Ising Model on Twisted Lattice and Holographic RG flow

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The partition function of the two-dimensional Ising model is exactly obtained on a lattice with a twisted boundary condition. The continuum limit of the model off the critical temperature is found to give the mass-deformed Ising conformal field theory (CFT) on the torus with the complex structure $\tau$. We find that the renormalization group (RG) flow of the mass parameter can be holographically described in terms of the three-dimensional gravity including a scalar field with a simple nonlinear kinetic function and a quadratic potential.

1. Introduction

The AdS/CFT correspondence[1, 2, 3] has been giving valuable informations for various field theories especially in the strong coupling regions. In particular, the correspondence between the classical gravities and the large $N$ (gauge) field theories has been most intensively studied. Among many tests on this correspondence, the exactly soluble models in low dimensions, say, the two-dimensional conformal field theories, are expected to shed light. The relation of AdS gravity with the Virasoro algebra of CFT is based on the pioneering work of Brown and Henneaux[4], and the minimal conformal field theories in two dimensions have been discussed in relation to the AdS gravity in three dimensions[5, 6]. Subsequently extensive analyses have been done for the $W_N$ minimal conformal field theories and many evidences are presented for the relevance of the higher spin field theories together with gravity for these minimal models in the large $N$ limit[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

As for the recent progress in the stronger statement of the AdS/CFT correspondence, that is, the correspondence between the “quantum” gravity or the string theory and the finite $N$ (gauge) field theory, the finite $N$ effect of the one-dimensional supersymmetric gauge theory with 16 supercharges (the finite $N$ BFSS matrix model[23]) has been directly examined using the Monte Carlo simulation[24, 25], where the $\alpha'$-corrections to the Type IIA supergravity [26] are reproduced from the gauge theory. This strongly motivates to regard field theories in some category as candidates of the quantum gravity at least in the semi-classical meaning.
In fact, in Ref. [27], it has been proposed that the two-dimensional minimal models are holographically dual to three-dimensional quantum gravities. In particular, the simplest minimal model, namely the Ising conformal field theory, is conjectured to be dual with the three-dimensional Euclidean pure quantum gravity.

We here briefly review the argument in Ref. [27]. Let us consider the three-dimensional quantum gravity with negative cosmological constant. The “quantum” here means that we integrate over all the possible three-dimensional metric. In this path-integral, the boundary of the three-dimensional geometry is fixed to the 2-torus with the complex structure $\tau$, and thus the partition function of the gravity is a function of $\tau$ and $\bar{\tau}$. The important fact is that the smooth classical solutions of the three-dimensional pure gravity are restricted to the locally AdS geometries, which are obtained by the $SL(2, \mathbb{Z})$ transformation from the thermal AdS geometry. Thus, if we can use the semi-classical approach, the partition function of the quantum gravity is obtained by summing up the classical contributions of each solution with some quantum corrections. The expansion parameter of the saddle point approximation is the inverse of the quantity,

$$c = \frac{3L}{2G_N}, \quad (1.1)$$

where $L$ is the AdS radius and $G_N$ is the Newton constant. The quantity $c$ equals to the central charge of the two-dimensional Virasoro algebra which appears at the boundary of the asymptotically AdS geometry[4]. Therefore the semi-classical approximation is usually expected to work only for large $c$ region. The most important assumption in Ref. [27] is that the partition function can be evaluated by summing up the classical geometries even in the strong coupling region, $c \sim 1$. Once this assumption is accepted, the partition function can be written as

$$Z_{\text{grav}}(\tau, \bar{\tau}) = \sum_{\gamma \in SL(2,\mathbb{Z})/\Gamma_c} Z_{\text{vac}}(\gamma\tau, \gamma\bar{\tau}), \quad (1.2)$$

where $\Gamma_c$ is a subgroup of $SL(2, \mathbb{Z})$ which does not change the topology of the three-dimensional geometry and $Z_{\text{vac}}(\tau, \bar{\tau})$ is the contribution from the thermal AdS geometry. The points are that $Z_{\text{vac}}(\tau, \bar{\tau})$ can be calculated explicitly and that the symmetry $\Gamma_c$ is enhanced at specific values of $c < 1$ where the summation of (1.2) becomes a finite sum. Note that these values exactly equal to the central charges allowed for the two-dimensional minimal models. In particular, when $c = \frac{1}{2}$, the partition function of the gravity (1.2) reproduces that of the two-dimensional $c = \frac{1}{2}$ conformal field theory, that is, the Ising conformal field theory. This result suggests the duality between the $c = \frac{1}{2}$ minimal model and the three-dimensional quantum pure gravity.

This argument actually gives the correspondence between the $c = \frac{1}{2}$ conformal fixed point of the theory space of the two-dimensional field theory and the three-dimensional quantum gravity. Then it is natural to expect that there will be a gravity description at least in the
vicinity of the conformal fixed point in the theory space of the two-dimensional quantum field theories. In the context of the AdS/CFT correspondence, the renormalization group (RG) flow of a coupling constant in the field theory can be identified with the classical trajectory of the corresponding bulk field in the asymptotically AdS geometry, which is called the holographic RG [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] (see also [39]). In this scheme, the radial coordinate of the AdS gravity can be identified with the RG parameter and the AdS boundary corresponds to the conformal fixed point. In the case of the Ising model, although the conformal fixed point corresponds to the “quantum” gravity, we can expect that the central assumption in Ref. [27], namely, only the classical solutions contribute to the partition function of the quantum gravity, still works at least in the vicinity of the AdS solution even after adding an additional field in the bulk theory. Then we can examine the RG structure around the $c = 1/2$ conformal fixed point using the traditional technique of the holographic RG.

The purpose of our paper is to work out the exact solution of the Ising model partition function on the lattice corresponding to the torus with the generic complex structure $\tau$, and study the corresponding holographic RG structure of the continuum theory around the conformal fixed point. Usually the partition function of Ising model in two dimensions is obtained for rectangular lattices with the periodic boundary condition, and one obtains the Ising conformal field theory on a rectangular torus ($\tau = i$) by taking the continuum limit at the critical temperature [40, 41, 42]. However, the most general torus has a complex structure with the parameter $\tau$, representing the shape of the torus. In the literature, we have found no explicit solution of the Ising model partition function on the lattice corresponding to the torus with the generic complex structure, although there have been works to obtain finitized conformal spectrum of Ising model on manifold of various topology using corner-transfer matrix, Yang-Baxter technique, or thermodynamic Bethe Ansatz[43, 44, 45, 46]. We explicitly compute the partition function of the Ising model on the twisted lattice, that is, a lattice with such a boundary condition that the “space” position is shifted to some amount when one goes around the “time” direction, and show that it ends up with the torus with the complex structure in the continuum limit. By taking the continuum limit at off-critical temperature with an appropriate scaling, we identify the deviation parameter as the mass of the Ising conformal field theory on the torus with the complex structure. We also work out the classical solution of the three-dimensional Einstein gravity with a single scalar field. We find that a simple nonlinear kinetic function and a simple quadratic potential for the scalar field can capture the RG flow from the Ising field theory with the central charge $c = 1/2$ at the ultraviolet fixed point towards the $c = 0$ case at the infrared by using the technique of the holographic RG via the Hamilton-Jacobi equation of the gravity.
This paper is organized as follows: In the next section, the partition function of the two-dimensional Ising model is obtained on the twisted lattice representing the discretized version of the torus with the complex structure. In section 3, the continuum limit of the partition function is obtained retaining the deviation from critical temperature, which results in a mass term for a free Majorana fermion. In section 4, the holographic description of the renormalization group flow is worked out for the massive Majorana fermion in terms of the Hamilton-Jacobi equation. Section 5 is devoted to a summary of our results and a discussion. Some technical details in computing the partition function of the Ising model is summarized in Appendix A. Some details of partition function of two-dimensional massive Majorana fermion on the torus is given in Appendix B.

2. Partition function of the 2D Ising model on the twisted lattice

Let us consider the 2D Ising model on a rectangular lattice with the size $n \times m$. There is a "spin" $s(x) = \pm 1$ at each site $x = (x_1, x_2)$ ($x_1 = 1, \ldots, n$, $x_2 = 1, \ldots, m$) and the Hamiltonian of the system is given by

$$H = -J_1 \sum_x s(x)s(x + \hat{1}) - J_2 \sum_x s(x)s(x + \hat{2}),$$

(2.1)

where $J_1$ ($J_2$) and $\hat{1}$ ($\hat{2}$) are the coupling constant and the unit vector in the "space" ("time") direction, respectively. As for the boundary condition, we impose the periodic boundary condition to the space direction and the twisted or shifted boundary condition with an integer parameter $p \in \mathbb{Z}$ to the time direction, namely, the space position is shifted by $p$.

Fig. 1 The spin degrees of freedoms are on the sites of the $n \times m$ lattice. In our case, we take the usual periodic boundary condition for the $x_1$ direction but we adopt the twisted boundary condition for $x_2$, namely, we identify $(x_1 + p, x_2 + m)$ with $(x_1, x_2)$. 

condition to the space direction and the twisted or shifted boundary condition with an integer parameter $p \in \mathbb{Z}$ to the time direction, namely, the space position is shifted by $p$.
when one goes around the time direction (see Fig. 1):

\[ s(x_1 + n, x_2) = s(x_1, x_2), \quad s(x_1 + p, x_2 + m) = s(x_1, x_2). \]  

(2.2)

In the following, we evaluate the partition function,

\[ Z_p \equiv \sum_{\{s(x) = \pm 1\}} e^{-\beta H}, \]  

(2.3)

under this boundary condition.

Let us first define the matrices with the size of \(2^n\),

\[ S^i_k \equiv 1_2 \otimes \cdots \otimes \sigma^i \otimes \cdots \otimes 1_2, \]  

(2.4)

where the Pauli matrices \(\sigma^i (i = 1, 2, 3)\) are placed at the \(k\)-th position \((k = 1, \ldots, n)\). Using these matrices, the transfer matrix of this system is written as

\[ T = (2 \sinh 2\alpha) V_a V_b, \]  

(2.5)

with

\[ V_a \equiv \prod_{k=1}^{n} \exp\left(\tilde{\alpha} S^1_k\right), \quad V_b \equiv \prod_{k=1}^{n} \exp\left(b S^3_k S^3_{k+1}\right), \]  

(2.6)

where \(a\) and \(b\) are given by \(a \equiv \beta J_1\) and \(b \equiv \beta J_2\) and \(\tilde{\alpha}\) is defined by

\[ \sinh 2\tilde{\alpha} = \frac{1}{\sinh 2\alpha}. \]  

(2.7)

Note that the system is in the ordered phase for \(\tilde{\alpha} < b\) and is in the disordered phase for \(\tilde{\alpha} > b\). We further define the "shift matrix" \(\Sigma\) whose components are explicitly given by

\[
\begin{cases}
\Sigma_{k,2k-1} = \Sigma_{2^{n-k}+k,2k} = 1, \quad (k = 1, \ldots, 2^{n-1}) \\
0, \quad \text{(others)}
\end{cases}
\]  

(2.8)

We see that \(\Sigma\) has the property,

\[ \Sigma^{-1} S^i_k \Sigma = S^i_{k-1}, \quad (S^i_{n+1} \equiv S^i_1). \]  

(2.9)

We can then write the partition function in Eq.(2.3) with the boundary condition in Eq.(2.2) as

\[ Z = \text{Tr}\left(T^m \Sigma^p\right). \]  

(2.10)

In order to estimate the transfer matrix, we here construct the Dirac matrices of \(\text{Spin}(2n)\) as

\[ \Gamma_{2k-1} \equiv S^1_1 S^1_2 \cdots S^1_{k-1} S^3_k, \quad \Gamma_{2k} \equiv S^1_1 S^1_2 \cdots S^1_{k-1} S^2_k, \quad (k = 1, \ldots, n) \]  

(2.11)

and

\[ U \equiv S^1_1 S^1_2 \cdots S^1_n = i^n \Gamma_1 \cdots \Gamma_n, \]  

(2.12)
which satisfy the Clifford algebra,
\[
\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu,\nu}, \quad \{\Gamma_\mu, U\} = 0, \quad U^2 = 1 \quad (\mu, \nu = 1, \cdots, 2n)
\] (2.13)

As usual, the generators of $Spin(2n)$ in the chiral and anti-chiral representations are defined as
\[
J_{\mu\nu}^\pm \equiv \frac{i}{4}[\Gamma_\mu, \Gamma_\nu]U_\pm,
\] (2.14)

where
\[
U_\pm \equiv \frac{1}{2}(1 \pm U).
\] (2.15)

Using them, we can also divide $V_a$, $V_b$ and $\Sigma$ into the chiral and anti-chiral sectors as $V_a^\pm \equiv V_a U_\pm$, $V_b^\pm \equiv V_b U_\pm$ and $\Sigma_\pm \equiv \Sigma U_\pm$. In the following, we use the notation,
\[
H_\pm \equiv (V_a^\pm)^{1/2} V_b^\pm (V_a^\pm)^{1/2}.
\] (2.16)

For $\Sigma_\pm$, we can easily see
\[
(V_a^\pm)^{-1/2} \Sigma_\pm (V_a^\pm)^{1/2} = \Sigma_\pm.
\] (2.17)

Then the partition function Eq.(2.3) can be written as
\[
Z = (2 \sinh 2a)^m n^2 \text{Tr} \left( H_+^m \Sigma_+^p U_+ + H_-^m \Sigma_-^p U_- \right) \\
= (2 \sinh 2a)^m \left\{ \text{Tr}_+ \left( H_+^m \Sigma_+^p \right) + \text{Tr}_- \left( H_-^m \Sigma_-^p \right) \right\},
\] (2.18)

where $\text{Tr}_\pm$ denote the trace over the chiral and anti-chiral sectors of the spin representation of $Spin(2n)$, respectively.

In evaluating (2.18), the following fact is useful: Suppose that $A \in Spin(2n; \mathbb{C})$ in the fundamental representation is transformed into the “canonical form” by $T \in O(2n)$ as
\[
T^T A T = \bigoplus_{k=1}^{n} e^{i \theta_k J_{2k-1,2k}} = \bigoplus_{k=1}^{n} R(-i\theta_k), \quad (\theta_k \in \mathbb{C})
\] (2.19)

where $J_{\mu\nu}$ are the generators of $Spin(2n; \mathbb{C})$ in the fundamental representation and $R(\theta)$ is the two-dimensional rotation matrix with the (complex) angle $\theta$,
\[
R(\theta) \equiv \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}.
\] (2.20)

Then we can write $\text{Tr}_\pm(A)$ as
\[
\text{Tr}_\pm(A) = \frac{1}{2} \left( \prod_{k=1}^{n} 2 \cosh \frac{\theta_k}{2} \pm \det(T) \prod_{k=1}^{n} 2 \sinh \frac{\theta_k}{2} \right).
\] (2.21)

Thus, we first express $H_+^m \Sigma_+^p$ in the fundamental representation and then transform them into the canonical form using appropriate matrices $T_\pm \in O(2n)$. 

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The computation along this strategy is straightforward and we summarize it in Appendix A. As a preparation to show the result, we define the quantity $\gamma_I$ ($I = 1, \cdots, 2n$) as the positive solution of the equation,

$$\cosh \gamma_I = \cosh 2\bar{a} \cosh 2b - \cos \left( \frac{\pi I}{n} \right) \sinh 2\bar{a} \sinh 2b,$$

(2.22)

and

$$\tilde{\gamma}_I = \gamma_I - \frac{i\pi p I}{mn}.$$

(2.23)

Note that $\tilde{\gamma}_I$ satisfy the relation:

$$e^{m\tilde{\gamma}_{2n-I}} = e^{m\tilde{\gamma}_I}.$$  

(2.24)

In addition, we will often use $\gamma_0 = \gamma_{2n}$ in the following. Combining the above consideration and the results (A24) and (A25), we obtain the partition function of the two-dimensional Ising model in the ordered phase with the twisted boundary condition in Eq. (2.2):

$$Z = \frac{1}{2} (2 \sinh 2a) \sum_{i=1}^{4} Z_i,$$

(2.25)

with

$$Z_1 = R_1 \prod_{r=1}^{\frac{n}{2}} \left| 2 \cosh \left( \frac{m}{2} \tilde{\gamma}_{2r-1} \right) \right|^2 \equiv \left( \prod_{k=1}^{n} e^{m \tilde{\gamma}_{2k-1}} \right) P_1,$$

$$Z_2 = R_2 \prod_{r=1}^{\frac{n}{2}} \left| 2 \sinh \left( \frac{m}{2} \tilde{\gamma}_{2r-1} \right) \right|^2 \equiv \left( \prod_{k=1}^{n} e^{m \tilde{\gamma}_{2k-1}} \right) P_2,$$

$$Z_3 = 2 \cosh \left( \frac{m}{2} \gamma_0 \right) R_3 \prod_{r=1}^{\frac{n-1}{2}} \left| 2 \cosh \left( \frac{m}{2} \tilde{\gamma}_{2r} \right) \right|^2 \equiv \left( \prod_{k=1}^{n} e^{m \tilde{\gamma}_{2k}} \right) \cdot (1 + e^{-m\gamma_0}) P_3,$$

$$Z_4 = 2 \sinh \left( \frac{m}{2} \gamma_0 \right) R_4 \prod_{r=1}^{\frac{n-1}{2}} \left| 2 \sinh \left( \frac{m}{2} \tilde{\gamma}_{2r} \right) \right|^2 \equiv \left( \prod_{k=1}^{n} e^{m \tilde{\gamma}_{2r}} \right) \cdot (1 - e^{-m\gamma_0}) P_4,$$

(2.26)

where we have used the reflection property (2.24) and introduced

$$R_1 = \begin{cases} 1 & (n : \text{even}) \\ 2 \cosh \left( \frac{m}{2} \gamma_n \right) & (n : \text{odd}, \ p : \text{even}) \end{cases}, \quad R_2 = \begin{cases} 1 & (n : \text{even}) \\ 2 \sinh \left( \frac{m}{2} \gamma_n \right) & (n : \text{odd}, \ p : \text{even}) \end{cases},$$

$$R_3 = \begin{cases} 1 & (n : \text{odd}) \\ 2 \cosh \left( \frac{m}{2} \gamma_n \right) & (n : \text{even}, \ p : \text{odd}) \end{cases}, \quad R_4 = \begin{cases} 1 & (n : \text{odd}) \\ 2 \sinh \left( \frac{m}{2} \gamma_n \right) & (n : \text{even}, \ p : \text{odd}) \end{cases}.$$

(2.27)
and

\[ P_1 = \left( \prod_{r=1}^{\lfloor \frac{n}{2} \rfloor} |1 + e^{-m\tilde{g}_{2r-1}}|^2 \right) \left( 1 + (-1)^p \delta_{(-1)^n, 1} e^{-m\gamma_n} \right), \]

\[ P_2 = \left( \prod_{r=1}^{\lfloor \frac{n}{2} \rfloor} |1 - e^{-m\tilde{g}_{2r-1}}|^2 \right) \left( 1 - (-1)^p \delta_{(-1)^n, 1} e^{-m\gamma_n} \right), \]

\[ P_3 = \left( \prod_{r=1}^{\lfloor \frac{n}{2} \rfloor} |1 + e^{-m\tilde{g}_{2r}}|^2 \right) \left( 1 + (-1)^p \delta_{(-1)^n, 1} e^{-m\gamma_n} \right), \]

\[ P_4 = \left( \prod_{r=1}^{\lfloor \frac{n}{2} \rfloor} |1 - e^{-m\tilde{g}_{2r}}|^2 \right) \left( 1 - (-1)^p \delta_{(-1)^n, 1} e^{-m\gamma_n} \right). \]

(2.28)

The result Eq.(2.26) constitutes our new result of partition function of the two-dimensional Ising model on a twisted lattice that gives the discretized version of torus with the complex structure

\[ \tau = \tau_1 + i\tau_2, \quad \tau_1 \equiv \frac{p}{n}, \quad \tau_2 \equiv \frac{m}{n}. \]

(2.29)

whose continuum limit gives the torus with the complex structure \( \tau \) as described in the next section.

3. Continuum limit of the partition function

We next consider the continuum limit, that is, the limit of \( m, n, p \to \infty \) with fixing the ratios \( \tau_1, \tau_2 \) in Eq.(2.29). In taking this limit, we also tune the coupling constant so that the theory properly reaches to a continuum theory around the critical point. For simplicity, we consider the case of \( J_1 = J_2 \), that is, \( a = b \equiv K \). Under this condition, we defined the parameter \( \mu \) through the relation

\[ \frac{\mu^2}{4n^2} = \frac{1}{2} \left( \sinh 2K + \frac{1}{\sinh 2K} \right) - 1. \]

(3.1)

Recalling that the critical temperature \( K^* \) is given by \( \sinh 2K^* = 1 \), the parameter \( \mu \) expresses a deviation from the critical temperature for a finite \( n \) [42]. Note that we take the continuum limit \( n \to \infty \) with fixing \( \mu = O(1) \). This means that the temperature approaches to the critical value as \( K - K^* = O(n^{-1}) \) in taking the continuum limit.

In this parametrization, \( \gamma_I \) can be expressed as

\[ \cosh \gamma_I = \frac{\mu^2}{2n^2} + 1 + 2 \sin^2 \frac{\pi I}{2n}. \]

(3.2)

Since we are interested in the continuum limit, only the region \( I \ll n \) is relevant. Then, in such a region, \( \gamma_I \) can be expanded for \( n \gg 1 \) as

\[ \gamma_I = \frac{2\pi}{n} \sqrt{\left( \frac{\mu}{2\pi} \right)^2 + \left( \frac{I}{2} \right)^2} + O(n^{-3}). \]

(3.3)
Substituting it to Eq.(2.28) and taking the limit of \( n \to \infty \), \( P_1, \ldots, P_4 \) become

\[
\begin{align*}
\lim_{n \to \infty} P_1 &= \prod_{k=0}^\infty \left| 1 + e^{2\pi i(k+\frac{1}{2})\left(\tau_1 + i\tau_2 \sqrt{1 + \left(\frac{\mu}{(2k+1)\pi}\right)^2}\right)} \right|^2, \\
\lim_{n \to \infty} P_2 &= \prod_{k=0}^\infty \left| 1 - e^{2\pi i(k+\frac{1}{2})\left(\tau_1 + i\tau_2 \sqrt{1 + \left(\frac{\mu}{(2k+1)\pi}\right)^2}\right)} \right|^2, \\
\end{align*}
\]

where we have used the reflection property of \( \tilde{\gamma}_I \) (2.24). We can also show

\[
\lim_{n \to \infty} e^{-\tau_{2}\gamma_0} = e^{-\gamma_{2}\mu},
\]

and

\[
\lim_{n \to \infty} e^{\frac{\pi i}{2}\tilde{\gamma}_I} = e^{\pi \tau_2 \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\mu}{2\pi}\right)^2} - i\frac{\pi}{2} \gamma_I}.
\]

Combining Eqs.(3.4), (3.5) and (3.6), we see that the partition function in the continuum limit can be expressed as

\[
Z = C \sum_{i=1}^{4} Z_{i}^{\text{cont.}},
\]

where \( C \) is an irrelevant constant and \( Z_{i}^{\text{cont.}} \equiv \lim_{n \to \infty} Z_i \) are given by

\[
\begin{align*}
Z_{1}^{\text{cont.}} &= \left( \prod_{k \in \mathbb{Z}} e^{\pi \tau_2 \sqrt{k^2 + \left(\frac{\mu}{2\pi}\right)^2}} \right)^\infty \left| 1 + e^{2\pi i(k+\frac{1}{2})\left(\tau_1 + i\tau_2 \sqrt{1 + \left(\frac{\mu}{(2k+1)\pi}\right)^2}\right)} \right|^2, \\
Z_{2}^{\text{cont.}} &= \left( \prod_{k \in \mathbb{Z}} e^{\pi \tau_2 \sqrt{k^2 + \left(\frac{\mu}{2\pi}\right)^2}} \right)^\infty \left| 1 - e^{2\pi i(k+\frac{1}{2})\left(\tau_1 + i\tau_2 \sqrt{1 + \left(\frac{\mu}{(2k+1)\pi}\right)^2}\right)} \right|^2, \\
Z_{3}^{\text{cont.}} &= \left( \prod_{k \in \mathbb{Z}} e^{\pi \tau_2 \sqrt{k^2 + \left(\frac{\mu}{2\pi}\right)^2}} \right)^\infty \left| 1 + e^{-\tau_{2}\mu} \prod_{k=1}^{\infty} \right| 1 + e^{2\pi i(k+\frac{1}{2})\left(\tau_1 + i\tau_2 \sqrt{1 + \left(\frac{\mu}{(2k+1)\pi}\right)^2}\right)} \right|^2, \\
Z_{4}^{\text{cont.}} &= \left( \prod_{k \in \mathbb{Z}} e^{\pi \tau_2 \sqrt{k^2 + \left(\frac{\mu}{2\pi}\right)^2}} \right)^\infty \left| 1 - e^{-\tau_{2}\mu} \prod_{k=1}^{\infty} \right| 1 - e^{2\pi i(k+\frac{1}{2})\left(\tau_1 + i\tau_2 \sqrt{1 + \left(\frac{\mu}{(2k+1)\pi}\right)^2}\right)} \right|^2, \\
\end{align*}
\]

which exactly agree* with the result in Eq.(B10) of continuum field theory of a free massive Majorana fermion

\[
Z_{1}^{\text{cont.}} = Z_{\frac{1}{2}, \frac{1}{2}}, \quad Z_{2}^{\text{cont.}} = Z_{0, \frac{1}{2}}, \quad Z_{3}^{\text{cont.}} = Z_{\frac{1}{2}, 0}, \quad Z_{4}^{\text{cont.}} = Z_{0, 0},
\]

*The prefactor in Eq.(2.25) gives a non-universal overall factor in the continuum limit.
where the labels $\mu, \nu = 0, \frac{1}{2}$ of $Z_{\mu,\nu}$ specify the boundary condition of the fermion as described in Appendix B. This means that the continuum limit of the two-dimensional Ising model with the boundary condition in Eq. (2.2) is the two-dimensional massive fermion theory on the torus. In particular, the combination $\frac{p}{n} + i \frac{m}{n}$ becomes the complex structure $\tau = \tau_1 + i\tau_2$ of the continuum torus as shown in Eq. (2.29), and the parameter $\mu$ in Eq. (3.1) representing the deviation from the critical temperature is nothing but the mass parameter of the fermion in the continuum limit.

Before closing this section, it is worth looking at the RG structure of the Ising model. Fixing parameters $\tau_1$ and $\tau_2$ in Eq. (2.29) which specify the geometry of the torus, the parameter space of the Ising model is spanned by $(n, \mu)$, and the procedure of taking the continuum limit is nothing but taking the limit of $n \to \infty$ with fixing $\mu$. As we have seen, after taking this limit, the parameter $\mu$ is precisely identical to the mass parameter of the two-dimensional massive free fermion and the limit $\mu \to 0$ corresponds to the massless Majorana fermion, that is, $c = \frac{1}{2}$ CFT. Therefore the flow parametrized by the parameter $\mu$ starting from the conformal fixed point exactly equals to the mass deformation of the $c = \frac{1}{2}$ CFT (Fig. 2).

**Fig. 2** The parameter space of the 2D Ising model with the condition $p = \tau_1 n$ and $m = \tau_2 n$. The curved lines express the RG flows of an irrelevant operator associated with $\frac{1}{n}$. After taking the continuum limit, the parameter $\mu$ is identical to the mass of the two-dimensional free fermion.

### 4. Holographic description of the RG flow

#### 4.1. Holographic RG flow of single scalar field

In this section, we holographically describe the RG flow starting from the conformal fixed point along the parameter $\mu$.

In the spirit of the AdS/CFT correspondence, the source (coupling constant) of an operator in the boundary field theory is identified with a field in the bulk gravity. The value of the
coupling constant of the boundary field theory varies towards infrared (IR) as a RG flow. On the other hand, the classical solution of the bulk gravity provides a trajectory of the bulk field along the radial direction of an asymptotic AdS geometry. This trajectory is regarded as the RG flow of the coupling constant of the boundary field theory away from a CFT at ultraviolet (UV) fixed point. As mentioned in the introduction, the two-dimensional $c = \frac{1}{2}$ CFT is conjectured to be dual to the “quantum” pure gravity under the assumption that the path integral over the metric of 3D space-time is localized to the classical solutions, that is, the BTZ black holes. We here assume that the same is true at least in the neighborhood of the CFT fixed point; the classical solutions are dominant even in the presence of additional fields in evaluating the partition function of the quantum gravity.

In the analysis of the previous section, we found that there are two independent flows from the conformal fixed point parametrized by $\mu$ and $1/n$. We expect that the parameter $1/n$ in the boundary theory corresponds to certain discretized version of bulk gravity, which is difficult to work out at present. In the following, we explore the continuum bulk gravity to study the RG flow of the parameter $\mu$, which can be identified as the mass of the fermion of the boundary field theory.

Since the parameter $\mu$ couples to the operator $\bar{\Psi}\Psi$ at the boundary field theory, it is plausible to assume that the corresponding field in the gravity is a real scalar field $\phi$, which may be considered as the minimum number of degrees of freedom to describe the RG flow of a single parameter $\mu$. We thus consider the following action of three dimensional Euclidean gravity with a real scalar field $\phi$: \[^{1}\]

$$
S = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left[ -R + V(\phi) + \frac{1}{2} g^{\mu\nu} K(\phi) \partial_\mu \phi \partial_\nu \phi \right], \tag{4.1}
$$

where $G_N$ is the Newton constant of three-dimensional gravity, $g_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$) is the metric, $R$ is the Ricci scalar, $g = \det(g_{\mu\nu})$, and $K(\phi)$ is a function of $\phi$, which describes the nonlinearity of the kinetic term of scalar field $\phi$. Since the AdS$_3$ geometry with the radius $L$ should be a solution of this system when $\phi = 0$, we demand $V(\phi)$ to satisfy

$$
V(\phi = 0) = -\frac{2}{L^2}. \tag{4.2}
$$

Since $\phi$ should have a nonsingular kinetic term at least for small $\phi$, we further require that $K(\phi)$ is regular at $\phi = 0$. Without loss of generality, we can fix

$$
K(\phi = 0) = 1 \tag{4.3}
$$

by choosing the normalization of the field $\phi$. Our final task is to determine functions $V(\phi)$ and $K(\phi)$ by requiring the solutions $\phi, g_{\mu\nu}$ to describe the holographic RG flow of the Ising model off critical temperature.

\[^{1}\] We here simply omit to write the boundary terms of the gravity action.
4.2. Solution of the gravity corresponding to the mass deformation

Since we are interested in the evolution of the scalar field along the radial direction of an asymptotic AdS geometry, we set the following ansatz for the metric and the scalar field:

\[ ds^2 = e^{2h(r)}dr^2 + \frac{r^2}{L^2} \sum_{i=1,2} dx_i^2, \quad \phi = \phi(r), \quad (4.4) \]

where \( x_i (i = 1, 2) \) express the two-dimensional transverse directions, and \( r \in [0, \infty) \) is the radial coordinate, which may be regarded as the Euclidean time. We have also assumed that the functions \( h(r) \) and \( \phi(r) \) depend only on \( r \). Note that we have fixed the gauge by setting \( g_{ij} = \frac{r^2}{L^2} \delta_{ij} \) in Eq.(4.4). Since the geometry is asymptotically anti de Sitter space, \( e^{2h} \) must be expandable around \( r = \infty \) as

\[ e^{2h} = \frac{L^2}{r^2} + O(r^{-4}). \quad (4.5) \]

In this setup, the independent field equations are given by\(^\dagger\)

\[ \frac{1}{2} K(\phi) \dot{\phi}(r)^2 = \frac{1}{r^2} + \frac{\dot{h}(r)}{r}, \quad (4.6) \]

\[ V(\phi) = e^{-2h(r)} \left( -\frac{1}{r^2} + \frac{\dot{h}(r)}{r} \right), \quad (4.7) \]

where the dot \( \dot{\cdot} \) expresses the derivation with respect to \( r \). Our first task is to obtain \( h(r) \) and \( \phi(r) \) for given fixed functions \( K(\phi) \) and \( V(\phi) \) by solving Eqs. (4.6) and (4.7). Using the boundary condition in Eq.(4.5), we can integrate Eq.(4.6) to obtain \( h(r) \) in terms of \( \phi(r) \) and \( K(\phi) \) as

\[ e^{2h(r)} = \frac{L^2}{r^2} e^{-\int_{s}^{\infty} s K(\phi(s)) (\dot{\phi}(s))^2 ds}. \quad (4.8) \]

The standard way to solve the equations would be plugging this to Eq.(4.7) and solve the obtained single equation of \( \phi(r) \) for given \( K(\phi) \) and \( V(\phi) \). However we proceed the opposite way: We first fix the behavior of \( \phi(r) \) from a physical requirement and determine the relation between \( K(\phi) \) and \( V(\phi) \).

Since the mass deformation keeps the theory free, the scaling of the mass parameter should be trivial. Then we can assume

\[ \mu(a) = \frac{a_0}{a} \mu_0, \quad (4.9) \]

where \( a \) is the length scale of the (boundary) field theory and \( \mu_0 \) is the mass at the reference length scale \( a_0 \). We should also recall that, in the context of the holographic RG, the radial coordinate \( r \) is identified with the RG parameter, namely the “Euclidean time”. The evolution of a bulk field along this radial direction is identified with the RG flow of the corresponding coupling constant. In our choice of gauge in Eq.(4.4), the radial coordinate \( r \)

\(^\dagger\) The field equation of the scalar matter \( \phi \) can be derived from those of the metric.
can be identified as the length scale $a$ of the two-dimensional boundary field theory. Then the behavior of the field $\phi(r)$ can be regarded as the scaling behavior of the corresponding coupling constant [37, 38]. In the present context, we are looking for such a solution that corresponds to the mass parameter $\mu$ of the free fermion theory which exactly behaves as Eq.(4.9). To achieve this goal, we should fix the solution of the scalar field as a function of radial coordinate $r$ (length scale parameter)

$$\phi(r) = \frac{L}{r} \phi_0 \equiv \tilde{\phi}(r)$$  \hspace{1cm} (4.10)

with a constant $\phi_0$, instead of solving $\phi(r)$ for given fixed $K(\phi)$ and $V(\phi)$. Eq.(4.10) allows us to change a variable from $r$ to $\phi$, and Eq.(4.8) becomes

$$e^{2h(r)} = \frac{L^2}{r^2} e^{-\int_0^{\phi(r)} xK(x)dx}.$$  \hspace{1cm} (4.11)

Substituting Eqs.(4.10) and (4.11) into Eq.(4.7), we obtain a relation between $K(\phi)$ and $V(\phi)$

$$V(\phi) = -\frac{2}{L^2} \left(1 - \frac{\phi^2}{4}K(\phi)\right) e^{\int_0^{\phi} xK(x)dx}.$$  \hspace{1cm} (4.12)

This is a necessary condition for the bulk scalar field $\phi$ to become the holographic dual to the mass parameter of the free fermion.

### 4.3. Holographic RG flow in terms of Hamilton-Jacobi equation

Even after imposing the exact scaling behavior of the mass parameter in Eqs.(4.9) and (4.10), we still have a freedom to choose one function of $\phi$ in the action, either $K(\phi)$ or $V(\phi)$, in order to represent the holographic RG flow of our system. In fact, with the requirement of the scalar field behavior $\phi(r)$ in Eq.(4.10), the bulk geometry is uniquely determined by Eq.(4.8) for any choice of either $K(\phi)$ or $V(\phi)$ which are related by Eq.(4.12) as a consequence of field equations.

In order to fix it, let us further consider the holographic RG structure of this system based on the Hamilton-Jacobi equation of the bulk gravity [37, 38]:

$$G_{ij;kl} \left(\frac{1}{\sqrt{g}} \frac{\delta S_{cl}}{\delta g_{ij}} \right) \left(\frac{1}{\sqrt{g}} \frac{\delta S_{cl}}{\delta g_{kl}} \right) + \frac{1}{2K(\phi)} \left(\frac{1}{\sqrt{g}} \frac{\delta S_{cl}}{\delta \phi} \right)^2 = V(\phi) - R + \frac{K(\phi)}{2} g^{ij} \partial_i \phi \partial_j \phi,$$  \hspace{1cm} (4.13)

where $G_{ij;kl}$ is defined by $G_{ij;kl} \equiv g_{ik}g_{jl} - g_{ij}g_{kl}$, and $R$ is the two-dimensional scalar curvature constructed from $g_{ij}(x)$. The classical action $S_{cl} = S_{cl}[g_{ij}(x), \phi(x)]$ is a functional of the boundary values of the metric $g_{ij}(x)$ ($i,j = 1, 2$) and the scalar field $\phi(x)$ on the two-dimensional surface at a specific value of the radial coordinate, say $r = r_0$, and is obtained by substituting the classical solution into the bulk action in Eq.(4.1).

The momentum constraint of the bulk gravity insures that the classical action is invariant under the diffeomorphism of the two-dimensional boundary. We can then expand the classical
action $S_{\text{cl}}$ in powers of derivatives\(^\dagger\) in the transverse directions:

$$
S_{\text{cl}} = \int d^2 x \sqrt{g} (W(\phi) + \cdots),
$$

(4.14)

where $\cdots$ includes terms with the derivatives of $\phi$ and the two-dimensional curvature tensors, which vanish when $\phi$ is independent of $x^i$ and the transverse geometry is flat. The Hamilton-Jacobi equation in Eq.(4.13) can also be expanded in powers of derivatives, and gives the following relation among $W(\phi)$, $K(\phi)$ and $V(\phi)$ at the leading order of the expansion in powers of derivatives[37, 38]

$$
V(\phi) = -\frac{1}{2} W(\phi)^2 + \frac{1}{2K(\phi)} W'(\phi)^2.
$$

(4.15)

By combining this relation with Eq.(4.12), we can determine $W(\phi)$ in terms of $K(\phi)$ as

$$
W(\phi) = \frac{2}{L} \exp \left( \frac{1}{2} \int_0^\phi x K(x) dx \right).
$$

(4.16)

Also, repeating the argument given in Refs. [37, 38], we can obtain the $\beta$-function of $\phi$ as

$$
\beta(\phi) = \frac{2}{K(\phi)} \frac{W'(\phi)}{W(\phi)},
$$

(4.17)

and the holographic $c$-function

$$
c(\phi) = \frac{3}{G_N} \frac{1}{W(\phi)},
$$

(4.18)

which is a monotonically decreasing function when the coefficient of the kinetic term $K(\phi)$ is positive definite. Note that the $c$-function (4.18) is defined as the coefficient of the scalar curvature appearing in evaluating the expectation value of the energy-momentum tensor. This kind of the $c$-function has also been proposed using functional renormalization group equation[48].

The gauge/gravity correspondence asserts that the classical action of the bulk gravity is regarded as the (regularized) free energy of the dual boundary field theory. In our case, we have considered the three-dimensional gravity with a single bulk scalar field $\phi$ as a candidate of the dual description of the massive two-dimensional free fermion. Therefore one may naively expect that $e^{-S_{\text{cl}}}$ should be equal to the partition function of the two-dimensional massive free fermion in Eq.(3.7), when $\phi$ is independent of the transverse coordinates $x^i$ and the transverse geometry is flat. However, we should recall that the Ising field theory corresponds to the three-dimensional “quantum” gravity [5, 6]. The three-dimensional gravity with appropriate boundary conditions possesses the conformal symmetry with the central charge $\frac{1}{2}$ at the boundary[4]. In Ref. [27], as mentioned in Introduction, the three-dimensional pure gravity is considered. It has been show that the integration over all the

\(^\dagger\) In Refs.[37, 38], the authors divide the classical action into the local and the non-local parts as $S[\phi(x), g_{ij}(x)] = S_{\text{loc}}[\phi(x), g_{ij}(x)] + \Gamma[\phi(x), g_{ij}(x)]$, and show that the non-local part $\Gamma$ satisfies the RG equation of the boundary field theory. In this paper, we further expand $\Gamma[\phi(x), g_{ij}(x)]$ in powers of derivatives.
possible three-dimensional metric with fixing boundary condition is localized to the classical
three-dimensional geometries (BTZ black holes) and the partition function of the quantum
gravity turns out to that of the Ising field theory by relying on the boundary conformal
symmetry. Our analysis in this paper is based on the same assumption that this quasi-semi-
classical approach still makes sense even after adding the mass term to the boundary field
theory. Therefore, the partition function of the massive fermion as a sum of functions of
the moduli parameter \( \tau \) in Eq.(3.7) should be obtained after summing up all the possible
classical geometries.

However it is actually hard to carry out this procedure explicitly, since we look at only one
solution of the classical field equations and the conformal symmetry is broken by introducing
nonvanishing values of the scalar field \( \phi \) to represent the mass term of the boundary fermion.
However it is still possible to determine the scalar potential \( V(\phi) \) of the bulk gravity as
follows. We observe that the boundary condition of the fermion becomes irrelevant and that
the partition function becomes \( \tau \)-independent when the geometry of the boundary becomes
\( \mathbb{R}^2 \). In this case, the partition function becomes extremely simple and we can expect that
every solution gives the same contribution to the partition function even if there are several
solutions in the bulk with the same boundary condition. When the geometry is \( \mathbb{R}^2 \), the free
energy of the massive fermion is given by

\[
\mathcal{F} = \int \frac{d^2 p}{(2\pi)^2} \log \left( p^2 + \mu^2 \right),
\]  

(4.19)

where \( \mu \) is the mass parameter. In order for the free energy in Eq.(4.19) to be well-defined,
we need to regularize both UV and IR divergences. Although the result depends on details
of the regularization, the regularized free energy in general takes the form

\[
\mathcal{F}_{\text{reg}} = a + b\mu^2 + c\mu^2 \log \mu,
\]  

(4.20)

where \( a, b \) and \( c \) are some constants.

As mentioned above, we regard the scalar field \( \phi \) at the boundary as the mass parameter
of the boundary fermion and we identify the regularized free energy with the classical action
\( S_{\text{cl}} \) in Eq.(4.14) at the boundary which is obtained from the bulk three-dimensional gravity.
In addition, \( W(\phi) \) in Eq.(4.14) is proportional to the free energy because \( \phi \) does not have
\( x^i \)-dependence. Therefore \( W(\phi) \) can be expressed as

\[
W(\phi) = A + B\phi^2 + C\phi^2 \log \phi,
\]  

(4.21)

where \( A, B \) and \( C \) are some constants. Using Eqs.(4.16) and (4.15), we can determine \( K(\phi) \)
and \( V(\phi) \) in the following.

Suppose \( C \neq 0 \), we observe that \( W(\phi) \) is dominated by the last term \( C\phi^2 \log \phi \) at small
values of \( \phi \). This implies that the RG flow at \( \phi \to 0 \) is discontinuous with the AdS solution
in Eq.(4.16) when \( C \neq 0 \). Since we identify the evolution of \( \phi \) from \( r = 0 \) to \( r = \infty \) along the
radial direction (the Euclidean "time evolution") as the RG flow from the UV fixed point to IR, this behavior is not acceptable as the RG flow of the mass parameter. Therefore, we should adopt a regularization scheme in the holographic renormalization group by imposing the condition

\[ c = C = 0. \]  

(4.22)

Note that the dimensional regularization realizes this condition for example. With this choice of \( W(\phi) \), we obtain \( K(\phi) \) and \( W(\phi) \) from Eq.(4.16) and the normalization condition Eq.(4.3),

\[ K(\phi) = \frac{1}{1 + \frac{1}{4} \phi^2}, \]  

and

\[ W(\phi) = \frac{2}{L} \left( 1 + \frac{1}{4} \phi^2 \right). \]  

(4.24)

Incidentally, we obtain the \( \beta \)-function (4.17) as

\[ \beta(\phi) = \phi, \]  

(4.25)

and the c-function from (4.18) as

\[ c(\phi) = \frac{1}{2} + \frac{\phi^2}{2}. \]  

(4.26)

The beta-function (4.25) is consistent with the notion that the bulk scalar field \( \phi \) corresponds to the mass parameter of the boundary field theory, and the c-function (4.26) is a monotonically decreasing function of \( \phi \in [0, \infty) \) from \( \frac{1}{2} \) to 0 as expected. Substituting this result into Eq.(4.12), we obtain

\[ V(\phi) = -\frac{2}{L^2} \left( 1 + \frac{\phi^2}{4} \right), \]  

(4.27)

which is monotonically decreasing \( V \to -\infty \) as \( \phi \to \infty \) (see fig.3). This is also consistent with the expectation that mass perturbation deforms the \( c = 1/2 \) Ising CFT to flow to a system with less degrees of freedom as dictated by the c-theorem. Namely it is likely to flow to nothing \( (c = 0) \) in the IR.

### 5. Summary and Discussion

In this article, we have obtained the partition function of the Ising model on Euclidean two-dimensional lattice with the twisted boundary condition representing the torus with a complex structure \( \tau \) in a discretized version. We have taken an appropriate scaling limit to obtain a continuum limit for the torus with the complex structure \( \tau \), retaining a deviation from the critical temperature. The resulting continuum partition function agrees with that of the mass-deformed Ising CFT, namely the continuum field theory of massive Majorana fermion on the torus with the complex structure \( \tau \). We have also discussed the RG flow of the Ising model off critical temperature in terms of the three-dimensional AdS gravity.
The shape of the potential corresponding to the flow from the $c = \frac{1}{2}$ CFT to the infrared by the mass deformation. The point $\phi = 0$ corresponds to the AdS boundary with the radius $L$ and $\phi = \infty$ corresponds to $r = 0$.

together with a scalar field $\phi$ representing the RG flow of the mass parameter. By using the Hamilton-Jacobi equation, we have found that a simple nonlinear kinetic function $K(\phi)$ in Eq.(4.23) and a simple potential $V(\phi)$ in Eq.(4.27) for the scalar field $\phi$ describes the expected RG flow from $c = 1/2$ CFT at the UV region towards $c = 0$ at the IR.

Let us comment on the relation of our results to those in Ref.[47] where a similar setup has been used to look for solutions interpolating two AdS geometries corresponding to two different conformal field theories. They assumed the canonical kinetic term $K(\phi) = 1$ for the scalar field $\phi$, which is achieved by a field redefinition from our case. They required that the metric component $G_{tt}$ has a double zero at the horizon $r_0$ (IR), at which the metric becomes another AdS geometry in addition to an AdS geometry at $r = \infty$ (UV). Near $r \to \infty$ at UV, their solution exhibits a chirally asymmetric Virasoro algebra, where the excitation spectra of the left and right Virasoro algebras, $L_0$ and $\bar{L}_0$ are different. This peculiar behavior comes from their Ansatz with the non-vanishing metric component $G_{t\varphi}$, where $t, \varphi$ are coordinates other than $r$ corresponding to our $x_1, x_2$ in Eq.(4.4). We have chosen a different ansatz with $G_{t\varphi} = 0$ in Eq.(4.4) to obtain chirally symmetric Virasoro algebras, as in the Ising model.

We have used the Hamilton-Jacobi equation to describe the RG flow of partition functions of the Ising model on the torus. There are more detailed informations such as multi-point correlation functions or (equivalently) partition functions on higher genus Riemann surfaces. In principle, one should be able to describe these data, which are worth studying. However, it is likely that one needs to overcome the problems associated with the strong coupling, or quantum effects in gravity in order to describe these data quantitatively in the presence of matter fields.
Recent applications of AdS/CFT to condensed matter physics provide many interesting insights into possible phase structures of strongly coupled system. In particular, introducing periodicity into the system to mimic lattice structures seems to be a crucial feature of these applications\cite{51, 52, 53, 54, 55}. Since many realistic models of statistical physics or condensed matter physics are built on some discretized system, it is desirable to obtain discretized models as the boundary of some discretized model for the bulk in discussing the AdS/CFT correspondence. We hope that the continuum space-time would emerge by taking the continuum limit of the discretized system in the bulk and the gravity would be realized as a sort of a cooperative phenomenon. This ambitious objective is of course still quite hard to achieve. How to obtain a discretized version of (quantum) gravity realizing the given discretized model on the boundary is the key question which is worth pursuing. With this philosophy in mind and taking our result quite optimistic, we may expect that the usual two-dimensional Ising model would be a candidate of the discrete boundary model related to a discrete version of quantum gravity in the above sense, and our result may be a first step toward this direction, since the deviation parameter $\mu$ from the critical temperature is actually included in the original discretized Ising model. However our discussion still remained within the continuum theory. In this sense, it would be a great step if one finds a gravity description of the RG flow parametrized by $\frac{1}{n}$ in Fig. 2. Such an attempt is interesting in its own right, and furthermore can provide a concrete starting point to incorporate lattice structures into the AdS/CFT correspondence, which may play a vital role in condensed matter physics applications.

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**A. Transformation of $H_\pm$ and $\Sigma_\pm$**

Let us express the generators of $\text{Spin}(2n)$ in the fundamental representation as

$$
\left( \hat{J}_{\mu \nu} \right)_{\rho \sigma} = i (\delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho}) \cdot (1 \leq \mu < \nu \leq 2n) \quad (A1)
$$

Let us consider $H_\pm$ and $\Sigma_\pm$ defined in (2.16) in the fundamental representation, which are matrices with the size of $2n$ and we write them as $\hat{H}_\pm$ and $\hat{\Sigma}_\pm$, respectively, in the following.
They are written as

$$\hat{H}_\pm = \begin{pmatrix} x & y & 0 & \cdots & 0 & \mp y^\dagger \\ y^\dagger & x & y & 0 & \cdots & 0 \\ 0 & y^\dagger & x & y & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mp y & 0 & \cdots & y^\dagger & x \end{pmatrix}, \quad \hat{\Sigma}_\pm = \alpha_\pm \begin{pmatrix} 0 & 0 & 0 & \cdots & \mp 1_2 \\ 1_2 & 0 & 0 & \cdots & 0 \\ 0 & 1_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1_2 & 0 \end{pmatrix}, \tag{A2}$$

where $x$ and $y$ are $2 \times 2$ matrices,

$$x \equiv \left( \begin{array}{cc} \cosh 2\tilde{a} \cosh 2b & i \sinh 2\tilde{a} \cosh 2b \\ -i \sinh 2\tilde{a} \cosh 2b & \cosh 2\tilde{a} \cosh 2b \end{array} \right), \quad y \equiv \left( \begin{array}{cc} -\frac{1}{2} \sinh 2\tilde{a} \sinh 2b & -i \sinh^2 \tilde{a} \sinh 2b \\ i \cosh^2 \tilde{a} \sinh 2b & -\frac{1}{2} \sinh 2\tilde{a} \sinh 2b \end{array} \right), \tag{A3}$$

and the overall factor $\alpha_\pm$ in the definition of $\hat{\Sigma}_\pm$ is given by

$$\alpha_+ = \begin{cases} 1 & (n : \text{even}) \\ -i & (n : \text{odd}) \end{cases}, \quad \alpha_- = \begin{cases} -i & (n : \text{even}) \\ 1 & (n : \text{odd}) \end{cases}, \tag{A4}$$

which are necessary in order to reproduce (2.8) in the spin representation.

Our goal in this appendix is explicitly transforming the matrices $\hat{H}_\pm$ and $\hat{\Sigma}_\pm$ into this canonical form. Then we can easily estimate $\text{Tr}_\pm H^{m}_\pm \Sigma^{p}_\pm$ in the spin representation by using (2.21).

We first introduce matrices $\Omega_\pm \in SO(2n)$ whose $(i,j)$ blocks are given by

$$(\Omega_+)_ij = \frac{1}{\sqrt{n}} R \left( \frac{2\pi i(j - 1/2)}{n} \right), \quad (\Omega_-)_ij = \frac{1}{\sqrt{n}} R \left( \frac{2\pi ij}{n} \right), \quad (i,j = 1, \cdots, n) \tag{A5}$$

respectively, where $R(\theta)$ is defined in (2.20). The similarity transformations of $H_\pm$ and $\Sigma_\pm$ by $\Omega_\pm$ become

$$\Omega_+^T \hat{H}_+ \Omega_+ = \left( \begin{array}{cccc} M_1 & & & \\ M_2 & & & \\ & \ddots & & \\ N_{3n-1} & & & \end{array} \right),$$

$$\Omega_+^T \hat{\Sigma}_+ \Omega_+ = \left( \begin{array}{ccc} R \left( -\frac{\pi}{n} \right) & & \\ & R \left( -\frac{3\pi}{n} \right) & \\ & \ddots & \\ & & R \left( \frac{2\pi}{n} \right) \end{array} \right).$$

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and

\[
\Omega^T \hat{H} \Omega = \begin{pmatrix}
M_2 & N_{2n-2} & 0 \\
M_4 & N_{2n-4} & \\
\vdots & \vdots & \ddots \\
N_4 & M_{2n-4} & \\
N_2 & M_{2n-2} & 0 \\
0 & 0 & M_0
\end{pmatrix},
\]

\[
\Omega^T \hat{\Sigma} \Omega = \begin{pmatrix}
R \left( -\frac{2\pi}{n} \right) & R \left( -\frac{4\pi}{n} \right) & \cdots \\
R \left( \frac{4\pi}{n} \right) & R \left( \frac{2\pi}{n} \right) & \\
& & & & 1_2
\end{pmatrix},
\]

where \( M_I \) and \( N_I \) \((I = 1, \cdots, 2n)\) are matrices,

\[
M_I \equiv \begin{pmatrix} A_I & iB_I \\ -iB_I & A_I \end{pmatrix}, \quad N_I \equiv \begin{pmatrix} iC_I & 0 \\ 0 & -iC_I \end{pmatrix},
\]

\[(A7)\]

with

\[
A_I = \cosh 2\tilde{a} \cosh 2b - \cos \left( \frac{\pi I}{n} \right) \sinh 2\tilde{a} \sinh 2b,
\]

\[
B_I = \sinh 2\tilde{a} \cosh 2b - \cos \left( \frac{\pi I}{n} \right) \cosh 2\tilde{a} \sinh 2b,
\]

\[
C_I = \sin \left( \frac{\pi I}{n} \right) \sinh 2b.
\]

\[(A9)\]

Note that \( I \) runs odd (even) numbers for the matrices with the index + (−). It is easy to see that \( M_I \) and \( N_I \) satisfy

\[
M_{2n-I} = M_I, \quad N_{2n-I} = -N_I.
\]

\[(A10)\]

Since \( A_I, B_I \) and \( C_I \) satisfy

\[
A_I^2 - B_I^2 - C_I^2 = 1,
\]

\[(A11)\]

we can uniquely determine the parameters \( \gamma_I > 0, \theta_I \in [0, \frac{\pi}{2}] \) and \( \varepsilon_I = \pm 1 \) by

\[
A_I \equiv \cosh \gamma_I, \quad B_I \equiv \varepsilon_I \sinh \gamma_I \cos \theta_I, \quad C_I \equiv \pm \sinh \gamma_I \sin \theta_I,
\]

\[(A12)\]

where the sign in the definition of \( C_I \) takes + for \( 1 \leq I \leq n \) and − for \( n + 1 \leq I \leq 2n \). Note that the \( \gamma_I \) appearing in \( (A12) \) is the same one defined in (2.22). We also note that
$N_0 = N_n = 0$ and $M_0$ and $M_n$ are given by

$$M_0 = R(\pm i\gamma_0), \quad M_n = R(-i\gamma_n), \quad (A13)$$

where the sign appearing in the expression of $M_0$ takes $+$ in the disordered phase ($\tilde{a} > b$) and $-$ in the ordered phase ($\tilde{a} < b$).

We can rearrange the matrices (A6) and (A7) by permuting the elements properly. To this end, we introduce the matrices which generate the transportations,

$$T_{i,j} \equiv \begin{pmatrix} 1 & 2(i-1) \\ 0 & 2 \\ \vdots & 1 \\ 1 & 2 \\ \vdots & 0 \\ 1 & 2(n-j) \end{pmatrix}, \quad (1 \leq i < j \leq n) \quad (A14)$$

and the cyclic rotations,

$$C_{i,j} \equiv T_{j-1,i}T_{j-2,j-1} \cdots T_{i,i+1}. \quad (A15)$$

Then by defining

$$S_+ = \begin{cases} C_{2,n}C_{4,n} \cdots C_{n-2,n} & (n : even) \\ C_{n-1,n}C_{2,n-1}C_{4,n-1} \cdots C_{n-3,n-1} & (n : odd) \end{cases}, \quad (A16)$$

we obtain

$$(\Omega_+S_+)^T \hat{H}_+ (\Omega_+S_+) = \begin{cases} \bigoplus_{r=1}^{n/2} X_{2r-1} & (n : even) \\ \bigoplus_{r=1}^{n/2} X_{2r-1} \oplus M_n & (n : odd) \end{cases}, \quad (A17)$$

and

$$(\Omega_+S_+)^T \hat{\Sigma}_+ (\Omega_+S_+) = \begin{cases} \bigoplus_{r=1}^{n/2} R_4 \left( \frac{2r-1}{n} \pi \right) & (n : even) \\ -i \left( \bigoplus_{r=1}^{n/2} R_4 \left( \frac{2r-1}{n} \pi \right) \oplus (-1_2) \right) & (n : odd) \end{cases}, \quad (A17)$$
and

\[
(\Omega_- S_-)^T \hat{H}_- (\Omega_- S_-) = \begin{cases} 
\bigoplus_{r=1}^{n/2} X_{2r} \oplus M_n \oplus M_0 & (n : \text{even}) \\
\bigoplus_{r=1}^{n/2} X_{2r} \oplus M_0 & (n : \text{odd})
\end{cases},
\]

\[
(\Omega_- S_-)^T \hat{\Sigma}_- (\Omega_- S_-) = \begin{cases} 
-i \bigoplus_{r=1}^{n/2} R_4 \left( \frac{2r}{n} \pi \right) \oplus (-1_2) \oplus 1_2 & (n : \text{even}) \\
\bigoplus_{r=1}^{n-1} R_4 \left( \frac{2r-1}{n} \pi \right) \oplus 1_2 & (n : \text{odd})
\end{cases},
\]

where \(X_I\) and \(R_4(\theta)\) are 4 \times 4 matrices defined by

\[
X_I \equiv \begin{pmatrix} M_I & -N_I \\ N_I & M_I \end{pmatrix}, \quad R_4(\phi) \equiv R(-\phi) \oplus R(\phi).
\]

The matrices \(X_I\) and \(R_4(\phi)\) are simultaneously transformed into the canonical forms as

\[
P_I^T X_I P_I = R(-i\epsilon_I \gamma_I) \oplus R(-i\epsilon_I \gamma_I), \quad P_I^T R_4(\phi) P_I = R(-\phi) \oplus R(\phi),
\]

using the 4 \times 4 matrices,

\[
P_I \equiv \begin{pmatrix} \cos \frac{\theta_I}{2} & 0 & 0 & \epsilon_I \sin \frac{\theta_I}{2} \\ 0 & \cos \frac{\theta_I}{2} & \epsilon_I \sin \frac{\theta_I}{2} & 0 \\ 0 & -\epsilon_I \sin \frac{\theta_I}{2} & \cos \frac{\theta_I}{2} & 0 \\ -\epsilon_I \sin \frac{\theta_I}{2} & 0 & 0 & \cos \frac{\theta_I}{2} \end{pmatrix}.
\]

Thus we define the matrices,

\[
P_+ \equiv \bigoplus_{r=1}^{n/2} P_{2r-1} \quad (n : \text{even}) \quad P_- \equiv \bigoplus_{r=1}^{n/2} P_{2r} \oplus 1_4 \quad (n : \text{even})
\]

\[
\bigoplus_{r=1}^{n-1} P_{2r-1} \oplus 1_2 \quad (n : \text{odd}) \quad \bigoplus_{r=1}^{n-1} P_{2r} \oplus 1_2 \quad (n : \text{odd})
\]

We finally consider the combinations,

\[
T_\pm \equiv \Omega_\pm S_\pm P_\pm,
\]

which transform \(H_\pm^{\Sigma_\pm}\) in the fundamental representation into the canonical forms, respectively:
\[ n: \text{even} \]
\[
\begin{align*}
T^T_+ (\hat{H}^m \hat{\Sigma}^p_+) T_+ \\
&= \bigoplus_{r=1}^{n/2} \left( R(-im\epsilon_{2r-1}\gamma_{2r-1} - \frac{2r-1}{n} p\pi) \oplus R(-im\epsilon_{2r-1}\gamma_{2r-1} + \frac{2r-1}{n} p\pi) \right), \\
T^T_- (\hat{H}^- m \hat{\Sigma}^p_-) T_- \\
&= (-i)^p \left\{ \bigoplus_{r=1}^{n-2} \left( R(-im\epsilon_{2r}\gamma_{2r} - \frac{2r}{n} p\pi) \oplus R(-im\epsilon_{2r}\gamma_{2r} + \frac{2r}{n} p\pi) \right) \\
&\quad \oplus R(-im\gamma_n + p\pi) \oplus R(\pm im\gamma_0) \right\},
\end{align*}
\]
\(\text{(A24)}\)

\[ n: \text{odd} \]
\[
\begin{align*}
T^T_+ (\hat{H}^m \hat{\Sigma}^p_+) T_+ \\
&= (-i)^p \left\{ \bigoplus_{r=1}^{n-1} \left( R(-im\epsilon_{2r-1}\gamma_{2r-1} - \frac{2r-1}{n} p\pi) \oplus R(-im\epsilon_{2r-1}\gamma_{2r-1} + \frac{2r-1}{n} p\pi) \right) \\
&\quad \oplus R(-im\gamma_n + p\pi) \right\}, \\
T^T_- (\hat{H}^- m \hat{\Sigma}^p_-) T_- \\
&= \bigoplus_{r=1}^{n-1} \left( R(-im\epsilon_{2r}\gamma_{2r} - \frac{2r}{n} p\pi) \oplus R(-im\epsilon_{2r}\gamma_{2r} + \frac{2r}{n} p\pi) \right) \oplus R(\pm im\gamma_0),
\end{align*}
\]
\(\text{(A25)}\)

where we have used (A13). We can easily see \(\det T_\pm = 1\). These results motivate to introduce (2.23). Note that, when we consider the continuum limit, we should approach to the critical temperature from the ordered phase. Thus the sign appearing in \(R(\pm im\gamma_0)\) is chosen as +.

**B. Partition function of 2D massive fermion on the torus**

Let us consider 2-torus with periods \(\omega_1, \omega_2 \in \mathbb{C}\) and free Majorana fermion with mass \(M\) on it:
\[
S = \frac{1}{2} \int d^2 x \Psi^T D \Psi, 
\]
\(\text{(B1)}\)

where \(\Psi\) is a two-component spinor, \(\Psi = (\psi, \overline{\psi})^T\) and \(D\) is the Dirac matrix given by
\[
D = \begin{pmatrix} \partial_1 + i\partial_2 & M \\ -M & \partial_1 - i\partial_2 \end{pmatrix}. 
\]
\(\text{(B2)}\)

In the following, we evaluate the partition function,
\[
Z = \int d\Psi e^{-S} = \text{Pf} D, 
\]
\(\text{(B3)}\)
where PfD denotes the Pfaffian of the Dirac operator $D$.

Let $\{k_1, k_2\} = \{-i\omega_2/A, i\omega_1/A\}$ denote the dual vectors corresponding to $\{\omega_1, \omega_2\}$, where $A = \text{Im}(\omega_2\bar{\omega}_1)$ is the area of the torus. The plane-wave on the torus is given by

$$u_{mn}(z, \bar{z}) \equiv e^{2\pi i(-n\text{Re}(k_1)+m\text{Re}(k_2))} = e^{\frac{2\pi i}{A}[-i(m\omega_1+n\omega_2)z+i(m\omega_1+n\omega_2)\bar{z}]}, \quad (B4)$$

where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ and $m, n$ take integer or half integer, $m \in \mathbb{Z} + \mu$ and $n \in \mathbb{Z} + \nu$, $(\mu, \nu = 0$ or $1/2)$, depending on the periodicity of the fermion. Recall that the eigenvalue of the Laplacian $-(\partial_1^2 + \partial_2^2) = -4\partial\bar{\partial}$ is

$$\lambda_{mn} = \frac{(2\pi)^2}{A^2}|m\omega_1 + n\omega_2|^2 = N^2|m + \tau n|^2, \quad (B5)$$

where $N = \frac{|2\pi \omega_1|}{A}$ and

$$\tau \equiv \omega_2/\omega_1 = \tau_1 + i\tau_2. \quad (\tau_1, \tau_2 \in \mathbb{R}) \quad (B6)$$

Then the partition function in Eq.(B3) is expressed as

$$Z = \sum_{\mu, \nu=0, 1/2} Z_{\mu, \nu}, \quad (B7)$$

with

$$Z_{\mu, \nu} = \prod_{m,n \in \mathbb{Z}} N\sqrt{|m + n\tau + \mu + \nu\tau|^2 + (M/N)^2}. \quad (B8)$$

Rescaling the mass parameter $M$ as

$$\mu \equiv \frac{2\pi M}{\tau_2 N} \quad (B9)$$

and using the zeta function regularization, $Z_{\mu, \nu}$ are estimated as as

$$Z_{0, 0} = e^{\pi \tau_2 \sum_{n \in \mathbb{Z}} \sqrt{n^2 + \left(\frac{\mu}{2\pi}\right)^2}} \left(1 - e^{-\tau_2 \mu}\right) \prod_{n=1}^{\infty} \left[1 - e^{-2\pi in\tau_2}\left(\tau_1 + i\tau_2\sqrt{1 + \left(\frac{\mu}{2\pi n}\right)^2}\right)^2\right],$$

$$Z_{0, 1} = e^{\pi \tau_2 \sum_{n \in \mathbb{Z}} \sqrt{(n+\frac{1}{2})^2 + \left(\frac{\mu}{2\pi}\right)^2}} \prod_{n=0}^{\infty} \left[1 - e^{-\pi i(2n+1)\tau_2}\left(\tau_1 + i\tau_2\sqrt{1 + \left(\frac{\mu}{(2n+1)\pi}\right)^2}\right)^2\right],$$

$$Z_{\frac{1}{2}, 0} = e^{\pi \tau_2 \sum_{n \in \mathbb{Z}} \sqrt{n^2 + \left(\frac{\mu}{2\pi}\right)^2}} \left(1 + e^{-\tau_2 \mu}\right) \prod_{n=1}^{\infty} \left[1 + e^{-2\pi in\tau_2}\left(\tau_1 + i\tau_2\sqrt{1 + \left(\frac{\mu}{2\pi n}\right)^2}\right)^2\right],$$

$$Z_{\frac{1}{2}, 1} = e^{\pi \tau_2 \sum_{n \in \mathbb{Z}} \sqrt{(n+\frac{1}{2})^2 + \left(\frac{\mu}{2\pi}\right)^2}} \prod_{n=0}^{\infty} \left[1 + e^{-\pi i(2n)\tau_2}\left(\tau_1 + i\tau_2\sqrt{1 + \left(\frac{\mu}{(2n+1)\pi}\right)^2}\right)^2\right].$$

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We should note that the exponents of the overall factors in Eq. (B10) logarithmically diverge;

\[
\pi \tau_2 \sum_{n \in \mathbb{Z}} \sqrt{n^2 + \left( \frac{\mu}{2\pi} \right)^2} = -\frac{\pi \tau_2}{6} + \frac{\tau_2 \mu^2}{2\pi} \zeta(1) + \mathcal{O}(\mu^4),
\]

\[
\pi \tau_2 \sum_{n \in \mathbb{Z}} \sqrt{(n + \frac{1}{2})^2 + \left( \frac{\mu}{2\pi} \right)^2} = \frac{\pi \tau_2}{12} + \frac{\tau_2 \mu^2}{4\pi} \zeta(1) + \mathcal{O}(\mu^4).
\]

(B11)

Thus, precisely speaking, we implicitly regard that the logarithmic divergence \( \zeta(1) \) is properly regularized in Eq. (B10).

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