a cosmological constant, which has \( w = -1 \), many other possibilities have been proposed, including an evolving scalar field (quintessence) [4–8], a scalar field with a non-standard kinetic term (k-essence) [9–17], or simply an arbitrary barotropic fluid with a pre-determined form for \( p(\rho) \), such as the Chaplygin gas and its various generalizations [18–25].

Lacking a definite model for the dark energy (aside from the perennial favorite cosmological constant), it is interesting to determine what can be derived from fairly general assumptions about the nature of the dark energy. In particular, a plausible assumption about the dark energy is that it obeys the weak energy condition (WEC). If \( T_{\mu\nu} \) is the energy–momentum tensor of the dark energy, then the weak energy condition states that

\[
T_{\mu\nu} t^\mu t^\nu \geq 0, \quad (2)
\]

where \( t^\mu \) is any timelike vector. In a Friedman–Robertson–Walker universe, this reduces to a condition on the density and pressure:

\[
\rho \geq 0, \quad (3)
\]

and

\[
\rho + p \geq 0. \quad (4)
\]

Although models in which the dark energy violates the WEC are not inconsistent with current observations (as first pointed out by Caldwell [26]), there are good reasons to believe that the WEC is satisfied [27–30].

The WEC has already been used previously to constrain the expansion history of the universe [31–37]. These previous studies, however, all applied the WEC constraints (Eqs. (3) and (4)) to the total cosmological fluid. Similarly, Schuecker et al. [38] applied several other energy conditions to the total cosmological fluid. In this Letter, instead, we assume a model consisting of matter plus a fluid obeying the WEC, and we determine the corresponding constraints that this places on the expansion history.
In Section 2, we derive the limits that can be placed on $H(z)$ and $r(z)$ from the WEC. In Section 3, we apply these constraints to analyses of the observations. Our results are discussed in Section 4.

2. Consequences of the weak energy condition

We assume a flat Friedman–Robertson–Walker model, containing a pressureless matter component and a dark energy component obeying the WEC. The radiation contribution to the density at late times is negligible and can be neglected. The matter density includes both baryonic and dark matter, and it scales with redshift $z$ as

$$\rho_M = \rho_{M0}(1+z)^3,$$

where the 0 subscript will refer throughout to present-day values. The WEC imposes two conditions on the dark energy density: first, that

$$\rho_{DE} \geq 0,$$

at any redshift (a restatement of Eq. (3)) and second, that redshift-dependent Hubble parameter

$$H(z)=\frac{\dot{a}}{a} \equiv \frac{\dot{r}}{r},$$

Eq. (10), but the converse is not true. Although Eq. (12) gives the stronger limit, we include both limits because current data can provide some estimates for $\ddot{H}(z)$ (as in the next section), but are too poor to provide any limits on $\dddot{H}(z)$. By construction, the standard $\Lambda$CDM model saturates both limits. Eq. (12) was previously introduced by Sahni and Starobinsky in the context of quintessence models [39] and by Boisseau et al. in the examination of scalar-tensor models [40].

Now consider the coordinate distance $r(z)$, defined by

$$r(z) = \int_0^z \frac{dz'}{H(z')}.$$

The coordinate distance is important because it is directly related to the luminosity distance, $d_L$, through $d_L = c(1+z)r(z)/H_0$, and it is $d_L$ which is measured in supernova redshift surveys. Hence, a considerable effort has been put into designing parametrizations for $r(z)$ to fit to the supernova data [41–48]. The purpose of many of these investigations is to go from a best-fit form for $r(z)$ to the potential for an underlying quintessence model.

The derivative of Eq. (13) gives

$$r'(z) = \frac{1}{H(z)},$$

which, when combined with Eq. (10), yields

$$r'(z) \leq H_0^{-1} [\Omega_M (1+z)^3 + (1-\Omega_M)]^{-1/2}.$$

Similarly, Eq. (12) yields

$$-r''(z) \leq \frac{3}{2} H_0^2 \Omega_M (1+z)^2.$$

Eqs. (15) and (16) give the dark-energy WEC constraint on the coordinate distance. As for the limits on $H(z)$, Eq. (16) implies (15), but the converse is not true.

3. Comparison with observations

Consider first our limits on the evolution of $H(z)$. Estimates of $H(z)$ were derived by Simon, Verde, and Jimenez [49] using passively evolving galaxies; these limits have been used to constrain cosmological parameters in dark energy models [50]. A second approach to deriving $H(z)$ based on the Supernova data has been explored in Refs. [51] and [52]. We will use the results of [49] in our discussion here.

Eq. (10) can be rewritten as

$$\Omega_M \leq \frac{\ddot{H}(z)^2 - 1}{(1+z)^3 - 1}.$$

Thus, a single value of $\ddot{H}(z)$ can provide an upper bound on $\Omega_M$. Since our bound applies to any dark energy model satisfying the WEC, we do not attempt to fit any particular model (as was done in Ref. [50]); rather, we calculate the upper bound individually for each $H(z)$ measurement. The upper bound on $\Omega_M$ depends on $\dddot{H}(z)$, so our results will naturally be sensitive to the value of $H_0$. Following Ref. [50] we consider two priors for $H_0$: $H_0 = 73 \pm 3$ km s$^{-1}$ Mpc$^{-1}$ from...
statistics analysis in Ref. [54]. Our results are shown in Figs. 1 and 2 (where all error bars are 1-sigma). For $z = 1.53$, where the error bars are smallest, we get the tightest upper bound on $\Omega_M$: $\Omega_M \leq 0.18 \pm 0.05$ for $H_0 = 73 \pm 3$ km s$^{-1}$ Mpc$^{-1}$, and $\Omega_M \leq 0.21 \pm 0.06$ for $H_0 = 68 \pm 4$ km s$^{-1}$ Mpc$^{-1}$. (For a very different approach, see Ref. [55].)

Next we consider the consequences of the limits on the coordinate distance $r(z)$ given by Eqs. (15) and (16). The quantity observers actually measure is the apparent magnitude $m(z)$, given by

$$m(z) = M + 5 \log_{10} (D_L(z)),$$

where $D_L$ is the Hubble-free luminosity distance,

$$D_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')} ,$$

and $\mathcal{M}$ is the magnitude zero-point offset, which depends on the absolute magnitude $M$ as

$$\mathcal{M} = M + 5 \log_{10} \left( \frac{H_0^{-1}}{\text{Mpc}} \right) + 25 = M - 5 \log_{10} h + 42.38 .$$

The distance modulus given by the SNIa data is defined as

$$\mu(z) = m(z) - M = 5 \log_{10} (D_L(z)) + \mu_0 ,$$

where $\mu_0 = 42.38 - 5 \log_{10} h$.

As an example, we now choose a representative fitting function, translate it into a fitting function for $r(z)$, and then determine how the WEC constrains the parameters of the fitting function. In this Letter, we use the fitting function for $\mu$, first introduced by Padmanabhan and Choudhury [47], given by

$$\mu_{fit} = \mu_0 + 5 \log_{10} \left[ \frac{z(1 + w_1 z)}{(1 + w_2 z)} \right] ,$$

where $\mu_0$, $w_1$ and $w_2$ are the three independent fitting parameters. By comparing Eqs. (21) and (22), we can write

$$D_L = \frac{z(1 + w_1 z)}{(1 + w_2 z)} ,$$

so that

$$r(z) = \frac{1}{H_0} \frac{z(1 + w_1 z)}{(1 + w_2 z)} .$$

Since we have an analytic form for $r(z)$, we use Eq. (16) alone to determine the values of $w_1$ and $w_2$ which violate the WEC; any $r(z)$ which satisfies Eq. (16) will automatically satisfy Eq. (15). We require Eq. (16) to be satisfied for $z < 2$, the range over which the supernova data extend.

The allowed region for $w_1$ and $w_2$ is displayed in Fig. 3. As expected, the constraints are tighter for larger values of $\Omega_M$, but rather surprisingly, the excluded region in parameter space is rather insensitive to the assumed value of $\Omega_M$ (and a significant region of parameter space is excluded even in the limit $\Omega_M \to 0$). Also in Fig. 3, we display the confidence regions for $w_1$ and $w_2$ from the supernova data, with no assumptions about the equation of state for the dark energy. We use 60 Essence supernovae [56], 57 SNLS supernovae [57] and 45 nearby supernovae. We have also included the new data release of 30 SNe Ia detected by HST and classified as the Gold sample by Riess et al. [58]. The combined data set can be found in Ref. [59]. Clearly, the best-fit values for $w_1$ and $w_2$ lie slightly outside of the allowed region for $\Omega_M > 0.3$. Not too much should be read into this: given that $\Lambda$CDM models provide a good fit to all current observations, and such models saturate our bounds, we would expect the best fit parameters to lie near the boundary of the excluded region. The important point is that Fig. 3 shows how one can use our constraints on $r(z)$ to eliminate regions of parameter space for which the dark energy violates the WEC.

Of course, it is also possible to parametrize $w$ as a function of $z$ and use this parametrization to derive a form for $r(z)$ (see,
e.g., Ref. [44]); in this case the WEC is trivially satisfied as long as the parametrization for \( w(z) \) forces \( 1 + w(z) \) to be nonnegative.

### 4. Discussion

We have examined how the WEC constrains the redshift evolution of both the Hubble parameter \( H(z) \) and coordinate distance \( r(z) \). The constraint on \( H(z) \) can be combined with observations of \( H(z) \) to put upper bounds on \( \Omega_M \). While the scatter in these estimated upper limits is large, as are the errors, the important point is that these are generic upper limits, independent of the nature of the dark energy (as long as it satisfies the weak energy condition). Improved measurements, particularly of \( H(z) \), will strongly improve this upper bound. The constraints on \( r(z) \) do not provide similarly useful limits, but they can be applied to any parametrization of \( r(z) \) to eliminate in advance any regions of parameter space in which the dark energy violates the weak energy condition.

What happens if the WEC is violated by the dark energy? If one allows for arbitrary evolution, then there are clearly no constraints on \( r(z) \) and \( H(z) \). An intermediate case, which provides weaker limits than the ones we have discussed, is when the WEC applies to the total fluid (matter and dark energy together) [31–37]. In this case, for example, there is no limit corresponding to Eq. (10), while the limit corresponding to Eq. (12) becomes \( H(z)H'(z) \geq 0 \). Thus, this version of the WEC provides no bound on \( \Omega_M \), although it does constrain the evolution of \( H(z) \). Applying the WEC to the total fluid also yields constraints on \( r(z) \); these are discussed in detail in Ref. [32]. These constraints are weaker but more general than the constraints we have obtained by applying the WEC to the dark energy alone.

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Fig. 3. The region in the \( w_1, w_2 \) plane allowed by the weak energy condition for the parametrization of \( r(z) \) given in Eq. (24). The region between the two sets of curves is allowed. Right-hand boundaries are given for \( \Omega_M = 0.2 \) (dashed), \( \Omega_M = 0.3 \) (dotted) and \( \Omega_M = 0.4 \) (dot-dash). Left-hand boundary (solid) is independent of \( \Omega_M \). The two ellipses are the 1\( \sigma \) and 2\( \sigma \) contours obtained by fitting equation (22) to the supernova data without the weak energy constraint.
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