A FLOW APPROACH TO THE GENERALIZED LOEWNER-NIRENBERG PROBLEM OF THE $\sigma_k$-RICCI EQUATION

Abstract. We introduce a flow approach to the generalized Loewner-Nirenberg problem (1.5) – (1.7) of the $\sigma_k$-Ricci equation on a compact manifold ($M^n, g$) with boundary. We prove that for initial data $u_0 \in C^4(M)$ which is a subsolution to the $\sigma_k$-Ricci equation (1.5), the Cauchy-Dirichlet problem (3.1) – (3.3) has a unique solution $u$ which converges in $C^4_{loc}(M^\circ)$ to the solution $u_\infty$ of the problem (1.5) – (1.7), as $t \to \infty$.

1. Introduction

Let ($M^n, g$) be a compact Riemannian manifold with boundary of dimension $n \geq 3$. Denote $M^\circ$ to be the interior of $M$. In [10], we considered the Cauchy-Dirichlet problem of the Yamabe flow which starts from a positive subsolution of the Yamabe equation (1.1) and converges in $C^2_{loc}(M^\circ)$ to the solution to the Loewner-Nirenberg problem

\begin{align}
4(n-1) \Delta u - R_g u - n(n-1)u^{\frac{n+2}{n-2}} = 0, & \quad \text{in } M^\circ, \label{1.1} \\
n u(p) \to \infty, & \quad \text{as } p \to \partial M, \label{1.2}
\end{align}

which is originally studied by Loewner and Nirenberg [15] on Euclidean domains, and later by Aviles and McOwen [2] [3] on general compact manifolds with boundary. A signature feature of our flow is that it preserves the solution $u(\cdot, t)$ as a sub-solution to the Yamabe equation for $t > 0$.

In this paper, we extend this approach to study the generalized Loewner-Nirenberg problem for the fully nonlinear equation studied in [5] and [7].

Definition 1.1. For $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, \ldots, n$, we define the elementary symmetric functions as

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k},$$

and define the cone

$$\Gamma_+^k = \{ \Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, \forall j \leq k \},$$

which is the connected component of the set $\{ \sigma_k > 0 \}$ containing the positive definite cone on $\mathbb{R}^n$. We also define $\Gamma_-^k = -\Gamma_+^k$. For a symmetric $n \times n$ matrix $A$, $\sigma_k(A)$ is defined to be $\sigma_k(\Lambda)$ with $\Lambda = (\lambda_1, \ldots, \lambda_n)$ the eigenvalues of $A$.

The $\sigma_k$-scalar curvature equation is introduced in [18]. Let ($M^n, g$) be a smooth compact Riemannian manifold with boundary of dimension $n \geq 3$. Denote $\text{Ric}_g$ as the Ricci tensor of $g$.
In [7], for any $k = 1, \ldots, n$, the authors studied the Dirichlet boundary value problem of the $\sigma_k$ equation of $-Ric\bar{g}$, in seek of a conformal metric $\bar{g} = e^{2u}g$ such that $\bar{Ric}\bar{g} \in \Gamma_k$ and

\begin{align}
\sigma_k(-Ric\bar{g}) \equiv \sigma_k(-\bar{g}^{-1}Ric\bar{g}) = \tilde{\beta}_{k,n} \text{ in } M,
\end{align}

(1.3)

\begin{align}
u_0 = 0 \text{ on } \partial M,
\end{align}

(1.4)

where $\tilde{\beta}_{k,n} = (n - 1)^k \left(\frac{n}{k}\right)$, or equivalently, we have the $\sigma_k$-Ricci equation

\begin{align}
\sigma_k(\tilde{\nabla}^2 u) = \tilde{\beta}_{k,n}e^{2u}
\end{align}

(1.5)

where

\begin{align}
\tilde{\nabla}^2 u = -Ric\bar{g} + (n - 2)\nabla^2 u + \Delta u g + (n - 2)(|du|^2 g - du \otimes du).
\end{align}

(1.6)

A more interesting result in [7] is that they generalized the Loewner-Nirenberg problem to the $\sigma_k$-Ricci equation (1.5) (see also [5]). They proved that there exists a unique solution $u_k$ to (1.5) with the property that

\begin{align}
u_k(p) \to +\infty
\end{align}

(1.7)

uniformly as $p \to \partial M$; moreover,

\begin{align}
\lim_{p \to \partial M}[u_k(p) + \log(r(p))] = 0
\end{align}

(1.8)

as $p \to \partial M$, where $r(p)$ is the distance of $p$ to $\partial M$. Notice that in [5] Guan gave an alternative approach to similar results, using metrics of negative Ricci curvature in the conformal class constructed in [16] as the background metric. In comparison, the argument in [7] uses a general background conformal metric and concludes the existence of a prescribed $\sigma_k$-Ricci curvature metric of negative Ricci curvature. In this paper, we give a flow approach to the generalized Loewner-Nirenberg problem to the $\sigma_k$-Ricci equation (1.5) starting from a sub-solution to (1.5), with a background metric of negative Ricci curvature in the conformal class. In particular, we introduce the Cauchy-Dirichlet problem (3.1) – (3.3) of the $\sigma_k$-Ricci curvature flow.

In order to get the lower bound control of the blowing up ratio near the boundary, we need to assume that the boundary data $\phi$ could not go to infinity too slowly as $t \to \infty$.

**Definition 1.2.** We call a function $\xi(t) \in C^1([0, \infty))$ a low-speed increasing function if $\xi(t) > 0$ for $t \geq 0$, $\lim_{t \to \infty} \xi(t) = \infty$, and there exist two constants $T > 0$ and $\tau > 0$ such that for $t \geq T$,

\begin{align}
\xi'(t) \leq \tau.
\end{align}

(1.9)

Here are some examples of low-speed increasing functions: $t^\alpha$ for some $0 < \alpha < 1$, $\log(t)$, and finitely many composition of log functions: $\log \circ \log \circ \ldots \circ \log(t)$ for $t > 0$ large, etc.

**Theorem 1.3.** Assume $(M^n, g)$ $(n \geq 3)$ is a compact manifold with boundary of $C^{4,\alpha}$, and $(M, g)$ is either a compact domain in $\mathbb{R}^n$ or with Ricci curvature $Ric_g \leq -\delta_0 g$ for some $\delta_0 \geq (n - 1)$. Assume $u_0 \in C^{4,\alpha}(M)$ is a subsolution to (1.5) satisfying (3.6) at the points $x \in \partial M$ where $\nu(x) = 0$ for the function $\nu$ defined in (3.5). Also, assume $\phi \in C^{4+\alpha,2+\frac{\alpha}{2}}(\partial M \times [0, T_0])$ for all $T_0 > 0$, $\phi|_{\partial M \times [0, +\infty)}$ and $\phi$ satisfies the compatible condition (3.4) with $u_0$. Moreover, assume that there exist a low-speed increasing function $\xi(t)$ satisfying (1.9) for some $T > 0$ and $\tau > 0$, and a constant $T_1 > T$ such that $\phi(x, t) \geq \log(\xi(t))$ for $(x, t) \in \partial M \times [T_1, \infty)$. Then there exists a unique solution $u \in C^{4,2}(M \times [0, +\infty))$ to the Cauchy-Dirichlet problem
such that $u \in C^{4+\alpha, 2+\frac{\alpha}{2}}(M \times [0, T])$ for all $T > 0$. Moreover, the solution $u$ converges to a solution $u_\infty$ to the equation (1.5) locally uniformly on $M^\circ$ in $C^4$, and

$$\lim_{x \to \partial M} (u_\infty(x) + \log(r(x))) = 0,$$

where $r(x)$ is the distance of $x$ to $\partial M$.

Notice that our assumption on the boundary data $\phi$ and the speed that $\phi \to \infty$ as $t \to \infty$ is pretty general. When $u_0$ is a solution to (1.5) in a neighborhood of $\partial M$, then (3.6) holds automatically, while the condition (3.6) disappears when $u_0$ is a strict sub-solution to (1.5) in a neighborhood of $\partial M$. For instance, for any sub-solution $u_0$ to (1.5), $u_0 - C$ is a strict sub-solution for any constant $C > 0$. For the long time existence of the flow, one needs to establish the global a priori estimates on the solution $u$ up to $C^2$-norm: both the boundary estimates and the interior estimates, starting from the $L^\infty$ control by the maximum principle and heavily depending on the monotonicity of $u$ and the control of $u_t$. In particular, $u_t \geq 0$ and hence $u(\cdot, t)$ is a sub-solution to (1.5) for any $t \geq 0$, which together with the uniform interior upper bound control makes the convergence possible and gives a natural lower bound of $u$. For the convergence of the flow, we have to give the uniform interior $C^2$-estimates on $u$ which is independent of $t > 0$. Finally the asymptotic boundary behavior near the boundary as $t \to \infty$ is established, which implies that the limit function is a solution to the generalized Loewner-Nirenberg problem. Many of the barrier functions in these estimates can be viewed as a parabolic version of those in [7] and [5]. This flow approach works well for the Loewner-Nirenberg problem of more general nonlinear equations in [5].

**Corollary 1.4.** Assume $(M^\alpha, g)$ is a compact manifold with boundary of $C^{4,\alpha}$. Then there exists a sub-solution $u_0$ to (1.5) and a $\sigma_k$-Ricci curvature flow $g(t) = e^{2t}g$ starting from $g_0 = e^{2t_0}g$ and satisfying (3.1) and the Cauchy-Dirichlet condition (3.2)–(3.3) with some boundary data $\phi$ such that $g(t)$ converges to $g_\infty = e^{2u_\infty}g$ locally uniformly in $C^4$ as $t \to +\infty$, where $u_\infty$ is the unique generalized Loewner-Nirenberg solution to (1.5) i.e., $u_\infty(x) \to \infty$ as $x \to \partial M$. Moreover,

$$\lim_{x \to \partial M} (u_\infty(x) + \log(r(x))) = 0.$$

**Proof.** As discussed in Section 2 by [16] there exists a metric in the conformal class $[g]$ of $C^{4,\alpha}$, which is still denoted as $g$ such that $\text{Ric}_g < -(n-1)g$. If $M$ is a Euclidean domain, we can alternatively just choose $g$ to be the Euclidean metric. We then take $g$ as the background metric. Now we choose a sub-solution $u_0$ to (1.5) such that $u_0$ satisfies (3.6) on the boundary. For instance, if $(M, g)$ is a sub-domain in Euclidean space, we choose $u_0$ to be either the global sub-solution constructed in [7] (just take $\eta(s) = s$ for the subsolution $u$ in Section 2) for the constants $A$ and $p$ large, or the solution to (1.5) with $u_0 = 0$ on $\partial M$ obtained in [7] or [5]. For general $(M, g)$, with the background metric satisfying $\text{Ric}_g < -(n-1)g$, we can either take $u_0$ to be the solution to (3.1) with $u_0 = 0$ on $\partial M$ obtained in [7] or [5], or use the global sub-solution constructed in Section 2 or $u_0 = v - 1$ where $v \in C^{4,\alpha}(M)$ is any sub-solution of (1.5) and hence $u_0$ is a strict sub-solution (with “>” instead of “=” in (3.1)). Then we construct the boundary data $\phi \in C^{4,2}(\partial M \times [0, \infty))$ satisfying the compatible condition (3.4) at $t = 0$ such that $\phi_0 \in C^{4+\alpha, 2+\frac{\alpha}{2}}(\partial M \times [0, T])$ for any $T > 0$, $\phi_t \geq 0$ on $\partial M \times [0, \infty)$ and $\phi(x, t) \geq \xi(t)$ on $\partial M \times [0, T]$ for some $T > 0$, where $\xi(t)$ is a low-speed increasing function in Definition 1.2. Now we consider the solution to the Cauchy-Dirichlet boundary value problem (3.1)–(3.3). Therefore, by Theorem 1.3 we have the required conclusion. \qed
One can easily adapt this approach to the convergence of a $\sigma_k$-Ricci curvature flow to the solution to the Dirichlet boundary value problem of (1.5).

**Corollary 1.5.** Assume $(M^n, g)$ is a compact manifold with boundary of $C^{4,\alpha}$. Let $\varphi_0 \in C^{4,\alpha}(\partial M)$. Then there exists a sub-solution $u_0$ to (1.5) and a $\sigma_k$-Ricci curvature flow $g(t) = e^{2u(t)}g$ starting from $g_0 = e^{2u_0}g$ and satisfying (3.1) and some Cauchy-Dirichlet condition such that $g(t)$ converges to $g_{\infty} = e^{2u_{\infty}}g$ uniformly in $C^4$ as $t \to +\infty$, where $u_{\infty}$ is the unique solution to (1.5) such that $u_{\infty} = \varphi_0$ on $\partial M$.

Recently, in [4] the authors studied a more general fully nonlinear equations with less restriction on regularity and convexity on the nonlinear structures on smooth domains in Euclidean space and obtained a unique continuous viscosity solution, which is locally Lipschitz in the interior and shares the same blowing up ratio with the solution to the Loewner-Nirenberg problem near the boundary.

The paper is organized as follows: In Section 2, we construct a global sub-solution in $C^{4,\alpha}(M)$ to the $\sigma_k$-Ricci equation (1.5). In Section 3 we formulate the maximum principle, show the monotonicity of the flow and give the global a priori estimates of the solution for the long time existence of the flow. In Section 4 we first prove the long time existence of the flow based on the a priori estimates in Section 3 and then we give the uniform interior estimates of the solution independent of $t$, and establish the asymptotic behavior of the solution near the boundary (see Lemma 4.4) and prove Theorem 1.3. Finally we give a proof of Corollary 1.5.

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### 2. A Global Subsolution to (1.5)

We now construct a global subsolution $u \in C^{4,\alpha}(M)$ to the homogeneous Dirichlet boundary value problem (1.4) – (1.5). Recall that in [7], the authors constructed a global subsolution with singularity at the cut locus of the distance function to some point, which serves as a global uniform lower bound of the solution. We modify it to a smooth function in order to avoid complicated argument on the cut locus in our setting. Let $(M^n, g)$ be a compact Riemannian manifold with boundary of $C^{4,\alpha}$. We extend the manifold to a new manifold with boundary $M_1 = M \cup (\partial M \times [0, \epsilon_0])$ for some small constant $\epsilon_0$ with $\partial M = \partial M \times [0]$ and extend $g$ to a $C^{4,\alpha}$ metric on $M_1$. One can construct a conformal metric $h \in [g]$ of $C^{4,\alpha}$ with $\text{Ric}_h < 0$ on $M_1$, which always exists by the proof in [16]. Without loss of generality, we take $h$ as the background metric and still denote $h$ as $g$ in $M_1$, with $\text{Ric}_g \leq -\delta_0 g$ for some constant $\delta_0 > 0$ in $M$. In fact by scaling we assume $\text{Ric}_g \leq -\delta_0 g$ with $\delta_0 > (n-1)$ large in $M$.

Notice that there exist two small constants $0 < \epsilon_1 < \delta$ such that $\text{dist}(x, \partial M_1) > 2\epsilon_1 + 4\delta$ for $x \in \partial M$, and also $\epsilon_1 + 2\delta$ is less than the injectivity radius of any point $q$ in the tubular neighborhood of $\partial M$

$$\Omega = \{x \in M_1 \mid \text{dist}_g(x, \partial M) \leq \epsilon_1 + 2\delta\},$$

with $\text{dist}_g(x, \partial M)$ distance function to $\partial M$, and moreover for $x \in \Omega$, the distance $\text{dist}_g(x, \partial M)$ is realized by a unique point $x_1 \in \partial M$ through a unique shortest geodesic connecting $x$ and $x_1$, which is orthogonal to $\partial M$ at $x_1$. For any $x_0 \in \partial M$, we pick up the point $\bar{x} \in M_1 \setminus M$ on the geodesic starting from $x_0$ along the outer normal vector of $\partial M$ so that $\text{dist}_g(x_0, \bar{x}) = \epsilon_1$. We define the distance function $r(x) = \text{dist}_g(x, \bar{x})$ for $x \in M_1$. In particular, $r(x_0) = \epsilon_1$ and $r$ is
smooth for \( r \leq 2\delta + \epsilon_1 \). It is clear that \( r(x) \geq r(x_0) \) for any \( x \in M \) and the equality holds if and only if \( x = x_0 \).

Now for a fixed \( x_0 \in \partial M \) and the corresponding point \( \bar{x} \), we can choose the subsolution in the following way: We let \( A > 0 \) and \( p > 0 \) be two large constant to be determined so that

\[
N = A[(-\delta + r(x_0))^{-p} + r(x_0)^{-p}]
\]
is large, and we define a convex function \( \eta \in C^5(\mathbb{R}) \), so that

\[
\eta(s) = \eta(A(2\delta + r(x_0))^{-p} - r(x_0)^{-p})) \text{ for } s \leq A [(2\delta + r(x_0))^{-p} - r(x_0)^{-p}], \text{ and}
\]

\[
\eta(s) = s, \text{ for } s \geq A [(\delta + r(x_0))^{-p} - r(x_0)^{-p}].
\]

It is clear that \( \eta'(s) \geq 0 \) and \( \eta''(s) \geq 0 \), for \( s \in \mathbb{R} \). Now we define

\[
\underline{u}(x) = \eta(A (r(x)^{-p} - r(x_0)^{-p})),
\]

and hence \( \underline{u} \in C^{4,\alpha}(M) \). We claim that we can choose uniform large constants \( A > 0 \) and \( p > 0 \) independent of \( x_0 \in \partial M \) so that \( \underline{u} \) is a subsolution. First, we give the calculation

\[
\nabla \underline{u}(x) = -Ap' \eta^{-p-1} \nabla r,
\]

\[
\nabla_i \nabla_i \underline{u}(x) = A^2 p^2 \eta'' r^{-2p-2} \nabla_i r \nabla_j r + p(p+1)Ap' \eta'^{-p-2} \nabla_i r \nabla_j r - pAp' \eta'^{-p-1} \nabla_i \nabla_j r
\]

\[
= A^2 p^2 \eta'' r^{-2p-2} \nabla_i r \nabla_j r + Ap^{-p-2} \eta'[(p+1) \nabla_i r \nabla_j r - r \nabla_i \nabla_j r],
\]

\[
\Delta \underline{u}(x) = A^2 p^2 \eta'' r^{-2p-2} \vert \nabla r \vert^2 + Ap(p+1)\eta'^{-p-2} \vert \nabla r \vert^2 - Ap\eta'^{-p-1} \Delta r
\]

\[
= A^2 p^2 \eta'' r^{-2p-2} + Ap^{-p-2} \eta'[(p+1) - r\Delta r],
\]

It is clear that for given \( \delta > \epsilon_1 > 0 \), we can choose \( p > 0 \) such that, for any \( x \in M \) such that \( r(x) \leq 2\delta + r(x_0) \), we have that \( (p+1) - r\Delta r > 0 \), where \( p > 0 \) is independent of the choice of \( x_0 \in \partial M \). In fact, we choose \( p > 0 \) large so that the matrix

\[
[(p+1 - r\Delta r)g_{ij} - (n-2)r \nabla_i \nabla_j r]
\]
is positive for \( x \in M \) such that \( r(x_0) \leq r(x) \leq 2\delta + r(x_0) \). Therefore,

\[
(n-2) \nabla^2 \underline{u}(x) + \Delta \underline{u}(x) g
\]
is always non-negative on \( M \). Since \(-Ric > \delta_0 g\) with some constant \( \delta_0 > (n-1) \) on \( M \) and

\[
|du(x)|^2 g - du(x) \otimes du(x)
\]
is semi-positive, we have that for \( 0 \leq s \leq 1 \),

\[
\nabla^2 \underline{u}(x) \equiv sg - (1-s)Ric_g + (n-2) \nabla^2 \underline{u}(x) + \Delta \underline{u}(x) + (n-2)(|du(x)|^2 g - du(x) \otimes du(x))
\]

\[
\geq (s + (1-s)\delta_0)g \geq g.
\]

By the definition of \( \eta \),

\[
\underline{u}(x) \leq \eta(A((r(x_0) + \delta)^{-p} - r(x_0)^{-p})) = A((r(x_0) + \delta)^{-p} - r(x_0)^{-p})
\]

for \( r(x) \geq \delta + r(x_0) \). Now \( A > 0 \) and \( p > 0 \) is chosen to be large so that

\[
A((r(x_0) + \delta)^{-p} - r(x_0)^{-p}) < -\frac{1}{2} \log((n-1)),
\]

and hence

\[
\sigma_n(g^{-1} \nabla^2 \underline{u}) \geq \sigma_n(\delta_0) = 1 > \beta_{n,n} e^{2\underline{u}}
\]
for \( x \in M \) with \( r(x) \geq \delta + r(x_0) \). On the other hand, for \( x \in M \) with \( r(x) \leq \delta + r(x_0) \), we have
\[
\eta(A(r(x)^{-p} - r(x_0)^{-p})) = A(r(x)^{-p} - r(x_0)^{-p}),
\]
\[
\eta'(A(r(x)^{-p} - r(x_0)^{-p})) = 1,
\]
\[
\eta''(A(r(x)^{-p} - r(x_0)^{-p})) = 0,
\]
and hence, as discussed in \[7\], for \( A > 0 \) and \( p > 0 \) large,
\[
(2.3) \quad \bar{\nabla}^2 u(x) > (n - 1)g,
\]
for \( x \in M \) with \( r(x) \leq \delta + r(x_0) \), where the term \((2.1)\) serves as the main controlling positive term. Since \( u \leq 0 \), we have \( u \in C^{4,\alpha}(M) \) is a subsolution to the \( \sigma_n \) equation when \( r(x) \leq \delta + r(x_0) \) and hence a sub-solution on \( M \) by \([2.2]\), with \( u \leq 0 \) on \( \partial M \). Let \( S_k = \sigma_k(\bar{\nabla}^2 u)\left(\frac{n}{k}\right)^{-1} \) for \( 1 \leq k \leq n \).

By Maclaurin’s inequality,
\[
S_1 \geq S_{\frac{1}{2}} \geq .. \geq S_{\frac{1}{k}} \geq .. \geq S_n,
\]
which implies that a subsolution to the \( \sigma_n \) equation is a subsolution to the \( \sigma_k \) equation for \( 1 \leq k \leq n \), while a supersolution of the \( \sigma_1 \) equation such that \( \bar{\nabla}^2 u \in \Gamma_\kappa \) is a supersolution to the \( \sigma_k \) equation for \( 1 \leq k \leq n \). In particular, \( u \) serves as a subsolution to the \( \sigma_k \) equations and a uniform lower bound of the solutions to the homogeneous Dirichlet boundary value problem for \( 1 \leq k \leq n \). Moreover, by \([2.2], (2.3)\) and the fact \( u \leq 0 \) on \( \partial M \), we have
\[
(2.4) \quad \sigma_k(\bar{\nabla}^2 u) > \beta_{k,n} e^{2k\phi}
\]
on \( M \). Recall that \( A > 0 \) and \( p > 0 \) are independent of \( x_0 \in \partial M \). This proves the claim. Therefore, we have constructed a strict sub-solution \( u \in C^{4,\alpha}(M) \) to \([1.5]\) and \( u \leq 0 \) on \( M \).

3. A priori estimates for the \( \sigma_k \)-Ricci curvature flow

On a compact Riemannian manifold \((M^n, g)\) with boundary \( \partial M \) of \( C^{4,\alpha} \). We denote \( M^\circ \) the interior of \( M \). If \((M, g)\) is a bounded domain in the Euclidean space \( \mathbb{R}^n \), we choose the natural extension \((M_1, g_1)\) which is a small tubular neighborhood of \( M \) in \( \mathbb{R}^n \), and the global subsolution used in \([7]\) has no singularity in \( M \). For general compact Riemannian manifold \((M^n, g)\) with boundary, with the extension \((M_1, g_1)\) in Section \(2\) we choose \( g_1 \) (and hence \( g \) on \( M \)) to be the conformal metric which has \( -\text{Ric}_{g_1} \geq \delta_0 g_1 \) with \( \delta_0 > n - 1 \).

For \( k = 1, ..., n \), we consider the Cauchy-Dirichlet problem of the \( \sigma_k \)-Ricci curvature flow
\[
(3.1) \quad 2ku_t = \log(\sigma_k(\bar{\nabla}^2 u)) - \log(\beta_{k,n}) - 2ku, \quad \text{on } M \times [0, +\infty),
\]
\[
(3.2) \quad u|_{t=0} = u_0,
\]
\[
(3.3) \quad u|_{\partial M} = \phi, \quad t \geq 0,
\]
where \( u_0 \in C^{4,\alpha}(M) \) is a subsolution to the \( \sigma_k \)-Ricci equation \([1.5]\), \( \bar{\nabla}^2 u \) is defined in \([1.6]\), and \( \phi \in C^{4+\alpha,2+\frac{\alpha}{2}}(\partial M \times [0, T]) \) for all \( T > 0 \), and moreover, \( \phi \) satisfies \( \phi_t \geq 0 \) for \( t \geq 0 \), \( \phi(t) \rightarrow +\infty \) as \( t \rightarrow +\infty \). To guarantee that the solution \( u \) to the Cauchy-Dirichlet problem of \([3.1]\) satisfies
\( u \in C^{4,\alpha} (M \times [0, T_0]) \) for some \( T_0 > 0 \), we need the compatible condition

\[
\begin{align*}
\left\{ \begin{array}{l}
u_0(x) = \phi(x, 0), \text{ for } x \in \partial M, \\
2k\phi_\ell(x, 0) = \log(\sigma_k(\nabla^2 u_0)(x)) - \log(\bar{\beta}_{k,n}) - 2k u_0(x), \text{ for } x \in \partial M, \\
2k\phi_n(x, 0) = L_0(v(x)), \text{ for } x \in \partial M,
\end{array} \right.
\end{align*}
\]

where the function \( v \in C^2(M) \) is

\[
v(x) \equiv \frac{1}{2k} (\log(\sigma_k(\nabla^2 u_0)(x)) - \log(\bar{\beta}_{k,n}) - 2k u_0(x))
\]

and \( L_0 \) is the linear operator

\[
L_0(\varphi) = \frac{\bar{T}_{ij}^{k-1}}{\sigma_k(\nabla^2 u_0)} [(n - 2) \nabla_i \nabla_j \varphi + \Delta \varphi g_{ij} + (n - 2)(2g^{km} \nabla_k u_0 \nabla_m \varphi g_{ij} - \nabla_i \varphi \nabla_j u_0 - \nabla_i u_0 \nabla_j \varphi)] - 2k \varphi,
\]

for any \( \varphi \in C^2(M) \), where \( \bar{T}_{ij}^{k-1} \) is the \((k-1)\)-th Newton transformation of \( \nabla^2 u_0 \), which is positive definite. In order to find boundary data \( \phi \in C^{4,\alpha} (\partial M \times [0, \infty)) \) compatible with \( u_0 \) such that \( \phi_t \geq 0 \) on \( \partial M \times [0, \infty) \), we need to assume that for the subsolution \( u_0 \in C^{4,\alpha} (M) \),

\[
(\sigma_k(\nabla^2 u_0) > \bar{\beta}_{k,n} e^{2k u_0}
\]

for all \( x \in \partial M \). For instance, the global subsolution \( u \) we constructed in Section 2 by (2.4).

Another example is \( u_0 = \varphi - C \), with \( \varphi \) a sub-solution of (1.5) and \( C > 0 \) a constant and hence, \( u_0 \) is a strict sub-solution of (1.5) on \( M \). Also, if \( u_0 \in C^{4,\alpha} (M) \) is a solution to (1.5), then \( v = 0 \) on \( M \) and hence (3.6) holds automatically. When \( u_0 \) is a solution to (1.5) with \( u_0 = 0 \) on \( \partial M \) as obtained in (7) and (5), we can choose the boundary data \( \phi = \phi(t) \in C^3 \) such that \( \phi(0) = \phi'(0) = \phi''(0) = 0 \) and \( \phi'(t) \geq 0 \) for \( t \geq 0 \). For a given constant \( T > 0 \), we call a function \( u \in C^2 (M \times [0, T]) \) a sub-solution (super-solution) of (3.1) if \( \nabla^2 u \in \Gamma_k^+ \) and \( u \) satisfies the inequality with "\( \leq \)" ("\( \geq \)") instead of "\( = \)" in (3.1). Notice that sub-solution and super-solution are defined similarly for (1.5).

We now prove a maximum principle, which serves as a comparison theorem for later use.

**Lemma 3.1.** Let \( u \) and \( v \) be sub- and super- solutions to (3.1), with \( u \leq v \) on \( \partial M \times [0, T) \) and \( M \times [0, T) \), then we have \( u \leq v \) on \( M \times [0, T) \).

**Proof.** The proof is a modification of the maximum principle of \( \sigma_k \)-Ricci equation in [7]. We argue by contradiction. Let \( \xi = u - v \). Assume that there exist \( 0 < t_1 < T \) and \( x \in M^o \) such that

\[
\xi(x, t_1) = \sup_{M \times [0, t_1]} \xi > 0.
\]

Then we have at \( (x, t_1) \),

\[
\tilde{u}_t \geq v_t, \quad \nabla \tilde{u} = \nabla v,
\]

\[
\nabla^2 (v - \tilde{u}) \geq 0,
\]

7
and hence
\[ \tilde{\nabla}^2 \tilde{u} + \mathcal{V} = \tilde{\nabla}^2 v \]
with \( \mathcal{V} = (n - 2) \nabla^2 (v - \tilde{u}) + \Delta (v - \tilde{u}) g \geq 0 \), which implies that \( \sigma_k(\tilde{\nabla}^2 \tilde{u}) \leq \sigma_k(\tilde{\nabla}^2 v) \), and hence
\[ 2k\tilde{u}_t - \log(\sigma_k(\tilde{\nabla}^2 \tilde{u})) \geq 2kv_t - \log(\sigma_k(\tilde{\nabla}^2 v)) \]
at \((x, t_1)\). On the other hand, the function \( \tilde{u} = u - \xi(x, t_1) \) is a strict sub-solution to (3.1) on \( M \times [0, T) \):
\[ 2k\tilde{u}_t = 2ku_t \leq \log(\sigma_k(\tilde{\nabla}^2 u)) - \log(\tilde{\beta}_{k,n}) - 2ku < \log(\sigma_k(\tilde{\nabla}^2 \tilde{u})) - \log(\tilde{\beta}_{k,n}) - 2\tilde{u}. \]
By the definition of sub- and super-solutions, we have at \((x, t_1)\),
\[ 2k\tilde{u}_t - \log(\sigma_k(\tilde{\nabla}^2 \tilde{u})) < - \log(\tilde{\beta}_{k,n}) - 2k\tilde{u} = - \log(\tilde{\beta}_{k,n}) - 2kv \leq 2kv_t - \log(\sigma_k(\tilde{\nabla}^2 v)), \]
which is a contradiction. This proves the lemma. \( \square \)

Based on the fact that the initial data \( u_0 \) is a subsolution of (1.5) and the boundary data \( \phi \) is increasing in \( t \), we have the monotonicity lemma.

**Lemma 3.2.** Assume that \( u_0 \in C^3(M) \) is a subsolution to the \( \sigma_k \)-Ricci equation (1.5), and \( u \in C^{3,2}(M \times [0, T]) \) is a solution to (3.1) for some \( T > 0 \). Assume that \( u(x, t) = \phi(x, t) \) for any \((x, t) \in \partial M \times [0, T)\) and \( \frac{\partial \phi}{\partial \nu} \geq 0 \) on \( \partial M \times [0, T) \). Then \( u_t \geq 0 \) in \( M \times [0, T) \). In particular, \( u \) is increasing along \( t \geq 0 \). Moreover, we have upper bound estimates for \( u_t \) on \( M \times [0, T) \).

**Proof.** Let \( v = u_t \). We take derivative of \( t \) on both sides of the equation (3.1) to have
\[
(3.7) \quad 2kv_t = \frac{1}{\sigma_k(\tilde{\nabla}^2 u)} \tilde{\nabla}^{ij}_{k-1}[(n - 2)\nabla_i \nabla_j v + \Delta v g_{ij} + (n - 2)(2g^{km}u_k v_m g_{ij} - v_i u_j - u_i v_j)] - 2kv,
\]
where \( \tilde{\nabla}^{ij}_{k-1} \) is the \((k - 1)\)-th Newton transformation of \( \tilde{\nabla}^2 u \), which is positive definite since \( \tilde{\nabla}^2 u \in \Gamma_k^+ \). Recall that \( u_0 \) is a subsolution of (1.5), by the equation (3.1) we have that \( v(x, 0) \geq 0 \) for \( x \in M \). Also, \( v(x, t) = \phi(x, t) \geq 0 \) for \((x, t) \in \partial M \times [0, T) \). We will use maximum principle to obtain that \( v \geq 0 \) on \( M \times [0, T) \). Otherwise, assume that there exists \( x_0 \in M^0 \) and \( t_1 \in (0, T) \) such that
\[
v(x_0, t_1) = \inf_{M \times [0, t_1]} v < 0,
\]
then at \((x_0, t_1)\), we have that
\[
v_t \leq 0, \quad \nabla v = 0, \quad \nabla^2 v \geq 0, \quad v < 0,
\]
and hence
\[
v_t \leq 0, \quad \frac{1}{\sigma^k(\tilde{\nabla}^2 u)} \tilde{\nabla}^{ij}_{k-1}[(n - 2)\nabla_i \nabla_j v + \Delta v g_{ij} + (n - 2)(2g^{km}u_k v_m g_{ij} - v_i u_j - u_i v_j)] - 2kv > 0,
\]
at \((x_0, t_1)\), contradicting with the equation (3.7). Therefore, \( v = u_t \geq 0 \) on \( M \times [0, T) \). In particular, \( u \) is a sub-solution to (1.5) for each \( t > 0 \).

Similarly, assume \( v(x_0, t_1) = \sup_{M \times [0, t_1]} v > 0 \) for some \((x_0, t_1) \in M^0 \times (0, T) \). Then at \((x_0, t_1)\),
\[
v_t \geq 0, \quad \frac{1}{\sigma^k(\tilde{\nabla}^2 u)} \tilde{\nabla}^{ij}_{k-1}[(n - 2)\nabla_i \nabla_j v + \Delta v g_{ij} + (n - 2)(2g^{km}u_k v_m g_{ij} - v_i u_j - u_i v_j)] - 2kv < 0,
\]
contradicting with the equation (3.7). Therefore, combining with (3.1) at $t = 0$, we have

$$v(x, t) = u_t(x, t) \leq \max \left\{ \frac{1}{2k} \sup_M \left[ \log(\sigma_1(\nabla^2 u_0)) - \log(\tilde{\beta}_{k,n}) - 2ku_0 \right], \sup_{\partial M \times [0,t]} \phi \right\}$$

for any $(x, t) \in M \times [0, T)$. By integration, we have

$$u(x, t) = u_0(x) + \int_0^t u_t(x, s) ds$$

$$\leq u_0(x) + t \max \left\{ \frac{1}{2k} \sup_M \left[ \log(\sigma_1(\nabla^2 u_0)) - \log(\tilde{\beta}_{k,n}) - 2ku_0 \right], \sup_{\partial M \times [0,t]} \phi \right\},$$

for any $(x, t) \in M \times [0, T)$; on the other hand, by monotonicity, $u(x, t) \geq u_0(x)$. Hence, we obtain the upper and lower bound estimates for $u$ on $M \times [0, T)$.

\[\square\]

We then give the boundary $C^1$ estimates on $u$.

**Lemma 3.3.** Assume $(M^n, g)$ is a compact manifold with boundary of $C^{4,\alpha}$, and $(M, g)$ is either a compact domain in $\mathbb{R}^n$ or with Ricci curvature $\text{Ric}_g \leq -\delta_0 g$ for some $\delta_0 \geq (n-1)$. Let $u \in C^4(M \times [0, T_0])$ be a solution to the Cauchy-Dirichlet problem (3.1) for some $T_0 > 0$. Assume $u_0 \in C^{4,\alpha}(M)$ is a subsolution to (1.5) satisfying (3.6) at the points $x \in \partial M$ where $v(x) = 0$. Also, assume $\phi \in C^{4+\alpha,2+\frac{\alpha}{2}}(\partial M \times [0,T_1])$ for all $T_1 > 0$, $\phi_t(x, t) \geq 0$ on $\partial M \times [0, +\infty)$ and $\phi$ satisfies the compatible condition (3.4) with $u_0$. Then we have the boundary gradient estimates of $u$ i.e., there exists a constant $C = C(T_0) > 0$ such that

$$|\nabla u(x, t)| \leq C$$

for $(x, t) \in \partial M \times [0, T_0)$.

**Proof.** By the Dirichlet boundary condition, tangential derivatives of $u$ on $\partial M \times [0, t_0]$ is controlled by the tangential derivatives of the boundary data $\phi$ and hence, for the boundary gradient estimates of $u$, we only need to control $|\frac{\partial u}{\partial n}|$ with $n$ the outer normal vector field of $\partial M$.

Since $\nabla^2 u \in \Gamma^+_{k}$, we will show the lower bound of $\frac{\partial u}{\partial n}$ based on the control of $\sup_{M \times [0, T_0]} |u|$ as Guan’s argument in Lemma 5.2 in [5]. Indeed, we have

$$\text{tr}(\nabla^2 u) = 2(n-1)[\Delta u + \frac{(n-2)}{2} |\nabla u|^2 - \frac{1}{2(n-1)} R_g] \geq 0,$$

where $R_g \leq 0$ since $\text{Ric}_g \leq 0$. Let $\nu = e^{\frac{\alpha}{2} u}$. Then we have

$$[\Delta \nu - \frac{n-2}{4(n-1)} R_g \nu] \geq 0.$$

Let $m = \sup_{M \times [0, T_0]} |u|$, which is bounded by the proof of Lemma 3.2. For any $t > 0$, let $\tilde{\nu} = \tilde{\nu}(x, t)$ be the solution to the Dirichlet boundary value problem of the linear elliptic equation

$$\Delta \tilde{\nu} = \frac{n-2}{4(n-1)} R_g e^{\frac{\alpha}{2} m}, \quad \text{in} \ M,$$

$$\tilde{\nu}(x, t) = e^{\frac{\alpha}{2} \phi(x,t)}, \quad p \in \partial M.$$
Then by continuity, for any $T > 0$, there exists a uniform constant $C = C(T) > 0$, such that
\[ \sup_{(x,t) \in \partial M \times [0,T]} \frac{\partial}{\partial n} \tilde{v} \leq C(T) < +\infty. \]
For $t < T_0$, we have
\[ \Delta \tilde{v}(x,t) \leq \frac{n-2}{4(n-1)} R_g v(x,t) \leq \Delta v(x,t), \quad \forall x \in M, \]
\[ \tilde{v}(x,t) = v(x,t), \quad x \in \partial M. \]
By maximum principle, $v(x,t) \leq \tilde{v}(x,t)$ in $M$ and since $v(x,t) = \tilde{v}(x,t)$ for $(x,t) \in \partial M \times [0, T_0)$, we have
\[ \frac{\partial}{\partial n} v \geq \frac{\partial}{\partial n} \tilde{v} \geq -C \]
for some uniform constant $C = C(T_0) > 0$ on $\partial M \times [0, T_0)$, and hence
\[ \frac{\partial}{\partial n} u \geq \frac{2}{n-2} e^{-\frac{\rho^2}{2}} \frac{\partial}{\partial n} \tilde{v} \geq -\frac{2}{n-2} C(T_0) e^{-\frac{\rho^2}{2}} \sup_{M \times [0,T_0]} |u| \]
for $(x,t) \in \partial M \times [0, T_0)$. This gives a uniform lower bound of $\frac{\partial}{\partial n} u$ on $\partial M \times [0, T_0)$.

Now we give upper bound estimates on $\frac{\partial}{\partial n} u$. Let $(M_1, g_1)$ be either a small tubular neighborhood of $(M, g)$ in $\mathbb{R}^n$, or an extension of $(M, g)$ as in Section [2] respectively. For any $x_0 \in \partial M$, let $\bar{x} \in M_1 \setminus M$ be as in Section [2] and $r(x)$ be the distance function to $\bar{x}$ in $M_1$ for $x \in M_1$. Let $\delta_1 > 0$ be a small constant such that $\delta_1 < \delta$ with $\delta > 0$ defined in Section [2]. Define the domain $U = \{x \in M, r(x) \leq r(x_0) + \delta_1\}$, with its boundary $\partial U = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0 = U \cap \partial M$ and $\Gamma_1 = \{x \in M | r(x) = r(x_0) + \delta_1\}$. Since $2\delta + r(x_0)$ is less than the injectivity radius at $\bar{x}$, $r(x)$ is smooth in $U$. For given $T > 0$, we extend $\phi$ to a $C^{4+\alpha,2+\beta}$ function on $U \times [0, T]$ for any $T > 0$ so that $\phi(x, 0) = u_0(x)$ for $x \in U$. Define the function
\[ u(x,t) = \phi(x,t) + A\left( \frac{1}{r(x)^p} - \frac{1}{r(x_0)^p} \right), \]
on $U \times [0, T]$, with two large constants $A > 0$ and $p > 0$ to be determined. We will choose $A = A(T)$ and $p = p(T)$ large so that $u$ is a barrier function that controls the lower bound of $u$ on $U \times [0, T]$. Direct computations lead to
\[ u_t = \phi_t, \]
\[ \nabla u = \nabla \phi - A p r^{-p-1} \nabla r, \]
\[ \nabla_i \nabla_j u = \nabla_i \nabla_j \phi + A p (p + 1) r^{-p-2} \nabla_i r \nabla_j r - A p r^{-p-1} \nabla_i \nabla_j r \]
\[ \Delta u = \Delta \phi + A p (p + 1) r^{-p-2} |\nabla r|^2 - A p r^{-p-1} \Delta r = \Delta \phi + A p (p + 1) r^{-p-2} - A p r^{-p-1} \Delta r. \]
By continuity, there exist constants $C_1 > 0$ and $C_2 = C_2(T) > 0$ such that $|\nabla^2 r| + |\Delta r| \leq C_1$ in $U$ and $|\nabla \phi| + |\nabla^2 \phi| + |\Delta \phi| \leq C_2$ in $U \times [0, T]$. We have the calculation
\[ (\tilde{\nabla}^2 u)_{ij} = -Ric_{ij}(g) + (n-2) [\nabla_i \nabla_j \phi + A p (p + 1) r^{-p-2} \nabla_i r \nabla_j r - A p r^{-p-1} \nabla_i \nabla_j r] \]
\[ + [\Delta \phi + A p (p + 1) r^{-p-2} - A p r^{-p-1} \Delta r] g_{ij} + (n-2) |\nabla u|^2 g_{ij} - \nabla_i \nabla_j u. \]
Since $-Ric_g \geq 0$ and the matrix $(\nabla_i r \nabla_j r)$ and the last term are semi-positive, we have
\[ (\nabla^2 u)_{ij} \geq (n-2) [\nabla_i \nabla_j \phi - A p r^{-p-1} \nabla_i \nabla_j r] + [\Delta \phi - A p r^{-p-1} \Delta r] g_{ij} + A p (p + 1) r^{-p-2} g_{ij}, \]
and hence for any \( N_1 > 0 \) and \( A > 0 \), there exists a constant \( p_0 = p_0(T, N_1, A) > 0 \), such that for \( p > p_0 \), we have
\[
(\nabla^2 u)_{ij} \geq N_1 g_{ij}
\]
on \( U \times [0, T] \). Let
\[
N_1 \geq \beta_{n,n}^4 e^{2\sup_{\partial \Omega} |\phi|} + 2\sup_{\partial \Omega} |\phi|.
\]
Then we have
\[
\log(\sigma_n(\nabla^2 u)) \geq \log(N_1^n) \geq 2n\phi + \log(\beta_{n,n}) + 2n\phi \geq 2n\phi + \log(\beta_{n,n}) + 2n\phi
\]
on \( U \times [0, T] \). Therefore, \( u \) is a subsolution of the \( \sigma_n \)-Ricci curvature flow. By Maclaurin’s inequality, \( u \) is a subsolution of the \( \sigma_k \)-Ricci curvature flow for any \( 1 \leq k \leq n \). By definition, we know that \( u \leq u \) on \( \Gamma_0 \times [0, T_0] \). On \( \Gamma_1 \times [0, T_0] \), \( u \) and \( \phi \) has uniform upper and lower bounds, and hence we can choose \( A \) and \( p \) large enough so that \( u < u \) on \( \Gamma_1 \times [0, T_0] \). Also, we have
\[
\bar{u}(x, 0) \leq \phi(x, 0) = u_0(x)
\]
for \( x \in U \). By maximum principle in Lemma 3.1, we have that
\[
u \geq \bar{u}
\]
in \( U \times [0, T_0] \). Since \( u(x_0, t) = \phi(x_0, t) = u(x_0, t) \), we have
\[
\frac{\partial}{\partial n} u \leq \frac{\partial}{\partial n} \bar{u}
\]
at \((x_0, t)\) for \( t \in [0, T_0] \), where \( n \) is the unit outer normal vector of \( \partial M \) at \( x_0 \). Notice that the constants used here can be chosen uniformly for all \( x_0 \in \partial M \) and hence, there exists a unique constant \( m_1 = m_1(T_0) > 0 \), such that \( \frac{\partial}{\partial n} u \leq m_1 \) on \( \partial M \times [0, T_0] \). Therefore, we have the \( C^1 \) estimates of \( u \) at points on \( \partial M \) i.e., there exists a constant \( C = C(T_0) > 0 \) such that
\[
|\nabla u(x, t)| \leq C
\]
for \((x, t) \in \partial M \times [0, T_0) \). \( \square \)

Now we give the \( C^1 \) estimates of \( u \) on \( M \times [0, T_0] \).

**Lemma 3.4.** Let \((M, g)\) and \( u \in C^4(M \times [0, T_0]) \) be as in Lemma 3.3 Then there exists a constant \( C = C(T_0) > 0 \) such that
\[
|\nabla u(x, t)| \leq C
\]
for \((x, t) \in M \times [0, T_0] \).

**Proof.** The interior gradient estimate is relatively standard, and here we modify the argument in [11] (see also [8]). By Lemma 3.2 there exist two constants \( -\infty < \beta_1 < \beta_2 < +\infty \) depending on \( T_0 \) such that \( \beta_1 \leq u \leq \beta_2 \) on \( M \times [0, T_0] \). We define a function
\[
\xi(x, t) = (1 + \frac{|\nabla u|^2}{2})e^{\eta(u)},
\]
where
\[
\eta(s) = C_1(C_2 + s)^p
\]
is a function on \( s \in [\beta_1, +\infty) \) with constants \( C_2 > -\beta_1, C_1 > 0 \) and \( p > 0 \), depending only on \( T_0, \beta_1 \) and \( \beta_2 \), to be determined. Suppose that there exists \( x_0 \in M^o \) and \( t_0 \in (0, T_0) \) such that

\[
\xi(x_0, t_0) = \sup_{M \times [0, t_0]} \xi.
\]

We take geodesic normal coordinates \((x^1, \ldots, x^n)\) centered at \( x_0 \in M \) such that \( \Gamma^m_{ij}(x_0) = 0 \), \( g_{ij}(x_0) = \delta_{ij} \). Then we have at \((x_0, t_0)\),

\begin{align*}
(3.9) \quad & \xi_{x_i} = e^{\eta(u)} [u_{x_i,x_i} u_{x_i} + (1 + \frac{1}{2} u_{x_i} u_{x_i}) \eta'(u) u_{x_i}] = 0, \\
(3.10) \quad & \xi_i = e^{\eta(u)} [u_{x_i,x_i} u_{x_i} + (1 + \frac{1}{2} u_{x_i} u_{x_i}) \eta'(u) u_i] \geq 0,
\end{align*}

\[
0 \geq \xi_{x_i x_j} = \left[ \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} g^{ab} u_{x_i} u_{x_b} + u_{x_i x_j,x_j} u_{x_i} + u_{x_i x_j,x_i} u_{x_j} + \eta'(u) u_{x_i x_j} u_{x_i} u_{x_j} + \frac{1}{2} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} g_{ab} u_{x_i} u_{x_b} + \frac{1}{2} \eta'(u) u_{x_i} u_{x_j} \right] e^{\eta(u)}
\]

\[
+ \left( 1 + \frac{1}{2} |\nabla u|^2 \right) (\eta'(u))^2 u_{x_i} u_{x_j} + \left( 1 + \frac{1}{2} |\nabla u|^2 \right) \eta''(u) u_{x_i} u_{x_j} + \left( 1 + \frac{1}{2} |\nabla u|^2 \right) \eta'(u) u_{x_i x_j} e^{\eta(u)}
\]

\[
= \left[ \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} g^{ab} u_{x_i} u_{x_b} + u_{x_i x_j,x_j} u_{x_i} + u_{x_i x_j,x_i} u_{x_j} + \left( 1 + \frac{1}{2} |\nabla u|^2 \right) \eta''(u) u_{x_i} u_{x_j} + \left( 1 + \frac{1}{2} |\nabla u|^2 \right) \eta'(u) u_{x_i} u_{x_j} \right] e^{\eta(u)}.
\]

where the last identity is by (3.9). Notice that the tensor

\[
\tilde{Q}_{ij} = \frac{1}{\sigma_k(\nabla^2 u)} ((n - 2)(\tilde{T}_{k-1})_{ij} + g_{ab} (\tilde{T}_{k-1})_{ab ij}),
\]

is positive definite. Therefore, at \((x_0, t_0)\),

\[
0 \geq \left[ \frac{1}{(1 + \frac{1}{2} |\nabla u|^2)} \left( \tilde{Q}_{ij} u_{x_i} u_{x_j} + \frac{1}{2} \tilde{Q}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} g^{ab} u_{x_i} u_{x_b} + \tilde{Q}_{ij} u_{x_i} u_{x_j} \right) + (\eta''(u) - (\eta'(u))^2) \tilde{Q}_{ij} u_{x_i} u_{x_j} + \eta'(u) \tilde{Q}_{ij} u_{x_i} u_{x_j} \right] e^{\eta(u)}.
\]

By definition, at \((x_0, t_0)\) we have

\[
\nabla^2 u = -Ric_x + (n - 2) u_{x_i,x_i} + \Delta u \delta_{ij} - (n - 2) u_{x_i} u_{x_j} + (n - 2) |\nabla u|^2 \delta_{ij},
\]

and hence by the identity \( \tilde{T}_{ij} (\nabla^2 u)_{ij} = k \sigma_k (\nabla^2 u) \) and the equation (3.1), we obtain

\[
\tilde{Q}_{ij} u_{x_i} u_{x_j} = \frac{1}{\sigma_k(\nabla^2 u)} \left[ \tilde{T}_{ij} (\nabla^2 u)_{ij} + \tilde{T}_{ij} (Ric_{ij} + (n - 2) u_{x_i} u_{x_j} - (n - 2) |\nabla u|^2 \delta_{ij}) \right]
\]

\[
= \frac{1}{\sigma_k(\nabla^2 u)} \left[ k \tilde{\beta}_{k,n} e^{2k u + 2k u} + \tilde{T}_{ij} (Ric_{ij} + (n - 2) u_{x_i} u_{x_j} - (n - 2) |\nabla u|^2 \delta_{ij}) \right],
\]

at \((x_0, t_0)\). Now take derivative of \( x_i \) on both sides of (3.1), and we have at \((x_0, t_0)\),

\[
2k u_{x_i} = \frac{1}{\sigma_k(\nabla^2 u)} \tilde{T}_{ab} - \frac{\partial}{\partial x_i} Ric_{ab} + (n - 2) u_{x_a,x_b, x_i} - (n - 2) \frac{\partial}{\partial x_i} \Gamma^m_{ab} u_{x_m} + (u_{x_m,x_n,x_i} - \frac{\partial}{\partial x_i} \Gamma^c_{mn} u_{x_c}) g_{ab} + (n - 2) (2 u_{x_m,x_n} g_{ab} - u_{x_m,x_n} g_{ab} - u_{x_m,x_n} g_{ab}) - 2k u_{x_i},
\]

12
and hence at \((x_0, t_0)\), for \(1 \leq a \leq n\),
\[
\bar{Q}_{ij}u_{x_ia,x_jx_a} = 2k(u_{x_a} + u_{x_a}) + \frac{1}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}[(n - 2)(-2u_{x_a}u_{x_a}g_{ij} + u_{x_a}u_{x_j}) + u_{x_a}u_{x_j} + \frac{\partial}{\partial x_a} \Gamma^m_{ij}u_{x_a})
\]
\[+ \frac{\partial}{\partial x_a} \Gamma^m_{ij}u_{x_a}g_{ij} + \frac{\partial}{\partial x_a} \Gamma_{ij}].\]

Now contracting this equation with \(\nabla u\) we have at \((x_0, t_0)\),
\[
\bar{Q}_{ij}u_{x_ia,x_jx_a} = 2k(u_{x_a}u_{x_a} + u_{x_a}u_{x_a}) + \frac{1}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}[(n - 2)(-2u_{x_a}u_{x_a}u_{x_a}g_{ij} + 2u_{x_a}u_{x_a}u_{x_a})
\]
\[+ \frac{\partial}{\partial x_a} \Gamma^m_{ij}u_{x_a}u_{x_a} + \frac{\partial}{\partial x_a} \Gamma^m_{ij}u_{x_a}g_{ij} + u_{x_a} + \frac{\partial}{\partial x_a} \Gamma_{ij}]
\]
\[\geq \frac{1}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}[(n - 2)(2(1 + \frac{1}{2} |\nabla u|^2)\eta'(u)(u_{x_a}u_{x_a}g_{ij} - u_{x_a}u_{x_j}) + \frac{\partial}{\partial x_a} \Gamma^m_{ij}u_{x_a})
\]
\[+ \frac{\partial}{\partial x_a} \Gamma^m_{ij}u_{x_a}g_{ij} + u_{x_a} \frac{\partial}{\partial x_a} \Gamma_{ij}] + 2k(u_{x_a}u_{x_a} - (1 + \frac{1}{2} |\nabla u|^2)\eta'(u)u_t),\]

where the last inequality is by (3.9) and (3.10). Substituting (3.12) and (3.13) to (3.11), we have
\[
0 \geq \frac{1}{(1 + \frac{1}{2} |\nabla u|^2)[2k(|\nabla u|^2 - (1 + \frac{1}{2} |\nabla u|^2)\eta'(u)u_t] + \frac{\tilde{T}_{ij}}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}u_{x_a}u_{x_j} + \frac{\tilde{T}_{ij}}{\sigma_k(\nabla^2 u)}(\eta'' - (\eta')^2 + (n - 2)\eta') |\nabla u|^2 \sum_i \tilde{T}_{ii}
\]
\[+ \frac{\tilde{T}_{ij}}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}Ric_{ij} - 2k\eta' u_t
\]
\[+ \frac{1}{(1 + \frac{1}{2} |\nabla u|^2)[2k(|\nabla u|^2 + \frac{\tilde{T}_{ij}}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}u_{x_a}u_{x_j} + \frac{\tilde{T}_{ij}}{\sigma_k(\nabla^2 u)} \sigma_k(\nabla^2 u) \tilde{T}_{ij}Ric_{ij}u_{x_a}u_{x_j} + \tilde{Q}_{ij}u_{x_a}u_{x_a}u_{x_a}u_{x_j} + \tilde{Q}_{ij}R_{iajb}u_{x_a}u_{x_j}].\]

Recall that \(u\) and \(u_t\) are uniformly bounded from above and blow on \(M \times [0, T_0]\) by Lemma 3.2, and so is the term
\[
\frac{1}{\sigma_k(\nabla^2 u)} = \bar{p}^{-1}_k e^{-2k\nu - 2k u}.
\]

Since \(\bar{T}_{k-1}^{(1)}\) and \(\tilde{Q}_{k-1}^{(1)}\) are positively definite, we have at \((x_0, t_0)\),
\[
0 \geq \frac{e^{2k\nu}}{\sigma_k(\nabla^2 u)}(\eta'' - (\eta')^2 - \eta') \bar{T}_{ij}u_{x_a}u_{x_j} + \frac{e^{2k\nu}}{\sigma_k(\nabla^2 u)}(\eta'' - (\eta')^2 + (n - 2)\eta') |\nabla u|^2 \sum_i \bar{T}_{ii}
\]
\[\geq C - \frac{1}{\sigma_k(\nabla^2 u)} = \bar{p}^{-1}_k e^{-2k\nu - 2k u},\]

13
with the constant \(C > 0\) depending on \(T_0\), \(\sup_{\partial M \times [0, T_0]} (|\phi| + |\phi'|)\), \(\sup_M \log(\sigma_k(\nabla^2 u_0))\), \(\sup_M |u_0|\), \(\sup_M (|\text{Ric}_g| + |\nabla \text{Ric}_g|)\) and \(\sup_{\beta_1 \leq s \leq \beta_2} |\eta'(s)|\). By the definition of \(\eta\), we have \(\eta' > 0\), and
\[
\eta'' - (\eta')^2 - \eta' = C_1 p(C_2 + s)^{p-2}[(p - 1) - C_1 p(C_2 + s)^p - (C_2 + s)].
\]
For \(\beta_1 \leq s \leq \beta_2\), we choose \(C_2 = 1 - \beta_1\), \(p > 0\) large and then choose \(C_1 > 0\) small so that
\[
\eta'' - (\eta')^2 \geq C_1 p,
\]
and hence at \((x_0, t_0)\)
\[
|\nabla u|^2 \sum_{i} \tilde{T}_{ii} \leq \frac{\sigma_k(\nabla^2 u)}{C_1 p e^{2k_0}} (C + C \sum_i \tilde{T}_{ii}) = \frac{1}{C_1 p} \tilde{\beta}_{k,n} e^{2k_0} (C + C \sum_i \tilde{T}_{ii}) \leq \tilde{C}(1 + \sum_i \tilde{T}_{ii}),
\]
where the constant \(\tilde{C} > 0\) depends on \(T_0\), \(\sup_{\partial M \times [0, T_0]} (|\phi| + |\phi'|)\), \(\sup_M \log(\sigma_k(\nabla^2 u_0))\), \(\sup_M |u_0|\), \(\sup_M (|\text{Ric}_g| + |\nabla \text{Ric}_g|)\) and \(\sup_{\beta_1 \leq s \leq \beta_2} |\eta'(s)|\). Recall that
\[
\sum_i \tilde{T}_{ii} = (n - k + 1) \sigma_{k-1}(\nabla^2 u) \geq (n - k + 1) \binom{n}{k-1} (\binom{n}{k})^{-1} \sigma_k(\nabla^2 u) \tilde{\beta}_{k,n} e^{2k_0 + 2k_0} \geq C,
\]
for some uniform constant \(C = C(T_0) > 0\), where we have used the Maclaurin’s inequality and the uniform lower bound of \(u\) and \(u_t \geq 0\). Therefore,
\[
|\nabla u|^2 \leq \tilde{C}(1 + \frac{1}{C}).
\]
This combining with the boundary \(C^1\) estimates completes the proof of the gradient estimates of \(u\) on \(M \times [0, T_0]\).

Now we consider the \(C^2\) estimates on \(u\) at the points on \(\partial M \times [0, T_0]\).

**Lemma 3.5.** Let \((M, g)\) and \(u \in C^4(M \times [0, T_0])\) be as in Lemma 3.3 Then there exists a constant \(C = C(T_0) > 0\) such that
\[
|\nabla^2 u| \leq C
\]
on \(\partial M \times [0, T_0]\).

**Proof.** We use the indices \(e_i, e_j\) to refer to the tangential vector fields on \(\partial M\) and \(n\) the outer normal vector field. Notice that we have obtained the uniform bounds
\[
\sup_{\partial M \times [0, T_0]} (|u| + |\nabla u|) \leq K,
\]
for some constant \(K > 0\) on \(\partial M \times [0, T_0]\). By definition, we immediately have the control on the second order tangential derivatives
\[
\sup_{\partial M \times [0, T_0]} |\nabla_i \nabla_j u| \leq C
\]
on \(\partial M \times [0, T_0]\) with some constant \(C > 0\) depending on \(K\) and \(\sup_{\partial M \times [0, T_0]} (|\phi| + |\nabla \phi| + |\nabla^2 \phi|)\) where \(\nabla^2 \phi\) means the second order tangential derivatives of \(\phi\) on \(\partial M\). We extend \(\phi\) to a function
in $C^{4,2}(U \times [0, +\infty))$ still denoted as $\phi$ such that $\phi \in C^{4+a,2+\frac{a}{2}}(M \times [0, T])$ for any $T > 0$ and $\phi(x, 0) = u_0(x)$ for $x \in M$.

We now estimate the mixed second order derivatives $|\nabla_n \nabla_i u|$ with $n$ the normal vector field on $\partial M$. Let $(M_1, g_1)$ be the extension of $(M, g)$ as in Section 2. Let $\delta > \epsilon_1 > 0$ be the small constants in Section 2. For any $x_0 \in \partial M$, let $\bar{x}$ be the point with respect to $x_0$ as defined in Section 2. Define the exponential map $\text{Exp} : \partial M \times [-\epsilon_1 - 2\delta, \epsilon_1 + 2\delta] \rightarrow M_1$ such that $\text{Exp}_q(s)$ is the point along the geodesic starting from $q \in \partial M$ in the normal direction of $\partial M$ of distance $|s|$ to $q$. Here we take the inner direction to be positive i.e., $\text{Exp}_q(s) \in M^\circ$ when $s > 0$. In particular, $\bar{x} = \text{Exp}_{x_0}(-\epsilon_1)$. Notice that $\text{Exp} : \partial M \times [-\epsilon_1 - 2\delta, \epsilon_1 + 2\delta]$ is a diffeomorphism to its image. In fact we can choose $\epsilon_1 + 2\delta < \epsilon$ where $\epsilon$ is strictly less than the lower bound of injectivity radius of each point in the thin $(\epsilon_1 + 2\delta)$-neighborhood $\Omega$ of $\partial M$. We now use the Femi coordinate in a small neighborhood $V_{x_0} = B_{\epsilon}(x_0)$ of $x_0$ in $M_1$: Let $(x^1, ..., x^{n-1})$ be a geodesic normal coordinate centered at $x_0$ on $(\partial M, g|_{\partial M})$. We take $(x^1(q), ..., x^{n-1}(q), x^n)$ as the coordinate of the point $\text{Exp}_q(x^n)$ in $V_{x_0}$. Define the distance function $r(x) = \text{dist}(x, \bar{x})$ for $x \in M_1$. Denote $U = \{x \in M | r(x) \leq \delta + r(x_0)\}$, $\Gamma_0 = U \cap \partial M$ and $\Gamma_1 = \{x \in M | r(x) = \delta + r(x_0)\}$. By our choice of the small constant $\epsilon_1 + 2\delta$, we have $\Gamma_0 \subseteq V_{x_0}$ and hence $\frac{\partial}{\partial x^i}$ ($i < n$) is a tangential derivative of $\partial M$ on $\Gamma_0$. It is clear that $r(x)$ is smooth on $U$. The metric has the orthogonal decomposition $g = d(x^n)^2 + g_{x^n}$ in $U$ and we have $\Gamma_{ab}(x_0) = 0$ for $a, b, c \in \{1, 2, ..., n\}$. For $i \in \{1, ..., n-1\}$, taking derivative of $\frac{\partial}{\partial x^i}$ on both sides of (3.11) we have

$$0 = -2ku_{ix_i} - 2ku_{x_i} + \frac{1}{\sigma_k(\nabla^2 u)}\bar{T}_{ab}[-\nabla_i \text{Ric}_{ab} + (n - 2)\nabla_i \nabla_a u + \nabla_i \Delta u g_{ab}]
+ 2(n - 2)(\nabla_i \nabla_c u \nabla_c g_{ab} - \nabla_i \nabla_a u \nabla_i u).$$

(3.15)

Now we commute derivatives to have

$$\nabla_i \nabla_a \nabla_i u = \nabla_a \nabla_i u_{x_i} + \text{Rm} * \nabla_i u,$nabla_i \Delta u = \Delta u_{x_i} + \text{Rm} * \nabla_i u,$

where the terms $\text{Rm} * \nabla_i u$ are contractions of some Riemannian curvature terms and $\nabla_i u$. Define the linearized operator $L$ acting on $\varphi$ as

(3.16) \quad \quad \quad L(\varphi) \equiv \frac{1}{\sigma_k(\nabla^2 u)}\bar{T}_{ab}[(n - 2)\nabla_a \nabla_b \varphi + \Delta \varphi g_{ab} + 2(n - 2)(\nabla \varphi, \nabla u > g_{ab} - \nabla_a \varphi \nabla_b u)]
- 2k\varphi_i - 2k\varphi.$$

Therefore, by (3.15) we have

$$|L(u_{x_i})| = \frac{1}{\sigma_k(\nabla^2 u)}|\bar{T}_{ab}(-\nabla_i \text{Ric}_{ab} + (\text{Rm} * \nabla_i u))| \leq C \sum_i \bar{T}_{ii}(1 + |\nabla u|)
\leq C \sum_i \bar{T}_{ii},$$

(3.17)
for some constant \( C > 0 \) depending on \( \sup_M |Rm| \), the lower bound of \( u_t + u \) and the upper bound of \( |\nabla u| \) on \( M \times [0, T_0) \), which has been uniformly controlled. Recall that by (3.14), we have

\[
\sum_i \bar{T}_{ii} \geq C
\]

for some uniform constant \( C = C(T_0) > 0 \), and hence direct calculation leads to the bound

(3.18)

\[
|L(\phi_{x_i})| \leq C \sum_i \bar{T}_{ii} + C \leq C \sum_i \bar{T}_{ii},
\]

on \( U \times [0, T_0) \), where \( C > 0 \) in the inequalities are uniform constants depending on \( T_0, k, n, \sup_{M \times [0, T_0]} (|u| + |u_t| + |\nabla u|) \) and \( \sup_{U \times [0, T_0]} (|\phi_{x_i}| + |\phi_{x_i}t| + |\nabla \phi_{x_i}| + |\nabla^2 \phi_{x_i}|) \). Define the function \( v = u_{x_i} - \phi_{x_i} \) in \( U \times [0, T_0) \). Now by (3.17) and (3.18) we have

\[
|L(v)| \leq C \sum_i \bar{T}_{ii},
\]

for some uniform constant \( C = C(T_0) > 0 \). Also, \( v = 0 \) on \( \Gamma_0 \).

Now let

\[
\xi(x) = \frac{1}{r(x)^p} - \frac{1}{r(x_0)^p}
\]

for \( x \in U \), where \( p > 0 \) is a constant depending on \( T_0 \) to be determined. Following the calculation in Section 2 we have that for \( p = p(T_0) > 0 \) large,

\[
(n - 2)|\nabla^2 \xi| + \Delta \xi g \geq \frac{p^2}{4} r^{-p-2} g.
\]

Since \( \xi \leq 0, |\nabla u| \) is uniformly bounded from above and \( u_t + u \) is uniformly bounded from blow, we choose \( p = p(T_0) > 0 \) large so that

\[
L(\xi) \geq \frac{1}{\beta \epsilon n e^{2k_d + 2k_d}} \left[ \frac{p^2}{4} r^{-p-2} - C|\nabla u||\nabla \xi|| \sum_i \bar{T}_{ii} - 2k \xi \right]
\]

\[
\geq \frac{1}{C} \left( \frac{p^2}{4} r^{-p-2} - Cpr^{-p-1} \right) \sum_i \bar{T}_{ii} \geq \frac{p^2}{8C} r^{-p-2} \sum_i \bar{T}_{ii}
\]

\[
\geq |L(v)|
\]

on \( U \times [0, T_0) \) for some uniform constant \( C = C(T_0) > 0 \). Now we take \( p > 0 \) even larger so that \( \xi < -|v| \) on \( \Gamma_1 \times [0, T_0) \) and hence, \( \xi \leq -|v| \) on \( \partial U \times [0, T_0) \). Recall that

\[
\xi(x) \leq 0 = v(x, 0)
\]

for \( x \in M \), we have by maximum principle,

\[
\pm v(x, t) \geq \xi(x)
\]

for \( (x, t) \in U \times [0, T_0) \). Since \( v(x_0, t) = \xi(x_0) = 0 \), we have for \( i = 1, \ldots, n - 1, \)

\[
|\nabla_n u_{x_i}(x_0, t)| \leq |\nabla_n \phi_{x_i}(x_0, t)| + |\nabla_n v_{x_i}(x_0, t)| \leq |\nabla_n \phi_{x_i}(x_0, t)| + \nabla_n \xi(x_0) \leq C,
\]
for any \((x_0, t) \in \partial M \times [0, T_0]\) with some uniform constant \(C = C(T_0) > 0\) independent of the choice of \((x_0, t) \in \partial M \times [0, T_0]\), where \(\nabla_n\) is the outer normal derivative at \(x_0 \in \partial M\). For the second order normal derivative \(\nabla^2_n u\), since \(\text{tr}(\nabla^2 u) \geq 0\), i.e.

\[2(n - 1)\Delta u + (n - 2)(n - 1)|\nabla u|^2 - R_g \geq 0,\]

by the estimates on the other second order derivatives, \(\nabla^2_n u\) is bounded from below and we still need to derive an upper bound of \(\nabla^2_n u\). Orthogonally decompose the matrix \(\nabla^2 u\) at \(x_0 \in \partial M\) in normal and tangential directions. By the previous estimates we have

\[
\nabla^2 u = \begin{pmatrix} (n - 1)u_{nn} & 0 \\ 0 & u_{nn}g|_{\partial M} \end{pmatrix} + O(1)
\]

with the term \(|O(1)| \leq C\) for some uniform constant \(C = C(T_0) > 0\) and hence, as the term \(u_{nn} \rightarrow +\infty\), we have

\[
s \sigma_k(\nabla^2 u) = (u_{nn})^k (\Lambda_{k,n} + o(1)) \rightarrow +\infty,
\]

where \(\Lambda_{k,n}\) is a positive constant. On the other hand, recall that

\[
0 < \frac{1}{C} \leq \sigma_k(\nabla^2 u) = \beta_{k,n}e^{2\kappa_0 + 2\kappa_1} \leq C,
\]

for some uniform constant \(C = C(T_0) > 0\) on \(M \times [0, T_0]\) and hence, we have that there exists a uniform constant \(C = C(T_0) > 0\) such that \(\nabla^2_n u(x_0) \leq C\). Notice that the constant \(C\) here is independent of the choice of \(x_0 \in \partial M\). This completes the boundary \(C^2\) estimates of \(u\).

\[\square\]

**Proposition 3.6.** Let \((M, g)\) and \(u \in C^4(M \times [0, T_0])\) be as in Lemma \([3.3]\). Then there exists a constant \(C = C(T_0) > 0\) such that for any \((x, t) \in M \times [0, T_0]\) we have \(\|
abla^2 u(x, t)\| \leq C\).

**Proof.** The proof is a modification of Proposition 3.3 in \([11]\), see also \([8]\). We have obtained the global \(C^1\) estimates and boundary \(C^2\) estimates on \(u\). Now suppose the maximum of \(\|
abla^2 u\|\) is achieved at a point in the interior.

Denote \(S(TM)\) the unit tangent bundle of \((M, g)\). We define a function \(h : S(TM) \times [0, T_0) \rightarrow \mathbb{R}\), such that

\[
h(x, e_x, t) = (\nabla^2 u + m|\nabla u|^2 g)(e_x, e_x),
\]

for any \(x \in M, t \in [0, T_0)\) and \(e_x \in ST_xM\), with \(m > 1\) a constant to be fixed. Suppose there exist \((q, t_1) \in M^\circ \times [0, T_0)\) and a unit tangent vector \(e_q \in ST_qM\) such that

\[h(q, e_q, t_1) = \sup_{S(TM) \times [0, t_1]} h.
\]

Notice that on \(S(TM) \subseteq S(TM_1)\) (here \((M_1, g_1)\) is the extension of \((M, g)\) as in Section \([2]\), we can find a uniform constant \(C' > 0\) and a uniform small constant \(\delta_0 > 0\) such that for any \(x \in M\) and any \(e_x \in T_xM_1, e_x\) can be extended to a unit vector field \(e\) on \(B_{\delta_0}(x) \subseteq M_1\) such that \(\nabla e(x) = 0\) and \(\|
abla^2 e(x)\| \leq C'\) at this point \(x\). Take the geodesic normal coordinates \((x^1, \ldots, x^n)\) at \(q\), and hence we have \(\Gamma^e_{ab}(q) = 0\) and \(g_{ij}(q) = \delta_{ij}\). By rotating, we assume \(\nabla^2 u = u_q\) is
diagonal at $q$ and $e_q = \frac{\partial}{\partial x^i}$ at $(q,t_1)$. Let the unit vector field $e = \sum_i \xi^i \frac{\partial}{\partial x^i}$ be the extension of $e_q$ on $B_{\delta_0}(q)$ with $\nabla e(q) = 0$ and $|\nabla^2 e(q)| \leq C'$. We have

$$\xi^1(q) = 1, \quad \xi^i(q) = 0, \quad i \geq 1, \quad \text{and} \quad \frac{\partial}{\partial x} \xi^i(q) = 0, \quad i, j = 1, \ldots, n.$$ 

It is clear that the fact $\nabla^2 u \in \Gamma^+$ and the uniform bound of $|\nabla u|$ on $M \times [0, T_0]$ imply that there exists a uniform constant $C > -\infty$ such that $\nabla^2_1 u > C$ at $(q, t_1)$. Now we define a function $\tilde{h}$ in a small neighborhood $U \times [t_1 - \epsilon, t_1 + \epsilon]$ of $(q, t_1)$ such that

$$\tilde{h}(x,t) = (\nabla^2 u + m|\nabla u|^2) (e,e) = \xi^i \xi^j (u_{,ij} - \Gamma^a_{ij} u_{,a}) + m|\nabla u|^2.$$ 

Since $\tilde{h}$ achieves its maximum in $U \times [t_1 - \epsilon, t_1]$, we have that at $(q, t_1)$,

$$\frac{\partial}{\partial t} \tilde{h} = u_{,t} + 2m u_{,e_t} \geq 0, \quad \frac{\partial}{\partial x^i} \tilde{h} = u_{,ix} + 2mu_{,e_x} \geq 0, \quad \frac{\partial^2}{\partial x^i \partial x^j} \tilde{h} = u_{,ij} + 2\sigma_{ij}u_{,e} + 2\sigma_{ij}u_{,e}u_{,ij} \
0 \geq \tilde{h}_{,x^i} = u_{,x^i, x} + \frac{\partial^2}{\partial x^i \partial x^j} \Gamma^a_{ij} u_{,e} - \frac{\partial^2}{\partial x^i \partial x^j} \Gamma^a_{ij} u_{,x} + m \frac{\partial^2}{\partial x^i \partial x^j} \sigma_{ij} u_{,e} + 2 \frac{\partial^2}{\partial x^i \partial x^j} \sigma_{ij} u_{,e} u_{,i} + 2 \frac{\partial^2}{\partial x^i \partial x^j} \sigma_{ij} u_{,e} u_{,i}.$$ 

Differentiating equation (3.21) with respect to $x^a$ yields

$$2ku_{,e_t} + 2ku_{,e} = \frac{1}{\sigma_k(\nabla^2 u)} \tilde{T}_k[-\nabla_a Ric_{ij} + (n - 2)\nabla_a \nabla^2_1 u + (\Delta u)_{,a} g_{ij} \
+ (n - 2)(\nabla_a \nabla_b u \nabla_c g_{ij} - 2\nabla_a \nabla_b u \nabla_c u)].$$ 

Define the function $F(r_{ij}) = \log(\sigma_k(r_{ij}))$ on $\Gamma^+$. Differentiating (3.1) twice, we obtain

$$2k\nabla^2 u = (\frac{\partial^2}{\partial r_{ab} \partial r_{ij}}) \nabla_1(\nabla^2 u)_{ab} \nabla_1(\nabla^2 u)_{ij} + \frac{1}{\sigma_k(\nabla^2 u)} \tilde{T}_k[-\nabla^2_1 Ric_{ij} + (n - 2)\nabla^2_1 u + \nabla^2_1 (\Delta u) g_{ij} \
+ 2(n - 2)(\nabla^2_1 u, \nabla u) + \nabla_1 \nabla_a u \nabla_1 \nabla_a u g_{ij} - \nabla_1 \nabla_1 \nabla_1 u + \nabla_1 \nabla_1 \nabla_1 u)] - 2k\nabla^2_1 u \
\leq \frac{1}{\sigma_k(\nabla^2 u)} \tilde{T}_k[2n(2 - n)(\nabla^2_1 u, \nabla u) + \nabla_1 \nabla_a u \nabla_1 \nabla_a u g_{ij} - \nabla_1 \nabla_1 \nabla_1 u + \nabla_1 \nabla_1 \nabla_1 u)] - 2k\nabla^2_1 u,$$
since $F$ is concave on $\Gamma^+_{X}$. In particular, at $(q, t_1)$ we rewrite these two derivatives as

\begin{equation}
2k(u_{x^t} + u_{x^t}) = \bar{Q}_{ij}(u_{x^t x^t}) - \frac{\partial}{\partial x^i} \Gamma^b_{ij} u_{x^b} + \frac{\bar{T}_{ij}}{\sigma_k(\nabla_2 u)}[-\nabla_a \text{Ric}_{ij} + 2(n - 2)(u_{x^t x^t}, u_{x^t}, g_{ij} - u_{x^t} u_{x^t})],
\end{equation}

and hence combining with (3.21), we have

\begin{align*}
0 & \geq \bar{Q}_{ij}(\frac{\partial^2}{\partial (x^i)^2} \Gamma^a_{ij} u_{x^a} + 2 \frac{\partial}{\partial x^i} \Gamma^a_{ij} u_{x^a x^i} + 2 \frac{\partial}{\partial x^i} \Gamma^a_{ij} u_{x^a x^i} - \frac{\partial^2}{\partial x^i \partial x^j} \Gamma^a_{ij} u_{x^a} - 2 \frac{\partial}{\partial x^j} \Gamma^a_{ij} u_{x^a} - 2 \frac{\partial}{\partial x^j} \Gamma^a_{ij} u_{x^a x^j}) \\
& \quad - \frac{\bar{T}_{ij}}{\sigma_k(\nabla_2 u)}[2(n - 2)(u_{x^t x^t}, u_{x^t, x^t} - \frac{\partial}{\partial x^i} \Gamma^b_{ij} u_{x^b} u_{x^a}, u_{x^t, x^t} - \frac{\partial}{\partial x^i} \Gamma^b_{ij} u_{x^b} u_{x^a} + u_{x^t, x^t}), u_{x^t, x^t}) - \nabla^2 \text{Ric}_{ij}] + m \bar{Q}_{ij}(2(n - 2)(u_{x^t x^t}, u_{x^t}, g_{ij} - u_{x^t} u_{x^t})]) \\
& \quad + 2mu_{x^t}(2k u_{x^t} + 2k u_{x^t}) + \frac{\bar{T}_{ij}}{\sigma_k(\nabla_2 u)}(\nabla_2 \text{Ric}_{ij}) - 4kmu_{x^t} u_{x^t} \\
& \quad + 2k(u_{x^t, x^t} + u_{x^t, x^t}) + 2 \bar{Q}_{ij}(\frac{\partial^2}{\partial x^i \partial x^j} u_{x^a x^i} x^j).
\end{align*}

Therefore, by (3.19) and (3.20) we have

\begin{align*}
0 & \geq \bar{Q}_{ij}(\frac{\partial^2}{\partial (x^i)^2} \Gamma^a_{ij} u_{x^a} + 2 \frac{\partial}{\partial x^i} \Gamma^a_{ij} u_{x^a x^i} + 2 \frac{\partial}{\partial x^i} \Gamma^a_{ij} u_{x^a x^i} - \frac{\partial^2}{\partial x^i \partial x^j} \Gamma^a_{ij} u_{x^a} - 2 \frac{\partial}{\partial x^j} \Gamma^a_{ij} u_{x^a} - 2 \frac{\partial}{\partial x^j} \Gamma^a_{ij} u_{x^a x^j}) - 4kmu_{x^t} u_{x^t} \\
& \quad + 2k(u_{x^t, x^t} + u_{x^t, x^t}) + 2 \bar{Q}_{ij}(\frac{\partial^2}{\partial x^i \partial x^j} u_{x^a x^i} x^j) + 2 \bar{Q}_{ij}(\frac{\partial^2}{\partial x^i \partial x^j} u_{x^a x^i} x^j).
\end{align*}
By assumption, we have at \((q, t_1), u_{x_i} \leq u_{x_i}^i\) for \(i \geq 2\) and \(u_{x_i} = 0\) for \(i \neq j\). Recall that there exists a unique \(C > -\infty\) on \(M \times (0, T_0)\) such that \(u_{x_i}^1 = \nabla^2_1 u > C\) at \((q, t_1)\) and hence, we have

\[
0 \geq - C - C u_{x_i} - (1 + m)(C u_{x_i} + C) \sum_i \tilde{T}_{ii} + \frac{1}{\sigma_i(\tilde{\nabla}^2 u)}(2(m - 2(n - 2))u_{x_i}^2 \sum_i \tilde{T}_{ii}
+ 2(n - 2)(1 + m)u_{x_i} u_{x_i} \tilde{T}_{ij}]
\geq - C - C u_{x_i} - (1 + m)(C u_{x_i} + C) \sum_i \tilde{T}_{ii} + \frac{1}{\sigma_i(\tilde{\nabla}^2 u)}(2(m - 2(n - 2))u_{x_i}^2 \sum_i \tilde{T}_{ii},
\]

where \(C > 0\) is a uniform constant on \(M \times (0, T_0)\) depending on \(k, n, C', (M, g)\) and

\[
\sup_{M \times (0, T_0)} (|u| + |u_1| + |\nabla u| + |Rm| + |\nabla Rm| + |\nabla^2 Ric|).
\]

Now take \(m\) to be a constant strictly larger than \((n - 2)\). Recall that \(\sigma_i(\tilde{\nabla}^2 u)\) is uniformly bounded from above and below. On the other hand, by (3.14), \(\sum_i \tilde{T}_{ii} > C\) for some uniform constant \(C > 0\) on \(M \times (0, T_0)\), and hence we obtain that there exists a uniform constant \(C > 0\) on \(M \times (0, T_0)\), such that

\[
u_{x_i} \leq C
\]

at \((q, t_1)\). Therefore, combining with the boundary \(C^2\) estimates, we have that there exists a uniform constant \(C > 0\) on \(M \times (0, T_0)\), such that

\[
|\nabla^2 u| \leq C
\]

on \(M \times (0, T_0)\).

Remark. Here we give a way to extend the unit vector \(e_q\) at \(q \in M \subseteq M_1\) in Proposition 3.6 to a unit vector field \(e\) in a neighborhood of \(q\) with \(|\nabla^2 e(q)| \leq C'\) for some \(C' > 0\) independent of \(q \in M\). Under the normal coordinates \((x^1, \ldots, x^n)\) in \(B_\delta(q)\) at \(q, \Gamma_{ij}^m(0) = 0\) and \(g_{ij}(0) = \delta_{ij}\). Let \(\tilde{e}(x) = \frac{\partial}{\partial x_i}\) for \(x \in B_\delta(0)\), where \(\delta > 0\) is less than the uniform lower bound of the injectivity radius of the points \(q \in M\) in \((M_1, g_1)\). Let \(e(x) \equiv \xi^i \frac{\partial}{\partial \bar{x}^i} = \frac{\tilde{e}(x)}{\sqrt{\bar{g}}(x)}\) for \(x \in B_\delta(q)\). Since

\[
\nabla_i \xi^i\big|_{x=0} = \frac{\partial \tilde{e}^j}{\partial \bar{x}^i} = 0
\]

at \(x = 0\) (at \(q\)), we have

\[
\nabla_i \xi^i = \frac{\partial (\xi^j)}{\partial \bar{x}^i} = \frac{\partial \tilde{e}^j}{\partial \bar{x}^i} - \frac{\partial \tilde{e}^a}{\partial \bar{x}^i} \tilde{e}^j = 0,
\]

at the point \(q\). Therefore, the extension \(\xi\) of \(e_q\) in \(B_\delta(q)\) is a unit vector field with \(\nabla_i \xi^i(q) = 0\). It is easy to see that there exists a uniform constant \(C > 0\) depending on the lower bound of the injectivity radius and upper bound of the norm of the curvature for points in \(M\) in \((M_1, g_1)\), such that \(|\nabla^2 \xi(q)| \leq C\), for the extension \(e\) of \(e_q\) defined above.
4. Convergence of the $\sigma_k$-Ricci curvature flow

Now we can prove the long time existence of the flow.

**Theorem 4.1.** Assume $(M^n, g)$ is a compact manifold with boundary of $C^{4,\alpha}$, and $(M, g)$ is either a compact domain in $\mathbb{R}^n$ or with Ricci curvature $\text{Ric} \leq -\delta_0 g$ for some $\delta_0 \geq (n-1)$. Assume $u_0 \in C^{4,\alpha}(M)$ is a subsolution to (1.5) satisfying (5.6) at the points $x \in \partial M$ where $v(x) = 0$. Also, assume $\phi \in C^{4+\alpha/2,\frac{\alpha}{2}}(\partial M \times [0, T_1])$ for all $T_1 > 0$, $\phi(x, t) \geq 0$ on $\partial M \times [0, +\infty)$ and $\phi$ satisfies the compatible condition (5.4) with $u_0$. There exists a unique solution $u \in C^{4,2}(M \times [0, +\infty))$ to the Cauchy-Dirichlet problem (3.1) such that $u \in C^{4+\alpha/2,\frac{\alpha}{2}}(M \times [0, T])$ for all $T > 0$, and the equation (3.1) is uniformly parabolic in $t \in [0, T]$ for any $T > 0$.

**Proof.** Since $u_0$ is a subsolution to (1.5), the equation is strictly parabolic at $t = 0$. By the compatibility condition of $\phi$ and $u_0$, the implicit function theorem yields that there exists $T_0 > 0$ such that the flow is parabolic on $M \times [0, T_0)$ and the Cauchy-Dirichlet problem has a unique solution $u \in C^{4,2}(M \times [0, T_0))$ such that $u \in C^{4+\alpha/2,\frac{\alpha}{2}}(M \times [0, t_1])$ for any $t_1 \in (0, T_0)$. Recall that

$$\sigma_k(\nabla^2 u) = \beta_{k,n} e^{2ku+2ka} \geq \beta_{k,n} e^{2ku},$$

with the right hand side increasing by Lemma 3.2. Also, Lemma 3.2 gives the uniform upper and lower bounds of $u$ on $M \times [0, T_0)$. By the a priori estimates in Lemma 3.4 and Proposition 3.6, we have $\nabla^2 u \in \Gamma^+_k$ and the equation is uniformly parabolic, and hence Krylov Theorem for fully nonlinear parabolic equations yields uniform $C^2,\alpha T_0(M)$ estimates on $u$ with some constant $0 < \alpha T_0 < 1$ for $t \in [0, T_0)$, see [9]. In turn the Schauder estimates yield uniform $C^{4+\alpha/2,\frac{\alpha}{2}}$ estimates on $u$ in $M \times [0, T_0)$. Also, these a priori estimates apply to $u$ on $M \times [0, T]$ for any $T > 0$ with the corresponding constants depending on $T$, and classical parabolic equation theory applies to extend the flow to $M \times [0, +\infty)$ and $u \in C^{4+\alpha/2,\frac{\alpha}{2}}(M \times [0, T])$ for all $T > 0$. This completes the proof of the long time existence of the flow.

□

To show the convergence of the flow, we establish the $C^1$ and $C^2$ interior estimates on $u$ based on the bound $\sup_{U \times [0, +\infty]} |u|$ for any compact subset $U \subseteq M^\circ$.

**Lemma 4.2.** Assume $u \in C^{4,2}(M \times [0, +\infty))$ is a solution to the Cauchy-Dirichlet boundary value problem of the equation (1.5) with $u_0 \geq 0$. Assume that for any compact subset $U \subseteq M^\circ$, there exists a constant $C_0 = C_0(U) > 0$ such that

$$|u| \leq C_0$$

on $U \times [0, +\infty)$. Also, for some $T > 0$, we assume that there exists a constant $C = C(T) > 0$ such that

$$|u| + |\nabla u| \leq C(T)$$

on $M \times [0, T]$. Then for a point $q_1 \in M^\circ$, there exists a constant $C_1 > 0$ depending on $B_2(q_1)$, $C_0(B_2(q_1))$ and $C(T)$ such that

$$|\nabla u| \leq C_1$$

on $B_2(q_1) \times [0, +\infty)$, where $r$ is the distance of $q_1$ to $\partial M$. 21
Proof. It is a modification of the interior estimates in [5]. For any \( T_1 > T \), we consider the function

\[
F(x, t) = \mu(x) we^{f(u)}
\]

on \( B_r(q_1) \times [0, T_1] \), where \( w = \frac{|\nabla u|^2}{2} \), and \( \mu \in C^2_0(B_{\frac{q}{2}}(q_1)) \) is a cut-off function such that

\[
(4.1) \quad \mu = 1 \text{ on } B_{\frac{q}{2}}(q_1), \quad 0 \leq \mu \leq 1, \quad |\nabla \mu| \leq b_0 \mu^2, \quad |\nabla^2 \mu| \leq b_0,
\]

for some \( b_0 > 0 \) as defined in [5], and \( f(u) \) is to be determined later. By the assumption of the lemma, if \( F(x, t) \) achieves its maximum on \( B_{\frac{q}{2}}(q_1) \times [0, T_1] \) at a point \((x_0, t_0) \in B_{\frac{q}{2}}(q_1) \times [0, T] \), then \( F(x, t) \) is uniformly bounded and hence

\[
|\nabla u| \leq C
\]

on \( B_{\frac{q}{2}}(q_1) \times [0, T_1] \) with a constant \( C > 0 \) independent of \( T_1 \). So from now on, we assume that there exists \((x_0, t_0) \in B_{\frac{q}{2}}(q_1) \times (T, T_1) \) such that

\[
F(x_0, t_0) = \sup_{B_{\frac{q}{2}}(q_1) \times [0, T_1]} F.
\]

We choose the normal coordinate \((x^1, ..., x^a)\) at \( x_0 \). Then at \((x_0, t_0)\), we have

\[
(4.2) \quad \frac{W_i}{w} + f' u_i \geq 0,
\]

\[
(4.3) \quad \frac{\nabla \mu}{\mu} + \frac{\nabla w}{w} + f' \nabla u = 0,
\]

\[
(4.4) \quad \tilde{T}_{ij} \left[ \frac{\nabla_i \nabla_j \mu}{\mu} - \frac{\nabla_i \nabla_j w}{w} \right] - \nabla_i \nabla_j \left( \frac{\nabla_i \nabla_j w}{w^2} \right) + f'' \nabla_i \nabla_j u + f''' \nabla_i \nabla_j \mu \leq 0.
\]

By (4.3) we have

\[
\tilde{T}_{ij} \frac{\nabla_i \nabla_j w}{w^2} \leq 3 \tilde{T}_{ij} \frac{\nabla_i \nabla_j \mu}{\mu^2} + \frac{3}{2} (f'')^2 \tilde{T}_{ij} \nabla_i u \nabla_j u,
\]

and hence plugging this inequality and the definition of \( w \) into (4.4) we have

\[
\frac{1}{w} \tilde{T}_{ij} \nabla^2 m \nabla^2 m u + \tilde{T}_{ij} \left( \frac{\nabla^2 \mu}{\mu} - 4 \frac{\nabla_i \nabla_j \mu}{\mu^2} \right) + \frac{1}{w} \tilde{T}_{ij} \nabla_i \nabla_j \nabla m u \nabla m u
\]

\[
+ f'' \tilde{T}_{ij} \nabla^2 m u + \left( f''' - \frac{3}{2} (f'')^2 \right) \tilde{T}_{ij} \nabla_i u \nabla_j u \leq 0.
\]

Dropping the non-negative first term, changing the order of derivatives for the third order derivative term and by our choice of \( \mu \), we have at \((x_0, t_0)\),

\[
\frac{1}{w} \tilde{T}_{ij} \nabla_i \nabla_j m u \nabla m u + f'' \tilde{T}_{ij} \nabla^2 m u + \left( f''' - \frac{3}{2} (f'')^2 \right) \tilde{T}_{ij} \nabla_i u \nabla_j u \leq \left( \frac{C}{\mu} + \frac{C}{2 w^{-1} |\nabla u|^2} \right) \sum_i \tilde{T}_{ii}
\]

\[
= C \left( \frac{1}{\mu} + 1 \right) \sum_i \tilde{T}_{ii},
\]
for some uniform constant $C > 0$ depending on $b_0$ and $\sup |Rm|$ on $B_{\frac{3}{2}}(q_1)$. Similar argument yields

$$\frac{1}{w} \nabla_m \Delta u \nabla_m u + f' \Delta u + \left(f'' - \frac{3}{2}(f')^2\right) |\nabla u|^2 \leq C \left(\frac{1}{\mu} + 1\right).$$

Combining these two inequalities and the equation (3.15), we have

$$2k(u_{x_i}u_{x_i} + |\nabla u|^2)\sigma_k(\nabla^2 u) - \tilde{T}_{ab} \nabla_i u(-\nabla_i \text{Ric}_{ab} + 2(n - 2)(\nabla^2 u \nabla g_{ab} - \nabla^2 u \nabla b))$$

$$\leq -w[(n - 2)(f' \tilde{T}_{ij} \nabla^2 u + (f'' - \frac{3}{2}(f')^2) \tilde{T}_{ij} \nabla_i u \nabla_j u) + (f' \Delta u + (f'' - \frac{3}{2}(f')^2) |\nabla u|^2) \sum_i \tilde{T}_{ii}]$$

$$+ w\left(\frac{C}{\mu} + C\right) \sum_i \tilde{T}_{ii}.$$

Substituting (4.2), (4.3) and the following identity into this inequality

$$\tilde{T}_{ab} \nabla_i u = \tilde{T}_{ab}(-\text{Ric}_{ab} + (n - 2)\nabla^2 u + \Delta g_{ab} + (n - 2)(\nabla^2 g_{ab} - \nabla a u \nabla b)) = k \sigma_k(\nabla^2 u),$$

we have at $(x_0, t_0)$,

$$2k(-f'u_i w + |\nabla u|^2)\sigma_k(\nabla^2 u) - C |\nabla u| \sum_i \tilde{T}_{ii}$$

$$+ 2(n - 2)w \tilde{T}_{ij}[(\frac{\nabla_i \nabla_j u}{\mu} + f' |\nabla u|^2) g_{ij} - (\frac{\nabla_i \mu \nabla_j u}{\mu} + f' \nabla_i u \nabla_j u)]$$

$$\leq -w[(n - 2)(f'' - \frac{3}{2}(f')^2) \tilde{T}_{ij} \nabla_i u \nabla_j u + (f'' - \frac{3}{2}(f')^2) |\nabla u|^2 \sum_i \tilde{T}_{ii}]$$

$$- kw f' \sigma_k(\nabla^2 u) + f' w \tilde{T}_{ab}(-\text{Ric}_{ab} + (n - 2)(\nabla^2 g_{ab} - \nabla a u \nabla b)) + w\left(\frac{C}{\mu} + C\right) \sum_i \tilde{T}_{ii}.$$
for some \( C > 0 \) depending on \( n \), \( \sup(|Rm| + |\nabla Ric|) \) and \( b_0 \), where we have used the Cauchy inequality and the constant \( b_2 > 0 \) is to be determined. Now we take
\[
f(u) = (2 + u - \inf_{B_{2N}(q_1) \times (0, +\infty)} u)^{-N}
\]
for some constant \( N > 1 \) to be fixed. Therefore,
\[
-N2^{-N-1} \leq f' = -N(2 + u - \inf_{B_{2N}(q_1) \times (0, +\infty)} u)^{-N-1} \leq -N(2 + \text{osc} u)^{-N-1} < 0,
\]
\[
f'' - \frac{3}{2}(f')^2 + 3(n-2)f' = \frac{N[(N+1) - N(2 + u - \inf_{B_{2N}(q_1) \times \mathbb{R}^n} u)^{-N} - 3(n-2)(2 + u - \inf_{B_{2N}(q_1) \times \mathbb{R}^n} u)]}{(2 + u - \inf_{B_{2N}(q_1) \times \mathbb{R}^n} u)^{-N-2}} \geq N(2 + u - \inf_{B_{2N}(q_1) \times (0, +\infty)} u)^{-N-2}[(1 - 2^{-N})N + 1 - 3(n-2)(2 + \text{osc} u)]
\]
where
\[
\text{osc} u = \sup_{B_{2N}(q_1) \times (0, +\infty)}(u - \inf_{B_{2N}(q_1) \times (0, +\infty)} u) \leq 2 \sup_{B_{2N}(q_1) \times (0, +\infty)} |u|.
\]

Now we take \( N > 1 \) large so that
\[
f'' - \frac{3}{2}(f')^2 + 3(n-2)f' > 0,
\]
and take \( b_2 = (n-2)N(2 + \text{osc} u)^{-N-1} \), and hence,
\[
\frac{2k}{C}(-f'\sigma_t + 2 + \frac{1}{2}f''\bar{\sigma} \beta_{k,n} e^{2k(u+t)} + |\nabla u|^2 \sum_i \bar{T}_{ii}) \leq C(1 + \frac{1}{b_2})(\frac{1}{\mu} + 1) \sum_i \bar{T}_{ii}
\]
(4.5)

for some \( C > 0 \) depending on \( n \), \( \sup |u| \), \( \sup(|Rm| + |\nabla Ric|) \) and \( b_0 \). Notice that if \( u_t < \frac{1}{2} \), since \( u_t \geq 0 \), and \( u \) and \( f'(u) \) are uniformly bounded, we have for some uniform constant \( C > 0 \),
\[
|\nabla u|^2 \sum_i \bar{T}_{ii} \leq C(1 + \frac{1}{b_2})(\frac{1}{\mu} + 1) \sum_i \bar{T}_{ii} + C.
\]
On the other hand, by (3.14),
\[
\sum_i \bar{T}_{ii} \geq (n-k+1) \left( \frac{n}{k-1} \right) \left( \frac{n}{k} \right)^{-1} \beta_{k,n} e^{2kn+2kn} \frac{1}{\mu^3} \geq C
\]
for a uniform \( C > 0 \) depending on \( \sup |u| \), and hence we have
\[
\mu |\nabla u|^2 \leq C
\]
at \((x_0, t_0)\) for some uniform constant \( C > 0 \) depending on \( n \), \( \sup |u| \), \( \sup(|Rm| + |\nabla Ric|) \) and \( b_0 \), independent of \( T_1 \). For the case \( u_t \geq \frac{1}{2} \) at \((x_0, t_0)\), the first term in (4.5) is positive and hence
\[
|\nabla u|^2 \sum_i \bar{T}_{ii} \leq C(1 + \frac{1}{b_2})(\frac{1}{\mu} + 1) \sum_i \bar{T}_{ii}.
\]
and again we have
\[ \mu|\nabla u|^2 \leq C \]
at \((x_0, t_0)\) for some uniform constant \( C > 0 \) depending on \( n, \sup |u|, \sup(|Rm| + |\nabla Ric|) \) and \( b_0 \), independent of \( T_1 \). Therefore, by the arbitrary choice of \( T_1 > T \),
\[ F(x, t) \leq F(x_0, t_0) \leq 2Ce^{-2N} \]
for \((x, t) \in [0, +\infty)\). In particular,
\[ |\nabla u(x, t)| \leq C \]
for \((x, t) \in B^*_x(q_1) \times [0, +\infty)\), for some uniform constant \( C > 0 \) depending on \( n, \sup_{B^*_x(q_1) \times [0, +\infty)} |u| \), \( \sup(|Rm| + |\nabla Ric|) \), \( b_0 \) and \( B^*_x(q_1) \). Therefore, for any compact subsets \( U \) and \( U_1 \) such that \( U \subseteq U_1 \subseteq U_1^c \subseteq M^c \), there exists a uniform constant \( C > 0 \) depending on \( U, \sup_{U \times [0, +\infty)} |u| \) and \( \sup(|Rm| + |\nabla Ric|) \) such that
\[ |\nabla u(x, t)| \leq C + \sup_{U \times [0, T]} |\nabla u| \]
for \((x, t) \in U \times [0, +\infty)\).

Based on the interior \( C^1 \) estimates, the interior \( C^2 \) estimates are relatively easy modifications of the \( C^2 \) estimates in Proposition 3.6.

**Lemma 4.3.** Assume \( u \in C^{4,2}(M \times [0, +\infty)) \) is a solution to the Cauchy-Dirichlet boundary value problem of the equation \( (1.5) \) with \( u_t \geq 0 \). Assume that for any compact subset \( U \subseteq M^c \), there exists a constant \( C_0(U) > 0 \) such that
\[ |u| \leq C_0 \]
on \( U \times [0, +\infty) \). Also, for some \( T > 0 \), we assume that there exists a constant \( C = C(T) > 0 \) such that
\[ |\nabla^2 u| \leq C(T) \]
on \( M \times [0, T] \). Then for a point \( q_1 \in M^c \), there exists a constant \( C' > 0 \) depending on \( B^*_x(q_1) \), \( C_0(B^*_x(q_1)) \) and \( \sup_{B^*_x(q_1) \times [0, +\infty)} |\nabla u| \) such that
\[ |\nabla^2 u| \leq C' \]
on \( B^*_x(q_1) \times [0, +\infty) \), where \( r \) is the distance of \( q_1 \) to \( \partial M \).

**Proof.** For any \( T_1 > T \), we consider the function \( H : S(TM) \times [0, T_1) \to \mathbb{R} \) such that
\[ H(x, e_x, t) = \mu(x)h(x, e_x, t) \]
for \( x \in M, e_x \in ST_x M \) and \( t \geq 0 \), where \( h \) is defined in the proof of Proposition 3.6 and \( \mu \in C_0^2(B_{\frac{1}{2}}(q_1)) \) satisfies (4.11) for some constant \( b_0 > 0 \). By continuity, there exists a point \( (q, t_0) \in B_{\frac{1}{2}}(q_1) \times [0, T_1] \) and \( e_q \in ST_q M \), such that

\[
H(q, e_q, t_0) = \sup_{STM \times [0, T_1]} \mu(x) h(x, e_x, t).
\]

If \( t_0 \leq T \), then by assumption, \(|\nabla^2 u|\) and hence \( H \) are well controlled. Therefore, we assume that \( t_0 > T \). The same as in Proposition 3.6, we choose the normal coordinates \((x^1, ..., x^u)\) at \( q \) so that \( e_q = \frac{\partial}{\partial x^i} \) and we extend \( e_q \) to a unit vector field \( e = \xi^i \frac{\partial}{\partial u^i} \) in the neighborhood of \( q \) in the same way. We define the function

\[
\tilde{H}(x, t) = H(x, e(x), t) = \mu(x) \tilde{h}(x, t) = \mu(x)(\xi^i \xi^j \nabla_i \nabla_j u + m|\nabla u|^2)
\]

in a neighborhood of \((q, t_0)\), for some constant \( m > 1 \) to be fixed. Therefore, at \((q, t_0)\), we have

\[
\begin{align*}
\tilde{h}_i &= \nabla_1 \nabla_i u + 2m \nabla_a u \nabla_a u \geq 0, \\
\frac{\nabla \mu}{\mu} + \frac{\nabla \tilde{h}}{\tilde{h}} &= 0, \\
\tilde{T}_{ij} &= \frac{\nabla^2 \mu}{\mu} - \frac{\nabla_i \mu \nabla_j \mu}{\mu^2} + \frac{\nabla_i \tilde{h} \nabla_j \tilde{h}}{\tilde{h}^2} + \nabla_j \xi^a \nabla^2 \xi^a u \leq 0, \\
\Delta \mu &= \frac{\nabla \mu}{\mu}^2 + \frac{\nabla \tilde{h}}{\tilde{h}}^2 + \frac{\nabla \tilde{h}}{\tilde{h}}^2 + \Delta \xi^a \nabla^2 \xi^a u \leq 0.
\end{align*}
\]

Direct calculation and changing order of derivatives yield at \((q, t_0)\),

\[
\begin{align*}
\nabla_i \tilde{h} &= \nabla_1 \nabla_i \nabla u + Rm \nabla u + 2m \nabla_a u \nabla_a u, \\
\nabla_j \nabla_i \tilde{h} &= \nabla_1 \nabla_j \nabla_i u + \nabla Rm \nabla u + Rm \nabla^2 u + 2m \nabla a u \nabla_i u \nabla_a u + \nabla^2 a u \nabla_a u \nabla_i u + Rm \nabla u \nabla u, 
\end{align*}
\]

and hence combining these inequalities at the maximum point \((q, t_0)\) we have

\[
\begin{align*}
\tilde{T}_{ij}[(n-2)\nabla_1 \nabla_i \nabla_j u + \nabla_1 \nabla_1 \Delta u_{ijij}] \\
\leq \tilde{T}_{ij}[(n-2)\nabla_j \tilde{h} + \Delta \tilde{h}_{ij}] - 2m[(n-2)\tilde{T}_{ij} \nabla a u \nabla a u + \nabla a \Delta u \nabla a u \sum_i \tilde{T}_{ii}] \\
- 2m[(n-2)\tilde{T}_{ij} \nabla_{ja} u \nabla_{ia} u + \nabla_{ia} u \nabla_{ja} u \sum_i \tilde{T}_{ii}] + (C + C|\nabla^2 u|) \sum_i \tilde{T}_{ii} \\
\leq - \tilde{h} \tilde{T}_{ij}[(n-2)(\frac{\nabla^2 \mu}{\mu} - 2 \frac{\nabla_i \mu \nabla_j \mu}{\mu^2})] + (\frac{\Delta \mu}{\mu} - 2 \frac{|\nabla \mu|^2}{\mu^2}) g_{ij} + (C + C|\nabla^2 u|) \sum_i \tilde{T}_{ii} \\
- 2m[(n-2)\tilde{T}_{ij} \nabla a u \nabla a u + \nabla a \Delta u \nabla a u \sum_i \tilde{T}_{ii}] - 2m[(n-2)\tilde{T}_{ij} \nabla_{ja} u \nabla_{ia} u + \nabla_{ia} u \nabla_{ja} u \sum_i \tilde{T}_{ii}] \\
\leq - 2m[(n-2)\tilde{T}_{ij} \nabla a u \nabla a u + \nabla a \Delta u \nabla a u \sum_i \tilde{T}_{ii}] - 2m[(n-2)\tilde{T}_{ij} \nabla_{ja} u \nabla_{ia} u + \nabla_{ia} u \nabla_{ja} u \sum_i \tilde{T}_{ii}] \\
+ C(1 + (1 + \frac{1}{\mu})|\nabla^2 u|) \sum_i \tilde{T}_{ii},
\end{align*}
\]
where $C$ depends on $\sup |Rm|$, $b_0$, $\sup_{B_{T_1}(q_1) \times (0, \infty)} |\nabla u|$ and the uniform upper bound of $|\nabla^2 e(q)|$ (see Proposition 3.6), and hence combining this inequality with the two inequalities (5.22) we have

$$2k(\nabla_{i1}^2 u + \nabla_{i1}^2 u)\sigma_k(\nabla^2 u) - 2(n - 2)\bar{T}_{ii}[\nabla_{i1}^2 u\nabla_{i1}^2 u - \nabla_{i1}^2 u\nabla_{i1}^2 u]
\leq -4km(\nabla_{a1} u, \nabla_{a1} u + |\nabla u|^2)\sigma_k(\nabla^2 u) - 2m[(n - 2)\bar{T}_{ij} \nabla_{j1} u\nabla_{i1}^2 u + \nabla_{b1} u\nabla_{b1} u \sum_i \bar{T}_{ii}]
\leq C(1 + m + (1 + m + \frac{1}{\mu})|\nabla^2 u|) \sum_i \bar{T}_{ii}.$$ 

Plugging in (4.6) and (4.7), we have

$$2k\nabla_{i1}^2 u\sigma_k(\nabla^2 u) - 2(n - 2)\bar{T}_{ii}[\nabla_{i1}^2 u\nabla_{i1}^2 u - \nabla_{i1}^2 u\nabla_{i1}^2 u]
\leq -4km|\nabla u|^2\sigma_k(\nabla^2 u) - 2m[(n - 2)\bar{T}_{ij} \nabla_{j1} u\nabla_{i1}^2 u + \nabla_{b1} u\nabla_{b1} u \sum_i \bar{T}_{ii}]
\leq C(1 + m + (1 + m + \frac{1}{\mu})|\nabla^2 u|) \sum_i \bar{T}_{ii}.$$ 

Since $\nabla_{i1}^2 u(q, t_0) = 0$ for $i \geq 2$ by the choice of coordinates as in Proposition 3.6 and

$$\nabla_{i1}^2 u(q, t_0) \geq \nabla_{i1}^2 u(q, t_0)$$

for $i \geq 2$, and hence we have

$$2k(\nabla_{i1}^2 u + 2m|\nabla u|^2)\sigma_k(\nabla^2 u) + (2m - 2(n - 2))\nabla_{i1}^2 u\nabla_{i1}^2 u \sum_i \bar{T}_{ii}
\leq C(1 + m + n(1 + m + \frac{1}{\mu})|\nabla_{i1}^2 u|) \sum_i \bar{T}_{ii}.$$ 

We take $m$ large and use the equation (5.1) to obtain

$$2k(\nabla_{i1}^2 u + 2m|\nabla u|^2)\bar{\beta}_{k1} e^{2k(u + \mu)} + \nabla_{i1}^2 u\nabla_{i1}^2 u \sum_i \bar{T}_{ii}
\leq C(1 + (1 + \frac{1}{\mu})|\nabla_{i1}^2 u|) \sum_i \bar{T}_{ii},$$

for some uniform $C > 0$ independent of $T_1$, and hence if $\nabla_{i1}^2 u(q, t_0) > 1$, the first term in this inequality is positive and since $\sum_i \bar{T}_{ii}$ is uniformly bounded from below by (5.14), we have

$$\mu \nabla_{i1}^2 u(q, t_0) \leq C,$$

for some uniform constant $C > 0$ independent of $T_1$, and hence

$$\tilde{H} \leq C$$

in $B_{2\rho}(q_1) \times [0, T_1]$ with $C > 0$ independent of $T_1$; while if $\nabla_{i1}^2 u(q, t_0) \leq 1$, we trivially have the uniform upper bound of $\tilde{H}$ by its definition and the bound of $|\nabla u|$ on $B_{2\rho}(q_1) \times [0, \infty)$. By the arbitrary choice of $T_1 > T$, $\tilde{H}$ has a uniform upper bound on $B_{2\rho}(q_1) \times [0, \infty)$. In particular,

$$\nabla_{i1}^2 u \leq C,$$
in $B_2(q_1) \times [0, \infty)$. Since $\overline{\nabla^2 u} \in \Gamma^+_k$, and $|\nabla u|$ is uniformly bounded in $B_2(q_1)$, we have that there exists a uniform constant $\alpha > -\infty$ such that

$$\Delta u \geq \alpha,$$

and hence

$$|\nabla^2 u| \leq n^3 (C + |\alpha|),$$
on $B_2(q_1) \times [0, \infty)$. This completes the proof of the lemma. □

Now we prove the convergence of the flow and the asymptotic behavior near the boundary as $t \to \infty$.

Proof of Theorem 1.3. Long time existence of the solution $u$ has been obtained in Theorem 4.1, and we only need to consider the convergence of $u$ and its asymptotic behavior near the boundary as $t \to \infty$.

First we establish the uniform upper bound estimates on $u$ on any given compact subset of $M^e$. By the Maclaurin’s inequality, $u$ is a subsolution to the $\sigma_1$-Ricci curvature flow (3.1). By the maximum principle for $\sigma_1$-Ricci curvature flow in Lemma 3.1, to get the upper bound of $u$, it suffices to find a super-solution to the scalar curvature equation i.e., (1.5) with $k = 1$ satisfying (1.7) near $\partial M$. Direct application of Lemma 5.2 in [7], where a sequence of super-solutions to the scalar curvature equation on corresponding small geodesic balls blowing up on the boundary was constructed, yields the upper bound of $u$:

$$\limsup_{x \to \partial M} [u(x, t) + \log(r(x))] \leq 0,$$

uniformly for all $t > 0$; and moreover, for any compact subset $U \subseteq M^e$, there exists a constant $C > 0$ depending on $U$ such that $u(x, t) \leq C$ for all $(x, t) \in U \times [0, +\infty)$. Here is an alternative argument: by maximum principle for $\sigma_1$-Ricci curvature flow in Lemma 3.1, $u(x, t) \leq u_{LN}(x)$, for $(x, t) \in M^e \times [0, \infty)$, where $u_{LN}$ is the solution to the Loewner-Nirenberg problem of the constant scalar curvature equation on $M$. Recall that

$$u_{LN}(x) \leq -\log(r(x)) + o(1)$$

near the boundary,

with $o(1) \to 0$ as $x \to \partial M$, see in [15][14][11] for instance.

By Lemma 5.2, $u(x, t)$ is increasing along $t > 0$ and hence

$$u_0(x) \leq u(x, t) \leq u_{LN}(x)$$

for $(x, t) \in M^e \times [0, +\infty)$. Or just use the super-solution to (1.5) on a small ball centered at $x$ constructed in Lemma 5.2 in [7] instead of $u_{LN}$. Therefore, $u(x, t)$ converges as $t \to \infty$ for any $x \in M^e$. By Lemma 4.2 and Lemma 4.3, we have that for any compact subsets $U \subseteq U_1 \subseteq M^e$ with $U \subseteq U_1^\circ$, there exists a constant $C > 0$ such that

$$|\nabla u| + |\nabla^2 u| \leq C$$
in $U_1 \times [0, \infty)$ and hence, the equation (3.1) is uniformly parabolic and by (3.1), $u_t$ has a uniform upper bound on $U_1 \times [0, \infty)$. By Krylov’s Theorem and the classical Schauder estimates, we
have that there exists a uniform constant $C > 0$ depending on $U_1$ such that
\[ \|u\|_{C^{1,0}(U \times [0, \infty))} \leq C, \]
and
\[ (4.8) \quad \|u\|_{C^{1,0}(U)} \leq C, \]
for all $t \geq 0$. Since $u$ increases and has uniform upper bound in $U$, by the Harnack inequality of the linear uniformly parabolic equation (3.7) for $u$, we have
\[ v = u_t \to 0 \]
uniformly on $U$ as $t \to +\infty$. Therefore, $u(x, t) \to u_\infty(x)$ uniformly for $x \in U$ as $t \to +\infty$. By the uniform bound (4.8) and the interpolation inequality, we have
\[ u(x, t) \to u_\infty(x) \]
in $C^4(U)$ as $t \to \infty$. By the arbitrary choice of the compact subset $U \subseteq M$, we have that $u_\infty$ is a solution to (1.5) in $M$.

Now we consider the lower bound of $u$ near the boundary. Applying Lemma 4.4 to be proved later, we have that there exist $\delta_1 > 0$ small and $T_2 > T_1$, such that
\[ u(x, t) \geq -\log(r(x) + \epsilon(t)) + w(x) \]
for $x \in M$ with $r(x) \leq \delta_1$ and $t \geq T$, where $w(x) \leq 0$ with $w|_{\partial M} = 0$ and $\epsilon(t) \to 0$ as $t \to +\infty$. By the upper and lower bound estimates on $u$ near the boundary, we have
\[ u_\infty(x) + \log(r(x)) \to 0 \]
uniformly as $x \to \partial M$. \qed

We will show the lower bound of the asymptotic behavior of $u$ near the boundary as $t \to \infty$, for which we need $\phi$ to increase not too slowly.

**Lemma 4.4.** Let $(M, g)$, $u_0$, $\phi$, $T_1 > T$ and $u$ be as in Theorem 7.3. Let $r(x)$ be the distance function of $x \in M$ to the boundary $\partial M$. Then there exist $\delta_1 > 0$ small and $T_2 > T_1$, such that
\[ u(x, t) \geq -\log(r(x) + \epsilon(t)) + w(x) \]
for $x \in M$ with $r(x) \leq \delta_1$ and $t \geq T_2$, where $\epsilon = \xi(t)^{-1}$ and $w$ is a function of $C^2$ where $r(x) \leq \delta_1$ such that $w(x) \leq 0$ with $w|_{\partial M} = 0$.

**Proof.** Let $\delta_1 > 0$ be a small constant to be fixed. Define the exponential map $\text{Exp} : \partial M \times [0, \delta_1] \to M$ such that $\text{Exp}_q(s) \in M$ is the point on the geodesic starting from $q \in \partial M$ in the direction of inner normal vector with distance $s$ to $q$. $\delta_1$ is chosen small so that $\text{Exp}$ is a diffeomorphism to the image. Define
\[ U_{\delta_1} = \{\text{Exp}_q(s) \mid (q, s) \in \partial M \times [0, \delta_1]\}. \]
The metric has the orthogonal decomposition
\[ g = ds^2 + g_s, \]
with $g_s$ the restriction of $g$ on $\Sigma_s = \{z \in M \mid r(z) = s\}$ for $0 \leq s \leq \delta_1$. Define the function
\[ u(x, t) = -\log(r(x) + \epsilon(t)) + w(x) \]
for \((x, t) \in U_{\delta_1} \times [T, +\infty)\) where

\[
w(x) = A \left( \frac{1}{r(x) + \delta} - \frac{1}{\delta^p} \right)
\]

with constants \(A > 0, p > 1\) large and \(\delta > 0\) small to be determined. By definition, we have

\[
(4.9) \quad -\frac{\epsilon'(t)}{\epsilon(t)^2} = \xi'(t) \leq \tau
\]

for \(t \geq T\). Let \(\tilde{r}(x, t) = r(x) + \epsilon(t)\). For any \(x_0 \in U_{\delta_1}^0\), let \(\{e_1, ..., e_n\}\) be an orthonormal basis at \(x_0\) such that \(e_1 = \frac{\partial}{\partial r}\). The same calculation as in Lemma 5.1 in [7] yields

\[
\nabla^2 u = -Ric_g + (n-2)\nabla^2 w + \Delta g + (n-2)(w')^2 \begin{pmatrix} 0 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 1 \\
\end{pmatrix}
\]

\[
+ \frac{1}{\tilde{r}^2} \begin{pmatrix} (n-1)g - 2(n-2)\tilde{r}w' & 0 & \cdots & 0 \\
0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 1 \\
\end{pmatrix} - \tilde{r}((n-2)\nabla^2 r + \Delta \gamma) \nabla^2 r + \Delta \gamma
\]
on \( U_{\delta_1} \). Therefore, if we also assume \( Ap \geq 8n\tau \), then we obtain
\[
\log(\det(\nabla^2 \varphi)) - \log(\tilde{\beta}_{n,n}) - 2nu \geq \log(\det(\frac{1}{r^2}\Phi)) - \log(\tilde{\beta}_{n,n}) - 2nu
\]
\[
\geq \log(\tilde{r}^{-2n}(1 + Ap\tilde{r})) - 2nu
\]
\[
\geq \log(1 + Ap\tilde{r}) \geq \log(1 + 8n\tau\tilde{r}),
\]
and hence for \( \tilde{r} \leq (8n\tau)^{-1} \) and \( t \geq \max\{T, T'\} \), by (4.9) we have
\[
(4.10) \quad \log(\det(\nabla^2 \varphi)) - \log(\tilde{\beta}_{n,n}) - 2nu \geq 4n\tau\tilde{r} \geq -2n\epsilon' \tilde{r} = 2nu.
\]
Since \( \lim_{t \to \infty} \epsilon(t) = 0 \), we take \( T_2 \geq \max\{T_1, T'\} \) such that \( \epsilon(t) \leq (16n\tau)^{-1} \) for \( t \geq T_2 \) and let
\[
\delta_1 < \min\{(16n\tau)^{-1}, (20\gamma)^{-1}\}.
\]
We will choose \( A \) and \( p \) large so that \( \underline{u} \) gives a lower bound of \( u \) on \( U_{\delta_1} \times [T_2, \infty) \). Notice that \( \partial U_{\delta_1} = \Sigma_{\delta_1} \cup \partial M \). By assumption we have
\[
\underline{u}(x, t) = \log(\xi(t)) \leq \phi(x, t)
\]
for \( (x, t) \in \partial M \times [T_2, \infty) \). On \( \Sigma_{\delta_1} \), since \( u \) is increasing, we have \( u(x, t) \geq u_0(x) \). Notice that there exists \( A_1 > 0 \) such that for \( A \geq A_1 \) and any \( p \geq 1 \), we have
\[
-\log(\delta_1) + A((\delta_1 + \delta)^{-p} - \delta^{-p}) < \inf_{\Sigma_{\delta_1}} u_0,
\]
and hence we have on \( \Sigma_{\delta_1} \times [T_2, \infty) \),
\[
\underline{u} \leq u.
\]
Finally, we consider the control on \( U_{\delta_1} \times \{T_2\} \). Since \( \underline{u}(\cdot, T_2), u(\cdot, T_2) \in C^1(M) \) and \( \underline{u} \leq u = \phi \)
on \( \partial M \times \{T_2\} \), there exist \( A_2 > 0 \) and \( p_2 > 0 \) such that for \( A \geq A_2 \) and \( p \geq p_2 \), we have
\[
\underline{u} \leq u
\]
on \( U_{\delta_1} \times \{T_2\} \).

In summary, we assume
\[
Ap \geq \max\{K_0, 8n\tau\}, \quad p \geq \max\{1, p_0, p_2\}, \quad A \geq \max\{A_0, A_1, A_2\},
\]
\[
\delta + \delta_1 < 1, \quad \delta_1 < \min\{(16n\tau)^{-1}, (20\gamma)^{-1}\},
\]
and \( \delta_1 > 0 \) is small so that \( \text{Exp} \) is a diffeomorphism. Therefore, \( \underline{u} \) is a sub-solution to (3.1) for \( k = n \) by (4.10) and hence a sub-solution to (3.1) for \( 1 \leq k \leq n \) on \( U_{\delta_1} \times [T_2, \infty) \), by Maclaurin’s inequality; moreover,
\[
\underline{u} \leq u, \quad \text{on} \quad (\partial M \bigcup \Sigma_{\delta_1}) \times [T_2, \infty) \bigcup U_{\delta_1} \times \{T_2\}.
\]
Therefore, by the maximum principle in Lemma (3.1) we have
\[
u(x, t) \geq \underline{u}(x, t) = -\log(r(x) + \epsilon(t)) + A((r(x) + \delta)^{-p} - \delta^{-p})
\]
on \( U_{\delta_1} \times [T_2, \infty) \).
\[\square\]
Proof of Corollary 3.5. The equation (1.5) is conformally covariant, and hence it is equivalent to consider the case when the background metric $g$ is the Euclidean metric when $(M, g)$ is a domain in the Euclidean space, while $g \in C^{4,\alpha}$ is chosen to be a metric constructed in [16] (see Section 2 in the present paper) such that $\text{Ric}_g \prec -(n - 1)g$ in the conformal class for a general manifold $(M, g)$. Let $u_0 = \underline{u} + \min\{0, \inf_{\partial M} \varphi_0\}$ with $\underline{u}$ a sub-solution constructed in Section 2 for $A > 0$ and $p > 0$ large, and when $(M, g)$ is a domain in Euclidean space one can just take $\underline{u}$ be the global sub-solution in [7] (just take the function $\eta(s) = s$ for the sub-solution $\underline{u}$ in Section 2) with $A > 0$ and $p > 0$ large. Then $u_0$ is a strict sub-solution near the boundary with $u_0 < \varphi_0$ on $\partial M$ and hence, we can construct the boundary data $\phi \in C^{4,0.2}((\partial M \times [0, \infty))$ satisfying the compatible condition (3.4) at $t = 0$ such that $\phi_t \geq 0$ on $\partial M \times [0, \infty)$ and $\phi(x, t) \rightarrow \varphi_0(x)$ uniformly in $C^{4,\alpha'}(\partial M)$ as $t \rightarrow \infty$ for some $0 < \alpha' < 1$.

Consider the Cauchy-Dirichlet boundary value problem (3.1) - (3.3). It is clear that Lemma 3.1 and Lemma 3.2 still hold true. Recall that by Maclaurin’s inequality $u$ is a sub-solution to the $\sigma_1$-Ricci curvature flow (3.1). On the other hand, for the $\sigma_1$-Ricci equation (1.5), which is the Yamabe equation, classical variational methods yield a unique minimizing solution $u_1$ to the Dirichlet boundary value problem with $u_1 = \varphi_0$ on $\partial M$, see [17]. By Lemma 3.1 for the $\sigma_1$-Ricci curvature flow, we have $u(x, t) \leq u_1(x)$ for $(x, t) \in M \times [0, \infty)$ and hence we have a uniform upper bound of $u$. Also, the a priori $C^2$ estimates from Lemma 3.3 to Proposition 3.6 hold with uniform bound of $\|u(\cdot, t)\|_{C^2(M)}$ independent of $t > 0$. By Theorem 4.1 we have the long time existence of the unique solution $u$. Things are even better in this case: there exists a uniform constant $C > 0$ such that for any $T > 0$,

$$\|u\|_{C^{4,0.2}(M \times [T, T+1])} \leq C,$$

by Krylov’s Theorem and the standard Schauder estimates. Remark that here we do not need the locally uniformly interior estimates.

By (4.11), there exists a sequence $t_j \rightarrow \infty$, such that $u(x, t_j) \rightarrow u_\infty(x)$ in $C^4(M)$ for some $u_\infty \in C^{4,\alpha}(M)$ as $t_j \rightarrow \infty$. By monotonicity of $u$, $u(x, t) \rightarrow u_\infty(x)$ uniformly for $x \in M$ as $t \rightarrow \infty$. By (4.11) and the interpolation inequality, we have $u(x, t) \rightarrow u_\infty(x)$ uniformly in $C^4(M)$ as $t \rightarrow \infty$ and hence, $u_\infty = \varphi_0$ on $\partial M$. Since $u_t \geq 0$ satisfies the linear uniformly parabolic equation (3.7), by Harnack inequality, $u_t \rightarrow 0$ locally uniformly in $M^\circ$ as $t \rightarrow \infty$ and hence, $u_\infty$ is a solution to (1.5). This completes the proof of the corollary.

□

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