AKS scheme for face and Calogero-Moser-Sutherland type models

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Abstract

We give the construction of quantum Lax equations for IRF models and difference versions of Calogero-Moser-Sutherland models introduced by Ruijsenaars. We solve the equations using factorization properties of the underlying face Hopf algebras/elliptic quantum groups. This construction is in the spirit of the Adler-Kostant-Symes method and generalizes our previous work to the case of face Hopf algebras/elliptic quantum groups with dynamical R-matrices.

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1 Introduction

Face Hopf algebras \cite{1} have been found to be the algebraic structure that underlies some particularly interesting integrable models of statistical physics. They generalize Hopf algebras and quantum groups. Another closely related generalization of quantum groups, the so-called elliptic quantum groups, were introduced by Felder \cite{2} in the context of IRF (face) models \cite{3}. Face Hopf algebras and elliptic quantum groups play the same role for face models as quantum groups do for vertex models. The R-matrices are now replaced by dynamical R-matrices, which first appeared in the context of Liouville string field theory \cite{4}; they can be understood as a reformulation of the Boltzmann weights in Baxter’s solutions of the face-type Yang-Baxter equation. A partial classification of the dynamical R-matrices is given in \cite{5}. Recently another type of integrable quantum systems, known as Ruijsenaars models \cite{6}, and their various limiting cases have been shown to be connected to quantum groups and elliptic quantum groups \cite{7, 8, 9, 10, 11} through different approaches.

The Calogero-Moser-Sutherland class of integrable models describe the motion of particles on a one-dimensional line or circle interacting via pairwise potentials that are given by Weierstrass elliptic functions and their various degenerations. The simplest case is an inverse $r^2$ potential. The Ruijsenaars-Schneider model is a relativistic generalization, whose quantum mechanical version, the Ruijsenaars model, is the model that we are interested in here.

The Hamiltonian of the Ruijsenaars model for two particles with coordinates $x_1$ and $x_2$ has the form

$$H = \left\{ \frac{\theta(\frac{c\eta}{2} - \lambda)}{\theta(-\lambda)} t_1^{(\lambda)} + \frac{\theta(\frac{c\eta}{2} + \lambda)}{\theta(\lambda)} t_2^{(\lambda)} \right\},$$

where $\lambda = x_1 - x_2$. Here $c \in \mathbb{C}$ is the coupling constant, $\eta$ is the relativistic deformation parameter, the $\theta$-function is given in (27); we have set $\hbar = 1$. 

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The Hamiltonian acts on a wave function as
\[ \mathcal{H} \psi(\lambda) = \frac{\theta(\frac{\eta}{2} - \lambda)}{\theta(-\lambda)} \psi(\lambda - \eta) + \frac{\theta(\frac{\eta}{2} + \lambda)}{\theta(\lambda)} \psi(\lambda + \eta), \]

the \( t_i^{(\lambda)} \) that appear in the Hamiltonian are hence shift-operators in the variable \( \lambda \); in the present case of two particles they generate a one-dimensional graph:

```
--- t_2 --- t_1 ---
  \lambda
```

Relative to a fixed vertex \( \lambda \) the vertices of this graph are at points \( \eta \cdot \mathbb{Z} \in \mathbb{R} \), the (ordered) paths connect neighboring vertices and are hence intervals in \( \mathbb{R} \). There are two paths per interval, one in positive, one in negative direction.

Now consider a \( N \)-particle Ruijsenaars model: The graph relative to a fixed \( \lambda \) is then a \( N - 1 \) dimensional hyper-cubic lattice. Relative to \( \lambda \in \mathbb{R}^{N-1} \) the vertices are at points \( (\eta \cdot \mathbb{Z})^{\times(N-1)} \). This picture can obviously be generalized and in the following we shall consider an arbitrary ordered graph.

**Remark:** For the Ruijsenaars system \( \lambda \) can take any value in \( \mathbb{R}^{N-1} \), so we are a priori let to a huge graph that consists of a continuous family of disconnected graphs. Since the graphs are disconnected it is sufficient to consider one. Later we will in fact write all expressions with respect to one fixed vertex \( \lambda \). This will lead to equations containing explicit shifts on the graph and thus to dynamical \( R \)-matrices. Apart from the physical interpretation there is a priori no reason to restrict the values of particle-coordinates at the level of the shift operator to \( \mathbb{R} \). We can and shall everywhere in the following take \( \mathbb{C} \) instead of \( \mathbb{R} \).

### 1.1 Face algebras

The Hilbert space of the model will be build from vector spaces on paths of fixed length. As we shall argue in the following, the operators on this Hilbert
space can be chosen to be elements of a Face Algebra $F$ (or weak $C^*$-Hopf algebras [12, 13], which is essentially the same with a $*$-structure).

There are two commuting projection operators $e^i, e_i \in F$ for each vertex of the graph (projectors onto bra’s and ket’s corresponding to vertex $i$):

\[ e_i e_j = \delta_{ij} e_i, \quad e_i e^j = \delta_{ij} e^i, \quad \sum e_i = \sum e^i = 1. \]  

(1)

$F$ shall be equipped with a coalgebra structure such that the combination $e^i_j \equiv e^i e_j = e^j e^i$ is a corepresentation:

\[ \Delta(e^i_j) = \sum_k e^i_k \otimes e^k_j, \quad \epsilon(e^i_j) = \delta_{ij}. \]  

(2)

It follows that $\Delta(1) = \sum_k e_k \otimes e^k \neq 1 \otimes 1$ (unless the graph has only a single vertex) – this is a key feature of face algebras (and weak $C^*$-Hopf algebras).

So far we have considered matrices with indices that are vertices, $i.e.$ paths of length zero. In the given setting it is natural to also allow paths of fixed length on a finite oriented graph $G$ as matrix indices. To illustrate this, here are some paths of length 3 on a graph that is a square lattice, $e.g.$ on part of a graph corresponding to a particular Ruijsenaars system:

We shall use capital letters to label paths. A path $P$ has an origin (source) $\cdot P$, an end (range) $P \cdot$ and a length $\#P$. Two paths $Q, P$ can be concatenated to form a new path $Q \cdot P$, if the end of the first path coincides with the start of the second path, $i.e.$ if $Q \cdot = \cdot P$ (this explains our choice of notation).
The important point is, that the symbols $T_B^A$, where $\#A = \#B \geq 0$, with relations

$$\Delta (T_B^A) = \sum_{A'} T_B^{A'} \otimes T_B^{A'} \quad (#A = #A' = #B) \quad (3)$$

$$\epsilon (T_B^A) = \delta_{AB} \quad (4)$$

$$T_B^A T_D^C = \delta_{A,C} \delta_{B,D} T_B^{A} C \quad (5)$$

span an object that obeys the axioms of a face algebra. Relations (3) and (4) make $T_B^A$ a corepresentation; (3) is the rule for combining representations. The axioms of a face algebra can be found in [1, 14].

**Pictorial representation:**

$$T_B^A \sim \begin{array}{c}
A \\
B
\end{array} \quad T_B^A T_D^C \sim \begin{array}{cc}
A & C \\
B & D
\end{array} \quad \Delta T_B^A = \sum_{A'} T_B^{A'} \otimes T_B^{A'} \sim \begin{array}{c}
A \\
B
\end{array}$$

The dashed paths indicate the $F$-space(s), their orientation is from lower to upper “index”. Inner paths are summed over.

Often it is convenient not to consider a particular representation, i.e. $T$-matrices corresponding to paths of a given fixed length but rather an abstract, universal $T$. Heuristically one can think of the universal $T$ as an abstract matrix or group element, but it does in fact simply provide an alternative to the usual Hopf algebra notation: $T$, $T_1 \otimes T_1$, $T_1 T_2$, e.g. correspond to the identity map, coproduct map and multiplication map of $F$ respectively. (More details are given in the appendix.) Keeping this in mind we shall—without loss of generality—nethertheless use a formal notation that treats $T$
as if it was the canonical element $T_{12}$ of $U \otimes F$, where $U$ is the dual of $F$ via the pairing $\langle \cdot, \cdot \rangle$.

$$\langle T, f \rangle = f, \quad \langle T_1 \otimes T_1, f \rangle = \Delta f, \quad \langle T_1 T_2, f \otimes g \rangle = fg; \quad f, g \in F$$

A face Hopf algebra has an anti-algebra and anti-coalgebra endomorphism, called the antipode and denoted by $S$ or—in the universal tensor formalism—by $\tilde{T}$: $\langle \tilde{T}, f \rangle = S(f)$. The antipode satisfies some compatibility conditions with the coproduct that are given in the appendix.

Remark: In the limit of a graph with a single vertex a face Hopf algebra is the same as a Hopf algebra. Ordinary matrix indices correspond to closed loops in that case.

### 1.2 Boltzmann weights

By dualization we can describe a coquasitriangular structure of $F$ by giving a quasitriangular structure for $U$. The axioms for a quasitriangular face algebra are similar to those of a quasitriangular Hopf algebra; there is a universal $R \in U \otimes U$ that controls the non-cocommutativity of the coproduct in $U$ and the non-commutativity of the product in $F$,

$$RT_1 T_2 = T_2 T_1 R, \quad \tilde{R} T_2 T_1 = T_1 T_2 \tilde{R}, \quad \tilde{R} \equiv (S \otimes \text{id})(R),$$

however the antipode of $R$ is no longer inverse of $R$ but rather

$$\tilde{R} R = \Delta(1), \quad R \tilde{R} = \Delta'(1).$$

The numerical “$R$-matrix” obtained by contracting $R$ with two face corepresentations is given by the face Boltzmann weight $W$:

$$\langle R, T^A_B \otimes T^C_D \rangle = R^{AC}_{BD} \equiv W(C^B_A D) \sim C^B_A.$$
The pictorial representation makes sense since the Boltzmann weight is zero unless $C \cdot A$ and $B \cdot D$ are valid paths with common source and range as will be discussed in more detail below. Also note that $T^A_B \mapsto \langle R, T^A_B \otimes T^C_D \rangle$ is a representation of the matrix elements of $(T^A_B)$ while $T^C_D \mapsto \langle R, T^A_B \otimes T^C_D \rangle$ is an anti-representation. Consistent with our pictorial representation for the $T$-matrices we see that the orientation of the paths in $F$-space remain the same for the first case but are reversed for the latter:

\[ A \sim B \xrightarrow{\text{rep.}} C \quad B \xrightarrow{\text{anti-rep.}} D \sim C \]

Definition: For $f \in F$ we can define two algebra homomorphisms $F \rightarrow U$:

\[ R^+(f) = \langle R, f \otimes \text{id} \rangle, \quad R^-(f) = \langle \bar{R}, \text{id} \otimes f \rangle \]  

(8)

Yang-Baxter Equation. As a consequence of the axioms of a quasitriangular face Hopf algebra $R$ satisfies the Yang-Baxter Equation

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad \in \quad U \otimes U \otimes U . \]  

(9)

Contracted with $T^A_B \otimes T^C_D \otimes T^E_F$ this expression yields a numerical Yang-Baxter equation with the following pictorial representation [3]:

The inner edges are paths that are summed over. Moving along the outer edges of the hexagon we will later read of the shifts in the Yang-Baxter equation for dynamical $R$-matrices.

So far we have argued heuristically that the Ruijsenaars system naturally leads to graphs and face algebras. The formal relation between face Hopf
algebras and oriented graphs it is established by a lemma of Hayashi (Lemma 3.1 of [14]) which says that any right or left comodule $M$ of a face Hopf algebra $F$ decomposes as a linear space to a direct sum $M = \oplus_{i,j} M_{ij}$ with indices $i, j$ running over all vertices. So we can naturally associate paths from $j$ to $i$ to any pair of indices such that $M_{ij} \neq \emptyset$. So we may speak (and we really do in the paper) of the vectors of a comodule or a module as of paths. As the dual object $U$ to a face Hopf algebra is again a face Hopf algebra characterized by the same set of vertices [14], the same applies to its comodules. It is convenient to choose the orientation of paths appearing in the decomposition of a comodule of $U$ (and hence in the module of $F$) opposite to the convention that one uses in the case of $F$.

Particularly we have for any matrix corepresentation $T^A_B$ of $F$ with symbols $A, B$ used to label some linear basis in $M$, the linear span $\langle T^A_B \rangle$ of all $T^A_B$ decomposes as linear space (bicomodule of the face Hopf algebra) as a direct sum

$$\bigoplus_{i,j,k,l} \langle e^i e_j T^A_B e^k e_l \rangle = \langle T^A_B \rangle$$

i.e. a sum over paths with fixed starting and ending vertices. The upper indices $i$ and $k$ fix the beginning and the end of the path $A$ and the lower indices $j$ and $l$ fix the beginning and the end of the path $B$.

Let us assume that the matrix elements $T^A_B$ of a corepresentation of $F$ act in a module of paths that we shall label by greek characters $\alpha, \beta$, etc. The definition of the dual face Hopf algebra implies that the matrix element $(T^A_B)^\alpha_\beta$ is nonzero only if $\cdot \alpha = \cdot B$, $\cdot A = \alpha \cdot$, $B \cdot = \beta \cdot$, and $A \cdot = \beta \cdot$, i.e. if paths $\alpha \cdot A$ and $B \cdot \beta$ have common starting and endpoints. This justifies the pictorial representation used in the paper. It also follows immediately that in the case of a coquasitriangular Face Hopf algebra $\langle \mathcal{R}, T^A_B \otimes T^C_D \rangle = R^{AC}_{BD} \equiv W(c^R_A D)$ is zero unless $\cdot C = \cdot B$, $B \cdot = \cdot D$, $C \cdot = \cdot A$ and $A \cdot = D \cdot$. 7
To make a contact with the Ruijsenaars type of models, we have to assume that the (coquasitriangular) face Hopf algebra $F$ is generated by the matrix elements of some fundamental corepresentation of it. We shall postulate the paths of the corresponding corepresentation to be of length 1. The paths belonging to the $n$-fold tensor product of the fundamental corepresentation are then by definition of length $n$. Taking an infinite tensor product of the fundamental corepresentation we get a graph that corresponds to the one generated by the shift operators of the related integrable model.

In the next section we are going to formulate a quantum version of the so-called Main Theorem which gives the solution by factorization of the Heisenberg equations of motion. For this construction $F$ needs to have a coquasitriangular structure – this will also fix its algebra structure.

2 Quantum factorization

The cocommutative functions in $F$ are of particular interest to us since they form a set of mutually commutative operators. We shall pick a Hamiltonian from this set. Cocommutative means that the result of an application of comultiplication $\Delta$ is invariant under exchange of the two resulting factors. The typical example is a trace of the T-matrix. The following theorem gives for the case of face Hopf algebras what has become known as the “Main theorem” for the solution by factorization of the equations of motion \cite{15, 16, 17, 18, 19}. The following theorem is a direct generalization of our previous results for Hopf algebras/Quantum Groups \cite{20, 21}:

**Theorem 2.1 (Main theorem for face algebras)**

(i) The set of cocommutative functions, denoted $I$, is a commutative sub-algebra of $F$. 

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(ii) The Heisenberg equations of motion defined by a Hamiltonian $H \in I$ are of Lax form

$$i \frac{dT}{dt} = [M^\pm, T], \quad (10)$$

with $M^\pm = 1 \otimes H - m_\pm \in U_\pm \otimes F$, $m_\pm = R^\pm(H,(2)) \otimes H,(1)$; see (8).

(iii) Let $g_\pm(t) \in U_\pm \otimes F$ be the solutions to the factorization problem

$$g_\pm^{-1}(t)g_\pm(t) = \exp(it(m_+ - m_-)) \in U \otimes F, \quad (11)$$

then

$$T(t) = g_\pm(t)T(0)g_\pm(t)^{-1} \quad (12)$$

solves the Lax equation (11); $g_\pm(t)$ are given by

$$g_\pm(t) = \exp(-it(1 \otimes h)) \exp(it(1 \otimes h - M^\pm(0))) \quad (13)$$

and are the solutions to the differential equation

$$i \frac{dg_\pm(t)}{dt} = M^\pm(t)g_\pm(t), \quad g_\pm(0) = 1. \quad (14)$$

Proof: The proof is similar to the one given in [21] for factorizable quasitriangular Hopf algebras. Here we shall only emphasize the points that are different because we are now dealing with face algebras. An important relation that we shall use several times in the proof is

$$\Delta(1)T_1T_2 = T_1T_2 = T_1T_2\Delta(1). \quad (15)$$

(Note that the generalization is not trivial since now $\Delta(1) \neq 1 \otimes 1$ in general.)

(i) Let $f, g \in I \subset F$, then $fg \in I$. Let us show that $f$ and $g$ commute:
\[ fg = \langle T_1 T_2, f \otimes g \rangle = \langle \Delta(1) T_1 T_2, f \otimes g \rangle = \langle \tilde{R} R T_1 T_2, f \otimes g \rangle = \langle T_2 T_1 R \tilde{R}, f \otimes g \rangle = \langle T_2 T_1 \Delta'(1), f \otimes g \rangle = \langle T_2 T_1, f \otimes g \rangle = gf. \]

\( \tilde{R} \equiv (S \otimes \text{id})(R) \) can be commuted with \( T_2 T_1 R \) in the fifth step because \( f \) and \( g \) are both cocommutative.

(ii) We need to show that \( [R^\pm(\mathcal{H}(2)) \otimes \mathcal{H}(1), T] = 0 \). This follows from the cocommutativity of \( \mathcal{H} \) and (3):

\[ \langle T_1 R_{21}^\pm T_2, \mathcal{H} \otimes \text{id} \rangle = \langle R_{21}^\pm T_1 T_2, \mathcal{H} \otimes \text{id} \rangle = \langle T_2 T_1 R_{21}^\pm, \mathcal{H} \otimes \text{id} \rangle. \]

(iii) We need to show that \( [m_+, m_-] = 0 \); then the proof of [2] applies.

\[
\begin{align*}
m_+ m_- &= \langle T_1 R_{13} T_2 \tilde{R}_{32}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle \equiv \langle T_1 T_2 R_{13} \tilde{R}_{32}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle \\
&= \langle \tilde{R}_{12} T_2 T_1 R_{12} R_{13} \tilde{R}_{32}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle = \langle \tilde{R}_{12} T_2 T_1 \tilde{R}_{32} R_{13} R_{12}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle \\
&= \langle R_{12} \tilde{R}_{12} T_2 T_1 \tilde{R}_{32} R_{13}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle = \langle \Delta'(1) T_2 T_1 \tilde{R}_{32} R_{13}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle \\
&= \langle T_2 \tilde{R}_{32} T_1 R_{13}, \mathcal{H} \otimes \mathcal{H} \otimes \text{id} \rangle = m_- m_+.
\end{align*}
\]

Remark: The objects in this theorem \((M^\pm, m_\pm, g_\pm(t), T(t))\) can be interpreted (a) as elements of \( U \otimes F \), (b) as maps \( F \to F \) or (c), when a representation of \( U \) is considered, as matrices with \( F \)-valued matrix elements.

### 3 Dynamical operators

In the main theorem we dealt with expressions that live in \( U \otimes F \) (and should be understood as maps from \( F \) into itself). In this section we want to write expressions with respect to one fixed vertex. Like we mentioned in the introduction we are particularly interested in the action of the Hamiltonian with respect to a fixed vertex. Since the Hamiltonian is an element of \( F \), we shall initially fix the vertex in this space; due to the definition of the dual face Hopf algebra \( U \) this will also fix a corresponding vertex in that space.
We shall proceed as follows: We will fix a vertex in the $T$-matrix with the help of $e^\lambda, e_\lambda \in F$ and will also introduce the corresponding universal $T(\lambda)$. Next we will construct dynamical $R$-matrices $R^{\pm}(\lambda)$ as $R^{\pm}(T(\lambda))$ – this is an algebra homomorphism – and will give the Yang-Baxter and $RTT$ equations with shifts as an illustration. Finally we shall plug everything into the main theorem.

Convention for corepresentation $T$ with respect to a fixed vertex: $T(\lambda)^A_B$ is zero unless the range (end) of path $A$ is equal to the fixed vertex $\lambda$. Such a $T$ will map the vector space spanned by vectors $v^A$ with $A \cdot = \lambda$ fixed to itself.

With the help of the projection operator $e^\lambda \in F$ we can give the following explicit expression:

\[
T(\lambda)^A_B = T^A_B e^\lambda \sim \begin{array}{c}
A \\
\lambda \\
B
\end{array}
\]  \hspace{1cm} (16)

The universal $T(\lambda)$ is an abstraction of this and is defined analogously.

\[
T_{12}(\lambda) = T_{12}(e^\lambda)_2 \sim \begin{array}{c}
\lambda
\end{array}
\]  \hspace{1cm} (17)

**Coproducts of $T$.** Expressions for the coproducts $\Delta_1 T(\lambda)$ and $\Delta_2 T(\lambda)$ follow either directly from the definition of $T(\lambda)$ or can be read of the corresponding pictorial representations.

(i) Coproduct in $F$-space [3]:

\[
\Delta_2 T(\lambda) = T_{12}(\lambda)T_{13}(\lambda - h_2) \sim \begin{array}{c}
\lambda \\
\lambda - h_2
\end{array}
\]  \hspace{1cm} (18)

The shift operator $h$ in $F$-space that appears here is

\[
h_{(F)} = \sum_{\eta,\mu} (\mu - \eta) e^\mu_\eta \in F, \]  \hspace{1cm} (19)
where we assume some appropriate (local) embedding of the vertices of the underlying graph in $\mathbb{C}^n$ so that the difference of vertices makes sense.

Proof: $\Delta_2 T_{12}(e^\lambda)_2 = \sum_\eta T_{12}(e^\lambda)_{22}(e^\eta)_3 = \sum_{\eta,\mu} T_{12}(e^\lambda)_{22}(e^\mu)_2 T_{13}(e^\eta)_3 = \sum_{\eta,\mu} T_{12}(\lambda)(e^\mu)_2 T_{13}(\lambda + \eta - \mu) = T_{12}(\lambda)T_{13}(\lambda - h_2)$. In the second and third step we used $e^\lambda e^\mu \propto \delta_{\lambda,\mu}$.

(ii) Coproduct in $U$-space: (gives multiplication in $F$)

\[
\Delta_1 T(\lambda) = T_{13}(\lambda - h_2)T_{23}(\lambda) \sim \begin{array}{c}
\lambda - h_2 \\
\lambda
\end{array} \quad \text{(20)}
\]

This time we have a shift operator in $U$-space:

\[
h(U) = \sum_{\eta,\mu} (\mu - \eta) E_\mu E_\eta \in U. \quad \text{(21)}
\]

The proof is a little more involved than the one given for the $F$-case above and uses $(e^\eta)_2 T_{12} = (E^\eta)_1 T_{12}$ and $T_{12}(e^\mu)_2 = (E_\mu)_1 T_{12}$ which follow from $T\tilde{T}T = T$, $\tilde{T}T\tilde{T} = \tilde{T}$ and $TT = \sum_\xi E^\xi \otimes e^\xi$.

Dynamical $R$-matrix. Using the fact that $f \mapsto R^+(f) \equiv \langle R, f \otimes \text{id} \rangle$ is an algebra-homomorphism we define

\[
R_{12}(\lambda) \equiv \langle R, T_1(\lambda) \otimes T_2 \rangle \quad \text{(22)}
\]

The numerical $R$-matrix is defined analogously: $R(\lambda)_{AC/BD} = \langle R, T(\lambda)_B^A \otimes T_D^C \rangle$. In the pictorial representation this fixes the one vertex of $R$ that is only endpoint to paths.

Dynamical $RTT$-equation. From the coproduct $\Delta_1 T$ and $R\Delta(x) = \Delta'(x)R$ for all $x \in U$ follows [2]:

\[
R_{12}(\lambda)T_1(\lambda - h_2)T_2(\lambda) = T_2(\lambda - h_1)T_1(\lambda)R_{12}(\lambda - h_3) \quad \text{(23)}
\]
Pictorially:

\[
R_{12} \lambda = R_{13} T_2 R_{23} (\lambda - h_2) R_{13} (\lambda - h_1) R_{12} (\lambda - h_3)
\]

Shifts \(h_1, h_2\) are in \(U\)-space, shift \(h_3\) is in \(F\)-space. Twice contracted with \(R\) the dynamical \(RTT\)-equation yields the dynamical Yang-Baxter equation:

\[
R_{12}(\lambda) R_{13}(\lambda - h_2) R_{23}(\lambda) = R_{23}(\lambda - h_1) R_{13}(\lambda) R_{12}(\lambda - h_3)
\]

\((24)\)

Pictorially:

\[
Pictorially:
\]

Hamiltonian and Lax operators in dynamical setting

The Hamiltonian \(H\) should be a cocomutative element of \(F\). In the case of the Ruijsenaars model it can be chosen to be the trace of a \(T\)-matrix, \(i.e.\) the \(U\)-trace of \(T\) in an appropriate representation \(\rho: H = \text{tr}^{(\rho)}_1 T_1\). This can be written as a sum over vertices \(\lambda\) of operators that act in the respective subspaces corresponding to paths ending in the vertex \(\lambda\):

\[
H = \sum_{Q, \#Q \text{ fixed}} T^Q_Q = \sum_{\lambda} H(\lambda)
\]

\((25)\)

with \(H(\lambda) = He^\lambda = \sum T(\lambda)^Q_Q\). The pictorial representation of the Hamiltonian is two closed dashed paths \((F\text{-space})\) connected by paths \(Q\) of fixed length that are summed over. In \(H(\lambda)\) the end of path \(Q\) is fixed. When we
look at a representation on Hilbert space the paths $Q$ with endpoint $\lambda$ that appear in the component $\mathcal{H}(\lambda)$ of the Hamiltonian $\mathcal{H}$ will shift the argument of a state $\psi(\lambda)$ corresponding to the vertex $\lambda$ to a new vertex corresponding to the starting point of the path $Q$. In the next section we will see in detail how this construction is applied to the Ruijsenaars system. For this we will have to take a representation of the face Hopf algebra $F$. We shall then denote the resulting Lax operator by $L$. The Hamiltonian will contain a sum (coming from the trace) over shift operators.

For the Lax operators it is convenient to fix two vertices $\lambda$ and $\mu$ corresponding to paths from $\mu$ to $\lambda$. (We did not need to do this for the Hamiltonian since it acts only on closed paths.) The $T$ that appears in the Lax equation (10) becomes

$$T_{12}(\lambda, \mu) = T_{12}(e_{\mu} e^{\lambda})_2 = (E_{\lambda})_1 T_{12}(E_{\mu})_1;$$

it should be taken in some representation of $F$ on the appropriate Hilbert space. On space $U$ we are interested in a finite-dimensional representation that is going to give us matrices. Both is done in the next section and we shall call the resulting operator $L(\lambda, \mu)$. Similarly we proceed with the other two Lax operators $M^\pm$:

$$M^\pm(\lambda, \mu) \equiv (E_{\lambda} \otimes 1) M^\pm(E_{\mu} \otimes 1) = E_{\lambda} \delta_{\lambda, \mu} \otimes \mathcal{H} - m^\pm(\lambda, \mu).$$

(Recall that $E_{\lambda} E_{\mu} = E_{\lambda} \delta_{\lambda, \mu}$. If we define $M^\pm_{012} = T_{02}(1 - R^0_{01})$ then $M^\pm_{12} = \langle M^\pm_{012}, \mathcal{H} \otimes \text{id}^2 \rangle$ and we can write dynamical Lax equations in an obvious notation as

$$i \frac{dT(\lambda, \mu)}{dt} = \left\langle M^\pm_{012}(\lambda, \lambda - h_0) T_{12}(\lambda - h_0, \mu) - T_{12}(\lambda, \mu - h_0) M^\pm_{012}(\mu - h_0, \mu), \mathcal{H} \otimes \text{id}^2 \right\rangle$$

(26)

where $h_0 = \sum_{\alpha, \beta} (\alpha - \beta) E^\beta_{\alpha}$ is the shift operator in the space contracted by the Hamiltonian. If the Hamiltonian is a trace we shall find a sum over shifts.
Remark: There is another possible choice of conventions for the fixed vertex. We could work with \( T_{12}\{\nu\} = (e_\nu)_2T_{12} \) instead of \( T_{12}(\lambda) = T_{12}(e^\lambda)_2 \). This would have fixed the lower left vertex in the pictorial representation of \( T^A_B \) and the vertex that is starting-point for all paths in \( R\{\nu\} \). The dynamical equations would of course look a little different from the ones that we have given.

4 The Lax Pair

Here we give as an example the Lax pair for the case of the \( N \)-particle quantum Ruijsenaars model. We may think of the face Hopf algebra \( F \) as of the elliptic quantum group associated to \( sl(N) \) introduced by Felder [2]. We will continue to use the the symbol \( F \) for it. In that case there is an additional spectral parameter entering all relations in the same way as it is in the case ordinary quantum groups.

Let \( h \) be the Cartan subalgebra of \( sl(N) \) and \( h^* \) its dual. The actual graph related to the elliptic quantum group \( F \) is \( h^* \sim \mathbb{C}^{N-1} \). This is the huge graph of the remark made in the introduction. However it decomposes into a continuous family of disconnected graphs, each one isomorphic to \(-\eta.\Lambda\), the \(-\eta \in \mathbb{C} \) multiple of the weight lattice \( \Lambda \) of \( sl(N) \), and we can restrict ourselves to this one component for simplicity. Correspondingly the shifts in all formulas are rescaled by a factor \(-\eta\). The space \( h^* \sim \mathbb{C}^{N-1} \) itself will be considered as the orthogonal complement of \( \mathbb{C}^N = \oplus_{i=1,\ldots,N} \mathbb{C}\varepsilon_i \), \( \langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij} \) with respect to \( \sum_{i=1,\ldots,N} \varepsilon_i \). We write the orthogonal projection \( \varepsilon_i = \varepsilon_i - \frac{1}{N} \sum_k \varepsilon_k \) for the generator of \( h^* \); \( \langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij} - 1/N \). The points of the lattice \( \eta.\Lambda \) will be denoted by greek characters \( \lambda, \mu \), etc.

The elliptic quantum group is defined by the matrix elements of the “fundamental corepresentation”. This is described with the help of the paths of length 1 in the following way: Let us associate a one-dimensional linear space
$V_{\rho,\lambda} \sim \mathbb{C}\eta_\epsilon k$ and a corresponding path of length 1 to any pair $\lambda, \rho \in \eta\Lambda$, such that $\rho - \lambda = \eta\epsilon_k$, for some $k = 1, \ldots, N$. We let $V_{\rho,\lambda} = \emptyset$ for all other pairs of vertices. The vector space $V$ of the fundamental corepresentation is formed by all paths of length 1

$$V = \bigoplus_{\rho,\lambda} V_{\rho,\lambda} \sim \bigoplus_{\lambda,\rho} \xrightarrow{\rho} \xrightarrow{\lambda} = \bigoplus_{\lambda,\rho} \xrightarrow{\lambda + \eta\epsilon_i} \xrightarrow{\lambda}.$$

As all spaces $V_{\rho,\lambda}$ are at most one-dimensional, we can characterize the numerical $R$-matrix $R^{AC}_{BD}$ in the fundamental corepresentation by just four indices referring to the vertices of the “square” defined by paths $A, B, C, D$ in the case of nonzero matrix element $R^{AC}_{BD}$. Let us set $\cdot B = \cdot C = \nu, \cdot D = B \cdot = \mu, D \cdot = A \cdot = \lambda$ and $\cdot A = C \cdot = \rho$ and also

$$R^{AC}_{BD} = W(C^B_A D) \equiv W\left(\begin{array}{c c}
\nu & \mu \\
\rho & \lambda 
\end{array}\right),$$

Then the non-zero Boltzmann weights as given by \[22\] are: ($i \neq j$)

$$W\left(\begin{array}{c c}
\lambda + 2\eta\epsilon_i & \lambda + \eta\epsilon_i \\
\lambda + \eta\epsilon_i & \lambda 
\end{array}\right) = 1 \sim \xrightarrow{\nu} \xrightarrow{\lambda},$$

$$W\left(\begin{array}{c c}
\lambda + \eta(\epsilon_i + \epsilon_j) & \lambda + \eta\epsilon_i \\
\lambda + \eta\epsilon_i & \lambda 
\end{array}\right) = \theta(u)\theta(-u + \lambda_{ij}) \theta(u + \eta)\theta(\lambda_{ij}) \sim \xrightarrow{\nu} \xrightarrow{\lambda},$$

and

$$W\left(\begin{array}{c c}
\lambda + \eta(\epsilon_i + \epsilon_j) & \lambda + \eta\epsilon_j \\
\lambda + \eta\epsilon_i & \lambda 
\end{array}\right) = \theta(u)\theta(\eta + \lambda_{ij}) \theta(u + \eta)\theta(\lambda_{ij}) \sim \xrightarrow{\nu} \xrightarrow{\lambda}.$$

Here $\lambda_{ij} \equiv \lambda_i - \lambda_j = \langle \lambda, \epsilon_i - \epsilon_j \rangle$, $\theta(u)$ is the Jacobi theta function

$$\theta(u) = \sum_{j \in \mathbb{Z}} e^{\pi i(j + \frac{1}{2})^2 \tau + 2\pi i(j + \frac{1}{2})(u + \frac{1}{2})},$$

$u \in \mathbb{C}$ is the spectral parameter and $\tau$ is the elliptic modulus parameter. The matrix element $T^A_B(\lambda|u)$ (now depending also on the spectral parameter $u$, which also enters all previous expressions in the standard way) in the fundamental corepresentation is also uniquely determined by the value of

\[27\]
the vertices that fix the length-1 paths $A$ and $B$. Let us use the following notation for it $(i, j = 1, \ldots, N)$

\[ T_A^B(\lambda|u) = \sum_{\mu} L_j^i(\mu, \lambda|u)e_\mu \sim \begin{array}{c} \lambda + \eta_\epsilon_j, \\ \mu + \eta_\epsilon_j \end{array} \begin{array}{c} A \\ B \end{array} \]

where $\mu = B$, $\lambda = A$, $\mu + \eta_\epsilon_j = \cdot B$, $\lambda + \eta_\epsilon_i = \cdot A$.

As the lattice $\eta.\Lambda$ that we consider is just one connected component of a continuous family of disconnected graphs, the vertices $\lambda, \mu, \ldots$ are allowed to take any values in $\mathbb{C}^{(N-1)}$. This will be assumed implicitly in the rest of this section. Now we need to specify the appropriate representation of $F$ which can be read of from [7]. We can characterize it by its path decomposition. To any pair of vertices $\lambda, \mu$ we associate a one-dimensional vector space $\tilde{V}_{\lambda, \mu} \sim \mathbb{C}$ (path from $\mu$ to $\lambda$). The representation space $\tilde{V}$ is then

\[ \tilde{V} = \bigoplus_{\lambda, \mu \in h^*} \tilde{V}_{\lambda, \mu}. \]

The matrix element $T_A^B(\lambda|u) \equiv L_j^i(\lambda, \mu|u)$ for fixed $A, B$ and hence also with fixed $i, j, \lambda, \mu$ is obviously non-zero only if restricted to act from $\tilde{V}_{\lambda, \mu}$ to $\tilde{V}_{\lambda+\eta_\epsilon_i, \mu+\eta_\epsilon_j}$ in which case it acts as multiplication by

\[ L_j^i(\lambda, \mu|u) = \frac{\theta(cN + u + \lambda_i - \mu_j)}{\theta(u)} \prod_{k \neq i} \frac{\theta(cN + \lambda_k - \mu_j)}{\theta(\lambda_k - \lambda_i)}. \]  

Here we used notation $\lambda_i = \langle \lambda, \epsilon_i \rangle$ for $\lambda \in h^*$. $c \in \mathbb{C}$ will play the role of coupling constant.

The Hamiltonian is chosen in accordance with Section 3 as $\mathcal{H} = \sum_{P} T_P^P$, i.e. the trace in the “fundamental” corepresentation. In accordance with the discussion of the previous sections it is non-zero only when acting on the diagonal subspace (closed paths)

\[ H = \bigoplus_{\lambda \in h^*} H_\lambda \equiv \bigoplus_{\lambda \in h^*} \tilde{V}_{\lambda, \lambda} \]
of \( \tilde{V} \). So this is the actual state space of the integrable system under consideration. The Lax operator \( M^{\pm} \) acts from \( \tilde{V}_{\lambda, \mu} \) to \( \tilde{V}_{\lambda + \eta \epsilon_i, \mu + \eta \epsilon_j} \) as multiplication by \( M^{\pm l}_{k} (\lambda, \mu | u, v) \) with
\[
(1 \otimes \mathcal{H})^{l}_{k} (\lambda, \mu | v) = \delta_{\lambda, \mu} \delta_{k}^{l} \frac{\theta(\frac{c}{N} + v)}{\theta(v)} \prod_{j \neq i} \frac{\theta(\frac{c}{N} + \lambda_{j,i})}{\theta(\lambda_{j,i})}.
\] (29)

\[
m^{+ l}_{k} (\lambda, \mu | u, v) = \delta_{\epsilon_{i} + \epsilon_{k}, \epsilon_{j} + \epsilon_{l}} \delta_{\lambda, \mu + \eta \epsilon_{j}} \sum_{i,j} L^{i}_{j} (\mu + \eta \epsilon_{k}, \mu | v) W \left( \frac{\lambda + \eta \epsilon_{i}}{\mu + \eta \epsilon_{k}}, \frac{\lambda}{\mu} \right) W \left( \frac{\lambda_{j,i}}{\lambda_{j,i}} \right).
\] (30)

\( m^{-} \) is given by a similar formula with the inverse Boltzmann weight.

**Dynamical Lax Equation**
\[
\frac{dL^{i}_{k} (\lambda, \mu | u)}{dt} = \sum_{j, \nu} M^{\pm j}_{k} (\lambda, \nu | u, v) L^{i}_{j} (\nu, \mu | u) - L^{i}_{j} (\lambda, \nu | u) M^{\pm j}_{k} (\nu, \mu | u, v)
\]

The Hamiltonian \( \mathcal{H} \) maps the component \( H_{\lambda} \sim \mathbb{C}.|\lambda| \) into the component \( H_{\lambda + \eta \epsilon_{i}} \sim \mathbb{C}.|\lambda + \eta \epsilon_{i}| \). Obviously \( \tilde{V} \) can be understood as the complex vector space of all functions in \( \lambda, \mu \) as well as \( H \) can be understood as the complex vector space of all functions in \( \lambda \). In that case the Hamiltonian \( \mathcal{H} \) given by (29) is proportional to a difference operator in variable \( \lambda \in h^{*} \)
\[
\mathcal{H} \propto \sum_{i} t^{(\lambda)}_{i} \prod_{j \neq i} \frac{\theta(\frac{c}{N} + \lambda_{j,i})}{\theta(\lambda_{j,i})},
\] (31)
where \( t^{(\lambda)}_{i} \) has an obvious meaning of the shift operator by \( -\eta \epsilon_{i} \) in the variable \( \lambda \). This is equivalent [23] to the Ruijsenaars Hamiltonian [3]. It follows from [7] that in the same way we can obtain the higher order Hamiltonians concerning traces in properly fused “fundamental representations”.

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Appendix

For many reasons it is very convenient to use a formalism based on the so-called universal $T$. The expressions formally resemble those in a matrix representation but give nevertheless general face Hopf algebra statements. This greatly simplifies notation but also the interpretation and application of the resulting expressions. When we are dealing with quantizations of the functions on a group we need to keep track both of the non-commutativity of the quantized functions and the residue of the underlying group structure. Both these structures can be encoded in algebraic relations for the universal $T$ which easily allows to control two non-commutative structures. In fact $T$ can be regarded as a universal group element. Universal tensor expressions can formally be read in two ways: Either as “group” operations (or rather operations in $U$) or as the corresponding pull-back maps in the dual space. This simplifies the heuristics of “dualizing and reversing arrows” and allows us to keep track of the classical limit. Example: Multiplication in $U$, $x \otimes y \mapsto xy$: The corresponding pull-back map in the dual space $F$ is the coproduct $\Delta$. Both operations are summarized in the same universal expression $T_{12} T_{13}$.

Sometimes $T$ can be realized as the canonical element $U \otimes F$, but we do not need to limit ourselves to these cases an will instead define $T$ as the identity map from $F$ to itself and will use the same symbol for the identity map $U \to U$.

For the application of this identity map to an element of $F$ we shall nevertheless use the same bracket notation as we would for a true canonical
element in the finite case, i.e. \( f \equiv \langle \text{id}, f \rangle = \langle T_{12}, f \otimes \text{id} \rangle \). This notation is very convenient—like inserting the unity in quantum mechanics. The identity map on a product is given by \( ab \mapsto a \cdot b \) or

\[
\langle T_{12}, ab \otimes \text{id} \rangle = \langle T_{13}T_{23}, a \otimes b \otimes \text{id} \rangle = a \cdot b
\]

We shall write \( \Delta_1 T_{12} = T_{13}T_{23} \) to express this fact—this is hence the universal-T notation for the multiplication map in \( F \) (and the coproduct map in \( U \)). The coproduct map in \( F \) (multiplication map in \( U \)) is \( T_{12}T_{13} \):

\[
f \mapsto \langle T_{12}T_{13}, f \otimes \text{id} \otimes \text{id} \rangle = \Delta(f)
\]

The antipode map is \( \tilde{T} \): \( f \mapsto S(f) \), the contraction map \( x \otimes 1 \) with \( x \in U = F^* \) maps \( f \in F \) to \( 1 \langle x, f \rangle \), the counit map is \( 1 \otimes 1 \), etc. For face algebras the counit is not an algebra homomorphism but rather \( \Delta_1 F = \sum_k e_k \otimes e_k \) and \( \Delta_1 U = \sum_k E_k \otimes E_k \), where \( \langle E_k, f \rangle = \epsilon(f e_k) \) and \( \langle E^k, f \rangle = \epsilon(e_k f) \). Therefore \( \epsilon(ab) = \sum_k \epsilon(a e_k)\epsilon(e_k b) \). This is one of the face algebra axioms. It differs from the corresponding ordinary Hopf algebra axiom (there: \( \epsilon(ab) = \epsilon(a)\epsilon(b) \)). Some important relations involving the coproduct and antipode maps are

\[
\tilde{T}T = \sum_i E_i \otimes e_i, \quad TT = \sum_i E^i \otimes e^i, \quad \tilde{T}T\tilde{T} = \tilde{T}, \quad T\tilde{T}T = T.
\]

In the pictorial representation this relations imply that the vertices of the paths in \( U \) and \( F \) that form \( T \) match to give a closed “square”. They also give us two ways to fix the four vertices of \( T \):

\[
(1 \otimes e_k^i)T(1 \otimes e_j^i) = (E_j^i \otimes 1)T(E_i^k \otimes 1) \sim \begin{array}{c}
v_i \downarrow \quad \quad \quad \downarrow v_j \\
E_i^k \quad \quad \quad E_j^i
\end{array}
\]

Further useful relations can be obtained from this by summing over some of the indices and using \( \sum_i e^i_j = e_j, \sum_j e_j = 1 \), etc.
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