Analytic knots, satellites and the 4-ball genus

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Abstract

Call a smooth knot (or smooth link) in the unit sphere in $\mathbb{C}^2$ analytic (respectively, smoothly analytic) if it bounds a complex curve (respectively, a smooth complex curve) in the complex ball. Let $K$ be a smoothly analytic knot. For a small tubular neighbourhood of $K$ we give a sharp lower bound for the 4-ball genus of an arbitrary analytic link $L$ contained in it.

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1 Introduction

Let $K$ be a smooth knot in the 3-sphere $S^3$. A smooth knot or smooth link $L$ contained in a tubular neighbourhood $N(K) \subset S^3$ of $K$ is a satellite of $K$ if it is not isotopic to $K$ and not contained in a 3-ball inside $N(K)$. Satellites have been considered since long. They play a role in the problem which knot complements admit hyperbolic structure and received recent interest from the point of view of invariants of knots and links. In the following we always consider knots and links to be smooth and oriented. Associate to the pair $(K, L)$ an entire number $w_K(L)$ in the following way. Consider a projection $pr : N(K) \to K$. The image $pr(L) \subset K$ is homologous to $n \cdot [K]$ for an entire number $n$. Put $w_K(L) = n$ and call $n$ the winding number of $L$ with respect to $K$.

The pattern $L$ of $L$ gives more precise information on the satellite $L$. It is defined as follows. Denote by $U$ a standard realization of the unknot. For instance, identify $S^3$ with the unit sphere $\partial B^2$ in $\mathbb{C}^2$. Let $H = \{z_2 = 0\}$ be the first coordinate line in $\mathbb{C}^2$ and let $U$ be the unknot $U = \partial B^2 \cap H$ oriented as boundary of a complex disc. Consider a tubular neighbourhood $N(U) \subset S^3$ of $U$. Trivialise both, $N(K)$ and $N(U)$, by Seifert framing, i.e. by a transversal vector field on the knot which points in the direction of a smooth oriented surface which is contained in the sphere and bounded by the knot. Such a surface is called a Seifert surface. Note that the trivialization does not depend on the choice of the Seifert surface. Consider a diffeomorphism $\varphi_K : N(K) \to N(U)$ which maps Seifert framing to Seifert framing. The pattern $L$ of the satellite $L$ is the free isotopy class of $\varphi_K(L)$.

A classical paper of Schubert [13] relates the genus of a knot to the genus of its satellites. The (smooth) genus $g(K)$ of a knot (or link) $K$ is the minimal genus among smooth oriented surfaces in $S^3$ bounded by $K$. Schubert’s theorem is the following.

For a satellite knot $L$ of a knot $K \subset S^3$ with $n = w_K(L)$ the following inequality for the genera holds

$$g(L) \geq |n| g(K).$$

Moreover,

$$g(L) \geq |n| g(K) + g(L).$$

Identify again $S^3$ with the unit sphere $\partial B^2$ in $\mathbb{C}^2$. We consider links (or knots) which are obtained as the transverse intersection of $\partial B^2$ with a relatively closed complex curve $\tilde{X}$ in a neighbourhood of the closed unit ball $\overline{B^2}$. We always consider an analytic link in the sphere $\partial B^2$ oriented as boundary of a complex curve in the unit ball. Following Rudolph [11] we call such links analytic, and smoothly analytic if the complex curve $\tilde{X} \cap \overline{B^2}$ bounded by the link is smooth.

Since $H^2(\mathbb{B}^2, \mathbb{Z}) = 0$ the complex curve is the zero locus $\{z \in B^2 : f(z) = 0\}$ of an analytic function in a neighbourhood of the closed ball. The function $f$
can be uniformly approximated on $\mathbb{B}^2$ by a polynomial, which gives an isotopic link in $\partial \mathbb{B}^2$ that bounds a piece of an algebraic hypersurface. If the curve $\{z \in \mathbb{B}^2 : f(z) = 0\}$ is singular its genus is defined to be the genus of its smooth perturbation $\{z \in \mathbb{B}^2 : f(z) = \varepsilon\}$ for generic small enough numbers $\varepsilon$. We will always consider analytic links oriented as boundaries of complex curves in the ball.

We are interested in the (smooth) 4-ball genus $g_4(L)$ of a knot (or link) $L$, called also slice genus. This is the minimal genus among smooth oriented surfaces embedded into $\mathbb{B}^2$ and bounded by $L$. (If $L$ is a link and the surface is not connected we mean the sum of the genera of the connected components.) Always $g_4(L) \leq g(L)$ but $g_4(L)$ may be strictly smaller than $g(L)$. The 4-ball genus gives a lower bound for the unknotting number of a knot, the smallest number of crossing changes needed to unknot the knot. The class of analytic knots is interesting from the point of view of knot invariants: for them half the Rasmussen invariant and also the $\tau$-invariant are equal to the 4-ball genus of the knot.

By a deep theorem of Kronheimer and Mrowka the 4-ball genus of an analytic knot is realized on the complex curve bounded by it. Respectively, for a link which bounds a connected complex curve its 4-ball genus is realized by the genus of this curve.

The following theorem holds. Let as above $n = w_K(L)$ denote the winding number of a satellite $L$ with respect to a knot $K$.

**Theorem 1.** Let $K$ be a smoothly analytic knot in $\partial \mathbb{B}^2$. There exists a tubular neighbourhood $N(K) \subset \partial \mathbb{B}^2$ of $K$ such that for any analytic link $L \subset N(K)$ the number $n = w_K(L)$ is non-negative and the following statements hold.

1. If $n$ is positive then the following lower bound for the 4-ball genus holds
   $$g_4(L) \geq ng_4(K) - (n - 1). \quad (3)$$

2. If $L$ is itself a knot then
   $$g_4(L) \geq ng_4(K) - \left\lceil \frac{n - 1}{2} \right\rceil. \quad (4)$$

($\lceil x \rceil$ denotes the largest integer not exceeding the real number $x$).

For $n = 1$ the statements are true also if $K$ bounds a singular curve.

The estimates are sharp in the following sense.

3. For each smoothly analytic knot $K$, each natural number $n$ and any tubular neighbourhood $N(K) \subset \partial \mathbb{B}^2$ of $K$ there exists an analytic link $L \subset N(K)$ such that equality in (3) is obtained.
4. Further, for each smoothly analytic knot $K$ there is a smoothly analytic knot $K_1$ which is smoothly isotopic to $K$ with the following property. For any tubular neighbourhood $N(K_1) \subset \partial \mathbb{B}^2$ of $K_1$ and for each natural number $n$ there exists an analytic knot $L \subset N(K_1)$ such that equality in (4) is obtained for the knot $K_1$ and the knot $L$. In general the original knot $K$ does not have this property.

The condition of analyticity of $K$ and $L$ cannot be removed. Indeed, for any knot $K$ (in particular, for an analytic knot $K$) the connected sum $L$ with its mirror is a satellite of $K$ with $n = w_K(L) = 1$ and $g_4(L) = 0$.

We do not know how big the tubular neighbourhood $N(K)$ in the theorem can be chosen, in particular, we do not know whether for $n = 1$ the statement is true for analytic links in any tubular neighbourhood of an analytic knot. For $n > 1$ we do not know sharp estimates of the 4-ball genus of satellites of knots which bound singular complex curves.

Some satellites are especially simple and useful. They are defined in terms of closed braids. Recall the following definitions. Let $C_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$ be the configuration space of $n$ particles which move in the plane without collision. The symmetrized configuration space $C_n(\mathbb{C})/S_n$ is its quotient by the action of the symmetric group $S_n$. Each point in $C_n(\mathbb{C})/S_n$ can be be considered as unordered tuple of $n$ points and can be identified with the monic polynomial whose collection of zeros equals this unordered tuple. The space of monic polynomials of degree $n$ without multiple zeros is denoted by $\mathfrak{P}_n$, the space of all monic polynomials of degree $n$ is denoted by $\overline{\mathfrak{P}}_n$. The set of coefficients of polynomials in $\mathfrak{P}_n$ is equal to $\mathbb{C}^n \setminus \{D_n = 0\}$, where $D_n$ is the discriminant, i.e. $D_n$ is a polynomial on $\mathbb{C}^n$ which vanishes exactly if the monic polynomial with these coefficients has multiple zeros.

Recall that a geometric braid with base point $E_n \in \mathfrak{P}_n$ can be considered as a continuous map of the interval $[0, 1]$ into $\mathfrak{P}_n$ with initial and terminating point equal to $E_n$. A braid with base point $E_n$ is an isotopy class of geometric braids with this base point. Such braids form a group which is isomorphic to a group $B_n$ with $n - 1$ generators, denoted by $\sigma_1, \ldots, \sigma_{n-1}$, and finitely many relations. The group is called Artin group.

A braid is quasi-positive if it is the product of conjugates of the standard generators $\sigma_i$ of $B_n$ (here conjugates of inverses of these generators are not allowed as factors).

We call an oriented closed curve $\hat{L}$ in $S^1 \times D^2$ a closed geometric braid if the projection to $S^1$ is orientation preserving on $\hat{L}$. The circle $S^1$ is assumed to be oriented, $D^2$ is a disc of real dimension 2. The number of preimages of a point is called the number of strands. A closed braid is a free isotopy class of closed geometric braids. It is well-known that free isotopy classes of closed geometric braids on $n$ strands (for short, closed geometric $n$-braids) are in one to one correspondence to conjugacy classes in the Artin group $B_n$ of braids on $n$ strands (see e.g. [2]).

The notion of closed geometric braids is sometimes used in a more special situation, namely, by a closed geometric braid one means an oriented closed
Let $\tilde{L}$ in $\partial B^2 \setminus \{z_1 = 0\}$ for which $d \arg z_1 | \tilde{L} > 0$.

Note that alternatively a geometric braid with base point $E_n$ can be considered as a collection of $n$ disjoint arcs in $[0,1] \times D^2$ which join the collection $\{1\} \times E_n$ in the top $\{1\} \times D^2$ with the "identical" collection $\{0\} \times E_n$ in the bottom $\{0\} \times D^2$ and is such that for each arc the canonical projection to $[0,1]$ is a homeomorphism. Identifying top and bottom we obtain a closed geometric braid in $S^1 \times D^2$, called the closure of the geometric braid.

S. Orevkov pointed out that for $n > 1$ the statement of Theorem 1 does not extend to the situation of two closures of quasi-positive geometric braids, one being a satellite of the other [10]. (For convenience of the reader details are given below in example 3.)

Let a tubular neighbourhood $N(K)$ of the knot $K$ be the image of a diffeomorphism from $S^1 \times D^2$ onto $N(K)$ so that $K$ is the image of $S^1 \times \{0\}$. If a link $L$ in $N(K)$ is the image of a closed geometric braid on $n$ strands in $S^1 \times D^2$ then $L$ is called an $n$-cable of $K$.

The following theorem describes the links $L$ which may appear in the situation of Theorem 1. Let as above $K \subset \partial B^2$ be an analytic knot, and let $L \subset \partial B^2$ be an analytic link. Denote by $X$, respectively by $Y$, the complex curves in $B^2$ bounded by $K$, respectively bounded by $L$. Denote by $\mathbb{D}$ the unit disc in the complex plane.

**Theorem 2.** Let $K$ be a smoothly analytic knot in $\partial B^2$. There exists a tubular neighbourhood $N(K) \subset \mathbb{C}^2$ of $K$ such that $\Omega_1 = \mathbb{B}^2 \setminus N(K)$ is a strictly pseudo-convex ball whose boundary is a smooth deformation of the sphere $\partial \mathbb{B}^2$ and for any analytic link $L \subset N(K) \cap \partial B^2$ the following holds.

1. (Cobordism.) $K$ is cobordant to a knot $K_1 \subset \partial \Omega_1$ of the same 4-ball genus $g_4(K_1) = g_4(K)$ as $K$. If $n = w_k(L) > 0$ then $L$ is cobordant to an $n$-cable $L_1 \subset \partial \Omega_1$ of $K_1$. If $n = w_k(L) = 0$, then $L$ is cobordant to the empty set.

2. (Holomorphic coverings.) There is a tubular neighbourhood $\mathcal{N}_1(X_1)$ of $X_1 = X \cap \Omega_1$ in $\Omega_1$, which contains $Y_1 = Y \cap \Omega_1$ and a diffeomorphism $\phi$ of $X_1 \times \mathbb{D}$ onto $\mathcal{N}_1(X_1)$ which is holomorphic outside a neighbourhood of $\partial X_1 \times \mathbb{D}$. The diffeomorphism maps $\partial X_1 \times \mathbb{D}$ onto a tubular neighbourhood of $K_1$ in $\partial \Omega_1$ and induces Seifert framing on the tubular neighbourhood.

Equip $\phi^{-1}(Y_1)$ with the complex structure which is the pull back of the complex structure of $Y_1$ by the mapping $\phi^{-1}$. The smooth manifold with boundary $\phi^{-1}(Y_1)$ is smoothly embedded into $X_1 \times \mathbb{D}$. Moreover, the embedding is holomorphic outside a neighbourhood of $\phi^{-1}(\partial Y_1)$ and for the canonical projection $P_{X_1} : X_1 \times \mathbb{D} \to X_1$ the restriction $p \overset{\text{def}}{=} P_{X_1} | \phi^{-1}(Y_1) : \phi^{-1}(Y_1) \to X_1$ is a branched holomorphic covering outside a neighbourhood of $\phi^{-1}(\partial Y_1)$ and in a neighbourhood of the critical points of $p$ and is a topological covering outside this set.

3. (Patterns of analytic closed braids in $N(K_1)$.) The pattern $\mathcal{L}_1$ of $L_1$
corresponds to the conjugacy class in the braid group $B_n$ of the product of a quasi-positive braid $w \in B_n$ with $g = g_4(K)$ commutators in $B_n$.

4. (Realization of patterns as analytic links.) Let $L$ be a pattern as described in statement 3. Then for each analytic knot there exists an isotopic analytic knot $K \subset \partial B^2$ such that the following holds. For all natural numbers $n$ and any a priori given tubular neighbourhood of $K$, the pattern $L$ can be realized by an analytic link contained in this neighbourhood.

In statement 3 for the strictly pseudoconvex ball $\Omega$ the pattern of a link $L_1 \subset \partial \Omega$ contained in a tubular neighbourhood of a knot $K_1 \subset \partial \Omega$ is defined as in the case $\Omega = B^2$.

There is a continuous decreasing family $\Omega_t, t \in [0,1]$, of strictly pseudoconvex balls $\Omega_t$ and a continuous family $\psi_t : \partial \Omega_t \to \partial B^2$ such that $\Omega_0 = B^2, \psi_0 = \text{id}$ and $\psi_t(K) = K_t \overset{\text{def}}{=} X \cap \partial \Omega_t$. So, one can “identify” $L_1$ with a closed $n$-braid in a tubular neighbourhood of $K$ in $\partial B^2$ (for short, with an $n$-cable of $K$ in $\partial B^2$).

The class of cobordisms $L \cup (-L_1)$ which may occur in the first part of the theorem is rich, see below for examples. The pattern $L$ is not necessarily quasi-positive. For connected quasi-positive patterns the following lemma holds. Denote as before by $\varphi_K$ a diffeomorphism from a tubular neighbourhood of a knot $K$ onto a tubular neighbourhood of the unknot $U$ such that $\varphi_K$ preserves Seifert framing.

**Lemma 1.** Let $K \subset \partial B^2$ be an analytic knot. Suppose $L'$ is the closure of a quasi-positive geometric $n$-braid and is connected. Then after an isotopy of $K$ the isotopy class of $\varphi_K^{-1}(L')$ can be represented by an analytic knot $L$ contained in an a priori given tubular neighbourhood $N(K)$ of $K$ and

$$g_4(L) = n g_4(K) + g_4(L').$$

The original motivation for Theorem 1 concerns the case $n = 1$ and strictly pseudoconvex domains instead of the ball $B^2$, and is related to the following fact ($[\mathbb{H}]$) and to the question below. Let $\Omega$ be a strictly pseudoconvex domain in a two-dimensional Stein manifold. Then each element $e$ of the fundamental group of the boundary $\pi_1(\partial \Omega)$ whose representatives are contractible in $\Omega$ can be represented by the boundary of an immersed analytic disc in $\Omega$.

**Question.** What is the minimal self-intersection number among analytic discs whose boundary represents $e$? In particular, let $\Omega = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + z^5 = \varepsilon\} \cap \mathbb{B}^3$ be the natural Stein filling of the Poincaré sphere. ($\varepsilon > 0$ is a small positive number, $\mathbb{B}^3$ is the unit ball in $\mathbb{C}^3$.) The loops $\{x = \varepsilon^{\frac{1}{2}}\} \cap \partial \Omega$, $\{y = \varepsilon^{\frac{1}{3}}\} \cap \partial \Omega$, $\{z = \varepsilon^{\frac{1}{5}}\} \cap \partial \Omega$ bound immersed analytic discs in $\Omega$. Do they minimize the self-intersection numbers among analytic discs whose boundaries represent the respective element of $\pi_1(\partial \Omega)$?
The question concerns the complex structure of $\Omega$ rather than its Stein homotopy type.

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2 Examples

The first two examples describe cobordisms which occur in part 1 of Theorem 2. The third example shows that the statement of Theorem 1 does not extend to closures of quasi-positive geometric braids $L$ that are closed 2-cables of an analytic knot $K$. The example is due to S. Orevkov [10].

Example 1. (Twisted Whitehead doubles (winding number zero).)

Consider the analytic discs $\{ z_2 = 0 \} \cap \mathbb{B}^2$ and $\{ z_1 = 0 \} \cap \mathbb{B}^2$. They intersect at the origin. The union of their boundaries forms the Hopf link. Apply an automorphism of the closed ball which maps the Hopf link to an analytic link $L$ which is the union of two circles and is contained in a 3-ball which is a subset of a small tubular neighbourhood $N(K)$ of a given analytic knot $K$. Join a point $p$ on one circle by a Legendrian arc in $N(K)$ with a point $q$ on the other circle. Choose the arc without self-intersections and without intersection points with the circles other than the endpoints. Moreover, the Legendrian arc is chosen to be the longer part of a loop representing a generator of the fundamental group of $N(K)$. Consider a partition of the Legendrian arc into small closed arcs with pairwise disjoint interior. For each small arc we take an analytic disc on $\mathbb{B}^2$ (which extends to a complex curve in a neighbourhood of the closed ball) such that its boundary lies on the sphere and passes through the two endpoints of the arc. See figure 1. Take an analytic function $f$ in a neighbourhood of $\overline{\mathbb{B}^2}$ whose zero set is the union of all these discs with the complex curve bounded by the link $L$. For a suitable small $\varepsilon$ the set $\{ f = \varepsilon \} \cap \mathbb{B}^2$ is a smooth complex curve with connected boundary. Part of the boundary approximates a compact subset of $L \setminus (\{ p \} \cup \{ q \})$, the other part consists of two arcs close to the Legendrian arc. The two arcs are traveled "in opposite direction."
Example 2. (Sum of two analytic links.)

Take two analytic links $L_1$ and $L_2$ in the tubular neighbourhood $N(K)$ of an analytic knot $K$. Join a point $p$ in one of the links $L_1$ with a point $q$ in the other link $L_2$ by a Legendrian arc in $N(K)$. The Legendrian arc is chosen without self-intersections and with interior disjoint from the two links. As in example 1 we find a complex curve $X$ in $\mathbb{B}^2$ which “approximates” the union of the complex curves bounded by the links and the Legendrian arc. See figure 2. The boundary $\partial X$ is connected. Part of it approximates a compact subset of $(L_1 \cup L_2) \setminus \{p\} \cup \{q\}$ the other part consists of two arcs close to the Legendrian arc, the two arcs traveled in opposite direction. One of the links can be taken to be an analytic $n$-cable of $K$, the second link may be obtained from an arbitrary analytic knot in $\partial \mathbb{B}^2$ by an automorphism of $\mathbb{B}^2$ which maps the second link to a 3-ball contained in $N(K)$.

Example 3. (Closures of quasi-positive geometric braids that are 2-cables of closures of quasi-positive geometric braids, see [10], corollary 2.15.)

Consider the following braid $\Delta_2^n \circ \sigma_{n-1} \circ \ldots \circ \sigma_1$ in the braid group $B_n$. Here $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators and $\Delta_n$ is Garside’s half-twist. (By induction $\Delta_0 = \Delta_1 = 1, \Delta_n = \sigma_1 \circ \ldots \circ \sigma_{n-1} \circ \Delta_{n-1}$.) The braid is positive (i.e., a word containing only generators, not their inverses), hence quasi-positive. The free isotopy class of its closure is represented by a smoothly analytic knot $K_1 \subset \partial \mathbb{B}^2$ ([11]). (The closure of $\Delta_2^n$ is a link with $n$ connected components. Hence, the closure of the considered braid is connected.) The (smooth) complex curve $X_1$ bounded by $K_1$ is (after adjusting near $\partial \mathbb{B}^2$) a branched holomorphic
covering of the disc with number of branch points \( b_1 \) equal to the degree of the braid (i.e., it is equal to the sum of exponents of generators of \( \mathcal{B}_n \) in the word). Hence, \( b_1 \) is equal to \( n \cdot (n - 1) + n - 1 = n^2 - 1 \). The genus \( g_4(X_1) \) equals 
\[
\frac{b_1 - n + 1}{2} = \frac{n^2}{2} + O(n)
\]
by the Riemann-Hurwitz relation.

Orevkov proved \( [10] \), Corollary 2.15, that for large \( n \) and \( N \leq \frac{8}{3} n^2 + O(n) \) the braid \( \sigma_1^{-N} \Delta_2^2 \in \mathcal{B}_{2n} \) is quasi-positive. Its closure is a 2-cable of \( \Delta_2^2 \). A modification of this example provides a quasi-positive braid in \( \mathcal{B}_{2n} \) whose closure is a 2-cable of \( K_1 \) and is connected. Indeed, consider \( \sigma_1^{-N} \circ c(\sigma_{n-1}) \circ \ldots \circ c(\sigma_1) \circ \Delta_2^2 \) for odd \( N = \frac{8}{3} n^2 + O(n) \). Here \( c(\sigma_j) \overset{\text{def}}{=} \sigma_{2j} \circ \sigma_{2j-1} \circ \sigma_{2j+1} \circ \sigma_{2j} \), \( j = 1, \ldots, n - 1 \). See figure 3 for \( n = 3 \). Denote the closure of this braid by \( K_2 \). \( K_2 \) bounds a quasi-positive surface (see [3] or the explanation after the statement of Lemma 4) with number of branch points \( b_2 \) equal to the degree of the braid, i.e., equal to \( (2n)^2 - \frac{8}{3} n^2 + O(n) = \frac{4}{3} n^2 + O(n) \), and hence of genus \( \frac{2}{3} n^2 + O(n) \). Hence, for large \( n \) the lower bound for the 4-ball genus is different from that in the case of analytic satellites in small neighbourhoods of smoothly analytic knots.

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Figure 3.
3 Reduction to the case of \( n \)-cables and holomorphic coverings

Theorem 1 can be formulated in the more general situation when the ball \( B^2 \) is replaced by a relatively compact strictly pseudoconvex domain \( \Omega \) in a Stein surface \( \tilde{\Omega} \) (for short, when \( \Omega \) is a Stein domain). In this section we will work in this more general setting.

**Lemma 2.** (Deformation of tubular neighbourhoods of curves on strictly pseudoconvex boundaries to Levi-flat hypersurfaces) Let \( \tilde{X} \) be a complex curve in \( \tilde{\Omega} \) (maybe, singular) which intersects \( \partial\Omega \) transversely along a knot \( K \). There are arbitrarily small tubular neighbourhoods \( N_{\partial} \subset \partial\Omega \) of \( K \) with the following properties. The boundary \( T = \partial N_{\partial} \) bounds a real-analytic Levi-flat hypersurface \( \tilde{N}_{\Omega} \subset \Omega \) which intersects \( \tilde{X} \) transversely along a simple closed curve \( K' \). Moreover, the projection \( N_{\Omega} \rightarrow K' \) defines a smooth fiber bundle with fibers being analytic discs \( \Delta_z, z \in K' \), with boundary \( \partial \Delta_z \) in \( T \subset \partial\Omega \). The union \( N_{\Omega} \cup T \cup \tilde{N}_{\Omega} \) bounds an open subset \( A \) of \( \Omega \) which is diffeomorphic to \( S^1 \times b^3 \) (a real 3-dimensional ball), and \( \Omega \setminus \overline{A} \) is diffeomorphic to \( \Omega \). Moreover, \( K \cup (-K') \) bounds an annulus \( \tilde{X} \cap A \) on \( \tilde{X} \).

Such tubular neighbourhoods \( N_{\partial} \) will be called good. The objects and the notation of Lemma 2 will be used in the following propositions and their proofs.

**Proposition 1.** (Cobordism) Let \( \Omega \) be a Stein domain on a Stein surface \( \tilde{\Omega} \) and let \( X \) be a complex curve in \( \tilde{\Omega} \) (maybe, singular) which intersects \( \partial\Omega \) transversely along a knot \( K \). Let \( L \) be the transverse intersection of \( \partial\Omega \) with a complex curve \( \tilde{Y} \) in \( \tilde{\Omega} \) such that \( L \) is contained in a good neighbourhood \( N_{\partial} \) of \( K \) and has winding number \( w_K(L) = n \) around \( K \). Let \( A \) be as in Lemma 2, and let \( N'_{\tilde{\Omega}} \) be the set of points of \( N_{\tilde{\Omega}} \) which have distance to the boundary \( \partial\Omega \) exceeding \( \epsilon \). Then for small \( \epsilon > 0 \) and any strictly pseudoconvex domain \( \Omega_1 \), \( \Omega \setminus A \subset \Omega_1 \subset \Omega \), whose boundary \( \partial\Omega_1 \) contains a part \( \tilde{N}'_{\tilde{\Omega}} \), which is sufficiently \( C^2 \)-close to \( N'_{\tilde{\Omega}} \), the following holds.
Either \( n = 0 \) and then \( Y \cap \partial \Omega_1 = \emptyset \), or \( n > 0 \). In the latter case (maybe, after a small perturbation of \( \partial \Omega_1 \)) \( L_1 = Y \cap \partial \Omega_1 \) is an \( n \)-cable (in \( \partial \Omega_1 \)) of \( K_1 = X \cap \partial \Omega_1 \).

**Lemma 3. (Tubular neighbourhood of knots and of complex curves)**

Let \( X \subset \Omega \) be the zero set \( X = \{ f = 0 \} \) of an analytic function in \( \Omega \) such that \( X \) intersects \( \partial \Omega \) transversely along a knot \( K \). For a complex curve \( Y \) on \( \Omega \) intersecting \( \partial \Omega \) transversely along a link \( L \), the inclusion \( L \subset \{ |f| < a \} \cap \partial \Omega \) for some positive number \( a \) implies the inclusion \( Y = Y \cap \Omega \subset \{ |f| < a \} \cap \Omega \).

More information can be obtained in the case when the gradient of \( f \) does not vanish on \( X \cap \Omega \). Let \( a > 0 \) be a small number. Then, possibly after taking for \( \Omega \) a smaller Stein surface with \( \Omega \Subset \tilde{\Omega} \), the gradient of \( f \) does not vanish in a neighbourhood of the subset \( \{ |f| \leq a \} \) of \( \tilde{\Omega} \) and there exists a holomorphic vector field \( V^f = (V_1^f, V_2^f) \) near this set such that \( \frac{\partial}{\partial z_1} f \cdot V_1^f + \frac{\partial}{\partial z_2} f \cdot V_2^f = 1 \).

For a domain \( X \subset \tilde{\Omega} \) with \( \tilde{X} \cap \Omega \subset X \) and for small enough \( a \) the flow of \( V^f \) defines a biholomorphic mapping \( \phi^f \) from \( X \times a \mathbb{D} \) onto a tubular neighbourhood \( T_\phi(X) \) of \( X \). We obtain a trivial holomorphic fiber bundle \( T_\phi(X) \to X \) with fiber \( \phi^f(\{p\} \times a \mathbb{D}) \) over the point \( p \in X \). We will always consider the tubular neighbourhood \( T_\phi(X) \) as total space of this bundle. If \( a \) is small enough then \( \phi^f(X \times a \mathbb{D}) \cap \Omega \) is compact.

For relatively closed complex curves \( \tilde{X} \) and \( \tilde{Y} \) in \( \tilde{\Omega} \) we put \( X_1 = \tilde{X} \cap \Omega_1 \), \( Y_1 = \tilde{Y} \cap \Omega_1 \) with \( \Omega_1 \) as in Proposition 1. We use previous notation.

We will now make a smooth deformation of the fiber bundle \( T_\phi(X) \) fixing it outside a neighbourhood of \( \partial \Omega_1 \) so that a tubular neighbourhood of \( \partial X_1 \) on \( \partial \Omega_1 \) is the union of fibers. We are interested in the restriction of the new fiber bundle to a tubular neighbourhood \( N_1(X_1) \) in \( \Omega_1 \).

**Proposition 2. (Deformation of disc bundles)** Suppose for some \( a > 0 \) the gradient of \( f \) does not vanish on the subset \( \{ |f| < a \} \) of \( \tilde{\Omega} \), and the set \( \{ |f| < a \} \cap \partial \Omega \) is contained in a good neighbourhood \( N_\Omega \subset \partial \Omega \) of \( K \). Let \( A \) be as in Lemma 2, let \( \epsilon > 0 \) be small, and let \( \Omega_1, \Omega \setminus A \subset \Omega \subset \tilde{\Omega} \), be a strictly pseudoconvex domain whose boundary \( \partial \Omega_1 \) contains a part \( \tilde{N}_\Omega^f \), which is sufficiently \( C^2 \)-close to the set \( N_\Omega^f \) of Proposition 1.

Let \( L \) be a smoothly analytic link, \( L = \tilde{Y} \cap \partial \Omega \) for a relatively closed complex curve in \( \tilde{\Omega} \). Assume that \( L \subset \{ |f| < a \} \), and, hence, \( Y_1 = Y \cap \Omega_1 \subset \{ |f| < a \} \).

Then there exists a diffeomorphism \( \phi \) of \( X_1 \times a \mathbb{D} \) onto a tubular neighbourhood \( N_1(X_1) \) of \( X_1 \) in \( \Omega_1 \), such that \( \phi^{-1}(N_1(X_1) \cap \partial \Omega_1) = \partial X_1 \times a \mathbb{D} \). The mapping \( \phi \) is holomorphic on \( X' \times a \mathbb{D} \) for an open subset \( X' \) of \( X_1 \) which is diffeomorphic to \( X_1 \) (and, hence, to \( X \)). Given any neighbourhood of \( K \) in \( \mathbb{C}^2 \) one can take the number \( a > 0 \) so small that the set \( X \setminus X' \) can be assumed to be contained in this neighbourhood. If \( \Omega \) (and, hence, also \( \Omega_1 \)) is a strictly pseudoconvex ball then the diffeomorphism induces Seifert framing on a tubular neighbourhood of \( K_1 = \partial X_1 \) in \( \partial \Omega_1 \).

Equip \( \phi^{-1}(Y_1) \) with the complex structure which is the pull back of the complex structure of \( Y_1 \) by the mapping \( \phi^{-1} \). The smooth manifold with boundary \( \phi^{-1}(Y_1) \) is smoothly embedded into \( X_1 \times \mathbb{D} \). Moreover, the embedding is
holomorphic outside a neighbourhood of $\phi^{-1}(\partial Y_1)$. The mapping $\phi$ can be chosen so that for the canonical projection $P_{\tilde{X}_1} : \tilde{X}_1 \times \mathbb{D} \to \tilde{X}_1$ the restriction $p = P_{\tilde{X}_1} | \phi^{-1}(\tilde{Y}_1) : \phi^{-1}(\tilde{Y}_1) \to \tilde{X}_1$ is a branched holomorphic covering outside a neighbourhood of $\tilde{\phi}^{-1}(\partial Y_1)$ and in a neighbourhood of each critical point and $p$ is a topological covering outside this set.

**Proof of Lemma 2.** Near each boundary point the domain $\Omega$ is strictly convex in suitable holomorphic coordinates, hence there exists a smooth fiber bundle over the curve $K$ with fiber over each $z \in K$ a holomorphic disc embedded into $\bar{\Omega}$ which is complex tangent to $\partial \Omega$ at the point $z$ and does not meet $\partial \Omega$ otherwise. Moreover, we may assume that the union of the discs form a smooth Levi-flat surface such that a neighbourhood of $K$ on $\partial \Omega$ can be considered as the graph over this surface of a function with non-degenerate quadratic form in the directions of the holomorphic discs. Note that the holomorphic discs are transversal to $X$. Choose a trivialization and consider a $C^1$ approximation of the smooth bundle by a holomorphic disc bundle over a neighbourhood $v$ of $K$ on $\tilde{X}$. If $v$ is small enough the holomorphic disc through each $z \in v \cap \Omega$ intersects $\Omega$ along a connected simply connected set $\Delta_z$. Choose a real analytic oriented loop $K'$ in $v \cap \Omega$ such that $K \cup (-K')$ bounds an annulus on $\tilde{X}$. Put $N_{\Omega} = \bigcup_{z \in K'} \Delta_z, T = \partial N_{\Omega} \subset \partial \Omega$ and let $N_\theta$ be the connected component of $\partial \Omega \setminus T$ containing $K$.

**Proof of Proposition 1.** Let $N_\Omega$ be the Levi-flat hypersurface associated to the good tubular neighbourhood $N_\theta$ of $K$. Take a small neighbourhood $u \subset X$ of $K' \stackrel{\text{def}}{=} N_\Omega \cap X$. Since $L = \tilde{Y} \cap \partial \Omega \subset N_\theta$ (hence $\tilde{Y}$ does not meet $T = \bigcup_{z \in K'} \partial \Delta_z$ where the $\Delta_z$ are as in the proof of Lemma 2) the intersection number of $Y$ with $\Delta_z$ is constant for $z \in u$. The holomorphic projection $p : u_\Delta \stackrel{\text{def}}{=} \bigcup_{z \in u} \Delta_z \to u$ defines a (branched) holomorphic covering $p | Y \cap u_\Delta \to u$.

Deforming $K'$ and shrinking $u$ we may assume that the latter covering is unramified. Orient $K'$ as boundary of $X \cap (\Omega \setminus \bar{A})$ and orient $L' = Y \cap N_\Delta$ as boundary of $Y \cap (\Omega \setminus \bar{A})$. Since a disc $\Delta_z$ is either contained in $A$ or does not meet $A$, the projection $p$ maps $A \cap u_\Delta$ into $A \cap X$ and $u_\Delta \setminus A$ into $X \setminus \bar{A}$. Hence $p | Y$ maps the side $Y \setminus \bar{A}$ of $L'$ on $Y$ to the side $X \setminus \bar{A}$ of $K'$ on $X$, in other words $p | L' : L' \to K'$ is orientation preserving if $L'$ and $K'$ are oriented as boundaries of complex curves in $\Omega \setminus \bar{A}$.

The intersection number of $Y$ with each disc $\Delta_z$ equals $w_K(L) = n$. This follows from the fact that $L$ and $L'$ cobound the set $Y \cap A$ (after a small perturbation we may assume that this set is a smooth manifold). Indeed, for a neighbourhood of $\bar{A}$ which is diffeomorphic to $K \times b^3$ let $e^{2\pi i s}$, $s \in [0, 1]$, be the parameter in the direction of $K$. Integrate the form $ds$ along $L$ and along $L'$ and apply Stokes’ theorem.

Hence, for any small $\epsilon > 0$ and any strictly pseudoconvex domain $\Omega_1$, $\Omega \setminus A \subset \Omega_1 \subset \Omega$ whose boundary $\partial \Omega_1$ contains a part $N_\Omega^\prime$, which is sufficiently $C^2$-close
to the set $\tilde{N}^r_\Omega$, the statement of the proposition holds. \hfill \Box

**Proof of Lemma 3.** The inclusion $Y \subset \{|f| < a\} \cap \Omega$ follows from the maximum principle applied to $f \mid Y$ and the fact that $\partial Y \subset \{|f| < a\} \cap \partial \Omega$. \hfill \Box

**Proof of Proposition 2.** Denote by $X'$ the relatively compact domain on $\tilde{X}$ which is bounded by $K' = \tilde{X} \cap N^r_\Omega$. Since the discs $\Delta_z$ constituting $N^r_\Omega$, and the fibers $\phi^J(\{z\} \times a \mathbb{D})$ are transversal to $\mathcal{X}$ the biholomorphic mapping $\phi^J : \mathcal{X} \times D \rightarrow B_{\mathcal{A}}(\mathcal{X})$ can be changed near $K' \times a \mathbb{D}$ so that one obtains a smooth mapping $\tilde{\phi} : \mathcal{X} \times a \mathbb{D} \rightarrow \tilde{\Omega}$ with the following properties.

The mapping $\tilde{\phi}$ maps $\mathcal{X} \times a \mathbb{D}$ diffeomorphically onto its image and maps each disc $\{z\} \times a \mathbb{D}$, $z \in \mathcal{X}$, onto a holomorphic disc, so that we obtain a smooth fiber bundle with fibers being holomorphic discs. Further, $\tilde{\phi} = \phi^J$ on $X'' \times a \mathbb{D}$ for a relatively compact domain $X'' \subset X'$ which is diffeomorphic to $X'$. Hence, $\tilde{\phi}$ is holomorphic on $X'' \times a \mathbb{D}$. If $a$ is small the complement $X' \setminus X''$ can be chosen to be contained in a small neighbourhood of $\partial X_1$. Moreover, the fiber over each point $z \in K'$ coincides with a relatively compact subset of $\Delta_z$ and the curve $Y$ does not meet the boundaries $\bigcup_{z \in \mathcal{X}'} \tilde{\phi}(\{z\} \times a \partial \mathbb{D})$ of the disc fibers over $\tilde{X}'$.

Let $\tilde{P}_{\mathcal{X}}$ be the bundle projection $\tilde{P}_{\mathcal{X}}(\tilde{\phi}(x, \zeta)) = x$, $x \in \mathcal{X}$, $\zeta \in \mathbb{D}$. $\tilde{\phi}$ may be chosen to be holomorphic near fibers containing critical points of $\tilde{P}_{\mathcal{X}} | Y_1$.

If $\partial \Omega_1$ contains a part $\tilde{N}^r_\Omega$, which is sufficiently $C^2$-close to $N^r_\Omega$, we may smoothly change $\tilde{\phi}$ on $(\tilde{X}_1 \setminus \mathcal{X}') \times a \mathbb{D}$ to obtain a smooth mapping $\phi$ which maps $\tilde{X}_1 \times a \mathbb{D}$ diffeomorphically onto a tubular neighbourhood of $\tilde{X}_1$ in $\Omega_1$, and maps $\partial X_1 \times a \mathbb{D}$ diffeomorphically onto a tubular neighbourhood of $K_1$ in $\partial \Omega_1$ and is holomorphic on $X'' \times a \mathbb{D}$.

The preimage $\phi^{-1}(Y_1) \subset \tilde{X}_1 \times a \mathbb{D}$ is a smooth manifold with boundary. Equip its interior with the complex structure which equals the pull back via $\phi$ of the complex structure of $Y_1$. Since $\phi = \phi^J$ on $X'' \times a \mathbb{D}$ the choice of $\phi$ implies that $p = \tilde{P}_{\mathcal{X}_1} | \phi^{-1}(Y_1) : \phi^{-1}(Y_1) \rightarrow \tilde{X}_1$ is a branched holomorphic covering outside a neighbourhood of $\phi^{-1}(\partial Y_1)$ and in a neighbourhood of each critical point, and a topological covering outside this set.

Consider now the case when $\Omega = \mathbb{B}^2$, hence $\Omega_1$ is a strictly pseudoconvex ball. With notation as before, the trivialization of $B_{\mathcal{A}}(\mathcal{X})$ is given, for example, by the normal vector field on $\mathcal{X}$ that points in the direction of positive values of the function $f$. The set $\{f > 0\} \cap \partial \Omega_1$ is, after a small generic perturbation which fixes the part of the set which is contained in a neighbourhood of $K_1$, a Seifert surface for $K_1$. Hence the trivialization on the tubular neighbourhood $\phi(\partial X_1 \times a \mathbb{D})$ of $K_1$ defined by $\phi$ coincides with Seifert framing. \hfill \Box

The following lemma is needed for the proofs of the theorems. It provides an isotopy of an arbitrary smoothly analytic knot to a smoothly analytic knot with more convenient properties.
Lemma 4. (Isotopy to closed geometric braids) Let $K$ be a smoothly analytic knot and let $\epsilon$ and $\delta$ be small positive numbers. Then there exists an isotopy of $K$ to a smoothly analytic knot $K' = X \cap \partial B^2$, where $X$ is a relatively closed curve in $(1 + \epsilon)\mathbb{D} \times \delta \mathbb{D}$ such that $\partial X$ is a subset of $(1 + \epsilon)\partial \mathbb{D} \times \delta \mathbb{D}$. The branched covering map $p : X \to (1 + \epsilon)\mathbb{D}$ has branch locus in $(1 - \epsilon)\mathbb{D}$.

The lemma is a consequence of the following facts. Following [3] a quasi-positive surface in $\mathbb{D} \times \mathbb{C}$ is a smooth proper embedding $\iota$ of a Riemann surface into $\mathbb{D} \times \mathbb{C}$ which is holomorphic in a neighbourhood of the critical points of the canonical projection $P_1 : \mathbb{D} \times \mathbb{C} \to \mathbb{D}$, and has the following properties. The composition $P_1 \circ \iota$ is a branched holomorphic covering in a neighbourhood of the critical points of $P_1$ and is an orientation preserving topological covering (unramified) outside neighbourhoods of these points.

Rudolph [11] proved the following statement. A link $L$ in $\partial \mathbb{D} \times \mathbb{C}$ is the closure of a quasi-positive geometric braid if it bounds a quasipositive surface in $\partial \mathbb{D} \times \mathbb{C}$. This happens if $L$ is isotopic in $\partial \mathbb{D} \times \mathbb{C}$ to the boundary of a complex curve.

Bennequin [1] proved that any oriented link in $\partial B^2$ which is positively transverse to the complex tangent lines is isotopic through links with the same property (for short, it is transverse isotopic) to a closed braid (in $\partial B^2 \setminus \{z_1 = 0\}$).

Sketch of proof of Lemma 4. Let $X$ be the complex curve in the ball bounded by $K$. Put $K_1 = K$ and let $K_t$ be a transverse isotopy to a knot $K_2$ which is the closure of a braid. Consider the set

$$X' = X \cup \bigcup_{t \in [1,2]} tK_t \cup \bigcup_{t \in [2,\infty]} tK_2.$$

Here $tK_t \overset{\text{def}}{=} \{tz : z \in K_t\}$. Boileau and Orevkov [3] proved that, after smoothing, the set $X' \cap ((2 + 2\epsilon)\partial \mathbb{D} \times \mathbb{C})$ is a quasi-positive surface in $(2 + 2\epsilon)\partial \mathbb{D} \times \mathbb{C}$. By Rudolph’s theorem the link $L_0 \overset{\text{def}}{=} X' \cap ((2 + 2\epsilon)\partial \mathbb{D} \times \mathbb{C})$ is isotopic in $(2 + 2\epsilon)\partial \mathbb{D} \times \mathbb{C}$ to the boundary $L_1 \overset{\text{def}}{=} \partial X'$ of a complex curve $X'$ such that the projection $X' \to (2 + 2\epsilon)\mathbb{D}$ has branch locus in $(2 - 2\epsilon)\mathbb{D}$. The isotopy provides a family of links $L_t$, $t \in [0,1]$, in $X' \cap ((2 + 2\epsilon)\partial \mathbb{D} \times \mathbb{C})$. Given $\delta > 0$, by contraction in the $z_2$-direction the isotopy can be chosen so that $L_1 = \partial X'$ (and hence also the complex curve $X'$) is contained in $(2 + 2\epsilon)\partial \mathbb{D} \times 2\delta \mathbb{D}$.

Put $K' = \frac{1}{2}(X' \cap 2\partial B^2)$. It remains to prove that $K_2$ is isotopic to $K'$ (since $K$ is isotopic to $K_2$). Note that for a positive constant $\beta$ we have the inclusion $K_2 \subset \{|z_1| > \beta\}$. Consider the part $2\partial B^2 \cap \{|z_1| > 2\beta\}$ of the sphere of radius 2. Assign to each point $z$ in this set the point of intersection of the ray $\{tz : t > 1\}$ with the set $\{|z_1| = 2 + 2\epsilon\} \times \mathbb{C}$. We obtain a diffeomorphism $F$ of $2\partial B^2 \cap \{|z_1| > 2\beta\}$ onto a subset of $\{|z_1| = 2 + 2\epsilon\} \times \mathbb{C}$. Consider the inverse $F^{-1}(L_t)$, $t \in [0,1]$, of Rudolph’s isotopy. This is an isotopy of links in $2\partial B^2$ with $F^{-1}(L_0) = 2K_2$. If $\delta$ is small enough then $F^{-1}(L_1) = F^{-1}(\partial X')$. 

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approximates \( X' \cap 2\partial B^2 \) well enough and, hence, is isotopic to it. The lemma is proved.

Let \( X \) be a smooth 2-manifold (or a smooth 2-manifold with boundary). We call an embedding of a smooth 2-manifold (or of a smooth 2-manifold with boundary) \( Y \) into the product \( X \times D \) an \( n \)-horizontal embedding if for the natural projection \( P_X : X \times D \to X \) the restriction \( P_X | Y : Y \to X \) is an \( n \)-covering (smooth and unramified).

**Proposition 3.** Let \( \epsilon \) and \( \delta \) be small positive numbers. Suppose an open Riemann surface \( X \) with smooth connected boundary is holomorphically embedded into \((1 + \epsilon)D \times \delta D\) in such a way that for some \( \epsilon > 0 \) the mapping \( P_1 | X : X \to (1 + \epsilon)D \) is a simple branched covering with branch locus in \((1 - \epsilon)D\). Suppose \( i : \bar{Y} \to \bar{X} \times D \) is a smooth \( n \)-horizontal embedding of the closure of an open Riemann surface \( Y \) into \( \bar{X} \times D \). Then the following statements hold.

1. **(Isotopy of \( n \)-horizontal embeddings to holomorphic embeddings)**
   There exists a simply connected smoothly bounded domain \( D \subset \mathbb{D} \) containing the branch locus of \( P_1 | X \) such that the following holds. Put \( X = (P_1 | X)^{-1}(D) = X \cap (D \times \mathbb{C}) \) and \( Y = Y \cap (X \times D) \). The embedding of the two-manifold \( Y \) into \( X \times D \) is isotopic through horizontal embeddings to a holomorphic embedding of a Riemann surface into \( X \times D \).

2. **(Free isotopy classes of boundary links)**
   Denote by \( w_Y \in B_n \) a braid whose conjugacy class corresponds to the free isotopy class of the boundary link \( \partial Y \subset \partial X \times D \). Let \( w \in B_n \) be a quasi-positive braid. Then there exists a domain \( D_1 \subset \mathbb{D} \), such that with \( X_1 = (P_1)^{-1}(D_1) \) there exists a smooth embedding \( i : \tilde{Y}_1 \to \tilde{X}_1 \times D \) of the closure \( \tilde{Y}_1 \) of an open Riemann surface \( \tilde{Y}_1 \) into the disc bundle which is holomorphic on the Riemann surface \( \tilde{Y}_1 \) and such that \( P_{\tilde{X}} | \tilde{Y}_1 \) is a branched \( n \)-covering of \( X_1 \) and the isotopy class of the boundary link \( \partial Y_1 \subset \partial X_1 \times D \) corresponds to the conjugacy class of \( w \circ w_Y \).

The number of branch points equals the degree of the braid \( w \).

Notice that the complex curves \( X \) and \( X' \) are diffeomorphic. Shrinking \( \delta \) by a contraction in the \( z_2 \)-direction we may also assume that \( X' \) and \( X' \cap B^2 \) are diffeomorphic.

Postpone the proof of Proposition 3.

The following lemma relates the boundaries of the embedded surfaces in Proposition 3 to analytic links which are cables of analytic knots.

**Lemma 5.** **(Boundary links of horizontally embedded surfaces and cables of knots)** Let \( \epsilon > 0 \) and \( \delta > 0 \) be sufficiently small numbers, and let \( X' \) be a Riemann surface which is holomorphically embedded into \((1 + \epsilon)D \times \delta D\) so
that $P_1|\mathcal{X}:\mathcal{X}\to (1+\varepsilon)\mathbb{D}$ is a branched covering with branch locus contained in $(1-\varepsilon)\mathbb{D}$.

Suppose $\mathcal{X}$ has smooth boundary. Denote by $K$ the knot $K = \mathcal{X} \cap \partial \mathbb{B}^2$.

Let $D \subset \mathbb{D}$ be a simply connected smoothly bounded domain containing the branch locus of $P_1|\mathcal{X}$ (so that $X = \mathcal{X} \cap (D \times \mathbb{C})$ is diffeomorphic to $\mathcal{X}$). Suppose there is an open Riemann surface $Y$ and a holomorphic embedding $i : Y \to T_0(X)$ into a small tubular neighbourhood of $X$ such that for the projection $P_X : T_0(X) \to X$ the mapping $p = P_X \circ i$ is a branched holomorphic $n$-covering.

Then there exists an isotopy of $K$ in $\partial \mathbb{B}^2$ to a smoothly analytic knot $\tilde{K}$ and there exists an analytic link $L$ which is an $n$-cable of $\tilde{K}$ contained in an a priori given tubular neighbourhood of $\tilde{K}$ with the following property. The pattern of $L$ equals the isotopy class of $\partial Y$ in $T_0(\partial X)$ and $L$ bounds a complex curve $Y$ in $\mathbb{B}^2$ which is diffeomorphic to $Y$.

Proof of Lemma 5. Let $D_t$, $t = [0, 1]$, be a continuous decreasing family of simply connected smoothly bounded domains with $D_0 = \mathbb{D}$ and $D_1 = D$. Let $\alpha$ be a continuous function on $[0, 1]$ with $\alpha(0) = 1$ and $\alpha(t) > 1$ for $t > 0$. Denote by $\omega_t : D_t \to \alpha(t)\mathbb{D}$ the conformal mapping with $\omega_t'(0) > 0$. The $\omega_t$ depend continuously on $t$.

Let $t \to s(t)$, $t \in [0, 1]$, be a continuous decreasing positive function with $s(0) = 1$ which takes sufficiently small values for $t$ away from 0. For $z = (z_1, z_2) \in D_t \times \mathbb{C}$, $t \in [0, 1]$, we put $G_t(z) = (\omega_t(z_1), s(t) \cdot z_2)$. The mapping $G_t|\mathcal{X} \cap (D_t \times \mathbb{C})$ is a conformal map onto a (relatively closed) complex curve $X_t$ in $\alpha(t)\mathbb{D} \times \partial \mathbb{D}$. If $\delta$ is small enough then for suitable choices of the functions $s$ and $\alpha$ and $t$ the intersection of each $X_t$ with $\partial \mathbb{B}^2$ is transversal, and, hence, $K_t = X_t \cap \partial \mathbb{B}^2$, $t \in [0, 1]$, is a transversal isotopy. Moreover, if $s(1) > 0$ is small then the subset $\tilde{X} = X_1 \cap \mathbb{B}^2$ of $X_1$ is close to $X_1$ and is diffeomorphic to $X_1$. Put $\tilde{K} = \partial \tilde{X}$.

We may assume that $a > 0$ is sufficiently small and $Y$ is identified with an embedded submanifold of $T_0(X)$. The Riemann surface $Y_1 = G_1(Y)$ is conformally equivalent to $Y$ and embedded into the tubular neighbourhood $G_1(T_0(X))$ of $X_1$. For a suitable choice of the function $s$ and small $\alpha$ the Riemann surface $\tilde{Y} = Y_1 \cap \mathbb{B}^2$ is diffeomorphic to $Y_1$ and the boundary $\tilde{L} \defeq \partial \tilde{Y} \subset \partial \mathbb{B}^2$ is an $n$-cable of $\partial X$ contained in the small tubular neighbourhood $G_1(T_0(X)) \cap \partial \mathbb{B}^2$ of $\tilde{K}$. Moreover, the free isotopy class of $\partial \tilde{Y}$ in $G_1(T_0(X))$ equals the pattern of $\tilde{L}$.

Proof of Lemma 1. After an isotopy we may assume that $K = \mathcal{X} \cap \partial \mathbb{B}^2$ for a smooth complex curve $\mathcal{X}$ which is holomorphically embedded into $(1+\varepsilon)\mathbb{D} \times \delta \mathbb{D}$ for small positive numbers $\varepsilon$ and $\delta$. Moreover, the mapping $P_1|\mathcal{X}:\mathcal{X}\to (1+\varepsilon)\mathbb{D}$ is a branched covering with branch locus in $(1-\varepsilon)\mathbb{D}$.

For a small positive $a$ we identify the tubular neighbourhood $T_0(\mathcal{X})$ with the trivial holomorphic disc bundle $\mathcal{X} \times a \mathbb{D}$. Put $\mathcal{Y} = \mathcal{X} \times E \subset \partial \mathcal{X} \times a \mathbb{D}$ for a subset $E$ of $a \mathbb{D}$ consisting of $n$ points. The boundary $\partial \mathcal{Y} \subset \partial \mathcal{X} \times a \mathbb{D}$ represents the trivial conjugacy class of closed $n$-braids in $\partial \mathcal{X} \times a \mathbb{D}$.
Let \( w \in \mathcal{B}_n \) be a braid whose conjugacy class equals the pattern \( \mathcal{L}' \) of \( L' \). By lemma 4 we may assume after an isotopy that \( L' \) is contained in a small neighbourhood of the unknot \( U \) and \( L' = \partial \mathbb{B}^2 \cap Y' \) for a complex curve \( Y' \) in a neighbourhood of \( \mathbb{B}^2 \) such that \( P_1 | Y' : Y' \to (1 + \varepsilon)\mathbb{D} \) is a branched \( n \)-covering with branch locus in \( (1 - \varepsilon)\mathbb{D} \). Then \( g_4(L') = g(Y') \). Denote by \( B' \) the degree of \( w \).

Apply Proposition 3 to \( \mathcal{X}, \mathcal{Y} \), the trivial braid \( w_{\mathcal{Y}} \), and the quasipositive braid \( w \in \mathcal{B}_n \). We obtain a smoothly bounded domain \( D \subset \mathbb{D} \), a Riemann surface \( X_1 = (P_1)^{-1}(\mathbb{D}) \) and a smooth embedding of the closure \( Y_1 \) of an open Riemann surface \( Y_1 \) into \( \mathcal{X} \times a\mathbb{D} \) which is holomorphic on \( Y_1 \) and such that \( P_1 | Y_1 \) is a branched holomorphic covering of \( X_1 \) with number of branch points equal to \( B' \). The surface \( X_1 \) is diffeomorphic to \( \mathcal{X} \), and the surface \( Y_1 \) is diffeomorphic to \( \mathcal{Y} \). The isotopy class of the boundary link \( \partial Y_1 \subset \partial X_1 \times a\mathbb{D} \) corresponds to the conjugacy class of \( w_{\mathcal{Y}} \circ w = w \). In other words, the free isotopy class of \( \partial Y_1 \subset \partial X_1 \times \mathbb{D} \) equals the pattern \( \mathcal{L}' \).

Lemma 4 provides an isotopy of \( K \) to a smoothly analytic knot \( \tilde{K} \), which bounds a Riemann surface \( \tilde{X} \subset \mathbb{B}^2 \) that is diffeomorphic to \( \mathcal{X} \), and an \( n \)-cable \( \tilde{L} \) in a small tubular neighbourhood of \( \tilde{K} \), which bounds a Riemann surface \( \tilde{Y} \) that is diffeomorphic to \( Y_1 \). Moreover, the pattern of \( \tilde{L} \) is \( \mathcal{L}' \).

The formula for the 4-ball genus follows from the Riemann-Hurwitz relation. Indeed, the Riemann-Hurwitz relation for branched coverings of open Riemann surfaces is the following.

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be connected open Riemann surfaces with smooth boundaries and let \( p : Y \to X \) be an \( n \)-fold branched covering (smooth and orientation preserving at regular points and holomorphic near branch points, without branch points on the boundary). Then

\[
\chi(Y) = n \cdot \chi(X) - B.
\]

Here \( \chi \) is the Euler characteristic and \( B \) is the number of branch points (counted with multiplicity). Denote by \( k(\mathcal{X}) \) the number of boundary components of a Riemann surface \( \mathcal{X} \). Then

\[
\chi(\mathcal{X}) = 2 - 2g(\mathcal{X}) - k(\mathcal{X}).
\]

Hence, in our situation

\[
\chi(\tilde{Y}) = n \chi(X_1) - B' = n \chi(\tilde{X}) - B'.
\]

We used that \( X_1 \) is diffeomorphic to \( \tilde{X} \) and \( Y_1 \) is diffeomorphic to \( \tilde{Y} \):

Since \( 1 - 2g_4(L') = \chi_4(Y') = n \chi((1 + \varepsilon)\mathbb{D}) - B' = n - B' \), we obtain

\[
1 - 2g(\tilde{Y}) = n(1 - 2g(\mathcal{X})) - B' = -2ng(\mathcal{X}) + 1 + 2g_4(L'),
\]

hence

\[
g_4(\tilde{L}) = g(\tilde{Y}) = ng(\mathcal{X}) + g_4(L') = ng_4(\tilde{K}) + g_4(L').
\]
For the proof of Proposition 3 it will be convenient to work with the following terminology.

A continuous mapping from a smooth manifold (or a smooth manifold with boundary) \(X\) into the set of all monic polynomials \(\mathbb{P}_n\) of degree \(n\) is a quasi-polynomial of degree \(n\). It can be written as function in two variables \(x \in X, \zeta \in \mathbb{C}\), i.e. \(P(x, \zeta) = a_0(x) + a_1(x)\zeta + \ldots + a_{n-1}(x)\zeta^{n-1} + \zeta^n\), for continuous functions \(a_j, j = 1, \ldots, n\), on \(X\). If the image of the map is contained in the space \(\mathbb{P}_n\) of monic polynomials of degree \(n\) without multiple zeros, it is called separable. Two separable quasi-polynomials are isotopic if there is a continuous family of separable quasi-polynomials joining them. For a separable quasi-polynomial, considered as a function \(\mathcal{P}\) on \(X \times \mathbb{C}\), its zero set \(\mathcal{S}_\mathcal{P} = \{(x, \zeta) \in X \times \mathbb{C}, \mathcal{P}(x, \zeta) = 0\}\) is a surface which is \(n\)-horizontally embedded into \(X \times \mathbb{C}^n\). Vice versa, each \(n\)-horizontally embedded 2-manifold in \(X \times \mathbb{C}\) corresponds to a separable quasi-polynomial of degree \(n\) on \(X\). An isotopy of separable quasi-polynomials (i.e. a family of separable quasi-polynomials depending continuously on a parameter in \([0, 1]\)) is equivalent to an isotopy of \(n\)-horizontally embedded manifolds.

**Proof of Proposition 3.** For the proof of assertion 1 we consider a simple smooth arc \(\Gamma\) (i.e. a diffeomorphic image of the closed unit interval) in the disc \(\mathbb{D}\) which passes once through each point of the branch locus \(E\) of the covering \(X' \to (1 + \epsilon)\mathbb{D}\). Let \(\Gamma_X = X' \cap (P_1)^{-1}(\Gamma) \subset X\) and let \(\mathcal{P}\) be the mapping from \(X'\) into \(C_n(\mathbb{C}) \cong \mathbb{P}_n \cong \mathbb{C}^n \setminus \{D_n = 0\} \subset \mathbb{C}^n\) defined by the embedding of \(\bar{Y}\) into \(X' \times \mathbb{D}\). The subset \(\Gamma_X\) of \(X\) has no interior point. Hence, by Mergelyan’s Theorem the restriction \(\mathcal{P} | \Gamma_X\) can be approximated uniformly on \(\Gamma_X\) by an analytic mapping of a neighbourhood \(X' \subset X\) of \(\Gamma_X\) into \(\mathbb{C}^n\). After perhaps shrinking \(X'\), we may assume that the restriction \(P_1 | X'\) is a branched covering onto a domain \(D \subset \mathbb{D}\). The set \(\Gamma_X\) is a deformation retract of \(X'\). After shrinking \(X'\) we may assume that \(X'\) is diffeomorphic to \(X\).

If the approximation is good enough then, after, perhaps, shrinking \(X'\) again, the image of \(X'\) under the approximating mapping is contained in the symmetrized configuration space. We obtain a holomorphic mapping of \(X'\) into the symmetrized configuration space which is isotopic to \(\mathcal{P} | X'\) through smooth mappings into symmetrized configuration space. The isotopy of mappings defines an isotopy of \(n\)-horizontal embeddings into \(X' \times \mathbb{D}\) which join the embedding of \(Y \cap (X' \times \mathbb{D})\) with a holomorphic \(n\)-horizontal embedding of a complex curve into \(X' \times \mathbb{D}\). We proved assertion 1.

For the proof of assertion 2 we need Lemma 6 below. The following construction prepares its statement. Let \(X'\) be an open Riemann surface with smooth connected boundary. Take any smoothly bounded domain \(X_0 \subset X\) which is a strong deformation retract of \(X\), and take simply connected smoothly bounded domains \(X_j \Subset X' \setminus X_0, j = 1, \ldots, k\), with pairwise disjoint closure. Consider simple smooth pairwise disjoint arcs \(\gamma_j : [0, 1] \to X', j = 1, \ldots, k\), such that for each \(j\) the interior of \(\gamma_j\) is contained in \(X' \setminus \bigcup_{j=0}^k \bar{X}_j\) and \(\gamma_j\) joins a boundary point of \(X_0\) with boundary point of \(X_j\). Consider disjoint “rectangles”
around the $\gamma_j$, i.e. for some $\varepsilon > 0$ the $R_j$ are diffeomorphic mappings of $(-\varepsilon, \varepsilon) \times [0, 1]$ with image in $X$, which are holomorphic on the open set $(-\varepsilon, \varepsilon) \times (-1, 1)$ and such that $R_j|\{0\} \times [0, 1]$ equals $\gamma_j$ and the sides $R_j((-\varepsilon, \varepsilon) \times \{0\})$ and $R_j((-\varepsilon, \varepsilon) \times \{1\})$ are contained in $\partial X_0$, respectively in $\partial X_1$. Suppose the domain $X_0 \cup \bigcup_{j=1}^k X_k \cup \bigcup_{j=1}^k R_j((-\varepsilon, \varepsilon) \times [0, 1])$ has smooth boundary (see fig. 5). Then it is again a deformation retract of $X$. We call any domain of the above described type a thickening of $\bigcup_{j=0}^k X_k \cup \bigcup_{j=1}^k \gamma_j$.

Denote by $P_X : X \times D \to X$ the canonical projection to the first factor. Consider a branched holomorphic covering $p$ of $X$ by an open Riemann surface $Y$ and a holomorphic embedding $i$ of $Y$ into the disc bundle $X \times D$ such that $P_X \circ i = p$ (for short $i$ lifts $p$). The following Lemma 6 provides the embedding of another open Riemann surface $Y$ into the holomorphic disc bundle over a deformation retract $X$ of $X$ such that the embedding lifts a branched covering with "more" branch points and has the following property: the isotopy class of the embedding of the boundary $\partial Y$ into $\partial X \times D$ differs from that of $\partial Y \subset \partial X \times D$ by a prescribed quasi-positive braid.

**Lemma 6. (Adding branch points)** Let $X$ and $Y$ be open Riemann surfaces with smooth boundary. Suppose the boundary of $X$ is connected. Let $i : Y \to X \times D$ be a smooth embedding which is holomorphic on $Y$ and such that for the canonical projection $P_X : X \times D \to X$ the restriction $P_X|Y$ is a (simple branched or unbranched) holomorphic $n$-covering of $X$. Denote by $w_Y \in B_n$ a braid whose conjugacy class corresponds to the isotopy class of the boundary link $\partial Y \subset \partial X \times D$. Let $w \in B_n$ be a quasi-positive braid of degree $m$. Let $X_0$ be as above. Suppose $X \setminus X_0$ does not intersect the branch locus of $P_X|Y$. 

Figure 5.
Then for \( k \geq m \) and for each collection \( \mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_k, \gamma_1, \ldots, \gamma_k \), as above there exists a thickening \( X \subset \mathcal{X} \) of \( \bigcup_{j=0}^{k} \mathcal{X}_j \cup \bigcup_{j=1}^{k} \gamma_j \) and a holomorphic embedding of an open Riemann surface \( i : Y \to X \times \mathbb{D} \) into the disc bundle over \( X \) such that \( P_\mathcal{X} \mid Y \) is a simple branched \( n \)-covering of \( X \) whose branch locus contains exactly one point in each member of a chosen collection of \( m \) of the sets \( \mathcal{X}_j \), \( j = 1, \ldots, k \), and may contain also some points in \( \mathcal{X}_0 \). Moreover, the isotopy class of the boundary link \( \partial Y \subset \partial X \times \mathbb{D} \) corresponds to conjugacy class of the braid \( w \circ w_\gamma \in \mathcal{B}_n \). The same conclusion holds for any thickening of \( \bigcup_{j=0}^{k} \mathcal{X}_j \cup \bigcup_{j=1}^{k} \gamma_j \) which is contained in \( X \).

**End of Proof of assertion 2 of Proposition 3.** Let \( D \) be the domain chosen in assertion 1, let \( d_j, j = 1, \ldots, \deg w \), be pairwise disjoint smoothly bounded open topological discs contained in \( \partial D \), and let \( \Gamma_j, j = 1, \ldots, n \), be pairwise disjoint arcs with interior contained in \( \partial (D \cup \bigcup d_j) \) such that \( \Gamma_j \) joins a boundary point of \( D \) with a boundary point of \( d_j \). Let \( D_1 \subset D \) be a small enough thickening of \( D \cup \bigcup (d_j \cup \Gamma_j) \). Then \( (P_1)^{-1}(D_1) \) is a small thickening of \( (P_1)^{-1}(D \cup \bigcup (d_j \cup \Gamma_j)) \). Apply Lemma 5, where we take \( \mathcal{X}_0 = (P_1)^{-1}(D) \subset \mathcal{X} \), \( \mathcal{X}_j, j = 1, \ldots, k \), running over all components of the preimages under \( P_1 \) of the \( d_j \) for all \( i \), and \( \gamma_j, j = 1, \ldots, k \), (with the respective label) running over all components of the preimages of all \( \Gamma_j \). Choose the collection of \( m = \deg w \) of the \( \mathcal{X}_j \)'s so that \( P_1 \) is injective on the union of the sets of the collection.

Lemma 5 implies statement 2 of Proposition 3. □

**Proof of Lemma 6.** Denote by \( \mathcal{A} \) the compact set \( \mathcal{A} = \bigcup_{j=0}^{k} \mathcal{X}_j \cup \bigcup_{j=1}^{k} \gamma_j \). Let \( \mathcal{C} \) be the (oriented) closed curve which is contained in the boundary \( \partial \mathcal{A} \) and surrounds \( \mathcal{A} \) counterclockwise (see fig. 5). \( \mathcal{C} \) is obtained from \( \partial \mathcal{X}_0 \) (traveled counterclockwise) in the following way: for each \( j = 1, \ldots, k \), we cut \( \partial \mathcal{X}_j \) at the starting point \( q_j \) of \( \gamma_j \) and insert the curve \( \gamma_j \) followed by \( \partial \mathcal{X}_j \) traveled from the endpoint of \( \gamma_j \) surrounding \( \mathcal{X}_j \) counterclockwise, followed by \( \gamma_j^{-1} \) (\( \gamma_j \) with inverse orientation).

Consider the closed geometric braids \( \partial \mathcal{Y} \subset \partial \mathcal{X} \times \mathbb{C} \) and \( \mathcal{Y} \cap (\partial \mathcal{X}_0 \times \mathbb{C}) \). Let \( Q \) be a point in \( \partial \mathcal{X} \) and let \( q = q_l \in \partial \mathcal{X}_0 \) be the initial point of the arc \( \gamma_1 \). Put \( E^n_{Q} = P_\mathcal{C} \left( \mathcal{Y} \cap \{Q\} \times \mathbb{C} \right) \) and \( E^n_a = P_\mathcal{C} \left( \mathcal{Y} \cap \{q\} \times \mathbb{C} \right) \). Here \( P_\mathcal{C} : \mathcal{X} \times \mathbb{C} \to \mathbb{C} \) is the canonical projection.

Denote by \( \Gamma \) an arc whose interior parametrizes \( \partial \mathcal{X} \setminus \{Q\} \) (with orientation induced from the orientation of \( \partial \mathcal{X} \) as boundary of \( \mathcal{X} \)) and whose endpoints are equal to \( Q \). Let \( \Gamma_0 \) be the respective object for \( \partial \mathcal{X}_0 \) and the point \( q \). The closed geometric braid \( \partial \mathcal{Y} \) defines a continuous map from \( \Gamma \) to \( \Psi_n \) with base point \( E^n_{Q} \), the closed geometric braid \( \mathcal{Y} \cap (\partial \mathcal{X}_0 \times \mathbb{C}) \) defines a continuous map from \( \Gamma_0 \) to \( \Psi_n \) with base point \( E^n_a \).

Let \( \gamma \) be a closed arc with interior contained in \( \mathcal{X} \setminus \left( \bigcup_{j=0}^{k} \mathcal{X}_j \cup \bigcup_{j=1}^{k} \gamma_j \right) \), with
initial point \( q \) and with terminating point \( Q \). The embedding of \( \tilde{Y} \) into \( \tilde{X} \times \mathbb{C} \) defines a continuous map from the curve \( \Gamma_0 \) into \( \mathcal{P}_n \) with endpoints equal to \( E_n^\gamma \), i.e., it defines a geometric braid with base point \( E_n^\gamma \). In the same way we obtain a continuous map of \( \gamma + \Gamma + (-\gamma) \) to \( \mathcal{P}_n \) which is another geometric braid with base point \( E_n^\gamma \). (The sum of pathes means, that we first walk along \( \gamma \) until we reach its terminating point, which is the initial point of \( \Gamma \), then along \( \Gamma \), then along \( -\gamma \), which means \( \gamma \) equipped with orientation opposite to the orientation of \( \gamma \).) After identifying the pathes \( \Gamma_0 \) and \( \gamma + \Gamma + (-\gamma) \) by a homeomorphism, the two geometric braids are isotopic, since \( \tilde{X} \backslash \mathcal{X}_0 \) does not contain points in the branch locus of the projection \( P_X \mid Y \). Denote the isotopy class of these braids by \( w'_Y \). Having in mind a homomorphism between \( \mathcal{B}_n \) and the group of isotopy classes of geometric braids with base point \( E_n^\gamma \), write \( w'_Y \overset{\text{def}}{=} w_1^{-1} \circ w_Y \circ w_1 \) for a braid \( w_1 \in \mathcal{B}_n \).

Pick \( m \) of the \( \mathcal{X}_j \), \( j \geq 1 \), and label them in the order we meet the initial point \( q_j \) of \( \gamma_j \) when traveling along \( \partial \mathcal{X}_0 \) in the direction of orientation of \( \partial \mathcal{X}_0 \) as boundary of \( \mathcal{X}_0 \), starting from \( q \). For transparency of the proof we assume \( m = k \). The braid \( w \) is quasi-positive. Hence, we may write \( w' \overset{\text{def}}{=} w_1 \circ w \circ (w_1)^{-1} = (v_1)^{-1} \sigma_{\xi}, v_1 \circ \ldots \circ (v_m)^{-1} \sigma_{\xi_n} v_m \) for \( v_j \in \mathcal{B}_n, \ j = 1, \ldots, m \).

Consider the \( n \) points in \( E_n^\gamma \) and label them by \( \xi_1, \ldots, \xi_n \). Put \( \varrho = \min\{|\xi_i - \xi_j| : i \neq j\} \). For each \( j \) we pick a point \( \eta_j \in \mathcal{X}_j \) and consider the holomorphic curve \( \mathcal{Z}_j = \{(z_1, z_2) \in \mathcal{X}_j \times \mathbb{C} : \alpha_j (z_2 - \xi_j)^2 = z_1 - \eta_j \} \). Here \( \alpha_j \) is a constant chosen so that \( \mathcal{Z}_j \) is contained in \( \mathcal{X}_j \times \{|z_2 - \xi_j| < \frac{\varrho}{2}\} \). Consider the surfaces \( \mathcal{Y}_j = \mathcal{Z}_j \cup \bigcup_{j \neq j', j \neq j} \{(z_1, z_2) : z_2 = \xi, z_1 \in \mathcal{X}_j', j = 1, \ldots, m \}. \) Each \( \mathcal{Y}_j \) is a Riemann surface which has \( m - 1 \) connected components and smooth boundary.

The boundary \( \partial \mathcal{Y}_j \) corresponds to the conjugacy class of braids which contains \( \sigma_{\xi_j} \). The projection \( P_X \mid \mathcal{Y}_j \rightarrow \mathcal{X}_j \) is a simple branched covering with a single point \( \eta_j \) in the branch locus. The embedding of the closure \( \overline{\mathcal{Y}_j} \) into \( \overline{\mathcal{X}_j} \times \mathbb{C} \) corresponds to a quasi-polynomial, denoted \( \mathcal{P}_j(z, \zeta), z \in \mathcal{X}_j, \zeta \in \mathbb{C} \), such that the polynomial \( \zeta \rightarrow \mathcal{P}_j(\eta_j, \zeta), \zeta \in \mathbb{C} \), has a double zero, and the restriction of the quasi-polynomial to \( \mathcal{X}_j \setminus \{\eta_j\} \) is separable.

Let \( Q_1 \) be the terminating point of \( \gamma_1 \). Denote by \( \Gamma_1 \) a closed arc whose interior parametrizes \( \partial \mathcal{X}_1 \setminus \{Q_1\} \) with orientation induced from the orientation of \( \partial \mathcal{X}_1 \) and whose endpoints are equal to \( Q_1 \). Define a mapping from \( \overline{\mathcal{X}_0} \cup \gamma_1 \cup \overline{\mathcal{X}_1} \) into the space \( \overline{\mathcal{P}_n} \) of all monic polynomials of degree \( n \) as follows. For \( z_1 \in \overline{\mathcal{X}_0} \) we let the map be equal to the monic polynomial with zeros \( \overline{\mathcal{Y}} \cap \{(z_1) \times \mathbb{C}\} \), and for \( z_1 \in \overline{\mathcal{X}_1} \) we let the map be equal to the monic polynomial with zeros \( \overline{\mathcal{Y}} \cap \{(z_1) \times \mathbb{C}\} \). We extend the map continuously by a map from \( \gamma_1 \) to the space \( \overline{\mathcal{P}_n} \) of polynomials without multiple roots so that the following holds. The induced mapping from \( \gamma_1 + \Gamma_1 + (-\gamma_1) + \Gamma_0 \) to \( \mathcal{P}_n \) with base point \( E_n^\gamma \) represents the geometric braid \( (v_1)^{-1} \sigma_{\xi}, v_1 \circ w'_Y \). (We use the same homomorphism between \( \mathcal{B}_n \) and the group of isotopy classes of geometric braids with base point \( E_n^\gamma \) as before.)

Continue by induction. Let \( Q_2 \) be the terminal point of \( \gamma_2 \). Denote by \( \Gamma_2 \) an arc whose interior parametrizes \( \partial \mathcal{X}_2 \setminus \{Q_2\} \) with orientation induced from the orientation of \( \partial \mathcal{X}_2 \) and whose endpoints are equal to \( Q_2 \). Denote by \( \Gamma_0 \) the
path obtained by walking along $\partial X_0$ according to its orientation from $q = q_1$ until the initial point $q_2$ of the curve $\gamma_2$. Consider the continuous map from $X_0 \cup \gamma_1 \cup X_1 \cup \gamma_2 \cup X_2$ into the space $\mathbb{P}_n$, which is equal to the previous map on $X_0 \cup \gamma_1 \cup X_1$, which is equal to the monic polynomial with zeros $\gamma_2 \cap \{z_2\} \times \mathbb{C}$ for $z_2 \in X_2$, and which is continuously extended to $\gamma_2$ so that the following holds. The induced mapping from $(\gamma_1 + \Gamma_1 + (-\gamma_1)) + (\Gamma_0^1 + \gamma_2 + \Gamma_2 + (-\gamma_2)) + (\Gamma_1^0) + \Gamma_0$ has image in $\mathbb{P}_n$ and represents the braid $(v_1)^{-1} \sigma_{\ell_1} v_1 \circ (v_2)^{-1} \sigma_{\ell_2} v_2 \circ w'_{\gamma}$.

By induction we obtain a continuous map from $A = X_0 \cup \bigcup_{j=1}^m (\gamma_j \cup X_j)$ to $\mathbb{P}_n$ which is holomorphic on $\bigcup_{j=0}^m X_j$. Moreover, the induced map on the curve $C$ defines a geometric braid corresponding to the conjugacy class of $(v_1)^{-1} \sigma_{\ell_1} v_1 \circ \ldots \circ (v_m)^{-1} \sigma_{\ell_m} v_m \circ w'_{\gamma} = w_1 w w_1^{-1} w_1 w_2 w_1^{-1} = w_1 w w_2 w_1^{-1}$.

The mapping from $A$ to $\mathbb{P}_n$ corresponds to a quasi-polynomial of degree $n$ on $A$ denoted by $P$, $P(z, \zeta) = \sum_{m=0}^n a_m(z) \zeta^m$, $z \in A$, $\zeta \in \mathbb{D}$. Here $a_n \equiv 1$. The coefficients $a_m$ are continuous functions on $A$ which are analytic on the interior

$\text{Int } A = \bigcup_{j=0}^k X_j$. The polynomial $\zeta \to P(z, \zeta)$ has multiple zeros exactly if $z$ is in the branch locus of $P_X | \gamma$. This happens for some points in $X_0$ and for the points $\eta_j \in X_j$, $j = 1, \ldots, m$. Denote by $Y'$ the zero set of the quasi-polynomial in $A \times \mathbb{C}$.

By the Mergelyan theorem on open Riemann surfaces ([17], Corollary 4) each coefficient $a_m$, $m = 0, 1, \ldots, n - 1$, can be uniformly approximated by a holomorphic function $a_\ell_m$ in a neighbourhood $\tilde{A}$ of $A$ on $X$. Denote $\tilde{P}(z, \zeta) = \sum_{m=0}^n \tilde{a}_m(z) \zeta^m$, $z \in \tilde{A}$, $\zeta \in \mathbb{D}$. Here $\tilde{a}_n = a_n \equiv 1$. We may assume that $0$ is a regular value of $\tilde{P}$. If the approximation is good enough then for each $z \in \partial A$ the polynomial $\tilde{P}(z, \cdot)$ has $n$ distinct roots which are close to the roots of $P(z, \cdot)$.

There is a thickening $X \subset \tilde{A}$ of $\bigcup_{j=0}^k X_j \cup \bigcup_{j=1}^k \gamma_j$ such that $\tilde{P}(z, \cdot)$ has no multiple roots for $z$ in the closure of each rectangle $R_j$ added to build the thickening. Put $Y = \{(z, \zeta) \in \tilde{X} \times \mathbb{D} : \tilde{P}(z, \zeta) = 0\}$. The interior $Y$ is holomorphically embedded into $X \times \mathbb{D}$ and the projection $P_X | Y : Y \to X$ extends to a neighbourhood of $Y$ as branched covering with branch locus in $\bigcup_{j=0}^k X_j \subset X$. In particular, if $\tilde{X}$ is close enough to $A$, then the closed geometric braids $\partial Y \subset \partial X \times \mathbb{C}$ and $Y \cap (C \times \mathbb{C})$ are free isotopic (after identifying $\partial X$ and $C$ by a homeomorphism), and, hence, they correspond to the same conjugacy class in $B_n$, namely to the conjugacy class of $w \circ w'_{\gamma}$. The same conclusion holds for $X$ replaced by any thickening of $\bigcup_{j=0}^k X_j \cup \bigcup_{j=1}^k \gamma_j$ contained in $X$. The lemma is proved. \hfill \Box

Note that the proof of Lemma 6 provides also a proof of one of the implications of Rudolph’s theorem. Indeed, let $L$ be the closure of a quasipositive braid. Put $X = \mathbb{D}$, let the $X_j$ be as in the lemma and consider a constant mapping from $X_0$ to $\mathbb{P}_n$. The lemma provides a simply connected domain $X \subset \mathbb{D}$ and a
holomorphic embedding of a Riemann surface $Y$ into $X \times \mathbb{C}$ such that $Y \to X$ is a simple branched covering and $Y$ has smooth boundary $\partial Y$ such that the embedding $\partial Y \to \partial X \times \mathbb{C}$ defines a closed geometric braid which corresponds to the conjugacy class of the quasi-positive braid. To obtain a complex curve in $\mathbb{D} \times \mathbb{C}$ with boundary contained in $\partial \mathbb{D} \times \mathbb{C}$ and isotopic to $L$, we map $X$ conformally onto $\mathbb{D}$.

For the proofs of the theorems we need the following known proposition.

**Proposition 4.** Let $X$ be a connected smooth surface or a connected smooth surface with boundary. Let $\pi_1(X, x_0)$ be the fundamental group of $X$ with a given base point $x_0$. The following statements hold.

1. There is a one-to-one correspondence between homomorphisms $\Psi : \pi_1(X, x_0) \to S_n$ and unramified $n$-coverings $p : Y \to X$ with given label of points in the fiber $p^{-1}(x_0)$.

2. There is a one-to-one correspondence between homomorphisms $\Phi : \pi_1(X, x_0) \to B_n$ and isotopy classes of separable quasi-polynomials with fixed value $E_n$ at $x_0$ and given label of points in $E_n$. The quasi-polynomial lifts a covering $p$ iff the homomorphism $\Phi$ lifts the homomorphism $p_* : \pi_1(X, x_0) \to S_n$ corresponding to $p$, i.e. $p_* = \tau_n \circ \Phi$ for the canonical homomorphism $\tau_n : B_n \to S_n$.

3. The connected components of the covering space $Y$ of an unramified holomorphic $n$-covering $p : Y \to X$ are in bijective correspondence to the orbits of $p_* (\pi_1(X, x_0))$ in $S_n$ on the set consisting of $n$ points. In particular, $Y$ is connected iff $p_* (\pi_1(X, x_0))$ acts transitively.

4. Suppose $X$ is a 2-manifold with boundary. Suppose the boundary $\partial X$ is connected and the base point $x_0$ is contained in the boundary. Denote by $\{\partial X\}$ the element of the fundamental group $\pi_1(X, x_0)$ which is represented by traveling along $\partial X$ in the direction of orientation as boundary of $X$ starting from $x_0$. (So, $\{\partial X\}$ is the product of $g$ commutators of suitable generators of the fundamental group.) Let $p : Y \to X$ be an unramified $n$-covering and let $P(x, \zeta)$ be a separable quasi-polynomial lifting it, i.e. the covering is equivalent to $P_{\mathbb{D}} : \mathbb{S}_{\mathbb{P}} \to X$.

Then the connected components of the boundary $\partial Y$ correspond to the orbits of the single even permutation $p_* (\{\partial X\})$. $p_* (\{\partial X\})$ is the product of $g$ commutators in $S_n$. Here $g$ is the genus of $X$.

Moreover, the free isotopy class of the boundary link $\partial \mathbb{S}_{\mathbb{P}} \subset \partial X \times \mathbb{C}$ is a closed geometric braid representing the conjugacy class of the product of $g$ commutators in $B_n$.

Notice that a conjugate of a commutator is again a commutator.
4 Proof of the theorems

The following proposition is a direct consequence of the Riemann-Hurwitz relation and is used for the proof of Theorem 1.

**Proposition 5.** Let \( \mathcal{X} \) be a connected open Riemann surface with smooth connected boundary and let \( \mathcal{Y} \) be an open Riemann surface which has connected components \( \mathcal{Y}_1, \ldots, \mathcal{Y}_m \). Let \( p : \mathcal{Y} \to \mathcal{X} \) be a smooth map which is a finite unramified covering outside a finite collection of critical points and is a branched holomorphic covering on the preimage of a neighbourhood of each point in the branch locus. Denote by \( k_j \) the number of boundary components of \( \mathcal{Y}_j \), by \( n_j \) the multiplicity of the covering \( p |_{\mathcal{Y}_j} \) and by \( B_j \) the number of its branch points (counted with multiplicity). Put \( k = \sum_{j=1}^{m} k_j \), \( B = \sum_{j=1}^{m} B_j \). Let \( \mathcal{Y} \) be a Riemann surface which contains \( \mathcal{Y} \). Then

\[
g(\mathcal{Y}) \geq g(\mathcal{Y}) \geq ng(\mathcal{X}) - (n - 1) \tag{5}
\]

If \( \mathcal{Y} \) has connected boundary then

\[
g(\mathcal{Y}) \geq g(\mathcal{Y}) + k - m \geq ng(\mathcal{X}) - \left\lfloor \frac{n-k}{2} \right\rfloor \geq ng(\mathcal{X}) - \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{6}
\]

If \( n \) is odd then in (6) all relations may be equalities only if \( k = 1 \) (i.e. if \( \mathcal{Y} \) and its boundary \( \partial \mathcal{Y} \) are connected) and the covering \( \mathcal{Y} \to \mathcal{X} \) is unramified.

If \( n \) is even in (6) all relations may be equalities only if \( k = 1 \) or \( k = 2 \).

**Proof.** To obtain (5) we use for each \( j \) the Riemann-Hurwitz relation

\[
2 - 2g(\mathcal{Y}_j) - k_j = n_j (1 - 2g(\mathcal{X})) - B_j.
\]

Consider the sum over all \( j \) taking into account the relations

\[
m \geq 1, \quad \sum_{j=1}^{m} n_j = n, \quad k_j \leq n_j, \quad B = \sum_{j=1}^{m} B_j \geq 0,
\]

we obtain

\[
2g(\mathcal{Y}) = \sum_{j=1}^{m} 2g(\mathcal{Y}_j) = n(2g(\mathcal{X}) - 1) + B + \sum_{j=1}^{m} (2 - k_j) =
\]

\[
2ng(\mathcal{X}) - n + B + 2m - k \geq n \cdot 2g(\mathcal{X}) + 2(1 - n), \tag{7}
\]

and, hence, (5).

Suppose \( \mathcal{Y} \) has connected boundary (hence, \( \mathcal{Y} \) is connected itself). Then \( \mathcal{Y} \setminus \mathcal{Y} \) contains \( m - 1 \) disjoint simple arcs (i.e. diffeomorphic images of the interval \([0, 1]\)) with endpoints on \( \partial \mathcal{Y} \) whose interiors are contained in \( \mathcal{Y} \setminus \mathcal{Y} \) such that the union of \( \mathcal{Y} \) with these arcs is connected. Attaching to \( \mathcal{Y} \) the union of small neighbourhoods of these arcs does not change the genus. Hence, there is a connected open subset \( \mathcal{Y}_1 \) of \( \mathcal{Y} \) which contains \( \mathcal{Y} \) and the union of the arcs such that \( \mathcal{Y}_1 \) has the same genus \( g(\mathcal{Y}_1) = g(\mathcal{Y}) \) as \( \mathcal{Y} \).
The Riemann surface $Y_1$ has $k - (m - 1)$ boundary components. Since $Y$ has connected boundary there are $k - m$ disjoint simple arcs, each with endpoints on $\partial Y_1$ and interior contained in $Y \setminus \bar{Y}_1$ such that the union of $\partial Y_1$ with these arcs is connected. Attach consecutively disjoint small neighbourhoods of the arcs. Each attachment increases the genus by one. Hence, $Y$ contains an open subset $Y_2$, which contains $Y_1$ and the union of the arcs, and has genus $g(Y_2) = g(Y) + k - m$. Hence, be 

$$g(Y_2) = n g(X) - \frac{n - B - k}{2}.$$ 

Since the right hand side of this equation is an integral number there is at least one branch point if $n - k$ is odd. We obtain 

$$g(Y) \geq g(Y_2) = n g(X) - \frac{n - B - k}{2} \geq n g(X) - \frac{n - k}{2}.$$ 

Since $k \geq 1$ we obtain \[\text{(7)}\].

It is clear that for odd $n$ equality $g(Y) = n g(X) - \frac{n - 1}{2}$ can be attained only if $k = 1$ and the covering is unramified (i.e. $B = 0$). If $n$ is even the equality can be attained only for $k = 1$ or $k = 2$ (thus $Y$ may have one or two connected components). The proposition is proved. □

Proof of Theorem 1. Proposition 1 implies immediately that $n \geq 0$, and provides the 4-ball genus estimate for $n = 1$. Let $n > 1$, and let $X_1 = \Omega_1 \cap \bar{X}$, $Y_1 = \Omega_1 \cap \bar{Y}$, and $\phi$ be as in Proposition 2. Put $X' = X_1$, and put $Y' = \phi^{-1}(Y_1)$. Let $Y$ be equipped with the pull back of the complex structure of $Y_1$. Let $p : Y = \phi^{-1}(Y_1) \to X' = X_1$ be the covering considered in Proposition 2.

Apply Proposition 5 to the chosen Riemann surfaces $X'$ and $Y$ and to the Riemann surface $Y' \defeq B^2 \cap \bar{Y}$. Since $g(Y) = g_k(L)$, and since $X'$ is diffeomorphic to $X$ and $g(X) = g_k(K)$, the formula \[\text{(5)}\] gives statement (1) of the theorem (the lower bound for the general case of a link $L$).

If $L$ is a knot, hence, i.e. $\partial Y$ is connected, inequality \[\text{(6)}\] with $g_k(L) = g(Y)$ and $g_k(K) = g(X)$ gives statement 2. The lower bound for the 4-ball genus in Theorem 1 is proved.

The sharpness of the estimate for the general case of links $L$ (statement 3) is an immediate consequence of the following lemma.

**Lemma 7.** Let $X$ be an open Riemann surface with smooth connected boundary. For any natural $n$ there exists an unbranched holomorphic $n$-covering $Y$ of $X$ such that $Y$ is connected and has $n$ boundary components. Moreover, there is a holomorphic embedding of $Y$ into the disc bundle $X \times \mathbb{D}$ that lifts the covering map.

Indeed, let $K$ be a smoothly analytic knot, i.e. $K = \partial \mathbb{B}^2 \cap X$ for a complex curve $X = \{z \in \mathbb{C}^2 : f(z) = 0\}$. We may assume that $X \cap \mathbb{B}^2$ is diffeomorphic to $X$. Identify the disc bundle $X \times \mathbb{D}$ with a small tubular neighbourhood of $X$ in $\mathbb{C}^2$ and consider the Riemann surface $Y$ of Lemma 7 to
be an embedded submanifold of the tubular neighbourhood. For these $X$ and $Y$ equality in the Riemann-Hurwitz relation (7) is obtained, since the covering is unbranched, $Y$ is connected and the number of boundary components of $Y$ is maximal. Let $L = Y \cap \partial B^2$. $X$ is diffeomorphic to $X \cap B^2$. If the tubular neighbourhood is small enough then $Y$ is diffeomorphic $Y \cap B^2$. Hence, equality (3) for the 4-ball genus of $K = X \cap \partial B^2$ and $L = Y \cap \partial B^2$ follows. Statement 3 is proved.

Proof of Lemma 7. Let $F : \mathbb{C} \to \mathbb{C}$ be the complex linear mapping $F(\zeta) = e^{2\pi i n} \zeta$. The $n$-th iterate $F^n$ is the identity. The fundamental group $\pi_1(X, x_0)$ is a free group on $g = g(X)$ generators. Denote by $\langle F \rangle$ the group of self-homeomorphisms of $\mathbb{C}$ generated by $F$. Let $\Phi : \pi_1(X, x_0) \to \langle F \rangle$ be a homomorphism which assigns the element $F$ to one of the generators of the fundamental group, and assigns to each of the other generators any element of the group $\langle F \rangle$.

By proposition 4 the image $\Phi(\partial X)$ is the identity.

Let $\widetilde{X}$ be the universal covering of $X$. The fundamental group $\pi_1(X, x_0)$ acts on $\widetilde{X} \times \mathbb{C}$ as follows.

$$\widetilde{X} \times \mathbb{C} \ni (x, \zeta) \mapsto (\gamma(x), \Phi(\gamma)(\zeta)), \ \gamma \in \pi_1(X, x_0).$$

(8)

The action is free and properly discontinuous. Hence, the mapping

$$p : \widetilde{X} \times \mathbb{C} \to \mathcal{E} \overset{\text{def}}{=} \widetilde{X} \times \mathbb{C} / \pi_1(X, x_0)$$

(9)

is a holomorphic covering map. It defines a holomorphic fiber bundle over $X$ with fiber being a line, and with transition functions being complex linear. Since $X$ is open the holomorphic line bundle is trivial. The covering map $\Phi$ respects a holomorphic foliation on $\mathcal{E}$, namely the trivial foliation with leaves $\widetilde{X} \times \{\zeta\}, \zeta \in \mathbb{C}$. The map $F$ permutes the points of the set $E = \{1, e^{2\pi i n}, \ldots, e^{2\pi i(n-1)}\}$ along a cycle of length $n$. Hence, the covering map $p$ maps the set $\widetilde{X} \times E$ to a single leaf. This leaf is a Riemann surface which we denote by $Y$ and identify with a surface which is $n$-horizontally embedded into the trivial bundle $\mathcal{E} \times \mathbb{C}$. The projection $P_X : \mathcal{E} \times \mathbb{C} \to \mathcal{E}$ restricts to $Y$ as unramified covering. The covering corresponds to a homomorphism $\Psi : \pi_1(X, x_0) \to \langle s \rangle$, where $s = (12 \ldots n)$ is a cycle of length $n$, and $\langle s \rangle$ is the subgroup of the symmetric group generated by $s$. Hence, $\Psi(\{\partial X\}) = \text{id}$, and $Y$ has $n$ boundary components by Proposition 4.

Sharpness of the bound for the case when $L$ is also required to be a knot (statement 4) is obtained as follows. Let $K$ be an analytic knot with $g_4(k) = g$. Using the isotopy provided by Lemma 4 we may assume that $K = X \cap \partial B^2$ for a smooth complex curve $X$ contained in $(1 + \epsilon)\mathbb{D} \times \partial \mathbb{D}$ for small positive numbers $\epsilon$ and $\delta$ such that $P_1 \mid X : X \to (1 + \epsilon)\mathbb{D}$ is a branched covering with branch locus in $(1 - \epsilon)\mathbb{D}$.

If $n$ is odd there exists an unbranched holomorphic covering $p : Y \to X$ such that $Y$ has connected boundary. This follows immediately from Proposition 4 and a theorem of Ore which says that each even permutation is a
commutator. For convenience of the reader we provide the following simple examples on commutators. Example 1 together with statements 1 and 4 of Proposition 4 provide the required unbranched covering. Indeed, take the covering corresponding to the homomorphism \( \Psi : \pi_1(\mathcal{X}, x_0) \to S_n \) for which \( \Psi(a_1) = s_1, \Psi(a_2) = s_2, \psi(a_j) = \text{id}, j = 2, \ldots, 2g \), for a suitable choice of generators \( a_j \) of \( \pi_1(\mathcal{X}, x_0) \) and for the permutations \( s_1 \) and \( s_2 \) of Example 1. Example 2 will be used below.

**Example 1.** Suppose \( n \) is an odd number, \( n = 2m + 1 \). Consider the following two permutations \( s_1 = (23) \ldots (2m 2m + 1) \) and \( s_2 = (12)(34) \ldots (2m − 1 2m) \) in \( S_n \). Then the commutator \( s = [s_1, s_2] \) is a cycle of order \( n \). (See fig. 6a for \( n = 7 \).)

**Example 2.** Let \( n = 2m \) be an even number. Consider the permutations \( s_1 = (23) \ldots (2m − 2 2m − 1) \) and \( s_2 = (12) \ldots (2m − 1 2m) \) in \( S_n \). The commutator \( s = [s_1, s_2] \) is the disjoint union of two cycles of order \( n/2 \). Note that the subgroup \( \langle s_1, s_2 \rangle \) of \( S_n \) generated by \( s_1 \) and \( s_2 \) acts transitively on \( \{1, 2, \ldots, n\} \). (See fig. 6b for \( n = 8 \).)

![fig. 6a](image1)

![fig. 6b](image2)

**Figure 6.**

Consider the holomorphic mapping \( i = (p, f) : \mathcal{Y} \to \mathcal{X} \times a \mathbb{D} \). Here \( f \) is a bounded holomorphic function and \( p \) is the covering map. The mapping \( i \) is an immersion. It is an embedding of \( p^{-1}(\mathcal{X}_0) \) into \( \mathcal{X}_0 \times a \mathbb{D} \) for a domain \( \mathcal{X}_0 \subset \mathcal{X} \) iff \( f \) separates the points of the fiber \( p^{-1}(z) \) for each \( z \in \mathcal{X}_0 \). Choose the holomorphic function \( f : \mathcal{Y} \to a \mathbb{D} \) so that it separates all points of the fibers \( p^{-1}(z_j) \) for all points \( z_j \) in the branch locus of \( P_1 \mid \mathcal{X} \). This is possible since \( \mathcal{X} \) is an open Riemann surface with smooth connected boundary, hence it has bounded holomorphic functions with prescribed values at finitely many points. From the identity theorem applied to \( f \circ \varphi_j \) for the local inverses \( \varphi_j \) of \( p \) it follows that the function \( f \) separates points of fibers for all \( z \in \mathcal{X} \) not belonging to a discrete subset \( \Lambda \) of \( \mathcal{X} \). The disc \( \overline{\mathbb{D}} \) contains only finitely many points of \( P_1(\Lambda) \). Join them by pairwise disjoint simple smooth arcs with points in \( \partial \mathbb{D} \). Remove small neighbourhoods of the arcs from \( \mathbb{D} \) so that we obtain a simply
connected smoothly bounded domain $D \subset \mathbb{D}$. We may assume that $D$ contains the branch locus of $P_1 \mid X$, in other words, $X_1 = X \cap (D \times \mathbb{C})$ is diffeomorphic to $X$. Identify the disc bundle $X_1 \times a \mathbb{D}$ with a small enough tubular neighbourhood of $X_1$ we obtain an embedding of $Y_1 = p^{-1}(X_1)$ into the tubular neighbourhood of $X_1$ such that $Y_1$ is diffeomorphic to $Y$. By the Riemann-Hurwitz relation we obtain

$$1 - 2g(Y_1) = \chi(Y_1) = n \cdot \chi(X_1) = n(1 - 2g(X_1)),$$

hence, since $X_1$ is diffeomorphic to $X$ and $Y_1$ is diffeomorphic to $Y$, we obtain

$$g(Y) = n \cdot g(X) - \frac{n - 1}{2}.$$

Lemma 5 provides a further isotopy to an analytic knot $K$ and an analytic knot $L$ which is an $n$-cable of $K$ contained in a small tubular neighbourhood of $K$ such that equality in relation 1 holds for the 4-ball genus bound is attained.

It remains to consider the case when $n$ is even. We may assume that $X$ is as in the case of odd $n$. There exists an unbranched holomorphic covering $p : Y \to X$ with $Y$ connected and with boundary consisting of two connected components. This follows from statements 1 and 4 of Proposition 4 and Example 2 above. (See fig. 6b for the case $n = 8$.) By statement 2 of Proposition 4 there is a separable quasipolynomial $P$ that lifts the covering. Hence, $Y = \mathcal{S}_p$ for the zero set of the quasipolynomial and the covering map $p$ equals $P_X \mid Y : Y \to X$. Shrinking $X$ we may assume that $X$ and $Y$ have smooth boundary and $Y$ is smoothly $n$-horizontally embedded into $X \times \mathbb{D}$.

By statement 1 of Proposition 3 there exists a simply connected smoothly bounded domain $D \subset \mathbb{D}$ such that the following holds. The set $X_1 = X \cap (D \times \mathbb{C})$ is diffeomorphic to $X$ and the unbranched $n$-covering $Y_1 = p^{-1}(X_1)$ of $X_1$ is diffeomorphic to $Y$. Moreover, the embedding $Y_1 \to X_1 \times a \mathbb{D}$ is isotopic to a holomorphic embedding of a Riemann surface $Y$ into the disc bundle, $i : Y \to X_1 \times a \mathbb{D}$, with $P_X \circ i = p$.

Let $w_{Y_1} \in B_n$ represent the conjugacy class corresponding to the isotopy class of $\partial Y_1$ in $\partial X_1 \times a \mathbb{D}$. Take for $w$ a conjugate of a generator of $B_n$ which permutes two strands of $w_{Y_1}$ corresponding to different connected components of $\partial Y_1$ (in other words, the closure of $w \circ w_{Y_1}$ defines a connected closed braid). Apply statement 2 of Proposition 3. We obtain a domain $X_1 \subset X$ of the form $X_1 = P_{\Gamma}^{-1}(D_1)$ for a smoothly bounded simply connected domain $D_1$, $D \subset D_1 \subset \mathbb{D}$, such that $X_1$ is diffeomorphic to $X$. Also we obtain an embedding of an open Riemann surface $i : Y_1 \to X_1 \times a \mathbb{D}$ into the disc bundle such that $P_X \mid Y_1$ is a branched holomorphic covering with a single branch point and with connected boundary $\partial Y_1 \subset \partial X_1 \times a \mathbb{D}$ which determines a closed geometric braid which (after identifying $\partial X_1$ with $\partial X_1$ by a homeomorphism) represents the conjugacy class of $w \circ w_{Y_1}$.

By the Riemann-Hurwitz relation

$$1 - 2g(Y_1) = n(1 - 2g(X_1)) - 1,$$
hence
\[ g(Y_1) = ng(X_1) - \frac{n - 2}{2}. \]

Since \( X_1 \) is diffeomorphic to \( \mathcal{X} \), and, hence, to \( \mathcal{X} \cap \partial \mathbb{B}^2 \), we have \( g_4(K) = g(X_1) \). Lemma 5 gives a further isotopy of \( K \) to a smoothly analytic knot again denoted by \( K \) and an analytic knot \( L \) in a small neighbourhood of \( K \) which is an \( n \)-cable of \( K \) such that \( g_4(L) = g(Y_1) \). The first part of statement 4 is proved.

The following example proves the last part of statement 4. Embed the standard punctured torus holomorphically into \( \mathbb{C} \) using the Weierstraß \( \wp \)-function:

\[ (\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})) / (\mathbb{Z} + i\mathbb{Z}) \ni \zeta \rightarrow (\wp(\zeta), \wp'(\zeta)) \in \mathbb{C}^2. \quad (10) \]

Denote the image of the embedding by \( X \). Let \( R \) be a large positive number. The intersection \( X_R = \mathcal{X} \cap R\mathbb{B}^2 \) is a torus with a hole. If \( R \) is large then \( X_R \) contains a domain \( \mathcal{R} \) which is adjacent to \( \partial X_R \) and is conformally equivalent to an annulus of conformal module larger than \( \frac{r}{2 \log \frac{r_1}{r_2}} \). (Recall that for \( 0 \leq r_1 < r_2 \leq \infty \) the conformal module of the round annulus \( \{ r_1 < |z| < r_2 \} \) in the complex plane equals \( \frac{1}{2 \pi} \log \frac{r_1}{r_2} \).) Put \( X = \frac{1}{\pi}X_R \subset \mathbb{B}^2 \) for a number \( R \) with this property. Let \( K = \partial X \subset \partial \mathbb{B}^2 \). Then \( K \) is a smoothly analytic knot.

Suppose for any \( a > 0 \) there exists an analytic knot \( L \) contained in the tubular neighbourhood \( \partial \mathbb{B}^2 \cap T_a(\mathcal{X}) \), such that \( n = w_K(L) = 3 \) and equality is obtained in the 4-ball genus estimate \( \mathcal{L} \) for \( K \) and \( L \). Let \( Y \) be the complex curve bounded by \( L \). Apply Lemma 2 and Propositions 1 and 2. Let \( \Omega_1, X_1, \) and \( Y_1 \) be as in Proposition 2, so that \( X_1 \) is diffeomorphic to \( X \) and close to \( X \).

Let \( \phi : X_1 \times a\mathbb{D} \rightarrow \mathcal{N}_1(X_1) \) be as in Proposition 2 a diffeomorphism onto a small enough neighbourhood \( \mathcal{N}_1(X_1) \) of \( X_1 \). The diffeomorphism \( \phi \) is holomorphic on \( X' \times a\mathbb{D} \) for a domain \( X' \subset X_1 \) which is diffeomorphic to \( X_1 \). If \( \mathcal{N}_1(X_1) \) is small enough then \( X' \) is close to \( X_1 \). Since \( X_1 \) is close to \( X \) the set \( \mathcal{R}' \equiv X' \cap \mathcal{R} \) is conformally equivalent to an annulus of conformal module close to that of \( \mathcal{R} \), in particular the conformal module of \( \mathcal{R}' \) is larger than \( \frac{r}{2 \log \frac{r_1}{r_2}} \).

Moreover, let \( (\hat{\phi})^{-1}(Y_1) \) be equipped with complex structure as in Proposition 2. Then \( Y' = (\hat{\phi})^{-1}(Y_1) \cap (X' \times a\mathbb{D}) \) is holomorphically embedded into \( X' \times a\mathbb{D} \).

Apply Proposition 5 with \( \mathcal{X} = X' \) and \( \mathcal{Y} = Y' \). Since \( n = 3 \) is odd, \( L = \partial Y \) is connected and equality holds in \( \mathcal{L} \), by Proposition 5 the Riemann surface \( \mathcal{Y} = Y' \) has connected boundary and the covering is unramified. In other words, the Riemann surface \( \mathcal{Y}' = (\phi)^{-1}(Y') \) is \( n \)-horizontally embedded into \( X' \times a\mathbb{D} \). The embedding defines a holomorphic map of \( X' \) to \( \mathcal{P}_n \). Its restriction to \( \mathcal{R}' \) is a holomorphic map into \( \mathcal{P}_n \), which represents the free isotopy class of \( L' = \partial Y' \).

Since the conformal module of \( \mathcal{R}' \) is large, by Lemma 8.3 of \[5\] the class of \( L' \) is the conjugacy class of a pure braid, i.e. \( L' \) cannot be connected. The contradiction proves the last part of statement 4 of Theorem 1. Theorem 1 is proved. \( \square \)
Proof of Theorem 2. Statement 1 follows from Proposition 1. Statement 2 is a consequence of Proposition 1 and Proposition 2.

Prove statement 3. Let \( X_1 \) and \( Y_1 \) be as in statement 2. The pattern \( L_1 \) equals the free isotopy class of \( \partial Y_1 \) in \( N(\partial X_1) \). Apply Proposition 2. Put \( X = X_1, Y = (\hat{\phi})^{-1}(Y_1) \) (with the conformal structure being the pullback of the structure on \( Y_1 \)). Let \( p : Y \rightarrow X \) be the \( n \)-covering which appeared in proposition 2 (which is a branched holomorphic covering outside a a neighbourhood of \( \partial Y \) and in a neighbourhood of each critical point and is unramified outside this set).

If the covering \( p \) is unramified then by Proposition 4 the link \( L_1 \) is the conjugacy class of a product of \( g = g(X_1) = g_4(K) \) commutators in \( \mathcal{B}_n \).

Consider now the general case. Let \( d \subset X \) be a smoothly bounded simply connected domain which contains the branch locus of \( p \) such that \( X \setminus d \) is diffeomorphic to \( X \) (in particular, \( \partial X \cap \partial d \neq \emptyset \)). (For example, one can take for \( d \) the union of the following sets: suitable neighbourhoods of simple disjoint arcs joining a critical value of \( p \) with a boundary point of \( X \), and a suitable simply connected part of a collar of \( \partial X \) in \( X \).) Put \( Y_d \defeq p^{-1}(d) \) and use the following notation: \( X_{Cd} \defeq X \setminus d \) and \( Y_{Cd} \defeq p^{-1}(X_{Cd}) \).

Take a base point \( q \) in \( X \) which is a boundary point of \( X \cap \partial d \) (in particular, \( q \in \partial d \cap \partial X \)). Choose the point \( E_d = P_\partial(Y \cap \{\{ q \} \times \mathbb{D}) \) as base point in the symmetrized configuration space.

Let \( \Gamma_d \) be an arc whose interior parametrizes \( \partial X_d \setminus \{ q \} \) and whose two endpoints are equal to \( q \). Respectively, we denote by \( \Gamma_{Cd} \) an arc whose interior parametrizes \( \partial X_{Cd} \setminus \{ q \} \) and whose two endpoints are equal to \( q \). Both arcs are equipped with orientation induced by orienting \( \partial X_d \), or \( \partial X_{Cd} \), respectively, as boundaries of the domains \( X_d \), and \( X_{Cd} \), respectively.

The \( n \)-horizontal embeddings \( \partial Y_d \subset \partial X_d \times \mathbb{C}, \) and \( \partial Y_{Cd} \subset \partial X_{Cd} \times \mathbb{C} \) respectively, define continuous mappings from \( \Gamma_d \) into \( \mathcal{P}_n \), and from \( \Gamma_{Cd} \) into \( \mathcal{P}_n \), respectively. Identifying \( \mathcal{B}_n \) with the group of isotopy classes of geometric braids with base point \( E_d \) we obtain (after identifying \( \Gamma_d \) and \( \Gamma_{Cd} \) with the unit interval) two braids \( w_d \) and \( w_{Cd} \). Since \( d \) is simply connected the braid \( w_d \) is quasi-positive by Rudolph’s theorem. Since the covering over \( X_{Cd} \) is unramified the braid \( w_{Cd} \) is a product of \( g \) commutators in \( \mathcal{B}_n \). (See statement 4 of Proposition 4.)

Let \( \Gamma_X \) be an arc whose interior parametrizes \( \partial X \setminus \{ q \} \) (with orientation induced from the orientation of \( \partial X \) as boundary of \( X \)) and whose two endpoints are equal to \( q \). The isotopy class of the continuous mapping from \( \Gamma_X \) to \( \mathcal{P}_n \) which is defined by the \( n \)-horizontal embedding \( \partial Y \subset \partial X \times \mathbb{C} \) is equal (after identification of the curves \( \Gamma_d + \Gamma_{Cd} \) and \( \Gamma_X \) ) to the braid \( w_d \circ w_{Cd} \). Statement 3 is proved.

It remains to prove statement 4. By Lemma 4 after an isotopy we are in the situation when the knot is equal to \( K_1 \defeq X \cap \partial \mathbb{D}^2 \) for a smooth complex curve \( X \) in a small neighbourhood of \( \{ z_2 = 0 \} \) such that for a small positive number \( \varepsilon \) the mapping \( P_1 : X \rightarrow (1 + \varepsilon) \mathbb{D} \) is a branched covering with branch locus in \( (1 - \varepsilon) \mathbb{D} \). We may assume that \( X \) has smooth boundary.

Let \([\alpha_1, \beta_1] \circ \ldots \circ [\alpha_g, \beta_g], \alpha_j, \beta_j \in \mathcal{B}_n \) for \( j = 1, \ldots, g \), be a product of \( g =
$g(\mathcal{X})$ commutators. By statements 2 and 4 of Proposition 4 there exists a smooth $n$-horizontal embedding of a smooth surface with boundary $\mathcal{Y}$, $\mathcal{Y} \rightarrow \mathcal{X} \times \mathbb{C}$, such that the embedding of the boundary $\partial \mathcal{Y} \rightarrow \partial \mathcal{X} \times \mathbb{C}$ corresponds to the conjugacy class of the afore mentioned product of commutators. Indeed, let $\mathcal{Y}$ be the zero set of the quasi-polynomial which corresponds to the homomorphism $\Phi$ for which $\Phi(a_j) = \alpha_j$, $\Phi(b_j) = \beta_j$ for suitable generators $a_j$ and $b_j$, $j = 1, \ldots, g$, of the fundamental group.

Let $w \in \mathcal{B}_n$ be a quasipositive braid such that the pattern $L$ is the conjugacy class of the braid $w \circ [\alpha_1, \beta_1] \circ \ldots \circ [\alpha_g, \beta_g]$. Take $X_1 \subset \mathcal{X}$ and $Y_1 \subset \mathcal{Y}$ as in Proposition 3, so that, in particular, $\partial Y_1 \subset \partial X_1 \times \mathbb{C}$ corresponds to the conjugacy class of the braid $w \circ [\alpha_1, \beta_1] \circ \ldots \circ [\alpha_g, \beta_g]$.

Then by Lemma 5 the conjugacy class $L$ of $w \circ [\alpha_1, \beta_1] \circ \ldots \circ [\alpha_g, \beta_g]$, can be realized by an analytic link contained in an a priory given neighbourhood of a knot $K \subset \partial \mathbb{B}^2$ which is isotopic to $K_1$.

Theorem 2 is proved. \qed
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