ON THE GROUP-THEORETICAL APPROACH TO RELATIVISTIC WAVE EQUATIONS FOR ARBITRARY SPIN

L. Nanni

Formulating a relativistic equation for particles with arbitrary spin remains an open challenge in theoretical physics. In this study, the main algebraic approaches used to generalize the Dirac and Kemmer–Duffin equations for arbitrary-spin particles are investigated. It is proved that an irreducible relativistic equation formulated using spin matrices satisfying the commutation relations of the anti-de Sitter group leads to inconsistent results, mainly as a consequence of the violation of unitarity and the appearance of a mass spectrum that does not reflect the physical reality of elementary particles. However, the introduction of subsidiary conditions resolves the problem of unitarity and restores the physical meaning of the mass spectrum. The equations obtained by these approaches are solved and the physical nature of the solutions is discussed.

Keywords: relativistic wave equation, higher spin, anti-de Sitter group, irreducible representations of Lorentz group

DOI: 10.1134/S0040577921110027

1. Introduction

Formulating the Klein–Gordon equation [1] for spin-zero particles and the Dirac equation [2] for spin-half particles, together with discovering the positron [3] and neutron [4], gave strong impetus to the development of generalized relativistic equations for particles with arbitrary spin. Physicists of that period were convinced that other particles with different masses and spins would soon be discovered, although theoretical physics at the time had no theory that could predict the existence of such particles. The first to confront this issue was Majorana in 1932 [5], although his main goal was to prove that the negative-energy solutions of the Dirac equation were not physically acceptable. Even though Majorana’s goal turned out to be futile (a few months after the publication of his work, the positron was discovered), his research led to the first infinite-dimensional representation of the homogeneous Lorentz group. In that work, which was a precursor to his symmetric theory of the electron and positron published in 1937 [6], Majorana used group theory for the first time, which in the 1930s was still largely considered part of pure mathematics. The Majorana equation provides a wide spectrum of solutions, including space-like ones. Moreover, for time-like solutions, an unexpected discrete mass spectrum arises. We argue that, apart from its mathematical interest, the Majorana equation can be useful for investigating the nature of particle masses [7] and for studying composite quantum systems with many degrees of freedom [8]–[10].

Majorana’s work was rediscovered in the second half of the 1930s by Wigner [11] and Pauli [12]. It is also believed that Dirac read Majorana’s paper before publishing the results of his study on relativistic wave
equations for arbitrary spin, although he does not mention Majorana in his paper [13]. In fact, in his work, Dirac constructs the spin matrices following a Lie-algebra-based approach that resembles the approach used by Majorana in 1932, even though Dirac presents it using a more sophisticated formalism (that of spinors and tensors). Bhabha, to whom Pauli reported Majorana’s work in the early 1940s, in 1945 published an article on the formulation of a relativistic equation for arbitrary spins by faithfully following Majorana’s theory, except for the requirement of indeterminacy of the sign of energy [14]. In doing so, Bhabha recovered the solutions with negative energy (antiparticle states), eliminating the mathematical complexity arising from the need to solve an infinite system of differential equations. Bhabha’s theory remains one of the most promising approaches to describing particles with spin greater than one. Unsurprisingly, the equations of Kemmer and Duffin [15], [16] for spin-one particles, of Rarita and Schwinger [17] for spin-three-halves particles, and of Bargmann and Wigner [18] for particles with arbitrary spin can be obtained from that of Bhabha by introducing suitable subsidiary conditions [19].

During the last decades, the Standard Model of particle physics has evolved to one of the most precise theories in physics, describing the properties and interactions of fundamental particles in various experiments with high accuracy. However, it has some shortcomings from both the experimental and theoretical standpoints. There is no approved mechanism for the generation of masses of the fundamental particles, in particular for massive neutrinos. Besides, the standard model does not explain the observations of dark matter in the universe. Moreover, the gauge couplings of the three forces in the standard model do not unify, implying that a fundamental theory combining all forces cannot be formulated. In this scenario, we address the relativistic theory of particles with arbitrary spin as an answer to these questions, emphasizing the problem of the mass spectrum of baryons and mesons.

In this paper, we investigate and compare (in reverse chronological order) the different physical and mathematical approaches used by Bhabha, Dirac, and Majorana to formulating relativistic wave equations. This allows us to understand better why a generalized relativistic equation cannot be obtained strictly within the framework of the Lie group of symmetries of spacetime, but instead requires the introduction of subsidiary conditions that lead to results for which an appropriate physical interpretation is needed in terms of known elementary particles. Within each approach, we solve the obtained equations using complex analysis when algebraic methods lead to excessive computational difficulties. We find that the generalized equations are nothing but finite or infinite systems of linear equations, each related to a given representation of the Lorentz group. The properties of the solutions obtained are also investigated. Finally, we propose some physical interpretations of the mass spectra, considering both their correlation with Sidharth’s empirical formula and the possibility that they are related to composite systems formed by low-spin elementary particles interacting at a scale not far above their masses.

2. The Bhabha approach

Bhabha formulated a linear equation of the type

\[(i\hbar \Gamma^0 \partial_0 - i\hbar c \Gamma^\mu \partial_\mu - \chi)\psi = 0 \] (1)

where \(\Gamma^\nu (\nu = 0, 1, 2, 3)\) are finite square matrices that satisfy the commutation relations of the Lorentz group in five dimensions. This group is recognized as the anti-de Sitter group \(O(3, 2)\) and can be obtained by adding a timelike direction to the four-dimensional Minkowski spacetime [20]. \(O(3, 2)\) has two timelike directions and three spacelike directions and is therefore not a spacetime in the ordinary sense (a Lorentzian manifold with one temporal and three spatial dimensions). However, the hypersurface defined by the equation \(x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 = R^2\) is a spacetime. It has a constant positive curvature and (after a renormalization) reproduces the Minkowski spacetime when the curvature tends to zero. In this group, boosts and spatial
rotations are transformations that leave the fifth axis unchanged. Therefore, the irreducible representations of the $\Gamma^\nu$ matrices can be obtained from the irreducible representations of the $O(3,2)$ group. The wave function $\psi(x,\sigma)$ is a function of the spacetime coordinate $x = (ct, x, y, z)$ and the spin coordinate $\sigma$. The term $\chi$ in (1) is related to an unspecified rest-mass energy and its physical meaning is clarified later in this paper. The explicit form of this energy is given by the usual formula $\chi = mc^2$, but as discussed in Sec. 5, the mass $m$ depends on both the particle spin and a constant. Bhabha required no other subsidiary conditions for his theory.

Equation (1) must transform in a covariant manner under any transformations of the homogeneous Lorentz group. Letting $\Lambda^\nu_\mu$ denote a Lorentz transformation, we can rewrite (1) as

$$[i\hbar\Gamma^0(\Lambda^{-1})\partial_t - i\hbar c\Gamma^\mu(\Lambda^{-1})\partial_\mu - \chi]\psi = 0$$

which, on rearrangement, gives

$$[i\hbar(\Gamma^0\Lambda^{-1})(\Lambda\partial_t) - i\hbar c(\Gamma^\mu\Lambda^{-1})(\Lambda\partial_\mu) - \chi]\psi = 0.$$  (2)

If we set

$$\Gamma^\nu\Lambda^{-1} = \Lambda\Gamma^\nu = (\Gamma^\nu)', \quad \Lambda\partial_\nu = \partial_\nu\Lambda^{-1} = (\partial_\nu)',$$

then Eq. (1) becomes

$$[i\hbar(\Gamma^0)'(\partial_t)' - i\hbar c(\Gamma^\mu)'(\partial_\mu)' - \chi]\psi = 0.$$  (3)

Therefore, the Lorentz transformation $\Lambda^\nu_\mu$ transforms the differential four-vector operator $\partial_\nu$ in $(\partial_\nu)'$ and the spin four-vector operator $\Gamma^\nu$ in $(\Gamma^\nu)'$. This means that there exists an operator $U$ such that

$$\langle \Gamma^\nu \rangle' = U\Gamma^\nu U^{-1} \implies \Gamma^\nu = \Lambda^\nu_\mu U\Gamma^\nu U^{-1}.$$  (4)

The operators $U$ form a representation of the Lorentz group (which is a Lie group, i.e., a group of transformations that depend continuously on some parameters) of dimension $d$, whose kernel is given, for infinitesimal transformations, by the six generators $J_1 = (J_1, J_2, J_3)$ and $K = (K_1, K_2, K_3)$ that satisfy the commutation relations [21]

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k,$$  (5)

where $\epsilon_{ijk}$ is the Levi-Civita three-index symbol. In relations (5) we recognize the quantum algebra of angular momentum. Therefore, the generators $J_i$ and $K_i$ are infinitesimal rotations of Minkowski spacetime (i.e., they differ little from the identity), thereby allowing the elements of the Lie group to be written in parametric form as $g(\alpha) = e^{i\alpha_aT^a}$, where $\alpha_a$ are real numbers and $T^a$ are the aforementioned generators. The dimension $d$ of the representation and therefore the dimension of the matrices $\Gamma^\nu$ depend on the particle spin. By this algorithm, using the spin matrices for $s_0 = 1/2$ and $s_0 = 1$, we construct all those for spin $s = s_0 + 1$. We now discuss this method in greater detail.

It is known that a Lorentz representation in the anti-de Sitter space is characterized by two positive numbers [22], the first of which is the spin $s$ and the second, denoted by $m$, is $m = s, s-1, s-2, \ldots, 0,$ where $m = 0$ occurs only for an integer spin. Therefore, the number $m$ represents the projection of the spin along a given direction, such as the $z$ axis. Each pair $(s, m)$ identifies an irreducible representation of the $O(3,2)$ group whose dimension is given by [14]

$$d_5(s, m) = \frac{2}{3}\left(s + \frac{3}{2}\right)\left(m + \frac{1}{2}\right)(s - m + 1)(s + m + 2).$$  (6)
If we project the five-dimensional anti-de Sitter space along the fifth axis, then we obtain a four-dimensional real space whose orthogonal group $O(3,1)$ has an irreducible representations of dimension [14]

$$d_4(s, m) = \begin{cases} 
2(s - m + 1)(s + m + 1), & \text{if } m \neq 0, \\
(s + 1)(s + 1), & \text{if } m = 0. 
\end{cases}$$ (9)

Equations (8) and (9) are tensorial representations of the $O(3,2)$ group obtained by an appropriate two-row Young diagrams. This (pictorial) formalism establishes a one-to-one correspondence with irreducible representations of the symmetric group over the complex numbers [23]. Therefore, having set the spin $s$, we obtain $s+1/2$ equations of type (1) if $s$ is half of an odd integer (half-odd integer) or $s+1$ equations if $s$ is an integer. These equations, which are nothing but linear systems of differential equations, are formed by $d_4(s, m)$ numbers of single coupled differential equations. For instance, if $s = 1$, then we have two equations of type (1), the first formed by $d_4(1, 1) = 6$ first-order differential equations and the second by $d_4(1, 0) = 4$ first-order differential equations.

Given that Bhabha modified the Majorana theory by reintroducing the negative-energy solutions, it is clear that the matrices $\Gamma^\nu$ are $d_4(s, m) \times d_4(s, m)$ square matrices, formed from $\frac{1}{2}d_4(s, m) \times \frac{1}{2}d_4(s, m)$ spin matrices. The latter are obtained from the angular momentum ladder operators constructed using suitable linear combinations of the generators $J$ and $K$ [23], [24]. Also, the matrices representing the six generators have dimensions that depend on the choice of spin. In particular, they are diagonal block matrices of the type

$$J_{ij}(s) = \text{diag}[J_{ij}(s, s), J_{ij}(s, s - 1), \ldots], \quad K_{ij}(s) = \text{diag}[K_{ij}(s, s), K_{ij}(s, s - 1), \ldots].$$

The components of the spin matrices forming the $\Gamma^\nu$ operators are

$$
\begin{align*}
(S_x)_{ij} &= \frac{\hbar}{2}(\delta_{i,i+1} + \delta_{i+1,i})\sqrt{(s+1)(i+j+1) - ij}, \\
(S_y)_{ij} &= \frac{\hbar}{2}(\delta_{i,i+1} - \delta_{i+1,i})\sqrt{(s+1)(i+j-1) - ij}, \\
(S_z)_{ij} &= \begin{cases} 
\hbar \delta_{i,j}(s + 1 - i), & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}
\end{align*}
$$

(10)

where the indices $i$ and $j$ run from zero to $2s+1$. The matrices $S_x$ and $S_y$ have nontrivial elements on the two diagonals parallel to the main one, while the matrix $S_z$ has nontrivial elements only on the main diagonal.

The obtained results clarify why the term $\chi$ in (1) is not defined unambiguously, at least not for spin-half and spin-one particles. In fact, the commutation relations of the anti-de Sitter group [25] for $s > 1$ lead to a $\Gamma^0$ matrix with more than two distinct eigenvalues. These commutation relations can be written as

$$[\Gamma^m, \Gamma^n] = \kappa I^{mn},$$

(11)

where $I^{mn}$ is a generator of the anti-de Sitter group and $\kappa$ is a numerical constant. As suggested by Bhabha, this constant can be removed by multiplying by $\kappa^{-1/2}$, but this change is reflected in the value of the rest-mass energy $\chi$. In fact, rewriting (1) in the center-of-mass reference frame,

$$(i\hbar \Gamma^0 \partial_t - \chi)\psi = 0,$$ (12)

and accounting for the fact that the matrix $\Gamma^0$ has distinct eigenvalues, which depend on the spin, we obtain a discrete mass spectrum. In particular, if $s$ is an integer, then we obtain $2s$ distinct eigenvalues, half of which
are positive and the other half are negative, whereas if \( s \) is a half-odd integer, then we obtain \( 2s + 1 \) distinct eigenvalues, half of which are positive and the other half are negative. Therefore, for a fixed particle spin, the Bhabha equation returns a discrete mass spectrum formed by fractional values of \( \chi \), of which those with a positive (resp. negative) sign are attributed to particle (resp. antiparticle) states.

We can summarize the foregoing with an example. For a particle with spin \( s = 3/2 \), we have two equations (systems) of type (1), namely, two distinct irreducible representations of dimensions \( d_4(3/2, 3/2) = 8 \) and \( d_4(3/2, 1/2) = 12 \). This means that the first system comprises eight single differential equations and the second, twelve. For the mass spectrum, we have four values: \( m = \pm 2\chi/3 \) and \( m = \pm 2\chi \). The first pair of masses is attributed to the irreducible representation of dimension \( d_4(3/2, 3/2) \) and the second, to the representation of dimension \( d_4(3/2, 1/2) \).

Whatever the chosen spin, the Bhabha equation is equivalent to a finite system of differential equations whose coefficient matrix is nonsingular. In fact, all spin matrices \( \Gamma^{\nu} \) are invertible. Therefore, the system admits only one solution, which can be calculated using Cramer’s rule. For free particles, the solutions are wave functions of the type

\[
\psi = \varphi(E, p)e^{\pi i(p_x x + p_y y + p_z z - Et)/\hbar},
\]

where \( \varphi(E, p) \) is a spinor whose nontrivial components are suitable combinations of the four-momentum components \((E/c, p_x, p_y, p_z)\), while \( E \) is the particle energy, whose value depends on the spin,

\[
E^2 = p^2 c^2 + \left(\frac{\chi}{s}\right)^2.
\]

where \( p^2 \) is the dot product of the Euclidean vector \( p = (p_x, p_y, p_z) \). Therefore, the energy \( E \) depends on the spin via the term \( \chi \). Substituting (13) in (1) and evaluating all the derivatives, we obtain a system of algebraic equations in which the unknowns are the spinor components \( \varphi_\nu \):

\[
[i\hbar(\Gamma^{\nu^\prime})^0 \varphi_\nu \partial_\chi - i\hbar c \Gamma^{\rho\nu^\prime})^0 \varphi_\nu \partial_\rho - \chi \varphi_\nu]e^{\pi i(p_x x + p_y y + p_z z - Et)/\hbar} = 0.
\]

Using Cramer’s rule, we obtain all the values of \( \varphi_\nu \), namely, the complex numbers

\[
0, 1, \alpha \left(\frac{cp_z}{E + \chi/s}\right)^n, \beta \left(\frac{c(p_x \pm ip_y)}{E + \chi/s}\right)^n,
\]

where \( n = s \) for integer spin and \( n = s + 1/2 \) for half-odd integer spin, while \( \alpha \) and \( \beta \) are constant numbers from among the elements of the spin matrices.

For clarity, we reconsider the example of a spin 3/2 particle. For the particle state whose spin is up and has a the z-component equal to 3/2, the spinor \( \varphi(E, p) \) can be written as

\[
\varphi(E, p) = (1, 0, 0, \varphi_5, \varphi_6, \varphi_7, \varphi_8)^T e^{\pi i(p_x x + p_y y + p_z z - Et)/\hbar}.
\]

Substituting this spinor in the \( s = 3/2 \) Bhabha equation corresponding to the irreducible representation of dimension \( d_4(3/2, 3/2) = 8 \), we obtain

\[
\varphi_5 = \varphi_6 \propto \left(\frac{cp_z}{E + 2\chi/3}\right)^2, \quad \varphi_7 = \varphi_8 \propto \left(\frac{c(p_x \pm ip_y)}{E + 2\chi/3}\right)^2.
\]

This procedure must be repeated for all other possible particle–antiparticle states with spin up and spin down. The main inconsistency of Bhabha’s theory arises from the fact that the representations of the homogeneous Lorentz group are finite and therefore nonunitary, at least for all spin values greater than one. Furthermore, each component of the wave function by itself does not satisfy the Klein–Gordon equation, which it should in any relativistic quantum theory. In addition to these inconsistencies, we are also dealing with a mass spectrum that cannot be interpreted on the basis of current knowledge of particle physics. We postpone our discussion of these aspects to Sec. 6.
3. The Dirac approach

Dirac constructed a relativistic wave equation for spin greater than 1/2 such that the representations of the Lorentz group were unitary and each component of the wave function by itself satisfied the Klein–Gordon equation [13]. The main problem with this approach is that unitary representations require spin matrices that do not ensure the Lorentz invariance of the equation. Dirac overcame this difficulty by the ad hoc introduction of two auxiliary matrices, although these complicate the mass spectrum that is obtained, which depends on two parameters rather than just one.

To formulate his theory, Dirac wrote the generic spin operator as a six-vector of components \( (S_{xt}, S_{xy}, S_{xz}, S_{yt}, S_{yz}, S_{zt}) \), in units of \( \hbar \). These components satisfy the angular momentum commutation relations \([26], [27]\). From these components, it is possible to construct six generators \( \alpha_\mu \) and \( \beta_\mu \), \( \mu = 1, 2, 3 \), that behave like independent angular momenta in \( \mathbb{R}^3 \):

\[
\alpha_\mu = \frac{1}{2}(S_{jk} - iS_{kt}), \quad \beta_\mu = \frac{1}{2}(S_{jk} + iS_{kt}),
\]

where the six operators are obtained by cyclic permutations of \( i, j, k \) from \( x \) to \( z \). Operators (18) satisfy the commutation relations

\[
\alpha_i \alpha_j - \alpha_j \alpha_i = i\alpha_k, \quad \beta_i \beta_j - \beta_j \beta_i = i\beta_k,
\]

in addition to the following relations involving the squares of the individual components:

\[
\sum_{\mu=1}^{3} \alpha_\mu^2 = k(k+1), \quad \sum_{\mu=1}^{3} \beta_\mu^2 = l(l+1).
\]

Here, \( k \) and \( l \) are integers or half-odd integers and are the two parameters that characterize the representations of the homogeneous Lorentz group. As expected, they coincide with those obtained from Bhabha’s theory. Equations (20) show that \( \alpha_\mu \) and \( \beta_\mu \) are square matrices of respective dimensions \( k(k+1) \times k(k+1) \) and \( l(l+1) \times l(l+1) \). From these matrices, we construct two new operators

\[
A = \begin{pmatrix}
\alpha_x & \alpha_x - i\alpha_y \\
\alpha_x + i\alpha_y & -\alpha_x
\end{pmatrix}, \quad B = \begin{pmatrix}
\beta_x & \beta_x - i\beta_y \\
\beta_x + i\beta_y & -\beta_x
\end{pmatrix}
\]

that satisfy

\[
A(A + 1) = k(k+1), \quad B(B + 1) = l(l+1)
\]

and whose eigenvalues are \( k, k-1, \ldots, -k \) and \( l, l-1, \ldots, -l \), respectively. The dimension of the matrices \( A \) and \( B \) are respectively \( 2(2k + 1) \times 2(2k + 1) \) and \( 2(2l + 1) \times 2(2l + 1) \), i.e., they have twice as many elements as the matrices \( \alpha_\mu \) and \( \beta_\mu \). In this, Dirac faithfully followed the approach used to formulate his equation for spin-1/2 particles, where the matrices \( \alpha_\mu \) have twice the dimension of the Pauli matrices from which they are formed [2].

The next step is to find a unitary transformation \( U \) that makes the matrices \( A \) and \( B \) diagonal. We note that no requirement has been imposed regarding the algebraic nature of the matrices \( \alpha_\mu \) and \( \beta_\mu \). Specifically, it is not required that these matrices be Hermitian (which would ensure the unitarity of the representations of the Lorentz group). The transformation that diagonalizes the matrices \( A \) and \( B \) is

\[
U^{-1}AU = \text{diag}(k, \ldots, k, -k-1, \ldots, -k-1), \quad \text{q times, p times}
\]

\[
U^{-1}BU = \text{diag}(l, \ldots, l, -l-1, \ldots, -l-1), \quad \text{q times, p times}
\]

(23)
where \( q \) and \( p \) are the multiplicities of the eigenvalues of \( A \) and \( B \). Dirac proved that his theory was consistent only if
\[
m = 2(k + 1), \ 2(l + 1), \quad n = 2k, \ 2l.
\] (24)
where \( m \) is the number of columns of \( A \) and \( n \) is the number of columns of \( B \). Relations (24) satisfy the constraint \( m + n = 2(2k + 1) \) and ensure that the matrices \( \alpha_z \) and \( \beta_z \) have respective eigenvalues \( k, \ldots, -k \) and \( l, \ldots, -l \). We have thus obtained explicit forms for the matrices \( \alpha_z \) and \( \beta_z \) through which, using Eqs. (19) and (20), we obtain those for the matrices \( \alpha_x, \alpha_y \) and \( \beta_x, \beta_y \).

We now have all the algebraic tools for writing the relativistic equation for a particle of spin \( s \). Because the matrices \( A \) and \( B \) have eigenvalues with different multiplicities, Dirac split the wave function \( \psi \) into two parts, \( \psi_A \) and \( \psi_B \), of which the former has \((2k + 1)\times 2l\) components and the latter \((2l + 1)\times 2k\) components. The wave functions \( \psi_A \) and \( \psi_B \) must satisfy the coupled equations
\[
(i\hbar \partial_t - i\hbar c\alpha^\mu \partial_\mu)\psi_A = m'c^2 \delta \psi_B, \quad (i\hbar \partial_t - i\hbar c\beta^\mu \partial_\mu)\psi_B = m''c^2 \gamma \psi_A,
\] (25)
where \( \delta \) and \( \gamma \) are nonsingular matrices of respective dimensions \( 2k \times 2k \) and \( 2l \times 2l \). These matrices have been introduced in an ad hoc manner so as to have the Klein–Gordon equation satisfied. In fact, obtaining \( \psi_B \) from the first equation in (25) and replacing it in the second one, we arrive at
\[
(-\hbar^2 \partial_t^2 + \hbar^2 \beta^\mu \alpha_\mu \partial_\mu)\psi_A = m'm''c^4 \gamma \delta \psi_A
\] (26)
which is the Klein–Gordon equation if we impose the conditions \( \beta^\mu \alpha_\mu = \mathbb{1} \) and \( \gamma \delta = \mathbb{1} \). These algebraic constraints represent the subsidiary conditions needed to make the relativistic equation consistent with the typical energy–momentum relation. The term \( m'm''c^4 \) can only have the meaning of the squared rest mass energy and, because the matrices \( A \) and \( B \) have been constructed, depends on the two parameters \( k \) and \( l \).

In fact, setting \( m'm''c^4 = \chi^2 \) (to use the same formalism as in Bhabha’s theory), we easily obtain the relations
\[
m'c^2 = \chi \left(\frac{k}{l}\right)^{1/2}, \quad m''c^2 = \chi \left(\frac{l}{k}\right)^{1/2}.
\] (27)
Therefore, the two mass spectra are dual to each other. It follows from (27) that the matrices \( \delta \) and \( \gamma \) can be chosen as \( \text{diag}[(k/l)^{1/2}] \) and \( \text{diag}[(l/k)^{1/2}] \). In this way, their product trivially gives the unit matrix, as Eq. (26) requires.

Compared with Bhabha’s theory, the mass spectrum that arises from (25) is characterized by two parameters. For instance, a particle with the rest energy \( \chi \) and spin \( s = 3/2 \) has the mass states
\[
\begin{array}{c|c|c}
\frac{3}{2} & \frac{3}{2} & \implies \psi_A \to m' = \frac{\chi}{c^2}, \quad \psi_A \to m'' = \frac{\chi}{c^2} \\
\frac{3}{2} & \frac{1}{2} & \implies \psi_A \to m' = \sqrt{3} \frac{\chi}{c^2}, \quad \psi_A \to m'' = \frac{\chi}{\sqrt{3}c^2} \\
\frac{3}{2} & -\frac{3}{2} & \implies \psi_A \to m' = -\frac{\chi}{c^2}, \quad \psi_A \to m'' = -\frac{\chi}{c^2} \\
\frac{3}{2} & -\frac{1}{2} & \implies \psi_A \to m' = \frac{\chi}{\sqrt{3}c^2}, \quad \psi_A \to m'' = -\frac{\chi}{\sqrt{3}c^2} \\
\end{array}
\] (28)
where the negative values refer to antiparticle states. As expected, with the introduction of the matrices \( \delta \) and \( \gamma \) into the mass terms in (25), the obtained mass spectrum is twice that of Bhabha’s theory. The number of differential equations that comprise the coupled system (25), on the other hand, remains unchanged compared with the case \( s = 3/2 \) addressed in the previous section.
The system of coupled equations (25) can be solved easily using Cramer’s rule, assuming that the coefficient matrix is invertible. However, the spinors \( \psi_A \) and \( \psi_B \) are to be further split into two parts, because each of them contains terms referring to two distinct mass values. Remaining with the example of \( s = 3/2 \), we can write

\[
\psi_A = \psi'_A \left( \pm \frac{\sqrt{3} \chi}{c^2} \right) \oplus \psi''_A \left( \pm \frac{\sqrt{3} \chi}{c^2} \right), \quad \psi_B = \psi'_B \left( \pm \frac{\chi}{c^2} \right) \oplus \psi''_B \left( \pm \frac{\sqrt{3} \chi}{c^2} \right),
\]

where the explicit forms of \( \psi'_A \), \( \psi''_A \), \( \psi'_B \), and \( \psi''_B \) are

\[
\psi'_A = (\varphi_1, \varphi_2)^T e^{\mp \sqrt{3}(p_x+q_y+q_z-E't)/\hbar}, \quad \psi''_A = (\varphi_3, \varphi_4)^T e^{\mp \sqrt{3}(p_x+q_y+q_z-E't)/\hbar},
\]

\[
\psi'_B = (\varphi_5, \varphi_6)^T e^{\mp \sqrt{3}(p_x+q_y+q_z-E''t)/\hbar}, \quad \psi''_B = (\varphi_7, \varphi_8)^T e^{\mp \sqrt{3}(p_x+q_y+q_z-E''''t)/\hbar}.
\]

The energies \( E', \ldots, E''' \) are given by

\[
E' = E''' = (p^2 c^2 + \chi^2)^{1/2}, \quad E'' = [p^2 c^2 + (\sqrt{3} \chi)^2]^{1/2}, \quad E''' = [p^2 c^2 + (\chi/\sqrt{3})^2]^{1/2}.
\]

Successively substituting (31) in (30), (30) in (29), and (29) in (25) gives two systems, each of four algebraic equations, from which we obtain the eight spinor components. Overall, the method of solving (25) is similar to that discussed for the Bhabha equation, only a little more laborious. Depending on the state under consideration, the nontrivial components of the four-spinor have a form similar to (17). For instance, for the spin-up state \( (3/2, 1/2) \), we have

\[
\varphi_5 \propto \left( \frac{cp_x}{E + \chi} \right)^2, \quad \varphi_6 \propto \left( \frac{c(p_x \pm ip_y)}{E + \sqrt{3} \chi} \right)^2,
\]

\[
\varphi_7 \propto \left( \frac{cp_x}{E + \chi} \right)^2, \quad \varphi_8 \propto \left( \frac{c(p_x \pm ip_y)}{E + \frac{\chi}{\sqrt{3}}} \right)^2.
\]

4. The Majorana approach

Majorana’s approach was aimed at eliminating the negative-energy solutions that appear in the Dirac equation for spin-1/2 particles [5]. This was in 1932, and when Majorana formulated his theory, the positron had yet to be discovered (or at least the news of its discovery had not yet arrived in Italy). Majorana considers these solutions unphysical and occurring because the eigenvalues of the Dirac energy operator are square roots of positive real numbers. Majorana’s aim, therefore, was to find a relativistic equation that admits only positive-frequency solutions and that, for slow motions, reduces to the Schrödinger equation. The only way to achieve this goal is to eliminate the uncertainty in the sign of the energy, by formulating an equation that admits only a single root. Algebraically, this is possible by considering infinite-component wave functions. The Majorana equation can be written as (1), but with \( \Gamma'' \) obtained as infinite sequences of finite spin matrices. These matrices provide an infinite-dimensional representation of the homogeneous Lorentz group. Therefore, the root of the equation is a wave function with an infinite number of components that cannot be split into finite spinors, because is not possible to decide a priori to which spin a component corresponds.
The starting point of the Majorana approach is to require that the quadratic form \( \psi^\ast \Gamma^0 \psi \) be positive definite and that it transform covariantly under the action of the elements of the homogeneous Lorentz group. With these constraints, Majorana determined the structure of all the matrices \( \Gamma^\nu \).

In this section, we do not intend to investigate the algebraic method developed by Majorana: other authors have published detailed studies of the scientific importance of Majorana’s work \cite{24, 25, 28–31}. Rather, we want to explicitly determine and investigate all the possible solutions: time-like, light-like, and space-like. In fact, unlike Bhabha’s and Dirac’s equations, the Majorana equation allows superluminal solutions with a continuous mass spectrum, a peculiarity attributed to the requirement of a positive-definite \( \psi^\ast \Gamma^0 \psi \) term.

We write the Majorana equation as

\[
(i \hbar \partial_0 - i \hbar c \Gamma^\mu \partial_\mu - \chi)\psi = 0,
\]

where, as usual, \( \chi \) is an undefined rest-mass energy. The wave function \( \psi \) depends on the spacetime coordinates and on the infinite components of the spin space, which transform under a unitary representation of the \( O(3, 1) \) group generated by the Lie operators \( J \) and \( K \) \cite{5}. The structure of the matrices \( \Gamma^0 \) and \( \Gamma^\mu \) is very similar to that of Bhabha matrices, with the nontrivial components of \( \Gamma^0 \) lying along the main diagonal and those of \( \Gamma^\mu \) being blocks along one of the two diagonals. In particular, the nontrivial components of the matrix \( \Gamma^0 \) are given by \( s + 1/2 \). For convenience, we rewrite (33) as

\[
i \hbar \partial_0 \psi = H \psi, \quad H = i \hbar \sum_{\mu=1}^{3} \Gamma^\mu + \chi (\Gamma^0)^{-1}.
\]

Here, the matrix \((\Gamma^0)^{-1}\) has eigenvalues given by \( 1/(s+1/2) \) with multiplicity \( 2s+1 \). Regarding the structure of the other three matrices, \( \Gamma^3 \) has nontrivial elements on the main diagonal, given by \( (m, m - 1, \ldots, -m) \), while \( \Gamma^2 \) and \( \Gamma^3 \) have nontrivial elements on the two diagonals parallel to the main one.

Taking the wave function as the plane wave (13), where the spinor \( \varphi(E, p) \) has an infinite number of components and the energy \( E \) is a function of the spin \( s \), and considering the case of a half-odd integer spin, we find that the Hamiltonian matrix takes a form in which the spin matrices with progressively increasing values develop along the main diagonal, giving the typical block structure

\[
\begin{pmatrix}
\chi - cp_z & c(p_x + ip_y) & 0 & \ldots & \ldots & 0 & 0 \\
c(p_x - ip_y) & \chi + cp_z & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & \chi - 3cp_z & \sqrt{3}c(p_x + ip_y) & 0 & \ldots & 0 \\
\vdots & \vdots & \sqrt{3}c(p_x - ip_y) & \chi - 3cp_z & c(p_x + ip_y) & \vspace{3pt} \vdots & 0 \\
\vdots & \vdots & 0 & c(p_x - ip_y) & \chi + 3cp_z & \vspace{3pt} \vdots & 0 \\
0 & 0 & 0 & 0 & \sqrt{3}c(p_x - ip_y) & \chi + 3cp_z & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix}.
\]

By inserting this matrix in (34), we obtain a system of infinite algebraic equations with an infinite number of components. This matrix is clearly nonsingular. This means that \( H \) has a nonzero determinant, a necessary but not sufficient condition for an infinite system to admit only one root. To ensure the uniqueness of the
solution, the vector of coefficients, whose components are $i\hbar \partial_t \psi_i = E_i$, must be bounded \[32\]. The Majorana equation satisfies this condition, given that the rest-mass energy decreases with increasing spin:

$$E^2(s) = p^2 c^2 + \left(\frac{\chi}{s + 1/2}\right)^2.$$ \[35\]

We have thus proved that (34) admits only one solution. However, the spinor components can be calculated through finite truncations \[32\]. Although laborious, this method is justified by the fact that the spinor components are given by

$$\psi_s(\mu) \propto \left[\pm \frac{c(p_x \pm ip_y)}{E + \chi}\right]^{s+1/2},$$

$$\psi_s(\nu) \propto \left[\frac{c(p_x \pm ip_y)}{E + \chi}\right]^{s+1/2},$$ \[36\]

where the first line holds for half-odd integer spin and the second line for integer spin. The indices $\mu$ and $\nu$ are related by $\nu = \mu + 1$. For time-like solutions, where the particle velocity is always less than the speed of light, the quantities in square brackets in (36) are less than unity and therefore, because these quantities are raised to the power of either $s + 1/2$ or $s + 1$, the spinor components become progressively smaller as the spin $s$ increases. This explains why it is possible to solve the infinite system of equations by finite truncations.

By choosing a cutoff beyond which spinor components (36) are deemed insignificant, we can approximate the infinite system as a finite one that can be solved easily using Cramer’s rule. However, this approach cannot be used for light-like and space-like solutions because the quantities in square brackets are then equal to or greater than unity. This means that with increasing spin, the components in (36) retain the same order of magnitude (light-like solutions) or increase progressively (space-like solutions).

Given the difficulty in solving the Majorana equation with algebraic methods, we use an analytic approach (although other methods based on the quaternionic algebra are effective options \[32\]) based on the fact that once a solution for a particular configuration has been obtained, it is possible to obtain all the others via Lorentz transformations.

We suppose that $\psi(p_\nu)$ is a particular solution and seek the solution for another configuration. Then, proceeding similarly to Sec. 2, we find that there exists a matrix $U(\Lambda)$ that depends on the Lorentz transformation and relates the two configurations under consideration such that

$$\psi'(p_\nu) = U(\Lambda)\psi(p_\nu) \implies \psi(p_\nu) = U^{-1}(\Lambda)\psi'(p_\nu).$$ \[37\]

Substituting this in (33) gives

$$(i\hbar \Gamma^0 \partial_t - i\hbar c \Gamma^\mu \partial_\mu - \chi)U^{-1}(\Lambda)\psi'(p_\nu) = 0,$$ \[38\]

whence, by multiplying the left-hand side by $U(\Lambda)$, we obtain

$$(i\hbar U \Gamma^0 U^{-1} \partial_t - i\hbar c U \Gamma^\mu U^{-1} \partial_\mu - \chi)\psi(p_\nu) = 0.$$ \[39\]

Because $U \Gamma^\nu U^{-1}$ is equal to $\Lambda_{\nu}^\nu$, Eq. (39) shows that $\psi'(p_\nu)$ also satisfies (33).

We first consider the case of time-like solutions. The simplest configuration from which to obtain a particular solution is that in the center-of-mass reference frame, where $E = mc^2$ and $p = 0$. The Majorana equation then becomes

$$(i\hbar \Gamma^0 \partial_t - \chi)\psi = 0 \implies m = \frac{\chi}{c^2(s + 1/2)},$$ \[40\]
because the eigenvalues of $\Gamma^0$ are $s + 1/2$. As anticipated above, the time-like solutions have a discrete mass spectrum. To understand the physical nature of these solutions, we introduce a representation of the anti-de Sitter group $O(3, 2)$ through complex analysis [34]–[37]

$$z = x + iy, \quad \bar{z} = x - iy, \quad \partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}. \quad (41)$$

Using these transformations, we can write the four-vector $\Gamma^\nu$ as

$$\Gamma^\nu = \left( \frac{1}{2} (z\bar{\partial} - \partial\bar{z}), -\frac{1}{4} (z\bar{\partial} + z\partial), \frac{i}{4} (z\bar{\partial} + z\partial), -\frac{1}{2} (z\bar{\partial} + \partial\bar{z}) \right). \quad (42)$$

With this representation, we have moved to a two-dimensional space where operators (41), normalized so as to be dimensionless, take the form

$$z = \frac{x + iy}{\lambda_0}, \quad \bar{z} = \frac{x^2 + y^2}{\lambda_0^2}, \quad \partial\bar{\partial} = \frac{1}{4} \lambda_0^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad (43)$$

where $\lambda_0 = (x^2 + y^2)^{1/2}$ is a unit length. Substituting the time component of (42) in (40) and using (43), we obtain

$$\left[ -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} \mu \omega^2 \lambda^2 - E \right] \psi = 0, \quad (44)$$

where

$$\omega = \frac{2\hbar}{\mu \lambda_0}, \quad E = \frac{2\hbar^2}{\mu \lambda_0^2}. \quad (45)$$

Equation (44) is the Schrödinger equation of a two-dimensional harmonic oscillator in an attractive parabolic potential. Therefore, the solution of the Majorana equation in the center-of-mass reference frame is an infinite set of harmonic oscillators with the reduced mass given by

$$\mu = \frac{1}{c^2 (s + 1/2)(s + 1/2) + 1} \left( \frac{\chi}{s + 1/2} + \frac{\chi}{(s + 1/2) + 1} \right)^{-1} = \frac{\chi}{2c^2(s + 1)}. \quad (46)$$

As the spin increases, the frequency and energy of the oscillator increase. Therefore, the solution of the Majorana equation is a function with an infinite number of components given by

$$\psi_s = \hbar(x, y) \exp \left[ -\frac{\chi \omega (x^2 + y^2)}{4\hbar c^2 (s + 1)} \right]. \quad (47)$$

The spin can be written as $2s(n_x + n_y)$, where $n_x$ and $n_y$ are the quantum numbers (integer numbers) of the two-dimensional oscillator. The function $\hbar(x, y)$ depends on the quantum numbers $n_x$ and $n_y$ and is a polynomial function of the two spatial coordinates [37].

This approach shows that although the mass spectrum decreases with increasing spin, the mass states represented by the harmonic oscillators become progressively more energetic. This would seem to contradict experimental results, according to which lighter elementary particles are more stable than heavier ones (for example, the muon is heavier than the electron but rapidly decays into lighter and very stable particles). This contradiction would seem to compromise the application of the Majorana model to elementary particles, but at the same time suggests that it could be useful for the study of systems of interacting particles, not interpretable by the Standard Model, currently unknown or only hypothesized (such as dark matter or exotic baryons such as pentaquarks). If we had used the purely algebraic method of solution, this aspect...
of fundamental importance for the theory would not have emerged, and its absence would have led to a misleading interpretation of the mass spectrum.

We now consider the massive light-like solutions. The best configuration for the particular solution in this case is that in which the momentum has its nonzero component along the $z$ axis. The Majorana equation then becomes

$$(i\hbar\Gamma^0\partial_t \pm i\hbar c \Gamma^3 \partial_z - \chi)\psi = 0.$$  \hspace{1cm} (48)

Because the free particle is traveling at the speed of light, it is convenient to use the parameterization $(i\hbar\partial_t)\psi = (i\hbar c \partial_z)\psi = \hbar \omega$. Assuming that the matrices $\Gamma^0$ and $\Gamma^3$ are diagonal, it is easy to obtain the energy spectrum associated with (48) as

$$\hbar \omega = \frac{\chi}{(s + 1/2) \pm \delta},$$  \hspace{1cm} (49)

where $\delta$ are the eigenvalues of $\Gamma^3$. The physical meaning of the particular solution becomes clear if we rewrite (48) using the complex operators

$$\left[ -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{2\hbar\chi}{\mu \omega \lambda_0^2} \right] \psi = 0,$$

$$\left( \hbar \omega \frac{x^2 + y^2}{\lambda_0^2} - \chi \right) \psi = 0,$$  \hspace{1cm} (50)

where the first and second equations respectively correspond to the $+$ and $-$ signs in front of the $\Gamma^3$ term in (48) and $\omega = 2\hbar / \mu \lambda_0^2$. The first equation in (50) is the Schrödinger equation for a particle in a constant potential and with zero total energy. As usual, the spin dependence is contained in the reduced mass, as in the time-like case. Because a light-like particle cannot have zero energy, the obtained result can be interpreted by assuming that the first equation in (50) describes the motion of a composite system whose total energy is zero. The second equation, on the other hand, can be rewritten as

$$(x^2 + y^2)\psi = \frac{\lambda_0^2 \chi}{\hbar \omega} \psi \implies x^2 + y^2 = \frac{\lambda_0^2 \chi}{\hbar \omega}.$$  \hspace{1cm} (51)

This is the Cartesian equation of a circle of radius $\lambda_0^2 \chi / \omega$, which implies that the motion of the harmonic oscillator associated with each spin is constrained to lie on a circle.

Finally, we consider space-like solutions, choosing the configuration to be that in which the particle has zero energy (a reference frame with infinite velocity) and only a $z$ component of spatial momentum. The Majorana equation then becomes

$$(i\hbar c \Gamma^3 \partial_z - \chi)\psi = 0 \implies m^2 = -\left( \frac{\chi}{\delta c^2} \right)^2,$$  \hspace{1cm} (52)

where $\chi$ is an imaginary rest energy. In this case also, the physical meaning of the solution becomes clear when we rewrite (52) using complex operators (43):

$$\left[ -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{2} \mu \omega^2 x^2 - E \right] \psi = 0.$$  \hspace{1cm} (53)

This equation is completely analogous to (44) except for the presence of a repulsive parabolic potential. Therefore, the space-like solutions are energetically unstable but still possible. If instead the particle is subluminal with momentum $p_z$, then we obtain an equation completely similar to (53) but with an attractive parabolic potential. This proves that the time-like solutions, unlike the tachyon ones, are stable.
5. Mass quantization

The connection between the masses of elementary particles and the symmetries of the Lorentz group has long been a topic of debate [38]–[40]. According to Varlamov [41], the classification of relativistic wave equations is based on the interlocking representations of the Lorentz group. A system of interlocking representations is associated with a system of eigenvector subspaces of the energy operator. Such a correspondence allows defining the matter spectrum such that each level of this spectrum presents some state of an elementary particle. The theories investigated in this paper lead to a discrete mass spectrum, the correctness of which can be assessed only after a choice of the rest energy $\chi$ has been made. This is a feature shared by all quantum theories dealing with representations of the Lorentz group [40]. In our opinion, this is due to the difficulty in finding a unitary representation of the Lorentz group without introducing subsidiary conditions or other physical or mathematical constraints. Recently, an empirical formula has been proposed by Sidharth for the mass spectrum of baryons and mesons. It has the form [41], [42]

$$m = 137p\left(q + \frac{1}{2}\right),$$  \hspace{1cm} (54)

where 137 is the pion mass in MeV/c$^2$, while $p$ and $q$ are positive integer numbers. Formula (54) reproduces the entire mass spectrum of known baryons and mesons with errors never exceeding 3%. Sidharth suggested that the numbers $p$ and $q$ could be quantum numbers of harmonic oscillators, without giving a satisfactory proof. Following this idea and considering that the time-like solutions of the Majorana equation can be interpreted in terms of the energy spectrum of harmonic oscillators with varying reduced mass, we can think of correlating formula (54) with the expressions in (45).

The wave equations for particles with arbitrary spin, in fact, are suitable for describing composite systems formed by elementary particles [8], [25]. In the center-of-mass frame, the Majorana oscillator energy can be rewritten as

$$E = m(s)c^2 = \frac{2\hbar^2}{\mu\lambda_0^2} \implies m(s) = \frac{2\hbar^2}{c^2\mu\lambda_0^2},$$  \hspace{1cm} (55)

and, on substituting for $\mu$ from (46), we obtain the Majorana mass spectrum

$$m(s) = \frac{2\hbar^2(s + 1)}{\chi\lambda_0^2}. \hspace{1cm} (56)$$

By equating this expression and formula (54), we obtain an algebraic relation between the spin $s$ and the positive integer numbers $p$ and $q$, which in addition gives physical meaning to the rest energy $\chi$:

$$s = pq + \frac{1}{2}p - 1, \quad \chi = \frac{4\hbar^2}{137\lambda_0^2}. \hspace{1cm} (57)$$

The value of the constant $\chi$ is $1.13687 \cdot 10^{-27}$ MeV/$\lambda_0^2$. If we assume that the order of magnitude of $\lambda_0$ is no greater than $10^{-15}$ m (i.e., the average dimension of a nucleus), then $\chi$ is of the order of 1 GeV. From the first expression in (57), we see that a half-odd integer spin can be obtained only if $p$ is an odd positive integer. By combining all the possible values of $p$ and $q$, all spin sequences are obtained. Majorana’s theory, however, does not provide the numerical value of the constant $\chi$, which must be calculated using experimental data. This is also the case for the Standard Model, where, for example, the coupling constants are obtained from experimental data. In this framework, the mass of the pion acquires the meaning of a physical constant from which the entire mass spectrum of baryons and mesons can be obtained.
We note that the constant $\chi$ depends on the particular theory under consideration. In fact, we can also apply complex operators (44) to the Bhabha and Dirac equations, obtaining finite sets of two-dimensional harmonic oscillators whose reduced mass depends on the algebra of the representation of the Lorentz group. For instance, in the Bhabha theory, where the energy spectrum is given by $\chi/s$, the reduced mass becomes $\mu = \chi/(2s + 1)$, which leads to the formula $\chi = 2\hbar^2/137\lambda_0^2$ and the spin $s = pq/2 + p/4 - 1/2$. In the case of the Dirac theory, things become somewhat more complicated because we have two dual mass spectra and therefore two values of $\chi$.

The interpretation we have given to the mass spectrum could be somewhat speculative and forced, but the logical connection between the empirical formula and the fact that the relativistic equations can always be reformulated as those of quantum harmonic oscillators is evident. Therefore, our speculation still has a robust scientific basis.

We can think more simply of the mass spectra that arise from the representations of the Lorentz group as possible energy states that a particle can have, just as the hydrogen atom can occupy states of increasing energy. The fact that these states have not yet been detected experimentally could be due to their high energy, far from the energy scale of current experimental techniques. Another interpretation could be that these states with higher spins become strongly interacting at a scale not far above their mass. This means that such particles could arise as composites of other, lower-spin particles. For instance, in quantum chromodynamics (QCD), there are spin-9/2 hadrons that are explained as high-spin excitations of baryons made out of three elementary spin-1/2 particles [43], [44].

6. Concluding discussion

We have investigated the main algebraic approaches to formulating the equations for an arbitrary spin. All the other equations relating to particles with a well-defined spin, such as the Dirac, Kemmer–Duffin, and Rarita–Schwinger equations, are nothing other than particular cases of one of these generalized equations. The algebraic structure of these equations is the same: what changes is the form of the spin matrices that characterize them. In fact, depending on the initial hypotheses and the physical and mathematical constraints imposed, finite or infinite sets of linear equations are obtained, whose solutions may or may not satisfy the Klein–Gordon equation and which lead to discrete mass spectra that depend on the spin. Each equation of the set is related to a specific representation of the homogeneous Lorentz group, which is unitary only if suitable subsidiary conditions are added to the commutation relations of the spin operators. This suggests that the algebraic apparatus with which the generalized equation should be formulated is perhaps not complete, although it is known that the representations of the Lorentz group are deeply connected with the physical nature of particles.

The structure of the mass spectra obtained from the relativistic equations previously described does not reflect the reality of known particles. The symmetries of spacetime and the subsidiary conditions imposed by Bhabha, Dirac, and Majorana on their theories are not sufficient to construct a model capable of describing particles with any spin. However, by relating the mass energy term that unites these equations with an empirical formula that describes the spectrum of masses of all the particles known to date (in this work, we have chosen that of Sidharth, which is among the most promising in terms of having just a small deviation between theoretical and measured values), it is possible to determine the value of the term $\chi$ that characterizes each of the equations under consideration. This relation occurs through the mass of the pion, which in the framework we are studying assumes the meaning of a physical constant and which together with the spin value allows the complete mass spectrum of the known particles to be obtained. It represents the subsidiary condition that permits us to reconsider the old relativistic equations within the framework of modern particle physics. With them, in fact, it will be possible to obtain field theories that could reveal the existence of new possible particles that have yet to be observed, or new forms of interaction of matter.
Other interpretations of the mass spectra can be given, in which it is assumed that each particle can occupy higher-spin states with energies that are not yet accessible to experimental observation, or in which the higher-spin states arise in composite systems with many degrees of freedom.

Conflicts of interest. The author declares no conflicts of interest.

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