GEOMETRY AND DUALITY IN SUPERSYMMETRIC $\sigma$-MODELS

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Abstract

The Supersymmetric Dual Sigma Model (SDSM) is a local field theory introduced to be nonlocally equivalent to the Supersymmetric Chiral nonlinear $\sigma$-Model (SCM), this dual equivalence being proven by explicit canonical transformation in tangent space. This model is here reconstructed in superspace and identified as a chiral-entwined supersymmetrization of the Dual Sigma Model (DSM). This analysis sheds light on the boson-fermion symphysis of the dual transition, and on the new geometry of the DSM.

1 Introduction and Conclusions

A long-standing broad question in field theory involves the equivalence of field theories which may appear very different. Historically, physicists have been comfortable with local changes of field variables (as in the Higgs mechanism) which correspond to point transformations in classical mechanics: these automatically preserve the canonical structure of the theory, i.e. the Poisson Brackets, and the canonical commutators upon quantization—all of which may also be addressed equivalently in the conventional functional integral formalism.

A subtler issue arises, however, when nonlocal transformations are considered, which link two local field theories. As a rule, nonabelian (t-)duality transformations in two dimensions (popular in string culture [1, 2, 3]), which broadly map gradients to curls, hence waves to solitons, are of this type: equivalence results when these are canonical of the non-trivial type that mixes canonical momenta with field gradients, which results in nonlocal maps. Conversely, failure of such transformations to preserve the canonical structure leads to striking inequivalences in such theories, (cf. the PCM, a double limit contraction of the WZWN $\sigma$-model, in [§ § §]).

1 i.e. preserve canonical commutation relations. Canonical transformations in quantum mechanics underlie Dirac’s path integral formulation [§], and have been discussed extensively in field theory, e.g. [§ § §].
Unfortunately, so far, there is no systematic theory of such transformations in field theory, and most discussions merely abstract and change notations on a handful of examples. The only examples available are in two dimensions: Sine-Gordon/Thirring [10]; CM/DSM, [8, 9]; and, finally, SCM/SDSM [11]. The second system was first introduced via first-order functional integral manipulations [12] and the equivalence was shown to be canonical [8] at the classical level, and argued [8] to extend to the quantized theory; moreover, a credible case has been made for complete equivalence of the respective S-matrices of the two theories [15].

The third system was introduced in [11] and illustrates some intricacies of the nonlocal canonical equivalence best, by dint of symphysis, which is the conjoining in the transition maps of fermion bilinears with bosons to yield transformed bosons, and of fermions with bosons to yield transformed fermions. It was first constructed via a generating functional Ansatz [11] and first-order lagrangean quadrature of the type [12], and not as the supersymmetrization of the bosonic DSM (which, unlike the CM, contains torsion). However, we demonstrate here that the same model may also be constructed by judiciously supersymmetrizing the bosonic DSM: in the past [13, 17], we have provided a general procedure of supersymmetrizing any given torsionful manifold, such as that of the DSM. Thus, a superfield construction of the same model from a different starting point sheds new light on the supersymmetry realization at work.

Here, we discuss more explicitly detailed aspects of the SDSM, with special emphasis on the special realization of supersymmetry and the superfields controlling it, to shed light on the intriguing chiral entwining structure at work. We review the bosonic case in Section 2; we then review the supersymmetric theories including tangent space supersymmetric actions in Section 3; we then summarize ref [11] by way of introduction of the SDSM, in Section 4. In Section 5, we proceed to rederive the SDSM via superfield extension of the DSM and Fridling-Jevicki-type quadrature in superspace, and explain and support special features observed before which appeared accidental. As a guide to insight of the field theory results, a brief review of canonical transformations of classical mechanics from the more modern, Poisson-Bracket-based point of view is provided in Appendix A. Subtle relations between curved manifold and tangent space in the dual theory, specifying a new inhomogeneous geometry are illustrated in Appendix B. Every effort is made to stay consistent to the conventions of ref [11]; we correct a typographical error in the part of the action involving a factor of 3/8 in the quartic fermion interaction.

2 Review of the Bosonic Theories

Recall the standard bosonic nonlinear Chiral Model (CM) on $O(4) \simeq O(3) \times O(3) \simeq SU(2) \times SU(2)$. In geometrical language,

$$\mathcal{L}_{CM} = \frac{1}{2} g_{ab} \partial_{\mu} \varphi^a \partial^{\mu} \varphi^b,$$

(2.1)

where $g_{ab}$ is the metric on the field manifold (three-sphere). Explicitly, with group elements parameterized as $U = \varphi^0 + i \tau^j \varphi^j$, $(j = 1, 2, 3)$, where $(\varphi^0)^2 + \varphi^2 = 1$, and $\varphi^2 \equiv \sum_j (\varphi^j)^2$, we may resolve $\varphi^0 = \pm \sqrt{1 - \varphi^2}$, to obtain

$$g_{ab} = \delta^{ab} + \varphi^a \varphi^b / (1 - \varphi^2),$$

(2.2)

and hence

$$\mathcal{L}_{CM} = \frac{1}{2} \left( \delta^{ij} + \frac{\varphi^i \varphi^j}{1 - \varphi^2} \right) \partial_{\mu} \varphi^i \partial^{\mu} \varphi^j = \frac{1}{2} J_{\mu}^i J^{i \mu},$$

(2.3)

\footnote{Ref [13] has introduced objections to this identification, based on observation of the respective effective actions at two loops, supported by [14]. We care to suppose that, with proper appreciation of the new underlying geometry, some form of identification may eventually go through.}
where we re-expressed the action in terms of tangent quantities through the use of either left- or right-invariant vielbeine (e.g. \([\ref{1}]\))—either choice yields this Sugawara current-current form. Choosing left-invariant dreibeine gives \(V + A\) currents which are vectors on the tangent space. In terms of the above explicit coordinates,

\[
U^{-1} \partial_{\mu} U = -i \tau^{i} J_{\mu}^{i},
\]

\[
J_{\mu}^{i} = -v_{a}^{i} \partial_{\mu} \varphi^{a},
\]

where

\[
v_{a}^{j} = \sqrt{1 - \varphi^{2}} g_{aj} + \varepsilon^{ajb} \varphi^{b}.
\]

Note that these currents are pure gauge, or curvature-free, such that

\[
\varepsilon^{\mu \nu} \left( \partial_{\mu} J_{\nu}^{i} + \varepsilon^{ijk} J_{\mu}^{j} J_{\nu}^{k} \right) = 0.
\]

This is canonically equivalent to the Dual \(\sigma\)-Model [2] (DSM)\(^3\) with torsion and a new geometry:

\[
\mathcal{L}_{DSM} = \frac{1}{1 + 4 \Phi^{2}} \left( \frac{1}{2} \left( \delta^{ij} + 4 \Phi^{i} \Phi^{j} \right) \partial_{\mu} \Phi^{i} \partial_{\mu} \Phi^{j} - \varepsilon^{\mu \nu} \varepsilon^{ijk} \Phi^{i} \partial_{\mu} \Phi^{j} \partial_{\nu} \Phi^{k} \right). \tag{2.8}
\]

This new geometry is explained in Appendix B, and, in particular, how the tangent- and curved-space indices of the field/coordinates \(\Phi^{i}\) in this geometry are indistinguishable.

The generator for a canonical transformation relating \(\varphi\) and \(\Phi\) at any fixed time is the tangent space functional \(\mathcal{F}[\Phi, \varphi] = \int_{-\infty}^{\infty} dx \, \Phi^{i} \Phi^{j}[\varphi]\). Specifically,

\[
\mathcal{F}[\Phi, \varphi] = \int_{-\infty}^{\infty} dx \, \Phi^{i} \left( \sqrt{1 - \varphi^{2}} \frac{\partial}{\partial x} \varphi^{i} + \varepsilon^{ijk} \varphi^{j} \frac{\partial}{\partial x} \varphi^{k} \right). \tag{2.9}
\]

Although we originally constructed \(F\) in the hamiltonian framework by the indirect reasoning reviewed below, its structure is also evident within the lagrangean framework as follows. Treating \(J\) as independent variables in \(\mathcal{L}_{CM} = \frac{1}{2} J_{\mu}^{j} J^{\mu \mu} \), impose, with ref [12], the pure gauge condition on the currents by adding a Lagrange multiplier term, \(\mathcal{L}_{\lambda} = \Phi^{j} \varepsilon^{\mu \nu} (\partial_{\mu} J_{\nu}^{j} + \varepsilon^{jkl} J_{\mu}^{j} J_{\nu}^{l})\). Then, complete a square for the \(J^{i}\)’s and eliminate them from the dynamics (integrate them out) in favor of the DSM. But to do this, one first writes \(\mathcal{L}_{\lambda} = \partial_{\mu} \Phi^{j} \varepsilon^{\mu \nu} J_{\nu}^{i} - \varepsilon^{\mu \nu} J_{\nu}^{i} \partial_{\mu} \Phi^{j} + \varepsilon^{jk} \varepsilon^{\mu \nu} \Phi^{i} J_{\mu}^{j} J_{\nu}^{k}\). The total divergence term, integrated over a world-sheet with a constant time boundary, gives our generating functional relating the CM to the DSM\(^4\).

The conjugate momentum of \(\Phi^{i}\) specified by the generating functional is

\[
\Pi_{i} = \frac{\delta \mathcal{F}[\Phi, \varphi]}{\delta \dot{\Phi}^{i}} = \sqrt{1 - \varphi^{2}} \frac{\partial}{\partial x} \dot{\varphi}^{i} - \dot{\varphi}^{i} \frac{\partial}{\partial x} \left( \sqrt{1 - \varphi^{2}} \right) + \varepsilon^{ijk} \dot{\varphi}^{j} \frac{\partial}{\partial x} \dot{\varphi}^{k} = \left( \sqrt{1 - \varphi^{2}} \delta^{ij} + \frac{\dot{\varphi}^{i} \dot{\varphi}^{j}}{\sqrt{1 - \varphi^{2}}} - \varepsilon^{ijk} \dot{\varphi}^{k} \right) \frac{\partial}{\partial x} \dot{\varphi}^{j} = J_{i}^{\lambda}. \tag{2.10}
\]

\(^3\)This model was also considered in the last of ref [1] and the combination of [13].

\(^4\)Roughly speaking, the Lagrange multiplier dual field \(\Phi\) characterizes the ratio of normal magnitudes of the Sugawara lagrangian and the zero-curvature constraint, respectively, in the function space of currents, upon extremization. Recall constrained extremization of a surface \(f(x, y) - \lambda g(x, y)\) via the Lagrange multiplier \(\lambda\). Vanishing variation specifies that the constraint surface \(g(x, y) = 0\) touches the surface \(z = f(x, y)\) at the contact point \((x_{0}, y_{0}, z_{0} = f(x_{0}, y_{0}))\). The section plane \(z = z_{0}\) intersects the respective surfaces at two curves which are tangent to each other, at the point of contact: the normals to these two curves on this plane are parallel, the ratio of their magnitudes being \(\lambda\).
The conjugate of $\varphi^i$ is

$$\varpi_i = -\frac{\delta F[\Phi, \varphi]}{\delta \varphi^i} = \left( \frac{1}{\sqrt{1 - \varphi^2}} \delta^{ij} + \frac{\varphi^i \varphi^j}{\sqrt{1 - \varphi^2}} + \varepsilon^{ijk} \varphi^k \right) \frac{\partial}{\partial x} \Phi^j + \frac{2}{\sqrt{1 - \varphi^2}} \left( \varphi^i \Phi^j - \Phi^i \varphi^j \right) \frac{\partial}{\partial x} \varphi^j, \quad (2.11)$$

To solve for the fields themselves, e.g. $\Phi[\varphi]$, their canonical momenta may be eliminated through substitution for $\Pi_i$ and $\varpi_i$, in terms of $\partial \varphi^j / \partial t$ and $\partial \Phi^j / \partial t$, as follows from $\mathcal{L}_1$ and $\mathcal{L}_3$:

$$\Pi_i = \frac{1}{1 + 4\Phi^2} \left( (\delta^{ij} + 4\Phi^i \Phi^j) \frac{\partial}{\partial t} \Phi^j + 2\varepsilon^{ijk} \Phi^j \frac{\partial}{\partial x} \Phi^k \right), \quad \varpi_i = \left( \delta^{ij} + \frac{\varphi^i \varphi^j}{1 - \varphi^2} \right) \frac{\partial}{\partial t} \varphi^j. \quad (2.12)$$

However, the resulting transformation laws are complicated and nonlocal, as illustrated at the end of this section. Instead, it is relatively more instructive to simply identify the conserved, curvature-free current in the two theories, consistently to the above, an identification which will turn out to be local. It is then straightforward to exploit the current-current form of the respective Hamiltonian densities which will thus likewise identify.

Now, then, in the DSM, what is the conserved, curvature-free current? In contrast to the PCM, where it was essentially forced to be a topological current, here a topological current by itself will not suffice; neither will a conserved Noether current. (Under isospin transformations, $\delta \Phi^i = \varepsilon^{ijk} \Phi^j \omega^k$, the Noether current of $\mathcal{L}_3$ is $I^i = \delta \mathcal{L}_3 / \delta (\partial_t \omega^i)$ so $I^i = \varepsilon^{ijk} \Phi^j \Pi_k$, but it is not curvature-free.) Instead, the conserved, curvature-free current $\mathcal{J}^i[\Phi, \Pi] = J^i[\varphi, \varpi]$ (identified with $J^i[\Phi]$ of the CM) is a mixture of the Noether isocurrent and a topological current: $\mathcal{J}^i = 2I^i - \varepsilon^{ijk} \partial_i \Pi_k$, so that $\mathcal{J}^1 = \Pi_i$. Both conservation and curvature-freedom now hold on-shell, for the on-shell identified current $\mathcal{J}^i$.

$$\mathcal{J}^i = \frac{-1}{1 + 4\Phi^2} \left( (\delta^{ij} + 4\Phi^i \Phi^j) \varepsilon^{ij} \partial_i \Phi^j + 2\varepsilon^{ijk} \Phi^j \partial_k \Phi^k \right). \quad (2.13)$$

Nevertheless, the following canonical identifications of currents can be shown \cite{8} to hold off-shell:

$$\mathcal{J}^i_1 \equiv \Pi_i = \left( \frac{1}{\sqrt{1 - \varphi^2}} \delta^{ij} + \frac{\varphi^i \varphi^j}{\sqrt{1 - \varphi^2}} - \varepsilon^{ijk} \varphi^k \right) \frac{\partial}{\partial x} \varphi^j \equiv J^1_i, \quad (2.14)$$

$$\mathcal{J}^i_0 \equiv -\frac{\partial}{\partial x} \Phi^i - 2\varepsilon^{ijk} \Phi^j \Pi_k = -\sqrt{1 - \varphi^2} \varpi_i - \varepsilon^{ijk} \varphi^j \varpi_k \equiv J^0_i. \quad (2.15)$$

These two relations combine to integrate to the explicit nonlocal map \cite{3},

$$\Phi(x) \cdot \tau = U^{-1}(x) U(0) \Phi(0) \cdot \tau U^{-1}(0) U(x) + U^{-1}(x) \left( \int_0^x dy \, iU(y) \partial_y U^{-1}(y) \right) U(x) \quad . \quad (2.16)$$

The dual character of this nonlocal transition is manifest in the weak field limit. It bears repeating that canonical commutators of such complicated nonlocal expressions are simple and conventional, precisely because the transformation is canonical: e.g. equal time commutators of two expressions of this type at different space points $x$ and $z$ vanish, as these expressions are essentially local, despite formidable appearances!

Substituted into the Sugawara-current-current Hamiltonians, these locally identified currents \cite{2.14,2.15} further lead to mutual local identification of the respective Hamiltonian densities,

$$\mathcal{H}_{CM} = J_0 J_0 + J_1 J_1 = \mathcal{J}_0 \mathcal{J}_0 + \mathcal{J}_1 \mathcal{J}_1 = \mathcal{H}_{DSM} \quad . \quad (2.17)$$

\footnote{The equations of motion are necessary, as Hamilton’s equations have already been utilized in the elimination of the canonical momenta from the subsequent expressions.}
In terms of the dreibeine of the new dual geometry (discussed in Appendix B), the currents of the DSM read
\[ J^\mu_j = -(V_{(ja)} \varepsilon^{\mu\nu} \partial_\nu \Phi^a + V_{[ja]} \partial^\mu \Phi^a). \] (2.18)

The reader may contrast these currents with those of a WZWN model on a group manifold. The currents for the latter are obtained by keeping the vielbein intact: \[ J^\mu_j = -v^j_a (\partial_\mu \phi^a + \eta \varepsilon_{\mu\nu} \partial^\nu \phi^a). \] (For the CM, \( \eta = 0 \)). Also note that the DSM dreibein already appeared in the above action, \( L_{DSM} = \frac{1}{2} V^j_a \left( \partial_\mu \Phi^a \partial^\mu \phi^a + \varepsilon^{\mu\nu} \partial_\mu \Phi^a \partial_\nu \Phi^a \right) \).

### 3 Supersymmetric Theories

In a direct application of the general construction for supersymmetric \( \sigma \)-models with torsion [17, 11] (whose conventions we use), the supersymmetric extensions of the two bosonic models above through the addition of Majorana fermions, the SCM [18] without torsion, and the \( sdsm \) with torsion, are readily read off. In following sections, we canonically transform between the two. We first review this in component formalism [11] but in the next section we will provide a complementary picture in superspace which illuminates and confirms our construction.

The SCM is
\[ L_{SCM} = \frac{1}{2} \left( g_{ab} \partial_\mu \varphi^a \partial_\mu \varphi^b + i g_{ab} \overline{\psi}^a \gamma^\mu \psi^b + \frac{1}{6} R_{abcd} \overline{\psi}^a \psi^c \overline{\psi}^d \gamma^\mu \psi^b \right) \] (3.1)

where
\[ D_\mu \psi^b = \partial_\mu \psi^b + \Gamma^b_{cd} \partial_\mu \varphi^c \psi^d, \quad \Gamma^a_{bc} = \varphi^a \partial_{(b} \varphi^c \partial_{c)} \partial \psi^a, \quad R_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc} = R_{cdab}. \] (3.2)

\[ g^{ab} = \delta^{ab} - \varphi^a \varphi^b = (\delta^{ab} + \varepsilon^{ab} \varphi^a), \quad \varphi^a = \sqrt{1 - \varphi^2} \delta^{ab} \delta^{bc}, \quad \psi^a = \sqrt{1 - \varphi^2} g_{ab} + \varepsilon^{ab} \varphi^b. \] (3.3)

The Cartan-Maurer relations of the Dreibeine merit recall:
\[ \frac{1}{2} \left( \partial_a v^j_b - \partial_b v^j_a \right) = \varepsilon^{ijkl} v^k_a v^l_b = \varepsilon_{jab} + \frac{\delta^{ja} \varphi^b - \delta^{jb} \varphi^a}{\sqrt{1 - \varphi^2}}. \] (3.4)

The conserved currents now consist of the previous bosonic terms augmented by spinor bilinears.
\[ C^i_\mu = J^i_\mu + K^i_\mu, \quad J^i_\mu = -v^j_a \partial_\mu \varphi^a, \quad K^i_\mu = \frac{i}{2} \varepsilon^{ijkl} v^j_a v^k_b \gamma^a \gamma^b \varphi^l. \] (3.5)

The last term, explicitly, is given by the above results for the Cartan-Maurer relations,
\[ K^i_\mu = \frac{1}{2} i \varepsilon^{ab} \varphi^a \gamma_\mu \varphi^b - i \frac{\varphi^a \overline{\psi}^a \gamma_\mu \varphi^b}{\sqrt{1 - \varphi^2}}. \] (3.6)

This is to be expected: The tangent-space left-invariant spinor of (A.51) specified in ref [17],
\[ \chi^j = v^j_a \psi^a, \] (3.7)
transforms as \( \delta \chi^j = \varepsilon^{jkl} \xi^k \chi^l \) under a full \( V + A \) transformation. The right rotation in tangent space transforms linearly, and the spinor’s contribution to the corresponding current is that of a conventional...
isorotation. Recall (ref [17], (A.52), (A.29)) that now the lagrangean simplifies significantly, to a mere function of tangent-space spinors and currents:

$$\mathcal{L}_{SCM} = \frac{1}{2} \left( J_j^\mu J^\mu j + i \bar{\chi}^j \slashed{\partial} \chi^j + i \varepsilon^{jkl} \bar{\chi}^j \gamma_5 \chi^k + \frac{1}{4} \left( \bar{\chi} \chi \right)^2 \right).$$

(3.8)

This tangent space formulation is at the heart of the canonical transformation, as will become apparent in the following section.

The supersymmetry transformation laws are

$$\delta \chi^j = i \slashed{J}^j \epsilon - \frac{1}{2} \varepsilon^{jkl} (\gamma_5 \varepsilon \bar{\chi}^k \gamma_5 \chi^l + \gamma_\mu \varepsilon \bar{\chi}^k \gamma^\mu \chi^l), \quad \delta J_j^\mu = - \bar{\varepsilon} \left( \partial_\mu \chi^j + \varepsilon^{jkl} J_\mu^k \chi^l \right).$$

(3.9)

The bosonic generating functional can then be re-presented as

$$F[\Phi, \varphi] = \int_{-\infty}^{\infty} dx \Phi^{i} v^{i}_{a} \partial_{1} \varphi^{a}.$$ 

(3.10)

The above supersymmetry transformations follow from the general case:

$$\delta \varphi = \bar{\epsilon} \psi, \quad \delta \psi = (\mathfrak{F} - i \slashed{\partial} \varphi) \epsilon, \quad \delta \mathfrak{F} = - i \bar{\epsilon} \bar{\partial} \psi,$$

(3.11)

where we use the conventions of ref [17] eq (5.7),

$$2\mathfrak{F}^{a} = \Gamma^{a}_{bc} \bar{\psi} \gamma^{c} - S^{a}_{bc} \bar{\psi} \gamma^{c},$$

(3.12)

where the torsion $S$ vanishes here—but, of course, not for the $L_{SDSM}$ model, below.

In this case, the SCM equations of motion, written covariantly, directly lead to conservation of the supercurrent of ref [18]:

$$s_\mu = - i \bar{\psi} \gamma_\mu \varphi^a g_{ab} \psi^b = i \slashed{J}^j \gamma_\mu \chi^j.$$ 

(3.13)

Correspondingly, we may express the lagrangean in terms of conserved vector currents.

$$\mathcal{L}_{SCM} = \frac{1}{2} \left( C_j^\mu C^\mu j + i \bar{\chi}^j \slashed{\partial} \chi^j + \frac{3}{4} \left( \bar{\chi}^j \chi^j \right)^2 \right),$$

(3.14)

with supersymmetry transformations

$$\delta \chi^j = i C_j^\mu \epsilon - 2 \gamma_\mu \varepsilon C_k^{jkl} (\bar{\chi}^k \gamma_\mu \chi^l), \quad \delta C_j^\mu = - \bar{\epsilon} \left( \partial_\mu \chi^j + \varepsilon^{jkl} C_\mu^k \gamma_\mu \chi^l \right).$$

(3.15)

Taking $n = 3/2$ in ref [17], the conventional supersymmetrization $L_{SDSM}$ of the DSM (which, unlike the SCM, contains torsion) is readily seen to be

$$L_{SDSM} = \frac{1}{2} \left( G_{ab} \partial_\mu \Phi^{a} \partial^{\mu} \Phi^{b} + i G_{ab} \slashed{\partial} \Phi^{a} \psi^{b} + E_{ab} \varepsilon^{\mu\nu} \partial_\mu \Phi^{a} \partial_\nu \Phi^{b} + \frac{1}{8} R_{abcd} \bar{\psi}^{a} (1 + \gamma_\mu) \psi^{b} (1 + \gamma_\nu) \psi^{d} \right),$$

(3.16)

with a new geometry elaborated in Appendix B. The curly-covariant derivative on the fermions is

$$\mathcal{D}_\mu \psi^{a} = \partial_\mu \psi^{a} + \Gamma^{a}_{bc} \psi^{b} \partial_\mu \psi^{c} - S^{a}_{bc} \psi^{b} \varepsilon_{\mu\nu} \partial^{\nu} \psi^{c},$$

(3.17)

in terms of the tensors of the new geometry specified in Appendix B.
4 Canonical Equivalence of the Supersymmetric Models

If these two supersymmetric theories are canonically equivalent like their bosonic limits, a generating functional in tangent space for such a canonical transformation is needed. Taking into consideration dimensional consistency, Lorentz-invariance, and a good free-field limit, we posit the Ansatz for the supersymmetric theory in tangent space relating $\varphi$ and $\chi$ at any fixed time to $\Phi$ and $X$ (the bosons and fermions of the dual theory):

$$F[\Phi, X, \varphi, \chi] = \int dx \left( \Phi^j J^1_j [\varphi] - \frac{i}{2} \overline{X^j \gamma^1 \chi} \right) = \int dx \left( \Phi^i (\sqrt{1 - \varphi^2} \frac{\partial}{\partial x} \varphi^i + \varepsilon^{ijk} \varphi^j \varphi^k) - \frac{i}{2} \overline{X^j \gamma^1 \chi} \right).$$

(4.1)

Classically, the canonical conjugate to $\chi$,

$$\pi_\chi \equiv \delta L_{SCM} / \delta \partial_0 \chi = -i \chi^\dagger / 2$$

(4.2)
is obtained from $F$ as

$$- \delta F / \delta \chi = -i X^\dagger \gamma_p / 2,$$

(4.3)
where $\gamma_p \equiv \gamma^0 \gamma^1$. So under the canonical transformation

$$\chi^j = \gamma_p X^j.$$

(4.4)

Likewise, the momentum conjugate to $X$ is

$$\delta F / \delta X = -i \chi^\dagger \gamma_p / 2,$$

(4.5)
leading to

$$\pi_X \equiv -i X^\dagger / 2,$$

(4.6)
which specifies part of the dual lagrangean.

This chiral rotation of the fermions reflects the duality transition of their bosonic superpartners, whose gradients map to curls in the weak field limit, as already noted. (For a mathematically subtle interpretation of tangent space canonical transformation in terms of Cartan's equivalence problem see ref [19].)

As a result of (4.4), the equal-time anticommutation relations for Majorana spinors in tangent space,

$$\{ \chi^j(x), \chi^k(y) \} = \{ X^j(x), X^k(y) \} = 2 \delta^{ijk} \delta(x - y),$$

(4.7)
are preserved in the above transformation, thus confirming its identification as canonical.

To handle the bosons, it may be advantageous to resort to the first order Lagrangean quadrature mentioned before. Adding the customary pure-gauge-enforcing Lagrange multiplier to the tangent space SCM and integrating by parts leads to

$$L_{SCM\lambda} = \frac{1}{2} \left( J^j \gamma^1 J^1_j - 2 \varepsilon^{\mu \nu} J^j_\mu \partial_\nu \Phi^j + \varepsilon^{ijk} \partial_\mu \gamma^i \partial_\nu \gamma^j \partial_\tau \gamma^k + \frac{1}{4} (\bar{\chi} \chi)^2 \right),$$

(4.8)
where

$$M^{\mu \nu}_{ab} \equiv g^{\mu \nu} \delta_{ab} + 2 \varepsilon^{\mu \nu} \varepsilon_{abc} \Phi^c,$$

(4.9)
with an inverse which satisfies $M^{\mu \nu}_{ab} (N^\lambda_\nu)_{bc} = g^{\mu \lambda} \delta_{ac}$:

$$N^{\mu \nu}_{ab} = g^{\mu \nu} G_{ab} + \varepsilon^{\mu \nu} E_{ab}. $$

(4.10)
The crucial transition bridge to the SDSM relies on the bosonic current encountered in section 2,
\[ J_{ij}^\mu = -N_{ij}^{\mu
u} \varepsilon_{\nu\lambda} \partial^\lambda (\Phi_j^k - \frac{1}{1 + 4\Phi^2} ((\delta_{ij} + 4\Phi^i_j^k)\varepsilon_{\mu\nu} \partial_\nu \Phi_j^i + 2\varepsilon_{ijk} \Phi_i^j \partial^\mu \Phi^k)), \] (4.11)
which conjoins with the fermionic bilinear component into
\[ J^{\mu\nu} = -N_{ij}^{\mu
u} (\varepsilon_{\nu\lambda} \partial^\lambda (\Phi_j^k) - \frac{i}{2} \varepsilon_{bcd} \gamma_\nu \chi^d) = J^{\mu j} + N_{jk}^{\mu\nu} K_\nu^k, \] (4.12)
which follows from varying \( J \) in the first order lagrangean \( L_{SCML} \). (Note from \[1.4\], \( K_{j}^\nu |X| = K_{j}^\nu [X] \).) This is the on-shell relation linking \( J \) to \( \Phi \). Below, this is shown to be derivable from the generating functional \( F_s \).

The lagrangean resulting from Fridling-Jevicki-type \[2\] substitution for \( J \), or, equivalently, completing the quadratic in \( L_{SCML} \) and integrating the shifted \( J_s \) out, is \[1\]:
\[ L_{SDSM} = \frac{1}{2} \chi^j_\partial \chi^j + \frac{1}{8} (\chi_\chi)^2 - \frac{1}{2} (\varepsilon_{\mu\lambda} \partial^\lambda (\Phi_j^k - \frac{2}{1 + 4\Phi^2} (\delta_{ij} + 4\Phi^i_j^k) \varepsilon_{\mu\nu} \partial_\nu \Phi_j^i + 2\varepsilon_{ijk} \Phi_i^j \partial^\mu \Phi^k)) \]
\[ = \frac{i}{2} \chi^j_\partial \chi^j + \frac{i}{2} \varepsilon_{ijk} \chi_j \chi_k + (\chi_\chi^j + \varepsilon_{ijk} \chi_j \chi_k) - \frac{2}{1 + 4\Phi^2} \varepsilon_{ijk} \chi_j \chi_k (K_{ij}^k + 2\chi^{ij} \chi_k \chi(k^j)) \] (4.13)
This result forces the variation \( \delta/\delta \Phi^a \) of the generating functional to be just the \( \varphi \) bosonic current \( J \), while the \( \chi^j = \gamma_p \chi^{ij} \)'s are regarded as independent variables. As a result, the generating functional automatically yields
\[ \Pi = J^1 = J^1 + (N \cdot K)^1. \] (4.14)

We now use the variation w.r.t. \( \varphi \) to match the timelike components as well. The arguments of the bosonic model ref \[8\] eqs (3.11, 3.12), which connect timelike components of currents) remain exactly as they were,
\[ -\frac{\partial}{\partial x} \Phi^i - 2\varepsilon_{ijk} \Phi_j^k \Pi_k = -\sqrt{1 - \varphi^2} \varpi_i - \varepsilon_{ijk} \varphi^j \varpi_k \] (4.15)
since both sides are equal to
\[ -\frac{\partial}{\partial x} \Phi^i + 2\Phi^j \left( \varphi^j \frac{\partial}{\partial x} \varphi^i - \varphi^i \frac{\partial}{\partial x} \varphi^j \right) + 2\varepsilon_{ijk} \Phi_j^k \left( \varphi^k \frac{\partial}{\partial x} \sqrt{1 - \varphi^2} - \sqrt{1 - \varphi^2} \frac{\partial}{\partial x} \varphi^k \right); \]
but now \( \Pi \) contains an additional fermionic piece beyond its bosonic component, as seen above. And likewise \( \varpi \):
\[ \varpi^i = \text{bosonic} \quad + K_0^j (\sqrt{1 - \varphi^2} \delta^i_j + \varphi^i \varphi^j / \sqrt{1 - \varphi^2} + \varepsilon_{ij} \varphi^j) \] (4.16)
These then introduce fermionic current pieces in the above eq (4.15),
\[ J_0^i - 2\varepsilon_{ijk} \Phi_j^k (N \cdot K)^1 = J_0^i - K_0^1 \] (4.17)
As a direct consequence,
\[ J^0 = J^0 + (N \cdot K)^0, \] (4.18)
and the above pivotal identification of currents (4.12) holds. Note how duality is implemented on these manifolds: \textit{Bosons in one theory contain both bosons and fermions in the dual theory.}

Further note that the conserved current is
\[ J^{\mu j} + N_{jk}^{\mu\nu} K_\nu^k + K^{\mu j}. \] (4.19)
The supercurrent for the DSM is
\[ \mathcal{S}_\mu = i \gamma_\mu (\mathcal{J}^j + \mathcal{N} \cdot K^j) \gamma_\mu X^j, \] (4.20)
which holds by the symphysis identification of currents (4.12) above. Consequently, supercharges identify, and whence hamiltonians (which are their squares), energy-momentum tensors, and all such quantities conventionally constrained by supersymmetry.

From the structure of this supercurrent, it then follows that this is a “chirally twisted” realization of supersymmetry:
\[ \delta \Phi^a = \bar{\epsilon}(X^a + 2 \gamma_\mu \epsilon^{abk} \Phi^b X^k), \] (4.21)
which preserves the parity of the original \( X \), and thus of the action. This is the combination entering into the spinors of the \( \mathcal{L}_{SDSM} \) model (the fermion entering in the respective superfield of the next section) for comparison with the SDSM:
\[ \Psi^a = X^a + 2 \gamma_\mu \epsilon^{abj} \Phi^b X^j = (V^{(aj)} + \gamma_\mu V^{[aj]})X^j. \] (4.22)
\[ (1 + \gamma_\mu)\Psi^a = (1 + \gamma_\mu)V^{aj}X^j, \] (4.23)
\[ (1 - \gamma_\mu)\Psi^a = (1 - \gamma_\mu)V^{aj}X^j, \] (4.24)
where \( V^{ai} \) is the transpose of \( V^{ai} \). Consequently, its inverse \( \bar{V}_{ai} = G_{ai} - E_{ai} \) is also the transpose of \( V_{ai} \). As a result,
\[ X_j = (V_{(aj)} + \gamma_\mu V^{[aj]}) \Psi^a. \] (4.25)
This last relation strongly echoes eq (2.13), which finds its explanation in the superfield formulation below.

It turns out that \( \mathcal{L}_{SDSM} = \mathcal{L}_{sdsm} \). The pure bosonic piece of \( \mathcal{L}_{SDSM} \) above is, naturally, \( \mathcal{L}_{DSM} \). To further match with the pieces of \( \mathcal{L}_{sdsm} \) quadratic and quartic in fermion fields, respectively, we need check the following two relations.

For the function \( W \) in \( \frac{1}{2} \bar{X} \gamma^m \Phi^m W_{km} X^k \), one needs show
\[ \epsilon^{ikj} G_{jm} \gamma_\mu + \epsilon^{ikj} E_{jm} = 2(-\delta^{ai} \gamma_\mu + 2 \epsilon^{ain} \Phi^n)G_{ab} \epsilon^{bkm} + (\delta^{ai} - 2 \gamma_\mu \epsilon^{ain} \Phi^n)(\Gamma_{abm} - \gamma_\mu S_{abm})(\delta^{bk} - 2 \gamma_\mu \epsilon^{bks} \Phi^s), \] (4.26)
which, indeed, holds.

Secondly, for the fermion quartic terms, use is made of the group-Fierz identities
\[ (\Phi \cdot \bar{X} \cdot X \cdot \Phi)(\epsilon^{jkl} \Phi^i \Phi^j \gamma_\mu X^k) = (\Phi^2 \bar{X} \cdot X)(\epsilon^{jkl} \Phi^i \Phi^j \gamma_\mu X^k) = \frac{1}{2} \Phi^2 \epsilon^{jkl} (\Phi \cdot \bar{X} X^l)(\Phi^i \gamma_\mu X^k), \] (4.27)
to prove
\[ (\bar{X}^i X^i) \left( \frac{3}{8} (\bar{X}^j X^j) - \frac{2}{(1 + 4 \Phi^2)}(\Phi^i \Phi^j \Phi^k \gamma_\mu X^k) + \frac{1}{(1 + 4 \Phi^2)}(\epsilon^{jkl} \Phi^i \Phi^j \gamma_\mu X^k) \right) = \]
\[ = \frac{1}{16} R_{abcd} \bar{\Psi}^a (1 + \gamma_\mu) \Psi^c \Psi^b (1 + \gamma_\mu) \Psi^d = \frac{1}{16} R_{abcd} \bar{V}^a_j \gamma_\mu V^b_i \gamma_\mu V^d_i \gamma_\mu V^j (1 + \gamma_\mu) X^i X^j (1 + \gamma_\mu) X^l. \] (4.28)
Consequently, supersymmetrization of the mutually dual bosonic models produces mutually dual theories, as demonstrated:
\[ \begin{array}{ccl}
CM & \xrightarrow{\text{susy}} & SCM \\
\downarrow \text{dual} & & \downarrow \text{dual} \\
DSM & \xrightarrow{\text{susy}} & SDSM = DSM.
\end{array} \] (4.29)

In these variables, the supercurrent for the DSM now reads,
\[ \mathcal{S}_\mu = i \Phi^a \gamma_\mu G_{ab} \Psi^b + i \gamma_\mu \gamma^\nu \gamma_\mu (N \cdot K)^{\nu j} (V_{(aj)} + \gamma_\mu V^{[aj]}) \Psi^a. \] (4.30)
which may now be compared to eqn (3.13).
5 Superspace Formulation

General superfield formulations for supersymmetric \(\sigma\)-models with torsion are given in ref \[17\]. Recall

\[
D = \frac{\partial}{\partial \theta} - i \partial \theta, \quad \overline{D} = - \frac{\partial}{\partial \theta} + i \partial \theta, \quad Q = \frac{\partial}{\partial \theta} + i \partial \theta.
\]  

(5.1)

The scalar superfield for the supersymmetric chiral model, SCM, is

\[
\varphi^a = \psi^a + \frac{1}{2} \theta \sigma^a
\]

(5.2)

and has superderivatives

\[
D \varphi^a = \psi^a + \left(\sigma^a - i \partial \varphi^a\right) \theta + \frac{i}{2} \theta \partial \bar{\psi} \varphi^a,
\]

\[
\overline{D} \varphi^a = \bar{\psi}^a + \bar{\theta} \left(\sigma^a + i \partial \varphi^a\right) - \frac{i}{2} \partial \theta \overline{\bar{\psi} \varphi^a}.
\]

(5.3)

From this, one may construct the bilinear

\[
\overline{D} \varphi^a \gamma_\mu \varphi^b = \overline{\psi} \gamma_\mu \psi^b + \overline{\psi} \gamma_\mu \left(\sigma^b - i \partial \varphi^b\right) \theta - \overline{\psi} \gamma_\mu \left(\sigma^a - i \partial \varphi^a\right) \theta
\]

\[
+ \partial_{\mu} \left[\frac{i}{2} \overline{\psi} \gamma_\mu \psi^b - \varphi^a \partial_{\mu} \varphi^b\right] \varepsilon^{\mu \nu \theta}.
\]

(5.4)

The corresponding superfield for the dual theory is

\[
\Phi^a = \Phi^a + \theta \Psi^a + \frac{1}{2} \theta \eta^a
\]

(5.5)

hence

\[
D \Phi^a = \Psi^a + \eta^a \theta + \frac{i}{2} \partial \theta \Phi^a.
\]

(5.6)

The dual transition between two field theories, however, is normally effected in tangent space. To address tangent space in superspace, start from the chiral element superfield

\[
G = U(x) \left(1 + i \theta \chi \cdot \tau + \frac{\theta \partial \theta}{2} (i Z \cdot \tau + i \chi \cdot \tau)\right),
\]

(5.7)

a most general Ansatz, such that, for unconstrained \(\chi\) and \(Z\),

\[
G^\dagger \ G = 1.
\]

(5.8)

Recalling \(U^{-1} \partial_{\mu} U = -i \tau \cdot J_{\mu}\), obtain its superspace analog \[20\], the spinorial current superfield,

\[
G^{-1} \overline{D} \overline{G} = i \chi \cdot \tau + (i Z \cdot \tau - \partial \cdot \tau) \theta - \frac{i}{2} \gamma_\mu \theta \varepsilon^{ijkl} \tau^i \gamma^j \gamma^k \gamma^l - \frac{i}{2} \gamma_\mu \theta \varepsilon^{ijkl} \tau^i \gamma^l \gamma^k \gamma^j
\]

\[
+ \frac{\theta \partial \theta}{2} (2i \varepsilon^{ijkl} \tau^j Z^k \chi^l - \partial \chi \cdot \tau - 2 \varepsilon^{ijkl} \partial \cdot \tau \chi^l + i \chi \cdot \tau \partial \chi)\ ,
\]

(5.9)

notably valued in the SU(2) Lie algebra (unlike \(G\)). Consequently, the tangent space SCM action is specified in superspace by

\[
\mathcal{L}_{sfield} = \frac{1}{8} \int d^2 \theta \text{Tr} \overline{D} \overline{G} \overline{DG} = \frac{1}{2} \left(Z^2 + J_\mu \cdot J^\mu + i \chi \cdot \partial \chi + i \varepsilon^{ijkl} \chi^i \partial^j \chi^l + \frac{1}{4} (\overline{\chi} \cdot \chi)^2\right).
\]

(5.10)
Thus, $Z = 0$ on-shell, matching the component supersymmetry transformations on the tangent space objects of the SCM exhibited in eq (3.9). The variational result for the (right-) conservation law in superspace is thus

$$\int d^2 \theta \ \overline{D}(G^{-1}D) = -2i\partial_{\mu} \left( J^\mu \cdot \tau + \frac{i}{2} \epsilon^{ijkl} \tau^j \gamma^k \chi^l \right), \quad (5.11)$$

which again identifies with the conserved current.

Now consider the superspace curvature identity (20)

$$D\gamma^p (G^{-1}D) + (G^{-1}D) \gamma^p (G^{-1}D) = 0. \quad (5.12)$$

This is an identity. But, if the pure gauge form of $J_\mu$ is not noted/utilized in the current superfield (only 6), now dubbed for this purpose $\mathcal{J}$, it turns to a constraint instead, with all components vanishing identically, except for the $\theta \theta$ term which consists of (only) the zero-curvature constraint for $J_\mu$:

$$D\gamma^p \mathcal{J} + \mathcal{J} \gamma^p (G^{-1}D) = i \theta \theta \tau^j \epsilon_{\mu \nu} \left( \partial_\mu J^\nu + \epsilon^{jkl} J^k \gamma^l \right) = 0. \quad (5.13)$$

As a consequence, enforcing this constraint through a Lagrange-multiplier superfield $\Phi$ in an appendage to the superspace action eq (5.10):

$$L_{\text{sfield}} = -\frac{1}{4} \int d^2 \theta \ Tr \left( \frac{1}{2} \mathcal{J} + i \Phi \cdot \tau \left( D\gamma^p \mathcal{J} + \mathcal{J} \gamma^p \right) \right) \quad (5.14)$$

will only involve $\Phi^i$ but not $\Psi$ or $Y$, as already observed in practice in the component calculation in the previous section. Thus, Fridling-Jevicki-type (12) quadrature in superspace will lead to a superfield formulation of the SDSM, below.

Even though the fermions $\Psi$ appeared to be projected out and thus superfluous “gauge freedom” components of the original $\Phi$ superfield, they emerge in the final answer below (20). Superspace integration by parts yields the usual quadratic,

$$= -\frac{1}{4} \int d^2 x \int d^2 \theta \ Tr \left( i \mathcal{J} \gamma^p D\Phi \cdot \tau + i \Phi \cdot \tau \mathcal{J} \gamma^p \right) + \frac{1}{2} \mathcal{J} \mathcal{J} \quad (5.15)$$

$$= -\frac{1}{4} \int d^2 x \int d^2 \theta \left( 2i \mathcal{J} \gamma^p D\Phi^j + \mathcal{J} \mathcal{M}^{jk} \mathcal{J}^k \right)$$

$$= \frac{1}{4} \int d^2 x \int d^2 \theta \overline{D}\Phi \mathcal{N}^{jk} D\Phi^k - (\mathcal{J} - \overline{D}\Phi \gamma^p \mathcal{N}^{mn} \gamma^m \mathcal{N}^{nj}) \mathcal{M}^{jk} (\mathcal{J}^k + i \mathcal{N}^{kn} \gamma^p \overline{D}\Phi^n) , \quad (5.16)$$

and a total divergence term discussed below, where

$$\mathcal{M}^{jk} = \delta^{jk} - 2\gamma^p \epsilon^{jkl} \Phi^l . \quad (5.16)$$

This is the dreibein-twisted-chiral structure observed before, and it is inherited by its inverse

$$\mathcal{N}^{jk} = G_{jk}(\Phi) - \gamma_p E_{jk}(\Phi) , \quad (5.17)$$

also already encountered in the previous section. The customary elimination of

$$J^k = -i \mathcal{N}^{kj} \gamma^p \overline{D}\Phi^j , \quad (5.18)$$

That is, we did assume in the current superfield the nontrivial positioning imposed on the fermions and the auxiliary fields, which thus satisfy the equation identically.
identifies, to $O(\theta^0)$, $\chi_j = -\gamma_p(G_{jk} - \gamma_p E_{jk})\Psi^k$ encountered in eq (4.25); and the further terms are expected to yield the current symphysis formulas. The algebraic equations for the auxiliary fields result from substitution into the superspace action,

$$\int \mathcal{L}_{\text{field}} = \frac{1}{4} \int d^2x \int d^2\theta \left( D\Phi^i N^{jk} D\Phi_k \right) = \frac{1}{4} \int d^2x \int d^2\theta \left( D\Phi^i (G^{jk} - \gamma_p E^{jk}) D\Phi_k \right). \quad (5.19)$$

This action automatically coincides with $\mathcal{L}_{\text{sdsm}}$, as it is the superfield formulation of a $\sigma$-model with torsion possessing a given geometry, (cf., e.g., eqn (5.3) of ref [17]), in this case this new geometry already discussed.

The generating functional $F$ with spinors should emerge out of the total divergence term resulting from the above superspace integration by parts:

$$-\frac{1}{4} \int d^2x \int d^2\theta \text{Tr} D\gamma_p (i\Phi \cdot \tau J) = \int d^2x \int d^2\theta \left( \frac{\theta}{2} \partial^\nu \varepsilon_{\nu\mu} (J^\mu j \Phi^j + \frac{i}{2} \bar{\chi} j \gamma^\mu \Psi^j - \frac{i}{2} \bar{\chi} j \gamma^\mu \varepsilon^{jkl} \Phi^k \chi^l), \right. \quad (5.20)$$

i.e.

$$= \int dx \left( J^1 j \Phi^j + \frac{i}{2} \bar{\chi} j \gamma^1 \Psi^j - \frac{i}{2} \bar{\chi} j \gamma^1 \varepsilon^{jkl} \Phi^k \chi^l \right). \quad (5.21)$$

However, superfield formulations of general $\sigma$-models with torsion always contain a total divergence of a fermion bilinear term in the reduction from superspace, cf. eqn (5.8) in [17]. In this case, this extra term contributes $-\frac{i}{2} \bar{\Psi} j \gamma^1 \varepsilon^{jkl} \Phi^k \Psi^l/(1 + 4\Phi^2)$ to the above, to produce the generating functional

$$F' = \int dx \left( J^1 j \Phi^j + \frac{i}{2} \bar{\chi} j \gamma^1 \Psi^j - \frac{i}{2} \bar{\chi} j \gamma^1 \varepsilon^{jkl} \Phi^k \chi^l - \frac{i}{2} \bar{\Psi} j \gamma^1 \varepsilon^{jkl} \Phi^k \Psi^l \right). \quad (5.22)$$

Variation with respect to $\bar{\chi}$ yields $\bar{\pi}_\chi$, hence

$$\gamma_p \chi^j = \Psi^j - 2\varepsilon^{jkl} \Phi^k \chi^l, \quad (5.23)$$

so that

$$\gamma_p \chi^j = (V_{\langle aj} + \gamma_p V_{\langle aj}) \Psi^a, \quad (5.24)$$

the right-hand-side fermion actually being $X$ as introduced in (4.25). Note that we never really introduced tangent space fermions for the dual theory in superspace—they emerge as an output. Moreover, variation with respect to $\bar{\Psi}$ yields $-\bar{\pi}_\Psi$ for $\mathcal{L}_{\text{sdsm}}$, hence

$$\gamma_p G_{jk} \Psi^k = \chi^j + E_{jk} \Psi^k, \quad (5.25)$$

and hence

$$\gamma_p \chi^j = G_{jk} \Psi^k - \gamma_p E_{jk} \Psi^k = X^j, \quad (5.26)$$

consistently to the above.

Eliminating $\Psi's$ in favor of $X's$ in $F'$ yields

$$F' = \int dx \left( J^1 j \Phi^j + \frac{i}{2} \bar{\chi} j \gamma^1 X^j - \frac{i}{2} (\bar{\chi}^j + \bar{X}^j \gamma^p) \gamma^1 \varepsilon^{jkl} \Phi^k \chi^l - \gamma^p X^l \right). \quad (5.27)$$

This provides an alternate route to the same theory, and the final term vanishes given the above results of the transformation.
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A Appendix: Overview of Canonical Transformations in Mechanics

We review some rarely emphasized aspects of canonical transformations in classical mechanics: we take as our starting point invariance of Poisson Brackets (PB), instead of the more conventional preservation of Hamilton’s equations. Poisson Brackets are suitable for eventual quantization, \( \{u, v\} \rightarrow \frac{1}{\hbar}[u, v] \), as they turn into canonical commutators. Extension to field theory merely involves arraying a continuum of modes, and transcription of sums into integrals, as exemplified in the text.

Poisson Brackets can be defined as

\[
\{u, v\}_\text{qp} = \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial q}.
\]  

(A.1)

PB’s are antisymmetric; linear; they obey the Jacobi identity (from associativity of the underlying operators); and convert by the chain rule:

\[
\{u, v\}_\text{QP} = \{u, v\}_\text{qp} \{q, p\}_\text{QP}.
\]  

(A.2)

Now the transformations

\[
(q, p) \rightarrow (Q(q, p), P(q, p))
\]  

(A.3)

are called canonical when they yield trivial Jacobians,

\[
\{Q, P\}_\text{qp} = 1 \quad \text{hence} \quad \{q, p\}_\text{QP} = 1,
\]  

(A.4)

so that they preserve the Poisson Brackets (“canonical invariants”) of their functions. Equivalently,

\[
\{q, p\} = \{Q, P\},
\]  

(A.5)

in any basis. Following Poincaré, the measure of phase-space area/volume is seen to be preserved,

\[
dQdP = dqdp \{Q, P\}.
\]  

(A.6)

For instance, point transformations (which generalize to usual local field redefinitions in field theory) are

\[
Q = J(q), \quad P = p/J'(q) \quad \Rightarrow \quad \{J(q), p/J'(q)\}_\text{qp} = 1.
\]  

(A.7)

Now consider the transformation generated by the hybrid function \( F_2(q, P) \):

\[
p = \frac{\partial F_2(q, P)}{\partial q}, \quad Q = \frac{\partial F_2(q, P)}{\partial P},
\]  

(A.8)
where, in principle, the first equation can be inverted to produce \( P(q,p) \), which is then substituted into the second to yield \( Q \) as a function of \( q,p \). (The point transformation just illustrated is generated by \( F_2 = PJ(q,p) \).

This transformation is canonical, seen explicitly as follows.

Define \( p = \partial F_2(q,P)/\partial q \big|_P = F' \) and \( Q = \partial F_2(q,P)/\partial P \big|_q = F'' \), contradistinguishing the arguments being differentiated. Now switch basis to \( q,p \), and consider \( P \) as a function of \( q,p \), partly specified via the partial differential equation \( \partial p/\partial p = 1 = F'P,q \). It is then straightforward to show in this \( q,p \) basis that

\[
1 = \{q, F'\} = \{F', P\}, \tag{A.9}
\]

since the middle expression equals 1 of the l.h.s. by the above differential equation, and, by the same token, the r.h.s. also equals

\[
(F''P,q + F'P_p)P_q = 1. \tag{A.10}
\]

Infinitesimally, this transformation is also easily seen to be canonical, as follows. Introduce a generating function infinitesimally expanded around the identity through an expansion parameter \( w \):

\[
F_2(q,P) = qP - wG(q,P). \tag{A.11}
\]

The \( O(w^0) \) piece is the identity, as

\[
p = P - w\frac{\partial G(q,P)}{\partial q}, \quad Q = q - w\frac{\partial G(q,P)}{\partial P}. \tag{A.12}
\]

To leading order in \( w \), one can substitute \( G(q,p) \) for \( G(q,P) \), so that

\[
Q - q = -w\frac{\partial G(q,p)}{\partial p} + O(w^2) = w\{G,q\}_{qp} + O(w^2), \tag{A.13}
\]

\[
P - p = w\frac{\partial G(q,p)}{\partial q} + O(w^2) = w\{G,p\}_{qp} + O(w^2). \tag{A.14}
\]

Then it is easy to see, by the Jacobi identity, that this transformation is canonical to \( O(w^2) \):

\[
\{q + w\{G,q\}, p + w\{G,p\}\} = 1 + w\{G,\{q,p\}\} = 1. \tag{A.15}
\]

Actually, the full, exponentiated transformation

\[
Q = e^{w\{G,q\}}q \equiv q + w\{G,q\} + w^2\{G,\{G,q\}\}/2! + ... \tag{A.16}
\]

\[
P = e^{w\{G,p\}}p \equiv p + w\{G,p\} + w^2\{G,\{G,p\}\}/2! + ... \tag{A.17}
\]

is canonical to all-orders in \( w \),

\[
Y(w) \equiv \{Q,P\} = 1, \tag{A.18}
\]

for essentially the same reason.

**Proof:** Note that, from the very definition of the Hadamard exponential operator above, for any \( w \)-independent \( a \), such as \( q \) or \( p \),

\[
\frac{d}{dw}(e^{w\{G,a\}}) = \{G, e^{w\{G,a\}}\}. \tag{A.19}
\]

Consequently, by the Jacobi identity:

\[
\frac{dY(w)}{dw} = \{\{G,Q\}, P\} + \{Q,\{G,P\}\} = \{G,Y(w)\}. \tag{A.20}
\]
Now \( Y(w) = 1 + \sum_{n=1}^{\infty} w^n y_n \) yields the recursion
\[
y_{n+1} = \{G, y_n\}, \tag{A.21}
\]
and all the \( y_{n>0} = 0 \) by induction based on \( y_0 = 1 \). Consequently \( Y(w) = 1 \). \( \blacksquare \)

Note this is a standard integrated Lie-evolution, but it is still hard to fully connect to the unfolded general \( F_2 \).

The other three types of canonical transformation can be obtained through the trivial symplectic reflection, which is also canonical:
\[
(q, p) \mapsto (Q = -p, P = q) \quad \text{since} \quad \{-p, q\} = 1, \tag{A.22}
\]
and thus may combine with other canonical transformations to yield more canonical transformations, by the chain rule of PBs. Applied to the variables \( Q, P \) of the above \( F_2 \), it produces the canonical transformation, generated by the same functional form, now called \( F_1 \):
\[
p = \frac{\partial F_1(q, Q)}{\partial q}, \quad P = -\frac{\partial F_1(q, Q)}{\partial Q}, \tag{A.23}
\]
and likewise for the other two types.

The Field Theoretical construction in this article on the \( \sigma \)-model is of the \( F_1(q, Q) \) type, but it could be trivially converted into the \( F_2(q, P) \) type by this trivial symplectic reflection (readily generated by \( F_1(q, Q) = qQ \)). The classical mechanics analog of our transformation is \( F_1(q, Q) = Q J(q) \), which then goes by the above symplectic reflection to
\[
F_2(q, P) = -PJ(q), \tag{A.24}
\]
s.t.
\[
-\frac{\partial F_2(q, P)}{\partial P} = Q, \quad -\frac{\partial F_2(q, P)}{\partial q} = p, \tag{A.25}
\]
thus a point transformation:
\[
Q = J(q) \tag{A.26}
\]
\[
p = P \frac{\partial J(q)}{\partial q}, \quad \text{hence} \quad P = \frac{p}{\partial J(q)/\partial q}. \tag{A.27}
\]
In other words, the canonical transformation used, in more conventional \( F_2 \) language, is simply a transition from the \( q \)'s to the \( J(q) \)'s with the standard point determinant scaling for the momenta to preserve the PB's ,
\[
\{J(q), p \frac{1}{\partial J(q)/\partial q}\} = 1. \tag{A.28}
\]
In field theory, it is a transition from \( \varphi \) to \( J_1(\varphi) \). But interchange of \( \Phi \)'s and \( \Pi \)'s in the Hamiltonian of the DSM, yields something unconventional, involving space derivatives of the \( \Pi \)'s.

Now, how is the identity transformation generated by \( F_1(q, Q) \), and why should one choose to base the discussion on \( F_2 \), in the first place? Some awkward features of the \( F_1(q, Q) \) generating function have been pointed out by Schwinger [23]. The inverse canonical transformation is, evidently,
\[
F_1(Q, q) = -F_1(q, Q). \tag{A.29}
\]

The composition of two successive transformations \( (q, p) \mapsto (\tilde{q}, \tilde{p}) \mapsto (Q, P) \) is simply generated by the the \textit{sum} of the respective generating functions,
\[
W(q, Q) = F_1(q, \tilde{q}) + F_1(\tilde{q}, Q), \tag{A.30}
\]
where each term generates the respective piece of the total transformation, and the dependence on the intermediate point (“superfluous variable”) vanishes between the two in the total transformation,

$$\frac{\partial W(q, Q)}{\partial \bar{q}} = 0.$$  \hfill (A.31)

But, by the above defined inverse, the identity should be then generated by $W(q, Q) = 0$. This singularity of the generator can be made more palatable by retaining the superfluous variable $\bar{q}$ which serves as a Lagrange multiplier:

$$W(q, Q) = \bar{q}(q - Q),$$  \hfill (A.32)

hence

$$\frac{\partial W}{\partial \bar{q}} = 0 \implies q = Q,$$  \hfill (A.33)

$$\frac{\partial W}{\partial q} = \bar{q} = -\frac{\partial W}{\partial Q} \implies p = P.$$  \hfill (A.34)

*Motion is a canonical transformation*, of the above infinitesimal $F_2$ type. When $G$ is chosen to be the hamiltonian $H$, and $w = -dt$,

$$dq = dt \{q, H\}_{qp}, \quad dp = dt \{p, H\}_{qp},$$  \hfill (A.35)

which comprise Hamilton’s Equations,

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q,$$  \hfill (A.36)

dictating incompressible flow of the phase fluid, $\partial \dot{q}/\partial q + \partial \dot{p}/\partial p = 0$. (As seen above, such a transformation readily exponentiates.) Consequently, for any function $f(q, p)$,

$$\{f(q, p), H\}_{qp} = \frac{df}{dt},$$  \hfill (A.37)

Now consider some arbitrary canonical transformation to $Q, P$, and take $f$ to be $Q$ and $P$, respectively; switching PB basis by virtue of the canonical nature of the transformation, one sees directly that

$$\dot{Q} = \{Q, H\}_Q = \partial H/\partial P, \quad \dot{P} = \{P, H\}_P = -\partial H/\partial Q.$$  \hfill (A.38)

That is, any canonical transformation also preserves Hamilton’s Equations of motion:

$$q_t \xrightarrow{H} q_T \xleftarrow{F} q_t \xrightarrow{H} q_T.$$  \hfill (A.39)

Motion is also generated by the Action Integral $S$ (Hamilton’s principal function), but on the classical path, via an $F_1$ transformation; this is the one utilized by Dirac in his celebrated quantum Hamilton-Jacobi functional integral \[3\]:

$$p_t = \frac{\delta \int_T^t d\tau L}{\delta q_t} \quad p_T = -\frac{\delta \int_T^t d\tau L}{\delta q_T}.$$  \hfill (A.40)

The intermediate-times variables $q(t)$ are not arbitrary, but must be specified by the equations of motion. To appreciate this, recall that the action integral may be effectively regarded as a sum of infinitesimal transformation generators of type $F_1$, namely $dt L(q + \frac{Q}{2}, \frac{q - Q}{dt})$; thus

$$\delta \int_T^t d\tau L = \int_T^t d\tau \delta q(\frac{\delta L}{\delta q} - \frac{d}{dt} \frac{\delta L}{\delta q_t} + \frac{\delta L}{\delta q_t} \delta q_t - \frac{\delta L_T}{\delta q_T} \delta q_T).$$  \hfill (A.41)
Vanishing of the first term in parenthesis (the Euler-Lagrange equations of motion) to yield the above result is dictated by the requirement of independence from the (intermediate times) superfluous variables. That is, the requirement of continuous canonical transformation for motion underlies the classical variational principle. Dirac [4] discovered that in Quantum Mechanics the generator must be exponentiated and the superfluous variables must be integrated over instead—the above classical path is then only the contribution to leading order in \( \bar{\hbar} \).

B Appendix: Explicit Geometry of the Dual Models

The dual sigma model is not a WZWN model on a group manifold. The geometry of the DSM is described in detail by the following, in the conventions of ref [17].

\[
G_{ab} = V^j_a V^j_b = \frac{1}{1+4\Phi^2} \left( \delta_{ab} + 4\Phi^a\Phi^b \right),
\]

\[
\det G = 1/(1+4\Phi^2)^2,
\]

\[
E_{ab} = \frac{1}{1+4\Phi^2} \left( -2\varepsilon^{abc}\Phi^c \right),
\]

\[
V^j_a = G_{aj} + E_{aj} = \frac{1}{1+4\Phi^2} \left( \delta_{aj} + 4\Phi^a\Phi^j - 2\varepsilon^{ajc}\Phi^c \right),
\]

\[
\det V^j_a = \sqrt{\det G} = (1+4\Phi^2)^{-1}.
\]

\[
V^{aj} = \delta_{aj} - 2\varepsilon^{ajc}\Phi^c,
\]

\[
G^{-1}_{ab} = V^{aj}V^{bj} = \left( 1 + 4\Phi^2 \right) \delta_{ab} - 4\Phi^a\Phi^b.
\]

Note that, in this remarkable geometry, base- and target-space indices are innocuously interchangeable for our choice of coordinates (\( \Phi^i \)'s), since \( \Phi^a = G_{ab}\Phi^b = \Phi^a, \Phi^h = \Phi^aV^a_h \).

Connections are obtained in the usual way:

\[
\Gamma_{abc} = \frac{1}{2} \left( \partial_b G_{ac} + \partial_c G_{ab} - \partial_a G_{bc} \right)
= \frac{4}{1+4\Phi^2} \left( \Phi^a\delta_{bc} + \Phi^bG_{ac} - \Phi^bG_{ac} - \Phi^cG_{ab} \right),
\]

\[
\Gamma^a_{bc} = \frac{4}{1+4\Phi^2} \left( \Phi^a\delta_{bc} + \Phi^bG_{ac} - \Phi^b\delta_{ac} - \Phi^c\delta_{ab} \right)
= \frac{16\Phi^a\Phi^b\Phi^c}{(1+4\Phi^2)^2} + 8\Phi^a\delta_{bc} \left( 1 + 4\Phi^2 \right)/(1+4\Phi^2)^2 - \frac{4\Phi^b\delta_{ac} + 4\Phi^c\delta_{ab}}{(1+4\Phi^2)^2},
\]

\[
S_{abc} = \frac{1}{2} \left( \partial_a E_{bc} + \partial_b E_{ca} + \partial_c E_{ab} \right)
= \frac{(3 + 4\Phi^2)}{(1+4\Phi^2)^2} \left(-\varepsilon^{abc}\right).
\]

Note that the DSM dreibein does not satisfy Cartan-Maurer equations. Rather,

\[
\partial_a V^j_b - \partial_b V^j_a + 4\varepsilon^{jkl}V^k_a V^l_b = -4\frac{1-4\Phi^2}{(1+4\Phi^2)^2} \left( \Phi^a\delta_{bj} - \Phi^b\delta_{aj} \right) - \frac{16}{(1+4\Phi^2)^2} \left( \Phi^b\varepsilon^{ajc}\Phi^c - \Phi^a\varepsilon^{bjc}\Phi^c \right).
\]
where
\[
\varepsilon^{jkl} V^k_a V^l_b = \frac{-2}{1 + 4\Phi^2} \left( \Phi^a \delta_{bj} - \Phi^b \delta_{aj} \right) + \frac{1}{1 + 4\Phi^2} \varepsilon^{abj}.
\] (B.12)

Alternatively,
\[
\partial_b V^j_a - \partial_a V^j_b = \frac{-4}{1 + 4\Phi^2} \varepsilon^{abc} G_{cj} - 4\varepsilon^{abc} S_{cjd} \Phi^d.
\] (B.13)

Contrast these to the corresponding relations for the CM,
\[
\partial_a v^j_b - \partial_b v^j_a = 2\varepsilon^{jkl} v^k_a v^l_b = 2\varepsilon^{abj} + \frac{-2}{\sqrt{1 - \varphi^2}} \left( \varphi^a \delta_{bj} - \varphi^b \delta_{aj} \right).
\] (B.14)

However, note that, in contrast to group manifolds,
\[
S_{abc} = \frac{3 + 4\Phi^2}{1 + 4\Phi^2} \varepsilon^{jkl} V^k_a V^l_b.
\] (B.15)

So, e.g.,
\[
\partial_a G_{bj} - \partial_b G_{aj} = 2S_{abc} V^{[cj]}.
\] (B.16)

Hence
\[
\Gamma_{[abc]} = S_{abd} V^{[de]}.
\] (B.17)

The geometry follows from direct, albeit lengthy computation (using Maple, http://daisy.waterloo.edu/):
\[
D_a S_{abc} = \frac{16}{(1 + 4\Phi^2)^3} \Phi^d \varepsilon_{abc},
\] (B.18)

\[
R^a_{bcd} = \frac{4}{(1 + 4\Phi^2)^3} \left( \frac{3 + 12\Phi^2 + 16\Phi^4}{(1 + 4\Phi^2)^4} \left( \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right) \right)
+ \left( 12 + 16\Phi^2 \right) \Phi^b \left( \Phi^c \delta^{ad} - \Phi^d \delta^{ac} \right),
\] (B.19)

\[
S_{fcd} S_{eb}^f - S_{fac} S_{db}^f = \frac{(3 + 4\Phi^2)^2}{(1 + 4\Phi^2)^4} \left( \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right),
\] (B.20)

\[
S_{dfe}^a S_{eb}^f - S_{cef} S_{db}^f = \frac{(3 + 4\Phi^2)^2}{(1 + 4\Phi^2)^3} \left( \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right).
\] (B.21)

Thus, the torsionful Riemann tensor amounts to
\[
\mathcal{R}^a_{bcd} = R^a_{bcd} - S_{dfe}^a S_{eb}^f + S_{cef} S_{db}^f + D_v S_{bd}^a - D_d S_{bc}^a
= \frac{3}{(1 + 4\Phi^2)} \left( \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right)
+ \frac{4}{(1 + 4\Phi^2)^2} \left( \Phi^b \left( \Phi^c \delta^{ad} - \Phi^d \delta^{ac} \right) + \frac{16}{(1 + 4\Phi^2)^3} \Phi^a \left( \Phi^d \delta^{bc} - \Phi^c \delta^{bd} \right) \right)
+ \frac{16}{(1 + 4\Phi^2)^3} \left( \Phi^e \varepsilon_{abd} - \Phi^d \varepsilon_{abc} \right) + \frac{64}{(1 + 4\Phi^2)^3} \Phi^f \Phi^j \left( \Phi^d \varepsilon_{bcf} - \Phi^c \varepsilon_{bdf} \right). \] (B.22)
Actually, curly-R without a raised index is a little easier to look at, since it has the usual antisymmetries under interchange of pairs of indices, but not the symmetry under interchange of pairs of pairs that the torsionless curvatures have.

\[
R_{abcd} = \frac{3}{(1 + 4\Phi^2)^2} \left( \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} \right)
+ \frac{4}{(1 + 4\Phi^2)^3} \left( \Phi^a \left( \Phi^d\delta^{bc} - \Phi^c\delta^{bd} \right) - \Phi^b \left( \Phi^d\delta^{ac} - \Phi^c\delta^{ad} \right) \right)
+ \frac{16}{(1 + 4\Phi^2)^3} \left( \Phi^c\varepsilon_{abd} - \Phi^d\varepsilon_{abc} \right).
\] (B.24)

There is an alternate route to this torsionful curvature. The spin connection [17] is worked out to be

\[
\Omega_{a}^{ij} = \frac{V^{bi} D_a V^j_b}{(1 + 4\Phi^2)^2} \equiv V^{bi} \left( D_a V^j_b + S_{abc} V^{cj} \right)
= \frac{1}{(1 + 4\Phi^2)^2} \left[ (14 + 24\Phi^2) \left( \delta^{a\hat{i}}\Phi^{\hat{j}} - \delta^{a\hat{i}}\Phi^{\hat{j}} \right) + (5 + 4\Phi^2) \left( \varepsilon^{a\hat{i}j} + 4\Phi^a\varepsilon^{ijm}\Phi^m \right) \right].
\] (B.25)

This is not the spin connection on any group manifold. On the one hand, first differentiate the spin connection and then combine it with its square, antisymmetrically, to yield the curly-curvature:

\[
\partial_a \Omega_{b}^{ij} - \partial_b \Omega_{a}^{ij} + \Omega_{a}^{ik} \Omega_{b}^{kj} - \Omega_{b}^{ik} \Omega_{a}^{kj} = \frac{1}{(1 + 4\Phi^2)^3} \left( 3 \times (1 + 4\Phi^2)^2 \left( \delta^{b\hat{j}}\delta^{a\hat{i}} - \delta^{b\hat{i}}\delta^{a\hat{j}} \right)
- 16 \times (4\Phi^2 + 5) \left( \Phi^a \left( \delta^{b\hat{j}}\Phi^{\hat{i}} - \delta^{b\hat{i}}\Phi^{\hat{j}} \right) - \Phi^b \left( \delta^{a\hat{j}}\Phi^{\hat{i}} - \delta^{a\hat{i}}\Phi^{\hat{j}} \right) \right)
- 2 \times (16\Phi^4 + 24\Phi^2 - 11) \left( \Phi^a\varepsilon^{b\hat{j}j} - \Phi^b\varepsilon^{a\hat{j}j} \right) \right).
\] (B.26)

On the other hand, for our dreibein

\[
(\mathcal{R}^{ij})_{ab} = V^{ci} V^{dj} R_{cdab} = (1 + 4\Phi^2) \mathcal{R}_{ijab} + 4 \left( \Phi^i \mathcal{R}_{jmb} - \Phi^j \mathcal{R}_{imab} \right) \Phi^m + \left( \varepsilon^{cd\hat{i}}\Phi^{\hat{j}} - \varepsilon^{cd\hat{j}}\Phi^{\hat{i}} \right) \mathcal{R}_{cdab}, \quad (B.27)
\]

which gives explicit agreement with the RHS of the previous expression (providing some algebraic checks) so

\[
(\mathcal{R}^{ij})_{ab} = \partial_a \Omega_{b}^{ij} - \partial_b \Omega_{a}^{ij} + \Omega_{a}^{ik} \Omega_{b}^{kj} - \Omega_{b}^{ik} \Omega_{a}^{kj}, \quad (B.28)
\]
as it should be.

Recall, for three-dimensional manifolds, the Weyl tensor vanishes (even when torsion is present, as in the model at hand), permitting us to re-express the Riemann tensor in terms of Ricci and scalar curvatures:

\[
R_{abcd} = \frac{1}{2} \left( G_{ad} G_{bc} - G_{ac} G_{bd} \right) \mathcal{R} + G_{ac} \mathcal{R}_{bd} - G_{bd} \mathcal{R}_{ac} - G_{bc} \mathcal{R}_{ad} \quad (B.29)
\]

\[
= \left( \mathcal{R}_{ac} - \frac{1}{4} G_{ac} \mathcal{R} \right) G_{bd} - \left( \mathcal{R}_{ad} - \frac{1}{4} G_{ad} \mathcal{R} \right) G_{bc} + G_{ac} \left( \mathcal{R}_{bd} - \frac{1}{4} G_{bd} \mathcal{R} \right) - G_{bd} \left( \mathcal{R}_{ac} - \frac{1}{4} G_{ac} \mathcal{R} \right). \quad (N.B. \ a, b, c = 1, 2, 3 \ only.)
\]

On the other hand, for three-dimensional manifolds, the issue of conformal flatness is decided not by the Weyl tensor, but rather by the Cotton tensor [22], obtained by taking derivatives of the Ricci and scalar curvature combinations exhibited in the last equality. Without torsion, the Cotton tensor is defined as

\[
C_{abc} = D_c \left( R_{ab} - \frac{1}{4} G_{ab} \mathcal{R} \right) - D_b \left( R_{ac} - \frac{1}{4} G_{ac} \mathcal{R} \right). \quad (B.30)
\]
The manifold is conformally equivalent to a flat space iff $C_{abc} = 0$. It is straightforward to check that $C_{abc} = 0$ for the dual sigma model. (This is also true, rather obviously, for the usual chiral model.)

Along these lines, it is interesting to note the same linear combination of Ricci and scalar curvature appears in the quartic spinor terms for a general supersymmetric model defined on a three-dimensional manifold. Taking into account the Majorana property of the spinors, and making Fierz rearrangements, gives

$$\mathcal{R}_{abcd} \overline{\Psi}^a (1 + \gamma_p) \Psi^b \overline{\Psi}^c (1 + \gamma_p) \Psi^d = 4G_{ab} \left( \mathcal{R}^b_{\cd} - \frac{1}{4} G_{cd} \mathcal{R} \right) \overline{\Psi}^c (1 + \gamma_p) \Psi^d$$

$$= 4G_{ab} \left( \mathcal{R}_{(cd)} - \frac{1}{4} G_{cd} \mathcal{R} \right) \overline{\Psi}^c \Psi^d + 4G_{ab} \overline{\Psi}^a \Psi^b \mathcal{R}_{[cd]} \overline{\gamma_p \Psi}^d .$$

(N.B. $a, b, c = 1, 2, 3$ only.)

Another way to say this, valid for higher dimensional manifolds, is to write the quartic term in general cases using the Weyl tensor:

$$C_{abcd} = \mathcal{R}_{abcd} + \frac{1}{2} (G_{ac} \mathcal{R}_{bd} - G_{ad} \mathcal{R}_{bc} - G_{bd} \mathcal{R}_{ac} + G_{bc} \mathcal{R}_{ad} .$$

$$= \mathcal{R}_{abcd} \overline{\Psi}^a (1 + \gamma_p) \Psi^b \overline{\Psi}^c (1 + \gamma_p) \Psi^d =$$

$$= C_{abcd} \overline{\Psi}^a (1 + \gamma_p) \Psi^b \overline{\Psi}^c (1 + \gamma_p) \Psi^d + 4G_{ab} \overline{\Psi}^a \Psi^b \left( \mathcal{R}_{cd} - \frac{1}{4} G_{cd} \mathcal{R} \right) \overline{\Psi}^c (1 + \gamma_p) \Psi^d .$$

Specification of the curly curvature now allows evaluation of the curly-Ricci tensor, crucial to the computation of the $\beta$-function to one loop [17]:

$$\mathcal{R}_{(bd)} = \frac{2}{(1 + 4\Phi^2)^2} \left( (3 + 16\Phi^4) \delta^{bd} - 4 (3 + 32\Phi^2 + 16\Phi^4) \Phi^b \Phi^d \right) .$$

There is also the antisymmetric part of the curly-Ricci tensor,

$$\mathcal{R}_{[bd]} = D_a S_{bd} = \frac{16}{(1 + 4\Phi^2)^3} \epsilon^{bd} \Phi_f = -8 (\det G) E_{ab} .$$

(We disagree here with the corresponding expressions of ref [13], and agree with [4]; we are grateful to L. Palla for communicating to us this result before publication). Finally, there is the scalar curly-curvature:

$$\mathcal{R} = \frac{2}{(1 + 4\Phi^2)^2} (3 - 4\Phi^2)^2 = 2 \left( \frac{4}{1 + 4\Phi^2} - 1 \right)^2 ,$$

with small and large field limits, 18 and 2, respectively. I.e. the curvature starts as a sphere of $18/r^2$, goes to zero at $\Phi^2 = 3/4$, and stabilizes asymptotically to $2/r^2$.

It goes without saying that full appreciation of this new geometry should hold the key to renormalization of the dual models beyond one loop.

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