ALTERNATIVE THEOREM OF NAVIER-STOKES EQUATIONS IN \( \mathbb{R}^3 \)

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Abstract. We consider Cauchy problem of the incompressible Navier-Stokes equations with initial data \( u_0 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \). There exist a maximum time interval \([0, T_{\text{max}}]\) and a unique solution \( u \in C([0, T_{\text{max}}]; L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)) \) \((\forall p > 3)\). We find one of function class \( S_{\text{regular}} \) defined by scaling invariant norm pair such that 
\[ T_{\text{max}} = \infty \] provided \( u_0 \in S_{\text{regular}} \). Especially, \( \| u_0 \|_{L^p} \) is arbitrarily large for any \( u_0 \in S_{\text{regular}} \) and \( p > 3 \). On the other hand, the alternative theorem is proved. It is that either 
\[ T_{\text{max}} = \infty \] or 
\[ T_{\text{max}} \in (T_l, T_r) \]. Especially, \( T_r < T_{\text{max}} < \infty \) is disappearing. Here the explicit expressions of \( T_l \) and \( T_r \) are given. This alternative theorem is one kind of regular criterion which can be verified by computer. If \( T_{\text{max}} = \infty \), the solution \( u \) is regular for any \((t, x) \in (0, \infty) \times \mathbb{R}^3 \). As \( t \to \infty \), the solution is decay. On the other hand, lower bound of blow up rate of \( u \) is obtained again provided \( T_{\text{max}} \in (T_l, T_r) \).

1. Introduction

In this paper, we study Cauchy problem of the incompressible Navier-Stokes (NS) equations in \( \mathbb{R}^3 \),
\begin{align*}
(1.1) & 
\quad u_t - \Delta u + (u \cdot \nabla)u + \nabla P = 0, \\
(1.2) & 
\quad \nabla \cdot u = 0, \\
(1.3) & 
\quad u|_{t=0} = u_0,
\end{align*}
where \( u = u(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x)) \) and \( P = P(t, x) \) stand for the unknown velocity vector field of fluid and its pressure, \( u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x)) \) is the given initial velocity vector field satisfying \( \nabla \cdot u_0 = 0 \). Here \( \partial_{x_j} \) denotes by \( \partial_j \) \((j = 1, 2, 3)\).

For the mathematical setting of this problem, we introduce Hilbert space
\[ H(\mathbb{R}^3) = \{ u \in (L^2(\mathbb{R}^3))^3 | \nabla \cdot u = 0 \} \]
edowed with \( (L^2(\mathbb{R}^3))^3 \) norm (resp. scalar product denoted by \( (\cdot, \cdot) \)). For simplicity of presentation, space \( (L^q(\mathbb{R}^3))^3 \) (resp. \( (H^m(\mathbb{R}^3))^3 \)) denotes by \( L^q \) (reps. \( H^m \)), where \( q \geq 1 \) and \( m \geq 0 \).

Taking divergence of the equation \( (1.1) \), we get
\begin{align*}
(1.4) & 
\quad \sum_{j,k=1}^{3} \partial_j \partial_k \{ u^j u^k \} + \Delta P = 0,
\end{align*}

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in the sense of distributions. Then one has

\[(1.5) \quad P = R_j R_k (u^j u^k) = \sum_{j,k=1}^{3} R_j R_k (u^j u^k).\]

Here \( R_j \) \((j = 1, 2, 3)\) is Riesz operator \[14\], and

\[(1.6) \quad \hat{R}_j w(\xi) = i\xi_j \hat{w}(\xi), \quad w \in \mathcal{S}(\mathbb{R}^3),\]

where Fourier transform of \( w \) denotes by \( \hat{w} \). In what follows we use the usual convention and sum over the repeated indices.

Thus the system \((1.1)-(1.2)\) can be rewritten as follows

\[(1.7) \quad u_t - \Delta u + (u \cdot \nabla) u + \nabla R_j R_k (u^j u^k) = 0,\]

in the sense of distributions. Taking divergence of the equation \((1.7)\), one has

\[\partial_t (\nabla \cdot u) - \Delta (\nabla \cdot u) = 0.\]

For the solution \( u \) of equation \((1.7)\) with \((1.3)\), \( \nabla \cdot u = 0 \) provided \( \nabla \cdot u_0 = 0 \). The equations \((1.7)-(1.3)\) are employed to solve the problem \((1.1)-(1.3)\).

There is a large literature studying the incompressible Navier-Stokes equations. In 1934 Leray \[29\] proved that there exists a global weak solution to the problem \((1.1)-(1.3)\) with initial data in \( L^2 \). In 1951 Hopf \[19\] extended this result to bounded smooth domain. Moreover Leray-Hopf weak solutions satisfy energy inequality \[46\]

\[(1.8) \quad \| u(t, \cdot) \|^2_{L^2} + 2 \int_0^t \| \nabla u(\tau, \cdot) \|^2_{L^2} d\tau \leq \| u_0 \|^2_{L^2}, \quad \forall t > 0.\]

The uniqueness and regularity of Leray-Hopf weak solution is a famous open question. Numerous regularity criteria were proved \[12, 15, 24, 25, 27, 36, 43, 49\]. Serrin type criteria states as follows.

**Proposition 1.1 (Regularity Criteria).** Let \( u \) be Leray-Hopf weak solution of the problem \((1.1)-(1.3)\) and \( u \in L^q(0, T; L^p(\mathbb{R}^3)) \). Then \( u \) is regular on \((0, T]\) provided \( \frac{2}{q} + \frac{3}{p} = 1 \) and \( p \in [3, \infty] \).

Local existence and uniqueness of mild solution or strong solution were established \[4, 17, 21, 22, 23, 52\] with initial data in \( L^p(\mathbb{R}^3) \), \( p > 3 \).

**Proposition 1.2 (Local Mild Solution).** Let \( u_0 \in L^p(\mathbb{R}^3) \) and \( p > 3 \). Then there exist \( T = T(\| u_0 \|_{L^p}) > 0 \) and a unique solution \( u \) of the problem \((1.1)-(1.3)\) such that \( u \in C([0, T]; L^p(\mathbb{R}^3)) \).

Besides the local-posedness, the lower bounds of possible blowup solutions were considered \[8, 9, 11, 15, 29, 37\]. The concentration phenomena of possible blowup solutions was studied \[30\].

It is well-known that the equations \((1.7)\) is scaling-invariant in the sense that if \( u \) solves \((1.7)\) with initial data \( u_0 \), so does \( u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \) with initial data \( \lambda u_0(\lambda x) \). A space \( X \) defined on \( \mathbb{R}^3 \) is so-called to be critical provided \( \| u_0 \|_X = \| \lambda u_0(\lambda \cdot) \|_X \) for any \( \lambda > 0 \). \( L^q(\mathbb{R}^3) \) is one of critical spaces. For the initial data in critical spaces, the posedness of global solution of the equations \((1.7)\) is obtained \[5, 10, 26, 35\] with small initial data. The regularity criterion was established \[12, 13, 24, 31, 42\]. On the other hand, the ill-posedness was showed \[2, 14, 50, 53\].
It is also studied that solutions of the problem (1.1)–(1.3) are in various function spaces [6, 16, 18, 20, 26, 45]. Partial regularity of suitable weak solutions was established [3, 28, 32, 35, 39, 48, 51]. Non-existence of self-similar solutions was proved [34, 47].

We denote by $T_{\text{max}}$ the supremum of all $T > 0$ so that the mild solution $u$ exists in time interval $[0, T]$, i.e. the solution $u \in C([0, T_{\text{max}}); L^p(\mathbb{R}^3))$. The goal of this paper is to discuss whether $T_{\text{max}} = \infty$ or not. Now let us state the main results.

**Theorem 1.3** (Global Existence and Uniqueness of $L^p$ Solution). Let the initial data $u_0 \in L^3(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $p > 3$ and $\nabla \cdot u_0 = 0$. Assume that $u_0$ satisfies

\[
\|u_0\|_{L^3} \leq \frac{1}{2^\frac{2p-4}{p} (1 + 3C_\infty^p)}.
\]

Then there exists a unique global solution $u$ of the problem (1.1)–(1.3) such that $u \in C([0, \infty); L^p(\mathbb{R}^3))$.

**Theorem 1.4** (Alternative Theorem). Let the initial data $u_0 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $\nabla \cdot u_0 = 0$. For any $p \in (3, \infty]$, there exist $T_{\text{max}} > 0$ and a unique solution $u \in C([0, T_{\text{max}}); L^p(\mathbb{R}^3))$ of the problem (1.1)–(1.3). Moreover $T_{\text{max}}$ is independent of $p$, $u \in C([0, T_{\text{max}}); H'(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)) \cap L^2(0, T_{\text{max}}; H^1(\mathbb{R}^3))$, and the following energy inequality is satisfied,

\[
\|u(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s, \cdot)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2, \quad \forall 0 \leq t < T_{\text{max}}.
\]

For any $q \in [1, 3]$ and $p \in (3, \infty]$, let us define

\[
T_l = T_l(p, q) = \left\{ \frac{8pC_3}{p-3} (1 + 3C_\infty^p) \right\}^\frac{2p-4}{p+3}.
\]

\[
T_r = T_r(p, q) = \left\{ \frac{8pC_3}{p-3} (1 + 3C_\infty^p) \right\}^\frac{2p-4}{p-q}.
\]

Here constant $C_\infty$ is defined in (2.9), $C_0 = C_0(p, q)$ is defined in Lemma 3.1 and $C_3 = C_3(p, p/2)$ is defined in Lemma 3.2.

Then we have

1. $T_{\text{max}} = \infty$ if there exists one pair $(p, q)$ such that $p \in (3, \infty]$, $q \in [1, 3]$ and $T_r(p, q) \leq T_l(p, q)$.

On the other hand, either $T_{\text{max}} = \infty$, or $T_l < T_{\text{max}} \leq T_r$. Especially, $T_r < T_{\text{max}} < \infty$ is disappearing.

2. As $t \to \infty$, the solution $u$ is decay provided $T_{\text{max}} = \infty$. The decay rate is as follows,

\[
\|u(t, \cdot)\|_{L^p} \leq \frac{1 - \left\{ \frac{8pC_3}{p-3} (1 + 3C_\infty^p) \right\}^\frac{2p-4}{p+3}}{t^\frac{2p-4}{p(q+4)}} C_0 \|u_0\|_{L^p} t^{-\frac{3-q}{2p}}, \quad \forall t > T_r.
\]

Moreover this solution $u(t, x)$ is regular for any $(t, x) \in (0, \infty) \times \mathbb{R}^3$. 
(3) This solution $u$ blows up provided $T_l < T_{\text{max}} \leq T_r$. Moreover the lower bound of blow up rate is as follows,

$$
\|u(t, \cdot)\|_{L^p} \geq \frac{C_b}{(T_{\text{max}} - t)^{\frac{p-3}{2p}}}, \quad C_b = \frac{p - 3}{8pC_3(1 + 3C_{\infty}^p)}, \quad \forall t \in [0, T_{\text{max}}).
$$

Some remarks of Theorem 1.3 and Theorem 1.4 are needed.

The explicit expressions of all constants in Theorem 1.3 and Theorem 1.4 are given. These explicit expressions is necessary and convenient for numerical analysis.

The disappearance of $T_r < T_{\text{max}} < \infty$ is important for solving the problem (1.1)–(1.3) by numerical calculation. In fact, we can claim that the solutions are global, regular and unique, as soon as the time interval of numerical solutions is over $[0, T_r]$. In this sense, we think that the alternative property proved in Theorem 1.4 is one kind of regularity criterion which can be verified by computer.

The condition (1.13) is equivalent to

$$
Q_{\lambda}^p(u_0) = \|u_0\|_{L^p} \|u_0\|_{L^q}^{\frac{2p}{2q}} \leq \frac{1}{K_0},
$$

where constant

$$
K_0 = \left\{ \frac{8pC_3}{p - 3} \left(1 + 3C_{\infty}^p\right) \right\}^{\frac{2p}{p+2}} + \frac{2q}{p-3} C_0^{\frac{2p}{2q}}
$$

$$
= \left\{ \frac{p}{p-3} 2^{3-\frac{7}{p}} e^{-\frac{1}{2p} \frac{p+2}{2p}} \left(1 + 3C_{\infty}^p\right) \right\}^{\frac{2p}{p+2}} + \frac{2q}{p-3} (4\pi)^{\frac{3(p-q)}{p(3-q)}}.
$$

We have one important observation that quantity $Q_{\lambda}^p(u)$ is scaling invariant, i.e. $Q_{\lambda}^p(u_\lambda) = Q_{\lambda}^p(u)$. Thus we call the quantity $Q_{\lambda}^p(\cdot)$ to be the scaling invariant norm pair.

Moreover the assumption (1.13), i.e. (1.16), dose not mean that the initial data $u_0$ is small. In fact, if there exists $\lambda_0 > 0$ such that $u_{0\lambda_0}(x)$ satisfies (1.16), then $u_{0\lambda}$ satisfies (1.16) for any $\lambda > 0$. Now let $\lambda \to \infty$ and $\lambda \to 0$, one has

$$
\|u_{0\lambda}\|_{L^p} \to \infty, \quad \lambda \to \infty, \quad \forall p > 3,
$$

$$
\|u_{0\lambda}\|_{L^q} \to 0, \quad \lambda \to \infty, \quad \forall q \in [1, 3),
$$

$$
\|u_{0\lambda}\|_{L^p} \to 0, \quad \lambda \to 0, \quad \forall p > 3,
$$

$$
\|u_{0\lambda}\|_{L^q} \to \infty, \quad \lambda \to 0, \quad \forall q \in [1, 3).
$$

Here $u_{0\lambda}(x) = \lambda u_0(\lambda x)$.

For example, take scalar function $\phi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$ and

$$
\phi(y) = \begin{cases} 
1, & |y| \leq 1, \\
0, & |y| \geq 2.
\end{cases}
$$

Define vector function

$$
\psi(x) = (\psi^1(x), \psi^2(x), \psi^3(x)) = (\partial_{x_3}\phi(r), -\partial_{x_3}\phi(r), \partial_{x_2}\phi(r) - \partial_{x_4}\phi(r)).
$$

Here $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is obvious that $\nabla \cdot \psi = 0$. Select initial data

$$
\psi = \lambda \phi(\psi) \big|_{y = \lambda x},
$$

where $\alpha$ and $\lambda$ are positive constants. Thanks the quantity $Q_{\lambda}^p(\cdot)$ is the scaling invariant norm pair, we can take $\alpha > 0$ large enough such that $u_0$ satisfies the conditions (1.16).
and \((1.13)\) for any \(\lambda > 0\). Then fix \(\alpha\), let \(\lambda \to \infty\) and \(\lambda \to 0\). It is easy to verify that \(u_0\) satisfies

\[
\|u_0\|_{L^p} \to \infty, \quad \text{as} \quad \lambda \to \infty, \quad \forall p > 3,
\]
\[
\|u_0\|_{L^q} \to 0, \quad \text{as} \quad \lambda \to \infty, \quad \forall q \in [1, 3), \tag{1.19}
\]
\[
\|u_0\|_{L^p} \to 0, \quad \text{as} \quad \lambda \to 0, \quad \forall p > 3,
\]
\[
\|u_0\|_{L^q} \to \infty, \quad \text{as} \quad \lambda \to 0, \quad \forall q \in [1, 3). \tag{1.20}
\]

Similarly, norm \(\| \cdot \|_{L^3}\) is scaling invariant. For Theorem 1.3 and the condition \((1.9)\), we have the same remarks as above. Especially, the condition \((1.9)\) does not mean that the initial data \(u_0\) is small.

Let us define function class

\[
S_{\text{regular}} = \{ u_0 | u_0 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \ u_0 \ \text{satisfies (1.1)}, \text{ or there exist one } p \in (3, \infty) \ \text{and one } q \in [1, 3) \ \text{such that } u_0 \ \text{satisfies (1.10)} \}.
\]

Thus there exists a unique global solution \(u\) of the problem \((1.1)-(1.3)\) with initial data \(u_0 \in S_{\text{regular}}\) such that \(u \in C([0, \infty); H(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L^q(0, \infty); H^1(\mathbb{R}^3))\) for any \(p > 3\). Moreover this solution is regular for any \((t, x) \in (0, \infty) \times \mathbb{R}^3\).

Optimal estimate constants, such as decay rate, are not the main topics of this paper. More decay results can be found in [7, 33, 40, 41], etc.

Comparing \((1.15)\) with the results in references [29, 15], etc., the blow up rate \(\frac{p-2}{ap}\) is same. The only difference is the explicit expression of constant \(C_b\). From this explicit expression, it is clear that the methods of this paper don’t work for solutions in critical space, i.e. \(u \in C([0, T_{\max}], L^3(\mathbb{R}^3))\).

Some estimates of heat semigroup are the key ingredients in the proof of Theorem 1.3 and Theorem 1.4. Let

\[
G(t, x) = (4\pi t)^{-3/2} \exp\left\{-\frac{|x|^2}{4t}\right\},
\]
\[
G_j(t, x) = \partial_j G(t, x) = -(4\pi t)^{-3/2} \frac{x_j}{2t} \exp\left\{-\frac{|x|^2}{4t}\right\},
\]
\[
G_{klj}(t, x) = R_k R_l G_j(t, x),
\]
\[
G(t)u_0 = G(t, \cdot) * u_0 = (4\pi t)^{-3/2} \int_{\mathbb{R}^3} \exp\left\{-\frac{|y|^2}{4t}\right\} u_0(x-y) dy,
\]
\[
G_j(t)u_0 = G_j(t, \cdot) * u_0 = -(4\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{y_j}{2t} \exp\left\{-\frac{|y|^2}{4t}\right\} u_0(x-y) dy,
\]
\[
G_{klj}(t)u_0 = G_{klj}(t, \cdot) * u_0 = \int_{\mathbb{R}^3} G_{klj}(t, y) u_0(x-y) dy,
\]

where \(R_k\) is Riesz operator with respect to \(x \in \mathbb{R}^3\). Then the equations \((1.1)-(1.3)\) are equivalent to

\[
u^j(t, x) = G(t)u^j_0 - \int_0^t \{G_m(t-\tau)(u^m u^j) + G_{klj}(t-\tau)(u^k u^l)\} d\tau, \quad j = 1, 2, 3.
\]
The following estimates
\begin{equation}
\|G(t)u_0\|_{L^p} \leq Ct^{-\frac{3(p-q)}{2pq}}\|u_0\|_{L^q}, \quad \forall t > 0, \ 1 < q < p, \tag{1.29}
\end{equation}
\begin{equation}
\|G_m(t)u_0\|_{L^p} \leq Ct^{-\frac{3p-3q+pq}{2pq}}\|u_0\|_{L^q}, \quad \forall t > 0, \ 1 < q < p, \tag{1.30}
\end{equation}
\begin{equation}
\|G_{klj}(t)u_0\|_{L^p} \leq Ct^{-\frac{3p-3q+pq}{2pq}}\|u_0\|_{L^q}, \quad \forall t > 0, \ 1 < q < p, \tag{1.31}
\end{equation}
are used in the proof of Theorem 1.3 and Theorem 1.4.

The plan of this paper is as follows. Section 2 is devoted to prove that Riesz transform \(R_a\) is \((L^\infty, L^\infty)\) operator in sub-space \(H^s\) \((s > \frac{3}{2})\) of \(L^\infty\). Section 3 is devoted to show some linear estimates. Section 4 is devoted to prove the main theorems.

The symbol \(C\) denotes a generic positive constant, its value may change from line to line.

2. Riesz transforms in sub-space of \(L^\infty\)

This section is devoted to prove that Riesz transform \(R_a\) is \((L^\infty, L^\infty)\) operator in sub-space \(H^s\) \((s > \frac{3}{2})\) of \(L^\infty\). Let us recall briefly the classical construction of Littlewood-Paley decomposition in \(\mathbb{R}^3\). Select even (or radial) function \(\psi \in C_0^\infty(\mathbb{R}^3)\) such that
\[
0 \leq \psi \leq 1, \quad \psi(\xi) = 1 \text{ for } |\xi| \leq \frac{1}{2}, \quad \psi(\xi) = 0 \text{ for } |\xi| \geq 1.
\]
Define \(\phi(\xi) = \psi(\xi/2) - \psi(\xi)\). Then we have
\[
supp \phi \subseteq \{1/2 \leq |\xi| \leq 2\}
\]
and the following partition of unity
\[
1 = \psi(\xi) + \sum_{j \geq 0} \phi(2^{-j}\xi), \quad \forall \xi \in \mathbb{R}^3.
\]
We first consider \(w \in H^s(\mathbb{R}^3)\) \((s > \frac{3}{2})\). Functions \(S_0w, \ w_{-1}\) and \(w_j\) are defined as follows
\[
w_{-1} = S_0w = \psi(D)w, \quad w_j = \phi(2^{-j}D)w,
\]
i.e.
\[
\hat{w}_{-1}(\xi) = \hat{S_0}w(\xi) = \psi(\xi)\hat{w}(\xi), \quad \hat{w}_j(\xi) = \phi(2^{-j}\xi)\hat{w}(\xi).
\]
Now we obtain the Littlewood-Paley decomposition
\[
w = S_0w + \sum_{j \geq 0} w_j, \quad w \in H^s(\mathbb{R}^3).
\]
Let
\[
S_jw = \sum_{k=-1}^{j-1} w_k, \quad j \geq 1.
\]
Then
\begin{equation}
\|w - S_jw\|_{H^s} \to 0, \quad \text{as } j \to \infty, \tag{2.1}
\end{equation}
and
\begin{equation}
\|w - S_jw\|_{L^\infty} \leq C\|w - S_jw\|_{H^s} \to 0, \quad \text{as } j \to \infty. \tag{2.2}
\end{equation}
For Riesz transform \(R_a\), we have the following lemma.
Lemma 2.1 (\(L^\infty\) Estimate of Riesz transform). Let \(w \in H^s(\mathbb{R}^3)\) \((s > \frac{3}{2})\). Then there exists a positive constant \(C_\infty\) such that
\[
(2.3) \quad \|R_aw\|_{L^\infty} \leq C_\infty\|w\|_{L^\infty}, \quad a = 1, 2, 3.
\]
For any \(w \in L^q(\mathbb{R}^3), 2 \leq q < \infty\), one has
\[
(2.4) \quad \|R_aw\|_{L^q} \leq C_\infty^{1-2/q}\|w\|_{L^q}, \quad a = 1, 2, 3.
\]
For any \(w \in L^q(\mathbb{R}^3), 1 < q < 2\), we have
\[
(2.5) \quad \|R_aw\|_{L^q} \leq C_\infty^{2/q-1}\|w\|_{L^q}, \quad a = 1, 2, 3.
\]
Proof. Applying the Littlewood-Paley decomposition of \(w\), we have
\[
R_aw = R_0w + \sum_{j \geq 0} R_aw_j.
\]
Employing (2.1) \[(2.2)\], we get that
\[
(2.6) \quad \|R_aw - R_0w\|_{H^s} \to 0, \quad \text{as } j \to \infty, \quad \text{and}
\]
\[
(2.7) \quad \|R_aw - R_0w\|_{L^\infty} \leq C\|R_0w - R_0w\|_{H^s} \to 0, \quad \text{as } j \to \infty.
\]
Now we use the idea of \([1]\) to estimate \(\|R_0w\|_{L^\infty}\). Note that
\[
\hat{R_0w}(\xi) = \frac{i\xi_a}{|\xi|} \hat{\psi}(2^{-j}\xi)\hat{w}(\xi), \quad \xi = (\xi_1, \xi_2, \xi_3).
\]
Let \(\Psi_a(\xi) = \frac{i\xi_a}{|\xi|} \hat{\psi}(\xi)\) and the inverse Fourier transform of \(\Psi_a\) denote by \(\check{\Psi}_a\). Then
\[
R_0w = \Psi_a(2^{-j}D)w = \Phi * w,
\]
where \(\Phi(x) = 2^{3j}\check{\Psi}_a(2^jx)\), and
\[
(2.8) \quad \int_{\mathbb{R}^3} |\Phi(x)|dx = \int_{\mathbb{R}^3} |\check{\Psi}_a(y)|dy = C_a < \infty.
\]
Define
\[
(2.9) \quad C_\infty = \max_{a=1,2,3} C_a.
\]
Therefore we have
\[
(2.10) \quad \|R_0w\|_{L^\infty} \leq C_\infty\|w\|_{L^\infty}, \quad a = 1, 2, 3.
\]
Letting \(j \to \infty\), we obtain the estimate (2.3).
Noting the fact that \(C_0^\infty(\mathbb{R}^3)\) is dense in \(L^q(\mathbb{R}^3), 1 < q < \infty\), and
\[
(2.11) \quad \|R_0w\|_{L^2} = \|w\|_{L^2},
\]
applying equality (2.11), the estimate (2.3) and Riesz-Thorin interpolation theorem, we can prove the inequality (2.4).
For any \(w \in L^q(\mathbb{R}^3), 1 < q < 2\), \(\phi \in C_0^\infty(\mathbb{R}^3)\), one has
\[
(R_aw, \phi) = (w, -R_aw \phi) \leq \|w\|_{L^q}\|R_aw\|_{L^{q'}} \leq C_\infty^{1-2/q'}\|w\|_{L^q}\|\phi\|_{L^{q'}},
\]
where \(\frac{1}{q} + \frac{1}{q'} = 1\). Therefore the inequality (2.5) is proved. \(\square\)
3. Linear Estimates

This section is devoted to establishing some estimates of the semigroups $G(t)$, $G_m(t)$ and $G_{klj}(t)$, which are used in next section.

**Lemma 3.1.** For the semigroup $G(t)$, we have the following estimates
\[
\|G(t)u_0\|_{L^p} \leq \|u_0\|_{L^p}, \quad \forall t > 0, \quad 1 \leq p \leq \infty, \tag{3.1}
\]
\[
\|G(t)u_0\|_{L^p} \leq (4\pi t)^{-\frac{3(p-1)}{4p}}\|u_0\|_{L^1}, \quad \forall t > 0, \quad 1 \leq p \leq \infty, \tag{3.2}
\]
\[
\|G(t)u_0\|_{L^p} \leq (4\pi t)^{-\frac{3(p-2)}{2p}}\|u_0\|_{L^{p'}}, \quad \forall t > 0, \quad 2 \leq p \leq \infty, \tag{3.3}
\]
\[
\|G(t)u_0\|_{L^p} \leq C_0 t^{-\frac{3(p-q)}{2pq}}\|u_0\|_{L^q}, \quad \forall t > 0, \quad 1 < q < p. \tag{3.4}
\]
Here constant $C_0 = C_0(p,q) = (4\pi)^{-\frac{3(p-q)}{2pq}}$.

*Proof.* By simple calculation, we have
\[
\int_{\mathbb{R}^3} G(t,x)dx = 1,
\]
\[
G(t,x)dx \leq (4\pi t)^{-\frac{3}{2}}.
\]
Thus
\[
\|G(t)u_0\|_{L^p} \leq \|G(t,\cdot)\|_{L^1}\|u_0\|_{L^p} \leq \|u_0\|_{L^p}, \tag{3.5}
\]
\[
\|G(t)u_0\|_{L^\infty} \leq \|G(t,\cdot)\|_{L^1}\|u_0\|_{L^1} \leq (4\pi t)^{-\frac{3}{2}}\|u_0\|_{L^1}.
\]
The estimate (3.1) is proved.

Taking (3.1) with $p = 1$ and (3.5), and using Riesz-Thorin interpolation, we can prove estimate (3.2).

Taking (3.1) with $p = 2$ and (3.5), and using Riesz-Thorin interpolation, we can prove estimate (3.3).

Using Riesz-Thorin interpolation with respect to (3.1) and (3.2), we can prove the estimate (3.4). \[\square\]

**Lemma 3.2.** For the semigroup $G_m(t)$ ($m = 1, 2, 3$), we have the following estimates
\[
\|G_m(t)u_0\|_{L^p} \leq \left(\pi t\right)^{-\frac{1}{2}}\|u_0\|_{L^p}, \quad \forall t > 0, \quad 1 \leq p \leq \infty, \tag{3.6}
\]
\[
\|G_m(t)u_0\|_{L^p} \leq C_1 t^{-\frac{4p-3}{2p}}\|u_0\|_{L^1}, \quad \forall t > 0, \quad 1 \leq p \leq \infty, \tag{3.7}
\]
\[
\|G_m(t)u_0\|_{L^p} \leq C_2 t^{-\frac{2p-4}{p}}\|u_0\|_{L^{p'}}, \quad \forall t > 0, \quad 2 \leq p \leq \infty, \tag{3.8}
\]
\[
\|G_m(t)u_0\|_{L^p} \leq C_3 t^{-\frac{3p-3q+pq}{2pq}}\|u_0\|_{L^q}, \quad \forall t > 0, \quad 1 < q < p. \tag{3.9}
\]
Here constants
\[
C_1 = 2^{-\frac{7(p-1)}{2p}} \frac{e^{-\frac{p-1}{2p}}}{\pi^\frac{3p-2}{2p}},
\]
\[
C_2 = 2^{-\frac{7(p-2)}{2p}} \frac{e^{-\frac{p-2}{2p}}}{\pi^\frac{3p-4}{2p}},
\]
\[
C_3 = C_3(p,q) = 2^{-\frac{7(p-q)}{2pq}} \frac{e^{-\frac{p-q}{2pq}}}{\pi^\frac{2p-2q+pq}{2pq}}.
\]
Proof. By simple calculation, we have

\[ \int_{\mathbb{R}^3} |G_m(t,x)| dx = (\pi t)^{-\frac{1}{2}}, \]

(3.10)

\[ \|G_m(t,\cdot)\|_{L^\infty} \leq 2^{-\frac{9}{2}} e^{-\frac{1}{2} \pi^{-\frac{2}{3}} t^{-2}}. \]

Thus

\[ \|G_m(t)u_0\|_{L^p} \leq \|G_m(t,\cdot)\|_{L^1} \|u_0\|_{L^p} \leq (\pi t)^{-\frac{1}{2}} \|u_0\|_{L^p}, \]

(3.12)

The estimate (3.6) is proved.

Taking (3.6) with \( p = 1 \) and (3.12), and using Riesz-Thorin interpolation, we can prove estimate (3.7).

Taking (3.6) with \( p = 2 \) and (3.12), and using Riesz-Thorin interpolation, we can prove estimate (3.8).

Using Riesz-Thorin interpolation with respect to (3.6) and (3.7), we can prove the estimate (3.9). \( \square \)

**Lemma 3.3.** For the semigroup \( G_{klj}(t) (k,l,j = 1,2,3) \), we have the following estimates

\[ \|G_{klj}(t)u_0\|_{L^p} \leq C_4(\pi t)^{-\frac{1}{2}} \|u_0\|_{L^p}, \quad \forall t > 0, \quad 1 < p \leq \infty, \]

(3.13)

\[ \|G_{klj}(t)u_0\|_{L^p} \leq C_4 C_1 t^{-\frac{4p-3}{2p}} \|u_0\|_{L^1}, \quad \forall t > 0, \quad 1 < p \leq \infty, \]

(3.14)

\[ \|G_{klj}(t)u_0\|_{L^p} \leq C_5 C_2 t^{-\frac{2p-3}{p}} \|u_0\|_{L^{p'}} , \quad \forall t > 0, \quad 2 \leq p \leq \infty, \]

(3.15)

\[ \|G_{klj}(t)u_0\|_{L^p} \leq C_5 C_3 t^{-\frac{3p-3+pq}{2pq}} \|u_0\|_{L^q}, \quad \forall t > 0, \quad 1 < q < p. \]

(3.16)

Here

\[ C_4 = \begin{cases} C_{\infty}^p, & 1 < p < 2, \\ C_{2p-4}^p, & p \geq 2. \end{cases} \]

\[ C_5 = \begin{cases} C_{\infty}^p, & 1 < p < 2, \\ C_{2p-4}^p, & p \geq 2 > q, \\ C_{2q-4}^q, & q \geq 2, \end{cases} \]

Proof. Note that

\[ G_{klj}(t)u_0 = \{R_k R_l G_j(t)\}u_0 = G_j(t)\{R_k R_l u_0\}, \]

(3.17)

where \( R_k \) is Riesz transformation.

Using formula (3.17), Lemma 2.1 and Lemma 3.2, we can prove this lemma. \( \square \)
4. Proof of Main Theorems

This section is devoted to prove the main theorems.
First let us review Proposition\cite{122}

**Lemma 4.1 (\(L^p\) Local Solution).** Let the initial data \(L^p(\mathbb{R}^3)\), \(\nabla \cdot u_0 = 0\) and \(p > 3\). Then there exist \(T_{\text{max}} = T_{\text{max}}(p)\) and a unique solution \(u\) of the problem \((1.1) - (1.3)\) such that \(u \in C([0, T_{\text{max}}); L^p(\mathbb{R}^3))\) and

\[
\lim_{t \to 0^+} ||u(t, \cdot) - u_0||_{L^p} = 0.
\]
If \(T_{\text{max}} < \infty\), one has

\[
\sup_{t \to T_{\text{max}}} \lim_{t \to T_{\text{max}}} ||u(t, \cdot)||_{L^p} = \infty.
\]

**Proof.** Let

\[B = \left\{ u = (u^1, u^2, u^3) \mid \nabla \cdot u = 0, \ u \in C([0, T]; L^p(\mathbb{R}^3)), \right\} \]

For any \(u \in B\), we define mapping \(M\) as follows

\[Mu = (Mu^1, Mu^2, Mu^3)\]

\[Mu^j = G(t)u^j_0 - \int_0^t \{G_m(t - \tau)(u^m u^j) + G_{klj}(t - \tau)(u^k u^l)\} d\tau, \ j = 1, 2, 3.\]

Using \((3.1)\), we have

\[||G(t)u^j_0||_{L^p} \leq ||u^j_0||_{L^p}.\]

Applying \((3.9)\) \((3.13)\) with \(q = \frac{p}{2}\), one has

\[
\left| \int_0^t \{G_m(t - \tau)(u^m u^j) + G_{klj}(t - \tau)(u^k u^l)\} d\tau \right|_{L^p} 
\leq C_3 \int_0^t (t - \tau)^{-\frac{p+3}{2p}} \left( ||u(\tau, \cdot)||_{L^p} ||u^j(\tau, \cdot)||_{L^p} \right) + C_3^p \||u(\tau, \cdot)||_{L^p}^2 \right) d\tau.
\]

Here \(C_3 = C_3(p, p/2)\). Thus we have

\[
\max_{0 \leq t \leq T} ||Mu(t, \cdot)||_{L^p} \leq ||u_0||_{L^p} + \frac{2pC_3}{p - 3} \left( 1 + 3C_3^p \right) T^{\frac{p - 3}{2p}} \left( \max_{0 \leq t \leq T} ||u(t, \cdot)||_{L^p} \right)^2
\]

\[
\leq ||u_0||_{L^p} + \frac{8pC_3}{p - 3} \left( 1 + 3C_3^p \right) T^{\frac{p - 3}{2p}} ||u_0||_{L^p}^2.
\]

Let

\[T \leq \left\{ \frac{8pC_3}{p - 3} \left( 1 + 3C_3^p \right) ||u_0||_{L^p} \right\}^{-\frac{2p}{p - 3}}.\]

Then \(Mu \in B\) for any \(u \in B\).

Similarly, for any \(u, v \in B\), we have

\[
\max_{0 \leq t \leq T} ||Mu(t, \cdot) - Mv(t, \cdot)||_{L^p}
\]

\[
\leq \frac{2pC_3}{p - 3} \left( 1 + 3C_3^p \right) T^{\frac{p - 3}{2p}} \left( \max_{0 \leq t \leq T} ||u(t, \cdot)||_{L^p} + ||v(t, \cdot)||_{L^p} \right) \max_{0 \leq t \leq T} ||u(t, \cdot) - v(t, \cdot)||_{L^p}
\]

\[
\leq \frac{8pC_3}{p - 3} \left( 1 + 3C_3^p \right) T^{\frac{p - 3}{2p}} ||u_0||_{L^p} \max_{0 \leq t \leq T} ||u(t, \cdot) - v(t, \cdot)||_{L^p}.
\]
Let
\begin{equation}
T = T_0 = \left\{ \frac{8pC_3}{\theta(p-3)} \left( 1 + 3C\infty^p \right) \right\}^{-\frac{2p}{p-3}}, \quad 0 < \theta < 1.
\end{equation}

Then
\[
\max_{0 \leq t \leq T} \| Mu(t, \cdot) - M v(t, \cdot) \|_{L^p} \leq \theta \max_{0 \leq t \leq T} \| u(t, \cdot) - v(t, \cdot) \|_{L^p}, \quad 0 < \theta < 1, \quad \forall u, v \in B,
\]
and $M : B \to B$ is contraction mapping.

Therefore there exits a unique fixed point $u \in B$ of the mapping $M$, i.e., there exits a unique solution $u \in C([0, T_0]; L^p(\mathbb{R}^3))$ of the problem (1.1)–(1.3). Especially we have
\begin{equation}
\max_{0 \leq t \leq T_0} \| u(t, \cdot) \|_{L^p} \leq 2\| u_0 \|_{L^p},
\end{equation}
and
\begin{equation}
\| u(t, \cdot) - u_0 \|_{L^p} \leq \| u(t, \cdot) - G(t)u_0 \|_{L^p} + \| G(t)u_0 - u_0 \|_{L^p} \to 0, \quad t \to 0^+.
\end{equation}

Taking $t = T_0$ as initial time and $u(T_0, x)$ as initial data, instead of $t = 0$ and $u_0$ respectively, we resolve the problem (1.1)–(1.3) again. Using the above arguments, we prove that there exits a unique solution $u \in C([0, T_0 + T_1]; L^p(\mathbb{R}^3))$ of the problem (1.1)–(1.3). Here
\begin{equation}
T_1 = \left\{ \frac{18pC_3}{p-3} \left( 1 + 3C\infty^p \right) \right\}^{-\frac{2p}{p-3}}, \quad \theta = \frac{2}{3},
\end{equation}
\begin{equation}
\max_{0 \leq t \leq T_1} \| u(T_0 + t, \cdot) \|_{L^p} \leq 3\| u_0 \|_{L^p}.
\end{equation}

The existence interval $[0, T_0 + T_1]$ is the extension of $[0, T_0]$. Repeating the above procedure again and again, we extend the existence interval $[0, T]$ as large as possible. Let
\begin{equation}
T_{max} = T_{max}(p) = \sup \{ T \mid \text{there exits a unique solution } u \in C([0, T]; L^p(\mathbb{R}^3)) \}.
\end{equation}

If $T_{max} < \infty$ and
\[
\sup_{t \to T_{max}} \| u(t, \cdot) \|_{L^p} \leq U_0 < \infty,
\]
let
\begin{equation}
T_2 = \left\{ \frac{18pC_3}{p-3} \left( 1 + 3C\infty^p \right) U_0 \right\}^{-\frac{2p}{p-3}}.
\end{equation}

Taking $t = T_{max} - \frac{1}{2}T_2$ as initial time and $u(T_{max} - \frac{1}{2}T_2, x)$ as initial data, instead of $t = 0$ and $u_0$ respectively, we resolve the problem (1.1)–(1.3) again. Using the above arguments, we prove that there exits a unique solution $u \in C([0, T_{max} + \frac{1}{2}T_2]; L^p(\mathbb{R}^3))$ of the problem (1.1)–(1.3), and
\begin{equation}
\max_{0 \leq t \leq T_2} \| u(T_{max} - \frac{1}{2}T_2 + t, \cdot) \|_{L^p} \leq 3U_0.
\end{equation}

This is contradictory with the definition of $T_{max}$. Thus (4.2) is valid. \qed
4.1. Proof of Theorem 1.3

Proof. Using (4.28) (3.3) (3.9) (3.16), we have

\[
\begin{align*}
\max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p} & \leq C_0 T^{-\frac{p-3}{2p}} \|u_0\|_{L^3} \\
& + \frac{2pC_3}{p-3} (1 \mp 3C_\infty^p) T^{\frac{p-3}{2p}} \{ \max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p} \}^2, \quad \forall T \geq 0.
\end{align*}
\]

(4.12)

Here \( C_0 = C_0(p, 3) \) and \( C_3 = C_3(p, p/2) \). Let

\[
Z(T) = T^{\frac{p-3}{2p}} \max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p}, \quad Q_2(Z) = C_0 \|u_0\|_{L^3} - Z + \frac{2pC_3}{p-3} (1 \mp 3C_\infty^p) Z^2.
\]

Then (4.12) implies that

\[
Q_2(Z) \geq 0, \quad \forall T \geq 0.
\]

Employing (1.9), we have

\[
1 - \frac{8pC_3}{p-3} (1 \mp 3C_\infty^p) C_0 \|u_0\|_{L^3} > 0.
\]

Then equation \( Q_2(Z) = 0 \) has two solutions

\[
Z_1 = 1 - \{1 - \frac{8pC_3}{p-3} (1 \mp 3C_\infty^p) C_0 \|u_0\|_{L^3}\}^{\frac{1}{2}}, \\
Z_2 = 1 + \{1 - \frac{8pC_3}{p-3} (1 \mp 3C_\infty^p) C_0 \|u_0\|_{L^3}\}^{\frac{1}{2}}.
\]

Thus we have

\[
[0, \infty) = [0, Z_1] \cup (Z_1, Z_2) \cup [Z_2, \infty), \\
Q_2(Z) \geq 0, \quad \forall Z \in [0, Z_1] \cup [Z_2, \infty), \\
Q_2(Z) < 0, \quad \forall Z \in (Z_1, Z_2).
\]

Since \( Z(0) = 0 \) and \( Q_2(Z(0)) = C_0 \|u_0\|_{L^3} \geq 0 \), one has \( Z(0) \in [0, Z_1] \). Thanks \( Z(t) \) and \( Q_2(Z(t)) \) are continuous with respect to \( t \), \( Z(t) \) can not arrive at \( [Z_2, \infty) \) from \( [0, Z_1] \) by jumping over interval \( (Z_1, Z_2) \). Hence we have

\[
(4.14) \quad Z(t) \leq Z_1, \quad \forall t \geq 0.
\]

Combining (4.2) and (4.14), one gets that \( T_{max} = \infty \).

Theorem 1.3 is proved.

\[ \square \]

4.2. Proof of Theorem 1.4

Proof. Since \( L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \subset L^p(\mathbb{R}^3) \) is dense for any \( p \in (1, \infty) \). By Lemma 4.1, there exists a unique solution \( u \in C([0, T_{max}); L^p(\mathbb{R}^3)) \) of the problem (1.1)–(1.3) for any \( 3 < p \leq \infty \).
Applying (1.28) (3.1) (3.9) (3.16), we have
\[
\max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p}
\]
\[
\leq \|u_0\|_{L^p} + \frac{2pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) T^{\frac{p-3}{4p}} \left\{ \max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p} \right\}^2, \quad \forall T \geq 0,
\]
where \(p > 3\), \(C_3 = C_3(p, p/2)\) and \(q = \frac{p}{2}\) in (3.9) (3.16).

Let
\[
Y(T) = T^{\frac{p-3}{4p}} \max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p},
\]
\[
Q_1(Y) = \|u_0\|_{L^p} - T^{\frac{p-3}{4p}} Y + \frac{2pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) Y^2.
\]

Then (4.15) implies that
\[
Q_1(Y) \geq 0, \quad \forall T \geq 0.
\]

Let
\[
T_{10} = \left\{ \frac{8pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) \|u_0\|_{L^p} \right\}^{-\frac{2p}{p-3}}.
\]

If \(T < T_{10}\), the equation \(Q_1(Y) = 0\) has two solutions
\[
Y_1 = \frac{1 - \left\{1 - \frac{8pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) T^{\frac{p-3}{4p}} \|u_0\|_{L^p} \right\}^{\frac{1}{2}}}{\frac{4pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) T^{\frac{p-3}{4p}}},
\]
\[
Y_2 = \frac{1 + \left\{1 - \frac{8pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) T^{\frac{p-3}{4p}} \|u_0\|_{L^p} \right\}^{\frac{1}{2}}}{\frac{4pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) T^{\frac{p-3}{4p}}}.
\]

Thus we have
\[
[0, \infty) = [0, Y_1] \cup (Y_1, Y_2) \cup [Y_2, \infty),
\]
\[
Q_1(Y) \geq 0, \quad \forall Y \in [0, Y_1] \cup [Y_2, \infty),
\]
\[
Q_1(Y) < 0, \quad \forall Y \in (Y_1, Y_2).
\]

Since \(Y(0) = 0\), \(Q_1(Y(0)) = 0\), one has \(Y(0) \in [0, Y_1]\). Thanks \(Y(t)\) and \(Q_1(Y(t))\) are continuous with respect to \(t\), \(Y(t)\) can not arrive at \([Y_2, \infty)\) from \([0, Y_1]\) by jumping over interval \((Y_1, Y_2)\). Thus we have
\[
Y(t) \leq Y_1, \quad \forall t < T_{10}.
\]

Combining (4.2) and (4.17), one gets \(T_{\max} > T_{10}\).

Using (1.28) (3.4) (3.9) (3.16), we have
\[
\max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p} \leq C_0 T^{-\frac{3(p-q)}{4pq}} \|u_0\|_{L^q} + \frac{2pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) T^{\frac{p-3}{4p}} \left\{ \max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p} \right\}^2, \quad \forall T \geq 0,
\]
where \(1 \leq q < 3\), \(p > 3\), \(C_0 = C_0(p, q)\), \(C_3 = C_3(p, p/2)\).

Let
\[
Z(T) = T^{\frac{3p+pq-6q}{4pq}} \max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^p},
\]
\[
Q_2(Z) = C_0 \|u_0\|_{L^q} - T^{-\frac{3}{4q}} Z + \frac{2pC_3}{p-3} \left(1 + 3C_{\infty}^{p-3}\right) Z^2.
\]
Then \( (4.18) \) implies that
\[
Q_2(Z) \geq 0, \quad \forall T \geq 0.
\]
Let
\[
T_{20} = \left\{ \frac{8pC_3}{p-3} \left( 1 + 3C_\infty^{\frac{2p-4}{p}} \right) C_0 \| u_0 \|_{L^q} \right\}^{\frac{2q}{3-q}}.
\]
If \( T > T_{20} \), the equation \( Q_2(Z) = 0 \) has two solutions
\[
Z_1 = T^{\frac{3-q}{4q}} - \left\{ T^{\frac{3-q}{4q}} - \frac{8pC_3}{p-3} \left( 1 + 3C_\infty^{\frac{2p-4}{p}} \right) C_0 \| u_0 \|_{L^q} \right\}^{\frac{1}{2}},
\]
\[
Z_2 = T^{\frac{3-q}{4q}} + \left\{ T^{\frac{3-q}{4q}} - \frac{8pC_3}{p-3} \left( 1 + 3C_\infty^{\frac{2p-4}{p}} \right) C_0 \| u_0 \|_{L^q} \right\}^{\frac{1}{2}}.
\]
Thus we have
\[
[0, \infty) = [0, Z_1] \cup (Z_1, Z_2) \cup [Z_2, \infty),
\]
\[
Q_2(Z) \geq 0, \quad \forall Z \in [0, Z_1] \cup [Z_2, \infty),
\]
\[
Q_2(Z) < 0, \quad \forall Z \in (Z_1, Z_2).
\]
Since \( Z(0) = 0 \) and \( Q_2(Z(0)) = C_0 \| u_0 \|_{L^2} \geq 0 \), one has \( Z(0) \in [0, Z_1] \). Thanks \( Z(t) \) and \( Q_2(Z(t)) \) are continuous with respect to \( t \), \( Z(t) \) can not arrive at \([Z_2, \infty)\) from \([0, Z_1]\) by jumping over interval \((Z_1, Z_2)\). If \( T_{\text{max}} > T_{20} \), we have
\[
(4.20) \quad Z(t) \leq Z_1, \quad \forall t \in [0, T_{\text{max}}).
\]
Combining \((4.2)\) and \((4.20)\), one gets that \( T_{\text{max}} > T_{20} \) implies \( T_{\text{max}} = \infty \).
Hence \( T_{\text{max}} = \infty \) provided \( T_{20} \leq T_{10} \).
On the other hand, either \( T_{\text{max}} = \infty \), or \( T_{10} < T_{\text{max}} \leq T_{20} \). Especially, \( T_{20} < T_{\text{max}} < \infty \) is disappearing.
Since
\[
\| u_0 \|_{L^2} \leq \| u_0 \|_{L^1} \| u_0 \|_{L^\infty},
\]
thus Leray-Hopf weak solution is existent. By the uniqueness of mild solution in \( L^p(\mathbb{R}^3) \) \( (p > 3) \), this solution \( u \) satisfies the energy inequality \((1.10)\) and
\[
(4.21) \quad u \in C([0, T_{\text{max}}); H(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)) \cap L^2(0, T_{\text{max}}; H^1(\mathbb{R}^3)).
\]
Thanks the regularity criteria Proposition \((1.1)\) this solution \( u(t, x) \) is regular for any \((t, x) \in (0, \infty) \times \mathbb{R}^3 \) provided \( T_{\text{max}} = \infty \).
Since \( u \in C([0, T_{\text{max}}(p)); H(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)) \), by Riesz-Thorin interpolation one has
\[
\| u(t, \cdot) \|_{L^r} \leq \| u(t, \cdot) \|_{L^\frac{2(p-2)}{p-2}} \| u(t, \cdot) \|_{L^\frac{p(r-2)}{r-2}}, \quad \forall r \in (3, p).
\]
Thus \( u \in C([0, T_{\text{max}}(p)); H(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)) \) and \( T_{\text{max}}(r) \geq T_{\text{max}}(p) \) for any \( r \in (3, p) \). If \( T_{\text{max}}(p) < \infty \) and \( T_{\text{max}}(r) > T_{\text{max}}(p) \), then
\[
C_{rp} = \max_{0 \leq t \leq T_{\text{max}}(p)} \| u(t, \cdot) \|_{L^r} < \infty.
\]
Let $\alpha > 0$ small enough and $T_\alpha = T_{\max}(p) - \alpha$. Taking $t = T_\alpha$ as initial time and $u(T_\alpha, x)$ as initial data, instead of $t = 0$ and $u_0$ respectively, we resolve the problem \((1.1) - (1.3)\) again. Therefore we have

\[
    u^i(T_\alpha + t, x) = G(t)u^i(T_\alpha, x) - \int_0^t \{G_m(t - \tau)(u^m u^j) + G_{klj}(t - \tau)(u^{k,u^j})\}(T_\alpha + \tau, x)d\tau, \quad \forall t \geq 0, \quad j = 1, 2, 3.
\]

Using \((4.22)\), \((3.4)\), \((3.9)\), \((3.16)\), we have

\[
    \omega(T) = T^{\frac{3(p-r)}{2rp}} T \max \|u(T_\alpha + t, \cdot)\|_{L^p},
\]

\[
    Q_3(\omega) = C_0C_{rp} - T^{\frac{3-r}{p}} \omega + \frac{2pC_3}{p-3} (1 + 3C_\infty^p) \omega^2.
\]

Then \((4.23)\) implies that

\[
    Q_3(\omega) \geq 0, \quad \forall T \geq 0.
\]

Let

\[
    T_{30} = \left\{ \frac{8pC_3}{p-3} (1 + 3C_\infty^p) C_0 C_{rp} \right\}^{\frac{1}{2p-4}}.
\]

Since $r > 3$, if $T < T_{30}$, the equation $Q_3(Z) = 0$ has two solutions

\[
    \omega_1 = \frac{1 - \left\{ 1 - T^{\frac{r-3}{p-3}} \frac{8pC_3}{p-3} (1 + 3C_\infty^p) C_0 C_{rp} \right\}^{\frac{1}{2p-4}}}{\frac{4pC_3}{p-3} (1 + 3C_\infty^p) T^{\frac{r-3}{4p}}}.
\]

\[
    \omega_2 = \frac{1 + \left\{ 1 - T^{\frac{r-3}{p-3}} \frac{8pC_3}{p-3} (1 + 3C_\infty^p) C_0 C_{rp} \right\}^{\frac{1}{2p-4}}}{\frac{4pC_3}{p-3} (1 + 3C_\infty^p) T^{\frac{r-3}{4p}}}.
\]

Thus we have

\[
    [0, \infty) = [0, \omega_1] \cup (\omega_1, \omega_2) \cup [\omega_2, \infty),
\]

\[
    Q_3(\omega) \geq 0, \quad \forall Z \in [0, \omega_1] \cup [\omega_2, \infty),
\]

\[
    Q_3(\omega) < 0, \quad \forall Z \in (\omega_1, \omega_2).
\]

Since $\omega(0) = 0$ and $Q_3(\omega(0)) = C_0 C_{rp} \geq 0$, one has $\omega(0) \in [0, \omega_1]$. Thanks $\omega(t)$ and $Q_3(\omega(t))$ are continuous with respect to $t$, $\omega(t)$ can not arrive at $[\omega_2, \infty)$ from $[0, \omega_1]$ by jumping over interval $(\omega_1, \omega_2)$. Thus we have

\[
    \omega(T) \leq \omega_1, \quad \forall T \in [0, T_{30}).
\]
Choose $\alpha = \frac{1}{3} T_{30}$, the estimate (4.25) means that

$$\|u(T_{\text{max}}(p) - \frac{1}{3} T_{30} + t, \cdot)\|_{L^p} \leq 1 - \{1 - t^{\frac{p-3}{2p}} \frac{8pC_1}{p-3} (1 + 3C_\infty^p) C_0 C_{rp}\}^\frac{1}{2} \frac{4pC_1}{p-3} (1 + 3C_\infty^p) t^{\frac{p-3}{2p}}, \forall t \in [0, T_{30}).$$

This is contradictory with (4.2). Thus

$$T_{\text{max}}(r) = T_{\text{max}}(p), \quad \forall r, p \in (3, \infty].$$

If $T_{10} < T_{\text{max}} \leq T_{20}$, let $\epsilon > 0$ and $T_\epsilon = T_{\text{max}} - \epsilon$. Taking $t = T_\epsilon$ as initial time and $u(T_\epsilon, x)$ as initial data, instead of $t = 0$ and $u_0$ respectively, we resolve the problem (1.1)–(1.3) again. Thus we have

$$u^j(T_\epsilon + t, x) = G(t) u^j(T_\epsilon, x) - \int_0^t \{G_\epsilon(t - \tau)(u^m u^j) + G_{klj}(t - \tau)(u^k u^l)\} (T_\epsilon + \tau, x) d\tau, \quad \forall t \in [0, \epsilon), \quad j = 1, 2, 3.$$  \hfill (4.27)

Using (4.27) (3.1) (3.9) (3.16), we have

$$\max_{0 \leq s \leq t} \|u(T_\epsilon + s, \cdot)\|_{L^p} \leq \|u(T_\epsilon, \cdot)\|_{L^p} + \frac{2pC_3}{p-3} (1 + 3C_\infty^p) t^{\frac{p-3}{2p}} \max_{0 \leq s \leq t} \|u(T_\epsilon + s, \cdot)\|_{L^p}^2, \quad \forall t \in [0, \epsilon),$$  \hfill (4.28)

where $p > 3$ and $C_3 = C_3(p, p/2)$.

Let

$$W(t) = t^{\frac{p-3}{4p}} \max_{0 \leq s \leq t} \|u(T_\epsilon + s, \cdot)\|_{L^p},$$

$$Q_4(W) = \|u(T_\epsilon, \cdot)\|_{L^p} - t^{\frac{p-3}{4p}} W + \frac{2pC_3}{p-3} (1 + 3C_\infty^p) W^2.$$  \hfill (4.29)

Then (4.28) implies that

$$Q_4(W) \geq 0, \quad \forall t \in [0, \epsilon).$$  \hfill (4.29)

Now we have

$$\frac{d}{dW} Q_4(W) = \frac{4pC_3}{p-3} (1 + 3C_\infty^p) W - t^{\frac{p-3}{4p}}.$$  \hfill (4.30)

Thus $Q_4(W)$ takes minimum at point $W = W_0$, where

$$W_0 = \frac{p - 3}{4pC_3 (1 + 3C_\infty^p) t^{\frac{p-3}{4p}}}.$$  \hfill (4.31)

(4.29) means that

$$Q_4(W_0) \geq 0, \quad \forall t \in [0, \epsilon).$$  \hfill (4.30)

(4.30) is equivalent to

$$\|u(T_\epsilon, \cdot)\|_{L^p} \geq \frac{p - 3}{8pC_3 (1 + 3C_\infty^p) t^{\frac{p-3}{4p}}}.$$  \hfill (4.31)
Thus we have

$$
\|u(t, \cdot)\|_{L^p} \geq \frac{p - 3}{8p C_3 \left(1 + 3C_{1, \infty}^p\right)} \left(T_{\text{max}} - t\right)^{\frac{p - 3}{2p}}, \quad \forall t \in [0, T_{\text{max}}).
$$

(4.32)

Theorem 1.4 is proved.

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