Exponential Decay for Small Non-Linear Perturbations of Expanding Flat Homogeneous Cosmologies *

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November 2, 2021

Abstract

It is shown that during expanding phases of flat homogeneous cosmologies all small enough non-linear perturbations decay exponentially. This result holds for a large class of perfect fluid equations of state, but notably not for very “stiff” fluids as the pure radiation case.

1 Introduction

On expanding phases of nearly homogeneous cosmological models it is believed that small perturbations are pulled apart and so washed away. Thus, if they were the only type of perturbations present they would decay to zero, making the model more and more homogeneous as time elapses. This intuitive picture of homogenization, basic in all arguments used on inflationary models, has so far not been rigorously justified beyond the corresponding one on the linearized equations off a homogeneous background, the corresponding nonlinear case of Newtonian cosmological models [1], or the Electro-Yang-Mills-Vacuum case using conformal methods, [2].

The main mathematical difficulty to tackle this problem has been the absence of a set of variables in which Einstein’s equations coupled to a fluid could at the same time have a well posed formulation and a linearization off a homogeneous expanding background where all eigenvalues have negative real part, for in that case, as we shall argue in §3, decay of small (but nonlinear) perturbations can be shown to follow quite easily. As we shall show in §2 this problem has been recently overcome by using a novel set of Lagrangian variables, in which the time component of a frame variable follows the fluid four-vector under evolution, [3]. Thus, using these variables we are able to show exponential decay of small non-linear perturbations for a wide range of equation of state.

*Work supported by CONICOR, CONICET, and SeCyT, UNC
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In the next section we introduce Friedrich’s system for the Einstein-Euler system of equations in Lagrangian variables. Since this system has a well posed initial value problem (it is symmetric hyperbolic) we know local in time solutions exist, and so the only question is whether they decay or not if they are sufficiently close to an homogeneous expanding cosmology. To answer this we look at the linearization of this system off a homogeneous cosmological model, and display the conditions under which the real part of the eigenvalues of the system are negative definite. For the flat case, the only one considered in detail, they turn out to be only conditions on the equation of state, and the only one which seems to be a real restriction within the class of equations of state usually considered is condition (39), which limits the sound speed of the fluid to be less than one third of the speed of light. This rules out from consideration the stiff or pure radiation perfect fluids. As noted by several colleagues, perhaps this limiting case could be treated using the conformal Einstein’s equations [2]. It seems that one can not go beyond that limit with the present approach and thus one is led to wonder whether this limitation has a physical origin, in which case it should be explored in more detail.

In §3 we complete the assertion on decay by giving a general proof of exponential decay for symmetric hyperbolic systems whose eigenvalues have negative definite real part, that is for system as the one we are considering. This theorem is a simple adaptation to the case under consideration of a more general class of results which are known in the literature on non-linear decay, for a resent review of them see [6].

We present the results for flat backgrounds, for non-flat ones similar results should hold as long as the expansion is large enough, but further work is needed to fully understand how big this expansion has to be in terms of invariant quantities, in particular how negative the cosmological constant has to be, so we don’t elaborate in detail here.

2 The evolution equations

We use the evolution system for the Einstein-Euler equations as defined in [3]. In that system one introduces a set of orthonormal frame vectors \( \{ u^a, e^a_i \} \), \( i = 1..3 \), \( u^a u^b g_{ab} = 1 \), \( e^a_i e^b_j g_{ab} = -\delta_{ij} \). In accordance with the Lagrangian coordinate scheme, the time-like component of the frame is chosen to be the four-velocity of the fluid, (so in general it is not surface orthogonal), and the other frame vectors are chosen to be dragged along the integral curves of \( u^a \), that is along the flow lines, using Fermi transport, that is, \( \Gamma^0_{i k} = 0 \). The remaining derivatives of the frame vectors are grouped into two set of scalars, \( a^i := \Gamma^i_{00} \), and \( \chi_{ij} := -\Gamma^0_{ij} \), they shall form part of the evolution system. Note that \( \chi_{ij} \) is not symmetric, for \( u^a \) is in general not surface orthogonal.

Part of the fluid equations are implied by the evolution equations for the components of the acceleration, \( a^i \), and a constraint which we shall indicate latter. The rest of them, and the only one surviving as such is the conservation equation for the energy density \( \rho := T_{ab} u^a u^b \), equation (39).

To complete the evolution system one adds part of the Bianchi identities to obtain evolution equations for the components of the electric and magnetic decompositions of the Weyl tensor. The resulting system is,
\[ \partial_t e^\mu_i = -\chi^i_j e^\mu_j - a_i e^\mu_0 \]  
\[ \partial_t \Gamma^j_{ik} = -\Gamma^j_{lk} \chi^l_i - 2\chi^i_{[a|k]} h^{jl} - B_{ip} \epsilon^{pj} k \]  
\[ \partial_t a_i + \nu^2 D^j_k \chi^k_i = - (\chi^i_j + \beta \delta^i_j + \nu^2 (\chi^i_i - \chi^j_j)) a_j \]  
\[ \partial_t \chi_{ij} - D_j a_i = -\chi^l_i \chi_{lj} - a_i a_j + 2\nu^2 \chi \chi_{[ij]} \]  
\[ \partial_t \rho + \nu^2 D^k \chi^k_{ij} = \chi_{(i} E_{j)k} + 2 \chi^k_{i} E_{jk} - h_{ij} \chi^k E_{kl} - 2 \chi E_{ij} \]  
\[ \partial_t B_{ij} - D_k E_{i(i} \epsilon_{j)k} = \chi_{(i} B_{j)k} - \chi B_{ij} - \chi_{kl} B_{pq} \epsilon^{pl} (\epsilon^{qk} l) \]

where

\[ \beta := \frac{\rho + p}{\nu^2} \frac{\partial^2 p}{\partial \rho^2} - \nu^2. \]  

This system is symmetric hyperbolic when the speed on sound \( \nu^2 \) is positive, and so in that case it has a well posed initial value formulation. Thus there is an energy estimate and local in time solutions exist and are unique for smooth enough initial data sets in \( L^2 \). In what follows we shall use implicitly the existence and uniqueness of these solutions.

For simplicity we are considering here only homentropic fluids. In the more general case one needs further evolution equation for the entropy and its space derivatives, see [3]. The evolution equation for the entropy just asserts it is conserved along integral curves of the fluid four-vector, the evolution equation for the space derivatives of the entropy, assert consequently that they decay at the precise rate at which the expansion smooths things out. Since these evolution equations are not coupled at the principal part level, one can always treat them as we shall treat the frame in \( \S 3.1 \) and obtain decay properties without having to impose further conditions on the system.

In addition to these evolution equations there are a number of constraint equations, the usual ones that arise in general relativity, plus some connecting the different fields, for instance those which guarantee that the \( \chi_{ij} \) are part of the connection coefficients of the frame. Here we give a brief discussion on the constraint problem, but do not go in details, for the aim of this work is to study the time behavior of solutions once initial data for them –satisfy the constraint– are given.

To solve the constraints one would follow the usual procedure for frame equations, and so we only remark on the differences which appear in this case. One first solve for a three metric and extrinsic curvature the usual scalar and vector constraints with some boundary or asymptotic conditions. Here we consider either an initial surface whose background is a flat 3-torus, or a flat \( R^3 \). In the case of a 3-torus we consider periodic boundary conditions \[ \text{[4]} \).

\[ ^1 \text{Here the perturbation of the mass density has to be adjusted so that there can be solutions.} \]
case ($R^3$) we consider isolated perturbations in the sense that one requires the perturbations to decay to zero asymptotically and are of finite energy. In the general case the perturbations in $R^3$ decay too slowly for the frame perturbations to be in $L^2$, for there is a mass perturbation, and so one must use the variant of the decay theorem discussed in §3.1. One can also tune the mass density perturbation so that the total mass perturbation vanishes, but these seems to restrict the space of allowed perturbations. For this subclass of perturbations one can use the standard decay theorems given in §3. Once this is done one then provides a lapse-shift pair which in this case can be thought of as the 3+1 decomposition of the fluid four vector. Thus one has a four metric, and one knows, using the field equations, all space and time derivatives of that metric at the initial surface. One then chooses a frame for that metric so that the time-like component of that frame coincides with the fluid four-vector.

Most of the time derivatives of that frame are fixed from the metric condition and so are given by the extrinsic curvature of the surface, the rest, amounting to frame rotations and accelerations are fixed by the gauge condition imposed (Fermi transport) on the frame and from the acceleration, which is given by the Euler’s equation, (14), which here is part of the constraints. Thus we have all derivatives of the frame at the initial surface and so can compute the frame coefficients and from them all frame related quantities which enter as dynamical variables. They satisfy automatically the rest of the constraints. In the same way one can compute the Weyl tensor at the surface and its frame components, and so complete the set of initial data.

One does not need for this case to prove that the constraints are propagated for this follows from general arguments, see for instance [3]. Indeed suppose initial data is given satisfying the constraint, and suppose for contradiction that at some point $p$ inside the domain of dependence of the initial slice the constraints cease to be satisfied. Then we can take a small neighborhood of that point and inside it, to the past of it, construct a space-like slice so that $p$ is in its domain of dependence. In that slice the smooth induced initial data satisfies the constraints and so we can generate a local solution of Einstein’s equation using, say, the harmonic gauge. That solution satisfies the constraints along evolution. Furthermore we can construct locally a frame whose time-like component points along the fluid and whose other components are propagated using Fermi transport. The proof of this involves only the theory of ordinary differential equations. Thus we have a solution of Einstein’s equations in the gauge of our propagation equations and so it is a solution to them, furthermore all constraints are satisfied, even at $p$. But the solutions to the evolution equations are unique, and so we reach a contradiction, implying the constraints must propagate correctly inside the domain of dependence. The constraint propagation has also been checked explicitly, [5].

In the above system it appears $\nu^2$ in the denominator, so it is not immediate that this equations behave nicely in the limit when this quantity –the sound speed– goes to zero, and for many equations of state this happens when the density goes to zero. Here we are not interested in reaching the limit where the expansion makes the density go to zero, but nevertheless it is interesting to obtain bounds which are uniform in $\nu^2$ in the whole region where it becomes small. An easy way to do this is to rescale the acceleration with a factor $\nu$ and so get a system whose coefficients are bounded even in the limit $\rho \to 0$. Defining $\tilde{a}_i = \frac{1}{\nu} a_i$, and multiplying equation (3) by $\frac{1}{\nu}$ we get,
We take as background solution a Friedmann–Robertson-Walker Universe, namely,

2.1 The background solution

\[ \partial_t e_i^a = -\chi_i^j e_j^a - \nu \dot{\alpha}_i e_0^a \]  \hspace{1cm} (9)

\[ \partial_t \Gamma_{ik}^j = -\Gamma_{lk}^i \chi_j^l - 2\nu \chi_{ij} \dot{\alpha}_k h_j^i - B_{ip} \epsilon_{pj}^k \]  \hspace{1cm} (10)

\[ \partial_t \dot{\alpha}_i + \nu D_k \chi_i^k = -\left( \chi_i^j + \tilde{\beta} \chi_i^j + \nu^2 (\chi_i^j - \chi_j^i) \right) \ddot{\alpha}_j \]  \hspace{1cm} (11)

\[ \partial_t \chi_{ij} - \nu D_j \dot{\alpha}_i = -\chi_i^l \chi_{lj} - (\gamma + \nu^2) \dot{\alpha}_i \dot{\alpha}_j + 2
\nu^2 \chi \chi_{ij} \]  \hspace{1cm} (12)

\[ -E_{ij} - \frac{\kappa_1 p}{3} + p) h_{ij} - \frac{\Lambda}{3} h_{ij} \]  \hspace{1cm} (13)

\[ \partial_t E_{ij} + D_k B_{i(j} \epsilon_{k)i} = \chi_{ij} E_{j} + 2\chi_{ij} E_{ij} - h_{ij} \chi_{kl} E_{kl} - 2\chi E_{ij} \]  \hspace{1cm} (14)

\[ + 2 \nu \dot{\alpha}_i \epsilon_{ik} (i \dot{B}_j) - \frac{\kappa}{2} (\rho + p) (\chi_{ij} - \frac{1}{3} \chi h_{ij} \]  \hspace{1cm} (15)

\[ \partial_t B_{ij} - D_k E_{i(j} \epsilon_{k)i} = \chi_{ij} E_{j} - \chi_{ij} B_{ij} - \chi_{kl} B_{pq} \epsilon_{i(j} e_{k)} (i \epsilon_{k} q) \]  \hspace{1cm} (16)

where \( \gamma (\rho) := \frac{(\rho + p)}{2\rho} \) and \( \tilde{\beta} := \beta - \gamma = -3 \gamma - \nu^2 \), to get these equations we have used the evolution equation for \( \rho \), and one of the constraint equations for the system, namely one of the fluid equations in the Euler description,

\[ D_j p = (\rho + p) a_j \]  \hspace{1cm} (16)

This system is symmetric hyperbolic even for \( \nu = 0 \), and so well adapted for the case in which the mass density tends to zero. But to have a well posed system one also needs that the coefficient of the system be smooth. It can be seen that for many realistic equation of state \( \gamma \) is bounded as a function of \( \rho \), but in general it is not differentiable. For instance, for an equation of state of the form \( p = k \rho^\frac{\gamma}{k} \), \( 1 \leq \gamma \leq \frac{4}{3} \) second derivatives are not bounded when \( \rho \to 0 \), unless the relation between \( p \) and \( \rho \) becomes linear.\(^2\) But notice that using the evolution equation for \( \rho \) one can see that time derivatives of \( \gamma \) are bounded \( \gamma \) is a contribution of an extra factor \( (\rho + p) \). Similarly, using \( \gamma \), one can see that the space derivatives of \( \gamma \) are also bounded.\(^3\) So, in general one can not proceed to make a priori estimates for derivatives of the fields, as needed for showing existence– for equation \( \gamma \) is known only a posteriori to hold. But nevertheless one can make a posteriori estimates on the solution, for if we know that the constraint are satisfied we can, using \( \gamma \), obtain bounds on the space derivatives of the solutions to any desired order. This shall be enough for our purposes of obtaining uniform (in \( \nu^2 \)) estimates in regions where solutions are known to exist.

2.1 The background solution

We take as background solution a Friedmann–Robertson-Walker Universe, namely,

\[ ds^2 = dt^2 - \frac{a_0(t)^2}{\Omega_2} (dx^2 + dy^2 + dz^2), \]  \hspace{1cm} (17)

\(^2\)For an equation of state of the form \( p = k \rho^\gamma \) one has, \( \gamma (\rho) = \frac{\gamma - 1}{2} (1 + k \rho^{\gamma - 1}) \)

\(^3\)For the case \( p = k \rho^\gamma \) one can see that all derivatives are polynomial expressions in \( \rho^\frac{\gamma - 1}{2} \), and so regular, as long as \( \gamma \geq 1 \).
with \( \omega = 1 + kr^2/4 \), and \( a_0(t) \) solution of,

\[
\left( \frac{\dot{a}_0}{a_0} \right)^2 + \frac{k}{a_0^2} - \frac{\kappa \rho_0}{3} + \frac{\Lambda}{3} = 0,
\]

where \( k = 1, -1, 0 \) determines the curvature of the surface, \( \Lambda \leq 0 \) is the cosmological constant, and \( \rho_0 \) is the background energy density.

The background density satisfies,

\[
\dot{\rho}_0 = -3h_0(\rho_0 + p_0),
\]

where \( h_0 = \frac{\dot{a}_0}{a_0} \), is the Hubble function and \( p_0 = p_0(\rho_0) \) is the pressure, which is assumed to be a function of the density only. We shall assume that either \( \Lambda < 0 \) or that we are in a time period where \( h_0 > \bar{h}_0 > 0 \), for some constant \( \bar{h}_0 \).

It is easy to check that all evolution and gauge equations are satisfied if in the above coordinates one takes the following frame:

\[
0 u^\mu = 0 e_0^\mu = (1, 0, 0, 0), \quad 0 e_i^\mu = \left( 0, \frac{\omega}{a_0} \delta_i^\mu \right)
\]

The corresponding fields for this frame take the form:

\[
\begin{align*}
0 \Gamma_{ijk} &= \frac{k \omega}{2a_0} (-h_{jk} x^i + \delta_i^j x_k), \\
0 \Gamma_{ij}^k &= -n_0 \chi_{ij} = -h_0 \ h_{ij} \\
\text{all others} &= 0.
\end{align*}
\]

(20)

where \( x^i = 0 e_i^\mu x^\nu = \frac{\omega}{a_0} \delta_i^\mu x^\nu \).

The proper time of co-moving observers would be taken to be the time with respect to which we assert the exponential decay result. Given any perturbed space-time this time is no longer an invariant, but it is so up to order \( \varepsilon \), that is up to the size of the perturbation. Furthermore, while the expansion is producing an exponential decay the difference between this time and the proper time of geodesic observers is uniformly bounded and goes to zero as \( \varepsilon \to 0 \).

### 2.2 The linearized equations

The linearized equations are,

\[
\begin{align*}
\dot{e}_i^\mu &= -h_0 \ e_i^\mu - \chi_{ij} \ e_j^\mu - \nu_0 \ a_i \delta_0^\mu - \nu_0 \ a_0 \delta_i^\mu, \\
\dot{\Gamma}_{jk}^i &= -h_0 \ 1 \Gamma_{jk}^i - \epsilon \ l_k \ 1 B_{jl} \\
\dot{\chi}_{ij} &= -2h_0(1 \chi_{ij} + 3 \nu_0 \Delta \chi_{ij}) - 1 E_{ij} - \kappa \ 6 \ (1 - 3 \nu_0^2) h_{ij} \rho_1 \\
\dot{B}_{ij} &= -3h_0 \ (1 \ E_{m(i)} e_{j)} - k m) \\
\dot{E}_{ij} &= -3h_0 \ (1 \ E_{m(i)} e_{j)} - k m) \\
\dot{\rho}_1 &= -3h_0 \ (1 + \nu_0^2) \rho_1 - (\rho_0 + p_0) \ 1 \chi
\end{align*}
\]
where

\[
\alpha = (1 - 3\nu^2 + 9\rho + p \frac{\partial^2 p}{\partial \rho^2})
\]

\[
= 1 - 3\nu_0^2 - 9\gamma,
\]

## 2.3 Negativity of the eigenvalues

We notice that our system has the following form,

\[
u_t = A^a(u)D_a u + (B_0 + \varepsilon B_1(u))u,
\]

where the matrix \( A^a(u) \) is symmetric, in the sense that there exists a strictly positive symmetric bilinear form \( H(u) \), smooth in \( u \), such that \( H(u)(A^a(u)) - (A^a(u))^\dagger H(u) = 0 \).

Due to this property it follows quite easily that the eigenvalues of the system have strictly negative real part for small enough \( \varepsilon \) if and only if \( HB_0 + B_0^\dagger H \) is a negative definite bilinear form. In the next section we shall show that the strict negativity of the eigenvalues suffices to show exponential decay.

It can be easily seen that a family of bilinear forms making the system symmetric-hyperbolic is given by,

\[
\langle (e, \Gamma, a, \chi, E, B, \rho), H_0(e, \Gamma, a, \chi, E, B, \rho) \rangle >
\]

\[
= + C^2_e l_{\mu\nu} h^{ij} e^i\rho^j + C^2_\Gamma h^{ij} \rho^i \rho^j
\]

\[
+ C^2_\chi h^{ji} h^{km} \Gamma^i_k \Gamma^j_m - C^2_\chi h^{ij} \tilde{a}_i \tilde{a}_j
\]

\[
+ C^2_h h^{im} h^{jn} (\chi^{ST}_{ij} \chi^{ST}_{mn} + \chi^{A}_{ij} \chi^{A}_{mn}) + \frac{1}{9} \chi^2
\]

\[
+ h^{im} h^{jn} E_{ij} E_{mn} + h^{im} h^{jn} B_{ij} B_{mn} + C^2_\rho \rho^2
\]

where \( \chi^{ST}_{ij} (\chi^{A}_{ij}) \) denote the symmetric trace free, and the antisymmetric parts of \( \chi_{ij} \) respectively, \( l_{\mu\nu} := \sigma_{\mu\nu} - 2\theta \sigma_{\mu0} \sigma_{\nu0} \), and \( C_e, C_\Gamma, C_\chi, \) and \( C_\rho \) are constants which can take any nonzero value, reflecting the fact that the bilinear form which symmetrizes the system is not unique. These constants shall be determined bellow in such a way as to maximize the range on which the negativity of \( B_0 \) holds. Notice that \( H(u) = H_0 \), a bilinear form which only depends on the background solution and so it is constant in space directions.

Using the values of \( B_0 \) from the linearized equations we obtain,

\[
\frac{1}{2h_0} < (e, \Gamma, a, \chi, E, B, \rho), (H_0 B_0 + B_0^\dagger H_0)(e, \Gamma, a, \chi, E, B, \rho) >
\]

\[
= -C^2_e l_{\mu\nu} h^{ij} e^i\rho^j + C^2_\Gamma h^{ij} \rho^i \rho^j
\]

\[
+ C^2_\chi h^{ij} \tilde{a}_i \tilde{a}_j - 2C^2_\chi [(1 - 3\nu_0^2)(\chi^A)^2 + (\chi^{ST})^2] + \frac{1}{9} \chi^2
\]

\[
- 3(E)^2 - 3(B)^2 - 3C^2_\rho (1 + \nu_0^2) \rho^2
\]

\[
- \frac{\nu_0}{h_0} C^2_e l_{\mu\nu} h^{ij} \chi_i e^j e^\nu + \frac{1}{h_0} C^2_e l_{\mu\nu} h^{ik} \chi_i e^j e^\nu
\]

\[
- \frac{\nu_0}{h_0} C^2_\Gamma h^{ij} \tilde{a}_i \tilde{a}_j e^\nu + \frac{1}{h_0} C^2_\Gamma h^{ij} \chi_i e^j e^\nu
\]
Thus negativity of the eigenvalues follows if we can choose the constants on the diagonal of $H_0$ so that:

\[-\frac{1}{h_0}C_e^2 B_{ij}\xi^i_kh_{ip}h^{jm}h^{kn}\Gamma_{mn} \]
\[-\frac{1}{h_0}[C^2 + \frac{k}{2}(\rho_0 + p_0)]E^{ST} \]
\[-\frac{1}{h_0}\kappa C^2(1 - 3\nu_0^2) + C^2_ρ(\rho_0 + p_0)\rho \]

We now choose the constants in such a way to obtain the desired bounds. The first and the second can always be satisfied by choosing $C_e$ small enough. In fact, from the arguments used for the case where the frame perturbations are not square integrable one sees that these two conditions are superfluous. Note that the second is implied by the first if the fifth inequality is taken into account. The third can be always satisfied by choosing $C_Γ$ small enough. The fourth inequality is a real constraint on the equation of state of the fluid, as is the fifth, which in particular rules out from our considerations pure radiation Universes, both are very similar. The sixth inequality can be maximized choosing $C^2_χ = \kappa(\rho_0 + p_0)/2$, and the seventh, choosing $C_ρ = \frac{\kappa}{2\sqrt{3}}$. Both give similar conditions, the first gives,

\[\frac{1}{4}\frac{C^2}{\alpha^2 C^2_χ} < h_0^2 \quad (35)\]
\[\frac{1}{4}\frac{\nu_0^2 C^2}{\alpha^2 C^2_χ} < h_0^2 \quad (36)\]
\[\frac{1}{4}\frac{C^2}{C^2_Γ} < h_0^2 \quad (37)\]
\[3\nu_0^2 - 9\frac{\rho_0 + p_0}{2\nu_0^2} \frac{\partial^2 p}{\partial\rho^2} < 1 \quad (\alpha > 0) \quad (38)\]
\[3\nu_0^2 < 1 \quad (39)\]
\[\frac{1}{2\sqrt{6}}[C_χ + \frac{\kappa(\rho_0 + p_0)}{2C_χ}] < h_0 \quad (40)\]
\[\frac{1}{\sqrt{2\sqrt{1 + \nu_0^2}}}[\kappa(1 + 3\nu_0^2)C_χ + (\rho_0 + p_0)C_ρ] < h_0 \quad (41)\]

Using now (18), and again (39) to estimate $p_0 < \rho_0/3$ in the second condition we get,

\[\frac{k}{a(t)^2} < \frac{\kappa\rho_0}{3\Lambda} \quad (45)\]
Since the right hand side is positive, while \( k = -1, 0, 1 \), we see that this is only a condition for the case of closed cosmologies, namely \( k = 1 \). In that case it says,

\[
1 = k < \frac{1}{2} \left( \frac{2M_0}{R_0} - \frac{2}{3} \Lambda R_0^2 \right)
\]

(46)

where \( M_0, \) and \( R_0 \) are the mass and radius of the Universe. Thus slowly expanding Universes with mass over radius close to closure and small cosmological constant cannot be treated with our method.

We summarize this in the following Lemma:

**Lemma 1** If the equation of state is such that conditions (38–39) are satisfied, and for the case \( k = 1 \) condition (46) is also satisfied, then \( H_0 B_0 + B_0^\dagger H_0 \) is negative definite.

### 3 The Energy Argument

In this section we recall the standard argument leading to asymptotic stability of solutions from conditions on the non-principal part of a symmetric hyperbolic system.

We begin considering a system of the form:

\[
u_t = (A_0^u + \varepsilon A_1^u(u)) D_au + (B_0 + \varepsilon B_1(u)) u
\]

(47)

with:

1. The matrices \( A_0^u, \) and \( B_0 \) are constant in space and time directions.
2. Symmetric Hyperbolicity: there exists a strictly positive symmetric bilinear form \( H = H(u) = H_0 + \varepsilon H_1(u) \) smooth in \( u \), such that
   \[
   H(u)(A_0^u + \varepsilon A_1^u(u)) - (A_0^u + \varepsilon A_1^u(u))^\dagger H(u) = 0.
   \]
3. Decay condition: The matrix \( B_0 \) is negative definite, namely
   \[
   < u, (H_0 B_0 + B_0^\dagger H_0) u > \leq -\delta < u, H_0 u >, \text{ for some } \delta > 0.
   \]

**Remarks:**

1. Perturbations of flat Friedmann-Robertson-Walker space-times can almost be cast in this form by splitting the solution as a background solution plus a small part, \( u_E := u_0 + \varepsilon u \), and noticing that the background solution (out of which one constructs \( A_0^u, \) and \( B_0 \)) is constant in space directions. In this case, the background solution, as well as \( A_0^u, \) \( B_0, \) and \( H_0 \) depend on time, and so we shall latter modify the present argument accordingly.
2. For our system \( H = H_0 \), for it can be chosen only to depend on the background fields, and on the tetrad components of the metric, which by definition are constant (in space) scalars.
3. For our system, and only for convenience in notation, we shall later use derivatives along the frame directions and not \( D_a \).
4. The topology of the initial space can be either compact, $T^3$, say, or open $R^3$. In the presence of constraints the open case might be uninteresting for there could be too few solutions of the constraints which are in the required function spaces for showing existence. We shall make latter a variant of the standard theorems which requires less stringent conditions on the initial data. We shall always assume we have solutions of the constraint equations in the required functional spaces.

We now define an energy vector by:

$$E^a := (\partial_t)^a < u, Hu > + < u, HA^a u >,$$

(48)

and compute,

$$\nabla_a E^a = < u_t, Hu > + < u, Hu_t > + < u, H^a u >
+ < D_a u, HA^a u > + < u, HA^a D_a u > + < u, D_a (HA^a) u >$$

(49)

$$= 2\Re < u, H(u_t + A^a D_a u) > + < u, (H_t + D_a (HA^a)) u >$$

$$= 2\Re < u, H_0 B_0 u > + \varepsilon [2\Re < u, (H_0 + \varepsilon H_1) B_1 u >
+ < u, ((H_1)_t + D_a (H_0 A^a_1 + H_1 A^a_0 + \varepsilon H_1 A^a_1)) u >]$$

Similar expressions hold for space derivatives of $u$, (i.e. tangent to the family of hypersurfaces) since due to the constancy along space and time directions of $A^a_0$ and $B_0$, they satisfy similar equations with same zeroth order (in $\varepsilon$) terms.

Thus, considering a $E^p_a$ sum of all the $E^a$’s for $u$ and all its derivatives up to order $p \geq 3$, integrating in space and time its divergence, using Gauss theorem, and (for simplicity) taking the limit when the integration region shrinks to zero on the time direction we get,

$$\frac{d}{dt}(E^p) \leq -\frac{\delta}{2} E^p + \varepsilon F(E^p),$$

(50)

where

$$E^p(u) := \int_{\Sigma_\tau} E^p n_a(u) d\Sigma,$$

(51)

$n_a$ is the normal to a family of space-like hypersurfaces parametrized with $\tau$, and where we have used standard Sobolev and Gagliardo-Niremberg-Moser estimates to control all the nonlinear terms.

Thus, given an initial value for $u$ such that its $E^p$ norm is finite we can choose $\varepsilon$ so that the right hand side of the above inequality is negative, and so we conclude that the $E^p$ norm of $u$ at future times can not become larger that its initial value. Furthermore if the function $F(\cdot)$ can be chosen to approach zero at least linearly then for small enough $\varepsilon$ there is an exponential decay. But the above mentioned condition in $F(\cdot)$ follows directly from the smoothness assumptions on the coefficients of the equations system, thus we have:

**Theorem 1** Given any initial data $u_0$ whose $E^p$ norm is finite, then there exists $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$ a solution exists and it decays exponentially to zero.
For the case at hand one must be a bit more careful for the frame chosen is not surface orthogonal, and furthermore the matrices $A^0_i$ are not time independent. To control the first problem we shall use the background homogeneity hypersurfaces for which the zeroth component of the background frame is surface-orthogonal, thus the contributions due to the non-surface-orthogonality of the frame to the energy difference will be of order $\varepsilon$. We call that time $\tau$ ($= t_0$), and correspondingly choose $n_a := n e_a^0$. We also introduce the matrix $A^i$ so that the principal part becomes

$$u_t = (A^0_0 + \varepsilon A^0_i(u))e_i(u) + (\hat{\mathcal{B}}_0 + \varepsilon \hat{\mathcal{B}}_1(u))u$$

(52)

For the case at hand, namely the flat case, this change is trivial, and in particular $\hat{\mathcal{B}}_0 = B_0$.

Defining now the energy four-vector as

$$\hat{E}^a := e_0^a < u, H u > + e_i^a < u, H A^i u > := e_0^a \hat{E}^0 + e_i^a \hat{E}^i,$$

(53)

we get a similar expression for eqn. [3].

$$\nabla_a \hat{E}^a = < u_t, H u > + < u, H u_t > + < u, H_t u >$$

$$+ < u, H A^i e_i(u) > + < u, e_i(H A^i)u > + < e_i(u), H A^i u >$$

$$+ \hat{E}^0 \nabla_a e_0^a + \hat{E}^i \nabla_a e_i^a$$

$$= 2 \Re < u, H (u_t + A^i e_i(u)) > + < u, (H_t + e_i(H A^i))u >$$

$$+ \hat{E}^0 \chi + \hat{E}^i (\Gamma^j_{ji} - a_i)$$

$$= 2 \Re < u, H_0 B_0 u > + \hat{E}^0 \chi + \varepsilon (2 \Re < u, (H_0 + \varepsilon H_1)B_1 u >$$

$$+ < u, ((H_1)_t + e_i(H_0 A^i + H_1 A^i + \varepsilon H_1 A^i))u > - \hat{E}^i (\Gamma^j_{ji} - a_i)$$

(54)

where we have used that $a_i = 0$, and for the flat case $\gamma_{0i}^j = 0$. The only new zeroth order term is $\hat{E}^0 \chi = 3 h_0 \hat{E}^0$. Since, $\partial_\mu (a_0^2 \sqrt{-g} \hat{E}^\mu) = a_0^{-3} \sqrt{-g} \nabla_a \hat{E}^a = a_0^{-3} \sqrt{-g} \hat{E}^0$, we get

$$\frac{1}{a_0^{-3} \sqrt{-g}} \partial_\mu (a_0^{-3} \sqrt{-g} \hat{E}^\mu)$$

$$= < u_t, H u > + < u, H u_t > + < u, H_t u > + < u, H A^i e_i(u) >$$

$$+ < u, e_i(H A^i)u > + < e_i(u), H A^i u > + \hat{E}^i (\Gamma^j_{ji} - a_i)$$

$$= 2 \Re < u, H (u_t + A^i e_i(u)) > + < u, (H_t + e_i(H A^i))u >$$

$$+ \hat{E}^i (\Gamma^j_{ji} - a_i)$$

$$= 2 \Re < u, H_0 B_0 u > + \varepsilon (2 \Re < u, (H_0 + \varepsilon H_1)B_1 u >$$

$$+ < u, ((H_1)_t + e_i(H_0 A^i + H_1 A^i + \varepsilon H_1 A^i))u > - \hat{E}^i (\Gamma^j_{ji} - a_i)$$

(55)

Thus applying Gauss theorem to the coordinate divergence in the above expression, and noticing that $a_0^{-3} \sqrt{-g} = 1 + O(\varepsilon)$ we get the desired energy estimated as for the four dimensional flat case treated above. Notice that the new volume element is time independent to zeroth order, and so the constant on the Sobolev embedding and on other estimates can be taken to be also time independent.
Theorem 2 If the equation of state of the fluid is such that conditions (28) are satisfied, then given any initial data perturbation of an expanding flat homogeneous cosmology, \( u_0 \) whose \( E_p \) norm is finite, there exists \( \varepsilon_0 > 0 \) such that for all \( 0 \leq \varepsilon \leq \varepsilon_0 \) a solution exists and it decays exponentially to zero.

3.1 A variant for the case in which \( e_t^{\mu} \) is not in \( L^2 \).

In some cases it is not possible to have perturbations satisfying the constraint equations and having the \( L^2 \) norm of \( e_t^{\mu} \) finite, this is because in the open \((R^3)\) flat background case there could arise the need of considering massive perturbations, and they do not decay at infinity sufficiently fast. In that case we can only have, \( D_\alpha e_t^{\mu} \in L^2 \). There are several ways to deal with this problem, and perhaps the best treatment is the one in [6]. For the present problem a machinery such as in [7] is not necessary, for the decay properties can be established locally (pointwise) for part of the equations. The idea is not to look at the Sobolev norms of \( e_t^{\mu} \), but rather directly at its maximum norm of it, this is possible because the time derivative of the frame does not have any space derivative, it is just an ordinary differential equation. Thus, if we call by \( e \) the frame variables, and by \( u \) all the others we have a system of the form,

\[
\begin{align*}
e_t &= -h_0 e_t + f_u(u) + \varepsilon f_e(e, u) \\
u_t &= (A_0^\alpha + \varepsilon A_1^{\alpha}(e, u))D_\alpha u + (B_0 + \varepsilon B_1(u))u
\end{align*}
\]

where here the matrices \( A^\alpha \), and \( B \) are not the same as before, but are the corresponding restrictions. The usual theory of energy bounds using the Gagliardo-Nirenberg-Moser estimates, (see for instance Proposition 3.7, and note below it, of [8]) give us for the second equation, and the corresponding equations for the space derivatives of both, \( e \), and \( u \), the following inequality,

\[
\frac{d}{dt}(E_p(u) + E_{p-1}(De)) \leq -\delta(E_p(u) + E_{p-1}(De)) + \varepsilon F(E_p(u), E_{p-1}(De), ||e||_{C^0}),
\]

for some smooth \( F \) with \( F(0, \cdot, \cdot) = 0 \), since the negativity of the restricted \( B_0 \) as well as for the whole \( B_0 \) (needed for the estimates on the system of space derivatives \((De, Du)\)) follows as before. Note that the \( ||De||_{C^0} \) is bounded by the \( E_{p-1}(De) \) norm if \( p \geq 3 \).

While for the first equation we get, at each point \( q \) of the base manifold,

\[
|e(q, t)|_t \leq -\delta|e(q, t)| + G_u(E_p(u)) + \varepsilon G_e(E_p(u), |e(q, t)|),
\]

With \( G_u \), and \( G_e \), smooth, polynomially bounded, functions of all its arguments, and with \( G_u(0) = 0 \), and \( G_e(\cdot, 0) = 0 \). In fact in our case \( G_u \) is a linear function of the \( E_p(u) \) norm.

To show exponential decay for \( \varepsilon \) small enough we assume, for contradiction, that given initial data \((e_0, u_0)\) at \( t = 0 \) there exists a time \( T^* > 0 \) given by

\[
T^* = \inf_{T > 0} \left\{ \begin{array}{l} E_p(u(T)) = E_p(u(0))(1 + \Delta^2), \text{ or} \\
E_{p-1}(De(T)) = E_{p-1}(De(0))(1 + \Delta^2), \text{ or} \\
||e||_{C^0}(T) = ||e||_{C^0}(0) + \frac{C\varepsilon}{\delta(E_p(u(0)))} \end{array} \right\}
\]
where \( C(E_p(u)) \) is a function to be defined below, and \( \Delta \) is some non-zero constant. We shall show that \( T^* = \infty \), and from the proof it also follows that the solution not only exists for all times, but in fact decays exponentially.

Given the initial data we choose \( \varepsilon_G \) small enough so that

\[
\varepsilon_G F(E_p(u(0))(1 + \Delta^2), E_{p-1}(De(0))(1 + \Delta^2), ||e||_{C^0(0)} + \frac{C(E_p(u(0)))}{\delta}) \leq (61)
\]

\[
\frac{\delta}{2}(E_p(u(0)) + E_{p-1}(De(0)))
\]

then it is clear that \((u, De)\) will decay as \( e^{-\frac{\delta}{2}} \) as long as \( T \leq T^* \)

We now turn to the frame equation and choose \( \varepsilon_L \) so that

\[
\varepsilon_L G_e(E_p(u(0))(1 + \Delta^2), ||e||_{C^0(0)} + \frac{C(E_p(u(0)))}{\delta}) \leq (62)
\]

\[
\frac{\delta}{2}(||e||_{C^0(0)} + \frac{C(E_p(u(0)))}{\delta})
\]

then we have, as long as \( T \leq T^* \)

\[
|e(q, t)|_t \leq -\frac{\delta}{2}|e(q, t)| + G_u(E_p(u)),
\]

and from this it follows that

\[
||e||_{C^0(T)} \leq (||e||_{C^0(0)} + \frac{\tilde{C}}{\delta})e^{-\frac{\delta}{2}},
\]

where \( \tilde{C} = G_u(E_p(u)) \). Thus we see that taking \( C = \tilde{C} \), and \( \varepsilon = \min\{\varepsilon_G, \varepsilon_L\} \)

we obtain that \( T^* = \infty \), and so the solution exists for all times and furthermore decays exponentially. Thus we have proven a similar Theorem as the one above for the case where the \( L^2 \) norm of the frame is not bounded, but rather its \( C^0 \) norm is.

Acknowledgements:
I thank H. Friedrich for pointing to me the possibility of using his description of the Einstein-Euler system for this particular problem, and for several enlightening discussions. I also thank Uwe Brauer and Gabriel Nagy for pointing to aspects of earlier versions of this work which needed further refinements or corrections.

References

[1] U. Brauer, A. Rendal, and O. A. Reula, “The cosmic no-hair theorem and the nonlinear stability of homogeneous Newtonian cosmological models,” Class. Quant. Grav. 11 (1994), no. 9, 2283–2296, http://xxx.lanl.gov/gr-qc/9403050

[2] H. Friedrich “On the Global Existence and the Asymptotic Behavior of Solutions to the Einstein-Maxwell-Yang-Mills Equations,” J. Diff. Geometry 34 (1991), 275-345.
[3] H. Friedrich “On the evolution equations for gravitating ideal fluid bodies in general relativity,” Phys. Rev. D 57 (1998), 2317-.

[4] H. Friedrich “Personal Communication”.

[5] Reula, O.A. “Hyperbolic Methods for Einstein’s Equations,” Living Reviews in Relativity, http://www.livingreviews.org/Articles/Volume1/1998-3reula), 1998.

[6] Kreiss, H-O, and Lorenz, Jens “Stability for time-dependent differential equations,” Acta Numerica (1998), 203-285.

[7] Choquet-Bruhat, Y., Christodoulou, D. and Francaviglia, M. “Cauchy data on a manifold,” Ann. Inst. Henri Poincaré XXIX, no. 3, (1978) 241-255.

[8] Taylor, M. E. “Partial Differential Equations III” Applied Mathematical Sciences 117, Springer, 1997.