Symmetric Mahler’s conjecture for the volume product in the three dimensional case

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Abstract

In this paper, we prove Mahler’s conjecture concerning the volume product of centrally symmetric convex bodies in $\mathbb{R}^n$ in the case where $n = 3$. More precisely, we show that for every three dimensional centrally symmetric convex body $K \subset \mathbb{R}^3$, the volume product $|K| |K^o|$ is greater than or equal to $32/3$.

1 Introduction

1.1 Mahler’s conjecture for the volume product

A convex body in $\mathbb{R}^n$ is a compact convex set in $\mathbb{R}^n$ with nonempty interior. Denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$. A convex body $K \in \mathcal{K}^n$ is said to be centrally symmetric if it satisfies $K = -K$. We denote by $\mathcal{K}^n_0$ the set of all $K \in \mathcal{K}^n$ which are centrally symmetric.

Let $K \in \mathcal{K}^n$ be a convex body. The interior of $K$ is denoted by $\text{int } K$. For $z \in \text{int } K$, the polar body of $K$ with respect to $z$ is defined by

$$K^z = \{y \in \mathbb{R}^n; (y - z) \cdot (x - z) \leq 1 \text{ for any } x \in K\},$$

where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$. Denote by $|K|$ the $n$-dimensional volume of $K$ in $\mathbb{R}^n$. Then the volume product of $K$ is defined by

$$\mathcal{P}(K) := \min_{z \in \text{int } K} |K| |K^z|.$$ (1)

Note that this quantity is an affine invariant, i.e., $\mathcal{P}(AK) = \mathcal{P}(K)$ for any invertible affine map $A : \mathbb{R}^n \to \mathbb{R}^n$. It is well-known that for each $K \in \mathcal{K}^n_0$ the minimum of $\mathcal{P}$ is attained at the unique point $z$ on $K$, which is called Santaló point of $K$ (see...
For a centrally symmetric convex body $K \in \mathcal{K}_0^n$, the Santaló point of $K$ is nothing but the origin $O$. In the following, the polar of $K$ with respect to $O$ is denoted by $K^\circ$.

The upper bound for $\mathcal{P}(K)$ is well-known as Blaschke-Santaló inequality (see [Sa]) and the bound is attained only for ellipsoids (see [MP]). In contrast, the sharp lower bound estimate is fairly difficult and it remains as a longstanding open problem since 1939 as follows.

**Conjecture** (Symmetric Mahler’s conjecture; [Ma2]). Any $K \in \mathcal{K}_0^n$ satisfies that

$$\mathcal{P}(K) = |K| |K^\circ| \geq \frac{4^n}{n!}.$$  \hfill (2)

This is trivial for $n = 1$ and was proved by Mahler himself for $n = 2$ [Ma1]. There are many alternative proofs and the characterization of the equality case for $n = 2$ (see, e.g., [Me], [Sc1 Section 10.7], and references therein). However, the case where $n \geq 3$ is still widely open. We solve affirmatively symmetric Mahler’s conjecture in the three dimensional case.

**Theorem 1.1.** For any three dimensional centrally symmetric convex body $K \in \mathcal{K}_0^3$, we have

$$|K| |K^\circ| \geq \frac{32}{3},$$

with equality if and only if either $K$ or $K^\circ$ is a parallelepiped.

Further, we exhibit known partial results about the conjecture. An asymptotic lower bound of the volume product was first given by Bourgain and Milman [BM] and the best known constant is due to Kuperberg [K]. Meanwhile, the conjecture itself has been proved for very restricted cases; for zonoids by Reisner [R, R2] (see also [GMR]) and for unconditional bodies by Saint-Raymond [SR]. Note that the equality of (2) is attained by an $n$-cube, for instance. In [NPRZ], the estimate (2) was confirmed for $K \in \mathcal{K}_0^n$ sufficiently close to the unit $n$-cube in the Banach-Mazur distance. For other partial results, see e.g., [BF], [FMZ], [LR], [RSW].

There is another question when $K \in \mathcal{K}_0^n$ is not necessarily assumed to be centrally symmetric.

**Conjecture** (Non-symmetric Mahler’s conjecture). Any $K \in \mathcal{K}_n$ satisfies that

$$\mathcal{P}(K) = \min_{z \in \text{int} K} |K| |K^z| \geq \frac{(n + 1)^{n+1}}{(n!)^2}.$$  \hfill (2)

This was proved by Mahler for $n = 2$ (see [Ma1]). The alternative proof of Mahler’s result by Meyer [Me] is notable. Indeed, we were able to extract an important idea for attacking to the three dimensional symmetric case from [Me]. We first improve and simplify the idea and adapt it to the symmetric case (see Section 2). This method is applicable to not only the two dimensional case but also the three dimensional case. Note that non-symmetric Mahler’s conjecture remains open for $n \geq 3$, see e.g., [BF], [FMZ], [KR], [RSW].
1.2 An application to Viterbo’s conjecture

Recently, a surprising connection between symmetric Mahler’s conjecture and a conjecture in the field of symplectic geometry was discovered. In 2000, Viterbo posed in [V] an isoperimetric-type conjecture for symplectic capacities of convex bodies in $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0$. A symplectic capacity $c$ is a symplectic invariant which assigns a non-negative real number to each of symplectic manifolds of dimension $2n$. The Hofer-Zehnder capacity $c_{HZ}$ is one of the important symplectic capacities, which is related to Hamiltonian dynamics on symplectic manifolds. For details, see [AKO] or a foundational book [HZ].

**Conjecture** (Viterbo [V]). For any symplectic capacity $c$ and any convex body $\Sigma \in \mathcal{K}^{2n}$,

$$\frac{c(\Sigma)}{c(B^{2n})} \leq \left( \frac{\text{vol}(\Sigma)}{\text{vol}(B^{2n})} \right)^{1/n}$$

holds, where $B^{2n}$ denotes the $2n$-dimensional unit ball and $\text{vol}(\Sigma)$ denotes the symplectic volume of $\Sigma$.

This conjecture is unsolved even in the case of $n = 2$. Note that $c(B^{2n}) = \pi$. In [AKO], Artstein-Avidan, Karasev, and Ostrover calculated the Hofer-Zehnder capacity of a convex domain $K \times K^o \subset \mathbb{R}^{2n}$ ($K \in \mathcal{K}_0^{2n}$) as follows.

**Theorem 1.2** ([AKO], Theorem 1.7). For any centrally symmetric convex body $K \in \mathcal{K}_0^{2n}$, we have

$$c_{HZ}(K \times K^o) = 4.$$  

Using it, they gave the following remarkable observation that Viterbo’s conjecture implies symmetric Mahler’s conjecture (see [AKO] Section 1]).

$$\frac{4^n}{\pi^n} = \frac{c_{HZ}(K \times K^o)^n}{\pi^n} \leq \frac{\text{vol}(K \times K^o)}{\pi^n/n!} = \frac{|K||K^o|}{\pi^n/n!}.$$  

Conversely, if we assume that symmetric Mahler’s conjecture holds, then Theorem 1.2 implies the inequality (3) for the case that $c = c_{HZ}$ and $\Sigma = K \times K^o$. Therefore, Theorem 1.1 immediately yields

**Corollary 1.3.** Let $K \subset \mathbb{R}^3$ be a centrally symmetric convex body. Then the inequality (3) holds for $\Sigma = K \times K^o \subset (\mathbb{R}^6, \omega_0)$ with respect to the Hofer-Zehnder capacity $c_{HZ}$.

Of course, due to the result of [Ma1], the inequality (3) also holds for $\Sigma = K \times K^o \subset (\mathbb{R}^4, \omega_0)$, where $K \in \mathcal{K}_0^2$, with respect to $c_{HZ}$.

1.3 Organization of the paper and our method

Our approach to the conjecture started with a trial to adapt the proof of the two dimensional non-symmetric case ([Me]) to the symmetric case. We then found a key idea to estimate $\mathcal{P}(K)$ effectively.
The case where \( n = 2 \) contains the essence of the idea in a primitive way. Moreover that is useful to understand the strategy of our proof of the three dimensional case. Let \( K \in \mathcal{K}_0^3 \). First, by using coordinate axes, we divide \( K \) into four parts \( K_1, K_2, K_3(= -K_1), K_4(= -K_2) \). Here we may assume that \( |K_1| = |K_2| \). Next, we consider the corresponding decomposition of \( K^0 \). Roughly speaking, the division \( K^0 = K^0_1 \cup K^0_2 \cup K^0_3 \cup K^0_4 \) is obtained by using the following two lines. One is the line through the origin \( O \) and a point \( B^0 \) on \( \partial K^0 \) with the maximal \( x \)-coordinate. Another is the line through \( O \) and a point \( C^0 \) on \( \partial K^0 \) with the maximal \( y \)-coordinate. Then, for any point \( P \in K^0 \), the areas \( |K^0_i| \) \( (i = 1, 2) \) are estimated from below by the signed area of the quadrilaterals \( OB^0PC^0 \) and \( OC^0P(-B^0) \), respectively. These estimates give two test points \( S_1, S_2 \) on \( K \). The same observation on \( K \) yields further two test points \( R_1, R_2 \) on \( K^0 \). By paring \( R_i \) and \( S_i \) \( (i = 1, 2) \), we easily get \( \mathcal{P}(K) \geq 8 \). Including the equality case we exhibit the details in Section 2.

From Section 3, we begin the proof of the three dimensional case. We first divide \( K \) into eight parts \( K_i \) \( (i = 1, \ldots , 8) \) as in the two dimensional case. However, compare to the two dimensional case, to make the corresponding decomposition of \( K^0 \) is not so simple. The first ingredient of our proof of Theorem 1.1 is to concentrate only on the class of convex bodies \( K \in \mathcal{K}_0^3 \) which are strongly convex and \( C^\infty \) boundary. Schneider’s approximation procedure (Proposition 6.4) guarantees that the problem is reduced to considering only this class. For a convex body \( K \) in the class, we can define a diffeomorphism \( \partial K \to \partial K^0 \), which yields the corresponding eight pieces decomposition \( K^0_i \) of \( K^0 \) from that of \( K \). The precise setting is explained in Section 3.2.

The second ingredient is that only a few test points give a sharp estimate of \( \mathcal{P}(K) \), provided \( K \) has an additional symmetry. When \( n = 2 \), the symmetry we need is \( |K_1| = |K_2| \), although this is not an essential assumption. When \( n = 3 \), we have to control not only the volume of \( K_i \) but also the areas of boundary faces of \( K_i \) in the three coordinate planes. In Section 3.3 we give a sharp estimate of \( \mathcal{P}(K) \) from below under the condition that \( K \) has “good” symmetries. The condition is \( \left( 15 \right) \). Under this condition, the process of estimating \( \mathcal{P}(K) \) is regarded as a direct generalization of the two dimensional case. A (signed) volume comparison inequality (Lemma 3.1) which yields test points on \( K \) and \( K^0 \) is in Section 3.3, of which proof is given in Section 3.2.

How can we release \( K \) from the very strong condition \( \left( 15 \right) \)? Note that \( \mathcal{P}(K) \) is invariant under linear transformations of \( \mathbb{R}^3 \). The third ingredient is to make the most of the freedom of the action of linear transformations. As a test case, in Section 4 we prove Theorem 1.1 for \( K \in \mathcal{K}_0^3 \) equipped with an extra symmetry with respect to a hyperplane through \( O \) (Proposition 4.1). In this case, by a linear transformation \( A \), we can deform \( K \) into \( AK \) which satisfies the condition \( \left( 15 \right) \) by means of the intermediate value theorem. However, for a general \( K \) to find a linear transformation \( A \) such that \( AK \) satisfies the condition \( \left( 15 \right) \) is highly nontrivial.

Sections 5 and 6 which are the technical part of the present paper, are devoted to find an appropriate linear transformation \( A \) for each \( K \in \mathcal{K}_0^3 \) which is strongly convex with smooth boundary \( \partial K \). We proceed as follows. First, we define \( K(\theta, \phi, \psi) \in \mathcal{K}_0^3 \) as the image of the initial \( K \) under the action of \( SO(3) \). (Here, \( \theta, \phi, \psi \) mean rotation angles with \( x, y, z \)-axes, respectively.) Next, in Section 5.2...
we define a linear transform \( A \) such that \( AK(\theta, \phi, \psi) \) satisfies the condition (19) (Proposition 5.3), which is a partial condition of (15). Finally, to get the full condition (15) for \( AK(\theta, \phi, \psi) \), we introduce in Section 5.3 three smooth functions \( F(K), G(K), H(K) \) on a contractible region \( D \subset \mathbb{R}^3 \) with the coordinates \( \theta, \phi, \psi \). These functions are defined by the volume of pieces of the eight part decomposition of \( AK(\theta, \phi, \psi) \) and the resulting two dimensional quantities. Then the problem to find \( AK \) which satisfies the desired condition (15) is reduced to the existence of a zero \( (\theta, \phi, \psi) \) of the map \( (F,G,H): D \rightarrow \mathbb{R}^3 \). Here, if \( (F,G,H) \) has no zero on \( \partial D \), then we can define a map

\[
\mathcal{F} = \frac{(F,G,H)}{\sqrt{F^2 + G^2 + H^2}} : \partial D \rightarrow S^2.
\]

From Section 5.4 to 5.8, we discuss properties of the map \( (F,G,H) \) in order to calculate the degree of the map \( \mathcal{F} \). The centrally symmetric convex body \( AK(\theta, \phi, \psi) \) equips with additional symmetries under rotations by specific angles around \( x, y, z \)-axesises. These induces certain symmetries of the functions \( F, G, \) and \( H \) (Lemmas 5.4, 5.8, 5.10 and 5.12). The results are summarized in Proposition 5.14 and essential for the degree calculation in the next section.

In Section 6, we actually prove that \( \deg \mathcal{F} \neq 0 \) after a suitable perturbation of this map \( \mathcal{F} \). This immediately implies the existence of a zero of the map \( (F,G,H) \). Note that the smoothness of \( \partial K \) also has enough merit to the calculation of \( \deg \mathcal{F} \). The construction of the above perturbation of \( \mathcal{F} \) is technical, so it is discussed in Appendix A. Anyway the perturbed \( \mathcal{F} \) equips with \( (\pm 1, 0, 0) \) as regular values. This fact is crucial for the calculation of \( \deg \mathcal{F} \). Indeed, it enables us to reduce the calculation of \( \deg \mathcal{F} \) to the calculation of the winding number of a map \( \mathcal{G} : S^1 \rightarrow S^1 \) (Proposition 6.2), which is fairly accessible. We see that the winding number of \( \mathcal{G} \) is odd. This reduction process is explained in Section 6.2 and the calculation of the winding number of the map \( \mathcal{G} \) is carried out in Section 6.3. Consequently, the existence of a zero of the map \( \mathcal{G} \) ensures that \( \mathcal{P}(K) \geq 32/3 \) for any convex body \( K \in K^3_0 \) which is strongly convex with \( C^\infty \) boundary. Finally, Schneider’s approximation theorem implies that the inequality holds for all \( K \in K^3_0 \).

In Section 7, we determine the equality condition (Propositions 7.3 and 7.4). First, for a general \( K \in K^3_0 \) with \( \mathcal{P}(K) = 32/3 \), we find a linear transformation \( A \) such that \( AK \) satisfies the condition (15). Indeed, there exists a sequence \( \{K_n\}_{n \in \mathbb{N}} \) such that \( \lim_{n \to \infty} K_n = K \), each \( K_n \in K^3_0 \) is strongly convex with smooth boundary. By the results in Sections 5 and 6 there exists \( \{A_n\}_{n \in \mathbb{N}} \) such that each \( A_n K_n \) satisfies the condition (15). As the limit of a subsequence, we get the required \( A \). Moreover, from \( \mathcal{P}(K) = 32/3 \), we can show that \( P_i(K^\circ) = (Q_i(K))^2 \) and \( |Q_i(K)||P_i(K^\circ)| = 8 \) where \( Q_i \) is the cross section with a coordinate plane and \( P_i \) is the projection to the coordinate plane \( (i = 1, 2, 3) \). Thus, the result of the two dimensional case (Proposition 2.2) implies that \( Q_i(K) \) and \( P_i(K^\circ) \) are parallelograms. Next, we extend the sharp estimate in Section 3.5. Then we see \( K \) is polyhedron with at most 20 vertices. Finally, by case analysis and calculating the dual faces of the vertices, we prove that either \( K \) or \( K^\circ \) is a parallelepiped.
2 The two dimensional case

In this section, we give a simple proof of Mahler’s theorem for the two dimensional case. The essence of the idea originates from [Me]. The goal of this section is the following

**Theorem 2.1** ([Ma1], [R2]). Let \( K \in \mathcal{K}_0^2 \). Then \( \mathcal{P}(K) \geq 8 \) with equality if and only if \( K \) is a parallelogram.

To prove it, for \( K \in \mathcal{K}_0^2 \), we divide \( K \) into the following four pieces:

\[
\begin{align*}
K_1 & := K \cap \{(x, y) \in \mathbb{R}^2; x \geq 0, y \geq 0\}, \\
K_2 & := K \cap \{(x, y) \in \mathbb{R}^2; x \leq 0, y \geq 0\}, \\
K_3 & := -K_1 = K \cap \{(x, y) \in \mathbb{R}^2; x \leq 0, y \leq 0\}, \\
K_4 & := -K_2 = K \cap \{(x, y) \in \mathbb{R}^2; x \geq 0, y \leq 0\}.
\end{align*}
\]

We denote by polygon \( \{P_1, \ldots, P_m\} \) the star-shaped polygon in \( \mathbb{R}^2 \) consists of successive vertices \( P_1, \ldots, P_m \) in the counterclockwise order and the edges \( P_1P_2, \ldots, P_{m-1}P_m, P_mP_1 \).

**Proof of Theorem 2.1.** Since the volume product \( \mathcal{P}(K) \) is invariant under rotations of \( K \) around the origin \( O = (0,0) \), we may assume that \( |K_1| = |K_2| \) owing to the intermediate value theorem. By the assumption that \( K \) is centrally symmetric, this means that

\[
|K_1| = |K_2| = \frac{|K|}{4}. \tag{4}
\]

Moreover, the condition (4) is preserved under scaling by a diagonal matrix. Therefore, for any \( K \in \mathcal{K}_0^2 \), there exists a linear transformation \( A \) such that \( AK \in \mathcal{K}_0^2 \) satisfies the assumptions of the following Proposition 2.2. It asserts that

\[
\mathcal{P}(K) = \mathcal{P}(AK) \geq 8.
\]

Moreover, if \( \mathcal{P}(K) = 8 \), then \( AK \) is a square. Hence \( K \) is a parallelogram. \( \Box \)

**Proposition 2.2.** Let \( K \in \mathcal{K}_0^2 \). Suppose that \( |K_i| = |K|/4 \) \( (i = 1, 2), (1,0), (0,1) \in \partial K \). Then \( \mathcal{P}(K) \geq 8 \) holds. In addition, when \( \mathcal{P}(K) = 8 \), there exists a constant \( a \in (-1,1) \) such that

\[
K = \text{conv} \left\{ \frac{\pm 1}{1 + a^2} (1 - a, 1 + a), \frac{\pm 1}{1 + a^2} (-1 - a, 1 - a) \right\}, \tag{5}
\]

\[
K^o = \text{conv} \{ \pm (1,a), \pm (-a,1) \}.
\]

Especially, \( K \) and \( K^o \) are squares, \( |K| = 4/(1 + a^2) \), and \( |K^o| = 2(1 + a^2) \).

**Proof.** We put \( B := (1,0), C := (0,1) \). Under the assumptions, \( \pm B = (\pm 1,0) \) are the intersection points of \( \partial K \) and the \( x \)-axis, and \( \pm C = (0, \pm 1) \) are the intersection points of \( \partial K \) and the \( y \)-axis. By the definition of the polar \( K^o \), we have

\[
\pm u = \pm B \cdot (u,v) \leq 1, \quad \pm v = \pm C \cdot (u,v) \leq 1 \text{ for any } (u,v) \in K^o.
\]
Hence, \( K^\circ \subset [-1,1] \times [-1,1] \).

We first consider the case where \((1,1) \in K^\circ \). In this case, by the definition of \( K^\circ \), we have
\[
x + y = (x,y) \cdot (1,1) \leq 1 \text{ for any } (x,y) \in K.
\]
On the other hand, since \( K_1 \) and \( K_2 \) are convex,
\[
K_1 \supset \text{conv}\{(0,0),(1,0),(0,1)\}, \quad K_2 \supset \text{conv}\{(0,0),(0,1),-(1,0)\}. \tag{6}
\]
Thus \( K_1 = \text{conv}\{(0,0),(1,0),(0,1)\} \) holds. Especially, \(|K_1| = 1/2\). Since \(|K_1| = |K_2|\) by the assumption, we have \(|K_2| = 1/2\). Combining it with \((6)\), we obtain \( K_2 = \text{conv}\{(0,0),(0,1),-(1,0)\} \). Since \( K \) is centrally symmetric,
\[
K = \text{conv} \{\pm(1,0),\pm(0,1)\} \tag{7}
\]
holds. Hence \( K^\circ = [-1,1] \times [-1,1] \), \( \mathcal{P}(K) = 8 \), and \((5)\) holds with \( a = 1 \). Similarly, if \((-1,1) \in K^\circ \), then we get the same conclusion. Thus, since \( K \) is centrally symmetric, it is sufficient to consider the case where
\[
K^\circ \subset [-1,1] \times [-1,1] \setminus \{\pm(1,1),\pm(-1,1)\}. \tag{8}
\]
Since \( B, C \in \partial K \), by \((8)\) and the definition of \( K^\circ \), there exist points \( B^\circ = (1, b), C^\circ = (c, 1) \in K^\circ \) with \( b, c \in (-1, 1) \) satisfying that \( B \cdot B^\circ = 1 \) and \( C \cdot C^\circ = 1 \). (Note that these points are not necessarily unique in general.) The point \( B^\circ \) (resp. \( C^\circ \)) is characterized as a point with the maximal \( u\)- (resp. \( v\))-coordinate among all the points in \( K^\circ \). The points \( B^\circ \) and \( C^\circ \) enable us to divide \( K^\circ \) into the following:
\[
K^\circ_1 := K^\circ \cap \{(u,v) \in \mathbb{R}^2; v \geq bu, cv \leq u\}, \quad K^\circ_3 := -K^\circ_1,
\]
\[
K^\circ_2 := K^\circ \cap \{(u,v) \in \mathbb{R}^2; v \geq bu, cv \geq u\}, \quad K^\circ_4 := -K^\circ_2.
\]
Now we shall find test points to estimate \( \mathcal{P}(K) \). For any \( P = (u,v) \in K^\circ \), the sum of the signed area of the triangle \( O\) \( B^\circ \) \( P \) and that of the triangle \( O\) \( P\) \( C^\circ \) is less than or equal to \( |K^\circ_1| \), that is,
\[
\frac{1}{2} \begin{vmatrix} 1 & b \\ u & v \end{vmatrix} + \frac{1}{2} \begin{vmatrix} 1 & u \\ c & 1 \end{vmatrix} \leq |K^\circ_1|.
\]
This inequality is equivalent to
\[
(u,v) \cdot \frac{1}{2|K^\circ_1|}(1-b,1-c) \leq 1.
\]
Since \( K = (K^\circ)^\circ = \{(x,y) \in \mathbb{R}^2; (u,v) \cdot (x,y) \leq 1 \text{ for any } (u,v) \in K^\circ\} \), we have
\[
S_1 := \frac{1}{2|K^\circ_1|}(1-b,1-c) \in K.
\]
More precisely, \( S_1 \in K_1 \) since \( b, c \in (-1,1) \). A similar argument for the piece \( K^\circ_2 \) yields
\[
S_2 := \frac{1}{2|K^\circ_2|}(-1-b,1+c) \in K.
\]
Note that \( S_2 \subset K_2 \). Repeating the same arguments for the convex bodies \( K_1 \) and \( K_2 \), we have two test points in \( K^\circ \):

\[
R_1 := \frac{1}{2|K_1|} (1, 1) = \frac{2}{|K|} (1, 1), \quad R_2 := \frac{1}{2|K_2|} (-1, 1) = \frac{2}{|K|} (-1, 1).
\]

By the definition of \( K^\circ \), we have \( R_1 \cdot S_1 \leq 1 \), \( R_2 \cdot S_2 \leq 1 \), that is,

\[
(1, 1) \cdot (1 - b, 1 - c) \leq |K||K_1^\circ|, \quad (-1, 1) \cdot (-1 - b, 1 + c) \leq |K||K_2^\circ|.
\]

Hence, we obtain

\[
\mathcal{P}(K) = |K||K^\circ| = 2|K|(|K_1^\circ| + |K_2^\circ|) \geq 2 ((2 - b - c) + (2 + b + c)) = 8.
\]

Hereafter, let us determine the equality condition. Now suppose that \( \mathcal{P}(K) = 8 \).

Since \( K \supset \text{conv}\{\pm (1, 0), \pm (0, 1)\} \), we have \( |K| \geq 2 \). If \( |K| = 2 \), then we get \( (7) \) and \( K^\circ = [-1, 1] \times [-1, 1] \). It contradicts to \( (7) \). Thus \( |K| > 2 \) holds. We put

\[
\tilde{K}_1 := \text{polygon} \{O, B, S_1, C\}, \quad \tilde{K}_2 := \text{polygon} \{O, C, S_2, -B\}.
\]

Then, \( \tilde{K}_i \subset K_i \) for each \( i = 1, 2 \), because \( K_i \) is convex. Moreover, since

\[
|\tilde{K}_1| = \frac{1}{4|K_1^\circ|} \begin{vmatrix} 1 & 0 & 1 - b & 1 - c \\ 1 - b & 1 - c & 0 & 1 \end{vmatrix} = \frac{2 - b - c}{4|K_1^\circ|},
\]

\[
|\tilde{K}_2| = \frac{1}{4|K_2^\circ|} \begin{vmatrix} 0 & 1 & 1 - b & 1 + c \\ -1 - b & 1 + c & -1 & 0 \end{vmatrix} = \frac{2 + b + c}{4|K_2^\circ|},
\]

we have

\[
\mathcal{P}(K) = 2 (|K||K_1^\circ| + |K||K_2^\circ|) = 2 (4|K_1^\circ||K_1^\circ| + 4|K_2^\circ||K_2^\circ|) \\
\geq 2 \left( 4|\tilde{K}_1||K_1^\circ| + 4|\tilde{K}_2||K_2^\circ| \right) = 2 ((2 - b - c) + (2 + b + c)) = 8. \quad (9)
\]

The assumption \( \mathcal{P}(K) = 8 \) means that \( (9) \) holds with equality. Hence,

\[
|K_1| = |\tilde{K}_1|, \quad |K_2| = |\tilde{K}_2|, \quad |K_1^\circ| = \frac{2 - b - c}{|K|}, \quad |K_2^\circ| = \frac{2 + b + c}{|K|}. \quad (10)
\]

Since \( \tilde{K}_i \subset K_i \) \( (i = 1, 2) \), we obtain \( \tilde{K} = K_i \) \( (i = 1, 2) \). Consequently, we have

\[
K = \text{polygon} \{B, S_1, C, S_2, -B, -S_1, -C, -S_2\}.
\]

We repeat a similar argument for \( K^\circ \). We put

\[
\tilde{K}^\circ := \text{polygon} \{B^\circ, R_1, C^\circ, R_2, -B^\circ, -R_1, -C^\circ, -R_2\}.
\]

Then, \( \tilde{K}^\circ \subset K^\circ \) because \( K^\circ \) is convex. Moreover, since \( b, c \in (-1, 1) \), \( \tilde{K}^\circ \) is a simple polygon. Thus, we have

\[
\frac{|\tilde{K}^\circ|}{2} = \frac{1}{|K|} \begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} + \frac{1}{|K|} \begin{vmatrix} 1 & 1 \\ c & 1 \end{vmatrix} + \frac{1}{|K|} \begin{vmatrix} c & 1 \\ -1 & 1 \end{vmatrix} + \frac{1}{|K|} \begin{vmatrix} -1 & 1 \\ -1 & -b \end{vmatrix} = \frac{4}{|K|}.
\]
which implies
\[ 8 = \mathcal{P}(K) = |K| |K^\circ| \geq |K| |\tilde{K}^\circ| = 8. \]

Consequently, we obtain
\[ K^\circ = \tilde{K}^\circ = \text{polygon} \{ B^\circ, R_1, C^\circ, R_2, -B^\circ, -R_1, -C^\circ, -R_2 \}. \]

Next, we consider the dual faces of the edges of \( K \). Since the line segment \( BS_1 \) is a part of an edge of \( K \), its dual face is a vertex of \( K^\circ \). By a direct calculation (see Section 7.1), the dual face of \( BS_1 \) is
\[ \left( 1, \frac{2|K^\circ_1| - 1 + b}{1 - c} \right), \]
which is a vertex of \( K^\circ \). By (11),
\[ \left( 1, \frac{2|K^\circ_1| - 1 + b}{1 - c} \right) \in \{ \pm B^\circ, \pm R_1, \pm C^\circ, \pm R_2 \} = \left\{ \pm (1, b), \pm \frac{2}{|K|} (1, 1), \pm (c, 1), \pm \frac{2}{|K|} (-1, 1) \right\}. \]

Since \( c \in (-1, 1) \) and \( |K| > 2 \), we have
\[ \left( 1, \frac{2|K^\circ_1| - 1 + b}{1 - c} \right) = (1, b). \]

Hence \( |K^\circ_1| = (1 - bc)/2 \). Similarly, the dual face of the line segment \( CS_2 \) is
\[ \left( 1, \frac{2|K^\circ_2| - 1 + b}{1 + b} \right) \in \{ \pm B^\circ, \pm R_1, \pm C^\circ, \pm R_2 \} = \left\{ \pm (1, b), \pm \frac{2}{|K|} (1, 1), \pm (c, 1), \pm \frac{2}{|K|} (-1, 1) \right\}. \]

Since \( b \in (-1, 1) \) and \( |K| > 2 \), we have
\[ \left( 1, \frac{2|K^\circ_2| - 1 + b}{1 + b} \right) = (c, 1). \]

Hence \( |K^\circ_2| = (1 - bc)/2 \) and
\[ |K^\circ_2| = \frac{1 - bc}{2} = |K^\circ_1|. \]

Moreover, \( c = -b \) holds by (10). Then, we have
\[ |K^\circ_1| = |K^\circ_2| = \frac{2}{|K|} = \frac{1 + b^2}{2}, \]
\[ S_1 = \frac{1}{1 + b^2} (1 - b, 1 + b), \quad S_2 = \frac{1}{1 + b^2} (-1 - b, 1 - b). \]

We see that three points \( S_1, (1, 0) \), and \( -S_2 \) are on the line \( x + by = 1 \). Similarly, \( S_1, (0, 1) \), and \( S_2 \) are on the line \( -bx + y = 1 \). Consequently, we obtain
\[ K = \text{conv} \{ \pm S_1, \pm S_2 \} = \text{conv} \left\{ \pm \frac{1}{1 + b^2} (1 - b, 1 + b), \pm \frac{1}{1 + b^2} (-1 - b, 1 - b) \right\}, \]
which is a square with \( |K| = 4/(1 + b^2) \). Then
\[ K^\circ = \text{conv} \{ \pm (1, b), \pm (-b, 1) \} \]
is also a square with \( |K^\circ| = 2(1 + b^2) \).
3 A sharp estimate for the three dimensional volume product

3.1 Preliminaries

To begin with, we summarize some notations and terminologies about convex bodies in arbitrary dimension. Let $K \in \mathbb{K}^n$ be an $n$-dimensional convex body with $O \in \text{int} K$. The Minkowski gauge $\mu_K$ of $K$ is defined by

$$\mu_K(x) := \min\{\lambda \geq 0; x \in \lambda K\}$$

for $x \in \mathbb{R}^n$, and the radial function $\rho_K$ of $K$ is defined by

$$\rho_K(x) := \max\{\lambda \geq 0; \lambda x \in K\} = 1/\mu_K(x)$$

for $x \in \mathbb{R}^n \setminus \{O\}$. Note that $\rho_K(x) = 1$ on $\partial K$. For a point $x \in \mathbb{R}^n$, denote by $-x$ the opposite point of $x$ with respect to the origin $O$. And we abbreviately use the same symbol $x$ to represent the position vector of $x$.

For a curve $c(t) (0 \leq t \leq 1)$ in $\mathbb{R}^n$, denote by $\tilde{c}$ the curve which has the same image as $c$ with the reverse orientation, that is,

$$\tilde{c}(t) = c(1 - t) \quad (0 \leq t \leq 1).$$

Let $c, d$ be smooth curves in $\mathbb{R}^n$. Suppose that the terminal point of $c$ coincides with the initial point of $d$. Then denote by $c \cup d$ the piecewise smooth curve in $\mathbb{R}^n$ which starts at $c(0)$ and ends at $d(1)$.

Now we examine the correspondence of the boundary points between a convex body $K$ and its polar body $K^\circ$, when $K$ is a strongly convex body defined below. Assume that the boundary $\partial K$ is a $C^\infty$-hypersurface in $\mathbb{R}^n$. Consider a smooth function

$$h_K(x) = \frac{1}{2} \mu^2_K(x)$$

on $\mathbb{R}^n$. The condition that $x \in \partial K$ is equivalent to $h_K(x) = 1/2$, and hence $\partial K$ is the level set $h_K^{-1}(1/2)$ and the gradient $\nabla h_K(x)$ is an outward normal vector at $x \in \partial K$.

We say that $K$ is strongly convex if $h_K$ is a strongly convex function, that is, the Hessian matrix $D^2 h_K(x)$ is positive definite for each $x \in S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$.

Assume that $K$ is strongly convex. By the definition of polar, for every $y \in \partial K^\circ$ there exists the unique point $x \in \partial K$ such that $y \cdot x = 1$. This correspondence gives rise to the map $\hat{\Lambda} : \partial K^\circ \rightarrow \partial K$ defined by $\hat{\Lambda}(y) := x$. Since the gradient $\nabla h_K(x)$ at the point $x$ is the same direction as $y$, we have

$$\nabla h_K(x) = a(y) y$$

with a positive function $a$ on $\partial K^\circ$. Since $\nabla h_K$ is 1-homogeneous, Euler’s formula implies

$$1 = y \cdot x = \frac{\nabla h_K(x)}{a(y)} \cdot x = \frac{2h_K(x)}{a(y)}.$$
This means that the condition $x \in \partial K$ is equivalent to $a(y) = 1$, so that
\[ \nabla h_K(x) = y \quad \text{for} \quad y \in \partial K^\circ. \]

From $\hat{\Lambda}(y) = x$, we find that a smooth map $\nabla h_K|_{\partial K} : \partial K \to \partial K^\circ$ is the inverse of $\hat{\Lambda}$. We define $\Lambda := \nabla h_K|_{\partial K}$.

**Claim.** $\Lambda : \partial K \to \partial K^\circ$ is a $C^\infty$-diffeomorphism.

**Proof.** Let us consider a function $h_K((\nabla h_K)^{-1}(y)) \in C^\infty(\mathbb{R}^n \setminus \{0\})$. For $y \neq 0$, since $y/\mu_K(y) \in \partial K^\circ$, $h_K$ is 2-homogeneous, and $(\nabla h_K)^{-1}$ is 1-homogeneous, we have
\[
\begin{align*}
h_K\left((\nabla h_K)^{-1}(y)\right) &= h_K\left((\nabla h_K)^{-1}\left(\mu_K(y)\frac{y}{\mu_K(y)}\right)\right) \\
&= \frac{\mu_{K^\circ}^2(y)}{2}h_K\left(\hat{\Lambda}\left(y\frac{1}{\mu_K(y)}\right)\right) \\
&= \frac{\mu_{K^\circ}^2(y)}{2} = h_{K^\circ}(y),
\end{align*}
\]

where we used the fact that $\hat{\Lambda} = (\nabla h_K)^{-1}|_{\partial K^\circ} : \partial K^\circ \to \partial K$. Consequently, we have
\[
h_{K^\circ}(y) = h_K \circ (\nabla h_K)^{-1}(y) \quad \text{for} \quad y \in \mathbb{R}^n \setminus \{0\}.
\]

Since $h_K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $h_{K^\circ} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and the hypersurface $\partial K^\circ = h_{K^\circ}^{-1}(1/2)$ is of class $C^\infty$. Thus, $\hat{\Lambda} : \partial K^\circ \to \partial K$ is a $C^\infty$-diffeomorphism. \qed

Note that by (12), $K^\circ$ is smooth strongly convex if $K$ is smooth strongly convex.

### 3.2 Convex bodies in $\mathbb{R}^3$

From now on, we focus on centrally symmetric convex bodies in $\mathbb{R}^3$. Denote by $\hat{\mathcal{K}}$ the set of all centrally symmetric convex bodies $K \in \mathcal{K}_0^3$ which are strongly convex with smooth boundary $\partial K$. Throughout this paper, for a convex body $K \in \mathcal{K}_0^3$ we fix the orientation on $K$ induced from the natural orientation of $\mathbb{R}^3$. From it the orientation of the boundary $\partial K$, that of any domain $S$ on $\partial K$ and that of $\partial S$ (that is, a closed curve on $\partial K$) are induced, respectively. For a point $P = (p_1, p_2, p_3)$ in $\mathbb{R}^3$ we write $-P = (-p_1, -p_2, -p_3)$. If $K \in \hat{\mathcal{K}}$ and $P \in K$, then $-P \notin K$. For $K \in \hat{\mathcal{K}}$ and any distinct two points $P, Q \in \partial K$ with $P \neq -Q$, let us introduce an oriented curve from $P$ to $Q$ on the boundary $\partial K$ defined by
\[
C_K(P, Q) := \{\rho_K((1 - t)P + tQ)((1 - t)P + tQ); \ 0 \leq t \leq 1\}.
\]

We call it an oriented segment on $\partial K$. The polar body of $K \in \hat{\mathcal{K}}$ is given by
\[
K^\circ = \{(u, v, w) \in \mathbb{R}^3; (u, v, w) \cdot (x, y, z) \leq 1 \text{ for all } (x, y, z) \in K\},
\]

which is also an element of $\hat{\mathcal{K}}$ from the argument in Section 3.1. Note that $O \in \text{int } K^\circ$ and $(K^\circ)^\circ = K$. 

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Let $K \in \hat{K}$. First, we decompose the convex body $K$ into eight pieces. Using the radial function $\rho_K$ we take three points

$$A := \rho_K((1,0,0))(1,0,0), B := \rho_K((0,1,0))(0,1,0), C := \rho_K((0,0,1))(0,0,1)$$

on the boundary $\partial K$. These points lie in the $x$-axis, $y$-axis, and $z$-axis, respectively, so we can write $A = (a_1,0,0), B = (0,b_2,0), C = (0,0,c_3)$ with $a_1,b_2,c_3 > 0$. Since $K$ is centrally symmetric, we have $-A, -B, -C \in \partial K$.

We consider six oriented segments on $\partial K$ as follows:

$$d := C_K(B,C), \quad e := C_K(C,-B),$$

$$f := C_K(C,A), \quad g := C_K(A,-C),$$

$$h := C_K(A,B), \quad i := C_K(B,-A).$$

Then another six segments $-d, -e, -f, -g, -h$, and $-i$ on $\partial K$ are automatically defined. Note that $\pm d, \pm e$ are in the $yz$-plane, $\pm f, \pm g$ are in the $zx$-plane, and $\pm h, \pm i$ are in the $xy$-plane. We denote by $Q_i(K)$ ($i = 1,2,3$) the sections of $K$ by the $yz$-plane, $zx$-plane, and $xy$-plane, respectively. They are strongly convex sets in each plane and closed curves $d \cup e \cup -d \cup -e, f \cup g \cup -f \cup -g$, and $h \cup i \cup -h \cup -i$ are the boundaries of $Q_i(K)$ ($i = 1,2,3$), respectively. Hence these closed curves mutually do not intersect except for $\pm A, \pm B, \pm C$.

For an oriented simple closed curve $c$ on $\partial K$ denote by $S_K(c) \subset \partial K$ the piece of surface enclosed by $c$ with the orientation compatible with that of $c$. In this paper, we always equip $S_K(c)$ with this orientation. We denote by $O \ast S_K(c)$ the cone over $S_K(c)$:

$$O \ast S_K(c) = \{ \lambda u; u \in S_K(c) \text{ and } 0 \leq \lambda \leq 1 \}.$$
Using this notation, we divide $K$ into the following eight pieces:

$$
K_1 := O \ast S_K(d \cup f \cup h), \quad K_2 := O \ast S_K(d \cup i \cup -g),
$$
$$
K_3 := O \ast S_K(\tilde{e} \cup \tilde{g} \cup -h), \quad K_4 := O \ast S_K(e \cup -i \cup \tilde{f}),
$$
$$
K_5 := O \ast S_K(-e \cup \tilde{h} \cup g), \quad K_6 := O \ast S_K(-\tilde{e} \cup -f \cup \tilde{i}),
$$
$$
K_7 := O \ast S_K(-d \cup -\tilde{h} \cup -f), \quad K_8 := O \ast S_K(-d \cup \tilde{g} \cup -\tilde{i}).
$$

The volume of each $K_i$ ($i = 1, 2, \ldots, 8$) is denoted by $|K_i|.$

Next, we decompose the polar body $K^o$ into eight pieces associated to the above decomposition of $K.$ Using the map $\Lambda : \partial K \rightarrow \partial K^o$ defined in Section 3.1, we put three points and six curves on $\partial K^o$ as follows:

$$
A^o := \Lambda(A), \quad B^o := \Lambda(B), \quad C^o := \Lambda(C);
$$
$$
d^o := \Lambda(d), \quad e^o := \Lambda(e), \quad f^o := \Lambda(f), \quad g^o := \Lambda(g), \quad h^o := \Lambda(h), \quad i^o := \Lambda(i).
$$

From the definition of the polar body and the strong convexity of $K^o$, the point $A^o$ (resp. $B^o, C^o$) is the unique point with the maximal $u$-coordinate (resp. $v$-coordinate, $w$-coordinate) among all the points in $K^o.$ Since $K^o$ is centrally symmetric, we have $-A^o, -B^o,-C^o \in \partial K^o$ and $-d^o, -e^o, -f^o, -g^o, -h^o, -i^o \in \partial K^o.$

Since $\Lambda : \partial K \rightarrow \partial K^o$ is a $C^\infty$-diffeomorphism, smooth closed curves $d^o \cup e^o \cup -d^o \cup -e^o, f^o \cup g^o \cup -f^o \cup -g^o,$ and $h^o \cup i^o \cup -h^o \cup -i^o$ on $\partial K^o$ are mutually do not intersect except for $\pm A^o, \pm B^o, \pm C^o.$ Thus we can define a decomposition of $K^o$ into the following eight pieces

$$
K_1^o := O \ast S_{K^o}(d^o \cup f^o \cup h^o), \quad K_2^o := O \ast S_{K^o}(d^o \cup i^o \cup -g^o),
$$
$$
K_3^o := O \ast S_{K^o}(\tilde{e}^o \cup \tilde{g}^o \cup -h^o), \quad K_4^o := O \ast S_{K^o}(e^o \cup -i^o \cup \tilde{f}^o),
$$
$$
K_5^o := O \ast S_{K^o}(-e^o \cup \tilde{h}^o \cup g^o), \quad K_6^o := O \ast S_{K^o}(-\tilde{e}^o \cup -f^o \cup \tilde{i}^o),
$$
$$
K_7^o := O \ast S_{K^o}(-d^o \cup -\tilde{h}^o \cup -f^o), \quad K_8^o := O \ast S_{K^o}(-d^o \cup \tilde{g}^o \cup -\tilde{i}^o),
$$

which corresponds to the decomposition of $K$ described before. The volume of $K_i^o$ ($i = 1, 2, \ldots, 8$) is denoted by $|K_i^o|.$

Denote by $P_i$ ($i = 1, 2, 3$) the orthogonal projections to the $vw$-plane, $wu$-plane and $uv$-plane, respectively. Since $K^o \subset \mathbb{R}^3$ is strongly convex, the projections $P_i(K^o)$ ($i = 1, 2, 3$) of $K^o$ are compact strongly convex domains in each plane.

**Claim.** The boundaries $\partial(P_i(K^o))$ ($i = 1, 2, 3$) coincide with smooth simple closed curves $P_1(d^o \cup e^o \cup -d^o \cup -e^o), P_2(f^o \cup g^o \cup -f^o \cup -g^o),$ and $P_3(h^o \cup i^o \cup -h^o \cup -i^o)$, respectively.

**Proof.** Fix a point $Q \in \partial(Q_1(K)).$ Since $Q$ lies on the set

$$
Q_1(d \cup e \cup (-d) \cup (-e)) = d \cup e \cup (-d) \cup (-e),
$$

$Q$ is on one of the four segments in the right-hand side. If $Q \in d$, then there exists $t \in [0, 1]$ such that $Q = d(t).$ By the definitions of $d(t)$ and $d^o(t)$, we have $d(t) = Q_1(d(t)) = (0, d_2(t), d_3(t)), d^o(t) = (d_1(t), d_2^o(t), d_3^o(t)), P_1(d^o(t)) = (0, d_2^o(t), d_3^o(t)),$ and

$$
1 = d(t) \cdot d^o(t) = d_2(t) d_2^o(t) + d_3(t) d_3^o(t) = Q_1(d(t)) \cdot P_1(d^o(t)).
$$
Combining it with the conditions \( d(t) = Q_1(d(t)) \in \partial(Q_1(K)) \) and \( P_1(d^o(t)) \in P_1(K^o) \), we find that \( P_1(d^o(t)) \in \partial P_1(K^o) \). Considering other cases similarly, we have
\[
P_1(d^o \cup e^o \cup (-d^o) \cup (-e^o)) \subset \partial(P_1(K^o)).
\]
On the other hand, for \( P \in \partial(P_1(K^o)) \), there exists a point \( Q \) in \( \partial(Q_1(K)) = Q_1(d \cup e \cup (-d) \cup (-e)) \) such that \( Q \cdot P = 1 \). The point \( Q \) is represented, for instance, by \( d(t) \) for some \( t \in [0, 1) \). By the definition, \( \tilde{P} := d^o(t) = \Lambda(d(t)) \) is the unique point on \( \partial K \) such that \( Q \cdot \tilde{P} = 1 \). Moreover, by the strict convexity of \( K^o \), we have \( P_1(\tilde{P}) = P \). Hence,
\[
\partial(P_1(K^o)) \subset P_1(d^o \cup e^o \cup (-d^o) \cup (-e^o))
\]
also holds.

\[ \square \]

### 3.3 A volume estimate of a piece of \( K^o \) from below

The next task is to estimate \( |K^o_i| \) \( (i = 1, 2, \ldots, 8) \) from below. Consider \( K^o_i = O \ast S_{K^o}(d^o \cup f^o \cup h^o) \), for instance. The boundary \( \partial K^o_i \) is the union of the cone
\[
O \ast (d^o \cup f^o \cup h^o)
\]
and \( K^o_i \cap \partial K^o \). For any point \( P = (u, v, w) \in \mathbb{R}^3 \), the signed volume of the cone
\[
P \ast (O \ast (d^o \cup f^o \cup h^o)) = \{(1 - \lambda)P + \lambda \xi \in \mathbb{R}^3; \xi \in O \ast (d^o \cup f^o \cup h^o), \ 0 \leq \lambda \leq 1\}
\]
is described as
\[
\frac{1}{6} \int_0^1 \begin{vmatrix}
P \\
d^o(t)
\end{vmatrix}
+ \begin{vmatrix}
P \\
f^o(t)
\end{vmatrix}
+ \begin{vmatrix}
P \\
h^o(t)
\end{vmatrix}
\right) dt,
\]
where \( d^o(t) = (d^o_1(t), d^o_2(t), d^o_3(t)) \) and \( \left| \cdot \right| \) denotes the determinant for a square matrix.

Here we put
\[
\begin{align*}
\overline{d}_1 := & \int_0^1 \begin{vmatrix} d^o_1(t) \\ d^o_2(t) \\ d^o_3(t) \end{vmatrix} dt, \\
\overline{d}_2 := & \int_0^1 \begin{vmatrix} d^o_2(t) \\ d^o_3(t) \\ d^o_1(t) \end{vmatrix} dt, \\
\overline{d}_3 := & \int_0^1 \begin{vmatrix} d^o_3(t) \\ d^o_1(t) \\ d^o_2(t) \end{vmatrix} dt,
\end{align*}
\]
and
then we have
\[ \int_0^1 \frac{P}{d^\circ(t)} \, dt = (\overrightarrow{d_1^\circ}, \overrightarrow{d_2^\circ}, \overrightarrow{d_3^\circ}) \cdot (u, v, w). \]

For simplicity, we put \( \overrightarrow{d^\circ} := (\overrightarrow{d_1^\circ}, \overrightarrow{d_2^\circ}, \overrightarrow{d_3^\circ}) \). Similarly, we define \( \overrightarrow{e^\circ}, \overrightarrow{f^\circ}, \overrightarrow{g^\circ}, \overrightarrow{h^\circ} \), and \( \overrightarrow{v^\circ} \). Then the signed volume \( |g| \) of the cone \( P^\circ \) \((O \ast (d^\circ \cup f^\circ \cup h^\circ))\) is represented as \( (\overrightarrow{d^\circ} + \overrightarrow{f^\circ} + \overrightarrow{h^\circ}) \cdot (u, v, w)/6 \).

If the cone \( K_1^\circ \) is convex in \( \mathbb{R}^3 \) and \( P \in K_1^\circ \), then the quantity \( |g| \) coincides with the volume \( |P^\circ \) \((O \ast (d^\circ \cup f^\circ \cup h^\circ))\)| and it is less than \( |K_1^\circ| \). In case \( K_1^\circ \subset \mathbb{R}^3 \) is not convex or \( P \in K^\circ \) is not in \( K_1^\circ \), the comparison of the signed volume \( (13) \) with \( |K_1^\circ| \) is a nontrivial problem. However, under the condition that \( P \) lies in the convex set \( K^\circ \) the following holds, which is a key lemma to estimate \( P(K) \).

**Lemma 3.1.** Let \( K^\circ \in \hat{K} \). For any point \( P = (u, v, w) \in K^\circ \), we have
\[ (\overrightarrow{d^\circ} + \overrightarrow{f^\circ} + \overrightarrow{h^\circ}) \cdot (u, v, w) \leq 6|K_1^\circ|. \]

By the definition of polar, we obtain
\[ \frac{1}{6|K_1^\circ|} (\overrightarrow{d^\circ} + \overrightarrow{f^\circ} + \overrightarrow{h^\circ}) \in K. \]

For other \( K_i^\circ \)’s similar estimates hold (see the table below).

The proof of Lemma 3.1 is somewhat long, so we will prove it in the next subsection. To discuss other \( K_i^\circ \)’s we need further preparations. For the curve \( d^\circ \) on \( \partial K^\circ \) defined by \( d^\circ(t) = d^\circ(1-t) \) \((0 \leq t \leq 1)\), we have
\[ (d^\circ) \cdot (u, v, w) = \int_0^1 \frac{P}{(d^\circ(1-t))} \, dt = \int_0^1 \frac{P}{d^\circ(1-t)} \, dt = \overrightarrow{d^\circ} \cdot (u, v, w) \]
for any \( P = (u, v, w) \in \mathbb{R}^3 \). Moreover, since
\[ (d^\circ) \cdot (u, v, w) = \int_0^1 \frac{P}{-d^\circ(t)} \, dt = \overrightarrow{d^\circ} \cdot (u, v, w), \]
we obtain
\[ (d^\circ) = -\overrightarrow{d^\circ}, \quad (-d^\circ) = \overrightarrow{d^\circ}, \quad (-\overrightarrow{d}) = \overrightarrow{d^\circ}. \]

We can do the same observation for \( K_2^\circ, K_3^\circ, \) and \( K_4^\circ \). Since \( K^\circ \) is centrally symmetric, the volume of \( K_2^\circ, K_3^\circ, K_4^\circ, \) and \( K_5^\circ \) equals that of \( K_3^\circ, K_4^\circ, K_1^\circ, \) and \( K_2^\circ \), respectively.

Summarizing the above arguments, we obtain

| \( K_i^\circ \) = \( O \ast S_{K^\circ}(d^\circ \cup f^\circ \cup h^\circ) \) | point contained in \( \hat{K} \) |
|---|---|
| \( K_1^\circ = O \ast S_{K^\circ}(d^\circ \cup f^\circ \cup h^\circ) \) | \( S_1 := \frac{1}{6|K_1^\circ|} (\overrightarrow{d^\circ} + \overrightarrow{f^\circ} + \overrightarrow{h^\circ}) \) |
| \( K_2^\circ = O \ast S_{K^\circ}(d^\circ \cup i^\circ \cup -g^\circ) \) | \( S_2 := \frac{1}{6|K_2^\circ|} (\overrightarrow{-d^\circ} + \overrightarrow{g^\circ} + \overrightarrow{v^\circ}) \) |
| \( K_3^\circ = O \ast S_{K^\circ}(e^\circ \cup -\overrightarrow{g^\circ} \cup -h^\circ) \) | \( S_3 := \frac{1}{6|K_3^\circ|} (\overrightarrow{-e^\circ} - \overrightarrow{g^\circ} + \overrightarrow{h^\circ}) \) |
| \( K_4^\circ = O \ast S_{K^\circ}(e^\circ \cup -i^\circ \cup f^\circ) \) | \( S_4 := \frac{1}{6|K_4^\circ|} (\overrightarrow{e^\circ} - \overrightarrow{f^\circ} + \overrightarrow{v^\circ}) \) |
Furthermore, similar arguments for the convex body $K$ give the following results.

Note that the calculation is easier than the case of the polar $K^\circ$, because $K_i$ ($i = 1, 2, \ldots, 8$) are convex.

| $K_1 = O * S_K(d \cup f \cup h)$ | $R_1 := \frac{1}{6|K_1|} (d + f + h)$ |
| $K_2 = O * S_K(d \cup i \cup -g)$ | $R_2 := \frac{1}{6|K_2|} (-d + \bar{g} + \bar{i})$ |
| $K_3 = O * S_K(\bar{e} \cup -g \cup -h)$ | $R_3 := \frac{1}{6|K_3|} (-\bar{e} - \bar{g} + \bar{h})$ |
| $K_4 = O * S_K(e \cup -i \cup \bar{f})$ | $R_4 := \frac{1}{6|K_4|} (\bar{e} - \bar{f} + \bar{i})$ |

In Section 3.5, these eight points $R_i, S_i$ ($i = 1, 2, 3, 4$) will be effectively used as test points to estimate $P(K)$ from below.

### 3.4 The proof of Lemma 3.1

Lemma 3.1 is a direct consequence of the following

**Proposition 3.2.** Let $K \in \hat{K}$. Let $c$ be a piecewise smooth, oriented, simple closed curve on $\partial K$. Let $S_K(c) \subset \partial K$ be a piece of surface enclosed by the curve $c$. Then for any point $P \in K$, the inequality

$$\frac{1}{6} \int_c (P \times r) \cdot dr \leq |O * S_K(c)|$$

holds.

The proof will involve a number of steps. First we consider the case of a “triangle” in the following sense. Let $A_1, A_2, A_3 \in \partial K$ be mutually distinct points with $(A_1 \times A_2) \cdot A_3 > 0$. For $(i, j) = (1, 2), (2, 3), (3, 1)$, denote by $C_{i,j}$ the oriented segment $C_K(A_i, A_j)$ on $\partial K$, which is parametrized by

$$l_{i,j}(t) := \rho_K((1 - t)A_i + tA_j)((1 - t)A_i + tA_j) \quad (0 \leq t \leq 1).$$

Each $C_{i,j}$ is also in the plane which through three points $O, A_i$, and $A_j$. Let us consider the domain $S_K(c)$ on $\partial K$ enclosed by $c := C_{1,2} \cup C_{2,3} \cup C_{3,1}$. We call this $S_K(c)$ a triangle on the smooth convex surface $\partial K$.

Putting $S_{i,j} := O * C_{i,j}$ and $D_{i,j} := P * S_{i,j}$, then

$$\frac{1}{6} \int_{C_{i,j}} (P \times r) \cdot dr = \frac{1}{6} \int_0^1 (P \times l_{i,j}(t)) \cdot l_{i,j}'(t) \, dt = \frac{1}{6} \int_0^1 \left| \frac{P}{l_{i,j}'(t)} \right| \, dt$$

is the signed volume of $D_{i,j}$. Let $|D_{i,j}|$ denote its absolute value. Note that all $D_{i,j}$ mutually do not intersect their interiors.

In this setting, the following lemma holds.

**Lemma 3.3.** Let $S_K(c)$ be a triangle on $\partial K$. For any point $P \in K$, we have

$$\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{2,3}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{3,1}} (P \times r) \cdot dr \leq |O * S_K(c)|.$$
Proof. Fix $P \in K$. Since $A_1, A_2, A_3$ are linearly independent as vectors in $\mathbb{R}^3$, there exist $t_1, t_2, t_3 \in \mathbb{R}$ uniquely such that $P = t_1 A_1 + t_2 A_2 + t_3 A_3$. It suffices to check the formula for the following four cases.

Case: $t_1, t_2, t_3 \geq 0$; Since $P$ is contained in $O * S_K(c)$, the signed volume of $D_{i,j}$ is non-negative for any $(i, j)$. Moreover, $D_{1,2} \cup D_{2,3} \cup D_{3,1} \subset O * S_K(c)$. Hence, we obtain

$$
\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{2,3}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{3,1}} (P \times r) \cdot dr
= |D_{1,2}| + |D_{2,3}| + |D_{3,1}| \leq |O * S_K(c)|.
$$

Case: One $t_i$ is negative and the others are non-negative; Without loss of generality, we may assume that $t_1, t_2 \geq 0$ and $t_3 < 0$. Taking the sign of the signed volume of $D_{i,j}$ into account, we have

$$
\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{2,3}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{3,1}} (P \times r) \cdot dr
= -|D_{1,2}| + |D_{2,3}| + |D_{3,1}|.
$$

Since $K$ is convex, the cone $D_{2,3}$ is divided by $S_{1,2}$ into the following two parts:

$$
D^b_{2,3} := D_{2,3} \cap (O * S_K(c)), \quad D^a_{2,3} := \overline{D_{2,3} \setminus D^b_{2,3}},
$$

where $\overline{D}$ denotes the closure of a subset $D \subset \mathbb{R}^3$. Note that the intersection $D^a_{2,3} \cap D^b_{2,3}$ has measure zero in $\mathbb{R}^3$. Similarly, we can divide $D_{3,1}$ into two parts $D^a_{3,1}$ and $D^b_{3,1}$.

Since $D_{1,2}$ contains $D^a_{2,3}$ and $D^a_{3,1}$, and $O * S_K(c)$ contains $D^b_{2,3}$ and $D^b_{3,1}$, we have

$$
|D^a_{2,3}| + |D^a_{3,1}| \leq |D_{1,2}|, \quad |D^b_{2,3}| + |D^b_{3,1}| \leq |O * S_K(c)|.
$$

Summarizing, we obtain

$$
-|D_{1,2}| + |D_{2,3}| + |D_{3,1}| \leq -|D^a_{2,3}| - |D^a_{3,1}| + |D_{2,3}| + |D_{3,1}|
= |D^b_{2,3}| + |D^b_{3,1}| \leq |O * S_K(c)|,
$$

which confirms the formula.

Case: Two $t_i, t_j$ are negative and the other is non-negative; We may assume that $t_1, t_2 < 0$ and $t_3 \geq 0$. In this case, from the sign of the signed volume of $D_{i,j},$

$$
\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{2,3}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{3,1}} (P \times r) \cdot dr
= |D_{1,2}| - |D_{2,3}| - |D_{3,1}|
$$

holds. Taking the position of $P$ and the convexity of $K$ into consideration, $D_{1,2}$ is divided by $S_{2,3} \cup S_{3,1}$ into

$$
D^b_{1,2} := D_{1,2} \cap (O * S_K(c)) \text{ and } D^a_{1,2} := \overline{D_{1,2} \setminus D^b_{1,2}}.
$$
Since $D_{1,2} \subset D_{2,3} \cup D_{3,1}$ and $D_{1,2}^{b} \subset O \ast S_{K}(c)$, we have

$$|D_{1,2}| - |D_{2,3}| - |D_{3,1}| = |D_{1,2}^{a}| + |D_{1,2}^{b}| - |D_{2,3}| - |D_{3,1}| \leq |D_{2,3}| + |D_{3,1}| + (O \ast S_{K}(c)) - |D_{2,3}| - |D_{3,1}| = |O \ast S_{K}(c)|,$$

which also justifies the formula.

**Case:** $t_{1}, t_{2}, t_{3} < 0$; In this final case,

$$\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{2,3}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{4,1}} (P \times r) \cdot dr = -|D_{1,2}| - |D_{2,3}| - |D_{3,1}| \leq 0 < |O \ast S_{K}(c)|$$

holds, so trivially the formula holds. \qed

From Lemma 3.3, we obtain the following

**Lemma 3.4.** Let $A_{1}, \ldots, A_{m} \in \partial K$ be mutually distinct points on $\partial K$. For $i = 1, \ldots, m$, let $C_{i+1} = C_{K}(A_{i}, A_{i+1})$ be a segment on $\partial K$ which satisfies that $A_{i} \neq -A_{i+1}$, where $A_{m+1}$ means $A_{1}$. Suppose that $c := C_{1,2} \cup C_{2,3} \cup \cdots \cup C_{m-1,m} \cup C_{m,m+1}$ is an oriented simple closed curve on $\partial K$. Then for the “polygon” $S_{K}(c)$ on $\partial K$ enclosed by the curve $c$, we have

$$\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr \leq |O \ast S_{K}(c)| \quad \text{for any } P \in K.$$

**Proof.** We first consider the case where $m = 4$ and $S_{K}(c)$ is “small” in the sense that $S_{K}(c) = S_{K}(C_{1,2} \cup C_{2,3} \cup C_{3,4} \cup C_{4,1})$ is decomposable into two triangles on $\partial K$, for instance, $S_{K}(C_{1,2} \cup C_{2,4} \cup C_{4,1})$ and $S_{K}(C_{2,3} \cup C_{3,4} \cup C_{4,2})$. Then the cone $O \ast S_{K}(c)$ is divided into

$$K_{a} := O \ast S_{K}(C_{1,2} \cup C_{2,4} \cup C_{4,1}) \quad \text{and} \quad K_{b} := O \ast S_{K}(C_{2,3} \cup C_{3,4} \cup C_{4,2}).$$

We apply Lemma 3.3 to $K_{a}$ and $K_{b}$, then

$$\frac{1}{6} \int_{C_{1,2}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{2,4}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{4,1}} (P \times r) \cdot dr \leq |K_{a}|,$n

$$\frac{1}{6} \int_{C_{2,3}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{3,4}} (P \times r) \cdot dr + \frac{1}{6} \int_{C_{4,2}} (P \times r) \cdot dr \leq |K_{b}|$$

hold. Since $C_{4,2}$ has the reverse orientation of $C_{2,4}$, we obtain

$$\frac{1}{6} \int_{C_{1,2} \cup C_{2,3} \cup C_{3,4} \cup C_{4,1}} (P \times r) \cdot dr \leq |K_{a}| + |K_{b}| = |O \ast S_{K}(c)|.$$

For general $m \in \mathbb{N}$, adding extra vertices on $S_{K}(c)$ if necessary, we can decompose $S_{K}(c)$ into finitely many triangles and apply the above argument inductively. \qed

**Proof of Proposition 3.2.** First, we approximate the piecewise smooth simple closed curve $c$ by polygonal lines $c_{m}$ as in Lemma 3.4. Then we get the inequality in Proposition 3.2 for $c_{m}$. Next, we have only to take a limit of it. \qed
3.5 A sharp estimate

In this subsection, we give a sufficient condition of deducing inequality (2) (see the condition (15) below). In the following sections, for any $K \in \hat{K}$ we carefully select a linear transformation $\mathcal{A}$ such that $\mathcal{A}K \in \hat{K}$ satisfies this condition.

From the arguments in Section 3.3, we know that $R_i \in K^o$ and $S_i \in K$ ($i = 1, 2, 3, 4$). By the definition of polar, we obtain $R_i \cdot S_i \leq 1$. In other words,

$$
\begin{align*}
(\bar{d} + \bar{f} + \bar{h}) \cdot (\bar{d}^o + \bar{f}^o + \bar{h}^o) &\leq 36|K_1||K^o_1|, \\
(-\bar{d} + \bar{g} + \bar{i}) \cdot (-\bar{d}^o + \bar{g}^o + \bar{i}^o) &\leq 36|K_2||K^o_2|, \\
(-\bar{e} - \bar{g} + \bar{h}) \cdot (-\bar{e}^o - \bar{g}^o + \bar{h}^o) &\leq 36|K_3||K^o_3|, \\
(\bar{e} - \bar{f} + \bar{i}) \cdot (\bar{e}^o - \bar{f}^o + \bar{i}^o) &\leq 36|K_4||K^o_4|.
\end{align*}
$$

(14)

Here, assume that

$$|K_1| = |K_2| = |K_3| = |K_4|, \quad \bar{d} = \bar{e}, \quad \bar{f} = \bar{g}, \quad \bar{h} = \bar{i}.
$$

(15)

We now check that this condition implies symmetric Mahler’s conjecture for the three dimensional case. Indeed, the condition (15) yields $|K_i| = |K|/8$, and hence equations (14) and (15) implies that

$$
\begin{align*}
(\bar{d} + \bar{f} + \bar{h}) \cdot (\bar{d}^o + \bar{f}^o + \bar{h}^o) &\leq \frac{9}{2}|K||K^o_1|, \\
(-\bar{d} + \bar{f} + \bar{h}) \cdot (-\bar{d}^o + \bar{f}^o + \bar{h}^o) &\leq \frac{9}{2}|K||K^o_2|, \\
(-\bar{d} - \bar{f} + \bar{h}) \cdot (-\bar{e}^o - \bar{g}^o + \bar{h}^o) &\leq \frac{9}{2}|K||K^o_3|, \\
(\bar{d} - \bar{f} + \bar{h}) \cdot (\bar{e}^o - \bar{f}^o + \bar{i}^o) &\leq \frac{9}{2}|K||K^o_4|.
\end{align*}
$$

It then follows from these inequalities that

$$
\begin{align*}
\frac{9}{4}|K||K^o| &= \frac{9}{2}|K||K^o_1| + |K^o_2| + |K^o_3| + |K^o_4| \\
&\geq \bar{d} \cdot (\bar{d}^o + \bar{f}^o + \bar{h}^o) - (-\bar{d}^o + \bar{g}^o + \bar{i}^o) - (-\bar{e}^o - \bar{g}^o + \bar{h}^o) + (\bar{e}^o - \bar{f}^o + \bar{i}^o) \\
&\quad + \bar{f} \cdot (\bar{d}^o + \bar{f}^o + \bar{h}^o) + (-\bar{d}^o + \bar{g}^o + \bar{i}^o) - (-\bar{e}^o - \bar{g}^o + \bar{h}^o) - (\bar{e}^o - \bar{f}^o + \bar{i}^o) \\
&\quad + \bar{h} \cdot (\bar{d}^o + \bar{f}^o + \bar{h}^o) + (-\bar{d}^o + \bar{g}^o + \bar{i}^o) + (-\bar{e}^o - \bar{g}^o + \bar{h}^o) + (\bar{e}^o - \bar{f}^o + \bar{i}^o) \\
&= 2\bar{d} \cdot (\bar{d}^o + \bar{e}^o) + 2\bar{f} \cdot (\bar{f}^o + \bar{g}^o) + 2\bar{h} \cdot (\bar{h}^o + \bar{i}^o).
\end{align*}
$$

Let us examine the first term in the last line. Recalling the definition of $\bar{d}^o = (\bar{d}^o_1, \bar{d}^o_2, \bar{d}^o_3)$, its $u$-component $\bar{d}^o_1$ is twice the area of the convex region $O \ast P_1(d^o)$ in the $vw$-plane. Similarly, $\bar{e}^o_1$ is twice $|O \ast P_1(e^o)|$. Since $K^o \in \hat{K}$, the convex domain $P_1(K^o)$ in the $vw$-plane is also centrally symmetric, and hence

$$
\bar{d}^o_1 + \bar{e}^o_1 = 2(|O \ast P_1(d^o)| + |O \ast P_1(e^o)|) = |P_1(K^o)|.
$$
Again, since the curve $d \subset K$ lies in the $yz$-plane, we have $\overrightarrow{d} = (d_1, 0, 0)$, where

$$\overrightarrow{d}_1 = \int_0^1 \begin{vmatrix} d_2(t) & d_3(t) \\ d_2'(t) & d_3'(t) \end{vmatrix} dt.$$  

Here $\overrightarrow{d}_1$ is nothing but twice the area of the convex region $O * d$ in the $yz$-plane. From the assumption that $\overrightarrow{d} = \overrightarrow{e}$ in the condition (15), we have $\overrightarrow{d}_1 = \frac{|Q_1(K)|}{2}$. Therefore, the first term $2\overrightarrow{d} \cdot (\overrightarrow{d} + \overrightarrow{e})$ is equal to $|Q_1(K)| |P_1(K^\circ)|$. Applying the same argument to the second and the third terms, we have

$$f_2 + g_2 = |P_2(K^\circ)|, \quad h_3 + i_3 = |P_3(K^\circ)|$$

and

$$2\overrightarrow{f} = (0, |Q_2(K)|, 0), \quad 2\overrightarrow{h} = (0, 0, |Q_3(K)|).$$

Consequently, we obtain

$$\frac{9}{4} |K| |K^\circ| \geq |Q_1(K)| |P_1(K^\circ)| + |Q_2(K)| |P_2(K^\circ)| + |Q_3(K)| |P_3(K^\circ)|.$$  

It is well-known that convex domains $Q_i(K)$ and $P_i(K^\circ)$ are polar bodies of each other (e.g. [GMR, p. 274]). By Mahler’s theorem [Ma1] (see also [R], [GMR], [MR], and Section 2), we have $|Q_i(K)| |P_i(K^\circ)| \geq 4^i / 2! = 8 (i = 1, 2, 3)$. Thus

$$|K| |K^\circ| \geq \frac{4}{9} \times 8 \times 3 = \frac{32}{3} = \frac{4^3}{3!}$$

holds. We conclude that the Mahler conjecture is true for $K \in \hat{K}$ satisfying the condition (15).

## 4 The case with a hyperplane symmetry

From the conclusion of the previous section, it is important to find out when the condition (15) holds. We start with the following simple observation.

**Example.** Consider the case that $K$ is symmetric with respect to the $yz$-plane, $zx$-plane, and $xy$-plane (more generally, its image $\mathcal{A}K$ by a linear transformation $\mathcal{A}$ is symmetric with respect to these three planes). In this case, the condition (15) automatically holds, and hence we have $\mathcal{P}(K) \geq 32/3$. This fact is the three dimensional case of the result by Saint-Raymond [SR].

From now on, suppose that $K \in \hat{K}$ is symmetric with respect to a plane. Since the volume product is invariant under linear transformations of $K$, we may assume that the hyperplane of symmetry is the $yz$-plane. In this case, the area of $O * d$ and that of $O * (-g)$ coincide. Taking the signs into consideration, $\overrightarrow{f} = \overrightarrow{g}$ holds. Similarly, $\overrightarrow{h} = \overrightarrow{g}$ holds. Concerning the volume of $K_i$ ($i = 1, \ldots, 8$), we have

$$|K_1| = |K_2| = |K_8|, \quad |K_4| = |K_3| = |K_5|.$$
Consequently, the condition \([\text{(15)}]\) holds, provided \(K\) satisfies
\[
\overrightarrow{d} = \overrightarrow{e}, \quad |K_1| = |K_4|.
\] (16)

For a given convex body \(K \in \hat{K}\), we consider the image of \(K\) by the following linear transformation:
\[
A(\alpha)X(\theta) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

For \(K \in \hat{K}\), we introduced vectors \(\overrightarrow{d} = (d_1, d_2, d_3)\), etc., in Section 3.3. From now on, to clarify the dependence on the convex body, we denote the vector \(\overrightarrow{d}\) for \(A(\alpha)X(\theta)K \in \hat{K}\) by \(\overrightarrow{d}(A(\alpha)X(\theta)K)\), etc. Similarly, for the image \(A(\alpha)X(\theta)K \in \hat{K}\) of a convex body \(K\), we denote by \((A(\alpha)X(\theta)K)_i\) \((i = 1, \ldots, 8)\) the decomposition in Section 3.2.

First, since \(A(\alpha)\) and \(X(\theta)\) keep the symmetry of \(K\) with respect to the \(yz\)-plane, we have
\[
\overrightarrow{d}(A(\alpha)X(\theta)K) = \overrightarrow{d}(A(\alpha)X(\theta)K),
\]
\[
\overrightarrow{e}(A(\alpha)X(\theta)K) = \overrightarrow{e}(A(\alpha)X(\theta)K),
\]
\[
|\langle A(\alpha)X(\theta)K_1 \rangle| = |\langle A(\alpha)X(\theta)K_2 \rangle| = |\langle A(\alpha)X(\theta)K_5 \rangle|,
\]
\[
|\langle A(\alpha)X(\theta)K_3 \rangle| = |\langle A(\alpha)X(\theta)K_4 \rangle| = |\langle A(\alpha)X(\theta)K_7 \rangle|
\]
for any \(\alpha, \theta \in \mathbb{R}\). Next, for a given \(\theta \in \mathbb{R}\) we can choose \(\alpha = \alpha(\theta) \in \mathbb{R}\) to satisfy
\[
|\langle A(\alpha)X(\theta)K_1 \rangle| = |\langle A(\alpha)X(\theta)K_4 \rangle|,
\]
see Proposition 5.3 below in the special case that \(\Phi(A(\alpha)X(\theta)K) = \Psi(A(\alpha)X(\theta)K) = \pi/2\). 

\(|\Delta_1(A(\alpha)X(\theta)K)| = |\Delta_2(A(\alpha)X(\theta)K)|, |\Delta_3(A(\alpha)X(\theta)K)| = |\Delta_4(A(\alpha)X(\theta)K)|\)
for \(X(\theta)K\). Note that such \(\alpha\) is uniquely determined for each \(\theta\), and \(\alpha\) depends on \(\theta\) continuously. Moreover, we can easily see that there exists some \(\theta_0\) such that \(\alpha(\theta_0) = 0\). In this setting, we obtain \(\alpha(\theta_0 + \pi/2) = 0\) and
\[
\overrightarrow{d}(A(\alpha(\theta_0 + \pi/2))X(\theta_0 + \pi/2)K) = \overrightarrow{d}(X(\theta_0 + \pi/2)K) = \overrightarrow{e}(A(\alpha(\theta_0))X(\theta_0)K),
\]
\[
\overrightarrow{e}(A(\alpha(\theta_0 + \pi/2))X(\theta_0 + \pi/2)K) = \overrightarrow{e}(X(\theta_0 + \pi/2)K) = \overrightarrow{d}(A(\alpha(\theta_0))X(\theta_0)K).
\] (17)

Next, we put
\[
F(\theta) := \overrightarrow{d}(A(\alpha(\theta))X(\theta)K) - \overrightarrow{e}(A(\alpha(\theta))X(\theta)K).
\]

Since \(\alpha\) is continuous with respect to \(\theta\), \(F\) is continuous on \(\mathbb{R}\). By (17), we have \(F(\theta_0 + \pi/2) = -F(\theta_0)\). By using the intermediate value theorem, there exists \(\theta_1 \in [\theta_0, \theta_0 + \pi/2]\) such that \(F(\theta_1) = 0\), which yields
\[
\overrightarrow{d}(A(\alpha(\theta_1))X(\theta_1)K) = \overrightarrow{e}(A(\alpha(\theta_1))X(\theta_1)K).
\]
Consequently, \(A(\alpha(\theta_1))X(\theta_1)K \in \hat{K}\) satisfies the condition (16), and hence (15). Since \(A(\alpha(\theta_1))X(\theta_1)\) is a linear transformation of \(\mathbb{R}^3\), we obtain
\[
\mathcal{P}(K) = \mathcal{P}(A(\alpha(\theta_1))X(\theta_1)K) \geq \frac{32}{3}.
\]
Thus, we have proved the following
Proposition 4.1. Let $K \in \mathcal{K}$ be a three dimensional centrally symmetric convex body which is symmetric with respect to a plane. Then $\mathcal{P}(K) \geq \frac{32}{3}$ holds.

5 The general case

5.1 Notations

In order to prove [2] for general $K \in \mathcal{K}_0^3$, we first consider some linear transform of $K \in \hat{K}$. For this purpose, we introduce some notations. For $\theta, \phi, \psi \in \mathbb{R}$, $X(\theta)$, $Y(\phi)$, $Z(\psi)$ denote rotations in $\mathbb{R}^3$ as follows:

$$X(\theta) = \begin{pmatrix} 1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta \end{pmatrix}, \quad Y(\phi) = \begin{pmatrix} \cos \phi & 0 & \sin \phi \\
0 & 1 & 0 \\
-\sin \phi & 0 & \cos \phi \end{pmatrix},$$

$$Z(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1 \end{pmatrix}.$$ Hereafter, we put $K(\theta, \phi, \psi) := X(\theta)Y(\phi)Z(\psi)K$. Each $\Delta_i(K)$ denotes the part of $K$ included in the $i$-th octant:

$$\Delta_1(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \geq 0, y \geq 0, z \geq 0 \},$$

$$\Delta_2(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \geq 0, y \geq 0, z \leq 0 \},$$

$$\Delta_3(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \geq 0, y \leq 0, z \geq 0 \},$$

$$\Delta_4(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \geq 0, y \leq 0, z \leq 0 \},$$

$$\Delta_5(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \leq 0, y \geq 0, z \geq 0 \},$$

$$\Delta_6(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \leq 0, y \geq 0, z \leq 0 \},$$

$$\Delta_7(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \leq 0, y \leq 0, z \geq 0 \},$$

$$\Delta_8(K) := K \cap \{ (x, y, z) \in \mathbb{R}^3 ; x \leq 0, y \leq 0, z \leq 0 \}.$$

As in Section [3, 2] we put

$$A(K) := \rho_K((1, 0, 0))(1, 0, 0), B(K) := \rho_K((0, 1, 0))(0, 1, 0), C(K) := \rho_K((0, 0, 1))(0, 0, 1)$$

and

$$d(K) := C_K(B(K), C(K)), \quad e(K) := C_K(C(K), -B(K)),$$

$$f(K) := C_K(C(K), A(K)), \quad g(K) := C_K(A(K), -C(K)),$$

$$h(K) := C_K(A(K), B(K)), \quad i(K) := C_K(B(K), -A(K)).$$

Then we see that $\bar{d}(K)$ is calculated as

$$\left( \int_0^1 \left| \begin{array}{cc} d_2(K)(t) & d_3(K)(t) \\ d_3(K)'(t) & d_4(K)(t) \end{array} \right| dt, \int_0^1 \left| \begin{array}{cc} d_3(K)(t) & d_4(K)(t) \\ d_4(K)'(t) & d_5(K)(t) \end{array} \right| dt, \int_0^1 \left| \begin{array}{cc} d_4(K)(t) & d_5(K)(t) \\ d_5(K)'(t) & d_6(K)(t) \end{array} \right| dt \right)

= 2(|P_1(O \ast d(K))|, |P_2(O \ast d(K))|, |P_3(O \ast d(K))|)

= 2(|O \ast d(K)|, 0, 0),$$

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where \( P_1, P_2, \) and \( P_3 \) denote projections to the \( yz \)-plane, \( zx \)-plane, and \( xy \)-plane, respectively. Similarly, we get

\[
\begin{align*}
\tilde{d}(K) &= 2 (|O \ast d(K)|, 0, 0), \\
\tilde{f}(K) &= 2 (0, |O \ast f(K)|, 0), \\
\tilde{h}(K) &= 2 (0, 0, |O \ast h(K)|),
\end{align*}
\]

Similarly, we get

\[
\begin{align*}
\tilde{e}(K) &= 2 (|O \ast e(K)|, 0, 0), \\
\tilde{g}(K) &= 2 (0, |O \ast g(K)|, 0), \\
\tilde{i}(K) &= 2 (0, 0, |O \ast i(K)|).
\end{align*}
\]

Note that

\[
\begin{align*}
\Delta_1(K) &= O \ast S_K (d(K) \cup f(K) \cup h(K)), \\
\Delta_2(K) &= O \ast S_K (d(K) \cup i(K) \cup -g(K)), \\
\Delta_3(K) &= O \ast S_K (e(K) \cup -g(K) \cup -h(K)), \\
\Delta_4(K) &= O \ast S_K (e(K) \cup -i(K) \cup f(K))
\end{align*}
\]

and

\[
\begin{align*}
O \ast d(K) &= K \cap \{(x, y, z) \in \mathbb{R}^3; x = 0, y \geq 0, z \geq 0\}, \\
O \ast e(K) &= K \cap \{(x, y, z) \in \mathbb{R}^3; x = 0, y \leq 0, z \geq 0\}, \\
O \ast f(K) &= K \cap \{(x, y, z) \in \mathbb{R}^3; x \geq 0, y = 0, z \geq 0\}, \\
O \ast g(K) &= K \cap \{(x, y, z) \in \mathbb{R}^3; x \geq 0, y = 0, z \leq 0\}, \\
O \ast h(K) &= K \cap \{(x, y, z) \in \mathbb{R}^3; x \geq 0, y \geq 0, z = 0\}, \\
O \ast i(K) &= K \cap \{(x, y, z) \in \mathbb{R}^3; x \leq 0, y \geq 0, z = 0\}.
\end{align*}
\]

Therefore, the condition (15) is equivalent to the following equations:

\[
\begin{align*}
|\Delta_1(K)| &= |\Delta_2(K)| = |\Delta_3(K)| = |\Delta_4(K)|, \\
|O \ast d(K)| &= |O \ast e(K)|, \\
|O \ast f(K)| &= |O \ast g(K)|, \\
|O \ast h(K)| &= |O \ast i(K)|.
\end{align*}
\] (18)

According to the discussion in Section 3.5, we obtain

**Proposition 5.1.** If \( K \in \hat{K} \) satisfies the condition (18), then

\[
P(K) \geq \frac{32}{3}.
\]

### 5.2 Linear transform

In this section, we choose a linear transform \( \mathcal{A} = \mathcal{A}(K) \) (depends on \( K \)) which satisfies

\[
\begin{align*}
|O \ast f(\mathcal{A}K)| &= |O \ast g(\mathcal{A}K)|, \\
|O \ast h(\mathcal{A}K)| &= |O \ast i(\mathcal{A}K)|, \\
|\Delta_1(\mathcal{A}K)| + |\Delta_2(\mathcal{A}K)| &= |\Delta_3(\mathcal{A}K)| + |\Delta_4(\mathcal{A}K)|,
\end{align*}
\] (19)

where \( \mathcal{A}K = \mathcal{A}(K)K \). For the purpose, we use the following spherical coordinates:

\[
P(\alpha, \beta) = (\cos \alpha, \sin \alpha \cos \beta, \sin \alpha \sin \beta) \in S^2.
\]

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First, we choose the angle $\Theta(K) \in (0, \pi)$ which satisfies that
\[
\int_{0}^{\Theta(K)} d\beta \int_{0}^{\pi} \rho_{K}^{3}(P(\alpha, \beta)) \sin \alpha d\alpha = \int_{\Theta(K)}^{\pi} d\beta \int_{0}^{\pi} \rho_{K}^{3}(P(\alpha, \beta)) \sin \alpha d\alpha. \tag{20}
\]
Since the left-hand side is increasing and the right-hand side is decreasing with respect to $\Theta(K)$, we see that $\Theta(K)$ is uniquely determined. Actually, we have the following

**Lemma 5.2.** For each $K \in \hat{K}$, there exists the unique $\Theta(K) \in (0, \pi)$ satisfying the relation \[(20)\]. Moreover, $\Theta(\theta, \phi, \psi) := \Theta(K(\theta, \phi, \psi))$ is smooth with respect to $(\theta, \phi, \psi) \in \mathbb{R}^{3}$.

**Proof.** We put
\[
I(\Theta, \theta, \phi, \psi) := \int_{0}^{\Theta} d\beta \int_{0}^{\pi} \rho_{K(\theta, \phi, \psi)}^{3}(P(\alpha, \beta)) \sin \alpha d\alpha - \int_{\Theta}^{\pi} d\beta \int_{0}^{\pi} \rho_{K(\theta, \phi, \psi)}^{3}(P(\alpha, \beta)) \sin \alpha d\alpha.
\]
Since $K \in \hat{K}$, by the definition of $K(\theta, \phi, \psi)$, we see that $I$ is smooth in $\mathbb{R}^{4}$. On the other hand, since $\rho_{K}(P(\alpha, \beta)) > 0$, we have
\[
I(0, \theta, \phi, \psi) = -\int_{0}^{\pi} d\beta \int_{0}^{\pi} \rho_{K(\theta, \phi, \psi)}^{3}(P(\alpha, \beta)) \sin \alpha d\alpha < 0,
\]
\[
I(\pi, \theta, \phi, \psi) = \int_{0}^{\pi} d\beta \int_{0}^{\pi} \rho_{K(\theta, \phi, \psi)}^{3}(P(\alpha, \beta)) \sin \alpha d\alpha > 0,
\]
\[
\frac{\partial I}{\partial \Theta}(\Theta, \theta, \phi, \psi) = 2 \int_{0}^{\pi} \rho_{K(\theta, \phi, \psi)}^{3}(P(\alpha, \Theta)) \sin \alpha d\alpha > 0.
\]
By the intermediate value theorem, there exists the unique $\Theta(\theta, \phi, \psi) = \Theta(K(\theta, \phi, \psi))$ such that $I(\Theta(\theta, \phi, \psi), \theta, \phi, \psi) = 0$. Moreover, by the implicit function theorem, we see that $\Theta(\theta, \phi, \psi)$ is smooth in $\mathbb{R}^{3}$.

Next, we introduce the quantities $\Phi(K) \in (0, \pi)$ and $\Psi(K) \in (0, \pi)$ defined by
\[
\int_{0}^{\Phi(K)} \rho_{K}^{2}(P(\alpha, 0)) d\alpha = \int_{0}^{\Phi(K)} \rho_{K}^{2}(P(\alpha, 0)) d\alpha,
\]
\[
\int_{0}^{\Psi(K)} \rho_{K}^{2}(P(\alpha, \Theta(K))) d\alpha = \int_{0}^{\Psi(K)} \rho_{K}^{2}(P(\alpha, \Theta(K))) d\alpha.
\]
Similarly as $\Theta(K)$, the above $\Phi(K)$ and $\Psi(K)$ are uniquely determined for each $K \in \hat{K}$, and $\Phi(\theta, \phi, \psi) := \Phi(K(\theta, \phi, \psi))$ and $\Psi(\theta, \phi, \psi) := \Psi(K(\theta, \phi, \psi))$ are smooth functions on $\mathbb{R}^{3}$. Now, we define $\mathcal{A} = \mathcal{A}(K)$ as
\[
\mathcal{A}(K) := \begin{pmatrix}
\frac{1}{\tan(\Theta(K))} & \frac{1}{\tan(\Theta(K))} \\
\sin(\Theta(K)) \tan(\Psi(K)) & \frac{1}{\tan(\Theta(K))} \\
0 & 0 \\
\frac{1}{\tan(\Theta(K))} & 1
\end{pmatrix}^{-1}.
\]

**Proposition 5.3.** For every $K \in \hat{K}$, the condition \[(19)\] holds for $\mathcal{A}K$ which is the image of $K$ by the above linear transform.
Proof. Denote by $\chi_A$ the characteristic function:

$$\chi_A(P) = \begin{cases} 1 & \text{if } P \in A, \\ 0 & \text{if } P \notin A. \end{cases}$$

By the definition of $\Delta_i(K)$, we have

$$|\Delta_1(\mathcal{A}K)| + |\Delta_2(\mathcal{A}K)| - |\Delta_3(\mathcal{A}K)| - |\Delta_4(\mathcal{A}K)|$$

$$= \int_{\{\hat{y}>0, \hat{z}>0\}} \chi_{\mathcal{A}K}(\tilde{P}) d\tilde{x}d\tilde{y}d\tilde{z} - \int_{\{\hat{y}<0, \hat{z}>0\}} \chi_{\mathcal{A}K}(\tilde{P}) d\tilde{x}d\tilde{y}d\tilde{z}$$

$$= \int_{\{\hat{y}>0, \hat{z}>0\}} \chi_{\mathcal{A}^{-1}\mathcal{P}} d\tilde{x}d\tilde{y}d\tilde{z} - \int_{\{\hat{y}<0, \hat{z}>0\}} \chi_{\mathcal{A}^{-1}\mathcal{P}} d\tilde{x}d\tilde{y}d\tilde{z}$$

$$= \int_{\{y-z/\tan(\Theta(K))>0, z>0\}} \chi_{\mathcal{A}K}(P) dxdydz - \int_{\{y-z/\tan(\Theta(K))<0, z>0\}} \chi_{\mathcal{A}K}(P) dxdydz,$$

where we used the substitution $P = \mathcal{A}^{-1}\tilde{P}$, that is

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \mathcal{A}P = \begin{pmatrix} x - \frac{y}{\tan(\Phi(K))} - \frac{z}{\tan(\Psi(K))} \tan(\Theta(K)) & \frac{y}{\tan(\Phi(K))} \\ y - \frac{z}{\tan(\Phi(K))} & \frac{z}{\tan(\Phi(K))} \end{pmatrix}.$$ 

By using the polar coordinates $x = r \cos \alpha$, $y = r \sin \alpha \cos \beta$, $z = r \sin \alpha \sin \beta$, we have

$$y - \frac{z}{\tan(\Theta(K))} = r \sin \alpha \cos \beta - \frac{r \sin \alpha \sin \beta}{\tan(\Theta(K))} = z \left( \frac{1}{\tan \beta} - \frac{1}{\tan(\Theta(K))} \right).$$

Under the condition $r > 0$ and $\sin \alpha > 0$, $z > 0$ if and only if $0 < \beta < \pi$. Thus we have

$$y - \frac{z}{\tan(\Theta(K))} > 0, z > 0 \text{ if and only if } 0 < \beta < \Theta(K),$$

$$y - \frac{z}{\tan(\Theta(K))} < 0, z > 0 \text{ if and only if } \Theta(K) < \beta < \pi.$$ 

Therefore, by Lemma 5.2, we have

$$|\Delta_1(\mathcal{A}K)| + |\Delta_2(\mathcal{A}K)| - |\Delta_3(\mathcal{A}K)| - |\Delta_4(\mathcal{A}K)|$$

$$= \frac{1}{3} \int_0^{\Theta(K)} d\beta \int_0^\pi \rho_3^K(P(\alpha, \beta)) \sin \alpha d\alpha - \frac{1}{3} \int_0^{\pi} d\beta \int_0^{\pi} \rho_3^K(P(\alpha, \beta)) \sin \alpha d\alpha$$

$$= 0.$$ 

Next, since

$$|O * f(\mathcal{A}K)| = \int_{\{\tilde{x}>0, \tilde{y}=0, \tilde{z}>0\}} \chi_{\mathcal{A}K}(\tilde{P}) d\tilde{x}d\tilde{z},$$

$$|O * g(\mathcal{A}K)| = \int_{\{\tilde{x}>0, \tilde{y}=0, \tilde{z}<0\}} \chi_{\mathcal{A}K}(\tilde{P}) d\tilde{x}d\tilde{z} = \int_{\{\tilde{x}<0, \tilde{y}=0, \tilde{z}>0\}} \chi_{\mathcal{A}K}(\tilde{P}) d\tilde{x}d\tilde{z},$$

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by the substitution $P = A^{-1} \hat{P}$, we have

$$
|O \ast f(AK)| - |O \ast g(AK)|
= \int_{\{ \hat{x}>0, \hat{y}=0, \hat{z}>0 \}} \chi_{AK}(\hat{P}) \, d\hat{x} d\hat{z} - \int_{\{ \hat{x}<0, \hat{y}=0, \hat{z}>0 \}} \chi_{AK}(\hat{P}) \, d\hat{x} d\hat{z}
= \int_{\{ x-z/(\sin(\Theta(K)) \tan(\Psi(K)))>0, y=0, z=0 \}} \chi_K(A^{-1} \hat{P}) \, dx dz - \int_{\{ x-z/(\sin(\Theta(K)) \tan(\Psi(K)))<0, y=0, z=0 \}} \chi_K(A^{-1} \hat{P}) \, dx dz.
$$

By using the above polar coordinates, if $y = z/\tan(\Theta(K))$ and $z > 0$, then $\beta = \Theta(K) \in (0, \pi)$. Since

$$
x - \frac{z}{\sin(\Theta(K)) \tan(\Psi(K))} = r \sin \alpha \left( \frac{1}{\tan \alpha} - \frac{1}{\tan(\Psi(K))} \right),
$$

we see that $x - z/(\sin(\Theta(K)) \tan(\Psi(K))) > 0$ if and only if $0 < \alpha < \Psi(K)$, and $x - z/(\sin(\Theta(K)) \tan(\Psi(K))) < 0$ if and only if $\Psi(K) < \alpha < \pi$. Thus, by the definition of $\Psi(K)$, we obtain

$$
|O \ast f(AK)| - |O \ast g(AK)|
= \sin(\Theta(K)) \left( \int_{\{ r>0, \Psi(K)<\alpha<\pi \}} \chi_K(P(\alpha, \Theta(K))) r \, dr d\alpha - \int_{\{ r>0, \Psi(K)<\alpha<\pi \}} \chi_K(P(\alpha, \Theta(K))) r \, dr d\alpha \right)
= \sin(\Theta(K)) \left( \frac{1}{2} \int_0^{\Psi(K)} \rho_1^2(P(\alpha, \Theta(K))) \, d\alpha - \frac{1}{2} \int_0^{\Psi(K)} \rho_1^2(P(\alpha, \Theta(K))) \, d\alpha \right) = 0.
$$

Finally, we show $|O \ast h(AK)| = |O \ast i(AK)|$ similarly. Since

$$
|O \ast h(AK)| = \int_{\{ \hat{x}>0, \hat{y}>0, \hat{z}=0 \}} \chi_{AK}(\hat{P}) \, d\hat{x} d\hat{y},
|O \ast i(AK)| = \int_{\{ \hat{x}<0, \hat{y}>0, \hat{z}=0 \}} \chi_{AK}(\hat{P}) \, d\hat{x} d\hat{y},
$$

by the substitution $P = A^{-1} \hat{P}$, we have

$$
|O \ast h(AK)| - |O \ast i(AK)|
= \int_{\{ \hat{x}>0, \hat{y}>0, \hat{z}=0 \}} \chi_K(A^{-1} \hat{P}) \, d\hat{x} d\hat{y} - \int_{\{ \hat{x}<0, \hat{y}>0, \hat{z}=0 \}} \chi_K(A^{-1} \hat{P}) \, d\hat{x} d\hat{y}
= \int_{\{ x-y/\tan(\Phi(K))>0, y>0, z=0 \}} \chi_K(P) \, dx dy - \int_{\{ x-y/\tan(\Phi(K))<0, y>0, z=0 \}} \chi_K(P) \, dx dy.
$$

By using the same polar coordinates, for the case $y > 0$, $z = 0$, we have $\beta = 0$ and

$$
x - \frac{y}{\tan(\Phi(K))} = r \sin \alpha \left( \frac{1}{\tan \alpha} - \frac{1}{\tan(\Phi(K))} \right),
$$

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Hence, $x - y / \tan(\Phi(K)) > 0$ if and only if $0 < \alpha < \Phi(K)$, and $x - y / \tan(\Phi(K)) < 0$ if and only if $\Phi(K) < \alpha < \pi$. Thus, by the definition of $\Phi(K)$, we obtain

$$|O \ast h(AK)| - |O \ast i(AK)| = \int_{\{r > 0, 0 < \alpha < \Phi(K)\}} \chi_K(P(\alpha, 0)) r \, dr \, d\alpha - \int_{\{r > 0, \Phi(K) < \alpha < \pi\}} \chi_K(P(\alpha, 0)) r \, dr \, d\alpha = \frac{1}{2} \int_0^\pi \rho_K^2(P(\alpha, 0)) \, d\alpha - \frac{1}{2} \int_{\Phi(K)}^\pi \rho_K^2(P(\alpha, 0)) \, d\alpha = 0.$$  

\[\square\]

### 5.3 Definition of a smooth map $(F, G, H)$

For simplicity, we denotes $AK := A(K)K$ and $AK(\theta, \phi, \psi) := A(K(\theta, \phi, \psi)) K(\theta, \phi, \psi) = A(X(\theta)Y(\phi)Z(\psi)K)X(\theta)Y(\phi)Z(\psi)K$.

For every $K \in \hat{K}$, we define real numbers $F, G, H$ by

$$F(K) := |O \ast d(AK)| - |O \ast e(AK)|,$$

$$G(K) := |\Delta_1(AK)| + |\Delta_3(AK)| - |\Delta_2(AK)| - |\Delta_4(AK)|,$$

$$H(K) := |\Delta_1(AK)| + |\Delta_4(AK)| - |\Delta_2(AK)| - |\Delta_3(AK)|,$$

and introduce the following functions on $\mathbb{R}^3$:

$$F(\theta, \phi, \psi) := F(AK(\theta, \phi, \psi)),$$

$$G(\theta, \phi, \psi) := G(AK(\theta, \phi, \psi)),$$

$$H(\theta, \phi, \psi) := H(AK(\theta, \phi, \psi)).$$

If we can find a zero $(\theta, \phi, \psi)$ of $(F, G, H)$, then $AK(\theta, \phi, \psi)$ satisfies the condition [18] by Proposition 5.3. To show the existence of such a zero, we consider the following region in $\mathbb{R}^3$:

$$D = \{(\theta, \phi, \psi) \in \mathbb{R}^3; 0 \leq \phi \leq \pi, 0 \leq \psi \leq \pi, 0 \leq \theta \leq \pi - \Theta(0, \phi, \psi)\}.$$  

Note that $\Theta$ is a smooth function introduced in the previous subsection and $0 < \Theta(0, \phi, \psi) < \pi$. We divide the boundary $\partial D$ into the following six parts:

$$M_1 := \{(0, \phi, \psi) \in \mathbb{R}^3; 0 \leq \phi \leq \pi, 0 \leq \psi \leq \pi\},$$

$$M_2 := \{(\pi - \Theta(0, \phi, \psi), \phi, \psi) \in \mathbb{R}^3; 0 \leq \phi \leq \pi, 0 \leq \psi \leq \pi\},$$

$$M_3 := \{(\theta, 0, \psi) \in \mathbb{R}^3; 0 \leq \psi \leq \pi, 0 \leq \theta \leq \pi - \Theta(0, 0, \psi)\},$$

$$M_4 := \{(\theta, \pi, \psi) \in \mathbb{R}^3; 0 \leq \psi \leq \pi, 0 \leq \theta \leq \pi - \Theta(0, \pi, \psi)\},$$

$$M_5 := \{(\theta, \phi, 0) \in \mathbb{R}^3; 0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi - \Theta(0, \phi, 0)\},$$

$$M_6 := \{(\theta, \phi, \pi) \in \mathbb{R}^3; 0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi - \Theta(0, \phi, \pi)\}.  

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Then, we have
\[ \partial D = M_1 \cup \cdots \cup M_6 \]
and \( M_i \cap M_j \) is empty or a curve for each \( i \neq j \). In the case where \( (F, G, H) \) has no zeros on \( \partial D \), we can use the degree of a map
\[ \mathcal{F} := \frac{(F, G, H)}{\sqrt{F^2 + G^2 + H^2}} : \partial D \to S^2 \]
to find a zero in the interior of \( D \). In order to calculate the degree of \( \mathcal{F} \), in the rest of this section, we show the identities \([25],[26],\) and \([27]\) in Proposition 5.14 below.

5.4 Some formulas

We prepare some formulas.

**Lemma 5.4.** For \( K \in \hat{K} \) and \( \alpha, \beta, \gamma \in \mathbb{R} \),
\begin{align*}
\rho_{(X(\gamma)K)}(P(\alpha, \beta)) &= \rho_K(P(\alpha, \beta - \gamma)), \\
\rho_K(P(\alpha, \beta \pm \pi)) &= \rho_K(P(\pi - \alpha, \beta)).
\end{align*}

**Proof.** By the definition of \( \mu_K \), we have
\[ \mu_{(X(\gamma)K)}(P) = \min\{t \geq 0; P \in tX(\gamma)K\} = \min\{t \geq 0; X(-\gamma)P \in tK\} = \mu_K(X(-\gamma)P). \]

By the definition of \( \rho_K \), we get
\[ \rho_{(X(\gamma)K)}(P) = \rho_K(X(-\gamma)P). \]

Moreover, for \( P = P(\alpha, \beta) \in S^2 \),
\[ X(-\gamma)P(\alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \cos \beta \\ \sin \alpha \sin \beta \end{pmatrix} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \cos(\beta - \gamma) \\ \sin \alpha \sin(\beta - \gamma) \end{pmatrix} = P(\alpha, \beta - \gamma), \]
which proves the first equation.

Next, by a direct calculation, we have \( P(\alpha, \beta \pm \pi) = -P(\pi - \alpha, \beta) \). Since \( K \) is centrally symmetric, we obtain
\[ \rho_K(P(\alpha, \beta \pm \pi)) = \rho_K(-P(\pi - \alpha, \beta)) = \rho_K(P(\pi - \alpha, \beta)). \]

**Lemma 5.5.** \( \Theta(K) + \Theta(X(\pi - \Theta(K))K) = \pi. \)
Proof. For \( \beta_0, \beta_1 \in \mathbb{R} \), by Lemma 5.4 we have

\[
\int_{\beta_0}^{\beta_1} d\beta \int_0^\pi \rho_k^3(P(\alpha, \beta)) \sin \alpha \, d\alpha = \int_{\beta_0}^{\beta_1} d\beta \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\alpha, \beta)) \sin \alpha \, d\alpha \]
\[
= \int_{\beta_0 + \pi - \Theta(K)}^{\beta_1 + \pi - \Theta(K)} d\beta \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\alpha, \tilde{\beta})) \sin \alpha \, d\alpha, \tag{21}
\]

where we used the substitution \( \tilde{\beta} = \beta + \pi - \Theta(K) \). Applying (21) with \((\beta_0, \beta_1) = (0, \Theta(K)), (\Theta(K), \pi)\) to the definition (20) of \( \Theta(K) \), we obtain

\[
\int_0^\pi \tilde{\beta} \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\alpha, \tilde{\beta})) \sin \alpha \, d\alpha = \int_0^{2\pi - \Theta(K)} \tilde{\beta} \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\alpha, \tilde{\beta})) \sin \alpha \, d\alpha \]
\[
= \int_0^{\pi - \Theta(K)} \tilde{\beta} \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\alpha, \tilde{\beta} + \pi)) \sin \alpha \, d\alpha \]
\[
= \int_0^{\pi - \Theta(K)} \tilde{\beta} \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\pi - \alpha, \tilde{\beta})) \sin \alpha \, d\alpha \]
\[
= \int_0^{\pi - \Theta(K)} \tilde{\beta} \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\tilde{\alpha}, \tilde{\beta})) \sin (\pi - \tilde{\alpha}) (-d\tilde{\alpha}) \]
\[
= \int_0^{\pi - \Theta(K)} \tilde{\beta} \int_0^\pi \rho_{X(\pi - \Theta(K))}^3(P(\tilde{\alpha}, \tilde{\beta})) \sin \tilde{\alpha} \, d\tilde{\alpha},
\]

where we used Lemma 5.4 and the substitution \( \tilde{\alpha} = \pi - \alpha \). Since \( \pi - \Theta(K) \in (0, \pi) \), the uniqueness of \( \Theta(X(\pi - \Theta(K))) \) implies \( \Theta(X(\pi - \Theta(K))) = \pi - \Theta(K) \) from (20).

\section{5.5 Identities relating to the rotation \( X(\pi - \Theta(K)) \) and \( X(\pi) \)}

Here we examine the behavior of the functions \( F, G, \) and \( H \) under the rotation \( X \) of \( K \).

\textbf{Lemma 5.6.} Put \( L = X(\pi - \Theta(K))K \). Then

\[ \Theta(L) = \pi - \Theta(K), \quad \Phi(L) = \pi - \Psi(K), \quad \Psi(L) = \Phi(K). \]

Proof. For simplicity, we put \( \tilde{\Theta} = \Theta(L), \Theta = \Theta(K), \tilde{\Phi} = \Phi(L), \Phi = \Phi(K), \tilde{\Psi} = \Psi(L), \Psi = \Psi(K) \). Note that \( \tilde{\Theta} = \pi - \Theta \) by Lemma 5.5, which means the first formula.

By the definition of \( \tilde{\Phi} \), we have

\[
\int_0^\Phi \rho_L^2(P(\alpha, 0)) \, d\alpha = \int_0^\pi \rho_L^2(P(\alpha, 0)) \, d\alpha.
\]

By Lemma 5.4 we get

\[
\rho_L^2(P(\alpha, 0)) = \rho_{X(\pi - \Theta)}^2(K)(P(\alpha, 0)) = \rho_k^2(P(\alpha - \pi)) = \rho_k^2(P(\pi - \alpha, \Theta))
\]

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\[
\int_0^\Phi \rho_L^2(P(\alpha, 0)) \, d\alpha = \int_0^\Phi \rho_K^2(P(\pi - \alpha, \Theta)) \, d\alpha = \int_{\pi - \Phi}^\pi \rho_K^2(P(\tilde{\alpha}, \Theta)) \, d\tilde{\alpha},
\]
where we used the substitution \( \tilde{\alpha} = \pi - \alpha \). Similarly,

\[
\int_\Phi^\pi \rho_L^2(P(\alpha, 0)) \, d\alpha = \int_\Phi^\pi \rho_K^2(P(\pi - \alpha, \Theta)) \, d\alpha = \int_0^{\pi - \Phi} \rho_K^2(P(\tilde{\alpha}, \Theta)) \, d\tilde{\alpha}.
\]

Consequently, we have

\[
\int_{\pi - \Phi}^\pi \rho_L^2(P(\alpha, 0)) \, d\alpha = \int_{\pi - \Phi}^\pi \rho_K^2(P(\alpha, 0)) \, d\alpha.
\]

By the uniqueness of \( \Psi = \Psi(K) \), we see \( \Psi = \pi - \tilde{\Phi} \). Similarly, by the definition of \( \tilde{\Psi} \),

\[
\int_0^\Psi \rho_L^2(P(\alpha, 0)) \, d\alpha = \int_\Psi^\pi \rho_K^2(P(\alpha, 0)) \, d\alpha
\]
holds. By Lemma 5.4, we have

\[
\rho_L^2(P(\alpha, \tilde{\Theta})) = \rho_{X(\pi - \Theta)K}(P(\alpha, \pi - \Theta)) = \rho_K^2(P(\alpha, 0)).
\]

Thus we get

\[
\int_0^\Psi \rho_K^2(P(\alpha, 0)) \, d\alpha = \int_\Psi^\pi \rho_K^2(P(\alpha, 0)) \, d\alpha.
\]

By the uniqueness of \( \Phi = \Phi(K) \), we obtain \( \tilde{\Psi} = \Phi \).

**Lemma 5.7.**

\[
F(X(\pi - \Theta)K)K = -F(K),
\]

\[
G(X(\pi - \Theta)K)K = -H(K),
\]

\[
H(X(\pi - \Theta)K)K = G(K).
\]

**Proof.** We use the same notations as in Lemma 5.6. Putting \( \tilde{A} = A(L) \), \( A = A(K) \), we have

\[
\tilde{A}L = \tilde{A}X(\pi - \Theta)K = \left( \tilde{A}X(\pi - \Theta)A^{-1} \right)AK.
\]

By a direct calculation using Lemma 5.6,

\[
(\tilde{A}X(\pi - \Theta)A^{-1})^{-1} = AX(\Theta - \pi)\tilde{A}^{-1}
\]

\[
= \begin{pmatrix} 1 & 1/\tan \Phi & 1/(\sin \Theta \tan \Psi) \\ 0 & 1 & 1/\tan \Theta \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Theta - \pi) & -\sin(\Theta - \pi) \\ 0 & \sin(\Theta - \pi) & \cos(\Theta - \pi) \end{pmatrix}
\]

\[
\times \begin{pmatrix} 1 & 1/\tan \tilde{\Phi} & 1/(\sin \tilde{\Theta} \tan \tilde{\Psi}) \\ 0 & 1 & 1/\tan \tilde{\Theta} \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/\sin \Theta \\ 0 & -\sin \Theta & 0 \end{pmatrix}.
\]
Thus, for any $\tilde{P} = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, we have
\[
\chi_{AL}(\tilde{P}) = \chi_{A_X(\pi-\Theta)A^{-1}AK}(\tilde{P}) \\
= \chi_{AK}(A_X(\pi-\Theta)A^{-1}\tilde{P}) \\
= \chi_{AK}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 / \sin \Theta & 0 \\ 0 & -\sin \Theta & 0 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}\right) = \chi_{AK}\left(\begin{pmatrix} \tilde{x} \\ \tilde{y} / \sin \Theta \\ -\tilde{z} \end{pmatrix}\right).
\]

By the substitution $x = \tilde{x}$, $y = \tilde{y} / \sin \Theta$, $z = -\tilde{z} \sin \Theta$,
\[
|\Delta_1(\tilde{A}L)| = \int_{\{\tilde{x} > 0, \tilde{y} > 0, \tilde{z} > 0\}} \chi_{AL}(\tilde{P}) \, d\tilde{x} d\tilde{y} d\tilde{z} \\
= \int_{\{x > 0, y > 0, z > 0\}} \chi_{AK}(P) \, dx dy dz = |\Delta_5(\tilde{A}K)| = |\Delta_3(\tilde{A}K)|,
\]
where the last equality comes from the fact that $AK$ is centrally symmetric. Similarly, we obtain
\[
|\Delta_2(\tilde{A}L)| = \int_{\{\tilde{x} < 0, \tilde{y} > 0, \tilde{z} > 0\}} \chi_{AL}(\tilde{P}) \, d\tilde{x} d\tilde{y} d\tilde{z} \\
= \int_{\{x < 0, y > 0, z > 0\}} \chi_{AK}(P) \, dx dy dz = |\Delta_6(\tilde{A}K)| = |\Delta_4(\tilde{A}K)|,
\]
\[
|\Delta_3(\tilde{A}L)| = \int_{\{\tilde{x} < 0, \tilde{y} < 0, \tilde{z} > 0\}} \chi_{AL}(\tilde{P}) \, d\tilde{x} d\tilde{y} d\tilde{z} \\
= \int_{\{x < 0, y < 0, z > 0\}} \chi_{AK}(P) \, dx dy dz = |\Delta_2(\tilde{A}K)|,
\]
\[
|\Delta_4(\tilde{A}L)| = \int_{\{\tilde{x} < 0, \tilde{y} < 0, \tilde{z} > 0\}} \chi_{AL}(\tilde{P}) \, d\tilde{x} d\tilde{y} d\tilde{z} \\
= \int_{\{x < 0, y < 0, z > 0\}} \chi_{AK}(P) \, dx dy dz = |\Delta_1(\tilde{A}K)|.
\]
Since $d\tilde{y} d\tilde{z} = dy dz$, we also obtain
\[
|O \ast d(\tilde{A}L)| = \int_{\{\tilde{x} = 0, \tilde{y} > 0, \tilde{z} > 0\}} \chi_{AL}(\tilde{P}) \, d\tilde{y} d\tilde{z} \\
= \int_{\{x = 0, y > 0, z > 0\}} \chi_{AK}(P) \, dy dz = |O \ast e(\tilde{A}K)| = |O \ast e(\tilde{A}K)|,
\]
\[
|O \ast e(\tilde{A}L)| = \int_{\{\tilde{x} = 0, \tilde{y} < 0, \tilde{z} > 0\}} \chi_{AL}(\tilde{P}) \, d\tilde{y} d\tilde{z} \\
= \int_{\{x = 0, y < 0, z > 0\}} \chi_{AK}(P) \, dy dz = |O \ast d(\tilde{A}K)|.
\]
Lemma 5.8.

\[ F(L) = |O \ast d(\bar{L})| - |O \ast e(\bar{L})| = |O \ast e(\bar{L})| - |O \ast d(\bar{L})| \]
\[ G(L) = |\Delta_3(\bar{L})| + |\Delta_4(\bar{L})| - |\Delta_1(\bar{L})| - |\Delta_2(\bar{L})| \]
\[ H(L) = |\Delta_1(\bar{L})| + |\Delta_4(\bar{L})| - |\Delta_2(\bar{L})| - |\Delta_3(\bar{L})| \]

Therefore,

\[ X = X(\pi)K = X(2\pi - \pi)K = X(2\pi - \Theta(L) - \Theta(K))K = X(\pi - \Theta(L))X(\pi - \Theta(K))K = X(\pi - \Theta(L))L, \]

and so we can apply Lemmas 5.6 and 5.7 to \( X(\pi)K = X(\pi - \Theta(L))L \). By Lemma 5.6 we have

\[ \Theta(X(\pi)K) = \Theta(K), \quad \Phi(X(\pi)K) = \pi - \Theta(K), \quad \Psi(X(\pi)K) = \pi - \Psi(K), \]

It follows from Lemma 5.7 that

\[ F(X(\pi)K) = F(X(\pi - \Theta(L))L) = -F(L) = F(K), \]
\[ G(X(\pi)K) = G(X(\pi - \Theta(L))L) = -H(L) = -G(K), \]
\[ H(X(\pi)K) = H(X(\pi - \Theta(L))L) = G(L) = -H(K). \]

Lemma 5.9. Put \( L = Y(\pi)K \). Then

\[ \Theta(L) = \pi - \Theta(K), \quad \Phi(L) = \pi - \Phi(K), \quad \Psi(L) = \Psi(K). \]
Proof. For simplicity, we put \( \tilde{\Theta} = \Theta(L) \), \( \Theta = \Theta(K) \), \( \tilde{\Phi} = \Phi(L) \), \( \Phi = \Phi(K) \), \( \tilde{\Psi} = \Psi(L) \), \( \Psi = \Psi(K) \). We first claim that

\[
\rho_L(P(\alpha, \beta)) = \rho_K(P(\alpha, \pi - \beta)).
\] (22)

Actually, \( Y(-\pi)P(\alpha, \beta) = -P(\alpha, \pi - \beta) \) holds by a direct calculation. Thus we have

\[
\rho_L(P(\alpha, \beta)) = \rho_Y(\pi)P(\alpha, \beta) = \rho_K(Y(\pi)P(\alpha, \beta)) = \rho_K(-P(\alpha, \pi - \beta)).
\]

Since \( K \) is centrally symmetric, the claim (22) holds.

By (22) and the definition of \( \tilde{\Theta} \),

\[
\int_{\tilde{\Theta}}^\pi d\beta \int_0^\pi \rho^3_K(P(\alpha, \pi - \beta)) \sin \alpha d\alpha = \int_{\tilde{\Theta}}^\pi d\beta \int_0^\pi \rho^3_K(P(\alpha, \pi - \beta)) \sin \alpha d\alpha.
\]

By the substitution \( \tilde{\beta} = \pi - \beta \),

\[
\int_{\tilde{\Theta}}^\pi d\beta \int_0^\pi \rho^3_K(P(\alpha, \tilde{\beta})) \sin \alpha d\alpha = \int_{\tilde{\Theta}}^\pi d\beta \int_0^\pi \rho^3_K(P(\alpha, \tilde{\beta})) \sin \alpha d\alpha.
\]

By the definition and uniqueness of \( \Theta \), we get \( \Theta = \pi - \tilde{\Theta} \). By the definition of \( \tilde{\Phi} \),

\[
\int_{\tilde{\Phi}}^\pi \rho^2_L(P(\alpha, 0)) d\alpha = \int_{\tilde{\Phi}}^\pi \rho^2_K(P(\alpha, 0)) d\alpha.
\]

By Lemma 5.4, \( \rho_K(P(\alpha, \pi - \beta)) = \rho_K(P(\pi - \alpha, -\beta)) \) holds. It then follows from (22) that

\[
\int_{\tilde{\Phi}}^\phi \rho^2_L(P(\alpha, 0)) d\alpha = \int_{\tilde{\Phi}}^\phi \rho^2_K(P(\pi - \alpha, 0)) d\alpha = \int_{\pi - \tilde{\Phi}}^\phi \rho^2_K(P(\alpha, 0)) d\alpha = \int_{\pi - \tilde{\Phi}}^\phi \rho^2_K(P(\alpha, 0)) d\alpha,
\]

where we used the substitution \( \tilde{\alpha} = \pi - \alpha \). Therefore, by the definition of \( \Phi \), \( \Phi = \pi - \tilde{\Phi} \). Similarly, by the definition of \( \tilde{\Psi} \),

\[
\int_{\tilde{\Psi}}^\Psi \rho^2_L(P(\alpha, \tilde{\Theta})) d\alpha = \int_{\tilde{\Psi}}^\Psi \rho^2_L(P(\alpha, \tilde{\Theta})) d\alpha.
\]

By (22) and \( \Theta = \pi - \tilde{\Theta} \), we have

\[
\rho^2_L(P(\alpha, \tilde{\Theta})) = \rho^2_K(P(\alpha, \pi - \tilde{\Theta})) = \rho^2_K(P(\alpha, \Theta)).
\]

Thus

\[
\int_{\tilde{\Psi}}^\Psi \rho^2_K(P(\alpha, \Theta)) d\alpha = \int_{\tilde{\Psi}}^\Psi \rho^2_K(P(\alpha, \Theta)) d\alpha,
\]

which means that \( \Psi = \Psi \).

\[\square\]

Lemma 5.10.

\[
F(Y(\pi)K) = -F(K), \quad G(Y(\pi)K) = -G(K), \quad H(Y(\pi)K) = H(K).
\]
Proof. We use the same notations as in Lemma 5.9. Putting $\tilde{A} = A(L), A = A(K)$, we have

$$\tilde{A}L = \tilde{A}Y(\pi)K = (\tilde{A}Y(\pi)A^{-1})AK.$$  

Thus

$$\chi_{\tilde{A}L}(P) = \chi_{AK}((\tilde{A}Y(\pi)A^{-1})^{-1}P) = \chi_{AK}(\tilde{A}Y(-\pi)\tilde{A}^{-1}P).$$  

By Lemma 5.9 and a direct calculation,

$$\tilde{A}Y(-\pi)\tilde{A}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Thus, for any $\tilde{P} = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, we have

$$|\Delta_1(\tilde{A}L)| = \int_{\{\tilde{x} > 0, \tilde{y} > 0, \tilde{z} > 0\}} \chi_{\tilde{A}L}(\tilde{P}) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z}$$

$$= \int_{\{\tilde{x} > 0, \tilde{y} > 0, \tilde{z} > 0\}} \chi_{AK}(P) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z} = |\Delta_6(AK)| = |\Delta_4(AK)|,$$

where we used the substitution $x = -\tilde{x}, y = \tilde{y}, z = -\tilde{z}$. Similarly, we have

$$|\Delta_2(\tilde{A}L)| = \int_{\{\tilde{x} < 0, \tilde{y} > 0, \tilde{z} > 0\}} \chi_{\tilde{A}L}(\tilde{P}) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z}$$

$$= \int_{\{\tilde{x} > 0, \tilde{y} > 0, \tilde{z} < 0\}} \chi_{AK}(P) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z} = |\Delta_5(AK)| = |\Delta_3(AK)|,$$

$$|\Delta_3(\tilde{A}L)| = \int_{\{\tilde{x} < 0, \tilde{y} < 0, \tilde{z} > 0\}} \chi_{\tilde{A}L}(\tilde{P}) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z}$$

$$= \int_{\{\tilde{x} > 0, \tilde{y} > 0, \tilde{z} < 0\}} \chi_{AK}(P) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z} = |\Delta_6(AK)| = |\Delta_4(AK)|,$$

$$|\Delta_4(\tilde{A}L)| = \int_{\{\tilde{x} < 0, \tilde{y} < 0, \tilde{z} > 0\}} \chi_{\tilde{A}L}(\tilde{P}) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z}$$

$$= \int_{\{\tilde{x} > 0, \tilde{y} < 0, \tilde{z} < 0\}} \chi_{AK}(P) \tilde{d}\tilde{x}\tilde{y}\tilde{d}\tilde{z} = |\Delta_7(AK)| = |\Delta_1(AK)|.$$
Therefore,
\[
F(L) = |O * d(\bar{A}L)| - |O * e(\bar{A}L)| \\
= |O * e(\bar{A}K)| - |O * d(\bar{A}K)| = -F(K),
\]
\[
G(L) = |\Delta_1(\bar{A}L)| + |\Delta_3(\bar{A}L)| - |\Delta_2(\bar{A}L)| - |\Delta_4(\bar{A}L)| \\
= |\Delta_4(\bar{A}K)| + |\Delta_2(\bar{A}K)| - |\Delta_3(\bar{A}K)| - |\Delta_4(\bar{A}K)| = -G(K),
\]
\[
H(L) = |\Delta_1(\bar{A}L)| + |\Delta_4(\bar{A}L)| - |\Delta_2(\bar{A}L)| - |\Delta_3(\bar{A}L)| \\
= |\Delta_4(\bar{A}K)| + |\Delta_1(\bar{A}K)| - |\Delta_3(\bar{A}K)| - |\Delta_2(\bar{A}K)| = H(K).
\]

\[\square\]

5.7 Identities relating to the rotation \( Z(\pi) \)

Lemma 5.11. Put \( L = Z(\pi)K \). Then
\[
\Theta(L) = \pi - \Theta(K), \quad \Phi(L) = \Phi(K), \quad \Psi(L) = \pi - \Psi(K).
\]

Proof. We put \( \tilde{\Theta} = \Theta(L) \), \( \tilde{\Theta} = \Theta(K) \), \( \tilde{\Phi} = \Phi(L) \), \( \tilde{\Phi} = \Phi(K) \), \( \tilde{\Psi} = \Psi(L) \), \( \Psi = \Psi(K) \). Since \( Z(-\pi)P(\alpha, \beta) = P(\pi - \alpha, \pi - \beta) \), we have
\[
\rho_L(P(\alpha, \beta)) = \rho_{Z(\pi)K}(P(\alpha, \beta)) = \rho_K(Z(-\pi)P(\alpha, \beta)) = \rho_K(P(\pi - \alpha, \pi - \beta)).
\]

It follows from Lemma 5.4 that
\[
\rho_L(P(\alpha, \beta)) = \rho_K(P(\pi - \alpha, \pi - \beta)) = \rho_K(P(\pi - \alpha, \beta)). \quad (23)
\]

By (23) and the definition of \( \tilde{\Theta} \), we have
\[
\int_0^\tilde{\Theta} d\beta \int_0^\pi \rho_K^{\beta}(P(\pi - \alpha, \pi - \beta)) \sin \alpha d\alpha = \int_\tilde{\Theta}^\pi d\beta \int_0^\pi \rho_K^{\beta}(P(\pi - \alpha, \pi - \beta)) \sin \alpha d\alpha.
\]

By the substitution \( \tilde{\alpha} = \pi - \alpha, \tilde{\beta} = \pi - \beta \),
\[
\int_{\pi - \tilde{\Theta}}^\pi d\tilde{\beta} \int_0^\pi \rho_K^{\beta}(P(\tilde{\alpha}, \tilde{\beta})) \sin \tilde{\alpha} d\tilde{\alpha} = \int_0^{\pi - \tilde{\Theta}} d\tilde{\beta} \int_0^\pi \rho_K^{\beta}(P(\tilde{\alpha}, \tilde{\beta})) \sin \tilde{\alpha} d\tilde{\alpha}.
\]

By the definition of \( \Theta \), we get \( \Theta = \pi - \tilde{\Theta} \). Furthermore, we have
\[
\int_0^\Phi \rho_K^2(P(\alpha, 0)) d\alpha = \int_0^\Phi \rho_L^2(P(\alpha, 0)) d\alpha = \int_\Phi^\pi \rho_L^2(P(\alpha, 0)) d\alpha = \int_\Phi^\pi \rho_K^2(P(\alpha, 0)) d\alpha.
\]

By the definition of \( \Phi \), we obtain \( \Phi = \tilde{\Phi} \). Finally, the definition of \( \tilde{\Psi} \) means
\[
\int_0^{\tilde{\Psi}} \rho_L^2(P(\alpha, \tilde{\Theta})) d\alpha = \int_{\tilde{\Psi}}^\pi \rho_L^2(P(\alpha, \tilde{\Theta})) d\alpha.
\]

By (23) and \( \Theta = \pi - \tilde{\Theta} \), we obtain
\[
\rho_L^2(P(\alpha, \tilde{\Theta})) = \rho_K^2(P(\pi - \alpha, \pi - \tilde{\Theta})) = \rho_K^2(P(\pi - \alpha, \Theta)).
\]

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Thus
\[
\int_0^\Psi \rho_\pi^2(P(\alpha, \Theta)) \, d\alpha = \int_0^\Psi \rho_\pi^2(P(\pi - \alpha, \Theta)) \, d\alpha = \int_{\pi-\Psi}^\Psi \rho_\pi^2(P(\tilde\alpha, \Theta)) \, d\tilde\alpha,
\]
\[
\int_0^\Psi \rho_{\tilde\pi}^2(P(\alpha, \Theta)) \, d\alpha = \int_0^\Psi \rho_{\tilde\pi}^2(P(\pi - \alpha, \Theta)) \, d\alpha = \int_{\pi-\Psi}^\Psi \rho_{\tilde\pi}^2(P(\tilde\alpha, \Theta)) \, d\tilde\alpha.
\]
Therefore, \( \pi - \tilde\Psi = \Psi \).

**Lemma 5.12.**

\[
F(Z(\pi)K) = -F(K), \quad G(Z(\pi)K) = G(K), \quad H(Z(\pi)K) = -H(K).
\]

**Proof.** We use the same notations as in Lemma 5.11. Putting \( \tilde{A} = A(L) \), \( A = A(K) \), we have

\[
\tilde{A}L = \tilde{A}Z(\pi)K = (\tilde{A}Z(\pi)A^{-1})AK.
\]

Then
\[
\chi_{\tilde{A}L}(P) = \chi_{AK}((\tilde{A}Z(\pi)A^{-1})^{-1}P) = \chi_{AK}(AZ(-\pi)\tilde{A}^{-1}P).
\]

By Lemma 5.11 and a direct calculation,

\[
AZ(-\pi)\tilde{A}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Thus, for any \( \tilde{P} = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3 \), we see

\[
|\Delta_1(\tilde{A}L)| = \int_{\{\tilde{x}>0, \tilde{y}>0, \tilde{z}>0\}} \chi_{\tilde{A}L}(\tilde{P}) \, d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int_{\{\tilde{x}>0, \tilde{y}>0, \tilde{z}>0\}} \chi_{AK} \left( \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \right) \, d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int_{\{\tilde{x}>0, \tilde{y}>0, \tilde{z}>0\}} \chi_{AK} \left( \begin{pmatrix} -\tilde{x} \\ -\tilde{y} \\ \tilde{z} \end{pmatrix} \right) \, d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int_{\{x>0, y>0, z>0\}} \chi_{AK}(P) \, dxdydz = |\Delta_3(AK)|,
\]

where we used the substitution \( x = -\tilde{x}, y = -\tilde{y}, z = \tilde{z} \). Similarly, we obtain

\[
|\Delta_2(\tilde{A}L)| = \int_{\{x>0, y>0, z>0\}} \chi_{\tilde{A}L}(\tilde{P}) \, d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int_{\{x>0, y>0, z>0\}} \chi_{AK}(P) \, dxdydz = |\Delta_4(AK)|,
\]

\[
|\Delta_3(\tilde{A}L)| = \int_{\{x<0, y<0, z>0\}} \chi_{\tilde{A}L}(\tilde{P}) \, d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int_{\{x>0, y>0, z>0\}} \chi_{AK}(P) \, dxdydz = |\Delta_1(AK)|,
\]

\[
|\Delta_4(\tilde{A}L)| = \int_{\{x<0, y<0, z>0\}} \chi_{\tilde{A}L}(\tilde{P}) \, d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int_{\{x<0, y<0, z>0\}} \chi_{AK}(P) \, dxdydz = |\Delta_2(AK)|.
\]

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Moreover, we have

\[ |O * d(\bar{AL})| = \int_{\{y,z>0\}} \chi_{\bar{AL}}(\bar{P}) \, dydz \]

\[ = \int_{\{y,z>0\}} \chi_A L(\bar{P}) \, dydz = |O * e(\bar{AK})|, \]

\[ |O * e(\bar{AL})| = \int_{\{z>0\}} \chi_{\bar{AL}}(\bar{P}) \, dydz = |O * d(\bar{AK})| \]

Therefore,

\[ F(L) = |O * d(\bar{AL})| - |O * e(\bar{AL})| \]

\[ = |O * e(\bar{AK})| - |O * d(\bar{AK})| = -F(K), \]

\[ G(L) = |\Delta_1(\bar{AL})| + |\Delta_3(\bar{AL})| - |\Delta_2(\bar{AL})| - |\Delta_4(\bar{AL})| \]

\[ = |\Delta_3(\bar{AK})| + |\Delta_1(\bar{AK})| - |\Delta_4(\bar{AK})| - |\Delta_2(\bar{AK})| = G(K), \]

\[ H(L) = |\Delta_1(\bar{AL})| + |\Delta_4(\bar{AL})| - |\Delta_2(\bar{AL})| - |\Delta_3(\bar{AL})| \]

\[ = |\Delta_3(\bar{AK})| + |\Delta_2(\bar{AK})| - |\Delta_4(\bar{AK})| - |\Delta_1(\bar{AK})| = -H(K). \]

\[ \square \]

5.8 Key identities on \( \partial D \)

By using the lemmas prepared in the above subsections, we show Proposition 5.14 below, which states key identities on \( \partial D \) for the degree calculation in the next section. To begin with, we introduce the map \( \Gamma \) defined by

\[ \Gamma_\psi(\theta) := \pi - \Theta(\theta, 0, \psi) + \theta. \]

**Lemma 5.13.** \( \Gamma \) satisfies the following properties:

(i). \( \theta \mapsto \Gamma_\psi(\theta) \) is increasing.

(ii). \( \Gamma_\psi([0, \pi - \Theta(0, 0, \psi)]) = [\pi - \Theta(0, 0, \psi), \pi] \).

(iii). \( \Gamma_\psi, (\Gamma_\psi)^{-1} \) is smooth with respect to \( \theta \) and \( \psi \).

**Proof.** By Lemma 5.2, \( \Gamma_\psi(\theta) \) is smooth with respect to \( \theta \) and \( \psi \). We claim that

\[ \frac{\partial \Gamma_\psi}{\partial \theta}(\theta) = -\frac{\partial \Theta}{\partial \theta}(\theta, 0, \psi) + 1 > 0. \]  \hspace{1cm} (24)

If the inequality (24) holds, then the properties (i), (ii), and (iii) are easily verified. Indeed, (i) follows immediately. Note that \( \Gamma_\psi(0) = \pi - \Theta(0, 0, \psi) \). By Lemma 5.5 we have

\[ \Gamma_\psi(\pi - \Theta(0, 0, \psi)) = \pi - \Theta(\pi - \Theta(0, 0, \psi), 0, \psi) + \pi - \Theta(0, 0, \psi) \]

\[ = \pi - \Theta(X(\pi - \Theta(0, 0, \psi))K(0, 0, \psi)) + \pi - \Theta(0, 0, \psi) = \pi, \]
which proves (ii), because $\Gamma_\psi$ is increasing by (i). Moreover, $\Gamma_\psi^{-1}$ is smooth by the implicit function theorem.

Therefore, it suffices to prove (24). By the definition of $\Theta$,

$$0 = \int_{\Theta(\theta,0,\psi)}^{\Theta(\theta,\theta,\psi)} d\beta \int_0^\pi \rho_K^3(\theta,\theta,\psi)(P(\alpha,\beta)) \sin \alpha d\alpha$$

By Lemma 5.4, we have

$$\rho_K(0,0,\psi)(P(\alpha,\beta)) = \rho_K(0,0,\psi)(P(\alpha,\beta - \theta)).$$

Thus, by the substitution $\tilde{\beta} = \beta - \theta$, we obtain

$$0 = \int_{\Theta(\theta,0,\psi)}^{\Theta(\theta,\theta,\psi)} d\tilde{\beta} \int_0^\pi \rho_K^3(\theta,\theta,\psi)(P(\alpha,\tilde{\beta})) \sin \alpha d\alpha$$

By differentiating it with respect to $\theta$,

$$0 = 2 \left( \frac{\partial \Theta}{\partial \theta} (\theta, 0, \psi) - 1 \right) \int_0^\pi \rho_K^3(\theta,0,\psi)(P(\alpha,\Theta(\theta,0,\psi) - \theta)) \sin \alpha d\alpha$$

$$+ \int_0^\pi \rho_K^3(\theta,0,\psi)(P(\alpha,-\theta)) \sin \alpha d\alpha + \int_0^\pi \rho_K^3(\theta,0,\psi)(P(\alpha,\pi - \theta)) \sin \alpha d\alpha,$$

so that

$$- \frac{\partial \Theta}{\partial \theta} (\theta, 0, \psi) + 1$$

$$= \frac{\int_0^\pi \rho_K^3(\theta,0,\psi)(P(\alpha,-\theta)) \sin \alpha d\alpha + \int_0^\pi \rho_K^3(\theta,0,\psi)(P(\alpha,\pi - \theta)) \sin \alpha d\alpha}{2 \int_0^\pi \rho_K^3(\theta,0,\psi)(P(\alpha,\Theta(\theta,0,\psi) - \theta)) \sin \alpha d\alpha} > 0.$$

It completes the proof.

**Proposition 5.14.** Between $(0, \phi, \psi) \in M_1$ and $(\pi - \Theta(0, \phi, \psi), \phi, \psi) \in M_2$, the following formulas hold:

$$F(\pi - \Theta(0, \phi, \psi), \phi, \psi) = -F(0, \phi, \psi),$$

$$G(\pi - \Theta(0, \phi, \psi), \phi, \psi) = -H(0, \phi, \psi),$$

$$H(\pi - \Theta(0, \phi, \psi), \phi, \psi) = G(0, \phi, \psi).$$

(25)

Between $\tilde{\theta}, 0, \psi) \in M_3$ and $(\theta, \pi, \psi) \in M_4$, the following formulas hold:

$$F(\theta, \pi, \psi) = F(\tilde{\theta}, 0, \psi),$$

$$G(\theta, \pi, \psi) = -H(\tilde{\theta}, 0, \psi),$$

$$H(\theta, \pi, \psi) = -G(\tilde{\theta}, 0, \psi).$$

(26)
Proof. First, we show (25). We have

\[ F(\theta, \phi, \pi) = -F(-\theta, -\phi, 0) = F(\theta, \pi - \phi, 0), \]
\[ G(\theta, \phi, \pi) = G(-\theta, -\phi, 0) = -G(\theta, \pi - \phi, 0), \]
\[ H(\theta, \phi, \pi) = -H(-\theta, -\phi, 0) = -H(\theta, \pi - \phi, 0). \]  

(27)

Noting \( \Theta(\phi, \pi, \psi) = \Theta(K(\phi, \pi, \psi)) \), Lemma 5.7 implies

\[ F(\pi - \Theta(\theta, \phi, \psi) + \theta, \phi, \psi) = F(\theta, \phi, \psi), \]
\[ G(\pi - \Theta(\theta, \phi, \psi) + \theta, \phi, \psi) = -H(\theta, \phi, \psi), \]
\[ H(\pi - \Theta(\theta, \phi, \psi) + \theta, \phi, \psi) = G(\theta, \phi, \psi). \]  

(28)

Putting \( \theta = 0 \), we get (25).

Next, let us consider the case \((\theta, \pi, \psi) \in M_4\). Since \( X(\theta)Y(\pi) = Y(\pi)X(-\theta), \)
\( Y(\phi)Z(\pi) = Z(\pi)Y(-\phi), \) and \( X(\theta)Z(\pi) = Z(\pi)X(\theta) \), we have

\[ K(\theta, \pi, \psi) = X(\theta)Y(\pi)Z(\psi)K \]
\[ = Y(\pi)X(-\theta)Y(0)Z(\psi)K \]
\[ = Y(\pi)K(-\theta, 0, \psi). \]

By Lemma 5.10 we have

\[ F(\theta, \pi, \psi) = -F(-\theta, 0, \psi), \quad G(\theta, \pi, \psi) = -G(-\theta, 0, \psi), \quad H(\theta, \pi, \psi) = H(-\theta, 0, \psi). \]

Since \( F, G, H \) are \( 2\pi \)-periodic with respect to \( \theta \), we use Lemma 5.8 to obtain

\[ F(-\theta, 0, \psi) = F(\pi + \pi - \theta, 0, \psi), \]
\[ G(-\theta, 0, \psi) = G(\pi + \pi - \theta, 0, \psi), \]
\[ H(-\theta, 0, \psi) = H(\pi + \pi - \theta, 0, \psi). \]

Here, by the definition of \( M_4 \), \( 0 \leq \theta \leq \pi - \Theta(0, \pi, \psi) \) holds, and hence \( \Theta(0, \pi, \psi) \leq \pi - \theta \leq \pi \). On the other hand, by Lemma 5.9 we have

\[ \Theta(0, \pi, \psi) = \pi - \Theta(0, 0, \psi). \]  

(29)

Thus \( \pi - \Theta(0, 0, \psi) \leq \pi - \theta \leq \pi \). For each \( \pi - \theta \in [\pi - \Theta(0, 0, \psi), \pi] \), putting \( \tilde{\theta} = (\Gamma_{\psi})^{-1}(\pi - \theta) \in [0, \pi - \Theta(0, 0, \psi)] \) \[ \text{[see Lemma 5.13]}, \text{by using (28), we get} \]
\[ -F(\pi - \theta, 0, \psi) = -F(\Gamma_{\psi}(\tilde{\theta}), 0, \psi), \]
\[ G(\pi - \theta, 0, \psi) = G(\Gamma_{\psi}(\tilde{\theta}), 0, \psi), \]
\[ -H(\pi - \theta, 0, \psi) = -H(\Gamma_{\psi}(\tilde{\theta}), 0, \psi). \]
which imply (26). Finally, we consider the case \( (\theta, \phi, \pi) \in M_6 \). Note that
\[
K(\theta, \phi, \pi) = X(\theta)Y(\phi)Z(\pi)K
= X(\theta)Z(\pi)Y(-\phi)K
= Z(\pi)X(-\theta)Y(-\phi)Z(0)K
= Z(\pi)K(-\theta, -\phi, 0),
\]
\[
K(-\theta, -\phi, 0) = X(-\theta)Y(2\pi - \phi)Z(0)K
= Y(\pi)X(\theta)Y(\pi - \phi)Z(0)K
= Y(\pi)K(\theta, \pi - \phi, 0).
\]

Thus, by Lemmas 5.12 and 5.10 we obtain
\[
F(\theta, \phi, \pi) = -F(-\theta, -\phi, 0) = F(\theta, \pi - \phi, 0),
G(\theta, \phi, \pi) = G(-\theta, -\phi, 0) = -G(\theta, \pi - \phi, 0),
H(\theta, \phi, \pi) = -H(-\theta, -\phi, 0) = -H(\theta, \pi - \phi, 0).
\]

Since \( K(0, \phi, \pi) = Z(\pi)Y(\pi)K(0, \pi - \phi, 0) \), by the definition of \( \Theta \), we have
\[
\Theta(0, \phi, \pi) = \Theta(0, \pi - \phi, 0).
\] (30)

It means that \( (\theta, \phi, \pi) \in M_6 \) (\( \theta \in [0, \pi - \Theta(0, \phi, \pi)] \)) if and only if \( (\theta, \pi - \phi, 0) \in M_5 \) \( (\theta \in [0, \pi - \Theta(0, \pi - \phi, 0)] \)). Hence (27) is verified.

\section{Calculation of degree}  
\subsection{Setting}  
We use new coordinates \((s, \phi, \psi)\) instead of \((\theta, \phi, \psi)\), where
\[
s = \frac{\theta}{\pi - \Theta(0, \phi, \psi)}.
\]

For the new coordinates \((s, \phi, \psi)\), we see that
\[
M_1 = \{0\} \times [0, \pi] \times [0, \pi], \quad M_3 = [0, 1] \times \{0\} \times [0, \pi], \quad M_5 = [0, 1] \times [0, \pi] \times \{0\},
M_2 = \{1\} \times [0, \pi] \times [0, \pi], \quad M_4 = [0, 1] \times [\pi] \times [0, \pi], \quad M_6 = [0, 1] \times [0, \pi] \times \{\pi\}.
\]

We define
\[
\hat{F}(s, \phi, \psi) := F((\pi - \Theta(0, \phi, \psi))s, \phi, \psi),
\hat{G}(s, \phi, \psi) := G((\pi - \Theta(0, \phi, \psi))s, \phi, \psi),
\hat{H}(s, \phi, \psi) := H((\pi - \Theta(0, \phi, \psi))s, \phi, \psi).
\]

Note that
\[
D = \{(s, \phi, \psi) \in \mathbb{R}^3; 0 \leq s \leq 1, 0 \leq \phi \leq \pi, 0 \leq \psi \leq \pi \}
\]
and \((\hat{F}, \hat{G}, \hat{H}) \in C^\infty(D, \mathbb{R}^3)\).
Then \((\hat{F}, \hat{G}, \hat{H})\) is a triplet of continuous functions on \(\partial D = M_1 \cup \cdots \cup M_6\), of class \(C^{\infty}\) in the interior of each \(M_i\) for \(i = 1, \ldots, 6\). Hence, \((\hat{F}, \hat{G}, \hat{H})\) can be regarded as a piecewise smooth vector field on \(\partial D\). It follows from Proposition 5.14, (29), and (30) that \((\hat{F}, \hat{G}, \hat{H})\) satisfies the identities:

\[
\begin{align*}
\hat{F}(1, \phi, \psi) &= -\hat{F}(0, \phi, \psi), \\
\hat{G}(1, \phi, \psi) &= -\hat{H}(0, \phi, \psi), \\
\hat{H}(1, \phi, \psi) &= \hat{G}(0, \phi, \psi)
\end{align*}
\]

for \(\phi, \psi \in [0, \pi]\) and

\[
\begin{align*}
\hat{F}(s, \pi, \psi) &= \hat{F}(T_\psi(s), 0, \psi), \\
\hat{G}(s, \pi, \psi) &= -\hat{H}(T_\psi(s), 0, \psi), \\
\hat{H}(s, \pi, \psi) &= -\hat{G}(T_\psi(s), 0, \psi)
\end{align*}
\]

for \(s \in [0, 1], \psi \in [0, \pi]\) and

\[
\begin{align*}
\hat{F}(s, \phi, \pi) &= \hat{F}(s, \pi - \phi, 0), \\
\hat{G}(s, \phi, \pi) &= -\hat{G}(s, \pi - \phi, 0), \\
\hat{H}(s, \phi, \pi) &= -\hat{H}(s, \pi - \phi, 0)
\end{align*}
\]

for \(s \in [0, 1], \phi \in [0, \pi]\), where \(T_\psi : [0, 1] \to [0, 1]\) is defined by

\[T_\psi(s) := \frac{1}{\pi - \Theta(0, 0, \psi)} \Gamma^{-1}_\psi (\pi - \Theta(0, 0, \psi)s).\]

By Lemma 5.13, \(T_\psi(s)\) is smooth with respect to \(\psi \) and \(s\), decreasing with respect to \(s\), \(T_\psi(0) = 1\), and \(T_\psi(1) = 0\).

As we mentioned in Section 5.3 if we can find a zero \((\theta, \phi, \psi)\) in \(D\) of the vector field \((F, G, H)\), then the equations (18) in Section 5.1 are satisfied and the proof of 3-dimensional symmetric Mahler conjecture is completed. In the following two subsections, we consider the case that \((F, G, H)\) has no zeros on the boundary \(\partial D\). Now, assume that \((F, G, H) \neq 0\) on \(\partial D\), that is, \((\hat{F}, \hat{G}, \hat{H}) \neq 0\) on \(\partial D\). Then we shall calculate the degree of

\[
\mathcal{F} := \frac{\langle \hat{F}, \hat{G}, \hat{H} \rangle}{\sqrt{\hat{F}^2 + \hat{G}^2 + \hat{H}^2}} : \partial D \to S^2.
\]

### 6.2 Reduction

For notational convenience, in Sections \[6.2, 6.3\] and Appendix \[A\] we denote the above \(\hat{F}, \hat{G}, \) and \(\hat{H}\) by \(F, G,\) and \(H\), respectively, and consider

\[
\mathcal{F} := \frac{\langle F, G, H \rangle}{\sqrt{F^2 + G^2 + H^2}} : \partial D \to S^2.
\]

In order to calculate \(\deg \mathcal{F}\), we first need to modify the map \(\mathcal{F}\) so as to satisfy the property that \((\pm 1, 0, 0)\) are its regular values.
Proposition 6.1. There exists a map \( \tilde{F} : \partial D \to S^2 \subset \mathbb{R}^3 \) satisfying the following properties (i)–(iv):

(i). \( \tilde{F} \) is homotopic to \( F \). Especially, \( \deg \tilde{F} = \deg F \) holds.

(ii). \( \tilde{F}(\partial M_i) \not\ni (\pm 1,0,0) \) for \( i = 1, \ldots, 6 \).

(iii). \( (\pm 1,0,0) \) are regular values of \( \tilde{F} \).

(iv). \( \tilde{F} \) satisfies (31), (32), and (33).

We give a proof of Proposition 6.1 in Appendix A. Owing to Proposition 6.1, in what follows we may assume that

\[ F(\partial M_i) \not\ni (\pm 1,0,0) \] for \( i = 1, \ldots, 6 \), and \( (\pm 1,0,0) \) are regular values of \( F \). Note that \( F(s,\phi,\psi) = (\pm 1,0,0) \) if and only if \( G(s,\phi,\psi) = H(s,\phi,\psi) = 0 \) and \( \text{sgn} \, F(s,\phi,\psi) = \pm 1 \). Hence, the inverse image of \( (\pm 1,0,0) \) by \( F|_{M_i} \) corresponds to the zeros of the vector field \((G,H)\) on \( M_1 \). By the condition (34), \((G,H) \neq (0,0)\) on \( \partial M_1 \) and the map

\[ \mathcal{G} := \frac{(G,H)}{\sqrt{G^2+H^2}} : \partial M_1 \to S^1 \]

is well-defined.

Proposition 6.2. For the above maps \( F : \partial D \to S^2 \) and \( \mathcal{G} : \partial M_1 \to S^1 \subset \mathbb{R}^2 \), we have

\[ \deg F = w(\mathcal{G},(0,0)) \pmod{2}, \]

where \( \deg F \) is the Brouwer-Kronecker degree, and \( w(\mathcal{G},(0,0)) \) is the winding number.

Proof. Since \((1,0,0)\) is a regular value of \( F \), we have the representation

\[ \deg F = \sum_{P \in F^{-1}(1,0,0)} \text{sgn} \, \det (dF)_P. \]

Since the inverse image \( F^{-1}(1,0,0) \) is a finite set, we find out the parity of \( \deg F \) by counting the elements. For each \( i = 1, \ldots, 6 \), denote by \( k_i \) the cardinality of \( F^{-1}(1,0,0) \cap \text{int} \, M_i \), that is

\[ k_i := \# \{ (s,\phi,\psi) \in \text{int} \, M_i ; F(s,\phi,\psi) = (1,0,0) \}. \] (35)

Similarly \( k_- \) denotes the cardinality of \( F^{-1}(-1,0,0) \cap \text{int} \, M_1 \), that is

\[ k_- := \# \{ (s,\phi,\psi) \in \text{int} \, M_1 ; F(s,\phi,\psi) = (-1,0,0) \}. \] (36)

By (34), we have

\[ \deg F = \sum_{P \in F^{-1}(1,0,0)} \text{sgn} \, \det (dF)_P = k_1 + \cdots + k_6 \pmod{2}. \]
If \( P \in \mathcal{F}^{-1}(1,0,0) \cap M_4 \), that is \( P = (s, \pi, \psi) \in \mathcal{F}^{-1}(1,0,0) \) for some \( s \in [0,1] \) and \( \psi \in [0,\pi] \), then for \( \tilde{P} = (T_\psi(s),0,\psi) \in M_3 \) we have \( \tilde{P} \in \mathcal{F}^{-1}(1,0,0) \) by (32). Since \( (s, \pi, \psi) \mapsto (T_\psi(s),0,\psi) \) is a \( C^\infty \)-diffeomorphism from \( \text{int} M_4 \) to \( \text{int} M_3 \), we see \( k_4 \leq k_3 \). Using the same argument for \( P \in M_3 \cap \mathcal{F}^{-1}(1,0,0) \), we obtain \( k_3 \leq k_4 \), and hence
\[
k_3 = k_4.
\]
Similarly, using the correspondence between \( M_5 \) and \( M_6 \) by (33), we get
\[
k_5 = k_6.
\]
Furthermore, by (31) the set \( \mathcal{F}^{-1}(1,0,0) \cap M_2 \) exactly corresponds to \( \mathcal{F}^{-1}(-1,0,0) \cap M_1 \), which yields
\[
k_2 = k_-.
\]
Consequently, we obtain
\[
k_1 + \cdots + k_6 = k_1 + k_- + 2(k_3 + k_5)
\]
and
\[
\deg \mathcal{F} = k_1 + k_- \pmod{2}.
\]
By (35) and (36), \( k_1 + k_- \) can be viewed as the number of zeros of the vector field \((G,H)\) on \( M_1 \). Note that \( P \in \text{int} M_1 \) is a regular point of \((G,H)\) with \((G,H)(P) = (0,0)\) if and only if \( P \in \text{int} M_1 \) is a regular point of \( \mathcal{F} \) with \( \mathcal{F}(P) = (\pm 1,0,0) \). Since \((G,H) \neq (0,0)\) on \( \partial M_1 \), \((0,0)\) is a regular value of \((G,H)\) because \((\pm 1,0,0)\) are regular values of \( \mathcal{F} \). Thus, we have
\[
d((G,H),M_1,(0,0)) = \sum_{(G,H)(P)=(0,0)} \text{sgn} \det(d(G,H))_P = k_1 + k_- \pmod{2},
\]
where the left-hand side is the Euclidean degree. Moreover, it is known that
\[
d((G,H),M_1,(0,0)) = w((G,H),(0,0)),
\]
where the right-hand side is the winding number of \((G,H) : \partial M_1 \to \mathbb{R}^2 \setminus \{(0,0)\}\) around \((0,0)\). Since \((G,H) : \partial M_1 \to \mathbb{R}^2 \setminus \{(0,0)\}\) is homotopic to
\[
\mathcal{G} = \frac{(G,H)}{\sqrt{G^2 + H^2}} : \partial M_1 \to S^1 \subset \mathbb{R}^2 \setminus \{(0,0)\},
\]
we have
\[
w((G,H),(0,0)) = w(\mathcal{G},(0,0)).
\]
It completes the proof.

\[\square\]

6.3 Calculation of \( w(\mathcal{G},(0,0)) \)

Proposition 6.2 reduces the problem to the following

**Proposition 6.3.** \( w(\mathcal{G},(0,0)) \) is odd.
Proof. To calculate the winding number, we represent each point $P_t$ on $\partial M_1$ as

$$P_t = \begin{cases} 
(0, t, 0) & \text{if } 0 \leq t \leq \pi, \\
(0, \pi, t - \pi) & \text{if } \pi \leq t \leq 2\pi, \\
(0, 3\pi - t, \pi) & \text{if } 2\pi \leq t \leq 3\pi, \\
(0, 0, 4\pi - t) & \text{if } 3\pi \leq t \leq 4\pi,
\end{cases}$$

which parametrizes $\partial M_1$. Put

$$\mathcal{G}(P_t) = (\tilde{G}(P_t), \tilde{H}(P_t)) := \frac{(G(P_t), H(P_t))}{\sqrt{G^2(P_t) + H^2(P_t)}} = (\cos s, \sin s).$$

Then $s$ is a piecewise smooth function of $t$. We have

$$w(\mathcal{G}, (0, 0)) = \frac{1}{2\pi} \int_0^{4\pi} W(t) \, dt,$$

where $W(t) := \tilde{G}(P_t) \dot{H}'(P_t) - \dot{G}'(P_t) \tilde{H}(P_t)$. First, by (31) and (32), we have

$$G(0, \pi, \psi) = -H(1, 0, \psi) = -G(0, 0, \psi),$$
$$H(0, \pi, \psi) = -G(1, 0, \psi) = H(0, 0, \psi),$$
$$(G^2 + H^2)(0, \pi, \psi) = (G^2 + H^2)(0, 0, \psi).$$

Thus

$$\tilde{G}(0, \pi, \psi) = -\tilde{G}(0, 0, \psi), \quad \tilde{H}(0, \pi, \psi) = \tilde{H}(0, 0, \psi),$$

which yield that, for $3\pi \leq t \leq 4\pi$,

$$\tilde{G}(P_t) = \tilde{G}(0, 0, 4\pi - t) = -\tilde{G}(0, \pi, 4\pi - t),$$
$$\tilde{H}(P_t) = \tilde{H}(0, 0, 4\pi - t) = \tilde{H}(0, \pi, 4\pi - t).$$

So, we obtain

$$\int_{3\pi}^{4\pi} W(t) \, dt$$
$$= \int_{3\pi}^{4\pi} \tilde{G}(0, \pi, 4\pi - t) \dot{H}_{\psi}(0, \pi, 4\pi - t) - \dot{G}_{\psi}(0, \pi, 4\pi - t) \tilde{H}(0, \pi, 4\pi - t) \, dt$$
$$= -\int_{2\pi}^{3\pi} \tilde{G}(0, \pi, \tau - \pi) \dot{H}_{\psi}(0, \pi, \tau - \pi) - \dot{G}_{\psi}(0, \pi, \tau - \pi) \tilde{H}(0, \pi, \tau - \pi) \, d\tau,$$

where we used the substitution $4\pi - t = \tau - \pi$. On the other hand, for $\pi \leq t \leq 2\pi$, we have $\tilde{G}(t) = \tilde{G}(0, \pi, t - \pi), \tilde{H}(t) = \tilde{H}(0, \pi, t - \pi)$. So, we obtain

$$\int_{\pi}^{2\pi} W(t) \, dt$$
$$= \int_{\pi}^{2\pi} \tilde{G}(0, \pi, t - \pi) \dot{H}_{\psi}(0, \pi, t - \pi) - \dot{G}_{\psi}(0, \pi, t - \pi) \tilde{H}(0, \pi, t - \pi) \, dt.$$

Hence

$$\int_{3\pi}^{4\pi} W(t) \, dt = \int_{\pi}^{2\pi} W(t) \, dt.$$
Next, by (33), we have
\[ G(0, \phi, \pi) = -G(0, \pi - \phi, 0), \]
\[ H(0, \phi, \pi) = -H(0, \pi - \phi, 0), \]
\[ (G^2 + H^2)(0, \phi, \pi) = (G^2 + H^2)(0, \pi - \phi, 0). \]

Thus
\[ \tilde{G}(0, \phi, \pi) = -\tilde{G}(0, \pi - \phi, 0), \quad \tilde{H}(0, \phi, \pi) = -\tilde{H}(0, \pi - \phi, 0), \tag{37} \]
which yield that, for \( 2\pi \leq t \leq 3\pi \),
\[ \tilde{G}(P_t) = \tilde{G}(0, 3\pi - t, \pi) = -\tilde{G}(0, t - 2\pi, 0), \]
\[ \tilde{H}(P_t) = \tilde{H}(0, 3\pi - t, \pi) = -\tilde{H}(0, t - 2\pi, 0). \]

Therefore,
\[
\int_{2\pi}^{3\pi} W(t) \, dt \\
= \int_{2\pi}^{3\pi} \tilde{G}(0, t - 2\pi, 0) \tilde{H}_\phi(0, t - 2\pi, 0) - \tilde{G}_\phi(0, t - 2\pi, 0) \tilde{H}(0, t - 2\pi, 0) \, dt \\
= \int_{2\pi}^{3\pi} \tilde{G}(0, \tau, 0) \tilde{H}_\phi(0, \tau, 0) - \tilde{G}_\phi(0, \tau, 0) \tilde{H}(0, \tau, 0) \, d\tau,
\]
where we used the substitution \( \tau = t - 2\pi \). On the other hand, for \( 0 \leq t \leq \pi \),
\[ \tilde{G}(P_t) = \tilde{G}(0, t, 0), \quad \tilde{H}(P_t) = \tilde{H}(0, t, 0). \]

Thus,
\[
\int_{0}^{\pi} W(t) \, dt = \int_{0}^{\pi} \tilde{G}(0, t, 0) \tilde{H}_\phi(0, t, 0) - \tilde{G}_\phi(0, t, 0) \tilde{H}(0, t, 0) \, dt.
\]

Hence,
\[
\int_{2\pi}^{3\pi} W(t) \, dt = \int_{0}^{\pi} W(t) \, dt.
\]

Consequently, we get
\[
\int_{0}^{4\pi} W(t) \, dt = 2 \int_{0}^{2\pi} W(t) \, dt.
\]

Since \( \tilde{G}(P_t) = \cos s \), \( \tilde{H}(P_t) = \sin s \), and \( W(t) = \frac{ds}{dt} \), we have
\[ 2 \int_{0}^{2\pi} W(t) \, dt = 2 \int_{s_0}^{s_1} ds = 2(s_1 - s_0), \]
where \( s_0 \) and \( s_1 \) are determined by
\[
\begin{pmatrix} \cos s_0 \\ \sin s_0 \end{pmatrix} = \begin{pmatrix} \tilde{G}(P_0) \\ \tilde{H}(P_0) \end{pmatrix}, \quad \begin{pmatrix} \cos s_1 \\ \sin s_1 \end{pmatrix} = \begin{pmatrix} \tilde{G}(P_{2\pi}) \\ \tilde{H}(P_{2\pi}) \end{pmatrix},
\]
respectively. By (37), we see
\[
\begin{pmatrix} \tilde{G}(P_{2\pi}) \\ \tilde{H}(P_{2\pi}) \end{pmatrix} = \begin{pmatrix} \tilde{G}(0, \pi, \pi) \\ \tilde{H}(0, \pi, \pi) \end{pmatrix} = - \begin{pmatrix} \tilde{G}(0, 0, 0) \\ \tilde{H}(0, 0, 0) \end{pmatrix} = - \begin{pmatrix} \tilde{G}(P_0) \\ \tilde{H}(P_0) \end{pmatrix}.
\]

This means that there exists \( m \in \mathbb{Z} \) such that \( s_1 = s_0 + (2m + 1)\pi \). Hence
\[
\int_{0}^{4\pi} W(t) \, dt = 2(2m + 1)\pi,
\]
which yields \( w(\mathcal{G}, (0, 0)) = 2m + 1 \). \qed
6.4 Proof of the main theorem (inequality part)

Let us prove the inequality part of Theorem 1.1. Note that a convex body $K \in \mathcal{K}^3_0$ can be approximated by an element of $\hat{\mathcal{K}}$. The following is a direct consequence of a general approximation procedure for convex bodies by R. Schneider [Sc2, pp. 438].

**Proposition 6.4.** For any centrally symmetric convex body $K \in \mathcal{K}^3_0$ and any $\varepsilon > 0$, there exists a convex body $L \in \mathcal{K}^3$ with the following properties:

(a) $\delta(K, L) < \varepsilon$,

(b) $L$ has $C^\infty$ support function and is strongly convex,

(c) $L$ is also centrally symmetric,

where $\delta$ denotes the Hausdorff distance on $\mathcal{K}^3$. In particular, $L \in \hat{\mathcal{K}}$.

Since the volume is continuous with respect to the Hausdorff distance, it suffices to show the inequality (2) for $K \in \hat{\mathcal{K}}$.

Fix $K \in \hat{\mathcal{K}}$. We show the following

**Claim.** $(\hat{F}, \hat{G}, \hat{H})$ has a zero on $D$.

**Proof.** In the case where $(\hat{F}, \hat{G}, \hat{H})$ has a zero on $\partial D$, the claim trivially holds. If not,

$$\mathcal{F} = \frac{(\hat{F}, \hat{G}, \hat{H})}{\sqrt{\hat{F}^2 + \hat{G}^2 + \hat{H}^2}} : \partial D \to S^2$$

is well-defined and deg $\mathcal{F}$ is odd by Propositions 6.2 and 6.3. By, for example, [OR, pp. 157–158, Propositions 4.4 and 4.6], $(\hat{F}, \hat{G}, \hat{H})$ has a zero in $D$. □

We denote the zero of $(\hat{F}, \hat{G}, \hat{H})$ by $(s_0, \phi_0, \psi_0)$. Putting $\theta_0 = (\pi - \Theta(0, \phi_0, \psi_0))s_0$, it means that $(F, G, H) = (0, 0, 0)$ at $(\theta_0, \phi_0, \psi_0)$. Thus the convex body

$$AK(\theta_0, \phi_0, \psi_0) = A(X(\theta_0)Y(\phi_0)Z(\psi_0)K)X(\theta_0)Y(\phi_0)Z(\psi_0)K$$

satisfies the key equality (15). Since $A(X(\theta_0)Y(\phi_0)Z(\psi_0)K)X(\theta_0)Y(\phi_0)Z(\psi_0)$ is a linear transformation, by Proposition 5.1 we obtain

$$\mathcal{P}(K) = \mathcal{P}(AK(\theta_0, \phi_0, \psi_0)) \geq \frac{32}{3}.$$ 

Consequently, every $K \in \hat{\mathcal{K}}$ satisfies (2). Thus we complete the proof of the Mahler conjecture for any centrally symmetric convex body $K \in \mathcal{K}^3_0$.

7 The equality case

In this section, we show the equality part of Theorem 1.1. That is,

**Theorem 7.1.** Let $K$ be a centrally symmetric convex body in $\mathbb{R}^3$ with $\mathcal{P}(K) = 32/3$. Then, $K$ or $K^o$ is a parallelepiped.
7.1 Dual face

Let $K \in \mathcal{K}_0^3$ be a centrally symmetric convex polytope in $\mathbb{R}^n$. Then $K^\circ$ is also a centrally symmetric convex polytope. For a face $F$ of $K$, its dual face $F^\circ \subset K^\circ$ is defined by

$$F^\circ := \{Q \in K^\circ; P \cdot Q = 1 \text{ for any } P \in F\}.$$ 

For a $k$-dimensional face $F$, the dimension of $F^\circ$ is $n - k - 1$.

Assume $K \in \mathcal{K}_0^3$ is a polygon. If a face $F \subset K$ is a line segment which contains two points $(a, b)$ and $(c, d)$, then its dual face $F^\circ$ is a vertex of $K^\circ$. Putting $F^\circ = \{(\alpha, \beta)\}$, by the definition of $F^\circ$, we obtain that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

Hence,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d - b \\ a - c \end{pmatrix}.$$ 

Assume $K \in \mathcal{K}_0^3$ is a polytope. If a 2-dimensional face $F$ of $K$ contains three points $(x_i, y_i, z_i)$ ($i = 1, 2, 3$) which do not on the same straight line, then its dual face $F^\circ$ is a vertex of $K^\circ$. Putting $F^\circ = \{(u, v, w)\}$, we similarly obtain

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

Hence,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

7.2 Inequality

For a general convex body $K$, we need the inequality similarly as Lemma 3.3. As in Section 3.4 let $A_i$ ($i = 1, 2, 3$) be points with $A_1 \cdot (A_2 \times A_3) > 0$, and consider $C_{i,j} = \mathcal{C}_K(A_i, A_j)$ for $(i, j) = (1, 2), (2, 3), (3, 1)$.

**Lemma 7.2.** Assume $K \in \mathcal{K}_0^3$. Let $\mathcal{S}_K(C_{1,2} \cup C_{2,3} \cup C_{3,1})$ be a triangle on $\partial K$. For any $P \in K$, we have

$$\frac{1}{3} \left( P \cdot \overline{C_{1,2}} + P \cdot \overline{C_{2,3}} + P \cdot \overline{C_{3,1}} \right) \leq |O \ast \mathcal{S}_K(C_{1,2} \cup C_{2,3} \cup C_{3,1})|,$$ 

where $\overline{C_{i,j}} = (C_{i,j,1}, C_{i,j,2}, C_{i,j,3})$ and $C_{i,j,k} = \text{sgn}((A_i \times A_j)_k)|O \ast P_k(C_{i,j})|$. 

**Proof.** Since (38) depends on $K$ continuously, by the Hausdorff approximation, we may assume $K \in \hat{K}_0$. By Lemma 3.3 it is sufficient to show that

$$\int_{C_{1,2}} (P \times r) \cdot dr = 2P \cdot \overline{C_{1,2}}.$$ 

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The curve $C_{1,2}$ is parametrized by $l(t) := \rho_K((1-t)A_1 + tA_2)((1-t)A_1 + tA_2)$ $(t \in [0, 1])$. By a direct calculation,
\[ l(t) \times l'(t) = \rho_K((1-t)A_1 + tA_2)((1-t)A + tA_2) \times ((\nabla \rho_K((1-t)A_1 + tA_2) \cdot (A_2 - A_1))((1-t)A_1 + tA_2) + \rho_K((1-t)A_1 + tA_2)(A_2 - A_1)) = \rho_K^2((1-t)A_1 + tA_2)(A_1 \times A_2). \]

Thus, we have
\[ \int_{C_{1,2}} (P \times r) \cdot dr = P \cdot \int_0^1 (l(t) \times l'(t)) \, dt = P \cdot (A_1 \times A_2) \int_0^1 \rho_K^2((1-t)A_1 + tA_2) \, dt \]
Since the absolute value of $(\int_0^1 l \times l' \, dt)_k$ is equal to $2|O \ast P_k(C_{1,2})|$ and $\rho_K((1-t)A_1 + tA_2) > 0$, we obtain [39].

### 7.3 Linear transformation of $K \in K_0^3$

To prove Theorem [7.1] for a general centrally symmetric convex body $K$, we need to find a transformation $A$ such that $AK$ satisfies [18].

**Proposition 7.3.** Let $K \in K_0^3$ be a convex body which satisfies $\mathcal{P}(K) = 32/3$. Then there exists a linear transformation $B \in GL(3)$ such that $L = BK$ satisfies $\mathcal{P}(L) = 32/3$, the condition [18], and $\pm(1, 0, 0), \pm(0, 1, 0), \pm(0, 0, 1) \in \partial K$.

**Proof.** Fix $K \in K_0^3$. By Proposition [6.4], there exists $(K_m)_{m=1}^\infty \subset \hat{K}$ such that $K_m \to K$ as $m \to \infty$. By the results in Sections [5] and [6] for each $m \in \mathbb{N}$, there exist a linear transformation $B_m = A(K_m(\theta_m, \phi_m, \psi_m))X(\theta_m)Y(\phi_m)Z(\psi_m)$ such that $B_mK_m$ satisfies the condition [18]. Hence,
\[ |\Delta_i(B_mK_m)| = \frac{|B_mK_m|}{8} \quad (i = 1, \ldots, 4), \]
\[ d(B_mK_m) = \tau(B_mK_m), \quad f(B_mK_m) = \gamma(B_mK_m), \quad h(B_mK_m) = \theta(B_mK_m). \]

Note that $(\theta_m)_{m=1}^\infty, (\phi_m)_{m=1}^\infty, (\psi_m)_{m=1}^\infty \subset [0, \pi]$ are bounded sequences.

**Claim.** $(A(K_m))_{m=1}^\infty$ and $(A(K_m)^{-1})_{m=1}^\infty$ are bounded sequences.

**Proof.** By the definition of $A$ in Section [7.2] it is sufficient to show that
\[ \left(\frac{1}{\tan \Theta_m}\right)_{m=1}^\infty, \quad \left(\frac{1}{\tan \Phi_m}\right)_{m=1}^\infty, \quad \left(\frac{1}{\sin \Theta_m \tan \Psi_m}\right)_{m=1}^\infty \]
are bounded, where $\Theta_m = \Theta(\theta_m, \phi_m, \psi_m), \Phi_m = \Phi(\theta_m, \phi_m, \psi_m), \Psi_m = \Psi(\theta_m, \phi_m, \psi_m)$.

For simplicity, we put $\hat{K} := K_m(\theta_m, \phi_m, \psi_m)$. Since $(K_m)_{m=1}^\infty$ converges to $K$, there exists $r > 1$ independent of $m$ such that
\[ B \left(0, \frac{1}{r}\right) \subset K_m \subset B(0, r), \]

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where $B(0,r)$ is the open ball with center 0 and radius $r$. Since each $\tilde{K}^m$ is the image of $K_m$ by a rotation, the same inequality holds for $\tilde{K}^m$. Hence,

$$\frac{1}{r} \leq \rho_{\tilde{K}^m}(P) \leq r \quad \text{if} \quad P \in S^2 \subset \mathbb{R}^3.$$ 

Thus, by the definition (20) of $\Theta$, we have

$$2\Theta_m \frac{1}{r^3} \leq 2(\pi - \Theta_m)r^3, \quad 2\Theta_m r^3 \geq 2(\pi - \Theta_m) \frac{1}{r^3}.$$ 

Hence,

$$\frac{1}{r^3 + 1/r^3} \pi \leq \Theta_m \leq \frac{r^3}{r^3 + 1/r^3} \pi.$$ 

Since

$$0 < \frac{1}{r^3 + 1/r^3} < \frac{r^3}{r^3 + 1/r^3} < 1,$$

we obtain that $(1/\tan \Theta_m)_{m=1}^\infty$ and $(1/\sin \Theta_m)_{m=1}^\infty$ are bounded. Similarly, by the definitions of $\Phi$ and $\Psi$, we have

$$\frac{1}{r^2 + 1/r^2} \pi \leq \Phi_m \leq \frac{r^2}{r^2 + 1/r^2} \pi, \quad \frac{1}{r^2 + 1/r^2} \pi \leq \Psi_m \leq \frac{r^2}{r^2 + 1/r^2} \pi.$$ 

Hence, the claim holds. \qed

Thus, taking subsequence if necessary, there exists a linear transformation $B \in GL(3)$ such that \( \lim_{m \to \infty} B_m = B \) in $GL(3)$, and $\lim_{m \to \infty} B_m K_m = B K$ in the Hausdorff distance. Thus $\lim_{m \to \infty} \rho_{B_m K_m} = \rho_{B K}$ uniformly in $S^2 \subset \mathbb{R}^3$. Since $\Delta_i(B_m K_m)$ and $\overline{d}(B_m K_m)$, etc. are expressed by the integral of $\rho_{B_m K_m}$ on closed subsets on $S^2$, taking as $m \to \infty$ in (40), we obtain that

$$|\Delta_i(BK)| = \frac{|BK|}{8} \quad (i = 1, \ldots, 4),$$

$$\overline{d}(BK) = \overline{e}(BK), \quad \overline{f}(BK) = \overline{g}(BK), \quad \overline{h}(BK) = \overline{h}(BK).$$

Moreover, since each $B_m K_m$ satisfies (18), we obtain

$$\frac{9}{4} |B_m K_m||\langle B_m K_m \rangle^o|$$

$$\leq |Q_1(B_m K_m)||P_1(\langle B_m K_m \rangle^o)| + |Q_2(B_m K_m)||P_2(\langle B_m K_m \rangle^o)| + |Q_3(B_m K_m)||P_3(\langle B_m K_m \rangle^o)|$$

as in Section 3.5. Since $\lim_{m \to \infty}(B_m K_m) = (BK)^o$ in the Hausdorff distance,

$$\frac{9}{4} \mathcal{P}(BK) = \frac{9}{4} |BK||\langle BK \rangle^o|$$

$$\geq |Q_1(BK)||P_1(\langle BK \rangle^o)| + |Q_2(BK)||P_2(\langle BK \rangle^o)| + |Q_3(BK)||P_3(\langle BK \rangle^o)|$$

holds. By Theorem 2.1 $|Q_i(BK)||P_i(\langle BK \rangle^o)| \geq 8 \quad (i = 1, 2, 3).$ Since $\mathcal{P}(BK) = \mathcal{P}(K) = 32/3$, we obtain $|Q_i(BK)||P_i(\langle BK \rangle^o)| = 8 \quad (i = 1, 2, 3)$. If necessary, we choose a diagonal matrix $C$ to satisfy that $\pm (1,0,0), \pm (0,1,0), \pm (0,0,1) \in \partial (C BK)$. Then, $B = CB$ satisfies the required conditions. \qed
7.4 The equality case

Proposition 7.4. Let $L \in \mathcal{K}_0^3$ with $\mathcal{P}(L) = 32/3$. Assume that $L$ satisfies the condition $(\text{LS})$, $\pm (1, 0, 0), \pm (0, 1, 0), \pm (0, 0, 1) \in \partial L$, and $|Q_i(L)||P_i(L^\circ)| = 8$ $(i = 1, 2, 3)$. Then, $L$ or $L^\circ$ is a parallelepiped.

Proof. For each $i = 1, 2, 3$, we have $(Q_i(L))^\circ = P_i(L^\circ)$ and $|Q_i(L)||P_i(L^\circ)| = 8$. Moreover, $(\text{LS})$ holds. Thus, we can apply Proposition 2.2 to each $Q_i(L)$. As a result, there exist $a, b, c \in (-1, 1)$ such that

$$Q_1(L) = \text{conv}\left\{ \frac{\pm 1}{1 + a^2}(0, 1 - a, 1 + a), \frac{\pm 1}{1 + a^2}(0, -1 - a, 1 + a) \right\},$$

$$P_1(L^\circ) = \text{conv}\left\{ \pm (0, 1, a), \pm (0, -a, 1) \right\},$$

$$Q_2(L) = \text{conv}\left\{ \frac{\pm 1}{1 + b^2}(1 + b, 0, 0), \frac{\pm 1}{1 + b^2}(1 - b, 0, -1 - b) \right\},$$

$$P_2(L^\circ) = \text{conv}\left\{ \pm (b, 0, 1), \pm (1, 0, -b) \right\},$$

$$Q_3(L) = \text{conv}\left\{ \frac{\pm 1}{1 + c^2}(1 + c, 1 + c, 0), \frac{\pm 1}{1 + c^2}(-1 - c, 1 - c, 0) \right\},$$

$$P_3(L^\circ) = \text{conv}\left\{ \pm (1, c, 0), \pm (-c, 1, 0) \right\}.$$

Hence,

$$|Q_1(L)| = \frac{4}{1 + a^2}, \quad |Q_2(L)| = \frac{4}{1 + b^2}, \quad |Q_3(L)| = \frac{4}{1 + c^2}.$$

Using $\Delta_i$ in Section 5.1, we put $L_i := \Delta_i(L) \ (i = 1, 2, 3, 4)$. Then

$$|L_1 \cap \{ x = 0 \}| = \frac{1}{1 + a^2}, \quad |L_1 \cap \{ y = 0 \}| = \frac{1}{1 + b^2}, \quad |L_1 \cap \{ z = 0 \}| = \frac{1}{1 + c^2}.$$

It follows from Lemma 7.2 that

$$\frac{|L|}{8} = |L_1| \geq \frac{1}{3} \left( \frac{1}{1 + a^2}, \frac{1}{1 + b^2}, \frac{1}{1 + c^2} \right) \cdot (x, y, z) \quad (42)$$

for any $(x, y, z) \in L$. Hence, $(8/3|L|)(1/(1 + a^2), 1/(1 + b^2), 1/(1 + c^2)) \in L^\circ$ holds.

Similarly, we obtain that

$$\frac{8}{3|L|} \left( \frac{\pm 1}{1 + a^2}, \frac{\pm 1}{1 + b^2}, \frac{\pm 1}{1 + c^2} \right) \in L^\circ. \quad (43)$$

On the other hand, since $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) \in L,$

$$(\pm 1, 0, 0) \cdot (u, v, w) = \pm u \leq 1,$$

$$(0, \pm 1, 0) \cdot (u, v, w) = \pm v \leq 1,$$

$$(0, 0, \pm 1) \cdot (u, v, w) = \pm w \leq 1$$

hold for any $(u, v, w) \in L^\circ$. Hence,

$$L^\circ \subset [-1, 1] \times [-1, 1] \times [-1, 1]. \quad (44)$$

Now we determine the shapes of $L$ and $L^\circ$. Without loss of generalities, it suffices to consider the following four cases: (i) $a, b, c \in (-1, 1)$, (ii) $a, b \in (-1, 1)$, $c = 1$, (iii) $a = b = 1$, $c \in (-1, 1)$, (iv) $a = b = c = 1$. 

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**Case (i) a, b, c \in (-1, 1);** By the assumption \((0, 0, 1) \in \partial L\), we have \(L^o \cap \{w = 1\} \neq \emptyset\). By (44),

\[
P_1(L^o \cap \{w = 1\}) = P_1(L^o) \cap \{w = 1\} = \{(0, -a, 1)\},
\]

\[
P_2(L^o \cap \{w = 1\}) = P_2(L^o) \cap \{w = 1\} = \{(b, 0, 1)\}
\]

hold. Thus, \(L^o \cap \{w = 1\} = \{(b, -a, 1)\}\). Similarly, we have

\[
L^o \cap \{u = \pm 1\} = \{\pm(1, c, -b)\},
\]

\[
L^o \cap \{v = \pm 1\} = \{\pm(-c, 1, a)\},
\]

\[
L^o \cap \{w = \pm 1\} = \{\pm(b, -a, 1)\}.
\]

(45)

By (44), we obtain

\[
\pm(1, c, -b), \pm(-c, 1, a), \pm(b, -a, 1) \in \partial L^o.
\]

(46)

Note that the points in (46) are mutually distinct, because \(a, b, c \in (-1, 1)\). Moreover, these six points are in different faces of the cube \([-1, 1]^3\). We use these points to divide \(L^o\).

The line segment connecting from \((1, c, -b)\) to \((-c, 1, a)\) is on \(\partial L^o\). Indeed, if \(Q = s(1, c, -b) + (1 - s)(-c, 1, a)\) is an interior point of \(L^o\) for some \(s \in (0, 1)\), then there exists a open ball \(B(Q, \epsilon)\) with the center \(Q\) and radius \(\epsilon > 0\) such that \(B(Q, \epsilon) \subset L^o\). Thus \(P_3(B(Q, \epsilon)) \subset P_3(L^o)\), and \(P_3(Q)\) is an interior point of \(P_3(L^o)\). On the other hand, by (44), the line segment connecting from \((1, c, 0)\) to \((-c, 1, 0)\) is a part of \(\partial P_3(L^o)\). It is a contradiction. Therefore, \(C_L((1, c, -b), (-c, 1, a))\) is the line segment connecting from \((1, c, -b)\) to \((-c, 1, a)\). Similarly, we see that the segments connecting \((b, -a, 1)\), \((1, c, -b)\), \((-c, 1, a)\) cyclically is a triangle on \(\partial L^o\), because \(((b, -a, 1) \times (1, c, -b)) \cdot (-c, 1, a) = 1 + a^2 + b^2 + c^2 > 0\). We denote the triangle by \(T((b, -a, 1), (1, c, -b), (-c, 1, a))\). Put

\[
L^o_1 := O \ast T((b, -a, 1), (1, c, -b), (-c, 1, a)),
\]

\[
L^o_2 := O \ast T((b, -a, 1), (-c, 1, a), (-1, c, -b)),
\]

\[
L^o_3 := O \ast T((b, -a, 1), (-1, c, -b), (-c, 1, a)),
\]

\[
L^o_4 := O \ast T((-b, -a, 1), (1, c, -b), (-c, 1, a)).
\]

Then, by the symmetry of \(L\), we have \(|L| = 2(|L^o_1| + |L^o_2| + |L^o_3| + |L^o_4|)|. Applying Lemma 7.2 to \(L^o_1\), we have

\[
|L^o_1| \geq \frac{1}{6} \begin{vmatrix} u & v & w \\ b & -a & 1 \\ c & -b \\ \end{vmatrix} + \frac{1}{6} \begin{vmatrix} u & v & w \\ 1 & c & -b \\ -c & 1 & a \\ \end{vmatrix} + \frac{1}{6} \begin{vmatrix} u & v & w \\ -c & 1 & a \\ b & -a & 1 \\ \end{vmatrix}
\]

\[
= \frac{1}{6} (1 + a^2 + ab + ac + b - c, 1 + b^2 + ab + bc + c - a, 1 + c^2 + ac + bc + a - b) \cdot (u, v, w)
\]

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for any \((u,v,w)\in L^\circ\). Similarly, we obtain
\[
|L_2^\circ| \geq \frac{1}{6} \begin{vmatrix} u & v & w \\ b & -c & 1 \\ -c & 1 & a \end{vmatrix} + \frac{1}{6} \begin{vmatrix} u & v & w \\ -1 & -c & b \\ b & -a & 1 \end{vmatrix} = \frac{1}{6}(-1 - a^2 + ab + ac + b - c, 1 + b^2 - ab + bc - c - a, 1 + c^2 - ac + bc + a + b) \cdot (u,v,w),
\]
\[
|L_3^\circ| \geq \frac{1}{6} \begin{vmatrix} u & v & w \\ b & -a & 1 \\ -1 & -c & b \end{vmatrix} + \frac{1}{6} \begin{vmatrix} u & v & w \\ c & -1 & -a \\ b & -a & 1 \end{vmatrix} = \frac{1}{6}(-1 - a^2 - ab + ac + b + c, -1 - b^2 - ab + bc - c - a, 1 + c^2 - ac - bc - a + b) \cdot (u,v,w),
\]
\[
|L_4^\circ| \geq \frac{1}{6} \begin{vmatrix} u & v & w \\ b & -a & 1 \\ c & -1 & -a \end{vmatrix} + \frac{1}{6} \begin{vmatrix} u & v & w \\ c & -1 & -b \\ b & -a & 1 \end{vmatrix} = \frac{1}{6}(1 + a^2 - ab + ac + b + c, -1 - b^2 + ab + bc + c - a, 1 + c^2 + ac - bc - a - b) \cdot (u,v,w)
\]
for any \((u,v,w)\in L^\circ\). Therefore, the four points
\[
S_1 := \frac{1}{6|L_1^\circ|}(1 + a^2 + ab + ac + b - c, 1 + b^2 + ab + bc + c - a, 1 + c^2 + ac + bc + a - b),
\]
\[
S_2 := \frac{1}{6|L_2^\circ|}(-1 - a^2 + ab + ac + b - c, 1 + b^2 - ab + bc - c - a, 1 + c^2 - ac + bc + a + b),
\]
\[
S_3 := \frac{1}{6|L_3^\circ|}(-1 - a^2 - ab + ac + b + c, -1 - b^2 - ab + bc - c - a, 1 + c^2 - ac - bc - a + b),
\]
\[
S_4 := \frac{1}{6|L_4^\circ|}(1 + a^2 - ab + ac + b + c, -1 - b^2 + ab + bc + c - a, 1 + c^2 + ac - bc - a - b)
\]
are contained in \(L\). Combining the fact that \(S_1 \in L\) with (12), we have
\[
|L_1| \geq \frac{1}{18|L_1^\circ|} \left( 3 + \frac{ab + ac + b - c}{1 + a^2} + \frac{ab + bc + c - a}{1 + b^2} + \frac{ac + bc + a - b}{1 + c^2} \right).
\]
(47)
Similarly, we obtain
\[
|L_2| \geq \frac{1}{18|L_2^\circ|} \left( 3 - \frac{ab + ac + b - c}{1 + a^2} - \frac{ab + bc - c - a}{1 + b^2} + \frac{ac + bc + a + b}{1 + c^2} \right),
\]
\[
|L_3| \geq \frac{1}{18|L_3^\circ|} \left( 3 - \frac{-ab + ac + b + c}{1 + a^2} - \frac{-ab + bc - c - a}{1 + b^2} + \frac{-ac + bc - a + b}{1 + c^2} \right),
\]
\[
|L_4| \geq \frac{1}{18|L_4^\circ|} \left( 3 + \frac{-ab + ac + b + c}{1 + a^2} - \frac{ab + bc + c - a}{1 + b^2} + \frac{ac - bc - a - b}{1 + c^2} \right).
\]
Thus,
\[
|L_1||L_1^\circ| + |L_2||L_2^\circ| + |L_3||L_3^\circ| + |L_4||L_4^\circ| \geq \frac{12}{18} = \frac{2}{3}.
\]
On the other hand, since the assumption [18] for \(L\) means \(|L_1| = |L|/8\), we have
\[
|L_1||L_1^\circ| + \cdots + |L_4||L_4^\circ| = \frac{|L|}{8} (|L_1^\circ| + |L_2^\circ| + |L_3^\circ| + |L_4^\circ|) = \frac{|L||L^\circ|}{16} = \frac{32}{3} \cdot \frac{1}{16} = \frac{2}{3}.
\]
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Thus, the above inequality holds with equality. Hence (47) holds with equality. That is, (42) for $S_1$ holds with equality. Taking the proof of Lemma 3.3 (Lemma 7.2) into account, we have

$$S_1 \in \{(x, y, z); x \geq 0, y \geq 0, z \geq 0\},$$

$$L_1 = \text{conv}\{L_1 \cap \{x = 0\}, L_1 \cap \{y = 0\}, L_1 \cap \{z = 0\}, S_1\}.$$  

Hence $S_1 \in \partial L$. We put $S_1 = (\alpha, \beta, \gamma)$ for simplicity.

Now, we consider the case $S_1 \notin \{x = 0\}$. Then

$$\text{conv}\left\{\left(\frac{0, 1 - a, 1 + a}{1 + a^2}, \frac{0, -1 - a, 1 - a}{1 + a^2}\right), S_1\right\}$$

is a part of a face of $L$. Its dual face is a vertex of $L^\circ$. Then the dual face is

$$\begin{pmatrix}
0 & (1 - a)/(1 + a^2) & (1 + a)/(1 + a^2) \\
0 & -(1 + a)/(1 + a^2) & (1 - a)/(1 + a^2) \\
\alpha & \beta & \gamma
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
\frac{-\gamma + a\beta + 1}{\alpha} \\
-a \\
1
\end{pmatrix}.
$$

Comparing it with (45), we obtain

$$\left(\frac{-\gamma + a\beta + 1}{\alpha}, -a, 1\right) = (b, -a, 1).$$

Hence $-\gamma + a\beta + 1 = b\alpha$. Then,

$$6|L_1^\circ| = 6|L_1^\circ|(b\alpha - a\beta + \gamma)$$

$$= b(1 + a^2 + ab + ac + b - c) - a(1 + b^2 + ab + bc + c - a) + 1 + c^2 + ac + bc + a - b$$

$$= 1 + a^2 + b^2 + c^2.$$  

(48)

On the other hand, the volume of $\text{conv}\{O, (1, c, -b), (-c, 1, a), (b, -a, 1)\} \subset L_1^\circ$ is

$$\frac{1}{6}
\begin{vmatrix}
1 & c & -b \\
-1 & 1 & a \\
b & -a & 1
\end{vmatrix}
= \frac{1}{6}(1 + a^2 + b^2 + c^2).$$

Hence,

$$L_1^\circ = \text{conv}\{O, (1, c, -b), (-c, 1, a), (b, -a, 1)\}.$$  

(49)

Next, we consider the case $S_1 \in \{x = 0\}$. Since $S_1 \in \partial L$, we obtain $S_1 \in \partial Q_1(L)$. Since $S_1$ is in the first octant, $S_1$ is on the line segment connecting from $(0, 1 - a, 1 + a)/(1 + a^2)$ to $(0, 0, 1)$, or on the line segment connecting from $(0, 0, 1)$ to $(0, 1 - a, 1 + a)/(1 + a^2)$. For the former case, we have

$$S_1 \cdot (b, -a, 1) = (1 - s)(0, 1 - a, 1 + a)/(1 + a^2) \cdot (b, -a, 1) + s(0, 0, 1) \cdot (b, -a, 1) = 1,$$

which yields $b\alpha - a\beta + \gamma = 1$, and $6|L_1^\circ| = 1 + a^2 + b^2 + c^2$ holds as (45). Thus, we obtain (49) as well. For the latter case, we have

$$S_1 \cdot (-c, 1, a) = (1 - s)(0, 1, 0) \cdot (-c, 1, a) + s(0, 1 - a, 1 + a)/(1 + a^2) \cdot (-c, 1, a) = 1.$$
Since $\alpha = 0$, we obtain $\beta + a\gamma = 1$. Thus
\[
6|L_0^o| = 6|L_1^o| (\beta + a\gamma)
\]
\[
= 1 + b^2 + ab + bc + c - a + a(1 + c^2 + ac + bc + a - b)
\]
\[
= 1 + a^2 + b^2 + c^2 + c(1 + a^2 + ab + ac + b - c)
\]
\[
= 1 + a^2 + b^2 + c^2 + 6c|L_0^o| \alpha
\]
\[
= 1 + a^2 + b^2 + c^2,
\]
which yields (19) as well.

By repeating the above arguments for $L_2^o, L_3^o, L_4^o$, we obtain
\[
L^o = \text{conv} \{ \pm (1, c, -b), \pm (-c, 1, a), \pm (b, -a, 1) \}.
\]

Thus, $L^o$ is a centrally symmetric octahedron. Then,
\[
L = \text{conv} \{ \pm S_1, \pm S_2, \pm S_3, \pm S_4 \}
\]
is a parallelepiped.

**Case (ii) $a, b \in (-1, 1)$, $c = 1$**; By (11), we have
\[
Q_3(L) = \text{conv} \{ \pm (1, 0, 0), \pm (0, 1, 0) \}, \quad P_3(L^o) = [-1, 1] \times [-1, 1] \times \{0\}.
\]

Thus, we obtain
\[
P_3(L^o \cap \{ u = 1 \}) = P_3(L^o) \cap \{ u = 1 \} = \{1\} \times [-1, 1] \times \{0\}.
\]

Hence, for any $v \in [-1, 1]$, there exists $w(v) \in [-1, 1]$ such that $(1, v, w(v)) \in L^o$ and
\[
(1, v, w(v)) \in L^o \cap \{ u = 1 \}.
\]

On the other hand, combining (11) with $b \in (-1, 1)$, we have
\[
P_2(L^o \cap \{ u = 1 \}) = \{(1, 0, -b)\}.
\]

Consequently, we obtain $w(v) = -b$ ($v \in [-1, 1]$) and
\[
\{1\} \times [-1, 1] \times \{-b\} \subset L^o.
\]

Especially, we have $(1, -1, -b), (1, 1, -b) \in L^o$. Combining (11) with $a \in (-1, 1),
\[
P_1((1, 1, -b)) = (0, 1, -b) \in P_1(L^o) \cap \{ v = 1 \} = \{(0, 1, a)\},
\]
\[
P_1((1, -1, -b)) = (0, -1, -b) \in P_1(L^o) \cap \{ v = -1 \} = \{(0, -1, -a)\}.
\]

Hence $-b = a = -a$. Thus, $a = b = 0$. Especially, by (50), we obtain
\[
\{1\} \times [-1, 1] \times \{0\} \subset L^o.
\]

By the symmetry and convexity of $L^o$, we obtain
\[
[-1, 1] \times [-1, 1] \times \{0\} \subset L^o.
\]
Since $a = b = 0$, similarly as the above,

\[ P_1(L^\circ \cap \{ w = 1 \}) = P_1(L^\circ) \cap \{ w = 1 \} = \{(0, 0, 1)\}, \]
\[ P_2(L^\circ \cap \{ w = 1 \}) = P_2(L^\circ) \cap \{ w = 1 \} = \{(0, 0, 1)\}, \]

which implies that $(0, 0, 1) \in L^\circ$. In summary,

\[ \text{conv} \{ [-1,1] \times [-1,1] \times \{0\}, \pm(0,0,1) \} \subset L^\circ. \] (51)

Now, since $a = b = 0$, by (43), we have

\[ \frac{8}{3|L|} \left( 1,1,\frac{1}{2} \right) \in L^\circ. \]

Since

\[ P_2 \left( \frac{8}{3|L|} \left( 1,1,\frac{1}{2} \right) \right) = \frac{8}{3|L|} \left( 1,0,\frac{1}{2} \right) \in P_2(L^\circ) = \text{conv} \{ \pm(1,0,0), \pm(0,0,1) \}, \]

we obtain

\[ \frac{8}{3|L|} \left( 1 + \frac{1}{2} \right) = \frac{4}{|L|} \leq 1. \]

Since $P(L) = 32/3$, we have

\[ |L^\circ| = \frac{32}{3|L|} \leq \frac{8}{3}. \] (52)

On the other hand, by (51),

\[ |L^\circ| \geq \text{conv} \{ [-1,1] \times [-1,1] \times \{0\}, \pm(0,0,1) \} = \frac{8}{3} \]

holds. Therefore, $|L^\circ| = 8/3$ and

\[ L^\circ = \text{conv} \{ [-1,1] \times [-1,1] \times \{0\}, \pm(0,0,1) \}. \]

Hence, $L^\circ$ is a centrally symmetric octahedron and

\[ L = \text{conv} \{ (\pm1,0,\pm1), (0,\pm1,\pm1) \} \]

is a parallelepiped.

**Case (iii) $a = b = 1$, $c \in (-1,1)$;** By (111), we have

\[ P_1(L^\circ) = \{0\} \times [-1,1] \times [-1,1], \quad P_2(L^\circ) = [-1,1] \times \{0\} \times [-1,1], \]
\[ P_3(L^\circ) = \text{conv} \{ \pm(1,c,0), \pm(-c,1,0) \}. \]

Since $c \in (-1,1)$, we see

\[ P_2(L^\circ \cap \{ u = 1 \}) = \{1\} \times \{0\} \times [-1,1], \]
\[ P_3(L^\circ \cap \{ u = 1 \}) = \{(1,c,0)\}. \]

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Hence
\[ \{1\} \times \{c\} \times [-1, 1] \subset L^\circ. \]
By the symmetry of \( L^\circ \), we obtain
\[ \{-1\} \times \{-c\} \times [-1, 1] \subset L^\circ. \]
Similarly,
\[ P_1(L^\circ \cap \{v = 1\}) = \{0\} \times \{1\} \times [-1, 1], \]
\[ P_3(L^\circ \cap \{v = 1\}) = \{(-c, 1, 0)\}. \]
Thus
\[ \{-c\} \times \{1\} \times [-1, 1] \subset L^\circ. \]
By the symmetry of \( L^\circ \), we get
\[ \{c\} \times \{-1\} \times [-1, 1] \subset L^\circ. \]
By the convexity of \( L^\circ \),
\[ \text{conv} \{\pm(1,c,1), \pm(-c,1,1), \pm(1,c,-1), \pm(-c,1,-1)\} \subset L^\circ. \]
On the other hand, we have \( L^\circ \subset [-1, 1] \times [-1, 1] \times [-1, 1] \) by (41). Combining it with \( P_3(L^\circ) = \text{conv} \{\pm(1,c,0), \pm(-c,1,0)\} \), we obtain
\[ L^\circ \subset \text{conv} \{\pm(1,c,1), \pm(-c,1,1), \pm(1,c,-1), \pm(-c,1,-1)\}. \]
Thus,
\[ L^\circ = \text{conv} \{\pm(1,c,1), \pm(-c,1,1), \pm(1,c,-1), \pm(-c,1,-1)\} \]
is a parallelepiped.

**Case (iv) \( a = b = c = 1 \);**

By (41), we have
\[ Q_1(L) = \text{conv} \{\pm(0,1,0), \pm(0,0,1)\}, \]
\[ P_1(L^\circ) = \{0\} \times [-1, 1] \times [-1, 1], \]
\[ Q_2(L) = \text{conv} \{\pm(1,0,0), \pm(0,0,1)\}, \]
\[ P_2(L^\circ) = [-1, 1] \times \{0\} \times [-1, 1], \]
\[ Q_3(L) = \text{conv} \{\pm(1,0,0), \pm(0,1,0)\}, \]
\[ P_3(L^\circ) = [-1, 1] \times [-1, 1] \times \{0\}. \]
Recall that \( L^\circ \subset [-1, 1] \times [-1, 1] \times [-1, 1] \). If \((1,1,1) \in L^\circ \), then we have
\[ (1,1,1) \cdot (x,y,z) = x + y + z \leq 1 \]
for any \((x,y,z) \in L\). On the other hand, \((1,0,0),(0,1,0),(0,0,1) \in L\). By the definition of \( L_1 \), we obtain
\[ L_1 = \text{conv} \{(1,0,0), (0,1,0), (0,0,1)\}. \]
Moreover, since
\[ |L_2| = \frac{|L|}{8} = |L_1| = \frac{1}{6}, \]
\((-1, 0, 0), (0, 1, 0), (0, 0, 1) \in L, \) and the definition of \(L_2, \) we have
\[ L_2 = \text{conv}\ \{(-1, 0, 0), (0, 1, 0), (0, 0, 1)\}. \]

By similar arguments for \(L_3\) and \(L_4,\) we obtain
\[ L = \text{conv}\ \{\pm(1, 0, 0), \pm(0, 1, 0), \pm(0, 0, 1)\}. \]

Thus
\[ L^o = [-1, 1] \times [-1, 1] \times [-1, 1] \]
is a parallelepiped. Similarly, if one of the eight points \((\pm 1, \pm 1, \pm 1)\) is in \(L^o, \) then \(L\) is an octahedron and \(L^o\) is a parallelepiped.

Thus, from now on we may assume that \((\pm 1, \pm 1, \pm 1) \notin L^o.\) Using the characterization of \(Q_i(L)\) \((i = 1, 2, 3),\) by Lemma \[7.2\] we have
\[ \frac{|L|}{8} = \frac{|L_1|}{1} \geq \frac{1}{6} (1, 1, 1) \cdot (x, y, z) \]
for any \((x, y, z) \in L.\) Similar arguments about \(L_2, \ldots, L_8\) imply
\[ \frac{4}{3|L|}(\pm 1, \pm 1, \pm 1) \in L^o. \]

Next, by the characterization of \(P_i(L^o)\) \((i = 1, 2, 3),\) there exists \(\alpha_i, \beta_i, \gamma_i \in (-1, 1)\) \((i = 1, 2)\) such that
\[
A_1^o := (\alpha_1, 1, 1), \\
B_1^o := (1, \beta_1, 1), \\
C_1^o := (1, 1, \gamma_1), \\
A_2^o := (\alpha_2, -1, 1), \\
B_2^o := (1, \beta_2, -1), \\
C_2^o := (-1, 1, \gamma_2),
\]
are in \(L^o.\) We consider three planes \(HOA_1^oA_2^o, \ HOB_1^oB_2^o, \) and \(HOC_1^oC_2^o\) determined by \(O, A_1^o, A_2^o, O, B_1^o, B_2^o; \) and \(O, C_1^o, C_2^o,\) respectively. We divide \(L^o\) by these planes into 8 pieces \(L_i^o\) \((i = 1, \ldots, 8).\) By the convexity of \(L^o,\) each line segment connecting any two points in \(\{\pm A_1^o, \pm A_2^o, \pm B_1^o, \pm B_2^o, \pm C_1^o, \pm C_2^o\}\) is in \(L^o.\) Thus \(HOA_1^oA_2^o, \ HOB_1^oB_2^o, \)
and \(HOC_1^oC_2^o\) contain \(\text{conv}\ \{\pm A_1^o, \pm A_2^o\}, \) \(\text{conv}\ \{\pm B_1^o, \pm B_2^o\}, \) and \(\text{conv}\ \{\pm C_1^o, \pm C_2^o\},\)
respectively. We put
\[
D^o = (1, d_2, d_3) := B_1^oB_2^o \cap C_1^o(-C_2^o), \\
E^o = (e_1, 1, e_3) := C_1^oC_2^o \cap A_1^o(-A_2^o), \\
F^o = (f_1, f_2, 1) := A_1^oA_2^o \cap B_1^o(-B_2^o).
\]

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Similarly, we obtain that
\[
|L_0| \geq \frac{2}{|L|} \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & f_1 & 2 \\
\beta_1 & 1 & 1
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & \beta_1 & 1 \\
1 & f_1 & 2
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & \beta_1 & 1 \\
1 & f_1 & 2
\end{array} \right]
\]
\[
= \frac{2}{|L|} \left( 6 - 2\alpha_1 - 2\beta_1 - 2\gamma_1 - d_2 - d_3 - e_1 - e_3 - f_1 - f_2 + \gamma_1d_2 + \beta_1d_3 + \gamma_1e_1 + \alpha_1e_3 + \beta_1f_1 + \alpha_1f_2 \right)
\]

Similarly, we obtain that
\[
|L_2| \geq \frac{2}{|L|} \left[ \begin{array}{ccc}
-1 & 1 & 1 \\
f_1 & f_2 & 1 \\
\alpha_1 & 1 & 1
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
-1 & 1 & 1 \\
\alpha_1 & 1 & 1 \\
e_1 & 1 & e_3
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
-1 & 1 & 1 \\
e_1 & 1 & e_3 \\
\alpha_1 & 1 & 1
\end{array} \right]
\]
\[
= \frac{2}{|L|} \left( 6 + 2\alpha_1 + 2\beta_2 - 2\gamma_2 + d_2 + d_3 + e_1 - e_3 + f_1 - f_2 + \beta_2d_3 - \alpha_2e_3 - \beta_2f_1 - \alpha_2f_2 - \gamma_2d_2 - \gamma_2e_1 \right)
\]

\[
|L_3| \geq \frac{2}{|L|} \left[ \begin{array}{ccc}
-1 & 1 & 1 \\
f_1 & f_2 & 1 \\
-\beta_2 & 1 & 1
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
-1 & 1 & 1 \\
-\beta_2 & 1 & 1 \\
-e_1 & 1 & e_3
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
-1 & 1 & 1 \\
-e_1 & 1 & e_3 \\
\alpha_2 & 1 & 1
\end{array} \right]
\]
\[
= \frac{2}{|L|} \left( 6 + 2\alpha_2 - 2\beta_2 + 2\gamma_1 - d_2 + d_3 - e_1 + e_3 + f_1 + f_2 - \beta_2d_3 + \alpha_2e_3 - \beta_2f_1 - \alpha_2f_2 - \gamma_1d_2 - \gamma_1e_1 \right)
\]

\[
|L_4| \geq \frac{2}{|L|} \left[ \begin{array}{ccc}
1 & -1 & 1 \\
f_1 & f_2 & 1 \\
\alpha_2 & 1 & 1
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
1 & -1 & 1 \\
\alpha_2 & 1 & 1 \\
-e_1 & 1 & -e_3
\end{array} \right] + \frac{2}{|L|} \left[ \begin{array}{ccc}
1 & -1 & 1 \\
-e_1 & 1 & -e_3 \\
1 & -1 & -\gamma_2
\end{array} \right]
\]
\[
= \frac{2}{|L|} \left( 6 - 2\alpha_2 + 2\beta_1 + 2\gamma_2 + d_2 - d_3 + e_1 + e_3 - f_1 + f_2 - \beta_1d_3 - \alpha_2e_3 - \beta_1f_1 - \alpha_2f_2 + \gamma_2d_2 + \gamma_2e_1 \right)
\]

Taking these sums, we obtain
\[
\frac{|L_0|}{2} = |L_1^0| + |L_2^0| + |L_3^0| + |L_4^0| \geq \frac{16}{3|L|}
\]
By (31), we have Proof of Lemma A.1. 

Remark A.2. More generally, if a vector field of the origin. We put 0 ≤ ζ(0), π, π ≤ 1, (±1, ±1, ±1) /∈ L° does not occur, and L° = [−1, 1] × [−1, 1] × [−1, 1]. Hence L° is a parallelepiped. □

Proof of Theorem 7.1 Combining Proposition 7.3 and Proposition 7.4, the conclusion follows immediately. □

A Proof of Proposition 6.1

To prove Proposition 6.1 we have to modify the map F defined on ∂D. We consider ∂D as a hexahedron, and deform F near each vertices of ∂D and deform near each edges of ∂D. We introduce some auxiliary functions satisfy (31), (32), and (33). We first check that G2 + H2 has same value at vertices.

Lemma A.1. (G2 + H2) takes same value at eight vertices on ∂D; (0, 0, 0), (0, π, 0), (0, 0, π), (0, 0, 0), (1, 0, 0), (1, π, 0), (1, π, π), (1, 0, π).

Remark A.2. More generally, if a vector field (0, G, H) satisfies (31), (32), and (33), then G2 + H2 takes same value at the eight vertices.

Proof of Lemma A.1. By (31), we have

\[
(G^2 + H^2)(1, 0, 0) = (G^2 + H^2)(0, 0, 0),
\]
\[
(G^2 + H^2)(1, π, 0) = (G^2 + H^2)(0, π, 0),
\]
\[
(G^2 + H^2)(1, π, π) = (G^2 + H^2)(0, π, π),
\]
\[
(G^2 + H^2)(1, 0, π) = (G^2 + H^2)(0, 0, π).
\]

By (32), we also obtain

\[
(G^2 + H^2)(0, π, 0) = (G^2 + H^2)(1, 0, 0),
\]
\[
(G^2 + H^2)(0, π, π) = (G^2 + H^2)(1, 0, π).
\]

By (33),

\[
(G^2 + H^2)(0, 0, π) = (G^2 + H^2)(0, π, 0).
\]

Thus we see the conclusion. □

To modify the vector field (F, G, H) on D ⊂ R3, we introduce a cut-off function ζ(s, φ, ψ). Let ζ : D → R be a smooth function such that, radially symmetric with respect to the origin, 0 ≤ ζ ≤ 1, ζ(0, 0, 0) = 1, and supp ζ is a small neighborhood of the origin. We put

\[
G_0(s, φ, ψ) := ζ(s, φ, ψ) - ζ(s, π - φ, ψ)
\]
\[- ζ(s, π - φ, π - ψ) + ζ(s, φ, π - ψ),
\]
\[
H_0(s, φ, ψ) := ζ(T_φ^{-1}(s), φ, ψ) - ζ(T_φ(s), π - φ, ψ)
\]
\[- ζ(T_φ(s), π - φ, π - ψ) + ζ(T_φ^{-1}(s), φ, π - ψ).
\]
Lemma A.3. \((0, G_0, H_0)\) satisfies \((31), (32), \text{ and } (33)\).

Proof. Since \(\text{supp } \zeta\) is a small neighborhood of the origin \((0, 0, 0)\), we see \(\zeta(1, \phi, \psi) = \zeta(s, \pi, \psi) = \zeta(s, \phi, \pi) = 0\). Combining them with the formulas of Lemma A.5 below, we can check that \((0, G_0, H_0)\) satisfies \((31), (32), \text{ and } (33)\) directly as follows:

\[
G_0(0, \phi, \psi) = \zeta(0, \phi, \psi) - \zeta(0, \pi - \phi, \psi) - \zeta(0, \pi - \phi, \pi - \psi) + \zeta(0, \phi, \pi - \psi),
\]

\[
H_0(0, \phi, \psi) = \zeta(T^{-1}_\psi(0), \phi, \psi) - \zeta(T_\psi(0), \pi - \phi, \psi) - \zeta(T_\psi(0), \pi - \phi, \pi - \psi) + \zeta(T^{-1}_\psi(0), \phi, \pi - \psi) = 0,
\]

\[
G_0(1, \phi, \psi) = \zeta(1, \phi, \psi) - \zeta(1, \pi - \phi, \psi) - \zeta(1, \pi - \phi, \pi - \psi) + \zeta(1, \phi, \pi - \psi) = 0,
\]

\[
H_0(1, \phi, \psi) = \zeta(T^{-1}_\psi(1), \phi, \psi) - \zeta(T_\psi(1), \pi - \phi, \psi) - \zeta(T_\psi(1), \pi - \phi, \pi - \psi) + \zeta(T^{-1}_\psi(1), \phi, \pi - \psi) = \zeta(0, \phi, \psi) - \zeta(0, \pi - \phi, \psi) - \zeta(0, \pi - \phi, \pi - \psi) + \zeta(0, \phi, \pi - \psi),
\]

\[
G_0(T_\psi(s), 0, \psi) = \zeta(T_\psi(s), 0, \psi) - \zeta(T_\psi(s), \pi - 0, \psi) - \zeta(T_\psi(s), \pi - 0, \pi - \psi) + \zeta(T_\psi(s), 0, \pi - \psi) - \zeta(T_\psi(s), 0, \psi) + \zeta(T_\psi(s), 0, \pi - \psi) = \zeta(s, 0, \psi) + \zeta(s, 0, \pi - \psi),
\]

\[
H_0(T_\psi(s), 0, \psi) = \zeta(T^{-1}_\psi(T_\psi(s)), 0, \psi) - \zeta(T_\psi(T_\psi(s)), \pi - 0, \psi) - \zeta(T_\psi(T_\psi(s)), \pi - 0, \pi - \psi) + \zeta(T^{-1}_\psi(T_\psi(s)), 0, \pi - \psi) = \zeta(s, 0, \psi) + \zeta(s, 0, \pi - \psi),
\]

\[
G_0(s, \pi, \psi) = \zeta(s, \pi, \psi) - \zeta(s, \pi - \pi, \psi) - \zeta(s, \pi - \pi, \pi - \psi) + \zeta(s, \pi, \pi - \psi) = -\zeta(s, 0, \psi) - \zeta(s, 0, \pi - \psi),
\]

\[
H_0(s, \pi, \psi) = \zeta(T^{-1}_\psi(s), \pi, \psi) - \zeta(T_\psi(s), \pi - \pi, \psi) - \zeta(T_\psi(s), \pi - \pi, \pi - \psi) + \zeta(T^{-1}_\psi(s), \pi, \pi - \psi) = -\zeta(T_\psi(s), 0, \psi) - \zeta(T_\psi(s), 0, \pi - \psi),
\]

\[
G_0(s, \pi - \phi, 0) = \zeta(s, \pi - \phi, 0) - \zeta(s, \pi - (\pi - \phi), 0) - \zeta(s, \pi - (\pi - \phi), \pi - 0) + \zeta(s, \pi - \phi, \pi - 0) = \zeta(s, \pi - \phi, 0) - \zeta(s, \phi, 0),
\]

\[
H_0(s, \pi - \phi, 0) = \zeta(T^{-1}_0(s), \pi - \phi, 0) - \zeta(T_0(s), \pi - (\pi - \phi), 0) - \zeta(T_0(s), \pi - (\pi - \phi), \pi - 0) + \zeta(T^{-1}_0(s), \pi - \phi, \pi - 0) = \zeta(T^{-1}_0(s), \pi - \phi, 0) - \zeta(T_0(s), \phi, 0),
\]

\[\text{60}\]
\begin{align*}
G_0(s, \phi, \pi) &= \zeta(s, \phi, \pi) - \zeta(s, \pi - \phi, \pi) \\
&\quad - \zeta(s, \pi - \phi, \pi - \pi) + \zeta(s, \phi, \pi - \pi) \\
&= -\zeta(s, \pi - \phi, 0) + \zeta(s, \phi, 0), \\
H_0(s, \phi, \pi) &= \zeta(T^{-1}_\pi(s), \phi, \pi) - \zeta(T_\pi(s), \pi - \phi, \pi) \\
&\quad - \zeta(T_\pi(s), \pi - \phi, \pi - \pi) + \zeta(T^{-1}_\pi(s), \phi, \pi - \pi) \\
&= -\zeta(T_\pi(s), \pi - \phi, 0) + \zeta(T^{-1}_\pi(s), \phi, 0).
\end{align*}

\square

Fix \( \epsilon > 0 \). We also introduce cut-off functions \( \zeta_1(s, \phi, \psi) \), \( \zeta_2(s, \phi, \psi) \), and \( \zeta_3(s, \phi, \psi) \) such that

\[ \zeta_1 = 1 \text{ on } [\epsilon, 1 - \epsilon] \times \{0\} \times \{0\}, \]
\[ \zeta_2 = 1 \text{ on } \{0\} \times [\epsilon, \pi - \epsilon] \times \{0\}, \]
\[ \zeta_3 = 1 \text{ on } \{0\} \times \{0\} \times [\epsilon, \pi - \epsilon], \]

each \( \text{supp} \zeta_i \ (i = 1, 2, 3) \) is contained in \( \epsilon/2 \)-neighborhood of \( [\epsilon, 1 - \epsilon] \times \{0\} \times \{0\} \), \( \{0\} \times [\epsilon, \pi - \epsilon] \times \{0\} \), and \( \{0\} \times \{0\} \times [\epsilon, \pi - \epsilon] \) respectively, and \( 0 \leq \zeta_i \leq 1 \ (i = 1, 2, 3) \). We put

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix}(s, \phi, \psi) := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(s, \phi, \psi) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\psi(s), \pi - \phi, \psi) + \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T^{-1}_\psi(s), \phi, \pi - \psi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(s, \pi - \phi, \pi - \psi),
\]

\[
\begin{pmatrix} G_2 \\ H_2 \end{pmatrix}(s, \phi, \psi) := \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \zeta_2(s, \phi, \psi) + \begin{pmatrix} -\alpha_2 \\ -\beta_2 \end{pmatrix} \zeta_2(s, \pi - \phi, \pi - \psi) + \begin{pmatrix} -\beta_2 \\ \alpha_2 \end{pmatrix} \zeta_2(T_\psi^{-1}(s), \phi, \psi) + \begin{pmatrix} \beta_2 \\ -\alpha_2 \end{pmatrix} \zeta_2(T_\psi(s), \pi - \phi, \pi - \psi),
\]

\[
\begin{pmatrix} G_3 \\ H_3 \end{pmatrix}(s, \phi, \psi) := \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} \zeta_3(s, \phi, \psi) + \begin{pmatrix} -\beta_3 \\ -\alpha_3 \end{pmatrix} \zeta_3(T_\psi(s), \pi - \phi, \psi) + \begin{pmatrix} -\alpha_3 \\ -\beta_3 \end{pmatrix} \zeta_3(T_\psi^{-1}(s), \phi, \psi),
\]

where we choose the \( \epsilon \) sufficiently small with \( 0 < \epsilon < 1/2 \), and \( \alpha_i, \beta_i \ (i = 1, 2, 3) \) are positive real numbers. Then, we can assume that \( \text{supp} G_i \cap \text{supp} G_j = \emptyset \) and \( \text{supp} H_i \cap \text{supp} H_j = \emptyset \) for \( i \neq j \).

**Lemma A.4.** For each \( i = 1, 2, 3 \), the vector field \((0, G_i, H_i)\) on \( D \) satisfies \((31), \ (32), \) and \((33)\).

**Proof.** By the definition of \( \zeta_i \), we have

\[
\zeta_i(1, \cdot, \cdot) = \zeta_i(\cdot, \pi, \cdot) = \zeta_i(\cdot, \cdot, \pi) = 0 \quad (i = 1, 2, 3),
\]
\[
\zeta_i(0, \cdot, \cdot) = \zeta_2(\cdot, 0, \cdot) = \zeta_3(\cdot, \cdot, 0) = 0.
\]
Combining them with the formulas of Lemma A.5 below, we can check that \((0, G_1, H_1)\) satisfies (31), (32), and (33) directly as follows:

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix} (0, \phi, \psi) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(0, \phi, \psi) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\psi(0), \pi - \phi, \psi) \\
+ \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_\psi^{-1}(0), \phi, \pi - \psi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(0, \pi - \phi, \pi - \psi) \\
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix} (1, \phi, \psi) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(1, \phi, \psi) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\psi(1), \pi - \phi, \psi) \\
+ \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_\psi^{-1}(1), \phi, \pi - \psi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(1, \pi - \phi, \pi - \psi) \\
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix} (T_\psi(s), 0, \psi) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(T_\psi(s), 0, \psi) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\psi(T_\psi(s)), \pi, \psi) \\
+ \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_\psi^{-1}(T_\psi(s)), 0, \pi - \psi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(T_\psi(s), \pi, \pi - \psi) \\
= \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(T_\psi(s), 0, \psi) + \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(s, 0, \pi - \psi),
\]

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix} (s, \pi, \psi) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(s, \pi, \psi) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\psi(s), 0, \psi) \\
+ \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_\psi^{-1}(s), \pi, \pi - \psi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(s, 0, \pi - \psi) \\
= \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\psi(s), 0, \psi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(s, 0, \pi - \psi),
\]

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix} (s, \pi - \phi, 0) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(s, \pi - \phi, 0) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_0(s), \phi, 0) \\
+ \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_0^{-1}(s), \pi - \phi, \pi) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(s, \phi, \pi) \\
= \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(s, \pi - \phi, 0) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_0(s), \phi, 0),
\]

\[
\begin{pmatrix} G_1 \\ H_1 \end{pmatrix} (s, \phi, \pi) = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \zeta_1(s, \phi, \pi) + \begin{pmatrix} -\beta_1 \\ -\alpha_1 \end{pmatrix} \zeta_1(T_\pi(s), \pi - \phi, \pi) \\
+ \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_\pi^{-1}(s), \phi, 0) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(s, \pi - \phi, 0) \\
= \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \zeta_1(T_\pi^{-1}(s), \phi, 0) + \begin{pmatrix} -\alpha_1 \\ -\beta_1 \end{pmatrix} \zeta_1(s, \pi - \phi, 0).
\]

Similarly, \((0, G_i, H_i)\) \((i = 2, 3)\) also satisfy (31), (32), and (33). □
Here, we show the lemma used in the above lemmas.

**Lemma A.5.** \( T_0^{-1}(s) = T_\pi(s), \ T_\pi^{-1}(s) = T_0(s) \).

**Proof.** Applying Lemma 5.6 to \( K = K(\theta, \phi, \psi) \), we see that

\[
\Theta(\pi - \Theta(\theta, \phi, \psi) + \theta, \phi, \psi) + \Theta(\theta, \phi, \psi) = \pi.
\]

Combining (29) with the definition of \( T_\psi \), we have

\[
\Gamma_\psi (\Theta(0, \pi, \psi)T_\psi(s)) = \pi - \Theta(0, 0, \psi)s.
\]

By the definition of \( \Gamma_\psi \),

\[
\pi - \Theta(\Theta(0, \pi, \psi)T_\psi(s), 0, \psi) + \Theta(0, \pi, \psi)T_\psi(s) = \pi - \Theta(0, 0, \psi)s.
\]

Put \( \Theta_0 := \Theta(0, 0, 0) \) and \( \Theta_1 := \Theta(0, \pi, 0) \). Then, by (30) and (29),

\[
\Theta_0 = \Theta(0, \pi, \pi), \quad \Theta_1 = \Theta(0, 0, \pi) = \pi - \Theta_0.
\]

Thus, we obtain

\[
\begin{align*}
\pi & - \Theta(\Theta_1 T_0(s), 0, 0) + \Theta_1 T_0(s) = \pi - \Theta_0 s, \quad (55) \\
\pi & - \Theta(\Theta_0 T_\pi(t), 0, \pi) + \Theta_0 T_\pi(t) = \pi - \Theta_1 t, \quad (56)
\end{align*}
\]

for \( s, t \in [0, 1] \), and such \( T_0(s), T_\pi(t) \in [0, 1] \) is uniquely determined, since \( \Gamma_\psi : [0, \Theta(0, \pi, \psi)] \to [\pi - \Theta(0, 0, \psi), \pi] \) is bijective by Lemma 5.13.

On substituting \((\theta, \phi, \psi) = (\Theta_0 T_\pi(t), 0, \pi)\) into (54), we have

\[
\Theta\left(\pi - \Theta(\Theta_0 T_\pi(t), 0, \pi) + \Theta_0 T_\pi(t), 0, \pi\right) + \Theta(\Theta_0 T_\pi(t), 0, \pi) = \pi.
\]

By (56) and Lemmas 5.8 and 5.11, the first term of the left-hand side is

\[
\Theta\left(\pi - \Theta(\Theta_0 T_\pi(t), 0, \pi) + \Theta_0 T_\pi(t), 0, \pi\right) = \Theta(\pi - \Theta_1 t, 0, \pi)
\]

\[
= \Theta(X(\pi)X(-\Theta_1 t)Z(\pi)K)
\]

\[
= \Theta(X(\pi)X(\Theta_1 t)K K)
\]

\[
= \Theta(Z(\pi)X(\Theta_1 t)K)
\]

\[
= \pi - \Theta(X(\Theta_1 t)K)
\]

\[
= \pi - \Theta(\Theta_1 t, 0, 0).
\]

By (56),

\[
\Theta(\Theta_0 T_\pi(t), 0, \pi) = \Theta_0 T_\pi(t) + \Theta_1 t.
\]

Consequently, we get

\[
\pi - \Theta(\Theta_1 t, 0, 0) + \Theta_1 t = \pi - \Theta_0 T_\pi(t).
\]

By (55) with \( s = T_\pi(t) \), the uniqueness asserts that \( t = T_0(T_\pi(t)) \).
Proof of Proposition 6.1. For

\[ F \mapsto \mathcal{F} = \frac{(\tilde{F}, \tilde{G}, \tilde{H})}{\sqrt{\tilde{F}^2 + \tilde{G}^2 + \tilde{H}^2}} : \partial D \to S^2, \]

we consider the following conditions:

(C1) \( \mathcal{F} \) is homotopic to \( F \).

(C2) \( \tilde{F}^2 + \tilde{G}^2 + \tilde{H}^2 \neq 0 \) on \( \partial D \).

(C3) \( (\tilde{F}, \tilde{G}, \tilde{H}) \) satisfies the identities (31), (32), and (33).

(C4) \( \tilde{G}^2 + \tilde{H}^2 \neq 0 \) at the vertices of \( \partial D \).

(C5) \( \tilde{G}^2 + \tilde{H}^2 \neq 0 \) on \( \partial M_i \) (\( i = 1, \ldots, 6 \)).

We shall construct \( \mathcal{F} \) such that the conditions (C1)–(C5) hold and \((\pm 1, 0, 0) \in S^2\) are regular values of \( \mathcal{F} \).

First, we put

\[ \mathcal{F}_0 := \frac{(F, G_0, H_0)}{\sqrt{F^2 + G_0^2 + H_0^2}} := \frac{(F, G + \alpha_0 G_0, H + \alpha_0 H_0)}{\sqrt{F^2 + (G + \alpha_0 G_0)^2 + (H + \alpha_0 H_0)^2}} : \partial D \to S^2, \]

where we take sufficiently small \( \alpha_0 > 0 \) if \( (G^2 + H^2)(0, 0, 0) = 0 \), take \( \alpha_0 = 0 \) if \( (G^2 + H^2)(0, 0, 0) \neq 0 \). Then

\[ (\tilde{G}_0^2 + \tilde{H}_0^2)(0, 0, 0) = (G(0, 0, 0) + \alpha_0)^2 + (H(0, 0, 0))^2 \neq 0. \] (57)

By Lemma A.3, \((0, G_0, H_0)\) satisfies (31), (32), and (33). By (57) and Remark A.2, (C4) holds for \((F, G_0, H_0)\).

Since \( F^2 + G^2 + H^2 \neq 0 \) on \( \partial D \) by the assumption, for sufficiently small \( \alpha_0 \), \( F^2 + (G + \alpha_0 G_0)^2 + (H + \alpha_0 H_0)^2 \neq 0 \) on \( \partial D \). Thus, \( \mathcal{F}_0 \) satisfies the conditions (C1) and (C2). By Lemma A.3, \( \mathcal{F}_0 \) satisfies the conditions (C3). Consequently, (C1)–(C4) hold for \( \mathcal{F}_0 \).

Second, by continuity, there exist \( \delta_0 > 0 \) and \( \delta_1 > 0 \) such that \( \sqrt{\tilde{G}_0^2 + \tilde{H}_0^2} \geq \delta_1 \) on \( \delta_0 \)-neighborhood of the vertices of \( D \). For \( \alpha_i \) and \( \beta_i \) above with \( \alpha_i^2 + \beta_i^2 \leq \delta_1/4 \), on \( \delta_0 \)-neighborhood of the vertices, we see \( |(G_i, H_i)| \leq \sqrt{\alpha_i^2 + \beta_i^2} \leq \delta_1/4 \). Thus

\[ |(\tilde{G}_0 + G_1 + G_2 + G_3, \tilde{H}_0 + H_1 + H_2 + H_3)| \geq |(\tilde{G}_0, \tilde{H}_0)| - \sum_{i=1}^{3} |(G_i, H_i)| \geq \delta_1/4 > 0 \]
on \( \delta_0 \)-neighborhood of the vertices.

Thus, the points on \( \partial M_i \) with \((\tilde{G}_0 + G_1 + G_2 + G_3, \tilde{H}_0 + H_1 + H_2 + H_3) = (0, 0)\) are contained in the following 12 lines: \( [\delta_0/2, 1 - \delta_0/2] \times \{0, \pi\} \times \{0, \pi\}, \{0, 1\} \times [\delta_0/2, \pi - \delta_0/2] \times \{0, \pi\}, \{0, 1\} \times \{0, \pi\} \times [\delta_0/2, \pi - \delta_0/2] \). Moreover, on the 12 lines, \((G_i, H_i) \) \((i = 1, 2, 3)\) are constants. Consider the image \( U \) of \((G + \alpha_0 G_0, H + \alpha_0 H_0)\) on the 12 lines. Since \( U \) is the union of 12 curves of class \( C^\infty \), the 2-dimensional
Lebesgue measure of $U$ is 0. Therefore, there exists $\alpha_i, \beta_i$ such that $|\alpha_i|$ and $|\beta_i|$ are small, and
\[(\pm \alpha_i, \pm \beta_i) \cap U = \emptyset, (\pm \beta_i, \pm \alpha_i) \cap U = \emptyset\]
for $i = 1, 2, 3$. In this setting, on the 12 lines, we obtain
\[(\tilde{G}_0 + G_1 + G_2 + G_3, \tilde{H}_0 + H_1 + H_2 + H_3) \neq (0, 0).
Thus, we put $\tilde{G}_1 := \tilde{G}_0 + G_1 + G_2 + G_3, \tilde{H}_1 := \tilde{H}_0 + H_1 + H_2 + H_3,$ and
\[\mathcal{F}_1 := \frac{(F, \tilde{G}_1, \tilde{H}_1)}{\sqrt{F^2 + \tilde{G}_1^2 + \tilde{H}_1^2}} : \partial D \to S^2\]
for sufficiently small $|\alpha_i|$ and $|\beta_i|$. Then $\mathcal{F}_1$ satisfies (C1)–(C5).

Next, we will construct a deformation of $\mathcal{F}_1$ which satisfies that $(\pm 1, 0, 0)$ are its regular values on $M_1 \cup M_2$.

Since $(\tilde{G}_1, \tilde{H}_1) \neq (0, 0)$ on $\partial M_1$, there exists $\delta_2 > 0$ such that
\[\sqrt{\tilde{G}_1^2 + \tilde{H}_1^2} \geq \delta_2 \text{ on } V,
\]
where $V \subset M_1$ is some small neighborhood of $\partial M_1$. By Sard’s theorem, there exist positive real numbers $\alpha_4, \beta_4$ with $\sqrt{\alpha_4^2 + \beta_4^2} < \delta_2/2$ such that $(\alpha_4, \beta_4)$ is a regular value of $(\tilde{G}_1, \tilde{H}_1)$ on $M_1$, and $(-\beta_4, \alpha_4)$ is a regular value of $(\tilde{G}_1, \tilde{H}_1)$ on $M_2$. We note that, since $|\alpha_4|$ and $|\beta_4|$ are small, $F \neq 0$ at the regular points $(\tilde{G}_1, \tilde{H}_1)^{-1}(\alpha_4, \beta_4)$ on $M_1$ and $(\tilde{G}_1, \tilde{H}_1)^{-1}(-\beta_4, \alpha_4)$ on $M_2$. Let $\zeta_4(\phi, \psi) \in C_0^\infty(\int M_1, [0, 1])$ be a cut-off function such that $\zeta_4 = 1$ on $M_1 \setminus V$. We put
\[(G_4, H_4)(s, \phi, \psi) := \begin{cases} (\alpha_4 \zeta_4(\phi, \psi), \beta_4 \zeta_4(\phi, \psi)) & \text{if } (s, \phi, \psi) \in M_1, \\ (-\beta_4 \zeta_4(\phi, \psi), \alpha_4 \zeta_4(\phi, \psi)) & \text{if } (s, \phi, \psi) \in M_2, \\ (0, 0) & \text{if } (s, \phi, \psi) \in M_3 \cup \cdots \cup M_6. \end{cases}\]
Then $(0, G_4, H_4)$ satisfies (31), (32), and (33). Thus, we define $\hat{G}_2 := \tilde{G}_1 - G_4, \hat{H}_2 := \tilde{H}_1 - H_4,$ and
\[\hat{\mathcal{F}}_2 := \frac{(F, \hat{G}_2, \hat{H}_2)}{\sqrt{F^2 + \hat{G}_2^2 + \hat{H}_2^2}} : \partial D \to S^2.
\]
Then, $\hat{\mathcal{F}}_2$ satisfies (C1)–(C5). On $V(\subset M_1)$, we obtain
\[|\langle \hat{G}_2, \hat{H}_2 \rangle| \geq |\langle \tilde{G}_1, \tilde{H}_1 \rangle| - |\langle G_4, H_4 \rangle| \geq \frac{\delta_2}{2} > 0.
\]
So, there exist no zeros of $(\hat{G}_2, \hat{H}_2)$ on $V$. On $M_1 \setminus V$, we have
\[\langle \hat{G}_2, \hat{H}_2 \rangle = (0, 0) \text{ if and only if } (\tilde{G}_1, \tilde{H}_1) = (\alpha_4, \beta_4).
\]
Thus, by the definition of $(\alpha_4, \beta_4)$, $(0, 0)$ is a regular value of $(\hat{G}_2, \hat{H}_2)$ on $M_1 \setminus V$, and $F \neq 0$ at the regular values. Therefore, $(\pm 1, 0, 0)$ are regular values of $\hat{\mathcal{F}}_2$ on $M_1$. Similarly, we obtain that $(\pm 1, 0, 0)$ are regular values of $\hat{\mathcal{F}}_2$ on $M_2$.

Moreover, we can modify $\hat{\mathcal{F}}_2$ similarly with respect to $M_3, \ldots, M_6$. Then the modified $\mathcal{F}$ satisfies the conditions (i)–(iv) in Proposition 6.1.
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