Chiral phases for massive fermions and spinor classifications

Cheng-Yang Lee

Center for Theoretical Physics, College of Physical Science and Technology, Sichuan University, Chengdu, 610064, China
Email: cylee@scu.edu.cn

Abstract

We explore the physics of regular spinors in the Lounesto classification. These spinors are constructed by introducing two chiral phases. One is a degree of freedom present in choosing the $\gamma^\mu$ matrices that leaves the Lorentz generators invariant. Another is a degree of freedom allowed by the massive spin-half field that is unconstrained by Poincaré symmetry and locality. By choosing appropriate values of the two phases, we obtain all the classes of regular spinors. For the Dirac fermions, both phases are equal. We argue that the chiral phase of the Standard Model fermions is a new set of physical parameter. We find, the lepton chiral phases cannot be measured within the weak interaction. For quarks, their phases are conjectured to be identical. In general, to measure the phases between different fermionic generations would require interactions beyond the Standard Model.

1 Introduction

The Dirac and Weyl fields successfully describe the massive and massless fermions of the Standard Model (SM). From first principle, they are constructed from the spin-half unitary irreducible representations of the Poincaré group [1]. Therefore, the SM fermions can be classified as either Dirac or Weyl [1].

Despite the success of the SM, it also has some shortcomings. Firstly, it predicts the neutrinos to be massless while experiments suggest that they are massive. Secondly, it does not include dark matter. More generally, it is reasonable to expect the existence of new fermions beyond the SM. We are entitled to ask - are all fermions Dirac or Weyl? The Lounesto spinor classification [2] provides an intriguing perspective to this question. In the four-dimensional Minkowski space-time, there are six classes of spinors and they are divided into the regular (1st-3rd class) and singular (4th-6th) spinors (the definitions are given in the next section). The

1 Majorana fermion is a special case of the Dirac fermion by identifying particles with anti-particles.
Dirac spinors belong to the 2nd class. ELKO (a complete set of Majorana spinors) and its generalisation belongs to the 4th-5th class \[3–6\]. The Weyl spinors belongs to the 6th class.

The fermionic fields for Dirac and Weyl spinors describe the massive and massless fermions in the SM. The fermionic fields for ELKO describe a mass dimension one fermion with renormalisable quartic self-interaction making it a potential dark matter candidate \[3–10\]. In light of existing works, a natural question arises - What are the physics of the 1st and 3rd class? By generalising the 1st-3rd class spinor solutions found in \[11–14\], we show that the fermionic fields constructed from each classes of spinors are physically different.

By re-examining the construction of the Dirac field, we show that the phase introduced in \[11–14\] appear naturally and is unconstrained by Poincaré symmetry and locality. We also discover an additional chiral phase degree of freedom for the spin-half generators of the Lorentz transformations. Specifically, under a change of basis for the \(\gamma^\mu\) matrices via the chiral transformation, the Lorentz generators remain invariant. By choosing appropriate values of the two phases, we obtain all the regular spinors.

The interplay of the two phases give rises to new physics on two fronts. Firstly, different classes of spinors and fermionic fields are physically distinct. Secondly, even within the same class, there are new physics. The second point may seem surprising at first. We will discuss this in detail focusing on the Dirac fermions and potential physics beyond the SM.

The paper is organised as follows. In sec. 2, we review the Lounesto spinor classification and re-examine the construction of the massive spin-half field. We show that both the Lorentz generators and the Dirac spinors have a chiral phase degree of freedom that are unconstrained by Poincaré symmetry. In sec. 3, we explore under what circumstances these phases may be measured and the physical ramifications for the SM and beyond.

## 2 Chiral phases and spinor classification

We start by reviewing the Lounesto spinor classification. To do so, we need to choose the \(\gamma^\mu\) matrices and construct the bi-linear covariants for spinors. In the chiral representation,

\[
\gamma^\mu = \begin{pmatrix} O & \sigma^\mu \\
\tilde{\sigma}^\mu & O \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & O \\
O & I \end{pmatrix}
\]

where \(\sigma^\mu = (I, \sigma)\) and \(\tilde{\sigma}^\mu = (I, -\sigma)\). The Lorentz generators are given by

\[
J^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu].
\]
The choice of (1) and (2) means that the top and bottom Weyl spinors are left- and right-handed respectively. The first chiral degree of freedom comes from (1) and (2). We note, $J^{\mu\nu}$ is invariant under the chiral transformation

$$\gamma_a^\mu = e^{-i\gamma^5/2} \gamma^\mu e^{i\gamma^5/2}, \quad \gamma_a^5 = \gamma^5. \quad (3)$$

Since (3) is a change of basis, one would naively conclude that physics is independent of the phase so choosing $\alpha = 0$ is justified. However, there are subtleties involved that is related to the Lounesto classification and when there are multiple fermions. These issues are discussed below. The bilinear covariants used for the classifications are

$$\Omega_1 = \overline{\psi}_a \psi, \quad \Omega_2 = \overline{\psi}_a \gamma_5^a \psi, \quad K^\mu = \overline{\psi}_a \gamma_a^\mu \gamma^\psi \psi, \quad S^{\mu\nu} = \overline{\psi}_a \gamma_a^\mu \gamma_a^\nu \psi \quad (4)$$

where $\overline{\psi}_a \equiv \psi^\dagger \gamma_0^a$ is the spinor dual. For later convenience, we also define $\overline{\psi} \equiv \psi^\dagger \gamma^0$. The Lounesto classes are [2]

Regular spinors

1. $\Omega_1 \neq 0, \Omega_2 \neq 0$.
2. $\Omega_1 \neq 0, \Omega_2 = 0$.
3. $\Omega_1 = 0, \Omega_2 \neq 0$.

Singular spinors

4. $\Omega_1 = 0, \Omega_2 = 0, K^\mu \neq 0, S^{\mu\nu} \neq 0$.
5. $\Omega_1 = 0, \Omega_2 = 0, K^\mu = 0, S^{\mu\nu} \neq 0$.
6. $\Omega_1 = 0, \Omega_2 = 0, K^\mu \neq 0, S^{\mu\nu} = 0$.

Taking the Dirac spinors from [1] with $\overline{\psi} = \psi^\dagger \gamma^0$, explicit calculations show that they belong to the 2nd class. The 4th class is a complete set of Majorana spinors also known as ELKO and the 5th class is obtained by a one parameter transformation of ELKO [5, 6]. The fermionic field constructed from ELKO yields a mass dimension one field. For more details see [3–10] and references therein. In [2], Lounesto actually identified the 1st-3rd class to be the Dirac spinors. In our opinion, this is incorrect. As we will show, fermionic fields constructed from different classes of regular spinors are physically different.

In [11–14], it was shown that by introducing an additional phase to the Dirac spinors, one obtains the 1st-3rd class by choosing appropriate values for the phase. The construct of [11–14] can be understood as follows, given a left-handed Weyl spinor $\phi(p)$, it follows that $e^{i\beta} \phi(-p)$
where $\beta$ is a constant phase transforms as a right-handed spinor. Therefore, we obtain a four-component spinor of the form

$$\psi(p) = \begin{bmatrix} \phi(p) \\ e^{i\beta}\phi(-p) \end{bmatrix}$$

and it satisfies

$$\left(\gamma^\mu p_\mu - me^{-i\beta} \right)\psi(p) = 0. \quad (6)$$

By choosing the appropriate value of $\beta$, we obtain the 1st-3rd classes (here, the spinor dual is taken to be $\psi^* = \psi^\dagger \gamma^0$). When $e^{i\beta} \neq 1$, the spinors do not belong to the 2nd class and there are new physics. The reason is that these solutions differ from the Dirac spinors and the observables for certain interactions have phase-dependence.

Before we explore the physics of the regular spinors, we re-examine the above construct from first-principle. The massive spin-half field with appropriate normalisation is given by

$$\psi(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \sum_\sigma \left[ e^{-ip \cdot x} u(p, \sigma) a(p, \sigma) + e^{ip \cdot x} v(p, \sigma) b^\dagger(p, \sigma) \right]. \quad (7)$$

Rotation invariance demands the spinors to take the form

$$u(0, \frac{1}{2}) = \begin{pmatrix} c_+ \\ 0 \\ c_- \end{pmatrix}, \quad u(0, -\frac{1}{2}) = \begin{pmatrix} 0 \\ c_+ \\ c_- \end{pmatrix}, \quad v(0, \frac{1}{2}) = \begin{pmatrix} 0 \\ d_+ \\ 0 \end{pmatrix}, \quad v(0, -\frac{1}{2}) = -\begin{pmatrix} d_+ \\ 0 \\ d_- \end{pmatrix} \quad (8)$$

where $c_\pm$ and $d_\pm$ are coefficients to be determined. In [1], they are fixed by the demand of parity conservation and locality. These demands can in fact be satisfied without fixing all the coefficients. In other words, there exists an additional degree of freedom for the fermionic field. Without the loss of generality, we fix the spinors to have unit norm $|c_\pm|^2 = |d_\pm|^2 = 1$ and take

$$c_+ = e^{i\beta_u} c_- \quad d_+ = e^{i\beta_v} d_-$$

where $\beta_u$ and $\beta_v$ are constant phases. We define the dual spinors to be

$$\overline{u}_a(p, \sigma) \equiv u^\dagger(p, \sigma) \gamma_a^0, \quad \overline{v}_a(p, \sigma) \equiv v^\dagger(p, \sigma) \gamma_a^0. \quad (10)$$

By computing the equal-time anti-commutator $\{\psi(t, x), \overline{u}_a(t, y)\}$, we find that locality demands $e^{i\beta_u} = -e^{-i\beta_v}$ and hence $\beta_v = \beta_u + (2n + 1)\pi$ where $n$ is an integer. Here, we take $n = 0$ and

\[\text{For the Majorana fermionic field } \psi_M \equiv |a\rangle \rightarrow |b\rangle, \text{ locality requires an additional constraint } c_2^2 = d_2^2. \text{ The resulting equal-time anti-commutator of interest is } \{\psi_M(t, x), \psi_M(t, y)\} = -ie^{i\beta} \gamma^2 \delta^3(x - y).\]
\[ \beta_u = \beta \] so the spin-sums are

\[ \sum_\sigma u(p, \sigma) \overline{u}_\alpha(p, \sigma) = \gamma^\mu \rho_\mu + e^{i \gamma^5 (\alpha - \beta)} m, \quad (11) \]
\[ \sum_\sigma v(p, \sigma) \overline{v}_\alpha(p, \sigma) = \gamma^\mu \rho_\mu - e^{i \gamma^5 (\alpha - \beta)} m. \quad (12) \]

The scalar and pseudo-scalar inner-products are

\[ \overline{u}_\alpha(p, \sigma) u(p, \sigma') = -\overline{v}_\alpha(p, \sigma) v(p, \sigma') = 2m \delta_{\sigma \sigma'} \cos(\alpha - \beta) \quad (13) \]

and

\[ \overline{u}_\alpha(p, \sigma) \gamma^5 u(p, \sigma') = -\overline{v}_\alpha(p, \sigma) \gamma^5 v(p, \sigma') = 2im \delta_{\sigma \sigma'} \sin(\alpha - \beta). \quad (14) \]

From (13) and (14), by choosing different values of \( \alpha, \beta \), we obtain the different classes of regular spinors. Taking \( (\alpha - \beta) \in [0, 2\pi) \), we find

\[ (\alpha - \beta) = \begin{cases} 
\text{others} & \text{1st class} \\
0, \pi & \text{2nd class} \\
\frac{\pi}{2}, \frac{3\pi}{2} & \text{3rd class} 
\end{cases} \quad (15) \]

Since the spinor solutions are \( \alpha \)-independent, the field equation and Lagrangian are also \( \alpha \)-independent. They are given by

\[ \left[ i \gamma^\mu \partial_\mu - me^{i \gamma^5 \beta} \right] \psi(x) = 0 \quad (16) \]

and

\[ \mathcal{L} = \overline{\psi} \left[ i \gamma^\mu \partial_\mu - me^{i \gamma^5 \beta} \right] \psi. \quad (17) \]

The phase between \( \overline{\psi}_\alpha \) and \( \gamma^\mu_\alpha \) cancels so the free Lagrangian is \( \alpha \)-independent. But in general, interactions constructed from \( \overline{\psi}_\alpha \) and \( \psi \) can still have phase dependence.

While the two phases arise from chiral transformation, they are independent of each other. The \( \alpha \) phase originates from a freedom in choosing the \( \gamma^\mu \) matrices that leaves the Lorentz generators invariant. The \( \beta \) phase is a degree of freedom allowed by the fermionic fields which is unconstrained by Poincaré symmetry and locality. Having identified the two chiral phases, we would like to explore their physical consequences. In the context of the classification, the standard Dirac spinors correspond to the choice \( \alpha = \beta = 0 \) so they belong to the second Lounesto class. But from (15), spinors of the second class can be obtained by setting \( \alpha = \beta \) but without requiring them to vanish so it is a generalisation to the Dirac spinors.\(^3\)

\(^3\)The other choice \( \alpha = \beta + \pi \) is related to \( \alpha = \beta \) by replacing \( m \rightarrow -m \) so they are physically equivalent.
we will study the construct without fixing the two phases. In sec. 3.2, we will study the phase dependence of Dirac fermions in the SM.

We start with the discrete symmetries. Let \( C, P \) and \( T \) be the charge-conjugation, parity and time-reversal transformations respectively. Their actions on a single massive spin-half fermionic state \( |p, \sigma\rangle \) and anti-fermionic state \( |\bar{p}, \sigma\rangle \) are the usual ones \[1\]

\[
\begin{align*}
C|p, \sigma\rangle &= \delta_C|\bar{p}, \sigma\rangle, \\
C|\bar{p}, \sigma\rangle &= \delta_C|p, \sigma\rangle, \\
P|p, \sigma\rangle &= \delta_P|p, -\sigma\rangle, \\
P|\bar{p}, \sigma\rangle &= \delta_P|\bar{p}, -\sigma\rangle, \\
T|p, \sigma\rangle &= \delta_T(-1)^{1/2-\sigma}|p, -\sigma\rangle, \\
T|\bar{p}, \sigma\rangle &= \delta_T(-1)^{1/2-\sigma}|-\bar{p}, -\sigma\rangle
\end{align*}
\]

where \( \delta_{C,P,T} \) and \( \overline{\delta}_{C,P,T} \) are the intrinsic phases. Here, we will just provide the relevant identities, the resulting transformations and the intrinsic phase relations that must be imposed to preserve these symmetries. For more details, a similar analysis can be found in \[1\]. Starting with charge-conjugation, we use the identity

\[
u(p, \sigma) = \left( \frac{c_\sigma}{d^*_\sigma} \right) \gamma^2 \nu^*(p, \sigma)
\]

and obtain

\[
C\psi(t, x)C^{-1} = i e^{i \beta} \delta_C \left( \frac{c_\sigma}{d^*_\sigma} \right) \gamma^2 \psi^*(t, x), \quad \delta_C^* = \overline{\delta}_C.
\]

For parity, we use the identities

\[
e^{-i \gamma^5 \gamma^0} u(p, \sigma) = u(-p, \sigma), \quad e^{-i \gamma^5 \gamma^0} v(p, \sigma) = -v(-p, \sigma)
\]

and obtain

\[
P\psi(t, x)P^{-1} = e^{-i \gamma^5 \gamma^0} \delta^*_P \psi(t, -x), \quad \delta^*_P = -\overline{\delta}_P.
\]

Under time-reversal, the relevant identities are

\[
u^*(p, \sigma) = (-1)^{1/2-\sigma} e^{-i \beta} \left( \frac{d_\sigma}{c^*_\sigma} \right) e^{i \gamma^5 \beta} (i \gamma^0 \gamma^2 \gamma^5) \nu(-p, -\sigma),
\]

\[
u^*(p, \sigma) = (-1)^{1/2-\sigma} e^{-i \beta} \left( \frac{d_\sigma}{c^*_\sigma} \right) e^{i \gamma^5 \beta} (i \gamma^0 \gamma^2 \gamma^5) \nu(-p, -\sigma).
\]

The result is

\[
T\psi(t, x)T^{-1} = -i e^{-i \beta} \delta_T \left( \frac{c_\sigma}{d^*_\sigma} \right) e^{i \gamma^5 \beta} (\gamma^0 \gamma^2 \gamma^5) \psi(-t, x), \quad \delta_T^* = \left( \frac{c_\sigma}{d^*_\sigma} \right) \left( \frac{d_\sigma}{c^*_\sigma} \right) \overline{\delta}_T.
\]
Finally, the CPT transformation is\(^4\)
\[
(CPT)\psi(t, x)(CPT)^{-1} = \left(\delta_C \delta_P \delta_T \frac{c_-}{d_-}\right)^* \gamma^5 \psi^*(-t, -x).
\] (26)

We find that in general, the discrete transformations for \(\psi\) are dependent on the phase \(\beta\) and the ratios of spinor coefficients. Nevertheless, they are all conserved. Similarly, the Lagrangian also preserves \(C\), \(P\) and \(T\) in the sense that it transforms as a scalar.

# 3 Measuring the phases

In the previous section, we have identified two degrees of freedom for the fermionic field that appears in the form of chiral phases. An important question is – *are these phases measurable?* The answer is yes. We first show that there are interactions whose observables are dependent on \(\alpha\) and \(\beta\). Later, we explore the physical ramifications of the phases for the SM fermions.

## 3.1 Single fermionic interactions

We start with interactions of a single fermion. To determine the phase dependence of the interactions, it is convenient to define the standard Dirac field \(\Psi(x)\) with \(\alpha = \beta = 0\)
\[
\Psi(x) \equiv (2\pi)^{-3/2} \int \frac{d^3 p}{\sqrt{2E}} \sum_\sigma \left[ e^{-i p \cdot x} U(p, \sigma) a(p, \sigma) + e^{i p \cdot x} V(p, \sigma) b^\dagger(p, \sigma) \right]
\] (27)

where
\[
U(0, \frac{1}{2}) = c_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U(0, -\frac{1}{2}) = c_- \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad V(0, \frac{1}{2}) = d_- \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad V(0, -\frac{1}{2}) = d_- \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\] (28)

with \(|c_-|^2 = |d_-|^2 = 1\).\(^5\) This is related to \(\psi\) by
\[
\psi(x) = e^{i\beta/2} e^{-i\gamma^5 \beta/2} \Psi(x)
\] (29)

whose the Lagrangian is given by \([17]\). One could argue that \(\beta\) is physically irrelevant because it can be removed by performing the inverse transformation of (29) but this is incorrect. If \(\beta\) is

\(^4\)For Majorana fermions, the above transformations remain the same. The intrinsic charge-conjugation is real \(\tilde{\delta}_C = \delta_C^*\). The intrinsic parity is imaginary \(\tilde{\delta}_P = -\delta_P\). The intrinsic time-reversal satisfies \(\tilde{\delta}_T = (c_-/d_-)(d_-/c_-)^*\delta_T^*\).

\(^5\)The Dirac spinors given in \([1]\) are obtained by setting \(c_- = 1\) and \(d_- = -1\).
irrelevant, then the observables should be $\beta$-independent and the Dirac Lagrangian should be invariant under chiral transformation. But neither statements are true. Firstly, the Lagrangian has chiral symmetry only for massless fermions. Secondly, as we will show, for certain interactions, the observables are functions of $\alpha$ and $\beta$.\footnote{Alternatively, one could argue that $\beta$ is irrelevant since the Dirac Lagrangian does not have chiral symmetry. We find this argument to be unsatisfactory. In sec.\textsuperscript{2}, we have shown that $\beta$ is an intrinsic phase that appears in the spinor solutions that are unconstrained by Poincaré symmetry and locality. Therefore, the most general field equation and Lagrangian should have $\beta$ dependence.}

We start with the Yukawa interaction of the form
\[ \mathcal{H}_Y = g \overline{\psi}_a e^{i\gamma^5(\beta - \alpha)} \psi \phi \]  
for coupling $g$ and real scalar $\phi$. Here, we have inserted the matrix $e^{i\gamma^5(\beta - \alpha)}$ between $\overline{\psi}_a$ and $\psi$. This ensures that when considering the Yukawa interaction for massless fermions, a scalar field with a non-vanishing vacuum expectation value would generate the desired mass term in (17). Using (29), we find $\overline{\psi}_a e^{i\gamma^5(\beta - \alpha)} \psi = \overline{\Psi} \Psi$ so $\mathcal{H}_Y$ is phase-independent. We can also have a Yukawa interaction of the form
\[ \mathcal{H}_Y' = g' \overline{\psi}_a \psi \phi. \]  
In this case, we find $\overline{\psi}_a \psi = \overline{\Psi} e^{i\gamma^5(\alpha - \beta)} \Psi$ so there is a phase dependence. The total scalar decay to fermion and anti-fermion of momentum $p_1$ and $p_2$ for $\mathcal{H}_Y'$ is then proportional to
\[ \text{tr} \left[ \left( \frac{\not{p} + me^{-i\gamma^5 \beta}}{2} \right) e^{i\gamma^5(\alpha - \beta)} \left( \frac{\not{p} - me^{-i\gamma^5 \beta}}{2} \right) e^{i\gamma^5(\beta - \alpha)} \right] = 4 \left[ p_1 \cdot p_2 - m^2 \cos(2(\beta - \alpha)) \right]. \]  
For $\alpha = \beta$, we have the standard result for Dirac fermions. But when $\alpha \neq \beta$, the decay rate is different. In particular, for $\beta = \alpha + \pi/4$, the mass squared term contribution identically vanish.

### 3.2 Multiple fermionic interactions and the Standard Model

We have now shown that the two chiral phases can have non-trivial effects on observables. This result is not surprising since by choosing different values of phases, the spinor inner-products and spin-sums are different. Therefore, fermionic fields constructed from each of the three classes of regular spinors are physically distinct. Now, we will go one step further by showing that even within the same class, the chiral phases can give rise to non-trivial physics. To see this, we will focus on the Dirac spinors because of its relevance to the SM but the argument apply to all regular spinors.

The standard Dirac fermions in the SM corresponds to the choice $\alpha = \beta = 0$. But according to (15), it is not necessary to fix the values of the phases. Therefore, we propose a generalisation
to the Dirac fermion by choosing \( \alpha = \beta \) but not requiring them to vanish. Here, we take the definition to mean that the phase of the \( \gamma^\mu \) matrix is fixed to the phase \( \beta \) of the spinor solution. The inner-products and spin-sums are

\[
\bar{u}_\beta(p, \sigma)u(p', \sigma') = -\bar{v}_\beta(p, \sigma)v(p', \sigma') = 2m\delta_{\sigma\sigma'},
\]

(33)

\[
\bar{u}_\beta(p, \sigma)\gamma^5u(p, \sigma') = \bar{v}_\beta(p, \sigma)\gamma^5v(p, \sigma') = 0
\]

(34)

and

\[
\sum_\sigma u(p, \sigma)\bar{u}_\beta(p, \sigma) = \gamma^\mu p_\mu + mI,
\]

(35)

\[
\sum_\sigma v(p, \sigma)\bar{v}_\beta(p, \sigma) = \gamma^\mu p_\mu - mI.
\]

(36)

Therefore, this is a change of basis of the \( \gamma^\mu \) matrix so one would naively conclude that the phase can take arbitrary value with no physical consequences. This conclusion would indeed be true for a single fermion but it may not be true when there are multiple fermions. To see this, let us consider two Dirac fermions described by \( \psi \) and \( \lambda \). By definition, the phase relations are

\[
\alpha_\psi = \beta_\psi, \quad \alpha_\lambda = \beta_\lambda
\]

(37)

but \( \beta_\psi \) and \( \beta_\lambda \) are not necessarily equal. In other words, the phase may be specie-dependent.

We note, by taking \( \beta_\psi \) and \( \beta_\lambda \) to be different does not amount to a mismatch in the choice of spinor basis because the Lorentz generators obtained from \( \gamma^\mu_\psi \) and \( \gamma^\mu_\lambda \) are the same. Extending the argument to all regular spinors, this means that fermionic fields belonging of the same class may have different phases. Below, we explore the physical consequences for the Dirac fermions.

A renormalisable phase-dependent interaction is given by

\[
\mathcal{H}_Y(\beta_\psi, \beta_\lambda) = \frac{g}{2} \left( \bar{\psi}_\beta_\psi \lambda + \bar{\lambda}_\beta_\lambda \psi \right) \phi + \text{h.c.}
\]

(38)

where

\[
\bar{\psi}_{\beta_\psi \beta_\lambda} \equiv \frac{1}{2} \gamma^0_\beta_\psi + \gamma^0_\beta_\lambda, \quad \bar{\lambda}_{\beta_\psi \beta_\lambda} \equiv \frac{1}{2} \gamma^1_\beta_\psi + \gamma^0_\beta_\lambda.
\]

(39)

At tree-level, the amplitude for scalar decay is proportional to

\[
M_{(\bar{\psi}_\lambda)(\phi)} \sim \left| \bar{u}_{\beta_\psi}(p, \sigma_\psi) + \bar{v}_{\beta_\lambda}(p, \sigma_\psi) \right| u(p, \sigma_\lambda)
\]

(40)

so the total decay rate is proportional to

\[
|M_{(\bar{\psi}_\lambda)(\phi)}|^2 \sim \text{tr} \left\{ \sum_{\sigma_\psi, \sigma_\lambda} v(p, \sigma_\psi) \left[ \bar{u}_{\beta_\psi}(p, \sigma_\psi) + \bar{v}_{\beta_\lambda}(p, \sigma_\psi) \right] u(p, \sigma_\lambda) \left[ \bar{u}_{\beta_\psi}(p, \sigma_\lambda) + \bar{v}_{\beta_\lambda}(p, \sigma_\lambda) \right] \right\}.
\]

(41)
The spin-sums are

\[
\sum_{\sigma \lambda} u(p, \sigma \psi) \bar{u}_\beta \varphi(p, \sigma \alpha) = \gamma \beta_\alpha \cdot p + m_\lambda I, \tag{42}
\]

\[
\sum_{\sigma \lambda} u(p, \sigma \psi) \bar{u}_\beta \varphi(p, \sigma \alpha) = \gamma \beta_\psi \cdot p + e^{-i\gamma_5 (\beta_\lambda - \beta_\psi)} m_\lambda I, \tag{43}
\]

\[
\sum_{\sigma \psi} v(p, \sigma \psi) \bar{v}_\beta \varphi(p, \sigma \lambda) = \gamma \beta_\psi \cdot p - m_\psi, \tag{44}
\]

\[
\sum_{\sigma \psi} v(p, \sigma \psi) \bar{v}_\beta \varphi(p, \sigma \lambda) = \gamma \beta_\lambda \cdot p - e^{-i\gamma_5 (\beta_\psi - \beta_\lambda)} m_\psi. \tag{45}
\]

Substituting the spin-sums into (41), we obtain

\[|M(\varphi \lambda)(\varphi)|^2 \sim \left[ 1 + \cos(\beta_\lambda - \beta_\psi) \right] \left[ p_\lambda \cdot p_\psi - m_\lambda m_\psi \right]. \tag{46}\]

Since \(|M(\varphi \lambda)(\varphi)|^2\) is proportional to the total decay rate summed over \(\sigma \lambda\) and \(\sigma \psi\), all the contributing terms are positive-definite. Therefore, when \((\beta_\lambda - \beta_\psi) = n\pi/2\) for some integer \(n\), the scalar decay rate is identically zero for all \(\sigma \lambda\) and \(\sigma \psi\).

We have presented a Yukawa interaction where relative phase give rises to non-trivial effects but, such an interaction is not part of the SM. For instance, taking \(\psi, \lambda\) and \(\phi\) to be the electron, muon and Higgs field then (38) is a flavor-violating interaction beyond the SM and is strongly constrained by experiments. To explore the physical consequences of the chiral phases within the SM fermions, we shall confine ourselves to the weak interaction and minimally extend the SM by taking the neutrinos to be massive. In this case, the neutral-current interactions are phase-independent because it is of the generic form \(\bar{\psi} \gamma^\mu (I - \gamma^5) \psi\) so the phase cancels. On the other hand, the charged-current interactions are functions of fermionic fields of different species so they may have phase dependence. For definitiveness, we consider the first generation lepton interaction

\[
\mathcal{L}^{(CC)}(x) = -\frac{g}{2\sqrt{2}} \bar{\nu}_e(x) \gamma^\mu (I - \gamma^5) e(x) W_\mu(x) + \mathrm{h.c.} \tag{47}
\]

with

\[
\nu_e(x) \equiv \sum_{i=1}^{3} U_{ei} v_i(x) \tag{48}
\]

where \(U\) is the 3 \times 3 unitary mixing matrix and \(v_i\) are the three massive neutrino fields of mass \(m_i\). Let \(V_i\) and \(E\) be the corresponding Dirac fields for \(v_i\) and \(e\) respectively so

\[
\nu_i(x) = e^{i\beta_i/2} e^{-i\gamma_5 \beta_i/2} V_i(x), \tag{49}
\]

\[
e(x) = e^{i\beta_e/2} e^{-i\gamma_5 \beta_e/2} E(x). \tag{50}
\]
Substituting (48-50) into (47), we obtain
\[ L^{(CC)}(x) = -\frac{g^2}{2\sqrt{2}} \sum_{i=1}^{3} e^{i(\beta_e - \beta_i)} U_{ei} \overline{V}_i(x) \gamma^\mu (I - \gamma^5) E(x) W_\mu(x). \] (51)

For each interacting term, the relative phase \((\beta_e - \beta_i)\) is global so they can be ignored. This result also apply to leptons of the second and third generation. Therefore, the lepton phases \(\beta_\ell\) and \(\beta_i\) are not measurable in the weak interaction.

Next, we study the quark chiral phases by computing the neutron \(\beta\) decay rate. The transition from neutron to proton is equivalent to the transition from down quark. Therefore, the quark phases may be replaced by the neutron and proton phase
\[ \beta_d \equiv \beta_n, \quad \beta_u \equiv \beta_p. \] (52)

Explicit computation yields (see app. A)
\[ \frac{d\Gamma_{\text{avg}}}{dE_e} = \frac{d\Gamma^{(\text{SM})}_{\text{avg}}}{dE_e} + \frac{G_F^2|V_{ud}|^2}{2\pi^3} (f^2 - g^2) \left[ 1 - \cos(\beta_n - \beta_p) \right] (m_n - m_p - E_e)^2 p_e E_e. \] (53)

Here, the relative phase makes a non-trivial contribution because \(f \neq g\). The lepton chiral phases have no effects on the interactions so we may ignore the neutrino masses. Integrating
over the electron energy, the neutron lifetime is

\[
\tau_n = \tau_n^{(\text{SM})} \left\{ 1 + \frac{(f^2 - g^2) \left[ 1 - \cos(\beta_n - \beta_p) \right]}{f^2 + 3g^2} \right\}^{-1}
\]

(54)

where \(\tau_n^{(\text{SM})}\) is the SM result given by

\[
\tau_n^{(\text{SM})} = \frac{2\pi^3}{G_F^2 |V_{ud}|^2 m_e^5 F} (f^2 + 3g^2)^{-1}
\]

(55)

and

\[
F = \int_{m_e}^{m_n-m_p} dE_e \frac{(m_n - m_p - E_e)^2 E_e |p_e|}{m_e^5}.
\]

(56)

For \(\beta_n \neq \beta_p\), the deviation to the SM prediction can be as large as 25% (see fig. 1). Taking \(\tau_n^{(\text{SM})}\) to be the experimentally measured neutron mean lifetime 879.4 ± 0.6 s [15], we see that if \(\beta_n \neq \beta_p\), then \(|\beta_n - \beta_p|\) must be very fine tuned. Therefore, it is reasonable to conclude that neutron and proton have the same chiral phase \(\beta_n = \beta_p\). Equivalently, \(\beta_u = \beta_d\).

The large deviation from SM prediction comes from the relative phase that appears in the mass matrix. As a result, when evaluating the traces of the spin-sums with the form factors taken into account, the contributions from the mass term with respect to the energy-momentum terms are not suppressed (see app. A). Neutron decay only involve quarks of the first generation. But in general, the Cabbibo-Kobayashi-Maskawa matrix induces interactions between quarks of different generations so the quark chiral phases can make considerable contributions. Therefore, to conform with the SM, we conjecture all quarks to have the same chiral phase. As for leptons, the phases of \(\nu_\ell\) and \(\ell\) are not measurable in the weak interaction so they can in principle be different. These are summarised in table 1. The important question is, can the SM fermions have non-vanishing chiral phases and how they can be measured.

| Fermionic doublets | Chiral phase |
|-------------------|-------------|
| \((\nu_\ell, \ell)\) | \(\beta_\ell, \beta_i\) |
| \((u, d), (c, s), (t, b)\) | \(\beta_Q\) |

Table 1: Chiral phases for lepton and quark doublets. We take the neutrinos to be massive so \(\beta_i\) is the phase for the mass eigenstates. We conjecture all quarks to have the same chiral phase.

4 Conclusions

The Lounesto classification has provided an impetus to study quantum field theories of different spinor classes. Here, we have explored the physics of two chiral phases for massive
fermions. The first phase $\alpha$ comes from the chiral transformation on $\gamma^\mu$ that leaves the Lorentz generators invariant while the second phase $\beta$ associated with the spinor solutions found in \footnote{13} is a freedom allowed by Poincaré symmetry and locality. By choosing the values of $\alpha$ and $\beta$, we may obtain the regular spinors.

The regular spinors and their fermionic fields can be studied without fixing the phases $\alpha$ and $\beta$. We find, for certain types of Yukawa interactions, the observables are functions of $(\beta - \alpha)$. Therefore, different regular spinors give rise to different physics. All massive fermions of the SM are all constructed from the 2nd class. In other words, fermionic fields constructed from the 1st and 3rd classes are not part of the SM.

A natural generalisation to the Dirac fermions is obtained by allowing the phases to be equal $\alpha = \beta$ but not necessarily zero. One would naively assume that there are no new physics but this may not be true. Given two Dirac fields $\psi$ and $\lambda$, we would have $\alpha_\psi = \beta_\psi$ and $\alpha_\lambda = \beta_\lambda$ but there are no reasons to demand $\beta_\psi$ and $\beta_\lambda$ to be equal. From \footnote{15}, we note that the regular spinors are classified without having to fix the values of the phases. Instead, it suffices to specify the relations between $\alpha$ and $\beta$. Therefore, our argument apply to all fermionic fields constructed from regular spinors. That is, there can be relative phases between fermions of the same class.

We studied the SM weak interaction by taking all leptons and quarks to have different chiral phases a priori. We find, the leptons and neutrinos phases are not measurable so they can in principle be different. As for quarks, there is a phase dependence for neutron $\beta$ decay but the deviations from the SM prediction is substantial and cannot naturally satisfy the experimental constraints. Therefore, we must have $\beta_n = \beta_p$ or equivalently $\beta_d = \beta_u$. Based on this result and taking quark mixing into account, we conjecture all quarks to have the same chiral phase (see \footnote{1}). To determine whether SM fermions have non-vanishing chiral phases requires interactions beyond the SM such as \footnote{38}. These phase-dependent interactions are in general flavor-violating but are strongly constrained by experiments. Beyond the SM, flavor-violating interactions can for instance be generated from two Higgs doublet models \footnote{16}–\footnote{18}. Nevertheless, it is intriguing that the SM fermions have a chiral phase degree of freedom which may be specie-dependent. We believe that these phases make up a new set of physical parameters for the SM fermions and measuring them constitute an important direction in the search for physics beyond the SM.

## A Neutron $\beta$ decay

We use the effective four-fermion interaction of the SM to compute neutron beta decay

\[ \mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left[ \bar{e} \gamma^\mu (1 - \gamma^5) \nu_e \right] \left[ V_{ud} \bar{u} \gamma_\mu (f - \gamma^5 g) d \right]. \]
where $f = 1$, $g = 1.2756$ \footnote{The Particle Data Group value differs from our convention by a minus sign.} and $V$ is the Cabbibo-Kobayashi-Maskawa matrix. We take the neutrinos to be massive so

$$v_e(x) = \sum_i U_{ei} \nu_i(x)$$

where $U$ is a $3 \times 3$ unitary mixing matrix and $\nu_i$ is the massive neutrino field of mass $m_i$ so the decay process is $n \rightarrow p e \bar{\nu}_i$. Later, we will show that the neutrino masses can be ignored. We write the tree-level scattering amplitude for neutron decay as

$$M(p, \bar{\nu}, e) = N \mu M_\mu$$

where

$$N \mu = \bar{u}(p, \sigma_e)\gamma_\mu(I - \gamma^5)u(p_i, \sigma_i)$$

$$M_\mu = \frac{G_F}{\sqrt{2}} \frac{V_{ud}}{(2\pi)^3 (16E_e E_i E_p m_n)^{1/2}} \bar{u}(p_p, \sigma_p)\gamma_\mu(f - \gamma^5 g)u(p_n, \sigma_n).$$

The total transition probability for neutron beta decay is given by

$$\sum_{\text{spins}} |N \mu M_\mu|^2 = \left(\sum_{\sigma_e \sigma_i} N \mu N^{\dagger}_{\sigma_i}\right)\left(\sum_{\sigma \mu \sigma_n} M_\mu M^{\dagger}_{\sigma_n}\right).$$

Terms on the right-hand side of (62) evaluate to

$$\sum_{\sigma_e \sigma_i} N \mu N^{\dagger}_{\sigma_i} = \text{tr} \left[ (p_e + e^{i\gamma^5 \beta_e} m_e)\gamma_\mu(I - \gamma^5)(p_i - e^{i\gamma^5 \beta_i} m_i)\gamma^\nu(I - \gamma^5) \right]$$

$$= 8 \left[-ie^{\alpha_\beta \mu \nu} (p_i)_\alpha (p_e)_\beta - \eta^{\mu \nu}(p_i \cdot p_e) + (p^\mu_i p^\nu_e + p^\nu_i p^\mu_e) \right],$$

and

$$\sum_{\sigma_p \sigma_n} M_\mu M^{\dagger}_{\sigma_n} = \frac{G^2_F |V_{ud}|^2}{(2\pi)^6 (32E_e E_i E_p E_n)} \text{tr} \left[ (p_p + e^{i\gamma^5 \beta_p} m_p)\gamma_\mu(f - \gamma^5 g)(p_n + e^{i\gamma^5 \beta_n} m_n)\gamma^\nu(f - \gamma^5 g) \right]$$

$$= \frac{G^2_F |V_{ud}|^2}{(2\pi)^6 (8E_e E_i E_p E_n)} \left\{ -2if g e^{\alpha_\beta \mu \nu} (p_n)_\alpha (p_p)_\beta + (f^2 + g^2)(p^\mu_n p^\nu_p + p^\mu_p p^\nu_n) \right. + \eta^{\mu \nu} \left[ (f^2 - g^2)m_n m_p \cos(\beta_n - \beta_p) - (f^2 + g^2)(p_n \cdot p_p) \right] \right\}. \quad (64)$$

From the lepton sector, the trace is independent of $\beta_e$ and $\beta_i$ so the decay rate does not depend on the lepton phases. Therefore, we will ignore the neutrino masses and replace $E_i$ and $p_i$
with $E_e$ and $p_\nu$ respectively. For the quark sector, the phase dependence arises because $f \neq g$. Substituting (63) and (64) into (62), we obtain

$$\sum_{\text{spins}} |N^\mu M_{\mu} |^2 = \frac{2 G_F^2 |V_{ud}|^2}{(2\pi)^6 (E_e E_\nu E_p E_n)} \left[ (f + g)^2 (p_n \cdot p_\nu) (p_p \cdot p_e) + (f - g)^2 (p_n \cdot p_e) (p_p \cdot p_\nu) - (f^2 - g^2) m_n m_p (p_e \cdot p_\nu) \cos(\beta_n - \beta_p) \right]. \quad (65)$$

Taking the neutron to be at rest and ignoring proton recoil $|p_p| \sim 0$, (65) simplifies to

$$\sum_{\text{spins}} |N^\mu M_{\mu} |^2 = \frac{2 G_F^2 |V_{ud}|^2}{(2\pi)^6} \left[ 2(f^2 + g^2) - (f^2 - g^2) \cos(\beta_n - \beta_p) + \frac{p_\nu \cdot p_e}{E_e E_\nu} (f^2 - g^2) \cos(\beta_n - \beta_p) \right]. \quad (66)$$

In this case, the decay rate is

$$d\Gamma = (2\pi) \sum_{\text{spins}} |N^\mu M_{\mu} |^2 \delta(m_n - m_p - E_e - E_\nu) d^3 p_\nu d^3 p_e. \quad (67)$$

Substituting (65) into (67) and perform the relevant integrations, the average decay rate is

$$\frac{d\Gamma_{\text{avg}}^{\text{SM}}}{dE_e} = \frac{G_F^2 |V_{ud}|^2}{\pi^3} \left[ (f^2 + g^2) - \frac{1}{2} (f^2 - g^2) \cos(\beta_n - \beta_p) \right] (m_n - m_p - E_e)^2 p_\nu E_e$$

$$= \frac{d\Gamma_{\text{avg}}^{\text{SM}}}{dE_e} + \frac{G_F^2 |V_{ud}|^2}{2\pi^3} (f^2 - g^2) \left[ 1 - \cos(\beta_n - \beta_p) \right] (m_n - m_p - E_e)^2 p_\nu E_e. \quad (68)$$

**Acknowledgements**

I would like to thank R. J. Bueno Rogerio, J. M. Hoff da Silva, C. H. Coronado Villalobos for correspondence; Niels Gresnigt, Takaaki Nomura and Cong Zhang for discussions. I am grateful to the hospitality offered by the Xi’an Jiaotong-Liverpool University where part of this work was completed.

**References**

[1] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6, 2005.
[2] P. Lounesto, *Clifford algebras and spinors*, Lond.Math.Soc.Lect.Note Ser. **286** (2001) 1–338.

[3] D. V. Ahluwalia and D. Grumiller, *Spin half fermions with mass dimension one: Theory, phenomenology, and dark matter*, JCAP **0507** (2005) 012, [hep-th/0412080].

[4] D. V. Ahluwalia and D. Grumiller, *Dark matter: A Spin one half fermion field with mass dimension one?*, Phys. Rev. D **72** (2005) 067701, [hep-th/0410192].

[5] R. T. Cavalcanti, *Classification of Singular Spinor Fields and Other Mass Dimension One Fermions*, Int. J. Mod. Phys. D **23** (2014), no. 14 1444002, [arXiv:1408.0720].

[6] C.-Y. Lee, *Fermionic degeneracy and non-local contributions in flag-dipole spinors and mass dimension one fermions*, Eur. Phys. J. C **81** (2021), no. 1 90, [arXiv:1809.0438].

[7] D. V. Ahluwalia, C.-Y. Lee, and D. Schritt, *Elko as self-interacting fermionic dark matter with axis of locality*, Phys. Lett. B **687** (2010) 248–252, [arXiv:0804.1854].

[8] D. V. Ahluwalia, C.-Y. Lee, and D. Schritt, *Self-interacting Elko dark matter with an axis of locality*, Phys. Rev. D **83** (2011) 065017, [arXiv:0911.2947].

[9] D. V. Ahluwalia, *The theory of local mass dimension one fermions of spin one half*, Adv. Appl. Clifford Algebras **27** (2017), no. 3 2247–2285.

[10] D. Ahluwalia, *Mass dimension one fermions*. Cambridge monographs on mathematical physics, Cambridge University Press, 2019.

[11] R. J. Bueno Rogerio, *Constraints on mapping the Lounesto’s classes*, Eur. Phys. J. C **79** (2019), no. 11 929, [arXiv:1911.0238].

[12] R. J. Bueno Rogerio, *Subliminal aspects concerning the Lounesto’s classification*, Eur. Phys. J. C **80** (2020), no. 4 299, [arXiv:1911.0850].

[13] R. J. Bueno Rogerio, J. M. Hoff da Silva, and C. H. Coronado Villalobos, *Regular spinors and fermionic fields*, Phys. Lett. A **402** (2021) 127368, [arXiv:2010.0859].

[14] R. J. B. Rogerio, *Spin-half fermions endowed with bosonic traces – Towards phases and classes*, [arXiv:2104.0706].

[15] **Particle Data Group** Collaboration, P. A. Zyla et al., *Review of Particle Physics*, PTEP **2020** (2020), no. 8 083C01.

[16] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, M. Sher, and J. P. Silva, *Theory and phenomenology of two-Higgs-doublet models*, Phys. Rept. **516** (2012) 1–102, [arXiv:1106.0034].

[17] T. Nomura and P. Sanyal, *Lepton specific two-Higgs-doublet model based on a U(1)_X gauge symmetry with dark matter*, Phys. Rev. D **100** (2019), no. 11 115036, [arXiv:1907.0271].
[18] T. Nomura, *Phenomenology of two Higgs doublet model with flavor dependent U(1) symmetry*, *Springer Proc. Phys.* **248** (2020) 175–182.