Optimal Control of Timed Event Graphs with Resource Sharing and Output-Reference Update

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1. INTRODUCTION

Timed event graphs (TEGs) constitute a subclass of timed Petri nets, being characterized by the fact that each place has precisely one upstream and one downstream transition and all arcs have weight 1. They are well suited to model timed discrete-event systems exhibiting synchronization and delay phenomena. In some idempotent semirings, like the max-plus and min-plus algebras, it is possible to represent the dynamic behavior of TEGs by linear models (see Baccelli et al. (1992) for a thorough coverage), which can serve as a basis for performance evaluation as well as for control. In this context, optimal control normally refers to a just-in-time philosophy: given an output-reference specifying, say, a desired production schedule, the aim is to determine the latest possible way to fire the input transitions while guaranteeing that the output ones fire not later than required. In industrial applications, for example, this amounts to satisfying customer demand while minimizing internal stocks. For a tutorial introduction to this control framework, the reader may refer to Hardouin et al. (2018).

In some applications, it may be necessary to update the reference for the system’s output during run-time, for instance when customer demand is increased and a new production objective must be considered. In Menguy et al. (2000), a strategy has been presented to optimally update the input in face of such changes in the output-reference. Systems of practical interest often involve limited resources that are shared among different users (subsystems). As examples, one can think of a railway network where single-track segments are used by multiple trains, or of computational tasks competing for the use of a fixed number of processors. TEGs do not allow for concurrency or choice and hence are inapt to model such resource-sharing phenomena. Overcoming this limitation has motivated several efforts in the literature. In Corrêa et al. (2009), constraints due to resource sharing are translated into additional inequalities in the system model. Addad et al. (2012) model conflicting TEGs by max-plus time-varying equations; the models are restricted to safe conflict places. Boussahel et al. (2016) relax the safety hypothesis on the conflict places and study cycle time evaluation on conflicting TEGs with multiple shared resources. In Moradi et al. (2017), the modeling and control of a number of TEGs that share multiple resources is addressed. Obviously, because of resource sharing, the overall system is no longer a TEG. Under a prespecified priority policy, the authors show how to compute the optimal (just-in-time) input for each subsystem with respect to its individual output-reference.

In this paper, we propose a formal method to obtain the optimal control inputs in face of changes in the output-references for TEGs that share resources under a given priority policy, thus merging the results from Menguy et al. (2000) with those of Moradi et al. (2017). To the best of our knowledge, this problem has not been previously handled in the literature. Prospective applications include emergency call centers (as studied, e.g., in Allamigeon et al. (2015)), where the arrival of high-priority calls may render it necessary to reschedule the answers to lower-priority ones. We consider a set of TEGs operating under optimal schedules with respect to their individual output-references and to the priority policy; supposing the output-reference of one or more of the subsystems is updated during run-time, we show how to optimally update all their inputs so that their outputs are as close as possible to the corresponding new references and the priority policy is still observed. In case the performance limitation of the

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subsystems, combined with the limited availability of the resources, make it impossible to respect some of the new references, we also provide the optimal way to relax such references so that the ultimately obtained inputs lead to tracking them as closely as possible.

The examples presented along this paper serve solely the purpose of illustrating and helping elucidating the results. Due to space limitations, we do not present a more comprehensive example. The proposed method can, however, be applied to larger, more general systems of practical relevance (see Section 5.3 for further comments).

The paper is organized as follows. Section 2 summarizes well-known facts on idempotent semirings. In Section 3, we adapt existing results on the control of TEGs with output-reference update to the idempotent semiring used in this paper. Section 4 provides an overview of previous results on modeling and control of TEGs with shared resources. The major purpose of these three sections is making the paper as self-contained as possible. In Section 5, the main contributions of the paper are presented; namely, we formulate and solve the problem of determining the optimal control inputs for TEGs with shared resources in face of changes in the output-references. Section 6 presents the conclusions and final remarks.

2. PRELIMINARIES

In this section, we present a summary of some basic definitions and results on idempotent semirings and timed event graphs; for an exhaustive discussion, the reader may refer to Baccelli et al. (1992). We also touch on some topics from residuation theory and control of TEGs (see Blyth and Janowitz (1972) and Hardouin et al. (2018), respectively).

2.1 Idempotent semirings

An idempotent semiring \( \mathcal{D} \) is a set \( \mathcal{D} \) endowed with two binary operations, denoted \( \oplus \) (sum) and \( \otimes \) (product), such that: \( \oplus \) is associative, commutative, idempotent (i.e., \( (\forall a \in \mathcal{D}) a \oplus a = a \)), and has a neutral (zero) element, denoted \( \varepsilon \); \( \otimes \) is associative, distributes over \( \oplus \), and has a neutral (unit) element, denoted \( e \); the element \( \varepsilon \) is absorbing for \( \otimes \) (i.e., \( (\forall a \in \mathcal{D}) a \otimes \varepsilon = \varepsilon \)). As in conventional algebra, the product symbol \( \otimes \) is often omitted. An order relation can be defined over \( \mathcal{D} \) by

\[
(\forall a, b \in \mathcal{D}) \quad a \preceq b \iff a \oplus b = b.
\]

Note that \( \varepsilon \) is the bottom element of \( \mathcal{D} \), as \( (\forall a \in \mathcal{D}) \varepsilon \preceq a \).

An idempotent semiring \( \mathcal{D} \) is complete if it is closed for infinite sums and if the product distributes over infinite sums. For a complete idempotent semiring, the top element is defined as \( \top = \bigoplus_{x \in \mathcal{D}} x \), and the greatest lower bound operation, denoted \( \land \), by

\[
(\forall a, b \in \mathcal{D}) \quad a \land b = \bigoplus_{x \land y \leq b} x.
\]

\( \land \) is associative, commutative, and idempotent, and we have \( a \land b = b \iff a \preceq b \iff a \land b = a \).

Example 1. The set \( \mathbb{Z}^+ \) of \( \mathbb{Z} \cup \{ -\infty, +\infty \} \), with the minimum operation as \( \min \) and conventional addition as \( \oplus \), forms a complete idempotent semiring called min-plus algebra, denoted \( \mathbb{Z}_{\min} \), in which \( \varepsilon = +\infty \), \( e = 0 \), and \( \top = -\infty \). Note that in \( \mathbb{Z}_{\min} \) we have \( 2 \oplus 5 = 2 \), so \( 5 \leq 2 \); the order is reversed with respect to the conventional order over \( \mathbb{Z} \).

A mapping \( \Pi : \mathcal{D} \to \mathcal{C} \), with \( \mathcal{D} \) and \( \mathcal{C} \) two idempotent semirings, is isitone if \( (\forall a, b \in \mathcal{D}) a \preceq b \Rightarrow \Pi(a) \preceq \Pi(b) \).

Remark 2. The composition of two isotone mappings is isitone.

Remark 3. Let \( \Pi \) be an isitone mapping over a complete idempotent semiring \( \mathcal{D} \), and let \( Y = \{ x \in \mathcal{D} \mid \Pi(x) = x \} \) be the set of fixed points of \( \Pi \). \( \bigoplus_{y \in Y} \Pi(y) \) is the least (resp. greatest) fixed point of \( \Pi \).

Algorithms exist (e.g., Hardouin et al. (2018)) which allow to compute, in a finite number of steps, the least and greatest fixed points of isotone mappings over complete idempotent semirings, provided such fixed points are finite.

In a complete idempotent semiring \( \mathcal{D} \), the Kleene star operator on \( a \in \mathcal{D} \) is defined as \( a^* = \bigoplus_{i \geq 0} a^i \), with \( a^0 = e \).

Remark 4. The implicit equation \( x = ax \ominus b \) over a complete idempotent semiring admits \( x = a^*b \) as least solution (see Baccelli et al. (1992)).

2.2 Semirings of formal power series

Let \( s = \{ s(t) \}_{t \in \mathbb{Z}} \) be a sequence over \( \mathbb{Z}_{\min} \). The \( \delta \)-transform of \( s \) is a formal power series in \( \delta \) with coefficients in \( \mathbb{Z}_{\min} \) and exponents in \( \mathbb{Z} \), defined by

\[
s = \bigoplus_{t \in \mathbb{Z}} s(t) \delta^t.
\]

We denote both the sequence and its \( \delta \)-transform by the same symbol, as no ambiguity will occur. Since

\[
s \otimes \delta = \bigoplus_{t \in \mathbb{Z}} s(t) \otimes \delta^{t+1} = \bigoplus_{t \in \mathbb{Z}} s(t - 1) \otimes \delta^t,
\]

multiplication by \( \delta \) can be seen as a backward shift operation.

Definition 5. The set of formal power series in \( \delta \) with coefficients in \( \mathbb{Z}_{\min} \) and exponents in \( \mathbb{Z} \), with addition and multiplication defined by

\[
\begin{align*}
s &+ s' = \bigoplus_{t \in \mathbb{Z}} (s(t) \otimes s'(t)) \delta^t, \\
s &\otimes s' = \bigoplus_{t \in \mathbb{Z}} \left( \bigoplus_{\tau \in \mathbb{Z}} (s(\tau) \otimes s'(t - \tau)) \right) \delta^t,
\end{align*}
\]

is a complete idempotent semiring, denoted \( \mathbb{Z}_{\min}[\delta] \). Note that the order in \( \mathbb{Z}_{\min}[\delta] \) is induced by the order in \( \mathbb{Z}_{\min} \), i.e., \( s \preceq s' \iff (\forall t \in \mathbb{Z}) s(t) \preceq s'(t) \).

In this paper we will use sequences to represent the number of firings of transitions in TEGs, so that, e.g., \( s(t) \) represents the accumulated number of firings of a transition up to time \( t \). Such sequences are clearly nonincreasing (in the order of \( \mathbb{Z}_{\min} \)), meaning their \( \delta \)-transforms obey \( s(t - 1) \geq s(t) \) for all \( t \). We will henceforth refer to such series \( s \) as counters.

Definition 6. The set of counters (i.e., nonincreasing power series) in \( \mathbb{Z}_{\min}[\delta] \) is a complete idempotent semiring, named \( \mathbb{Z}_{\min}[s, \delta] \), with zero element \( s_0 \) given by \( s_0(t) = \varepsilon \) for all \( t \), unit element \( s_e \) given by \( s_e(t) = e \) for \( t \leq 0 \) and
$s_c(t) = \varepsilon$ for $t > 0$, and top element $s_\top$ given by $s_\top(t) = \top$ for all $t$. We will denote this semiring by $\Sigma$, for brevity.

Counters can be represented compactly by omitting terms $s(t)\delta^i$ whenever $s(t) = s(t + 1)$. For example, a counter $s$ with $s(t) = e$ for $t \leq 3$, $s(t) = 1$ for $4 \leq t \leq 7$, $s(t) = 3$ for $8 \leq t \leq 12$, and $s(t) = 6$ for $t \geq 13$ can be written $s = e\delta^0 \oplus 1\delta^1 \oplus 3\delta^2 \oplus 6\delta^3 \oplus 6\delta^{+\infty}$.

### 2.3 TEG models in idempotent semirings

Timed event graphs (TEGs) are timed Petri nets in which each place has exactly one upstream and one downstream transition and all arcs have weight 1. With each place $p$ is associated a holding time, representing the minimum time every token needs to spend in $p$ before it can contribute to the firing of its downstream transition. In a TEG, we can distinguish input transitions (those that are not affected by the firing of other transitions), output transitions (those that do not affect the firing of other transitions), and internal transitions (those that are neither input nor output transitions). In this paper, we will limit our discussion to SISO TEGs, i.e., TEGs with only one input and one output transition, which we denote respectively by $u$ and $y$; internal transitions are denoted by $x_i$. An example of a SISO TEG is shown in Fig. 1.

A TEG is said to be operating under the earliest firing rule if every transition fires as soon as it is enabled.

With each transition $x_i$, we associate a sequence $\{x_i(t)\}_{t \geq 0}$, for simplicity denoted by the same symbol, where $x_i(t)$ represents the accumulated number of firings of $x_i$ up to and including time $t$. Similarly, we associate sequences $\{u(t)\}_{t \geq 0}$ and $\{y(t)\}_{t \geq 0}$ with transitions $u$ and $y$, respectively. In $\mathbb{Z}_{\text{min}}$, the number of firings of transition $x_i$ of the TEG from Fig. 1 follows, under the earliest firing rule,

$$(\forall t \in \mathbb{Z}_{\text{min}}) \quad x_i(t) = u(t) \oplus 2\delta^2 x_i(t - 2) ,$$

which, through the $\delta$-transform, can be expressed in $\Sigma$ as

$$x_i = u \oplus 2\delta^2 x_i .$$

We can obtain similar relations for $x_2$ and $y$ and, defining the vector $x = [x_1 x_2]$, write

$$x = \begin{bmatrix} \varepsilon & 2\delta^2 \\ \varepsilon & \varepsilon \varepsilon \delta^0 \end{bmatrix} x \oplus \begin{bmatrix} \varepsilon & \varepsilon \delta^0 \end{bmatrix} u ,$$

$$y = \begin{bmatrix} \varepsilon & \varepsilon \delta^0 \end{bmatrix} x .$$

In general, a TEG can be described by implicit equations over $\Sigma$ of the form

$$x = Ax \oplus Bu ,$$

$$y = Cx .$$

From Remark 4, the least solution of (1) is given by

$$y = CA^*Bu ,$$

where $G = CA^*Bu$ is the transfer function of the system. For instance, for the system from Fig. 1 we obtain the (scalar) transfer function $G = e\delta^3(2\delta^3)^*$.

### 2.4 Residuation theory

Residuation theory provides, under certain conditions, greatest (resp. least) solutions to inequalities such as $f(x) \preceq b$ (resp. $f(x) \succeq b$).

### 2.5 Optimal control of TEGs

Assume that a TEG to be controlled is modeled by equations like (1) and that an output-reference $z$ is given. Under the just-in-time paradigm, we aim at firing the input transition $u$ the least possible number of times while guaranteeing that the output transition $y$ fires, by each time instant, at least as many times as specified by $z$. In other words, we seek the greatest (in the order of $\mathbb{Z}_{\text{min}}$) $u$ such that $y = G \otimes u \succeq z$. Based on (2) and Remark 9, the solution is directly obtained by

$$u_{\text{opt}} = G^{\otimes z} .$$

**Example 11.** From the TEG for Fig. 1, suppose it is required that transition $y$ fires once at time $t = 43$, twice at $t = 47$, and three times at $t = 55$, meaning the accumulated number of firings of $y$ should be $e (= 0)$ for $t \leq 42$, 1 for $43 \leq t \leq 46$, 3 for $47 \leq t \leq 54$, and 6 for $t \geq 55$. This is represented by the output-reference $z = e\delta^{42} \oplus 1\delta^{46} \oplus 3\delta^{54} \oplus 6\delta^{+\infty}$. Applying (3), we get $u_{\text{opt}} = e\delta^{38} \oplus 1\delta^{44} \oplus 2\delta^{51} \oplus 3\delta^{54} \oplus 4\delta^{53} \oplus 5\delta^{54} \oplus 6\delta^{+\infty}$, and the corresponding optimal output is $y_{\text{opt}} = G \otimes u_{\text{opt}} = e\delta^{41} \oplus 1\delta^{51} \oplus 2\delta^{54} \oplus 3\delta^{57} \oplus 4\delta^{54} \oplus 5\delta^{57} \oplus 6\delta^{+\infty}$. One can verify that

![Fig. 1. A SISO TEG, with input $u$ and output $y$.](image1)

![Fig. 2. Optimal schedule obtained in Example 11; the gray bars represent the operation of the system, and the dashed bars are the delays imposed by the resource.](image2)
$y_{opt} \preceq z$. Interpreting the places between $x_1$ and $x_2$ as the operation of the system (top) and a double-capacity resource (bottom), the optimal schedule obtained above can be displayed in a chart as shown in Fig. 2, where each row corresponds to one instance of the resource.

3. OPTIMAL CONTROL OF TEGS WITH OUTPUT-REFERENCE UPDATE

The material of this section is a dual version, adapted to the point of view of counters, of the results from Menguy et al. (2000).

It is plausible to consider that the reference for the output of a system may be updated during run-time, for instance when customer demand is increased and a new production objective must be taken into account. For a system like the one from Example 11, let reference $z$ be updated to a new one, $z'$, at time $T$. The problem at hand is to find the input $u_{opt}$ which optimally tracks $z'$ without, however, changing the inputs given up to time $T$. Define the mapping $r_T : \Sigma \to \Sigma$

$$[r_T(u)](t) = \begin{cases} u(t), & if \ t \leq T; \\ \varepsilon, & if \ t > T. \end{cases} \quad (4)$$

Our objective can then be restated as follows: find the greatest element $u_{opt}$ of the set

$$\mathcal{F} = \{ u \in \Sigma \mid G \otimes u \preceq z' \text{ and } r_T(u) = r_T(u_{opt}) \},$$

where $u_{opt}$ is the optimal input with respect to reference $z'$, computed as in (3). The following theorem provides, given that certain conditions are met, a way to compute this greatest element.

**Theorem 12.** (Menguy et al. (2000)) Let $\mathcal{D}$ and $\mathcal{C}$ be complete idempotent semirings, $f_1, f_2 : \mathcal{D} \to \mathcal{C}$ residuated mappings, and $c_1, c_2 \in \mathcal{C}$. If the set

$$\mathcal{S} = \{ x \in \mathcal{D} \mid f_1(x) \preceq c_1 \text{ and } f_2(x) = c_2 \}$$

is nonempty, we have $\bigoplus_{x \in \mathcal{S}} x = f_1^\#(c_1) \vee f_2^\#(c_2)$. \hfill \(\Box\)

An obvious correspondence between $\mathcal{F}$ and $\mathcal{S}$ can be established by taking $\mathcal{D}$ as $\Sigma$, $f_1$ as $L_G$ (which is well known to be residuated — see Remark 9), $c_1$ as $z'$, $f_2$ as $r_T$, and $c_2$ as $r_T(u_{opt})$.

**Remark 13.** Mapping $r_T$ as defined in (4) is residuated, with

$$[r_T^\#(u)](t) = \begin{cases} u(t), & if \ t \leq T; \\ u(T), & if \ t > T. \end{cases}$$

In fact, $r_T^\#$ is clearly isotone and we have $1$ $r_T \circ r_T^\# = r_T \preceq Id_\Sigma$ and $r_T^\# \circ r_T = r_T^\# \preceq Id_\Sigma$, so the conditions from Theorem 8 are fulfilled. \hfill \(\Box\)

Hence, if set $\mathcal{F}$ is nonempty, Theorem 12 provides the desired solution $u_{opt}$. In general, however, $\mathcal{F}$ can be empty. Considering the set

$$\tilde{\mathcal{F}} = \{ u \in \Sigma \mid r_T(u) = r_T(u_{opt}) \},$$

it is easy to see that

$$y \overset{\text{def}}{=} \bigwedge_{u \in \tilde{\mathcal{F}}} u = r_T(u_{opt}).$$

1. Note that the order $\preceq$ on $\Sigma$ induces an order, for simplicity also denoted $\preceq$, on the set of mappings over $\Sigma$: for any such mappings $\Theta_1, \Theta_2$, one has $\Theta_1 \preceq \Theta_2 \Leftrightarrow (\forall x \in \Sigma) \Theta_1(x) \preceq \Theta_2(x)$.\hfill \(\Box\)

Moreover, $r_T \circ r_T = r_T$ implies $u \in \tilde{\mathcal{F}}$, as $r_T(u) = r_T(r_T(u_{opt})) = r_T(u_{opt})$, and $z$ is isotone (since $L_G$ is residuated), we have

$$F \neq \emptyset \Rightarrow G \otimes u \preceq z'. \quad (4)$$

Since the condition $r_T(u) = r_T(u_{opt})$ cannot be relaxed, in case $G \otimes u \preceq z'$ we must increase $z'$; more precisely, we wish to find the least counter $z'' \succeq z'$ such that

$$F_{\tilde{\mathcal{F}}} = \{ u \in \Sigma \mid G \otimes u \preceq z'' \text{ and } r_T(u) = r_T(u_{opt}) \}$$

is not empty. The following result provides the answer.

**Proposition 14.** The least counter $z'' \succeq z'$ such that $F_{\tilde{\mathcal{F}}} \neq \emptyset$ is $z'' = z' + (G \otimes u)$. \hfill \(\Box\)

**Proof.** For $z'' = z' + (G \otimes u)$, we have $u \in F_{\tilde{\mathcal{F}}}$, therefore $F_{\tilde{\mathcal{F}}} \neq \emptyset$. Hence $z'' \succeq z'$ such that $F_{\tilde{\mathcal{F}}} \neq \emptyset$, and take any $v \in F_{\tilde{\mathcal{F}}}$. Clearly $v \in \tilde{\mathcal{F}}$ and hence $u \preceq v$, as $L_G$ is isotone, we have $G \otimes u \preceq G \otimes v \preceq z''$, implying $z'' = z' + (G \otimes u) \preceq z' + z'' = 2z''$. \hfill \(\Box\)

Applying Theorem 12 and recalling that $r_T^\# \circ r_T = r_T^\#$, we obtain

$$u_{opt} = G^\#(z' + (G \otimes u)) \cup r_T^\#(u_{opt}).$$

Note that in case $F \neq \emptyset$ we have $z'' = z' + (G \otimes u) = z'$.\hfill \(\Box\)

**Example 15.** For the system from Example 11 (Fig. 1) operating according to the optimal input obtained for output-reference $z$, suppose that at time $T = 42$ a new demand is received: three firings of $y$ are now required at $t = 54$ (instead of at $t = 55$). This translates to $z' = 5752 \otimes 1646 \otimes 3533 \otimes 535 \otimes 65^\infty$. In this case one can verify that set $\mathcal{F}$ is empty, so we seek the least $z'' \succeq z'$ such that $F_{\tilde{\mathcal{F}}} \neq \emptyset$, according to Proposition 14. With $u = r_T(u_{opt}) = 5752 \otimes 1646 \otimes 2542 \otimes 5353 \otimes 535 \otimes 65^\infty$, we obtain $z'' = z' + (G \otimes u) = 5752 \otimes 1646 \otimes 3533 \otimes 535 \otimes 65^\infty$, which is the reference we will effectively track. From (5), we get $u_{opt} = 5752 \otimes 1646 \otimes 2542 \otimes 3533 \otimes 4550 \otimes 5551 \otimes 65^\infty$, and hence $u'_{opt} = 5752 \otimes 1646 \otimes 2542 \otimes 3533 \otimes 4550 \otimes 5551 \otimes 65^\infty$. The updated optimal schedule is shown in Fig. 3, to be interpreted as explained in Example 11. \hfill \(\Box\)

4. MODELING AND OPTIMAL CONTROL OF TEGS WITH RESOURCE SHARING

We now turn our attention to systems in which a number of TEGs $S^1, \ldots, S^K$ share a resource, as illustrated in Fig. 4. $H^k$ represents the internal dynamics of $S^K$; $\beta$ may, in general, be a TEG (or, in simple cases, just a single place) describing the capacity of the resource as well as the minimal delay between release and allocation events. Clearly, the overall system is no longer a TEG. For simplicity, let us assume that there is only one shared resource (with arbitrary capacity) and that input transitions ($u^r$) are connected to resource-allocation transitions ($x^r$) via a single place with zero delay and no initial tokens, the same...
Fig. 4. A number of TEGs with a shared resource.

**Fig. 5.** A join and a fork structure.

being true for the connection between resource-release transitions \( x_R^k \) and output transitions \( y^k \). The extension to more general cases is straightforward; for details, the reader is referred to Moradi et al. (2017), on which this section is mainly based.

### 4.1 Modeling of TEGs with shared resources

It is not possible to model systems exhibiting such resource-sharing phenomena by linear equations like (1). In order to express the relationship among counters \( s_1, s_2 \in \Sigma \), written \( s_1 \odot s_2 \), is the counter defined as follows:

\[
(\forall t \in \mathbb{Z}) \quad (s_1 \odot s_2)(t) = s_1(t) \odot s_2(t).
\]

This operation is commutative, distributes over \( \oplus \) and \( \Lambda \), has neutral element \( e^A_{+\infty} \), and \( s_e \) is absorbing for it (i.e., \( (\forall s \in \Sigma) \ s_e \odot s = s_e \)).

Consider a join structure (i.e., a place with two or more incoming transitions) as shown in Fig. 5. At any time instant \( t \), the accumulated number of firings of \( \gamma \) cannot exceed that of \( \lambda_1 \) and \( \lambda_2 \) combined, which translates to \( \lambda_1 \odot \lambda_2 \geq \gamma \).

Similarly, for a fork structure (i.e., a place with two or more outgoing transitions) such as the one shown in Fig. 5, the accumulated number of firings of \( \gamma_1 \) and \( \gamma_2 \) combined can never exceed that of \( \lambda \), meaning \( \lambda \leq \gamma_1 \odot \gamma_2 \).

Generalizing these ideas allows us to write, for the system from Fig. 4,

\[
x_R \odot \cdots \odot x_R \leq \alpha_1 \quad \text{and} \quad \alpha_2 \leq x_A^1 \odot \cdots \odot x_A^K\nonumber
\]

which, combined with \( \beta \odot \alpha_1 \leq \alpha_2 \), leads to

\[
\beta \odot \left( \bigotimes_{k=1}^{K} x_R^k \right) \leq \bigotimes_{k=1}^{K} x_A^k.
\]

### 4.2 Optimal control of TEGs with resource sharing

For a system like the one from Fig. 4, competition for the resource is, in general, going to make it impossible for all subsystems to concurrently follow a just-in-time schedule with respect to their individual output-references. One way to settle the dispute is introducing a priority policy among the subsystems. We henceforth assume, without loss of generality, that subsystem \( S_k^k \) has higher priority than \( S_k^{k+1} \), for all \( k \in \{1, \ldots, K-1\} \). The priority policy is based on a simple rule: for each \( k \in \{2, \ldots, K\} \) and for all \( j \in \{1, \ldots, k-1\} \), \( S_k^k \) cannot interfere with the performance of \( S_j^j \).

Let the input-output behavior of each \( S_k^k \), ignoring all other subsystems, be described by \( y^k = G^k \otimes u^k \) — which, according to the assumptions made above, is equivalent to \( x_R^k = G^k \otimes x_A^k \) — and assume that corresponding references \( z^k \) are given. The subsystem with highest priority, \( S_1^1 \), is free to use the resource at will; therefore, we can effectively neglect all other subsystems and simply compute its optimal input by \( u_1^{\text{opt}} = x_1^{\text{opt}} = G^1 z_1 \) (cf. Section 2.5).

For \( S_2^2 \), we must compute the optimal input under the restriction that the optimal behavior of \( S_1^1 \) is unchanged; based on (6), this means we must respect

\[
\beta \odot (x_1^{\text{opt}} \odot x_2^2) \leq x_1^{\text{opt}} \odot x_2^2. \tag{7}
\]

In fact, we want to determine the greatest \( x_2^2 \) — and thus also the corresponding \( u_2^2 \) — fulfilling both \( G^2 \otimes u_2^2 \leq z_2^2 \) and (7); seeing that (7) implies

\[
x_2^{\text{opt}} \odot x_2^2 \leq \beta (x_2^{\text{opt}} \odot x_2^2), \tag{8}
\]

the following result comes in handy.

**Proposition 17.** (Hardouin et al. (2008)) For any \( a \in \Sigma \), the mapping \( \Pi_a: \Sigma \rightarrow \Sigma, x \mapsto a \odot x \), is residuated. For any \( b \in \Sigma, \Pi_b(x), \) denoted \( b \ominus a \), is the greatest \( x \in \Sigma \) such that \( a \odot x \leq b \).

From Proposition 17, inequality (8) leads to

\[
x_2^{\text{opt}} \leq (\beta (x_2^{\text{opt}} \odot x_2^2)) \ominus x_2^{\text{opt}}.
\]

which, combined with \( x_2^2 = G^2 \otimes x_2^2 \leq z_2^2 \), implies

\[
x_2^{\text{opt}} \leq G^2 k \left[ (\beta (x_2^{\text{opt}} \odot x_2^2)) \ominus x_2^{\text{opt}} \right] \land G^2 k z_2^2. \tag{9}
\]

The greatest \( x_2^2 \) satisfying (9), \( x_2^{\text{opt}} \), is the greatest fixed point (provided it exists) of the mapping \( \Phi^2: \Sigma \rightarrow \Sigma, \Phi^2(x_2^2) = G^2 k \left[ (\beta (x_2^{\text{opt}} \odot x_2^2)) \ominus x_2^{\text{opt}} \right] \land G^2 k z_2^2 \). \tag{10}

As \( \Phi^2 \) can be verified to be isotope (see Remark 2), Remark 3 ensures the existence of its greatest fixed point, which yields the desired optimal solution \( x_2^{\text{opt}} = (w_2^{\text{opt}}) \).

Using the same procedure, we obtain, for each \( k \),

\[
x_A^k \leq G^k k \left[ (\beta (\bigotimes_{i=1}^{k-1} x_i^{\text{opt}}) \odot x_A^k) \ominus \bigotimes_{i=1}^{k-1} x_i^{\text{opt}} \right] \land G^k k z^k \tag{11}
\]

and, defining a mapping \( \Phi^k \) by analogy with (10), its greatest fixed point provides \( x_k^{\text{opt}} \) and, therefore, also \( y_k^{\text{opt}} \).

**Example 18.** Consider the system from Fig. 7, where subsystems \( S_1^1 \) and \( S_2^2 \) share a resource with capacity 2. \( S_1^1 \), including the resource and ignoring \( S_2^2 \), is the system from...
Example 11, whose transfer function is \( G^1 = \varepsilon \delta^{3}(255)^s \) (cf. Section 2.3). For \( S^2 \), we obtain \( G^2 = \varepsilon \delta^{3}(255)^s \). In this example, \( \beta = 255^3 \). The references \( z^1 = \varepsilon \delta^{42} \oplus 1 \delta^{36} \oplus 3 \delta^{42} \oplus 6 \delta^{45} \oplus 5 \delta^{54} \oplus 6 \delta^{45} \oplus 5 \delta^{54} \oplus 3 \delta^{45} \oplus 6 \delta^{45} \oplus 5 \delta^{54} \) are given. As \( S^1 \) has the highest priority, we can simply compute \( u^1_{\text{opt}} = x^1_{\text{opt}} \), which is the same counter as \( u_{\text{opt}} \) obtained in Example 11. To determine \( x^2_{\text{opt}} \), we follow the procedure described in this section. Computing the greatest fixed point of \( \Phi^2 \) as in (10), we get \( x^2_{\text{opt}} = \varepsilon \delta^{27} \oplus 1 \delta^{31} \oplus 2 \delta^{34} \oplus 3 \delta^{45} \oplus 4 \delta^{54} \oplus 5 \delta^{63} \oplus 6 \delta^{63} \oplus 5 \delta^{63} \oplus 3 \delta^{63} \oplus 2 \delta^{63} \oplus 1 \delta^{63} \) (\( = u^2_{\text{opt}} \)), as shown in Fig. 6. Because the availability of the resource for \( S^2 \) is subject to the operation of \( S^1 \), the firings of \( y^2 \) have to be given considerably earlier than \( y^1 \); this, however, the latest they can be given so as to respect \( z^2 \) without interfering with \( S^1 \). ⊗

4.3 Supplementary remarks

**Proposition 19.** (Adapted from Hardouin et al. (2008)) Let \( \hat{\Sigma} = \{ s \in \Sigma \mid (s \in \Sigma) \} \). For any \( a \in \hat{\Sigma} \), the mapping \( \Pi_a : \hat{\Sigma} \rightarrow \Sigma, x \mapsto a \circ x \), is dually residuated. For any \( b \in \hat{\Sigma} \), \( \Pi_b(a) \), denoted \( b \circ a \), is the least \( x \in \hat{\Sigma} \) such that \( a \circ x \geq b \).

**Proof.** For an arbitrary \( a \in \hat{\Sigma} \), we have \( (\forall t \in \hat{\Sigma}) a(t) \otimes \top = \top \), therefore \( \Pi_a(s_\top) = a \circ s_\top = s_\top \). Moreover, since \( \circ \) distributes over \( \wedge \) (cf. Def. 16), for any \( A \subseteq \Sigma \) it holds that \( \Pi_a(\bigwedge_{x \in A} x) = a \circ (\bigwedge_{x \in A} x) = \bigwedge_{x \in A} (a \circ x) = \bigwedge_{x \in A} \Pi_a(x) \). The result then follows from Theorem 10. ⊓⊔

**Remark 20.** (Hardouin et al. (2008)) Given two counters \( x_1, x_2 \in \Sigma \), the series \( s \in \Sigma^\infty \) defined by \( (\forall t \in \hat{\Sigma}) s(t) = x_1(t) \oplus x_2(t) \) is not necessarily a counter; \( x_1 \otimes x_2 \) is the least counter greater than or equal to \( x_1 \oplus x_2 \).

5. OPTIMAL CONTROL OF TEGS WITH RESOURCE SHARING AND OUTPUT-REFERENCE UPDATE

In this section, as the main the contribution of this paper, we incorporate the ideas discussed in Section 3 to the class of systems studied in Section 4 by showing how to determine the optimal (just-in-time) control inputs in face of changes in the output-references for TEGs that share resources under a given priority policy. We again emphasize that, in this setting, the overall system is not a TEG. The assumptions made in Section 4 are still in place.
Proposition 23. \( \bigwedge_{T_k} = \bigwedge \{ x \in \Sigma \mid T^k(x) = x \} \in \tilde{F}^k \).

**Proof.** Any \( x_k^k \in \Sigma \) such that \( T^k(x_k^k) = x_k^k \) satisfies
\[
(\beta \odot (H^k_R \odot (G^k \odot x_k^k) \odot L^k_R)) \odot^b (H^k_A \odot L^k_A) \preceq x_k^k
\]
and, by consequence, also (\( \ast \)). According to Remark 3, \( \bigwedge_{T_k} \) is a fixed point of \( T_k \), therefore (\( \ast \)) holds for \( x_k^k = \bigwedge_{T_k} \), and it suffices to prove that \( r_T(\bigwedge_{T_k}) = r_T(x_k^k) \).

\( \bigwedge_{T_k} \) being a fixed point of \( T_k \) implies \( \bigwedge_{T_k} \preceq r_T(x_k^k) \), so
\[
r_T(\bigwedge_{T_k}) \preceq r_T(r_T(x_k^k)) = r_T(x_k^k) \text{.}
\]

Moreover, \( r_T^f(x_k^k) \) is a fixed point of \( T_k \), as can be seen from the following argument. Since we assume \( x_k^k \) is to be given for each \( i \in \{1, \ldots, k-1\} \), according to (\( \ast \)) we know \( x_k^{(k-1)} \) fulfills
\[
(\beta \odot (H^k_R \odot (G^{(k-1)} \odot x_{k}^{(k-1)})) \odot L^{(k-1)}_A) \preceq (H^k_A \odot r_T^f(x_k^{(k-1)}) \odot L^k_A)
\]
which, in turn, implies
\[
(\beta \odot (H^k_R \odot (G^k \odot x_k^k) \odot L^k_R)) \odot^b (H^k_A \odot L^k_A) \preceq r_T^f(x_k^k) \text{.}
\]

This, together with the fact that \( r_T^f(x_k^k) \preceq r_T(x_k^k) \), implies \( T^k(r_T^f(x_k^k)) = r_T^f(x_k^k) \). Hence, \( \bigwedge_{T_k} \preceq r_T^f(x_k^k) \) and, as \( r_T \) is isotope and \( r_T \circ r_T = r_T \), we have \( r_T(\bigwedge_{T_k}) \preceq r_T(r_T^f(x_k^k)) = r_T(x_k^k) \), which concludes the proof. \( \square \)

As clearly \( \tilde{F}^k \subseteq \{ x \in \Sigma \mid T^k(x) = x \} \), from Proposition 23 we conclude that
\[
\begin{align*}
x_k^k \text{ def } & = \bigwedge_{x \in \tilde{F}^k} x = \bigwedge_{T_k} \text{.} \\
\end{align*}
\]

**Proposition 24.** The least counter \( z_k^{\wedge} \) is such that \( F_{z_k^{\wedge}} \neq \emptyset \) is \( z_k^{k'} = z_k^{\wedge} \odot (G^k \odot x_k^k) \).

**Proof.** Taking \( z_k^{\wedge} = z_k^{\wedge} \odot (G^k \odot x_k^k) \), it can be readily checked that \( z_k^k \in F_{z_k^{\wedge}} \), therefore \( F_{z_k^{\wedge}} \neq \emptyset \); the proof then proceeds by direct analogy with that of Proposition 14. \( \square \)

Now, define the mapping \( \Psi^k : \Sigma \to \Sigma \),
\[
\Psi^k(x) = z_k^{\wedge} \wedge \big[ (\beta \odot (H^k_A \odot x \odot L^k_A)) \odot^b (H^k_R \odot L^k_R) \big] \text{,}
\]
with \( z_k^{\wedge} = z_k^{\wedge} \odot G^k \odot x_k^k \). Then, we can write
\[
F_{z_k^{\wedge}} = \{ x \in \Sigma \mid G^k \odot x \preceq \Psi^k(x) \text{ and } r_T(x) = r_T(x_k^k) \} \text{.}
\]

Based on Theorem 12 and recalling that \( r_T^f \circ r_T = r_T^f \), we conclude that \( x_k^k \) is the greatest \( x \in \Sigma \) such that \( x \leq G^k \odot \Psi^k(x) \), and, equivalently, \( x = G^k \odot \Psi^k(x) \) \( \wedge \)
Fig. 8. Updated optimal schedules for $S^1$ and $S^2$ obtained in Example 25; the gray and black bars represent the operation of $S^1$ and $S^2$, respectively, whereas the dashed bars are the delays imposed by the resource.

$$r^*_{1}(x^*_{A_{1\text{opt}}}) \wedge x.$$ Finally, $x^*_{A_{2\text{opt}}}$ can be computed as the greatest fixed point of the (isotone) mapping $\Gamma^k : \Sigma \rightarrow \Sigma$,

$$\Gamma^k(x) = G^k \Psi^k(x) \wedge r^*_{1}(x^*_{A_{1\text{opt}}}) \wedge x. \quad (14)$$

**Example 25.** Consider the system from Example 18 (Fig. 7), with $S^1$ and $S^2$ both operating under the obtained optimal schedules. Now, suppose new references $z^1 = e^3_{32} \oplus 13_{41} \oplus 28_{46} \oplus 35_{54} \oplus 66_{5\infty}$ and $z^2 = z^2$ are received at time $T = 33$. Observing the priority policy, we start by updating the input of $S^1$. In this case, we have $L^1_A = r^*_{1}(x^*_{A_{1\text{opt}}}) = e^3_{27} \oplus 13_{31} \oplus 28_{5\infty}$ and $L^2_A = G^2 \oplus r^*_{1}(x^*_{A_{2\text{opt}}}) = e^{32}_{32} \oplus 13^{36} \oplus 28_{3\infty}$. Defining $\mathcal{F}^1$ as in (11), one can check that in this case $\mathcal{F}^1 \neq \emptyset$; then, $z^1_{\text{new}} = z^1_{\text{old}}$ and we can directly look for the greatest fixed point of $\Gamma^1$ (defined as in (14)), which is $x^*_{A_{1\text{opt}}} = e^3_{27} \oplus 13_{31} \oplus 28_{5\infty}$ and $d^2_{31} \oplus 35_{54} \oplus 66_{5\infty}(= u_{opt})$. Then, $x^*_{A_{2\text{opt}}} = e_{31}^{31} \oplus 13_{41} \oplus 28_{46} \oplus 35_{49} \oplus 45_{54} \oplus 66_{5\infty}(= y_{opt})$. We now proceed to update $x^2$: with $\mathcal{H}^2_A = x^*_{A_{2\text{opt}}}$ and $\mathcal{H}^2_R = x^*_{R_{2\text{opt}}}$ in this case $\mathcal{F}^2 = \emptyset$, so we look for the least $z^2_{\text{new}} \leq z^2_{\text{old}}$ such that $z^2_{\text{new}} \neq \emptyset$. According to (13), we obtain $z^2_{\text{new}} = e^2_{27} \oplus 13_{31} \oplus 28_{5\infty}$ and, from Proposition 24, $z^2_{\text{new}} = z^2_{\text{old}} \oplus (G^2 \oplus x^*_{A_{2\text{opt}}}) = e^3_{27} \oplus 15_{40} \oplus 28_{51} \oplus 35_{5\infty}$. Computing the greatest fixed point of $\Gamma^2$ then yields $x^*_{A_{2\text{opt}}} = e^2_{27} \oplus 13_{31} \oplus 28_{5\infty} \oplus 35_{5\infty} \oplus 66_{5\infty}(= u_{opt})$ and $x^*_{R_{2\text{opt}}} = e^{32}_{32} \oplus 15_{40} \oplus 28_{51} \oplus 35_{5\infty}(= y_{opt})$. These updated optimal schedules are shown in Fig. 8. See that $x^*_{A_{2\text{opt}}} \neq x^*_{A_{2\text{opt}}}$ even though $z^2_{\text{new}} = z^2_{\text{old}}$ (cf. Remark 22).

5.3 Extension to more general cases

The results in this section are developed under the assumption made in the beginning of Section 4. However, they can be readily extended to more general cases; namely, the same method can be applied to the case of multiple shared resources, and the simplifying assumptions on the connection between input and internal transitions, as well as between internal and output ones, can be dropped. For such generalizations in the framework of TEGs with shared resources but without output-reference update (Section 4), the reader may consult Moradi et al. (2017). The explicit generalization of the main results from this paper and a comprehensive case study are subjects for future work.

6. CONCLUSION

This paper solves the problem of ensuring that a number of TEGs competing for the use of shared resources operate optimally (in a just-in-time sense) even in face of changes in their output-references. The proposed method assumes a prespecified priority policy on the component TEGs, and the optimal inputs are computed under the rule that the operation of lower-priority subsystems cannot interfere with the performance of higher-priority ones. We also study the case in which the limited availability of the resources renders it impossible to respect the updated output-reference for one or more of the subsystems. In this case, we show how to relax such references in an optimal way so that the ultimately obtained inputs lead to tracking them as closely as possible.

The results are illustrated through simple examples; exploiting the generality of the method and applying it to a larger, more practically-motivated case study, as well as investigating the flexibilization of the priority policy, are subjects for future work.

**REFERENCES**

Aded, B., Amari, S., and Lesage, J.-J. (2012). Networked conflicting timed event graphs representation in (max,+) algebra. *Discrete Event Dynamic Systems*, 22(4), 429–449.

Allamigeon, X., Boff, V., and Gaubert, S. (2015). Performance evaluation of an emergency call center: tropical polynomial systems applied to timed petri nets. In *Formal Modeling and Analysis of Timed Systems (FORMATS 2015)*. Springer.

Baccelli, F., Cohen, G., Olser, G.J., and Quadrat, J.P. (2009). *Synchronization and Linearity: an Algebra for Discrete Event Systems*. Wiley.

Blyth, T. and Janowitz, M. (1972). *Residuation Theory*. Pergamon press.

Boussahel, W., Amari, S., and Kara, R. (2016). Analytic evaluation of the cycle time on networked conflicting timed event graphs in the (max,+) algebra. *Discrete Event Dynamic Systems*, 26(4), 561–581.

Corrêa, A., Abbas-Turki, A., Bouyekhf, R., and El Moudni, A. (2009). A dioid model for invariable resource sharing problems. *IEEE Transactions on Systems, Man, and Cybernetics*, 39(4), 770–781.

Hardouin, L., Cottenceau, B., Lagrange, S., and Le Corrone, E. (2008). Performance analysis of linear systems over semiring with additive inputs. In *9th International Workshop on Discrete Event Systems (WODES)*. Göteborg, Sweden.

Hardouin, L., Cottenceau, B., Shang, Y., and Raisch, J. (2018). Control and state estimation for max-plus linear systems. *Foundations and Trends in Systems and Control*, 6(1), 1–116.

Menguy, E., Boinond, J.L., Hardouin, L., and Ferrer, J.L. (2000). Just-in-time control of timed event graphs: update of reference input, presence of uncontrollable input. *IEEE Transactions on Automatic Control*, 45(11), 2155–2159.

Moradi, S., Hardouin, L., and Raisch, J. (2017). Optimal control of a class of timed discrete event systems with shared resources, an approach based on the hadamard product of series in dioids. In *56th IEEE Conference on Decision and Control (CDC)*. Melbourne, Australia.