Shadowing by non-uniformly hyperbolic periodic points and uniform hyperbolicity

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Abstract

We prove that, under a mild condition on the hyperbolicity of its periodic points, a map \(g\) which is topologically conjugated to a hyperbolic map (respectively, an expanding map) is also a hyperbolic map (respectively, an expanding map). In particular, this result gives a partial positive answer for a question asked by A Katok, in a related context.

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1. Introduction

Since Smale proposed the notion of a uniformly hyperbolic dynamical system, the theory and results obtained by dynamists around the world have described many of its features, from the structural and measure-theoretical points of view.

Nevertheless, the study of conditions for a non-uniformly expanding (NUE) map to expand is not well understood, as regards the few results concerning the subject. One of these results is the remarkable theorem of Mañe [5], valid for invariant sets without critical points for interval maps. Outside this setting, not much is known, and it is by itself an interesting point of research. In particular, the study of non-uniform expanding rates and conditions over a given set of points and their relations with uniform expanding behaviour appears in several recent papers [3, 9, 10]. Let us briefly describe some of these results.

We say that a local diffeomorphism \(f\) is NUE on a set \(X\), if there exists \(\eta < 0\) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| [Df(j(f(x))^{-1}]^{-1} \| \leq \eta < 0 \quad \text{for all } x \in X.
\]
In [3], the authors proved that any local diffeomorphism in a compact manifold admitting non-uniform expansion at a set of total probability, i.e. with full measure for any invariant measure, is in fact an expanding map. Similar results hold for diffeomorphisms.

From Oseledets [7], one knows that if $\mu$ is an invariant measure for a $C^1$ map $f$, then the number

$$\lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \|Df^nx(x)v\|$$

is defined in a set of total probability and it is called the Lyapunov exponent at $x$ in the direction $v$. In [9], the author proved that if $f$ is a local diffeomorphism such that all Lyapunov exponents are positive then it is, in fact, an expanding map and he also obtained the results for diffeomorphisms admitting continuous splitting (see theorems 9 and 16).

Here, we consider the universe of systems without critical points which are topologically conjugated to expanding maps. In such a context, a necessary and sufficient condition for a system to be expanding is just that it is NUE on the set of periodic points. Therefore, as a main result, we prove that a local diffeomorphism topologically conjugated to an expanding map is itself an expanding map if, and only if, it is NUE on the set of periodic points. We also obtain a similar result for dynamics with non-uniformly hyperbolic (NUH) periodic points conjugated to an uniformly hyperbolic map.

**Theorem A.** Let $g : M \to M$ be a $C^2$-class local diffeomorphism on a compact manifold $M$. Suppose that $g$ is topologically conjugated to an expanding $C^1$ map $f$. If $g$ is NUE on the set $\text{Per}(g)$ of periodic points, then $g$ is an expanding map.

**Remark 1.** We observe that the condition NUE on the periodic points is not enough to assure that the map $g$ is expanding, even if we assume that $g$ is topologically conjugated to an expanding map. It is a standard matter that the map $z \to z^2$, defined on the circle, is topologically conjugated to a map with criticalities satisfying the NUE condition on the periodic points. See figure 1.

In theorem A, due to the fact that we are dealing with maps that are local diffeomorphisms, we avoid examples as in remark 1.

For diffeomorphisms, the existence of a continuous splitting of $M$ plays a similar role. In [10], the authors exhibit an example of a non-hyperbolic horseshoe such that the splitting is continuous over the periodic points and all Lyapunov exponents are non-zero and bounded away from zero. In particular, some condition of continuity of the splitting in the closure of the periodic points is necessary. In order to state our results in the invertible case, we need the following definition.

**Definition 2 (NUH set).** Let $g : M \to M$ be a diffeomorphism on a compact manifold $M$. We say that an invariant set $S \subset M$ is a NUH set or, simply, NUH, iff

1. there is an $Dg$-invariant splitting $T_S M = E^c \oplus E^u$,
2. there exists $\eta < 0$ and an adapted Riemannian metric for which any point $p \in S$ satisfies

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg^j(p)\|_{E^c(p)} \leq \eta$$

and

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg^j(p)\|_{E^u(p)} \leq \eta.$$
We also recall here the notion of hyperbolic set:

**Definition 3.** Let $\Lambda$ be an invariant set for a $C^1$ diffeomorphism $f$ of a manifold $M$. We say $\Lambda$ is a hyperbolic set if there is a continuous splitting $T\Lambda M = E^s \oplus E^u$ which is $Tf$-invariant ($Tf(E^s) = E^s$, $Tf(E^u) = E^u$) and for which there are constants $c > 0$, $0 < \lambda < 1$, such that

$$\|Tf^n|_{E^s}\| < c\lambda^n, \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n, \quad \forall n \in \mathbb{N}.$$ 

For the diffeomorphism case, we have two slightly different results.

**Theorem B.** Let $g : M \to M$ be a $C^2$ diffeomorphism on a compact manifold $M$, and let $\Lambda \subset M$ be a compact invariant set. Suppose that $g|\Lambda$ is topologically conjugated to a $C^1$ diffeomorphism $f$ restricted to a set $\Lambda$, hyperbolic for $f$. If the set $\text{Per}(g)$ of periodic points of $g$ is NUH and $T_{\text{Per}(g)} M = E^s \oplus E^u$ is a dominated splitting, then $\Lambda$ is a hyperbolic set for $g$.

**Theorem C.** Let $g : M \to M$ be a $C^2$ diffeomorphism on a compact manifold $M$, and let $\Lambda \subset M$ be a compact invariant set. Suppose that $g|\Lambda$ is topologically conjugated to a $C^1$ diffeomorphism $f$ restricted to a set $\Lambda$, hyperbolic for $f$. If the set $\text{Per}(g)$ of periodic points of $g$ is NUH, and $T_{\text{Per}(g)} M = E^s \oplus E^u$ has a continuous extension to a splitting on $T_{\text{Per}(g)} M$, then $\Lambda$ is a hyperbolic set for $g$.

In fact, theorem B is a consequence of theorem C. Nevertheless, the hypotheses in B are easier to verify.

**Remark 4.** With the same proof, all results in this paper are valid if the derivative $Dg$ is just Hölder continuous. Without modifying the proofs, we can also exchange $f$ by a shift in the hypotheses of our theorems, obtaining the same hypotheses.
Definition 5 (Shadowing by periodic points). Let \( f : M \to M \) be a map and \( \hat{\Lambda} \subset M \) be a compact \( g \)-invariant set. We say that \( (f, \hat{\Lambda}) \) has the shadowing by periodic points property if given \( \epsilon > 0 \) there exists a periodic point \( p \in M \) with period \( n \) such that \( d(f^n(p), f^j(x)) < \epsilon \), \( \forall 0 \leq j \leq n \). In this case, we say that the orbit of \( p \) \( \epsilon \)-shadows the orbit segment \( \{x, \ldots, f^n(x)\} \).

If \( \hat{\Lambda} \subset M \) is a hyperbolic, compact invariant set for a diffeomorphism \( f \), then the classical theory of hyperbolic systems implies that \( (f, \hat{\Lambda}) \) has the shadowing by periodic points property (see proposition 8.5 in [6]). The same is also valid for any system which is topologically conjugated to \( f \). Shadowing by periodic points is the key ingredient in the proofs of the theorems A, B, and C we stated earlier. Therefore, as a consequence of their proofs, we also obtain the following (more general) results.

Theorem D. Let \( g : M \to M \) be a \( C^2 \) local diffeomorphism on a compact manifold \( M \). Suppose there exists an invariant compact set \( \hat{\Lambda} \subset M \) such that \( (g, \hat{\Lambda}) \) has the shadowing by periodic points property. If \( g \) is NUE on the set \( \text{Per}(g) \) of periodic points, then \( g \) is an expanding map on \( \hat{\Lambda} \).

Theorem E. Let \( g : M \to M \) be a \( C^2 \) diffeomorphism on a compact manifold \( M \), and let \( \Lambda \subset M \) be a compact \( g \)-invariant set. Suppose that \( (g, \Lambda) \) has the shadowing by periodic points property. If the set \( \text{Per}(g) \) of periodic points of \( g \) is NUH and \( T_{\text{Per}(g)} M = E^s \oplus E^u \) is a dominated splitting, then \( \Lambda \) is a hyperbolic set for \( g \).

Theorem F. Let \( g : M \to M \) be a \( C^2 \) diffeomorphism on a compact manifold \( M \), and let \( \Lambda \subset M \) be a compact \( g \)-invariant set. Suppose that \( (g, \Lambda) \) has the shadowing by periodic points property. If the set \( \text{Per}(g) \) of periodic points of \( g \) is NUH and \( T_{\text{Per}(g)} M = E^s \oplus E^u \) has a continuous extension to a splitting on \( T_{\text{Per}(g)} M \), then \( \Lambda \) is a hyperbolic set for \( g \).

2. The endomorphism case: NUE periodic set

During this section, \( g : M \to M \) will always be a \( C^2 \)-local diffeomorphism which is topologically conjugated to a \( C^1 \) expanding endomorphism.

We start by stating the following remark on the properties of a periodic set which is NUH set (see definition in section 1).

Remark 6. Given a point \( p \in \text{Per}(g) \), let us set \( t = t(p) := \text{period}(p) \). In such a case, the following equivalence is immediate: \( S := \text{Per}(g) \) is NUE iff there exists \( \varsigma < 1 \) such that for each periodic point \( p \), \( \prod_{j=0}^{t(p)-1} \| [Dg]^{(g^j(p))^{-1}} \| < \varsigma^{t(p)} \).

For the sequence, we give a simplified definition for the case of local diffeomorphisms of the notion introduced by [4].

Definition 7 (Hyperbolic time for local diffeomorphisms). Let \( z \in M \) be a regular point. We say that \( k \in \mathbb{N} \) is a \( \varsigma \)-hyperbolic time for \( z \) if, for \( i = 1, \ldots, k \), holds
\[
\prod_{j=1}^{i} \| [Dg]^{(g^{(j-1)(z))^{-1}}} \| \leq \varsigma^i.
\]

Lemma 8. Suppose that \( g \) is topologically conjugated to an expanding map \( f \). Let \( x \) be a recurrent, regular point of \( g \). If \( \text{Per}(g) \) is NUE, then all Lyapunov exponents of \( x \) are positive.
Proof. Let $\delta > 0$ such that, given any ball $B(z, \delta)$, the corresponding inverse branches of $g$ are well-defined diffeomorphisms. Let $\zeta = e^n, \eta < 0$ as in the definition of NUE, on page 75, $\zeta < \zeta' < 1$ fixed, and let $\epsilon > 0$ such that $(\sqrt{\zeta'})^{-1} - \epsilon > 1$. Since $x$ is a regular point, there is $n_0 \in \mathbb{N}$ such that

$$(\zeta_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\zeta_j + \epsilon)^n \cdot \|v_j\| \quad \forall v_j \in E_j, \quad \forall n \geq n_0.$$

where $E_j$ are the Lyapunov eigenspaces and $\log(\zeta_j)$ are their respective Lyapunov exponents.

Now, by Pliss’ lemma [8], there exists $n_1 > n_0$ such that for any point $y$ and $n > n_1$ for which $\prod_{j=0}^{n} \|Dg^n(\eta_j(y))^{-1}\|^{-1} \geq \zeta^{-n}$ holds, then $y$ has at least $n_0$-hyperbolic times less than $n$.

We fix $0 < \delta' < \delta$ such that

$$\|Dg^{-1}(y)\| \leq \frac{1}{\sqrt{\zeta}}\|Dg^{-1}(z)\|, \quad \forall z, y; d(z, y) < \delta'.$$

where $g^{-1}$ is an inverse branch for $g$.

We set $0 < \delta'' < \delta'$ such that if $g^{-n}$ is an arbitrary composition of $n$ inverse branches for $g$, then $\text{diam}(g^{-n}(B(z, \delta''))) < \delta'$, $\forall z \in M, \forall n \in \mathbb{N}$. This occurs because it is valid for the hyperbolic system $f$ to which $g$ is conjugated.

As $x$ is a recurrent point, we set $n_2 \geq n_1$ a return time such that a neighbourhood $V_x \subset B(x, \delta'')$ of $x$ is taken by $g^n$ onto $B(x, \delta')$.

Therefore, writing $G := (g^n|_{V_x})^{-1}, G : B(x, \delta'') \rightarrow V_x \subset B(x, \delta')$ has a fixed point $p \in V_x$, which is a periodic point of period $n_2$ for $g$. By hypothesis, $p$ is a hyperbolic periodic point for which we have

$$\prod_{j=0}^{n_2-1} \|Dg^n(\eta_j(p))^{-1}\|^{-1} \geq \|\zeta^{-n_2}\| \Rightarrow \|DG(p)\| \leq \|\zeta^{-n_2}\|.$$

By our choice of $n_1$ and the equation above, there exists a $\zeta'$-hyperbolic time $n_0 < n' < n_2$ for $p$.

Due to lemma 2.7 in [4] (see also proposition 2.23 in [2]), $n'$ is also a $\sqrt{\zeta'}$-hyperbolic time for $x$. In particular, this implies that

$$\|Dg^n(x) \cdot v\| \geq \sqrt{\zeta'}^{-n'} \|v\|, \quad \forall v \in T_p M.$$

Therefore, $\zeta_j \geq \sqrt{\zeta'}^{-1} - \epsilon > 1, \forall j$. This means that all Lyapunov exponents of $x$ are greater than 1. \qed

We note that the set of Oseledet’s regular, recurrent points is a total probability set, due to Oseledet’s theorem and Poincaré’s recurrence theorem. This means that such a set has measure equal to 1 for any $g$-invariant probability measure. So, for any $g$-invariant measure, lemma 8 implies that all Lyapunov exponents are positive. Therefore, our theorem A is obtained by applying lemma 8 to the following result.

Theorem 9 ([9]). Let $f : M \rightarrow M$ be a $C^1$ local diffeomorphism on a compact Riemannian manifold. If the Lyapunov exponents of every $f$ invariant probability measure are positive, then $g$ is uniformly expanding.

3. The diffeomorphism case: NUH periodic set

Now we treat the case when $f$ is a diffeomorphism. Throughout this section, we suppose that the periodic set $\text{Per}(g)$ is NUH (see definition 2, on page 76).
The following remark is analogous to remark 6 for the diffeomorphism case.

**Remark 10.** We note that the set of periodic points $\text{Per}(g)$ is NUH iff there exists $\zeta < 1$ such that for each periodic point $p$ with period $t(p)$, $\prod_{j=0}^{t(p)-1} \|Dg|_{E^c}(g^j(p))\|^{-1} < \zeta^{t(p)}$ and $\prod_{j=0}^{t(p)-1} \|Dg|_{E^u}(g^j(p))\| < \zeta^{t(p)}$.

Before we state and prove the next lemma let us introduce some notation. Given a periodic point $p \in M$, we denote the cone over $E^s(p)$ by $E^s(c_p)$ and that for each periodic point $p \in M$, the set of period points $\text{Per}(g)$.

**Remark 11.** Let $g : M \rightarrow M$ be a $C^2$ diffeomorphism and $\Lambda \subset M$ be some compact $g$-invariant set. Suppose that $g|\Lambda$ is topologically conjugated to a $f|\Lambda$, where $\Lambda$ is a hyperbolic set for $f$. Let $x$ be a recurrent, regular point of $g$. Suppose that $\text{Per}(g)$ is NUH and that the splitting $T_{\text{Per}(g)} = E^s \oplus E^u$ has a continuous extension to $T_{\text{Per}(g)} M = E^s \oplus E^u$. Then all Lyapunov exponents of $x$ are non-zero.

**Lemma 12.** Let $g : M \rightarrow M$ be a $C^2$ diffeomorphism and $\Lambda \subset M$ be some compact $g$-invariant set. Suppose that $g|\Lambda$ is topologically conjugated to a $f|\Lambda$, where $\Lambda$ is a hyperbolic set for $f$. Let $x$ be a recurrent, regular point of $g$. Suppose that $\text{Per}(g)$ is NUH and that the splitting $T_{\text{Per}(g)} = E^s \oplus E^u$ has a continuous extension to $T_{\text{Per}(g)} M = E^s \oplus E^u$. Then all Lyapunov exponents of $x$ are non-zero.

**Proof.** Let $\zeta = \epsilon^0$, $\eta < 0$ as in definition 2, $\zeta < \zeta^\prime < 1$ fixed, and let $\epsilon > 0$ such that $(\sqrt[2]{\zeta^\prime})^{-1} - \epsilon > 1$. Since $x$ is a regular point, there is $n_0 \in \mathbb{N}$ such that

$$((\zeta_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\zeta_j + \epsilon)^n \cdot \|v_j\|, \quad \forall v_j \in E_j, \quad n \geq n_0$$

and

$$((\zeta_j - \epsilon)^\eta \cdot \|v_j\| > \|Dg^\eta(x) \cdot v_j\| > (\zeta_j + \epsilon)^\eta \cdot \|v_j\|, \quad \forall v_j \in E_j, \quad n \geq n_0,$$

where $E_j$ are the Lyapunov eigenspaces and $\log(\zeta_j)$ are their respective Lyapunov exponents. We denote by $E^s(x)$ (respectively, $E^u(x)$) the space spanned by the Lyapunov eigenspaces with negative (respectively, positive) Lyapunov exponents. $E^\epsilon(x)$ will denote the Lyapunov eigenspace corresponding to an eventual zero Lyapunov exponent.

Let us prove that the dimension of the space $E^\epsilon(x)$ corresponding to the negative Lyapunov exponents of $x$ is equal to or greater than the dimension of the stable space of any periodic point. An analogous result will obviously hold for $E^\epsilon(x)$. Therefore, we conclude that $T_x M = E^s(x) \oplus E^u(x)$ and that all Lyapunov exponents of $x$ are non-zero.

By using charts, due to the uniform continuity of $Dg$, we can fix $0 < a < 1$ and $0 < \delta'$ such that if $z$ is periodic,

$$\|Dg(y) \cdot v\| \leq \frac{1}{\sqrt{s}} \|Dg|_{E^\epsilon(z)}(z)\| \cdot \|v\|, \quad \forall y \in B(z, \delta'), \quad v \in C^n_a(z).$$
Due to the continuity of $E^1$, we can assume a small enough such that each cone $C^\prime_s(z)$ contains $E^1(y)$ or all points $y \in B(z, \delta) \cap \text{Per}(g)$.

Now, by Pliss’ lemma in [8], there exists $n_1 > n_0$ such that any point $z \in \text{Per}(g)$ for which we have $\prod_{j=0}^{n_1} \|Dg|_{E^1(g^{-n_1+1}(z))}\| \leq \varsigma^r$; for some $n \geq n_1$, then $z$ has, at least, $n_0 \varsigma^{-n}$-hyperbolic times less than $n$.

As $x$ is a recurrent point (also for $g^{-1}$), we set $n_2 \geq n_1$ a return time for $g^{-1}$ such that there exists one periodic point $p$ with period $n_2$ that $\delta/3$-shadows the orbit segment $\{x, g^{-1}(x), \ldots, g^{-n_2}(x)\}$. Such periodic point exists because $g|_{\Lambda}$ is conjugated to a diffeomorphism $f|_{\Lambda}$ which is shadowed by periodic points (see proposition 8.5 in [6]).

By hypothesis, $p$ is a hyperbolic periodic point for which we have

$$\prod_{j=0}^{n_1-1} \|Dg|_{E^\prime_j(g^j(p))}\| \leq \|\varsigma^{n_2}\|.$$ 

By our choice of $n_1$ and the equation above, there exists a $\varsigma$-hyperbolic time $n_0 < n' < n_2$ for $p$.

Due to proposition 2.23 in [2], $n'$ is also a $\sqrt{\varsigma}$-hyperbolic time for $x$. More precisely, this means that

$$\prod_{j=0}^{n_1-1} \|Dg|_{E^\prime_j(g^{-n'+1}(x))}\| \leq \sqrt{\varsigma},$$

since the space $E^1(g^{-n'+1}(x)) \subset C^\prime_s(g^{-n'}(p))$.

In particular, this implies that

$$\|Dg^{-n'}(x) \cdot v\| \geq \sqrt{\varsigma}^{-n'} \|v\|, \quad \forall v \in E^1(x). \quad (3)$$

This implies that the dimension of the negative Lyapunov exponents space $E^{\ast -}(x)$ is at least the dimension of $E^1(x)$ which equals the dimension of $E^{\ast -}(p)$.

Applying the same arguments above to $E^{\ast -}(x)$ we conclude that the dimension of the positive Lyapunov exponents space at $x$ is at least the dimension of $E^{\ast +}(p)$ and this concludes the lemma.

For the following results we recall here the definition of dominated splitting.

**Definition 13 (Dominated splitting).** Let $f : M \to M$ be a diffeomorphism on a compact manifold $M$ and let $X \subset M$ be an invariant subset. We say that a splitting $T_X M = E \oplus \dot{E}$ is a dominated splitting iff

1. the splitting is invariant by $Df$, which means that $Df(E(x)) = E(f(x))$ and $Df(\dot{E}(x)) = \dot{E}(f(x))$,
2. there exist $0 < \lambda < 1$ and some $l \in \mathbb{N}$ such that for all $x \in X$

$$\sup_{v \in E, ||v|| = 1} \{||df^l(x)v||\} \cdot \inf_{v \in \dot{E}, ||v|| = 1} \{||df^l(x)v||\}^{-1} \leq \lambda.$$

A priori, we do not require dominated splitting to be continuous. However, they always remains so.

**Lemma 14.** Let $f : M \to M$ be a diffeomorphism on a compact manifold $M$. Let $X \subset M$ be some $f$-invariant set. Suppose there exists some invariant dominated splitting $T_X M = E \oplus \dot{E}$. Then, such splitting is continuous in $T_X M$ and unique since we fix the dimensions of $E, \dot{E}$. Moreover, it extends uniquely and continuously to a splitting of $T_\Sigma M$. 

Proof. By replacing $f$ by an iterate, there is no loss of generality in supposing that $l \in \mathbb{N}$ in definition 13 equals to 1. We start by constructing an invariant dominated splitting on $T_{\hat{x}}M$. Let $\mathcal{O}(x)$ be an orbit contained in $\hat{X}$. Our construction will be dependent on some choices. We choose one representative of $\mathcal{O}(x)$, for example $x$. Let us also choose some $(x_n)$, $x_n \in X$, $x_n \to x \in M$, as $n \to \infty$. Let $v_1^n, \ldots, v_s^n \in E(x_n)$, \(\hat{v}_{s+1}^n, \ldots, \hat{v}_m^n \in \hat{E}(x_n)\) be orthonormal bases of $E(x_n)$ and $\hat{E}(x_n)$, respectively. The domination property is equivalent to
\[
\left\| Df(x_n) \sum_{j=1}^s \alpha_j v_j^n \right\| \cdot \left\| Df(x_n) \sum_{i=s+1}^m \beta_i \hat{v}_i^n \right\|^{-1} \leq \lambda < 1,
\]
for any convex combinations $\sum_{j=1}^s \alpha_j v_j^n$, $\sum_{i=s+1}^m \beta_i \hat{v}_i^n$. Replacing by some convergent subsequence, if necessary, there is no loss in supposing that $(v_1^1, \ldots, v_s^1)$, $v_1^1, \ldots, v_s^1 \in T_{x_n}M$ (respectively, $(\hat{v}_{s+1}^1, \ldots, \hat{v}_m^1)$) is the limit of the sequence $(v_1^n, \ldots, v_s^n)$ (respectively, of the sequence $(\hat{v}_{s+1}^n, \ldots, \hat{v}_m^n)$). Since the domination property is a closed condition,
\[
\left\| Df(x) \sum_{j=1}^s \alpha_j v_j \right\| \cdot \left\| Df(x) \sum_{i=s+1}^m \beta_i \hat{v}_i \right\|^{-1} \leq \lambda < 1
\]
holds.

Now, we write $G$ for the Gram–Schmidt operator (which takes a linearly independent set of vectors on an orthonormal set of vectors spanning the same vector space). Given any iterate $y = f^k(x)$, $k \in \mathbb{Z}$, then $f^k(x_n) \to y$ and
\[
G \circ (Df^k(x_n)\hat{v}_1^n, \ldots, Df^k(x_n)\hat{v}_m^n) \to G \circ (Df^k(x)v_1^n, \ldots, Df^k(x)v_s^n),
\]
\[
G \circ (Df^k(x)\hat{v}_{s+1}^n, \ldots, Df^k(x)\hat{v}_m^n) \to G \circ (Df^k(x)\hat{v}_{s+1}^1, \ldots, Df^k(x)\hat{v}_m^1),
\]
as $n \to \infty$. Writing $(w_1^k, \ldots, w_s^k) := G \circ (Df^k(x)v_1^n, \ldots, Df^k(x)v_s^n)$ and $(\hat{w}_{s+1}^k, \ldots, \hat{w}_m^k) := G \circ (Df^k(x)\hat{v}_{s+1}^n, \ldots, Df^k(x)\hat{v}_m^n)$, $k \in \mathbb{Z}$, the same calculations above show that $T_{\hat{x}}M = \text{span}\{w_1^k, \ldots, w_s^k\} \oplus \text{span}\{\hat{w}_{s+1}^k, \ldots, \hat{w}_m^k\} =: E(y) \oplus \hat{E}(y)$ is a dominated splitting.

Moreover, it is clear that
\[
Df(f^k(x))\text{span}\{w_1^k, \ldots, w_s^k\} = \text{span}\{w_{s+1}^k, \ldots, w_{m+1}^k\}
\]
and
\[
Df(f^k(x))\text{span}\{\hat{w}_{s+1}^k, \ldots, \hat{w}_m^k\} = \text{span}\{\hat{w}_{s+1}^k, \ldots, \hat{w}_{m+1}^k\},
\]
which implies that it is an invariant splitting.

Note that since the dominated splitting condition is a closed condition, if we prove that there exists a unique dominated splitting with the same dimensions of the splitting we constructed, it will be automatically continuous. This is because, given $x_n \to x \in X$, any convergent sequences of orthonormal bases of $E(x_n), \hat{E}(x_n)$ will converge to orthonormal bases of dominated spaces in $T_{\hat{x}}M$ which due to the uniqueness will necessarily be $E(x), \hat{E}(x)$.

The argument to prove uniqueness is the following. Suppose that we have two invariant dominated splittings $T_{\hat{x}}M = E \oplus \hat{E}, T_{\hat{x}}M = E \oplus \hat{E}'$. Fix an arbitrary $x \in \hat{X}$. By replacing $f$ with some positive iterate $f^l$, there is no loss of generality in supposing that the domination condition is valid for $l = 1$ on both splittings. The domination condition yields
\[
\|df|_{E(x)}\| \left( \inf_{v \in \hat{E}, \|v\| = 1} \|Df(x)v\| \right)^{-1} \leq \lambda
\]
and 
\[ \|df\|_{E'(x)} \left( \inf_{v \in E', \|v\| = 1} \|Df(x)v\| \right)^{-1} \leq \lambda. \]

Let us show that \( E(x) = E'(x) \). Note that if \( E(x) \subset E'(x) \) (or vice versa), as the spaces have the same dimension, they should be the same. So, let us suppose by contradiction that there exist \( v \in E(x) \setminus E'(x) \) and \( v' \in E'(x) \setminus E(x) \). We then write \( v = v_E + \hat{v}_E \), with \( v_E \in E' \), \( \hat{v}_E \notin E' \) and \( v_E \neq 0 \). This last inequality, together with the invariance of the splittings, implies that 
\[ Df^n(x) \cdot v = \alpha_n v'_E + \beta_n \hat{v}'_E, \]
where \( v'_E \) and \( \hat{v}'_E \) are unitary vectors, respectively, in \( E'(f^n(x)) \) and \( \hat{E}'(f^n(x)) \) (which dominates \( E'(f^n(x)) \)), as we take \( n \) sufficiently big. This implies that, for all \( n \in \mathbb{N} \) sufficiently big, there exists \( v_n = Df^n(x) \cdot v / \|Df^n(x) \cdot v\| \in E(f^n(x)) \) such that
\[ \|\hat{v}'_E\| < \tilde{\lambda} < 1, \]
where \( \gamma_n = f^n(x) \). Now, we repeat the same argument for \( v' \in E'(x) \setminus E(x) \), and again for all \( n \) sufficiently big, we obtain unitary vectors \( v'_n \in E'(y_n) \) such that
\[ \|\hat{v}'_E\| < \tilde{\lambda} < 1. \]
Therefore, we have
\[ \|df\|_{E(y_n)} \cdot \|Df(y_n)v_n\|^{-1} < \tilde{\lambda} < 1, \]
which is a contradiction. \( \square \)

**Lemma 15.** Suppose that \( g \) is topologically conjugated to a hyperbolic map \( f \). Let \( x \) be a recurrent, regular point of \( g \). Suppose that \( \text{Per}(g) \) is NUH and that the splitting \( T_{\text{Per}(g)}M = E_{cs} \oplus E_{cu} \) is a dominated splitting. Then all Lyapunov exponents of \( x \) are non-zero.

**Proof.** The proof is a direct consequence of lemmas 12 and 14. By lemma 14, the invariant dominated splitting over \( T_{\text{Per}(g)}M \) extends to a unique continuous invariant (dominated) splitting over \( \overline{T_{\text{Per}(g)}M} \). So, we are under the hypotheses of lemma 12, which allows us to conclude that all Lyapunov exponents of any recurrent point \( x \in M \) are non-zero. \( \square \)

By the same arguments as in the expanding case (see paragraph below the proof of lemma 8), our theorem C is obtained applying lemma 12 to the following result.

**Theorem 16 ([9]).** Let \( f : M \rightarrow M \) be a \( C^1 \) diffeomorphism on a compact Riemannian manifold, with a positively invariant set \( \Lambda \) for which the tangent bundle has a continuous splitting \( T\Lambda M = E^{cu} \oplus E^{cs} \). If \( f \) has positive Lyapunov exponents in the \( E^{cu} \) direction and negative Lyapunov exponents in the \( E^{cs} \) direction on a set of total probability, then \( f \) is uniformly hyperbolic.

Theorem B is obtained as an immediate consequence of lemma 15 and theorem C. The other results (theorems D, E, F) are a direct consequence of the proofs of theorems A, B and C, respectively.
4. On a question of A Katok

A Katok has conjectured that a $C^{1+}$ system which is Hölder conjugated to an expanding map (respectively, an Anosov diffeomorphism) is also expanding (respectively, is also an Anosov diffeomorphism).

Note that, under the hypotheses of such a conjecture, the periodic points of the $g : M \to M$ are hyperbolic, with uniform bounds for the eigenvalues of iterate of $Dg$ in the period of such points. This is proven below.

First, we consider the expanding case. Let $p$ be a periodic point of period $t$ of $g$. Then, $h(p)$ is a periodic point of period $t$ of $f$. Let us call $f^{-1}$ the inverse branch of $f$, defined on a neighbourhood of the orbit of $h(p)$, for which $h(p) = \hat{p}$ is a periodic point of period $t$. Analogously, let us call $g^{-1}$ the inverse branch of $g$ for which $p$ is a periodic point of period $t$. Since $f$ is an expanding map, there are $0 < \lambda < 1$ and $\delta > 0$ such that

$$d(f^{-1}(\hat{x}), f^{-1}(\hat{y})) \leq \lambda^j d(\hat{x}, \hat{y}), \quad \forall j \in \mathbb{N}, \quad \forall \hat{x}, \hat{y} \in B(\hat{p}, \delta).$$

As an immediate consequence of the $C^\alpha$ conjugation $h$ there exists $\delta > 0$ such that

$$d(g^{-j}(x), g^{-j}(y)) \leq (\hat{\lambda}^\sigma)^j K^{1+\alpha} d(x, y)^{\alpha^j}, \quad \forall j \in \mathbb{N}, \quad \forall x, y \in B(p, \delta).$$

and

$$d(g^{-j}(x), g^{-j}(y)) \leq (\hat{\lambda}^\sigma)^j K^{1+\alpha} g^\sigma, \quad \forall j \in \mathbb{N}, \quad \forall x, y \in B(p, \delta). \quad (4)$$

**Proposition 17.** Let $B(x_0, r) \subset M$ and $G : \overline{B(x_0, r)} \to B(x_0, r)$ be a class $C^1$ local diffeomorphism such that $G(x_0) = x_0$ and for some $0 < \lambda < 1$ and $0 < \beta < 1$

$$d(G^n(x), G^n(y)) \leq \lambda^n d(x, y)^{\beta}, \quad \forall x, y \in B(x_0, r).$$

Then all eigenvalues of $DG(x_0)$ are equal or less than $\lambda$.

**Proof.** Using charts, there is no loss of generality in supposing that $M$ is a euclidean space and $x_0 = 0$. By contradiction, suppose there exists an invariant splitting $\mathbb{R}^n = E^s + E^u$, an adapted norm $\|x\| = \|(x_s, x_u)\| = \max\{\|x_s\|, \|x_u\|\}$ and $\sigma > \lambda$ such that

$$\|DG(0) \cdot x_s\| \leq \lambda \cdot \|x_s\|, \quad \forall x_s \in E^s,$$

$$\|DG(0) \cdot x_u\| \geq \sigma \cdot \|x_u\|, \quad \forall x_u \in E^u.$$  

Let $\epsilon > 0$ such that $\lambda + \epsilon < \sigma - \epsilon$ and take $\theta = (\lambda + \epsilon)/(\sigma - \epsilon)$.

Therefore, there is $\tilde{r} \leq r$ such that if we write

$$G(x) = DG(0) \cdot x + \rho(x),$$

then $\|\rho(x)\| < \epsilon \|x\|, \forall x, \|x\| < \tilde{r}$. We define a central cone

$$V_\epsilon := \{(x_s, x_u) ; \|x_s\| \leq \theta \|x_u\|\}.$$  

By the hypothesis, there exists $\tilde{r} \leq \tilde{r}$ such that $G^n(B(0, \tilde{r})) \subset B(0, \tilde{r})$, $\forall n \in \mathbb{N}$. So, let us iterate $x \in B(0, \tilde{r}) \cap V_\epsilon$ (we write $x^n = G^n(x)$). We obtain

$$\|x^n_s\| \geq \sigma \|x^0_s\| - \epsilon \|x^0_u\| \geq (\sigma - \epsilon) \|x^0_s\|$$

and

$$\|x^n_s\| \leq \lambda \|x^0_s\| + \epsilon \|x^0_u\| \leq (\lambda + \epsilon) \|x^0_s\|.$$
This implies that
\[ \|x^1_c\| \leq \frac{\lambda + \epsilon}{\sigma - \epsilon} \|x^1_s\|. \]
In particular, if \( x \in B(0, \tilde{r}) \cap V_c \) then \( G(x) \in V_c \).
Therefore proceeding inductively, we obtain
\[ \|x^n_c\| = \|x^n_s\| \geq (\sigma - \epsilon)^n \|x^0_c\| = (\sigma - \epsilon)^n \|x^0\|. \]
This contradicts the hypothesis, which implies that
\[ \|x^n\| \leq \text{const} \cdot \lambda^n, \quad \forall n \in \mathbb{N}. \]
As \( \epsilon > 0 \) is arbitrary, we conclude that any eigenvalue of \( DG(0) \) is less than \( \lambda \).

The above proposition implies our assertion that if a map \( g \) is Hölder conjugated to an expanding map (respectively, Anosov) then all periodic points have only non-zero Lyapunov exponents, and such exponents are uniformly bounded away from zero. However, up to now we do not know if, for example, the mild uniformity given by a Hölder conjugation, plus the conjugation itself, implies that \( \text{Per}(g) \) is NUE.

Nevertheless, as a direct consequence of the last section, we obtain that such a conjecture is valid in the case that \( \text{Per}(g) \) is NUE (respectively, for Anosov, if \( \text{Per}(g) \) is NUH with dominated splitting).

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References

[1] Castro A 2002 Backward inducing and exponential decay of correlations for partially hyperbolic attractors Israel J. Math. 130 29–75
[2] Castro A 2004 Fast mixing for attractors with mostly contracting central direction Ergod. Theory Dyn. Syst. 24 17–44
[3] Alves J, Araújo V and Saussol B 2003 On the uniform hyperbolicity of some nonuniformly hyperbolic maps Proc. Am. Math. Soc. 131 1303–9
[4] Alves J, Bonatti C and Viana M 2000 SRB measures for partially hyperbolic systems whose central direction is mostly expanding Invent. Math. 140 351–98
[5] Mañé R 1985 Hyperbolicity, sinks and measure in one-dimensional dynamics Commun. Math. Phys. 100 495–524
[6] Shub M 1987 Global Stability of Dynamical Systems (Berlin: Springer)
[7] Oseledets V I 1968 A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems Trans. Moscow Math. Soc. 19 197–231
[8] Pliss V 1972 On a conjecture due to Smale Diff. Uravnenija 8 262–8
[9] Cao Y 2003 Non-zero Lyapunov exponents and uniform hyperbolicity Nonlinearity 16 1473–9
[10] Cao Y, Luzzato S and Rios I 2006 Some non-hyperbolic systems with strictly non-zero Lyapunov exponents for all invariant measures: horseshoes with internal tangencies Discrete Contin. Dyn. Syst. 15 61–71