In his recent letter [1], Braak suggested that a regular spectrum of the Rabi model was given by the zeros of a transcendental function \(G_\pm(x)\) (cf Eqs. (3)-(5) of Ref. [1]) and highlighted the role of the discrete \(\mathbb{Z}_2\)-symmetry, or parity, in determining \(G_\pm(x)\). We show here to the contrary that one can define a transcendental function \(F_0(x)\) and obtain the regular spectrum of the Rabi model as the zeros of \(F_0(x)\) (see Fig. 1) without ever making use of the underlying \(\mathbb{Z}_2\)-symmetry of the model.

In the latter case the task amounts to finding an entire function \(\phi(z) = \sum_{n=0}^{\infty} c_n z^n\) which belongs to the Bargmann space of analytical functions \(\mathcal{B}\) [2] and which coefficients \(c_n\) satisfy a three-term recurrence relation (Eq. (4) of Ref. [1]; Eqs. (A.6), (A.8) of Ref. [2])

\[
y_{n+1} + a_n y_n + b_n y_{n-1} = 0, \quad (1)
\]

with \(a_n = -f_n(n+1)\), where \(f_n(x)\) is given by Eq. (5) of [1] (cf Eq. (A.8) of Ref. [2]), and \(b_n = 1/(n+1)\). Since \(a_n \to -\omega/(2g)\) and \(b_n \to 0\) in the limit \(n \to \infty\), there exist, in virtue of the theorem by Perron (Theorem 2.2 in Ref. [3]), two linearly independent solutions of the recurrence. One of the two solutions guaranteed by Perron’s theorem is the so-called minimal solution that satisfies \(\lim_{n \to \infty} y_{n+1}/y_n = 0\) [3]. On taking for \(c_n\) the minimal solution, \(\phi(z)\) would belong to \(\mathcal{B}\) irrespective of the value of \(x\) and of the parameters \(g\), \(\Delta\), and \(\omega\) of the Rabi model [1]. The ratios of subsequent terms \(c_{n+1}/c_n\) of the minimal solution are related to continued fractions (cf Theorem 1.1 due to Pincherle in Ref. [2])

\[
r_n = \frac{c_{n+1}}{c_n} = \frac{-b_{n+1}}{a_{n+1}} \frac{b_{n+2}}{a_{n+2}} \frac{b_{n+3}}{a_{n+3}} \cdots \quad (2)
\]

Now follows a point of crucial importance. Mathematical theorems on three-term recurrence relations in Ref. [2] are derived assuming \(n \geq 1\) in [1]. On the other hand, physical problems require the recurrence (1) to be also valid for \(n = 0\) [2]. Given the requirement of analyticity (\(c_{-k} \equiv 0\) for \(k > 1\)), the three-term recurrence (1) reduces for \(n = 0\) to an equation involving mere two-terms and imposes that \(r_0 = c_1/c_0 = -a_0 = f_0(x)\). However, \(r_0\) has been unambiguously fixed by [2], while taking into account (1) for \(n \geq 1\) only [3], and in general \(r_0 \neq f_0(x)\) [2]. It turns out that Eq. (1) for \(n = 0\) plays for the three-term recurrence (1) a role analogous to the boundary condition imposed on the solutions of the Schrödinger equation which enforces quantization of energy levels. The expansion coefficients \(c_0\) and \(c_1\) of entire function \(\phi \in \mathcal{B}\) defined by the minimal solution of

![Figure 1](http://link.aps.org/supplemental/10.1103/PhysRevLett.107.100401/10.1103/PhysRevLett.107.100401). Comment on “Integrability of the Rabi model”

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In his recent letter [1], Braak suggested that a regular spectrum of the Rabi model was given by the zeros of a transcendental function \(G_\pm(x)\) (cf Eqs. (3)-(5) of Ref. [1]) and highlighted the role of the discrete \(\mathbb{Z}_2\)-symmetry, or parity, in determining \(G_\pm(x)\). We show here to the contrary that one can define a transcendental function \(F_0(x)\) and obtain the regular spectrum of the Rabi model as the zeros of \(F_0(x)\) (see Fig. 1) without ever making use of the underlying \(\mathbb{Z}_2\)-symmetry of the model.

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