Analyticity of the susceptibility function for unimodal Markovian maps of the interval

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Abstract

We study the expression (susceptibility)

\[ \Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_I \rho(x) X(x) \frac{d}{dx}(A(f^n x)), \]

where \( f \) is a unimodal Markovian map of the interval \( I \), \( \rho = \rho_f \) is the corresponding absolutely continuous invariant measure and \( A \) is a \( C^1 \) function defined on \( I \). We show that \( \Psi(\lambda) \) is analytic near \( \lambda = 1 \), where \( \Psi(1) \) is formally the derivative of \( \int_I \rho(\text{d}x)A(x) \) with respect to \( f \) in the direction of the vector field \( X \).

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In a previous note [Ru] the susceptibility function was analysed for some examples of maps of the interval. The purpose of the present note is to give a concise treatment of the general unimodal Markovian case (assuming \( f \) real analytic). We hope that it will similarly be possible to analyse maps satisfying the Collet–Eckmann condition. Eventually, as explained in [Ru], application of a theorem of Whitney [Wh] should prove differentiability of the map \( f \mapsto \rho_f \) restricted to a suitable set.

Set-up

Let \( I \) be a compact interval of \( \mathbb{R} \) and \( f : I \to I \) be real analytic. We assume that there is \( c \) in the interior of \( I \) such that \( f'(c) = 0 \), \( f'(x) > 0 \) for \( x < c \), \( f'(x) < 0 \) for \( x > c \) and

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Replacing $I$ by a possibly smaller interval, we assume that $I = [a, b]$ where $a = f^2(c)$, $b = f(c)$. We assume that the postcritical orbit $P = \{ f^n c : n \geq 1 \}$ is finite: $P = \{ p_1, \ldots, p_m \}$; in particular, $f$ is Markovian. We shall assume that $f$ is \textit{analytically expanding} in the sense of assumption A below; in particular, the periodic orbits of $f$ are assumed to be repelling, and therefore $c$ cannot be periodic. We also assume that $f$ is topologically mixing (this can always be achieved by replacing $I$ by a smaller interval and $f$ by some iterate $f^N$).

**Theorem.** Under the above conditions, and assumption A stated later, there is a unique $f$-invariant probability measure $\rho$ absolutely continuous with respect to Lebesgue measure on $I$. If $X$ is real analytic on $I$ and $A \in C^1(I)$, then
\[
\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_I \rho(dx) X(x) \frac{d}{dx} A(f^n x)
\]
extends to a meromorphic function in $C$, without poles on $\{ \lambda : |\lambda| = 1 \}$.

**Change of variable**

The finite set $\{ c \} \cup P$ decomposes $I$ into $m$ subintervals $I_j$, with $2m$ endpoints (we ‘double’ the endpoints of consecutive subintervals, distinguishing between a – endpoint at the right of an interval and a + endpoint at the left). Note that $\eta = \{ I_j : j = 1, \ldots, m \}$ is a Markov partition for the map $f$. Consider the critical values of $f^n$. Then for large $n > 0$, the set of critical values will be stabilized and is always $P$. We define \textit{polar} endpoints as follows:

1. $p \in P$ is a polar $-$ endpoint of an interval in $\eta$ if $p$ is a local maximum value of $f^n$ for $n$ large.
2. $v \in P$ is a polar $+$ endpoint of an interval in $\eta$ if $p$ is a local minimum value of $f^n$ for $n$ large.

Every $p \in P$ is a polar $-$ or $+$ endpoint and may be both, $c$ is a nonpolar endpoint on both sides.

We define now an increasing continuous map $\sigma : I \to R$ so that $J = \sigma I$ is a compact interval. We write $\sigma I_j = J_j$ for $1 \leq j \leq m$; denote by $\omega$ the inverse of $\sigma$. We assume that $\omega|J_j$ extends to a holomorphic function in a complex neighbourhood of $J_j$ for $1 \leq j \leq m$ and that for $q \in \{ c \} \cup P$, $\omega$ has the property
\[
\omega(\sigma q \pm \xi) = \omega(\sigma q) \pm \frac{\xi^2}{2} + O(\xi^4)
\]
if $q$ is a $\pm$ polar endpoint, and
\[
\omega(\sigma q \pm \xi) = \omega(\sigma q) \pm \xi + O(\xi^2)
\]
if $q$ is a nonpolar endpoint. (We should really consider disjoint copies of the $I_j$ and $J_j$, and disjoint neighbourhoods of these in $C$ or in a Riemann surface two-sheeted near polar endpoints. This would lead to notational complications that we prefer to omit.) Applications of this singular change of coordinate have been used in [Ji1, BJR, Ru]; the reference [Ji2] contains some more relevant study regarding the method of singular change of coordinates in one-dimensional dynamical systems. The reader is encouraged to compare this method with orbifold metrics in [Th, chapter 13]. Another relevant application of this method in complex dynamical systems can be found in [DH]. From now on we shall say that $\sigma q$ is a $\pm$ polar (nonpolar) endpoint if $q$ is $\pm$ polar (nonpolar).
The dynamical system viewed after the change of variable

For any two intervals $I_j, I_k \in \eta$ with $f I_j \supset I_k$, we define
\[ \psi_{jk} = \sigma \circ (f|I_j)^{-1} \circ (\omega|I_k). \]

Note that the $\psi_{jk}$ are restrictions of inverse branches of $g = \sigma \circ f \circ \omega : J \to J$ to intervals in $\omega \eta$. The function $\psi_{jk} : J_k \to J_j$ extends holomorphically to a complex neighbourhood of $J_k$. Indeed, note that $(f|I_j)^{-1}$ is holomorphic except if $I_j$ is one of the two intervals around $c$, in which case the singularity is corrected by $\omega|J_n$, where $J_n$ is the rightmost interval in $\omega \eta$. In other cases $\omega|J_k$ cancels the singularity of $\sigma|I_j$ by our definition of $\omega$. (Note that $\psi_{jk}'(x) \geq 0$ or $\leq 0$ on $J_k$ and may vanish only at an interval endpoint.)

**Assumption A.** Each $J_k$, for $k = 1, \ldots, m$, has a bounded open connected neighbourhood $U_k$ in $C$ such that $\psi_{jk} : J_k \to J_j$ extends to a continuous function $\psi_{jk} : \bar{U}_k \to C$ holomorphic in $U_k$ and with $\psi_{jk} \bar{U}_k \subset U_j$.

One checks that the sets $U_k$ can be assumed to be in $\epsilon$-neighbourhoods of the $J_k$. Also, assumption A implies that periodic points for $g$ are strictly repelling. The smoothness of $\omega$, $\sigma$ in the interior of subintervals shows that the same property holds for $f$, apart from interval endpoints where we however also assume the property to hold:

the periodic orbits of $f$ are strictly repelling.

Markovian graph

Consider the Markov partition $\eta = \{ I_j \}$. Let us write $j > k$ ($j$ covers $k$) if $f I_j \supset I_k$ (we allow $j > j$). This defines a directed graph with vertex set $\{ 1, \ldots, m \}$ and oriented edges $(j, k)$ for $j > k$. Since we have assumed our dynamical system $f$ to be topological mixing, our graph is also mixing in the sense that there is $N \geq 1$ such that for all $j, k \in \{ 1, \ldots, m \}$ we have $j > \cdots > k$ ($N$ edges) corresponding to $f^N I_j \supset I_k$.

Transfer operators

For a function $\Phi = (\Phi_j)$ defined on $\sqcup J_j$, we write
\[
(\mathcal{L} \Phi)_k(z) = \sum_{j \succ k} \sgn(j) \psi_{jk}'(z) \Phi_j(\psi_{jk} z),
\]
\[
(\mathcal{L}_0 \Phi)_k(z) = \sum_{j \succ k} \sgn(j) \Phi_j(\psi_{jk} z),
\]
where $\sgn(j)$ is $+1$ if $\psi_{jk}$ is increasing on $J_k$ and $-1$ if $\psi_{jk}$ is decreasing on $J_k$. If $H$ is the Hilbert space of functions on $\sqcup_{j \in \mathbb{L}} \bar{U}_j$ which are square integrable with respect to Lebesgue measure, and have holomorphic restrictions to the $U_j$, then $\mathcal{L}$ and $\mathcal{L}_0$ acting on $H$ are holomorphy improving, hence compact and of trace class.

Properties of $\mathcal{L}$ (refer to [B])

For $x \in J_k$ we have
\[
(\mathcal{L} \Phi)_k(x) = \sum_{j \succ k} |\psi_{jk}'(x)| \Phi_j(\psi_{jk} x),
\]
hence $\Phi \geq 0$ implies $\mathcal{L}\Phi \geq 0$ ($\mathcal{L}$ preserves positivity) and
\[
\int_j dx (\mathcal{L}\Phi)(x) = \sum_k \int_{J_k} dx (\mathcal{L}\Phi)_k(x) = \sum_j \int_{J_j} dx \Phi_j(x) = \int_j dx \Phi(x)
\]
($\mathcal{L}$ preserves mass). Using mixing one finds that $\mathcal{L}$ has a simple eigenvalue $\mu_0 = 1$ corresponding to an eigenfunction $\sigma_0 > 0$. The other eigenvalues $\mu_\ell$ satisfy $|\mu_\ell| < 1$ and their (generalized) eigenfunctions $\sigma_\ell$ satisfy $\int_{J_\ell} dx \sigma_\ell(x) = 0$. If we normalize $\sigma_0$ by $\int_{J_0} dx \sigma_0(x) = 1$, then $\sigma_0(x)dx$ is the unique $g$-invariant probability measure absolutely continuous with respect to Lebesgue measure on $J$. In particular, $\sigma_0(x)dx$ is ergodic.

Let now, $H_1 \subset H$ consist of those $\Phi = (\Phi_k)$ such that the derivative $\Phi'_1$ vanishes at the (polar) endpoints $\sigma a, \sigma b$ of $J$ and such that at the common endpoint $\sigma q (q \in \{c\} \cup \{\sigma a, \sigma b\})$ of two subintervals we have equality on both sides of a quantity which is either
- the value of $\Phi$ for a nonpolar endpoint or
- the value of $\pm \Phi'$ for a polar $\pm$ endpoint.

We note that $\mathcal{L}H_1 \subset H_1$ (this requires a case by case discussion). Furthermore $\sigma_0 \in H_1$ (take $\phi \in H$ such that $\phi \geq 0$, $\int_{J_0} dy \phi(y) = 1$ and $\phi, \phi'$ vanishes at subinterval endpoints; then $\phi \in H_1$ and $\sigma_0 = \lim_{n \to \infty} \mathcal{L}^n \phi \in H_1$).

**Evaluating $\Psi(\lambda)$**

The image $\rho(dx) = \rho(x)dx$ of $\sigma_0(y)dy$ by $\omega$ is the unique $f$-invariant probability measure absolutely continuous with respect to Lebesgue measure on $I$. We have
\[
\rho(x) = \sigma_0(\sigma x)\sigma'(x).
\]
Consider now the expression
\[
\Psi(\lambda) = \sum_{n=0}^\infty \lambda^n \int_J \rho(dx) X(x) \frac{d}{dx} A(f^n x),
\]
where we assume that $X$ extends to a holomorphic function in a neighbourhood of each $J_k$ and takes the same value on both sides of common endpoints of intervals in $\eta$ (continuity). Also assume that $A \in C^1(I)$. For sufficiently small $|\lambda|$, the series defining $\Psi(\lambda)$ converges. Writing $B = A \circ \omega$ ($B$ has piecewise continuous derivative) and $x = \omega y$ we have
\[
X(x) \frac{d}{dx} A(f^n x) = X(\omega y) \frac{1}{\omega'(y)} \frac{d}{dy} B(g^n y),
\]
hence
\[
\Psi(\lambda) = \sum_{n=0}^\infty \lambda^n \int_J dy \sigma_0(y) \frac{X(\omega y)}{\omega'(y)} \frac{d}{dy} B(g^n y).
\]
Defining $Y(y) = \sigma_0(y)X(\omega y)/\omega'(y)$, we see that $Y$ extends to a holomorphic function in a neighbourhood of each $J_k$, which we may take to be $U_k$, except for a simple pole at each polar endpoint of $J_k$. Since $\sigma_0 \in H_1$, the properties assumed for $\omega$ imply that also $(X \circ \omega) \times \sigma_0 \in H_1$.

Note that near a nonpolar subinterval endpoint $\sigma q$
\[
\omega'(\sigma q \pm \xi) = 1 + O(\xi)
\]
and near a polar endpoint
\[
\omega'(\sigma q \pm \xi) = \xi + O(\xi^3).
\]
Therefore
\[ Y(\sigma q \pm \xi) = A^\pm \frac{1}{\xi} + B^\pm + O(\xi), \]
where \( B^+ = B^- \) for the two sides of \( \sigma q \) and \( B^+ = 0 \) at the left endpoint \( \sigma a \) of \( J \), \( B^- = 0 \) at the right endpoint \( \sigma b \) of \( J \). We may write
\[ \int_J dy \sigma_0(y) X(\omega y) \frac{d}{dy} B(g^n y) = \int_J dy Y(y) g'(y) \cdots g'(g^{n-1} y) B'(g^n y) \]
where \( L_0 \) has been defined above, and we have thus
\[ \Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_J ds(L_0^n Y)(s)B'(s). \]

**Properties of \( L_0 \)**

We let now \( H_0 \subset H \) be the space of functions \( \Phi = (\Phi_k) \) vanishing at the endpoints \( \sigma a, \sigma b \) of \( J \) and such that the values of \( \Phi \) on both sides of common endpoints of intervals \( J_k \) coincide (continuity). Therefore \( L_0 H_0 \subset H_0 \).

There is a periodic orbit \( \gamma_1, \ldots, \gamma_p \) (with \( g' \gamma_i = \gamma_{i+1(\mod p)} \)) of polar endpoints where \( \gamma_{ja} \) is the \( \pm \) endpoint of some subinterval \( J_{k(\alpha)} \). Choose \( P_a \) to be 0 on subintervals different from \( J_{k(\alpha)} \) and to be holomorphic on a complex neighbourhood of \( J_{k(\alpha)} \) except at \( \gamma_a \). Also assume that
\[ P_a(\gamma_a \pm \xi) = \frac{1}{\xi} + O(\xi) \]
and that \( P_a \) vanishes at the endpoint of \( J_{k(\alpha)} \) different from \( \gamma_a \). Then
\[ L_0 P_a - |f'(\gamma(\alpha))|^{1/2} P_{a+1(\mod p)} \in H_0. \]

Therefore \( L_0^p P_1 - \Lambda P_1 = u \in H_0 \) where \( \Lambda = \prod_{a=1}^{p} |f'(\gamma(\alpha))|^{1/2} > 1 \). Since the spectrum of \( L \) acting on \( H \) is contained in the closed unit disk and since the derivative \( u' \) is in \( H \), we may define \( v = (L_0^p - \Lambda)^{-1} u' \in H \). Since \( \int_J dy u'(y) = 0 \) we also have \( \int_J dy v(y) = 0 \) and we can take \( w \in H_0 \) such that \( u' = v \). We have thus
\[ ((L_0^p - \Lambda)w)' = (L_0^p - \Lambda)w' = (L_0^p - \Lambda)w = u' \]
so that \( (L_0^p - \Lambda)w = u \) (there is no additive constant of integration since \( (L_0^p - \Lambda)w \) and \( u \) are in \( H_0) \). Finally
\[ (L_0^p - \Lambda)(P_1 - w) = 0. \]

There is thus a \( L_0 \)-invariant \( p \)-dimensional vector space spanned by vectors \( P_a - w_a \) with \( w_a \in H_0 \), such that the spectrum of \( L_0 \) restricted to this space consists of eigenvalues \( \omega_\ell \) with
\[ \omega_\ell = \Lambda^{1/p} e^{2\pi i \ell/p} = \left| \prod_{a=1}^{p} f'(\gamma_a) \right|^{1/2} e^{2\pi i \ell/p} \]
for \( \ell = 0, \ldots, p - 1 \).

For the postcritical but nonperiodic polar points \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_a \) define \( \tilde{P}_a \) like \( P_a \) above, with \( \gamma \) replaced by \( \tilde{\gamma} \). For each \( \beta \) there is \( \alpha = \alpha(\beta) \) with
\[ L_0^p (\tilde{P}_\beta - \Lambda_{\beta} P_a) \in H_0 \]
with some \( \Lambda_{\beta} \neq 0 \), hence
\[ L_0^p (\tilde{P}_\beta - \Lambda_{\beta}(P_a - w_a)) = \tilde{Y}_\beta \in H_0. \]
Poles of $\Psi(\lambda)$

We may now write

$$Y = Y_0 + Y_1 + Y_2,$$

where

$$Y_0 \in H_0,$$

$$Y_1 = \sum_{u=1}^{p} c_u (P_u - w_u),$$

$$Y_2 = \sum_{\beta=1}^{q} \tilde{c}_{\beta} (\tilde{P}_{\beta} - \tilde{\Lambda}_\beta (P_{u(\beta)} - w_{u(\beta)}))$$

and there is a corresponding decomposition $\Psi_1(\lambda) = \Psi_0(\lambda) + \Psi_1(\lambda) + \Psi_2(\lambda)$. Here $\Psi_1(\lambda)$ is a sum of terms $C_\ell(\lambda - \omega_\ell)$ where $\omega_\ell = \Lambda^{1/p} \times \text{th root of unity}$; $\Psi_2(\lambda)$ is polynomial of degree $q - 1$ in $\lambda$ plus $\Lambda^{1/q} \sum_{\beta=1}^{p} \tilde{c}_{\beta} \tilde{\Lambda}_\beta (\Psi_0(\lambda))$ where $\tilde{\Psi}_\beta$ is obtained if we replace $Y$ by $\tilde{Y}_\beta$ in the definition of $\Psi$. The poles of $\Psi(\lambda)$ are thus those of $\Psi_1(\lambda)$ at the $\omega_\ell$ and those of $\Psi_0(\lambda)$ and $\tilde{\Psi}_\beta(\lambda)$. The discussion is the same for $\Psi_0$ and the $\tilde{\Psi}_\beta$, we shall thus only consider $\Psi_0$. Since $Y_0 \in H_0$ and $L_0 H_0 \subset H_0$ we have

$$\Psi_0(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{J} \text{ds} (L_0^n Y_0)(s) B(s) = - \sum_{n=0}^{\infty} \lambda^n \int_{J} \text{ds} (L_0^n Y_0')(s) B(s)$$

$$= - \sum_{n=0}^{\infty} \lambda^n \int_{J} \text{ds} (L_0^n Y_0')(s) B(s).$$

It follows that $\Psi_0(\lambda)$ extends meromorphically to $\mathbb{C}$ with poles at the $\mu_{k}^{-1}$. We want to show that the residue of the pole at $\mu_{0}^{-1} = 1$ vanishes. Since $\int_{J} \text{d} y_0(y) = 0$ for $k \geq 1$, the coefficient of $\sigma_0$ in the expansion of $Y_0'$ is proportional to

$$\int_{J} \text{d} y_0'(y) = Y(\sigma b) - Y(\sigma a) = 0$$

because $Y_0 \in H_0$. Therefore $\Psi_0(\lambda)$ is holomorphic for $|\lambda| = 1$ and the same holds for the $\tilde{\Psi}_\beta(\lambda)$, concluding the proof of the theorem. In fact we know that the poles of $\Psi(\lambda)$ are located at $\mu_{k}^{-1}$ for $k \geq 1$ and at $\omega_{\ell}^{-1}$ for $\ell = 0, \ldots, p - 1$, so that $|\mu_{k}^{-1}| > 1, |\omega_{\ell}^{-1}| < 1$.

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