RANK 3 ARITHMETICALLY COHEN-MACALAY BUNDLES OVER HYPERSURFACES

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ABSTRACT. Let $X$ be a smooth projective hypersurface of dimension $\geq 5$ and let $E$ be an arithmetically Cohen-Macaulay bundle on $X$ of any rank. We prove that $E$ splits as a direct sum of line bundles if and only if $H^i(X, \wedge^2 E) = 0$ for $i = 1, 2, 3, 4$. As a corollary this result proves a conjecture of Buchweitz, Greuel and Schreyer for the case of rank 3 arithmetically Cohen-Macaulay bundles.

1. Introduction

We work over an algebraically closed field of characteristic 0. Let $\{X, \mathcal{O}_X(1)\} \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree $d$. We say a vector bundle on $X$ is split if it can be written as a direct sum of line bundles. We say that it is indecomposable if it can not be written as a direct sum of vector bundles of strictly smaller rank.

An arithmetically Cohen-Macaulay (ACM) vector bundle $E$ on $X$ is a locally free sheaf satisfying

$$H^i(X, E) := \oplus_{k \in \mathbb{Z}} H^i(X, E(k)) = 0 \text{ for } i = 1, \ldots, n-1$$

Some of the reasons why the study of ACM bundles is important are:

- On projective space, ACM bundles are precisely the bundles which are direct sum of line bundles [Horrocks1964].
- By semicontinuity, ACM bundles form an open set in any flat family of vector bundles over $X$.
- The $n$’th syzygy of a resolution of any vector bundle on $X$ by split bundles, is an arithmetically Cohen-Macaulay bundle [Eisenbud1981].
- These sheaves correspond to maximal Cohen-Macaulay modules over the associated coordinate ring [Beauville2000].

When $d > 1$ there always exist indecomposable arithmetically Cohen-Macaulay bundles see e.g. [KRR2007] for low dimensional construction and [BGS1987] for a construction for higher dimensional hypersurfaces. The following conjecture forms the basis of research done in the direction of investigating the splitting behaviour of ACM bundles over hypersurfaces:
Conjecture (Buchweitz, Greuel and Schreyer [BGS1987]): Let \( X \subset \mathbb{P}^n \) be a hypersurface. Let \( E \) be an ACM bundle on \( X \). If \( \text{rank } E < 2^e \), where \( e = \left\lfloor \frac{n-2}{2} \right\rfloor \), then \( E \) splits. (Here \([q]\) denotes the largest integer \( \leq q \).

\[ \Box \]

This conjecture can not be strengthened further as the authors constructed an indecomposable ACM bundle of rank \( 2^e \) in op. cit.

For rank 2 ACM bundles, the conjecture follows from [Kleppe1978]. Generic behaviour for rank 2 case is also well understood when \( n \geq 4 \) and we refer the reader to [CM2002], [CM2005], [KRR2007], [KRR2007(2)], [Ravindra2009] and to the reference cited in these articles. For lower dimensional case, we refer the reader to [Madonna1998], [Madonna2000], [Faenzi2008], [CF2009] and [CH2011]. The result for rank 2 bundles was generalized to complete intersections in [BR2010].

For rank 3 ACM bundles the conjecture predicts splitting for \( n \geq 5 \) dimensional hypersurfaces. We proved a weaker version in [Tripathi2015]. In this article, we prove the conjecture for rank 3 arithmetically Cohen-Macaulay bundles.

**Theorem 1.1.** Let \( X \) be a smooth hypersurface of dimension \( \geq 5 \). Let \( E \) be a rank 3 arithmetically Cohen-Macaulay bundle over \( X \). Then \( E \) is a split bundle.

This result follows as a corollary from the main result of this article - a splitting criterion for ACM bundles of any rank.

**Theorem 1.2.** Let \( X \) be a smooth hypersurface of dimension \( \geq 5 \). Let \( E \) be an arithmetically Cohen-Macaulay vector bundle on \( X \) of any rank. Then \( E \) splits if and only if \( H^i(X, \wedge^2 E) = 0 \) for \( i = 1, 2, 3, 4 \).

2. Preliminaries

In this section, we will recall some standard facts about arithmetically Cohen-Macaulay bundles over hypersurfaces.

Let \( X \subset \mathbb{P}^{n+1} \) be a degree \( d \) smooth hypersurface given by homogeneous polynomial \( f = 0 \). Let \( E \) be an ACM bundle of rank \( r \) on \( X \). By Serre’s duality, \( E^\vee \) is also ACM.

For notational ease, we will use \( \sim \) to denote a vector bundle on \( \mathbb{P}^{n+1} \). By Hilbert’s syzygy theorem, being a coherent sheaf on \( \mathbb{P}^{n+1} \), \( E \) admits a finite length minimal free resolution

\[
0 \to \tilde{F}_t \to \tilde{F}_{t-1} \to \ldots \to \tilde{F}_1 \to \tilde{F}_0 \to E \to 0
\]

where \( \tilde{F}_i \) are direct sums of the form \( \oplus_j \mathcal{O}_{\mathbb{P}^{n+1}}(a_j) \). By minimality of the resolution and the ACM condition on \( E \), the first syzygy \( \tilde{K} = \text{Ker}(\tilde{F}_0 \to E) \) is an ACM bundle on \( \mathbb{P}^{n+1} \) and therefore is a split bundle by Horrocks’s criterion. Thus the minimal free resolution of \( E \) on \( \mathbb{P}^{n+1} \) is of the form

\[
0 \to \tilde{F}_1 \xrightarrow{\phi} \tilde{F}_0 \to E \to 0
\]
Localizing at the generic point, one checks that the ranks of $\tilde{F}_1$ and $\tilde{F}_0$ are same. Restricting the above resolution to $X$ gives,

$$0 \to \text{Tor}_1^{\mathcal{O}_{\mathbb{P}^{n+1}}}(E, \mathcal{O}_X) \to \tilde{F}_1 \to \tilde{F}_0 \to E \to 0$$

where one computes the $\text{Tor}$ term by tensoring $0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{\times f} \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$ with $E$ to get $\text{Tor}_1^{\mathcal{O}_{\mathbb{P}^{n+1}}}(E, \mathcal{O}_X) = E(-d)$ as multiplication by $f$ vanishes on $X$. Thus the above four term sequence breaks up as

$$0 \to \tilde{F}_0(-d) \to \tilde{F}_1 \to \tilde{F}_0 \to 0 \tag{2}$$
$$0 \to E(-d) \to \tilde{F}_1 \to E \to 0 \tag{3}$$

where $\tilde{F}_i = \tilde{F}_i \otimes \mathcal{O}_X$ are split bundles over $X$ of rank $m$ and $E^\sigma := \text{Ker}(\tilde{F}_0 \to E)$ is an arithmetically Cohen-Macaulay bundle on $X$.

We state the following facts (without proof) about matrix factorization theory of Eisenbud and the connection between $E$ and $E^\sigma$. We choose a matrix (with homogeneous polynomial entries) to represent the map $\phi : \tilde{F}_1 \to \tilde{F}_0$ and henceforth we will use the symbol $\phi$ interchangeably to represent either the matrix or the map. Then

1. There exists an injective map $\psi : \tilde{F}_0(-d) \to \tilde{F}_1$ such that $\phi \psi = \psi \phi = f \cdot 1$ where $1$ denotes the identity matrix.
2. $\text{Coker}(\psi) = E^\sigma$ and $E$ is indecomposable if and only if $E^\sigma$ is indecomposable.
3. $0 \to \tilde{F}_0(-d) \to \tilde{F}_1 \to E^\sigma \to 0$ is a minimal free resolution of $E^\sigma$.

For details, we refer to section 6 of [Eisenbud1981] and section 2 of [CH2011].

**Lemma 2.1.** Let $f$ be any homogeneous (perhaps reducible) polynomial of degree $d$. Let $X = V(f) \subset \mathbb{P}^{n+1}$ be the vanishing set. Suppose $\mathcal{F}$ be any coherent sheaf on $X$ which admits a free resolution on $\mathbb{P}^{n+1}$ of the form

$$0 \to \tilde{F}_1 \to \tilde{F}_0 \to \mathcal{F} \to 0$$

where $\tilde{F}_i$ are direct sum of line bundles on $\mathbb{P}^{n+1}$. Then $\mathcal{F}$ is a reflexive sheaf on $X$.

**Proof.** We apply $\text{Hom}(-, \mathcal{O}_{\mathbb{P}^{n+1}})$ on the resolution of $\mathcal{F}$ to get

$$0 \to \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \to \tilde{F}_0^\vee \to \tilde{F}_1^\vee \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \to 0$$

First term vanishes. To compute the $\mathcal{E}xt$ term, we apply $\text{Hom}(\mathcal{F}, -)$ on

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$$

to get

$$0 \to \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \to \text{Hom}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}})(-d) \xrightarrow{\times f}$$

Here the first term vanishes as before and the last map (multiplication by $f$) vanishes as the sheaves are supported on $X$. Thus we get $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \cong \mathcal{F}^\vee(d)$ and a resolution
of $\mathcal{F}^\vee$ on $\mathbb{P}^{n+1}$ as

$$
0 \to \tilde{F}_0^\vee (-d) \to \tilde{F}_1^\vee (-d) \to \mathcal{F}^\vee \to 0
$$

(4)

Applying the whole process once again to the above resolution of $\mathcal{F}^\vee$ we get the following resolution of $\mathcal{F}'\vee$

$$
0 \to \tilde{F}_1 \to \tilde{F}_0 \to \mathcal{F}'\vee \to 0
$$

Comparing with the resolution of $\mathcal{F}$, one gets the claim. \qed

Given a short exact sequence of vector bundles $0 \to E_1 \to E_2 \to E_3 \to 0$ on a variety $X$, there exists a resolution of the $k$'th exterior power $\wedge^k E_3$,

$$
0 \to S^k E_1 \to S^{k-1} E_1 \otimes \wedge^1 E_2 \to \ldots \wedge^k E_2 \to \wedge^k E_3 \to 0
$$

(5)

Dually, we also have a resolution of $k$'th symmetric power,

$$
0 \to \wedge^k E_1 \to \wedge^k E_2 \to \wedge^{k-1} E_2 \otimes S^1 E_3 \to \ldots \wedge^1 E_2 \otimes S^{k-1} E_3 \to S^k E_3 \to 0
$$

(6)

For details we refer the reader to [BE1975].

3. A COKERNEL SHEAF

Suppose rank $\tilde{F}_0 = \text{rank} \tilde{F}_1 = m$. Fix any integer $k \leq \min\{\text{rank}(E), \text{rank}(E^\sigma)\}$. Let $X_k = V(f^k)$ denote the scheme-theoretic $k$'th thickening of $X \subset \mathbb{P}^{n+1}$.

We consider the $k$'th exterior power of the map $\phi : \tilde{F}_1 \to \tilde{F}_0$ in equation (1) and denote the cokernel sheaf by $\mathcal{F}_k$

$$
0 \to \wedge^k \tilde{F}_1 \xrightarrow{\wedge^k \phi} \wedge^k \tilde{F}_0 \to \mathcal{F}_k \to 0
$$

(7)

The following lemma states some properties of the sheaf $\mathcal{F}_k$. Our proof is similar to that in section 2 of [KRR2007] where the case when $E$ is a rank 2 ACM bundle and $k = 2$ was studied.

**Lemma 3.1.**  
1. $\mathcal{F}_k$ is a coherent $\mathcal{O}_{X_k}$-module where $X_k$ is the thickened hypersurface defined scheme theoretically by $f^k$.
2. $\tilde{F}_k := \mathcal{F}_k \otimes \mathcal{O}_X$ is a vector bundle on $X$ of rank $\binom{m}{k} - \binom{m-r}{k}$
3. $\mathcal{F}_k$ is an ACM and reflexive sheaf on $X_k$.

**Proof.** First two claims can be verified locally. By localising on $X$, one can assume that equation (1) looks like

$$
0 \to \mathcal{O}_p^\oplus m \xrightarrow{\phi} \mathcal{O}_p^\oplus m \to E_p \to 0
$$

and the matrix $\phi$ is given by the $m \times m$ diagonal matrix

$$
\{f, \ldots, f, 1, \ldots, 1\}
where \( f \) appears \( r = \text{rank}(E) \) times and 1 appears \( m - r \) times on the diagonal. Then locally the matrix \( \wedge^k \phi \) is the diagonal matrix 
\[
\{ f^k, \ldots, f^k, f^{k-1}, \ldots, f^{k-1}, f, 1, 1, \ldots, 1 \}
\]

where \( f^{k-i} \) appears \( \binom{r}{k-i} \binom{m-r}{i} \) times on the diagonal. In particular, locally \( F_k \) is of the form
\[
\mathcal{O}_{X_k}^\oplus(k) \oplus \mathcal{O}_{X_{k-1}}^\oplus(k-1) \binom{r}{1} \oplus \cdots \oplus \mathcal{O}_{X_{k-i}}^\oplus(k-i) \binom{m-r}{i} \oplus \cdots \oplus \mathcal{O}_{X}^\oplus(k) \binom{m-r}{k-1}.
\]

This proves the first claim and also that \( \bar{F}_k = F_k \otimes \mathcal{O}_X \) is a vector bundle on \( X \). Claim about the rank is verified by the above local description of \( F_k \) and the combinatorial identity
\[
\binom{m}{k} = \sum_i \binom{r}{i} \binom{m-r}{k-i}.
\]

By equation (7), one easily sees that \( F_k \) is an ACM sheaf on \( X \). Lemma 2.1 completes the proof by showing that \( F_k \) is a reflexive sheaf. 

We now restrict sequence (7) to \( X \)
\[
0 \rightarrow Tor_{P_{n+1}}^1(F_k, \mathcal{O}_X) \rightarrow \wedge^k \tilde{F}_1 \rightarrow \wedge^k \bar{F}_0 \rightarrow \bar{F}_k \rightarrow 0 \tag{8}
\]

This is a sequence of vector bundles and the \( Tor \) term is a vector bundle of same rank as \( \bar{F}_k \). In fact, the map \( F_1 \rightarrow F_0 \) factors via \( E^\sigma \), therefore by functoriality of exterior product, the map \( \wedge^k \tilde{F}_1 \rightarrow \wedge^k \bar{F}_0 \) factors via \( \wedge^k E^\sigma \) and the sequence [5] breaks up as
\[
0 \rightarrow Tor_{P_{n+1}}^1(F_k, \mathcal{O}_X) \rightarrow \wedge^k \tilde{F}_1 \rightarrow \wedge^k E^\sigma \rightarrow 0 \tag{9}
\]
and
\[
0 \rightarrow \wedge^k E^\sigma \rightarrow \wedge^k \bar{F}_0 \rightarrow \bar{F}_k \rightarrow 0 \tag{10}
\]

Thus the \( Tor \) term appears as the first term in the filtration of \( k \)'th exterior power of \( \bar{F}_1 \) derived from the sequence \( 0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0 \). We can say more,

**Lemma 3.2.** \( Tor_{P_{n+1}}^1(F_k, \mathcal{O}_X) \simeq \bar{F}_k^\vee(-kd) \)

**Proof.** We consider the \( k \)'th exterior power of the minimal resolution of \( E^\vee \) given by sequence [4]
\[
0 \rightarrow (\wedge^k \tilde{F}_0^\vee)(-kd) \rightarrow (\wedge^k \bar{F}_1^\vee)(-kd) \rightarrow \mathcal{F}_k' \rightarrow 0 \tag{11}
\]
where \( \mathcal{F}_k' \) is defined by the sequence. Restricting to \( X \) gives
\[
0 \rightarrow Tor_{P_{n+1}}^1(F_k', \mathcal{O}_X) \rightarrow (\wedge^k \tilde{F}_0^\vee)(-kd) \rightarrow (\wedge^k \bar{F}_1^\vee)(-kd) \rightarrow \bar{F}_k' \rightarrow 0
\]

As in lemma 3.1 one can verify (by looking at the exterior power matrix locally) that \( \bar{F}_k' \) is a vector bundle and thus above is a exact sequence of vector bundles. So we can
dualize (and then twist by $-kd$) to get:

$$0 \rightarrow \mathcal{E}^\vee_k(-kd) \rightarrow \wedge^k \mathcal{F}_1 \rightarrow \wedge^k \mathcal{F}_0 \rightarrow \text{Tor}^1(\mathcal{O}_X, \mathcal{F}_k)^\vee(-kd) \rightarrow 0 \quad (12)$$

Comparing with equation (11), we get

$$\text{Tor}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \mathcal{E}^\vee_k(-kd) \quad (13)$$

We complete the proof by showing that $\mathcal{F}_k' \cong \mathcal{F}_k^\vee$. Applying $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^{n+1}})$ to sequence (11) and simplifying as in the proof of Lemma 2.1, we get

$$0 \rightarrow \wedge^k \mathcal{F}_1 \rightarrow \wedge^k \mathcal{F}_0 \rightarrow \mathcal{F}_k' \rightarrow 0 \quad (14)$$

Comparing this with the sequence (11) and using the fact that by Lemma 2.1, $\mathcal{F}_k', \mathcal{F}_k^\vee$ are both reflexive sheaves, we get that $\mathcal{F}_k' \cong \mathcal{F}_k^\vee$.

**Lemma 3.3.** There exists a short exact sequence

$$0 \rightarrow \wedge^k E(-kd) \rightarrow \text{Tor}^1_{\mathbb{P}}(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \text{Tor}^1_{X,k}(\mathcal{F}_k, \mathcal{O}_X) \rightarrow 0$$

**Proof.** We restrict the sequence (7) to $X_k$ to get a free $\mathcal{O}_{X_k}$-resolution of $\mathcal{F}_k$

$$\cdots \rightarrow \wedge^k F_1(-kd) \rightarrow \wedge^k F_0(-kd) \rightarrow \wedge^k F_1 \rightarrow \wedge^k F_0 \rightarrow \mathcal{F}_k \rightarrow 0$$

Tensoring this resolution with $\mathcal{O}_X$ gives a complex from which we get

$$\text{Tor}^1_{X,k}(\mathcal{F}_k, \mathcal{O}_X) \cong \frac{\ker(\wedge^k F_1 \rightarrow \wedge^k F_0)}{\text{im}(\wedge^k F_0(-kd) \rightarrow \wedge^k F_1)} \quad (15)$$

To compute $\ker(\wedge^k F_1 \rightarrow \wedge^k F_0)$, we tensor the sequence (7) with $\mathcal{O}_X$ to get

$$\ker(\wedge^k F_1 \rightarrow \wedge^k F_0) \cong \text{Tor}^1_{\mathbb{P}}(\mathcal{F}_k, \mathcal{O}_X)$$

For the $\text{im}(\wedge^k F_0(-kd) \rightarrow \wedge^k F_1)$ term, we note that the map $\bar{F}_0(-d) \rightarrow \bar{F}_1$ factors via $E(-d)$ so by functoriality of wedge power,

$$\text{im}(\wedge^k F_0(-kd) \rightarrow \wedge^k F_1) \cong \wedge^k E(-kd)$$

This completes the proof of the lemma. \qed

### 3.1. A short exact sequence.

Let $\mathcal{F}$ be any coherent $\mathcal{O}_{X_k}$-module. The inclusions $X_{k-1} \hookrightarrow \mathbb{P}^{n+1}$ and $X \hookrightarrow X_k$ induces following short exact sequences

$$0 \rightarrow \mathcal{O}_{X_{k-1}}(-d) \rightarrow \mathcal{O}_{X_k} \rightarrow \mathcal{O}_X \rightarrow 0 \quad (16)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-(k-1)d) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{X_{k-1}} \rightarrow 0 \quad (17)$$

Tensoring both sequences with $\otimes_{\mathbb{P}} \mathcal{F}$, we get

$$0 \rightarrow \text{Tor}^1_{\mathbb{P}}(\mathcal{F}, \mathcal{O}_{X_{k-1}}(-d)) \rightarrow \mathcal{F}(-kd) \rightarrow \text{Tor}^1_{\mathbb{P}}(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{F}|_{X_{k-1}}(-d) \rightarrow \mathcal{F} \rightarrow \overline{\mathcal{F}} \rightarrow 0 \quad (18)$$

$$0 \rightarrow \text{Tor}^1_{\mathbb{P}}(\mathcal{F}, \mathcal{O}_{X_{k-1}}) \rightarrow \mathcal{F}(-(k-1)d) \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_{X_{k-1}} \rightarrow 0 \quad (19)$$
Similarly, tensoring sequence $\langle 16 \rangle$ with $\otimes_X \mathcal{F}$, we get
\[
0 \to Tor^1_{X_k}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{F}|_{X_{k-1}}(-d) \to \mathcal{F} \to \mathcal{F} \to 0 \quad (20)
\]
Comparing sequences $\langle 18 \rangle$ and $\langle 20 \rangle$ gives
\[
0 \to Tor^1_F(\mathcal{F}, \mathcal{O}_{X_{k-1}})(-d) \to \mathcal{F}(-kd) \to Tor^1_F(\mathcal{F}, \mathcal{O}_X) \to Tor^1_{X_k}(\mathcal{F}, \mathcal{O}_X) \to 0 \quad (21)
\]
Lemma 3.4. With notations as above,
\[
\text{Ker}[Tor^1_F(\mathcal{F}, \mathcal{O}_X) \to Tor^1_{X_k}(\mathcal{F}, \mathcal{O}_X)] \cong \text{Ker}[\mathcal{F}(-d) \to \mathcal{F}|_{X_{k-1}}(-d)]
\]
Proof. Twist the sequence $\langle 19 \rangle$ by $-d$ and compare it with the sequence $\langle 21 \rangle$. \qed

Proposition 3.5. There exists a short exact sequence
\[
0 \to \wedge^k \mathcal{E}(-(k - 1)d) \to \mathcal{F}_k \to \mathcal{F}_k|_{X_{k-1}} \to 0
\]
Proof. Follows from Lemma 3.3 and by putting $\mathcal{F} = \mathcal{F}_k$ in Lemma 3.4. \qed

4. Proof of the theorem

We now apply above results for $k = 2$.

Proposition 4.1. Let $E$ be an ACM bundle on a smooth hypersurface of dimension $\geq 3$. Then $\wedge^2 E$ is ACM if and only if $\wedge^2 E^\sigma$ is ACM.

Proof. Assume that $\wedge^2 E$ is ACM. For $k = 2$, we get following short exact sequences for $E$ (sequence $\langle 10 \rangle$ and the sequence from Lemma $\langle 3.5 \rangle$)
\[
0 \to \wedge^2 E^\sigma \to \wedge^2 \mathcal{F}_0 \to \mathcal{F}_2 \to 0 \quad (22)
\]
\[
0 \to \wedge^2 E(-d) \to \mathcal{F}_2 \to \mathcal{F}_2 \to 0 \quad (23)
\]
Comparing sequences $\langle 22 \rangle$, $\langle 23 \rangle$ and using the fact that $\wedge^2 \mathcal{F}_0, \mathcal{F}_2$ are all ACM, we get $H^i_\ast(\wedge^2 E^\sigma) = 0$ when $i = 2, \ldots n - 1$ where $n = \dim(X)$.

To prove the vanishing for $i = 1$, we note that $E^\vee$ is also ACM and $E^{\vee \sigma} \cong E^{\sigma \vee}(-d)$, e.g. by lemma 2.5 of [CH2011]. Therefore the same proof shows that $H^i_\ast(\wedge^2 (E^{\sigma \vee})) = 0$ when $i = 2, \ldots n - 1$. Applying Serre’s duality completes the proof. \qed

We now prove our main result,

Proof of Theorem 1.2 Suffices to show one direction. Assume $H^i_\ast(X, \wedge^2 E) = 0$ for $i = 1, 2, 3, 4$. Consider the composition of sequences $\langle 5 \rangle$ and $\langle 6 \rangle$:
\[
0 \to \wedge^2 E(-2d) \to \wedge^2 \mathcal{F}_1 \to \mathcal{F}_1 \otimes E^\sigma \to E^\sigma \otimes \mathcal{F}_0 \to \wedge^2 \mathcal{F}_0 \to \wedge^2 E \to 0
\]
One concludes that $H^i(X, \wedge^2 E(k)) = H^{i+4}(X, \wedge^2 E(k - 2d))$ for $i = 1, \ldots n - 5$. Thus $\wedge^2 E$ is ACM. By Lemma 4.1, $\wedge^2 E^\sigma$ is also ACM. We consider sequence $\langle 3 \rangle$
\[
0 \to S^2 E(-d) \to E(-d) \otimes \mathcal{F}_1 \to \wedge^2 \mathcal{F}_1 \to \wedge^2 E^\sigma \to 0
\]
This gives \( H_i^v(S^2E) = 0 \) when \( i = 3, \ldots n - 1 \). Since \( \wedge^2E \) is ACM implies \( \wedge^2E^\vee \) is also ACM, we do a dual analysis to get \( H_i^v(S^2E^\vee) = 0 \) when \( i = 3, \ldots n - 1 \). Applying Serre’s duality and combining this with the vanishing for \( S^2E \), we get that when \( n - 3 \geq 2 \) then \( S^2E \) is also ACM.

Thus when \( \dim(X) \geq 5 \), \( E \otimes E = \wedge^2E \oplus S^2E \) is ACM which by Theorem 5.3 implies that \( E \) is split.

\begin{remark}
We note that the statement \( \wedge^2E \) is ACM implies \( E \otimes E \) is ACM is tight in the dimension. For a counterexample in lower dimension, consider any rank 2 indecomposable ACM vector bundle on a hypersurface of dimension 4. Then \( \wedge^2E \) is ACM but \( E \otimes E \cong E \otimes E^\vee(t) \) can not be ACM for otherwise \( H^2_v(X, E \text{nd}(E)) = 0 \) and hence in particular, by lemma 2.2 of [KRR2007], \( E \) is split which contradicts the indecomposability of \( E \).
\end{remark}

5. \( E \otimes E \) is ACM implies \( E \) is split

Let \( f \in R = k[x_0, x_1, \ldots, x_{n+1}] \) be a homogeneous irreducible polynomial of positive degree. Let \( S = R/(f) \) and \( X = \text{Proj}(S) \) be the corresponding hypersurface.

We state the following result without proof

\begin{lemma}
Let \( E \) be a vector bundle on \( X \). Let \( M = H^0_v(X, E) \) be corresponding graded \( S \)-module. Then \( E \) splits if \( M \) is a free \( S \)-module.
\end{lemma}

Following result is Theorem 3.1 in [HW1994]

\begin{theorem}[Huneke-Weigand]
Let \((R,m)\) be an abstract hypersurface and let \( M, N \) be \( R \)-modules, at least one of which has constant rank. If \( M \otimes_R N \) is a maximal Cohen-Macaulay \( R \)-module then either \( M \) or \( N \) is free.
\end{theorem}

The corresponding version for vector bundles is of course not true as every vector bundle on a planar curve is ACM (vacuously) and there exists indecomposable vector bundles on various planar curves. Though for our need, the following corollary suffices.

\begin{theorem}[Corollary to Theorem 5.2]
Let \( X = \text{Proj}(S) \) be a hypersurface of dimension \( \geq 3 \). Let \( E \) be an ACM vector bundle on \( X \). Further assume that \( E \otimes E \) is ACM. Then \( E \) splits.
\end{theorem}

\begin{proof}
We consider a minimal resolution of \( E \) on \( X \)
\[
0 \to E^\sigma \to \bar{F}_0 \to E \to 0 \tag{24}
\]
and
\[
0 \to E(-d) \to \bar{F}_1 \to E^\sigma \to 0 \tag{25}
\]

Proof. We consider a minimal resolution of \( E \) on \( X \)
Where $\tilde{F}_0, \tilde{F}_1$ are direct sum of line bundles. Tensoring sequence (24) with $E$ and sequence (25) with $E^\sigma$ and using the fact that $E \otimes E$ is ACM, we deduce that $E \otimes E^\sigma$ is ACM. Thus there exists a short exact sequence of graded $S$-modules:

$$0 \rightarrow H^0_*(E^\sigma \otimes E) \rightarrow H^0_*(\tilde{F}_0 \otimes E) \rightarrow H^0_*(E \otimes E) \rightarrow 0$$

Here we are using the fact that $\dim(X) \geq 3$. Sequence (24) yields the following right exact sequence

$$H^0_*(E^\sigma) \otimes H^0_*(E) \rightarrow H^0_*(\tilde{F}_0) \otimes H^0_*(E) \rightarrow H^0_*(E) \otimes H^0_*(E) \rightarrow 0$$

Thus we get the following commutative diagram

$$
\begin{array}{cccccc}
H^0_*(E^\sigma) & \otimes & H^0_*(E) & \otimes & H^0_*(E) & \otimes & H^0_*(E) \\
\downarrow & & & & & & \downarrow \\
H^0_*(E^\sigma \otimes E) & \otimes & H^0_*(\tilde{F}_0 \otimes E) & \otimes & H^0_*(E \otimes E) & \otimes & 0
\end{array}
$$

where the all vertical maps are naturally defined. Middle map is an equality because $\tilde{F}_0$ is a split bundle. By Snake’s lemma, $\phi_1$ is a surjective map.

Similarly we get following commutative diagram from the sequence (25)

$$H^0_*(E(-d)) \otimes H^0_*(E) \rightarrow H^0_*(\tilde{F}_1) \otimes H^0_*(E) \rightarrow H^0_*(E^\sigma) \otimes H^0_*(E) \rightarrow 0$$

$$
\begin{array}{cccccc}
H^0_*(E(-d) \otimes E) & \otimes & H^0_*(\tilde{F}_1 \otimes E) & \otimes & H^0_*(E^\sigma \otimes E) & \otimes & 0
\end{array}
$$

By Snake’s lemma $\phi_2$ is surjective. In turn this implies that $\phi_1$ is injective and hence $H^0_*(E) \otimes H^0_*(E) \rightarrow H^0_*(E \otimes E)$ is an isomorphism. Thus $H^0_*(E) \otimes H^0_*(E)$ is a maximal Cohen-Macaulay module and we can apply Theorem 5.2 to conclude that $H^0_*(E)$ is free and therefore $E$ splits.

\begin{proof}[Proof of Theorem 1.1] The perfect pairing $E \times \wedge^2 E \mapsto \wedge^3 E = O_X(e)$ induces an isomorphism $\wedge^2 E \cong E^\sigma(e)$. By Serre’s duality then $\wedge^2 E$ is ACM and hence we can apply Theorem 1.2.
\end{proof}

6. Acknowledgement

We thank Suresh Nayak for pointing out a crucial mistake in an earlier version. We thank Jishnu Biswas and Girivaru Ravindra for constant support and useful conversations.

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