On Graded Quasi-Prime Submodules

Khaldoun Al-Zoubi
Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan
e-mail: kfzoubi@just.edu.jo

Rashid Abu-Dawwas
Department of Mathematics, Yarmouk University, Irbid, Jordan
e-mail: rrashid@yu.edu.jo

Abstract. Let $G$ be a group with identity $e$. Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. In this paper, we introduce the concept of graded quasi-prime submodules and give some basic results about graded quasi-prime submodules of graded modules. Special attention has been paid, when graded modules are graded multiplication, to find extra properties of these submodules. Furthermore, a topology related to graded quasi-prime submodules is introduced.

1. Introduction

Graded prime submodules of graded modules over graded commutative rings have been introduced and studied in [2, 5]. Here we introduce the concept of graded quasi-prime submodules and we investigate some properties of graded quasi-prime submodules of graded modules over graded commutative rings and consider some conditions under which a graded quasi-prime submodule of a graded module is graded prime. Also, the behavior of graded quasi-prime submodules under localization is studied. Furthermore, we introduce a topology on the set of graded quasi-prime submodules and some properties of this topology are given.

Before we state some results, let us introduce some notation and terminologies. Let $G$ be a group with identity $e$ and $R$ be a commutative ring. Then $R$ is a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by $(R, G)$. The elements of $R_g$ are called homogeneous of degree $g$ where $R_g$ is the additive subgroup of $R$ indexed

* Corresponding Author.
Received November 29, 2013; revised August 23, 2014; accepted October 2, 2014.
2010 Mathematics Subject Classification: 13A02, 16W50.
Key words and phrases: graded quasi-prime submodules, graded prime submodules.

259
by \( g \in G \). If \( x \in R \), then \( x \) can be written uniquely as \( \sum_{g \in G} x_g \), where \( x_g \) is the component of \( x \) in \( R_g \). Moreover, \( h(R) = \bigcup_{g \in G} R_g \). Let \( I \) be an ideal of \( R \). Then \( I \) is called a graded ideal of \( (R, G) \) if \( I = \bigoplus_{g \in G} (I \cap R_g) \). Thus, if \( x \in I \), then \( x = \sum_{g \in G} x_g \) with \( x_g \in I \). An ideal of a \( G \)-graded ring need not be \( G \)-graded (see Example 2.4 in [1]).

Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring and let \( I \) be a graded ideal of \( R \). Then the quotient ring \( R/I \) is also a \( G \)-graded ring. Indeed, \( R/I = \bigoplus_{g \in G} (R/I)_g \) where \( (R/I)_g = \{ x + I : x \in R_g \} \). For the simplicity, we will denote the graded ring \( (R, G) \) by \( R \). Let \( R \) be a \( G \)-graded ring and \( M \) an \( R \)-module. We say that \( M \) is a \( G \)-graded \( R \)-module (or \( G \)-graded \( R \)-module) if there exists a family of subgroups \( \{M_g\}_{g \in G} \) of \( M \) such that \( M = \bigoplus_{g \in G} M_g \) (as abelian groups) and \( R_g M_h \subseteq M_{gh} \) for all \( g, h \in G \). Here, \( R_g M_h \) denotes the additive subgroup of \( M \) consisting of all finite sums of elements \( r_g s_h \) with \( r_g \in R_g \) and \( s_h \in M_h \). Also, we write \( h(M) = \bigcup_{g \in G} M_g \) and the elements of \( h(M) \) are called homogeneous. Let \( M = \bigoplus_{g \in G} M_g \) be a graded \( R \)-module and \( N \) a submodule of \( M \). Then \( N \) is called a graded submodule of \( M \) if \( N = \bigoplus_{g \in G} (N \cap M_g) \). In this case, \( N_g \) is called the \( g \)-component of \( N \). Moreover, \( M/N \) becomes a \( G \)-graded \( R \)-module with \( g \)-component \( (M/N)_g = (M_g + N)/N \) for \( g \in G \). Let \( R \) be a \( G \)-graded ring and \( S \subseteq h(R) \) be a multiplicatively closed subset of \( R \). Then the ring of fraction \( S^{-1}R \) is a graded ring which is called the graded ring of fractions. Indeed, \( S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g \) where \( (S^{-1}R)_g = \{ r/s : r \in R, s \in S \text{ and } g = (\deg s)^{-1}(\deg r) \} \). Let \( M \) be a graded module over a \( G \)-graded ring \( R \) and \( S \subseteq h(R) \) be a multiplicatively closed subset of \( R \). The module of fractions \( S^{-1}M \) over a graded ring \( S^{-1}R \) is a graded module which is called the module of fractions, if \( S^{-1}M = \bigoplus_{g \in G} (S^{-1}M)_g \) where \( (S^{-1}M)_g = \{ m/s : m \in M, s \in S \text{ and } g = (\deg s)^{-1}(\deg m) \} \). We write \( h(S^{-1}R) = \bigcup_{g \in G} (S^{-1}R)_g \) and \( h(S^{-1}M) = \bigcup_{g \in G} (S^{-1}M)_g \). Consider the graded homomorphism \( \eta : M \to S^{-1}M \) defined by \( \eta(m) = m/1 \). For any graded submodule \( N \) of \( M \), the submodule of \( S^{-1}M \) generated by \( \eta(N) \) is denoted by \( S^{-1}N \). Similar to non graded case, one can prove that \( S^{-1}N = \{ \beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S \} \) and that \( S^{-1}N \neq S^{-1}M \) if and only if \( S \cap (N :_RM) = \phi \). If \( K \) is a graded submodule of an \( S^{-1}R \)-module \( S^{-1}M \), then \( K \cap M \) will denote the graded submodule \( \eta^{-1}(K) \) of \( M \). Moreover, similar to the non graded case one can prove that \( S^{-1}(K \cap M) = K \). For more details, one can refer to [4].

### 2. Some Properties of Graded Quasi-Prime Submodules

In this section, we define the graded quasi-prime submodules and give some of their basic properties.

**Definition 2.1.** A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to
be a graded quasi-prime if whenever $K_1$ and $K_2$ are graded submodules of $M$ with $K_1 \cap K_2 \subseteq N$, either $K_1 \subseteq N$ or $K_2 \subseteq N$.

The following lemma is known, but we write it here for the sake of references.

**Lemma 2.2.** [3, Lemma 2.1] Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then the following hold:

(i) If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \cap J$ are graded ideals.

(ii) If $N$ is a graded submodule of $M$, $r \in h(R)$, $x \in h(M)$ and $I$ is a graded ideal of $R$, then $Rx, IN$ and $rN$ are graded submodules of $M$.

(iii) If $N$ and $K$ are graded submodules of $M$, then $N + K$ and $N \cap K$ are also graded submodules of $M$ and $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of $R$.

(iv) Let $\{N_\lambda\}$ be a collection of graded submodules of $M$. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodules of $M$.

Recall that a proper graded submodule $N$ of a graded $R$-module $M$ is said to be graded irreducible if for each graded submodules $K_1$ and $K_2$ of $M$, $N = K_1 \cap K_2$ implies that either $N = K_1$ or $N = K_2$.

**Theorem 2.3.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded submodule of $M$. If $N$ is a graded quasi-prime submodule of $M$, then $N$ is a graded irreducible submodule of $M$.

**Proof.** Assume that $N$ is a graded quasi-prime submodule of $M$ and $K_1, K_2$ are graded submodules of $M$ such that $N = K_1 \cap K_2$. Since $N$ is a graded quasi-prime submodule and $K_1 \cap K_2 \subseteq N$, we have either $K_1 \subseteq N$ or $K_2 \subseteq N$ and hence either $N = K_1$ or $N = K_2$. Thus $N$ is a graded irreducible submodule.

**Theorem 2.4.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded quasi-prime submodule of $M$. If $V$ is a graded submodule contained in $N$, then $N/V$ is a graded quasi-prime submodule of $M/V$.

**Proof.** Let $K_1$ and $K_2$ be graded submodules of $M$ such that $(K_1/V) \cap (K_2/V) \subseteq N/V$. Then $K_1 \cap K_2 = (K_1 + V) \cap (K_2 + V) \subseteq N + V = N$. Since $N$ is a graded quasi-prime submodule, either $K_1 \subseteq N$ or $K_2 \subseteq N$. It follows that either $K_1/V \subseteq N/V$ or $K_2/V \subseteq N/V$. Thus $N/V$ is a graded quasi-prime submodule.

In the following theorem, we give a characterization of graded quasi-prime submodules.

**Theorem 2.5.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a proper graded submodule of $M$. Then the following statements are equivalent.

(i) $N$ is a graded quasi-prime submodule of $M$. 

(ii) For every pair of elements \( m, m' \in h(M) \) such that \( mR \cap m'R \subseteq N \), either \( m \in N \) or \( m' \in N \).

Proof. (i) \( \Rightarrow \) (ii) This follows from Lemma 2.2(2) and the definition of graded quasi-prime.

(ii) \( \Rightarrow \) (i) Let \( K_1 \) and \( K_2 \) be graded submodules of \( M \) such that \( K_1 \cap K_2 \subseteq N \) and \( K_1 \not\subseteq N \). Then there exists an element \( k_h \in (K_1 \cap h(M)) \setminus N \). Now, let \( g \in G \) and set \( m = \sum_{g \in G} m_g \in K_2 \). Then for all \( g \in G \), \( k_hR \cap m_gR \subseteq K_1 \cap K_2 \subseteq N \). By our assumption, we obtain \( m_g \in N \). So \( m \in N \), which indicates that \( K_2 \subseteq N \). Thus \( N \) is a graded quasi-prime submodule of \( M \).

Recall that a graded \( R \)-module \( M \) is called graded multiplication if for each graded submodule \( N \) of \( M \), \( N = (N :_R M)M \), (see [5, Definition 2]). Also, a proper graded submodule \( N \) of a graded \( R \)-module \( M \) is called graded prime submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \), either \( r \in (N :_r M) \) or \( m \in N \), (see [2, Definition 2.2]). The following result provides some conditions under which a graded prime submodule is graded quasi-prime.

**Theorem 2.6.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded multiplication \( R \)-module and \( N \) a graded submodule of \( M \). If \( N \) is a graded prime submodule of \( M \), then \( N \) is a graded quasi-prime.

Proof. Assume that \( N \) is a graded prime and let \( K_1, K_2 \) be graded submodules of \( M \) such that \( K_1 \cap K_2 \subseteq N \) but \( K_1 \not\subseteq N \) and \( K_2 \not\subseteq N \). Since \( M \) is a graded multiplication, \( K_1 = J_1M \) and \( K_2 = J_2M \) for some graded ideals \( J_1 \) and \( J_2 \) of \( R \). So there are \( j_1 \in J_1 \cap h(R) \), \( j_2 \in J_2 \cap h(R) \) and \( m_1, m_2 \in h(M) \) such that \( j_1m_1 \notin N \) and \( j_2m_2 \notin N \). Since \( N \) is a graded prime submodule and \( j_1j_2m_1 \in K_1 \cap K_2 \subseteq N \), we conclude that \( j_2 \in (N :_R M) \), i.e., \( j_2M \subseteq N \). So \( j_2m_2 \in N \), a contradiction. Thus \( N \) is graded quasi-prime.

**Lemma 2.7.** Let \( R \) be a \( G \)-graded ring and \( M \) a faithful graded multiplication \( R \)-module. Then \( \bigcap_{\alpha \in \Delta} (I_\alpha M) = \left( \bigcap_{\alpha \in \Delta} I_\alpha \right)M \) where \( I_\alpha \) is a graded ideal of \( R \).

Proof. See [5, Theorem 8].

A proper graded ideal \( P \) of a graded ring \( R \) is said to be graded quasi-prime if for graded ideals \( J_1 \) and \( J_2 \) of \( R \), the inclusion \( J_1 \cap J_2 \subseteq P \) implies that either \( J_1 \subseteq P \) or \( J_2 \subseteq P \).

**Theorem 2.8.** Let \( R \) be a \( G \)-graded ring, \( M \) a faithful graded multiplication \( R \)-module and \( N \) a graded submodule of \( M \). Then \( N \) is a graded quasi-prime submodule of \( M \) if and only if \( (N :_R M) \) is a graded quasi-prime ideal of \( R \).

Proof. \( \Rightarrow \) Assume that \( N \) is a graded quasi-prime submodule. By Lemma 2.2(iii), \( (N :_R M) \) is a graded ideal. Let \( J_1 \) and \( J_2 \) be graded ideals of \( R \) such that \( J_1 \cap J_2 \subseteq \)
and hence \((N :_R M), i.e., (J_1 \cap J_2)M \subseteq N\). By Lemma 2.7, we have \((J_1 \cap J_2)M = (J_1M) \cap (J_2M) \subseteq N\). Since \(N\) is a graded quasi-prime submodule of \(M\), either \(J_1M \subseteq N\) or \(J_2M \subseteq N\) and so either \(J_1 \subseteq (N :_R M)\) or \(J_2 \subseteq (N :_R M)\). Thus \((N :_R M)\) is a graded quasi-prime ideal of \(R\).

\((\Leftarrow)\) Assume that \((N :_R M)\) is a graded quasi-prime ideal of \(R\) and let \(K_1, K_2\) be graded submodules of \(M\) such that \(K_1 \cap K_2 \subseteq N\). Then \((K_1 \cap K_2 :_R M) \subseteq (N :_R M)\) and hence \((K_1 :_R M) \cap (K_2 :_R M) \subseteq (N :_R M)\). Since \((N :_R M)\) is a graded quasi-prime ideal of \(R\), either \((K_1 :_R M) \subseteq (N :_R M)\) or \((K_2 :_R M) \subseteq (N :_R M)\). Since \(M\) is a graded multiplication, we conclude that either \(K_1 = (K_1 :_R M)M \subseteq (N :_R M)M = N\) or \(K_2 = (K_2 :_R M)M \subseteq (N :_R M)M = N\). Thus \(N\) is a graded quasi-prime submodule of \(M\).

The graded radical of a graded ideal \(I\), denoted by \(Gr(I)\), is the set of all \(x = \sum_{g \in G} x_g \in R\) such that for each \(g \in G\) there exists \(n_g > 0\) with \(x_g^{n_g} \in I\). Note that if \(r\) is a homogeneous element of \(R\), then \(r \in Gr(I)\) if and only if \(r^n \in I\) for some \(n \in \mathbb{N}\), (see [7, Definition 2.1]). Recall that a proper graded ideal \(P\) of \(R\) is said to be a graded prime ideal if whenever \(r, s \in h(R)\) with \(rs \in P\), then either \(r \in P\) or \(s \in P\), (see [7]). The following theorem shows the relationship between graded prime submodules and graded quasi-prime submodules.

**Theorem 2.9.** Let \(R\) be a \(G\)-graded ring, \(M\) a faithful graded multiplication \(R\)-module and \(N\) a graded submodule of \(M\) such that \(Gr((N :_R M) = (N :_R M)\). Then \(N\) is a graded quasi-prime submodule if and only if it is graded prime.

**Proof.** \((\Rightarrow)\) Assume that \(N\) is a graded quasi-prime submodule. By Theorem 2.8, \((N :_R M)\) is a graded quasi-prime ideal of \(R\). First, we show that \((N :_R M)\) is a graded prime ideal. Let \(I_1, I_2\) be graded ideals of \(R\) such that \(I_1I_2 \subseteq (N :_R M)\). Hence by [7, Proposition 2.4], we conclude that \(I_1 \cap I_2 \subseteq Gr(I_1 \cap I_2) \subseteq Gr(I_1I_2) \subseteq Gr((N :_R M)) = (N :_R M)\). Since \((N :_R M)\) is a graded quasi-prime ideal, either \(I_1 \subseteq (N :_R M)\) or \(I_2 \subseteq (N :_R M)\). So \((N :_R M)\) is a graded prime ideal by [7, Proposition 1.2]. It follows that \(N\) is a graded prime submodule of \(M\) by [5, Corollary 3].

\((\Leftarrow)\) Theorem 2.6.

The following results study the behavior of graded quasi-prime submodules under localization.

**Theorem 2.10.** Let \(N\) be a graded submodule of a graded \(R\)-module \(M\) and \(S \subseteq h(R)\) be a multiplicatively closed subset of \(R\). If \(S^{-1}N\) is a graded quasi-prime submodule of \(S^{-1}M\), then \(S^{-1}N \cap M\) is a graded quasi-prime submodule of \(M\).

**Proof.** Assume that \(S^{-1}N\) is a graded quasi-prime submodule and let \(K_1, K_2\) be graded submodules of \(M\) such that \(K_1 \cap K_2 \subseteq S^{-1}N \cap M\). It is easy to see that \(S^{-1}K_1 \cap S^{-1}K_2 \subseteq S^{-1}N\). Since \(S^{-1}N\) is a graded quasi-prime, either \(S^{-1}K_1 \subseteq S^{-1}N\) or \(S^{-1}K_2 \subseteq S^{-1}N\) and hence either \(K_1 \subseteq S^{-1}N \cap M\) or \(K_2 \subseteq S^{-1}N \cap M\). Thus \(S^{-1}N \cap M\) is a graded quasi-prime submodule.
Recall that a proper graded submodule $N$ of a graded $R$-module $M$ is said to be a graded primary submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in Gr((N :_R M))$ (see [5, Definition 6]).

**Lemma 2.11.** Let $N$ be a graded submodule of a graded $R$-module $M$ and $S \subseteq h(R)$ be a multiplicatively closed subset of $R$ such that $Gr((N :_R M)) \cap S = \phi$. If $N$ is a graded primary submodule of $M$, then $S^{-1}N \cap M = N$.

**Proof.** Let $x = \sum_{g \in G} x_g \in S^{-1}N \cap M$. Then for all $g \in G$, there are elements $n_{g} \in N \cap h(M)$ and $s \in S$ such that $x_{g} = \frac{m_{g}}{s}$. Hence there exists $t \in S$ such that $stx_{g} = tn_{g} \in N$. Since $N$ is a graded primary submodule and $Gr((N :_R M)) \cap S = \phi$, $x_{g} \in N$. So $x \in N$, which shows that $S^{-1}N \cap M \subseteq N$. The opposite inclusion is obvious. Thus $S^{-1}N \cap M = N$. \hfill $\Box$

**Theorem 2.12.** Let $N$ be a graded primary submodule of a graded $R$-module $M$ and $S \subseteq h(R)$ be a multiplicatively closed subset of $R$ such that $Gr((N :_R M)) \cap S = \phi$. If $N$ is a graded quasi-prime submodule of $M$, then $S^{-1}N$ is a graded quasi-prime submodule of $S^{-1}M$.

**Proof.** Assume that $N$ is a graded quasi-prime submodule of $M$ and let $K_{1}$, $K_{2}$ be graded submodules of $S^{-1}M$ such that $K_{1} \cap K_{2} \subseteq S^{-1}N$. Then $(K_{1} \cap M) \cap (K_{2} \cap M) \subseteq S^{-1}N \cap M$. By Lemma 2.11, $S^{-1}N \cap M = N$. Since $N$ is a graded quasi-prime submodule, either $K_{1} \cap M \subseteq N$ or $K_{2} \cap M \subseteq N$. So either $K_{1} = S^{-1}(K_{1} \cap M) \subseteq S^{-1}N$ or $K_{2} = S^{-1}(K_{2} \cap M) \subseteq S^{-1}N$. Thus $S^{-1}N$ is graded quasi-prime. \hfill $\Box$

3. Topology on the Graded Quasi-Prime Submodules

In this section, we introduce a topology on the set of graded quasi-prime submodules and some properties of this topology are given.

If $R$ is a $G$-graded ring and $M$ is a graded $R$-module, we consider $qSpec_{g}(M)$ which is the set of all graded quasi-prime submodules of $M$. We call $qSpec_{g}(M)$, the graded quasi-prime spectrum of $M$. For each subset $A \subseteq h(M)$, let $qV_{g}(A) = \{ P \in qSpec_{g}(M) : A \subseteq P \}$.

**Theorem 3.1.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then the following hold:

(i) For each subset $A \subseteq h(M)$, $qV_{g}(A) = qV_{g}(N)$, where $N$ is the graded submodule of $M$ generated by $A$.

(ii) $qV_{g}(0) = qSpec_{g}(M)$ and $qV_{g}(M) = \phi$.

(iii) If $\{ N_{\alpha} \}_{\alpha \in \Delta}$ is a family of graded submodules of $M$, then $\bigcap_{\alpha \in \Delta} qV_{g}(N_{\alpha}) = qV_{g}(\sum_{\alpha \in \Delta} N_{\alpha})$. 


For every pair \( N \) and \( K \) of graded submodules of \( M \), \( qV_g(N \cap K) = qV_g(N) \cup qV_g(K) \).

**Proof.** (i) – (iii) Clear.

(iv) Let \( N \), \( K \) be any graded submodules of \( M \) and \( P \in qV_g(N \cap K) \). Then \( N \cap K \subseteq P \). Since \( P \) is a graded quasi-prime submodule, either \( N \subseteq P \) or \( K \subseteq P \), i.e., \( P \in qV_g(N) \) or \( P \in qV_g(K) \). Hence \( qV_g(N \cap K) \subseteq qV_g(N) \cup qV_g(K) \). Other side of the inclusion is obvious. Thus \( qV_g(N \cap K) = qV_g(N) \cup qV_g(K) \).

Let \( q\zeta_g(M) = \{ qV_g(N) : N \) is a graded submodule of \( M \} \). Then \( q\zeta_g(M) \) contains the empty set and \( q\text{Spec}_g(M) \). Also, \( q\zeta_g(M) \) is closed under arbitrary intersections and finite unions. Therefore, \( q\zeta_g(M) \) satisfies the axioms for the closed sets of the unique topology \( q\tau_g \) on \( q\text{Spec}_g(M) \). Then the topology \( q\tau_g(M) \) on \( q\text{Spec}_g(M) \) is called the quasi-Zariski topology. Let \( X = q\text{Spec}_g(M) \). For every subset \( S \) of \( h(M) \), define \( X_S = X - qV_g(S) \). In particular, if \( S = \{ a \} \), then we denote \( X_S \) by \( X_a \).

**Theorem 3.2.** Let \( M \) be a graded \( R \)-module. Then the set \( \{ X_a : a \in h(M) \} \) is a basis for the quasi-Zariski topology on \( X \).

**Proof.** Let \( U \) be a non-void open subset of \( X \). Then \( U = X - qV_g(N) \) for some graded submodule \( N \) of \( M \). Assume that \( N \) is generated by \( A \subseteq h(M) \). Then \( U = X - qV_g(N) = X - qV_g(\bigcup_{a \in A} \{ a \}) = X - \bigcap_{a \in A} qV_g(a) = \bigcup_{a \in A} (X - qV_g(a)) = \bigcup_{a \in A} X_a. \)

For each graded submodule \( N \) of a graded \( R \)-module \( M \), we consider \( q\text{Gr}_M(N) = \{ P : P \) is a graded quasi-prime submodule of \( M \) containing \( N \} \).

**Lemma 3.3.** Let \( N \) be a graded submodule of a graded \( R \)-module \( M \). Then the following hold:

(i) \( qV_g(N) = qV_g(q\text{Gr}_M(N)) \).

(ii) For each graded submodule \( K \) of \( M \), \( qV_g(K) \subseteq qV_g(N) \) if and only if \( q\text{Gr}_M(N) \subseteq q\text{Gr}_M(K) \).

**Proof.** Clear

Recall that a topological space is said to be Noetherian if its closed sets satisfy the descending chain condition. Also, recall that a graded \( R \)-module \( M \) is called graded Noetherian if it is satisfies the ascending chain condition on graded submodules of \( M \).

**Theorem 3.4.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. If \( M \) is graded Noetherian, then \( q\text{Spec}_g(M) \) is a Noetherian topological space.

**Proof.** Let \( \cdots \subseteq qV_g(N_3) \subseteq qV_g(N_2) \subseteq qV_g(N_1) \) be a descending chain of closed subsets of \( q\text{Spec}_g(M) \), where \( \{ N_k \}_{k=1}^{\infty} \) is a family of graded submodules of \( M \).
By Lemma 3.3, we have $q\text{Gr}_M(N_1) \subseteq q\text{Gr}_M(N_2) \subseteq q\text{Gr}_M(N_3) \subseteq \cdots$. Since $M$ is graded Noetherian, there exists a positive integer $k$ such that $q\text{Gr}_M(N_k) = q\text{Gr}_M(N_{k+i})$ for each $i = 1, 2, 3, \ldots$. By Lemma 3.3, we conclude that

$$qV_g(N_k) = qV_g(q\text{Gr}_M(N_k)) = qV_g(q\text{Gr}_M(N_{k+i})) = qV_g(N_{k+i})$$

for all $i = 1, 2, 3 \ldots$ Thus $q\text{Spec}_g(M)$ is a Noetherian topological space. \hfill\box

**Acknowledgments.** The authors wish to thank sincerely the referees for their valuable comments and suggestions.

**References**

[1] R. Abu-Dawwas and M. Ali, *Comultiplication modules over strongly graded rings*, Int. J. Pure Appl. Math., 81(5) (2012), 693-699.

[2] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci., 33(1) (2006), 3-7.

[3] F. Farzalipour and P. Ghasvand, *On the union of graded prime submodules*, Thai. J. Math., 9(1) (2011), 49-55.

[4] C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory*, Mathematical Library 28, North Holand, Amsterdam, 1982.

[5] K. H. Oral, U. Tekir and A. G. Agargun, *On graded prime and primary submodules*, Turk. J. Math., 35 (2011), 159-167.

[6] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turk. J. Math., 28 (2004), 217-229.

[7] M. Refai, M. Hailat, and S. Obiedat, *Graded radicals and graded prime spectra*, Far East J. Math. Sci. (FJMS), Part I, (2000), 59-73.