MEAN WIDTH INEQUALITIES OF SECTIONS AND PROJECTIONS OF
CONVEX BODIES FOR ISOTROPIC MEASURES

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Abstract. In this paper, we establish mean width inequalities of sections and projections
of convex bodies for isotropic measures with complete equality conditions, which extends the
recent work of Alonso-Gutiérrez and Brazitikos. Different from their approach, our proof
is based on the approach developed by Lutwak, Yang and Zhang, by using the Ball-Barthe
inequality, the mass transportation, and the isotropic embedding.

1. Introduction

In extremal problems related to the mean width of convex bodies, Euclidean balls, cubes
(or cross-polytopes), and simplices are usually the extremizers. An important example is the
Urysohn inequality (e.g., [51, p. 382]) which expresses the geometric fact that Euclidean balls
minimize the mean width of convex bodies of given volume. Another example is the mean
width inequality [9, 49, 50] that simplices have the extremal mean width of convex bodies in
the John (Löwner) position, whereas cubes and cross-polytopes are the extremizers in the
symmetric cases. For more information about the mean width, see, e.g., [2, 3, 14–16, 21, 25,
26, 38, 47]. In this paper, we will focus on the mean width of lower-dimensional sections
and projections of convex bodies. The question of estimating sections and projections of
convex bodies is always attractive, because it is not easy to give their sharp bounds (even
for volumes) for general convex bodies without any additional assumption. But for some
special convex bodies (e.g., cubes, $l^p_n$-balls, and simplices) and special positions (e.g., John
and isotropic positions), some remarkable results for the lower dimensional sections and
projections have been intensively investigated, for example, [1, 4, 5, 11, 13, 17–20, 22, 29–31, 33,
36, 37, 45, 46, 48, 53].

To introduce the mean width of a convex body, let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the scalar product
and Euclidean norm in $n$-dimensional Euclidean space $\mathbb{R}^n$. Denote by $B^n_2$ and $S^{n-1}$ the
Euclidean unit ball and its boundary in $\mathbb{R}^n$, respectively. The volume of $B^n_2$ is solely written
by $\omega_n$. We also denote by $B^n_1$ and $B^n_\infty$ the unit cross-polytope (the $l^1_n$-ball) and the unit cube
(the $l_\infty^n$-ball) in $\mathbb{R}^n$. A convex body $K$ in $\mathbb{R}^n$ is a compact convex set with non-empty interior.
Its support function $h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined for $x \in \mathbb{R}^n$ by $h_K(x) = \max \{ \langle x, y \rangle : y \in K \}$. The
mean width of $K$ is defined by

$$W(K) = \frac{1}{n\omega_n} \int_{S^{n-1}} (h_K(u) + h_K(-u))du = \frac{2}{n\omega_n} \int_{S^{n-1}} h_K(u)du,$$

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where $du$ is the $(n-1)$-dimensional Hausdorff measure (that coincides with the spherical Lebesgue measure in this case) with total mass $n\omega_n$. If convex body $K$ contains the origin $o$ in its interior, then its polar body $K^\circ$ is defined by $K^\circ = \{x \in \mathbb{R}^n : (x,y) \leq 1 \text{ for all } y \in K\}$.

To state the mean width inequality, we shall first introduce the concept of John and Löwner positions. It is well-known that there is a unique maximal volume ellipsoid (called the John ellipsoid) contained in a convex body $K$. We say that $K$ is in the John (Löwner) position if the John (Löwner) ellipsoid is the Euclidean unit ball $B_n$. In 1948, Fritz John [32] showed that a convex body $K$ is in the John (Löwner) position if and only if for some $m \geq n$ there are unit vectors $(u_i)^m_1$ on the boundary of $K$ and positive numbers $(c_i)^m_1$ satisfying

$$\sum_{i=1}^{m} c_i u_i = 0 \quad (1.1)$$

and $$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \quad (1.2)$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the space spanned by $u_i$ and $I_n$ is the identity map on $\mathbb{R}^n$. Notice that $(u_i)^m_1$ are not concentrated on any hemisphere due to condition (1.1) and condition (1.2) guarantees that the $(u_i)^m_1$ do not all lie close to a proper subspace of $\mathbb{R}^n$. Let $\Delta_n$ denote the regular simplex in $\mathbb{R}^n$ inscribed into $B_n^\circ$. Its polar $\Delta_n^\circ$ is still a regular simplex with inradius 1. Obviously, the unit cross-polytope $B_n^\circ$ and $\Delta_n$ are in the Löwner position. The unit cube $B^\circ_\infty$ and the polar $\Delta^\circ_n$ are in the John position.

An important application of the John (Löwner) position due to Ball [5,6] is that condition (1.2) can be perfectly combined with the Brascamp-Lieb inequality, which nowadays is called as the geometric Brascamp-Lieb inequality. Later, the reverse Brascamp-Lieb inequality was proposed and established by Barthe [10]. The geometric Brascamp-Lieb inequality and its dual play an important role in establishing reverse (affine) isoperimetric inequalities, that usually have simplices or, in the symmetric case, cubes and their polar s, as extremals. The work of Ball and Barthe has motivated a series of new studies (see, e.g., [1,5,7,9,12,15,24,27,28,34,42,44,52]). In particular, by the geometric Brascamp-Lieb inequality and its dual, Barthe [9], Schmuckenschläger [50] established the following mean width inequalities (see also Schechtman and Schmuckenschläger [49]).

**Theorem 1.1.** Let $K$ be a convex body in $\mathbb{R}^n$. If $K$ is in Löwner position, then $W(K) \geq W(\Delta_n)$, with equality if and only if $K = \Delta_n$. If $K$ is in John position, then $W(K) \leq W(\Delta_n^\circ)$, with equality if and only if $K = \Delta_n^\circ$.

In symmetric cases, if $K$ is in Löwner position, then $W(K) \geq W(B^\circ_1)$, with equality if and only if $K = B^\circ_1$. If $K$ is in John position, then $W(K) \leq W(B^\infty_\infty)$, with equality if and only if $K = B^\infty_\infty$.

Let $G_{n,k}$ be the Grassmann manifold of $k$-dimensional linear subspaces in $\mathbb{R}^n$, $1 \leq k \leq n$. Denote by $P_H$ the orthogonal projection onto the subspace $H \in G_{n,k}$. Denote by $\Delta_k$ the $k$-dimensional regular simplex inscribed into $B^k_2$ in $\mathbb{R}^k$ and by $\Delta_k^\circ$ the polar of $\Delta_k$ (where the polar operation is taken in $\mathbb{R}^k$). We still denote the mean width of a convex body in $\mathbb{R}^k$ by $W(\cdot)$. Very recently, Alonso-Gutiérrez and Brazitikos [1] gave some estimates for the mean width of sections and projections of convex bodies in the John (Löwner) position as follows. The case $k = n$ of Theorem 1.2 was proved in [9,49,50].
Let $H \subseteq G_{n,k}$ and $K$ be a convex body in $\mathbb{R}^n$. If $K$ is in Löwner position, then $W(P_H K) \geq \sqrt{\frac{k}{n}} W(\triangle_k)$. If $K$ is in John position, then $W(K \cap H) \leq C \frac{n}{k} \sqrt{\frac{\log n}{\log k}} W(\triangle_k)$, where $C$ is an absolute constant.

Furthermore, let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. If $K$ is in Löwner position, then $W(P_H K) \geq \sqrt{\frac{k}{n}} W(B^k)$. If $K$ is in John position, then $W(K \cap H) \leq \sqrt{\frac{k}{n}} W(B^k)$.

A finite Borel measure $\mu$ on $S^{n-1}$ is said to be isotropic if
\[
\int_{S^{n-1}} u \otimes u d\mu(u) = I_n. \tag{1.3}
\]
Note that it is impossible for an isotropic measure to be concentrated on a proper subspace of $\mathbb{R}^n$. Denote by $\text{supp} \mu$ the support set of $\mu$. The centroid of the measure $\mu$ is defined as
\[
\frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} u d\mu(u).
\]
The measure $\mu$ is said to be even if it assumes the same value on antipodal sets. Condition (1.3) reduces to (1.2) if the isotropic measure $\mu$ is of the form $\sum_{i=1}^{m} c_i \delta_{u_i}$ or $(1/2) \sum_{i=1}^{m} (c_i \delta_{u_i} + c_i \delta_{-u_i})$ on $S^{n-1}$ ($\delta_x$ stands for the Dirac mass at $x$).

The aim of this paper is to establish the following mean width inequalities of sections and projections of convex bodies for isotropic measures with complete equality conditions.

**Theorem 1.3.** Let $\mu$ be an isotropic measure on $S^{n-1}$ whose centroid is at the origin, and let $K$ be a convex body in $\mathbb{R}^n$ such that $K$ contains the convex hull of $\text{supp} \mu$; i.e., $K \supseteq \text{conv}\{\text{supp} \mu\}$. For any given $H \subseteq G_{n,k}$, we have $W(P_H K) \geq \sqrt{\frac{k}{n}} W(\triangle_k)$ with equality if and only if $P_H K = \sqrt{\frac{k}{n}} \triangle_k$.

Furthermore, if $\mu$ is even isotropic, then for any given $H \subseteq G_{n,k}$, $W(P_H K) \geq \sqrt{\frac{k}{n}} W(B^k)$ with equality if and only if $P_H K = \sqrt{\frac{k}{n}} B^k$, and $W(K^\circ \cap H) \leq \sqrt{\frac{k}{n}} W(B^k)$ with equality if and only if $K^\circ \cap H = \sqrt{\frac{k}{n}} B^k$.

The case $k = n$ of Theorem 1.3 was established in [38]. When $\mu$ is discrete, Theorem 1.3 is stated in Theorem 1.2 (without equality conditions), and reduces to Theorem 1.1 for $k = n$.

Note that the proof of Theorem 1.3 is based on the approach developed by Lutwak, Yang, and Zhang [41][43], which extends the work of Ball [6] and Barthe [9][10] by using the techniques of mass transportation, isotropic embedding, and the Ball–Barthe inequality. For more applications about this approach, see e.g., [34][40][52].

We also note that the proof of the asymmetric case highly relies on Lemma 3.3 which is inspired from the idea of Alonso-Gutiérrez and Brazitikos [1].

The rest of this paper is organized as follows: In Section 2 some of the basic notations and preliminaries are listed. Section 3 provides some lemmas. The proof of Theorem 1.3 is presented in Sections 4 and 5.

## 2. Notations and Preliminaries

We list some basic facts about convex bodies. As general references we recommend the books of Gardner [23] and Schneider [51].
We shall use $\| \cdot \|$ to denote the standard Euclidean norm on $\mathbb{R}^n$. Write $\{e_1, \ldots, e_n\}$ for the standard orthonormal basis of $\mathbb{R}^n$. A convex body $K$ in $\mathbb{R}^n$ is a compact convex set with non-empty interior whose support function, $h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined for $x \in \mathbb{R}^n$ by $$h_K(x) = \max \{ \langle x, y \rangle : y \in K \},$$ where $\langle x, y \rangle$ denotes the standard inner product of $x$ and $y$ in $\mathbb{R}^n$. If convex body $K$ contains the origin $o$ in its interior, then its polar body $K^o$ is defined by $$K^o = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K \}.$$ The Minkowski functional $\| \cdot \|_K$ of $K$ is defined by $$\|x\|_K = \min \{ t > 0 : x \in tK \}.$$ (2.1) In this case, $$\|x\|_K = h_K(o)(x).$$ (2.2) Obviously, $h_K$ is homogeneous of degree 1 if $K$ contains the origin in its interior. Integrating it with respect to the Gaussian probability measure $\gamma_n$ with the density $\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\|x\|^2/2}$, by polar coordinates, we have $$\int_{\mathbb{R}^n} h_K(x)d\gamma_n(x) = \int_0^\infty r^n \frac{e^{-\frac{r^2}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{S^{n-1}} h_K(u)dudu$$ $$= \frac{2c_n}{n\omega_n} \int_{S^{n-1}} h_K(u)du = c_n W(K),$$ (2.3) where $c_n = \frac{\Gamma(n+1)}{\sqrt{2\pi}n^{\frac{n}{2}}}$. We note that the mean width of $K^o$ is equivalent to an important quantity in the local theory of Banach spaces—the $\ell$-norm of $K$ (up to a factor); i.e., by (2.4), $$\ell(K) := \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x) = \int_{\mathbb{R}^n} h_K^o(x)d\gamma_n(x) = c_n W(K^o).$$ Moreover, $$W(K) = \frac{1}{c_n} \int_{\mathbb{R}^n} \|x\|_{K^o} d\gamma_n(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \left( \int_0^{\|x\|_{K^o}} dt \right) d\gamma_n(x)$$ $$= \frac{1}{c_n} \int_0^\infty \int_{\mathbb{R}^n} 1_{\{\|x\|_{K^o} > t\}}(x)d\gamma_n(x)dt = \frac{1}{c_n} \int_0^\infty [1 - \gamma_n(tK^o)]dt,$$ (2.5) where $1_A(\cdot)$ is the characteristic function of a set $A \subset \mathbb{R}^n$.

If $\mu$ is an isotropic measure on $S^{n-1}$ whose centroid is at the origin, then taking the trace in (1.3) yields $$\mu(S^{n-1}) = n.$$ (2.6) Moreover, for any $H \in G_{n,k}$, $$I_H = \int_{S^{n-1} \setminus H^\perp} P_H u \otimes P_H ud\mu(u) = \int_{S^{n-1} \setminus H^\perp} \frac{P_H u}{\|P_H u\|} \otimes \frac{P_H u}{\|P_H u\|} \|P_H u\|^2 d\mu(u),$$ (2.7) where $I_H$ denotes the identity in $H$, and $$\int_{S^{n-1} \setminus H^\perp} P_H ud\mu(u) = P_H \left( \int_{S^{n-1}} ud\mu(u) \right) = o.$$ (2.8)
Taking the trace in (2.7) yields

$$\int_{S^{n-1} \setminus H} \|P_H u\|^2 d\mu(u) = k. \quad (2.9)$$

The following Ball-Barthe inequality was established by Lutwak, Yang, and Zhang [41], extending the discrete case due to Ball and Barthe [10].

**Lemma 2.1.** Let $1 \leq k \leq n$ and $H \in G_{n,k}$. If $\nu$ is an isotropic measure on $S^{n-1} \cap H$ and $f$ is a positive continuous function on $\text{supp} \, \nu$, then

$$\det \int_{S^{n-1} \cap H} f(w)w \otimes w d\nu(w) \geq \exp \left\{ \int_{S^{n-1} \cap H} \log f(w) d\nu(w) \right\}, \quad (2.10)$$

with equality if and only if $f(w_1) \cdots f(w_k)$ is constant for linearly independent unit vectors $w_1, \ldots, w_k \in \text{supp} \, \nu$.

If $\nu$ is an isotropic measure on $S^{n-1} \cap H$, then let $L_\nu$ denote the space equipped with the standard $L_p$ norm of the function $f : S^{n-1} \cap H \rightarrow \mathbb{R}$, with respect to $\nu$; i.e., for $1 \leq p < \infty$

$$\|f\|_{\nu,p} = \left( \int_{S^{n-1} \cap H} |f(w)|^p d\nu(w) \right)^{\frac{1}{p}}.$$

Obviously, for $f \in L_1(\nu)$, $\int_{S^{n-1} \cap H} w f(w) d\nu(w) \in \mathbb{R}^k$. We shall need the following lemma due to Lutwak, Yang, and Zhang [43].

**Lemma 2.2.** If $\nu$ is an isotropic measure on $S^{n-1} \cap H$ and $f \in L_2(\nu)$, then

$$\left\| \int_{S^{n-1} \cap H} w f(w) d\nu(w) \right\| \leq \left( \int_{S^{n-1} \cap H} f(w)^2 d\nu(w) \right)^{1/2}.$$

If $\mu$ is an isotropic measure on $S^{n-1}$ whose centroid is at the origin, then we can define the convex body $C$ in $\mathbb{R}^n$ as the convex hull of the support of $\mu$; i.e.,

$$C = \text{conv}\{\text{supp} \, \mu\}.$$

Since $\text{supp} \, \mu$ is not contained in a closed hemisphere of $S^{n-1}$, the convex body $C$ must contain the origin in its interior. Then the polar body $C^\circ$ of $C$ is given by

$$C^\circ = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq 1 \text{ for all } u \in \text{supp} \, \mu \}.$$

Obviously, $C \subseteq B^n_2$ and $B^n_2 \subseteq C^\circ$. For $H \in G_{n,k}$, we have

$$C^\circ \cap H = \left\{ y \in H : \langle y, u \rangle \leq 1 \text{ for all } u \in \text{supp} \, \mu \right\} = \left\{ y \in H : \langle y, P_H u \rangle \leq 1 \text{ for all } u \in \text{supp} \, \mu \setminus H^\perp \right\}, \quad (2.11)$$

and

$$P_H C = P_H (\text{conv} \{ \text{supp} \, \mu \}) = \text{conv} \{ P_H u : u \in \text{supp} \, \mu \} = \text{conv} \{ P_H u : u \in \text{supp} \, \mu \setminus H^\perp \}, \quad (2.12)$$

where $H^\perp$ is the orthogonal complement of $H$. The following identity is easy to check (or see [23, 0.38]):

$$C^\circ \cap H = (P_H C)^\circ, \quad (2.13)$$

where the polar operation on the right is taken in $H$. 
If $K$ is a convex body in $\mathbb{R}^n$ such that $K \supset C$, then we have $P_H K \supset P_H C$ and $K^\circ \cap H \subseteq C^\circ \cap H$. Thus, to prove Theorem 1.3 it suffices to prove the following theorem.

**Theorem 2.3.** Let $H \in G_{n,k}$ and let $\mu$ be an isotropic measure on $S^{n-1}$ whose centroid is at the origin. Then $W(P_H C) \geq \sqrt{\frac{n}{n-1}} W(\triangle_k)$ with equality if and only if $P_H C = \sqrt{\frac{n}{n-1}} \triangle_k$.

If $\mu$ is even isotropic, then $W(P_H C) \geq \sqrt{\frac{n}{n-1}} W(B_k^1)$ with equality if and only if $P_H C = \sqrt{\frac{n}{n-1}} B_k^1$, and $W(C^\circ \cap H) \leq \sqrt{\frac{n}{n-1}} W(B_k^\infty)$ with equality if and only if $C^\circ \cap H = \sqrt{\frac{n}{n-1}} B_k^\infty$.

3. Some Lemmas

Define a function $v : S^{n-1} \to S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by

$$v(u) = \left( -\frac{\sqrt{n}}{\sqrt{n+1}} u, \frac{1}{\sqrt{n+1}} \right), \quad u \in S^{n-1}. \quad (3.1)$$

For $H \in G_{n,k}$, define the $(k+1)$-dimensional subspace $H' = H \times \mathbb{R} \subset \mathbb{R}^{n+1}$ by

$$H' = \text{span}\{H, e_{n+1}\}. \quad (3.2)$$

If $\mu$ is an isotropic measure on $S^{n-1}$ whose centroid is at the origin, then for each $u \in \text{supp} \, \mu$, define $w(u) \in S^n \cap H'$ by

$$w(u) = \frac{P_{H'} v(u)}{\|P_{H'} v(u)\|} = \frac{1}{\|P_{H'} v(u)\|} \left( -\frac{\sqrt{n}}{\sqrt{n+1}} P_H u, \frac{1}{\sqrt{n+1}} \right), \quad (3.3)$$

where $v(u)$ is defined in (3.1). Let $\nu$ be the Borel measure on $S^n \cap H'$ defined by

$$\int_{S^n \cap H'} g(w) d\nu(w) = \frac{n+1}{n} \int_{S^{n-1}} g\left( \frac{P_{H'} v(u)}{\|P_{H'} v(u)\|} \right) \|P_{H'} v(u)\|^2 d\mu(u), \quad (3.4)$$

for any continuous function $g : S^n \cap H' \to \mathbb{R}$. The following lemma shows that the measure $\nu$ induced by (3.4) is also isotropic on $S^n \cap H'$. In other words, the continuous map $P_{H'} v(\cdot) : S^{n-1} \to H'$ is an isotropic embedding of $\mu$. The concept of isotropic embedding was originated from Ball [6] and introduced by Lutwak, Yang, and Zhang [44] (see also [52]).

**Lemma 3.1.** If $\mu$ is an isotropic measure on $S^{n-1}$ whose centroid is at the origin, then the measure $\nu$ induced by (3.4) is isotropic on $S^n \cap H'$. Moreover,

$$\int_{S^n \cap H'} w(w, e_{n+1}) d\nu(w) = e_{n+1}. \quad (3.5)$$

**Proof.** Let $z = (x, r) \in H' = H \times \mathbb{R} \subset \mathbb{R}^{n+1}$. From (3.4), (3.1), (2.8), and (2.6), we get

$$\int_{S^n \cap H'} \langle z, w \rangle^2 d\nu(w) = \frac{n+1}{n} \int_{S^{n-1}} \left( \langle x, r \rangle, \left( -\frac{\sqrt{n}}{\sqrt{n+1}} P_H u, \frac{1}{\sqrt{n+1}} \right) \right)^2 \|P_{H'} v(u)\|^2 d\mu(u)$$

$$= \frac{n+1}{n} \int_{S^{n-1}} \langle x, r \rangle^2 \|P_{H'} v(u)\|^2 d\mu(u)$$

$$= \frac{n+1}{n} \int_{S^{n-1}} \langle x, r \rangle^2 \|P_{H'} v(u)\|^2 d\mu(u)$$

$$= \int_{S^{n-1}} \langle x, P_H u \rangle^2 d\mu(u) - \frac{2r}{\sqrt{n}} \int_{S^{n-1}} \langle x, P_H u \rangle d\mu(u) + \frac{r^2}{n} \int_{S^{n-1}} d\mu(u)$$

$$= |x|^2 + r^2$$
which implies that the measure $\nu$ on $S^n \cap H'$ is isotropic.

Moreover, by (3.3), (3.1), and (2.8),
\[
\langle z, \int_{S^n \cap H'} w \langle w, e_{n+1} \rangle d\nu(w) \rangle = \sqrt{n+1} \frac{1}{n} \int_{S^{n-1}} \langle x, r \rangle P_H u d\mu(u) = \sqrt{n+1} \frac{1}{n} \nu(S^n \cap H') = \frac{n(k+1)}{n+1}.
\]

as desired. \hfill \square

Since $\nu$ is an isotropic measure on $S^n \cap H'$, it follows that
\[
\int_{S^n \cap H'} w \otimes w d\nu(w) = I_{H'}.
\]

Taking the trace in the above equation yields
\[
\nu(S^n \cap H') = k + 1,
\]

or, by (3.4),
\[
\int_{S^{n-1}} \|P_{H'} v(u)\|^2 d\mu(u) = \frac{n}{n+1} \int_{S^n \cap H'} d\nu(w) = \frac{n}{n+1} \nu(S^n \cap H') = \frac{n(k+1)}{n+1}.
\]

To prove Lemma 3.3, the following Prékopa-Leindler inequality will be needed. We offer a simple proof based on Gardner [23, Theorem 7.1].

**Lemma 3.2.** Let $\sigma$ be a Borel probability measure on $S^{n-1}$, and let $(f_u(\cdot))_{u \in \text{supp} \sigma}$ be a family of nonnegative integrable functions on $\mathbb{R}$ such that, for any functions $(\tau_u(\cdot))_{u \in \text{supp} \sigma} : \mathbb{R} \to \mathbb{R}$ being integrable with respect to $\sigma$ and for a nonnegative integrable function $h$ on $\mathbb{R}$,
\[
h\left(\int_{S^{n-1}} \tau_u(r) d\sigma(u)\right) \geq \exp \left(\int_{S^{n-1}} \log f_u(\tau_u(r)) d\sigma(u)\right).
\]

Then,
\[
\int_{\mathbb{R}} h(r) dr \geq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u(r) dr\right) d\sigma(u)\right).
\]

**Proof.** It is sufficient to prove the assertion for continuous, positive, integrable functions; the general case then follows by standard measure-theoretic arguments. For any fixed $u \in \text{supp} \sigma$, let
\[
\int_{\mathbb{R}} f_u(r) dr = F_u > 0.
\]

For $t \in (0,1)$, define $\phi_u : (0,1) \to \mathbb{R}$ by
\[
\frac{1}{F_u} \int_{-\infty}^{\phi_u(t)} f_u(r) dr = t.
\]
The functions $\phi_u$ are increasing and differentiable, so the differential of $\phi_u$ gives
\[
\frac{f_u(\phi_u(t))\phi_u'(t)}{F_u} = 1.
\]
The function $\phi_u'$ are continuous and therefore bounded on the sphere. Let
\[
T(t) = \int_{S^{n-1}} \phi_u(t)d\sigma(u).
\]
By dominated convergence and the arithmetic-geometric means inequality, we have
\[
T'(t) = \int_{S^{n-1}} \phi_u'(t)d\sigma(u) \geq \exp\left(\int_{S^{n-1}} \log \phi_u'(t)d\sigma(u)\right)
\]
\[
= \exp\left(\int_{S^{n-1}} \log \left(\frac{F_u}{f_u(\phi_u(t))}\right)d\sigma(u)\right).
\]
Therefore, applying (3.8) with $\tau_u = \phi_u$ yields
\[
\int_{\mathbb{R}} h(r)dr \geq \int_{0}^{1} h(T(t))T'(t)dt
\]
\[
\geq \int_{0}^{1} h\left(\int_{S^{n-1}} \phi_u(t)d\sigma(u)\right) \exp\left(\int_{S^{n-1}} \log \left(\frac{F_u}{f_u(\phi_u(t))}\right)d\sigma(u)\right)dt
\]
\[
\geq \int_{0}^{1} \exp\left(\int_{S^{n-1}} \log f_u(\phi_u(t))d\sigma(u)\right) \exp\left(\int_{S^{n-1}} \log \left(\frac{F_u}{f_u(\phi_u(t))}\right)d\sigma(u)\right)dt
\]
\[
= \int_{0}^{1} \exp\left(\int_{S^{n-1}} \log F_u d\sigma(u)\right)dt
\]
\[
= \exp\left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u d\sigma(u)\right)\right),
\]
as desired. \qed

To prove the asymmetric case, the following crucial lemma, inspired by the idea of Alonso-Gutiérrez and Brazitikos [1], will be needed.

**Lemma 3.3.** Let $\lambda$ be an arbitrary real number, $H \in G_{n,k}$, and let $H'$ be defined in (3.2). If
\[
\int_{0}^{\infty} e^{-\frac{r^2}{2} + (\lambda - \sqrt{n+1})r} \gamma_k\left(\frac{r}{\sqrt{n}}(C^o \cap H)\right)dr
\]
\[
\leq (2\pi)^{-\frac{k}{2}} \exp\left(\int_{S^n \cap H'} \log \left(\int_{0}^{\infty} e^{-\frac{r^2}{2} + (\lambda - \sqrt{n+1})r} dr\right) d\nu(w)\right),
\]
then we have,
\[
W(P_{HC}) \geq \sqrt{\frac{k}{n}} W(\triangle_k),
\]
where $\triangle_k$ is the $k$-dimensional regular simplex inscribed into $B^k_2$ in $H$.

**Proof.** Observe that inequality (3.9) is equivalent to
\[
\int_{0}^{\infty} e^{-\frac{(r-(\lambda - \sqrt{n+1})r)^2}{2}} \gamma_k\left(\frac{r}{\sqrt{n}}(C^o \cap H)\right)dr
\]
\[
\leq (2\pi)^{-\frac{k}{2}} \exp\left(\int_{S^n \cap H'} \log \left(\int_{0}^{\infty} e^{-\frac{(r-(\lambda - \sqrt{n+1})r)^2}{2}} dr\right) d\nu(w)\right).
\]

(3.10)
In fact, it is easy to see that
\[ \int_0^\infty e^{-\frac{r^2}{2} + (\lambda - \sqrt{n+1})r} \gamma_k \left( \frac{r}{\sqrt{n}} (C^0 \cap H) \right) dr = e^{(\lambda - \sqrt{n+1})^2} \int_0^\infty e^{-\frac{(r-(\lambda - \sqrt{n+1})e_{n+1})^2}{2}} \gamma_k \left( \frac{r}{\sqrt{n}} (C^0 \cap H) \right) dr. \]

On the other hand, by (3.4), (3.3), and (2.6),
\[
\begin{align*}
\exp \left( \int_{S^n \cap H'} \log \left( \int_0^\infty e^{-\frac{r^2}{2} + (\lambda - \sqrt{n+1})e_{n+1})r} \, d\nu(w) \right) \right) &= \exp \left( \int_{S^n \cap H'} \log \left( \int_0^\infty e^{-\frac{(r-(\lambda - \sqrt{n+1})e_{n+1})^2}{2}} \, d\nu(w) \right) \right) \\
&= \exp \left( \int_{S^n \cap H'} \frac{1}{2} \left( \lambda - \sqrt{n+1} \right)^2 \, d\nu(w) \right)
\end{align*}
\]
\[= e^{(\lambda - \sqrt{n+1})^2} \exp \left( \int_{S^n \cap H'} \log \left( \int_0^\infty e^{-\frac{(r-(\lambda - \sqrt{n+1})e_{n+1})^2}{2}} \, d\nu(w) \right) \right).\]

Canceling the factor \(e^{(\lambda - \sqrt{n+1})^2}\) yields inequality (3.10).

Note that by (3.7)
\[\frac{1}{k+1} \int_{S^n \cap H'} \, d\nu(w) = 1.\]

By Jensen’s inequality, for any integrable functions \(\tau_w\) with respect to \(\nu\), we obviously have,
\[\frac{1}{k+1} \int_{S^n \cap H'} \frac{\langle \tau_w(r) - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \rangle^2}{2} \, d\nu(w) \ge \frac{\left( \frac{1}{k+1} \int_{S^n \cap H'} \tau_w(r) - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \rangle \, d\nu(w) \right)^2}{2},\]
and thus,
\[\exp \left( - \frac{1}{k+1} \int_{S^n \cap H'} \frac{\langle \tau_w(r) - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \rangle^2}{2} \, d\nu(w) \right) \le \exp \left( - \frac{\left( \frac{1}{k+1} \int_{S^n \cap H'} \tau_w(r) - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \rangle \, d\nu(w) \right)^2}{2} \right). \quad (3.11)\]

For \(r \in \mathbb{R}^+\), let
\[h(r) = \left( - \frac{\left( \frac{1}{k+1} \int_{S^n \cap H'} r - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \rangle \, d\nu(w) \right)^2}{2} \right),\]
and 

\[ f_w(r) = \exp \left( -\frac{(r - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle)^2}{2} \right). \]

Then by (3.11) and Lemma 3.2 we obtain

\[
\exp \left( \frac{1}{k+1} \int_{S^n \cap H'} \log \left( \int_0^\infty \exp \left( -\frac{(r - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle)^2}{2} \right) dr \right) d\nu(w) \right) \\
\leq \int_0^\infty \exp \left( -\left( \frac{1}{k+1} \int_{S^n \cap H'} r - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle d\nu(w) \right)^2 \right) dr. \tag{3.12}
\]

Let 

\[ \alpha_\lambda = \left( \frac{\lambda}{\sqrt{n+1}} - 1 \right), \quad \beta = \frac{1}{\sqrt{k+1}} \int_{S^n \cap H'} \langle w, \sqrt{n+1}e_{n+1} \rangle d\nu(w). \]

Then by (3.4), the Cauchy-Schwartz inequality, (2.6), and (3.7), we get

\[
\beta = \frac{n+1}{n\sqrt{k+1}} \int_{S^{n-1}} \| P_{H'} v(u) \| d\mu(u) \\
\leq \frac{n+1}{n\sqrt{k+1}} \left( \int_{S^{n-1}} d\mu(u) \right)^{\frac{1}{2}} \left( \int_{S^{n-1}} \| P_{H'} v(u) \|^2 d\mu(u) \right)^{\frac{1}{2}} \\
= \frac{\sqrt{n+1}}{\sqrt{n+1}}. \tag{3.13}
\]

Therefore, by (3.12), we arrive at

\[
\exp \left( \int_{S^n \cap H'} \log \left( \int_0^\infty e^{-\frac{(r - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle)^2}{2}} dr \right) d\nu(w) \right) \\
\leq \left( \int_0^\infty \exp \left( -\frac{r - \frac{\alpha_\lambda \beta}{\sqrt{k+1}}}{2} \right) dr \right)^{k+1} \\
= \left( \int_0^\infty \exp \left( -\frac{r - \frac{\alpha_\lambda \beta}{\sqrt{k+1}}}{2} \right) dr \right)^{k+1} \\
= \prod_{i=1}^{k+1} \int_{[0, \infty)} \exp \left( -\frac{x_i - \frac{\alpha_\lambda \beta}{\sqrt{k+1}}}{2} \right) dx_i \cdots dx_{k+1} \\
= \int_{[-\frac{\alpha_\lambda \beta}{\sqrt{k+1}}, \infty)^{k+1}} e^{-\frac{|x|^2}{2}} dx,
\]

where \( x = (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \). Notice that, for the direction \( v_0 = \left( \frac{1}{\sqrt{k+1}}, \ldots, \frac{1}{\sqrt{k+1}} \right) \),

\[
\left\{ x \in \left[ -\frac{\alpha_\lambda \beta}{\sqrt{k+1}}, \infty \right)^{k+1} : \langle x, v_0 \rangle = r \right\} \\
= \left\{ x \in \left[ -\alpha_\lambda \beta, \infty \right)^{k+1} : \left( \frac{x}{\sqrt{k+1}}, v_0 \right) = r \right\} \\
= \left\{ x \in [0, \infty)^{k+1} : \left( \frac{x}{\sqrt{k+1}}, v_0 \right) = r + \alpha_\lambda \beta \right\} - \alpha_\lambda \beta (1, \ldots, 1)
\]
the change of variables
\[ \int_{-\alpha\lambda\beta \sqrt{k+1}} \mathcal{L} \, dx \]
By the rotation invariance of Gaussian measures, we may integrate in the direction \(v_0\), make the change of variables \(\langle x, v_0 \rangle = r\), and integrate in the hyperplane \(\{x \in \mathbb{R}^{k+1} : \langle x, v_0 \rangle = r\}\) as follows.
\[
\int_{-\alpha\lambda\beta \sqrt{k+1}} e^{-\frac{r^2}{2}} \, dx
= \int_{-\alpha\lambda\beta \sqrt{k+1}} e^{-\frac{\|P_{v_0} x\|^2 - \|P_{v_0} x\|^2}{2}} \, dx
= (2\pi)^{\frac{k}{2}} \int_{-\alpha\lambda\beta \sqrt{k+1}} \gamma_k \left( \frac{x}{\sqrt{k+1}} : \langle x, v_0 \rangle = r \right) e^{-\frac{r^2}{2}} \, dr
= (2\pi)^{\frac{k}{2}} \int_{-\alpha\lambda\beta \sqrt{k+1}} \gamma_k \left( (r + \alpha\lambda\beta \sqrt{k+1}) \mathcal{L} \right) \, dr
= (2\pi)^{\frac{k}{2}} \int_{0}^{\infty} e^{-\frac{(r - \alpha\lambda\beta \sqrt{k+1} \mathcal{L})^2}{2}} \gamma_k \left( \frac{r}{\sqrt{k+1}} \mathcal{L} \right) \, dr,
\]
where we use the fact that \(\gamma_k(\sqrt{k+1}\mathcal{L} \{e_1, \ldots, e_{k+1}\}) = \gamma_k(\mathcal{L})\).

Thus, we have proved that
\[
\int_{0}^{\infty} e^{-\frac{(r - \alpha\lambda\beta \sqrt{k+1} \mathcal{L})^2}{2}} \gamma_k \left( \frac{r}{\sqrt{n}(\mathcal{L})} \right) \, dr \leq \int_{0}^{\infty} e^{-\frac{(r - \alpha\lambda\beta \sqrt{k+1} \mathcal{L})^2}{2}} \gamma_k \left( \frac{r}{\sqrt{k+1} \mathcal{L}} \right) \, dr.
\]
Applying the above inequality to \(-\alpha\lambda\beta\) also gives
\[
\int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \gamma_k \left( \frac{r}{\sqrt{n}(\mathcal{L})} \right) \, dr \leq \int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \gamma_k \left( \frac{r}{\sqrt{k+1} \mathcal{L}} \right) \, dr,
\]
or, equivalently,
\[
\int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \gamma_k \left( \frac{|r|}{\sqrt{n}(\mathcal{L})} \right) \, dr \leq \int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \gamma_k \left( \frac{|r|}{\sqrt{k+1} \mathcal{L}} \right) \, dr.
\]
Therefore, for any \(\alpha\lambda \in \mathbb{R}\),
\[
\int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \gamma_k \left( \frac{|r|}{\sqrt{n}(\mathcal{L})} \right) \, dr \leq \int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \gamma_k \left( \frac{|r|}{\sqrt{k+1} \mathcal{L}} \right) \, dr,
\]
which is equivalent to for all \(\alpha\lambda \in \mathbb{R}\),
\[
\int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \left( 1 - \gamma_k \left( \frac{|r|}{\sqrt{n}(\mathcal{L})} \right) \right) \, dr \geq \int_{-\infty}^{0} e^{-\frac{(r + \alpha\lambda\beta \sqrt{k+1})^2}{2}} \left( 1 - \gamma_k \left( \frac{|r|}{\sqrt{k+1} \mathcal{L}} \right) \right) \, dr.
\]
Integrating with respect to \(\alpha\lambda\), we get
\[
\frac{1}{\sqrt{n} + 1} \int_{-\infty}^{0} \left( 1 - \gamma_k \left( \frac{|r|}{\sqrt{n}(\mathcal{L})} \right) \right) \, dr \geq \frac{1}{\beta} \int_{-\infty}^{0} \left( 1 - \gamma_k \left( \frac{|r|}{\sqrt{k+1} \mathcal{L}} \right) \right) \, dr.
\]
Equivalently,
\[
\frac{1}{\sqrt{n+1}} \int_0^\infty \left(1 - \gamma_k \left( r \left( \frac{C^\circ \cap H}{r} \right) \right) \right) dr \geq \frac{1}{\beta} \int_0^\infty \left(1 - \gamma_k \left( \frac{r \Delta_k^0}{\sqrt{k}} \right) \right) dr,
\]
or
\[
\sqrt{\frac{n}{n+1}} \int_0^\infty \left(1 - \gamma_k \left( r (C^\circ \cap H) \right) \right) dr \geq \sqrt{\frac{k}{\beta}} \int_0^\infty \left(1 - \gamma_k \left( r \Delta_k^0 \right) \right) dr,
\]
Thus, by (2.13), (2.5), and (3.13), we obtain
\[
W(P_H C) = W((C^\circ \cap H) \circ) \geq \frac{1}{\beta} \sqrt{\frac{(n+1)k}{n}} W(\Delta_k) \geq \sqrt{k} W(\Delta_k).
\]

\[\square\]

4. The asymmetric case

**Theorem 4.1.** Let \( H \in G_{n,k} \) and let \( \Delta_k \) be a regular \( k \)-simplex inscribed in \( B_k^2 \). If \( \mu \) is an isotropic measure on \( S^{n-1} \) whose centroid is at the origin, then
\[
W(P_H C) \geq \sqrt{\frac{k}{n}} W(\Delta_k),
\]
where \( C = \text{conv}\{\text{supp } \mu\} \). There is equality if and only if \( P_H C = \sqrt{\frac{k}{n}} \Delta_k \).

**Proof.** Since \( \mu \) is an isotropic measure on \( S^{n-1} \) whose centroid is at the origin, it follows from Lemma 3.1 that the measure \( \nu \) induced by (3.4) is isotropic on \( S^n \cap H' \), where \( H' = \text{span}\{H, e_{n+1}\} \).

Let \( \lambda \) be an arbitrary real number. For \( w \in \text{supp } \nu \), let
\[
G_{\lambda,w} = \int_0^\infty e^{-\frac{s^2}{2} + \langle w, (\lambda - \sqrt{n+1}) e_{n+1} \rangle s} ds.
\]
Define the smooth strictly increasing function \( \phi_{\lambda,w}: (0, \infty) \to \mathbb{R} \) by
\[
\frac{1}{G_{\lambda,w}} \int_0^t e^{-\frac{s^2}{2} + \langle w, (\lambda - \sqrt{n+1}) e_{n+1} \rangle s} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\phi_{\lambda,w}(t)} e^{-s^2} ds.
\]
Then \( \phi'_{\lambda,w} > 0 \) and differentiating both sides of the above identity yields, for all \( t > 0 \),
\[
-\frac{t^2}{2} + \langle w, (\lambda - \sqrt{n+1}) e_{n+1} \rangle t = \log G_{\lambda,w} - \log \sqrt{\pi} - \phi_{\lambda,w}(t)^2 + \log \phi'_{\lambda,w}(t).
\]

Define an open cone \( \mathcal{D} \subset H' = H \times \mathbb{R} \) by
\[
\mathcal{D} = \bigcup_{r > 0} \left( \frac{r}{\sqrt{n}} \text{relint}(C^\circ \cap H) \right) \times \{r\}.
\]
It is easy to see that \( z = (x, r) \in \mathcal{D} \) if and only if \( \frac{\sqrt{n}}{r} x \in \text{relint}(C^\circ \cap H) \). By the definition (2.11) of \( C^\circ \cap H \), it is equivalent to
\[
\langle x, P_H u \rangle < \frac{r}{\sqrt{n}} \quad \text{for all } u \in \text{supp } \mu.
\]
From (3.3) we have
\[
\langle z, w(u) \rangle = \left( (x,r), \frac{1}{\|P_H v(u)\|} \left( -\frac{\sqrt{n}}{\sqrt{n+1}} P_H u, \frac{1}{\sqrt{n+1}} \right) \right).
\]
\[\frac{1}{\|P_Hv(u)\|} \left( - \frac{\sqrt{n}}{\sqrt{n+1}} (x, P_Hu) + \frac{r}{\sqrt{n+1}} \right) > 0,\]

for \(u \in \text{supp} \mu\). Then we obtain that

\[z \in \mathcal{D} \iff \langle z, w(u) \rangle > 0 \quad \text{for all} \quad u \in \text{supp} \mu. \quad (4.3)\]

For the isotropic measure \(\nu\) on \(S^n \cap H'\) induced by (3.4), define a transformation \(T_1 : \mathcal{D} \to H'\) by

\[T_1 z = \int_{S^n \cap H'} w\phi_{\lambda,w}(\langle z, w \rangle) d\nu(w), \quad (4.4)\]

for each \(z \in \mathcal{D}\). The differential of \(T_1\) is given by

\[dT_1(z) = \int_{S^n \cap H'} \phi'_{\lambda,w}(\langle z, w \rangle) w \otimes w d\nu(w), \quad (4.5)\]

for each \(z \in \mathcal{D}\). Thus for each \(y \in H'\),

\[\langle y, dT_1(z)y \rangle = \int_{S^n \cap H'} \phi'_{\lambda,w}(\langle z, w \rangle) \langle w, y \rangle^2 d\nu(w).\]

Since \(\phi'_{\lambda,w} > 0\) and \(\nu\) is not concentrated on the subspace \(H'\), it follows that the matrix \(dT_1(z)\) is positive definite for each \(z \in \mathcal{D}\). Hence, the mean value theorem shows that \(T_1 : \mathcal{D} \to H'\) is injective.

In light of (4.3), let \(t = \langle z, w \rangle\) for \(z \in \mathcal{D}\) and \(w \in \text{supp} \nu\). Integrating (4.2) with respect to \(\nu\) and by the fact that \(\nu(S^n \cap H') = k + 1\), we get

\[\exp \int_{S^n \cap H'} \left( - \frac{\langle z, w \rangle^2}{2} + \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \langle z, w \rangle \right) d\nu(w) = \exp \int_{S^n \cap H'} \left( \log \frac{G_{\lambda,w}}{\sqrt{\pi}} - \phi_{\lambda,w}(\langle z, w \rangle)^2 + \log \phi'_{\lambda,w}(\langle z, w \rangle) \right) d\nu(w) = \left( \frac{1}{\sqrt{\pi}} \right)^{k+1} A_\lambda \exp \int_{S^n \cap H'} \left( - \phi_{\lambda,w}(\langle z, w \rangle)^2 + \log \phi'_{\lambda,w}(\langle z, w \rangle) \right) d\nu(w), \quad (4.6)\]

where

\[A_\lambda = \exp \int_{S^n \cap H'} \log G_{\lambda,w} d\nu(w).\]

Integrating (4.6) on \(\mathcal{D}\), from (3.6), the Ball-Barthe inequality (2.10), (4.5), Lemma 2.2 and making the change of variables \(y = T_1 z\), we have

\[\int_{\mathcal{D}} \exp \left( - \int_{S^n \cap H'} \frac{\langle z, w \rangle^2}{2} - \langle w, (\lambda - \sqrt{n+1})e_{n+1} \rangle \langle z, w \rangle \right) d\nu(w) dz \]

\[= \left( \frac{1}{\sqrt{\pi}} \right)^{k+1} A_\lambda \int_{\mathcal{D}} \exp \left( - \int_{S^n \cap H'} \phi_{\lambda,w}(\langle z, w \rangle)^2 - \log \phi'_{\lambda,w}(\langle z, w \rangle) d\nu(w) \right) dz \]

\[= \left( \frac{1}{\sqrt{\pi}} \right)^{k+1} A_\lambda \int_{\mathcal{D}} \exp \left( - \int_{S^n \cap H'} \phi_{\lambda,w}(\langle z, w \rangle)^2 d\nu(w) \right) \exp \left( \int_{S^n \cap H'} \log \phi'_{\lambda,w}(\langle z, w \rangle) d\nu(w) \right) dz \]

\[\leq \left( \frac{1}{\sqrt{\pi}} \right)^{k+1} A_\lambda \int_{\mathcal{D}} \exp \left( - \int_{S^n \cap H'} \phi_{\lambda,w}(\langle z, w \rangle)^2 d\nu(w) \right) |dT_1(z)| dz\]
\[
\leq \left( \frac{1}{\sqrt{n}} \right)^{k+1} A_\lambda \int_D e^{-|T_1 z|^2} |dT_1(z)| dz \\
\leq \left( \frac{1}{\sqrt{n}} \right)^{k+1} A_\lambda \int_{\mathbb{R}^{k+1}} e^{-|y|^2} dy \\
= A_\lambda.
\]

On the other hand, since \( \nu \) is isotropic on \( S^n \cap H' \), from (3.5) and the definition of \( D \), we obtain, for \( z = (x, r) \in D \),
\[
\int_D \exp \left( - \int_{S^n \cap H'} \left( \frac{\langle z, w \rangle^2}{2} - \langle w, (\lambda - \sqrt{n} + 1)e_{n+1} \rangle \langle z, w \rangle \right) d\nu(w) \right) dz \\
= \int_D \exp \left( - \int_{S^n \cap H'} \left( \frac{\langle z, w \rangle^2}{2} - (\lambda - \sqrt{n} + 1) \langle z, w, e_{n+1} \rangle w \right) d\nu(w) \right) dz \\
= \int_D \exp \left( - \int_{S^n \cap H'} \frac{\langle z, w \rangle^2}{2} d\nu(w) + (\lambda - \sqrt{n} + 1) \langle z, \int_{S^n \cap H'} w \langle w, e_{n+1} \rangle d\nu(w) \rangle \right) dz \\
= \int_D \exp \left( - |z|^2 + (\lambda - \sqrt{n} + 1) \langle z, e_{n+1} \rangle \right) dz \\
= \int_0^\infty \int_{\text{relint}(C^n \cap H)} e^{-\frac{x^2}{2} + (\lambda - \sqrt{n} + 1)r} e^{-\frac{r^2}{2}} dxdr \\
= (2\pi)^{\frac{k}{2}} \int_0^\infty e^{-\frac{x^2}{2} + (\lambda - \sqrt{n} + 1)r} \gamma_k \left( \frac{r}{\sqrt{n}} (C^n \cap H) \right) dr.
\]

Therefore, we prove that
\[
\int_0^\infty e^{-\frac{x^2}{2} + (\lambda - \sqrt{n} + 1)r} \gamma_k \left( \frac{r}{\sqrt{n}} (C^n \cap H) \right) dr \leq (2\pi)^{-\frac{k}{2}} A_\lambda. \tag{4.7}
\]

which, by (4.11) and Lemma 3.3 implies that
\[
W(P_H C) \geq \sqrt{\frac{k}{n} W(\Delta_k)}. \tag{4.8}
\]

Next, we will show that the equality of the above inequality holds if and only if \( P_H C = \sqrt{\frac{k}{n} \Delta_k} \). Actually, suppose that there is equality in (4.8). Since \( \nu \) is isotropic on \( S^n \cap H' \), and hence not concentrated on a proper subspace of \( H' \), there exists a basis
\[
\{w_1, \ldots, w_{k+1} : w_i \in \text{supp } \nu, i = 1, \ldots, k + 1 \}
\]
of \( H' \). We will show that \( \text{supp } \nu = \{w_1, \ldots, w_{k+1} \} \). Assume that \( \{w_0, w_2, \ldots, w_{k+1} \} \) is another basis of \( H' \), where \( w_0 \in \text{supp } \nu \) and \( w_0 = a_1 w_1 + \cdots + a_{k+1} w_{k+1} \) such that at least one coefficient, say \( a_1 \), is not zero. Then the equality conditions of the Ball-Barthe inequality (2.10) imply that
\[
\phi'_{\lambda, w_1}(\langle z, w_1 \rangle) \cdots \phi'_{\lambda, w_{k+1}}(\langle z, w_{k+1} \rangle) = \phi'_{\lambda, w_0}(\langle z, w_0 \rangle) \phi'_{\lambda, w_2}(\langle z, w_2 \rangle) \cdots \phi'_{\lambda, w_{k+1}}(\langle z, w_{k+1} \rangle),
\]
for all \( z \in D \). Since \( \phi'_{\lambda, w} > 0 \), we have
\[
\phi'_{\lambda, w_1}(\langle z, w_1 \rangle) = \phi'_{\lambda, w_0}(\langle z, w_0 \rangle),
\]
Therefore, the assumption of equality implies that
\[ \sqrt{\phi''_\lambda (\langle z, w_1 \rangle)} w_1 = \phi''_\lambda (\langle z, w_0 \rangle) w_0, \]
for all \( z \in \mathcal{D} \), which implies that \( w_0 = w_1 \) since \( \phi''_\lambda (\langle z, w_1 \rangle) \neq 0 \) for some \( z \in \mathcal{D} \). By (3.3), we see that \( \nu \) is supported inside an open hemisphere on \( S^n \cap H' \), which gives \( w_0 = w_1 \).

Therefore, the assumption of equality implies that
\[ \text{supp } \nu = \{ w_1, \ldots, w_{k+1} \}. \]
Since \( \nu \) is isotropic, it follows that \( \{ w_1, \ldots, w_{k+1} \} \) is an orthogonal basis of \( H' \) (see, e.g., (3.3)). Thus,
\[ \langle w_i, w_j \rangle = 0, \quad 1 \leq i \neq j \leq k + 1. \]
That is, by (3.3),
\[ \langle P_H u_i, P_H u_j \rangle = -\frac{1}{n}, \]
whenever \( P_H u_i \neq P_H u_j \), where \( u_i, u_j \in \text{supp } \mu \setminus H^\perp \), which is equivalent to
\[ \langle P_H \sqrt{n/k} u_i, P_H \sqrt{n/k} u_j \rangle = -\frac{1}{k}, \quad (4.9) \]
whenever \( P_H u_i \neq P_H u_j \), where \( u_i, u_j \in \text{supp } \mu \setminus H^\perp \). Meanwhile, the Cauchy-Schwarz inequality and (3.7) yield that the equality of inequality (3.13) holds if and only if \( \|P_H v(u)\| = \sqrt{k+1}/n+1 \). Thus, by (3.3), we have \( \|P_H u\| = \sqrt{k/n} \) for each \( u \in \text{supp } \mu \setminus H^\perp \), which means that \( P_H \sqrt{\frac{k}{n}} u \) is an unit vector in \( H \) for each \( u \in \text{supp } \mu \setminus H^\perp \). Recall that
\[ \sqrt{\frac{n}{k}} P_H C = \sqrt{\frac{n}{k}} \text{conv} \{ P_H u : u \in \text{supp } \mu \setminus H^\perp \} \]
\[ = \text{conv} \{ P_H \sqrt{\frac{n}{k}} u : u \in \text{supp } \mu \setminus H^\perp \}. \]
Together with (4.9), we have \( \sqrt{k/n} P_H C \) is a regular simplex \( \Delta_k \) inscribed in \( S^{k-1} \subset H \), which gives that \( P_H C = \sqrt{k/n} \Delta_k \).

On the other hand, if \( P_H C = \sqrt{k/n} \Delta_k \), then by the definition of \( P_H C \),
\[ (P_H \sqrt{\frac{n}{k}} u)_{u \in \text{supp } \mu \setminus H^\perp} \]
are all unit vectors in \( H \) (4.10) with identity (4.9). By (3.3), we must have
\[ \langle w_i, w_j \rangle = 0, \quad 1 \leq i \neq j \leq k + 1. \]
Thus, \( \{ w_1, \ldots, w_{k+1} \} = \text{supp } \nu \) is an orthogonal basis of \( H' \). From the fact that \( \sum_{i=1}^{k+1} w_i \otimes w_i = I_{k+1} \) and \( \langle w_i, w_j \rangle = \delta_{ij} \), we see that equalities in the Ball-Barthe inequality and Lemma 2.2 hold. Since \( T_1 : \mathcal{D} \to \mathbb{R}^{k+1} \) is globally 1-1, it follows that equality in (4.7) holds. Meanwhile, (4.10) implies that \( \|P_H u\| = \sqrt{k/n} \) and thus \( \|P_H v(u)\| = \sqrt{k+1/n+1} \), a constant for each \( u \in \text{supp } \mu \setminus H^\perp \). Therefore, by the equality conditions of Jensen’s inequality, equality in (3.11), and hence in (3.12), holds. The desired result immediately follows.
Theorem 4.1 can be stated in terms of the $\ell$-norm of convex bodies by taking account of (2.4) and (2.13).

**Theorem 4.2.** Let $H \in G_{n,k}$ and let $\triangle_k$ be a regular $k$-simplex inscribed in $B_2^k$. If $\mu$ is an isotropic measure on $S^{n-1}$ whose centroid is at the origin, then

$$\ell(C^\circ \cap H) \geq \sqrt{\frac{n}{k}} \ell(\triangle_k^\circ),$$

where $C = \text{conv}\{\text{supp } \mu\}$. There is equality if and only if $C^\circ \cap H = \sqrt{\frac{n}{k}} \triangle_k^\circ$.

The case $k = n$ of Theorem 4.2 was proved in [38].

Let $K$ be a convex body in $\mathbb{R}^n$ in the Löwner position. Then by the John theorem, there exist $m \geq n$ contact points of $K$ and $S^{n-1}$ forming an isotropic measure $\mu$; i.e., $\text{supp } \mu = \{u_1, \ldots, u_m\}$. So we have

$$K \supseteq C = \text{conv}\{u_1, \ldots, u_m\},$$

which implies $P_H K \supseteq P_H C$. Therefore, by Theorem 4.1, we obtain that

$$W(P_H K) \geq W(P_H C) \geq \sqrt{\frac{k}{n}} W(\triangle_k),$$

with equality if and only if $P_H K = \sqrt{\frac{k}{n}} \triangle_k$. This inequality was proved by Alonso-Gutiérrez and Brazitikos [1], and by Schmuckenschläger [50] for $k = n$.

5. **The symmetric case**

For $H \in G_{n,k}$, if $\mu$ is an isotropic measure on $S^{n-1}$, then we define the Borel measure $\bar{\mu}$ on $S^{n-1} \cap H$ by

$$\bar{\mu}(A) = \int_{S^{n-1} \cap H} 1_A \left( \frac{P_H u}{\|P_H u\|} \right) \|P_H u\|^2 d\mu(u)$$

(5.1)

for Borel sets $A \subset S^{n-1} \cap H$. Thus, for an arbitrary $y \in H$, we have

$$\int_{S^{n-1} \cap H} (y \cdot w)^2 d\bar{\mu}(w) = \int_{S^{n-1} \cap H} \left( y \cdot \frac{P_H u}{\|P_H u\|} \right)^2 \|P_H u\|^2 d\mu(u)$$

$$= \int_{S^{n-1}} (y \cdot u)^2 d\mu(u) = \|y\|^2.$$

Hence, $\bar{\mu}$ is isotropic on $S^{n-1} \cap H$, and

$$\bar{\mu}(S^{n-1} \cap H) = \int_{S^{n-1} \cap H} \|P_H u\|^2 d\mu(u) = k.$$  

(5.2)

**Theorem 5.1.** If $H \in G_{n,k}$ and $\mu$ is an even isotropic measure on $S^{n-1}$, then

$$W(P_H C) \geq \sqrt{\frac{k}{n}} W(B_1^k).$$

with equality if and only if $P_H C = \sqrt{\frac{k}{n}} B_1^k$. 
Proof. Let $\tau > 0$ be a real number, and for $u \in \text{supp} \mu$, let

$$
\Gamma_{\tau,u} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} 1\left[-\frac{x}{\|H u\|}, \frac{x}{\|H u\|}\right] e^{-\frac{x^2}{2}} ds.
$$

(5.3)

Define the smooth strictly increasing function $\phi_{\tau,u} : \left(-\frac{\tau}{\|H u\|}, \frac{\tau}{\|H u\|}\right) \to \mathbb{R}$ by

$$
\frac{1}{\Gamma_{\tau,u}\sqrt{2\pi}} \int_{-\infty}^{t} 1\left[-\frac{x}{\|H u\|}, \frac{x}{\|H u\|}\right] e^{-\frac{x^2}{2}} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\phi_{\tau,u}(t)} e^{-s^2} ds.
$$

Then $\phi'_{\tau,u} > 0$, and differentiating both sides gives

$$
\log 1\left[-\frac{x}{\|H u\|}, \frac{x}{\|H u\|}\right](t) - \frac{t^2}{2} = \log(\sqrt{2\Gamma_{\tau,u}}) - \phi_{\tau,u}(t)^2 + \log \phi'_{\tau,u}(t),
$$

(5.4)

for all $t \in \left[-\frac{\tau}{\|H u\|}, \frac{\tau}{\|H u\|}\right]$.

Since $\mu$ is an even isotropic measure on $S^{n-1}$, by (2.11), for $\tau > 0$, we have

$$
\text{relint}(\tau(C^o \cap H)) = \{ x \in H : |\langle x, P_H u \rangle| < \tau \text{ for all } u \in \text{supp} \mu \setminus H^\perp \}
$$

$$
= \left\{ x \in H : \left| \langle x, \frac{P_H u}{\|P_H u\|} \rangle \right| < \frac{\tau}{\|P_H u\|} \right\} \text{ for all } u \in \text{supp} \mu \setminus H^\perp.
$$

Then for each $x \in \text{relint}(\tau C^o)$

$$
\exp \left( \int_{S^{n-1}\setminus H^\perp} \log 1\left[-\frac{x}{\|H u\|}, \frac{x}{\|H u\|}\right] \left( \langle x, \frac{P_H u}{\|P_H u\|} \rangle \right) \|P_H u\|^2 d\mu(u) \right) = 1.
$$

(5.5)

Define $T : \text{relint}(\tau(C^o \cap H)) \to H$ by

$$
Tx = \int_{S^{n-1}\setminus H^\perp} P_H u \phi_{\tau,u} \left( \langle x, \frac{P_H u}{\|P_H u\|} \rangle \right) \|P_H u\| d\mu(u).
$$

(5.6)

Note that for all $x \in \text{relint}(\tau(C^o \cap H))$ and all $u \in \text{supp} \mu \setminus H^\perp$, $\langle x, \frac{P_H u}{\|P_H u\|} \rangle$ is in the domain of $\phi_{\tau,u}$. It follows that

$$
dT(x) = \int_{S^{n-1}\setminus H^\perp} P_H u \otimes P_H u \phi'_{\tau,u} \left( \langle x, \frac{P_H u}{\|P_H u\|} \rangle \right) d\mu(u).
$$

(5.7)

Since $\phi'_{\tau,u} > 0$, the matrix $dT(x)$ is positive definite for each $x \in \text{relint}(\tau(C^o \cap H))$. Hence, the transformation $T : \text{relint}(\tau(C^o \cap H)) \to H$ is injective.

The following two inequalities are the direct consequences of Lemmas 2.1 and 2.2

$$
\exp \left( \int_{S^{n-1}\setminus H^\perp} \log \phi'_{\tau,u} \left( \langle x, \frac{P_H u}{\|P_H u\|} \rangle \right) \|P_H u\|^2 d\mu(u) \right) \leq |dT(x)|,
$$

(5.8)

and

$$
\int_{S^{n-1}\setminus H^\perp} \phi_{\tau,u} \left( \langle x, \frac{P_H u}{\|P_H u\|} \rangle \right)^2 \|P_H u\|^2 d\mu(u) \geq \|Tx\|^2.
$$

(5.9)

From (2.7) we get, for $x \in H$,

$$
e^{-\frac{|u|^2}{2}} = \exp \left( -\frac{1}{2} \int_{S^{n-1}\setminus H^\perp} \langle x, P_H u \rangle^2 d\mu(u) \right).
$$

(5.10)
Therefore, by (5.5), (5.4), (2.9), (5.8), (5.9), and making the change of variable $y = T x$, we have

\[
\gamma_k(\tau(C^o \cap H)) = \int_{\text{relint}(\tau(C^o \cap H))} d\gamma_k(x)
\]

\[
= (2\pi)^{-\frac{k}{2}} \int_{\text{relint}(\tau(C^o \cap H))} e^{-\frac{\|x\|^2}{2}} dx
\]

\[
= (2\pi)^{-\frac{k}{2}} \int_{\text{int}(\tau(C^o \cap H))} \exp \left( \int_{S^{n-1} \cap H^\perp} \left( \log 1 + \sum_{i=1}^k \left( \left\langle x, \frac{P_H u}{\|P_H u\|} \right\rangle \right)^2 \|P_H u\|^2 d\mu(u) \right) \right) dx
\]

\[
\leq \pi^{-\frac{k}{2}} \int_{\text{relint}(\tau(C^o \cap H))} \exp \left( \int_{S^{n-1} \cap H^\perp} \log(\Gamma_{\tau,u}) \|P_H u\|^2 d\mu(u) \right)
\]

\[
\times \exp \left( \int_{S^{n-1} \cap H^\perp} -\phi_{\tau,u} \left( \left\langle x, \frac{P_H u}{\|P_H u\|} \right\rangle \right)^2 \|P_H u\|^2 d\mu(u) \right) dx
\]

\[
= \exp \left( \int_{S^{n-1} \cap H^\perp} \log(\Gamma_{\tau,u}) \|P_H u\|^2 d\mu(u) \right)
\]

\[
= \exp \left( \int_{S^{n-1} \cap H^\perp} \log \left( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} 1 \left( \left[ -\frac{\tau}{\|P_H u\|}, \frac{\tau}{\|P_H u\|} \right] \right) e^{-\frac{s^2}{2}} ds \right) \|P_H u\|^2 d\mu(u) \right)
\]

\[
= \exp \left[ \frac{1}{k} \int_{S^{n-1} \cap H^\perp} \log \left( \gamma_1 \left( \frac{\tau}{\|P_H u\|} [-e_1, e_1] \right) \right) \|P_H u\|^2 d\mu(u) \right]^k.
\]

Since $\gamma_1$ is log-concave and $\frac{1}{k} \|P_H u\|^2 d\mu(u)$ is a probability measure on $S^{n-1} \cap H^\perp$ (by (2.9)), by Hölder's inequality, we have

\[
\exp \left[ \frac{1}{k} \int_{S^{n-1} \cap H^\perp} \log \left( \gamma_1 \left( \frac{\tau}{\|P_H u\|} [-e_1, e_1] \right) \right) \|P_H u\|^2 d\mu(u) \right]^k
\]

\[
\leq \gamma_1 \left( \frac{1}{k} \int_{S^{n-1} \cap H^\perp} \frac{\tau}{\|P_H u\|} [-e_1, e_1] \|P_H u\|^2 d\mu(u) \right)^k.
\]
where $B^k_\infty$ is a unit cube in $\mathbb{R}^k$. Meanwhile, by H"older’s inequality, (2.6), and (2.9), we also have

$$\frac{1}{k} \int_{S^{n-1} \setminus H^\perp} \| P_H u \| d\mu(u) \leq \frac{1}{k} \left( \int_{S^{n-1} \setminus H^\perp} d\mu(u) \right)^{\frac{1}{2}} \left( \int_{S^{n-1} \setminus H^\perp} \| P_H u \|^2 d\mu(u) \right)^{\frac{1}{2}} \leq \frac{1}{k} \left( \int_{S^{n-1}} d\mu(u) \right)^{\frac{1}{2}} \left( \int_{S^{n-1} \setminus H^\perp} \| P_H u \|^2 d\mu(u) \right)^{\frac{1}{2}} = \sqrt{n k} \cdot \sqrt{\frac{n}{k}} \quad (5.11)$$

Hence, we obtain

$$\gamma_k (\tau (C^\circ \cap H)) \leq \gamma_k \left( \tau \sqrt{\frac{n}{k} B^k_\infty} \right),$$

which implies that

$$\int_0^\infty (1 - \gamma_k (\tau (C^\circ \cap H))) d\tau \geq \int_0^\infty \left( 1 - \gamma_k \left( \tau \sqrt{\frac{n}{k} B^k_\infty} \right) \right) d\tau.$$

for all $\tau > 0$. Thus, by (2.4) and (2.5), we finally get

$$W(P_H C) \geq W \left( \sqrt{\frac{k}{n}} B^k_\infty \right) = \sqrt{\frac{k}{n}} W(B^k_\infty).$$

Assume that equality holds in the above inequality. Since $\mu$ is even isotropic, it follows that $\bar{\mu}$ defined in (5.1) is also even isotropic on $S^{n-1} \cap H$, and thus $\bar{\mu}$ is not concentrated on any subsphere of $S^{n-1} \cap H$. Then there exist linearly independent $w_1, \ldots, w_k \in \text{supp } \bar{\mu}$ such that $\{ \pm w_1, \ldots, \pm w_k \} \subseteq \text{supp } \bar{\mu}$, where $w_i = \frac{\mu_{u_i}^n}{\| P_H u_i \|}$, $u_i \in \text{supp } \mu \setminus H^\perp$, $i = 0, 1, \ldots, k$. The argument, as the same as the proof in Theorem 4.1, yields that $\{ w_1, \ldots, w_k \} = \left\{ \frac{\mu_{u_{i_1}}}{\| P_H u_{i_1} \|}, \ldots, \frac{\mu_{u_{i_k}}}{\| P_H u_{i_k} \|} \right\}$ is in fact an orthogonal basis in $S^{n-1} \cap H$. Moreover, the equality conditions of H"older’s inequality imply that equality in (5.11) holds if and only if $\| P_H u \| = \sqrt{\frac{k}{n}}$ for all $u \in \text{supp } \mu \setminus H^\perp$ and $\mu$ must concentrate on $S^{n-1} \setminus H^\perp$. It means that $P_H \sqrt{\frac{k}{n}} u$ is an unit vector in $H$ for each $u \in \text{supp } \mu \setminus H^\perp$. Thus,

$$\sqrt{\frac{n}{k}} P_H C = \sqrt{\frac{n}{k}} \text{conv} \left\{ P_H u : u \in \text{supp } \mu \setminus H^\perp \right\} = \text{conv} \left\{ P_H \sqrt{\frac{n}{k}} u : u \in \text{supp } \mu \setminus H^\perp \right\} = \text{conv} \left\{ \pm P_H \sqrt{\frac{n}{k}} u_{i_1}, \ldots, \pm P_H \sqrt{\frac{n}{k}} u_{i_k} \right\},$$

which is $B^k_1$ in $H$.

The if part is easy to check, and we omit the details. \qed
When $\mu$ is a discrete even isotropic measure, the case $k = n$ of Theorem 5.1 was proved by Schechtman and Schmuckenschläger [49].

**Theorem 5.2.** If $H \in G_{n,k}$ and $\mu$ is an even isotropic measure on $S^{n-1}$, then

$$W(C^o \cap H) \leq \sqrt{n/k}W(B_k^\infty).$$

with equality if and only if $C^o \cap H = \sqrt{n/k}B_k^\infty$.

**Proof.** From (2.7) we have

$$x = \int_{S^{n-1}\setminus H^\perp} \langle x, P_H u \rangle P_H u d\mu(u)$$

for all $x \in H$. Define a zonoid $Z$ in $H$ by

$$h_Z(x) = \int_{S^{n-1}\setminus H^\perp} |\langle x, P_H u \rangle| d\mu(u).$$

From the definition of the polar body, it follows that

$$Z^o = \{x \in H : h_Z(x) \leq 1\}.$$

Define a closure as

$$M = \text{cl}\left\{ \int_{S^{n-1}\setminus H^\perp} \langle x, P_H u \rangle P_H u d\mu(u) : H : h_Z(x) \leq 1 \right\}.$$

It is easily shown that $M$ is a convex body in $H$ and $M = Z^o$. (The polarity is taken in $H$.) Since $\mu$ is an even isotropic measure on $S^{n-1}$, from the definition of support function and (2.12), we have, for $y \in H$,

$$h_M(y) = \sup_{h_Z(x) \leq 1} |\langle y, \int_{S^{n-1}\setminus H^\perp} \langle x, P_H u \rangle P_H u d\mu(u) \rangle|$$

$$= \sup_{h_Z(x) \leq 1} \int_{S^{n-1}\setminus H^\perp} \langle x, P_H u \rangle \langle y, P_H u \rangle d\mu(u)$$

$$\leq \sup_{h_Z(x) \leq 1} \int_{S^{n-1}\setminus H^\perp} |\langle x, P_H u \rangle| |\langle y, P_H u \rangle| d\mu(u)$$

$$\leq \sup_{u \in \text{supp}\mu \setminus H^\perp} |\langle y, P_H u \rangle|$$

$$= h_{P_HC}(y),$$

where $P_HC = \text{conv}\{P_H u : u \in \text{supp}\mu \setminus H^\perp\}$. Hence $M \subseteq P_HC$. Therefore, for $x \in H$,

$$h_{C^o \cap H}(x) = h_{(P_HC)^o}(x) \leq h_{M^o}(x) = \int_{S^{n-1}\setminus H^\perp} |\langle x, P_H u \rangle| d\mu(u).$$

Recall that if $K$ is a convex body in $H$, then by (2.3),

$$W(K) = \frac{1}{c_k} \int_H h_K(x) d\gamma_k(x).$$
Integrating with respect to the Gaussian measure, using the Fubini theorem, the rotational invariance of Gaussian measure, and (5.11), we get,

\[
W(C^o \cap H) = \frac{1}{c_k} \int_H h_{C^o \cap H}(x) d\gamma_k(x)
\]

\[
\leq \frac{1}{c_k} \int_H \int_{S^{n-1} \setminus H^\perp} |\langle x, P_H u \rangle| d\mu(u) d\gamma_k(x)
\]

\[
= \frac{1}{c_k} \int_{S^{n-1} \setminus H^\perp} \int_H |\langle x, P_H u \rangle| d\gamma_k(x) d\mu(u)
\]

\[
= \frac{1}{c_k} \int_{S^{n-1} \setminus H^\perp} \int_H |\langle x, e_1 \rangle| d\gamma_k(x) \|P_H u\| d\mu(u)
\]

\[
= \frac{1}{c_k} \int_{S^{n-1} \setminus H^\perp} |\langle x, e_1 \rangle| d\gamma_k(x) \int_{S^{n-1} \setminus H^\perp} \|P_H u\| d\mu(u)
\]

\[
= \frac{1}{k} W(B^k_\infty) \int_{S^{n-1} \setminus H^\perp} \|P_H u\| d\mu(u)
\]

\[
\leq \sqrt{\frac{n}{k}} W(B^k_\infty).
\]

Now we study the equality case. Assume that \(W(C^o \cap H) \leq \sqrt{\frac{n}{k}} W(B^k_\infty)\). Then we must have

\[
h_{C^o \cap H}(x) = \int_{S^{n-1} \setminus H^\perp} |\langle x, P_H u \rangle| d\mu(u),
\]

for all \(x \in \mathbb{R}^n\). Thus, for \(v \in \text{supp} \mu \cap H\), we get

\[
h_{C^o \cap H}(v) = \int_{S^{n-1} \setminus H^\perp} |\langle v, P_H u \rangle| d\mu(u).
\]

Notice that for \(v = P_H v \in \partial(P_H C)\) (the boundary of \(P_H C\)) and

\[
h_{C^o \cap H}(v) = \|v\|_{P_H C} = \min\{t > 0 : v \in t(P_H C)\} = 1.
\]

Since \(|\langle v, P_H u \rangle| \leq 1\), it follows from (2.7) that

\[
1 = \int_{S^{n-1}} |\langle v, P_H u \rangle| d\mu(u) \geq \int_{S^{n-1}} |\langle v, P_H u \rangle|^2 d\mu(u) = |v|^2 = 1.
\]

Necessarily, \(|\langle v, P_H u \rangle| = 1\), or \(|\langle v, P_H u \rangle| \in \{-1, 1\}\) for arbitrary \(v \in \text{supp} \mu \cap H\) and \(u \in \text{supp} \mu \setminus H^\perp\). This means that \(\|P_H u\| = 1\) and \(\mu\) concentrates on \(H \cup H^\perp\). Since \(\mu\) is even and not concentrated on a proper subspace of \(\mathbb{R}^n\), there must exist orthogonal unit vectors \(v_1, \ldots, v_k\) in \(H\) such that

\[
\text{supp} \mu \cap H = \{\pm v_1, \ldots, \pm v_k\}.
\]

Hence, by (5.11),

\[
\mu(S^{n-1} \cap H) = \mu(S^{n-1} \setminus H^\perp) = \sqrt{nk}.
\]
Since $\mu$ is isotropic, it follows that $\mu(\{\pm v_i\}) = \sqrt{\frac{n}{k}}$, $i = 1, \ldots, k$, and therefore $C^o \cap H = \sqrt{\frac{n}{k}} B_k^\infty$.

The if part is obvious by the definition of mean width. □

Theorems 5.1 and 5.2 can be stated in terms of the $\ell$-norm of convex bodies by taking account of (2.4) and (2.13).

Theorem 5.3. If $H \in G_{n,k}$ and $\mu$ is an even isotropic measure on $S^{n-1}$, then
\[
\ell(C^o \cap H) \geq \sqrt{\frac{n}{k}} \ell(B_k^\infty) \quad \text{with equality if and only if} \quad C^o \cap H = \sqrt{\frac{n}{k}} B_k^\infty.
\]

The case $k = n$ of Theorem 5.3 was proved in [38], and when $\mu$ is a discrete even isotropic measure, this situation was remarked by Schechtman and Schmuckenschläger [49], and proved by Barthe [9].

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