Finite-time and exact Lyapunov dimension of the Henon map.

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This work is devoted to further consideration of the Hénon map with negative values of the shrinking parameter and the study of transient oscillations, multistability, and possible existence of hidden attractors. The computation of the finite-time Lyapunov exponents by different algorithms is discussed. A new adaptive algorithm for the finite-time Lyapunov dimension computation in studying the dynamics of dimension is used. Analytical estimates of the Lyapunov dimension using the localization of attractors are given. A proof of the conjecture on the Lyapunov dimension of self-excited attractors and derivation of the exact Lyapunov dimension formula are revisited.

I. INTRODUCTION

In 1963, American mathematician and meteorologist Edward Lorenz published an article in which he investigated an approximate model of the fluid convection in a two-dimensional layer and discovered its irregular behavior [52]. The analysis of this behavior was closely related to the phenomenon of chaos, namely to the sensitive dependence on the initial conditions and the existence in the phase space of the system of a chaotic attractor with a complex geometrical structure. Later, this three-dimensional system was called the Lorenz system, and so far it has great scientific interest [48, 63, 64].

In 1969, French mathematician and astronomer, Michel Hénon, showed that the essential properties of the Lorenz system (i.e., folding and shrinking of volumes) can be preserved by means of a specially chosen sequence of approximating maps [32]. The Poincaré map of the Lorenz system was chosen as the reference, resulting in the following map

\[ \varphi(x, y) = (1 + y - ax^2, bx), \]

where \( a > a_0 = -\frac{(b-1)^2}{4} \) (folding parameter), and \( b \in (0,1) \) (shrinking parameter) are parameters of mapping. One usually considers an equivalent form of the Hénon map (e.g. [9, 29, 33])

\[ \varphi(x, y) = (a + by - x^2, x), \]

obtained from (1) by changing the coordinates \( x := ax, y := \frac{b}{b}y \). Along with positive values of parameter \( b \), considered in [32], later on one also studies the Hénon system with negative values \( b \in (-1,0] \) [25, 31].

The Hénon map and its various generalizations have attracted the attention of researchers with its comparative simplicity and the ability to model its dynamics without integrating differential equations (see, e.g. [4, 5, 23, 27–29, 61]). Despite being introduced initially as a theoretical transformation, it also has several physical interpretations [7, 31]. Further, we will study Hénon map in the form (2) with \( b \neq 0, \ |b| < 1 \).

II. TRANSIENT OSCILLATIONS, ATTRACTORS AND MULTISTABILITY

Consider a dynamical system with discrete time \((\{\varphi^n\}_{n \geq 0}, (U, \subseteq \mathbb{R}^2, || \cdot ||))\) generated by the recurrence equation with map (2)

\[ u(t+1) = \varphi(u(t)), \quad u(0) = u_0 \in U, \quad t \in \mathbb{N}_0, \]

where \( \varphi^t = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{t \text{ times}} \), \( \varphi^0 = \text{id}_{\mathbb{R}^2} \), and \( u(t) = (x(t), y(t)) \) with \( ||u_0|| = \sqrt{x_0^2 + y_0^2} \).

The equilibria \( O_\pm = (x_\pm, x_\pm) \) of the system (3) exist when \( a > a_0 \). Here

\[ x_\pm = \frac{1}{2} (b - 1 \pm \sqrt{(b - 1)^2 + 4a}). \]

The 2 \times 2 Jacobian matrix is defined as follows:

\[ J(u_0) = J((x_0, y_0)) = D\varphi((x_0, y_0)) = \begin{pmatrix} -2x_0 & b \\ 1 & 0 \end{pmatrix}, \]

where \( |\det J(u_0)| = |b| < 1 \), and has the eigenvalues \( \lambda \pm(u_0) = -x_0 \pm \sqrt{b + x_0^2} \), and eigenvectors \( \nu^\pm(u_0) = \begin{pmatrix} -x_0 \pm \sqrt{b + x_0^2} \\ 1 \end{pmatrix} \). One can check that equilibrium \( O_- \) is always unstable and \( O_+ \) is unstable when \( a > a_1 = \frac{(b-1)^2}{4} \).

In the seminal work [32] for a fixed value \( b = 0.3 \) it was shown numerically that for \( a < a_0 \) and \( a > a_3 \approx 1.55 \) any trajectory \( \varphi^t(u_0) \) tends to infinity for all \( u_0 \in \mathbb{R}^2 \). For \( a_0 < a < a_3 \), depending on the initial point \( u_0 \), the trajectory \( \varphi^t(u_0) \) either tends to infinity, or reaches the attractor that is a stable equilibrium for \( a_0 < a < a_1 \), a periodic orbit for \( a_1 < a < a_2 \approx 1.06 \), and a nontrivial chaotic attractor for \( a_2 < a < a_3 \). These numerical experiments imply that there is no global attractor in the Hénon system for \( U = \mathbb{R}^2 \).

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Similar self-excited attractor can be also obtained for negative values of parameter $b$ (e.g. for $a = 2.1$, $b = -0.3$ in [25, 31]).

In [19] the multistability and possible existence of hidden attractor in the Hénon system $a = 1.49$, $b = -0.138$ were studied by the perpetual point method [18, 57]. For these values of parameters $|\lambda^-(O_-)| > 1$, $|\lambda^-(O_-)| < 1$ and from the initial point

$$u_0 = O_- + \delta \frac{\nu^+(O_-)^*}{||\nu^+(O_-)||}, \quad \delta \leq 0.01$$

on the unstable manifold of the saddle $O_-$, defined by the eigenvector $\nu^+(O_-)$, a chaotic attractor in Fig. 2a can be visualized. Thus, the chaotic attractor obtained in [19] is a self-excited attractor with respect to $O_-$. In addition, a self-excited periodic attractor can be visualized from vicinity of $O_+$. See coexisting self-excited periodic and chaotic attractors and their basins of attraction in Fig. 2b.

In [27] the multistability in the Hénon system was studied for positive parameters. In Fig. 4a, for parameters $a = 0.98$, $b = 0.4415$ there are three co-existing self-excited attractors: period-8 orbit self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_+ + \delta \frac{\nu^-(O_+)^*}{||\nu^-(O_+)||}$, $\delta = 10^{-4}$), period-12 orbit self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_- + \delta \frac{\nu^-(O_-)^*}{||\nu^-(O_-)||}$, $\delta = 0.1$), and period-20 orbit self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_- + \delta \frac{\nu^-(O_-)^*}{||\nu^-(O_-)||}$, $\delta = 0.1$). In Fig. 4b, for parameters $a = 0.972$, $b = 0.4575$ there are three co-existing attractors: period-12 orbit self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_- + \delta \frac{\nu^+(O_-)^*}{||\nu^+(O_-)||}$, $\delta = 0.01$), period-16 orbit self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_- + \delta \frac{\nu^+(O_-)^*}{||\nu^+(O_-)||}$, $\delta = 0.1$), last but not least, in Fig. 4c, for parameters $a = 0.97$, $b = 0.466$ there are three co-existing self-excited attractors: period-8 orbit self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_- + \delta \frac{\nu^+(O_-)^*}{||\nu^+(O_-)||}$, $\delta = 0.01$), and two chaotic attractors each one self-excited with respect to $O_{\pm}$ (used initial data $u_0 = O_- + \delta \frac{\nu^{-}(O_+)^*}{||\nu^{-}(O_+)||}$, $\delta = 10^{-7}$, and $u_0 = O_- + \delta \frac{\nu^{-}(O_+)^*}{||\nu^{-}(O_+)||}$, $\delta = 0.1$, respectively). In [23, 24] the coexistence of periodic orbits is studied near the critical cases when $b \rightarrow \pm 1$ and the map becomes less and less dissipative. The possible existence of hidden chaotic attractors in the Hénon map requires further investigation.

In the numerical computation of trajectory over a finite-time interval it is often difficult to distinguish a sustained oscillation from a transient oscillation (a transient set in the phase space, e.g. chaotic or quasi periodic, which can nevertheless persist for a long time) [30, 41]. Thus, a similar to the above classification can be introduced for the transient sets.

**FIG. 1:** Chaotic Hénon attractor (blue) for the parameters $a = 1.4$, $b = 0.3$ is self-excited with respect to both equilibria $O_{\pm}$ (red); the basin of attraction (orange).

**Definition 1 ([40, 47, 49, 50]):** An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of a stationary state (an equilibrium), otherwise, it is called a hidden attractor.

For a self-excited attractor its basin of attraction is connected with an unstable equilibrium and, therefore, self-excited attractors can be localized numerically by the standard computational procedure in which after a transient process a trajectory, started in a neighborhood of an unstable equilibrium (e.g., from a point of its unstable manifold), is attracted to the state of oscillation and then traces it. Thus, self-excited attractors can be easily visualized (e.g. the classical Lorenz and Rössler attractors can be visualized by a trajectory from a vicinity of unstable zero equilibrium).

For $a = 1.4$, $b = 0.3$ the equilibria $O_{\pm}$ are saddles ($|\lambda^-(O_{\pm})| < 1$, $|\lambda^+(O_{\pm})| > 1$) and one can visualize the classical Hénon attractor [32] (see Fig. 1) from the $\delta$-vicinity of $O_{\pm}$ using trajectories with the initial data

$$u_0 = O_{\pm} + \delta \frac{\nu^+(O_{\pm})^*}{||\nu^+(O_{\pm})||}, \quad \delta = 0.1.$$
Definition 2 ([10, 13]) A transient oscillating set is called hidden if it does not involve and attract trajectories from a small neighborhood of equilibria; otherwise, it is called self-excited.

In the Hénon system (3) it is possible to observe the long-lived transient chaotic sets for $b = 0.3$ and $a \in (1.29, 1.3)$. For example, for $a = 1.29915$ it is possible to localize a self-excited transient chaotic set with respect to the saddle $O_+$ ($|\lambda^+(O_+)| < 1$, $|\lambda^-(O_+)| > 1$)
using the initial data

\[ u_0 = O_+ + \delta \frac{v^-(O_+)^*}{\|v^-(O_+)\|} \]

which persists for \( \approx 7300 \) iterations and after that contracts into a period 7 limit cycle.

In order to distinguish an attracting chaotic set (attractor) from a transient chaotic set in numerical experiments, one can consider a grid of points in a small neighborhood of the set and check the attraction of corresponding trajectories towards the set. Various examples of hidden transient chaotic sets localization are discussed, e.g., in [10, 13–15, 67].

In [25], for parameters \( a > 0 \) and \( b \in (0, 1) \), it is suggested an analytical bounded localization of attractors in the Hénon map by the set \( \mathcal{B} = \mathcal{M} \setminus (Q \cup R_1 \cup R_2) \), where

\[
\begin{align*}
M &= \{ x, y \mid x < m, y < m + a \}, \quad m = \frac{a(1+2b)}{1-b} > 0, \\
Q &= \{ x, y \mid x < r, y < 0 \}, \quad r = \frac{1+\sqrt{4a}}{2} < 0, \\
R_1 &= \{ x, y \mid x < -\sqrt{b(m+a)} + a - r, \ y \leq m + a \}, \\
R_2 &= \{ x, y \mid x \leq m, \ y < \frac{1}{b}(a + \sqrt{a + bm - r}) \}.
\end{align*}
\]

A similar set can be considered for negative values of \( b \).

Further by \( K \) we denote a bounded closed invariant set, e.g., a maximum attractor with respect to the set of all nondivergent points from \( \mathcal{B} \) (i.e. \( u_0 \in \mathcal{B} : \limsup_{t \to \infty} |\varphi^t(u_0)| \neq \infty \)).

III. FINITE-TIME LYAPUNOV DIMENSION AND ALGORITHMS FOR ITS COMPUTATION

The concept of the Lyapunov dimension was suggested in the seminal paper by Kaplan & Yorke [34] and later it has been developed and rigorously justified in a number of papers. Nowadays, various approaches to the Lyapunov dimension definition are used (see, e.g. [22, 42]).

Below we consider the concept of the finite-time Lyapunov dimension [35, 39], which is convenient for carrying out numerical experiments with finite time.

Let a nonempty closed bounded set \( K \subset \mathbb{R}^2 \) be invariant with respect to dynamical system generated by (3) \( \{ \varphi^t \}_{t \geq 0} \), i.e. \( \varphi^t(K) = K \) for all \( t \geq 0 \) (e.g. \( K \) is an attractor). Further we use compact notations for the finite-time local Lyapunov dimension: \( \dim_{t,u}(t,u_0) = \dim_{t,u}(\varphi^t,u_0) \), the finite-time Lyapunov dimension: \( \dim_{t,K}(t,K) = \dim_{t,K}(\varphi^t,K) \), and for the Lyapunov dimension: \( \dim_{t,K} = \dim_{t,K}(\{ \varphi^t \}_{t \geq 0},K) \).

Consider linearization of system (3) along the solution \( u(t,u_0) = \varphi^t(u_0), u_0 \in \mathbb{R}^2 \):

\[ v(t+1) = J(u(t,u_0))v(t), \quad v(0) = v_0 \in \mathbb{R}^2, \quad t \in \mathbb{N}_0. \]

Consider a fundamental matrix \( \Phi(t,u_0) \) of solutions of linearized system (6) such that \( \Phi(0,u_0) = I \), i.e.

\[ \Phi(t,u_0) = J(u(t-1,u_0))J(u(t-2,u_0)) \cdots J(u(1,u_0))J(u_0). \]

Then for any solution \( v(t,v_0) \) of (6) with the initial data \( v(0,v_0) = v_0 \) we have

\[ v(t,v_0) = \Phi(t,u_0)v_0, \quad u_0, v_0 \in \mathbb{R}^2. \]

Let \( \sigma_i(t,u_0) = \sigma_i(\Phi(t,u_0)), \ i = 1,2 \) be the singular values of \( \Phi(t,u_0) \) (i.e. \( \sigma_1(t,u_0) > 0 \) and \( \sigma_i(t,u_0) \) are the eigenvalues of the symmetric matrix \( \Phi(t,u_0)^T \Phi(t,u_0) \) with respect to their algebraic multiplicity), ordered so that \( \sigma_1(t,u_0) \geq \sigma_2(t,u_0) > 0 \) for any \( t \) and \( u_0 \). Consider the ordered set of the finite-time Lyapunov exponents at the point \( u_0 \) for \( t > 0 \):

\[ \text{LE}_{1,2}(t,u_0) = \frac{1}{t} \ln \sigma_{1,2}(t,u_0), \ \text{LE}_1(t,u_0) \geq \text{LE}_2(t,u_0). \]

Consider the Kaplan-Yorke formula [34] with respect to the ordered set \( \lambda_1 \geq \cdots \geq \lambda_n \):

\[ d^{\text{KY}}(\{\lambda_i\}_{i=1}^n) = j + \sum_{i=1}^j \lambda_i, \quad j = \max\{m: \sum_{i=1}^m \lambda_i \geq n \}. \]
For the ordered set of finite-time Lyapunov exponents \( \{ LE_i(t,u_0) \}_{i=1}^m \) and \( j(t,u_0) = \max\{ m : \sum_{i=1}^n LE_i(t,u_0) \geq 0 \} \) we get
\[
d^{KY}(\{LE_i(t,u_0)\}_{i=1}^2) = \begin{cases} 
0, & j(t,u_0) = 0 \\
1 + \frac{LE_i(t,u_0)}{|LE_i(t,u_0)|}, & j(t,u_0) = 1 \\
2, & j(t,u_0) = 2 
\end{cases}
\]
Then for a certain point \( u_0 \) and invariant closed bounded set \( K \) the finite-time local Lyapunov dimension \([35,39]\) is defined as
\[
dim_L(t,u_0) = d^{KY}(\{LE_i(t,u_0)\}_{i=1}^2)
\]
and the finite-time Lyapunov dimension is as follows
\[
dim_L(t,K) = \sup_{u_0 \in K} \dim_L(t,u_0) \tag{10}
\]
In this approach the use of Kaplan-Yorke formula (9) can be rigorously justified by the Douady-Oesterlé theorem \([17]\), which implies that for any fixed \( t > 0 \) the Lyapunov dimension of the map \( \varphi^t \) with respect to a closed bounded invariant set \( K \), defined by (10), is an upper estimate of the Hausdorff dimension of the set \( K \):
\[
dim_H K \leq \dim_L(t,K).
\]

A. Adaptive algorithm for the computation of the finite-time Lyapunov dimension

To compute the finite-time Lyapunov exponents \((8)\) one has to find the fundamental matrix \( \Phi(t,u_0) \) of \((6)\) from the following variational equation
\[
u(s+1,u_0) = \varphi^-\psi(u(s,u_0)), \quad \nu(0,u_0) = u_0, \quad s = 0,1,..,t-1
\]
\[
\Phi(s+1,u_0) = J(u(s,u_0))\Phi(s,u_0), \quad \Phi(0,u_0) = I,
\]
and its Singular Value Decomposition (SVD)
\[
\Phi(t,u_0) = U(t,u_0)\Sigma(t,u_0)V^*(t,u_0),
\]
where \( U(t,u_0)V(t,u_0) \equiv I \), \( V(t,u_0) = \frac{1}{\ln||v_1(t,u_0)||} \), \( \Sigma(t,u_0) = \text{diag}\{\sigma_1(t,u_0),\sigma_2(t,u_0)\} \) is a diagonal matrix composed by the singular values of \( \Phi(t,u_0) \), and compute the finite-time Lyapunov exponents \( LE_{1,2}(t,u_0) \) from \( \Sigma(t,u_0) \) as in \((8)\).

To avoid the exponential growth of values in the computation, we use the QR factorization and treppinization routine:
\[
\Phi(t,u_0) = J(u(t-1,u_0)) \cdots J(u(1,u_0))J(u(0,u_0)) = \\
= J(u(t-1,u_0)) \cdots J(u(1,u_0))Q^p_1 R^p_1 = \\
\equiv Q^R_{1} R^R_{1} \cdots R^R_{1},
\]
Then matrix \( \Sigma(t,u_0) = U^*(t,u_0)\Phi(t,u_0)V(t,u_0) \) can be approximated by sequential QR decomposition of the product of matrices:
\[
\Sigma^0 := \Phi(t,u_0)^*Q^p_1 = (R^0_1)^* \cdots (R^0_1)^* Q^1_1 R^1_1 \cdots R^1_1,
\]
\[
\Sigma^1 := (Q^0_1)^*\Phi(t,u_0)Q^1_1 = (R^1_1)^* \cdots (R^1_1)^* Q^2_1 R^2_1 \cdots R^2_1,
\]
\[
\vdots
\]
\[
\Sigma^p := \begin{cases} 
(\sum_{i=1}^{2s/2}\Phi(t,u_0)^*U^p, \text{ (p is even)} \\
(\sum_{i=1}^{2s/2}\Phi(t,u_0)^*V^p, \text{ (p is odd)}
\end{cases} = \\
= (R^p_1)^* \cdots (R^p_1)^* \sigma_1 \sigma_2,
\]
where \( U^p := Q^0_1 Q^2_1 \cdots Q^{2(s/2)}_1, V^p := Q^1_1 Q^3_1 \cdots Q^{2(s/2)-1}_1 \) and \([59,62]\)
\[
\sigma_i = R^p_i[i,i] \cdots R^p_i[i,i] \rightarrow \sigma_i(t,u_0), \quad i=1,2.
\]
For a large \( t \) the convergence can be very rapid \([62, p. 44]\) (e.g. \( p = 1 \) is taken in \([62, p. 44]\) for the Lorenz system with the classical parameters).

For the study of dynamics of the finite-time Lyapunov exponents \([39]\) we can adaptively choose \( p \) for \( s=1,..,t \) so as to obtain a uniform estimate with respect to \( s \):
\[
p(\delta) = p(s,\delta) : \max_{i=1,2} \left[ \frac{1}{s} \ln \sigma_i^{p(s)}(s,u_0) - \frac{1}{s} \ln \sigma_i^{p(s-1)}(s,u_0) \right] < \delta
\]
Thus, the finite-time Lyapunov exponents can be approximated\(^2\) as
\[
LE_i(t,u_0) \approx LE_i^{p(s)}(t,u_0) = \frac{1}{t} \sum_{s=1}^{t} \ln R^p_s[i,i]. \tag{13}
\]
For the Hénon system \((3)\) with canonical parameters \( a = 1.4, b = 0.3 \), using the described above adaptive algorithm with \( \delta = 10^{-3} \), we calculate the finite-time Lyapunov exponents of system (3) for the initial condition \( x_0 = (a/b, \frac{c}{b}) \) and \( t = 20 \).

\(^2\) In Benettin’s algorithm \([6]\) the so-called finite-time Lyapunov characteristic exponents (LCEs) \([53]\), which are the exponential growth rates of norms of the fundamental matrix columns \( \Phi(t,u_0) = \{v_1(t,u_0),v_2(t,u_0)\} \): \( LCE_{1,2}(t,u_0) = \frac{1}{t} \ln||v_1(t,u_0)|| \), are computed by \((13)\) with \( p = 0 \): \( LCE_{2,2}(t,u_0) \approx LCE_{2,2}(t,u_0) = \frac{1}{t} \sum_{i=1}^{t} \ln R^p[i,i] \). The following artificial analytical example demonstrates possible differences between LCEs and LCEs: the matrix \([35,39]\)
\[
R(t) = \begin{pmatrix} 1 & e^{at} - e^{-at} \\
0 & 1 \end{pmatrix}
\]
has \( LCE_{1,2}(t) = \pm |a| \), \( LCE_{1,2}(t) = \frac{1}{t} \ln\left((e^{at} - e^{-at})^2 + 1\right) \frac{1}{2} \in (0, \text{a}) \), \( LCE_{2,2}(t) \equiv 0 \); The approximation by Benettin’s algorithm becomes worse with increasing time: \( LCE_{1,2}(t) \rightarrow 0 \equiv LCE_{1,2}(t), \) \( LCE_{2,2}(t) \rightarrow +\infty \). Remark that the notions of LCEs and LCEs often do not differ (see, e.g. Eckmann & Ruelle \([20, p.620,p.650]\), Wolf et al. \([65, p.290,291]\), and Abarbanel et al. \([1, p.1363,p.1364]\), e.g. relying on ergodicity; however, the computations of LCEs by (13) and LCEs by \( LE_i^{p(s)}(t,u_0) \) may give non relevant results.
local Lyapunov dimension $\dim_L(t, u_0)$, where $u_0$ is the initial point of the trajectory $u(t, u_0)$ that localizes the self-excited attractor (see Fig. 1). The comparison of the graphics for the adaptive algorithm and the algorithm with $p = 0$ is presented in Fig. 6. For $t = 1000$ we obtain $\dim_L(1000, u_0) \approx 1.253$. The corresponding numerical routine implemented in MATLAB is presented in Appendix VI.

Applying the statistical physics approach and assuming the ergodicity (see, e.g. [26, 34, 42]), the Lyapunov dimension of attractor $\dim_L K$ is often estimated by the local Lyapunov dimension $\dim_L(t, u_0)$, corresponding to a “typical” trajectory, which belongs to the attractor: $\{u(t, u_0), t \geq 0\}$, $u_0 \in K$, and its limit value $\lim_{t \to +\infty} \dim_L(t, u_0)$. However, rigorous check of ergodicity for the Henon system with a particular value of the parameters is a challenging task (see, e.g. [4, 5]). See, also related discussions in [3][12][54][66, p.9][55, p.19], and the works [37, 46] on the Perron effects of the largest Lyapunov exponent sign reversals. For example, consider parameters $a = 0.97, b = 0.466$ (see Fig. 4c). In this case for $u_0 = O_+ + \delta_1 \frac{\nu^+(O_+)}{||\nu^+(O_+)||}$, $\delta_1 = 10^{-4}$ and $u_0^2 = O_+ - \delta_2 \frac{\nu^-(O_-)}{||\nu^-(O_-)||}$, $\delta_2 = 0.1$ after a transient process during $[0, T_{trans} = 10^5]$ we get initial points $u_0^1$ and $u_0^2$, respectively, and compute finite-time Lyapunov exponents and finite-time local Lyapunov dimension for the time interval $[0, T = 10^4]$ by the adaptive algorithm with $\delta = 10^{-8}$: dynamics of the finite-time Lyapunov exponents and finite-time local Lyapunov dimension is presented in Fig. 5, finally we have

\[
\text{LE}_1(T, u_0^1) \approx 0.01198317, \quad \dim_L(T, u_0^1) \approx 1.01545113, \\
\text{LE}_1(T, u_0^2) \approx 0.01405416, \quad \dim_L(T, u_0^2) \approx 1.01807321.
\]

In one of the pioneering works by Yorke et al. [26, p.190] the exact limit values of finite-time Lyapunov exponents, if they exist and are the same for all $u \in K$, are called the absolute ones, and it is noted that the absolute Lyapunov exponents rarely exist. Remark that while

**FIG. 5:** Evolution of $\text{LE}_1(t, u_0^1)$ (red) and $\text{LE}_1(t, u_0^2)$ (blue), computed along the trajectories $u(t, u_0^1)$ and $u(t, u_0^2)$ for $t \in [1, 10000]$ in the Henon system (3) with parameters $a = 0.97, b = 0.466$.

**FIG. 6:** Evolution of the finite-time local Lyapunov dimension $\dim_L(t, u_0)$ computed with $p = 0$ (blue) and by the adaptive algorithm (red). The corresponding values in yellow box shows the number of iterations $p(t, \delta)$, which are necessary to meet the tolerance $\delta = 10^{-8}$.
the time series obtained from a physical experiment are assumed to be reliable on the whole considered time interval, the time series, obtained numerically from mathematical dynamical model, can be reliable on a limited time interval only due to computational errors. Also, if the trajectory belongs to a transient chaotic set (see, e.g. Fig. 3), which can be (almost) indistinguishable numerically from sustained chaos, then any very long-time computation may be insufficient to reveal the limit values of the finite-time Lyapunov exponents and finite-time Lyapunov dimension (see Fig. 7)[39].

FIG. 7: Evolution of the finite-time local Lyapunov dimension $\dim_{t}(t, u_0)$ (red) computed via the adaptive algorithm along the transient chaotic set for parameters $a = 1.29915$, $b = 0.3$ and $t \in [1, 10000]$, and tolerance $\delta = 10^{-8}$. For $t \in [1, 7300]$ the behavior seems to be chaotic.

Thus, to get a reliable numerical estimation of the Lyapunov dimension of attractor $K$ we localize the attractor $K \subset K^{\varepsilon}$, consider a grid of points $K^{\varepsilon}_{\text{grid}}$ on $K^{\varepsilon}$, and find the maximum of the corresponding finite-time local Lyapunov dimensions for a certain time interval $[0, T]$:

$$\dim_{t} K \leq \dim_{t} K \approx \inf_{t \in [0, T]} \max_{u \in K^{\varepsilon}_{\text{grid}}} \dim_{t}(t, u)$$

$$= \inf_{t \in [0, T]} \max_{u \in K^{\varepsilon}_{\text{grid}}} \left( \frac{j(t, u) + \text{LE}_{j(t, u)}(t, u)}{\text{LE}_{j(t, u)}(t, u)} \right) \leq \text{max}_{u \in K^{\varepsilon}_{\text{grid}}} \dim_{t}(T, u) \approx \dim_{t}(T, K). \quad (14)$$

Additionally, we can consider a set $K^{\varepsilon}_{\text{grid}}$ of $N(\varepsilon)$ random points in $K^{\varepsilon}$, where $N(\varepsilon)$ is the number points in $K^{\varepsilon}_{\text{grid}}$: If the maximum of the finite-time local Lyapunov dimensions for $K_{\text{grid}}^{\varepsilon}$ and $K_{\text{grid}}^{\varepsilon}$ are different, i.e. $\max_{u \in K^{\varepsilon}_{\text{grid}}} \dim_{t}(\varphi^{t}, u) - \max_{u \in K^{\varepsilon}_{\text{grid}}} \dim_{t}(\varphi^{t}, u) > \delta$, then we decrease $\varepsilon$. This may help to improve reliability of the result and at the same time to ensure its repeatability.

IV. THE LYAPUNOV DIMENSION: ANALYTICAL ESTIMATIONS AND EXACT VALUE

To estimate the Hausdorff dimension of invariant closed bounded set $K$, one can use the map $\varphi^{t}$ with any time $t$ (e.g., $t = 0$ leads to the trivial estimate $\dim_{t} K \leq 2$), and, thus, the best estimation is

$$\dim_{t} K \leq \inf_{t \geq 0} \dim_{t}(t, K).$$

The following property:

$$\inf_{t \geq 0} \sup_{u \in K} \dim_{t}(t, u_0) = \lim_{t \to +\infty} \inf_{u \in K} \dim_{t}(t, u_0), \quad (15)$$

allows one to introduce the Lyapunov dimension of $K$ as [35]

$$\dim_{t} K = \lim_{t \to +\infty} \sup_{u \in K} \dim_{t}(t, u_0) \quad (16)$$

and get an upper estimation of the Hausdorff dimension:

$$\dim_{t} K \leq \dim_{t} K.$$

Recall that a set with noninterger Hausdorff dimension is referred as a fractal set [20].

In contrast to the finite-time Lyapunov dimension (10), the Lyapunov dimension (16)\(^3\) is invariant under smooth change of coordinates [35, 36]. This property and a proper choice of smooth change of coordinates may significantly simplify the computation of the Lyapunov dimension of dynamical system. Consider an effective analytical approach, proposed by Leonov [35, 43, 45], for estimating the Lyapunov dimension. In the work [35] it is shown how this approach can be justified by the invariance of the Lyapunov dimension of compact invariant set with respect to the special smooth change of variables $w = h(u)$ with $Dh(u) = e^{V(u)(j+s)^{-1}} S$, where $V(u)$ is a

---

\(^3\) This definition can be reformulated via the singular value function $\omega_{d}(\varphi^{t}(u)) = \sigma_{1}(t, u) \cdots \sigma_{d}(t, u)\sigma_{d+1}(t, u)^{d-|d|}$, where $d \in [0, n]$ and $|d|$ is the largest integer less or equal to $d$: $\dim_{t}(t, u) = \max\{d \in [0, n] : \omega_{d}(\varphi^{t}(u)) \geq 1\}$ and $\dim_{t} K = \lim_{t \to +\infty} \sup_{u \in K} \dim_{t}(t, u)$. Another approach to the introduction of the Lyapunov dimension of dynamical system was developed by Constantin, Eden, Foiaş, and Temam [11, 21, 22]. They consider $\omega_{d}(\varphi^{t}(u))^{1/t}$ instead of $\omega_{d}(\varphi^{t}(u))$ and apply the theory of positive operators to prove the existence of a critical point $u_{0}^{d}$ (which may be not unique), where the corresponding global Lyapunov dimension achieves maximum (see [21]): $\dim_{t}((\varphi^{t})_{1}^{\geq d_{0}}) = \inf\{d \in [0, n] : \lim_{t \to +\infty} \max_{u \in K} \ln \omega_{d}(\varphi^{t}(u))^{1/t} < 0\} = \inf\{d \in [0, n] : \lim_{t \to +\infty} \sup_{u \in K} \ln \omega_{d}(\varphi^{t}(u_{0}))^{1/t} < 0\} = \dim_{t}((\varphi^{t})_{1}^{\geq d_{0}})$, and, thus, rigorously justify the usage of the local Lyapunov dimension $\dim_{t}((\varphi^{t})_{1}^{\geq d_{0}}, u)$.
continuous scalar function and $S$ is a nonsingular matrix. Let $\sigma_i(J(u_0))$, $i = 1, 2$ be the singular values of $J(u_0)$ (i.e. the square roots of the eigenvalues of the symmetrized Jacobian matrix $J(u_0)^*J(u_0)$), ordered so that $\sigma_1(J(u_0)) \geq \sigma_2(J(u_0)) > 0$ for any $u_0 \in K$.

**Theorem 1** ([35]) If there exist a real $s \in [0, 1]$, a continuous scalar function $V(u)$, and a nonsingular $2 \times 2$ matrix $S$ such that

$$\sup_{u_0 \in K} \left( \ln \sigma_1(J(u_0)) + s \ln \sigma_2(J(u_0)) + (V(\varphi(u_0)) - V(u_0)) \right) < 0,$$

(17)

then

$$\dim_H K \leq \dim_L K < 1 + s.$$  

To avoid numerical localization of attractor, we can consider estimation (17) e.g. by the absorbing set or the whole phase space.

In [8, 56] it is demonstrated how a technique similar to the above can be effectively used to derive constructive upper bounds of the topological entropy of dynamical systems.

If we consider $V = 0$ and $S = I$, then we get the Kaplan-Yorke formula with respect to the ordered set of logarithms of the singular values of the Jacobian matrix: $d_L^{\KY} = \left( \ln \sigma_i(u, S) \right)_{i=1}^2$, and its supremum on the set $K$ gives an upper estimation of the finite-time Lyapunov dimension. This is a generalization of ideas, discussed e.g. in [17, 60], on the Hausdorff dimension estimation by the eigenvalues of symmetrized Jacobian matrix.

The Jacobian (4) of the Hénon map (2) has the following singular values:

$$\sigma_1(x) = \frac{1}{2} \left( \sqrt{4x^2 + (1 + |b|)^2} + \sqrt{4x^2 + (1 - |b|)^2} \right) \geq 1,$$

$$\sigma_2(x) = \frac{|b|}{\sigma_1(x)} \leq |b|.$$  

These expressions give the following estimation:

$$\dim_H K \leq \dim_L K \leq \sup_{(x, y) \in B} \left( 1 + \frac{1}{1 - \frac{\ln |b|}{\ln \sigma_1(x)}} \right).$$

The maximum of the right-hand side value is determined by the maximum value of $x^2$ (or $|x|$) on $B$. In [33], for canonical parameters $a = 1.4$, $b = 0.3$ it was considered the square $(x, y) \in [-1.8, 1.8] \times [-1.8, 1.8]$ that gives the estimation

$$\dim_H K \leq \dim_L K \leq 1 + \frac{1}{1 - \frac{\ln 0.3}{\ln \sigma_1(1.8)}} \approx 1.523.$$  

Using Feit’s analytical localization (5) we can get

$$\dim_H K \leq \dim_L K \leq$$

$$\leq 1 + \left( 1 - \frac{\ln b}{\ln \sigma_1(\max \{-r,m,\sqrt{b(m+a)+a-r}\})} \right)^{-1}.$$  

For parameters $a = 1.4$, $b = 0.3$ we obtain $\dim_H K \leq \dim_L K \leq 1.5319$.

Remark, that if the Jacobian matrix $J(u_{eq})$ at one of the equilibria has simple real eigenvalues: $|\lambda_1(u_{eq})| \geq |\lambda_2(u_{eq})|$, then the invariance of the Lyapunov dimension with respect to linear change of variables implies [35] the following

$$\dim_{L} u_{eq} = d_{LY}^{\KY} \left( \left\{ |\lambda_i(u_{eq})| \right\}_{i=1}^2 \right).$$

(18)

If the maximum of local Lyapunov dimensions on the B-attractor, involving all equilibria, is achieved at equilibrium point: $\dim_{L} u_{eq}^{cr} = \max_{u_0 \in K} \dim_{L} u_0$, then this allows one to get analytical formula of the exact Lyapunov dimension. In general, a conjecture on the Lyapunov dimension of self-excited attractor [35, 38] is that the Lyapunov dimension of typical self-excited attractor does not exceed the Lyapunov dimension of one of unstable equilibria, the unstable manifold of which intersects with the basin of attraction and visualize the attractor.

Following [44] for the Hénon system (3) with parameters $a > -\frac{(b-1)^2}{4}$ and $|b| < 1$ we can consider

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{|b|} \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{|b|} \end{pmatrix}, \ s \in [0, 1).$$

In this case we have

$$S J((x,y)) S^{-1} = \begin{pmatrix} -2x & \sqrt{|b|} \\ \sqrt{|b|} & 0 \end{pmatrix},$$

and

$$\sigma_1((x, y), S) = (\sqrt{x^2 + |b| + |x|}, \sigma_2((x, y), S) = \frac{|b|}{\sigma_1((x, y), S)}.$$  

If we take $V((x, y)) = \gamma(1 - s)(x + by)$, then condition (17) with $j = 1$ and

$$s > s^* = \frac{1}{1 - \frac{\ln |b|}{\ln \sigma_1((x, x), S)}},$$

is satisfied for all $(x, y) \in \mathbb{R}^2$ and we do not need any localization of the set $K$ in the phase space. By (18) and (9), at the equilibrium point $u_{eq}^{cr} = (x_-, x_-)$ we get

$$\dim_{L} \left\{ \phi^t \right\}_{t \geq 0}, (x_-, x_-) =$$

$$= \dim_{L}^{\KY} \left( \left\{ |\lambda_i(x_-, x_-)\right\}_{i=1}^2 \right) = 1 + s^*.$$  

Therefore, for a bounded invariant set $K \ni (x_-, x_-)$ (e.g. maximum B-attractor) we have [44]

$$\dim_{L} \left\{ \phi^t \right\}_{t \geq 0}, K^{B} = \dim_{L} \left\{ \phi^t \right\}_{t \geq 0}, (x_-, x_-) =$$

$$= 1 + \frac{1}{1 - \frac{\ln |b|}{\ln \sigma_1((x, x), S)}}.$$

4 This term was suggested by Doering et al. in [16].
Here for \( a = 1.4 \) and \( b = 0.3 \) we have \( \dim L\{\{x^4\}_{t>0}, K^B\} = 1.495 \ldots \). Embedding of the attractor into three-dimensional phase space (see attractors of generalized Hénon map in \([2, 28]\)) increases the Lyapunov dimension by one. 

Using the above approach one can obtain the Lyapunov dimension formulas for invariant sets of other discrete systems (see, e.g. \([51, 58]\)).

\section{Conclusion}

In this work the Hénon map with positive and negative values of the shrinking parameter is considered and transient oscillations, multistability and possible existence of hidden attractors are studied. A new adaptive algorithm of the finite-time Lyapunov dimension computation is used for studying the dynamics of the dimension. Analytical estimate of the Lyapunov dimension using localization of attractors is given. The proof of the conjecture on the Lyapunov dimension of self-excited attractors and derivation of the exact Lyapunov dimension formula are extended to negative values of the parameters.

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VI. APPENDIX: MATLAB CODE

Listing 1: henonMap.m – function defining the Hénon map.

```matlab
function out = henonMap(x, a, b)
out = zeros(6,1);
out(1) = a + b * x(2) - x(1)^2;
out(2) = x(1);
out(3:6) = [-2 * x(1), b; 1, 0];
end
```

Listing 2: qr_pos.m – function implementing the QR decomposition with positive diagonal elements in R.

```matlab
function [Q, R] = qr_pos(A)
[Q, R] = qr(A);
D = diag(sign(diag(R)));
Q = Q * D; R = D * R;
end
```

Listing 3: treppeniterationQR.m – function implementing the treppeniteration QR decomposition for product of matrices.

```matlab
function [Q, R] = treppeniterationQR(matFact)
[-, dimOde, nFactors] = size(matFact);
R = zeros(dimOde, dimOde, nFactors);
Q = eye(dimOde, dimOde);
for jFactor = nFactors : -1 : 1
    C = matFact(:, :, jFactor) * Q;
    [Q, R(:, :, jFactor)] = qr_pos(C);
end
end
```

Listing 4: computeLEsDiscrTol.m – function implementing the LEs numerical computation via the approximation of the singular values matrix with adaptively chosen number of iterations.

```matlab
function [t, LEs, svdIterations] = computeLEsDiscrTol(extMap, initPoint, nFactors, LEsTol)
dimMap = length(initPoint);
dimExtMap = dimMap * (dimMap + 1);
initCond = extMapSolution(1 : dimMap);
t = 1 : 1 : nFactors;
LEs = zeros(nFactors, dimMap);
svdIterations = zeros(nFactors, 1);
for iFactor = 1 : nFactors
    currFactorization = fundMat(:, :, nFactors-iFactor+1);
currSvdIteration = 1;
    LEsWithinTol = false;
    while ~LEsWithinTol
        currSvdIteration = currSvdIteration + 1;
        accumLEs = zeros(1, dimMap);
        for jFactor = 1 : iFactor
            accumLEs = accumLEs + log(diag(currFactorization(:, :, jFactor))');
        end
        LEs(iFactor, :) = accumLEs / iFactor;
    end
    LEsWithinTol = all(abs(LEs(iFactor, :) - prevLEs) < LEsTol);
end
end
```

Listing 5: kaplanYorkeFormula.m – function implementing the local Lyapunov dimension calculation via Kaplan-Yorke formula.

```matlab
function LD = kaplanYorkeFormula(LEs)
end
```
% Initialization of the local Lyapunov
dimension:
LD = 0;

% Number of LCEs:
nLEs = length(LEs);

% Sorted LCEs:
sortedLEs = sort(LEs, 'descend');

% Main loop:
leSum = sortedLEs(1);
if ( sortedLEs(1) > 0 )
    for i = 1 : nLEs-1
        if sortedLEs(i+1) ~= 0
            LD = i + leSum / abs(sortedLEs(i+1));
            leSum = leSum + sortedLEs(i+1);
            break;
        end
    end
end
end
end
end

Listing 6: henonLD.m – script with application of the described numerical procedure for LCEs computation to the Hénon system.

function henonLD
    % Canonical parameters
    a = 1.4; b = 0.3;
    function out = J( x, a, b )
        out = [-2 * x(1), b; 1, 0];
    end

    % Equilibrium
    S1 = 1/2*((b-1) + sqrt((b-1)^2 + 4*a));
    [V1, D1] = eig(J([S1, S1], a, b));
    D1 = diag(D1);
    IX1 = find(abs(D1) > 1);
    delta = 1e-3;
    initPoint = [S1, S1] + delta* V1(:, IX1(1)) / norm(V1(:, IX1(1)));
    % Parameters for numerical procedure
    nFactors = 1000;
    LEsTol = 1e-8;

    % LCEs computation
    [t, LEs, svdIterations] = computeLEsDiscrTol(@(x) henonMap( x, a, b), initPoint, nFactors, LEsTol);
    % LD computation
    LD = cellfun(@kaplanYorkeFormula, num2cell(LEs, 2));
    % Plotting
    figure(1);
    plot(t, LEs(:, 1), 'Color', 'red');
    plot(t, LEs(:, 2), 'Color', 'blue');
    hold off;
    grid on; axis on;
    xlabel('t'); ylabel('LE')

    figure(2);
    plot(t, LD, 'Color', 'green');
    hold off;
    grid on; axis on;
    xlabel('t'); ylabel('LD')