On Superplanckian Scattering on the Brane

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Abstract

The multidimensional space-time with \((D-4)\) compact extra space dimensions and SM fields confined on four-dimensional brane is considered. The elastic scattering amplitude of two particles interacting by gravitational forces is calculated at superplanckian energies. A particular attention is paid to a proper account of zero (massless) graviton mode. The renormalized Born pole is reproduced in the eikonal amplitude which makes a leading contribution at small momentum transfers. This singular part of the amplitude coincides with well-known \(D\)-dimensional amplitude taken at \(D \to 4\). The expression for a contribution from massive graviton modes to the eikonal is derived, and it asymptotics in the impact parameter are calculated. Our formula gives correct four-dimensional result at \(R_c \to 0\), where \(R_c\) is the radius of the higher dimensions, contrary to formulae obtained in recent papers on collisions of particles living on the brane. The results are also compared with those obtained previously for the scattering of the bulk fields in flat higher dimensions.

1 Introduction

In the 4-dimensional space-time the gravity is very weak as compared with the interactions of the Standard Model (SM) fields. Namely, the Newton constant is equal to \(G_N = M_{Pl}^{-2}\), where \(M_{Pl} = 1.2 \cdot 10^{19}\) GeV is the Planck mass, while the electroweak scale is about \(m_{EW} \sim 10^3\) GeV. In order to

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explain a huge ratio of two physical scales in nature, $M_{Pl}/m_{EW}$, a scheme with additional space dimensions with a flat metric has been proposed \[1\] (in what follows, referred to as a ADD model). All $d$ extra dimensions are compact with the radius $R_c$. In other words, the space-time is $R^4 \times M_d$, where $M_d$ is a $d$-dimensional manifold of the volume $R_c^d$. If $R_c^{-1} \ll m_{EW}$, a gravitational potential will get negligible corrections at distances $r \gg R_c$.

Let $M_D$ be a fundamental Planck scale in $D$-dimensional theory ($D = 4 + d$). Then it can be shown \[1\] that

$$M_{Pl}^2 = R_c^d M_D^{2+d}.$$ (1)

One can get $M_D \sim 1$ TeV, if the compactification radius $R_c$ is large enough. The radius $R_c$ depends on $d$ and it ranges from 1 mm to 1 fm if $d$ runs from 2 to 6. Since $R_c \gg m_{EW}^{-1}$, all Standard Model (SM) gauge and matter fields are to be confined to a 3-dimensional brane embedded into the $(3 + d)$-dimensional space (gravity alone lives in the bulk).

From the point of view of the 4-dimensional space-time, there arise a Kaluza-Klein (KK) tower of massive graviton modes, $G_{\mu\nu}^{(n)}$, with masses

$$m_n = \frac{\sqrt{n^2}}{R_c}, \quad n^2 = n_1^2 + n_2^2 + \ldots + n_d^2,$$ (2)

where $n$ defines the KK excitation level. So, a mass splitting is $\Delta m \sim R_c^{-1}$ and we have an almost continuous spectrum of the gravitons.

The interaction of the gravitons with the SM fields is described by the Lagrangian \[1\]

$$\mathcal{L} = -\frac{1}{\bar{M}_{Pl}} G_{\mu\nu}^{(n)} T^{\mu\nu},$$ (3)

where $\mu, \nu = 0, 1, 2, 3$ and $\bar{M}_{Pl} = M_{Pl}/\sqrt{8\pi}$ is the reduced Planck mass. One can conclude from \[3\] that the coupling of both massless and massive graviton is universal and very small ($\sim 1/\bar{M}_{Pl}$). However, the multiplicity of the KK states produced in high energy collisions is huge and it is equal to $(\sqrt{s} R_c)^d$, where $\sqrt{s}$ is the collision energy. A typical cross-section of a process involving the production of the KK graviton excitations with masses $m_n \leq \sqrt{s}$ is suppressed only by the scale $M_D$:

$$\sigma_{KK} \sim \frac{s^{d/2}}{M_D^{d+2}}.$$ (4)
So, the ADD model can be tested at future hadronic colliders and at $e^+ e^-$ linear colliders in the range of TeV energies (the Planck regime). There is a lot of papers on a collider phenomenology within the framework of the extra dimensions. An interested reader can find references in reviews [2].

The collisions in the transplanckian regime ($\sqrt{s} \gg M_D$) were considered in a variety of papers in the framework of the string theory [3], in the eikonal approximation of the reggeized graviton exchange [4] as well as in different 4-dimensional approaches [5,6]. The equivalence of various schemes has been demonstrated in [7]. Note, in papers [3,4] a collision of the bulk fields in the $D$-dimensional flat space-time was considered.

In Refs. [8] an estimate of high-energy gravitational cross sections of hadrons have been made. The contribution from the KK excitations of the graviton changes $t$-channel propagator ($-t$)$^{-1}$ by

$$\frac{1}{-t} \rightarrow \sum_{n_1^2+\ldots+n_d^2 \geq 0} \frac{1}{-t + \sum_{i=1}^d \frac{n_i^2}{R_c^2}}.$$  \hspace{1cm} (5)

It was argued in [8], that the range $n \lesssim n_{\text{max}} \sim M_s R_c$, where $M_s$ is a quantum gravity (string) scale, makes a dominant contribution to hadronic cross sections. Using a replacement ($d > 2$)

$$\sum_{n_1^2+\ldots+n_d^2 \geq 0} \frac{1}{-t + \sum_{i=1}^d \frac{n_i^2}{R_c^2}} \rightarrow R_c^2 \int d^d \Omega \int_0^{n_{\text{max}}} dn n^{d-3},$$  \hspace{1cm} (6)

the following results for the total cross section has been obtained (after unitarization):

$$\sigma_{\text{tot}}(s) \simeq \frac{4\pi s}{M_D^2}.$$  \hspace{1cm} (7)

However, papers [8] have unjustified approximations and incorrect estimates, as it was pointed out in Ref. [9]. In particular, the presence of the massless exchange quantum (zero mode of the graviton) should result in infinite elastic and total hadronic cross sections, contrary to equality (7). For strong interactions without gravitational forces, the upper (Froissart) bound for $\sigma_{\text{tot}}(s)$ is modified by an additive term $(\pi r_c/m_\pi) \ln s$, if the extra dimensions are compactified onto a circle with the radius $r_c$ [10].
Recently, results on a collision of the brane particles, which interact by graviton forces in the ADD model with the compact extra dimensions, have been presented in [11].

We are confident, however, that the massless graviton mode was not taken properly into account in both [8] and [11]. In particular, we cannot agree with a conclusion that long-range forces are completely hidden by the interactions of the massive graviton excitations, coming from the extra dimensions [11]. That is why, in the present paper we calculate the scattering amplitude for two particles confined on the brane, by separating massless graviton contribution from massive graviton effects from the very beginning.

In the next Section we review briefly the results on the scattering in both $D$ flat dimensions [3, 4] and four flat dimensions [5, 6]. In the beginning of Section 3 we consider the approach proposed in [11]. The rest of the Section 3 is devoted to calculations of the eikonal amplitude. In the last Section we discuss our results and compare them with the results obtained by other authors mentioned in the paper. In the Appendix technical details of our calculations are presented.

2 Transplanckian collision in the bulk

In this Section we remind some results on transplanckian collisions in models with the extra dimensions. As was mentioned in the Introduction, the transplanckian regime has been analyzed in details in the string theory. The string theory has the fundamental classical constant $\alpha'$, its inverse being the string tension. Since the leading graviton trajectory is at $\alpha(t) = 2 + (\alpha'/2)t$, one expects that at high $s$ graviton exchange will dominate light-string scattering amplitude for any number of loops.

The transplanckian regime is characterized by a strong effective coupling $\alpha_G(s) = G_D s$ ($G_D = M_D^{-(2+d)}$ is a $D$-dimensional Newton constant). In Refs. [3] four-string scattering amplitude was calculated in the kinematical region

$$\alpha' s \gtrsim (M_D \sqrt{\alpha})^{d+2} \gg 1, \quad |t| \gtrsim \alpha'^{-1}, \quad \alpha'|t| \gtrsim (\alpha' R_c)^{-2}. \quad (8)$$

Inequalities (8) mean that the tree amplitude is large ($G_D s \alpha'^{-d/2} \gg 1$), while the loop expansion parameter, $G_D \alpha'^{-1-d/2}$, is small. Due to the second
restriction on \( t \) in (8), compactified momenta are not noticeably excited. In terms of an impact parameter \( b \), the limitations look like

\[
b > \lambda_s, \quad b > R_G(s),
\]

where \( \lambda_s = \sqrt{2\alpha'\hbar} \) is a fundamental quantum length in the string theory and \( R_G(s) \approx (2G_D\sqrt{s})^{1/(d+1)} \) is a gravitational radius.

The leading contribution to the scattering amplitude at the impact parameter \( b \) has all powers of \( \alpha_G(s) \) and it is the same in all approaches at \( b \gg \lambda_{Pl}, R_G(s) \), where \( \lambda_{Pl} = (\hbar G_D)^{1/(d+2)} \) is the Planck length.

The amplitude is of a classical (eikonal) form. For large \( b \) the eikonal function is given by (10)

\[
\chi(b, s) \equiv \chi_{ACV}(b, s) \approx \left( \frac{\tilde{b}_c}{b} \right)^d + i\pi^2 \frac{G_D s \alpha^{-d/2}}{(\pi \ln s)^{d/2+1}} \exp \left( -\frac{b^2}{4\alpha' \ln s} \right),
\]

where \( \tilde{b}_c = [\alpha_G(s)2\pi^{-d/2}\Gamma(1+d/2)]^{1/d} \). As one can see, \( \chi_{ACV}(b, s) \) has both real and imaginary part. The former has a power-like behavior in \( b \), while the latter decreases exponentially at \( b \gg 2\alpha' \ln s \). Correspondingly, at small \( t \) (namely, at \( |t| \ll \alpha_G(s)^{-2/d} \)) the amplitude has the asymptotics

\[
A_{ACV}^{\text{eik}}(s, t) \sim A_B(s, t)
+ i \, \text{const} \frac{(16\pi\alpha_G(s))^2 s}{d(d-2)} \left[ -|t|^{d/2-1} + (16\pi G_D s)^{2/d-1} \right].
\]

Here

\[
A_B(s, t) = \frac{8\pi \alpha_G(s) s}{-t}
\]

is the Born amplitude. Thus, the Born term dominates at small \( t \).

For large \( t \) (\( |t| \gg \tilde{b}_c^{-2} \)) the amplitude has the following behavior (3):

\[
A_{ACV}^{\text{eik}}(s, t) \sim \frac{8\pi \alpha_G(s) s}{-t} \exp(i\phi_D)
\times \left( 4\pi (\tilde{b}_c \sqrt{|t|})^d \right)^{-d/2(d+1)}.
\]

The \( D \)-dimensional phase

\[
\phi_D = \frac{d+1}{d} \left( G_D s 2\pi^{-d/2} \Gamma(1+d/2)|t|^{d/2} \right)^{1/(d+1)}
\]

is the Born amplitude. Thus, the Born term dominates at small \( t \).

For large \( t \) (\( |t| \gg \tilde{b}_c^{-2} \)) the amplitude has the following behavior (3):

\[
A_{ACV}^{\text{eik}}(s, t) \sim \frac{8\pi \alpha_G(s) s}{-t} \exp(i\phi_D)
\times \left( 4\pi (\tilde{b}_c \sqrt{|t|})^d \right)^{-d/2(d+1)}.
\]
has a pole at $D = 4$. So, the limit $D \to 4$ is completely non-perturbative due to the divergent (Coulomb) phase.

In Ref. [4] the same result (13) was obtained by summing multiple reggeized graviton exchange in the eikonal approximation. Although Regge behavior is present at each order, it is absent in the final result (13). It is important to note that the magnitude of the scattering amplitude is defined by a single non-reggeized graviton exchange (in full analogy with the case of Coulomb scattering dominated by a single photon exchange).

The scattering amplitude may be directly calculated in four dimensions [5, 6] (see also [13]):

$$A^{eik}_{HVV}(s, t) = A_B(s, t) \frac{\Gamma(1 - i\alpha_G(s))}{\Gamma(1 + i\alpha_G(s))} \left( \frac{4\mu_{IR}^2}{-t} \right)^{-i\alpha_G(s)},$$

or it can be obtained from the $D$-dimensional expression by taking the limit $D \to 4$ [3, 4]. The quantity $\mu_{IR}$ in (15) is an infrared cutoff. It arises in the limit $D \to 4$, when the pole $(D - 4)^{-1}$ is interpreted as the logarithm of $\mu_{IR}$ [4]. If the amplitude is calculated as a sum of soft gravitons with a small mass $m_{grav}$, this cutoff is related to a graviton mass, $\mu_{IR} = (1/2)m_{grav}e^\gamma$, where $\gamma$ is the Euler constant.

Let us note, the eikonal amplitude in quantum electrodynamics can be obtained from (15) by a simple replacement $-\alpha_G(s) \to \alpha_{em} = e_1e_2/4\pi$, where $e_{1, 2}$ are electric charges of colliding massless charged particles [14]. In such a case, $\mu_{IR}$ is proportional to a regulating photon "mass" [14].

At large $z$, $|\arg z| < \pi$, the $\Gamma$-function has an asymptotics $\Gamma(z) = \sqrt{2\pi}e^{-z}e^{(z-1/2)\ln z}[1 + O(z^{-1})]$ [15]. Then we obtain from (15) that in four dimensions (see also [4])

$$A^{eik}(s, t)\big|_{\alpha_G(s) > 1} \simeq -iA_B(s, t) \left( \frac{4\mu_{IR}^2}{-t} \right)^{-i\alpha_G(s)} \times \exp \left[-2i\alpha_G(s)(\ln \alpha_G(s) - 1)\right].$$

In the next Section we will consider the case when colliding particles are confined to the brane, with the gravity living in the bulk. Another difference will be that the compactified momenta become essential, contrary to the approach considered in this Section.
3 Transplanckian collision on the brane

In the ADD model, all SM fields live on the $(1+3)$-dimensional brane embedded in the $D$-dimensional space-time. Thus, their collisions are also confined to the brane. In particular, the impact parameter space is two-fold. On the other hand, in the transplanckian region, where the collision energy $\sqrt{s}$ is much larger than the fundamental gravity scale $M_D$, but a momentum transfer $t$ is small, the scattering of 4-dimensional particles is dominated by the exchange of $D$-dimensional gravitons.

The (elastic) scattering of two (different) massless particles living on the brane in the kinematical region

$$\sqrt{s} \gg M_D, \quad s \gg -t$$

was recently considered in Refs. [11, 12]. The ladder diagrams contributing to a nonreggeized graviton exchange in $t$-channel were summed in the eikonal approximation [11]. From the point of view of a 4-dimensional observer, the massless bulk graviton is represented by the tower of massive gravitons [2]. Since the higher space dimensions are compactified with the radius $R_c$, one has a sum in a (quantized) momentum transfer in the extra dimensions $q_{\perp}^{(n)} = n/R_c$ instead of an integral in $d^{D-4}q_{\perp}$. The Born amplitude is, therefore, of the form

$$A^B(s, t) = G_N s^2 \sum_{n_1^2 + \ldots + n_d^2 \geq 0} \frac{1}{-t + \sum_{i=1}^{d} \frac{n_i^2}{R_c^2}}. \quad (18)$$

Here and in what follows the reduced gravitational constant, $\bar{G}_N = \bar{M}_P^{-2}$ is always assumed [3]. For simplicity, we will write $G_N$ instead of $\bar{G}_N$ (and, correspondingly, $G_D$ instead of $\bar{G}_D$). Thus, to compare our results this those of Refs. [3, 4, 5, 6], one will have to use a substitution $G_N \rightarrow 8\pi G_N$.

In Ref. [11] the following replacement

$$\sum_{n_1^2 + \ldots + n_d^2 \geq 0} \frac{1}{-t + \sum_{i=1}^{d} \frac{n_i^2}{R_c^2}} \rightarrow \int d^d\Omega \int_0^\infty dl^{d-1} \frac{1}{-t + \frac{l^2}{R_c^2}} \quad (19)$$

was made by assuming that $R_c$ is large (compare with [3]). As a result, it was obtained:

$$A^B_{GRW}(s, t) = \pi^{d/2} \Gamma(1 - d/2) \left(\frac{s}{M_D^2}\right)^2 \left(\frac{-t}{M_D^2}\right)^{d/2-1}. \quad (20)$$
At one and higher-loop levels it is ladder diagram that makes a leading contribution to the amplitude. The sum of all such diagrams results in the eikonal representation for the amplitude [11]:

$$A_{eik}^{GRW}(s, t) = -2is \int d^2b_\perp e^{iq_\perp b_\perp} \left(e^{i\chi_{GRW}(b_\perp)} - 1\right), \quad (21)$$

with the eikonal given by

$$\chi_{GRW}(b_\perp) = \frac{1}{2s} \int \frac{d^2q_\perp}{(2\pi)^2} e^{-iq_\perp b_\perp} A^B(s, q_\perp^2). \quad (22)$$

After a substitution of the Born amplitude (20) in equation (22), one gets [11]

$$\chi_{GRW}(b) = \left(\frac{b_c}{b}\right)^d, \quad (23)$$

where

$$b_c = \left[\frac{s (4\pi)^{d/2-1} \Gamma(d/2)}{2M_D^{d+2}}\right]^{1/d} \equiv 2\sqrt{\pi} R_c \left[\frac{G_N s \Gamma(d/2)}{8\pi}\right]^{1/d}. \quad (24)$$

At $d \to 0$, the eikonal $\chi_{GRW}(b)$ (23) has an expansion:

$$\chi_{GRW}(b) \bigg|_{d \to 0} \approx \frac{G_N s}{8\pi} \left[\frac{2}{d} + \ln \left(\frac{2R_c}{b}\right)^2 + \Psi(1) + \ln \pi\right], \quad (25)$$

where $\Psi(z)$ is the $\Psi$-function [15].

At all intermediate steps (19)-(22), the number of the extra dimension $d$ was regarded as a (non integer) parameter. The final expression (23) has no divergences in $d$ at $d > 0$, although the Born amplitude has simple poles at $d = 2, 4, \ldots$ (because of the $\Gamma$-function in (20)).

In [12] the conclusion was made that “even for $q = 0$, the scattering amplitude is dominated by $b \sim b_c$ and not by $b = \infty$, as opposed to the Coulomb case. This result follows from the different dimensionalities of the space on which the scattered particles and exchange graviton live”.

We will show that this statement is not correct and that the Born amplitude survives after summation of the KK-excitations of the graviton and does contribute to the eikonal. In its turn, this means that long-range forces (Coulomb singularity) still presents in the scattering of brane particles.
Indeed, for \( d = 1 \) series (18) converges, and it has a pole \( t^{-1} \) corresponding to a zero massless mode of the graviton. It would be strange to expect that long-range forces do present for \( d = 1 \), but disappear when the gravity live in more than one extra dimensions. For \( d \geq 2 \), sum (18) is divergent and it needs a regularization. Following [11], we will use the dimensional regularization, by considering \( d \) to be non-integer at intermediate steps of our calculations. The final result will be well-defined for all \( d \geq 0 \).

Although the change “summation in \( n \)” \( \rightarrow \) “integration in \( dn \)” (where \( n \) labels the KK excitation level of the graviton) is justified at \( R_c \sqrt{|t|} \gg 1 \), it should be done more accurately than it was dealt with in (6) and (19). The crucial point is that a contribution from the zero (massless) graviton mode must be isolated before replacement (19):

\[
A_B(s, t) = G_N s^2 \sum_{n_1^2 + \ldots + n_d^2 \geq 1} \frac{1}{-t + m_n^2}. \tag{26}
\]

In the case of large extra dimensions, when the mass splitting is small \( (\Delta m = 1/R_c) \), we can write

\[
\sum_{n_1^2 + \ldots + n_d^2 \geq 1} \frac{1}{-t + m_n^2} \to R_c^d \int d^d \Omega \int_{R_c^{-1}}^{\infty} d m m^{d-1} \frac{1}{-t + m^2} \tag{27}
\]

(strictly speaking, the inequality \( \sqrt{|t|} R_c \gg 1 \) must be satisfied). Then equation (26) can be recast as follows:

\[
A_B(s, t) = G_N s^2 \left[ 1 + \left( \sqrt{|t|} R_c \right)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{(\sqrt{|t|} R_c)^{-1}}^{\infty} dy y^{d-1} \frac{1}{1 + y^2} \right].
\]

\[
\equiv A_0^B(s, t) + A_{mass}^B(s, t) \tag{28}
\]

The integral in the RHS of (28), representing the contribution from the massive gravitons, can be calculated and rewritten in the form

\[
A_{mass}^B(s, t) = G_N s^2 \left( \sqrt{|t|} R_c \right)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{(\sqrt{|t|} R_c)^{-1}}^{\infty} dy y^{d-1} \frac{1}{1 + y^2}
\]

\[
= G_N s^2 R_c^2 \frac{\pi^{d/2}}{\Gamma(d/2)(1 - d/2)} \left( \frac{1}{2} - \frac{d}{2}; \frac{1}{2} - \frac{d}{2}; t R_c^2 \right). \tag{29}
\]
where \( _2F_1(\alpha_1, \alpha_2; \beta_1; z) \) is the hypergeometric function \cite{15}, and we have taken into account the relation

\[
R_c = \frac{1}{M_D} \left( \frac{M_P}{M_D} \right)^{2/d} = \left( \frac{G_D}{G_N} \right)^{1/d}.
\]

Thus, \( A^B_{\text{mass}}(s, t) \) converges at \( t \to 0 \).

We see from \cite{29} that \( A^B_{\text{mass}}(s, t) = 0 \) for \( d = 0 \). In such a case, it is the 4-dimensional massless graviton that contributes to \( A^B(s, t) \) \cite{28}, as it should be.

In order to get an asymptotics of the Born amplitude at large \( t \), we use the equivalent expression for \( A^B_{\text{mass}}(s, t) \):

\[
A^B_{\text{mass}}(s, t) = \frac{G_N s^2}{-t} \pi^{d/2} \left[ \Gamma(1 - d/2) \left( -t R_c^2 \right)^{d/2} \right.

- \frac{1}{\Gamma(1 + d/2)} \frac{1}{2} \left. _2F_1 \left( 1, \frac{d}{2}; 1 + \frac{d}{2}; \frac{1}{t R_c^2} \right) \right].
\]

As it follows from \cite{31}, large \( t \) behavior of \( A^B_{\text{mass}}(s, t) \) is similar to that of \( A^B_{\text{GRW}}(s, t) \) \cite{20}:

\[
A^B_{\text{mass}}(s, t) \bigg|_{R_c^2 | t | \gg 1} \approx \pi^{d/2} \Gamma(1 - d/2) \left( \frac{s}{M_D^2} \right)^2 \left( \frac{-t}{M_D^2} \right)^{d/2-1}

\times \left[ 1 - \frac{1}{\Gamma(1 + d/2) \Gamma(1 - d/2)} \left( -t R_c^2 \right)^{-d/2} \right].
\]

It is convenient to divide "massive" part of the eikonal,

\[
\chi_{\text{mass}}(b) = \frac{1}{2s} \int \frac{d^2 q_{\perp}}{(2\pi)^2} A^B_{\text{mass}}(s, t)

= \frac{1}{4\pi s} \int_0^\infty q_{\perp} dq_{\perp} J_0(bq_{\perp}) A^B_{\text{mass}}(s, -q_{\perp}^2),
\]

into two parts:

\[
\chi_{\text{mass}} \equiv \chi^{(1)}_{\text{mass}}(b) + \chi^{(2)}_{\text{mass}}(b).
\]

Here \( \chi^{(1)}_{\text{mass}}(b) \equiv \chi_{\text{GRW}}(b) \) \cite{23} and

\[
\chi^{(2)}_{\text{mass}}(b) = -G_N s \frac{\pi^{d/2-1}}{8\Gamma(1 + d/2)} I(b),
\]

\[10\]
where \( I(b) \) is given by the integral

\[
I(b) = \int_{0}^{\infty} \frac{dx}{x} J_0 \left( \frac{b}{R_c \sqrt{x}} \right) \frac{b^2}{4R_c^2} F_1 \left( \frac{1}{2}, 1 + \frac{d}{2}; 1 + \frac{d}{2}; -x \right).
\] (36)

The integral in (36) can not be directly expressed in terms of algebraic or special functions. But we will be able to calculate its behavior in impact parameter at both large and small \( b \), if we define \( I(b) \) as a limit

\[
I = \lim_{\epsilon \to 0} I_\epsilon,
\] (37)

where we have introduced

\[
I_\epsilon(b) = \int_{0}^{\infty} dx x^{-1+\epsilon} J_0 \left( \frac{b}{R_c \sqrt{x}} \right) \frac{b^2}{4R_c^2} F_1 \left( \frac{1}{2}, 1 + \frac{d}{2}; 1 + \frac{d}{2}; -x \right).
\] (38)

The integral in (38) is well defined at \(-3/4 < \text{Re} \, \epsilon < 1, \text{Re} \, d/2\) (we assume that \text{Re} \, d > 0). Thus, the limit \( \lim_{\epsilon \to 0} I_\epsilon \) exists. Moreover, \( I_\epsilon \) is a table integral (see formula 2.21.4.6 from [16]):

\[
I_\epsilon(b) = \frac{1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} \left[ - \left( \frac{b^2}{4R_c^2} \right)^\epsilon F_3 \left( \frac{d}{2}, 1 + \frac{d}{2}, 1 + \epsilon, 1 + \epsilon; \frac{b^2}{4R_c^2} \right) \right.
\]
\[\left. + \frac{d}{d-2\epsilon} \Gamma^2(1+\epsilon) F_2 \left( 1 + \frac{d}{2} - \frac{d}{2} - \epsilon, 1; \frac{b^2}{4R_c^2} \right) \right],
\] (39)

where \( \gamma_F(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \) is the generalized hypergeometric function [15].

At \( b \ll R_c \), we immediately get from (37) and (38) \((d > 0)\):

\[
I(b) \bigg|_{b \ll R_c} \simeq \lim_{\epsilon \to 0} \epsilon \left\{ - \left( \frac{b^2}{4R_c^2} \right)^\epsilon \left[ 1 + \frac{1}{1+\epsilon} \frac{d}{d+2} \left( \frac{b}{2R_c} \right)^2 \right] \right.
\]
\[\left. + \frac{d}{d-2\epsilon} \Gamma^2(1+\epsilon) \right\}
\]
\[= 2 \left\{ \ln \left( \frac{2R_c}{b} \right) \left[ 1 + \frac{d}{d+2} \left( \frac{b}{2R_c} \right)^2 \right] + \frac{1}{d} + \Psi(1) \right\}.
\] (40)
The region \( b \gg R_c \) is much more difficult to analyze. The asymptotics of \( I(b) \) is calculated in the Appendix and the result is the following:

\[
I(b) \bigg|_{b \gg R_c} \simeq \left( \frac{2R_c}{b} \right)^d \frac{\Gamma(1 + d/2)}{\Gamma(1 - d/2)} \times \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ -\frac{\Gamma(1 - d/2)}{\Gamma(1 - d/2 + \epsilon)} + \frac{\Gamma(d/2 - \epsilon)}{\Gamma(d/2)} \right]
= \left( \frac{2R_c}{b} \right)^d \frac{d}{2} \Gamma(1 + \frac{d}{2}) \frac{\cos \pi \frac{d}{2}}{2}. \tag{41}
\]

From formulae (28), (34) and (35) it follows that

\[
\chi(b) = \chi_0(b) + \chi_{\text{mass}}(b), \tag{42}
\]

where

\[
\chi_0(b) = \frac{1}{4\pi s} \int_0^\infty q_\perp dq_\perp J_0(bq_\perp) A^B_0(s, -q_\perp^2) \tag{43}
\]

and

\[
\chi_{\text{mass}}(b) = \left( \frac{b_c}{b} \right)^d - G_N s \frac{\pi^{d/2-1}}{8\Gamma(1 + d/2)} I(b). \tag{44}
\]

The asymptotics of \( I(b) \) at small and large \( b \) are calculated above (see (40), (41)). The zero mode graviton contribution to the Born amplitude is

\[
A^B_0(s, t) = \frac{G_N s^2}{-t}. \tag{45}
\]

Correspondingly, the eikonal amplitude is represented by the expression

\[
A^{\text{eik}}(s, t) = -4\pi is \int_0^\infty b db J_0(b\sqrt{-t}) \left[ e^{i(\chi_0(b) + \chi_{\text{mass}}(b))} - 1 \right]. \tag{46}
\]

The massless graviton contribution to the eikonal (43) is divergent due to the Coulomb-like pole in \( t \) (45). In order to regularize it, let us assume that the colliding particles are confined to a \((4 + \delta)\)-dimensional brane with \( \delta > 0 \), while the gravity propagates in \((4 + \delta + d)\) dimensions. Then instead of equation (43) we will have

\[
\chi_0(b, \delta) = \frac{1}{2s} \int \frac{d^{2+\delta}q_\perp}{(2\pi)^{2+\delta}} e^{ibq_\perp} A^B_0(s, -q_\perp^2, \delta), \tag{47}
\]
where $A_0^B(s, t, \delta) = G_{4+\delta} s^2/|t|$ and $G_{4+\delta}$ is a gravitational constant in $(4+\delta)$ dimensions. The “massive” part of the eikonal, $\chi_{mass}(b, \delta)$, is analogously determined via $A_{mass}(s, t, \delta)$.

We define the (eikonal) amplitude of the scattering in four dimensions as a limit

$$A^{eik}(s, t) = \lim_{\delta \to 0} A^{eik}(s, t, \delta),$$

where the eikonal amplitude,

$$A^{eik}(s, t, \delta) = -2is \int d^{2+\delta} b e^{ibq_{\perp}} \left[ e^{i(\chi_0(b, \delta) + \chi_{mass}(b, \delta))} - 1 \right],$$

can be rewritten by adding and subtracting the “massless” term:

$$A^{eik}(s, t, \delta) = -2is \int d^{2+\delta} b e^{ibq_{\perp}} \left[ e^{i\chi_0(b, \delta)} - 1 \right] - 2is \int d^{2+\delta} b e^{ibq_{\perp}} \left[ e^{i\chi_0(b, \delta)} - 1 \right],$$

It is easy to check that $\chi_{mass}(t, \delta)$ is non-singular at $\delta = 0$. As for $\chi_0(t, \delta)$, at small $\delta$ we get from (47)

$$\chi_0(b, \delta) \bigg|_{\delta \to 0} = \frac{G_N s}{4\pi} \left[ \frac{1}{\delta} - \ln(b M_{Pl}) + O(\delta) \right].$$

The first integral in the RHS of (50) converges at $b = 0$ ($\chi_{mass}(b) \sim A b^{-d} + B \ln(1/b)$, if $b \to 0$), and it is well-defined at $b = \infty$, if $d > 2$ ($\chi_{mass}(b) \sim C b^{-d}$, if $b \to \infty$).

The second integral in the RHS of (50) is well-known. Let us put

$$\frac{1}{\delta} = \ln \left( \frac{M_{Pl}}{\mu_{IR}} \right),$$

where $\mu_{IR}$ is an infrared regulator at $\delta \to 0$. Then this integral is given by the expression for $A_{HV V}^{eik}(s, t)$ (15) with a replacement $\alpha_G \to \alpha_G/8\pi$ (or, equivalently, $G_N \to G_N/8\pi$, see our remarks after formula (18)). As a result, we obtain

$$A^{eik}(s, t) = \left( \frac{A_{IR}^2}{-t} \right)^{-i\alpha_G(s)/8\pi} \left\{ A_0^B(s, t) \frac{\Gamma(1 - i\alpha_G(s)/8\pi)}{\Gamma(1 + i\alpha_G(s)/8\pi)} \right\}$$

$$-4\pi i \int_0^\infty dx \frac{s}{-t} x e^{-\alpha_G(s)/4\pi} J_0(x) \left[ \exp \left( i\chi_{mass} \left( \frac{x}{\sqrt{|t|}} \right) \right) - 1 \right].$$
Here $\chi_{\text{mass}}(b)$ is defined by formula (44) and $A_0^B(s, t)$ is the singular part of the Born amplitude (45).

It follows directly from (33) and (29) that

$$\chi_{\text{mass}}(b) \bigg|_{d \to 0} \to 0. \quad (54)$$

As can be seen from (25), (35) and (40), the small $b$ behavior of $\chi_{\text{mass}}(b)$ is consistent with (54). Large $b$ asymptotics of $\chi_{\text{mass}}(b)$ also obeys this limit (see equation (55)). Thus, by taking the limit $d \to 0$ in (53), we reproduce the well-known four-dimensional result (15) derived in Refs. [5, 6].

Introducing a 4-dimensional phase

$$\phi_4 = \frac{G_N s}{8\pi} \ln \left( \frac{-t}{4\mu^2 R_c} \right), \quad (55)$$

we get our final result:

\begin{align*}
A^{\text{eik}}(s, t) &= s e^{i\phi_4} \left\{ \frac{G_N s}{-t} \frac{\Gamma(1 - iG_N s/8\pi)}{\Gamma(1 + iG_N s/8\pi)} - 16\pi R_c^2 \sqrt{|t|} \right\} \\
&\quad \times \int_0^\infty dz \, z^{1 - iG_N s/4\pi} J_0(2R_c \sqrt{|t|} z) \left[ e^{i\chi_{\text{mass}}(z)} - 1 \right], \quad (56)
\end{align*}

with

$$\chi_{\text{mass}}(z) \bigg|_{z \ll 1} \simeq G_N s \frac{\pi^{d/2 - 1} \Gamma(d/2)}{8} z^{-d} - G_N s \frac{\pi^{d/2 - 1} \Gamma(1 + d/2)}{4\Gamma(1 + d/2)}$$

$$\times \left\{ \ln \left( \frac{1}{z} \right) \left[ 1 + \frac{d}{d + 2} z^2 \right] + \frac{1}{d} + \Psi(1) \right\} \quad (57)$$

and

$$\chi_{\text{mass}}(z) \bigg|_{z \gg 1} = G_N s \frac{\pi^{d/2 - 1} \Gamma(d/2)}{4} z^{-d} \sin^2 \left( \frac{\pi d}{4} \right). \quad (58)$$

We have introduced a dimensionless variable $z = b/2R_c$. The amplitude $A^{\text{eik}}(s, t)$ is well-defined for all $d \geq 0$.

Our expression (56) has infinite phase (55). It was shown many years ago [17] that in quantum gravity each different particle pair in the initial (or final) state contributes a divergent phase factor to the $S$-matrix.

Note, the second term in (56) is regular at $t = 0$. Thus, for small $t$ (namely, at $|t|R_c^2 \ll 1$) the main contribution to the eikonal amplitude comes
from the Born pole (the first term in (56)). It is interesting that this term does not depend on either the compactification radius $R_c$ or the number of the extra dimensions $d$. In other words, it is entirely 4-dimensional.

Large $t$ behavior is determined by small values of variable $z$ in integral (56). Taking into account the asymptotics of $\chi_{mass}(z)$ at $z \approx 0$ (57), one can obtain by using stationary-phase technique ($d > 0$):

$$A e^{ik(s,t)} \Bigg|_{|t|R_2 \gg 1} \simeq -4\pi i s e^{i\phi_4} \frac{1}{|t|} \frac{2^{G_N s/4\pi}}{\sqrt{1 + d}} \times \left[ \left( d(b_c \sqrt{|t|})^d \right)^{1/(d+1)} \right]^{1 - i G_N s/4\pi} \exp \left[ i(1 + d) \left( \frac{b_c \sqrt{|t|}}{d} \right)^{d/(d+1)} \right]. \tag{59}$$

Formula (56) has correct physical limits. In a case, when the compactification radius $R_c$ tends to zero and, consequently, the KK graviton excitations become very heavy ($m_n \to \infty$, see (2)) and decouple from the brane particles, they make no contribution to the amplitude (since $\chi_{mass} \to 0$), but a renormalized Born amplitude still present in (56). The same is true for $d \to 0$ (no extra dimensions and, consequently, no massive gravitons are present in the nature).

On the other hand, the expression for $A_{GRW} e^{ik}(s,t)$ obtained in Ref. [11] (see formulae (21), (23)-(24)), results in an incorrect value, $A_{GRW} e^{ik}(s,t) = 0$, in the limit $R_c \to 0$.

4 Discussions

In the present paper the scattering amplitude of two brane particles, interacting by the gravity forces, is calculated in the ADD model in the eikonal approximation. A particular attention is payed to the account of the contribution from the massless graviton mode, contrary to a technique used in Ref. [11]. Our main results is formula (56), where $\chi_{mass}(b)$ represents the contribution from the KK graviton modes with $n \geq 1$. The expression for $\chi_{mass}(b)$ and its asymptotic behavior are presented by equations (44), (36) and (57), (58), respectively. Our formula (56) gives correct four-dimensional result at both $D \to 4$ and $R_c \to 0$.

It is interesting to compare our results with those, describing a collision of two bulk particles in $D$ dimensions with $D > 4$. First of all, $\chi_{ACV} e^{ik}$ has an imaginary part, while our $\chi e^{ik}$ does not. The imaginary part appears in $\chi_{ACV} e^{ik}$, when one sums multiple exchange of reggeized gravitons [10].
The asymptotics of our eikonal at large impact parameter \((58)\) coincides with the real part of \(\chi^{\text{eik}}_{ACV}(10)\), up to a constant depending on the number of the extra dimensions.

The \(D\)-dimensional eikonal amplitude, \(A^{\text{eik}}_{ACV}(11)\) has the nonrenormalized Born pole at \(t = 0\). In the brane amplitude \((56)\) the renormalized Born pole is reproduced. This singular part makes a leading contribution at small momentum transfers and it coincides with the \(D\)-dimensional amplitude taken at \(D \to 4\).

To summarize, the presence of the compact extra dimensions do not influence small \(t\) behavior of the scattering amplitude, if a collision takes place on the \((1+3)\)-dimensional brane. This is easy to understand, since, from the point of view of four dimensions, the higher space dimensions supply us with the KK tower of massive exchange quanta (in our case, massive gravitons). These new massive quanta can not hide the long-range forces originated from the massless graviton.

Accounting for the contributions of the massive gravitons results in an additive term, which is important at large and intermediate \(t\) (59). Note, that the eikonal depends, in general, on the ratio \(b/2R_c\) (it is of the form \(\chi(b) \sim (b_c/b)^d\) only at \(b \to 0, \infty\)). Therefore, the relevant dimensionless parameters for the amplitude are \(G_N s\) and \(R_c^2|t|\) (but not \(b_c^2|t|\)). Remind that in the flat \((4 + d)\) dimensions the eikonal amplitude is given in terms of the dimensional parameter \(G_D s|t|^{d/2}\) (see formulae in Section 2).

Let us stress again, we disagree with the statement of Refs. [11,12] saying that the brane amplitude in the eikonal approximation is dominated by the region \(b \sim b_c\), and it has no Coulomb part (see our comments in the text after formula (25)). The accurate account of the massless graviton mode shows that this is not the case, and \(A^{\text{eik}}\) has both “massless” (Coulomb) and “massive” term (56).

In Ref. [12] the quantity \(\chi_{GRW}(b)\) from [11] was applied for calculations of di-jet differential cross sections at the LHC energy in a kinematical region where gravity dominates. For not small \(t\), the “massless” part of the eikonal can be neglected. Nevertheless, the “massive” term of the eikonal, representing the contribution from the KK graviton modes, is given by more complicated expression than that derived in [11]. In particular, asymptotic behavior of \(\chi(b)\) (58) differs from the corresponding asymptotics for \(\chi_{GRW}(23)\) by a factor 2 \(\sin^2(\pi d/4)\). It seems, that the estimates for the di-jet differential cross sections should be reconsidered.
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Appendix

In this Appendix we will calculate the asymptotics of the RHS of equation (39) at large values of $b/2R_c$. The quantity under consideration, $I(b)$, is represented in terms of the generalized hypergeometric functions, $1F2(\alpha_1; \beta_1, \beta_2; z)$ and $2F3(\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; z)$ (37), (39). Let us introduce the notations

$$pF_q\left(\alpha_p \left| z \right.\right) \equiv pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z). \quad (A.1)$$

$$\Gamma(\alpha_p) \equiv \Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p). \quad (A.2)$$

Sometimes we will write simply $pF_q(z)$.

To solve the problem, we need to use a full asymptotic expansion of $pF_{p+1}(z)$ at large $z$ [18]:

$$pF_{p+1}\left(\alpha_p \left| \frac{z^2}{4} \right.\right) \sim \frac{\Gamma(\beta_{p+1})}{\Gamma(\alpha_p)} \left\{ K_{p,p+1} \left[ \left(\frac{1}{2}z\right)^2 \right] + L_{p,p+1} \left[ \left(\frac{1}{2}ze^{i\pi} \right)^2 \right] \right\}, \quad (A.3)$$

$|z| \to \infty$, arg $z = 0$. Let us remind that $z = b/R_c > 0$ and $p = 1, 2$ in our case. The function $K_{p,p+1}(z)$ in (A.3) is given by the series in inverse powers of variable $z$ [18]:

$$K_{p,p+1}\left[ \left(\frac{1}{2}z\right)^2 \right] = \frac{1}{2^{2\gamma+1}\sqrt{\pi}} e^{\frac{2\gamma}{2\pi}} z^{-2\gamma} \sum_{k=0}^{\infty} d_k z^{-k}, \quad d_0 = 1, \quad (A.4)$$

with

$$\gamma = \frac{1}{2} \left( \frac{1}{2} + \sum_{n=1}^{p} \alpha_n - \sum_{n=1}^{p+1} \beta_n \right). \quad (A.5)$$
Thus, $K_{p,p+1}(z)$ increases exponentially in $z$ at $z \to +\infty$, that can result in an exponential rise of $I_\epsilon$ [39] in the impact parameter $b$. However, we will show now that it is not the case due to a complete cancellation of terms, proportional to $K_{1,2}(z)$ and $z^r K_{2,3}(z)$ in [39].

Let us note first that $\gamma = -3/4$ and $\gamma = -3/4 - \epsilon$, respectively, for $1F_2(d/2 - \epsilon; 1 + d/2 - \epsilon, 1; z)$ and $2F_3(d/2, 1; 1 + d/2, 1 + \epsilon, 1 + \epsilon; z)$. For $p = 1$, coefficients $d_k$ in (A.4) obey recursion formulae [18]:

$$2(k + 1) d_{k+1}^{(p=1)} = [3k^2 + 2k(1 + C_1 - 3B_1) + 4D_1] d_k^{(p=1)}$$

$$-(k - 2\gamma - 1)(k - 2\gamma + 1 - 2\beta_1)(k - 2\gamma + 1 - 2\beta_2) d_{k-1}^{(p=1)}$$

$$= A_k^{(p=1)} d_k^{(p=1)} + A_{k-1}^{(p=1)} d_{k-1}^{(p=1)}, \quad (A.6)$$

where we have introduced the notations

$$B_1 = \sum_{n=1}^{p} \alpha_n, \quad C_1 = \sum_{n=1}^{p+1} \beta_n, \quad (A.7)$$

$$B_2 = \sum_{n=2}^{p} \sum_{m=1}^{n-1} \alpha_n \alpha_m, \quad C_2 = \sum_{n=2}^{p+1} \sum_{m=1}^{n-1} \beta_n \beta_m, \quad (A.8)$$

$$D_1 = C_2 - B_2 + \frac{1}{4}(B_1 - C_1)(3B_1 + C_1 - 2) - \frac{3}{16}. \quad (A.9)$$

For $p = 2$, recursion relations look like [18]:

$$2(k + 1) d_{k+1}^{(p=2)} = [5k^2 + 2k(3 + B_1 - 3C_1 - 10\gamma) + 4D_1] d_k^{(p=2)}$$

$$-[4k^2 - 6k^2(C_1 + 4\gamma) + 2k(24\gamma^2 + 12\gamma C_1 + C_1 + 4C_2 - 1)$$

$$-32\gamma^3 - 24\gamma^2 C_1 - 4\gamma(C_1 + 4C_2 - 1) + 2C_1 - 4C_2 - 8C_3 - 1] d_{k-1}^{(p=2)}$$

$$+(k - 2\gamma - 2)(k - 2\gamma - 2\beta_1)(k - 2\gamma - 2\beta_2)(k - 2\gamma - 2\beta_3) d_{k-2}^{(p=2)}$$

$$= A_k^{(p=2)} d_k^{(p=2)} + A_{k-1}^{(p=2)} d_{k-1}^{(p=2)} + A_{k-2}^{(p=2)} d_{k-2}^{(p=2)}, \quad (A.10)$$

where

$$C_3 = \beta_1 \beta_2 \beta_3 \quad (A.11)$$

and other quantities are defined as before (A.7)-(A.9).

By making a replacement $k \to k - 1$ in (A.6), we get

$$2k d_{k}^{(p=1)} = [3(k - 1)^2 + 2(k - 1)(1 + C_1 - 3B_1) + 4D_1] d_{k-1}^{(p=1)}$$

$$-(k - 2\gamma - 2)(k - 2\gamma - 2\beta_1)(k - 2\gamma - 2\beta_2) d_{k-2}^{(p=1)}. \quad (A.12)$$
Let us now rewrite equation (A.6) in the form
\[ 2(k + 1) d_{k+1}^{(p=1)} = [\tilde{A}_k^{(p=1)} - A_k^{(p=2)}] d_k^{(p=1)} + A_k^{(p=2)} d_{k-1}^{(p=1)} + \tilde{A}_{k-1}^{(p=1)} d_{k-1}^{(p=1)} \] (A.13)
and substitute \( d_k^{(p=1)} \) from (A.12) into the first term in the RHS of (A.13). Then we obtain:
\[ 2(k + 1) d_{k+1}^{(p=1)} = A_k^{(p=1)} d_k^{(p=1)} + A_k^{(p=1)} d_{k-1}^{(p=1)} + A_k^{(p=1)} d_{k-2}^{(p=1)}. \] (A.14)

By direct calculations one can check that \( A_k^{(p=1)} = A_k^{(p=2)} \), namely:
\[
\begin{align*}
A_k^{(p=1)} &= A_k^{(p=2)} = 5k^2 + k(5 - 2n + 8\epsilon) + \frac{13}{4} - 2n + 4\epsilon, \\
A_{k-1}^{(p=1)} &= A_{k-1}^{(p=2)} = -4k^3 + 3k^2(n - 4\epsilon) + k(-1 + 4n\epsilon - 8\epsilon^2) + \frac{1}{4}(n - 4\epsilon), \\
A_{k-2}^{(p=1)} &= A_{k-2}^{(p=2)} = \frac{1}{16}(2k - 1)^2(2k - 1 + 4\epsilon) \\
&\quad \times (2k - 1 - 2n + 4\epsilon). \quad (A.15)
\end{align*}
\]

In other words, we have shown that both \( d_k^{(p=1)} \) and \( d_k^{(p=2)} \) obey the same recursion formulae, and \( d_k^{(p=1)} = d_k^{(p=2)} \) for all \( k \) in series (A.4).

Therefore, the \( K \)-functions do not contribute to expansion (A.3) in our case, and we have
\[
I_\epsilon(b) = \frac{d}{2\epsilon} \Gamma(1 + \epsilon)\Gamma(1 + \epsilon) \left[ -\left( \frac{b^2}{4R_c^2} \right)^\epsilon \right.
\times L_{2,3} \left( \begin{array}{c} d/2, 1 \\ 1 + d/2, 1 + \epsilon, 1 + \epsilon \end{array} \left| \frac{b^2}{4R_c^2} \right. \right] + L_{1,2} \left( \begin{array}{c} d/2 - \epsilon, /, 1 \\ 1 + d/2 - \epsilon, /, 1 \end{array} \left| \frac{b^2}{4R_c^2} \right. \right]. \quad (A.16)
\]

Note, the \( L \)-function is related to the Meijer \( G \)-function \[18\]:
\[
L_{p,p+1} \left( \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_{p+1} \end{array} \bigg| z \right) = G_{p+1,1}^{p,0} \left( \frac{1}{z} \bigg| \begin{array}{c} 1, \beta_1, \ldots, \beta_{p+1} \\ \alpha_1, \ldots, \alpha_p \end{array} \right). \quad (A.17)
\]
In its turn, the $G$-function can be represented as a series in generalized hypergeometric functions of an inverse power of $z$:

\[
L_{p,p+1}(z) = \sum_{n=1}^{p} L_{n,p}(z),
\]

(A.18)

where

\[
L_{n,p+1}(z) = z^{-\alpha_n} \frac{\Gamma(\alpha_n)\Gamma(\alpha_p - \alpha_n)}{\Gamma(\beta_{p+1} - \alpha_n)} \\
\times \binom{p+2}{p+1} F_{p+1} \left( \frac{\alpha_n, 1 + \alpha_n - \beta_{p+1}}{1 + \alpha_n - \alpha_p} \left| -\frac{1}{z} \right. \right) \]

(A.19)

(the superscript * means that the term with $\alpha_n = \alpha_p$ is not included in a product of the $\Gamma$-functions).

As a result, we arrive at the expression for $I_\epsilon$, which is convenient for analyzing its large $b$ behavior:

\[
I_\epsilon(b) = \frac{1}{\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(1 + \epsilon)\Gamma(1 + d/2)}{\Gamma(1 - d/2 + \epsilon)} \\
\times \left\{ -\left( \frac{b^2}{4R_c^2} \right)^{\epsilon-1} \frac{\Gamma(d/2 - 1)}{\Gamma^2(\epsilon)\Gamma^2(d/2)} \right\} \\
\times 4F_1 \left( 1, 1 - \frac{d}{2}, 1 - \epsilon, 1 - \epsilon; 2 - \frac{d}{2}; -\frac{4R_c^2}{b^2} \right) \\
+ \left( \frac{b^2}{4R_c^2} \right)^{\epsilon-d/2} \frac{1}{\Gamma(1 - d/2 + \epsilon)} \\
\times \left[ -\frac{\Gamma(1 - d/2)}{\Gamma(1 - d/2 + \epsilon)} + \frac{\Gamma(d/2 - \epsilon)}{\Gamma(d/2)} \right]. \]

(A.20)

Up to now, we did not consider parameter $\epsilon$ to be small. Finally, a desired asymptotics looks like

\[
I(b) \bigg|_{b \gg R_c} = \lim_{\epsilon \to 0} I_\epsilon \bigg|_{b \gg R_c} = \left( \frac{2R_c}{b} \right)^{d} \frac{\Gamma(1 + d/2)}{\Gamma(1 - d/2)} \\
\times \frac{\Gamma(1 - d/2)}{\Gamma(1 - d/2 + \epsilon)} + \frac{\Gamma(d/2 - \epsilon)}{\Gamma(d/2)}. \]

(A.21)

By expanding the RHS of equality (A.21) in $\epsilon$ and taking the limit $\epsilon \to 0$, we derive an asymptotic formula for $I(b)$ presented in the text.
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