Subleading Correction to Statistical Entropy for BMPV Black Hole*

Nabamita Banerjee

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, INDIA
Email: nabamita@mri.ernet.in

ABSTRACT: We study higher derivative corrections to the statistical entropy function and the statistical entropy for five dimensional BMPV black holes by doing the asymptotic expansion of the partition function. This enables us to evaluate entropy for a large range of charges, even out of Cardy (Farey tail) limit.

KEYWORDS: Higher derivative, Degeneracy, Statistical Entropy.

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1. Introduction

Counting of 1/4 BPS dyonic states in four dimensional $\mathcal{N} = 4$ supersymmetric string theories has been studied in great detail in last few years\cite{1,2,3,4,5,6,7,8}. We now have a good understanding of the degeneracy formula, its moduli dependence and the wall crossing formulae. Large charge asymptotic expansion of these degeneracy formulae exactly capture the dyonic black hole entropy including certain subleading corrections due to higher derivative corrections to the supergravity.

Five dimensional spinning (BMPV) black holes\cite{9} is a close cousin of the four dimensional dyonic black hole. These black holes were first constructed in\cite{10}, as a spinning generalization of\cite{11}. These are charged, spinning, 5-dimensional black holes with constant dilation and constant moduli in type $IIB$ theory on $K3 \times S^1$. The microscopic configurations of these black holes can be described as $p$-solitonic states ($Dp$-branes) in type $IIB$ theory on $K3 \times S^1$, for $p = 1, 3, 5$. The states also carry certain momenta along the $S^1$ circle and angular momenta along the non compact directions. This microscopic description is much similar to that of the four dimensional dyonic black holes. In fact, when described in terms of D-branes, the BMPV black hole consists of $D1$-$D5$-$p$ system, whereas the four dimensional dyonic black hole in addition has a KK monopole background. It is therefore natural to study BMPV black hole entropy in terms of the four dimensional dyon degeneracy.
formula without KK monopole contribution. A general degeneracy formula for D1-D5-p system is then easy to write down. However, we are interested in finding out subleading correction to the BMPV black hole entropy due to higher derivative terms in the effective action. Higher derivative correction to five dimensional black holes has been computed in [11] and their entropy has been computed [12]. In this paper we will take a different approach to this problem. Determination of subleading correction is done in a most effective fashion using the statistical entropy function and the effective action formalism. Using the statistical entropy function one can write down a one dimensional effective action, and using the Feynman diagram technique one can obtain systematic large charge asymptotic expansion of the statistical entropy. This method correctly reproduces subleading correction to the entropy of four dimensional 1/4 BPS dyonic black holes.

This particular feature of the statistical entropy function gives the motivation to compute similar subleading correction to the five dimensional BMPV black hole. The exactness of the statistical entropy (or the statistical entropy function) suggests that we can evaluate the entropy (or the entropy function) to any order.

The rests of the paper is divided into three sections. In the first section, we present a different form of the degeneracy function of the five dimensional BMPV black holes, based on the degeneracy of the four dimensional dyonic black holes. In the next section we discuss the first subleading \(O(Q^0)\) correction to the statistical entropy function and statistical entropy of these black holes. In the last section, we have some discussions on our results.

As this paper was being written a paper [13] appeared in the arXiv that discusses similar issues.

2. Degeneracy Function For 5-dimensional BMPV Black Holes

In this section, we rewrite the degeneracy function of the BMPV black holes in a different form. The microscopic description for this black hole is a particular D-brane configuration in type IIB theory compactified on \(K3 \times S^1\). This contains \(Q_1\) number of D1 branes, \(Q_5\) number of D5 branes, \(-n\) units of momenta along \(S^1\) circle and angular momenta \(J_1\) and \(J_2\) along the non-compact spatial directions. This configuration, however does not contain any D3 branes. For extremal black holes, the corresponding microscopic configuration requires the modulus of the two angular momenta to be same \(|J_1| = |J_2| = J\). The microscopic computation for the leading entropy was first done in [9]. We will write \(\alpha'\) (inverse string tension) exact degeneracy function for this configuration, from the knowledge of the degeneracy function of 4-dimensional dyonic black holes in \(\mathcal{N} = 4\) supersymmetric string theory. Here we will sketch in brief how the degeneracy function was obtained for these 4-dimensional black holes [14], [15].
2.1 Degeneracy Function of 4-dimensional Dyonic Black holes

Let us consider type IIB theory compactified on $K3 \times S^1 \times \tilde{S}^1$. Following a chain of duality transformations, one can look at the same theory as a heterotic string theory compactified on $T^6$. These theories have dyonic black-hole solutions. Let us consider a specific configuration in this compactified type IIB theory:

- $Q_1$ number of D1-branes wrapped along $S^1$,
- $Q_5$ number of D5-brane wrapped along $K3 \times S^1$,
- a single Kaluza-Klein monopole associated with $\tilde{S}^1$ circle,
- $-n$ units of momentum along $S^1$ direction and $J$ units of angular momentum along $\tilde{S}^1$ direction. In the dual Heterotic picture, this represents dyonic black hole solutions. If we stay in a region of the moduli space where the type IIB theory is weakly coupled, the partition function of the entire system can be obtained by considering three weakly interacting sources:

1. the relative motion of the D1-brane in the plane of D5-brane, carrying certain momenta $-L$ along $S^1$ and $J' \tilde{S}^1$ directions,

2. the center of mass motion of D1-D5 system in the KK-monopole background carrying momenta $-l_0$ along $S^1$ and $j_0$ along $\tilde{S}^1$ directions,

3. excitations of the KK-monopole carrying $-l_0'$ momentum along $S^1$,

with $n = L + l_0 + l_0'$ and $J = J' + j_0$ being the sum of momenta along $S^1$ and $\tilde{S}^1$ directions respectively. Hence, in the weak coupling limit, the partition function $f(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ of the configuration can be expressed as,

$$f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = -\frac{1}{64} \left( \sum_{Q_1, L, J'} (-1)^J' d_{D1}(Q_1, L, J') e^{2\pi i (\tilde{\rho}Q_1/\sqrt{N} + \tilde{\sigma}L + \tilde{v}J')} \right) \left( \sum_{l_0, j_0} (-1)^{j_0} d_{CM}(l_0, j_0) e^{2\pi i l_0 \tilde{\rho} + 2\pi i j_0 \tilde{v}} \right) \left( \sum_{l'_0} d_{KK}(l'_0) e^{2\pi il'_0 \tilde{\rho}} \right),$$

(2.1)

where $d_{D1}(Q_1, L, J')$ is the degeneracy of source (1), $d_{CM}(l_0, j_0)$ is the degeneracy associated with source (2) and $d_{KK}(l'_0)$ denotes the degeneracy associated with source (3). The factor of 1/64 accounts for the fact that a single quarter BPS supermultiplet has 64 states. Evaluating these three pieces separately, the full partition function of the system looks like,

$$f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = e^{-2\pi i (\tilde{\rho} + \tilde{v})} \prod_{k' \in \mathbb{Z}, l \in \mathbb{Z}, j \in \mathbb{Z}} \left( 1 - e^{2\pi i (\tilde{\rho}k' + \tilde{\sigma}l + \tilde{v}j)} \right)^{-c(4l' - j^2)}. \quad (2.2)$$
Then we define the degeneracy function $\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ and degeneracy of states $d(\vec{Q}, \vec{P})$ as,

$$f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \frac{e^{2\pi i \tilde{\sigma}}}{\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})},$$

$$d(\vec{Q}, \vec{P}) = (-1)^{Q-P+1} h \left( \frac{1}{2} Q^2, \frac{1}{2} P^2, Q \cdot P \right),$$ (2.3)

where $(\vec{Q}, \vec{P})$ are the charge vectors carried by the black holes:

$$Q = \begin{pmatrix} 0 \\ -n \\ 0 \\ -1 \end{pmatrix}, \quad P = \begin{pmatrix} Q_5(Q_1 - Q_5) \\ -J \\ Q_5 \\ 0 \end{pmatrix},$$ (2.4)

and $h(m, n, p)$ are the coefficients of Fourier expansion of the function $1/\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$:

$$\frac{1}{\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})} = \sum_{m, n, p} g(m, n, p) e^{2\pi i (m \tilde{\rho} + n \tilde{\sigma} + p \tilde{v})}.$$ (2.5)

At this point it is worth noting from equations (2.2) and (2.5) that the power series gets a contribution $e^{-2\pi i \tilde{v}}(1 - e^{-2\pi i \tilde{v}})^{-1}$ from $k' = l = 0$ term and one can expand the series either in $e^{-2\pi i \tilde{v}}$ or in $e^{2\pi i \tilde{v}}$. There is an ambiguity in the expansion, we will come back to this point in the last section.

To evaluate the degeneracy of a state associated with charges $(\vec{Q}, \vec{P})$, we need to invert equation (2.5) as,

$$d(\vec{Q}, \vec{P}) = (-1)^{Q-P+1} \int_{\mathcal{C}} d\tilde{\rho} d\tilde{\sigma} d\tilde{v} e^{-\pi i (\tilde{\rho} Q^2 + \tilde{\sigma} P^2 + 2\tilde{v} Q \cdot P)} \frac{1}{\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})},$$ (2.6)

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labeled by $(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$, given by

$$\tilde{\rho}_2 = M_1, \quad \tilde{\sigma}_2 = M_2, \quad \tilde{v}_2 = -M_3, \quad 0 \leq \tilde{\rho}_1 \leq 1, \quad 0 \leq \tilde{\sigma}_1 \leq 1, \quad 0 \leq \tilde{v}_1 \leq 1.$$ (2.7)

$M_1, M_2$ and $M_3$ are large but fixed positive numbers with $M_3 << M_1, M_2$. The choice of the $M_i$'s is determined from the requirement that the Fourier expansion is convergent in the region of integration.

The $\mathcal{N} = 4$ supersymmetric string theories discussed above are invariant under $O(6, 22, \mathbb{Z})$ T-duality and $SL(2, \mathbb{Z})$ S-duality symmetry. The T-duality invariants are given as,

$$Q^2 = 2n, \quad P^2 = 2Q_5(Q_1 - Q_5), \quad Q \cdot P = J.$$ (2.8)

The function $\tilde{\Phi}$ actually behaves as a modular form of weight $k = 10$ under the S-duality group $SL(2, \mathbb{Z})$. 
2.2 Degeneracy for BMPV Black Holes

Here we will compute the degeneracy function for BMPV black holes from our knowledge of the degeneracy function of 4-dimensional black holes we studied in the last section. Comparing with the four-dimensional black hole, we will treat the microscopic configuration of the BMPV black holes to be same as the one considered in 4-dimensional case except for the following changes. The radius of the $\tilde{S}^1$ circle is infinite and therefore the KK-monopole sector is replaced by $\mathbb{R}^4$. We will again work in a region of the moduli space where IIB theory is weakly coupled. The partition function of this configuration will only get contribution from the source (1) of section (2.1), i.e., the relative motion of the D1-branes in the plane of D5-branes. Hence, we have,

$$f_{\text{bmpv}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = -\frac{1}{64} \left( \sum_{Q_1, L, J'} (-1)^J d_{D1}(Q_1, L, J') e^{2\pi i(\tilde{\sigma} Q_1/N + \tilde{\rho} L + \tilde{v} J')} \right)$$

$$= \prod_{k', l, j \in \mathbb{Z}} \left( 1 - e^{2\pi i(\tilde{\sigma} k' + \tilde{\rho} l + \tilde{v} j)} \right)^{-c(4k' - j^2)}. \quad (2.9)$$

Following the steps given in equations (2.3), (2.5) and (2.6), we define the degeneracy function and degeneracy of states for the BMPV black hole. The degeneracy function is given as,

$$\tilde{\Phi}_{\text{bmpv}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \frac{\tilde{\Phi}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}{G(\tilde{\rho}, \tilde{v})}, \quad (2.10)$$

where,

$$G(\tilde{\rho}, \tilde{v}) = 64e^{2\pi i(\tilde{\rho} + \tilde{v})}(1 - e^{-2\pi i\tilde{v}})^2 \prod_{n=1}^{\infty} (1 - e^{2\pi in\tilde{\rho}})^{20}(1 - e^{2\pi i(n\tilde{\rho} + \tilde{v})})^2$$

$$= (1 - e^{2\pi i(n\tilde{\rho} - \tilde{v})})^2. \quad (2.11)$$

Here the function $G(\tilde{\rho}, \tilde{v})$ basically captures the degeneracy of the KK-monopole sector and the center of mass motion of the D1-D5 system in KK-monopole background for four dimensional dyonic black holes.

3. Correction to The Statistical Entropy Function

Similar to the 4-dimensional black hole, we define the degeneracy of states for the BMPV black holes as,

$$d(Q, P) = (-1)^{Q-P+1} \int_C d\tilde{\rho} d\tilde{\sigma} d\tilde{v} e^{-\pi i(\tilde{\rho} Q^2 + \tilde{\sigma} P^2 + 2\tilde{v} Q - P)} \frac{1}{\tilde{\Phi}_{\text{bmpv}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}. \quad (3.1)$$
The statistical entropy for the system is then given as,

$$S_{\text{stat}} = \ln d(\vec{Q}, \vec{P}) . \quad (3.2)$$

One can evaluate the integral (3.1) by saddle point method and estimate the statistical entropy for the system. We will take a different approach to estimate the entropy. From the integral (3.1), we will first evaluate a function $\Gamma_{\text{stat}}$ analogous to black hole entropy function. This function is called the statistical entropy function. The statistical entropy is then obtained as the value of this function at its extrema. This can be done by following two steps:

- The $v$ integral is done by residue methods. The function $\Phi_{\text{bmpv}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ has a zero at

$$\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v} = 0 . \quad (3.3)$$

Near this pole the function $\Phi_{\text{bmpv}}$ behaves as,

$$\Phi_{\text{bmpv}}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = (2v - \rho - \sigma)^k v^2 \frac{g(\rho) g(\sigma)}{G(\rho, \sigma, v)} , \quad (3.4)$$

where

$$\rho = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2}{\tilde{\rho}}, \quad \sigma = \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v} - 1)^2}{\tilde{\rho}}, \quad v = \frac{\tilde{\rho}\tilde{\sigma} - \til{v}^2 + \til{v}}{\til{\rho}} , \quad (3.5)$$

$k$ is related to the rank $r$ of the gauge group via the relation

$$r = 2k + 8 , \quad (3.6)$$

and $g(\tau)$ is a known function which depends on the details of the theory. Typically it transforms as a modular function of weight $(k + 2)$ under a certain subgroup of the $SL(2, \mathbb{Z})$ group. In the $(\rho, \sigma, v)$ variables the pole at (3.3) is at $v = 0$. Near this pole, the integrand looks like, $v^{-2} F(\rho, \sigma, v) \hat{G}(\rho, \sigma, v)$ where,

$$F(\rho, \sigma, v) = \frac{(2v - \rho - v)^{(-k-3)}}{g(\rho) g(\sigma)} e^{-i\pi \left( \frac{2^p q_{2p+q}^2}{2^q p + \omega_{2^p q + 2^q p}} + \frac{2(v-\rho)^2}{2^q p} Q P \right)} ,$$

$$\hat{G}(\rho, \sigma, v) = 64 e^{2\pi i \frac{1+q}{2^q p}} (1 - e^{-2\pi i \frac{v-\rho}{2^q p - \omega}})^2 \prod_{n=1}^{\infty} (1 - e^{2\pi i \frac{n}{2^q p - \omega}})^2 (1 - e^{2\pi i \frac{n(q+1)(v-\rho)}{2^q p + \omega}})^2 (1 - e^{2\pi i \frac{n(q+1)(v+\rho)}{2^q p + \omega}})^2 . \quad (3.7)$$

After doing the $v$ integral using the above relation, (3.1) takes the form,

$$e^{S_{\text{stat}}(\vec{Q}, \vec{P})} \equiv d(\vec{Q}, \vec{P}) \sim \int \frac{d^2 \tau}{\tau^2} e^{-\Phi_{\text{bmpv}}(\tau)} , \quad (3.8)$$
where \( \tau_1 \) and \( \tau_2 \) are two complex variables, related to \( \rho \) and \( \sigma \) via
\[
\rho \equiv \tau_1 + i\tau_2, \quad \sigma \equiv \tau_1 - i\tau_2, \quad (3.9)
\]
and the effective action \( F_{bmpv} \) is given as,
\[
F_{bmpv}(\vec{\tau}) = F(\vec{\tau}) - \ln \hat{G}(\vec{\tau}) - \ln \left(1 + \frac{f\hat{G}'}{Gf'}(\vec{\tau})\right)
\]
\[
F(\vec{\tau}) = -\left[\frac{\pi}{2\tau_2} |Q - \tau P|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k + 2) \ln(2\tau_2) + \ln \left\{K_0 \left(2(k + 3) + \frac{\pi}{\tau_2}|Q - \tau P|^2\right)\right\}\right],
\]
\[
K_0 = \text{constant}. \quad (3.10)
\]
The function \( F(\vec{\tau}) \) is actually the effective action for 4-dimensional black holes. The function \( \hat{G}(\vec{\tau}) \) and \( f(\vec{\tau}) \) are same as \( \hat{G}(\rho, \sigma, v) \) and \( F(\rho, \sigma, v) \) in (3.7) respectively, evaluated at \( v = 0 \) and expressed as functions of \( \vec{\tau} \). Here ’ means derivative with respect to \( v \) evaluated at \( v = 0 \). We give the expressions for the function \( \hat{G}(\vec{\tau}) \) here for later use:
\[
\hat{G}(\vec{\tau}) = -64e^{-\frac{\pi}{\tau_2}(1-\tau_1)}(1 + e^{-\frac{\pi}{\tau_2}\tau_1})^2 \prod_{n=1}^{\infty} \left(1 - e^{-\frac{\pi}{\tau_2}n}\right)^2 \prod_{n=1}^{\infty} \left(1 + e^{-\frac{\pi}{\tau_2}(n+\tau_1)}\right)^2 (1 + e^{-\frac{\pi}{\tau_2}(n+\tau_1)})^2. \quad (3.11)
\]

Next we evaluate (3.8) by considering it to be a zero dimensional field theory with fields \( \tau, \bar{\tau} \) (or equivalently \( \tau_1, \tau_2 \)) and action \( F_{bmpv}(\vec{\tau}) - 2\ln \tau_2 \). We apply background field method technique to obtain the statistical entropy function. In this method, we do an asymptotic expansion of the action around a fixed background point \( \vec{\tau}_B \), which is not the saddle point of the action. This expansion is valid for
\[
Q^2 > 0 \quad P^2 > 0 \quad Q^2 P^2 > (Q \cdot P)^2. \quad (3.12)
\]
The statistical entropy function (to a certain order in charges) is then given as a sum of all 1PI vacuum diagrams (required to that order) in this zero dimensional field theory.

We now want to evaluate the four derivative, i.e., \( O(Q^0) \) correction to the statistical entropy. The last term in (3.11) is of \( O(Q^{-2n}, n \geq 1) \). Similarly, the last term in \( F(\vec{\tau}) \) also goes as \( O(Q^{-2n}, n \geq 1) \). Hence, up to order \( Q^0 \), these terms will not contribute. The first term in \( F(\vec{\tau}) \) is \( O(Q^2) \), while the second term of \( F(\vec{\tau}) \) and \( F_{bmpv}(\vec{\tau}) \) are \( O(Q^0) \). Therefore the first term needs to be expanded up to one loop, whereas the other two terms are required at tree level.
Taking all these issues in to account, we find the statistical entropy function up to order $Q^0$ as,

$$
\Gamma^\text{stat}_{\text{bmpv}}(\vec{\tau}_B) = \Gamma_0(\vec{\tau}_B) + \Gamma_1(\vec{\tau}_B) - \ln \hat{G}(\vec{\tau}_B)
$$

$$
\Gamma_0(\vec{\tau}_B) = \frac{\Pi}{2T_B} |Q - \tau_B P|^2 \sim O(Q^2)
$$

$$
\Gamma_1(\vec{\tau}_B) = \ln g(\tau_B) + \ln g(-\bar{\tau}_B) + (k + 2) \ln(2\tau_B) - \ln(4\pi K_0) \sim O(Q^0).
$$

(3.13)

4. Correction to Statistical Entropy

The statistical entropy of the system can be obtained by extremizing the function $\Gamma^\text{stat}_{\text{bmpv}}$ and evaluating it at its extrema. It is an straightforward exercise to check that for evaluating the entropy up to order $Q^0$, it is enough to compute $\Gamma^\text{bmpv}$ at the extrema of $\Gamma_0$, given as,

$$
(\tau_0)_1 = \frac{Q \cdot P}{P^2}, \quad (\tau_0)_2 = \frac{\sqrt{Q^2P^2 - (Q \cdot P)^2}}{P^2}.
$$

(4.1)

Correction to this extrema due to $\Gamma_1$ will give $O(Q^{-2})$ correction to the entropy. The expressions for the corrected entropy is,

$$
S^\text{stat}_{\text{bmpv}} = \Gamma^\text{stat}_{\text{bmpv}}(\vec{\tau}_0).
$$

(4.2)

Here, we give the approximate statistical entropies $S_{\text{stat}}^{(0)} = S^{(0)}$ calculated using the ‘tree level’ statistical entropy function, $S_{\text{stat}}^{(1)} = S^{(0)} + S^{(1)}$ calculated using the ‘tree level’ and ‘one loop’ statistical entropy function in a tabular form.

| $Q^2$ | $P^2$ | $Q \cdot P$ | $d(Q, P)$ | $S^\text{stat}$ | $S_{\text{stat}}^{(0)}$ | $S_{\text{stat}}^{(1)}$ |
|-------|-------|-------------|------------|----------------|----------------|----------------|
| 2     | 2     | 0           | 5424       | 8.59          | 6.28           | 8.12           |
| 4     | 4     | 0           | 2540544    | 14.74         | 12.57          | 14.40          |
| 6     | 6     | 0           | 1254480000 | 20.95         | 18.85          | 20.69          |
| 6     | 6     | 1           | 991591800  | 20.71         | 18.59          | 20.46          |
| 6     | 6     | 2           | 483665920  | 20.00         | 17.77          | 19.76          |
| 6     | 6     | -1          | 991591800  | 20.71         | 18.59          | 20.46          |
| 6     | 6     | -2          | 483665920  | 20.00         | 17.77          | 19.76          |
We find that the asymptotic expansion of the statistical entropy is in good agreement with the exact entropy of the system. The agreement is better for higher values of charges. This is expected because asymptotic expansion accurate for large charges but starts deviating for small values of charges.

5. Degeneracy for More General 5D Black Holes

The above analysis can easily be generalized to all five-dimensional CHL black holes (for 4D CHL dyonic black holes see [15]). These are black holes in the theory obtained by compactifying Heterotic and type IIB string theory compactified on \( \frac{T^4 \times S^1}{\mathbb{Z}_N} \), where the \( \mathbb{Z}_N \) group involves \( \frac{1}{N} \) units of shift along the \( S^1 \) circle and an order \( N \) transformation on \( T^4 \). This transformation is chosen such that the theory preserves \( \mathcal{N} = 4 \) supersymmetry. The partition function of these dyons is given by,

\[
f(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = e^{-2\pi i(\tilde{\alpha}\tilde{\rho}+\tilde{v})} \prod_{b=0}^{N-1} \prod_{r=0}^{\infty} \prod_{k',l \in \mathbb{Z}_r, j \in \mathbb{Z}_b} \left( 1 - e^{2\pi i(\tilde{\alpha}k' + \tilde{\rho}l + \tilde{v}j)} \right) - \sum_{s=0}^{N-1} e^{-2\pi is/N} c_b^{r,s}(4k' - j^2)
\]

(5.1)

where, \( c_b^{(r,s)}(4k' - j^2) \) are some constants that can be obtained from the elliptic genus of the theory. Here also we can eliminate the contribution from the KK-monopole sector and get the degeneracy function for the generic five-dimensional black holes as,

\[
\tilde{\Phi}_{bmpv}(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \frac{\Phi(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}{G(\tilde{\rho}, \tilde{v})},
\]

(5.2)

where,

\[
G(\tilde{\rho}, \tilde{v}) = 64e^{2\pi i(\tilde{\alpha}\tilde{\rho}+\tilde{v})}(1 - e^{-2\pi i\tilde{v}})^2 \prod_{n=1}^{\infty}(1 - e^{2\pi in\tilde{\rho}}) - \sum_{s=0}^{N-1} e^{-2\pi is/N} c_0^{(0,s)}(0) - \sum_{n=0}^{N-1} e^{-2\pi in/\mathcal{N}} c_1^{(0,s)}(-1)(1 - e^{2\pi i(n\tilde{\rho} - \tilde{v})}) - \sum_{s=0}^{N-1} e^{-2\pi is/N} c_1^{(0,s)}(1).
\]

(5.3)

With these modified expressions, one can proceed to compute first subleading correction to the entropy of these general black holes. For these orbifolded theories, the rank of the gauge group \( r \) reduces and accordingly the number \( k \) defined in (3.6) changes. Our previous analysis, corresponding to \( r = 28 \) and \( k = 10 \) goes through in all these cases. One can also produce an explicit chart for systematic corrections to statistical entropy as we have in the previous section for these black holes, while there are quantitative changes, qualitative behaviour remains the same.

6. Discussion

We studied the four-derivative \( (O(Q^0)) \) correction to the statistical entropy function and the statistical entropy by doing asymptotic expansion of the statistical entropy
function. This expansion is valid in the limit \( \{3.12\} \), but is different from the Cardy limit (or Fareytail limit \([16],[17]\)), in our case \( Q^2 (= n) \) and \( P^2(Q_1 Q_5) \) can be of same order whereas the Cardy limit corresponds to \( n \gg Q_1 Q_5 \).

We find that the exact statistical degeneracy computed around the saddle point \( \nu = 0 \), is independent of the sign of \( Q \cdot P \). It is worthwhile to compare this with the four-dimensional black holes. In 4D case, the exact degeneracy changes as the sign of \( Q \cdot P \) change. This jump in the degeneracy is related to the issue of walls of marginal stability as discussed in details in \([18],[19]\). As pointed out below (2.5), there is an extra zero at \( \bar{v} = 0 \) in \( \bar{\Phi} \), compared to \( \Phi_{bmpv} \). Because of this pole, there is an ambiguity in the Fourier expansion and we get the jump in degeneracy for two signs of \( Q \cdot P \). Physically, it is related to the dynamics of the KK-monopole. However, this sector is absent in the 5D BMPV black holes. For this five dimensional black holes, we do not have any walls of marginal stability associated with this particular zero of the function \( \Phi_{bmpv} \).

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