SPIKKED INSTANTONS FROM INTERSECTING D-BRANES

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The moduli space of spiked instantons that arises in the context of the BPS/CFT correspondence [N1] is realised as the moduli space of classical vacua, i.e. low-energy open string field configurations, of a certain stack of intersecting D1-branes and D5-branes in Type IIB string theory. The presence of a constant B-field induces an interesting dynamics involving the tachyon condensation.

1 Introduction

Gauge theories in four dimensions with \( \mathcal{N} = 2 \) supersymmetry occupy a special place in a quantum field theorist’s landscape. On the one hand, the large amount of supersymmetry severely constrains the dynamics in the form of non-renormalisation theorems, existence of strong-weak dualities etc. On the other hand, it is weak enough to permit interesting non-perturbative effects like confinement, interacting infrared fixed points and so on. Such implications have been explored extensively in the past decades, starting with the seminal work of [SW1, SW2, AD, G] to note a few, with connections being uncovered to diverse areas such as quantum integrable systems [NS], two dimensional conformal field theories [LMN, N2, NO, N1, AGT] and so on.

One of the main features of \( \mathcal{N} = 2 \) supersymmetric gauge theories is the existence of certain sectors in the space of observables which do not receive perturbative quantum corrections beyond one loop, the BPS observables. It turns out that, due to \( \mathcal{N} = 2 \) supersymmetry, the full set of non-perturbative corrections to these observables can be expressed as integrals over instanton moduli space. Using the Ω-background [N2, NO], one calculates these integrals directly by working equivariantly with respect to various symmetries acting on the instanton moduli space (as in [MNS, LNS]), especially the rotational symmetry inherited from four dimensional spacetime.

With the above technique at hand, one can study transitions in the gauge theory between configurations of different instanton number. Observables which encode information about

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such non-perturbative transitions can be expressed in terms of integrals over generalised instanton moduli spaces, known as the moduli space of spiked instantons, first considered in [N3]. Considerations of the structure of these observables (the $qq$-characters) leads to far-reaching consequences such as the BPS/CFT correspondence [N1, N3, N4].

String theory has provided more than one way of constructing large classes of $\mathcal{N} = 2$ supersymmetric gauge theories [DM, KKV, W1]. One such class is the class of quiver gauge theories [DM] which can be engineered by considering the gauge theory on a stack of D3-branes located at a singularity of ADE type. Instantons in this gauge theory have an alternate description as D-instantons bound to the D3-branes [D1]. It is then very natural to consider a state with arbitrary number of D-instantons some of which are bound to the D3-branes and also processes in which the number of bound D-instantons can change.

The simplest case of the non-perturbative transitions alluded to above can be realised by adding an auxiliary stack of D3-branes intersecting with the original stack at a point. This setup gives rise to the moduli space of crossed instantons. The transitions then correspond to some D-instantons escaping from the worldvolume of one stack of D3-branes onto the other stack. The full non-perturbative partition function of the gauge theory on this second stack of D3-branes is then interpreted as an observable in the original gauge theory that encodes information about these transitions. This observable can be written as the expectation value of a local operator inserted at the point of intersection in the original gauge theory [N3].

The analysis can be suitably generalised to include surface defects in the original gauge theory which correspond to including further stacks of D3-branes which intersect the original stack on codimension two hypersurfaces. The moduli space of bound states of the D-instantons with the various stacks of D3-branes is the moduli space of spiked instantons. Calculating suitable integrals over this moduli space would then provide us with a wealth of information regarding non-perturbative transitions in $\mathcal{N} = 2$ gauge theories.

In the present paper, we obtain the moduli space of spiked instantons by studying the low-energy field theory on the worldvolume of D1-branes probing a configuration of intersecting D5-branes which preserves $\mathcal{N} = (0, 2)$ supersymmetry on the common two dimensional intersection (this is T-dual to the setup described above). The precise configuration is described in Section 2 below. In order to bind the D1-branes to the D5-branes, we turn on a small constant NSNS $B$-field along the D5-branes. In Section 3, we study the amount of supersymmetry preserved by the D-brane configuration in the presence of such a background. The next step is to study the spectrum of open strings stretching between various pairs of D-branes in order to extract the low-energy theory on the D1-branes. This is done is Section 4. The salient feature is the presence of tachyons for generic values of the $B$-field.

In Section 5, we consider a stack of D1-branes two stacks of D5-branes intersecting on along the D1-brane worldvolume and obtain the low-energy theory on the D1-branes (the analogous setup without $B$-field was considered in [T]). We calculate the various interaction terms in the low-energy effective action by considering 3-point open string amplitudes on the disk and
obtain the moduli space of crossed instantons as the classical vacuum moduli space of the low-
energy theory. In Section 6, we generalise to the spiked case. We also provide two appendices,
the first of which has details regarding the propagation of open strings in a constant $B$-field
and the vertex operators for various states in the open string spectrum. Appendix B has
details about $\mathcal{N} = (0, 2)$ superspace that is relevant for the spiked instanton story.

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2 Preliminaries

Write the ten dimensional spacetime $\mathbb{R}^{1,9} \simeq \mathbb{R}^{1,1} \times \mathbb{R}^8$ as $\mathbb{R}^{1,1} \times \mathbb{C}^4$ by choosing a complex
structure on the $\mathbb{R}^8$. Let $\mathbf{4} = \{1, 2, 3, 4\}$ be the set of coordinate labels of the $\mathbb{C}^4$. There
are six 2-planes $\mathbb{C}^2_A$ that sit inside the $\mathbb{C}^4$ labelled by an index $A \in \mathbf{6} = \left(\frac{\mathbf{4}}{2}\right)$ i.e. the set
of unordered pairs of numbers in $\mathbf{4}$. Explicitly, $\mathbf{6} = \{(12), (13), (14), (23), (24), (34)\}$. We
consider a system of D-branes which consists of $k$ D1-branes spanning $\mathbb{R}^{1,1}$ and $n_A$ D5-branes
spanning $\mathbb{R}^{1,1} \times \mathbb{C}^2_A$ with $A \in \mathbf{6}$. Here onwards, $\mathbb{R}^{1,1}$ refers to the common $1 + 1$ dimensional
intersection of the D-brane configuration and is taken to be along the $x^0, x^9$ directions.

The D1-D5 system for spiked instantons.

| $\mathbb{R}^{1,9}$ | 1 2 | 3 4 | 5 6 | 7 8 | 9 0 |
|-------------------|-----|-----|-----|-----|-----|
| $\mathbb{C}^4 \times \mathbb{R}^{1,1}$ | $z^1$ | $z^2$ | $z^3$ | $z^4$ | $x$ | $t$ |
| D1                |     |     |     |     | $\times$ | $\times$ |
| $D5_{(12)}$       | $\times$ | $\times$ | $\times$ |     | $\times$ | $\times$ |
| $D5_{(13)}$       | $\times$ |     | $\times$ |     |     | $\times$ |
| $D5_{(14)}$       |     | $\times$ |     | $\times$ |     | $\times$ |
| $D5_{(23)}$       |     |     | $\times$ | $\times$ |     | $\times$ |
| $D5_{(24)}$       | $\times$ |     |     | $\times$ | $\times$ | $\times$ |
| $D5_{(34)}$       |     |     |     |     | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
Type IIB string theory has two supersymmetry parameters $\epsilon$ and $\tilde{\epsilon}$ which are Majorana-Weyl spinors of the same chirality (say left-handed). That is,

$$\Gamma_c \epsilon = \epsilon, \quad \Gamma_c \tilde{\epsilon} = \tilde{\epsilon} \quad \text{where} \quad \Gamma_c = \Gamma^1 \cdots \Gamma^9 \Gamma^0 \quad \text{and} \quad (\Gamma_c)^2 = 1. \quad (2.1)$$

The presence of a Dp-brane gives the following constraint on the supersymmetry parameters:

$$\tilde{\epsilon} = \frac{1}{(p+1)!} \epsilon_{\mu_0 \mu_1 \cdots \mu_p} \Gamma^{\mu_0 \mu_1 \cdots \mu_p} \epsilon. \quad (2.2)$$

Here, $\mu_0, \ldots, \mu_p$ take $p+1$ values corresponding to the spacetime extent of the Dp-brane and $\Gamma^{\mu_0 \mu_1 \cdots \mu_p}$ is the totally antisymmetrised product of $p+1$ $\Gamma$-matrices. Suppose the spatial extent of the Dp-brane is along $\{x^{i_1}, \ldots, x^{i_p}\}$ with $i_1 < i_2 < \cdots < i_p$. Then, the Levi-Civita symbol $\epsilon$ is normalised such that $\epsilon_{i_1 i_2 \cdots i_p 0} = +1$. In the presence of D1-branes along $\mathbf{R}^{1,1}$ and D5-branes along $\mathbf{R}^{1,1} \times \mathbf{C}^2_{(12)}$, the constraints are $\tilde{\epsilon} = \Gamma^9 \epsilon$ and $\tilde{\epsilon} = \Gamma^{123490} \epsilon$ which give

$$\Gamma^{1234} \epsilon = \epsilon. \quad (2.3)$$

Since $\Gamma^{1234}$ squares to identity and is traceless, half of the sixteen real components of $\epsilon$ are set to zero. This leaves us with a total of eight independent supersymmetry parameters for the D1-D5$_{(12)}$ system. Including all six stacks of D5-branes gives us the following constraints on $\epsilon$:

$$\Gamma^{1234} \epsilon = \epsilon, \quad \Gamma^{1256} \epsilon = \epsilon, \quad \Gamma^{1278} \epsilon = \epsilon, \quad \Gamma^{3456} \epsilon = \epsilon, \quad \Gamma^{3478} \epsilon = \epsilon, \quad \Gamma^{5678} \epsilon = \epsilon. \quad (2.4)$$

It is easy to check that these constraints are not all compatible with each other. For example, combining the first and fourth constraints above gives us (using that $(\Gamma^{34})^2 = -\mathbb{1}$),

$$\Gamma^{1256} \epsilon = -\epsilon, \quad (2.5)$$

which is in contradiction with the second constraint in that the two respective eigenspaces of $\Gamma^{1256}$ do not overlap. Thus, the above D-brane system is not supersymmetric. Is it stable?

The zero-point energy in the R sector is always zero due to worldsheet supersymmetry. The NS sector zero-point energy $E_0$ for a Dp-Dp' system

$$E_0 = -\frac{1}{2} + \frac{\nu}{8}, \quad (2.6)$$

where $\nu$ is the number of ND+DN boundary conditions among the eight non-lightcone directions.

- For D1-D1 strings and D5$_A$-D5$_A$ strings, we have $\nu = 0$ and $E_0 = -\frac{1}{2}$ but this state is removed by GSO projection and one has a ground state with zero energy.
- The D1-D5$_A$ system has $\nu = 4$ and a four-fold degenerate ground state with $E_0 = 0$. The GSO projection removes half of the states.
- The D5$_{(ac)}$-D5$_{(bc)}$ system with $a \neq b$ has $\nu = 4$ and hence $E_0 = 0$. Again, the GSO projected ground state is two-fold degenerate.
The D5A-D5A system has \( \nu = 8 \) and \( E_0 = +\frac{1}{2} \). Here, \( \overline{A} \) refers to the complement of \( A \) in the set \( 4: \overline{A} = 4 \setminus A \). For example, for \( A = (12) \), we have \( \overline{A} = (34) \). The lowest energy state after GSO projection is eight-fold degenerate and has positive energy. These states do not survive in the low-energy limit and only the zero-energy states from the R sector survive. Such a fermionic ground state is allowed by chiral supersymmetry. Indeed, the above configuration preserves \( \mathcal{N} = (0,8) \) supersymmetry on the intersection. The massive states from the NS sector have superpartners which are first excited states in the R sector (see Section 4.3 below).

Thus, after GSO projection, the NS sector zero-point energies are either zero or positive implying that the system is stable.

There is a different system of D1-branes and D5-branes which does preserve some fraction of supersymmetry. We impose the maximal set of constraints which are compatible with each other. One such set is given by

\[
\Gamma^{1234} \epsilon = \epsilon, \quad \Gamma^{1256} \epsilon = -\epsilon, \quad \Gamma^{1278} \epsilon = \epsilon, \quad \Gamma^{3456} \epsilon = \epsilon, \quad \Gamma^{3478} \epsilon = -\epsilon, \quad \Gamma^{5678} \epsilon = \epsilon. \quad (2.7)
\]

The constraints with a negative sign correspond to \( \overline{D5} \)-branes. Only three of the above six constraints are independent, preserving one-sixteenth of the 32 supercharges. Thus, a configuration of D1-branes with four stacks of D5-branes and two stacks of \( \overline{D5} \)-branes in the above arrangement preserves **two supercharges**. Next, we would like to study the effects of turning on a constant NSNS B-field background.

### 3 Supersymmetry in a constant B-field background

Consider a constant NSNS B-field background of the form:

\[
2\pi \alpha' B_{12} = b_1, \quad 2\pi \alpha' B_{34} = b_2, \quad 2\pi \alpha' B_{56} = b_3, \quad 2\pi \alpha' B_{78} = b_4. \quad (3.1)
\]

This choice of B-field preserves the SO(2)\(^4\) rotational symmetry of the above intersecting D-brane system. Such a symmetry is essential for eventually considering the generalisation to the \( \Omega \)-background.

We first state our conventions and introduce some notation.

- Introduce the variables \( v_a, a \in 4 \) with

\[
e^{2\pi i v_a} = \frac{1 + i b_a}{1 - i b_a}, \quad b_a = \tan \pi v_a, \quad -\frac{1}{2} < v_a < \frac{1}{2}. \quad (3.2)
\]

The limits \( v_a \to \pm \frac{1}{2} \) correspond to \( b_a \to \pm \infty \).

- For each \( A \in 6 \), let \( \Gamma_A = \Gamma^{2a-1} \Gamma^{2a} \Gamma^{2b-1} \Gamma^{2b} \) for \( A = (ab) \).
• Choose the following representation for the $\Gamma$-matrices. This representation corresponds to a particular choice of the cocycle operators for open string vertex operators given in [KLLSW]. See Appendix A for more details.

\[
\begin{align*}
\Gamma^1 &= \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 , & \Gamma^7 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 , \\
\Gamma^2 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 , & \Gamma^8 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 , \\
\Gamma^3 &= \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 , & \Gamma^9 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 , \\
\Gamma^4 &= -\sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 , & \Gamma^{10} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes (-i\sigma_2) , \\
\Gamma^5 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 , & \Gamma_c &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes 1 , \\
\Gamma^6 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 , & C_- &= e^{3\pi i/4} \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 .
\end{align*}
\]

The chirality matrices in $C^2_A$ are chosen to be $\Gamma_c(C^2_A) = \Gamma_A$ where $\Gamma_A$ is defined above and the chirality matrix in $R^{1,1}$ is $\Gamma_c(R^{1,1}) = -\Gamma^0\Gamma^9$.

• The $32_{\mathbb{C}}$ dimensional spinor representation can then be constructed by considering simultaneous eigenvectors $|\pm, \pm, \pm, \pm, \pm\rangle$ of $-i\Gamma^{12}, -i\Gamma^{34}, -i\Gamma^{56}, -i\Gamma^{78}$ and $-\Gamma^{09}$, and using the linear combinations $-i\Gamma^1 \pm \Gamma^2, \ldots, -i\Gamma^7 \pm \Gamma^8, \Gamma^0 \pm \Gamma^9$ as raising and lowering operators respectively. The basis of the representation is then given by the 32 vectors $|\pm, \pm, \pm, \pm, \pm\rangle$. The left-handed (right-handed) $16_{\mathbb{C}}$ subspace of $\Gamma_c$ is then spanned by the subset of the above with even (odd) number of negative signs.

Next, we study the amount of supersymmetry preserved in the presence of a constant $B$-field for the two different D-brane systems considered above.

**No anti D5-branes**

In the presence of a constant $B$-field of the form (3.1), the constraint arising from the stack of D5$_A$ branes becomes

\[
\tilde{\epsilon} = \Gamma^{09} R_A \epsilon , \tag{3.3}
\]

where $R_A$ is given by

\[
R_A = \exp \left( \sum_{a \in A} \theta_a \Gamma^{2a-1} \Gamma^{2a} \right) , \tag{3.4}
\]

with $\theta_a := \pi(v_a + \frac{1}{2})$. Combining this with the constraint $\tilde{\epsilon} = \Gamma^{09} \epsilon$ from the D1-branes, we get

\[
R_A \epsilon = \epsilon \quad \text{for every} \quad A \in \mathbf{6} . \tag{3.5}
\]

Let $r(\theta) := \exp(i\sigma_3 \theta)$. Then, we have

\[
\begin{align*}
R_{(12)} &= r(\theta_1) \otimes r(\theta_2) \otimes 1 \otimes 1 \otimes 1 , & R_{(13)} &= r(\theta_1) \otimes 1 \otimes r(\theta_3) \otimes 1 \otimes 1 , \\
R_{(14)} &= r(\theta_1) \otimes 1 \otimes 1 \otimes r(\theta_4) \otimes 1 , & R_{(23)} &= 1 \otimes r(\theta_2) \otimes r(\theta_3) \otimes 1 \otimes 1 , \\
R_{(24)} &= 1 \otimes r(\theta_2) \otimes 1 \otimes r(\theta_4) \otimes 1 , & R_{(34)} &= 1 \otimes 1 \otimes r(\theta_3) \otimes r(\theta_4) \otimes 1 . \tag{3.6}
\end{align*}
\]
The equations $R_A \epsilon = \epsilon$ have a solution if, for some choice of signs,

$$\exp (\pm i \theta_a \pm i \theta_b) = 1 \quad \text{with} \quad 0 \leq \theta_a \leq \pi \quad \forall \quad a \in 4 .$$  
(3.7)

By inspection, it can be seen that the above set of equations has no solution except when all the $\theta_a = 0$ or when all the $\theta_a = \pi$. This corresponds to $v_a \to \pm \frac{1}{2}$, or equivalently, $b_a \to \pm \infty$. Thus, turning on a finite $B$-field of the above form does not make the brane configuration supersymmetric.

**Four D5-branes and two anti D5-branes**

Next, let us consider the configuration of D5-branes and D5-branes that preserves two supercharges. The constraints on $\epsilon$ can then be succinctly written as

$$\Gamma_A \epsilon = \epsilon_{A \overline{A}} \epsilon ,$$  
(3.8)

where $\overline{A} = 4 \setminus A$. For example, when $A = (12)$, we have $\overline{A} = (34)$ and $\epsilon_{A \overline{A}} = \epsilon_{1234} = +1$ and when $A = (13)$ we have $\epsilon_{1324} = -1$. When we turn on the above constant $B$-field, the equations become

$$R_A \epsilon = \epsilon_{A \overline{A}} \epsilon .$$  
(3.9)

This gives the conditions

$$\exp (\pm i \theta_a \pm i \theta_b) = \epsilon_{ab \overline{ab}} \quad \text{with} \quad A = (ab) , \quad \overline{A} = (\overline{ab}) \quad \text{for} \quad A \in 6 .$$  
(3.10)

These equations have solutions corresponding to finite $B$ only when $\theta_a = \frac{\pi}{2}$ for all $a \in 4$ with some choice of signs. This corresponds to $v_a = 0$ which is the zero $B$-field point.

**Stability**

The question of stability arises in this situation too. First we observe that supersymmetry is completely lost about the original vacuum for a non-zero finite value of the constant $B$-field. Thus, stability is no longer guaranteed. Secondly, a constant $B$-field background typically introduces instability in the form of tachyons in the D-brane spectrum.

In some situations, e.g. the D1-D5 system, the effects of the $B$-field can be accommodated by turning on a Fayet-Iliopoulos parameter in the low energy effective action. The tachyon instability leads to the system transitioning to a nearby vacuum at which point supersymmetry is restored.

We shall see that something similar happens in the spiked scenario as well, with some differences. To study the stability we need to derive the spectrum of open strings in the presence of D-branes in a constant $B$-field background.
4 Spectrum of Dp-Dp’ strings

The boundary conditions for an open string are modified in the presence of a B-field. Let the
worldsheet bosons and fermions along $C^4$ be resp. $Z^a(\sigma, \tau)$ and $\Psi^{\pm a} (\sigma, \tau)$, $a \in 4$. Neumann
boundary conditions along $C_a$ are modified to (cf. Appendix A):

Twisted(T): $\partial_{++} Z^a = e^{-2\pi i v_a} \partial_{--} Z^a$, $\Psi^{a+} = e^{2\pi i v_a} \Psi^{a-}$. (4.1)

Neumann and Dirichlet boundary conditions are obtained by setting $v_a = 0$ and $v_a \to \frac{1}{2}$
respectively. Sending one of the $v_a$’s to $-\frac{1}{2}$ would give Dirichlet boundary conditions on an
anti D-brane. Consider the more general boundary conditions with $-\frac{1}{2} \leq \mu, \nu \leq \frac{1}{2}$:

$\partial_{++} Z = e^{-2\pi i \nu} \partial_{--} Z$, $\Psi^+ = e^{2\pi i \nu} \Psi^-$ at $\sigma = 0$,
$\partial_{++} Z = e^{-2\pi i \mu} \partial_{--} Z$, $\Psi^+ = \pm e^{2\pi i \mu} \Psi^-$ at $\sigma = \pi$, (4.2)

The low-energy spectrum for this system has been worked out in Appendix A. We summarise
the results here.

1. **Non-integer modes:** The worldsheet boson $Z$ has moding $Z + \theta$ with $\theta = \mu - \nu$. The R sector fermions have the same moding as $Z$ due to rigid supersymmetry on the
worldsheet and the NS sector fermions have moding $Z + \epsilon$ with $\epsilon = \theta + \frac{1}{2} = \mu - \nu + \frac{1}{2}$.

2. **Excitations:** The zero-point energy in the NS sector is given by

$$E_0 = \frac{1}{8} - \frac{1}{2} |\theta| - \frac{1}{2} |\theta|$$. (4.3)

The first excited state in the NS sector has energy $E_0 + |\epsilon|$ or $E_0 + |1 - \epsilon|$ when $-\frac{1}{2} \leq \epsilon \leq \frac{1}{2}$
and $\frac{1}{2} < \epsilon < \frac{3}{2}$ respectively.

The zero-point energy in the R sector vanishes due to rigid supersymmetry on the
worldsheet. The first excited state in the R sector has energy $|\theta|$ for $0 \leq |\theta| \leq \frac{1}{2}$ and
$1 - |\theta|$ for $\frac{1}{2} \leq |\theta| \leq 1$.

3. **Spectral flow:** When $\epsilon$ crosses the integer $s$ from the left ($s = 0$ or 1), the state with
energy $s - \epsilon$ becomes negative and enters the Dirac sea and the state $\epsilon - s$ crosses into
the positive energy region. The raising and lowering roles of the NS fermion operators
$d_s$ and $d_s^\dagger$ are interchanged. Using $d_s^\dagger d_s = -d_s d_s^\dagger + 1$, we see that the number operator changes by one unit $N_d \to N_d + 1$. This changes the sign of the parity operator $(-)^{FNS} := (-1)^{N_d}$ and the GSO projectors $\frac{1}{2}(1 \pm (-)^{FNS})$ are consequently interchanged. A similar
phenomenon occurs in the R sector when $\theta$ crosses 0.

4.1 D1-D1 strings

The open strings satisfy NN boundary conditions along $R^{1,1}$ and DD boundary conditions
along $C^4$. The worldsheet bosons have momentum zero modes along $R^{1,1}$ and none along $C^4$
and hence all the states are supported along $R^{1,1}$. 
**NS sector:** There are no zero modes for the NS fermions and the NS zero-point energy is $-\frac{1}{2}$. The NS fermion oscillators $d_1^{\mu \dagger}$, $\mu = 0, 9$ and $d_2^{a \dagger}$, $a \in \mathbb{A}$ raise the energy by $\frac{1}{2}$. The oscillators $d_1^{a \dagger}$ gives rise to two states which are the components of a gauge field $v_{\pm}(x,t)$ while the four complex oscillators $d_2^{a \dagger}$ create four states in the adjoint of $U(k)$ corresponding to complex scalars $B_a(x,t)$. Assigning the NS vacuum a fermion number $F_{NS} = -1$, the GSO projection with projector $\frac{1}{2}(1 + (-)F_{NS})$ projects out the vacuum while retaining the zero-energy states.

**R sector:** The R sector has ten zero modes thus giving a real 32 dimensional ground state transforming in the adjoint of $U(k)$. The fermion parity $(-)^F_R$ on the zero modes is then $(-)^F_R = \Gamma^{1-8}\Gamma^{90} = \Gamma_{c}(R^{1,9})$. The GSO projection with $\frac{1}{2}(1 + (-)^F_R)$ gives a left-handed fermion in $1 + 9$ dimensions which splits up into eight right-handed and eight left-handed fermions in $1 + 1$ dimensions.

We decompose the spacetime scalars and fermions into representations of $SO(C^2_A) \times SO(C^2_{\overline{A}})$ using $\Gamma_{c}(R^{1,9}) = \Gamma_{c}(R^{1,1})\Gamma_{c}(C^2_A)\Gamma_{c}(C^2_{\overline{A}})$. Writing each $SO(4)$ as $SU(2) \times SU(2)$ with $\alpha, \alpha', \alpha''$, denoting the fundamentals of the four $SU(2)$'s, we have

$$
\text{Scalars : } X_{\alpha\alpha} \oplus X_{\alpha'\alpha'},
\text{Fermions : } \lambda_{\alpha\alpha'} \oplus \lambda_{\alpha''},
$$

with reality conditions $\lambda_{\alpha\alpha'} = -\varepsilon^{\alpha \beta} \varepsilon^{\alpha' \beta'} \lambda_{\alpha'}$ and so on for the fermions.

### 4.2 D1-D5$_A$ strings

For a D1-D5$_A$ string the boundary conditions are DT for $a \in A$ and DD for $a \in \overline{A}$. These boundary conditions imply $Z + v_a - \frac{1}{2}$ moding for the bosons $Z^a$ with $a \in A$ and $Z$ moding for $a \in \overline{A}$. The R fermions have the same moding as the bosons and the NS fermions have moding $Z + v_a$ for $a \in A$ and $Z + \frac{1}{2}$ for $a \in \overline{A}$. Since the string is orientable, states from different orientations are distinct and have to be combined together in order to form a CPT invariant spectrum.

**NS sector:** Let $A = (ab)$. The NS zero-point energy is given by $-\frac{1}{2}(|v_a| + |v_b|)$. For $v_a$ and $v_b$ close to zero, the oscillators with lowest positive energy are from the NS fermions and increase energy by $|v_a|$ and $|v_b|$. The first four states in the NS sector have the energies

$$
\frac{1}{2}(\pm|v_a| \pm |v_b|) \text{ or equivalently, } \frac{1}{2}(\pm v_a \pm v_b) .
$$

When either of $v_a$ and $v_b$ crosses zero, the sign of $(-)^F_{NS}$ is flipped (cf. point 3 above). It is then easy to see that states which have definite values of $(-)^F_{NS}$ are $\frac{1}{2}(\pm v_a \pm v_b)$ rather than $\frac{1}{2}(\pm|v_a| \pm |v_b|)$. We assign $(-)^F_{NS} = -1$ to the state with energy $-\frac{1}{2}(v_a + v_b)$ and choose the GSO projector to be $\frac{1}{2}(1 + (-)^F_{NS})$. The states with energies $\pm\frac{1}{2}(v_a + v_b)$ are projected out and the states that remain are

$$
-\frac{1}{2}(v_a - v_b), \frac{1}{2}(v_a - v_b) .
$$
These states transform in the \((k, \bar{\pi}_A)\) of \(U(k) \times U(n_A)\). The string with opposite orientation furnishes two more states with the same energy and which transform in the \((\bar{k}, \pi_A)\) of \(U(k) \times U(n_A)\). Thus, we get two complex scalars \(\phi^1\) and \(\phi^2\) in the bifundamental of \(U(k) \times U(n_A)\) with masses given by

\[
m^2 = \pm \frac{1}{2\alpha'}(v_a - v_b) .
\]  

(4.7)

In the limit \(v_a, v_b \to 0\), the two states become degenerate. We also have \((-)^{F_{\text{NS}}} = \Gamma_A = \Gamma_c(C^2_A)\), which implies that the above GSO projection results in a left-handed spinor \(\phi^0\) in \(C^2_A\). These constitute the two complex scalars of a \(\mathcal{N} = (4,4)\) bifundamental hypermultiplet in \(R^{1,1}\).

The two complex NS fermions along \(C^2_A\) have energy \(\frac{1}{2}\). The oscillators associated to worldsheet bosons along \(C^2_A\) have energies \(\frac{1}{2} \pm v_a, \frac{1}{2} \pm v_b\).

**R sector:** The zero-point energy vanishes in the \(R\) sector. There are six zero modes from fermions along \(R^{1,1} \times C^2_A\), which give an eight-dimensional ground state consisting of spinors \(|\alpha', \pm\rangle\) and \(|\alpha', \pm\rangle\) where \(+(-)\) indicates left(right)-handed spinors in \(R^{1,1}\) and \(\alpha'\)(') left(right)-handed spinors in \(C^2_A\). The fermion parity operator \((-)^{F_{\text{R}}}\) is given by \((-)^{F_{\text{R}}} = \Gamma_A \Gamma_{90} = \Gamma_c(R^{1,1})\Gamma_c(C^2_A)\). The GSO projection with \(\frac{1}{2}(1 - (-)^{F_{\text{R}}}\) retains the states that satisfy \(\Gamma_c(R^{1,1}) = \pm 1\), \(\Gamma_c(C^2_A) = \mp 1\). Together with the states from the oppositely oriented string, we thus have spinors \(\zeta_{\alpha''}^- = -\zeta_\alpha^+\) and \(\lambda_{\alpha''}^+ = \lambda_\alpha^-\). They transform in the \((k, \bar{\pi}_A)\) of \(U(k) \times U(n_A)\) and constitute the fermionic part of the \(\mathcal{N} = (4,4)\) bifundamental hypermultiplet in \(R^{1,1}\).

The first few single-oscillator excitations in the \(R\) sector come from fermionic and bosonic oscillators along \(C^2_{(ab)}\) with energies \(\frac{1}{2} \pm v_a, \frac{1}{2} \pm v_b\). The oscillators in the other directions raise the energy by 1. After GSO projection, the states from worldsheet bosons acting on \(|\alpha', -\rangle\) and \(|\alpha', +\rangle\) and states from R fermions acting on \(|\alpha', +\rangle\) and \(|\alpha', -\rangle\) are retained.

**D1-D5\(\bar{A}\) strings:** The above analysis carries through but the modings become \(Z + \frac{1}{2} - v_a, Z + v_b - \frac{1}{2}\) for \(Z_a\) and \(Z_b\) respectively. Also, the GSO projection is carried out with the opposite projectors. The NS sector states remaining after GSO projection have energies \(\pm \frac{1}{2}(v_a + v_b)\) and in the limit \(v_a, v_b \to 0\) form a right-handed spinor in \(C^2_A\). The \(R\) sector fermions which are right(left)-handed in \(R^{1,1}\) are right(left)-handed in \(C^2_A\).

**4.3 D5\(\bar{A}\)-D5\(\bar{\Xi}\) strings**

The boundary conditions are \(\text{TD}\) for \(a \in A\) and \(\text{DT}\) for \(a \in \bar{A}\). These imply the following modings for the bosons and R fermions:

\[
Z + \frac{1}{2} - v_a \text{ for } a \in A \text{ and } Z + v_\pi - \frac{1}{2} \text{ for } a \in \bar{A} .
\]  

(4.8)

The NS fermions have \(Z - v_a\) and \(Z + v_\pi\) moding respectively.
**NS sector:** The zero point energy in the NS sector is then

\[ \frac{1}{2} - \frac{1}{2} \sum_{a \in 4} |v_a| . \]  

(4.9)

The lowest excitation energies in the NS sector are \(|v_a|\) for \(a \in 4\). The first few states are then

\[ \frac{1}{2}(1 \pm v_1 \pm v_2 \pm v_3 \pm v_4) . \]  

(4.10)

We assign \((-)^{F_{NS}} = -1\) to the state with energy \(\frac{1}{2}(1 - (v_1 + v_2 + v_3 + v_4))\). GSO projection with \(\frac{1}{2}(1 + (-)^{F_{NS}})\) removes states with an even number of negative signs. The remaining states are

\[ \frac{1}{2}[1 \pm (v_1 - v_2 - v_3 - v_4)] , \frac{1}{2}[1 \pm (v_1 + v_2 + v_3 - v_4)] , \frac{1}{2}[1 \pm (v_1 + v_2 - v_3 + v_4)] , \frac{1}{2}[1 \pm (v_1 - v_2 + v_3 + v_4)] . \]  

(4.11)

For small enough \(|v_a|\), the above energies are all positive: there is no tachyon or massless state in the NS sector. There is another copy of these states from the string with opposite orientation. Together, they form eight massive complex scalars that transform in the \((n_A, \overline{n_A})\) of \(U(n_A) \times U(n_B)\).

**R sector:** The ground state energy in the R sector is zero as always. The only zero modes are the ones along \(R^{1,1}\) and we denote them by \(\Gamma^0\) and \(\Gamma^9\). We have \((-)^{F_R} = \Gamma^{90} = \Gamma_c(R^{1,1})\). Assign \((-)^{F_R} = -1\) for the ground state \(|R\rangle\) and define

\[ g = \frac{\Gamma^9 + \Gamma^0}{\sqrt{2}}, \quad g^\dagger = \frac{\Gamma^9 - \Gamma^0}{\sqrt{2}} . \]  

(4.12)

Acting on \(|R\rangle\) with \(g^\dagger\) provides another state of zero energy but with \((-)^{F_R} = +1\). The GSO projection with \(\frac{1}{2}(1 + (-)^{F_R})\) retains \(g^\dagger|R\rangle\) which is a left-handed fermion. Together with a similar state from the oppositely oriented string, this fermion transforms in the bifundamental of \(U(n_A) \times U(n_B)\).

For small \(v_a\), the first two sets of single-oscillator excitations for the worldsheet bosons and R fermions come from the \(C^4\) directions and have energy \(\frac{1}{2} \mp v_a\). The GSO projection keeps the eight states obtained from the worldsheet bosons acting on \(g^\dagger|R\rangle\) and the eight states from R fermions acting on \(|R\rangle\). Together with states from the oppositely oriented string, they form four right- and left-moving fermions with mass-squared \(\frac{1}{2} \mp v_a\) and four right- and left-moving fermions with mass-squared \(\frac{1}{2} - v_a\).

In the limit \(v_a \rightarrow 0\), the eight right-moving and eight left-moving fermions become degenerate and the eight right-movers are in fact the superpartners of the scalars from the NS sector.

### 4.4 D5\(_{\langle ca \rangle}\)-D5\(_{\langle cb \rangle}\) strings

Here \(C^2_{(ca)}\) and \(C^2_{(cb)}\) share a common \(C_c\). Let the remaining direction be \(C_d\). The boundary conditions are now \(TT\) for \(Z^c\), \(TD\) for \(Z^a\), \(DT\) for \(Z^b\) and \(DD\) for \(Z^d\). The modings are
Z for $Z^c$ and $Z^d$, $Z + \frac{1}{2} - v_a$ for $Z^a$ and $Z + v_b - \frac{1}{2}$ for $Z^b$. The R fermions have the same modings and the NS fermions have the modings shifted by $\frac{1}{2}$. The modings are the same as for a D1-$\overline{D}5_{(ab)}$ system. The worldsheet bosons have momentum and position zero modes along $R^{1,1} \times C_c$. Hence all the states will be supported on the four dimensional space $R^{1,1} \times C_c$.

**NS sector:** The zero-point energy is $-\frac{1}{2}(|v_a| + |v_b|)$ and the lowest-lying excitation energies are $|v_a|$ and $|v_b|$. Thus, the lowest energy states are $\frac{1}{2}(\pm v_a \pm v_b)$. We assign $(-)^{F_{NS}} = -1$ to the state $-\frac{1}{2}(v_a + v_b)$ and perform GSO projection with $\frac{1}{2}(1 - (-)^{F_{NS}})$ to get the states

$$-\frac{1}{2}(v_a + v_b), \frac{1}{2}(v_a + v_b).$$

(4.13)

After including states from the oppositely oriented string, these give two complex scalars $\sigma^1$, $\sigma^2$ which transform as $(n_{(ca)}, \overline{n}_{(cb)})$ with masses $m^2 = \pm \frac{1}{2m}(v_a + v_b)$. In the limit $v_a, v_b \rightarrow 0$, the two scalars are massless and combine into a right-handed spinor in $C^2_{(ab)}$ since $(-)^{F_{NS}} = \Gamma_c(C^2_{(ab)})$. These constitute the bosonic part of a $N = 2$ hypermultiplet in $R^{1,1} \times C_c$.

**R sector:** The worldsheet fermions along $R^{1,1} \times C^2_{(cd)}$ are integer moded, giving six zero modes and an eight dimensional ground state. The fermion parity operator is given by $(-)^{F_R} = \Gamma_c(C^2_{(cd)})\Gamma^{90} = \Gamma_c(R^{1,1} \times C_c)\Gamma_c(C_d)$ where $\Gamma_c(R^{1,1} \times C_c) = i\Gamma^{2c-1}\Gamma^{2e}\Gamma^9\Gamma^0$ and $\Gamma_c(C_d) = -i\Gamma^{2d-1}\Gamma^{2d}$. We use the GSO projector $\frac{1}{2}(1 - (-)^{F_R})$ to get a left-handed fermion $\lambda$ and a right-handed fermion $\overline{\chi}$ in $R^{1,1} \times C_c$ with $\Gamma_c(C_d) = \pm 1$ respectively. These constitute the fermionic part of a $N = 2$ hypermultiplet.

**D5$_{(ca)}$-$\overline{D}5_{(cb)}$ strings:** For this case, the GSO projections are reversed. The NS sector states then have energies $\pm \frac{1}{2}(v_a - v_b)$ and form a left-handed spinor in $C^2_{(ab)}$ in the limit $v_a, v_b \rightarrow 0$. The R sector fermions have opposite eigenvalues under $\Gamma_c(C_d)$.

## 5 Crossed instantons

We first consider the simpler configuration of crossed instantons: $k$ D1-branes along $R^{1,1}$, $n$ D5-branes along $R^{1,1} \times C^2_{(12)}$ and $n'$ D5-branes along $R^{1,1} \times C^2_{(34)}$. This setup preserves four supercharges organised into $N = (0, 4)$ supersymmetry on the two dimensional intersection $R^{1,1}$. This setup has been studied in the context of AdS$_3$ holography by [T, GMMS] and others. Another place where $N = (0, 4)$ supersymmetry appears is the ADHM sigma model [W3] which has a stringy realisation as a D1-D5-D9 brane system [D2]. More recently, the authors in [PSY] explore a class of $N = (0, 4)$ superconformal theories obtained by compactifying M5-branes on four-manifolds of the form $P^1 \times \mathcal{C}$ where $\mathcal{C}$ is a Riemann surface with punctures.

We are interested in studying the bound states of D1-branes with the crossed D5-branes above with the constant B-field background in (3.1). As we have seen in the previous section, there are generically tachyons in the spectrum and supersymmetry is broken. We are interested
in the end point of tachyon condensation [Se, A, GS] and the all-important question: is supersymmetry restored at the end point of the condensation?

We shall find that for a particular locus in the space of $B$-fields, the supersymmetry breaking can be described by a Fayet-Iliopoulos term in the low-energy theory. For small values of $B$-field, we can then study the condensation of the tachyons in the low-energy effective theory. The relevant low-energy degrees of freedom are those of a supersymmetric $U(k)$ gauge theory interacting with various matter multiplets supported on $R^{1,1}$. In particular, we freeze the supersymmetric gauge degrees of freedom supported on the D5-branes to their classical vacuum expectation values.

**Note:** The above D5-brane system without the D1-branes has been studied in great detail by many authors, notably by [IKS]. There are chiral fermions (the field $\lambda$ below) in the 1 + 1 dimensional intersection arising from the strings stretching between the two stacks of D5-branes. The chiral fermions render the gauge theories on the intersection anomalous and the degrees of freedom in the bulk of the D5-branes are necessary to cancel these anomalies via the anomaly inflow mechanism. Since we have frozen these gauge degrees of freedom, these issues are not immediately relevant to our analysis below. In our case, the low-energy theory on the intersection has $U(n) \times U(n')$ as rigid symmetries.

The spacetime Lorentz group $SO(1, 9)$ is broken down to $SO(1, 1) \times SO(4) \times SO(4)'$. The low energy theory on $R^{1,1}$ has the internal rigid symmetry group $SO(4) \times SO(4)' \times U(n) \times U(n')$. It will be useful to write $SO(4) \times SO(4)' = SU(2)_L \times SU(2)_R \times SU(2)'_L \times SU(2)'_R$ with the indices $(\alpha, \dot{\alpha}, \alpha', \dot{\alpha}')$ denoting the fundamental representations of the respective $SU(2)$s.

The sixteen components of the left-handed spinor $\epsilon$ can be written in terms of spinors which have definite chirality under each of $SO(1, 1)$, $SO(4)$ and $SO(4)'$ as follows:

$$\epsilon = \eta^{\alpha \dot{\alpha}}_L \oplus \eta^{\dot{\alpha} \alpha}_R \oplus \eta^{\dot{\beta} \dot{\alpha}}_R \oplus \eta^{\beta \alpha}_L. \quad (5.1)$$

The subscripts indicate chirality in 1 + 1 dimensions. We see that the product of the three chiralities is +1 which agrees with $\epsilon$ being left-handed in 9 + 1 dimensions. There must also be a reality condition on each of the $\eta$’s that arises from the Majorana condition on $\epsilon$. Since the fundamental representation of $SU(2)$ is pseudoreal, the $\eta$’s are in a real representation of the corresponding $SU(2) \times SU(2)$. In other words, we have

$$\eta^{\alpha \dot{\alpha}}_R = -\varepsilon^{\alpha \beta} \epsilon_{\beta \gamma} \eta^{\beta \gamma}_R \quad \text{and so on.} \quad (5.2)$$

(To check this, write $\eta^{\alpha \dot{\alpha}}_R = \eta_m (\sigma^m)_{\alpha \dot{\alpha}}$ for some dummy real 4-vector $\eta_m$ with $\sigma^m = (\sigma^1, \sigma^2, \sigma^3, i \mathbb{1})$, $\varepsilon^{12} = \varepsilon^{12}' = +1$ and $\varepsilon^{\alpha \beta} \epsilon_{\beta \gamma} = -\delta^{\alpha \beta} \epsilon_{\beta \gamma} = -\delta^{\alpha \beta} \epsilon_{\beta \gamma}$.)

The constraints on $\epsilon$ due to the above configuration of branes are $\Gamma^{1234} \epsilon = \epsilon$ and $\Gamma^{5678} \epsilon = \epsilon$, which means $\epsilon$ has to be left-handed in both $C^2_{(12)}$ and $C^2_{(34)}$. Thus, there are four real left-handed supersymmetry parameters $\eta^{\alpha \dot{\alpha}}_L$ corresponding to supersymmetry in the left-moving sector: we have $N = (0, 4)$ supersymmetry in the 1 + 1 dimensional intersection $R^{1,1}$. The
R-symmetry of the $\mathcal{N} = (0, 4)$ supersymmetry algebra is $\text{SU}(2)_L \times \text{SU}(2)'_L$ and the parameters $\eta^{\alpha \dot{\alpha}}$ transform as a bispinor under this R-symmetry. We denote $\eta^{\alpha \dot{\alpha}}$ as $\eta^{\alpha' \dot{\alpha}'}$ or equivalently $\eta^{\alpha \dot{\alpha}'}$ in the sequel.

5.1 Low-energy spectrum and $\mathcal{N} = (0, 2)$ decomposition

We write the low-energy action in $\mathcal{N} = (0, 2)$ superspace by choosing a particular $\mathcal{N} = (0, 2)$ subalgebra of the $\mathcal{N} = (0, 4)$ supersymmetry algebra. See Appendix B for a description of $\mathcal{N} = (0, 2)$ superspace.

We choose the $\mathcal{N} = (0, 2)$ subalgebra generated by $\eta^{11'} := \eta^+$ and $-\eta^{22'} = \eta^{11'} = \eta^+$ (this will be the subalgebra preserved by the spiked instanton configuration). The supercoordinates are $\theta^+$ and $\bar{\theta}^+$. The R-symmetry $U(1)_L$ of the left-moving supersymmetry is generated by $F_L := F_L + F_R + F'_L + F'_R = F_{34} + F_{78}$ where $F_L = \frac{1}{2}(-F_{12} + F_{34})$, $F_R = \frac{1}{2}(F_{12} + F_{34})$, $F'_L = \frac{1}{2}(-F_{56} + F_{78})$ and $F'_R = \frac{1}{2}(F_{56} + F_{78})$. In our conventions, $\eta^+ = \eta^{11'}$ has charges $F_{12} = F_{56} = -\frac{1}{2}$ and $F_{34} = F_{78} = \frac{1}{2}$ giving $F_L = F'_L = +1/2$ and $F_R = F'_R = 0$ and hence a charge of +1 under $U(1)_L$. The $\mathcal{N} = (0, 2)$ content of the various multiplets from $D_p$-$D_p'$ strings are summarised in Table 1. The various fields are displayed with indices that indicate their $\text{SO}(4) \times \text{SO}(4)'$ representations.

Note: In order to avoid too many indices on the fields, the scalar component of a chiral multiplet $\Phi$ will be denoted by the same letter and the right-handed spin-$\frac{1}{2}$ component by $\zeta_\Phi$ in the sequel. Also, the left-handed spin-$\frac{1}{2}$ component of a Fermi superfield $\Lambda_a$ will be denoted as $\lambda_a$ where $a$ is an index that runs over all Fermi superfields in the theory. For example, the chiral multiplet $\tilde{J}$ in Table 1b has components $\tilde{\phi}^{2 \dagger}$ and $\tilde{\zeta}_J^{1 \dagger}$ which will be alternatively referred to as $\tilde{J}$ and $\zeta_\tilde{J}$ respectively. The left-moving fermionic component Fermi superfield $\Lambda_{\tilde{J}}$ will be denoted as $\lambda_{\tilde{J}}$.

5.2 Tachyons and Fayet-Iliopoulos terms

We are interested in generalising the above setup to one with a constant $B$-field of the form (3.1). We have seen in the analysis of the open string spectrum that there are tachyons of mass-squared

$$\frac{1}{2 \alpha'} |v_1 - v_2|, \quad \frac{1}{2 \alpha'} |v_3 - v_4|,$$

(5.3)

in the $D1$-$D5$ and $D1$-$D5'$ spectra (the on-shell formula is $m^2 = -k^2$). In our conventions these correspond to the fields $I$, $\tilde{I}$ for $(v_1 - v_2), (v_3 - v_4) > 0$ and $J$, $\tilde{J}$ for $(v_1 - v_2), (v_3 - v_4) < 0$. The system is no longer supersymmetric about the original vacuum (where all the vacuum expectation values are set to zero) due to the presence of tachyons. Can this supersymmetry breaking be interpreted as an $F$-term or $D$-term breaking?

Let us study the simpler problem $k$ D1-branes along $\mathbf{R}^{1,1}$ and $n$ D5-branes along $\mathbf{R}^{1,1} \times \mathbf{C}^2$. The low-energy effective action for the D5-branes contains the following coupling to the
Table 1: Various $\mathcal{N} = (0, 2)$ multiplets for the crossed instanton system.

(a) **D1-D1 strings**

| (0, 4) multiplet | Fields | (0, 2) multiplets |
|------------------|--------|------------------|
| Vector | $v_-$ ; $\lambda^\alpha\alpha'$ | Vector $V = (v_- ; \lambda_1^{11'})$, Fermi $A_2 = (\lambda_2^{12'})$ |
| Standard hyper | $X^{\alpha\bar{\alpha}} ; \zeta^{\alpha\bar{\alpha}}_+$ | Chiral $B_1 = (X^{11} ; \zeta_+^{22})$, Chiral $B_2 = (X^{12} ; \zeta_+^{23})$ |
| Twisted hyper | $X^{\alpha'\bar{\alpha}'} ; \zeta^{\alpha'\bar{\alpha}'}_+$ | Chiral $B_3 = (X^{11'} ; \zeta_+^{22'})$, Chiral $B_4 = (X^{12'} ; \zeta_+^{23'})$ |
| Fermi | $\lambda^{\alpha'\bar{\alpha}'}$ | Fermi $A_3 = (\lambda_+^{11'})$, Fermi $A_4 = (\lambda_+^{12'})$ |

(b) **D1-D5$_{(12)}$ strings**

$I, \Lambda_I$ transform in the $(k, \nu)$ of $U(k) \times U(n)$ while $J, \Lambda_J$ transform in the $(\overline{k}, \nu)$.

| (0, 4) multiplet | Fields | (0, 2) multiplets |
|------------------|--------|------------------|
| Standard hyper | $\phi^\alpha ; \zeta^\alpha_+$ | Chiral $I = (\phi^1 ; \zeta^2_+)$, Chiral $J = (\phi^{2\dagger} ; \zeta^{1\dagger}_+)$ |
| Fermi | $\lambda^{\alpha}_-$ | Fermi $\Lambda_I = (\lambda_-^{1\dagger})$, Fermi $\Lambda_J = (\lambda_-^{2\dagger})$ |

(c) **D1-D5$_{(34)}$ strings**

$\tilde{I}, \tilde{\Lambda}_I$ transform in the $(k, \nu')$ of $U(k) \times U(n')$ while $\tilde{J}, \tilde{\Lambda}_J$ transforms in the $(\overline{k}, \nu')$.

| (0, 4) multiplet | Fields | (0, 2) multiplets |
|------------------|--------|------------------|
| Twisted hyper | $\tilde{\phi}^{\alpha'} ; \zeta^{\alpha}_+$ | Chiral $\tilde{I} = (\tilde{\phi}^{1'} ; \zeta^{2\dagger}_+)$, Chiral $\tilde{J} = (\tilde{\phi}^{2\dagger'} ; \zeta^{1\dagger}_+)$ |
| Fermi | $\tilde{\lambda}^{\alpha}_+$ | Fermi $\tilde{\Lambda}_I = (\lambda^{1\dagger}_+)$, Fermi $\tilde{\Lambda}_J = (\lambda^{2\dagger}_+)$ |

(d) **D5$_{(12)}$-D5$_{(34)}$ strings**

$\Lambda$ transforms in the $(n, \nu')$ of $U(n) \times U(n')$.

| (0, 4) multiplet | Fields | (0, 2) multiplets |
|------------------|--------|------------------|
| Fermi | $\lambda_-$ | Fermi $\Lambda = (\lambda_-)$ |
(pullback of the) 2-form RR gauge field $C_2$:

$$
\frac{e_5}{2} \int_{\mathbb{R}^{1,1} \times C^2_A} C_2 \wedge \text{Tr} (\mathcal{F} \wedge \mathcal{F}) ,
$$

(5.4)

where $\mathcal{F} := 2\pi \alpha' (F - B)$ with $F$ the $U(n)$ field strength on the stack of D5-branes and $B$ the (pullback of the) NSNS $B$-field. The charge quantum $e_5$ is related to the D5-brane tension as $e_5 = T_5$ by virtue of its BPS nature and is given by

$$
e_5 = \frac{1}{g_s \sqrt{\alpha'}} (2\pi \sqrt{\alpha'})^5 .
$$

(5.5)

Let us consider a situation where $C_2$ is supported along $\mathbb{R}^{1,1}$, $F$ along $C^2_A$ and $B = 0$. Then, the above coupling becomes

$$S_{WZ,2} = e_1 k \int_{\mathbb{R}^{1,1}} C_2 , \quad \text{with} \quad k = \frac{1}{8\pi^2} \int_{C^2_A} \text{Tr} F \wedge F .
$$

(5.6)

Here, $e_1 = (g_s 2\pi \alpha')^{-1}$ is the D1-brane charge quantum. The coupling $k$ is the familiar instanton number of a $U(n)$ instanton in $C^2_A$. The above form of coupling implies that instantons of charge $k$ in the $U(n)$ gauge theory on the D5-branes induce D1-brane charge $e_1 k$ on the worldvolume. This was first realised in [D1].

A constant $B$-field along $C^2_A$ does a similar job and induces a D1-brane charge density

$$J_1 = \frac{N e_1}{8\pi^2} B \wedge B = \frac{N e_1}{8\pi^2} \frac{b_a b_b}{(2\pi \alpha')^2} \text{dVol}(C^2_A) ,
$$

(5.7)

The instability is qualitatively different for different ranges of the $B$-field values [SW3]. Let $C^2_A$ have the standard orientation. When $v_a$ and $v_b$ have opposite signs, $J_1$ is negative and corresponds to induced anti D1-branes. The tachyon in the NS sector then corresponds to the standard D1-D$\overline{1}$ tachyon. The condensation of this tachyon results in the annihilation of part of the D1-D$\overline{1}$ charge density and results in an excited state of the D5-brane with excitation energy proportional to the tachyon mass $m^2 = \frac{1}{2\alpha'} |v_a - v_b|$. When $v_a$ and $v_b$ have the same sign, the charge density is positive and corresponds to induced D1-branes. For $v_a \neq v_b$, tachyon condensation corresponds to the external D1-brane dissolving into the D5-brane and forming a bound state with the induced D1-branes (the Higgs branch of the D1-D5 system). The point with $v_a = v_b \neq 0$ corresponds to a self-dual $B$-field in which case the tachyon disappears and the D1-D5 system forms a bound state at threshold.

In either of these scenarios, one can describe these tachyon masses as arising from FI terms in the low energy effective action, at least for small values of $v_a - v_b$.

In the present situation of crossed instantons, Fayet-Iliopoulos terms arise as vacuum expectation values of auxiliary fields in the adjoint representation of $U(k)$. We have one real auxiliary field $D$ and one complex auxiliary field $G_2$ in the $N = (0, 4)$ vector multiplet, two complex
auxiliary fields $G_3$ and $G_4$ from the $\mathcal{N} = (0, 4)$ Fermi multiplets $\Lambda_3$ and $\Lambda_4$. The FI terms then correspond to the following $J$-terms in the $\mathcal{N} = (0, 2)$ action:

$$S_{\text{FI}} = -\frac{1}{\sqrt{2}} \text{Im} \int d^2x \text{D} + \text{Tr} \left\{ -\sqrt{2}tF_+ + b_2\Lambda_2 + b_3\Lambda_3 + b_4\Lambda_4 \right\},$$

$$= \int d^2x \text{Tr} \left\{ \frac{\theta}{2\pi} v_0 + rD + \text{Re}(b_2G_2 + b_3G_3 + b_4G_4) \right\}. \quad (5.8)$$

$t = \frac{\theta}{2\pi} + ir$ is the complexified Fayet-Iliopoulos parameter where $\theta$ is the two dimensional $\theta$-angle and $r$ is the real FI parameter. The components of the field strength Fermi multiplet $\mathcal{F}_-$ are given by

$$\lambda_{11}' := -(\mathcal{F}_-)_1, \quad D + iv_0 := (\nabla_+ \mathcal{F}_-)_1 . \quad (5.9)$$

From the SO(4) × SO(4)' properties of the Fermi multiplets in table 1a, it is easy to see that all FI terms except $r$ break the SO(2)$^4$ rotational symmetry that is preserved by the $B$-field in (3.1). Hence, only a non-zero $r$ could possibly account for the effect of such a $B$-field. The terms in the action involving $D$ are

$$\text{Tr}_k \left( \frac{1}{g^2} D^2 - \sum_{a \in \mathbb{A}} [B_a, B_a] D - IDI^\dagger + J^\dagger DJ - \bar{I}D\bar{I}^\dagger + \bar{J}^\dagger D\bar{J} + rD \right), \quad (5.10)$$

which gives the field equation

$$\frac{1}{g^2} D = \sum_{a \in \mathbb{A}} [B_a, B_a] + II^\dagger - J^\dagger J + \bar{I}\bar{I}^\dagger - \bar{J}\bar{J}^\dagger - r \cdot \mathbbm{1}_k . \quad (5.11)$$

The contribution to the Lagrangian from the $D$-terms is $-\frac{1}{2g^2} \text{Tr}_k D^2$ where $D$ substituted with its field equation. There are various quartic interaction terms along with the following mass terms for $I, \bar{I}, J$ and $\bar{J}$:

$$-\frac{g^2}{2} \text{Tr}_k \left( -rII^\dagger - r\bar{I}\bar{I}^\dagger + rJ^\dagger J + r\bar{J}\bar{J}^\dagger \right) . \quad (5.12)$$

As we can see, the mass-squared of $I$ and $\bar{I}$ are equal to $-\frac{g^2}{2}r$ and those of $J$ and $\bar{J}$ are equal to $+\frac{g^2}{2}r$. For small $B$-field, these answers could also be obtained via three-point disk amplitudes a la [HK] with two open string vertices (e.g. $I, I^\dagger$) and one closed string vertex corresponding to $B_{\mu\nu}$. The computation involves contributions from the boundary of the moduli space of the disk with three punctures (two punctures on the boundary and one in the bulk) and will be reported in a forthcoming paper [NP].

Comparing this with (5.3), we see that the $B$-fields must be related to each other and $r$ as

$$\frac{1}{2\pi g_s \alpha'^2} (v_1 - v_2) = \frac{1}{2\pi g_s \alpha'^2} (v_3 - v_4) = -r . \quad (5.13)$$

We thank Ashoke Sen for a discussion on this point.
Here, we have used that the coupling constant $g^2$ is given in terms of $\alpha'$ and the closed string coupling $g_s$ as $g^2 = 2\pi g_s \alpha'$. Thus, for the low-energy effective action to be supersymmetric, the constant $B$-field must satisfy

$$v_1 - v_2 = v_3 - v_4 .$$

We restrict our attention to constant $B$-field backgrounds satisfying the above constraint since these are amenable to an analysis using standard ideas of $D$-type supersymmetry breaking.

### 5.3 Yukawa couplings

So far, we have determined the minimally coupled kinetic terms and the masses coming from $D$-term interactions in the low energy effective theory. The remaining terms describing the dynamics are the $E$-terms and $J$-terms for the various Fermi multiplets. A simple way to obtain these is to look at the Yukawa couplings in the theory. Recall from Appendix B that Yukawa terms for a Fermi superfield $\Psi$ are of the general form

$$E_{\Psi} = +\bar{\psi}^a \frac{\partial E_a}{\partial \phi_j} \zeta_{j+} , \quad \text{and} \quad J_{\Psi} = -\frac{\partial J_a}{\partial \phi_j} \zeta_{j+} \psi_{a-} .$$

We obtain these terms in the low-energy effective action by computing 3-point string amplitudes on the disk. The idea is to look for non-zero amplitudes that involve only fields in the chiral multiplets but not their complex conjugates i.e. the fields in the chiral multiplets displayed in Table 1.

A general open string vertex operator in a constant $B$-field background has the form

$$V_{\lambda}(k, z) = \omega(\lambda) c(z) B(z) e^{\lambda H(z)} e^{2ik \cdot X(z)} c_{\lambda} .$$

Here, $\lambda$ is a weight in the covariant lattice $D_2 \oplus D_2 \oplus \Gamma_{1,1}$ corresponding to the spacetime symmetry $SO(1, 1) \times SO(4) \times SO(4')$ and $c_{\lambda}$ is the associated cocycle operator. $B(z)$ is the appropriate product of boundary condition changing operators for the worldsheet bosons. The weights for the various fields and the boundary condition changing operators for the worldsheet bosons are derived in Appendix A and summarised in Tables 2 and 3.

The rest of the notation is quite standard: $c(z)$ is the coordinate ghost, $H(z)$ is a 6-dimensional vector containing the five bosons that bosonise the ten worldsheet fermions and the sixth boson being the one that bosonises the superconformal ghosts, $k = (k^0, k^9)$ is the 1 + 1 dimensional momentum and $X = (X^0, X^9)$ are the worldsheet bosons corresponding to the 1 + 1 dimensional intersection. $\omega(\lambda)$ is an a priori undetermined c-number phase.

The general structure of a 3-pt function with open string vertex operators in the canonical
Table 2: Covariant weights for the vertex operators arising from D1-D1 strings. In our conventions, a left-handed spinor \( \psi^\alpha \) of SO(4) is specified by the weights \( \psi^{\alpha=1} = (-, +) \), \( \psi^{\alpha=2} = (+, -) \) and a right-handed spinor \( \psi^{\dot{\alpha}} \) by \( \psi^{\dot{\alpha}=1} = (-, -) \), \( \psi^{\dot{\alpha}=2} = (+, +) \).

| State            | Field | U(1) \( \ell \) | \( D_2 \oplus D_2 \oplus \Gamma_{1,1} \) weight |
|------------------|-------|-----------------|-----------------------------------------------|
| D1-D1 vector     | \( v_{\pm \pm} \) | 0               | \( (0, 0) \oplus (0, 0) \oplus (\mp 1; -1) \) |
| D1-D1 scalars    | \( X^{11}, B_1 \) | 0               | \( (-1, 0) \oplus (0, 0) \oplus (0; -1) \)    |
|                  | \( X^{12}, B_2 \) | 1               | \( (0, 1) \oplus (0, 0) \oplus (0; -1) \)    |
|                  | \( X^{11'}, B_3 \) | 0               | \( (0, 0) \oplus (-1, 0) \oplus (0; -1) \)    |
|                  | \( X^{12'}, B_4 \) | 1               | \( (0, 0) \oplus (0, 1) \oplus (0; -1) \)    |
| D1-D1 gauginos   | \( \lambda^{11'}, f \) | 1               | \( (-, +) \oplus (1, -) \oplus (+; -) \)    |
|                  | \( \lambda^{12'}, \lambda_2 \) | 0               | \( (-, +) \oplus (+, -) \oplus (+; -) \)    |
|                  | \( \lambda^{11'}, \lambda_3 \) | -1              | \( (-, -) \oplus (-, +) \oplus (+; -) \)    |
|                  | \( \lambda^{22'}, \lambda_4 \) | 0               | \( (-, -) \oplus (+, +) \oplus (+; -) \)    |
|                  | \( \zeta^{1}, \zeta_1 \) | -1              | \( (-, -) \oplus (1, -) \oplus (+; -) \)    |
|                  | \( \zeta^{2}, \zeta_2 \) | 0               | \( (+, +) \oplus (1, -) \oplus (-; -) \)    |
|                  | \( \zeta^{21'}, \zeta_3 \) | -1              | \( (+, -) \oplus (+, +) \oplus (-; -) \)    |
|                  | \( \zeta^{22'}, \zeta_4 \) | 0               | \( (+, -) \oplus (+, +) \oplus (-; -) \)    |

Table 3: Covariant weights for D1-D5\(_{(12)}\), D1-D5\(_{(34)}\) and D5\(_{(12)}\)-D5\(_{(34)}\) strings.

| State                  | Field | U(1) \( \ell \) | \( D_2 \oplus D_2 \oplus \Gamma_{1,1} \) weight |
|------------------------|-------|-----------------|-----------------------------------------------|
| D1-D5\(_{(12)}\) bosons | \( \phi^1, I \) | \( \frac{1}{2} - v_2 \) | \( (v_1 - \frac{1}{2}, v_2 + \frac{1}{2}) \oplus (0, 0) \oplus (0; -1) \) |
|                        | \( \phi^{21}, J \) | \( \frac{1}{2} + v_2 \) | \( (v_1 - \frac{1}{2}, v_2 + \frac{1}{2}) \oplus (0, 0) \oplus (0; -1) \) |
| D1-D5\(_{(12)}\) fermions | \( \zeta^{11}, \zeta_1 \) | \( \frac{1}{2} + v_2 \) | \( (v_1, v_2) \oplus (1, -) \oplus (-; -) \) |
|                        | \( \zeta^{22}, \zeta_2 \) | \( \frac{1}{2} - v_2 \) | \( (v_1, v_2) \oplus (1, -) \oplus (-; -) \) |
|                        | \( \lambda^{11}, \lambda_1 \) | \( \frac{1}{2} + v_2 \) | \( (v_1, v_2) \oplus (+, +) \oplus (+; -) \) |
|                        | \( \lambda^{22}, \lambda_2 \) | \( \frac{1}{2} - v_2 \) | \( (v_1, v_2) \oplus (+, +) \oplus (+; -) \) |
| D1-D5\(_{(34)}\) bosons | \( \phi^{11}, \bar{I} \) | \( \frac{1}{2} - v_4 \) | \( (0, 0) \oplus (v_3 - \frac{1}{2}, v_4 + \frac{1}{2}) \oplus (0; -1) \) |
|                        | \( \phi^{22}, \bar{J} \) | \( \frac{1}{2} + v_4 \) | \( (0, 0) \oplus (v_3 - \frac{1}{2}, v_4 + \frac{1}{2}) \oplus (0; -1) \) |
| D1-D5\(_{(34)}\) fermions | \( \zeta^{11}, \bar{\zeta}_1 \) | \( \frac{1}{2} - v_4 \) | \( (+, -) \oplus (v_3, v_4) \oplus (-; -) \) |
|                        | \( \zeta^{22}, \bar{\zeta}_2 \) | \( \frac{1}{2} - v_4 \) | \( (+, -) \oplus (v_3, v_4) \oplus (-; -) \) |
|                        | \( \lambda^{11}, \bar{\lambda}_1 \) | \( \frac{1}{2} + v_4 \) | \( (+, +) \oplus (v_3, v_4) \oplus (+; -) \) |
|                        | \( \lambda^{22}, \bar{\lambda}_2 \) | \( \frac{1}{2} - v_4 \) | \( (+, +) \oplus (v_3, v_4) \oplus (+; -) \) |
| D5\(_{(12)}\)-D5\(_{(34)}\) fermions | \( \lambda_-, \lambda \) | \( v_2 - v_4 \) | \( (v_1, v_2) \oplus (-v_3, -v_4) \oplus (+; -) \) |
A few comments are in order:

1. The phase prefactor \( \prod_{i<j} e^{i\pi\lambda_i - \lambda_j} \) in the last expression is due to the cocycle operators \( c_{\lambda_i} \) commuting across the vertex operators \( e^{\lambda_j H} \). Here, \( M \) is a \( 6 \times 6 \) matrix whose form is given in Appendix A. These phases are crucial for obtaining the correct low-energy Yukawa couplings.

2. For the case of crossed instantons, all the \( E \)-terms and \( J \)-terms turn out to be quadratic in the superfields. Looking at (5.15), it is easy to see that there will be two different amplitudes that arise from the same \( E \)- or \( J \)-term. We get relations between the phases \( \omega(\lambda) \) by equating the coefficients of these two amplitudes.

3. The correlators are non-zero only when the spacetime momenta add up to zero, the \( D_2 \oplus D_2 \oplus \Gamma_{1,1} \) weights add up to \((0,0,0,0; -2)\) with the first five entries signifying \( \text{SO}(4) \times \text{SO}(4)' \times \text{SO}(1,1) \) invariance and the \(-2\) indicating that the superconformal anomaly on the disk is soaked up.

4. When the correlators are non-zero, it can be shown that the different contributions to the exponent of \( x_i - x_j \) coming from the coordinate ghosts, the BCC operators for the worldsheet bosons, the vertex operators for the worldsheet fermions and the vertex operators for the \( \mathbb{R}^{1,1} \) directions all add up to zero. This shows that the correlator is independent of the points of insertion of the vertex operators as it should be due to \( \text{SL}(2, \mathbb{R}) \) invariance.

The \( E \)-term and \( J \)-term Yukawa couplings for the various Fermi multiplets are as follows:

**The D1-D1 Fermi multiplets \( \Lambda_2, \Lambda_3, \Lambda_4 \)**

\[
\mathcal{J}^{\Lambda_2} = \omega(B_2)\omega(\zeta_4)\omega(\lambda_2) e^{-\frac{\pi i}{4}} \text{Tr}_k \left( [B_2, \zeta_4]_2 + [\zeta_3, B_4]_2 \right) \\
+ \omega(\tilde{I})\omega(\tilde{\zeta}_4)\omega(\lambda_2) e^{\frac{\pi i}{4}(-1-2\nu_3)} \text{Tr}_k \left( \tilde{I} \tilde{\zeta}_2 \lambda_2 + \tilde{\zeta}_4 \tilde{J} \lambda_2 \right),
\]

\[
\mathcal{E}_{\Lambda_2} = \omega(\tilde{\chi}_2)\omega(B_1)\omega(\zeta_2) i \text{Tr}_k \left( \tilde{\chi}_2[B_1, \zeta_2] + \tilde{\chi}_2[\zeta_1, B_2] \right) \\
+ \omega(\tilde{\chi}_2)\omega(\zeta_4)\omega(J) e^{\frac{\pi i}{4}(-3+6\nu_1)} \text{Tr}_k \left( \tilde{\chi}_2 \zeta_4 J + \tilde{\chi}_2 I \zeta_4 \right). \tag{5.18}
\]

\[
\mathcal{J}^{\Lambda_3} = \omega(B_2)\omega(\zeta_4)\omega(\lambda_3) e^{-\frac{\pi i}{4}} \text{Tr}_k \left( [B_2, \zeta_4]_3 + [\zeta_2, B_4]_3 \right),
\]

\[
\mathcal{E}_{\Lambda_3} = \omega(\tilde{\chi}_3)\omega(B_1)\omega(\zeta_3) i \text{Tr}_k \left( \tilde{\chi}_3[B_1, \zeta_3] + \tilde{\chi}_3[\zeta_1, B_3] \right). \tag{5.19}
\]
\[ \mathcal{J}_{\Lambda_4} = \omega(B_2)\omega(\zeta_3)\omega(\lambda_4) \ \e^{\frac{i\pi}{4}} \ \text{Tr}_k \ (\{B_2, \zeta_3\lambda_1 + [\zeta_2, B_3]\lambda_4\}), \]
\[ \mathcal{E}_{\Lambda_4} = \omega(\bar{\lambda}_4)\omega(B_1)\omega(\zeta_4) \ i \ \text{Tr}_k \ (\bar{\lambda}_4[B_1, \zeta_4] + \bar{\lambda}_4[\zeta_1, B_4]). \]  

(5.20)

Relations:
\[ \frac{\omega(B_3)\omega(\zeta_4)}{\omega(\zeta_3)\omega(B_1)} = \frac{\omega(B_1)\omega(\zeta_2)}{\omega(\zeta_1)\omega(B_2)} = i, \quad \frac{\omega(I)\omega(\zeta_J)}{\omega(\zeta_J)\omega(I)} = \frac{\omega(\bar{I})\omega(\bar{\zeta}_J)}{\omega(\bar{\zeta}_J)\omega(\bar{I})} = \e^{-\frac{i\pi}{4}}, \]
\[ \frac{\omega(B_2)\omega(\zeta_4)}{\omega(\zeta_2)\omega(B_4)} = 1, \quad \frac{\omega(B_2)\omega(\zeta_3)}{\omega(\zeta_2)\omega(B_3)} = \frac{\omega(B_4)\omega(\zeta_1)}{\omega(\zeta_4)\omega(B_1)} = -i. \]  

(5.21)

The D1-D5\textsubscript{(12)} Fermi multiplets \(\Lambda_I, \Lambda_J\)
\[ \mathcal{J}^{\Lambda_I} = \omega(B_3)\omega(\zeta_J)\omega(\lambda_I) \ i \ \text{Tr}_k \ (B_3 \zeta_J \lambda_I + \zeta_3 I \lambda_J), \]
\[ \mathcal{E}_{\Lambda_I} = \omega(\bar{\lambda}_I)\omega(\zeta_J)\omega(B_4) \ e^{\frac{i\pi}{4}(4+8v_1)} \ \text{Tr}_k \ (\bar{\lambda}_I \zeta_J B_4 + \bar{\lambda}_J \zeta_J \lambda_I). \]  

(5.22)

Relations:
\[ \frac{\omega(I)\omega(\zeta_J)}{\omega(\zeta_J)\omega(B_3)} = \e^{\frac{i\pi}{4}(-1+2v_1)}, \quad \frac{\omega(J)\omega(\zeta_J)}{\omega(\zeta_J)\omega(B_4)} = \e^{\frac{i\pi}{4}(2+2v_1)}, \]
\[ \frac{\omega(I)\omega(\zeta_J)}{\omega(\zeta_J)\omega(B_4)} = \e^{\frac{i\pi}{4}(1+2v_1)}, \quad \frac{\omega(J)\omega(\zeta_J)}{\omega(\zeta_J)\omega(B_3)} = \e^{\frac{i\pi}{4}(2v_1)}. \]  

(5.24)

The D1-D5\textsubscript{(34)} Fermi multiplets \(\bar{\Lambda}_I, \bar{\Lambda}_J\)
\[ \mathcal{J}^{\bar{\Lambda}_J} = \omega(B_1)\omega(\bar{\zeta}_J)\omega(\bar{\lambda}_J) \ e^{\frac{i\pi}{4}(-6+4v_3-4v_4)} \ \text{Tr}_k \ (B_1 \bar{\zeta}_J \bar{\lambda}_J + \bar{\zeta}_1 \bar{\lambda}_J \bar{\lambda}_J), \]
\[ \mathcal{E}_{\bar{\Lambda}_J} = \omega(\bar{\lambda}_J)\omega(\bar{\zeta}_J)\omega(B_2) \ e^{\frac{i\pi}{4}(4v_3+4v_4)} \ \text{Tr}_k \ (\bar{\lambda}_J \bar{\zeta}_J B_2 + \bar{\lambda}_J \bar{\zeta}_J \bar{\lambda}_J). \]  

(5.25)

Relations:
\[ \frac{\omega(I)\omega(\bar{\zeta}_J)}{\omega(\bar{\zeta}_J)\omega(B_1)} = \e^{\frac{i\pi}{4}(-1+2v_3)}, \quad \frac{\omega(J)\omega(\bar{\zeta}_J)}{\omega(\bar{\zeta}_J)\omega(B_2)} = \e^{\frac{i\pi}{4}(2+2v_3)}, \]
\[ \frac{\omega(I)\omega(\bar{\zeta}_J)}{\omega(\bar{\zeta}_J)\omega(B_2)} = \e^{\frac{i\pi}{4}(1+2v_3)}, \quad \frac{\omega(J)\omega(\bar{\zeta}_J)}{\omega(\bar{\zeta}_J)\omega(B_1)} = \e^{\frac{i\pi}{4}(2v_3)}. \]  

(5.27)

The D5\textsubscript{(12)}-D5\textsubscript{(34)} Fermi multiplet \(\Lambda\)
\[ \mathcal{J}^{\Lambda} = \omega(\bar{\zeta}_J)\omega(I)\omega(\lambda) \ e^{\frac{i\pi}{4}(-2-2v_1-4v_2+4v_3+4v_4-4v_1v_3)} \ \text{Tr}_k \ (I \bar{\zeta}_J + \zeta_I \bar{\lambda}_J), \]
\[ \mathcal{E}_{\Lambda} = \omega(\bar{\lambda}_J)\omega(J)\omega(\bar{\zeta}_J) \ e^{\frac{i\pi}{4}(2+6v_1+4v_2-4v_3-4v_4+4v_1v_2)} \ \text{Tr}_k \ (\bar{\zeta}_J \bar{\lambda}_J + \bar{\lambda}_J \bar{\zeta}_J). \]  

(5.28)
Relations:
\[ \frac{\omega(I) \omega(\bar{J})}{\omega(\zeta_I) \omega(\bar{J})} = e^{i\pi(1+2v_1-2v_3)} , \quad \frac{\omega(J) \omega(\bar{I})}{\omega(\zeta_J) \omega(I)} = e^{i\pi(1+2v_1-2v_3)} . \] (5.29)

The above couplings can be massaged into a nicer form by using the relations between the phases and choosing a convenient value for the independent phases. In other words, we redefine the fields by absorbing the independent phases into the corresponding fields. From the relations, it is easy to see that there is only one independent ratio of the form \( \frac{\omega(\phi)}{\omega(\zeta_I)} \) where \((\phi, \zeta_I)\) form a \( \mathcal{N} = (0, 2) \) chiral superfield. The phases corresponding to the Fermi superfields are not determined from the above relations. Setting \( \frac{\omega(B_1)}{\omega(\zeta_I)} = c \), we have
\[ \frac{\omega(B_1)}{\omega(\zeta_I)} = \frac{\omega(B_2)}{\omega(\zeta_3)} = \bar{c} , \quad \frac{\omega(B_2)}{\omega(\zeta_2)} = \frac{\omega(B_4)}{\omega(\zeta_4)} = -i\bar{c} , \]
\[ \frac{\omega(I)}{\omega(\zeta_I)} = e^{i\pi(1-2v_1)} \bar{c} , \quad \frac{\omega(J)}{\omega(\zeta_J)} = e^{i\pi(2v_1)} \bar{c} , \quad \frac{\omega(\bar{I})}{\omega(\zeta_I)} = e^{i\pi(1-2v_3)} \bar{c} , \quad \frac{\omega(\bar{J})}{\omega(\zeta_J)} = e^{i\pi(2v_3)} \bar{c} . \] (5.30)

We now make a particular choice for the independent phases:
\[ c = e^{\frac{3i\pi}{8}} , \quad \omega(B_a) = 1 , \quad \omega(I) = \omega(J) = e^{-\frac{i\pi}{4}(1+2v_1)} , \quad \omega(\bar{I}) = \omega(\bar{J}) = e^{\frac{i\pi}{4}(1+2v_3)} , \]
\[ \omega(\lambda_2) = \omega(\lambda_4) = e^{-\frac{i\pi}{4}} , \quad \omega(\lambda_3) = e^{\frac{3i\pi}{8}} , \quad \omega(\lambda_f) = e^{-\frac{3i\pi}{8}} e^{i\pi v_1} , \quad \omega(\lambda_l) = e^{-\frac{i\pi}{8}} e^{-i\pi (v_1 + 2v_2)} , \]
\[ \omega(\bar{\lambda}_j) = e^{\frac{i\pi}{8}} e^{i\pi (v_3 + v_4)} , \quad \omega(\bar{\lambda}_l) = e^{\frac{3i\pi}{8}} e^{-i\pi (v_3 + v_4)} , \quad \omega(\lambda) = e^{-\frac{3i\pi}{8}} e^{i\pi (v_1 + v_2 - v_3 - v_4 + v_1 v_2)} . \] (5.31)

With the above choice of phases, the \( J \)- and \( E \)-terms are given by
\[ J^{A_2} = [B_3, B_4] + \bar{I} \bar{J} , \quad E_{A_2} = [B_1, B_2] + IJ , \]
\[ J^{A_3} = [B_2, B_4] , \quad E_{A_3} = [-B_1, B_3] , \quad J^{A_4} = [B_2, B_3] , \quad E_{A_4} = [B_1, B_4] , \]
\[ J^{A_J} = B_3 I , \quad E_{A_J} = J B_4 , \quad J^{A_I} = -J B_3 , \quad E_{A_I} = B_4 I , \]
\[ J^{A_\bar{J}} = -B_1 \bar{I} , \quad E_{A_\bar{J}} = -\bar{J} B_2 , \quad J^{A_I} = \bar{J} B_1 , \quad E_{A_I} = -B_2 \bar{I} , \]
\[ J^A = \bar{J} I , \quad E_A = -J \bar{I} . \] (5.32)

The identity \( \text{Tr}_k J \cdot E = 0 \): We have
\[ \text{Tr}_k \{ ([B_3, B_4] + \bar{I} \bar{J})([B_1, B_2] + IJ) - [B_2, B_4][B_1, B_3] + [B_2, B_3][B_1, B_4] + B_3 I J B_4 - B_4 I J B_3 + B_1 \bar{I} \bar{J} B_2 - B_2 \bar{I} \bar{J} B_1 - I J \bar{I} J \} = 0 . \] (5.33)

Thus, \( \text{Tr}_k J \cdot E = 0 \) is indeed satisfied and the low-energy effective action is indeed \( \mathcal{N} = (0, 2) \) supersymmetric. Since the action is also covariant with respect to the \( SU(2)_L \times SU(2)'_L \) R-symmetry, it is \( \mathcal{N} = (0, 4) \) supersymmetric as well. This is the same result that is obtained in [T] for the case of zero \( B \)-field.
5.4 The crossed instanton moduli space

The bosonic potential energy $U$ is

$$U = \frac{g^2}{2} \text{Tr} D^2 + \sum_a |E_a|^2 + \sum_a |J^a|^2 ,$$  

(5.34)

with the auxiliary field $D$ substituted with its field equation in (5.11). The minima of the potential can be obtained by solving the equations $D = 0$, $E_a = 0$ and $J^a = 0$. We relabel $I, J \rightarrow I_{12}, J_{12}$ and $\bar{I}, \bar{J} \rightarrow I_{34}, J_{34}$ in anticipation of the spiked instanton scenario. The vacuum moduli space is then defined by the following equations upto a $U(k)$ gauge transformation:

$D$-terms:

$$D^\alpha \cdot r \cdot 1_k = \sum_{a=1}^4 [B_a, B_a^\dagger] + I_{12}I_{12}^\dagger - J_{12}J_{12} - J_{34}J_{34}^\dagger - r \cdot 1_k = 0 .$$  

(5.35)

$J$-terms:

$$\mu^{C}_{34} = [B_3, B_4] + I_{34}J_{34} = 0 , \quad \mu^{C}_{24} = [B_2, B_4] = 0 , \quad \mu^{C}_{23} = [B_2, B_3] = 0 , \quad \sigma^{C}_{3,12} = B_3I_{12} = 0 , \quad \tilde{\sigma}^{C}_{3,12} = -J_{12}B_3 = 0 , \quad \sigma^{C}_{1,34} = -B_1I_{34} = 0 , \quad \sigma^{C}_{1,34} = J_{34}B_1 = 0 , \quad \Upsilon^{C}_{12} = J_{34}I_{12} = 0 .$$  

(5.36)

$E$-terms:

$$\mu^{C}_{12} = [B_1, B_2] + I_{12}J_{12} = 0 , \quad \mu^{C}_{13} = [B_1, B_3] = 0 , \quad \mu^{C}_{14} = [B_1, B_4] = 0 , \quad \sigma^{C}_{1,12} = B_1I_{12} = 0 , \quad \tilde{\sigma}^{C}_{1,12} = J_{12}B_1 = 0 , \quad \sigma^{C}_{2,34} = -B_2I_{34} = 0 , \quad \sigma^{C}_{2,34} = -J_{34}B_2 = 0 , \quad \Upsilon^{C}_{34} = -J_{12}I_{34} = 0 .$$  

(5.37)

Symmetries

Note that the above equations are invariant under $U(k) \times U(n) \times U(n')$ transformations. The crossed instanton moduli space is then defined by the solutions of the above equations modulo $U(k)$ gauge transformations. The group $P(U(n) \times U(n')) \cong \frac{U(n) \times U(n')}{U(1)_c}$, where $U(1)_c$ is the common centre of $U(n) \times U(n')$, remains a global symmetry on the moduli space. These are the framing rotations described in [N4].

There are additional symmetries from the $SU(2)_L \times SU(2)_R \times SU(2)'_L \times SU(2)'_R$ arising from rotations of the transverse $\mathbb{R}^8$. To see how many of these symmetries are preserved by the vacuum moduli space, we first form real combinations of the holomorphic equations above:

$$s_A := \mu^{C}_A + \varepsilon_A^\pi (\mu^{C}_A)^\dagger = 0 , \quad \text{for} \quad A \in \bar{\mathfrak{g}} ,$$  

$$\sigma^{\pi A} := \sigma^{\pi A}_C + \varepsilon^{\pi A}_B (\tilde{\sigma}^{C}_B)^\dagger = 0 , \quad \text{for} \quad A \in \bar{\mathfrak{g}} , \quad \pi \in \bar{\mathfrak{a}} ,$$  

$$\Upsilon^C_A := \Upsilon^C_A - \varepsilon_A^\pi (\Upsilon^C_A)^\dagger = 0 \quad \text{for} \quad A \in \bar{\mathfrak{g}} .$$  

(5.38)
Using the $SO(4) \times SO(4)'$ transformation properties of the fields in Table 1 it is easy to see that the equations with $r = 0$ preserve a diagonal subgroup $SU(2)_{\Delta}$ of the R-symmetry $SU(2)_{L} \times SU(2)'_{L}$. The equations $\mu^{R}$, $s_{12}$ and $s_{34}$ form a triplet and the other real equations are invariant under $SU(2)_{\Delta}$.

For $r \neq 0$, the subgroup $SU(2)_{\Delta}$ is broken down to its maximal torus $U(1)_{\Delta}$ which is the R-symmetry $U(1)_{L}$ of the $\mathcal{N} = (0, 2)$ subalgebra that was chosen above. The factors $SU(2)_{R} \times SU(2)'_{R}$ survive as spectator symmetries. Hence, the total global symmetry on the crossed instanton moduli space is

$$P \left( U(n) \times U(n') \right) \times SU(2)_{R} \times SU(2)'_{R} \times U(1)_{\Delta} \ .$$

(5.39)

**Note:** The vacuum moduli space for $r = 0$ splits up into many distinct branches corresponding to the Coulomb branch, the two Higgs branches (with the D1’s binding to either of the D5-branes) and mixed branches [T]. Once a non-zero $r$ is introduced, the D1-branes bind necessarily to some stack of D5-branes and the moduli space becomes connected. Turning on $r$ also has the effect of reducing the global symmetries as we saw above. It would be interesting to repeat the R-charge analysis of [T] in this case and explore the infrared limit of this $\mathcal{N} = (0, 4)$ gauge theory and its $\mathcal{N} = (0, 2)$ spiked generalisation along the lines of [SiWi1, SiWi2].

### 6 Spiked instantons

Consider the crossed instanton setup of D1-D5$_{(12)}$-D5$_{(34)}$ branes. Let us choose the $B$-field such that $v_{1}v_{2} \leq 0$ and $v_{3}v_{4} \leq 0$. This ensures that the tachyons are of D1-D1 type. In this region of the space of $B$-fields, the tachyon mass can never be zero unless the $v$’s are zero.

Let us introduce a stack of D5$_{(23)}$-branes to the mix. In order to realise a symmetric situation where the instability here is also of D1-D1 type, we need $v_{2}v_{3} \leq 0$. This implies that $v_{1}v_{3} \geq 0$ and $v_{2}v_{4} \geq 0$. Suppose we next add the two stacks of five branes along $R^{1,1} \times C^{2}_{(13)}$ and $R^{1,1} \times C^{2}_{(13)}$. The constraints $v_{1}v_{3} \geq 0$ and $v_{2}v_{4} \geq 0$ and the requirement that the tachyons should be D1-D1 tachyons automatically force these stacks to be made of anti D5-branes! We thus have the following configuration of D5-branes and anti D5-branes:

$$D5_{(12)} , D5_{(34)} , D5_{(23)} , D5_{(14)} , \overline{D5}_{(13)} , \overline{D5}_{(24)} \ .$$

(6.1)

This is the same configuration of six stacks of D5-branes which preserves two supercharges when the $B$-field is dialled to zero. Though two the six stacks of D5-branes are composed of antibranes, the D1-branes bind to the various stacks of D5-branes and $\overline{D5}$-branes in a symmetric fashion in that all the tachyonic instabilities are of the brane-antibrane type.

One may again enquire as to whether an FI term in the low-energy effective action can accommodate the effect of the constant $B$-field of the form (3.1). The masses of the tachyons

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for the various D1-D5 (and D1\(\overline{D5}\)) strings can be read off from the derivation of the open string spectrum in Section 4:

\[
\frac{1}{2\alpha'}|v_1 - v_2|, \quad \frac{1}{2\alpha'}|v_3 - v_4|, \quad \frac{1}{2\alpha'}|v_2 - v_3|, \\
\frac{1}{2\alpha'}|v_1 - v_4|, \quad \frac{1}{2\alpha'}|v_1 + v_3|, \quad \frac{1}{2\alpha'}|v_2 + v_4|.
\] (6.2)

Repeating the analysis in the crossed case, we see that the field equation for the auxiliary field \(D\) becomes

\[
D = \sum_{a \in \mathcal{A}} [B_a, B_a^\dagger] + \sum_{A \in \mathcal{B}} (I_A J_A^\dagger - J_A I_A^\dagger) - r \cdot 1_k.
\] (6.3)

giving rise to the same mass-squared \(|r|\) to all the tachyons. Thus, the \(B\)-field values must satisfy

\[
v_1 = -v_2 = v_3 = -v_4,
\] (6.4)

in order to be accounted for by the real FI parameter in the low-energy theory.

The presence of the extra four stacks of D5-branes gives rise to additional terms in the \(E\)-terms and \(J\)-terms for the Fermi multiplets \(\Lambda_3\) and \(\Lambda_4\). There are also additional Fermi multiplets from the open strings stretching between D1-branes and these stacks of D5-branes and \(\overline{D5}\)-branes. Repeating the disk amplitude calculation as above, one get the following equations:

1. The real moment map:

\[
\mu_R - r \cdot 1_k := \sum_{a \in \mathcal{A}} [B_a, B_a^\dagger] + \sum_{A \in \mathcal{B}} (I_A J_A^\dagger - J_A I_A^\dagger) - r \cdot 1_k = 0.
\] (6.5)

2. For \(A = (ab) \in \mathcal{B}\) with \(a < b\),

\[
\mu^C_A := [B_a, B_b] + I_A J_A = 0.
\] (6.6)

3. For \(A \in \mathcal{B}, \ \overline{A} = \mathcal{A} \setminus A\) and \(\overline{a} \in \overline{A}\),

\[
\sigma^C_{\overline{a}A} := B_{\overline{a}} I_A = 0, \quad \tilde{\sigma}^C_{\overline{a}A} := J_A B_{\overline{a}} = 0.
\] (6.7)

4. For \(A \in \mathcal{B}, \ \overline{A} = \mathcal{A} \setminus A\),

\[
\gamma^C_A := J_\overline{A} I_A = 0.
\] (6.8)

**Symmetries**

The symmetries of the above equations can be obtained in a similar way to the crossed instanton case. The total global symmetry is given by

\[
P \left( \bigotimes_{A \in \mathcal{B}} U(n_A) \right) \times U(1)^3,
\] (6.9)

where \(U(1)^3\) is a maximal torus of \(SU(4)\), the isometry group of the transverse \(\mathbb{C}^4\) which preserves some fraction of supersymmetry.
Folded branes

The above equations arise from considering D1-D1 strings, D1-D5\(_A\) strings and D5\(_A\)-D5\(_A\) strings. There are also additional equations that result from the interaction of D1-branes with states from open strings stretching between D5\(_A\) and D5\(_B\) with \(A = (ac)\) and \(B = (bc)\) i.e. two stacks of D5-branes that have a line \(C_c\) in common. This is the setup of folded branes and once we throw in D1-branes, the classical moduli space of vacua is called the moduli space of folded instantons.

The open string spectrum for this case was analysed in Section 2 and there we saw that there were tachyons in the NS sector with mass \(\frac{1}{2}|v_a \pm v_b|\) for D5-D5 strings and D5-D5 strings respectively. Note that this is opposite to the masses one obtains for D1-D5 and D1-D5 strings. Thus, for the configuration of branes in (6.1) it is easy to see that the spectrum of tachyon masses is precisely the same as in (6.2). With the constraint in (6.4), all tachyons have the same mass-squared which is equal to \(\frac{1}{\alpha'}|v_1|\).

All the states arising from such strings are supported over the four dimensional subspace \(R^{1,1} \times C_c\) with a constant B-field tan \(\pi v_c\) along \(C_c\) which makes the space non-commutative. It has been conjectured in [N3, N4] that the interaction of these states with the states supported on \(R^{1,1}\) gives rise to an additional (infinite) set of equations of the form

\[
\Upsilon_{A,B,j} = J_A(B_c)^{j-1} I_B = 0, \quad \text{for} \quad j = 1, 2, \ldots
\]  

(6.10)

The stringy derivation of these equations requires a deeper analysis of the non-commutative theory on \(R^{1,1} \times C_c\) and also the calculation of \((j+2)\)-point disk amplitudes. These calculations shall be reported in a forthcoming paper [NP].
A Open strings in a constant $B$-field

We follow the treatment of background gauge fields in [ACNY]. Consider an open string propagating in flat spacetime in the presence of a constant $B$-field with components only along spatial directions. The $\mathcal{N} = (1,1)$ supersymmetric worldsheet theory is formulated in terms of the superfield $X^\mu$ with components

$$X^\mu := X^\mu_1, \quad \psi^\mu_\pm := (D_\pm X^\mu)_1, \quad F^\mu = (D_+ D_- X^\mu)_1,$$

where the $1$ sets all the Grassmann coordinates to zero. Our conventions are such that $D^2_\pm = i \partial_{\pm}, \{D_+, D_\pm\} = 0$ and $\partial_{\pm} = \frac{1}{2} (\partial_r \pm \partial_\sigma)$. The supersymmetric action is given by

$$S = \frac{1}{\pi \alpha'} \int d\tau d\sigma \, D_+ D_- \left\{ (G_{\mu\nu} + 2\pi \alpha' B_{\mu\nu}) \, D_+ X^\mu D_- X^\nu \right\},$$

$$= \frac{1}{\pi \alpha'} \int d\tau d\sigma \, (G_{\mu\nu} + 2\pi \alpha' B_{\mu\nu}) \left( \partial_{++} X^\mu \partial_{--} X^\nu - F^\mu F^\nu + i \psi^\nu_- \partial_{++} \psi^\mu_- + i \psi^\mu_+ \partial_{--} \psi^\nu_+ \right),$$

$$= \frac{1}{\pi \alpha'} \int d\tau d\sigma \, G_{\mu\nu} \left( \partial_{++} X^\mu \partial_{--} X^\nu - F^\mu F^\nu + i \psi^\nu_- \partial_{++} \psi^\mu_- + i \psi^\mu_+ \partial_{--} \psi^\nu_+ \right) +$$

$$- \frac{1}{\pi \alpha'} \int d\tau \, 2\pi \alpha' B_{\mu\nu} \left[ 2(\partial_\tau X^\mu) X^\nu + i \psi^\nu_- \psi^\mu_- + i \psi^\mu_+ \psi^\nu_+ \right]_{\sigma=0} + \text{total } \tau\text{-derivative} \ .$$

The boundary conditions that result from the Euler-Lagrange variation are

$$\left( G_{\mu\nu} \partial_\sigma X^\mu + 2\pi \alpha' B_{\mu\nu} \partial_\tau X^\mu \right) \delta X^\nu \bigg|_{\sigma=0} = 0 \ ,$$

$$\delta \psi^\mu_-(G_{\mu\nu} - 2\pi \alpha' B_{\mu\nu}) \psi^\nu_+ \bigg|_{\sigma=0} = \delta \psi^\mu_+(G_{\mu\nu} + 2\pi \alpha' B_{\mu\nu}) \psi^\nu_- \bigg|_{\sigma=0} ,$$

and the same ones for $\sigma = \pi$ as well. We assume that the metric $G_{\mu\nu}$ is the standard Minkowski metric and choose a coordinate system such that the constant $B$-field is in block diagonal form:

$$2\pi \alpha' B = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} .$$

If the metric contains off-diagonal components, it is in general not possible to cast the $B$-field in the above form since the metric and $B$-field preserve different subgroups of $\text{GL}(1,9)$. In such a coordinate system, the above analysis reduces to that of an open string in $\mathbb{R}^2$ with a constant $B$-field $\frac{1}{2\pi \alpha'} dx^1 \wedge dx^2$. Let us first study the boundary conditions on the worldsheet bosons. Writing $b = \tan \pi v$ and $Z := \frac{1}{\sqrt{2}}(X^1 + iX^2)$ the boundary conditions become

$$(\partial_\sigma Z + i2\pi \alpha' B \partial_\tau Z) \delta \overline{Z} = 0 \ .$$

Thus, we can have two types of boundary conditions at each end:

**Dirichlet (D)**: $\delta Z = 0$ i.e. $Z = z_0 \in \mathbb{C}$ ,

**Twisted (T)**: $\partial_\sigma Z + i2\pi \alpha' B \partial_\tau Z = 0$ .

(A.6)
The twisted boundary conditions can also be written as \( \partial_{++}Z = e^{-2\pi i\nu} \partial_{--}Z \) with \( 2\pi \alpha' B = \tan \pi v \). Dirichlet boundary conditions are realised also by letting \( v \to \infty \). In order to accommodate all types of boundary conditions at both ends, we introduce the more general boundary conditions

\[
\begin{align*}
\partial_{++}Z &= e^{-2\pi i\nu} \partial_{--}Z, \quad \text{at } \sigma = 0, \\
\partial_{++}Z &= e^{-2\pi i\mu} \partial_{--}Z, \quad \text{at } \sigma = \pi. 
\end{align*}
\] (A.7)

The boundary conditions with \( B \)-field can be realised by taking \( \nu = \mu, \mu = \frac{1}{2} \) for the TD case and \( \nu = \frac{1}{2}, \mu = \nu \) for the DT case. The solution to the \( Z \) field equation consists of independent left-moving and right-moving waves:

\[
Z(\tau, \sigma) = Z_L(\tau + \sigma) + Z_R(\tau - \sigma),
\] (A.8)

with the mode expansions

\[
\begin{align*}
Z_L &= \frac{1}{2} Z_L + \frac{\ell^2}{2} p_L(\tau + \sigma) + \frac{\ell}{2} \sum_{k \neq 0} \alpha_{L, k} e^{-ik(\tau + \sigma)}, \\
Z_R &= \frac{1}{2} Z_R + \frac{\ell^2}{2} p_R(\tau - \sigma) + \frac{\ell}{2} \sum_{k \neq 0} \alpha_{R, k} e^{-ik(\tau - \sigma)}. 
\end{align*}
\] (A.9)

Here, \( \ell \) is the string length. The boundary conditions relate the modes in \( Z_L \) and \( Z_R \) as

\[
\begin{align*}
p_L &= e^{-2\pi i\nu} p_R, \quad \alpha_{L, k} = e^{-2\pi i\nu} \alpha_{R, k}, \\
p_L &= e^{-2\pi i\mu} p_R, \quad \alpha_{L, k} e^{-ik\pi} = e^{-2\pi i\mu} e^{ik\pi} \alpha_{R, k}. 
\end{align*}
\] (A.10)

For \( \nu \neq \mu \) we get \( p_L = p_R = 0 \) and

\[
e^{2\pi i(k - \mu + \nu)} = 1 \implies k \in \mathbb{Z} + \mu - \nu. \] (A.11)

Let \( z_0 = \frac{1}{2}(z_L + z_R), \theta = \mu - \nu \) and \( \theta_n = n + \theta \). The mode expansion for \( Z \) becomes

\[
Z(\tau, \sigma) = z_0 + \ell \left[ \sum_{m=1}^{\infty} \frac{\alpha_m}{\theta_m} f_m(\tau, \sigma) + \sum_{n=0}^{\infty} \frac{\beta_n}{\theta_n} f_n(\tau, \sigma) \right],
\]

with \( f_n(\tau, \sigma) = e^{-i\pi\nu} e^{-i\theta_n \tau} \cos[\theta_n \sigma + \pi\nu] \). (A.12)

**Note:** For \( \theta = 0 \), there will be no \( \beta_0 \) term above but there will be a momentum zero-mode \( \ell^2 p_R e^{-i\pi\nu} (\tau \cos \pi \nu - i\sigma \sin \pi \nu) =: \ell^2 \mathcal{P}(\tau - ib\sigma) \) where \( b = \tan \pi \nu \). The mode expansion becomes

\[
Z(\tau, \sigma) = z_0 + \ell^2 \mathcal{P}(\tau - ib\sigma) + \ell \sum_{m=1}^{\infty} \left[ \frac{\alpha_m}{m} f_m(\tau, \sigma) - \frac{\beta_m}{m} f_{-m}(\tau, \sigma) \right], \quad \text{for } \theta = 0.
\]

We focus on the \( \theta \neq 0 \) case. The functions \( \varphi_n(\sigma) := \cos[\theta_n \sigma + \pi \nu] \) satisfy the completeness relation:

\[
\int_0^{\pi} d\sigma \left[ (\theta_m + \theta_n) + b \delta(\sigma) - b' \delta(\pi - \sigma) \right] \varphi_m(\sigma) \varphi_n(\sigma) = \pi \theta_m \delta_{mn}. \] (A.13)
These imply the following completeness relation for the functions $f_n(\tau, \sigma) = e^{-i\theta_n \tau} \varphi_n(\sigma)$:

$$\int_0^\pi d\sigma \, f_m \left[ i \partial_\tau + b \delta(\sigma) - b' \delta(\pi - \sigma) \right] f_n = \pi \theta_m \delta_{mn}, \quad (A.14)$$

and for the constant mode $z_0$:

$$\int_0^\pi d\sigma \left[ i \partial_\tau + b \delta(\sigma) - b' \delta(\pi - \sigma) \right] f_n = 0. \quad (A.15)$$

Using the above relations one can invert the formula for $Z$ to obtain

$$z_0 = \frac{1}{b - b'} \int d\sigma \left[ i \partial_\tau Z + (b \delta(\sigma) - b' \delta(\pi - \sigma)) Z \right],$$

$$\ell \alpha_m = \int \frac{d\sigma}{\pi} \left[ i \bar{f}_m \partial_\tau + \left( \theta_m + b \delta(\sigma) - b' \delta(\pi - \sigma) \right) \bar{f}_m Z \right],$$

$$\ell \beta_n^\dagger = \int \frac{d\sigma}{\pi} \left[ i f_{-n} \partial_\tau + \left( \theta_{-m} + b \delta(\sigma) - b' \delta(\pi - \sigma) \right) \bar{f}_{-n} Z \right]. \quad (A.16)$$

To quantise the system we impose the following equal-time commutation relations:

$$[P(\tau, \sigma), Z(\tau, \sigma')] = -i\hbar \delta(\sigma - \sigma'), \quad [\bar{P}(\tau, \sigma), \bar{Z}(\tau, \sigma')] = -i\hbar \delta(\sigma - \sigma'). \quad (A.17)$$

The conjugate momentum $P(\tau, \sigma)$ is given by

$$P(\tau, \sigma) = \frac{\partial \mathcal{L}}{\partial (\partial_\tau Z(\tau, \sigma))} = \frac{1}{2\pi \alpha'} \left[ \partial_\tau Z(\tau, \sigma) - \frac{ib'}{2} Z(\tau, \pi) + \frac{ib}{2} \bar{Z}(\tau, 0) \right]. \quad (A.18)$$

In terms of $Z(\tau, \sigma)$ and $P(\tau, \sigma)$ the zero mode and oscillators are given by

$$z_0 = \frac{1}{b - b'} \int d\sigma \left[ 2\pi i \alpha' \bar{P} + \left( \frac{b}{2} \delta(\sigma) - \frac{b'}{2} \delta(\pi - \sigma) \right) Z \right],$$

$$\ell \alpha_m = \int \frac{d\sigma}{\pi} \left[ 2\pi i \alpha' \bar{f}_m \bar{P} + \left( \theta_m + \frac{b}{2} \delta(\sigma) - \frac{b'}{2} \delta(\pi - \sigma) \right) \bar{f}_m Z \right],$$

$$\ell \beta_n^\dagger = \int \frac{d\sigma}{\pi} \left[ 2\pi i \alpha' f_{-n} \bar{P} + \left( \theta_{-m} + \frac{b}{2} \delta(\sigma) - \frac{b'}{2} \delta(\pi - \sigma) \right) f_{-n} Z \right]. \quad (A.19)$$

Setting $2\alpha' = \ell^2$ and using the above completeness relations, we get

$$[z_0, z_n] = \frac{\ell^2}{b - b'}, \quad [\alpha_m, \alpha_m^\dagger] = (m + \theta) \delta_{mn}, \quad [\beta_n, \beta_n^\dagger] = (n - \theta) \delta_{nn'}. \quad (A.20)$$

**Worldsheet fermions**

In 1+1 dimensions, right(left)-handed spinors are left(right)-moving on-shell and supersymmetry relates left-movers to left-movers and right-movers to right-movers:

$$\delta Z = -i\epsilon^+ \Psi^- + i\epsilon^- \Psi^+, \quad \delta \Psi^+ = -2\partial_- Z \epsilon^- + \mathcal{F} \epsilon^-, \quad \delta \Psi^- = 2\partial_+ Z \epsilon^+ + \mathcal{F} \epsilon^+, \quad (A.21)$$

where we have introduced the complex combinations $\Psi^\pm = \frac{1}{\sqrt{2}}(\psi^{1\pm} + i\psi^{2\pm})$ and $\mathcal{F} = \frac{1}{\sqrt{2}}(F^1 + iF^2)$. The presence of a boundary reduces the supersymmetry by half by imposing a relation
between the parameters: \( \epsilon^+ = \pm \epsilon^- \). One can always impose \( \epsilon^+ = -\epsilon^- \) at one end, say \( \sigma = 0 \). On the other end, two choices are possible and they correspond to the R and NS sectors:

\[
\sigma = \pi : \quad \begin{cases} \\
\epsilon^+ = -\epsilon^- & \text{Ramond} , \\
\epsilon^+ = \epsilon^- & \text{Neveu-Schwarz} . 
\end{cases} \tag{A.22}
\]

It is evident that rigid supersymmetry is present only in the R sector and that it has only one parameter \( \epsilon = \epsilon^+ = -\epsilon^- \). The boundary condition on \( \Psi^\pm \) corresponding to \( \partial_{++} Z = e^{-2\pi i\nu} \partial_{--} Z \) at \( \sigma = 0 \) is given by

\[
\Psi^+ = e^{2\pi i\nu} \Psi^- \quad \text{at} \quad \sigma = 0 . \tag{A.23}
\]

Similarly, the boundary condition at \( \sigma = \pi \) is

\[
\text{At} \quad \sigma = \pi : \quad \begin{cases} \\
\Psi^+ = e^{2\pi i\mu} \Psi^- & \text{R sector} , \\
\Psi^+ = -e^{2\pi i\mu} \Psi^- & \text{NS sector} . 
\end{cases} \tag{A.24}
\]

In order to write down the mode expansions, we combine \( \Psi^+(\tau - \sigma) \) and \( \Psi^-(\tau + \sigma) \) on \( 0 \leq \sigma \leq \pi \) into one field \( \Psi \) on the double interval \( -\pi \leq \sigma \leq \pi \) such that

\[
\Psi(\tau + \sigma) = \begin{cases} \\
\Psi^+(\tau + \sigma) & -\pi \leq \sigma \leq 0 , \\
e^{-2\pi i\nu} \Psi^-(\tau + \sigma) & 0 \leq \sigma \leq \pi .
\end{cases} \tag{A.25}
\]

Treating \( -\pi \leq \sigma \leq \pi \) as an angular variable we see that \( \Psi \) is continuous at \( \sigma = 0 \) by virtue of (A.23) and twisted-periodic across \( \sigma = \pi \) due to (A.24): \( \Psi(\pi) = \pm e^{2\pi i(\nu-\mu)} \Psi(-\pi) \). The mode expansion for \( \Psi(\tau + \sigma) \) in the R sector is

\[
\Psi_R(\tau + \sigma) = \frac{\ell}{2} \left[ \sum_{m=1}^{\infty} a_m e^{-i\theta_m(\tau+\sigma)} + \sum_{n=0}^{\infty} b_n^* e^{-i\theta_{-n}(\tau+\sigma)} \right] , \tag{A.26}
\]

and in the NS sector is

\[
\Psi_{NS}(\tau + \sigma) = \frac{\ell}{2} \left[ \sum_{r=1}^{\infty} c_r e^{-i\epsilon_r(\tau+\sigma)} + \sum_{s=0}^{\infty} d_s^* e^{-i\epsilon_{-s}(\tau+\sigma)} \right] , \tag{A.27}
\]

where \( \epsilon = \theta + \frac{1}{2} = \mu - \nu + \frac{1}{2} \) and \( \epsilon_n = \epsilon + n \). The action for the doubled \( \Psi(\tau + \sigma) \) is

\[
S[\Psi] = \frac{2i}{\pi \alpha'} \int d\tau \int_{-\pi}^{\pi} d\sigma \bar{\Psi} \partial_{\tau} \partial_{\sigma} \Psi . \tag{A.28}
\]

The boundary term for the fermions in (A.2) measures the jump in \( \bar{\Psi}^+ \Psi^+ + \bar{\Psi}^- \Psi^- \) between two boundary components \( (\sigma = 0, \pi) \) of the worldsheet. Since \( \bar{\Psi} \Psi \) is periodic on the double interval, such a boundary term is absent in the above action. The conjugate momentum is then

\[
\Pi(\tau + \sigma) = \frac{\partial L}{\partial \dot{\Psi}} = -\frac{2i}{\pi \alpha'} \bar{\Psi}(\tau + \sigma) . \tag{A.29}
\]

The correct equal-time anticommutation relation follows from Dirac's constrained Hamiltonian formalism:

\[
\{ \Pi(\tau + \sigma) , \Psi(\tau + \sigma') \} = -\frac{i}{2} \delta(\sigma - \sigma') \implies \{ \bar{\Psi}(\tau + \sigma) , \Psi(\tau + \sigma') \} = \frac{\pi \alpha'}{4} \delta(\sigma - \sigma') . \tag{A.30}
\]
Using the completeness relations and \(2\alpha' = l^2\), we get
\[
\{a_{m}, a_{m'}^\dagger\} = \delta_{mm'}, \quad \{b_{n}, b_{n'}^\dagger\} = \delta_{nn'}, \quad \{c_{r}, c_{r'}^\dagger\} = \delta_{rr'}, \quad \{d_{s}, d_{s'}^\dagger\} = \delta_{ss'}.
\] (A.31)

The expression for \(L_0\) in the R and NS sectors is given by
\[
L_0^{(R)} - E_0 = \sum_{m=1}^{\infty} \alpha_m a_m + (m + \theta) a_m^\dagger a_m + \sum_{n=0}^{\infty} \beta_n b_n + (n - \theta) b_n^\dagger b_n,
\]
\[
L_0^{(NS)} - E_0 = \sum_{m=1}^{\infty} \alpha_m a_m + \sum_{r=1}^{\infty} (r + \epsilon) c_r^\dagger c_r + \sum_{n=0}^{\infty} \beta_n b_n + \sum_{s=0}^{\infty} (s - \epsilon) d_s^\dagger d_s.
\] (A.32)

Recall that \(\epsilon = \mu - \nu + \frac{1}{2}\). Since \(|\mu|, |\nu| < \frac{1}{2}\), we have \(-\frac{1}{2} < \epsilon < \frac{2}{3}\). The first few states of the spectrum in the NS sector for different ranges of \(\epsilon\) are as in Table 4. Observe that

| Table 4: Spectral flow in the NS sector |
|----------------------------------------|
| (a) \(-\frac{1}{2} < \epsilon < 0\) | (b) \(0 < \epsilon < \frac{1}{2}\) | (c) \(\frac{1}{2} < \epsilon < 1\) | (d) \(1 < \epsilon < \frac{3}{2}\) |
| \(E - E_0\) | NS | \(E - E_0\) | NS | \(E - E_0\) | NS | \(E - E_0\) | NS |
| \(-\epsilon\) | \(d_0^\dagger\) | \(\epsilon\) | \(d_0\) | \(-\epsilon + 1\) | \(d_1^\dagger\) | \(-\epsilon + 1\) | \(d_1\) |
| \(\epsilon + 1\) | \(c_1^\dagger\) | \(-\epsilon + 1\) | \(d_1^\dagger\) | \(\epsilon\) | \(d_0\) | \(-\epsilon + 2\) | \(d_2^\dagger\) |
| \(-\epsilon + 1\) | \(d_1^\dagger\) | \(\epsilon + 1\) | \(c_1^\dagger\) | \(-\epsilon + 2\) | \(d_2^\dagger\) | \(\epsilon\) | \(d_0\) |
| \(\epsilon + 2\) | \(c_2^\dagger\) | \(-\epsilon + 2\) | \(d_2^\dagger\) | \(\epsilon + 1\) | \(c_1^\dagger\) | \(-\epsilon + 3\) | \(d_3^\dagger\) |

as we dial up \(\epsilon\), negative energy states from the Dirac sea cross the zero-point energy and become positive energy states. The first excited state in the NS sector has energy \(|\epsilon|\) or \(|1 - \epsilon|\) depending on whether \(-\frac{1}{2} < \epsilon < \frac{1}{2}\) or \(\frac{1}{2} < \epsilon < \frac{3}{2}\). A similar analysis can be made for the R sector and the results are in Table 5.

| Table 5: Spectral flow in the R sector |
|----------------------------------------|
| (a) \(-1 < \theta < -\frac{1}{2}\) | (b) \(-\frac{1}{2} < \theta < 0\) | (c) \(0 < \theta < \frac{1}{2}\) | (d) \(\frac{1}{2} < \theta < 1\) |
| \(E\) | R | \(E\) | R | \(E\) | R | \(E\) | R |
| \(\theta + 1\) | \(a_1^\dagger\) | \(-\theta\) | \(b_1^\dagger\) | \(\theta\) | \(b_0\) | \(-\theta + 1\) | \(b_1^\dagger\) |
| \(-\theta\) | \(b_0^\dagger\) | \(\theta + 1\) | \(a_0^\dagger\) | \(-\theta + 1\) | \(b_1^\dagger\) | \(\theta\) | \(b_0\) |
| \(\theta + 2\) | \(a_2^\dagger\) | \(-\theta + 1\) | \(b_1^\dagger\) | \(\theta + 1\) | \(a_1^\dagger\) | \(-\theta + 2\) | \(b_2^\dagger\) |
| \(-\theta + 1\) | \(b_1^\dagger\) | \(\theta + 2\) | \(a_2^\dagger\) | \(-\theta + 2\) | \(b_2^\dagger\) | \(\theta + 1\) | \(a_1^\dagger\) |

The zero-point energies for a complex boson and a complex fermion with moding \(Z + v + \frac{1}{2}\), \(|v| \leq \frac{1}{2}\) are
\[
\frac{1}{24} - \frac{v^2}{2}, \quad \frac{1}{24} + \frac{v^2}{2}
\]
respectively. (A.33)
The complex boson $Z$ has moding $Z + \theta$ and so do the fermions in the R sector. This is a consequence of rigid supersymmetry on the worldsheet in the R sector. Thus, the zero-point energy in the R sector vanishes. The fermions in the NS sector have moding $Z + \epsilon$ and the total zero-point energy in the NS sector is given by

$$E_0 = \frac{1}{24} - \frac{(|\theta| - \frac{1}{2})^2}{2} - \frac{1}{24} + \frac{(|\epsilon - \frac{1}{2}| - \frac{1}{2})^2}{2},$$

$$= \frac{1}{8} - \frac{1}{2}||\theta| - \frac{1}{2}|.$$  \hspace{1cm} (A.34)

Since $[z, \bar{z}] = \ell^2 b - b'$ and $[z, L_0] = 0$, we can build an infinite tower of states (given that the $Z$ direction is non-compact) from each $L_0$ eigenstate. Thus, each state in the spectrum is infinitely degenerate and furnishes a representation of the Heisenberg algebra of $z$ and $\bar{z}$.

### A.1 Boundary condition changing operators

We map the strip $-\infty < \tau < \infty$, $0 \leq \sigma \leq \pi$ to the upper half plane $H = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$ by first Wick-rotating $\tau = -it$ and using the map $z = \exp(t + i\sigma)$. In particular, the boundary at $\sigma = 0, \pi$ is mapped to $z = \mathcal{Z} > 0$ and $< 0$ respectively. Consider a complex boson $Z$ with $TT'$ boundary conditions. Using $\partial_{++} = iz\partial$ and $\partial_{--} = i\bar{z}\partial$, we can write the corresponding boundary conditions on $H$:

$$\partial Z = e^{-2\pi i\nu} \bar{Z} \quad \text{for} \quad z = \mathcal{Z} > 0,$$

$$\partial Z = e^{2\pi i\mu} \bar{Z} \quad \text{for} \quad z = e^{2\pi i} \mathcal{Z} < 0.$$  \hspace{1cm} (A.35)

with $-\frac{1}{2} \leq \mu, \nu \leq \frac{1}{2}$. We define bulk chiral currents $J(z) = i\partial Z$, $\bar{J}(\mathcal{Z}) = i\bar{\partial}Z$ using the modes in (A.12):

$$J(z) = i\partial Z(z) = -\frac{i\ell}{2} \sum_{n=1}^{\infty} \alpha_n e^{-2\pi i\nu} z^{-1-\theta_n} - \frac{i\ell}{2} \sum_{m=0}^{\infty} \beta_m^{\dagger} e^{-2\pi i\nu} z^{-1-\theta_m},$$

$$\bar{J}(\mathcal{Z}) = i\bar{\partial}Z(\mathcal{Z}) = -\frac{i\ell}{2} \sum_{n=1}^{\infty} \alpha_n \mathcal{Z}^{-1-\theta_n} - \frac{i\ell}{2} \sum_{m=0}^{\infty} \beta_m^{\dagger} \mathcal{Z}^{-1-\theta_m},$$  \hspace{1cm} (A.36)

where $\theta = \mu - \nu$ and $\theta_n = \theta + n$. Since the modes are not integers, we need to specify a branch cut: we choose it to be at $-\infty < z \leq 0$. We also define the hermitian conjugate currents

$$J^\ast(z) := z^{-1} \bar{J}(\mathcal{Z}^{-1}), \quad \bar{J}^\ast(\mathcal{Z}) := \mathcal{Z}^{-1} \bar{J}(z^{-1}).$$  \hspace{1cm} (A.37)

The gluing conditions for the currents are then:

$$J(z) = e^{-2\pi i\nu} \bar{J}(\mathcal{Z}) \quad \text{for} \quad z = \mathcal{Z} > 0,$$

$$J(z) = e^{2\pi i\mu} \bar{J}(\mathcal{Z}) \quad \text{for} \quad z = e^{2\pi i} \mathcal{Z} < 0,$$

$$J^\ast(z) = e^{2\pi i\nu} \bar{J}^\ast(\mathcal{Z}) \quad \text{for} \quad z = \mathcal{Z} > 0,$$

$$J^\ast(z) = e^{-2\pi i\mu} \bar{J}^\ast(\mathcal{Z}) \quad \text{for} \quad z = e^{2\pi i} \mathcal{Z} < 0.$$  \hspace{1cm} (A.38)
The gluing conditions allow us to extend the domain of definition of the currents to the full z-plane by employing the doubling trick:

\[
J(z) = -\frac{i\ell}{2} \sum_{n \geq 1} \alpha_n e^{-2\pi i\nu z^{-1} - \theta_n} - \frac{i\ell}{2} \sum_{m \geq 0} \beta^*_m e^{-2\pi i\nu z^{-1} - \theta_{-m}},
\]

\[
J^*(z) = \frac{i\ell}{2} \sum_{n \geq 1} \alpha^*_n e^{2\pi i\nu z^{-1} + \theta_n} + \frac{i\ell}{2} \sum_{m \geq 0} \beta_m e^{2\pi i\nu z^{-1} + \theta_{-m}},
\]

(A.39)

The doubled stress tensor \(T(z)\) is given by

\[
T(z) = \lim_{w \to z} \frac{4}{\ell^2} \left( J(w) J^*(z) - \frac{\ell^2}{4(w-z)^2} \right).
\]

(A.40)

The change in boundary conditions from \(\mu\) to \(\nu\) at \(z = 0\) and vice-versa at \(z = \infty\) can be interpreted as there being present boundary condition changing operators (BCC) \(\sigma(0)\) and \(\sigma^+(\infty)\) where \(\sigma^+\) is the operator conjugate to \(\sigma\). The conformal dimension of \(\sigma\) is obtained from the one-point function of \(T(z)\). Following the treatment in [DFMS, FGRS] we first define \(J = J_+ + J_-\) where \(J_+\) contains only annihilation operators. We have, for \(0 < \theta < 1\),

\[
J_+(w) = -\frac{i\ell}{2} \sum_{n \geq 1} \alpha_n e^{-2\pi i\nu w^{-1} - \theta_n} - \frac{i\ell}{2} \beta^*_0 e^{-2\pi i\nu w^{-1} - 1},
\]

(A.41)

and for \(-1 < \theta < 0\) the last term is absent. Next, we compute

\[
J(w) J^*(z) - \frac{\ell^2}{4(w-z)^2} = J_-(w) J^*(z) + J^*(z) J_+(w) + [J_+(w), J^*(z)] - \frac{\ell^2}{4(w-z)^2},
\]

(A.42)

\[
= J_-(w) J^*(z) + J^*(z) J_+(w) + \frac{\ell^2}{4} \partial_z \left( \frac{z^\nu}{w} \right)^{(*)} \frac{1}{w-z},
\]

where the exponent \((*)\) is \(\theta\) for \(0 < \theta < 1\) and \(1 + \theta\) for \(-1 < \theta < 0\). This finally gives

\[
T(z) = \frac{|\theta| (1 - |\theta|)}{2z^2} + \frac{4}{\ell^2} J_-^*(z) J_+(z) + \frac{4}{\ell^2} J_+^*(z) J_-(z),
\]

(A.43)

which gives the one-point function

\[
\langle T(z) \rangle = \frac{|\theta| (1 - |\theta|)}{2z^2}.
\]

(A.44)

This can be interpreted as there being two BCC operators \(\sigma, \sigma^+\) inserted resp. at \(z = 0\) and \(z = \infty\) with conformal weight \(h_\sigma = \frac{|\theta|(1-|\theta|)}{2}\). Their two-point function is

\[
\langle \sigma(x_1) \sigma^+(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2h_\sigma}}.
\]

(A.45)

We next describe BCC operators for the worldsheet fermions \(\Psi^\pm\). Since the fermions have conformal dimension \(\frac{1}{2}\), we should include a Jacobian factor \(z^{-1/2}\) while mapping them from the strip to the upper half-plane. We employ the doubling trick to directly write the R and NS fermions on the full z-plane in the R and NS sectors:

\[
\Psi_R(z) = \frac{\ell}{2} \sum_{n \geq 1} a_n z^{- \epsilon_n} + \frac{\ell}{2} \sum_{m \geq 0} b^*_m z^{-\epsilon_m},
\]

\[
\Psi_{NS}(z) = \frac{\ell}{2} \sum_{r \geq 1} c_r z^{-\theta_r - 1} + \frac{\ell}{2} \sum_{s \geq 0} d^*_s z^{-\theta_s - 1}.
\]

(A.46)
with $\epsilon = \theta + \frac{1}{2}$. In order to describe the BCC operators, we first bosonise $\Psi(z)$ by introducing an antihermitian scalar $H(z)$:

$$\Psi(z) = e^{H(z)}, \quad \Psi^*(z) = e^{-H(z)} \quad \text{with} \quad \langle H(w)H(z) \rangle = \log(w - z) .$$  \hspace{1cm} (A.47)

The normal ordering symbol $:\ :$ is omitted in the above definition for the sake of brevity. Now, consider the OPE of the operator $e^{-\theta H(x)}$ with $\Psi$:

$$\Psi(z)e^{-\theta H(0)} \sim z^{\theta} e^{(1-\theta)H(0)} + \cdots .$$  \hspace{1cm} (A.48)

First, we notice that as $z \to e^{2\pi i} z$, $\Psi$ picks up a phase $e^{-2\pi i \theta}$, which matches with the monodromy of $\Psi_{NS}$. Further, we observe that the right-hand side of the first OPE is regular as $z \to 0$ for $\theta < 0$. This requires that $\Psi_{NS}(z)$ annihilate the state $e^{-\theta H(0)}|0\rangle$ in the limit $z \to 0$ where the state $|0\rangle$ is the $\text{SL}(2,\mathbb{R})$-invariant vacuum. From Table 4, we see that the NS vacuum $|\text{ns}\rangle$ has the same properties for $-\frac{1}{2} < \theta < 0$. Thus, we can identify the state $e^{-\theta H(0)}|0\rangle$ with the NS vacuum for this range of $\theta$:

$$e^{-\theta H(0)}|0\rangle = |\text{NS}\rangle \quad \text{for} \quad -\frac{1}{2} < \theta < 0 .$$  \hspace{1cm} (A.49)

For the range $-1 < \theta < -\frac{1}{2}$, we see from Table 4 that $\Psi_{NS}$ annihilates the excited state $d^0_0|\text{NS}\rangle$. Thus, for this range of $\theta$ one has to identify $e^{-\theta H(0)}|0\rangle$ with the first excited state:

$$e^{-\theta H(0)}|0\rangle = d^0_0|\text{NS}\rangle \quad \text{for} \quad -1 < \theta < -\frac{1}{2} .$$  \hspace{1cm} (A.50)

The NS vacuum is obtained by applying $d_0$ to the above and $d_0$ is contained in the Hermitian conjugate field $\Psi^*(z)$, defined as

$$\Psi^*(z) := z^{-1} \overline{\Psi(z^{-1})} .$$

This gives

$$\Psi^*_{NS}(z) := \frac{\ell}{2} \sum_{r \geq 1} c_r^\dagger z^{\theta r} + \frac{\ell}{2} \sum_{s \geq 0} d_s z^{\theta - s} .$$  \hspace{1cm} (A.51)

Thus, the operator corresponding to the NS vacuum for this range of $\theta$ is obtained by fusing $\Psi^*$ with $e^{-\theta H(x)}$:

$$\Psi^*(z)e^{-\theta H(0)} \sim z^{\theta} e^{-(1+\theta)H(0)} + \cdots .$$  \hspace{1cm} (A.52)

Thus the NS vacuum is to be identified with the operator on the right hand side:

$$e^{-(1+\theta)H(0)}|0\rangle = |\text{NS}\rangle \quad \text{for} \quad -1 < \theta < -\frac{1}{2} .$$  \hspace{1cm} (A.53)

Similarly, for $\theta > 0$ there are two cases $0 < \theta < \frac{1}{2}$ and $\frac{1}{2} < \theta < 1$ for which the operators corresponding to $|\text{NS}\rangle$ are $e^{-\theta H(x)}$ and $e^{(1-\theta)H(x)}$ respectively. The same analysis can be made for the R sector as well. We summarise the results in Table 6. We designate the operator corresponding to the NS and R vacua as $s^{\text{NS}}(x)$ and $s^{\text{R}}(x)$ respectively. These shall be the BCC operators for the respective sectors. Also observe that the operators $s^{\text{NS},\text{R}}$ always have the
Vacuum states are degenerate, so either limit gives the same spectrum. BCC operators are for either of the two limits limiting cases of NN, the vacuum corresponds to \( \theta^\pm H(x) \), the first two excited states corresponding to \( e^{\pm H} \) and \( e^{\mp H/2} \). The two states are degenerate, so either limit gives the same spectrum.

For DN boundary conditions, we have \( \mu = 0 \) and \( \nu = \frac{1}{2} \) giving \( \theta = -\frac{1}{2} \) and \( \epsilon = 0 \). In the NS sector, the ground state and the first excited state are degenerate, corresponding to the operators \( e^{\pm H/2} \). In the R sector, the ground state is the SL(2,R)-invariant vacuum and the first two excited states corresponding to \( e^{\pm H} \) are degenerate.

For ND boundary conditions, we have \( \mu = \frac{1}{2} \) and \( \nu = 0 \) giving \( \theta = \frac{1}{2} \) and \( \epsilon = 1 \). The discussion on the NS and R sector states is identical to the DN case.

For the TD case, we have \( \mu = \frac{1}{2} \) and \( \nu = v \) giving \( \theta = \frac{1}{2} - v \) and \( \epsilon = 1 - v \). The range of \( \theta \) is \( 0 \leq \theta \leq 1 \), giving the ground BCC operator \( e^{(1-\epsilon)H(x)} \) in the R sector and \( e^{-\theta H(x)} \) in the NS sector for \( 0 \leq \theta \leq \frac{1}{2} \) and \( \frac{1}{2} \leq \theta \leq 1 \) respectively.

For the DT case, we have \( \mu = v \) and \( \nu = \frac{1}{2} \) giving \( \theta = v - \frac{1}{2} \) and \( \epsilon = v \). The ground BCC operators are \( e^{-\epsilon H(x)} \) in the R sector and \( e^{-(1+\theta)H(x)} \) in the NS sector for \( -1 \leq \theta \leq -\frac{1}{2} \) and \( -\frac{1}{2} \leq \theta \leq 0 \) respectively.

| Ground BCC operator | \(-1 < \theta < -\frac{1}{2}\) | \(-\frac{1}{2} < \theta < 0\) | \(0 < \theta < \frac{1}{2}\) | \(\frac{1}{2} < \theta < 1\) |
|---------------------|-----------------|-----------------|-----------------|-----------------|
| \(s^{\text{NS}}(x)\) | \(e^{-\theta H(x)}\) | \(e^{-(1+\theta)H(x)}\) | \(e^{-\theta H(x)}\) | \(e^{(1-\theta)H(x)}\) |
| \(s^{\text{R}}(x)\) | \(e^{-\epsilon H(x)}\) | \(e^{-\epsilon H(x)}\) | \(e^{(1-\epsilon)H(x)}\) | \(e^{(1-\epsilon)H(x)}\) |

| Excited BCC operator | \(-1 < \theta < -\frac{1}{2}\) | \(-\frac{1}{2} < \theta < 0\) | \(0 < \theta < \frac{1}{2}\) | \(\frac{1}{2} < \theta < 1\) |
|---------------------|-----------------|-----------------|-----------------|-----------------|
| \(t^{\text{NS}}(x)\) | \(e^{-\theta H(x)}\) | \(e^{-(1+\theta)H(x)}\) | \(e^{(1-\theta)H(x)}\) | \(e^{-\theta H(x)}\) |
| \(\bar{t}^{\text{NS}}(x)\) | \(e^{-(2+\theta)H(x)}\) | \(e^{(1-\theta)H(x)}\) | \(e^{(1-\theta)H(x)}\) | \(e^{(2-\theta)H(x)}\) |
| \(t^{\text{R}}(x)\) | \(e^{-(1+\epsilon)H(x)}\) | \(e^{(1-\epsilon)H(x)}\) | \(e^{-\epsilon H(x)}\) | \(e^{(2-\epsilon)H(x)}\) |
| \(\bar{t}^{\text{R}}(x)\) | \(e^{(1-\epsilon)H(x)}\) | \(e^{-\epsilon H(x)}\) | \(e^{(2-\epsilon)H(x)}\) | \(e^{-\epsilon H(x)}\) |
A.2 The covariant lattice

Consider the following linear combinations of the holomorphic (left-moving) part of the world-sheet fermions:

\[ \Psi^{\pm e_a} = \frac{\psi^{2a-1} \pm i\psi^{2a}}{\sqrt{2}}, \quad a \in 4, \quad \Psi^{\pm e_5} = \frac{\psi^9 \pm \psi^0}{\sqrt{2}}. \]  

(A.54)

Here \( e_m, m = 1, \ldots, 5, \) are unit vectors \((e_m)_i = \delta_{im}\) of the \( D_5 \) weight lattice. Along with their negatives, they form the weights of the vector representation of \( \text{so}(1,9) \). Under complex conjugation the fermions behave as follows:

\[ (\Psi^{+ e_a})^* = \Psi^{- e_a}, \quad (\Psi^{\pm e_5})^* = \Psi^{\mp e_5}. \]  

(A.55)

In order to bosonize these with the correct properties under complex conjugation, we introduce antihermitian scalars \( H_a(z) \) and a hermitian scalar \( H_5(z) \) which satisfy

\[ \langle H_m(z)H_n(w) \rangle = \delta_{mn} \log(z - w), \quad m, n = 1, \ldots, 5. \]  

The bosonised versions of the fermions are

\[ \Psi^{\pm e_m}(z) := e^{\pm H_m(z)} c_{\pm e_m}, \quad m = 1, \ldots, 5. \]  

(A.56)

The object \( c_{\pm e_m} \) is a cocycle operator which is defined \([KLLSW]\) in terms of the fermion number operators \( N_m \) as

\[ c_{\pm e_m} := (-)^{N_1 + \cdots + N_{m-1}}. \]  

(A.57)

These ensure that the fermions \( \Psi^{\pm e_m}, \Psi^{\pm e_n} \) for \( m \neq n \) anticommute after bosonisation. In a broader context, these fermions are used to construct the currents whose modes satisfy the commutation relations of the affine Kač-Moody algebra \( \hat{D}_5 \). The commutation relations between the generators corresponding to the roots of the Lie algebra involve certain 2-cocycles. In order to obtain these 2-cocycles correctly via bosonised vertex operators, we need to include the above cocycle operators in the definition of the vertex operators. These 2-cocycles were first treated by Bardakci and Halpern \([BH]\) in the physics literature and by Frenkel and Kač \([FK]\), and Segal \([S]\) in the mathematics literature.

In terms of the bosons \( H_m \), the number operators are given by \( N_m := (\partial H_m)_0 \) where \((\partial H_m)_0\) are the zero modes of \( \partial H_m \). The bosons \( H_m \) have the following mode expansion \([PR]\):

\[ H_m(z) = h_m + N_m \log z + \sum_{k \neq 0} \frac{\alpha_{m,k}}{k} z^{-k}. \]  

(A.58)

The Hermitian conjugate field \( H^*_m(z) \) is defined as follows:

\[ H^*_m(z) := \overline{H_m(\bar{z}^{-1})}. \]  

(A.59)

Since \( H_a \) are antihermitian and \( H_5 \) is hermitian, the modes satisfy

\[ (h_a)^\dagger = -h_a, \quad (N_a)^\dagger = N_a, \quad (\alpha_{a,k})^\dagger = \alpha_{a,k}, \quad a \in 4, \quad (h_5)^\dagger = h_5, \quad (N_5)^\dagger = -N_5, \quad (\alpha_{5,k})^\dagger = -\alpha_{5,k}. \]  

(A.60)
These properties will be required in the discussion on the cocycle operators. Next, we discuss superconformal ghosts. The contribution due to these have to be included appropriately in each vertex operator to ensure that operator products are mutually local. One bosonises the superconformal ghosts $\beta, \gamma$ using a hermitian scalar field $\varphi$ with $\langle \varphi(z)\varphi(w) \rangle = -\log(z-w)$ and two fermions $\xi(z)$ and $\eta(z)$ with $\langle \xi(z)\xi(w) \rangle = (z-w)^{-1}$:

$$
\beta(z) := e^{-\varphi(z)}\partial \xi(z) \ , \quad \gamma(z) := \eta(z)e^{\varphi(z)} .
$$

(A.61)

The fermions $\xi$ and $\eta$ are further bosonised as

$$
\xi(z) = e^{\zeta(z)} \ , \quad \eta(z) = e^{-\zeta(z)} .
$$

(A.62)

with $\zeta(z)$ a hermitian scalar with $\langle \zeta(z)\zeta(w) \rangle = \log(z-w)$. The superghost picture number operator $N_6$ is given by the zero mode of $\partial \zeta - \partial \varphi$. Under conjugation, it satisfies

$$
(N_6)^\dagger = -N_6 - Q = -N_6 - 2 ,
$$

(A.63)

where $Q = 2$ is the background charge of the $\beta\gamma$ CFT. The picture charge of $e^{q_\varphi(z)}$ is $q$ and its conformal dimension is $-\frac{1}{2}q(g+Q)$. The operator conjugate to $e^{q_\varphi}$ is $e^{-(g+Q)\varphi}$ and it also has conformal dimension $-\frac{1}{2}q(g+Q)$.

In the canonical ghost picture, vertex operators in the NS sector acquire a factor of $e^{-\varphi}$ and those in the R sector a factor of $e^{-\varphi/2}$. The integer and half-integer exponents are correlated with the integer and half-integer modes for the NS and R fermions on the doubled plane. The integer and half-integer ghost numbers can be interpreted as belonging to a $D_1$ weight lattice which can then be combined with the spacetime $D_5$ weights to get a covariant lattice $\Gamma_{5,1}$. The lattice $\Gamma_{5,1}$ is Lorentzian since $\langle \varphi(z)\varphi(w) \rangle = -\log(z-w)$ as opposed to $\langle H_m(z)H_n(w) \rangle = \delta_{mn} \log(z-w)$. Writing $H_6 := -\varphi$, we have $\langle H_\mu(z)H_\nu(w) \rangle = \eta_{\mu\nu} \log(z-w)$ with $\eta_{66} = -1, \eta_{6m} = 0$ and $\eta_{mn} = \delta_{mn}$. A general vertex operator in the (worldsheet) fermionic sector is then given by

$$
V_\lambda(z) = e^{\lambda \cdot H(z)}c_\lambda .
$$

(A.64)

where $\lambda$ is a weight in the covariant lattice $\Gamma_{5,1}$, $c_\lambda$ is the cocycle operator corresponding to $\lambda$ and the dot product $\lambda \cdot H$ is with respect to the Lorentzian metric $\eta_{\mu\nu}$. We give a formula for $c_\lambda$ below. The $\Gamma_{5,1}$ weights $\lambda$ with $\lambda_6 = -1, -\frac{1}{2}, -\frac{3}{2}$ directly correspond to physical states whose mass-squared is given by

$$
\alpha' m^2 = \frac{1}{2} \lambda^2 + \lambda \cdot e_\varphi - 1 ,
$$

(A.65)

where $e_\varphi$ is the unit vector $(0,0,0,0,0;1)$. The term $\lambda \cdot e_\varphi$ arises due to the background charge of the $\beta\gamma$ CFT. The other $\Gamma_{5,1}$ weights do not correspond directly to physical states but linear combinations of the corresponding vertex operators correspond to physical operators with picture charge different from the canonical ones.
A.3 The D1-D5_A-D5_{\overline{A}} system

Consider the D1-D5_A-D5_{\overline{A}} system. The spacetime Lorentz symmetry SO(1, 9) is broken down to SO(4) × SO(4)^′ × SO(1, 1) with spacetime now being the 1 + 1 dimensional intersection R^{1,1}_1 of the D-branes. The worldsheet fermion contribution to the total vertex operator can thus be described by (A.64) where \( \lambda \) is now a weight in the covariant lattice \( D_2 \oplus D_2 \oplus \Gamma_{1,1} \).

In the presence of a constant \( B \)-field, the weights \( \lambda \) have to be generalised to include entries which are neither integral nor half-integral. The precise weights can be obtained by following the procedure outlined in the previous sections.

An (unintegrated) open string vertex operator with only fermionic oscillators then has the form

\[
V_\lambda(k, z) = \omega(\lambda) \, c(z) \, B(z) \, e^{\lambda \cdot H(z)} \, e^{2ik \cdot X(z)} c_\lambda ,
\]

where \( \omega(\lambda) \) is an arbitrary c-number phase, \( c(z) \) is the coordinate ghost, \( B(z) \) is the product of the appropriate BCC operators for the worldsheet bosons and \( k \) is the 1 + 1 dimensional spacetime momentum. We have suppressed Chan-Paton factors. The mass formula for a state with weight \( \lambda \) becomes

\[
\alpha' m^2(\lambda) = -\alpha' k^2 = \frac{1}{2} \lambda^2 + \lambda \cdot e_\varphi - 1 + \sum_{\sigma \in B} h_\sigma .
\]

The notation \( \sum_{\sigma \in B} \) indicates the summation of the conformal dimensions \( h_\sigma \) of bosonic BCC operators \( \sigma(z) \) present in \( \Sigma(z) \) above.

The weights of the vertex operators for the case with \( A = (12), \overline{A} = (34) \) are recorded in Tables 2 and 3 in the main text. Let us calculate the masses of some of the states using the above machinery. For the D1-D5_{(12)} states \((\phi^1, \phi^2)\) we have BCC operators for the worldsheet bosons \( Z^1 \) and \( Z^2 \) with conformal dimensions \( \frac{1}{2} (\frac{1}{4} - v_1^2) \) and \( \frac{1}{2} (\frac{1}{4} - v_2^2) \) respectively. We thus get

\[
\alpha' m^2(\phi^1) = \frac{1}{2} \left[ (v_1 + \frac{1}{2})^2 + (v_2 - \frac{1}{2})^2 - 1 \right] + 1 - \frac{1}{2} \left[ (\frac{1}{4} - v_1^2) + (\frac{1}{4} - v_2^2) \right]
\]

\[
= \frac{1}{2} (v_1 - v_2).
\]

\[
\alpha' m^2(\phi^2) = \frac{1}{2} \left[ (v_1 - \frac{1}{2})^2 + (\frac{1}{2} + v_2)^2 - 1 \right] + 1 - \frac{1}{2} \left[ (\frac{1}{4} - v_1^2) + (\frac{1}{4} - v_2^2) \right]
\]

\[
= -\frac{1}{2} (v_1 - v_2).
\]

Observe that as \( v_a \to 0, \) both the states become massless and combine into a left-handed spinor \( \phi^a \). Lastly, let us calculate the mass of the D5_{(12)}-D5_{(34)} fermion:

\[
\alpha' m^2(\lambda) = \frac{1}{2} \left[ \frac{1}{4} + \sum_{a \in \Delta} (v_a^2) - \frac{1}{4} \right] + \frac{1}{2} - 1 + \frac{1}{2} \sum_{a \in \Delta} (\frac{1}{4} - v_a^2) = 0 .
\]

These formulas match with the worldsheet oscillator derivation in Section 4.
A.4 Cocycle operators

We follow the treatment in [KLLSW] and write the cocycle operators $c_\lambda$ as follows:

$$c_\lambda := \exp \left( i \pi M_{\rho \sigma} \lambda^\rho N^\sigma \right), \quad \text{(A.71)}$$

where $N_\sigma$ is the vector of number operators $(N_1, \ldots, N_6)$ and $M_{\mu \nu}$ is the matrix

$$M_{\mu \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 \\ +1 & +1 & 0 & 0 & 0 & 0 \\ -1 & +1 & -1 & 0 & 0 & 0 \\ +1 & +1 & +1 & +1 & 0 & 0 \\ -1 & -1 & -1 & -1 & +1 & 0 \end{pmatrix}. \quad \text{(A.72)}$$

The indices $\sigma, \rho$ are raised and lowered using the indefinite metric $\eta_{\mu \nu}$. The OPE between two vertex operators $V_\lambda(z)$ and $V_{\lambda'}(z)$ then becomes

$$V_\lambda(z)V_{\lambda'}(w) \sim (z-w)^{\lambda \cdot \lambda'} e^{i \pi M_{\rho \sigma} N^\rho N^\sigma} V_{\lambda + \lambda'}(w) + \cdots. \quad \text{(A.73)}$$

The signs in the matrix $M$ are chosen to obtain the correct charge conjugation matrices in the OPEs

$$S^A(z)S^B(w) \sim (z-w)^{-1} C^{AB} + \cdots, \quad \hat{S}^A(z)\hat{S}^B(w) \sim -i(z-w)^{-1} C^{A\hat{B}} + \cdots, \quad \text{(A.74)}$$

where $S^A, \hat{S}^B$ are the $9+1$ dimensional left- and right-handed spinor vertex operators in the canonical ghost picture. They are given by the $D_{5,1}$ weights $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2})$ with even and odd number of minus signs respectively. The corresponding $\Gamma$-matrices are obtained from the OPE

$$\psi^\mu(z)S^A(w) \sim (z-w)^{-1} (\Gamma^\mu)^{A\hat{B}} S_{\hat{B}}^{(-\frac{1}{2})}(w) + \cdots, \quad \text{(A.75)}$$

where $\psi^\mu(z)$ is the $9+1$ dimensional vector vertex operator from the NS sector with $D_{5,1}$ weight $(0, \ldots, 0, \pm 1, 0, \ldots, 0; -1)$ and $S_{\hat{B}}^{(-\frac{1}{2})}$ is the operator that is conjugate to the operator $S_{\hat{B}}^{(-\frac{1}{2})}$ in the canonical ghost picture. We obtain the following helicity representation for the $\Gamma$-matrices and the charge conjugation matrix from the above OPEs [KLLSW]:

$$\Gamma^1 = \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \quad \Gamma^7 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1,$$
$$\Gamma^2 = \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \quad \Gamma^8 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1,$$
$$\Gamma^3 = \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1, \quad \Gamma^9 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1,$$
$$\Gamma^4 = -\sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1, \quad \Gamma^0 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes (i\sigma_2),$$
$$\Gamma^5 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1, \quad \Gamma^c = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3,$$
$$\Gamma^6 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1, \quad C_\omega = e^{3\pi i/4} \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2.$$

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A.5 CPT conjugate vertex operators

In the calculation of the Yukawa couplings arising from the various $E$-terms and $J$-terms, the $J$-term couplings involve the right-moving fermions while the $E$-term couplings involve the conjugate right-moving fermions. Hence we need vertex operators for CPT conjugate states. The transformation of the cocycle operators under CPT are quite intricate and must be handled with care.

Recall that the $H_a$ are antihermitian and $H_5, H_6$ are hermitian. The number operators satisfy:

$$ (N_a)^\dagger = N_a \ , \ a \in \mathbf{4} , \ (N_5)^\dagger = -N_5 , \ (N_6)^\dagger = -N_6 - 2 . \quad (A.76) $$

Spacetime CPT is implemented as Hermitian conjugation on the vertex operators. The operator $e^{\lambda H}$ thus transforms as

$$ (e^{\lambda H})^\dagger = (e^{\lambda_a H_a + \lambda_5 H_5 - \lambda_6 H_6})^\dagger = e^{-\lambda_a H_a + \lambda_5 H_5 - \lambda_6 H_6} =: e^{\lambda^* \cdot H} , \quad (A.77) $$

where we have defined $\lambda^* := (-\lambda_a, \lambda_5; \lambda_6)$ to be the CPT conjugate weight. The cocycle operator $c_\lambda$ transforms as

$$ (c_\lambda)^\dagger = \exp \left[ -i\pi \lambda \cdot M \cdot N^\dagger \right] = \exp \left[ -i\pi (\lambda \cdot M)_b N_b + i\pi (\lambda \cdot M)_5 N_5 \right] , $$$$ = \exp \left[ i\pi (-\lambda_a M_{ab} N_b - \lambda_5 M_{5a} N_a + \lambda_6 M_{6a} N_a - \lambda_6 M_{65} N_5) \right] , $$$$ = \exp \left[ i\pi (\lambda^* \cdot M)_a N_a + i\pi (\lambda^* \cdot M)_5 N_5 \right] \times \exp \left[ -2i\pi (\lambda_5 M_{5a} N_a - \lambda_6 M_{6a} N_a) \right] , $$$$ = c_{\lambda'} \ e^{-\pi \left( 2(\lambda_5 + \lambda_6) \Sigma_b N_b \right)} =: c_{\lambda'} , \quad (A.78) $$

where we have defined $\lambda' = (-\lambda_a - \lambda_5 - 2\lambda_6; \lambda_6) = \lambda^* - 2(\lambda_5 + \lambda_6)e_5$. Thus, the CPT conjugate of the operator $V_\lambda(k, z)$ in (A.66) is given by

$$ \tilde{V}_\lambda(k, z) = \omega(\lambda)^\dagger \ c(z) \ B^\dagger(z) \ (c_\lambda)^\dagger \ (e^{\lambda H(z)})^\dagger \ e^{-2ik \cdot X(z)} $$

$$ = \omega(\lambda)^\dagger \ e^{i\pi \lambda^* \cdot M \cdot \lambda'} \ c(z) \ B^\dagger(z) \ e^{\lambda^* \cdot H(z)} c_{\lambda'} \ e^{-2ik \cdot X(z)} . \quad (A.79) $$

B $\mathcal{N} = (0, 2)$ superspace

A 1+1 dimensional theory with $\mathcal{N} = (0, 2)$ supersymmetry has two supercharges $(Q_+, \bar{Q}_+)$ in the left-moving sector. The corresponding supersymmetry parameters are left-handed spinors $(\epsilon^+, \bar{\epsilon}^+)$. The Dirac equation in 1+1 dimensions imposes that left(right)-handed fermions are right(left)-movers on-shell. A scalar has both left- and right-moving parts. The left-moving part of the scalar will then have a superpartner fermion which is left-moving on-shell and hence right-handed. Thus, a scalar multiplet has a scalar and a right-handed fermion as its on-shell
degrees of freedom. A fermion which is right-moving on-shell (and hence left-handed) can form a multiplet on its own under such a supersymmetry. We next describe these multiplets and their gauged versions in superspace.

\( \mathcal{N} = (0, 2) \) superspace is described by coordinates \( (x^{\pm\pm}, \theta^+\!, \bar{\theta}^+) \) where \( \theta^+ \) and \( \bar{\theta}^+ \) are left-handed spinors. The corresponding supercovariant derivatives are denoted by \( (\partial_{++}, D_+, \bar{D}_+) \). They satisfy the algebra

\[
D_+^2 = \bar{D}_+^2 = 0, \quad \{D_+, \bar{D}_+\} = 2i\partial_{++}.
\]

(B.1)

We would like to study constrained superfields of the form \( D_+^c (\cdot) = 0 \). The natural complex structure of the supercovariant derivatives then imposes a complex structure on the space of constrained superfields. There are three kinds of \( \mathcal{N} = (0, 2) \) superfields that will be important for us: Vector, Chiral, Fermi. Before we study these representations, let us briefly discuss the representation theory of SO(1,1): the Lorentz group in 1+1 dimensions.

### B.1 Representations of SO(1,1)

The group SO(1,1) is abelian and has a single boost generator \( T_{01} \) with group element

\[
g(\lambda) = \exp(\lambda T_{01}), \quad \lambda \in \mathbb{R}.
\]

All irreducible representations are one dimensional. The representation on coordinates \( x^\mu, \mu = 0, 1 \), is given by \( T_{01} = -x_0 \partial_1 + x_1 \partial_0 \). In terms of lightcone coordinates \( x^{\pm\pm} = x^0 \pm x^1 \) where \( x^{++} \) is left-moving and \( x^{--} \) is right-moving, we have

\[
g(\lambda) \cdot x^{\pm\pm} = e^{\pm\lambda} x^{\pm\pm}.
\]

(B.2)

The vector representation mimics the above transformation rule: \( v^{\pm\pm} \to e^{\mp\lambda} v^{\pm\pm} \).

The spinor representations are the basic representations of the double cover Spin(1,1) with \( T_{01} = \frac{1}{2} \rho_0 \rho_1 \) where \( \rho^\mu \) are 1 + 1 dimensional Dirac matrices. Let \( \rho_c = -\rho^0 \rho^1 = \rho_0 \rho_1 \) and define left-handed and right-handed spinors \( v^+ \) and \( v^- \) to satisfy \( \rho_c v^\pm = \pm v^\pm \). We then have \( T_{01} = \frac{1}{2} \rho_c \) and \( v^\pm \) transform as

\[
g(\lambda) \cdot v^\pm = e^{\pm\frac{\lambda}{2}} v^\pm.
\]

(B.3)

(Observe that the product \( v^\pm w^\pm \) transforms in the same way as \( x^{\pm\pm} \).) We raise and lower the indices using the totally antisymmetric \( \varepsilon \)-symbol with \( \varepsilon_{+-} = +1 = \varepsilon^{+-} \):

\[
v^+ = \varepsilon^{+-} v^- = v_-, \quad v^- = \varepsilon^{-+} v_+ = -v_+.
\]

(B.4)

We thus conclude that an irreducible representation of SO(1,1) is an object with some number of + signs \( v^{++---} \) (left-moving) or some number of − signs \( w^{-----} \) (right-moving).

**Note:** The Berezin differentials \( d\theta^+, d\bar{\theta}^+ \) transform as

\[
d\theta^+ \to e^{-\frac{1}{2}\lambda} d\theta^+, \quad d\bar{\theta}^+ \to e^{-\frac{1}{2}\lambda} d\bar{\theta}^+.
\]

Thus, the most general superspace action is of the form

\[
\int d^2 x \, d\theta^+ d\bar{\theta}^+ K_{--} + \int d^2 x \, (d\theta^+ \mathcal{W}_- - \text{h.c.}),
\]

(B.5)
where \( K_- \) and \( W_- \) are functions of the various superfields in the theory with \( K_- \) unconstrained and \( D_+ W_- = 0 \). Equivalently, one can write
\[
\int d^2x D_+ D_+ K_- + \int d^2x (D_+ W_- - \text{h.c.}) \ , \tag{B.6}
\]
upto total \( \partial_{++} \) derivative terms.

### B.2 Chiral

A chiral superfield \( \Phi \) is a Lorentz scalar and satisfies
\[
\begin{align*}
D_+ \Phi &= 0 \quad \text{and} \quad \sqrt{2} \zeta_+ := (D_+ \Phi)_1 \ .
\end{align*}
\tag{B.7}
\]
The object \( D_+ D_+ \Phi \) contains nothing new: \( (D_+ D_+ \Phi)_1 = 2i \partial_{++} \phi \). Thus, this multiplet contains a scalar \( \phi \) and a right-handed fermion \( \zeta \). The free action is
\[
S_{\text{chiral}} = -\frac{i}{2} \int d^2x D_+ D_+ \Phi \partial_{--} \Phi = \int d^2x \left( -\overline{\psi} \partial_{++} \psi - i \sqrt{2} \zeta_+ \partial_{--} \zeta_+ \right) \ . \tag{B.8}
\]

### B.3 Fermi

A Fermi superfield \( \Psi_- \) is a left-handed spinor and satisfies \( \overline{D}_+ \Psi_- = \sqrt{2} E(\Phi) \) where \( E(\Phi) \) is a holomorphic function of the chiral multiplets \( \Phi_i \) in the theory. \( \Psi_- \) has components
\[
\begin{align*}
\psi_- &= (\Psi_-)_1 \ , \quad -\sqrt{2} G := (D_+ \Psi_-)_1 \ , \\
(D_+ \overline{D}_+ \Psi_-)_1 &= 2 \frac{\partial E}{\partial \phi_i} \zeta_{+,i} \ .
\end{align*}
\tag{B.9}
\]
The two-derivative action for \( \Psi_- \) is
\[
S_{\text{Fermi}} = \frac{1}{2} \int d^2x D_+ \overline{D}_+ \Psi_- \Psi_- = \int d^2x \left( -i \overline{\psi} \partial_{++} \psi + |G|^2 - |E(\phi)|^2 + \overline{\psi} \frac{\partial E}{\partial \phi_i} \zeta_{+,i} + \frac{\partial E}{\partial \phi_j} \zeta_{+,j} \psi_- \right) \ . \tag{B.10}
\]
We see that the left-handed fermion \( \psi_- \) satisfies \( \partial_{++} \psi_- = 0 \) for \( E = 0 \) and hence is right-moving on-shell.

### B.4 Potential terms

Let \( \Phi_i \) collectively denote all the \((0,2)\) chiral multiplets in the theory and \( \Psi_a \) the \((0,2)\) Fermi multiplets (we suppress the Lorentz index on \( \Psi_a \)). We have already seen the \( E \)-term previously when we discussed kinetic terms. We can also write a superpotential, also known as "\( J \)-term" in \((0,2)\) literature:
\[
S_J = -\frac{1}{\sqrt{2}} \int d^2x D_+ (J^a(\Phi_i) \Psi_a) - \text{h.c.} \ ,
\]
\[
= \int d^2x \left( J^a(\phi_i) G_a + \overline{G}^a \overline{J}_a(\phi) - \frac{\partial J^a}{\partial \phi_j} \zeta_{j+} \psi_{-a} - \overline{\psi}_{-a} \frac{\partial \overline{J}_a}{\partial \phi_j} \zeta_{j+} \right) \ . \tag{B.11}
\]
Invariance of the above term under $\mathcal{N} = (0, 2)$ supersymmetry requires $\nabla_+(\Psi_a J^a) = 0$. This implies

$$0 = E_a J^a =: E \cdot J . \quad (B.12)$$

This constraint is necessary for the action to be $\mathcal{N} = (0, 2)$ supersymmetric. If the action for a theory can be written in $(0, 2)$ superspace but the above constraint is violated, then the theory is only $(0, 1)$ supersymmetric.

### B.5 Vector

Suppose we have some matter fields $\Upsilon$ transforming under a rigid symmetry $\Upsilon \rightarrow e^{iK} \Upsilon$ with $K = K^a T_a$ an hermitian parameter. We choose hermitian generators $T_a$ with $\text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$ in the fundamental representation. We gauge this symmetry by introducing gauge-covariant supercovariant derivatives $\nabla_+, \overline{\nabla}_+$ and $\nabla_{++}$ which transform as $\nabla \rightarrow e^{iK} \nabla e^{-iK}$ under gauge transformations. The superspace constraints are

$$\nabla_+^2 = 0 , \quad \nabla_+^2 = 0 , \quad \text{and} \quad \{\nabla_+ , \nabla_+ \} = 2i \nabla_{++} . \quad (B.13)$$

The non-trivial curvatures are given by

$$\mathcal{F} = [\nabla_+, \nabla_-] , \quad \mathcal{F}_- = [\overline{\nabla}_+, \nabla_-] , \quad \overline{\mathcal{F}}_- = [\nabla_+, \nabla_-] . \quad (B.14)$$

The Bianchi identities give

$$\nabla_+ \mathcal{F} = \nabla_+ \mathcal{F} = 0 , \quad \nabla_+ \mathcal{F}_- = 0 = \nabla_+ \overline{\mathcal{F}}_- . \quad (B.15)$$

The components of the above field strengths are given by

$$\lambda_- := -(\mathcal{F}_-) , \quad v_{01} := \mathcal{F}_- , \quad D + iv_{01} := (\nabla_+ \mathcal{F}_-) . \quad (B.16)$$

The gauge action is given by

$$S_{\text{gauge}} = \frac{1}{2g^2} \int d^2x \, D_+ \overline{D}_+ \text{Tr} \, \mathcal{F}_- \overline{\mathcal{F}}_- ,$$

$$= \frac{1}{g^2} \int d^2x \, \text{Tr} \left( \frac{1}{2} v_{01}^2 - i \lambda_- D_{++} \lambda_- + \frac{1}{2} D^2 \right) . \quad (B.17)$$

The chirality constraint for a chiral superfield $\Phi$ in a complex representation of the gauge group becomes $\nabla_+ \Phi = 0$ and the components are defined to be

$$\phi := \Phi_1 , \quad \sqrt{2} \zeta_+ := (\nabla_+ \Phi)_1 . \quad (B.18)$$

The minimally coupled action is

$$S_{\text{chiral}} = -\frac{i}{2} \int d^2x \, D_+ \overline{D}_+ \overline{\Phi} \nabla_- \Phi ,$$

$$= \int d^2x \left( -\overline{\Phi} D_\mu \phi - i \zeta_+ D_{-+} \zeta_+ + i \sqrt{2} \phi \lambda_- \zeta_+ - i \sqrt{2} \overline{\zeta}_+ \phi - \overline{\phi} D \phi \right) . \quad (B.19)$$
Similarly, the constraint for a Fermi superfield $\Psi_a$ in some representation of the gauge group becomes $\nabla_+ \Psi_a = \sqrt{2} E_a(\Phi)$. The minimally coupled action is

$$S_{\text{Fermi}} = \frac{1}{2} \int d^2 x D_+ \overline{D}_+ \Psi^a \Psi_a,$$

$$= \int d^2 x \left( -i \overline{\psi}^- D_+ \psi_a - \mathcal{G}^a G_a - E_a E^a + \overline{\psi}^- \frac{\partial E_a}{\partial \phi_j} \zeta_j + \frac{\partial \overline{E}^a}{\partial \phi^j} \overline{\zeta}_j \psi_a \right). \quad (B.20)$$

### B.6 Holomorphic representation

The constraints $\nabla_+^2 = \overline{\nabla}_+^2 = 0$ can be solved by introducing a complex Lie algebra valued superfield $\Omega = \Omega^a T_a$ called the prepotential:

$$\nabla_+ = e^{-i\Omega} D_+ e^{i\Omega} := D_+ + i \Gamma_+ , \quad \overline{\nabla}_+ = e^{-\overline{\Omega}} \overline{D}_+ e^{i\overline{\Omega}} := \overline{D}_+ - i \Gamma_+ . \quad (B.21)$$

where we have defined the spinor connections $\Gamma_+$ and $\Gamma_+$. We also define $\nabla_{\pm \mp} := D_{\pm \mp} + i \Gamma_{\pm \mp}$. The gauge transformation $\nabla \rightarrow e^{iK} \nabla e^{-iK}$ can be reproduced by assigning the following transformation rule for $\Omega$:

$$e^{i\Omega} \rightarrow e^{i\Omega} e^{-i\Lambda}, \quad e^{-\overline{\Omega}} \rightarrow e^{-\overline{\Omega}} e^{i\Lambda}. \quad (B.22)$$

The above solution has additional gauge invariances:

$$e^{i\Omega} \rightarrow e^{i\Lambda} e^{i\Omega}, \quad e^{-\overline{\Omega}} \rightarrow e^{i\overline{\Lambda}} e^{-\overline{\Omega}}. \quad (B.23)$$

where $\Lambda$ is a Lie algebra valued chiral superfield $D_+ \Lambda = 0$. One can use the hermitian $K$ to gauge away the hermitian part of $\Omega$. Equivalently, we look at the $K$-inert hermitian object

$$e^V := e^{i\Omega} e^{-i\overline{\Omega}}, \quad \text{with} \quad e^V \rightarrow e^{i\overline{\Lambda}} e^{i\overline{V}} e^{-i\overline{\Lambda}}. \quad (B.24)$$

(In the gauge where $\Omega = -\overline{\Omega}$, we have $V = 2i\overline{\Omega}$.)

One can go to the holomorphic representation via a (non-unitary) change of basis $\nabla \rightarrow e^{i\overline{\Omega}} \nabla e^{-i\overline{\Omega}}, \quad \nabla \rightarrow e^{i\overline{\Omega}} \nabla$ for a general matter superfield $\Upsilon$. The spinor derivatives become $\nabla_+ = e^{-V} D_+ e^V, \quad \overline{\nabla}_+ = \overline{D}_+ e^V$ which gives

$$i \Gamma_+ = e^{-V} (D_+ e^V), \quad \Gamma_+ = 0 , \quad (B.25)$$

thus justifying the name holomorphic. In this representation, the chirality constraint becomes $D_+ \Upsilon = 0$. All the derivatives are $K$-inert but transform under $\Lambda$ as $\nabla \rightarrow e^{i\Lambda} \nabla e^{-i\Lambda}$ with $D_+ \Lambda = 0$ and the connections transform as

$$i \delta \Gamma_+ = -i \nabla_+ \Lambda, \quad \delta \Gamma_+ = 0 . \quad (B.26)$$

The components of $\Gamma_+$ are $\gamma_+ := (\Gamma_+)_1, \quad v_{++} := \frac{1}{2} (\overline{\nabla}_+ \Gamma_+) \parallel$ of which $\gamma_+$ can be set to zero using the gauge transformation above. The same gauge freedom gives

$$\delta v_{++} = \frac{1}{2} (\overline{\nabla}_+ \delta \Gamma_+) \parallel = - \frac{1}{2} (\overline{\nabla}_+ \{ \overline{\nabla}_+ , \nabla_+ \} \Lambda) \parallel = -i \overline{\nabla}_+ \lambda , \quad (B.27)$$

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which is the usual transformation for a non-abelian gauge field \( v_{++} \). The final constraint \( \{ \nabla_+, \nabla_+ \} = 2i \nabla_{++} \) in fact gives \( 2i \Gamma_+ = \nabla_+ \Gamma_+ \) whose bosonic part is \( 2v_{++} \). The curvatures are given by

\[
F_+ = [\nabla_+, \nabla_-] , \quad \tilde{F}_- = [\nabla_+, \nabla_-] .
\]

(B.28)

The superspace Lagrangians for the chiral, Fermi and vector multiplets in the holomorphic representation are \( \Phi e^V \nabla_- \Phi \), \( \Psi e^V \bar{\Psi} \ould\) and \( F_+ \tilde{F}_- \) respectively.

### B.7 Duality exchanging \( E \leftrightarrow J \)

Consider a Fermi superfield \( \Psi_a \) satisfying \( \nabla_+ \Psi_a = \sqrt{2} E_a \). The most general action with \( J \)-term is

\[
S[\Psi_a] = -\frac{1}{2} \int d^2x D_+ \bar{D}_+ \bar{\Psi}^a \Psi_a - \frac{1}{\sqrt{2}} \int d^2x \{ D_+ \Psi_a J^a + \bar{D}_+ \bar{\Psi}^a \bar{J}_a \} .
\]

(B.29)

The kinetic term can be reproduced from the following first order action for \( \Psi_a \) by integrating out the unconstrained Grassmann superfield \( \Lambda_a \):

\[
S[\Lambda_a, \Psi_a] = \frac{1}{2} \int d^2x D_+ \bar{D}_+ \{ \bar{\Lambda}^a \Lambda_a - \Psi_a \bar{\Lambda}^a - \Lambda_a \bar{\Psi}^a \} - \frac{1}{\sqrt{2}} \int d^2x \{ D_+ \Psi_a J^a + \bar{D}_+ \bar{\Psi}^a \bar{J}_a \} .
\]

(B.30)

Instead, we could integrate out \( \Psi_a \). To do this, we push in \( D_+ \) in the Lagrange multiplier term \( \Psi_a \bar{\Lambda}^a \) (and appropriately for its complex conjugate) to get

\[
S[\Lambda_a, \Psi_a] = \frac{1}{2} \int d^2x D_+ \{ -\sqrt{2} E_a \bar{\Lambda}^a + \Psi_a \nabla_+ \bar{\Lambda}^a - \sqrt{2} \bar{\Psi}_a J^a \} - \text{h.c.} .
\]

(B.31)

Integrating out \( \Psi_a \) gives \( \nabla_+ \bar{\Lambda}^a = \sqrt{2} J^a \). Relabelling \( \bar{\Lambda}^a = \Psi'^a \), we have \( \nabla_+ \Psi'^a = \sqrt{2} J^a \) and the action

\[
S[\Psi'_a] = -\frac{1}{2} \int d^2x D_+ \bar{D}_+ \Psi'_a \Psi'^a - \frac{1}{\sqrt{2}} \int d^2x \{ D_+ (\Psi'^a E_a) + \bar{D}_+ (\bar{\Psi}'_a \bar{E}^a) \} .
\]

(B.32)

\textbf{Note:} The new Fermi multiplet \( \Psi'^a \) transforms in the conjugate representation of the various symmetry groups in the theory as compared to \( \Psi_a \).

### B.8 Reduction to \( \mathcal{N} = (0, 1) \) superspace

We study the reduction to \( \mathcal{N} = (0, 1) \) superspace. The derivatives \( D_+, \bar{D}_+ \) are written as

\[
D_+ = \frac{D + iQ}{\sqrt{2}} , \quad \bar{D}_+ = \frac{D - iQ}{\sqrt{2}} , \quad \implies \quad D = \frac{D_+ + \bar{D}_+}{\sqrt{2}} , \quad Q = \frac{D_+ - \bar{D}_+}{\sqrt{2}i} .
\]

(B.33)

with

\[
Q^2 = D^2 = i\partial_{++} , \quad \{ D, Q \} = 0 .
\]

(B.34)
Here, $D$ is the $\mathcal{N} = (0,1)$ real supercovariant derivative and $Q$ serves as the (non-manifest) generator of the extra supersymmetry. The $(0,2)$ chiral and Fermi superfields (and their anti-chiral counterparts) become $(0,1)$ scalar and Fermi superfields with components

$$
\begin{align*}
(\Phi_i) &= \phi_i , & (D\Phi_i) &= \zeta_{i+} , & (\bar{\Phi}^i) &= \bar{\phi}^i , & (D\bar{\Phi}^i) &= -\bar{\zeta}_{+} , \\
(\Psi_a) &= \psi_{a-} , & (D\Psi_a) &= -(G_a - E_a) , & (\bar{\Psi}^a) &= \bar{\psi}^a , & (D\bar{\Psi}^a) &= -(\bar{G}^a - \bar{E}^a) .
\end{align*}
$$

(B.35)

In particular, we see that the complex structure on the space of fields imposed by the superspace derivatives $D_+$ and $D_-$ is lost upon reduction to $(0,1)$ superspace. The above $(0,1)$ superfields transform as follows under the extra supersymmetry:

$$
Q \Phi_i = -iD\Phi_i , \quad Q \Psi_a = -iD\Psi_a + 2iE_a .
$$

(B.36)

The $(0,1)$ actions are then given by

$$
S_{\text{chiral}} = -i \int d^2x J^a \bar{E}_a - \int d^2x D (\bar{G}^a \Psi_a - \bar{E}_a \Psi_a - \bar{G}^a \Psi_a) 
$$

(B.37)

The $J$-term becomes the $\mathcal{N} = (0,1)$ superpotential

$$
S_J = \int d^2x J^a E_a - \int d^2x J^a \Psi_a + \text{h.c.} = - \int d^2x J^a \Psi_a + \text{h.c.} .
$$

(B.38)

The first term would have prevented us from writing the $(0,2)$ superpotential in $(0,1)$ superspace but it vanishes due to the identity $J \cdot E = 0$ that is required for $\mathcal{N} = (0,2)$ supersymmetry. The total action for the Fermi superfield is then

$$
\int d^2x D (\bar{\Psi}^a D\Psi_a - (\bar{E}^a + J^a) \Psi_a - \bar{G}^a (E_a + J_a)) .
$$

(B.39)

As we can see, the $E$- and $J$- terms are on equal footing in $(0,1)$ superspace and they can be written as a single $(0,1)$ superpotential term $D ((J^a + E^a) \Psi_a)$. The bosonic potential that arises from such a superpotential term is

$$
\int d^2x \sum_a |J^a + \bar{E}^a|^2 .
$$

(B.40)

### B.9 $(2,2) \rightarrow (0,2)$

We shall be schematic here and details can be found in Section 6 of [W2]. A twisted chiral superfield $\Sigma$ satisfies $\nabla_+ \Sigma = \nabla_- \Sigma = 0$. The $(0,2)$ decomposition is then a chiral and a Fermi multiplet:

$$
\Sigma := \Sigma_+ , \quad \text{with} \quad \nabla_+ \Sigma = 0 , \\
\bar{\Sigma} := \frac{1}{\sqrt{2}} (\nabla_- \Sigma) , \quad \text{with} \quad \nabla_+ \bar{\Sigma} = 0 .
$$

(B.41)
where \( | \) indicates that we have set \( \theta^- = \bar{\theta}^- = 0 \). All other combinations of the supercovariant derivatives acting on \( \Sigma \) are either zero or derivatives of the above fields. The \((2,2)\) field strength is a special case of a twisted chiral multiplet: \( 2\sqrt{2} \Sigma = \{ \nabla_+, \nabla_- \} \). The complex scalar \( \sigma \) now sits in a separate \((0,2)\) chiral multiplet \( \Sigma \) and \( \tilde{\Sigma} \) is the familiar \((0,2)\) field strength \( F_- \) (upto a factor of \(-i/2\)).

A \((2,2)\) chiral superfield \( \Phi \) satisfies \( \nabla_+ \Phi = \nabla_- \Phi = 0 \). The \((0,2)\) decomposition is then

\[
\Phi := \Phi_1 \quad \text{with} \quad \nabla_+ \Phi = 0 , \\
\Phi_- := \frac{1}{\sqrt{2}} (\nabla_- \Phi) , \quad \text{with} \quad \nabla_+ \Phi_- = \frac{1}{\sqrt{2}} (\nabla_+, \nabla_-) \Phi = 2 \Sigma \Phi ,
\]

(8.42)

where \( \Sigma \) is the \((0,2)\) chiral multiplet that contains the complex scalar \( \sigma \). Thus, a \((2,2)\) chiral multiplet \( \Phi \) splits into a \((0,2)\) chiral \( \Phi \) and a Fermi \( \Phi_- \) which has an \( E \)-term \( E_{\Phi_-} = \sqrt{2} \Sigma \Phi \).

A \((2,2)\) superpotential \( W(\Phi_i) \) gives rise to \((0,2)\) superpotential \( W(\Phi_i, \Phi_i^-) \) after the \( D_- \) in the measure has been pushed into the action:

\[
\int D_{+} D_{-} W(\Phi_i) = \sqrt{2} \int D_{+} \frac{\partial W}{\partial \Phi_i} \nabla_- \Phi_i = \sqrt{2} \int D_{+} \frac{\partial W}{\partial \Phi_i} \Phi_i^- ,
\]

(8.43)

giving a \( J \)-term \( J^i = -2 \frac{\partial W}{\partial \Phi_i} \). The constraint \( \nabla_+(J^i \Phi_i^-) = 0 \) becomes

\[
\frac{\partial W}{\partial \Phi_i} \Sigma \Phi_i = 0 \, ,
\]

(8.44)

which is nothing but the condition of gauge invariance of \( W(\Phi) \).

\[ \square \]

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