Entanglement entropy and correlations in loop quantum gravity

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Black hole entropy is one of the few windows toward the quantum aspects of gravitation and its study over the years have highlighted the holographic nature of gravity. At the non-perturbative level in quantum gravity, promising explanations are being explored in terms of the entanglement entropy between regions of space. In the context of loop quantum gravity, this translates into the analysis of the correlations between regions of the spin network states defining the quantum state of geometry of space. In this paper, we explore a class of states, motivated by results in condensed matter physics, satisfying an area law for entanglement entropy and having non-trivial correlations. We highlight that entanglement comes from holonomy operators acting on loops crossing the boundary of the region.

I. INTRODUCTION

One of the most fascinating prediction of General Relativity, Einstein’s classical theory of gravitation, is the existence of black holes, a region of space-time where nothing not even light can escape. Seen originally as a simple non-physical curiosity, the mathematical theory of black holes is now a fully grown subject. A stationary black hole appears in fact to be quite a simple object defined entirely by its mass, angular momentum and charge (although this must be put in contrast with the recent work underlining the existence of classical hairs for black holes in relation to gravity’s soft modes \([1, 2]\)). Moreover, a straight analogy can be drawn between black holes dynamics predicted by Einstein’s equation and thermodynamics. In particular, a notion of entropy is associated to a black hole \([3]\) which, in presence of quantum fields, is related to the area of the event horizon by the Hawking formula \([4]\) and has lead to the holographic principle relating geometric quantities and entropy in quantum gravity \([5]\).

In the context of loop quantum gravity (see \([8, 10]\)), black hole entropy was mainly studied from the isolated horizon concept which assumes boundary conditions at the classical level \([11]\). From a purely quantum perspective, this classical input should be removed. Instead, the quantum route is to compute an entanglement entropy between a bipartite partition of the spin network state into an inside/outside regions. For instance, in the 3d Riemannian BF formulation of gravity, such an entanglement (after a suitable regularization) satisfies an holographic behavior for the flat state \([12]\) (see also \([13, 14]\) for a more general treatment in loop gravity and lattice gauge theory beyond the flat state). Those calculations have strong similarities with those in some spin models like the toric code model useful for quantum computation purposes \([15, 17]\). Here we would like to push further this similarities and propose to study a class of test states having holographic properties and non trivial correlations.

A very active line of research in this direction is tensor network renormalisation techniques applied to the context of the AdS/CFT correspondence for quantum gravity. Indeed tensor network states turned out to be a tremendously efficient ansatz to study holography in quantum gravity \([18]\), especially when looking at the holographic entanglement entropy \([19, 20]\). In particular, multi-scale entanglement renormalization ansatz (MERA) \([21]\) are especially promising as they appear to be variational ansatz of conformal field theory ground states and allow for a lattice realization of the AdS/CFT correspondence \([22, 24]\). Moreover, they have opened a fecund interaction between quantum gravity, quantum information and quantum computing.

Here, going in a similar direction although without using the MERA tools, our goal is to better understand the structure of correlations in loop quantum gravity (the interested reader can nevertheless find in \([25]\) a first application of tensor network techniques to loop quantum gravity and spin network states). The motivation behind our study is twofold. The first one is to have a clearer understanding of the physical states solving all the constraints of canonical quantum general relativity. It is expected they should have non trivial correlations mapping to the two points correlation functions of gravitons at the classical limit and that they should satisfy an area law. We will introduce an ansatz for quantum states as superpositions of loop states with loops of arbitrary sizes with weights scaling for instance with the loop area, their perimeter and their number. The second motivation is related to the definition and action of the Hamiltonian constraint of the theory which is still under active research. Identifying quantum states, with well-behaved correlations (both holographic and admitting nice 2-point correlations) would give great insights toward the proper form and action of the quantum dynamics implementing Einstein equation. Our work can be seen as complementary to the study of entanglement on spin network states built from local Hamiltonian as in condensed matter models developed in \([26, 28]\).

The present paper is structured as follows. Section \([11]\)
reviews the basic features of the toric code model that is then used to define the proper class of spin network states. The entanglement entropy between a partition (the system is a closed region) of the spin network is evaluated and we show that it scales as the number of degrees of freedom of the boundary in Section IV. Loops crossing the boundary are seen to be at the origin of this entanglement. Section IV discusses a necessary generalization for the correlations to be non trivial and for the entanglement entropy to scale as the area.

II. THE TORIC CODE MODEL

The toric code model is a topological model of spin $1/2$ living on the links of a general 2D lattice. The anyonic structure of the excitations makes it useful for fault-tolerant quantum computation [13, 14]. This model can be shown to be equivalent to a $BF$ theory on the discrete group $Z_2$, a highly interesting fact since gravity can be formulated as a (constrained) $BF$ theory.

Considering a bipartite partition of the lattice, it was found that the entanglement entropy between those regions were proportional to the boundary area (plus a topological term) which is reminiscent of the holographic character of the excitations makes it useful for fault-tolerant quantum computation [13, 14]. This model can be shown to be equivalent to a $BF$ theory on the discrete group $Z_2$, a highly interesting fact since gravity can be formulated as a (constrained) $BF$ theory.

As mentioned above, one of the interesting result of the toric code model is the area scaling law of the entanglement entropy for the ground state between a bipartite partition of the lattice [16]. Consider a region $S$ and its exterior $E$, the global system being in a ground state, see Fig I. The reduced density matrix of the region $S$ is needed to obtain the entropy. The loop structure coming from the state is composed of three kinds of loops, those contained completely in $S$ or $E$ and those belonging to both. We then have the following result (with a detailed proof in annex [16]): the entanglement entropy associated to a given region $S$ whose (contractible) frontier possesses $n_{SE}$ degrees of freedom in the ground state is

$$ S = n_{SE} - 1 $$

This entropy is proportional to the number of degrees of freedom on the boundary and scales as the area. The minus one is a topological contribution and is model dependent in some sense [17].

What we intend to do now is the study in the context of loop quantum gravity the class of wavefunction having the same loop structure as the one for the toric code model and study the entanglement entropy and correlations they contain.

III. DEFINITION AND PROPERTIES

A. The loop decomposition

The holographic principle is one of the few accepted feature every quantum theory of gravity should have. Simply stated, it says that volume degrees of freedom of a region of spacetime are encoded on some degrees of freedom on the boundary [17]. We saw that the entanglement
entropy of the toric code model has this same behavior. The purpose here is then quite simple: we want to adapt the ground state structure of this model for the spin-network state on a given random 2d lattice with edge degrees of freedom fixed to the fundamental spin 1/2 excitation (this condition can be relaxed by choosing any spin j).

FIG. 1. Illustration of one possible loop structure appearing in the superposition defining the Kitaev state motivated by the ground state structure of the toric code model. Three kind of loops are distinguished when a subsystem is chosen and only loops crossing the boundary give a non zero entanglement.

The natural SU(2) gauge invariant object in loop quantum gravity is the holonomy, here \( \chi_{1/2} (\prod_{e \in \mathcal{L}} g_e) \) for a given loop \( \mathcal{L} \) and group element \( g_e \in \text{SU}(2) \) for each edge. We thus define by analogy the state which has the same loop structure than \( \mathcal{L} \). In fact, canonical model of statistical physics such as the Ising model or \( O(N) \) models, hints toward adding new amplitude contribution like a perimeter \( P(\mathcal{C}) \) contribution \( \gamma^P(\mathcal{C}) \) or/and a number of loops \( N(\mathcal{C}) \) contribution \( \beta^N(\mathcal{C}) \). So the natural general states we are interested in are given by the wave function

\[
\psi_{\alpha,\beta,\gamma}(g_e) = \sum_{\mathcal{C}} \alpha^{A(\mathcal{C})} \beta^{N(\mathcal{C})} \gamma^{P(\mathcal{C})} \prod_{e \in \mathcal{L}} \chi_{1/2} (\prod_{e \in \mathcal{L}} g_e) \tag{6}
\]

A given configuration \( \mathcal{C} \) is composed of non intersecting loops \( \mathcal{L} \) having no links in common while \( A(\mathcal{C}), N(\mathcal{C}) \) and \( P(\mathcal{C}) \) are respectively the total number of loop and the perimeter of the configuration and \( \alpha, \beta, \gamma \in \mathbb{C} \) are complex amplitudes.

Our goal is to study this class of states, the scaling law of the entanglement entropy between a partition of the spin network and then the correlation two point functions between spins of different edges. We start with the simple case \( \beta = \gamma = 1 \) and see that the entropy scales as expected with the number of degrees of freedom of the boundary. However, the correlations will appear to be topological, motivating the introduction of a more general class of states with amplitudes function of the perimeter of the loops or their number.

In fact, we could have first thought of a simpler state constructed as a product of all holonomies of each plaquette (with a potential contribution from a boundary for a finite size graph) as

\[
\psi(g_e) = \prod_p \chi_{1/2} (\prod_{e \in \mathcal{P}} g_e) \chi_{1/2} (\prod_{e \in \mathcal{B}} g_e) \tag{7}
\]

At first sight, it would appear that such a state would display some non trivial correlations. However this is not the case both for the holonomy and spin two point functions in the infinite size limit. We won’t dwell on this state in the core of this paper, see annex \( \text{A} \) for more details.

B. Behavior under coarse-graining

The state \( \psi_{\alpha,\beta,\gamma} \) have very nice properties under some coarse-graining procedures due the very particular loop structure we chose. For a graph \( \Gamma \), one procedure is to simply eliminate a link \( e_0 \) constructing the new graph \( \Gamma \setminus e_0 \) and another is to pinch the link to a node defining the pinched graph \( \Gamma - e_0 \).

From the wave function \( \psi^{\Gamma}_{\alpha,\beta,\gamma} \), the pure elimination of a link is done by a simple average. Here, since the loops composing the state are always non overlapping, the integration over \( e_0 \) amounts to remove all loops containing it.

\[
\int_{\text{SU}(2)} \psi^{\Gamma}_{\alpha,\beta,\gamma}(g_{e_0}, g_e) \, dg_{e_0} = \psi^{\Gamma \setminus e_0}_{\alpha,\beta,\gamma}(g_e) \tag{7}
\]

Thus the coarse-grained state corresponds exactly to the state on the coarse-grained graph \( \Gamma \setminus e_0 \). We have a stability under this coarse-graining procedure.

FIG. 2. The pinch coarse-graining method is an invariant procedure only for the case \( \gamma = 1 \), meaning the state doesn’t contain perimeter information.

The second method is to pinch the link. This is done by imposing the holonomy on \( e_0 \) to be equal to the identity. The coarse-grained state is \( \psi_{\alpha,\beta,\gamma}^{\Gamma}(g_e) \big|_{g_{e_0}=\text{id}} \). For a given loop containing \( e_0 \), pinching the link doesn’t change the area or the number of loops, but only its perimeter. Separating configuration containing the link \( e_0 \) or not, forming respectively the sets \( \mathcal{C}_0 \) and \( \mathcal{C} \setminus e_0 \), we have

\[
\psi^{\Gamma}_{\alpha,\beta,\gamma}(g_e) \big|_{g_{e_0}=\text{id}} = \sum_{\mathcal{C}_0} \alpha^{A(\mathcal{C})} \gamma^{P(\mathcal{C})} \prod_{e \in \mathcal{L}} \beta \chi_{1/2} (\prod_{e \in \mathcal{L}} g_e) + \gamma \sum_{\mathcal{C}_0} \alpha^{A(\mathcal{C})} \beta^{P(\mathcal{C})} \prod_{e \in \mathcal{L}} \chi_{1/2} (\prod_{e \in \mathcal{L}} g_e) \tag{7}
\]
The invariance under coarse-graining is recovered at the condition that \( \gamma = 1 \), meaning that the perimeter of the loops doesn’t matter: \( \psi_{\alpha,\beta,\gamma}^{T}(g_{e}) \bigg|_{g_{\alpha}=1} = \psi_{\alpha,\beta,\gamma}^{T-g_{\alpha}}(g_{e}) \).

### IV. ENTANGLEMENT ENTROPY

#### A. Entanglement entropy

The next step is to compute the entanglement (Von Neumann) entropy \( S = \text{tr} (\rho_{S} \ln \rho_{S}) \) between a bipartite partition of the graph. The system \( S \) of interest will be a bounded connected region and the rest of the graph forms the environment \( E \) whose degrees of freedom are traced out. The boundary group elements will be by convention incorporated into the system and won’t be traced over. For simplicity, we will restrict the evaluation of the entropy for \( \beta = \gamma = 1 \).

To compute the entropy of \( S \), we need its reduced density matrix defined as

\[
\rho_{S}(g_{e}, g_{e}') = \int \psi_{\alpha}(g_{e} \in S, h_{e} \in S, g_{e} ' \in S) \psi_{\alpha}(h_{e} \in S, g_{e} ' \in S) \, dh_{e} \in S
\]

(8)

The method to evaluate the entropy \( S = - \text{tr} (\rho_{S} \ln \rho_{S}) \) is based on the replica trick [29]. Computing the successive power of the reduced density matrix \( \rho_{S}^{n} \), \( n \in \mathbb{N} \), we then obtain the entropy by \( S = - \frac{2 \text{tr} \rho_{S}^{2}}{\text{tr} \rho_{S}} \bigg|_{n=1} \).

The first step is to compute the reduced density matrix. Denoting respectively \( C_{S}, C_{E} \) and \( C_{SE} \) the loops belonging to \( S, E \) or both, we have (see annex [B2])

\[
\chi_{1/2}(L_{S}(g)) \chi_{1/2}(L_{S}'(g')) \prod_{L_{SE} \in C_{SE}} \left( \frac{1}{2} \chi_{1/2}(L_{SE}(g,g')) \right)
\]

(9)

with \( \mathcal{N}(\alpha) \) the norm and \( \mathcal{N}_{E}(\alpha) = \sum_{C_{E}} |\alpha|^{2A(C_{E})} \) the factor coming out of the partial trace on the environment. We see that two contributions appear, one with only loops in \( S \) and another coming from loops crossing the boundary. This last term is responsible for the entanglement between \( S \) and \( E \).

Figure 3 shows an example of a configuration appearing in the reduced density matrix. To understand simply the form of \( \rho_{S} \), let’s imagine we have only two loops configuration, one copy for the bra and ket of the density matrix. Each configuration is composed of non overlapping and non intersecting loops. Nonetheless, each copy can overlap since their are independent. Now, tracing out the \( E \) degrees of freedom imposes that the parts in \( E \) from each copies to be exactly the same, otherwise the average gives zero. Complications come from loop crossing the boundary. The average of the \( E \) part of crossing loops gives a contribution of the from \( \int \text{SU}(2) \chi_{1/2}(gh) \chi_{1/2}(g'h) \, dh = \frac{1}{2} \chi_{1/2}(gg'^{-1}) \). This is at the origin of the boundary holonomies in [9]. Now considering again all the allowed configurations, we see that for a given bulk/boundary plaquette choice like in Fig 3 their is a huge redundancy coming from the \( E \) plaquettes. After the partial trace, this leads to the overall \( \mathcal{N}_{E}(\alpha) \) prefactor.

The next step is to compute the successive power and take the trace. At the end a simple formula remains,

\[
\text{tr} \rho_{S}^{n}(g, g') = \left( \frac{\mathcal{N}_{E}(\alpha)}{\mathcal{N}(\alpha)} \right)^{n} \mathcal{N}_{S}^{n-1}(\alpha) \sum_{C_{S} \cup C_{SE}} \frac{|\alpha|^{2A(C_{S} \cup C_{SE})}}{(4^{n-1}) \# C_{SE}}
\]

(10)

with \( n_{SE} \) the number of degrees of freedom at the boundary and \( f(|\alpha|^{2}) = \ln (1 + |\alpha|^{2}) - \ln (|\alpha|^{2}) \). This formula is quite general and is valid for an arbitrary graph.
as long as the loop structure of the spin network state is the same.

B. Boundary degrees of freedom - Purification

We came to understand that the entanglement in the subsystem $S$ prepared in the state $|\psi\rangle$ can be traced back to loops crossing the boundary. In fact, we can understand the state $|\psi\rangle$ as resulting from tracing out additional boundary degrees of freedom. This idea goes in the same spirit as recent studies on local subsystems in gauge theories and gravity [3, 4].

To purify the state, consider at each puncture a new degree of freedom, for instance a new fictitious edge. We work in the extended Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_c^\otimes N$ with $\mathcal{H}_c$ the Hilbert space associated to the new edge, $N$ the total number of puncture and $\mathcal{H}_S$ the Hilbert space of the system. We then construct a pure state as superposition of loops in the bulk and paths joining pairs of punctures, see for instance FIG[3] For a path $P$, we use the holonomy (properly oriented)

$$\chi_{1/2}(P) = \chi_{1/2}(h_{s,t_0})$$

with $h_{s,t_0}$ associated to the pair of boundary degrees of freedom, the source and target of the path respectively. The reduced density matrix [6] can then be purified by considering the state $|\psi_{SB}\rangle$ (B for boundary) with wavefunction

$$\psi_{SB}(g) = \sum_c \alpha^{A(c)} \beta^{N(c)} \gamma^{P(c)} \prod_{L \in C} \chi_{1/2}(L) \prod_{P \in C} \chi_{1/2}(P)$$

(11)

Then $\rho_{SB} = \text{tr}(|\psi_{SB}\rangle \langle \psi_{SB}|)$. We have purified the reduced density matrix of the system. One could argue that their are many ways to purify a quantum state and consequently why it has something to say about entanglement, correlations and entropy.

C. On correlations

This holographic behavior is a good sign for this class of states to be good candidates for physical states solutions of the Hamiltonian constraint of loop quantum gravity. What’s more, for physical solutions, we expect the correlations between geometrical observables to be non trivial. This is where the limit $\beta = \gamma = 1$ fails. Indeed, the spin (or holonomies) two points correlation functions are topological in the sense that they do not depend on the graph distance between the edges.

Let’s look for instance at the spin two points functions $\langle \hat{j}_e \hat{j}_{e'} \rangle - \langle \hat{j}_e \rangle \langle \hat{j}_{e'} \rangle$. This spin operator is defined by its action on a spin network state with the help of the Peter-Weyl theorem 

$$\langle \hat{j}_e \psi(g, g_e) = \sum_{j_e} (2j_e + 1) j_e \int \chi_{j_e} (g_e h^{-1}) \psi(g, h) \, dh.$$ 

The method to evaluate the averages goes as follows. First we have only the spin 1/2 component of the average that gives a non zero contribution, so that we have

$$\langle \hat{j}_e \rangle = \frac{\int_{SU(2)} \chi_{1/2}(g_e h^{-1}) \psi(g, g_e) \, dg_e \, dg_e}{N(a)}$$

$$= \sum_{c \in \mathcal{C}} \alpha^{A(c)} \beta^{A(c')} \int_{SU(2)} \chi_{1/2}(g_e h^{-1})$$

$$\times \prod_{L \in C} \chi_{1/2}(h_e, g) \prod_{L' \in C'} \chi_{1/2}(g_e, g) \, dg_e \, dg_e$$

(12)

We integrate over $g_e$. If $g_e \notin \mathcal{C'}$ the integral gives zero. Otherwise we have simply $\int_{SU(2)} \chi_{1/2}(g_e h^{-1}) \chi_{1/2}(g_e, h) \, dg_e = \frac{1}{2} \chi_{1/2}(h h_e)$ ; substitute $h_e$ for $g_e$ with a factor one half. Finally, denoting by $C'_e$ a configuration of loops containing the link $e$

$$\langle \hat{j}_{e'} \rangle = \frac{1}{2N(a)} \sum_{c, c'_e} \alpha^{A(c)} \beta^{A(c'_e)}$$

$$\prod_{L \in C, L' \in C'} \int_{SU(2)} \chi_{1/2}(g_e) \chi_{1/2}(g_e) \prod_{e \in L} g_e \prod_{e \in L'} g'_e \, dg_e$$

$$= \frac{1}{2N(a)} \sum_{c_e} \alpha^{A(2c_e)} = \frac{1}{(1 + \alpha^2)^2}$$

(13)

The explicit evaluation of $\langle \hat{j}_e \hat{j}_{e'} \rangle$ follows the same steps. Distinguishing the two cases when the spins belong to the same loop or not, see FIG[4] we have respectively

$$\langle \hat{j}_e \hat{j}_{e'} \rangle = \frac{1}{4} \frac{|\alpha|^2}{(1 + |\alpha|^2)^2} \text{ and } \langle \hat{j}_e \hat{j}_{e'} \rangle = \frac{|\alpha|^2}{(1 + |\alpha|^2)^2}.$$ 

(14)

In both cases, the correlation $\langle \hat{j}_e \hat{j}_{e'} \rangle = \langle \hat{j}_e \rangle \langle \hat{j}_{e'} \rangle$ is not in any way a function of the distance between the edges which is particularly clear when the edges don’t belong to the same loop where the correlation is strictly zero.

From the structure of state, we should have naively expected the correlations to scale in some way as the graph distance between the edges $e$ and $e'$. This is in fact not
the case since the averages counts every loops meeting the edges in a democratic way (both configuration in Fig. 2 give the same correlations). Introducing a contribution to the amplitude proportional for instance to the number of loops can be a solution to this issue. The limit $\beta = \gamma = 1$ has thus to be reconsidered to account for non trivial correlations.

D. Example

![Diagram](image)

FIG. 5. Illustrative example for the evaluation of the entanglement entropy for a two loops state.

As an illustrative example, consider a two loops state whose wave function is $\psi(h_S, h_b, h_E) = 1 + \alpha \chi(h_S h_b) + \alpha \chi(h_E h_b^{-1}) + \alpha^2 \chi(h_S h_E)$. The reduced density matrix, obtained by taking two copies of the state and tracing out over the environment has the form

$$\rho_S(h_b, h_S, h_b', h_S') = \int \psi^*(h_S', h_b', h_E) \psi(h_S, h_b, h_E) \, dh_E = 1 + \alpha \chi(h_S h_b) + \alpha \chi(h_S h_b') + \alpha^2 \chi(h_S h_E) + \alpha^2 \chi(h_S h_E')$$

$$+ \frac{1}{2} \left[ \chi(h_b h_b'^{-1}) + \alpha \chi(h_b h_E^{-1}) + \alpha \chi(h_b h_E'^{-1}) + \alpha^2 \chi(h_S h_E') \right]$$

(15)

The computation of the successive power of the reduced density matrix and the trace in then straightforward. We have $\rho_S^2(g, g') = \frac{1 + |\alpha|^2}{N(\alpha)} \left( 1 + \frac{|\alpha|^2}{N(\alpha)} \right)$ and the entanglement entropy follows formula (16).

V. FINDING NON TRIVIAL CORRELATIONS

A. Distinguishing loops

We saw in the last section why correlations were trivial for the restricted state studied for entanglement entropy. This was coming from the fact that their was non distinction between loops passing through both links or not, see Figure 2. To understand how the solution comes about, let’s look first at a simpler state constructed as the superposition of single loop holonomy

$$\psi(g_e) = \sum_{\mathcal{L}} \alpha^{|\mathcal{L}|} \chi_{1/2} \left( \prod_{e \in \mathcal{L}} g_e \right)$$

(16)

This term is the first non trivial term of \(\mathcal{E}\) in an expansion of $\beta$ (for $\gamma = 1$). The spin two points correlation function is straightforwardly evaluated as

$$\langle \hat{J}_e \hat{J}_{e'} \rangle = \int \chi_{1/2} \left( g_e h_E^{-1} \right) \chi_{1/2} \left( g_{e'} h_E^{-1} \right)$$

$$\psi(h_e, h_{e'}, g) \psi(g_e, g_{e'}, g) \, dg d h_{e', e'} d g_{e, e'}$$

$$= \frac{1}{N} \left( \frac{N}{N_S N_E} \right)$$

(17)

with $N(e, e') = \sum_{\mathcal{L} e e'} |\alpha|^{|\mathcal{L} e e'|}$ is a sum over all loops passing through both edge $e$ and $e'$. Now in this case, the correlations will scale non trivially on the minimal area between the edges since we must consider loops passing through both links at the same time. Here is the main difference between this state and the previous one.

We can go even further and analyze the entanglement entropy of a local region for this state. In fact, the computation is completely similar to the one presented in IV. However, the entanglement entropy doesn’t follow an area law, doesn’t even scale as a function of the boundary degrees of freedom. Indeed, the area scaling came from the term $\ln \left( \frac{N}{N_S N_E} \right)$ and the multiplicative nature of $N = N_S N_E N_S E$ whereas for $\mathcal{E}$ $N$ is additive. Thus the same contribution $\ln \left( \frac{N}{N_S N_E} \right)$ is not only a function of the boundary degrees of freedom.

B. The proposal

The previous discussions show that two ingredients are necessary to obtain states with non trivial correlations and an entanglement entropy for a localized region to scale as the area of the boundary (at least to be a function of boundary degrees of freedom). Area law entanglement entropy came from the loop structure of the toric code model, more precisely all configurations of non intersecting and overlapping loops enter the superposition. Non trivial correlations came on the other hand from the fact that a clear distinction between loops passing by both edges $e$ and $e'$ and those that do not was made. The solution presents itself when we come back to our original state \(\mathcal{E}\) with amplitude scaling as the area and the number of loops (we omit the perimeter contribution since it can obstruct coarse-graining invariance),

$$\psi_{\alpha, \beta}(g_e) = \sum_{\mathcal{L}} \alpha^{|\mathcal{L}|} \chi_{1/2} \left( \prod_{e \in \mathcal{L}} g_e \right)$$

(18)

The two requirements are here met. The $\beta$ factor corresponds exactly to a number of loops contribution.
It is straightforward to generalize the following discussion to an arbitrary superposition of holonomies \( f(g) = \sum_{j=1/2}^\infty p_j \chi_j(g) \); the formal expressions remains the same as before.

Let’s review its features. Concerning correlations, we can now distinguish a dominant term in the two points function. Indeed, what rendered the correlations topological initially was that their was no distinction between the cases when the edges \( e \) and \( e' \) belong to the same loop or different ones. With the additional \( \beta \) contribution, we can now pinpoint a dominant term which is the one with minimal area and only one loop connecting the edges. Denoting by \( L_x \) and \( L_y \) the horizontal and vertical graph distance respectively connecting two given links \( e \) and \( e' \), the number of such minimal loops is \( (\min(A_{xy})) \). Thus, the dominant contribution to the spin correlations is

\[
\langle \hat{b}_e \hat{b}_e' \rangle = \frac{1}{4} |\beta|^2 |\alpha|^2 A_{\min} \left( \frac{A_{\min}}{L_y} \right) + o(|\beta|^2, |\alpha|^2 A_{\min}) \quad (19)
\]

\[
= \frac{1}{4} |\beta|^2 |\alpha|^2 A_{\min} \left( \frac{A_{\min}}{L_y} \right) \left( \frac{L_x - L_y}{2A_{\min}} \right)^2 \frac{1}{\sqrt{A_{\min} \pi / 2}} \quad (20)
\]

The correlations are now non topological. The correlations are maximum when the number of minimal paths joining the edges is maximal. This can be seen as entropic competition between the number of paths linking the edges \( e \) and \( e' \) and an energetic term \( |\alpha|^2 A_{\min} \). The more connected the edges are the more correlated they are. In the light of the distance from correlation point of view \([30, 31]\), the edges get closer when more different minimal paths of the graph exist.

The entropy can be obtained following the same steps as in Section IV by computing the successive power of the reduced density matrix and by employing the replica formula for the Von Neumann entropy. We have in the end an entanglement entropy function only of the boundary degrees of freedom,

\[
S = \ln(N_{SE}(\alpha, \beta)) - \frac{1}{N(\alpha, \beta)} \sum_{C_{SE}} [A(C_{SE}) \ln (|\alpha|^2)]
\]

\[
+ N(C_{SE}) \ln \left( \frac{|\beta|^2}{4} \right) |\alpha|^2 A(C_{SE}) |\beta|^2 N(C_{SE})
\]

(21)

In the special case where \( \alpha = 1 \) and \( |\beta| = 2 \), we have a very simple expression for the entanglement entropy,

\[
S = \ln(N_{SE}(\alpha, \beta))
\]

(22)

So in the end, we see that the state \([18]\) is a good candidate to be a physical state, at least mirrors some features the true physical solution of the Hamiltonian constraint might be, since it as correlations that are function of some measure of distance in the graph through the minimal area and as an area law scaling entanglement entropy.

VI. CONCLUSION

In this paper, we introduced a class of states in loop quantum gravity whose entanglement entropy for a bounded region scales as the area of the boundary (number of degrees of freedom) and whose correlation functions between distant spins are non trivial. Its structure is motivated by a condensed matter model, Kitaev’s toric code model, where the ground state can be seen as a gas of loops on the lattice. Our ansatz mimics this structure, being defined as a superposition of non intersecting loops of arbitrary size. To each configuration, an amplitude function of the area, the perimeter or the number of loops is considered.

We showed that indeed the entanglement entropy of a region scales as the area of its boundary using the replica trick method. The source of entanglement is seen to be exclusively due to loops crossing the boundary and the fact that the entanglement depends only on boundary degrees of freedom depends on the configuration structure. This analysis serves also to illustrate extended Hilbert space ideas coming from research on local subsystems in gauge and gravity theories by seeing it as a clever way to purify a state. On the side of the correlations, their non triviality come from the fact that some loops are distinct from the other. What’s more, we showed that correlations grow as the number of minimal path joining the two spins is larger.

The idea behind those kind of investigations is to have a clearer understanding of the physical states of quantum gravity solving, ideally, all the constraints of the theory. From there we could infer the structure of the Hamiltonian constraints they are solution of pointing toward the structure of the true quantum Hamiltonian constraint. Indeed, constructing a good Hamiltonian constraint is still under active research and we expect those retro-engineering studies to open new perspectives.

At the end of the day, the goal would be to weave the standard loop quantum gravity techniques for designing quantum states of geometry by the action of holonomy operators and volume excitations with the MERA vision of local unitaries and (dis-)entangling operations, in order to understand the structure of (local) holographic states in (loop) quantum gravity.
Appendix A: A first naive approach

We start our analysis of correlations on the simplest state on an oriented finite size regular square lattice. This state $|\psi\rangle$ called in the following loop state is defined through its wave-function in the following way:

$$
\psi(g_e) = \prod_p \chi_1/2 \left( \prod_{e \in p} g_e \right) \chi_{1/2} \left( \prod_{e \in \partial} g_e \right)
$$

(A1)
This state is constructed with spin 1/2 holonomies of each loop which are \( n_p \) in number in the bulk and a boundary term. Each spin of the bulk, living on a link of the graph, belongs to two holonomies. Only spin 0 and 1 are thus allowed in the bulk. On the contrary, a boundary spin belongs to only one holonomy and thus can only be a spin 1/2. Up to now, this state is unnormalized. Its norm is \( \langle \psi | \psi \rangle = 1 + \frac{1}{3^p} \). As the number of loops tends to infinity, the state becomes normalized.

Let us now study the correlations structure of this state. First of all, we study the holonomy correlations. The average value of the holonomy \( \chi_j (g_p) \) over the loop state is

\[
\langle \chi_j (g_p) \rangle = \frac{\delta_{j2} \frac{1}{3^p} + \delta_{j1} (1 + \frac{2}{3^p}) + \delta_{j0} (1 + \frac{1}{3^p})}{\langle \psi | \psi \rangle} \rightarrow_{n_p \rightarrow + \infty} \delta_{j0} + \delta_{j1}
\]

As the number of loops tends to infinity, the average value of the holonomy is one for spin 1 and 0, zero for any other value which is quite pleasing following the structure of the state. For the correlations, we have to distinguish between adjacent loops and distant loops. The large \( n_p \) limit reads

\[
\langle \chi_j (g_p) \chi_k (g_{p'}) \rangle = \begin{cases} \langle \chi_j (g_p) \rangle \langle \chi_k (g_{p'}) \rangle & \text{if } (p, p') \\ \frac{1}{2} (\delta_{j0} + \delta_{j1}) (2 \delta_{k0} + \delta_{k1}) & \text{if not} \end{cases}
\]  

(A3)

Holonomies of distant loops are thus completely decorrelated. For adjacent loops, the two points function is slightly modified. The loop state appears to be completely topological with respect to holonomies.

We proceed by the analysis of the spin correlation function. First of all, we precise the action of the spin operator of a given link \( e \) on a given wave function \( \psi (g) \). Thanks to the Peter-Weyl theorem, we have, denoting by \( d_{jE} = 2j_E + 1 \),

\[
\left( \hat{j}_e \psi \right) (g, g_e) = \sum_{j_e} d_{j_e} \int \chi_{j_e} (g_e h^{-1}) \psi (g, h) dh
\]

(A4)

For the particular state we are considering, only the spin 0 and 1 contribute. For the average value of the spin on a given link \( e \), only the spin 1 part gives a non zero contribution. In the large \( n_p \) limit, we have

\[
\langle \psi | \hat{j}_e | \psi \rangle = \frac{3}{4}
\]

(A5)

This result can be understood quite easily as the probability to have a spin 1 on the link \( e \), being the ratio between the dimension of the spin 1 Hilbert space and the dimension of the total Hilbert space comprising spin 0 and 1. As for the two points function, we are exactly in the same situation as we were for the holonomies. We have

\[
\langle \psi | \hat{j}_e \hat{j}_{e'} | \psi \rangle = \frac{3}{4} \langle \psi | \hat{j}_e | \psi \rangle \langle \psi | \hat{j}_{e'} | \psi \rangle
\]

(A6)

except for spins belonging to the same loop in which case the correlations are a simple constant for every situations. Once again the conclusion is that the loop state state is topological.

Thus, such a simple state is not a good candidate for our purpose. The problem seems to come from the fact that only spins belonging to the same loop see each other. This suggests the idea to construct a state with more extended loops and a more complex structure. Having also in mind that a state of the gravitational field should satisfy the holographic principle, a particular model used in condensed matter and quantum information, the toric code model, will allow us to shed some lights on a possible good loop structure.

Appendix B: Proofs

1. Entanglement in toric code model

**Proposition:** The entanglement entropy associated to a given region \( S \) whose (contractible) frontier possesses \( n_{SE} \) degrees of freedom in the ground state is

\[
S = n_{SE} - 1
\]
The reduced density matrix of $S$ is obtained by tracing out the $E$ degrees of freedom. The most transparent way to do so is to really distinguish the loops belonging to $S$, $E$ or both, as illustrated on fig. 7. The respective sets are denoted $C_S$, $C_E$ and $C_{SE}$ respectively. Then

$$\rho_S = \sum_{C, C'} \otimes_{\gamma \in C'} |1_{e \in S}, 0_{e \notin S}\rangle \langle 1_{e' \in S}, 0_{e' \notin S}| \prod_{|e \in E, 0_{e \notin E}|} |1_{e' \in E}, 0_{e' \notin E}|$$

$$= \frac{4}{2^{n_p+1}} \left( \sum_{C_E} 1 \right) \sum_{C_S, C'_{SE}} \otimes_{\gamma \in C_S \cup C_{SE}} |1_{e \in \gamma}, 0_{e \notin \gamma}\rangle \langle 1_{e \in \gamma}, 0_{e \notin \gamma}|$$

$$= \frac{2^{n_E}}{2^{n_p-1}} \sum_{C_S \cup C_{SE}} \otimes_{\gamma \in C_S \cup C_{SE}} |1_{e \in \gamma}, 0_{e \notin \gamma}\rangle \langle 1_{e \in \gamma}, 0_{e \notin \gamma}|$$

(B1)

The sets $(C_S, C_E, C_{SE})$ are defined in the core of the paper. We first check that the trace of this density matrix is equal to one.

$$\text{tr} \rho_S = \frac{2^{n_E}}{2^{n_p-1}} \left( \sum_{C_S, C_{SE}} 1 \right)$$

(B2)

The counting is done by choosing $k$ plaquettes among $S$ or the boundary without choosing choosing the whole set of plaquette of the boundary (such a choice is equivalent to choosing loops that belong only to $S$ and/or $E$). Thus $\sum_{k} \binom{n_p+n_{SE}-1}{k} = 2^{n_p+n_{SE}-1}$. We this wa conclude that the reduced density matrix trace to unity.

To determine the entropy, the method is to look at the squared density matrix. Explicitly we have

$$\rho_S^2 = \left( \frac{2^{n_E}}{2^{n_p-1}} \right)^2 \sum_{C_S \cup C_{SE}} \sum_{C'_S \cup C'_{SE}} \otimes_{\gamma \in C_S \cup C_{SE}} |1_{e \in \gamma}, 0_{e \notin \gamma}\rangle \langle 1_{e \in \gamma}, 0_{e \notin \gamma}| \prod_{\gamma' \in C'_S \cup C'_{SE}} \langle \gamma' \in C'_S \cup C'_{SE} | \gamma' \in C_S \cup C_{SE} \rangle$$

$$= \left( \frac{2^{n_E}}{2^{n_p-1}} \right)^2 \left( \sum_{C_S \cup C_{SE}} \sum_{C'_S \cup C'_{SE}} \otimes_{\gamma \in C_S \cup C_{SE}} |1_{e \in \gamma}, 0_{e \notin \gamma}\rangle \langle 1_{e \in \gamma}, 0_{e \notin \gamma}| \prod_{\gamma' \in C'_S \cup C'_{SE}} \langle \gamma' \in C'_S \cup C'_{SE} | \gamma' \in C_S \cup C_{SE} \rangle \right)$$

$$= \left( \frac{2^{n_E}}{2^{n_p-1}} \right)^2 \left( \sum_{C_S \cup C_{SE}} \sum_{C'_S \cup C'_{SE}} \otimes_{\gamma \in C_S \cup C_{SE}} |1_{e \in \gamma}, 0_{e \notin \gamma}\rangle \langle 1_{e \in \gamma}, 0_{e \notin \gamma}| \prod_{\gamma' \in C'_S \cup C'_{SE}} \langle \gamma' \in C'_S \cup C'_{SE} | \gamma' \in C_S \cup C_{SE} \rangle \right)$$

$$\delta(C'_S-C_S)\delta(C_{SE}-C_{SE})$$

$$= \frac{2^{n_E+n_S}}{2^{n_p-1}} \rho_S = \frac{1}{2^{n_p-1}} \rho_S$$

(B3)

2. Entanglement Kitaev state

Proof. We consider a bipartite partition $S$, $E$ of the lattice and we want to evaluate the entanglement entropy between those two regions. We’ll make the assumption that the boundary of $S$ is given by contractible loop on the dual lattice for more simplicity.
The full density matrix reads

$$\rho_{SE} = \frac{1}{N(\alpha)} \sum_{C, C'} \pi^{A(C')} \alpha^{A(C)} \prod_{\mathcal{L} \in C, \mathcal{L}' \in C'} \chi_{1/2}(\mathcal{L}) \chi_{1/2}(\mathcal{L}')$$  \hspace{1cm} (B4)$$

with $N(\alpha)$ the (squared) norm of the state. Once again, we distinguish the loops belonging to $S$, $E$ or both, writing respectively $C_S$, $C_E$ and $C_{SE}$. The reduced density matrix for $S$ is obtained by tracing out the $E$ degrees of freedom. So

$$\rho_S = \frac{1}{N(\alpha)} \sum_{C_S \cup C_{SE}} \pi^{A(C_S)} \alpha^{A(C_E)} \prod_{\mathcal{L}_S, \mathcal{L}'_S} \chi_{1/2}(\mathcal{L}_S) \chi_{1/2}(\mathcal{L}'_S) \int \prod_{\mathcal{L}_E, \mathcal{L}'_E} \chi_{1/2}(\mathcal{L}_E) \chi_{1/2}(\mathcal{L}'_E) \, dg_{E}$$

$$\times \int \prod_{\mathcal{L}_{SE}, \mathcal{L}'_{SE}} \chi_{1/2}(\mathcal{L}_{SE}) \chi_{1/2}(\mathcal{L}'_{SE}) \, dg_{E}$$

\hspace{1cm} (B5)$$

This last integral is the one that creates entanglement between the bulk and the exterior region. The integration over $E$ gives rise to loops crossing the boundary of the region. The first integral can be done straightforwardly and along with the area contribution gives an overall factor from the environment $N_E(\alpha)$. We then have the final form of the reduced density matrix

$$\rho_S(g, g') = \frac{N_E(\alpha)}{N(\alpha)} \sum_{C_S \cup C_{SE}} \pi^{A(C_S) \cup A(C_{SE})} \alpha^{A(C_S) \cup A(C_{SE})} \prod_{\mathcal{L}_S \in C_S, \mathcal{L}_S' \in C_S} \chi_{1/2}(\mathcal{L}_S(g)) \chi_{1/2}(\mathcal{L}'_S(g')) \prod_{\mathcal{L}_{SE} \in C_{SE}} \left( \frac{1}{2} \chi_{1/2}(\mathcal{L}_{SE}(g, g')) \right)$$

\hspace{1cm} (B6)$$

To be more precise, $C_{SE}$ is the set of loops in $S$ passing on (an even number of) punctures of the boundary. From the very construction of the state and the trace procedure the sets satisfy $C_S \cap C_{SE} = \emptyset$ and $C_S' \cap C_{SE} = \emptyset$. First of all, we can check that its trace is equal to one

$$\text{tr} \rho_S = \frac{N_E(\alpha)}{N(\alpha)} \sum_{C_S \cup C_{SE}} \pi^{A(C_S) \cup A(C_{SE})} \alpha^{A(C_S) \cup A(C_{SE})} \prod_{\mathcal{L}_S \in C_S, \mathcal{L}_S' \in C_S} \delta(\mathcal{L}_S - \mathcal{L}'_S) \prod_{\mathcal{L}_{SE} \in C_{SE}} \left( \frac{1}{2} \right)$$

$$= \frac{N_E(\alpha)}{N(\alpha)} \sum_{C_S, C_{SE}} |\alpha|^{2A(C_S)+2A(C_{SE})} \frac{n_S + n_{SE}}{k} = 1$$

\hspace{1cm} (B7)$$

In order to calculate the entropy, we use the replica trick. We need then to evaluate $\text{tr} \rho_S^2$. Let’s look at the square of the reduced density matrix first,

$$\rho_S^2(g, g') = \left( \frac{N_E(\alpha)}{N(\alpha)} \right)^2 \sum_{C_S \cup C_{SE}, C_S' \cup C_{SE}', \tilde{C}_S \cup \tilde{C}_{SE}, \tilde{C}_S' \cup \tilde{C}_{SE}'} \pi^{A(C_S' \cup C_{SE}')} \alpha^{A(C_S \cup C_{SE})} \alpha^{A(\tilde{C}_S \cup \tilde{C}_{SE})} \alpha^{A(\tilde{C}_S' \cup \tilde{C}_{SE}')}$$

$$\times \prod_{\mathcal{L}_S} \chi_{1/2}(\mathcal{L}_S(g)) \left[ \int \chi_{1/2}(\mathcal{L}'_S(h)) \chi_{1/2}(\tilde{\mathcal{L}}_S(h)) \, dh \right] \chi_{1/2}(\tilde{\mathcal{L}}'_S(g'))$$

$$\times \prod_{\mathcal{L}_{SE}} \left[ \frac{1}{2} \int \chi_{1/2}(\mathcal{L}_{SE}(g, h)) \chi_{1/2}(\mathcal{L}_{SE}(h, g')) \, dh \right]$$

$$= \left( \frac{N_E(\alpha)}{N(\alpha)} \right)^2 \mathcal{N}_S(\alpha) \sum_{C_S \cup C_{SE}} \pi^{A(C_S) \cup A(C_{SE})} \alpha^{A(C_S) \cup A(C_{SE})} \prod_{\mathcal{L}_S \in C_S, \mathcal{L}_S' \in C_S} \chi_{1/2}(\mathcal{L}_S(g)) \chi_{1/2}(\mathcal{L}'_S(g'))$$

$$\times \prod_{\mathcal{L}_{SE} \in C_{SE}} \left( \frac{1}{2} \chi_{1/2}(\mathcal{L}_{SE}(g, g')) \right)$$

\hspace{1cm} (B8)$$
By recursion we then obtain simply
\[
\rho_S^n(g, g') = \left( \frac{N_E(\alpha)}{N(\alpha)} \right)^n \mathcal{N}_S^{-1}(\alpha) \sum_{\mathcal{C}_S \cup \mathcal{C}_{SE}} \alpha^{A(\mathcal{C}_S \cup \mathcal{C}_{SE})} |\alpha|^{2(n-1)A(\mathcal{C}_{SE})} \prod_{\mathcal{L}_S \in \mathcal{C}_S} \chi_{1/2}(\mathcal{L}_S(g)) \chi_{1/2}(\mathcal{L}_S'(g')) \times \prod_{\mathcal{L}_S \in \mathcal{C}_{SE}} \left( \frac{1}{2^{n-1}} \chi_{1/2}(\mathcal{L}_S(g, g')) \right)
\]
\[\text{(B9)}\]
\[
\text{tr } \rho^n_S(g, g') = \left( \frac{N_E(\alpha)}{N(\alpha)} \right)^n \mathcal{N}_S^{-1}(\alpha) \sum_{\mathcal{C}_S \cup \mathcal{C}_{SE}} \frac{\alpha^{A(\mathcal{C}_S \cup \mathcal{C}_{SE})} |\alpha|^{2n}}{(4^{n-1})^{\# \mathcal{L}_{SE}}} \]
\[\text{(B10)}\]
Now using the replica trick 
\[
S_\alpha = n_{SE} f(|\alpha|^2) + \frac{2 \ln 2}{(1 + |\alpha|^2)^n_{SE}} \sum_{\mathcal{C}_{SE}} \# \mathcal{L}_{SE} |\alpha|^{2A(\mathcal{C}_{SE})}
\]
\[\text{(B11)}\]
with 
\[
f(|\alpha|^2) = \ln (1 + |\alpha|^2) - \frac{|\alpha|^2}{1 + |\alpha|^2} \ln (|\alpha|^2).
\]
