Rényi Entropy of Multivariate Autoregressive Moving Average Control Systems

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Abstract. The Rényi entropy is an important measure of the information; it is proposed by Rényi in the context of entropy theory as a generalization of Shannon entropy. We study in detail this measure of multivariate autoregressive moving average (ARMA) control systems. The characteristic function of output process is represented from the terms of its residual characteristic functions. Simple expression to compute the Rényi entropy for the output process of these systems is derived. In addition, we investigate the covariance matrix for finding the upper bound of the entropy. Finally, we present three separate examples that serve to illustrate the behavior of information in a multivariate ARMA control system where the control and noise distributed as Gaussian, Cauchy and Laplace processes.

1 Introduction

It is well known that the models of ARMA processes are strong tools for understanding, analyzing, representing and perhaps predicting future values of a very wide range of phenomena [1]. Andel in [2] was one of the first authors whose were interested by determining the distributions of univariate AR(p) process $x(t) = a_1 x(t-1) + w(t)$. He derived the characteristic function of this process in the following form $\phi_{x(t)}(s) = \prod_{j=0}^{p} \phi_{w(t)}(a_j^1 s)$, where, $\phi_{x(t)}(s), \phi_{w(t)}$ are the characteristic functions of $x(t)$ and $w(t)$ respectively. Priestley [3] represented the characteristic function of univariate AR(2) of its residual characteristic functions. The behavior of univariate AR(1) process is studied by Sim in [4], where he used the Gamma and Weibull as perturbations. Abid [1] derived a general formula to represent the characteristic function of univariate ARMA(p,q) process. In multivariate analysis, an ARMA control system plays a major role in many applications such as speech processing [5-7], biological signal processing, electroencephalogram classification EEG [8], electrocardiogram classification ECG [9], geophysical exploration [10-11], communication systems [12] and in the other applications. Our study model consists three parts, the first one is an autoregressive (AR), the second part is control process and moving average (MA) is the third part. This model is usually then referred to as the multivariate ARMA control system [13].

On the other hand, in 1948, Shannon [14] presented measure to quantify the uncertainty of an event. Rényi in 1961 [15] generalized this measure for probability distribution which means the sensitive to the fine details of a density function. Recently important work Abid and Hamee in [16] considered the problem of deriving the Shannon entropy of the univariate ARMA process. The main idea was to find the expression of the characteristic function of output process in terms of its residual characteristic functions and then solved the problem of writing causal function parameters in terms of corresponding univariate ARMA(p,q) process. Gibson in [17] studied the Shannon entropy power and mutual information of univariate AR(p) process. In this paper, we discuss the information in a model of multivariate ARMA control process according to Rényi measure. The explicit expression of Rényi entropy of this model is derived. By using some conditions such as stationary, stability and independence of processes, we have derived the upper bound of entropy. The covariance matrix of these
processes can be calculated. In addition, the formula of the characteristic function of output process from its residual characteristic functions is found. The remainder of this paper is organized as follows: In Section II, we begin with a preliminary material of the measure of information (Rényi entropy). Section III. provides a description of multivariate ARMA control systems. Asymptotic expression for the characteristic function of output process of these systems is derived in Section IV., we propose the formula of Rényi entropy in Section V. In Section VI., we find the upper bound of Rényi entropy. Examples to illustrate this study are given in Section VII. Finally, the conclusion of this paper is shown in Section VIII.

2 Preliminary Material

In this section, we introduce some basic definitions, notations and lemmas related to entropy theory. Some notations are introduced for convenience: $I_d$ represents an $d \times d$ identity matrix, $M_{d}$, $MC_{d}$, $ML_{d}$ represent multivariate normal, Cauchy and Laplace distributions, respectively in the space $R^d$ and $A'$ is the transpose of the matrix $A$.

Definition 1. [14] Let $X$ be a continuous random vector in $R^d$ with probability density function $f(x; \theta)$. Then the Shannon entropy is defined as follows:

$$H(X; \theta) = -E \left( \ln(f(x; \theta)) \right)$$  \hspace{1cm} (1)

Definition 2. [15] An $\alpha$-th-order Rényi entropy of a continuous random vector $X \in R^d$ with probability density function $f(x; \theta)$ is defined as follows:

$$R_{\alpha}(X; \theta) = \begin{cases} \frac{1}{1-\alpha} \ln(E(f(x; \theta)^{\alpha-1})), & 0 < \alpha < \infty, \alpha \neq 1 \\ -E(\ln(f(x; \theta))), & \alpha = 1 \end{cases}$$  \hspace{1cm} (2)

Lemma 1. [19] If a random vector $X \in R^d$ has zero mean and covariance matrix $S_X = E(XX')$ (not necessarily normal), then the following inequality is accomplished

$$H(X; S_X) \leq \frac{1}{2} \ln(\det(2\pi \exp(1) S_{2}))$$  \hspace{1cm} (3)

with equality if and only if $X \sim MN_d(0, S_X)$.

Lemma 2. [20] If a random vector $X \in R^d$ has covariance matrix $S_X = E(XX')$, then the upper bound of Rényi entropy of $X$ is shown in terms of $S_X$ as follows:

$$R_{\alpha}(X; S_X) \leq C_d(\alpha) + \frac{1}{2} \ln(\det(S_{2}))$$  \hspace{1cm} (4)

where,

$$C_d(\alpha) = \frac{d}{2} \ln \left( \frac{(\pi(\alpha(d+2)-d))}{\alpha-1} \right) + \frac{1}{\alpha-1} \ln \left( \frac{(\alpha(d+2)-d)}{2\alpha} \right) + \ln \left( \frac{\Gamma(\frac{\alpha}{\alpha-1})}{\Gamma(\frac{\alpha(d+2)-d}{2(\alpha-1)})} \right), \alpha > 1$$

$$= \frac{d}{2} \ln \left( \frac{(\pi(\alpha(d+2)-d))}{1-\alpha} \right) - \frac{\alpha}{1-\alpha} \ln \left( \frac{(\alpha(d+2)-d)}{2\alpha} \right) - \ln \left( \frac{\Gamma(\frac{\alpha}{1-\alpha})}{\Gamma(\frac{\alpha(d+2)-d}{2(1-\alpha)})} \right), \frac{d}{d+2} < \alpha < 1$$

$$\frac{d}{2} \ln(2\pi \exp(1)), \alpha = 1$$

Lemma 3. [21] If $X \sim MN_d(\mu, S)$ for some positive definite matrix $S$ then the Rényi entropy of $X$ can be written as follows:

$$R_{\alpha}(X; \mu, S) = \begin{cases} \frac{1}{2} \ln(\det(2\pi S)) - \frac{d}{2(1-\alpha)} \ln(\alpha), & 0 < \alpha < \infty, \alpha \neq 1 \\ \frac{1}{2} \ln(\det(2\pi \exp(1) S)), & \alpha = 1 \end{cases}$$

Lemma 4. The Rényi entropy of multivariate Cauchy random vector $X \sim MC_d(\mu, S)$ is in the following form

$$R_{\alpha}(X; \mu, S) = \begin{cases} \ln \left( \frac{\alpha^{-d/2} \det(4\pi S)^{1/2}}{\sqrt{\pi}} \Gamma(\frac{d+1}{2}) \right), & 0 < \alpha < \infty, \alpha \neq 1 \\ \ln \left( \frac{\det(4\pi \exp(2) \pi S)^{1/2}}{\sqrt{\pi}} \Gamma(\frac{d+1}{2}) \right), & \alpha = 1 \end{cases}$$

Proof: The proof is directed from the definition 2.

3 System Description

The multivariate ARMA control system has been introduced by the work of Bao et al. in [13], which it can be mathematically described by a stochastic control system such as the following form:

$$x(t) = \sum_{i=1}^{p} A_i x(t-i) + \sum_{j=0}^{r} B_j u(t-j) + \sum_{k=0}^{q} D_k w(t-k)$$

$$x(0) = 0$$

where, $x(t)$ is the system output state, $u(t)$ is the system input vector, $w(t)$ is the white noise vector with mean zero. $A_i \in R^{d \times d}, B_j \in R^{d \times d}, D_k \in R^{d \times d}, B_0 = I_d, D_0 = I_d$. Assume
that $E(u(t)u'(s)) = \begin{cases} S_u, & s = t \\ 0, & s \neq t \end{cases}$, $E(w(t)w'(s)) = \begin{cases} S_w, & s = t \\ 0, & s \neq t \end{cases}$ and the both processes $u(t)$, $w(s)$ are independent for any $s$, $t$. Rewriting the equation (8) by using the lag operator $L^s x(t) = x(t - j)$, we give

$$(I_d - A(L))x(t) = B(L)u(t) + D(L)w(t) \quad (9)$$

where,

$$A(L) = A_1L + A_2L^2 + \cdots + A_pL^p$$
$$B(L) = B_1L + B_2L^2 + \cdots + B_rL^r$$
$$D(L) = D_1L + D_2L^2 + \cdots + D_qL^q$$

**Definition 3.** [22] The output process $x(t)$ of the control system in (8) is said to be stable if and only if $\det((I - A(z))^{-1}) \neq 0$ for all complex numbers $z$, $|z| < 1$.

Assume that the process $x(t)$ is stable, then the equation (9) can be written as

$$x(t) = (I - A(L))^{-1}B(L)u(t) + (I - A(L))^{-1}D(L)w(t) \quad (10)$$

To use the notations

$$M(L) = (I - A(L))^{-1}B(L)$$
$$M'(L) = (I - A(L))^{-1}D(L)$$

where, $M(L) = \sum_{j=0}^{\infty} M_j L^j$ and $M'(L) = \sum_{i=0}^{\infty} M'_i L^i$, then the equation (10) appears as follows:

$$x(t) = \sum_{j=0}^{\infty} M_j L^j u(t) + \sum_{i=0}^{\infty} M'_i L^i w(t) \quad (11)$$

Now, we compute the matrices $M_j$, $j = 0,1,2,\ldots$, $M'_i$, $i = 0,1,2,\ldots$, by Multiplying both sides of the equation (11) by the operator $(I_d - A(L))$, to get

$$(I_d - A(L))x(t) = (I_d - A(L))\sum_{j=0}^{\infty} M_j L^j u(t) + (I_d - A(L))\sum_{i=0}^{\infty} M'_i L^i w(t)$$

Consequently, if $M_0 = I_d$ and $M'_0 = I_d$ then the last equation can be written as

$$(I_d - A(L))x(t) = \left(I_d + \sum_{i=1}^{\infty} \left(M_i - \sum_{j=1}^{i} A_j M_{i-j}\right) L^i\right) u(t) + \left(I_d + \sum_{i=1}^{\infty} \left(M'_i - \sum_{j=1}^{i} A_j M'_{i-j}\right) L^i\right) w(t) \quad (12)$$

Again, if $A_j = 0$ for $j > p$, $B_i = 0$ for $i > r$, $D_i = 0$ for $i > q$, then equation (8) can be re write in the following form

$$(I_d - A(L))x(t) = \sum_{i=0}^{\infty} B_i L^i u(t) + \sum_{i=0}^{\infty} D_i L^i w(t) \quad (13)$$

By comparing the coefficients in equations (12) and (13), we obtain

$$M_i = B_i + \sum_{j=1}^{i} A_j M_{i-j} \quad (14)$$
$$M'_i = D_i + \sum_{j=1}^{i} A_j M'_{i-j} \quad (15)$$

### 4 Characteristic Function

Based on the equivalent formula in (11) of multivariate ARMA control process in (8) the representation of characteristic function from its residual characteristic functions as follows

$$\varphi_{x(t)}(s) = E\left(\exp \left(is' \sum_{j=0}^{\infty} M_j u(t - j) + \sum_{k=0}^{\infty} M'_k w(t - k)\right)\right) = \prod_{j=0}^{\infty} E\left(\exp \left(is'M_j u(t - j)\right)\right)$$

Here, $i = \sqrt{-1}$, the last equation is obtained from the independent conditions. Hence,

$$\varphi_{x(t)}(s) = \prod_{j=0}^{\infty} \varphi_{M_j u(t-j)}(s) \varphi_{M'_j w(t-j)}(s) \quad (16)$$

where, $\varphi_{M_j u(t-j)}$, $\varphi_{M'_j w(t-j)}$ represent the characteristic functions of $M_j u(t - j)$ and $M'_j w(t - j)$ respectively.

### 5 Rényi Entropy of Multivariate ARMA Control Process

In this section, we drive the expression of Rényi entropy of multivariate ARMA control process in the system (8) when its distributions multivariate normal, Cauchy and Laplace.

1. **Multivariate Normal Distribution**

Assume that the control and white noise processes are both distributed as zero mean multivariate normal with covariance matrices $S_u$ and $S_w$ respectively and they satisfy the independent conditions. Then from the equation (16) the characteristic function of process $x(t)$ can be written as

$$M_i = B_i + \sum_{j=1}^{i} A_j M_{i-j} \quad (14)$$
$$M'_i = D_i + \sum_{j=1}^{i} A_j M'_{i-j} \quad (15)$$
Clearly the distribution of the process \( x(t) \) is multivariate normal with zero mean and covariance matrix \( \Sigma \).

\[
\text{Cov}(x(t)) = \sum_{j=0}^{\infty} (M'_j S_w M'_j') s
\]

Therefore, the Rényi entropy of process \( x(t) \) in control system (8) is in the following form

\[
R_{\alpha}(X(t)) = \ln \left( \frac{1}{\pi^{d/2}} \sqrt{\det(\Sigma)} \right)^{1/2} \Gamma \left( \frac{d+1}{2} \right), \quad \alpha \neq 1
\]

\[
= \ln \left( \frac{(\det(4\exp(2\pi D)))^{1/2} \Gamma \left( \frac{d+1}{2} \right)}{\sqrt{\pi}} \right), \quad \alpha = 1
\]

2. **Multivariate Cauchy Distribution**

In the second case, we take the control and white noise processes as both distributed as zero mean multivariate Cauchy with scale matrices \( S_u \) and \( S_w \) respectively and satisfy the independent conditions. The characteristic function of any stationary process \( y(t) \) which distributed multivariate Cauchy with scale parameter \( S_y \) is in the following form [23-24]

\[
\varphi_{y(t)}(s) = \exp \left\{ is^t \mu - \sqrt{s^t S_y s} \right\}
\]

Therefore, from equation (16) the characteristic function of output process \( x(t) \) in control system (8) can be written as

\[
\varphi_{x(t)}(s) = \exp \left( -\sum_{j=0}^{\infty} \left( \sqrt{s^t M'_j S_u M'_j s} + \sqrt{s^t M'_j S_w M'_j s} \right) \right)
\]

But both matrices \( S_u \) and \( S_w \) are positive definite, then they can be written as \( S_u = L_u L_u^t \) and \( S_w = L_w L_w^t \) respectively, where \( L_u, L_w \in \mathbb{R}^{d \times d} \). Therefore, the characteristic function of \( x(t) \) appears as follows

\[
\varphi_{x(t)}(s) = \exp \left( -\sum_{j=0}^{\infty} \left( \sqrt{s^t K_j K_j^t s} + \sqrt{s^t K_j^* K_j^{*t} s} \right) \right)
\]

where, \( K_j = M'_j L_u \cdot K_j^* = M'_j L_w, j = 0,1,2, ...\)

Assume that the matrices \( K_j K_j^t, K_j^* K_j^{*t}, j = 0,1,2,... \) are proportional one to another then there exists a matrix \( D \) such that [24]

\[
\sqrt{s^t D s} = \sum_{j=0}^{\infty} \left( \sqrt{s^t K_j K_j^t s} + \sqrt{s^t K_j^* K_j^{*t} s} \right)
\]

This assumption, gives us the following result

\[
\varphi_{x(t)}(s) = \exp \left( -\sqrt{s^t D s} \right)
\]

Clearly, under the above conditions the process \( x(t) \) distributed Cauchy with scale matrix \( D \). Consequently, lemma 3. gives the Rényi entropy of process \( x(t) \) in the following form, for \( 0 < \alpha < \infty \)

\[
R_{\alpha}(X(t)) = \ln \left( \frac{1}{\sqrt{\pi} \sqrt{\det(4\exp(2\pi D))}} \right)^{1/2} \Gamma \left( \frac{d+1}{2} \right), \quad \alpha \neq 1
\]

\[
= \ln \left( \frac{(\det(4\exp(2\pi D)))^{1/2} \Gamma \left( \frac{d+1}{2} \right)}{\sqrt{\pi}} \right), \quad \alpha = 1
\]

3. **Multivariate Laplace Distribution**

As the third case, we take both the control and white noise stationary processes distributed multivariate Laplace with zero mean, covariance matrices \( S_u \) and \( S_w \) respectively, which belong to the set of \( d \times d \) positive definite matrices and also they satisfy the independence assumptions.

The characteristic function of any random vector \( y \) in \( \mathbb{R}^{d} \) which distributed multivariate Laplace with covariance matrix \( S_y \) is in the following form: [25]

\[
\varphi_{y}(s) = \frac{1}{1 + \frac{1}{2} s^t S_y s}
\]

Therefore, the characteristic function of \( x(t) \) can be written according to equation (16) as follows

\[
\varphi_{x(t)}(s) = \prod_{j=0}^{\infty} \left( \frac{1}{1 + \frac{1}{2} s^t K_j K_j^t s} \right)
\]

The formula in (27) does not assign any traditional probability distribution of process \( x(t) \), so one can use the uniqueness relation between the probability density and its characteristic function to get on the distribution of \( x(t) \). In fact, there is no analytical expression for the Rényi entropy of \( x(t) \) in control system (8) when it’s component processes \( u(t), w(t) \), therefore we consider the upper bound of this entropy exist.

6. **Upper Bound of Rényi Entropy**

According to determine the upper bound of Rényi entropy for the output process of stochastic control system in (8), we need to present expression of the covariance matrix of this process. In this section, we derive the formula of the covariance matrix to give us the upper bound of Rényi entropy for a multivariate ARMA control process which its distribution is difficult to determine. From the
representation of process \(x(t)\) in equation (11) and the independent conditions in this paper, it can be seen that the expectation of matrices \(E(u(t)x'(s)) = 0\) and \(E(w(t)x'(s)) = 0\) for \(t < s\). Therefore, for any \(\tau > \max\{r, q\}\), the autocovariance function of \(x(t)\) can be represented as follows

\[
\Phi(\tau) = \sum_{i=1}^{\infty} A_i \Phi(\tau - i) \tag{28}
\]

where, \(\Phi(\tau) = E(x(t)x'(t-\tau))\). Hence, the above relation is used to compute the matrices \(\Phi(\tau)\) for \(\tau = p, p+1, \ldots\) when \(p < \max\{r, q\}\). Now, we shall discuss the case when \(\tau = \max\{r, q\}\); first, we must rewrite the equation (8) in the following representation

\[
\tilde{X}(t) = \Theta \tilde{X}(t-1) + \tilde{U}(t) + \tilde{W}(t) \tag{29}
\]

where,

\[
\tilde{X}(t) = [x'(t), x'(t-1), \ldots, x'(t-p+1), u'(t), u'(t-1), \ldots, u'(t-r+1), w'(t), w'(t-1), \ldots, w'(t-q+1)]
\]

represents the information vector, the parameter matrix is

\[
\begin{bmatrix}
\Theta_{11} & \Theta_{12} \\
\Theta_{21} & \Theta_{22}
\end{bmatrix}
\]

\[
\Theta_{11} =
\begin{bmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
0 & 0 & \ldots & A_p
\end{bmatrix}_{dp \times dp}
\]

\[
\Theta_{12} =
\begin{bmatrix}
B_1 & \ldots & B_r & D_1 & \ldots & D_q
\end{bmatrix}_{dp \times d(p+r+q)}
\]

\[
\Theta_{21} =
\begin{bmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{bmatrix}_{dp \times d(r+q)}
\]

\[
\Theta_{22} =
\begin{bmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{bmatrix}_{dp \times d(r+q)}
\]

\[
\tilde{W}(t) =
\begin{bmatrix}
w(t) & dp \times 1 \\
\vdots & \vdots \\
0 & dr \times 1
\end{bmatrix}
\]

\[
\tilde{U}(t) =
\begin{bmatrix}
u(t) & dp \times 1 \\
\vdots & \vdots \\
0 & dq \times 1
\end{bmatrix}
\]

\[
\tilde{Y}(t) =
\begin{bmatrix}
y(t) & dp \times 1 \\
\vdots & \vdots \\
0 & dq \times 1
\end{bmatrix}
\]

Lemma 5. The process \(\tilde{X}(t)\) in the control system (29) is stable if and only if the process \(X(t)\) in control system (8) is stable.

**Proof:** for any complex number \(z\), we have

\[
\det(I_{d(p+r+q)} - \Theta z) = \det(I_{dp} - \Theta_{11} z) \det(I_{d(r+q)} - \Theta_{22} z)
\]

we apply the representation in equation (11) on equation (29) to get

\[
\tilde{X}(t) = \sum_{j=0}^{\infty} \Theta^j \tilde{U}(t-j) + \sum_{i=0}^{\infty} \Theta^i \tilde{W}(t-i) \tag{30}
\]

Let \(\tilde{I}_1, \tilde{I}_2\) be the \(d \times d(p+r+q)\), \(d(p+r+q) \times d\) and \(d(p+r+q) \times d\) respectively matrices such that

\[
I = [I_d \ 0 \ 0 \cdots 0], \quad \tilde{I}_1 = [I_d \ \text{dp} \times d \ \text{dr} \times d \ \text{dq} \times d], \quad \tilde{I}_2 = [I_d \ \text{dp} \times d \ \text{dr} \times d \ \text{dq} \times d]
\]

Multiplying equation (30) by \(\tilde{I}_1\) we obtain

\[
x(t) = \sum_{j=0}^{\infty} I \Theta^j \tilde{U}(t-j) + \sum_{i=0}^{\infty} I \Theta^i \tilde{W}(t-i)
\]

Hence,

\[
x(t) = \sum_{j=0}^{\infty} I \Theta^j \tilde{U}(t-j) + \sum_{i=0}^{\infty} I \Theta^i \tilde{W}(t-i)
\]

But , if we compare the above equation with equation in (11) then we get, \(M_i' = I \Theta^j \tilde{U}(t-j)\) for \(j = 0, 1, 2, \ldots\) and \(M_i'' = I \Theta^i \tilde{W}(t-i)\) for \(j, i = 0, 1, 2, \ldots\) This impels that the control systems in (8) and (29) are equivalent. This complete the proof of this lemma. Now, Multiplying by \(\tilde{X}'(t)\), \(\tau = 0, 1, 2, \ldots\) and taking the expectation for both sides of equation (29) , we obtain

\[
E(\tilde{X}(t)\tilde{X}'(t-\tau)) = \Theta E(\tilde{X}(t-1)\tilde{X}'(t-\tau)) + E(\tilde{X}'(t)\tilde{X}'(t-\tau))
\]

If we take \(\tau = 0\) and use independent assumptions of system in (8) then the covariance matrix \(\Phi(0)\) of \(\tilde{X}(t)\) is given as follows:

\[
\Phi(0) = \Theta \Phi(0) + S_{\tilde{U}} + S_{\tilde{W}} \tag{31}
\]

Again, if we take \(\tau = 1\) then

\[
\Phi(1) = \Theta \Phi(0) \tag{32}
\]

Substituting equation (32) in equation (31) give us

\[
\Phi(0) = \Theta \Phi(0) \Theta' + S_{\tilde{U}} + S_{\tilde{W}}
\]

The vec. operator for both sides of above equation, gives
Using the property of vec operator \( (I_{d(p+r+q)})^2 - \Theta \Theta' \) vec \( (\Phi(0)) \), we get
\[
\text{vec}
\left( \Phi(0) \right) = \text{vec}(\Theta \Phi(0) \Theta') + \text{vec}(S_{\Theta}) + \text{vec}(S_{\Theta'}) \\
\text{vec}
\left( \Phi(0) \right) = \Theta \Theta' \text{vec}(\Phi(0)) \\
\frac{1}{(I_{d(p+r+q)})^2 - \Theta \Theta'} \text{vec}(S_{\Theta}) + \text{vec}(S_{\Theta'}) \\
\text{vec}
\left( \Phi(0) \right) = \left( I_{d(p+r+q)} \right)^2 - \Theta \Theta' \text{vec}(S_{\Theta}) + \frac{1}{(I_{d(p+r+q)})^2 - \Theta \Theta'} \text{vec}(S_{\Theta'}) \\
\text{vec}
\left( \Phi(0) \right) = \left( I_{d(p+r+q)} \right)^2 - \Theta \Theta' \text{vec}(S_{\Theta}) + \frac{1}{(I_{d(p+r+q)})^2 - \Theta \Theta'} \text{vec}(S_{\Theta'}) \text{ (33)}
\]
After computing vec \( (\Phi(0)) \) from the equation (33), we can collect \( \Phi(0), \Phi(1), ..., \Phi(p-1) \) from the representation
\[
\Phi(0) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \\
\text{such that } S_{11} = \begin{bmatrix} S_u & 0 & \ldots & 0 \\ 0 & S_u & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_u \end{bmatrix},
\]
\[ S_{12} = \begin{bmatrix} S_w & 0 & \ldots & 0 \\ 0 & S_w & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_w \end{bmatrix} \]
\[
\Phi_{12} = [k_u_{w0}, k_u_{w1}, ..., k_u_{wq}], k_u_{w0} = \begin{bmatrix} E(x(t)u(t)) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, k_u_{w1} = \begin{bmatrix} E(x(t)u(t-1)) \\ E(x(t-1)u(t)) \\ \vdots \\ 0 \end{bmatrix}, ..., k_u_{wr} = \begin{bmatrix} E(x(t)u(t-r+1)) \\ E(x(t-1)u(t-r+1)) \\ \vdots \\ E(x(t-p)u(t-r+1)) \end{bmatrix} \\
\]
\[
\Phi_{22} = \begin{bmatrix} E(x(t)w'(t-1)) \\ E(x(t-1)w'(t-1)) \\ \vdots \\ E(x(t-p)w'(t-1)) \end{bmatrix}
\]
\[ k_w_{11} = \begin{bmatrix} E(x(t)w'(t-1)) \\ E(x(t-1)w'(t-1)) \\ \vdots \\ E(x(t-p)w'(t-1)) \end{bmatrix}, k_w_{r+1} = \begin{bmatrix} E(x(t)(r+1)) \\ E(x(t-1)(r+1)) \\ \vdots \\ E(x(t-p)(r+1)) \end{bmatrix} \]

Consequently, the upper bound for the Rényi entropy of multivariate ARMA control process in (8) is given in following form:
\[
\text{Cupper} = C_d(\alpha) + \frac{1}{2} \ln(\text{det}(\Phi(0))) \text{ (34)}
\]
where, \( C_d \) is defined in equation (5).

7 EXAMPLES

In this section, the examples are given to show that the formulas of characteristic function in (16) and upper bounds in (34) are effective.

Example 1. Consider three dimensional ARMA control system as follows
\[
\begin{align*}
x(t) &= A_1 x(t-1) + u(t) + B_1 u(t-1) \\
&+ w(t) + D_1 w(t-1)
\end{align*}
\]
\[ x(0) = 0 \]

wherein, \( u(t) \in \mathbb{R}^3 \) is a zero mean stationary Gaussian process with covariance matrix \( S_u \) which represents the system input vector, \( w(t) \in \mathbb{R}^3 \) is the stationary Gaussian white noise process with mean zero, covariance matrix \( S_w \). Both processes \( u(t) \) and \( w(t) \) satisfy the independence assumptions. Consider the parameters as follows
\[ A_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0.1 & 0.1 & 0.3 \\ 0 & 0.2 & 0.3 \end{bmatrix}, D_1 = I_3, \]
\[ S_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.74 \end{bmatrix}, S_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}. \]

The characteristic polynomial of this process is
\[
\text{det}(I_3 - A_1 z) = (1 - 0.5z)(1 - 0.4z - 0.03z^2)
\]
Therefore, the roots of this polynomials are 2, 2.1525, -15.4858. Clearly, the length of all roots greater than 1. Hence, the output process \( x(t) \) is stable. To compute the matrices \( \mathbf{M}_1' \) and \( \mathbf{M}_1'' \) from equations (14) and (15)
Equation (19) gives the values for Rényi entropy of process \( x(t) \) as follow:

\[
R_{\alpha}(X(t)) = \left\{ \ln \left( 34.1884 \alpha^{-\frac{3}{\alpha-1}} \right) \right\}, \quad 0 < \alpha < \infty, \alpha \neq 1
\]

Rényi entropy is upper bounded by equation (34) as follows:

Firstly, to compute the \( \vec{\Phi}(0) \) in equation (33) and then we collect the matrices \( \Phi(0), \Phi(1), ..., \Phi(p-1) \) from the representation \( \vec{\Phi}(0) = [\Phi_{11}, \Phi_{12}, ..., \Phi_{p-12}] \) and so that we get on the covariance matrix \( \Phi(0) \) of \( X(t) \) in the following form:

\[
\Phi(0) = \begin{bmatrix}
5.1700 & 0.3765 & 0.0443 \\
0.3765 & 0.9241 & 1.3560 \\
0.0443 & 1.3560 & 2.8917
\end{bmatrix}
\]

Therefore, from equation (34), we get on the upper bound

\[
C_{\text{Upper}} = C_3(\alpha) + 0.6860
\]

**Example 2.** Consider ARMA control system in example 1. with the following assumptions: \( u(t) \in \mathbb{R}^3 \) is a zero mean stationary Cauchy process with scale matrix \( S_u \) which it represents the system input vector, \( w(t) \in \mathbb{R}^3 \) is the stationary Cauchy white noise process with mean zero and scale matrix \( S_w \). Both processes \( u(t) \) and \( w(t) \) satisfy the independent assumptions. consider that the parameters as follows:

\[
A_1 = \begin{bmatrix}
0.5 & 0 & 0 \\
0 & 0.5 & 0 \\
0 & 0 & 0.5
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0.3 & 0 & 0 \\
0 & 0.3 & 0 \\
0 & 0 & 0.3
\end{bmatrix},
\]

\[
D_1 = \begin{bmatrix}
0 & 0.25 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad S_u = \begin{bmatrix}
0.25 & 0 & 0 \\
0 & 1.0 & 0 \\
0 & 0 & 0.5
\end{bmatrix},
\]

\[
S_w = \begin{bmatrix}
0.25 & 0 & 0 \\
0 & 1.0 & 0 \\
0 & 0 & 0.5
\end{bmatrix}
\]

The characteristic polynomial of this process is \( \det(I_3 - A_1 z) = (1 - 0.5 z)^3 \), all roots of this polynomials are \( 2 \) then the length of all roots greater than 1. Therefore, the output process \( x(t) \) is stable. Now to compute the matrices \( \mathcal{M}_i \) and \( \mathcal{M}_j^\ast \) from equations (14) and (15):

\[
\mathcal{M}_0 = \begin{bmatrix}
0.4 & 0 & 0 \\
0 & 0.4 & 0 \\
0 & 0 & 0.4
\end{bmatrix}, \quad \mathcal{M}_1 = \begin{bmatrix}
0 & 0.2 & 0 \\
0 & 0.2 & 0 \\
0 & 0.2 & 0
\end{bmatrix},
\]

\[
\mathcal{M}_2 = \begin{bmatrix}
0.1 & 0 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0.1
\end{bmatrix}, \quad \mathcal{M}_3 = \begin{bmatrix}
0.05 & 0 & 0 \\
0 & 0.05 & 0 \\
0 & 0 & 0.05
\end{bmatrix}, ...
\]

\[
\mathcal{M}_0^* = \begin{bmatrix}
0.75 & 0 & 0 \\
0 & 0.75 & 0 \\
0 & 0 & 0.75
\end{bmatrix}
\]

\[
\mathcal{M}_1^* = \begin{bmatrix}
0.375 & 0 & 0 \\
0 & 0.375 & 0 \\
0 & 0 & 0.375
\end{bmatrix}
\]

\[
\mathcal{M}_2^* = \begin{bmatrix}
0.1875 & 0 & 0 \\
0 & 0.1875 & 0 \\
0 & 0 & 0.1875
\end{bmatrix}
\]

\[
\mathcal{M}_3^* = \begin{bmatrix}
0.0938 & 0 & 0 \\
0 & 0.0938 & 0 \\
0 & 0 & 0.0938
\end{bmatrix}
\]

\[
L_u = \begin{bmatrix}
0.3162 & 0 & 0 \\
0 & 0.7071 & 0 \\
0 & 0 & 0.7071
\end{bmatrix}, \quad L_w = \begin{bmatrix}
0.3162 & 0 & 0 \\
0 & 0.7071 & 0 \\
0 & 0 & 0.7071
\end{bmatrix}
\]

\[
K_0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad K_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_4 = \begin{bmatrix}
0.025 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad K_5 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_7 = \begin{bmatrix}
0.75 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
K_0^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad K_1^* = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* K_2^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* K_2^* K_3^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* K_2^* K_3^* K_4^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* K_2^* K_3^* K_4^* K_5^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* K_2^* K_3^* K_4^* K_5^* K_6^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
K_0^* K_1^* K_2^* K_3^* K_4^* K_5^* K_6^* K_7^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Obviously, the proportional condition of matrices \( K_j K_j^* \), \( j = 0, 1, 2, ... \) is hold. Then,

\[
D = a^2 * S_u
\]

where,

\[
a = (0.4 + 0.2 + 0.1 + 0.05 + 0.025 + ... + 0.75 + 0.375 + 0.1875 + 0.0938 + ...)
\]

Therefore, the values of Rényi entropy of process \( x(t) \) are...
Example 3. In this example, we take the processes \(u(t)\) and \(w(t)\) distributed multivariate Laplace with the same conditions and parameters in example 1. Then the upper bound for Rényi of \(X(t)\) is given as follows:

\[C_{\text{Upper}} = C_3(\alpha) + 0.6860\]

8 Conclusions

In this paper, we have derived the formula of the characteristic function of multivariate ARMA control process, then we used this formula as a tool to find the expression of Rényi entropy. As special case, three models (Gaussian, Cauchy and Laplace) are discussed. Also the covariance matrix and upper bound of entropy are computed.

9 References

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