On completing a measurement model by symmetry

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Abstract

An appeal for symmetry is made to build established notions of specific representation and specific nonlinearity of measurement (often called “model error”) into a canonical linear regression model. Additive components are derived from the trivially complete model \( M = m \). Factor analysis and equation error motivate corresponding notions of representation and nonlinearity in an errors-in-variables framework, with a novel interpretation of terms. It is suggested that a modern interpretation of correlation involves both linear and nonlinear association.

1 Introduction

A regression model provides an imprecise framework of inference, but not just because we happen to employ exploratory measurements, whose causes are only partially understood. Notions of measurement error and model error are both in play. Notably, Kruskal (1988) warns us to be cautious of what we might describe as correlated model error. Yet how is caution expressed in a model? We address the related question of whether additive models are complete with respect to the measurements that they are a model of. One expression that we propose to include is equation error (Fuller 1987; Carroll and Ruppert 1996), which is the notion that a linear relationship between measurements is ad hoc, even in the absence of measurement error. Prior expressions have taken equation error as a single identifiable term (Kipnis et al. 1999; Carroll et al. 2006); here, we take it in parts, as components of two separate terms.

We also consider whether the common distinction between a sample and a population has an expression in model building. This is the notion of specific representation, where we are exploring measurements whose causal support is limited, perhaps in part by design. It seems tautological that every sample is imperfect, and for every imperfect sample there is representation error, but this says little about the difference between two or more imperfect subsets. Thus, we draw on factor analysis (Kline 2015) to motivate a more symmetric treatment of two specific representations. The resulting expression is taken as part of a single identifiable term, but we have reason to interpret equation error and representation error as partially overlapping.

An appeal is made to explicitly accommodate notions of nonlinearity and representation in a linear model (Lehmann 2008). The errors-in-variables model (Fuller 2006; Dunn 2011) provides a canonical framework. From the outset, symmetry is called on to yield a division of terms that is specific to our measures, and seems complete and consistent with an exploration of what we seek to measure. The next section describes this model building exercise and the notions that we ascribe to each term. Conclusions are given in the final section, with a comment on distance correlation (Edelmann et al. 2021) and challenges of an errors-in-variables solution.

2 Model building

We begin with the final form of the model that we wish to derive. Calibrated (\( C \)) and uncalibrated (\( U \)) measurements are on the left hand side (LHS), and on the right hand side (RHS) are shared truth (\( t \)) and its additive (\( \alpha_U \)) and multiplicative (\( \beta_U \)) agreement in \( U \). Included are additive shared error (\( \epsilon \)) and an unshared residual (\( \epsilon_C \) or \( \epsilon_U \)), as in

\[
\begin{align*}
C &= t + \epsilon_C \\
U &= \alpha_U + \beta_U t + \epsilon + \epsilon_U.
\end{align*}
\]

The theoretical richness of this model follows from our interpretation of its terms, but at this point, all we have is a measurement model. It seems asymmetric to take \( U \) to be causal of \( C \), following a traditional approach (Fuller 1987). Instead, we make a more abstract claim that the RHS terms represent causes of both \( C \) and \( U \), but as yet, descriptors like “calibrated”, “truth”, “error”, and “residual” are undefined. By “shared error”, for instance, we only refer...
to a RHS duplication of $\epsilon$ (where “shared”, “correlated”, and “associated” are synonyms), without an explicit reference to causality, such as that $C$ and $U$ are independently measured.

Following Fuller (2006), the corresponding classic regression equation is $U = \alpha_U + \beta_U C + \epsilon$ and the canonical errors-in-variables model is obtained by writing $\epsilon$ as part of $\epsilon_C$ and $\epsilon_U$. We depart further from a traditional approach by assuming that each term on the RHS may represent not just causes that we can explore more easily by $C$ and $U$, but also those we can’t, or don’t want to explore in either of them. Thus, all variables are simultaneously explanatory and non-explanatory, and a strict functional (fixed explanatory variable) or structural (random explanatory variable) distinction is difficult to apply. However, it is convenient to think of the explorable (functional) causes as fixed signal, the more hidden (structural) causes as random noise, and that a partially different mixture of causes has led to $C$ and $U$.

Because each term on the RHS of a model captures signal and noise, the trivial model $M = m$ is also a complete model. By complete, we mean that under any model partitioning, $m$ remains the union of all partitioned components. Assuming that approximate solutions are identifiable, we opt to divide $M$ and $m$ into well motivated additive components to build (1). We take $C$ and $U$ to be repeated measurements (two subsets of $M$) that each represent the same population both nonlinearly and imperfectly, with somewhat different causal supports. Otherwise, the division of $M$ is unspecified.

A model is still complete if an additive division of $m$ is made that separates the correlated and uncorrelated parts of $M$, as in

\begin{align*}
C &= s_C + \epsilon_C \\
U &= s_U + \epsilon_U.
\end{align*}

(2)

Again, signal and noise are both assumed to exist in the shared ($s_C$ and $s_U$) and unshared ($\epsilon_C$ and $\epsilon_U$) terms. It is perhaps unusual to accommodate signal in $\epsilon_C$ and $\epsilon_U$, at least for the classic regression equation (Lehmann 2008), but this is an established interpretation in factor analysis, where it is called the “specific” contribution (Kline 2015). Similarly, we claim that $C$ and $U$ have specific causal supports and representations of the same population. With different specific representations, it would be a formal loss of generality to exclude signal from what is unshared in either of them. Moreover, it follows by symmetry that instead of just two or three components, the most general accommodation is four: shared and unshared signal and noise. With a division of measurements $M$ into the subsets $C$ and $U$, eight components are implied on the RHS of (2), but with signal and noise combined, the model $m$ is only separated into four terms.

A complete division of $m$ can also be made by separating the linear and nonlinear parts of $M$, as in

\begin{align*}
C &= t + n_C \\
U &= \alpha_U + \beta_U t + n_U.
\end{align*}

(3)

Again, signal and noise are both assumed to exist in the linear ($t$ and $\alpha_U + \beta_U t$) and nonlinear ($n_C$ and $n_U$) terms. As in (2), it may be unusual to accommodate signal in $n_C$ and $n_U$, but this is an established notion in the errors-in-variables literature, where it is usually expressed as a single term called “equation error” (Fuller 1987; Carroll and Ruppert 1996; Kipnis et al. 1999; Carroll et al. 2006). Here, we claim that $C$ and $U$ are both nonlinear measures of the same population. To say otherwise, and thereby exclude signal from what may be nonlinear about either of them, would be a formal loss of generality. Again, it follows by symmetry that the most general accommodation is four components: linear and nonlinear signal and noise. The RHS of (3) is thus taken as four terms, each with signal and noise components.

As a basis for exploring causality by $C$ and $U$, we interpret (1) as a division of $m$ into terms that capture linear association ($t$ and $\alpha_U + \beta_U t$), nonlinear association ($\epsilon$), and a lack of association ($\epsilon_C$ and $\epsilon_U$). As in (2) and (3), this is a complete division of additive terms if the RHS includes all four signal and noise components in each of $C$ and $U$; these are the linear and nonlinear, correlated and uncorrelated components. By $\alpha_U$, $\beta_U$, and $t$, the models (1) and (3) provide a generic form of the linear association between $C$ and $U$. Because linear association is generic, $\epsilon$ is both shared and nonlinear by definition. As in (2), $\epsilon$ is required to provide a generic form of complete association. It follows that the residual terms ($\epsilon_C$ and $\epsilon_U$) capture completely any lack of association between $C$ and $U$, including any linear and nonlinear unassociated components that may exist.

When equation error and representation error are aspects of $M$ that are missing from an incomplete model, then we would expect to describe them as “model error”, but in a complete additive model, they are proper expressions of signal (Kruskal 1988). Equation error typically refers to the nonlinear signal of $\epsilon$ and $\epsilon_C$, or $\epsilon$ and $\epsilon_U$. Representation error typically refers to the signal of $\epsilon_C$ or $\epsilon_U$. Thus, it seems difficult to express equation error as a single term, or to distinguish between equation error and representation error.
3 Conclusions

An appeal for symmetry is made in deriving and interpreting a complete additive measurement model. A complete model is one that includes all components under any partitioning, even if this partitioning is only notional. For example, there is no explicit partitioning of signal and noise, which are expected to be somewhat different in two set of measures. Other partitionings are motivated by “model error”, or an incomplete accommodation of the notions of equation error (Carroll and Ruppert 1996) and representation error. One novel aspect is that when complementary notions of specific nonlinearity and specific representation (Kline 2015) are taken to define measurements from the outset, a symmetric model formulation highlights signal components that are not easy to recognize and accommodate as signal.

This exercise provides a useful characterization of descriptors like “calibrated”, “truth”, and “error”. If truth and signal are taken as synonyms, then it seems more appropriate to describe a variable like $t$ as linear than as truth. If error and noise are synonyms, then (1) can be called both an error model and a signal model. Regarding the identity of $C$ as a calibration reference, this is formally undefined, even if a familiar set of measurements is available. Strictly speaking, calibration only applies to corresponding components of association (e.g., linear calibration only applies to the linear association terms), and not to all signal components. The notion that there is signal in every component of (1) follows from specific representation, which is what Haraway (1988) describes as “situated knowledges”.

Specific nonlinearity of $C$ and $U$ is another novel aspect, whose consequence is that variance, covariance, and correlation all include (by way of $\epsilon$) a nonlinear signal component, which is what Pearson (1902) describes as “personal”. Specific nonlinearity is irrelevant without a division of $M$, but our interpretation of (1) permits nonlinear association between any two measurement subsets. Of course, this is not inconsistent with the practice of using an appropriate measurement unit, if one is available. Mahalanobis (1940) questions the unit, and its function in allowing us to avoid systematic differences in measurement. We often expect our choice of unit to do just that. Even if we choose a unit so as not to contribute to nonlinear association, however, a contribution could still arise out of differences in the remaining causal supports of $C$ and $U$. The use of an appropriate unit might be expected to minimize differences in causal support, but not necessarily eliminate them.

Finally, we acknowledge a longstanding need to identify the magnitude of $\epsilon$ (by any interpretation). Conventional regression model solutions often neglect this term using what Kruskal (1988) and Lehmann (2008) describe as the “assumption of independence”. Following Edelmann et al. (2021), however, a modern definition of dependence does not place any constraint on correlation. In practice, this means that $\epsilon$ cannot be neglected on the basis that independence seems natural or obvious. The presence of $\epsilon$ might imply a solution of (1) that is more ambitious than for the errors-in-variables model (Fuller 2006; Dunn 2011), but this is not necessarily the case (Pólya 1973). There is the prospect that if $M$ includes more than two repeated samples ($C$ and $U$), a numerical solution of (1) is possible using a complete model of the longitudinal samples near $C$ and $U$.

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