Topological Electropoles in 4+1 Dimensional EYMCS Theory

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ABSTRACT

A class of explicit exact solutions of Einstein Yang Mills Chern Simons (EYMCS) theory corresponding to topological solitons carrying non-Abelian topological electric charge is obtained. This verifies a conjecture made in Ref.[1,2] regarding the stabilization of the corresponding charged configurations in the theory without gravity.

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As is well known, the instanton solution\(^3\) of Yang-Mills theory in 4 Euclidean dimensions can be viewed as a static topological soliton of YM theory in 5 Minkowski dimensions. One also has (“caloron”\(^4\)) solutions when one dimension is compact (i.e on \(M^4 \times S^1\)) and these can serve\(^5\) as a paradigm for investigating the phenomenology of solitons in theories with extra compact dimensions. Coupling the theory to fermions can induce (Abelian and Non-Abelian) charges on the solitons due to the phenomenon of fermion number fractionalization and its generalizations\(^6\). The bosonic effective action which takes account of this phenomenon involves the addition of the terms \(\epsilon^{MNPQ} A_M F_{NL} F_{PQ}^A\) and \(\omega_5 (A^A_M)\) (with coefficients proportional to the number of fermions) to the YM action. Here \(A_M, A_M^A\) are the Electromagnetic and YM gauge potentials and \(F_{MN}^A\) the YM field strength while \(\omega_5\) is the five dimensional Chern-Simons term. These terms are also part of the \(d = 5\) Maxwell-Einstein-Yang-Mills supergravity action\(^11\).

Thus the study of solutions of YMCS theory in \(4 + 1\) dimensions is well motivated. In Ref.[5,1,2] we showed that the structure of the field equations implies that the instantonic soliton, when trivially embedded in \(SU(N)\) picks up a topological non-Abelian charge in the direction of the diagonal generators of \(SU(N)\) (other than isospin: thus for \(SU(3)\) the charge is in the “8” direction). This charge has an gauge covariant meaning in the sense that the charged configuration transforms as a \(^NC_2\) multiplet (thus a triplet or antitriplet for \(SU(3)\)) of \(SU(N)\).

Unfortunately the repulsive electrostatic energy of these configurations makes them unstable towards growth of of the (arbitrary) scale parameter (\(\rho\)) of the instanton configuration. As noted in Ref.[2] this is in accord with the extension of Deser’s theorem\(^7\) to the YMCS theory. Recall that this theorem relies only upon the structure of the energy momentum tensor to show that there are no regular finite energy static solutions of the pure YM field equations which have non zero electric field in five (non-compact) Minkowski dimensions. Since the CS term ,
being metric independent, does not contribute to the stress tensor the proof goes through unchanged. Thus, on $M^5$, YMCS theory has no charged solutions. On the other hand if one dimension is compact the same arguments yield only a much weaker integral constraint. Moreover the compactification of of one dimension implies that one component of the gauge field behaves as a scalar in 4 dimensions and can give rise to attractive forces that can counterbalance the electrostatic repulsion. Another natural possibility is to couple the theory to gravity especially since the Newtonian and Coulombic interactions have the same structure. In fact following our conjecture in Ref.[2] we were able to show numerically that if the Newtonian and Coulombic energies of the flat space solution were cancelled by tuning the couplings then EYMCS theory (under a spherically symmetric ’t Hooft ansatz) yielded essentially the flat space (i.e self dual) solution for the YM fields. Deviation from this fine tuned relation between the gauge and gravitational couplings resulted in unacceptable (infinite energy) asymptotic behaviour. This is natural since a net attraction or repulsion of a configuration with arbitrary scale should drive the scale to zero or infinity. However the miracle of an exact force balance for an extended object that is a solution of a highly nonlinear theory requires more than the apocryphae of numerical analysis to inspire belief! It is thus pleasing that using a method developed in Ref.[8] for a closely related model (namely 5 dimensional EYM theories with Abelian CS tems plus the $\epsilon AGG$ coupling mentioned above ) we can obtain exact explicit charged solitonic solutions of the EYMCS theory. In this letter we give solutions for $SU(3)$ EYMCS theory. Generalizations to $SU(N)$ and the inclusion of electromagnetism are straightforward and will be reported elsewhere.

We begin with the $SU(3)$ EYMCS action:

$$S = \int d^5x \left( -\frac{1}{16\pi G_5} ER + E(\frac{1}{2g_5^2} tr(F_{MN}F^{MN}) + \frac{iN_f}{48\pi^2} \omega_5) \right)$$

$$\omega_5 = \epsilon^{MNLQP} \ tr(\partial_M A_N(\partial_L A_P A_Q + \frac{3}{2} A_L A_P A_Q) + \frac{3}{5} A_M A_N A_L A_P A_Q) \right) \right)$$

(1)
Where $E = \sqrt{-\det(g_{MN})}$, $g_{MN}$ is the five dimensional metric and $G_5, g_5$ are the gravitational and gauge couplings in 5 dimensions with mass dimensions $-3$ and $-1/2$ respectively. Our conventions for gravitational quantities are those of Weinberg$^9$ while those for YM fields are as below:

\[ A_M = A^A_M \frac{\lambda^A}{2i} \quad \text{tr}\lambda^A\lambda^B = 2\delta^{AB} \]

\[ F_{MN} = F^A_{MN} \frac{\lambda^A}{2i} = \partial_{[M} A_{N]} + [A_M, A_N] \]

\[ D_M = \partial_M + [A_M, \quad] \]

\[ A, B, \ldots = 1, \ldots, 8; \quad M, N, \ldots = 0, 1, 2, 3, 4 \]

\[ \mu, \nu, \ldots = 1, 2, 3, 4; \quad a, b, \ldots = 1, 2, 3 \]

We choose a static metric of the Majumdar-Papapetrou$^{12,13}$ form:

\[ ds^2 = -\frac{1}{(B^2)} dt^2 + B dx^\mu dx^\mu \]  

\[ (3) \]

here $dx^\mu dx^\mu$ is the Euclidean 4 metric which is appropriate$^{13}$ when trying to find solutions in which the gravitational attraction and Coulombic repulsion of a system of charges is in balance. In Ref.$[8]$ this form of the metric was also shown to be one appropriate for finding solutions of $d = 5$ supergravity theories that saturate a generalized Bogomolny$^{14}$ bound on the mass of charged field configurations.

The field equations are ($\gamma = N_f/(48\pi^2)$):

\[ G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R = -8\pi G_5 T_{MN} \]

\[ D_M(EF^{MNA}) = -\frac{3\gamma g_5^2}{8} \epsilon^{NMLPQ} \text{tr}(\lambda^A F_{ML} F_{PQ}) \]

\[ (4) \]

and $T_{MN}$ is the stress tensor:

\[ T_{MN} = \frac{1}{g_5^2} (F^A_{ML} F^{AL}_M - \frac{1}{4} g_{MN} F^A_{LP} F^{ALP}) \]

\[ (5) \]
The CS term gives rise\(^2\) to a source term for the YM field equations which is proportional to the sum of the non-Abelian topological current:

\[
T^{NA} = \frac{3\gamma}{2} \varepsilon^{NMPQL} \partial_M tr(\lambda^A (A_P \partial_L A_Q + \frac{2}{3} A_P A_L A_Q)) \tag{6}
\]

\[
\partial_M T^{MA} \equiv 0
\]

and the \(SU(3)\) Noether current derived from the CS term.

We first recall the argument concerning charge induction\(^1,\!^2\) for the reader’s convenience. The constraint equation for static configurations is:

\[
D_\mu (E D^\mu A^{0A}) = -\frac{3\gamma g_5^2}{4} tr(\lambda^A F_{\mu\nu} \tilde{F}_{\mu\nu}) \tag{7}
\]

where \(\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}\). For a configuration with nonzero Pontryagin density arising from static magnetic fields in the \(SU(2)\) sector labelled by \(a\), it is clear that the rhs is non zero (only) for \(A = 8\). Thus it is inconsistent to set \(A_8 = 0\) for such static configurations. The minimum consistent static ansatz is to take \(\{A_\mu^a, A_0^8\}\) to be nonzero. With the metric (3) eqn(7) becomes:

\[
\partial_\mu (B^2 \partial_\mu A_0^8) = -(\sqrt{3} g_5^2 \frac{8}{3} F_{\mu\nu} a^a_{\mu\nu}) \tag{8}
\]

We now carry over the arguments of Ref.[8] to the present case: For a flat space (anti) self-dual field configuration (i.e \(F^a_{\mu\nu} = \pm \tilde{F}^a_{\mu\nu}\)) the non-trivial space components of the YM field equations reduce (using the flat space Bianchi identity) to:

\[
\partial_\mu (\pm \frac{1}{B} + \sqrt{3} g_5^2 \frac{A_0^8}{2} A_0^8) \tilde{F}^a_{\mu\nu} = 0 \tag{9}
\]

Thus if we choose \((k = \frac{2}{\sqrt{3} g_5^2})\):

\[
A_0^8 = \pm \frac{k}{B} \tag{10}
\]
all the YM field equations are solved provided (8) is satisfied. Using eqn.(10), eqn.(8) reads:

\[ \partial_{\mu}\partial_{\mu} B = \pm \frac{1}{4k^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \]  

(11)

It remains to show that if one fine tunes the couplings then the Einstein equations also reduce to (11). First note that due to flat space (anti) self-duality and the conformal structure of the space sector of the metric the magnetic fields \( F_{\mu\nu}^a \) contribute only to \( T_{00} \); all other components are identically zero. Thus the Einstein equations become (\( \delta = (\frac{4\pi G_5 k^2}{g_5^2} - \frac{3}{4}) \)) (no sum over \( \lambda \)):

\[ \frac{3}{2} \frac{\partial^2 B}{B^4} + \delta(\frac{\partial_{\mu} B}{B})^2 = \mp \frac{2\pi G_5}{g_5^2 B^4} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \]

\[ \delta((\frac{\partial_{\mu} B}{B})^2 - 2(\frac{\partial_{\lambda} B}{B})^2) = 0 \]

\[ \delta \partial_{\mu} B \partial_{\lambda} B = 0 \]

thus provided we choose \( \delta = 0 \) i.e

\[ \frac{16\pi G_5 k^2}{3g_5^2} = 1 \]  

(13a)

or, equivalently,

\[ g_5^2 = 16\pi (\frac{4\pi^2 G_5}{N_f^2})^{\frac{1}{3}} \]  

(13b)

one finds that the Einstein equations reduce to the single equation (11) ! Thus, like the electrostatic potential, the static gravitational potential B also has the Pontryagin density as its source. This is essentially the reason for the possibility of a force balance. Hence one only needs to solve one flatspace Poisson equation with a flatspace (anti) selfdual configuration of nonzero Pontryagin index as its source to obtain charged topological soliton solutions of EYMCS theory ! Equation (13) is the force balance condition i.e it ensures that the effects of gravitational attraction
and electrostatic repulsion on the self-dual configuration cancel exactly. Notice that spherical symmetry was nowhere assumed.

It remains to solve for the potential function $B$. We shall restrict ourselves here to completely regular solutions. Solutions with singularities correspond to black holes with a specified charge to mass ratio superimposed on the regular solutions found here and will be discussed elsewhere. Consider a non-singular (anti) self dual flat space YM field configuration localized in a region $V$ of extent $R$. Then the general solution of (11) is

$$B = \alpha_1 \pm \frac{1}{16\pi^2 k^2} \int d^4y \frac{1}{(x-y)^2} F_{\mu\nu}(y) \tilde{F}_{\mu\nu}(y)$$

(14)

From the regularity of the field strengths it is clear that $B$ is everywhere non-singular. Moreover for $r >> R$ one has

$$B = \alpha_1 + \frac{2|\nu|}{k^2 r^2} + O(1/r^3)$$

(15)

where

$$\nu = \frac{1}{32\pi^2} \int d^4xF_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)$$

(16)

is the flat space Pontryagin index of the configuration.

To identify the constant $\alpha_1$ and the mass and charge of the soliton one defines as usual a new coordinate $\tilde{r} = \sqrt{Br}$ so that the metric becomes

$$ds^2 = -\frac{1}{B^2} dt^2 + \left(1 - \frac{\tilde{r} B'(\tilde{r})}{2B}\right)^2 d\tilde{r}^2 + \tilde{r}^2 d\Omega_3$$

(17)

where $d\Omega_3$ the metric on the unit 3-sphere. For any localized system this metric should go over to the metric on $M^5$ as $\tilde{r} \to \infty$. Then $B \to 1$ and $r \to \tilde{r}$. Since one has for a localized system in 5 dimensions

10
\[ g_{00} = \frac{-1}{B^2} \rightarrow -1 + \frac{8MG_5}{3\pi} \frac{1}{r^2} + O\left(\frac{1}{r^3}\right) \quad (18) \]

where \( M \) is the ADM mass, we immediately get:

\[ \alpha_1 = 1 \quad M = \frac{8\pi^2 |\nu|}{g_5^2} \quad (19) \]

Furthermore the asymptotic form of the electric field:

\[ F^{0\mu} = \frac{Q \bar{x}^\mu}{2\pi^2 r^4} \quad (20) \]

gives for the electrostatic charge \( Q^8 \):

\[ Q^8 = -\frac{\nu N_f g_5^2}{4\sqrt{3}} \quad (21) \]

If we restrict ourselves to (anti) self-dual configurations which can be obtained via the 't Hooft\(^3\) ansatz:

\[ A_\mu = -\bar{\eta}_{a\mu\nu} \partial_\nu ln \Pi(x^\mu) \quad (22) \]

we can obtain completely explicit solutions for all fields in terms of the \( x^\mu \) coordinates. The self-duality condition reduces to solving the equation \( \Pi^{-1} \partial^2 \Pi = 0 \) for the “superpotential” \( \Pi(x^\mu) \). The general solution corresponding to \( N \) instantons with scale factors \( \rho_i \) and locations \( z_i^\mu \) is:

\[ \Pi = 1 + \sum_{i=1}^{N} \frac{\rho_i^2}{(x - z_i)^2} \quad (23) \]

Although in this ansatz the gauge potentials are singular the field strengths are everywhere regular and one can always transform to a non-singular gauge. Note also that eqn.(11) is invariant under \( SU(2) \) gauge transformations while \( SU(3) \) gauge transformations give rise to the \( SU(3) \) multiplet structure described in Ref.[2].
To obtain the explicit form of the solution one need only remark that for the solutions (23) one can write:

\[ (F_{\mu\nu}^a)^2 = 2\partial^2 (\partial_\lambda \log(\Pi))^2 = F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \] (24)

Thus the relevant solution of (11) is:

\[ B = 1 + \sum_{i=1}^{N} \frac{\beta_i^2}{(x - z_i^2)^2} - \frac{1}{2k^2}(\partial_\lambda \log(\Pi))^2 \] (25)

The constants \(\{\beta_i, z_i^\mu\}\) are fixed by demanding regularity as \(x^\mu \to z_i^\mu\):

\[ z_i^\mu = z_i^\mu, \quad \beta_i = \frac{2}{k^2} \] (26)

Thus for instance for \(N = 1\) one has:

\[ B = 1 + \frac{2(r^2 + 2\rho^2)}{k^2(r^2 + \rho^2)^2} \] (27)

Clearly B is everywhere regular and nonzero unless \(\rho = 0\) so that there are no horizons.

It is also easy to obtain explicit solutions when the asymptotic space is \(M^4 \times S^1\). One merely modifies the the metric (3) to \((m = 1, 2, 3, R \text{ is the radius of and } \theta \text{ the angle on } S^1)\)

\[ ds^2 = -\frac{1}{(B^2)}dt^2 + B(dx^m dx^m + R^2 d\theta^2) \] (28)

and uses the periodic superpotential for N calorons.

In conclusion, using the technique of Ref.[8], we have shown that gravity can stabilize the solitonic configurations of YMCS theory on \(M^5\) and \(M^4 \times S^1\) which carry a topological non-abelian electric charge and were introduced in Ref[1-3]. Our solutions suggest that the derivation of Bogomolny type bounds for solitons
in Einstein Maxwell theories (with non-minimal terms) given in Ref[8] may be extendable to a bound on the mass of solitons in terms of the Casimir operator. Our solutions are of interest in the context of Kaluza-Klein theories. The program\textsuperscript{5,1,2} of building a simple paradigm for the phenomenology of higher dimensional solitons by carrying out the collective quantization of the global gauge zero modes of these solitons can now proceed.

**Note Added**

The expert reader will notice that our solution follows by embedding the gauge group $SU(2) \times U(1)$ of Ref.[8] in $SU(3)$. However, as explained in Ref.[2] the vacuum of the theory does not break $SU(3)$ and the gauge covariance of the field equations implies that one obtains solitons that are *triplets* of $SU(3)$. In this way an essentially non-Abelian structure is present. The particular embedding chosen corresponds to a $SU(3)$ eigenstate in which the isospin quantum number is zero. The solutions of Ref.[8] are in turn closely related to those of Ref.[15]. A detailed comparison will be given elsewhere.

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