Telephone-cord instabilities in thin smectic capillaries

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Abstract

Telephone-cord patterns have been recently observed in smectic liquid crystal capillaries. In this paper we analyse the effects that may induce them. As long as the capillary keeps its linear shape, we show that a nonzero chiral cholesteric pitch favors the SmA*-SmC* transition. However, neither the cholesteric pitch nor the presence of an intrinsic bending stress are able to give rise to a curved capillary shape.

The key ingredient for the telephone-cord instability is spontaneous polarization. The free energy minimizer of a spontaneously polarized SmA* is attained on a planar capillary, characterized by a nonzero curvature. More interestingly, in the SmC* phase the combined effect of the molecular tilt and the spontaneous polarization pushes towards a helicoidal capillary shape, with nonzero curvature and torsion.

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1 Introduction

Telephone-cord instabilities in carbon films have been identified as processes which allow the material to relax its residual stress [1]. Recently, similar helicoid fibers have been observed in bent-shaped liquid crystals (the so-called banana liquid crystals) [2]. In these observations, the telephone-cord instability occurs in the smectic phase, and a central role is played by the spontaneous polarization, which characterizes the banana molecules [4]. In fact, although some nematics are polar liquids [5], the first well-known liquid crystals exhibiting significant spontaneous polarizations are smectic-C*. In this phase, the molecules are tilted with respect to the layer normal, and thus break the mirror symmetry [7].

The local polarization vector of the helical phases of a smectic-C* is perpendicular to both the director and the layer normal. Consequently, it is free to rotate in the plane of the layers giving a zero polarization average over one pitch. The electrooptic effects of the SmC* phase emerge with the unwinding of the helix by surface stabilization, and result in homogeneous spontaneous polarization throughout the sample. These homogeneous director states give rise to the ferroelectric SmC* phases. For a much more detailed description of the role of polarization in liquid crystals, we refer the reader to the book by Lagerwall [10], and more precisely to Sections 4.9-4.10, 5.4-5.6, 6.1-6.2, and 12.2-12.5 therein.

In this paper we analyze how polarization and chirality may influence the ground state shapes of thin filaments. Our main results deal with a smectic liquid crystal endowed with a nonzero spontaneous polarization. In the SmA* phase we find curved planar configurations (the capillary axis is bent with nonzero curvature) that have lower energy than straight ones. More interestingly, in the SmC* phase the ground state configuration has also a nonzero torsion. We derive analytical relations linking the curvature and the torsion of the ground state shapes to the material parameters.

Another interesting issue, stemming from experimental evidence on telephone cord instabilities and reflected in our results, is the fact that the mechanism for the capillary to decrease its energy is by bending and twisting. For other geometries, the mechanism for energy minimization may be the formation of domains [4][10]. However, domain formation is often coupled with the creation of energetically expensive boundary defects. The defect energy favors the changes in material geometry that we describe in our analysis.

The plan of the paper is as follows. In Section 2 we present and discuss the model and the free energy functional. Section 3 is devoted to linear capillary shapes: in it we show how a nonzero cholesteric pitch may anticipate the SmA-SmC transition. In Sections 4 and 5 we analyse the curved domains. In the former we prove that neither the cholesteric pitch nor the intrinsic bending stress are able to bend the axis of the smectic capillary. In the latter we determine the curved shapes induced by the spontaneous polarization: they are planar or three-dimensional depending on the SmA*-SmC* phase of the liquid crystal. In the concluding section we collect and discuss the above results.

2 Free energy functional

We consider a liquid crystal occupying a curvilinear cylinder $\Omega$. The domain is thus the set of points which lie within a maximum distance $r$ from a smooth curve $c : [0, \ell] \to \mathbb{R}^3$ (to be determined):

$$\Omega = \{ P \in \mathbb{R}^3 : P = c(s) + \xi e , \text{ for some } s \in [0, \ell], \xi \in [0, r], \text{ and } e \cdot e = 1 \}. \quad (2.1)$$

Let $N$, $B$ be the normal and binormal unit vectors on $c$ (the unit tangent $T$ completes an orthogonal basis), and let $\kappa, \tau$ denote the curvature and torsion along the same curve. Given a point $P \in \Omega$, the arc-length $s \in [0, \ell]$ identifies its projection on $c$, while $\xi \in [0, r]$ yields its distance from $c$. Let $\theta \in [0, 2\pi)$ be the angle that the unit vector $e$ in (2.1) determines with $N$.

We show in Appendix A1 that the coordinate set $(s, \xi, \theta)$ is well defined as long as $\Omega$ is sufficiently thin: $r < \min_{s \in [0, \ell]} \kappa^{-1}(s)$. 

According to the experimental conditions in which smectic helices have been observed \[2, 3\], we consider a freely suspended capillary, immersed in an isotropic fluid that does not interact with the surface director. Thus, free-boundary conditions will be imposed on both the nematic and the smectic variables. However, an anchoring energy will be necessary in order to take into account the surface director. Accordingly, we will assume throughout our calculations that all fields depend only on the arc-length \( s \).

### 2.1 Nematic energy

We introduce the angles \( \alpha, \varphi \) to identify the director orientation as

\[
\mathbf{n} = \cos \alpha \mathbf{T} + \sin \alpha \cos \varphi \mathbf{N} + \sin \alpha \sin \varphi \mathbf{B}.
\]

Let us also define for future use the unit-vectors

\[
\mathbf{n}_\perp := -\sin \varphi \mathbf{N} + \cos \varphi \mathbf{B} \quad \text{and} \quad \mathbf{n}_3 := -\sin \alpha \mathbf{T} + \cos \alpha \cos \varphi \mathbf{N} + \cos \alpha \sin \varphi \mathbf{B}.
\]

Together with \( \mathbf{n} \), they complete another orthogonal basis \(( \mathbf{n} \wedge \mathbf{n}_3 = \mathbf{n}_\perp ; \) we arbitrarily define \( \varphi = 0 \) when \( \alpha = 0 \). We have (see again Appendix A1 for technical details):

\[
\nabla \mathbf{n} = \left( \frac{\alpha' + \kappa \cos \varphi}{1 - \kappa \xi \cos \vartheta} \mathbf{n}_3 + \frac{(q - \tau) \sin \alpha - \kappa \cos \alpha \sin \varphi}{1 - \kappa \xi \cos \vartheta} \mathbf{n}_\perp \right) \otimes \mathbf{T},
\]

where a prime denotes differentiation with respect to the arc-length \( s \). The Frank free energy density is given by \[11\]:

\[
\sigma_F(\alpha, \varphi) = K_1 (\text{div} \, \mathbf{n})^2 + K_2 (\mathbf{n} \cdot \text{curl} \, \mathbf{n} + q_{ch})^2 + K_3 |\mathbf{n} \wedge \text{curl} \, \mathbf{n} + \mathbf{v}_0|^2
\]

\[
+ (K_2 + K_4) \left( \text{tr}(\nabla \mathbf{n})^2 - (\text{div} \, \mathbf{n})^2 \right)
\]

\[
= \frac{(\alpha' + \kappa \cos \varphi)^2}{(1 - \kappa \xi \cos \vartheta)^2} \left( K_1 \sin^2 \alpha + K_3 \cos^2 \alpha \right)
\]

\[
+ K_2 \left( q_{ch} - \frac{\sin \alpha}{1 - \kappa \xi \cos \vartheta} \left( (q - \tau) \sin \alpha - \kappa \cos \alpha \sin \varphi \right) \right)^2
\]

\[
+ K_3 \left( b_0 \sin \alpha - \frac{\tau \sin \alpha - \kappa \cos \alpha \sin \varphi}{1 - \kappa \xi \cos \vartheta} \right)^2,
\]

where \( q_{ch} \) is the cholesteric pitch and \( \mathbf{v}_0 = b_0 \mathbf{T} \wedge \mathbf{n} = b_0 \sin \alpha \mathbf{n}_\perp \) is the intrinsic bending stress.

### 2.2 Smectic energy

Let \( \psi(s) = \rho(s) e^{i\omega(s)} \) be the smectic order parameter \[11\], so that

\[
\nabla \psi = \frac{\rho' + i \omega'}{1 - \kappa \xi \cos \vartheta} \mathbf{T},
\]

and \( \mathbf{T} \) is also the normal to the smectic layers. The smectic part of the free energy density is given by:

\[
\sigma_{sm}[\rho, \omega, \alpha] = C_\perp |\nabla \psi - i q_{sm} \psi \mathbf{n}|^2 + C_a |\mathbf{n} \cdot (\nabla \psi - i q_{sm} \psi \mathbf{n})|^2 + \zeta(\rho)
\]

\[
= C_\perp \left[ \frac{\rho'^2}{(1 - \kappa \xi \cos \vartheta)^2} + \rho^2 \left( \frac{\omega'}{1 - \kappa \xi \cos \vartheta} - q_{sm} \cos \alpha \right)^2 + q_{sm}^2 \rho^2 \sin^2 \alpha \right]
\]

\[
+ C_a \left[ \frac{\rho^2 \cos^2 \alpha}{(1 - \kappa \xi \cos \vartheta)^2} + \rho^2 \left( \frac{\omega' \cos \alpha}{1 - \kappa \xi \cos \vartheta} - q_{sm} \right)^2 \right] + \zeta(\rho),
\]
where \( C_a := C \parallel - C \perp \), \( q_{sm} \) is the smectic pitch, and \( \zeta \) is a scalar potential depending on the degree of smectic order.

The smectic energy (2.2) does not rule out the possibility of the elastic constants \( C \parallel, C \perp \) being different (\( C_a \neq 0 \)). The \( C_a \)-term may be neglected when dealing with SmA materials \[12, 13\], but it is necessary to keep it in the free energy when tilted phases come into play. The SmA phase may become unstable when \( C \perp < 0 \) \[14\], and in that case a higher-order term should be included in the free energy to ensure the functional to be positive definite. However, we do not need to insert extra terms in the free energy, since we will prove that the transition to a tilted phase can be induced by the cholesteric pitch, even in the presence of a positive \( C \perp \). We remark that the free energy density (2.2) remains positive-defined even if \( C \perp \) is negative, provided \( \alpha \) is not too large. In fact, \( \sigma_{sm} \geq 0 \) whenever

\[
C_a \cos^2 \alpha + C \perp = C \parallel \cos^2 \alpha + C \perp \sin^2 \alpha > 0 \quad \text{i.e.} \quad C \perp \geq 0 \quad \text{or} \quad \tan^2 \alpha < -\frac{C \parallel}{C \perp}.
\]

### 2.3 Spontaneous polarization

One important difference between polar smectics and solids is the freedom of the polarization vector to rotate in the layer plane in the former (\( P \) is a Goldstone variable) as opposed to taking specific values determined by the solid lattice \[10, 15, 16\]. Because of this vectorial symmetry the energy density of the field \( P \) contains, together with a term of the form \( |\nabla P|^2 \) which penalizes interfaces in the material, a term proportional to \((\text{div } P)^2\):

\[
\sigma_{pol}[P] = G_1 (\text{div } P)^2 + G |\nabla P|^2 + \mathcal{G}(|P|).
\]  

In (2.3), \( G \) denotes a scalar potential which determines the polarization intensity \( |P| \). When the permanent molecular polarization is not sufficiently strong to self-interact, this term avoids the onset of a spontaneous polarization. This is why we will insert the potential (2.3) only in Section 5, when we will be dealing with spontaneously polarized materials.

A complete description of the polarization energy density can be found in \[17\]. We remark that, in materials with strong permanent polarization, the \( G_1 \)-term can also take the different form \((\text{div } P - c_0)^2\), where \( c_0 \) can be either positive or negative. This reflects the preference of the material for a specific sign of the polarization. However, in the following we will restrict our attention to the case \( c_0 = 0 \). We also neglect the nonlocal Coulombian interaction of the polarization with the self-field.

### 2.4 Anchoring energy

The presence of a nonzero polarization induces a surface charge in the capillary, which in turn requires an opposite charge layer in the surrounding fluid. This boundary effect can be taken into account through an effective anchoring energy, which depends on the polarization \[18, 19\]:

\[
\sigma_{anch}[P] = \omega_P P \cdot (1 - p \cdot \nu)
\]

where \( P \) and \( p \) respectively denote the intensity and the direction of the polarization vector \( P \), \( \nu \) is the outer normal at the external surface, and \( \omega_P \) is an effective anchoring strength. The anchoring potential above may favor either homeotropic or planar anchoring for the polarization vector, depending on the sign of \( \omega_P \).

### 3 Linear shapes

We first consider a linear smectic capillary, in the absence of spontaneous polarization. In this section we show that the presence of a non-zero cholesteric pitch may induce a SmA-SmC transition in the ground state configuration, even if \( C \perp > 0 \).
Let $\kappa = \tau \equiv 0$. The free energy density $\sigma = \sigma_F + \sigma_{sm}$ simplifies into

\[
\sigma[\alpha, \varphi, \rho, \omega] = (K_1 \sin^2 \alpha + K_3 \cos^2 \alpha) \alpha'^2 + K_2 \left(q_{ch} - \varphi' \sin^2 \alpha\right)^2 \\
+ K_3 \sin^2 \alpha \left(b_0 - \varphi' \cos \alpha\right)^2 + C_\alpha \left[\rho^2 \cos^2 \alpha + \rho^2 (\omega' \cos \alpha - q_{sm})^2\right] \\
+ C_\perp \left[\rho^2 + \rho^2 (\omega' - q_{sm} \cos \alpha)^2 + q_{sm}^2 \rho^2 \sin^2 \alpha\right] + \zeta(\rho).
\]

(3.1)

The Euler-Lagrange equations associated to the the free energy density (3.1), with respect to the variables $\varphi$ and $\omega$, can be easily integrated once to yield

\[
\frac{\partial \sigma}{\partial \varphi'} = c_1 \quad \text{and} \quad \frac{\partial \sigma}{\partial \omega'} = c_2,
\]

with $c_1$ and $c_2$ constants along the capillary. The free-boundary conditions require $c_1 = 0$ and $c_2 = 0$, and thus

\[
\varphi' = \frac{K_2 q_{ch} + K_3 b_0 \cos \alpha}{K_2 \sin^2 \alpha + K_3 \cos^2 \alpha} \quad \text{and} \quad \omega' = \frac{C_\parallel q_{sm} \cos \alpha}{C_\parallel \cos^2 \alpha + C_\perp \sin^2 \alpha}.
\]

Furthermore, the free energy density (3.1) is minimized if $\alpha' \equiv 0$ and $\rho' \equiv 0$ (which is allowed by the free-boundary conditions). When these requirements are satisfied, the free energy density depends only on the constant values of $\alpha$ and $\rho$:

\[
\sigma(\alpha_0, \rho_0) = \frac{K_2 K_3 (q_{ch} \cos \alpha_0 - b_0 \sin^2 \alpha_0)^2}{K_2 \sin^2 \alpha_0 + K_3 \cos^2 \alpha_0} + \frac{C_\parallel C_\perp \rho_0 q_{sm}^2 \sin^2 \alpha_0}{C_\parallel \cos^2 \alpha_0 + C_\perp \sin^2 \alpha_0} + \zeta(\rho_0).
\]

(3.2)

The smectic A phase ($\alpha_0 = 0$) is always associated to a stationarity point of (3.2). However, it becomes unstable even when $C_\perp > 0$, provided that

\[
C_\perp q_{sm}^2 \rho_0^2 < \frac{K_2}{K_3} q_{ch} (K_2 q_{ch} + 2b_0 K_3).
\]

(3.3)

In fact,

\[
\sigma(\alpha_0, \rho_0) = \sigma(0, \rho_0) + \left(C_\perp q_{sm}^2 \rho_0^2 - \frac{K_2 q_{ch} (K_2 q_{ch} + 2b_0 K_3)}{K_3}\right) \alpha_0^2 + O(\alpha_0^4) \quad \text{as} \quad \alpha_0 \to 0.
\]

Figure 4 shows that, when condition (3.3) applies, the preferred angle moves with continuity from $\alpha_0 = 0$. An exceptional situation arises when $b_0 = 0$ and $C_\perp = C_\parallel$ (bold plot of the right panel). In that case the optimal value of $\alpha_0$ jumps from 0 to $\frac{\pi}{2}$ when $q_{ch}$ exceeds $q_{sm}$. In all other cases, the SmA-SmC transition induced by the cholesteric pitch is second-order.

4 Bent domains

Let us now consider a general shape, with $\kappa, \tau \neq 0$, in the absence of spontaneous polarization. In this section, we prove that the combined effect of intrinsic bending stresses and/or chirality do not induce shape transitions towards curved domains.

The ground state configuration of the free energy density $\sigma := \sigma_F + \sigma_{sm}$ is still characterized
The minimum value of this energy is obtained when \( \kappa = 0 \). To prove this assertion we notice that \( f \) is monotonically increasing*. In particular, it is always greater than \( f(0) = 1 \). Furthermore, it is possible to write the free energy functional as:

\[
\mathcal{F} = A \int_0^\ell ds \left[ f(kr) \left(K_{13}A^2 + K_{23}\Phi^2 + \bar{C}\omega^2\right) - 2(K_{23}q_1\Phi + \bar{C}\omega) + K_{23}q_2^2 + C_{\parallel \parallel}q_0^2 \right].
\]

The Euler-Lagrange equation for \( \omega \) and the free-boundary conditions yield \( \omega' \equiv \omega_{\text{opt}}' = \frac{\bar{C}}{f(kr)} \).

If we insert it in the free energy, we arrive at

\[
\mathcal{F} = A \int_0^\ell ds \left( f(kr) \left(K_{13}A^2 + K_{23}\Phi^2 - 2K_{23}q_1\Phi + 2K_{23}q_2^2 - \frac{\bar{C}\omega^2}{f(kr)} + C_{\parallel \parallel}q_0^2 \right) \right).
\]

The minimum value of this energy is obtained when \( \kappa = 0 \). To prove this assertion we notice that \( f \) is monotonically increasing*. In particular, it is always greater than \( f(0) = 1 \). Furthermore, it is possible to write the free energy functional as:

\[
\mathcal{F} = A \int_0^\ell ds \left( f(kr)K_{13}A^2 + (f(kr) - 1)K_{23}\Phi^2 + \bar{C}\omega^2 \frac{f(kr) - 1}{f(kr)} + K_{23}(\Phi - q_1)^2 + \frac{K_{23}^2}{K_{23}} (q_{\text{ch}} \cos \alpha - b_0 \sin^2 \alpha)^2 + \frac{C_{\parallel \parallel}q_0^2 q_0^2}{C_{\parallel \parallel} \cos^2 \alpha + C_{\perp} \sin^2 \alpha} \right). \tag{4.1}
\]

All the terms depending on the curvature (that is, all terms appearing in the first row of (4.1)) are minimized if \( \kappa = 0 \), and thus the ground state shape of \( \Omega \) is linear. When this is the case, the search for the energy minimizer may proceed as in Section 3.

*The function \( f \) is monotonically increasing since \( f'(x) = 2 \left( 2 - x^2 - 2\sqrt{1-x^2} \right) / \left( x^3 \sqrt{1-x^2} \right) > 0 \) \( \forall x \in (0, 1) \).
5  Polarization-induced transitions

We now focus attention on spontaneously polarized liquid crystals. First, we insert in the free-energy functional the terms \( \sigma_{\text{pol}} \) and \( \sigma_{\text{anch}} \) introduced in Section 2. Furthermore, the intrinsic bending in the \( K_3 \)-term is to be replaced by a term \( \lambda \mathbf{P} \), proportional to the polarization vector. We identify \( \mathbf{P} \) through the angles \( \beta, \phi \) as follows:

\[
\mathbf{P} = P \mathbf{p} = P \left( \sin \beta \mathbf{n} + \cos \beta \cos \phi \mathbf{n}_\perp + \cos \beta \sin \phi \mathbf{n}_3 \right).
\]

We will show that these changes induce a spontaneous curvature in the shape of a smectic-\( A^* \) capillary, and both a curvature and a torsion in a smectic-\( C^* \).

5.1 Bent smectic-\( A^* \)

In a smectic-\( A \) the liquid crystal molecules are orthogonal to the layers. Let then \( \alpha \equiv 0 \). We assume that the potentials \( \zeta \) and \( G \) are strong enough to fix the values of \( \rho \equiv \text{const}., \rho_0 \) and \( P \equiv \text{const}., P_0 = \lambda_0 / \lambda, \) where \( \lambda_0 \) has the dimensions of an inverse length\(^1\). In order to simplify notations, we put \( \zeta(\rho_0) = G(P_0) = 0 \). Finally, we define the quantities \( \Gamma := G P_0^2 \) and \( \Gamma_1 := \kappa G_1 P_0^2 \), having the dimensions of nematic elastic constants.

The bulk free energy density \( \sigma_b = \sigma_F + \sigma_{\text{sm}} + \sigma_{\text{pol}} \) now reads as:

\[
\sigma_b = K_2 q_{\text{ch}}^2 + K_3 \left( \lambda_0 - \frac{\kappa}{1 - \kappa \xi \cos \vartheta} \right)^2 + \frac{2 \kappa \lambda_0}{1 - \kappa \xi \cos \vartheta} (1 - \cos \beta \sin \phi) + \frac{C_{\parallel P_0^2}}{1 - \kappa \xi \cos \vartheta} q_{\text{sm}}^2
+ \frac{(\beta' - \kappa \sin \phi)^2 \cos^2 \beta}{(1 - \kappa \xi \cos \vartheta)^2} + \frac{\Gamma_1}{\Gamma} \left( (\beta' - \kappa \sin \phi)^2 + \left( (\phi' + \tau) \cos \beta + \kappa \sin \beta \cos \phi \right)^2 \right)
+ \left( \frac{\phi'}{\phi''} + \frac{\cos \beta}{\cos \phi} \right) \frac{f(kr)}{f(kr)} - \frac{1}{f(kr)} \frac{\int \sigma_{\text{anch}} r (1 - kr \cos \vartheta) d\vartheta + \int \sigma_b \frac{\xi d\xi}{1 - \kappa \xi \cos \vartheta}}{\int \sigma_b \frac{\xi d\xi}{1 - \kappa \xi \cos \vartheta}} = A \left[ \frac{2 \omega_p P_0}{r} + \kappa \omega_p P_0 \cos \beta \sin \phi \right.

\]

To prove that the spontaneous polarization bends the smectic-\( A^* \), it suffices to find a curved configuration possessing a smaller free energy than the linear one. We begin by noticing that in the linear case (\( \kappa = \tau = 0 \) the free energy per unit length \( \mathcal{F} \) is minimized when \( \beta \) and \( \phi \) assume any constant value. When this is the case, the optimal value for the free energy is

\[
\mathcal{F}_{\text{opt}} \bigg|_{\kappa, \beta = 0} = A \ell \left( \frac{2 \omega_p P_0}{r} + K_2 q_{\text{ch}}^2 + K_3 \lambda_0^2 \right).
\]

\(^1\)The following arguments can be generalized to the case of non-uniform \( P, \rho \), but we skip those quite longer proofs to shorten our presentation.
We are looking for a configuration with a free energy lower than (5.2). To this aim, we focus on curves with constant curvature and torsion. Both the \( K \)-term and the anchoring energy are minimized if \( \mathbf{P} \) lies in the plane orthogonal to the layer normal (\( \beta = 0 \)). Furthermore, the free energy density is minimized if the polarization vector lies parallel or anti-parallel to the principal normal \( \mathbf{N} \), depending on whether \( K_3 \) is greater or smaller than \( \omega_p/(2\lambda) \). In the following, we assume that \( K_3 \geq \omega_p/(2\lambda) \). In this case the minimization process requires \( \mathbf{p} = \mathbf{N} \) (i.e., \( \phi = \frac{\pi}{2} \)). Nevertheless, the considerations below would stand in the case \( K_3 < \omega_p/(2\lambda) \), provided we choose \( \phi = -\frac{\pi}{2} \).

With the choices above, the free energy depends only on the particular values chosen by \( \kappa \) and \( \tau \), and can be written as

\[
\frac{F_{\text{opt}}(\kappa, \tau)}{A\ell} = A_0 - 2A_1\kappa r + A_2(\kappa r)^2 f(\kappa r) - \frac{A_3}{f(\kappa r)} + B_1 f(\kappa r)(\tau r)^2,
\]

where the signs are chosen in a way such that the \( A \)'s and \( B \)'s are non-negative:

\[
\begin{align*}
A_0 &= K_2\ell_{\text{ch}}^2 + K_3\lambda_0^2 + C_\parallel\rho_0 q_{\text{sm}}^2 + 2\omega_p P_0/r \\
A_1 &= \left(K_3 - \frac{\omega_p}{2\lambda}\right) \frac{\lambda_0}{r} \\
A_2 &= (K_3 + \Gamma + \Gamma_1)/r^2 \\
A_3 &= C_\parallel\rho_0 q_{\text{sm}}^2 \\
B_1 &= \Gamma/r^2.
\end{align*}
\]

The free energy \( F_{\text{opt}} \) is clearly minimized when \( \tau = 0 \) (plane curve). On the contrary, the minimum of \( F_{\text{opt}} \) is attained when the curvature has a strictly positive value, since

\[
\frac{F_{\text{opt}}(\kappa, 0)}{A\ell} = (A_0 - A_3) - 2A_1\kappa r + \left(A_2 + \frac{1}{3} A_3\right)(\kappa r)^2 + O(\kappa r)^4 \quad \text{as } \kappa r \to 0.
\]

We remark that \( F_{\text{opt}} \) possesses a unique minimum as a function of \( \kappa \). Indeed, the condition \( \frac{\partial}{\partial \kappa} F_{\text{opt}} = 0 \) is equivalent to

\[
A_2 \left(2(\kappa r)f(\kappa r) + (\kappa r)^2 f'(\kappa r)\right) + \frac{A_3f'(\kappa r)}{f^2(\kappa r)} = 2A_1
\]

and this equation has one and only one root, since the function at the left-hand side vanishes when \( \kappa r \to 0 \), is everywhere strictly increasing and diverges when \( \kappa r \to 1^- \). Let \( x := \kappa r \). Equation (5.3) can be written as

\[
2xf(x) + x^2f'(x) + \frac{\xi f'(x)}{f^2(x)} = \frac{\lambda_0}{\lambda_0^*},
\]

with

\[
\xi := \frac{A_3}{A_2} = \frac{C_\parallel\rho_0 q_{\text{sm}}^2 r^2}{K_3 + \Gamma + \Gamma_1} \quad \text{and} \quad \lambda_0^* := \frac{A_2\lambda_0}{2A_1} = \frac{K_3 + \Gamma + \Gamma_1}{(2K_3 - \omega_p/\lambda)r}.
\]

Figure 2 shows how the solutions of (5.3) depend on \( \lambda_0 \) (which is proportional to the intensity of the spontaneous polarization) for three different values of the dimensionless parameter \( \xi \). In the absence of spontaneous polarization the curvature is null. Then, it increases monotonically with \( \lambda_0 \). When the spontaneous polarization makes \( \lambda_0 \) much greater than its reference value \( \lambda_0^* \), the curvature approaches its maximum allowed value \( r^{-1} \). The curvature increases more rapidly when \( \xi \) is small, that is, in thinner capillaries.

### 5.2 Helicoidal smectic-C*

In this final section we study how the spontaneous polarization may induce a telephone-cord transition in a smectic-C* capillary. We focus on a particular, even if quite common, case. We
Figure 2: The preferred curvature of the axis of the smectic-A* capillary increases with the spontaneous polarization. The inverse length $\lambda_0$ is proportional to $|P|$, and its reference value $\lambda_0^*$ is defined in (5.5). From top to bottom, the graphs correspond to $\xi = 0, 1, 10$.

assume that the smectic part of the free energy is able to fix the opening angle of the smectic-C* cones to a fixed value: $\alpha \equiv \alpha_0$. Furthermore, we assume that the spontaneous polarization of the liquid crystal molecules is always orthogonal to the director direction: $\beta \equiv 0$, and determines a constant angle with respect to the principal normal of the capillary ($\phi \equiv \text{const}$). The former of these assumptions holds, for example, when we are analyzing a system of banana molecules, in which the polarization is induced by the curvature of the liquid crystal rods, and it is always orthogonal to $n$.

Even under the above simplifying assumptions, the bulk free energy density to be minimized is still quite cumbersome to handle:

$$
\sigma_b = \sigma_{sm}(\alpha_0) + K_1 \frac{\kappa^2 \cos^2 \varphi \sin^2 \alpha_0}{(1 - \kappa \xi \cos \vartheta)^2} + K_2 \left( q_{ch} - \frac{\sin \alpha_0}{1 - \kappa \xi \cos \vartheta} \left( (\varphi' - \tau) \sin \alpha_0 - \kappa \cos \alpha_0 \sin \varphi \right) \right)^2
+ \frac{\Gamma_1}{(1 - \kappa \xi \cos \vartheta)^2} \left[ - \left( (\varphi' - \tau) \sin \alpha_0 \cos \phi + \kappa \cos \varphi \sin \phi - \kappa \cos \alpha_0 \sin \varphi \cos \phi \right) \cos \alpha_0
+ \left( (\varphi' - \tau) \cos \alpha_0 + \kappa \sin \alpha_0 \sin \varphi \right) \cos \phi \sin \alpha_0 \right]^2
+ \frac{\Gamma}{(1 - \kappa \xi \cos \vartheta)^2} \left[ \left( (\varphi' - \tau) \sin \alpha_0 \cos \phi + \kappa \cos \varphi \sin \phi - \kappa \cos \alpha_0 \sin \varphi \cos \phi \right)^2
+ \left( (\varphi' - \tau) \cos \alpha_0 + \kappa \sin \alpha_0 \sin \varphi \right)^2 \right],
$$

where $\sigma_{sm}$ represents the smectic part, which fixes the value of $\alpha_0$. The above expression simplifies if we introduce the quantities $x := \{x_i, i = 1, 2, 3\}$, $A := \{A_{ij} : i, j = 1, 2, 3\}$, $b := \{b_i, i = 1, 2, 3\}$,
and \( c \in \mathbb{R} \), defined as:

\[
\begin{align*}
&x_1 := \frac{\kappa \cos \varphi}{1 - \kappa \xi \cos \theta}, \\
&x_2 := \frac{(\varphi' - \tau) \sin \alpha_0 - \kappa \cos \alpha_0 \sin \varphi}{1 - \kappa \xi \cos \theta}, \\
&x_3 := \frac{(\varphi' - \tau) \cos \alpha_0 + \kappa \sin \alpha_0 \sin \varphi}{1 - \kappa \xi \cos \theta}, \\
&A_{11} := K_1 \sin^2 \alpha_0 + K_3 \cos^2 \alpha_0 + (\Gamma + \Gamma_1 \cos^2 \alpha_0) \sin^2 \phi, \\
&A_{22} := K_2 \sin^2 \alpha_0 + K_3 \cos^2 \alpha_0 + (\Gamma + \Gamma_1 \cos^2 \alpha_0) \cos^2 \phi, \\
&A_{33} := \Gamma + \Gamma_1 \sin^2 \alpha_0 \cos^2 \phi, \\
&A_{12} = A_{21} := (\Gamma + \Gamma_1 \cos^2 \alpha_0) \sin \phi \cos \phi, \\
&A_{13} = A_{31} := -\Gamma \sin \alpha_0 \cos \alpha_0 \sin \phi \cos \phi, \\
&A_{23} = A_{32} := -\Gamma \sin \alpha_0 \cos \alpha_0 \cos^2 \phi, \\
&b_1 := K_3 \lambda_0 \sin \phi \cos \alpha_0, \quad b_2 := K_2 q_{ch} \sin \alpha_0 + K_3 \lambda_0 \cos \phi \cos \alpha_0, \quad b_3 := 0, \\
&c := \sigma_{sm}(\alpha_0) + K_2 q_{ch}^2 + K_3 \lambda_0^2
\end{align*}
\]

which allow to write

\[
\sigma_b = \mathbf{x} \cdot \mathbf{Ax} - 2 \mathbf{b} \cdot \mathbf{x} + c.
\]

When \( \beta = 0 \), the scalar product between the polarization direction and the outside normal to \( \partial \Omega \) is given by:

\[
\mathbf{p} \cdot \mathbf{v} = \cos \alpha_0 \sin \phi \cos(\theta - \varphi) + \cos \phi \sin(\theta - \varphi).
\]

Thus, the integration of the anchoring energy across the section orthogonal to the axis of the capillary yields:

\[
\int_0^{2\pi} \sigma_{anch} r (1 - \kappa r \cos \theta) d\theta = A \left[ 2 + \kappa r \left( \cos \alpha_0 \sin \phi \cos \varphi + \cos \phi \sin \varphi \right) \right] \frac{\omega_p P_0}{r}.
\]

We now specialize our study to the small curvature (or thin capillary) regime \( \kappa r \ll 1 \). In this case we can neglect the correction to 1 in the denominators of the \( x_i \)'s, and the integration of the bulk free energy density over the transverse section simply corresponds to a multiplication by \( A \). This allows us to derive an analytic expression for the free energy minimizer. In fact, in this case the total free energy can be written as:

\[
\frac{\mathcal{F}}{A} = \mathbf{x} \cdot \mathbf{Ax} - 2 \mathbf{b} \cdot \mathbf{x} + \tilde{c}, \quad (5.6)
\]

provided we define the \( \tilde{b}_i \)'s and \( \tilde{c} \) as follows:

\[
\begin{align*}
&\tilde{b}_1 := \left( K_3 - \frac{\omega_p}{2\lambda} \right) \lambda_0 \sin \phi \cos \alpha_0, \\
&\tilde{b}_2 := K_2 q_{ch} \sin \alpha_0 + \left( K_3 - \frac{\omega_p}{2\lambda} \right) \lambda_0 \cos \phi \cos \alpha_0, \\
&\tilde{b}_3 := \frac{\omega_p P_0}{2} \cos \phi \sin \alpha_0, \\
&\tilde{c} := \sigma_{sm}(\alpha_0) + K_2 q_{ch}^2 + K_3 \lambda_0^2 + \frac{2\omega_p P_0}{r}.
\end{align*}
\]

The functional (5.6) is minimized with respect to the possible values assumed by the \( x_i \)'s when

\[
\mathbf{x} = \mathbf{x}_{opt} := A^{-1} \mathbf{b}.
\]

(5.7)

(The symmetric matrix \( A \) is positive definite because of the positivity of the elastic free energy density). When this is the case, the free energy takes the value

\[
\frac{\mathcal{F}_{opt}}{A} = \hat{c} - \mathbf{b} \cdot A^{-1} \mathbf{b}. \quad (5.8)
\]
The $x_i$'s obtained from (5.7) fix the constant values of $\varphi$, $\kappa$, and $\tau$. Indeed, in the thin capillary limit $\kappa r \ll 1$, and setting $\varphi' = 0$, the definition of the $x_i$'s can be written as

\begin{align*}
x_{\text{opt}, 1} &= \kappa \cos \varphi \\
x_{\text{opt}, 2} &= -\tau \sin \alpha_0 - \kappa \cos \varphi \\
x_{\text{opt}, 3} &= -\tau \cos \alpha_0 + \kappa \sin \alpha_0 \sin \varphi,
\end{align*}

which can be inverted to obtain:

\begin{align*}
\kappa &= \sqrt{x_{\text{opt}, 1}^2 + (x_{\text{opt}, 3} \sin \alpha_0 - x_{\text{opt}, 2} \cos \alpha_0)^2} \\
\tau &= -\frac{x_{\text{opt}, 2} \sin \alpha_0 + x_{\text{opt}, 3} \cos \alpha_0}{x_{\text{opt}, 1}} \quad \text{and} \\
\varphi &= \arctan \frac{x_{\text{opt}, 3} \sin \alpha_0 - x_{\text{opt}, 2} \cos \alpha_0}{x_{\text{opt}, 1}}.
\end{align*}

Thus, at this stage, $F_{\text{opt}}$ in (5.8) still depends on the constant value attained by $\phi$, the angle that identifies the polarization direction. Only the minimization of $F_{\text{opt}}$ with respect to $\phi$ yields the complete description of the ground state configuration.

In order to illustrate the result of this minimization procedure, we conclude this section by analyzing in detail two particular cases. In both of them the optimal shape of the smectic capillary turns out to be a three-dimensional helix, characterized by non-null values of both the curvature and the torsion of its axis.

### 5.2.1 1-constant approximation

Let us consider the particular case in which

\begin{align*}
K_1 = K_2 = K_3 = \Gamma = \omega_P/\lambda =: K \quad \text{and} \quad \Gamma_1 = 0,
\end{align*}

while keeping the thin capillary regime $\lambda_0 r \ll 1$. The optimal shape of the capillary axis depends on the tilt angle $\alpha_0$ of the smectic-C* molecules and on the cholesteric pitch $q_{\text{ch}}$. Figure 3 illustrates the results. The right panel (displaying the torsion) proves the three-dimensional character of the capillary axis. The torsion is enhanced by the presence of a cholesteric pitch. However, a non-zero $q_{\text{ch}}$ is not a necessary ingredient to obtain three-dimensional shapes. In fact, if we add $q_{\text{ch}} = 0$ to (5.8), we can derive an analytical expression for the optimal shape for all values of $\alpha_0$:

\begin{align*}
\kappa_{\text{opt}}|_{q_{\text{ch}}=0} = \frac{3 \cos 2\alpha_0 - 1}{8} \lambda_0 \quad \text{and} \quad \tau_{\text{opt}}|_{q_{\text{ch}}=0} = -\frac{3}{8} \sin 2\alpha_0 \lambda_0.
\end{align*}

On the contrary, the role played by $\lambda_0$ (*i.e.*, the spontaneous polarization) is crucial. The ratio between either $\kappa$, $\tau$ and $\lambda_0$ is finite. Thus, both $\kappa$ and $\tau$ vanish when $\lambda_0$ does so. This observation is consistent with the results presented in Section 4, where we have proved that the optimal capillary shape is linear if the spontaneous polarization is null. Figure 3 is also coherent with the result derived in Section 5.1 for a smectic-A*: in the limit $\alpha \to 0$, the torsion vanishes while the curvature does not.

### 5.2.2 Small tilt angle

Let us now analyse in more detail the small-$\alpha_0$ limit. If the bend elastic constant prevails again over the effective anchoring ($K_1 \geq \omega_P/(2\lambda)$), we obtain:

\begin{align*}
\kappa_{\text{opt}} = \frac{K_3 - \omega_P/(2\lambda)}{K_3 + \Gamma + \Gamma_1} \lambda_0 + O(\alpha_0), \quad \tau_{\text{opt}} = -\frac{K_3}{\Gamma} \left(1 - \frac{K_3 - \omega_P/(2\lambda)}{K_3 + \Gamma + \Gamma_1}\right) \lambda_0 \alpha_0 + O(\alpha_0^2) \quad \text{as} \quad \alpha_0 \to 0.
\end{align*}

These results display the same qualitative features of the 1-constant solution analysed above. The preferred curvature becomes different from zero as soon as the spontaneous polarization appears,
even when $\alpha_0$ vanishes. On the contrary, the torsion vanishes when either $\lambda_0$ or $\alpha_0$ do so. However, a new and interesting result stems from the computation of the optimal free energy up to $O(\alpha^2)$. We obtain:

$$\frac{F_{\text{opt}}}{A\ell} = \sigma_{\text{sm}}(\alpha_0) + \left[ K_3 \lambda_0^2 - \frac{(K_3 - \frac{\omega P}{K_3})^2 \lambda_0^2}{K_3 + \Gamma + \Gamma_1} + K_2 q_{\text{ch}}^2 + \frac{2\omega P \lambda_0}{\lambda r} \right] - \frac{2K_2 (K_3 - \frac{\omega P}{K_3}) \lambda_0 q_{\text{ch}} \alpha_0}{K_3 + \Gamma + \Gamma_1} + O(\alpha_0^2).$$

(5.10)

The minus sign in front of the first-order term in $F_{\text{opt}}$ is crucial. It implies that it is possible to gain free energy by tilting the director with respect to the layer normal. This result holds even if $\sigma_{\text{sm}}(\alpha_0)$ pushes towards the smectic-A state, because in that case $\sigma_{\text{sm}}$ is minimum when $\alpha_0 = 0$, so that it does not contribute to the $O(\alpha_0)$-term we are discussing. The structure of the first-order term in $\alpha_0$ shows that this instabilization of the smectic-A* phase is a combined effect of both the spontaneous polarization and the cholesteric pitch. Once $\alpha_0$ becomes non-null, a non-zero value of the torsion becomes preferred and the ground-state configuration of the smectic-C* becomes helicoidal.

6 Discussion

The present theoretical study proves that telephone-cord instabilities are to be expected in smectic-C* liquid crystals. We have derived the ground state configurations and the preferred shapes of a thin smectic capillary, possibly endowed with spontaneous polarization. Having in mind the experimental conditions in which these instabilities have been already observed, we have imposed free boundary conditions at the external surface of the capillary for both the nematic and the smectic variables. Nevertheless, a boundary energy has been inserted in the free energy functional to take into account polarization effects on the surrounding liquid.

As long as the spontaneous polarization is absent, the preferred capillary shape remains linear, as we prove in Section 4. In this case, our analysis (Section 3) proves that a non-null cholesteric pitch may induce a SmA-SmC transition, even if the elastic constant $C_{\perp}$ is positive. Figure 1 shows how the optimal value of the tilt angle $\alpha$ depends on the cholesteric pitch for several different values of the elastic constants and the intrinsic bending stress.

In Section 5 we have focused on spontaneously polarized smectic liquid crystals. We have found evidence for possible circular smectic-A* and helicoidal smectic-C* capillaries. Figure 2 shows how the curvature of a smectic-A* capillary is expected to increase with the spontaneous polarization. Figure 3 displays both the curvature and the torsion as a function of the tilt angle, for several
For any differentiable real function \( \Psi : \mathbb{R} \rightarrow \mathbb{R} \), further, \( s \) depends only on the arc-length where a comma denotes differentiation with respect to the indicated variable. In particular, if \( \Psi \) coordinate system \((s, \xi, \vartheta)\), since \( r < 1 \) for all \( s \in [0, \ell] \), since \( \kappa \) is non-negative by construction. The volume of \( \Omega \) is:

\[
\text{Vol}(\Omega) = \int_0^\ell ds \int_0^r d\xi \int_0^{2\pi} d\vartheta \left( 1 - \kappa \xi \cos \vartheta \right) = \pi r^2 \ell.
\]

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Appendix A1: Cylindrical-curvilinear coordinates

Let \( \Omega \) be the domain defined in (2.1), \( \{T, N, B\} \) the intrinsic frame associated to \( c \), and \( (s, \xi, \vartheta) \) the coordinates introduced in (2.1). Let further \( e_\xi, e_\vartheta \) be the unit vectors defined as:

\[
e_\xi := \cos \vartheta N + \sin \vartheta B \quad \text{and} \quad e_\vartheta := -\sin \vartheta N + \cos \vartheta B.
\]

We thus obtain

\[
\dot{e}_\xi = \left( \dot{\vartheta} - \tau \right) e_\vartheta - \kappa \cos \vartheta T \quad \text{and} \quad \dot{e}_\vartheta = -\left( \dot{\vartheta} - \tau \right) e_\xi + \kappa \sin \vartheta T.
\]

For any differentiable real function \( \Psi : \mathbb{R}^3 \rightarrow \mathbb{R} \) we have:

\[
\nabla \Psi = \frac{\Psi_s + \tau \Psi_\vartheta}{1 - \kappa \xi \cos \vartheta} T + \frac{\Psi_\xi}{\xi} e_\xi + \frac{\Psi_\vartheta}{\xi} e_\vartheta,
\]

where a comma denotes differentiation with respect to the indicated variable. In particular, if \( \Theta \) depends only on the arc-length \( s \), \( \nabla \Theta(s) = \frac{1}{1 - \kappa \xi \cos \vartheta} \frac{d\Theta}{ds} T \).

Furthermore,

\[
\nabla T = \frac{\kappa \cos \vartheta}{1 - \kappa \xi \cos \vartheta} e_\xi \otimes T - \frac{\kappa \sin \vartheta}{1 - \kappa \xi \cos \vartheta} e_\vartheta \otimes T,
\]

\[
\nabla e_\xi = -\frac{\kappa \cos \vartheta}{1 - \kappa \xi \cos \vartheta} T \otimes T + \frac{1}{\xi} e_\vartheta \otimes e_\vartheta,
\]

\[
\nabla e_\vartheta = \frac{\kappa \sin \vartheta}{1 - \kappa \xi \cos \vartheta} T \otimes T - \frac{1}{\xi} e_\xi \otimes e_\vartheta,
\]

\[
\nabla B = \frac{\tau}{1 - \kappa \xi \cos \vartheta} N \otimes T \quad \text{and}
\]

\[
\nabla N = -\frac{\kappa}{1 - \kappa \xi \cos \vartheta} T \otimes T - \frac{\tau}{1 - \kappa \xi \cos \vartheta} B \otimes T.
\]

The volume element in \( \Omega \) is given by \( dv = \xi |1 - \kappa \xi \cos \vartheta| \, ds \, d\xi \, d\vartheta \), so that the curvilinear coordinate system \((s, \xi, \vartheta)\) is well defined as long as \(|1 - \kappa \xi| > 0\), which implies \( \kappa(s) \, r < 1 \) for all \( s \in [0, \ell] \), since \( \kappa \) is non-negative by construction. The volume of \( \Omega \) is:

\[
\text{Vol}(\Omega) = \int_0^\ell ds \int_0^r d\xi \int_0^{2\pi} d\vartheta (1 - \kappa \xi \cos \vartheta) = \pi r^2 \ell.
\]
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