On Landsberg general \((\alpha, \beta)\)-metrics with a conformal 1-form

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Abstract

In this paper, we study almost regular Landsberg general \((\alpha, \beta)\)-metrics in Finsler geometry. The corresponding equivalent equations are given. By solving the equations, we give the classification of Landsberg general \((\alpha, \beta)\)-metrics under the condition that \(\beta\) is closed and conformal to \(\alpha\). Under this condition, we prove that regular Landsberg general \((\alpha, \beta)\)-metrics must be Berwaldian when the dimension is greater than two. For the almost regular case, the classification also is given and some new non-Berwaldian Landsberg metrics are found.

Keywords: Finsler metric; Landsberg metric; Berwald metric; general \((\alpha, \beta)\)-metric; Landsberg curvature.

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1 Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction [5]. More colorful than the Riemannian case, there are some non-Riemannian quantities in Finsler geometry, such as the Berwald curvature, the Landsberg curvature, Douglas curvature, S-curvature and etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian[10]. A Finsler metric is called a Berwald metric if its Berwald curvature vanishes and is called a Landsberg metric if its Landsberg curvature vanishes. On Berwald manifold (manifold with Berwald metric), all the tangent spaces with the induced norm are linearly isometric to each other. The local structures of Berwald metrics have been studied by Z.I. Szabó [17]. As we known that all Berwald metrics are Landsberg metrics. This is because that on Landsberg manifold (manifold with Landsberg metric), the tangent spaces with the induced norm are all isometric. It is a long existing open problem in Finsler geometry:

\textbf{Whether or not any Landsberg metric is a Berwald metric ?}

Matsumoto called it the most important unsolved problem in Finsler geometry in 2004. Due to the many unsuccessful attempts made for finding non-Berwald Landsberg metrics, D. Bao called them "unicorns". In order to answer this problem, it is natural to study and construct non-Riemannian Landsberg metrics. Some geometers considered this problem and found this problem is still open when the metric is regular [3, 6, 11, 18, 19]. Thus, there is no regular Landsberg metric found which is non-Berwaldian till now. However, in the almost regular case when the metric is allowed to be singular in some directions, the question has a satisfied answer. In 2006, G. Asanov proved that his metrics arising from Finslerian General Relativity are actually Landsberg metrics but not Berwald metrics [1, 2]. After that, Z. Shen obtained a two-parameter family of Landsberg metrics including Asanov’s examples [13]. Then it is meaningful to ask how to find more Landsberg metrics which are non-Berwaldian even in the almost regular case?

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The metrics obtained by G. Asanov and Z. Shen are belong to \((\alpha, \beta)\)-metrics which are defined by the following form

\[ F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}, \]

where \(\phi(s)\) is a \(C^\infty\) function, \(\alpha = \sqrt{a_{ij}(x)y^iy^j}\) is a Riemannian metric and \(\beta = b_i(x)y^i\) is a 1-form. Randers metric is the simplest non-Riemannian Finsler metric in the form \(F = \alpha + \beta\) [12]. It is well-known that a Randers metric is a Landsberg metric if and only if it is a Berwald metric [10,15]. In 2009, Z. Shen proved that a regular \((\alpha, \beta)\)-metric is a Landsberg metric if and only if it is a Berwald metric and constructed a family of almost regular non-Berwald Landsberg \((\alpha, \beta)\)-metrics [13].

To study the "unicorns" problem and find more Landsberg metrics, its natural to consider a more general metric

\[ F = \alpha \phi(\beta^2, s), \quad s = \frac{\beta}{\alpha}, \]

where \(\phi(s)\) is a \(C^\infty\) function, \(\alpha = \sqrt{a_{ij}(x)y^iy^j}\) is a Riemannian metric and \(\beta = b_i(x)y^i\) is a 1-form, \(b := \|\beta_x\|_\alpha\). It is easy to see that \(F\) is the spherically symmetric metric when \(\alpha = |y|\) is the Euclidean metric and \(\beta = (x, y)\) is the Euclidean inner production. Obviously, \(\beta\) is closed and conformal to \(\alpha\) for all spherically symmetric metrics, i.e.,

\[ b_{ij} = ca_{ij}, \quad (1.1) \]

where \(c = c(x)\) is a scalar function. When \(c = 0\), then \(\beta\) is said to be parallel to \(\alpha\). Some papers studied general \((\alpha, \beta)\)-metrics under this condition [7,8,13,20,22]. Thus, it is interesting to study the problems when \(\beta\) is closed and conformal to \(\alpha\).

In this paper, we mainly consider the "unicorns" problem. We find the equivalent equations of the Berwald and Landsberg general \((\alpha, \beta)\)-metrics under the condition (1.1). Based on these equivalent equations, a more refined characterization is obtained for the Landsberg general \((\alpha, \beta)\)-metrics. For Berwald metrics, we get the following theorem.

**Theorem 1.1** Let \(F = \alpha \phi(\beta^2, s)\) be a general \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold with \(n \geq 3\). Suppose that \(\beta\) is closed and conformal to \(\alpha\), i.e.,

\[ b_{ij} = ca_{ij}, \quad (1.2) \]

where \(c = c(x) \neq 0\) is a scalar function. If \(F\) is a Berwald metric, then it must be a Riemannian metric.

When \(c = c(x) = 0\), it is obvious that \(F\) is a Berwald metric with the same geodesic coefficients of \(\alpha\). In [21], H. Zhu studied the general \((\alpha, \beta)\)-metrics with isotropic Berwald curvature. She obtained the equivalent equations and find the solutions. In fact, the solutions are just Riemannian metrics. Thus the above Theorem 1.1 is based on Zhu’s solutions.

In [23], M. Zohrehvand and H. Maleki studied the regular case when \(\beta\) is closed and conformal to \(\alpha\) and found that Landsberg general \((\alpha, \beta)\)-metrics must be Berwaldian in this case. However, they didn’t consider the almost regular case. A function \(F = \alpha \phi(\beta^2, s)\) is called an almost regular general \((\alpha, \beta)\)-metric if \(\beta\) satisfies that \(\|\beta\|_\alpha \leq b_0, \forall x \in M\). If general \((\alpha, \beta)\)-metrics are allowed to be singular in two extremal directions, then there are some non-trivial solutions. An almost regular general \((\alpha, \beta)\)-metric \(F = \alpha \phi(\beta^2, s)\) might be singular in the two extremal direction \(y \in T_xM\) with \(\beta(x, y) = \pm b_0\alpha(x, y)\). In Proposition 1, we give the equivalent equations of almost regular Landsberg general \((\alpha, \beta)\)-metrics (non-Riemannian) (1.14) and (1.15). To determine the Landsberg metrics, the efficient way is to solve these equations. Actually, we obtain the following result about the general solutions of (1.14) and (1.15).

**Theorem 1.2** Let \(F = \alpha \phi(\beta^2, s)\) be an almost regular non-Riemannian general \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold with \(n \geq 3\). Suppose that \(\beta\) is closed and conformal to \(\alpha\), i.e.,

\[ b_{ij} = ca_{ij}, \quad (1.3) \]

where \(c = c(x) \neq 0\) is a scalar function. Then \(F\) is a Landsberg metric if and only if the function \(\phi = \phi(b^2, s)\) is given by

\[ \phi = \lambda_3 e^{\int_0^\infty \frac{2t+\sqrt{2t^2-t^4+2t^2-s_1+s_2+2s_3s_4}}{2s_1-1+2s_2-s_3-b_0(t^2-t^4)+s_4} dt} \quad (1.4) \]
where \( c_1 = c_1(b^2), \lambda_0 = \lambda_0(b^2), \lambda_1 = \lambda_1(b^2) \) and \( \lambda_3 = \lambda_3(b^2) > 0 \) are \( C^\infty \) functions of \( b^2 \) satisfy the following condition,

\[
\int_0^s A_1(b^2, t) dt = \frac{1}{2b^4s^2}\left\{ b^4(b^2 - s^2)(2\lambda_0 + s^2\lambda_2) - 2sc_1(b^2 - s^2)\frac{b^4}{s} - b^4 \right\} \lambda_3 A(b^2, s) + \left[ b^4 s(\lambda_2 s^2 + 2\lambda_1 + 2\lambda_0) + 2c_1(b^2 - s^2)^2 \right] \lambda_3 - 2b^4 s\lambda_3^2 \right\},
\]

where

\[
A(b^2, t) = \frac{2c_1 \sqrt{b^2 - t^2} + nb^2t\lambda_1 + nb^2t\lambda_0}{2c_1 \sqrt{b^2 - t^2} + nb^2t\lambda_1 + nb^2t - 2b^2(b^2 - t^2)\lambda_0}, \quad A_1(b^2, t) = \partial A(b^2, t) / \partial b^2.
\]

Moreover, \( F \) is not a Berwald metric if and only if \( c_1 \neq 0 \).

In condition (1.1), \( \beta \) can be expressed by [14] when \( \alpha \) is of constant sectional curvature. One can give the explicit expression of \( \phi \) by choosing suitable functions \( c_1(b^2), \lambda_0(b^2), \lambda_1(b^2) \) and \( \lambda_3(b^2) \). When \( c_1(b^2), \lambda_0(b^2), \lambda_1(b^2), \lambda_3(b^2) \) are constants, then \( \phi \) is just (4) in [13]. Thus, some explicit examples can be constructed. In Section 6, we give a new almost regular Landsberg metric.

By Theorem 1.2 when \( c_1 = 0 \), \( F \) is a Berwald metric. In fact,

\[
\phi = \lambda_3 e^{\int_0^s \frac{s^2+2s\lambda_1+2s^2\lambda_0}{2c_1 \sqrt{b^2 - t^2} + nb^2t\lambda_1 + nb^2t - 2b^2(b^2 - t^2)\lambda_0} dt} = \lambda_3 \sqrt{s^2 + 1 - 2(b^2 - s^2)} \lambda_0.
\]

Then \( F = \alpha \phi \) is a Riemannian metric.

**Corollary 1.3** Let \( F = \alpha \phi(b^2, \frac{\partial}{\partial x}) \) be a regular general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold with \( n \geq 3 \). Suppose that \( \beta \) is closed and conformal to \( \alpha \), i.e.,

\[
b_{ij} = c a_{ij},
\]

where \( c = c(x) \neq 0 \) is a scalar function. Then \( F \) is a Landsberg metric if and only if \( F \) is a Riemannian metric.

Then it is natural to consider the general case when \( \beta \) is not conformal to \( \alpha \). It would be studied in our next paper.

## 2 Berwald curvature and Landsberg curvature

For a Finsler metric \( F = F(x, y) \) on a manifold \( M \), the spray \( G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \) is a vector field on \( TM \), where \( G^i = G^i(x, y) \) are defined by

\[
G^i = \frac{1}{4} g^{ij} \left\{ [F^2]_{x^m y^j} y^m - [F^2]_{x^i} \right\},
\]

where \( g_{ij} := \frac{1}{2} [F^2]_{y^i y^j} \) and \( (g^{ij}) := (g_{ij})^{-1} \).

The following lemma is to ensure the positivity of the general \((\alpha, \beta)\)-metric [20].

**Lemma 2.1** (20) Let \( M \) be an \( n \)-dimensional manifold. \( F = \alpha \phi(b^2, \frac{\partial}{\partial x}) \) is a Finsler metric on \( M \) for any Riemannian metric \( \alpha \) and 1-form \( \beta \) with \( \| \beta \|_\alpha < b_0 \) if and only if \( \phi = \phi(b^2, s) \) is a positive \( C^\infty \) function satisfying

\[
\phi - s \phi_\beta > 0, \quad \phi - s \phi_\beta + (b^2 - s^2) \phi_\beta > 0,
\]

when \( n \geq 3 \) or

\[
\phi - s \phi_\beta + (b^2 - s^2) \phi_\beta > 0,
\]

when \( n = 2 \), where \( s \) and \( b \) are arbitrary numbers with \( |s| \leq b < b_0 \).

In this paper, we denote the covariant derivative of the 1-form \( \beta \) with respect to the Riemannian metric \( \alpha \) by \( b_{ij} \). Moreover, for simplicity, let

\[
r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad r_{00} = r_{ij} y^i y^j, \quad s^i = a^i j s_{jk} y^k,
\]
Proposition 2.2

Let

\[ r_i = b^j r_{ji}, \quad s_i = b^j s_{ji}, \quad r_0 = r_i y^i, \quad s_0 = s_i y^i, \quad r^i = a^{ij} r_j, \quad s^i = a^{ij} s_j, \quad r = b^i r_i, \]

The Berwald tensor \( B = B^i_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l \) is defined by

\[
B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.
\]

A Finsler metric is called a *Berwald metric* if \( B^i_{jkl} = 0 \), i.e., the spray coefficients \( G^i = G^i(x, y) \) are quadratic in \( y \in T_x M \) at every point \( x \in M \).

The Landsberg tensor \( L = L_{i}d x^i \otimes dx^j \otimes dx^k \) is defined by

\[
L_{jkl} := -\frac{1}{2} F F_{y^j} [G^i]_{y^k y^l}. \quad (2.4)
\]

A Finsler metric is called a *Landsberg metric* if \( L_{jkl} = 0 \). Clearly, any Berwald metric is a Landsberg metric. The *mean Landsberg curvature* \( J = J_k dx^k \) is defined by

\[
J_k = g^{ij} L_{ijkl}.
\]

A Finsler metric is called a *weak Landsberg metric* if \( L_{jkl} = 0 \). Some results about weak landsberg metric can be found in [4, 9].

The geodesic coefficients \( G^i \) of a general \((\alpha, \beta)\)-metric \( F = \alpha \phi(b^2, \frac{r}{y}) \) were given in [20] as the following

\[
G^i = G^i_\alpha + Py^i + Q^i, \quad (2.5)
\]

where

\[
P = \left\{ \Theta(-2\alpha Q s_0 + r_00 + 2\alpha^2 Rr) + \alpha \Omega(r_0 + s_0) \right\} a^{-1},
\]

\[
Q^i = \alpha Q s_0^i - \alpha^2 R(r^i + s^i) + \left\{ \Psi(-2\alpha Q s_0 + r_00 + 2\alpha^2 Rr) + \alpha \Pi(r_0 + s_0) \right\} b_i,
\]

where

\[
Q = \frac{\phi_2}{\phi - s\phi_2}, \quad \Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_22}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_22)}, \quad \Psi = \frac{\phi_2}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_22)},
\]

\[
R = \frac{\phi_1}{\phi - s\phi_2}, \quad \Pi = \frac{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_22)}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_22)}, \quad \Omega = \frac{\phi_2}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \Pi.
\]

Obviously, if \( \beta \) is parallel with respect to \( \alpha \) \((r_{ij} = 0 \text{ and } s_{ij} = 0)\), then \( P = 0 \) and \( Q^i = 0 \). In this case, \( G^i = G^i_\alpha \) are quadratic in \( y \), and \( F \) is a Berwald metric.

In the following propositions, we give the Berwald curvature of general \((\alpha, \beta)\)-metrics and the Landsberg curvature of general \((\alpha, \beta)\)-metrics. In [21], H. Zhu obtained the expression of \( B^i_{jkl} \). Here we give a different version expressed by \( h_{ijk}, h_{ij} \) and our \( X \) is \( E \) in [21]. Here \( X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad H_1 = \frac{\partial}{\partial H}, \quad H_2 = \frac{\partial}{\partial y}, \) and so on.

**Proposition 2.2** Let \( F = \alpha \phi(b^2, s) \) be a general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold with \( n \geq 2 \). Suppose that \( \beta \) satisfies \( (1.1) \), then the Berwald curvature of \( F \) is given by

\[
B^i_{jkl} = \frac{1}{\alpha^4} \left\{ (H_2 - sH_{22})[h_{jk} h_l + h_{jl} h_k + h_{kl} h_j + H_{222} h_j h_k h_l] b^i \right. \]

\[- \frac{1}{\alpha^2} \left\{ (X - sX_2)[h_{jk} h_l + h_{jl} h_k + h_{kl} h_j] + X_{22}[h_j h_k y_l + h_j y_k h_l] + h_j h_k y_l \right\} b^i \]

\[+ sX_{22}[h_{jk} h_l + h_{jl} h_k + h_{kl} h_j] - X_{222} h_j h_k h_l \right\} y^i + \frac{1}{\alpha^4} \left[ (X - sX_2) h_{kl} + X_{22} h_j h_k \right] \delta^i_j \]

\[+ \frac{1}{\alpha^3} \left[ (X - sX_2) h_{jl} + X_{22} h_j h_l \delta^i_k \right] \delta^j_l + \frac{1}{\alpha^3} \left[ (X - sX_2) h_{jk} + X_{22} h_k h_j \right] \delta^i_k, \quad (2.6)
\]

where

\[
h_j = ab_j - sy_j, \quad h_{jk} = \alpha^2 a_{jk} - y_j y_k,
\]
\[ X = \frac{\phi_2 + 2s\phi_1}{2\phi} - H \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}, \]  
(2.7)

\[ H = \frac{\phi_2 - 2(\phi_1 - s\phi_2)}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_2)}. \]  
(2.8)

**Proof:** By (2.4), we get

\[ r_{00} = \alpha^2, \quad r_0 = c\beta, \quad r = cb^2, \quad r^i = cb^i, \quad s^i = 0, \quad s^i_0 = 0, \quad s_0 = 0. \]  
(2.9)

Substituting (2.9) into (2.5) yields

\[ G^i = G^i_\alpha + \alpha(\Theta(1 + 2b^2R) + s\Omega)y^i + \alpha^2(\Psi(1 + 2b^2R) + s\Pi - R)b^i = G^i_\alpha + \alpha Xy^i + \alpha^2 Hb^i. \]  
(2.10)

where \( X \) and \( H \) are given by (2.7) and (2.8) respectively. By (2.3) and a direct computation, we obtain (2.6). Q.E.D.

Based on (2.6) we can give the Landsberg curvature.

**Proposition 2.3** Let \( F = \alpha \phi(b^2, s) \) be a general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold with \( n \geq 2 \). Suppose that \( \beta \) satisfies (2.1), then the Landsberg curvature of \( F \) is given by

\[ L_{jkl} = -\frac{\rho}{6\alpha^2} \left\{ h_j h_k C_1 + h_j h_l C_k + h_k h_l C_j + 3E_j h_{kl} + 3E_k h_{jl} + 3E_l h_{jk} \right\}, \]  
(2.11)

where

\[ h_j = \alpha b_j - sy_j, \quad h_{jk} = \alpha^2 a_{jk} - y_j y_k, \quad \rho = \phi(\phi - s\phi_2), \]

\[ C_j = \alpha^2 \left\{ [b^2Q + s]H_{222} + [1 + sQ]X_{22} + 3QX_{22} \right\} h_j, \]  
(2.12)

\[ E_j = \alpha^2 \left\{ [b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] \right\} h_j, \]  
(2.13)

\[ X = \frac{\phi_2 + 2s\phi_1}{2\phi} - H \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}, \]  
(2.14)

\[ H = \frac{\phi_2 - 2(\phi_1 - s\phi_2)}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_2)}. \]  
(2.15)

**Proof:** Substituting \( F = \alpha \phi(b^2, s) \) into (2.4) yields

\[ L_{jkl} = -\frac{1}{2} \alpha \phi(\alpha \phi y)[G^i]_{y_i y_j y_k} y_i = -\frac{1}{2} \rho(\alpha Q b_i + y_i) B_{jkl}^i, \]  
(2.16)

where \( \rho = \phi(\phi - s\phi_2) \). By plugging (2.10) into (2.10) and a direct computation, we obtain (2.11). Q.E.D.

### 3 Berwald \((\alpha, \beta)\)-metric

In this section, we are going to give the equivalent equations of Berwald general \((\alpha, \beta)\)-metrics first. Then Theorem 1.1 can be proved.

**Lemma 3.1** Let \( F = \alpha \phi(b^2, s) \) be a non-Riemannian general \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( n \geq 2 \). Suppose \( J = J(b^2, s) \) and \( M = M(b^2, s) \) are arbitrary \( C^\infty \) functions, then the following facts hold:

(i) \( (n \geq 3) \) \( h_j h_k h_i J + h_j h_l h_i M + h_l h_k h_i M + h_k h_l h_i M = 0 \) if and only if \( J = 0 \) and \( M = 0 \);

(ii) \( (n = 2) \) \( h_j h_k h_i J + h_j h_l h_i M + h_l h_k h_i M + h_k h_l h_i M = 0 \) if and only if \( (b^2 - s^2)J + 3M = 0 \).
Proof: (i) \((n \geq 3)\)  
"Necessity": Contracting \(h_jh_kh_lJ + h_jh_kh_lM + h_jh_kh_lM = 0\) with \(b^j\), \(b^k\) and \(b^l\) yields 
\[(b^2 - s^2)J + 3M = 0.\] (3.1)  
Let \(\omega_{jk} := (b^2 - s^2)h_{jk} - h_{j}h_{k}\). Then (3.1) is equivalent to the following equation: 
\[\omega_{jk}h_lM + \omega_{jl}h_kM + \omega_{kl}h_jM = 0.\] (3.2)  
Contracting (3.2) with \(b^l\) yields 
\[\alpha(b^2 - s^2)\omega_{jk}M = 0.\] (3.3)  
Contracting (3.3) with \(a^{jk}\) yields 
\[a^{jk}\alpha(b^2 - s^2)\omega_{jk}M = (n - 2)\alpha^3(b^2 - s^2)^2M = 0.\] (3.4)  
Since \(n \geq 3\), we obtain \(M = 0\). Then \(J = 0\) by (3.1).  
"Sufficiency": It is obvious.  
(ii) \((n = 2)\)  
"Necessity": Contracting \(h_jh_kh_lJ + h_jh_kh_lM + h_jh_kh_lM + h_jh_kh_lM = 0\) with \(b^j\), \(b^k\) and \(b^l\) yields 
\[(b^2 - s^2)J + 3M = 0.\] (3.5)  
"Sufficiency": By \(3M = -(b^2 - s^2)J\), we get 
\[h_jh_kh_lJ + h_jh_kh_lM + h_jh_kh_lM + h_jh_kh_lM = \frac{1}{3}\left\{[h_jh_k - (b^2 - s^2)h_{jk}]h_l + [h_kh_l - (b^2 - s^2)h_{kl}]h_j + [h_lh_j - (b^2 - s^2)h_{lj}]h_k\right\}J.\] 
When \(n = 2\), \((b^2 - s^2)h_{jk} - h_jh_k = 0\). Then we get 
\[h_jh_kh_lJ + h_jh_kh_lM + h_jh_kh_lM + h_jh_kh_lM = 0.\] Q.E.D.  
Based on Lemma 3.1 we can prove the following proposition.  

**Proposition 3.2** Let \(F = \alpha\phi(b^2, \frac{\Delta}{n})\) be a general \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold with \(n \geq 3\). Suppose that \(\beta\) satisfies 
\[b_{ij} = \alpha a_{ij},\] where \(c = c(x) \neq 0\) is a scalar function. Then \(F\) is a Berwald metric if and only if \(\phi = \phi(b^2, \frac{\Delta}{n})\) satisfies 
\[H_2 - sH_{22} = 0, \quad X - sX_2 = 0.\] (3.6)  
**Proof**: Contracting (2.6) with \(b_i\) and contracting (2.6) with \(y_l\) respectively yields 
\[\frac{1}{\alpha^2}\left\{h_jh_kh_l\tilde{J} + h_jh_kh_l\tilde{M} + h_jh_lh_k\tilde{M} + h_kh_jh_l\tilde{M}\right\} = B_{jkli}^i b_i\] (3.7)  
and 
\[\frac{1}{\alpha^3}\left\{h_jh_kh_l\tilde{J} + h_jh_kh_l\tilde{M} + h_jh_lh_k\tilde{M} + h_kh_jh_l\tilde{M}\right\} = B_{jkli}^i y_i,\] (3.8)  
where 
\[\tilde{J} = b^2H_{222} + sX_{222} + 3X_{22}, \quad \tilde{M} = X - sX_2 - s^2X_{22} + b^2(H_2 - sH_{22}).\] (3.9)  
\[\tilde{J} = X_{222} + sH_{222}, \quad \tilde{M} = s(H_2 - sH_{22}) - sX_{22}.\] (3.10)  
"Necessity": By Lemma 3.1 \(B_{jkli}^i = 0\) if and only if \(\tilde{J} = \tilde{M} = \tilde{J} = \tilde{M} = 0\). Then by (3.10), 
\[H_2 - sH_{22} = X_{22}.\] (3.11)
Contracting (3.15) with \( J \) yields
\[
X - sX_2 = -(b^2 - s^2)X_{22}. \tag{3.12}
\]
\( \bar{J} - s\bar{J} = 0 \) is equivalent to
\[
(b^2 - s^2)H_{222} = -3X_{22}. \tag{3.13}
\]
Contracting \( B^i_{jkl} \) for \( i \) and \( j \), we get
\[
B^m_{mkl} = \frac{1}{\alpha^3} \left\{ (H_2 - sH_{22})|2h_kh_l + (b^2 - s^2)h_{kl}| + (b^2 - s^2)H_{222}h_kh_l \right. \\
\left. - X_{22}h_kh_l + (n + 2)[(X - sX_2)h_{kl} + X_{22}h_kh_l]\right\} = 0, \tag{3.14}
\]
Plugging (3.11), (3.12) and (3.13) into (3.14) yields
\[
B^m_{mkl} = \frac{1}{\alpha^3} \left\{ X_{22}|2h_kh_l + (b^2 - s^2)h_{kl}| - 3X_{22}h_kh_l - X_{22}h_kh_l + (n + 2)[-(b^2 - s^2)X_{22}h_kl + X_{22}h_kh_l]\right\} = 0, \tag{3.15}
\]
Contracting (3.15) with \( a^k \) yields
\[
\frac{1}{\alpha}(b^2 - s^2)(n - n^2 + 1)X_{22} = 0. \tag{3.16}
\]
Because \( n - n^2 + 1 \neq 0 \), it follows from (3.10) that
\[
X_{22} = 0. \tag{3.17}
\]
Plugging (3.17) into (3.11) and (3.12) yields
\[
H_2 - sH_{22} = 0, \quad X - sX_2 = 0. \tag{3.18}
\]
"Sufficiency": By (3.6), we get
\[
X_{22} = 0, \quad X_{222} = 0, \quad H_{222} = 0. \tag{3.19}
\]
Plugging (3.6) and (3.19) into (2.6) yields \( B^i_{jkl} = 0 \). Q.E.D.

**Proof of Theorem 1.1**: In [21], H. Zhu gave the general solutions of (3.6). Some of them are just Riemannian metrics and the others can be expressed by
\[
\phi = \varphi \left( \frac{s^2}{e^{\int (\frac{1}{2t} - b^2 t)db^2} + \int te^{\int (\frac{1}{2t} - b^2 t)db^2}db^2} \right) e^{\int (\frac{1}{2b^2 t - \frac{1}{2t}})db^2 s},
\]
where \( \varphi \) is any positive differentiable function and \( t \) is a smooth function of \( b^2 \). It is easy to see that \( \phi \) is an odd function in \( s \). When \( s = 0, \phi = 0 \). Thus it is excluded. Thus there is no non-Riemannian Berwald general \((\alpha, \beta)\)-metric when the dimension \( n \geq 3 \). Q.E.D.

### 4 Landsberg \((\alpha, \beta)\)-metric

In this section, we are going to give the sufficient and necessary conditions of a general \((\alpha, \beta)\)-metric to be a Landsberg metric. By definition, a Finsler metric is called a Landsberg metric if \( L_{jkl} = 0 \). First we prove a equivalent condition of \( L_{jkl} = 0 \). Note that
\[
h_jb^j = \alpha(b^2 - s^2), \quad h_jy^j = 0, \quad C_jy^j = 0, \quad E_jy^j = 0,
\]
\[
h_{jk}b^k = \alpha h_j, \quad h_{jk}y^k = 0, \quad h_{jk}b^j b^k = \alpha^2(b^2 - s^2).
\]
Let
\[
C : = b^jC_j = \alpha^3(b^2 - s^2) \left\{ [b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} \right\},
\]
\[
E : = b^jE_j = \alpha^3(b^2 - s^2) \left\{ [b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] \right\}.
\]
Lemma 4.1 Let $F = \alpha \phi(b^2, s)$ be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $n \geq 2$. Suppose that $\beta$ satisfies (1.1), then the following hold:

(i) $(n \geq 3)$ $L_{jkl} = 0$ if and only if $C_j = 0$ and $E_j = 0$;

(ii) $(n = 2)$ $L_{jkl} = 0$ if and only if $(b^2 - s^2)C_j + 3E_j = 0$.

Proof: (i) $(n \geq 3)$.

"Necessity": Assume that $L_{jkl} = 0$. Then by (2.11), we get

$$h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3E_j h_{jl} + 3E_k h_{jl} + 3E_l h_{jk} = 0. \quad (4.1)$$

Contracting (4.1) with $b^i$, $b^k$ and $b^l$ yields

$$(b^2 - s^2)C + 3E = 0. \quad (b^2 - s^2)C + 3E = 0. \quad (4.2)$$

Let $\omega_{jk} := (b^2 - s^2)h_{jk} - h_j h_k$. By (4.2), (4.1) is equivalent to the following equation:

$$\omega_{jk} C_l + \omega_{jl} C_k + \omega_{kl} C_j = 0. \quad (4.3)$$

Contracting (4.3) with $b^l$ yields

$$\omega_{jk} C = 0. \quad (4.4)$$

Noting $a^{kl} \omega_{kl} = (n - 2)\alpha^2(b^2 - s^2)$ and contracting (4.4) with $a^{jk}$ yields

$$a^{jk} \omega_{jk} C = (n - 2)\alpha^2(b^2 - s^2)C = 0. \quad (4.5)$$

Since $n \geq 3$, we obtain $C = 0$. Contracting (4.3) with $a^{kl}$ yields

$$-2\alpha C h_j + (n - 1)\alpha^2(b^2 - s^2)C_j = 0.$$

Then $C_j = 0$ and $E_j = 0$ by (4.3).

"Sufficiency": It is obvious by (2.11).

(ii) $(n = 2)$.

"Necessity": Assume that $L_{jkl} = 0$. Then by (2.11), we get

$$h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3E_j h_{jl} + 3E_k h_{jl} + 3E_l h_{jk} = 0. \quad (4.5)$$

Contracting (4.5) with $b^i$, $b^k$ and $b^l$ yields

$$(b^2 - s^2)C + 3E = 0. \quad (b^2 - s^2)C + 3E = 0. \quad (4.2)$$

Contracting (4.5) with $b^k$ and $b^l$ yields

$$(b^2 - s^2)C_j + 3E_j = 0. \quad (4.3)$$

"Sufficiency": Plugging $3E_j = -(b^2 - s^2)C_j$ into (2.11), we get

$$L_{jkl} = -\frac{\rho}{6\alpha^3} \left\{ h_j h_k C_l + h_j h_l C_k + h_k h_l C_j - (b^2 - s^2)C_j h_{kl} - (b^2 - s^2)C_k h_{jl} - (b^2 - s^2)C_l h_{jk} \right\} \quad (4.6)$$

$$= -\frac{\rho}{6\alpha^3} \left\{ h_j h_k -(b^2 - s^2)h_{jk} C_l + [h_j h_l -(b^2 - s^2)h_{jl}] C_k + [h_k h_l -(b^2 - s^2)h_{kl}] C_j \right\}.$$
Lemma 4.2 Let $F = \alpha \phi(b^2, s)$ be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $n \geq 2$. Suppose that $\beta$ satisfies

$$b_{ij} = c a_{ij},$$

(4.7)

where $c = c(x) \neq 0$ is a scalar function. If one of the following holds:

(i) $(n \geq 3)$ The function $\phi = \phi(b^2, \frac{\beta}{\alpha})$ satisfies

$$[b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] = 0$$

(4.8)

and

$$[b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} = 0.$$  

(4.9)

(ii) $(n = 2)$ The function $\phi = \phi(b^2, \frac{\beta}{\alpha})$ satisfies

$$(b^2 - s^2)[b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} + 3\phi(2Q + s)(H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] = 0.$$  

(4.10)

then $F = \alpha \phi(b^2, s)$ is a Landsberg metric.

Proof: When the dimension $n \geq 3$, by the assumption (4.8) and (4.9), we get

$$C_j = \alpha b^2 \left\{ [b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} \right\} h_j = 0,$$

$$E_j = \alpha b^2 \left\{ [b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] \right\} h_j = 0.$$

By Lemma 4.1 we obtain $L_{jkl} = 0$.

When the dimension $n = 2$, by (4.10) $\times c \alpha^2 h_j$, we get

$$(b^2 - s^2)\alpha b^2 \left\{ [b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} \right\} h_j$$

$$+ 3\alpha b^2 \left\{ [b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] \right\} h_j = 0.$$  

(4.11)

Which is equivalent to

$$(b^2 - s^2)C_j + 3E_j = 0.$$  

(4.12)

By Lemma 4.1 we obtain $L_{jkl} = 0$.

Now we can prove the following proposition.

Proposition 4.3 Let $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold with $n \geq 2$. Suppose that $\beta$ satisfies

$$b_{ij} = c a_{ij},$$

(4.13)

where $c = c(x) \neq 0$ is a scalar function. Then $F$ is a Landsberg metric if and only if one of the following holds:

(i) When the dimension $n \geq 3$, the function $\phi = \phi(b^2, \frac{\beta}{\alpha})$ satisfies

$$[b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] = 0,$$

(4.14)

$$[b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} = 0.$$  

(4.15)

(ii) When the dimension $n = 2$, the function $\phi = \phi(b^2, \frac{\beta}{\alpha})$ satisfies

$$(b^2 - s^2)[b^2Q + s]H_{222} + [1 + sQ]X_{222} + 3QX_{22} + 3\phi(2Q + s)(H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] = 0.$$  

(4.16)

Proof: The sufficiency is proved by Lemma 4.2. We only need to prove the necessity. When $n \geq 3$, by Lemma 4.1 we get $C_j = 0$ and $E_j = 0$. Then $\phi = \phi(b^2, s)$ satisfies (4.14) and (4.15) by (2.12) and (2.13). When $n = 2$, by Lemma 4.1 we get $(b^2 - s^2)C_j + 3E_j = 0$. Then by (2.12) and (2.13), we get (4.16). Q.E.D.
5 Proof of Theorem 1.2

In this section, we give the general solutions of (4.14) and (4.15) to prove Theorem 1.2. The following lemmas are needed in our proof.

Lemma 5.1 If a $C^\infty$ function $\phi = \phi(b^2, s)$ satisfies $Q = 0$ (i.e. $\frac{\phi_2}{\phi - s\phi_2} = 0$), then

$$\phi = f(b^2),$$

where $f(b^2)$ is an arbitrary $C^\infty$ function of $b^2$.

Proof: By the assumption,

$$Q = \frac{\phi_2}{\phi - s\phi_2} = 0.$$

Then $\phi_2 = 0$. Q.E.D.

Lemma 5.2 If a positive $C^\infty$ function $\phi = \phi(b^2, s)$ satisfies $Q - sQ_2 = 0$, then

$$\phi(b^2, s) = g(b^2)\sqrt{1 + h(b^2) s^2},$$

(5.1)

where $g(b^2) > 0$, $h(b^2)$ are $C^\infty$ functions of $b^2$.

Proof: Differentiating $Q - sQ_2 = 0$ with respect to $s$ yields $-sQ_2 = 0$. Then we get

$$Q_2 = 0.$$ (5.2)

Integrating (5.2) with respect to $s$ yields

$$Q = h(b^2)s + f(b^2),$$

(5.3)

where $h(b^2)$ and $f(b^2)$ are $C^\infty$ functions of $b^2$. Substitute (5.3) into $Q - sQ_2 = 0$ yields $f(b^2) = 0$. Then

$$Q = h(b^2)s.$$ (5.4)

By $Q = \frac{\phi_2}{\phi - s\phi_2}$, we get

$$\frac{\phi_2}{\phi} = \frac{Q}{1 + sQ}.$$ (5.5)

Which is equivalent to

$$[\ln \phi]_2 = \frac{Q}{1 + sQ}.$$ (5.6)

Integrating the above equation with respect to $s$ yields

$$\phi = g(b^2)e^{\int\frac{Q}{1 + sQ}ds}.$$ (5.7)

Substitute (5.4) into (5.7) yields (5.1). Q.E.D.

Lemma 5.3 Let $\phi = \phi(b^2, s)$ be a positive $C^\infty$ function. Then $\phi$ satisfies (4.14) and (4.15) if and only if the function $\phi = \phi(b^2, s)$ satisfies

$$X - sX_2 = \frac{c_1}{\sqrt{b^2 - s^2}},$$

(5.8)

$$H_2 - sH_{22} = -\frac{c_3}{(b^2 - s^2)^2},$$

(5.9)

where $c_1 = c_1(b^2)$ is a $C^\infty$ function of $b^2$.  

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Proof: "Necessity": Differentiating (4.14) with respect to \( s \) yields

\[
\begin{align*}
[b^2 Q + s](-sH_{222}) + [b^2 Q + 1](H_2 - sH_{222}) - [Q + sQ]sX_{22} \\
- [1 + sQ]X_{22} - [1 + sQ]sX_{222} + Q_2[X - sX_2] + Q[-sX_{22}] = 0.
\end{align*}
\] (5.10)

By (4.15) \( \times s + (5.10) \),
\[
[b^2 Q + 1](H_2 - sH_{222}) - [1 + s^2 Q_2]X_{22} + Q_2[X - sX_2] = 0.
\] (5.11)

By Lemma 5.2, we get \( Q - sQ_2 \neq 0 \) because \( F \) is a non-Riemannian metric by the assumption. Hence, it follows from (5.12)
\[
X_{22} = H_2 - sH_{22}.
\] (5.13)

Differentiating (5.13) with respect to \( s \) yields
\[
X_{222} = -sH_{222}.
\] (5.14)

Plugging (5.13) and (5.14) into (4.15) yields
\[
Q\{(b^2 - s^2)H_{222} + 3(H_2 - sH_{22})\} = 0.
\] (5.15)

By Lemma 5.1, we get \( Q \neq 0 \). Then it follows from (5.15) that
\[
(b^2 - s^2)H_{222} + 3(H_2 - sH_{22}) = 0.
\] (5.16)

Plugging (5.13) into (4.14) yields
\[
Q\{(b^2 - s^2)X_{22} + (X - sX_2)\} = 0.
\] (5.17)

By Lemma 5.1, we get \( Q \neq 0 \). By (5.17), we get
\[
(b^2 - s^2)X_{22} + (X - sX_2) = 0.
\] (5.18)

In fact, (5.16) and (5.18) can be solved. (5.16) is equivalent to
\[
\frac{-sH_{222}}{H_2 - sH_{22}} = \frac{3s}{b^2 - s^2}.
\]

Which can be written in
\[
\left[ \ln |H_2 - sH_{22}| \right]_2 = \left[ -\frac{3}{2} \ln (b^2 - s^2) \right]_2.
\]

Integrating the above equation with respect to \( s \) yields
\[
H_2 - sH_{22} = \frac{c_2}{(b^2 - s^2)^{3/2}},
\] (5.19)

where \( c_2 = c_2(b^2) \) is a \( C^\infty \) function of \( b^2 \).

By a similar argument for (5.18), we get
\[
X - sX_2 = \frac{c_1}{\sqrt{b^2 - s^2}},
\] (5.20)

where \( c_1 = c_1(b^2) \) is a \( C^\infty \) function of \( b^2 \).

Plugging (5.19) and (5.20) into (5.13) yields
\[
\frac{c_1}{(b^2 - s^2)^{3/2}} + \frac{c_2}{(b^2 - s^2)^{3/2}} = 0.
\] (5.21)

Then
\[
c_2 = -c_1.
\] (5.22)
Substitute (5.22) into (5.19) yields
\[ H_2 - sH_{22} = -\frac{c_1}{(b^2 - s^2)^2}. \] (5.23)

If \( \phi = \phi(b^2, s) \) satisfies (5.8), (5.9) and \( c_1 \neq 0 \), then \( F = \alpha\phi(b^2, s) \) is an almost regular Landsberg metric. This metric might be singular in two directions \( y \in T_xM \) with \( \beta(x, y) = \pm \alpha(x, y) \).

"Sufficiency". It is obvious by plugging (5.8), (5.9) back to (4.14) and (4.15). By a direct computation, we get
\[
[b^2Q + s](H_2 - sH_{22}) - [1 + sQ]sX_{22} + Q[X - sX_2] \\
= -[b^2Q + s] - \frac{c_1}{(b^2 - s^2)^2} + [s + s^2Q] - \frac{c_1}{(b^2 - s^2)^2} + Q \frac{c_1}{\sqrt{b^2 - s^2}} = 0,
\] (5.24)

\[
[b^2Q + s]H_{22} + [1 + sQ]X_{22} + 3QX_2 \\
= [b^2Q + s] - \frac{3c_1}{(b^2 - s^2)^2} - [1 + sQ] - \frac{3sc_1}{(b^2 - s^2)^2} - 3Q \frac{c_1}{(b^2 - s^2)^2} = 0.
\] (5.25)

Q.E.D.

**Proof of Theorem 1.2**. By Proposition 4.3, \( \phi \) satisfies (4.14) and (4.15). Then \( \phi \) satisfies (5.8) and (5.9) by Lemma 5.3. Thus we only need to solve (5.8) and (5.9) for \( \phi \). Substituting (5.8), (5.9) back to (5.18), (5.16) yields
\[ X_{22} = -\frac{c_1}{(b^2 - s^2)^2}, \]
\[ H_{222} = \frac{3c_1}{(b^2 - s^2)^2}. \]

Then by integrating \( X_{22} \) and \( H_{222} \) with respect to \( s \), we get
\[ X = \frac{c_1 \sqrt{b^2 - s^2}}{b^2} + \lambda_1 s + k_0, \] (5.26)
\[ H = \frac{1}{2} \lambda_2 s^2 - \frac{c_1 s \sqrt{b^2 - s^2}}{b^2} + k_1 s + \lambda_0, \] (5.27)

where \( c_1 = c_1(b^2), k_0 = k_0(b^2), k_1 = k_1(b^2), \lambda_0 = \lambda_0(b^2), \lambda_1 = \lambda_1(b^2) \) and \( \lambda_2 = \lambda_2(b^2) \) are \( C^\infty \) functions of \( b^2 \).

Substituting (5.26) and (5.27) into (5.8) and (5.9) yields \( k_0 = 0 \) and \( k_1 = 0 \). Then we get
\[ X = \frac{c_1 \sqrt{b^2 - s^2}}{b^2} + \lambda_1 s, \] (5.28)
\[ H = \frac{1}{2} \lambda_2 s^2 - \frac{c_1 s \sqrt{b^2 - s^2}}{b^2} + \lambda_0, \] (5.29)

where \( c_1 = c_1(b^2), \lambda_0 = \lambda_0(b^2), \lambda_1 = \lambda_1(b^2) \) and \( \lambda_2 = \lambda_2(b^2) \) are \( C^\infty \) functions of \( b^2 \).

By (2.14) and (2.15), we get
\[ 2\phi X - \phi_2 - 2s\phi_1 + 2H \left[s\phi + (b^2 - s^2)\phi_2\right] = 0, \] (5.30)
\[ 2H \left[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\right] - \phi_{22} + 2(\phi_1 - s\phi_1) = 0. \] (5.31)

Differentiating (5.30) with respect to \( s \) yields
\[ 2\phi_2 X + 2\phi X_2 - \phi_2 - 2s\phi_{12} + 2H_2[s\phi + (b^2 - s^2)\phi_2] + 2H[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}] = 0. \] (5.32)

To cancel the term involving \( \phi_{12} \), we consider (5.32) - (5.31). Then we get
\[ \phi_2 X + \phi X_2 - 2\phi_1 + H_2[s\phi + (b^2 - s^2)\phi_2] = 0. \] (5.33)

(5.33) \times s - (5.30) yields
\[ \left\{sX + sH_2(b^2 - s^2) + 1 - 2H(b^2 - s^2)\right\}\phi_2 + \left\{sX_2 + s^2H_2 - 2X - 2sH\right\}\phi = 0. \] (5.34)
Plugging (5.28) and (5.29) into (5.34) yields

$$\left\{ 2sc_1\sqrt{b^2 - s^2 + b^2s^2}, 1 + b^2 - 2b^2(b^2 - s^2), \lambda_0 \right\} \phi_2 - \left\{ 2c_1\sqrt{b^2 - s^2 + b^2s}, 1 + 2b^2s, \lambda_0 \right\} \phi = 0.$$  \hspace{0.5cm} (5.35)

Which is equivalent to

$$\left\{ 2sc_1\sqrt{b^2 - s^2 + b^2s^2}, 1 + b^2 - 2b^2(b^2 - s^2), \lambda_0 \right\} (\ln \phi) = \left\{ 2c_1\sqrt{b^2 - s^2 + b^2s}, 1 + 2b^2s, \lambda_0 \right\}.$$  \hspace{0.5cm} (5.36)

Then by a direct computation, we get

$$\phi = \lambda_3 e^{\int_0^s \frac{2c_1\sqrt{b^2 - t^2 + b^2t\lambda_3 + 2b^2t\lambda_0}}{2c_1\sqrt{b^2 - t^2 + b^2t\lambda_1 + b^2 - 2b^2(b^2 - t^2)\lambda_0}} dt}.$$  \hspace{0.5cm} (5.37)

where $c_1 = c_1(b^2)$, $\lambda_0 = \lambda_0(b^2)$, $\lambda_1 = \lambda_1(b^2)$ and $\lambda_3 = \lambda_3(b^2) > 0$ are $C^\infty$ functions of $b^2$. It can be seen that $\phi(b^2, s)$ might be singular in two directions $y \in T_x M$ with $\beta(x, y) = \pm b\alpha(x, y)$. To prove the sufficiency, we only need substitute (5.37) into (5.30). In fact, $c_1(b^2)$, $\lambda_0(b^2)$, $\lambda_1(b^2)$ and $\lambda_3(b^2)$ are not arbitrary. They should satisfy the following condition,

$$\int_0^s A_1(b^2, t) dt = \frac{1}{2b^3s\lambda_3} \left\{ \left[ b^4(b^2 - s^2)(2\lambda_0 + s^2\lambda_2) - 2sc_1(b^2 - s^2)^2 - b^4\right] \lambda_3 A(b^2, s) + \left[ b^4s\lambda_2 s^2 + 2\lambda_1 + 2\lambda_0 \right] + 2c_1(b^2 - s^2)^2 \right\} - 3 - 2b^4s\lambda_3,$$  \hspace{0.5cm} (5.38)

where

$$A(b^2, t) = \frac{2c_1\sqrt{b^2 - t^2 + b^2t\lambda_1 + b^2 - b^2(b^2 - t^2)\lambda_0}}{2c_1\sqrt{b^2 - t^2 + b^2t\lambda_1 + b^2 - b^2(b^2 - t^2)\lambda_0}}, \quad A_1(b^2) = \frac{\partial A(b^2, t)}{\partial b^2}.$$  \hspace{0.5cm} (5.39)

Plugging (5.28) and (5.29) into $G^i$ yields

$$G^i = G^i_0 + ca Xy^i + ca^2 Hb^i = G^i_0 + c \left\{ \left[ c_1 \frac{\sqrt{b^2\alpha^2 - \beta^2}}{b^2} + \lambda_1 \beta \right] y^i + \left[ \frac{1}{2} \lambda_2 \beta^2 - c_1 \frac{\sqrt{b^2\alpha^2 - \beta^2}}{b^2} + \lambda_0 \alpha^2 \right] b^i \right\}.$$  

Thus if $c_1 \neq 0$, then $G^i$ are not quadratic in $y$. Hence $F$ is not a Berwald metric. Q.E.D.

6 Explicit Example

In this section, we construct a new Landsberg metric by Theorem 1.2. In order to find explicit metric, we need to choose suitable $c_1(b^2)$, $\lambda_0(b^2)$, $\lambda_1(b^2)$ and $\lambda_3(b^2)$.

Differentiating (1.5) with respect to $s$ yields

$$c(b^2)s^2 - b^2\sqrt{b^2 - s^2}\tau(b^2)s - b^2c(b^2) = 0,$$  \hspace{0.5cm} (6.1)

where

$$c(b^2) = \left[ 2(2b^2\lambda_0 + b^4\lambda_2 + b^2\lambda_1 + 1)\lambda_0 - \lambda_1 + b^2(4\lambda_0 - \lambda_2) \right] c_1 + 2(1 - 2b^2\lambda_0)c_1', \hspace{0.5cm} (6.2)$$

$$\tau(b^2) = (2b^2\lambda_0 - 1)\lambda_1^2 + \left[ 2b^2(b^2\lambda_2 + 2\lambda_0)\lambda_0 + 2b^2\lambda_0 - b^2\lambda_2 \right] \lambda_1 + 2b^2(\lambda_2 - \lambda_1)\lambda_0 + 4\lambda_0^2 - \lambda_2 + \lambda_1 + 2\lambda_0. \hspace{0.5cm} (6.3)$$

Because $s$ is a variable in (6.1), we get

$$c(b^2) = 0, \quad \tau(b^2) = 0.$$  \hspace{0.5cm} (6.4)

From the above equation, if we set the two functions in $\lambda_0$, $\lambda_1$, $\lambda_2$, $c_1$, then the other functions can be determined by (6.4). Then we can get $\lambda_3$ by (1.5).
Example 6.1  Put $\lambda_0(b^2) = 0$, $\lambda_1(b^2) = 0$ in (6.4), we get $\lambda_2 = 0$, $c_1 = k$ ($k$ is a constant). Then we can get $\lambda_3 = m$ ($m$ is a constant). Let $c_1 = 1$, $\lambda_3 = 1$, then

$$\phi = e^{\int_0^s \frac{2\sqrt{b^2-s^2}}{2s^2} dt}.$$  \hspace{1cm} (6.5)

To ensure the positivity of $F$, $\phi$ should satisfy Lemma 2.1. In the following we prove that $\phi$ satisfies Lemma 2.1.

Differentiating (6.5) with respect to $s$ yields

$$\phi_2 = e^{\int_0^s \frac{2\sqrt{b^2-s^2}}{2s^2} dt} \left[ \frac{2\sqrt{b^2-s^2}}{2s^2} \right].$$ \hspace{1cm} (6.6)

Differentiating (6.6) with respect to $s$ yields

$$\phi_{22} = e^{\int_0^s \frac{2\sqrt{b^2-s^2}}{2s^2} dt} \left[ \frac{2b^2s}{\sqrt{b^2-s^2}(2s^2+b^2)} \right].$$

Then

$$\phi - s\phi_2 = e^{\int_0^s \frac{2\sqrt{b^2-s^2}}{2s^2} dt} \left[ 1 - \frac{2s\sqrt{b^2-s^2}}{2s^2+b^2} \right] > 0,$$

$$\phi - s\phi_2 + (b^2-s^2)\phi_{22} = e^{\int_0^s \frac{2\sqrt{b^2-s^2}}{2s^2} dt} \left[ 1 - \frac{4s^2(b^2-s^2)+4b^2s\sqrt{b^2-s^2}}{4s^4(b^2-s^2) + 4b^2s\sqrt{b^2-s^2} + b^4} \right] > 0.$$

Thus $F = \alpha \phi$ is an almost regular Landsberg (non-Berwaldian) metric when $\beta$ is closed and conformal to $\alpha$.

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