An upper bound for the $Z$-spectral radius of adjacency tensors

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Abstract
Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices with degree sequence $\Delta_1 = d_1 \geq \cdots \geq d_n = \delta$. In this paper, in terms of degree $d_i$, we give a new upper bound for the $Z$-spectral radius of the adjacency tensor of $\mathcal{H}$. Some examples are given to show the efficiency of the bound.

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1 Introduction
Let $\mathcal{A} = (a_{i_1 \cdots i_n})$ be an $m$th order $n$-dimensional real square tensor, $x$ be a real $n$-vector. Then we define the following real $n$-vector:

$$\mathcal{A}x^{m-1} = \left( \sum_{i_2, \cdots, i_m=1}^n a_{i_2 \cdots i_m} x_2 \cdots x_m \right)_{1 \leq i \leq n}, \quad x^{[m-1]} = (x_1^{m-1})_{1 \leq i \leq n}.$$  

If there exist a real vector $x$ and a real number $\lambda$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then $\lambda$ is called an $H$-eigenvalue of $\mathcal{A}$ and $x$ is called an eigenvector of $\mathcal{A}$ associated with $\lambda$ [1, 2]. If there exist a real vector $x$ and a real number $\lambda$ such that

$$\mathcal{A}x^{m-1} = \lambda x, \quad x^T x = 1,$$

then $\lambda$ is called a $Z$-eigenvalue of $\mathcal{A}$ and $x$ is called an eigenvector of $\mathcal{A}$ associated with $\lambda$.

You can see more about the eigenvalues of tensors in [3–7].

Let $\mathcal{H}$ be a hypergraph with a vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H}) = \{e_1, e_2, \ldots, e_t\}$. If every edge of $\mathcal{H}$ contains exactly $k$ distinct vertices, then $\mathcal{H}$ is called a $k$-uniform hypergraph. The degree of a vertex $i$ in $\mathcal{H}$ is the number of edges incident with $i$, denoted by $d_i$. If $d_i = d$ for any $i \in V(\mathcal{H})$, then the hypergraph $\mathcal{H}$ is called a regular hypergraph. Recently, the spectral radii of hypergraphs have been studied in [8, 9].

Let $\{i_1, \ldots, i_k\} \in E(\mathcal{H})$ mean that there is an edge containing $k$ distinct vertices $i_1, \ldots, i_k$. Then the adjacency tensor $A(\mathcal{H}) = (a_{i_1 \cdots i_k})$ of a hypergraph $\mathcal{H}$ is a $k$th order $n$-dimensional tensor.
tensor with entries:

$$a_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, \ldots, i_k\} \in E(\mathcal{H}), \\ 0, & \text{otherwise}. \end{cases}$$

Let $D(\mathcal{H}) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the degree diagonal tensor of the graph $\mathcal{H}$. Then the tensor $Q(\mathcal{H}) = D(\mathcal{H}) + A(\mathcal{H})$ is called the signless Laplacian tensor of the hypergraph $\mathcal{H}$. The largest modulus of the $Z$-eigenvalues of the adjacency tensor $A(\mathcal{H})$ is denoted by $\rho_Z(\mathcal{H})$, which is called the $Z$-spectral radius of the adjacency tensor $A(\mathcal{H})$.

For a $k$-uniform hypergraph $\mathcal{H}$, let $\Delta = d_1 \geq \cdots \geq d_n = \delta$ be the degree sequence of the hypergraph $\mathcal{H}$. In 2013, Xie and Chang [8] presented the following upper bound for the largest $Z$-eigenvalues $\rho_Z(\mathcal{H})$ of adjacency tensors:

$$\rho_Z(\mathcal{H}) \leq \Delta.$$ (1)

In this paper, we give a new upper bounds in terms of degree $d_i$ for the $Z$-spectral radius of hypergraphs, which improves the bound as shown in (1). Then we give some examples to compare these bounds for $Z$-spectral radius of hypergraphs.

2 Preliminaries

Some basic definitions and useful results are listed as follows.

**Definition 2.1** ([10]) The tensor $A$ is called reducible if there exists a nonempty proper index subset $J \subset \{1, 2, \ldots, n\}$ such that $a_{i_1 \cdots i_m} = 0$, $\forall i_1 \in J$, $\forall i_2, \ldots, i_m \notin J$. If $A$ is not reducible, then we call $A$ to be irreducible.

**Definition 2.2** Let $A$ be an $m$-order and $n$-dimensional tensor. We define $\sigma(A)$ the $Z$-spectrum of $A$ by the set of all $Z$-eigenvalues of $A$. Assume $\sigma(A) \neq \emptyset$, then the $Z$-spectral radius of $A$ is denoted by

$$\rho_Z(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \}.$$ 

The concept of weakly symmetric was first introduced and used by Chang, Pearson, and Zhang [11] in order to study the following Perron–Frobenius theorem for the $Z$-eigenvalue of nonnegative tensors.

**Lemma 2.1** ([11]) Let $A = (a_{i_1 \cdots i_m})$ be a weakly symmetric nonnegative tensor, then the spectral radius $\rho_Z(A)$ is a positive $Z$-eigenvalue with a nonnegative $Z$-eigenvector $x$. Furthermore, if $A$ is irreducible, $x$ is positive.

$|A|$ means that $(|A|)_{i_1 \cdots i_m} = |a_{i_1 \cdots i_m}|$. Two useful lemmas are given as follows.

**Lemma 2.2** Let $A$ and $B$ be two weakly symmetric and irreducible tensors of order $m$ and dimension $n$. If $B$ and $B - |A|$ are nonnegative, then $\rho_Z(B) \geq \rho_Z(|A|)$.

**Proof** Let $y$ be the eigenvector associated with $\beta$, where $\beta$ is a $Z$-eigenvalue of $A$. Then we can get

$$|\beta||y| = |Ay^{m-1}| \leq |A||y^{m-1}| \leq |B||y^{m-1}|.$$
By Theorem 4.7 of [11], we have
\[ \rho(Z(B)) = \max_{y \geq 0} \min_{y_i > 0} \frac{(B|y|^{m-1})_i}{|y_i|} \geq \min_{y_i > 0} \frac{(B|y|^{m-1})_i}{|y_i|} \geq |\beta|. \]

Then
\[ \rho(Z(B)) \geq \rho(Z(|A|)). \] \( \square \)

**Lemma 2.3** Let \( \{A_k\} \) be a sequence of nonnegative, weakly symmetric tensors of order \( m \) and dimension \( n \), and \( A_k - A_{k+1} \) be nonnegative for each positive integer \( k \). Then
\[ \lim_{k \to \infty} \rho(Z(A_k)) = \rho(Z(\lim_{k \to \infty} A_k)). \]

**Proof** Let \( A = \lim_{k \to \infty} A_k \). Since \( A_k - A_{k+1} \) is nonnegative, by Lemma 2.2, we know that \( \{\rho(Z(A_k))\} \) is a monotone decreasing sequence with a lower bound \( \rho(Z(A)) \). So \( \lim_{k \to \infty} A_k \) exists and
\[ \lambda = \lim_{k \to \infty} \rho(Z(A_k)) \geq \rho(Z(A)). \]

Since \( \{A_k\} \) is nonnegative, weakly symmetric, then there exists a nonnegative vector \( x^{(k)} \) such that \( A_k(x^{(k)})^{m-1} = \rho(Z(A_k))x^{(k)} \) and \( x^{(k)}T x^{(k)} = 1 \). Then \( \{x^{(k)}\} \) is a bounded sequence, it has a convergent subsequence \( \{y_i\} \). Suppose that \( y = \lim_{k \to \infty} y_i \). By \( A_k y_i^{m-1} = \rho(Z(A_k))y_i \), we get \( A y^{m-1} = \lambda y \). So \( \lambda \) is an eigenvalue of \( A \). Since \( \lambda \leq \rho(Z(A)) \), we have \( \rho(Z(A)) = \lambda \). \( \square \)

**3 The Z-spectral radius of tensors and hypergraphs**

In this section, let \( r_i(A) = \sum_{i_2, \ldots, i_m = 1}^n |a_{i_2 \ldots i_m}| - |a_{i_1|} | \), we give some bounds on the Z-spectral radius of tensors and hypergraphs.

**Theorem 3.1** Let \( A \) be weakly symmetric nonnegative tensors of order \( m \) and dimension \( n \). Then
\[ \rho(Z(A)) \leq \max_{a_{1 \ldots i_m} \neq 0} \left\{ \prod_{j=1}^m r_j^m (A) \right\}. \]

**Proof** Case 1. If \( A \) is irreducible, by Lemma 2.1, let \( u = (u_i) \) be the positive eigenvector associated with the largest Z-eigenvalues \( \rho(Z(A)) \) of \( A \). Then
\[ Au^{m-1} = \rho(Z(A))u. \]

Let \( u_\alpha = \max\{u_{i_1} \cdots u_{i_m} : a_{i_1\ldots i_m} \neq 0, 1 \leq i_1, \ldots, i_m \leq n\} \), then
\[ \rho(Z(A))u_\alpha^2 = \sum_{i_2, \ldots, i_m = 1}^n a_{i_2 \ldots i_m} u_{i_1} u_{i_2} \cdots u_{i_m} \]
\[ = \sum_{a_{i_2 \ldots i_m} \neq 0} a_{i_2 \ldots i_m} u_{i_1} u_{i_2} \cdots u_{i_m} \]
\[ \leq r_1(A)u_\alpha. \] (2)
Suppose that \( u_\alpha = u_{j_1} \cdots u_{j_m} \). Then, from (2), we can get
\[
\rho_Z(A) u_{j_1}^2 \leq r_{j_1}(A) u_\alpha,
\]
\[
\vdots
\]
\[
\rho_Z(A) u_{j_m}^2 \leq r_{j_m}(A) u_\alpha.
\]
Then, by \( u_{\alpha}^m \leq u_2^2 \), we have
\[
\prod_{i=1}^{m} \rho_Z^m(A) u_{j_i}^2 \leq u_{\alpha}^m \prod_{i=1}^{m} r_{j_i}(A) \leq u_2^2 \prod_{i=1}^{m} r_{j_i}(A).
\]
Therefore,
\[
\rho_Z(A) \leq \max_{a_{1 \cdots m} \neq 0} \left\{ \prod_{j=1}^{m} r_{j_i}(A) \right\}.
\]

Case 2. If \( A \) is reducible. Let \( T = (t_{i_1i_2 \cdots i_m}), t_{i_1i_2 \cdots i_m} = 1 \) for all \( 1 \leq i_1, i_2, \ldots, i_m \leq n \). Then \( A + \epsilon T \) is an irreducible nonnegative tensor for any chosen positive real number \( \epsilon \). Now we substitute \( A + \epsilon T \) for \( A \), respectively, in the previous case. When \( \epsilon \to 0 \), the result follows by the continuity of \( \rho_Z(A + \epsilon T) \).

By Theorem 3.1, a bound on the \( Z \)-spectral radius of a uniform hypergraph is obtained, we also compare the bound with the result in (1).

**Theorem 3.2** Let \( H \) be a \( k \)-uniform hypergraph on \( n \) vertices with the degree sequence \( \Delta = d_1 \geq \cdots \geq d_n = \delta \). Then
\[
\rho_Z(H) \leq \max_{[i_1 \cdots i_k] \in E(H)} \left\{ \prod_{j=1}^{k} d_{j_i}^{\frac{1}{k}}(A) \right\}.
\]

**Proof** Case 1. \( A(H) \) is irreducible. In this case, by Lemma 2.1, there exists a positive eigenvector corresponding to the spectral radius \( \rho_Z(H) \). Then, by Theorem 3.1, we have
\[
\rho_Z(H) \leq \max_{[i_1 \cdots i_k] \in E(H)} \left\{ \prod_{j=1}^{k} d_{j_i}^{\frac{1}{k}}(A) \right\}.
\]

Case 2. If \( A(H) \) is reducible. Let \( T = (t_{i_1i_2 \cdots i_k}), t_{i_1i_2 \cdots i_k} = 1 \) for all \( 1 \leq i_1, i_2, \ldots, i_k \leq n \). Then \( A(H) + \epsilon T \) is an irreducible nonnegative tensor for any chosen positive real number \( \epsilon \). Now we substitute \( A(H) + \epsilon T \) for \( A(H) \), respectively, in the previous case. When \( \epsilon \to 0 \), the result follows by the continuity of \( \rho_Z(A(H) + \epsilon T) \).

**Remark** Obviously, we can get
\[
\max_{[i_1 \cdots i_k] \in E(H)} \left\{ \prod_{j=1}^{k} d_{j_i}^{\frac{1}{k}}(A) \right\} \leq \Delta.
\]
That is to say, our bound in Theorem 3.2 is always better than the bound in (1).
Table 1 Upper bounds for the hypergraphs $H_1$ and $H_2$

|       | (1) | (3) |
|-------|-----|-----|
| $H_1$ | 3   | 3 2 |
| $H_2$ | 3   | 3 2 |

We now show the efficiency of the new upper bound in Theorem 3.2 by the following examples.

**Example 1** Consider 3-uniform hypergraph $H_1$ with a vertex set $V(H_1) = \{1, 2, 3, 4, 5, 6, 7\}$ and an edge set $E(H_1) = \{e_1, e_2, e_3\}$, where $e_1 = \{1, 2, 3\}$, $e_2 = \{1, 4, 5\}$, $e_3 = \{1, 6, 7\}$.

**Example 2** Consider 3-uniform hypergraph $H_2$ with a vertex set $V(H_2) = \{1, 2, 3, 4, 5, 6, 7\}$ and an edge set $E(H_2) = \{e_1, e_2, e_3\}$, where $e_1 = \{1, 6, 7\}$, $e_2 = \{2, 6, 7\}$, $e_3 = \{3, 6, 7\}$.

From Table 1, we can find that bound (3) is always better than (1).

4 Conclusion
In this paper, we get a new bound for the $Z$-spectral radius of tensors. As applications, in terms of the degree sequence $d_i$, we obtain a new bound for the $Z$-spectral radius of hypergraphs, which is always better than the bound in [8]. We list two examples to show the efficiency of our new bound.

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Competing interests
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Authors’ contributions
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