THE ŁOJASIEWICZ EXPONENT OF NONDEGENERATE SURFACE SINGULARITIES

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Abstract. We give some estimations of the Łojasiewicz exponent of nondegenerate surface singularities in terms of their Newton diagrams. We also give an exact formula for the Łojasiewicz exponent of such singularities in some special cases. The results are stronger than Fukui inequality [8]. It is also a multidimensional generalization of the Lenarcik theorem [13].

1. Introduction

Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a holomorphic function in an open neighborhood of \( 0 \in \mathbb{C}^n \) and \( \sum_{\nu \in \mathbb{N}^n} a_{\nu} z^\nu \) be the Taylor expansion of \( f \) at \( 0 \). We define \( \Gamma_+ := \text{conv}\{v + \mathbb{R}^n_+: a_{\nu} \neq 0\} \subset \mathbb{R}^n \) and we call it the Newton diagram of \( f \). Let \( u \in \mathbb{R}^n_+ \setminus \{0\} \). Put \( l(u, \Gamma_+) := \inf \{ \langle u, v \rangle : v \in \Gamma_+ \} \) and \( \Delta(u, \Gamma_+) := \{ v \in \Gamma_+ : \langle u, v \rangle = l(u, \Gamma_+) \} \). We say that \( S \subset \mathbb{R}^n_+ \) is a face of \( \Gamma_+ \) if \( S = \Delta(u, \Gamma_+) \) for some \( u \in \mathbb{R}^n_+ \setminus \{0\} \). The vector \( u \) is called a primitive vector of \( S \). It is easy to see that \( S \) is a closed and convex set and \( S \subset \text{Fr}(\Gamma_+) \), where \( \text{Fr}(A) \) denotes the boundary of \( A \). One can prove that a face \( S \subset \Gamma_+ \) is compact if and only if all coordinates of its primitive vector \( u \) are positive. We call the family of all compact faces of \( \Gamma_+ \) the Newton boundary of \( f \) and we denote it by \( \Gamma(f) \). Denote by \( \Gamma_k \) the set of all compact \( k \)-dimensional faces of \( \Gamma(f) \), \( k = 0, \ldots, n-1 \). For every compact face \( S \subset \Gamma(f) \) define the polynomial \( f_S := \sum_{\nu \in S} a_{\nu} z^\nu \). We say that \( f \) is nondegenerate on a face \( S \subset \Gamma(f) \) if the system of equations \( (f_S)_x^i = \cdots = (f_S)_x^n = 0 \) has no solution in \( (\mathbb{C}^*)^n \), where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). We say that \( f \) is nondegenerate in the Kouchnirenko sense (shortly nondegenerate) if it is nondegenerate on each face of \( \Gamma(f) \). We say that \( f \) is a singularity if
is a nonzero holomorphic function in some open neighborhood of the origin such that \( f(0) = 0, \nabla f(0) = 0 \), where \( \nabla f = (f'_z, \ldots, f'_{zn}) \). We say that \( f \) is an isolated singularity if \( f \) is a singularity, which has an isolated critical point at the origin, i.e., \( \nabla f(z) \neq 0 \) for \( z \neq 0 \) near 0.

Let \( i \in \{1, \ldots, n\}, n \geq 2 \).

**Definition 1.1.** We say that \( S \in \Gamma^{n-1}(f) \subset \mathbb{R}^n \) is an exceptional face with respect to the axis \( OX_i \) if one of its vertices is at a distance 1 to the axis \( OX_i \) and the remaining vertices define an \((n-2)\)-dimensional face which lies in one of the coordinate hyperplanes including the axis \( OX_i \).

![Fig. 1: An exceptional face \( S \) with respect to the axis \( OX_3 \)](image)

We say that \( S \in \Gamma^{n-1}(f) \) is an exceptional face of \( f \) if there exists \( i \in \{1, \ldots, n\} \) such that \( S \) is an exceptional face with respect to the axis \( OX_i \). Denote by \( E_f \) the set of all exceptional faces of \( f \). We call the face \( S \in \Gamma^{n-1}(f) \) unexceptional of \( f \) if \( S \notin E_f \).

**Definition 1.2.** We say that the Newton diagram of \( f \) is convenient if it has nonempty intersection with every coordinate axis.

**Definition 1.3.** We say that the Newton diagram of \( f \) is nearly convenient if its distance to every coordinate axis does not exceed 1.

For a hyperplane \( L \) supporting a face \( S \in \Gamma(f) \) define the number \( m(L) := \max_{i=1}^{n} x_i(L) \), where \( x_1(L), \ldots, x_n(L) \) are the nonzero coordinates of intersections of the hyperplane \( L \) with the coordinate axes. Moreover, if \( S \in \Gamma^{n-1}(f) \), the hyperplane supporting \( S \) is uniquely determined. Hence for
$S \in \Gamma^{n-1}(f)$ we can define $x_i(S) := x_i(L)$, $i = 1, \ldots, n$ and $m(S) := m(L)$, where $L$ is the hyperplane supporting $S$. It is easy to see that

$$x_i(S) = \frac{l(u, \Gamma_+(f))}{u_i}, \quad i = 1, \ldots, n,$$

where $u$ is a primitive vector of $S$. It is easy to check that the “near convenience” of the Newton diagram is a necessary condition for $f$ to be an isolated singularity. For a singularity $f$ such that $\Gamma^{n-1}(f) \neq \emptyset$, define

$$m_0(f) := \max_{S \in \Gamma^{n-1}(f)} m(S).$$

It is easy to see that in the case of convenient $\Gamma_+(f)$ the number $m_0(f)$ is equal to the maximum of coordinates of the intersection points of the sum of all Newton boundary faces with the union of all axes.

**Remark 1.4.** A definition of $m_0(f)$ for all singularities (even for $\Gamma^{n-1}(f) = \emptyset$), can be found in [8]. In the case $\Gamma^{n-1}(f) \neq \emptyset$ both definitions are equivalent.

Let $f = (f_1, \ldots, f_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic mapping having an isolated zero at the origin. Define the number

$$l_0(f) := \inf \left\{ \alpha \in \mathbb{R}_+ : \exists C > 0 \exists r > 0 \forall \|z\| < r \|f(z)\| \geq C\|z\|^\alpha \right\}$$

and call it the Lojasiewicz exponent of the mapping $f$. There are formulas and estimations of the number $l_0(f)$ under some nondegeneracy conditions on $f$ (see [1], [2], [4], [13], [15], [18]).

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. Define the number $\mathcal{L}_0(f) := l_0(\nabla f)$ and call it the Lojasiewicz exponent of singularity $f$. Now we give known properties of the Lojasiewicz exponent (see [14]):

(a) $\mathcal{L}_0(f)$ is a rational number.

(b) $\mathcal{L}_0(f) = \sup \left\{ \frac{\ord \nabla f(\phi(t))}{\ord \phi(t)} : 0 \neq \phi(t) \in \mathbb{C}\{t\}^n, \phi(0) = 0 \right\}$.

(c) The infimum in the definition of the Lojasiewicz exponent is attained for $\alpha = \mathcal{L}_0(f)$.

(d) $s(f) = [\mathcal{L}_0(f)] + 1$, where $s(f)$ is the degree of $C^0$-sufficiency of $f$ (see [5], [19]).

Lenarcik [13] gave the formula for the Lojasiewicz exponent for singularities of two variables, nondegenerate in Kouchnirenko sense, in terms of its Newton diagram (for other formulas in the two-dimensional case see [6], [7]).

**Theorem 1.5** [13]. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an isolated nondegenerate singularity and $\Gamma^1(f) \setminus E_f \neq \emptyset$. Then

$$\mathcal{L}_0(f) = \max_{S \in \Gamma^1(f) \setminus E_f} m(S) - 1.$$
Remark 1.6. In the two-dimensional case one can prove that for isolated singularities such that $\Gamma^1(f) \setminus E_f = \emptyset$, we have $L_0(f) = 1$.

In the multidimensional case we have the following upper bound for $L_0(f)$, which was given by T. Fukui in 1991 (without removing any faces).

**Theorem 1.7** [8]. Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated nondegenerate singularity. Then

\begin{equation}
L_0(f) \leq m_0(f) - 1.
\end{equation}

We improve the Fukui inequality and simultaneously generalize the Lenarcik result (in a weak form) to the 3-dimensional case (by removing exceptional faces).

Denote by $AB$ the segment joining two different points $A, B \in \mathbb{R}^n$. We consider the following segments in $\mathbb{R}^3$:

$I_k^1 = (0, 1, 1)(k, 0, 0) \quad I_k^2 = (1, 0, 1)(0, k, 0) \quad I_k^3 = (1, 1, 0)(0, 0, k),

k \in \{2, 3, \ldots \}.

Put $J := \{ I_j^k : j = 1, 2, 3 \quad k = 2, 3, \ldots \}$. Every segment $I$ of this family intersects exactly one coordinate axis exactly at one point. Denote by $m(I)$ the nonzero coordinate of this point (equal to $k$). We give now the main result, which is an improvement of the above Theorem 1.7 for $n = 3$.

**Theorem 1.8** (main result). Let $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated and nondegenerate singularity.

1. \quad If $\Gamma^2(f) = \emptyset$ or $\Gamma^2(f) = E_f$, then there exists exactly one segment $I \in J \cap \Gamma^1(f)$ and $L_0(f) = m(I) - 1$.

2. \quad If $\Gamma^2(f) \setminus E_f \neq \emptyset$, then

\begin{equation}
L_0(f) \leq \max_{S \in \Gamma^2(f) \setminus E_f} m(S) - 1.
\end{equation}

The proof is given in Section 4.

Let us recall that if $(v_1, \ldots, v_n)$ is a sequence of $n$ rational numbers (called **weights**) such that $v_i \geq 2$ for $i = 1, \ldots, n$, then a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called **weighted homogeneous** of type $(v_1, \ldots, v_n)$ if it is a linear combination of monomials $c_{\alpha_1, \ldots, \alpha_n}z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with $\alpha_1/v_1 + \cdots + \alpha_n/v_n = 1$. In [12] there was given a formula for the Lojasiewicz exponent of weighted homogeneous surface singularities in terms of their weights.

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Theorem 1.9 [12]. Let \( f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) be an isolated weighted homogeneous singularity of type \((v_1, v_2, v_3)\), then
\[
\mathcal{L}_0(f) = \max_{i=1}^3 v_i - 1.
\]
(in the real case see [10]).

Since in the 3-dimensional case the weights are topological invariants of weighted homogeneous singularities (see [21]), we get from the above formula that the Lojasiewicz exponent is a topological invariant of such singularities. We can reformulate this result in terms of the Newton diagram as follows.

Theorem 1.10. Let \( f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) be an isolated weighted homogeneous singularity.

1. If \( \Gamma^2(f) = \emptyset \) or \( \Gamma^2(f) \) consist of one exceptional face, then there exists exactly one segment \( I \in \mathcal{J} \cap \Gamma^1(f) \) and \( \mathcal{L}_0(f) = m(I) - 1 \).

2. If \( \Gamma^2(f) \) consists of one unexceptional face \( S \), then \( \mathcal{L}_0(f) = m(S) - 1 \).

The main result of this paper is a transfer of the above theorem to the nondegenerate case. In 2010 the paper by Tan, Yau, Zuo [20] appeared, in which Theorem 1.9 was given in an analogous form for \( n \)-variables, \( n > 3 \), but their proof is false (i.e. the proof of their Proposition 3.4). Some results for weighted homogeneous singularities in the \( n \)-dimensional case were also given by Bivià-Ausina and Encinas ([3], [4]).

In Section 2 we give auxiliary lemmas and properties. In Section 3 we prove the lemma about the choice of an unexceptional face. The proof of the main theorem is given in Section 4. We present examples and open problems in Section 5.

2. Auxiliary lemmas and properties

Put \( A - 1_i := A - (0, \ldots, 1, \ldots, 0) \), \( i = 1, \ldots, n \), for every nonempty \( A \subset \mathbb{R}^n \). Denote by \( \mathcal{O}^n \) the local ring of germs of holomorphic functions in \( n \)-variables at \( 0 \in \mathbb{C}^n \), \( n \in \mathbb{N}_+ \). Let \( f \in \mathcal{O}^n \), \( f(z) = \sum_{\nu \in \mathbb{N}^n} a_{\nu} z^\nu \) in some open neighborhood of the origin. Let us define the set \( \text{supp} f := \{ \nu \in \mathbb{N}^n : a_{\nu} \neq 0 \} \) and call it the support of \( f \). Denote by \( \text{in} f \) the initial form of \( f \) and by \( \text{ord} f \) the order of \( f \). Let \( w = (w_1, \ldots, w_n) \in \mathbb{N}_+ \). Define the number \( \text{ord}_w f := \inf \{ \nu_1 w_1 + \cdots + \nu_n w_n : \nu = (\nu_1, \ldots, \nu_n) \in \text{supp} f \} \) and call it the order of \( f \) with respect to \( w \). The polynomial, which is the sum of such monomials \( a_{\nu_1, \ldots, \nu_n} z_1^{\nu_1} \cdots z_n^{\nu_n} \), for which \( \nu_1 w_1 + \cdots + \nu_n w_n = \text{ord}_w f \), are the initial form of \( f \) with respect to \( w \), and we denote it by \( \text{in}_w f \). Now, we give two simple and useful properties. Because the proofs are easy we omit them.
That there exists a monomial of in parametrization such that \( \phi, \ldots, n = 0 \), singularity \( (1) \) supporting hyperplane to \( \Gamma \). They show that nondegenerate singularity is a “near generic” isolated singularity, therefore by Property 2.1 we can find a monomial in which the variable \( z_i \) appears, then

\[
(\text{in}_w f)'_{z_i} = \text{in}_w f'_{z_i}.
\]

Moreover, if \( L \) is the supporting hyperplane to \( \Gamma_+(f) \) such that \( w \perp L \), then \( L - 1 \) is a supporting hyperplane to \( \Gamma_+(f') \).

The following lemma will be useful in the proof of Lemma 2.9.

**Lemma 2.3.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), \( n \geq 2 \), be a singularity and \( \phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n \) be a parameterization such that \( \phi(0) = 0 \), \( \phi_i \neq 0 \), \( i = 1, \ldots, n \), and \( w := (\text{ord}_i \phi_i)_{i=1}^n \). Let

\[
K := \{ i \in \{1, \ldots, n \} : f'_{z_i} \circ \phi = 0 \} \neq \emptyset.
\]

Then for the face \( S := \Delta(w, \Gamma_+(f)) \in \Gamma(f) \) we have \( (f_S)'_{z_i} \circ \text{in}_w f = 0 \) for \( i \in K \).

**Proof.** Put \( J := \{ j \in K : S \subset \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j = 0 \} \} \). Then for every \( i \in K \setminus J \) we can find a monomial in \( \text{in}_w(f) \) in which the variable \( z_i \) appears. Therefore by Property 2.2 we get \( (\text{in}_w f)'_{z_i} = \text{in}_w f'_{z_i} \) for \( i \in K \setminus J \). Therefore by Property 2.1a we get for \( i \in K \setminus J \)

\[
0 = \text{in}_w f'_{z_i} \circ \phi = (\text{in}_w f)'_{z_i} \circ \phi = (f_S)'_{z_i} \circ \phi.
\]

On the other hand \( (f_S)'_{z_i} \circ \phi = 0 \), for \( i \in J \). \( \square \)

The following corollaries are direct consequences of the above lemma. They show that nondegenerate singularity is a “near generic” isolated singularity.

**Corollary 2.4.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), \( n \geq 2 \), be a singularity and \( \phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n \) be a parameterization such that \( \phi(0) = 0 \), \( \phi_i \neq 0 \), \( i = 1, \ldots, n \). If \( \nabla f \circ \phi = 0 \), then there exists a face \( S \in \Gamma(f) \) such that \( \nabla (f_S) \circ \text{in}_w f = 0 \), thus \( f \) is degenerate on the face \( S \).

**Corollary 2.5.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), \( n \geq 2 \), be a nondegenerate singularity. If \( \phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n \) is a parameterization such that \( \phi(0) = 0 \), \( \phi_i \neq 0 \), \( i = 1, \ldots, n \), then \( \nabla f \circ \phi \neq 0 \).
Example 2.6. The assumptions \( \phi_i \neq 0, i = 1, \ldots, n \), are necessary in the above corollaries. Indeed, let \( f(z_1, z_2, z_3) = z_1(z_2 + z_3) \) and \( \phi(t) = (0, t, -t) \). It is easy to check that \( f \) is a nondegenerate singularity and \( \nabla f \circ \phi = 0 \).

Now we give a simple property which is needed in the proof of the next property.

**Property 2.7.** Let \( f \in \mathcal{O}^n \), \( f(0) = 0 \). Then \( l(u, \Gamma_+(f)) \geq \min_{i=1}^n u_i \).

The next useful property will be often used in the next part of the paper.

**Property 2.8.** Let \( f \in \mathcal{O}^n \), \( f(0) = 0 \), and \( L \) be a supporting hyperplane to a compact face of \( \Gamma_+(f) \). Then \( m(L) \geq 1 \). Moreover, if \( f_{z_i}'(0) = 0 \) for some \( i \in \{1, \ldots, n\} \) and \( L - 1_i \) supports a compact face of \( \Gamma_+(f_{z_i}') \), then \( m(L - 1_i) \geq 1, m(L) \geq 2 \) and \( m(L - 1_i) \leq m(L) - 1 \).

Because the proofs of the above properties are easy we omit them. The next lemma is important in the second part of the proof of Theorem 1.8. It shows a method to find an upper bound of the Lojasiewicz exponent of nondegenerate singularity in terms of its Newton diagram.

**Lemma 2.9.** Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be a nondegenerate singularity. Let \( \phi = (\phi_i)_{i=1}^n \in \mathbb{C}[t]^n \) be a parametrization such that \( \phi(0) = 0, \phi_i \neq 0, i = 1, 2, \ldots, n \), and let \( L \) be the supporting hyperplane to \( \Gamma_+(f) \) such that \( w := (\text{ord } \phi_i)_{i=1}^n \perp L \). Then \( \nabla f \circ \phi \neq 0 \) and

\[
\frac{\text{ord } (\nabla f \circ \phi)}{\text{ord } \phi} \leq m(L) - 1.
\]

**Proof.** By Corollary 2.5 we get that \( \nabla f \circ \phi \neq 0 \). From our assumption \( w \) has positive coordinates, so \( L \) is a supporting hyperplane to a compact face \( S \in \Gamma(f) \). Set

\[
J := \{ i \in \{1, \ldots, n\} : (f_S)'_{z_i} \circ \phi \neq 0 \}, \quad K := \{ i \in \{1, \ldots, n\} : f'_{z_i} \circ \phi \neq 0 \}.
\]

Since \( f \) is a nondegenerate singularity, we have \( J \neq \emptyset \). By Lemma 2.3, \( J \subset K \). Therefore by Property 2.8 and Property 2.1,

\[
\frac{\text{ord } (\nabla f \circ \phi)}{\text{ord } \phi} = \min_{i \in K} \frac{\text{ord } (f'_{z_i} \circ \phi)}{\text{ord } \phi} \leq \min_{i \in J} \frac{\text{ord } (f'_{z_i} \circ \phi)}{\text{ord } \phi} = \min_{i \in J} \frac{\text{ord}_w f'_{z_i}}{\min_{i=1}^n w_i} = \min_{i \in J} m(L - 1_i) \leq m(L) - 1. \quad \square
\]

The following well known property says that the Newton boundary of the restriction \( f|_{\{z_{k+1} = \cdots = z_n = 0\}} \) is the restriction of the Newton boundary of \( f \) to the set \( \{x_{k+1} = \cdots = x_n = 0\} \subset \mathbb{R}^n \).
Property 2.10. Let \( f \in \mathcal{O}^n, \ n \geq 2 \). Assume that \( g(z_1, \ldots, z_k) := f(z_1, \ldots, z_k, 0, \ldots, 0) \in \mathcal{O}^k, \ k < n, \) is a nonzero germ. Then

\[
\Gamma(g) = \{ S \in \Gamma(f) : S \subset \{ x_{k+1} = \cdots = x_n = 0 \} \}.
\]

The following classical property is required in the next part of the paper.

Property 2.11. Let \( f \in \mathcal{O}^n, \ n \geq 3, \) be an isolated singularity. Then \( f \) is an irreducible germ in \( \mathcal{O}^n \).

Proof. Suppose to the contrary that \( f = gh, \) where \( g \) and \( h \) are non-invertible in \( \mathcal{O}^n \). Then \( f'_z = g'_z h + h'_z g, \ i = 1, 2, \ldots, n. \) Hence \( V(g, h) \subset V(\nabla f). \) Since \( g \) and \( h \) are non-invertible, therefore \( V(g, h) \neq \emptyset, \) because \( 0 \in V(g, h). \) Hence by Corollary 8 [9, p. 81] and as \( n \geq 3, \) we have that \( \dim V(g, h) \geq 1. \) Therefore \( \dim V(\nabla f) \geq 1, \) which is impossible, since \( \nabla f \) has an isolated zero at \( 0. \)

Remark 2.12. There exist reducible isolated singularities of two variables, e.g. \( f(z_1, z_2) = z_1 z_2. \)

The following corollary is a direct consequence of Property 2.11.

Corollary 2.13. Let \( f \in \mathcal{O}^n, \ n \geq 3, \) be an isolated singularity. Then

\[
\{ x \in \mathbb{R}^n : x_i = 0 \} \cap \Gamma_+(f) \neq \emptyset, \ i = 1, \ldots, n.
\]

The last lemma says that in some additional conditions the Milnor number is equal to the Łojasiewicz exponent.

Lemma 2.14 (see [16]). Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be an isolated singularity. Then \( \mathcal{L}_0(f) \leq \mu_0(f). \) If additionally \( \text{rank} [f^n_{z_i z_j}]_{i,j=1}^n(0) \geq n - 1, \) then \( \mathcal{L}_0(f) = \mu_0(f). \)

3. A lemma about the choice of an unexceptional face

The following lemma associates a suitable unexceptional face of \( f \) to every coordinate axis in \( \mathbb{R}^3. \) It turns out to be the main tool in the proof of Part 2\(^0 \) of the main result.

Lemma 3.1 (the choice of an unexceptional face). Let \( f \in \mathcal{O}^3 \) be an isolated singularity such that \( \Gamma^2(f) \setminus E_f \neq \emptyset. \) Then for every axis \( OX_i, \ i = 1, 2, 3, \) there exists a face \( S_i \in \Gamma^2(f) \setminus E_f \) such that at least one of the two following conditions is true:

i) there exists a point \( W \in OX_i \) which is a vertex of the face \( S_i, \)

ii) there exist \( j, k \in \{ 1, 2, 3 \} \setminus \{ i \}, \ j \neq k \) and vertices \( W \in OX_i X_j \) such that their distance to the axis \( OX_i \) equals 1 and \( Y \in OX_i X_k \) such that the segment \( WY \) is an edge of the face \( S_i. \)
Before we pass to the proof we give some properties, lemmas and auxiliary facts. We begin with a simple property of the vertices of the Newton boundary.

Property 3.2. Let \( f \in \mathcal{O}^n \) and \( A \in \Gamma^0(f) \). Then \((A + \mathbb{R}_+^n) \cap \Gamma^0(f) = \{A\} \).

The following property says that segments joining vertices which lie “properly near” to the coordinate axes are edges. Denote by \( x_i(A) \) the \( i \)-coordinate of the point \( A \in \mathbb{R}^n \).

Property 3.3. Let \( f \in \mathcal{O}^3 \) be a singularity and \( \{i, j, k\} \) be a permutation of the set \( \{1, 2, 3\} \). Suppose that there exists a point \( W \in \Gamma^0(f) \cap OX_iX_k \) at a distance 1 to the axis \( OX_i \) and \( \Gamma^0(f) \cap OX_iX_j \neq \emptyset \). If \( Y \in \Gamma^0(f) \cap OX_iX_j \) is the point with the smallest distance to the axis \( OX_i \), then \( WY \in \Gamma^1(f) \).

Proof. Without loss of generality we may assume that \( i = 1, j = 2, k = 3 \). If \( Y \in OX_1 \), then \( WY \subset OX_1X_3 \) and by Property 2.10, \( WY \in \Gamma^1(g) \subset \Gamma^1(f) \), where \( g = f|_{\{z_2=0\}} \). If \( Y \notin OX_1 \), then to get the assertion it suffices to find a supporting plane to \( \Gamma_+(f) \) along the segment \( WY \). To this end we first observe that planes going through \( WY \) can intersect the axis \( OX_1 \) arbitrary far away. Therefore we can choose a vector \( u \) and a plane \( L : \langle u, x \rangle = b > 0 \) going through \( WY \) such that \( 1 < x_3(L) < 2 \) and \( x_2(Y) < x_2(L) < x_2(Y) + 1 \) and \( P = (0, 1, 1) \) lie above \( L \) (see Fig. 2). Since every coordinate of the point from \( \text{supp} f \) is an integer, we get \( L \cap \Gamma_+(f) = WY \) and there are no points of \( \text{supp} f \) below the plane \( L \). \( \square \)

Recall that we have already defined the family \( \mathcal{J} = \{ I^k_j : j = 1, 2, 3, k = 2, 3, \ldots \} \).
PROPOSITION 3.4. Let \( f \in \mathcal{O}^3 \). Then the set \( J \cap \Gamma^1(f) \) is empty or consists of one element. Moreover, if \( J \cap \Gamma^1(f) \neq \emptyset \), then either \( \Gamma^2(f) = \emptyset \) or \( \Gamma^2(f) = E_f \neq \emptyset \).

PROOF. Suppose that \( J \cap \Gamma^1(f) \neq \emptyset \). Without loss of generality we may assume that \( I^k_3 \in J \cap \Gamma^1(f) \) for some \( k \in \{2, 3, \ldots \} \). Let \( A = (1, 1, 0) \), \( B = (0, 0, k) \) be the vertices of the segment \( I^k_3 \). By Property 3.2, \( (A + \mathbb{R}_+^3) \cap \Gamma^0(f) = \{A\} \). Since the vertices of \( \Gamma^0(f) \) have integer coordinates, therefore \( \Gamma^0(f) \setminus \{A\} \subset (OX_1X_3 \cup OX_2X_3) \). If \( \Gamma^0(f) = \{A, B\} \), then \( \Gamma^2(f) = \emptyset \) and \( \Gamma^1(f) = \{I^k_3\} \). Otherwise \( \Gamma^2(f) \neq \emptyset \) and joining \( A \) with points of \( \Gamma^0(f) \cap OX_1X_3 \) and with points of \( \Gamma^0(f) \cap OX_2X_3 \) we get each face of \( \Gamma^2(f) \) and they are all exceptional. Observe that in this case we have \( \Gamma^1(f) \cap J = \{I^k_3\} \).

LEMMA 3.5. Let \( f \in \mathcal{O}^3 \) be an isolated singularity and \( \{i, j, k\} \) be a permutation of the set \( \{1, 2, 3\} \). Moreover, let \( S \) be a nonempty family of all the exceptional faces with respect to the axis \( OX_i \). Suppose that they all have a common vertex \( W \in OX_iX_j \) at a distance 1 to the axis \( OX_i \) and their remaining vertices lie in the plane \( OX_iX_k \), and let \( Y \) be the vertex with the smallest distance to the axis \( OX_k \). Then the segment \( WY \) is either an edge of some unexceptional face of \( f \) or \( WY \in J \) (see Fig. 3).

Fig. 3: \( WY \) is a common edge of an exceptional face \( S \) and an unexceptional face \( T \)

PROOF. Without loss of generality we may assume that \( i = 1, j = 2, k = 3 \). From the assumption the segment \( WY \) is an edge of an exceptional face \( S \in S \). Now, suppose that \( WY \) is not an edge of any other face \( T \in \Gamma^2(f) \). In particular \( WY \) is not an edge of any unexceptional face. Then by near convenience of \( \Gamma_+(f) \) we have \( W = (1, 1, 0) \) and since \( \Gamma(f) \cap OX_2X_3 \neq \emptyset \) (see Corollary 2.13), we get \( Y \in OX_3 \). Hence \( WY \in J \). Now, suppose
that $\overline{WY}$ is also an edge of a face $T \in \Gamma^2(f)$, $T \neq S$. If $T \in \Gamma^2(f) \setminus E_f$, then we get the statement. Suppose that $T$ is exceptional with respect to an axis different from $OX_1$. For $W \not\in OX_1X_3$ and $W \not\in OX_2X_3$, then $\overline{WY}$ could not be an edge of an exceptional face with respect to $OX_3$. Therefore $T$ is exceptional with respect to the axis $OX_2$. Since $\overline{WY} \not\subset OX_1X_2$ and $\overline{WY} \not\subset OX_2X_3$, one of the vertices ($W$ or $Y$) is at a distance 1 to the axis $OX_2$. As $Y \in OX_1X_3$ and $f$ is a singularity, $Y$ cannot be such a point. Therefore $W$ is at a distance 1 to the axis $OX_2$ and moreover $Y \in OX_2X_3$. Hence $W = (1, 1, 0)$ and $Y \in OX_3$, $\overline{WY} \in \mathcal{J}$. □

Directly by Proposition 3.4 and Lemma 3.5 we get the following lemma, which turns out to be the key in the proof of Lemma 3.1.

**Lemma 3.6.** Let $f \in \mathcal{O}^3$ be an isolated singularity such that $\Gamma^2(f) \setminus E_f \neq \emptyset$. Moreover, let $\{i, j, k\}$ be a permutation of the set $\{1, 2, 3\}$ and $S$ be a nonempty family of all the exceptional faces with respect to the axis $OX_i$. Suppose that they all have a common vertex $W \in OX_iX_j$ at a distance 1 to the axis $OX_i$ and their remaining vertices lie in the plane $OX_iX_k$, and let $Y$ be the vertex with the smallest distance to the axis $OX_k$. Then the segment $\overline{WY}$ is an edge of some unexceptional face of $f$ (see Fig. 3).

Denote by $\#F$ the number of elements in a finite set $F$. The following proposition says what the Newton boundary of isolated singularities which have no 2-dimensional faces, looks like.

**Proposition 3.7.** Let $f \in \mathcal{O}^3$ be an isolated singularity. If $\Gamma^2(f) = \emptyset$, then there exists $I \in \mathcal{J}$ such that $\Gamma^1(f) = \{I\}$.

**Proof.** If $\Gamma^2(f) = \emptyset$, then of course $\Gamma^1(f)$ consists of only one segment $I = \overline{AB}$ and $\Gamma^0(f) = \{A, B\}$. Let $i \in \{1, 2, 3\}$. Denote by $N_i$ the set of vertices which lie on the axis $OX_i$ or at a distance 1 to it. By near convenience of $\Gamma_+(f)$ we get that $N_i \neq \emptyset$. If $N_i$ are pairwise disjoint, then $\#\Gamma^0(f) \geq 3$, a contradiction. Hence there exist $j, k \in \{1, 2, 3\}$, $j \neq k$ such that $N_j \cap N_k \neq \emptyset$. Without loss of generality we may assume that $j = 1$ and $k = 2$. Let $W \in N_1 \cap N_2$. Since $f$ is a singularity, we get $W \not\in OX_m$, $m = 1, 2, 3$. Hence $W = (1, 1, 0) \in \{A, B\}$. Without loss of generality we may assume that $W = A$. Then by Corollary 2.13 we get that $B \in OX_3$. Since $f$ is a singularity, we have $k := x_3(B) \geq 2$. Summing up $I = I^k_3 \in \mathcal{J}$. □

Now, we give a simple condition to decide when all 2-dimensional faces of the Newton boundary are exceptional or there is no any 2-dimensional face (see Theorem 1.8, 1$^0$).

**Theorem 3.8.** Let $f \in \mathcal{O}^n$ be an isolated singularity. Then

$$(\Gamma^2(f) = \emptyset \text{ or } \Gamma^2(f) = E_f \neq \emptyset) \iff \Gamma^1(f) \cap \mathcal{J} \neq \emptyset.$$
Proof. “⇒”. If $\Gamma^2(f) = \emptyset$, then by Proposition 3.7 we get that $\Gamma^1(f) \cap J \neq \emptyset$. Now, suppose that $\Gamma^2(f) = E_f \neq \emptyset$. Let $S \in E_f$. Without loss of generality we may assume that $S$ is exceptional with respect to the axis $OX_3$. Let $S$ be the family of all exceptional faces with respect to the axis $OX_3$. There exists a vertex $W$ at a distance 1 to the axis $OX_3$ which is a common vertex of this family. Without loss of generality we suppose that $W \in OX_1X_3$, the remaining vertices lie in the plane $OX_2X_3$, and let $Y$ be the one with the smallest distance to the axis $OX_2$. Then by Lemma 3.5, $WY \in J$.

“⇐”. It is a direct consequence of Proposition 3.4.

Proof of Lemma 3.1. Let $i \in \{1, 2, 3\}$. Without loss of generality we may assume that $i = 3$. By near convenience of $\Gamma_+(f)$ there exists a vertex which lies on $OX_3$ or at a distance 1 to it. If there exists a vertex which lies on the axis $OX_3$, then we will denote it by $W_3$. If there exists a vertex of the Newton boundary at a distance 1 to the axis $OX_3$ which lies on the plane $OX_iX_3$, then we will denote it by $W_i$, $i = 1, 2$. We have the following cases.

1. There exists a vertex $W_3$ and there are no vertices $W_1$ and $W_2$. If $W_3$ is a vertex of some unexceptional face, then the condition i) is fulfilled for this face. Otherwise it is a vertex of some exceptional face $T$. Since there are no vertices $W_1$ and $W_2$, it is an exceptional face with respect to the axis $OX_1$ or $OX_2$. Without loss of generality we may assume that it is exceptional with respect to $OX_1$. Then there exists a vertex $B \in OX_1X_2$ at a distance 1 to the axis $OX_1$. By Lemma 3.6 the segment $BW_3$ is an edge of some unexceptional face $S_3$, so the condition i) is fulfilled for this face (Fig. 4, 1).

2. Now, suppose that there exists the vertex $W_1$ or $W_2$. Without loss of generality we may assume that there exists the vertex $W_1$. By Corol-
We get that $\Gamma_+ (f) \cap OX_2 X_3 \neq \emptyset$. Let $Y \in OX_2 X_3 \cap \Gamma^0 (f)$ be the vertex with the smallest distance to the axis $OX_3$. By Property 3.3, the segment $\overline{W_1 Y} \in \Gamma^1 (f)$. If it is an edge of some unexceptional face, then the condition ii) is fulfilled for this face. Otherwise it is an edge of some exceptional face. We have the following cases.

a) $Y \notin OX_2$ and $Y \notin OX_3$. Then the segment $\overline{W_1 Y}$ can not be an edge of any exceptional face with respect to $OX_2$ or $OX_1$. Therefore it is an edge of some exceptional face $T$ with respect to the axis $OX_3$.

If $x_2 (Y) > 1$ then by Lemma 3.6 there exists a vertex $A \in OX_2 X_3$ such that the segment $\overline{AW_1}$ is an edge of some unexceptional face $S_3$, so the condition ii) is fulfilled for this face (Fig. 4, 20a).

If $x_2 (Y) = 1$ then $Y = W_2$. Hence by Lemma 3.6 there exists $i \in \{1, 2\}$ and a vertex $A_i \in OX_i X_3$ such that the segment $\overline{A_i W_{3-i}}$ is an edge of some unexceptional face, thus the condition ii) is fulfilled for this face.

b) $Y \in OX_2$. Since $Y$ is the one with the smallest distance to the axis $OX_3$, there are no other vertices on the plane $OX_2 X_3$. If the segment $\overline{W_1 Y}$ is an edge of some unexceptional face, the condition ii) is fulfilled for this face. Otherwise the segment $\overline{W_1 Y}$ is an edge of some exceptional face with respect to $OX_1$. Hence $W_1$ is at a distance 1 to the axis $OX_1$. So $W_1 = (1, 0, 1)$ and $\overline{W_1 Y} = I_k^2$, where $k = x_2 (Y)$, which by Proposition 3.4 is not possible (Fig. 4, 20b).

c) $Y \in OX_3$. Then $Y = W_3$. If the segment $\overline{W_1 W_3}$ is an edge of some unexceptional face, the condition ii) is fulfilled for this face. Otherwise the segment $\overline{W_1 W_3}$ is an edge of some exceptional face with respect to the axis $OX_1$ or $OX_3$.

If it is an edge of the exceptional face with respect to $OX_1$, then there exists a vertex $B \in OX_1 X_2$ at a distance 1 to the axis $OX_1$. By Lemma 3.6 the segment $\overline{BW_3}$ is an edge of some unexceptional face, thus the condition i) is fulfilled for this face.

If it is an edge of the exceptional face with respect to $OX_3$, then there exists a vertex $W_2 \in OX_2 X_3$ at a distance 1 to the axis $OX_3$. Hence by Lemma 3.6 there exists $i \in \{1, 2\}$ and a vertex $A_i \in OX_i X_3$ such that the segment $\overline{A_i W_{3-i}}$ is an edge of some unexceptional face. Thus the condition ii) is fulfilled for this face. □

4. Proof of the main result

Proof of Theorem 1.8. 10 If $\Gamma^2 (f) = \emptyset$ or $\Gamma^2 (f) = E_f \neq \emptyset$, then by Theorem 3.8 and Proposition 3.4 there exists exactly one segment $I \in \Gamma^1 (f) \cap \mathcal{J}$. Without loss of generality we can assume that $I = I_k^3$ for some $k \in \{2, 3, \ldots\}$. Then $(1, 1, 0) \in \text{supp} (f)$, hence we can find a monomial of the form $az_1 z_2$, $a \neq 0$ in the Taylor expansion of $f$. Therefore $f(z_1 z_2) = a \neq 0$. Observe that $(2, 0, 0) \notin \text{supp} (f)$ or $(0, 2, 0) \notin \text{supp} (f)$. Otherwise the point

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(1, 1, 0) would be in the interior of the segment \((2, 0, 0)(0, 2, 0)\), which would contradict \((1, 1, 0) \in \Gamma^0(f)\). So \(f''_{z_1 z_1}(0) = 0\) or \(f''_{z_2 z_2}(0) = 0\). Summing up,

\[
\text{rank} \left[ \frac{\partial^2 f}{\partial z_i \partial z_j} \right]_{i,j=1}^3 (0) \geq 2.
\]

Then by Lemma 2.14 we get that \(\mathcal{L}_0(f) = \mu_0(f)\). We have the following cases.

a) The Newton diagram \(\Gamma_0^+(f)\) is convenient. Then \(\Gamma^2(f) \neq \emptyset\), so by assumption \(\Gamma^2(f) = E_f\). Since \(f\) is nondegenerate, by Kouchnirenko theorem ([11], Theorem I) the Milnor number \(\mu_0(f)\) is equal to the Newton number \(\nu(f)\). By definition

\[
\nu(f) = 3!V_3 - 2!V_2 + V_1 - 1,
\]

where \(V_3\) is the volume of the set \(\mathbb{R}^3_+ \setminus \text{int (}\Gamma_0^+(f)\)\), and \(V_k, k = 1, 2\) are \(k\)-dimensional Lebesgue measures of the intersection of this set and sum of linear subspaces of dimension \(k\) spanned by the coordinate axes.

It is not difficult to check that \(\nu(f) = k - 1 = m(I) - 1\) (see Fig. 5). Summing up we get

\[
\mathcal{L}_0(f) = \mu_0(f) = \nu(f) = m(I) - 1,
\]

which finishes the proof in this case.

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b) If $\Gamma_+(f)$ is not convenient, then we deform $f$ to get an isolated singularity, which has the Newton diagram convenient. To this end we define a new singularity

$$g(z_1, z_2, z_3) := f(z_1, z_2, z_3) + \alpha_1 z_1^v + \alpha_2 z_2^v,$$

with $\alpha_i = 0$, if $\Gamma_+(f) \cap OX_i \neq \emptyset$ or else $\alpha_i = 1$, $i = 1, 2$. We choose a number $v \in \{2, 3, \ldots\}$ to fulfill the following conditions:

i) $\text{ord}(\nabla g - \nabla f) > \mathcal{L}_0(f)$,

ii) if $E_f \neq \emptyset$, then $v > \max_{S \in E_f} m(S)$,

iii) if $\Gamma^2(f) = \emptyset$, then $v > m(I)$.

Since $(0, 0, m(I)) \in \text{supp}(f)$, by definition of $g$ we get that $\Gamma_+(g)$ is convenient. Since $\text{ord}(\nabla g - \nabla f) > \mathcal{L}_0(f)$, by Lemma 1.4 in [17], $g$ is an isolated singularity and

$$\mathcal{L}_0(g) = \mathcal{L}_0(f).$$

Moreover $\Gamma(g) = \Gamma(f) \cup T$, where $T$ is the set of faces $S \in \Gamma(g) \setminus \Gamma(f)$ such that $(v, 0, 0) \in S$ (if $\alpha_1 = 1$) or $(0, v, 0) \in S$ (if $\alpha_2 = 1$). Observe that the two-dimensional faces of $T$ are exceptional faces of $g$. Since $\Gamma^2(f) = \emptyset$ or $\Gamma^2(f) = E_f$ we get that $\Gamma^2(g) = E_q$. It is easy to check that for every face $S \in T$ there exists $i \in \{1, 2, 3\}$ such that $(f_S)_{z_i}$ is a monomial. Therefore $g$ is nondegenerate on every face $S \in T$. Then by the nondegeneracy of $f$ and by the equality $\Gamma(g) = \Gamma(f) \cup T$ we get that $g$ is nondegenerate. Hence by the proof of Case a) used for $g$ we have that $\mathcal{L}_0(g) = m(I) - 1$. Summing up by (7) we get that

$$\mathcal{L}_0(f) = \mathcal{L}_0(g) = m(I) - 1.$$

20 If $\Gamma^2(f) \setminus E_f \neq \emptyset$, by Lemma 3.1 we choose the face $S_i \in \Gamma^2(f) \setminus E_f$ for every axis $OX_i$, $i = 1, 2, 3$, such that at least one of the two conditions is fulfilled:

i) there exists a point $W \in OX_i$ which is a vertex of the face $S_i$,

ii) there exist $j, k \in \{1, 2, 3\} \setminus \{i\}$, $j \neq k$ and vertices $W \in OX_iX_j$ at a distance 1 to the axis $OX_i$ and $Y \in OX_iX_k$ such that the segment $\overline{WY}$ is an edge of the face $S_i$.

We show that $\mathcal{L}_0(f) \leq \max_{i=1}^3 m(S_i) - 1$. Suppose to the contrary that

$$\mathcal{L}_0(f) > \max_{i=1}^3 m(S_i) - 1.$$

By Property b) of the Lojasiewicz exponent (after Remark 1.4) there exists a parameterization $\phi = (\phi_i)_{i=1}^3 \in \mathbb{C}\{t\}^3$, $\phi(0) = 0$ such that

$$\frac{\text{ord}(\nabla f \circ \phi)}{\text{ord} \phi} > \max_{i=1}^3 m(S_i) - 1.$$
We have the following cases.

a) \( \phi_i \neq 0, i = 1, 2, 3 \). Denote by \( L \) the supporting hyperplane to \( \Gamma_+(f) \) such that \( L \perp (\text{ord } \phi_i)_{i=1}^3 \). Then by Lemma 2.9

\[
\frac{\text{ord} (\nabla f \circ \phi)}{\text{ord } \phi} \leq m(L) - 1.
\]

This and inequality (8) show that \( m(L) > \max_{i=1}^3 m(S_i) \). Without loss of generality we can assume that \( m(L) = x_1(L) \). Hence we get that

(9) \( x_1(L) > m(S_1) \geq x_1(T) \),

where \( T \) is the plane determined by the face \( S_1 \). By the inequality (9) the condition i) for the face \( S_1 \) is not possible. Thus condition ii) is fulfilled for this face and without loss of generality we may assume that \( j = 3 \) in this condition. Then there are vertices \( W \in OX_1X_3 \) at a distance 1 to the axis \( OX_1 \) and \( Y \in OX_1X_2 \) such that the segment \( WY \) is an edge of the face \( S_1 \) (Fig. 6). We shall show that there exists a plane \( K \parallel L \) which supports \( \Gamma_+(f'_{z_3}) \) exactly at one point \( W - 1 \in OX_1 \) and \( m(K) \leq m(S_1) - 1 \).

Fig. 6: An unexceptional face \( S_1 \)

For \( i = 2, 3 \) we will denote by \( l_i \) the line \( L \cap OX_1X_i \) and by \( \alpha_i \) the acute angle between the line \( l_i \) and the axis \( OX_1 \), and by \( \beta_i \) the acute angle between the line \( T \cap OX_1X_i \) and the axis \( OX_1 \). Since \( L \) is a supporting plane to \( \Gamma_+(f) \), \( W \) lies on the line \( l_3 \) or above it and \( Y \) lies on the line \( l_2 \) or
above it. Therefore by (9) we get that \( \alpha_i < \beta_i, \ i = 2, 3 \). Now, let \( K \parallel L \) be the plane such that \( W - 1_3 \in K \). Since the set \( \text{supp} \ f \) lies in the plane \( T \) or above it, \( \text{supp} \ f'_{z_3} \) lies in the plane \( T - 1_3 \) or above it. For \( \alpha_i < \beta_i, \ i = 2, 3 \) and \( K \parallel L \), then \( x_i(K) < x_i(T - 1_3), \ i = 2, 3 \), moreover \( \text{supp} \ f'_{z_3} \), except the point \( (W - 1_3) \in K \), lies above the plane \( K \). Therefore the plane \( K \) supports \( \Gamma_+(f'_{z_3}) \) exactly at one point \( W - 1_3 \in OX_1 \). Moreover \( m(K) \leq m(T - 1_3) \). This and Property 2.8 show that

\[
(10) \quad m(K) \leq m(T - 1_3) \leq m(T) - 1 = m(S_1) - 1.
\]

Summing up by inequality (10) and by Property 2.1b we have

\[
\frac{\text{ord} (\nabla f \circ \phi)}{\text{ord} \phi} \leq \frac{\text{ord} (f'_{z_3} \circ \phi)}{\text{ord} \phi} = m(K) \leq m(S_1) - 1,
\]

which leads to a contradiction with inequality (8).

b) There exists \( i \in \{1, 2, 3\} \) such that \( \phi_i = 0 \). Without loss of generality we can assume that \( i = 1 \). Hence \( \phi = (0, \phi_2, \phi_3) \). Denote \( \phi_0 = (\phi_2, \phi_3) \). We represent the singularity \( f \) in the form

\[
f(z_1, z_2, z_3) = g(z_2, z_3) + z_1 h(z_2, z_3) + z_1^2 h_2(z_1, z_2, z_3),
\]

where \( g, h \in O^2, \ h_2 \in O^3 \). Since \( f \) is an isolated singularity, we have \( g \neq 0 \) (see Property 2.11) and thus \( \Gamma(g) \neq \emptyset \). Moreover \( g(0) = h(0) = 0, \ \nabla g(0) = 0 \), hence \( g \) is a singularity (not necessarily isolated). For if \( f \) is nondegenerate and \( \Gamma(g) = \{ S \in \Gamma(f) : S \subset \{x_1 = 0\}\} \) (see Property 2.10), \( g \) is nondegenerate. Summing up, \( g \) is a nondegenerate singularity.

Suppose first that \( \phi_2 = 0 \) and \( \phi_3 \neq 0 \) (the case \( \phi_3 = 0 \) and \( \phi_2 \neq 0 \) is analogous). It is easy to observe that in each Case i) or ii) by the choice of the face \( S_3 \) and the vertex \( W \), there exists \( i \in \{1, 2, 3\} \) such that \( W - 1_i \in OX_3 \). Then from Property 2.8 we have

\[
(11) \quad \frac{\text{ord} (\nabla f \circ \phi)}{\text{ord} \phi} \leq \frac{\text{ord} (f'_{z_3} \circ \phi)}{\text{ord} \phi} = \frac{x_3(W - 1_i) \text{ord} \phi_3}{\text{ord} \phi_3} = x_3(W - 1_i)
\]

\[
\leq m(T - 1_i) \leq m(T) - 1,
\]

where \( T \) is the supporting plane to the face \( S_3 \). The last inequality contradicts the inequality (8).

Now, suppose that \( \phi_2 \neq 0 \) and \( \phi_3 \neq 0 \). Denote \( w = (\text{ord} \phi_2, \text{ord} \phi_3) \). Consider the unique supporting line \( l \subset OX_2X_3 \) to \( \Gamma_+(g) \subset OX_2X_3 \) such that \( w \perp l \). Then by Lemma 2.9 we have

\[
\frac{\text{ord} (\nabla f \circ \phi)}{\text{ord} \phi} \leq \frac{\text{ord} (f'_{z_2} \circ \phi, f'_{z_3} \circ \phi)}{\text{ord} \phi} = \frac{\text{ord} (\nabla g \circ \phi_0)}{\text{ord} \phi_0} \leq m(l) - 1.
\]
This and inequality (8) show that \( m(l) > \max_{i=1}^{3} m(S_i) \). Without loss of generality we may assume that \( m(l) = x_3(l) \). Hence we obtain that

\[
(12) \quad x_3(l) > m(S_3) = m(T) \geq x_3(T).
\]

Then for the face \( S_3 \) the condition i) can not be true. Therefore the condition ii) holds for it. Choose \( j, k \in \{1, 2\}, j \neq k \) and vertices \( W_j \in OX_3X_j \) at a distance 1 from the axis \( OX_3 \) and \( Y \in OX_3X_k \) such that the segment \( W_jY \) is an edge of the face \( S_3 \). We shall show that there is a line \( k_j \subset OX_2X_3, k_j \parallel l \), which supports \( \Gamma_+(f'_{z_j}(0, z_2, z_3)) \subset OX_2X_3 \) exactly at one point \( W_j - 1_j \) and \( m(k_j) \leq m(S_3) - 1 \).

\[\text{Fig. 7: } j = 1\]

Denote \( r_j = T \cap \{x_1 = 2 - j\} \) and let \( A \) be the vertex of the edge \( W_jY \) which lies on the plane \( OX_2X_3 \). Let \( \alpha \) be the acute angle between the line \( l \) and the axis \( OX_3 \) and \( \beta \) the acute angle between the line \( T \cap OX_2X_3 \) and the axis \( OX_3 \). Since \( l \) is a supporting line to \( \Gamma_+(g) \), \( A \) lies on the line \( l \) or above it (in Fig. 7 and Fig. 8 \( A \in l \)). This and (12) show that \( \alpha < \beta \).
Now, consider the line $k_j \parallel l$ such that $W - 1_j \in k_j$. The set $\text{supp } f$ lies on the plane $T$ or above it. Hence the set $\text{supp } f'_{z_j}(0, z_2, z_3)$ lies on the line $r_j - 1_j$ or above it. Since $\alpha < \beta$ and $k_j \parallel l$, we have $x_2(k_j) < x_2(r_j - 1_j)$, moreover $\text{supp } f'_{z_j}(0, z_2, z_3)$ except the point $(W - 1_j) \in k_j$, lies above the line $k_j$. Therefore the line $k_j$ supports $\Gamma_+(f'_{z_j}(0, z_2, z_3))$ exactly at one point $W_j - 1_j$. Moreover $m(k_j) \leq m(r_j - 1_j)$. Then by Property 2.8 we get that

$$m(k_j) \leq m(r_j - 1_j) \leq m(T - 1_j) \leq m(T) - 1 = m(S_3) - 1. \tag{13}$$

Summing up, by inequality (13) and by Property 2.1b,

$$\frac{\text{ord } (\nabla f \circ \phi)}{\text{ord } \phi} \leq \frac{\text{ord } (f'_{z_j} \circ \phi)}{\text{ord } \phi} \leq \frac{\text{ord } (f'_{z_j}(0, \phi_0))}{\text{ord } \phi_0} = m(k_j) \leq m(S_3) - 1,$$

which leads to a contradiction with inequality (8). Therefore we obtain

$$\mathcal{L}_0(f) \leq \max_{i=1}^3 m(S_i) - 1 \leq \max_{S \in \Gamma^2(f) \setminus E_f} m(S) - 1,$$

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which completes the proof of part $2^0$. $\square$

5. Examples and open problems

Now, we give examples to illustrate the main result of this paper (Theorem 1.8). The first example illustrates part $1^0$ of Theorem 1.8. For the singularity of this example we have $\Gamma^2(f) = E_f$ and $L_0(f) < \max_{S \in E_f} m(S) - 1$.

**Example 5.1.** Let $f(z_1, z_2, z_3) := z_1^3 + z_2^3 + z_3^2 + z_1 z_2$. In this case $\Gamma^2(f) = E_f$, thus, under part $1^0$ of Theorem 1.8 we have that $I_3^3 \in \Gamma^1(f) \cap J$ and $L_0(f) = m(I_3^3) - 1 = 1 < 2 = \max_{S \in E_f} m(S) - 1$.

The next example also illustrates part $1^0$ of Theorem 1.8. For the singularity in this example we have $\Gamma^2(f) = \emptyset$ and $\Gamma^1(f) = \{I_2^3\}$.

**Example 5.2.** Let $f(z_1, z_2, z_3) := z_2^3 + z_1 z_3 + z_1^2 z_3^2$. In this case $\Gamma^2(f) = \emptyset$. Thus, under part $1^0$ of Theorem 1.8 we have that $I_2^3 \in \Gamma^1(f) \cap J$ and $L_0(f) = m(I_2^3) - 1 = 2$.

The last example illustrates part $2^0$ of Theorem 1.8. It shows that for the singularity in this example, the estimate obtained from this part of the theorem is optimal, i.e. in formula (5) we have the equality.

**Example 5.3.** Let $f(z_1, z_2, z_3) := z_1^3 + z_2^3 + z_1 z_3^4 + z_2 z_3^4 + z_3^{20}$. In this case $\Gamma^2(f) = \{S_1, S_2\}$, $S_1 = \text{conv} \{ (3,0,0), (0,3,0), (1,0,4), (0,1,4) \}$, $S_2 = \text{conv} \{ (1,0,4), (0,1,4), (0,0,20) \}$, $E_f = \{S_2\}$. Hence from part $2^0$ of Theorem 1.8 we get that $L_0(f) \leq m(S_1) - 1 = 5$ (by Fukui theorem [8] we would get $L_0(f) \leq \max(m(S_1), m(S_2)) - 1 = 19$). In this example one can show that $L_0(f) = 5$.

The above example suggests that the following is true.

**Conjecture 5.4.** Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $n \geq 2$, be an isolated and nondegenerate singularity such that $\Gamma^{n-1}(f) \setminus E_f \neq \emptyset$. Then

$$L_0(f) = \max_{S \in \Gamma^2(f) \setminus E_f} m(S) - 1. \tag{14}$$

Certainly there arises the question how to generalize part $1^0$ of Theorem 1.8 to the $n$-dimensional case for $n > 3$.

**Problem 5.5.** Characterize isolated singularities in $n$-variables, $n > 3$, for which $\Gamma^{n-1}(f) = \emptyset$ or $\Gamma^{n-1}(f) = E_f$, and give the formula for the Lojasiewicz exponent of such singularities.
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