Research Article

Qing Zhang*

A local converse theorem for U(2, 2)

DOI: 10.1515/forum-2016-0135
Received June 13, 2016; revised November 10, 2016

Abstract: Let $F$ be a $p$-adic field and let $E/F$ be a quadratic extension. In this paper, we prove the local converse theorem for generic representations of $U_{E/F}(2, 2)$ if $E/F$ is unramified or the residue characteristic of $F$ is odd. Our method is purely local and analytic, and the same method also gives the local converse theorem for $Sp_q(F)$ and $Sp_q(E)$ if the residue characteristic of $F$ is odd.

Keywords: Gamma factors, Howe vectors, local converse theorem

MSC 2010: 11F70, 22E50

Communicated by: Freydoon Shahidi

Introduction

Let $F$ be a $p$-adic field and let $E/F$ be a quadratic extension. Let $G = U_{E/F}(n, n)$, and let $\psi$ be a generic character of its maximal unipotent subgroup. Let $\pi$ be a $\psi$-generic irreducible smooth representation of $G$ and let $\tau$ be a generic irreducible representation of $GL_m(E)$. Then one can define a gamma factor $\gamma(s, \pi \times \tau, \psi)$. The local converse problem asks if one can determine the representation $\pi$ uniquely if one knows enough information of the gamma factors $\gamma(s, \pi \times \tau, \psi)$ for various twists. More precisely, we have the following conjecture.

Conjecture 0.1 (Local converse conjecture, see [22, Conjecture 6.3]). Let $\pi, \pi'$ be two $\psi$-generic irreducible admissible representations of $G(F)$ with the same central character. If $\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi' \times \tau, \psi)$ for all irreducible supercuspidal representations $\tau$ of $GL_k(E)$ with $k \leq n$, then $\pi_1 \cong \pi_2$.

Remark. Conjecture 6.3 of [22] is stated for all classical groups. Here for simplicity, we only consider the case $G = U_{E/F}(n, n)$. On the other hand, there is no central character assumption in [22, Conjecture 6.3]. In the $GL_n$ case, the equality of the central character is in fact a consequence of the equality of the gamma factors twisting by $GL_1$, see [23, Corollary 2.7]. Thus it is natural to expect that this is also true in the classical group case.

In this paper, we confirm the above conjecture in the case when $n = 2$ and $E/F$ is unramified or $E/F$ is ramified but the residue characteristic of $F$ is not 2. More precisely, we have the following:

Theorem (Local Converse Theorem for $U(2, 2)$, Theorem 5.6). Suppose that $E/F$ is unramified or $E/F$ is ramified but the residue characteristic of $F$ is odd. Let $\pi, \pi'$ be two $\psi$-generic irreducible admissible representations of $U_{E/F}(2, 2)(F)$. If $\gamma(s, \pi \times \eta, \psi) = \gamma(s, \pi' \times \eta, \psi)$ and $\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi' \times \tau, \psi)$ for all quasi-characters $\eta$ of $E^\times$ and all irreducible generic admissible representations $\tau$ of $GL_2(E)$, then $\pi \cong \pi'$.

The $\gamma$-factors used here are of Rankin–Selberg type, which are defined from local functional equation using Shimura type integrals [4]. In [24], the Rankin–Selberg $\gamma$-factors for the group $Sp_{2n}$ are studied. In particular, it is proved in [24] that the Rankin–Selberg gamma factors are the same as the local gamma factors defined using Langlands-Shahidi method. Unfortunately, the unitary group case is not included in [24]. Since the local zeta integrals used in the unitary case to define the gamma factors are totally parallel to the symplectic...
case, it is natural to believe that the gamma factors used here are multiplicative and are the same as the Langlands–Shahidi local gamma factors up to a normalizing factor. Once we know the gamma factors are multiplicative, it suffices to twist by the quasi-characters of $E^\times$ and supercuspidal representations of $\text{GL}_2(E)$ in the above local converse theorem.

We also mention that, based on the same methods, we can also prove the local converse theorems for $\text{Sp}_4(F)$ and $\overline{\text{Sp}}_4(F)$ when $F$ is a $p$-adic field with odd residue characteristic.

Our proof of the local converse theorem is based on detailed analysis on the partial Bessel functions associated with Howe vectors. In [2, 3], E.M. Baruch proved local converse theorem for $\text{GSp}_4$ and $U(2, 1)$ using Howe vectors. In [30], we proved the stability of the Rankin–Selberg gamma factors for $\text{Sp}_{2n}$ and $\overline{\text{Sp}}_{2n}$ using Baruch’s methods, and remarked that this method might be used to prove the local converse theorem for $\text{Sp}_{2n}$, $\overline{\text{Sp}}_{2n}$ and $U(n, n)$ once we can extend a stability property of partial Bessel functions associated with Howe vectors (see [30, Theorem 3.11]) to the most general case. In this paper, we illustrate how to get such a local converse theorem in the small rank case. In fact, in the case $n = 2$, the stability property of partial Bessel functions associated with Howe vectors can be checked directly because the Weyl group of $U(2, 2)$ is small, see Proposition 2.5. Using a similar method, the local converse theorem for $U(1, 1)$ is proved in [29].

We expect our method can be used to give local converse theorems for more general groups.

In this paper, we also construct a new local gamma factor $\gamma'(s, \pi \times \eta, \psi)$ for a generic representation $\pi$ of $U_{E/F}(2, 2)$ and a quasi-character $\eta$ of $E^\times$. This new gamma factor is defined by Hecke type local zeta integrals, which are easier to handle than the Shimura type integrals. This construction can be extended to the case $U(n, n) \times \text{GL}_m$ when $m < n$. But it is not known whether this new local zeta integrals come from global zeta integrals. We show that the new local gamma factors $\gamma'(s, \pi \times \eta, \psi)$ can be used to replace $\gamma(s, \pi \times \eta, \psi)$ in the local converse theorem, see Theorem 6.7.

The paper is organized as follows. In Section 1, we review the definition of $\gamma$-factors for $U_{E/F}(2, 2) \times \text{GL}_m(E)$ with $k \leq 2$, which is totally analogous to the symplectic case as treated in [24]. In Section 2, we review the definition of Howe vectors and a stability property of Howe vectors. We constructed some sections of induced representations in Section 3 which will be used in the later calculation. In Section 4 and Section 5, we consider the gamma factors twisting by $\text{GL}_1$ and $\text{GL}_2$ and finish the proof of the local converse theorem when $E/F$ is unramified. In Section 6, we construct a new gamma factor $\gamma'(s, \pi \times \eta, \psi)$ for a generic representation $\pi$ of $U(2, 2)$ and a quasi-character $\eta$ of $E^\times$. We also show that this new gamma factor can replace the old gamma factor in the local converse theorem. In Section 7, we give a brief account of the proof of the local converse theorem in the case $E/F$ is ramified and the residue characteristic of $F$ is odd. We also consider the local converse theorem for $\text{Sp}_4(F)$ and $\overline{\text{Sp}}_4(F)$ when $F$ is a local field with odd residue characteristic in Section 7.

**Notations**

Let $E/F$ be a quadratic extension of local fields and let $\epsilon_{E/F}$ be the local class field theory character of $F^\times$. Denote the nontrivial Galois action by $x \mapsto \bar{x}$ for $x \in E$. Let $\mathcal{O}_E$ (resp. $\mathcal{O}_F$) be the ring of integers of $E$ (resp. $F$), and let $\mathcal{P}_E$ (resp. $\mathcal{P}_F$) be the maximal ideal of $\mathcal{O}_E$ (resp. $\mathcal{O}_F$). Let $q_E = |\mathcal{O}_E/\mathcal{P}_E|$ and $q_F = |\mathcal{O}_F/\mathcal{P}_F|$. Let $E^1 = \{x \in E^\times : xx = 1\}$.

**U(2, 2) and its subgroups**

Let $G = U_{E/F}(2, 2)$ be the isometry group of the Hermitian form defined by

$$s = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix},$$

where $1_2$ is the $2 \times 2$ identity matrix. Explicitly,

$$G(F) = \{g \in \text{GL}_4(E) : gs^t g = s\}.$$
When the field extension $E/F$ is understood, we will ignore it from the notation, and write $U(2, 2)$ instead of $U_{E/F}(2, 2)$. Let $P = MN$ be the Siegel parabolic subgroups with Levi subgroup

$$M(F) = \left\{ m(a) := \begin{pmatrix} a & \alpha^{-1} \\ \alpha & 1 \end{pmatrix} : a \in \text{GL}_2(E) \right\},$$

and unipotent subgroup

$$N(F) = \left\{ n(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \text{Herm}_2(F) \right\},$$

where $\text{Herm}_2(F) = \{ x \in \text{Mat}_{2 \times 2}(E) : x = x^* \}$. Let $T$ be the maximal torus in $M$. A typical element of $T$ is of the form $t(a_1, a_2) = \text{diag}(a_1, a_2, \alpha_1^{-1}, \alpha_2^{-1}) \in T$ with $a_1, a_2 \in E^\times$. We also use the notation

$$t(a) := t(a, 1), \quad a \in E^\times.$$

Let $U$ be the maximal unipotent subgroup defined by

$$U = \left\{ m(u)n : u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(E) : n \in N \right\}.$$

Let $Z$ be the center of $G$. Then $Z = \{ tz, z : z \in E^1 \} \cong E^1$. Denote $B = TU$, a Borel subgroup of $G$.

From the isomorphism $M \cong \text{GL}_2(E)$, we view $\text{GL}_2(E)$ as a subgroup of $G$. In $\text{GL}_2(E)$, we denote by $B^{(2)} = T^{(2)} N^{(2)}$ the upper triangular Borel subgroup with $T^{(2)}$ the torus and $N^{(2)}$ be the upper triangular unipotent subgroup.

**Roots and Weyl group**

The group $G$ has two simple roots defined by

$$\alpha(t(a_1, a_2)) = \frac{a_1}{a_2}, \quad \beta(t(a_1, a_2)) = a_2 \alpha_2, \quad a_1, a_2 \in E^\times.$$

The positive roots are $\Sigma^+ = \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta \}$. Let $s_\alpha$ be the simple reflection of $\alpha$ and let $s_\beta$ be the simple reflection of $\beta$. The Weyl group $W$ of $G$ is $\{ 1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\beta s_\alpha s_\beta, s_\alpha s_\beta s_\alpha s_\beta \}$. We denote

$$w_0 = (s_\alpha s_\beta)^2, \quad w_1 = s_\beta s_\alpha s_\beta, \quad w_2 = s_\alpha s_\beta s_\alpha.$$

Let

$$\hat{s}_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{s}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\hat{s}_\alpha$ (resp. $\hat{s}_\beta$) is a representative of $s_\alpha$ (resp. $s_\beta$).

We take $\hat{w}_1 = \hat{s}_\beta \hat{s}_\alpha \hat{s}_\beta, \hat{w}_2 = \hat{s}_\alpha \hat{s}_\beta \hat{s}_\alpha$ and $\hat{w}_0 = (\hat{s}_\alpha \hat{s}_\beta)^2$. Then we can check that

$$\hat{w}_0 = \begin{pmatrix} -I_2 \\ I_2 \end{pmatrix}.$$

We have the relation

$$s_\alpha(\beta) = 2\alpha + \beta, \quad s_\beta(\alpha) = \alpha + \beta.$$

For a root $\gamma$, let $U_\gamma$ be the one parameter subgroup associated to $\gamma$. Let $x_\gamma : F \to U_\gamma$ or $x_\gamma : E \to U_\gamma$ be the corresponding isomorphism. For example,

$$U_\alpha = \left\{ \begin{pmatrix} 1 & r \\ 1 & 1 - r \end{pmatrix} : r \in E \right\}, \quad U_{-\alpha-\beta} = \left\{ \begin{pmatrix} 1 & 1 \\ r & 1 \end{pmatrix} : r \in E \right\}.$$
1 Local gamma factors for $U(2, 2) \times \text{Res}_{E/F}(GL_k)$, $k = 1, 2$

1.1 Weil representations of $U(1, 1)$

Let $W$ be the 2-dimensional skew-Hermitian space with skew-Hermitian structure defined by

$$(w_1, w_2)_W = w_1 \begin{pmatrix} 1 \\
-1
\end{pmatrix}^t w_2,$$

where $w_1, w_2 \in W$ are viewed as row vectors. Let $G_1 = U(1, 1) = U(W)$ be the isometry group of $W$, i.e.,

$$G_1(F) = \left\{ g \in GL_2(E) : g \begin{pmatrix} 1 \\
-1
\end{pmatrix}^t \begin{pmatrix} 1 \\
-1
\end{pmatrix} = \begin{pmatrix} 1 \\
-1
\end{pmatrix} \right\}.$$

In the group $G_1$, we will use the following notations:

$$m_1(a) = \begin{pmatrix} a \\
\bar{a}
\end{pmatrix}^{-1} \quad \text{for } a \in E^\times, \quad n_1(x) = \begin{pmatrix} 1 & x \\
0 & 1
\end{pmatrix} \quad \text{for } x \in F.$$

Let $M_1$ be the subgroup of $G_1$ consisting elements of the form $m_1(a), a \in E^\times$ and let $N_1$ be the subgroup of $G_1$ consisting elements of the form $n_1(x), x \in F$. Let $B_1 = M_1 N_1$, which is a Borel subgroup of $G_1$.

The skew-Hermitian space $W_{E/F}$ can be viewed as a symplectic space over $F$, with the symplectic form $\langle \cdot, \cdot \rangle$ defined by $\text{tr}_{E/F}(\langle \cdot, \cdot \rangle)_W$. We then have an embedding $G_1 \to \text{Sp}(W)$. Let $\mu$ be a character of $E^\times$ such that $\mu|_F = \epsilon_E/F$; then there is a splitting $s_\mu : G_1 \to \text{Sp}(W)$, see [15].

Let $H(W)$ be the Heisenberg group associated to $W$. Explicitly, $H(W) = W \oplus F$ with the product (written additively) defined by

$$(w_1, t_1) + (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}(w_1, w_2)_W), \quad w_1, w_2 \in W, \quad t_1, t_2 \in F.$$

Let $J_W = U(W) \ltimes H(W)$, which is called the Fourier-Jacobi group associated to $W$ in the literature, see [9] for example. Let $\psi$ be an additive character of $F$; then we have a Weil representation $\omega_\psi$ of the group $\text{Sp}(W) \ltimes H(W)$, which can be realized on the space $S(E)$. For a character $\mu$ of $E^\times$ such that $\mu|_F = \epsilon_E/F$, we then have a Weil representation $\omega_{\mu, \psi}$ of $J_W$ by the embedding $J_W \to \text{Sp}(W) \ltimes H(W)$ induced by $s_\mu$.

We will view $G_1$ as a subgroup of $G = U(2, 2)$ by the embedding

$$g = \begin{pmatrix} a & b \\
c & d
\end{pmatrix} \mapsto \begin{pmatrix} 1 & a \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & b \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & c \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & d \\
0 & 1
\end{pmatrix}.$$  

Let $H$ be the subgroup of $G$ consisting elements of the form

$$[x, y, z] = m\begin{pmatrix} 1 & x \\
0 & 1
\end{pmatrix} n\begin{pmatrix} z & y \\
0 & \bar{z}
\end{pmatrix}, \quad x, y \in E, \quad z \in F.$$

We can check that the map

$$H(W) \to H, \quad ([x, y, t] \mapsto [x, y, t - \frac{1}{2} \text{tr}(xy)]$$

defines an isomorphism, and $G_1 \cdot H \cong J_W = U(W) \ltimes H(W)$.

Thus we get a Weil representation $\omega_{\mu, \psi}$ of $G_1 \cdot H$ on $S(E)$. The following formulas hold:

$$\omega_{\mu, \psi}((0, 0, z), [x, 0, 0])\phi(\xi) = \psi(z + \text{tr}(\xi y))\phi(x + \xi), \quad \omega_{\mu, \psi}(m_1(a))\phi(\xi) = \mu(a)|a|^{1/2}\phi(\xi a), \quad \omega_{\mu, \psi}(n_1(b))\phi(\xi) = \psi(b\xi)\phi(\xi), \quad \omega_{\mu, \psi}(w^1)\phi(\xi) = \epsilon_\psi \int_E \psi(- \text{tr}(\xi y))\phi(y) dy$$  \hspace{1cm} (1.1)

for $\phi \in S(E), \xi \in E$, where $\epsilon_\psi$ is certain Weil index (we do not need its precise definition here), $w^1 = (\begin{pmatrix} -1 & 0 \\
0 & -1
\end{pmatrix})$ is the unique nontrivial Weyl element of $G_1$ and $dy$ is the Haar measure on $E$ such that this Fourier transform is self-dual.
1.2 Weil representations of $U(2, 2)$

Let $\psi$ be a nontrivial additive character of $F$, let $\mu$ be a character of $E^\times$ such that $\mu|_{F^\times} = \epsilon_{EF}$ as above. Let $U(1)$ be the isometry group of the 1-dimensional Hermitian space $E$ with the Hermitian form $(x, y) = xy$. Then we have a Weil representation $\omega_{\mu, \psi}$ of the pair $G \times U(1)$ on $S(E^2)$, where $E^2 = E \oplus E$. We have the familiar formulas:

$$\omega_{\mu, \psi}(h) \Phi(x) = \Phi(h^{-1}x), \quad \Phi \in S(E^2), \quad h \in U(1), \ x \in E^2,$$

$$\omega_{\mu, \psi}(m(a)) \Phi(x) = \mu(\det(a))|\det(a)|^{1/2} \Phi(xa), \quad a \in \text{GL}_2(E),$$

$$\omega_{\mu, \psi}(n(b)) \Phi(x) = \psi(xb^\dagger \bar{x}) \Phi(x), \quad b \in \text{Herm}_2(F),$$

$$\omega_{\mu, \psi}(w_0) \Phi(x) = \gamma_\psi \int_{\tilde{E}^2} \Phi(y) \psi(-tr_{EF}(x^\dagger y)) \, dy,$$

where $x = (x_1, x_2) \in E^2$ is viewed as a row vector and $\gamma_\psi$ is certain Weil index.

As a representation of $G = U(2, 2)$, $\omega_{\mu, \psi}$ is not irreducible. Let $\chi$ be a character of $E^1$, and let $S(E^2, \chi)$ be the subspace of $S(E^2)$ such that $\Phi(xz) = \chi(z) \Phi(x)$ for all $z \in U(1)$ and $x \in E^2$. Then $S(E^2, \chi)$ is invariant under the action of $G$. Denote this representation by $\omega_{\mu, \psi, \chi}$. Then $\omega_{\mu, \psi, \chi}$ is an irreducible representation of $G$, and

$$\omega_{\mu, \psi} = \bigoplus_{\chi \in \tilde{E}^1} \omega_{\mu, \psi, \chi},$$

where $\tilde{E}^1$ denotes the dual group of $E^1$.

1.3 Induced representation and intertwining operator

For a quasi-character $\eta$ of $E^\times$, and a complex number $s \in \mathbb{C}$, let $\eta_s$ be the character of $E^\times$ defined by

$$\eta_s(a) = \eta(a)|a|_E^s.$$

We consider the (normalized) induced representation $\text{Ind}_{B_1}^{G_1}(\eta_{s-1/2})$. By definition, $\text{Ind}_{B_1}^{G_1}(\eta_{s-1/2})$ consists smooth complex valued functions $f_s$ on $G_1$ such that

$$f_s(n_1(b)m_1(a)g) = \eta_s(a)f_s(g), \quad b \in F, \ a \in E^\times, \ g \in G_1.$$

There is an intertwining operator $M(s) : \text{Ind}_{B_1}^{G_1}(\eta_{s-1/2}) \to \text{Ind}_{B_1}^{G_1}(\eta_{s+1/2})$ defined by

$$M(s)(f_s)(g) = \int_{N_1} f_s(w^4ng) \, dn,$$

where $\eta^* = \bar{w}^4 \eta = \bar{\eta}^{-1}$. It is well known that the intertwining operator $M(s)$ is well-defined for $\Re(s) \gg 0$ and can be meromorphically continued to all $s \in \mathbb{C}$.

Let $(\tau, V_\tau)$ be an irreducible smooth representation of $\text{GL}_2(E)$; we consider the induced representation

$$I(s, \tau) = \text{Ind}_{B_1}^{G_1}(|\det|_E^s)^{1/2}.$$

We now fix a nontrivial additive character $\psi_E$ of $E$ such that $\psi_{EF} = 1$. For example, we can take a nontrivial additive character $\psi$ of $F$ and a pure imaginary element $e \in E$, and then define $\psi_E(x) = \psi(\text{tr}(ex))$. Given such a character $\psi_E$ of $E$, which is also viewed as a character of the unipotent subgroup $N^{(2)}$ of $\text{GL}_2(E)$, we fix a Whittaker functional $\lambda \in \text{Hom}_{\mathbb{C}}(N^{(2)}, \psi_E)$. A section $\xi_s \in I(s, \tau)$, defines a map $G \to V_\tau$. We consider the $\mathbb{C}$-valued function $f_{\xi_s}(g)$ on $G$ defined by

$$f_{\xi_s}(g) = \lambda(\xi_s(g)).$$

Note that $\xi_s$ satisfies the relation

$$\xi_s(nm(a)g) = |\det(a)|_E^{s+1/2} \tau(a) \xi_s(g), \quad n \in N, \ a \in \text{GL}_2(E).$$

Thus we get

$$f_{\xi_s}(nm(n_1)g) = \psi_E(n_1)f_{\xi_s}(g), \quad n \in N, \ n_1 \in N^{(2)}.$$
Recall that \( w_1 = s_\beta s_\alpha s_\beta \), which is the unique element in \( \mathbf{W} \) such that \( w_1(a) \) is positive and simple and \( w_1(\beta) < 0 \). We view \( \tau \) as a representation of \( M \) and define a representation \( \tau^* := \tilde{w}_1 \tau \) of \( M \) on the same space \( V_\tau \) by conjugation of \( \tilde{w}_1 \), i.e.,

\[
(\tilde{w}_1 \tau)(m(a)) = \tau(\tilde{w}_1^{-1} m(a) \tilde{w}_1) = \tau(m(\tilde{J}_2 t^{-1} \tilde{a}^{-1} \tilde{J}_2)),
\]

where \( \tilde{J}_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \). Note that for \( n_1 = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \in N(2,1) \), we have

\[
\tau^*(n_1) = \tau\left( \begin{pmatrix} 1 & -x/2 \\ 0 & 1 \end{pmatrix} \right)
\]

and by our choice of \( \psi_E \), we have \( \psi_E(x) = \psi_E(-x) \). Thus the fixed Whittaker functional \( \lambda : V_\tau \to \mathbb{C}[\psi_E] \) for \( \tau \)
gives a \( \psi_E \) Whittaker functional for \( \tau^* \), i.e., \( \lambda \in \text{Hom}_{W(\tau^*)}(\tau^*, \psi_E) \).

We consider the normalized induced representation

\[
I(1-s, \tau^*) := \text{Ind}_{P}^{G}(\tau^* | \det|^{-s+1/2}).
\]

For a section \( \xi_{1-s} \in I(1-s, \tau^*) \), we can define \( f_{\xi_{1-s}} \) similar as above and we also have

\[
f_{\xi_{1-s}}(nm(n_1)g) = \psi_E(n_1)f_{\xi_{1-s}}(g).
\]

Consider the standard intertwining operator

\[
M(s) : I(s, \tau) \to I(1-s, \tau^*), \quad M(s)(\xi_\alpha)(g) = \int \xi_\alpha(n^{-1}w_1 g) dn,
\]

which is absolutely convergent for \( \text{Re}(s) > 0 \) and can be meromorphically continued to all \( s \in \mathbb{C} \).

### 1.4 The local zeta integral and gamma factor

For \( g \in G \), we denote \( j(g) = s_\alpha g s_\alpha \). We fix a nontrivial additive character \( \psi \) (resp. \( \psi_E \)) of \( F \) (resp. \( E \)) such that \( \psi_{E|F} = 1 \) and define a character \( \psi_U \) of \( U \) by

\[
\psi_U\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \psi(-x) \psi(b).
\]

Let \( \pi \) be an irreducible smooth \( \psi_U \)-generic representation of \( G \), and let \( \mathcal{W}(\pi, \psi_U) \) be the space of \( \psi_U \) Whittaker functions on \( \pi \). Let \( \eta \) be a quasi-character of \( E^* \), and \( \tau \) be an irreducible smooth representation of \( GL_2(E) \).

For \( W \in \mathcal{W}(\pi, \psi_U), \phi_1 \in S(E), \phi_2 \in S(E^2), f_3 \in I(s, \eta), \xi_\alpha \in I(s, \tau) \), we consider the following integrals:

\[
\Psi(W, \phi_1, f_3) = \int_{N_{\xi_\alpha}(G)} \int_{E} W(j(xa(x)g))(\omega_{\mu, \psi}^{-1}(g)\phi_1(x)) dx df_3(g) \ dg
\]

and

\[
\Psi(W, \phi_2, f_3) = \int_{U \cap \xi_\alpha} \int_{G} W(g) \omega_{\mu, \psi}^{-1}(g)\phi_2(e_2) f_3(g) \ dg,
\]

where \( e_2 \) is the row vector \((0, 1) \in E^2 \) and \( f_{\xi_\alpha} \) is defined in the previous section. It is easy to see the above integrals are well-defined formally. Using a standard estimate of the Whittaker function \( W \), one can show that the above integrals are absolutely convergent when \( \text{Re}(s) > 0 \) and in fact define rational functions of \( q_E^{-s} \), see [4, Proposition 6.4].

**Remark 1.1.** The above local zeta integrals were first studied by Gelbart and Piatetski–Shapiro in the case \( Sp_{2n} \times GL_m \) and \( U(n, n) \times Res_{E/F}(GL_m) \) in [12]. In the symplectic group case, Ginzburg, Rallis and Soudry [13, 14] extended the construction to the case \( Sp_{2n} \times GL_m \) for general \( m \). The above integrals in the unitary group cases are considered in [4].
Proposition 1.2. There are meromorphic functions $\gamma(s, \pi \times (\mu \eta), \psi_U)$, and $\gamma(s, \pi \times (\mu \tau), \psi_U)$ such that

$$\Psi(W, \phi_1, M(s)f_s) = \gamma(s, \pi \times (\mu \eta), \psi_U)$$

and

$$\Psi(W, \phi_2, M(s)\xi_s) = \gamma(s, \pi \times (\mu \tau), \psi_U)$$

for all $W \in \mathcal{W}(\pi, \psi_U), \phi_1 \in S(E), \phi_s \in S(E^2), f_s \in I(s, \eta)$ and $\xi_s \in I(s, \tau)$.

We remark that the gamma factor we used is non-normalized, i.e., we do not consider the normalization of the intertwining operators. For the normalizing process of gamma factors for symplectic groups, see [24].

Proof. The local functional equation follows from the uniqueness of the Fourier–Jacobi models [9, 27]. See [24] for some details of the proof in the $\text{Sp}_{2n}$ case. \qed

2 Howe vectors

In the following sections, we will follow Baruch’s method given in [2, 3], to give a proof of the local converse theorem for generic representations of $U(2,2) = U_{E/F}(2,2)$ when $E/F$ is unramified.

One main tool of Baruch’s method is Howe vectors, which are used to define partial Bessel functions. In this section, we give a review of the Howe vectors in our case following [2, 3].

From this section till the end of Section 5, we assume the quadratic extension $E/F$ is unramified.

2.1 Howe vectors

Let $p_F$ be a uniformizer of $F$, which also can be viewed as a uniformizer of $E$ since $E/F$ is unramified.

Let $\psi$ (resp. $\psi_E$) be an unramified additive character of $F$ (resp. $E$). As in Section 1, we require that $\psi_E$ is trivial on $F$. From these data, we have defined a character $\psi_U$ of $U$ in Section 1.4.

For a positive integer $m$, we consider the congruence subgroup $K_m = (1 + \text{Mat}_{4 \times 4}(\mathcal{O}_E^m)) \cap G$. Define a character $\tau_m$ on $K_m$ by

$$\tau_m(k) = \psi_E(-p_F^{-2m}k_{12})\psi(\frac{1}{2}p_F^{-2m}\tau_{E/F}(k_{24})) \quad \text{for } k = (k_{ij}) \in K_m.$$ 

It is easy to see that $\tau_m$ is indeed a character on $K_m$.

Let

$$d_m = t(p_F^{-3m}, p_F^{-m}) \in G,$$

and $H_m = d_m K_m d_m^{-1}$. Then

$$H_m = \left( \begin{array}{ccc} 1 + \mathcal{O}_E^m & \mathcal{O}_E^{-m} & \mathcal{O}_E^{-5m} \\ \mathcal{O}_E^{3m} & 1 + \mathcal{O}_E^m & \mathcal{O}_E^{-3m} \\ \mathcal{O}_E^{5m} & \mathcal{O}_E^{3m} & 1 + \mathcal{O}_E^m \end{array} \right) \cap G.$$

We define a character $\psi_m$ on $H_m$ by

$$\psi_m(j) = \tau_m(d_m^{-1}j d_m)$$

for $j \in H_m$. Let $U_m = U \cap H_m$; we can check that $\psi_m|_{U_m} = \psi_U|_{U_m}$.

Now let $(\pi, V_\pi)$ be a $\psi_U$-generic irreducible smooth representation of $G = U(2,2)$. We fix a Whittaker functional for $\pi$ and thus for $\psi \in V_\pi$, there is an associated Whittaker function $W_\psi$. Let $\psi \in V_\pi$ be a vector such that $W_\psi(1) = 1$. For $m \geq 1$, as Baruch defined in [2, 3], we consider

$$v_m = \frac{1}{\text{Vol}(U_m)} \int_{U_m} \psi_U(u)^{-1} \pi(u)\nu \, du.$$

Let $C = C(\nu)$ be an integer such that $\nu$ is fixed by $K_C$. 

Q. Zhang, A local converse theorem for $U(2, 2)$ — 1477
**Lemma 2.1.** We have:

1. $W_{v_m}(1) = 1$.
2. If $m \geq C$, we have $\pi(j)v_m = \psi_m(j)v_m$ for all $j \in H_m$.
3. If $k \leq m$, then
   \[ v_m = \frac{1}{\text{Vol}(U_m)} \int_{U_m} \psi_U(u)^{-1} \pi(u)v_k \, du. \]

**Proof.** The proof is given in [2] in the general case. Although [2] is not published, the proof in the general case is in fact the same as the U(2, 1) case, which can be found in [3].

From Lemma 2.1 (2), we have
\[ W_{v_m}(ugj) = \psi_U(u)\psi_m(j)W_{v_m}(j) \quad \text{for all } u \in U, j \in H_m, m \geq C. \tag{2.1} \]

The vectors $\{v_m\}_{m \geq C}$ are called Howe vectors and $W_{v_m}$ are the associated partial Bessel functions.

**Lemma 2.2.** The following statements hold.

1. For $m \geq C$, $t \in T$, if $W_{v_m}(t) \neq 0$, then $\alpha(t) \in 1 + \mathfrak{p}^m$ and $\beta(t) \in 1 + \mathfrak{p}_F^m$, where $\alpha(t) = a_1/a_2$ and $\beta(t) = a_2\bar{a}_2$ for $t = (a_1, a_2) \in T$.
2. For $w = s_{a_1}s_{a_2}, s_\beta s_{a_1}$ or $s_\beta s_{a_2}$, we have $W_{v_m}(tw) = 0$ for all $t \in T$ and $m \geq C$.

**Proof.** (1) For $x \in \mathfrak{p}^{-m}$, consider the element $x_\alpha(x) \in U_m$. We have
\[ tx_\alpha(x) = x_\alpha(\alpha(t)x) t. \]

Thus by Lemma 2.1 or equation (2.1), we have
\[ \psi_m(x_\alpha(x))W_{v_m}(t) = \psi_U(x_\alpha(\alpha(t)x))W_{v_m}(t). \]

If $W_{v_m}(t) \neq 0$, we get $\psi_m(x_\alpha(x)) = \psi_U(x_\alpha(\alpha(t)x))$, or $\psi_E(x) = \psi_E(\alpha(t)x)$. Since this is true for all $x \in \mathfrak{p}^{-m}$ and $\psi_E$ is unramified, we get $\alpha(t) - 1 \in \mathfrak{p}^m$, or $\alpha(t) \in 1 + \mathfrak{p}^m$. A similar argument shows that $\beta(t) \in 1 + \mathfrak{p}_F^m$. This proves (1).

(2) Note that $w$ which is given in the condition sends a simple root $y$ to a positive non-simple root $w(y)$. In fact, we have $s_\alpha(\beta) = 2\alpha + \beta, s_\beta(\alpha) = \alpha + \beta, s_a s_\beta(\alpha) = \alpha + \beta$ and $s_\beta s_a(\beta) = 2\alpha + \beta$. Take $r$ such that $x_\chi(r) \in U_m$, by equation (2.1), we have
\[ \psi_U(r)W_{v_m}(tw) = W_{v_m}(twx_\chi(r)) = W_{v_m}(x_\omega(y)(ry(t))tw) = W_{v_m}(tw). \]

We can take $r$ such that $\psi_U(t) \neq 1$. Thus $W_{v_m}(tw) = 0$. \hfill \square

**Corollary 2.3.** For $m \geq C$, and $t = (a_1, a_2) \in T$, if $W_{v_m}(t) \neq 0$, then
\[ a_1/a_2 \in 1 + \mathfrak{p}^m \quad \text{and} \quad a_2 \in E^1(1 + \mathfrak{p}_F^m). \]

**Proof.** By Lemma 2.2, we have $a_1/a_2 = \alpha(t) \in 1 + \mathfrak{p}^m$ and $a_2\bar{a}_2 \in 1 + \mathfrak{p}_F^m$. Since $E/F$ is unramified, we have $\text{Nm}_{E/F}(1 + \mathfrak{p}^m) = 1 + \mathfrak{p}_F^m$, see [25, Chapter V, Section 2]. Since $\text{Nm}(a_2) = a_2\bar{a}_2 \in 1 + \mathfrak{p}_F^m$, there exists an element $b \in 1 + \mathfrak{p}_F^m$ such that $b\bar{b} = a_2\bar{a}_2$. It is clear that $a_2/b \in E^1$, and thus $a_2 \in E^1(1 + \mathfrak{p}_F^m)$. \hfill \square

We fix the following notations:

**Notation.** Let $(\pi, V_\pi)$ and $(\pi', V_{\pi'})$ be two irreducible admissible $\psi_U$-generic representations of $G$ such that $\omega_\pi = \omega_{\pi'}$, where $\omega_\pi$ is the central character of $\pi$. Let $v \in V_\pi$ and $v' \in V_{\pi'}$ such that $W_{v}(1) = W_{v'}(1) = 1$. We have defined Howe vectors $v_m, v'_m$ for $m \geq C$, where $C = C(v, v')$ is an integer such that $v$ is fixed by $\pi(K_C)$ and $v'$ is fixed by $\pi'(K_C)$.

**Corollary 2.4.** If $m \geq L$, we have $W_{v_m}(b) = W_{v'_m}(b)$ for all $b \in B$.

**Proof.** For $b \in B$, we can write $b = ut$ with $u \in U, t \in T$. Since
\[ W_{v_m}(ut) = \psi_U(u)W_{v_m}(t) \quad \text{and} \quad W_{v'_m}(ut) = \psi_U(u)W_{v'_m}(t), \]
it suffices to prove that $W_{v_n}(t) = W'_{v_n}(t)$ for all $t \in T$. Suppose $t = (a_1, a_2)$. By Corollary 2.3, if $a_2 \not\in E^1(1 + P^m_E)$ or $a_1 a_2^{-1} \not\in 1 + P^m_E$, we have $W_{v_n}(t) = 0 = W'_{v_n}(t)$. Now suppose that $a_2 \in E^1(1 + P^m_E)$ and $a_1 a_2^{-1} \in 1 + P^m_E$. We can write $a_2 = za_2' a_2^{-1}$ with $z \in E^1, a_2' \in 1 + P^m_E$. Then
\[ t = t(z, z) t(a_2' a_2^{-1}, a_2'). \]

Since $t(z, z)$ is in the center $Z$ of $G$, we have
\[ W_{v_n}(t) = \omega_{m}(z) W_{v_n}(t(a_2' a_2^{-1}, a_2')). \]

Since $a_2' \in 1 + P^m_E$, $a_2' a_2^{-1} \in 1 + P^m_E$, we have $W_{v_n}(t(a_2' a_2^{-1}, a_2')) \in H_m$. Thus by Lemma 2.1 (2),
\[ W_{v_n}(t) = \psi_{m}(t(a_2' a_2^{-1}, a_2')) W_{v_n}(1) = W_{v_n}(1) = 1. \]

It follows that $W_{v_n}(t) = \omega_{m}(z)$. The same argument shows that $W_{v_n'}(t) = \omega_{m'}(z)$. Since $\omega_{m} = \omega_{m'}$, we finally get $W_{v_n}(t) = W_{v_n'}(t)$. \hfill \Box

### 2.2 A stability property of the Whittaker functions associated with Howe vectors

We recall the Bruhat order on the Weyl group $W$, see [18] for example. An element $w \in W$ can be written as a product of simple roots in a minimal length, say $w = s_{\gamma_1} \cdots s_{\gamma_l}$ with $l = \text{length}(w)$. We say $w' \leq w$ if $w'$ can be written as a product of sub-expression, i.e.,
\[ w' = s_{\gamma_i} \cdots s_{\gamma_k}, \]
with $1 \leq i_1 < i_2 < \cdots < i_k \leq l$. This definition is independent of the choice of the minimal expression of $w$. We say $w' < w$ if $w' \leq w$ and $w' \neq w$.

For $w \in W$, denote $U_w^+ = \{ u \in U : uu^{-1} w^{-1} \in U \}$ and $U_w^- = \{ u \in U : uu^{-1} w \in U \}$. Then
\[ U_w^- = \bigcap_{y \in \Sigma_w^+} U_y \quad \text{and} \quad U_w^+ = \bigcap_{y \in \Sigma_w^-} U_y, \]
where $\Sigma_w^+ = \{ y \in \Sigma^+, w(y) > 0 \}$ and $\Sigma_w^- = \{ y \in \Sigma^+, w(y) < 0 \}$. Recall that $U_{w, m} = U_w^- \cap H_m$.

#### Proposition 2.5. Given $w \in W$. Let $a_t, 0 \leq t \leq l(w)$, be a sequence of integers with $a_0 = 0$ and $a_t \geq t + a_{t-1}$ for all $t$ with $1 \leq t \leq l(w)$. Let $m$ be an integer such that $m \geq 4^{a_0} C$.

1. If $W_{v_n}(tw') = W'_{v_n}(tw')$ for all $w' \leq w, k \geq 4^{a_0} C$ and $t \in T$, then
\[ W_{v_n}(tuw_{w}) = W'_{v_n}(tuw_{w}) \]
for all $u \in U_w^-$.

2. If $W_{v_n}(tw') = W'_{v_n}(tw')$ for all $w' \leq w, k \geq 4^{a_0} C$ and $t \in T$, then
\[ W_{v_n}(g) = W'_{v_n}(g) \]
for all $g \in BwB$.

#### Remark. We can take the sequence $a_t = t^2$ as in [2]. We can also take the sequence $a_0 = 0, a_1 = 1, a_2 = 3, a_3 = 6$ and $a_4 = 10$ in our case.

#### Proof. This is essentially [2, Lemma 6.2.6]. Since [2] is not published, we explain more about the proof. For $w \leq w_2 = s_{\alpha_1} s_{\alpha_2}$, Proposition 2.5 is proved in a more general setting in Theorem 3.11 [30], which in fact justified an ambiguity in the original proof of [2, Lemma 6.2.6]. Now we consider the case when $w = w_1 = s_{\beta} s_{\alpha} s_{\beta}$ and $w = w_0 = (s_{\alpha} s_{\beta})^2$. We order the set $\Sigma'$ by height, i.e., we denote $\gamma_1 = \alpha$, $\gamma_2 = \beta$, $\gamma_3 = \alpha + \beta$, $\gamma_4 = 2\alpha + \beta$. Here the order of $\alpha$ and $\beta$ are not important.

From the proof of [30, Theorem 3.11], to prove our proposition for $w = w_0$ or $w_1$, it suffices to check the following Claim (*):
Claim (\(*\)). Suppose that \(w = w_1 \) or \(w_0\). Given \(g = tw_{\alpha}(r_k) \cdots x_{\gamma}(r_1) \in G\) with \(t \in T\), \(r_i \neq 0\) and \(w(y_i) < 0\), where the subscript of \(y\) is decreasing, i.e., \(k > k - 1 \geq \cdots \geq i\), then \(g x_{-\gamma}(-r_i^{-1}) \in B w B\) for some \(w' \) where \(w' < w\).

To prove this claim, we need to use the Chevalley relation \(x_{\gamma}(r_i) x_{-\gamma}(-r_i) \in s_{\gamma} B\), where \(s_{\gamma}\) is the reflection associated with \(\gamma\). For this relation, see [26].

We will check Claim (\(*\)) case by case. First suppose that \(i = 1\) or 2, so that \(\alpha_i\) is simple. Then by the above Chevalley relation, we have
\[
g x_{-\gamma}(-r_i^{-1}) = tw_{\alpha}(r_i) \cdots x_{\gamma}(r_1) s_{\gamma} b \quad \text{for some } b \in B.
\]
Using the relation \(s_{\gamma}^{-1} x_{\gamma} s_{\gamma} = x_{\gamma(y)}(r_i)\), we get
\[
g x_{-\gamma}(-r_i^{-1}) = tw s_{\gamma} x_{\gamma(y)}(r_i) \cdots x_{\gamma(y+1)}(r_{i+1}) b.
\]
Since \(s_{\gamma(y)} > 0\), we get \(x_{\gamma(y)}(r_i) \in U\), and thus
\[
g x_{-\gamma}(-r_i^{-1}) \in B w s_{\gamma} B.
\]
Now the assertion follows since \(w' = w s_{\gamma} < w\) by the assumption \(w(y_1) < 0\).

Next we consider the case \(i = 3\), so that \(g = tw x_{2 \alpha}(r_2) x_{\alpha}(r_3)\). We can check that \(s_{\alpha+\beta} = s_{\beta} s_{\alpha} s_{\beta} = w_1\).

Using the above Chevalley relation, we can get
\[
g x_{-\alpha}(r_3^{-1}) \in B w U_{2 \alpha + \beta} w_1 B.
\]
Note that \(w_1 = w_i^{-1}\) and \(w_1 U_{2 \alpha + \beta} w_1 = U_{w_1(2 \alpha + \beta)} = U_{- \beta} \subset B s_{\beta} B\), thus we get
\[
g x_{-\alpha}(r_3^{-1}) \in B w w_1 B s_{\beta} B.
\]
If \(w = w_1\), then \(g x_{-\alpha}(r_3^{-1}) \in B w B\) with \(w' = s_{\beta} < w_1\). If \(w = w_0\), then \(w w_1 = s_{\alpha}\), and thus
\[
g x_{-\alpha}(r_3^{-1}) \in B s_{\alpha} B s_{\beta} B = B s_{\alpha} s_{\beta} B.
\]
The assertion follows with \(w' = s_{\alpha} s_{\beta} < w_0\).

Finally, we consider the case \(i = 4\), so that \(g = tw x_{2 \alpha}(r_4)\). We have \(s_{2 \alpha + \beta} = s_{\alpha} s_{\beta} s_{\alpha} = w_2\). Thus from the Chevalley relation, we get
\[
g x_{-\alpha}(r_4^{-1}) \in B w w_2 B.
\]
It suffices to check that \(w' = w w_2 < w\). In fact, we have \(w_1 w_2 = s_{\alpha} s_{\beta} < w_1\), and \(w_0 w_2 = s_{\beta} < w_0\). The proof of Claim (\(*\)) and hence the proposition is complete. \(\square\)

As a direct consequence of Proposition 2.5, Lemma 2.2 (2) and Corollary 2.4, we have the following:

**Corollary 2.6.** Let \(\alpha\) be a sequence as in Proposition 2.5.

1. If \(w_1 = 1\), \(s_{\alpha} \), \(s_{\beta} \), \(s_{\alpha} s_{\beta} \), \(s_{\beta} s_{\alpha}\), we have
\[
W_{v_m}(g) = W_{v_m}(g)
\]
   for all \(g \in B w B\) and \(m \geq 4^{a(v \omega)} C\).

2. If \(w = w_1, w_2\), we have
\[
W_{v_m}(t w u_w) = W_{v_m}(t w u_w)
\]
   for \(u_w \in U_w = U_{w, m} \) for \(m \geq 4^{a(v \omega)} C\) and all \(t \in T\).

### 3 Induced representations and intertwining operator

In this section, we will construct some sections in the induced representations \(I(s, \eta)\) and \(I(s, \tau)\) for a given quasi-character \(\eta\) of \(E^*\) and an irreducible smooth representation \(\tau\) of \(GL_2(E)\). Since the construction for sections in \(I(s, \eta)\) is quite similar in the case \(I(s, \tau)\) and the proof is easier, we only write down the statement and the proof in the case \(I(s, \tau)\).
Let \( \tilde{N} \) be the opposite of \( N \), i.e., \( \tilde{N} \) consists of matrices of the form

\[
\hat{n}(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \text{Herm}_2(F).
\]

Recall that \( N^{(2)} \) is the unipotent subgroup of \( \text{GL}_2(E) \), and under the embedding \( \text{GL}_2(E) \cong M \hookrightarrow G \), we have \( N^{(2)} \cong U_a \).

Let \( X \) be an open compact subgroup of \( N \). For \( x \in X, i > 0 \), define \( A(x, i) = \{ \hat{n} \in \tilde{N} : \hat{n} x \in P \cdot \tilde{N}_i \} \).

**Lemma 3.1.** The following statements hold.

1. For any positive integer \( c \), there exists an integer \( i_1 = i_1(X, c) \) such that for all \( i \geq i_1, x \in X \) and \( \hat{n} \in A(x, i) \), we can write

\[
\hat{n} x = n \hat{m}(a) \hat{n}_0
\]

with \( n \in N, \hat{n}_0 \in \tilde{N}_i \) and \( a \in K^{(2)}_c := 1 + \text{Mat}_{2 \times 2}(P^c_E) \).

2. There exists an integer \( i_0 = i_0(X) \) such that \( A(x, i) \) is independent of \( x \) for all \( i \geq i_0 \), i.e., \( A(x_1, i) = A(x_2, i) \) for all \( x_1, x_2 \in X \). In fact, we can choose \( i_0 = i_0(X) \) such that \( A(x, i) = \tilde{N}_i \) for all \( i \geq i_0 \). Here \( \tilde{N}_i = H_i \cap \tilde{N} \).

In the \( \text{GL}_n \) case, this is [2, Lemma 4.1]. The proof in our case is similar.

**Proof.** Since \( X \) is compact, there is a constant \( D_X \) such that \( |x_{ij}| < D_X \), for all \( x = n((x_{ij})) \in X \subset N \).

For \( x \in X, \hat{n} \in A(x, i) \), we assume that \( \hat{n} x = p \hat{y}^{-1} \) with \( p \in P, \hat{y}^{-1} \in \tilde{N}_i \). We have

\[
\hat{n}^{-1} p = x \hat{y}
\]

Let

\[
\hat{n}^{-1} = \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix} = \hat{n}(\hat{b}), \quad p = n(b') \hat{m}(a) \quad \text{for} \quad \hat{b}, b' \in \text{Herm}_2(F), \ a \in \text{GL}_2(E).
\]

Then

\[
\hat{n}^{-1} p = \hat{n}(\hat{b}) n(b') \hat{m}(a) = \begin{pmatrix} a \\ b a \\ b' a^{-1} \end{pmatrix} \begin{pmatrix} b' + 1 \end{pmatrix} a^{-1}.
\]

On the other hand, if we assume \( x = n(x_0) \) and \( \hat{y} = \hat{n}(\tilde{y}_0) \) with \( x_0, \tilde{y}_0 \in \text{Herm}_2(F) \), then we have

\[
x \hat{y} = \begin{pmatrix} 1 & x_0 \\ 1 & \tilde{y}_0 \end{pmatrix} = \begin{pmatrix} 1 + x_0 \tilde{y}_0 & x_0 \\ \tilde{y}_0 & 1 \end{pmatrix}.
\]

From \( \hat{n}^{-1} p = x \hat{y} \), we get

\[
a = 1 + x_0 \tilde{y}_0
\]

and

\[
\hat{b} = \hat{y} a^{-1} = \tilde{y}_0 (1 + x_0 \tilde{y}_0)^{-1},
\]

since the entries of \( x_0 \) are bounded and the entries of \( \tilde{y}_0 \) go to zero as \( i \to \infty \). Thus for any positive integer \( c \), we can take \( i_1 = i_1(X, c) \) such that if \( i \geq i_1 \), we have

\[
a = 1 + x_0 \tilde{y}_0 \in K^{(2)}_c.
\]

Thus (1) follows.

To prove (2), we write \( \hat{b} = (\hat{b}_{jk}), \ x_0 = (x_{jk}) \) and \( \tilde{y}_0 = (\tilde{y}_{jk}) \), \( j, k = 1, 2 \). By Cramer’s rule, we have

\[
(1 + x_0 \tilde{y}_0)^{-1} = \det(1 + x_0 \tilde{y}_0)^{-1} \begin{pmatrix} 1 + x_{21} \tilde{y}_{12} + x_{22} \tilde{y}_{22} & -x_{11} \tilde{y}_{12} - x_{12} \tilde{y}_{22} \\ -x_{21} \tilde{y}_{11} - x_{22} \tilde{y}_{21} & 1 + x_{11} \tilde{y}_{11} + x_{12} \tilde{y}_{21} \end{pmatrix}.
\]

From \( \hat{b} = \tilde{y}_0 (1 + x_0 \tilde{y}_0)^{-1} \), we can solve that

\[
\hat{b} = \det(1 + x_0 \tilde{y}_0)^{-1} \begin{pmatrix} \tilde{y}_{11} + \det(\tilde{y}_0)x_{22} & \tilde{y}_{12} - \det(\tilde{y}_0)x_{12} \\ \tilde{y}_{21} - \det(\tilde{y}_0)x_{21} & \tilde{y}_{22} + \det(\tilde{y}_0)x_{11} \end{pmatrix}.
\]
Note that \( \hat{y} \in \hat{N}_i \), i.e.,

\[
\hat{y}_0 \in \left( \frac{g_{7i}^T}{g_{5i}^T} \right).
\]

We have \( \det(\hat{y}_0) \in \mathcal{P}_E^{101} \). We can choose \( i_2 = i_2(X) \) large enough such that for \( i \geq i_2 \), we have

\[
q_{E}^{-10i} D_{X} \leq q_{E}^{-7i}.
\]

Then

\[
\det(\hat{y}_0)_{xjk} \in \mathcal{P}_E^{7i}, \quad i \geq i_2,
\]

for all \( j, k = 1, 2 \). Now we take \( i_0(X) = \max\{i_2(X), i_1(X, 1)\} \). Then for \( i \geq i_0(X) \), we have

\[
\det(1 + x_0\hat{y}_0) \in 1 + \mathcal{P}_E \subset \mathcal{O}_E^X.
\]

From equations (3.1) and (3.2), we get

\[
\hat{b} \in \left( \frac{g_{7i}^T + g_{7i}^T}{g_{5i}^T + g_{7i}^T} \right) \subset \left( \frac{g_{7i}^T}{g_{5i}^T} \right),
\]

Thus \( \hat{n}^{-1} = \hat{u}(\hat{b}) \in \hat{N}_i \), and hence \( \hat{n} \in \hat{N}_i \). This shows \( A(x, i) \subset \hat{N}_i \).

We will show that every element \( \hat{y} \in \hat{N}_i \) is contained in \( A(x, i) \) for \( i \) large enough. As above, we write \( \hat{y} = \hat{u}(\hat{y}_0) \) and \( x = \mathfrak{m}(x_0) \). We first notice that, for \( i \) large enough, we can assume \( \det(1 + \hat{y}_0x_0) \neq 0 \). From this, it is easy to check that \( x_0 \in \mathcal{P}_E \), and thus we can write \( \hat{y}x = px \) for \( p \in P, \hat{n} \in \hat{N} \). A similar argument as above will show that \( \hat{n} \in \hat{N}_i \) for \( i \) large. In fact, we only have to switch the role of \( \hat{y} \) and \( \hat{n} \) in the above argument. Thus \( \hat{y} \in A(x, i) \) for \( i \) large. This finishes the proof of (2).

For \( i > 0 \) and a vector \( v \in V_{\tau} \), we consider a function \( \xi^{s, \varepsilon}_{s} : G \times V_{\tau} \rightarrow V_{\tau} \) defined by

\[
\xi^{s, \varepsilon}_{s}(g) = \begin{cases} 
\left| \det(a) \right|^{s/2} \tau(a)^{\varepsilon}, & \text{if } g = \mathfrak{m}(a)\hat{n}, \ a \in \mathcal{O}_E^X, \ \hat{n} \in \hat{N}_i, \\
0, & \text{otherwise}.
\end{cases}
\]

**Lemma 3.2.** There is an integer \( i_2 = i_2(e) \) such that if \( i \geq i_2 \), \( \xi^{s, \varepsilon}_{s} \) defines an element in \( I(s, \tau) = \text{Ind}_{\mathcal{P}}^{G}(\tau_{s-1/2}) \), where \( \tau_{s-1/2} = \tau|\det|^{s-1/2} \).

**Proof.** It is clear that \( \xi^{s, \varepsilon}_{s}(\mathfrak{m}(a)g) = \delta_{p}^{1/2}(a)\tau_{s-1/2}(a)\xi^{s, \varepsilon}_{s}(g) \), where \( \delta_{p} \) is the modulus character of the Siegel parabolic subgroup \( P \). We need to check that for \( i \) large enough, there is an open compact subgroup \( H_{i, e} < G \) such that

\[
\xi^{s, \varepsilon}_{s}(gh) = \xi^{s, \varepsilon}_{s}(g) \quad \text{for all } h \in H_{i, e}.
\]

Let \( c > 0 \) be an integer such that \( c \) is fixed by \( K_{c}^{(2)} = 1 + \text{Mat}_{2, 2}(\mathcal{P}_E) \subset \text{GL}(E) \) under \( \tau \). Recall that \( K_{c} \) is used to denote the congruence subgroup \( G \cap (1 + \text{Mat}_{4, 2}(\mathcal{P}_E)) \) of \( G \). Let \( i_2(e, \tau) = \max\{c, i_0(N \cap K_{c}), i_1(N \cap K_{c}, e)\} \), where \( i_0 \) and \( i_1 \) are defined in Lemma 3.1, and \( N_{c} = N \cap f_{c} \) is defined in Section 3. By Lemma 3.1, we have \( A(x, i) = \hat{N}_i \) for \( x \in N_{c}, i \geq i_2(e, \tau) \).

We take \( H_{i, e} = K_{7i} \). We have the decomposition \( K_{7i} = (\hat{N} \cap K_{7i})(M \cap K_{7i})(N \cap K_{7i}) \). For \( h \in \hat{N} \cap K_{7i} \subset \hat{N}_i \), it is clear that \( \xi^{s, \varepsilon}_{s}(gh) = \xi^{s, \varepsilon}_{s}(g) \). For \( h = \mathfrak{m}(a_0) \in M \cap K_{7i} \) with \( a_0 \in K_{7i}^{(2)} = 1 + \text{Mat}_{2, 2}(\mathcal{P}_E) \), and \( \hat{n} \in \hat{N}_i \), we have

\[
\mathfrak{m}(a_0)^{-1} \hat{n} \mathfrak{m}(a_0) \in \hat{N}_i
\]

by an explicit calculation.

From this we can check that \( g \in P\hat{N}_i \) if and only if \( \mathfrak{m}(a_0) \in P\hat{N}_i \). Moreover,

\[
\xi^{s, \varepsilon}_{s}(\mathfrak{m}(a)\hat{n}\mathfrak{m}(a_0)) = \xi^{s, \varepsilon}_{s}(\mathfrak{m}(aa_0)\mathfrak{m}(a_0^{-1})\hat{n}\mathfrak{m}(a_0)) = |\det(aa_0)|^{s/2} \tau(aa_0)^{\varepsilon} = |\det(a)|^{s/2} \tau(a)^{\varepsilon} = \xi^{s, \varepsilon}_{s}(\mathfrak{m}(a)\hat{n}),
\]

where we used \( i > c \) and hence \( \tau(a_0)^{\varepsilon} = e \) for \( a_0 \in K_{7i}^{(2)} \).

Next, we assume that \( h \in N \cap K_{7i} \subset N \cap K_{c} \). By the above lemma and our assumption, we have

\[
A(h, i) = A(h^{-1}, i) = \hat{N}_i.
\]

In particular, for \( \hat{n} \in \hat{N}_i \), we have \( \hat{n} \mathfrak{h} \in P \cdot \hat{N}_i \) and \( \hat{n} \mathfrak{h}^{-1} \in P \cdot \hat{N}_i \). Thus \( g \in P\hat{N}_i \) if and only if \( gh \in P\hat{N}_i \). Moreover,
by Lemma 3.1 (1), we have \( nh = n_0 m(a_0) \tilde{n}_0 \) with \( a_0 \in K_c^{(2)} \), then we have

\[
\xi_{s,s}^{i,e} (nm(a) \tilde{n}h) = \xi_{s,s}^{i,e} (nm(a) n_0 m(a^{-1}) m(a a_0) \tilde{n}_0)
\]

\[
= | \det(a a_0) |^{s + 1 / 2} \tau(a a_0) e = | \det(a) |^{s + 1 / 2} \tau(a) e = \xi_{s,s}^{i,e} (nm(a) \tilde{n}).
\]

This shows that \( \xi_{s,s}^{i,e} (gh) = \xi_{s,s}^{i,e} (g) \) for all \( h \in K_{7l} \). The proof is complete. \( \square \)

Consider the function

\[
f_{s,s}^{i,e}(g) := f_{s,s}^{i,e}(g) = \lambda(\xi_{s,s}^{i,e}(g)),
\]

where \( \lambda \) is a fixed Whittaker functional of \( \tau \). Then \( \sup f_{s,s}^{i,e} = P \cdot \tilde{N} \) and we have

\[
f_{s,s}^{i,e} (nm(a) \tilde{n}) = | \det(a) |^{s + 1 / 2} W_{s}^{(2)}(a),
\]

(3.3)

where \( W_{C}^{(2)}(a) = \lambda(\tau(a) e) \) is the Whittaker function of \( \tau \) associated to the vector \( e \in V_{\tau} \).

Now we consider \( \xi_{1-s}^{i,e} = M(s) \xi_{s,s}^{i,e} \in I(1 - s, \tau^*) \). By definition, we have

\[
\tilde{\xi}_{s,s}^{i,e} (g) = \int_{N} \xi_{s,s}^{i,e} (w_{1}^{-1} ng) \ dn.
\]

Let \( X \) be an open compact subgroup of \( N \); we evaluate \( \tilde{\xi} \) at \( \tilde{w}_{1}x \) for \( x \in X \).

**Proposition 3.3.** There is an integer \( I = I(X, e) \) such that for \( i \geq I \), we have \( \tilde{\xi}_{1-s}^{i,e}(w_{1}x) = \text{vol}(\tilde{N}) e \) for all \( x \in X \).

**Proof.** We have

\[
\tilde{\xi}_{s,s}^{i,e} (w_{1}x) = \int_{N} \xi_{s,s}^{i,e} (w_{1}^{-1} n w_{1}x) \ dn.
\]

Again, let \( c \) be a positive integer such that \( e \) is fixed by \( \tau(K_{c}^{(2)}) \). Let \( I = \max(i_{0}(X), i_{1}(X, e)) \). By the definition of \( \xi_{s,s}^{i,e} \) and Lemma 3.1, for \( i \geq I \), we have \( \xi_{s,s}^{i,e} (w_{1}^{-1} n w_{1}x) \neq 0 \) if and only if \( w_{1}^{-1} n w_{1}x \in P \cdot \tilde{N} \) if and only if \( \tilde{w}_{1}^{-1} n \tilde{w}_{1} \in \tilde{N} \). By part (1) of Lemma 3.1, if \( \tilde{w}_{1}^{-1} n \tilde{w}_{1} \in A(x, i) = \tilde{N} \), we have

\[
\tilde{w}_{1}^{-1} n \tilde{w}_{1} \times n' m(a) \tilde{n} \quad \text{with} \quad n' \in N, \ a \in K_{c}^{(2)}, \ \tilde{n} \in \tilde{N}.
\]

Since for \( a \in K_{c}^{(2)} \), we have \( \tau(a) e = e \) and \( | \det(a) | = 1 \), we get

\[
\xi_{s,s}^{i,e} (w_{1}^{-1} n w_{1}x) = \begin{cases} | \det(a) |^{s + 1 / 2} \tau(a) e = e, & \text{if} \ \tilde{w}_{1}^{-1} n \tilde{w}_{1} \in \tilde{N}, \\ 0, & \text{otherwise}. \end{cases}
\]

Thus

\[
\tilde{\xi}_{s,s}^{i,e} (w_{1}x) = \text{vol}(\tilde{N}) e.
\]

This proves the assertion. \( \square \)

Let \( j_{1-s}^{i,e}(g) = f_{s,s}^{i,e}(g) = \lambda(\xi_{1-s}^{i,e}(g)) \). Then

\[
j_{1-s}^{i,e} (nm(a) w_{1}x) = \text{vol}(\tilde{N}) | \det(a) |^{3 / 2 - s} W_{s}^{(2)}(a), \quad i > R(X, e), \ x \in X.
\]

(3.4)

Here \( W^{(2)} \) denotes the Whittaker function for the representation \( \tau^{*} = w_{1} \tau \) of \( M \cong \text{GL}_{2}(E) \), see Section 1.

### 4 Twisting by characters of \( E^{x} \)

We keep the notations of Section 2. In particular, \( E/F \) is unramified, \( \psi \) is an unramified additive character of \( F \), and \( \pi, \pi' \) are two \( \psi_{U} \)-generic irreducible smooth representations of \( U(2, 2)(F) \) with the same central character. We also fixed \( v \in V_{\pi}, v' \in V_{\pi'} \) and a positive integer \( C = C(v, v') \). We have defined Howe vectors \( v_{m}, v'_{m} \). Let \( \mu \) be a fixed character of \( E^{x} \) such that \( \mu |_{E^{x}} = \epsilon_{E/F} \).
Proposition 4.1. If \( \gamma(s, \pi \times \mu \eta, \psi) = \gamma(s, \pi' \times \mu \eta, \psi) \) for all quasi-characters \( \eta \) of \( F^s \), then

\[
W_{\gamma_m}(t \omega_2) = W_{\gamma_{m'}}(t \omega_2)
\]

for all \( t \in T, m \geq 4^6 C \). Recall that \( \omega_2 = s_0 s_b s_d \).

Proof. An almost identical argument as the proof of [30, Theorem 4.4] will give us the following:

\[
\gamma(s, \pi \times \mu \eta, \psi) - \gamma(s, \pi' \times \mu \eta, \psi) = e_{\psi^{-1}} q_F^{5k} q_E^{4k} \int_{E^s} (W_{\gamma_m}(t(a) \omega_2) - W_{\gamma_{m'}}(t(a) \omega_2)) \mu(a) \eta_{-s-1/2}^{-1}(a) \ da
\]

(4.1)

for \( k = 4^6 C \). Recall that \( e_{\psi^{-1}} \) is the Weil index appeared in the Weil representation formula (1.1),

\[
t(a) = \text{diag}(a, 1, a^{-1}, 1), \quad \eta_{-s-1}(a) = \eta(a)|a|^{-s-1}.
\]

In fact, in the \( Sp_{2n} \) case, equation (4.1) (with a little bit modification on the constant term \( q_F^{5k} q_E^{4k} \) and the exponent \( -s - 1/2 \)) is [30, equation (4.4)] with \( k = 4^{4(w_0)} C = 4^9 C \), where we used the sequence \( a_t = t^2 \) in Proposition 2.5. It is easy to see that this is true for \( k = 4^6 C \) by choosing the sequence \( a_0 = 0, a_1 = 1, a_2 = 3, a_3 = 6 \) and \( a_6 = 10 \).

By the assumption on \( \gamma \)-factors and equation (4.1), we have

\[
\int_{E^s} (W_{\gamma_m}(t(a) \omega_2) - W_{\gamma_{m'}}(t(a) \omega_2)) \mu(a) \eta_{-s-1/2}^{-1}(a) \ da \equiv 0
\]

for any quasi-character \( \eta \) of \( F^s \). By the inverse Mellin transformation, we get

\[
W_{\gamma_m}(t(a) \omega_2) = W_{\gamma_{m'}}(t(a) \omega_2) \quad \text{for all } a \in E^s.
\]

From this, we can prove \( W_{\gamma_m}(t \omega_2) = W_{\gamma_{m'}}(t \omega_2) \) for all \( t = (a, b) \) as follows. Take \( r \in \mathfrak{Z}_F^{-k} \), so that \( x(a) \in U_k \). From equation (2.1) and the relation

\[
t \omega_2 x_B(r) = t x_B(r) \omega_2 = x_B(b^2 r) t \omega_2,
\]

we get \( \psi(r) W_{\gamma_m}(t \omega_2) = \psi(b^2 r) W_{\gamma_m}(t \omega_2) \). Thus, if \( W_{\gamma_m}(t \omega_2) \neq 0 \), we have \( \psi(r) = \psi(b^2 r) \) for all \( r \in \mathfrak{Z}_F^{-k} \), which implies \( b^2 \in 1 + \mathfrak{Z}_F^{k} \). Thus we get \( b = e b_1 \) with \( b_1 \in 1 + \mathfrak{Z}_F^{k} \) and \( e \in E_1^s \) (see the proof of Corollary 2.3). Thus we can write \( t = e t_1 b_1 t_1 \), with \( a_1 = e a \) and \( t_1 = t(1, b_1) \). Since

\[
t \omega_2 = e t_1 a_1 \ t_1 \omega_2 = e t_1 a_1 \ t_1 \omega_2
\]

and \( t_1 \in H_k \), we get

\[
W_{\gamma_m}(t \omega_2) = \omega_{\pi}(e) W_{\gamma_m}(t(a) \omega_2),
\]

by equation (2.1). Here \( \omega_{\pi} \) is the central character of \( \pi \). Since \( \pi \) and \( \pi' \) have the same central character, we get \( W_{\gamma_m}(t \omega_2) = W_{\gamma_{m'}}(t \omega_2) \) for all \( t \in T \). By Lemma 2.1 (3) and Proposition 2.5, it is easy to check that

\[
W_{\gamma_m}(t \omega_2) = W_{\gamma_{m'}}(t \omega_2) \quad \text{for all } m \geq k = 4^6 C, \ t \in T.
\]

This concludes the proof.

□

Proposition 4.2. Suppose that \( \gamma(s, \pi \times \mu \eta, \psi) = \gamma(s, \pi' \times \mu \eta, \psi) \) for all quasi-characters \( \eta \) of \( F^s \). Then for \( m \geq 4^6 C, a \in GL_2(E) \) and \( n \in N - N_m \), we have

\[
W_{\gamma_m}(m(a) \omega_1 n) = W_{\gamma_{m'}}(m(a) \omega_1 n).
\]

Recall that \( N_m = N \cap H_m \).

Proof. The proof is similar to the proof of [30, Theorem 3.11] and the method is due to Baruch [2]. We give the details here.
By Corollary 2.6, we have $W_{\psi_1}(g) = W_{\psi_1}(g)$ for all $g \in BwB$ with $w \in W$ with length($w$) $\leq 2$ and $k \geq 4^{(1/w)^2}C$. From the condition and Proposition 4.1, we get $W_{\psi_1}(tw_2) = W_{\psi_1}(tw_2)$ for $t \in T$, $k \geq 4^6C$. Thus we have $W_{\psi_4}(g) = W_{\psi_4}(g)$ for $g \in BwB$ and $k \geq 4^6C$.

For $n \in N$, we can write $n = x_{\alpha_2 \beta}(r_1) x_{\alpha_2 \beta}(r_2) x_{\alpha_2 \beta}(r_1)$ for $r_1, r_2 \in F$, $r_2 \in E$. Denote $y_3 = 2 \alpha + \beta$, $y_2 = \alpha + \beta$, $y_1 = \beta$. Let $j (1 \leq j \leq 3)$ be the first index such that $x_{\beta}(r_j) \notin N_m$. Then it suffices to show that

$$W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) = W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)).$$

We first show equation (4.2) under the assumption $x_{\beta}(r_j) \notin N_m$. Let $x_j \in \mathbb{R}_+$ be the first index such that $x_{\beta}(r_j) \notin N_m$. Then it suffices to show that

$$W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) = \frac{1}{\text{vol}(U_m)} \int_{U_m} W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) u \psi_u^{-1}(u) \, du.$$

There is a similar formula for $W_{\psi_4}$, and thus it suffices to show that

$$W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) = W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) \quad \text{for all } u \in U_m. \quad (4.2)$$

Write $u = x_{a}(s_1) x_{\beta}(s_3) x_{\beta}(s_2) x_{\beta}(s_1)$. From the Chevalley relation,

$$[x_a, x_{\beta}] \in \prod_{s \geq 1} U_{s a + y_i},$$

see [26, Chapter 3, relation (R2) above Lemma 21], we get

$$x_{\beta}(r_3) \ldots x_{\beta}(r_j) x_{a}(s_1) x_{\beta}(s_3) x_{\beta}(s_2) x_{\beta}(s_1)$$

with $r_j' = r_j$. Since $N$ is abelian, we get

$$x_{\beta}(r_3) \ldots x_{\beta}(r_j) u = x_{a}(s_1) x_{\beta}(s_3) x_{\beta}(s_2) x_{\beta}(s_1),$$

with $r_j' = r_j + s_k$ for $k \geq j$. In particular, we have $|r_j| = |r_j|$ since $x_{\beta}(r_j) \notin N_m$ and $x_{\beta}(s_j) \in N_m$. Since $w_1(a) = a$, we can write

$$m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j) u = m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)$$

for some $a_j \in \text{GL}_2(E)$. We first show equation (4.2) under the assumption $x_{\beta}(r_j) \notin N_m$. Under this assumption, it suffices to show that

$$W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) = W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)).$$

by equation (2.1). Since $x_{\beta}(r_j) \notin N_m$, we have $r_j \notin \mathfrak{g}^{-(2\text{ht}(y_j) - 1)m}$, see the structure of $H_m$ given in Section 2, and thus $-1/r_j \in \mathfrak{g}^{(2\text{ht}(y_j) - 1)m} \subset \mathfrak{g}^{(2\text{ht}(y_j) - 1)k}$ since $m \geq 3k$. We then get $x_{\beta}(-1/r_j) \in \mathfrak{h}$, since $|r_j| = |r_j|$, we have $x_{\beta}(-1/r_j) \in \mathfrak{h}$ too. By equation (2.1), we have

$$W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)) = W_{\psi_4}(m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j)).$$

There is a similar relation for $W_{\psi_4}$. Thus it suffices to show the following:

**Claim.** We have $m(a) x_{\beta}(r_3) \ldots x_{\beta}(r_j) x_{\beta}(-1/r_j) \in BwB$ for some $w \in W - \{w_0, w_1\}$.

We will check Claim (•) case by case. We need to use the following Chevalley relation:

$$x_{\beta}(r) x_{\beta}(-1/r) \in s_\beta B,$$

where $s_\beta$ is the reflection defined by the root $\beta$ (not necessarily simple), see [26, Chapter 3, relation (R3)].

If $j = 1$, we have $y_1 = \beta$ and thus $x_{\beta}(r_1) x_{\beta}(-1/r_1) = s_\beta b$ for some $b \in B$. Thus

$$m(a) x_{\beta}(r_3) x_{\beta}(r_2) x_{\beta}(r_1) x_{\beta}(-1/r_1) \in Mw_1 U_{2a + \beta} U_{a + \beta} S_\beta B.$$

Since $s_\beta(2a + \beta) > 0$, $s_\beta(a + \beta) > 0$, we get $Mw_1 U_{2a + \beta} U_{a + \beta} S_\beta B < Mw_1 S_\beta B = M_\beta S_\beta b B$. Since $M \subset B \cup B a B$, we get $M_\beta S_\beta B \subset B_\beta S_\beta B \cup B a B S_\beta B$. Since $B a B S_\beta S_\beta a B = B_\beta S_\beta S_\beta a B$ by [26, Lemma 25], the assertion follows.
Next, we consider the case \( j = 2 \). In this case \( s_{y_2} = s_{a+\beta} = w_1 \), and thus

\[
\mathbf{m}(a)w_1x_{y_2}(r_2)x_{y_1}(r_2)x_{a+\beta}(-1/r_2) \in Mw_1U_{2a+\beta}w_1B.
\]

Since \( w_1U_{2a+\beta}w_1 = U_{w(2a+\beta)} = U_{-\beta} \subset BS_\beta B \), we get

\[
Mw_1U_{2a+\beta}w_1B \subset MBs_\beta B \subset BS_\beta B \cup BS_\beta B = BS_\beta B \cup BS_\beta B.
\]

The assertion follows.

Finally, we check Claim (\(
\ast
\)) when \( j = 3 \). In this case \( s_{y_3} = s_{2a+\beta} = s_{a}s_\beta s_{a} = w_2 \). Thus

\[
\mathbf{m}(a_1)w_1x_{y_3}(r_3)x_{y_2}(r_2)x_{a+\beta}(-1/r_3) \in Mw_1w_2B.
\]

It is easy to check that \( w_1w_2 = s_{a}s_\beta \) and \( MS_\beta s_\beta B \subset BS_\beta B \cup BS_\beta B \). Thus the assertion follows.

The proof of Claim (\(
\ast
\)) and hence the proposition is finished if \( x_{y_{j-1}}(r_{j-1}) \ldots x_{y_1}(r_1) \in N_k \). Otherwise, there will be an integer \( i \) with \( 1 \leq i \leq j-1 \) such that \( x_{y_{j-i}}(r_{j-i}) \ldots x_{y_1}(r_1) \in N_k \) but \( x_{y_j}(r_j) \notin N_k \). We just repeat the above process by taking an integer \( k_1 \) such that \( 3k_1 \leq k < 4k_1 \) and then reduce everything to \( W_{y_i} \). This process stops after at most three steps, and in each step we get a integer \( k_i \) which satisfies \( k_i \geq \frac{1}{2}k_{i-1} \geq \frac{1}{4}k_2 \geq 4^6 \gamma \). Each step goes through since \( W_{y_i}(t\hat{\omega}) = W_{y_i'}(t\hat{\omega}) \) for all \( t \in T, w \in \mathbf{W} - \{w_0, w_1\} \) and \( k \geq 4^6 \gamma \). The proof is complete. \( \square \)

5 Twisting by representations of \( \text{GL}_2(E) \)

We will fix our notation as in Section 2. In particular, \( E/F \) is an unramified extension of \( p \)-adic fields.

5.1 Howe vectors for Weil representations

Let \( \mu \) be a character of \( E^\times \) such that \( \mu|_{F^\times} = e_{E/F} \); hence we have the Weil representation \( \omega_{\mu, \psi^{-1}} \) of \( \text{U}(2, 2)(F) \) on \( \mathfrak{S}(E^2) \). Recall that \( \psi \) is an unramified character of \( F \). Given an integer \( m \), we consider the function \( \Phi^m \in \mathfrak{S}(E^2) \) defined by

\[
\Phi^m(x, y) = \text{Char}_{p^{3m}}(x) \cdot \text{Char}_{1+p^m}(y),
\]

where for a subset \( A \subset E \), \( \text{Char}_A \) denotes the characteristic function of \( A \).

Proposition 5.1. The following statements hold.

1. For \( n \in N_m \), we have

\[
\omega_{\mu, \psi^{-1}}(n)\Phi^m = \psi^{-1}_U(n)\Phi^m.
\]

2. For \( \hat{n} \in \hat{N}_m \), we have

\[
\omega_{\mu, \psi^{-1}}(\hat{n})\Phi^m = \Phi^m.
\]

Proof. (1) For \( n = (b) \in N_m \), we have

\[
\omega_{\psi^{-1}, \mu}(b)\Phi^m(x) = \psi^{-1}(xb^i\hat{x})\Phi^m(x).
\]

Write \( x = (x_1, x_2) \) and \( b = (b_{ij}) \); then

\[
xb^i\hat{x} = b_{11}x_1\hat{x}_1 + b_{12}x_1\hat{x}_2 + b_{21}x_2\hat{x}_1 + b_{22}x_2\hat{x}_2.
\]

For \( (x_1, x_2) \in \text{supp}(\Phi^m) \), we get \( x_1 \in \text{supp}(\Phi^m) \) and \( x_2 \in 1 + \text{supp}(\Phi^m) \). From \( n \in N_m \), we get \( b_{11} \in \mathcal{P}^{-5m}_E \), \( b_{12} = \hat{b}_{21} \in \mathcal{P}^{-3m}_E \) and \( b_{22} \in \mathcal{P}^{-m}_E \). Thus

\[
xb^i\hat{x} \equiv b_{22} \text{ mod } \mathcal{O}_F.
\]

Hence

\[
\omega_{\psi^{-1}, \mu}(b)\Phi^m = \psi^{-1}(b_{22})\Phi^m = \psi^{-1}_U(n)\Phi^m.
\]
Let $\Phi = \omega(w_0)\Phi^m$ temporarily. We have

$$
\Phi'(x_1, x_2) = \omega(w_0)\Phi^m(x_1, x_2) = \gamma_\psi \int_{E^2} \Phi^m(y)\psi^{-1}(-\text{tr}_{E/F}(x^t\bar{y})) dy = : \gamma_\psi^{-1}f_1(x_1)f_2(x_2),
$$

where

$$
f_1(x_1) = \int_{\mathfrak{p}_E^m} \psi(\text{tr}_{E/F}(\bar{y}_1x_1)) dy_1 = q_E^{-3m} \text{Char}_{E^0}(x_1),
$$

$$
f_2(x_2) = \int_{1 + \mathfrak{p}_E^m} \psi(\text{tr}(\bar{u}x_2)) du = q_E^{-m}\psi(\text{tr}(x_2)) \text{Char}_{E^0}(x_2).
$$

A similar argument as above shows that $\Phi'$ is fixed by $\mathfrak{n}(b)$, and thus

$$
\omega(\tilde{n})\Phi^m = \omega(w_0^{-1})\omega(n)\omega(w_0)\Phi^m = \omega(w_0^{-1})\Phi' = \omega(w_0^{-1}w_0)\Phi^m = \Phi^m.
$$

This completes the proof. \hfill \Box

**Lemma 5.2.** Let $m$ be a positive integer. For $a = (a_{21}, a_{22}) \in \text{GL}_2(E)$ with $|a_{21}| \leq q_E^m$ and $|a_{22}| \leq q_E^3$, we have

$$
\omega_{\mu, \psi^{-1}}(w_0)\Phi^m(e_2 a) = \gamma_\psi^{-1}q_E^{-m}\psi(\text{tr}_{E/F}(a_{22})).
$$

**Proof.** We have

$$
\omega_{\mu, \psi^{-1}}(w_0)\Phi^m(e_2 a) = \omega_{\mu, \psi^{-1}}(w_0)\Phi^m((a_{21}, a_{22}))
$$

$$
= \gamma_\psi^{-1} \int_{E^2} \Phi^m(y_1, y_2)\psi(\text{tr}(a_{21}\bar{y}_1 + a_{22}\bar{y}_2)) dy_1 dy_2
$$

$$
= \gamma_\psi^{-1} \int_{\mathfrak{p}_E^m} \psi(\text{tr}(a_{21}\bar{y}_1)) dy_1 \int_{1 + \mathfrak{p}_E^m} \psi(\text{tr}_{E/F}(a_{22}\bar{y}_2)) dy_2
$$

$$
= \gamma_\psi^{-1}q_E^{-m}\psi(\text{tr}_{E/F}(a_{22})).
$$

Here we used $\psi(\text{tr}(a_{21}\bar{y}_1)) = 1$ for $a_{21} \in \mathfrak{p}_E^{-3m}$ and $y_1 \in \mathfrak{p}_E^m$, and $\psi(\text{tr}_{E/F}(a_{22}\bar{y}_2)) = \psi(\text{tr}_{E/F}(a_{22}))$ for $a_{22} \in \mathfrak{p}_E^{-m}$ and $y_2 \in 1 + \mathfrak{p}_E^m$. \hfill \Box

### 5.2 Twisting by GL$_2$

Before we go to the proof of the local converse theorem, we recall the following result of Jacquet–Shalika.

Let $\phi$ be a smooth complex valued function on $GL_2(E)$ such that $\phi(ug) = \psi_E^{-1}(u)\phi(g)$ for all $u \in N^{(2)}$ and $g \in GL_2(E)$.

**Proposition 5.3 (Jacquet–Shalika).** Suppose that for each integer $m$, the set $g \in GL_2(E)$ such that $|\text{det}(g)| = q_E^m$ and $\phi(g) \neq 0$ is contained in a compact set modulo $N^{(2)}$. Let

$$
A(\phi, W^{(2)}, s) = \int_{N^{(2)} \cdot GL_2(E)} \phi(a)W^{(2)}(a)|\text{det}(a)|^s da.
$$

For every irreducible generic representation $\tau$ and every $W^{(2)} \in W(\tau, \psi_E)$, assume that the integral $A(\phi, W^{(2)}, s)$ converges absolutely in some half plane which might depend on $W^{(2)}$. If $A(\phi, W^{(2)}, s) = 0$ for all irreducible generic representation $\tau$, all $W^{(2)} \in W(\tau, \psi_E)$ and all $s$ when it is absolutely convergent, then $\phi(a) \equiv 0$.

This is a corollary of [21, Lemma 3.2]. For an argument that [21, Lemma 3.2] implies the present form of Proposition 5.3, one can see [7, Corollary 2.1], for example.
We are back to the notation of Section 2. Let us repeat part of them to avoid ambiguity. We are given an unramified character $\psi$ of $F$ and an unramified character $\psi_E$ of $E$ such that $\psi_E|_F$ is trivial. From them, we defined a character $\psi_U$ of $U$ such that $\psi_U|_{U_0} = \psi^{-1}_E$ and $\psi_U|_{U_0'} = \psi$. We are given two $\psi_U$-generic representations $\pi$ and $\pi'$ of $G = U(2, 2)$ such that $\omega_\pi = \omega_{\pi'}$. For a vector $v \in V_\pi$ with $W_v(1) = 1$, and an integer $m > 0$, we defined the Howe vector $v_m$. Similarly, we have $v' \in V_{\pi'}$ and $v'_m$. We fixed an integer $C$ such that $v$ (resp. $v'$) is fixed by $K_C$ under $\pi$ (resp. $\pi'$).

**Lemma 5.4.** The following statements hold.

1. Let $a = (a_{11}, a_{12}, a_{21}, a_{22}) \in \text{GL}_2(E)$. If $W_v(m(a)w_1) \neq 0$, then $|a_{22}|_E \leq q_F^{7m/2}$ and $|a_{21}|_E \leq q_F^{3m} = q_E^{3m/2}$.
2. For each $k$, the set $X_k = \{ a \in \text{GL}_2(E) : |\det(a)| = q_E^k, W_v(m(a)w_1) \neq 0 \}$ is compact modulo $N^{(2)}$.

**Proof.** (1) Given $r \in \mathfrak{g}_F^m$, we have

\[
m(a)w_1x_{-2a \cdot \beta}(r) = m(a)x_\beta(r)w_1 = n(b)m(a)w_1,
\]

where

\[
b = \begin{pmatrix} a_{12}a_{11}r & a_{12}a_{22}r \\ a_{22}a_{11}r & a_{22}a_{22}r \end{pmatrix}.
\]

Since $x_{-2a \cdot \beta}(r) \in I_m$, we get

\[W_v(m(a)w_1) = \psi(a_{22}a_{22}r)W_v(m(a)w_1).
\]

If $W_v(m(a)w_1) \neq 0$, we get $\psi(a_{22}a_{22}r) = 1$ for all $r \in \mathfrak{g}_F^m$. Thus $a_{22}a_{22} \in \mathfrak{g}_F^{7m}$. Hence

\[|a_{22}|_E = |a_{22}a_{22}|_F \leq q_F^{7m/2} = q_E^{7m/2}.
\]

For the second part, we take $r \in \mathfrak{g}_F^{3m}$, we have the relation

\[
m(a)w_1x_{-\beta}(r) = m(a)x_{2a + \beta}(r)w_1 = n(b)m(a)w_1
\]

with

\[
b = \begin{pmatrix} a_{11}a_{11}r & a_{11}a_{21}r \\ a_{21}a_{11}r & a_{21}a_{21}r \end{pmatrix}.
\]

Since $x_{-\beta}(r) \in I_m$, we have

\[W_v(m(a)w_1) = \psi(a_{21}a_{21}r)W_v(m(a)w_0).
\]

If $W_v(m(a)w_1) \neq 0$, we get $\psi(a_{21}a_{21}r) = 1$ for all $r \in \mathfrak{g}_F^{3m}$, thus $a_{21}a_{21} \in \mathfrak{g}_F^{3m}$. Hence

\[|a_{21}|_E = |a_{21}a_{21}|_F \leq q_F^{3m} = q_E^{3m/2}.
\]

(2) From the Iwasawa decomposition of $GL_2$, it suffices to show that the set

\[Y_k := \{ \text{diag}(a_1, a_2) \in GL_2(F) : |a_1a_2| = q_E^k, W_v(m(\text{diag}(a_1, a_2))w_1) \neq 0 \}
\]

is compact. Take $r \in \mathcal{O}_F$, we have $x_a(-r) \in H_m$ and $\psi_U(x_a(r)) = 1$. Write $t = t(a_1, a_2)$. From the relation

\[t \overline{w}_1x_a(-r) = t x_a(r)w_1 = x_a(a_1r/a_2)t \overline{w}_1
\]

we get $\psi_U(x_a(a_1r/a_2)W_v(t\overline{w}_1) = W_v(t\overline{w}_1)$. Thus if $W_v(t\overline{w}_1) \neq 0$, we get $a_1/a_2 \in \mathcal{O}_F$. By (1), we get

\[Y_k \subset \{ \text{diag}(a_1, a_2) : |a_1a_2| = q_E^k, a_1/a_2 \in \mathcal{O}_F, a_2 \in \mathfrak{g}_F^{-7m} \}.
\]

Given $\text{diag}(a_1, a_2) \in Y_k$, we obtain $|a_2|_E \leq q_F^m$ and $|a_1|_E = |a_2|^{-1}_E q_F^{k} \geq q_F^{-k - 7m}$. Since $a_1/a_2 \in \mathcal{O}_F$, it follows that $|a_1|_E \leq |a_2|_E$. Then we have

\[q_F^{2k - 7m} \leq |a_1|_E \leq |a_2|_E \leq q_F^{7m}.
\]

Thus $Y_k$ is compact. \qed
Proposition 5.5. Let $\mu$ be a fixed character such that $\mu|_{E^c} = e_{E/F}$. Suppose that
1. $\gamma(s, \pi \times \mu \eta, \psi) = \gamma(s, \pi' \times \mu \eta, \psi)$ for all quasi-character $\eta$ of $E^c$,
2. $\gamma(s, \pi \times \mu \tau, \psi) = \gamma(s, \pi' \times \mu \tau, \psi)$ for all irreducible representations $\tau$ of $GL_2(E)$.

Then we have
$$W_v(tw) = W_{v'}(tw)$$
for all $t \in T$, $m \geq 4^3 C$ and $w = w_1$ or $w = w_0$.

Proof. The proof is quite similar to the proof of Proposition 4.1.

Let $m = 4^3 C$ and let $k = \frac{3m}{2} > m$. We have defined $\Phi^k \in \mathcal{H}(E^2)$. For simplicity, we will write $\omega_{\mu, \psi \cdot 1}$ as $\omega$.

For a vector $\epsilon \in V_r$, we take an integer $i$ such that $i \geq \max\{k = 3m/2, i_2(\epsilon), I(N_m, \epsilon)\}$. See Lemma 3.2 and Proposition 3.3 for the definition of $i_2(\epsilon)$ and $I(N_m, \epsilon)$. By Lemma 3.2, we have defined a section $\xi_{i\epsilon}^i \in I(s, \tau)$ in Section 3.

Let $W = W_{v_m}$ or $W_{v_m}$. We will compute the integral $\Psi(W, \xi_{i\epsilon}^i, \Phi^k)$. This integral is defined over $U \setminus G$ and we will take the integral on the open dense subset $U \setminus NM\tilde{N} = N^{(2)} \setminus GL_2(E) \times \tilde{N}$. For $g = n\tilde{m}(a)\tilde{n}$, the Haar measure can be taken as $dg = |\det(a)|^{1/2} d\tilde{m} \tilde{n}$. By the definition of $\xi_{i\epsilon}^i$ and equation (3.3), for $Re(s) > 0$, we have
$$\Psi(W, \xi_{i\epsilon}^i, \Phi^k) = \int_{U \setminus G} W(g)f_{\xi_{i\epsilon}^i}(g)(\omega(g))\Phi^k(\epsilon_2) \, dg$$
$$= \int_{N^{(2)} \setminus GL_2(E) \times \tilde{N}} W(\tilde{m}(a)\tilde{n})f_{\xi_{i\epsilon}^i}\tilde{(\tilde{m}(a)\tilde{n})}\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2)|\det(a)|^{-2} \, d\tilde{m} \tilde{n}$$
$$= \int_{N^{(2)} \setminus GL_2(E) \times \tilde{N}} W(\tilde{m}(a)\tilde{n})|\det(a)|^{s-3/2} W_e^{(2)}(a)\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2) \, d\tilde{m} \tilde{n}.$$

Since $i \geq k > m$, it follows that $\tilde{N}_i \subset \tilde{N}_k \subset \tilde{N}_m$. Thus for $\tilde{n} \in \tilde{N}_i$, we have $W(\tilde{m}(a)\tilde{n}) = W(\tilde{m})$ and $\omega(\tilde{n})\Phi^k = \Phi^k$ by Lemma 2.1 and Proposition 5.1. Thus we have
$$\Psi(W, \xi_{i\epsilon}^i, \Phi^k) = \rm{vol}(\tilde{N}_i) \int_{E \setminus N^{(2)} \setminus GL_2(E)} W(\tilde{m}(a)) W_e^{(2)}(a)|\det(a)|^{s-1} \mu(\det(a))\Phi^k(\epsilon_2) \, da,$$
where $E^1$ embeds into $T^{(2)}_r \setminus GL_2(E)$ diagonally.

Since $M \subset B \cup Bs_uB$, and we have showed that $W_{v_m}(g) = W_{v_m}(g)$ for $g \in B \cup Bs_uB$ in Corollary 2.6, we get
$$\Psi(W_{v_m}, \xi_{i\epsilon}^i, \Phi^k) = \Psi(W_{v_m}, \xi_{i\epsilon}^i, \Phi^k). \quad (5.1)$$

Next, we compute the integral $\Psi(W, \xi_{i\epsilon}^i, \Phi^k)$ for $i \geq 1, s = M(s)(\xi_{i\epsilon}^i)$, we will take this integral on the open dense set $U \setminus NMw_1U \setminus U \setminus G$. We have
$$\Psi(W, \xi_{i\epsilon}^i, \Phi^k) = \int_{N^{(2)} \setminus GL_2(E) \times N} W(\tilde{m}(a)\tilde{n})f_{\xi_{i\epsilon}^i}\tilde{(\tilde{m}(a)\tilde{n})}(\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2)|\det(a)|^{-2} \, da \, dn$$
$$= \int_{N^{(2)} \setminus GL_2(E) \times N} W(\tilde{m}(a)\tilde{n})f_{\xi_{i\epsilon}^i}\tilde{(\tilde{m}(a)\tilde{n})}(\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2)|\det(a)|^{-2} \, dn$$
$$+ \int_{N^{(2)} \setminus GL_2(E) \times (N-N_m)} W(\tilde{m}(a)\tilde{n})f_{\xi_{i\epsilon}^i}\tilde{(\tilde{m}(a)\tilde{n})}(\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2)|\det(a)|^{-2} \, dn$$
$$=: I + II.$$

Since $i \geq I(N_m, \epsilon)$, we have
$$f_{\xi_{i\epsilon}^i}(\tilde{m}(a)\tilde{n}) = \rm{vol}(\tilde{N}_i)|\det(a)|^{3/2-5} \tilde{W}_e^{(2)}(a), \quad i \geq I(N_m, \epsilon), \quad n \in N_m,$$
see Proposition 3.3 and equation (3.4).

For $n \in N_m \subset N_i$, by Lemma 2.1 and Proposition 5.1 (2), we have
$$W(\tilde{m}(a)\tilde{n}) = \psi_U(n) W(\tilde{m}(a)\tilde{n}), (\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2) = \psi_U^{-1}(n) (\omega(\tilde{m}(a)\tilde{n})\Phi^k(\epsilon_2).$$
Thus the term I becomes

$$\text{vol}(\hat{N}) \text{ vol}(N_m) \int_{N(2) \backslash \text{GL}_2(E)} W(m(a)\hat{w}_1)\hat{W}_e(a)|\text{det}(a)|^{-s-1/2}(\omega(m(a)\hat{w}_1)\Phi^k)(e_2) \; da.$$ 

By assumption (1) and Proposition 4.2, we get $W_{v_m}(m(a)\hat{w}_1 n) = W_{v_m}(m(a)\hat{w}_1 n)$ for $n \in N - N_m$. Thus the term II for $W = W_{v_m}$ and for $W = W_{v_m}$ are the same. For $\text{Re}(s) \ll 0$, we then get

$$\Psi(W_{v_m}, \xi_m, \Phi^k) - \Psi(W_{v_m}, \xi_m, \Phi^k) = \text{vol}(\hat{N}) \text{ vol}(N_m) \int_{E \backslash N(2) \backslash \text{GL}_2(E)} (W_{v_m}(m(a)\hat{w}_1) - W_{v_m}(m(a)\hat{w}_1)) \cdot (\omega(m(a)\hat{w}_1)\Phi^k)(e_2)\hat{W}_e(a)|\text{det}(a)|^{-s-1/2} \; da.$$ 

By equation (5.1), equation (5.2) and the local functional equation, we get

$$d_m(\gamma(s, \pi, \omega, \psi^{-1}, \chi), r) - \gamma(s, \pi', \omega, \psi^{-1}, \chi, r) = \int_{N(2) \backslash \text{GL}_2(E)} (W_{v_m}(m(a)\hat{w}_1) - W_{v_m}(m(a)\hat{w}_1)) (\omega(m(a)\hat{w}_1)\Phi^k)(e_2)\hat{W}_e(a)|\text{det}(a)|^{-s-1/2} \; da,$$

where $d_m = \Psi(W, \xi_m, \Phi^k) \text{ vol}(N_m)^{-1} \text{ vol}(\hat{N})^{-1}$, which is independent of $i$.

By our assumption on $\gamma$-factors, we have

$$\int_{N(2) \backslash \text{GL}_2(E)} (W_{v_m}(m(a)\hat{w}_1) - W_{v_m}(m(a)\hat{w}_1)) (\omega(m(a)\hat{w}_1)\Phi^k)(e_2)\hat{W}_e(a)|\text{det}(a)|^{-s-1/2} \; da \equiv 0.$$ 

By Proposition 5.3 and Lemma 5.4 (2), we get

$$(W_{v_m}(m(a)\hat{w}_1) - W_{v_m}(m(a)\hat{w}_1)) (\omega(m(a)\hat{w}_1)\Phi^k)(e_2)\hat{W}_e(a)|\text{det}(a)|^{-s-1/2} \; da = 0. \quad (5.3)$$

For $t = t(a_1, a_2) \in T$, we have

$$t(a_1, a_2)\hat{w}_1 = m\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\hat{w}_0.$$ 

By Lemma 5.2, for $|a_2| \leq q_E^{\frac{3k}{4}}$, we have $(\omega(t\hat{w}_1)\Phi^k)(e_2) \neq 0$. By equation (5.3), we get

$$W_{v_m}(t\hat{w}_1) = W_{v_m'}(t\hat{w}_1) \quad \text{for all } t = t(a_1, a_2) \text{ with } |a_2| \leq q_E^{\frac{3k}{4}}. \quad (5.4)$$

From Lemma 5.4 (1), we get

$$W_{v_m}(t\hat{w}_1) = W_{v_m'}(t\hat{w}_1) = 0 \quad \text{if } |a_2| > q_E^{\frac{7m}{2}}. \quad (5.5)$$

Since $3k > 7m/2$, by equation (5.4) and equation (5.5), we get

$$W_{v_m}(t\hat{w}_1) = W_{v_m'}(t\hat{w}_1) \quad \text{for all } t \in T.$$

On the other hand, we have

$$t(a_1, a_2)\hat{w}_0 = m(a')\hat{w}_1 \quad \text{with } a' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \text{GL}_2(E).$$

By Lemma 5.2, for $|a_2| \leq q_E^{\frac{k}{4}}$, we have

$$(\omega(t(a_1, a_2)\hat{w}_0)\Phi^k)(e_2) \neq 0.$$ 

By equation (5.3), we get

$$W_{v_m}(t(a_1, a_2)\hat{w}_0) = W_{v_m}(m(a')\hat{w}_1) = W_{v_m}(m(a')\hat{w}_1) = W_{v_m}(t(a_1, a_2)\hat{w}_0), \quad \text{if } |a_2| \leq q_E^{\frac{k}{4}}. \quad (5.6)$$
By Lemma 5.4 (1), we have
\[ W_{v_n}(t(a_1, a_2)w_0) = 0 = W_{v_n}(t(a_1, a_2)w_0) \text{ if } |a_2|_E > q_E^{3|m|/2} = q_E^{3}. \tag{5.7} \]
From equation (5.6) and equation (5.7), we get
\[ W_{v_n}(tw_0) = W_{v_n}(tw_0) \text{ for all } t \in T. \]
Thus we proved \( W_{v_n}(tw_1) = W_{v_n}(tw_1) \) and \( W_{v_n}(tw_0) = W_{v_n}(tw_0) \) for all \( t \in T \) and \( m = 4^9L \). The same is true for \( m \geq 4^9L \) by Lemma 2.1 (3) and Proposition 2.5. This finishes the proof. \( \square \)

**Theorem 5.6** (Local Converse Theorem for \( U_{E/F}(2, 2) \) when \( E/F \) is unramified). Assume that \( E/F \) is unramified. Let \( \pi \) and \( \pi' \) be two \( \psi_U \)-generic irreducible smooth representations of \( U_{E/F}(2, 2) \) with the same central character. If
\[ \gamma(s, \pi \times (\mu \eta), \psi_U) = \gamma(s, \pi' \times (\mu \eta), \psi_U) \text{ and } \gamma(s, \pi \times (\mu t), \psi_U) = \gamma(s, \pi' \times (\mu t), \psi_U) \]
for all quasi-character \( \eta \) of \( E^* \) and all irreducible generic smooth representation \( \tau \) of \( GL_2(E) \), then \( \pi \equiv \pi' \).

**Proof.** If \( \psi_U \) is unramified, by Proposition 5.5 and Proposition 2.5, we get
\[ W_{v_n}(g) = W_{v_n}(g) \]
for all \( g \in G \). Thus \( \pi \equiv \pi' \) by the uniqueness of the Whittaker functional. If \( \psi_U \) is not unramified, we only have to modify the above proof a little bit. \( \square \)

### 6 A new local zeta integral for \( U(2, 2) \times Res_{E/F}(GL(1)) \)

The local zeta integrals for \( U(2, 2) \times Res_{E/F}(GL(1)) \) considered in Section 1 are considered in [4]. These local integrals come from global zeta integrals.

In this section, we consider a new local zeta integral for \( U(2, 2) \times Res_{E/F}(GL(1)) \) in the generic case and prove the local functional equation and hence the existence of the \( \gamma \)-factors. We also prove that this new gamma factor can be used to obtain the local converse theorem. The advantage of the new local zeta integral is that it does not involve the Weil representation so that it is simpler than the old one. But it is not clear if the new local zeta integral comes from a global zeta integral.

#### 6.1 A new local zeta integral for \( U(2, 2) \times Res_{E/F}(GL(1)) \) and the local functional equation

Let \( F \) be a \( p \)-adic field and let \( E/F \) be a quadratic field extension. Let \( G = U(2, 2)(F) \) and let \( \pi \) be an irreducible admissible smooth representation of \( G \). Let \( \psi \) (resp. \( \psi_E \)) be a fixed nontrivial additive character of \( F \) (resp. \( E \)). We require that \( \psi_{E/F} \) is trivial. We define a generic character \( \psi_U \) on \( U \) as before, i.e., \( \psi_U|U_n = \psi_E^{-1} \) and \( \psi_U|U_{p} = \psi \).

We assume \( \pi \) is \( \psi_U \)-generic. Let \( \mathcal{W}(\pi, \psi) \) be the set of Whittaker functions of \( \pi \). For \( W \in \mathcal{W}(\pi, \psi) \), and a quasi-character \( \eta \) of \( E^* \), we consider the integral
\[ \Psi(s, W, \eta) = \int_{E^*} W(t(a))\eta(a)|a|_E^{s-3/2} \, da. \]
Recall that \( t(a) = m(\text{diag}(a, 1)) \).

We consider the integral
\[ \overline{\Psi}(1-s, W, \eta^*) = \int_{E^*} \int_{E^*} W(t(a)\omega_2(x_1+\beta(x_2+\beta(y))) \, dx \, dy \, \eta^*(a)|a|^{-s-1/2} \, da, \]
where \( \eta^*(a) = \eta(a^{-1}) \).
Lemma 6.1. The integrals \( \Psi(s, W, \eta) \) is convergent when \( \text{Re}(s) \gg 0 \) and defines a meromorphic function of \( q_E^{-s} \). Similarly the integral \( \overline{\Psi}(1 - s, W, \eta^*) \) is convergent when \( \text{Re}(s) \ll 0 \) and also defines a meromorphic function of \( q_E^{-s} \).

Proof. Let \( W \) be a Whittaker function of \( \pi \). By [4, Lemma 4.3], we have
(1) the function \( a \mapsto W(a) \) on \( E^\times \) is supported on a open compact subset \( \Omega \subset E \).
On the other hand, as [4, Lemma 4.1], one can show that
(2) for \( a \in E^\times \), the function \( (x, y) \mapsto W(t(a)w_2x_{a^s}\beta(x)x_{2a^s}\beta(y)) \) has compact support which is independent of \( a \).

From (1), we can check that the local zeta integral \( \Psi(s, W, \eta) \) is absolutely convergent for \( \text{Re}(s) \geq 0 \) and defines a meromorphic function of \( q_E^{-s} \). From (1) and (2), we can check that the integral \( \overline{\Psi}(1 - s, W, \eta^*) \) is absolutely convergent for \( \text{Re}(s) \leq 0 \) and defines a meromorphic function of \( q_E^{-s} \). See [4, Section 4] for more explanations in similar situations.

We next consider the local functional equation of these kind integrals.

Consider the linear form \( W \mapsto B(W) = \overline{\Psi}(1 - s, W, \eta^*) \) on \( \mathcal{W}(\pi, \psi_U) \).

Lemma 6.2. Let \( \psi_N \) be the character of \( N \) defined by \( \psi_N = \psi_U|_N \). Then we have
\[
B(\pi(n)W) = \psi_N(n)B(W), \quad n \in N,
\]
and
\[
B(\pi(t(a))W) = \eta^{-1}(a)|a|^{3/2 - s}B(W), \quad a \in E^\times.
\]

Proof. This follows from a straightforward matrix calculation, and we omit the details.

Proposition 6.3. Except for finite number of \( q^{-s} \), up to a scalar there is at most one linear functional \( A \) on \( \mathcal{W}(\pi, \psi_U) \) such that
\[
A(\pi(n)W) = \psi_N(n)A(W)
\]
and
\[
A(\pi(t(a))W) = \eta^{-1}(a)|a|^{-s+3/2}A(W).
\]

Proof. The first condition says that \( A \in \text{Hom}_{\mathcal{C}}(\pi_N, \psi_N) \). The twisted Jacquet module \( \pi_N, \psi_N \) defines a representation of the mirabolic subgroup \( P_2 \subset \text{GL}_2 \), where
\[ P_2 = \left\{ \begin{pmatrix} a & x \\ 1 & 1 \end{pmatrix} \right\} \subset \text{GL}_2(E) \]
The second condition says that
\[
A \in \text{Hom}_{\overline{T}(\mathcal{C})}(\pi_N, \psi_N), \quad \eta^{-1} \cdot |^{-s+3/2} \]
where \( \overline{T}(\mathcal{C}) = \{ \text{diag}(a, 1) \} \subset \text{GL}_2(E) \). As before, let \( N^{(2)} \) be the unipotent subgroup of \( P_2 \). As a representation of \( P_2 \), we have the exact sequence
\[
0 \rightarrow \text{ind}_{N^{(2)}}^{P_2}(\pi_N, \psi_N) \rightarrow \pi_{N^{(2)}}, \psi_E \rightarrow \pi_N, \psi_N \rightarrow 0.
\]
Note that \( (\pi_N, \psi_N)|_{N^{(2)}} \) has dimension 1 by the uniqueness of Whittaker functionals, and \( (\pi_N, \psi_N)|_{N^{(2)}} \) has finite dimension by the following proposition, Proposition 6.4. The above sequence can be written as
\[
0 \rightarrow \text{ind}_{\tilde{N}}^{P_2}(\psi_E) \rightarrow \pi_{N^{(2)}}, \psi_E \rightarrow \pi_N, \psi_N \rightarrow 0.
\] (6.1)

In the proof of the local functional equation for local zeta integral of \( GL_2(E) \) in [19], Jacquet and Langlands showed that
\[
\text{Hom}_{\overline{T}(\mathcal{C})}(\text{ind}_{\tilde{N}}^{P_2}(\psi), \eta^{-1} \cdot |^{-s+3/2})
\]
has dimension 1. Since \( (\pi_N, \psi_N)|_{N^{(2)}} \) has finite number of \( q^{-s} \), we have \( \text{dim} \text{Hom}_{\overline{T}(\mathcal{C})}(\pi_N, \psi_N), \eta^{-1} \cdot |^{-s+3/2}) \leq 1 \) by the exact sequence (6.1).
Proposition 6.4 (Kazhdan). Let \((\pi, V)\) be an irreducible smooth representation of \(G\), and let \(\theta\) be the character on \(U\) defined by
\[
\theta \begin{pmatrix} 1 & x \\ 1 & \tilde{x} \end{pmatrix} \begin{pmatrix} 1 & b_{11} & b_{12} \\ 1 & b_{12} & b_{22} \\ -\tilde{x} & 1 & 1 \end{pmatrix} = \psi(b_{22}).
\]
Then the twisted Jacquet module \(V_{U, \theta}\) has finite dimension.

In the \(GL_n\) case, this is a theorem of Kazhdan, [5, Theorem 5.21]. The proof in our case is similar and we omit the details.

**Corollary 6.5.** There is a meromorphic function \(y'(s, \pi, \eta)\) such that
\[
\Psi(1-s, \pi, \eta^*) = y'(s, \pi \times \eta, \psi_U)\Psi(s, W, \eta) \quad \text{for all} \ W \in \mathcal{W}(\pi, \psi_U).
\]

**Proof.** Consider the linear functional \(W \mapsto A(W) = \Psi(s, W, \eta)\) on \(\mathcal{W}(\pi, \psi_U)\). It is clear that
\[
A(\pi(n) W) = \psi_N(n) A(W), \quad n \in N,
\]
and
\[
A(\pi(t(a)) W) = \eta^{-1}(a)|a|^{3s-5} A(W).
\]
Now the assertion follows from Lemma 6.2 and Proposition 6.3.

### 6.2 Proof of the local converse theorem using the new \(\gamma\)-factor

Now we are going to prove the local converse theorem using the new gamma factor \(y'(s, \pi \times \eta, \psi)\). We recall our notations from Section 2. We assume that \(E/F\) is unramified and we are given two \(\psi_U\)-generic representation \(\pi, \pi'\) of \(G = U(2, 2)(F)\) with the same central character. For \(v \in \pi\) (resp \(v' \in \pi'\)) with \(W_v(1)\) (resp \(W_{v'}(1)\) = 1), we have defined Howe vectors \(v_m\) (resp \(v'_m\)) for positive integers \(m\). Let \(C\) be an integer such that \(v\) and \(v'\) are both fixed by \(K_C\).

**Proposition 6.6.** If \(y'(s, \pi \times \eta, \psi_U) = y'(s, \pi' \times \eta, \psi_U)\) for all quasi-characters \(\eta\) of \(F^*\), then
\[
W_{v_m}(t \bar{w}_2) = W_{v'_m}(t \bar{w}_2)
\]
for \(t \in T\), and all \(m \geq 4^6 C\).

**Proof.** Let \(m \geq 4^6 C\). By Corollary 2.3, we have \(W_{v_m}(t(a)) = 0\) for all \(a \notin 1 + J^*_E\). By Lemma 2.1, we have \(W_{v_m}(t(a)) = 1\) for all \(a \in 1 + J^*_E\). Thus we have
\[
\Psi(s, W_{v_m}, \eta) = \int_{E^*} W(t(a) \bar{w}_2 x_{a+\beta}(x) x_{2a+\beta}(y)) \, dx \, dy \, \eta^*(a) |a|^{-s-1/2} \, da = \int_{1 + J^*_E} \eta(a) \, da.
\]
The same calculation works for \(W_{v'_m}\). Thus we have
\[
\Psi(s, W_{v_m}, \eta) = \Psi(s, W_{v'_m}, \eta).
\]
Let \(W = W_{v_m}\) or \(W_{v'_m}\). We have
\[
\Psi(1-s, W, \eta^*) = \int_{E^*} \int_{E^*} W(t(a) \bar{w}_2 x_{a+\beta}(x) x_{2a+\beta}(y)) \, dx \, dy \, \eta^*(a) |a|^{-s-1/2} \, da
\]
\[
= \int_{E^*} \int_{\mathcal{H}^*_E} W(t(a) \bar{w}_2 x_{a+\beta}(x) x_{2a+\beta}(y)) \, dx \, dy \, \eta^*(a) |a|^{-s-1/2} \, da
\]
\[
+ \int_{E^*} \int_{\mathcal{H}^*_E} W(t(a) \bar{w}_2 x_{a+\beta}(x) x_{2a+\beta}(y)) \, dx \, dy \, \eta^*(a) |a|^{-s-1/2} \, da
\]
\[
=: I + II.
\]
For $x \in \mathcal{O}_E^{-3m}$, $y \in \mathcal{O}_F^{-5m}$, we have $x_{a_2}{(x)}x_{2a_2}{(y)} \in H_m$, and thus

$$W(t(a)\bar{w}_2x_{a_2}{(x)}x_{2a_2}{(y)}) = W(t(a)\bar{w}_2),$$

by Lemma 2.1 or equation (2.1). Then the term I becomes

$$\text{vol}(\mathcal{O}_E^{-3m})\text{vol}(\mathcal{O}_F^{-5m}) \int_{E^x} W(t(a)\bar{w}_2)\eta^*(a)|a|^{-s-1/2} \, da.$$ 

By Corollary 2.6, we have

$$W_{\nu_m}(t(a)\bar{w}_2x_{a_2}{(x)}x_{2a_2}{(y)}) = W_{\nu_m}(t(a)\bar{w}_2x_{a_2}{(x)}x_{2a_2}{(y)})$$

for $x \not\in \mathcal{O}_E^{-3m}$ or $y \not\in \mathcal{O}_F^{-5m}$. Thus the term II for $W = W_{\nu_m}$ and for $W = W_{\nu_m}$ are the same. Thus

$$\bar{\Psi}(1 - s, W_{\nu_m}, \eta^*) - \bar{\Psi}(1 - s, W_{\nu_m}, \eta^*) = \text{vol}(\mathcal{O}_E^{-3m})\text{vol}(\mathcal{O}_F^{-5m}) \int_{E^x} (W_{\nu_m}(t(a)\bar{w}_2) - W_{\nu_m}(t(a)\bar{w}_2))\eta^*(a)|a|^{-s-1/2} \, da. \quad (6.3)$$

By equation (6.2), equation (6.3) and the local functional equation, i.e., Corollary 6.5, we have

$$d_m(y'(s, \pi \times \eta, \psi_U) - y'(s, \pi' \times \eta, \psi_U)) \int_{E^x} (W_{\nu_m}(t(a)\bar{w}_2) - W_{\nu_m}(t(a)\bar{w}_2))\eta^*(a)|a|^{-s-1/2} \, da,$$

where $d_m = \Psi(s, W_{\nu_m}, \eta)$. Thus by the assumption, we get

$$\int_{E^x} (W_{\nu_m}(t(a)\bar{w}_2) - W_{\nu_m}(t(a)\bar{w}_2))\eta^*(a)|a|^{-s-1/2} \, da \equiv 0,$$

which implies that $W_{\nu_m}(t(a)\bar{w}_2) = W_{\nu_m}(t(a)\bar{w}_2)$. The rest of the proof is the same as the proof of Proposition 4.1.

From the proof of Theorem 5.6, we see that it can be restated as follows:

**Theorem 6.7** (Local Converse Theorem for generic representations of unramified $\text{U}(2, 2)$). Suppose that $E/F$ is unramified. Let $\pi$ and $\pi'$ be two $\psi_U$-generic irreducible representations of $G$ with the same central character. Suppose that

$$y'(s, \pi \times \eta, \psi_U) = y'(s, \pi' \times \eta, \psi_U), \quad y(s, \pi \times \tau, \psi_U) = y(s, \pi \times \tau, \psi_U)$$

for all quasi-characters $\eta$ of $E^x$ and all irreducible generic representations $\tau$ of $\text{GL}_2(E)$. Then $\pi \equiv \pi'$.

## 7 Concluding remarks

### 7.1 When $E/F$ is ramified

In this subsection, we give a brief account on how to modify the method we used in previous sections for unramified $E/F$ so that it adapts when $E/F$ is ramified and the residue characteristic of $F$ is odd. We also explain the technical difficulty of our approach when $E/F$ is ramified and $p = 2$. See [29] for similar situation in the $\text{U}(1, 1)$ case.

We assume that the field extension $E/F$ is ramified. Let $p$ be the characteristic of the residue field of $F$. Let $\mathcal{O}_E$ (resp. $\mathcal{O}_F$) be the maximal ideal of $\mathcal{O}_F$ (resp. $\mathcal{O}_F$), and let $p_F$ (resp. $p_F$) be a prime element of $E$ (resp. $F$). Then we have $\mathcal{O}_E = \mathcal{O}_F$ and $p_F = p_{\mathcal{O}_F}^d$ for some $u \in \mathcal{O}_F$.

Let $\mathcal{O}_E/F = \mathcal{O}_E^d$ be the different of $E/F$ for some positive integer $d$.

Let $x \in \mathcal{O}_E$ be an element such that $\mathcal{O}_E = \mathcal{O}_F[x]$. Let $v_E$ be the discrete valuation of $E$. Define $v_E = v_E(x - x)$. The integer $i_{E/F}$ is independent of choice of $E/F$. It is known that $i_{E/F} = 1$ if $p \neq 2$, and $i_{E/F} \geq 2$ if $p = 2$, see [25, Chapter IV].
Let $\psi_U$ be the unramified generic character of $U$ as in Section 2. In the ramified case, we need to change the definition of Howe vectors a little bit. The reason is that $\mathcal{P}_E \cap F \neq \mathcal{P}_F$. To remedy this, we basically need to replace $\mathcal{P}_E$ in Section 3 by $\mathcal{P}_F \cdot \mathcal{O}_E = \mathcal{P}_E^2$. We give some details on this. We define $J_m = d_m K_{2m} d_m^{-1}$, where $d_m = t(p_{2m}, p_{3m})$ and $K_{2m} = (1 + \text{Mat}_{s_{44}}(p_{2m}))$ are defined in Section 3. Define the character $\psi_m$ on $J_m$ in the same way as in Section 3. We still have $\psi_m|_{U_m} = \psi|_{U_{2m}}$.

Let $(\pi, V)$ be a $\psi_U$-generic representation of $G = U(2, 2)$. Starting from a vector $v \in V$, we define the Howe vectors $v_m$ in the same way. Let $C$ be an integer such that $v$ is fixed by $K_{2C}$, then Lemma 2.1 still holds.

Part (1) of Lemma 2.2 should be modified to:

**Lemma 7.1.** Let $t = t(a_1, a_2) \in T$. If $W_{v_m}(t) \neq 0$, then $a_1/a_2 \in 1 + \mathcal{P}_E^{2m}$ and $a_2 \in 1 + \mathcal{P}_F^m$.

Now assume that $(\pi, V)$ and $(\pi', V')$ are two $\psi_U$-generic representations of $G = U_{E/F}(2, 2)$ with the same central character as in Section 2. We take $v \in V$, $v' \in V'$ and define Howe vectors $v_m, v'_m$. Let $C$ be an integer such that $v, v'$ are fixed by $K_{2C}$.

**Lemma 7.2.** If the residue characteristic $p$ of $F$ is not 2, then $W_{v_m}(t) = W_{v'_m}(t)$ for all $t \in T$.

**Proof.** Let $t = t(a_1, a_2) \in T$ with $a_1, a_2 \in E^\times$. If $W_{v_m}(t) \neq 0$, then $a_1/a_2 \in 1 + \mathcal{P}_E^{2m}$ and $a_2 \in 1 + \mathcal{P}_F^m$ by Lemma 7.1. If $p \neq 2$, we have $\text{Nm}_{E/F}(1 + \mathcal{P}_E^{2m}) = 1 + \mathcal{P}_F^m$ by [25, Corollary 3, Section 3, Chapter V]. From this, we get $a_2 \in E^1(1 + \mathcal{P}_E^{2m})$. Notice that $t(a_1, a_2) \in H_m$ for $a_1/a_2 = 1 + \mathcal{P}_E^{2m}$, the following proof is the same as in the unramified case.

**Remark 7.3.** If $p = 2$, then [25, Corollary 3, Section 3, Chapter V] says that

$$\text{Nm}_{E/F}(1 + \mathcal{P}_E^{2m-1}) = 1 + \mathcal{P}_E^m$$

for $m \geq i_{E/F}$. Thus from $a_2 \in 1 + \mathcal{P}_E^m$, we only get $a_2 \in E^1(1 + \mathcal{P}_E^{2m-1})$ for $m \geq i_{E/F}$. Note that

$$T \cap H_m = t(1 + \mathcal{P}_E^{2m}, 1 + \mathcal{P}_E^{2m})$$

and thus $t(a_1, a_2) \notin H_m$ for some $a_2 \in 1 + \mathcal{P}_E^{2m-1}$ and $a_1$ with $a_1/a_2 = 1 + \mathcal{P}_E^{2m}$, since $i_{E/F} > 1$ in the case $p = 2$. Thus we cannot conclude that $W_{v_m}(t(a_1, a_2)) = W_{v'_m}(t(a_1, a_2))$ from Lemma 2.1 (2) or equation (2.1). This is the reason that we need to exclude the case $p = 2$.

With slightly modification, one can check easily that the proof of the local converse theorem goes through when $E/F$ is ramified and the residue characteristic of $F$ is odd. We omit the details.

In the case $E/F$ is ramified and $p = 2$, one still expects that the local converse theorem holds for $U_{E/F}(2, 2)$. Our approach does not go through due to the technique difficulty which is discussed in Remark 7.3.

### 7.2 Local converse theorem for $\text{Sp}_4(F)$ and $\tilde{\text{Sp}}_4(F)$

The gamma factors for generic irreducible smooth representations (resp. genuine generic irreducible smooth representations) of $\text{Sp}_4(F)$ (resp. $\tilde{\text{Sp}}_4(F)$) are studied in [24]. The local zeta integrals in these cases are defined in the same manner as the $U_{E/F}(2, 2)$ case as we considered before. Thus, with similar argument, we can obtain the local converse theorem in these cases, i.e., we have:

**Theorem 7.4.** Let $F$ be a $p$-adic local field with odd residue characteristic. Let $\pi$ and $\pi'$ be two $\psi_U$-generic irreducible smooth representations (resp. genuine $\psi_U$-generic irreducible smooth representations of) $\text{Sp}_4(F)$ (resp. $\tilde{\text{Sp}}_4(F)$) with the same central character. If

$$\gamma(s, \pi \times \eta, \psi) = \gamma(s, \pi' \times \eta, \psi) \quad \text{and} \quad \gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi' \times \tau, \psi),$$

for all quasi-characters $\eta$ of $F^\times$ and all irreducible generic smooth representations of $GL_2(F)$, then $\pi \cong \pi'$.

Here we remark that, in Theorem 7.4, the fixing generic character $\psi_U$ is crucial because the maximal split torus acts on the generic characters with more than one orbits.
Remark 7.5. The above version of a local converse theorem for Spₙ(F) for generic square integrable representations is now known without the restriction of the residue characteristic of F and the central character. In fact, given an arbitrary p-adic field F, suppose that we have two 𝜨ᵤ- generic irreducible generic square integrable representations π, π′ of Spₙ(F) such that

\[ y(s, \pi \times \eta, \psi) = y(s, \pi' \times \eta, \psi) \quad \text{and} \quad y(s, \pi \times \tau, \psi) = y(s, \pi' \times \tau, \psi) \]

for all quasi-characters η of F∞ and all irreducible generic smooth representations τ of GL₂(F). Using the local Langlands correspondence for Spₙ (see [10, 11]), we get two Langlands parameters

\[ \phi, \phi' : WD(F) \to SO₅(C) \subset GL₅(C). \]

Now apply the local Langlands correspondence for GLₙ (see [16, 17]), we get two irreducible representations σ, σ′ of GLₙ(F). From the fact that π (resp. π′) is square integrable, one can check that σ (resp. σ′) is generic. In fact, since π is square integrable, one can find a number field F and a place v such that F_v = F, and a generic cuspidal (automorphic) representation Π of Spₙ(F) such that Π_v = π. Let \( \Pi_v \) be the generic transfer of Π as in [8], which is a generic cuspidal representation of GL_v. Let \( \tilde{\sigma} = \Pi_v \), which is a generic irreducible representation of GL_v(F). Then one can check that \( \sigma = \tilde{\sigma} \) using the fact that each construction above preserves local gamma factors (in fact, in [28], S. Takeda gives a proof that the generic transfer constructed in [8] is functorial with respect to local Langlands correspondence for more general groups). Thus σ is generic.

Remark 7.5. The above version of a local converse theorem for Spₙ(F) for generic square integrable representations is now known without the restriction of the residue characteristic of F and the central character. In fact, given an arbitrary p-adic field F, suppose that we have two 𝜨ᵤ- generic irreducible generic square integrable representations π, π′ of Spₙ(F) such that

\[ y(s, \pi \times \eta, \psi) = y(s, \pi' \times \eta, \psi) \quad \text{and} \quad y(s, \pi \times \tau, \psi) = y(s, \pi' \times \tau, \psi) \]

for all quasi-characters η of F∞ and all irreducible generic smooth representations τ of GL₂(F). Using the local Langlands correspondence for Spₙ (see [10, 11]), we get two Langlands parameters

\[ \phi, \phi' : WD(F) \to SO₅(C) \subset GL₅(C). \]

Now apply the local Langlands correspondence for GLₙ (see [16, 17]), we get two irreducible representations σ, σ′ of GLₙ(F). From the fact that π (resp. π′) is square integrable, one can check that σ (resp. σ′) is generic. In fact, since π is square integrable, one can find a number field F and a place v such that F_v = F, and a generic cuspidal (automorphic) representation Π of Spₙ(F) such that Π_v = π. Let \( \Pi_v \) be the generic transfer of Π as in [8], which is a generic cuspidal representation of GL_v. Let \( \tilde{\sigma} = \Pi_v \), which is a generic irreducible representation of GL_v(F). Then one can check that \( \sigma = \tilde{\sigma} \) using the fact that each construction above preserves local gamma factors (in fact, in [28], S. Takeda gives a proof that the generic transfer constructed in [8] is functorial with respect to local Langlands correspondence for more general groups). Thus σ is generic.

In each step π ↦→ φ ↦→ σ, the gamma factors are preserved. Thus we get

\[ y(s, \sigma \times \eta, \psi) = y(s, \sigma' \times \eta, \psi) \quad \text{and} \quad y(s, \sigma \times \tau, \psi) = y(s, \sigma' \times \tau, \psi) \]

for all quasi-characters η of F∞ and all irreducible generic smooth representations τ of GL₂(F). From the Jacquet’s conjecture for the local converse problem for p-adic GLₙ, which is recently proved for prime n in [1] and for general n in [6, 20], we get σ ≅ σ′. Since the local Langlands correspondence for GLₙ is bijective, we get φ = φ′ (up to equivalence). Thus we get that π and π′ are in the same L-packet defined in [10]. In [10], it is shown that in each L-packet, there is at most one 𝜨ᵤ- generic representation. Thus we get π ≅ π′.

Acknowledgment: I would like to thank my advisor Professor James W. Cogdell for his constant encouragement, generous support and countless hours he spent on this work. I thank Professor Joseph Hundley for pointing out the reference [4], The influence of E. M. Baruch’s thesis [2] on this paper should be evident for the reader. I would like to express my appreciation to Professor Baruch for his pioneer work. I am also very grateful for the anonymous referee for pointing out a mistake of a remark in the first draft and for his/her careful reading of this paper.

References

[1] M. Adrian, B. Liu, S. Stevens and P. Xu, On the Jacquet conjecture on the local converse problem for p-adic GLₙ, Represent. Theory 20 (2016), 1–13.
[2] E. M. Baruch, Local factors attached to representations of p-adic groups and strong multiplicity one, Ph.D. thesis, Yale University, 1995.
[3] E. M. Baruch, On the Gamma factors attached to representations of U(2, 1) over a p-adic field, Israel J. Math. 102 (1997), 317–345.
[4] A. Ben-Artzi and D. Soudry, L-functions for Uₘ × RₓGLₙ (n ≤ [\frac{m}{2}]), in: Automorphic Forms and L-functions I. Global Aspects (Rehovot/Aviv 2006), Contemp. Math. 488, American Mathematical Society, Providence (2009), 13–59.
[5] J. Bernstein and A. V. Zelevinski, Representations of the group GL(n, F), where F is non-archimedian local field, Russian Math. Surveys 31 (1976), no. 3, 1–68.
[6] J. Chai, Bessel functions and local converse conjecture of Jacquet, preprint (2016), https://arxiv.org/abs/1601.05450; to appear in J. Eur. Math. Soc. (JEMS).

---

1 Thanks due to Shuichiro Takeda for communicating with me the following argument.
[7] J. J. Chen, The $n \times (n - 2)$ local converse theorem for $GL(n)$ over a $p$-adic field, *J. Number Theory* **120** (2006), 193–205.

[8] J. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro and F. Shahidi, Functoriality for the classical groups, *Pub. Math. Inst. Hautes Études Sci.* **99** (2004), 163–233.

[9] W. Gan, B. Gross and D. Prasa, Symplectic local root numbers, central critical $L$-values, and restriction problems in the representation theory of classical groups, *Astérisque* **346** (2012), 1–109.

[10] W. Gan and S. Takeda, The local Langlands conjecture for $Sp(4)$, *Int. Math. Res. Not. IMRN* **2010** (2010), no. 15, 2987–3038.

[11] W. Gan and S. Takeda, The local Langlands conjecture for $GSp(4)$, *Ann. of Math. (2)* **173** (2011), 1841–1882.

[12] D. Ginzburg, S. Rallis and D. Soudry, Periods, poles of $L$-functions and symplectic-orthogonal theta liftings, *J. Reine Angew. Math.* **487** (1997), 85–114.

[13] M. Harris, S. Kudla and W. Sweet, Theta dichotomy for unitary groups, *J. Amer. Math. Soc.* **9** (1996), 961–1004.

[14] G. Henniart, Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps $p$-adique, *Invent. Math.* **139** (2000), 439–455.

[15] J. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math. 29, Cambridge University Press, Cambridge, 1990.

[16] H. Jacquet and R. Langlands, Automorphic Forms on $GL(2)$, Lecture Notes in Math. 114, Springer, Berlin, 1970.

[17] H. Jacquet and B. Liu, On the local Converse Theorem for $p$-adic $GL_n$, preprint (2016), https://arxiv.org/abs/1601.03656; to appear in *Amer. J. Math.*

[18] D. Jiang and C.-F. Nien, On the local Langlands conjecture and related problems over $p$-adic local fields, in: *Proceedings to the 6th International Congress of Chinese Mathematicians* (Taipei 2013), Higher Education Press, Beijing (2016).

[19] E. Kaplan, Complementary results on the Rankin–Selberg gamma factors of classical groups, *J. Number Theory* **146** (2015), 390–447.

[20] J. P. Serre, Local Fields, Grad. Texts in Math. 67, Springer, New York, 1979.

[21] R. Steinberg, Lectures on Chevalley Groups, Yale University, New Haven, 1967.

[22] B. Sun, Multiplicity one theorems for Fourier–Jacobi models, *Amer. J. Math.* **134** (2012), 1655–1678.

[23] Q. Zhang, A local converse theorem for $U(1, 1)$, preprint (2015), http://arxiv.org/abs/1508.07062; to appear in *Int. J. Number Theory*.

[24] Q. Zhang, Stability of Rankin–Selberg gamma-factors for $Sp_{2n}$, $\tilde{Sp}_{2n}$ and $U(n, n)$, preprint (2015), http://arxiv.org/abs/1511.03713; to appear in *Int. J. Number Theory*. 