Abstract. We consider one-dimensional excited random walks (ERWs) with i.i.d. markovian cookie stacks in the non-boundary recurrent regime. We prove that under diffusive scaling such an ERW converges in the standard Skorokhod topology to a multiple of Brownian motion perturbed at its extrema (BMPE). All parameters of the limiting process are given explicitly in terms of those of the cookie markov chain at a single site. While our results extend the results in [DK12] (ERWs with boundedly many cookies per site) and [KP16] (ERWs with periodic cookie stacks), the approach taken is very different and involves coarse graining of both the ERW and the random environment changed by the walk. Through a careful analysis of the environment left by the walk after each “mesoscopic” step, we are able to construct a coupling of the ERW at this “mesoscopic” scale with a suitable discretization of the limiting BMPE. The analysis is based on generalized Ray-Knight theorems for the directed edge local times of the ERW stopped at certain stopping times and evolving in both the original random cookie environment and (which is much more challenging) in the environment created by the walk after each “mesoscopic” step.

1. Introduction and the main result

1.1. Introduction. Over the past several decades, a number of different one-dimensional self-interacting random walks have been studied through what may be called a “Ray-Knight” approach. It was observed that for these walks the joint distributions of edge local times have the structure of a Markov chain, and by analyzing this Markov chain one is able to obtain information about the original self-interacting random walk. Examples of this approach are numerous and include [KKS75, Tot94, Tot95, Tot96, BS08a, BS08b, TV08, KZ08, Pin10, KM11, DK12, Pet12, KZ13, KZ14, MPV14, DK15, KOS16, CdHPP16, KP17, PT17, HLSH18, Tra18].

We refer to this line of thought as a “Ray-Knight” approach in reference to the Ray-Knight theorems for Brownian motion which give a description of the local time profiles of a standard Brownian motion stopped when the local time at a fixed site exceeds a fixed level. The Ray-Knight theorems describe these local time profiles (viewed as processes in the spatial coordinate) as a gluing together of certain diffusion processes. In fact, for several models of self-interacting random walks one can prove that the Markov chains which correspond to the directed (or undirected) edge local times of the walk have scaling limits which are diffusion processes. This was first noticed by Tóth in [Tot94, Tot95, Tot96] and, more recently, found to be true for other models, [KZ08, KM11, KP17, PT17]. Yet the goal had now become different, namely, to study properties of the original process from information about its local times and not the other way around as in the classical Ray-Knight theorems.

Regarding scaling limits of self-interacting random walks, the Ray-Knight approach is easier to use when the process is transient, i.e. when with probability one it goes to $+\infty$ (or $-\infty$) as the time tends to infinity, see [KKS75, BS08b, KZ08, KM11, KP17, PT17, Tra18]. This is because the Ray-Knight information on local times can be readily used to deduce limiting distributions for
the hitting times of the random walk, and if the walk is transient to the right then by inverting
the role of time and space one can deduce a limiting distribution for the running maximum of the
walk. If one can also control the distance between the walk and its running maximum, then one
obtains a limiting distribution for the walk. On the other hand, proving the existence of a scaling
limit through the Ray-Knight approach when the walk is recurrent (in the sense that it returns to
the starting point infinitely often) is a more delicate task. In the aforementioned series of papers,
Tóth introduced generalized Ray-Knight theorems and showed how to exploit them to show the
convergence in distribution of the endpoint of a class of rescaled “recurrent” self-interacting random
walks along a sequence of random geometric times independent of the walk. For one particular
model, Mountford, Pimentel and Valle [MPV14] were able to obtain additional estimates that
allowed them to prove the convergence of one dimensional distributions of the walk with Tóth’s
method. Even in this case, however, characterization of multi-dimensional limiting distributions
using this “roadmap” seems out of reach.

In this paper, we show how a Ray-Knight approach can be used for a particular self-interacting
random walk model (excited random walks with markovian cookie stacks) to prove not just the
convergence of finite dimensional distributions but a full functional limit theorem. Our method
is completely different from that of Tóth in that instead of “inverting” the Ray-Knight theorems
to get information on the distribution of the endpoint of the walk, we use information from the
Ray-Knight-type results to construct a coupling of the walk with the conjectured scaling limit (a
Brownian motion perturbed at its extrema). It is also completely different from methods used in
[DK12, DK15, KPT16, HLSH18] for variants of this model where the random walk was decomposed in
a natural way into two parts, a martingale and an accumulated drift, each of which contributed the
considering part of a similar decomposition of the limiting process. We refer to [KPT16 Section
5] for a discussion as to why the same kind of decomposition cannot work for the general model
considered in the current paper. The main approach in this paper is robust in the sense that it
could, in theory, be applied to other self-interacting random walks as long as one can prove the type
of Ray-Knight theorems for the walk that are needed. Since there are a number of self-interacting
random walks for which similar (but weaker) Ray-Knight theorems have been proved but for which
full limiting distributions have not yet been obtained (e.g., [Tóth96, T18]), it may be possible to
adapt our techniques to get functional limit theorems for these random walks as well.

1.2. Excited random walks with markovian cookie stacks. Excited random walks (ERW),
sometimes also called cookie random walks, are a model of self-interacting random walks where the
transition probabilities of the walk depend on the local time of the walk at the present site. This
model was first introduced by Benjamini and Wilson in [BW03] where the transition probabilities
were only different on the first visit to a site (only a single excitation at each site). The model was
then generalized in [Zer05] and [KZ08] to include multiple excitations at each site and to allow for
randomness in the excitation environment.

For one-dimensional ERW, the model is described as follows. A cookie environment is an element
\( \omega = \{ \omega_x(j) \}_{x \in \mathbb{Z}, j \geq 1} \in (0,1)^{\mathbb{Z} \times \mathbb{N}} \). Given a fixed cookie environment \( \omega \) we can then construct a
random walk \( \{ X_n \}_{n \geq 0} \) as follows. The walk starts at \( X_0 = 0 \) and then when at the site \( x \) for the
\( j \)-th time steps to the right with probability \( \omega_x(j) \) or to the left with probability \( 1 - \omega_x(j) \). That
is, letting \( P_\omega \) denote the law of the process in the cookie environment \( \omega \) we have
\[
P_\omega(X_{n+1} = X_n + 1 \mid X_0, X_1, \ldots, X_n) = \omega_{X_n} \left( \sum_{i=0}^{n} 1_{\{ X_i = X_n \}} \right).
\]
The distribution \( P_\omega \) of the walk in a fixed environment is called the quenched law. We will assume
that the cookie environment \( \omega \) is chosen randomly according to some distribution \( \mathbb{P} \) on cookie
environments so that the annealed law of the walk \( P \) is defined by averaging the quenched law with
respect to \( \mathbb{P} \). That is \( P(\cdot) = \mathbb{E}[P_\omega(\cdot)] \).
The “cookie” terminology for these walks dates back to [Zer05] and comes from the following interpretation of the walk. Each site has a (possibly infinite) stack of cookies initially at that site. The random walker then always eats the top remaining cookie at his current location; the cookie induces some excitation/drift to the walker which determines the law of his next step. If there is a finite $M < \infty$ for which $\omega_x(j) = 1/2$ for all $x \in \mathbb{Z}$ and $j > M$ then we say that there are only $M$ cookies per site and the walker takes steps which are equally likely to the right or left when at a site where all the cookies are already eaten. With this cookie terminology we will refer to $\omega_x(j)$ as the $j$-th cookie at site $x$ and $\omega_x = \{\omega_x(j)\}_{j \geq 1}$ as the cookie stack at site $x$.

To give some additional structure to the model we need to describe the distribution of the cookie environment $\mathbb{P}$. We will assume that a cookie stack at each site is generated by an independent copy of a finite state Markov chain.

**Assumption 1.** There is a function $p : \{1, 2, \ldots, N\} \rightarrow (0, 1)$ such that $\omega_x(j) = p(R_j^x)$, $j \in \mathbb{N}$, where $\{R_j^x\}_{j \geq 1}$, $x \in \mathbb{Z}$, are i.i.d. Markov chains on $\{1, 2, \ldots, N\}$ with transition matrix $K$ and initial distribution $\eta$. The Markov chain $\{R_j^x\}_{j \geq 1}$ has a unique stationary distribution $\mu$ and $\bar{p} := \sum_{i=1}^N \mu(i)p(i) = \frac{1}{2}$.

The assumption of markovian cookie stacks was first made in [KP17] where it was shown that a number of asymptotic behaviors of the walk (such as recurrence/transience, ballistic behavior, and limiting distributions for the transient cases) can be explicitly characterized. If the condition $\bar{p} = 1/2$ is dropped, then clearly the random walk should have some asymptotic drift to the right/left. In fact, in [KP17] it was shown that if $\bar{p} \neq 1/2$ then the walk has a non-zero limiting speed and satisfies a CLT for a limiting distribution under the annealed measure $P$. However, if $\bar{p} = 1/2$ then the behavior can be much more varied. For instance, the walk can be either recurrent or transient depending (in a complicated but explicit way) on the parameters of the model.

**Theorem 1.1** ([KP17]). There exist two parameters $\theta^+$ and $\theta^-$ which characterize the recurrence/transience of the excited random walk as follows.

1. If $\theta^+ > 1$ then $P(\lim_{n \to \infty} X_n = +\infty) = 1$.
2. If $\theta^- > 1$ then $P(\lim_{n \to \infty} X_n = -\infty) = 1$.
3. If $\max\{\theta^+, \theta^-\} \leq 1$ then $P(\lim\inf_{n \to \infty} X_n = -\infty, \lim\sup_{n \to \infty} X_n = +\infty) = 1$.

**Remark 1.2.** It was shown in [KP17] that the parameter $\theta^+$ can be written as an explicit function $\theta^+ = \Theta(\eta, K, p(\cdot))$ of the parameters $\eta$, $K$ and $p(\cdot)$. Moreover, $\theta^- = \Theta(\eta, K, 1 - p(\cdot))$ is given by the same function but with $p(\cdot)$ replaced by $1 - p(\cdot)$. In the present paper, the parameters $K$ and $p(\cdot)$ will always be fixed, but we will at times be interested in cookie environments with different initial cookie distributions. Thus, for any distribution $\eta'$ on $\{1, 2, \ldots, N\}$ we will write $\theta^+(\eta')$ and $\theta^-(\eta')$ for $\Theta(\eta', K, p(\cdot))$ and $\Theta(\eta', K, 1 - p(\cdot))$, respectively. In the special case where $\eta' = \eta$ from Assumption [1] we will just write $\theta^\pm$ instead of $\theta^\pm(\eta)$.

**Remark 1.3.** It can be shown from the explicit formulas for $\theta^\pm$ (see Section [2.1]) that $\theta^+ + \theta^- < 1$ so that Theorem [1.1] gives a complete characterization of recurrence and transience for excited random walks with markovian cookie stacks [KP17] Section 4].

**Remark 1.4** ($M$ cookies per stack). In a particular case when the Markov chain $\{R_j^x\}_{j \geq 1}$ has an absorbing state $a \in \{1, 2, \ldots, N\}$ (which is unique by Assumption [1]) with $p(a) = 1/2$ and reaches it by the $M$-th step with probability 1, that is when

1. $P(\omega_x(j) = 1/2, \forall j > M) = 1$, the formulas for $\theta^\pm$ have a particularly simple form, namely, $\theta^+ = -\theta^- = \delta$ where

$$
\delta = \sum_{j=1}^M \mathbb{E}[2\omega_0(j) - 1].
$$
For additional examples we refer to [KPI7, Section 1.4].

In addition to the criteria for recurrence/transience stated in Theorem 1.1, the paper [KPI7] also contains characterizations of ballisticity (non-zero limiting linear speed) and limit laws in the transient cases. These results generalized some of those that had been proved earlier in [Zer05, BS05a, BS05b, KZ08, KMT08] for ERWs with \( M \) cookies per stack. A notable omission, however, was the limiting behavior in the recurrent case when \( \max\{\theta^+, \theta^-\} < 1 \). This is the focus of the present paper.

1.3. Main results. In the case when there are \( M \) cookies per stack and cookies stacks are i.i.d., functional limit theorems for recurrent ERW were first obtained by Dolgopyat, [Dol11], and Dolgopyat and Kosygina, [DK12]. Before stating their and our results we need the following definition.

**Definition 1.5.** For any \( \alpha, \beta < 1 \), a Brownian motion \((\alpha, \beta)\)-perturbed at its extrema \(((\alpha, \beta)\text{-BMPE})\) is a process \( \{W(t)\}_{t \geq 0} \) started at \( W(0) = 0 \), continuous in \( t \), and solving the functional equation

\[
W(t) = B(t) + \alpha \sup_{s \leq t} W(s) + \beta \inf_{s \leq t} W(s),
\]

where here and throughout the paper \( \{B(t)\}_{t \geq 0} \) is a standard one-dimensional Brownian motion.

While it is not obvious that the functional equation (3) has a solution, it was shown in [PW97, CD99] that for all \( \alpha, \beta < 1 \) there is a pathwise unique continuous solution and it is adapted to the filtration of \( B \). In the special case when \( \alpha = 0 \) or \( \beta = 0 \) the solution can be made explicit. For instance, if \( \beta = 0 \) then as shown in [CPY98, p. 242]

\[
W(t) = B(t) + \frac{\alpha}{1 - \alpha} B^\ast(t), \quad \text{where } B^\ast(t) = \sup_{s \leq t} B(s).
\]

In the theorem below and throughout the remainder of the paper the symbol \( \xrightarrow{J_1} \) will denote convergence in distribution with respect to the Skorokhod \( J_1 \) topology.

**Theorem 1.6 ([DK12]).** Suppose that \( \omega_x, x \in \mathbb{Z} \), are i.i.d., \((1)\) holds, and \( \mathbb{P}(\omega_x(j) \in (0, 1) \forall j \in \{1, 2, \ldots, M\}) > 0 \). Let \( \{X_n\}_{n \geq 0} \) be an ERW in this cookie environment and \( \delta \) be given by \((2)\). Then the following statements hold with respect to the averaged measure \( \mathbb{P} \).

1. If \( \delta \in (-1, 1) \), then \( \left\{ \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right\}_{t \geq 0} \xrightarrow{J_1} \{W(t)\}_{t \geq 0} \), where \( W \) is a \((\delta, -\delta)\text{-BMPE})\.

2. If \( \delta \in \{-1, 1\} \), then there exists a constant \( a > 0 \) such that \( \left\{ \frac{AX_{\lfloor nt \rfloor}}{a \sqrt{n \log n}} \right\}_{t \geq 0} \xrightarrow{J_1} \{B^\ast(t)\}_{t \geq 0} \).

**Remark 1.7.** Note that the limit in the boundary cases \( \delta \in \{-1, 1\} \) is somewhat surprising since the ERW is recurrent but the scaling limit is transient. In the non-boundary cases, it is not hard to see that BMPE is a reasonable scaling limit. Indeed, since there are only \( M \)-cookies per site it is natural to expect that the scaling limit should be a process that behaves like a Brownian motion when not near the running minimum or maximum and experiences some sort of additional drift when at the minimum or maximum.

In this paper we show that the results of Theorem 1.6 can be extended to the case of markovian cookie stacks. Both theorems below hold with respect to the averaged measure \( \mathbb{P} \).

**Theorem 1.8.** If \( \max\{\theta^+, \theta^-\} < 1 \), then \( \left\{ \frac{X_{\lfloor nt \rfloor}}{a \sqrt{n \log n}} \right\}_{t \geq 0} \xrightarrow{J_1} \{W(t)\}_{t \geq 0} \) where \( W \) is a \((\theta^+, \theta^-)\text{-BMPE})\.

**Theorem 1.9.** If \( \theta^+ = 1 \) then there exists a constant \( a > 0 \) such that \( \left\{ \frac{X_{\lfloor nt \rfloor}}{a \sqrt{n \log n}} \right\}_{t \geq 0} \xrightarrow{J_1} \{B^\ast(t)\}_{t \geq 0} \). Similarly, if \( \theta^- = 1 \) then the above statement holds with \(-X_{\lfloor nt \rfloor}\) in place of \( X_{\lfloor nt \rfloor} \).
We have separated the statements of the scaling limits in the boundary and non-boundary cases because the proof techniques are completely different. In fact, the proof of the scaling limits for recurrent ERW in the boundary case ($\theta^+ = 1$ or $\theta^- = 1$) is exactly the same as that in [DK12] for the case of $M$ cookies per stack and depends only on certain tail estimates for the directed edge local time processes that have already been obtained for the case of Markov cookie stacks. See [KP16] p. 8 and [KP17] Theorem 2.7 for further details.

The proof of Theorem 1.8, on the other hand, is quite different from previous cases and thus is the focus of the remainder of the paper. As we have noted above, BMPE was already shown to be the scaling limit of ERW with $M$ cookies per stack, but there have also been a few other self-interacting random walks which have been shown to converge to BMPE. We list all cases we are aware of below.

1. Random walk with partial reflection at extrema [Dav96]. In this walk the random walk has a drift when at its running maximum/minimum and jumps to the left/right with equal probability otherwise. This walk is clearly a discrete analog of the BMPE.
2. ERW with $M$ cookies per stack with $\delta \in (-1, 1)$. As noted above this was proved in [DK12].
3. ERW with periodic cookie stacks with $\max\{\theta^+, \theta^-\} < 1$. This special case of Theorem 1.8 was proved in [KP16].
4. Broken rotor walk [HLSH18]. This walk, though not described as such in the original paper, can be seen as an ERW with Markovian cookie stacks where the Markov chain is a two state Markov chain with transition matrix $K = \begin{pmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{pmatrix}$ and where the cookie values are degenerate in that $p(1) = 1$ and $p(0) = 0$ (that is, the behavior of the walk is deterministic given the realization of the cookie environment).

In all of these previous papers, the proof followed the same general strategy. First, one proves that the random walk can be approximated by a martingale plus a linear combination of the running maximum and minimum of the walk. Next, one proves that the martingale term in this approximation converges to Brownian motion under diffusive scaling. Finally, one proves tightness for the random walk process under diffusive scaling and from this concludes that any scaling limit must satisfy a functional equation like (3) in the definition of BMPE. This strategy does not seem to work for the current model, at least not without involving an intermediate scale and an additional control on the environment. As mentioned in the introduction, a more detailed discussion of the problems arising when implementing this approach can be found in [KP16] Section 5.

1.4. Ideas of the proof. The main idea of our proof is to use information on the local time processes to determine the movement of the ERW on a macroscopic scale. For a BMPE this is understood through the Ray-Knight type theorems proved in [CPY98]. For a $(\theta^+, \theta^-)$-BMPE $W$ let \{\ell_{x,t}^W\}_{x \in \mathbb{R}, t \geq 0}$ be the local time process of $W$, and if $\tau_x^W = \inf\{t \geq 0 : W(t) = x\}$ is the hitting time of $x \in \mathbb{R}$ then it was shown in [CPY98] Theorem 3.4] that \{\ell_{x,\tau_x^W}^W\}_{x \geq 1}$ is a gluing together of two Bessel squared processes; that is, \{\ell_{x,\tau_x^W}^W\}_{x \in [-1, 0]} is a Bessel squared process of dimension $2(1 - \theta^-)$ started at 0 and \{\ell_{x,\tau_x^W}^W\}_{x \geq 0}$ is a Bessel squared process of dimension $2\theta^+$ which is killed when reaching zero. See Figure 1. From this Ray-Knight theorem for BMPE we can deduce some information about macroscopic behavior of $W$. For instance, the event that $W$ exits the interval $(-1, 1)$ to the left is equal to the event that the local time process \{\ell_{x,\tau_x^W}^W\}_{x \geq 1}$ dies out somewhere in $(0, 1)$. Moreover, when this event happens, the location where the local time process dies out is equal to the running maximum of $W$ by time $\tau_{-1}$ and the area under the curve of $x \mapsto \ell_{x,\tau_x^W}^W$ is equal to the time for $W$ to exit the interval $(-1, 1)$. A similar analysis of the local time profile at time $\tau_1^W$ can be used to determine the distribution of the exit time and the running minimum of $W$ when the process exits $(-1, 1)$ to the right.
The above explains how one can describe the initial macroscopic behavior of a BMPE using the Ray-Knight theorems for BMPE. However, understanding the macroscopic behavior of the BMPE at later times is a little more complicated because the BMPE $W$ is not a Markov process. Nevertheless, if we define

$$I(t) = \inf_{s \leq t} W(s) \quad \text{and} \quad S(t) = \sup_{s \leq t} W(s), \quad t \geq 0,$$

to be the running minimum and maximum of $W$ respectively, then $\{(I(t), W(t), S(t))\}_{t \geq 0}$ is a Markov process. Suppose that at time $t$ we have $(I(t), W(t), S(t)) = (w + w, w, w + \bar{w})$ for some $w \in \mathbb{R}$ and $w \leq 0 \leq \bar{w}$ and we want to know the probability that $W$ will subsequently exit the interval $(w - 1, w + 1)$ to the left. By the Markov property and translation invariance of Brownian motion we can then consider the process started at $(I(0), W(0), S(0)) = (w, 0, \bar{w})$ (that is, started with artificial non-zero minimum and maximum) and use the local time profiles stopped at times $\tau^{-}_W$ or $\tau^{+}_W$ as before. However, in this case since the minimum and maximum are not initially zero the distributions of the local time profiles are different. In this case (see, for example, [CDH00 Proposition 2.1]) if we start from $(I(0), W(0), S(0)) = (w, 0, \bar{w})$ then $\{\ell^{W}_x, \tau^{-}_W\}_{x \geq -1}$ is a gluing together of (up to) 4 squared Bessel processes of (1) dimension $2(1 - \theta^{-})$ on the interval $[-1, w \vee -1]$, (2) dimension 2 on the interval $[w \vee -1, 0]$, (3) dimension 0 on the interval $[0, \bar{w}]$, and (4) dimension $2\theta^{+}$ on the interval $[\bar{w}, \infty)$. See Figure 1.

**Figure 1.** On the left is a graphical representation of the Ray-Knight theorem for a standard $(\theta^{+}, \theta^{-})$-BMPE stopped when the process first reaches $-1$. On the right is a graphical representation of the Ray-Knight theorem for a $(\theta^{+}, \theta^{-})$-BMPE started with initial condition $(w, 0, \bar{w}) = (-0.5, 0, 0.5)$ and stopped when the process first reaches $-1$.

One of the key results of the present work is a set of generalized Ray-Knight theorems for the ERW on a “mesoscopic” scale. More precisely, we first fix an $\varepsilon \in (0, 1)$ and define stopping times $\{T^{\varepsilon, n}_k\}_{k \geq 0}$ for the ERW by

$$T^{\varepsilon, n}_0 = 0, \quad T^{\varepsilon, n}_k = T^{\varepsilon, n}_{k, \varepsilon} \wedge T^{\varepsilon, n}_{k, -\varepsilon}, \quad \text{where} \quad T^{\varepsilon, n}_{k, \pm} = \inf\{j > T^{\varepsilon, n}_{k-1} : X_j - X_{T^{\varepsilon, n}_{k-1}} = \pm \varepsilon \sqrt{n}\}, \quad k \in \mathbb{N}.
$$

(We refer to $\sqrt{n}$ as the macroscopic scale for the ERW and $\varepsilon \sqrt{n}$ as the mesoscopic scale since we will later take $\varepsilon \to 0$.) First of all, we show that the local time profile of the ERW when it first reaches $[-\varepsilon \sqrt{n}, \varepsilon \sqrt{n}]$, converges when scaled by $[\varepsilon \sqrt{n}]$ to a concatenation of Bessel squared processes of generalized dimension $2(1 - \theta^{-})$ and $2\theta^{+}$ just as in the Ray-Knight Theorems for BMPE. This then allows us to couple the first step of the induced mesoscopic walk $X^{\varepsilon, n}_{T^{\varepsilon, n}_1}$ with the first macroscopic step of a BMPE.

Yet the most challenging and technical part of the paper is in an extension of this coupling via a Ray-Knight approach to subsequent steps of the induced mesoscopic walk. In order to do this, we
need rather strong control on the distribution of the remaining cookie environment at the stopping times $T^{ε,n}_k$. That is, while initially the distribution of first cookies was independent with marginal $η$ at each site, after the walk has run for a long time the distribution of the next cookie to be used at sites within the range of the walk is no longer $η$ and no longer necessarily independent for different sites. However, we are able to approximate the distribution of next cookies in a convenient way. There are two distributions $π^+$ and $π^−$, which we can explicitly identify (see Section 2.1 and [KP17, Lemma 3.2 and (37)]), such that the next cookie distribution is approximately i.i.d. $π^−$ between the running minimum and the current location, approximately i.i.d. $π^+$ between the current location and the running maximum, and i.i.d. $η$ outside of the range of the walk. Moreover, recalling that the parameters $θ^+ = θ^+(η)$ and $θ− = θ^−(η)$ depend on the initial distribution $η$ of first cookies and since it follows from [KP17, Corollary 3.5 and equation (38)] that $θ^+(π^−) = 0$ and $θ^−(π^−) = 0$, from this we are able to show that the local time process of the ERW after time $T^{ε,n}_{k,−}$ and up until time $T^{ε,n}_{k,+}$ can be approximated by a concatenation of Bessel squared processes of dimensions $2(1−θ−), 2, 0,$ and $2θ^+$ just as in the case of the BMPE shown on the right in Figure 1. A similar result can be obtained for the local time process between times $T^{ε,n}_{k,−}$ and $T^{ε,n}_{k,+}$.

Remark 1.10. One can, in fact, check using the definitions of the distributions $π^+$ and $π^−$ that the initial distribution of cookies that are independent and distributed according to $π^−$ on $[−∞, −1]∩Z$, $π^+$ on $[1,∞]∩Z$, and $1/2π^− + 1/2π^+ = µ$ at 0 is stationary for the cookie environment seen from the walker. That is, if this is the distribution of the initial first cookies then at any later time the remaining next cookies, shifted so that the current location of the random walk is taken to the origin, has the same distribution. We did not use this fact in our proof, nor are we able to even see how it could be used to prove convergence of the ERW to a BMPE. However, it may be possible to use Kipnis-Varadhan techniques to prove that the path of an ERW with this stationary initial configuration of cookies converges in distribution to a Brownian motion. Again, since $θ^+(π^+) = 0$ and $θ^−(π^−) = 0$ this is consistent with what would be expected from our main results since a $(0,0)$-BMPE is just a standard Brownian motion.

Remark 1.11. It is interesting to note, and somewhat surprising, that while there is an asymmetry in the cookie environment in the interior of the range of the walk (approximately distribution $π^−$ to the left and $π^+$ to the right) this asymmetry is not seen in the scaling limit which behaves like a Brownian motion in the interior of its range. We note, however, that the steps of the walk in the interior of the range are still highly correlated and this is reflected in the presence of the scaling parameter $a$ in the statement of Theorem 1.8 which in general is not equal to 1.

1.5. Outline of the paper. The paper is organized as follows. In Section 2 we define branching-like processes (BLPs) and recall from [KP17] their fundamental properties. These processes are essential to describe the behavior of local times of ERWs and to apply a Ray-Knight approach. Section 3 discusses some basic properties of BMPEs, including the Ray-Knight Theorems and couplings of BMPEs started from slightly different initial conditions. In Section 4 we construct various discretizations of BMPEs which will be used in Section 7 for coupling with our ERW.

Section 5 for the most part, discusses diffusion approximations for the local times which are needed to relate exiting probabilities of ERWs to those of BMPEs. It establishes “classical” results where the Markov chains that generate the cookie stacks initially have product distribution. This is then extended to the case when the initial values of the Markov chains are regular in a scale that is small compared to the macroscopic scale. Section 6 concerns the regularity of cookies environments in two ways. Firstly, we prove that throughout time scale of order $n$ the next states of the cookie Markov chains are to scale $n^{1/4}$ distributed like (in a crude averaging sense) $π^+$-product measure between the current position of the ERW and the current maximum and like $π^−$-product measure between the current minimum and the current position of the ERW. Secondly, we show that at each time $T^{ε,n}_k$ the distribution of the next states of the cookie Markov chains around points
(\{\varepsilon \sqrt{n} Z\} \setminus \{X_{T_k^{\varepsilon,n}}\}) is very close to appropriate product measures in a total variation sense. These two results permit us to argue that the past does not play too big a role in the future at macroscopic level.

Thereafter the paper works to implement the argument that \{n^{-1/2}X_{T_k^{\varepsilon,n}}\}_{k \geq 1} evolves like a discretized BMPE and that the times \(T_k^{\varepsilon,n}\) are well-controlled. More precisely, Section 4, drawing on diffusion approximations and the “environmental” results of Section 6, constructs a coupling of our ERW and discretized BMPE, while the final Section 8 establishes a law of large numbers for \(\{T_k^{\varepsilon,n}\}_{k \geq 1}\), which enables us to pass from the discretized process to the general renormalized process and complete the proof of Theorem 1.8.

The proofs of many results that are of a technical nature and that are easy to believe are placed in an appendix, since the reader may wish to omit them on a first pass.

1.6. Notation. For the convenience of the reader, we collect here some notation that will be used throughout the paper.

- We write \(x_+\) for \(x \lor 0\) and \(Z_+\) for \(\mathbb{N} \cup \{0\}\).
- For any \(a < b\) we will let \([a, b) = [a, b) \cap \mathbb{Z}\). Similarly, we will use \([a, \infty)\) for \([a, \infty) \cap \mathbb{Z}\).
- We write \(\|\mu_1 - \mu_2\|_{TV}\) for the total variation distance between two measures \(\mu_1\) and \(\mu_2\). For two random variables \(V\) and \(U\), \(d_{TV}(V, U)\) will denote the total variation distance between their distributions.
- We denote by \(P_\gamma\) the averaged probability measure when the first cookies are i.i.d. with marginal distribution \(\gamma\). We shall typically drop the subscript and write \(P\) instead of \(P_\eta\) if \(\gamma = \eta\), the original initial distribution of the first cookies.
- The local time of the ERW at \(x\) by time \(n\) is given by

\[
\mathcal{L}(0, x) = 0, \quad \mathcal{L}(n, x) = \sum_{j=0}^{n-1} \mathbb{1}_{\{X_j = x\}}, \quad n \in \mathbb{N}, \quad x \in \mathbb{Z}.
\]

- For a stochastic process \(Z = (Z_n)_{n \geq 0}\) and \(a \in \mathbb{R}\) we define the hitting times

\[
\tau_a^Z = \inf\{n \geq 0 : Z_n \geq a\}, \quad \sigma_a^Z = \inf\{n \geq 0 : Z_n \leq a\}, \quad \sigma_a^{Z_{m,a}} = \inf\{n \geq m : Z_n \leq a\},
\]

with \(\inf \emptyset = \infty\). For instance \(\sigma_{X_m}^\varepsilon \wedge \tau_{m}^\varepsilon\) will denote the exit time of the excited random walk from the interval \((-m, m)\). A similar definition will apply to hitting times of processes in continuous time. We shall occasionally drop the superscript whenever there is no ambiguity about which process we are talking about.

- With mild abuse of terminology we shall refer to \(\{R_1^x\}_{x \in \mathbb{Z}}\) as “the first cookies”. The expression “the first cookies at time \(\tau\)” will refer to the collection \(\{R_{\mathcal{L}(x,\tau)+1}^x\}_{x \in \mathbb{Z}}\) for a stopping time \(\tau\) and will denote the next states of the cookie Markov chains at time \(\tau\).

2. The branching-like processes (BLPs)

In this section we introduce four Markov chains \(U^+, U^-, V^+, \) and \(V^-\) taking values on \(\mathbb{Z}_+\) which are useful in analyzing excited random walks. We will refer to these Markov chains as the branching-like processes (BLPs) due to a similarity in structure to Galton-Watson branching process (or branching processes with migration). We will first describe the transition probabilities of the four BLPs and then give a brief description of their relation to the directed edge local times of excited random walks.

From this point on we will shift the meaning of cookie and cookie environments. Henceforth, the cookie at site \(x\) at time \(k\) will refer to \(R_{\mathcal{L}(x,k)+1}^x\). In particular, given time \(k\), the first cookie environment for the ERW will refer to the variables \(\{R_{\mathcal{L}(x,k)+1}^x\}_{x \in \mathbb{Z}}\). Of course, the distinction between the former usage of cookie and the present and future usage is moot if \(p : \{1, 2, \ldots, N\} \rightarrow (0, 1)\) is injective. We note that while the (present sense) cookie environment shifted by the current
location of the ERW, is always a Markov chain, in general the former cookie environment need not have this property. Given the first cookie at site $x$ at time $k$, the evolution of \( \{R^x_{\ell(x,n)+1}\}_{n \geq k} \) is independent of the past history of $X$ (again unlike the cookie evolution in the previous sense).

We will describe the distribution of the four BLPs given the distribution of the first cookies \( \{R^x_1\}_{x \in [\ell,r]} \) on an interval \([\ell, r]\) \(\subset \mathbb{Z}\). The distribution of the first cookies can either be deterministic or random with independence over the sites (e.g., \( \{R^x_1\}_{x \in \mathbb{Z}} \) can be i.i.d. with distribution $\eta$). Given the distribution of the first cookies on \([\ell, r]\), we can construct the BLP $U^+$ as follows. First, we generate the remainder of the environment $\omega_x(j) = p(R^x_j)$, $j \geq 1$, at each site $x \in [\ell, r]$ by letting \( \{R^x_j\}_{j \geq 1} \) be a realization of the Markov chain in Assumption 1 but with $R^x_1$ having the prescribed initial distribution. The realizations of the Markov chains at different sites are independent. Next, given the entire cookie environment on \([\ell, r]\), we let \( \{\xi^x_j\}_{x \in [\ell,r],j \geq 1} \) be a family of independent Bernoulli random variables with $\xi^x_j \sim \text{Ber}(\omega_x(j))$. Finally, we let the BLP $U^+$ started with initial value $U^+_0 = m \in \mathbb{Z}_+$ be defined as follows.

\[
U^+_0 = m, \quad U^+_i = \inf \left\{ k \geq 0 : \sum_{j=1}^{k+U^+_i-1} (1 - \xi^x_{j+i}) = U^+_{i-1} \right\} \quad \text{for } i \in \{1, 2, \ldots, r - \ell\}.
\]

That is, $U^+_i$ is the number of “successes” before the $U^+_{i-1}$th “failure” in the sequence of Bernoulli trials \( \{\xi^x_{i+j}\}_{j \geq 1} \). The BLP $V^+$ is defined similarly, but instead we have

\[
V^+_0 = m, \quad V^+_i = \inf \left\{ k \geq 0 : \sum_{j=1}^{k+V^+_i-1+1} (1 - \xi^x_{j+i}) = V^+_{i-1} + 1 \right\} \quad \text{for } i \in \{1, 2, \ldots, r - \ell\},
\]

so that $V^+_i$ is the number of successes before the $(V^+_{i-1} + 1)$-th failure in the sequence \( \{\xi^x_{j+i}\}_{j \geq 1} \). The BLPs $U^-$ and $V^-$ are constructed similarly but reversing the role of “successes” and “failures” and using the cookie stacks from right to left instead. That is, given the initial values of $U^+_0$ or $V^+_0$ we let

\[
U^-_i = \inf \left\{ k \geq 0 : \sum_{j=1}^{k+U^-_{i-1}} \xi^x_{j-i} = U^-_{i-1} \right\} \quad \text{for } i \in \{1, 2, \ldots, r - \ell\},
\]

and

\[
V^-_i = \inf \left\{ k \geq 0 : \sum_{j=1}^{k+V^-_{i-1}+1} \xi^x_{j-i} = V^-_{i-1} + 1 \right\} \quad \text{for } i \in \{1, 2, \ldots, r - \ell\}.
\]

Before giving the connection of the BLPs with the excited random walk, we first mention some properties of the BLPs that we will use throughout the paper.

1. Because the Markov chains \( \{R^x_j\}_{j \geq 1} \) are independent at different sites, it follows that all four of the BLPs $U^\pm$ and $V^\pm$ are Markov chains. For general first cookie conditions the BLPs are time inhomogeneous Markov chains with the transition probabilities at different times depending on the distribution of the first cookies at different sites, but if the initial distribution \( \{R^x_1\}_{x} \) is i.i.d. then the BLPs are time homogeneous Markov chain.

2. The processes $U^+$ and $U^-$ have 0 as an absorbing state. In contrast, $V^+$ and $V^-$ are irreducible Markov chains on $\mathbb{Z}_+$.

3. The BLPs all have a natural monotonicity property with respect to the initial condition. If $Z$ and $Z'$ are two instances of the same BLP started from the same first cookie environments but with different initial conditions $Z_0 = k < k' = Z'_0$, then our construction above provides
a coupling such that $Z_i \leq Z'_i$ for all $i$ (as long as both processes use the same Bernoulli random variables $\{\xi^i_k\}_{x,j}$).

We now explain the connection of the BLPs to the study of excited random walks. For any $\ell \geq 1$ and $x \geq -\ell$ let

$$E^{(x,-\ell)}_x = \sum_{i=0}^{\sigma X_{x,-\ell} - 1} 1 \{X_i = x, X_{i+1} = x+1\}$$

be the number of steps the ERW takes from $x$ to $x + 1$ prior to the first visit to $-\ell$. Then, it can be seen that the sequence $\{E^{(x,-\ell)}_x\}_{x \geq -\ell}$ has the same distribution as a concatenation of a $V^+$ and $U^+$ process. More precisely,

- $(E^{(x,-\ell)}_{x+\ell}, E^{(x,-\ell)}_{x+\ell+1}, \ldots, E^{(x,-\ell)}_{x-1}, E^{(x,-\ell)}_x)$ has the same distribution as $(V^+_0, V^+_1, \ldots, V^+_\ell)$ started with $V^+_0 = 0$ and using the cookie environment on the interval $[0, \ell]$.

- Given $E^{(x,-\ell)}_x = \sigma$, the sequence $(E^{(x,-\ell)}_{x-\ell}, E^{(x,-\ell)}_{x-\ell+1}, \ldots, E^{(x,-\ell)}_{x-1}, E^{(x,-\ell)}_x)$ has the same distribution as $(U^+_0, U^+_1, U^+_2, \ldots)$ started with $U^+_0 = \sigma$ and using the cookie environment on the interval $[0, \ell]$.

(See [KZ08] or [KP17] for more details.) Let $\{Z_i\}_{i \geq 0}$ denote the concatenation of the above $V^+$ and $U^+$ processes. This connection of the ERW with the BLPs allows us to restate an exit distribution problem for the ERW as a question about the process $Z$. Indeed, one sees that the random walk exits the interval $(-\ell, \ell)$ at $-\ell$ if and only if the process $\{E^{(x,-\ell)}_x\}_{x \geq \ell}$ dies out before $x = \ell$. Therefore, letting $\sigma Z_{\ell,0} = \inf\{i \geq \ell : Z_i = 0\}$ we have that

$$P(\sigma X_{x,-\ell} < \tau X_{\ell}) = P(\sigma Z_{\ell,0} < 2\ell).$$

Moreover, since $\sigma X_{x,-\ell} = \ell + 2 \sum_{x \geq -\ell} E^{(x,-\ell)}_x$, it follows that conditioned on the event $\{\sigma X_{x,-\ell} < \tau X_{\ell}\}$ the exit time $\sigma X_{x,-\ell} \land \tau X_{\ell}$ for the ERW has the same distribution as $\ell + 2 \sum_{i=0}^{\sigma Z_{\ell,0} - 1} Z_i$ conditioned on the event $\{\sigma Z_{\ell,0} < 2\ell\}$.

In this paper we will often be interested in similar exit distribution and exit time problems for the ERW but conditioned on some knowledge of the walk up to a certain time. For instance, suppose the random walk has already evolved for some amount of time $T$ (either a deterministic time or a stopping time for the walk) and that we know by this time the maximum and minimum are $S = \max_{k \leq T} X_k$ and $I = \min_{k \leq T} X_k$, the current position of the walk is $X_T = z \in [I, S]$, and we also know the values of the next cookies to be used at all sites $x \in [I, S]$ that have been visited thus far. Given all this information, we wish to know after time $T$ whether the walk will reach $z - \ell$ or $z + \ell$ first. This can be translated to a problem about concatenated BLPs as follows. Let $\{Z_k\}_{k=0}^{2\ell}$ be a concatenation of BLPs such that

- $(Z_0, Z_1, \ldots, Z_\ell)$ is a $V^+$ process started from $V^+_0 = 0$ and using the remaining first cookie environment on $[z, -\ell, z]$.

- Given $Z_\ell = \sigma$, the process $(Z_{\ell+1}, Z_{\ell+2}, \ldots, Z_{2\ell})$ is a $U^+$ process started from $U^+_0 = \sigma$ and using the remaining first cookie environment on $[z, z + \ell]$.

As above, questions about the exit distribution and exit time of the walk after time $T$ and until hitting $z - \ell$ or $z + \ell$ can be translated to questions about the concatenated BLP $Z$.

To illustrate how the BLPs $U^+$ and $V^+$ arise in connection with the study of ERW. The BLPs $U^-$ and $V^-$ arise in a somewhat similar manner. For instance, let $D^{(x,-\ell)}_x = \sum_{i=0}^{\tau X_{x,-\ell} - 1} 1 \{X_i = x, X_{i+1} = x+1\}$ be the number of steps from $x$ to $x - 1$ prior to the walk first reaching $\ell$. Then, one can see that the process $(D^{(x,-\ell)}_x, D^{(x,-\ell)}_{x-1}, \ldots, D^{(x,-\ell)}_0)$ has the same distribution as the concatenation of a $V^-$ process and

\footnote{Implicitly we are using here that under the assumptions of this paper the walk is recurrent. Thus, for $\mathbb{P}$-a.e. cookie environment $\omega$ we have that $\mathbb{P}_\omega(\sigma^X_{x,-\ell} < \infty) = 1$.}
a $U^-$ process. One can use this concatenated BLP process to study the probability the ERW exits an interval to the right and the distribution of the time it takes the walk to exit an interval on the event that it exits to the right.

Because arguments involving the BLPs $U^-$ and $V^-$ are symmetric to those involving $U^+$ and $V^+$, we will give all proofs only for the processes $U^+$ and $V^+$.

2.1. **Parameters associated to the BLPs.** The parameters $\theta^+$ and $\theta^-$ which appear in the statement of the main results are defined in terms of the BLPs. We close this section by giving the description of these parameters along with several other related parameters that will be used throughout the paper. While explicit formulas for all parameters discussed below can be found in [KP17, equation (37)], we restrict our attention here to the probabilistic definition of these parameters in terms of the BLPs.

Let $r^+ = (r^+(i))_{1 \leq i \leq N}$ and $r^- = (r^-(i))_{1 \leq i \leq N}$ be the vectors with entries

\[
(8) \quad r^+(i) = \lim_{n \to \infty} E[U^+_i - n | U^+_0 = n, R_1^+ = i], \quad \text{and} \quad r^-(i) = \lim_{n \to \infty} E[U^-_i - n | U^-_0 = n, R_1^- = i].
\]

That is, $r^\pm(i)$ gives the limit of the expected “drift” of the first step of the process $U^\pm$ when the first cookie to be used is of type $i$ and the BLP is started from a very large initial value $U_0^\pm = n$. Next let

\[
(9) \quad \nu = \lim_{n \to \infty} \frac{\text{Var}(U^+_n | U^+_0 = n, R_1^+ = i)}{n} = \lim_{n \to \infty} \frac{\text{Var}(U^-_n | U^-_0 = n, R_1^- = i)}{n}.
\]

The proof that the limits in (8) and (9) exist and that the limits in (9) are equal and do not depend on the distribution of the first cookie can be found in [KP17]. Moreover, it was shown in [KP17, Proposition 4.3 and Lemma 4.4] that these parameters have the following relation:

\[
(10) \quad r^+(i) + r^-(i) = \frac{\nu}{2} - 1, \quad \forall i \in \{1, 2, \ldots, N\}.
\]

Finally, the parameters $\theta^+$ and $\theta^-$ are defined by

\[
(11) \quad \theta^+ = \theta^+(\eta) = \frac{2\eta \cdot r^+}{\nu} \quad \text{and} \quad \theta^- = \theta^-(\eta) = \frac{2\eta \cdot r^-}{\nu}.
\]

Note that the equations (10) and (11) imply that $\theta^+ + \theta^- = 1 - \frac{\nu}{2}$ and, thus, $\theta^+ + \theta^- < 1$. The relevance of the parameters $\theta^+$ and $\theta^-$ is that the BLPs have scaling limits which are Bessel squared processes, and the parameters $\theta^\pm$ identify the generalized “dimension” of these Bessel squared processes (this will be detailed further in Sections 5.1 and 5.3).

It should be noted that $r^+$, $r^-$ and $\nu$ depend only on the transition matrix $K$ and the function $p(\cdot)$ which appear in the description of markovian cookie stacks in Assumption 4. The parameters $\theta^+$ and $\theta^-$, however, depend not only on $K$ and $p(\cdot)$ but also on the initial distribution $\eta$ of the first cookies.

Finally, we will introduce two distributions $\pi^+$ and $\pi^-$ which depend only on $K$ and $p(\cdot)$ and play an important role in this paper. Let $\pi^\pm = (\pi^\pm(i))_{1 \leq i \leq N}$ be defined by

\[
\pi^\pm(i) = \lim_{n \to \infty} P\left(R_{U^\pm_0 + n}^\pm = i | U_0^\pm = n, R_1^\pm = i'\right).
\]

In words, using the sequence of Bernoulli random variables $\{\xi_j\}_{j \geq 1}$, the distributions $\pi^+$ and $\pi^-$ give the limiting distribution of the next value in the underlying Markov chain $\{R_j^\pm\}_{j \geq 1}$ immediately following the $n$-th “failure” or “success,” respectively, as $n \to \infty$.

The relevance of the distributions $\pi^\pm$ is that they are good approximations for the distribution of the remaining first cookie environment at sites with a large local time (which is most sites in the range). Indeed, for a site $x$ within the range of the walk but to the right of the current location,
since the last step from that site was to the left (corresponding to a “failure” in a sequence \(\{\xi^z_j\}_{j \geq 1}\)), the probability that the remaining first cookie at \(x\) is of “type \(i\)” can be approximated by \(\pi^+(i)\).

Since we will at times be using the BLPs in cookie environments which have first cookie distributions which are approximately \(\pi^+\) or \(\pi^-\), it is important to note what the parameters \(\theta^+\) and \(\theta^-\) are with these distributions on the first cookies. It was shown in [KPL17] Corollary 3.5 that \(\pi^+ \cdot \pi^+ = 0\) and \(\pi^- \cdot \pi^- = 0\). Substituting these equations into \([11]\) we get \(\theta^+(\pi^+) = 0\) and \(\theta^-(\pi^-) = 0\).

3. Brownian motion perturbed at extrema: preliminaries

Recall the notation \(\Xi\). Though in the introduction a BMPE had initial value 0, henceforth the process triple \((I, W, S) := \{(I(t), W(t), S(t))\}_{t \geq 0}\) is a Markov process which can be considered starting from any initial state \((w, w, \overline{w})\), \(w \leq w \leq \overline{w}\). When BMPE starts from \((0, 0, 0)\) we shall call it a standard BMPE. We remark that a standard \((\alpha, \beta)\)-BMPE, \([R]\) inherits the scaling property of the Brownian motion: for every \(c > 0\) the process \(\{cW(c^{-2}t)\}_{t \geq 0}\) is a standard BMPE with the same parameters.

For reference convenience we shall use \((\theta^+, \theta^-)\)-BMPE in place of \((\alpha, \beta)\)-BMPE and assume throughout this section that \(\theta^+\) and \(\theta^-\) are arbitrary real numbers strictly less than 1. In later sections \(\theta^+\) and \(\theta^-\) will be fixed as parameters of our ERW but in this section we only require that \(\theta^+, \theta^- < 1\).

3.1. Exit probabilities. We shall define for \(a \in \mathbb{R}, w \leq w \leq \overline{w}\),
\[
\tau_a(w, w, \overline{w}) = \inf\{t > 0 : W(t) = a \mid (I(0), W(0), S(0)) = (w, w, \overline{w})\}
\]
and for \(\varepsilon > 0\)
\[
\tau(\varepsilon, w, w, \overline{w}) := \tau_{w-\varepsilon}(w, w, \overline{w}) \wedge \tau_{w+\varepsilon}(w, w, \overline{w}).
\]
These are respectively the first time \(W\) hits \(a\) and the first time \(W\) exits \((w - \varepsilon, w + \varepsilon)\) given that it started at \((w, w, \overline{w})\). We also drop the arguments \(w, w, \overline{w}\) whenever \((w, w, \overline{w}) = (0, 0, 0)\). Clearly, when \(w - \varepsilon \leq w \leq \overline{w} + \varepsilon\), \(P(W(\tau, w, w, \overline{w})) = w \pm \varepsilon = 1/2\). In many other cases, these probabilities can also be computed explicitly. The following lemma can be found in \([PW97]\) Proposition 4(iii)] (see also Proposition 3 in \([PW97]\)).

**Lemma 3.1.** Let \(W\) be a standard \((\theta^+, \theta^-)\)-BMPE. Then for \(a < 0 < b\)
\[
P(\tau_a < \tau_b) = \frac{1}{B(1-\theta^+, 1-\theta^-)} \int_0^{\frac{b}{b-a} \theta^+} (1-t)^{-\theta^+} dt,
\]
where \(B(\cdot, \cdot)\) is the beta function

**Corollary 3.2.** Let \(W\) be an \((\theta^+, \theta^-)\)-BMPE starting from \((w, 0, \overline{w})\).
If \(w \leq a < 0 \leq b \leq \overline{w}\), then
\[
P(\tau_a(w, 0, \overline{w}) < \tau_b(w, 0, \overline{w})) = \frac{-a}{b-a} \left( \frac{b-a}{w-a} \right)^{\theta^+}.
\]
If \(a \leq w < b \leq \overline{w}\) then
\[
P(\tau_a(w, 0, \overline{w}) < \tau_b(w, 0, \overline{w})) = \frac{b}{b-a} \left( \frac{b-a}{b-w} \right)^{\theta^-}.
\]

**Proof.** We shall prove the second statement, the first one is obtained in a symmetric way.

Observe that to reach \(a\) before \(b\) the process \(W\) has to reach \(w\) before \(b\) and that between \(w\) and \(b\) the process \(W\) behaves simply as a standard Brownian motion. Using this observation and the Markov property of the triple \((I, W, S)\) we get
\[
P(\tau_a(w, 0, \overline{w}) < \tau_b(w, 0, \overline{w})) = \frac{b}{b-w} P(\tau_a(w, w, \overline{w}) < \tau_b(w, w, \overline{w})).
\]
Next, note that the last probability is equal to the probability that a standard \((0, \theta^-)\)-BMPE reaches \(a - w\) before \(b - w\). Applying Lemma 3.1 we obtain the desired result.

\[\square\]

### 3.2. Ray-Knight theorems for BMPE.

As we described in the introduction, to approximate exit probabilities and the exit time of our ERW from an interval we shall use an approach based on edge local time BLPs. In this subsection we discuss the continuous counterpart of these results in more detail.

As noted in the Introduction (see, for example, [CPY98]) the local times of \((\theta^+, \theta^-)\)-BMPE satisfy analogs of the first and second Ray-Knight theorems. These theorems involve squared Bessel processes of generalized dimensions \(d \in \mathbb{R}\) which we shall denote by BESQ\(^d\). BESQ\(^d\) process starting at \(y \geq 0\) is a unique strong solution of the SDE (see, for example, [RY99, Chapter XI] for \(d \geq 0\) and [GJY03, Section 3] for \(d < 0\))

\[\tag{14}\]
\[y(t) = y + dt + 2 \int_0^t \sqrt{|y(s)|} dB(s).\]

When \(d \geq 0\) and \(y(0) \geq 0\), the solution \(y(s)\) of the above equation is always non-negative, and the absolute value in (14) can be simply dropped. We recall that when \(d \geq 2\) the process \(\{y(t)\}_{t \geq 0}\) with \(y(0) \geq 0\) is strictly positive for all \(t > 0\) with probability 1. When \(d < 2\) then with probability 1 the process hits zero in finite time. Up to this time, \(\tau_0^y\), we also can drop the absolute value even if \(d < 0\).

In this paper we will start with \(y(0) \geq 0\) and stop the process with \(d \leq 0\) at time \(\tau_0^y\). This means that we are always in the setting when \(|y(s)| = y(s)\) in (14). However, for convenience we often use \((y(s))_+\) instead. With this choice, when \(d \leq 0\), after time \(\tau_0^y\) the process continues degenerately as \(y(s + \tau_0^y) = ds \leq 0\) for all \(s \geq 0\). We continue to refer to solutions of

\[y(t) = y + dt + 2 \int_0^t \sqrt{(y(s))_+} dB(s),\]

for any \(d \in \mathbb{R}\) as BESQ\(^d\). This definition coincides with (14) for all \(d \geq 0\) and \(y \geq 0\) and for all \(d < 0\) and \(y \geq 0\) up to \(\tau_0^y\).

Denote by \(\{(\ell^W_x)_{x \in \mathbb{R}, t \geq 0}\}\) the jointly continuous family of local times of \((\theta^+, \theta^-)\)-BMPE \(W\). The starting triple for \(W\) will not be reflected in the notation. It will be given explicitly in each case.

The following proposition states Ray-Knight theorems for BMPE in the most convenient form for our purposes.

**Proposition 3.3** ([CDH00], Proposition 2.1). Let \(W\) be a \((\theta^+, \theta^-)\)-BMPE starting from \((w, w, \bar{w})\), \(0 \leq w \leq w \leq \bar{w}\), and \(\tau_0^W = \inf\{t \geq 0 : W(t) = 0\}\). Then the local time process \(\{\ell^W_x, \tau_0^W\}_{x \geq 0}\) has the same law as \(\{y(x \wedge \sigma^y_{w,0})\}_{x \geq 0}\) where \(\{y(x)\}_{x \geq 0}\) is the unique strong solution of the equation

\[y(x) = 2 \int_0^x \sqrt{(y(s))_+} dB(s) + \int_0^x (2(1 - \theta^-)1_{\{0 \leq s \leq w\}} + 21_{\{w \leq s \leq w\}} + 2\theta^+1_{\{s \geq \bar{w}\}}) ds.\]

In words, the process \(\{\ell^W_x, \tau_0^W\}_{x \geq 0}\) is an inhomogeneous Markov process which is a BESQ\(^{2(1 - \theta^-)}\) on \([0, w]\), a BESQ\(^2\) on \([w, w]\), a BESQ\(^0\) on \((w, \bar{w})\) and a BESQ\(^{2\theta^+}\) on \([\bar{w}, \infty)\), absorbed at its first zero after \(w\).

This proposition immediately implies the following statement. Its discrete version, Lemma 5.19 (“concatenation lemma”), is one of the main tools of this paper.

**Corollary 3.4.** Let \(W\) be a \((\theta^+, \theta^-)\)-BMPE starting from \((w, w, \bar{w})\), \(0 \leq w \leq w \leq \bar{w}\) and \(\{y(x)\}_{x \geq 0}\) be the process defined in Proposition 3.3. Then

(i) for any \(b > w\), \(P(\tau_0^W(w, w, \bar{w}) < \tau_b^W(w, w, \bar{w})) = P(\sigma^y_{w,0} < b)\);
(ii) for any $b > \overline{w}$ and any interval $J \subseteq (\overline{w}, b)$

\[ P(\tau_0(w, w, \overline{w}) < \tau_b(w, w, \overline{w}), \ S(\tau_0(w, w, \overline{w})) \in J) = P(\sigma_{w,0}^{\overline{y}} \in J). \]

Remark 3.5. Note that (ii) and (i) imply that

\[ P(\tau_0(w, w, \overline{w}) < \tau_b(w, w, \overline{w}), \ S(\tau_0(w, w, \overline{w})) = \overline{w}) = P(\sigma_{w,0}^{\overline{y}} \leq \overline{w}). \]

Remark 3.6. Part (i) of this corollary and known facts about the distribution of BESQ processes can be used to derive Lemma 3.1.

### 3.3. Coupling of BMPEs with different initial data.

We shall need the following coupling result about Brownian motions perturbed only at one extremum. For definiteness, we assume that the perturbation is at the maximum, i.e. we shall consider BMPEs with $\theta^- = 0$. In this case, the process $\{(W(t), S(t))\}_{t \geq 0}$ is markovian.

**Lemma 3.7.** Let $\{(W_i(t), S_i(t))\}_{t \geq 0}$, $i = 1, 2$, be Brownian motions perturbed at the maximum, i.e. BMPE with parameters $\theta^- = 0$ and $\theta^+ \in (-\infty, 1)$. Suppose that $(W_i(0), S_i(0)) \in \{(w, \overline{w}) \in [-1, 1]^2 : w \leq \overline{w}\}$, $i = 1, 2$. There exists a coupling such that for some $C_1 > 0$ uniformly over all initial conditions in $\{(w, \overline{w}) \in [-1, 1]^2 : w \leq \overline{w}\}$

\[ P((W_1(1), S_1(1)) = (W_2(1), S_2(1)) \text{ and } W_i(t) \in [-4, 4] \forall t \in [0, 1], \ i = 1, 2) \geq C_1. \]

The proof of this lemma is given in the Appendix.

**Corollary 3.8.** Let $\{(W_i(t), S_i(t))\}_{t \geq 0}$, $i = 1, 2$, be as in Lemma 3.7. There exists a coupling and nontrivial constants $C_2$ and $K_0$ (which do not depend on the initial conditions $(W_i(0), S_i(0)) = (w_i, \overline{w}_i) \in [-1, 1]^2$) such that for all $R \geq 4$ outside probability $C_2 R^{-\frac{1}{K_0}}$

\[ (W_i(t), S_i(t)) = (W_j(t), S_j(t)) \ \forall t > \rho_R, \]

where $\rho_R = \inf\{t \geq 0 : \max\{|W_i(t)|, |W_j(t)|\} \geq R\}$.

**Proof.** Let $\tau_{\text{coup}} = \inf\{t \geq 0 : (W_1(t), S_1(t)) = (W_2(t), S_2(t))\}$ be the coupling time of the BMPEs, and for any choice of $w_i \leq \overline{w}_i$, $i = 1, 2$ and $R > 0$ let

\[ a_R(w_1, \overline{w}_1, w_2, \overline{w}_2) = P(\tau_{\text{coup}} > \rho_R) \mid (W_i(0), S_i(0)) = (w_i, \overline{w}_i), \ i = 1, 2. \]

If for any $r > 0$ we denote $\Delta_r = \{(w, \overline{w}) \in [-r, r]^2 : w \leq \overline{w}\}$, then it follows easily from Lemma 3.7 that

\[ \sup_{(w_1, \overline{w}_1), (w_2, \overline{w}_2) \in \Delta_1} a_4(w_1, \overline{w}_1, w_2, \overline{w}_2) \leq 1 - C_1. \]

Note that since a rescaled BMPE is again a (time changed) BMPE it follows that $a_R(w_1, \overline{w}_1, w_2, \overline{w}_2) = a_{cR}(c w_1, c \overline{w}_1, c w_2, c \overline{w}_2)$ for any $c > 0$. In particular, this implies that

\[ \sup_{(w_1, \overline{w}_1), (w_2, \overline{w}_2) \in \Delta_r} a_{4r}(w_1, \overline{w}_1, w_2, \overline{w}_2) \leq 1 - C_1. \]

The coupling in Lemma 3.7 gives a coupling of the two processes up until time $\rho_4$. If the coupling constructed in Lemma 3.7 doesn’t succeed by this time (i.e., if $\tau_{\text{coup}} > \rho_4$) then from time $\rho_4$ to $\rho_{16}$ we can use a rescaled version of the coupling from Lemma 3.7 to obtain

\[ a_{16}(w_1, \overline{w}_1, w_2, \overline{w}_2) = E \left[ a_{16}(W_1(\rho_4), S_1(\rho_4), W_2(\rho_4), S_2(\rho_4)) \mathbf{1}_{\{\tau_{\text{coup}} > \rho_4\}} \mid (W_i(0), S_i(0)) = (w_i, \overline{w}_i), \ i = 1, 2 \right] \]

\[ \leq \left( \sup_{(w_1', \overline{w}_1'), (w_2', \overline{w}_2') \in \Delta_4} a_{16}(w_1', \overline{w}_1', w_2', \overline{w}_2') \right) a_4(w_1, \overline{w}_1, w_2, \overline{w}_2) \]

\[ \leq (1 - C_1)^2. \]
Similarly, we can show that
\[
\sup_{(w_1, \overline{w}_1, w_2, \overline{w}_2) \in \Delta_1} a_k^*(w_1, \overline{w}_1, w_2, \overline{w}_2) \leq (1 - C_1)^k, \quad \forall k \geq 1.
\]
Therefore, if \(4^k \leq R < 4^{k+1}\) we have
\[
\sup_{(w_1, \overline{w}_1, w_2, \overline{w}_2) \in \Delta_1} a_R^*(w_1, \overline{w}_1, w_2, \overline{w}_2) \leq (1 - C_1)^k \leq (1 - C_1)^\log_4(R) - 1 = (1 - C_1)^{-1} R^{-\log_4(1 - C_1)}.
\]
This completes the proof of the corollary with \(C_2 = (1 - C_1)^{-1}\) and \(\frac{1}{K_0} = \log_4\left(\frac{1}{1 - C_1}\right)\).

\[\square\]

4. Discretizations of BMPE

Recall that the standard BMPE has Brownian scaling, that is for every \(\varepsilon > 0\)
\[
\{(I(t), W(t), S(t))\}_{t \geq 0} \overset{\text{Law}}{=} \{(\varepsilon I(\varepsilon^{-2} t), \varepsilon W(\varepsilon^{-2} t), \varepsilon S(\varepsilon^{-2} t))\}_{t \geq 0}.
\]

4.1. Basic BMPE-walk. Our first step will be to define a natural sequence of random walks \(\{(I_k, W_k, S_k)\}_{k \geq 0}\) which after rescaling converges to BMPE. Set \(\tau_0 = 0\) and let
\[
(I_k, W_k, S_k) = (I(\tau_k), W(\tau_k), S(\tau_k)), \quad \tau_{k+1} := \inf\{t > \tau_k : |W(t) - W(\tau_k)| = 1\}, \quad k \in \mathbb{N}_0.
\]
The walk \((I_k, W_k, S_k), k \geq 0\), is markovian and can be also be constructed directly by specifying its transition probabilities. Set \((I_0, W_0, S_0) = (0, 0, 0)\) and define the transition probabilities as follows.

- “In the bulk”, i.e. on the set \(\{I_k + 1 \leq W_k \leq S_k - 1\}\),
  \[
  W_{k+1} = W_k \pm 1 \text{ with equal probabilities, } I_{k+1} = I_k, \, S_{k+1} = S_k.
  \]
- “At the extrema”, i.e. on \(\{S_k - W_k < 1\} \cup \{W_k - I_k < 1\}\),
  \[
  P(W_{k+1} = W_k + 1 \mid I_k, W_k, S_k) = 1 - P(W_{k+1} = W_k - 1 \mid I_k, W_k, S_k)
  = P(W(\tau(1, I_k, W_k, S_k)) = W_k + 1).
  \]

Now we need to see what happens to the extrema. This is easy when \(k > 0\) and the end point of the walk, \(W_k\), lands outside of \([I_k, S_k]\). But if \(k = 0\) or if the walk is, say, close to the max but “decides” to jump to the left, the new max should be chosen according to the distribution of the BMPE. More precisely, for every Borel set \(B\)
\[
P(I_{k+1} \in B \mid I_k, W_k, S_k) = P(I(\tau(1, I_k, W_k, S_k) \in B));
\]
\[
P(S_{k+1} \in B \mid I_k, W_k, S_k) = P(S(\tau(1, I_k, W_k, S_k) \in B).
\]

**Proposition 4.1.** Let \((I_0, W_0, S_0) = (0, 0, 0) = (I(0), W(0), S(0))\). Then for each \(T > 0\)
\[
(15) \quad \sup_{0 \leq s \leq T} \left| \varepsilon W(\varepsilon^{-2}s) - \varepsilon W[\varepsilon^{-2}s] \right| \overset{P}{\to} 0 \quad \text{as } \varepsilon \to 0.
\]
Here \(\overset{P}{\to}\) denotes the convergence in probability.

The proof of this proposition is given in the Appendix.

4.2. Modified BMPE-walk. For our coupling it will be convenient to have a slightly modified discretization which allows for small shifts of the running extrema. For each \(\varepsilon > 0\) we shall need a process \(\{(\hat{I}_k, \hat{W}_k, \hat{S}_k)\}_{k \geq 0}\) adapted to some filtration \((\mathcal{F}_k^\varepsilon)_{k \geq 0}\) and satisfying properties listed below. Since we shall be coupling this process with a rescaled ERW, the filtration will contain information about both processes. At this time we shall make only the necessary specifications and shall not describe the filtration. Our goal is to show that if we have a family of processes indexed by \(\varepsilon\) which satisfies the properties listed below then after rescaling this family converges weakly to a BMPE.
We wish that the evolution of \( \{(\tilde{I}_k^\varepsilon, \tilde{W}_k^\varepsilon, \tilde{S}_k^\varepsilon)\}_{k \geq 0} \) be close to that of the walk \( \{(I_k, W_k, S_k)\}_{k \geq 0} \) when \( \varepsilon \) is small. To describe this “closeness” we divide up the interval \((-1, 1)\) into \(2L = 2\lfloor \varepsilon^{-3(K_0+1)} \rfloor\) equal intervals with disjoint interior of length \(\varepsilon^{-1}\),

\[
(-1, 1) = \left( \bigcup_{\ell = -L}^{L} \left( \frac{\ell}{L}, \frac{\ell + 1}{L} \right) \right) =: \bigcup_{0 < |\ell| \leq L} J_\ell.
\]

Here \(K_0\) is the constant from Lemma 3.8. Then \(x, x + 1 = \left( \bigcup_{\ell = 1}^{L} J_\ell \right) + x\) where \(J + x\) is simply the translation of set \(J\) by \(x\).

**Properties of \( \{(\tilde{I}_k^\varepsilon, \tilde{W}_k^\varepsilon, \tilde{S}_k^\varepsilon)\}_{k \geq 0} \).** We shall require that the process \( \{(\tilde{I}_k^\varepsilon, \tilde{W}_k^\varepsilon, \tilde{S}_k^\varepsilon)\}_{k \geq 0} \) satisfy the following three conditions.

(i) (Starting point) \( \tilde{I}_0^\varepsilon = \tilde{W}_0^\varepsilon = \tilde{S}_0^\varepsilon = 0 \).

(ii) (Steps of the walk) \( \tilde{W}_k^\varepsilon + 1 \in \{ \tilde{W}_k^\varepsilon - 1, \tilde{W}_k^\varepsilon + 1 \} \) with probability 1 for all \( k \geq 0 \) and

\[
P(\tilde{W}_{k+1}^\varepsilon = \tilde{W}_k^\varepsilon + 1 \mid F_k^\varepsilon) = 1 - P(\tilde{W}_{k+1}^\varepsilon = \tilde{W}_k^\varepsilon - 1 \mid F_k^\varepsilon) = P(W(\tau(1, \tilde{I}_k^\varepsilon, \tilde{W}_k^\varepsilon, \tilde{S}_k^\varepsilon)) = \tilde{W}_k^\varepsilon + 1).
\]

where the right hand side probabilities are those for the BMPE (see notation \([12], [13]\)).

(iii) (“Choice” of extrema) If \( \tilde{I}_{k+1}^\varepsilon \leq \tilde{W}_{k+1}^\varepsilon \leq \tilde{S}_{k+1}^\varepsilon \) then \( \tilde{I}_{k+1}^\varepsilon = \tilde{I}_k^\varepsilon \) and \( \tilde{S}_{k+1}^\varepsilon = \tilde{S}_k^\varepsilon \). The definitions when \( \tilde{W}_k^\varepsilon \) is close to its minimum or maximum are more complicated. Set

\[
x_k = \tilde{W}_k^\varepsilon, \quad a_k = \tilde{I}_k^\varepsilon - \tilde{W}_k^\varepsilon, \quad b_k = \tilde{S}_k^\varepsilon - \tilde{W}_k^\varepsilon.
\]

Note that the first step is special as both \(|a_0| < 1\) and \(b_0 < 1\) so that we need to choose a new minimum and a new maximum. For all other steps only one extremum might need to be changed. Suppose for definiteness that \(b_k < 1\) and we need to determine a new maximum. If \(b_k < 1\) and the walk moves to the right then the new maximum is simply \(x_k + 1\) and as in (ii)

\[
P(\tilde{S}_{k+1}^\varepsilon = x_k + 1 \mid F_k^\varepsilon) = P(\tilde{W}_{k+1}^\varepsilon = \tilde{W}_k^\varepsilon + 1 \mid F_k^\varepsilon) = P(W(\tau(1, a_k, 0, b_k)) = 1).
\]

If the walk moves to the left then we first choose an intermediate index \(\ell_k\) according to probabilities

\[
P(\ell_k = \ell) = P(S(\tau(1, a_k, 0, b_k)) \in J_\ell \mid W(\tau(1, a_k, 0, b_k)) = -1), \quad \ell \in \{1, 2, \ldots, L\}.
\]

Given \(F_k^\varepsilon\) and the index \(\ell_k\) we “pick” the value of \(\tilde{S}_{k+1}^\varepsilon\) within \((J_\ell_k \cup J_{\ell_k+1}) \cap [0, 1]) + x_k\) arbitrarily, provided that \(\tilde{S}_{k+1}^\varepsilon\) is \(F_{k+1}^\varepsilon\)-measurable, i.e. the process remains adapted to the filtration \(F_k^\varepsilon\), \(k \geq 0\).

When we define the coupling of \(\tilde{W}_k^\varepsilon\) with our rescaled ERW we shall construct a particular version of \(\tilde{W}_k^\varepsilon\) explicitly.

**Theorem 4.2.** There exists a constant \(C > 0\) such that for all sufficiently small \(\varepsilon > 0\) the following holds: If \(\tilde{W}_k^\varepsilon\) satisfies (i)-(iii) above then we can couple \(\{\tilde{W}_k^\varepsilon\}_{k \geq 1}\) with a basic BMPE-walk \(\{W_k\}_{k \geq 1}\) so that for any \(K > 0\) we have

\[
P \left( \tilde{W}_k^\varepsilon = W_k, \forall k \leq K\varepsilon^{-2} \right) \geq 1 - CK\varepsilon.
\]

Since rescaling implies that \(\varepsilon W(\varepsilon^{-2}t) \overset{\text{law}}{\rightarrow} \{W(t)\}_{t \geq 0}\), then Proposition 4.1 and Theorem 4.2 immediately imply the following corollary.

**Corollary 4.3.** If \(\tilde{W}_k^\varepsilon\) satisfies (i)-(iii) above then for every \(\varepsilon > 0\) there exists a BMPE \(W_k^\varepsilon\) such that for all \(\delta, T > 0\)

\[
\lim_{\varepsilon \to 0} P \left( \sup_{0 \leq s \leq T} \left| W_k^\varepsilon(s) - \tilde{W}_k^\varepsilon(\varepsilon^{-2}s) \right| > \delta \right) = 0.
\]
Proof of Theorem 4.2: Our proof will in fact prove the following stronger statement than (17). We will show that with probability at least $1 - CK\varepsilon$ we have for all $k \leq K\varepsilon^{-2}$

$$\tilde{W}_k^\varepsilon = W_k, \quad [S_k] = [\tilde{S}_k^\varepsilon], \quad [I_k] = [\tilde{I}_k^\varepsilon], \quad |\tilde{S}_k^\varepsilon - S_k| \leq 2L^{-1}, \quad \text{and} \quad |\tilde{I}_k^\varepsilon - I_k| \leq 2L^{-1}. \tag{18}$$

For the first step of the coupling, we use a single BMPE stopped when exiting $(-1, 1)$ to generate both $(I_1, W_1, S_1)$ and $(\tilde{I}_1^\varepsilon, \tilde{W}_1^\varepsilon, \tilde{S}_1^\varepsilon)$. If the BMPE exits to the left so that $I_1 = W_1 = -1$ and $S_1 \in J_k$ for some $0 < \ell < L$ then we let $\tilde{I}_1^\varepsilon = \tilde{W}_1^\varepsilon = -1$ and choose $\tilde{S}_1^\varepsilon \in (J_\ell \cup J_{\ell + 1}) \cap [0, 1)$ in some way that is $F_1^{\varepsilon}$-measurable. Thus, we can give a coupling so that (18) holds for $k = 1$ with probability 1.

For later steps, we suppose (18) holds for some $k \geq 1$. If we are “in the bulk” (i.e., if $I_k + 1 \leq W_k \leq S_k - 1$), then in the next step of the walk the minimums and maximums remain unchanged while the walks both move to the right or left with equal probabilities. That is, if (18) holds at time $k$ when the walk is in the bulk then (18) will also hold at time $k + 1$. It remains to show how we can couple the walks when (18) holds for some $k \geq 1$ and we are “at the extrema.” Without loss of generality we may assume that $I_k + 1 \leq W_k$ and $W_k > S_k - 1$ so that we are near the maximum. Let $x = S_k - W_k$ and $\tilde{x} = \tilde{S}_k^\varepsilon - \tilde{W}_k^\varepsilon$, and note that our assumptions are that $x, \tilde{x} \in (0, 1)$ and $|x - \tilde{x}| < 2L^{-1}$, and assume without loss of generality that $x \leq \tilde{x}$. We now consider a coupling of two Brownian motions perturbed at the maximum $W$ and $\tilde{W}$ with initial conditions $(W(0), \tilde{W}(0)) = (0, x)$ and $(\tilde{W}(0), \tilde{W}(0)) = (0, \tilde{x})$ until they reach either $-1$ or 1. We let $W(t) = \tilde{W}(t)$ for all $t \leq \tau_{-1,x} = \inf\{s : W(s) \in (-1, x]\}$. If $W(\tau_{-1,x}) = -1$ then our coupling is complete. On the other hand, if $W(\tau_{-1,x}) = x$ then we still need to describe the remainder of the coupling. We consider two cases.

Case I: $x > 1 - \varepsilon^3$. Given that the processes $W$ and $\tilde{W}$ reached $x \geq 1 - \varepsilon^3$ before $-1$, it follows from Corollary 3.2 that the probability they will reach 1 before $-1$ is at least $1 - C\varepsilon^3$ for some $C > 0$ depending only on $\Theta$. Case II: $x < 1 - \varepsilon^3$. If $\tilde{x} > x$ then the processes $W$ and $\tilde{W}$ may no longer be exactly coupled after reaching $x$. However, since $|\tilde{x} - x| \leq 2L^{-1}$ it follows from Corollary 3.8 that we can couple them so that both processes and the maximums join together again before exiting the interval $[x - \varepsilon^3, x + \varepsilon^3]$ with probability at least $1 - C_3(\varepsilon^3L^{-1})^{-1/5} \geq 1 - C\varepsilon^3$.

In either case, we have shown that we can create a coupling so that outside of probability $C\varepsilon^3$ the processes exit out of the same side of the interval $(-1, 1)$ and that at this time the maximums are either unchanged or both changed to the same value. We then apply this to the BMPE-walks by using the process $W$ to generate $W_{k+1}$ and $S_{k+1}$ and $\tilde{W}$ (plus additional randomness which is $F_{k+1}^{\varepsilon}$-measurable) to generate $\tilde{W}_{k+1}^\varepsilon$ and $\tilde{S}_{k+1}^\varepsilon$ so that with probability at least $1 - C\varepsilon^3$ we have $W_{k+1} = \tilde{W}_{k+1}^\varepsilon$ and $|S_{k+1} - \tilde{S}_{k+1}^\varepsilon| \leq 2L^{-1}$.

We have therefore shown that if (18) holds for some $k \geq 1$ then with probability at least $1 - C\varepsilon^3$ it again holds for $k + 1$. This is enough to show that (18) holds for all $k \leq K\varepsilon^{-2}$ with probability at least $1 - CK\varepsilon$, and this finishes the proof of the theorem. \hfill \Box

5. Toolbox

We would like to argue that our ERW $X = \{X_n\}_{n \geq 0}$ considered only at the stopping times \(T_{k}^{c,n}\) described in Section 4.2 and scaled down by $\lceil \varepsilon \sqrt{n} \rceil$ behaves essentially as a modified BMPE-walk described in Section 4.2. The important issue here is that the ERW moves in a random environment and the environment is modified by the walk. In this section we shall collect a number of results concerning BLPs in random environments which will be helpful as long as we know that the cookie environment is “good” in some way. We begin this section with several definitions which will be used to quantify exactly what we mean by “good.”
Recall the parameters \( r^\pm = (r^\pm(1), \ldots, r^\pm(N)) \) introduced in Section 2.1 and that that \( R_j^2 \), \( j \in \mathbb{N} \) is the cookie Markov chain at site \( x \) with values in \( \{1, 2, \ldots, N\} \) so that \( \omega_x(j) = p(R_j^2) \in (0, 1) \) is the probability that the ERW jumps to the right after the \( j \)-th visit to \( x \).

**Definition 5.1.** Let \( \alpha \in (0, 1) \), \( m \in \mathbb{N} \), and \( \rho \in \mathbb{R} \). The first cookies \( \{R_1^2\}_{j \in \mathbb{Z}} \) are said to be \((m^\alpha, \rho)\)-good on a discrete interval \( I \) if for every discrete subinterval \( J \subset I \) of length \([m^\alpha]\)

\[
\frac{1}{m^\alpha} \sum_{z \in J} r^+(R_1^2) - \rho \leq \frac{1}{\ln m}, \quad \text{or, equivalently (by (10)),}
\]

\[
\frac{1}{m^\alpha} \sum_{z \in J} r^-(R_1^2) - \left( \nu - 1 - \rho \right) \leq \frac{1}{\ln m}.
\]

We shall say that the family of first cookie environments is \( m^\alpha\)-good on some interval \( I \) if there is a constant \( \rho \) for which it is \((m^\alpha, \rho)\)-good.

The relevance of the above definition is that if the first cookie environment in \( I \) is (approximately) i.i.d. with marginal \( \eta^\prime \), then we expect the interval to be \((m^\alpha, \rho)\)-good with \( \rho = \eta^\prime \cdot r^+ \). In particular, we expect intervals in the initial cookie environment (which is i.i.d. \( \eta \)) to be \((m^\alpha, \frac{\nu \cdot r^+}{2})\)-good, whereas if an interval has first cookie environments which are approximately i.i.d. \( \pi^+ \) or \( \pi^- \) then we expect the interval to be \((m^\alpha, 0)\)-good or \((m^\alpha, \frac{\nu}{2} - 1)\)-good, respectively.

**Definition 5.2.** Given \( x > 0 \), \( \varepsilon > 0 \), and \( a, b \in \mathbb{R} \), \( a < b \), a first cookie environment on the interval \([a, b] \) is said to be \( x\)-lifting from the left (resp. right) if for a \( V^+ \) (resp. \( V^- \)) process which uses the environment with these first cookies for generations \( 1, 2, \ldots, [b] - [a] + 1 \) (resp. \( [b] - [a] + 1, \ldots, 2, 1 \)) and starts with 0 particles in generation 0

\[
P(\tau_x^{V^+} \leq b - a) \geq 1 - \varepsilon^3 \quad \text{(resp. } P(\tau_x^{V^-} \leq b - a) \geq 1 - \varepsilon^3)\).
\]

**Definition 5.3.** Given \( x > 0 \), \( \varepsilon > 0 \), and \( a, b \in \mathbb{R} \), \( a < b \), a first cookie environment on the interval \([a, b] \) is said to be \( x\)-grounding from the left (resp. right) if for a \( U^+ \) (resp. \( U^- \)) process which uses the environment with these first cookies for generations \( 1, 2, \ldots, [b] - [a] + 1 \) (resp. \( [b] - [a] + 1, \ldots, 2, 1 \)) and starts with \([x] \) particles in generation 0

\[
P(\sigma_0^{U^+} \leq b - a) \geq 1 - \varepsilon^3 \quad \text{(resp. } P(\sigma_0^{U^-} \leq b - a) \geq 1 - \varepsilon^3)\).
\]

### 5.1. The “full” diffusion approximation in product environments.

In this subsection we extend the results of [KP17, Lemma 6.1] to either the convergence on \( D([0, \infty)) \) (for \( V^+ \) processes with positive drifts) or the convergence up to the first hitting time of 0. The diffusion approximations of the BLPs here and throughout the paper will generally be of the form

\[
dY(t) = D(t) \, dt + \sqrt{\nu(Y(t))} \, dB(t),
\]

where the constant \( \nu > 0 \) is the parameter which was defined earlier in (9) and the drift \( D(t) \) is a nonrandom piecewise constant function of time depending on the particular BLP being considered \((U^\pm, V^\pm, \text{or concatenation of those})\) and the distribution of the first cookies.

We note that if \( Y \) is defined as in (19) and \( D(t) \equiv D \) then the process \( \frac{Y(t)}{\nu} \) is a BESQ process of generalized dimension \( 4D/\nu \). Weaker versions of diffusion approximation for BLPs with initial cookie distributions i.i.d. \( \eta \) were proved earlier in [KP17] where the drift \( D(t) \equiv \eta \cdot r^\pm \) in the case of \( U^\pm \) and \( D(t) \equiv 1 + \eta \cdot r^\pm \) in the case of \( V^\pm \).

**Definition 5.4.** We shall say that a family of stochastic processes \( Z^m = \{Z_k^m\}_{k \geq 0} \), \( m \in \mathbb{N} \), admits an approximation by a BESQ process of generalized dimension \( \delta \in \mathbb{R} \) if \( \forall \theta > 0 \) and \( \forall Y > \delta \) the rescaled processes \( m^{-1} Z^m_{[mt] + \sigma_\delta} \) with \( m^{-1} Z_0^m \rightarrow y \) converge weakly in the standard \((J_1)\) Skorokhod topology to a positive multiple of BESQ\(\delta \) process \( Y(t \wedge \sigma_\delta) \) with \( Y(0) = y \).
In terms of the above definition, and recalling the relations \cite{Fel71} and \cite{KP17}, the arguments in \cite{Fel71} and \cite{KP17} show that if the initial cookie distribution is i.i.d. \( \eta \) then the BLPs \( U^+, U^-, V^+, \) and \( V^- \) admit approximation by a BESQ processes of generalized dimensions \( 2\theta^+, 2\theta^-, 2(1 - \theta^-) \) and \( 2(1 - \theta^+) \), respectively.

Since we are assuming in this paper that \( \max\{\theta^+, \theta^-\} < 1 \), the dimensions of the BESQ processes associated to \( V^\pm \) are strictly positive and the dimensions of the BESQ processes associated to \( U^\pm \) are strictly less than \( 2 \).

**Theorem 5.5** (Diffusion approximation in i.i.d. environments). Assume that the cookie environment is i.i.d. with marginal \( \eta \).

(1) Suppose that \( 4\nu^{-1}(1 + \eta \cdot r^+) = 2(1 - \theta^-) > 0 \) and consider a sequence of rescaled BLPs
\[
Y_m(t) := m^{-1} V_{m,|mt|}^+, \ t \geq 0,
\]
with initial distributions \( \kappa_m, Y_m(0) \sim \kappa_m. \) If \( \kappa_m \xrightarrow{m \to \infty} \kappa \) then
\[
\{Y_m(t)\}_{t \geq 0} \xrightarrow{J_1} \{Y(t)\}_{t \geq 0},
\]
where \( \{Y(t)\}_{t \geq 0} \) is the solution of \cite{Fel71} with \( D(t) \equiv 1 + \eta \cdot r^+ \) and \( Y(0) \sim \kappa \).

(2) Suppose that \( 4\nu^{-1}(\eta \cdot r^+) = 2\theta^- < 2 \) and consider a sequence of rescaled BLPs \( Y_m(t) := m^{-1} U_{m,|mt|}^+, \ t \geq 0 \), with initial distributions \( \kappa_m, Y_m(0) \sim \kappa_m. \) If \( \kappa_m \xrightarrow{m \to \infty} \kappa \) then
\[
\{Y_m(t)\}_{t \geq 0} \xrightarrow{J_1} \{Y(t \wedge \sigma_0^Y)\}_{t \geq 0},
\]
where \( \{Y(t)\}_{t \geq 0} \) is the solution of \cite{Fel71} with \( D(t) \equiv \eta \cdot r^+ \) and \( Y(0) \sim \kappa \). Moreover,
\[
\sigma_0^Y \Rightarrow \sigma_0^Y \text{ as } m \to \infty.
\]

**Remark 5.6.** Part (2) also holds for the process \( V^+ \) if \( 4\nu^{-1}(1 + \eta \cdot r^+) = 2(1 - \theta^-) < 2 \) provided that we replace the drift of the \( Y \) process in that part with \( 1 + \eta \cdot r^+ \) and \( Y_m(t) \) with \( Y_m(t \wedge \sigma_0^Y) \).

The proof needs practically no changes.

The proof of this theorem is standard and, for convenience of the reader, is given in the Appendix.

5.2. **Lifting from 0 and driving to extinction in i.i.d. environments.** The next two lemmas are stated for the \( V^+ \) process with parameter \( \theta^- < 1 \). Similar statements with identical proofs hold for the \( V^- \) process with parameter \( \theta^+ < 1 \).

**Lemma 5.7.** If the environment is i.i.d. \( \eta' \) where \( \theta^-(\eta') < 1 \) and \( V^+ \) is the BLP with \( V_0^+ = 0 \) then
\[
\lim_{\delta \to 0} \limsup_{m \to \infty} P \left( V_m^+ \leq \delta m \right) = 0.
\]

**Proof of Lemma 5.7.** The proof is based on the Dynkin-Lamperti theorem for renewal processes with infinite expectation, \cite[XIV.3, p. 472]{Fel71}.

Assume first that \( \theta^- \in (0, 1) \) so that the diffusion approximation \( Y = \{Y(t)\}_{t \geq 0} \) is a multiple of a BESQ\(^{2(1 - \theta^-)}\) of dimension strictly between 0 and 2. Let \( N_m = \sum_{k=1}^m 1_{\{V_k^+ = 0\}} \) and \( \sigma_i \) be the end of the \( i \)-th lifetime of \( V^+ \). Random variables \( \sigma_1, \sigma_2, \ldots \) are i.i.d. finite random variables with infinite expectation.\footnote{The lifetime \( \sigma_1 \) has infinite expectation for \( \theta^- < 1 \). For \( \theta^- < 0 \) also the probability that \( \sigma_1 = \infty \) is positive.} Then Dynkin-Lamperti theorem states that \( (m - \sigma_{N_m})/m \) and \( (\sigma_{N_{m+1}} - m)/m \) converge in distribution to random variables with explicit densities supported on \((0, 1)\) and \((0, \infty)\) respectively.

Proof of Lemma 5.7. The proof is based on the Dynkin-Lamperti theorem for renewal processes with infinite expectation, \cite[XIV.3, p. 472]{Fel71}.

Assume first that \( \theta^- \in (0, 1) \) so that the diffusion approximation \( Y = \{Y(t)\}_{t \geq 0} \) is a multiple of a BESQ\(^{2(1 - \theta^-)}\) of dimension strictly between 0 and 2. Let \( N_m = \sum_{k=1}^m 1_{\{V_k^+ = 0\}} \) and \( \sigma_i \) be the end of the \( i \)-th lifetime of \( V^+ \). Random variables \( \sigma_1, \sigma_2, \ldots \) are i.i.d. finite random variables with infinite expectation.\footnote{The lifetime \( \sigma_1 \) has infinite expectation for \( \theta^- < 1 \). For \( \theta^- < 0 \) also the probability that \( \sigma_1 = \infty \) is positive.} Then Dynkin-Lamperti theorem states that \( (m - \sigma_{N_m})/m \) and \( (\sigma_{N_{m+1}} - m)/m \) converge in distribution to random variables with explicit densities supported on \((0, 1)\) and \((0, \infty)\) respectively.
Given \( \varepsilon > 0 \), we can find an \( s > 0 \) such that \( P(\sigma_{N_m+1} - m \leq sm) < \varepsilon \) for all sufficiently large \( m \). Then
\[
P(V^+_m \leq \delta m) \leq P(V^+_m \leq \delta m, \sigma_{N_m+1} - m > sm) + P(\sigma_{N_m+1} - m \leq sm)
\leq P(\sigma_{N_m+1} - m > sm \mid V^+_m \leq \delta m) + P(\sigma_{N_m+1} - m \leq sm)
\leq P(\sigma_1 > sm \mid V^+_0 = \delta m) + P(\sigma_{N_m+1} - m \leq sm).
\]
Going from the second line to the third we used monotonicity of the BLP in the initial number of particles and Markov property. Taking a limit as \( m \to \infty \) we see that
\[
\limsup_{m \to \infty} P(V^+_m \leq \delta m) \leq P^Y(\tau_0 > s \mid Y(0) = \delta) + \varepsilon = P^Y(\tau_0 > \delta^{-2}s \mid Y(0) = 1) + \varepsilon.
\]
Finally, letting \( \delta \to 0 \) and using the fact that \( P^Y(\tau_0 = \infty \mid Y(0) = 1) = 0 \) for \( \theta^- \in (0,1) \) we get
\[
\lim_{\delta \to 0} \limsup_{m \to \infty} P(V^+_m \leq \delta m) \leq \varepsilon.
\]
Now we can let \( \varepsilon \to 0 \) and get the result for the case when \( \theta^- \in (0,1) \).

If \( \theta^- \leq 0 \) then the process \( V^+ \) can be coupled with a “smaller” process (corresponding to \( \theta^- \in (0,1) \)). The coupling can be done by adding one or more cookies of strength \( \min_{i \leq N(p_i)} \) before the first cookie in each stack (or a geometric number of these with an appropriate success probability). Details of how such a coupling can be constructed can be found in [KP17, Section 5.1]. This will complete the proof of the lemma by comparison. \( \square \)

Next we show that BLPs which evolve in environments close to i.i.d. and which admit an approximation by a BESQ process of dimension less than 2 will become extinct very soon after becoming macroscopically small.

**Lemma 5.8.** Let \( Z \) be a BLP in an i.i.d. cookie environment. Assume that it admits an approximation by a BESQ process of dimension strictly less than 2. Then for all \( \delta, \varepsilon > 0 \) there is a \( \delta' > 0 \) such that for all sufficiently large \( m \)
\[
P(\sigma_0 > \delta m \mid Z_0 \leq \delta' m) < \varepsilon.
\]

*Proof.* The proof is the same as that of (5.5) in [KM11]. \( \square \)

### 5.3. BLPs in \( m^{1/4} \)-good environments.

In this section we extend the diffusion approximation of BLPs (and some of the resulting consequences) from the case where the first cookie environments are i.i.d. to the weaker condition of \( (m^{1/4}, \rho) \)-good. The cost of this relaxation is that we will not be able to get convergence of the hitting time of 0 as in (20). Nevertheless, we will be able to get enough control on this hitting time (Lemma 5.13) for our applications later.

A number of the results in this section hold for more than one of the four different BLPs (\( U^\pm \) or \( V^\pm \)). Thus, if a result holds for one or more of these BLPs we will state the result in terms of a generic BLP \( Z \) and will state which of the four BLPs \( Z \) can be (if no restrictions are made it is assumed that \( Z \) can be any of the four BLPs). Also, if the result concerns one of the BLPs using the cookie environment on the interval \([a,b]\), we will always assume that if the BLP \( Z \) is either \( U^+ \) or \( V^+ \) then the cookie stacks are used to generate successive generations of the BLP from left to right whereas if the BLP is either \( U^- \) or \( V^- \) then the cookie stacks are used from right to left.

**Theorem 5.9** (Diffusion approximation in \( m^{1/4} \)-good environments). Suppose that for some \( \rho \in \mathbb{R}, T > 0 \) the first cookies are \( (m^{1/4}, \rho) \)-good on intervals \([0,mT]\) for all sufficiently large \( m \). Fix an arbitrary \( \delta > 0 \) and consider a sequence of rescaled BLPs \( Y^*_m(t) := m^{-1}Z^m_{|mt| \wedge \sigma_{sm}}, t \in [0,T], \) with initial distributions \( Y^*_m(0) \sim \kappa_m \). If \( \kappa_m \Rightarrow \kappa \) then
\[
\{Y^*_m(t)\}_{0 \leq t \leq T} \xrightarrow{f_1} \{Y(t \wedge \sigma_{s})\}_{0 \leq t \leq T},
\]
where \( \{Y(t)\}_{t \geq 0} \) is the solution of (19) with \( Y(0) \sim \mathcal{U} \) and where \( D(t) \) is a constant equal to \( \rho \) for \( U^+ \), \( \rho + 1 \) for \( V^+ \), \( \nu/2 - 1 - \rho \) for \( U^- \), and \( \nu/2 - \rho \) for \( V^- \).

The proof of this theorem is given in the Appendix.

The diffusion approximation in Theorem 5.9 guarantees the convergence as long as the processes stay macroscopically away from zero. Nevertheless, when the limiting diffusion process is a \( \text{BESQ}^0 \) process, the diffusion approximation can be extended to all times (see Corollary 5.12). This will follow from Theorem 5.9 together with the following lemma which says that when the BLP becomes “macroscopically small” and then it cannot become “macroscopically much larger” during a fixed macroscopic time period.

**Lemma 5.10.** Let \( Z^m \) be a BLP using the cookies on the interval \([0, m]\), and suppose that the first cookies on intervals \([0, m]\) are \((m^{1/4}, \rho)\)-good where the parameter \( \rho \) is such that the family \( Z^m \), \( m \in \mathbb{N} \), admits an approximation by a \( \text{BESQ} \) process of dimension 0. Then \( \forall \varepsilon > 0, \forall \delta > 0 \) there is a \( \delta' \in (0, \delta) \) such that for all sufficiently large \( m \)

\[
P(\max_{j \leq m} Z_j^m \geq \delta m \mid Z_0^m \leq \delta' m) < \varepsilon.
\]

**Remark 5.11.** We will only apply Lemma 5.10 in the case of the BLP \( U^+ \) or \( U^- \). Due to Theorem 5.9, we see that the condition that the approximating \( \text{BESQ} \) process is of dimension 0 if \( \rho = 0 \) in the case of \( U^+ \) or \( \rho = \nu/2 - 1 \) in the case of \( U^- \).

**Proof.** Recall that \( \tau_{Z_j^m}^2 \) is the first entrance time of the process \( Z \) to the interval \([x, \infty)\). It is notationally convenient to prove an equivalent statement, namely, that \( \forall \varepsilon > 0, \forall L > 1 \) there is a \( k \in \mathbb{N} \) such that for all sufficiently large \( m \)

\[
P(\tau_{Z_j^m}^2 \leq 2^k L m \mid Z_0^m = m) < \varepsilon.
\]

The equivalence can be easily seen from the following relabeling (from the last expression to the original): \( m \rightarrow \delta' m \), \( 2^k \rightarrow \delta/\delta' \), \( L \rightarrow 1/\delta \).

Our proof is based on comparison of \( Z_j^m \), \( j \geq 0 \), with a modified process \( \tilde{Z}_j^m \), \( j \geq 0 \), and a diffusion approximation. The process \( \tilde{Z}_j^m \) coincides with \( Z^m \) up until \( \sigma_m^{Z_m} \) at which it resets to \( m \). After the reset it continues as a “fresh copy” of \( Z^m \) but in the environment shifted by \( \sigma_m^{Z_m} \) and \( \tau_{Z_j^m}^2 \), and so on. Let \( T_k^m, i \in \mathbb{N} \), be the sequence of waiting times between consecutive resets of \( Z_j^m \) and \( N_k^m \) be the total number of resets until the first reset from the upper boundary inclusively. Then by construction and monotonicity of BLPs \( \tilde{Z}_j^m \geq Z_j^m \) for all \( j < \sum_{i=1}^{N_k^m} T_k^m \) and, therefore,

\[
P(\tau_{Z_j^m}^2 \leq 2^k L m \mid Z_0^m = m) \leq P\left(\sum_{i=1}^{N_k^m} T_k^m \leq 2^k L m \mid Z_0^m = m\right).
\]

We conclude that it is enough to show that \( \forall \varepsilon > 0 \) and \( \forall L > 1 \) there is a \( k \in \mathbb{N} \) such that for all sufficiently large \( m \)

\[
P\left(\sum_{i=1}^{N_k^m} T_k^m \leq 2^k L m \mid Z_0^m = m\right) < \varepsilon.
\]

By Theorem 5.9 the process \( m^{-1} Z^m \) admits an approximation by the zero dimensional \( \text{BESQ} \) process \( Y \) with the starting point \( Y(0) = 1 \) up to the time \( \sigma_{1/2}^Y \). Note that

\[
P(\sigma_{1/2}^Y < \tau_{2^k}^Y \mid Y(0) = 1) = 1 - (2^{k+1} - 1)^{-1}, \quad \forall k \in \mathbb{N}.
\]

We shall also consider a right-continuous process \( \tilde{Y}_k(t), t \geq 0 \), which coincides with \( Y \) up to the time \( \sigma_{1/2}^Y \) and \( \tau_{2^k}^Y \), jumps to 1 at time \( \sigma_{1/2}^Y \) and \( \tau_{2^k}^Y \) and continues to follow a “fresh copy” of \( Y \) until it
again hits the boundary of $[1/2, 2^k]$ at which time $\bar{Y}_k$ resets to 1, and so on. Let $T_{k,i}, i \in \mathbb{N}$, be a sequence of waiting times between consecutive jumps of $\bar{Y}_k$. Random variables $T_{k,i}, i \in \mathbb{N}$, are i.i.d. and have the same distribution as $\sigma_{1/2}^Y \land \tau_{2^k}^Y$. Denote by $N_k$ be the number of jumps of $\bar{Y}$ until the first jump down from $2^k$ to 1 inclusively. By construction, $N_k$ has a geometric distribution on $\mathbb{N}$ with parameter $(2^{k+1} - 1)^{-1}$. Given an arbitrary $\varepsilon > 0$ and $L > 0$ we shall first show that there is a $k \in \mathbb{N}$ such that

$$P \left( \sum_{i=1}^{N_k} T_{k,i} \leq 2^k L \right) \leq \varepsilon / 2 \tag{22}$$

and then argue that (21) holds by Theorem 5.9.

For any fixed $\alpha \in (0, \varepsilon / 4)$, $k_0 \in \mathbb{N}$, and $k \geq k_0 \lor \log_2 \frac{1}{\alpha}$ we have

$$P \left( \sum_{i=1}^{N_k} T_{k,i} \leq 2^k L \right) \leq P \left( \sum_{i=1}^{\alpha 2^k} T_{k,i} \leq 2^k L, N_k \geq \lfloor \alpha 2^k \rfloor \right) + P(N_k < \lfloor \alpha 2^k \rfloor) \tag{23}$$

Centering, we get that

$$P \left( \sum_{i=1}^{N_k} T_{k,i} \leq 2^k L \right) \leq P \left( \frac{1}{\lfloor \alpha 2^k \rfloor} \sum_{i=1}^{\lfloor \alpha 2^k \rfloor} (T_{k,i} - E(T_{k,i})) \leq - \left( E(T_{k,0,1}) - \frac{L 2^k}{\lfloor \alpha 2^k \rfloor} \right) \right) + \frac{\alpha 2^k}{2^{k+1} - 1}$$

Applying the optional stopping theorem to the local martingale $Y(t) \ln Y(t) - t$ we obtain

$$E(T_{k,0,1}) = \frac{\ln 2(2^{k_0} - 2^{k_0} + 1)}{2^{k_0+1} - 1} \quad \text{and} \quad \lim_{k_0 \to \infty} \frac{E(T_{k,0,1})}{(2 - 1 \ln 2)k_0} = 1.$$ 

Thus, we can choose $k_0$ so that $2L / \alpha < E(T_{k,0,1}) / 2$ and conclude by the weak law of large numbers that (22) holds for all sufficiently large $k$.

Return now to the process $\bar{Z}^{m,k}$. By Theorem 5.9 and the continuous mapping theorem

$$\frac{T_{k,i}^m}{m} \Rightarrow T_{k,i}, \forall i, k \in \mathbb{N}. \tag{24}$$

Since the cookie stacks, given the first cookies, are independent and $\bar{Z}^m$ has the strong Markov property, $\{T_{k,i}^m\}_{i \in \mathbb{N}}$ is a sequence of independent random variables while $\{T_{k,i}\}_{i \in \mathbb{N}}$ is an i.i.d. sequence. Therefore, for each fixed $n \in \mathbb{N}$ we also have that

$$\sum_{i=1}^{n} \frac{T_{k,i}^m}{m} \Rightarrow \sum_{i=1}^{n} T_{k,i}. \tag{25}$$

Next, we claim that $N_k^m \Rightarrow N_k$. Indeed, denoting by $p_{k,i}^m$ the probability that the $i$-th reset of $\bar{Z}^{m,k}$ is from the upper boundary we have, again by Theorem 5.9, that $\forall k, n \in \mathbb{N}$

$$P(N_k^m > n) = \prod_{i=1}^{n} (1 - p_{k,i}^m) \Rightarrow \left(1 - \frac{1}{2^{k+1} - 1}\right)^n = P(N_k > n).$$

Repeating the same steps for $T_{k,i}^m$ and $N_{k}^m$ as in [23] and using the weak convergence results (24) and (25) we conclude that with the same choice of $k_0$ and $k$ as above the inequality (21) holds for all sufficiently large $m$.

Corollary 5.12. Assume the conditions of Lemma [5.10] and consider a sequence of rescaled BLPs $Y_m(t) := m^{-1}U_{[m]}^+, t \in [0,T]$, with $Y_m(0) \sim \kappa_m$. If $\kappa_m \Rightarrow \kappa$ then

\[
\{Y_m(t)\}_{0 \leq t \leq T} \xrightarrow{\mathcal{D}} \{Y(t)\}_{0 \leq t \leq T},
\]

where $\{Y(t)\}_{t \geq 0}$ solves (19) with $D(t) \equiv 0$ and $Y(0) \sim \kappa$.

For the proof of Corollary 5.12 we refer to Remark A.2.

Note that Corollary 5.12 does not imply that the stopping times $\sigma_0^Y_m$ converge in distribution to $\sigma_0^Y$. However, our environments have additional properties which will allow us to get more information.

Assumption 2. Let $c_m \in \mathbb{Z}_+/m$ and $c_m \rightarrow c \in [0,1]$. Assume that the first cookie environments are $(m^{1/4}, 0)$-good on $[0, c_m m]$ and are i.i.d. with marginal $\eta$ on $[c_m m, m]$. Suppose also that if $c = 1$, then the interval $[(1 - \delta_1)m, m]$ is $\delta_2m$-grounding for some $\delta_1, \delta_2 > 0$.

Lemma 5.13. Fix $\varepsilon > 0$ and an arbitrary sequence $y_m \in \mathbb{Z}_+/m$, $m \in \mathbb{N}$. Let $\{Z^m_k\}_{k \leq m}$, $m \in \mathbb{N}$, be $U^+$ processes starting at $y_m m$ in first cookie environments on $[0, m]$ satisfying Assumption 3. Let $Y_m(t), t \in [0,1]$ be a solution of (19) with $D(t) = (\eta \cdot r^+)1_{\{t > c\}}$ and $Y_m(0) = y_m$. If $\delta_1, \delta_2 > 0$ are sufficiently small then there is an $m_0 = m_0(\varepsilon, c, \delta_1, \delta_2)$ which is independent of the choice of sequence $y_m$, $m \in \mathbb{N}$, such that for all $m \geq m_0$

\[
|P(\sigma_0^{Z^m} < m) - P(\sigma_0^Y < m)| < 4\varepsilon^3.
\]

Proof. To reduce notation we write $g(x) = P(\sigma_0^Y < 1 | Y(0) = x)$ where $Y$ solves the same equation as all the $Y_m$. The function $g$ is decreasing and it is easily seen that $g(x)$ is continuous and tends to zero as $x \rightarrow \infty$. Thus we can set $g(\infty) = 0$ and choose $0 < y^1 < y^2 < \ldots < y^R < y^{R+1} = \infty$ so that $g(y^i) - g(y^{i+1}) < \varepsilon^3$ for all $i \in \{1, 2, \ldots, R\}$. For each $y^i$, $i \in \{1, 2, \ldots, R\}$, we define a sequence $y^i_m = \left[\frac{m y^i}{m}\right] \in \left[\frac{y^i}{m}, \frac{y^i + 1}{m}\right)$, $m \in \mathbb{N}$. We claim that it is sufficient to show that there is an $m_0 = m_0(\varepsilon, c, \delta_1, \delta_2)$ such that for all $m \geq m_0$

\[
|P(\sigma_0^{Z^m} < m | Z^m_0 = y^i_m m) - g(y^i)| < 3\varepsilon^3
\]

for all $i \in \{1, 2, \ldots, R\}$.

Indeed, suppose (27) holds and $y_m \in \mathbb{Z}_+/m$. Then $y^i_m \leq y_m < y^{i+1}$ for some $i \in \{1, 2, \ldots, R\}$. Since for our choice of the sequence $y^i_m$ we have

\[
y^i \leq y^i_m \leq y_m \leq y^{i+1} - \frac{1}{m} < y^{i+1} \leq y^i_{m+1},
\]

by monotonicity we get

\[
P(\sigma_0^{Z^m} < m | Z^m_0 = y^i_m m) \leq P(\sigma_0^{Z^m} < m | Z^m_0 = y^i m) < g(y^i) + 3\varepsilon^3 < g(y^i) + 4\varepsilon^3,
\]

\[
P(\sigma_0^{Z^m} < m | Z^m_0 = y^i m) \geq P(\sigma_0^{Z^m} < m | Z^m_0 = y^{i+1} m) > g(y^{i+1}) - 3\varepsilon^3 > g(y^i) - 4\varepsilon^3,
\]

and (26) follows. Thus, we need only to prove (27).

Fix an $i \in \{1, 2, \ldots, R\}$ and let $Y$ be a solution of (19) with $Y(0) = y^i$ and $Z$ be the $U^+$ process with $Z_0^m = y^i_m m$ evolving in the cookie environment on $[0, m]$.

Case $c < 1$. For any $\delta_3 > 0$ we have

\[
|P(\sigma_0^{Z^m} < m) - P(\sigma_0^Y < 1)| = |P(\sigma_0^{Z^m} \geq m) - P(\sigma_0^Y > 1)|
\]

\[
\leq P(\sigma_0^{Z^m} \geq m, \sigma_{\delta_3 m}^{Z^m} \leq c m) + P(\sigma_0^Y > 1, \sigma_{\delta_3}^Y < c)
\]

\[
+ |P(\sigma_0^{Z^m} \geq m, \sigma_{\delta_3 m}^{Z^m} > c m) - P(\sigma_0^Y > 1, \sigma_{\delta_3}^Y > c)|.
\]
We claim that we can choose $\delta_3$ small enough so that all three terms in the right hand side of (28) are small when $m$ is large. We shall treat the first and the last terms since the second term can be dealt with like the first but is simpler.

That the third term tends to zero is an immediate consequence of Theorem 5.9 and then Theorem 5.5 and the fact that the boundary of set $\{\sigma_0^Y > 1, \sigma_3^\delta < c\}$ has probability zero.

For the first term in (28), we choose $\delta_4$ so that $P(\sigma_0^Y \geq 1 \mid Y(c) = \delta_4) < \epsilon^3/4$. Then we fix $\delta_3$ so that $(\delta_3, \delta_4, \epsilon^3/4)$ are as $(\delta', \delta, \epsilon)$ for Lemma 5.10 Next we note that

$$\{\sigma_0^Z \geq m, \sigma_3^Z \leq c m \} \subseteq \{\sigma_3^m \leq c m, Z_{c m} \geq \delta_4 m\} \cup \{Z_{c m} \leq \delta_4 m, \sigma_0^Z \geq m\}.$$  

By Lemma 5.10 the first event has probability less than $\epsilon^3/4$ by our choice of $\delta_3$ and $\delta_4$ for $m$ large, while the probability of the second event is similarly bounded by Theorem 5.5 and our choice of $\delta_1$.

Case $c = 1$. For any $\delta_3 > 0$

$$|P(\sigma_0^Y < 1) - P(\sigma_3^Y < 1 - \delta_1)| \leq P(\sigma_0^Y \geq 1, \sigma_3^Y < 1 - \delta_1) + P(\sigma_3^Y \in (1 - \delta_1, 1))$$

and similarly with $Y$ replaced by $m^{-1}Z^m$.

$$\lim_{m \to \infty} P(\sigma_3^m \leq m(1 - \delta_1)) = P(\sigma_3^Y < 1 - \delta_1),$$

it will suffice to show that the two terms on the right are bounded appropriately for process $Y$ and for $Z^m$ for $m$ large provided $\delta_1$ and then $\delta_3$ are well chosen. For the second term for $Y$ we first choose $\delta_1$ so small that $P(\sigma_0^Y \in (1 - \delta_1, 1 + \delta_1)) < \epsilon^3/10$ and then choose $\delta_3$ so small that $P(\sigma_0^Y < 1 \mid Y(0) = \delta_3) \geq 1/2$. The strong Markov property then gives the bound

$$P(\sigma_0^Y \in (1 - \delta_1, 1 + \delta_1)) \geq P(\sigma_0^Y \in (1 - \delta_1, 1 + \delta_1) \mid \sigma_3^Y \in (1 - \delta_1, 1)) P(\sigma_3^Y \in (1 - \delta_1, 1)) \geq 1/2 P(\sigma_3^Y \in (1 - \delta_1, 1)),$$

from which we conclude that $P(\sigma_3^Y \in (1 - \delta_1, 1)) < \epsilon^3/5$. This bound applies for $Z^m$ when $m$ is large by Theorem 5.9.

For the first term we treat $Z^m$ as the argument for $Y$ is similar but simpler. Decreasing $\delta_3$ if necessary we ensure that $(\delta_3, \delta_2, \epsilon^3/5)$ are as $(\delta', \delta, \epsilon)$ for Lemma 5.10 Then

$$\{\sigma_0^Z \geq m, \sigma_3^Z \leq (1 - \delta_1)m\} \subseteq \{\sigma_3^Z \leq (1 - \delta_1)m, Z_{(1 - \delta_1)m} \geq \delta_2 m\} \cup \{Z_{(1 - \delta_1)m} \leq \delta_2 m, \sigma_0^Z \geq m\}.$$  

The probability of the first set is less than $\epsilon^3/5$ for $m$ large by Theorem 5.10 while of the last is less than $\epsilon^3$ by our grounding hypothesis for the cookie environments. This latter bound can be reduced arbitrarily for $Y$.

The following lemma expresses a simple coupling result which leads to Corollary 5.17 below.

**Lemma 5.14.** For every $\epsilon > 0$ there exists a $\delta' > 0$ such that a BESQ$^2$ process $Y$ beginning at space-time point $(y, t) \in [0, \delta']^2$ and a BESQ$^2$ process $Z$ beginning at $(0, 0)$ can be coupled together so that with probability at least $1 - \epsilon^3/5$ there exists a $\sigma \in (t, \epsilon^5)$ such that

(i) $Y(s) = Z(s)$ for all $s \in [\sigma, \infty]$;

(ii) $(\sup_{t \leq s \leq \sigma} Y(s)) \lor (\sup_{t \leq s \leq \sigma} Z(s)) \leq \epsilon^5$.

The next lemma provides a basic coupling of two BESQ$^2$ processes. Lemma 5.14 follows from it by a simple scaling argument in the same way that Corollary 3.8 follows from Lemma 3.7. For the coupling we will make the processes independent until the first time that they meet and then equal thereafter.
Lemma 5.15. Let \( Y \) and \( Z \) be independent BESQ\(^2\) processes beginning at space-time points \((y_0, t_0) \in [0, 1]^2\) and \((z_0, s_0) \in [0, 1]^2\) respectively and \( \sigma = \inf\{ s > s_0 \lor t_0 : Z(s) = Y(s) \text{ or } Z(s) \lor Y(s) = 2 \} \land 2 \). Then
\[
c := \inf_{(y_0, t_0, z_0, s_0) \in [0, 1]^4} P(\sigma = \inf\{ s > s_0 \lor t_0 : Z(s) = Y(s) \}) > 0.
\]

Proof. We suppose without loss of generality that \( s_0 < t_0 \) and let
\[
A_1 = \left\{ \max_{s_0 \leq u \leq 3/2} Z(u) < 3/2, Z(3/2) > 4/3 \right\};
\]
\[
A_2 = \left\{ \max_{t_0 \leq u \leq 3/2} Z(u) < 3/2, \max_{t_0 \leq u \leq 3/2} Y(u) < 4/3, \max_{3/2 < u \leq 2} Y(u) < 2, Y(2) > 5/3 \right\}.
\]
It is clear that \( P(A_1) \) and \( P(A_2 \mid A_1) \) are bounded away from zero uniformly over \((y_0, t_0, z_0, s_0) \in [0, 1]^4\). Noticing that \( A_1 \cap A_2 \subseteq \{ \sigma = \inf\{ s > s_0 \lor t_0 : Z(s) = Y(s) \} \} \) completes the proof. \( \square \)

The next statement is a consequence of [KP17] Lemma 6.3.

Lemma 5.16. Let \( V_0^+ = 0 \). Then for every fixed \( \delta > 0 \) uniformly over first cookie environments
\[
\lim_{m \to \infty} P(\tau_{\delta m}^{V^+} \leq \delta m, V_{\tau_{\delta m}^{V^+}} \geq \delta m + m^{2/3}) = 0.
\]

The following corollary is immediate given Theorem 5.9, Lemma 5.14, Lemma 5.16 and the fact that BESQ\(^2\) processes do not return to zero.

Corollary 5.17. Given \( \varepsilon > 0 \), parameters \( \delta_1, \delta_2 > 0 \) can be chosen so small that for any sequence of first cookie environments on \([0, m]\) which are

(i) \( (m^{1/4}, \nu/2 - 1) \)-good,
(ii) \( \delta_2 m \)-lifting from the left on \([0, \delta_1 m]\)

and any fixed \( s \in [\varepsilon^5, 1] \), every distributional limit point of \( \{ m^{-1} V_{[s m]}^+ \}_{m \geq 1} \) with \( V_0^+ = 0 \) evolving in this environment must be within \( 3 \varepsilon^3/2 \) (in total variation distance) of the law of \( Y(s) \) where \( Y \) is a solution of (19) with \( D \equiv \nu/2 \) and \( Y(0) = 0 \), i.e. a \( \frac{\nu}{4} \) BESQ\(^2\) process.

Proposition 5.18. Denote by \( \{ Z_k^m \}_{0 \leq k \leq 2m} \) a concatenation of a \( V^+ \) process on \([0, m]\) starting at 0 and a \( U^+ \) process on \([m, 2m]\). Given \( \varepsilon \in (0, \varepsilon_0) \) for some \( \varepsilon_0 \) fixed small, parameters \( \delta_1, \delta_2 > 0 \) can be chosen so small that for any sequence of first cookie environments on \([0, 2m]\) which are

(i) \( (m^{1/4}, \nu/2 - 1) \)-good on \([0, m]\) and \( (m^{1/4}, 0) \)-good on \([m, 2m]\);
(ii) \( \delta_2 m \)-lifting from the left on \([0, \delta_1 m]\) and \( \delta_2 m \)-grounding from the left on \([2m - \delta_1 m, 2m]\),

every limit point of \( m^{-2} \sum_{k=0}^{2m-1} Z_k^m \mathbb{1}_{\{ Z_{2m-1}^m = 0 \}} \) must be of the form
\[
\int K(z, \cdot) \lambda_1(dz)
\]
where \( K(\cdot, \cdot) \) is a probability kernel satisfying \( K(z, [z - \varepsilon^8, z + \varepsilon^8]) = 0 \) for all \( z \) and \( \lambda_1 \) is a probability measure within \( 8 \varepsilon^3 \) (in total variation distance) from the law that is one half \( \delta_0 \) plus one half the law of \( \frac{\nu}{4} \) times the time for the standard Brownian motion to exit \((-1, 1)\).

Proof. Throughout the proof \( Y \) will be a solution of (19) with \( D(t) = \frac{\nu}{4} \mathbb{1}_{\{t < 1\}} \) with initial condition \( Y(0) = 0 \). The previous results in this section will allow us to approximate certain probabilities for the process \( Z^m \) in terms of corresponding probabilities involving the process \( Y \). Also, since \( Y \) is \( \frac{\nu}{4} \) times a concatenation of a standard BESQ\(^2\) process and BESQ\(^0\) process, it follows easily from the Ray-Knight Theorems that \( \int_0^2 Y(s) ds \mathbb{1}_{\{ \sigma_1, 0 < 2 \}} \) has the law that is one half \( \delta_0 \) plus one half the law of \( \frac{\nu}{4} \) times the time for the standard Brownian motion to exit \((-1, 1)\).
We fix a small $\delta_3 > 0$ and write $\Sigma^m := \left( \sum_{k=0}^{2m-1} Z^m_k \right) \mathbb{1}_{\{Z_{2m-1}^m = 0\}}$ as $\Sigma^a + \Sigma^b + \Sigma^c$ where

$$\Sigma^m_a = \sum_{k=0}^{e^m} Z^m_k \mathbb{1}_{\{\sigma_{m,0}<2m\}}, \quad \Sigma^m_b = \sum_{k=e^m+1}^{\sigma_{m,m\delta}} Z^m_k \mathbb{1}_{\{\sigma_{m,0}<2m\}}, \quad \Sigma^m_c = \sum_{k=\sigma_{m,m\delta}+1}^{2m-1} Z^m_k \mathbb{1}_{\{\sigma_{m,0}<2m\}}.$$

Step 1. We claim that if $\delta_3, \delta_1$ are fixed sufficiently small, then outside of probability $\varepsilon^3/4$, (i) $\Sigma^m_a/m^2$ and $\Sigma^m_c/m^2$ are less than $\varepsilon^8/4$ for all $m$ large.

(ii) $\int_0^{\varepsilon^5} Y(s) ds \mathbb{1}_{\{\sigma_{1,0}<2\}}$ and $\int_{\varepsilon^5}^{2} Y(s) ds \mathbb{1}_{\{\sigma_{1,0}<2\}}$ are less than $\varepsilon^8/4$ for all $m$ large.

We first consider (i). The bound for $\Sigma^m_a$ is easily seen, since by monotonicity in the initial number of particles we have for all sufficiently small $\varepsilon$

$$P(\Sigma^m_a > (\varepsilon^8/4)m^2) \leq P(\tau_{\varepsilon^4 m}^Z \leq \varepsilon^5 m \mid Z_0^m = 0) \leq P(\tau_{\varepsilon^4 m}^Z \leq \varepsilon^5 m \mid Z_0^m = \varepsilon^5 m),$$

and it follows from Theorem 5.9 that the last probability is at most $\varepsilon^3/4$ for $m$ large enough. Similarly, if the the event $\{\Sigma^m_c \geq (\varepsilon^8/4)m^2\}$ occurs then after time $\sigma_{m,m\delta}$ the process $Z^m$ goes above $(\varepsilon^8/4)m$ before time $2m$, and by Lemma 5.10 the probability of this is less than $\varepsilon^3/4$ if $\delta_3$ is chosen sufficiently small (depending on $\varepsilon$). This finishes the proof of (i). The proof of the bounds for (ii) is similar (but simpler since we do not need to go through the diffusion approximation steps).

Step 2. We claim that the total variation distance between any limit point of $\Sigma^m_b/m^2$ and $\int_{\varepsilon^5}^{2} Y(s) ds \mathbb{1}_{\{\sigma_{1,0}<2\}}$ is less than $7\varepsilon^3/2$. We begin by introducing some notation. Let $\Sigma_a$, $\Sigma_b$, and $\Sigma_c$ be the analogs of $\Sigma_a^m$, $\Sigma_b^m$, and $\Sigma_c^m$, respectively for the process $Y$ in place of $Z^m$, that is

$$\Sigma_a = \int_0^{\varepsilon^5} Y(s) ds \mathbb{1}_{\{\sigma_{1,0}<2\}}, \quad \Sigma_b = \int_{\varepsilon^5}^{\sigma_{1,\delta_3}} Y(s) ds \mathbb{1}_{\{\sigma_{1,0}<2\}}, \quad \Sigma_c = \int_{\sigma_{1,\delta_3}}^{2} Y(s) ds \mathbb{1}_{\{\sigma_{1,0}<2\}}.$$

Also, let $\tilde{\Sigma}_b = \left( \sum_{k=\varepsilon^m+1}^{\sigma_{m,m\delta}} Z^m_k \right) \mathbb{1}_{\{\sigma_{m,m\delta}<2m\}}$ and $\tilde{\Sigma}_b = \int_{\varepsilon^5}^{\sigma_{1,\delta_3}} Y(s) ds \mathbb{1}_{\{\sigma_{1,\delta_3}<2\}}$.

Let $V$ be a weak subsequential limit of $\Sigma^m_b/m^2$. We can then take a further subsequence (which, for ease of notation, we will keep denoting by $m \in \mathbb{N}$) along which

- $(\Sigma^m_b/m^2, \Sigma^m_c/m^2)$ converges in distribution to a random vector $(V, \tilde{V})$,
- $Z^m_{\varepsilon^m/m}$ converges in distribution to a random variable $\zeta$,
- and $Z^m_{\varepsilon^m/m}$ converges in distribution to a random variable $\zeta$.

Note that it follows from Corollary 5.17 that $d_{TV}(\zeta_1, Y(\varepsilon^5)) < 3\varepsilon^3/2$ and $d_{TV}(\zeta_2, Y(1)) < 3\varepsilon^3/2$. Then, using Theorem 5.9 and the fact that $Z^m_{\varepsilon^m/m} \Rightarrow \zeta$ we can conclude that the distribution of $\tilde{V}$ is given by

$$P(\tilde{V} \in A) = \int P(\tilde{\Sigma}_b \in A \mid Y(\varepsilon^5) = z) P(\zeta_1 \in dz).$$

From this it follows that $d_{TV}(\tilde{V}, \tilde{\Sigma}_b) \leq d_{TV}(\zeta_1, Y(\varepsilon^5)) \leq 3\varepsilon^3/2$. Next, note that

$$d_{TV}(\Sigma_b, \tilde{\Sigma}_b) \leq P(\Sigma_b \neq \tilde{\Sigma}_b) \leq P(\Sigma_{1,\delta_3}^Y < 2 - \delta_1, \sigma_{1,0}^Y > 2) + P(\sigma_{1,\delta_3}^Y \in (2 - \delta_1, 2)).$$

We first choose $\delta_1$ so small that $\sup_{y>0} P(\sigma_{1,0}^Y \in (2 - \delta_1, 2 + \delta_1) \mid Y(1) = y) < \varepsilon^3/10$. Next we choose $\delta_3$ depending on $\delta_1$ so that

$$P(\sigma_{1,\delta_3}^Y < 2 - \delta_1, \sigma_{1,0}^Y > 2) \leq P(\sigma_{1,0}^Y > \delta_1 \mid Y(1) = \delta_3) < \varepsilon^3/10.$$
This implies first of all that

$$\sup_{y \geq 0} P(\gamma_1 \in (2 - \delta_1, 2) \mid Y(1) = y) < \frac{\varepsilon^3}{5},$$

and, therefore, $d_{TV}(\Sigma_b, \tilde{\Sigma}_b) < 2\varepsilon^3/5$. In a similar manner,

$$P(\Sigma^m_b \neq \tilde{\Sigma}^m_b) = P\left(\sigma^m_{\delta_3 m} < 2m \leq \sigma^m_{\delta_0 m}\right) \leq P\left(\sigma^m_{\delta_3 m} \leq (2 - \delta_1)m, \sigma^m_{\delta_0 m} \geq 2m\right) + P\left(\sigma^m_{\delta_3 m} \in ((2 - \delta_1)m, 2m)\right).$$

By the argument at the end of the proof of Lemma 5.13 we can bound the first probability above by $6\varepsilon^3/5$ for $\delta_3$ small and $m$ large enough, while Theorem 5.9 together with (30) imply that the second probability can also be bounded above by $\varepsilon^3/5$ as $m \to \infty$. Then using the joint convergence $(\Sigma^m_b / m^2, \tilde{\Sigma}^m_b / m^2) \Rightarrow (V, \tilde{V})$, we get that

$$d_{TV}(V, \tilde{V}) \leq P(V \neq \tilde{V}) \leq \liminf_{m \to \infty} P(\Sigma^m_b \neq \tilde{\Sigma}^m_b) \leq \frac{7\varepsilon^3}{5}.$$  

Combining the above estimates we conclude that

$$d_{TV}(V, \Sigma_b) \leq d_{TV}(V, \tilde{V}) + d_{TV}(\tilde{V}, \tilde{\Sigma}_b) + d_{TV}(\tilde{\Sigma}_b, \Sigma_b) < \frac{7\varepsilon^3}{2}.$$

**Step 3.** Now any weak limit of $\{\Sigma^m\}_{m \geq 1}$ can be written as $U + V$ where $(U, V)$ is a weak limit (possibly along a further subsequence) of $\{(\Sigma^m_a + \Sigma^m_c)/m^2, \Sigma^m_b / m^2\}_{m \geq 1}$. Moreover, as we have shown in Steps 1 and 2,

- $V$ has law within $7\varepsilon^3/2$ of the law of $\int_{\varepsilon^3}^{\varepsilon^3/4} Y(s) ds 1_{\{\sigma_1, o < 2\}}$ in total variation distance.
- $U \leq \varepsilon^8/2$ outside of probability $\varepsilon^3/2$.

By Step 1(ii) and Step 2, we can adjoin a positive variable $U'$ so that $U' + V$ has law within $7\varepsilon^3/2$ in total variation distance of that of $\int_{0}^{2} Y(s) ds 1_{\{\sigma_1, o < 2\}}$ and $U' \leq \varepsilon^8/2$ outside of probability $4\varepsilon^3$. Indeed, let $(U', V)$ have joint distribution defined by the regular conditional probability

$$P(U' \in A \mid V = x) = P(\Sigma_a + \Sigma_c \in A \mid \Sigma_b = x).$$

This implies first of all that

$$d_{TV}\left(U' + V, \int_{0}^{2} Y(s) ds 1_{\{\sigma_1, o < 2\}}\right) \leq d_{TV}(V, \Sigma_b) \leq \frac{7}{2} \varepsilon^3,$$

and secondly that

$$\left| P\left(U' > \frac{\varepsilon^8}{2}\right) - P\left(\Sigma_a + \Sigma_c > \frac{\varepsilon^8}{2}\right)\right| \leq \frac{7}{2} \varepsilon^3$$

and then Step 1(ii) implies that $P(U' > \varepsilon^8/2) \leq 7\varepsilon^3/2 + \varepsilon^3/2 = 4\varepsilon^3$.

From this we see that we can write the limiting distribution of $\{\Sigma^m\}_{m \geq 1}$ along this subsequence as that of $(V + U') + U - U'$. We now introduce the following kernels or sub-kernels

- $L$ denotes the regular conditional probability of $U - U'$ given $V + U'$;
- $L^\varepsilon$ denotes the sub-kernel $L^\varepsilon(x, D) = L(x, D \cap [-\varepsilon^8, \varepsilon^8])$;
- $H(x, D) = L(x, D - x)$, and $H^\varepsilon(x, D) = L^\varepsilon(x, D - x)$.

Thus, if $\gamma$ denotes the law of $V + U'$, then $\int H(x, \cdot) \gamma(dx)$ is the law of $U + V$ and

$$P(U + V \in A) = \int H^\varepsilon(x, A) \gamma(dx) + \int (H - H^\varepsilon)(x, A) \gamma(dx).$$
The second term is a measure which we denote by $\gamma_b$. Note that $\gamma_b(\mathbb{R}) = P(|U - U'| > \varepsilon^8) < 9\varepsilon^3/2$.

We then take $\lambda_1$ to be the measure $\gamma_a + \gamma_b$, where $\gamma_a$ is the measure that is absolutely continuous with respect to $\gamma$ with $\frac{d\gamma_a}{d\gamma}(x) = H^\varepsilon(x, \mathbb{R})$. Direct calculation then yields that

$$\lambda_1(A) = P(U' + V \in A, |U - U'| \leq \varepsilon^8) + P(U + V \in A, |U - U'| > \varepsilon^8),$$

from which one obtains both that $\lambda_1$ is a probability measure and that the total variation distance between $\lambda_1$ and $\gamma$ is at most $P(|U - U'| > \varepsilon^8) \leq 9\varepsilon^3/2$. Since the total variation distance between $\gamma$ and $\int_{s_1 \leq 2} Y(s)ds \mathbb{1}_{|s_1| < 2}$ is at most $7\varepsilon^3/2$ we then get that the total variation distance between $\lambda_1$ and $\int_{s_1 \leq 2} Y(s)ds \mathbb{1}_{|s_1| < 2}$ is at most $8\varepsilon^3$. Finally, the proof is completed by letting $K$ be the kernel

$$K(x, A) = \frac{H^\varepsilon(x, A)}{H^\varepsilon(x, \mathbb{R})} \frac{d\gamma_a}{d\lambda_1}(x) + \mathbb{1}_{\{x \in A\}} \frac{d\gamma_b}{d\lambda_1}(x),$$

since one can check easily that $\int H(x, \cdot) \gamma(dx) = \int K(x, \cdot) \lambda_1(dx)$ and $K(x, [x - \varepsilon^8, x + \varepsilon^8[) = 0$. \hfill \Box

5.4. Concatenation lemma. The walk can go from the bulk to a previously untouched territory and then back to the bulk. For this reason we need to consider BLPs in an environment, which is possibly a concatenation of the one our walk created in the bulk and the original i.i.d. environment. Lemma 5.19 below says, loosely speaking, that if the environment is “good” then our mesoscopic walk moves right or left and “chooses” its extrema with probabilities close of those of the basic BMPE-walk as given in Corollary 3.4. This lemma addresses all situations: the very first step of the mesoscopic walk, transitions between the bulk and the boundary, and steps in the bulk.

Let $0 \leq w \leq \varepsilon \leq 1 \leq \varepsilon < b = 2$ as well as small values $0 < \delta < \delta_1 < \varepsilon^3$ be fixed and $w_m, \overline{w}_m, \overline{b}_m, m \in \mathbb{Z}/m$ converge to $w, \varepsilon, \varepsilon, \varepsilon$ respectively as $m \to \infty$. We shall consider a BLP $\{Z^m_k\}_{0 \leq k \leq b_m}$ which evolves as a $V^+$ process on $[0, w_m m]$ and as a $U^+$ process afterwards. In particular, $Z^m$ has an immigrant in each generation up to $[w_m m]$ and, thus, $0$ becomes an absorbing state only after $[w_m m]$.

Next we define which environments are considered “good” by imposing three conditions. In Section 6 we shall show that they are satisfied with high probability.

**Assumption 3.** The following properties hold for all sufficiently large $m$.

(i) Either $\varepsilon \geq \varepsilon^3$ and the first cookies on $[0, w_m m]$ are i.i.d. with marginal $\eta$ or $w = w_m = 0$ and the first cookie environment on $[0, \varepsilon m]$ is $\delta_2$-lifting from the left.

(ii) The first cookie environment on $[w_m m, w_m m]$ is $(m^{1/4}, \nu/2 - 1)$-good and the first cookie environment on $[w_m m, \overline{w}_m m]$ is $(m^{1/4}, 0)$-good.

(iii) Either $\varepsilon \leq 2 - \varepsilon^3$ and the first cookies on $[w_m m, b_m m]$ are i.i.d. with marginal $\eta$ or $w = 2$ and the first cookie environment on $[b_m - \varepsilon_1 m, b_m m]$ is $\delta_2$-grounding from the left.

To clarify the meaning of the above conditions, let us mention that the first step of the mesoscopic walk corresponds to $w = w = \varepsilon = 1$. The case when $w = 0$ and $\varepsilon = 2$ corresponds to steps in the bulk. The other cases are when the step is at the boundary, and then the interval $(w, \varepsilon)$ represents the bulk region on the macroscopic scale.

**Lemma 5.19 (Concatenation lemma).** There is a constant $K$ such that for every $\varepsilon > 0$ and all first cookie environments satisfying Assumption 3 with sufficiently small $0 < \delta_2 < \delta_1 < \varepsilon^3$, the following statements hold.

(i) For all $m$ sufficiently large, $|P(\sigma_{w_m m, 0} \leq b_m m) - P(\tau_{w}(w, \varepsilon, \varepsilon) \leq \tau_{w}(w, \varepsilon, \varepsilon))| \leq K\varepsilon^3$.

(ii) Let $\overline{w} \leq 2 - \varepsilon^3$, intervals $J_\ell$, $1 \leq \ell \leq L$, be as in (16), and choose $\varepsilon_0$ (which might depend on $m$) so that $\overline{w}_m \in J_{\varepsilon_0} + 1$. Then for all sufficiently large $m$

$$\sum_{\ell > \varepsilon_0 + 1} |P(m^{-1} \sigma_{w_m m, 0} \in J_{\ell + 1}) - P(S(\tau_{w}(w, \varepsilon, \varepsilon)) \in J_{\ell + 1})| < K\varepsilon^3.$$
Remark 5.20. Since \( \{\sigma_{w,m,0} \leq b_m m\} = \{\exists \ell \in [\ell_0, L] : m^{-1}\sigma_{w,m,0} \vee \overline{w}_m \in J_{\ell} + 1\} \), (i) and (ii) imply that for some (possibly different) constant \( K \) and all sufficiently large \( m \)

\[
|P(m^{-1}\sigma_{w,m,0} \vee \overline{w}_m \in (J_{\ell_0} \cup J_{\ell_0+1}) + 1) - P(S(\tau_0(w, w, \overline{w})) \in (J_{\ell_0} \cup J_{\ell_0+1}) + 1)| \leq Ke^3.
\]

Part (ii) deals with the maximum \( S \) for a BMPE when \( \overline{w} \leq 2 - \varepsilon^3 \). The argument given applies equally to the minimum \( I \) when \( \overline{w} \geq \varepsilon^3 \).

\textbf{Proof.} We begin with part (i) and let \( \{Y(t)\}_{0 \leq t \leq 2} \) be a solution of \( (19) \) with

\[
D(t) = \frac{\nu}{2}((1 - \theta)1_{\{0 \leq t < \overline{w}\}} + 1_{\{w \leq t \leq 2\}} + \theta^+ 1_{\{\overline{w} \leq t \leq 2\}}) \quad \text{and} \quad Y(0) = 0.
\]

The process \( \{Y(t)\}_{0 \leq t \leq 2} \) is a constant multiple of \( \{y(t)\}_{0 \leq t \leq 2} \) from Proposition 3.3. Therefore, by part (i) of Corollary 3.4 we have that \( P(\tau_0(w, w, \overline{w}) < \tau_0(w, w, \overline{w})) = P(\sigma_{w,0}^Y < b) \). Let \( g(x) := P(\sigma_{w,0}^Y < b \mid Y(w) = x) \). It is clear that \( g \) is continuous and

\[
P(\tau_0(w, w, \overline{w}) < \tau_0(w, w, \overline{w})) = E[g(Y(w))],
\]

while by Lemma 5.13

\[
|P(\sigma_{w,m,0} \leq b_m m \mid Z_{w,m}^m) - g(m^{-1}Z_{w,m}^m)| < 4e^3.
\]

So our proof for (i) comes down to showing that for \( m \) large

\[
(32) \quad |E[g(Y(w))] - E[g(m^{-1}Z_{w,m}^m)]| < Ke^3
\]

for a universal \( K \). If \( \overline{w} = 0 \) then \( (32) \) is an immediate consequence of Corollary 5.17. If \( \overline{w} \geq \varepsilon^3 \), then by Theorem 3.5

\[
\{m^{-1}Z_{\lfloor mt \rfloor}^m\}_{0 \leq t \leq \overline{w}_m} \overset{J_1}{\Longrightarrow} \{Y(t)\}_{0 \leq t \leq \overline{w}}.
\]

Moreover, by Theorem 5.9 and the fact that a BESQ\(^2\) process a.s. does not hit 0,

\[
\{m^{-1}Z_{\lfloor mt \rfloor}^m\}_{\overline{w}_m \leq t \leq \overline{w}_m} \overset{J_1}{\Longrightarrow} \{Y(t)\}_{\overline{w} \leq t \leq 1}
\]

In particular, \( m^{-1}Z_{w,m}^m \Longrightarrow Y(1) \) and \( (32) \) holds. Part (i) of the lemma is proven.

The proof of part (ii) splits into two cases according to whether \( \overline{w} \geq \varepsilon^3 \) or \( \overline{w} = 0 \). The first case is easier (though essentially the same) so we content ourselves with the second case.

It follows from Corollary 5.12 and Corollary 5.17 that any limit point \( \tilde{Y} \) of \( \{m^{-1}Z_{\lfloor mt \rfloor}^m\}_{w,m \leq t \leq 2} \) solves the same equation \( (19) \) as \( Y \) on time domain \([1, 2]\) and that the law of \( \tilde{Y}(1) \) is within \( 3\varepsilon^3/2 \) of the law of \( Y(1) \) in total variation norm. The closeness of the laws of \( Y(1) \) and \( \tilde{Y}(1) \) and the Markov property imply that

\[
\sum_{t > \ell_0 + 1} |P(\sigma_{1,0}^\tilde{Y} \in J_{t} + 1) - P(\sigma_{1,0}^Y \in J_{t} + 1)| \leq \frac{3}{2} \varepsilon^3.
\]

A slight subtlety, arising from the weakness of the conclusion of Corollary 5.12 compared to Theorem 5.5 is that we cannot claim that \( m^{-1}\sigma_{w,m,0} \Rightarrow \sigma_{1,0}^\tilde{Y} \). However, given the power of Theorem 5.5 we can assert that \( m^{-1}\sigma_{w,m,0} \vee \overline{w}_m \) converges to \( \sigma_{1,0}^\tilde{Y} \vee \overline{w} \). This and the fact that the law of \( \sigma_{1,0}^\tilde{Y} \vee \overline{w} \) has no atoms in \((\overline{w}, 2]\) permits us to conclude that for any fixed sufficiently small and all \( m \) sufficiently large

\[
\sum_{t \in [t_0 + 2, L]} |P(m^{-1}\sigma_{w,m,0} \vee \overline{w}_m \in J_{t} + 1) - P(\sigma_{1,0}^Y \vee \overline{w} \in J_{t} + 1)|
\]

\[
= \sum_{t \in [t_0 + 2, L]} |P(m^{-1}\sigma_{w,m,0} \in J_{t} + 1) - P(\sigma_{1,0}^Y \in J_{t} + 1)| \leq 2\varepsilon^3.
\]
Noting that for every $\ell \in [\ell_0 + 1, L]$, $P(\sigma_{1,0}^Y \in J_\ell + 1) = P(S(\tau_0(w, w, \overline{w})) \in J_\ell + 1)$ completes the proof.

6. Environmental issues

The applicability of our tools from previous sections depends on whether the environment is "good" in some way. Maintaining the desired properties of the environment as the walk moves from one mesoscopic site to another is crucial for our arguments. In this section we shall prove some important properties of the cookie environment modified by the walk. This will allow us to couple our rescaled “mesoscopic” ERW with a modified BMPE-walk $W^\tau$ and establish the desired functional limit theorem.

We note that for a fixed $\varepsilon > 0$ the scaling parameter $m$ in Section 5 is roughly of order $\varepsilon^2 \sqrt{n}$. This is why $m^{1/4}$-goodness of the first cookie environment becomes $n^{1/8}$-goodness in this section.

6.1. $n^{1/8}$-goodness of the environment.

**Lemma 6.1.** For $n \geq 1$ and $K < \infty$, let $A_{n,K}$ be the event that at every time $k$ until exiting the interval $[-K\sqrt{n}, K\sqrt{n}]$ the remaining first cookie environment on the interval $[X_k, S_k]$ is $(n^{1/8}, 0)$-good and the remaining first cookie environment on the interval $[I_k, X_k]$ is $(n^{1/8}, \nu - 1)$-good. If $\max\{\theta^+, \theta^-, \nu\} < 1$, then $\lim_{n \to \infty} P(A_{n,K}) = 1$ for any $K < \infty$.

Before giving the proof of Lemma 6.1 we state the following simple corollary which follows from the fact that the walk doesn’t exit the interval $[-k\varepsilon\sqrt{n}, k\varepsilon\sqrt{n}]$ before the stopping time $T_{k,n}^\varepsilon$.

**Corollary 6.2.** Let $\max\{\theta^+, \theta^+, \nu\} < 1$. For any fixed $\varepsilon > 0$ and $k, n \geq 1$ let $A_{k,n}^\varepsilon$ be the event that at time $T_{k,n}^\varepsilon$ the remaining first cookie environment on the interval $[X_k^\varepsilon, S_k^\varepsilon]$ is $(n^{1/8}, 0)$-good and the remaining first cookie environment on the interval $[I_k^\varepsilon, X_k^\varepsilon]$ is $(n^{1/8}, \nu - 1)$-good. Then, $\lim_{n \to \infty} P(A_{k,n}^\varepsilon) = 1$, for any $k \geq 1$.

**Remark 6.3.** The intuition behind Lemma 6.1 is that after the walk has taken a large number of steps, the remaining first cookie environment of the sites to the right (resp. left) of the present location up to the running maximum (resp. running minimum) are approximately independent and distributed according to $\pi^+$ (resp. $\pi^-$).

The proof of Lemma 6.1 will rely on some preliminary estimates regarding the BLP.

**Lemma 6.4** (Lemma 3.6 in [KP16]). If $\theta^+ < 1$, then for any $0 < \alpha < \beta$ there exist constants $C, c > 0$ such that

$$\sup_{j \geq 0} P \left( \sum_{i=0}^{\sigma_{U^+}^\varepsilon - 1} 1\{U_i^+ < m^\alpha\} > m^\beta \mid U_0^+ = j \right) \leq Ce^{-cm^\beta - \alpha}, \quad m \geq 1.$$  

Similar statements hold for the BLPs $U^-$, $V^+$, and $V^-$ if the assumption $\theta^+ < 1$ is replaced by $\theta^- < 1$, $\theta^- > 0$, and $\theta^+ > 0$, respectively.

**Remark 6.5.** Note that while [KP16] was written for excited random walks in periodic cookie stacks, the proof of the above lemma in this paper relied only on some facts concerning the BLPs that were also proved for excited random walks with markovian cookie stacks in [KP17].

Lemma 6.4 controls the time spent by a BLP below a certain level before reaching level zero. While zero is an absorbing point for $U^\pm$, it is not absorbing for $V^\pm$, and we will at times need to control the time spent by these processes below some level on a fixed time interval. The following lemma accomplishes this. It is similar to Lemma 3.8 in [KP16], but the statement here is more flexible for the applications we need. Moreover, the proof below corrects an error in the proof of Lemma 3.8 in [KP16].
Lemma 6.6. If $\theta^- < 1$, $\alpha \in (0, 1 - (\theta^- \vee 0))$ and $\beta \in ((\theta^- \vee 0) + \alpha, 1)$, then there exist constants $C, c, r > 0$ such that

$$\sup_{j \geq 0} P\left( \sum_{i \leq m} 1_{\{V_i^+ < m^\alpha\}} > m^\beta \left| V_0^+ = j \right. \right) \leq C e^{-cm^r}.$$ 

A similar statement holds for the process $V^- \text{ if } \theta^- \text{ is replaced everywhere above by } \theta^+.$

Proof. Since the probability in the statement of the lemma is non-decreasing in $j$, we need only to prove the inequality when $j = 0$. Also, we will assume that $\theta^- \in (0, 1)$ since if $\theta^- \leq 0$ we can couple it to another BLP which has parameter $\theta^- \in (0, 1)$ and which is always less than or equal to $V^+$ (see Lemma 5.1 in [KP17]).

Now, fix some $\gamma \in (\theta^- - \alpha, \beta - \alpha)$ (note that this is possible by the assumptions on $\alpha$ and $\beta$). Then, if the event $\{\sum_{i \leq m} 1_{\{V_i^+ < m^\alpha\}} > m^\beta\}$ occurs, either

1. the process $V^+$ returns to 0 at least $\lceil m^\gamma \rceil$ times in the first $m$ steps of the Markov chain,
2. or in one of the first $\lceil m^\gamma \rceil$ excursions from 0 of the process $V^+$ it stays below $m^\alpha$ for at least $m^{\beta - \gamma}$ steps.

The first of these events implies that each of the first $\lceil m^\gamma \rceil$ excursions from 0 lasts at most $m$ steps and is thus its probability is bounded above by

$$\left( 1 - P(\sigma_0^V > m | V_0^+ = 0) \right)^{\lceil m^\gamma \rceil} \leq (1 - cm^{-\theta^-})^{m^\gamma} \leq e^{-cm^{\gamma - \theta^-}},$$

where the first inequality follows from known tail asymptotics for $\sigma_0^V$ when $\theta^- > 0$; see [KP17 Theorem 2.7]. On the other hand, by Lemma 6.4, the probability of the second event does not exceed

$$m^\gamma P\left( \sum_{i < \sigma_0^V} 1_{\{V_i^+ < m^\alpha\}} > m^{\beta - \gamma} \left| V_0^+ = 0 \right. \right) \leq C m^\gamma e^{-cm^{\beta - \alpha - \gamma}}.$$ 

Choosing $r \in (0, (\beta - \alpha - \gamma) \wedge (\gamma - \theta^-))$ we have that both events considered above have at least a stretched exponential decay in $m$. \qed

We will also need the following lemma which gives control on the number of times any site can be visited before exiting a fixed interval.

Lemma 6.7. If $\max\{\theta^+, \theta^-\} < 1$, then

$$\lim_{r \to \infty} \lim_{m \to \infty} \sup_{x \to \infty} P \left( \max_{|x| \leq m} \mathcal{L}(\tau_m^X \wedge \sigma_{-m}^X, x) > rm \right) = 0.$$ 

Proof. Clearly it is enough to prove an upper bound on

$$P \left( \max_{x \in [-m, 0]} \mathcal{L}(\tau_m^X \wedge \sigma_{-m}^X, x) > rm \right) \leq P \left( \max_{x \in [-m, 0]} \mathcal{L}(\sigma_{-m}^X, x) > rm \right),$$

as a similar argument will control the local time to the right of the origin. To this end, recall from [7] that $\mathcal{E}_x^m$ is the number of steps right from $x$ before time $\sigma_{-m}^X$ and note that

$$\mathcal{L}(\sigma_{-m}^X, x) = \mathcal{E}_x^m + \mathcal{E}_{x-1}^m + 1, \quad \text{for all } -m < x \leq 0.$$ 

Since $(\mathcal{E}_m^m, \mathcal{E}_{m+1}^m, \ldots, \mathcal{E}_{m-1}^m, \mathcal{E}_0^m)$ has the same distribution as the BLP $(V_0^+, V_1^+, \ldots, V_{m-1}^+, V_m^+)$ started with $V_0^+ = 0$, then

$$P \left( \max_{x \in [-m, 0]} \mathcal{L}(\sigma_{-m}^X, x) > rm \right) \leq P \left( \max_{i \leq m} V_i^+ > rm/3 | V_0^+ = 0 \right).$$
Finally, it follows from the diffusion approximation in Lemma 5.5 that the probability on the right converges to 0 as first \( m \to \infty \) and then \( r \to \infty \).

**Proof of Lemma 6.7.** We begin by introducing some new notation that will be used in this proof. For \( x \in \mathbb{Z} \) and \( m \geq 0 \) let \( \tau_{x,m} \) be the stopping time of the \((m+1)\)-st visit of the ERW \( X \) to location \( x \). That is, \( \tau_{x,m} = \inf \{ k \geq 0 : \sum_{i \leq k} 1_{\{X_i = x\}} = m + 1 \} \). Also, \( y \in \mathbb{Z} \) let \( \mathcal{E}_y^{(x,m)} \) and \( \mathcal{D}_y^{(x,m)} \) be the number of steps right and left from \( x \), respectively, prior to time \( \tau_{x,m} \). That is,

\[
\mathcal{E}_y^{(x,m)} = \sum_{n=0}^{\tau_{x,m} - 1} 1_{\{X_n = y, X_{n+1} = y+1\}} \quad \text{and} \quad \mathcal{D}_y^{(x,m)} = \sum_{n=0}^{\tau_{x,m} - 1} 1_{\{X_n = y, X_{n+1} = y-1\}}.
\]

In the proof below we will use the following facts concerning these the directed edge local times. First of all, we note that

\[
(33) \quad \mathcal{D}_y^{(x,m)} = \mathcal{E}_{y-1}^{(x,m)} + 1_{\{x < y\}} \quad \text{for} \quad x < y.
\]

Secondly, the process \( \{\mathcal{E}_{y}^{(x,m)}\}_{y \geq x} \) has the same distribution as a BLP or concatenation of BLPs. If \( x \geq 0 \) then this is a \( U^+ \) process using the cookie environment on \([x, \infty)\) but if \( x < 0 \) then it is a concatenation of a \( V^+ \) process using the cookie environment on \([x, 0] \) with a \( U^+ \) process using the cookie environment on \([0, \infty)\) (see Section 2.2 of [KP16] for more details on this connection with BLPs).

Using the above notation, for any \( n \geq 1 \) and \( K, K' < \infty \) let \( \tilde{A}_{K,K',n} \) be the event that at every (random) time \( \tau_{x,m} \) with \( |x| \leq K \sqrt{n} \) and \( m \leq K' \sqrt{n} \) the remaining first cookie environment is \( (n^{1/8}, 0) \)-good on \([X_{\tau_{x,m}}, S_{\tau_{x,m}} \wedge K \sqrt{n}] \) and and \( (n^{1/8}, \frac{\sqrt{2}}{2} - 1) \)-good on \([S_{\tau_{x,m}} \wedge -K \sqrt{n}, X_{\tau_{x,m}}] \). Since

\[
P(A_{n,K}^c) \leq P(\tilde{A}_{n,K,K',n}^c) + P\left( \max_{|x| \leq K \sqrt{n}} \mathcal{L}(\tau_{K \sqrt{n}}^x \wedge \sigma_{K \sqrt{n}}^x, x) > K' \sqrt{n} \right),
\]

and since Lemma 6.7 implies that the second term on the right can be made arbitrarily small for \( n \) large by taking \( K' \) large enough, it is enough to show that \( \lim_{n \to \infty} P(\tilde{A}_{n,K,K',n}^c) = 0 \) for all \( K, K' < \infty \).

Now, for the remainder of the proof, we’ll only prove that the remaining first cookie environments are \((n^{1/8}, 0)\)-good to the right of the current location using the directed edge local times \( \mathcal{E}_{y}^{(x,m)} \) and the corresponding BLPs \( U^+ \) and \( V^+ \). The proof that the first cookie environments are \((n^{1/8}, \frac{\sqrt{2}}{2} - 1)\)-good to the left of the current location is similar using \( \mathcal{D}_{y}^{(x,m)} \) and the BLPs \( U^- \) and \( V^- \). For \( K < \infty \), \( |x| \leq K \sqrt{n} \), and \( m, n \geq 1 \) define

\[
B_{K,n}^{(x,m)} = \left\{ \sum_{y \in J} r_{y}^+ \left( \mathcal{P}_{\lambda(\tau_{x,m}^y)}^{y} \right) 1_{\{y \leq S_{x,m}\}} > \frac{n^{1/8}}{\ln n}, \text{for some } J \subset \lceil x, K \sqrt{n} \rceil \text{ with } |J| = \lfloor n^{1/8} \rfloor \right\}.
\]

Then,

\[
(34) \quad P(\tilde{A}_{n,K,K',n}^c) \leq \sum_{|x| \leq K \sqrt{n}} P\left( B_{K,n}^{(x,m)} \right).
\]

Thus it remains only to bound the probabilities \( P(\tilde{B}_{K,n}^{(x,m)}) \). To this end, first let for \( y \geq x \)

\[
r_{y}^{x,m} = \mathbb{E} \left[ r_{y}^+ \left( \mathcal{P}_{\lambda(\tau_{x,m}^y)}^{y} \right) \right] \quad \text{where} \quad \mathcal{G}_{z}^{x,m} = \sigma(\mathcal{E}_{u}^{(x,m)}, R_{j}^{u}, x \leq u \leq z \text{ and } j \geq 1).
\]

That is, \( \mathcal{G}_{x-1}^{x,m} \) is the trivial \( \sigma \)-field and \( \mathcal{G}_{z}^{x,m}, z \geq x \), contains all the information about the number of steps right from sites in \([x, z]\) and all of the cookies in the stacks in \([x, z]\). With this notation
we have

\begin{equation}
    P(B_{K,n}^{(x,m)}) \leq \sum_{J \subset [x, K\sqrt{n}], |J| = \lfloor n^{1/8} \rfloor} \sum_{y \in J} P \left( \sum_{y \in J} \left\{ r^+ \left( R_y^y \mathcal{L}(\tau_{x,m}, y) + 1 \right) - r_y^x \right\} 1\{y \leq S_{x,m}\} > \frac{n^{1/8}}{2 \ln n} \right)
\end{equation}

(36)

\begin{equation}
    + P \left( \exists J \subset [x, K\sqrt{n}], |J| = \lfloor n^{1/8} \rfloor : \sum_{y \in J} r_y^x 1\{y \leq S_{x,m}\} > \frac{n^{1/8}}{2 \ln n} \right)
\end{equation}

For the first term on the right, first note that (33) implies that \( \mathcal{L}(\tau_{x,m}, y) = \mathcal{E}_y^{(x,m)} + \mathcal{E}_{y-1}^{(x,m)} + 1 \{x < y \leq 0\} \) so that the terms in braces are \( \mathcal{G}_y^{x,m} \)-measurable. Secondly, note that if \( y \leq 0 \) then \( 1\{y \leq S_{x,m}\} = 1 \) whereas if \( y > 0 \) then \( \{y \leq S_{x,m}\} = \{\mathcal{E}_{y-1}^{(x,m)} \geq 1\} \). In either case we have that \( 1\{y \leq S_{x,m}\} \) is \( \mathcal{G}_{y-1}^{x,m} \)-measurable, and thus the sums inside the first probability on the right are martingale difference sums with bounded increments. Therefore, it follows from Azuma’s inequality that

\[ \text{the sum in (35) } \leq \sum_{J \subset [x, K\sqrt{n}], |J| = \lfloor n^{1/8} \rfloor} e^{-c \frac{n^{1/4}}{\ln n} |J|} \leq C Kn^{1/2} e^{-\frac{n^{1/8}}{4 \ln n}}. \]

To bound the probability in (36), note first of all that \( R_y^y \mathcal{L}(\tau_{x,m}, y) + 1 \) represents the next cookie to be used at \( y \) after time \( \tau_{x,m} \). If \( y > x \) then the last visit to \( y \) prior to \( \tau_{x,m} \) resulted in a step to the left. Since \( D_y^{(x,m)} \) is the number of steps left from \( y \) prior to \( \tau_{x,m} \), we have that the distribution of \( R_y^y \mathcal{L}(\tau_{x,m}, y) + 1 \) conditioned on \( D_y^{(x,m)} = \ell \) is equal to the distribution of the next cookie in a stack after the \( \ell \)-th step left, and this distribution is known to converge to \( \pi^+ \) exponentially fast in \( \ell \) (see [KP17 Section 3]). Since \( D_y^{(x,m)} = \mathcal{E}_{y-1}^{(x,m)} + 1 \{x < y \leq 0\} \) is \( \mathcal{G}_{y-1}^{x,m} \)-measurable and \( \pi^+ \cdot r^+ = 0 \), this implies that there are constants \( C, c > 0 \) such that

\[ |r_y^x| = |r_y^x - \pi^+ \cdot r^+| \leq C e^{-c \mathcal{E}_{y-1}^{(x,m)}}, \quad \text{for all } y > x. \]

Therefore, we have that for any \( \alpha > 0 \)

\[ \left| \sum_{y \in J} r_y^x 1\{y \leq S_{x,m}\} \right| \leq C + \sum_{y \in J \setminus \{x\}} C e^{-c \mathcal{E}_{y-1}^{(x,m)}} 1\{y \leq S_{x,m}\}, \]

\[ \leq C \left( 1 + |J| e^{-c n^\alpha} + \sum_{y \in J \setminus \{x\}} 1\{\mathcal{E}_{y-1}^{(x,m)} < n^\alpha\} 1\{y \leq S_{x,m}\} \right). \]

Using this we obtain that for \( \alpha > 0, \beta < \frac{1}{8} \) and \( n \) sufficiently large

(36)

\[ \leq P \left( \exists J \subset [x + 1, K\sqrt{n}], |J| = \lfloor n^{1/8} \rfloor : C \sum_{y \in J} 1\{\mathcal{E}_{y-1}^{(x,m)} < n^\alpha\} 1\{y \leq S_{x,m}\} > \frac{n^{1/8}}{4 \ln n} \right) \]

\[ \leq \sum_{x < z \leq 0} P \left( \sum_{y = z} 1\{\mathcal{E}_{y-1}^{(x,m)} < n^\alpha\} > n^\beta \right) + P \left( \sum_{y \leq 0} 1\{\mathcal{E}_{y-1}^{(x,m)} < n^\alpha\} 1\{y \leq S_{x,m}\} > n^\beta \right) \]

\[ \leq |x| \sup_{j \geq 1} \left( \sum_{i=0}^{n^{1/8}} 1\{V_i^+ < n^\alpha\} > n^\beta | V_0^+ = j \right) + \sup_{j \geq 0} \left( \sum_{i=0}^{n_0^+ - 1} 1\{U_i^+ < n^\alpha\} > n^\beta | U_0^+ = j \right), \]
where the last inequality follows from connection of \( \{ \xi_{y,m}^{x} \}_{y \geq x} \) with the BLPs \( U^{+} \) and \( V^{+} \) noted at the beginning of the proof. If we then choose \( \alpha \in (0, 1 - (2/3)^{1/4}) \) and \( \beta \in (2/3 + \alpha, 1/2) \) we can apply Lemmas 6.4 and 6.6 to bound the last line above by \( C(|x| + 1)e^{-c n^{1/8}} \) for some constants \( C, c, r > 0 \). Applying this, together with the bound on the sum in (35), we obtain that

\[
(34) \leq C(2K \sqrt{n} + 1)K' \sqrt{n} \left( Kn^{1/2}e^{-c \sqrt{n}} + (1 + K)\sqrt{n}e^{-c n^{1/8}} \right),
\]

for \( n \) large enough. Since the right side vanishes as \( n \to \infty \) for any \( K, K' < \infty \), this completes the proof of Lemma 6.1.

6.2. Lifting and grounding properties of the environment. We shall show that in a small neighborhood of every mesoscopic site except for the site occupied by the walk, the environment is locally close to i.i.d. in an appropriate equilibrium. This property is preserved with probability close to 1 as the walk moves from one mesoscopic site to another for any fixed (possibly very large) number of steps (order \( \epsilon^{-2} \)). The important consequence of this is that the environment around every mesoscopic site in the bulk will have lifting and grounding properties (see Definitions 5.2 and 5.3) which together with \( n^{1/8} \)-goodness will allow us to use our diffusion approximations (i.e. versions of generalized Ray-Knight theorems). We start with several definitions.

Definition 6.8. Given \( \epsilon > 0, \delta_1 \in (0, \epsilon^3), \delta_2 > 0, \) and \( k \in \mathbb{Z}_+, \) for the ERW \( X \) stopped at time \( T_{\epsilon,n}^{e,n} \), the first cookie environment on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] \) is said to be bulk regular for \( (\delta_1, \delta_2) \) if

- \( I_{T_{\epsilon,n}^{e,n}} \leq X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n} \) and \( X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n} \leq S_{T_{\epsilon,n}^{e,n}}; \)
- on \( [X_{T_{\epsilon,n}^{e,n}}, X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] \) the first cookie environment is \( (n^{1/8}, 0) \)-good and on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}}] \) the first cookie environment is \( (n^{1/8}, \nu/2 - 1) \)-good;
- on \( [X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}, - \delta_1 \epsilon \sqrt{n}], X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] \) the first cookie environment is \( \delta_2 \epsilon \sqrt{n} \)-lifting from the right and \( \delta_2 \epsilon \sqrt{n} \)-grounding from the left; on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n} + \delta_1 \epsilon \sqrt{n}] \) the first cookie environment is \( \delta_2 \epsilon \sqrt{n} \)-lifting from the left and \( \delta_2 \epsilon \sqrt{n} \)-grounding from the right.

Definition 6.9. Given \( \epsilon > 0, \delta_1 \in (0, \epsilon^3), \delta_2 > 0, \) and \( k \in \mathbb{Z}_+, \) for the ERW \( X \) stopped at time \( T_{\epsilon,n}^{e,n} \), the first cookie environment on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] \) is said to be \( S \)-regular for \( (\delta_1, \delta_2) \) if

- \( S_{T_{\epsilon,n}^{e,n}} \in [X_{T_{\epsilon,n}^{e,n}}, X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] - \epsilon^{4} \sqrt{n}] \); on \( [X_{T_{\epsilon,n}^{e,n}}, S_{T_{\epsilon,n}^{e,n}}] \) the first cookie environment is \( (n^{1/8}, 0) \)-good and on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}}] \) the first cookie environment is \( (n^{1/8}, \nu/2 - 1) \)-good;
- on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n} + \delta_1 \epsilon \sqrt{n}] \) the first cookie environment is \( \delta_2 \epsilon \sqrt{n} \)-lifting from the left and \( \delta_2 \epsilon \sqrt{n} \)-grounding from the right.

The notion of \( I \)-regular for \( (\delta_1, \delta_2) \) on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] \) first cookie environment is defined in a symmetric manner.

We shall say that the first cookie environment on \( [X_{T_{\epsilon,n}^{e,n}} - \epsilon \sqrt{n}, X_{T_{\epsilon,n}^{e,n}} + \epsilon \sqrt{n}] \) is regular for \( (\delta_1, \delta_2) \) if it is either bulk regular, \( S \)-regular, or \( I \)-regular.

Definition 6.10. Given \( \epsilon > 0, \delta_1 \in (0, \epsilon^3), \delta_2 > 0, \) we shall write \( \delta_1 \approx \delta_2 \) if for all sufficiently large \( n, \)

\[
P_{\pi^+} \left( T_{\delta_2 \epsilon \sqrt{n}}^{-} \leq \delta_1 \epsilon \sqrt{n} \right) \geq 1 - \epsilon^6, \quad P_{\pi^+} \left( \sigma_{0}^{U^+} \leq \delta_1 \epsilon \sqrt{n} \mid U_{0}^{+} = \delta_2 \epsilon \sqrt{n} \right) \geq 1 - \epsilon^6,
\]

\[
P_{\pi^-} \left( T_{\delta_2 \epsilon \sqrt{n}}^{+} \leq \delta_1 \epsilon \sqrt{n} \right) \geq 1 - \epsilon^6, \quad P_{\pi^-} \left( \sigma_{0}^{U^-} \leq \delta_1 \epsilon \sqrt{n} \mid U_{0}^{-} = \delta_2 \epsilon \sqrt{n} \right) \geq 1 - \epsilon^6.
\]
Remark 6.11. In the proof below, we will need the fact that one can always find parameters $\delta_1$ and $\delta_2$ that are sufficiently small and in the relation $\delta_1 \lesssim \delta_2$. To see this, recall that $\theta^+(\pi^-) = 0$ and $\theta^- (\pi^-) = 0$. Then it follows from Lemmas 5.7 and 5.8 (and the discussion at the beginning of Section 5.1) that the conditions in Definition 6.10 hold if $\delta_2$ is sufficiently small compared to $\delta_1$.

Remark 6.12. Note that the events in Definition 6.10 are closely related to the definitions of lifting and grounding first cookie environments in Definitions 5.2 and 5.3. In particular, if $\delta \lesssim \delta_2$ then a first cookie environment on $[0, [\delta_1 \varepsilon \sqrt{n}]]$ with $\pi^+$-product measure will be $\delta_2 \varepsilon \sqrt{n}$-lifting from the right and $\delta_2 \varepsilon \sqrt{n}$-grounding from the left with probability at least $1 - 2 \varepsilon^6 / \varepsilon^3 = 1 - 2 \varepsilon^3$.

Lemma 6.13. Given an $\varepsilon > 0$ let $\delta_1 \lesssim \delta_2$ and $\delta_1$ be sufficiently small. Suppose that for some $k \in \mathbb{Z}_+$ we have $X_{T_{k+1}^\varepsilon,n} = x$ and the environment on $[x - [\varepsilon \sqrt{n}], x + [\varepsilon \sqrt{n}]]$ is regular for $(\delta_1, \delta_2)$. Then there is a $C > 0$ not depending on $\varepsilon$ such that for $n$ large, outside of probability $C \varepsilon^3$,

1. on the event $\{T_{k+1}^\varepsilon,n - T_{k+1}^\varepsilon,n \}$ the first cookie environment on $[x - [\delta_1 \varepsilon \sqrt{n}], x]$ is $\delta_2 \varepsilon \sqrt{n}$-lifting from the right and $\delta_2 \varepsilon \sqrt{n}$-grounding from the left;
2. on the event $\{T_{k+1}^\varepsilon,n - T_{k+1}^\varepsilon,n \}$ the first cookie environment on $[x, x + [\delta_1 \varepsilon \sqrt{n}]]$ is $\delta_2 \varepsilon \sqrt{n}$-lifting from the left and $\delta_2 \varepsilon \sqrt{n}$-grounding from the right.

The proof relies on the following two lemmas.

Lemma 6.14. There is a constant $C_4$ such that uniformly over all first cookie environments satisfying the conditions of Lemma 6.13 for all sufficiently large $n$,

$$C_4 \leq P(T_{k+1}^\varepsilon,n - T_{k+1}^\varepsilon,n \mid X_{T_{k}^\varepsilon,n} = x, R_1^{\varepsilon,n}, y \in [x - [\varepsilon \sqrt{n}], x + [\varepsilon \sqrt{n}]) \leq 1 - C_4.$$ 

Proof. By Lemma 5.19 it is enough to show that uniformly in $\varepsilon \in (0, 1)$ and $\pi \in (1, 2)$ the probability $P(\tau_0(\varepsilon, 1, \pi) < \tau_2(\varepsilon, 1, \pi))$ is bounded away from 0 and 1. By symmetry of the problem, we only have to argue that this probability is uniformly bounded away from 0. By Proposition 3.3 we must simply show that for the process $y(\cdot)$ and $w = 1$ the probability of $y(\cdot)$ hitting zero in $(1, 2)$ is bounded away from zero as $\pi$ and $\varepsilon$ vary. But this follows easily by noting that the minimum is achieved with $\varepsilon$ equal to 0 or 1 and $\pi$ equal to 1 or 2. $\square$

Lemma 6.15. Assume the conditions of Lemma 6.13. There is a $\delta_3 = \delta_3(\varepsilon, \theta^-) > 0$ such that with probability $1 - 2 \varepsilon^3$ for $n$ large enough, the process $V^+$ (resp. $V^-$) starting with $0$ particles in generation $0$ and using the first cookie environment on $[x - [\varepsilon \sqrt{n}], x]$ for generations $1, 2, \ldots, [\varepsilon \sqrt{n}]$ (resp. $[x, x + [\varepsilon \sqrt{n}]] - 1$) for generations $[\varepsilon \sqrt{n}], [\varepsilon \sqrt{n}], \varepsilon \sqrt{n}, 1]$ satisfies

$$V_j^+ (\text{resp. } V_j^-) \geq \delta_3 \sqrt{n}, \quad \forall j \in [\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}, \varepsilon \sqrt{n}].$$

Proof. We shall only consider the process $V^+$, the other case follows by a symmetric reasoning.

Step 1. We start with the case when the first cookie environment on $[x - [\varepsilon \sqrt{n}], x + [\varepsilon \sqrt{n}]]$ is either bulk regular or S-regular for $(\delta_1, \delta_2)$. To simplify the notation and without loss of generality we shall assume that $x = [\varepsilon \sqrt{n}]$ and consider the process $V^+$ on $[0, [\varepsilon \sqrt{n}]]$. Given our assumptions on the first cookie environment, we have $\tau_{\delta_2 \varepsilon \sqrt{n}, \pi} V^+ \leq [\delta_1 \varepsilon \sqrt{n}]$ outside of probability $\varepsilon^3$. By the strong Markov property and monotonicity of BLPS with respect to the initial number of particles we have that, conditional on $G_{\delta_2 \varepsilon \sqrt{n}, \pi}^+ := \sigma(V_j^+, j \leq \tau_{\delta_2 \varepsilon \sqrt{n}, \pi} V^+)$, $V^+_{\delta_2 \varepsilon \sqrt{n}, \pi + \ell}$ will be stochastically larger than the BLP $Z^\varepsilon$, $\ell = 0, 1, \ldots, [\varepsilon \sqrt{n}] - [\delta_2 \varepsilon \sqrt{n}]$ which starts with $[\delta_2 \varepsilon \sqrt{n}]$ particles in generation 0 and evolves in the environment on $[[\tau_{\delta_2 \varepsilon \sqrt{n}, \pi} V^+ + 1, [\varepsilon \sqrt{n}]]$ for generations $1, 2, \ldots, [\varepsilon \sqrt{n}] - \tau_{\delta_2 \varepsilon \sqrt{n}, \pi}^V$. Without loss of generality we can extend the process $Z^\varepsilon$ to the full interval $[0, [\varepsilon \sqrt{n}]]$ by choosing the environment on $[[\varepsilon \sqrt{n}] - \tau_{\delta_2 \varepsilon \sqrt{n}, \pi}^V + 1, [\varepsilon \sqrt{n}]]$ to be in $\pi^+$ product measure. By Theorem 5.9 and the fact that the environment is assumed to be either bulk- or S-regular, for each $\delta \in (0, \delta_2/2)$
the processes
\[
\frac{Z_\varepsilon^+, t \in [0, 1],}{\varepsilon \sqrt{n}},
\]
converge weakly as \( n \to \infty \) to a constant multiple of a BESQ\(^2 \), \( Y(t \wedge \sigma_\delta), t \in [0, 1], \) with \( Y(0) = \delta_2 > 0 \). We can choose \( \delta_3 = \delta_3(\varepsilon, \delta_2) > 0 \) so that
\[
P \left( \min_{t \in [0, 1]} Y(t) > \delta_3 \right) > 1 - \frac{\varepsilon^3}{2}.
\]
Then outside of probability \( \varepsilon^3 \) for \( n \) large \( Z_\varepsilon^+ \geq \delta_3 \sqrt{n} \) for all \( \ell \in [0, \varepsilon \sqrt{n}] \). By stochastic domination we conclude that for the same \( \delta_3 \) and all \( n \) large \( V_j^+ \geq \delta_3 \sqrt{n} \) for all \( j \in [[\varepsilon \sqrt{n}] - [\varepsilon_1 \varepsilon \sqrt{n}], [\varepsilon \sqrt{n}]] \) as claimed.

**Step 2.** Suppose now that the first cookie environment on \([x - \varepsilon \sqrt{n}], x + \varepsilon \sqrt{n}]\) is I-regular for \((\delta_1, \delta_2)\) and \( \delta_1 \) is sufficiently small. Then \( x \leq 0 \). However, after an appropriate shift we may again assume that \( x = \varepsilon \sqrt{n} \) so that \( I_{T_k} \in [\varepsilon^4 \sqrt{n}, \varepsilon \sqrt{n}] \). The process \( V_j^+ \) will be evolving in the product environment with marginal \( \eta \) for \( j \in [0, I_{T_{k,n}} - 1] \) and then for \( j \in [I_{T_{k,n}}, \varepsilon \sqrt{n}] \) will use the environment modified by the walk. Recall that the first cookies on the latter interval are a part of the information known at time \( T_{k,n} \). By the regularity assumption, \( I_{T_{k,n}} \geq [\varepsilon^4 \sqrt{n}] \) so that we can use Theorem 5.5 at least on the time interval \([0, \varepsilon] \). The diffusion approximation of Theorem 5.5 is a \( \xi BESQ^{2(1-\theta^-)} \) process \( Y \) with \( Y(0) = 0 \). Since \( 2(1-\theta^-) > 0 \), by scaling properties of BESQ processes we get that
\[
\inf_{t \in [\varepsilon^4, 1]} P(Y(t) > \delta) = \inf_{t \in [\varepsilon^4, 1]} P(tY(1) > \delta) = P(Y(1) > \delta) = P(Y(1) > \delta_{-4}) \to 1 \quad \text{as} \quad \delta \to 0.
\]
Therefore, given \( \varepsilon > 0 \), we can find \( \delta = \delta(\varepsilon, \theta^-) > 0 \) such that \( \inf_{t \in [\varepsilon^4, 1]} P(Y(t) > \delta) > 1 - \varepsilon^3/2 \).

Since the “switch point” from the original product environment to the environment modified by the walk, \( I_{T_{k,n}} \), is a part of the information given at time \( T_{k,n} \) and since \( Y(n^{-1/2}(I_{T_{k,n}} - 1)) \) is with probability at least \( 1 - \varepsilon^3/2 \), we get by Theorem 5.5 that
\[
P \left( V^+_{I_{T_{k,n}} - 1} \geq [\delta \sqrt{n}] \right) \geq 1 - \varepsilon^3 \quad \text{for all sufficiently large} \ n.
\]

Next we shall choose \( \delta_3 \). Let \( s_n := n^{-1/2}(I_{T_{k,n}} - 1) \). Using the fact that BESQ\(^2 \) process \( \tilde{Y} \) with \( \tilde{Y}(0) = \delta \) a.s. does not hit zero we can find a \( \delta_3 = \delta_3(\varepsilon, \delta) \in (0, \delta) \) such that \( P(\sigma_{\tilde{Y}} > 1 | \tilde{Y}(0) = \delta) \geq 1 - \varepsilon^3/4 \). The requirement for \( \delta_1 = \delta_1(\varepsilon, \theta^-) \) to be sufficiently small comes from the fact that we do not have any control on how close \( I_{T_{k,n}} \) is to \( \varepsilon \sqrt{n} \). It could happen that \( I_{T_{k,n}} \in [[\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}], [\varepsilon \sqrt{n}]] \). We know that (37) holds and we need to show that
\[
P \left( V^+_{I_{T_{k,n}} - 1} \geq [\delta_3 \sqrt{n}] \right) \geq 1 - 2\varepsilon^3 \quad \text{for all sufficiently large} \ n.
\]
By our choice of \( \delta_3 \) the process \( \tilde{Y} \) with \( \tilde{Y}(s_n) > \delta \) stays above \( \delta_3 \) on \([s_n, s_n + 1] \) with probability at least \( 1 - \varepsilon^3/4 \). We shall choose \( \delta_1 = \delta_1(\varepsilon, \delta, \delta_3) \) so that on the event \( \{Y(s_n) \geq \delta\} \) the process \( Y \) stays above \( \delta_3 \) on \([s_n - \delta_1 \varepsilon, s_n] \) with probability at least \( 1 - \varepsilon^3/4 \). Thus, we let \( \delta_1 = \delta_1(\varepsilon, \delta_3) > 0 \) be so small that
\[
\max_{y \geq 0} P \left( \min_{t \in [0, \delta_1 \varepsilon]} Y(t) \leq \delta_3, Y(\delta_1 \varepsilon) \geq \delta | Y(0) = y \right) \leq P \left( \tau_{\delta_3}^Y < \delta_1 \varepsilon | Y(0) = \delta_3 \right) < \varepsilon^3/4.
\]
Note that \( \delta, \delta_3, \delta_1 \) depend only on \( \varepsilon \) and \( \theta^- \). Theorems 5.5, Theorem 5.9 and our choice of \( \delta, \delta_3, \delta_1 \) give (38).
Proof of Lemma 6.13. It is enough to show (1). Apart from Lemmas 6.14 and 6.15 we shall use the fact that the auxiliary Markov chain which keeps track of the next cookie in the stack after each successive “failure” in the corresponding sequence Bernoulli trials converges to its equilibrium distribution $\pi^+$ exponentially fast (see (18) on p. 1472 of [KP17]).

Without loss of generality we shall assume that $x = [\varepsilon \sqrt{n}]$. Note that at sites visited by the walk by time $T_{k+1}^{\varepsilon,n}$, the first cookies are non-random while on any unvisited interval they are in the initial product measure with marginal $\eta$. In all cases the first cookie distribution on $[0, [\varepsilon \sqrt{n}]]$ is a (possibly degenerate) product measure. Given the conditions imposed on the environment and the ERW at time $T_k^{\varepsilon,n}$, consider the event

$$A = \{ \text{at time } T_{k+1}^{\varepsilon,n} \text{ the first cookie environment on } [[\varepsilon \sqrt{n}] - [\varepsilon \delta_1 \sqrt{n}, [\varepsilon \sqrt{n}]] \text{ is}
\begin{array}{c}
either not \delta_2 \varepsilon \sqrt{n}\text{-lifting from the right or not } \delta_2 \varepsilon \sqrt{n}\text{-grounding from the left.}
\end{array}$$

We can estimate the probability of $A$ by considering a BLP $V^+$ from Lemma 6.15 which uses the cookie environment created by the walk on $[1, [\varepsilon \sqrt{n}]]$ up to time $T_k^{\varepsilon,n}$ for generations 1, 2, ..., $[\varepsilon \sqrt{n}]$.

**Step 1.** By Lemma 6.15 if $\sigma = \inf\{j \geq [\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}] : V_j^+ < \delta_3 \sqrt{n} \}$ then $P(\sigma \leq [\varepsilon \sqrt{n}]) \leq 2\varepsilon^3$ for all sufficiently large $n$.

**Step 2.** For $j \in [[\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}], [\varepsilon \sqrt{n}]]$, let $\psi_{j,k}$ be the state of the cookie Markov chain at site $j$ at time $T_k^{\varepsilon,n}$,

$$\psi_{j,k} = R^i_{L(T_k^{\varepsilon,n},j)+1},$$

and $\psi_{j,k+1}^-$ be the state of the cookie Markov chain at site $j$ at time $T_{k+1}^{\varepsilon,n}$,

$$\psi_{j,k+1}^- = R^i_{L(T_{k+1}^{\varepsilon,n},j)+1}.$$

Then by (18) on p. 16 of [KP17], for every $i \in \mathcal{R}$

$$\left| P\left(\psi_{j,k+1}^- = i \mid \psi_{j,k}, V_{j-1}^+ = k, \psi_{j,k+1}^-, 0 \leq j \right) - \pi^+(i) \right| \leq c_7 e^{-c_8 k}, \quad \forall k \in \mathbb{N}.$$ 

Summing up over $i \in \mathcal{R}$ and using induction over $j \in [[\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}] + 1, [\varepsilon \sqrt{n}]]$ we conclude that on the event $\{\sigma > [\varepsilon \sqrt{n}]\}$ for all sufficiently large $n$ the total variation distance between the joint distribution of $\{\psi_{j,k+1}^-\}_{[\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}] < j \leq [\varepsilon \sqrt{n}]}$ and a $\pi^+$-product measure is at most $N([\delta_1 \varepsilon \sqrt{n}] + 1)c_7 e^{-c_8 \delta_1 \varepsilon \sqrt{n}} \leq \varepsilon^3$.

**Step 3.** By remark 6.12 and the assumption that $\delta_1 \prec \delta_2$, the probability that a first cookie environment on the interval $[[\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}], [\varepsilon \sqrt{n}]]$ sampled from the $\pi^+$-product measure will be either not $\delta_2 \varepsilon \sqrt{n}$-lifting from the right or not $\delta_2 \varepsilon \sqrt{n}$-grounding from the left is at most $2\varepsilon^6 / \varepsilon^3 = 2\varepsilon^3$ for all $n$ large.

Adding up the probabilities from Steps 1–3 we conclude that given that the first cookie environment at time $T_k^{\varepsilon,n}$ satisfies all conditions of the lemma, the probability of event $A$ does not exceed $5\varepsilon^3$ for all sufficiently large $n$.

**Step 4.** Finally, we have to also condition on the event $\{T_{k+1}^{\varepsilon,n} = T_k^{\varepsilon,n}\}$. We know by Lemma 6.14 that the probability of this event is at least $C_4 > 0$ uniformly overall environments satisfying the conditions of Lemma 6.13. Therefore, conditioning on $\{T_{k+1}^{\varepsilon,n} = T_k^{\varepsilon,n}\}$ we get that the probability of $A$ is less than or equal to $5\varepsilon^3 / C_4$.

**Lemma 6.16.** Under the assumptions of Lemma 6.13 there is a constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$ and all $n \geq n_0(\varepsilon)$ outside of probability $C\varepsilon^{2.9}$

$$\max_{T_k^{\varepsilon,n} \leq i \leq T_{k+1}^{\varepsilon,n}} X_i - \min_{T_k^{\varepsilon,n} \leq i \leq T_{k+1}^{\varepsilon,n}} X_i \leq 2[\varepsilon \sqrt{n}] - [\varepsilon^3 \sqrt{n}].$$
Proof. Without loss of generality we shall assume that \((I_{T_k^ε,n}, X_{T_k^ε,n}, S_{T_k^ε,n}) = (m_k^ε,n, 0, M_k^ε,n)\) for some integers \(m_k^ε,n \leq 0 \) and \(M_k^ε,n \geq 0\). Let
\[
\tau = \inf\{n \geq T_k^ε,n : |X_n| = |ε\sqrt{n}| - |ε^4\sqrt{n}|\}.
\]
Assume for definiteness that \(X_\tau = -|ε\sqrt{n}| + |ε^4\sqrt{n}|\). Heuristically, if subsequent to \(τ\) the walk \(X\) were a simple symmetric random walk on spatial interval \([-|ε\sqrt{n}|, 0]\) starting from \([ε\sqrt{n}], 0\) then by gambler’s ruin considerations \(X\) would (outside of probability of order \(ε^3\)) hit \(-|ε\sqrt{n}|\) before 0. Given the nature of our problem, we recast this in terms of upcrossings: outside of this order of probability we do not expect an upcrossing to 0 between time \(τ\) and \(T_{k+ε,n}\). Since the cookie environment equilibrates very fast, these simple heuristics happen to be almost correct.

We will consider, as usual, upcrossings from \(-|ε\sqrt{n}| + i\) made between times \(T_k^ε,n\) and \(T_{k+ε,n}^{-}\). We will decompose these as the sum of upcrossings between \(T_k^ε,n\) and \(τ\) and “additional” upcrossings made afterwards. We will show that the number of these additional upcrossings (outside probability of order \(ε^2\)) becomes small and stays small until it becomes 0 before \(i = |ε\sqrt{n}|\). Below we denote by \(C\) possibly different positive constants.

**Step 1.** Consider the BLP \(V^+\) which starts with 0 particles in generation 0 and uses the first cookie environment recorded at time \(T_k^ε,n\) on \([-|ε\sqrt{n}| + 1, 0]\) for generations \([1, |ε\sqrt{n}|]\). Denote by \(V^+\) the same type of process, but let it instead use the first cookie environment on \([-|ε\sqrt{n}| + |ε^4\sqrt{n}| + 1, 0]\) for generations \([1, |ε\sqrt{n}| - |ε^4\sqrt{n}|]\). This process gives the number of upcrossings from \(-|ε\sqrt{n}| + |ε^4\sqrt{n}| + i\) by time \(τ\).

We embed \(\tilde{V}^+\) into \(V^+\) and denote by \(\tilde{U}^+\) the number of “additional” upcrossings. Namely, we observe that the number of particles in generation \(j \geq |ε^4\sqrt{n}|\) of the process \(V^+\) is equal to the number of particles of \(\tilde{V}^+\) in generation \(j - |ε^4\sqrt{n}|\) plus the number of particles of the process \(\tilde{U}^+\) in generation \(j - |ε^4\sqrt{n}|\), where \(\tilde{U}^+\) uses the environment created by the ERW at time \(τ\) on \([-|ε\sqrt{n}| + |ε^4\sqrt{n}|, 0]\) for generations \([1, |ε\sqrt{n}| - |ε^4\sqrt{n}|]\). In short,
\[
V_j^+ = \tilde{V}_{j-|ε^4+\sqrt{n}|}^+ + \tilde{U}_{j-|ε^4\sqrt{n}|}^+, \quad j \geq |ε^4\sqrt{n}|.
\]
Note that \(\tilde{V}_0^+ = 0\) and \(\tilde{U}^+\) starts with \(V^+_{|ε^4\sqrt{n}|}\) particles in generation 0. Our goal is to show that with large probability the process \(\tilde{U}^+\) dies out before \(j = |ε\sqrt{n}|\).

**Step 2.** We consider process \(V^+\). According to Lemma 6.15 there is a \(δ_3 > 0\) such that with probability at least \(1 - 2ε^3\)
\[
V_j^+ \geq δ_3\sqrt{n}, \quad \forall j \in \{||ε\sqrt{n}| - δ_1\sqrt{n}|, |ε\sqrt{n}|]\.
\]
The process \(V^+\) is dominated by the process which starts with \(|ε^4\sqrt{n}|\) particles in generation 0 and uses the same environment and the same coin tosses. Using an appropriate diffusion approximation (depending on whether \(m_k^ε,n \leq -|ε\sqrt{n}|\) or \(m_k^ε,n \geq -|ε\sqrt{n}| + |ε^4\sqrt{n}|\)) we can say that with probability at least \(1 - Cε^3\) for all sufficiently large \(n\) the process \(V^+\) will have no more than \(|ε^3\sqrt{n}|\) particles in generation \(|ε^4\sqrt{n}|\), provided that \(ε\) was fixed sufficiently small. That is outside of probability \(Cε^3\) we have that \(V_j^+_{|ε^4\sqrt{n}|} = U_j^+ \leq |ε^3\sqrt{n}|\).

**Step 3.** We consider \(\tilde{U}^+\). Our aim is to show that outside of probability \(Cε^2\) it becomes small in time \(|ε\sqrt{n}|/2\). By Step 2 and monotonicity of BLPs in the initial number of particles, it will be sufficient to analyze the same process but starting from a larger value \(|ε^3\sqrt{n}|\). We shall denote this process by \(\tilde{U}^+\). By Theorem 5.9 for any fixed \(δ' > 0\), processes \(|ε\sqrt{n}|^{-1} \tilde{U}^+_{|ε\sqrt{n}|}^s s \in [0, \frac{1}{2}]\), \(n \in \mathbb{N}\), converge in distribution as \(n \to \infty\) to a \(\frac{1}{2}\)BESQ \(0\) process \(Y\) starting at \(ε^2\) and stopped on hitting \(δ'\). By scaling and tail decay of extinction probabilities (see, for example, [KM11, Lemma 3.3]),
\[
P(σ_k^Y > 1/2 \mid Y(0) = ε^2) \leq P(σ_0^Y > 1/2 \mid Y(0) = ε^2) = P(σ_0^Y > 1/(2ε^2) \mid Y(0) = 1) < Cε^2.
\]
So for any $\delta' > 0$ and $n$ large enough, $P(\sigma_{[\delta' \sqrt{n}]}^\uparrow > [\varepsilon \sqrt{n}]/2) < C\varepsilon^{2.9}$. We now fix $\delta'$ so that $(\delta', \delta_3/2)$ are as $(\delta', \delta)$ for Lemma 5.10. We conclude that outside of probability $C\varepsilon^{2.9}$

$$\bar{U}_n^+ \leq \bar{U}_n^+ \leq \delta_3 \sqrt{n}/2 \quad \text{for} \quad [\varepsilon \sqrt{n}]/2 \leq k \leq [\varepsilon \sqrt{n}] - [\varepsilon^4 \sqrt{n}].$$

**Step 4.** The last inequality, (40) and (39) imply that outside of probability $C\varepsilon^{2.9}$, for $n$ large

$$\bar{V}_n^+ \geq \delta_3 \sqrt{n}/3 \quad \text{for} \quad k \leq [\varepsilon \sqrt{n}] - [\delta_1 \varepsilon \sqrt{n}] - [\varepsilon^4 \sqrt{n}], [\varepsilon \sqrt{n}] - [\varepsilon^4 \sqrt{n}].$$

This is enough to argue exactly as in the proof of Lemma 6.13 that outside of probability $C\varepsilon^{2.9}$ for $n$ large the first cookie environment on this interval is $\delta_2 \sqrt{n}$-grounding from the right at time $\tau$. Therefore, we can conclude that with probability $1 - C\varepsilon^{2.9}$ the process $\bar{U}_n^+$ will die out by generation $[\varepsilon \sqrt{n}] - [\varepsilon^4 \sqrt{n}]$ (assuming as we may that $\delta_3/2 < \delta_2 \varepsilon$). In other words, after hitting $- [\varepsilon \sqrt{n}] + [\varepsilon^4 \sqrt{n}]$ the ERW will hit $[- \varepsilon \sqrt{n}]$ before $[\varepsilon \sqrt{n}] - [\varepsilon^4 \sqrt{n}]$ (even before 0) with probability $1 - C\varepsilon^{2.9}$. \qed

7. **COUPLING OF THE RESCALED MESOSCOPIC WALK AND BMPE-WALK**

Let $W_{k}^{\varepsilon,n} = \frac{X_{k}^{\varepsilon,n}}{[\varepsilon \sqrt{n}]}$, $k \in \mathbb{Z}_+$, be a re-scaled mesoscopic walk taking values in $\mathbb{Z}$ and

$$I_{k}^{\varepsilon,n} = \min_{j \leq T_{k}^{\varepsilon,n}} \frac{X_{j}^{\varepsilon,n}}{[\varepsilon \sqrt{n}]}, \quad S_{k}^{\varepsilon,n} = \max_{j \leq T_{k}^{\varepsilon,n}} \frac{X_{j}^{\varepsilon,n}}{[\varepsilon \sqrt{n}]}, \quad k \in \mathbb{Z}_+,$

be its running minimum and maximum respectively. The walk $\{(I_{k}^{\varepsilon,n}, W_{k}^{\varepsilon,n}, S_{k}^{\varepsilon,n})\}_{k \geq 0}$ is non-markovian. It depends on the ERW path in a random cookie environment. To make it into a Markov process we have to retain some information about the environment at each mesoscopic step. Let

$$\mu_{k}^{\varepsilon,n,x} = \begin{cases} \eta, & \text{if } k = 0 \text{ or } x \not\in [I_{k}^{\varepsilon,n}, S_{k}^{\varepsilon,n}]; \\ \delta_i, & \text{where } i = R_{\mathcal{L}(T_{k}^{\varepsilon,n}, x)+1} \text{ otherwise.} \end{cases}$$

In words, for $k = 0$ or if a site has not been visited before time $T_{k}^{\varepsilon,n}$ we set the distribution of the first cookie at that site to $\eta$. For each site that has been visited before time $T_{k}^{\varepsilon,n}$ we record the next state of the cookie Markov chain (and, thus, fix the first cookie in the stack) at this site at time $T_{k}^{\varepsilon,n}$. Now the process

$$\{(X_{k}^{\varepsilon,n}, G_{x \in \mathbb{Z}} \mu_{k}^{\varepsilon,n,x})\}_{k \geq 0} := \{(I_{k}^{\varepsilon,n}, W_{k}^{\varepsilon,n}, S_{k}^{\varepsilon,n}), G_{x \in \mathbb{Z}} \mu_{k}^{\varepsilon,n,x})\}_{k \geq 0}$$

is a Markov process, since the information collected at each step is sufficient to generate the next.

We now describe a coupling between $X^{\varepsilon,n}$ and a modified BMPE $X^{\varepsilon} := (I^{\varepsilon}, W^{\varepsilon}, S^{\varepsilon})$ defined in Section 4.2. For each $n$ we have to use a different version of BMPE-walk, $X^{\varepsilon,n}$, which is indicated by an additional superscript $n$. In this coupling we will, in particular, address the filtration $\mathcal{F}_k^{\varepsilon,n}$ associated to our discrete time process. We take $X^{\varepsilon,n}$ as a primary object and use its randomness (plus auxiliary, independent randomness) to define the coupling. Our description will detail how to construct $X^{\varepsilon,n}$ in full but we will talk of the coupling being “broken” for certain time indices. This term will signify that from this point the two processes are no longer close (or that we do not expect them to be close).

Our goal is to couple $X^{\varepsilon,n}$, with the process $\tilde{X}^{\varepsilon,n}$ so that if the coupling is not broken by step $k$, then $X_{j}^{\varepsilon,n} = \tilde{X}_{j}^{\varepsilon,n}$ for each $0 \leq j \leq k$. We repeat that in describing $\tilde{X}^{\varepsilon,n}$, we must also describe the filtration $\mathcal{F}_k^{\varepsilon,n}$ to which it is adapted. We will certainly have that for each $k$ the filtration $\mathcal{F}_k^{\varepsilon,n}$ contains the $\sigma$-algebra generated by our ERW up to time $T_{k}^{\varepsilon,n}$. We will also suppose (after enlarging the probability space if need be) that for each $k$, $\mathcal{F}_k^{\varepsilon,n}$ contains a number of i.i.d. uniform random variables independent of the ERW $X$ and its cookie environment. These uniform random variables will be used to generate the evolution of $\tilde{X}^{\varepsilon,n}$ once the coupling is broken: if the coupling
is broken at step \( k \), then \( \tilde{X}_{k+1}^{\varepsilon,n} \) is generated corresponding to a BMPE with initial conditions \( \tilde{X}_k^{\varepsilon,n} \) using these additional uniform random variables.

We begin with \( k = 0 \) by setting \( X_0^{\varepsilon,n} = \tilde{X}_0^{\varepsilon,n} = (0, 0, 0) \) and saying that at step \( k = 0 \) the coupling is unbroken.

### 7.1. First step.

This step is for the coupling between \( X_1^{\varepsilon,n} \) and \( \tilde{X}_1^{\varepsilon,n} \) but it introduces ideas that will be used later in coupling near extrema. This step is special as it is the only step when both extrema will change.

Recall that intervals \( J_\ell \) were defined in (10). We compute for each \( \ell \) the probabilities \( p_{1,\ell}^n \) where

\[
p_{1,\ell}^n = P(A_{1,\ell}^n) := P(W_{1,\ell}^{\varepsilon,n} = 1, I_{1,\ell}^{\varepsilon,n} \in J_\ell) \quad \text{for} \quad -\ell = 1, 2, \ldots, L;
\]

\[
p_{1,\ell}^n = P(A_{1,\ell}^n) := P(W_{1,\ell}^{\varepsilon,n} = -1, S_{1,\ell}^{\varepsilon,n} \in J_\ell) \quad \text{for} \quad \ell = 1, 2, \ldots, L.
\]

We also compute the corresponding probabilities for a BMPE, \((I, W, S)\), starting from \((0, 0, 0)\):

\[
q_{1,\ell} = P(W(\tau(1, 0, 0, 0)) = 1, I(\tau(1)) \in J_\ell) \quad \text{for} \quad -\ell = 1, 2, \ldots, L;
\]

\[
q_{1,\ell} = P(W(\tau(1, 0, 0, 0)) = -1, S(\tau(1)) \in J_\ell) \quad \text{for} \quad \ell = 1, 2, \ldots, L.
\]

The triple \((\tilde{I}_{1,\ell}^{\varepsilon,n}, \tilde{W}_{1,\ell}^{\varepsilon,n}, \tilde{S}_{1,\ell}^{\varepsilon,n})\) is obtained by utilizing the maximal coupling of probability measures \(\{p_{1,\ell}^n\}_{0<|\ell|\leq L}\) and \(\{q_{1,\ell}\}_{0<|\ell|\leq L}\). More precisely, if the event \(A_\ell^n\) occurs for the ERW then with probability \(1 - \frac{n_\ell}{p_{1,\ell}^n}\) we let

\[
(\tilde{I}_{1,\ell}^{\varepsilon,n}, \tilde{W}_{1,\ell}^{\varepsilon,n}, \tilde{S}_{1,\ell}^{\varepsilon,n}) = (I_{1,\ell}^{\varepsilon,n}, W_{1,\ell}^{\varepsilon,n}, S_{1,\ell}^{\varepsilon,n}) = \begin{cases} (1^{\varepsilon,n}, 1, 1) & \text{if} \ -\ell = 1, 2, \ldots, L \\ (-1, -1, 1^{\varepsilon,n}) & \text{if} \ \ell = 1, 2, \ldots, L, \end{cases}
\]

and if the above has not yet determined \((\tilde{I}_{1,\ell}^{\varepsilon,n}, \tilde{W}_{1,\ell}^{\varepsilon,n}, \tilde{S}_{1,\ell}^{\varepsilon,n})\) (which is true with probability \(\sum_{0<|\ell|\leq L} (q_{1,\ell} - p_{1,\ell}^n)\)) then we use auxiliary independent randomness to determine \((\tilde{I}_{1,\ell}^{\varepsilon,n}, \tilde{W}_{1,\ell}^{\varepsilon,n}, \tilde{S}_{1,\ell}^{\varepsilon,n})\) so that

\[
(\tilde{I}_{1,\ell}^{\varepsilon,n}, \tilde{W}_{1,\ell}^{\varepsilon,n}, \tilde{S}_{1,\ell}^{\varepsilon,n}) = \begin{cases} \left(\frac{\ell + 1}{L}, 1, 1\right) & \text{if} \ -\ell = 1, 2, \ldots, L \\ (-1, -1, \frac{\ell + 1}{L}) & \text{if} \ \ell = 1, 2, \ldots, L, \end{cases}
\]

with probability \(\frac{(q_{1,\ell} - p_{1,\ell}^n)_+}{\sum_{0<|\ell|\leq L} (q_{1,\ell} - p_{1,\ell}^n)_+}\).

Note that this coupling is such that

\[
P(\tilde{W}_{1,\ell}^{\varepsilon,n} = 1, \tilde{S}_{1,\ell}^{\varepsilon,n} = 1, \tilde{I}_{1,\ell}^{\varepsilon,n} \in J_\ell) = q_{1,\ell} \quad \text{for} \quad -\ell = 1, 2, \ldots, L;
\]

\[
P(\tilde{W}_{1,\ell}^{\varepsilon,n} = -1, \tilde{I}_{1,\ell}^{\varepsilon,n} = -1, \tilde{S}_{1,\ell}^{\varepsilon,n} \in J_\ell) = q_{1,\ell} \quad \text{for} \quad \ell = 1, 2, \ldots, L,
\]

and such that \(X_{1,\ell}^{\varepsilon,n} = \tilde{X}_{1,\ell}^{\varepsilon,n}\) with probability at least

\[
1 - \sum_{0<|\ell|\leq L} (p_{1,\ell}^n - q_{1,\ell})_+ = 1 - \frac{1}{2} \sum_{0<|\ell|\leq L} |p_{1,\ell}^n - q_{1,\ell}|.
\]

### Definition 7.1.

We say that the coupling is broken after step \( j \) if

(41) \( \tilde{X}_j^{\varepsilon,n} \neq X_j^{\varepsilon,n} \) or

(42) \( W_j^{\varepsilon,n} - W_{j-1}^{\varepsilon,n} = -1 \) and \( \max_{T_{j-1}^{\varepsilon,n} \leq i < T_j^{\varepsilon,n}} (X_i - X_{T_{j-1}^{\varepsilon,n}}) > (1 - \varepsilon^3) \varepsilon \sqrt{n} \) or

(43) \( W_j^{\varepsilon,n} - W_{j-1}^{\varepsilon,n} = 1 \) and \( \min_{T_{j-1}^{\varepsilon,n} \leq i < T_j^{\varepsilon,n}} (X_i - X_{T_{j-1}^{\varepsilon,n}}) < (-1 + \varepsilon^3) \varepsilon \sqrt{n} \).

Once the coupling is broken it remains so subsequently.
Note that conditions (42) and (43) ensure that the coupling is broken on the $j$-th step if the walk goes very close to the right (or left) endpoint of $[X_{T_{j-1}^0} - \epsilon \sqrt{n}, X_{T_{j-1}^0} + \epsilon \sqrt{n}]$ but then ultimately reaches the left (or right) endpoint first. Therefore, if the coupling is unbroken after step $j$ then the environment in an $\epsilon^3$-neighborhood of each integer point in the range of $W_{\epsilon,n}^k$ up to time $j$ except for $W_{\epsilon,n}^k$ and $W_{\epsilon,n}^{j-1}$ remains unchanged by the $j$-th step of the walk $W_{\epsilon,n}^k$ and thus preserves any lifting or grounding properties. Because of this, Lemma 6.13 will allow us to get, with high probability, lifting and grounding properties at all sites of $[\epsilon \sqrt{n}]Z$ other than the position of the walk at time $T_{k,n}^\epsilon$ (see Lemma 7.3 below).

7.2. Steps after the first. We now pass to the coupling for the $k+1$-th step given that the $k$-th step has been completed. As already stated, if the coupling is broken before or at step $k$, then $(\tilde{I}_{k+1}^\epsilon, \tilde{W}_{k+1}^\epsilon, \tilde{S}_{k+1}^\epsilon)$ is chosen independently using auxiliary uniform random variables independent of the cookie process. So in the following we assume that the coupling is unbroken. We note that (unlike in the first step) for $\epsilon$ fixed $\{I_k^\epsilon, S_k^\epsilon\} \not\subset (W_{k,n}^\epsilon - 1, W_{k,n}^\epsilon + 1)$.

Steps in the bulk: We first give the coupling in the case $\{I_k^\epsilon, S_k^\epsilon\} \cap (W_{k,n}^\epsilon - 1, W_{k,n}^\epsilon + 1) = \emptyset$. We let

$$p_k^n = P(W_{k,n}^\epsilon k+1 = W_{k,n}^\epsilon k + 1 \mid \mathcal{F}_k^n),$$

and note that the corresponding probability for the BMPE is exactly $\frac{1}{2}$ since we are away from the extremes. If $W_{k,n}^\epsilon k+1 = W_{k,n}^\epsilon k - 1$, then we take

$$(\tilde{I}_{k+1}^\epsilon, \tilde{W}_{k+1}^\epsilon, \tilde{S}_{k+1}^\epsilon) = (I_k^\epsilon, W_{k,n}^\epsilon k - 1, S_k^\epsilon) \text{ with probability } 1 \wedge \frac{1/2}{p_k^n},$$

If $W_{k,n}^\epsilon k+1 = W_{k,n}^\epsilon k + 1$, then we take

$$(\tilde{I}_{k+1}^\epsilon, \tilde{W}_{k+1}^\epsilon, \tilde{S}_{k+1}^\epsilon) = (I_k^\epsilon, W_{k,n}^\epsilon k + 1, S_k^\epsilon) \text{ with probability } 1 \wedge \frac{1/2}{p_k^n}. $$

If $(\tilde{I}_{k+1}^\epsilon, \tilde{W}_{k+1}^\epsilon, \tilde{S}_{k+1}^\epsilon)$ is undefined we use an auxiliary uniform random variable to define it so that it satisfies properties (i)-(iii) of Section 4.2. Then we check if the coupling is broken (see Definition 7.1).

Steps at the boundary: It remains to detail the coupling if $\{I_k^\epsilon, S_k^\epsilon\} \cap (W_{k,n}^\epsilon k - 1, W_{k,n}^\epsilon k + 1) \neq \emptyset$. We suppose that

$$\{I_k^\epsilon, S_k^\epsilon\} \cap (W_{k,n}^\epsilon k - 1, W_{k,n}^\epsilon k + 1) = \{S_k^\epsilon\}$$

and omit details for the other case. For notational clarity and to emphasize the congruence with the first step, we translate the space so that $W_{k,n}^\epsilon k = W_{k,n}^\epsilon 0 = 0$ and $S_k^\epsilon = S_k^\epsilon [0, 1 - \epsilon^3].$

We divide up $(-1, 1)$ into the same intervals (16) as in the first step and find $\ell_k$ such that $S_k^\epsilon \in J_{\ell_k}$. Note that since $S_k^\epsilon < 1 - \epsilon^3$, we know that $J_{\ell_k+1} \subset (0, 1).$ We shall join $J_{\ell_k}$ and $J_{\ell_k+1}$ to form a single interval which we shall again call $J_{k+1}$. Then we compute for $\ell \geq \ell_k + 1$

$$p_{k,\ell} = P(W_{k,n}^\epsilon k+1 = -1, S_{k+1}^\epsilon \in J_{\ell} \mid \mathcal{F}_k^n);$$

$$q_{k,\ell} = P(W(\tau(1, -1, 0, S_k^n)) = -1, S(\tau(1, -1, 0, S_k^n)) \in J_{\ell});$$

where $(I, W, S)$ is a BMPE with initial condition $(I_0, W_0, S_0) = (-1, 0, S_k^n)$. We also compute

$$p_k^n = P(W_{k,n}^\epsilon k+1 = 1 \mid \mathcal{F}_k^n) \text{ and } q_k = P(W(\tau(1, -1, 0, S_k^n)) = 1).$$

If $W_{k,n}^\epsilon k+1 = W_{k,n}^\epsilon k + 1$, then we take $\tilde{W}_{k+1}^\epsilon = \tilde{W}_{k+1}^\epsilon + 1$ with probability $1 \wedge \frac{q_k}{p_k^n}$ and in this case

$$((\tilde{I}_{k+1}^\epsilon, \tilde{W}_{k+1}^\epsilon, \tilde{S}_{k+1}^\epsilon) = (I_k^\epsilon, W_{k,n}^\epsilon k + 1, W_{k,n}^\epsilon k + 1).$$

If $W_{k,n}^\epsilon k+1 = W_{k,n}^\epsilon k - 1$ and $S_{k+1}^\epsilon \in J_{\ell_k}$, $\ell \in [\ell_k + 1, L]$, then with probability $1 \wedge \frac{q_{k,\ell}}{p_{k,\ell}}$, we take

$$((\tilde{I}_{k+1}^\epsilon, \tilde{W}_{k+1}^\epsilon, \tilde{S}_{k+1}^\epsilon) = (I_k^\epsilon, W_{k,n}^\epsilon k - 1, S_{k+1}^\epsilon).$$
If at this point \((I_{k+1}^{ε,n}, W_{k+1}^{ε,n}, \tilde{S}_{k+1}^{ε,n})\) is undefined, we use the auxiliary independent randomness in a similar manner as on the first step. That is, we let \((I_{k+1}^{ε,n}, W_{k+1}^{ε,n}, \tilde{S}_{k+1}^{ε,n})\) equal
\[
(I_{k}^{ε,n}, W_{k}^{ε,n} - 1, \frac{ℓ - 1}{L}), \quad \text{with probability } \frac{(q_{k,ℓ} - p_{k,ℓ}^n)}{(q_k - p_k^n) + \sum_{0<|v'|\leq L(q_{1,v'} - p_{1,v}')^+} + \sum_{0<|v'|\leq L(q_{1,v'} - p_{1,v}')^+}.
\]
for \(ℓ \in [ℓ_k + 1, L]\) and
\[
(I_{k}^{ε,n}, W_{k}^{ε,n} + 1, W_{k}^{ε,n} + 1), \quad \text{with probability } \frac{(q_{k} - p_{k}^n)}{(q_k - p_k^n) + \sum_{0<|v'|\leq L(q_{1,v'} - p_{1,v}')^+} + \sum_{0<|v'|\leq L(q_{1,v'} - p_{1,v}')^+}.
\]

7.3. The Coupling Theorem. Having constructed the coupling, we can now state the main result of this section.

**Theorem 7.2.** For every \(K > 0\)
\[
\lim_{ε \to 0} \liminf_{n \to ∞} P \left( \inf\{k \geq 0 : X_k^{ε,n} \neq \tilde{X}_k^{ε,n} \} \geq \frac{K}{ε^2} \right) = 1.
\]

*Proof.* Fix an arbitrary \(K > 0\). We shall make a list of conditions on the ERW path and on the first cookie environments at each mesoscopic step which will ensure that the coupling is preserved with high probability. These conditions involve two additional parameters \(δ_1, δ_2 > 0\) which will depend only on \(ε\) and which we shall choose later. For now it is enough to say that \(δ_1\) and \(δ_2\) are chosen so that \(δ_1 \leq δ_2\) and are small enough so that Lemmas 5.19, 6.13 and 6.16 can be applied. The conditions to be satisfied at each step \(j\) are as follows.

(Ei) The interval \([ε√n)(W_j^{ε,n} - 1), [ε√n)(W_j^{ε,n} - 1) + [εδ_1√n]\] is \(δ_2ε√n\)-lifting from the left and \(δ_2ε√n\)-grounding from the right.

(Eii) The interval \([ε√n)(W_j^{ε,n} + 1) - [εδ_1√n), [ε√n)(W_j^{ε,n} + 1]\] is \(δ_2ε√n\)-lifting from the right and \(δ_2ε√n\)-grounding from the left.

(Eiii) The first cookie environment is \((n^{1/8}, ν/2 - 1)\)-good on \([ε√n)(I_j^{ε,n} \lor (W_j^{ε,n} - 1), [ε√n)(W_j^{ε,n}]\) and is \((n^{1/8}, 0)\)-good on \([ε√n)(W_j^{ε,n}, [ε√n)(W_j^{ε,n} + 1 \land S_j^{ε,n})]\).

We remark that for \(j = 0\) the condition (Eiii) is vacuous, and we shall agree that it automatically holds. Let
\[
β^{ε,n} := \inf\{j \geq 0 : \text{ at least one of conditions (Ei)-(Eiii) above does not hold for } j\}
\]
and \(τ^{ε,n}\) be the step at which the coupling breaks down, i.e.
\[
τ^{ε,n} := \inf\{j \geq 1 : \text{ at least one of (41)-(43) does not hold for } j\}.
\]
We have also agreed that at time 0 the coupling is unbroken. This implies that \(τ^{ε,n} > 0\). Using this notation we can say that
\[
P \left( \inf\{k \geq 0 : X_k^{ε,n} \neq \tilde{X}_k^{ε,n} \} > \frac{K}{ε^2} \right) \geq P \left( τ^{ε,n} > \frac{K}{ε^2}, β^{ε,n} > \frac{K}{ε^2} \right).
\]

As a first step toward controlling the probability on the right, we need the following lemmas

**Lemma 7.3.** There exists a constant \(C_5 > 0\) such that for every \(ε > 0\) and \(n\) large enough that
\[
P (\text{Conditions (Ei) and (Eii) hold for } j = k \mid τ^{ε,n} > k - 1, β^{ε,n} > k - 1) \geq 1 - C_5ε^3,
\]
for all \(k \geq 1\).

*Proof.* If \(τ^{ε,n} > k - 1\) and \(β^{ε,n} > k - 1\), then the remaining first cookie environment on \([X_{T_{k-1}}^{ε,n} - [ε√n], X_{T_k}^{ε,n} - [ε√n]\] is “regular” as defined in the Definitions 6.8 and 6.9. Thus the conclusion of Lemma 7.3 follows directly from Lemma 6.13. □
Lemma 7.4. There exists a constant $C_6 > 0$ such that for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon, \delta_1(\varepsilon), \delta_2(\varepsilon))$ such that for all $n \geq n_0$ and all $k \in \mathbb{N}$

\[ P(\tau^{\varepsilon,n} > k \mid \beta^{\varepsilon,n} > k - 1, \tau^{\varepsilon,n} > k - 1) \geq 1 - C_6\varepsilon^{2.9}. \]

Proof. The validity of inequality (45) has to be checked for three different cases:

1. the first step, i.e. $k = 1$;
2. $k \geq 2$ and $\{I_{k-1}^{\varepsilon,n}, S_{k-1}^{\varepsilon,n}\} \cap [W_{k-1}^{\varepsilon,n} - 1, W_{k-1}^{\varepsilon,n} + 1] = \emptyset$;
3. $k \geq 2$ and $\{I_{k-1}^{\varepsilon,n}, S_{k-1}^{\varepsilon,n}\} \cap [W_{k-1}^{\varepsilon,n} - 1, W_{k-1}^{\varepsilon,n} + 1] \neq \emptyset$.

Case (1). Let $k = 1$. Recall that $\tau^{\varepsilon,n} > 0$ and we start with $\lambda_0^{\varepsilon,n} = \tilde{\lambda}_0^{\varepsilon,n} = (0, 0, 0)$ and an i.i.d. cookie environment with the marginal distribution $\eta$. The probability that the coupling breaks down at the first step is bounded above by

\[
\frac{1}{2} \max_{0 \leq |t| \leq L} |p_{1,t} - p_{1,t}^n| + P(W_1^{\varepsilon,n} = 1, I_1^{\varepsilon,n} < -1 + \varepsilon^3) + P(W_1^{\varepsilon,n} = -1, S_1^{\varepsilon,n} > 1 - \varepsilon^3).
\]

We started with a product measure, so all conditions of the concatenation lemma (Lemma 5.19 and Lemma 6.16) hold. Therefore, there exists an $n_1 = n_1(\varepsilon, \delta_1(\varepsilon), \delta_2(\varepsilon))$ such that all the terms above are bounded by a constant multiple of $\varepsilon^{2.9}$ for $n \geq n_1$. Thus, (45) is satisfied in this case with $n_0 = n_1$ for some $C_6 = C_{6,1}$.

Case (2). Let $k \geq 2$, $\beta^{\varepsilon,n} > k - 1$, $\tau^{\varepsilon,n} > k - 1$, and $\{I_{k-1}^{\varepsilon,n}, S_{k-1}^{\varepsilon,n}\} \cap [W_{k-1}^{\varepsilon,n} - 1, W_{k-1}^{\varepsilon,n} + 1] = \emptyset$. Then the probability that the coupling breaks down at step $k$ does not exceed

\[
\frac{1}{2} |p_k^n - \frac{1}{2}| + P(W_k^{\varepsilon,n} - W_{k-1}^{\varepsilon,n} = 1, I_k^{\varepsilon,n} < -1 + \varepsilon^3) + P(W_k^{\varepsilon,n} - W_{k-1}^{\varepsilon,n} = -1, S_k^{\varepsilon,n} - W_{k-1}^{\varepsilon,n} > 1 - \varepsilon^3).
\]

Since the coupling hasn’t broken by the $(k-1)$-th step, the remaining first cookie environment in $[X_{\tau^{\varepsilon,n} - \lfloor \sqrt{n} \rfloor}, X_{\tau^{\varepsilon,n} + \lceil \sqrt{n} \rceil}]$ satisfies the conditions of the Lemma 5.19 and 6.16. Thus, the above sum does not exceed $C_{6,2}\varepsilon^{2.9}$ for all $n \geq n_2(\varepsilon, \delta_1(\varepsilon), \delta_2(\varepsilon))$, where $C_{6,2}$ does not depend on either $\varepsilon$ or $k \geq 2$.

Case (3). Let $k \geq 2$, $\beta^{\varepsilon,n} > k - 1$, $\tau^{\varepsilon,n} > k - 1$, and $\{I_{k-1}^{\varepsilon,n}, S_{k-1}^{\varepsilon,n}\} \cap [W_{k-1}^{\varepsilon,n} - 1, W_{k-1}^{\varepsilon,n} + 1] = \{S_{k-1}^{\varepsilon,n}\}$. The other case is symmetric and we shall not give details.

Under the above assumptions, the probability that the coupling breaks down at step $k$ does not exceed

\[
\frac{1}{2} |p_k^n - q_k| + \frac{1}{2} \max_{1 \leq \ell \leq L} |q_{k,\ell} - p_{k,\ell}^n| + P(W_k^{\varepsilon,n} - W_{k-1}^{\varepsilon,n} = 1, I_k^{\varepsilon,n} < -1 + \varepsilon^3) + P(W_k^{\varepsilon,n} - W_{k-1}^{\varepsilon,n} = -1, S_k^{\varepsilon,n} - W_{k-1}^{\varepsilon,n} > 1 - \varepsilon^3).
\]

Again, since the coupling has not yet been broken we can apply Lemmas 5.19 and 6.16 to conclude that this sum does not exceed $C_{6,3}\varepsilon^{2.9}$ for all $n \geq n_3(\varepsilon, \delta_1(\varepsilon), \delta_2(\varepsilon))$, where $C_{6,3}$ does not depend on either $\varepsilon$ or $k \geq 2$.

This completes the proof of (45) with $C_6 = \max\{C_{6,1}, C_{6,2}, C_{6,3}\}$ and $n_0 = \max\{n_2, n_3\}$ for $k \geq 2$. \qed

We will next use Lemmas 7.3 and 7.4 to obtain a lower bound on (44). In particular, we will show that for every $\varepsilon > 0$ and every $n \geq n_0 = n_0(\varepsilon, \delta_1(\varepsilon), \delta_2(\varepsilon), k)$ we have

\[ P(\tau^{\varepsilon,n} > k, \beta^{\varepsilon,n} > k) \geq (1 - C_7\varepsilon^{2.9})^{k+1}, \quad \text{where } C_7 = C_5 + C_6 + 1. \]

We will prove (46) by induction.

Base case: $k = 0$. Since $\tau^{\varepsilon,n} > 0$ by definition and since (Eiii) is vacuous at step 0, we need only check that conditions (Ei) and (Eii) hold. Using Lemmas 5.7 and 5.8 and choosing $\delta_1$ and $\delta_2$ appropriately (depending on $\varepsilon$) we have that $P(\tau^{\varepsilon,n} > 0, \beta^{\varepsilon,n} > 0) > 1 - \varepsilon^3$ for all $n$ large.
Induction step: $k \geq 1$. We will assume that (46) holds for $k-1$. Next, first of all that
\[
P(\tau_{\varepsilon,n} > k, \beta_{\varepsilon,n} > k)
= P(\tau_{\varepsilon,n} > k-1, \beta_{\varepsilon,n} > k-1) P(\tau_{\varepsilon,n} > k, \beta_{\varepsilon,n} > k \mid \tau_{\varepsilon,n} > k-1, \beta_{\varepsilon,n} > k-1)
\geq P(\tau_{\varepsilon,n} > k-1, \beta_{\varepsilon,n} > k-1)
\times \{P(\tau_{\varepsilon,n} > k \mid \tau_{\varepsilon,n} > k-1, \beta_{\varepsilon,n} > k-1) + P(\beta_{\varepsilon,n} > k \mid \beta_{\varepsilon,n} > k-1, \tau_{\varepsilon,n} > k-1) - 1\}^2
\geq (1 - C_7 \varepsilon_{2.9})^k \left\{-C_8 \varepsilon_{2.9} + P(\beta_{\varepsilon,n} > k \mid \beta_{\varepsilon,n} > k-1, \tau_{\varepsilon,n} > k-1)\right\},
\]
where the last inequality holds by the induction assumption and Lemma 7.4 for $n$ large enough (depending on $\varepsilon$ and $k$). For the last probability in the braces on the right, Lemma 7.3 controls the (conditional) probability that conditions (Ei) and (Eii) hold and Corollary 6.2 controls the (unconditional) probability that condition (Eiii) holds. More precisely, since the event $A_{k,\varepsilon,n}$ in the statement of Corollary 6.2 implies that condition (Eiii) holds then for $n$ large enough
\[
P(\beta_{\varepsilon,n} > k \mid \beta_{\varepsilon,n} > k-1, \tau_{\varepsilon,n} > k-1) \geq 1 - C_5 \varepsilon_{3} - P\left((A_{k,\varepsilon,n})^c \mid \beta_{\varepsilon,n} > k-1, \tau_{\varepsilon,n} > k-1\right)
\geq 1 - C_5 \varepsilon_{3} - \frac{P((A_{k,\varepsilon,n})^c)}{(1 - C_7 \varepsilon_{2.9})^k},
\]
and since $P((A_{k,\varepsilon,n})^c) \to 0$ as $n \to \infty$ (by Corollary 6.2) it follows that the right side is larger than
\[
(1 - (C_5 + 1)^3) \geq (1 - (C_5 + 1)^{2.9}) \text{ for } n \text{ large enough (again depending on } k \text{ and } \varepsilon). \]
Applying this to (47) finishes the proof of (46).

Finally, applying (46) to (44) we obtain that
\[
\lim_{\varepsilon \to 0} \liminf_{n \to \infty} P\left(\inf\{k \geq 0: X_{k,\varepsilon,n} \neq \bar{X}_{k,\varepsilon,n}\} > \frac{K}{\varepsilon^2}\right) \geq \lim_{\varepsilon \to 0} (1 - C_7 \varepsilon_{2.9})^{[K/\varepsilon^2]+1} = 1.
\]
This completes the proof of Theorem 7.2.

8. Time control and the proof of Theorem 1.8

The previous section established that the embedded process $\{W_{\varepsilon,n}\}_{k \geq 0}$ is close to a modified BMPE walk. From Section 4 we know that modified walks converge to BMPE. To complete the proof of Theorem 1.8 we just have to show a law of large numbers for the variables $T_{\varepsilon,n}^k$, $k \geq 0$.

Lemma 8.1. For each $K, h > 0$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and all $n \geq n_0(\varepsilon)$
\[
P\left(\sup_{k < \varepsilon^{-2}K} \left|T_{\varepsilon,n}^k - \frac{\nu}{2} kn_{\varepsilon^2}\right| > hn\right) < h.
\]

Let us assume for the moment that this lemma holds and give a proof of Theorem 1.8

Proof of Theorem 1.8. From Theorem 7.2 we know that for all $K, \delta > 0$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and all sufficiently large $n$, with probability at least $1 - \delta$ we have
\[
(I_{\varepsilon,n}^k, W_{\varepsilon,n}^k, S_{\varepsilon,n}^k) = (\bar{T}_{\varepsilon,n}^k, \bar{W}_{\varepsilon,n}^k, \bar{S}_{\varepsilon,n}^k), \quad 0 \leq k < \varepsilon^{-2}K,
\]
where $(\bar{T}_{\varepsilon,n}^k, \bar{W}_{\varepsilon,n}^k, \bar{S}_{\varepsilon,n}^k)$ is a modified BMPE-walk. This and Corollary 4.3 imply that there exists a family of BMPEs $\{W_{n,\varepsilon}(t)\}_{t \geq 0}$ such that for all $T, \delta > 0$ there exists $\varepsilon_0 > 0$ such that
\[
\forall \varepsilon \in (0, \varepsilon_0) \exists n_0(\varepsilon) \text{ such that } P\left(\sup_{t \leq T} |\varepsilon W_{\varepsilon,n}^k - W_{n,\varepsilon}(t)| > \delta\right) < \delta \text{ for all } n \geq n_0(\varepsilon).
\]
To complete the proof it is enough to replace $\varepsilon W_{[\varepsilon^{-2} t]}$ with $\hat{W}^{\varepsilon,n}(t) := \varepsilon W_{k_t}$ where $k_t = k_t(n, \varepsilon)$ is such that $T_{k_t}^{\varepsilon,n} \leq \frac{\varepsilon}{2} tn < T_{k_t+1}^{\varepsilon,n}$. Indeed, for all large $n$ the process $X_{\lfloor tn/2 \rfloor}/\sqrt{n}$ always stays within $\varepsilon$ of $\varepsilon X_{T_{k_t}^{\varepsilon,n}/\lfloor \varepsilon\sqrt{n} \rfloor} = \varepsilon W_{k_t}$, and if we know that
\begin{equation}
\mathbb{P}\left( \sup_{t \in T} |\hat{W}^{\varepsilon,n}(t) - W_{n,\varepsilon}(t)| > \delta \right) < \delta \quad \text{for all } n \geq n_0(\varepsilon),
\end{equation}
then we have the convergence claimed in Theorem 1.8. To see that (49) holds we simply note that
$$|\hat{W}^{\varepsilon,n}(t) - W_{n,\varepsilon}(t)| = |\varepsilon W_{k_t} - W_{n,\varepsilon}(t)| \leq |\varepsilon W_{k_t} - W_{n,\varepsilon}(k_t \varepsilon^2)| + |W_{n,\varepsilon}(k_t \varepsilon^2) - W_{n,\varepsilon}(t)|,$$
where both terms in the right hand side are controlled by (48), Lemma 8.1, and path continuity of BMPE.

**Proof of Lemma 8.1.** Just as in the proof of Proposition 4.1 we argue that the increments in the bulk are dominant and increments at extremes are negligible. Thus in analyzing the bulk increments we must be more precise, whereas a reasonable bound on increments at the extremes will meet our purpose.

To improve legibility, we drop $\varepsilon$ and $n$ from the notation and write $H_i = T_i^{\varepsilon,n} - T_{i-1}^{\varepsilon,n}, i \in \mathbb{N}$. We wish to use the law of large numbers for i.i.d. random variables but the $\{H_i/n\}_{i \geq 1}$ are neither identically distributed nor independent (even in the limit as $n$ tends to infinity). As a first step to address this, we separate out the $H_i$ according to whether the walk is in the bulk or at an extreme at time $T_{i-1}^{\varepsilon,n}$. Accordingly, we set $\mathcal{B}$ as the set of indices $i < K/\varepsilon^2$ such that $I_{T_{i-1}^{\varepsilon,n}} + [\varepsilon \sqrt{n}] \leq X_{T_{i-1}^{\varepsilon,n}} \leq S_{T_{i-1}^{\varepsilon,n}} - [\varepsilon \sqrt{n}]$, set $\mathcal{S}$ to be those $i < K/\varepsilon^2$ for which $S_{T_{i-1}^{\varepsilon,n}} < X_{T_{i-1}^{\varepsilon,n}} + [\varepsilon \sqrt{n}]$, and $\mathcal{I}$ for the remainder, that is those $i < K/\varepsilon^2$ for which $I_{T_{i-1}^{\varepsilon,n}} > X_{T_{i-1}^{\varepsilon,n}} - [\varepsilon \sqrt{n}]$.

The random variables $\{H_i/(n \varepsilon^2)\}_{i \in \mathcal{B}}$ are still not proven to be i.i.d., even in a limit as $n$ tends to infinity. But they are “close” to i.i.d. random variables whose law is that of the time for a variance $2/\nu$ Brownian motion, starting at 0, to leave $(-1, 1)$.

To show our convergence it will be enough to show that $\forall K, h \in (0, \infty)$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $n \geq n_0(\varepsilon)$
\begin{equation}
\mathbb{P}\left( |\mathcal{S} \cup \mathcal{I}| > \frac{2h}{\nu \varepsilon^2} \right) < h
\end{equation}
\begin{equation}
\mathbb{P}\left( \sum_{i \in \mathcal{S} \cup \mathcal{I}} H_i > hn \right) < h, \quad \text{and}
\end{equation}
\begin{equation}
\mathbb{P}\left( \sup_{k < \varepsilon^{-2} K} \left| \sum_{j \in [1, k] \cap \mathcal{B}} \left( H_j - \frac{1}{2} \nu \varepsilon^2 n \right) \right| > hn \right) < h.
\end{equation}

We expect (50) to hold since it should be the case that a negligible fraction of steps for the embedded process $W_{k_t}^{\varepsilon,n}$ are at the extremes. To make this precise, note that with high probability using Theorems 4.2 and 7.2 we can couple the embedded process with a BMPE walk. Then (50) follows by showing that almost surely a BMPE walk spends a negligible fraction of time at its extremes. This fact about BMPE walks is proved in (62) in the Appendix. It remains now to prove (51) and (52).

**Step 1.** We begin with (51). We will show the inequality
$$\mathbb{P}\left( \sum_{i \in \mathcal{S}} H_i > hn \right) < h.$$
The analogous inequality with $S$ replaced by $T$ is proved similarly and so is not explicitly treated.

As before we introduce BLP $\{Z^i_k\}_{k \geq 0}$ where $Z^i_k$ is the number of jumps from $X_{T^i_{k-1}} - \lfloor \varepsilon \sqrt{n} \rfloor + k$ to $X_{T^i_{k-1}} - \lfloor \varepsilon \sqrt{n} \rfloor + k + 1$ in time interval $(T^i_{k-1}, T^i_k)$ where $T^i_{k-1}$ is defined in (6). (So with reasonable probability $T^i_{k-1} = T^i_k$ and with reasonable probability it is definitely larger.) If $T^i_{k-1} = T^i_k$ then

$$H_i = T^i_{k-1} - T^i_k = 2 \sum_{k=1}^{2\lfloor \varepsilon \sqrt{n} \rfloor} Z^i_k + \lfloor \varepsilon \sqrt{n} \rfloor,$$

otherwise it is less than the right-hand side. So it is enough to show that for $\varepsilon < \varepsilon_0$ and $n > n_0(\varepsilon)$

$$P\left( \sum_{i \in S} \sum_{k=1}^{2\lfloor \varepsilon \sqrt{n} \rfloor} Z^i_k > hn \right) < h.$$

We write $\Sigma_i := \sum_{k=1}^{2\lfloor \varepsilon \sqrt{n} \rfloor} Z^i_k$, and we will use the trivial inequality

$$\Sigma_i \leq 2\lfloor \varepsilon \sqrt{n} \rfloor \max_{1 \leq k \leq 2\lfloor \varepsilon \sqrt{n} \rfloor} Z^i_k,$$

together with Corollary A.7 to bound $\Sigma_i$ above. For any $K' \in (0, \infty)$ and $\varepsilon > 0$, Corollary A.7 implies that for $i \in S$ (and $n$ sufficiently large), given $F_{T^i_{k-1}}$, on the set that the coupling has not been broken,

$$\Sigma_i 1_{\{\Sigma_i \leq 2K'\lfloor \varepsilon \sqrt{n} \rfloor^2\}} \text{ is stochastically dominated by } 2\lfloor \varepsilon \sqrt{n} \rfloor^2 \zeta,$$

where $P(\zeta \geq x) = 2C_{12}e^{-C_{11}x/2} \wedge 1$. It is important to note that the law of $\zeta$ does not depend on $\varepsilon$ or $K'$. Given $h > 0$, we fix $K'$ so that

$$2C_{12}e^{-C_{11}K'/2} < \frac{\varepsilon^2 h}{4K}.$$

Then for $n \geq n_0(K')$ (by Corollary A.7) we have

$$P\left( \forall i \in S: \Sigma_i = \Sigma_i 1_{\{\Sigma_i \leq 2K'\lfloor \varepsilon \sqrt{n} \rfloor^2\}} \right) \geq 1 - \frac{h}{4}.$$

We choose $\alpha > 0$ so that $4\alpha K E[\zeta] < h$. By the weak law of large numbers there exists $\varepsilon_1 > 0$ so that for all $N \geq \alpha K / \varepsilon_1^2$,

$$P\left( \frac{1}{N} \sum_{j=1}^{N} \zeta_j > 2E[\zeta] \right) < \frac{h}{4}, \text{ where the } \zeta_j \text{ are i.i.d. copies of } \zeta.$$

As noted in the proof of (50) above, it follows from Theorems 4.2 7.2 and (62) in the Appendix, that there exists $\varepsilon_2 > 0$ so that for $\varepsilon < \varepsilon_2$ and $n \geq n_0(\varepsilon)$,

$$P(|S| > \alpha K / \varepsilon^2) < \frac{h}{4}.$$

Finally, let $\varepsilon_3 > 0$ be such that for $\varepsilon < \varepsilon_3$, the probability that the coupling breaks down before time $K / \varepsilon^2$ is less than $h/4$ for $n$ sufficiently large.

We are now ready to prove inequality (51). Choose $\varepsilon_0 < \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$. Given $\varepsilon < \varepsilon_0$ we have $n_0 = n_0(\varepsilon)$ so that for $n \geq n_0$

(i) $P(\text{coupling breaks down before } K / \varepsilon^2) < h/4$;
(ii) $P(\exists i \in S: \Sigma_i > 2K'\lfloor \varepsilon \sqrt{n} \rfloor^2) < h/4$;
(iii) $P(|S| > \alpha K / \varepsilon^2) < h/4$. 


Then for \( n \geq n_0 \), outside probability \( h/4 \) by (ii) we have

\[
\sum_{i \in S} \Sigma_i = \sum_{i \in S} \Sigma_i I_{\{\Sigma_i \leq 2K'|\epsilon \sqrt{n}|^2\}},
\]

which (if the coupling has not broken down) is stochastically dominated by \( 2|\epsilon \sqrt{n}|^2 \sum_{j=1}^{\vert S \vert} \zeta_j \) where \( \{\zeta_j\}_{j \geq 1} \) is a sequence of independent copies of \( \zeta \). By (iii), outside of a further set of probability \( h/4 \), the last expression is bounded stochastically by \( 2|\epsilon \sqrt{n}|^2 \sum_{j=1}^{\alpha K'/\epsilon^2} \zeta_j \). Finally by \( \lfloor \frac{1}{2} \rfloor \), excluding a final set of probability \( h/4 \), we have that this sum is less than

\[
4\alpha K \epsilon^{-2} E\zeta |\epsilon \sqrt{n}|^2 \leq \epsilon n
\]

by our choice of \( \alpha \). This completes Step 1.

**Step 2.** We now turn to the inequality \( \lfloor \frac{1}{2} \rfloor \). We write \( H_i = H_{i,-} + H_{i,+} \) where \( H_{i,-} = H_i 1_{\{T_{i,n}^{\epsilon,-} = T_{i,n}^{\epsilon,+}\}} \). It is enough to show that for all \( K, h > 0, \exists \epsilon_0 \) so that for each \( \epsilon < \epsilon_0, \exists n_0 = n_0(\epsilon) \) so that

\[
P\left( \sup_{k < K/\epsilon^2} \left| \sum_{i \in [1,k] \cap B} (H_{i,+} - \nu \epsilon^2 n/4) \right| > \epsilon n \right) < h \quad \text{for all} \quad n \geq n_0.
\]

As the proofs are identical, we just treat the sum of \( H_{i,-} \).

As in Step 1, for \( k \in [0, 2|\epsilon \sqrt{n}|] \) we define \( Z_k^i \) as the number of jumps from \( X_{T_{i-1}^{\epsilon,-} - [\sqrt{\epsilon n} + k]} \) to \( X_{T_{i-1}^{\epsilon,+} - [\sqrt{\epsilon n} + k]} + 1 \) in time interval \( (T_{i-1}^{\epsilon,-}, T_{i-1}^{\epsilon,+}) \). We note that if \( Z_k^i = 0 \) for some \( k \in [\lfloor \sqrt{\epsilon n} \rfloor, 2 \lfloor \sqrt{\epsilon n} \rfloor - 1] \), then \( T_{i-1}^{\epsilon,-} = T_{i-1}^{\epsilon,+} \) and we have

\[
H_i = H_{i,-} = T_{i-1}^{\epsilon,+} - T_{i-1}^{\epsilon,-} = 2 \sum_{k=0}^{2|\epsilon \sqrt{n}| - 1} Z_k^i + |\epsilon \sqrt{n}|.
\]

Or, restating, \( H_{i,-} = \left( 2 \sum_{k=0}^{2|\epsilon \sqrt{n}| - 1} Z_k^i + |\epsilon \sqrt{n}| \right) 1_{\{\sigma_{\lfloor \epsilon \sqrt{n} \rfloor}^{\epsilon} \leq 2|\epsilon \sqrt{n}|\}} \). Therefore, it suffices to show that

\[
P\left( \sup_{k < K/\epsilon^2} \left| \sum_{i \in [1,k] \cap B} (\Sigma_i^n - \nu /8) \right| > \epsilon^{-2} h \right) < h \quad \text{for all} \quad n \geq n_0,
\]

where

\[
\Sigma_i^n := \frac{1}{|\epsilon \sqrt{n}|^2} \sum_{k=0}^{2|\epsilon \sqrt{n}| - 1} Z_k^{i} 1_{\{\sigma_{\lfloor \epsilon \sqrt{n} \rfloor}^{\epsilon} \leq 2|\epsilon \sqrt{n}|\}}
\]

and this will do. We will exploit Proposition 5.18.

**Definition 8.2.** Given a law \( \lambda_0 \) on \( \mathbb{R} \) with \( \ell = \int x \lambda_0(dx) \) well defined and finite, we define \( \mathcal{H}_{\delta,\epsilon} \) to be the collection of laws on \( \mathbb{R}, \lambda \), such that \( \lambda \) can be written as \( \lambda = \int K(z, \cdot) \lambda_1(dz) \) where

(i) \( K(\cdot, \cdot) \) is a probability kernel satisfying \( K(z, [z - \delta, z + \delta]) = 0 \) for all \( z \) and

(ii) \( \|\lambda_0 - \lambda_1\|TV < 8\epsilon^3 \).

We say that a sequence of random variables \( \Xi_1, \Xi_2, \ldots, \Xi_N \) is \( \mathcal{H}_{\delta,\epsilon} \)-chain if the law of \( \Xi_1 \) is in \( \mathcal{H}_{\delta,\epsilon} \) and for \( 1 < j \leq N \), the conditional law of \( \Xi_j \) given \( \Xi_1, \Xi_2, \ldots, \Xi_{j-1} \) is in \( \mathcal{H}_{\delta,\epsilon} \).

We will also need following lemma and corollary.

**Lemma 8.3.** For an \( \mathcal{H}_{\delta,\epsilon} \)-chain \( \Xi_i \), \( 1 \leq i \leq N \), taking only finitely many values and all \( c > 0 \)

\[
P\left( \sup_{j \leq N} \left| \sum_{i=1}^{j} \Xi_i - j\ell \right| \geq c \right) \leq P\left( \sup_{j \leq N} \left| \sum_{i=1}^{j} \zeta_i - j\ell \right| \geq c - N\delta \right) + 8N\epsilon^3.
\]

where \( \zeta_i \), \( 1 \leq i \leq N \), are i.i.d. random variables with law \( \lambda_0 \).
Remark 8.4. The assumption that the random variables take only finitely many values is simply an artificial condition that suits our purposes and avoids measurability issues.

Proof. We claim that an $\mathcal{H}_{\delta,\varepsilon}$-chain $\Xi_i$, $1 \leq i \leq N$, can be coupled with i.i.d. $\zeta_i$, $1 \leq i \leq N$, with law $\lambda_0$ so that $P(|\xi_i - \Xi_i| \geq \delta) \leq 8\varepsilon^3$ for all $1 \leq i \leq N$. We then note that

$$\left\{ \sup_{j \leq N} \left| \sum_{i=1}^{j} \xi_i - j\ell \right| \geq c \right\} \subset \left\{ \sup_{j \leq N} \left| \sum_{i=1}^{j} \zeta_i - j\ell \right| \geq c - N\delta \right\} \bigcup \left\{ \bigcup_{j=1}^{N} \{ |\zeta_j - \Xi_j| \geq \delta \} \right\}$$

The conclusion is now simply an application of the union bound. So it remains to establish the claim. The coupling is based on finding, given some $\lambda \in \mathcal{H}_{\delta,\varepsilon}$, a measure $\nu$ on $\mathbb{R}^2$ having respective marginals $\lambda$ and $\lambda_0$ and such that $\nu(\{(x,y) : |x-y| > \delta\}) < 8\varepsilon^3$. The existence of such a measure is shown by Lemma A.8. Given $\lambda_j$, the conditional law of $\Xi_j$ given $\Xi_1, \Xi_2, \ldots, \Xi_{j-1}$, we take $\nu_j$ to be the corresponding coupled law on $\mathbb{R}^2$. Then for $L_j$, the regular conditional kernel for $y$ (the second coordinate) given $x$ under law $\nu_j$, we choose $\zeta_j$ according to probability $L_j(\Xi_j, \cdot)$ using an auxiliary uniform random variable in the usual manner.

The following corollary is a direct consequence of Lemma 8.3 and convergence in distribution (see [EK86], Section 3, Theorem 1.2).

Corollary 8.5. For a fixed positive integer $N$ let $(\Xi_1^{n}, \Xi_2^{n}, \ldots, \Xi_N^{n})_{n \geq 1}$ be a sequence of finite valued random vectors in $\mathbb{R}^N$ such that every distributional limit point of $\Xi_1^n$ and of the conditional probability of $\Xi_j^n$ given $\Xi_1^n, \Xi_2^n, \ldots, \Xi_{j-1}^n$, $1 \leq j \leq N$, as $n \to \infty$ (considered as a probability on $\mathbb{R}$) is in $\mathcal{H}_{\delta,\varepsilon}$. Then for all $c > 0$

$$\limsup_{n \to \infty} P \left( \sup_{j \leq N} \left| \sum_{i=1}^{j} \Xi_i^n - j\ell c \right| \geq c \right) \leq P \left( \sup_{j \leq N} \left| \sum_{i=1}^{j} \zeta_i - j\ell \right| \geq c - N\delta \right) + 8N\varepsilon^3$$

where $\zeta_i$, $i \geq 1$, are i.i.d. random variables with law $\lambda_0$.

To apply this corollary we restrict our attention to the event that the coupling does not break down before $\varepsilon^{-2}K$. This event has probability at least $1 - h/4$ for all sufficiently small $\varepsilon$. We fix such an $\varepsilon$ and enumerate the points in $[1, \varepsilon^{-2}K] \cap \mathcal{B}$ by $j \in [1, N]$ so that $N = |\mathcal{B}| \leq \varepsilon^{-2}K$. Next we let $\Xi_j^n$, $j \in [1, N]$, be equal to the corresponding $\Sigma_j^n$, $i \in [1, \varepsilon^{-2}K] \cap \mathcal{B}$. Then by Proposition 5.18 the sequence $(\Xi_1^n, \Xi_2^n, \ldots, \Xi_N^n)_{n \geq 1}$ satisfies the conditions of Corollary 8.5 with $\delta = \varepsilon^8$ and $\lambda_0$ equal to one half $\delta_0$ plus one half the law of $\nu/4$ times the time for the standard Brownian motion to exit $(-1,1)$ (note that this gives $\ell = \nu/8$). Choosing $c = \varepsilon^{-2}h$ we arrive at (58) provided that $\varepsilon = \varepsilon(K, h)$ was chosen sufficiently small. \hfill \Box

APPENDIX A.

A.1. Proofs of facts regarding BMPE.

Proof of Lemma 3.7. Step 1. We shall restate the question in terms of Brownian motion and its running maximum $B^s(t) = \max_{0 \leq s \leq t} B(s)$. Note that by [CPY98, p. 242]

$$W(t) = B(t) + \frac{\theta^+}{1 - \theta^+} B^s(t)$$

is a pathwise unique solution of the equation $W(t) = B(t) + \theta^+ S(t)$ with $S(0) = W(0) = 0$. To allow for non-zero initial data we may assume that $B_i(t), i = 1, 2$, are defined for $t \in [-1, \infty)$ and that

$$B_i(0) = W_i(0) - \theta^+ S_i(0), \quad B^*_{i}(0) = \max_{t \in [-1,0]} B_i(t) = (1 - \theta^+) S_i(0), \quad i = 1, 2.$$
Then
\[ W_i(t) = B_i(t) + \frac{\theta^+}{1 - \theta^+} B_i^+(t) \]
is a solution of \( W_i(t) = B_i(t) + \theta^+ S_i(t) \) for \( t \geq 0 \) with the given initial pair \((W_i(0), S_i(0))\), \( i = 1, 2 \).

We conclude that
\[
\begin{bmatrix}
W_i(t) \\
S_i(t)
\end{bmatrix} = \frac{1}{1 - \theta^+} \begin{bmatrix}
1 - \theta^+
& \theta^+

0
& 1
\end{bmatrix}
\begin{bmatrix}
B_i(t) \\
B_i^+(t)
\end{bmatrix}, \quad i = 1, 2,
\]
and, thus,
\[(B_1(1), B_1^+(1)) = (B_2(1), B_2^+(1)) \Rightarrow (W_1(1), S_1(1)) = (W_2(1), S_2(1)).\]

Moreover, if \((B_i(t), B_i^+(t)) \in [-K, K]^2\) for all \( t \in [-1, 1] \) then
\[ S_i(0) \vee \max_{t \in [0,1]} |W_i(t)| \leq K \left(1 + \frac{[\theta^+]^2}{1 - \theta^+}\right), \quad i = 1, 2. \]

**Step 2.** We shall now couple two pairs of Brownian motions and their running maxima. Without loss of generality we can shift one starting point to the origin and assume that \((B_1(0), B_1^+(0)) = (0, b_1), (B_2(0), B_2^+(0)) = (a_2, b_2)\), where \((0, b_1), (a_2, b_2) \in \{(x, y) : x \leq y\}\). Note that for \( b_1 > 0 \) the distribution of \((B_1(1), B_1^+(1))\) is not absolutely continuous as the line \( y = b_1 \) carries a positive measure. But \((B_1(1), B_1^+(1))\) has a density on \( \{y > b_1\}\). A similar remark applies to the other pair.

Denote by \( \mu_{0,b_1} \) and \( \mu_{a_2,b_2} \) the absolutely continuous parts of distributions of \((B_1(1), B_1^+(1))\) and \((B_2(1), B_2^+(1))\) respectively. Then there is a \( c_0 > 0 \) such that for \( b_1, |a_2|, |b_2| \leq c_0 \)
\[ \|\mu_{0,b_1}\|_{TV} \geq \frac{9}{10}, \quad \|\mu_{a_2,b_2}\|_{TV} \geq \frac{9}{10}, \quad \|\mu_{0,b_1} - \mu_{a_2,b_2}\|_{TV} \leq \frac{1}{5}. \]

Next we choose \( r_0 > c_0 \) such that for a standard Brownian motion \( B(\cdot) \)
\[ P\left( \max_{0 \leq t \leq 1} |B(t)| > r_0 - c_0 \right) \leq \frac{1}{10} \]
and let \( \mu_{a,b}^{r_0} \) denote the distribution of \((B(1), B^+(1))\) with \( B(0) = a, B^+(0) = b, -c_0 \leq a \leq b \leq c_0, \)
rericted to \( \mathbb{R} \times (b, \infty) \), and killed upon leaving \([-r_0, r_0]^2\). Then for \( b_1, |a_2|, |b_2| \leq c_0 \)
\[ \|\mu_{0,b_1}^{r_0}\|_{TV} \geq \frac{9}{10}, \quad \|\mu_{a_2,b_2}^{r_0}\|_{TV} \geq \frac{9}{10}, \quad \|\mu_{0,b_1}^{r_0} - \mu_{a_2,b_2}^{r_0}\|_{TV} \leq \frac{1}{5} \leq \frac{1}{10}. \]

Thus, for \( b_1, |a_2|, |b_2| \leq c_0 \) we can couple \((B_1(t), B_1^+(t))\) and \((B_2(t), B_2^+(t))\) so that \((B_1(1), B_1^+(1)) = (B_2(1), B_2^+(1))\) and \( \max_{t \in [0,1]} |B(t)| \leq r_0, i = 1, 2, \) with probability \( 4/5 - 3/10 = 1/2 \).

From Steps 1 and 2 we conclude that there are constants \( c_1 > 0, \ r_1 \in (c_1, \infty) \) such that \((W_1(0), S_1(0)) = (0, \overline{w}_1)\) and \((W_2(0), S_2(0)) = (w_2, \overline{w}_2)\) with \( |\overline{w}_1|, |w_2|, |\overline{w}_2| \leq c_1 \) then there is a coupling such that with probability \(1/2\)
\[ (W_1(1), S_1(1)) = (W_2(1), S_2(1)) \quad \text{and} \quad \max_{t \in [0,1]} |W_i(t)| \leq r_1, \ i = 1, 2. \]

**Step 3.** Let \( A_1 \) be the event that
\[ \begin{array}{l}
(i) \ W_i(1/2), S_i(1/2) \in [3/2, 2], \ i = 1, 2, \ \\
(ii) \ S_i(0) \vee \max_{t \in [0,1/2]} |W_i(t)| \leq 2, \ i = 1, 2.
\end{array} \]

Note that without loss of generality we can assume that \( r_1 \geq \sqrt{2} \) so that \( 1/2 + 1/r_1^2 \leq 1 \). It is easy to see that \( P(A_1) \geq p_1 \) for some \( p_1 = p_1(c_1, r_1) > 0 \) uniformly over \( W_i(0), S_i(0) \in [-1, 1], \ i = 1, 2. \)

Let \( A_2 \) be the event that \((W_1(t), S_1(t))\) and \((W_2(t), S_2(t))\), \( t \in [1/2, 1/2 + 1/r_1^2] \), are coupled as above and scaled accordingly so that
\[ (W_1(1/2+1/r_1^2), S_1(1/2+1/r_1^2)) = (W_2(1/2+1/r_1^2), S_2(1/2+1/r_1^2)) \quad \text{and} \quad \max_{1/2 \leq t \leq 1/2 + 1/r_1^2} |W_i(t)| \leq 3. \]

By Steps 1, 2, and scaling, the conditional probability of \( A_2 \) given \( A_1 \) is \( 1/2 \) uniformly over \( W_i(1/2), S_i(1/2), i = 1, 2 \). Once we have the coupling, we note that the probability that over
the leftover time period \([1/2 + 1/\tau_1^2, 1]\) the coupled processes do not exit \([-4, 4]\) is strictly positive. This finishes the proof.

\[\square\]

**Proof of Proposition 4.1.** Let \(\hat{W}(t) = \varepsilon W(\varepsilon^{-2} t)\), \(\tau_0^\varepsilon = 0\), and 
\[
\tau_k^\varepsilon = \inf \{s > \tau_{k-1}^\varepsilon : |\hat{W}(s) - \hat{W}(\tau_{k-1}^\varepsilon)| = \varepsilon\}, \quad k \in \mathbb{N}.
\]

With this notation, establishing (15) is equivalent to showing

\[
\sup_{0 \leq s \leq T} |\hat{W}(s) - \hat{W}(\tau_{[\varepsilon^{-2} s]}^\varepsilon)| \xrightarrow{P} 0 \quad \text{as } \varepsilon \to 0.
\]

As \(\hat{W}(s), s \geq 0\), is pathwise continuous (and its law does not depend on \(\varepsilon\)) (60) is implied by

\[
\sup_{0 \leq s \leq T} |s - \tau_{[\varepsilon^{-2} s]}^\varepsilon| \xrightarrow{P} 0 \quad \text{as } \varepsilon \to 0.
\]

In turn this is equivalent to showing that for each \(0 < T < \infty\),

\[
\sup_{1 \leq K \leq \varepsilon^{-2} T} \left| \sum_{k=1}^K (\tau_k^\varepsilon - \tau_{k-1}^\varepsilon - \varepsilon^2) \right| \xrightarrow{P} 0 \quad \text{as } \varepsilon \to 0.
\]

Again by scaling, we see that (61) is equivalent to \((\tau_k, I_k, W_k, S_k)\) were defined in Section 4.1

\[
\sup_{1 \leq K \leq N} \left| \frac{1}{N} \sum_{k=1}^K (\tau_k - \tau_{k-1} - 1) \right| \xrightarrow{P} 0 \quad \text{as } N \to \infty.
\]

To this end, first note that when \(W_k\) is in the bulk (that is when \(I_k + 1 \leq W_k \leq S_k - 1\)) then \(\tau_{k+1} - \tau_k\) has the same distribution as the exit time of a standard Brownian motion from \((-1, 1)\). On the other hand, if \(W_k\) is at the extreme (either \(S_k < W_k + 1\) or \(I_k > W_k - 1\)) then the distribution of \(\tau_{k+1} - \tau_k\) depends on the specific values of \(S_k - W_k\) or \(S_k - I_k\). However, using the representation in (59) we infer that for all \(k \geq 1\) the distribution of \(\tau_{k+1} - \tau_k\) given \(F_k = \sigma(W(t) : t \leq \tau_k)\) is stochastically dominated by

\[
\inf \{t > 0 : W(t) - S(t) = -2\} \xrightarrow{[59]} \inf \{t > 0 : B(t) - \max_{s \leq t} B(s) = -2\}.
\]

In particular, this implies that the conditional mean and variance of \(\tau_k - \tau_{k-1}\) given \(F_{k-1}\) are uniformly bounded. That is, there exist constants \(A, B < \infty\) such that

\[
t_k := E[(\tau_k - \tau_{k-1} | F_{k-1})] \leq A \quad \text{and} \quad E[(\tau_k - \tau_{k-1} - t_k)^2 | F_{k-1}] \leq B < \infty.
\]

Then, it follows from Doob’s martingale inequality that for any \(\delta > 0\)

\[
P \left( \sup_{1 \leq K \leq N} \left| \frac{1}{N} \sum_{k=1}^K (\tau_k - \tau_{k-1} - t_k) \right| \geq \delta \right) \leq \frac{1}{\delta^2} E \left[ \left( \frac{1}{N} \sum_{k=1}^N (\tau_k - \tau_{k-1} - t_k) \right)^2 \right] \leq \frac{B}{\delta^2 N}.
\]

Thus, it remains only to show that

\[
\sup_{1 \leq K \leq N} \left| \frac{1}{N} \sum_{k=1}^K (t_k - 1) \right| \xrightarrow{P} 0.
\]

However, since \(t_k \equiv 1\) when \(W_k\) is at the extreme and is uniformly bounded otherwise, it is enough to show that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{1}\{(I_k, W_k, S_k) \text{ is at the extreme} \} = 0, \quad P\text{-a.s.}
\]
It’s enough only to consider the right extremes (that is, when \( S_k < W_k + 1 \)) since the left extremes can be handled similarly. We’ll show that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\{W_k \geq 1, S_k < W_k + 1\}} = 0, \quad P\text{-a.s.}
\]  

(62)

The proof of this will rely on the following facts.

- If \( W_k = m \geq 1 \) and \( S_k < m + 1 \), the probability (conditioned on \( W(t) \) for \( t \leq \tau_k \)) that \( W_{k+1} = m+1 \) is at least \( p_-(1/2) \land (1/2)^{1-\theta^+} > 0 \) and at most \( p_+ = (1/2) \lor (1/2)^{1-\theta^+} < 1 \). This follows from Corollary 3.2.

- If \( W_k = m \geq 1 \) and \( S_k \geq m + 1 \), the probability that \( W_{k+1} = m + 1 \) is exactly 1/2.

First of all, for any \( m \geq 1 \) let \( \chi_m = \sum_{k=1}^{\infty} \mathbb{1}_{\{W_k = m, S_k < m+1\}} \) be the total number of times a right extreme occurs and the BMPE-walk is at location \( m \). It is easy to see that the sequence \( \{\chi_m\}_{m \geq 1} \) is i.i.d. Moreover, since whenever \( W_k = m \) is at the extreme, the probability that the next step is to the right is at least \( p_- \) and so \( \chi_m \) is stochastically dominated by a \( \text{Geom}(p_-) \) random variable. In particular, \( E[\chi_1] < \infty \). Thus,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \chi_m = E[\chi_1] < \infty, \quad P\text{-a.s.}
\]

(63)

Next, for \( n \geq 0 \) let \( \rho_n = \inf\{k \geq 0 : W_k = n\} \) be the time it takes for the walk \( W_k \) to reach \( n \) for the first time. It is easy to see that \( \rho_{n+1} - \rho_n \) stochastically dominates the time it takes the Markov chain on \( \{0, 1, \ldots, n, n+1\} \) shown in Figure 2 to step from \( n \) to \( n + 1 \).

![Figure 2](image)

**Figure 2.** The above Markov chain behaves like a simple symmetric random walk at \( x = 1, 2, \ldots, n - 1 \), an asymmetric simple random walk at \( x = n \), and reflects to the right at \( x = 0 \).

Let \( \{\gamma_n\}_{n \geq 0} \) be a sequence of independent random variables where for each \( n \) the random variable \( \gamma_n \) has the distribution of the time for the Markov chain in Figure 2 to cross from \( n \) to \( n + 1 \). Then \( \rho_n \) stochastically dominates \( \sum_{k=0}^{n-1} \gamma_k \) and thus

\[
\lim_{n \to \infty} \rho_n = \infty, \quad P\text{-a.s.}
\]

(64)

Finally, we are ready to prove (62). For each \( \alpha_k \geq 1 \) there is a unique \( n \geq 0 \) such that \( S_N \in \{n, n+1\} \) and note that \( S_N \in \{n, n+1\} \) is equivalent to \( \rho_n \leq N < \rho_{n+1} \). Therefore, on the event \( \{\rho_n \leq N < \rho_{n+1}\} \) we have

\[
\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\{W_k \geq 1, S_k < W_k + 1\}} \leq \frac{1}{N} \sum_{m=1}^{n} \chi_m \leq \left( \frac{n}{\rho_n} \right) \left( \frac{1}{n} \sum_{m=1}^{n} \chi_m \right).
\]

Since \( n \to \infty \) as \( N \to \infty \), we have that (62) follows from (63) and (64). \(\square\)

Note that the random variables \( \{\gamma_k\}_{k \geq 0} \) are independent and \( \gamma_{n+1} \) stochastically dominates \( \gamma_n \). Moreover, for \( n \in \mathbb{N} \) by an easy recursion computation, \( E[\gamma_n] = \frac{1}{p_+} + \frac{1-p_+}{p_+} (2n-1) \to \infty \) as \( n \to \infty \).
A.2. Proofs of diffusion approximation results for BLPs.

Proof of Theorem 5.5.1 (1) The proof of this part is very similar to the one of [KPT17, Lemma 7.1] and is based on [EK86] Theorem 4.1, p. 354. First of all, the martingale problem for

\[ A = \left\{ \left( f, G f = \frac{\nu}{2} x + \frac{\partial^2}{\partial x^2} + D \frac{\partial}{\partial x} \right) : f \in C_c^\infty(\mathbb{R}) \right\} \]

on \( C_0[0, \infty) \) is well-posed by [EK86, Corollary 3.4, p. 295] and the fact that the existence and distributional uniqueness hold for solutions of (19) with arbitrary initial distributions.

Define \( A_m(t) \) and \( B_m(t) \) for all \( t \geq 0 \) by

\[ A_m(t) := \frac{1}{m^2} \sum_{k=1}^{\lfloor mt \rfloor} \text{Var}(V_{m,k}^+ | V_{m,k-1}^+) \quad \text{and} \quad B_m(t) := \frac{1}{m} \sum_{k=1}^{\lfloor mt \rfloor} E[V_{m,k}^+ - V_{m,k-1}^+ | V_{m,k-1}^+]. \]

Then for each \( m \in \mathbb{N} \) the processes \( M_m(t) := Y_m(t) - B_m(t) \) and \( M_m^2(t) - A_m(t), t \geq 0 \), are martingales with respect to the natural filtration of \( V_m^+ \).

Recall that \( r_m = m^{-1} \tau_m \). To apply the cited theorem we only need to check that for all \( T, r > 0 \) the following five conditions hold:

\begin{align*}
&\lim_{m \to \infty} E \left[ \sup_{t \leq T \wedge r_m} |Y_m(t) - Y_m(t^-)|^2 \right] = 0. \\
&\lim_{m \to \infty} E \left[ \sup_{t \leq T \wedge r_m} |B_m(t) - B_m(t^-)|^2 \right] = 0. \\
&\lim_{m \to \infty} E \left[ \sup_{t \leq T \wedge r_m} |A_m(t) - A_m(t^-)| \right] = 0. \\
&\sup_{t \leq T \wedge r_m} \left| B_m(t) - (1 + \eta \cdot \mathbf{r}^+) t \right| \xrightarrow{P} 0, \quad m \to \infty. \\
&\sup_{t \leq T \wedge r_m} \left| A_m(t) - \nu \int_0^t (Y_m(s))_+ \, ds \right| \xrightarrow{P} 0, \quad m \to \infty.
\end{align*}

Recalling the construction of the BLP \( V^+ \) in terms of the Bernoulli trials \( \{\xi_j^+\}_{x \geq 0, j \geq 1} \) as in Section 2, let \( G_j^k \) be the number of “successes” between the \((i - 1)\)-th and \( i \)-th “failure” in the sequence of Bernoulli trials \( \{\xi_j^k\}_{j \geq 1} \) so that

\begin{equation}
V_{m,k}^+ = \sum_{j=1}^{V_{m,k-1}^+ + 1} G_j^k = V_{m,k-1}^+ + 1 + \sum_{j=1}^{V_{m,k-1}^+ + 1} (G_j^k - 1).
\end{equation}

Using this representation for the \( V^+ \) processes, condition (65) states that for every \( T, r > 0 \)

\[ \lim_{m \to \infty} \frac{1}{m^2} E \left[ \max_{1 \leq k \leq (Tm) \wedge r_m} \left| \sum_{j=1}^{V_{m,k-1}^+ + 1} (G_j^k - 1) \right|^2 \right] = 0, \]

\footnote{A more detailed discussion of \( (19) \) can be found immediately following (3.1) in [KZ14].}
where \( \tau_{r_m}^{V^+} = \inf \{ k \geq 0 : V^+_{m,k} \geq r m \} \). To see that it holds we write

\[
\frac{1}{m^2} E \left[ \max_{1 \leq k \leq (Tm) \land \tau_{r_m}^{V^+}} \sum_{j=1}^{V^+_{m,k-1}+1} (G^k_j - 1)^2 \right] \leq \frac{1}{m^2} E \left[ \max_{1 \leq k \leq Tm} \max_{1 \leq \ell \leq r m+1} \left| \sum_{j=1}^{\ell} (G^k_j - 1) \right|^2 \right]
\]

\[
= \frac{1}{m^2} \sum_{y=0}^\infty P \left( \max_{1 \leq k \leq Tm} \max_{1 \leq \ell \leq r m+1} \left| \sum_{j=1}^{\ell} (G^k_j - 1) \right| > y \right)
\]

\[
\leq \frac{r^{3/2}}{\sqrt{m}} + (rT + 1) \sum_{y \geq (r m)^{3/2}} \max_{1 \leq \ell \leq r m+1} P \left( \left| \sum_{j=1}^{\ell} (G^k_j - 1) \right| > \sqrt{y} \right).
\]

Finally we apply Lemma A.1 from [KP17] to get that the expression in the last line does not exceed

\[
\frac{r^{3/2}}{\sqrt{m}} + rT \sum_{y \geq (r m)^{3/2}} C \left( \exp \left\{ -c \left( \frac{y}{\sqrt{y} \lor (8 r m)} \right) \right\} + \exp \{ -c \sqrt{y} \} \right)
\]

\[
\leq \frac{r^{3/2}}{\sqrt{m}} + rT \sum_{y \geq (r m)^{3/2}} C \left( \exp \left\{ -c \left( \frac{y}{\sqrt{y} \lor (8 y^{2/3})} \right) \right\} + \exp \{ -c \sqrt{y} \} \right) \to 0 \text{ as } m \to \infty.
\]

Conditions (66) and (67) follow from Propositions 4.1 and 4.2 of [KP17] respectively. Indeed, by [KP17 Proposition 4.1] for some \( c_1, c_2 > 0, \) all and \( n \geq 0 \)

\[ |E[V^+_1 \mid V^+_0 = n] - n - (1 + \eta \cdot r^+)| \leq c_1 e^{-c_2 n}. \]

Using the Markov property and the fact that \( V^+_{m,k-1} \leq r m \) for \( k \leq \tau_{r_m}^{V^+} \) we get

\[
\lim_{m \to \infty} E \left[ \sup_{t \leq T \land \tau_{r_m}^{V^+}} |B_m(t) - B_m(t^-)|^2 \right] = \lim_{m \to \infty} \frac{1}{m^2} E \left[ \max_{1 \leq k \leq (Tm) \land \tau_{r_m}^{V^+}} \left( E[V^+_{m,k} - V^+_{m,k-1} \mid V^+_{m,k-1}] \right)^2 \right]
\]

\[
\leq \lim_{m \to \infty} \frac{1}{m^2} E \left[ \max_{1 \leq k \leq (Tm) \land \tau_{r_m}^{V^+}} \left( E[V^+_{m,k} \mid V^+_{m,k-1}] - V^+_{m,k-1} - (1 + \eta \cdot r^+) \right)^2 \right]
\]

\[
\leq \lim_{m \to \infty} \frac{c_1^2}{m^2} E \left[ \max_{1 \leq k \leq (Tm) \land \tau_{r_m}^{V^+}} e^{-2c_2 V^+_{m,k-1}} \right] = 0,
\]

Similarly, by [KP17 Proposition 4.2] there is a \( c_3 > 0 \) such that \( |\text{Var}(V^+_1 \mid V^+_0 = n) - \nu n| \leq c_3 \) for all \( n \geq 0 \). Therefore,

\[
\lim_{m \to \infty} E \left[ \sup_{t \leq T \land \tau_{r_m}^{V^+}} |A_m(t) - A_m(t^-)| \right] = \lim_{m \to \infty} \frac{1}{m^2} E \left[ \max_{1 \leq k \leq (Tm) \land \tau_{r_m}^{V^+}} \text{Var}(V^+_{m,k} \mid V^+_{m,k-1}) \right]
\]

\[
\leq \lim_{m \to \infty} \frac{(\nu rm + c_3)}{m^2} = 0.
\]
To check condition (68), note that
\[
\sup_{t \leq T \wedge \tau_{Y^m}} |B_m(t) - (1 + \eta \cdot r^+ t)|
\leq \frac{1 + \eta \cdot r^+}{m} + \sup_{1 \leq k \leq (Tm) \wedge \tau_{Y^m}} \sum_{j=1}^{k} E\left[ V_{m,j}^+ - V_{m,j-1}^+ \mid V_{m,j-1}^+ \right] - (1 + \eta \cdot r^+)
\leq \frac{1 + \eta \cdot r^+}{m} + \frac{c_1}{m} \sum_{j=1}^{(Tm) \wedge \tau_{Y^m}} e^{-c_2 V_{m,j-1}^+} \leq \frac{c_4}{m} + \frac{c_1}{m} \sum_{j=1}^{Tm} \{V_{m,j-1}^+ \leq m\alpha\}.
\]

By Lemma 6.6, for any \(\alpha \in (0, 1 - \theta^-)\) the last expression goes to 0 in probability as \(m \to \infty\), and we have shown that condition (68) holds.

Finally, to check condition (69) note that
\[
\sup_{t \leq T \wedge \tau_{Y^m}} \left| A_m(t) - \nu \int_0^t (Y_m(s))_+ ds \right|
\leq \max_{1 \leq k \leq (Tm) \wedge \tau_{Y^m}} \left[ \frac{1}{m^2} \sum_{j=1}^{k} \text{Var}(V_{m,j}^+ \mid V_{m,j-1}^+) - \nu \sum_{j=1}^{k} V_{m,j-1}^+ \right] + \nu \sum_{j=1}^{Tm} V_{m,j-1}^+
\leq \max_{1 \leq k \leq (Tm) \wedge \tau_{Y^m}} \left( \frac{1}{m^2} \sum_{j=1}^{k} \text{Var}(V_{m,j}^+ \mid V_{m,j-1}^+) - \nu V_{m,j-1}^+ \right) + \frac{\nu}{m^2} \sum_{j=1}^{Tm} V_{m,j-1}^+
\leq \frac{c_3 T + \nu}{m} \to 0
\]
as \(m \to \infty\). This completes the proof of condition (69) and thus also the proof of part (1).

(2) The process convergence part of the argument is based on [Bil99, Theorem 3.2] which we state below for the reader’s convenience.

**Theorem A.1.** (Bil99, Theorem 3.2) Let \((S, d)\) be a metric space. Suppose that \(Y_{m,\ell}, Y_m, Y^{(\ell)}\) \((m, \ell \in \mathbb{N})\) and \(Y^{(\infty)}\) are \(S\)-valued random variables such that \(Y_{m,\ell}\) and \(Y_m\) are defined on the same probability space with probability measure \(P^m\) for all \(m, \ell \in \mathbb{N}\). If \(Y_{m,\ell} \Rightarrow Y^{(\ell)} \Rightarrow Y^{(\infty)}\) and
\[
\lim_{\ell \to \infty} \limsup_{m \to \infty} P^m(d(Y_{m,\ell}, Y_m) > \varepsilon) = 0
\]
for each \(\varepsilon > 0\), then \(Y_m \Rightarrow Y^{(\infty)}\).

**Remark A.2.** The proof of Corollary 5.12 repeats the argument below word for word on the space \(D([0, T])\) with the metric \(d_T^\circ\) (see Bil99, p.166 and (12.16)) and use Lemma 5.10 instead of Lemma 5.8.

In addition to processes \(Y_m\) and \(Y\) defined in the statement, for \(\delta := 1/\ell > 0\) we let \(Y_{m,\ell}(t) = m^{-1}U_{m,\ell}^{(\delta)}(t) = Y(t \wedge \sigma_\delta),\ Y^{(\ell)}(t) = Y(t \wedge \sigma_0),\ t \geq 0,\) and work in the space \(D([0, \infty))\) with the \(J_1\) metric \(d_\infty^\circ\) (see [Bil99, (16.4)])). From [KP17, Lemma 8.1] or, alternatively, by repeating essentially word for word the proof of part (1), we know that \(\forall \ell \in \mathbb{N}, Y_{m,\ell} \Rightarrow Y^{(\ell)}\) for all \(m \to \infty\). Moreover, \(Y^{(\ell)} \Rightarrow Y^{(\infty)}\) as \(\theta^+ < 1\). Indeed, using the properties of BESQ\(^d\) with \(d < 2\) we have \(\forall \varepsilon > 0\)
\[
P\left( \sup_{t \geq 0} |Y(t \wedge \sigma_\delta) - Y(t \wedge \sigma_0)| > \varepsilon \right) \leq P\left( \sup_{t \geq \sigma_\delta} Y(t \wedge \sigma_0) > \varepsilon \right)
\leq P(\tau^Y_{\varepsilon/2} < \sigma^Y_0 \mid Y(0) = \delta) \to 0\]as \(\delta \to 0\).\footnote{Lemma 6.1 is stated and proved in [KP17] for the processes \(V^\circ\) with deterministic initial conditions but it holds with the same proof for the other 3 processes and random initial distributions.}
We are left to check the last condition of Theorem A.1. For all $\delta \in (0, \varepsilon/2)$ and $r > 0$ we have that

\[
P^m (d^\infty (Y_{m, \ell}, Y_m) > \varepsilon) \leq P \left( \sup_{k \geq m \delta} U_{m,k}^+ \geq \varepsilon m/2 \right) \leq P \left( \sup_{k \geq m \delta} U_{m,k}^+ \geq \varepsilon m/2 \mid U_0^+ = |\delta m| \right)
\]

\[
P \left( \tau_{\varepsilon m/2} < \sigma_0^U \mid U_0^+ = |\delta m| \right) \leq P \left( \tau_{\varepsilon m/2} \leq rm \mid U_0^+ = |\delta m| \right) + P \left( \sigma_0^U > rm \mid U_0^+ = |\delta m| \right).
\]

By Lemma A.3 (see below) and Lemma 5.8 we can control the last two probabilities and conclude that

\[
\lim_{\ell \to \infty} \limsup_{m \to \infty} P^m (d^\infty (Y_{m, \ell}, Y_m) > \varepsilon) = 0.
\]

By Theorem A.1, $Y_m \Rightarrow Y(\infty)$ as claimed.

We are left to show (20). By the continuous mapping theorem, [KZ14, Lemma 3.3], and the a.s. continuity of $Y$ we have that $\sigma_0^Y \to Y(\infty)$ and $\sigma_0^Y \Rightarrow \sigma_0^Y$. To use Theorem A.1 again, we need to estimate $P(\sigma_0^Y - \sigma_0^m > \varepsilon \mid Y_0^m)$. By the strong Markov property and monotonicity in the starting point, this probability does not exceed $P(\sigma_0^Y > \varepsilon m \mid U_0^+ = |\delta m|)$ which converges to 0 as $\delta \to 0$ by Lemma 5.8. Thus, $\sigma_0^Y \Rightarrow \sigma_0^Y$. □

The proof of Theorem 5.9 depends on several facts which we shall state and prove first. Recall that $\max\{\theta^+, \theta^-\} < 1$. The BLP $Z$ below can be any of the BLPs $U^\pm$ and $V^\pm$.

**Lemma A.3.** For all $T, \varepsilon > 0$ there is an $L > 0$ such that for an arbitrary fixed selection of the first cookies and for all $m \in \mathbb{N}$

\[
P \left( \max_{k \leq T m} Z_k^m \leq L m \mid Z_0 = m \right) > 1 - \varepsilon.
\]

**Proof.** By Propositions 3.1, 3.6, 4.1, 4.2 of [KP17] we have that for all $k \in \mathbb{N}$

\[
|E[Z_k^m \mid Z_{k-1}^m] - Z_k^m| \leq \gamma; \quad E[(Z_k^m)^2 \mid Z_{k-1}^m] \leq (Z_{k-1}^m)^2 + \alpha Z_{k-1}^m + \beta,
\]

where constants $\alpha, \beta, \gamma$ do not depend on $k, m$ or a choice of the first cookies. If we set

\[
b_k := E[(Z_k^m)^2 \mid Z_0^m = m], \quad a_k := E[Z_k^m \mid Z_0^m = m],
\]

then estimates (71) imply that

\[
a_k \leq m + \gamma k, \quad b_k \leq b_{k-1} + \alpha \gamma (k - 1) + \alpha m + \beta.
\]

We conclude that

\[
E[Z_k^m \mid Z_0^m = m] \leq m + \gamma k, \quad E[(Z_k^m)^2 \mid Z_0^m = m] \leq k(\alpha m + \beta) + \frac{1}{2} \alpha \gamma k (k - 1).
\]

Let $M_0^m = m, M_k^m := Z_k^m - \sum_{j=1}^k E[Z_j^m - Z_{j-1}^m \mid Z_{j-1}^m], k \in \mathbb{N}$. Then $M_k^m, k \geq 0$, is a martingale with respect to its natural filtration. Since $|M_{T m}^m - Z_{T m}^m| \leq |\gamma T m|$, we have that

\[
E[(M_{T m}^m)^2] \leq 2 E[(Z_{T m}^m)^2 \mid Z_0^m = m] + 2 (\gamma T m)^2 \leq C(\alpha, \beta, \gamma, T) m^2.
\]

By the maximal inequality, for $L > \gamma T$ and all $m \in \mathbb{N},$

\[
P \left( \max_{k \leq T m} Z_k^m \geq m L \right) \leq P \left( \max_{k \leq T m} |M_k^m| \geq m (L - \gamma T) \right) \leq \frac{4 E[(M_{T m}^m)^2]}{(L - \gamma T)^2 m^2} \leq \frac{4 C(\alpha, \beta, \gamma, T)}{(L - \gamma T)^2}.
\]

We can choose $L$ large enough to ensure that the last expression is less than $1 - \varepsilon$. □
Lemma A.4. For each \( m \in \mathbb{N} \) let \( Z^m \) be one of the four kinds of BLPs and \( Z^m_0 \leq K \) for some \( K > 0 \). Fix \( \varepsilon > 0 \) and define
\[
Y^\varepsilon_m := \frac{Z^m_{\lfloor tm \rfloor}}{m}, \quad \tilde{Y}^\varepsilon_m := \frac{Z^m_{\lfloor tm^{3/4}m^{1/4} \rfloor}}{m}, \quad t \geq 0.
\]
Then uniformly over all first cookie environments for every \( T, \delta > 0 \)
\[
P \left( \sup_{0 \leq t \leq T} |\tilde{Y}^\varepsilon_m - Y^\varepsilon_m| > \delta \right) \to 0 \quad \text{as} \quad m \to \infty.
\]

Proof. Let \( A_L \) be the event that \( \max_{j \leq TM} Z^m_j \leq Lm \). By Lemma A.3 given an arbitrary \( \varepsilon' > 0 \), there is an \( L \) such that \( P(A_L) > 1 - \varepsilon' \). Denote by \( B_k \) the event
\[
\{ \forall j \in [1, m^{1/4}] : |Z^m_{j+1} - Z^m_j| \leq m^{3/5} \}.
\]
Then by Lemma A.1 from [KP17] there are \( c, C > 0 \) such that
\[
P(B^c_k \cap A_L) \leq Cm^{1/4}e^{-cm^{3/5}/L}.
\]
We conclude that
\[
P \left( \sup_{0 \leq t \leq T} |\tilde{Y}^\varepsilon_m - Y^\varepsilon_m| > \delta \right) \leq P \left( \bigcup_{k \leq Tm^{3/4}} (B^c_k \cap A_L) \right) + P(A^c_L) \leq CTme^{-cm^{3/5}/L} + \varepsilon'.
\]
Since \( \varepsilon' \) was arbitrary, the proof is complete. \( \square \)

The proof of the following lemma is identical to the one of Lemma 7.1 in [KP17], and is, thus, omitted.

Lemma A.5. Let \( D \in \mathbb{R}, \nu > 0 \), and \{\( Y(t) \}_{t \geq 0} \) be a solution of (19) with \( D(t) \equiv D \) and \( Y(0) \sim \kappa \). Let (time-inhomogeneous countable) Markov chains \( Z^m_k := \{Z^m_k\}_{k \geq 0} \) with values in \( \mathbb{R} \) satisfy the following conditions:

1. for each \( T, r > 0 \) there is a deterministic function \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( g(x) \to 0 \) as \( x \to \infty \),

\[
\max_{1 \leq k \leq (Tn) \wedge (\tau^m_{n+1})} \left| E[Z^m_k - Z^m_{k-1} \mid Z^m_{k-1}] - D \right| \leq g(n);
\]

\[
\max_{1 \leq k \leq (Tn) \wedge (\tau^m_{n+1})} \left| \text{Var}(Z^m_k \mid Z^m_{k-1}) \right| \leq g(n)
\]

for some sequence \( \{N_n\}_{n \in \mathbb{N}} \), \( N_n \to \infty, N_n = o(n) \) as \( n \to \infty \);

2. for each \( T, r > 0 \)

\[
E \left[ \max_{1 \leq k \leq (Tn) \wedge (\tau^m_{n+1})} (Z^m_k - Z^m_{k-1})^2 \right] = o(n^2) \quad \text{as} \quad n \to \infty.
\]

Set \( Y_n(t) = n^{-1}Z^m_{[nt]}, t \geq 0 \), and assume that \( Y_n(0) \sim \kappa_n \) where \( \kappa_n \Rightarrow \kappa \). Then \( Y_n \xrightarrow{n \to \infty} Y \).

Now we have all ingredients for the proof of Theorem 5.9

Proof of Theorem 5.9. We give a detailed proof only for the case \( Z^m_j := V^+_m, j \geq 0 \), but the same proof works for the other BLPs.

We start by modifying our process \( \{V^+_m\}_{j \geq 0} \). Let \( N_m \in \mathbb{N} \) satisfy \( N_m \to \infty \) and \( N_m = o(m^{3/4}) \) as \( m \to \infty \). We define \( \hat{V}^+_m = V^+_m \) and recalling the representation in (70) for \( V^+_m \) we get
\[
(72) \quad \hat{V}^+_m = \hat{V}^+_m + 1 + \sum_{t=1}^{(V^+_m, j-1) + 1 \wedge [Nm^{1/4}]} (G^j_t - 1).
\]
Note that the modified process is identical to our original process \( \{V^+_{m,j}\}_{j \geq 0} \) up to the first entrance time in the interval \((-\infty, N_m m^{1/4})\). Given the conditions of our theorem, it is enough to prove the result for the modified process. For convenience of the reader, we state the expectation and variance estimates for \( \tilde{V}^+_m \) (Propositions 4.1 and 4.2 from \[\text{[KP17]}\]). For all \( m, j \in \mathbb{N} \)

\[
(73) \quad |E[\tilde{V}^+_m - \tilde{V}^+_{m-1} \mid \tilde{V}^+_m - (r^+(R^+_1) + 1)]| \leq c_{12} e^{-c_{13}(\tilde{V}^+_m - \nu N_m m^{1/4})} \leq c_{12} e^{-c_{13} N_m m^{1/4}} =: \varepsilon_m;
\]

\[
(74) \quad |\text{Var}(\tilde{V}^+_m \mid \tilde{V}^+_{m-1}) - \nu(\tilde{V}^+_m - (r^+(R^+_1) + 1))| \leq c_{14}.
\]

We are planning to apply Lemma \[\text{A.5}\] to the process \( Z^n_k := m^{-1/4}\tilde{V}^+_{m,(k-1)m^{1/4}} \) with \( n = [m^{3/4}] \) and then conclude by Lemma \[\text{A.3}\]. We just need to check the conditions of Lemma \[\text{A.5}\].

**Step 1.** Given the first cookies on \( [(k-1)m^{1/4}, km^{1/4}] \), we get by the properties of conditional expectation and \[(73)\] that

\[
\left| E \left[ \tilde{V}^+_{m,(k-1)m^{1/4}} - \tilde{V}^+_{m,(k-1)m^{1/4}} \mid \tilde{V}^+_{m,(k-1)m^{1/4}} \right] - \sum_{j=([k-1)m^{1/4}]+1}^{[km^{1/4}]} (r^+(R^+_1) + 1) \right| \leq \sum_{j=([k-1)m^{1/4}]+1}^{[km^{1/4}]} E \left[ \left| \tilde{V}^+_{m,j} - \tilde{V}^+_{m-1} \mid \tilde{V}^+_m - (r^+(R^+_1) + 1) \right| \tilde{V}^+_{m,(k-1)m^{1/4}} \right] \leq \varepsilon_m m^{1/4}.
\]

Recalling the meaning of the condition that the first cookie environment is \((m^{1/4}, \rho)\)-good we see that for all \( m \) and \( k \)

\[
(75) \quad \left| \frac{1}{m^{1/4}} E \left[ \tilde{V}^+_{m,(k-1)m^{1/4}} - \tilde{V}^+_{m,(k-1)m^{1/4}} \mid \tilde{V}^+_{m,(k-1)m^{1/4}} \right] - (\rho + 1) \right| \leq \frac{1}{\ln m} + \varepsilon_m.
\]

**Step 2.** Our next task is to deal with conditional variance over intervals \([[(k-1)m^{1/4}, km^{1/4}]\) for \( k \leq T m^{3/4} \land \tau_{rm} \) with arbitrary fixed \( T, r > 0 \). We want to show that

\[
(76) \quad \max_{1 \leq k \leq T m^{3/4} \land \tau_{rm}} \left| \text{Var}(\tilde{V}^+_{m,(k-1)m^{1/4}} \mid \tilde{V}^+_{m,(k-1)m^{1/4}}) - \nu[m^{1/4}](\tilde{V}^+_{m,(k-1)m^{1/4}} \land \nu N_m m^{1/4}) \right| = o(N_m m^{1/2}),
\]

where \( \tau_{rm} \) is the first time the process \( \tilde{V}^+_{m,(k-1)m^{1/4}}, k \geq 0 \), enters \((rm, \infty)\).

Fix an arbitrary \( m \in \mathbb{N} \) and \( k, 1 \leq k \leq T m^{3/4} \land \tau_{rm} \). To simplify the notation, we shall use \( V_j \) instead of \( \tilde{V}^+_{m,(k-1)m^{1/4} + j} \) and \( V_j \) instead of \( \tilde{V}^+_{m,(k-1)m^{1/4} + j} \) instead of \( V_j \land N_m m^{1/4} \) for \( j \in [0, m^{1/4}] \). We shall also write \( E_0[\cdot] \) and \( \text{Var}_0(\cdot) \) instead of \( E[\cdot \mid V_0] \) and \( \text{Var}(\cdot \mid V_0) \).

With this notation, the \( k \)-th term in \[(76)\] can be estimated as follows:

\[
(77) \quad \left| \text{Var}_0(V_{m^{1/4}}) - \nu[m^{1/4}] V_{0+} \right| \leq \sum_{j=1}^{m^{1/4}} \left| \text{Var}_0(V_j) - \text{Var}_0(V_{j-1}) - \nu V_{0+} \right|.
\]

We shall show that for \( N_m \) such that \( N_m / m^{3/5} \to \infty \) (retaining the property that \( N_m = o(m^{3/4}) \)) each term in the above sum is \( o(N_m m^{1/4}) \) as \( m \to \infty \).
First we apply the conditional variance formula (conditioning on \( \mathcal{F}_{j-1} \) and using the Markov property to replace \( \mathcal{F}_{j-1} \) with \( V_{j-1} \)) and get that

\[
(78) \quad |\text{Var}_0(V_j) - \text{Var}_0(V_{j-1}) - \nu V_{0+}| = |E_0[\text{Var}(V_j | V_{j-1})] + \text{Var}_0(E(V_j | V_{j-1})) - \text{Var}_0(V_{j-1}) - \nu V_{0+}| \\
\leq |E_0[\text{Var}(V_j | V_{j-1}) - \nu V_{j-1+}]| + \nu|E_0(V_{j-1+} - V_{0+})| \\
+ |\text{Var}_0((E[V_j | V_{j-1}] - V_{j-1}) + V_j - \text{Var}_0(V_{j-1})|.
\]

We know from (73) that \( |E[V_j | V_{j-1}] - V_{j-1}| \leq \alpha \) for some constant \( \alpha \). Note that if \( |Y| \leq \alpha \) then \( \text{Var}(Y) \leq \alpha^2 \) and

\[
|\text{Var}(X + Y) - \text{Var}(X)| \leq \alpha^2 + 2\alpha \sqrt{\text{Var}(X)}.
\]

Applying this inequality with \( X = V_{j-1} \) and \( Y = E[V_j | V_{j-1}] - V_{j-1} \) to the last term of (78) and using (74) to estimate the first term we obtain for some constant \( C_8 > 0 \)

\[
|\text{Var}_0(V_j) - \text{Var}_0(V_{j-1}) - \nu V_{0+}| \leq C_8 + \nu|E_0(V_{j-1+} - V_{0+})| + 2\alpha \sqrt{\text{Var}_0(V_{j-1})}.
\]

Let

\[
(79) \quad B_k = \{ \forall j \in [1, m^{1/4}], |V_j - V_{j-1}| \leq m^{3/5} \}.
\]

Since we are considering only \( k \leq T m^{3/4} \wedge (\tau_{rm} + 1) \), we can assume that \( V_0 \leq rm \). Then by Lemma A.1 from [KP17] there are \( c, C > 0 \) such that

\[
P(B_k^c) \leq C m^{1/4} e^{-cm^{3/5}/(16\epsilon)}.
\]

Recall that \( N_m/m^{3/5} \to \infty \) and \( N_m = o(m^{3/4}) \) as \( m \to \infty \). If \( V_0 \geq N_m m^{1/4} \) then on the set \( B_k \)

\[
|V_{j+} - V_{0+}| = |V_{j+} - V_0| \leq m^{1/4} m^{3/5} = o(N_m m^{1/4}),
\]

and if \( V_0 < N_m m^{1/4} \) then on \( B_k \)

\[
|V_{j+} - V_{0+}| = |V_{j+} - [N_m m^{1/4}]| \leq m^{1/4} m^{3/5} 1_{\{V_{j+} > N_m m^{1/4}\}} = o(N_m m^{1/4}).
\]

Using these estimates we get

\[
|\text{Var}_0(V_j) - \text{Var}_0(V_{j-1}) - \nu V_{0+}|
\leq C_8 + \nu|E_0[(V_{j-1+} - V_{0+}) 1_{\{B_k\}}]| + \nu|E_0[(V_{j-1+} - V_{0+}) 1_{\{B_k^c\}}]| + 2\alpha \sqrt{\text{Var}_0(V_{j-1})}
\leq o(N_m m^{1/4}) + \nu \sqrt{E_0[(V_{j-1+} - V_{0+})^2] P(B_k^c)} + 2\alpha \sqrt{E_0[(V_{j-1} - V_0)^2]}.
\]

Now we observe that

\[
(V_{j-1+} - V_{0+})^2 \leq 3((V_{j-1+} - V_{j-1})^2 + (V_{j-1} - V_0)^2 + (V_0 - V_{0+})^2),
\]

where \( 0 \leq V_{i+} - V_i \leq N_m m^{1/4} \) for all \( i \). Taking into account a stretched exponential decay of \( P(B_k^c) \) we arrive at the inequality

\[
|\text{Var}_0(V_j) - \text{Var}_0(V_{j-1}) - \nu V_{0+}| \leq o(N_m m^{1/4}) + 2(\nu + \alpha) \sqrt{E_0[(V_{j-1} - V_0)^2]}.
\]
To bound the last term, we let \( j \in \llbracket 1, m^{1/4} \rrbracket \) and use (73), (74) to obtain

\[
E_0[(V_j - V_0)^2] \leq j \sum_{i=1}^j E_0 \left[ (V_i - V_{i-1})^2 \right] = j \sum_{i=1}^j E_0 \left[ E \left[ (V_i - V_{i-1})^2 \mid V_{i-1} \right] \right]
\]

\[
\leq j \sum_{i=1}^j E_0 \left[ \text{Var}(V_i - V_{i-1} \mid V_{i-1}) + \nu V_{i-1} \right] + j \sum_{i=1}^j E_0 \left[ E \left[ V_i - V_{i-1} \mid V_{i-1} \right]^2 \right] + j \nu \sum_{i=1}^j E_0[V_{i-1}]
\]

\[
\leq C_9 m^{1/2} + j \nu \sum_{i=1}^j (E_0[V_{i-1} - V_j] + E_0[V_j - V_0]) + \nu m^{1/2} V_0 = O(m^{3/2}),
\]

where \( C_9 \) is some fixed constant appropriately larger than \( c_{14} \). This implies that the right hand side of (77) is \( o(N_m m^{1/4}) \) and, thus, completes the proof of (76).

**Step 3.** We need to show that

\[
E \left[ \max_{1 \leq k \leq T m^{3/4} \wedge (\tau_m + 1)} \left( \tilde{V}_{m,[km^{1/4}]}^+ - \tilde{V}_{m,[(k-1)m^{1/4}]}^+ \right)^2 \right] = o(m^2).
\]

Let \( B_k \) be defined as in (79). Then the right hand side of the above expression is equal to

\[
E \left[ \max_{1 \leq k \leq T m^{3/4} \wedge (\tau_m + 1)} \left( \tilde{V}_{m,[km^{1/4}]}^+ - \tilde{V}_{m,[(k-1)m^{1/4}]}^+ \right)^2 \mathbb{1}_{\{B_k\}} \right]
\]

\[
\leq (m^{3/5+1/4})^2 + T m^{3/4} \max_{1 \leq k \leq T m^{3/4} \wedge (\tau_m + 1)} E \left[ \left( \tilde{V}_{m,[km^{1/4}]}^+ - \tilde{V}_{m,[(k-1)m^{1/4}]}^+ \right)^2 \mathbb{1}_{\{B_k\}} \right]
\]

\[
\leq o(m^2) + T m^{3/4} \max_{1 \leq k \leq T m^{3/4} \wedge (\tau_m + 1)} \left( E \left[ \left( \tilde{V}_{m,[km^{1/4}]}^+ - \tilde{V}_{m,[(k-1)m^{1/4}]}^+ \right)^4 \right] \right)^{1/2} \left( P(B_k^c) \right)^{1/2}.
\]

Given the stretched exponential decay of the last probability, any polynomial in \( m \) bound on the 4-th moment above will suffice.

Fix an arbitrary \( k, 1 \leq k \leq T m^{3/4} \wedge (\tau_m + 1) \) and recall our shortcut notation from the previous step. For each \( j \in \llbracket 1, m^{1/4} \rrbracket \), using the representation in (72) together with Lemma A.3 from [KP17] we can obtain that

\[
E \left[ (V_j - V_{j-1})^4 \right] = E \left[ E \left[ (V_j - V_{j-1})^4 \mid V_{j-1} \right] \right]
\]

\[
\leq C_{10} E \left[ ((V_{j-1} + 1) \vee N_m m^{1/4})^2 \right] \leq C_{10} E \left[ V_{j-1}^2 \right] + o(m^2).
\]

Finally, by (80),

\[
E[V_{j-1}^2 \mid V_0] \leq 2E[(V_j - V_0)^2 \mid V_0] + 2V_0^2 \leq O(m^{3/2}) + 2(rm)^2.
\]

Collecting all these estimates we get a desired polynomial bound, and we are done.

**Step 4.** Estimates (75), (76), and (81) imply that the process \( Z_k^n = m^{-1/4} \tilde{V}_{m,[km^{1/4}]}^+ \) with \( n = \lfloor m^{3/4} \rfloor \) satisfies the conditions of Lemma A.5 with \( D = 1 + \rho \). An application of Lemma A.5 and Lemma A.4 completes the proof. \( \square \)

**A.3. Other results needed.** In the proof of Lemma 8.1 we need some large deviation estimates for the supremum of a concatenation of BLPs. We show this below as a corollary of an analogous result for concatenation of BESQ processes.

**Lemma A.6.** Let \((Y(t))_{t \geq 0} \) be a solution of

\[
dY(t) = D(t) \, dt + \sqrt{\nu(Y(t))} \, dB(t), \quad 0 \leq t \leq T, \quad Y(0) = y \in (0, T),
\]

where

\[
\nu(y) = C_0 \left( \frac{1}{y^2} \right), \quad y > 0,
\]

...
where \( \nu > 0 \) and \( D : [0, T] \to \mathbb{R} \) is a piecewise constant non-random function bounded above by some \( d > 0 \). Then there exist \( C_{11}, C_{12} > 0 \) (which depend on \( d \) and \( \nu \) but not on \( y \) and \( T \)) such that

\[
P \left( \sup_{t \leq T} Y(t) \geq xT \right) \leq C_{12}e^{-C_{11}x} \quad \text{for all } x \geq 0.
\]

**Proof.** Without loss of generality we can assume that \( x \geq 2 \). By the comparison theorem for one-dimensional SDEs the process \( 4Y/\nu \) is stochastically dominated by a BESQ\([4d/\nu]\)(4\( y/\nu \)) process. The last process is just \( 4y/\nu \) plus the sum of squares of \([4d/\nu]\) independent one-dimensional Brownian motions. Therefore, the probability in question does not exceed

\[
P \left( \max_{t \leq T} \sum_{i=1}^{\lceil 4d/\nu \rceil} B_i^2(t) \geq \frac{4(Tx - y)}{\nu} \right) \leq \left\lfloor \frac{4}{} \right\rfloor P \left( \max_{t \leq T} |B(s)| \geq \sqrt{\frac{2Tx}{\nu\lceil 4d/\nu \rceil}} \right) \leq C_{12}e^{-C_{11}x}.
\]

\[\square\]

**Corollary A.7.** For \( m \in \mathbb{N} \) let \( \{Z_j^m\}_{j \geq 0} \) be a BLP starting from 0 that is the concatenation of \( V^+ \) and then two \( U^+ \) processes on 3 intervals \( I_1, I_2, I_3 \) where \( I_1 \cup I_2 \cup I_3 = [0, 2\varepsilon m] \) and assume that the first cookie environment on \( I_1 \) is \((m^{1/4}, \frac{\nu}{2} - 1)\)-good, the first cookie environment on \( I_2 \) is \((m^{1/4}, 0)\)-good) and the first cookie environment on \( I_3 \) is i.i.d. with distribution \( \eta \).

Then for \( C_{11}, C_{12} \) as in Lemma A.6 we have that for every \( K < \infty \), there exists \( m_0(K) < \infty \) such that

\[
P \left( \sup_{j \leq 2\varepsilon m} Z_j^m \geq 2\varepsilon mx \right) \leq 2C_{12}e^{-C_{11}x} \quad \text{for all } m \geq m_0(K) \text{ and } x \leq K.
\]

**Proof.** We fix \( K \in (0, \infty) \). Though the interest in the corollary is for BLPs starting at value 0, by monotonicity of these processes, it is enough to show the desired result for BLPs satisfying \( Z_0^m = \lfloor \varepsilon m \rfloor \). We argue by contradiction and suppose that the result is not true. This implies the existence of a sequence \( \{m_k\}_{k \geq 0} \), intervals \( I_{jk}, I_{jk}^m \), and \( P_{mk} \) partitioning \([0, 2\varepsilon m]\) and suitable \( m_k \) indexed environments satisfying the stated hypotheses on these intervals so that the stated probability bound is violated for all \( k \). Taking a subsequence if needed we may suppose that, in the obvious sense, that the intervals \( I_{jk}^m \) divided by \( \varepsilon m_k \) converge to intervals \( I_j \) for \( j = 1, 2, 3 \).

In the following, to avoid a burdensome notation, we write \( m_k \) as \( m \). It is sufficient to show that under these conditions the claimed probability bounds hold.

By Theorem 5.9, Corollary 5.12 and then Theorem 5.5 the processes \( \{m^{-1}Z_j^m\}_{j \geq 0} \) converge weakly to a concatenation of a \( \frac{\nu}{4} \) BESQ\( ^2 \) process starting at \( \varepsilon \) (on interval \( I_1 \)) with a \( \frac{\nu}{4} \) BESQ\( ^0 \) process on \( I_2 \) and then a \( \frac{\nu}{4} \) BESQ\( ^{2\theta_1} \) process on \( I_3 \). Note that for the interval \( I_1 \), Theorem 5.9 suffices since a BESQ\( ^2 \) process starting at \( \varepsilon \) never hits zero. Lemma A.6 is applicable to this limit process, and we get that for every \( x \geq 0 \), \( \limsup_{m \to \infty} P(\sup_{j \leq 2\varepsilon m} Z_j^m \geq 2\varepsilon mx) \leq C_{12}e^{-C_{11}x} \).

To complete the proof we take \( 0 = x_0 < x_1 < \ldots < x_r = K \) so that \( \forall i, x_i - x_{i-1} < \delta \) where \( e^{-C_{11}\delta} < 3/2 \). For \( m \) sufficiently large and all \( x_i, i \in [0, r], \) we have \( P(\sup_{j \leq 2\varepsilon m} Z_j^m \geq 2\varepsilon mx_i) \leq \frac{4}{3} C_{12}e^{-C_{11}\delta x} \), and so for such \( m \) by monotonicity

\[
\forall x \leq K, \ P \left( \sup_{j \leq 2\varepsilon m} Z_j^m \geq 2\varepsilon mx \right) \leq \frac{4}{3} C_{12} e^{-C_{11}(x - \delta)} \leq 2C_{12}e^{-C_{11}x} \quad \square
\]

Finally, we need the following general lemma about couplings which is used in the proof of Lemma 8.3 For this, recall the definition of the family of probability measures \( \mathcal{H}_{\delta, \varepsilon} \) in Definition 8.2

**Lemma A.8.** For every \( \lambda \in \mathcal{H}_{\delta, \varepsilon} \) there is a coupling \( \nu \) of probability measures \( \lambda \) and \( \lambda_0 \) such that \( \nu(\{(x, y) \in \mathbb{R}^2 : |x - y| > \delta \}) < 8e^3 \).
Proof. We shall construct a random vector \((\zeta, \zeta^{(0)}, \zeta^{(1)})\) with respective marginal distributions \(\lambda, \lambda_0, \lambda_1\) so that \(P(|\zeta - \zeta^{(0)}| > \delta) < 8\varepsilon^3\). Then \(\nu\) is the joint distribution of \((\zeta, \zeta^{(0)})\).

Recall that \(\lambda \in \mathcal{H}_{\delta, \varepsilon}\) can be represented as \(\lambda = \int K(z, \cdot) \lambda_1(\text{d}z)\) with \(K\) and \(\lambda_1\) satisfying the conditions in Definition 8.2. Let \(\nu_0\) be a maximal coupling of \(\lambda_0\) and \(\lambda_1\) and \((\zeta^{(0)}, \zeta^{(1)})\) be a random vector with distribution \(\nu_0\). Then

\[
\nu_0(\{(y, z) \in \mathbb{R} : y \neq z\}) = P(\zeta^{(0)} \neq \zeta^{(1)}) = \|\zeta^{(0)} - \zeta^{(1)}\|_{TV} < 8\varepsilon^3.
\]

Denote the regular conditional probability distribution of \(\zeta^{(0)}\) given \(\zeta^{(1)} = z\) by \(K_0(z, \cdot)\). We construct \((\zeta, \zeta^{(0)}, \zeta^{(1)})\) as follows.

- draw \(\zeta^{(1)}\) according to \(\lambda_1\);
- given \(\zeta^{(1)} = z\), draw \(\zeta\) from \(K(z, \cdot)\) and \(\zeta^{(0)}\) from \(K_0(y, \cdot)\) independently from each other.

We have

\[
P(|\zeta - \zeta^{(0)}| > \delta) = P(|\zeta - \zeta^{(1)}| > \delta, \zeta^{(0)} = \zeta^{(1)}) + P(|\zeta - \zeta^{(0)}| > \delta, \zeta^{(0)} \neq \zeta^{(1)})
\]

\[
\leq P(|\zeta - \zeta^{(1)}| > \delta) + P(\zeta^{(0)} \neq \zeta^{(1)}) = \int K(z, [z - \delta, z + \delta]) \lambda_1(\text{d}z) + \nu_0(\{(y, z) \in \mathbb{R} : y \neq z\}) < 8\varepsilon^3.
\]

\[\square\]

References

[Bill99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience publication.

[BS08a] Anne-Laure Basdevant and Arvind Singh. On the speed of a cookie random walk. *Probab. Theory Related Fields*, 141(3-4):625–645, 2008.

[BS08b] Anne-Laure Basdevant and Arvind Singh. Rate of growth of a transient cookie random walk. *Electron. J. Probab.*, 13:no. 26, 811–851, 2008.

[BW03] Itai Benjamini and David B. Wilson. Excited random walk. *Electron. Comm. Probab.*, 8:86–92 (electronic), 2003.

[CD99] L. Chaumont and R. A. Doney. Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion. *Probab. Theory Related Fields*, 113(4):519–534, 1999.

[CDH00] L. Chaumont, R. A. Doney, and Y. Hu. Upper and lower limits of doubly perturbed Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, 36(2):219–249, 2000.

[CdHPP16] F. Caravenna, F. den Hollander, N. Pétrélis, and J. Poisat. Annealed scaling for a charged polymer. *Math. Phys. Anal. Geom.*, 19(1):Art. 2, 87, 2016.

[CPTY98] Philippe Carmona, Frédérique Petit, and Marc Yor. Beta variables as times spent in \([0, \alpha]\) by certain perturbed Brownian motions. *J. London Math. Soc. (2)*, 58(1):239–256, 1998.

[Dav96] Burgess Davis. Weak limits of perturbed random walks and the equation \(Y_t = B_t + \alpha \sup\{Y_s : s \leq t\} + \beta \inf\{Y_s : s \leq t\} \). *Ann. Probab.*, 24(4):2007–2023, 1996.

[DK12] Dmitry Dolgopyat and Elena Kosygina. Scaling limits of recurrent excited random walks on integers. *Electron. Commun. Probab.*, 17:no. 35, 2012.

[DK15] Dmitry Dolgopyat and Elena Kosygina. Excursions and occupation times of critical excited random walks. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(1):427–450, 2015.

[Dol11] Dmitry Dolgopyat. Central limit theorem for excited random walk in the recurrent regime. *ALEA Lat. Am. J. Probab. Math. Stat.*, 8:259–268, 2011.

[Eck86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II*. Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[GJY03] Anja Göing-Jaeschke and Marc Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[HLSH18] Wilfried Huss, Lionel Levine, and Ecaterina Sava-Huss. Interpolating between random walk and rotor walk. *Random Structures Algorithms*, 52(2):263–282, 2018.

[KKS75] H. Kesten, M. V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math.*, 30:145–168, 1975.
[KM11] Elena Kosygina and Thomas Mountford. Limit laws of transient excited random walks on integers. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(2):575–600, 2011.

[KOS16] Gady Kozma, Tal Orenshtein, and Igor Shinkar. Excited random walk with periodic cookies. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1023–1049, 2016.

[KP16] Elena Kosygina and Jonathon Peterson. Functional limit laws for recurrent excited random walks with periodic cookie stacks. *Electron. J. Probab.*, 21:Paper No. 70, 24, 2016.

[KP17] Elena Kosygina and Jonathon Peterson. Excited random walks with Markovian cookie stacks. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(3):1458–1497, 2017.

[KZ08] Elena Kosygina and Martin P. W. Zerner. Positively and negatively excited random walks on integers, with branching processes. *Electron. J. Probab.*, 13:no. 64, 1952–1979, 2008.

[KZ13] Elena Kosygina and Martin Zerner. Excited random walks: results, methods, open problems. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 8(1):105–157, 2013.

[KZ14] Elena Kosygina and Martin P. W. Zerner. Excursions of excited random walks on integers. *Electron. J. Probab.*, 19:no. 25, 25, 2014.

[MPV14] Thomas Mountford, Leandro P. R. Pimentel, and Glauco Valle. Central limit theorem for the self-repelling random walk with directed edges. *ALEA Lat. Am. J. Probab. Math. Stat.*, 11(1):503–517, 2014.

[Pet12] Jonathon Peterson. Large deviations and slowdown asymptotics for one-dimensional excited random walks. *Electron. J. Probab.*, 17:no. 48, 48, 2012.

[Pin10] Ross G. Pinsky. Transience/recurrence and the speed of a one-dimensional random walk in a “have your cookie and eat it” environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(4):949–964, 2010.

[PT17] Ross G. Pinsky and Nicholas F. Travers. Transience, recurrence and the speed of a random walk in a site-based feedback environment. *Probab. Theory Related Fields*, 167(3-4):917–978, 2017.

[PW97] Mihael Perman and Wendelin Werner. Perturbed Brownian motions. *Probab. Theory Related Fields*, 108(3):357–383, 1997.

[RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[Tót94] Bálint Tóth. “True” self-avoiding walks with generalized bond repulsion on $\mathbb{Z}$. *J. Statist. Phys.*, 77(1-2):17–33, 1994.

[Tót95] Bálint Tóth. The “true” self-avoiding walk with bond repulsion on $\mathbb{Z}$: limit theorems. *Ann. Probab.*, 23(4):1523–1556, 1995.

[Tót96] Bálint Tóth. Generalized Ray-Knight theory and limit theorems for self-interacting random walks on $\mathbb{Z}^1$. *Ann. Probab.*, 24(3):1324–1367, 1996.

[Tra18] Nicholas F. Travers. Excited random walk in a Markovian environment. *Electron. J. Probab.*, 23:Paper No. 43, 60, 2018.

[TV08] Bálint Tóth and Bálint Vető. Self-repelling random walk with directed edges on $\mathbb{Z}$. *Electron. J. Probab.*, 13:no. 62, 1909–1926, 2008.

[Zer05] Martin P. W. Zerner. Multi-excited random walks on integers. *Probab. Theory Related Fields*, 133(1):98–122, 2005.

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