Nonlinear Thermal Instability in Compressible Viscous Flows Without Heat Conductivity

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Abstract. We investigate the thermal instability of a smooth equilibrium state, in which the density function satisfies Schwarzschild’s (instability) condition, to a compressible heat-conducting viscous flow without heat conductivity in the presence of a uniform gravitational field in a three-dimensional bounded domain. We show that the equilibrium state is linearly unstable by a modified variational method. Then, based on the constructed linearly unstable solutions and a local well-posedness result of classical solutions to the original nonlinear problem, we further construct the initial data of linearly unstable solutions to be the one of the original nonlinear problem, and establish an appropriate energy estimate of Gronwall-type. With the help of the established energy estimate, we finally show that the equilibrium state is nonlinearly unstable in the sense of Hadamard by a careful bootstrap instability argument.

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1. Introduction

Thermal (or convective) instability often arises when a fluid is heated from below. The classic example of this is a horizontal layer of fluid with its lower side hotter than its upper. The basic state is then one of rests with light and hot fluid below heavy and cool fluid. When the temperature difference across the layer is great enough, the stabilizing effects of viscosity and thermal conductivity are overcome by the destabilizing buoyancy, and an overturning instability ensues as thermal convection: hotter part of fluid is lighter and tends to rise as colder part tends to sink according to the action of the gravity force [8]. The phenomenon of thermal convection itself had been recognized by Rumford [43] and Thomson [47]. However, the first quantitative experiments on thermal instability and the recognition of the role of viscosity in the phenomenon are due to Benard [3], so the convection in a horizontal layer of a fluid heated from below is called Bénard convection.

The Bénard convection can be modeled by a (nonlinear) compressible Navier–Stokes–Fourier (simplified by NSF) equations, in which the coefficients of viscosity and heat conduction are non-zero, see [19] for example and the references cited therein. In 1906, Schwarzschild first derived the criterion for thermal stability of a fluid layer in hydrostatic equilibrium on the basis of Archimedes’ principle without considering the effects of viscosity and heat conduction [5,44]. Thus such criterion is named by Schwarzschild’s criterion. However, Schwarzschild’s argument is at most a suggestive one, since the buoyancy principle applies to the static, not to the perturbed state. Later the criterion was examined rigorously for the linearized Euler-Fourier system (in other words, the hydrodynamic equations for adiabatic motion, which are linearized for the case of small perturbations from equilibrium) by other authors, such as Lebowitz [32], Kaniel-Kovetz [30], Rosencrans [42] and so on.

Unfortunately, the rigorous derivation of Schwarzschild’s criterion for the compressible Euler–Fourier model (hyperbolic type) can not be applied to the compressible NSF model (hyperbolic–parabolic coupled...
Therefore, to investigate the effect of viscosity and heat conduction in the Bénard convection, we adopt the (nonlinear) Boussinesq approximation equations, in which the density is considered as a constant in all the terms of the equations except for the gravity term that is assumed to vary linearly with the temperature \([8]\). It should be noted that this approximation is reasonable, only if the thickness of the layer is small. Compared with the compressible NSF model, the Boussinesq model enjoys relatively good structure, and moreover, it is parabolic due to the absence of the continuity equation of mass conservation (hyperbolic type). Thus, based on the linearized Boussinesq model, Rayleigh first showed that instability would occur only when the adverse temperature gradient is so large that the Rayleigh number exceeds a certain critical value \([41]\). Rayleigh’s criterion rigorously verifies the destabilizing effect of buoyancy to the stabilizing effects of viscosity and heat conduction, please refer to the classical monograph \([6]\) for the details and other results involving the inhibitive effects of the rotation \([12,16]\) and the magnetic field \([13–15]\) on the thermal convection. Moreover, by the energy method and the bootstrap instability method, one can see that Rayleigh’s criterion on instability/stability in the Hadamard sense still holds for the nonlinear Boussinesq model, see \([21,29]\) for example. In addition, the mathematical theory of attractor bifurcation for two-dimensional Boussinesq model \([33]\) have been established Ma and Wang. However, the corresponding three-dimensional case is still an open problem.

When the density varies by many orders of magnitude (e.g., across a stellar convection zone \([45]\)), the Boussinesq model obviously fails. Hence, for the Bénard convection in compressible atmosphere, we must adopt the compressible NSF model, and expect to gain a complete understanding of the effects of density variation and compressibility. Next we briefly review some results on the Bénard convection based on the comprehensible NSF model (called compressible Bénard problem for simplicity). The results of linear stability and instability can be founded in \([26,46,48]\). Numerical simulations for the compressible Bénard convection are given in \([4,18]\) for two- and three-dimensional cases, respectively. Moreover, Gough et al. \([17]\) have noted that the stability or instability depends on the boundary conditions, the heat conduction coefficient and viscosity coefficient from the numerical computations. The first theoretical rigorous proof of nonlinear stability was given by Padula and Bollettmo \([40]\). Successively, Coscia and Padula further provided a computable critical number and showed a nonlinear stability result for the compressible Bénard problem whenever the Rayleigh number does not exceed the critical number \([7]\). Later, Aye and Nishida \([2]\) used the approach of a computer assisted proof to obtain the critical Rayleigh number for instability. Recently, Nishida et al. \([37,38]\) proved the existence of steady solutions of the compressible Bénard problem.

In \([19]\), Guidoboni and Padula showed the stability of the linearized compressible Bénard problem without thermal conductivity under the Schwarzschild’s stability condition. At present, it is still open whether the linear result obtained by Guidoboni and Padula can be generalized to the nonlinear case. However, in this article, we show that under Schwarzschild’s instability condition, the nonlinear compressible Bénard problem without thermal conductivity is unstable. We mention that Schwarzschild’s instability criterion can be derived on the basis of linearized energy method, however, our result first rigorously verifies the validity of Schwarzschild’s instability criterion for the nonlinear case. Next, we shall formulate our problem in details.

1.1. Rayleigh–Bénard problem

The motion of a three-dimensional (3D) compressible viscous fluid without heat conductivity in the presence of a uniform gravitational field in a bounded domain \(\Omega \subset \mathbb{R}^3\) with smooth boundary is governed by the following equations:

\[
\begin{aligned}
\rho_t + \text{div}(\rho v) &= 0, \\
\rho v_t + \rho v \cdot \nabla v + \nabla p &= \mu \Delta v + \mu_0 \nabla \text{div} v - \rho g e_3, \\
\rho e_t + \rho v \cdot \nabla e + p \text{div} v &= \mu |\nabla v + \nabla v^T|^2/2 + \varsigma (\text{div} v)^2.
\end{aligned}
\]
Here the unknowns $\rho := \rho(t,x)$, $v := v(t,x)$, $e := e(t,x)$ and $p := p(x,t)$ denote the velocity, specific internal energy and pressure of the compressible fluid respectively. $\mu > 0$ is the coefficient of shear viscosity, and $\mu_0 = \mu + \varsigma$ with $\varsigma$ being the bulk viscosity, satisfying $3\varsigma + 2\mu \geq 0$. $g > 0$ is the gravitational constant, $c_3 = (0,0,1)^T$ denotes the vertical unit vector. The superscript $T$ denotes the transposition.

As in [19], we assume the pressure satisfies the equations of state for idea gases [9], i.e.,
\begin{equation}
    p = R\rho T_e/M,
\end{equation}
where $M$ denotes the molar weight of the gas, $T_e$ the (absolute) temperature, and $R$ the ideal gas constant that is equal to the product of the Boltzmann constant and the Avogadro constant. The internal energy satisfies
\begin{equation}
    e = c_V T_e,
\end{equation}
where $c_V$ denotes the constant-volume specific heat per mole. Under the relations (1.2) and (1.3), the Eq. (1.1) are complete. If we add the dissipative term $\kappa \Delta T_e$ to the right hand side of (1.1)$_3$, then the resulting equations are the Navier–Stokes–Fourier equations with non-zero heat-conduction coefficient $\kappa$.

In this paper we consider the problem of nonlinear convective instability for the Eq. (1.1) around some equilibrium-state solution, the density in which satisfies Schwarzschild’s instability condition. Thus, we choose an (equilibrium-state) density profile $\bar{\rho} := \rho(x_3)$, which is independent of $(x_1,x_2)$ and satisfies
\begin{equation}
    \bar{\rho} \in C^4(\Omega), \quad \inf_{x \in \Omega} \bar{\rho} > 0,
\end{equation}
and Schwarzschild’s (instability) condition
\begin{equation}
    -\bar{\rho}'(x_3^0) < \frac{g\bar{\rho}^2(x_3^0)}{(1+a)\bar{p}(x_3^0)} \quad \text{for some } x_3^0 \in \{ x_3 | (x_1,x_2,x_3)^T \in \Omega \},
\end{equation}
where $\bar{\rho}' := d\bar{\rho}/dx_3$, $a := R/c_V M$ and $x_3^0$ denotes the third component of $x_0 \in \Omega$. We remark that the first condition in (1.4) guarantees that we can construct an unstable classical solution, while the second one in (1.4) prevents us from treating vacuum in the construction of unstable solutions. Schwarzschild’s condition results in the classical thermal instability as will be shown in Theorem 1.1 below. To clearly see the physical mechanism, next we reformulate (1.5) by the (equilibrium-state) temperature profile.

In view of the theory of first-order linear ODEs, for given $\bar{\rho}$ in (1.4) we can find a corresponding (equilibrium-state) internal energy profile $\bar{e}$ that only depends on $x_3$ and is unique up to a constant divided by $\bar{\rho}$, i.e.,
\begin{equation}
    \bar{e} = g(a\bar{\rho})^{-1}(C - F(\bar{\rho})),$$
\end{equation}
where $F(\bar{\rho})$ denotes some primitive function of $\bar{\rho}$. In view of (1.4), we can choose a sufficiently large constant $C$ such that $\bar{e} > 0$ on $\Omega$. Hence, we can construct an internal energy profile $\bar{e}$, such that
\begin{equation}
    \bar{e} > 0 \quad \text{on } \Omega, \quad \bar{e} \in C^4(\Omega), \quad \nabla \bar{p} = -\bar{p}ge_3 \quad \text{in } \Omega \quad \text{with } \bar{p} := a\bar{\rho}\bar{e}.
\end{equation}
Clearly, the function triple $(\bar{\rho},0,\bar{e})$ gives an equilibrium state solution to the system (1.1). By virtue of the equilibrium state (1.6), Schwarzschild’s condition (1.5) is equivalent to
\begin{equation}
    \bar{e}'(x_3^0) < -\frac{g}{(1+a)} \quad \text{for some } x_3^0,
\end{equation}
i.e., the convective condition in view of (1.3):
\begin{equation}
    \bar{T}_e(x_3^0) < -\frac{g}{(c_V + R/M)} \quad \text{for some } x_3^0.
\end{equation}
From (1.8) we can easily see why Schwarzschild’s condition leads to the convective instability from the physical viewpoint.

We mention that, by Mayer’s formula $c_p - c_V = R$, the convective condition reduces to
\begin{equation}
    \bar{T}_e(x_3^0) < -g/c_p \quad \text{for some } x_3^0
\end{equation}
for $M = 1$, where $c_p$ denotes the constant-pressure heat capacity per mole. In [19], Guidoboni and Padula considered a layer domain $\mathbb{R}^2 \times (0,l)$ with $\Theta_1$ and $\Theta_2$ being the temperatures of the lower and the upper
planes respectively, where $\Theta_1 > \Theta_2$. They chose the density profile $\bar{\rho} = \alpha \zeta^m$ and the temperature profile $\bar{T}_e = \beta \zeta$, where

$$\zeta := \frac{x_3}{l} + \frac{\Theta_1 - \Theta_2}{\beta l}, \quad m := \frac{g}{R \beta l} - 1 \text{ and } \beta := \frac{\Theta_1 - \Theta_2}{d}.$$ 

Thus, the condition (1.9) further reduces to the condition $\beta < g/c_p$, which is a special case of the convective condition (1.8). We mention that Schwarzschild’s condition has another version in terms of entropy, see [32,42] for example.

Now, we define the perturbation of $(\rho, v, e)$ around the equilibrium state $(\bar{\rho}, 0, \bar{e})$ by

$$\rho = \rho - \bar{\rho}, \quad u = v - 0, \quad \theta = e - \bar{e}.$$ 

Then, the triple $(\rho, u, \theta)$ satisfies the perturbed equations

$$\begin{aligned}
\rho_t + \text{div}((\rho + \bar{\rho})u) &= 0, \\
(\rho + \rho)u_t + (\rho + \bar{\rho})u \cdot \nabla u + a \nabla [(\rho + \bar{\rho})(\theta + \bar{e}) - \bar{\rho} \bar{e}] &= \mu \Delta u + \mu_0 \text{div}u - g \rho \bar{e}_3, \\
\theta_t + u \cdot \nabla (\theta + \bar{e}) + a(\theta + \bar{e}) \text{div}u &= \{\mu|\nabla u + \nabla u|^2/2 + \lambda (\text{div}u)^2\}/(\rho + \bar{\rho}).
\end{aligned} \tag{1.10}$$

To complete the statement of the perturbed problem, we specify the initial and boundary conditions:

$$\begin{aligned}
\rho_t + \text{div}((\rho + \bar{\rho})u) &= 0, \\
(\rho + \rho)u_t + (\rho + \bar{\rho})u \cdot \nabla u + a \nabla [(\rho + \bar{\rho})(\theta + \bar{e}) - \bar{\rho} \bar{e}] &= \mu \Delta u + \mu_0 \text{div}u - g \rho \bar{e}_3, \\
\theta_t + u \cdot \nabla (\theta + \bar{e}) + a(\theta + \bar{e}) \text{div}u &= \{\mu|\nabla u + \nabla u|^2/2 + \lambda (\text{div}u)^2\}/(\rho + \bar{\rho}).
\end{aligned} \tag{1.11}$$

We call the initial-boundary problem (1.10)–(1.12) the (compressible) Rayleigh–Bénard problem (without heat conduction). For classical solutions of the Rayleigh–Bénard problem, the initial data should further satisfy the compatibility condition on boundary:

$$\{(\rho_0 + \bar{\rho})u_0 \cdot \nabla u_0 + a \nabla [(\rho_0 + \bar{\rho})(\theta_0 + \bar{e}) - \bar{\rho} \bar{e}]\}_{|\partial \Omega} = (\mu \Delta u_0 + \mu_0 \text{div}u_0 - g \rho_0 \bar{e}_3)_{|\partial \Omega}.$$ 

Finally, if we linearize the Eq. (1.10) around the equilibrium state $(\bar{\rho}, 0, \bar{e})$, then the resulting linearized equations read as

$$\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + a \nabla (\rho \theta + \rho \bar{e}) &= \mu \Delta u + \mu_0 \text{div}u - g \rho \bar{e}_3, \\
\theta_t + \rho \bar{e}_3 + a \text{div}u &= 0.
\end{aligned} \tag{1.13}$$

Then the initial-boundary problem (1.11)–(1.13) is called the linearized (compressible) Rayleigh–Bénard problem (without heat conduction).

1.2. Main result

Before stating the main result of this paper, we explain the notations used throughout this paper. We always assume that the domain $\Omega$ is bounded with smooth boundary. For simplicity, we drop the domain $\Omega$ in Sobolve spaces and the corresponding norms as well as in integrands over $\Omega$, for example,

$$L^p := L^p(\Omega), \quad H^1 := W^{1,2}_0(\Omega), \quad H^k := W^{k,2}(\Omega), \quad \int := \int_{\Omega}.$$ 

In addition, a product space $(X)^n$ of vector functions is still denoted by $X$, for examples, the vector function $u \in (H^2)^3$ is denoted by $u \in H^2$ with norm $\|u\|_{H^2} := (\sum_{k=1}^3 \|u_k\|_{H^2}^2)^{1/2}$.

Now, we are able to state our main result on the nonlinear convective instability in the Rayleigh–Bénard problem (1.10)–(1.12).

**Theorem 1.1.** Under the assumptions (1.4)–(1.6), the equilibrium state $(\bar{\rho}, 0, \bar{e})$ is unstable in the Hadamard sense, that is, there are positive constants $A$, $m_0$, $\varepsilon$ and $\delta_0$, and functions $(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0, u_t) \in H^3$, such that for any $\delta \in (0, \delta_0)$ and the initial data

$$(\rho_0, u_0, \theta_0) := \delta(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0) + \delta^2(\bar{\rho}_0, u_t, \bar{\theta}_0) \in H^3,$$
there is a unique classical solution \((\rho, u, \theta)\) in \(C^0([0, T^{\max}), H^3)\) to the Rayleigh–Bénard problem (1.10)–(1.12) satisfying
\[
\|(u_1, u_2)(T^\delta)\|_{L^2}, \quad \|u_3(T^\delta)\|_{L^2} \geq \varepsilon
\] for some escape time \(T^\delta := \frac{1}{\Lambda} \ln \frac{2\varepsilon}{\max} \in (0, T^{\max})\), where \(T^{\max}\) denotes the maximal time of existence of the solution \((\rho, u, \theta)\), and \(u_i\) denotes the \(i\)-th component of \(u = (u_1, u_2, u_3)^T\).

**Remark 1.2.** Theorem 1.1 still holds for a horizontal periodic domain with finite height, i.e.,
\[
\Omega := \{x := (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid (x_1, x_2) \in \mathcal{T}, \ 0 < x_3 < l\} \quad \text{with} \quad l > 0,
\]
where \(\mathcal{T} := (2\pi L_1 \mathbb{T}) \times (2\pi L_2 \mathbb{T}), \mathbb{T} = \mathbb{R}/\mathbb{Z}\), and \(2\pi L_1, 2\pi L_2 > 0\) are the periodicity lengths. As mentioned before, it is still an open problem whether the initial-boundary problem (1.10)–(1.12) is stable, provided the density profile satisfies the stability condition
\[
- \bar{\rho}' > g\bar{\rho}^2/(1 + a)\bar{\rho} \text{ in } \Omega.
\] (1.15)

**Remark 1.3.** If the density profile satisfies
\[
\bar{\rho}' \geq 0,
\] which automatically implies Schwarzschild’s condition, then we can establish the instability of the perturbed density, i.e., Theorem 1.1 holds with \(\|\varrho(T^\delta)\|_{L^2} \geq \varepsilon\). The additional condition (1.16) is used to show
\[
\bar{\varrho}_0 := \text{div}(\bar{\varrho}\bar{n}_0) \not\equiv 0
\]
in the construction of a linear unstable solution (cf. (2.9)), where \((\bar{\varrho}_0, \bar{v}_0)\) is a solution to the time-independent system (2.1).

Next, we sketch the main idea in the proof of Theorem 1.1. The proof is broken up into three steps. Firstly, we make the following ansatz of growing mode solutions to the linearized problem:
\[
(\varrho(x, t), u(x, t), \theta(x, t)) = e^{\Lambda t}(\bar{\varrho}(x), \bar{v}(x), \bar{\theta}(x)) \quad \text{for some } \Lambda > 0
\] (1.17)
and deduce (1.13) thus into a time-independent PDE system on the unknown function \(\bar{v}\). Then we adapt and modify the modified variational method in [23] to the time-independent system to get a non-trivial solution \(\bar{v}\) with a sharp growth rate \(\Lambda\), which immediately implies that the linearized problem has an unstable solution in the form (1.17). This idea was used probably first by Guo and Tice to deal with an ODE problem arising in constructing unstable solutions of the linearized problem, and later adapted by other researchers to treat other linear instability problems of viscous fluids, see [25, 28]. Here we directly adapt this idea to the time-independent PDE system to avoid the use of the Fourier transform and to relax the restriction on domains. Secondly, we establish the energy estimates of Gronwall-type in \(H^3\)-norm. Similar (global in time) estimates were obtained for the compressible Navier–Stokes–Fourier equations with heat conductivity under the condition of small initial data and external forces [35, 36]. Here we have to modify the arguments in [35, 36] to deal with the compressible Navier–Stokes equations without heat conductivity. Namely, we deal with the sum \(\bar{\varepsilon} \varrho + \bar{\rho} \bar{v}\) as one term [see (3.34)] instead of dividing it into two terms in [36]; and we use the Eqs. (1.10)\(_1\) and (1.10)\(_2\) independently to control \(\|\varrho\|_{H^3}\) and \(\|\theta\|_{H^3}\) (i.e. Lemma 3.1), rather than coupling the equations together to control \(\|\varrho\|_{H^3}\) in [36]. With these modifications in techniques, we can get the desired estimates. Finally, we use the version of the bootstrap instability approach in [20] (the interested reader is referred to [10, 11, 22] for different versions of the bootstrap instability approach) to show Theorem 1.1, but have to circumvent two additional difficulties due to presence of boundary which do not appear for spatially periodic problems considered in [20]: (i) The idea of Duhamel’s principle on the solution operator for linear instability in [20] can not be directly applied to our boundary value problem here, since the nonlinear term in (1.10)\(_2\) does not vanish on boundary. To overcome this difficulty, we employ some specific energy estimates to replace Duhamel’s principle (see Lemma 4.2 on the error estimate for \(\|\varrho^d, u^d, \theta^d\|_{L^2}^2\)). (ii) On the boundary the initial data of the linearized Rayleigh–Bénard problem may not satisfy the compatibility condition imposed on the initial data of the corresponding nonlinear Rayleigh–Bénard problem (1.10)–(1.12). To circumvent this
difficulty, we employ the elliptic theory to construct initial data of the Rayleigh–Bénard problem that satisfy the compatibility condition and are close to the initial data of the linearized problem.

The rest of this paper is organized as follows. In Sect. 2 we construct unstable solutions of the linearized problem, while in Sect. 3 we deduce the nonlinear energy estimates. Section 4 is devoted to the proof of Theorem 1.1, and finally, in the appendix we give a proof of the sharp growth rate of solutions to the linearized problem in $H^2$-norm.

2. Linear instability

In this section, we adapt the modified variational method in [23] to construct a solution to the linearized equations (1.13) that has growing $H^3$-norm in time. We first make a solution ansatz (1.17) of growing normal mode. Substituting this ansatz into (1.13), one obtains the following time-independent system:

\[
\begin{cases}
\Lambda \tilde{\rho} + \text{div}(\tilde{\rho} \tilde{v}) = 0, \\
\Lambda \tilde{v} + a \nabla (\tilde{\rho} + \tilde{\rho} \tilde{\theta}) = \mu \Delta \tilde{v} + \mu_0 \nabla \text{div}\tilde{v} - g\tilde{\rho}e_3, \\
\Lambda \tilde{\theta} + \tilde{e} \tilde{v}_3 + a \tilde{\rho} \text{div}\tilde{v} = 0, \\
\tilde{v}|_{\partial\Omega} = 0.
\end{cases}
\] (2.1)

Eliminating $\tilde{\rho}$ and $\tilde{\theta}$, one has

\[
\begin{cases}
\Lambda^2 \tilde{\rho} + \nabla [g\tilde{\rho}e_3 - (1 + a)\tilde{\rho}\text{div}\tilde{v}] \\
= \Lambda \mu \tilde{\Delta} \tilde{v} + \Lambda \mu_0 \nabla \text{div}\tilde{v} + (g\tilde{\rho} \tilde{v}_3 + g\tilde{\rho} \text{div}\tilde{v})e_3, \\
\tilde{v}|_{\partial\Omega} = 0,
\end{cases}
\] (2.2)

where $\tilde{v}$ denotes the third component of $v$. In view of the basic idea of the modified variational method, we modify the boundary problem (2.2) as follows.

\[
\begin{cases}
\lambda^2 \tilde{\rho} + \nabla [g\tilde{\rho}e_3 - (1 + a)\tilde{\rho}\text{div}\tilde{v}] \\
= \lambda \mu \tilde{\Delta} \tilde{v} + \lambda \mu_0 \nabla \text{div}\tilde{v} + (g\tilde{\rho} \tilde{v}_3 + g\tilde{\rho} \text{div}\tilde{v})e_3, \\
\tilde{v}|_{\partial\Omega} = 0,
\end{cases}
\] (2.3)

where $\lambda := \lambda(s)$ depends on $s$. We remark that if $s = \Lambda$ is a fixed point of $\lambda(s)$ (i.e., $\lambda(\Lambda) = \Lambda$), then the problem (2.3) becomes (2.2).

Now, multiplying (2.3) by $\tilde{v}$ and integrating the resulting identity, we get

\[
\lambda^2 \int \tilde{\rho}|\tilde{v}|^2 dx = \int \{ g\tilde{\rho} \tilde{v}_3^2 + 2g\tilde{\rho} \tilde{v}_3 - (1 + a)\tilde{\rho}\text{div}\tilde{v}|\text{div}\tilde{v}\} dx - s \int (\mu|\nabla \tilde{v}|^2 + \mu_0 |\text{div}\tilde{v}|^2 ). 
\] (2.4)

We define

\[
E_1(\tilde{v}) = \int \{ g\tilde{\rho} \tilde{v}_3^2 + 2g\tilde{\rho} \tilde{v}_3 - (1 + a)\tilde{\rho}\text{div}\tilde{v}|\text{div}\tilde{v}\} dx,
\]

and

\[
E_2(\tilde{v}) = \int (\mu|\nabla \tilde{v}|^2 + \mu_0 |\text{div}\tilde{v}|^2 )dx.
\]

Then the standard energy functional for the problem (2.3) is given by

\[
E(\tilde{v}) := E_1(\tilde{v}, s) := E_1(\tilde{v}) - sE_2(\tilde{v})
\] (2.5)

with an associated admissible set

\[
\mathcal{A} := \left\{ \tilde{v} \in H_0^1 \mid J(\tilde{v}) := \int \tilde{\rho}|\tilde{v}|^2 dx = 1 \right\}.
\] (2.6)
Recalling (2.4), we can thus find \( \lambda \) by maximizing
\[
\lambda^2 := \sup_{\tilde{v} \in A} E(\tilde{v}). \tag{2.7}
\]
Obviously, \( \sup_{\tilde{v} \in A} E(\tilde{v}) < \infty \) for any \( s \geq 0 \).

Next we show that a maximizer of (2.7) exists and that the corresponding Euler–Lagrange equations are equivalent to (2.3).

**Proposition 2.1.** Assume that \((\bar{\rho}, \bar{e})\) satisfies (1.4) and (1.6), then for any but fixed \( s > 0 \), the following assertions hold.

1. \( E(\tilde{v}) \) achieves its supremum on \( A \).
2. Let \( \tilde{v}_0 \) be a maximizer and \( \lambda \) satisfy (2.7), then \( \tilde{v}_0 \in H^1 \) satisfies the boundary problem (2.3) and
\[
(\tilde{v}_0^1)^2 + (\tilde{v}_0^2)^2 \neq 0,
\]
where \( \tilde{v}_i^0 \) denotes the \( i \)-th component of \( \tilde{v}_0 \). In addition
\[
\text{div}(\tilde{v}_0) \neq 0, \quad \text{provided} \; \rho' \geq 0. \tag{2.9}
\]

**Proof.** (1) Let \( \tilde{v}_n \in A \) be a maximizing sequence, then \( E(\tilde{v}_n) \) is bounded from below. This fact together with (2.6) implies that \( \tilde{v}_n \) is bounded in \( H^1 \). So, there exists a \( \tilde{v}_0 \in H^1 \cap A \) and a subsequence (still denoted by \( v_n \) for simplicity), such that \( \tilde{v}_n \rightharpoonup \tilde{v}_0 \) weakly in \( H^1 \) and strongly in \( L^2 \). Moreover, by the lower semi-continuity, one has
\[
\sup_{\tilde{v} \in A} E(\tilde{v}) = \lim_{n \to \infty} \sup_{\tilde{v} \in A} E(\tilde{v}_n)
= \lim_{n \to \infty} \int \left[ g\rho'(\tilde{v}_3^2) + 2g\tilde{\rho}\tilde{v}_3^2 \text{div}\tilde{v}_n \right] \, dx
- \liminf_{n \to \infty} \int \left[ (1 + a)\tilde{\rho} \text{div}\tilde{v}_n \text{div}\tilde{v}_n + s(\mu|\nabla \tilde{v}_n|^2 + \mu_0|\text{div}\tilde{v}_n|^2) \right] \, dx
\leq E(\tilde{v}_0) \leq \sup_{\tilde{v} \in A} E(\tilde{v}),
\]
which shows that \( E(\tilde{v}) \) achieves its supremum on \( A \).

(2) To show the second assertion, we notice that since \( E(\tilde{v}) \) and \( J(\tilde{v}) \) are homogeneous of degree 2, (2.7) is equivalent to
\[
\lambda^2 = \sup_{\tilde{v} \in H^1_0} \frac{E(\tilde{v})}{J(\tilde{v}).} \tag{2.10}
\]
For any \( \tau \in \mathbb{R} \) and \( w \in H^1_0 \), we take \( \tilde{w}(\tau) := \tilde{v}_0 + \tau w \). Then (2.10) gives
\[
E(\tilde{w}(\tau)) - \Lambda^2 J(\tilde{w}(\tau)) \leq 0.
\]
If we set \( I(\tau) = E(\tilde{w}(\tau)) - \Lambda^2 J(\tilde{w}(\tau)) \), then we see that \( I(\tau) \in C^1(\mathbb{R}) \), \( I(\tau) \leq 0 \) for all \( \tau \in \mathbb{R} \) and \( I(0) = 0 \). This implies \( I'(0) = 0 \). Hence, a direct computation leads to
\[
\int_\Omega \{ s\mu \nabla \tilde{v}_0 : \nabla w + [s\mu_0 + (1 + a)\tilde{\rho}] \text{div}\tilde{v}_0 \text{div}w \} \, dx
= \int_\Omega [g\tilde{\rho} \text{div}\tilde{v}_0 e_3 + g\tilde{\rho}' \tilde{v}_0 e_3 - \nabla (g\tilde{\rho} \tilde{v}_0) - \Lambda^2 \tilde{\rho} \tilde{v}_0] \cdot \tilde{w} \, dx. \tag{2.11}
\]
which shows that \( \tilde{v} \) is a weak solution to the boundary problem (2.3). Recalling that \( 0 < \tilde{\rho} \in C^4(\tilde{\Omega}) \), \( \tilde{\rho} \in C^4(\tilde{\Omega}) \) and \( \tilde{v}_0 \in H^1(\tilde{\Omega}) \), by a bootstrap argument and the classical elliptic theory, we infer from the weak form (2.11) that \( \tilde{v}_0 \in H^4(\tilde{\Omega}) \).
Next we turn to the proof of (2.8) and (2.9) by contradiction. Suppose that \((\tilde{v}_1^0)^2 + (\tilde{v}_2^0)^2 \equiv 0\) or \(\text{div}(\tilde{\rho} \tilde{v}_0) \equiv 0\), then

\[
0 < \lambda^2 = \int \{g\tilde{\rho}'(\tilde{v}_3^0)^2 + [2g\tilde{\rho}\tilde{v}_3^0 - (1 + a)\tilde{p}\partial_{x_3} \tilde{v}_3^0]\partial_{x_3} \tilde{v}_3^0\} dx
- s \int (\mu |\nabla \tilde{v}_0|^2 + \mu_0 |\text{div} \tilde{v}_0|^2) dx
\]

or

\[
0 < \lambda^2 = \int \{g\tilde{\rho}'(\tilde{v}_3^0)^2 + [2g\tilde{\rho}\tilde{v}_3^0 - (1 + a)\tilde{p}\text{div} \tilde{v}_0]\} dx
- s \int (\mu |\nabla \tilde{v}_0|^2 + \mu_0 |\text{div} \tilde{v}_0|^2) dx
\]

(2.12)

which contradicts. Therefore, (2.8) and (2.9) hold. This completes the proof. 

Next, we want to show that there is a fixed point \(\Lambda\) such that \(\lambda(\Lambda) = \Lambda > 0\). To this end, we first give some properties of \(\alpha(s) := \sup_{\tilde{v} \in \mathcal{A}} E(\tilde{v}, s)\) as a function of \(s > 0\).

**Proposition 2.2.** Assume that \((\tilde{\rho}, \tilde{e})\) satisfies (1.4)–(1.6). Then the function \(\alpha(s)\) defined on \((0, \infty)\) enjoys the following properties:

1. \(\alpha(s) \in C_{\text{loc}}^{0,1}(0, \infty)\) is nonincreasing.
2. There are constants \(c_1, c_2 > 0\) which depend on \(g, \tilde{p}\) and \(\mu\), such that

\[
\alpha(s) \geq c_1 - sc_2.
\]

(2.14)

**Proof.** (1) Let \(\{\tilde{v}_{s_i}^n\}_{n=1}^{+\infty} \subset \mathcal{A}\) be a maximizing sequence of \(\sup_{\tilde{v} \in \mathcal{A}} E(\tilde{v}, s_i) = \alpha(s_i)\) for \(i = 1\) and 2. Then

\[
\alpha(s_1) \geq \limsup_{n \to \infty} E(\tilde{v}_{s_1}^n, s_1) \geq \liminf_{n \to \infty} E(\tilde{v}_{s_2}^n, s_2) = \alpha(s_2)
\]

for any \(0 < s_1 < s_2 < \infty\). Hence \(\alpha(s)\) is nonincreasing on \((0, \infty)\). Next we use this fact to show the continuity of \(\alpha(s)\).

Let \(I := [b, c] \subset (0, \infty)\) be a bounded interval. Noting that, by Cauchy–Schwarz’s inequality,

\[
E(\tilde{v}) \leq \int (g\tilde{\rho}' \tilde{v}_3^2 + 2g\tilde{\rho}\tilde{v}_3 \text{div} \tilde{v} - (1 + a)\tilde{p}\text{div} \tilde{v}) dx - (1 + a) \int \tilde{p}|\text{div} \tilde{v}|^2 dx
\]

\[
\leq g \left[ \left\| \frac{\tilde{\rho}'}{\tilde{\rho}} \right\|_{L^\infty} + \frac{g}{(1 + a)} \left\| \frac{\tilde{\rho}}{\tilde{p}} \right\|_{L^\infty} \right].
\]

Hence, by the monotonicity of \(\alpha(s)\) we have

\[
|\alpha(s)| \leq \max \left\{ |\alpha(b)|, g \left[ \left\| \frac{\tilde{\rho}'}{\tilde{\rho}} \right\|_{L^\infty} + \frac{g}{(1 + a)} \left\| \frac{\tilde{\rho}}{\tilde{p}} \right\|_{L^\infty} \right] \right\} := L < \infty
\]

(2.15)

for any \(s \in I\). On the other hand, for any \(s \in I\), there exists a maximizing sequence \(\{\tilde{v}_{s_i}^n\} \subset \mathcal{A}\) of \(\sup_{\tilde{v} \in \mathcal{A}} E(\tilde{v}, s)\), such that

\[
|\alpha(s) - E(\tilde{v}_{s_i}^n, s)| < 1.
\]

(2.16)
Making use of (2.5), (2.15) and (2.16), we infer that
\[
0 \leq \int (\mu |\nabla \tilde{v}|^2 + \mu_0 |\text{div}\tilde{v}|^2) \, dx
\]
\[
= \frac{1}{s} \int \{g\tilde{p}' |\tilde{v}_s|^2 + [2g\tilde{p}\tilde{v}_s - (1 + a)\tilde{p}\text{div}\tilde{v}_s] |\text{div}\tilde{v}_s|\} \, dx - \frac{E(\tilde{v}_s, s)}{s}
\]
\[
\leq \frac{1 + L}{b} + \frac{g}{b} \left[ \|\tilde{p}'\|_{L^\infty} + \frac{g}{(1 + a)} \|\tilde{p}\|_{L^\infty} \right] := K.
\]
Thus, for \( s_i \in I \ (i = 1, 2) \), we further find that
\[
\alpha(s_1) = \limsup_{n \to \infty} E(\tilde{v}_{s_1}, s_1)
\]
\[
\leq \limsup_{n \to \infty} E(\tilde{v}_{s_1}, s_2) + |s_1 - s_2| \limsup_{n \to \infty} \int_{\Omega} (\mu |\nabla \tilde{v}_s|^2 + \mu_0 |\text{div}\tilde{v}_s|^2) \, dx
\]
\[
\leq \alpha(s_2) + K|s_1 - s_2|.
\]

Reversing the role of the indices 1 and 2 in the derivation of the inequality (2.17), we obtain the same boundedness with the indices switched. Therefore, we deduce that
\[
|\alpha(s_1) - \alpha(s_2)| \leq K|s_1 - s_2|,
\]
which yields \( \alpha(s) \in C_{\text{loc}}^{0,1}(0, \infty) \).

(2) We turn to prove (2.14). First we should construct a function \( v \in H_0^1 \), such that
\[
E(v) > 0.
\]

In view of Schwarzschild’s condition, there is a cylinder
\[
\Xi_{x_0}^\delta := \left\{ x \in \mathbb{R}^3 \left| \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} \leq \delta, \ |x_3 - x_3^0| \leq \delta \right. \right\} \subset \Omega,
\]
such that
\[-\tilde{\rho}' < \frac{g\tilde{\rho}^2}{(1 + a)\tilde{p}} \text{ on } \Xi_{x_0}^\delta.
\]

Now, choose a smooth function \( f(z) \in H_0^1(\mathbb{R}) \), such that
\[
f(z) \begin{cases} > 0, & \text{for } |z| < \delta/2; \\
= 0, & \text{for } |z| \geq \delta/2
\end{cases}
\]

Then we define
\[
\tilde{v}(x) := \left( 0, \varphi(x_3) f \left( \sqrt{x_1^2 + x_2^2} \right), f(x_3) \partial_2 f \left( \sqrt{x_1^2 + x_2^2} \right) \right),
\]
where
\[
\varphi = \frac{g\bar{\rho}}{(1 + a)\bar{p}} f - f'.
\]

It is easy to check that \( v := \tilde{v}(x - x_0) \in H_0^1(\Omega) \) satisfies
\[
\int g \left[ \tilde{\rho}' + \frac{g\tilde{\rho}^2}{(1 + a)\tilde{p}} \right] v_3^2 \, dx > 0 \text{ and } \frac{g\tilde{\rho}v_3}{(1 + a)\tilde{p}} - \text{div}\tilde{v} = 0,
\]

which, together with the relation
\[
E(v) = \int \left\{ g \left[ \tilde{\rho}' + \frac{g\tilde{\rho}^2}{(1 + a)\tilde{p}} \right] v_3^2 - (1 + a)\tilde{p} \left[ \frac{g\tilde{\rho}v_3}{(1 + a)\tilde{p}} - \text{div}\tilde{v} \right]^2 \right\} \, dx,
\]

implies that (2.18).
With (2.18) to hand, one has
\[
\alpha(s) = \sup_{\tilde{v} \in \mathcal{A}} E(\tilde{v}, s) = \sup_{\tilde{v} \in H^3_0} \frac{E(\tilde{v}, s)}{J(\tilde{v})} = \frac{E(v, s)}{J(v)} \geq \frac{E_1(v)}{\int \tilde{v}^2 dx} - s \mu \int |\nabla v|^2 dx =: c_1 - sc_2
\]
for two positive constants \(c_1 := c_1(g, \bar{\rho})\) and \(c_2 := c_2(g, \mu, \bar{\rho})\). This completes the proof of Proposition 2.2.

Next we show that there exists a function \(\tilde{v}\) satisfying (2.2) with a grow rate \(\lambda\). Let \(\mathfrak{S} := \sup\{s \mid \alpha(\tau) > 0 \text{ for any } \tau \in (0, s)\}\). By virtue of Proposition 2.2, \(\mathfrak{S} > 0\); and moreover, \(\alpha(s) > 0\) for any \(s < \mathfrak{S}\). Since \(\alpha(s) = \sup_{\tilde{v} \in \mathcal{A}} E(\tilde{v}, s) < \infty\), we make use of the monotonicity of \(\alpha(s)\) to deduce that
\[
\lim_{s \to 0} \alpha(s) \text{ exists and the limit is a positive constant. (2.19)}
\]

On the other hand, by virtue of Poincaré’s inequality, there is a constant \(c_3\) dependent of \(g, \bar{\rho}\) and \(\Omega\), such that
\[
g \int (\bar{\rho}' \tilde{v}_3^2 + 2\bar{\rho} \tilde{v}_3 \text{div}\tilde{v}) \, dx \leq c_3 \int |\nabla \tilde{v}|^2 dx \quad \text{for any } \tilde{v} \in \mathcal{A}.
\]
Thus, if \(s > c_3/\mu\), then
\[
g \int (\bar{\rho}' \tilde{v}_3^2 + 2\bar{\rho} \tilde{v}_3 \text{div}\tilde{v}) \, dx - s \mu \int |\nabla \tilde{v}|^2 dx < 0 \quad \text{for any } \tilde{v} \in \mathcal{A},
\]
which implies that
\[
\alpha(s) \leq 0 \quad \text{for any } s > c_3/\mu.
\]
Hence \(\mathfrak{S} < \infty\), and moreover,
\[
\lim_{s \to \mathfrak{S}} \alpha(s) = 0. \quad (2.20)
\]

Now, employing a fixed-point argument, exploiting (2.19), (2.20), and the continuity of \(\alpha(s)\) on \((0, \mathfrak{S})\), we find that there exists a unique \(\Lambda \in (0, \mathfrak{S})\), such that
\[
\Lambda = \sqrt{\alpha(\Lambda)} = \sqrt{\sup_{\tilde{w} \in \mathcal{A}} E(\tilde{w}, \Lambda)} > 0. \quad (2.21)
\]

In view of Proposition 2.1, there is a solution \(\tilde{v} \in H^4\) to the boundary problem (2.3) with \(\Lambda\) constructed in (2.21). Moreover, \(\Lambda^2 = E(\tilde{v}, \Lambda), \tilde{v}_3^2 \neq 0\) and \(\tilde{v}_3 \neq 0\) by (2.21) and (2.5). In addition, \(\text{div}(\bar{\rho} \tilde{v}) \neq 0\) provided \(\bar{\rho}' \geq 0\). Thus we have proved

**Proposition 2.3.** Assume that \((\bar{\rho}, \bar{\epsilon})\) satisfies (1.4)–(1.6). Then there exists a \(\tilde{v} \in H^4\) satisfying the boundary problem (2.2) with a growth rate \(\Lambda > 0\) defined by
\[
\Lambda^2 = \sup_{\tilde{w} \in H^3_0(\Omega)} \frac{E_1(\tilde{w}) - \Lambda E_2(\tilde{w})}{\int \tilde{w}^2 dx}. \quad (2.22)
\]
Moreover, \(\tilde{v}\) satisfies \(\text{div}(\bar{\rho} \tilde{v}) \neq 0, \tilde{v}_3^2 \neq 0\) and \(\tilde{v}_3 \neq 0\). In particular, let \((\bar{\rho}, \tilde{v}, \tilde{\theta}) := -\text{div}(\bar{\rho} \tilde{v}), \bar{\epsilon} \tilde{v}_3 + \bar{\alpha} \text{div}\tilde{v})/\Lambda\), then \((\bar{\rho}, \tilde{v}, \tilde{\theta}) \in H^3\) satisfies (2.1). In addition, \(\bar{\rho} \neq 0\) provided \(\bar{\rho}' \geq 0\).

As a result of Proposition 2.3, one immediately gets the following linear instability.
Theorem 2.4. Assume that \((\bar{\rho}, \bar{v}, \bar{\theta})\) satisfies (1.4)–(1.6). Then the equilibrium state \((\bar{\rho}, 0, \bar{v}, \bar{\theta})\) is linearly unstable. That is, there exists an unstable solution

\[
(\rho, u, \theta) := e^{\Lambda t}(\bar{\rho}, \bar{v}, \bar{\theta})
\]

to the linearized Rayleigh–Bénard problem (1.11)–(1.13), such that \((\bar{\rho}, \bar{v}, \bar{\theta}) \in H^3\) and

\[
\|(u_1, u_2)(t)\|_{L^2} \text{ and } \|u_3(t)\|_{L^2} \to \infty \text{ as } t \to \infty,
\]

where the constant growth rate \(\Lambda\) and \((\bar{\rho}, \bar{v}, \bar{\theta})\) are constructed in Proposition 2.3. Moreover, \(\bar{\rho} \neq 0\) provided \(\bar{\rho}' \geq 0\).

Remark 2.5. Recently, Xi et.al. [49] further adopted the proof of Theorem 2.4 to give the unstable solutions for the corresponding linearized magnetic Bénard problem.

3. Nonlinear energy estimates

In this section, we derive some nonlinear energy estimates for the (nonlinear) Rayleigh–Bénard problem (1.10)–(1.12) and an estimate of Gronwall-type in \(H^3\)-norm, which will be used in the proof of Theorem 1.1 in the next section. To this end, let \((\rho, u, \theta)\) be a solution of the Rayleigh–Bénard problem, such that

\[
E(t) := E(\rho, u, \theta)(t) := \|(\rho, u, \theta)(t)\|_{H^3} \leq \delta_1^4,
\]

where \(\delta_1^4\) is sufficiently small. It should be noted that the smallness depends on the physical parameters in (1.10), and satisfies the following property by using the embedding \(H^3 \hookrightarrow L^{\infty}\):

\[
0 < \frac{\inf_{x \in \Omega}(\bar{\rho})}{2} \leq \rho(t, x) := \rho + \bar{\rho} \leq 2\sup_{x \in \Omega}\{\bar{\rho}\} \text{ for any } t \geq 0, \ \Omega \supseteq \Omega,
\]

where \(\rho\) and \(\bar{\rho}\) are constants. We remark here that these assumptions will be repeatedly used in what follows. Moreover, we assume that the solution \((\rho, u, \theta)\) possesses proper regularity, so that the procedure of formal calculations makes sense. For simplicity, we only sketch the outline and shall omit the detailed calculations. We remind that in the calculations that follow, we shall repeatedly use the Sobolev embedding theorem [39, Subsection 1.3.5.8], Young’s, Hölder’s and Poincaré’s inequalities, and the following interpolation inequality [1, Chapter 5]:

\[
\|f\|_{H^s} \lesssim \|f\|_{L^2}^{1-s} \|f\|_{H^{s+1}} \leq C_s \|f\|_{L^2} + \epsilon \|f\|_{H^{s+1}} \text{ for any constant } \epsilon > 0.
\]

In addition, we shall always use the following abbreviations in what follows.

\[
E_0 := E(\rho_0, u_0, N_0), \quad D^k := \{\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \partial_{x_3}^{k_3}\}_{k_1 + k_2 + k_3 = k},
\]

\[
\|gD^k f\|^2 := \sum_{k_1 + k_2 + k_3 = k} \|g \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \partial_{x_3}^{k_3} f\|^2 \text{ for any norm } \| \cdot \|,
\]

\[
\frac{d}{dt} := \partial_t + u \cdot \nabla \text{ denotes the material derivative},
\]

\[
L^0(\rho, u) := \partial_t + \partial^\rho u_3 + \partial^\rho \text{div} u, \quad N^0 := N^0(\rho, u) := -\text{div}(\rho u),
\]

\[
L^u := L^u(\rho, u, \theta) := \partial_t u + a(\bar{\rho} + \bar{\theta}) \rho \theta - \mu \Delta u - \mu_0 \text{div} u + g \varepsilon_3,
\]

\[
N^u := N^u(\rho, u, \theta) := -(\rho + \bar{\rho}) u \cdot \nabla u - \rho u_t - a \nabla \theta,
\]

\[
L^\theta := L^\theta(\rho, u, \theta) := \theta_t + \theta^\rho u_3 + a \text{div} u,
\]

\[
N^\theta := N^\theta(\rho, u, \theta) := [\mu/\nabla u + \nabla(u)^T]^{2}/2 + \lambda(\text{div} u)^2)/(\rho + \bar{\rho}) - u \cdot \nabla \theta - a \theta \text{div} u,
\]

\[
R(t) := \bigg(\bigg(\rho, u, \theta, \frac{d}{dt} (\bar{\rho} + \bar{\theta})\bigg)\bigg)^2_{H^2} + E(\|u\|_{H^3} + \|u\|^2_{H^4} + E^2),
\]

\(a \lesssim b\) means that \(a \leq Cb\) for some constant \(C > 0\),
where the constant $C$ may depend on some physical parameters in the perturbed equations (1.10). In particular, the perturbed equations can be written as the following non-homogenous form:

$$
L^ρ = N^ρ, \quad (3.2)
$$

$$
L^u = N^u, \quad (3.3)
$$

$$
L^θ = N^θ. \quad (3.4)
$$

In addition, we can use (3.2) and (3.4) to deduce that

$$
\text{div} u = -\frac{1}{\bar{\rho} + \bar{p}} \frac{d}{dt} (\bar{\rho} \bar{q} + \bar{p}) + (\bar{e}' \rho + \bar{p}' \theta) u_3 + \bar{p} (N^θ + u \cdot \nabla \theta) - (\bar{\rho} \bar{e})' u_3 - \bar{e} \rho \text{div} u.
$$

(3.5)

Thus (3.3) can be rewritten as follows, which will be used in the boundary estimates.

$$
L^u_{\text{new}} = N^u_{\text{new}}, \quad (3.6)
$$

where

$$
L^u_{\text{new}} := \bar{\rho} u_t - \mu \Delta u + g e_3 + \nabla \left[ \frac{\mu_0}{\bar{\rho} + \bar{p}} \frac{d}{dt} (\bar{\rho} \bar{q} + \bar{p}) + a \bar{e} \rho + a \bar{p} \theta \right]
$$

and

$$
N^u_{\text{new}} := N^u + \mu_0 \nabla \{(\bar{e}' \rho + \bar{p}' \theta) u_3 + \bar{p} (N^θ + u \cdot \nabla \theta) - (\bar{\rho} \bar{e})' u_3 - \bar{e} \rho \text{div} u / (\bar{\rho} + \bar{p}) \}.
$$

Next, we shall establish a series of lemmas which imply a priori estimates for the perturbed density, velocity and temperature.

### 3.1. Estimates on the whole domain

Firstly, we have the following estimate on the perturbed density and temperature.

#### Lemma 3.1

For $0 \leq k \leq 3$, it holds that

$$
\|(\rho, \theta)(t)\|_{H^k}^2 \lesssim \|(\rho, \theta)(0)\|_{H^k}^2 + \int_0^t \mathcal{E}(\|u\|_{H^{k+1}} + \mathcal{E}^2) dt.
$$

**Proof.** Using the identity

$$
\int_0^t \int D^k L^θ D^k θ dx dt = \int_0^t \int D^k N^θ D^k θ dx dt \quad \text{for } 0 \leq k \leq 3,
$$

we have

$$
\|D^k θ(t)\|_{L^2} = \|D^k θ(0)\|_{L^2} - \int_0^t \int D^k (\bar{e}' u_3 + a \bar{e} \text{div} u) D^k θ dx dt
$$

$$
- \int_0^t \int D^k (u \cdot \nabla θ + a \theta \text{div} u) D^k θ dx dt
$$

$$
+ \int_0^t \int D^k \{[\mu |\nabla u + \nabla (u)^T|^2/2 + \lambda (\text{div} u)^2] / (\rho + \bar{p})\} D^k θ dx dt
$$

Noting that $\mathcal{E} \leq 1$ and

$$
\int u \cdot \nabla D^k θ D^k θ dx = -\frac{1}{2} \int |D^k θ|^2 \text{div} u dx,
$$

hence it’s easy to get

$$
\|D^k θ(t)\|_{L^2} \lesssim \|D^k θ(0)\|_{L^2}^2 + \int_0^t \|D^k θ\|_{L^2}^2 (\|u\|_{H^{k+1}} + \mathcal{E}^2) dt. \quad (3.7)
$$
Similarly, using the identity
\[ \int_0^t \int D^k L^e D^k \varrho \text{d}x \text{d}\tau = \int_0^t \int D^k N^e D^k \varrho \text{d}x \text{d}\tau \quad \text{for } 0 \leq k \leq 3, \]
we arrive at
\[ \| \varrho(t) \|^2_{H^k} \lesssim \| D^k \varrho(0) \|^2_{L^2} + \int_0^t \| D^k \varrho \|_{L^2} (\| u \|_{H^{k+1}} + \mathcal{E}^2) \text{d}\tau. \tag{3.8} \]
Summing up (3.7) and (3.8), we immediately get the desired conclusion.

Secondly, we control the perturbed velocity.

**Lemma 3.2.** It holds that
\[ \| u \|^2_{H^3} \lesssim \| u_t \|^2_{H^1} + \| (\varrho, \theta) \|^2_{H^2} + \mathcal{E}^4. \]

**Proof.** Since the viscosity term in (3.3) defines a strongly elliptic operator on \( u \), we have for \( u \in H^k \cap H^1_0 \) (1 \( \leq k \leq 3 \)) that
\[ \| u \|^2_{H^k} \lesssim \| \mu \Delta u + \mu_0 \nabla \text{div} u \|^2_{H^{k-2}}. \tag{3.9} \]
Thus, applying (3.9) to the system
\[ - \mu \Delta u - \mu_0 \nabla \text{div} u = N_u - \rho u_t - g\rho e_3 - a \nabla (\overline{\varrho} + \overline{\theta}), \tag{3.10} \]
one immediately get the desired conclusion.

Thirdly, we bound the time-derivative of the perturbed velocity.

**Lemma 3.3.** It holds that
\[ \| (\varrho, \theta)_t \|^2_{H^k} \lesssim \| u \|^2_{H^{k+1}} + \mathcal{E}^4 \lesssim \mathcal{E}^2 \quad \text{for } 0 \leq k \leq 2, \tag{3.11} \]
\[ \| u_t(t) \|^2_{H^1} + \int_0^t \| u_{\tau \tau} \|^2_{L^2} \text{d}\tau \lesssim \| D^1 u_t |_{t=0} \|^2_{L^2} + \int_0^t (\| u \|^2_{H^2} + \mathcal{E}^4) \text{d}\tau, \tag{3.12} \]
\[ \| u_t \|^2_{H^2} \lesssim \| u_t \|^2_{H^3} + \| u \|^2_{H^3} + \mathcal{E}^4. \tag{3.13} \]

**Proof.** The inequality (3.11) follows directly from (3.2) and (3.4). By (3.3) we see that
\[ \| u_t \|^2_{H^1} \lesssim \| (\varrho, \theta) \|^2_{H^2} + \| u \|^2_{H^3} + \mathcal{E}^4 \lesssim \mathcal{E}^2. \tag{3.14} \]
On the other hand, noting that
\[ \int_0^t \int L^u u \cdot u_{\tau \tau} \text{d}x \text{d}\tau = \int_0^t \int N^u \cdot u_{\tau \tau} \text{d}x \text{d}\tau, \]
we have
\[
\begin{align*}
\mu \| \nabla u_t(t) \|^2_{L^2} + \mu_0 \| \text{div} u_t(t) \|^2_{L^2} &+ \int_0^t \| \sqrt{\rho} u_{\tau \tau} \|^2_{L^2} \text{d}\tau \\
= \mu \| \nabla u_t |_{t=0} \|^2_{L^2} + \mu_0 \| \text{div} u_t |_{t=0} \|^2_{L^2} &- \int_0^t \| a \nabla (\overline{\varrho} + \overline{\theta}) + g \rho e_3 \|_{L^2} \cdot u_{\tau \tau} \text{d}x \text{d}\tau \\
&- \int_0^t \| (\rho + \overline{\rho}) u \cdot \nabla u + \rho u_t + a \nabla (\overline{\rho} \overline{\theta}) \|_{L^2} \cdot u_{\tau \tau} \text{d}x \text{d}\tau \\
&\leq C \| D^1 u_t |_{t=0} \|_{L^2} + C \int_0^t \| (\varrho, \theta)_\tau \|^2_{H^1} + \| (\varrho_t, u_t, \theta_t) \|^2_{H^1} (\mathcal{E}^2 + \| \varrho_t \|^2_{H^1}) \text{d}\tau \\
&+ \frac{1}{2} \int_0^t \| \sqrt{\rho} u_{\tau \tau} \|^2_{L^2} \text{d}\tau + \| 2 \rho \|_{L^\infty} \int_0^t \| \sqrt{\rho} u_{\tau \tau} \|^2_{L^2} \text{d}\tau.
\end{align*}
\]
Lemma 3.4. For $\varrho$, and taking $\delta_1^0$ to be sufficiently small, we get
\[
\|w_t\|_{L^2} + \mu_0 \|\nabla w_t\|_{L^2} + \int_0^t \mu_\tau \|\nabla^2 w_{\tau\tau}\|_{L^2} \, d\tau \\
\leq \|D^1 w_t\|_{L^2} + \int_0^t \mu \|\nabla (\varrho, \theta)\|_{L^1} + \mu_0 \|\nabla (\varrho, u_\tau, \theta_\tau)\|_{L^1} \, d\tau.
\]
Thus, using (3.11) with $k = 1$, (3.14) and Poincaré’s inequality, we get (3.12).

Finally, taking the time derivative in (3.10), applying (3.9) to the resulting identity, and making use of (3.11) with $k = 1$ and (3.14), we obtain (3.13). This completes the proof of Lemma 3.3

3.2. Interior and boundary estimates

To begin with, we establish the interior estimates of higher-order mass derivatives of $\bar{\varrho} + \bar{\theta}$.

**Lemma 3.4.** For $1 \leq k \leq 3$, it holds that
\[
\|\chi_0 D^k (\varrho, u, \theta)(t)\|_{L^2}^2 + \int_0^t \left( \|\chi_0 D^{k+1} u\|_{L^2}^2 + \|\chi_0 D^k \frac{d}{dt} (\bar{\varrho} + \bar{\theta})\|_{L^2}^2 \right) \, d\tau \\
\lesssim \epsilon_0^2 + \int_0^t \mathcal{R} \, d\tau.
\]

**Proof.** Let $\chi_0$ be an arbitrary but fixed function in $C^\infty_0(\Omega)$. Then, we can deduce from (3.2)–(3.4) that
\[
\int_0^t \int \left( \frac{\alpha \chi_0^2 e}{\rho} |D^k \varrho| + \chi_0^2 |D^k u| + \frac{\chi_0^2 \tilde{\varrho}}{e} \right) \, dx \, d\tau
\]

which yields that
\[
\frac{1}{2} \int \left( \frac{\alpha \chi_0^2 e}{\rho} |D^k \varrho|^2 + \chi_0^2 |D^k u|^2 + \frac{\chi_0^2 \tilde{\varrho}}{e} |D^k \theta|^2 \right) \, d\tau
\]

\[
+ \int_0^t \left( \mu \|\chi_0 \nabla D^k u\|_{L^2} + \mu_0 \|\chi_0 D^k \nabla u\|_{L^2} \right) \, d\tau
\]

\[
= \frac{1}{2} \int \left( \frac{\alpha \chi_0^2 e}{\rho} |D^k \varrho(0)|^2 + \chi_0^2 |D^k u(0)|^2 + \frac{\chi_0^2 \tilde{\varrho}}{e} |D^k \theta(0)|^2 \right) \, dt
\]

\[- 2 \int_0^t \int \mu \chi_0 (\nabla \chi_0 \cdot \nabla D^k u) \cdot D^k u \, dx \, d\tau - \int_0^t \int \mu \chi_0 (\nabla \chi_0 D^k \nabla u) \cdot D^k u \, dx \, d\tau
\]

\[- \int_0^t \int \chi_0^2 |D^k (\tilde{\varrho} u_\tau) - \tilde{\varrho} D^k u_\tau| \cdot D^k u \, dx \, d\tau - \int_0^t \int \chi_0^2 |D^k \tilde{\varrho} u_\tau| \cdot D^k u \, dx \, d\tau
\]

\[- \int_0^t \int \left[ \frac{\alpha \chi_0^2 e}{\rho} D^k (\tilde{\varrho} u_3) \right] \, dx \, d\tau + \frac{\chi_0^2 \tilde{\varrho}}{e} D^k (\tilde{\varrho} u_3) \, dx \, d\tau
\]

\[+ 2a \int_0^t \int \chi_0 D^k (\bar{\varrho} + \bar{\theta}) \nabla \chi_0 \cdot D^k u \, dx \, d\tau
\]

\[+ \int_0^t \int \left[ \frac{\alpha \chi_0^2 e}{\rho} |D^k (\tilde{\varrho} D^k \nabla u) - \tilde{\varrho} D^k D^k \nabla u| \right] \, dx \, d\tau
\]

\[+ \int_0^t \int \left[ \frac{\alpha \chi_0^2 e}{\rho} |D^k (\tilde{\varrho} D^k \nabla u) - \tilde{\varrho} D^k D^k \nabla u| \right] \, dx \, d\tau
\]
\[ + \int_0^t \int \left( \frac{\alpha \rho \partial e}{\rho} D^2 N^0 D^0 \varrho + \chi_0^2 D^2 N^u \cdot D^0 u + \frac{\chi_0 \rho}{e} D^0 N^0 D^0 \theta \right) dx d\tau := L_1. \tag{3.15} \]

Using (3.5), we can infer that
\[ \| \chi_0 D^k \text{div} u \|_{L^2}^2 \geq \left\| \frac{\chi_0}{\rho e + \rho} D^k \frac{d}{dt}(\varepsilon \varrho + \rho \theta) \right\|_{L^2}^2 - \mathcal{R} \tag{3.16} \]

On the other hand, exploiting the facts
\[ - \int_0^t \chi_0^2 D^k u \cdot \nabla D^0 \theta D^0 \vartheta d\tau = \frac{1}{2} \int \left| D^0 \vartheta \right|^2 \text{div} \left( \frac{\chi_0^2 \rho}{e} u \right) dx \lesssim \mathcal{E}^3 \]

and
\[ - \int_0^t \chi_0^2 [\rho D^0 u + a D^0 \nabla (\rho \theta)] \cdot D^0 u d\tau \]
\[ = - \frac{1}{2} \int \chi_0^2 \rho \left| D^0 u \right|^2 dx + \frac{1}{2} \int \chi_0^2 \rho \left| D^0 u_0 \right|^2 dx + a \int_0^t \int D^0 (\rho \theta) \text{div} (\chi_0^2 D^0 u) dx d\tau \]
\[ + \frac{1}{2} \int_0^t \int \chi_0^2 (N^0 - \rho' u_3 - \rho \text{div} u) \left| D^0 u \right|^2 d\tau \]
\[ \lesssim \mathcal{E}^2_0 + \int_0^t \mathcal{R} d\tau + \left\| \frac{\varrho}{\rho} \right\|_{L^\infty} \int \chi_0^2 \rho \left| D^0 u \right|^2(t) dx, \]

it’s easy to derive that
\[ L_1 \leq C \mathcal{E}^2_0 + C \int_0^t \mathcal{R} d\tau + C \left\| \frac{\varrho}{\rho} \right\|_{L^\infty} \int \chi_0^2 \rho \left| D^0 u \right|^2 dx \]
\[ + \frac{1}{2} \int_0^t \left( \mu \| \chi_0 \nabla D^0 u \|_{L^2}^2 + \mu_0 \| \chi_0 D^0 \text{div} u \|_{L^2}^2 \right) d\tau. \tag{3.17} \]

Consequently, we get the desired conclusion from (3.15), (3.16) and (3.17) immediately. \( \Box \)

Next, let us establish the estimates near the boundary. Noting that \( \partial \Omega \) is smooth, similarly to that in [34,36], we choose a finite number of bounded open sets \( \{ O^j \}_{j=1}^N \) in \( \mathbb{R}^3 \), such that \( \bigcup_{j=1}^N O^j \supset \partial \Omega \). In each open set \( O^j \) we choose the local coordinates \( y = (y_1, y_2, y_3) \) as follows:

1. The surface \( O^j \cap \partial \Omega \) is the image of a smooth vector function \( z^j(y_1, y_2) = (z^j_1, z^j_2, z^j_3)(y_1, y_2) \) (e.g., take the local geodesic polar coordinate), satisfying \( |z^j_1| = 1, z^j_1 \cdot z^j_2 = 0 \), and \( |z^j_2| \geq \delta > 0 \), where \( \delta \) is some positive constant independent of \( 1 \leq j \leq N \).
2. Any \( x = (x_1, x_2, x_3) \in O^j \) is represented by
\[ x_i = \omega_i(y) := y_3 n_i(z^j(y_1, y_2)) + z^j_i(y_1, y_2) \text{ for } i = 1, 2, 3, \tag{3.18} \]
where \( (n_1, n_2, n_3)(z^j(y_1, y_2)) \) represents the internal unit normal vector at the point \( z^j(y_1, y_2) \) of the surface \( \partial \Omega \).

For the simplicity of presentation, we omit the subscript \( j \) in what follows. For \( k = 1, 2 \), we define the unit vectors
\[ \tilde{e}_1 = z_{y_1} \text{ and } \tilde{e}_2 = z_{y_2}/|z_{y_2}|. \]

An elementary calculation shows that the Jacobian \( J \) of the transform (3.18) is
\[ J = \omega_{y_1} \times \omega_{y_2} \cdot \tilde{n} = |z_{y_2}| + (\alpha |z_{y_2}| + \beta') y_3 + (\alpha \beta' - \beta \alpha') y_3^2, \tag{3.19} \]
where \( \tilde{n} = (n_1, n_2, n_3)(z^j(y_1, y_2)), \alpha = \tilde{n}_{y_1} \cdot \tilde{e}_1, \beta = \tilde{n}_{y_1} \cdot \tilde{e}_2, \alpha' = \tilde{n}_{y_2} \cdot \tilde{e}_1 \) and \( \beta' = \tilde{n}_{y_2} \cdot \tilde{e}_2 \). By (3.19), we find the transform (3.18) is regular by choosing \( y_3 \) so small that \( J \geq \delta/2 \). Therefore, the inverse function
of \( \omega(y) := (\omega_1, \omega_2, \omega_3)(y) \) exits, and we denote it by \( y = \omega^{-1}(x) \); moreover \( (y_1, y_2, y_3, x, x) \) make sense and can be expressed by, using a straightforward calculation,

\[
\begin{align*}
\partial_x y_1 &= \frac{1}{j}(\omega_{y_2} \times \omega_{y_3})_j = \frac{1}{j}(A\dot{e}_j^1 + B\dot{e}_j^2) =: a_{1j}, \\
\partial_x y_2 &= \frac{1}{j}(\omega_{y_3} \times \omega_{y_1})_j = \frac{1}{j}(C\dot{e}_j^3 + D\dot{e}_j^2) =: a_{2j}, \\
\partial_x y_3 &= \frac{1}{j}(\omega_{y_1} \times \omega_{y_2})_j = \bar{n}_j =: a_{3j},
\end{align*}
\]

(3.20)

where \( A = |z_{y_2}| + \beta' y_3, B = -y_3 \alpha', C = -\beta y_3, \bar{D} = 1 + \alpha y_3, \)

\[
J = A\bar{D} - BC \geq \delta/2
\]

and \( \bar{e}_j^m \) denotes the \( j \)-th component of \( \bar{e}_m \). Obviously, (3.20) gives

\[
\sum_{j=1}^{3} a_{3j}^2 |\bar{n}_j|^2 = 1, \quad a_{1j} a_{3j} = a_{2j} a_{3j} = 0, \quad J^2 = (AC + B\bar{D})^2 - (A^2 + B^2)(C^2 + \bar{D}^2)
\]

and

\[
\partial_x = a_{kj} \partial_{x_k},
\]

(3.22)

where we have used the Einstein convention of summing over repeated indices.

Thus, in \( O \), the three linear parts \( L^\theta, L^u_{\text{new}} = (L^u_{\text{new}}^1, L^u_{\text{new}}^2, L^u_{\text{new}}^3) \) and \( L^\theta \) in the local coordinates \((y_1, y_2, y_3)\) read as follows.

\[
\begin{align*}
L^\theta &= L^{\bar{\theta}} := \bar{\theta}_t + (a_{k3}\partial_{y_k} \bar{p}) \bar{u}_3 + \bar{p} a_{k3} \bar{e}_y, \\
L^u_{\text{new}} &= L^{\bar{u}}_{\text{new}} := \bar{p} a_{k3} \bar{e}_y + \frac{1}{j^2}[(A^2 + B^2)\bar{u}_y + 2AC + B\bar{D}] \bar{u}_{y_1} + (C^2 + \bar{D}^2) \bar{u}_{y_2} + J^2 \bar{u}_{y_3} + \text{less two order terms of } \bar{u}_y + g\bar{\theta} \bar{e}_3 + a_{ki} \left( \frac{\mu_0}{\bar{p} e} + \bar{G} + a\bar{\theta} + a\bar{\theta} \right)_{y_k}, \\
L^\theta &= L^{\bar{\theta}} := \bar{\theta}_t + (a_{k3}\partial_{y_k} \bar{e}) \bar{u}_3 + a\bar{e} a_{k3} \bar{e}_y,
\end{align*}
\]

where \((\bar{\theta}, \bar{e}, \bar{p}) := (\bar{\theta}, \bar{e}, \bar{p})|_{x = \omega(y)}, (\bar{\theta}, \bar{e}, \bar{\theta})(t, y) = (\theta, u, \theta)|_{x = \omega(y)}\), and

\[
\bar{G} := (\bar{e} \bar{\theta} + \bar{p} \bar{\theta})_t + u_{k} a_{ki} (\bar{e} \bar{\theta} + \bar{p} \bar{\theta})_{y_k}.
\]

Similarly, we define that

\[
\tilde{N}^{\tilde{\theta}} := N^\theta \quad \text{is written in the local coordinates},
\]

\[
\tilde{N}_{\text{new}}^T := N_{\text{new}}^u \quad \text{is written in the local coordinates},
\]

\[
\tilde{N}^{\tilde{\theta}} := N^u \quad \text{is written in the local coordinates}.
\]

With the notations above in hand, we can further rewrite the Eqs. (3.2), (3.4) and (3.6) in the local coordinates \( y \) as follows:

\[
\tilde{L}^\theta = \tilde{N}^{\tilde{\theta}}, \quad \tilde{L}^u_{\text{new}} = \tilde{N}_{\text{new}}^u, \quad \tilde{L}^\theta = \tilde{N}^{\tilde{\theta}}
\]

(3.23)

with initial and boundary conditions

\[
(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0) := (\bar{\theta}, \bar{u}, \bar{\theta})|_{t=0} = (\theta_0, u_0, \theta_0)|_{x = \omega(y)} \quad \text{in } \Omega
\]

and

\[
\bar{u}(t, y)|_{\partial \Omega \cap \{y_3 = 0\}} = 0 \quad \text{for any } t > 0,
\]

where \( \Omega = \{ y \mid y = w^{-1}(x), \ x \in O \cap \Omega \} \) and \( \tilde{L}^u_{\text{new}} = (\tilde{L}^u_{\text{new}}^1, \tilde{L}^u_{\text{new}}^2, \tilde{L}^u_{\text{new}}^3) \).

Let \( \chi \) be an arbitrary but fixed function in \( C_0^\infty(O) \) and \( \tilde{\chi} := \chi|_{x = \omega(y)} \). Obviously, \( \tilde{\chi} D^k y_1 y_2 \bar{u} = 0 \) on \( \partial \Omega \). Now, we control derivatives in the tangential directions.
Lemma 3.5. For $1 \leq k \leq 3$, it holds that
\[
\|\tilde{\chi}D_{y_1y_2}^k(\tilde{\varnothing}, \tilde{u}, \tilde{\theta})(t)\|_{L^2(\Omega)}^2 + \int_0^t \left( \|\tilde{\chi}D_{y_1y_2}^k D^1_y \tilde{u}\|_{L^2(\Omega)}^2 + \|\tilde{\chi}D_{y_1y_2}^k \tilde{G}\|_{L^2(\Omega)}^2 \right) \, dt \\
\lesssim \varepsilon_0^2 + \int_0^t R \, dt.
\]

Proof. To begin with, we deduce from (3.23) that
\[
\int_0^t \int_{\Omega} \left( \frac{a\tilde{\chi}\tilde{c}}{\rho} D_{y_1y_2}^k \tilde{L} \tilde{\varnothing} D_{y_1y_2}^k \tilde{\varnothing} + \tilde{\chi}^2 D_{y_1y_2}^k \tilde{L}_{\text{new}} \cdot D_{y_1y_2}^k \tilde{u} + \frac{\tilde{\chi}^2 \tilde{e}}{\varepsilon} D_{y_1y_2}^k \tilde{L} \tilde{\varnothing} D_{y_1y_2}^k \tilde{\varnothing} \right) \, dy \, dt \\
= \int_0^t \int_{\Omega} \left( \frac{a\tilde{\chi}\tilde{c}}{\rho} D_{y_1y_2}^k \tilde{N}_{\tilde{\varnothing}} D_{y_1y_2}^k \tilde{\varnothing} + \tilde{\chi}^2 D_{y_1y_2}^k \tilde{N}_{\text{new}} \cdot D_{y_1y_2}^k \tilde{u} + \frac{\tilde{\chi}^2 \tilde{e}}{\varepsilon} D_{y_1y_2}^k \tilde{N} \tilde{\varnothing} D_{y_1y_2}^k \tilde{\varnothing} \right) \, dy \, dt. \tag{3.24}
\]

In view of (3.5), we know that
\[
a_{k3} \tilde{u}_{y_k} = - \frac{1}{\tilde{\rho} \tilde{e} + \tilde{p}} G + \frac{(a_{k3} \tilde{c} y_k \tilde{\varnothing} + a_{k3} \tilde{p} y_k \tilde{\theta}) \tilde{u}_3 + \tilde{p} (\tilde{N} \tilde{\varnothing} + \tilde{u}_1 a_{k3} \tilde{\theta} y_k)(a_{k3} (\tilde{c} y_k \tilde{\theta} y_k) - a_{k3} (\tilde{p} y_k \tilde{\theta} y_k))}{\tilde{\rho} \tilde{e} + \tilde{p}} =: - \frac{1}{\tilde{\rho} \tilde{e} + \tilde{p}} G + L_2,
\]

By the formula of integration by parts, we have
\[
\int_{\Omega} \tilde{\chi}^2 D_{y_1y_2}^k \left[ a_{k1} \left( \frac{\mu_0}{\tilde{\rho} \tilde{e} + \tilde{p}} \tilde{G} \right) y_k \right] D_{y_1y_2}^k \tilde{u}_i \, dy \\
= \int_{\Omega} \frac{\mu_0 \tilde{\chi}^2}{(\tilde{\rho} \tilde{e} + \tilde{p})^2} |D_{y_1y_2}^k \tilde{G}|^2 \, dy + \int_{\Omega} \left( \text{less than } k + 1 \text{ order terms of } \tilde{G} \right) |D_{y_1y_2}^k L_2| \, dy \\
+ \int_{\Omega} \left( \text{less than } k + 1 \text{ order terms of } \tilde{G} \right) (\text{less than } k \text{ order terms of } \tilde{G}) \, dy \\
+ \int_{\Omega} \left( \text{less than } k \text{ order terms of } \tilde{G} \right) (\text{less than } k + 2 \text{ order terms of } \tilde{u}) \, dy \\
=: \int_{\Omega} \frac{\mu_0 \tilde{\chi}^2}{(\tilde{\rho} \tilde{e} + \tilde{p})^2} |D_{y_1y_2}^k \tilde{G}|^2 \, dy + L_3. \tag{3.25}
\]

Thus, similarly to (3.15), we can infer from the two equalities (3.24) and (3.25) that
\[
\frac{1}{2} \int_{\Omega} \left( \frac{a\tilde{\chi}\tilde{c}}{\rho} |D_{y_1y_2}^k \tilde{G}|^2 + \tilde{\chi}^2 \tilde{p} |D_{y_1y_2}^k \tilde{u}|^2 + \frac{\tilde{\chi}^2 \tilde{e}}{\varepsilon} |D_{y_1y_2}^k \tilde{\theta}|^2 \right)(t) \, dy \\
+ \int_0^t \int_{\Omega} \sum_{i=1}^3 [(A^2 + B^2) |D_{y_1y_2}^k \tilde{u}_i|^2 + 2(AC + BD) D_{y_1y_2}^k \tilde{u}_i D_{y_1y_2}^k \tilde{u}_i + (C^2 + D^2) |D_{y_1y_2}^k \tilde{u}_i|^2 + J^2 |D_{y_1y_2}^k \tilde{u}_i|^2] \, dy \, dt \\
= \frac{1}{2} \int_{\Omega} \left( \frac{a\tilde{\chi}\tilde{c}}{\rho} |D_{y_1y_2}^k \tilde{G}|^2 + \tilde{\chi}^2 \tilde{p} |D_{y_1y_2}^k \tilde{u}|^2 + \frac{\tilde{\chi}^2 \tilde{e}}{\varepsilon} |D_{y_1y_2}^k \tilde{\theta}|^2 \right) \, dt \\
- \int_0^t \int_{\Omega} \tilde{\chi}^2 |D_{y_1y_2}^k (\tilde{G} \tilde{u})| - \rho D_{y_1y_2}^k \tilde{u}_i \cdot D_{y_1y_2}^k \tilde{u}_i \, dy \, dt \\
+ \sum_{i=1}^3 \int_0^t \left( \text{less than } k + 1 \text{ order terms of } \tilde{G} \right) \, dy \, dt.
\]
Lemma 3.6. For $0 \leq k + l \leq 2$, it holds that
\[
\|\tilde{\chi}D^{k}_{y_{1}y_{2}}D^{l+1}_{y_{3}}(\tilde{e}\tilde{\rho} + \tilde{\rho}\tilde{\theta})(t)\|_{L^{2}(\tilde{\Omega})} \lesssim \|\varphi_{0}\|_{H^{3}} + \int_{0}^{t} \left( \|\tilde{\chi}D^{k}_{y_{1}y_{2}}D^{l}_{y_{3}}\tilde{\varphi}\|_{L^{2}(\tilde{\Omega})} + \|\tilde{\chi}D^{k}_{y_{1}y_{2}}\tilde{G}\|_{L^{2}(\tilde{\Omega})} \right) d\tau
\]
where we have defined $\tilde{h}_{1} = \tilde{\varphi}$ and $\tilde{h}_{2} = \tilde{\theta}$ for simplicity.

By virtue of (3.21), the matrix
\[
\begin{pmatrix}
A^{2} + B^{2} & AC + B\tilde{D} \\
AC + B\tilde{D} & C^{2} + \tilde{D}^{2}
\end{pmatrix}
\]
is strictly positive-defined, there thus exists a positive constant $\mu_{2}$ such that
\[
\frac{\mu_{2}}{\rho}D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{1}}|^{2} \leq (A^{2} + B^{2})|D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{1}}|^{2} + 2(AC + B\tilde{D})D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{1}}D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{2}} + (C^{2} + \tilde{D}^{2})|D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{2}}|^{2} + J^{2}|D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{3}}|^{2}.
\]
(3.27)

On the other hand, similarly to (3.17), it is easy to see that
\[
L_{4} \leq C\tilde{E}_{0}^{2} + C\int_{0}^{t} \tilde{R}\,d\tau + \frac{\mu_{2}}{2} \int_{0}^{t} \int_{\bar{\Omega}} \tilde{\chi}^{2}|D^{k}_{y_{1}y_{2}}D^{l}_{y}\tilde{u}|^{2} d\tau d\tau + \frac{1}{2} \int_{0}^{t} \int_{\bar{\Omega}} \frac{\mu_{0}\tilde{\chi}^{2}}{\tilde{\rho}^{2} + \tilde{\rho}^{2}}|D^{k}_{y_{1}y_{2}}\tilde{G}|^{2} d\tau d\tau + C\left\| \frac{\partial \tilde{\chi}}{\partial \tilde{\rho}} \right\|_{L^{\infty}(\tilde{\Omega})} \int_{0}^{t} \tilde{\chi}^{2}\tilde{\rho}|D^{k}_{y_{1}y_{2}}\tilde{u}|^{2}(t) d\tau,
\]
(3.28)

where we have defined $\tilde{E}(t) := \tilde{E}(\tilde{\varphi}, \tilde{u}, \tilde{\theta})(t) := \|(\tilde{\varphi}, \tilde{u}, \tilde{\theta})(t)\|_{H^{3}(\tilde{\Omega})}$, $\tilde{E}_{0} = \tilde{E}(0)$ and
\[
\tilde{R} := \left\| \frac{\partial \tilde{\varphi} + \tilde{\rho}\tilde{\theta}}{\partial t} \right\|_{H^{3}(\tilde{\Omega})}^2 + \tilde{E}(\|(\tilde{\varphi}, \tilde{u}, \tilde{\theta})(t)\|_{H^{3}(\tilde{\Omega})} = \|\tilde{u}\|_{H^{4}(\tilde{\Omega})} + \tilde{E}^{2}).
\]

Utilizing (3.28) and (3.27), we deduce from (3.26) that
\[
\|\tilde{\chi}D^{k}_{y_{1}y_{2}}\tilde{u}_{y_{1}}(t)\|_{L^{2}(\tilde{\Omega})} \lesssim \int_{0}^{t} \left( \|\tilde{\chi}D^{k}_{y_{1}y_{2}}D^{l}_{y}\tilde{u}\|_{L^{2}(\tilde{\Omega})}^{2} + \|\tilde{\chi}D^{k}_{y_{1}y_{2}}\tilde{G}\|_{L^{2}(\tilde{\Omega})}^{2} \right) d\tau
\]
\[
\lesssim \tilde{E}^{2}_{0} + \int_{0}^{t} \tilde{R}\,d\tau,
\]
(3.29)

Finally, using (3.18) and (3.19), we can obtain the following estimate by transformation of coordinates
\[
\|f\|_{H^{k}(\tilde{\Omega})} \lesssim \|f\|_{H^{k}}.
\]
(3.30)

In particular, we have $\tilde{E}_{0} \lesssim E_{0}$ and $\tilde{R} \lesssim \tilde{R}$. Consequently, we obtain the desired conclusion from (3.29). \(\Box\)

Next, we turn to the estimate of derivatives in the normal directions.

**Lemma 3.6.** For $0 \leq k + l \leq 2$, it holds that
\[
\|\tilde{\chi}D^{k}_{y_{1}y_{2}}D^{l+1}_{y_{3}}(\tilde{e}\tilde{\rho} + \tilde{\rho}\tilde{\theta})(t)\|_{L^{2}(\tilde{\Omega})} \lesssim \|\varphi_{0}\|_{H^{3}} + \int_{0}^{t} \left( \|\tilde{\chi}D^{k}_{y_{1}y_{2}}D^{l}_{y_{3}}\tilde{\varphi}\|_{L^{2}(\tilde{\Omega})} + \|\tilde{\chi}D^{k}_{y_{1}y_{2}}\tilde{G}\|_{L^{2}(\tilde{\Omega})} \right) d\tau
\]
where we have defined $\tilde{h}_{1} = \tilde{\varphi}$ and $\tilde{h}_{2} = \tilde{\theta}$ for simplicity.
Proof. Recalling that \( \sum_{i=1}^{3} \tilde{n}_t a_{ki} = \sum_{i=1}^{3} a_{3i} a_{ki} = 0, \sum_{i=1}^{3} a_{3i}^2 = 1 \) and the relation (3.20), we use the equations \( D_{y_3}^1 (\tilde{e} \tilde{L} \tilde{\theta} + \tilde{\rho} \tilde{L} \tilde{\theta} - \tilde{e} \tilde{N} \tilde{\theta} - \tilde{\rho} \tilde{N} \tilde{\theta}) = 0 \) and \( \tilde{n} \cdot (\tilde{E}_{\text{new}} - \tilde{N}_{\text{new}}^\tilde{n}) = 0 \), to arrive at

\[
\tilde{G}_{y_3} + \frac{\tilde{p} e}{\tilde{\theta}} \left[ (A \tilde{e}_1 + B \tilde{c}_2) \cdot \tilde{u}_{y_3 y_1} + (C \tilde{c}_1 + D \tilde{e}_2) \cdot \tilde{u}_{y_3 y_2} + J \tilde{n} \cdot \tilde{u}_{y_3 y_3} \right] \\
+ \text{less than two order terms of } \tilde{u}
\]

\[
= [\tilde{\rho} (\tilde{N} \tilde{\theta} + \tilde{u}_i a_{ki} \tilde{\theta}_{y_k}) - \tilde{e} \tilde{\theta} a_{ki} \tilde{u}_{y_k} + (a_{k3} \tilde{e}_{y_k} \tilde{\theta} - a_{k3} \tilde{\rho} \tilde{y}_k \tilde{\theta}) \tilde{u}_3]_{y_3}
\]

and

\[
\tilde{\rho} \tilde{n} \cdot \tilde{u}_t - \frac{\mu \tilde{n}}{\tilde{J}^2} \left[ (A^2 + B^2) \tilde{u}_{y_1 y_1} + 2 (AC + BD) \tilde{u}_{y_1 y_2} + (C^2 + D^2) \tilde{u}_{y_2 y_2} \right] \\
+ J^2 \tilde{u}_{y_3 y_3} + \text{less than two order terms of } \tilde{u} + g \tilde{e} \tilde{e}_3 \cdot \tilde{n}
\]

\[
+ \left[ \frac{\mu_0}{\tilde{p} \tilde{e} + \tilde{p}} \tilde{G} + a \tilde{e} \tilde{\theta} + a \tilde{p} \tilde{\theta} \right]_{y_3} = \tilde{n} \cdot \tilde{N}_{\text{new}}^\tilde{n}
\]

Eliminating \( \mu \tilde{n} \cdot \tilde{u}_{rr} \) from (3.31), we get

\[
\left[ \frac{(\mu + \mu_0)}{\tilde{p} e + \tilde{p}} \tilde{G} + a \tilde{e} \tilde{\theta} + a \tilde{p} \tilde{\theta} \right]_{y_3} = - \tilde{\rho} \tilde{n} \cdot \tilde{u}_t + \frac{\mu \tilde{n}}{\tilde{J}^2} \left[ (A^2 + B^2) \tilde{u}_{y_1 y_1} + 2 (AC + BD) \tilde{u}_{y_1 y_2} \right] \\
+ (C^2 + D^2) \tilde{u}_{y_2 y_2} - \frac{\mu}{\tilde{J}} \left[ (A \tilde{e}_1 + B \tilde{c}_2) \cdot \tilde{u}_{y_1 y_1} + (C \tilde{c}_1 + D \tilde{e}_2) \cdot \tilde{u}_{y_3 y_3} \right] \\
+ \text{less than two order terms of } \tilde{u} - g \tilde{e} \tilde{e}_3 \cdot \tilde{n} + \tilde{n} \cdot \tilde{N}_{\text{new}}^\tilde{n}
\]

\[
+ \frac{\mu}{\tilde{p} \tilde{e} + \tilde{p}} \left[ \tilde{\rho} (\tilde{N} \tilde{\theta} + \tilde{u}_i a_{ki} \tilde{\theta}_{y_k}) + (a_{k3} \tilde{e}_{y_k} \tilde{\theta} + a_{k3} \tilde{\rho} \tilde{y}_k \tilde{\theta}) \tilde{u}_3 - \tilde{e} \tilde{\theta} a_{ki} \tilde{u}_3 \right]_{y_3} := L_5.
\]

If we apply \( D_{y_1 y_2}^k D_{y_3}^l \) \( \{k + l = 0, \, 1, \, 2\} \) to (3.32), multiply then by \( \tilde{\chi}^2 D_{y_1 y_2}^k \tilde{G}_{y_3} + D_{y_1 y_2}^k D_{y_3}^{l+1} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta}) \) and integrate them, we get

\[
\frac{1}{2} \int_{\Omega} \left[ \frac{(\mu + \mu_0)}{\tilde{p} e + \tilde{p}} + a \right] \tilde{\chi}^2 D_{y_1 y_2}^k D_{y_3}^{l+1} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta})(t)^2 \, dy \\
+ \int_{0}^{t} \int_{\Omega} \left( a |\tilde{\chi} D_{y_1 y_2}^k D_{y_3}^{l+1} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta})|^2 + \frac{(\mu + \mu_0)}{\tilde{p} e + \tilde{p}} |\tilde{\chi} D_{y_1 y_2}^k D_{y_3}^{l+1} \tilde{G}_{y_3}|^2 \right) \, dy \, d\tau
\]

\[
= \frac{1}{2} \int_{\Omega} \left[ \frac{(\mu + \mu_0)}{\tilde{p} e + \tilde{p}} + a \right] \tilde{\chi}^2 D_{y_1 y_2}^k D_{y_3}^{l+1} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta}) \, dy \\
+ \int_{0}^{t} \int_{\Omega} \left( \text{less than } k + l + 1 \text{ order terms of } \tilde{G} + D_{y_1 y_2}^k D_{y_3}^{l+1} L_5 \right) \\
- \int_{0}^{t} \int_{\Omega} \left( \tilde{\chi}^2 D_{y_1 y_2}^k D_{y_3}^{l+1} [u_{ma_m j} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta})] - u_{ma_m j} D_{y_1 y_2}^k D_{y_3}^{l+1} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta}) \right) \, dy \, d\tau
\]

\[
= \int_{0}^{t} \int_{\Omega} \left( \frac{(\mu + \mu_0)}{\tilde{p} e + \tilde{p}} + a \right) D_{y_1 y_2}^k D_{y_3}^{l+1} (\tilde{e} \tilde{\theta} + \tilde{p} \tilde{\theta}) \, dy \, d\tau \\
\leq \frac{1}{2} \int_{0}^{t} \left[ \tilde{\chi}^2 u_{ma_m j} \left( a + \frac{(\mu + \mu_0)}{\tilde{p} e + \tilde{p}} \right) \right] \, dy \, d\tau
\]

\[:= L_6.\]
Noting that $H^2 \to L^\infty$ and $E \leq \delta^0_1$ is sufficiently small, thus it’s easy to estimate that

$$L_6 \lesssim \| (\tilde{\phi}_0, \tilde{\theta}_0) \|^2_{H^3(\tilde{\Omega})} + \int_0^t \left( | |D^{k+1}_y D^{l}_y D\tilde{u}|^2_{L^2(\tilde{\Omega})} + \tilde{R} \right) d\tau$$

$$+ \frac{1}{2} \int_0^t \int_{\tilde{\Omega}} \left( A |\tilde{\chi} D^{l+1}_y D\tilde{u}|^2 + \frac{\mu + \mu_0}{\rho \tilde{e} + \tilde{p}} |\tilde{\chi} D^{l+1}_y D\tilde{v}|^2 \right) dy d\tau,$$

and

$$\| \tilde{\chi} D^{l+1}_y (\tilde{\phi} + \tilde{p}) (t) \|^2_{L^2(\tilde{\Omega})}$$

$$+ \int_0^t \left( \| \tilde{\chi} D^{l+1}_y D\tilde{v} \|^2_{L^2(\tilde{\Omega})} + \| \tilde{\chi} D^{l+1}_y D\tilde{G} \|^2_{L^2(\tilde{\Omega})} \right) d\tau$$

$$\lesssim \| (\tilde{\phi}_0, \tilde{\theta}_0) \|^2_{H^3(\tilde{\Omega})} + \int_0^t \left( | |D^{k+1}_y D^{l}_y D\tilde{u}|^2_{L^2(\tilde{\Omega})} + \tilde{R} \right) d\tau,$$

which, together with (3.30), yields the desired conclusion. \square

Finally, we introduce the following lemma [34, Lemma 5.8] on the stationary Stokes equations to get the estimates on the tangential derivatives of both $u$ and $\tilde{\phi} + \tilde{\theta}$.

Lemma 3.7. Consider the Stokes problem

$$\begin{cases}
-\mu \Delta u + a \nabla \sigma = g, \\
div u = f, \\
u|_{\partial \Omega} = 0,
\end{cases}$$

where $f \in H^{k+1}$ and $g \in H^k$ ($k \geq 0$). Then the above problem has a solution $(\sigma, u) \in H^{k+1} \times H^{k+2} \cap H^1_0$ which is unique modulo a constant of integration for $\sigma$. Moreover, this solution satisfies

$$\| u \|^2_{H^{k+2}} + \| D\sigma \|^2_{H^k} \lesssim \| f \|^2_{H^{k+1}} + \| g \|^2_{H^k}.$$

Now we rewrite the perturbed equations as the Stokes problem:

$$\begin{cases}
-\mu \Delta u + a \nabla (\tilde{\phi} + \tilde{\theta}) = N u - \tilde{\rho} u_t - g \Phi e_3 + \mu_0 \nabla div u, \\
(\tilde{\rho} \tilde{e} + \tilde{p}) div u = \tilde{p} (N^\theta + u \cdot \nabla \theta) - \tilde{e}_g div u + (\tilde{e}^\theta + \tilde{p}^\theta) u_3 - \frac{d}{dt} (\tilde{e} \tilde{\rho} + \tilde{\theta}^\prime) u_3, \\
u|_{\partial \Omega} = 0,
\end{cases}$$

which, together with (3.18), implies that the following Stokes problem of $(\tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta})) |_{y = \omega^{-1}(x)}$:

$$\begin{cases}
-\mu \Delta \{ (\tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta})) |_{y = \omega^{-1}(x)} \} + a \nabla \{ (\tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta})) |_{y = \omega^{-1}(x)} \} \\
\text{less than } k + 2 \text{ order terms of } u \\
\text{+less than } k + 1 \text{ order terms of } \tilde{\rho}, \theta \text{ and } (N u - \tilde{\rho} u_t - g \Phi e_3) \\
+ \mu_0 \{ \tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta}) u \}_{|y = \omega^{-1}(x)} \text{ + less than } k \text{ order terms of } \nabla \text{div } u := L_7, \\
\text{div} \{ \tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta}) u \}_{|y = \omega^{-1}(x)} = \\
\{ \tilde{\chi} D^{l}_y \left[ \frac{d}{dt} (\tilde{e} \tilde{\rho} + \tilde{\theta}^\prime) (\tilde{e} \tilde{\Phi} + \tilde{p}) \right] - 1 \}_{|y = \omega^{-1}(x)} \text{ + less than } k \text{ order terms of } \frac{d}{dt} (\tilde{e} \tilde{\Phi} + \tilde{p}) + \tilde{p} \\
\{ \tilde{p} (N^\theta + u \cdot \nabla \theta) - \tilde{e}_g \text{div } u + (\tilde{e}^\theta + \tilde{p}^\theta) u_3 - (\tilde{e} \tilde{\rho} + \tilde{\theta}^\prime) u_3 \}_{|y = \omega^{-1}(x)} : = L_8, \\
\{ (\tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta}) u) \}_{|y = \omega^{-1}(x)} |_{\partial \Omega} = 0
\end{cases}$$

where $\nabla \text{div } u$ is written in local coordinates. Applying Lemma 3.7 to the above problem, we obtain

$$\| D^{2+l}_y (\tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta})) |_{y = \omega^{-1}(x)} \|^2_{L^2} + \| D^{l+1}_y (\tilde{\chi} D^{l}_y (\tilde{\phi} + \tilde{\theta})) |_{y = \omega^{-1}(x)} \|^2_{L^2}$$

$$\lesssim \| L_8 \|^2_{H^{l+1}} + \| L_7 \|^2_{H^l} \text{ for } 0 \leq l + k \leq 2,$$
which, combined with the estimates
\[
\|L_8\|_{H^{l+1}}^2 + \|L_7\|_{H^{l}}^2 \lesssim \left\| \chi(x)D_x^{l+1}(D_{y_1 y_2}^k \tilde{G})|_{y=\omega^{-1}(x)} \right\|_{L^2}^2 \\
+ \left\| \chi(x)D_x^l(D_{y_1 y_2}^k \tilde{U})|_{y=\omega^{-1}(x)} \right\|_{L^2}^2 + \mathcal{R},
\]
and
\[
\left\| \chi(x)D_x^l(D_{y_1 y_2}^k \tilde{U})|_{y=\omega^{-1}(x)} \right\|_{L^2}^2 \lesssim \left\| \chi(x)D_x^{l+1}(D_{y_1 y_2}^k \tilde{G})|_{y=\omega^{-1}(x)} \right\|_{L^2}^2 + \mathcal{R},
\]
yields
\[
\left\| \chi(x)D_x^{l+1}(D_{y_1 y_2}^k \tilde{U})|_{y=\omega^{-1}(x)} \right\|_{L^2}^2 + \left\| \chi(x)D_x^{l+1}(D_{y_1 y_2}^k \tilde{G})|_{y=\omega^{-1}(x)} \right\|_{L^2}^2 + \mathcal{R}.
\]
Consequently, by the transformation of coordinates (3.18), we easily obtain the following estimate:

**Lemma 3.8.** For $0 \leq l + k \leq 2$, we have
\[
\| \tilde{\chi} D_x^{l+1} D_{y_1 y_2}^k \tilde{u} \|^2_{L^2(\Omega)} + \| \tilde{\chi} D_x^{l+1} D_{y_1 y_2}^2 (\tilde{\epsilon} \tilde{\vartheta} + \tilde{\rho} \tilde{\theta}) \|^2_{L^2(\Omega)} \\
\lesssim \left\| \tilde{\chi} D_x^{l+1} D_{y_1 y_2}^k \tilde{G} \right\|^2_{L^2(\tilde{\Omega})} + \mathcal{R}.
\]

Now, we are able to establish the desired boundary estimate.

**Lemma 3.9.** For $0 \leq k \leq 3$, it holds that
\[
\sum_{k=0}^3 \int_0^t \left\{ \| \chi D^{k+1} u \|^2_{L^2} + \left\| \chi D^k \left[ \frac{d}{dt} (\tilde{\epsilon} \vartheta + \tilde{\rho} \tilde{\theta}) \right] \right\|_{L^2}^2 \right\} \, d\tau \lesssim \mathcal{E}_0^2 + \int_0^t \mathcal{R} \, d\tau.
\]

**Proof.** It suffices to show
\[
\int_0^t \left\{ \| \tilde{\chi} D_y^k D_{y_1 y_2}^l \tilde{u} \|^2_{L^2(\tilde{\Omega})} + \| \tilde{\chi} D_y^k \tilde{G} \|^2_{L^2(\tilde{\Omega})} \right\} \, d\tau \lesssim \mathcal{E}_0^2 + \int_0^t \mathcal{R} \, d\tau,
\]
from which we can immediately get the desired conclusion by transformation of coordinates.

We consider $D_y^k = D_{y_1 y_2}^l D_{y_1 y_2}^{k-l}$ for $1 \leq l \leq 3$. Then,
\[
\int_0^t \left\{ \| \tilde{\chi} D_y^l D_{y_1 y_2}^{k-l} D_{y_1 y_2}^l \tilde{u} \|^2_{L^2(\tilde{\Omega})} + \| \tilde{\chi} D_y^l D_{y_1 y_2}^{k-l} \tilde{G} \|^2_{L^2(\tilde{\Omega})} \right\} \, d\tau \\
\lesssim \int_0^t \left[ \| \tilde{\chi} D_y^l D_{y_1 y_2}^{k-l} \tilde{G} \|^2_{L^2(\tilde{\Omega})} + \mathcal{R} \right] \, d\tau \\
\lesssim \|(q_0, \theta_0)\|^2_{H^3} + \int_0^t \mathcal{R} \, d\tau + \sum_{0 \leq m \leq k-1} \int_0^t \| \tilde{\chi} D_y^m D_{y_1 y_2}^{k-m} D_y^l \tilde{u} \|^2_{L^2(\tilde{\Omega})} \, d\tau,
\]
where we have used Lemmas 3.6 and 3.8 for the first and second inequalities respectively. Obviously, if $m \geq 1$, we can continue to repeat the process above. Thus we conclude that
\[
\int_0^t \left\{ \| \tilde{\chi} D_y^{k+1} \tilde{u} \|^2_{L^2(\tilde{\Omega})} + \| \tilde{\chi} D_y^k \tilde{G} \|^2_{L^2(\tilde{\Omega})} \right\} \, d\tau \\
\lesssim \|(q_0, \theta_0)\|^2_{H^3} + \int_0^t \mathcal{R} \, d\tau + \int_0^t \| \tilde{\chi} D_y^k D_{y_1 y_2}^{l+1} \tilde{u} \|^2_{L^2(\tilde{\Omega})} \, d\tau \text{ for any } 0 \leq k \leq 3,
\]
which, together with Lemma 3.5, implies (3.33). \qed
3.3. Energy estimate of Gronwall-type

Now, we are able to establish the energy estimate of Gronwall-type. Putting Lemmas 3.4 and 3.9 together, we get
\[
\int_0^t \left\{ \|u\|_{H^4}^2 + \left\| \frac{d}{dt}(\bar{e}q + \bar{\rho}\theta) \right\|_{H^3}^2 \right\} \, d\tau \lesssim \mathcal{E}^2_0 + \int_0^t \mathcal{R} \, d\tau,
\]
where the right-hand side can be bounded as follows, using Lemma 3.3.
\[
\int_0^t \mathcal{R} \, d\tau \lesssim \mathcal{E}^2_0 + \int_0^t \left[ \|(q, \theta)\|_{H^2}^2 + \mathcal{E}(\|u\|_{H^3} + \|u\|_{H^4}^2) \right] \, d\tau.
\]
Hence,
\[
\int_0^t \left\{ \|u\|_{H^4}^2 + \left\| \frac{d}{dt}(\bar{e}q + \bar{\rho}\theta) \right\|_{H^3}^2 \right\} \, d\tau \lesssim \mathcal{E}^2_0 + \int_0^t \left[ \|(q, \theta)\|_{H^2}^2 + \mathcal{E}(\|u\|_{H^3} + \mathcal{E}^2) \right] \, d\tau.
\]
Using interpolation inequality and Young’s inequality, we further have
\[
\int_0^t \left\{ \|u\|_{H^4}^2 + \left\| \frac{d}{dt}(\bar{e}q + \bar{\rho}\theta) \right\|_{H^3}^2 \right\} \, d\tau \lesssim \mathcal{E}^2_0 + \int_0^t \left[ C_\epsilon \|(q, \theta)\|_{L^2}^2 + \mathcal{E}^2(\epsilon + \mathcal{E}) \right] \, d\tau,
\]
where the constant \( C_\epsilon \) depends on \( \epsilon \) and some physical parameters in (1.10). On the other hand, by Lemmas 3.1–3.2, (3.11) and (3.12), we find that
\[
\mathcal{E}^2(t) + \|(q, \theta)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \int_0^t \|u\|_{L^2}^2 \, d\tau
\]
\[
\lesssim \mathcal{E}^2_0 + \int_0^t \left[ \|u\|_{H^2}^2 + \mathcal{E}(\|u\|_{H^3} + \mathcal{E}^2) \right] \, d\tau
\]
\[
\lesssim \mathcal{E}^2_0 + \int_0^t \left[ C_\delta \|u\|_{L^2}^2 + C_\delta \|u\|_{H^4}^2 + \mathcal{E}^2(\delta + \mathcal{E}) \right] \, d\tau,
\]
where the constant \( C_\delta \) depends on \( \delta \) and some physical parameters in (1.10). Consequently, in view of the above inequality and (3.34), we obtain
\[
\mathcal{E}^2(t) + \|(q, \theta)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \int_0^t \|u\|_{L^2}^2 \, d\tau
\]
\[
\leq C \left\{ (1 + C_\epsilon) \mathcal{E}^2_0 + \int_0^t \left[ C_\delta (1 + C_\epsilon) \|(q, u, \theta)\|_{L^2}^2 + \mathcal{E}^2 \delta + C_\delta \epsilon(1 + C_\delta) \mathcal{E} \right] \, d\tau \right\}.
\]
(3.35)

In particular, \( \epsilon, \delta \) and \( \mathcal{E} \) can be chosen so small that \( C(\delta + C_\epsilon \epsilon + (1 + C_\delta) \mathcal{E}) < \Lambda \) later on.

Now, let us recall that the local existence and uniqueness of solutions to the perturbed equations (1.10) have been established in [31, Remark 6.1] for \( \bar{\rho} \) and \( \bar{\epsilon} \) being constants, while the global existence and uniqueness of small solutions to the perturbed equations (1.10) with heat conductivity have been shown in [35] for \( \bar{\rho}, \bar{\epsilon} \) being close to a constant state. By a slight modification in the proof of the local existence in [31] or [35], one can easily obtain the existence and uniqueness of a local solution \( (q, v, \theta) \in C^0([0, T], H^3) \) to the perturbed problem (1.10)–(1.12) for some \( T > 0 \). Moreover, this local solution satisfies the above \textit{a priori} estimate (3.35). Therefore, we arrive at the following conclusion:

**Proposition 3.10.** Assume that \( (\bar{\rho}, \bar{\epsilon}) \) satisfies (1.4) and (1.6). For any given initial data \( (q_0, u_0, \theta_0) \in H^3 \) satisfying the compatibility condition and
\[
\inf_{x \in \Omega} \{ q_0 + \bar{\rho}, \theta_0 + \bar{\epsilon} \} > 0,
\]
then there exist a \( T^{\max} > 0 \) and a unique solution \( (q, u, \theta) \in C^0([0, T^{\max}], H^3) \) to the Rayleigh–Bénard problem (1.10)–(1.12) satisfying
\[
\inf_{(0, T) \times \Omega} \{ q + \bar{\rho}, \theta + \bar{\epsilon} \} > 0.
\]
where $T^{\text{max}}$ denotes the maximal time of existence of the solution $(\rho, u, \theta)$. Moreover, there is a sufficiently small constant $\delta_1^0 \in (0, 1]$, such that if $E(t) \leq \delta_1^0$ on some interval $[0, T] \subset [0, T^{\text{max}})$, then the solution $(\rho, u, \theta)$ satisfies
\[
E^2(t) + \| (\rho, \theta) \|_{L^2}^2 + \| u_t(t) \|_{H^1}^2 + \int_0^t \| u_{tt}(\tau) \|_{L^2}^2 d\tau 
\leq C E_0^2 + \int_0^t \left( C \| (\rho, u, \theta) \|_{L^2}^2 + \Lambda E^2(\tau) \right) d\tau \quad \text{for any } t \in (0, T),
\]
where the constant $C$ only depends on $\delta_1^0$, $\Lambda$, $\Omega$ and the known physical parameters in (1.10).

4. Nonlinear instability

Now we are in a position to prove Theorem 1.1 by adopting and modifying the ideas in [20,27]. In view of Theorem 2.4, we can construct a (linear) solution
\[
(\rho^l, u^l, \theta^l) = e^{\Lambda t} (\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0)
\]
to the linearized problem (1.11)–(1.13) with the initial data $(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0) \in H^3$. Furthermore, this solution satisfies
\[
\| (\bar{u}_0^l, \bar{u}_0^2) \|_{L^2} \| \bar{u}_0^3 \|_{L^2} > 0,
\]
where $\bar{u}_0^i$ stands for the $i$-th component of $\bar{u}_0$ for $i = 1, 2$ and 3. In what follows, $C_1, \cdots, C_7$ will denote generic constants that may depend on $(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0)$, $\delta_1^0$, $\Lambda$, $\Omega$ and the known physical parameters in (1.10), but are independent of $\delta$.

Obvious, we can not directly use the initial data of the linearized equations (1.11)–(1.13) as the one of the associated nonlinear problem, since the linearized and nonlinear equations enjoy different compatibility conditions at the boundary. To get around this obstacle, we instead use the elliptic theory to construct initial data of the nonlinear equations problem which are close to the linear growing modes.

**Lemma 4.1.** Let $(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0)$ be the same as in (4.1). Then there exists a $\delta_2^0 \in (0, 1)$ depending on $(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0)$, such that for any $\delta \in (0, \delta_2^0)$, there is a $u_\tau$ which may depend on $\delta$ and enjoys the following properties:

1. The modified initial data
   \[
   (\bar{\rho}_0^\delta, \bar{u}_0^\delta, \bar{\theta}_0^\delta) = \delta(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0) + \delta^2(\bar{\rho}_0, u_\tau, \bar{\theta}_0)
   \]
satisfy $u_0^\delta|_{\partial \Omega} = 0$ and the compatibility condition:
   \[
   \left\{ (\bar{\rho}_0^\delta + \bar{\rho}) u_0^\delta \cdot \nabla u_0^\delta + a \nabla ((\bar{\rho}_0^\delta + \bar{\rho}) (\bar{\theta}_0^\delta + \bar{\epsilon}) - \bar{\rho} \bar{\epsilon}) - \mu \Delta u_0^\delta - \mu_0 \nabla \div u_0^\delta + g \bar{\theta}_0 e_3 \right\} |_{\partial \Omega} = 0.
   \]

2. $u_\tau$ satisfies the following estimate:
   \[
   \| u_\tau \|_{H^3} \leq C_1,
   \]
   where the constant $C_1$ depends on $\| (\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0) \|_{H^3}$ and other physical parameters, but is independent of $\delta$.

**Proof.** Notice that $(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0)$ satisfies
\[
\bar{u}_0|_{\partial \Omega} = 0, \quad [ a \nabla (\bar{\epsilon} \bar{\rho}_0 + \bar{\rho} \bar{\theta}_0) - \mu \Delta \bar{u}_0 - \mu_0 \nabla \div \bar{u}_0 + g \bar{\theta}_0 e_3 ] |_{\partial \Omega} = 0.
\]
Hence, if the modified initial data satisfy (4.3), then we expect $u_\tau$ to satisfy the following problem:
\[
\begin{cases}
\mu \Delta u_\tau + \mu_0 \nabla \div u_\tau - \delta^2 \bar{\rho}_0^{**} u_\tau \cdot \nabla u_\tau - \delta \bar{\rho}_0^{**} (u_0 \cdot \nabla u_\tau + u_\tau \cdot \nabla u_0) \\
= a \nabla (\bar{\epsilon} \bar{\rho}_0 + \bar{\rho} \bar{\theta}_0) + g \bar{\theta}_0 e_3 + \bar{\rho}_0^{**} \bar{u}_0 \cdot \nabla \bar{u}_0 - a \nabla \bar{\rho}_0 \bar{\theta}_0 := F(\bar{\rho}_0, \bar{u}_0, \bar{\theta}_0),
\end{cases}
\]
where $\bar{\rho}_0^\delta := (1 + \delta) \bar{\rho}_0$, $\bar{\theta}_0^\delta := (1 + \delta) \bar{\theta}_0$ and $\bar{\rho}_0^{**} := (\bar{\rho}_0 + \bar{\rho}) = (\delta + \delta^2) \bar{\rho}_0 + \bar{\rho}$. Thus the modified initial data naturally satisfy the compatibility condition.
Next we shall look for a solution $u_\tau$ to the boundary problem (4.4) when $\delta$ is sufficiently small. We begin with the linearization of (4.4) which reads as

$$
\mu \Delta u_\tau + \mu_0 \nabla \text{div} u_\tau = F(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0) + \delta^2 \varrho_0^* \nu \cdot \nabla v + \delta \varrho_0^*(\bar{u}_0 \cdot \nabla v + v \cdot \nabla \bar{u}_0)$$

with boundary condition

$$
u u_\tau|_{\Gamma} = 0.$$ (4.6)

Let $v \in H^3$, then it follows from the elliptic theory that there is a solution $u_\tau$ of (4.5)–(4.6) satisfying

$$
\|u_\tau\|_{H^3} \leq \|F(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0) + \delta^2 \varrho_0^* \nu \cdot \nabla v + \delta \varrho_0^*(\bar{u}_0 \cdot \nabla v + v \cdot \nabla \bar{u}_0)\|_{H^1}
$$

$$
\leq C_m(1 + \|(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0)\|_{H^2} + \delta^2 \|v\|_{H^2}^2).
$$

Now, we take $C_1 = C_m(2 + \|(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0)\|_{H^3}^2)$ and $\delta \leq \min\{C_1^{-1}, 1\}$. Then for any $\|v\|_{H^3}^2 \leq C_1$, one has

$$
\|u_\tau\|_{H^3} \leq C_1.
$$

Therefore we can construct an approximate function sequence $u^n_\tau$, such that

$$
\mu \Delta u^{n+1}_\tau + \mu_0 \nabla \text{div} u^{n+1}_\tau - \delta^2 \varrho_0^* u^n_\tau \cdot \nabla u^n_\tau - \delta \varrho_0^*(\bar{u}_0 \cdot \nabla u^n_\tau + u^n_\tau \cdot \nabla \bar{u}_0) = F(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0),
$$

and for any $n$,

$$
\|u^n_\tau\|_{H^3} \leq C_1, \quad \|u^{n+1}_\tau - u^n_\tau\|_{H^3} \leq C_2 \delta \|u^n_\tau - u^{n-1}_\tau\|_{H^3}
$$

for some constant $C_2$ independent of $\delta$ and $n$. Finally, we choose a $\delta$ sufficiently small so that $C_2 \delta < 1$, and then use a compactness argument to get a limit function which solves the nonlinear boundary problem (4.4). Moreover $\|u_\tau\|_{H^3} \leq C_1$. Thus we have proved Lemma 4.1.

Let $(\varrho_0^\delta, u_0^\delta, \theta_0^\delta)$ be constructed as in Lemma 4.1. Then there is a constant

$$
C_3 \geq \max\{1, \| (\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0) \|_{L^2}\}
$$

depending on $(\bar{\theta}_0, \bar{u}_0, \bar{\theta}_0)$, such that for any $\delta \in (0, \delta_0^3) \subset (0, 1)$,

$$
\mathcal{E}(\varrho_0^\delta, u_0^\delta, \theta_0^\delta) \leq C_3 \delta,
$$

where $\mathcal{E}$ is defined by (3.1). Recalling $\inf_{x \in \Omega} \{ \bar{\rho}, \bar{\varepsilon} \} > 0$ and the embedding theorem $H^2 \hookrightarrow L^\infty$, we can choose a sufficiently small $\delta$, such that

$$
\inf_{x \in \Omega} \{ \varrho_0^\delta + \bar{\rho}, \theta_0^\delta + \bar{\varepsilon} \} > 0.
$$

Hence, by virtue of Proposition 3.10, there is a $\delta_0^3 \in (0, \delta_0^3)$, such that for any $\delta < \delta_0^3$, there exists a unique local solution $(\varrho^\delta, u^\delta, \theta^\delta) \in C^0(0, T^\delta, H^3) \to (1.10)$ and (1.12), emanating from the initial data $(\varrho_0^\delta, u_0^\delta, \theta_0^\delta)$. Moreover, (4.7) holds for any $\delta$ satisfying $\mathcal{E}(\varrho_0^\delta, u_0^\delta, \theta_0^\delta) \leq C_3 \delta_0^3$. Let $C > 0$ and $\delta_0^3 > 0$ be the same constants as in Proposition 3.10 and $\delta_0 = \min\{\delta_0^3, \delta_0^0/C_3\}$. Let $\delta \in (0, \delta_0)$ and

$$
T^\delta = \frac{1}{A} \ln \frac{2\varepsilon_0}{\delta} > 0, \quad \text{i.e.,} \quad \delta e^{AT^\delta} = 2\varepsilon_0,
$$

where $\varepsilon_0 \leq 1$, independent of $\delta$, is sufficiently small and will be fixed later. In what follows, we denote $\mathcal{E}_\delta(t) := \mathcal{E}(\varrho^\delta, u^\delta, \theta^\delta)(t)$.

Define

$$
T^* = \sup \{ t \in (0, T^{max}) \mid \mathcal{E}_\delta(t) \leq C_3 \delta_0 \} > 0
$$

and

$$
T^{**} = \sup \{ t \in (0, T^{max}) \mid \| (\varrho^\delta, u^\delta, \theta^\delta) (t) \|_{L^2} \leq 2 \delta C_3 e^{AT^*} \} > 0,
$$

where $T^{max}$ denotes the maximal time of existence of the solution $(\varrho^\delta, u^\delta, \theta^\delta)$. Obviously, $T^*$ and $T^{**}$ may be finite, and furthermore,

$$
\mathcal{E}_\delta(T^*) = C_3 \delta_0 \quad \text{if} \ T^* < \infty, \quad \| (\varrho^\delta, u^\delta, \theta^\delta) (T^{**}) \|_{L^2} = 2 \delta C_3 e^{AT^{**}} \quad \text{if} \ T^{**} < T^{max}.
$$
Then for all \( t \leq \min\{T^*, T^{**}\} \), we deduce from the estimate (3.36) and the definition of \( T^* \) and \( T^{**} \) that
\[
E^2_\delta(t) + \|((\rho^\delta, \theta^\delta)_t)(t)\|_{H^2}^2 + \|u^\delta(t)\|_{H^3}^2 + \int_0^t \|u^\delta_t\|_{L^2}^2 d\tau \\
\leq C|E^2(\rho_0^\delta, u_0^\delta, \theta_0^\delta) + 2C_3^2 \delta^2 e^{2\Lambda t}/N| + \Lambda \int_0^t E^2_\delta(\tau)d\tau \\
\leq C_4 \delta^2 e^{2\Lambda t} + \Lambda \int_0^t E^2_\delta(\tau)d\tau
\]
for some constant \( C_4 > 0 \). Thus, applying Gronwall’s inequality, one concludes
\[
E^2_\delta(t) + \|((\rho^\delta, \theta^\delta)_t)(t)\|_{H^2}^2 + \|u^\delta(t)\|_{H^3}^2 + \int_0^t \|u^\delta_t\|_{L^2}^2 d\tau \leq C_5 \delta^2 e^{2\Lambda t} \tag{4.11}
\]
for some constant \( C_5 > 0 \).

Let \((\rho^d, u^d, \theta^d) = (\rho^\delta, u^\delta, \theta^\delta) - \delta(\rho^l, u^l, \theta^l)\). Noting that \((\rho^\delta_0^l, u_0^\delta, \theta_0^\delta) := \delta(\rho^l_0, u_0^l, \theta_0^l) \in H^3\), we find that \((\rho^d, u^d, \theta^d)\) satisfies the following error equations:
\[
\begin{align*}
\rho^d_t + \text{div}(\rho^d u^d) &= -\text{div}(\rho^\delta u^\delta) := N^\rho(\rho^\delta, u^\delta) := N_0^\rho, \\
\rho u^d_t + a \nabla(e^d + \bar{\rho} \rho^d) - \mu \nabla \text{div} u^d - \mu_0 \Delta u^d + g\rho^d e_3 &= (\rho^\delta + \bar{\rho}) u^\delta \cdot \nabla u^\delta - \rho^\delta u^\delta - a \nabla(\rho^\delta \theta^\delta) := N^u(\rho^\delta, u^\delta, \theta^\delta) := N_0^u, \\
\theta^d_t + \bar{e}^d u^d + a c \text{div} u^d &= (\rho^\delta + \bar{\rho})^{-1}[|\mu| \nabla u^\delta + \nabla(u^\delta)^T]/2 + \lambda(\text{div} u^d)^2 - u^\delta \cdot \nabla \theta^\delta - a \theta^\delta \text{div} u^\delta := N^\theta(\rho^\delta, u^\delta, \theta^\delta) := N_0^\theta,
\end{align*}
\tag{4.12}
\]
with initial data \((\rho^d(0), u^d(0), \theta^d(0)) = \delta^2(\rho_0^l, u_0^l, \theta_0^l)\) and boundary condition \(u^d|_{\partial\Omega} = 0\).

Next, we shall establish the error estimate for \((\rho^d, u^d, \theta^d)\) in \(L^2\)-norm.

**Lemma 4.2.** There is a constant \( C_6 \), such that for all \( t \leq \min\{T^\delta, T^*, T^{**}\} \),
\[
\|(\rho^d, u^d, \theta^d)(t)\|_{L^2}^2 \leq C_6 \delta^3 \theta^3 \Lambda t. \tag{4.13}
\]

**Proof.** We differentiate the linearized momentum equations (4.12)\_2 in time, multiply the resulting equations by \( u_i \) in \( L^2(\Omega) \), and use the equations (4.12)\_1 and (4.12)\_3 to deduce
\[
\frac{d}{dt} \int \left\{ \rho |u_i|^2 - g \rho |u_3|^2 + [(1 + a)\bar{\rho} \text{div} u^d - 2g\bar{\rho} u_3 |\text{div} u^d| \text{div} u^d \right\} dx \\
= -2\mu \int |\nabla u_i|^2 dx - 2\mu_0 \int |\text{div} u_i|^2 dx \\
+ 2 \int [\partial_i N_0^u - g N_0^g e_3 - a \nabla(\bar{e} N_0^g + \bar{\rho} N_0^g)] \cdot u_i dx.
\tag{4.14}
\]
Thanks to (2.22), one has
\[
\int \left\{ g \rho |u_3|^2 + [2g\bar{\rho} u_3 - (1 + a)\bar{\rho} \text{div} u^d] \text{div} u^d \right\} dx \\
\leq \Lambda \int (|\nabla u^d|^2 + \mu_0 |\text{div} u^d|^2) dx + \Lambda^2 \int \rho |u^d|^2 dx.
\]
Thus, integrating (4.14) in time from 0 to \( t \), we get
\[
\int |\sqrt{\rho} u^d(t)|^2_{L^2} + 2 \int_0^t (\mu |\nabla u^d|_{L^2}^2 + \mu_0 |\text{div} u^d|_{L^2}^2) d\tau \\
\leq T_1^0 + \Lambda^2 \int |\sqrt{\rho} u^d(t)|_{L^2} + \Lambda \mu |\nabla u^d(t)|_{L^2}^2 + \Lambda \mu_0 |\text{div} u^d(t)|_{L^2}^2 \\
+ 2 \int_0^t [\partial_i N_0^u - g N_0^g e_3 - a \nabla(\bar{e} N_0^g + \bar{\rho} N_0^g)] \cdot u_i dx d\tau, \tag{4.15}
\]
where

\[ I_1^0 \triangleq \left\{ \int \{ \tilde{\rho} |u_t^d| \leq 2 \} - g\tilde{\rho}'(u_0^d)^2 + \left[ (1 + \alpha)\tilde{\rho} \text{div} u - 2g\tilde{\rho}u_0^d |\text{div} u| \right] \, dx \right\} \bigg|_{t=0}. \]

Using Newton-Leibniz’s formula and Cauchy–Schwarz’s inequality, we find that

\[
\Lambda(\mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2) = I_2^0 + 2A \int_0^t \left( \sum_{1 \leq i, j \leq 3} \partial_x u_{ij}^d \partial_x u_{ij}^d \, dx + \mu_0 |\text{div} u^d(t)|_{L^2}^2 \right) \, dt 
\leq I_2^0 + \int_0^t (\mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2) \, dt + 2A \int_0^t (\mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2) \, dt, \tag{4.16}
\]

where \( I_2^0 = \Lambda(\mu \| \nabla u^d(0) \|_{L^2}^2 + \mu_0 |\text{div} u^d(0)|_{L^2}^2) \) and \( u_{ij}^d \) denotes the \( j \)-th component of \( u^d \). On the other hand,

\[
\Lambda \frac{d}{dt} \sqrt{\mu} u^d(t) \|_{L^2}^2 = \left( \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2 \right) \| \frac{\partial u^d(t)}{dt} \|_{L^2}^2 = \Lambda \frac{d}{dt} \sqrt{\mu} u^d(t) \|_{L^2}^2 + \frac{2}{\Lambda} \int_0^t (\mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2) \, ds + \frac{2I_0^0 + 2I_0^2}{\Lambda} + \int_0^t \left[ (a \nabla(\tilde{e}N_0^\delta + \tilde{\rho} N_0^\delta)) \cdot u^d \right] \, dx \, dt. \tag{4.17}
\]

Hence, putting (4.15)–(4.17) together, we obtain the differential inequality

\[
\frac{d}{dt} \| \sqrt{\rho} u^d(t) \|_{L^2}^2 + \mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2 \leq 2 \Lambda \left( \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u^d(t)|_{L^2}^2 \right) \| \frac{\partial u^d(t)}{dt} \|_{L^2}^2 + \frac{2I_0^0 + 2I_0^2}{\Lambda} + \int_0^t \left[ (a \nabla(\tilde{e}N_0^\delta + \tilde{\rho} N_0^\delta)) \cdot u^d \right] \, dx \, dt. \tag{4.18}
\]

Next, we control the last two terms on the right hand of (4.18). Noting that

\[
\delta e^{\lambda t} \leq 2 \varepsilon_0 \leq \varepsilon_0 \quad \text{for any } t \leq \min\{T^\delta, T^*, T^{**}\}, \tag{4.19}
\]

we utilize (4.11) and (4.1), Hölder’s inequality and Sobolev’s embedding theorem to infer that

\[
2 \int_0^t \left[ \int \left[ (\partial_t N_0^\delta - gN_0^\delta \tau - a \nabla(\tilde{e}N_0^\delta + \tilde{\rho} N_0^\delta)) \cdot u^d \right] \, dx \right] \, dt \leq \int_0^t \left( \| \nabla N_0^\delta \|_{H^1} + \| \partial_t N_0^\delta \|_{L^2} \right) \left( \| u^d \|_{H^1} + \| u^d \|_{L^2} \right) \, dt \leq \int_0^t \delta^3 e^{3\lambda t} + \delta^2 e^{2\lambda t} + \delta e^{\lambda t} \| u^d \|_{L^2} \delta e^{\lambda t} \, dt \leq \delta^3 e^{3\lambda t} + \delta^4 e^{4\lambda t} \leq \delta^3 e^{3\lambda t} \tag{4.20}
\]

and

\[
\frac{(I_0^0 + 2I_0^2)}{\Lambda} \leq \left( \left( \| \sqrt{\rho} u^d \|_{L^2}^2 + \mu \| \nabla u^d \|_{L^2}^2 + \mu_0 |\text{div} u^d| \|_{L^2}^2 \right) \right) \bigg|_{t=0} \leq \left( \left( \| \theta_1 \|_{L^2}^2 + \| u^d \|_{L^2}^2 + \| u^d \|_{L^2}^2 \right) \right) \bigg|_{t=0} \leq \delta t \left( \left( \| \hat{\theta}_0 \|_{L^2}^2 + \| u_r \|_{L^2}^2 \right) + \delta^2 e^{2\lambda t} + \delta^4 e^{4\lambda t} \| u^d \|_{L^2}^2 \right) \leq \delta^3 e^{3\lambda t}. \tag{4.21}
\]

Thus, substituting (4.21) and (4.20) into (4.18), we obtain

\[
\frac{\partial}{\partial_t} \| \sqrt{\rho} u^d(t) \|_{L^2}^2 + \mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u(t)|_{L^2}^2 \leq 2 \Lambda \left( \| \sqrt{\rho} u^d(t) \|_{L^2}^2 + \mu \| \nabla u^d(t) \|_{L^2}^2 + \mu_0 |\text{div} u(t)|_{L^2}^2 \right) \, dt \bigg| + C_\gamma \delta^3 e^{3\lambda t}. \]
Applying Gronwall’s inequality to the above inequality, one obtains

$$\|\sqrt{\rho}u^d(t)\|_{L^2}^2 + \int_0^t (\mu\|\nabla u^d\|_{L^2}^2 + \mu_0\|\text{div} u^d\|_{L^2}^2)\,dt \lesssim \delta^3 e^{3\Lambda t} + \delta^4 \|\sqrt{\rho}u^i\|_{L^2}^2 \lesssim \delta^3 e^{3\Lambda t}$$

(4.22)

for all $t \leq \min\{T^\delta, T^*\}$. Thus, making use of (4.15), (4.16) and (4.20)–(4.22), we deduce that

$$\frac{1}{\Lambda} \|\sqrt{\rho}u^d(t)\|_{L^2}^2 + \mu \|\nabla u^d(t)\|_{L^2}^2 + \mu_0 \|\text{div} u^d(t)\|_{L^2}^2 \leq \Lambda \|\sqrt{\rho}u^d(t)\|_{L^2}^2 + 2\Lambda \int_0^t (\mu\|\nabla u^d\|_{L^2}^2 + \mu_0\|\text{div} u^d\|_{L^2}^2)\,dt$$

(4.23)

$$+ \frac{2}{\Lambda} \int_0^t \int [\partial_t N^u - g N^\rho e_3 - a\nabla(\bar{e} N^\rho + \bar{\rho} N_{\delta}^\rho)] \cdot u^d\,dx\,dt + \frac{T_0^1 + 2T_0^2}{\Lambda} \lesssim \delta^3 e^{3\Lambda t},$$

which, together with Poincaré’s inequality and the estimates (4.22), yields

$$\|u^d(t)\|_{H^1}^2 + \|u^d(t)\|_{L^2}^2 + \int_0^t \|\nabla u^d\|_{L^2}^2\,dt \lesssim \delta^3 e^{3\Lambda t}.$$  

(4.24)

Finally, using the equations (4.12)₁ and (4.12)₂, and the estimates (4.19) and (4.24), we find that

$$\|(q^d, \theta^d)(t)\|_{L^2} \leq \delta^2 \|(\bar{\rho}_0, \theta_0)\|_{L^2} + \int_0^t \|(q^d, \theta^d)\|_{L^2}\,dt \lesssim \delta^2 + \int_0^t \||u^d|\|_{H^1} + \|(N_{\delta}^\rho, N_{\delta}^\theta)\|_{L^2}\,dt$$

$$\lesssim \delta^2 + \int_0^t (\delta^{\frac{3}{2}} e^{\frac{3}{2}A t} + \mathcal{E}_{\delta}(\tau))\,dt \lesssim \delta^{\frac{3}{2}} e^{\frac{3}{2}A t}.$$  

Putting the previous estimates together, we get (4.13) immediately. This completes the proof of Lemma 4.2.

Now, we claim that

$$T^\delta = \min \{T^\delta, T^*, T^{**}\},$$

(4.25)

provided that small $\varepsilon_0$ is taken to be

$$\varepsilon_0 = \min \left\{ \frac{C_3\delta_0}{4\sqrt{C_5}}, \frac{C_2^2}{8C_6}, \frac{m_0^2}{24C_6}, 1 \right\} > 0,$$

(4.26)

where we have denoted $m_0 = \min\{\|\bar{u}_{01}, \bar{u}_{02}\|_{L^2}, \|\bar{u}_{03}\|_{L^2}\} > 0$ due to (4.2).

Indeed, if $T^* = \min\{T^\delta, T^*, T^{**}\}$, then $T^* < \infty$. Moreover, from (4.11) and (4.8) we get

$$\mathcal{E}_{\delta}(T^*) \leq \sqrt{C_5}\delta e^{AT^*} \leq \sqrt{C_5}\delta e^{AT^*} = 2\sqrt{C_5}\varepsilon_0 < C_3\delta_0,$$

which contradicts with (4.9). On the other hand, if $T^{**} = \min\{T^\delta, T^*, T^{**}\}$, then $T^{**} < T^* \leq T^{max}$. Moreover, in view of (4.1), (4.8) and (4.13), we see that

$$\|(q^\delta, u^\delta, \theta^\delta)(T^{**})\|_{L^2} \leq \|(q^\delta, u^\delta, \theta^\delta)(T^{**})\|_{L^2} + \|(q^d, u^d, \theta^d)(T^{**})\|_{L^2} \leq \delta \|(q^d, u^d, \theta^d)(T^{**})\|_{L^2} + \sqrt{C_6}\delta^{3/2} e^{3AT^{**}/2}$$

$$\leq \delta C_3 e^{AT^{**}} + \sqrt{C_6}\delta^{3/2} e^{3AT^{**}/2} \leq \delta C_3 e^{AT^{**}} + \sqrt{C_6}\delta^{3/2} e^{3AT^{**}/2}$$

which also contradicts with (4.10). Therefore, (4.25) holds.
Since $T^\delta = \min\{T^\delta, T^*, T^{**}\}$, (4.13) holds for $t = T^\delta$, thus we again use (4.26) and (4.13) with $t = T^\delta$ to deduce that
\[
\|u_3^d(T^\delta)\|_{L^2} \geq \|u_0^d(T^\delta)\|_{L^2} - \|u_3^d(T^\delta)\|_{L^2} = \delta e^{\Lambda T^\delta} \|u_03\|_{L^2} - \|u_3^d(T^\delta)\|_{L^2} \\
\geq \delta e^{\Lambda T^\delta} \|u_03\|_{L^2} - \sqrt{C_0} e^{3/2} \|\Lambda T^\delta/2\|/2 \geq 2m_0\varepsilon_0 - \sqrt{C_0(2\varepsilon_0)^{3/2}} \geq m_0\varepsilon_0,
\]
where $u_3^d(T^\delta)$ denote the third component of $u^d(T^\delta)$. Similar, we also have
\[
\|(u_1^d, u_2^d)(T^\delta)\|_{L^2} \geq m_0\varepsilon_0.
\]
This completes the proof of Theorem 1.1 by defining $\varepsilon = m_0\varepsilon_0$. In addition, if $\rho' \geq 0$, then the function $\bar{\rho}_0$ constructed in (4.1) satisfies $\|\bar{\rho}_0\|_{L^2} > 0$. Thus we also obtain $\|\rho^d(T^\delta)\|_{L^2} \geq m_0\varepsilon_0$, if we define
\[
m_0 = \min\{\|\bar{\rho}_0\|_{L^2}, \|(\bar{u}_01, \bar{u}_02)\|_{L^2}, \|\bar{u}_03\|_{L^2}\} > 0.
\]
Hence, the assertion in Remark 1.3 holds.

5. Appendix

In this section we show that $\Lambda$ defined by (2.22) is the sharp growth rate for any solutions to the linearized problem (1.11)–(1.13). Since the density varies for a compressible fluid, the spectrums of the linearized solution operator are difficult to analyze in comparison with an incompressible fluid, and it is hard to obtain the largest growth rate of the solution operator in some Sobolev space in the usual way. Here we exploit energy estimates as in [24, 27] to show that $e^{\Lambda t}$ is indeed the sharp growth rate for $(\rho, u, \theta)$ in $H^2$-norm.

Proposition 5.1. Assume that the assumption of Theorem 2.4 is satisfied. Let $(\rho, u, \theta)$ solve the linearized Rayleigh–Bénard problem (1.11)–(1.13). Then, we have the following estimates.

\[
\|(\rho, \theta)(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \\
\leq Ce^{2\Lambda t}(\|(\rho_0, \theta_0)\|_{H^2}^2 + \|u_0\|_{H^2}^2),
\]

(5.1)

\[
\|(\rho, u, \theta)(t)\|_{H^2}^2 + \int_0^t \left[\|u\|_{H^2}^2 + \|(\varepsilon \rho_t + \bar{\rho}_0 t)\|_{H^2}^2\right] \, d\tau \\
\leq Ce^{2\Lambda t}(\|(\rho_0, u_0, \theta_0)\|_{H^2}^2)
\]

(5.2)

for any $t \geq 0$, where $\Lambda$ is constructed by (2.21), and the constant $C$ may depend on $g$, $\mu$, $\mu_0$, $\bar{e}$, $\bar{\rho}$, $\Lambda$ and $\Omega$.

Proof. The first estimate (5.1) can be shown by an argument similar to that in Lemma 4.2. In fact, following the process in the derivation of (4.18) and (4.23), we obtain the following two inequalities

\[
\frac{d}{dt}\|\sqrt{\rho} u(t)\|_{L^2}^2 + \mu\|\nabla u(t)\|_{L^2}^2 + \mu_0\|\text{div} u(t)\|_{L^2}^2 \\
\leq I_1 + 2\Lambda \left[\|\sqrt{\rho} u\|_{L^2}^2 + \int_0^t (\mu\|\nabla u\|_{L^2}^2 + \mu_0\|\text{div} u\|_{L^2}^2) \, d\tau\right]
\]

(5.3)

and

\[
\frac{1}{\Lambda}\|\sqrt{\rho} u(t)\|_{L^2}^2 + \mu\|\nabla u(t)\|_{L^2}^2 + \mu_0\|\text{div} u(t)\|_{L^2}^2 \\
\leq I_1 + \Lambda \|\sqrt{\rho} u(t)\|_{L^2}^2 + 2\Lambda \int_0^t (\mu\|\nabla u\|_{L^2}^2 + \mu_0\|\text{div} u\|_{L^2}^2) \, d\tau
\]

(5.4)
with
\[ I_1 = \left\{ 2(\mu \| \nabla u \|^2_{L^2} + \mu_0 \| \text{div} u \|^2_{L^2}) \ight. \\
+ \frac{1}{\Lambda} \int \left\{ \bar{\rho}|u_t|^2 - g\bar{\rho} u_3^2 + [(1 + a)\bar{\rho}\text{div} u - 2g\bar{\rho}u_3]\text{div} u \right\} dx \bigg|_{t=0}. \]

An application of Gronwall’s inequality to (5.3) implies that for any \( t \geq 0 \),
\[ \| \sqrt{\bar{\rho}}u(t) \|^2_{L^2} + \int_0^t (\mu \| \nabla u \|^2_{L^2} + \mu_0 \| \text{div} u \|^2_{L^2})d\tau \]
\[ \leq e^{2\Lambda t} \| \sqrt{\bar{\rho}}u_0 \|^2_{L^2} + \frac{I_1}{2\Lambda} (e^{2\Lambda t} - 1) \]
\[ \leq C e^{2\Lambda t}(\| (\rho_0, \theta_0) \|^2_{H^1} + \| u_0 \|^2_{H^2}), \]
which, together with Poincaré’s inequality and (5.4), results in
\[ \| u(t) \|^2_{H^1} + \| u_t(t) \|^2_{L^2} + \int_0^t \| \nabla u(s) \|^2_{L^2}ds \leq C e^{2\Lambda t}(\| (\rho_0, \theta_0) \|^2_{H^1} + \| u_0 \|^2_{H^2}). \]

Thus, using (1.13)_1 and (1.13)_2, we have
\[ \| (\rho, \theta)(t) \|^2_{L^2} \leq \| (\rho_0, \theta_0) \|^2_{L^2} + \int_0^t \| (\rho, \theta)_s(s) \|^2_{L^2}ds \]
\[ \leq \| (\rho_0, \theta_0) \|^2_{L^2} + (1 + a)\| (\bar{\rho}, \bar{\theta}) \|^2_{H^1} \int_0^t \| u(s) \|^2_{H^1}ds \]
\[ \leq C e^{\Lambda t}(\| (\rho_0, \theta_0) \|^2_{H^1} + \| u_0 \|^2_{H^2}). \]

Hence the estimate (5.1) follows from the above two estimates.

Finally, following the arguments in the proof of (3.35), one find that
\[ \| (\rho, u, \theta)(t) \|^2_{H^2} + \int_0^t \left[ \| u \|^2_{H^3} + \| (\bar{e} \rho_t + \bar{\rho} u_t) \|^2_{L^2} \right]d\tau \]
\[ \leq C \| (\rho_0, u_0, \theta_0) \|^2_{H^2} + \int_0^t \left[ C \| (\rho, u, \theta) \|^2_{L^2} + \Lambda \| (\rho, u, \theta) \|^2_{H^2} \right]d\tau, \]
which, combined with (5.1), gives (5.2) due to Gronwall’s inequality. This completes the proof. \( \square \)

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**Compliance with ethical standards**

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