DIAGONAL CUBIC FORMS AND THE LARGE SIEVE

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Abstract. Let $N(X)$ be the number of integral zeros $(x_1, \ldots, x_6) \in [-X, X]^6$ of $\sum_{1 \leq i \leq 6} x_i^3$. Works of Hooley and Heath-Brown imply $N(X) \ll \epsilon X^{3+\epsilon}$, if one assumes automorphy and GRH for certain Hasse–Weil $L$-functions. Assuming instead a natural large sieve inequality, we recover the same bound on $N(X)$. This is part of a more general statement, for diagonal cubic forms in $\geq 4$ variables, where we allow approximations to Hasse–Weil $L$-functions.

1. Introduction

Fix an integer $m \geq 4$. Fix integers $F_1, \ldots, F_m \in \mathbb{Z} \setminus \{0\}$ and let

$$F(x) := \sum_{1 \leq i \leq m} F_i x_i^3,$$

where $x = (x_1, \ldots, x_m)$. We are interested in the behavior, as $X \to \infty$, of the point count

$$N_F(X) := |\{x \in \mathbb{Z}^m \cap [-X, X]^m : F(x) = 0\}|.$$

Certain varieties, $V_{c,k}$, play a key role. For each $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ and field $k$, let

$$V_{c,k} := \{(\xi_1, \ldots, \xi_m) \in \mathbb{P}^{m-1}_k : \sum_{1 \leq i \leq m} F_i \xi_i^3 = \sum_{1 \leq i \leq m} c_i \xi_i = 0\},$$

where $\mathbb{P}^{m-1}_k$ is the projective space with coordinates $\xi_1, \ldots, \xi_m$ over $k$.

In the special case $F = \sum_{1 \leq i \leq 6} x_i^3$, with $m = 6$, we abbreviate $N_F(X)$ to $N(X)$. In this case, building on [Hoo86], the papers [Hoo97] and [HB98] each proved

(1.1) $N(X) \ll \epsilon X^{3+\epsilon}$,

assuming Hypothesis HW of [Hoo86, §6; HB98, §4] for the Hasse–Weil $L$-function of each smooth variety $V_{c,\mathbb{Q}}$ with $c \neq 0$. Unconditionally, by [Vau20, Theorem 1.2],

$$N(X) \ll \epsilon X^{7/2}/(\log X)^{5/2-\epsilon}$$

for $X \geq 2$, via methods stemming from work such as [Vau86, HT88, Bok93, BW10].

Hypothesis HW practically amounts to automorphy, plus the Grand Riemann Hypothesis (GRH). Automorphy remains open [Wan22, Appendix A]. Hooley suggests that a zero-density hypothesis would suffice in place of GRH [Hoo86, p. 51]. Following the usual paths laid out in [IK04, Theorem 10.4], a general such density hypothesis is provable assuming automorphy, a large sieve inequality, and progress on the Grand Lindelöf Hypothesis (GLH).

In the present paper, we show that a large sieve inequality by itself would imply (1.1). The precise large sieve inequality we need will be stated in §2, as Hypothesis 2.1.

Theorem 1.1. Suppose $m \in \{5, 6\}$. Assume Hypothesis 2.1. Then

(1.2) $N_F(X) \ll \epsilon X^{3(m-2)/4+\epsilon}$,

for all reals $X \geq 1$ and $\epsilon > 0$. 

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For $m = 6$, the exponent in (1.2) matches (1.1). In §2, we state a more general result, Theorem 2.7, valid for all $m \geq 4$. Our methods might also apply elsewhere \[\text{Wan22} \text{ §9.1}]. For instance, \[\text{Wan21} \text{ explains how one may hope to use the modularity of elliptic curves over } \mathbb{Q} \text{ to unconditionally produce an absolute constant } \delta > 0 \text{ such that}
\]
$$|\{a \in \mathbb{Z} : 1 \leq a \leq A\} \setminus \{x^2 + y^3 + z^3 : x, y, z \in \mathbb{Z}_{z \geq 0}\}| \ll A^{6/7-\delta}.$$ This would then improve on the existing bound $O_s(A^{6/7+\epsilon})$ due to Brüdern \[\text{Brü91}].

Conventions. We let $\mathbb{Z}_{\geq c} := \{n \in \mathbb{Z} : n \geq c\}$. We let $1_E := 1$ if a statement $E$ holds, and $1_E := 0$ otherwise. For integers $n \geq 1$, we let $\mu(n)$ denote the Möbius function.

We write $f \ll_S g$, or $g \gg_S f$, to mean $|f| \leq Cg$ for some coefficients $C, N, \epsilon > 0$. The implied constant $C$ is always allowed to depend on $m$ and $F$, in addition to $S$. We let $O_S(g)$ denote a quantity that is $\ll_S g$. We write $f \asymp_S g$ if $f \ll_S g \ll_S f$.

2. Framework and results

Let $\mathcal{D} := 3(\prod_{1 \leq i \leq m} F_i)^{2m-2} \in \mathbb{Z}$. For each $\mathbf{c} \in \mathbb{Z}^m$, let

$$\Delta(\mathbf{c}) := \mathcal{D} \prod_{(v_2, \ldots, v_m) \in \{1, -1\}^{m-1}} \left( (c_1^3/F_1)^{1/2} + \sum_{2 \leq i \leq m} v_i (c_i^3/F_i)^{1/2} \right) \in \mathbb{Z}.$$ (2.1)

For each field $k$ in which $\Delta(\mathbf{c})$ is invertible, the variety $V_{\mathbf{c}, k}$ is a smooth complete intersection, by the Jacobian criterion for smoothness. Let

$$S := \{\mathbf{c} \in \mathbb{Z}^m : \Delta(\mathbf{c}) \neq 0\}, \quad S(C) := S \cap [-C, C]^m.$$ (2.2)

For each $\mathbf{c} \in S$ and prime $p$, we define a local Euler factor $L_p(s, \mathbf{c})$, following Serre and Kahn \[\text{Ser70} \text{ §5.6}], \[\text{Kah20} \text{ §5.6}]. First, choose a prime $\ell \neq p$, and let

$$M(\mathbf{c}, \ell) := H^{m-3}(V_{\mathbf{c}, \mathbb{Q}}) / H^{m-3}(\mathbb{P}^{m-1}_{\mathbb{Q}}, \mathbb{Q}_\ell),$$

where $H^i(W, \mathbb{Q}_\ell)$ denotes the $i$th $\ell$-adic cohomology group of $W$. Let $M(\mathbf{c}, \ell)^{Ip} \subseteq M(\mathbf{c}, \ell)$ denote the group of inertia invariants of $M(\mathbf{c}, \ell)$. Let $\alpha_{\mathbf{c}, j}(p) \in \mathbb{C}$, for $1 \leq j \leq \dim M(\mathbf{c}, \ell)^{Ip}$, be the geometric Frobenius eigenvalues on $M(\mathbf{c}, \ell)^{Ip}$. Finally, let

$$\check{\alpha}_{\mathbf{c}, j}(p) := \frac{\alpha_{\mathbf{c}, j}(p)}{p^{(m-3)/2}}, \quad L_p(s, \mathbf{c}) := \prod_{1 \leq j \leq \dim M(\mathbf{c}, \ell)^{Ip}} (1 - \check{\alpha}_{\mathbf{c}, j}(p)p^{-s})^{-1}.$$ (2.3)

On multiplying over $p$, we obtain for each $\mathbf{c} \in S$ a global Hasse–Weil $L$-function

$$L(s, \mathbf{c}) := \prod_p L_p(s, \mathbf{c}) = \sum_{n \geq 1} \lambda_\mathbf{c}(n)n^{-s},$$ (2.4)

for some coefficients $\lambda_\mathbf{c}(n) \in \mathbb{C}$ defined by expanding the product over $p$. We now state Hypothesis 2.1. It asserts a large sieve inequality, (2.5), in a certain range.

Hypothesis 2.1. For all reals $C, N, \epsilon > 0$ with $N \leq C^3$, we have

$$\sum_{\mathbf{c} \in S(C)} \left| \sum_{n \leq N} v_n \lambda_\mathbf{c}(n) \right|^2 \ll \epsilon C^\epsilon \max(C^m, N) \sum_{n \leq N} |v_n|^2$$ (2.5)

for all vectors $(v_n)_{1 \leq n \leq N} \in \mathbb{C}[N]$. 

We now make some general comments on \( L(s, c) \). By [Kah20, §5.6.3, §5.6.4] and [Las17, Corollary 1.2], the factors \( L_p(s, c) \) are independent of the choice of \( \ell \), and we have
\[
|\hat{\alpha}_{c,j}(p)| \leq 1.
\]
By (2.6), the product and series in (2.4) converge absolutely for \( \Re(s) > 1 \).

We have \( \dim M(c, \ell)^l_p \leq \dim M(c, \ell) \ll_m 1 \) by [Kat01, Corollary of Theorem 3]. Therefore, by (2.6), we have \( \lambda_c(n) \ll n^\epsilon \) for all \( n \geq 1 \). Thus (2.5) is the large sieve inequality that one would naturally expect to hold. In fact, (2.5) could potentially hold in the range \( N \leq C^A \) for any constant \( A > 0 \). However, we will only need it in the range \( N \leq C^3 \).

The coefficients \( \lambda_c(n) \) can be interpreted geometrically, but it would take us too far afield to detail anything but the simplest case. For each \( c \in \mathbb{Z}^m \) and prime \( p \), let
\[
E_c(p) := \frac{|\{x \in \mathbb{P}_p^m : F(x) = c \cdot x = 0\}| - p^{n-2}}{p - 1}, \quad E^2_c(p) := \frac{E_c(p)}{p^{(m-3)/2}},
\]
where \( c \cdot x := \sum_{1 \leq i \leq m} c_i x_i \). If \( p \nmid \Delta(c) \), then \( M(c, \ell)^l_p = M(c, \ell) \) and
\[
(2.7) \quad \lambda_c(p) = \sum_{1 \leq j \leq \dim M(c, \ell)} \hat{\alpha}_{c,j}(p) = (-1)^{m-3} E^2_c(p),
\]
by (2.3) and the Grothendieck–Lefschetz trace formula.

We emphasize that our \( L \)-functions are normalized differently than in [Hoo86, HB98]. If \( H(s, c) \) is the \( L \)-function associated to \( V_{c, \mathbb{Q}} \) in [HB98, §4], then
\[
H(s + \frac{m-3}{2}, c) = L(s, c).
\]

**Proof framework.** We will analyze \( N_F(X) \) using the *delta method*, due to [DFI93, HB96]. This method features some complete exponential sums that we now recall. Let
\[
(2.8) \quad S_c(n) := \sum_{1 \leq a \leq n} \sum_{\gcd(a, n) = 1} e^{2\pi i (a F(x) + c x) / n}, \quad S^2_c(n) := \frac{S_c(n)}{n^{(m+1)/2}},
\]
for all \( c \in \mathbb{Z}^m \) and integers \( n \geq 1 \). It is known that \( S_c(n) \) is *multiplicative* in \( n \), meaning that \( S_c(1) = 1 \) and \( S_c(n_1 n_2) = S_c(n_1) S_c(n_2) \) whenever \( \gcd(n_1, n_2) = 1 \) [HB98, Lemma 4.1]. Thus \( S^2_c(n) \) is also multiplicative in \( n \). For each \( c \in \mathbb{Z}^m \), let
\[
\Phi(c, s) := \sum_{n \geq 1} S^2_c(n) n^{-s} = \prod_p \Phi_p(c, s),
\]
where \( \Phi_p(c, s) := \sum_{t \geq 0} S^2_c(p^t) p^{-ts} \). Ultimately, we will see that \( S^2_c(n) \) is related to \( \lambda_c(n) \) in a way that allows us to apply a large sieve inequality, like (2.5), to the delta method.

Before proceeding, we recall two basic definitions from the theory of Dirichlet series. For \( f, g : \mathbb{Z}_{\geq 1} \to \mathbb{C} \), the *Dirichlet convolution* \( f \ast g : \mathbb{Z}_{\geq 1} \to \mathbb{C} \) is defined by the formula
\[
(f \ast g)(n) := \sum_{ab = n} f(a) g(b).
\]
A Dirichlet series \( \sum_{n \geq 1} f(n) n^{-s} \) is said to be *invertible* if \( f(1) \neq 0 \), or equivalently, if there exists \( g : \mathbb{Z}_{\geq 1} \to \mathbb{C} \) with \( (f \ast g)(n) = 1_{n=1} \).
Our work is based on approximations of Dirichlet series. For each \( c \in S \), let \( \Psi(c, s) \) be an invertible Dirichlet series. The function \( c \mapsto \Psi(c, s) \), from \( S \) to the set of Dirichlet series, will be denoted simply by \( \Psi \). For each \( c \in S \) and integer \( n \geq 1 \), let
\[
b_c(n), \ a_c(n), \ a'_c(n)
\]
be the \( n^{-s} \) coefficients of the Dirichlet series
\[
\Psi(c, s), \ \Psi(c, s)^{-1}, \ \Phi(c, s)/\Psi(c, s),
\]
respectively. In terms of Dirichlet convolution, this means that
\[
(a_c * b_c)(n) = 1_{n=1}, \quad a'_c = S^c_n * a_c, \quad S^c_n = a'_c * b_c.
\]

For us, the following particular definition of approximation will be convenient.

**Definition 2.2.** Call \( \Psi \) an approximation of \( \Phi \) if the following three conditions hold:

1. If \( c \in S \), then \( b_c(n) \) is multiplicative in \( n \).
2. For all \( c \in S \), integers \( n \geq 1 \), and reals \( \epsilon > 0 \), we have
\[
\max(|b_c(n)|, |a'_c(n)|) \ll n^\epsilon \sum_{d\mid n} |S^c_n(d)|.
\]
3. For all \( c \in S \) and primes \( p \nmid \Delta(c) \), we have \( a'_c(p) \ll p^{-1/2} \).

**Theorem 2.3.** Suppose that for each \( c \in S \), we have
\[
\Psi(c, s) \in \left\{ \Phi(c, s), \ \prod_{p \mid \Delta(c)} \Phi_p(c, s), \ \prod_{p \mid \Delta(c)} L_p(s, c)^{(-1)^{m-3}}, \ L(s, c)^{(-1)^{m-3}} \right\}.
\]
Then \( \Psi \) is an approximation of \( \Phi \).

Theorem 2.3 provides natural examples of approximations. It will not be used until §8 so we defer the proof to that section. For the rest of §2, fix an approximation \( \Psi \) of \( \Phi \).

**Hypotheses.** Our main general result, Theorem 2.7, will assume that either of two specific hypotheses holds. Our first hypothesis is the following:

**Hypothesis 2.4.** For all reals \( C, N, \epsilon > 0 \) with \( N \leq C^3 \), we have
\[
\sum_{c \in S(C)} \left| \sum_{n \in I} b_c(n) \right|^2 \ll \epsilon C^\alpha \max(C^m, N) N
\]
for all real intervals \( I \subseteq (0, N] \).

The following two remarks may help to clarify the nature of this hypothesis.

1. If \( \Psi = L(s, c)^{-1} \), then Hypothesis 2.4 would easily follow from GRH. On the other hand, if \( \Psi = L(s, c) \), then Hypothesis 2.4 would follow from GLH plus a technical bound on \( |\{c \in S(C) : L(s, c) \text{ has a pole at } s = 1\}| \).
2. A density bound, namely \( |\{c \in S(C) : \sum_{n \in I} b_c(n) \geq N^\sigma\}| \ll \epsilon C^{m+\epsilon}/N^{2\sigma-1} \) for \( N \leq C^3 \) and \( \sigma \geq 1/2 \), would follow from Hypothesis 2.4. But \( C^{m+\epsilon}/N^{2\sigma-1} \) could be quite large even if \( N = C^3 \) and \( \sigma = 1 \). This is unlike in some density applications, e.g. [IK04, Theorem 10.5], where further input may be needed near \( \sigma = 1 \).
If $\Psi = L(s, c)^{-1}$, then Hypothesis 2.4 is perhaps unattractive in that $b_c(n)$ involves the Möbius function $\mu(n)$. We might thus wish to pass from $b_c(n)$ to $a_c(n)$. This is possible, to some extent, in the situation of the following definition:

**Definition 2.5.** Call $\Psi$ *standard* if for all $c \in S$, integers $n \geq 1$, and reals $\epsilon > 0$, we have

$$\max(|b_c(n)|, |a_c(n)|) \ll n^\epsilon.$$  

Let $\vartheta \in \{0, 1\}$ if $\Psi$ is standard, and let $\vartheta := 0$ if $\Psi$ is non-standard. Let

$$\gamma_c(n) := (1 - \vartheta) \cdot b_c(n) + \vartheta \cdot \mu(n)^2 a_c(n).$$

We now come to our main hypothesis: a large sieve inequality for $\gamma_c$, in a certain range.

**Hypothesis 2.6.** For all reals $C, N, \epsilon > 0$ with $N \leq C^3$, we have

$$\sum_{c \in S(c)} \left| \sum_{n \leq N} v_n \gamma_c(n) \right|^2 \ll \epsilon C^\epsilon \max(C^m, N) \sum_{n \leq N} |v_n|^2$$

for all vectors $(v_n)_{1 \leq n \leq N} \in \mathbb{C}^{|N|}$.

Again, some brief remarks may be helpful.

1. When $\vartheta = 1$, the factor $\mu(n)^2$ in (2.11) simply restricts us to square-free moduli $n$.
2. Hypothesis 2.6 remains open in general [Wan22, Remark 4.1.10].

**Results.** Fix a smooth, compactly supported function $w: \mathbb{R}^m \to \mathbb{R}$. Assume that

$$0 \notin \{ x \in \mathbb{R}^m : w(x) \neq 0 \}.$$  

For reals $X \geq 1$, let

$$N_{F, w}(X) := \sum_{x \in \mathbb{Z}^m} w(x/X) 1_{F(x) = 0}.$$  

If $m \geq 5$, then let $N_{F, w}'(X) := N_{F, w}(X)$. If $m = 4$, then let $Y$ denote the set of 2-dimensional rational vector spaces $L$ with $F|_L = 0$, and let

$$N_{F, w}'(X) := \sum_{x \in \mathbb{Z}^m \setminus (\bigcup_{L \in Y} L)} w(x/X) 1_{F(x) = 0}.$$  

**Theorem 2.7.** Assume Hypothesis 2.6 or Hypothesis 2.4. Then for some constant $c(F, w) \in \mathbb{R}$, we have

$$N_{F, w}'(X) - c(F, w) X^{m-3} \ll \epsilon X^{3(m-2)/4+\epsilon},$$

for all reals $X \geq 1$ and $\epsilon > 0$.

Note that $m, F, w$ are fixed. In other words, the implied constant in (2.16) is allowed to depend on $m, F, w$ in addition to $\epsilon$. Also, for numerical reference,

$$3(m - 2)/4 = 1.5 \cdot 1_{m=4} + 2.25 \cdot 1_{m=5} + 3 \cdot 1_{m=6} + \cdots.$$  

In particular, if $5 \leq m \leq 6$, then $m - 3 \leq 3(m - 2)/4$, and (2.16) simply says

$$N_{F, w}(X) \ll \epsilon X^{3(m-2)/4+\epsilon}.$$  

The rest of the paper is devoted to the proof of Theorems 1.1, 2.3, and 2.7. In §3, we reduce Hypothesis 2.4 to Hypothesis 2.6. In §§4–7, we recall the delta method for $N_{F, w}(X)$, then analyze parts of it unconditionally and parts of it using Hypothesis 2.4. In §8, we tie together the previous sections to complete the proofs.
3. A CONVERSION BETWEEN STANDARD COEFFICIENTS

In this section, we prove a useful consequence of Hypothesis 2.6. First, we record some standard lemmas that will be repeatedly used throughout the paper.

**Lemma 3.1.** Let $N, h \in \mathbb{Z}_{\geq 1}$. Then there are at most $O_h(N^{1/h})$ integers $n \in [N, 2N)$ such that $v_p(n) \geq h$ holds for all primes $p \mid n$.

**Proof.** This is classical; see e.g. [BG58]. □

To proceed, we need to introduce some notation. We write $u \mid v^\infty$ if there exists $k \in \mathbb{Z}_{\geq 1}$ with $u \mid v^k$. For an integer $c \neq 0$, we let $\text{sq}(c)$ (resp. cub$(c)$) denote the largest square-full (resp. cube-full) positive integer divisor of $c$. We also let $\text{sq}(0) := 0$.

**Lemma 3.2.** Let $N, R \in \mathbb{Z}_{\geq 1}$. Then there are at most $O_{\epsilon}(N^{\epsilon} R^\epsilon)$ positive integers $n \leq N$ with $n \mid R^\infty$.

**Proof.** We have

$$P \frac{n}{n} \mid R^\infty \leq \frac{N}{N} \leq P \frac{n}{n} \mid R^\infty \left(\frac{N}{n}\right) \leq N \epsilon \prod_{p \mid R} (1 - p^{-\epsilon})^{-1} \ll_{\epsilon} N \epsilon R^\epsilon.$$ □

**Lemma 3.3.** Let $N \in \mathbb{Z}_{\geq 1}$. Then the following hold:

1. We have

$$\sum_{n \leq N : n = \text{sq}(n)} n^{-1/2} \ll_{\epsilon} N^{\epsilon}.$$

2. We have

$$\sum |c| \leq N \text{sq}(c)^{1/2} \ll_{\epsilon} N^{1+\epsilon}.$$

3. For any $t \in \mathbb{R}$, we have

$$\sum_{1 \leq n \leq N} \text{cub}(n)^t \ll_{t, \epsilon} N^{\epsilon} \max(N, N^{1/3+t}).$$

**Proof.**

1: By the $h = 2$ case of Lemma 3.1 in dyadic intervals $n \in [2^k, 2^{k+1})$, we have

$$\sum_{n \leq N : n = \text{sq}(n)} n^{-1/2} \ll \sum_{0 \leq k \leq \log_2 N} (2^k)^{1/2} (2^k)^{-1/2} \ll_{\epsilon} N^{\epsilon}.$$

2: There are at most $N/d$ positive integers $n \leq N$ with $\text{sq}(n) = d$. Therefore,

$$\sum_{|c| \leq N} \text{sq}(c)^{1/2} = 2 \sum_{1 \leq n \leq N} \text{sq}(n)^{1/2} \leq 2 \sum_{d \leq N : d = \text{sq}(d)} \frac{N}{d} \cdot d^{1/2} \ll_{\epsilon} N^{1+\epsilon},$$

where the last inequality follows from (1).

3: There are at most $N/n_3$ positive integers $n \leq N$ with $\text{cub}(n) = n_3$. Thus

$$\sum_{1 \leq n \leq N} \text{cub}(n)^t \leq \sum_{n_3 \leq N : n_3 = \text{cub}(n_3)} \frac{N}{n_3} \cdot n_3^t \ll_{t, \epsilon} \sum_{0 \leq k \leq \log_2 N} (2^k)^{1/3} (2^k)^{-1} N \ll_{t, \epsilon} N^{\epsilon} \max(N, N^{1/3+t}),$$

by the $h = 3$ case of Lemma 3.1 in dyadic intervals $n_3 \in [2^k, 2^{k+1})$. □

**Proposition 3.4.** Fix an approximation $\Psi$ of $\Phi$. Assume Hypothesis 2.6. Then Hypothesis 2.4 holds.
Proof. First suppose \( \vartheta = 0 \). Then \( \gamma_c = b_c \) by (2.11). For \( C, N, I \) as in Hypothesis 2.4, the bound (2.12) with \( v_n := 1_{n \in I} \) thus trivially implies (2.10), as desired.

Now suppose \( \vartheta = 1 \). Then in particular, \( \Psi \) is standard. For the rest of the proof, let \( C, d, N \) denote positive variables. For integers \( d \) and intervals \( I \), let

\[
A_c(d, I) := \sum_{n \in I} 1_{\gcd(d, n) = 1} \mu(n) a_c(n).
\]

We have \( \gamma_c(n) = \mu(n)^2 a_c(n) \) by (2.11). Taking \( v_n := 1_{n \in I} 1_{\gcd(d, n) = 1} \mu(n) \) in (2.12), and observing that \( \mu(n)^3 = \mu(n) \), we find that Hypothesis 2.6 implies

\[
\sum_{c \in \mathcal{S}(C)} |A_c(d, I)|^2 \ll_C \max(C^m, N) N
\]

uniformly over reals \( C \), integers \( d \), reals \( N \leq C^3 \), and real intervals \( I \subseteq (0, N] \).

To proceed, we rewrite \( b_c(n) \) using multiplicativity. First, by (2.9), for primes \( p \) we have

\[
b_c(p) = -a_c(p).
\]

Furthermore, an integer \( n \geq 1 \) can be uniquely expressed in the form \( n_1 d \), where \( d \) is square-full, \( n_1 \) is coprime to \( d \), and \( n_1 \) is square-free. Therefore, for all \( n \geq 1 \), we have

\[
b_c(n) = \sum_{n_1 d = n} 1_{\gcd(d, n_1) = 1} \mu(n_1) a_c(n_1) \cdot 1_{d = \text{sq}(d)} b_c(d).
\]

We note here that \( \mu(n_1) \) is supported on square-free integers \( n_1 \).

Consider a real \( C \), a real \( N \leq C^3 \), and a real interval \( I \subseteq (0, N] \). Let \( B_c(I) := \sum_{n \in I} b_c(n) \). By (3.2), we have

\[
B_c(I) = \sum_{n_1 d = n \in I} 1_{\gcd(d, n_1) = 1} \mu(n_1) a_c(n_1) \cdot 1_{d = \text{sq}(d)} b_c(d)
\]

By the Cauchy–Schwarz inequality over \( d \), it follows that

\[
\sum_{c \in \mathcal{S}(C)} |B_c(I)|^2 \leq \sum_{c \in \mathcal{S}(C)} \left( \sum_{d \leq N : d = \text{sq}(d)} |b_c(d)|^2 d^{-1/2}\right) \left( \sum_{d \leq N : d = \text{sq}(d)} d^{1/2} |A_c(d, I/d)|^2 \right)
\]

by Lemma 3.3(1), since \( b_c(d) \ll_C d^\epsilon \) by Definition 2.5. Yet for all integers \( d \), we have

\[
\sum_{c \in \mathcal{S}(C)} |A_c(d, I/d)|^2 \ll_C \max(C^m, N/d) (N/d)
\]

by (3.1), since \( N/d \leq N \leq C^3 \) and \( I/d \subseteq (0, N/d] \). Plugging this into (3.3), we get

\[
\sum_{c \in \mathcal{S}(C)} |B_c(I)|^2 \ll_C N^\epsilon \sum_{d \leq N : d = \text{sq}(d)} d^{1/2} \max(C^m, N/d) (N/d)
\]

\[
\ll_C N^{2\epsilon} C^m \max(C^m, N) N,
\]

where the second inequality follows from Lemma 3.3(1) and the trivial bound \( \max(C^m, N/d) \leq \max(C^m, N) \). Thus (2.10) holds, uniformly over \( C, N, I \). \( \square \)
4. Delta method ingredients

Let $X \geq 1$. Assume \textit{[2.13]}, i.e. that $w$ is supported away from $0 \in \mathbb{R}^m$. Such an assumption is implicit in some of the integral estimates in \textit{[HB96, HB98]}. Set

$$Y := X^{(\deg F)/2} = X^{3/2}. \tag{4.1}$$

Fix $\epsilon_0 \in (0, 10^{-10}]$ and set

$$Z := Y/X^{1-\epsilon_0} = X^{1/2+\epsilon_0}. \tag{4.2}$$

Let $g_0(x) := \exp(-(1-x^2)^{-1})$ for $|x| < 1$, and $g_0(x) := 0$ for $|x| \geq 1$. Let

$$\varrho(x) := \frac{4g_0(4x - 3)}{\int_{y \in \mathbb{R}} g_0(y) \, dy}. \tag{4.3}$$

For $x > 0$ and $y \in \mathbb{R}$, let

$$h(x, y) := \sum_{j \geq 1} \frac{1}{x^j} \left( \varrho(x^j) - \varrho\left(\frac{|y|}{x^j}\right) \right).$$

This is precisely the function $h(x, y)$ defined in \textit{[HB96] §3}. For $c \in \mathbb{Z}^m$ and $n \geq 0$, let

$$I_c(n) := \int_{x \in \mathbb{R}^m} w(x/X) h(n/Y, F(x)/Y^2) e^{-2\pi i (c \cdot x/n)} \, dx. \tag{4.4}$$

Let $\|c\| := \max_{1 \leq i \leq m} |c_i|$. We now recall two standard results on the integral $I_c(n)$.

**Proposition 4.1** (\textit{[HB96] par. 1 of §7}). The functions $n \mapsto I_c(n)$ are supported on a range of the form $n \leq M_0(F, w) Y$, uniformly over $c \in \mathbb{Z}^m$, for some constant $M_0(F, w) > 0$.

**Lemma 4.2** (\textit{[HB98] (3.9)}). If $\|c\| \geq \epsilon_0 \geq 1$, then $I_c(n) \ll \epsilon_0 \|c\|^{-A}$, for all $A > 0$.

Proposition 4.1 and Lemma 4.2 together with the trivial bound $|S_c(n)| \leq n^{1+m}$, imply

$$Y^{-2} \sum_{n \geq 1} \sum_{\|c\| \geq Z} n^{-m} |S_c(n) I_c(n)| \ll \epsilon_0 A X^{-A}, \tag{4.5}$$

for all $A > 0$. Here $S_c(n)$ is defined as in \textit{[2.8]}. By \textit{[HB96] Theorem 2, (1.2)}], we have

$$N_{F,w}(X) = Y^{-2} \sum_{n \geq 1} \sum_{c \in \mathbb{Z}^m} n^{-m} S_c(n) I_c(n). \tag{4.6}$$

Equivalently, in terms of $S_c^2(n)$, we have

$$N_{F,w}(X) = X^{-3} \sum_{n \geq 1} \sum_{c \in \mathbb{Z}^m} n^{1-m/2} S_c^2(n) I_c(n). \tag{4.7}$$

In view of \textit{(4.3)}, analyzing $N_{F,w}(X)$ reduces to understanding the quantity

$$\Sigma_0 := X^{-3} \sum_{n \geq 1} \sum_{c \in [-Z, Z]^m} n^{1-m/2} S_c^2(n) I_c(n).$$

(Here $I_c(n) = I_c(n) 1_{n \leq M_0(F, w) Y}$. But it is more convenient to keep the factor $1_{n \leq M_0(F, w) Y}$ implicit, in order to allow for more flexible technique later on.)

We now recall some standard formulas for $S_c$ at primes $p$ and prime powers $p^l$.

**Proposition 4.3.** Say $p \nmid c$. Then $S_c^2(p) = E_c^2(p) + O(p^{-1/2})$. 

Proof. Let
\[ E(p) := \frac{\{|x \in \mathbb{F}_p^n : F(x) = 0\}| - p^{m-1}}{p-1}. \]
By [HB98, p. 680], we have \( S_c(p) = p^2E_c(p) - pE(p) \) and \( E(p) \ll p^{(m-2)/2} \). Thus
\[ S_c(p) = p^2E_c(p) + O(p^{m/2}). \]
Now divide by \( p^{(m+1)/2} \).

**Proposition 4.4.** Say \( p \nmid \Delta(c) \). Then \( S_c(p^l) = 0 \) for all integers \( l \geq 2 \).

**Proof.** This follows immediately from [HB98, Lemma 4.4].

Fix an approximation \( \Psi \) of \( \Phi \). Recall the definition of \( S \) from (2.2). For each \( c \in S \), we have \( S_c = a_c' * b_c \) by (2.9). The following result controls the coefficients \( a_c' \) and \( b_c \).

**Proposition 4.5.** Let \( c \in S \). Then \( a_c'(n) \) is multiplicative in \( n \). Moreover, for all primes \( p \) and integers \( k \geq 1 \), we have
\[
a_c'(p) \cdot 1_{p \nmid \Delta(c)} \ll p^{-1/2},
\]
\[
\max(|a_c'(p^k)|, |b_c(p^k)|) \ll p^{ke} + p^{ke} \sum_{d \mid p^k} |S_c^2(d)| \cdot 1_{p \mid \Delta(c)}.
\]

**Proof.** By (2.9), we have \( (a_c * b_c)(n) = \mathbb{1}_{n=1} \) and \( a_c' = S_c^2 * a_c \). Since \( b_c, S_c^2 \) are multiplicative, it follows that \( a_c, a_c' \) are too. It remains to bound \( a_c'(p^k), b_c(p^k) \). When \( p \nmid \Delta(c) \), there is nothing to prove, since condition (2) in Definition 2.2 already gives what we want. Now assume \( p \mid \Delta(c) \). Then condition (3) in Definition 2.2 gives \( a_c'(p) \ll p^{-1/2} \). On the other hand, \( E_c^2(p) \ll 1 \) by (2.7) and (2.6). Therefore, condition (2) in Definition 2.2 gives
\[ b_c(p^k), a_c'(p^k) \ll p^{ke} \sum_{d \mid p^k} |S_c^2(d)| \ll p^{ke}, \]
because \( S_c^2(p) = E_c^2(p) + O(p^{-1/2}) \ll 1 \) by Proposition 4.3 and \( S_c^2(p') \cdot 1_{l \geq 2} = 0 \) by Proposition 4.4. This completes the proof.

Let \( \omega(n) \) denote the number of distinct prime factors of \( n \). The following result, which is due to [Hoo86, HB98], gives a general pointwise bound on \( S_c^2(n) \).

**Proposition 4.6.** For some constant \( A_F > 0 \), we have
\[ n^{-1/2}|S_c^2(n)| \leq A_F^{\omega(n)} \prod_{1 \leq i \leq m} \gcd(cub(n)^2, \gcd(cub(n), sq(c_i))^3)^{1/12} \]
for all \( c \in \mathbb{Z}^m \) and integers \( n \geq 1 \).

**Proof.** By definition, \( S_c^2(n) = n^{(m+1)/2}S_c(n) \). Moreover, since \( F \) is diagonal, we have
\[ S_c(p^l) \ll_F p^{(1+m/2)} \prod_{1 \leq i \leq m} \gcd(cub(p^l), \gcd(cub(p^l), sq(c_i))^3)^{1/12}, \]
by [HB98, (5.1) and (5.2)] for \( l \geq 2 \) and [HB83, Lemma 11] for \( l = 1 \). The desired result follows immediately from the multiplicativity of \( S_c \).
We have stated Proposition 4.6 uniformly over \( c \in \mathbb{Z}^m \). We proceed to analyze the vectors \( c \) in sets based on which coordinates \( c_i \) are nonzero. For the rest of §4, we fix a set
\[
\mathcal{I} \subseteq \{1, 2, \ldots, m\}.
\]

Let
\[
\mathcal{R} := \{ c \in \mathbb{Z}^m \cap [-Z, Z]^m : 1_{c_i \neq 0} = 1_{i \in \mathcal{I}} \text{ for all } i \in \{1, 2, \ldots, m\} \}.
\]

By definition, if \( c \in \mathcal{R} \), then \( c_i \neq 0 \) if and only if \( i \in \mathcal{I} \).

Proposition 4.6 implies that for all \( c \in \mathcal{R} \) and integers \( n \geq 1 \), we have
\[
n^{-1/2}S^2_c(n) \ll n \cdot \text{cub}(n)^{(m-|\mathcal{I}|)/6} \prod_{i \in \mathcal{I}} \text{gcd}(\text{cub}(n), \text{sq}(c_i))^{1/4}.
\]

We will repeatedly use \((4.9)\) later in the present paper. We now turn to \( I_c(n) \).

**Lemma 4.7** ([HB96, HB98]). Assume \( |\mathcal{I}| \geq 1 \). Then uniformly over \( c \in \mathcal{R} \), reals \( n \geq 1 \), and integers \( k \in \{0, 1\} \), we have
\[
n^k(\partial/\partial n) I_c(n) \ll_{k, \epsilon} X^{m+\epsilon} \left( \frac{X\|c\|}{n} \right)^{1-(m+|\mathcal{I}|)/4} \prod_{i \in \mathcal{I}} \left( \frac{\|c\|}{|c_i|} \right)^{1/2}.
\]

**Proof.** By [HB98, Lemma 3.2], since \( F \) is diagonal, we have
\[
n^k(\partial/\partial n) I_c(n) \ll_{k, \epsilon} \left( \frac{X\|c\|}{n} \right)^{X^{m+\epsilon}} \prod_{1 \leq i \leq m} \min \left[ \left( \frac{n}{X|c_i|} \right)^{1/2}, \left( \frac{n}{X\|c\|} \right)^{1/4} \right]
\]
\[
\leq \left( \frac{X\|c\|}{n} \right)^{X^{m+\epsilon}} \prod_{i \in \mathcal{I}} \left( \frac{n}{X|c_i|} \right)^{1/2} \prod_{i \notin \mathcal{I}} \left( \frac{n}{X\|c\|} \right)^{1/4}.
\]

After writing \( (n/X|c_i|)^{1/2} = (n/X\|c\|)^{1/2} (|c_i|/\|c\|)^{1/2} \) in the final line of \((4.10)\), the desired inequality follows from the fact that \( 1 - |\mathcal{I}|/2 - (m - |\mathcal{I}|)/4 = 1 - (m + |\mathcal{I}|)/4 \).

For later convenience, we now make a definition: for \( c \in \mathbb{Z}^m \) and integers \( N \geq 1 \), let
\[
\|I_c\|_{1, \infty; N} := \sup_{n \in \mathbb{R}, N \leq n \leq 4N} \left( |I_c(n)| + |n(\partial/\partial n) I_c(n)| \right).
\]

In the rest of §4, we will concern ourselves only with \( c \in \mathcal{R} \) such that \( \Delta(c) \neq 0 \). If \( |\mathcal{I}| = 0 \), then no such \( c \) exist, because \( \mathcal{R} = \{0\} \) by \((4.8)\). Therefore, we may and do assume \( |\mathcal{I}| \geq 1 \) for the rest of §4. To proceed further, we break \( \mathcal{R} \) into dyadic pieces. For each \( i \in \mathcal{I} \), let \( C_i \in \{2^t : t \in \mathbb{Z}_{\geq 0}\} \) with \( 1 \leq C_i \leq 2Z \). Write
\[
\mathcal{C} := \{ c \in \mathcal{R} : |c_i| \in \{C_i, 2C_i\} \text{ for all } i \in \mathcal{I} \}, \quad C := \max_{i \in \mathcal{I}}(C_i).
\]

**Proposition 4.8.** Suppose \( N_0 \in \mathbb{Z}_{\geq 1} \) and \( N_0 \ll X^{O(1)} \). Then
\[
\sum_{c \in \mathcal{C} : \Delta(c) \neq 0} \left( \sum_{n_0 \in [N_0, 2N_0]} |a_c'(n_0)| \right)^2 \ll_{\epsilon} X^{\epsilon} N_0^{1+(m-|\mathcal{I}|)/3} \prod_{i \in \mathcal{I}} C_i.
\]
Proof. Consider an integer \( n_0 \in [N_0, 2N_0] \). If \( n_c := \prod_{p \mid \Delta(c)} p^{\nu_p(n_0)} \) and \( n_2 := \text{sq}(n_0/n_c) \), then Proposition 4.5 implies
\[
a'(n_0) = a'(\frac{n_0}{n_c n_2}) \cdot a'(n_2) \cdot a'(n_c)
\leq \epsilon \left( \frac{n_0}{n_c n_2} \right)^{1/2+\epsilon} \cdot n_2 \cdot |a'(n_c)|
\leq n_0^{-1/2+\epsilon} (n_c n_2)^{1/2} |a'(n_c)|.
\]
Since \( n_c \mid \Delta(c) \infty \) and \( n_2 \) is square-full, we find, upon summing over \( n_0 \), that
\[
\sum_{n_0 \in [N_0, 2N_0]} |a'(n_0)| \ll \epsilon \sum_{n_0 n_2 \leq 2N_0; n_c \mid \Delta(c) \infty, n_2 = \text{sq}(n_2)} \frac{N_0}{n_c n_2} \cdot N_0^{-1/2+\epsilon} (n_c n_2)^{1/2} |a'(n_c)|
\ll \epsilon N_0^{1/2+2\epsilon} \sum_{n_0 n_2 \leq 2N_0; n_c \mid \Delta(c) \infty} n_c^{-1/2} |a'(n_c)|
\ll \epsilon N_0^{1/2+2\epsilon} (N_0 C) \max_{n_c \leq 2N_0; n_c \mid \Delta(c) \infty} n_c^{-1/2} |a'(n_c)|,
\]
where we have used Lemma 3.3(1) to sum over \( n_2 \leq 2N_0/n_c \), and then used Lemma 3.2 to bound the sum over \( n_c \) by a maximum. Furthermore,
\[
\max_{n_c \leq 2N_0; n_c \mid \Delta(c) \infty} n_c^{-1/2} |a'(n_c)| \ll \epsilon N_0^{2\epsilon} \max_{d \leq 2N_0; d \mid \Delta(c) \infty} d^{-1/2} |S^2_c(d)|,
\]
since \( a'(n_c) \ll \epsilon n_c \sum_{d \mid n_c} |S^2_c(d)| \) by condition (2) in Definition 2.2. But
\[
\sum_{c \in C: \Delta(c) \neq 0} \max_{d \leq 2N_0; d \mid \Delta(c) \infty} d^{-1} |S^2_c(d)|^2 \leq \sum_{c \in C} \max_{d \leq 2N_0} d^{-1} |S^2_c(d)|^2
\ll \epsilon N_0^{(m-|I|)/3+2\epsilon} \sum_{c \in C} \prod_{i \in I} \text{sq}(c_i)^{1/2}
\]
by (4.9), since \( \gcd(\text{cub}(d), \text{sq}(c_i))^{1/4} \leq \text{sq}(c_i)^{1/4} \). Yet
\[
(4.13) \sum_{c \in C} \prod_{i \in I} \text{sq}(c_i)^{1/2} \ll \prod_{i \in I} C_i^{1+\epsilon},
\]
by Lemma 3.3(2). Proposition 4.8 follows upon combining the previous four displays. \( \Box \)

We are now prepared to prove a crucial bound for \( 5 \).

Lemma 4.9. Suppose \( N_0, N \in \mathbb{Z}_{\geq 1} \) and \( N_0, N \ll X^{O(1)} \). Let
\[
Q_c = \|I_c\|_{1,\infty; N} \sum_{n_0 \in [N_0, 2N_0]} |a'(n_0)|.
\]
Then
\[
\left( \sum_{c \in C: \Delta(c) \neq 0} Q_c \right)^{1/2} \ll \epsilon X^{m+\epsilon} N_0^{1/2+(m-|I|)/6} (X/N)^{1-(m+|I|)/4} \max[Z^{1+|I|-m}/4, 1].
\]
Proof. With notation as in Proposition 4.8, consider an element $c \in C$. Then by (4.12), we have $|c_i| \geq C_i$ for all $i \in I$, whence $\|c\| \geq C$. Now (4.11) and Lemma 4.7 imply
\[
\|I_c\|_{1,\infty;N} \ll_{\varepsilon} X^{m+\varepsilon}(XC/N)^{1-(m+|I|)/4} \prod_{i \in I} (C/C_i)^{1/2},
\]
since $|I| \geq 1$. By Proposition 4.8, it follows that
\[
\sum_{c \in C: \Delta(c) \neq 0} Q_c^2 \ll_{\varepsilon} X^{2m+3\varepsilon} N_0^{1+(m-|I|)/3} (XC/N)^{2-(m+|I|)/2} \prod_{i \in I} C.
\]
By (4.12) we have $1 \leq C \leq Z$, since $1 \leq C_i \leq Z$ for all $i$. The quantity $C^{2-(m+|I|)/2} \prod_{i \in I} C = C^{2+(|I|-m)/2}$ is maximized either at $C = Z$ or $C = 1$, so we conclude that
\[
\sum_{c \in C: \Delta(c) \neq 0} Q_c^2 \ll_{\varepsilon} X^{2m+4\varepsilon} N_0^{1+(m-|I|)/3} (X/N)^{2-(m+|I|)/2} \max[Z^{2+(|I|-m)/2}, 1].
\]
Summing over all possibilities for $C$, we get
\[
\sum_{c \in C: \Delta(c) \neq 0} Q_c^2 \ll_{\varepsilon} X^{2m+4\varepsilon} N_0^{1+(m-|I|)/3} (X/N)^{2-(m+|I|)/2} \max[Z^{2+(|I|-m)/2}, 1].
\]
Lemma 4.9 follows upon taking a square root.

Having analyzed $I_c$ and $a_c'$ above, we now concentrate on $b_c$ for the rest of §4.

**Proposition 4.10.** Let the $C_i$, as well as $C$ and $C$, be as specified before Proposition 4.8. Suppose $N_1 \in \mathbb{Z}_{\geq 1}$ and $N_1 \ll X^{O(1)}$. Then
\[
\sum_{c \in C: \Delta(c) \neq 0} \left( \sum_{n_1 \in [N_1, 2N_1]} |b_c(n_1)| \right)^2 \ll_{\varepsilon} X^e N_1^{\max(2,1+(m-|I|)/3)} \prod_{i \in I} C_i.
\]

**Proof.** We mimic the proof of Proposition 4.8. Consider an integer $n_1 \in [N_1, 2N_1]$. If $n_c := \prod_{p|\Delta(c)} p^{\nu_p(n_1)}$, then by Proposition 4.5 and the multiplicativity of $b_c$, we have
\[
b_c(n_1) = b_c(n_1/n_c) b_c(n_c) \ll_{\varepsilon} (n_1/n_c)^e |b_c(n_c)| \leq n_1^e |b_c(n_c)|.
\]
Upon summing over $n_1$, then,
\[
\sum_{n_1 \in [N_1, 2N_1]} |b_c(n_1)| \ll_{\varepsilon} \sum_{n \leq 2N_1: n \in \Delta(c) = 0} \frac{N_1}{n_e} |b_c(n_e)| \ll_{\varepsilon} N_1^{1+2e} C^e \max_{n_1 \leq 2N_1} n_1^{-1} |b_c(n)|
\]
by Lemma 3.2. Condition (2) in Definition 2.2 implies
\[
\max_{n_1 \leq 2N_1} n_1^{-1} |b_c(n)| \ll_{\varepsilon} N_1^{2e} \max_{n_1 \leq 2N_1} n_1^{-1} |S^2_c(n)|.
\]
But by (4.9), we have
\[
\sum_{n \leq 2N_1} \max_{n \leq 2N_1} n^{-2} |S^2_c(n)|^2 \ll_{\varepsilon} N_1^{2e} \max_{n_1 \leq 2N_1} n_1^{-1+(m-|I|)/3} \sum_{c \in C: i \in I} \|c_i\|^{1/2}.
\]
The desired result follows upon combining the last three displays with (4.13).
Lemma 4.11. Suppose \( N_1 \in \mathbb{Z}_{\geq 1} \) and \( N_1 \ll X^{O(1)} \). Then
\[
\sum_{c \in \mathbb{R} : \Delta(c) \neq 0} \left( \sum_{n_1 \in [N_1, 2N_1]} |b_c(n_1)| \right)^2 \ll_{\epsilon} X^{\epsilon} N_1^{\max(2, 1 + (m - |I|)/3)} |I|.
\]

Proof. This follows from Proposition 4.10 upon summing over all possibilities for \( C \). \( \square \)

We need the following lemma in \( \S 5 \). Let
\[
(4.14) \quad \beta := 1 + 10 \cdot M_0(F, w) \ll 1.
\]

Lemma 4.12. Assume Hypothesis 2.4. Then
\[
(4.15) \quad \sum_{c \in \mathbb{R} : \Delta(c) \neq 0} \left| \sum_{n_1 \in I} b_c(n_1) \right|^2 \ll_{\epsilon} \min \left( X^\epsilon Z^m N_1, X^\epsilon Z |I|^n N_1^{\max(2, 1 + (m - |I|)/3)} \right),
\]
for all positive integers \( N_1 \leq \beta Y \) and real intervals \( I \subseteq [N_1, 2N_1] \).

Proof. The bound \( X^\epsilon Z^m N_1 \) in (4.15) follows upon applying (2.10) with \( C = (2 \beta)^{1/3} Z \) and \( N = 2N_1 \). Meanwhile, \( X^\epsilon Z |I|^n N_1^{\max(2, 1 + (m - |I|)/3)} \) comes from Lemma 4.11. \( \square \)

5. Contribution from smooth hyperplane sections

Recall the key quantity \( \Sigma_0 \) from (4.6), involving a sum over \( c \in [-Z, Z]^m \). In this section, we concentrate on vectors \( c \in \mathcal{S}(Z) = \mathcal{S} \cap [-Z, Z]^m \), in the notation of (2.2). Let
\[
\Sigma_1 := X^{-3} \sum_{c \in \mathcal{S}(Z)} \sum_{n_1 \geq 1} n^{(1-m)/2} S_c^c(n) I_c(n).
\]
We will prove the following result:

Theorem 5.1. Assume Hypothesis 2.4. Then
\[
(5.1) \quad \Sigma_1 \ll_{\epsilon_0} X^{3(m-2)/4 + O(\epsilon_0)}.
\]

For each \( n \geq 1 \), we have \( S_c^c(n) = \sum_{n_0 | n_1 = n} a'_c(n_0) b_c(n_1) \), since \( S_c^c = a'_c * b_c \) by (2.9). Thus
\[
(5.2) \quad \Sigma_1 = X^{-3} \sum_{c \in \mathcal{S}(Z)} \sum_{n_0 \geq 1} a'_c(n_0) \sum_{n_1 \geq 1} (n_0 n_1)^{(1-m)/2} I_c(n_0 n_1) b_c(n_1).
\]
By Proposition 4.1, we have \( I_c(n) = 0 \) when \( n > \beta Y/10 \), where \( \beta \) is as in (4.14). Thus
\[
(5.3) \quad \Sigma_1 = X^{-3} \sum_{c \in \mathcal{S}(Z)} \sum_{(N_0, N_1) \in \mathcal{A}} \cap_{c, N_0, N_1} n_0 n_1 \leq \beta Y/10 \},
\]
where
\[
\mathcal{A} := \{(N_0, N_1) \in \{2^t : t \in \mathbb{Z}_{\geq 0}\}^2 : N_0 N_1 \leq \beta Y/10 \},
\]
\[
\cap_{c, N_0, N_1} := \sum_{n_0 \in [N_0 \cdot 2N_0]} a'_c(n_0) \sum_{n_1 \in [N_1 \cdot 2N_1]} (n_0 n_1)^{(1-m)/2} I_c(n_0 n_1) b_c(n_1).
\]
For convenience, let \( N := N_0 N_1 \), let \( B_c(J) := \sum_{n_1 \in J} b_c(n_1) \) for intervals \( J \), and let
\[
\varnothing_{c, n_0, N_1} := \sum_{n_1 \in [N_1 \cdot 2N_1]} \sum_{n_0 \in [N_0 \cdot 2N_0]} (n_0 n_1)^{(1-m)/2} I_c(n_0 n_1) b_c(n_1).
\]
Recall \( \|I_c\|_{1, \infty; N} \) from (4.11). We now have enough notation to state a key lemma:
Lemma 5.2. Let \((N_0, N_1) \in \mathcal{A}\). Then there exists a probability measure \(\nu = \nu_{N_0, N_1}\), supported on the real interval \([N_1, 2N_1]\), such that for all \(c \in \mathcal{S}\) and \(n_0 \in \mathbb{Z} \cap [N_0, 2N_0]\), we have
\[
\mathcal{V}_{e,n_0,N_1} \leq N^{(1-m)/2} \|I_e\|_{1,\infty;N} \int_{x \in [N_1, 2N_1]} |B_e([N_1, x])| \, d\nu(x).
\]

Proof. Let \(c \in \mathcal{S}\) and \(n_0 \in \mathbb{Z} \cap [N_0, 2N_0]\). For brevity, let \(I(n) = n^{(1-m)/2} I_e(n)\). Then
\[
\mathcal{V}_{e,n_0,N_1} = \sum_{n_1 \in [N_1, 2N_1]} I(n_0 n_1) : b_c(n_1).
\]

By partial summation over \(n_1\), it follows that
\[
|\mathcal{V}_{e,n_0,N_1}| \leq \|I(r)\|_{L^\infty([N_1, 2N_1])} |B_e([N_1, 2N_1])| + n_0 \|I'(r)\|_{L^\infty([N_1, 2N_1])} \sum_{k \in [N_1, 2N_1]} |B_e([N_1, k])|
\]
\[
\ll \|I(r)\|_{L^\infty([N_1, 2N_1])} |B_e([N_1, 2N_1])| + \frac{N}{N_1} \|I'(r)\|_{L^\infty([N_1, 2N_1])} \sum_{k \in [N_1, 2N_1]} |B_e([N_1, k])|,
\]
where \(\|f(r)\|_{L^\infty([N_1, 2N_1])} := \sup_{r \in [N_1, 2N_1]} |f(r)|\) for continuous functions \(f : [N, 4N] \to \mathbb{C}\). Here
\[
\max(\|I(r)\|_{L^\infty([N_1, 2N_1])}, N \|I'(r)\|_{L^\infty([N_1, 2N_1])}) \ll N^{(1-m)/2} \|I_e\|_{1,\infty;N}
\]
by (4.11). Finally, let
\[
\nu := \frac{1}{2} \left( \delta_{2N_1} + \frac{1}{N_1} \sum_{k \in [N_1, 2N_1]} \delta_k \right),
\]
where \(\delta_k\) is the Dirac measure supported on the singleton set \(\{k\}\). Then \(\nu\) is a probability measure supported on \([N_1, 2N_1]\). Also, the last three displays imply (5.4). \qed

Let \((N_0, N_1) \in \mathcal{A}\). Let \(\mathcal{I}\) and \(\mathcal{R}\) be as in (4.7) and (4.8), respectively. Since we are presently only interested in \(c \in \mathcal{S}\), we may and do assume \(|\mathcal{I}| \geq 1\). For each \(c \in \mathcal{S}\), we have
\[
|\mathcal{V}_{e,N_0,N_1}| \leq \sum_{n_0 \in [N_0, 2N_0]} |a'_e(n_0) \mathcal{V}_{e,n_0,N_1}|
\]
\[
\ll N^{(1-m)/2} \|I_e\|_{1,\infty;N} \sum_{n_0 \in [N_0, 2N_0]} |a'_e(n_0)| \int_{x \in [N_1, 2N_1]} |B_e([N_1, x])| \, d\nu(x),
\]
where the first and second inequality are justified by the triangle inequality and Lemma 5.2 respectively. Abbreviating \(B_e([N_1, x])\) to \(B_e(x)\) for convenience, we deduce that
\[
\sum_{e \in \mathcal{R} : \Delta(e) \neq 0} |\mathcal{V}_{e,N_0,N_1}| \ll \varepsilon X^{m+\varepsilon} Q_1 \left( \sum_{e \in \mathcal{R} : \Delta(e) \neq 0} \left( \int_{x \in [N_1, 2N_1]} |B_e(x)| \, d\nu \right)^2 \right)^{1/2}\]
by the Cauchy–Schwarz inequality and Lemma 4.9 where
\[
Q_1 := N^{(1-m)/2} N_0^{1/2+(m-|\mathcal{I}|)/6} (X/N)^{1-(m+|\mathcal{I}|)/4} \max[Z^{1+(|\mathcal{I}|-m)/4}, 1].
\]

Now, for the rest of \(\S\), we assume Hypothesis 2.4. We have
\[
\left( \int_{x \in [N_1, 2N_1]} |B_e(x)| \, d\nu \right)^2 \ll \int_{x \in [N_1, 2N_1]} |B_e(x)|^2 \, d\nu
\]
by the Cauchy–Schwarz inequality, so
\[
\left( \sum_{c \in \mathbb{R} : \Delta(c) \neq 0} \left( \int_{x \in [N_1, 2N_1]} |B_c(x)| \, dv \right)^2 \right)^{1/2} \ll \left( \int_{x \in [N_1, 2N_1]} \sum_{c \in \mathbb{R} : \Delta(c) \neq 0} |B_c(x)|^2 \, dv \right)^{1/2}
\]
\[\ll_{\epsilon} X^\epsilon Q_2 \]
by (4.15), where
\[
Q_2 := \min \left( Z^m N_1, Z^{m} N_1^{\max(2,1+(m-|I|)/3)} \right)^{1/2}.
\]

**Lemma 5.3.** We have \(Q_1 Q_2 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}\).

**Proof.** We split the proof into four cases.

Case 1: \(|I| = m\). Then \(Q_2 = (Z^m N_1)^{1/2}\), since \(|I| = m\) and \(N_1 \geq 1\). Therefore, \(Q_1 Q_2 = Q_3\), where
\[
Q_3 := Z^{m/2} N_1^{-1/2} \cdot N^{(1-m)/2} N_0^{1/2} (X/N)^{1-m/2} \max[Z, 1].
\]
But \(Q_3 = Z^{m/2} X^{1-m/2} \max[Z, 1]\), since \(N_1 N_0 = N\). By (4.2) we have \(Z = X^{1/2+\epsilon_0} \geq 1\), so
\[
Q_3 = X^{1-m/2} Z^{1+m/2} = X^{3/2-m/4+1+m/2} \epsilon_0.
\]
Thus \(Q_1 Q_2 = Q_3 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}\).

Case 2: \(|I| = m-1\) and \(N_1 \geq Z\). Then \(Q_2 = (Z^m N_1)^{1/2}\), by (5.7). Therefore, \(Q_1 Q_2 = Q_4\), where
\[
Q_4 := Z^{m/2} N_1^{-1/2} N_0^{(m-|I|)/6} (X/N)^{1-(m+|I|)/4} \max[Z^{1+(|I|-m)/4}, 1],
\]
since \(N_1 N_0 = N\). Since \((m - |I|)/6 \geq 0\) and \(N_0 = N/|N| \leq N/Z\), we have
\[
Q_4 \leq Z^{m/2} N_1^{-1/2} N_0^{(m-|I|)/6} (X/N)^{1-(m+|I|)/4} \max[Z^{1+(|I|-m)/4}, 1].
\]
The right-hand side of (5.9) is decreasing as a function of \(N\), because
\[
1 - m/2 + (m - |I|)/6 - 1 + (m + |I|)/4 = (|I| - m)/12 < 0.
\]
Since \(N \geq N_1 \geq Z\), it follows that
\[
Q_4 \leq Z^{m/2} X^{1-(m+|I|)/8} \max[Z^{1+(|I|-m)/4}, 1] 
\ll_{\epsilon_0} X^{O(\epsilon_0)} X^{1-(m+|I|)/8} \max[Z^{1+(|I|-m)/4}, 1],
\]
since \(Z = X^{1/2+\epsilon_0}\). But \(|I| = m - 1\), so
\[
X^{1-(m+|I|)/8} \max[X^{1/2+(|I|-m)/8}, 1] = \max[X^{3/2-m/4}, X^{9/8-m/4}] = X^{3/2-m/4}.
\]
Thus \(Q_1 Q_2 = Q_4 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}\).

Case 3: \(1 \leq |I| \leq m-2\). By (5.7), we have \(Q_2 \leq (Z^|I| N_1^{\max(2,1+(m-|I|)/3)})^{1/2}\). Since \(N_1 N_0 = N\), it follows that \(Q_1 Q_2 \leq Q_5\), where
\[
Q_5 := Z^{|I|/2} N_1^{\max(1,2,|I|)/6} N_1^{-m/2} N_0^{(m-|I|)/6} (X/N)^{1-(m+|I|)/4} \max[Z^{1+(|I|-m)/4}, 1].
\]
Since \(N_0 \geq 1\) and \(N_1 N_0 = N\), we have \(N_1^{\max(1,2,|I|)/6} N_0^{(m-|I|)/6} \leq N^{\max(1,2,|I|)/6}\). Thus
\[
Q_5 \leq Z^{|I|/2} N^{\max(1,2,|I|)/6} N_1^{-m/2} (X/N)^{1-(m+|I|)/4} \max[Z^{1+(|I|-m)/4}, 1].
\]
The right-hand side of (5.12) is weakly decreasing in \(N\), because
\[
\max(1/2, (m - |I|)/6) + 1 - m/2 - 1 + (m + |I|)/4 = \max(1/2, (m - |I|)/6) + (|I| - m)/4 
\leq 0,
\]
We begin with a slight extension of [Vau97, Lemma 4.9]. This yields the desired inequality, (5.1).

Thus

Taking \( t \) where

\[
\max(N, N) = \mathcal{Z}
\]

§ [HB96]. We roughly follow [HB96, par. 2]. Let

\[
Q_2 \leq Q_5 \ll \epsilon_0 X^{3/2-m+O(\epsilon_0)}.
\]

\[
Q_6 := Z^{1/2} N_1 \max(1/2, (m-|I|)/6) N_1^{1-m/2} (X/N_1) \max[Z^{1+(|I|-m)/4}, 1].
\]

But \( Q_6 \) is increasing in \( N_1 \), because

\[
\max(1/2, (m-|I|)/6) + 1 - m/2 - 1 + (m + |I|)/4 = 1/4 > 0,
\]

in view of the equality \(|I| = m - 1\) and \(|I| = m - 1\). Since \( N_1 \leq Z \) and \(|I| = m - 1\), it follows that

\[
Q_6 \ll \epsilon_0 X^{3/2-m+O(\epsilon_0)}.
\]

Thus \( Q_1 Q_2 \leq Q_5 \ll Q_6 \ll \epsilon_0 X^{3/2-m+O(\epsilon_0)} \).

**Remark 5.4.** Interestingly, the quantity \( Q_3 \) in (5.8) is constant over \((N_0, N_1) \in \mathcal{A}\).

By Lemma 5.3 the left-hand side of (5.5) is \( \ll \epsilon_0 X^{m+O(\epsilon_0)} X^{3/2-m/4} \). Upon summing over \((N_0, N_1) \in \mathcal{A}\) and the set of \( 2^m - 1 \) possible sets \( \mathcal{R} \), it follows from (5.3) that

\[
\Sigma_1 \ll \epsilon_0 X^{-3} X^{m+O(\epsilon_0)} X^{3/2-m/4} = X^{3(m-2)/4+O(\epsilon_0)}.
\]

This yields the desired inequality, (5.1).

### 6. Contribution from the Central Terms

Here we address the \( \epsilon = 0 \) contribution to (4.6), using the theory of \( I_0(n) \) developed in [HB96]. We roughly follow [HB96 §12, par. 2]. Let

\[
(6.1) \quad \Sigma_2 := X^{-3} \sum_{n \geq 1} n^{(1-m)/2} S_0^2(n) I_0(n).
\]

We begin with a slight extension of [Vau97, Lemma 4.9].

**Lemma 6.1.** If \( N \geq 1 \), then

\[
\sum_{n \in [N, 2N]} n^{-m} |S_0(n)| \ll \epsilon N^{(4-m)/3+\epsilon}.
\]

**Proof.** We have \( S_0(n) \ll \epsilon n^{1/2+\epsilon} \text{cup}(n)^{m/6} \) by Proposition 4.6. Thus

\[
n^{-m} S_0(n) \ll \epsilon n^{-m/2+\epsilon} \text{cup}(n)^{m/6}.
\]

Taking \( t = m/6 \) in Lemma 3.3(3), we get

\[
\sum_{n \in [N, 2N]} n^{-m} |S_0(n)| \ll \epsilon N^{1/m^{2}+\epsilon} \max(N, N^{1/3+m/6}) = N^{(4-m)/3+\epsilon},
\]

where we note that \( \max(N, N^{1/3+m/6}) = N^{1/3+m/6} \) because \( N \geq 1 \) and \( m \geq 4 \).
Lemma 6.1 implies, in particular, the familiar fact that the singular series

\[ \mathcal{S} := \sum_{n \geq 1} n^{-m} S_0(n) \]

converges absolutely for \( m \geq 5 \). It is also known that the real density

\[ \sigma_{\infty,w} := \lim_{\epsilon \to 0} (2\epsilon)^{-1} \int_{|F(x)| \leq \epsilon} w(x) \, dx \]

exists; see e.g. [HB96, Theorem 3]. Yet for all \( n \ll Y \), [HB96, Lemma 13] implies

\[ X^{-m} I_0(n) = \sigma_{\infty,w} + O((n/Y)^4), \]

for all \( A > 0 \). If \( m \geq 5 \), then via (6.4) with \( A = (m-4)/3 \), we get

\[
\sum_{n \leq M_0(F,w)Y} n^{-m} S_0(n) X^{-m} I_0(n) \\
= \sigma_{\infty,w} \sum_{n \leq M_0(F,w)Y} n^{-m} S_0(n) + \sum_{n \leq M_0(F,w)Y} O((n/Y)^{(m-4)/3} n^{-m} |S_0(n)|) \\
= \sigma_{\infty,w} \mathcal{S} + O(X^{(4-m)/3+\epsilon}),
\]

by Lemma 6.1 and (6.2). Also, by Proposition 4.1, we have \( I_0(n) = 0 \) for all \( n > M_0(F,w)Y \).

Since \( n^{-m} S_0(n) = n^{(1-m)/2} S_0^R(n) \) and \( Y = X^{3/2} \), it follows that if \( m \geq 5 \), then

\[ \Sigma_2 = X^{m-3} [\sigma_{\infty,w} \mathcal{S} + O(\epsilon X^{(4-m)/2+\epsilon})] = \sigma_{\infty,w} \mathcal{S} X^{m-3} + O(\epsilon X^{(m-2)/2+\epsilon}), \]

where \( \Sigma_2 \) is the quantity defined in (6.1). On the other hand, for all \( m \geq 4 \),

\[ \Sigma_2 \ll X^{m-3} \sum_{n \leq M_0(F,w)Y} n^{-m} |S_0(n)| \ll \epsilon X^{m-3+\epsilon}, \]

by Proposition 4.1 and Lemma 6.1, since \( I_0(n) \ll X^m \) by [HB96, Lemma 16].

7. Contribution from singular hyperplane sections

In this section, we study the quantity

\[ \Sigma_3 := X^{-3} \sum_{n \geq 1} \sum_{c \in [-Z,Z]^m: \Delta(c) = 0, \mathbf{c} \neq 0} n^{(1-m)/2} S_0^R(c) I_c(n). \]

We will prove the following result, extending work of Heath-Brown. Recall the definitions of \( N_{F,w}(X) \) and \( N'_{F,w}(X) \) from (2.14) and (2.15), respectively.

**Theorem 7.1.** If \( m \geq 5 \), then

\[ \Sigma_3 \ll \epsilon_0 \, X^{3(m-2)/4 + O(\epsilon_0)}. \]

If \( m = 4 \), then

\[ \Sigma_3 = N_{F,w}(X) - N'_{F,w}(X) + O(\epsilon_0 \, X^{3(m-2)/4 + O(\epsilon_0)}).
\]

The cases \( m = 4 \) and \( m = 6 \) of this result are due to Heath-Brown. For instance, the estimate (7.3) for \( m = 4 \) follows directly from [HB98, Lemmas 7.2 and 8.1], in view of the tail estimate (4.3). Therefore, we may and do assume \( m \geq 5 \), for the rest of this section.

We combine ideas from [Hoo86] and [HB98]. Let \( I \) and \( R \) be as in (4.7) and (4.8), respectively. Since we are only interested in \( \mathbf{c} \neq 0 \), we may and do assume \( |I| \geq 1 \). Let \( C \) and \( C' \) be as in (4.12), for some \( C_i \in \{2^t : t \in \mathbb{Z}_{\geq 0}\} \) with \( 1 \leq C_i \leq Z \).
By Proposition 4.1, the sum $\Sigma_3$ from (7.1) is supported on $n \leq M_0(F, w) Y$. Let

\begin{equation}
\Sigma_4 := X^{-3} \sum_{n \leq M_0(F, w) Y} \sum_{c \in C : \Delta(c) = 0} n^{(1-m)/2} \zeta_c(n) I_c(n).
\end{equation}

Now consider an element $c \in C$ with $\Delta(c) = 0$, assuming such a $c$ exists. Denote the nonempty fibers of the map $\mathcal{I} \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, $i \mapsto F_i c_i \equiv g_k \mod (\mathbb{Q}^\times)^2$ by

\[ \mathcal{I}(k) := \{ i \in \mathcal{I} : F_i c_i \equiv g_k \mod (\mathbb{Q}^\times)^2 \}, \]

for $1 \leq k \leq K$, say, where the $g_k$ are signed, nonzero square-free integers. Trivially, we have $\sum_{1 \leq k \leq K} |\mathcal{I}(k)| = |\mathcal{I}|$. For each $i \in \mathcal{I}(k)$, we may write

\begin{equation}
c_i = g_k F_i^{-1} e_i^2
\end{equation}

with $e_i \in \mathbb{Z}$. Moreover, by (2.1) and the $\mathbb{Q}$-linear independence of square roots of distinct square-free integers, we may choose the signs of the integers $e_i$ so that

\begin{equation}
\sum_{i \in \mathcal{I}(k)} F_i (e_i/F_i)^3 = 0.
\end{equation}

Since $c_i \neq 0$ implies $e_i \neq 0$ for all $i \in \mathcal{I}(k)$, we immediately deduce from (7.6) that

\begin{equation}
|\mathcal{I}(k)| \geq 2.
\end{equation}

We now prove a general lemma that will allow us, in Lemma 7.3, to exploit the structure uncovered in the previous paragraph.

**Lemma 7.2.** Let $J \in \mathbb{Z}_{\geq 2}$, let $d_1, \ldots, d_J \in \mathbb{Z}_{\geq 1}$, and let $G, E_1, \ldots, E_J \in \mathbb{R}_{>0}$. Then

\[ \sum_{1 \leq g \leq G : \mu(g)^2 = 1} \prod_{1 \leq i \leq J} \sum_{1 \leq e_i \leq E_i : d_i \mid \text{sq}(ge_i^2)} d_i^{1/2} \leq \prod_{1 \leq i \leq J} (2^{\omega(d_i)} G^{1/2} E_i). \]

**Proof.** By Hölder’s inequality over $g$, we may assume that $E_1 = \cdots = E_J = E$ and $d_1 = \cdots = d_J = d$, say. Let $S := \{ h \mid d \mid \mu(h)^2 = 1 \}$. Now consider integers $g, e \geq 1$ with $g$ square-free. Then $\text{sq}(ge^2) = \gcd(g, e) e^2$. Therefore, if $d \mid \text{sq}(ge^2)$, and we let $h := \gcd(g, e, d)$, then

\[ h \in S, \quad (d/h) \mid e^2, \]

whence $e$ is divisible by the integer $\prod_{p \mid (d/h)} p^{v_p(d/h)/2} \geq (d/h)^{1/2}$. Thus, given $h \in S$, the number of possible $e \in [1, E]$ is at most $E/(d/h)^{1/2}$. It follows that

\begin{equation}
\sum_{1 \leq e \leq E : d \mid \text{sq}(ge^2)} d^{1/2} \leq \sum_{h \in S} (d^{1/2} \cdot 1_{h \mid g} \cdot E/(d/h)^{1/2}) = \sum_{h \in S} (1_{h \mid g} \cdot h^{1/2} E),
\end{equation}

where $1_{h \mid g}$ is the indicator function for $h \mid g$. This completes the proof. 

**Remark.** The above inequality is tight for $d = 1$. For $d > 1$, we can use the M"{o}bius inversion formula to show that

\[ \sum_{1 \leq e \leq E : d \mid \text{sq}(ge^2)} d^{1/2} \leq \sum_{h \in S} (d^{1/2} \cdot 1_{h \mid g} \cdot E/(d/h)^{1/2}) = \sum_{h \in S} (1_{h \mid g} \cdot h^{1/2} E), \]

where $1_{h \mid g}$ is the indicator function for $h \mid g$. This completes the proof. 

**Theorem 7.3.** Let $J \in \mathbb{Z}_{\geq 2}$, let $d_1, \ldots, d_J \in \mathbb{Z}_{\geq 1}$, and let $G, E_1, \ldots, E_J \in \mathbb{R}_{>0}$. Then

\[ \sum_{1 \leq g \leq G : \mu(g)^2 = 1} \prod_{1 \leq i \leq J} \sum_{1 \leq e_i \leq E_i : d_i \mid \text{sq}(ge_i^2)} d_i^{1/2} \leq \prod_{1 \leq i \leq J} (2^{\omega(d_i)} G^{1/2} E_i). \]
for every square-free $g \geq 1$. By (7.8), and Hölder’s inequality over $h$, we get

$$\sum_{1 \leq g \leq G: \mu(g)^2 = 1} \left( \sum_{1 \leq e \leq E; \mu(g)^2 = 1} d^{1/2} \right)^J \leq \sum_{1 \leq g \leq G: \mu(g)^2 = 1} \left( \sum_{h \in S} (1_{h | g} \cdot h^{1/2} E) \right)^J \leq \sum_{1 \leq g \leq G: \mu(g)^2 = 1} |S|^{J-1} \sum_{h \in S} (1_{h | g} \cdot h^{1/2} E)^J \leq |S|^{J-1} \sum_{h \leq G} (G/h)(h^{1/2} E)^J \leq |S|^J G^{J/2} E^J,$$

where in the last step we note that $h^{J/2-1} \leq G^{J/2-1}$. This suffices, since $|S| = 2^{\omega(d)}$. □

**Lemma 7.3.** Let $n \geq 1$ be an integer. Then

$$\sum_{c \in \mathcal{C}: \Delta(c) \neq 0} n^{-1} |S_c(n)|^2 \ll_{\epsilon} n^{\epsilon} \text{cub}(n)^{(m-|\mathcal{I}|)/3} \prod_{i \in \mathcal{I}} C_i^{1/2+\epsilon}.$$  

**Proof.** Let $n_3 := \text{cub}(n)$. Fix a set $\mathcal{J} \subseteq \mathcal{I}$ with $|\mathcal{J}| \geq 2$. Let $G \in \{2^t : t \in \mathbb{Z}_{\geq 0}\}$, and let $E_i := (2F_i C_i/G)^{1/2}$ for each $i \in \mathcal{J}$. Let $\tau(\cdot)$ be the divisor function. Then

$$\sum_{|g| \in [G, 2G]: \mu(g)^2 = 1} \prod_{i \in \mathcal{J}} \sum_{|e_i| \leq (2F_i C_i/g)^{1/2}; \mu(g)^2 = 1} \text{gcd}(n_3, \text{sq}(gF_i^{-1}e_i^2))^{1/2} \leq 2^{1+|\mathcal{J}|} \sum_{g \in [G, 2G]: \mu(g)^2 = 1} \prod_{i \in \mathcal{J}} \sum_{|e_i| \leq E_i} \text{gcd}(n_3, \text{sq}(ge_i^2))^{1/2} \leq 2^{1+|\mathcal{J}|} \prod_{i \in \mathcal{J}} (\tau(n_3)^2 G^{1/2} E_i) \leq 2^{1+|\mathcal{J}|} (\tau(n_3)^2)^{|\mathcal{J}|} \prod_{i \in \mathcal{J}} (2F_i C_i)^{1/2},$$

where in the penultimate step we use Lemma 7.2 for each possible choice of divisors $d_i | n_3$, and we note that $2^{\omega(d_i)} \leq 2^{\omega(n_3)} \leq \tau(n_3)$. Moreover, if $G > \min_{i \in \mathcal{J}} (2F_i C_i)$, then

$$\sum_{|g| \in [G, 2G]: \mu(g)^2 = 1} \prod_{i \in \mathcal{J}} \sum_{|e_i| \leq (2F_i C_i/g)^{1/2}; \mu(g)^2 = 1} \text{gcd}(n_3, \text{sq}(gF_i^{-1}e_i^2))^{1/2} = 0,$$

since the sum over one of the variables $e_i \in \mathbb{Z} \setminus \{0\}$ is empty. On summing the penultimate display over all $G \in \{2^t : t \in \mathbb{Z}_{\geq 0}\}$ with $G \leq \min_{i \in \mathcal{J}} (2F_i C_i)$, we conclude that

$$(7.9) \quad \sum_{g \in \mathbb{Z} \setminus \{0\}: \mu(g)^2 = 1} \prod_{i \in \mathcal{J}} \sum_{|e_i| \leq (2F_i C_i/g)^{1/2}; \mu(g)^2 = 1} \text{gcd}(n_3, \text{sq}(gF_i^{-1}e_i^2))^{1/2} \ll_{|\mathcal{J}|, \epsilon} n_3^{\epsilon} \prod_{i \in \mathcal{J}} (2F_i C_i)^{1/2+\epsilon}.$$
Recall the constraints (7.5) and (7.7) on \( \{ \mathbf{c} \in \mathcal{C} : \Delta(\mathbf{c}) = 0 \} \). Applying (7.9) with \( \mathcal{J} = \mathcal{I}(k) \), for each \( k \in [1, K] \), and multiplying the resulting \( K \) inequalities, we get

\[
\prod_{1 \leq k \leq K} \sum_{g_{k} \in \mathbb{Z} \setminus \{0\}} \prod_{\mu((g_{k})^{2}) = 1} \sum_{i \in \mathbb{C}(k) \mid \mu \leq (2F_{i}C_{i}/|g_{k}|)^{1/2}, \ g_{k}F_{i}^{-1}c_{i}^{2} \in \mathbb{Z} \setminus \{0\}} \gcd(n_{3}, \text{sq}(g_{k}F_{i}^{-1}c_{i}^{2}))^{1/2} \ll \epsilon n_{3}^{\epsilon} \prod_{i \in \mathcal{I}} C_{i}^{1/2 + \epsilon},
\]

(7.10) since \( K \leq |\mathcal{I}| \leq m \), and the variables \( m, F_{i} \) are fixed. On summing (7.10) over all possible choices for the sets \( \mathcal{I}(k) \subseteq \mathcal{I} \), we deduce that

\[
\sum_{\mathbf{c} \in \mathcal{C}, \Delta(\mathbf{c}) = 0} \prod_{i \in \mathcal{I}} \gcd(n_{3}, \text{sq}(c_{i}))^{1/2} \ll \epsilon n_{3}^{\epsilon} \prod_{i \in \mathcal{I}} C_{i}^{1/2 + \epsilon}.
\]

Lemma 7.3 follows immediately from (4.9) and (7.11).

**Remark 7.4.** Interestingly, the proof of (7.11) uses the constraint (7.6) only through (7.7).

Taking \( n_{3} = 1 \) in (7.11) implies

\[
|\{ \mathbf{c} \in \mathcal{C} : \Delta(\mathbf{c}) = 0 \}| \ll \epsilon \prod_{i \in \mathcal{I}} C_{i}^{1/2 + \epsilon}.
\]

Therefore, Lemma 7.3 implies

\[
\sum_{\mathbf{c} \in \mathcal{C}, \Delta(\mathbf{c}) = 0} n^{-1/2} |S_{\mathbf{c}}(n)| \ll \epsilon n^{\epsilon} \text{cub}(n)^{(m-|\mathcal{I}|)/6} \prod_{i \in \mathcal{I}} C_{i}^{1/2 + \epsilon},
\]

(7.12) by the Cauchy–Schwarz inequality over \( \mathbf{c} \).

Let \( N \in \{ 2^{i} : t \in \mathbb{Z}_{\geq 0} \} \) with \( 1 \leq N \leq M_{0}(F, w)Y \). By Lemma 4.7 (7.12), and the \( t = (m - |\mathcal{I}|)/6 \) case of Lemma 3.3(3), the sum

\[
\Sigma_{5} := X^{-3} \sum_{n \in [N, 2N]} \sum_{\mathbf{c} \in \mathcal{C}, \Delta(\mathbf{c}) = 0} n^{(1-m)/2} |S_{\mathbf{c}}^{2}(n)I_{\mathbf{c}}(n)|
\]

satisfies the bound \( \Sigma_{5} \ll \epsilon X^{m-3+\epsilon}Q_{7} \), where

\[
Q_{7} := N^{1-m/2} (|XC/N|)^{1-(m+|\mathcal{I}|)/4} \max(N, N^{1/3+(m-|\mathcal{I}|)/6})^{C(\mathcal{I})/2} = X^{-((|\mathcal{I}|+m)/4) \max(N^{1+(|\mathcal{I}|-m)/4}, N^{1/3+(|\mathcal{I}|-m)/12})^{C(1+(|\mathcal{I}|-m)/4)}.
\]

Since \( N^{1+(|\mathcal{I}|-m)/4} = (N^{1/3+(|\mathcal{I}|-m)/12})^{3} \), we will analyze \( Q_{7} \) according to the sign of \( \mathbf{c} := 1 + (|\mathcal{I}|-m)/4 \).

**Case 1:** \( \mathbf{c} \leq 0 \). Then, since \( N, C \gg 1 \), we have

\[
Q_{7} \ll X^{1-(|\mathcal{I}|+m)/4} \leq X^{(6-m)/4},
\]

where the final inequality holds because \( X \geq 1 \) and \( |\mathcal{I}| \geq -2 \).

**Case 2:** \( \mathbf{c} \geq 0 \). Then, since \( N \ll Y \) and \( C \ll Z \), we have

\[
Q_{7} \ll X^{1-(|\mathcal{I}|+m)/4}Y^{1+(|\mathcal{I}|-m)/4}Z^{1+(|\mathcal{I}|-m)/4}.
\]

Plugging in (4.1) and (4.2), we get

\[
Q_{7} \ll_{\epsilon_{0}} X^{1-(|\mathcal{I}|+m)/4+O(\epsilon_{0})} (X^{2})^{1+(|\mathcal{I}|-m)/4} = X^{3+(|\mathcal{I}|-3m)/4+O(\epsilon_{0})}.
\]

Moreover, if \( |\mathcal{I}| \leq 2m - 6 \), then \( 3 + (|\mathcal{I}|-3m)/4 \leq (6-m)/4 \).
If \( m \geq 6 \), then \( 1 \leq |I| \leq m \leq 2m - 6 \), so regardless of what \( |I| \) is, it follows that

\[
\begin{align*}
\Sigma_3 \ll_{\epsilon_0} X^{m-3+\epsilon_0} Q_7 \\
\ll_{\epsilon_0} X^{m-3+\epsilon_0} X^{(6-m)/4+O(\epsilon_0)} \\
= X^{3(m-2)/4+O(\epsilon_0)},
\end{align*}
\]

whence by summing over all possibilities for \( N \) and \( C \) we get

\[
\Sigma_3, \Sigma_4 \ll_{\epsilon_0} X^{3(m-2)/4+O(\epsilon_0)},
\]

where \( \Sigma_3, \Sigma_4 \) are as defined in (7.1) and (7.4), respectively. This completes the proof of (7.2) for \( m \geq 6 \). For the rest of §7, we relinquish the previous definitions of \( C \) and \( C \).

For \( m = 5 \), we first show that a natural extension of [HB98, Lemma 7.1] holds.

**Lemma 7.5.** If \( 5 \leq m \leq 6 \) and \( C \gg 1 \), then \( \left| \{ c \in \mathbb{Z}^m \cap [-C, C]^m : \Delta(c) = 0 \} \right| \ll_{\epsilon} C^{m-3+\epsilon} \).

**Proof.** For \( m = 6 \), this follows directly from [HB98, Lemma 7.1]. Now suppose \( m = 5 \). A partition of \( m \) is an infinite, weakly decreasing sequence of nonnegative integers \( \lambda_1, \lambda_2, \ldots \), such that \( \sum_{k \geq 1} \lambda_k = m \). For any partition of \( m \), let

\[
e_k := 2 \cdot 1_{2 \leq \lambda_k \leq 4} + (\lambda_k - 2) \cdot 1_{\lambda_k \geq 5}
\]

for \( k \geq 1 \). Let \( \theta \) denote the maximum value of \( \frac{1}{2} \sum_{k \geq 1} e_k \) over all partitions of \( m \). By [HB98, p. 687], we have \( \left| \{ c \in \mathbb{Z}^m \cap [-C, C]^m : \Delta(c) = 0 \} \right| \ll_{\epsilon} C^{m+\epsilon} \).

Clearly \( \lambda_3 \leq [m/3] = 1 \), so \( e_k = 0 \) for all \( k \geq 3 \). If \( \lambda_2 \leq 1 \), then \( e_k = 0 \) for all \( k \geq 2 \), so \( \sum_{k \geq 1} e_k = e_1 \leq m - 2 \). If \( \lambda_2 \geq 2 \), then \( \lambda_1 \leq m - \lambda_2 \leq 3 \), so \( e_k \leq 2 \) for all \( k \geq 1 \), whence \( \sum_{k \geq 1} e_k = e_1 + e_2 \leq 4 \). In either case, \( \sum_{k \geq 1} e_k \leq 4 \). Therefore, \( \theta \leq 2 = m - 3 \).

We now recall a bound from [HB98] that is valid for all \( m \geq 4 \).

**Lemma 7.6.** Fix \( \epsilon > 0 \). Suppose \( 1 \ll N \ll X^{3/2} \) and \( 1 \ll C \ll X^{1/2+\epsilon} \). Let

\[
A = \sum_{N < q \leq 2N} \sum_{C < ||c|| \leq 2C : \Delta(c) = 0} q^{-m} S_\epsilon(q) I_{\epsilon}(q).
\]

Then there exist reals \( X_1, X_2, X_3 \gg 1 \) and an integer \( H \geq 1 \) such that \( X_1 X_2 X_3 \asymp N \) and

\[
A \ll \epsilon X^{m+4\epsilon} N^{-m} X_1^{1+m/2} X_2^{2/3+2m/3} X_3^{1+2m/3} H^{1/2} \left( \frac{N}{XC} \right)^{(m-2)/4} N_1 N_2(H),
\]

where in terms of the quantity \( \mathcal{D} = 3(\prod_{1 \leq i \leq m} F_i)^{2m-2} \) from \( \square \) we let

\[
\begin{align*}
N_1 := & \sum_{(q_1,q_2,q_3) : X_i < q_i \leq 2X_i} 1_{\text{cub}(q_1) = 1} 1_{\text{cub}(q_2) = \text{cub}(q_3)} 1_{q_3|\mathcal{D}|^\infty}, \\
N_2(H) := & \sum_{C < ||c|| \leq 2C} 1_{\text{H}(c) \Delta(c) = 0}. 
\end{align*}
\]

**Proof.** This is immediate from [HB98, pp. 688–689], from the definition of \( A \) on p. 688 to the definition of \( N_2(H) \) on p. 689. What Heath-Brown calls \( P \) (resp. \( X \)), we call \( X \) (resp. \( N \)). Moreover, in terms of Heath-Brown’s notation \( n \) and \( G \), our \( m \) and \( \Delta \) satisfy \( m = n \) and \( \Delta(c) = 3G(c) \). However, our \( C, q, c, X_1, X_2, X_3, H \) match Heath-Brown’s notation. \( \square \)
Applying Lemma 3.1 to $q_2$ and Lemma 3.2 to $q_3$, it is clear that

$$N_1 \ll_{\varepsilon} X_1 X_2^{1/3} X_3^\varepsilon.$$  

Now assume $5 \leq m \leq 6$. Then $N_2(H) = 0$ unless $H \leq 2C$, in which case

$$N_2(H) \ll_{\varepsilon} (C/H)^{m-3+\varepsilon}$$  

by Lemma 7.5. Plugging the last two displays into Lemma 7.6 with $\varepsilon := \varepsilon_0$, we get

$$A \ll_{\varepsilon_0} X^{m+O(\varepsilon_0)} N^{-m} X_1^{2+m/2} X_2^{1+2m/3} X_3^{1+2m/3} H^{1/2} \left( \frac{N}{XC} \right)^{(m-2)/4} \left( \frac{C}{H} \right)^{-m-3}. $$  

Since $m - 3 \geq 1/2$, we have $H^{1/2}(C/H)^{m-3} \leq C^{m-3}$. Moreover, $m \leq 6$ implies $2 + m/2 \geq 1 + 2m/3$, so $X_1^{2+m/2} X_2^{1+2m/3} X_3^{1+2m/3} \ll (X_1 X_2 X_3)^{2+m/2} \ll N^{2+m/2}$. Thus

(7.13)  

$$A \ll_{\varepsilon_0} X^{m+O(\varepsilon_0)} N^{2-m/2} \left( \frac{N}{XC} \right)^{(m-2)/4} C^{m-3}. $$  

Since $2 - m/2 + (m - 2)/4 = (6 - m)/4 \geq 0$ (resp. since $m - 3 \geq (m - 2)/4$), the right-hand side of (7.13) is weakly increasing in $N$ (resp. in $C$). Therefore

$$A \ll_{\varepsilon_0} X^{m+O(\varepsilon_0)}(X^{3/2})^{2-m/2}(X^{1/2})^{m-3} = X^{3m/4+3/2+O(\varepsilon_0)}.$$  

Summing over $1 \ll N = M_0(F,w)Y/2^{k_1}$ and $1 \ll C = Z/2^{k_2}$ with $k_1, k_2 \in \mathbb{Z}_{\geq 1}$, we get

$$\Sigma_3 \ll X^{-3} X^{3m/4+3/2+O(\varepsilon_0)} = X^{3(m-2)/4+O(\varepsilon_0)},$$

where $\Sigma_3$ is the quantity defined in (7.1). This completes the proof of (7.2).

8. Proof of main results

In this section, we first prove Theorem 2.7 because it builds directly on our work in §4,7 on the delta method. We then prove Theorem 2.3 using (2.6), (2.7), and Proposition 4.3. Finally, we combine Theorems 2.3 and 2.7 to prove Theorem 1.1.

Proof of Theorem 2.7. By Proposition 3.4 we see that Hypothesis 2.6 implies Hypothesis 2.4. Therefore, we may and do assume Hypothesis 2.4. Now recall the quantity $\Sigma_0$ from (4.6). By (4.5) and the tail estimate (4.3), we have

$$N_{F,w}(X) - \Sigma_0 \ll_{A,\varepsilon_0} X^{-A}.$$  

Case 1: $m = 4$. Then adding (5.1), (6.6), and (7.3) together, we get

$$\Sigma_0 = \Sigma_1 + \Sigma_2 + \Sigma_3 = N_{F,w}(X) - N_{F,w}^t(X) + O_{\varepsilon_0}(X^{3(m-2)/4+O(\varepsilon_0)}) + O_{\varepsilon_0}(X^{m-3+\varepsilon_0}).$$  

It follows that $N_{F,w}^t(X) \ll_{\varepsilon_0} X^{3(m-2)/4+O(\varepsilon_0)}$. Let $c(F,w) := 0$.

Case 2: $m \geq 5$. Then adding (5.1), (6.5), and (7.2) together, we get

$$\Sigma_0 = \Sigma_1 + \Sigma_2 + \Sigma_3 = c(F,w) X^{m-3} + O_{\varepsilon_0}(X^{3(m-2)/4+O(\varepsilon_0)}) + O_{\varepsilon_0}(X^{(m-2)/2+\varepsilon_0}),$$

where $c(F,w) := \sigma_{\infty,w} \mathcal{S}$. It follows that $N_{F,w}(X) - c(F,w) X^{m-3} \ll_{\varepsilon_0} X^{3(m-2)/4+O(\varepsilon_0)}$.

In each case, taking $\varepsilon_0 \to 0$ gives the desired result, (2.16).
Proof of Theorem 2.3. Let $c \in S$. Since $\Psi(c, s)$ has an Euler product, condition (1) in Definition 2.2 clearly holds. It remains to prove that conditions (2) and (3) hold.

Case 1: $\Psi(c, s) = \Phi(c, s)$. Then conditions (2) and (3) are trivial, since 

$$(b_e(n), a'_e(n)) = (S^2_e(n), 1_{n=1}).$$

Case 2: $\Psi(c, s) = \prod_{p \mid \Delta(c)} \Phi_p(c, s)$. Then conditions (2) and (3) are trivial, since 

$$(b_e(n), a'_e(n)) = (S^2_e(n) \cdot 1_{\gcd(n, \Delta(c)) = 1}, S^2_e(n) \cdot 1_{n \mid \Delta(c) \infty}).$$

Case 3: $\Psi(c, s) \in \{\prod_{p \mid \Delta(c)} L_p(s, c)^{(-1)^{m-3}}, L(s, c)^{(-1)^{m-3}}\}$. Then by (2.6), we have

$$(8.1) \quad b_e(n), a_e(n) \ll n^e.$$ But $a'_e = S^2_e \ast a_e$, by (2.9). Therefore, condition (2) holds. Furthermore, if $p \nmid \Delta(c)$, then $a_e(p) = -b_e(p)$ by (2.9) and $b_e(p) = (-1)^{m-3} \lambda_e(p) = E^3_e(p)$ by (2.7), so

$$a'_e(p) = S^2_e(p) + a_e(p) = S^2_e(p) - E^3_e(p) \ll p^{-1/2}$$

by Proposition 3.3. Therefore, condition (3) also holds.

Proof of Theorem 1.1. Let $\Psi := L(s, c)^{(-1)^{m-3}}$. Then $\Psi$ is an approximation of $\Phi$, by Theorem 2.3 Moreover, $\Psi$ is standard by (8.1) and Definition 2.5. Now let $\vartheta := 1$. Then $\gamma_e(n) = \mu(n)^m \lambda_e(n)$ by (2.11), since for all primes $p$ we have $a_e(p) = (-1)^{m-2} \lambda_e(p)$ by the definition of $a_e$. Upon plugging in $\mu(n)^m v_n$ for $v_n$ in Hypothesis 2.1, we immediately find that Hypothesis 2.6 holds. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a nonnegative, smooth, compactly supported function such that $\zeta(t) = 1$ for all $t \in [1, 4]$, and $\zeta(t) = 0$ for all $t \notin [1/2, 8]$. Let

$$w(x) := \zeta(\sum_{1 \leq i \leq m} x_i^2).$$

Then Theorem 2.7 implies $N_{F, w}(X) \ll X^{3(m-2)/4+\epsilon}$ for all $X \geq 1$. Since $w(x/2^k) = 1$ for all $x \in \mathbb{Z}^m$ in the annulus $4^k \leq \sum_{1 \leq i \leq m} x_i^2 \leq 4^{k+1}$, it follows that

$$N_F(X) - 1 = |\{x \in \mathbb{Z}^m \cap [-X, X]^m : F(x) = 0, x \neq 0\}|$$

$$\leq \sum_{0 \leq k \leq \log_4(4mX^2)} N_{F, w}(2^k)$$

$$\ll \epsilon \sum_{0 \leq k \leq \log_4(4mX^2)} (2^k)^{3(m-2)/4+\epsilon}$$

$$\ll (4mX^2)^{1/2}3^{(m-2)/4+\epsilon}$$

$$\ll X^{3(m-2)/4+\epsilon},$$

for all $X \geq 1$. This implies Theorem 1.1. \hfill \square

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