MEROMORPHIC HIGGS BUNDLES AND RELATED GEOMETRIES

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Abstract. The present note is mostly a survey on the generalised Hitchin integrable system and moduli spaces of meromorphic Higgs bundles. We also fill minor gaps in the existing literature, outline a calculation of the infinitesimal period map and review briefly some related geometries.

Keywords: Hitchin system, cameral covers, Donagi-Markman cubic, meromorphic Higgs bundles, integrable systems

Contents

1. Introduction
2. Meromorphic $G$-Higgs Bundles
3. Poisson Geometry
4. Cameral covers and the Hitchin map
5. The Infinitesimal Period Map
6. Some Related Geometries
References

1. Introduction

1.1. Integrable systems and complex geometry. Many moduli spaces arising in complex-algebraic or analytic geometry carry a symplectic or Poisson structure. The spaces considered in this survey are no exception. Let $G$ be a simple complex Lie group, $X$ a compact Riemann surface with canonical bundle $K_X = \Omega^1_X$ and $D$ a sufficiently positive effective divisor on $X$. Our exposition is built around the study of meromorphic, i.e., $K_X(D)$-valued, $G$-Higgs bundles on $X$ (Definition 2.1.) and their coarse moduli spaces $\text{Higgs}_{G,D}$. These spaces come with the additional structure of an algebraic completely integrable Hamiltonian system (ACIHS), known as the generalised (or ramified) Hitchin system.

Completely integrable Hamiltonian systems have long been an object of interest for both mathematicians and physicists. The last thirty years have brought significant advances in the study of their algebraic (and holomorphic) counterparts. This was stimulated by the development of new methods in abelian and non-abelian Hodge theory, complex dynamics and holomorphic symplectic geometry, Yang–Mills and Seiberg–Witten theories, and of course, the quest for understanding mirror symmetry in its various incarnations.

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The key difference between real and algebraic (or holomorphic) integrable systems is that abelian varieties (and complex tori) have moduli. Hence, after the removal of singular fibres, the structure morphism of the ACIHS is a $C^\infty$ torus fibration, which usually fails to be holomorphically locally trivial. It is thus important to understand the corresponding period map, or, less ambitiously, the differential of the latter.

1.2. Contents of the paper. We begin with a discussion of the moduli spaces $\text{Higgs}_{G,D}$ (§2) and their Poisson geometry (§3). Then in §4 we discuss cameral covers and the Hitchin map. Most of the results in these introductory sections are standard and based on [Bot95], [Mar94], [DM96] and [Mar00]. There are, however, a number of well-known (and used) extensions of results of Bottacin and Markman, for which we have not been able to locate a proper reference. For these we have included partial proofs, wherever appropriate.

One of our goals in this note is to outline a calculation of the infinitesimal period map of the generalised (ramified) Hitchin system. This is done in §5, and we refer to [BD14] for more details. In short, our main result in §5 is that the Balduzzi–Pantev formula ([Bal06], [DDP07]) holds along the maximal rank symplectic leaves of the generalised Hitchin system.

Admittedly, the ramified Hitchin system may seem very special, but we recall the folklore statement that all known ACIHS arise as special case of Hitchin’s. Some well-known examples are geodesic flows on ellipsoids (Jacobi–Moser–Mumford system), KP elliptic solitons, Calogero–Moser and elliptic Sklyanin systems. While some of these systems arise as complexifications of real CIHS, in general such complexifications do not give rise to ACIHS, since real Liouville tori need not “complexify well”. We direct the interested reader to the wonderful surveys [DM96] and [Mar00] for a detailed discussion and examples.

Apart from Higgs bundles, cameral covers and Prym varieties, there are several other geometric structures related to the space $\text{Higgs}_{G,D}$: special Kähler geometry, several flavours of Hodge theory, $tt^*$-geometry and Frobenius-like structures, to name a few. We devote our final section §6 to a very brief literature review and discussion of some of these structures.

1.3. Conventions and notation. In §§ 2, 3, 4 we alternate between the holomorphic and the algebraic viewpoint and emphasise the differences, whenever deemed important. For the proof of the main theorem in §5 we work in the holomorphic category. We fix the following two types of ingredients:

1. Geometric data:
   - a smooth, compact, connected Riemann surface $X$ of genus $g \geq 0$
   - a divisor $D \geq 0$ on $X$, with $K_X(D)^2$ very ample

2. Lie-theoretic data:
   - a simple complex Lie group $G$
   - Cartan and Borel subgroups $T \subset B \subset G$.

We denote by $Z$ or $Z(G)$ the centre of $G$. The twist of the canonical bundle of $X$ by $\mathcal{O}_X(D)$ will be denoted by $L := K_X(D)$. We shall also use the following – mostly standard – Lie-theoretic notation. The Lie algebras of the Cartan and Borel subgroups will be denoted, respectively, as $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, while $\mathcal{R}^+ \subset \mathcal{R} \subset \mathfrak{t}^\vee$ will denote the (positive) roots. We let $W = N_G(T)/T$ stand for the abstract Weyl group, which will be identified with its embeddings in $GL(\mathfrak{t}^\vee)$ and $GL(\mathfrak{t})$. Finally,
let \( l = \text{rk} \, \mathfrak{g} = \dim \mathfrak{t} \) be the rank of \( G \), and \( d_i \) \( (1 \leq i \leq l) \) the degrees of (any choice of) basic \( G \)-invariant polynomials on \( \mathfrak{g} \). For some of the calculations we will also use a fixed choice of generators \( \{ I_i \} \) of \( \mathbb{C}[\mathfrak{g}]^G \). We also fix an \( \text{Ad} \)-invariant symmetric bilinear form \( \text{Tr} \) on \( \mathfrak{g} \).

To these data one can associate two (closely related) families of abelian torsors parametrised by the *Hitchin base* \( \mathcal{B} = H^0(X, \mathfrak{t} \otimes_{\mathbb{C}} L/W) \cong H^0(X, \bigoplus_i L^{d_i}) \):

- a certain moduli space of meromorphic Higgs bundles on \( X \)
- a family of generalised Prym varieties for a family of (branched) \( W \)-Galois covers of \( X \).

Both (have connected components which) are ACIHS in the Poisson sense. The first family is known as the “generalised” or “ramified” Hitchin system (with singular fibres removed). The second family is the “abelianisation” of the first one, and is (locally on the base) isomorphic it. While globally different, they have the same infinitesimal period map, and we shall use the second family for our main Kodaira–Spencer calculation.

We remark that the Hitchin base \( \mathcal{B} \) depends on \( G \), but only via \( \mathfrak{g} \), and we write \( \mathcal{B}_{\mathfrak{g}} \) whenever it is important to emphasise this dependence. There are certain loci in \( \mathcal{B} \) for which we introduce special notation: the Zariski-open locus \( \mathcal{B} \subset \mathcal{B} \) of generic cameral covers, \( \mathcal{B}_0 \subset \mathcal{B} \) for the locus (25) of pluri-differentials vanishing along \( D \), and \( \mathcal{B} \subset \mathcal{B} \) for the base (26) of the integrable system, obtained by restricting the Hitchin map to a maximal rank symplectic leaf.

### 1.4. Acknowledgments

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### 2. Meromorphic \( G \)-Higgs Bundles

In this section we introduce our main objects of study: \( G \)-Higgs bundles on \( X \) with values in a vector bundle. Next, we discuss the main global properties of the coarse moduli space of \( K_X(D) \)-valued \( G \)-Higgs bundles. Finally, we study in more detail the locus in the moduli space, corresponding to Higgs bundles whose underlying principal bundle is regularly stable.

#### 2.1. \( L \)-valued \( G \)-Higgs bundles.

**The definition.** Higgs bundles come in various flavours, and we begin by reviewing one of the simplest variants of this notion: a Higgs bundle with values in a vector bundle.

**Definition 2.1.** Let \( V \) be a holomorphic (algebraic) vector bundle on \( X \). A holomorphic (algebraic) \( V \)-valued \( G \)-Higgs bundle on \( X \) is a pair \((P, \theta)\), consisting of a holomorphic (algebraic) principal \( G \)-bundle \( P \to X \) and a section \( \theta \in H^0(X, \text{ad}P \otimes V) \), called a Higgs field.

A \( V \)-valued Higgs bundle is also called a Higgs bundle with \( V \)-coefficients, the rationale being that for classical \( G \), the Higgs field can be represented in a local trivialisation by a matrix with coefficients in \( V \). This terminology follows [DG02],
where the authors separate the rôle of the coefficient object from that of the abstract (principal) Higgs bundle, see Definition 2.2, ibid. For Higgs bundles with other coefficients, e.g., abelian fibrations, see Part V, ibid.. We remark that $V$-valued Higgs bundles, especially when $V$ is a line bundle, are sometimes called $V$-twisted Higgs bundles. We avoid this terminology as potentially conflicting with the notion of a twisted Higgs bundle, understood as a Higgs field on a twisted bundle, the latter being a bundle on a $\mu_n$-banded gerbe over $X$.

Our coefficient bundle $V$ will be, almost exclusively, the line bundle $L = K_X(D)$. The only exception to this is section 6, where $V = T^*_B$, the cotangent bundle to a complex manifold, possibly non-compact and of dimension greater than one. In the case $D = 0$ our objects become $K_X$-valued $G$-Higgs bundles, which are the objects originally introduced by Hitchin in [Hit87a] (for $G = SL_2(\mathbb{C})$, $PGL_2(\mathbb{C})$) and [Hit87b] (for classical $G$).

An isomorphism $(P, \theta) \simeq (Q, \phi)$ between Higgs bundles is an isomorphism $f : P \cong Q$ between the $G$-bundles, which preserves the Higgs fields, i.e., $\text{ad} f(\theta) = \phi$.

The above notion of Higgs bundles is clearly functorial in the structure group. If $(P, \theta)$ is an $L$-valued $G$-Higgs bundle, and $\rho : G \to H$ a group homomorphism, then extension of structure group gives an $H$-bundle $P' = P \times^G H$. Moreover, we have a homomorphism $\text{ad} \circ \rho : G \to \text{Aut}(\mathfrak{h})$, and the associated bundle $P(\mathfrak{h}) = P \times^{\text{ad} \circ \rho} \mathfrak{h}$ is isomorphic to $\text{ad}P'$. Thus $\theta$ gives rise, via the homomorphism $\mathfrak{g} \to \mathfrak{h}$, to a Higgs field $\theta'$ on $P'$.

### 2.1.2. Families.

If $S$ is a complex manifold or complex space (respectively, an algebraic variety or scheme over $\mathbb{C}$), a family of $L$-valued $G$-Higgs bundles on $X$, parametrised by $S$ is a pair $(\mathscr{P}, \Theta)$, where $\mathscr{P} \to S \times X$ is a holomorphic (respectively, algebraic) principal $G$-bundle and $\Theta$ is a section (over $S \times X$) of $p_2^* L$. Equivalently, we think of $\Theta$ as being a section (over $S$) of $p_{S*}(\text{ad}\mathscr{P} \otimes p_2^* L)$, where $p_{S*} = p_1$ stands for the canonical projection. In the case of a pointed base $(S, o)$, $o \in S$ we call a family $(\mathscr{P}, \Theta)$ a deformation of $(P, \theta)$ if it is equipped with an isomorphism $(\mathscr{P}_o, \Theta_o) \simeq (P, \theta)$. There is also an obvious notion of isomorphism of families (and deformations).

In the sequel we suppress the distinction between the algebraic and the analytic case unless there is a danger of confusion. For vector bundles on curves this can be justified by the GAGA principle. In higher dimensions (e.g., for families) one should keep in mind that algebraic $G$-bundles are assumed to be isotrivial, i.e., trivial in étale topology, rather than in the Zariski topology, see [Ser58]. In case when $G = GL_n(\mathbb{C})$ (which we exclude) or when $G$ is reductive and the base is a curve, one can indeed use the Zariski topology, by a result of Springer ([Ste65]). Moreover, by a theorem of Drinfeld and Simpson ([DS95]) $G$-bundles on $S \times X$ are locally trivial in the product of the étale topology on $S$ and Zariski topology on $X$.

The following elementary example of a family of Higgs bundles will be needed in what follows. Consider $S = H^0(X, \text{ad}P \otimes L)$, where $P \to X$ is a $G$-bundle. Then the trivial family of $G$-bundles $\mathscr{P} = p_X^* P \to S \times X$ can be augmented to a family of Higgs bundles. Indeed,

$$
(1) \quad p_{S*}(\text{ad}\mathscr{P} \otimes p_X^* L) = \mathcal{O}_S \otimes H^0(X, \text{ad}P \otimes L) = T_S,
$$

and we take $\Theta \in H^0(S, p_{S*}(\text{ad}\mathscr{P} \otimes p_X^* L)) = H^0(S, T_S)$ to be the tautological section of $T_S$, i.e., the Euler vector field on $S$. 

2.2. Moduli Spaces.

2.2.1. Principal Bundles. Given a $G$-bundle $\pi : P \to X$ and a closed algebraic subgroup $R \subset G$, we obtain an associated $G/R$-bundle $\pi_R : P \times^G (G/R) = P/R \to X$ and a principal $R$-bundle $P \to P/R$, which we also denote by $P_R$, to avoid confusion. This relies on Proposition 3 of [Ser58], stating that $G \to G/R$ is a principal $R$-bundle (in the étale topology). In the analytic category, for $R \subset G$ a closed complex Lie subgroup, this follows from a theorem of Chevalley about existence of local analytic sections of the canonical projection. We then identify the set of $R$-reductions of $P$ with the set of sections $\Gamma(X, P/R)$ in the usual way: a section $\sigma : X \to P/R$ gives rise to an $R$-reduction $\sigma^* P_R = X \times_{P/R} P \subset P$.

Following Ramanathan ([Ram75], Definition 1.1), we say that a principal bundle $P \to X$ is stable (respectively, semi-stable) if, for every maximal parabolic subgroup $H \subset G$, and every $H$-reduction $\sigma : X \to P/H$, $\deg \sigma^* T_{\pi_H} > 0$ (resp. $\deg \sigma^* T_{\pi_H} \geq 0$). Here

$$T_{\pi_H} = \ker(d\pi_H) \subset T_{P/H}$$

stands for the relative tangent bundle of the morphism $\pi_H$ and is nothing but $P_H(g/h) = (P \times g/h)/H$. Equivalently (Lemma 2.1, ibid.), $P$ is (semi)-stable if for any reduction $\sigma : X \to P/H$ to a parabolic subgroup $H \subset G$, and any dominant character $\chi$ on $H$, one has $\deg \sigma^* (E_H \times \mathbb{C}^\times) < 0$ (respectively, $\leq 0$).

Ramanathan constructed a (coarse) moduli space $\text{Bun}_G$ of $S$-equivalence classes of semi-stable $G$-bundles (or isomorphism classes of poly-stable $G$-bundles). By Theorem 4.3 ([Ram75]), $\text{Bun}_G$ is a normal Hausdorff analytic space, with connected components $\text{Bun}_{G,c}$, indexed by the topological type $c \in \pi_1(G)$ of the bundle:

$$\text{Bun}_G = \coprod_{c \in \pi_1(G)} \text{Bun}_{G,c}.$$ 

Ramanathan also constructed the moduli space in the algebraic category, identifying $\text{Bun}_{G,c}$ as the analytification of a normal projective algebraic variety ([Ram96a], [Ram96b]). For a Tannakian construction of the moduli space, see [BS02].

By [Ram75], Proposition 3.2, if $P$ is stable, then $H^0(X, \text{ad}P) = 0$ and $\text{Aut} P$ is finite. More generally, in the case of a reductive – rather than simple – group $G$ one has $H^0(X, \text{ad}P) = \text{Lie} Z(G)$, ibid.. Since, if $g \geq 2$, every topological $G$-bundle admits some structure of a stable holomorphic $G$-bundle (Remark 5.3, ibid.), we see by a Hirzebruch–Riemann–Roch calculation that

$$\dim \text{Bun}_G = \dim G \cdot (g - 1).$$

For a discussion of low-genus cases, see [Gro57], [Ati57], [Tu93], [Las98], [FMW98].

The infinitesimal deformations of a semi-stable $G$-bundle $P$ are parametrised by $H^1(X, \text{ad}P)$, and, by Luna’s étale slice theorem, the GIT quotient

$$H^1(X, \text{ad}P) \sslash \text{Aut}(P)$$

is isomorphic to an étale neighbourhood of $[P] \in \text{Bun}_G$. There is a natural inclusion $Z(G) \subset \text{Aut}(P)$, and, for $g \geq 2$, the smooth locus of the open subvariety of stable bundles $\text{Bun}_{G,c}^{\text{st}} \subset \text{Bun}_G$ consists of the regularly stable bundles, i.e., those which satisfy $\text{Aut} P = Z(G)$, see [Ram75] or [BH12b], Proposition 2.3.
2.2.2. Higgs Bundles. Ramanathan’s definition of (semi-)stability also makes sense for Higgs bundles, provided one considers only parabolic reductions which “preserve the Higgs field”, in the following sense. Given a closed subgroup $R \subset G$ and an $R$-reduction $\sigma : X \to P/R$, we have a natural projection

$$\Pi_\sigma : \text{ad}P \to \text{ad} \sigma^* P_R.$$ 

If $\theta$ is a Higgs field on $P$, we say that an $R$-reduction $\sigma$ of $P$ is a Higgs reduction of $(P, \theta)$ if $\theta \in \ker(\Pi_\sigma \otimes \text{id})$. If $\sigma$ is a Higgs reduction, then the Higgs field $\theta$ on the $G$-bundle $P$ induces a Higgs field on the $R$-bundle $\sigma^* P_R$.

In this way, the choice of Higgs field singles out a class of Higgs reductions among all $R$-reductions. This class can be conveniently described in the approach from [BGO11], Definition 3.5. The projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{t}$ induces a bundle homomorphism

$$\eta : \pi^* \text{ad} P = P_R(\mathfrak{g}) \to T_{\pi R} = P_R(\mathfrak{g}/\mathfrak{t}),$$

and thus $\eta$ gives rise to a section

$$\eta(\pi^* \theta) \in H^0(P/R, T_{\pi R} \otimes \pi^* K_X(-D)) \subset H^0(P/R, T_{\pi R} \otimes \Omega^1_{P/R}(-D)).$$

The vanishing locus of this section determines a closed subscheme in $P/R$, the scheme of Higgs reductions of $(P, \theta)$. A reduction $\sigma : X \to P/R$ is a Higgs reduction precisely when its image is contained in the scheme of Higgs reductions. This scheme turns out to play an important rôle in studying $\Omega^1_X$-valued $G$-Higgs bundles on smooth projective varieties, but can be, in general, rather singular.

We say that $(P, \theta)$ is (semi-)stable if, for any Higgs reduction $\sigma : X \to P/H$ to a maximal parabolic $H \subset G$, $\deg T_{\pi_R} > 0$ (respectively, $\deg T_{\pi_R} \geq 0$). Suitably modifying Ramanathan’s construction, one obtains a quasi-projective coarse moduli space $\text{Higgs}_{G,D}$ of $S$-equivalence classes of semi-stable Higgs bundles.

When $D = 0$, $\text{Higgs}_{G,0}$ is in fact a partial compactification of $\text{Bun}^{\text{sm}}_G$ ([Hit87a], [Hit87b]). Moreover, when $D = 0$, it is known from [DP12, Lemma 4.2] (see [GPO14] for a different proof) that the connected components of the moduli space are indexed by $\pi_1(G)$, i.e., by the topological type of the $G$-bundle, underlying a Higgs bundle. This is expected to hold for arbitrary $D > 0$ (whenever the moduli space is non-empty), but there does not seem to exist a published statement to this effect. However, for each $c \in \pi_1(G)$ there exists an irreducible connected component, $\text{Higgs}_{G,D,c}$, characterised by the fact that it contains generic cameral covers, see 4.2 and [DM96], definition 4.9. As we shall see §4, there is a morphism (the Hitchin map) from $\text{Higgs}_{G,D}$ to a vector space $\mathcal{B}$, and a strict subvariety, $\Delta \subset \mathcal{B}$, such that the connected components of the fibres of $\text{Higgs}_{G,D,|B-\Delta} \to \mathcal{B} - \Delta$ are isomorphic to abelian varieties. These connected components of the Hitchin fibre are contained in the respective connected components $\text{Higgs}_{G,D,c}$. It seems presently unknown whether there exist connected components of $\text{Higgs}_{G,D}$, lying entirely over the discriminant locus $\Delta \subset \mathcal{B}$, but the arguments in [DP12], Lemma 4.2 seem to indicate that this is not the case.

For a discussion of $S$-equivalence, Harder–Narasimhan and Jordan–Hölder reductions in the case $D = 0$ (but possibly over higher-dimensional base), see [DP05], [G010], [BGO11].

Étale locally near $[(P, \theta)]$ the moduli space $\text{Higgs}_{G,D,c}$ is isomorphic to

$$\mathcal{H}^1(\mathcal{X}'_{(P,\theta)}) \parallel \text{Aut}(P, \theta),$$
where $\mathcal{X}^{\bullet}(P,\theta)$ is the deformation complex (10). A stable pair $(P,\theta)$ represents a smooth (regular) point in $\operatorname{Higgs}^{st}_{G,D=0}$ precisely when it is regularly stable, i.e., $\operatorname{Aut}(P,\theta) = Z(G)$, see [BR94] Theorem 3.1 or [Fal93]. Then by a Hirzebruch–Riemann–Roch calculation (see section 3.2, equation (12) ) one obtains

$$(2) \quad \dim \operatorname{Higgs}^{st}_{G,D} = \dim G \deg K_X(D).$$

In particular, for $D = 0$ (which, with our assumptions, implies $g \geq 2$) we have $\dim \operatorname{Higgs}^{st}_{G,D=0} = 2 \dim \operatorname{Bun}_G$. The moduli space $\operatorname{Higgs}^{st}_{G,D=0}$ is normal, with orbifold singularities at worst.

One can also construct the moduli space in the algebraic category, following a version of either Simpson’s ([Sim94] §4) or Nitsure’s ([Nit91] §5) construction. The former deals with $D = 0$, while the latter with $G = GL_n$. We should mention here that C. Simpson’s notion of semi-stability is not always equivalent to Ramanathan’s, see [BGO11], Remark 4.6. For a purely algebraic, GIT-free construction of the moduli space in the case $D = 0$, see [Fal93]. If willing to work only with everywhere regular Higgs fields, one can construct a moduli space via the spectral correspondence ([Don95], §5.4). We recall that $\mathfrak{g}^{\text{reg}} / \mathfrak{g}^{\text{reg}} \subset \mathfrak{g}$ is of codimension three and, since we are considering only (line bundle-valued) Higgs bundles on curves, being everywhere regular is a reasonable restriction.

As with principal bundles, one may prefer to fix a linear representation of $G$, and work with vector bundles with extra structure. An $L$-valued Higgs vector bundle is a pair $(E,\theta)$, consisting of a vector bundle $E$ and a section $\theta \in H^0(X, End_E \otimes L)$. For such pairs one defines stability using slope: $(E,\theta)$ is (semi-)stable, if, for every subbundle $F \subset E$, satisfying $\theta(F) \subset F \otimes L$, the inequality $\mu(F) < \mu(E)$, respectively $\mu(F) \leq \mu(E)$, holds; see [Hit87a], [Nit91]. When $\theta = 0$, this reduces to Mumford’s original notion of stability of vector bundles. We do not delve into a detailed comparison of the different notions of stability and the different constructions of moduli spaces, mainly because we are going to work exclusively with generic (regularly stable) Higgs bundles. We do, however, discuss briefly the behaviour of stability under group homomorphisms and compare the stability of a Higgs bundle with the stability of its adjoint bundle. In the next theorem, we relax slightly our usual assumptions and allow reductive structure groups.

**Theorem 2.1.** Let $X$ be a compact Riemann surface and $D$ an effective divisor on $X$, such that $H^0(X,K_X(D)) \neq (0)$.

1. Let $V$ be a finite-dimensional $\mathbb{C}$-vector space, $G = GL(V)$, and $P$ a (holomorphic) principal $G$-bundle, so that $adP = End(P \times^G V) = P \times^G End(V)$. Then $(P,\theta)$ is a stable (semi-stable) Higgs bundle if and only if $(P \times^G V,\theta)$ is a stable (semi-stable) Higgs vector bundle.

2. Let $\phi : G \to H$ be a surjective homomorphism between reductive (complex) algebraic groups, such that $\ker \phi \subset Z(G)$. Let $(P,\theta)$ be an $L$-valued $G$-Higgs bundle and $(P',\theta')$ the $L$-valued $H$-Higgs bundle, induced by $\phi$. Then $(P',\theta')$ is stable (semi-stable) if and only if $(P,\theta)$ is so.

3. An $L$-valued $G$-Higgs bundle $(P,\theta)$ is semi-stable if and only if the adjoint Higgs (vector) bundle $(adP, ad\theta)$ is semi-stable. If $(adP, ad\theta)$ is stable, then so is $(P,\theta)$. If $(P,\theta)$ is stable, $(adP, ad\theta)$ need not be stable, but is polystable.
(4) A Higgs bundle \((P, \theta)\) is semi-stable if and only if for any representation \(\phi : G \rightarrow \text{Aut}(V)\), such that \(\phi(Z_0(G)) \subset Z(\text{Aut}(V))\) the associated Higgs vector bundle is semi-stable.

**Proof:** Maximal parabolic subgroups \(H \subset GL(V)\) consist of automorphisms of \(V\), preserving a flag \((0) \neq U \subsetneq V\), \(V \cong U \oplus V/U\), and \(\sigma : X \rightarrow P/H\) is a Higgs reduction precisely when \(\theta\) preserves the vector bundle \(P(U) = P \times_G U \subset P(V)\). Moreover, \(\sigma^*T_{\pi_H} = \text{Hom}(P(U), P(V/U))\), and since
\[
\deg \text{Hom}(P(U), P(V/U)) = \text{rk } P(U) \text{rk } P(V/U) \left(\mu(P(V/U)) - \mu(U)\right),
\]
statement (1) is proved. See also [DP05] Lemma 7, [Ram75] Lemma 3.3 and [HM04], Corollary 1.

Statement (2) is proved as Proposition 7.1 in [Ram75]. The key point is that there is a one-to-one correspondence between the parabolic reductions of \(P\) and those of \(P'\). Indeed, one sees that the diagram
\[
\begin{array}{c}
1 \longrightarrow \ker \phi \longrightarrow G \xrightarrow{\phi} H \longrightarrow 1 \\
| \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \\
1 \longrightarrow \ker \phi \longrightarrow R \longrightarrow R' \longrightarrow 1
\end{array}
\]
induces, if \(\ker \phi \subset Z(G)\), maps between the corresponding cohomology groups
\[
\begin{array}{c}
H^1(X, \ker \phi(O_X)) \longrightarrow H^1(X, G(O_X)) \longrightarrow H^1(X, H(O_X)) \longrightarrow H^2(X, \ker \phi(O_X)), \\
| \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \\
H^1(X, \ker \phi(O_X)) \longrightarrow H^1(X, R(O_X)) \longrightarrow H^1(X, R'(O_X)) \longrightarrow H^2(X, \ker \phi(O_X))
\end{array}
\]
see [Gro55], 5.7.11. Here \(G(O_X)\) denotes the sheaf of germs of holomorphic maps from \(X\) to \(G\). Since this correspondence preserves Higgs reductions, (2) follows.

The first part of statement (3) is proved as in the case \(D = 0\), for which we refer to [AB01], Lemma 4.7, [DP05] Proposition 12 and [BGO11], Lemma 4.3 (i). Notice that in these arguments one can use statement (1) to pass from \(G\) to \(G^{ad} = G/Z(G)\).

The second part of statement (3) is proved by modifying the corresponding argument for principal bundles (e.g., Proposition 2 in [HM04]). Indeed, let \(H \subset G\) be a maximal parabolic and \(\mathfrak{h} = \text{Lie } H\). The short exact sequence of vector spaces
\[
0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0
\]
is a sequence of \(H\)-modules via the adjoint representation \(H \hookrightarrow G \rightarrow \text{Aut}(\mathfrak{g})\). Twisting with it the \(H\)-bundle \(P_H = P \rightarrow P/H\) gives a sequence of vector bundles on \(P/H\), which, when pulled back by an \(H\)-reduction \(\sigma : X \rightarrow P/H\), gives
\[
0 \longrightarrow \sigma^*(P_H \times_H \mathfrak{h}) \longrightarrow \text{ad } P \longrightarrow \sigma^*T_{\pi_H} \longrightarrow 0.
\]
Suppose that \((\text{ad } P, \text{ad } \theta)\) is a stable Higgs (vector) bundle. If the above reduction \(\sigma\) is a Higgs reduction, then \(\sigma^*(P_H \times_H \mathfrak{h}) \subset \text{ad } P\) is preserved by \(\text{ad } \theta\) and hence, by stability, \(\deg \sigma^*(P_H \times_H \mathfrak{h}) < \deg \text{ad } P\). But \(\deg \text{ad } P = 0\), since \(G\) is reductive and a choice of \(G\)-invariant bilinear form on \(\mathfrak{g}\) gives an isomorphism \(\text{ad } P \simeq \text{ad } P^\vee\). Hence \(\deg \sigma^*T_{\pi_H} > 0\).

For the final part of (3), see [AB01], Theorem 4.8. Examples of stable Higgs bundles which are not ad-stable (but strictly ad-semistable) exist already for principal bundles, i.e., when \(\theta = 0\). Moreover, such bundles always exist if \(\dim Z(G) > 0\).
For part (4), see [AAB00], Lemma 1.3 and [BGO11], Lemma 4.3 (ii). □

2.3. Over the locus of regularly stable bundles. By Theorem II.6 in [Fal93], if $X$ is of genus $g \geq 2$, one has $\text{Bun}_{G}^{rs} \neq \emptyset$. Moreover, if $g \geq 3$ or if $g \geq 2$ but $G \neq \text{PGL}_2$, then the codimension of the complement of $\text{Bun}_{G}^{rs}$ in $\text{Bun}_{G}^{ad}$ is at least two. In this section we assume this to be the case, and consider the Zariski open $\text{Higgs}_{G,D}^{o} \subset \text{Higgs}_{G,D}$, consisting of classes of pairs $(P, \theta)$, for which $[P] \in \text{Bun}_{G}^{rs}$. We sketch a direct construction of this locus as a vector bundle over $\text{Bun}_{G}^{rs}$ and discuss the existence of Poincaré family. Most of the constructions in this section are a natural generalisation of [Bot95], §§1, 3.

The short exact sequence

$$
1 \longrightarrow \text{Z}(G) \longrightarrow G \longrightarrow G^{ad} \longrightarrow 1
$$

gives rise to an exact sequence of pointed sets

$$
H^{1}(X, Z(G)) \longrightarrow H^{1}(X, G(O_{X})) \longrightarrow H^{1}(X, G^{ad}(O_{X})) \longrightarrow H^{2}(X, Z(G))
$$

and to a morphism $\pi : \text{Bun}_{G} \to \text{Bun}_{G}^{ad}$. On closed points the latter is given by $\pi([P]) = [P']$, where $P' = P \times G^{ad} = P/Z(G)$. Let us fix a topological type $c \in \pi_{1}(G)$ and consider the restriction

$$
\pi_{c} : \text{Bun}_{G,c}^{rs} \longrightarrow \text{Bun}_{G^{ad},c}^{rs},
$$

where $c' \in \pi_{1}(G^{ad})$ is the image of $c$ under the injection $\pi_{1}(G) \hookrightarrow \pi_{1}(G^{ad})$ induced by the covering space $G \to G^{ad}$. Notice that $\pi$ respects both stability (by Theorem 2.1) and minimality of automorphisms. By [BH12a], Corollary 6.9 (see also [BBN06], Theorem 1.1 if $c' = 0$) there exists a universal $G^{ad}$-bundle $\mathcal{S}' \to \text{Bun}_{G^{ad},c}^{rs} \times X$. Its pullback $\pi_{c}^{*}\mathcal{S}'$, $\pi_{c}' = (\pi_{c}, 1)$, could be called an adjoint Poincaré bundle, since $\mathcal{S}'_{\{E\} \times X} \cong E/Z(G)$, for $[E] \in \text{Bun}_{G,c}^{rs}$.

To incorporate Higgs fields, consider the adjoint bundle of $\pi_{c}^{*}\mathcal{S}'$, i.e., the vector bundle $\pi_{c}^{*}\text{ad}\mathcal{S}' = \text{ad}(\pi_{c}^{*}\mathcal{S}')$ on $\text{Bun}_{G,c}^{rs} \times X$. By semi-continuity and Grauert’s theorem, the quasi-coherent sheaf $\mathcal{F} = p_{1*}(\text{ad}(\pi_{c}^{*}\mathcal{S}') \otimes p_{2}^{*}L)$ is locally free of finite rank, and its total space $\text{tot} \mathcal{F} = \text{Spec} \text{Sym}^{*} \mathcal{F}^{\vee}$ is a vector bundle on $\text{Bun}_{G,c}^{rs}$.

Considering, for any $[E] \in \text{Bun}_{G,c}$, the diagram

$$
\begin{array}{ccc}
\{E\} \times X & \longrightarrow & \text{Bun}_{G,c}^{rs} \times X \\
\downarrow p_{1} & & \downarrow p_{1} \\
\{E\} & \longrightarrow & \text{Bun}_{G,c}^{rs}
\end{array}
$$

and using that $\text{ad}E = \text{ad}(E/Z(G))$, we obtain a canonical identification between the fibre of $\mathcal{F}$ over $[E]$ and the vector space of $L$-valued Higgs fields on $E/Z(G)$:

$$
\mathcal{F}_{[E]} = j_{E}^{*}\mathcal{F} = j_{E}^{*}p_{1*}(\text{ad}\pi_{c}^{*}\mathcal{S}' \otimes p_{2}^{*}L) = p_{1*}(\text{ad}E \otimes p_{2}^{*}L) = H^{0}(X, \text{ad}E \otimes L).
$$

We thus have identified tot $\mathcal{F}$ with $\text{Higgs}_{G,D,c}^{o}$. Notice that if $D = 0$, we have $\mathcal{F} \cong \Omega_{\text{Bun}_{G,c}^{rs}}^{1}$.

Finally, we turn to the question of existence of Poincaré family of Higgs bundles on $\text{Higgs}_{G,D,c}^{o}$. By general arguments (Luna’s étale slice theorem), such a family always exists locally in the étale (or analytic) topology. Using some recent results of Biswas–Hoffmann and Donagi–Pantev, we can say a bit more about the global or Zariski-local situation as well.
Given a non-empty $\mathbb{Z}$-open $U \subset \text{Higgs}^o_{G,D,c}$, an adjoint Poincaré family of $L$-valued $G$-Higgs bundles on $U$ is a pair $(\mathcal{Z}, \Theta)$, where $\mathcal{Z} \to U \times X$ is an adjoint Poincaré family of $G$-bundles, i.e., a $G^{\text{ad}}$-bundle, satisfying $\mathcal{Z} \mid_{\{(E, \theta)\} \times X} \simeq E/\mathbb{Z}(G)$, while $\Theta \in H^0(U, p_U^* (\text{ad} \mathcal{Z} \otimes p_X^* L))$ is a family of $L$-valued $G$-Higgs bundles over $U$, such that $\Theta \mid_{\{(E, \theta)\}} = \theta$.

We now introduce an extra piece of notation, following [BH12a]. Consider the coroot, cocharacter and coweight lattices in $\mathfrak{t}$:

$$\text{coroot}_G \subset \text{cochar}_G \subset \text{coweight}_G.$$ 

These lattices can be identified with $\text{Hom}(G_m, T^{\text{sc}})$, $\text{Hom}(G_m, T)$ and $\text{Hom}(G_m, T^{\text{ad}})$, respectively, where

$$T^{\text{sc}} \to T \to T^{\text{ad}}$$

are maximal tori in $G^{\text{sc}}, G$ and $G^{\text{ad}}$, respectively. Correspondingly, the fundamental groups of $G$ and $G^{\text{ad}}$ are

$$\pi_1(G) = \frac{\text{cochar}_G}{\text{coroot}_G} \subset \pi_1(G^{\text{ad}}) = \frac{\text{coweight}_G}{\text{coroot}_G}.$$

As one can see ([BH12a], Lemma 6.2), any even, $W$-invariant, integral, symmetric bilinear form on $\text{coroot}_G$ extends to a symmetric, $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on $\pi_1(G^{\text{ad}})$. These extensions generate a cyclic group

$$\Psi(G^{\text{ad}}) \subset \text{Hom}(\pi_1(G^{\text{ad}}) \otimes \mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

cf. Table 1 in [BH12a]. Consider the subgroup

$$\Psi'(G) = \left\{ b \in \Psi(G^{\text{ad}}) \mid b|_{\pi_1(G) \times \pi_1(G)} = 0 \right\} \subset \Psi(G^{\text{ad}})$$

of bilinear forms, vanishing on $\pi_1(G)$. With every element $c \in \pi_1(G)$ we associate an evaluation map

$$\text{ev}_G^c : \Psi'(G) \to \text{Hom} \left( \frac{\pi_1(G^{\text{ad}})}{\pi_1(G)}, \mathbb{Q}/\mathbb{Z} \right), \quad b \mapsto b(c).$$

More generally, Biswas and Hoffmann define an analogue of $\Psi'(G)$ for an arbitrary reductive group $G$ (Definition 6.4, ibid.). They tie the obstruction of the existence of Poincaré family on $\text{Bun}_{G,c}^{rs}$ with the cokernel of $\text{ev}_G^c$ (Theorem 6.8, ibid.). More concretely, the moduli stack of regularly stable bundles (of type $c$) is a $\mathbb{Z}(G)$-banded gerbe over $\text{Bun}_{G,c}^{rs}$, and the order of its class can be expressed via the order of finite group coker $\text{ev}_G^c$. For a related result in the Higgs setting, see Lemma 4.2 in [DP12].

**Theorem 2.2.** Let $G$ be a simple complex algebraic group, $c \in \pi_1(G)$ and $X$ a compact Riemann surface of genus $g \geq 3$ (or $g \geq 2$ and $G \neq \text{PGL}_2$). Then there exists an adjoint $L$-valued Poincaré $G$-Higgs bundle $(\mathcal{Z}, \Theta)$ over $\text{Higgs}^o_{G,D,c} \times X = \text{tot } \mathcal{F} \times X$. There exists a non-empty open subscheme $U \subset \text{Bun}_{G,c}^{rs}$ and an $L$-valued Poincaré $G$-Higgs bundle $(\mathcal{P}, \Theta)$ over $\text{tot } \mathcal{F} \subset \text{Higgs}^o_{G,D,c}$ if and only if $\text{coker } \text{ev}_G^c = 0$. If such an open $U$ exists, then the Poincaré family extends to all of $\text{Higgs}^o_{G,D,c}$ if and only if $G = G^{\text{ad}}$.

**Proof:**
Consider the diagram

\[
\begin{array}{ccc}
\text{tot } \mathcal{F} \times X & \xrightarrow{p_F} & \text{Bun}_{G,c}^{rs} \times X \\
p_1 & & p_1 \\
\text{tot } \mathcal{F} & \xrightarrow{p_F} & \text{Bun}_{G,c}^{rs} \\
\end{array}
\]

where \( p_F : \text{tot } \mathcal{F} \rightarrow \text{Bun}_{G,c}^{rs} \) is the bundle projection, \( p_F([P, \theta]) = [P] \). We have already seen that \( \pi'_c \mathcal{P}' \) is an adjoint Poincaré bundle. Now we pull it further back and set \( Q = (\pi'_c \circ p'_F) \ast \mathcal{P}' \rightarrow \text{tot } \mathcal{F} \times X \). To construct the Higgs field on it, recall that, as with any vector bundle, \( p'_F \mathcal{F} \) carries a tautological section, \( \lambda \in H^0(\text{tot } \mathcal{F}, p'_F \mathcal{F}) \). But then

\[
p'_F p_1 \ast (\pi'_c \ast \text{ad } \mathcal{P}' \otimes p'_F L) \simeq p_1 \ast (\pi'_c \ast \text{ad } \mathcal{P}' \otimes p'_F L)
\]

and we get the family of \( G^{ad} \)-Higgs bundles \( (\mathcal{Q}, \Theta) = ((\pi'_c \circ p'_F) \ast \mathcal{P}', \lambda) \), which is an adjoint Poincaré family.

Since \( \text{ad } \mathcal{Q} = \text{ad } (\mathcal{Q}/Z(G)) \), whenever there exists a Poincaré \( G \)-bundle on \( \text{Bun}_{G,c}^{rs} \times X \) (or an open subscheme thereof), we can pull the latter by \( p'_F \) to a \( G \)-bundle \( \mathcal{P} \rightarrow \text{tot } \mathcal{F} \times X \) and obtain a Poincaré family \( (\mathcal{P}, \lambda) \). Then the conditions for the existence of a regularly stable Poincaré \( G \)-bundle are given in Corollary 6.7, 6.9 of [BH12a] and Remark 6.10, ibid.

\[\square\]

3. Poisson Geometry

In this section we discuss the symplectic and Poisson aspects of the geometry of Higgs moduli. As a means of motivation, we start with \( K_X \)-valued Higgs bundles and recall the construction of the symplectic form on \( \text{Higgs}_{G,0,c}^{rs} \). Next we discuss the deformation theory of Higgs bundles, following Biswas and Ramanan, and describe the corresponding deformation complex. A Poisson bivector on a variety determines a morphism from the cotangent to the tangent sheaf of the latter. In our context, tangent spaces are expressed as hypercohomology groups of complexes of sheaves. We review duality for hypercohomology in 3.3 and define the Poisson bivector in 3.4. Finally, we review E.Markman’s approach to proving the integrability of the Poisson structure.

3.1. Symplectic Structure. One of the fundamental results in Hitchin’s seminal papers [Hit87a] and [Hit87b] is the discovery that \( \text{Higgs}_{G,D=0}^{rs} \) is holomorphic symplectic and carries the structure of an ACIHS, to be discussed later. Recall that, by definition, a quasi-projective algebraic variety is holomorphic symplectic if its smooth (regular) locus carries a symplectic structure, which extends to any desingularisation. In this subsection we review briefly the construction of the symplectic structure on \( \text{Higgs}_{G,D=0}^{rs} \) and then return to the general case \( D \neq 0 \) in the next subsection.

As we saw in (2.3), \( \text{Higgs}_{G,0,c}^{rs} \) contains a Zariski open subset \( \text{Higgs}_{G,0,c}^{rs} \) which can be identified with the total space of the vector bundle \( \mathcal{F} = p_1 \ast (\pi'_c \ast \mathcal{P}' \otimes p'_F K_X) \) on \( \text{Bun}_{G,c}^{rs} \). Furthermore, one can indentify \( \mathcal{F} \) with \( T \text{Bun}_{G,c}^{rs} \), the cotangent bundle to the smooth locus of the (coarse) moduli space of semi-stable \( G \)-bundles of topological type \( c \), and cotangent bundles carry a canonical symplectic structure.
Pointwise, at the class of a pair \((P, \theta)\), this identification can be done as follows. By Luna’s étale slice theorem, \(\text{Bun}_{G,c}^{rs}\) is, local-analytically near a regularly stable bundle \(P\), isomorphic to
\[
H^1(X, \text{ad}P) \parallel \text{Aut}(P) = H^1(X, \text{ad}P) \parallel Z(G) = H^1(X, \text{ad}P)
\]
and \(T_{[P]}\text{Bun}_{G,c}^{rs} = H^1(X, \text{ad}P)\). Next, the stability of \(P\) implies stability of the Higgs pair \((P, \theta)\) for any \(\theta \in H^0(X, \text{ad}P \otimes K_X)\). However, a choice of symmetric invariant bilinear form \(\text{Tr}\) on \(\mathfrak{g}\) (e.g., the Killing form) determines an isomorphism \(\text{ad}P = \text{ad}P^v\), which, when combined with Serre duality, gives an isomorphism \(H^0(X, \text{ad}P \otimes K_X) = H^1(X, \text{ad}P)^v\). Hence the (class of the) pair \((P, \theta)\) determines a point in \(T^v\text{Bun}_{G,c}^{rs}\).

The complement \(\text{Higgs}_{G,D=0}^{rs} \setminus T^v\text{Bun}_{G}^{sm}^{rs}\) is non-empty: there exist stable Higgs pairs with unstable underlying bundle. A concrete example is furnished by the uniformising (or Toda) Higgs bundle, see [Hit87a], Example 1.5, or by any Higgs bundle in the image of the Hitchin section ([Hit92]). As shown by these very examples, there are smooth points in this locus, i.e.,
\[
(\text{Higgs}_{G,D=0}^{rs} \setminus T^v\text{Bun}_{G}^{sm}^{rs})^{rs} \neq \emptyset,
\]
and we would like to extend the symplectic structure to the rest of \(\text{Higgs}_{G,D=0}^{rs}\).

By a variant of Schlessinger’s deformation theory developed in [BR94], the space of infinitesimal deformations of a Higgs bundle \((P, \theta)\) is \(H^1(\mathcal{E}_{(P, \theta)}^\bullet)\), where \(\mathcal{E}_{(P, \theta)}^\bullet\) is the Biswas–Ramanan complex
\[
\mathcal{E}_{(P, \theta)}^0 = \text{ad}P \xrightarrow{\text{ad} \theta} \text{ad}P \otimes K_X = \mathcal{E}_{(P, \theta)}^1.
\]
In fact, this is a very special case of Theorem 2.3, ibid. and we shall discuss the general case in subsection 3.2. If \([([P, \theta])] \in \text{Higgs}_{G,D=0}^{rs}\) then \(H^1(\mathcal{E}_{(P, \theta)}^\bullet) = T_{[P, \theta]}\text{Higgs}_{G,D=0}^{rs}\).

Being a (shifted) cone, the complex \(\mathcal{E}_{(P, \theta)}^\bullet\) is an extension of \(\text{ad}P\) by \(\text{ad}P \otimes K_X[-1]\) and the long exact sequence of hypercohomology gives a short exact sequence
\[
(0) \longrightarrow \ker h^0(\text{ad} \theta) \longrightarrow H^1(\mathcal{E}_{(P, \theta)}^\bullet) \xrightarrow{\text{pr}} \ker h^1(\text{ad} \theta) \longrightarrow (0).
\]
Here \(h^i(\text{ad} \theta) : H^i(\text{ad}P) \to H^i(\text{ad}P \otimes K_X)\) are the natural maps induced by \(\text{ad} \theta\). If \(P\) happens to be stable, equation (5) reduces to
\[
(0) \longrightarrow H^0(X, \text{ad}P \otimes K_X) \longrightarrow H^1(\mathcal{E}_{(P, \theta)}^\bullet) \xrightarrow{\text{pr}} H^1(X, \text{ad}P) \longrightarrow (0).
\]
Next, the combination of \(\text{Tr}\) and cup product pairing
\[
H^1(\mathcal{E}_{(P, \theta)}^\bullet) \otimes H^1(\mathcal{E}_{(P, \theta)}^\bullet) \to H^1(K_X) \cong \mathbb{C}
\]
induces a skew-symmetric bilinear form \(\omega_{(P, \theta)} \in \Lambda^2 \left( H^1(\mathcal{E}_{(P, \theta)}^\bullet)^v \right)\). As shown in [BR94], Theorem 4.3, this pairing gives rise to a symplectic form on \(\text{Higgs}_{G,D=0}^{rs}\) which coincides with the canonical symplectic form \(\omega_{\text{can}} = -d\lambda\) on \(T^v\text{Bun}_{G}^{sm}^{rs}\).

In terms of the deformation complex, the Liouville 1-form \(\lambda_{(P, \theta)} \in H^1(\mathcal{E}_{(P, \theta)}^\bullet)^v\) is given by
\[
\lambda_{(P, \theta)}(v) = \text{Tr} \text{pr}(v) \cap \theta.
\]
The symplectic form \( \omega \) determines (and is determined by) a map
\[
\omega : T_{Higgs,G,D=0} \longrightarrow T_{Higgs,G,D=0}^\vee.
\]
At \((P,\theta)\) this corresponds to the linear map \( H^1(\mathcal{E}_{(P,\theta)}^\bullet) \rightarrow H^1(\mathcal{E}_{(P,\theta)}^\bullet)^\vee \) determined by Grothendieck–Serre duality for hypercohomology. We are going to elaborate on this in the next subsection (see (13), (15)) where we address the case \( D \neq 0 \).

Finally, we recall how to express the symplectic form in Dolbeault terms. For that, one considers the (global sections of the total complex of the) Dolbeault resolution of the complex \( \mathcal{E}_{(P,\theta)}^\bullet \). The deformation theory of a stable Higgs pair is formal ([Sim92], Lemma 2.2). The Hermite–Yang–Mills metric on \((P,\theta)\) provides an embedding \( H^1(\mathcal{E}_{(P,\theta)}^\bullet) \subset A^1(adP) \), whose image consists of harmonic representatives of hypercohomology. In terms of the type decomposition
\[
A^1(adP) \simeq A^0(adP \otimes K_X) \oplus A^0,1(adP),
\]
the symplectic form is given by restricting the pairing
\[
\omega((\eta',\eta''),(\xi',\xi'')) = \int_X \text{Tr} \left( (\eta' \wedge \xi'' - \xi' \wedge \eta'') + (\eta' + \eta'') \wedge (\xi' + \xi'') \right)
\]
to the harmonic representatives of \( H^1(\mathcal{E}^\bullet) \).

This brings us back to Hitchin’s original motivation: if \( K \subset G \) is a maximal compact subgroup, and \( Q \subset P \) a \( K \)-reduction, then the space of holomorphic structures on \( Q \times K G \) is an affine space modelled on \( A^0,1(adQ_C) \). This torsor is canonically trivialised and identified with \( A^0,1(adP) \) by the holomorphic structure of \( P \). Then \( A^1(adP) \) can be thought of as the total space of the (weak) cotangent bundle to the space of holomorphic structures on \( Q \times K G \), and Hitchin’s original construction of \( Higgs_{G,D=0} \) was a kind of infinite-dimensional hyperkaehler Marsden–Weinstein reduction of the latter. The holomorphic symplectic form is thus a reduction of the canonical (weak) symplectic form on the product of a vector space with its (weak) dual.

3.2. Deformation theory.

3.2.1. One-parameter analytic deformations. The symplectic structure on the moduli space of \( K_X \)-valued \( G \)-Higgs bundles was defined in terms of the complex (4) which controls the infinitesimal deformations of the \( K_X \)-valued Higgs pair. For \( K_X(D) \)-valued Higgs bundles a similar complex exists. In fact, Biswas and Ramanan ([BR94]) have given a uniform description of the deformation theory of Higgs bundles with coefficients in an arbitrary vector bundle. Before turning to their theorem, which is concerned with infinitesimal deformations, we discuss the global case in the analytic category.

Let \( \rho : G \to \text{Aut} F \) be a linear representation of an algebraic group \( G \), \( P \) a principal \( G \)-bundle, \( P_F = P \times_G F \) the corresponding associated vector bundle and \( \phi \in \Gamma(X,\rho P) \). Consider now the analytifications of these objects. Let \( \Delta = \{ \epsilon : |\epsilon| < 1 \} \subset \mathbb{C} \) be the unit disk, and let \( \mathcal{P} \to X \times \Delta \) be a deformation of \( P \), i.e., a holomorphic \( G \)-bundle together with an isomorphism \( \mathcal{P}|_{X \times \{0\}} = P \). Let the section \( \Phi \in \Gamma(X \times \Delta, \mathcal{P} \times_G F) \) be a deformation of \( \phi \), i.e, \( \Phi|_{X \times \{0\}} = \phi \) under the above isomorphism of bundles. The section \( \Phi \) corresponds to a holomorphic
map \( \sigma_\Phi : \mathcal{P} \to F \), which is \( G \)-equivariant, i.e., \( R^*_g \sigma_\Phi = \rho(g^{-1}) \circ \sigma_\Phi \). Similarly, \( \phi \) corresponds to a \( G \)-equivariant map \( \sigma_\Phi : P \to F \), and \( \sigma_\Phi = \sigma_\Phi|_P \).

Fix an “admissible” cover \( \mathcal{U} = \{ U_i \} \) of \( X \), i.e., one for which the family \( \mathcal{P} \) is trivial over \( U_i \times \Delta \). Let us also fix trivialisations, i.e., \( G \)-bundle isomorphisms \( \Psi_i : \mathcal{P}_i \times \Delta \cong P_{U_i} \times \Delta = p^*_i P_{U_i} \), such that \( \Psi_i|_{\varepsilon = 0} = id \). Then the composition \( \Psi_{ij} = \Psi_i \circ \Psi_{ij}^{-1} \in \text{Aut}(P_{U_{ij}} \times \Delta) \) corresponds to a \( G \)-equivariant map \( \psi_{ij} : P_{U_{ij}} \times \Delta \to G \), defined by

\[
\Psi_{ij}(p, \varepsilon) = (p, \varepsilon) \cdot \psi_{ij}(p, \varepsilon) = (p \cdot \psi_{ij}(p, \varepsilon), \varepsilon).
\]

The \( G \)-equivariance of \( \psi_{ij} \) is with respect to the conjugation action, i.e., \( R^*_g \psi_{ij} = Ad(g^{-1}) \circ \psi_{ij} \), or, pointwise, \( \psi_{ij}(p \cdot g, \varepsilon) = g^{-1} \psi_{ij}(p, \varepsilon) g \).

Taking into account that \( \psi_{ij}(p, 0) = e \in G \), we see that the derivative

\[
\psi_{ij} = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \psi_{ij} = d\psi_{ij} \left( \frac{d}{d\varepsilon} \right) \bigg|_{\varepsilon=0} : P_{U_{ij}} \to \mathfrak{g}
\]

satisfies \( R^*_g \psi_{ij} = ad(g^{-1}) \circ \psi_{ij} \), and hence determines a section \( s_{ij} \in \Gamma(U_{ij}, \text{ad} P_{U_{ij}}) \).

The cocycle condition \( \Psi_{ij} \circ \Psi_{jk} \circ \Psi_{ki} = id \in \text{Aut}(P_{U_{ij}}) \) translates to

\[
\psi_{ij} \psi_{jk} \psi_{ki} = e : P_{U_{ijk}} \times \Delta \to G,
\]

which, in turn, gives \( s_{ij} + s_{jk} + s_{ki} = 0 \), i.e., \( \mathfrak{g} = (s_{ij}) \in \check{Z}^1_U(\text{ad} P) \).

It is then easy to see that the holomorphic maps

\[
\tau_i = \sigma_\Phi \circ \Psi_i^{-1} : P_{U_i} \times \Delta \to F
\]

are \( G \)-equivariant and satisfy \( \tau_j = \tau_i \circ \Psi_{ij} \), or, equivalently, \( \tau_j = \rho(\psi_{ji}) \circ \tau_i \). Differentiating this condition at \( \varepsilon = 0 \) gives that

\[
t_i = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \tau_i : P_{U_i} \to F
\]

satisfy \( t_j - t_i = \rho(\psi_{ji})(\phi) \), i.e. \( \delta^0 \phi = \rho(\phi)(\phi) \), where \( \delta^0 \) denotes the Čech differential.

The infinitesimal data associated to the deformation \( (\mathcal{P}, \Phi) \) is encoded in the pair \( (\mathfrak{g}, \mathfrak{g}) \), and it is not hard to trace how this data changes when we pass to an equivalent deformation.

### 3.2.2. The deformation functor

We turn now to infinitesimal deformations. Let \( G \), \( P \) and \( \rho \) be as before, and let \( V \) be a vector bundle on \( X \). Denote by \( \mathcal{F}(P, \rho) : \text{Art}_C \to \text{Sets} \) the formal deformation functor of the pair \( (P, \rho) \), where now \( \phi \in \Gamma(X, \rho P \otimes V) \). This is a functor from the category of Artin local \( C \)-algebras to the category of sets, which assigns to an algebra \( A \) the set of iso-classes of deformations of \( (P, \rho) \), parametrised by \( X \times \text{Spec} A \). In particular, \( \mathcal{F}(P, \rho)(\mathbb{C}[\varepsilon]/\varepsilon^2) \) is the space of infinitesimal deformation of the pair \( (P, \rho) \).

**Theorem 3.1** (Theorem 2.3, [BR94]). **There exists a canonical bijection**

\[
\mathcal{F}(P, \rho)(\mathbb{C}[\varepsilon]/\varepsilon^2) = \mathbb{H}^1(\mathcal{H}^{\mathbf{c}}_{(P, \rho)}),
\]

where \( \mathcal{H}^{\mathbf{c}}_{(P, \rho)} \) is the complex

\[
\mathcal{H}^{\mathbf{c}}_{(P, \rho)} = \text{adP} \xrightarrow{e(\phi)} \rho P \otimes V = \mathcal{H}^{\mathbf{1}}_{(P, \rho)},
\]

and \( e(\phi)(s) = \rho(s)(\phi) \).
For simplicity, we have not included $\rho$ or $V$ in the notation of the complex.

**Sketch of proof:** The theorem is proved by a direct infinitesimal calculation: if 
\[ \{U_i = \text{Spec } A_i\}_{i} \] is an affine cover of $X$, then $U_i[\varepsilon] = \text{Spec } (A_i \otimes \mathbb{C}[\varepsilon]/\varepsilon^2)$ is an affine cover of $X[\varepsilon] = X \times \mathbb{C}[\varepsilon]/\varepsilon^2$, and we can replace $\mathcal{X}^\bullet_{(P, \rho)}$ by its Čech resolution. Then elements of $H^1(\mathcal{X}^\bullet_{(P, \rho)})$ are identified with equivalence classes of pairs
\[ (s, \underline{t}) = ((s_{ij})_{ij}, (t_i)_{i}) \in \oplus_{ij} \operatorname{ad}P(U_{ij}) \bigoplus \oplus_i (\rho P \otimes V)(U_i), \]
which on double overlaps satisfy the two conditions
\begin{enumerate}
    \item $\delta^1 s = 0$
    \item $\phi(s_{ij}) = (\delta^0 \underline{t})_{ij}$.
\end{enumerate}

The restrictions $P_i = P|_{U_i}$ of $P$ determine trivial families $\mathcal{P}_i = p^*_i P_i$ on $U_i[\varepsilon]$. The first condition states that the automorphisms $(1 + s_{ij} \varepsilon) \in \text{Aut}(\mathcal{P}_i)$ glue these trivial families into a $G$-bundle $\mathcal{P}$ on $X[\varepsilon]$. The second condition guarantees that the local sections $(\phi + t_i \varepsilon)_i$ glue into a section of $\rho \mathcal{P} \otimes V$. This determines the map $H^1(\mathcal{X}^\bullet_{(P, \rho)}) \to \mathcal{F}(\mathbb{C}[\varepsilon]/\varepsilon^2)$. The map in the opposite direction is obtained by observing that any deformation of $P_i$ over $U_i[\varepsilon]$ is trivial. Hence, given a deformation $(\mathcal{P}, \Phi)$ of $(P, \rho)$ parametrised by $\text{Spec } \mathbb{C}[\varepsilon]/\varepsilon^2$, we can fix trivialisations and obtain the corresponding gluing data $\underline{s} = (s_{ij})$ and $\underline{t} = (t_i)$, where and $t_i \varepsilon = \Phi_i - p^* \phi|_{U_i[\varepsilon]}$. \hfill $\Box$

Eventually, we are interested in applying this theorem in the case $V = K_X(D)$ and $\rho(s) = -\text{ads}$. Then the complex $\mathcal{X}^\bullet_{(P, \theta)}$ takes the form
\begin{equation}
(10) \quad \xymatrix{ \text{ad}P \ar[r]^-{\text{ad} \theta} & \text{ad}P \otimes K_X(D), }
\end{equation}
and fits in the extension
\begin{equation}
(11) \quad \xymatrix{ 0 \ar[r] & \text{ad}P \otimes K_X(D)[-1] \ar[r] & \mathcal{X}^\bullet_{(P, \theta)} \ar[r] & \text{ad}P \ar[r] & 0. }
\end{equation}

We can use this exact sequence to calculate the dimension (2) of $\text{Higgs}_{G, D}$. Indeed, taking Euler characteristics gives
\[ -\chi(\text{ad}P \otimes K_X(D)) - \chi(\mathcal{X}^\bullet_{(P, \theta)}) + \chi(\text{ad}P) = 0, \]
and hence, by Hirzebruch–Riemann–Roch,
\begin{equation}
(12) \quad \dim \text{Higgs}_{G, D} = -\chi(\mathcal{X}^\bullet_{(P, \theta)}) = \dim \deg K_X(D),
\end{equation}
which is also nothing but $2 \dim \text{Bun}_G + \dim \deg D$.

By Luna’s étale slice theorem one can identify an étale (or analytic) neighbourhood of $(P, \theta)$ as
\[ H^1(\mathcal{X}^\bullet_{(P, \theta)}) \parallel \text{Aut}(P, \theta), \]
which for regularly stable pairs reduces to
\[ H^1(\mathcal{X}^\bullet_{(P, \theta)} \simeq T_{(P, \theta)} \text{Higgs}^r_{G, D, \varepsilon}. \]
3.3. **Digression on duality.** To handle the Poisson structure on $\text{Higgs}_{G,D,c}$ we need a small amount of duality theory (which was already used implicitly in (7)). Given a length-$(n + 1)$ complex of locally free sheaves
\[
(F^\bullet, d_\bullet) = \left( F^0 \xrightarrow{d_0} F^1 \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} F^n \right), \quad F^\bullet = \bigoplus_{k \in \mathbb{Z}} F^k[-k],
\]
let us denote by $\hat{F}^\bullet$ its (naive) dual complex, i.e., the Hom-complex (graded as usual) between the complex $F^\bullet$ and the complex $\mathcal{O}_X$ (concentrated in degree zero):
\[
\hat{F}^\bullet = \text{Hom}_{D^b(X)}(F, \mathcal{O}_X) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(F^{-k}, \mathcal{O}_X)[k].
\]
This complex is concentrated in degrees $(-n)$ to 0 and has differentials which are the duals of the respective differentials of $F^\bullet$:
\[
\delta_{-k} = d_{k-1}^\vee : \quad \hat{F}^{-k} = \text{Hom}(F^k, \mathcal{O}_X) \to \text{Hom}(F^{k-1}, \mathcal{O}_X) = \hat{F}^{k+1}.
\]
To place $\hat{F}^\bullet$ in non-negative degree we shift it by $n$ positions to the right and denote the new complex by $\hat{F}^\bullet$, i.e., $\hat{F}^\bullet = \hat{F}^\bullet[-n]$. Then Grothendieck–Serre duality (Theorem 3.12, \cite{Huy06}), which in this case is just Serre duality for hypercohomology, tells us that for all $i \in \mathbb{Z}$
\[
\mathbb{H}^i(\hat{F}^\bullet)^\vee = \mathbb{H}^{-i}(\hat{F}^\bullet \otimes K_X[1]) = \mathbb{H}^{n+1-i}(\hat{F}^\bullet \otimes K_X).
\]
The duality can be made explicit as follows. The contractions $F^k \otimes \text{Hom}(F^k, \mathcal{O}_X) \to \mathcal{O}_X$ give rise to a linear map
\[
\left( F^\bullet \otimes (\hat{F}^\bullet \otimes K_X) \right)_n = \bigoplus_{k=0}^n F^k \otimes \text{Hom}(F^k, \mathcal{O}_X) \otimes K_X \to K_X
\]
and hence to a morphism of complexes
\[
F^\bullet \otimes (\hat{F}^\bullet \otimes K_X) \longrightarrow K_X[-n].
\]
This is indeed a morphism of complexes, since the tensor product complex, being the total complex of a double complex, has differential obtained from the tensor product of the differentials of the two complexes, with alternating signs. The morphism (14) induces a map on cohomology
\[
\mathbb{H}^{n+1}(F^\bullet \otimes (\hat{F}^\bullet \otimes K_X)) \longrightarrow \mathbb{H}^{n+1}(K_X[-n]) = H^1(X, K_X) \simeq \mathbb{C}.
\]
Then the duality pairing corresponding to (13) is the composition of this map with the cup product pairing
\[
\mathbb{H}^i(F^\bullet) \otimes \mathbb{H}^{n+1-i}(\hat{F}^\bullet \otimes K_X) \longrightarrow \mathbb{H}^{n+1}(F^\bullet \otimes (\hat{F}^\bullet \otimes K_X)).
\]
The case of interest for us is the complex (10) which has length two ($n = 2$) and hence
\[
\mathbb{H}^i(\mathcal{X}_{(P, \theta)}^\bullet)^\vee = \mathbb{H}^{2-i}(\mathcal{X}_{(P, \theta)}^\bullet \otimes K_X)
\]
where
\[
\mathcal{X}_{(P, \theta)}^\bullet \otimes K_X : \quad \text{ad} P^\vee \otimes \mathcal{O}_X(-D) \xrightarrow{(\text{ad} \theta)^\vee} \text{ad} P^\vee \otimes K_X.
\]
Using the chosen Ad-invariant symmetric bilinear form on $\mathfrak{g}$, we identify $\mathcal{X}^\bullet_{(P, \theta)} \otimes K_X$ with the complex

$$\text{ad} P \otimes O_X(-D) \xrightarrow{-\text{ad} \theta} \text{ad} P \otimes K_X.$$

Remark 3.1. Naturally, changing the sign of the differential in the complex (9) gives an isomorphic complex, but we, nonetheless, make some comments on the sign choices involved. We use the (standard) convention for the 0-th Čech differential $(\delta^0_{ij})_{ij} = t_j - t_i$ and the (fairly standard) differential $e(\phi) + (-1)^{i+1} \delta^i : \hat{C}_{ij} \to \hat{C}_{i+1,j} \oplus \hat{C}_{i,j+1}$ for the Čech double complex. This forces our definition of $e(\phi)$ in Theorem 3.1 to differ by sign from the one in [BR94]. On the other hand, in equation (10) we have used a sign, corresponding to $-\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$, for the following reason. In view of duality, it would have been more natural if Higgs fields were defined using the co-adjoint representation, i.e., as sections of $(\text{ad} P)^\vee \otimes K_X$, as in [BR94], §4. In that case the deformation complex would satisfy $\mathcal{X}^\bullet_{(P, \theta)} \otimes K_X = \mathcal{X}^\bullet_{(P, 0)}$, without the need to choose an invariant symmetric bilinear form. Since we stick, however, to the standard definition of a Higgs field, and $\text{Tr}$ identifies $(\text{ad} \theta)^\vee$ with $-\text{ad} \theta$, the above sign choice is forced unto us.

**Proposition 3.1.** Let $(P, \theta)$ be a $K_X(D)$-valued $G$-Higgs bundle and $\mathfrak{U} = \{(U_i)\}$ an acyclic cover of $X$. Let $\alpha = ([\sigma \check{u}])$ and $\beta = ([\sigma \check{z}])$ be hypercohomology classes in $H^1(\mathcal{X}^\bullet_{(P, \theta)})$ and $H^1(\mathcal{X}^\bullet_{(P, \theta)} \otimes K_X)$, respectively, with Čech representatives

$$(\check{u}, \check{t}) \in \check{C}^1_{\mathfrak{U}}(\text{ad} P) \oplus \check{C}^0_{\mathfrak{U}}(\text{ad} P \otimes K_X(D))$$

and

$$(\check{z}, \check{r}) \in \check{C}^1_{\mathfrak{U}}(\text{ad} P(-D)) \oplus \check{C}^0_{\mathfrak{U}}(\text{ad} P \otimes K_X).$$

Then the duality pairing

$$H^1(\mathcal{X}^\bullet_{(P, 0)}) \otimes H^1(\mathcal{X}^\bullet_{(P, \theta)} \otimes K_X) \to H^1(X, K_X)$$

corresponding to (15) maps $\alpha \otimes \beta$ to $[\check{u}] \in H^1(X, K_X)$, where $\check{u} = (c_{ij})$ is given by

$$c_{ij} = \text{Tr}(t_i, \sigma_{ij}) - \text{Tr}(s_{ij}, \tau_j) \in K_X(U_{ij}).$$

**Proof:** In view of (15) and (14), to construct the explicit pairing we need to have an explicit description of cup product in hypercohomology. To simplify notation, set $F^\bullet = \mathcal{X}^\bullet_{(P, \theta)}$ and $G^\bullet = \mathcal{X}^\bullet_{(P, 0)} \otimes K_X$. Moreover, to avoid confusion, let us be explicit about tensor products of complexes, e.g., write $\text{tot}^\bullet(F^\bullet \otimes G^\bullet)$, rather than just $F^\bullet \otimes G^\bullet$. Finally, let us write $t\mathcal{C}^\bullet(F^\bullet)$ for the (total) Čech complex of $F^\bullet$ with respect to the cover $\mathfrak{U}$. This is the complex of vector spaces, whose $k$-th term is

$$t\mathcal{C}^k(F^\bullet) = \check{C}^k_{\mathfrak{U}}(F^0) \oplus \check{C}^{k-1}_{\mathfrak{U}}(F^1) = \oplus_{i_0 \ldots i_k} F^0(U_{i_0 \ldots i_k}) \bigoplus \oplus_{i_0 \ldots i_{k-1}} F^1(U_{i_0 \ldots i_{k-1}})$$

and whose $k$-th differential is

$$t\mathcal{C}^k(F^\bullet) \rightarrow t\mathcal{C}^{k+1}(F^\bullet).$$

Hypercohomology is computed as

$$H^n(F^\bullet) = H^n(t\mathcal{C}^\bullet(F^\bullet)),$$

and by Künneth formula we have

$$H^n(F^\bullet) \otimes H^m(G^\bullet) = H^n(t\mathcal{C}^\bullet(F^\bullet)) \otimes H^m(t\mathcal{C}^\bullet(G^\bullet)) = H^{n+m}(\text{tot}^\bullet(t\mathcal{C}^\bullet(F^\bullet) \otimes t\mathcal{C}^\bullet(G^\bullet))).$$

The cup product map

$$H^n(F^\bullet) \otimes H^m(G^\bullet) \to H^{n+m}(\text{tot}^\bullet(F^\bullet \otimes G^\bullet))$$
is induced by a morphism of complexes of vector spaces
\[ \text{tot}^\bullet (tC^\bullet (F) \otimes tC^\bullet (G)) \longrightarrow tC^\bullet (\text{tot} (F \otimes G)), \]
where, if \( \deg \alpha = n, \deg \beta = m \), one sets
\[ (\alpha \cup \beta)_{i_0...i_p} = \sum_{r=0}^p (-1)^{r(m-(p-r))} \alpha_{i_0...i_r} \otimes \beta_{i_r...i_p} \in \check{C}_d^\bullet (\text{tot}^{n+m-p} (F^\bullet \otimes G^\bullet)). \]

For a discussion of the sign one can consult Deninger ([Den95]), Deligne ([Del73]) or A.de Jong’s unpublished notes on algebraic de Rham cohomology.

If \( n = m = 1 \) and we consider an element in the image of the Künneth map, i.e., \( \alpha \otimes \beta \in tC^1(F^\bullet) \otimes tC^1(G^\bullet) \), then
\[ \alpha \cup \beta \in \check{C}^2_\Delta((F^\bullet \otimes G^\bullet)^0) \bigoplus \check{C}^1_\Delta((F^\bullet \otimes G^\bullet)^1) \bigoplus \check{C}^0_\Delta((F^\bullet \otimes G^\bullet)^2) \]
has a component of Čech degree 1 equal to
\[ (\alpha \cup \beta)_{i_0i_1} = \alpha_{i_0} \otimes \beta_{i_0i_1} - \alpha_{i_0i_1} \otimes \beta_{i_1} = t_{i_0} \otimes \sigma_{i_0i_1} - s_{i_0i_1} \tau_{i_1} \in \check{C}^1_\Delta((F^\bullet \otimes G^\bullet)^1) \]
Projecting onto this component and applying trace gives the claimed formula. \( \square \)

3.4. Poisson structure.

3.4.1. Generalities. A Higgs bundle is a decorated principal bundle: a pair, consisting of a principal bundle \( P \) and a section of the vector bundle \( \rho \rho P \otimes V \). From this viewpoint, it is only natural to ask whether the symplectic structure on \( \text{Higgs}_{G,D=0} \) persists when one varies the representation \( \rho \) or the coefficient bundle \( V \).

The symplectic structure was constructed from two ingredients: the identification
\[ \mathcal{F}_{(P,\theta)}(\mathbb{C}[\varepsilon]/\varepsilon^2) = \mathbb{H}^1(\mathcal{E}_{(P,\theta)}^\bullet) = T_{(P,\theta)} \text{Higgs}_{G,D=0}^{\text{ps}}, \]
due to Theorem 3.1, and the natural skew pairing (7), which gives the isomorphism (8). While the first ingredient makes sense in general (after replacing \( \mathcal{E}_{(P,\theta)}^\bullet \) with \( \mathcal{X}_{(P,\theta)}^\bullet \)), the second one relies substantially on the fact that the coefficient bundle is the dualising sheaf \( K_X \), and thus we cannot expect the moduli space to be symplectic for arbitrary \( V \). However, if \( V \simeq K_X(D), D > 0 \), then it turns out that an analogue of the dual of (8) still exists. While it may fail to be everywhere of maximal rank, it still satisfies the appropriate integrability condition. More precisely, Bottacin [Bot95] (for \( G = SL_n(\mathbb{C}) \) and \( G = GL_n(\mathbb{C}) \)) and Markman ([Mar94], [Mar00]) showed that, whenever nonempty, \( \text{Higgs}_{G,D,c} \) is a holomorphic Poisson variety.

We recall the definition of Poisson structure below but refer to [AG88], [Wei83] and [DM96] for details.

Let \( M \) be a smooth analytic (or quasi-projective algebraic) variety and \( \Pi \in H^0(M, \Lambda^2 T_M) \) a bivector (field). It determines a \( \mathbb{C} \)-linear skew-symmetric pairing \( \mathcal{O}_M \otimes \mathcal{O}_M \rightarrow \mathcal{O}_M \)
\[ (f, g) \longmapsto \{ f, g \} := (df \wedge dg) \Pi, \]
which is a \( \mathbb{C} \)-derivation in each entry. Hence to a (local) function \( f \in \mathcal{O}_M(U) \), \( U \subset M \), we can associate a \( \mathbb{C} \)-derivation of the \( \mathbb{C} \)-algebra \( \mathcal{O}_M(U) \), called its Hamiltonian vector field
\[ X_f = \{ f, \mu \} = df \wedge \mu \in T_M(U) = \text{Der}_{\mathbb{C}}(\mathcal{O}_M)(U). \]
We say that $\Pi$ is a Poisson structure if this pairing endows $\mathcal{O}_M$ with the structure of a sheaf of Lie algebras. The bracket $\{f, g\}$ is then called the Poisson bracket of $f$ and $g$. The variety $M$ is said to be Poisson if it admits a Poisson structure.

The Jacobi identity is equivalent to the requirement that the “adjoint representation” $f \mapsto \{f, \}$ be not only $\mathbb{C}$-linear, but also a homomorphism $\mathcal{O}_M \to \text{Der}_\mathbb{C}(\mathcal{O}_M)$ of sheaves of Lie algebras, i.e., $[X_f, X_g] = X_{\{f, g\}}$. This can also be phrased as the vanishing of the Schouten bracket of $\Pi$ with itself.

On a Poisson variety $(M, \Pi)$ one has an obvious sheaf homomorphism $\Psi : T_M \to T_M$, namely, $\alpha \mapsto \alpha \Pi$. It gives rise to a stratification of $M$ by submanifolds $M_k$, $k$-even, such that $\text{rk} \, \Psi|_{M_k} = k$. Then $(M_k, \Pi|_{M_k})$ is Poisson. The strata are further foliated ([Wei83]), local-analytically, by the $k$-dimensional integral leaves $S$ of the distribution $\Psi|_{M_k} \cap T_{M_k}$. If $S \subset M_k$ is a symplectic leaf, then $\Pi|_S$ is a symplectic structure on it. The leaves can also be identified as the level sets of the Casimir functions, i.e., the functions $f \in H^0(M, \mathcal{O}_M)$ with $X_f = 0$.

One of the best-known examples of Poisson structure is the Kostant–Kirillov Poisson structure. If $\mathcal{G}$ is a Lie group with Lie algebra $\mathfrak{g}$, then its (linear) dual $\mathfrak{g}^\vee$ carries a Poisson bracket

$$\{f, g\}_\alpha = \alpha([df, dg]).$$

The group $\mathcal{G}$ acts on $\mathfrak{g}^\vee$ via the coadjoint representation, and the coadjoint orbits are the symplectic leaves of the Kostant–Kirillov Poisson structure. The Casimir functions are the invariants $\mathbb{C}[\mathfrak{g}]^\mathcal{G} \subset H^0(\mathfrak{g}^\vee, \mathcal{O}_{\mathfrak{g}^\vee})$.

3.4.2. A bivector on $Higgs_{G,D}$ The moduli space $Higgs_{G,D,c}$ carries a canonical bivector $\Pi$ which can be described entirely in terms of homological algebra and is a natural candidate for a Poisson structure. We discuss it below, roughly along the lines of [Bot95] §3, [Mar94] §7.2 and [DM96], §5.4. Since these references deal exclusively with the case of $G = \text{GL}_n(\mathbb{C})$, we spell certain points in more detail, but see also [Mar00].

The canonical inclusion $s : \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ induces a morphism of complexes

$$I_s : \mathcal{X}^\bullet_{(P, \theta)} \otimes K_X \xrightarrow{\cdot s} \mathcal{X}^\bullet_{(P, \theta)}(-D) \xrightarrow{(s, 1)} \mathcal{X}^\bullet_{(P, \theta)}.$$  \hspace{1cm} (17)

Assuming that $(P, \theta)$ is regularly stable, the induced map on $\mathbb{H}^2$ gives

$$\Psi_{(P, \theta)} : \mathbb{H}^1(I_s) \xrightarrow{\cdot s} \mathcal{X}^\bullet_{(P, \theta)} \xrightarrow{(s, 1)} \mathcal{X}^\bullet_{(P, \theta)}.$$  \hspace{1cm} (18)

which is easily seen to be skew-adjoint. This map corresponds to an element

$$\Pi_{(P, \theta)} \in \Lambda^2 \left( \mathbb{H}^1(\mathcal{X}^\bullet_{(P, \theta)}) \right) \subset \mathbb{H}^1(\mathcal{X}^\bullet_{(P, \theta)}) \otimes \mathbb{C},$$  \hspace{1cm} (19)

which is our candidate for a Poisson bivector.

Let us give an explicit global description of the Poisson structure over the locus $Higgs_{G,D,c}^0 \subset Higgs_{G,D,c}^r$ (see 2.3), assuming $g \geq 2$. This locus is a vector bundle $\text{tot} \, \mathcal{F} = Higgs_{G,D,c}^0 \xrightarrow{p_F^F} \text{Bun}_{G,c}^r$, and so the isomorphism $p_F^F \mathcal{F}^\vee = \Omega^1_{\text{Higgs}/\text{Bun}}$ gives

$$\mathcal{F} = Higgs_{G,D,c}^0 \xrightarrow{p_F^F} \text{Bun}_{G,c}^r.$$  \hspace{1cm} (20)
for the relative tangent sequence on $\text{tot} \ F$. Since $F = p_1^*(\text{ad}(\pi_c^* \mathcal{P}')) \otimes p_*^2L)$, upon restriction to a point $(P, \theta) \in \text{tot} \ F$ this sequence becomes

\begin{equation}
(0) \rightarrow H^0(X, \text{ad}P \otimes L) \rightarrow \mathbb{H}^1(\mathcal{K}^\bullet_{(P, \theta)}) \rightarrow H^1(X, \text{ad}P) \rightarrow (0).
\end{equation}

This is nothing but the degree-$1$ piece of the cohomology sequence of $(11)$. If $D = 0$, this is the sequence $(6)$.

Consider again the diagram $(3)$ and recall that we have an adjoint Poincaré family of $G^{ad}$-Higgs bundles $(\mathcal{Q}, \Theta)$ on $\text{Higgs}^c_{G, D, c} \times X$. Here $\Theta \in H^0(\text{tot} \ F, p^*_2 \mathcal{F})$ is the tautological section and $\mathcal{Q} = (\pi_c' \circ p'_P)^* \mathcal{P}'$. We then have at our disposal the universal Biswas–Ramanan complex on $\text{Higgs}^c_{G, D, c} \times X$

\[ (22) \mathcal{K}^\bullet_{(\mathcal{Q}, \Theta)} : \text{ad} \mathcal{Q} \longrightarrow \text{ad} \mathcal{Q} \otimes p_*^2 \mathcal{L} \]

with its cone sequence

\[ (23) (0) \longrightarrow \text{ad} \mathcal{Q} \otimes p_*^2 \mathcal{L} [-1] \longrightarrow \mathcal{K}^\bullet_{(\mathcal{Q}, \Theta)} \longrightarrow \text{ad} \mathcal{Q} \longrightarrow (0). \]

The first (hyper-)derived image of $p_{1*}$ applied to $(22)$ gives the tangent sheaf

\[ T_{\text{Higgs}^c_{G, D, c}} = R^1 p_{1*} \mathcal{K}^\bullet_{(\mathcal{Q}, \Theta)}, \]

while applying $p_{1*}$ to $(23)$ gives the relative tangent sequence $(20)$. Similarly,

\[ T^\vee_{\text{Higgs}^c_{G, D, c}} = R^1 p_{1*} \left( \mathcal{K}^\bullet_{(\mathcal{Q}, \Theta)} \otimes p_*^2 \mathcal{K}_X \right). \]

Then we have the relative analogue of the map $(17)$,

\[ I_s : \mathcal{K}^\bullet_{(\mathcal{Q}, \Theta)} \otimes p_*^2 K_X \longrightarrow \mathcal{K}^\bullet_{(\mathcal{Q}, \Theta)}, \]

and

\[ (24) \Psi = R^1 p_{1*}(I_s) : T^\vee_{\text{Higgs}^c_{G, D, c}} \longrightarrow T_{\text{Higgs}^c_{G, D, c}} \]

determines a bivector $\Pi \in \Lambda^2(T_{\text{Higgs}^c_{G, D, c}})$ which restricts to $(19)$ at each pair $(P, \theta)$ for which $P$ is regularly stable.

### 3.4.3. Integrability of the Poisson bivector.

The bivector $\Psi$ does indeed determine a Poisson structure, but this is not easy to prove. In [Bot95] §4.6 (see also §4.2) this was done by a direct cocycle calculation rooted in the fact that the total space of the dual of a Lie algebroid carries a canonical Poisson structure.

Markman employed in ([Mar94], [Mar00]) a different strategy. He started by considering $G$-bundles with framing along the divisor $D$. The group of framings (level group) acts on this space and the action lifts to its cotangent bundle. Over a certain Zariski open subset of the latter the action is free and the quotient can be identified with an open subset of $\text{Higgs}^c_{G, D, c}$. By general properties of Marsden–Weinstein reduction, the canonical symplectic structure on the cotangent bundle to the moduli space of framed bundles descends to a Poisson structure on the reduced space. Markman then verified that this Poisson structure coincides with the one induced by the general hypercohomological argument above. Consequently, $\{\Pi, \Pi\} = 0$ everywhere. We review Markman’s construction in the next subsection.

### 3.5. Framed Bundles and Markman’s construction.
3.5.1. Jet Schemes. If $D \subset X$ is a (possibly non-reduced) divisor and $P \to X$ is a principal $G$-bundle, then data of framing of $P$ along $D$ is encoded in points of certain jet schemes of $G$. In order to make the exposition self-contained we recall here the definition and the explicit description of jet schemes of affine varieties.

To any given scheme $\mathcal{Y}$ (of finite type over $\mathbb{C}$) one can associate, for any $n \in \mathbb{N}$, the functor

$$\text{Hom}_{\text{Sch}}(\_ \times \text{Spec } \mathbb{C}[t]/t^{n+1}, \mathcal{Y}): \text{Sch}^\text{op}_{\mathbb{C}} \to \text{Sets}.$$ 

This functor is representable (EM09, §2) by a scheme $\mathcal{Y}_n$ (of finite type, over $\mathbb{C}$), known as the $n$-th jet scheme of $\mathcal{Y}$. It is easy to see that $\mathcal{Y}_0 = \mathcal{Y}$, that $\mathcal{Y}_1 = \text{tot } T_\mathcal{Y} = \text{Spec } \Omega^1_\mathcal{Y}$ and that there are natural maps $\mathcal{Y}_n \to \mathcal{Y}_{n-1}$. Notice that by definition, for any $\mathbb{C}$-algebra $A$ one has

$$\text{Hom}_{\text{Sch}}(\text{Spec } A[t]/t^{n+1}, \mathcal{Y}) = \text{Hom}_{\text{Sch}}(\text{Spec } A, \mathcal{Y}_n),$$

and in particular, $\mathcal{Y}_n(\mathbb{C})$ is identified with $\text{Hom}_{\text{Sch}}(\text{Spec } \mathbb{C}[t]/t^{n+1}, \mathcal{Y})$, the set of $n$-jets of paths into $\mathcal{Y}$.

We describe now $\mathcal{Y}_n$ for an embedded affine variety $\mathcal{Y} = \text{Spec } R \subset A^N$, where $R = \mathbb{C}[x_1, \ldots, x_N]/\mathfrak{a}$. For that we shall exhibit a $\mathbb{C}$-algebra $R_n$ together with an isomorphism

$$\text{Hom}_{\text{alg}}(R, A[t]/t^{n+1}) = \text{Hom}_{\text{alg}}(R_n, A),$$

functorial in $A$, and set $\mathcal{Y}_n = \text{Spec } R_n$. We consider first $\mathcal{Y} = A^N$, and claim that $A_n^N = A^{N(n+1)}$. Indeed, $\mathbb{C}$-algebra homomorphisms $\text{Spec } \mathbb{C}[x_1, \ldots, x_N] \to A$ are in bijection with matrices $\text{Mat}_{N \times (n+1)}(A)$, since a homomorphism $\phi$ is specified by $\phi(x_i) = \sum_{k=0}^{n} M_{ik} t^k$, $M_{ik} \in A$. But such matrices are also in bijection with algebra homomorphisms $\mathbb{C}[y_{10}, \ldots, y_{1n}, \ldots, y_{N0}, \ldots, y_{NN}] \to A$ via $M \mapsto \psi$, $\psi(y_{ik}) = M_{ik}$. Suppose next that $\mathfrak{n} = (f_1, \ldots, f_p) \neq (0)$. The homomorphisms $R \to A[t]/t^{n+1}$ are precisely those homomorphisms $\phi: \mathbb{C}[x_1, \ldots, x_N] \to A[t]/t^{n+1}$, which factor through the quotient, i.e., $f_i(\phi(x_1), \ldots, \phi(x_N)) = 0 \in A[t]/t^{n+1}$, $1 \leq i \leq p$. Expanding the latter gives $\sum_{k=0}^{n} g_{ik}((M_{ij})^{(k)} t^k = 0$, for some polynomials $g_{ik} \in \mathbb{C}[y_{10}, \ldots, y_{NN}]$. We then set $R_n = \mathbb{C}[y_{10}, \ldots, y_{1n}, \ldots, y_{NN}]/(g_{ik})$ and define $\psi: R_n \to A$ by $\psi(y_{ik}) = M_{ik}$ as before. The jet scheme $\mathcal{Y}_n \subset A^{N(n+1)} = A_n^N$ is cut out by the $g_{ik}$.

For non-affine $\mathcal{Y}$, the jet scheme is constructed by gluing the jet schemes of affine patches. As far as general properties of jet schemes go, we only mention that the natural maps $\mathcal{Y}_n \to \mathcal{Y}_{n-1}$ are $A_{\dim \mathcal{Y}}$-bundles, and in particular, the non-singularity of $\mathcal{Y} = \mathcal{Y}_0$ implies the non-singularity of $\mathcal{Y}_n$ for all $n \in \mathbb{N}$. Moreover, the assignment $\mathcal{Y} \mapsto \mathcal{Y}_n$ is functorial in $\mathcal{Y}$ and hence gives rise to an endofunctor of the category of schemes of finite type (over $\mathbb{C}$, or any algebraically closed field).

We are interested in jet schemes of (affine) algebraic groups. As it is easy to see, $(GL_N)_n = GL_N(\mathbb{C}[t]/t^{n+1})$, and by the above description, an affine embedding $G \subset GL_N(\mathbb{C})$ determines an embedding of the corresponding jet scheme $G_n \subset GL_N(\mathbb{C}[t]/t^{n+1})$.

3.5.2. Framed bundles. Let $D = \sum_{i=1}^{s} n_i q_i$ be a (sufficiently positive) divisor on $X$, $G$ the $\mathbb{C}$-scheme, corresponding to our simple group $G = G(\mathbb{C})$, and let $\widetilde{G}_D$ stand for the (group) scheme of maps from $D$ to $G$. We recall that it is defined as the $\mathbb{C}$-scheme representing the functor

$$\text{Hom}_{\text{Sch}}(\_ \times D, G): \text{Sch}^\text{op}_{\mathbb{C}} \to \text{Sets},$$

so, for any C-algebra, A, we have
\[
\text{Hom}_{\text{Sch}}(\text{Spec } A, \tilde{G}_D) = \text{Hom}_{\text{Sch}}(\text{Spec } A \times D, G) = \text{Hom}_{\text{Alg}}(H^0(O_G), A \otimes H^0(O_D)).
\]
We then see that the group of C-points is a product of the respective jet schemes of G, i.e.,
\[
\tilde{G}_D := \tilde{G}_D(\mathbb{C}) = \text{Hom}_{\text{Alg}}(H^0(O_G), H^0(O_D)) = \prod_{i=1}^s G_{n_i-1}.
\]
The level group is defined to be the quotient \( G_D = \tilde{G}_D/Z(G) \), where the centre \( Z(G) \) is embedded diagonally.

A framed bundle, sometimes also called a bundle with level-D structure is a pair \((P, \eta)\), where \( P \to X \) is a principal \( G \)-bundle and \( \eta \) is a trivialisation of \( P \) at \( D \), i.e., an isomorphism \( \eta : P\vert_D \cong D \times G \) of \( G \)-bundles. We define
\[
\text{Isom}((P_1, \eta_1), (P_2, \eta_2)) \subset \text{Isom}(P_1, P_2)
\]
as the set of isomorphisms \( f : P_1 \cong P_2 \), satisfying \( \eta_1 = \eta_2 \circ f_D \). There is a natural action of \( G_D \) on the set of isomorphism classes of framed bundles, namely,
\[
g \cdot [(P, \eta)] = [(P, \tilde{g} \circ \eta)], \quad \text{where } \tilde{g} \in \tilde{G}_D \text{ is a lift of } g \in G_D, \text{ and } \text{Stab}[(P, \eta)] = \text{Im}(\text{Aut}P \to \text{Aut}P_D)^{op}/Z(\tilde{G}).
\]
We denote by \( P = P(G, D, c) \) the smooth locus of the moduli space of isomorphism classes of stable framed \( G \)-bundles of topological type \( c \). We make some comments on the rôle of \( D \) in the definition of stability, and refer to [Mar94], [Mar00] and [Ses82] for more details.

Let \( \delta := \deg D \). A vector bundle \( E \to X \) is called \( \delta \)-stable, if for any proper sub-bundle \( F \subset E \), one has \( \frac{\deg F}{\deg E} < \frac{\deg E - \delta}{\deg E} \), and one defines similarly \( \delta \)-semistability. A framed vector bundle \((E, \eta)\) is (semi)stable, if \( E \) is \( \delta \)-(semi) stable. It is clear that if \( E \) is stable, then it is \( \delta \)-stable for any \( \delta \geq 0 \), and if \( E \) is semi-stable, it is \( \delta \)-stable for any \( \delta > 0 \). Seshadri in [Ses82] (part 4) constructed a projective coarse moduli space of semi-stable framed coherent sheaves. A framed \( G \)-bundle \((P, \eta)\) shall be called (semi-)stable, if \( \text{ad}P \) is \( \delta \)-(semi-)stable. By cocycle calculation it is not hard to see that the tangent space \( T_{P, (P, \eta)} = H^1(X, \text{ad}P \otimes O(-D)) \) of the infinitesimal deformations of \( P \) which preserve the framing, i.e., vanish along \( D \). Serre duality implies that \( \text{tot}(T^\vee P) \subset \text{tot}(\text{Isom}((P, \eta), (P, \theta))) \). Such triples, consisting of a Higgs bundle and a framing of the underlying \( G \)-bundle can be called framed Higgs bundles.

3.5.3. Symplectic reduction. The action of the level group on \( P \) lifts naturally to the \( T^\vee P \), and the lifted action is given by \( g \cdot [(P, \eta, \theta)] = [(P, \tilde{g} \circ \eta, \theta)] \). After considering the homomorphism \( \text{Aut}P \to \text{Aut}(\text{ad}P) \), Markman’s Lemma 6.7 ([Mar94], applied to \( \text{ad}P \) implies the stabilisers of the \( G_D \) action on \( P \) and \( T^\vee P \), are, respectively
\[
\text{Stab}[(P, \eta)] = \text{Aut}(P)^{op}/Z(\tilde{G}) \subset G_D
\]
and
\[
\text{Stab}[(P, \eta, \theta)] = \text{Aut}(P, \theta)^{op}/Z(\tilde{G}) \subset G_D.
\]
Correspondingly, the action of \( G_D \) on the loci \( P^\circ = \{[(P, \eta)] \mid [P] \in \text{Bun}_{G,c}^\circ \} \subset P \) and \( T^\vee P \) is free, and these are principal \( G_D \)-bundles over \( \text{Bun}_{G,c}^\circ \) and \( \text{Higgs}_{G,D,c}^\circ \), respectively.

Lifted actions on cotangent bundles give rise to a very special geometry, as we now recall, following [AG88] and [DM96]. Not only is the manifold \( M := \text{tot} T^\vee P \)
symplectic, but, moreover, the action $G_D \times M \to M$ is Poisson, and there exists a canonical $G_D$-equivariant moment map $\mu: M \to \mathfrak{g}_D^\vee := \text{Lie}G_D^\vee$. We explain these properties briefly.

Let $a: \mathfrak{g}_D \to H^0(M, T_M)$ denote the infinitesimal action map, assigning to each $\xi \in \mathfrak{g}_D$ the corresponding "fundamental vector field". The action of $G_D$ on $M$ is Hamiltonian if $\text{Im}(a)$ consists of Hamiltonian vector fields: for any $\xi \in \mathfrak{g}_D$, there is a global function $f \in H^0(\mathcal{O}_M)$, such that $a(\xi) = X_f$. The action is Poisson, if it is Hamiltonian and if the hamiltonian functions for the different $\xi \in \mathfrak{g}_D$ can be chosen compatibly, i.e., if there exists a Lie algebra homomorphism $H: \mathfrak{g}_D \to H^0(M, \mathcal{O}_M)$ and $a$ factors through it, giving $a(\xi) = X_{H(\xi)}$.

Dually, $H$ can be thought of as a moment map, i.e., a morphism $\mu: M \to \mathfrak{g}_D^\vee$. It is Poisson and $G_D$-equivariant. For the case that we consider -- a lifted action of a (connected) group on a cotangent bundle of a manifold -- there is a canonical moment map, [AG88]. Namely, if $(u, \theta) \in T_{\mathcal{P}, u}^\vee$, $\mu(u, \theta) = (d\rho_u)_\theta^\vee(\theta)$, where $\rho_u : G_D \to \mathcal{P}$ is the orbit map, $\rho_u(g) = g \cdot u$.

We can identify $\mathfrak{g}_D^\vee = \mathfrak{g}_D^\vee$ with $\mathfrak{g}_D^\vee \otimes H^0(K_X(D)|_D)$ via

$$H^0(\mathcal{O}_D) \otimes H^0(K_X(D)|_D) \longrightarrow H^0(K(X(D)|_D) \xrightarrow{\text{Res}} H^1(K_X) \simeq \mathbb{C},$$

where $\text{Res}$ is the (first) connecting homomorphism of the long exact cohomology sequence, associated to

$$(0) \longrightarrow K_X \longrightarrow K_X(D) \longrightarrow K_X(D)|_D \longrightarrow (0).$$

Then by ([Mar94], Proposition 6.12, [Mar00]), the moment map is explicated as

$$\mu([([P, \eta, \theta])])(A) = \text{Res}(A(\eta \circ \theta|_D \circ \eta^{-1})).$$

If the map $\mu$ were submersive and the quotient $M/G_D$ were to exist, it would carry a canonical Marsden–Weinstein Poisson structure, whose symplectic leaf through $m \in M$ would be

$$\mu^{-1}(\mathcal{O}_{\mu(m)})/G_D \simeq \mu^{-1}(\mu(m))/\text{Stab}(\mu(m)),$$

where $\mathcal{O}_{\mu(m)} \subset \mathfrak{g}_D^\vee$ is the coadjoint orbit through $\mu(m)$. Due to the presence of fixed points, however, this happens only on $M^o = (T^\vee \mathcal{P})^o$, and we have

$$\frac{(T^\vee \mathcal{P})^o}{\mathcal{P}} \xrightarrow{\mu} \frac{\mathfrak{g}_D^\vee}{G_D}. $$

Here $\mathfrak{g}_D^\vee \sslash G_D$ denotes, as usual, the GIT quotient, whose $\mathbb{C}$-points correspond to closures of orbits. We shall discuss and refine this picture in the next section.

The Poisson structure, obtained by reduction from $(T^\vee \mathcal{P})^o$ turns out to coincide with the one defined in terms of hypercohomology in (24), see [Mar94], Corollary 7.15., and the integrability of the former implies the integrability of the latter, which is defined on a bigger space.

4. Camera covers and the Hitchin map

In this section we review integrable system aspects of Higgs moduli. We start be recalling some Lie-theoretic background, namely, the adjoint quotient morphism
and its “global analogue” – the Hitchin map. Then we turn to \(L\)-valued cameral covers and discuss very briefly generalised Prym varieties and abelianisation.

4.1. Adjoint quotient and the Hitchin map.

4.1.1. The Adjoint Quotient. The group \(G\) acts naturally on the coordinate ring of \(\mathfrak{g}\) (via the coadjoint action) and by a theorem of Chevalley ([Che55]) the subalgebra of invariants \(\mathbb{C}[\mathfrak{g}]^G \subset \mathbb{C}[\mathfrak{g}]\) is isomorphic to a free algebra on \(l\) generators, which are homogeneous of degree \(d_j\), \(1 \leq j \leq l\). While the choice of generators is largely non-unique, the set of degrees \(\{d_j\}\) is determined by \(G\) and in fact has a topological significance – the Poincaré polynomial of \(G^{ad}\) is \(p_{G^{ad}}(t) = \prod_j(1 + t^{2d_i}-1)\). The inclusion \(\mathbb{C}[\mathfrak{g}]^G \subset \mathbb{C}[\mathfrak{g}]\) corresponds to a morphism (of affine varieties) \(\chi : \mathfrak{g} \to \mathfrak{g} / G = \text{Spec} \mathbb{C}[\mathfrak{g}]^G\). The \(\mathbb{C}\)-points of the GIT quotient \(\mathfrak{g} / G\) are closures of \(G\)-orbits in \(\mathfrak{g}\). A specific choice of \(I_j \in \text{Sym}^{d_j} \mathfrak{g}^*\) with \(\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[I_1, \ldots, I_l]\) determines an isomorphism \(\mathbb{C}^l \simeq \mathfrak{g} / G\) and identifies \(\chi\) with the map \(x \mapsto (I_1(x), \ldots I_l(x))\).

Without the choice of \(\{I_j\}\), \(\mathfrak{g} / G\) is not a vector space, but just an affine “cone” – a variety with a \(\mathbb{C}^\times\)-action and a single fixed point. The homothety action of \(\mathbb{C}^\times\) on \(\mathfrak{g}\) descends to an action on \(\mathfrak{g} / G \simeq \mathbb{C}^l\):

\[
t \cdot (b_1, \ldots, b_l) = (t^{d_1}b_1, \ldots, t^{d_l}b_l)
\]

and the morphism \(\chi\) is \(\mathbb{C}^\times\)-equivariant. It is also \(G\)-invariant by construction.

Having chosen Cartan and Borel subgroups \(T \subset B \subset G\), we can use the embedding \(W \subset GL(l)\) to gain another interpretation of \(\chi\). Indeed, the inclusion \(\mathbb{C}[t]^W \subset \mathbb{C}[t]\) gives rise to a quotient map \(t \to t/W\), a finite flat morphism of affine varieties. There exists a non-empty (Zariski-) open subset of \(t\) – the complement to the union of the root hyperplanes – on which the quotient map is an étale Galois cover with group \(W\). By a theorem Chevalley, the inclusion \(t \subset \mathfrak{g}\) determines an isomorphism \(\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[t]^W\), and, consequently, \(t/W \simeq \mathfrak{g} / G\). We can think then of the adjoint quotient map as a morphism between affine varieties (in fact, affine spaces) \(\mathfrak{g} \to t/W\).

Altogether, writing \(x^{ss}\) for the semi-simple part of \(x \in \mathfrak{g}\), we have the following descriptions of the morphism \(\chi\):

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\chi} & \mathfrak{g} / G \\
\downarrow{\chi} & \simeq & \downarrow{t/W} \\
x & \xrightarrow{G \cdot x} & (G \cdot x^{ss}) \cap t \\
& \simeq & (L(x))
\end{array}
\]

4.1.2. The Hitchin map. The adjoint quotient \(\chi\) induces a morphism of (total spaces of cone) bundles \(\text{ad}\mathcal{P} = P \times_{\text{Ad}} \mathfrak{g} \to P \times (\mathfrak{g} / G)\) which can be twisted with any \(\mathbb{C}^\times\)-torsor, since \(\chi\) is \(\mathbb{C}^\times\)-equivariant. In particular, twisting with \(L^X\) we get a morphism of affine varieties

\[
\chi_{X, \mathcal{P}} : H^0(X, \text{ad}\mathcal{P} \otimes L) \longrightarrow \mathcal{B} = H^0(X, t \otimes \mathcal{C} \cdot L/W) \simeq H^0(X, \bigoplus L^{d_i}).
\]

By the same token, if \(T\) is a complex manifold and \(\mathcal{P}\) a holomorphic principal \(G\)-bundle on \(T \times X\), \(\chi\) induces a morphism

\[
\chi_{T \times X, \mathcal{P}} : H^0(T \times X, \text{ad}\mathcal{P} \otimes p_2^* L) \longrightarrow H^0(T \times X, (t \otimes \mathcal{C} \cdot p_2^* L) / W) = H^0(T, \mathcal{O}_T) \otimes \mathcal{B}.
\]
Hence $\chi$ makes sense for families and gives rise to a morphism, $h$, from $\text{Higgs}_{G,D}$ to $\mathcal{B}$ by assigning to a $p^*_L L$-valued Higgs pair $(\mathcal{P}, \Theta)$ on $T \times X$ the section $\chi_{T \times X, \mathcal{P}}(\Theta) \in H^0(T, \mathcal{O}_T) \otimes \mathcal{B}$, which is nothing but a map $T \to \mathcal{B}$. This morphism $h$ is the Hitchin map, given on $\mathbb{C}$-points by $h([P, \theta]) = \chi_{X,P}(\theta)$, or, slightly informally, by $h([P, \theta]) = (I_1(\theta), \ldots, I_l(\theta))$, once the generators $\{I_k\}$ are fixed. In what follows, we are going to suppress all subscripts of $\chi$.

The restrictions of $h$ to the respective connected components are known to be proper morphisms

$$h_c : \text{Higgs}_{G,D,c} \to \mathcal{B}$$

which endow $\text{Higgs}_{G,D,c}$ with the structure of a Poisson ACIHS in the sense of [DM96], Definition 2.9. This means that $h_c$ is a proper flat morphism, which, away from a closed subvariety $\Delta \subseteq \mathcal{B}$ has Lagrangian fibres, isomorphic to abelian varieties. The Lagrangian condition in the algebraic Poisson context is understood generically: $Y \subset \text{Higgs}_{G,D,c}$ is Lagrangian, if there is a symplectic leaf $S \subset \text{Higgs}_{G,D,c}$, such that $Y \subset S$ and $Y \cap S \subset S$ is Lagrangian. For the proof we refer to [Mar94], [Mar00] and [Bot95], as well as [Don93], [Fal93], [Sco98] and [DG02], extending the work in [Hit87a] and [Hit87b]. The abelian varieties in question arise as generalised Prym varieties, associated to cameral or spectral covers, see also (4.2) and (4.3).

The foliation of $\text{Higgs}_{G,D,c}$ by closures of symplectic leaves is determined by the quotient $\mathcal{B}/B_0$, where

$$B_0 : = H^0(X, (t \otimes \mathbb{C} L/W) (-D)) \simeq H^0 \left( X, \bigoplus_{i=1}^l L^{d_i}(-D) \right) \subset \mathcal{B}.$$

We have the following diagram, which is a variant of [Mar94], Proposition 8.8:

and every $h_c$-fibre contains a unique leaf of maximal rank.

4.2. $L$-valued Cameral Covers. The Hitchin base $\mathcal{B}$ itself has modular interpretation – it parametrises cameral covers of $X$, as we will review now. The germ of the idea is already to be seen in Chevalley’s theorem: a $G$-conjugacy class in $\mathfrak{g}$ can be identified with a $W$-conjugacy class in $\mathfrak{t}$. Twisting the (ramified) $W$-cover $\chi : \mathfrak{t} \to \mathfrak{t}/W$ with $L$ we get a $W$-cover, $p$, of the total space of $t \otimes \mathbb{C} L/W \simeq \bigoplus_i L^{d_i}$,
and pulling that back by the evaluation morphism \( \text{ev} : \mathcal{B} \times X \to \text{tot } t \otimes C \mathcal{L}/W \),
\( \text{ev}(b,x) = b(x) \), we obtain a \( W \)-cover \( \mathcal{X} = \text{ev}^* \mathcal{P} \) of \( \mathcal{B} \times X \). This is the \textit{universal cameral cover}, and each restriction of \( \mathcal{X} \) to \( \{b\} \times X \) gives a cover \( \pi_b : \widetilde{X}_b \to X \).

Our cameral curves \( \widetilde{X}_b \) are all embedded in \( t \otimes C \mathcal{L} \) and inherit from it the \( W \)-action, which makes them ramified Galois covers (with group \( \mathcal{W} \)). It is worth mentioning that there is also an “abstract” notion of a cameral cover – one which does not involve the data of embedding into a vector bundle. Such covers are simply defined to be locally (étale or analytically) the pullback of the cover \( \chi \), see [DG02], Definition 2.6.

The singularities and ramification behaviour of cameral covers are fairly well controlled. Indeed, \( \chi \) a singular hypersurface \( t/W \), the zero locus of the discriminant
\[
\mathcal{D}_\chi = \prod_{\alpha \in \mathcal{R}} \alpha = (-1)^{|\mathcal{R}|/2} \prod_{\alpha \in \mathcal{R}^+} \alpha^2 = \mathcal{P}(\mathcal{L}) \in \mathcal{C}[\mathcal{L}] \simeq \mathcal{C}[t]^\mathcal{W} \subset \mathcal{C}[t].
\]

The singular points of that hypersurface are \( W \)-orbits of semi-simple elements, lying on more than one root hyperplane. For instance, if \( \mathfrak{g} = \mathfrak{sl}(\mathbb{C}) \), the discriminant hypersurfaces is a cuspidal cubic in \( t/W \simeq \mathbb{C}^2 \), the cusp being the orbit of the origin in \( t \simeq \mathbb{C}^2 \).

Every root \( \alpha \in t^\mathcal{V} \) gives a morphism of bundles \( t \otimes C \mathcal{L} \to \mathcal{L} \), which can be further pulled back to \( \text{tot } t \otimes C \mathcal{L}/W = \text{tot } t \otimes C \mathcal{L} \). Consequently, the discriminant \( \mathcal{D}_\chi \) gives a morphism (as varieties over \( X \)) between the total spaces of \( t \otimes C \mathcal{L} \) and \( \mathcal{L}^{\mathcal{R}} \), and that can also be pulled further up to \( \mathcal{D} \in \text{H}^{\mathcal{V}}(t \otimes C \mathcal{L}/W, q^* \mathcal{L}^{\mathcal{R}}) \), as indicated on the next diagram

\[
\begin{array}{ccc}
\widetilde{X}_b & \xrightarrow{\pi_b} & \mathcal{X} \\
\downarrow & \searrow \pi \downarrow & \searrow \text{tot } t \otimes C \mathcal{L} \\
\{b\} \times X & \xrightarrow{\text{ev}} & \text{tot } t \otimes C \mathcal{L}/W \xrightarrow{\text{ev} \mathcal{D}} \text{tot } q^* \mathcal{L}^{\mathcal{R}} \\
\downarrow & \nearrow q \downarrow & \nearrow \mathcal{D} \\
X & \xrightarrow{\pi} & \text{tot } t \otimes C \mathcal{L}/W \xrightarrow{\text{ev} \mathcal{D}} \text{tot } q^* \mathcal{L}^{\mathcal{R}}
\end{array}
\]

We denote by \( Z(\mathcal{D}) \subset \text{tot } t \otimes C \mathcal{L}/W \) the vanishing locus of this section.

The possible singularities of \( \widetilde{X}_b \) occur at the intersections of root hyperplanes, i.e., over points \( b \in \mathcal{B} \), where \( b(X) \) meets the singular locus of \( Z(\mathcal{D}) \). We shall call a cameral cover \textit{generic} if it is smooth with simple Galois ramification, i.e., all ramification points have ramification index one. That is, \( b \in \mathcal{B} \) is generic, if \( \text{ev}_b(X) \cap Z(\mathcal{D})^{\text{sing}} = \emptyset \) and \( \text{ev}_b(X) \cap Z(\mathcal{D})^{\text{sm}} \). We denote the locus of generic cameral covers by \( \mathcal{B} \), and \( \mathcal{B} \subset \mathcal{B} \) is a dense open subset.

4.3. Generalised Pryms and abelianisation. We recall here the definition of the generalised Prym variety, since the varieties used in the literature are not completely uniform. As before, let \( \Lambda_G := \text{cochar}_G \subset t \) be the cocharacter lattice. We have that \( \Lambda_G \simeq \text{Hom}(\mathbb{C}^\times, T) \) and \( \Lambda_G \otimes \mathbb{C}^\times \simeq T \). Donagi and Gaitsgory ([DG02]) introduce two abelian sheaves, \( \mathcal{T} \) and \( \mathcal{T}^\sigma \), on \( X \), associated with the cover \( \pi_o : \widetilde{X}_o \to X \). The sections of the sheaf \( \mathcal{T} \) on \( U \subset X \) are \( W \)-equivariant (holomorphic) maps \( \pi_o^{-1}(U) \to T \), i.e., \( \mathcal{T} = \pi_o^* \left( \Lambda_G \otimes \mathcal{O}_{\widetilde{X}_o} \right)^W \). We note that the \( W \)-action
on such maps incorporates both the $W$-action on $\tilde{X}_o$ and the $W$-action on $T$. In particular, an equivariant map must take the value $\pm 1$ on root hyperplanes. The sheaf $T$ is the subsheaf of $\mathcal{T}$, whose sections over $U$ are the sections in $\mathcal{T}(U)$, taking value $+1$ on $\tilde{X}_o \cap \{\alpha = 0\}$ for all roots $\alpha \in R$. Since we are assuming that $G$ is simple, one can see ([DP12], Lemma 3.3) that $T = \mathcal{T}$ if $G \neq B_1$. In the exceptional case, $\mathcal{T}/T$ is $\mathbb{Z}/2\mathbb{Z}$-torsion, supported at the branch points of $\pi_o$. Moreover, ibid., Claim 3.5, $H^1(X, \mathcal{T})$ and $H^1(X, T)$ are isogenous abelian varieties. The generalised Prym variety associated to the given cameral cover is $\text{Prym}_{X_o/X} := H^1(X, T)$. These varieties are isogenous to $\left(\Lambda_G \otimes_{\mathbb{Z}} H^1(\tilde{X}_o, \mathcal{O}^\times)^W\right)$, which is the set of $W$-invariant $T$-bundles on $\tilde{X}_o$.

We denote by $\text{Prym}_{X/o}$ the relative Prym fibration (over $\mathcal{B}$) associated with $X$. By the abelianisation theorem ([DG02]), $h^{-1}_o(o)$ is a $\text{Prym}^0_{X_o/X}$-torsor, and, moreover, $\text{Higgs}_{G,D,c}$ is a $\text{Prym}^0_{X/o}$-torsor. The two can be locally identified by choosing local sections (over $\mathcal{B}$).

We remark that Donagi and Gaitsgory describe explicitly spectral data, corresponding to $h^{-1}_o(o)$, i.e., the particular $\text{Prym}^0_{X_o/X}$-torsor, see [DG02], Theorem 6.4 and [DP12], Appendix A.1. We do not need this description in what follows, so will not dwell on it.

5. The Infinitesimal Period Map

5.1. The main theorem. We now have at hand all ingredients needed for stating the main result. Fix a point $o \in \mathcal{B}$ and a topological type $c \in \pi_1(G)$, such that $\text{Higgs}_{G,D,c} \neq \emptyset$. The base point corresponds to a maximal rank symplectic leaf $S$, whose closure in $\text{Higgs}_{G,D,c}$ is $h^{-1}_c(\{o\} + \mathcal{B}_0)$. In general, this closure is strictly bigger than the one in Markman’s construction, which takes place on the smaller locus $\text{Higgs}_{G,D,c}$ of Higgs pairs, having a regularly stable underlying bundle. The difference, however, is away from the generic locus of $\mathcal{B}$, to which we now restrict. We consider the set

$$B := (\{o\} + \mathcal{B}_0) \cap \mathcal{B} \subset \mathcal{B}$$

which supports an integrable system (in the symplectic sense), all of whose fibres are proper:

$$\text{Higgs}_{G,D,c} \supset S|_B = h^{-1}_c(B) \xrightarrow{h_B} B \ni o$$

where $h_B = h_c|_B$. Our main theorem is a statement about the infinitesimal period map of this family of abelian torsors.

Theorem A ([BD14]). There exists a natural isomorphism

$$T_{B,o} \simeq H^0(\tilde{X}_o, t \otimes \mathcal{O} K_{\tilde{X}_o})^W.$$

Let $Y_\xi \in T_{B,o}$ denote both the preimage of $\xi \in H^0(t \otimes \mathcal{O} K_{\tilde{X}_o})^W$ and the corresponding constant vector field. Under this isomorphism, the differential at $o \in B$ of the period map of $h_B : S|_B \rightarrow B$ is given by

$$c_o : H^0(\tilde{X}_o, t \otimes \mathcal{O} K_{\tilde{X}_o})^W \rightarrow \text{Sym}^2(H^0(\tilde{X}_o, t \otimes \mathcal{O} K_{\tilde{X}_o})^W)^\vee.$$
\[
c_o(\xi)(\eta, \zeta) = \frac{1}{2} \sum_{p \in \text{Ram}(\pi_o)} \text{Res}_p^2 \left( \pi_o \frac{\mathcal{L}_{Y_p}(\mathcal{O})}{\mathcal{D}} \right)_{\{o\} \times X} \eta \cup \zeta.
\]

5.2. Related results. We should note that several instances of Theorem A (and its reformulation, Theorem B in [BD14]) have already been established in the literature. First of all, for the usual Hitchin system \((D = 0)\) and \(G = SL_n(\mathbb{C})\) the formula appears in unpublished work of T. Pantev, while for the case \(G = SL_2(\mathbb{C})\) the formula can be found in [DDD+06], (47). Building on that, D. Balduzzi ([Bal06]) dealt with the case of a semi-simple structure group (still in the case \(D = 0\)). The same formula for the cubic is obtained (by a somewhat different method) in [HHI10]. When it comes to the generalised Hitchin system, the only similar result that we are aware of appears in the context of the Neumann oscillator, see [Hoe08]. There the base curve is \(\mathbb{P}^1\), while \(D = 2 \cdot \infty + \sum n_i q_i\) and \(G = SL_2(\mathbb{C})\).

5.3. Proof of the theorem. We break the argument into several steps.

5.3.1. Step 1: the isomorphism. Since \(\mathcal{B}\) is an open subset of an affine space modelled on \(\mathcal{B}_0\), we have a canonical isomorphism \(T_{\mathcal{B}, o} = \mathcal{B}_0\). This isomorphism, however, makes no reference to \(o \in \mathcal{B}\). So we restrict \(\mathcal{X}\) to \(\mathcal{B} \times X\), and consider the embedding \(\widetilde{X}_o \hookrightarrow \mathcal{X}|_{\mathcal{B} \times X}\). Setting \(N\) to be the normal bundle of the inclusion \(\iota_o \widetilde{X}_o \subset t \otimes_C L\) and \(r : t \otimes C L \to X\) to be the bundle projection, we get an isomorphism \(T_{\mathcal{B}, o} \simeq H^0(\widetilde{X}_o, N(-r^*D))^W\). We now claim that there is a natural isomorphism

\[
H^0(\widetilde{X}_o, N(-r^*D))^W \simeq H^0(\widetilde{X}_o, t \otimes_C K_{\widetilde{X}_o})^W
\]

For that, recall that for any vector space \(V\) there is a natural \(t\)-valued 2-form on \(V \oplus (V^\vee \otimes t)\), namely

\[
((x, \alpha \otimes s), (y, \beta \otimes t)) = \alpha(y)s - \beta(x)t.
\]

Here we could have replaced the Cartan subalgebra \(t\) by any vector space – for instance, replacing it by \(C\) would give the canonical symplectic form on \(V \oplus V^\vee\).

This form induces a \(t\)-valued 2-form on \(t \otimes_C K_X\), and, after twisting with \(D\), a meromorphic \(t\)-valued 2-form \(\omega_t \in H^0(t \otimes_C L, t \otimes_C \Omega^2_{t \otimes_C L}(r^*D))^W\). Contraction with \(\omega_t\) gives a sheaf homomorphism

\[
N \to t \otimes_C K_{\tilde{X}_o}(r^*D).
\]

While it is not necessarily an isomorphism, it does induce an isomorphism on invariant global sections. This follows from [Kij00], where (following the reasoning in [Hur97]), the author shows that the generalised Hitchin system satisfies the rank-2 condition of Hurtubise and Markman. That turns the statement into a special case of Proposition 2.11 in [HM98].

5.3.2. Step 2: Recasting the symplectic structure. The relative Prym fibration \(q_B : \text{Prym}_{X/B}^o \to \mathcal{B}\) is Lagrangian and under the local identifications \(\mathcal{S}|_{\mathcal{U}} \simeq \text{Prym}_{X/M}^o\), \(\mathcal{U} \subset \mathcal{B}\), the symplectic structures on both sides coincide. Indeed, for the abelian variety \(P_o = \text{Prym}_{X_o/X}^0\) we have

\[
T_{P_o} = H^1(\widetilde{X}_o, t \otimes_C \mathcal{O})^W \otimes_C \mathcal{O}_{P_o},
\]

and so, by Serre duality, for any \(\mathcal{L} \in P_o\) we get

\[
T_{P_o, \mathcal{L}} = T_{\mathcal{B}, o} \simeq H^0(\widetilde{X}_o, t \otimes_C K_{\widetilde{X}_o})^W.
\]
This gives the Lagrangian structure on the Prym fibration (restricted to \( B \)), and it coincides with the one obtained by Marsden–Weinstein reduction. For a very concrete description in the case \( X = P^1 \), see Theorem 1.10 in [Hur97].

5.3.3. Step 3: Reduction to a Kodaira–Spencer calculation. Any family \( h : \mathcal{H} \to B \) of polarised compact (connected) Kähler manifolds gives rise to a weight-1 polarised \( \mathbb{Z} \)-VHS \( (\mathcal{F}^*, \mathcal{F}^\mathbb{Z}_\ast, \nabla^G_M, S) \) on \( B \) with a period map \( \Phi \). By a theorem of Griffiths ([Gri68], part II, Theorem 1.27) we have that \( d\Phi_o = m^\vee \circ \kappa \), where \( \kappa : T_{B,o} \to H^1(\mathcal{H}_o, T) \) is the Kodaira–Spencer map and

\[
m^\vee : H^1(\mathcal{H}_o, T) \to H^1(\mathcal{H}_o, \mathcal{O}) \otimes H^0(\mathcal{H}_o, \Omega^1) \vee \simeq_S (T^1_o)^{\otimes 2}
\]

is induced by cup product \( H^1(T) \times H^0(\Omega^1) \to H^1(\mathcal{O}) \).

By a choice of local section we can replace the family \( S|_B \) by \( \mathcal{H} = \text{Prym}_{\tilde{X}_b}/B \) and from Step 2 we have that \( H^1(P_o, T_{P_o}) \simeq H^1(\tilde{X}_o, t \otimes \mathcal{O})^{W \otimes 2} \). Polarisation-preserving deformations are contained in

\[
\text{Sym}^2 H^1(\tilde{X}_o, t \otimes \mathcal{O})^{W \otimes 2} \simeq \text{Sym}^2 H^0(\tilde{X}_o, t \otimes K^2)^{W \otimes 2}.
\]

But by properties of cup product, \( m^\vee \) is dual to the multiplication map

\[
m : H^0(\tilde{X}_o, t \otimes K^{\otimes 2})^{W \otimes 2} \to H^0(\tilde{X}_o, t \otimes K^2) \to H^0(\tilde{X}_o, K^2).
\]

In this way, all data are expressed in terms of the family of cameral curves \( f = p_1 \circ \pi : \mathcal{X}|_B \to B \), and the Kodaira–Spencer maps coincide, since \( j_{\mathcal{B},B}^*T_{\text{Prym}/B} \simeq f_\ast T_{\mathcal{X}/B} \).

Moreover, the polarisation on the Pryms is determined by the polarisation on the \( \tilde{X}_b \). Hence the question of computing the infinitesimal period map of \( S|_B \to B \) is replaced with the same question, but for the family \( \mathcal{X}|_B \to B \).

Finally, since for a finite dimensional vector space \( V \) the natural isomorphism \( \text{Hom}(V^\vee, \text{Hom}(V^\vee, V)) = \text{Hom}(V^\vee \otimes 3, \mathbb{C}) \) is given by

\[
F \mapsto (Y \otimes \alpha \otimes \beta \mapsto \beta(F(Y)(\alpha))),
\]

we obtain the following

**Proposition 5.1.** The differential of the period map of \( h_B : S|_B \to B \) at \( o \) is given by

\[
c_o : H^0(\tilde{X}_o, t \otimes K^{\otimes 2})^{W \otimes 2} \to \text{Sym}^2 \left( H^0(\tilde{X}_o, t \otimes K^{\otimes 2})^{W \otimes 2} \right)^\vee,
\]

\[
c_o(\xi)(\eta, \zeta) = \frac{1}{2\pi i} \int_{\tilde{X}_o} \kappa(\eta \cup \zeta).
\]

where \( \kappa \) is the Kodaira–Spencer map of the family \( \mathcal{X}|_B \to B \) at \( o \in B \).

5.3.4. Step 4: Kodaira–Spencer calculation. If one is given a family of compact Kähler manifolds over a contractible base, then any holomorphic vector field on the base can be lifted to a smooth vector field on the total space of the family. In general, however, there does not exist a holomorphic lift. Locally, such lifts do exist, and the Kodaira–Spencer map measures the obstruction to the existence of a global one. More intrinsically, one can describe this map as a connecting homomorphism for the derived image of the projection morphism.

Indeed, getting back to our setup, we see that pushing forward the tangent sequence of \( f : \mathcal{X}|_B \to B \) gives a connecting homomorphism \( \delta : T_B \to \Omega^1 \otimes T_{\mathcal{X}_B}/B \).
For a contractible neighbourhood $\mathcal{U} \subset \mathcal{B}$ of $o \in \mathcal{B}$, the (global) Kodaira–Spencer map $\kappa$ over $\mathcal{U}$ is the induced map on global sections:

$$\kappa = \Gamma_\mathcal{U}(\delta) : \Gamma_\mathcal{U}(T_\mathcal{B}) \to \Gamma_\mathcal{U}(R^1 f_* T_{\mathcal{X}_\mathcal{B}/\mathcal{B}}) = H^1(\mathcal{X}_\mathcal{U}, T_{\mathcal{X}_\mathcal{U}/\mathcal{U}}),$$

while the (pointwise) Kodaira–Spencer map appearing in Griffiths’ theorem is $\kappa = \kappa_o : T_\mathcal{U} \to H^1(\breve{X}_o, T_{\breve{X}_o})$, obtained by passing to the fibre over $o$.

We compute $\kappa$ on a convenient Čech cover of $\mathcal{X}_\mathcal{B}$, thus following the original approach of Kodaira and Spencer, see [KS58], or [Kod86], Ch.4. Since we have a family of covers of a fixed target curve, $X$, we choose a covering which facilitates the handling of varying branch loci, as follows.

Let $\text{Ram}(\pi)$ be the ramification and branch loci of $\pi : \mathcal{X}_\mathcal{U} \to \mathcal{U} \times X$. We set $\mathcal{U} := \mathcal{X}_\mathcal{U} \setminus \text{Ram}(\pi)$ and introduce the cover $\mathcal{X}_\mathcal{U} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{V} \supset \text{Ram}(\pi)$ is a certain tubular neighbourhood, constructed as follows. Consider $\text{Bra}(\pi_o) = \{p_1, \ldots, p_N\} \subset X$, $N = |\mathcal{R}| \text{deg } L$, and choose an atlas $\{(U_j, z_j), j = 0 \ldots N\}$ of $X$, where $U_0 = X \setminus \text{Bra}(\pi_o)$, and $\{U_j \ni p_j\}$ non-intersecting open disks. For simplicity, we assume $\text{supp}(D) \cap \text{Bra}(\pi_o) = \emptyset$. Since $\mathcal{U} \subset \mathcal{B}$, by the genericity assumption $\mathcal{U} \times X \supset \text{Bra}(\pi)|_{\mathcal{U}} \to \mathcal{U}$ is an unramified $N : 1$ cover, and admits, by the implicit function theorem, local sections $c_j : \mathcal{U} \to X$, $1 \leq j \leq N$, such that $c_j(o) = p_j$, and $c_j(\mathcal{U}) \subset U_j$ (possibly after shrinking $\mathcal{U}$). We then define

$$\mathcal{V} := \pi^{-1} \left( \bigsqcup_{j \neq 0} \text{graph } c_j \right) \subset \mathcal{X}_\mathcal{U}.$$

This set has $\text{deg } L|\mathcal{R}||W|/2$ connected components, which we index as $\mathcal{V}_j$, and group them into $\mathcal{V}_{j\alpha} = \prod_k \mathcal{V}_{j\alpha}^k = \pi^{-1} (\text{graph } c_j) \cap \{(\alpha_l), l = 0 \ldots N\}$, $\mathcal{V} = \prod \mathcal{V}_{j\alpha}$. Let us note that the cover $\{\mathcal{U}, \mathcal{V}\}$ is good and that for the calculation of Čech 1-cochains we have to focus attention on $\pi^{-1} \left( \mathcal{U} \times \bigsqcup_{j \neq 0} U_j \right) \supset \mathcal{U} \cap \mathcal{V}$. We have local equations for $\mathcal{X}_{\mathcal{U} \times U_j}$ of the form

$$I_1(\alpha_1, \ldots, \alpha_l) = b_1(\beta, z),$$

$$\ldots,$$

$$I_l(\alpha_1, \ldots, \alpha_l) = b_l(\beta, z),$$

where $\beta$ are coordinates on $\mathcal{U}$, say obtained from a choice of basis on $\mathcal{B}_0$, and $z = z_j$. On every component of $\mathcal{U}$, we can use as (étale!) coordinates $(\beta, z)$, giving a parametrisation $\alpha_i = \theta_i^0(\beta, z)$ (after passing to a cover). This can then be carried to the other components by the $W$-action, $s_j \cdot g_i = g_i - n_{ij} g_j$. On the other hand, since $\text{supp}(D) \subset (X \setminus \bigsqcup_{j \neq 0} U_j)$, the local trivialisations of $K_X$ (provided by the atlas) turn roots into maps $t \otimes_C L_{U_j} \to C$. By genericity and WPT on $\mathcal{V}_{j\alpha}$, say $\alpha := \alpha_1$, we have

$$\alpha^2 = (z - c(\beta)) v(\beta, z),$$

$$\alpha_i = g_i(\alpha, z), \quad i \geq 2.$$  

Here the holomorphic function $v$ is non-zero along $\text{graph } (c) \subset \mathcal{U} \times U_j$. Inverting it, we have $z = \phi(\beta, \alpha) = e(\beta) + \alpha^2 u(\beta, \alpha)$, for some other holomorphic function $u$.

Starting with the (constant) vector field $Y = \partial_\beta \in \Gamma_\mathcal{U}(T_\mathcal{B})$, the cocycle $\{\kappa_\alpha(Y)\}$ representing $\kappa(Y)$ is given, on $\mathcal{V}_{j\alpha} \cap \mathcal{U}$ by the vertical vector field

$$\frac{\partial \alpha}{\partial \beta} \bigg|_{z = \phi(\beta, \alpha)} \frac{\partial}{\partial \alpha}.$$
By the implicit function theorem,
\[ \frac{\partial \alpha}{\partial \beta} \bigg|_{z = \phi(\beta, \alpha)} = \frac{\alpha \frac{\partial^2 \alpha^2}{\partial^2 \alpha}}{2} \bigg|_{z = \phi(\beta, \alpha)} = -\frac{\partial \beta c}{2\alpha u(\beta, \alpha)} + \frac{\alpha}{2} \frac{\partial \beta v}{\partial \beta} \bigg|_{z = \phi(\beta, \alpha)}. \]

Notice that along \( \alpha = 0 \) the first term has a pole, while the second has a zero. On the other hand, the “logarithmic derivative” of \( D \) along \( Y \) is
\[ \pi^* \frac{\partial \beta D}{\partial \beta} = \sum_{\alpha \in R^+} \frac{\partial \beta \alpha^2}{\alpha^2} \]
and so, up to a factor of 1/2, taking quadratic residues with respect to the latter has the same effect as contracting with \( \kappa_{\alpha z} \) (Y) and taking residues. Summing over all branch points, roots, and \( \mathbb{Z}/2\mathbb{Z} \) cosets in \( W \) completes the proof.

\[ \square \]

6. Some Related Geometries

6.1. The Donagi–Markman cubic condition. In this section we recall very briefly some structures, naturally related to the study of the infinitesimal period map for algebraic integrable systems. We also give references to recent developments in these areas.

As discussed in the introduction, many of the differences between real and algebraic (analytic) integrable systems are rooted in the fact that complex tori have moduli. One such striking difference arises when one considers a holomorphic family, \( h : \mathcal{C} \to \mathcal{B} \), of complex tori, with the property that \( 2 \dim \mathcal{B} = \dim \mathcal{C} \). In the smooth category, locally on \( \mathcal{B} \) there is always a compatible Lagrangian structure on the family, i.e., symplectic structure, for which the fibres are Lagrangian. In the holomorphic or algebraic context, however, there is a local obstruction to the existence of such structure, as discovered by Donagi and Markman. The presence of Lagrangian structure forces the infinitesimal period map to be a section of \( \text{Sym}^2 T_{\mathcal{B}}^\vee \), rather than just a section of \( T_{\mathcal{B}}^\vee \otimes \text{Sym}^2 T_{\mathcal{B}}^\vee \). After making the appropriate choices, this condition forces the period matrix to be locally a Hessian of a holomorphic function, known as holomorphic prepotential \( F : \mathcal{U} \to \mathbb{C} \).

We sketch now how this obstruction arises. Consider a contractible open set \( \mathcal{U} \subset \mathbb{C}^d \) and a holomorphic map \( \Phi : \mathcal{U} \to \mathbb{H}^d \subset \text{End}(\mathbb{C}^d) \), where \( \mathbb{H}^d \) is Siegel’s upper-half space of dimension \( d \). Let also \( \Omega = (1 | \Phi) : \mathcal{U} \to \text{Hom}(\mathbb{C}^{2d}, \mathbb{C}^d) \) be the map \( s \mapsto \Omega(s) = (1 | \Phi(s)) \). Finally, let \( \Gamma \simeq \mathbb{Z}^{2d} \) be the group of holomorphic automorphisms of \( \mathcal{U} \times \mathbb{C}^d \), generated by \( (s, z) \mapsto (s, z + \Omega(s)(e_j)), j = 1 \ldots 2d \). Then the quotient \( \mathcal{H} = \mathcal{U} \times \mathbb{C}^d / \Gamma \) is a family (over \( \mathcal{U} \)) of abelian varieties. We can happily endow \( \mathcal{U} \times \mathbb{C}^d \approx T_{\mathcal{U}}^\vee \) with the canonical symplectic structure, but this structure will not descend to the quotient, unless the sections \( s \mapsto (s, \Omega(s)(e_j)) \) of \( T_{\mathcal{U}}^\vee \), determined by the columns \( \Omega(e_j) \), happen to be Lagrangian. This happens precisely when the 1-forms \( \sum_k \Omega_{kj} ds_k \) are closed, i.e., \( \partial_i \Omega_{kj} = \partial_k \Omega_{ij} \). The latter implies that \( \Phi = \text{Hess} F \) for some holomorphic function \( F \), possibly after shrinking \( \mathcal{U} \). Correspondingly, \( c = d\Phi = \sum \partial^2_{ijk} F dx_i \cdot dx_j \cdot dx_k \).

We direct the reader to the beautiful expositions in [DM93], §1 and [DM96], §7 for a different version of this argument, discussion and applications. For the case of the generalised Hitchin system, our Theorem B in [BD14] contains a formula for the infinitesimal period map which makes it evident that \( c \) is a section of \( \text{Sym}^3 T_{\mathcal{B}}^\vee \).
6.2. Special Kähler and Seiberg–Witten geometry. The data of an ACIHS $h : \mathcal{H} \to \mathcal{B}$ naturally gives rise to a certain kind of Kähler geometry on $\mathcal{B}$. Indeed, the imaginary part of the period matrix $\text{Hess} \ F$ is symmetric and positive-definite, and hence can be used to define a Kähler metric. More intrinsically, away from the discriminant locus we have an identification between the vertical bundle (the direct image of the relative tangent bundle of $h$) and the cotangent bundle to $\mathcal{B}$. Choosing a local section of $h$ over $U \subset \mathcal{B}$ we can identify $\mathcal{H} \to \mathcal{B}$ with its relative Albanese fibration, i.e., identify $\mathcal{H}_U \to U$ with a family of polarised abelian varieties. The polarisation gives rise to a translation-invariant metric on each fibre, and we obtain a “semi-flat” metric on $\mathcal{B}$. Furthermore, the bundle of lattices can be used to define a flat connection on $T^\vee_B$, and hence on $T_B$.

This kind of differential-geometric structure is known as special Kähler geometry. Abstractly, one starts with a Kähler manifold $(M, I, \omega)$, $\mathcal{B} = (M, I)$, and considers a flat, symplectic, torsion-free connection $\nabla$ (on $T_M$), such that $d^\nabla I = 0$. The special Kähler geometry, arising from an integrable system carries an additional integrality property. Such structures were introduced by the physicists ([BCOV94]) for the study of vector multiplets in four-dimensional $N = 2$ SUSY. For a beautiful – and by now, classical – introduction to these structures, one can consult [Fre99]. The generalised Hitchin system depends on a large amount of input data, including the genus of $X$, the degree of the divisor $D$ and the group $G$. If one wants build realistic physical examples these discrete invariants are quite constrained. Donagi and Witten have discussed various proposals of this kind and made some tests in [DW96]. For a discussion of the relations between Seiberg–Witten theory, integrable systems in general and the generalised Hitchin system in particular, one can consult [Don98] and [Mar00]. For some recent developments one can consult the other articles in this volume.

6.3. VHS and $tt^*$-geometry. In the proof of Theorem A we have used explicitly several bits of elementary Hodge theory, and we have, on the other hand, indicated that the presence of an ACIHS on $\mathcal{B}$ is equivalent to the presence of an (integral) special Kähler structure. This is by no means a coincidence: one can show ([Her03], [BM09]) that the special Kähler data $(M, I, \omega, \nabla)$ is equivalent to the data of a weight-1 real polarised variation of Hodge structures on $T_{B,C}$. The latter means that we have a polarised, weight one, real VHS on $\mathcal{B}$, with Hodge flag $\mathcal{F} = T_B \subset \mathcal{F}^0$, where $\mathcal{F}^0 \otimes_{\mathcal{O}_B} \mathcal{O}_B^\infty = T_{B,C}$. This forces $\mathcal{F}^0$ to be a (holomorphic) extension of $T^\vee_B$ by $T_B$. To construct this extension from the special Kähler connection, one considers the type decomposition $\nabla = \nabla' + \nabla''$ and sets $\mathcal{F}_{\mathcal{F}^0} = \nabla''$. The Gauss–Manin connection of the VHS is the set to be $\nabla^{GM} = \nabla'$, the $(1,0)$-part of $\nabla$.

The kind of VHS appearing above falls within the context considered initially by C.Simpson in [Sim88]. Indeed, the second fundamental form of the Gauss–Manin connection $\nabla^{GM} : \mathcal{F}^0 \to \mathcal{F}^0 \otimes T^\vee_B$ is the induced $\mathcal{O}_B$-linear map $\mathcal{F}^1 \to (\mathcal{F}^0 / \mathcal{F}^1) \otimes T^\vee_B$. This can be identified with a $T_B^\vee$-valued Higgs field $\theta$ on the associated graded bundle $gr \mathcal{F}^1 = T_B \oplus T_B^\vee$. This is a nilpotent Higgs field, whose only (possibly) non-zero component is contained in $H^0(\mathcal{B}, T_B^\vee \otimes \mathcal{O}_B)$. This is precisely the Donagi–Markman cubic of the corresponding integrable system. Considering the Hermite–Yang–Mills equation for the Higgs vector bundle $(T_B \oplus T_B^\vee, \theta)$ one finds the $tt^*$-equation ([Fre99], 1.32) which is one of the main objects of interest in the original work of [BCOV94].
Thus, starting from an ACIHS, and considering its associated special Kähler geometry as a particular variation of Hodge structures on the base $B$, one stumbles upon a piece of non-abelian Hodge theory: a Higgs bundle on $T_B \oplus T_B^\vee$, for which the Hitchin–Kobayashi correspondence is tied with $tt^*$-geometry. These phenomena seem to be rooted in mirror symmetry and non-commutative geometry.

6.4. Bryant–Griffiths geometry and non-commutative Hodge structures.

The relation between Hodge theory, $tt^*$-geometry and mirror symmetry is extremely intricate and far from being completely understood. Here we indicate only the main points and give references to some of the literature.

One of the formulations of the mirror conjecture is in terms of Frobenius manifolds. On the $A$-side the Frobenius structure is obtained from the quantum cohomology product, while on the $B$-side it is phrased in terms of various extensions of the notion of Hodge structure. These extended structures seem to be most conveniently encapsulated in the notion of non-commutative Hodge structure, see [KKP08], [Sab11]. Variations of such structures on tangent bundles of manifolds give rise to $tt^*$-geometry.

When applied to the context of mirror symmetry of Calabi–Yau threefolds, the $B$-side Frobenius geometry can be expressed in elementary terms and related to Bryant–Griffiths ([BG83]) type geometry. We recall that the Bryant–Griffiths setup is the defining example of projective special Kähler geometry ($N = 2$ supergravity in physics), where the weight-one $\mathbb{R}$-VHS on $T_{B,\mathbb{C}}$ is refined to a weight-three VHS ([Fre99], §4). This relation between Frobenius-type structures and projective special Kähler geometry is discussed in great detail in [HHP10], but with a somewhat idiosyncratic nomenclature, originating in [Her03].

6.5. Large $N$ duality and ADE Hitchin systems. It turns out that if $G$ is a group of $ADE$ type, then the base $B_\mathfrak{g}$ of the usual ($D = 0$) Hitchin system does support the kind of Frobenius-like structures mentioned above. The relation between Hitchin systems and moduli of Calabi–Yau threefolds was discovered in [DDD+06] and [DDP07]. There the authors construct a family $\mathcal{X} \to B_\mathfrak{g}$ of surface-fibred quasi-projective Calabi–Yau threefolds, for which the family of intermediate Jacobians is isogenous to $\text{Prym}_{\mathfrak{g}/B}$ and the Yukawa cubic is identified with the Donagi–Markman cubic. The family of 3-folds $\mathcal{X} \to B_\mathfrak{g}$ arises as a family of surfaces over $B_\mathfrak{g} \times X$, and is constructed as follows. Let $\mathfrak{g}$ be a simple Lie algebra of type $ADE$, and $\Gamma \subset SL_2(\mathbb{C})$ a finite subgroup, corresponding to $\mathfrak{g}$ by the McKay correspondence. Let $X$ be a smooth curve of genus $g \geq 2$ and $V \to X$ a $\Gamma$-equivariant rank two vector bundle, with a fixed isomorphism $\text{det } V \simeq K_X$. To this data one associates the Calabi–Yau threefold $Y_0 = \text{tot } t \otimes \mathbb{C} K_X / \Gamma \to X$, all of whose fibres (over $X$) are ALE spaces of type $\Gamma$. Then, by work of B. Szendrői ([Sze04], [Sze08]), there exists a family of surfaces $Q \to \text{tot } t \otimes \mathbb{C} K_X / W$, uniquely characterised by two properties: that its restriction to the zero section (of $t \otimes \mathbb{C} K_X / W$) is isomorphic to $Y_0$ and that its restriction to a fibre is isomorphic to the universal unfolding of $\mathbb{C}^2 / \Gamma$. Then the family of threefolds is defined as $\mathcal{X} := \text{ev}^* Q$, where, as in §4.2, $\text{ev} : B_\mathfrak{g} \times X \to \text{tot } t \otimes \mathbb{C} K_X / W$ is the evaluation morphism.

In [HHP10] the corresponding Frobenius-type structures on $B_\mathfrak{g}$ are discussed.

We hope that our understanding of the Donagi–Markman cubic may be useful in unravelling some of the intricacies of the analogous story when $D \neq 0$. Judging
by the recent development in [CDDP15], however, that will require developing a considerable amount of new techniques.

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