Open quantum random walks and quantum Markov Chains on trees II: the recurrence

Farrukh Mukhamedov1,2 · Abdessatar Souissi3,4 · Tarek Hamdi3,5 · Amenallah Andolsi6

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Abstract
The present paper is a continuation of earlier appeared work, where we have constructed Quantum Markov Chains (QMC) associated with Open Quantum Random Walks and discussed a phase transition phenomena within the proposed scheme. In the current work, we define the recurrence and accessibility of within QMC over trees. Moreover, the recurrence property of QMC associated with Open Quantum Random Walks is investigated.

Keywords Open quantum random walks · Quantum Markov chain · Accessibility · Recurrence · Cayley tree

Mathematics Subject Classification 46L53 · 46L60 · 82B10

1 Introduction
The present paper is a continuation of earlier appeared work [25], where we have constructed Quantum Markov Chains (QMC) on trees associated with Open Quantum Random Walks (OQRW). Using such a construction, a phase transition phenomena first time detected a within the proposed scheme (see [23]). In this paper, we are going to define recurrence and accessibility of QMC over trees. Furthermore, the recurrence properties of QMC associated with OQRW is investigated. Notice that the mentioned problem has been investigated for discrete-time nearest-neighbor open quantum random walks on the integer line in [12]. However, in the present work, we focus on the recurrence problem associated with QMC, while in [12, 17, 18] the
recurrence has been treated with respect to the probability distribution defined by OQRW. In one dimensional setting, the recurrence problem had been paid attention by many authors (see for example [4, 5, 9, 13, 14, 17, 21]).

Notice that the theory of QMC, especially in the appearance of quantum walks, has generated a huge number of works [8, 11, 19, 20, 26, 27]. In [15] a novel approach has been proposed to explore quantum cryptography problems by employing of QMC. In the mentioned paper, mainly model checking technique was applied. Note that model-checking is an effective technique to check whether a system satisfies a desired property. The model-checking problem for a larger class of quantum systems quantum automata, quantum Markov chains was studied in [16] (see [31] for review). We think that the proposed approach could be adopted within QMC associated with OQRW [29, 30], but it will be discussed elsewhere.

2 Preliminaries

Let \( \Gamma_k = (V, E) \) be the semi-infinite Cayley tree of order \( k \) with root \( o \). Recall that, the vertices \( x \) and \( y \) are called nearest neighbors (denoted \( x \sim y \)) if they are joined through an edge (i.e. \( x, y \in E \)). A list \( x \sim x_1 \sim \cdots \sim x_{d-1} \sim y \) of vertices is called a path from \( x \) to \( y \). The distance (denoted \( d(x, y) \)) between two vertices \( x \) and \( y \) is the length of the shortest edge-path joining them.

Define

\[
W_n := \{ x \in V \mid d(x, o) = n \}, \quad \Lambda_n := \bigcup_{j \leq n} W_j, \quad \Lambda_{m,n} = \bigcup_{j=m}^n W_j.
\]

In what follows, a coordinate structure in \( \Gamma_k \) will be used. Namely, every vertex \( x \) (except for \( x^0 \)) of \( \Gamma_k \) has coordinates \((i_1, \ldots, i_n)\), here \( i_m \in \{1, \ldots, k\}, \ 1 \leq m \leq n \) and for the vertex \( x^0 \) we put \((0)\). The symbol \((0)\) constitutes level 0, and the sites \((i_1, \ldots, i_n)\) form the level \( n \) (i.e. \( d(x^0, x) = n \)) of the tree. By this structure, vertices \( x_{W_n}^{(1)}, x_{W_n}^{(2)}, \ldots, x_{W_n}^{(|W_n|)} \) of \( W_n \) can be represented (in the lexicographic order) as follows:

\begin{align*}
x_{W_n}^{(1)} & = (1, 1, \ldots, 1, 1), \quad x_{W_n}^{(2)} = (1, 1, \ldots, 1, 2), \quad \cdots \quad x_{W_n}^{(k)} = (1, 1, \ldots, 1, k), \\
x_{W_n}^{(k+1)} & = (1, 1, \ldots, 2, 1), \quad x_{W_n}^{(k+2)} = (1, 1, \ldots, 2, 2), \quad \cdots \quad x_{W_n}^{(2k)} = (1, 1, \ldots, 2, k), \\
& \vdots \\
x_{W_n}^{(|W_n|-k+1)} & = (k, k, \ldots, k, 1), \quad x_{W_n}^{(|W_n|-k+2)} = (k, k, \ldots, k, 2), \quad \cdots \quad x_{W_n}^{(|W_n|)} = (k, k, \ldots, k, k).
\end{align*}

Given a vertex \( x \in V \), by \( S(x) \) we denote the set of direct successors defined by

\[
S(x) := \{ y \in V : x \sim y \text{ and } d(y, o) > d(x, o) \}.
\]
In the coordinate form, such a set can be written as follows

\[ S(x) = \{(x, 1), (x, 2), \cdots, (x, k)\}. \]

Now, to each vertex \( x \), we associate a \( C^* \)-algebra of observables \( \mathcal{A}_x \) with identity \( 1_x \). It is assumed that the algebra \( \mathcal{A}_x \) has a trace \( \text{Tr} \). Given a bounded region \( V' \subset V \), we consider the algebra \( \mathcal{A}_{V'} = \bigotimes_{x \in V'} \mathcal{A}_x \). One has the following natural embedding

\[ \mathcal{A}_{\Lambda_n} \equiv \mathcal{A}_{\Lambda_n} \otimes 1_{W_{n+1}} \subset \mathcal{A}_{\Lambda_{n+1}}. \]

The algebra \( \mathcal{A}_{\Lambda_n} \) is then a subalgebra of \( \mathcal{A}_{\Lambda_{n+1}} \). Then the local algebra is defined by

\[ \mathcal{A}_{V; \text{loc}} := \bigcup_{n \in \mathbb{N}} \mathcal{A}_{\Lambda_n} \quad \text{(3)} \]

and the quasi-local algebra by

\[ \mathcal{A}_V := \overline{\mathcal{A}_{V; \text{loc}}}^{C^*}. \]

By \( S(\mathcal{A}_V) \) we denote the set of all states of \( \mathcal{A}_V \).

In what follows, by \( \text{Tr}_{\Lambda} : \mathcal{A}_V \to \mathcal{A}_{\Lambda} \) we mean normalized partial trace (i.e. \( \text{Tr}_{V}(1_V) = 1_{\Lambda} \), here \( 1_{\Lambda} = \bigotimes_{y \in \Lambda} 1_y \)), for any \( \Lambda \subseteq \text{fin} \) \( V \). For the sake of brevity, let us denote \( \text{Tr}_{\Lambda} : = \text{Tr}_{\Lambda_n} \), and \( \text{Tr}_{\chi} : = \text{Tr}_{\Lambda} \), if \( \Lambda = \{x\} \).

We notice that there are \( k \) natural shifts on the Cayley tree of order \( k \): for each \( x = (i_1, i_2, \cdots, i_n) \in \Lambda_n \) and \( j \in \{1, \ldots, k\} \)

\[ \alpha_j(x) = (j, x) = (j, i_1, i_2, \cdots, i_n) \in \Lambda_{n+1}. \quad \text{(4)} \]

Let \( g = (j_1, j_2, \cdots, j_N) \in V \), then one defines

\[ \alpha_g(x) := \alpha_{j_1} \circ \alpha_{j_2} \circ \cdots \circ \alpha_{j_N}(x) = (j_1, j_2, \cdots, j_N, i_1, i_2, \cdots, i_n). \]

The \( \alpha_j \)'s action on the algebra \( \mathcal{A}_V \) is given as follows:

\[ \alpha_j \left( \bigotimes_{x \in \Lambda_{\leq n}} a_x \right) := 1^{(o)} \otimes \bigotimes_{x \in \Lambda_{\leq n}} a_x^{(j, x)}. \quad \text{(5)} \]

The shift \( \alpha_j \) induces a \( * \)-isomorphism from \( \mathcal{A}_V \) into \( \mathcal{A}_{V_{(o, j)}} \). Let \( \alpha_j^{-1} \) its inverse isomorphism. For \( g \in V \), the map \( \alpha_g \) defines a \( * \)-isomorphism from \( \mathcal{A}_V \) into \( \mathcal{A}_{V_g} \) and its inverse isomorphism will be denoted by \( \alpha_g^{-1} \).

We recall that [3, 6] by a quantum Markov chain (QMC) on \( \mathcal{A}_V \) we mean a triplet \((\phi_o, (E_{\Lambda_n})_{n \geq 0}, (h_n)_n)\) of initial state \( \phi_o \in S(\mathcal{A}_o) \), a sequence of quasi-conditional
expectations\(^1\) \((E_{\Lambda_n})_n\) w.r.t. the triple \(\mathcal{A}_{\Lambda_{n-1}} \subseteq \mathcal{A}_{\Lambda_n} \subseteq \mathcal{A}_{\Lambda_{n+1}}\) and a sequence \(h_n \in \mathcal{A}_{W_{n,+}}\) of boundary conditions such that for each \(a \in \mathcal{A}_V\) the limit

\[
\varphi(a) := \lim_{n \to \infty} \phi_0 \circ E_{\Lambda_0} \circ E_{\Lambda_1} \circ \cdots \circ E_{\Lambda_n}(h_{n+1}^{1/2}a h_{n+1}^{1/2})
\]

exists in the weak*-topology and defines a state.

In the sequel, the state \(\varphi\) defined by (6) is also called quantum Markov chain. A QMC \(\varphi\) on \(\mathcal{A}_V\) is said to be tree-homogeneous if

\[
\varphi \circ \alpha_j = \varphi,
\]

for every \(j \in \{1, 2, \cdots, k\}\).

In the sequel, the trivial boundary condition \(h = 1\) will be considered. Then the associated tree-homogeneous quantum Markov chain \(\varphi\) is determined by the pair \(\varphi \equiv (\phi_o, \mathcal{E}) \equiv (\phi_o, \mathcal{E}, h = 1)\).\(^2\) In this notation, \(\mathcal{E}\) stands for a Markov transition expectation from \(\mathcal{A}_{(o)} \otimes \mathcal{A}_{(1)} \otimes \cdots \otimes \mathcal{A}_{(k)}\) into \(\mathcal{A}_{(o)}\). For each \(u\) by \(\mathcal{E}_u\) we denote the \(\alpha_u\)-shift of \(\mathcal{E}\) defined by

\[
\mathcal{E}_u = \alpha_u \circ \mathcal{E} \circ \alpha_u^{-1}.
\]

Clearly, \(\mathcal{E}_u\) is a transition expectation from \(\mathcal{A}_u \otimes \mathcal{A}_{(u,1)} \otimes \cdots \otimes \mathcal{A}_{(u,k)}\) into \(\mathcal{A}_u\). For each \(n \in \mathbb{N}\), we consider

\[
\mathcal{E}_{W_n} := \bigotimes_{u \in W_n} \mathcal{E}_u.
\]

One can see that \(\mathcal{E}_{W_n}\) is a Markov transition expectation from \(\mathcal{A}_{\Lambda_{[n,n+1]}}\) into \(\mathcal{A}_{W_n}\). Following [2, 24], we have the next result.

**Theorem 2.1** Let \(\varphi = (\phi_o, \mathcal{E})\) be a tree-homogeneous QMC. Then there exists a unique conditional expectation \(E_o\) from \(\mathcal{A}_V\) into \(\mathcal{A}_o\) characterized by

\[
E_o(a) = \mathcal{E}_o \left(a_o \otimes \mathcal{E}_{W_1} \left(a_{W_1} \otimes \cdots \otimes \mathcal{E}_{W_n} \left(a_{W_n} \otimes h_{n+1} \right)\right)\right),
\]

for all \(a = a_o \otimes a_{W_1} \otimes \cdots \otimes a_{W_n}\). Moreover, one has

\[
\varphi(\cdot) = \phi_o \circ E_o(\cdot).
\]

The forward Markov operator associated with \(\mathcal{E}_u\) is given by

\[
T_u(a) = \mathcal{E}_u(a \otimes I_{S(u)}), \quad a \in \mathcal{A}_u.
\]

\(^1\) A quasi-conditional expectation [11] is a completely positive identity preserving linear map \(E : \mathcal{A} \to \mathcal{B}\) such that \(E(ca) = cE(a)\), for all \(a \in \mathcal{A}, c \in \mathbb{C}\), where \(\mathbb{C} \subseteq \mathcal{B} \subseteq \mathcal{A}\). A Markov transition expectation from \(\mathcal{A}\) into \(\mathcal{B}\) is a completely positive identity preserving map, here \(\mathcal{B} \subseteq \mathcal{A}\)

\(^2\) We point out that the existence of other boundary conditions guides towards the phase transition problem within QMC scheme which was studied in [7, 22, 25].
While, there are $k$ backward Markov operators corresponding to the successors $(u, \ell)$, $j = 1, \ldots, k$ of $u$,

$$P_{u}^{(u, \ell)}(a) = \mathcal{E}_{u}(\mathbf{1}(u) \otimes a \otimes \mathbf{1}_{S(u) \setminus \{(u, \ell)\}}), \quad \forall a \in \mathcal{A}_{(u, \ell)}.$$ \hspace{1cm} (12)

For any ray $r = (u_{n})_{n}$, one defines

$$P_{u}^{m+n} = P_{u}^{m+1} \circ \cdots \circ P_{u}^{m+n-1}, \quad m, n \in \mathbb{N}. \hspace{1cm} (13)$$

The map $P_{u}^{m}$ defines a Markov operator from $\mathcal{A}_{u_{m}}$ into $\mathcal{A}_{u_{n}}$.

### 3 Recurrence of quantum Markov chains on trees

In the current section recurrence and weak recurrence for quantum Markov chains on trees will be investigated.

Now following [4, 28] given a projection $e \in \text{Proj}(\mathcal{A})$ and a ray $r = (u_{n})_{n} \in \text{Paths}(o, \infty)$, a stopping time $\tau_{e;r} = (\tau_{e;u_{n}})_{n}$ on the algebra $\mathcal{A}_{V}$, is defined as follows:

$$\tau_{e;o} = e^{(o)} \otimes \mathbf{1}_{V \setminus \{o\}}$$

$$\tau_{e;u_{1}} = e^{(o)} \otimes e^{(u_{1})} \otimes \mathbf{1}_{V \setminus \{u_{1}\}}$$

$$\vdots$$

$$\tau_{e;u_{n}} = e^{(o)} \otimes \cdots \otimes e^{(u_{n-1})} \otimes e^{(u_{n})} \otimes \mathbf{1}_{V \setminus \{u_{n}\}}$$

$$\tau_{e;u_{n};\infty} := e^{(o)} \otimes e^{(u_{1})} \otimes \cdots \otimes e^{(u_{n-1})} \otimes e^{(u_{n})} \otimes \mathbf{1}_{V \setminus \{u_{n}\}}, \hspace{1cm} (14)$$

where for each $a \in \mathcal{A}$ one has $a^{(u)} = \alpha_{u}(a)$. Put

$$\tau_{e;r;\infty} = \lim_{n \to \infty} \tau_{e;u_{n};\infty} = \bigotimes_{n \in \mathbb{N}} e^{(u_{n})}.$$ \hspace{1cm} (15)

**Definition 3.1** Let $\varphi = (\phi_{o}, \mathcal{E})$ be a tree-homogeneous quantum Markov chain. A projection $e \in \text{Proj}(\mathcal{A})$ is said to be

(i) $\mathcal{E}$–completely accessible if

$$E_{o}(\tau_{e;r;\infty}) := \lim_{n \to \infty} E_{o}(\tau_{e;u_{n};\infty}) = 0,$$ \hspace{1cm} (16)

for every ray $r = (x_{n})_{n}$.

(ii) $\varphi$–completely accessible if $\varphi(\tau_{e;r;\infty}) = 0$, for every ray $r = (x_{n})_{n}$.

(iii) $\mathcal{E}$-recurrent if $0 < \text{Tr}(\mathcal{E}(e \otimes \mathbf{1})) < \infty$ and one has

$$\frac{1}{\text{Tr}(\mathcal{E}(e \otimes \mathbf{1}))} \text{Tr} \left( E_{o} \left( \sum_{n \geq 0} e \otimes \tau_{e;x_{n}} \right) \right) = 1,$$ \hspace{1cm} (17)
for every ray \( r = (x_n)_n \).

(iv) \( \varphi \)-recurrent if \( \varphi(\alpha_0(e)) \neq 0 \) and

\[
\frac{1}{\varphi(\alpha_0(e))} \varphi \left( \sum_n e \otimes \tau_{e,x_n} \right) = 1,
\]

for every ray \( r = (x_n)_n \).

**Definition 3.2** Let \( \varphi = (\phi_o, \mathcal{E}) \) be a tree-homogeneous quantum Markov chain. Let \( e, f \in \text{Proj}(\mathcal{A}), e, f \neq 0 \). The projection \( f \) is

(i) \( \mathcal{E} \)-accessible from \( e \) (and we write \( e \rightarrow \mathcal{E} f \)) if for any ray \( r = (x_n)_n \) there exists \( m \in \mathbb{N} \) such that

\[
E_o \left( \alpha_0(e) \alpha_{x_m}(f) \right) \neq 0;
\]

(ii) \( \varphi \)-accessible from \( e \) (we denote it as \( e \rightarrow \varphi f \)) if for any ray \( r = (x_n)_n \) there exists \( m \in \mathbb{N} \) such that

\[
\varphi \left( \alpha_0(e) \alpha_{x_m}(f) \right) \neq 0.
\]

**Lemma 3.3** In the above notations:

\[
\sum_{n \geq 0} \tau_{e,x_n} = 1_{\mathcal{A}_V} - \tau_{e;r;\infty}.
\]

**Proof** see [28]

**Theorem 3.4** Let \( \varphi \equiv (\phi_o, \mathcal{E}) \) be a tree-homogeneous quantum Markov chain on \( \mathcal{A}_V \). Let \( e \in \text{Proj}(\mathcal{A}_V) \) be a projection

(i) \( e \) is \( \mathcal{E} \)-recurrent if and only if for any ray \( r = (x_n)_n \) one has

\[
\mathcal{E}(e \otimes E_o(\tau_{e;r;\infty})) = 0;
\]

(ii) \( e \) is \( \varphi \)-recurrent if and only if for any ray \( r = (x_n)_n \) one has

\[
\varphi(e \otimes \tau_{e;r;\infty}) = 0;
\]

(iii) \( e \) is \( \mathcal{E} \)-accessible from \( f \) if and only if for any ray \( r = (x_n)_n \) there exists \( m \in \mathbb{N} \) such that

\[
\mathcal{E}(e \otimes P_{x_1}^{x_m} T_{x_m} f) \neq 0;
\]

(iv) \( e \) is \( \varphi \)-accessible from \( f \) if and only if for any ray \( r = (x_n)_n \) there exists \( m \in \mathbb{N} \) such that

\[
\varphi(e \otimes P_{x_1}^{x_m} T_{x_m} f) \neq 0.
\]
Proof From Lemma 3.3 one has
\[ \sum_{n \geq 0} e \otimes \tau_{e; x_n} = e \otimes 1 - e \otimes \tau_{e; r; \infty}. \]

This leads to (i) and (ii).
One has
\[ \mathcal{E}_{W_n}(f(x_m) \otimes 1) = \mathcal{E}_{x_m}(f(x_m) \otimes 1) = T_{x_m}f \]
and
\[ \mathcal{E}_{W_n}(1) = P_{W_n}T_{x_m}f. \]

This proves (iii) and using (10) one gets (iv). \(\square\)

Corollary 3.5 Let \(\varphi \equiv (\varphi_0, \mathcal{E})\) be a tree-homogeneous quantum Markov chain. Any \(\mathcal{E}\)-recurrence projection is \(\varphi\)-recurrent. Conversely, if the initial state \(\varphi_0\) is faithful then any \(\varphi\)-recurrence projection is \(\mathcal{E}\)-recurrnet.

Proof Let \(e \in \text{Proj}(A)\) be a projection. For each \(\ell \in \{1, \ldots, k\}\), one has
\[ E_{o}\alpha(a) = \mathcal{E}(a \otimes E_{o}\alpha(a)), \quad \forall a \in A_0, \forall a \in A_V. \]

Then
\[ \varphi(e \otimes \tau_{e; r; \infty}) = \varphi(\alpha_0(e) \otimes \alpha_{x_1}(\tau_{e; r; \infty})) \]
\[ = \phi_0( \mathcal{E}(e \otimes E_{o}\alpha_{x_1}(\tau_{e; r; \infty}))) \]
\[ = \phi_0( \mathcal{E}(e \otimes E_{o}\tau_{e; r; \infty})). \]

Therefore, if \(\mathcal{E}(e \otimes E_{o}\tau_{e; r; \infty}) = 0\) then \(\varphi(e \otimes \tau_{e; r; \infty}) = 0\). This shows the first implication.

If the initial state \(\varphi_0\) is faithful, since \(\mathcal{E}(e \otimes E_{o}\tau_{e; r; \infty}) \geq 0\) then from the above computation, we have
\[ \varphi(e \otimes \tau_{e; r; \infty}) = 0 \Rightarrow \mathcal{E}(e \otimes E_{o}\tau_{e; r; \infty}) = 0. \]

This shows the converse direction, and finishes the proof. \(\square\)
4 Recurrence of QMC associated with OQRW

Let $\mathcal{H}$ and $\mathcal{K}$ be given two separable Hilbert spaces over the complex field $\mathbb{C}$. Let $\{|i\rangle\}_{i \in \Lambda}$ be an orthonormal basis of $\mathcal{K}$ indexed by a graph $\Lambda$ with almost-countable vertex set. The algebra of observable at a site $x \in V$ is considered to be $\mathcal{A}_x = \mathcal{A} := \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

In the language of OQRW [10] the Hilbert space $\mathcal{H}$ describes the internal degree of freedom of the quantum walker, while $\mathcal{K}$ describes the state space of the dynamics where the walk is done through the oriented graph $\Lambda$. The transition of the walker from a site $j$ to site $i$ is described by a bounded operator $B^j_i \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_{i \in \Lambda} B^i_j B^{i*}_j = 1_{\mathcal{B}(\mathcal{H})}.$$  \hspace{1cm} (24)

The operators $B^j_i$ act on $\mathcal{H}$ only, we dilate them as operators on $\mathcal{H} \otimes \mathcal{K}$ by putting

$$M^i_j = B^i_j \otimes |i\rangle \langle j| \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}).$$  \hspace{1cm} (25)

The operator $M^i_j$ encodes exactly the idea that while passing from $|j\rangle$ to $|i\rangle$ on the lattice, the effect is the operator $B^i_j$ on $\mathcal{H}$.

The initial density matrix of the dynamics is $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, of the form

$$\rho = \sum_{i \in \Lambda} \rho_i \otimes |i\rangle \langle i|; \quad \rho_i \in \mathcal{B}(\mathcal{H})^+.$$  \hspace{1cm} (26)

In what follows, for the sake of simplicity of calculations, we assume that $\rho_i \neq 0$ for all $i \in \Lambda$ (see [14, Remark 4.5] for other kind of initial states).

For a given initial state $\rho$ (see 26) the Open Quantum Random Walk (OQRW) is defined by the mapping $\mathcal{M}$, which has the following form

$$\mathcal{M}(\rho) = \sum_i \left( \sum_j B^i_j \rho_j B^{i*}_j \right) \otimes |i\rangle \langle i|.$$  \hspace{1cm} (27)

Let

$$A^i_j := \frac{1}{\text{Tr}(\rho_j)^{1/2}} \rho_j^{1/2} \otimes |i\rangle \langle j|, \quad i, j \in \Lambda.$$  \hspace{1cm} (28)

For each $u \in V$, one defines

$$K^i_{j (u, S(u))} := M^i_{j (u, S(u))} \otimes \bigotimes_{v \in S(u)} A^i_j \in \mathcal{A}_{\{u\} \cup S(u)}.$$  \hspace{1cm} (29)
The interaction of a vertex \( u \in V \) with its set of direct successors it described by

\[
K^{(u,S(u))} = \sum_{i,j} K_j^{(u,S(u))} \in \mathcal{A}_{[u] \cup S(u)}.
\]

Put

\[
\mathcal{E}_u(a) := \text{Tr}_{u}\left(K^{(u,S(u))} a K^{(u,S(u))}\right); \quad a \in \mathcal{A}_{[u] \cup S(u)}.
\] (30)

For each \( j, j' \in \Lambda \) we set

\[
\varphi_{jj'}(b) := \frac{1}{\text{Tr}(\rho_j)^{1/2} \text{Tr}(\rho_{j'})^{1/2}} \text{Tr}\left(\rho_j^{1/2} \rho_{j'}^{1/2} \otimes |j\rangle\langle j'| b\right); \quad \forall a \in \mathcal{A} \] (31)

One can see that \( \varphi_{jj'} \) is a linear functional on \( \mathcal{A} \). If \( j = j' \), we denote it simply denote \( \varphi_j \) instead of \( \varphi_{jj} \) one has

\[
\varphi_j(a) = \frac{1}{\text{Tr}(\rho_j)} \text{Tr}(\rho_j \otimes |j\rangle\langle j| a).
\] (32)

The functional \( \varphi_j \) is then, a state on \( \mathcal{A} \).

**Theorem 4.1** In the above notations, the map \( \mathcal{E}_u \) defines a Markov transition expectation from \( \mathcal{A}_{[u] \cup S(u)} \) into \( \mathcal{A}_u \) and

\[
\mathcal{E}_u(a_u \otimes a_{(u,1)} \otimes \cdots \otimes a_{(u,k)}) = \sum_{(i,j,j') \in \Lambda^3} M_j^i a_{(u,i)} M_j^i \prod_{\ell=1}^k \varphi_{jj'}(a_{(u,\ell)}). \] (33)

Moreover, the backward Markov operators associated with \( \mathcal{E}_u \) are given by

\[
P_u^{(u,\ell)}(a_{(u,\ell)}) = \sum_j \left( \mathbb{I}_{\mathcal{B}(\mathcal{H})} \otimes |j\rangle\langle j| \right) \varphi_j(a_{(u,\ell)}). \] (34)

The forward Markov operator associated with \( \mathcal{E}_u \) is given by

\[
T_u(a_u) = \sum_{ij} M_j^i a_u M_j^i, \] (35)

where \( a_u \in \mathcal{A} \) and \( a_{(u,\ell)} \in \mathcal{A}_{(u,\ell)} \) for each \( \ell \in \{1, \cdots, k\} \).
**Proof** The map $E_u$ (30), is clearly completely positive. Let $a = a_u \otimes a_{(u, 1)} \otimes \cdots \otimes a_{(u, k)}$. Taking into account (29) and (25) one gets

$$E_u(a) = \text{Tr}_{u}[\left( \sum_{(i, j) \in \Lambda^2} K^i_j \right) a \left( \sum_{(i, j) \in \Lambda^2} K^i_j \right)^*]$$

$$= \sum_{(i, j), (i', j') \in \Lambda^2} \text{Tr}_{u}[K^i_j (u, S(u)) a_u \otimes a_{(u, 1)} \cdots \otimes a_{(u, k)} K^{i'}_{j'} (u, S(u))^*]$$

$$= \sum_{(i, j), (i', j') \in \Lambda^2} \text{Tr}_{u1}\left( M^{i'}_{j'} (u) a_u M^{i'}_{j'} (u) \otimes \bigotimes_{\ell=1}^k (A^i_j a_{(u, \ell)} A'^i_{j'}) (u, \ell) \right)$$

$$= \sum_{(i, j), (i', j') \in \Lambda^2} M^{i'}_{j'} a_0 (u, 0) M^i_{j'} \prod_{\ell=1}^k \text{Tr}(A^i_j a_{(u, \ell)} A'^i_{j'}) \cdot$$

For each $\ell \in \{1, \ldots, k\}$, one has

$$\text{Tr}(A^i_j a_{(u, \ell)} A'^i_{j'}) = \text{Tr} \left( A'^i_{j'} A^i_j a_{(u, \ell)} \right)$$

$$= \frac{1}{\text{Tr}(\rho_j)^{1/2} \text{Tr}(\rho_{j'})^{1/2}} \text{Tr} \left( \rho_j^{1/2} \rho_{j'}^{1/2} \otimes |j'\rangle \langle j| a_{(u, \ell)} \right) \delta_{i, i'}$$

$$= \phi_{jj'}(a_{(u, \ell)}) \delta_{i, i'},$$

where $\delta_{i, i'}$ denotes the Kronecker symbol. This leads to (33). Moreover, one has

$$E_u(1_{(u, S(u))}) = \sum_{i, j} M^i_j M^i_j \prod_{\ell=1}^k \phi_{jj'}(1_{(u, \ell)})$$

$$= \sum_{i, j} M^i_j M^i_j = 1_u,$$

which yields that $E_u$ is a Markov transition expectation.

From (12) one has

$$P^{(u, \ell)}_{u} (a_{(u, \ell)}) = \sum_{i, j, j'} M^i_j M^i_{j'} \phi_{jj'}(a_{(u, \ell)}) \prod_{\ell' \neq \ell}^k \phi_{jj'}(1_{(u, \ell')})$$

$$= \sum_{i, j} M^i_j M^i_j \phi_j (a_{(u, \ell)})$$
\[
\sum_j \left( \sum_i B_j^i B_j^i \right) \varphi_j(a_{(u,\ell)})
\]
\[
= \sum_j \left( \mathbf{1}_H \otimes |j\rangle\langle j| \right) \varphi_j(a_{(u,\ell)}).
\]

The forward Markov operator (11) associated with \( E_u \) satisfies
\[
T_u(a_u) = \sum_{i,j,j'} M^i_j a_u M^i_j \prod_{\ell=1}^k \varphi_{jj'}(I_{(u,\ell)}) = \sum_{i,j} M^i_j a_u M^i_j.
\]

This completes the proof. \( \square \)

Now we are ready to build the conditional expectation \( E_{o|} \) corresponding to open quantum random walks by means of the transition expectations (30) and the quantum Markov chain \( \varphi \equiv (\phi_o, E) \), where
\[
E(a) := E_{o}(a) = \sum_{i,j} M^i_j a_o M^i_j \prod_{\ell=1}^k \varphi_j(a_{(o,\ell)}),
\]
for each \( a = a_o \otimes a_{(o,1)} \otimes \cdots \otimes a_{(o,k)}. \)

It is clear that for each \( u \in V \) the transition expectation \( E_u \) is a copy of \( E \) in the sense of (8). Using the proof of Theorem 4.1 one can establish the following result.

**Theorem 4.2** In the above notations, the conditional expectation \( E_{o|} \) associated with \( E \) through (9) has the following expression
\[
E_{o|}(a) = \sum_j \mathcal{M}_j(a_o) \prod_{u \in \Lambda_{[1,n]}} \psi_j(a_u),
\]
where
\[
\psi_j(b) = \frac{1}{\text{Tr}(\rho_j)} \sum_{i \in \Lambda} \text{Tr}\left( B_j^i \rho_j B_j^{i*} \otimes |i\rangle\langle i| \right), \quad \forall b \in A
\]
and \( a = \otimes_{u \in \Lambda_n} a_u \in A_{\Lambda_n}. \) Moreover, for any initial state \( \phi_o = \text{Tr}(\omega_o) \) the tree-homogeneous quantum Markov chain \( \varphi \equiv (\phi_o, E) \) is given by
\[
\varphi(a) = \sum_j \text{Tr}(\omega_o \mathcal{M}_j(a_o)) \prod_{u \in \Lambda_{[1,n]}} \psi_j(a_u),
\]
where
\[
\mathcal{M}_j(\cdot) = \sum_{i \in \Lambda} M_j^i \cdot M_j^i.
\]
Remark 4.3 We notice that in our previous work [25] the expression (39) defines a QMC associated with the disordered phase of the system that deals with phase transitions related to QMC on trees associated with OQRW.

Theorem 4.4 In the notations of Theorem 4.2, if e is a projection in \( \mathcal{A} \) such that

\[
p := \sup_{j \in \Lambda} \psi_j(e^\perp) < 1
\]

then e is \( \mathcal{E} \)-recurrent.

Proof Let \( r = (x_n)_n \) be a ray one the semi-infinite Cayley tree. Then

\[
E_{\tau_e}(\tau_{e, r; \infty}) \overset{(37)}{=} \sum_{j \in \Lambda} M_j^* \alpha_e(e^\perp) M_j \prod_{m=1}^{n} \psi_j(\alpha_{x_m}(e^\perp))
\]

\[
\overset{(38)}{=} \sum_{j \in \Lambda} M_j^* e^\perp M_j \left( \psi_j(e^\perp) \right)^n
\]

\[
\leq \sum_{j} M_j^* M_j p^n
\]

\[
= p^n.
\]

From (4.4) it follows that

\[
0 \leq E_{\tau_e}(\tau_{e, r; \infty}) = \lim_{n \to \infty} E_{\tau_e}(\tau_{e, x_n; \infty}) = 0.
\]

Thus \( E_{\tau_e}(\tau_{e, r; \infty}) = 0 \), and (20) implies that the projection e is \( \mathcal{E} \)-recurrent. \( \square \)

5 Examples

This section is devoted for illustration of the obtained results for the quantum Markov chains associated with OQRW.

Let \( \mathcal{H} = \mathcal{K} = \mathbb{C}^2 \). The algebra of observable at a site \( u \) is then \( \mathcal{A}_u = \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \equiv M_4(\mathbb{C}) \). Let \( \Lambda = \{1, 2\} \). The interactions are given by

\[
B_1^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad B_1^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_2^1 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad B_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( |a|^2 + |c|^2 = |b|^2 + |d|^2 = 1 \). Put

\[
p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
and

\[ |1 \rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |2 \rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Notice that \(|1 \rangle, |2 \rangle\) is an ortho-normal basis of \(\mathcal{K} \equiv \mathbb{C}^2\). In the sequel elements of \(\mathcal{B}(\mathcal{H})\) will be denoted by means of \(2 \times 2\) complex matrices, while elements of \(\mathcal{B}(\mathcal{K})\) will be written using Dirac notation \(|i \rangle \langle j|\).

Recall that any rank-1 projection in \(\mathbb{M}_2(\mathbb{C})\) has the form

\[ p(\varepsilon, z) = \begin{pmatrix} \varepsilon & z\sqrt{\varepsilon(1-\varepsilon)} \\ \overline{z}\sqrt{\varepsilon(1-\varepsilon)} & 1-\varepsilon \end{pmatrix}, \] (44)

where \(\varepsilon \in [0, 1], z \in \mathbb{C}\) with \(|z| = 1\). Then, let us consider the projection on \(\mathcal{A}\) having the following form

\[ e(\varepsilon, z, \xi) = p(\varepsilon, z) \otimes |\xi \rangle \langle \xi|, \]

where

\[ |\xi \rangle := \sum_{i \in \Lambda} \xi_i |i \rangle \in \mathcal{K} \]

being a unit vector. i.e. \(\sum_{i \in \Lambda} |\xi_i|^2 = 1\).

**Example 5.1** (\(E\)-recurrence) By (38) one computes

\[ \psi_j(e(\varepsilon, z, \xi)) = \frac{1}{\text{Tr}(\rho_j)} \sum_{i \in \Lambda} \text{Tr} \left( B^i_j \rho_j B^i_j \ast p(\varepsilon, z) \right) |\xi_i|^2. \]

Then, for \(\rho_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\), one finds

\[ \text{Tr} \left( B^1_1 \rho_j B^1_1 \ast p(\varepsilon, z) \right) = \varepsilon |a|^2, \]

\[ \text{Tr} \left( B^1_2 \rho_j B^1_2 \ast p(\varepsilon, z) \right) = 0, \]

\[ \text{Tr} \left( B^2_1 \rho_j B^2_1 \ast p(\varepsilon, z) \right) = \varepsilon |c|^2, \]

\[ \text{Tr} \left( B^2_2 \rho_j B^2_2 \ast p(\varepsilon, z) \right) = \varepsilon. \]
Hence,
\[
\psi_1(e(\varepsilon, z, \xi)) = \sum_{i \in \Lambda} \text{Tr}\left( B_1^i \rho_j B_1^{i*} p(\varepsilon, z) \right) |\xi_i|^2 = \varepsilon \left( |a|^2 |\xi_1|^2 + |c|^2 |\xi_2|^2 \right). \tag{45}
\]
\[
\psi_2(e(\varepsilon, z, \xi)) = \sum_{i \in \Lambda} \text{Tr}\left( B_2^i \rho_j B_2^{i*} p(\varepsilon, z) \right) |\xi_i|^2 = \varepsilon |\xi_2|^2. \tag{46}
\]
Consequently, Theorem 4.4 implies that \( e(\varepsilon, z, \xi) \perp \) is \( \mathcal{E} \)-recurrent whenever \( \varepsilon < 1 \). If \( \varepsilon = |a| = |\xi_1| = 1 \), the projection \( e(\varepsilon, z, \xi) \) becomes
\[
e(1, z, \xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes |1\rangle\langle 1|.
\]
Put
\[
f := e(1, z, \xi) \perp = 1_{M_2} \otimes |2\rangle\langle 2| + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes |1\rangle\langle 1|
\]
By (45) and (46) we find \( \psi_1(f^\perp) = 1 \) and \( \psi_2(f^\perp) = 0 \). It then follows from (37) that
\[
E_{o}(\tau_{f}; x_n; \infty) = M_1(f^\perp) = \sum_{i=1}^{2} M_i^{\dagger} f^\perp M_i^{\dagger} = B_1^{\dagger} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B_1 \otimes |1\rangle\langle 1| = f^\perp.
\]
Therefore,
\[
\mathcal{E}(f \otimes E_{o}(\tau_{f}; r; \infty)) = \mathcal{E}(f \otimes f^\perp) = M_1(f) \psi_1(f^\perp) = M_1(f) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes |1\rangle\langle 1| \neq 0.
\]
So, (22) implies that the projection \( f \) is not \( \mathcal{E} \)-recurrent. This means that the inequality in Theorem 4.4 is optimal.

**Example 5.2** (\( \mathcal{E} \)-accessibility) Recall that for \( \ell = 1, 2 \), a backward Markov operator is given by
\[
P_{u}(a_{(u, \ell)}) = \sum_{j=1}^{2} \left( 1_{B(H)} \otimes |j\rangle\langle j| \right) \varphi_j(a_{(u, \ell)}).
\]
Recall also that the forward Markov operator is given by

\[ T_u(a_n) = \sum_{ij} M_j a_n M_j^i. \]

Then,

\[ \mathcal{E} (e \otimes P_{x_0} T_{x_m} f) = \sum_j \psi_j(f) \mathcal{E} (e \otimes I \otimes |j \rangle \langle j|). \]

- Take \( e \in Proj(A) \) and \( f = e(\varepsilon, z, \xi) \), then using (45) and (46), we obtain

\[
\mathcal{E} (e \otimes P_{x_0} T_{x_m} e(\varepsilon, z, \xi)) = \sum_j \psi_j(e(\varepsilon, z, \xi)) \mathcal{E} (e \otimes I \otimes |j \rangle \langle j|)
\]

\[
= \varepsilon \left( |a|^2 |\xi_1|^2 + |c|^2 |\xi_2|^2 \right) \mathcal{E} (e \otimes I \otimes |1 \rangle \langle 1|) + \varepsilon |\xi_2|^2 \mathcal{E} (e \otimes I \otimes |2 \rangle \langle 2|)
\]

\[
= \varepsilon \left( |a|^2 |\xi_1|^2 + |c|^2 |\xi_2|^2 \right) \mathcal{M}_1(e) + |\xi_2|^2 \mathcal{M}_2(e)
\]

for any projection \( e \). In particular, one easily can see that there is no projection \( e \) which is \( \mathcal{E} \)-accessible from

\[ e(0, z, \xi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes |\xi \rangle \langle \xi|. \]

- Now, take

\[ f = \sigma^{x_{W_n}(1)} = I_{M_2} \otimes |1 \rangle \langle 1|, \]

where \( x_{W_n}(1) \) is defined by (1). Then

\[ \psi_1(\sigma^{x_{W_n}(1)}) = \text{Tr}(B_1^1 p B_1^{1*}) = |a|^2 \quad \text{and} \quad \psi_2(\sigma^{x_{W_n}(1)}) = \text{Tr}(B_2^1 p B_2^{1*}) = 0. \]

Hence,

\[ \mathcal{E} (e \otimes P_{x_1} T_{x_m} \sigma^{x_{W_n}(1)}) = |a|^2 \mathcal{E} \left( e \otimes (I \otimes |1 \rangle \langle 1|)^{(x_1)} \right) \]

\[ = |a|^2 \sum_i M_i^{1*} e M_i^1 \]

\[ = |a|^2 \mathcal{M}_1(e). \]

In particular, if \( |a| > 0 \), we infer

\[ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes |1 \rangle \langle 1| \]

is \( \mathcal{E} \)-accessible from \( \sigma^{x_{W_n}(1)} \), since \( \mathcal{M}_1(e_1) = e_1. \)
Example 5.3 ($\varphi$-accessibility) We notice that,

$$
\varphi \left( e \otimes P_{x_0}^{x_m} T_{x_m} f \right) = \sum_j \psi_j(f) \varphi \left( e \otimes 1 \otimes |j\rangle\langle j| \right),
$$

where

$$
\varphi \left( e \otimes 1 \otimes |j\rangle\langle j| \right) = \sum_k \text{Tr} \left( \omega_0 M_k(e) \right) \psi_k \left( 1 \otimes |j\rangle\langle j| \right)
$$

$$
= \sum_k \text{Tr} \left( \omega_0 M_k(e) \right) \psi_k \left( 1 \otimes |j\rangle\langle j| \right)
$$

$$
= \sum_k \text{Tr} \left( \omega_0 M_k(e) \right) \frac{\text{Tr}(B_{k}^j \rho_k B_{k}^{j*})}{\text{Tr}(\rho_k)}.
$$

Hence, for $|\alpha| > 0$ and

$$
\omega_0 = \begin{pmatrix}
(0) & (0) \\
(0) & (*)
\end{pmatrix}
$$

we deduce that

$$
e_1 = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \otimes |1\rangle\langle 1|
$$

is not $\varphi$-accessible from $\sigma^{x_{w_n}(1)}$, since

$$
\text{Tr} \left( \omega_0 M_1(e_1) \right) = \text{Tr} \left( \omega_0 e_1 \right) = 0 \quad \text{and} \quad M_2(e_1) = 0.
$$

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Authors and Affiliations

Farrukh Mukhamedov¹,² · Abdessatar Souissi³,⁴ · Tarek Hamdi³,⁵ · Amenallah Andolsi⁶

Tarek Hamdi
t.hamdi@qu.edu.sa

Amenallah Andolsi
amenallah.andolsi@fst.utm.tn

¹ Department of Mathematical Sciences, College of Science, United Arab Emirates University, 15551 Al-Ain, United Arab Emirates
² Institute of Mathematics Named After V.I.Romanovski, 4, University str., 100125 Tashkent, Uzbekistan
³ Department of Accounting, College of Business Management, Qassim University, Buraydah, Saudi Arabia
⁴ Preparatory Institute for Scientific and Technical Studies, Carthage University, 1054 Amilcar, Tunisia
⁵ Laboratoire d’Analyse Mathématiques et Applications LR11ES11, Université de Tunis El-Manar, Tunis, Tunisia
⁶ Nuclear Physics and High Energy Physics Research Unit, Faculty of Sciences of Tunis, University of Tunis El Manar, 2092 Tunis, Tunisia