Scattering of axial gravitational wave pulses by monopole black holes and QNMs: a semianalytic approach

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Abstract
We study scattering of short Gaussian pulses of axial gravitational waves by a spherically symmetric black hole that has swallowed one or more global monopoles. We qualitatively show how the response of the black hole to the impinging pulses depends both on the number of monopoles the black hole has swallowed and on the symmetry breaking scale of the model which gave rise to the monopoles. We use semianalytical methods to determine the corresponding quasinormal modes that get excited by the impinging pulses and that get imprinted in the black hole’s response to the pulses. When generalized to the case of rotating black holes, such modes are also expected to show up in various other dynamical processes such as the ringdown phase of a binary black hole merger in case at least one of the companion black holes of the binary has swallowed one or more global monopoles.

Keywords: monopoles, black holes, quasinormal modes, topological solitons, monopole black holes, scattering of gravitational waves

(Some figures may appear in colour only in the online journal)

1. Introduction
It has been known for a long time that black holes can be probed with short pulses of gravitational waves (see [1] for the pioneering work): when a gravitational wave pulse with a width that is comparable or less than the size of a black hole is scattered by the black hole, the impinging pulse excites the black hole’s (tensor) quasinormal modes which get then imprinted in the scattered outgoing wave pulse [1–3]. Since quasinormal modes only depend on the parame-
ters that characterize the black hole [4], this implies that such scattered pulses can be useful both to determine the values of parameters of a given black hole and to investigate whether or not a given black hole is completely characterized by the parameters that a theoretical model predicts.

Although many aspects about (tensor) quasinormal modes and the scattering of gravitational wave pulses by black holes or other compact objects have been studied in the literature (see e.g. [5–7] for reviews on quasinormal modes and [1, 3, 8, 9] for some works on scattering of wave pulses), to our knowledge, a theoretical analysis for the case of global monopole black holes is still missing. Such black holes can be formed when global monopoles that were produced in phase transitions in the early universe get swallowed by black holes or when overdense regions of matter that contain such monopoles collapse to black holes.

In this work we shall both determine tensor quasinormal modes of spherically symmetric monopole black holes and study the quasinormal mode contribution to the scattering of Gaussian axial gravitational wave pulses by such global monopole black holes. The aim is not to provide precise quantitative predictions but rather to show the qualitative behavior. Since quasinormal modes of a black hole not only show up in scattering setups but also in various other dynamical processes such as the ringdown phase of black hole binary mergers, our results also have implications for the understanding of such different processes whenever a global monopole black hole is involved. (Whereas we focus on the case of spherically symmetric black holes in this work, the application of our results to processes like the ringdown phase of a black hole binary merger would however require a generalisation of our analysis to the case of rotating black holes.)

The paper is organized as follows: in section 2, we review some aspects about global monopoles and global monopole black holes. In section 3, we determine tensor quasinormal modes of such black holes and investigate how they depend both on the symmetry breaking scale of the model that gave rise to the monopoles and on the number of monopoles that are inside of the black holes. In section 4, we study the scattering processes of the Gaussian axial gravitational wave pulses. In section 5, we conclude with a brief summary and discussion.

We use units in which \( c = \hbar = 1 \). For the metric we use the signature \((+,-,-,-)\). Although many numerical techniques exist, we mostly use known (semi-)analytical methods that we apply to the case of axial gravitational wave pulses scattered by monopole black holes. This requires several approximations to be made. As mentioned above, our results should therefore not be taken as precise quantitative predictions but are rather meant to show the correct qualitative behavior.

2. Global monopole black holes

2.1. Global monopoles

Global monopoles are topological defects that arise in models with a global symmetry \( A \) that is spontaneously broken down to a symmetry \( B \) in such a way that the second homotopy group \( \pi_2(A/B) \) is nontrivial. The simplest such model with a global symmetry \( O(3) \) that is spontaneously broken down to \( O(2) \), first mentioned by Polyakov [15], is described by the Lagrangian density

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1 However, some related works exist. Scattering and absorption of plane (scalar) waves by such black holes was studied in [10–12]. Scalar and spinor quasinormal modes in \( f(R) \) gravity were investigated in [13]. Quasinormal modes in the presence of quintessence were determined in [14].
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2. \tag{1}
\]

Here \( \phi^a \) is a scalar triplet field \((a = 1, 2, 3)\), \( \lambda \) a self-coupling constant and \( v \) the vacuum expectation value. Making a hedgehog ansatz in spherical coordinates,

\[
\phi^a = v h(r) n^a, \tag{2}
\]

where \( n^a \) is a unit vector in radial direction, and using boundary conditions \( h(0) = 0 \) and \( h(\infty) = \pm 1 \), one can find classical solutions of the corresponding equations of motion for \( h(r) \) which have a non-zero topological charge \([15]\)

\[
Q = \frac{1}{4\pi v^2} \oint_{S^2} \epsilon_{abc} \phi^a \partial_i \phi^b \partial_j \phi^c \, dx^i \wedge dx^j. \tag{3}
\]

Solutions with \( Q = 1 \) (corresponding to \( h(\infty) = 1 \)) are often referred to as ‘global monopoles’, whereas solutions with \( Q = -1 \) \((h(\infty) = -1)\) are known as ‘global antimonopoles’.

Global monopoles have an important property: Outside of the core of a monopole, where the solution-function \( h(r) \approx 1 \), the temporal–temporal component of the energy momentum tensor of the monopole takes the form \( T_{00} = \frac{v^2}{r^2} \). Therefore, the total energy \( E \) of one global monopole asymptotically scales with radial distance \( r \) as

\[
E = 4\pi \int_0^r T_{00} \, dv \sim M_c + 4\pi v^2 r \tag{4}
\]

and is thus linearly divergent. (Here \( M_c = \frac{\sqrt{\lambda}}{v} \) is the mass of the monopole core \([16]\).) This implies that, if one wants to consider finite energy objects in an infinite space, one cannot consider single isolated global monopoles. One could argue that there is no need for considering finite energy objects as long as the total number of global monopoles in our universe (with infinite total energy) is small enough such that their (finite) energy density does not overwhelm the universe. If one however wants to consider finite energy objects, there are two possible ways, already pointed out by Polyakov \([15]\), to obtain finite energy configurations from (1): the first possibility is to gauge the symmetry in (1). This gives rise to local finite energy magnetic monopoles (today well-known as ’t Hooft Polyakov monopoles \([15, 17]\)). In this work, we will however not consider such local magnetic monopoles. The second possibility is to consider (a network of) global monopole antimonopole pairs instead of single global monopoles or, in other words, an equal number of global monopoles and global antimonopoles. In this second case the divergences of the energies of the monopole and antimonopole of a pair cancel. The energy of one pair is then \( \sim 2M_c + 4\pi v^2 R \), where \( R \) is the distance between the monopole and the antimonopole of the pair. This leads to an attractive force \( F \) between the global monopole and the global antimonopole of a pair that does not depend on the distance between them, \( F = \partial_0 E = 4\pi v^2 \).

Although this has been a matter of debate in the literature \([18–20]\), global monopoles seem to be dynamically stable if they can freely move \([21]\). In particular, numerical simulations indicate that there is no instability that causes the field of a freely moving monopole antimonopole pair to collapse with no energy cost to a string \([22]\).

### 2.2. Black holes

If existent in our universe, global monopoles and global antimonopoles can happen to be inside of a black hole either when they are swallowed by the black hole or when an overdense region of matter that contains global monopoles and/or global antimonopoles collapses to a black hole.
In what follows we shall recall how these objects (global monopoles inside of black holes) are described theoretically in the spherically symmetric case.

When (1) is minimally coupled to Einstein gravity, one can find gravitating solutions of the Einstein field equations, \( G_{\mu\nu} = \frac{1}{M^2_P} T_{\mu\nu} \), both without event horizon (’gravitating monopoles’) and with event horizon (’monopole black holes’) [16]: far away from the monopole core where \( h(r) \approx 1 \), the energy momentum tensor for one global monopole takes the form

\[
T_{\mu\nu} = \text{diag} \left( \frac{v^2}{r^2}, \frac{v^2}{r^2}, 0, 0 \right),
\]

which implies anisotropic stress, \( p_r \neq p_t \), (with \( p_r \equiv -T_r^r \) the radial pressure and \( p_t \equiv -T^\theta_\theta \) the tangential pressure) and an energy density \( \rho \equiv T_{00} \) that is equal to minus the radial pressure \( p_r, \rho = -p_r \). A solution of the Einstein field equations with this energy momentum tensor (5) was found in [16] and is given by the line element

\[
ds^2 = A(r)dr^2 - A(r)^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \ d\phi^2 \right),
\]

with

\[
A(r) = 1 - \frac{v^2}{M^2_P} - \frac{2M}{M^2_P r}.
\]

Here \( m_P \) is the reduced Planck mass and \( M \) is a constant of integration. As indicated above, one often distinguishes two cases. The first case is the case where \( M \) is set only by the monopole energy momentum tensor, \( A(r) \equiv 1 - \frac{1}{M^2_P} \int T_{00} \, dr \). In that case \( M \) takes the value of \( M_c (8\pi)^{-1} \). Such configurations are referred to as ‘gravitating monopoles’$^2$. The second case is the case where \( M \) is not set by \( M_c \) but takes large positive values such that the configuration (6) has an event horizon and describes a large black hole (’monopole black hole’). The asymptotic form of the monopole black hole solution that we gave above (6), (8) is a good approximation for large monopole black holes (with the size of the black hole much larger than the core of the monopole), precise numerical solutions which are particularly important for intermediate regimes however also exist [30–33]. The stability of these objects was discussed in [34]. In what follows we shall use the asymptotic form of the global monopole black holes (5)–(7) generalized to the case where there are \( N \) global monopoles inside of the black hole (or \( N_m \) global antimonopoles and \( N + N_m \) global monopoles):

\[
T_{\mu\nu} = \text{diag} \left( \frac{N v^2}{r^2}, \frac{N v^2}{r^2}, 0, 0 \right),
\]

\[
A(r) = 1 - \frac{N v^2}{M^2_P} - \frac{2M}{M^2_P r}.
\]

$^2$It can be shown that \( M_c \) is a negative quantity leading to a repulsive Newton potential of the global monopole core [23]. Non-minimal couplings to gravity [24, 25] or an additional unbroken U(1) subgroup [26–29] can however give similar gravitating monopole solutions with attractive Newton potential of the core. Except of the small (repulsive) potential of the monopole core, there is no gravitational force that a gravitating global monopole exerts on the matter around it.
3. Quasinormal modes of monopole black holes

As other kinds of black holes, global monopole black holes possess characteristic quasinormal modes (see e.g. [5–7] for reviews on quasinormal modes). In this section, we shall determine the axial tensor quasinormal modes of monopole black holes with the help of semianalytical methods. We first derive the linearized Einstein field equations for the axial modes (odd parity modes) in Regge–Wheeler gauge. They reduce to a single second order differential equation for the perturbations that, as one can expect, goes to the Regge–Wheeler equation [35] in the limit \( v \to 0 \). In the next section we shall discuss one way of how to excite these quasinormal modes in a dynamical process.

3.1. Linearized Einstein field equations

In Regge–Wheeler gauge the perturbed line element can be written as [35]

\[
\text{d}s^2 = \text{d}s_0^2 + \text{d}s_{(\text{polar})}^2 + \text{d}s_{(\text{axial})}^2,
\]

where \( \text{d}s_0^2 \) is the line element of the background metric (6) with \( A(r) \) as defined in (9) and

\[
\text{d}s_{(\text{polar})}^2 \equiv H_0 A(r) e^{i m \phi} P_l (\cos \theta) \text{d}r^2 + 2 H_1 e^{i m \phi} P_l (\cos \theta) \text{d}r \text{d}t
\]

\[ + H_2 A(r)^{-1} e^{i m \phi} P_l (\cos \theta) \text{d}r^2 + r^2 K e^{i m \phi} P_l (\cos \theta) \text{d}\theta^2 \]

\[ + r^2 K \sin^2 \theta e^{i m \phi} P_l (\cos \theta) \text{d}\phi^2, \]

(11)

\[
\text{d}s_{(\text{axial})}^2 \equiv 2 h_0 e^{i m \phi} \sin \theta \partial_\theta P_l (\cos \theta) \text{d}t \text{d}\phi + 2 h_1 e^{i m \phi} \sin \theta \partial_\theta P_l (\cos \theta) \text{d}r \text{d}\phi.
\]

(12)

Here \( P_l \) are the standard Legendre polynomials of order \( l \). \( H_0, H_1, H_2, h_0 \) and \( h_1 \) are functions of only the temporal coordinate \( t \) and the radial coordinate \( r \). These expressions can be used to determine the perturbations of the Einstein tensor \( \delta G_{\mu\nu} \). The perturbations of the energy momentum tensor are given by

\[
\delta T_{\mu\nu} = \delta (\partial_\mu \phi^a \partial_\nu \phi^a + \partial_\mu \phi^a \delta (\partial_\nu \phi^a)) - h_{\mu\nu} \mathcal{L} - g_{\mu\nu} \delta \mathcal{L},
\]

(13)

where \( g_{\mu\nu} \) are the components of the background metric (6) and \( h_{\mu\nu} \) are the components of the metric perturbations (11), (12). We restrict our studies here to the axial modes. Since perturbations of the scalar field are polar, \( \delta \phi_{(\text{axial})} = 0 \), the axial modes of the perturbations of the energy momentum tensor \( \delta T_{\mu\nu}^{(\text{axial})} \) are, in our case of the global monopole with background energy momentum tensor (8), given by

\[
\delta T_{\mu\nu}^{(\text{axial})} = h_{\mu\nu}^{(\text{axial})} \frac{M^2}{r^2}.
\]

(14)

Here \( h_{\mu\nu}^{(\text{axial})} \) are the components of the axial metric perturbations (12). The non-vanishing components of the perturbed Einstein field equations for the axial modes, \( \delta G_{\mu\nu}^{(\text{axial})} = \frac{1}{M_p^2} \delta T_{\mu\nu}^{(\text{axial})} \), can for each frequency component be written as

\[
A(r) \partial_r \hat{h}_1 + i \nu A(r)^{-1} \hat{h}_0 + \frac{2}{r^2} \frac{M}{M_p^2} \hat{h}_1 = 0,
\]

(15)

\[ ^3 \text{We have determined the non-vanishing components for the axial modes of } \delta G_{\mu\nu} \text{ and provide them in appendix A.} \]
Figure 1. The effective potential (19) for $l = 2$ (left) and $l = 3$ (right) for the case $M = M_P$. The different colors represent different values of $N v^2$: The blue line is the Schwarzschild case $N v^2 = 0$, the yellow line is the case $N v^2 = 0.01 M^2_P$, the green line is $N v^2 = 0.04 M^2_P$ and the red line is $N v^2 = 0.09 M^2_P$.

\[ \frac{1}{r} A(r)^{-1} \left( 2iw \hat{h}_0 + rw^2 \hat{h}_1 - iw r \partial_r \hat{h}_0 \right) + \frac{2}{r^2} \hat{h}_1 - \frac{1}{r^2} (l(l+1)) \hat{h}_1 - \frac{2N v^2}{M_P^2 r^2} \hat{h}_1 = 0, \]  \hspace{1cm} (16)

\[ A(r) \partial_r^2 \hat{h}_0 + iw A(r) \partial_r \hat{h}_1 + \frac{2iw}{r} A(r) \hat{h}_1 - \frac{1}{r^2} (l(l+1)) \hat{h}_0 + \frac{4M}{M_P^2 r^3} \hat{h}_0 = 0, \]  \hspace{1cm} (17)

where $A(r)$ is defined as in (9) and $\hat{h}_0$ and $\hat{h}_1$ are the Fourier transforms of $h_0$ and $h_1$ and thus functions of the radial coordinate $r$ and of the frequency $w$. Here (15) is the \( \theta \phi \) component of the perturbed Einstein field equations, (16) the \( r \phi \) component and (17) the \( r \theta \) component. The last equation (17) is a consequence of (15) and (16). Similarly as for example in the case of a Schwarzschild black holes (see e.g. [35]) and in the case of a static perfect fluid star (see e.g. [36]), one can eliminate $\hat{h}_0$ from (16) by using (15). Then (16) reduces to the second order differential equation

\[ A(r) \partial_r \left( A(r) \partial_r \hat{\Psi} \right) + (w^2 - V_{eff}(r)) \hat{\Psi} = 0, \]  \hspace{1cm} (18)

where $\hat{\Psi} \equiv A(r) r^{-1} \hat{h}_1$ and

\[ V_{eff}(r) \equiv A(r) \left( \frac{l(l+1)}{r^2} - \frac{6M}{M_P^2 r^3} \right). \]  \hspace{1cm} (19)

The result (18) is a Schrödinger equation with the effective potential $V_{eff}$. Scattering of axial gravitational waves can therefore be studied by using techniques from one-dimensional quantum mechanical scattering theory. In the limit $v \to 0$, (18) goes to the Regge–Wheeler equation [35]. In order to see how the effective potential (19) qualitatively changes when $N v^2$ is increased, we plot it for $l = 2, l = 3$ and several values of $N v^2$ in figure 1. We chose three values close to 0 to see the qualitative change once the metric starts to differ from the Schwarzschild form.

3.2. Quasinormal modes

The wave equation (18) can be used to determine the corresponding quasinormal modes. By definition these are the eigenfunctions of (18) which are subject to the boundary conditions

\[ \hat{\Psi} \sim e^{-iwr}, \hspace{1cm} r^* \to -\infty, \]  \hspace{1cm} (20)
Note that it follows from (19) and the definition of \( r^* \) that \( V_{\text{eff}} \to 0 \) for \( r^* \to \pm \infty \).

Here \( r^* \) is the tortoise coordinate that is defined by \( \partial_{r^*} \equiv \Lambda(r) \partial_r \).

There exist several techniques, both numerical and semianalytical, to determine the corresponding complex eigenfrequencies. In this work, we shall use the semianalytical WKB method that yields the quasinormal mode frequencies to a good approximation [37], see e.g. [38] for a recent summary of the method. We determine quasinormalmode frequencies for \( l = 2 \) \( l = 3 \) modes both by using the approximation carried to third order beyond eikonal approximation (both for the \( l = 2 \) modes and for the \( l = 3 \) modes) [39] and by using the approximation carried out to eighth order (for the \( l = 2 \) modes) and to tenth order (for the \( l = 3 \) modes) combined with Padé approximations [40–42]. The latter is the semianalytical state-of-art technique to obtain very accurate results with the WKB method (see e.g. [38, 40, 41]).

When carrying out the approximation to third order, the quasinormal mode frequencies \( w_n \) \((n = 0, 1, 2, \ldots)\) are given by [39, 47]

\[
w_n^2 = V_0 + \sqrt{-2V_0^{(2)}} \Lambda(n) - i \left( n + \frac{1}{2} \right) \sqrt{-2V_0^{(2)}} (1 + \Omega(n)) ,
\]

where

\[
\Lambda(n) \equiv \frac{1}{8} \sqrt{-2V_0^{(2)}} \left( \frac{V_0^{(4)}}{V_0^{(2)}} \left( \frac{1}{4} + \left( n + \frac{1}{2} \right)^2 \right) - \frac{1}{36} \left( \frac{V_0^{(3)}}{V_0^{(2)}} \right)^2 \left( 7 + 60 \left( n + \frac{1}{2} \right)^2 \right) \right),
\]

\[
\Omega(n) \equiv \frac{-1}{2V_0^{(2)}} \left( \frac{5}{6912} \left( \frac{V_0^{(5)}}{V_0^{(2)}} \right)^4 \left( 77 + 188 \left( n + \frac{1}{2} \right)^2 \right) - \frac{1}{384} \left( \frac{V_0^{(3)}}{V_0^{(2)}} \right)^2 \left( 51 + 100 \left( n + \frac{1}{2} \right)^2 \right) + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0^{(2)}} \right)^2 \left( 67 + 68 \left( n + \frac{1}{2} \right)^2 \right) \right. 
\]

\[
\left. \times \left( 67 + 68 \left( n + \frac{1}{2} \right)^2 \right) + \frac{1}{288} \left( \frac{V_0^{(3)}}{V_0^{(2)}} \right)^2 \left( 19 + 28 \left( n + \frac{1}{2} \right)^2 \right) \right) \] \[
- \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0^{(2)}} \right) \left( 5 + 4 \left( n + \frac{1}{2} \right)^2 \right) .
\]

Here \( V_0^{(k)} \) stands for the \( k \)th derivative \( \frac{d^k V_{\text{eff}}}{dr_{\text{eff}}^k} \) evaluated at the peak \( r_0^* \) of the effective potential (19). When carrying out the approximation to higher order, in particular in the case when the approximation is combined with the Padé approximation (as proposed in [40–42]), the expressions can be found for example in the Mathematica code which is publicly available to download at [43] (see also [44] for the expressions up to sixth order).

Using these expressions, we have determined the dimensionless quasinormal mode frequencies \( w_n M_P^{-1} \) for \( l = 2, 3 \), \( n = 0, 1, 2, 3 \), \( M = M_P \) and several values of \( Nv^2 \). We list our results both for the third order and for the eighth order (tenth order respectively) in table 1 and plot...
them in figure 2. In the case of the eighth order (tenth order) results we have used the averaging technique over Padé approximations that is presented in [38, 43] for each quasinormal mode frequency. The results for $N \nu = 0$ correspond to the Schwarzschild case, they are very close to the exact numerical results for Schwarzschild black holes which were determined in [45, 46]. As one can see from the results in table 1 and figure 2, once $N \nu^2$ increases (while $l$ and $n$ are kept fixed), the real part of the quasinormal mode frequencies decreases whereas the imaginary part increases. This qualitative behavior of the modes $w_{n l}$ is both visible from the third order values as well as from the more precise higher order values.

| $l$ | $n$ | $\frac{N \nu^2}{M^2}$ | $\frac{\partial w_{n l}}{\partial M}$ (3rd order WKB) | $\frac{\partial^2 w_{n l}}{\partial M^2}$ (8th/10th order WKB) | $\nu(w_{n l})$ | $\lim_{M \to 0} M \frac{\partial w_{n l}}{\partial M}$ (at $w = w_{n l}$) |
|-----|-----|-----------------|--------------------|-----------------|-------------|-----------------|
| 2   | 0   | 0.3732 - i0.0892 | 0.3737 - i0.0890   | -3.9364 - i0.5679 | -1.4906 + i17.3224 |
| 2   | 0.01| 0.3686 - i0.0875 | 0.3691 - i0.0873   | -3.9526 - i0.5701 | -1.5056 + i17.6707 |
| 2   | 0.04| 0.3549 - i0.0825 | 0.3553 - i0.0823   | -4.0029 - i0.5767 | -1.5528 + i18.7642 |
| 2   | 0.09| 0.3320 - i0.0742 | 0.3324 - i0.0742   | -4.0927 - i0.5878 | -1.6399 + i20.8343 |
| 2   | 1   | 0.3460 - i0.2749 | 0.3468 - i0.2737   | -4.8442 - i0.9015 | -3.0148 + i15.8116 |
| 2   | 1.01| 0.3422 - i0.2695 | 0.3430 - i0.2684   | -4.8599 - i0.9076 | -3.0566 + i16.1422 |
| 2   | 1.04| 0.3307 - i0.2537 | 0.3314 - i0.2528   | -4.9086 - i0.9262 | -3.1881 + i17.1965 |
| 2   | 1.09| 0.3111 - i0.2284 | 0.3118 - i0.2278   | -4.9958 - i0.9577 | -3.4308 + i19.1882 |
| 2   | 2   | 0.3029 - i0.4711 | 0.3002 - i0.4766   | -5.8780 - i0.8873 | -3.6587 + i14.8384 |
| 2   | 2.01| 0.3003 - i0.4618 | 0.2976 - i0.4672   | -5.8925 - i0.8973 | -3.7111 + i15.1539 |
| 2   | 2.04| 0.2920 - i0.4344 | 0.2896 - i0.4395   | -5.9379 - i0.9225 | -3.8766 + i16.1598 |
| 2   | 2.09| 0.2776 - i0.3907 | 0.2761 - i0.3947   | -6.0193 - i0.9788 | -4.1845 + i18.0616 |
| 2   | 3   | 0.2475 - i0.6729 | 0.2427 - i0.7076   | -6.9994 - i0.6926 | -4.2870 + i14.1089 |
| 2   | 3   | 0.2463 - i0.6595 | 0.2419 - i0.6926   | -7.0127 - i0.7066 | -4.3478 + i14.4173 |
| 2   | 3   | 0.2424 - i0.6203 | 0.2391 - i0.6492   | -7.0542 - i0.7488 | -4.5396 + i15.4007 |
| 2   | 3   | 0.2347 - i0.5576 | 0.2326 - i0.5810   | -7.1294 - i0.8201 | -4.8947 + i17.2588 |
| 3   | 0   | 0.5993 - i0.9027 | 0.5994 - i0.9027   | -5.3151 - i0.6919 | -0.8220 + i16.8429 |
| 3   | 0   | 0.5910 - i0.9099 | 0.5912 - i0.9099   | -5.3393 - i0.6934 | -0.8325 + i17.1794 |
| 3   | 0   | 0.5663 - i0.8856 | 0.5664 - i0.8856   | -5.4141 - i0.6981 | -0.8658 + i18.2525 |
| 3   | 0   | 0.5256 - i0.7771 | 0.5258 - i0.7770   | -5.5471 - i0.7061 | -0.9274 + i20.2810 |
| 3   | 1   | 0.5824 - i0.2814 | 0.5826 - i0.2813   | -6.1904 - i1.2600 | -2.0103 + i16.1220 |
| 3   | 1   | 0.5745 - i0.2759 | 0.5748 - i0.2758   | -6.2141 - i1.2646 | -2.0400 + i16.4515 |
| 3   | 1   | 0.5511 - i0.2596 | 0.5513 - i0.2595   | -6.2876 - i1.2784 | -2.1339 + i17.5026 |
| 3   | 1   | 0.5124 - i0.2336 | 0.5126 - i0.2335   | -6.4183 - i1.3021 | -2.3080 + i19.4913 |
| 3   | 2   | 0.5532 - i0.4767 | 0.5517 - i0.4791   | -7.1576 - i1.4714 | -2.6138 + i15.3905 |
| 3   | 2   | 0.5461 - i0.4673 | 0.5446 - i0.4696   | -7.1805 - i1.4788 | -2.6560 + i15.7086 |
| 3   | 2   | 0.5247 - i0.4395 | 0.5234 - i0.4416   | -7.2514 - i1.5014 | -2.7990 + i16.7237 |
| 3   | 2   | 0.4893 - i0.3952 | 0.4882 - i0.3969   | -7.3776 - i1.5404 | -3.0398 + i18.6463 |
| 3   | 0   | 0.5157 - i0.6774 | 0.5120 - i0.6904   | -8.2036 - i1.4962 | -3.0162 + i14.8870 |
| 3   | 0   | 0.5095 - i0.6640 | 0.5058 - i0.6765   | -8.2254 - i1.5064 | -3.0652 + i15.1958 |
| 3   | 0   | 0.4908 - i0.6245 | 0.4873 - i0.6356   | -8.2930 - i1.5372 | -3.2204 + i16.1812 |
| 3   | 0   | 0.4596 - i0.5613 | 0.4563 - i0.5705   | -8.4137 - i1.5904 | -3.5110 + i18.0475 |
Figure 2. The axial tensor quasinormal modes $w_n$ for the case $M = M_P$ from Table 1 for $l = 2$ (left) and $l = 3$ (right). In both cases the quasinormal modes are plotted from $n = 0$ (points on the top right) to $n = 3$ (points on the bottom left). The different colors represent different values of $N v^2$: the blue points are the Schwarzschild quasinormal modes $N v^2 = 0$, the yellow points are the quasinormal modes for $N v^2 = 0.01 M_P^2$, the green points are the quasinormal modes for $N v^2 = 0.04 M_P^2$ and the red points are the quasinormal modes for $N v^2 = 0.09 M_P^2$.

4. Scattering of axial gravitational wave pulses by global monopole black holes

The quasinormal modes of a black hole get excited in many dynamical processes. One example of such a process is the scattering of a short wave pulse by the black hole: an impinging wave pulse with a width less than or comparable to the size of the black hole excites the quasinormal modes of the black hole which then get imprinted in the black hole’s response to the pulse [1–3, 48–50].

We shall study scattering of a Gaussian axial gravitational wave pulse by a monopole black hole which at some initial point of time $t_0$ takes the form

$$\Psi(t_0, r^*) = a e^{-b(r^*-x)^2}, \quad (25)$$

$$\frac{\partial \Psi}{\partial t} \bigg|_{t=t_0} = 0. \quad (26)$$

Here $a$ and $b$ are constants and $x$ is a point far away from the black hole at which the initial static Gaussian pulse is centered. The function $\Psi$ is the Fourier transform of the function $\hat{\Psi}$ that was introduced in (18). This kind of initial data was also used in other works (see e.g. [1, 3]). We consider the Cauchy problem (18), (26) and want to find the outgoing wave solutions for large $r$ in order to investigate how the response of the black hole to an impinging pulse with initial data (26) depends on $N v^2$. We focus on the quasinormal mode contribution to the solutions. This is the most characteristic contribution in such scattering setups for impinging pulses with a width smaller than or comparable to the size of the black hole. It dominates the response of the black hole at all but very early and very late times [48, 51]. The hope is that

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4 At very late and very early times, contributions from low frequencies (for very late times) and high frequencies (for very early times) dominate over the quasinormal mode contribution (see e.g. [48]). The late time contributions lead to a power-law tail of the outgoing wave [48, 51–54]. Following for example the procedure in [48] for a Schwarzschild black hole, it is straightforward to get the high and low frequency contributions also for the monopole black hole case. In this work, we only study the quasinormal mode contribution.
the study of this scattering process will give us a good understanding of the effects that global monopoles which were swallowed by a black hole can have on dynamical processes in which quasinormal modes of the black hole show up.

Let us first mention that in the setup which we are considering there is an event horizon at \( r^* \to -\infty \). Therefore, we demand the absence of outgoing waves when \( r^* \to -\infty \). Since \( V_{\text{eff}} \to 0 \) for \( r^* \to \pm \infty \), this implies that the relevant solutions of the wave equation (18) asymptotically scale as

\[
\tilde{\psi} \sim e^{-iwr^*}
\]

for \( r^* \to -\infty \) and as

\[
\tilde{\psi} \sim A_{\text{out}} e^{iwr^*} + A_{\text{in}} e^{-iwr^*}
\]

for \( r^* \to \infty \). Here \( A_{\text{out}} \) and \( A_{\text{in}} \) are complex coefficients that depend on \( w \). We want to find the quasinormal mode contribution to the solutions with initial data (26) for large \( r \). For this contribution, by definition (20), \( A_{\text{in}} = 0 \). There are several techniques that can be used to find such solutions. One option to obtain precise quantitative results would be to use numerical techniques. An alternative option which can be used in order to obtain qualitative results is to use analytical methods that require several approximations to be made. The latter is what we intend to do in this work. One analytic approach would be to try to use the WKB approximation that we have dealt with in section 3.2, to calculate the reflection and transmission coefficients\(^6\) with this method and then to Fourier transform back to real space as done for example in the pioneering work [1] (see e.g. [38, 55] for an application of a higher order WKB method along these lines). Another analytic approach which we shall take in this work is to use Greens functions [3, 51, 56, 57] (see also [49, 50]) and the phase integral method as we will recall in what follows and in appendix B. This is a well-known approach applicable in such contexts [3, 51, 56, 57] (see also [49, 50]). We shall follow the analysis in [3, 51, 56, 57] and apply it to our case of an axial gravitational wave pulse scattered by a monopole black hole. For completeness we also present most of the steps in the derivation that need no modification when compared to the derivations in [3, 51, 56, 57]. We mainly adopt the notation from [3].

Since \( \Psi \) vanishes for \( t < t_0 \) because of causality, one can use (18) to define a function \( \tilde{\Psi} \) by the integral transform

\[
\tilde{\Psi}(r^*, w) \equiv \int_{t_0}^{\infty} e^{iwt} \Psi(r^*, t) dt,
\]

which solves the differential equation

\[
\left( \partial_{r^*}^2 + w^2 - V_{\text{eff}}(r) \right) \tilde{\Psi}(r^*, w) = I(r^*, w),
\]

where

\[
I(r^*, w) \equiv e^{iw t_0} \left( iw \Psi(r^*, t_0) - \frac{\partial \Psi(r^*, t)}{\partial t} \bigg|_{t=t_0} \right) = e^{iw t_0} iw a e^{-b(r^* - x)^2}.
\]

The solution \( \tilde{\Psi}(r^*, w) \) can be written in terms of the Greens function \( G(r'^*, r^*) \) that solves

\(^6\) Here, as in scattering of one dimensional quantum mechanics, a reflection coefficient \( R \) can be defined as \( R \equiv \left| \frac{A_{\text{out}}}{A_{\text{in}}} \right|^2 \) and a transmission coefficient \( T \) as \( T \equiv \left| \frac{A_{\text{out}}}{A_{\text{in}}} \right|^2 \).
\begin{equation}
\left( \partial_{r^*}^2 + w^2 - V_{\text{eff}} \right) G(r^*, r^*) = \delta(r^* - r^*) \tag{32}
\end{equation}

as\textsuperscript{7}

\begin{equation}
\hat{\Psi}(r^*, w) = \int G(r^*, r^') I(r^*, w) dr^*. \tag{33}
\end{equation}

The Greens function can be expressed as linear combination of two independent solutions of the homogeneous equation (18), one, \( \hat{\Psi}^- \), that is a purely ingoing wave at the horizon and a linear combination of in- and outgoing waves at infinity and another one, \( \hat{\Psi}^+ \), that is a purely outgoing wave at infinity and a linear combination of in- and outgoing waves at the horizon. This means

\begin{equation}
\hat{\Psi}^- \sim e^{-iwr^*}, \quad \hat{\Psi}^+ \sim B_{\text{out}} e^{iwr^*} + B_{\text{in}} e^{-iwr^*} \tag{34}
\end{equation}

for \( r^* \to -\infty \) and

\begin{equation}
\hat{\Psi}^- \sim A_{\text{out}} e^{iwr^*} + A_{\text{in}} e^{-iwr^*}, \quad \hat{\Psi}^+ \sim e^{iwr^*} \tag{35}
\end{equation}

for \( r^* \to \infty \).

The solution (33) is then

\begin{equation}
\hat{\Psi}(r^*, w) = \hat{\Psi}^+ \int_{-\infty}^{r^*} \frac{J^-}{W} dr'^* + \hat{\Psi}^- \int_{r^*}^{\infty} \frac{J^+}{W} dr'^*, \tag{36}
\end{equation}

where \( W \) is the Wronskian

\begin{equation}
W \equiv \hat{\Psi}^- \partial_{r^*} \hat{\Psi}^+ - \hat{\Psi}^+ \partial_{r^*} \hat{\Psi}^- = 2iwA_{\text{in}}. \tag{37}
\end{equation}

For large \( r^* \) (36) can be approximated by

\begin{equation}
\hat{\Psi}(r^*, w) = \frac{J(w)}{2iwA_{\text{in}}} e^{iwr^*}, \tag{38}
\end{equation}

with

\begin{equation}
J(w) \equiv \int J^- dr'^*. \tag{39}
\end{equation}

Therefore, far away from the black hole the solution to (30) can be written in the time domain as

\begin{equation}
\Psi(r^*, t) = \frac{1}{4\pi i J_C} \int \frac{e^{-iw(t-r^*)}}{wA_{\text{in}}} J(w) dw, \tag{40}
\end{equation}

where the contour \( C \) is given for example in figure 1 in [3]. Since we are interested in the quasinormal mode contribution to (40) for which \( A_{\text{in}}(w) = 0 \) (20), we can approximate \( A_{\text{in}} \) around each quasinormal mode \( w_n \) (all values \( w_n \) for which \( A_{\text{in}}(w_n) = 0 \)) as \( A_{\text{in}} \approx (w - w_n)\epsilon_n \).

\textsuperscript{7}The boundary conditions of the Greens function are chosen such that surface terms disappear, see [3] for a more detailed discussion about this point.
Figure 3. The sum of the first four modes ($n = 0, 1, 2, 3$) of the quasinormal mode contribution to the response of the monopole black hole to an impinging Gaussian wave pulse for $x + r^* = 1000M_P^{-1}$, $t_0 = 0$, $a = M_P$, $b = M_P^2$, $l = 2$ and $M = M_P$. The different colors represent different values of $Nv^2$: the blue line is the Schwarzschild case $Nv^2 = 0$, the yellow line is the case $Nv^2 = 0.01M_P^2$, the green line is $Nv^2 = 0.04M_P^2$ and the red line is $Nv^2 = 0.09M_P^2$. As described in the text, these results were obtained from an analytic method which required some approximations to be made (rather than from a numerical calculation). They show the qualitative behavior and are not meant to be precise quantitative predictions.

Figure 4. The sum of the first four modes ($n = 0, 1, 2, 3$) of the quasinormal mode contribution to the response of the monopole black hole to an impinging Gaussian wave pulse for $x + r^* = 1000M_P^{-1}$, $t_0 = 0$, $a = M_P$, $b = M_P^2$, $l = 3$ and $M = M_P$. The different colors represent different values of $Nv^2$: the blue line is the Schwarzschild case $Nv^2 = 0$, the yellow line is the case $Nv^2 = 0.01M_P^2$, the green line is $Nv^2 = 0.04M_P^2$ and the red line is $Nv^2 = 0.09M_P^2$. As described in the text, these results were obtained from an analytic method which required some approximations to be made (rather than from a numerical calculation). They show the qualitative behavior and are not meant to be precise quantitative predictions.

where $\alpha_n \equiv \partial_{w_n}A_{\mu}(w_n)$. According to the residue theorem, the quasinormal mode contribution is given by\(^8\)

$$\Psi_Q(r^*, t) = -\frac{1}{2} \sum_n \frac{e^{-iw_n(t-r^*)}}{w_n\alpha_n}J(w_n). \quad (41)$$

\(^8\)Note that the contour $C$ encloses all the poles in the complex plane that correspond to the quasinormal modes, see e.g. figure 1 in [3].
Since quasinormal modes come in pairs \( w_n \) and \(-w_n^*\) [5, 7] (where the star denotes complex conjugation), this expression can be simplified to

\[
\Psi_Q(r^*, t) = -\text{Re} \left( \sum_n e^{-i\omega_n(t-r^*)} \frac{\omega_n}{w_n^2} J(w_n) \right).
\] (42)

In appendix B we show that

\[
\alpha_n = -2c \sqrt{\omega_n} \left. e^{-i\eta(w_n)} \left( \frac{\partial \gamma}{\partial w} \right) \right|_{w=w_n}
\] (43)

and that \( J(w_n) \) can for large \( r \) be approximated as

\[
J(w_n) \approx i c \sqrt{\omega_n} a \ e^{i\eta(w_n)} \sqrt{\frac{\pi}{b}} e^{i\omega_n(x+t_0)} \frac{\sqrt{\pi}}{b} e^{-i\omega_n^2b^2},
\] (44)

where the definitions for the functions \( \eta \) and \( \gamma \) are given in appendix B. Therefore, for large \( r \)

\[
\Psi_Q(r^*, t) \approx \frac{a}{2} \sqrt{\frac{\pi}{b}} \text{Re} \left( \sum_n i e^{2i\eta(w_n)} e^{-\frac{\omega_n^2}{4b}} \left. \left( \frac{\partial \gamma}{\partial w} \right)^{-1} \right|_{w=w_n} e^{-i\omega_n(t-t_0-x-r^*)} \right).
\] (45)

Both in the derivation we gave and in the final expression (45) there is no difference when compared to the Schwarzschild case [3]. The derivation in appendix B takes care of the fact that we are using a monopole black hole instead of a Schwarzschild black hole. This gives rise to different values for \( \eta(w_n) \) and \( \partial \omega \gamma|_{w=w_n} \) when compared to the values in the Schwarzschild case.

We calculated the values of \( \eta(w_n) \) and \( \partial \omega \gamma|_{w=w_n} \) for the quasinormal mode frequencies that we have determined in section 3.2. We present the results in table 1 and plot the corresponding quasinormal mode contribution to the responses of the black holes to impinging pulses with \( a = M_P \) and \( b = M_P^2 \) in figures 3 and 4. From these figures, one can therefore see how the quasinormal mode contribution to the response of a global monopole black hole to an impinging axial gravitational wave pulse with initial data (26) depends on \( N\nu^2 \). The plotted results are intended to show the qualitative behavior that we can expect and are not meant to be precise quantitative predictions. This is because we did not obtain these plots from a precise numerical analysis but rather used (semi-)analytic methods in the derivation which required several approximations to be made.

5. Discussion and outlook

We have studied axial tensor quasinormal modes of spherically symmetric black holes with \( N \) global monopoles inside. We showed how the quasinormal modes depend on \( N\nu^2 \) by calculating the quasinormal mode frequencies for global monopole black holes with several particular values of \( N\nu^2 \). (Here \( \nu \) is the symmetry breaking scale of the model that gave rise to the monopoles.) On top of that, for monopole black holes with these particular values of \( N\nu^2 \), we have studied one relatively simple dynamical process in which these quasinormal modes get excited, the scattering of short pulses of axial gravitational waves by the monopole black holes. We have determined the quasinormal mode contributions to the responses of the monopole black holes to the impinging pulses and in this way showed how the responses depend on \( N\nu^2 \). We used semianalytical methods instead of performing a full numerical analysis. Our
results are supposed to show the correct qualitative behavior rather than being precise quantitative predictions. Finally, we have mentioned that the quasinormal modes that we determined (the analogous modes in the case of rotating black holes respectively) are also expected to get excited and to show up in various other dynamical processes such as the ringdown phase of a black hole binary merger in case at least one of the companion black holes is a black hole with global monopoles inside.

In this work we did not consider polar (even parity) quasinormal modes. In the case of monopole black holes the analysis of the polar modes is considerably different from the axial mode analysis: whereas \( \delta \phi^{(\text{axial})} = 0 \), the polar perturbations of the scalar field are non-vanishing. The corresponding linearized Einstein field equations cannot be reduced to a single second order Zerilli-type [58] equation but lead to a set of coupled differential equations. We expect the polar mode spectrum to differ significantly from the axial mode spectrum. We leave a detailed investigation of the polar modes for future work.

In regard of the recent and potential future developments in the detection of gravitational waves, it is an interesting question whether or not the effects that we found in our theoretical analysis can have observable consequences, for example in the context of measurements of the ringdown phase of a binary black hole merger with gravitational wave interferometers. Since astrophysical black holes are typically rotating, a necessary step to quantitatively answer this question would be to generalize our analysis that was done for a spherically symmetric monopole black hole to the case of a rotating black hole. We expect that also in that case the black hole’s quasinormal modes will depend on \( Nv^2 \). When asking for potential observable manifestations in a measurement in which quasinormal modes show up, one should therefore ask what values of \( Nv^2 \) are realistic for black holes in our universe. Our analysis suggests that the larger \( Nv^2 \) in our universe can be the bigger potential effects can become. It however seems difficult to derive or even estimate realistic values of \( Nv^2 \) without having concrete models and simulations of the formation processes of global monopoles and monopole black holes at hand.

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Appendix A. Components of the axial perturbations of the Einstein tensor

In this appendix we provide the expressions for the components of the axial perturbations of the Einstein tensor, \( \delta G^{(\text{axial})}_{\mu\nu} \), that we have used when deriving (15)–(17). The background Einstein field equations are used when deriving these components. The functions \( \hat{h}_0 \) and \( \hat{h}_1 \) depend only on the radial coordinate \( r \) and on the frequency \( \omega \).

\[
\delta G^{(\text{axial})}_{\theta\phi} = \frac{1}{4} \left( -2iA(r)^{-1} \omega \hat{h}_0 - \frac{4M}{M^2} r \hat{h}_1 - 2A(r) \partial_r \hat{h}_1 \right) \\
\times e^{-i\omega t} \sin \theta \left( \partial_{\theta}^2 - \cot \theta \partial_{\theta} - \frac{1}{\sin \theta} \partial_{\phi}^2 \right) \left( P_l(\cos \theta) e^{im\phi} \right) (A.1)
\]
\[
\delta G_{r_i}^{(axial)} = \left( \frac{1}{2r} A(r)^{-1} \left( 2i \omega \hat{h}_0 + ru^2 \hat{h}_1 - iuv \partial_r \hat{h}_0 \right) - \frac{1}{2r^2} \left( l + 1 \right) \hat{h}_1 + \frac{1}{r^2} \hat{h}_0 \right) \sin \theta \partial_\theta P_i(\cos \theta) e^{-iuv + i\omega t} \]  
\tag{A.2}
\]

\[
\delta G_{\hat{r}_i}^{(axial)} = \left( \frac{1}{2} A(r) \partial_r^2 \hat{h}_0 + \frac{1}{2} iuvA(r) \partial_r \hat{h}_1 + \frac{iuv}{r} A(r) \hat{h}_1 - \frac{1}{2r^2} \left( l + 1 \right) \hat{h}_0 \right) + \frac{2M}{M_p r^2} \hat{h}_0 + \frac{N}{M_p^2 r^2} \hat{h}_0 \right) \sin \theta \partial_\theta P_i(\cos \theta) e^{-iuv + i\omega t} \]  
\tag{A.3}
\]

**Appendix B. Derivation of (43) and (44)**

In this appendix we derive (43) and (44) and give the expressions for \( \eta \) and \( \gamma \) that are used in (45). We shall use the phase integral method [3, 59–61] that was used in [3] to determine the quasinormal mode contribution to the response of a Schwarzschild black hole to a Gaussian scalar wave pulse. We follow the derivation in [3] and apply it to the case of axial gravitational quasinormal mode contribution to the response of a Schwarzschild black hole to a Gaussian scalar wave pulse. We mainly adopt the notation from [3].

As a first step, it is convenient to write the wave equation (18) that we want to solve in terms of the radial coordinate \( r \) instead of the tortoise coordinate \( r^* \). This gives

\[
\left( \partial_r^2 + R(r) \right) \hat{\Phi} = 0, \]  
\tag{B.1}
\]

where

\[
\hat{\Phi} \equiv \sqrt{A(r)} \hat{\Psi}, \]  
\tag{B.2}
\]

\[
R(r) \equiv \frac{1}{A(r)^2} \left( u^2 - V_{\text{eff}} + \frac{M^2}{M_p^2} + \frac{2M}{M_p r^3} A(r) \right). \]  
\tag{B.3}
\]

Since we demand the absence of outgoing waves at the event horizon, the relevant solutions to (B.1) asymptotically scale as

\[
\hat{\Phi} \sim \frac{rM_p^2 \left( 1 - \frac{\nu^2}{M_p^2} \right)}{2M} \left( \frac{1}{\nu \gamma} \right) \left( \frac{1}{\nu \gamma} \right)^{\nu \gamma} \frac{1}{M_p^2} e^{-i\omega t} \]  
\tag{B.4}
\]

for \( r \to \frac{2M}{M_p^2} \left( 1 - \frac{\nu^2}{M_p^2} \right) \) (see (27)) and as

\[
\hat{\Phi} \sim A_{\text{out}} \sqrt{1 - \frac{\nu^2}{M_p^2}} e^{i\frac{\nu}{m_p} \frac{1}{\nu \gamma}} \left( \frac{rM_p^2 \left( 1 - \frac{\nu^2}{M_p^2} \right)}{2M} \right) \frac{2\nu \gamma}{m_p^2} \]  
\tag{B.5}
\]

\[
+ A_{\text{in}} \sqrt{1 - \frac{\nu^2}{M_p^2}} e^{i\frac{\nu}{m_p} \frac{1}{\nu \gamma}} \left( \frac{rM_p^2 \left( 1 - \frac{\nu^2}{M_p^2} \right)}{2M} \right) \frac{-2\nu \gamma}{m_p^2} \]  
\tag{B.6}
\]
for $r \to \infty$ (see (28)).

As a next step one can find an approximate solution to (B.1) by using the phase-integral method [3, 59–61]. To lowest order approximation, the general solution of (B.1) can be written as a linear combination of the two functions $f_1$ and $f_2$ which are given by [3]

$$f_{1,2}(r, t_j) = \frac{1}{\sqrt{Q(r)}} e^{\pm i \int_{r'}^{r} Q(r') \, dr'}. \quad (B.7)$$

Here the sign '+' corresponds to $f_1$ whereas '-' corresponds to $f_2$. $t_j$ are the zeros of the function $Q^2(r)$. $Q^2(r)$ is given by

$$Q^2(r) \equiv R(r) - \frac{1}{4} \left( r - \frac{2M}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)} \right)^2. \quad (B.8)$$

The second term is usually added in these contexts for the approximation to match the scaling-behavior of the exact solution in the near-horizon regime [59]. This is because $R(r)$ has a second order pole at $r = \frac{2M}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)}$ which implies that the exact solution of (B.1) scales for $r \to \frac{2M}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)}$ as [59]

$$\hat{\Phi} \sim \left( r - \frac{2M}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)} \right)^{\alpha + 1}, \quad \hat{\Phi} \sim \left( r - \frac{2M}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)} \right)^{-\alpha}. \quad (B.9)$$

where $\alpha$ is defined by

$$\alpha (\alpha + 1) \equiv - \left( \frac{2M}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)} \right)^2 \left( \frac{1}{1 - \frac{N v^2 M^2_P}{w^2}} \right)^2 \left( \frac{M^2 P^2 \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)^2}{(2M)^4} \right). \quad (B.10)$$

It is easy to see that the phase integral approximation (B.7) matches this behavior when the second term in (B.8) is added.

Using (B.8), for large $r$ the functions $f_{1,2}(r, t_1)$ can be approximated as

$$f_{1,2}(r, t_1) \approx \sqrt{\frac{1}{w^2} \left( rM^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right) \right)^2 \left( M^2 P^2 \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)^2 \right) \frac{2M_P}{M^2_P \left( 1 - \frac{N v^2 M^2_P}{w^2} \right)} e^{\pm i \frac{N v^2 M^2_P}{w^2} + \eta(w)}}, \quad (B.11)$$

where
\[\eta(w) \equiv \int_{t_1}^{\infty} \left( Q(r) - w \left( 1 - \frac{2M}{M_P^2 r} - \frac{Nv^2}{M_P^2 r^2} \right) \right) \, dr \]

\[= - \frac{w}{1 - \frac{Nv^2}{M_P^2}} \left( t_1 + \frac{2M}{M_P^2} \ln \left( \frac{t_1 M_P^2 \left( 1 - \frac{Nv^2}{M_P^2} \right)}{2M} - 1 \right) \right). \tag{B.12}\]

We evaluated this integral to calculate \(\eta\) for the quasinormal mode frequencies \(w\), that we have determined in section 3.2. The results for \(\eta(w)\) are listed in table 1.

As discussed in [3], for large \(r\) the linear approximation of \(f_1\) and \(f_2\) that solves (B.1) is

\[\hat{\Phi} \approx c \left( -i \, e^{i\gamma} f_1(r, t_1) + (e^{i\gamma} + e^{-i\gamma}) f_2(r, t_1) \right), \tag{B.13}\]

where \(c\) is a normalisation constant and

\[\gamma \equiv \int_{t_2}^{t_1} Q \, dr. \tag{B.14}\]

Inserting (B.11) gives for large \(r\)

\[\hat{\Phi} \approx \frac{c}{\sqrt{w}} \left( -i \, e^{i\gamma + \eta} \left( \frac{r M_P^2}{2M} \right) \left( 1 - \frac{Nv^2}{M_P^2} \right) e^{-2\eta} \right) \left( e^{i\gamma} + e^{-i\gamma} \right) e^{-i\eta} \left( \frac{r M_P^2}{2M} \right) \left( 1 - \frac{Nv^2}{M_P^2} \right) e^{-2\eta} \left( \frac{r M_P^2}{2M} \right) \left( 1 - \frac{Nv^2}{M_P^2} \right) e^{-2\eta}. \tag{B.15}\]

When this expression is compared to (B.6), one obtains

\[A_{\text{out}} = \frac{ic}{\sqrt{w}} \, e^{i(\gamma + \eta)}, \tag{B.16}\]

\[A_{\text{in}} = \frac{c}{\sqrt{w}} \, e^{-i\eta} \left( e^{i\gamma} + e^{-i\gamma} \right). \tag{B.17}\]

Since \(A_{\text{in}} = 0\) for quasinormal modes, \(e^{-2i\eta(w_n)} = -1.\) Thus, for quasinormal mode frequencies \(w_n\)

\[\gamma(w_n) = \left( n + \frac{1}{2} \right) \pi. \tag{B.18}\]

By iterating this equation we have determined the values of \(\frac{\partial \gamma}{\partial w} \big|_{w=w_n}\) for the quasinormal mode frequencies that we have calculated in section 3.2. The results are listed in table 1. (Note also that, using (B.18) gives rise to another way to determine the quasinormal mode frequencies \(w_n\) that we have determined in section 3.2 [3, 59–61].)
Taking the derivative of (B.17) gives for \( \alpha_n \equiv \partial_\nu A^\nu(w_n) \) the expression

\[
\alpha_n = 2i \frac{e}{\sqrt{w_n}} e^{\left(\gamma(w_n) - \eta(w_n)\right)} \left( \frac{\partial \gamma}{\partial w} \right) \bigg|_{w = w_n}, \tag{B.19}
\]

which, using (B.18), gives (43).

Finally, the function \( J(w_n) \) that was defined in (39) with \( J \) as in (31) can be approximated at large \( r \) by using the asymptotic form (B.15) with quasinormal mode frequencies \( w_n \) inserted. Taking into account (B.18), the result is

\[
J(w_n) \approx \frac{ie^{i\omega_0}}{\sqrt{w_n}} \int dr^* \left(1 - \frac{Nv^2}{M^2}\right)^{-\frac{1}{2}} a e^{-b(r - x)^2} \sqrt{w_n} \left(1 - \frac{Nv^2}{M^2}\right), \tag{B.20}
\]

which yields the expression (44).

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