NORMAL SYSTEMS OVER ANR’S, RIGID EMBEDDINGS AND NONSEPARABLE ABSORBING SETS

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Abstract. Most of results of Bestvina and Mogilski [Characterizing certain incomplete infinite-dimensional absolute retracts, Michigan Math. J. 33 (1986), 291–313] on strong Z-sets in ANR’s and absorbing sets is generalized to nonseparable case. It is shown that if an ANR $X$ is locally homotopy dense embeddable in infinite-dimensional Hilbert manifolds and $w(U) = w(X)$ (where $'w'$ is the topological weight) for each open nonempty subset $U$ of $X$, then $X$ itself is homotopy dense embeddable in a Hilbert manifold. It is also demonstrated that whenever $X$ is an AR, its weak product $W(X, *) = \{ (x_n)_{n=1}^\infty \in X^\omega : x_n = * \text{ for almost all } n \}$ is homeomorphic to a pre-Hilbert space $E$ with $E \cong \Sigma E$. An intrinsic characterization of manifolds modelled on such pre-Hilbert spaces is given.

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In [28] Henderson has shown that a Z-set in a paracompact manifold $M$ modelled on a metrizable locally convex topological vector space $F$ such that $F^\omega \cong F$ is a strong Z-set in $M$. This result was used by Chapman [14] to generalize the results of Anderson and McCharen [3] on extending homeomorphisms between Z-sets of a manifold modelled on an infinite-dimensional Fréchet space. The homeomorphism extension theorem was applied in Toruńczyk’s original proof [44, 45] that every Fréchet space is homeomorphic to a Hilbert space. In his proof also strong Z-sets play important role. In the meantime it turned out that these sets are more applicable in the theory of incomplete ANR’s than Z-sets. With use of strong Z-sets several infinite-dimensional AR’s have been characterized, see e.g. [10], [6], [11], [17, 18], [34]. Strong Z-sets are therefore an important tool in studying ANR’s. We present here several theorems on strong Z-sets in (nonseparable) ANR’s which, in particular, generalize the results of Henderson [28] and Bestvina and Mogilski [10] and we use them to generalize most important facts on absorbing sets due to the latter authors. In their exposition and in [6] the second axiom of countability plays an important role and one may suggest that it is the point. In case of nonseparable ANR’s one has to
use different methods to prove that, e.g., being a strong Z-set is a local property. We show this by means of so-called normal systems, which turn out to sum up common features of the notion of a strong Z-set and the well-known strong discrete approximation property (and similar ones characterizing nonseparable Hilbert manifolds [44]). With use of normal systems and the so-called small maps approximation property (which discovers the hidden nature of normal systems), in short: SMAP, we show that if an ANR $X$ is locally homotopy dense embeddable in Hilbert manifolds and all its nonempty open subsets have the same topological weight, then $X$ itself is homotopy dense embeddable in a Hilbert manifold as well. SMAP for normal systems also enables us to shorten Toruńczyk’s original proof of the Hilbert space manifold characterization theorem, namely: Toruńczyk [44, Proof of 3.2, p. 256] in the final part of the proof of the theorem characterizing separable Hilbert manifolds among complete ANR’s by means of the strong discrete approximation property (briefly, SDAP; cf. [2]) argued that if a separable complete ANR $X$ has SDAP, then $X$ is locally a Hilbert manifold. However, he gave no explanation why SDAP is open hereditary, that is, if $X$ has SDAP, then all its open subsets also have SDAP (note that the limitation topology of $C(D, U)$ does not coincide with the topology of a subspace induced by the limitation topology of $C(D, X)$ if $U$ is open in $X$). We shall easily see, thanks to SMAP for normal systems, that Toruńczyk’s condition [44, (∗2), p. 253] is implied by SDAP.

The problem of investigation of nonseparable absorbing spaces was mentioned in the seminal paper [20] that greatly stimulated the development of the classical (separable) theory of absorbing spaces. So, in a sense, the recent paper resolves an old problem posed in the known list of problems [20]. Partial results in this direction were also obtained in 2003 by Sakai and Yaguchi [34].

Other topic, discussed in the paper, is related to the problem of classification of the weak products of AR’s (or, equivalently, absorbing sets for topological, closed hereditary, additive classes $\mathcal{C}$ such that $C_1 \times C_2 \in \mathcal{C}$ for all $C_1, C_2 \in \mathcal{C}$). We introduce rigid embeddings into normed spaces and by means of them we prove the main result of the paper which is new even in separable case and states that the weak product (defined in Abstract) of an arbitrary AR is homeomorphic to a pre-Hilbert space $E$ such that $E \cong \Sigma E$. This show that Corollary 5.4 of [10] which naturally generalizes Toruńczyk’s Factor Theorem [41, 42] (cf. [19]) is in fact equivalent to it. Finally, we give an intrinsic characterization of all nonzero pre-Hilbert spaces $E$ with $E \cong \Sigma E$: a metrizable space $X$ is homeomorphic to such a space iff $X$ is an AR and a $\sigma$-Z-space such that for each Z-set $K$ in $X$ the natural projection $(X \setminus K) \times X \to X \setminus K$ is a near-homeomorphism (i.e. it is approximable, in the limitation topology, by homeomorphisms).
1. Preliminaries

In this paper \(\mathbb{N}, I\) and \(Q\) denote the set of all nonnegative integers, the unit interval \([0, 1]\) and the Hilbert cube (i.e. \(Q \cong I^{\omega}\)), respectively. The letters \(X, Y, Z, K\), etc. stand for topological spaces. Following Banakh and Zarichnyy [7], we identify cardinals with the sets of ordinals of smaller size and endow them with the discrete topologies. By an ANR we mean a metrizable space which is an absolute neighbourhood retract for metrizable spaces. Compact and paracompact spaces are meant to be Hausdorff, in the opposite to normal spaces which are understood by us as having the property of separating closed disjoint sets. We write \(Y \cong Z\) iff \(Y\) and \(Z\) are homeomorphic. \(X^\omega\) stands for the countable infinite Cartesian power of \(X\), equipped with the Tichonov topology, and \(\text{cov}(X)\) is used to denote the collection of all open covers of \(X\). By a map we mean a continuous function. Whenever \(g\) is a map, \(\text{im } g\) and \(\overline{\text{im } g}\) stand for, respectively, the image of \(g\) and its closure. If \(A\) is a subset of \(X\), \(\text{int } A\) and \(\overline{A}\) denote the interior and the closure of \(A\) in the whole space \(X\). We use \(w(X)\) to denote the topological weight of \(X\).

If \(Y\) is paracompact, the space \(\mathcal{C}(X, Y)\) of all maps of \(X\) into \(Y\) in this paper is always equipped with the limitation topology. For definition and basic properties of this topology the Reader is referred to [44], [12]. The symbol \(B(f, \mathcal{U})\) (with \(f \in \mathcal{C}(X, Y)\) and \(\mathcal{U} \in \text{cov}(Y)\)) has the same meaning as in [44] and \(B(f, \mathcal{U})\) consists of all maps of \(X\) to \(Y\) which are \(\mathcal{U}\)-close to \(f\).

In the sequel we shall make use of the following powerful result.

1.1. **Lemma** (Michael [32], cf. [9, Proposition 4.1]). Let \(X\) be a paracompact space and \(\mathcal{W}\) a collection of some subsets of \(X\) which satisfies the following three conditions:

1. **(M1)** If \(A \in \mathcal{W}\) and \(U\) is an open subset of \(X\) contained in \(A\), then \(U \in \mathcal{W}\).
2. **(M2)** If \(U_1\) and \(U_2\) are open subsets of \(X\) and \(U_1, U_2 \in \mathcal{W}\), then \(U_1 \cup U_2 \in \mathcal{W}\).
3. **(M3)** If \(\{U_s\}_{s \in S}\) is a discrete (in \(X\)) collection of open subsets of \(X\) each of which is a member of \(\mathcal{W}\), then \(\bigcup_{s \in S} U_s \in \mathcal{W}\).

Then, \(X \in \mathcal{W}\) provided for every point \(a\) of \(X\) there is \(A \in \mathcal{W}\) such that \(a \in \text{int } A\).

For simplicity, every family (of subsets of a given topological space) which satisfies the properties (M1)–(M3) we call a *Michael collection*.

Recall that a space \(E\) is said to be a *neighbourhood extensor for a space \(X\)* iff every map from any closed subset of \(X\) into \(E\) is extendable to a map defined on some open subset of \(X\) (and with values in \(E\)). If \(E\) is a neighbourhood extensor for \(X\), every open subset of \(E\) is a neighbourhood extensor for each closed subset of \(X\), and \(X\) is normal.
provided $E$ is Hausdorff and has more than one point. A space $Y$ is called locally equiconnected (in short: LEC) iff there is an open in $Y \times Y$ neighbourhood $\Omega$ of the diagonal $\Delta_Y = \{(y, y): y \in Y\}$ and a map $\lambda: \Omega \times I \to Y$ such that $\lambda(y, y, t) = y$, $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ for each $(x, y) \in \Omega$ and $t \in I$. Such a map is called an equiconnecting function ([24]). Every ANR is LEC and there are examples of separable completely metrizable LEC spaces which are not ANR’s (see e.g. [13]). However, each LEC space is locally contractible and finite dimensional locally contractible metrizable spaces are ANR’s ([23]).

In the next section we shall need the following two properties of neighbourhood extensors, the proofs of which are left as exercises.

1.2. Lemma. Let $X$ and $Y$ be normal spaces such that $Y$ is a neighbourhood extensor for $X$ and let $A$ be a closed subset of $X$.

(A) If $f: A \to Y$ is a map such that $\text{im } f \subset V$ where $V$ is an open in $Y$ set contractible in its open neighbourhood $U \supset V$, then $f$ is extendable to a map of $X$ into $U$.

(B) (cf. [44, Lemma 1.3]) If, additionally, $Y$ is a paracompact LEC space, then the map $C(X, Y) \ni u \mapsto u|_A \in C(A, Y)$ is open.

Whenever we talk about the (topological) dimension, we mean the covering one. If $U$ is a family of subsets of a space $X$, $\text{ord}(U)$ stands for the order of $U$ and it is understood as a natural number or $\infty$. We say that $X$ is of finite-dimensional type (briefly, FDT) iff every open cover of $X$ (not necessarily finite) has a refinement (in $\text{cov}(X)$) of finite order. $X$ is said to be locally FDT if every point of $X$ has a (not necessarily open) neighbourhood which is FDT. Important examples of FDT [locally FDT] spaces are [locally] compact ones.

We denote by $\text{comp}(X)$ the least infinite cardinal $\alpha$ such that every open cover of $X$ has a subcover of cardinality less than $\alpha$. Similarly, $\text{comp}_l(X)$ is the least infinite cardinal $\alpha$ such that every point of $X$ has a (not necessarily open) neighbourhood $F$ such that $\text{comp}(F) \leq \alpha$. (Observe that $\text{comp}(X) = \aleph_0$ [comp$_l(X) = \aleph_0$] iff $X$ is [locally] compact.) The proofs of the following results are omitted. (Recall that a discrete subset of a topological space is a closed set whose topology is discrete.)

1.3. Lemma. Let $X$ be a paracompact space such that the set $X$ is infinite.

(I) $\text{comp}(X)$ is the least cardinal $\alpha$ with the following property: for every locally finite open cover $\{U_s\}_{s \in S}$ of $X$ consisting of non-empty sets, $\text{card } S < \alpha$.

(II) $\text{comp}(X)$ is the least cardinal $\alpha$ such that every discrete subset of $X$ has cardinality less than $\alpha$.

(III) If $X$ is metrizable, then either $X$ contains a discrete subset of cardinality $\omega(X)$ and then $\text{comp}(X)$ is the direct successor of
w(X), or each discrete subset of X is of cardinality less than w(X) and then \( \text{comp}(X) = w(X) \). What is more, in the second case there is a sequence of cardinals \( \alpha_0 < \alpha_1 < \ldots \) such that \( w(X) = \sup_{n\in\mathbb{N}} \alpha_n \).

1.4. **Lemma.** If \( X \) is metrizable and contains a closed set homeomorphic to \( X \times \mathbb{N} \), then \( X \) has a discrete subset of cardinality \( w(X) \).

We shall also involve some properties (the same as were used in [44]) of simplicial complexes with Whitehead’s weak topologies or the metric ones. By a (combinatorial) simplicial complex \( \mathcal{K} \) we shall always mean a complex whose vertices form an orthonormal system in some Hilbert space \( \mathcal{H} \), and its geometric realization \( |\mathcal{K}| \) will always be identified with the suitable subset of \( \mathcal{H} \). If \( |\mathcal{K}| \) is equipped with the weak topology, we shall write \( |\mathcal{K}|_w \). If it is equipped with the metric topology induced from the topology of \( \mathcal{H} \), we shall write \( |\mathcal{K}|_m \). The map \( |\mathcal{K}|_w \ni x \mapsto x \in |\mathcal{K}|_m \) is denoted by \( j_\mathcal{K} \). Adapting Toruńczyk’s proof of [44, Lemma 3.4] (cf. [21, Proof of Lemma 3.2]) one may show that

1.5. **Lemma.** Let \( X \) be a normal space, \( Y \) be an ANR, \( V \) an open subset of \( Y \), \( f : X \to V \) a map and let \( \mathcal{U} \in \text{cov}(Y) \). There is a simplicial complex \( \mathcal{K} \) and maps \( v : X \to |\mathcal{K}|_w \) and \( w : |\mathcal{K}|_m \to Y \) such that \( wj_\mathcal{K}v \in B(f,\mathcal{U}) \) and

- (SC1) \( \mathcal{K} \) is locally finite dimensional,
- (SC2) \( \mathcal{K} \) has less than \( \min(\text{comp}(X),\text{comp}(\bar{V})) \) vertices,
- (SC3) \( \dim \mathcal{K} \leq \min(\dim(X),\dim(\bar{V})) \),
- (SC4) \( \mathcal{K} \) is finite dimensional provided \( X \) is FDT or \( \bar{V} \) is FDT,
- (SC5) \( \mathcal{K} \) is locally finite provided \( V \) is separable.

For more information on simplicial complexes see e.g. [48], [22, 21], [31] or [9, II.§6].

2. **Small maps approximation property**

We begin with

2.1. **Definition.** For a subset \( B \) of a metrizable space \( Y \), let \( S_Y(X,B) \) be the collection of all maps \( g : X \to Y \) such that \( \overline{\text{im}} \ g \subset B \). Note that if \( B \) is open in \( Y \), \( S_Y(X,B) \) is open in \( \mathcal{C}(X,Y) \). Similarly, if \( \mathcal{B} \) is a family of subsets of \( Y \), \( S_Y(X,\mathcal{B}) \) stands for the union of all \( S_Y(X,B) \) with \( B \in \mathcal{B} \). The members of \( S_Y(X,\mathcal{B}) \) are said to be \( \mathcal{B} \)-small (in \( Y \)) maps. We write \( \mathcal{B} \)-small (in \( Y \)) instead of \( \{B\} \)-small. (It would be more appropriate to say ‘strongly small’.)

2.2. **Definition.** A subset \( D \) of \( \mathcal{C}(X,Y) \) (with paracompact \( Y \)) is said have small maps approximation property (in short: SMAP) iff there is \( \mathcal{U} \in \text{cov}(Y) \) such that \( S_Y(X,\mathcal{U}) \subset D \) (the closure taken in the limitation topology of \( \mathcal{C}(X,Y) \)).
We call a class \( \mathcal{T} \) of topological spaces closed hereditary (respectively open hereditary) if \( A \in \mathcal{T} \) for every closed (open) subset \( A \) of any member of \( \mathcal{T} \).

Utility of SMAP is explained in the following

2.3. Theorem. Let \( Y \) be a paracompact LEC space which is a neighbourhood extensor for a space \( X \). Let \( \mathcal{T} \) be the family of all closed subsets of \( X \). Suppose that \( \{ D_A \}_{A \in \mathcal{T}} \) is a collection such that

\[(D0) \text{ For each } A \in \mathcal{T}, D_A \text{ is an open subset of } C(A,Y).\]

\[(D1) \text{ If } B \in \mathcal{T}, A \text{ is a closed subset of } B \text{ and } g \in D_B, \text{ then } g\big|_A \in D_A.\]

\[(D2) \text{ If } A \in \mathcal{T} \text{ is the union of its two closed subsets } A_1 \text{ and } A_2 \text{ and } g \in C(A,Y) \text{ is such that } g\big|_{A_1} \in D_{A_1} \text{ for } j = 1, 2, \text{ then } g \in D_A.\]

\[(D3) \text{ If } A \in \mathcal{T} \text{ is the union of a discrete (in } A \text{) family } \{ A_i \}_{i \in I} \text{ of its closed subsets and } g \in C(A,Y) \text{ is such that } g\big|_{A_i} \in D_{A_i} \text{ for each } i \in I \text{ and the family } \{ g(A_i) \}_{i \in I} \text{ is discrete in } Y, \text{ then } g \in D_A.\]

Then the following conditions are equivalent:

(i) each \( D_A \) is a dense subset of \( C(A,Y) \),

(ii) \( D_X \) has SMAP.

Proof. We may assume that \( Y \) has more than one point. This implies that \( X \) is normal. Basically, we only need to show that (i) is implied by (ii).

Let \( W \) be the family of all open subsets of \( Y \) such that \( S_Y(A,U) \subset D_A \) for each \( A \in \mathcal{T} \). We shall show that \( W \) is a Michael collection. The point (M1) is clearly fulfilled and (M3) is left as an exercise. We pass to (M2). Let \( U_1, U_2 \in W, U = U_1 \cup U_2, A \in \mathcal{T}, f: A \to Y \) be \( U \)-small in \( Y \) and let \( V \in \text{cov}(Y) \). Take a star refinement \( G \in \text{cov}(Y) \) of \( V \). Let \( U^* \) be such an open subset of \( Y \) that \( \text{im} f \subset U^* \) and \( \overline{U^*} \subset U \). Since the sets \( U^* \setminus U_1 \) and \( U^* \setminus U_2 \) are closed and disjoint, there are two open sets \( U_1^* \) and \( U_2^* \) for which \( \overline{U^*} \setminus U_j \subset U_j^* \) \((j = 1, 2)\)

\[(2-1) \quad \overline{U_1^*} \cap \overline{U_2^*} = \emptyset.\]

Put \( B_j = U^* \setminus U_j^* \) \((j = 1, 2)\) and note that \( B_1 \) and \( B_2 \) are closed subsets of \( Y \) such that \( \text{im} f \subset B_1 \cup B_2 \) and \( B_j \subset U_j \) \((j = 1, 2)\). Further, take an open set \( U_0 \) for which \( B_2 \subset U_0 \) and \( \overline{U_0} \subset U_2 \). Now put \( A_j = f^{-1}(B_j) \in \mathcal{T}, f_1 = f|_{A_1} \to Y \) and \( G_1 = \{ G \cap U_0: G \in \mathcal{G} \} \cup \{ G \setminus B_2: G \in \mathcal{G} \} \in \text{cov}(Y) \). Observe that (by (2-1))

\[(2-2) \quad A \setminus A_1 \cap A \setminus A_2 = \emptyset.\]

By Lemma 1.2–(B), there is \( \mathcal{G}_1 \subset \text{cov}(Y) \) such that

\[(2-3) \quad B(f_1, \mathcal{G}_1 \setminus A_1, f, \mathcal{G}_1) \subset B(f_1, \mathcal{G}_1 \setminus A_1, f).\]

that is, for each \( \mathcal{G}_1 \)-close to \( f_1 \) map \( h_1: A_1 \to Y \) there is a \( \mathcal{G}_1 \)-close to \( f \) map \( h: A \to Y \) which extends \( h_1 \). Since \( U_1 \in W \) and \( \text{im} f_1 \subset U_1 \),
there is \( g'_1 \in D_{A_1} \) which is \( G'_1 \)-close to \( f_1 \). Thanks to (2-3) we may find \( g_1 \in C(A, Y) \) which is \( G_1 \)-close to \( f \) and extends \( g_1 \). This yields that
\[
(2-4) \quad g_1 \in B(f, G_1), \quad g_1 \big|_{A_1} \in D_{A_1}.
\]
Now put \( g'_2 = g_1 \big|_{A_1} \). By (2-4) and (D1), \( g'_2 \big|_{A_1 \cap A_2} \in D_{A_1 \cap A_2} \). We conclude from (D0) and the continuity of the map \( C(A_2, Y) \ni u \mapsto u \big|_{A_1 \cap A_2} \in C(A_1 \cap A_2, Y) \) that there is a refinement \( G_2 \in \text{cov}(Y) \) of \( G \) such that
\[
(2-5) \quad B(g'_2, G_2) \subset \{ h \in C(A_2, Y) : h \big|_{A_1 \cap A_2} \in D_{A_1 \cap A_2} \}.
\]
Let \( \lambda : \Omega \times I \to Y \) be an equiconnecting function (with \( \Omega \subset Y \times Y \)). Take a cover \( G'_2 \) of \( Y \) such that for each \( G'' \in G'_2 \),
\[
(2-6) \quad G' \times G' \subset \Omega \text{ and there is } G \in G_2 \text{ for which } \lambda(G' \times G' \times I) \subset G.
\]
(Notice that this implies that \( G'_2 \) refines \( G_2 \).) We infer from (2-4) and the definition of \( G_1 \) that \( \text{im } g'_2 \subset U_0 \) and thus \( g'_2 \) is \( U_2 \)-small in \( Y \). Since \( U_2 \) is a member of \( W \), there is a \( G'_2 \)-close to \( g'_2 \) map \( g_2 \in D_{A_2} \). Now using (2-2) and the assumption that \( X \) is normal take a map \( \mu : A \to I \) such that
\[
(2-7) \quad \mu \big|_{A \setminus A_j} \equiv 1 \quad \text{and} \quad \mu \big|_{A_j \setminus A_1} \equiv 0
\]
and define \( g : A \to Y \) as follows:
\[
g \big|_{A \setminus A_j} = g_2 \big|_{A \setminus A_j} \quad g \big|_{A_j \setminus A_1} = g_1 \big|_{A_j \setminus A_1}
\]
and \( g(a) = \lambda(g_1(a), g_2(a), \mu(a)) \) for \( a \in A_1 \cap A_2 \). (Note that the last formula makes sense because of (2-6) and the fact that \( g_2 \) is \( G'_2 \)-close to \( g_1 \big|_{A_2} \).) Thanks to (2-7), \( g \) is a well defined continuous function. What is more, by (D1) we have
\[
(2-8) \quad g \big|_{A \setminus A_j} \in D_{A \setminus A_j} \quad (j = 1, 2).
\]
Further, we conclude from (2-6) that
\[
(2-9) \quad g \big|_{A_2} \in B(g'_2, G_2).
\]
This, combined with (2-4) and the facts that \( G_2 \) refines \( G \) and \( G \) is a star refinement of \( V \), gives \( g \in B(f, V) \). So, to prove (M2), it suffices to show that \( g \in D_A \). But this follows from (2-9), (2-5), (2-8) and (D2).

We have shown that \( W \) is a Michael collection. Therefore, to prove that \( D_X \) is dense, it is enough (thanks to Lemma 1.1) to show that there is \( V \in \text{cov}(Y) \) such that \( V \subset W \). But LEC spaces are locally contractible and thus if \( U \in \text{cov}(Y) \) is such that
\[
(2-10) \quad S_Y(X, U) \subset D_X,
\]
there are \( V, D \in \text{cov}(Y) \) such that the family \( \{ D : D \in D \} \) refines \( U \) and each member of \( V \) is contractible in some element of \( D \). Now Lemma 1.2, (2-10) and (D1) imply that \( S_Y(A, V) \subset D_A \) for each \( A \in T \), which means that \( V \subset W \).
Finally, if $A \in \mathcal{T}$, then $Y$ is a neighbourhood extensor for $A$ (and $A$ is normal). Hence, by the above argument, $D_A$ is dense iff it has SMAP. But we have shown that $\mathcal{V} \subset \mathcal{W}$ for some $\mathcal{V} \in \text{cov}(Y)$, which gives SMAP for every closed subset of $X$.

2.4. **Definition.** Let $\mathcal{T}$ be a closed hereditary class of topological spaces and $Y$ be a paracompact space. A class $\{D_A\}_{A \in \mathcal{T}}$ which satisfies the conditions (D0)–(D3) appearing in the statement of Theorem 2.3 is said to be a normal system over $Y$. Whenever we deal with normal systems, the underlying class $\mathcal{T}$ is supposed to be closed hereditary.

As a simple consequence of Theorem 2.3 we obtain the following result (we omit its proof).

2.5. **Proposition.** Let $X$ and $Y$ be normal spaces such that $Y$ is a hereditary paracompact LEC space and it is a neighbourhood extensor for $X$. Let $\mathcal{T}$ and $\mathcal{O}$ be the families of all closed subsets of $X$ and of all open subsets of $Y$, respectively. Suppose that $\{D_{A,U}\}_{A \in \mathcal{T}}$ is such a collection that for every $U \in \mathcal{O}$ the family $\{D_A = D_{A,U}\}_{A \in \mathcal{T}}$ is a normal system over $U$ and

$$(D*) \ D_{X,Y} \cap S_Y(X,U) \subset D_{X,U}.$$  

Then each of the sets $D_{A,U}$ is dense in $C(A,U)$ provided $D_{X,Y}$ has SMAP.

2.6. **Remark.** Under the notation and assumptions of Theorem 2.3, the fact that $D_X$ has SMAP is equivalent to the following: $X$ may be covered by a finite family of closed sets $X_1, \ldots, X_n$ such that $D_{X_j}$ has SMAP for each $j$. This easily follows from Theorem 2.3, (D0)–(D2) and Lemma 1.2–(B).

Our next aim is to give equivalent conditions under which every member of a normal system $\{D_A\}_{A \in \mathcal{T}}$ over an ANR $Y$ is dense, when $\mathcal{T}$ is a rich class of topological spaces (such as compact, metrizable, of weight no greater than a fixed cardinal, of dimension no greater than a fixed natural number, etc.). This can be done by a simple adaptation of the concept of Toruńczyk [44].

For a normal system $\{D_A\}_{A \in \mathcal{T}}$ over a space $Y$ let us consider the following axioms:

(D4) If $A \in \mathcal{T}$, $B$ is a closed subset of $A$, $f \in C(A,Y)$ and $f|_B \in D_B$, then there is a closed subset $K$ of $A$ such that $f|_K \in D_K$ and $B \subset \text{int}_A K$.

(D5) If $A, B \in \mathcal{T}$ and $f \in C(A,B)$, then $D_B f \subset D_A$ (that is, $g \circ f \in D_A$ for each $g \in D_B$).

(D6) If $A, B \in \mathcal{T}$ and $A \cong B$, there is a homeomorphism $h: A \rightarrow B$ such that $D_B h = D_A$.

Note that (D6) follows from (D5) and it implies that if $A$ and $B$ are two homeomorphic members of $\mathcal{T}$, then $D_A$ is dense [has SMAP] iff
$D_B$ is dense [has SMAP]. A normal system satisfying the axiom (D4) is said to be strongly normal. If (D6) [(D5)] is fulfilled, we add the epithet topological [transitive] (thus we may talk about topological normal systems, transitive strongly normal systems, etc.). Usually normal systems are topological.

Before we formulate results on topological strongly normal systems, we will establish notation and terminology.

Let $H$ be a Hilbert space of dimension $\alpha > 0$ and let $E$ be an orthonormal basis of $H$. Fix $e \in E$. For a number $n \in \mathbb{N} \setminus \{0\}$, let $J_n(\alpha) [K_n(\alpha)]$ consists of all nonempty finite subsets $\sigma$ of $E$ such that $\text{card}(\sigma \cup \{e\}) \leq n + 1$ [card $\sigma \leq n + 1$]. It is clear that $|J_n(\alpha)|$ is an AR [ANR] of dimension $n$ and of weight $\max(\alpha, \aleph_0)$ and that every simplicial complex of dimension less than $n$ [no greater than $n$] which has at most $\alpha - 1$ (=$\alpha$ if $\alpha$ is infinite) vertices is isomorphic to a subcomplex of $J_n(\alpha)$ [of $K_n(\alpha)$].

Let us agree that $\tau$ is one of the topologies—weak or metric—for simplicial complexes and it is fixed. That is, whenever in the sequel appears a space of the form $|K|_\tau$, where $K$ is a simplicial complex, then $\tau$ always means ‘$w$’ or always means ‘$m$’. For simplicity, we say that a class $T$ is corelated to an ANR $Y$ if every member of $T$ is a normal space for which $Y$ is a neighbourhood extensor and for each $X \in T$ there is an open cover $V$ of $Y$ (depending on $X$) such that for every $V \in V$:

(i) $\text{comp}(\bar{V}) \leq \text{comp}(Y)$
(ii) $\bar{V}$ is FDT provided $Y$ is locally FDT,
(iii) for any map $f: X \to V$ and an open cover $\mathcal{U}$ of $Y$ there is a simplicial complex $K$ which witnesses Lemma 1.5 and such that $\mathcal{T}$ contains a space homeomorphic to $|K|_\tau$.

From now to the end of the section, we assume that $Y$ is an ANR, $\mathcal{T}$ a closed hereditary class corelated to $Y$ and $\mathfrak{D} = \{D_A\}_{A \in \mathcal{T}}$ is a transitive normal system over $Y$. Our purpose is to answer the question of when ($\star$) $D_X$ is dense for each space $X \in \mathcal{T}$.

The following is left as an exercise.

2.7. Lemma. (A) (cf. [44, Lemma 3.6]) Suppose $\mathfrak{D}$ is strongly normal. Let $K \in \mathcal{T}$ be such that $K \cong |K|_\tau$ where $K$ is a finite dimensional simplicial complex of dimension $n \in \mathbb{N}$ which has $\alpha > 0$ vertices. If $\mathcal{T}$ contains a space $Z \cong \mathbb{I}^n \times \alpha$ such that $D_Z$ has SMAP, then $D_K$ is dense in $C(K, Y)$.

(B) (cf. [44, part of the proof of Lemma 3.8]) If $D_K$ has SMAP for each space $K \in \mathcal{T}$ homeomorphic to a simplicial complex space, then ($\star$) is fulfilled.

(C) (cf. [44, proof of Lemma 3.8]) Let $K \in \mathcal{T}$ be a space homeomorphic to $|K|_\tau$ for some locally finite dimensional simplicial complex $K$ having $\alpha > 0$ vertices. If the class $\mathcal{T}$ contains a space $J(\alpha)$
homeomorphism to $\bigoplus_{n=1}^{\infty} |J_n(\alpha)|_{\tau}$ such that $D_{J(\alpha)}$ has SMAP, then $D_K$ is dense.

Let $\text{comp}(T \wedge Y) = \sup\{\min(\text{comp}(X), \text{comp}(Y)): X \in T\}$ and $\dim(T \wedge Y) = \sup\{\min(\dim(X), \dim(Y)): X \in T\}$. Lemma 2.7 yields the following

2.8. **Theorem.** In each of the following cases $(\ast)$ is fulfilled.

(I) $T$ consists of compact spaces or $Y$ is locally compact, and for each $m \in \mathbb{N} \cap [0, \dim(T \wedge Y)]$ there is $n \geq m$ such that $I^n \in T$ and $D_{I^n}$ has SMAP.

(II) $D$ is strongly normal, $T$ consists of FDT spaces or $Y$ is locally FDT and for each $m \in \mathbb{N} \cap [0, \dim(T \wedge Y)]$ and positive $\alpha < \text{comp}(T \wedge Y)$ there is $n \geq m$ such that $I^n \times \alpha \in T$ and $D_{I^n \times \alpha}$ has SMAP.

(III) $T$ consists of Lindelöf spaces or $Y$ is locally separable, $J = \bigoplus_{n=1}^{\infty} I^n \in T$ and $D_J$ has SMAP.

(IV) For each positive $\alpha < \text{comp}(T \wedge Y)$, $J(\alpha) = \bigoplus_{n=1}^{\infty} |J_n(\alpha)|_{\tau} \in T$ and $D_{J(\alpha)}$ has SMAP.

Since $Q \setminus \{\text{point}\} \cong Q \times [0, 1)$ ([15]), Theorem 2.3 and Theorem 2.8–(III) give

2.9. **Corollary.** If $X$ is separable, $Q \ast = Q \setminus \{\text{point}\} \in T$ and $D_Q$, has SMAP, then $D_Y$ is dense for each $Y \in \mathcal{T}$.

3. **Transitive strongly normal systems over special ANR’s**

In case of special ANR’s such as manifolds modelled on infinite-dimensional Hilbert spaces the points (III)–(IV) of Theorem 2.8 may be weakened as it will be done in this section.

We begin with the following result.

3.1. **Lemma.** Let $X$ be a paracompact space and $\{D_A\}_{A \in \mathcal{T}}$ a normal system over $X$. If $A \in \mathcal{T}$, $A = \bigcup_{s \in S} A_s$ where $\{A_s\}_{s \in S}$ is a discrete family of closed subsets of $A$, and $g \in \mathcal{C}(A, X)$ is such that $g|_{A_s} \in D_{A_s}$ for each $s \in S$ and the family $\{g(A_s)\}_{s \in S}$ is locally finite in $X$, then $g \in D_A$.

**Proof.** Let $\mathcal{W}$ be the collection of all open subsets $U$ of $X$ such that $g|_{g^{-1}(U)} \in D_{g^{-1}(U)}$. It is easy to show that $\mathcal{W}$ is a Michael collection and each point of $X$ has a neighbourhood belonging to $\mathcal{W}$. So, Lemma 1.1 finishes the proof. \(\Box\)

For spaces $X$ and $Y$, a discrete collection $\mathcal{B} = \{B_s\}_{s \in S}$ of closed subsets of $X$ and a subset $B$ of $X$, put

\[(3-1)\quad \mathcal{L}(B, Y; \mathcal{B}) = \{f \in \mathcal{C}(B, Y): \{f(B \cap B_s)\}_{s \in S} \text{ is locally finite in } Y\}.\]
It is not difficult to prove that if $Y$ is hereditary paracompact, $\mathcal{O}$ and $\mathcal{T}$ consist of all open subsets of $Y$ and closed subsets of $X$, respectively, and $D_{A,U} = L(A,U;B)$ for $A \in \mathcal{T}$ and $U \in \mathcal{O}$, then the axiom $(D\ast)$ is fulfilled and for each $U \in \mathcal{O}$ the system $\{D_A = D_{A,U}\}_{A \in \mathcal{T}}$ is strongly normal over $U$.

Following Banakh and Zarichnyy [7] (cf. [44, (*2), p. 253]), we say that an ANR $X$ has the countable locally finite approximation property (briefly, $\omega$-LFAP) if for every $U \in \text{cov}(X)$ there is a sequence of maps $\{f_n : X \to X\}_{n \in \mathbb{N}}$ such that each $f_n$ is $U$-close to $\text{id}_X$ and the family $\{f_n(X)\}_{n \in \mathbb{N}}$ is locally finite in $X$. Equivalently (using similar method as those in [44] or in the proof of Lemma 2.7), $X$ has $\omega$-LFAP iff the family $L(\oplus_{n \in \mathbb{N}}L_n, X; \{L_n\}_{n \in \mathbb{N}})$ has SMAP with $(\text{LF}_1)\{L_n\}_{n \in \mathbb{N}} = \{I_n^{n+1}\}_{n \in \mathbb{N}}$ provided $X$ is locally separable, $(\text{LF}_2)\{L_n\}_{n \in \mathbb{N}} = \{|J_n^{n+1}(\alpha)\}_{\tau \in \mathbb{N}}$ for every infinite $\alpha < \text{comp}(X)$.

Further, $X$ is said to have the $\kappa$-discrete $m$-cells property ([7], cf. [44, (1), p. 252]) if the family

$$\{f \in C(I^m \times \kappa, X) : \{f(I^m \times \{\beta\})\}_{\beta < \kappa} \text{ is discrete in } X\}$$

is dense in $C(I^m \times \kappa, X)$. Finally, $X$ has the strong discrete approximation property (briefly, SDAP; [2], cf. [44, Corollary 3.2]) if the family

$$\{f \in C(\oplus_{n \in \mathbb{N}}I^{n+1}, X) : \{f(I^{n+1})\}_{n \in \mathbb{N}} \text{ is discrete in } X\}$$

is dense in $C(\oplus_{n \in \mathbb{N}}I^{n+1}, X)$. These three concepts were used to characterize Hilbert manifolds ([44, 45]):

(H1) $X$ is a paracompact manifold modelled on $l^2$ iff $X$ is a locally separable completely metrizable ANR which has SDAP,

(H2) $X$ is a paracompact manifold modelled on a Hilbert space of dimension $\alpha > \aleph_0$ iff $X$ is a completely metrizable ANR of local weight $\alpha$ which has $\omega$-LFAP and has $\alpha$-discrete $n$-cells property for each $n \in \mathbb{N}$.

Note also that SDAP is equivalent to $\omega$-LFAP for locally separable ANR’s ([44], [16]) and that $\kappa$-discrete $m$-cells property (with infinite $\kappa$) is implied by its ‘locally finite version’ ([5, Lemma 4]), that is, we may replace the word ‘discrete’ in (3-2) by ‘locally finite’. Further, we infer from Proposition 2.5 that if $X$ has one of these three above defined properties, every open subset of $X$ has it as well. This explains that the condition [44, (*2), p. 253] is implied by SDAP (for locally separable $X$). This also shows that in the definitions of SDAP, $\omega$-LFAP and discrete cells approximation properties we may replace the word ‘dense’ by ‘has SMAP’.

Banakh [5] (see also [6, Theorem 3.1]) has proved that a connected ANR of weight $\alpha$ is homeomorphic to a homotopy dense subset of a Hilbert manifold iff it has $\omega$-LFAP and $\alpha$-discrete $n$-cells property for each $n$. Recall that a subset $A$ of an ANR $Y$ is homotopy dense in $Y$
iff there is a homotopy $H : Y \times I \to Y$ such that $H(y, 0) = y$ for each $y \in Y$ and $H(Y \times (0, 1)) \subset A$ ([6], [5]). Banakh’s result and the above comments show that

3.2. **Theorem.** If an ANR $X$ is locally homotopy dense embeddable in Hilbert manifolds and $w(U) = w(X)$ for each nonempty open subset $U$ of $X$, then $X$ itself is homotopy dense embeddable in a Hilbert manifold.

Now we shall develop the ideas of the previous section in case of ANR’s with $\omega$-LFAP or discrete cells properties. From now on, we assume that $Y$ is an ANR, $\mathcal{T}$ a closed hereditary class corelated to $Y$ and $\{D_A\}_{A \in \mathcal{T}}$ is a transitive strongly normal system over $Y$. Additionally, we assume that $\mathcal{T}$ contains the spaces $\oplus_{n \in \mathbb{N}} I^{n+1}$, and $I^m \times \alpha$ and $J(\alpha) = \oplus_{n \in \mathbb{N}} |J_{n+1}(\alpha)|_\tau$ for each positive $m \in \mathbb{N}$ and $\alpha < \text{comp}_1(Y)$.

3.3. **Proposition.** In each of the following cases $(\ast)$ is fulfilled.

(I) $Y$ has $\omega$-LFAP and $D_{I_n \times \alpha}$ has SMAP for each positive $n \in \mathbb{N}$ and $\alpha < \text{comp}_1(Y)$.

(II) $Y$ is locally separable and has SDAP and $D_{I_n}$ has SMAP for each $n \geq 1$.

**Proof.** The proofs of both the points are similar and therefore we shall only show (I). Let $\alpha < \text{comp}_1(Y)$. By Theorem 2.8, it suffices to prove that $D_{J(\alpha)}$ is dense. Let $f \in \mathcal{C}(J(\alpha), Y)$ and $\mathcal{U} \in \text{cov}(Y)$. Take a sequence $\{u_n : Y \to Y\}_{n \geq 1}$ such that $u_m \in B(\text{id}_Y, \mathcal{U})$ for each $m$ and the family $\{u_n(Y)\}_{n \geq 1}$ is locally finite in $Y$. By the assumption of (I) and Lemma 2.7-(A), $D_{|J_n(\alpha)|_\tau}$ is dense for each $n \geq 1$. So, there are maps $g_n \in D_{|J_n(\alpha)|_\tau}$ such that $g_n \in B(u_m f, \mathcal{U})$. Put $g = \bigcup_{n=1}^{\infty} g_n \in \mathcal{C}(J(\alpha), Y)$. By Lemma 3.1, $g \in D_{J(\alpha)}$. What is more, $g \in B(f, \text{st}(\mathcal{U}))$ (where st(\mathcal{U}) is the star of $\mathcal{U}$), which finishes the proof.

 Analogously one may show that

3.4. **Proposition.** If $Y$ has $\omega$-LFAP and $\alpha$-discrete $m$-cells property for every positive $m \in \mathbb{N}$ and $\alpha < \text{comp}_1(Y)$, and $D_{I_n}$ has SMAP for each $n \geq 1$, then $(\ast)$ is satisfied.

4. **Strong Z-sets**

Following Toruńczyk [39, 43, 44], we say that a closed subset $A$ of $X$ is a Z-set in $X$, if the set $\mathcal{C}(Q, X \setminus A)$ is dense in $\mathcal{C}(Q, X)$, or—equivalently—if $\mathcal{C}(I^n, X \setminus A)$ is dense in $\mathcal{C}(I^n, X)$ for each $n \geq 1$. (If $X$ is an ANR, this definition is equivalent to the original one by Anderson [1].) Similarly, $A$ is said to be a strong Z-set in $X$ iff for every $\mathcal{U} \in \text{cov}(X)$ there is a map $u : X \to X$ which is $\mathcal{U}$-close to $\text{id}_X$ and $A \cap \overline{\text{im} u} = \emptyset$ (cf. e.g. [11], [10], [17, 18]). In other words, $A = \hat{A}$ is a [strong] Z-set in $X$ iff $S_X(D, X \setminus A)$ is dense in $\mathcal{C}(D, X)$ where $D = Q \setminus [D = X]$. Not every Z-set in an ANR is a strong Z-set ([11,
Key example, p. 56]). However, combined results of Henderson [28] and Banakh [5] show that every $Z$-set in an ANR $X$ having $\omega$-LFAP and $\omega(X)$-discrete $n$-cells property for each $n$ is strong. We shall obtain this result independently of the theorems of Henderson and Banakh.

The following result is easy to prove.

4.1. Lemma. If $X$ is hereditary paracompact and $A$ its closed subset, $T$ is a class of all topological spaces and for $Y \in T$ and an open subset $U$ of $X$, $D_{W,U} = S_U(Y, U \setminus A)$, then $\{D_{Y,U}\}_{Y \in T}$ is a transitive strongly normal system over $U$ and $(D*)$ is fulfilled.

As an immediate consequence of the above lemma and the results of the previous sections, we obtain

4.2. Proposition. Let $X$ be an ANR and $A$ its closed subset.

(Z1) ([10], [6, Proposition 1.4.1]) If $X$ is locally compact, then every $Z$-set in $X$ is a strong $Z$-set.

(Z2) (cf. [6, Exercise 1.4.10]) If $X$ is locally separable, then $A$ is a strong $Z$-set in $X$ if and only if $S_X(\bigoplus_{n \in \mathbb{N}} I_+^{n+1}, X \setminus A)$ has SMAP, iff $S_X(\mathbb{Q} \setminus \{\text{point}\}, X \setminus A)$ has SMAP.

(Z3) If $X$ is locally FDT, then $A$ is a strong $Z$-set iff $S_X(I_+^{n} \times \alpha, X \setminus A)$ has SMAP for each positive $n \in \mathbb{N}$ no greater than $\dim X$ and each $\alpha < \text{comp}_l(X)$.

(Z4) A is a strong $Z$-set in $X$ iff $S_X(\bigoplus_{n \in \mathbb{N}} |J_{n+1}(\alpha)|, X \setminus A)$ has SMAP for each positive $\alpha < \text{comp}_l(X)$.

(Z5) ([10, Lemma 1.3]) If $A$ is a strong $Z$-set in $X$, then $U \cap A$ is a strong $Z$-set in $U$ for each open subset $U$ of $X$.

(Z6) (for separable $X$ see [10, Corollary 1.5]) If every point of $A$ has an open neighbourhood $U$ in $X$ such that $A \cap U$ is a strong $Z$-set in $U$, then $A$ is a strong $Z$-set in $X$.

(Z7) (cf. [10, Proposition 1.7] or [6, Proposition 1.4.3]) If $X$ is locally separable and has SDAP or if $X$ has $\omega$-LFAP and $\alpha$-discrete $m$-cells property for each positive $m \in \mathbb{N}$ and $\alpha < \text{comp}_l(X)$, then every $Z$-set in $X$ is strong.

The counterpart of (Z6) for $Z$-sets in its whole generality was first proved by Eells and Kuiper [25]. Their proof is based on the theorem of Whitehead [48] on weak homotopy equivalences and they worked with Anderson’s [1] definition of a $Z$-set. Here we gave an alternative proof of (Z6) for ‘Toruńczyk’s Z-sets’.

The property (Z5), by a simple use of SMAP, may be strengthened as follows: if $K$ is a strong $Z$-set in a metrizable space $Y$ and $X$ is an open subset of $Y$ such that $X$ is an ANR, then $K \cap X$ is a strong $Z$-set in $X$.

Following [6], we say that an ANR $X$ has the strong $Z$-set property iff every $Z$-set in $X$ is strong. (Z5) and (Z6) yield that the strong $Z$-set property is open hereditary (i.e. every open subset of an ANR $X$ has
the strong $Z$-set property provided so has $X$) and is local (if each point of $X$ has an open neighbourhood with the strong $Z$-set property, then $X$ has it as well). This facts will be used in the next section.

Bestvina and Mogilski [10] have proved that a $Z$-set being a strong $\sigma$-$Z$-set in a separable ANR is itself a strong $Z$-set. (This property were used by them to prove (Z6) for separable $X$.) We do not know whether the assumption of separability of the ANR in this statement may be omitted. The lack of such a property will force us in the next section to assume that ANR’s have the strong $Z$-set property.

5. Absorbing sets

In this section we generalize most important results of [10] to nonseparable case. All undefined symbols and notions have the same meaning as in [10] (after deleting all assumptions dealing with separability). In particular, a $C$-absorbing set is any space $X$ such that: $X$ is strongly $C$-universal and homotopy dense embeddable into a Hilbert manifold, $X \in C_\sigma$ and $X$ is a $\sigma$-$Z$-space (i.e. $X = \bigcup_{n=1}^{\infty} X_n$ for some sequence $(X_n)_{n=1}^{\infty}$ of $Z$-sets in $X$).

The proofs presented in [10] (see also [6]) of the following results (we quote only the most important ones) work also in nonseparable case (Chapman’s generalization [14] of Anderson’s-McCharen’s theorem [3] is needed).

5.1. Theorem. Let $C$ be a topological, closed hereditary and additive class of metrizable spaces and let $X$ be a metrizable space.

(SU1) ([10, Proposition 2.1]) If $X$ is strongly $C$-universal, so is every its open subset.

(SU2) ([10, Proposition 2.2]) If $C$ is also open hereditary, $X$ is an ANR having the strong $Z$-set property and for each $Z$-set $K$ in $X$ the space $X \setminus K$ is $C$-universal, then $X$ is strongly $C$-universal.

(SU3) ([10, Proposition 2.7]; for general proof see [6, Proposition 1.5.1]) If $X$ is an ANR having an open cover consisting of strongly $C$-universal sets, then $X$ itself is strongly $C$-universal.

(SU4) ([10, Proposition 2.6] or [6, Theorem 1.5.11]) If $X$ is strongly $C$-universal and $Y$ is an ANR such that $X \times Y$ has the strong $Z$-set property, then $X \times Y$ is strongly $C$-universal as well.

(SU5) ([10, Theorem 3.1] or [6, Theorem 1.6.3]) Two $C$-absorbing sets are homeomorphic iff they have the same homotopy type.

(SU6) ([10, Theorem 3.2]; $Z$-set Unknotting Theorem) If $X$ is $C$-absorbing, then every homeomorphism between two $Z$-sets in $X$ which is $U$-homotopic to the inclusion map for some $U \in \text{cov}(X)$ is extendable to a homeomorphism of the whole space $\text{st}(U)$-close to the identity map on $X$. 
If $X$ is an AR having more than one point, then the weak product
\[ W(X, \ast) = \{(x_n)_{n=1}^{\infty} \in X^\omega : x_n = \ast \text{ for almost all } n\} \]
(where $\ast \in X$ is a fixed 'basepoint') is $\mathcal{D}$-absorbing where $\mathcal{D}$ is the class of all spaces admitting closed embeddings into $W(X, \ast)$. The class $\mathcal{D}$ is topological, closed hereditary and additive. If $Y$ is also an AR, then $W(X, \ast)$ and $W(Y, \ast)$ are homeomorphic iff $X$ embeds as a closed subset of $W(Y, \ast)$ and $Y$ embeds as a closed subset of $W(X, \ast)$.

Bestvina and Mogilski have also proved that if a separable ANR $X$ is a $\sigma$-$Z$-space, has the strong $Z$-set property and is strongly $\mathcal{C}$-universal for a topological closed hereditary additive class $\mathcal{C}$, then it is also strongly $\mathcal{C}_\sigma$-universal (see [10, Proposition 2.3], the proof uses in its final part the second axiom of countability). We do not know whether the assumption of separability of $X$ may be omitted in this. However, below we prove its counterpart for absorbing sets (which, alternatively, may be used to prove the last claim of (SU7)). This result will be applied in the next section.

One of the most important results on separable absorbing sets, beside (SU5), states that in the definition of an absorbing set one may omit the assumption of homotopy dense embeddability into a Hilbert manifold—this is a consequence of [10, Lemma 1.9] and Banakh’s theorem [5]. It turns out that this is true also for nonseparable absorbing sets, which shows

5.2. Proposition. Let $\mathcal{C}$ be a topological, closed hereditary and additive class and $X$ be an ANR which is strongly $\mathcal{C}$-universal, has the strong $Z$-set property and is a $\sigma$-$Z$-space. If $X \in \mathcal{C}_\sigma$, then $X$ is $\mathcal{C}_\sigma$-absorbing.

Proof. For simplicity, put $\kappa = w(X)$. The proof of [10, Proposition 2.3] shows that the set of closed embeddings is dense in $\mathcal{C}(P, X)$ for any $P \in \mathcal{C}_\sigma$. Since $X \in \mathcal{C}_\sigma$, we get that $X \times \mathbb{N} \in \mathcal{C}_\sigma$ as well and thus the natural projection $X \times \mathbb{N} \to X$ is approximable by closed embeddings. This easily implies that $X$ has $\omega$-LFAP. What is more, by Lemma 1.4, $X$ contains a discrete subset of cardinality $\kappa$, say $A$. Now if $\{U_a\}_{a \in A}$ is a discrete family of open subsets of $X$ such that $a \in U_a$ for $a \in A$, there is a family $\{h_a : X \to X\}_{a \in A}$ of closed embeddings such that $\text{im } h_a \subset U_a$. This shows that $X$ contains a closed subset homeomorphic to $X \times A$ (namely $\bigcup_{a \in A} \text{im } h_a$) and therefore also the natural projection $X \times A \to X$ is approximable by closed embeddings. Hence $X$ satisfies the $\kappa$-discrete $m$-cells property for each $m$. So, thanks to Banakh’s theorem [5], $X$ is homotopy dense embeddable into a Hilbert manifold. Now we shall check that $X$ is $\mathcal{C}_\sigma$-universal. To prove this, it is enough to show that $Z$-embeddings form a dense subset of $\mathcal{C}(X, X)$. 

\[ W(X, \ast) = \{(x_n)_{n=1}^{\infty} \in X^\omega : x_n = \ast \text{ for almost all } n\} \]
Let \{u_\beta\}_{\beta<\kappa} be a dense subset of \mathcal{C}(Q, X). Further, let

\[ L = \bigoplus_{\beta<\kappa} \text{im } u_\beta \]

be the topological disjoint union of the images of the maps \( u_\beta, X' \) be the topological disjoint union of \( X \) and \( L \) and let \( v: L \to X \) be given by \( v(x) = x \). By the above argument, both the spaces \( L \) and \( X' \) are members of \( \mathcal{C}_\sigma \). Fix a map \( f: X' \to X \) and a cover \( \mathcal{U}_0 \in \text{cov}(X) \) and put \( f_0 = f \cup v: X' \to X \). Write \( X = \bigcup_{n=1}^{\infty} X_n \) where each \( X_n \) is (a strong) \( Z \)-set in \( X \), \( X_n \in \mathcal{C} \) and \( X_n \subset X_{n+1} \), and let \( d \) denote the metric of \( X \). Now arguing similarly as in the proof of [10, Proposition 2.3], construct sequences of closed embeddings \( \{f_n: X' \to X\}_{n=1}^{\infty} \) and covers \( \{\mathcal{U}_n\}_{n=1}^{\infty} \) of \( X \) such that for each \( j \geq 1 \),

(E1) \( f_j|_{X_{j-1}} \) is a \( Z \)-embedding,

(E2) \( f_j|_{X_{j-1}} = f_{j-1}|_{X_{j-1}} \) (with \( X_0 = \emptyset \)),

(E3) \( f_j \) is \( \mathcal{U}_j \)-close to \( f_{j-1}|_{X} \cup v: X' \to X \),

(E4) \( \mathcal{U}_j \) is a star refinement of \( \mathcal{U}_{j-1} \), mesh\( d(\mathcal{U}_j) < 2^{-j} \) and if \( g: X \to X \) is \( \mathcal{U}_j \)-close to \( f_{j-1}|_X \), then \( \overline{\text{im } g} \cap \bigcup_{k=1}^{\infty} f_k(L) = \emptyset \) (where \( \bigcup_{k=1}^{\infty} = \emptyset \)), and the formula \( h(x) = \lim_{n \to \infty} f_n(x) \) well defines a closed embedding \( h: X \to X \). Then \( h \) is \( \mathcal{U}_0 \)-close to \( f \) and \( h(X) \cap \bigcup_{n=1}^{\infty} f_n(L) = \emptyset \). But thanks to (E3) and (E4), the family

\[ \{f_n|_{\text{im } u_\beta} \circ u_\beta: n \geq 1, \beta < \kappa\} \]

is dense in \( \mathcal{C}(Q, X) \) and thus \( h(X) \) is a \( Z \)-set in \( X \).

To finish the proof, it suffices to apply the above argument for each open subset \( U \) of \( X \) (instead of \( X \)) and use (SU2). \( \square \)

The next two results will be used in the next section.

5.3. Corollary. Under the assumptions of Proposition 5.2, every \( \mathcal{F}_\sigma \) subset of \( X \) is closed embeddable in \( X \).

5.4. Proposition. Let \( X \) be a noncompact AR which contains a closed subset homeomorphic to \( X \times X \). Let \( Y \) be a metrizable space such that \( w(Y) < w(X) \). Each of the following two conditions is sufficient for the closed embeddability of \( Y \) in \( X \).

(A) Every point of \( Y \) has a neighbourhood (not necessarily open or closed) which is closed embeddable in \( X \).

(B) \( Y \) may be covered by a locally finite collection of its closed subsets each of which is closed embeddable in \( X \).

Proof. First assume that \( Y \) is the union of its two closed subsets \( Y_1 \) and \( Y_2 \) which are closed embeddable in \( X \). Since \( X \) is an AR, there is a map \( u_j: Y \to X \) such that \( u_j|_{Y_j} \) is a closed embedding. Further, take a map \( \lambda: Y \to I \) which is positive on \( Y \setminus Y_1 \) and negative on \( Y \setminus Y_2 \) (such a map exists because \( Y_1 \cap Y_2 \) is closed and \( G_\delta \) in both the spaces \( Y_1 \) and \( Y_2 \)). Now put \( u: Y \ni y \mapsto (u_1(y), u_2(y), \lambda(y)) \in X \times X \times I \). One
easily checks that $u$ is a closed embedding and that there is a closed embedding of $X \times X \times I$ into $X$ (because of the facts that $X \times X$ is closed embeddable in $X$ and $X$ contains an arc).

Now to prove the sufficiency of (A), let $W$ be the family of all open subsets $U$ of $Y$ such that $\bar{U}$ is closed embeddable in $X$. Thanks to Lemma 1.1, it is enough to prove that $W$ is a Michael collection. The property (M1) is immediate, (M2) follows from the first part of the proof, and to see (M3), use Lemma 1.4 and the fact that $X$ contains a closed copy of $X \times X$ (and therefore also a closed copy of $X \times w(X)$).

The sufficiency of (B) is implied by the one of (A) and the first part of the proof (with simple induction argument).

Let us call a class $C$ product if $C_1 \times C_2 \in C$ for each $C_1, C_2 \in C$.

For a metrizable space $X$, let $F(X)$ be the class of all metrizable spaces which are closed embeddable in $X^n \times I^n$ for some $n \in \mathbb{N}$. It is easily seen that $F(X)$ is topological, closed hereditary and product and that $F(X)$ coincides with the class of metrizable spaces admitting closed embeddings in $X^n$ for some $n \in \mathbb{N}$ provided $X$ contains an arc. What is more, the first paragraph of the proof of Proposition 5.4 shows that $F(X)$ is additive for an AR $X$. This facts will be used later.

5.5. Remark. In Final Remarks (on page 425) of [20] Dobrowolski and Mogilski give two classical examples of nonseparable absorbing sets, namely $l_2^2(A)$ and $\Sigma l_2^2(A)$ (where $A$ is an uncountable set). They also mention (SU5) in its full generality (compare Example 6.10).

6. Rigid embeddings

The main aim of this section is to prove that the weak product $W(X, \ast)$ (defined in (SU7)) for an arbitrary AR $X$ is homeomorphic to some pre-Hilbert space $E$ with $E \cong \Sigma E$, where

$$\Sigma E = \{(x_n)_{n=1}^{\infty} \in E^\omega : x_n = 0 \text{ for almost all } n\}.$$ 

To do this, we shall introduce and investigate certain types of embeddings. But first we establish notation. For $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ let $|t|_\infty = \max_j |t_j|$ and $|t|_p = \left(\sum_{j=1}^{n} |t_j|^p\right)^{\frac{1}{p}}$. Let $E$ be a real vector space. For $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in E^n$, put

$$s \cdot v = \sum_{j=1}^{n} s_j v_j.$$ 

More generally, if $h: D \to E$ is any function (with an arbitrary domain $D$) and $x = (x_1, \ldots, x_n) \in D^n$, then $s \cdot h(x)$ denotes the vector $\sum_{j=1}^{n} s_j h(x_j)$. If $y = (y_1, \ldots, y_n) \in D^n$ and $\sigma$ is a permutation of the set $\{1, \ldots, n\}$, $y_\sigma$ stands for the $n$-tuple $(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$. Finally, for a metric space $(X, d)$ and a natural number $n \geq 2$, $\delta_n: X^n \to [0, +\infty)$ is
a function defined by
\[ \delta_n(x) = \min\{d(x_j, x_k) : j \neq k\}, \quad x = (x_1, \ldots, x_n) \in X^n. \]
Additionally, we put \( \delta_1 \equiv 1 : X \to [0, +\infty) \). Now we are ready to put the following

6.1. Definition. Let \((X, d)\) be a metric space. A map \( h : X \to E \) where \((E, \| \cdot \|)\) is a normed space is said to be \textit{weakly separately rigid} (with respect to \( d \)) if for each natural number \( n \geq 1 \) and reals \( r > 0 \) and \( M \geq 1 \) there is a constant \( C = C(r, n, M) \) such that \( \|t \cdot h(x)\| \geq C \) whenever \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( x \in X^n \) are such that \( \min_j |t_j| \geq \frac{1}{M} \), \( |t|_{\infty} \leq M \) and \( \delta_n(x) \geq r \). If the constant \( C \) can be chosen independently of \( M \) (that is, \( C = C(r, n) \)) for each \( n \) and \( r \), the map \( h \) is said to be \textit{separately rigid} and if \( C \) depends only on \( r \), \( h \) is called \textit{almost rigid}. Finally, \( h \) is \textit{rigid} if for each \( r > 0 \) there is a constant \( C = C(r) \) such that \( \|t \cdot h(x)\| \geq C|t|_{\infty} \) for each \( t \in \mathbb{R}^n \) and \( x \in X^n \) with \( \delta_n(x) \geq r \), and any \( n \geq 1 \).

A subset \( A \) of a normed space \( E \) is said to be \textit{rigid}, \textit{almost rigid}, etc. if the inclusion map \( A \to E \) is (respectively) rigid, almost rigid, etc.

It is easy to see that a weakly separately rigid map is injective and its image is linearly independent. Note also that if a uniformly continuous map is rigid, almost rigid, separately rigid or weakly separately rigid, then so is its image. In the sequel we shall show that every weakly separately rigid map is an embedding whose image is closed in its own linear span. To see this, we need the following

6.2. Lemma. Let \( h : X \to E \) be a weakly separately rigid map of a metric space \((X, d)\) into a normed space \((E, \| \cdot \|)\) such that the linear span of \( \text{im} \ h \) coincides with \( E \). Let \( p \geq 1 \), \( r > 0 \), \( M \geq 1 \), \( t^{(n)} = (t_1^{(n)}, \ldots, t_p^{(n)}) \in [-M, -\frac{1}{M}]^p \cup [\frac{1}{M}, M]^p \subset \mathbb{R}^p \), \( x^{(n)} = (x_1^{(n)}, \ldots, x_p^{(n)}) \in X^p \) be such that \( \delta_p(x^{(n)}) \geq r \) and the sequence \( (t^{(n)} \cdot h(x^{(n)}))_{n=1}^{\infty} \) converges in \( E \). Then there is a sequence \( (\sigma_n)_{n=1}^{\infty} \) of permutations of the set \( \{1, \ldots, p\} \) such that both the sequences \( (t^{(n)}_{\sigma_n})_{n=1}^{\infty} \) and \( (x^{(n)}_{\sigma_n})_{n=1}^{\infty} \) converge in \( \mathbb{R}^p \) and \( X^p \) respectively.

\textbf{Proof.} Let us agree that \( \sum_{j=1}^{0} = 0 \). It suffices to prove that (under the assumptions of the lemma on \( t^{(n)} \) and \( x^{(n)} \)):

\( \bullet \) If \( \sum_{j=1}^{q} t_j^{(n)} h(x_j^{(n)}) \to \sum_{k=1}^{q} t_k h(x_k) \) for some \( q \geq 0 \), nonzero reals \( t_1, \ldots, t_q \) and distinct points \( x_1, \ldots, x_q \) of \( X \), then \( q = p \) and for some sequence \( (\sigma_n)_{n \geq 1} \) of permutations one has \( t^{(n)}_{\sigma_n(j)} \to t_j \) and \( x^{(n)}_{\sigma_n(j)} \to x_j \) for \( j = 1, \ldots, p \).

We shall show this by induction on \( p \), starting with \( p = 0 \). For \( p = 0 \) we only need to note that the image of \( h \) is linearly independent and thus \( q = 0 \). Now assume that we have proved \(( \bullet)\) for \( p - 1 \). The proof of \(( \bullet)\) for \( p \) is divided into three steps.
I. If $x_1^{(n)} \to x_1$, then $t_1^{(n)} \to t_1$.

Proof of I. We argue by contradiction. Suppose that $(t_1^{(n)})_{n \geq 1}$ does not converge to $t_1$. Passing to a subsequence, we may assume that $t_1^{(n)} \to s \neq t_1$. But then $t_1^{(n)} h(x_1^{(n)}) \to sh(x_1)$ and hence $\sum_{j=2}^p t_j^{(n)} h(x_j^{(n)}) \to (t_1 - s) h(x_1) + \sum_{j=2}^q t_j h(x_k)$. Now by the induction hypothesis we have $q = p - 1$ and $x_{t_n^{(n)}} \to x_1$ for some sequence of $l_n \in \{2, \ldots, p\}$. This yields $r \leq \delta p(x^{(n)}) \leq d(x_1^{(n)}, x_{t_n^{(n)}}) \leq d(x_1^{(n)}, x_1) + d(x_1, x_{t_n^{(n)}}) \to 0$, which is impossible.

II. There are a sequence of $l_n \in \{1, \ldots, p\}$ and $k \in \{1, \ldots, q\}$ such that $x_{t_n^{(n)}} \to x_k$.

Proof of II. Observe that $q > 0$, because $h$ is weakly separately rigid. Put $\alpha_n = \min\{d(x_j^{(n)}, x_k) : j \in \{1, \ldots, p\}, k \in \{1, \ldots, q\}\}$. It is enough to show that $\lim_{n \to \infty} \alpha_n = 0$. Suppose that the latter convergence does not hold. Passing to a subsequence, we may assume that $\alpha_n \geq c$ for all $n$ and some positive constant $c$. This implies that there is $\varepsilon > 0$ such that $\delta p+q(y^{(n)}) \geq \varepsilon$ for each $n$ where $y^{(n)} = (x_1^{(n)}, \ldots, x_p^{(n)}, x_1, \ldots, x_q) \in X^{p+q}$. Moreover, there is a constant $A \geq 1$ such that $s^{(n)} \in [-A, -\frac{1}{A}]^{p+q} \cup \frac{1}{A}, A]^{p+q} \subset \mathbb{R}^{p+q}$, where $s^{(n)} = (t_1^{(n)}, \ldots, t_p^{(n)}, -t_1, \ldots, -t_q)$. We have $s^{(n)} \cdot h(y^{(n)}) \to 0$ which denies the weak separate rigidity of $h$.

III. There is a sequence of $\sigma_n(1) \in \{1, \ldots, p\}$ such that $x_{\sigma_n(1)}^{(n)} \to x_1$.

Proof of III. As in the proof of II, it suffices to show that the sequence of $\gamma_n = \min_j d(x_j^{(n)}, x_1)$ tends to 0. Again, we argue by contradiction. Passing to a subsequence, applying II and after renumerating of $x^{(n)}$ (which depends on $n$), we may assume that
\[
\gamma_n \geq \varepsilon
\]
for all $n$ and a positive constant $\varepsilon$, and $x_1^{(n)} \to x_c$ for some $c \in \{2, \ldots, q\}$. By I, $t_1^{(n)} \to t_c$ and thus $\sum_{j=2}^p t_j^{(n)} h(x_j^{(n)}) \to \sum_{k \neq c} t_k h(x_k)$. So, we infer from the induction hypothesis that $q = p$ and $x_{j_n^{(n)}} \to x_1$ for some $j_n \in \{2, \ldots, p\}$, which contradicts (6-1).

Now (●) follows from III, I and the induction hypothesis. \hfill \Box

Applying the above lemma with $p = r = M = 1$ we obtain:

6.3. Corollary. Every weakly separately rigid map $h : X \to E$ such that $\lim h(X) = E$ is a closed embedding.

6.4. Proposition. Let $(X, d)$ be a nonempty metric space and let $\mathcal{C} = \mathcal{F}(X)$. If $h : X \to E$ is a weakly separately rigid map into a normed space $(E, \| \cdot \|)$ such that $\lim h(X) = E$, then $X \in \mathcal{F}(\Sigma E)$ and $\Sigma E \in \mathcal{C}_\sigma$.

Proof. Since the class $\mathcal{C}$ is product, we only need to check that $E \in \mathcal{C}_\sigma$ ($X \in \mathcal{F}(\Sigma E)$ thanks to Corollary 6.3). For (positive) natural numbers
\[ q = \sum \\text{of} \ l \]

observe that \( \Delta \) as a map of \( \Delta \times \Gamma \) is open in \( G \) and \( \Phi: \Delta \times \Gamma \rightarrow (t, x) \rightarrow t \bullet x \in G_{p,q} \).

observe that \( \Delta \) is a covering and therefore it is a local homeomorphism.

Take a small enough open cover \( W = \bigcup_{s \in S} W_s \) of \( G_{p,q} \) and a corresponding family \( \{V_s\}_{s \in S} \) of relatively open subsets of \( \Delta \times \Gamma \) such that \( \Phi \) restricted to each \( V_s \) is a homeomorphism of \( V_s \) onto \( U_s \). Next find an open cover \( W = \bigcup_{n=1}^{\infty} W_n \) of \( G_{p,q} \) such that \( W_n = \{W_{s,n}\}_{s \in S} \) is discrete in \( G_{p,q} \) and \( W_{s,n} \subset U_s \) for each \( s \in S \) and \( n \geq 1 \). Then the set \( D_n = \bigcup_{s \in S} W_{s,n} \) is open in \( G_{p,q} \) and is homeomorphic to \( \bigcup_{s \in S} (\Phi^{-1}(W_{s,n}) \cap V_s) \). we conclude from this that \( D_n \in \mathcal{E}_\sigma \) and thus also \( G_{p,q} \in \mathcal{E}_\sigma \) (since \( G_{p,q} = \bigcup_{n=1}^{\infty} D_n \) and each \( D_n \) is \( \mathcal{F}_\sigma \) in \( G_{p,q} \)).

we are now ready to prove

6.5. Theorem. If \( (X, d) \) is an absolute retract having more than one point and \( h: X \rightarrow E \) is a weakly separately rigid map into a normed space \( (E, ||\cdot||) \) such that \( \text{lin } h(X) = E \), then \( W(X, *) \cong \Sigma E \).

Proof. By (SU7) and Corollary 6.3, it is enough to show that \( E \) is closed embeddable in \( W(X, *) \). But this follows from Proposition 6.4, (SU7) and Proposition 5.2.

now we shall give an example of rigid embeddings.

6.6. Example. The following construction is due to Bessaga and Pełczyński \[8\] (or \[9, Proposition VI.7.1\]). Let \( (X, d) \) be a nonempty metric space. For each \( n \geq 1 \) take a locally finite partition of unity \( \{f_\lambda\}_{\lambda \in \Lambda_n} \) such that

(6-2) \[ f_\lambda(x) \cdot f_\lambda(y) = 0 \] whenever \( \lambda \in \Lambda_n \) and \( d(x, y) \geq 2^{-n} \)

and put \( g_\lambda = 2^{-n} f_\lambda \) (for \( \lambda \in \Lambda_n \)). Assuming that the sets \( \Lambda_1, \Lambda_2, \ldots \) of indices are pairwise disjoint, put \( \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n \) and define \( h: X \rightarrow l^2(\Lambda) \) = \{g: \Lambda \rightarrow \mathbb{R}||g||_2^2 = \sum_{\lambda \in \Lambda} g(\lambda)^2 < +\infty\} \) by \( (h(x))(\lambda) = \sqrt{g_\lambda(x)} \). Then \( h \) is continuous and \( \text{im } h \) is contained in the unit sphere of \( l^2(\Lambda) \). What is more, \( h \) is rigid: if \( p \geq 1 \), \( t = (t_1, \ldots, t_p) \in \mathbb{R}^p \), \( x = (x_1, \ldots, x_p) \in X^p \) and \( \delta_p(x) \geq 2^{-n} \), then \( \|t \bullet h(x)\|_2^2 \geq \sum_{\lambda \in \Lambda_n} 2^{-n}(\sum_{j=1}^{p} t_j \sqrt{f_\lambda(x_j)})^2 \) and (6-2) gives

\[ \|t \bullet h(x)\|_2^2 \geq 2^{-n} \sum_{\lambda \in \Lambda_n} \sum_{j=1}^{p} t_j^2 f_\lambda(x_j) = 2^{-n} \sum_{j=1}^{p} t_j^2 \sum_{\lambda \in \Lambda_n} f_\lambda(x_j) = 2^{-n} |t|_2^2. \]
This example shows that every metric space admits a rigid embedding into a Hilbert space.

Now we have

6.7. Theorem. For an arbitrary AR $X$, the weak product $W(X, \ast)$ is homeomorphic to some pre-Hilbert space $E$ such that $E \cong \Sigma E$.

Proof. Clearly, we may assume that $X$ has more than one point. By Example 6.6 and Theorem 6.5, there is a pre-Hilbert space $F$ with an inner product $\langle \cdot, - \rangle_F$ such that $W(X, \ast) \cong \Sigma F$. Let $E = \Sigma F$, i.e. $E$ is the set $\Sigma F$ with the norm $\| (x_n)_{n=1}^{\infty} \|_E = \sqrt{\sum_{n=1}^{\infty} (x_n, x_n)_F}$. There is a natural inner product in the space $E$ inducing the norm $\| \cdot \|_E$ and, e.g. by [42, Corollary 1.9], $E \cong \Sigma F$. So, $E \cong \Sigma E \cong W(X, \ast)$. □

It is easily seen by Toruńczyk’s characterization theorem for Hilbert manifolds [44, 45] that a connected metrizable space $X$ is a Hilbert manifold iff $X$ is a noncompact completely metrizable ANR such that the natural projection $X \times X \to X$ is approximable by closed embeddings. In similar spirit, with use of Theorem 6.7, we now characterize manifolds modelled on pre-Hilbert spaces $E$ with $E \cong \Sigma E$.

6.8. Proposition. Let $X$ be a connected nonempty metrizable space. Let $\Omega = X$ if $X$ is contractible and otherwise let $\Omega$ be the topological open or closed cone over $X$. The following conditions are equivalent:

(i) $X$ is homeomorphic to (an open subset of) some nonzero pre-Hilbert space $E$ with $E \cong \Sigma E$,

(ii) $X$ is a $C$-absorbing AR (ANR) for some topological, closed hereditary, additive and product class $C$,

(iii) $X$ is an AR (ANR) and a $\sigma$-$Z$-space such that for every $Z$-set $K$ in $X$ the natural projection $(X \setminus K) \times \Omega \to X \setminus K$ is a near-homeomorphism, i.e. it is approximable by homeomorphisms.

Proof. To prove that (iii) follows from (i) it suffices to apply a variation (cf. [29, Theorem 5]) of Schori’s theorem [35] (note that both the open and closed cones over $E$ are homeomorphic to $E$ for spaces $E$ as in (i)—see e.g. [27]— and thus if $X$ is an open subset of $E$, then both the open and closed cones over $X$ are factors of $E$, thanks to [41, 42] or [10, Corollary 5.4]).

To see that (i) is implied by (ii), first observe that the closed cone $\Omega$ of $X$ belongs to $C_\sigma$ and therefore $X \times W(\Omega, \ast) \in C_\sigma$ as well (since $C$ is product). Further, (SU4) yields that $X \times W(\Omega, \ast)$ is $C$-absorbing. So, $X$ and $X \times W(\Omega, \ast)$ are homeomorphic (thanks to (SU5)). Now use Theorem 6.7 and Toruńczyk’s Factor Theorem [41, 42] to finish the proof.

We pass to showing that (ii) follows from (iii). Let $C$ be the class of all topological spaces which are closed embeddable in $\Omega$. Clearly, $C$ is
topological and closed hereditary. By hypothesis, $X \times \Omega \cong X$ and thus also $X \times \Omega \times \Omega \cong X$. This yields that

\[(6-3) \quad \Omega \times \Omega \text{ is closed embeddable in } \Omega\]

(note that $X \in \mathcal{C}$). Moreover, $\Omega$ is noncompact because $X$ is a $\sigma$-$Z$-space. So, we infer from Proposition 5.4 that $\mathcal{C}$ is open hereditary and additive. Noticing that $\mathcal{C}$ is also product thanks to (6-3), we see that to finish the proof it suffices to show that $X$ is $\mathcal{C}$-absorbing. We shall do this using (SU2).

Let $K$ be a $Z$-set in $X$. For simplicity, put $U = X \setminus K$ and let $\pi : U \times \Omega \times \Omega \to U$ be the natural projection. It follows from the assumptions that $\pi$ is a near-homeomorphism. This, combined with Lemma 1.4, yields that the natural projection $U \times w(U) \to U$ is approximable by closed embeddings and therefore $U$ is homotopy dense embeddable in a Hilbert manifold (thanks to Banakh’s theorem [5]). Fix any $x_0 \in X$ and put $a = (x_0, 1) \in \Omega$. The set $\{a\}$ is a $Z$-set in $\Omega$ (because $\{x_0\}$ is a $Z$-set in $X$, which is implied by the fact that $X$ is a $\sigma$-$Z$-space). Now take a space $C \in \mathcal{C}$ and a map $f : C \to U$. There is a closed embedding $u : C \to \Omega$. We see that $v : C \ni x \mapsto (f(x), u(x), a) \in U \times \Omega \times \Omega$ is a $Z$-embedding such that $\pi \circ v = f$. So, since $\pi$ is a near-homeomorphism, $f$ is approximable by $Z$-embeddings. This shows that $U$ is $\mathcal{C}$-universal. Hence an application of (SU2) finishes the proof.

We do not know whether in the above result it suffices to check the point (iii) only for $K = \emptyset$ to obtain (i).

The technique of rigid embeddings enables us to give a simple proof of the following generalization of a special case of [6, Theorem 2.4.2]:

6.9. Proposition. Let $\mathcal{C}$ be a topological, closed hereditary, additive and product class of metrizable spaces such that $I \in \mathcal{C}$. There is a $\mathcal{C}$-absorbing pre-Hilbert space $F$ with $F \cong \Sigma F$ iff there is a space $X$ such that

(U1) $X \in \mathcal{C}_\sigma$ and
(U2) every member of $\mathcal{C}$ is closed embeddable in $X$.

Proof. The necessity is clear. To prove the sufficiency, embed rigidly the space $X$ into a pre-Hilbert space $E$ in such a way that $E$ coincides with the linear span of the image of the embedding and then apply Proposition 6.4 and (SU7).

6.10. Example. For a countable ordinal $\alpha > 0$ and an infinite cardinal $m$ let $M_\alpha(m)$ $[\mathcal{M}_\alpha^f(m)]$ and $A_\alpha(m)$ $[\mathcal{A}_\alpha^f(m)]$ be the class of all (metrizable) [finite dimensional] spaces of the absolute multiplicative and additive (respectively) Borelian class $\alpha$ and of weight no greater than $m$ (for definition and more on these classes see e.g. [37, 38], [26] or [30], where the Reader can find more references concerning the subject). It is easily seen that each of these classes is topological, closed hereditary, additive and product. Bestvina and Mogilski [10] have proved
that for an arbitrary \( \alpha \), there are an \( M_\alpha(\aleph_0) \)-absorbing AR and an \( A_\alpha(\aleph_0) \)-absorbing one, and Banakh, Radul and Zarichnyi have shown that also the classes \( M_\alpha(\aleph_0) \) and \( A_\alpha(\aleph_0) \) have absorbing AR’s (see [6, Theorem 2.4.9]). Note that the classes \( M_1 \) and \( A_1 \) consist of absolute \( G_\delta \)-spaces and absolute \( F_\sigma \)-spaces, respectively. It is well known that a metrizable space is of absolute \( G_\delta \)-class iff it is completely metrizable, and such a space is of absolute \( F_\sigma \)-class iff it is \( \sigma \)-locally compact ([36]), that is, it is the countable union of its locally compact closed subsets or, equivalently, the countable union of its locally compact closed subsets.

This easily implies that \( \Sigma l^2(m) \) is an \( M_1(m) \)-absorbing AR for each infinite cardinal \( m \) (where \( l^2(m) \) is the Hilbert space with an orthonormal basis of cardinality \( m \)), which was obtained e.g. by Toruńczyk [40]. Further, in [46] Tsuda has proved that for each natural \( n \) and infinite cardinal \( m \) there is a completely metrizable finite dimensional space \( T_n(m) \) of weight \( m \) such that every completely metrizable space \( X \) with \( \dim X \leq n \) and \( w(X) \leq m \) is closed embeddable in \( T_n(m) \). This implies that the topological disjoint union \( T(m) = \bigoplus_{n=1}^{\infty} T_n(m) \) of the spaces \( T_n(m) \) contains a closed copy of every member of \( C = M_1^f(m) \) and belongs to \( C_\sigma \). We conclude from this and Proposition 6.9 that \( M_1^f(m) \) has an absorbing AR. Analogous results hold for the classes \( A_1 \): the space \( l^2_f(m) = \) the linear span of a fixed orthonormal basis of \( l^2(m) \) is an \( A_1^f(m) \)-absorbing AR (cf. [47]). Indeed, it is clear that \( \Sigma l^2_f(m) \simeq l^2_f(m) \) and that \( I^n \times m \) is closed embeddable in \( l^2_f(m) \) for each natural \( n \). Since every finite dimensional locally compact metrizable space \( X \) of weight no greater than \( m \) is the union of its two closed subsets each of which is closed embeddable in \( I^n \times m \) (for some \( n \)), thus each such a space \( X \) is closed embeddable in \( l^2_f(m) \) and therefore—by Proposition 5.2—\( A_1^f(m) \subset F(l^2_f(m)) \). On the other hand, if \( Y \) is the closed cone over the topological disjoint union of the spaces \( I^n \times m \) (\( n = 1, 2, \ldots \) and \( m \) is fixed), then \( Y \) is \( \sigma \)-finite dimensional (i.e. \( Y \) is the countable union of its closed finite dimensional subsets) and \( \sigma \)-locally compact and \( W(Y, *) \) is therefore an \( A_1^f(m) \)-absorbing AR. So, (SU5) gives \( W(Y, *) \simeq l^2_f(m) \).

The latter method of finding an \( A_1^f(m) \)-absorbing AR works also for the classes \( A_1(m) \): it suffices to take \( W(Z, *) \) where \( Z \) is the closed cone over \( Q \times m \).

For ordinals \( \alpha \) greater than 1 (and uncountable cardinals \( m \)) and \( C \in \{ M_\alpha(m), M_\alpha^f(m), A_\alpha(m), A_\alpha^f(m) \} \) there are \( C \)-absorbing sets if only there are spaces \( X \) satisfying (U1) and (U2) (thanks to Proposition 6.9). The author however has no knowledge whether such spaces exist.

We end the paper with the following

6.11. **Proposition.** Every nonempty metric space is isometric to a subset \( A \) of some normed linear space \((E, || \cdot ||)\) such that for each
$n \geq 1$, $t \in \mathbb{R}^n$ and $a \in A^n$,

\[(6-4)\quad \|t \cdot a\| \geq \min\left(\frac{1}{2} \delta_n(a), 1\right) |t|_1,
\]

and if the metric of the given space is upper bounded by 2, $A$ may be taken to be contained in the unit sphere of $E$.

**Proof.** We shall improve Michael’s proof [33] (cf. [9, Theorem II.1.2]) of the Arens-Eells theorem [4]. Let $(X, d)$ be a metric space and $x_*$ an arbitrarily chosen element of $X$. If $d$ is upper bounded by 2, put $\psi \equiv 1: X \to [1, +\infty)$; otherwise let $\psi: X \ni x \mapsto d(x, x_*) + 1 \in [1, +\infty)$. Take $\omega \notin X$. We extend the metric $d$ to a metric on the set $\tilde{X} = X \cup \{\omega\}$ by putting $d(x, \omega) = \psi(x)$ for $x \in X$. Now let $\Gamma$ consists of all $d$-nonexpansive maps of $\tilde{X}$ into $\mathbb{R}$ which vanish at $\omega$. That is, $u: \tilde{X} \to \mathbb{R}$ belongs to $\Gamma$ iff $|u(z_1) - u(z_2)| \leq d(z_1, z_2)$ for all $z_1, z_2 \in \tilde{X}$ and $u(\omega) = 0$. Let $l^\infty(\Gamma)$ be the Banach space of all real-valued bounded functions on $\Gamma$ equipped with the supremum norm. For $x \in X$ denote by $\delta_x \in l^\infty(\Gamma)$ the evaluation map of $x$, i.e. $\delta_x(u) = u(x)$ for $u \in \Gamma$. It is easily seen that the map $h: (X, d) \ni x \mapsto \delta_x \in (l^\infty(\Gamma), \|\cdot\|)$ is isometric and that $\text{im } h$ is contained in the unit sphere of $l^\infty(\Gamma)$ if $d$ is upper bounded by 2. We shall check that (6-4) holds.

Let $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $x = (x_1, \ldots, x_n) \in X^n$ be such that $\delta_n(x) \geq r \in (0, 2]$. Observe that the map $u_0: \{x_1, \ldots, x_n\} \to \mathbb{R}$ given by $u_0(x_j) = \frac{t_j}{r} \text{sgn } t_j$ (where $\text{sgn}$ is the sign function) is $d$-nonexpansive and takes values in $[-1, 1]$. This yields that there is a map $u \in \Gamma$ which extends $u_0$. But then $\|\sum_{j=1}^n t_j \delta_{x_j}\| \geq \|\sum_{j=1}^n t_j u(x_j)\| = \frac{t}{2} |t|_1$. This, together with the fact that $h$ is isometric, gives (6-4). \qed

Having in mind Example 6.6 and Proposition 6.11, the following may be an interesting question:

*Is every metrizable space homeomorphic to a rigid subset of the unit sphere of some Hilbert space?*

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