Extraction of Resonances from Meson-Nucleon Reactions*

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Abstract

We present a pedagogical study of the commonly employed Speed-Plot (SP) and Time-delay (TD) methods for extracting the resonance parameters from the data of two particle coupled-channels reactions. Within several exactly solvable models, it is found that these two methods find poles on different Riemann sheets and are not always valid. We then develop an analytic continuation method for extracting nucleon resonances within a dynamical coupled-channel formulation of $\pi N$ and $\gamma N$ reactions. The main focus of this paper is on resolving the complications due to the coupling with the unstable $\pi \Delta, \rho N, \sigma N$ channels which decay into $\pi \pi N$ states. By using the results from the considered exactly solvable models, explicit numerical procedures are presented and verified. As a first application of the developed analytic continuation method, we present the nucleon resonances in some partial waves extracted within a recently developed coupled-channels model of $\pi N$ reactions. The results from this realistic $\pi N$ model, which includes $\pi N$, $\eta N$, $\pi \Delta$, $\rho N$, and $\sigma N$ channels, also show that the simple pole parametrization of the resonant propagator using the poles extracted from SP and TD methods works poorly.

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I. INTRODUCTION

The excited baryon and meson states couple strongly with the continuum states. Thus they are identified with the resonance states in hadron reactions. The spectra and decay widths of the hadron resonances reveal the role of the confinement and chiral symmetry of QCD in the non-perturbative region. Therefore, the extraction of the basic resonance parameters from reaction data is one of the important tasks in hadron physics. Ideally, it should involve the following steps:

1. Perform complete measurements of all independent observables of the reactions considered. For example, for pseudo-scalar meson photoproduction reactions one needs to measure 8 observables\[1\] : differential cross sections, three single polarizations $\Sigma$, $T$ and $P$, and four double polarizations $G$, $H$, $E$, and $F$.

2. Extract the partial-wave amplitudes (PWA) from the data. Here we need to solve a non-trivial practical problem since all observables are bi-linear combinations of PWA; i.e. $\sigma \sim f_{LS}^* f_{LS}$.

3. Extract the resonance parameters from the extracted PWA. Here the often employed methods are based on the Breit-Wigner form\[2\], Speed-Plot method of Hoehler\[3, 4\], and Time-Delay method of Wigner\[5, 6\]. A more sophisticated and rigorous method is to use the dispersion relations, K-matrix, and dynamical model to analytically continue the PWA to the complex energy plane on which the resonance poles and residues are determined. Extensive works based on these three models are reviewed in Ref.[7].

In reality, we do not have complete measurements for practically all meson-nucleon reactions. Even if the measurements are complete, the step 2 requires some model assumptions to solve the inverse bi-linear problem in extracting PWA. This model dependence must be taken into account in interpreting the extracted resonance parameters.

In this work, we focus on the step 3 in conjunction with the recent efforts in extracting the nucleon resonances from very extensive and high quality data of meson production reactions, as reviewed in Ref.[7]. The nucleon resonances $N^*, \Delta$ listed by the Particle Data Group\[8\] are mainly from the analysis of $\pi N$ scattering and pion photoproduction reactions. The Speed-Plot and Time-Delay methods are most often used in these analyses since they only require the PWA determined in the step 2. The purpose of this work is to examine the extent to which these two methods are valid and to develop an analytic continuation method within a recently developed dynamical model\[9\] of meson production reactions in the nucleon resonance region.

It is useful to first briefly recall how the resonances are defined in textbooks. By analytic continuation, the scattering T-matrix can be defined on the complex energy plane. Its analytical structure is well studied\[10, 11, 12, 13, 14\] for the non-relativistic two-body scattering. For the single-channel case, the scattering T-matrix is a single-valued function of momentum $p$ on the complex momentum $p$-plane, but is a double-valued function of energy $E$ on the complex energy $E$-plane because of the quadratic relation $p = |2mE|^{1/2} e^{i\phi_E/2}$. Therefore the complex $E$-plane is composed of two Riemann sheets. The physical ($p$) sheet is defined by specifying the range of phase $0 \leq \phi_E \leq 2\pi$, and the un-physical ($u$) sheet by $2\pi \leq \phi_E \leq 4\pi$. As illustrated in Fig. 1 the shaded area with Im$p > 0$ of the upper part of Fig. 1(a) corresponds to the physical $E$-sheet shown in Fig. 1(b). Similarly, the unphysical $E$-sheet shown in Fig. 1(c) corresponds to the Im$p < 0$ area in the lower part of Fig. 1(a).
FIG. 1: The complex momentum $p$-plane ((a)) and its corresponding complex energy $E$-plane which has a physical sheet ((b)) and unphysical sheet ((c)). Their correspondence is indicated by the same number. Solid squares (circles) represent the bound state (resonance) poles.

On the physical sheet, the only possible singularities are on the real $E$-axis : the bound state poles (solid square in Fig. 1(b)) below the threshold energy $E_{th}$ and the unitarity cut from $E_{th}$ to infinity. On the unphysical sheet, a pole (solid circle in Fig. 1(c)) on the lower half plane and $\text{Re } E_{pole} > E_{th}$ corresponds to a resonance. From unitarity and analyticity of S-matrix, each resonance pole has an accompanied pole, called conjugate pole, which exists on the upper half of the un-physical sheet. A resonance pole is due to the mechanism : an unstable system is formed and decay subsequently during the collision. The mathematical details of this interpretation can be found in textbooks, such as in chapter 8 of Goldberger and Watson[15].

For multi-channel case, the analytic structure of the scattering T-matrix becomes more complex [11, 12, 13, 14]. We postpone the discussion on this until section III where a two-channel Breit-Wigner form of scattering amplitude will be used to give a pedagogical explanation.

The essential point of the Time-Delay and Speed-Plot methods is that the resonance poles discussed above can be determined from the partial-wave amplitudes defined on the physical real energies. The concept of time delay was originally introduced by Eisenbud[5] and discussed by Wigner[6], Dalitz and Moorehouse[16], and Nussenzveig[17]. It is defined as the difference between the time in which a wave packet passes through an interaction region and the time spent by a free wave packet passing though the same distance. It was generalized to the time delay matrix in multi-channel problems and discussed further in terms of a lifetime matrix by Smith [18]. Kelkar et al.[19, 20] applied this multi-channel formulation to develop a practical Time-Delay method to extract hadron resonances.

The Speed-Plot method was developed by Hoehler[3, 4] to extract the nucleon resonances from the $\pi N$ partial-wave amplitudes. It is also based on the concept of time delay, but he identified resonances by ”speed” which is the absolute value of the energy derivative of the scattering amplitude. The speed is always positive by its definition while the time delay can be negative and becomes ”time advance” [21].
It is generally assumed that both the Speed-Plot and Time-Delay methods give approximate positions of the resonance poles which are rigorously defined by the analytic continuation of scattering amplitudes. However their validity is not clear at all. Moreover it is not clear which Riemann sheet the pole positions extracted from using these methods belong to. In this paper we analyze several exactly solvable models to clarify these questions. In particular, we find that these two methods give poles on different Riemann sheets.

We then develop an analytic continuation method for extracting the resonance parameters within a recently developed Hamiltonian formulation\[9\] of meson-nucleon reactions. We first establish the method by using several exactly solvable models. The main task here is to handle the singularities associated with the unstable particle channels such as \(\pi\Delta, \rho N\), and \(\sigma N\). We then apply the method to extract the nucleon resonances in some partial waves of \(\pi N\) scattering within the dynamical coupled-channels model developed in Ref.\[22\]. The analytical structure of the resonance propagator is also analyzed within this model.

In section II, we describe the formula for applying the Speed-Plot (SP) and Time-Delay (TD) methods. These two methods are then analyzed and tested in section III by using the commonly used two-channel Breit-Wigner form of S-matrix. In section IV, we develop an analytic continuation method by using several exactly solvable resonance models and further test the SP and TD methods. Section V is devoted to resolve the complications due to couplings with unstable particle channels. In section VI, we present the results from applying the developed analytic continuation method to extract the nucleon resonances in some partial waves within the \(\pi N\) model of Ref.\[22\]. A summary is given in section VII.

II. FORMULA FOR TIME-DELAY AND SPEED-Plot METHODS

The objective of the Time Delay (TD) and Speed-Plot (SP) methods is to determine the mass \(M_R\) and width \(\Gamma_R\) of a resonance from the S-matrix of reactions. In this section, we will not explain how these two methods were introduced, as briefly discussed in section I. Rather we only give their formula in practical applications.

For the single-channel elastic scattering case, the time delay of the outgoing wave packet with respect to the non-interacting wave packet in a given partial wave is defined\[6, 16, 17\] as

\[
\Delta t(E) = \text{Re} \left( -i \frac{1}{S(E)} \frac{dS(E)}{dE} \right),
\]

where the S-matrix is related to the phase shift \(\delta\) by

\[
S(E) = e^{2i\delta}.
\]

Eqs. (1) and (2) lead to a simple expression of time delay

\[
\Delta t = 2 \frac{d\delta}{dE}.
\]

The time-delay (TD) method is to find the resonance mass \(M_R\) by finding the maximum of the time delay

\[
\frac{d\Delta t}{dE} |_{E=M_R} = 0.
\]
As an ideal example, we evaluate the time delay for the S-matrix defined by the well known Breit-Wigner resonance formula

\[ S(E) = \frac{E - M_R - i\Gamma_R/2}{E - M_R + i\Gamma_R/2}, \]  

(5)

Eq. (5) then leads to

\[ \Delta t = \frac{\Gamma_R}{(E - M_R)^2 + \Gamma_R^2/4}, \]  

(6)

which obviously takes a maximum at \( E = M_R \) and hence \( M_R \) is defined as the resonance mass. It also gives the following simple physical interpretation of the width \( \Gamma_R \) in terms of the time delay of the wave packet passing through the interaction region

\[ \Delta t|_{E=M_R} = \frac{4}{\Gamma_R}. \]  

(7)

Here and in the rest of this paper, the normalization is chosen such that the S-matrix is related to the T-matrix by

\[ S(E) = 1 + 2iT(E). \]  

(8)

We then note that for the S-matrix Eq. (5), the width can also be expressed in terms of T-matrix as

\[ \frac{\Gamma_R}{2} = \left[ \frac{|T|}{\frac{dT}{dE}} \right]_{E=M_R}. \]  

(9)

In the analysis of \( \pi N \) scattering, Hoehler\[3, 4\] introduced a speed plot (SP) method. The speed is defined as

\[ \text{sp}(E) = \left| \frac{dT}{dE} \right|. \]  

(10)

Using Eqs. (2)-(3), Eq. (10) leads to

\[ \text{sp}(E) = \frac{|\Delta t|}{2}. \]  

(11)

Thus the speed is also related to the time delay of the wave packet. The SP method defines the resonance mass \( M_R \) by the maximum of the speed

\[ \frac{d}{dE}\text{sp}(E)|_{E=M_R} = 0. \]  

(12)

From Eqs. (4), (11), and (12), it is obvious that the speed plot and time delay will give the same resonance mass for the single-channel elastic scattering case.

For the multi-channel reactions with \( N \) open channels, the S-matrix becomes a \( N \times N \) matrix \( \hat{S}(E) \). Smith\[15\] introduced a life-time matrix \( \hat{Q} \) defined by

\[ \hat{Q}(E) = -i\hat{S}(E)\frac{d\hat{S}(E)}{dE}. \]  

(13)
The above equation can be considered as an extension of Eq. (1) of the single-channel case. The trace of the life-time matrix can be expressed in terms of eigen phases $\delta_i$ of the S-matrix:

$$\text{Tr} \hat{Q}(E) = 2 \frac{d}{dE} \sum_i \delta_i.$$  

(14)

The resonance mass $M_R$ is then obtained by finding

$$\frac{d\text{Tr} \hat{Q}(E)}{dE} \bigg|_{E=M_R} = 0.$$  

(15)

However, the eigen phases $\delta_i$ can be obtained only when we know all of the S-matrix elements associated with all open channels. In practice, only the elastic scattering amplitude and a few of the inelastic amplitudes can be extracted from the data. Therefore Eqs. (14)-(15) of Smith’s TD method cannot be used rigorously in practice.

Here we focus on the TD method used by Kelkar et al. (19, 20, 21). They defined the time delay for the channel $i$ only by the diagonal component $S_{ii}$ of the S-matrix and its derivative

$$\text{td}(E) = \text{Re} \left( -i \frac{1}{S_{ii}} \frac{dS_{ii}}{dE} \right).$$  

(16)

Obviously, this method is identical to Eq. (1) of the single-channel case except that the S-matrix element here is $S_{ii} = \eta e^{2i\delta}$ with $\eta$ denoting the inelasticity. Eq. (16) can be considered as an approximation of Eq. (13) by neglecting the inelastic channels in summing the intermediate states. In Refs. (19, 20, 21), the resonance mass $M_R$ is determined by the maximum of the time delay

$$\frac{d}{dE} \text{td}(E) \bigg|_{E=M_R} = 0,$$  

(17)

and the width $\Gamma_R$ by

$$\text{td}(M_R \pm \Gamma_R/2) = \text{td}(M_R)/2.$$  

(19)

The SP method for multi-channel case is simply to define the speed by the diagonal component $T_{ii}$ of the T-matrix

$$\text{sp}(E) = \left| \frac{dT_{ii}}{dE} \right|,$$  

(20)

and define the width $\Gamma_R$ by assuming that the T-matrix element can be parametrized as

$$T_{ii}(E) = T_b(E) + z(W) \frac{\Gamma_R/2}{E - M_R + i\Gamma_R/2}.$$  

(21)

Here $T_b$ is an non-resonant amplitude and the resonant amplitude is defined by the resonance mass $M_R$, the width $\Gamma_R$, and a complex residue $z(W)$. By assuming that the energy dependence of $T_b(W)$, $z(W)$, and $\Gamma_R$ can be neglected at energies near $M_R$, Eqs. (20) and (21) obviously satisfies Eq. (12) and lead to the following condition

$$\text{sp}(M_R \pm \Gamma_R/2) = \text{sp}(M_R)/2.$$  

(22)

Eqs. (20), (12) and (22) are used in applying the SP method to extract the resonance mass $M_R$ and width $\Gamma_R$ from the partial-wave amplitudes. Eq. (21) is not needed in practice, but is an essential assumption of SP method.
III. ANALYSIS OF SPEED- PLOT AND TIME- DELAY METHODS

To examine the SP and TD methods, we consider a commonly used two-channel Breit-Wigner (BW) amplitude which can be derived\cite{11, 14, 26, 27} from the analytical property of the S-matrix. To make the contact with what we will discuss in the rest of this paper, we will indicate here how this amplitude can be derived from a Hamiltonian formulation of meson-baryon reactions, such as that developed in Ref.\cite{9}.

It is sufficient to consider the simplest two-channel case with a non-relativistic two-particle Hamiltonian defined by

\[ H = H_0 + V. \]  

(23)

In the center of mass frame \( H_0 \) can be written as

\[ H_0 = \sum_i |i\rangle [m_{i1} + m_{i2} + \frac{p^2}{2\mu_i}] \langle i|, \]  

(24)

where \( m_{ik} \) is the mass of \( k \)-th particle in channel \( i \), and \( \mu_i = m_{i1}m_{i2}/(m_{i1} + m_{i2}) \) is the reduced mass. In each partial-wave, the S-matrix is a \( 2 \times 2 \) matrix and can be written

\[ S = \frac{1 - i\pi \rho K}{1 + i\pi \rho K}, \]  

(25)

where \( \rho \) is the density of state, and the K-matrix, which is also a \( 2 \times 2 \) matrix, is defined by the following Lippmann-Schwinger equation

\[ K(E) = V + V \frac{P}{E - H_0}K(E). \]  

(26)

Here \( P \) means taking the principal-value of the integration over the propagator.

We now consider the on-shell matrix element of the S-matrix Eq. (25). If the on-shell momentum is denoted as \( p_i \) for channel \( i \), we then have

\[ \langle i|1 \pm i\pi \rho K|j\rangle = \delta_{i,j} \pm i\pi \rho_i K_{i,j}, \]  

(27)

where \( \rho_i = p_i\mu_i \). The S-matrix element of the \( 1 \to 1 \) elastic scattering is then of the following explicit form

\[ S_{11} = \frac{(1 - i\pi \rho_1 K_{11})(1 + i\pi \rho_2 K_{22}) - \pi^2 \rho_1 \rho_2 K_{12}K_{21}}{(1 + i\pi \rho_1 K_{11})(1 + i\pi \rho_2 K_{22}) - \pi^2 \rho_1 \rho_2 K_{11}K_{22}}. \]  

(28)

If we assume that at energies near the resonance energy the K-matrix can be approximated as

\[ K_{ij} \sim \frac{g_ig_j}{E - M}, \]  

(29)

where \( M \) is a mass parameter which is a real number, Eq. (28) can then be written as

\[ S_{11} = \frac{E - M - ip_1\gamma_1 + ip_2\gamma_2}{E - M + ip_1\gamma_1 + ip_2\gamma_2}. \]  

(30)
Here we have defined $\gamma_i = \pi g_i^2 \mu_i > 0$. If we further assume that $\gamma_i$ is independent of scattering energy, Eq. (30) is the commonly used two-channel Breit-Wigner formula\cite{26, 27, 30}. In the rest of this section, we will follow these earlier works and treat $\gamma_1$ and $\gamma_2$ as energy independent parameters of the model.

Since the scattering T-matrix is related to the S-matrix by

$$S_{11}(E) = 1 + 2iT_{11}(E),$$

Eq. (30) leads to

$$T(E) = T_{11}(E)$$

$$= \frac{-\gamma_1 p_1}{E - M + i\gamma_1 p_1 + i\gamma_2 p_2}.\quad (32)$$

From nowon we use the notation $T(E)$ for the $1 \rightarrow 1$ amplitude $T_{11}(E)$.

A. Analytic Properties of the S-matrix

Within the two-channels Breit-Wigner model specified above, we will analyze in this subsection the analytic properties of the S-matrix on the complex energy E-plane. This will also allow us to explain clearly some terminologies which are commonly seen but often not explicitly explained in the literatures on resonance extractions.

The on-shell momenta $p_i$ for channel $i$ is defined by

$$E_i = \frac{p_i^2}{2\mu_i},$$

where

$$E_i = E - (m_{i1} + m_{i2}).\quad (34)$$

We can define the threshold variable $\Delta$ between two channels by

$$E_1 = \frac{p_2^2}{2\mu_2} + \frac{\Delta^2}{2\mu_1},$$

where

$$\frac{\Delta^2}{2\mu_1} = m_{21} + m_{22} - m_{11} - m_{12}$$

is the threshold energy of the second channel.

The momenta at poles of the S-matrix Eq. (30) can be determined by solving

$$E - M + ip_1\gamma_1 + ip_2\gamma_2 = 0.\quad (37)$$

By using Eqs. (33)-(36), the above equation can be written as

$$p_1^4 + ap_1^3 + bp_1^2 + cp_1 + d = 0,\quad (38)$$
where

\begin{align}
    a &= i4\mu_1\gamma_1, \\
    b &= 4\mu_1(\mu_2\gamma_2^2 - M - \mu_1\gamma_1^2), \\
    c &= -8i\mu_1^2M\gamma_1, \\
    d &= 4(\mu_1^2M^2 - \mu_1\mu_2\gamma_2^2\Delta^2).
\end{align}

Eq. (38) means that the Breit-Wigner amplitude Eq. (30) has four poles. Each pole is specified by two on-shell momenta \(P_\alpha = (p_{1\alpha}, p_{2\alpha})\) with \(\alpha = 1, 2, 3, 4\). The analytic properties of the S-matrix Eq. (30) depends on how these poles are located on the complex energy \(E\)-plane. As we explained in section I, the energy plane for each channel has two Riemann sheets because of the quadratic relation Eq. (33) between the momentum \(P\) specified by two on-shell momenta \(p\) and \(E\) with \(\gamma\).

The thresholds of both channels. From Eq. (37), we immediately notice that if \((\alpha_1, \alpha_2)\) are solutions \((E_{1\alpha_1, 2\alpha_2})\) of Eq. (38), they are denoted as \((E_{1\alpha_1, 2\alpha_2})\). Without losing generality, one can assume that one of the poles is in the range \((\Re p_1 > 0, \Re p_2 > 0)\) and the other in the range \((\Re p_1 < 0, \Re p_2 < 0)\). If the first pole \((p_{1a}, p_{2a})\) is in the region where \((\Re p_1 < 0, \Re p_2 > 0)\) and \((\Im p_{1a} < 0, \Im p_{2a} < 0)\), it is a pole, denoted as \(E_R\). Eq. (38) can be grouped into two pairs. In the following discussions, they are denoted as \((E_{a}, E_{a}^*)\) and \((E_{b}, E_{b}^*)\).

To be more specific, we now consider the case which is most relevant to our study of nucleon resonances. That is the \(\Re E_1 > 0\) and \(\Re E_2 > 0\) case that the poles are all above the thresholds of both channels. From Eq. (37), we immediately notice that if \((p_{1a}, p_{2a})\) with \(E = E_a\) is one of the solutions, \((-p_{1a}, -p_{2a})\) with \(E = E_a^*\) is also a solution. Therefore the four poles determined by Eq. (38) can be grouped into two pairs. In the following discussions, they are denoted as \((E_{a}, E_{a}^*)\) and \((E_{b}, E_{b}^*)\). Without losing generality, one can assume that one of the poles is in the range \((\Re p_1 > 0, \Re p_2 > 0)\) and the other in the range \((\Re p_1 < 0, \Re p_2 < 0)\). If the first pole \((p_{1a}, p_{2a})\) is in the region where \((\Re p_{1a} > 0, \Re p_{2a} > 0)\) and \((\Im p_{1a} < 0, \Im p_{2a} < 0)\), it is a pole, denoted as \(E_R\), on the \(uu\)-sheet of Fig. 2. This pole is usually called the resonance pole and is closer than other poles on \(up\) or \(pu\)-sheets to the physical \(pp\)-sheet, as will be explained later. In the Hamiltonian formulation considered in this work and the well developed collision theory, a resonance pole can be mathematically derived from the mechanism that an unstable system is formed and decay subsequently during the collision. The resonance pole \(E_R\) has an accompanied pole \(E_R^*\) at \((-p_{1a}^*, -p_{2a}^*)\) which is also on the \(uu\)-sheet as shown in Fig. 2. \(E_R^*\) is called the ‘conjugate pole’ of \(E_R\).

The second pole at \((p_{1b}, p_{2b})\) with \(\Im p_{1b} < 0\) and \(\Im p_{2b} > 0\) may be on the \(up\)-sheet (\(pu\)-sheet), depending on the parameters \(\gamma_1\) and \(\gamma_2\). This pole is called the shadow pole. A shadow pole on \(up\)-sheet and its conjugate pole are \(E_S\) and \(E_S^*\) in Fig. 2.

We note that in this simple BW model, the zeros of the S-matrix Eq. (30), where \(S_{11}(E) = 0\), is defined by its numerator

\[E - M - ip_1\gamma_1 + ip_2\gamma_2 = 0.\] (43)

The above equation can be cast into the form of Eq. (37) by simply replacing \(p_1\) by \(-p_1\). Thus solutions of Eq. (38), called the zeros of S-matrix, can be readily obtained from the solutions \((p_{1a}, p_{2a})\) and \((p_{1b}, p_{2b})\) of Eq. (38). They are \((-p_{1a}, p_{2a}^*)\) with \(E_a\) and \((p_{1b}^*, -p_{2b}^*)\) with \(E_a^*\) for \(\alpha = a, b\). The zero at \((-p_{1b}, p_{2b})\) is on the \(pp\)-sheet, denoted as \(E_{ZS}\) and \(E_{ZS}^*\) in Fig. 2. Similarly, the zero at \((-p_{1a}, p_{2a})\) is on the \(pu\)-sheet, shown as \(E_{ZR}\) together with its
FIG. 2: Poles and zeros of the simplified two-channel Breit-Wigner form ($\mu_1 = \mu_2$ and $\Delta = 0$) of S-matrix on the complex E-plane which has $pp$, $uu$, $up$ and $pu$ sheets. The open circles on the $uu$-sheet ((b)) are the resonance pole $E_R$ and its conjugate pole $E_R^*$. The open circles on the $up$-sheet ((c)) are the shadow pole $E_S$ and its conjugate pole $E_S^*$. The crossed on the $pp$-sheet ((a)) are the zero $E_{ZS}$ and its conjugate $E_{ZS}^*$ which are at the same energies of the shadow poles $E_S$ and $E_S^*$. The crossed on the $pu$-sheet ((d)) are the zero $E_{ZR}$ and its conjugate $E_{ZR}^*$ which are at the same energies of the resonance poles $E_R$ and $E_R^*$.

From the above analysis, it is clear that the poles and zeros of the S-matrix are closely related. Their locations on the 4 Riemann sheets can be conveniently displayed on one complex plane by introducing a variable $t$.

$$p_1 = \frac{\Delta^2 + t^2}{1 - t^2},$$

$$p_2 = 2\Delta \sqrt{\frac{\mu_2}{\mu_1}} \frac{t}{1 - t^2}.$$
FIG. 3: The poles and zeros of S-matrix shown in Fig. 2 are displayed on the complex \( t \)-plane defined by Eqs. (44) and (45).

lines. As seen in the derivations given above, this zero \( E_{ZS}^* \) is closely related to shadow pole \( E_S \). Thus the shadow poles can also be related to the observables. Of course, which pole is most important in determining the rapid energy dependence of observables also depends on the residues of the T-matrix at the pole positions.

In the next subsection, we will use a further simplified BW form to explain more clearly the close relations between the poles and zeros of the S-matrix. In particular we will see explicitly that the shadow pole \( E_S \) in Fig. 2 which is not on the same Riemann sheet as \( E_R \), can be located by the zeros of the S-matrix. More importantly, we will also see how the SP and TD methods work both analytically and numerically.

B. Positions of poles

For simplicity, we assume that the threshold energies of the two channels are the same and hence \( \mu \equiv \mu_1 = \mu_2 \) and \( \Delta = 0 \). Therefore we have simple relations between energy and momenta: \( E = p_1^2/2\mu = p_2^2/2\mu \) and \( p \equiv p_1 = \pm p_2 \). As discussed in the previous subsection, the four Riemann sheets are classified by the sign of the imaginary part of the momentum; namely, physical (unphysical) sheet is assigned by \( \Im p > 0 \) (\( \Im p < 0 \)). With
the simplification \( p \equiv p_1 = \pm p_2 \), we obviously have \( p_1 = p_2 = p \) on the \( pp \) and \( uu \)-sheets, and \( p_1 = -p_2 = p \) on \( uu \) and \( pu \)-sheets.

Let us start with the case of \( p_1 = p_2 = p \) for the \( pp \)-sheet or \( uu \)-sheet. The S-matrix element Eq. (30) of the first channel can be written as

\[
S_{11}(E) = \frac{E - M - i(\gamma_1 - \gamma_2)p}{E - M + i(\gamma_1 + \gamma_2)p}.
\]

It can be cast into the following more transparent form

\[
S_{11}(E) = \frac{(p + p_S)(p - p_S^*)}{(p - p_R)(p + p_R^*)},
\]

with

\[
p_R = \sqrt{2\mu M - (\mu \gamma_+)^2 - i\mu \gamma_+}, \tag{48}
\]
\[
p_S = \sqrt{2\mu M - (\mu \gamma_-)^2 - i\mu \gamma_-}, \tag{49}
\]

where

\[
\gamma_\pm = \gamma_1 \pm \gamma_2. \tag{50}
\]

Remembering that we consider \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \). To make use of Fig.2 in the following discussion, we consider the case that \( \gamma_1 > \gamma_2 \) and hence \( \gamma_\pm > 0 \) and both \( p_R \) and \( p_S \) defined in Eqs. (48)-(49) are associated with the unphysical \( u \)-sheet. For the case of \( \gamma_- < 0 \), \( p_S \) is associated with the physical \( p \)-sheet and the following presentation can be easily modified to account for this case.

Clearly, Eq. (47) means that the S-matrix has a pole at \( p_1 = p_2 = p = p_R \) on the \( uu \)-sheet with a resonance energy

\[
E_R = \frac{p_R^2}{2\mu} = M - \mu \gamma_+^2 - i\mu \gamma_+ \sqrt{\mu(2M - \mu \gamma_+^2)}. \tag{51}
\]

Its conjugate pole \( E_R^* \) is at \( p_1 = p_2 = p = -p_R^* \). The positions of \( E_R \) and \( E_R^* \) are shown in the upper right side of Fig.2. Eq. (17) also indicates that the zero of S-matrix is at \( p_1 = p_2 = -p_S \) which is on the \( pp \)-sheet because of \( \text{Im}(-p_S) > 0 \). The energy of this zero of S-matrix is

\[
E_{ZS} = \frac{p_S^2}{2\mu} = M - \mu \gamma_-^2 - i\mu \gamma_- \sqrt{\mu(2M - \mu \gamma_-^2)}. \tag{52}
\]

Its conjugate \( E_{ZS}^* \) is at \( p_1 = p_2 = p_S^* \). The positions of \( E_{ZS} \) and \( E_{ZS}^* \) are on the \( pp \)-sheet, as shown in the upper left side of Fig.2.

We next consider the \( p_1 = -p_2 = p \) case that the poles and zeros of the S-matrix are on the \( up \) or \( pu \)-sheets. The S-matrix Eq. (30) for this case then takes the following form

\[
S_{11}(E) = \frac{E - M - i(\gamma_1 + \gamma_2)p}{E - M + i(\gamma_1 - \gamma_2)p}. \tag{53}
\]
By comparing Eq. (46) and Eq. (53) and using the variables $p_R$ and $p_S$ defined by Eqs. (48) and (49), Eq. (53) can be written as

$$S_{11}(E) = \frac{(p + p_R)(p - p_R^*)}{(p - p_S)(p + p_S^*)}. \quad (54)$$

The above equation indicates that for the considered $\gamma_- > 0$, the S-matrix has a shadow pole at $p_1 = -p_2 = p = p_S$ on the up-sheet. Thus its position $E_S = p_S^2/(2\mu)$ is identical to $E_{ZS}$ of the zero of the S-matrix on the pp-sheet; namely

$$E_S = E_{ZS} = \frac{p_S^2}{2\mu} = M - \mu\gamma_-^2 - i\gamma_-\sqrt{\mu(2M - \mu\gamma_-^2)}. \quad (55)$$

This means that the shadow pole $E_S$ on the up-sheet can be found from searching for the zero $E_{ZS}$ of the S-matrix on the pp-sheet.

Eq. (51) also gives a zero of S-matrix at $p_1 = -p_2 = p = -p_R$ on $pu$-plane because $\text{Im}(-p_R) > 0$. Its energy $E_{ZR}$ is also identical to $E_R$ defined above

$$E_{ZR} = E_R = \frac{p_R^2}{2\mu} = M - \mu\gamma_+^2 - i\gamma_+\sqrt{\mu(2M - \mu\gamma_+^2)}. \quad (56)$$

The positions of $E_S$ and $E_{ZR}$ and their conjugates $E_S^*$ and $E_{ZR}^*$ are also in the lower parts of Fig.2.

From the above analysis for the $\gamma_- > 0$ case, we see that the energies of the resonance poles may be obtained by studying the poles of the S-matrix on the up-sheet and those of the shadow poles may be obtained from the zeros of the S-matrix on the pp-sheet. The analysis for the $\gamma_- < 0$ case is similar. Here we only mention that when $\gamma_- > 0$ is changed to $\gamma_- < 0$, the shadow poles $E_S$ and $E_S^*$ on the up-sheet move to the $pu$-sheet and zeros $E_{ZS}$ and $E_{ZS}^*$ will be on the uu-sheet.

Now let us examine how the SP and TD methods can find the poles defined by the above exact expressions of the two-channel BW amplitude. We first recall that in applying the SP and TD methods, the energy $E$ and momentum $p_i$ in the S-matrix are restricted on the positive real-axis. For the considered simplified case, we thus have $p_1 = p_2 = p$ and the $T$-matrix Eq.(32) then becomes

$$T(E) = \frac{-\gamma_1 p}{E - M + i\gamma_+ p}. \quad (57)$$

Our task is to examine whether the resonance mass $M_R$ and width $\Gamma_R$ found by applying the SP and TD methods on the expression Eq.(57) are close to the real and imaginary parts of the poles defined in the previous subsection.

According to Eqs. (12) and (21), the SP method finds the resonance mass $M_R$ by finding the maximum of the speed through the use of the condition

$$\left[ \frac{d}{dE} \left| \frac{dT}{dE} \right| \right]_{E=M_R} = 0. \quad (58)$$
With the T-matrix Eq. (57), Eq. (58) leads to the following equation

$$3M_R^3 + (3M + 2\mu\gamma^2)M_R^2 - (7M^2 - 6\mu M\gamma^2)M_R + M^3 = 0.$$  (59)

This equation can be written in the dimension-less form as

$$3\tilde{E}^3 + (3 + 2\alpha)\tilde{E}^2 - (7 - 6\alpha)\tilde{E} + 1 = 0,$$  (60)

with $\tilde{E} = M_R/M, \alpha = \mu\gamma^2/M_R$. With some inspections, one can see that Eq. (60) has real and positive $M_R$ solutions only in the $\alpha < 0.417$ region. Two of the three solutions are the maximum and minimum points of the speed, and the third one is less than 0. The SP method defines the maximum point of the speed as the resonance mass. We find that this solution can be expanded as

$$[M_R]_{SP} = M - \mu\gamma^2 + \frac{1}{4M}(\mu\gamma^2)^2 + O\left(\frac{(\mu\gamma^2)^3}{M^2}\right).$$  (61)

Clearly, $[M_R]_{SP}$ equals to the real part of Eq. (51) if we neglect the second and higher order terms in the expansion in powers of $\mu\gamma^2/M$. Therefore the SP method is accurate only under the condition that $\mu\gamma^2/M \ll 1$. Moreover, it is clear from the above equation that speed has no stationary point for $\mu\gamma^2/M > 0.417$ and therefore the SP method will fail to find the pole even if there is a pole within the model.

We next turn to discussing the width $\Gamma_R$ obtained by the SP method. It is evaluated by using Eq. (5). We find that it can also be expanded as

$$\frac{[\Gamma_R]_{SP}}{2} = \frac{[T]_{[M_R]_{SP}}}{[M_R]_{SP}} = \sqrt{\mu\gamma^2(2M - \mu\gamma^2)} \left[1 - \frac{\mu\gamma^2}{2M} + O\left(\left(\frac{\mu\gamma^2}{M}\right)^2\right)\right].$$  (62)

Here again, if we neglect higher order terms of $\mu\gamma^2/M$, the SP method can give the imaginary part of $E_R$ of the exact expression Eq. (51). As we have seen in Fig. 2, the two-channel BW S-matrix has two pairs of poles on the $uu$-sheet and the $up$-sheet. However the SP method can only find the pole $E_R$ on the $uu$-sheet.

From the above analysis, it is clear that the accuracy of SP method is controlled by $\mu\gamma^2/M$. We examine this by using an example with $\mu = \frac{m_N m_N}{m_N + m_\pi}, m_N = 938.5$ MeV and $m_\pi = 139.6$ MeV, and $M = m_N + m_\pi + 600$ MeV. In Fig. 3 the solid curves are the pole positions on the $uu$-sheet. They are obtained from evaluating the exact analytical formula Eq. (51) for $\gamma/\gamma_+ = 0.5$ and varying $\mu\gamma^2/M$ from 0.01 to 0.6. With the same parameters we then apply the SP method to search for pole $(M_R, -i\Gamma_R/2)$ from the amplitude Eq. (57) numerically by using Eqs. (9), (10) and (12). The obtained pole positions were the filled squares shown in Fig. 4. As expected, the SP method works very well for small $\mu\gamma^2/M$. However the pole position from SP starts to deviate from the exact results (solid curves) as $\mu\gamma^2/M$ increases. There is no filled squares in Fig. 4 in the region $\mu\gamma^2/M > 0.417$ because the SP cannot find a pole in this region. This is not because of the numerical accuracy of our calculation, but is the intrinsic limitation of the SP method as, discussed above.

We now turn to investigating the TD method. We apply Eq. (17) to search for the resonance mass $M_R$ from the amplitude Eq. (57). Since it is not clear how to interpret Kelkar’s prescription Eq. (15), we assume that the S-matrix element $S_{11}$ is of the following
FIG. 4: The $\mu\gamma^2/M$ dependence of the pole positions on the $uu$-sheet (solid curve) and $up$-sheet (dashed curve) of the simplified two-channels Breit-Wigner form ($\mu_1 = \mu_2, \Delta = 0$) calculated at $\gamma_-/\gamma_+ = 0.5$. The real (imaginary) parts of poles are shown in (a) (b) of the figure. The solid squares (solid circles) are the results obtained from using the SP (TD) method. Triangles in (b) are the widths of the poles obtained using Eq. (19).

The real (imaginary) parts of poles are shown in (a) (b) of the figure. The solid squares (solid circles) are the results obtained from using the SP (TD) method. Triangles in (b) are the widths of the poles obtained using Eq. (19).

IV. ANALYTIC CONTINUATION OF RESONANCE MODELS

With the analysis presented in the previous section, it is clear that the empirical partial-wave amplitudes determined from experimental data can not be blindly used to extract resonance parameters by using SP or TD methods. To make progress, one needs to construct...
a reaction model to fit the data and then extract the resonance parameters by analytic continuation within the model. In this paper, we focus on a dynamical model\cite{9} which accounts for the main features of meson production reactions in the nucleon resonance region. Our task in this section is to develop numerical methods for finding the resonance poles from such models which do not have analytical forms of their solutions. We will first consider the simplest one-channel and one-resonance case, then two-channels and one-resonance, and finally two-channels and two-resonances cases. All of these models are exactly solvable such that their poles are known analytically and the developed numerical methods can be tested.

A. One-channel, one-resonance

To be specific, we consider the two-particle reactions defined by the following well known isobar Hamiltonian in the center of mass frame

\[ H = H_0 + H', \] (66)

with

\[ H_0 = [E_1(p) + E_2(-p)] + |N_0\rangle M_0 \langle N_0|, \] (67)
\[ H' = |g\rangle \langle g|, \] (68)

where \( M_0 \) is the mass parameter of a bare particle \( N_0 \) which can decay into two particle states through the vertex interaction \( g \) in \( H' \), and \( E_i(p) = \sqrt{m_i^2 + p^2} \) is the energy of the \( i \)-th particle. The scattering operator is defined by

\[ t(E) = H' + H' \frac{1}{E - H_0 - H'} H', \] (69)

which leads to the following Lippmann-Schwinger equation for the scattering amplitudes in each partial-wave

\[ t(p', p; E) = v(p', p; E) + \int_{C_0} dq q^2 \frac{v(p', q; E)t(q, p; E)}{E - E_1(q) - E_2(q)}, \] (70)

where the integration path \( C_0 \) will be specified later. The interaction in Eq. (70) is

\[ v(p', p; E) = \frac{g(p')g(p)}{E - M_0}. \] (71)

Eqs. (70)-(71) leads to the following well known solution

\[ t(p', p; E) = \frac{g(p')g(p)}{E - M_0 - \Sigma(E)}, \] (72)

with

\[ \Sigma(E) = \int_{C_0} dp p^2 \frac{g^2(p)}{E - E_1(p) - E_2(p)}. \] (73)
From the analysis in the previous section, the resonance poles can be found from \( t(E) = t(p_0, p_0; E) \) on the unphysical Riemann sheet defined by \( \text{Im } p_0 < 0 \) with \( p_0 \) denoting the on-shell momentum

\[
E = \sqrt{m_1^2 + p_0^2 + \sqrt{m_2^2 + p_0^2}}.
\]  

(74)

Obviously \( p_0 \) is also the pole position of the propagator in Eq. (70) or Eq. (73).

The physical scattering amplitude at a positive energy \( E \) can be obtained from Eq. (70) or Eq. (72) by setting \( E \rightarrow E + i\epsilon \) with a positive \( \epsilon \rightarrow 0 \) and choosing the integration contour \( C_0 \) to be along the real-axis of \( p \) with \( 0 \leq p \leq \infty \). From Eq. (70) it is clear that \( t(E) \) has a discontinuity on the positive real \( E \)

\[
\text{Dis}(t(E)) = t(E + i\epsilon) - t(E - i\epsilon) = 2\pi i\rho(p_0)v(p_0, p_0)t(E),
\]

(75)

where \( \rho(p_0) = p_0 E_1(p_0)E_2(p_0)/E \). Thus the t-matrix has a cut running along the real positive \( E \). To find resonance poles, we need to find the solution of Eq. (70) on the unphysical sheet with \( \text{Im } p \leq 0 \) on which the pole \( p_0 \) of the propagator moves into the lower \( p \)-plane, as shown in (a) of Fig.5. From Eq. (75), it is clear that the solution of Eq. (70), with the contour \( C_0 \) chosen to be on the real-axis \( 0 \leq p \leq \infty \), will encounter the discontinuity and is not the solution on the unphysical sheet where we want to search for the resonance poles. It is well-known that this difficulty can be overcome by deforming the integration path to the contour \( C_1' \) shown in (a) of Fig.5. By this the pole will not cross the cut and the integral is analytically continued from real positive \( E \) to the lower half of the unphysical \( E \)-sheet with \( \text{Im } p_0 \leq 0 \). Obviously, the same solution can be obtained by choosing any contour which is below the pole position \( p_0 \), such as the contour \( C_1 \) of (b) of Fig.5.

With the solution of the form of Eq. (72), the numerical procedure of finding resonance poles is to solve

\[
E - M_0 - \Sigma(E) = 0.
\]

(76)
with

\[ \Sigma(E) = \int_{C_1} dp \frac{p^2}{E - E_1(p) - E_2(p)} \frac{g^2(p)}{E - E_1(p) - E_2(p)}, \]

for \( E \) on the unphysical Riemann sheet defined by \( \text{Im} p_0 \leq 0 \). To test this numerical procedure, let us consider the case that \( \Sigma(E) \) defined by Eq. (73) can be calculated analytically. Such an analytic form can be obtained by taking the non-relativistic kinematics \( E_1(p) + E_2(p) = m_1 + m_2 + p^2/(2\mu) \) with \( \mu = m_1m_2/(m_1 + m_2) \) and a monopole form for the vertex function

\[ g(p) = \frac{\lambda}{1 + p^2/\beta^2}, \]

where \( \beta \) is a cut-off parameter. The integration in \( \Sigma(E) \) of Eq. (73) can then be done exactly to give the following simple form

\[ \Sigma(E) = \frac{\pi \mu \beta^3 \lambda^2}{2(p_0 + i\beta)^2}, \]

where \( p_0 \) is defined by \( E = m_1 + m_2 + p_0^2/(2\mu) \). If the imaginary part of \( p_0 \) is positive (negative), it means that we choose the poles on physical (unphysical) sheet. Only the pole with \( \text{Im} p_0 \leq 0 \) on the unphysical sheet is called resonance, as discussed in the previous section.

The resonance poles on the unphysical sheet (\( \text{Im} p_0 \leq 0 \)) obtained from using Eq. (80) to solve Eq. (76) are the solid curve displayed in Fig. 6 using the parameters: \( m_1 = m_N = 938.5 \text{ MeV}, \ m_2 = m_\pi = 139.6 \text{ MeV}, \ M_0 = m_N + m_\pi + 600 \text{ MeV} \) and cutoff momentum \( \beta = 800 \text{ MeV} \), for a range of coupling constant \( 0 \leq \lambda \leq 0.04 \). We next solve Eqs. (76) and (78) by choosing contour \( C_1 \) illustrated in Fig. 5. The solutions are stable as far as the path is not too close to the pole. The solutions completely agree with the solid curve of the exact solution and hence are not displayed in Fig. 6. Here we also note that Eq. (80) also allows to calculate the poles by using the SP and TD methods. The results for the value of \( \lambda = 0.01, 0.02, 0.03, 0.04 \) are the solid squares in Fig. 6. As expected we confirm the previous findings that both SP and TD work well for the single channel case.

B. Two-channels, one-resonance

The formula for two-channels, one-resonance case can be easily obtained by extending the equations in the previous subsection to include channel label \( i = 1, 2 \). We thus have

\[ t_{ij}(p', p; E) = v_{ij}(p', p; E) + \sum_k \int_{C_0} dq q^2 \frac{v_{ik}(p', q)t_{kj}(q, p; E)}{E - E_{k1}(q) - E_{k2}(q)}, \]

where \( E_{kn}(p) = [m_{kn}^2 + p^2]^{1/2} \) with \( m_{kn} \) denoting the mass of \( n \)-th particle in channel \( k \), and

\[ v_{ij}(p', p; E) = g_i(p') \frac{1}{E - M_0} g_j(p). \]
FIG. 6: $\lambda$ dependence of the pole position of the one-channel, one-resonance model. The line represents the pole position of the t-matrix on the unphysical sheet. Solid squares are the pole positions extracted by using the SP and TD methods.

Eqs. (81)-(82) leads to

$$t_{ij}(p', p; E) = \frac{g_i(p')g_j(p)}{E - M_0 - \Sigma_1(E) - \Sigma_2(E)},$$

(83)

with

$$\Sigma_k(E) = \int_{C_0} dp p^2 \frac{g_k^2(p)}{E - E_{k1}(p) - E_{k2}(p)}.$$  

(84)

For the physical scattering amplitude at a positive energy $E$, Eq. (81) is solved by setting $E \rightarrow E + i\epsilon$ with a positive $\epsilon \rightarrow 0$ and choosing $C_0$ along the real axis $0 \leq p \leq \infty$.

With Eq. (83), the poles of the scattering amplitudes are defined by

$$E - M_0 - \Sigma_1(E) - \Sigma_2(E) = 0.$$  

(85)

The poles from solving the above equation can be on one of the four Riemann sheets, $pp$, $up$, $uu$, and $pu$, as explained in section III. The numerical procedure for finding the resonance poles on $uu$-sheet is to solve Eq. (85) for $E = E_{11}(p_{01}) + E_{12}(p_{01}) = E_{21}(p_{02}) + E_{22}(p_{02})$ with $\text{Im} p_{01} \leq 0$ and $\text{Im} p_{02} \leq 0$. The integration path $C_0$ is changed to $C_1$ shown in (b) of Fig. 5 to calculate both self-energies $\Sigma_1(E)$ and $\Sigma_2(E)$ of Eq. (81). Of course the contour $C_1$ for the integration over the momentum for $i$-th channel must be below the pole $p_{0i}$ defined by $E - E_{i1}(p_{0i}) - E_{i2}(p_{0i}) = 0$. Here we note that for finding the poles on $pu$-sheet ($up$-sheet), the contour $C_0$ is replaced by $C_1$ only for $\Sigma_2(E)$ ($\Sigma_1(E)$).
To test numerical procedures and to further test the SP and TD methods, let us consider again the non-relativistic kinematics \( E_{i1} (p) + E_{i2} (p) = \Theta_i + p^2/(2\mu_i) \) with \( \Theta_i = m_{i1} + m_{i2} \) and \( \mu_i = m_{i1} m_{i2}/(m_{i1} + m_{i2}) \). This will allow us to find the exact solutions by choosing the monopole form factor

\[
g_i(p) = \frac{\lambda_i}{1 + p^2/\beta_i^2}. \tag{86}
\]

We then have

\[
\Sigma_i (E) = \frac{\pi \mu_i \beta_i^3 \lambda_i^2}{2(p_i + i\beta_i)^2}. \tag{87}
\]

With Eq. (87), the poles defined by Eq. (85) can be found by solving algebraic equations. For numerical calculations, we consider a case similar to \( \pi N \) and \( \eta N \) with \( \lambda_1 = 0.02 \) for \( N_0 \to \pi N \) and a range of \( \lambda_2 = 0 - 0.02 \) for \( N_0 \to \eta N \). We see that when \( \lambda_2 \) is 0, which is the single-channel case, the \( uu \)-pole and the \( up \)-pole are on the same position. They then split as \( \lambda_2 \) increases.

We next evaluate Eq. (84) for \( E \) on the \( uu \)-sheet, \( up \)-sheet or \( pu \)-sheet by appropriately choosing the path \( C_0 \), as described above. The poles are found when the calculated \( \Sigma_1 (E) \) and \( \Sigma_2 (E) \) satisfy Eq. (85). We find that the poles obtained by this numerical procedure reproduce accurately the dash-dotted(\( uu \)-sheet), solid(\( up \)-sheet) and dashed(\( pu \)-sheet) curve in Fig.7 and hence are omitted there. Thus this analytic continuation method can be used in practice to find the resonance poles (i.e. poles on \( uu \)-sheet as defined in section III) for general case that \( \Sigma_i (E) \) can not be integrated analytically.

We now turn to examining the SP and TD methods. In Fig.7, we show that the SP method (open squares) reproduces the poles (dash-dotted curve) on the \( uu \)-sheet only at \( \lambda_2 \leq 0.015 \). When \( \lambda_2 \) is larger than 0.015 where the magnitude of \( \text{Im} \ E_i \) continues to increase, the speed has no maximum and the SP method fails to find the pole. This is another example showing that SP method has its limitation. On the other hand, the TD method (solid squares) can reproduce the poles on the \( up \)-sheet (solid curve) and \( pu \)-sheet (dashed curve) in the considered range of parameters. Here we see that the SP and TD find different poles which have different physical meanings in the Hamiltonian formulation. One can show\(^{15}\) that the poles on \( uu \)-sheet are due to the process that an unstable system is created and then decays during the collision and are called the resonance poles. The physical interpretations of the poles on \( pu \) and \( up \)-sheets remain to be developed.

It is interesting to point out here that the two-channel Breit-Wigner form analyzed in detail in the previous section can be derived from the two-channels, one-resonance model if the non-relativistic kinematics is used. To see this, we first write the non-relativistic relation between the S-matrix and the T-matrix

\[
S_{ij} (E) = \delta_{ij} + 2i T_{ij} (E), \tag{88}
\]

\[
T_{ij} (E) = -\pi \sqrt{\mu_i \mu_j} p_j l_{ij} (p_i, p_j; E), \tag{89}
\]

where \( p_i \) is the on-shell momentum in channel \( i \)

\[
p_i = \sqrt{2 \mu_i (E - \Theta_i)}. \tag{90}
\]
FIG. 7: $\lambda_2$ dependence of the pole positions on the $uu$-sheet (dash-dotted curve), $up$-sheet (solid curve) and $pu$-sheet (dashed curve) of the two-channels, one-resonance model. The open squares (solid squares) are obtained from using the SP (TD) method.

With the above and the analytic form Eq. (87) for $\Sigma_i(E)$, we can write the $1 \rightarrow 1$ elastic scattering amplitude of Eq. (89) as

$$T_{11}(p_1, p_1, E) = \frac{-p_1 \gamma_1(p_1)}{E - M(E) + ip_1 \gamma_1(p_1) + ip_2 \gamma_2(p_2)},$$

(91)

where $\gamma_i(p_i) = \pi \mu_i g_i^2(p_i) > 0$ and

$$M(E) = M^0 + \sum_k P \int p^2 dp \frac{g_k^2(p)}{E - m_{k1} - m_{k2} - \frac{p^2}{2\mu_k}}$$

(92)

where $P$ means taking the principal-value integration. By using Eq. (88), we then have the $1 \rightarrow 1$ elastic part of the S-matrix

$$S_{11} = \frac{E - M(E) - ip_1 \gamma_1(p_1) + ip_2 \gamma_2(p_2)}{E - M(E) + ip_1 \gamma_1(p_1) + ip_2 \gamma_2(p_2)}.$$  

(93)

If the $E$-dependence of $M(E)$ and $\gamma_i(p_i)$ are further neglected, Eqs. (91) and (93) are identical to what are usually called the two-channel Breit-Wigner resonant amplitude discussed in the previous section. Thus the conditions under which SP and TD are valid can be related now to the parameters of the vertex function $g_i(p)$ within this two-channels, one-resonance model.
C. Two-channels, two-resonances

For the two-channels, two-resonances case, the scattering amplitude is defined by the same Eq. (81), but with the following driving term

\[ v_{ij}(p', p; E) = g_{i1}(p') \frac{1}{E - M_1} g_{j1}(p) + g_{i2}(p') \frac{1}{E - M_2} g_{j2}(p). \]  

(94)

The scattering amplitude is then of the following form

\[ t_{ij}(p', p; E) = \sum_{\alpha, \beta} g_{i, \alpha}(p') [D^{-1}(E)]_{\alpha, \beta} g_{j, \beta}(p). \]  

(95)

The propagator \( D(E) \) in Eq. (95) is

\[ [D(E)]_{\alpha, \beta} = \left[ E - M_{\alpha} - \Sigma_{\alpha, \beta}(E) \right] \delta_{\alpha, \beta} \]  

(96)

with

\[ \Sigma_{\alpha, \beta}(E) = \sum_i \int_{C_0} dq q^2 \frac{g_{i, \alpha}(q) g_{i, \beta}(q)}{E - E_{i1}(q) - E_{i2}(q) + i\varepsilon}. \]  

(97)

The poles are defined by

\[ \text{Det} D(E) = \left[ E - M_1 - \Sigma_{11}(E) \right] \left[ E - M_2 - \Sigma_{22}(E) \right] - \Sigma_{12}(E) \Sigma_{21}(E) = 0. \]  

(98)

The numerical procedures of finding the resonance poles on the uu-sheet from Eqs. (97) and (98) are the same as that in the previous two subsections. Namely the path \( C_0 \) of Eq. (97) is set to be the path \( C_1 \) shown in Fig. 5 in evaluating the integrals for \( E \) on the uu-sheet where the on-shell momenta are \( \text{Im} p_i < 0 \) for \( i = 1, 2 \) channels. To test this, we again choose the non-relativistic kinematics and the monopole form factor like Eq. (86). The self energy \( \Sigma_{\alpha, \beta}(E) \) then takes the analytic form similar to Eq. (80). The condition Eq. (98) can then be expressed in a analytic form from which the pole positions on the unphysical sheet can be easily obtained.

We only state that the resulting poles on the unphysical sheets are reproduced by the numerical analytic continuation method described above. Instead our focus here is to further test the SP and TD methods for the situation that two resonances are close and could overlap. We again consider \( \pi N \) and \( \eta N \) channels and use the following form factor

\[ g_{i\alpha}(p) = \frac{\lambda_{i\alpha}}{1 + p^2/\beta_{i\alpha}^2}, \]  

(99)

where \( \alpha = 1, 2 \) denote the \( \alpha \)-th bare state with mass \( M_\alpha \). The four cut-off parameter \( \beta_{i\alpha} \) and four coupling constants \( \lambda_{i\alpha} \) are taken to be: \( \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 800 \text{ MeV}, \lambda_{11} = 0.005, \lambda_{12} = 0.01, \lambda_{21} = 0.003 \) and \( \lambda_{22} = 0.008 \). One of the bare masses \( M_1 \) is fixed in the calculations. \( M_2 \) is defined by

\[ M_2 = M_\pi + M_N + \tilde{M}_2, \]  

(100)

where \( \tilde{M}_2 \) is varied for examining how the poles move as \( M_2 \) moves away from the \( M_1 = M_\pi + M_N + 550 = 1628 \text{ MeV} \).
The pole positions are searched numerically by using the analytic continuation method described above. There are two poles on the \( uu \)-sheet and the other two on the \( up \)-sheet. As \( \tilde{M}_2 \) varies, these two poles will develop two trajectories. They are the crosses connected by the solid curves shown in Fig.8 for \( uu \)-sheet and Fig.9 for the \( up \)-sheet. According to the findings we made in section III, the poles, \((M_R, -i\Gamma_R/2)\), found by the SP (TD) method should be compared with the poles on \( uu \) (\( up \)) sheet of Fig. 8 (9). We now discuss these two comparisons.

We see from Fig. 8 that in the regions near \( \tilde{M}_2 = 700 \) MeV, the positions (Re \( E \)) of these two poles are far from each other and we find that SP (open squares connected by dashed lines near the point marked 700) works well. When \( \tilde{M}_2 \) is reduced to 600 MeV where the positions (Re \( E \)) of two poles move closer, the SP method can find only one pole (open squares connected by dashed line) near the top end of the trajectory on the right hand side. Apart from the points on the dashed lines, SP method fails to find poles close to the poles on the solid curves which are obtained numerically by the analytic continuation method.

The results for examining TD is shown in Fig.9. We see that TD can find two poles on the \( up \)-sheet in the considered range of \( \tilde{M}_2 = 600 - 700 \). The results are the open squares connected by dashed lines which are indistinguishable from the crosses connected by solid curves which were obtained by analytic continuation method. But TD obtains another two poles at \( \tilde{M}_2 = 600 \) and \( \tilde{M}_2 = 625 \), as indicated by the dashed line in the middle of Fig.9. The positions of these two poles are very close to those obtained from SP and they are interpreted as poles on \( uu \)-sheet. As we have discussed in section II, TD is sensitive to both zero and pole of the S-matrix on \( pp \) and \( uu \)-sheets. In this example, width of the poles on \( uu \)-sheet become comparable to the poles on \( up \)-sheet (zero on physical sheet) and hence TD could find pole on \( uu \)-sheet. The results shown in Fig. 8 and Fig. 9 further indicate the limitation of SP and TD methods.

The results from the above several models have shown that the TD method based on Eq. (16), where the phase of the elastic channel is used instead of the eigen phases discussed in Refs. [23, 24, 25], gives both the resonance poles on the \( uu \)-sheet and the zeros (shadow poles). Our findings could provide some information for investigating the differences between Refs. [23] and [34]. We also want to mention here that an improved SP method using higher order derivatives of the amplitudes was proposed in Ref. [35]. It may be interesting to compare this method with the TD method. While it could be interesting to address the questions concerning these recent developments, they are far from the main focus of this paper and will not be discussed further.

V. ANALYTIC CONTINUATION OF RESONANCE MODEL WITH UNSTABLE PARTICLE CHANNELS

For meson-baryon reactions, the nucleon resonances can decay into some unstable particle channels such as the \( \pi \Delta \), \( \rho N \), \( \sigma N \) considered in the model of Ref. [9]. Here we discuss the analytic continuation method to find resonance poles within such a reaction model.

It is sufficient to consider the one-channel and one resonance case. The scattering formula is then identical to that presented in subsection IV.A. The only difference is that one of the particles in the open channel can further decay into a two particle state. To be specific, let us consider the \( \pi \Delta \) channel. Within the same Hamiltonian formulation [9] used in the
FIG. 8: Pole positions (crosses connected by solid lines) on the $uu$-sheet of the two-channels, two-resonances model. The results from using the SP method are the open squares connected by the dashed lines. The numbers on the figure are the value of $\tilde{M}_2$.

FIG. 9: Pole positions (crosses connected by solid lines) on the $up$-sheet of the two-channels, two-resonances model. The results from using the TD method are the open squares connected by the dashed lines. The numbers on the figure are the value of $\tilde{M}_2$. 

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previous section, the scattering amplitude can then be written as

\[ t(p', p, E) = \frac{g_{N\pi\Delta}(p')g_{N\pi\Delta}(p)}{E - M_0 - \Sigma_{\pi\Delta}(E)} \]  

(101)

with

\[ \Sigma_{\pi\Delta}(E) = \int_{C_2} p^2 \, dp \frac{g_{N\pi\Delta}(p)}{E - E_\pi(p) - E_\Delta(p) - \Sigma(p, E)} \]  

(102)

where

\[ \Sigma_{\Delta}(p, E) = \int_{C_3} q^2 \, dq \frac{g_{\Delta\pi N}(q)}{E - E_\pi(p) - [(E_\pi(q) + E_N(q))^2 + p^2]^{1/2}} . \]  

(103)

To obtain the \( \pi\Delta \) self energy for complex \( E \), the analytic structure of the integrand of Eq. (102) should be examined first. The discontinuity of the \( \pi\Delta \) propagator in the integrand of Eq. (102) is the \( \pi\pi N \) cut along the real axis between \( \pm p_0 \) \(( -p_0 \leq p \leq p_0 )\) which is obtained by solving

\[ E = E_\pi(p_0) + [(m_\pi + m_N)^2 + p_0^2]^{1/2} . \]  

(104)

For finding the resonance poles on \( uu \)-sheet with \( \text{Im} \, p_0 \leq 0 \), the integration contour \( C_2 \) of Eq. (102) must be chosen below this cut which is the dashed line in Fig.10. There is also a singularity in the integrand of Eq. (102) at momentum \( p = p_x \), which satisfies

\[ E - E_\pi(p_x) - E_\Delta(p_x) - \Sigma_{\Delta}(p_x, E) = 0 . \]  

(105)

Physically, this singularity corresponds to the \( \pi\Delta \) two-body scattering state. For \( E \) with large imaginary part, \( p_x \) can be below the \( \pi\pi N \) cut as also indicated in Fig.10. Therefore the integration contour of momentum \( p \) must be chosen to be below the \( \pi\pi N \) cut (dashed line) and the singularity \( p_x \), such as the contour \( C_2 \) shown in Fig.10.

The singularity position \( q_0 \) of the propagator in Eq. (103) depends on spectator momentum \( p \)

\[ E - E_\pi(p) = [(E_\pi(q_0) + E_N(q_0))^2 + p^2]^{1/2} . \]  

(106)

Therefore the singularity \( q_0 \) moves along the dashed curve in Fig.11 when the momentum \( p \) varies along the path \( C_2 \) of Fig.10. To analytically continue \( \Sigma_{\Delta}(p, E) \) from positive energy \( E \) to the un-physical plane with \( \text{Im} \, p \leq 0 \), we need to choose the contour \( C_3 \) of Eq. (103) which must be below \( q_0 \). A possible contour \( C_3 \) is the solid curve in Fig.11.

To verify the numerical procedures described above, we again consider non-relativistic kinematics and monopole form factor. With the similar analytic form Eq. (80), we have

\[ \Sigma_{\Delta}(p, E) = \frac{\pi \mu_{\pi N} \beta^3_{\Delta\pi N}}{2(k + i\beta_{\Delta\pi N})^2} \]  

(107)

where

\[ \bar{k} = \left[ 2\mu_{\pi N} \left( E - 2m_\pi - m_N - \frac{p^2}{2\mu_{\pi N}} \right) \right]^{1/2} , \]  

(108)
FIG. 10: Contour $C_2$ for calculating the $\pi\Delta$ self energy on an unphysical sheet. See the text for the explanations of the dashed line and the singularity $p_x$.

FIG. 11: Contour $C_3$ for calculating the $\pi N$ self energy on the unphysical sheet. Dashed curve is the singularity $q_0$ of the propagator in Eq. (103), which depends on the spectator momentum $p$ on the contour $C_2$ of Fig 10.

with $\mu_{\pi N} = m_\pi (m_\pi + m_N)/(2m_\pi + m_N)$. With Eq. (107), we can solve Eq. (105) and verify its relation with $\pi N$ cut as discussed above and illustrated in Fig 10. Eq. (107) and the chosen monopole form factor also allow us to get

$$\Sigma_{\pi\Delta} = \int_{c_2} p^2 dp \frac{g_{N^*,\pi\Delta}^2}{(1 + p^2 / \beta_{N^*,\pi\Delta}^2)^2} \frac{1}{D_{\pi\Delta}(p, E)}$$

(109)

with

$$D_{\pi\Delta}(p, E) = E - m_\pi - m_\Delta - \frac{p^2}{2\mu_{\pi\Delta}} - \frac{\mu_{\pi N} g_{N^*,\pi N}^2 \beta_{N^*,\pi N}^3}{2(k + i\beta_{N^*,\pi N})^2}.$$  

(110)
FIG. 12: $\pi N^*$ Green function. The solid (dash-dotted) curve is the real (imaginary) part of exact Green function $G_{\pi N^*}(E) = 1/(E - M_0 - \Sigma_{\pi\Delta}(E))$. They are compared with the dashed (dotted) curve of the real (imaginary) part of the approximate Green function $G_{\pi N^*}^{\text{approx}}$.

Unfortunately, Eq. (109) can not be integrated out analytically for directly checking our numerical procedure for searching resonance poles.

We test our analytic continuation method by the following procedure. We calculate Eq. (109) numerically to find the pole position $E_R$ by solving $E_R - M_0 - \Sigma_{\pi\Delta}(E_R) = 0$ of the denominator of Eq. (101). With the parameters:

$$
\beta_{\pi N^*,\pi\Delta} = 800\text{MeV}, \quad g_{\pi N^*,\pi\Delta} = 0.02\text{MeV}^{-1/2},
\beta_{\Delta,\pi N} = 200\text{MeV}, \quad g_{\Delta,\pi N} = 0.05\text{MeV}^{-1/2},
M_0 = 650\text{MeV} + m_\pi + m_N,
$$

we find $E_R = (1679.1, -33.6i) \text{MeV}$. We then construct an approximate propagator

$$
G_{\pi N^*}^{\text{approx}}(E) = \frac{1}{E - E_R}.
$$

For the positive E, we find that $G_{\pi N^*}^{\text{approx}}(E)$ is in good agreement with the direct calculation of $G_{\pi N^*}(E) = 1/(E - M_0 - \Sigma_{\pi\Delta}(E))$ by using Eq. (109). The results are shown in Fig. 12. It is clear that the resonance pole found by our analytic continuation method can reproduce what is expected for a resonance propagator for real positive E. In this way our numerical procedure is justified and can be applied to solve Eqs. (101) - (103).

VI. RESONANCE MODEL WITH NON-RESONANT INTERACTIONS

With the numerical methods described above, we can proceed to extract the resonance poles within a coupled-channels model which also include non-resonant interactions. In this section, we explain how this can be done for the $\pi N$ model developed in Refs. [9, 22].

Recalling the formulations presented in Refs. [9, 22], the t-matrix considered is of the following form

$$
t_{ij}(p', p; W) = \tilde{t}_{ij}(p', p; W) + \sum_{\alpha, \beta} \tilde{\Gamma}_{i,\alpha}(p'; W) [D^{-1}(W)]_{\alpha,\beta} \tilde{\Gamma}_{j,\beta}(p; W).
$$
where \( i, j \) can be stable channels \( \pi N \) and \( \eta N \), or unstable channels \( \pi \Delta, \rho N, \) and \( \sigma N \), and \( \alpha \) denotes a bare resonant state with a mass \( M_\alpha \).

For extracting resonance poles, we now apply the methods presented in previous sections to choose appropriate contours for calculating various integrations on unphysical \( E \)-plane with \( \text{Im} \ p \leq 0 \). The non-resonant \( t \)-matrix \( \tilde{t}_{ij}(p'; p; E) \) is defined by the following couple-channel equations with the non-resonant potential \( v_{ij}(p', p) \),

\[
\tilde{t}_{ij}(p'; p; E) = v_{ij}(p', p) + \sum_k \int_{C_4} dq \q^2 \frac{\tilde{v}_{ik}(p', q)\tilde{t}_{kj}(q; p; E)}{E - E_k(q) - \Sigma_k(q, E) + i\varepsilon},
\]

(113)

where the contour is \( C_4 = C_1 \) (b) of Fig.5 for \( k = \pi N, \eta N \), and \( C_4 = C_2 \) (Fig.10) for \( k = \pi \Delta, \rho N, \sigma N \), and

\[
E_k(p) = \sqrt{m_{k_1}^2 + p^2} + \sqrt{m_{k_2}^2 + p^2}.
\]

(114)
The self-energies in Eq. (113) are \( \Sigma_{\pi N}(q, E) = \Sigma_{\eta N}(q, E) = 0 \) and \( \Sigma_{\pi \Delta}(q, E), \Sigma_{\rho N}(p, E), \) and \( \Sigma_{\sigma N}(q, E) \) are defined by the same Eq. (102) with appropriate changes of mass parameters and the choice of contour \( C_2 \) and \( C_3 \) shown in Figs. 10 and 11.

The dressed vertex \( \bar{\Gamma}_i(p; W) \) is determined by the bare vertex \( \Gamma_i(p) \) and the meson-baryon loop,

\[
\bar{\Gamma}_{i,\alpha}(p; E) = \Gamma_{i,\alpha}(p) + \sum_k \int_{C_4} dq \q^2 \frac{\tilde{t}_{ik}(p, q; E)\Gamma_k(q; E)}{E - E_k(q) - \Sigma_k(p, E) + i\varepsilon}.
\]

(115)
The resonance propagator \( D(E) \) in Eq. (112) is

\[
[D(E)]_{\alpha, \beta} = [E - M_\alpha]\delta_{\alpha, \beta} - \Sigma_{\alpha, \beta}(E),
\]

(116)
with

\[
\Sigma_{\alpha, \beta}(E) = \sum_k \int_{C_4} dq \q^2 \frac{\Gamma_{k,\alpha}(q)\bar{\Gamma}_{k,\beta}(q; E)}{E - E_k(q) - \Sigma_k(q, E) + i\varepsilon}.
\]

(117)

In Ref. [22], the above equations are solved on the real-axis by using the standard method of subtraction. Here we solve the equations by choosing contours indicated above. We first verify that our numerical results obtained here for positive real \( E \) agree with that of Ref. [22]. This establish our numerical procedure in this complex five-channel model.

Here we show the results for some of the \( S, P, D, \) and \( F \) partial waves of \( \pi N \) scattering within the model of Ref. [22]. We search the resonance poles by looking for zeros of the resonant propagator \( D(E) \) defined by Eq. (116). Our results from using the analytic continuation method are shown in the second column of Table 1. They are compared with those extracted by using SP and TD methods described in the previous sections as well as the values listed by PDG. The Breit-Wigner resonance parameters are also given by PDG, but are not considered here. We see in Table 1 that the SP method fails to find the first resonance pole in the \( S_{11} \) partial wave. We also see that while the real parts of the resonance poles from different approaches are within the ranges of PDG, the extracted imaginary parts can differ by as much as a factor of two or three.

The model of Ref. [22] is currently being improved by also fitting other data of \( \pi N \) and \( \gamma N \) reactions. For example, some progress has been made to also fit the data of \( \pi N \rightarrow \pi N \), \( \gamma N \rightarrow \pi N \), and \( \pi N \rightarrow \eta N \). We thus do not include the results for other partial waves in Table 1. Our purpose here is to simply demonstrate how the analytic continuation method works for a realistic model. The full resonance parameters, including the extracted
TABLE I: Resonance poles extracted from the $\pi N$ scattering amplitudes of Ref. [22]

| Analytic Continuation | Re   | Im   | Speed Plot | Re   | Im   | Time Delay | PDG           | Re   | Im   |
|-----------------------|------|------|------------|------|------|------------|---------------|------|------|
| S11                   | 1540 | -191 | -          | 1543 | -52  | 1490 ~ 1530 | 1543 ~ -125  | 1540 | -191 |
|                       | 1642 | -41  | 1644 -89   | 1645 | -61  | 1640 ~ 1670 | 1543 ~ -125  | 1640 | -41  |
| S31                   | 1563 | -95  | 1574 -67   | 1616 | -53  | 1590 ~ 1610 | 1590 ~ -60   | 1590 | -95  |
|                       | 1211 | -50  | 1212 -49   | 1212 | -49  | 1209 ~ 1211 | 1209 ~ -51   | 1209 | -50  |
| D13                   | 1521 | -58  | 1525 -57   | 1522 | -11  | 1505 ~ 1515 | 1505 ~ -60   | 1505 | -58  |
| F15                   | 1674 | -53  | 1671 -59   | 1683 | -24  | 1665 ~ 1680 | 1665 ~ -68   | 1665 | -53  |

residues and the relations to the Breit-Wigner parameters listed by PDG, extracted from our complete analysis of all $\pi N \gamma , N \rightarrow \pi N, \eta N, \pi \pi N$ will be reported elsewhere.

We now turn to discussing whether the extracted resonance pole $M$ can be used to evaluate the $N^*$ propagator defined by Eq. (116) for the physical positive $E$. Let us consider the $S_{31}$ case listed in Table I. Its $N^*$ propagator $G_{N^*} (E)$ can be written as

$$G_{N^*} (E) = \frac{1}{E - M_0 - \Sigma(E)}.$$  

(118)

$$\Sigma(E) = \Sigma_{\pi N} (E) + \Sigma_{\pi \Delta} (E) + \Sigma_{\rho N} (E).$$  

(119)

By using the analytic continuation methods described in the previous sections, the resonance energy $M = (1563 - i95) \text{ MeV}$ is found numerically by solving

$$M - M_0 - \Sigma(M) = 0.$$  

(120)

We now perform the Laurent expansion of $G_{N^*} (E)$ for real $E$ around the pole position $M$

$$G_{N^*} (E) = [(1 - \Sigma'(M))(E - M) - \frac{1}{2} \Sigma''(M)(E - M)^2 + \cdots]^{-1}$$

$$= \frac{1}{E - M} \left[ \frac{1}{1 - \Sigma'(M) - \frac{1}{2} \Sigma''(M)(E - M) + \cdots} \right]$$

$$= \frac{1}{E - M} \left[ \frac{1}{1 - \Sigma'(M)} + \frac{\Sigma''(M)(E - M)}{2(1 - \Sigma'(E))^2} + \cdots \right]$$

$$= \frac{1}{E - M} \cdot \frac{1}{1 - \Sigma'(M)} + \frac{\Sigma''(M)}{2(1 - \Sigma'(M))^2} + \cdots.$$  

(121)

The naive $1/(E - M)$ works well for a model studied in the previous section. However when the energy dependence of the self energy $\Sigma$ becomes large, two important modifications should be considered: (1) at the pole, the residue is not one but modified by the field renormalization factor $Z = 1/(1 - \Sigma'(M))$. (2) The second term of the last expression gives a constant term.

With the above expansion Eq. (121), we can introduce three different approximate forms for the $N^*$ propagator

$$G_{N^*}^{(0)} (E) = \frac{1}{E - M},$$  

(122)
FIG. 13: Comparisons of various resonance propagators: exact propagator $G_{N^*}(E)$ (cross), $G^{(0)}_{N^*}(E)$ (dashed curve), $G^{(1)}_{N^*}(E)$ (dash-dotted curve) and $G^{(2)}_{N^*}(E)$ (solid curve). The real (imaginary) parts are shown in the (a) ((b)) parts of the figure.

\[ G^{(1)}_{N^*}(E) = \frac{1}{E - M} \cdot \frac{1}{1 - \Sigma'(M)}, \tag{123} \]
\[ G^{(2)}_{N^*}(E) = \frac{1}{E - M} \cdot \frac{1}{1 - \Sigma'(M)} + \frac{\Sigma''(M)}{2(1 - \Sigma'(M))^2}. \tag{124} \]

In Fig. 13, we compare the above three approximate propagators with the exact result of $G_{N^*}(E)$ of Eq. (118). The simple $G^{(0)}_{N^*}(E)$ (dashed curves) of Eq. (122) are far from the exact green function $G_{N^*}(E)$ (cross) defined by Eq. (118). When the factor $1/(1 - \Sigma'(M))$ is included, we obtain the dash-dotted curves for $G^{(1)}_{N^*}(E)$. The solid curves are from $G^{(2)}_{N^*}(E)$.

We see that the constant term $\frac{\Sigma''(M)}{2(1 - \Sigma'(M))^2}$ of Eq. (124) mainly affects the imaginary part of the propagator.

Eq. (121) shows that the simple pole approximation $G^{(0)}_{N^*}(E) = \frac{1}{E - M}$, which can be cast into the usual Breit-Wigner form $1/(E - M_R + i\Gamma_R/2)$ with $M_R = \text{Re} M$ and $\Gamma_R = -2\text{Im} M$, works poorly. We also find that the pole parametrization of the resonant propagator using the poles extracted by the TD and SP methods also work poorly. For the considered $S_{31}$ case, the pole positions from using these two methods are: $E_{SP} = (1574, -67i)$ and $E_{TD} = (1616, -53i)$, as given in Table II. In Fig. 14, we compare the exact propagator (cross) $G_{N^*}(E)$ of Eq. (118) with the following two propagators

\[ G_{N^*,SP}(E) = \frac{1}{E - E_{SP}}, \tag{125} \]
\[ G_{N^*,TD}(E) = \frac{1}{E - E_{TD}}. \tag{126} \]

Clearly, phenomenological forms Eqs. (125)-(126) can not account for the complex coupled-channel resonant mechanisms.
FIG. 14: The resonant propagator $G_{N^*}(E)$ (cross) is compared with $G_{N^*,SP}(E)$ (solid curve) using the SP poles and $G_{N^*,TD}(E)$ (dashed curve) using the TD poles. The real (imaginary) parts are shown in the (a) ((b)) parts of the figure.

VII. SUMMARY

In this paper, we have presented a pedagogical study of the commonly used Speed-Plot (SP) and Time-Delay (TD) methods for extracting the resonance parameters from the empirically determined partial-wave amplitudes. Using a two-channel Breit-Wigner form of the S-matrix, we show that the poles extracted by using these two methods are on different Riemann sheets. The SP method can find resonance poles on the unphysical $uu$-sheet, while the TD method can find poles and zeros of S-matrix on $uu$ or $pp$-sheets and therefore its validity is sensitive to the poles on $up$ or $pu$-sheets. Furthermore, we also show numerically that these two methods can fail to find those poles. Our results support the previous findings that these two methods must be used with cautions in searching for nucleon resonances from the meson-nucleon reaction data in the region where the coupled-channel effects are important.

We then develop an analytic continuation method for extracting the resonance poles within a Hamiltonian formulation of meson-nucleon reactions. The main focus is on resolving the complications due to the coupling with the unstable $\pi\Delta$, $\rho N$, and $\sigma N$ channels which can decay into $\pi\pi N$ states. Explicit numerical procedures are presented and verified within several exactly solvable models. The results from these models are also used to further demonstrate the limitation of the SP and TD methods.

As a first application of the developed analytic continuation method, we present the results from analyzing the $S_{11}$, $S_{31}$, $P_{33}$, $D_{13}$ and $F_{15}$ amplitudes of the dynamical coupled-channels model of $\pi N$ reactions developed in Ref.\cite{22}. We also analyze the resonance propagators and show that the simple pole parametrization of the resonant propagator using the poles extracted from SP and TD methods works poorly.

With the progress made in this work, we can proceed to extract all nucleon resonance parameters within the model of Ref.\cite{23}. However, this can be done more accurately only when the coupling with the unstable $\pi\Delta$, $\rho N$ and $\sigma N$ channels are better determined by also fitting the two-pion production data. Our progress in this direction will be reported elsewhere.
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