Mathematical Models for Local Sensing Hashes

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Abstract

As data volumes continue to grow, searches in data are becoming increasingly time-consuming. Classical index structures for neighbor search are no longer sustainable due to the “curse of dimensionality”. Instead, approximated index structures offer a good opportunity to significantly accelerate the neighbor search for clustering and outlier detection and to have the lowest possible error rate in the results of the algorithms. Local sensing hashes is one of those. We indicate directions to mathematically model the properties of it.

1 Basic Definition

In a local sensing hashes we use a number \( m \) of alternative clusterings, and we refine for each of the \( m \) clusterings exactly one cell, in which the query point (origin) is located. Therefore, the selectivity of the query response corresponds to the intersection of the union of all \( m \) cells with the query ball. We start with the analysis of the case \( s = b = 1 \). We assume that each of the \( m \) cells is selected uniformly and independently such that the origin (query point) is located inside, so the upper boundary of each cell in each dimension is uniformly taken from the interval \([0, \frac{1}{2}]\) and the lower boundary is consequently from \([-\frac{1}{2}, 0]\).

The idea is to consider the convex, Voronoi-like cells of k-means as a \( d \)-dimensional grid of hyper-cubic cells which have a volume such that an expectation of \( n/k \) points are located within each. We denote the side length of these cells with \( b \). The query is modeled as a ball with perimeter \( s \), centered at origin. In the beginning, we assume that the ball is actually from a maximum metric such that the query is as well a hypercube with side length \( s \).

In the following we will use several definite integrals which we denote in a bit unusual way which helps for clarity we will write

\[
\int \ldots (0 \leq x \leq 1) \text{ instead of } \int_0^1 \ldots \, dx.
\]

We will need a few integrals throughout this paper:

\[
\int x + \frac{1}{2} (0 \leq x \leq 1) = \frac{3}{8}
\]

\[
\int \int (x + \frac{1}{2})(y + \frac{1}{2}) (0 \leq x, y \leq \frac{1}{2}) = \frac{2}{27}
\]

\[
\int \ldots \int (x_1 + \frac{1}{2}) \cdot \ldots \cdot (x_d + \frac{1}{2}) (0 \leq x_1, \ldots, x_d \leq \frac{1}{2}) = \left( \frac{3}{8} \right)^d
\]

\[
\int \int \min(x_1, x_2) + \frac{1}{2} (0 \leq x_1, x_2 \leq \frac{1}{2}) = \frac{5}{27}
\]

\[
\int \ldots \int \min(x_1, \ldots, x_d) + \frac{1}{2} (0 \leq x_1, \ldots, x_d \leq \frac{1}{2}) = d \cdot \int x^{d-1} \cdot (x + \frac{1}{2}) (0 \leq x \leq \frac{1}{2}) = \frac{2d + 1}{(d + 1) \cdot 2d + 1}
\]
\[
\int y - v \ (0 \leq y \leq \frac{1}{2}, -\frac{1}{2} \leq v \leq 0) = \frac{1}{8}
\]
\[
\int \ldots \int (y_1 - v_1) \cdot \ldots \cdot (y_d - v_d) \ (0 \leq y_1, \ldots, y_d \leq \frac{1}{2}, -\frac{1}{2} \leq v_1, \ldots, v_d \leq 0) = \left(\frac{1}{8}\right)^d
\]

And, finally, we can also solve the following combination:
\[
\int \ldots \int (\min(x_1, \ldots, x_d) + \frac{1}{2}) \cdot (y_1 - v_1) \cdot \ldots \cdot (y_m - v_m) \ (0 \leq x_1, \ldots, x_d, y_1, \ldots, y_m \leq \frac{1}{2}, -\frac{1}{2} \leq v_1, \ldots, v_m \leq 0)
\]

which gives:
\[
= \frac{2m + 1}{8^d \cdot (m + 1) \cdot 2^{m+1}}.
\]

**Definition 1.** \(p(m, \ell, d)\) is the (hyper-) volume of the hypercube representing the query which is occupied by at least \(\ell\) of the \(m\) cells (with \(\ell \leq m\)). Here the \(m\) cells are uniformly selected in \(\mathbb{R}^d\) such that the query point is inside the cell.

Note that \(p(m, 1, d)\) corresponds to the selectivity of the hashing provided that \(b\) and \(s\) are of equal size.

For the case \(m = \ell = 1\) we can easily derive the closed formula of \(p(1, 1, d)\). We simply have to form, in each quadrant, the (identical) expectation with which the query is occupied by the cell:
\[
p(1, 1, d) = 2^d \cdot \int \ldots \int (x_1 + \frac{1}{2}) \cdot \ldots \cdot (x_d + \frac{1}{2}) \ (0 \leq x_1, \ldots, x_d \leq \frac{1}{2}) = \left(\frac{4}{3}\right)^d
\]

The case \(m > 1, \ell = 1\) can be reduced to the case \(m = 1\):
\[
p(m, 1, d) = \binom{m}{1} \cdot p(1, 1, d) - \binom{m}{2} \cdot p(1, 2, d) + \binom{m}{3} \cdot p(1, 3, d) - \ldots \pm \binom{m}{m} \cdot p(1, m - 1, d)
\]

For the case \(m = 1, \ell = 2\), we start with the analysis of \(d = 1\) and construct this case with two variables \(x\) and \(y\). We want to estimate the area which is covered by \(x\) and \(y\), which is
\[
p(1, 2, 1) = \int \left\{ \begin{array}{ll}
1 - |x - y| & \text{if } x < 0 \text{ xor } y < 0 \\
1 - \max(|x|, |y|) & \text{otherwise}
\end{array} \right.
\]
\[
(-\frac{1}{2} \leq x, y \leq \frac{1}{2}) = \frac{7}{12}.
\]

This is an independent information in each dimension such that
\[
p(1, 2, d) = p(1, 2, 1)^d = \left(\frac{7}{12}\right)^d.
\]

For a generalization to \(\ell = 3\) we make a similar case distinction:
\[
p(1, 3, 1) = \int \ldots \int \left\{ \begin{array}{ll}
1 - \max(x, y, z) & \text{if } x, y, z \geq 0 \\
1 - \max(x, y) + z & \text{if } x, y \geq 0 \text{ and } z < 0 \\
\ldots & \ldots
\end{array} \right. = \frac{15}{32},
\]

where the first case occurs in 2 octants and the second case in the remaining 6 octants. We solve the first case by letting
\[
2 \cdot \int \ldots \int 1 - \max(x, y, z) = 6 \cdot \int (1 - x) \cdot x^2 = \frac{5}{32}
\]

and similarly the second case
\[
6 \cdot \int \ldots \int 1 - \max(x, y) + z = 12 \cdot \int (1 - x + z) \cdot x = \frac{5}{16}.
\]

For the general case, we consider \(\ell\) variables \(x_1, \ldots, x_\ell\). Again we consider a solution space where we distinguish if each variable is greater or less than 0. All variables are equivalent, thus we consider all quadrants of the solutions space equally where the same number of variables is < 0. There is a number of
\[
\binom{\ell}{i}
\]
quadrants of the solution space having \(i\) out of \(\ell\) variables < 0.

Therefore, we have
\[
p(1, \ell, 1) = \sum_{0 \leq i \leq \ell} \binom{\ell}{i} \cdot \int \ldots \int 1 - \max(x_1, \ldots, x_i) + \min(x_{i+1}, \ldots, x_\ell) \ (0 \leq x_1, \ldots, x_i \leq \frac{1}{2}, -\frac{1}{2} \leq x_{i+1}, \ldots, x_\ell \leq 0)
\]
2 Related Work

Approximate nearest neighbor search techniques can also be applied to the similarity join problem, however without guarantees on completeness and exactness of the result. There may be false positives as well as false negatives. Recently an approach [1] to Local Sens Hash is used on a representative point sample, to reduce the number of lookup operations. LSH is of interest in theoretical foundational work, where a recursive and cache-oblivious LSH approach [2] was proposed. The topic of approximate solutions for the similarity join is also an emerging field in deep learning [3]. There are approximative approaches which target low dimensional cases (spatial joins in 2–3 dimensions [4]) or higher (10–20) dimensional cases [5]. Very high-dimensional cases, with dimensions of 128 and above have been targeted with Symbolic Aggregate approXimation (SAX) techniques [6] to generate approximate candidates. SAX techniques rely on several indirect parameters like PAA size or the iSAX alphabet size.

3 Conclusion

As data volumes continue to grow, searches in data are becoming increasingly time-consuming. Classical index structures for neighbor search are no longer sustainable due to the “curse of dimensionality”. Instead, approximated index structures offer a good opportunity to significantly accelerate the neighbor search for clustering and outlier detection and to have the lowest possible error rate in the results of the algorithms. Local sensing hashes is one of those. We indicate directions to mathematically model the properties of it. There is still a lot research necessary with local sensing hashes.

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