GRIFFITHS INEQUALITIES FOR SOME O(n) CLASSICAL SPIN MODELS WITH \( n \geq 3 \)

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Abstract

The first and second Griffiths inequalities are proved for some classical O(n)-invariant spin models (including Euclidean quantum field theories) for any \( n \). The proof assumes a certain condition on an integral transform of the measure. Some examples are discussed.

1 Introduction

We consider in this paper classical spin models on a lattice \( \Lambda \), with an \( n \)-component vector spin \( s(x) = (s_1(x), \ldots, s_n(x)) \), assigned to each lattice site \( x \in \Lambda \). Such a model has the partition function

\[
Z[G] = \int \left\{ \prod_{x \in \Lambda} d^n s(x) \mu[s(x)] \right\} \exp \left[ \sum_{x,y \in \Lambda} J_{xy} s(x) \cdot s(y) + \sum_{x \in \Lambda} h_x \cdot s(x) \right], \tag{1.1}
\]

\(^1\)Work supported in part by National Science Foundation Grant 0070991 and PSC-CUNY Research Award 63502-00 32.
where $\mu(s) = \mu(s_1, \ldots, s_n) \geq 0$ is a nonnegative distribution on $\mathbb{R}^n$ and $J_{xy} = J_{yx}$ and $h_x$ are real nonnegative numbers for each pair $x, y \in \Lambda$. The expectation values are

$$
\langle s_{q_1}(x_1) \cdots s_{q_m}(x_m) \rangle = \frac{1}{Z[G]} \int \left\{ \prod_{x \in \Lambda} d^n s(x) \, \mu[s(x)] \right\} 
\times s_{q_1}(x_1) \cdots s_{q_m}(x_m) \exp \sum_{x,y \in \Lambda} J_{xy} \, s(x) \cdot s(y), \quad (1.2)
$$

Some special cases possess global $O(n)$ symmetry, but in general such a symmetry may not be present in (1.1) and (1.2). If the measure $\mu(s)$ depends only on $s \cdot s$ an $O(n)$ symmetry is realized through $s(x) \to Ms(x)$, where $M$ is an arbitrary $n \times n$ orthogonal matrix, $M^T = M^{-1}$ whose components are real.

The G.K.S. inequalities were first proved by Griffiths [3, 4, 5] and extended by Kelly and Sherman [6] for the Ising model, which is a special case of (1.1), (1.2) with $n = 1$. Ginibre proved the inequalities for the XY model, a case with $n = 2$ [7]. Ginibre’s method extends to other measures for $n = 1, 2$ [2, 7]. No one to date has succeeded in proving the second G.K.S. inequality for a model of the form (1.1), (1.2) for $n \geq 3$. Significantly, Sylvester showed that Ginibre’s method of proof fails for the $n \geq 3$ classical Heisenberg magnet (non-linear sigma models) [8]. We show that both G.K.S. inequalities are satisfied for a certain class of models with arbitrary $n$. We are able to find expressions for the measures for some of these models. As yet, the results have not been extended to familiar examples, such as Euclidean field theories with quartic interactions or the classical Heisenberg magnets. Nonetheless, it is significant that it is possible to prove the inequalities for any non-Abelian systems.

A particular limit of the potential yields $\mu[s(x)] = \delta[1 - s(x) \cdot s(x)]$, (1.1) and (1.2), which is the classical Heisenberg magnet. On a regular hypercubic lattice with translation-invariant couplings, such models are Euclidean quantum field theories with $\mu[s(x)] = e^{-aV[s(x) \cdot s(x)]}$, where $a$ is the lattice spacing for the potential function $V$ in the Lagrangian. The $O(n)$ nonlinear sigma model is the Euclidean quantum field theory which is a classical Heisenberg magnet.

The so-called second Griffiths inequality (Theorem 2 below) is proved here for measures satisfying a particular condition:

$$
\int d^n s \, \mu(s) \, e^{q \cdot s} = e^{\Xi(q \cdot q)} \quad (1.3)
$$

for any $q \in \mathbb{R}^n$ and for a function $\Xi(\cdot)$ which is analytic in $z = q \cdot q$ within a real interval $[0, q_0^2]$ for some $q_0 \neq 0$, and whose Taylor expansion in this interval has real nonnegative coefficients:

$$
\Xi(z) = \xi_0 + \xi_1 z + \xi_2 z^2 + \xi_3 z^3 + \cdots, \quad z \in [0, q_0^2], \quad \xi_1 \geq 0, \xi_2 \geq 0, \xi_3 \geq 0, \ldots \quad (1.4)
$$

We note that analyticity is not a particularly strong assumption; it is needed for the expansion of correlation functions (in $J_{xy}$ and $h_x$) to converge in a finite volume.
We note that (1.3) may be inverted, giving a measure $\mu(\cdot)$ in terms of any function $\Xi(\cdot)$ satisfying (1.4):
\[
\int \frac{d^n q}{(2\pi)^n} e^{\Xi(-q \cdot q)} e^{-i q \cdot s} = \mu(s) \tag{1.5}
\]

If $\Lambda$ is a regular hypercubic lattice of dimension $d$, lattice spacing $a$ and $J_{xy} = J_{yx}$
\[
J_{xy} = \begin{cases} 
\frac{1}{2g^2}, & \text{for } x, y \text{ nearest neighbors} \\
r, & \text{for } x = y \\
0, & \text{otherwise} 
\end{cases} \tag{1.6}
\]
then (1.1), (1.2) describes a lattice-regularized field theory with bare coupling constant $g$ and external current $h_x$. If the measure is of the form
\[
\mu(s) = e^{-\beta s \cdot s - \gamma_0 (s \cdot s)^2 + \ldots} \tag{1.7}
\]
the bare mass is
\[
m_0^2 = (-2d - 2rg^2 + 2g^2 \beta) a^{-2} \tag{1.8}
\]
and the bare quartic interaction is $\lambda_0 = a^{4-d} g^4 \gamma_0$. This is easily seen by writing
\[
\exp \frac{1}{2g^2} \sum_{x < y} s(x) \cdot s(y) = \exp \left[ -\frac{1}{4g^2} \sum_{x < y} \left\{ [s(x) - s(y)]^2 - 2s(x)^2 \right\} \right].
\]
\[
= \exp \left\{ -\frac{1}{2g^2} \sum_{x \in \Lambda} \sum_{\nu=1}^{d} \left[ s(x + \nu) - s(x) \right]^2 + \frac{d}{g^2} \sum_{x \in \Lambda} s(x)^2 \right\}.
\]
There can, of course, be higher-order bare interactions as well.

2 The G.K.S. inequalities

In this section, we discuss

Theorem 1: The correlation functions (1.2) of a lattice spin model on a finite lattice $\Lambda$ with
\[
\mu(s_1, \ldots, -s_l, \ldots, s_n) = \mu(s_1, \ldots, s_l, \ldots, s_n) \geq 0 . \tag{2.1}
\]
satisfy the inequality:
\[
\text{G.K.S. I: } \langle s(x) \cdot s(y) \rangle \geq 0 ,
\]
for all $x, y \in \Lambda$.

and

**Theorem 2**: The correlation functions (1.3) on a finite lattice $\Lambda$ satisfy the inequality:

$$G.K.S. \ II: \quad \langle s(x) \cdot s(y) \cdot s(u) \cdot s(v) \rangle - \langle s(x) \cdot s(y) \rangle \langle s(u) \cdot s(v) \rangle \geq 0,$$

for all $x, y, u, v \in \Lambda$, provided that the function $\mu(s)$ satisfies (1.3), (1.4).

Notice that (1.3) is a stronger condition than (2.1). The proof of the first theorem is quite well known and we include it purely for pedagogical reasons. Though both theorems are proved for a finite lattice, they will hold by continuity in the infinite-lattice limit, provided that the limit exists.

To prove the first theorem we use the following simple fact:

**Lemma 1**: For any choice of $\mu(\cdot)$ satisfying (2.1), the following integral is nonnegative:

$$I^{\alpha_1, \ldots, \alpha_n} = \int d^n s \mu(s) \left( (s_1)^{\alpha_1} \cdots (s_n)^{\alpha_n} \right) \geq 0,$$

for any nonnegative integers $\alpha_1, \ldots, \alpha_n$.

**Proof**: The hypothesis is true if any $\alpha_k, k = 1, \ldots, p$ is odd; for the integrand changes sign under $s_k \rightarrow -s_k$, and the integral vanishes by symmetry. Hence we assume that each $\alpha_k, k = 1, \ldots, n$ is even. Then the integral is

$$I^{\alpha_1, \ldots, \alpha_n} = 2^n \int d^n s \mu(s) \theta(s_1) \cdots \theta(s_n) \left( (s_1)^{\alpha_1} \cdots (s_n)^{\alpha_n} \right) \geq 0,$$

where $\theta(\cdot)$ is the usual step function: $\theta(w) = 1$ for real $w \geq 0$ and $\theta(w) = 0$ for real $w < 0$.

With this lemma established it is easy to find the

**Proof of Theorem 1**: For a finite lattice $\Lambda$ the Taylor expansions of (1.1) and (1.2) in the coefficients $J_{xy}$ and $h_x$ is convergent. The expansion is a nonnegative linear combination of products of integrals of the form (2.2), each of which is positive or zero. The hypothesis of Theorem 1 immediately follows.

Theorem 2 is a deeper result. We need another lemma to prove it.

**Lemma 2** (Ginibre’s Inequality): For a measure $\mu(\cdot)$, satisfying (1.3) and (1.4), the following integral converges and is nonnegative:

$$I^{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n} = \int d^n s \int d^n t \mu(s) \mu(t) \times (s_1 + t_1)^{\alpha_1} \cdots (s_n + t_n)^{\alpha_n} (s_1 - t_1)^{\beta_1} \cdots (s_n - t_n)^{\beta_n} \geq 0,$$

(2.3)
for any nonnegative integers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$.

**Proof**: Consider the expression

$$e^{\Xi[(g+j)^2] + \Xi[(g-j)^2]} = \int d^n s \int d^n t \; \mu(s) \mu(t) e^{g \cdot (s+t)} e^{j \cdot (s-t)}.$$

(2.4)

The right-hand side of (2.4) has the form in a neighborhood of $g = 0, j = 0$

$$e^{\Xi[(g+j)^2] + \Xi[(g-j)^2]} = \exp \left( 2\xi_0 + \xi_1 [(g+j)^2 + (g-j)^2] + \xi_2 \left[ ((g+j)^2)^2 + ((g-j)^2)^2 \right] + \cdots \right),$$

(2.5)

where $\xi_0 = \Xi(0)$. Now each term of the Taylor series in the exponent is a polynomial in $g^2, j^2$ and $g \cdot j$ with nonnegative coefficients. For expanding the $l$th term with the binomial formula yields

$$\xi_l \sum_{a=0}^{l} \binom{l}{a} \left[ (-2g \cdot j)^a (g^2 + j^2)^{l-a} + (2g \cdot j)^a (g^2 + j^2)^{l-a} \right].$$

(2.6)

All terms with negative coefficients are canceled in this expression.

Next observe that the right-hand side of (2.4) is the generating function for $f^{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n}$, as the derivatives of the right-hand side of (2.4) with respect to components of $g$ and $j$ yield the integrals (2.3). Hence these integrals exist by hypothesis. None of the derivatives with respect to components of $g$ and $j$ can be negative, by the argument in the previous paragraph, and the lemma is proved, if it assumed that $\Xi(z)$ is analytic in a neighborhood of $z = 0$.

We should mention at this point that Ginibre’s inequality is not true for the classical Heisenberg measure $\mu(s) = \delta(1 - s \cdot s)$, which does not satisfy the hypothesis of Lemma 2 [8].

**Proof of Theorem 2**: Following Ginibre [9], we write the connected two-point correlation as an expectation value over a distribution of two sets of $n$-component spins $s(x)$ and $t(x)$, each in $D^n$:

$$(Z[G]^2 \left[ \langle s(x) \cdot s(y) s(u) \cdot s(v) \rangle - \langle s(x) \cdot s(y) \rangle \langle s(u) \cdot s(v) \rangle \right]$$

$$= \int d[s]d[t] \exp \left\{ \sum_{w,z \in \Lambda} J_{wz} [s(w) \cdot s(z) + t(w) \cdot t(z)] + \sum_{w \in \Lambda} h_w \cdot [s(w) + t(w)] \right\}$$

$$\times [s(x) \cdot s(y) - t(x) \cdot t(y)] [s(u) \cdot s(v) - t(u) \cdot t(v) ],$$

(2.7)

where we have used the notation

$$\int d[s] = \int \left\{ \prod_{x \in \Lambda} d^n s(x) \; \mu[s(x)] \right\}.$$

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We show that the right-hand side of (2.7) is nonnegative. Note that
\[ s(x) \cdot s(y) + t(x) \cdot t(y) \]
\[ = \frac{1}{2} [s(x) + t(x)] \cdot [s(y) + t(y)] + \frac{1}{2} [s(x) - t(x)] \cdot [s(y) - t(y)], \quad (2.8) \]
and
\[ s(x) \cdot s(y) - t(x) \cdot t(y) \]
\[ = \frac{1}{2} [s(x) - t(x)] \cdot [s(y) + t(y)] + \frac{1}{2} [s(x) + t(x)] \cdot [s(y) - t(y)]. \quad (2.9) \]

Next, we expand the exponential in (2.7). Since \( J_{wz} \geq 0, h_z \geq 0, \) and using (2.8) and (2.9) we find that the expansion is a linear combination of products of integrals of the form (2.3) with nonnegative coefficients. By Lemma 2, the right-hand side of (2.7) is nonnegative.

3 Examples

Measures satisfying (1.3) and (1.4) can be constructed, at least in principle, using (1.5). The explicit evaluation of the Fourier integral is, unfortunately, usually impossible analytically. We note that the integral (1.5) is defined for a polynomial \( \Xi(z) = \xi_1 z + \xi_2 z^2 + \cdots + \xi_{2l+1} z^{2l+1}, \xi_1 \geq 0, \xi_2 \geq 0, \ldots, \xi_{2l+1} > 0. \) For the choice \( l = 0, \) the measure is the uninteresting Gaussian case.

We do not yet know whether systems satisfying (1.3), (1.4) display phase transitions and critical points in the thermodynamic limit. A simple argument using (1.5) reveals that \( \beta \geq 0 \) in (1.7). Consider the first and second derivatives of the measure \( \mu(\cdot) \) using (1.3) evaluated at \( s = 0: \)

\[ \frac{\partial}{\partial s_a} \mu(s) \bigg|_{s=0} = -i \int \frac{d^n q}{(2\pi)^n} q_a e^{\Xi(q)} = 0, \]

\[ \frac{\partial^2}{\partial s_a \partial s_b} \mu(s) \bigg|_{s=0} = -\frac{\delta_{ab}}{n} \int \frac{d^n q}{(2\pi)^n} q \cdot q \cdot e^{\Xi(q)}. \]

Since \( q \cdot q \cdot e^{\Xi(q)} \) is positive for \( q \cdot q > 0, \) the measure has a local maximum at \( s = 0. \) The possibility of \( \beta < 0 \) in (1.7) is thereby eliminated. This does not restrict the bare mass to be positive, as there is the negative contribution in (1.8) proportional to the dimension.
An obvious question is whether the measure of a scalar field theory with only a quartic interaction can satisfy (1.3), (1.4). We show that for $\beta = 0$

$$\mu(s) = e^{-\gamma_0 (s \cdot s)^2},$$

the answer is no. Using (1.3) to determine $\Xi(\cdot)$ as a power series in $\gamma_0^{-1/2}$, we find from equation (A.8) in the appendix

$$e^{\Xi(z)} = \frac{\pi^{n/2} \Gamma\left(\frac{n}{4}\right)}{2 \Gamma\left(\frac{n}{2}\right)} \gamma_0^{-n/4} \left[ 1 + \frac{\Gamma\left(\frac{n}{4} + \frac{3}{2}\right)}{2n \Gamma\left(\frac{n}{4}\right)} \gamma_0^{-1/2} z + \frac{1}{32(n+2)} \gamma_0^{-1} z^2 \right.$$ 

$$+ \frac{\Gamma\left(\frac{n}{4} + \frac{1}{2}\right)}{192n(n+4) \Gamma\left(\frac{n}{4}\right)} \gamma_0^{-3/2} z^3 + \cdots \]$$

Comparing this series with (1.4) gives for the first few coefficients

$$\xi_1 = \frac{\Gamma\left(\frac{n}{4} + \frac{3}{2}\right)}{2n \Gamma\left(\frac{n}{4}\right)} \gamma_0^{-1/2}, \quad \xi_2 = \left[ \frac{1}{32(n+2)} - \frac{\Gamma\left(\frac{n}{4} + \frac{3}{2}\right)}{8n^2 \Gamma\left(\frac{n}{4}\right)^2} \right] \gamma_0^{-1},$$

$$\xi_3 = \left[ -\frac{\Gamma\left(\frac{n}{4} + \frac{1}{2}\right)}{96n(n+4) \Gamma\left(\frac{n}{4}\right)} + \frac{\Gamma\left(\frac{n}{4} + \frac{3}{2}\right)^3}{24n^3 \Gamma\left(\frac{n}{4}\right)^3} \right] \gamma_0^{-3/2}.$$ 

It is relatively simple to compute these coefficients for even $n$. The author has checked that $\xi_2 > 0$ for $n = 2$ (an Abelian case) but $\xi_2 < 0$ for $n = 4, 6, 8, 10, 12$. Thus a simple quartic measure does not satisfy the conditions for our proof of the second G.K.S. inequality.

An alternative strategy is to start with a suitable choice for $\Xi(\cdot)$ and determine $\mu(\cdot)$ using (1.3). For the choice $\xi_1 = \xi_2 = \xi_4 = \xi_5 = \cdots = 0$:

$$e^{\Xi(h \cdot h)} = e^{\xi(h \cdot h)^3},$$

the formula (A.8) in the appendix may be used to show

$$\mu(s) = \frac{\pi^{n/2} \Gamma\left(\frac{n}{4}\right)}{3 \Gamma\left(\frac{n}{2}\right) \xi^{n/6}} \left[ 1 - \frac{\Gamma\left(\frac{n}{4} + \frac{3}{2}\right)}{2n \Gamma\left(\frac{n}{4}\right)} \xi^{-1/3} s \cdot s + \frac{\Gamma\left(\frac{n}{4} + \frac{3}{2}\right)}{8n(n+2) \Gamma\left(\frac{n}{4}\right)} \xi^{-2/3} (s \cdot s)^2 \right.$$ 

$$- \frac{1}{288(n+2)(n+4)} \xi^{-1} (s \cdot s)^3 + \cdots \right],$$

which is, as one would reasonably expect, a nonpolynomial interaction. Unfortunately this measure falls off more slowly than a Gaussian as $|s| \to \infty$. For field theory, this means that the potential cannot be approximated by a Ginzburg-Landau-type expression.
There are other possible choices of $\Xi(\cdot)$. For example, if $n = 3$, for an integer $c = 0, 1, 2, \ldots$
\[ e^{\Xi_c(h \cdot h)} = (-1)^c \frac{\partial^c}{\partial \omega^c} \left( \Omega^2 - h \cdot h \right)^0, \]
implies that
\[ \mu(s) = \frac{1}{4\pi} |s|^{c-1} e^{-\Omega|s|}. \]
One can check that $\Xi(z)$ has nonnegative derivatives. On a regular lattice with suitably translation-invariant couplings, such a model is a 3-component Euclidean quantum field theory, whose continuum limit has the Lagrangian (excluding the source term, which depends on $h_x$ on the lattice)
\[ L = \frac{1}{2g^2} (\partial s)^2 + \frac{m_0^2}{2g^2} s^2 + \omega|s| + (1 - c) \log |s|, \quad (3.1) \]
where $\omega$ is the limit of $\Omega a^{-d}$ as $a \to 0$. These 3-component field theories have non-polynomial interactions (even for $c = 1$). Though they are unconventional, (3.1) may be interesting models in their own right, as well as for the reason that they satisfy the second G.K.S. inequality.

4 Discussion

A major question suggested by the results of this paper is whether the method used here can be extended to models for which $\mu(s)$ falls off faster with large $s \cdot s$ than a Gaussian distribution. If not, another approach may be needed to prove the second G.K.S. inequality for such choices of $\mu(\cdot)$.

While the conditions (1.3), (1.4) are sufficient for the validity of the second G.K.S. inequality, it is by no means obvious that they are necessary. It may be that Ginibre’s inequality (Lemma 2.) is satisfied under weaker conditions.

The classical Heisenberg measure does not satisfy Ginibre’s inequality as formulated in reference [5]. We would like to suggest, however, that the situation may not be hopeless for this case. To obtain the classical Heisenberg magnet, all that is needed is that some measure $\mu(\cdot)$ depending on one or more parameters, satisfies the criteria (1.3), (1.4) and by taking some limit of these parameters, suitably adjusting $r$ in (1.6), keeping $g$ fixed
\[ \lim e^{(2d+2rg^2)s \cdot s} \mu(s) \longrightarrow \delta(s \cdot s - 1). \]
The expansions of correlation functions in the ferromagnetic couplings one obtains this way are different from those of Sylvester, who takes $\mu(s) = \delta(s \cdot s - 1)$ from the outset.

Acknowledgements: The author would like to thank Prof. David Tepper and Mr. Jing Xiao for discussions on Ginibre’s inequality.
Appendix: Expansions for integral transforms

To evaluate explicitly an integral of the form

\[ Y_L(P) = \int d^nQ \, e^{Q \cdot P} e^{-\alpha (Q \cdot Q)^L}, \quad (A.1) \]

where \( P \) and \( Q \) are vectors in \( \mathbb{R}^n \) and \( L \) is an integer, is difficult, except as a power series in \( P \cdot P \). We shall examine the terms of this power series in detail in this appendix and see whether it is convergent.

We wish to compute the integrals

\[ Y_{j_1, j_2, \ldots, j_{2l}} = \int d^nQ \, Q_{j_1} \cdots Q_{j_{2l}} \, e^{-\alpha (Q \cdot Q)^L}, \quad (A.2) \]

which occur in the expansion of \((A.1)\) in the components of \( P \). To evaluate \((A.2)\), we first find the integral

\[ Z_L^{j_i} = \int d^nQ \, (Q \cdot Q)^{j_i} e^{-\alpha (Q \cdot Q)^L} = \frac{\pi^{n/2} \Gamma \left( \frac{n+2l}{2L} \right)}{L \Gamma \left( \frac{n}{2} \right)} \alpha^{-\frac{n+2l}{2L}}, \quad (A.3) \]

To obtain this expression, we first integrated over the angles (yielding the standard result \( \int d\Omega = 2\pi^{n/2}/\Gamma(n/2) \)), then made a change of variable from \( q = |Q| \) to \( \alpha q^{2L} \).

Now this integral is obtained by contracting the indices \( j_1 \) with \( j_2, j_3 \) with \( j_4, \ldots, j_{2l-1} \) with \( j_{2l} \) with \( Y_{j_1, j_2, \ldots, j_{2l}} \) by \((A.2)\):

\[ Z_L^{j_i} = \delta^{j_1 j_2} \cdots \delta^{j_{2l-1} j_{2l}} Y_{j_1, j_2, \ldots, j_{2l}}, \quad (A.4) \]

where we have used the Einstein summation convention.

By symmetry, \( Y_{j_1, j_2, \ldots, j_{2l}} \) must have the form

\[ Y_{j_1, j_2, \ldots, j_{2l}} = A_L^i \left( \delta_{j_1 j_2} \delta_{j_3 j_4} \cdots \delta_{j_{2l-1} j_{2l}} + \text{all other contractions of index pairs} \right), \quad (A.5) \]

for some constant \( A_L^i \). There are \((2l-1)!! = (2l-1)(2l-3) \cdots 3 \cdot 1\) terms in parentheses on the right-hand side of \((A.5)\). We shall show that

\[ \delta^{j_1 j_2} \cdots \delta^{j_{2l-1} j_{2l}} \left( \delta_{j_1 j_2} \delta_{j_3 j_4} \cdots \delta_{j_{2l-1} j_{2l}} + \text{all other contractions of index pairs} \right) \]

\[ = \frac{(n+2l-2)!!}{(n-2)!!} = (n+2l-2)(n+2l-4) \cdots (n+2)n, \quad (A.6) \]

below. We can thereby evaluate \( A_L^i \) by using \((A.3)\), \((A.4)\) and \((A.5)\), obtaining

\[ Y_{j_1, j_2, \ldots, j_{2l}} = \frac{\pi^{n/2} \Gamma \left( \frac{n+2l}{2L} \right)}{L \Gamma \left( \frac{n}{2} \right)} \frac{(n-2)!!}{(n+2l-2)!!} \alpha^{-\frac{n+2l}{2L}}, \quad (A.7) \]
\[ \times \left( \delta_{j_1 j_2} \delta_{j_3 j_4} \cdots \delta_{j_{2l-1} j_{2l}} + \text{all other contractions of index pairs} \right) . \quad (A.7) \]

Next we prove \((A.6)\) by induction. The formula is trivial when \(l = 1\). Let us suppose that it is established for some particular value of \(l\) and show it for \(l + 1\). Consider

\[ \delta_{j_1 j_2} \cdots \delta_{j_{2l+1} j_{2l+2}} \left( \delta_{j_1 j_2} \delta_{j_3 j_4} \cdots \delta_{j_{2l+1} j_{2l+2}} + \text{all other contractions of index pairs} \right) . \]

The terms in the parentheses in this expression are of two types: i) Those containing \(\delta_{j_{2l+1} j_{2l+2}}\) and ii) those which do not. The sum of terms of type i) are, by hypothesis, \(\frac{n}{(n-2)!!} n\) (the factor \(n\) just comes from the sum over \(j_{2l+1} = j_{2l+2}\)). The terms of type ii) can be obtained by considering the left-hand side of \((A.6)\), uncontracting a pair of indices in the set \(\{ j_1, \ldots, j_{2l} \}\), contracting one member of this pair with \(j_{2l+1}\) and the other member of the pair with \(j_{2l+2}\). There are precisely \(2l\) ways to do this, and the sum of terms of type ii) is \(\frac{n}{(n-2)!!} \cdot 2l\). The sum of all terms of type i) and type ii) is therefore

\[ \frac{(n + 2l - 2)!!}{(n - 2)!!} (n + 2l) = \frac{[n + 2(l + 1) - 2]!!}{(n - 2)!!} , \]

establishing \((A.6)\) for \(l + 1\).

Putting these results together gives for \((A.1)\)

\[ Y_L(P) = \pi \frac{\alpha^{\frac{n}{2}}}{L \Gamma \left( \frac{n}{2} \right) \alpha^{\frac{3n}{4}}} + \pi \frac{\alpha^{\frac{n}{2}}}{L \Gamma \left( \frac{n}{2} \right) \alpha^{\frac{3n}{4}}} \sum_{l=1}^{\infty} \frac{(2l-1)!! \Gamma \left( \frac{n+2l}{2} \right)}{2l!(n+2l-2)!!} \alpha^{-\frac{3n}{4}} (P \cdot P)^l . \quad (A.8) \]

To test of the validity of \((A.8)\) let us take \(L = 1\). The relations

\[ r!! = \frac{r!}{2^{\frac{r-1}{2}}} , \quad r \text{ odd} , \]

and

\[ r!! = 2^r \left( \frac{r}{2} \right) ! , \quad r \text{ even} , \]

and

\[ \frac{\Gamma \left( \frac{n}{2} + l \right)}{\Gamma \left( \frac{n}{2} \right)} = \frac{\left( \frac{n}{2} + l - 1 \right) \Gamma \left( \frac{n}{2} + l - 1 \right)}{\Gamma \left( \frac{n}{2} \right)} = \cdots = \frac{\left( \frac{n}{2} + l - 1 \right) \left( \frac{n}{2} + l - 2 \right) \cdots \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \]

\[ = \frac{(n + 2l - 2)!!}{2^l(n - 2)!!} , \]

imply the familiar result:

\[ Y_1(P) = \pi \frac{\alpha^{-\frac{n}{4}}}{\alpha^{\frac{n}{4}}} \left[ 1 + \sum_{l=1}^{\infty} \frac{(2l-1)!!}{2^l(2l)!} \alpha^{-l} (P \cdot P)^l \right] = \frac{\pi n}{\alpha^\frac{n}{4}} \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{P \cdot P}{4\alpha} \right)^l . \]
\[
= \left( \frac{\pi}{\alpha} \right)^{\frac{n}{2}} e^{\frac{P\cdot P}{4\alpha}}. \quad (A.9)
\]

To establish convergence of (A.8), we compare the terms in the series for \( L > 1 \) to those for \( L = 1 \). Notice that for \( L > 1 \),
\[
\frac{(2l - 1)!!\Gamma\left(\frac{n+2l}{2L}\right)}{(2l)! (n + 2l - 2)!!} < \frac{(2l - 1)!!\Gamma\left(\frac{n+2l}{2}\right)}{(2l)! (n + 2l - 2)!!}.
\]
Since the series for \( L = 1 \) converges, so must that for \( L > 1 \).

References

[1] J. Ginibre: Commun. Math. Phys. 16 310-328 (1970)

[2] J. Glimm and A. Jaffe: Quantum Physics: A Functional Integral Point of View, Berlin: Springer-Verlag, sec. ed. 1987

[3] R.B. Griffiths: J. Math. Phys. 8, 478-483 (1967)

[4] R.B. Griffiths: J. Math. Phys. 8, 484-489 (1967)

[5] R.B. Griffiths: Commun. Math. Phys. 6, 121-127 (1967)

[6] D. Kelly and S. Sherman: J. Math. Phys. 9, 466-484 (1968)

[7] B. Simon: Functional Integration and Quantum Physics, New York: Academic Press 1979

[8] G.S. Sylvester: Commun. Math. Phys. 73, 105-114 (1980)