A NOTE ON TEISSIER PROBLEM FOR NEF CLASSES

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ABSTRACT. Teissier problem aims to characterize the equality case of Khovanskii-Teissier type inequality for (1, 1)-classes on a compact Kähler manifold. When each of the involved (1, 1)-classes is assumed to be nef and big, this problem has been solved by the previous works of Boucksom-Favre-Jonsson [1], Fu-Xiao [7] and Li [13]. In this note, we shall settle the case that the involved (1, 1)-classes are just assumed to be nef. We also extend the results to some settings where some of the (1, 1)-classes are not necessarily nef. By constructing examples, it is shown that our results are optimal.

1. Introduction

1.1. Khovanskii-Teissier inequalities and Teissier problem. Around the year 1979, Khovanskii and Teissier independently discovered deep inequalities in algebraic geometry, which are profound analogs of Alexandrov-Fenchel inequalities in convex geometry. There are many remarkable further developments on Khovanskii-Teissier type inequalities (see e.g. [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 17, 21] and references therein), among which we may recall the following one over an \( n \)-dimensional compact Kähler manifold \((X, \omega_X)\) as an example (see [4, 5, 8]), which will be closely related to our study here. In this note, when we write a class as \([\alpha]\), then \(\alpha\) always is a smooth representative of \([\alpha]\). Arbitrarily given \([\alpha_1],...,\alpha_n] \in H^{1,1}(X, \mathbb{R})\) and assume that the first \(n-1\) entries, \([\alpha_1],...,\alpha_{n-1}\], are nef, then the following Khovanskii-Teissier inequality holds:

\[
\left( \int_X \alpha_1 \wedge ... \wedge \alpha_{n-2} \wedge \alpha_{n-1} \wedge \alpha_n \right)^2 \geq \int_X \alpha_1 \wedge ... \wedge \alpha_{n-2} \wedge \alpha_{n-1}^2 \cdot \int_X \alpha_1 \wedge ... \wedge \alpha_{n-2} \wedge \alpha_n^2.
\] (1.1)

Given the above inequality, it is natural to consider

**Problem 1.1** (Teissier [17, 18]). Characterize the equality case in (1.1).

In fact, a classical result states that if further assume \([\alpha_1],...,\alpha_{n-1}\] are Kähler classes, then the equality in (1.1) holds if and only if \([\alpha_{n-1}]\) and \([\alpha_n]\) are proportional in \(H^{1,1}(X, \mathbb{R})\) (see e.g. [7, 12]). Moreover, if \([\alpha], [\beta] \in H^{1,1}(X, \mathbb{R})\) are nef and big, and for each \(k = 1, ..., n-1\), the equality in (1.1) holds for \([\alpha_1] = ... = [\alpha_k] = [\alpha]\) and \([\alpha_{k+1}] = ... = [\alpha_n] = [\beta]\), then \([\alpha]\) and \([\beta]\) are proportional, thanks to the works of Boucksom-Favre-Jonsson [11] when \([\alpha], [\beta]\) are rational and Fu-Xiao [7] when \([\alpha], [\beta]\) are transcendental. Li [13] further proved that if all the \(n\) entries \([\alpha_1],...,\alpha_{n}\] are nef and big, then the equality in (1.1) holds if and only if \([\alpha_1} \wedge ... \wedge \alpha_{n-2} \wedge \alpha_{n-1}\) and \([\alpha_1} \wedge ... \wedge \alpha_{n-2} \wedge \alpha_n]\) are proportional in \(H^{n-1,n-1}(X, \mathbb{R})\).

In this note, we consider the general case of Problem 1.1 that all the \([\alpha_j]\)'s are just assumed to be nef, dropping the bigness assumption.

1.2. Results on Teissier Problem 1.1. We now state our results on Problem 1.1.
**Theorem 1.2.** Let $X$ be an $n$-dimensional compact Kähler manifold, and $[\alpha_1], \ldots, [\alpha_{n-2}], [\alpha], [\beta] \in H^{1,1}(X, \mathbb{R})$. Assume either of the followings is satisfied.

(A1) $[\alpha_1], \ldots, [\alpha_{n-2}], [\alpha]$ are all nef and $[\beta]$ satisfies $\int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \beta^2 \geq 0$.

(A2) $[\alpha_1], \ldots, [\alpha_{n-2}], [\alpha]$ are all nef and $\int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha^2 > 0$.

Then

$$\left(\int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha \wedge \beta\right)^2 = \int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha^2 \cdot \int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \beta^2 \quad (1.2)$$

if and only if $[\alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha]$ and $[\alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \beta]$ are proportional.

**Example 1.3.**

1. The (A1) is satisfied if $[\beta]$ is also nef.
2. The (A2) is satisfied if $[\alpha_1], \ldots, [\alpha_{n-2}], [\alpha]$ are all nef and big, or if $[\alpha_1], \ldots, [\alpha_{n-2}]$ are nef classes with $[\alpha_1 \wedge \cdots \wedge \alpha_{n-2}] \neq 0$ and $[\alpha]$ is Kähler.

The following special case of Theorem 1.2(A1) may be of particular interest and applications.

**Theorem 1.4.** Let $X$ be an $n$-dimensional compact Kähler manifold, and $[\alpha_1], \ldots, [\alpha_{n-2}], [\alpha], [\beta] \in H^{1,1}(X, \mathbb{R})$ be nef classes. Then

$$\left(\int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha \wedge \beta\right)^2 = \int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha^2 \cdot \int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \beta^2 \quad (1.3)$$

if and only if $[\alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha]$ and $[\alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \beta]$ are proportional.

As an immediate application of Theorem 1.2(A1) (or 1.4), we can slightly extend Fu-Xiao’s result [7, Theorem 2.1, (1) and (6)] on nef and big classes to the case of nef classes.

**Corollary 1.5.** Let $X$ be an $n$-dimensional compact Kähler manifold, and $[\alpha], [\beta] \in H^{1,1}(X, \mathbb{R})$ are two nef classes with $[\alpha^k \wedge \beta^{n-k-1}] \neq 0$ for each $k = 0, 1, \ldots, n-1$. Write $s_k := \int_X \alpha^k \wedge \beta^{n-k}$, $k = 0, 1, \ldots, n$. If

$$s_k^2 = s_{k-1} \cdot s_{k+1} \text{ for every } k = 1, \ldots, n-1,$$

then $[\alpha^{n-1}]$ and $[\beta^{n-1}]$ are proportional.

Actually, by Theorem 1.2(A1) (or 1.4), the equalities in (1.4) imply that $[\alpha^k \wedge \beta^{n-k-1}]$’s are proportional, $k = 0, 1, \ldots, n-1$. In particular, $[\alpha^{n-1}]$ and $[\beta^{n-1}]$ are proportional, proving the desired result. This gives an alternative treatment for [7, Theorem 2.1, (1) and (6)] (also see [13, Corollary 3.11]).

The most general setting in the inequality (1.1) and Problem 1.1 assumes that $[\alpha_1], \ldots, [\alpha_{n-2}], [\alpha]$ are all nef and $[\beta]$ is arbitrary, while our Theorem 1.2 characterizes its equality under additional assumptions (i.e. $\int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \beta^2 \geq 0$ in (A1) and $\int_X \alpha_1 \wedge \cdots \wedge \alpha_{n-2} \wedge \alpha^2 > 0$ in (A2)). Then it is natural to wonder that are these additional assumptions in Theorem 1.2 necessary? Namely, is Theorem 1.2 optimal? We affirmatively answer this by Example 2.1 in the next section. Combining Theorem 1.2 and Example 2.1 settles Problem 1.1 completely.

On the other hand, comparing with the classical result stated below Problem 1.1 and results in [11, 7], it seems natural to ask: given the equality assumed in Theorem 1.2 above, can we make the stronger conclusion that $[\alpha]$ and $[\beta]$ are proportional? Of course this
question is interesting only when \([\alpha_1 \wedge ... \wedge \alpha_{n-2}] \neq 0\). The Example 2.3 in Section 2 shows that this is not always true. Therefore, we are naturally led to consider: given nef classes \([\alpha_1], ..., [\alpha_{n-2}]\), when can we conclude from the equality (1.2) that \([\alpha]\) and \([\beta]\) are proportional? This turns out to be intimately related to Hodge index theorem, whose definition is recalled as follows.

**Definition 1.6.** (Hodge index theorem [5, Section 4]; also [22, Definition 1.5]) For any \([\Omega] \in H^{n-2,n-2}(X, \mathbb{R}) := H^{n-2,n-2}(X, \mathbb{C}) \cap H^{2n-4}(X, \mathbb{R})\) and \([\eta] \in H^{1,1}(X, \mathbb{R})\), we define the primitive space with respect to \([\Omega], [\eta]\) by

\[
P = P([\Omega], [\eta]) := \{[\gamma] \in H^{1,1}(X, \mathbb{C}) | [\Omega] \wedge [\eta] \wedge [\gamma] = 0\}.
\]

Then we say \(([\Omega], [\eta])\) satisfies the Hodge index theorem if the quadratic form

\[
Q_{([\Omega], [\eta])}([\beta], [\gamma]) := \int_X [\Omega] \wedge \beta \wedge \gamma \tag{1.5}
\]

is negative definite on \(P([\Omega], [\eta])\).

Several known Hodge index theorems will be listed in Example 3.3. Here is our next theorem.

**Theorem 1.7.** Let \(X\) be an \(n\)-dimensional compact Kähler manifold. The followings are equivalent.

(1) for any \([\alpha_1], ..., [\alpha_{n-2}], [\alpha], [\beta]\) satisfying either of (A1), (A2) in Theorem 1.2,

\[
\left(\int_X [\alpha_1 \wedge ... \wedge [\alpha_{n-2} \wedge [\alpha] \wedge \beta]\right)^2 = \int_X [\alpha_1 \wedge ... \wedge [\alpha_{n-2} \wedge [\alpha] \wedge \beta]^2 \tag{1.5}
\]

if and only if \([\alpha]\) and \([\beta]\) are proportional.

(2) \(([\alpha_1 \wedge ... \wedge [\alpha_{n-2}], [\omega_X]])\) satisfies Hodge index theorem.

(3) there exists \([\theta] \in H^{1,1}(X, \mathbb{R})\) such that \(([\alpha_1 \wedge ... \wedge [\alpha_{n-2}], [\theta]])\) satisfies Hodge index theorem.

**Remark 1.8.** In the algebraic setting, a special (non-mixed) version of Theorem 1.2(A2) was contained in [14], whose idea can be modified to proved Theorem 1.2(A2) in the Kähler setting (we thank Jian Xiao for pointing this out to us); also see [13] for another argument in the case of all \(n(1,1)\)-classes being nef and big. As will be presented in Sections 3 and 4, our method in this note will take a different and slightly more general and abstract way, which enables us to unifiedly handle both items (A1) and (A2) of Theorem 1.2 as well as their extensions (see Section 4) in which some of the \((1,1)\)-classes are NOT necessarily nef. Moreover, Theorem 1.2 is somehow optimal, see Example 2.1.

**1.3. Organization.** The remaining part is organized as follows. In Section 2 we will exhibit several examples concerning the optimality of our results. In Section 3 some general properties of limits of pairs satisfying Hodge index theorem will be presented. In Section 4 we will prove results stated above. In Section 5 we discuss certain extensions of our results to more general settings where some of the \((1,1)\)-classes are NOT necessarily nef.

**2. Examples**

We first exhibit some examples, which illustrate the optimality of our results.
Example 2.1 (Theorem 1.2 is optimal). Assume $X$ is an $n$-dimensional compact Kähler manifold of $d := \dim H^{1,1}(X, \mathbb{R}) \geq 3$, and there is a holomorphic surjection $f : X \to S$ to a closed smooth Riemann surface $S$. Fix Kähler metrics $\omega$ on $X$ and $\chi$ on $Y$, and set $[\alpha] := [f^*\chi]$, which is a nef class on $X$. Note that $[\omega^{n-2} \wedge \alpha] \neq 0$ and
\[
\int_X \omega^{n-2} \wedge \alpha^2 = 0. \tag{2.1}
\]
Denote $P_1 := \{[\gamma] \in H^{1,1}(X, \mathbb{R}) | [\omega^{n-1} \wedge \gamma] = 0\}$ and $P_2 := \{[\gamma] \in H^{1,1}(X, \mathbb{R}) | [\omega^{n-2} \wedge \alpha \wedge \gamma] = 0\}$. Both $P_1$ and $P_2$ are $(d - 1)$-dimensional subspace in $H^{1,1}(X, \mathbb{R})$ (as $[\omega^{n-2} \wedge \alpha] \neq 0$). Because $d \geq 3$, by linear algebra we may fix a non-zero $[\beta] \in P_1 \cap P_2$. Then $[\beta]$ satisfies
\[
[\omega^{n-2} \wedge \alpha \wedge \beta] = 0 \text{ and } [\omega^{n-1} \wedge \beta] = 0 \tag{2.2}
\]
and
\[
\int_X \omega^{n-2} \wedge \beta^2 < 0. \tag{2.3}
\]
Note that given the second equality in (2.2) and the fact that $[\beta] \neq 0$, the (2.3) follows from the classical Hodge index theorem (see e.g. [20]). From (2.1) and (2.2), we obviously have the equality
\[
\left(\int_X \omega^{n-2} \wedge \alpha \wedge \beta\right)^2 = \int_X \omega^{n-2} \wedge \alpha^2 \cdot \int_X \omega^{n-2} \wedge \beta^2.
\]
However, we claim that $[\omega^{n-2} \wedge \alpha]$ and $[\omega^{n-2} \wedge \beta]$ are NOT proportional. Indeed, if $[\omega^{n-2} \wedge \beta] = c \cdot [\omega^{n-2} \wedge \alpha]$ for some $c \in \mathbb{R}$, then we conclude that
\[
\int_X \omega^{n-2} \wedge \beta^2 = c \cdot \int_X \omega^{n-2} \wedge \alpha \wedge \beta = 0,
\]
contradicting to (2.3). This example (see (2.1) and (2.3)) shows that Theorem 1.2 is optimal.

Remark 2.2. Combining Theorem 1.2 and Example 2.1 settles Problem 1.1 completely.

The next example shows that in general the conclusion in Theorems 1.2 (and 1.4) could not be improved to $[\alpha]$ and $[\beta]$ being proportional.

Example 2.3. $([\alpha]$ and $[\beta]$ may not be proportional) Let $f : X \to Y$ be holomorphic submersion over an $m$-dimensional compact Kähler manifold $Y$, $1 \leq m \leq n - 2$. For $j = 1, \ldots, m$, choose $\alpha_j = f^*\chi_j$, where $\chi_j$’s are Kähler metrics on $Y$, and $\alpha_{m+1}, \ldots, \alpha_{n-1}$ be Kähler metrics on $X$ and $\alpha_n = \alpha_{n-1} + f^*\chi$ with $\chi$ a Kähler metric on $Y$. Then $[\alpha_1 \wedge \ldots \wedge \alpha_{n-2}] \neq 0$ and the equality (1.2) holds with $[\alpha] = \alpha_{n-1}$ and $[\beta] = \alpha_n$, but $[\alpha_{n-1}]$ and $[\alpha_n]$ are NOT proportional. Acturally, if $[\alpha_{n-1}]$ and $[\alpha_n]$ are proportional, we can easily deduce that $[f^*\chi]$ is a Kähler or zero class, both of which are absurd.

Combining Theorem 1.7, one sees that $[f^*\chi_1 \wedge \ldots \wedge f^*\chi_m \wedge \alpha_{m+1} \wedge \ldots \wedge \alpha_{n-2}]$ does not satisfy Hodge index theorem (compare [22, Theorem 1.7]).
3. LIMITS OF PAIRS SATISFYING HODGE INDEX THEOREM

3.1. A generalized $m$-positivity. We first introduce the positivity condition used in the discussions.

**Definition 3.1.** ([22] Definition 1.1) Let $\Phi$ be a (strictly) positive $(m,m)$-form on $X$ and $\eta$ a real $(1,1)$-form on $X$. We say a real $(1,1)$-form $\alpha$ on $X$ is $(n-m)$-positive with respect to $(\Phi, \eta)$ if

$$\Phi \wedge \eta^{n-m-k} \wedge \alpha^k > 0$$

for any $1 \leq k \leq n-m$. In particular, the case that $(\Phi, \eta) = (\omega_X^m, \omega_X)$ gives the original $(n-m)$-positivity with respect to a fixed Kähler metric $\omega_X$, and in this case we say $\alpha$ is $(n-m)$-positive with respect to $\omega_X$.

**Definition 3.2** (generalized $m$-positivity cone). Fix an integer $m \leq n-2$. Assume $\omega_1, ..., \omega_m$ are Kähler metrics on $X$. Let $\Gamma \subset H^{1,1}(X, \mathbb{R})$ be the convex open cone consisting of $[\phi] \in H^{1,1}(X, \mathbb{R})$ that has a smooth representative $\phi$ which is $(n-m)$-positive with respect to $(\omega_1 \wedge ... \wedge \omega_m, \omega_X)$, and let $\Gamma$ be the closure of $\Gamma$ in $H^{1,1}(X, \mathbb{R})$. Note that $\Gamma$ contains the nef cone of $X$, as $\Gamma$ contains the Kähler cone of $X$.

3.2. Hodge index theorems. Let $\mathcal{H}$ be the set of pair $([\Omega], [\eta]) \in H^{n-2,n-2}(X, \mathbb{R}) \times H^{1,1}(X, \mathbb{R})$ satisfying the Hodge index theorem.

**Example 3.3** (Known elements in $\mathcal{H}$). Several elements in $\mathcal{H}$ have been pinpointed by classical and recent works.

1. If $\omega$ is a Kähler metric on $X$, then $([\omega^{n-2}], [\omega]) \in \mathcal{H}$ by the classical Hodge index theorem; if $\omega_1, ..., \omega_{n-1}$ are Kähler metrics, then $([\omega_1 \wedge ... \wedge \omega_{n-2}, \omega_{n-1}] \in \mathcal{H}$ by [[2], [5], [8]]; moreover, if $\omega$ is a Kähler metric and $\alpha_1, ..., \alpha_{n-m-1}$ are closed real $(1,1)$-forms which are $(n-m)$-positive with respect to $\omega$, then $([\omega^m \wedge \alpha_1 \wedge ... \wedge \alpha_{n-m-2}], [\alpha_{n-m-1}] \in \mathcal{H}$, thanks to Xiao [21].

2. Several abstract versions of Hodge index theorem have been discovered in [6] [15].

**Remark 3.4.** ([22] Remark 2.9) We remark some consequences of the Hodge index theorem, which should be well-known (see e.g. [5] [21]).

1. For any $([\Omega], [\eta]) \in \mathcal{H}$, we have the Hard Lefschetz and Lefschetz Decomposition Theorems (with respect to $([\Omega], [\eta])$) as follows:
   (a) The map $[\Omega] : H^{1,1}(X, \mathbb{C}) \to H^{n-1,n-1}(X, \mathbb{C})$ is an isomorphism;
   (b) The space $H^{1,1}(X, \mathbb{C})$ has a $Q$-orthogonal direct sum decomposition

$$H^{1,1}(X, \mathbb{C}) = P([\Omega], [\eta]) \oplus \mathbb{C}[\eta].$$

2. For any $([\Omega], [\eta]) \in \mathcal{H}$, we have the Khovanskii-Teissier type inequalities as follows.
   (c) For any closed real $(1,1)$-forms $\phi, \psi \in H^{1,1}(X, \mathbb{R})$ with $\phi$ 2-positive with respect to $(\Omega, \eta)$, then

$$\left( \int_X \Omega \wedge \phi \wedge \psi \right)^2 \geq \left( \int_X \Omega \wedge \phi^2 \right) \left( \int_X \Omega \wedge \psi^2 \right)$$

with equality if and only if $[\phi]$ and $[\psi]$ are proportional.
3.3. Boundary elements in $\overline{\mathcal{H}}$. In this subsection assume
\[(\Omega, [\eta]) = \lim(\Omega_i, [\eta_i]) \text{ in } H^{n-2,n-2}(X, \mathbb{R}) \times H^{1,1}(X, \mathbb{R}),\]
where $(\Omega_i, [\eta_i]) \in \mathcal{H}$; in other words, $(\Omega, [\eta]) \in \overline{\mathcal{H}}$, the closure of $\mathcal{H}$ in $H^{n-2,n-2}(X, \mathbb{R}) \times H^{1,1}(X, \mathbb{R})$.

We shall investigate the properties of such $(\Omega, [\eta])$, which slightly extends Section 4 and will be used in our later discussions.

Example 3.5 (Known elements on the boundary of $\overline{\mathcal{H}}$).
(1) ([3] Section 4) Let $[\alpha_1], ..., [\alpha_{n-2}]$ be nef $(1,1)$-classes with $[\alpha_1 \wedge ... \wedge \alpha_{n-2}] \neq 0$, and $\omega$ a Kähler metric. Note that both $[\alpha_1 \wedge ... \wedge \alpha_{n-2}] \wedge [\omega]$ and $[\alpha_1 \wedge ... \wedge \alpha_{n-2}] \wedge [\omega]^2$ can be represented by non-zero positive currents. Then
\[(\alpha_1 \wedge ... \wedge \alpha_{n-2}, [\omega]) \in \overline{\mathcal{H}} \text{ and } [\alpha_1 \wedge ... \wedge \alpha_{n-2}] \wedge [\omega^2] \neq 0.
\]
(2) Fix an integer $m \leq n - 2$. Assume $\omega_1, ..., \omega_m$ are Kähler metrics on $X$. Let $\Gamma \subset H^{1,1}(X, \mathbb{R})$ be the cone defined in Definition 3.2. Suppose $[\alpha_1], ..., [\alpha_{n-m-2}] \in \Gamma$ and set $[\Omega] := [\omega_1 \wedge ... \wedge \omega_m \wedge \alpha_1 \wedge ... \wedge \alpha_{n-m-2}]$. Assume $[\Omega] \neq 0$. Then for any Kähler metric $\omega$, by Example 3.3(2) we know $(\Omega, [\omega]) \in \overline{\mathcal{H}}$, as each $[\alpha_j]$ is a limit of elements in $\Gamma$. Moreover, for any $n - m - 2$ closed real $(1,1)$-forms $\tilde{\alpha}_1, ..., \tilde{\alpha}_{n-m-2}$ with $[\tilde{\alpha}_j] \in \Gamma$, by Gårding theory for hyperbolic polynomials (see e.g. [22] Theorem 2.2(1)), locally we have, for any $i, j$ and any Kähler metric $\tilde{\omega}$,
\[\omega_1 \wedge ... \wedge \omega_m \wedge \tilde{\alpha}_1 \wedge ... \wedge \tilde{\alpha}_{n-m-2} \wedge \sqrt{-1}dz^i \wedge d\bar{z}^j \wedge \sqrt{-1}dz^j \wedge d\bar{z}^j \geq 0\]
and
\[\omega_1 \wedge ... \wedge \omega_m \wedge \tilde{\alpha}_1 \wedge ... \wedge \tilde{\alpha}_{n-m-2} \wedge \tilde{\omega} \wedge \sqrt{-1}dz^j \wedge d\bar{z}^j > 0\]
from which, by limiting $\tilde{\alpha}_j$'s, one sees that both $[\Omega] \wedge [\omega]$ and $[\Omega] \wedge [\omega^2]$ can be represented by non-zero positive currents. Therefore, $[\Omega] \wedge [\omega^2] \neq 0$.

(3) Assume $[\alpha_1], ..., [\alpha_{n-2}], [\eta]$ are nef and big, and set $[\Omega] = [\alpha_1 \wedge ... \wedge \alpha_{n-2}]$. Then obviously $(\Omega, [\eta]) \in \overline{\mathcal{H}}$ and $[\Omega] \wedge [\omega^2] \neq 0$.

(4) We may particularly mention that, given the abstract versions of Hodge index theorem (see [10, 15]), we have some abstract elements on the boundary of $\overline{\mathcal{H}}$.

In general, $(\Omega, [\eta])$ may no longer satisfy the Hodge index theorem; however, it still has some good properties.

Lemma 3.6 (Lefschetz decomposition). Suppose $(\Omega, [\eta]) \in \overline{\mathcal{H}}$ and $[\Omega] \wedge [\eta^2] \neq 0$. Then any $[\beta] \in H^{1,1}(X, \mathbb{C})$ can be uniquely decomposed as
\[[\beta] = c \cdot [\eta] + [\gamma], \text{ with } c \in \mathbb{C} \text{ and } [\gamma] \in P([\Omega], [\eta]).\]

Proof. By definition,
\[(\Omega, [\eta]) = \lim(\Omega_i, [\eta_i]) \text{ in } H^{n-2,n-2}(X, \mathbb{R}) \times H^{1,1}(X, \mathbb{R})\]
for a family $(\Omega_i, [\eta_i]) \in \mathcal{H}$. Then by Remark 3.3 we know $[\beta] = c_i [\eta_i] + [\gamma_i]$ for some $c_i \in \mathbb{C}$ and $[\gamma_i] \in P([\Omega_i], [\eta_i])$. Therefore,
\[[\Omega_i] \wedge [\eta_i] \wedge [\beta] = c_i [\Omega_i] \wedge [\eta_i]^2.\]
Letting $i \to \infty$ and combining $[\Omega] \wedge [\eta^2] \neq 0$, we see that $c_i$’s are uniformly bounded, which in turn implies that $[\gamma_i]$’s are uniformly bounded. Up to passing to a subsequence, we may assume that $c_i \to c$ and $[\gamma_i] \to [\gamma]$. Therefore, 

$$[\beta] = c \cdot [\eta] + [\gamma].$$

Note that $[\gamma] \in P_{([\Omega], [\eta])}$, since $[\Omega] \wedge [\eta] \wedge [\gamma] = \lim \Omega_i \wedge [\eta_i] \wedge [\gamma_i] = 0$. We have proved the existence of the required decomposition.

To see the uniqueness, assume there exists another decomposition $[\beta] = c' \cdot [\eta] + [\gamma']$ for some $c' \in \mathbb{C}$ and $[\gamma'] \in P_{([\Omega], [\eta])}$, which gives that $(c - c')[\eta] = [\gamma'] - [\gamma] \in P_{([\Omega], [\eta])}$, i.e. $(c - c')[\Omega] \wedge [\eta] \wedge [\eta] = 0$. So $c = c'$ and $[\gamma] = [\gamma']$.

The proof is completed. 

\[\square\]

**Lemma 3.7.** Suppose $(\Omega, [\eta]) \in \overline{\mathcal{H}}$ and $[\Omega] \wedge [\eta] \neq 0$. Then the quadratic form 

$$Q_{(\Omega, [\eta])}([\beta], [\gamma]) := \int_X \Omega \wedge \beta \wedge \gamma$$

is semi-negative definite on $P_{([\Omega], [\eta])}$.

Note that here we only assume $[\Omega] \wedge [\eta] \neq 0$.

**Proof.** Write $(\Omega, [\eta]) = \lim (\Omega_i, [\eta_i])$ as above. Since $Q_{(\Omega_i, [\eta_i])}$ is negative definite on $P_{([\Omega_i], [\eta_i])}$, by continuity (here we need $[\Omega] \wedge [\eta] \neq 0$) we immediately conclude that $Q_{(\Omega, [\eta])}$ is semi-negative definite on $P_{([\Omega], [\eta])}$.

\[\square\]

**Lemma 3.8** (characterization of zero-eigenvector). Suppose $(\Omega, [\eta]) \in \overline{\mathcal{H}}$ and $[\Omega] \wedge [\eta^2] \neq 0$. Then for $[\gamma] \in P_{([\Omega], [\eta])}$, 

$$Q_{([\Omega], [\eta])}([\gamma], [\gamma]) = 0 \iff [\Omega] \wedge [\gamma] = 0.$$

**Proof.** It suffices to prove the “$\Rightarrow$” direction, as the other direction is trivial. The proof is similar to [5] Proposition 4.1. Since we are in a slightly more abstract setting, for convenience let’s present some details. Given $[\gamma] \in P_{([\Omega], [\eta])}$ with $Q_{(\Omega, [\eta])}([\gamma], [\gamma]) = 0$, to check that $[\Omega] \wedge [\gamma] = 0$, by Poincaré duality and the definition of $Q = Q_{(\Omega, [\eta])}$, it suffices to examine that $Q([\gamma], \cdot)$ is the zero functional on $H^{1,1}(X, \mathbb{C})$. On the other hand, note that $H^{1,1}(X, \mathbb{C}) = \mathbb{C}[\eta] + P_{(\Omega, [\eta])}$ by Lemma 3.6, therefore, to check $Q([\gamma], \cdot)$ is the zero functional on $H^{1,1}(X, \mathbb{C})$, it suffices to examine that $Q([\gamma], [\eta]) = 0$ and $Q([\gamma], \cdot) = 0$ on $P_{([\Omega], [\eta])}$. The former one is trivially true as $[\gamma] \in P_{([\Omega], [\eta])}$. To see the latter one, arbitrarily take $[\gamma'] \in P_{([\Omega], [\eta])}$. By Lemma 3.7 and $Q_{(\Omega, [\eta])}([\gamma], [\gamma]) = 0$, we have, for any $t \in \mathbb{R}$,

$$0 \geq Q([\gamma] + t[\gamma'], [\gamma] + t[\gamma']) = 2t \cdot Re(Q([\gamma], [\gamma'])) + t^2 \cdot Q([\gamma'], [\gamma'])$$

(3.1)

from which, by an elementary analysis, we conclude that $Re(Q([\gamma], [\gamma'])) = 0$. Similarly consider $Q(\sqrt{-1}[\gamma] + t[\gamma], \sqrt{-1}[\gamma] + t[\gamma'])$, we conclude that $Im(Q([\gamma], [\gamma'])) = 0$. Therefore, $Q([\gamma], [\gamma']) = 0$, as desired.

The proof is completed. 

\[\square\]

The above arguments can be applied to prove

**Lemma 3.9.** Suppose $(\Omega, [\eta]) \in \overline{\mathcal{H}}$ and $[\Omega] \wedge [\eta^2] \neq 0$. Then 

$$\int_X [\Omega] \wedge [\eta^2] \neq 0.$$
Proof. Indeed, since $H_{1,1}^{0,0}(X, \mathbb{R})$ is one-dimensional, $[\Omega] \wedge [\eta^2] = c \cdot [\omega_X^2]$ for some nonzero $c \in \mathbb{R}$. So $\int_X [\Omega] \wedge [\eta^2] \neq 0$.

Alternatively we may apply the above arguments to check this. Assume a contradiction that $\int_X [\Omega] \wedge [\eta^2] = 0$, i.e. $Q([\eta], [\eta]) = 0$. Obviously we also have $Q([\eta], \cdot) = 0$ on $P_{([\Omega], [\eta])}$.

Therefore, $Q([\eta], \cdot) = 0$ on $H^{1,1}(X, \mathbb{C})$, and hence $[\Omega] \wedge [\eta] = 0$, which contradicts to $[\Omega] \wedge [\eta^2] \neq 0$.

The proof is completed. 

Lemma 3.10. Suppose $([\Omega], [\eta]) \in \mathcal{H}$ and $[\Omega] \wedge [\eta^2] \neq 0$. Then for $[\gamma] \in H^{1,1}(X, \mathbb{C})$,

$$
\int_X [\Omega] \wedge [\eta] \wedge [\gamma] = 0 \iff [\gamma] \in P_{([\eta], [\eta])}.
$$

Proof. It suffices to prove the “$\Rightarrow$” direction, as the other direction is trivial. The proof is a simple application of Lemma 3.6. Indeed, by Lemma 3.6 we write $[\gamma] = c \cdot [\eta] + [\gamma']$ for some $c \in \mathbb{C}$ and $[\gamma'] \in P_{([\eta], [\eta])}$, then

$$
0 = \int_X [\Omega] \wedge [\eta] \wedge [\gamma] = c \cdot \int_X [\Omega] \wedge [\eta^2],
$$

which, combining Lemma 3.9, gives $c = 0$.

The proof is completed. 

4. Proofs of the results

Given the above preparations, now we are ready to prove our results stated in the introduction.

4.1. A general result. We first prove a general result for abstract elements on the boundary of $\mathcal{H}$.

Proposition 4.1. Suppose $([\Omega], [\eta]) \in \mathcal{H}$ and $[\Omega] \wedge [\eta^2] \neq 0$. Then for any $[\beta] \in H^{1,1}(X, \mathbb{R})$,

$$
\left( \int_X [\Omega] \wedge [\eta] \wedge [\beta] \right)^2 = \int_X [\Omega] \wedge [\eta^2] \int_X [\Omega] \wedge [\beta^2]. \tag{4.1}
$$

if and only if $[\Omega \wedge \eta]$ and $[\Omega \wedge \beta]$ are proportional.

Proof. Denote $Q = Q_{([\eta], [\eta])}$. Consider a function $f(t)$ for $t \in \mathbb{R}$:

$$
f(t) := Q([tn + \beta], [tn + \beta]) = t^2 Q([\eta], [\eta]) + 2t Q([\eta], [\beta]) + Q([\beta], [\beta]).
$$

May assume $Q([\eta], [\eta]) > 0$. The equality (4.1) implies that there exists exactly one $t_0 \in \mathbb{R}$ with $f(t_0) = 0$, and hence $f(t) > 0$ for any $t \neq t_0$. We separate the discussions into the following two cases.

Case 1: $Q([\beta], [\beta]) = 0$.

Then by an elementary analysis we conclude that $Q([\eta], [\beta]) = 0$, which implies $[\beta] \in P_{([\eta], [\eta])}$ by Lemma 3.11. Therefore, by Lemma 3.8 we see $[\Omega] \wedge [\beta] = 0$, and hence $[\Omega] \wedge [\eta]$ and $[\Omega] \wedge [\beta]$ are trivially proportional.

Case 2: $Q([\beta], [\beta]) \neq 0$.
Then \( Q(\beta, \beta) > 0 \). Up to rescaling \([\beta]\), we may assume \( Q(\beta, \beta) = Q([\eta], [\eta]) \). Moreover, up to changing \([\beta]\) to \(-[\beta]\), by (4.1) we may assume \( Q([\eta], \beta) = Q([\eta], \beta) = Q([\eta], [\eta]) \). Therefore, the unique zero of \( f \) is \( t_0 = -1 \), and we arrive at
\[
Q([\eta - \beta], [\eta - \beta]) = 0,
\]
and
\[
\int_X \Omega \wedge \eta \wedge (\eta - \beta) = 0.
\]
By Lemma 3.10 the latter implies
\[
[\eta - \beta] \in P([\Omega], [\eta]).
\]
Combining (1.2) and (4.3), by Lemma 3.7 we conclude that \([\Omega] \wedge [\eta - \beta] = 0\), i.e., \([\Omega] \wedge [\eta] \) and \([\Omega] \wedge [\beta] \) are proportional.

Proposition 4.1 is proved. \(\square\)

4.2. Proof of Theorem 1.2. We now prove Theorem 1.2.

Proof of Theorem 1.2. Given the setup of Theorem 1.2, set \([\Omega] := [\alpha_1] \wedge ... \wedge [\alpha_{n-2}]. \) May assume \([\Omega] \neq 0\).

Note that Item (A2) is already contained in Proposition 4.1 as a special case.

Next we look at Item (A1). Firstly we have \(((\Omega), [\omega_X]) \in \mathcal{F}^+ \) and \([\Omega] \wedge [\omega_X] \neq 0 \) (see Examples 3.5(1)). Denote \( Q := Q(\Omega, \omega_X) \) If \( \int_X \Omega \wedge \alpha^2 \neq 0 \), applying Proposition 4.1 gives the desired result. Now assume \( \int_X \Omega \wedge \alpha^2 = 0 \). Also assume \([\Omega] \wedge [\alpha] \neq 0 \) (otherwise the required result is trivially true). The equality (1.2) implies \( \int_X \Omega \wedge \alpha \wedge \beta = 0 \), therefore, \([\beta] \in P([\Omega], [\alpha]) \) and by Lemma 3.7
\[
\int_X \Omega \wedge \beta^2 \leq 0. \tag{4.4}
\]
On the other hand, by assumption, \( \int_X \Omega \wedge \beta^2 \geq 0 \). So we conclude that
\[
\int_X \Omega \wedge \beta^2 = 0. \tag{4.5}
\]
In this case, we have
\[
Q([s\alpha + t\beta], [s\alpha + t\beta]) = 0, \quad \forall s, t \in \mathbb{R}. \tag{4.6}
\]
Moreover, we can always fix \( s_0, t_0 \in \mathbb{R} \), at least one of which is non-zero, such that
\[
\int_X \Omega \wedge \omega_X \wedge (s_0\alpha + t_0\beta) = 0,
\]
which implies
\[
[s_0\alpha + t_0\beta] \in P([\Omega], [\omega_X]). \tag{4.7}
\]
by Lemma 3.10. Applying Lemma 3.8 with (4.6) and (4.7), we conclude \( [\Omega] \wedge [s_0\alpha + t_0\beta] = 0 \), proving the desired result.

Theorem 1.2 is proved. \(\square\)

Remark 4.2. In Li’s arguments in [13] Proof of Theorem 3.9] proving the case that all the involved classes \([\alpha_1], ..., [\alpha_{n-2}], [\alpha], [\beta] \) are nef and big, Shephard inequality (see [16] or [13, Theorem 3.5]) plays a crucial role (note that Shephard inequality also works when all the involved classes are nef). However, for the setting of Theorem 1.2 the last entry \([\beta] \) is no longer assumed to be nef, resulting that Shephard inequality can not be applied. Our
proof here provides an alternative proof for [13] Theorem 3.9 without using Shephard inequality.

4.3. Proof of Theorem [1.7]

Proof of Theorem [1.7] Note that the implication (2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1): Since \((\Omega, [\theta]) \in \mathcal{H}\), the map \([\Omega] \wedge \beta : H^{1,1}(X, \mathbb{C}) \to H^{n-1,n-1}(X, \mathbb{C})\) is an isomorphism (see Remark [3.3(a)]), and hence \([\Omega] \wedge [\alpha] + [\Omega] \wedge [\beta] = 0\) is proportional if and only if \([\alpha]\) and \([\beta]\) are proportional. Then applying Theorem 1.2 gives the result.

(1) \( \Rightarrow \) (2): Let’s only look at the Case (A1), as the other cases can be checked similarly. Denote \([\Omega] := \lbrack \alpha \wedge \ldots \wedge \alpha_{n-2}\] and \([\omega] := \lbrack \alpha \wedge \ldots \wedge \alpha_{n-2}\] are proportional. Then applying Theorem 1.2 gives the result. 

5. Extensions

In this section, we fix an integer \(m \leq n-2\) and \(m\) Kähler metrics \(\omega_1, \ldots, \omega_m\) on \(X\). Let \(\Gamma \subset H^{1,1}(X, \mathbb{R})\) be the convex open cone defined in Definition [3.2] and let \(\overline{\Gamma}\) be the closure of \(\Gamma\) in \(H^{1,1}(X, \mathbb{R})\). Arbitrarily take \([\alpha_1], \ldots, [\alpha_{m-2}], [\alpha] \in \overline{\Gamma}, [\beta] \in H^{1,1}(X, \mathbb{R})\) and set \([\Omega] := \lbrack \omega_1 \wedge \ldots \wedge \omega_m \wedge [\alpha] \wedge \ldots \wedge [\alpha_{m-2}]\], then we have

\[
\left( \int_X \Omega \wedge \alpha \wedge \beta \right)^2 \geq \int_X \Omega \wedge \alpha^2 \cdot \int_X \Omega \wedge \beta^2. \tag{5.1}
\]

The (5.1) is contained in [22] Theorem 1.6 and Theorem 2.10] (also see [3] [21] for the special case that \(\omega_1 = \ldots = \omega_m = \omega_X\) when each \([\alpha_j] \in \Gamma\); then taking a limit gives (5.1). Comparing with (1.1), \([\alpha_j]\)'s and \([\alpha]\) in (5.1) are not necessarily nef, however we pay the price that the \(\omega_j\)'s in (5.1) have been assumed to be Kähler (see Example 5.6 for a special case where none of the involved (1,1)-classes is Kahler). Given the inequality (5.1), it is also natural to consider the following generalized Teissier problem:

Problem 5.1 (Teissier, a generalized version). Characterize the equality case in (5.1).
5.1. Results on Problem 5.1. The above discussions in Sections 4 and 5 are valid for abstract elements on the boundary of \( \overline{F} \), and hence can be identically applied to study Problem 5.1 and give the following results.

**Theorem 5.2.** Suppose \([\alpha_1], \ldots, [\alpha_{n-m-2}], [\alpha], [\beta] \in H^{1,1}(X, \mathbb{R})\) and set \( [\Omega] := [\omega_1 \wedge \ldots \wedge \omega_m \wedge \alpha_1 \wedge \ldots \wedge \alpha_{n-m-2}] \). Assume either of the followings is satisfied.

\((C1)\) \([\alpha_1], \ldots, [\alpha_{n-m-2}], [\alpha] \in \hat{T}\) and \( \int_X \Omega \wedge [\beta]^2 \geq 0 \).

\((C2)\) \([\alpha_1], \ldots, [\alpha_{n-m-2}], [\alpha] \in T\), and \( \int_X \Omega \wedge [\alpha]^2 > 0 \).

Then

\[ \left( \int_X \Omega \wedge [\alpha] \wedge [\beta] \right) = \int_X [\Omega] \wedge [\alpha]^2 \cdot \int_X [\Omega] \wedge [\beta]^2 \]

if and only if \([\Omega \wedge [\alpha]]\) and \([\Omega \wedge [\beta]]\) are proportional.

**Remark 5.3.** Combining Theorem 5.2 and Example 2.1 settles Problem 5.1 completely.

**Remark 5.4.** The analogs of Theorem 1.4 and Corollary 1.5 can be carried out similarly; here we just mention the latter one. Assume \([\alpha], [\beta] \in \hat{T}\) with \( [\omega_1 \wedge \ldots \wedge \omega_m \wedge \alpha_k \wedge \beta_{n-m-1}] \neq 0 \) for \( k = 0, 1, \ldots, n - m - 1 \). Write \( s_k := \int_X [\omega_1 \wedge \ldots \wedge \omega_m \wedge \alpha_k \wedge \beta_{n-m-k}] \), \( k = 0, 1, \ldots, n - m \). If

\[ s_k^2 = s_{k-1} \cdot s_{k+1}, \quad k = 1, \ldots, n - m - 1, \]

then \([\omega_1 \wedge \ldots \wedge \omega_m \wedge \alpha_{n-m-1}]\) and \([\omega_1 \wedge \ldots \wedge \omega_m \wedge \beta_{n-m-1}]\) are proportional.

We also have the analog of Theorem 5.2 as follows.

**Theorem 5.5.** The followings are equivalent.

1. For any \([\alpha_1], \ldots, [\alpha_{n-2}], [\alpha], [\beta] \) (and then set \( [\Omega] := [\omega_1 \wedge \ldots \wedge \omega_m \wedge \alpha_1 \wedge \ldots \wedge \alpha_{n-m-2}] \)) satisfying either of \((C1), (C2)\) in Theorem 5.3

\[ \left( \int_X [\Omega] \wedge [\alpha] \wedge [\beta] \right) = \int_X [\Omega] \wedge [\alpha]^2 \cdot \int_X [\Omega] \wedge [\beta]^2 \]

if and only if \([\alpha]\) and \([\beta]\) are proportional.

2. \(([\Omega], [\omega_X])\) satisfies Hodge index theorem.

3. There exists \([\theta] \in H^{1,1}(X, \mathbb{R})\) such that \(([\Omega], [\theta])\) satisfies Hodge index theorem.

We end this note by considering a special case where none of the involved \((1,1)\)-classes is Kähler and some of the involved \((1,1)\)-classes are not necessarily nef.

**Example 5.6.** Let \( f : X^n \to Y^m \) be a holomorphic submersion, \( m < n \), and fix \( 2 < k < m \). Assume \([\eta_1], \ldots, [\eta_{n-m}]\) are nef \((1,1)\)-classes on \( X \), \( \chi_1, \ldots, \chi_{m-k}, \chi_Y \) Kähler metrics on \( Y \) and \( \alpha_1, \ldots, \alpha_{k-2}, \alpha_Y \) closed real \((1,1)\)-forms on \( Y \) which are \( k \)-positive with respect to \( (\chi_1 \wedge \ldots \wedge \chi_{m-k}, \chi_Y) \). Set \( \Omega_0 := \chi_1 \wedge \ldots \wedge \chi_{m-k} \wedge \alpha_1 \wedge \ldots \wedge \alpha_{k-2} \), and \( \Gamma_0 := \{ [\alpha] \in H^{1,1}(Y, \mathbb{R}) | [\alpha] \text{ is } 2 - \text{positive w.r.t. } (\Omega_0, \alpha_Y) \} \).

For any \( \epsilon > 0 \) and \([\alpha] \in \Gamma_0 \), by [22] Theorem 1.7 we know \(([\eta_1, \ldots, \eta_{n-m}, \eta_Y] \wedge f^* [\Omega_0], [f^* \alpha]\) satisfies Hodge index theorem, here \([\eta_1, \ldots, \eta_Y] := [\eta_1] + \epsilon [\omega_X] \). Now we set

\( \Omega := \eta_1 \wedge \ldots \wedge \eta_{n-m} \wedge f^* \Omega_0 = \eta_1 \wedge \ldots \wedge \eta_{n-m} \wedge f^* \chi_1 \wedge \ldots \wedge f^* \chi_{m-k} \wedge f^* \alpha_1 \wedge \ldots \wedge f^* \alpha_{k-2} \); note that in the definition of \([\Omega], [\eta_i]'s, [f^* \chi_i]'s, [f^* \alpha_i]'s\) are only nef (not necessarily Kähler) and \([f^* \alpha_i]'s\) are not necessarily nef. By an obvious limiting procedure we have the inequality:

\[ \left( \int_X \Omega \wedge f^* [\alpha] \wedge f^* [\beta] \right) \geq \int_X \Omega \wedge f^* [\alpha]^2 \cdot \int_X \Omega \wedge f^* [\beta]^2 \]
for any $[\alpha] \in \tilde{\Gamma}_0$ and $\beta \in H^{1,1}(X, \mathbb{R})$; furthermore, if either $\int_X \Omega \wedge \beta^2 \geq 0$ or $\int_X \Omega \wedge f^*\alpha^2 > 0$, then the equality holds if and only if $[\Omega \wedge f^*\alpha]$ and $[\Omega \wedge \beta]$ are proportional.

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