Note on Generalized Cuckoo Hashing with a Stash

Brice Minaud\(^1\) and Charalampos Papamanthou\(^2\)

\(^1\)Inria, ENS, PSL, CNRS, France
\(^2\)University of Maryland, USA

Abstract

Cuckoo hashing is a common hashing technique, guaranteeing constant-time lookups in the worst case. Adding a stash was proposed by Kirsch, Mitzenmacher, and Wieder at SICOMP 2010, as a way to reduce the probability of rehash. It has since become a standard technique in areas such as cryptography, where a superpolynomially low probability of rehash is often required. Another extension of cuckoo hashing is to allow multiple items per bucket, improving the load factor. That extension was also analyzed by Kirsch et al. in the presence of a stash. The purpose of this note is to repair a bug in that analysis. Letting \(d\) be the number of items per bucket, and \(s\) be the stash size, the original claim was that the probability that a valid cuckoo assignment fails to exist is \(O(n^{(1-d)(s+1)})\). We point to an error in the argument, and show that it is \(\Theta(n^{-d-s})\).

1 Introduction

Cuckoo hashing was introduced by Pagh and Rodler [PR04], and proceeds as follows. We wish to allocate a set \(S\) of \(n\) items into two tables \(T_1, T_2\) of size \(m = (1 + \varepsilon)n\) each, where \(\varepsilon > 0\) is an arbitrary constant. The construction is parametrized by two hash functions \(h_1 : S \rightarrow T_1\) and \(h_2 : S \rightarrow T_2\). Each item \(x \in S\) may be allocated either to \(h_1(x)\), or to \(h_2(x)\). Pagh and Rodler prove that a valid assignment (where no two items are assigned to the same location) exists with probability \(1 - O(n^{-\varepsilon})\) over the randomness of the hash functions.

In [KMW10], Kirsch, Mitzenmacher, and Wieder show that when adding a stash of constant size \(s\), the probability of failure becomes \(O(n^{-s})\), assuming uniformly random hash functions. They also consider a generalization of cuckoo hashing introduced by Dietzfelbinger and Weidling [DW07], where up to \(d\) items can be assigned to the same location in the table. In that setting, there is only a single table consisting of \(m = (1 + \varepsilon)n/d\) buckets, each of size \(d\). The benefit is that the load factor (i.e. the ratio of occupied space in the table after allocating all items) increases to \(1/(1 + \varepsilon)\), whereas it cannot exceed \(1/2\) using standard cuckoo hashing. Proposition 4.3 from [KMW10] claims the following.

**Proposition 1** ([KMW10] Proposition 4.3). For any constants \(\varepsilon > 0, d \geq 1 + \ln(1/\varepsilon)/(1 - \ln 2)\), and \(s \geq 0\), the probability that there exists no valid assignment for cuckoo hashing with buckets of size \(d\) and a stash of size \(s\) is \(O(n^{(s+1)(1-d)})\).

The proof of that proposition uses a statement of the form \(j \geq s + 1\), where \(j\) is the number of vertices in a certain graph. If \(j\) was the number of edges, the inequality would hold, but with \(j\) being the number of vertices, it appears to be incorrect.

In Section 3 we build a counter-example showing that the probability of failure must be \(\Omega(n^{-d-s})\). In Section 4 we then repair the proof, and show that the probability of failure is \(O(n^{-d-s})\) when \(d\) and \(s\) are constant, matching the counter-example. The result holds under the mild conditions \(\varepsilon < 1/4, s \leq m/(10\varepsilon^4)\). We also provide a bound for variable \(d\) and \(s\), which is of interest for cryptographic applications, where \(s\) is typically superconstant.
2 Notation

Fix an arbitrary constant $\varepsilon > 0$. We want to allocate $n$ items into $m = (1 + \varepsilon)n/d$ buckets, each of size $d \geq 1$. We are also able to send up to $s$ items to a separate stash. Let $S$ denote the set of items, and $V$ denote the set of buckets. Let $h_1, h_2$ be a pair of uniformly random functions $S \rightarrow V$. Each item $x \in S$ can be assigned to bucket $h_1(x)$, bucket $h_2(x)$, or to the stash.

We say that the two hash functions are suitable iff there exists a way to assign the items as above, so that no more than $d$ items are assigned to the same bucket, and no more than $s$ items are assigned to the stash. The goal is to upper-bound the probability that the hash functions fail to be suitable.

We also consider the associated cuckoo graph $G = (V, E)$, with edges $E = \{(h_1(x), h_2(x)) : x \in S\}$. This graph is undirected, may contain multiple copies of the same edge ($E$ is a multiset), and may contain loops. We say that the graph is suitable iff it is possible to remove at most $s$ edges, and orient the other edges, such that every vertex has outdegree at most $d$. (In some literature, including [DW07], edges are oriented the opposite way.) The hash functions are suitable iff the associated graph is suitable.

Let $\text{Fail}(n, \varepsilon, d, s)$ denote the probability that the graph fails to be suitable when inserting $n$ items into $m = (1 + \varepsilon)n/d$ buckets of size $d$, with a stash of size $s$, using uniformly random hash functions.

Throughout this note, $\varepsilon$ is viewed as a constant, as is standard in the analysis of cuckoo hashing schemes. In other words, $\text{Fail}$ is regarded as a function of $n, d, s$, for fixed $\varepsilon > 0$. Concretely, this means that the hidden constants in the $O()$, $\Omega()$, $\Theta()$ notation may depend on $\varepsilon$. In $\text{KMW10}$, $d$ and $s$ are also regarded as constants, with the same implication. Parts of our analysis will also view $d$ and $s$ as constant. Which quantities are constant will be explicit in proposition statements.

3 Lower bound on Fail

**Proposition 2.** Assume $\varepsilon > 0$, $d \geq 1$, $s \geq 0$ are constant. Then

$$\text{Fail}(n, \varepsilon, d, s) = \Omega(n^{-d-s}).$$

**Proof.** For $v \in V$, let $\mathcal{E}_v$ denote the event: there exists a subset of items $X \subseteq S$ of cardinality $d + s + 1$, such that $\forall x \in X, h_1(x) = h_2(x) = v$. Let $\mathcal{E} = \bigvee_{v \in V} \mathcal{E}_v$. If $\mathcal{E}_v$ holds, there can be no valid cuckoo assignment, since the bucket $v$ can hold at most $d$ items, and at most $s$ can be relocated to the stash. It follows that $\text{Fail}(n, \varepsilon, d, s) \geq \Pr(\mathcal{E})$. Our goal is to show $\Pr(\mathcal{E}) = \Omega(n^{-d-s})$.

**Claim 1.**

$$\Pr(\mathcal{E}_v) = \Omega(n^{-d-s-1}).$$

**Proof.** Since the hash functions are uniform, the probability that a given $x \in S$ satisfies $h_1(x) = h_2(x) = v$ is $m^{-2}$. $\mathcal{E}_v$ may be viewed as the event that there are at least $d + s + 1$ successes in a binomial experiment with $n$ trials, each with probability of success $m^{-2} = \Theta(n^{-2})$, whence

$$\Pr(\mathcal{E}_v) = \sum_{k \geq d+s+1} \binom{n}{k} m^{-2k} \left(1 - m^{-2}\right)^{n-k} \geq (1 - m^{-2})^n \sum_{k \geq d+s+1} \binom{n}{k} m^{-2k} \geq e^{n(-m^{-2} + O(m^{-4}))} \left(\frac{n}{(d + s + 1)m^2}\right)^{d+s+1} = e^{\Theta(1)} \Omega(n^{-d-s-1}) = \Omega(n^{-d-s-1}).$$

Consider the binary random variable $I_v$ that is equal to 0 if $\mathcal{E}_v$ occurs, 1 otherwise. Likewise, let $I$ denote the binary random variable that is equal to 0 iff $\mathcal{E} = \bigvee_v \mathcal{E}_v$ occurs. Observe $I = \prod_v I_v$. The next claim makes use of the notion of negative association [DR96].
Claim 2. The variables $I_v$ are negatively associated.

Proof. For $v \in V$, let $B_v = \{|x \in S : h_1(x) = h_2(x) = v\}$ denote the number of edges that loop in on $v$ in the cuckoo graph. Let $B' = \{|x \in S : h_1(x) \neq h_2(x)\}$ denote the number of edges that are not loops, so that $B' + \sum v B_v = n$. By [DR96] Theorem 13, the $B_v$’s together with $B'$ are negatively associated. A fortiori the $B_v$’s are negatively associated. We have $I_v = f(B_v)$ where $f$ is the non-increasing function that is equal to 1 if its input is strictly less than $d$ and 0 otherwise. Hence by [DR96] Proposition 7.2, the $I_v$’s are negatively associated.

By [DR96] Proposition 3, Claim 2 implies $\Pr(\mathcal{E}) = 1 - E(I) = 1 - E\left(\prod_v I_v\right) \geq 1 - \prod_v E(I_v) = 1 - (1 - \Pr(\mathcal{E}_v))^m$ for any fixed $v$. Using $m \Pr(\mathcal{E}_v) \to 0$ and $m \Pr(\mathcal{E}_v) = O\left(m^2 \Pr(\mathcal{E}_v)^2\right)$, we get $\Pr(\mathcal{E}_v) < \frac{1}{m}$, which concludes the proof.

4 Upper bound on Fail

In this section, we repair Proposition 3 as follows.

Proposition 3. Fix a constant $0 < \varepsilon \leq 0.25$. There exist constants $\alpha > 0$ and $\gamma < 1$, such that for every $d \geq 1 + \ln(1/\varepsilon)/(1 - \ln 2)$, and $1 \leq s \leq m/(10\varepsilon^4)$,

$$\mathbb{E}(\sum_{v \in V} I_v) = O\left(\left(\frac{\varepsilon}{m}\right)^{d+s} + \frac{2\varepsilon}{m} 2^{(d-1)} \left(\frac{\alpha_8}{m}\right)^{s-1} + m\gamma^m\right).$$

If $d$ and $s$ are constant, that expression is $O(n^{-d-s})$.

Proof. The proof follows [DW07] Proof of Theorem 1], but has to account for the addition of a stash. For $X \subseteq E$ a subset of edges, define $\Gamma(X) = \bigcup_{v \in X} \{h_1(x), h_2(x)\}$. Without a stash (i.e. $s = 0$), [DW07] observes that the graph is suitable iff $\forall X \subseteq S, |\Gamma(X)| \geq |X|/d$. If the condition fails, i.e. $|X|/d < |X|/d$, suitability fails “trivially” because there is not enough room to store the $|X|$ edges in $\Gamma(X)$. So the previous equivalence may be understood as saying: suitability holds if and only if it does not fail trivially on any subset of edges. This is still the case when adding a stash.

Claim 3. The graph $G$ is suitable iff for all $X \subseteq X, d|\Gamma(X)| + s \geq |X|$. 

Proof. If the condition $d|\Gamma(X)| + s \geq |X|$ fails for some subset $X$ of edges, then there is not enough room in $\Gamma(X)$ and the stash to store the edges in $X$. This shows that if $G$ is suitable, then the condition must hold for all $X$. Conversely, assume the graph is not suitable. We are going to build an $X$ such that the condition fails.

Define the overflow $ov(D)$ of a directed graph $D$ as the minimal number of edges to remove such that every vertex has outdegree at most $d$. Let $s'$ be the smallest integer such that the following statement holds: there exists a directed graph $D$ arising from orienting the edges of $G$ such that $ov(D) = s'$ (in other words, $s'$ is the minimum stash size, across all possible orientations of $G$). The fact that $G$ is not suitable translates to $s' > s$. Fix $D$ witnessing the previous statement. Consider the subset $Y$ of vertices of $D$ that have outdegree strictly more than $d$. Let $Y' \supseteq Y$ be the set of vertices that can be reached from $Y$ by following a directed path. Observe that the outdegree of every vertex $v$ in $Y'$ must be at least $d$. Otherwise, there would exist a directed path
from a vertex \( v \) in \( Y \) to some \( w \) in \( Y' \) with outdegree \(< d \). Flipping the direction of every edge along this path decreases the outdegree of \( v \) by 1, increases the outdegree of \( w \) by 1, and does not change the outdegree of intermediate vertices. Because \( v \) was over capacity (outdegree \( > d \)) and \( w \) was under capacity (outdegree \( < d \)), this means that after flipping the edges of the path, the overflow has decreased by 1. This would contradict the minimality of \( s' \). Let \( D' = (Y', X) \) be the subgraph of \( D \) induced by \( Y' \). Here, \( X \) is the set of edges of \( D \) whose endpoints are both in \( Y' \). Note that \( Y' = \Gamma(X) \). By construction, we have \( s' = \text{ov}(D) = \text{ov}(D') = \sum_{v \in Y'} \text{out}_{D'}(v) - d = |X| - d|Y'| = |X| - d\Gamma(X)|. \) Since \( s' > s, X \) witnesses \( d|\Gamma(X)| + s < |X| \).

By Claim 3 the probability that \( G \) is not suitable is equal to

\[
F := \Pr(\exists X \subseteq S : d|\Gamma(X)| + s < |X|).
\]

We adapt the proof of Theorem 1 in [DW07] Section 3.2. For \( 1 \leq j \leq m/(1 + \varepsilon) \), let \( F(j) \) be the probability that there exist a set \( Y \) of \( j \) vertices and a set \( X \) of \( dj + s + 1 \) edges that satisfy \( \Gamma(X) \subseteq Y \). Clearly,

\[
F \leq \sum_{1 \leq j \leq m/(1 + \varepsilon)} F(j).
\]

We split the sum into three cases, \( j = 1, 1 < j < T \), and \( T \leq j \leq m/(1 + \varepsilon) \), where \( T \) is a certain threshold whose value will be determined later. Note that this split is different from [DW07], which considers four cases. We require that \( T \leq m/(2e^4) \), so the split at \( T \) occurs within the range that is covered by case 2 from [DW07]. The remainder of case 2 from [DW07], as well as cases 3 and 4, will be treated together in our case 3.

Throughout the analysis, the relevant quantity is \( s + 1 \) rather than \( s \); to ease notation, it is convenient to define \( s' = s + 1 \). First, Eq. (13) from [DW07] becomes:

\[
F(j) \leq \binom{m}{j} \left( \frac{n_j}{m^2(d + s'/j)} \right)^{jd+s'} \left( \frac{n(m^2 - j^2)}{m^2(n - jd - s')} \right)^{n-jd-s'}
\]

Equation indices follow [DW07] when possible for ease of comparison, adding a prime, so (13') is the counterpart of (13) in [DW07]. Continuing from there and replacing \( n \) by \( dm/(1 + \varepsilon) \):

\[
F(j) \leq \frac{m^m}{j!(m-j)^{m-j}} \left( \frac{d}{d + s'/j} \right)^{jd+s'} \left( \frac{d(m^2 - j^2)}{(1 + \varepsilon)m(n - jd - s')} \right)^{dm/(1+\varepsilon)-jd-s'}
\]

\[
= \frac{m^m}{j!(m-j)^{m-j}} \left( \frac{d}{d + s'/j} \right)^{jd+s'} \left( \frac{m^2 - j^2}{(1 + \varepsilon)m(n - jd - s' - s'/d)} \right)^{dm/(1+\varepsilon)-jd-s'}
\]

\[
< (1 + \varepsilon)^{-jd-s'} \left( \frac{m^m}{j!(m-j)^{m-j}} \right)^{jd+s'} \left( \frac{m^2 - j^2}{(m - (1 + \varepsilon)(j + s'/d))} \right)^{dm/(1+\varepsilon)+j+s'/d}.
\]

Denote the expression on the right-hand side of (14) by \( f(j, \varepsilon) \). The goal is to bound \( F(j) \) or \( f(j, \varepsilon) \) for various ranges of \( j \).

**Case 1:** \( j = 1 \). By (13') we get:

\[
F(1) \leq m \left( \frac{d}{d + s} \right)^{d+s'} \left( \frac{1}{(1 + \varepsilon)m} \right)^{d+s'} \left( \frac{n - n/m^2}{n - d - s'} \right)^{n-d-s'}
\]

\[
< m \left( \frac{1}{(1 + \varepsilon)m} \right)^{d+s'} e^{d+s'-n/m^2} = O \left( \left( \frac{\varepsilon}{m} \right)^{d+s} \right).
\]

If \( s \) and \( d \) are constant, this is \( O(n^{-d-s}) \).

**Case 2:** \( 1 < j < T \leq m/2e^4 \). In [DW07], it is shown that \( \varepsilon \mapsto f(j, \varepsilon) \) decreases in their setting. This is done by computing the derivative of \( \log f \). Instead, we proceed as follows. It is not
necessary to show that \( \varepsilon \mapsto f(j, \varepsilon) \) decreases, what matters is only that \( F(j) \) can be upper-bounded by \( f(j, 0) \). Recall that the upper-bound \( F(j) \leq f(j, \varepsilon) \) was computed as a union bound as follows (see [DW07]):

\[
F(j) \leq \binom{m}{j} F'(j, \varepsilon) \leq f(j, \varepsilon)
\]

where \( F'(j, \varepsilon) \) denotes the probability that there exists \( X \subseteq S \) with \( |X| \geq jd + s + 1 \) such that \( \Gamma(X) \subseteq Y \), for any fixed subset \( Y \) of size \( j \). It is clear that \( \varepsilon \mapsto F'(j, \varepsilon) \) is decreasing because \( F'(j, \varepsilon) \) is the upper tail of a binomial experiment, and \( \varepsilon \) only decreases the probability of success of each trial in the experiment, while it does not affect any other parameter. It follows that \( F(j) \leq \binom{m}{j} F'(j, 0) \leq f(j, 0) \).

Using (14) we get:

\[
F(j) < f(j, 0) = m^{(1-d)j(d-1)+s'} (m-j)^j m^{-m} \binom{m^2-j^2}{m-j-s'/d} d^{(m-j-s'/d)} \\
= m^{(1-d)j(d-1)+s'} m^{-m} (1-j/m)^j m^{-m} d^{(m-j-s'/d)} \cdot \left( 1 + \frac{j}{m} + \frac{s'}{dm} \cdot \frac{1+j/m}{1-j/m-s'/d} \right) d^{(m-j-s'/d)} \\
< m^{(1-d)j(d-1)+s'} e^{(m-j)/m} d^{(m-j-s'/d)} \cdot \left( 1+j/m \cdot \frac{1+j/m}{1-j/m-s'/d} \right) d^{(m-j-s'/d)} \\
= \left( \frac{j}{m} \right)^{(d-1)j+s'} e^{(m-j)/m} d^{(m-j-s'/d)} \cdot \left( 1+j/m \cdot \frac{1+j/m}{1-j/m-s'/d} \right) d^{(m-j-s'/d)} \\
< \left( \frac{j}{m} \right)^{(d-1)j+s'} e^{(d+1)j} \left( \frac{1+j/m}{1-j/m-s'/d} \right)^{s'}. 
\]

Because \( j < T \leq m/(2e^4) \) and \( s \leq m/(10e^4) \), \( e^{(1+j/m)(1-j/m-s'/d)} \) is upper-bounded by an absolute constant \( C \). Reinjecting \( C \) we have:

\[
F(j) < \left( \frac{j}{m} \right)^{(d-1)j+s'} e^{(d+1)j} C^{s'} = \left( \frac{j}{m} \right)^{(d-1)j+s'} e^{(d+1)j} \left( \frac{Cj}{m} \right)^{s'}. 
\]

(17)

In the end, the expression is the same as in [DW07], with an additional term \((Cj/m)^{s'}\). Denote the rest of the expression by \( g(j) \), so that (17) becomes \( F(j) < g(j)(Cj/m)^{s'} \). It was already observed in [DW07] that in the relevant range \( 1 < j \leq m/(2e^4) \), \( g(j) \) is geometrically decreasing. More precisely, the ratio of one term to the previous term is upper-bounded by a constant \( < 1 \). It follows that \( g(2) \) dominates \( \sum_{2 \leq j < T} g(j) \). This yields:

\[
\sum_{2 \leq j < T} F(j) = O \left( g(2) \left( \frac{CT}{m} \right)^{s'} \right) = O \left( \left( \frac{2e^4}{m} \cdot \frac{2^{(d-1)}}{m} \right)^{s'} \left( \frac{CT}{m} \right)^{s'+1} \right). 
\]

Anticipating a little, \( T \) will be \( O(s) \), so if \( s \) and \( d \) are constant, this is \( O(n^{1-2d-s}) \).

**Case 3:** \( T \leq j \leq m/(2e^4) \). Observe that \( F(j) \) can only decrease with \( s \). The idea for this third case is that we upper-bound \( F(j) \) by its value if there is no stash. This allows us to reuse previous analysis.

As noted in [KMW10] Proposition 4.3, we have:

\[
\sum_{T \leq j \leq m/(2e^4)} F(j) = O \left( \left( \frac{T}{m} \cdot \frac{2^{(d-1)}}{m} \right)^{s'} \right) = O \left( \left( \frac{e^4}{m} \cdot \frac{T}{m} \right)^{s'} \right). 
\]

5
We want to choose $T \leq m/(2e^4)$ such that the above quantity is less than the bound we obtained for case 2 (up to a constant). That is, we want:

$$\left(\frac{e^3T}{m}\right)^{(d-1)T} \leq \left(\frac{2e}{m}\right)^{(d-1)} \left(\frac{CT}{m}\right)^{s'}.$$  

A sufficient condition for (*) is that the following two conditions hold (using the fact that for positive reals, if $x \leq a^2$ and $x \leq b^2$, then $x \leq ab$):

1. $$\left(\frac{e^3T}{m}\right)^T \leq \left(\frac{2e}{m}\right)^4$$  
2. $$\left(\frac{e^3T}{m}\right)^{(d-1)T} \leq \left(\frac{CT}{m}\right)^{2s+2}.$$  

For (C1), $T = 5$ is clearly suitable as $m \to \infty$. Moreover, elementary functional analysis shows $x \mapsto x^{c^x}$ is decreasing over the interval $(0,1/e)$ for any constant $c > 0$, so as long as $e^3T/m < 1/e$, the condition continues to hold for $T > 5$. In the end, to satisfy (C1), we derive the following sufficient conditions on $T$: $5 \leq T < m/e^4$. We already require $T \leq m/(2e^4)$, so the second inequality is for free.

For (C2), increasing $C$ if necessary, we can assume $C \geq e^3$, then setting $T \geq 2s + 2$ suffices.

In the end, we get $5 \leq T, 2s + 2 \leq T$. Since $s \geq 1$ it suffices to set $T = 5s$. We also required earlier that $T \leq m/(2e^3)$. (Like in [DW07], this is used implicitly in the argument for $g$ being geometrically decreasing in case 2.) It follows that we want $s \leq m/(10e^4)$, as required in the statement of the proposition.

With this choice of $T$, we get that the sum of $F(j)$’s for $T \leq j \leq m/(2e^4)$ is smaller than the bound from case 2, so we can safely ignore it. It remains to consider $j \geq m/(2e^4)$. This was already shown to be $O(m^\gamma)$ for some $\gamma < 1$ in [DW07].

**Acknowledgments**

The authors would like to thank Adam Kirsch, Michael Mitzenmacher, Udi Wieder, as well as Martin Dietzfelbinger, for their helpful comments.

**References**

[DR96] Devdatt Dubhashi and Desh Ranjan. Balls and bins: A study in negative dependence. *BRICS Report Series*, 3(25), 1996.

[DW07] Martin Dietzfelbinger and Christoph Weidling. Balanced allocation and dictionaries with tightly packed constant size bins. *Theoretical Computer Science*, 380(1-2):47–68, 2007.

[KMW10] Adam Kirsch, Michael Mitzenmacher, and Udi Wieder. More robust hashing: Cuckoo hashing with a stash. *SIAM Journal on Computing*, 39(4):1543–1561, 2010.

[PR04] Rasmus Pagh and Flemming Friche Rodler. Cuckoo hashing. *Journal of Algorithms*, 51(2):122–144, 2004.