I. INTRODUCTION

In quantum information and computation, the class of stabilizer circuits can be efficiently simulated by classical computers [1] using the stabilizer formalism [2]. Stabilizer circuits are composed solely of Hadamard (H), Phase (P), and controlled-NOT (CNOT) gates, defined as

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

and single-qubit Pauli measurements. A stabilizer circuit is called a Clifford circuit if it contains no measurements, and Hadamard, Phase, and CNOT gates are called Clifford gates. It is known that the Clifford gates, together with any non-Clifford gate, form a universal set for quantum computation [3]. Stabilizer circuits are especially important in fault-tolerant quantum computation (FTQC) for encoding, decoding, and error correction circuits [4–7], along with other applications, such as evaluation of the average gate fidelity via randomized benchmarking [8, 9], and efficient quantum simulations [10–14].

It is well known that any \( n \)-qubit (unitary) quantum circuit \( U \) of a certain level of the Clifford hierarchy can be implemented by gate teleportation [15], which requires a \( 2n \)-qubit ancilla state

\[ \ket{\Phi_U^n} = I \otimes U \left( \frac{\ket{00} + \ket{11}}{\sqrt{2}} \right)^{\otimes n}, \quad (1) \]

and is performed by a single step of Bell basis measurements followed by a controlled-correction circuit at a lower level of the Clifford hierarchy. For a Clifford circuit \( U \), \( \ket{\Phi_U^n} \) is a stabilizer state, which is a joint-\( (+1) \) eigenvector of \( 2n \) commuting Pauli operators, called stabilizer generators. The controlled-correction is simply a Pauli operator. Thus, the complexity of a Clifford circuit is dominated by the preparation of \( \ket{\Phi_U^n} \), which can be prepared by measuring the stabilizer generators on \( n \) EPR pairs (up to a Pauli correction).

At first sight, it seems as difficult to prepare such an ancilla state as to directly implement the circuit. However, in the case of FTQC, it is possibly easier to prepare specific known states for gate teleportation than to do gate operations on unknown states. One important example is the magic state distillation for the fault-tolerant implementation of non-Clifford gates [16, 17]. In some cases, it may even be impossible to do gate operations directly on the qubits. For example, for a FTQC scheme using multi-qubit quantum error-correcting codes [18–20], typically, its fault-tolerant logical Clifford gates, if they exist, are computationally difficult to find. Therefore we would like to investigate the implementation of stabilizer circuits by variants of gate teleportation in FTQC.

Consider an \( n \)-qubit Clifford circuit \( U \). Previously, Gottesman and Chuang showed that the ancillary state \( \ket{\Phi_U^n} \) can be fault-tolerantly prepared by a sequence of \( O(n) \) fault-tolerant Pauli operator measurements, with error correction and verification inserted between each two consecutive measurements [15]. This preparation is passive in that most of the procedure is error detection. We will show that to implement a Clifford circuit, it suffices to do \( O(1) \) gate teleportations with \( (\text{clean}) \) ancillas that are Calderbank-Shor-Steane (CSS) stabilizer states (up to single-qubit Clifford gate operations), and hence can be fault-tolerantly prepared [21, 22]. (A CSS state is defined by a set of stabilizer generators, each of which can be chosen to be the tensor product of identity and either \( X \) or \( Z \) Pauli operators.) These ancilla states are thus equivalent to two-colorable graph states [23].

Our idea is motivated by Clifford circuit synthesis [24, 25]. Aaronson and Gottesman showed that any Clifford circuit is equivalent to a circuit that contains \( 11 \) stages of computation in the sequence -H-C-P-C-P-C-H-P-C-P- [24], where -H-, -P-, and -C- stand for stages composed of only Hadamard,
Phase, and CNOT gates, respectively [26]. Recently, Maslov and Roetteler found that a Clifford circuit can be decomposed as a 9-stage sequence -C-P-C-P-H-P-C-P-C-. Therefore, it suffices to implement each of the -H-, -P-, and -C- stages of the 11-stage or 9-stage sequence for a Clifford circuit. In FTOQC, it is straightforward to combine Knill syndrome extraction [27] with gate teleportation [15], and clearly \( \Phi^U_0 \), where \( U \) is a -H-, -P-, or -C- circuit, is a CSS state up to single-qubit Clifford operations. Consequently, we can prepare the ancilla states for the 9-stage or 11-stage sequence by distillation [21, 22].

On the other hand, it is not so obvious how to combine Steane syndrome extraction [28] with gate teleportation. Since both Steane and Knill syndrome extraction have their own advantages, we would like also to derive a constant-depth gate teleportation procedure for Steane syndrome extraction. For example, we remark that measurements of logical Pauli operators can be implemented simultaneously with error correction in Steane syndrome extraction [28], and consequently we can have stabilizer circuits implemented solely with Steane syndrome extraction. Moreover, Steane syndrome measurements may lead to higher thresholds for certain CSS codes. In this paper, we can propose such a procedure for Steane syndrome extraction through a series of Pauli measurements that implement the 9-stage sequence with the help of appropriate ancilla states that are CSS states up to single-qubit Clifford operations. Again, these states can be fault-tolerantly prepared. We will discuss the procedure at the logical level: the underlying quantum error-correcting codes can be either single-qubit codes or multiple-qubit codes. If the underlying quantum error-correcting codes, we have a roughly constant resource overhead [21, 22].

The paper is organized as follows. We review preliminary material in Sec. II, including the stabilizer formalism and the representation of Clifford circuits. In Sec. III, we propose a method to measure arbitrary Pauli operators, and give conditions when several Pauli operators can be measured simultaneously. In Sec. IV, we explicitly show how to perform Clifford circuits via constant steps of Pauli measurements. Conclusions and discussion of the method are presented in Sec. VI.

II. PRELIMINARIES

A. Stabilizer formalism

The Hilbert space of a single qubit is the two-dimensional complex vector space \( \mathbb{C}^2 \) with an orthonormal basis \( \{ |0\rangle, |1\rangle \} \).

The Hilbert space of \( N \)-qubit states is hence \( \mathbb{C}^{2^N} \). Let \( \mathcal{P}_N = \mathbb{P}_1^\otimes N \) denote the \( N \)-fold Pauli group, where \( \mathcal{P}_1 = \{ \pm I, \pm iX, \pm iY, \pm iZ \} \), and \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \) and \( Y = iXZ \) are the Pauli matrices. Let \( X_j, Y_j, \) and \( Z_j \) act as single-qubit Pauli matrices on the \( j \)-th qubit and trivially elsewhere. We also introduce the notation \( X^a \), for \( a = a_1 \cdots a_N \in \mathbb{Z}_2^N \), to denote the operator \( \otimes_{j=1}^N X^{a_j} \) and let \( \text{supp}(a) = \{ j : a_j = 1 \} \). For \( a, b \in \mathbb{Z}_2^N \), denote the intersection of \( \text{supp}(a) \) and \( \text{supp}(b) \) by \( \mathcal{I}_{ab} \) and let \( \tau_{ab} = |\mathcal{I}_{ab}| \). An \( N \)-fold Pauli operator can be expressed as

\[
\hat{J} = \bigotimes_{j=1}^N X^a Z^b, \quad a, b \in \mathbb{Z}_2^N, \quad l \in \{ 0, 1, 2, 3 \}. \tag{2}
\]

Then \( (a \mid b) \) is called the binary representation of the Pauli operator \( i^l X^a Z^b \) up to an overall phase \( i^l \). In particular, \( \pm i^{\tau_{ab}} X^a Z^b \) has eigenvalues \( \pm 1 \). From now on we use the binary representation, and we may neglect the overall phase for simplicity when there is no ambiguity.

For two Pauli operators \( (a \mid b) \) and \( (e \mid f) \), one can define their symplectic inner product:

\[
(a \mid b)e = J_N(e \mid f)^t = \begin{cases} 0, & \{ X^a Z^b, X^e Z^f \} = 0, \\ 1, & \{ X^a Z^b, X^e Z^f \} = 0, \end{cases}
\]

where

\[
J_N = \begin{pmatrix} 0_N & I_N \\ I_N & 0_N \end{pmatrix},
\]

and \( 0_N \) and \( I_N \) are the identity and zero matrices of dimension \( N \), respectively. Here, \( M^t \) denotes the transpose of \( M \).

Consider a set of commuting Pauli operators \( \{ G_1, \ldots, G_s \} \) that does not generate \( I^{\otimes N} \). These Pauli operators generate an Abelian subgroup (stabilizer group) \( \mathcal{G} \) of \( \mathcal{P}_N \), and thus are called the stabilizer generators of \( \mathcal{G} \). Let \( S(\mathcal{G}) \) denote the \( 2^{N-s} \)-dimensional subspace of the \( N \)-qubit state space \( \mathbb{C}^{2^N} \) fixed by \( \mathcal{G} \), which is the joint-\((+1)\) eigenspace of \( G_1, \ldots, G_s \). Then for any \( |\psi\rangle \in S(\mathcal{G}) \), one has

\[
G|\psi\rangle = |\psi\rangle,
\]

for all \( G \in \mathcal{G} \). (Note that the overall phase of any \( G \in \mathcal{G} \) can be \( \pm 1 \) only.)

A set of \( s \) commuting \( N \)-fold Pauli operators has a binary representation as a matrix of the form:

\[
(A \mid B) = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_s & b_s \end{pmatrix}.
\]

Definition 1 (Symplectic partner). For a set of \( s \) commuting \( N \)-fold Pauli operators \( (A \mid B) \), its symplectic partner \( (E \mid F) \) is a set of \( s \) commuting \( N \)-fold Pauli operators satisfying the orthogonality relation with respect to the symplectic inner product:

\[
(A \mid B)J_N(E \mid F)^t = I_s.
\]

Note that if \( (A \mid B) \) is of rank less than \( N \), its symplectic partner is not unique.

B. Clifford circuits

An \( N \)-qubit Clifford circuit can be represented by a \( 2N \times 2N \) binary matrix with respect to the basis of the binary representation of Pauli operators in (2). For example, the idle circuit (no quantum gates) is represented by \( I_{2N} \), the \( 2N \times 2N \)
identity matrix. The representation of consecutive Clifford circuits $M_1, \ldots, M_j$ is their binary matrix product $M_1 \cdots M_j$.

The $N$-qubit Clifford circuits form a finite group, which, up to overall phases, is isomorphic to the binary symplectic matrix group defined as follows: [24]

**Definition 2 (Symplectic group).** The group of $2N \times 2N$ symplectic matrix over $\mathbb{Z}_2$ is defined as:

$$\text{Sp}(2N, \mathbb{Z}_2) \equiv \{ M \in \text{GL}(2N, \mathbb{Z}_2) : MJ_N M^t = J_N \}$$

under matrix multiplication.

In general, $M \in \text{Sp}(2N, \mathbb{Z}_2)$ has the form

$$M = \begin{pmatrix} Q & R \\ S & T \end{pmatrix},$$

where $Q$, $R$, $S$ and $T$ are $N \times N$ square matrices satisfying the following conditions:

$$QR^t = RQ^t, \quad ST^t = TS^t, \quad Q^t T + R^t S = I_N.$$  

(4)

In other words, $(Q|R)$ is a symplectic partner of $(S|T)$ by Def. 1. Unlike Ref. [24], here we omit the column vector that corresponds to the phases ($\pm 1$ only) of the operators. If needed, such overall phases can always be compensated by a single layer of gates consisting solely of $Z$ and $X$ gates [29] on some subsets of qubits [24, 25].

Let $C(j,l)$ denote a CNOT gate with control qubit $j$ and target qubit $l$. The actions of appending a Hadamard, Phase, or CNOT gate to a Clifford circuit $M$ can be described as follows:

1. A Hadamard gate on qubit $j$ exchanges columns $j$ and $N+j$ of $M$.
2. A Phase gate on qubit $j$ adds column $j$ to column $N+j$ (modulo 2) of $M$.
3. $C(j,l)$ adds column $j$ to column $l$ (modulo 2) of $M$ and adds column $N+j$ to column $N+l$ (modulo 2) of $M$.

Now, consider a $2^k$ dimensional subspace $S(G)$ of the $N$-qubit space, where $G$ has $k \leq N$ stabilizer generators. $S(G)$ encodes $k$ “logical” qubits. We focus on the effects of Clifford circuits on these logical qubits in the stabilizer formalism. Consider a set of matrices $C_G$ of the form:

$$\begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}.$$

(5)

Here, $(A|B)$ corresponds to the stabilizer generators of $G$: $(Q'|R')$ and $(S'|T')$ are $k \times 2N$ binary matrices orthogonal to $(A|B)$ with respect to the symplectic inner product, and which are symplectic partners of each other. They can be regarded as “logical operators” on $S(G)$. We define the following equivalence relation $R$ in $C_G$: Two matrices

$$C_1 = \begin{pmatrix} Q'_1 & R'_1 \\ S'_1 & T'_1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} Q'_2 & R'_2 \\ S'_2 & T'_2 \end{pmatrix},$$

are equivalent if (a) $(A_1|B_1)$ and $(A_2|B_2)$ generate the same stabilizer group $G$; and (b) $\left(\frac{Q'_1}{S'_1}, \frac{R'_1}{T'_1}\right)$ differs from $\left(\frac{Q'_2}{S'_2}, \frac{R'_2}{T'_2}\right)$ by multiplication of elements in $G$. Thus, there is a one-to-one correspondence between $C_G/R$ and $\text{Sp}(2k, \mathbb{Z}_2)$.

Therefore, $C_G/R$ captures the behavior of stabilizer circuits on $S(G)$. The circuit representation of Eq. (5) is called the generalized stabilizer form (GSF) of a stabilizer subspace throughout the paper. It will be used as the starting point of the discussion in the rest of the paper.

III. MEASUREMENTS OF PAULI OPERATORS

A. Measurement of an arbitrary single Pauli operator

We consider the measurement of an arbitrary Hermitian Pauli operator $\pm i^{a'b'} X^a Z^b$ on $N$ qubits, where $a, b \in \mathbb{Z}_2^N$.

$$|\Omega_{ab}\rangle = \frac{1}{\sqrt{2}} (I_N + i^{a'b'} X^a \otimes Z^b) |0\rangle \otimes |1\rangle.$$  

(6)

The measurement of $i^{a'b'} X^a Z^b$ can be realized by the circuit in Fig. 1 with two blocks of ancilla qubits, each containing $N$ qubits. The $2N$-qubit ancilla is prepared in the special state:

$$|\Omega_{ab}\rangle = \frac{1}{\sqrt{2}} (I_{2N} + i^{a'b'} X^a \otimes Z^b) |0\rangle \otimes |1\rangle.$$  

(7)

It is easy to see that it is a stabilizer state and thus, it can be prepared by a Clifford circuit.

Now we prove the functionality of the circuit in Fig. 1. We start with the joint state $|\psi\rangle |\Omega_{ab}\rangle$. After two transversal CNOTs, the state becomes

$$\frac{1}{\sqrt{2}} (|\psi\rangle |0\rangle \otimes |+\rangle) \otimes i^{a'b'} X^a Z^b |\psi\rangle X^a |0\rangle \otimes Z^b |+\rangle \otimes |\rangle.$$  

(7)

Let the measurement outcome of the $j$th qubit in the first and second blocks be $\nu_j^x$ and $\nu_j^z \in \{0, 1\}$, respectively. Then the joint output state is:
operators. Suppose the set of commuting Pauli operators to be
simultaneously by replacing it is easy to see that one can measure these operators simulta-
sensible, since measuring these operators in different time orders
useful when one wants to measure several Pauli operators
sible measurement of multiple Pauli operators. Here, we
ford circuit.

\[
\frac{1}{\sqrt{2}} \psi^N \bigotimes_{j=1} \left( I + \frac{(-1)^{i,j}}{2} X^a \right) |0\rangle \bigotimes_{j=1} \left( I + \frac{(-1)^{i,j}}{2} Z^b \right) |+\rangle + \frac{1}{\sqrt{2}} \left( I + \prod_{j \in \text{supp}(a)} (-1)^{i,j} \prod_{j \in \text{supp}(b)} (-1)^{i,j} \right) \psi^N \bigotimes_{j=1} \left( I + \frac{(-1)^{i,j}}{2} X^a \right) |0\rangle \bigotimes_{j=1} \left( I + \frac{(-1)^{i,j}}{2} Z^b \right) |+\rangle \bigotimes_{j=1} \left( I + \frac{(-1)^{i,j}}{2} Z^b \right) |+\rangle \bigotimes_{j=1} \left( I + \frac{(-1)^{i,j}}{2} Z^b \right) |+\rangle.
\]

(8)

which is the state after the measurement of \(i^{a,b} X^a Z^b\) on \(|\psi\rangle\) with measurement outcome \(\prod_{j \in \text{supp}(a)} (-1)^{i,j} \prod_{j \in \text{supp}(b)} (-1)^{i,j}\). Thus the circuit works as we claimed. This Pauli measurement is especially useful when one wants to measure several Pauli operators simultaneously, as we will see in the next subsection.

B. Simultaneous measurement of multiple Pauli operators

One may wish to measure several Pauli operators simulta-
nously. For a set of non-commuting operators this is not pos-
sible, since measuring these operators in different time orders
may lead to different final states even with the same measure-
ment outcomes. However, if the set of Pauli operators com-
mutes, this can be easily done by the circuit in Fig. 1. In this
paper, we restrict ourselves to a commuting set of \(d \leq n\) Pauli
operators. Suppose the set of commuting Pauli operators to be
measured is

\[\{X^{e_1} Z^{f_1}, \ldots, X^{e_d} Z^{f_d}\} \]

It is easy to see that one can measure these operators simulta-
nously by replacing \(|\Omega_{ab}\rangle\) in Fig. 1 with the following stabil-
izer state:

\[|\Omega_{EF}\rangle = \frac{1}{\sqrt{2^d}} \prod_{j=1} \left( I_{2N} + i^{a_j f_j} X^{e_j} \otimes Z^{f_j} \right) |0\rangle^{\otimes N} \otimes |+\rangle^{\otimes N}.\]

(9)

|\Omega_{EF}\rangle is also a stabilizer state and can be prepared by a Cliff-
donmeasuring multiple Pauli operators. Here, we con-
ider the special case when \(d = N - k\), the number of inde-
is the theory of the stabilizer formalism:

Lemma 1. Consider a circuit with GSF of the form Eq. (5)
and a set of \(N - k\) (independent) commuting Pauli operators:

\[\begin{pmatrix}
  e_1 & f_1 \\
  e_{N-k} & f_{N-k}
\end{pmatrix}
\]

If the following conditions are satisfied:

1. \((E|F) J_N (E|F)^t = 0_{N-k};\)
2. \((A|B) J_N (E|F)^t = I_{N-k};\)
3. \((Q' | R') J_N (E|F)^t = 0;\)
4. \((S' | T') J_N (E|F)^t = 0;\)

then the GSF of the circuit after the simultaneous measurements of \((E|F)\) becomes

\[\left( \frac{Q'}{S'} \frac{R'}{T'} \right).\]

The first two conditions state that \((E|F)\) is a symplectic partner of stabilizer generators \((A|B)\), while the third and forth imply that \((E|F)\) is orthogonal to the logical operators. Note that the measurement outcomes are encoded in the overall phases and hence are not explicitly shown in this discussion.

IV. CLIFFORD CIRCUITS VIA A CONSTANT NUMBER OF MEASUREMENT STEPS

In this section, we consider Clifford circuits consisting of \(n\) qubits. We provide a constructive proof to show that by intro-
ducing \(n\) extra auxiliary qubits, an arbitrary Clifford circuit can be implemented via a constant number of Pauli operator measurements, up to a permutation of qubits.

For clarity, we label the original \(n\) data qubits as \(Q_1, \ldots, Q_n\), and the auxiliary qubits as \(A_1, \ldots, A_n\). Now we have a total of \(N = 2n\) qubits in the order \(Q_1, \ldots, Q_n, A_1, \ldots, A_n\). Suppose that we want to implement a Clifford circuit \(C_{a_1 c_1, a_2 c_2}\) on \(Q_1, \ldots, Q_n\). As in Sec. II B, the GSF with \(n\) stabilizer generators (corresponding to the auxiliary qubits) can be written in the form of Eq. (5). If the initial state of the auxiliary qubits is \(|+\rangle^{\otimes n}\) or \(|0\rangle^{\otimes n}\), then we start with the GSF of the idle circuit:

\[I = \begin{pmatrix} 0_n & I_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ I_n & 0_n & 0_n & I_n \\ 0_n & I_n & 0_n & 0_n \end{pmatrix}\]

(10)

and end up with

\[C = \begin{pmatrix} 0_n & C_1 & 0_n & C_2 \\ 0_n & C_3 & 0_n & C_4 \\ I_n & 0_n & 0_n & I_n \\ 0_n & I_n & 0_n & I_n \end{pmatrix}\]

(11)

The goal is to find a sequence of Pauli measurements that transforms the initial circuits in Eq. (10) into the circuits in Eq. (11). Equivalently, one can start with Eq. (11) and reduce the matrix to the initial circuits through Pauli measurements.

A. 9-stage Clifford circuit decomposition

To find a sequence of Pauli measurements that implement a Clifford circuit, the first step is to decompose the Clifford cir-
cuit into simple stages, each of which only contains a single type of Clifford gates. It is known that any Clifford circuit has an equivalent circuit that contains an 11-stage computation as-

-H-C-P-C-P-C-H-P-C-P-C- [24]. Recently it was shown that a further reduction to a 9-stage computation -C-P-C-P-C-P-C- is possible [25]. We will consider the 9-stage decomposition in the following discussion. More specifically, one has the following result:

**Theorem 1** (Bruhat decomposition [25]). Any symplectic matrix \( M \) of dimension \( 2n \times 2n \) can be decomposed as

\[
M = M_{C}^{(1)} M_{P}^{(1)} M_{C}^{(2)} M_{P}^{(2)} M_{P}^{(3)} (\pi M_{C}^{(3)} \pi^{-1}) M_{P}^{(4)} (\pi M_{C}^{(4)} \pi^{-1}) \pi.
\]

(12)

Here, \( M_{C}^{(j)} \) are -C- stage matrices containing only CNOT gates \( C(q, r) \) such that \( q < r \); \( M_{P}^{(j)} \) and \( M_{H}^{(j)} \) represent matrices of -P- and -H- stages; \( \pi \) is a permutation matrix.

Compared to the 11-stage decomposition, the 9-stage decomposition has fewer stages, and it only requires CNOTs such that the index of the control qubit is less than the index of the target qubit. Recall that a symplectic matrix can be expressed as in Eq. (3). The corresponding symplectic matrix of a -C- stage with such CNOTs can be written as

\[
M_{C} = \begin{pmatrix} U & 0_n \\ 0_n & (U^*)^{-1} \end{pmatrix},
\]

(13)

where \( U \) is an invertible \( n \times n \) upper triangular matrix. As an example, a circuit of two consecutive CNOT gates is shown in Fig. 2 and its symplectic matrix is

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

There are approximately \( n^2/2 \) different such CNOT circuits [25]. As we will see soon, this property is particularly useful when one tries to implement -C- stages via Pauli measurements.

For a -P- stage, since \( P^4 = I_2 \), effectively there are three single-qubit gates: \( P, P^2 = Z \) and \( P^3 = PZ \). Note that we will postpone all the \( Z \) gates to the final stage, and thus the -P- layer consists of at most \( n \) individual Phase gates. Hence, the symplectic matrix of a -P- stage is in general of the form:

\[
M_{P} = \begin{pmatrix} I_n & \Lambda \\ 0_n & I_n \end{pmatrix},
\]

(14)

where \( \Lambda \) is a diagonal matrix.

Similar to the -P- stage, since \( H^2 = I_2 \), an -H- stage contains at most \( n \) individual \( H \) gates. The symplectic matrix of an -H- stage on an arbitrary set of \( m \) qubits can be written as

\[
M_{H, m} = \begin{pmatrix} 0 & 0 & I_m & 0 \\ 0 & I_{n-m} & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-m} \end{pmatrix},
\]

(15)

represents the Hadamard gates acting on \( Q_1, \ldots, Q_m \) and \( \pi' \) is some permutation matrix.

**B. The -P- stage**

As discussed above, a -P- stage only contains at most a single Phase gate acting on each qubit, and thus we discuss the effect of a single-qubit Phase gate on data qubit \( Q_j \). For \( m \) Phase gates acting on a subset of \( m \) qubits, such a procedure can be done simultaneously by Pauli measurements.

Consider a pair of qubits \( \{ A_j, Q_j \} \) with \( A_j \) in \( |0 \rangle \) state. The GSF of the idle circuit is:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

where the first and second columns correspond to \( A_j \) and \( Q_j \), respectively; the first two rows are the logical operators corresponding to \( Q_j \) and the third row represents the stabilizer generator corresponding to \( A_j \).

First, add the stabilizer row to both logical operator rows (which will give an equivalent GSF of the circuit), and measure operator \( (1 \ 1 \ 0 \ 1) \) or \( X_{A_j} Y_{Q_j} \). One obtains

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{pmatrix},
\]

which is equivalent to

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{pmatrix},
\]

by adding the stabilizer row to the first logical operator row. Next do the Pauli measurement \( (0 \ 0 \ 0 \ 1) \) or \( Z_{Q_j} \). One gets

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

After swapping \( A_j \) and \( Q_j \), the overall effect is a Phase gate on \( Q_j \) up to a Pauli correction depending on the measurement outcomes. The swap does not need to be done physically. Instead, one can just keep a record of it in software.

For the case of \( m \) Phase gates, since \( \{ X_{A_j} Y_{Q_j} \} | 1 \leq j \leq n \} \) and \( \{ Z_{Q_j} \} | 1 \leq j \leq n \} \) are commuting operator sets and the
measurements of \( \{Z_Q\} \) can be directly applied, it requires only two steps of Pauli measurement and one \( 4n \)-qubit ancilla state for a -P- stage. If Phase gates are applied to a set \( \mathcal{M} \) of qubits, then the required ancilla state is

\[
|\Omega_{\mathcal{M}}\rangle = \frac{1}{\sqrt{2^{2n}}} \prod_{j \in \mathcal{M}} (I_{2N} + i (X_j X_{j+n} \otimes Z_{j+n}) |0\rangle \otimes |+\rangle \otimes |+\rangle \otimes 2n. \tag{16}
\]

This state can be obtained by projecting \(|0\rangle \otimes |+\rangle \otimes 2n \) to the joint +1 eigenspace of \( \{X_j X_{j+n} \otimes Z_{j+n} \mid j \in \mathcal{M}\} \). Thus, it is stabilized by \( \{Z_j Z_{j+n} \otimes I_{2n}, Z_{j+n} \otimes X_j, X_j X_{j+n} \otimes Z_{j+n} \mid j \in \mathcal{M}\} \), which is a CSS state up to Hadamard gates on qubits \( \{j + n, j \in \mathcal{M}\} \) in the second ancilla block.

C. The -H-stage

Like the -P- stage, we consider only a single H on a data qubit. For a pair of qubits \( \{A_j, Q_j\} \) with \( A_j \) in \( |0\rangle \) state, the idle circuit is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

Adding the stabilizer row to the first row of the logical operator and then measuring \((1 0 | 0 1)\) or \(X_A Z_{Q_j}\), one obtains

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.
\]

Adding the stabilizer row to the second row of the logical operator and measuring \((0 1 | 0 0)\) or \(X_Q\), one gets

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

which is equivalent to

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

After swapping \( A_j \) and \( Q_j \), the overall effect is a Hadamard gate on \( Q_j \) with \( A_j \) in \(|+\rangle\) up to a Pauli correction depending on the measurement outcomes. (Again, this swap just needs to be recorded in software.)

Since \( \{X_A Z_{Q_j} \mid 1 \leq j \leq n\} \) and \( \{X_Q \mid 1 \leq j \leq n\} \) are both commuting sets, we need just two steps of Pauli measurements and one \( 4n \)-qubit ancilla state for an -H- stage. If Hadamard gates are applied to a set \( \mathcal{M} \) of qubits, the required ancilla state is

\[
|\Omega_{\mathcal{M}}\rangle = \frac{1}{\sqrt{2^{2n}}} \prod_{j \in \mathcal{M}} (I_{2N} + X_j \otimes Z_{j+n}) |0\rangle \otimes |\rangle \otimes 2n. \tag{17}
\]

It is easy to recognize that it is the same state one obtains after projecting \(|0\rangle \otimes |+\rangle \otimes 2n\) to the joint +1 eigenspace of \( \{X_j \otimes Z_{j+n} \mid j \in \mathcal{M}\} \). Thus, it is a CSS state (up to Hadamard gates) stabilized by \( \{X_j \otimes Z_{j+n}, Z_j \otimes X_{j+n} \mid j \in \mathcal{M}\} \).

D. The -C- stage

The set of measurement operators for a -C- stage is more complicated to find. We first introduce the following lemma that will be used later.

**Lemma 2.** Let \( L_1 \) be an \( n \times n \) lower triangular matrix with the diagonal elements being zeros. Suppose

\[
L = (I_n L_1).
\]

Then there exists a full-rank matrix \( L' = (L_2 L_3) \), where \( L_2 \) and \( L_3 \) are two \( n \times n \) lower triangular matrices, such that the rows of \( L' \) are linear combinations of rows of \( L \) and

\[
L' \begin{pmatrix} I_n \\ I_n \end{pmatrix} = L_2 + L_3 = I_n. \tag{18}
\]

**Proof.** Let \( l_j' \) denote the \( j \)th row vector of \( L' \) and \( c_p \) be the \( p \)th column vector of \( (I_n I_n)^t \). Equation (18) is equivalent to

\[
l_j' c_p = \delta_{jp}, \quad 1 \leq j, p \leq n, \tag{19}
\]

where \( \delta \) is the Kronecker delta function.

Let \( l_j \) denote the \( j \)th row vector of \( L \). Obviously, \( l_1 = (1, 0, \ldots, 0) \), satisfying \( l_1 c_p = \delta_{1p} \). Let \( l_1' = l_1 \).

It is easy to see that \( l_j c_p = 0 \) for \( p > j \), since \( L_1 \) is a lower triangular matrix. With all the diagonal elements of \( L_1 \) being 0, one has

\[
l_j c_j = 1. \tag{20}
\]

Define the set \( \mathcal{J}_j = \{p \mid l_j c_p = 1, p < j\} \). For \( j = 2, \ldots, n \), let

\[
l_j' = l_j + \sum_{p \in \mathcal{J}_j} l_p'. \tag{21}
\]

We also define a matrix \( L'(j) \) that contains the rows \( l_1', \ldots, l_j' \):

\[
L'(j) = \begin{pmatrix} l_1' \\ \vdots \\ l_j' \end{pmatrix}.
\]

Since \( L_1 \) is lower triangular, and the summation of \( l_p \) in Eq. (21) only counts the terms with \( p < j \), \( L'(j) \) can be written as

\[
L'(j) = \begin{pmatrix} L_2(j) \\ L_3(j) \end{pmatrix},
\]

where \( L_2(j) \) and \( L_3(j) \) are also lower triangular matrices. Eventually, we have \( L_2 = L_2(n) \) and \( L_3 = L_3(n) \).

It remains to prove Eq. (19). We prove this by induction. For \( j = 2 \), if \( L_2c_1 = 1 \), one has \( l_2' = l_2 + l_1 \). Thus \( l_2' c_1 = 0 \) and \( l_2' c_2 = 1 \), since \( l_1 c_1 = 1 \) and \( l_1 c_2 = 0 \). Also, \( l_2' c_p = 0 \) for \( p > 2 \) since \( L_2(2) \) and \( L_3(2) \) are lower triangular matrices. So \( l_2' c_p = \delta_{2p} \) holds for \( 1 \leq p \leq n \).

Now assume \( l_j' c_p = \delta_{jp} \), \( \ldots, l_j' c_p = \delta_{jp} \), holds. Then

\[
l_{j+1}' c_q = l_{j+1} c_q + \sum_{p \in \mathcal{J}_{j+1}} l_p' c_q.
\]
Consider $q < j + 1$ first. If $l_{j+1}c_q = 1$, then $q \in J_{j+1}$ and
\[ \sum_{p \in J_{j+1}} \nu_{pj}c_q = \sum_{p \in J_{j+1}} \delta_{pq} = 1. \]

Then $\nu_{j+1}c_q = 0$. If $l_{j+1}c_q = 0$, then $q \notin J_{j+1}$ and
\[ \sum_{p \in J_{j+1}} \nu_{pj}c_q = 0. \]
Again, $\nu_{j+1}c_q = 0$. When $q = j + 1$,
\[ l'_{j+1}c_q = l_{j+1}c_{j+1} = 1 \]
by Eq. (20). For $q > j + 1$, since $L_{j+1}^{(j+1)}$ and $L_{j+1}^{(j)}$ are both lower triangular, $l'_{j+1}c_q = 0$. Thus, $\nu_{j+1}c_p = \delta_{jp}$ holds for $1 \leq j, p \leq n$.

Now we are ready to show that any -C- stage containing $C(j, l)$ on $Q_1, \ldots, Q_n$ with $j < l$ can be implemented by a constant number of Pauli measurements. Unlike the case of -P- or -H- stage, we start from the GSF of an arbitrary -C- circuit with $A_1, \ldots, A_n$ in $|+\rangle^\otimes n$ state:
\[ \begin{pmatrix} 0_n & U_n & 0_n & 0_n \\ 0_n & 0 & 0_n & (U')^{-1} \\ I_n & 0_n & 0_n & 0_n \end{pmatrix}. \tag{22} \]

and try to reduce it to the idle circuit. Meanwhile, we will provide the reverse operations that will effectively implement the target CNOT circuit.

As mentioned before, $U$ is some invertible upper triangular matrix. The GSF is then equivalent to
\[ \begin{pmatrix} U + I_n & U_n & 0_n & 0_n \\ 0_n & 0 & 0_n & (U')^{-1} \\ I_n & 0_n & 0_n & 0_n \end{pmatrix} \tag{23} \]
since all the nonzero row vectors of $(U + I_n | 0_n | 0_n)$ can be generated by $(I_n | 0_n | 0_n)$ and we then add these vectors to the first row.

Since $U$ is of full rank, the diagonal elements of $U + I_n$ must be all zeros. Observe that $(0_n | 0_n | I_n \ (U')^{-1} + I_n)$ commutes with the logical operators and is a symplectic partner of the stabilizer generators. This can be checked by verifying that
\[ (I_n \ (U')^{-1} + I_n) (U + I_n \ U)^t = 0_n, \]
and
\[ (I_n | 0_n | 0_n) J_{2n} (0_n | 0_n | I_n \ (U')^{-1} + I_n)^t = I_{2n}. \]

According to Lemma 1, one can measure $n$ commuting Pauli operators $(0_n | 0_n | I_n \ (U')^{-1} + I_n)$ simultaneously. The GSF will then be transformed into
\[ \begin{pmatrix} U + I_n & U_n & 0_n & 0_n \\ 0_n & 0_n & I_n \ (U')^{-1} + I_n \end{pmatrix}. \tag{24} \]

(Meanwhile, we can perform the Pauli measurements $(I_n | 0_n | 0_n)$ to reverse the process (from Eq. (24) to Eq. (23)).)

Now, adding the third row of Eq. (24) to the second row, one can obtain an equivalent GSF
\[ \begin{pmatrix} U + I_n & U | 0_n & 0_n \\ 0_n & 0_n & I_n \ (U')^{-1} + I_n \end{pmatrix}. \tag{25} \]

Let $L = (I_n \ L_1)$, where $L_1 = (U')^{-1} + I_n$ is a lower triangular matrix with all the diagonal elements being 0. By Lemma 2, the GSF can be equivalently transformed into
\[ \begin{pmatrix} U + I_n & U | 0_n & 0_n \\ 0_n & 0_n & I_n \ (L_2 \ L_3) \ (U')^{-1} + I_n \end{pmatrix}, \tag{26} \]
where $(L_2 \ L_3) \ (I_n \ I_n)^t = I_n$. By Lemma 1 again, one can measure a set of $n$ Pauli operators $(I_n | I_n | 0_n \ 0_n)$ simultaneously and transform the GSF into
\[ \begin{pmatrix} U + I_n & U | 0_n & 0_n \\ 0_n & 0_n & I_n \ I_n \ 0_n \ 0_n \end{pmatrix}. \tag{27} \]

Meanwhile, measuring $(0_n | 0_n | L_2 \ L_3)$ will transfer the GSF of Eq. (27) into Eq. (26). Note that the measurement of $(0_n \ 0_n | L_2 \ L_3)$ is equivalent to measuring $(0_n \ 0_n | I_n \ (U')^{-1} + I_n)$.

Now, since the stabilizer generators in Eq. (27) are of the form $(I_n \ I_n | 0_n, 0_n)$, one can add $(U + I_n \ U + I_n | 0_n, 0_n)$ to the first row of Eq. (27), which equivalently reduces the GSF to:
\[ \begin{pmatrix} 0_n & I_n | 0_n \ 0_n & 0_n & I_n \ I_n \ 0_n \ 0_n \end{pmatrix}. \tag{28} \]

The final step is to eliminate the left-most $I_n$ in the second row of Eq. (28). This can be done by measuring $(0_n | 0_n | I_n \ 0_n)$ and adding the third row to the second. This will then transform the GSF into the second matrix in Eq. (10):
\[ \begin{pmatrix} 0_n & I_n | 0_n \ 0_n & 0_n & I_n \ 0_n \ 0_n \ 0_n \ I_n \ 0_n \ 0_n \end{pmatrix}. \tag{29} \]

Meanwhile, one can measure the set of $n$ Pauli operators $(I_n | I_n | 0_n, 0_n)$ to transform Eq. (29) to Eq. (28).

To reverse the whole procedure above and start from Eq. (29), we initially set $A_1, \ldots, A_n$ to $|0\rangle^\otimes n$ and perform the following three sets of Pauli measurements simultaneously:

1. $(I_n \ I_n | 0_n \ 0_n)$,
2. $(0_n \ 0_n | I_n \ (U')^{-1} + I_n)$,
3. $(I_n \ 0_n | 0_n \ 0_n)$.

The non-trivial measurements 1 and 2 need two $2n$-qubit CSS ancilla states (see Eq. (9)), which are
\[ |\Omega_{C_1}\rangle = \frac{1}{\sqrt{2^n}} \prod_{j=1}^{n} (I + X_j X_{j+n}) |0\rangle^\otimes 2n \]
and

$$|\Omega_{C_2}\rangle = \frac{1}{\sqrt{2^n}} \prod_{j=1}^{n} (I + Z^{u_j}) |+\rangle^{\otimes 2n},$$

(32)

where $u_j$ is the $j$th row of $(I_n^t (U_1^t)^{-1} + I_n^t)$. The first ancilla is actually an $n$-fold tensor product of Bell states. The second ancilla is the key resource state in our procedure to reduce the depth of -C- stage computation. Both of them are $2n$-qubit CSS states. The third step is a trivial bitwise measurement on $A_1, \ldots, A_n$ in the X basis and can be directly done without additional ancillas. The net effect is the desired -C- stage computation acting on $Q_1, \ldots, Q_n$, and the auxiliary qubits $A_1, \ldots, A_n$ are reset to $|+\rangle^{\otimes n}$ (up to $Z$ corrections). One can transfer $A_1, \ldots, A_n$ back into $|0\rangle^{\otimes n}$ or just keep them and start with $|+\rangle^{\otimes n}$ for the next stage. The procedure with $A_1, \ldots, A_n$ initially in $|+\rangle^{\otimes n}$ state for -C- stage is similar. As a conclusion, one has the following theorem:

**Theorem 2.** For a set of $2n$ qubits $A_1, \ldots, A_n, Q_1, \ldots, Q_n$, where $A_1, \ldots, A_n$ are initially in state $|0\rangle^{\otimes n}$ or $|+\rangle^{\otimes n}$, any -C- stage circuit containing only $C(j, l)$ with $j < l$ on $Q_1, \ldots, Q_n$ can be realized via three steps of Pauli measurements on these $2n$ qubits using two $2n$-qubit CSS states.

Note that the procedure to construct -C- stages via Pauli measurements in Theorem 2 also works with additional permutations on qubits $\{Q_1, \ldots, Q_n\}$. Thus, -C- stages with symplectic matrices of the form $\pi M C \pi^{-1}$ can also be computed using only three steps of measurements.

To implement an arbitrary Clifford circuit in the form of Eq. (12), one needs four -P- stages, four -C- stages, one -H- stage, and permutations of qubits (which can be done in software by keeping records). Overall, it requires five $4n$-qubit ancilla states of the form (9) and eight $2n$-qubit CSS states. Crucially, the ancilla states are all equivalent to CSS states up to single-qubit Clifford gate operations. This gives us the main result of the paper:

**Theorem 3.** Any Clifford circuit on $n$ qubits can be implemented up to a qubit permutation by $22$ steps of Pauli measurements, by introducing $n$ auxiliary qubits and preparing five $4n$-qubit stabilizer states and eight $2n$-qubit stabilizer state.

Like gate teleportation [15], this theorem implies that the gate complexity of the circuit is now completely dominated by the preparation of these CSS states. These resource states can be prepared using a stabilizer circuit of $O(n^2)$ gates with depth $O(n)$. On the other hand, note that these CSS states are equivalent to two-colorable graph states up to local Clifford operations. Thus, they can be approximated as non-degenerate ground states of two-body Hamiltonians [30, 31]. This fact may help to prepare these CSS states in an adiabatic manner.

Note that the total number of qubits used increases by a factor of five. The protocol can save the real computation time of stabilizer circuits by off-line preparation of CSS states.

**V. FAULT-TOLERANT ANCILLA STATES PREPARATION**

Quantum states are vulnerable to noise, which is the main obstacle to building large-scale quantum computers. The solution requires encoding the quantum state into some quantum error correcting code and performing fault-tolerant quantum computation; see [32] for details.

The Pauli measurement circuit in Fig. 1 is naturally compatible with Steane syndrome extraction when $2n$ qubits $A_1, \ldots, A_n, Q_1, \ldots, Q_n$ are encoded in an $[[n', k \geq 2n, d]]$ CSS code $Q$, while $4n$ additional qubits are also encoded into two blocks of the same code $Q$. That is to say, the syndrome measurements and Pauli measurements can be done simultaneously through the circuit in Fig. 1 at the logical level.

The ancillas states required in this paper, (16), (17), (31) and (32), are all CSS states up to single-qubit Clifford operations. Fortunately, when encoded in $Q$, these states can all be distilled fault-tolerantly using a structure based on classical error-correcting codes, with a constant overhead for the purpose of FTQC [22]. In such a scenario, one can fault-tolerantly compute Clifford circuits in $O(1)$ steps. In addition, the complexity of state preparation here is at the physical level rather than the logical level. Thus our method of Clifford circuit computation can have a speedup up by a factor of up to $O(d)$ compared to implementing the circuits directly on the data block in a code deformation manner.

**VI. DISCUSSION AND CONCLUSIONS**

In this paper, we proposed a method to compute Clifford circuits by a constant number of steps of Pauli measurement, which is suitable for Steane syndrome measurements. Consequently, the depth of the circuit is reduced to $O(1)$. It requires $n$ auxiliary qubits and preparation of five $4n$-qubit and eight $2n$-qubit CSS states.

The gate complexity is then completely transferred into the off-line preparation of these CSS states. The overall gate complexity, including the ancilla preparation, is $O(n^2)$ with a depth of $O(n)$. It seems initially at least as difficult as the direct implementation. However, preparing these known states is much simpler than implementing gate operations on unknown states. At the physical level, these states can be prepared via a gap-protected adiabatic process. When encoding in CSS codes, they can be prepared by distillation and post-selection, with a constant overhead in practice, which is especially interesting in the scenario of FTQC using multi-qubit CSS codes [20]. There, in general, one cannot find a fault-tolerant way to directly implement gates on logical qubits for a given multi-qubit CSS codes. By contrast, measurements of logical Pauli operators are very easy for any CSS codes by fault-tolerantly preparing logical CSS states (up to single-qubit Clifford operation) with only constant resource overhead [21, 22].

The method of this paper implies that the number of different encoded ancilla states can be reduced from $O(n^2 / \log n)$ to $O(1)$ for a given Clifford circuit, which can greatly simplify the fault-tolerant preparation procedure through high throughput distillation, if the number of Clifford circuits required is limited. This makes FTQC based on high rate multi-qubit error correcting codes more promising.
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