Zero Energy Solutions and Vortices in Schrödinger Equations

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Two-dimensional Schrödinger equations with rotationally symmetric potentials \( V_{\alpha}(\rho) = -a^2 g_{\alpha} \rho^{2(a-1)} \) with \( p = \sqrt{x^2 + y^2} \) and \( a \neq 0 \) are shown to have zero energy states contained in conjugate spaces of Gel'fand triplets. For the zero energy eigenvalue the equations for all \( a \) are reduced to the same equation representing two-dimensional free motions in the constant potential \( V_{\alpha} = -g_{\alpha} \) in terms of the conformal mappings of \( \zeta = z^a \) with \( z = x + iy \). Namely, the zero energy eigenstates are described by the plane waves with the fixed wave numbers \( k_x = \sqrt{mg/\hbar^2} \) in the mapped spaces. All the zero energy states are infinitely degenerate as same as the case of the parabolic potential barrier (PPB) shown in ref. 8. Following hydrodynamical arguments, we see that such states describe stationary flows round the origin, which are represented by the complex velocity potentials \( W = p_{a} z^{a} \), \( (p_{a} \) being a complex number) and their linear combinations create almost arbitrary vortex patterns. Examples of the vortex patterns in constant potentials and PPB are presented. We see that any vortices cannot be created in the linear combinations of the plane-wave solutions and the degenerate states which are not described by plane-waves play essential roles to create vortices. In the extension to three-dimensional problems with potentials being separable into \( 2 + 1 \) dimensions we show that the states in three dimensions have the same structure as the two-dimensional states with the zero energy but they can generally have non-zero total energies.

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I. INTRODUCTION

It is known that scattering states and unstable states like resonances are generally described by states in conjugate spaces of Gel'fand triplets (GT) \( \triangleright \). An example of such states for the parabolic potential barrier (PPB) \( V = -m\gamma^2 x^2/2 \) in one dimension has been studied by many authors \( [2-7] \). It has been shown that the one-dimensional PPB has pure imaginary energy eigenvalues \( \mp i(n+1/2)\hbar \gamma \) with \( n = 0, 1, 2, \cdots \), and the eigenfunctions are generalized functions in the conjugate space \( \mathcal{S}(\mathbb{R})^* \) of GT described by \( \mathcal{S}(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^* \), where \( \mathcal{S}(\mathbb{R}) \) and \( \mathcal{L}^2(\mathbb{R}) \), respectively, stand for a Schwartz space and a Lebesgue space \( \mathbb{L}^2 \). In general the energy eigenvalues \( \mathcal{E} \) of the conjugate spaces in GT are expressed by pairs of complex conjugates such that \( \mathcal{E} = \varepsilon \mp i\gamma \) with \( \varepsilon, \gamma \in \mathbb{R} \) and the states with the \( \mp \) sign, respectively, represent resonance-decay and resonance-formation processes. This pairing property of the energy eigenvalues indicates that states in higher-dimensional PPB possibly have zero energy eigenstates. In fact the PPB in two dimensions (generally in even dimensions) has zero energy eigenstates, which is included in the eigenvalues expressed by \( \mp i(n_x - n_y)\hbar \gamma \). We see that the zero energy states are obtained for zero and positive integers satisfying \( n_x = n_y \) and then they are infinitely degenerate. The zero energy states are interpreted as stationary flows round the centre of the PPB \( \triangleright \). Furthermore, following hydrodynamics, it has been also shown that some of such flows can be expressed by complex velocity potentials and various vortex structures appear in the linear combinations of the infinitely degenerate states. Considering that states in the conjugate spaces of GT are generally not normalizable but currents of those states are observable in quantum mechanics, the quantities observed in physical processes should be based on the probability currents such as currents in hydrodynamics. Hydrodynamical considerations will play a very important role in the investigations of quantum physics in GT. Hydrodynamical approach of quantum mechanics was vigorously investigated in the earlier stage of the development of quantum mechanics \( [9-16] \). Vortices were extensively examined by Hirschfelder \( [17-20] \) and a review article was written by Ghosh and Deb \( [21] \). It should be noted that such hydrodynamical idea is still useful in present-day quantum physics \( [22-23] \). Actually problems of vortices appear in many aspect of present-day physics such as vortex matters (vortex lattices) \( [24-25] \), vortices in non-neutral plasma \( [26-29] \) and Bose-Einstein gases \( [30-34] \) and so on. The vortex problems will hold a very important position in the hydrodynamical approach of quantum mechanics. As noted in ref. \( [3] \), the stationary flows in the two-dimensional PPB can create almost arbitrary patterns of vortices because of the infinite degeneracy. PPB potentials can be a good approximation to the repulsive forces that are very weak at the centre of the forces such that harmonic oscillators are a good approximation to the attractive forces being very weak at the centre. In fact PPB has been applied in some chemical problems \( [35-37] \) The PPB, however, is a very special potential and then it seems to be difficult that the results of PPB extend to more general potentials.

In this article we shall investigate stationary flows in more general types of potentials from the hydrodynamical point of view. Especially, we study stationary flows in two dimensions discussed in the PPB \( \triangleright \), because interesting quantities in hydrodynamics such as complex velocity potentials and vortices are definable in the dimensions. Furthermore it is well known that in two-
dimensional hydrodynamical problems conformal mappings can be a very strong tool to investigate velocities, complex velocity potentials and vortices [38-41]. Some solutions solved in a special aspect are possibly extend to others in terms of conformal mappings. Particularly we will pay attention to the stationary flows that are described by the zero energy eigenstates in the PPB [3]. Such zero energy states can also play an interesting role in statistical mechanics in GT, where a new type of entropy arises from the degeneracy of zero energy states. This fact that rotational symmetric potentials of the type \( V_a(\rho) = -a^2 g_a \rho^{2(a-1)/2} \) with \( \rho = \sqrt{x^2 + y^2} \) and \( a \neq 0 \) have the same zero energy solutions as those obtained in the PPB in two dimensions [8]. The other is vortex patterns which are known to be very powerful tools in two-dimensional hydrodynamical problems conformal mappings. We shall also see that the conformal mappings which are known to be very powerful tools in two-dimensional hydrodynamics become powerful tools also in the hydrodynamical approach of quantum mechanics and vortex patterns for all the potentials \( V_a(\rho) \) can be investigated in a very simple method.

We shall perform our considerations as following; in section 2 general property of Schrödinger equations with rotational symmetric potentials is investigated in terms of conformal mappings. In section 3 it is shown that, for zero energy solutions, all the equations in the mapped spaces can be reduced to the same equation describing free motions in a constant potential. This means that, as far as the zero energy eigenstates are concerned, all the symmetric potentials have the same solutions with the infinite degeneracy as those obtained in the PPB [3]. Boundary conditions are also discussed there. Following the considerations of the PPB, hydrodynamical arguments are performed and velocity, complex velocity potentials and vortices are investigated in section 4. In section 5 an extension of the argument to three-dimensional problems are carried out and vortices are studied in three dimensions. Non plane-wave solutions for the zero energy are briefly discussed in section 6. Remarks on non-zero energy solutions are presented in section 7. Some remarks and comments on the present work are given in section 8.

## II. CONFORMAL MAPPINGS OF SCHRÖDINGER EQUATIONS WITH SYMMETRIC POTENTIALS

We shall investigate the general structure of Schrödinger equations

\[
\frac{\partial}{\partial t} \Psi(t, x, y) = H \Psi(t, x, y),
\]

where the Hamiltonian \( H \) is described by rotational symmetric potentials in two-dimensional space \( (x, y) \). The eigenvalue problems with the energy eigenvalue \( \mathcal{E} \) are explicitly written as

\[
\left[ -\frac{\hbar^2}{2m} \Delta + V_a(\rho) \right] \psi(x, y) = \mathcal{E} \psi(x, y), \tag{1}
\]

where \( a \in \mathbb{R} \) \( (a \neq 0) \),

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]

\[
V_a(\rho) = -a^2 g_a \rho^{2(a-1)},
\]

with \( \rho = \sqrt{x^2 + y^2} \), \( m \) and \( g_a \) are, respectively, the mass of the particle and the coupling constant. Note here that \( V_a \) represents repulsive potentials for \( (g_a > 0, \ a > 1) \) and \( (g_a < 0, \ a < 1) \) and attractive potentials for \( (g_a > 0, \ a < 1) \) and \( (g_a < 0, \ a > 1) \). Since we investigate the equations in the conjugate spaces of GT, the energy eigenvalues \( \mathcal{E} \) of [3] are generally complex numbers.

Following the hydrodynamical argument [38-41], let us consider the following conformal mappings;

\[
\zeta_a = z^a, \quad \text{with} \quad z = x + iy. \tag{2}
\]

Note that the conformal mappings are singular at the origin except in the cases for \( a \) = positive integers, and the conformal mapping for \( a = 1 \) is trivial because nothing is changed by the mapping. We further notice that a complex factor \( A \) can be multiplied in the mappings such as \( \zeta = A z^a \), which will be discussed in the case for \( A = e^{i\alpha} \) with \( \alpha \in \mathbb{R} \). When we use the notation

\[
\zeta_a = u_a + iv_a,
\]

we see that

\[
u_a = \rho^a \cos \alpha \varphi, \quad v_a = \rho^a \sin \alpha \varphi, \tag{3}
\]
where \( u_a, v_a \in \mathbb{R} \) and \( \rho = \sqrt{x^2 + y^2}, \ \varphi = \arctan(y/x) \).

Using the notations
\[
\rho^2 = u_a^2 + v_a^2(= \rho^{2a}) \quad \text{and} \quad \varphi_a = a\varphi,
\]
we have
\[
u_a = \rho_a \cos \varphi_a \quad \text{and} \quad v_a = \rho_a \sin \varphi_a.
\]

In the \((u_a, v_a)\) plane the equations (1) are written down as
\[
a^2 \rho_a^{2(\alpha - 1)/\alpha} \left[ -\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = \mathcal{E} \psi(u_a, v_a),
\]
where
\[
\Delta_a = \frac{\partial^2}{\partial u_a^2} + \frac{\partial^2}{\partial v_a^2}.
\]

We can rewrite the equations as
\[
\left[ -\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = a^{-2} \mathcal{E} \rho_a^{2(1 - a)/a} \psi(u_a, v_a).
\]

Exchanging the second term of the left-hand side and the term of the right-hand side, we obtain
\[
\left[ -\frac{\hbar^2}{2m} \Delta_a - a^{-2} \mathcal{E} \rho_a^{2(1 - a)/a} \right] \psi(u_a, v_a) = g_a \psi(u_a, v_a).
\]

It is quite interesting that we can read this equations as follows; the eigenvalue problem for the potential \( V_a(\rho) \) in the \((x, y)\) plane given by (1) is replaced by the eigenvalue problem for the potential \( V_{1/a}(\rho_a) \) in the \((u_a, v_a)\) plane, where the roles of the eigenvalue \( \mathcal{E} \) and the coupling constant \( g_a \) are exchanged. We may consider that this relation represents a kind of duality between the energy and the coupling constant. From the relation we see that by solving the eigenvalues for the fixed \( \mathcal{E} \) in the \((u_a, v_a)\) plane we can determine the strength of the coupling constant \( g_a \) to reproduce the eigenvalue \( \mathcal{E} \) for the potential \( V_a(\rho) \) in the \((x, y)\) plane. We shall return to the relations between the problems for \( V_a \) in the \((x, y)\) plane and \( V_{1/a}(\rho_a) \) in the \((u_a, v_a)\) plane in section 7, because this theme is not the main subject of this section.

Here let us briefly comment on the conformal mappings
\[
\zeta_a = z^a.
\]
We see that the transformation maps the part of the \((x, y)\) plane described by \( 0 \leq \rho < \infty, \ 0 \leq \varphi < \pi/a \) on the upper half-plane of the \((u_a, v_a)\) plane for \( a > 0 \) and the lower half-plane for \( a < 0 \). Note here that the maps on the part of the \((u_a, v_a)\) plane with the angle \( \varphi_a = \varphi - \alpha \) can be carried out by using the conformal mappings
\[
\zeta_a(\alpha) = z^a e^{-i\alpha}.
\]

In the maps the variables
\[
u_a(\alpha) = \rho^a \cos(a\varphi - \alpha) \quad \text{and} \quad v_a(\alpha) = \rho^a \sin(a\varphi - \alpha)
\]
should be used. We also have the relations
\[
u_a(\alpha) = u_a \cos \alpha + v_a \sin \alpha \quad \text{and} \quad v_a(\alpha) = u_a \cos \alpha - u_a \sin \alpha.
\]

Of course, the relations \( u_a(0) = u_a \) and \( v_a(0) = v_a \) are obvious.

It will be better to comment on the meaning of the choice of the variables \( u_a \) and \( v_a \) given in (3). It is obvious that \( u_a \) and \( v_a \) are not suitable variables to represent the states having definite properties with respect to rotations, such as the states with definite angular momentum, in comparison with the polar coordinates \( \rho \) and \( \varphi \). In the following discussions, however, we will be interested only in the states describing stationary flows that are basic elements in hydrodynamics. In general the stationary flows, such as those in scattering problems, cannot be described by the states with definite angular momentum, because every stationary flow has the specific directions representing the incoming and out-going flows. (Examples of the stationary flows in PPB will be presented in section 4. See figs. 1 and 2.) Such stationary flows, of course, have no definite rotational symmetry except rotations with respect to some specific angles. We can understand such situations by considering the fact that the directions of the incoming flows are chosen by hand in scattering experiments. Actually it will be shown that the freedom of the phase \( \alpha \) in the conformal mapping (8) is related to such choices. (See section 3.) The choice of the variables \( u_a \) and \( v_a \) is, therefore, important in the following hydrodynamical approach, where the relations between the potentials \( V_a(\rho) \) with different values of \( a \) are studied. An explicit example of the difference between the choice of the polar coordinates and that of \( u_a \) and \( v_a \) has been shown in the case of PPB in section 3 of ref. [8].

III. ZERO ENERGY SOLUTIONS OF THE SCHRÖDINGER EQUATIONS

We shall here study the special solutions having zero energy eigenvalue \( \mathcal{E} = 0 \). As noted in section 1, energy eigenvalues in GT are generally complex and all energy eigenvalues appear as pairs of complex conjugates like \( \varepsilon = i\gamma \) \((\varepsilon, \gamma \in \mathbb{R})\) [1]. This indicates that, provided that a potential in one dimensional space has pure imaginary eigenvalues, the potentials extended in two dimensions possibly have zero energy states. This situation really occurs in parabolic potential barriers (PPB), that is, one dimensional PPB has pure imaginary eigenvalues.
we can represent all the solutions of (14) and (15) by
\[
\psi_0^\pm(u_a(\alpha)) \quad \text{and} \quad \phi_0^\pm(u_a(\alpha)) \quad \text{with} \quad -\pi < \alpha \leq \pi.
\] (17)

We also notice that the solutions \(\psi_0^\pm(u_a(\alpha))\) for \(g_a > 0\) are expressed by the plane waves with the fixed momentum \(p_a = \sqrt{2mg_a}\), whereas those \(\phi_0^\pm(u_a(\alpha))\) for \(g_a < 0\) are expressed by exponential growing or dumping functions. This difference is essential, because the plane-wave solutions can always be the states contained in the conjugate spaces of GT of which nuclear space is given by Schwartz space \(\mathbb{S}\), whereas the exponential growing functions such as \(e^{i\theta}\cos(\alpha\varphi - \alpha)\) with \(0 < \cos(\alpha\varphi - \alpha)\) cannot find a simple nuclear space for arbitrary values of \(a\). From now on we shall mainly discuss on the plane-wave solutions for \(g_a > 0\).

Let us summarize the main results in the cases of \(g_a > 0\).

1. All the potentials written by \(V_a(\rho)\) have zero energy eigenstates in GT.
2. All the solutions with the zero energy can be expressed by the plane-wave with the fixed momentum \(p_a = \sqrt{2mg_a}\) in the \((u_a, v_a)\) plane.
3. The zero energy solutions have an infinite freedom arising from the arbitrary angle \(\pi < a < \pi\) that corresponds to the freedom of the angle between the incoming particle and the \(x\) axis, which is given by \((\pi - a)/\alpha\).
4. In the case of the constant potential corresponding to \(a = 1\), though we have the same solutions obtained in the above arguments, their energy eigenvalues need not equal zero but the energies can take arbitrary values fulfilling the relation \(E + g_1 > 0\).

### B. Property of the zero energy states for \(2\pi\) rotation

In general eigenfunctions must have some definite properties with respect to the \(2\pi\) rotation of the azimuthal angle \(\phi\) in the original \((x, y)\) plane corresponding to statistical properties of the eigenstates, for instance, \(\psi(\theta, \phi + 2\pi) = \psi(\theta, \phi)\) for integer-spin states (bosonic states) and \(\psi(\theta, \phi + 2\pi) = -\psi(\theta, \phi)\) for \(\frac{1}{2}\)-spin states (fermionic states). We see that in the latter condition (fermionic condition) the eigenstates first return to the original states by \(4\pi\) rotation. It is known that states having new statistical properties which are different from Bose-Einstein and Fermi statistics appear in two dimensions and they are called anyon \([45-49]\). We may consider the eigenstates which first return to the original states by \(2\pi\) rotation \((l = \text{integers} \geq 3)\), of which condition in two dimensions is expressed by \(\psi^{\pm}(\phi + 2\pi) = e^{ik\pi/2}\psi^{\pm}(\phi)\) \((k = \text{positive integers} < l)\) corresponding to anyon states. The condition may be called anyonic condition. Here we shall study the construction of the eigenfunctions under some different conditions. In this argument we represent...
the eigenfunctions in terms of \( \psi_0^+(u_\alpha(\alpha)) \) and \( \phi_0^+(u_\alpha(\alpha)) \) given in [14] and [15].

(i) Cases for \( a = \pm n \) with \( n = 1, 2, 3, \cdots \)

Only the normal condition for the bosonic states (bosonic condition)

\[
\Psi_0^\alpha(\rho, \varphi(\alpha) + 2\pi) = \Psi_0^\alpha(\rho, \varphi(\alpha))
\]  

(18)

are available, because \( \varphi(\alpha) = \varphi + \alpha/a \) and then \( u_\alpha(\alpha) \) do not change in the addition of \( 2a\pi \) to their phases for the choices

\[
a = \pm n \quad \text{with} \quad n = 1, 2, 3, \cdots.
\]  

(19)

All the eigenfunctions \( \psi_0^+(u_\alpha(\alpha)) \) and \( \phi_0^+(u_\alpha(\alpha)) \) including the freedom of the arbitrary angle \( \alpha \) are satisfied by the bosonic condition.

(ii) Cases for \( a = \pm (2n - 1)/2 \) with \( n = 1, 2, 3, \cdots \)

Two types of the conditions

\[
\Psi_0^\alpha(\rho, \varphi(\alpha) + 2\pi; \pm) = \pm \Psi_0^\alpha(\rho, \varphi(\alpha); \pm)
\]  

(20)

are available. Since the addition of \( 2a\pi \) to the phase of \( u_\alpha(\alpha) \) changes only the signs of \( u_\alpha(\alpha) \) for the choices

\[
a = \pm (2n - 1)/2 \quad \text{with} \quad n = 1, 2, 3, \cdots,
\]  

we see that the linear combinations

\[
\psi_0^\alpha(u_\alpha(\alpha); \pm) = (\psi_0^+(u_\alpha(\alpha)) \pm \psi_0^-(u_\alpha(\alpha))) / \sqrt{2}
\]

\[
\phi_0^\alpha(u_\alpha(\alpha); \pm) = (\phi_0^+(u_\alpha(\alpha)) \pm \phi_0^-(u_\alpha(\alpha))) / \sqrt{2}
\]  

(21)

fulfill the \( \pm \) conditions, respectively.

(iii) Cases for rational numbers

Let us here study the case for rational numbers written by irreducible fractional numbers \( a_r = m/l \), which satisfies the relations \( 0 < a_r < 1 \). We can introduce generalized conditions such that

\[
\Psi_0^{\alpha_r}(\rho, \varphi(\alpha) + 2\pi; e^{i2k\pi/l}) = e^{i2k\pi/l}\Psi_0^{\alpha_r}(\rho, \varphi(\alpha); e^{i2k\pi/l}),
\]  

(22)

where \( k \) is zero or a positive integer being less than \( l \). It is transparent that the cases for \( (l = m = 1) \) and \( (l = 2, m = 1) \), respectively, correspond to the cases of (i) and (ii). The eigenfunctions satisfying the conditions are obtained as

\[
\psi_0^{\alpha_r}(u_\alpha(\alpha); \eta_k) = \sum_{n=0}^{l-1} (\eta_k)^{-n} \psi_0^+(u_\alpha(\alpha + 2na_r\pi)),
\]  

(23)

where \( 0 \leq k < l \) and

\[
\eta_k = e^{i2k\pi/l}.
\]

Since the phase \( \eta_k \) does not depend on the numerator \( m \) of the rational number \( a_r = m/l \), we see that the eigenfunctions for all \( a_r \) with the same denominator \( l \) can be expressed by the functions with the same property for the rotation of the angle \( 2\pi \) in the \((x, y)\) plane. We can, of course, describe the eigenfunctions for the case with \( g_a < 0 \) by replacing \( \psi_0^+(u_\alpha(\alpha + 2na_a\pi)) \) with \( \phi_0^+(u_\alpha(\alpha + 2na_a\pi)) \) in the right-hand side of (23).

Since all the rational numbers \( q \) are expressed by \( q = a_r + n \) with \( n = 0, \pm 1, \pm 2, \cdots \) and the addition of the integers to \( a_r \) does not change the above argument at all, the eigenfunctions for \( q \) satisfying the anyonic conditions are obtained by replacing \( a_r \) with \( q \).

At this moment it is hard to answer the question whether the choice of irrational numbers for \( a \) is physically meaningful or not.

Note here that all the solutions described by the plane waves fulfill the same condition with respect to the \( 2\pi \) rotation in the \((u_\alpha, v_\alpha)\) plane, i.e., the bosonic condition, because the addition of \( 2\pi \) to the angle \( \varphi_\alpha \) does not change \( u_\alpha(\alpha) \). We see that in order to represent the whole \((u_\alpha, v_\alpha)\) plane the double sheets of the \((x, y)\) planes (Riemann surface) are needed for the choice of \( a = \pm 1/2 \). In general for the choice of \( a_r = \pm 1/l \) the \( l \) sheets of the \((x, y)\) plane like \( l \) spiral sheets are required to cover the whole \((u_\alpha, v_\alpha)\) plane. We may consider that the case for \( a = 0 \) can be understood in the limit of \( l \to \infty \), where the infinite spiral sheets are needed in the \((x, y)\) plane.

C. Infinite degeneracy of the zero energy states

A kind of degeneracy arising from the angle of the incoming particle with respect to the \( x \) axis has been pointed out in section 2. We, however, see that the zero energy states have another type of infinite degeneracy that has been already solved in the two-dimensional PPB [8]. In the PPB the degeneracy arises from the pairing property of the energy eigenvalues given by \( \mp i(n + 1/2)\hbar \gamma \), that is, the energy eigenvalues of the type \( \mp i(n_x - n_y)\hbar \gamma \) appear in the two-dimensional PPB and hence the infinitely degenerate zero energy states are derived for all the cases satisfying \( n_x = n_y \). We see that the origin of the infinite degeneracy is due to the existence of the infinite number of resonances having the decay widths \( (n + 1/2)\hbar \gamma \) in the one-dimensional PPB and the coexistence of the resonance-formation and resonance-decay processes with equal probability in the two-dimensional PPB. The zero energy states are interpreted as the stationary flows expressed by the incoming flows corresponding to the formation process and the outgoing flows corresponding to the decay process, which will be shown in fig. 1 and fig. 2 of section 5. Let us see the degeneracy in (11) where the two-dimensional PPB is included. As an example we study the freedom for the
wavefunction \( \psi^\pm_0(u_a) \) given by (12). By putting the wavefunction \( f^\pm(u_a; v_a)\psi^\pm_0(u_a) \) into (11) where \( f^\pm(u_a; v_a) \) is a polynomial function of \( u_a \) and \( v_a \), we obtain the equation
\[
[\Delta_a \pm 2ik_a \frac{\partial}{\partial u_a}] f^\pm(u_a; v_a) = 0. \tag{24}
\]
As solved in ref. [8], a few examples of the functions \( f \) are given by
\[
\begin{align*}
f^0_0(u_a; v_a) &= 1, \\
f^1_0(u_a; v_a) &= 4k_a v_a, \\
f^2_0(u_a; v_a) &= 4(4k_a^2 v_a^2 + 1 \pm 4ik_a u_a).
\end{align*} \tag{25}
\]
In the two-dimensional PPB the functions are generally written by the multiple of the polynomials of degree \( n \), \( H_n^\pm(\sqrt{2k_2}x) \), such that
\[
f^\pm_n(u_2; v_2) = H_n^\pm(\sqrt{2k_2}x) \cdot H_n^\mp(\sqrt{2k_2}y), \tag{26}
\]
where \( x \) and \( y \) in the right-hand side should be considered as the functions of \( u_2 \) and \( v_2 \). Since the form of the equations (24) is the same for all \( a \), the solutions can be written by the same polynomial functions that are given in (24) for the PPB. That is to say, we can obtain the polynomials for arbitrary \( a \) by replacing \( u_2 \) and \( v_2 \) with \( u_a \) and \( v_a \) in (24). Note that the polynomials \( H_n^\pm(\xi) \) with \( \xi = \sqrt{m\gamma/\hbar}r \) are defined by the solutions for the eigenstates with \( \xi = \pm i(n+1/2)\hbar \gamma \) in one dimensional PPB of the type \( V(x) = -m\gamma^2 x^2/2 \) and they are written in terms of the Hermite polynomials \( H_n^\pm(\xi) \) as
\[
H_n^\pm(\xi) = e^{\pm in\pi/4} H_n(e^{\mp in\pi/4} \xi). \tag{27}
\]
(For details, see ref. [23].) It is remarkable that all the wavefunctions for arbitrary \( a \) can be represented by the same functions of the PPB in the \((u_a, v_a)\) plane. For \( \psi^0_0(v_a) \) we should take the polynomials \( f^0_n(v_a; u_a) \) in which the variables \( u_a \) and \( v_a \) are exchanged.

Let us here briefly comment on the boundary conditions discussed in the last subsection. Considering the relations \( u_a(\alpha + 2\pi) = u_a(\alpha) \) and \( v_a(\alpha + 2\pi) = v_a(\alpha) \), we can make the eigenstates satisfying the suitable boundary conditions by using the eigenfunctions \( \psi^\pm_{0n}(u_a(\alpha)) = f^\pm_n(u_a(\alpha); v_a(\alpha))\psi^\pm_0(u_a(\alpha)) \) in stead of \( \psi^\pm_0(u_a(\alpha)) \) in (3.7), (24) and (23). Note also that the eigenfunctions \( \psi^\pm_{0n}(u_a(\alpha)) \) for \( n \geq 1 \) does not describe plane waves and hence they cannot be normalized in terms of \( \delta \) functions. We essentially have to treat them as the eigenfunctions of the conjugate space \( \mathcal{S}(\mathbb{R}^2)^\times \) in GT that is expressed by
\[
\mathcal{S}(\mathbb{R}^2) \subset \mathcal{L}^2(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)^\times,
\]
where \( \mathcal{S}(\mathbb{R}^2) \) and \( \mathcal{L}^2(\mathbb{R}^2) \) are, respectively, Schwartz space and Lebesgue space in two dimensions. (For details, see [23].) We will see in the next section that this degeneracy plays an important role to investigate vortices.

Note that we can obtain the polynomials for the wavefunctions \( \phi^\pm_0(u_a) \) by replacing \( k_a \) with \( -ik_a \) in the polynomials derived from (24). We also easily see that in one dimension the equation corresponding to (24) does not bring any new freedom to the plane-wave solutions.

### IV. HYDRODYNAMICAL CONSIDERATIONS OF THE ZERO ENERGY STATES

In hydrodynamics conformal mappings are very powerful tools to understand structures of currents. Actually the important hydrodynamical ideas such as the property of complex velocity potentials, circulations of currents, strengths of vortices, strengths of sources and so forth do not change in the conformal mappings [38-41]. This fact means that we can simultaneously carry out the investigation of the hydrodynamical properties of the zero energy solutions for all the potentials \( V_a(\rho) \) in the mapped spaces, i.e., in the \((u_a, v_a)\) plane. Results for all the potentials with \( a \neq 0 \) can be obtained by the inverse transformations of the conformal mappings. In this section we shall study the zero energy states from a hydrodynamical viewpoint for the \( g_a > 0 \) cases, because the eigenstates for \( g_a < 0 \) represented by exponential growing or dumping functions do not describe any oscillating waves, which will be briefly discussed in section 6.

#### A. Currents and velocities

Though states in GT are generally not normalizable, the probability currents are observable in physical processes such as in scattering processes. We shall, therefore, study the currents and other quantities based on hydrodynamics. The probability density \( \rho(t, x, y) \) and the probability current \( j(t, x, y) \) of a state \( \psi(t, x, y) \) in non-relativistic quantum mechanics are defined by
\[
\begin{align*}
\rho(t, x, y) &\equiv |\psi(t, x, y)|^2, \\
j(t, x, y) &\equiv \text{Re} \left[ \psi^\dagger(t, x, y) \left( -i\hbar \nabla \right) \psi(t, x, y) \right]/m. \tag{29}
\end{align*}
\]
They satisfy the equation of continuity
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0. \tag{30}
\]
Following the analogue of the hydrodynamical approach [38-41], the fluid can be represented by the density \( \rho \) and the fluid velocity \( \mathbf{v} \). They satisfy Euler’s equation of continuity
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{31}
\]
Comparing this equation with (30), we are thus led to the following definition for the quantum velocity of a state 
\[ \psi(t, x, y), \]
\[
\mathbf{v} = \frac{j(t, x, y)}{\psi(t, x, y)} \]
(32)
in which \( j(t, x, y) \) is given by (29). Notice that \( \rho \) and \( j \) for the zero energy states do not depend on time \( t \).

Let us discuss them in the \((u_a, v_a)\) plane. All quantities \( O \) defined in the \((u_a, v_a)\) plane will be marked by the symbol 'hat' such as \( \hat{O} \) that can easily be transformed into the quantities in the \((x, y)\) plane. It is apparent that in the \((u_a, v_a)\) plane the currents of the plane waves \( \psi_0^\pm(u_a(\alpha)) \) are represented by the same form for all \( a \)
\[
\hat{j}_{u_a} = |N_a|^2 \hbar k_a \cos \alpha/m, \\
\hat{j}_{v_a} = |N_a|^2 \hbar k_a \sin \alpha/m. 
\]
(33)

Note here the following relations;
\[
u_a(\alpha) = u_a \cos \alpha + v_a \sin \alpha, \quad u_a(0) = u_a, \quad v_a(0) = v_a.
\]
When we represent the momentum in terms of the vector of the \((u_a, v_a)\) plane as
\[
\mathbf{p}_a = (\hbar k_a \cos \alpha, \hbar k_a \sin \alpha)
\]
for \( \psi_0^+(u_a(\alpha)) \), the currents are generally written by
\[
\hat{j} = |N_a|^2 \hat{p}_a/m. 
\]
(34)

Hence the velocities are given by
\[
\hat{\mathbf{v}} = \frac{\mathbf{p}_a}{m}. 
\]
(35)

Following the argument of hydrodynamics (for details, see Appendix of ref. [8]), we can introduce the complex velocity potential \( W \) as
\[
W_a = (\hat{p}_u - i\hat{p}_v)\zeta_a/m. 
\]
(36)
The velocity potential \( \Phi \) and the stream function \( \Psi \) can be introduced as same as those in hydrodynamics by
\[
W_a = \Phi + i\Psi,
\]
where they satisfy the following relations in the \((u_a, v_a)\) plane;
\[
\dot{v}_{u_a} = \frac{\partial \Phi}{\partial u_a} = \frac{\partial \Psi}{\partial v_a}, \quad \dot{v}_{v_a} = \frac{\partial \Phi}{\partial v_a} = -\frac{\partial \Psi}{\partial u_a}. 
\]
(37)

It is known that Cauchy-Riemann’s equations are satisfied by the velocity potential and the stream function.

The velocities of the \( u_a \) and \( v_a \) directions in the \((x, y)\) plane are given by
\[
v_{u_a} = \hbar_s \dot{v}_{u_a} \quad v_{v_a} = \hbar_s \dot{v}_{v_a}, 
\]
(38)
where \( \hbar_s = a(u_a^2 + v_a^2)^{(a-1)/2a} \). Hydrodynamics tells us that \( W_a \) describes corner flows with the angle \( \pi/a \) round the origin. For example, in the case of the PPB with \( a = 2 \) the plane waves in the \((u_a, v_a)\) plane, \( \psi_0^\pm(u_2) \), are expressed in fig. 1 and fig. 2.

**B. Vortices in the zero energy states**

In hydrodynamics vortices are very important objects. In quantum mechanics, since the velocity defined by (32) diverges at the zero points of the wavefunctions, the vortices generally appear at such nodal points of the wavefunctions [17-20]. The situation is, however, not so simple to determine the positions of vortices, because the vortices do not always appear at the points where the wavefunctions vanish, when the currents also vanish at...
the same points. Since the zero energy states have an infinite degeneracy and also the freedom of the angle $\alpha$, we will be able to create vortex patterns having arbitrary number of vortices at arbitrary positions. General study of quantized vortices is carried out in ref. [20]. We shall here discuss the vortex patterns in a few simple cases of the linear combinations in terms of the infinite degeneracy.

Let us study the vortex structures appearing in the linear combinations of two eigenstates constructed from (14) and (25). The following discussions are carried out in the $(u_a, v_a)$ plane, because the singularities of the velocity do not change in the conformal mappings except the singularity of the mappings at origin for $a \neq \text{positive integers}$. The general form of the linear combination of two states can be written as

$$\Psi = \psi_1 + \psi_2,$$  \hspace{1cm} (39)

where, since the two states are not normalized, the complex coefficients appearing in the linear combination are included in the two wavefunctions $\psi_1$ and $\psi_2$. The absolute square of $\Psi$ is evaluated as

$$|\Psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\text{Re}(\psi_1^\ast \psi_2).$$ \hspace{1cm} (40)

In general a component of the current of $\Psi$ is written as

$$\hat{j}_\mu = \frac{\hbar}{m} \text{Re}[\Psi^\ast (A_\mu \psi_1 + B_\mu \psi_2)], \hspace{1cm} (41)$$

where $\mu = u_a$ or $v_a$, and $A_\mu$ and $B_\mu$ are complex functions defined by

$$A_\mu = -i \frac{\partial \psi_1}{\partial \mu} \psi_1^{-1}, \hspace{1cm} B_\mu = -i \frac{\partial \psi_2}{\partial \mu} \psi_2^{-1}. \hspace{1cm} (42)$$

Let us study the nodal points of $|\Psi|^2$, where the vortices appear. We have

$$|\Psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2| \cos \theta,$$ \hspace{1cm} (43)

where $\theta$ denotes the phase between $\psi_1$ and $\psi_2$. It is trivial that nodal points appear when the following two conditions are fulfilled:

$$|\psi_1| = |\psi_2| \hspace{1cm} \text{and} \hspace{1cm} \cos \theta = -1. \hspace{1cm} (44)$$

We put the first relation into (43) and thus obtain

$$|\Psi|^2 = 2|\psi_1|^2(1 + \cos \theta). \hspace{1cm} (45)$$

Taking account of the same relation $|\psi_1| = |\psi_2|$, the current is written by

$$\hat{j}_\mu = \frac{\hbar}{m} |\psi_1|^2([|A_\mu| \cos \phi_A + |B_\mu| \cos \phi_B](1 + \cos \theta)$$

$$+ (|A_\mu| \sin \phi_A - |B_\mu| \sin \phi_B) \sin \theta], \hspace{1cm} (46)$$

where $\phi_A$ and $\phi_B$ are, respectively, the phases of $A_\mu$ and $B_\mu$. The velocity is evaluated as

$$\vec{v}_\mu = \frac{\hbar}{2m} \left[ (|A_\mu| \cos \phi_A + |B_\mu| \cos \phi_B)$$

$$+ (|A_\mu| \sin \phi_A - |B_\mu| \sin \phi_B) \sin \theta \right]. \hspace{1cm} (47)$$

We see that the second term in the bracket [...] diverges by de l’Hôpital’s theorem, when the second condition for the angle, $\cos \theta = -1$, is fulfilled. Thus we can obtain the condition for the divergence of the velocity

$$|A_\mu| \sin \phi_A - |B_\mu| \sin \phi_B \neq 0. \hspace{1cm} (48)$$

This equation means that the functions $A_\mu$ and $B_\mu$ must not be real and also the imaginary parts of $A_\mu$ and $B_\mu$ must not be equal at least for one of the components $\mu = u_a$ and $v_a$.

Now we can summarize the conditions for the determination of the vortex positions in the linear combinations of two wavefunctions $\psi_1$ and $\psi_2$ as follows:

(I) $|\psi_1| = |\psi_2|$,  
(II) $\theta = (2l-1)\pi$ with $l=\text{integers}$, ($\theta$ is the phase between $\psi_1$ and $\psi_2$),  
(III) $|A_\mu| \sin \phi_A - |B_\mu| \sin \phi_B \neq 0$, ($A_\mu$ and $B_\mu$ are defined in (12)).

Let us investigate the above conditions in a few simple examples.

Example (i): It is trivial that any linear combinations composed of the wavefunctions with the lowest polynomial (25) have no vortex, because the condition (III) is not fulfilled whereas nodal points satisfying the conditions (I) and (II) appear in the linear combinations.

Example (ii): The combination of the lowest polynomial and the second one such that

$$\Psi = \psi_{u_a}^+(u_a(\alpha)) - C \psi_{v_a}^+(v_a(0)|u_a(0))$$

has vortices at positions fulfilling the following conditions derived from (I) and (II):

$$u_a(0) = (-1)^n/4C|k_a|,$$

$$\theta + \theta_C = n \pi, \hspace{1cm} (n = \text{integers}), \hspace{1cm} (49)$$

where $\theta_C$ is the phase of $C$ and

$$\theta = k_a[u_a(0) - u_a(\alpha)]$$

$$= k_a[u_a(0)(1 - \cos \alpha) - v_a(0) \sin \alpha]. \hspace{1cm} (50)$$

Let us examine the relations (10) in two cases for $a = 1$ and 2, where $C$ is taken to be a real number, i.e., $\theta_C = 0$.

Case for $a = 1$: In this case we have $u_1(0) = x$ and $v_1(0) = y$ and then the relations are reduced to

$$y = (-1)^n/4C|k_1|,$$

$$x(1 - \cos \alpha) - y \sin \alpha = n \pi/k_1. \hspace{1cm} (51)$$
All vortices appear on the two lines $y = \pm 1/4|C|k_1$ parallel to the $x$ axis and they are at the cross points of the two lines and the lines $x = (n\pi + (-1)^n \sin \alpha / 4|C|)/k_1(1 - \cos \alpha)$ for $\alpha \neq 0$. The positions of vortices for $n = 0, \pm 1, \pm 2, \pm 3$ are presented in fig. 3, where $\alpha = \pi$ is taken. This situation is quite similar to the vortices called parallel vortex lines obtained in hydrodynamics.

![Diagram of vortex positions](image)

**FIG. 3.** Positions of vortices for $n = 0, \pm 1, \pm 2, \pm 3$ in a constant potential ($a = 1$), which are denoted by ●.

Case for $a = 2$ (PPB): Since the inverse transformation of the conformal mapping is described by the equations $u_2(0) = x^2 - y^2$ and $v_2(0) = 2xy$ in PPB [8], the relations are given by

\[
2xy = (-1)^n/4|C|k_2,
\]

\[
(x^2 - y^2)(1 - \cos \alpha) - 2xy \sin \alpha = n\pi/k_2.
\] (52)

Vortices appear at the cross points of $x^2 - y^2 = (n\pi + (-1)^n \sin \alpha / 4|C|)/k_2(1 - \cos \alpha)$ and $xy = (-1)^n/8|C|k_2$. The positions of two vortices for $n = 0$ and other four for $n = \pm 1$ are figured in fig. 4, where $\alpha = \pi$ is taken.

![Diagram of vortex positions](image)

**FIG. 4.** Positions of vortices for $n = 0, \pm 1$ in PPB ($a = 2$), which are denoted by ● for $n = 0$, ○ for $n = 1$ and ◦ for $n = -1$, respectively.

The vortices appear at the symmetric positions with respect to the origin, which are described by the cross points of the two equations:

\[
x^2 - y^2 = n\pi/2k_2, \quad xy = (-1)^n/8|C|k_2.
\] (53)

We can make so many variety of the vortex patterns by changing of the parameters, $\alpha$ and $C$, and the zero energy states in terms of the polynomials (52). Here we stress that, as shown in the above discussions, the higher polynomial solutions with $n \neq 0$ which are not described by the plane waves in the $(u_a, v_a)$ plane play the essential roles to create vortices.

Note here that the strength of vortex is characterized by the circulation $\Gamma$ that is represented by the integral round a closed contour $C$ encircling the vortex such that

\[
\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{s}
\] (54)

and it is quantized as

\[
\Gamma = 2\pi lh/m,
\] (55)

where the circulation number $l$ is an integer [18, 20, 23]. After simple but tedious calculations, we obtain that $l = -1$ for the vortices with $n = \text{even}$ and $l = 1$ for the vortices with $n = \text{odd}$.

Before closing this section we point out the fact that we can realize almost all of vortex patterns because of the infinite degeneracy of the zero energy solutions. The study of the vortex patterns will be carried out by determining the parameter $a$ (the type of potential) and by finding the best linear combination in terms of the infinitely degenerate zero-energy-states to describe the vortex patterns.
V. VORTICES IN THREE DIMENSIONS

We shall study vortices in three dimensions. It is obvious that the conformal mappings given in (3) cannot apply in three dimensions. Schrödinger equations in three dimensions are generally written as

\[-\frac{\hbar^2}{2m}(\Delta + \partial^2_{\partial z^2}) + V_a(x, y, z)\psi(x, y, z) = \mathcal{E}\psi(x, y, z), (56)\]

where \(\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2\). The equations, however, can be reduced to two-dimensional ones in cases where potentials are separable into \(2 + 1\) dimensions and the zero energy solutions are applicable to such cases. We shall discuss in the following three cases.

(I) Cases of free motions

Let us consider the cases where the motion of one direction (say \(z\)) is a free motion described by \(e^{ikz}\). In these cases the potentials can be written by two dimensional potentials such as \(V_a(\rho)\). In such cases we have

\[-\frac{\hbar^2}{2m}\Delta + V_a(\rho)]\psi(x, y)e^{ikz} = (\mathcal{E} - E_z)\psi(x, y)e^{ikz}, (57)\]

with \(E_z = \hbar^2k^2_z/2m\). The zero energy states in the two dimensions of \((x, y)\) appear when the relation

\[\mathcal{E} - E_z = 0 (58)\]

is fulfilled. It is trivial that the total energy \(\mathcal{E}\) must be positive because of \(E_z > 0\). We see that all the vortex patterns discussed in the last section appear in three-dimensional phenomena where the motion of one direction perpendicular to the vortex plane (the \((x, y)\) plane) is a free motion. Vortices in a constant magnetic field of \(z\) direction will be one of these cases.

(II) Cases of exponentials

We can present another cases where the motions perpendicular to the vortex plane are described by exponentials such that \(\psi(z) = e^{ikz}\). The equation (56) is written as

\[-\frac{\hbar^2}{2m}\Delta + V_a(\rho)]\psi(x, y)e^{ikz} = (\mathcal{E} + E_z)\psi(x, y)e^{ikz}. (59)\]

When the relation

\[\mathcal{E} + E_z = 0 (60)\]

is fulfilled, we have the same zero energy solutions and then we obtain the same vortex structures of the two dimensions. In these cases the total energy \(\mathcal{E}\) must apparently be zero or negative.

(III) Cases for separable potentials

Even if the potential of the \(z\) component is not constant potentials, the eigenvalue equations are separable into the \((x, y)\) and the \(z\)- components in the cases with potentials such that

\[V(x, y, z) = V_2(x, y) + V_1(z). (61)\]

When the eigenvalues of the \(z\) component are given by \(E_z\), the equations for the \((x, y)\) component become same as those of the cases (I) and (II).

In these three cases where the vortex plane \((x, y)\) and the other axis \((z)\) perpendicular to the vortex plane are completely separable and then the wavefunctions are written by the multiplicative forms such as \(\psi(x, y)\psi(z)\), all vortices are described by the axial type and the toroidal vortices do not appear \([21]\), because the positions of the vortices in the \((x, y)\) plane do not depend on \(z\).

Here we would like to note the construction of vortices by plane-wave solutions in the \((x, y)\) plane. Let us put the functions

\[\psi_0(x, y, z) = N_\alpha e^{i(k_x x + k_y y)}(e^{ik_z z} \text{ or } e^{ik_z z})\]

for \(k_x, k_y, k_z \in \mathbb{R}\) into (56), in which \(V_a(x, y, z) = 0\) is taken. In this case, however, the relations for zero energy \([58]\) and \([60]\) should not be taken but the different relations \(\mathcal{E} + E_z > 0\) must be required. Taking \(\hbar^2k^2_z/2m = \mathcal{E} + E_z\), we have the solutions same as those for the constant potential \(V_2 = -g_a\) in \([12]\), where \(g_a = \hbar^2k^2_z/2m\) and then \(k_x^2 + k_y^2 = k^2\). This fact implies that the parallel vortices discussed in section 4.2 are producible from the plane wave and the degenerate states with non-zero energy in three dimensions.

Real vortex phenomena \([24-31]\) appear in three dimensional spaces. Some of the vortex phenomena will be understood in the cases discussed above.

VI. SHORT NOTES ON ZERO ENERGY SOLUTIONS FOR \(G_\lambda < 0\)

As noted in the section 3.1, we have the zero energy solutions \(\phi_\alpha^\pm(\alpha \varphi) = M_\alpha e^{\pm i\varphi\lambda\alpha}\) of (15). In general they are unnormalizable in the \((x, y)\) plane. In some special cases, however, some of the four can be normalizable. For example, provided that the parameters \(a\) and \(\alpha\) are taken so as to fulfill the relation

\[\cos(a\varphi - \alpha > 0 \quad \text{for} \quad 0 \leq \varphi < 2\pi, (63)\]

\(\phi_\alpha(\alpha \varphi)\) can be normalizable. The relation can be fulfilled by the suitable choices of the parameters such that \(0 < a < 1/2\) and \((-1/2 - 2a)\pi < \alpha < \pi/2\). There
are, of course, different choices, when we take the different solutions from \( \phi_{\alpha}^\pm (u_\alpha(\alpha)) \). It is very hard to answer the question whether the choice of the solutions is physically meaningful or not. Such solutions, however, possibly have some meanings in phenomena limited in very special regions, provided that the solutions are used only in the limited regions and smoothly connected to other functions defined outside the regions. In fact the solutions are used for constructing the vortices from the plane-wave solutions in three-dimensional space. (See the argument of section 5.)

Note also here that the solutions \( \phi_{\alpha}^\pm (u_\alpha(\alpha)) = M_\alpha e^{\pm k_\alpha u_\alpha(\alpha)} \) have no current because they can be taken as real. The higher polynomial solutions with \( n \geq 2 \) of (25) or (26) can, however, have currents because they are generally complex. This means that we have a possibility for producing vortices from these solutions even if they will appear only in very limited regions.

VII. REMARKS ON NON-ZERO ENERGY SOLUTIONS

We shall briefly discuss the equation for non-zero energy given by (56)

\[
\left[-\frac{\hbar^2}{2m}\Delta_a - a^{-2}\mathcal{E}\, \rho^2(1-a)/a\right] \psi(u_a, v_a) = g_a \psi(u_a, v_a).
\]

As noted in section 2, this equation can be read as the equation for determining the strength of the coupling constant \( g_a \) of the original potential \( V_a = -a^2 g_a \rho^{2(a-1)} \) for the given energy \( \mathcal{E} \). We shall, however, discuss it from a different standpoint. If we can solve the eigenvalue problem for the potential of \( -a^{-2}\mathcal{E}\, \rho^2(1-a)/a \), we can obtain the eigenvalues of the original equation

\[
\left[-\frac{\hbar^2}{2m}\Delta - a^{-2} g_a \rho^{2(a-1)}\right] \psi(x, y) = \mathcal{E} \psi(x, y).
\]

Let us show one example for \( a = 1/2 \), where the original potential is written by

\[
V_{1/2}(\rho) = -\frac{1}{4}g_{1/2}\frac{1}{\rho} \quad \text{for} \quad g_{1/2} > 0.
\]

For real and negative eigenvalues (\( \mathcal{E} < 0 \)) the equation (64) can be understood as the two-dimensional harmonic oscillator with the spring constant \( k = 8|\mathcal{E}| \). The eigenvalues of the two-dimensional harmonic oscillator are well known as

\[
E_{n_x, n_y} = (n_x + n_y + 1)\hbar\omega,
\]

where \( n_x \) and \( n_y \) are zero or positive integers, and \( \omega = 2\sqrt{2|\mathcal{E}|}/m \). Thus we have the relation

\[
g_{1/2} = E_{n_x, n_y}.
\]

From this relation we obtain the eigenvalue \( \mathcal{E} \) as

\[
\mathcal{E} = -\frac{mg_{1/2}^2}{8(2N+1)^2\hbar^2},
\]

with \( N = (n_x + n_y)/2 \). We can directly confirm the eigenvalues by solving the original equation for the solutions \( \psi(x, y) = R(\rho)e^{\pm il\phi} \) (\( l_\pm = \text{integers} \)) that correspond to the symmetric solutions of the harmonic oscillator described by \( n_x = n_y \). We see that, provided that one of the eigenvalue problems can be solved, we can also obtain the eigenvalues of the other equation. It is interesting that harmonic oscillator (\( \rho^2 \)) and Coulomb type (\( \rho^{-1} \)) are mapped each other by the conformal mapping and there is a relation between the energy eigenvalues of the two potentials in two dimensions.

VIII. CONCLUDING REMARKS

We have shown that all Schrödinger equations with the symmetric potentials of the type \( V_a(\rho) \) in two dimensions can be reduced to the same equation with a constant potential for the zero energy eigenstates in terms of the conformal mappings and the states with the zero energy are in the infinite degeneracy. The degeneracy becomes not only the origin of the huge variety of vortex patterns but it will possibly be a interesting tool to investigate complicated problems of surface physics including boundaries as well. And the idea can be extended to phenomena in three dimensions. Particularly this scheme will become a powerful tool to vortex phenomena. Actually a vortex lattice solution has been found in this scheme [6]. We may expect that hydrodynamical approach in quantum mechanics presented here will open many interesting aspects in physics such as the investigation of vortex patterns [24-34]. We also note that the eigenstates satisfying the anyon boundaries will be applicable to problems in condensed matter physics such as vortex matters [24,25] and quantum Hall effects [45-49].

It should be noticed that some kinds of equations in hydrodynamics [38-41] are obtainable from the original eigenvalue equation (4) by changing parameters such as \( \hbar \) and mass \( m \). This means that the conformal mappings (4) are applicable to hydrodynamical problems in two dimensions and the infinite degeneracy can also take place. The analysis in terms of the functions obtained in this article will also become an interesting approach in many aspects of hydrodynamical problems.

Finally we would like to note that the infinite degeneracy of the zero energy solutions brings infinite variety to many-body systems with a fixed energy, which possibly becomes an origin of a new entropy different from the Boltzmann entropy [42-44]. The new entropy has nothing to do with the determination of usual temperatures in thermal equilibrium but the new freedom.
stored in the new entropy can be released in thermal non-equilibrium [43]. These considerations will also bring a very new aspect in statistical mechanics extended from Hilbert spaces to Gel’fand triplets [12,44].

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