The positive mass theorem in Kaluza–Klein picture

Tetsuya Shiromizu\textsuperscript{1,2,a}, Diego Soligon\textsuperscript{1}

\textsuperscript{1} Department of Mathematics, Nagoya University, Nagoya 464-8602, Japan
\textsuperscript{2} Kobayashi-Maskawa Institute, Nagoya University, Nagoya 464-8602, Japan

Received: 27 April 2022 / Accepted: 29 August 2022
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract We reconsider Schoen and Yau’s proof of the positive mass theorem from the extra-dimensional point of view, and we introduce a modified argument to prove the theorem in the Kaluza–Klein picture. We consider in this study an alternative condition to Jang’s equation, which makes the argument more physically intuitive.

1 Introduction

The positive mass theorem in general relativity asserts that for a nontrivial isolated physical system the total energy is nonnegative. It is fundamentally important as it guarantees the stability of space-time. Historically there have been two main ways to prove it. One of them, formulated by Witten [1], is inspired by supergravity and is easily accessible even by non-mathematicians. There has recently been some trial work based on this proof on the construction of dark energy theories compatible with the positive mass theorem [2–4], but these tend to be biased towards supergravity. Therefore, it would be interesting to address the same issue from another perspective, based on the other more mathematical proof of the positive mass theorem formulated by Schoen and Yau [5, 6], which has no relation to supergravity. This approach however could cause some difficulty for the rather convoluted mathematical techniques required and the absence of an immediate physical interpretation.

In this paper we propose a new method that follows a similar argument as in Schoen and Yau’s proof [6], which can be useful to address some physical problems, such as the stability of space-times with dark energy (as found for example in [2–4]). In fact, Schoen and Yau’s proof recalls some characteristics of the Kaluza–Klein picture of space-time [7, 8], an extra-dimensional theory first proposed as a possible candidate for a unified theory of gravity and electromagnetism. By reformulating the proof explicitly on a Kaluza–Klein space-time it is easier to obtain a physical interpretation.

The main idea of Ref. [6] in extending the Riemannian positive mass theorem [5] to the general case is to consider a function \( f \) on an initial data set \((\Sigma, q_{ab}, K_{ab})\), where \( q_{ab} \) and \( K_{ab} \) are the induced metric and the extrinsic curvature of \( \Sigma \), and take its graph \( \Sigma \times \mathbb{R} \). Equipped with the product metric \( dy^2 + q \), the trace of extrinsic curvature of \( \Sigma \times \mathbb{R} \) is supposed to be equal to \( \tilde{q}^{ab} K_{ab} \), a condition known as Jang’s equation [9]. It is noted that the induced metric on \( \Sigma \) can be deformed conformally to an asymptotically Euclidean metric with vanishing Ricci scalar, so that the Riemannian positive mass theorem [5, 10] can be applied.

We can then conclude by this result that the Arnowitt–Deser–Misner (ADM) mass of space-time is nonnegative.

The new approach we propose follows the same key steps, but instead of considering a function of an initial data set, we consider a function of space-time \( M \), whose graph is a hypersurface in \( M \times \mathbb{R} \), which can be regarded as a Kaluza–Klein space-time. In addition, the condition imposed by Jang’s equation is replaced by considering the existence of a marginally outer trapped surface (MOTS) in \( M \times \mathbb{R} \). Analogies between solutions of Jang’s equation and MOTS have already been considered [11]. The result is a new method of proving the positive mass theorem with an easy physical interpretation.

The rest of this paper is organized as follows. In Sect. 2 we give a brief review of Schoen and Yau’s 1981 proof for non-experts. In Sect. 3 we present a new way to prove the positive mass theorem in the Kaluza–Klein picture. Finally, we give a summary and a discussion of the result.

2 Brief review of the positive mass theorem

In this section, we will review Schoen and Yau’s 1981 proof of the positive mass theorem [6] (See also Ref. [12]). This part will be helpful for non-experts.

\(^a\) e-mail: shiromizu@math.nagoya-u.ac.jp (corresponding author)
We consider a \( n \)-dimensional asymptotically flat initial data set for a space-time \( (\Sigma, q_{ab}, K_{ab}) \), consisting of a \( n \)-dimensional manifold \( \Sigma \), a metric \( q_{ab} \) and the extrinsic curvature \( K_{ab} \), satisfying the constraint equations

\[
R - K_{ab}K^{ab} + K^2 = 2\rho
\]  

and

\[
\mathcal{D}_a K^a_b - \mathcal{D}_b K = J_b ,
\]

where \( R \) is the Ricci scalar of the metric \( q_{ab} \), \( \mathcal{D}_a \) is the covariant derivative with respect to \( q_{ab} \), \( \rho \) is the local mass density and \( J_b \) is the local current density. We assume that \( \rho \) and \( J_b \) obey the dominant energy condition

\[
\rho \geq \left( J^b J_b \right)^{1/2} .
\]

We then form the \((n + 1)\)-dimensional product manifold \( \hat{\Sigma} = \Sigma \times \mathbb{R} \) with metric \( \hat{h} \) defined by

\[
\hat{h} = dy^2 + q .
\]

In the above, we suppose that \( q \) does not depend on \( y \).

Given a function \( f \) on \( \Sigma \), we consider a hypersurface \( \Sigma \) in \( \hat{\Sigma} \) which is the graph of the function \( y = f(x^i) \), where \( x^i \) is the coordinate on \( \Sigma \) (See Fig. 1). Then the induced metric on \( \Sigma \) is

\[
\bar{q} = \hat{h} |_{y=f(x)} = (q_{ij} + \partial_i f \partial_j f)dx^i dx^j =: \bar{q}_{ij} dx^i dx^j .
\]

Hereafter we consider the ADM decomposition with respect to \( \Sigma \). The unit normal vector \( \bar{n}_a \) to \( \Sigma \) in \( \hat{\Sigma} \) is

\[
\bar{n}_a = a \bar{D}_a (y - f(x)) ,
\]

where \( \alpha \) is the lapse function and \( \bar{D}_a \) is the covariant derivative with respect to \( \bar{h} \). In the current setup, we have

\[
\alpha = (1 + (\mathcal{D} f)^2)^{-1/2} .
\]

The evolution equation for \( \bar{K} \) along the \( \bar{n} \)-direction is given by

\[
\alpha^{-1} \bar{D}^2 \alpha = -\bar{h} R_{ab} \bar{n}^a \bar{n}^b - \bar{\xi}_a \bar{K} - \bar{K}_{ab} \bar{K}^{ab} ,
\]

where \( \bar{h} R_{ab} \) is the Ricci tensor of \( \bar{h} \), \( \bar{D}_a \) is the covariant derivative with respect to the metric \( \bar{q} \) and \( \bar{K}_{ab} \) is the extrinsic curvature of \( \Sigma \).

The double trace of the Gauss equation with respect to \( \bar{\Sigma} \) gives us

\[
\bar{h} R - 2 \bar{h} R_{ab} \bar{n}^a \bar{n}^b = \bar{R} - \bar{K}^2 + \bar{K}_{ab} \bar{K}^{ab} ,
\]

where \( \bar{R} \) is the Ricci scalar of \( \bar{\Sigma} \). From the construction of \( \bar{h} \), we see that \( \bar{h} R = R \). Using Eq. (1), the equation above then becomes

\[
2 \bar{h} R_{ab} \bar{n}^a \bar{n}^b = 2\rho + K_{ab} K^{ab} - K^2 - \bar{K}_{ab} \bar{K}^{ab} + \bar{K}^2 - \bar{R} .
\]

Eqs. (8) and (10) imply

\[
2\rho = \bar{R} - K_{ab} K^{ab} + K^2 - \bar{K}^2 - \bar{K}_{ab} \bar{K}^{ab} - 2\xi_a \bar{K} - 2\alpha^{-1} \bar{D}^2 \alpha .
\]

Since

\[
\bar{D}_a (K^a_b - \delta^a_b K) = \mathcal{D}_a (K^a_b - q^a_b K) ,
\]

and direct calculation gives us

\[
\bar{n}^b \bar{D}_a (K^a_b - \delta^a_b K) = \bar{n}^a \bar{D}_a \bar{q}^b - 2 \bar{D}^a \ln \alpha K_{ab} \bar{n}^b - \bar{D}^a (\bar{q}^b_a K_{b} \bar{n}^c) - \bar{K} K_{ab} \bar{n}^a \bar{n}^b + K_{ab} \bar{K}^{ab} ,
\]

Eq. (2) implies

\[
\xi_a \bar{K} = \bar{D}^c (\bar{q}^b_a K_{b c} \bar{n}^c) + J_a \bar{n}^a - \bar{K} K_{ab} \bar{n}^a \bar{n}^b + K_{ab} \bar{K}^{ab} - 2\alpha^{-1} K_{ab} \bar{n}^b \bar{D}^a \alpha = 0 .
\]
Then, from Eqs. (11) and (14) we derive the key equation
\[
2(\rho - J_a \tilde{n}^a) = -2\tilde{D}^a X_a - 2|X_a|^2 q + 2\epsilon_n (K_{\tilde{n}} - \tilde{K}) - |K_{ab} - \tilde{K}_{ab}|_{\tilde{n}}^2 + \tilde{K}^2 + 2K_{ab} \tilde{n}^a \tilde{n}^b (\tilde{K}_{\tilde{n}} - \tilde{K}) ,
\]
where \( K_{\tilde{n}} = \tilde{q}^{ab} K_{ab} \) and
\[
X_a := \tilde{D}_a \ln \alpha + \tilde{q}_a ^b K_{cb} \tilde{n}^b .
\]
|\cdots|_{\tilde{n}} denotes the trace with respect to \( \tilde{q}_{ab} \). Now, if one can impose
\[
K_{\tilde{n}} = \tilde{K} ,
\]
Eq. (15) becomes
\[
2(\rho - J_a \tilde{n}^a) = -2\tilde{D}^a X_a - 2|X_a|^2 q + |K_{ab} - \tilde{K}_{ab}|_{\tilde{n}}^2 .
\]
Eq. (17) can be written in term of \( f \) as
\[
\alpha \tilde{q}^{ab} D_a D_b f = \tilde{q}^{ab} K_{ab}
\]
and it is called Jang’s equation. It has been shown that a solution to Eq. (19) exists when there are no apparent horizons \([6]\). If there is an apparent horizon, a more careful treatment is needed. However, the essence of the proof does not depend on the existence of apparent horizons. Therefore, for simplicity, we focus on the case in which a solution to Jang’s equation exists.\(^1\)

Let \( \phi \) be a function on \( \Sigma \). Let us multiply \( \phi^2 \) to Eq. (18) and integrate over \( \Sigma \). Using Eq. (17), we have
\[
\int_{\Sigma} \left[ 2(\rho - J_a \tilde{n}^a) - \tilde{R} \right] \phi^2 d\tilde{V} = \int_{\Sigma} \left[ -2\tilde{D}^a X_a \phi^2 - 2|X_a|^2 \phi^2 - |K_{ab} - \tilde{K}_{ab}|_{\tilde{n}}^2 \phi^2 \right] d\tilde{V}
\]
\[
= \int_{\Sigma} \left[ -2\phi \tilde{D}_a \phi - \tilde{D}_a \phi \right]_{\tilde{n}}^2 - |K_{ab} - \tilde{K}_{ab}|_{\tilde{n}}^2 + 2(\tilde{D} \phi)^2 \right] d\tilde{V}
\]
\[
\leq 2 \int_{\Sigma} (\tilde{D} \phi)^2 d\tilde{V} .
\]
We suppose that \( \phi \) satisfies
\[
(\tilde{D}^2 - \frac{n-2}{4(n-1)} \tilde{R}) \phi = 0
\]
and has the following asymptotic behaviour at infinity
\[
\phi = 1 - C/r^{n-2} + O(1/r^{n-1}) .
\]
Thus, by Eq. (20) and the dominant energy condition we have
\[
0 \leq 2 \int_{\Sigma} \left( \rho - J_a \tilde{n}^a \right) \phi^2 d\tilde{V} \leq \int_{\Sigma} (\tilde{R} \phi^2 + 2(\tilde{D} \phi)^2) d\tilde{V} = \frac{4(n-1)}{n-2} \int_{\Sigma} \phi \tilde{D}_a \phi d\tilde{S}^a - \frac{2n}{n-2} \int_{\Sigma} (\tilde{D} \phi)^2 d\tilde{V} ,
\]
hence
\[
0 \leq \frac{2n}{n-2} \int_{\Sigma} (\tilde{D} \phi)^2 d\tilde{V} \leq \frac{4(n-1)}{n-2} \int_{\Sigma} \phi \tilde{D}_a \phi d\tilde{S}^a = 64\pi C ,
\]
that is, \( C \geq 0 \).

Take the conformal transformation \( \tilde{q}_{ab} = \phi^{4/(n-2)} \tilde{q}_{ab} \). Equation (21) shows us that the Ricci scalar \( \tilde{R} \) of \( \tilde{q} \) vanishes. So, by the Riemannian positive mass theorem, the ADM mass \( \tilde{m} \) is nonnegative \([5, 10]\). Since \( m = \tilde{m} + 2C \), we see that \( m \geq 0 \). Here we used the asymptotic behaviour of \( \tilde{q}_{ij} \), that is,
\[
\tilde{q}_{ij} = \left( 1 + \frac{2}{n-2} \frac{m}{r^{n-2}} \right) \delta_{ij} + O(1/r^{n-1})
\]
where \( m \) is the ADM mass for \( (\Sigma, q) \).

We considered a \((n+1)\)-dimensional product manifold in the proof, which can be interpreted as a spacelike slice of a Kaluza–Klein space-time, that is a \((n+2)\)-dimensional space-time with \( n + 1 \) space dimensions. Jang’s equation is a key point in the proof: its meaning is nontrivial at first glance. Here, Jang’s equation can be written as
\[
\tilde{q}^{ab} (K_{ab} - \tilde{K}_{ab}) = \frac{1}{2} \tilde{q}^{ab} (\epsilon_n \tilde{q}_{ab} - \epsilon_n \tilde{q}_{ab}) = 0 .
\]
In consideration of the extra-dimensions construction, this expression suggests that the condition imposed by Jang’s equation can be replaced by imposing the vanishing of the null expansion. The analogy between Jang’s equation and the existence of a marginally outer trapped surface has been discussed extensively \([11]\).

---

\(^1\) If one is interested in the large-scale structure of space-time, the assumption that there is no apparent horizon is reasonable.
3 Proof of the positive energy theorem in a Kaluza–Klein picture

In this section we will slightly modify Schoen and Yau’s proof of the positive mass theorem [6, 12]. There are two main points in this new procedure. The first is that we shall consider the graph of a function on the full Lorentzian space-time instead of just on the Riemannian manifold corresponding to the space dimensions. The second is the imposition of a condition alternative to Jang’s equation. These are just slight modifications, but they allow for a better intuition of the physics behind the proof. Furthermore, they provide a useful connection to the technology developed in black hole studies [11, 13], so that it can be applied to the objects discussed here.

Let \( M \) be a \((n + 1)\)-dimensional Lorentzian space-time with metric \( g_{\mu \nu} \), and consider a \((n + 2)\)-dimensional product manifold \( \hat{M} = M \times \mathbb{R} \) equipped with the metric

\[
\hat{g} = ds^2 + g_{\mu \nu} dx^\mu dx^\nu ,
\]

where \( g_{\mu \nu} \) does not depend on \( y \). \((\hat{M}, \hat{g}_{ab})\) can be considered as a Kaluza–Klein space-time. Then, given a function \( f \) on \( M \) we can take a timelike hypersurface \( \hat{M} \) given by the graph \( y = f(x^\mu) \) (See Fig. 2). The metric induced on \( \hat{M} \) is

\[
\bar{g}_{\mu \nu} = g_{\mu \nu} + \partial_\mu f \partial_\nu f .
\]

Introducing a change of coordinate defined by

\[
\tilde{y} := y - f(x^\mu) ,
\]

the \((n + 2)\)-dimensional metric on \( \hat{M} \) is written as

\[
\hat{g} = d\tilde{y}^2 + 2\partial_\mu f d\tilde{y} dx^\mu + \bar{g}_{\mu \nu} dx^\mu dx^\nu .
\]

The unit normal vector to \( \hat{M} \) in \( \hat{M} \) is given by

\[
\tilde{n}_a = \hat{\nabla}_a \tilde{y} ,
\]

where \( \hat{\nabla}_a = (1 + g^{\alpha \beta} \partial_\alpha f \partial_\beta f)^{-1/2} \) and \( \hat{\nabla}_a \) is the covariant derivative with respect to \( \bar{g}_{ab} \). Here we suppose \( g^{\mu \nu} \partial_\mu f \partial_\nu f > 0 \) because \( \hat{M} \) is a timelike hypersurface.

Consider a spacelike hypersurface \((\hat{\Sigma}, \hat{g}_{ab}) \subset (\hat{M}, \bar{g}_{ab})\) given by the intersection of \( \hat{M} \) and the spacelike hypersurface \( \hat{\Sigma} \) with timelike unit normal vector \( \hat{r}^a \) in \( \hat{M} \), that is \( \hat{\Sigma} = \hat{M} \cap \hat{\Sigma} \) (See Fig. 2). Then, the metric is decomposed as

\[
\bar{g}_{ab} = \hat{g}_{ab} + \hat{n}_a \hat{n}_b = \hat{h}_{ab} - \hat{r}_a \hat{r}_b = \hat{q}_{ab} - \hat{\tau}_a \hat{\tau}_b ,
\]

where \( \hat{g}_{ab}, \hat{h}_{ab} \) and \( \hat{q}_{ab} \) are the induced metrics on \( \hat{M}, \hat{\Sigma} \) and \( \hat{\Sigma} \), respectively.

We denote the covariant derivatives \( \hat{D}_a \) and \( \hat{\nabla}_a \) with respect to \( \hat{h}_{ab} \) and \( \hat{q}_{ab} \), respectively. Then, we have following key equation [11, 13]

\[
\hat{n}^a \hat{D}_a \hat{\theta}_- = \frac{1}{2} \hat{\theta}_{-ab} \hat{\theta}_- + \frac{1}{2} \hat{\theta}_-^2 + (k(\hat{r})) \hat{\theta}_- + \hat{D}_a \hat{\tau}^a + \hat{\tau}_a \hat{\tau}^a - \frac{1}{2} \hat{\bar{R}} + \hat{G}_{ab} \hat{\tau}^a \hat{\tau}^b ,
\]

where

\[
\hat{\theta}_- := \hat{q}_{ab} \hat{\nabla}_a (\hat{r}_b - \hat{n}_b) ,
\]

\[
\hat{\theta}_{-ab} := \hat{q}_{a} \hat{q}_{b} \hat{\nabla}_c (\hat{r}_b - \hat{n}_b) ,
\]

\[
k(\hat{r}) := \hat{q}_{ab} \hat{D}_a \hat{n}_b .
\]
Using Eq. (46), by the Gauss theorem we have
\[ \kappa(\kappa) := \bar{n}^a \bar{n}^b \hat{\nabla}_a \bar{t}_b, \] (37)
\[ \bar{t}_a := \bar{g}^{ab} \hat{\nabla}_b \bar{t}_a + \hat{D}_a \ln \bar{\alpha}, \] (38)
\[ \bar{\ell}^a := \bar{t}^a - \bar{n}^a \] (39)
with a null vector \( \bar{\ell}^a \), and where \( \bar{R} \) is the Ricci scalar of \( \bar{g}_{ab} \) and \( \hat{G}_{ab} \) is the Einstein tensor for \( \hat{g}_{ab} \).

In the current setup, it is easy to see that
\[ \hat{g}_{ab} \bar{t}^a \bar{t}^b = \left( g_{ab} \right) \bar{t}^a \bar{t}^b, \] (40)
where \( g_{ab} \) is the Einstein tensor for \( g_{ab} \). Since \( \bar{t}^a \) is a null vector in \( \hat{M} \), Eq. (32) tells us
\[ g_{ab} \bar{t}^a \bar{t}^b = -(n_a \bar{t}^a)^2, \] (41)
so that \( g_{a\bar{b}} \bar{t}^b \) is a timelike vector in \( M \). In a similar way, \( g_{a\bar{b}} \bar{t}^b \) is also a timelike vector. Imposing the dominant energy condition for the Einstein equation on \( (M, g) \), Eq. (40) implies
\[ \hat{G}_{ab} \bar{t}^a \bar{t}^b \geq 0. \] (42)

Let us now introduce the condition that \( \bar{\Sigma} \) is required to satisfy, which corresponds to Jang’s equation in the original proof:
\[ \bar{\theta}_- = \bar{q}^{ab} \hat{\nabla}_a \bar{b} - \bar{q}^{ab} \hat{\nabla}_a \bar{d} = 0. \] (43)
In Jang’s equation as presented by [6], the first term is \( \bar{q}^{ab} \hat{\nabla}_a \bar{b} \), where \( \hat{\nabla}_a \bar{b} \) is treated as the pullback from \( \Sigma \) to \( \bar{\Sigma} \) (See Appendix A for more details on the distinction between the original proof and the one presented here). Note that Eq. (43) is satisfied by a marginally outer trapped surface [11]. Note also that \( \bar{\Sigma} \) is a non-compact, asymptotically flat surface. In terms of \( \bar{y} \), the equation becomes
\[ \bar{\alpha} \bar{q}^{ab} \hat{\nabla}_a \bar{b} \bar{y} = \bar{q}^{ab} \hat{\nabla}_a \bar{b}. \] (44)
This is an elliptic equation for \( \bar{y} \). The existence of a solution to Jang’s equation was proved by Schoen and Yau [6, 12] if there is no apparent horizon in \( \bar{\Sigma} \) and it was a fundamental part of their result. As for the existence of a solution to Eq. (44), the question will be left open in this paper and we will work under the assumption that it exists.

Once we impose Eq. (44), we see that on \( \bar{\Sigma} \)
\[ \bar{n}^a \hat{D}_a \bar{\theta}_- = 0. \] (45)
Since \( \bar{\Sigma} \) is not compact, this does not imply that \( \bar{\Sigma} \) is an apparent horizon, however one may regard \( \bar{\Sigma} \) as having a similar role to an apparent horizon from a technical point of view.

Let \( \varphi \) be a function over \( \bar{\Sigma} \), satisfying
\[ \left( \bar{D}^2 - \frac{n-2}{4(n-1)} \bar{R} \right) \varphi = 0, \] (46)
with asymptotic behaviour
\[ \varphi = 1 - C/r^{n-2} + O(1/r^{n-1}). \] (47)
As in the previous section, we multiply Eq. (33) by \( \varphi^2 \) and integrate over \( \bar{\Sigma} \), and by the dominant energy condition and Eq. (43),
\[ \int_{\bar{\Sigma}} \varphi^2 \left( \frac{1}{2} \bar{R} - \bar{t}_a \bar{t}^a - \hat{D}_a \bar{t}^a \right) d\bar{V} \geq 0. \] (48)
Using Eq. (46), by the Gauss theorem we have
\[ \int_{\bar{\Sigma}_{\infty}} \frac{2(n-1)}{n-2} \varphi \hat{D}_a \varphi - \varphi^2 \bar{t}_a d\bar{S} \geq \int_{\bar{\Sigma}} \left[ \frac{n}{2(n-2)} (\bar{D} \varphi)^2 + (\varphi \bar{t}_a - \hat{D}_a \varphi)^2 \right] d\bar{V} \geq 0. \] (49)
It is easy to see that \( \bar{t}_a = O(1/r^{2n-1}) \) does not contribute to the surface integral in the left-hand side, therefore
\[ C \geq 0. \] (50)

Now perform a conformal transformation given by
\[ \bar{\varphi} = \varphi^{4/(n-2)} \bar{\varphi}. \] (51)
Asymptotically at infinity
\[
\tilde{q}_{ij} = \left(1 + \frac{2}{n-2} \tilde{m} r^{-2} \right) \delta_{ij} + O(1/r^{n-1})
\]
and
\[
\bar{q}_{ij} = \left(1 + \frac{2}{n-2} \bar{m} r^{-2} \right) \delta_{ij} + O(1/r^{n-1}),
\]
where \(\bar{m}\) and \(\tilde{m}\) are the ADM masses for \(\bar{g}\) and \(\tilde{g}\), respectively. The conformal transformation tells us
\[
\bar{m} = \tilde{m} + 2C
\]
and the Ricci scalar of \((\Sigma, \tilde{g})\) vanishes,
\[
\tilde{R} = 0.
\]
Therefore we can apply the Riemannian positive mass theorem \([5, 10]\) to \((\Sigma, \tilde{g})\) so that
\[
\tilde{m} \geq 0.
\]
Hence, by Eq. (54)
\[
\bar{m} \geq 0.
\]
Since \(\tilde{q}_{ij} = q_{ij} + \partial_i f \partial_j f\) with \(\partial_i f = O(1/r^{n-1})\), the ADM mass, \(m\), of \((M, g)\) is equal to that of \((\bar{M}, \bar{g})\), that is, \(m = \bar{m}\). Thus, the ADM mass of space-time \((M, g)\) is nonnegative.

4 Summary and discussion

In this paper we propose an alternative proof of the positive mass theorem in a Kaluza–Klein picture. Instead of considering the graph of a function on a Riemannian manifold, we consider the graph of a function on the full Lorentzian space-time, which can be considered as a hypersurface on a Kaluza-Klein space-time. Jang’s equation is replaced by a condition directly related to the existence of a marginally trapped outer surface. Compared to the original proof, this proof provides a more direct physical intuition, paving the way for future consideration in the construction of dark energy models compatible with the positive mass theorem.

An open question that remains is the existence of a solution to Eq. (44), which was assumed in this paper. Another issue is the case in which there is an apparent horizon: that would need a careful treatment. Considering the similarity to Jang’s equation we would expect to be able to use a similar argument in this case.

Acknowledgements We would like to thank Tatsuya Morino for his presentation of Schoen and Yau’s proof of the positive mass theorem. We would also like to thank Prof. Sumio Yamada for the useful discussion on Jang’s equation. T. S. is supported by Grant-Aid for Scientific Research from Ministry of Education, Science, Sports and Culture of Japan (Nos. 16K05544, 17H01091). This work is supported in part by JSPS Bilateral Joint Research Projects (JSPS-NRF collaboration) “String Axion Cosmology”.

Data Availability Statement No Data associated in the manuscript.

Appendix A: ADM-decomposition

To see how Jang’s equation and the one proposed in this paper are explicitly distinct, consider the ADM decomposition of the metrics \(g\) of \(M\) and \(\bar{g}\) of \(\bar{M}\):
\[
g = g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + q_{ij}(dx^i + N_i dt)(dx^j + N_j dt),
\]
\[
\bar{g} = \bar{g}_{\mu\nu}dx^\mu dx^\nu = (g_{\mu\nu} + \partial_\mu f \partial_\nu f)dx^\mu dx^\nu = -\bar{N}^2 dt^2 + \bar{q}_{ij}(dx^i + \bar{N}_i dt)(dx^j + \bar{N}_j dt).
\]
The timelike unit normal vectors to \(\Sigma\) and \(\bar{\Sigma}\) are written as
\[
t^a = N^{-1}[(\partial_t)^a - N^i (\partial_i)^a]
\]
and
\[
\bar{t}^a = \bar{N}^{-1}[(\partial_t)^a - \bar{N}^i (\partial_i)^a].
\]
A direct comparison tells us
\[
- \bar{N}^2 + \bar{q}_{ij} \bar{N}^i \bar{N}^j = - N^2 + q_{ij} N^i N^j + \dot{f}^2,
\]

\(\odot\) Springer
\[ \bar{q}_{ij} \bar{N}^i = q_{ij} N^j + \hat{f} \hat{f}_i \] (63)

and

\[ \bar{q}_{ij} = q_{ij} + \hat{f}_i \hat{f}_j, \] (64)

where \( \hat{f} = \partial f \) and \( f_i = \partial_i f \).

Noting that the inverse of \( \bar{g}_{\mu\nu} \) is given by

\[ \bar{g}^{\mu\nu} = g^{\mu\nu} - \bar{\alpha}^2 \nabla_\mu f \nabla_\nu f, \] (65)

the relation between \( N^i \) and \( \bar{N}^i \) becomes

\[ \bar{N}^i = N^i - \bar{\alpha}^2 \nabla^i f f_N + \hat{f} \left( 1 - \bar{\alpha}^2 f_N^2 \right), \] (66)

where \( f_N = N^i f_i \). For \( N \) and \( \bar{N} \), we have

\[ \bar{N}^2 = N^2 + \bar{\alpha}^2 f_N^2 [1 + (\nabla f)^2 [\bar{\alpha}^2 (\nabla f)^2 - 2] + \hat{f}^2 [1 + (\nabla f)^2] [(\nabla f)^2 + \bar{\alpha}^2 f_N^4 (\bar{\alpha}^2 (\nabla f)^2 - 2)]. \] (67)

When

\[ \hat{f} = 0, \] (68)

since

\[ \bar{\alpha}^2 = (1 + (\nabla f)^2)^{-1/2}, \] (69)

Eqs. (66) and (67) are simplified to

\[ \bar{N}^i = N^i - \bar{\alpha}^2 \nabla^i f f_N \] (70)

and

\[ \bar{N}^2 = N^2 - \bar{\alpha}^2 f_N^2. \] (71)

Since \( r^a \) and \( \bar{r}^a \) are different quantities, \( \bar{q}^{ab} \hat{\nabla}_a \bar{r}_b \) and \( \bar{q}^{ab} \hat{\nabla}_a \bar{r}_b \) which appear in the paper are also different quantities.

References

1. E. Witten, Commun. Math. Phys. 80, 381 (1981)
2. M. Nozawa, T. Shiromizu, Phys. Rev. D 89(2), 023011 (2014)
3. M. Nozawa, T. Shiromizu, Nucl. Phys. B 887, 380 (2014)
4. B. Elder, A. Joyce, J. Khoury, A.J. Tolley, Phys. Rev. D 91(6), 064002 (2015)
5. R. Schoen, S.T. Yau, Commun. Math. Phys. 65, 45 (1979)
6. R. Schoen, S.T. Yau, Commun. Math. Phys. 79, 231 (1981)
7. T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.), 966–972 (1921)
8. O. Klein, Zeitschrift für Physik 37, 12 (1926)
9. P.S. Jang, J. Math. Phys. 19(5), 1152–1155 (1978)
10. R. Schoen and S. T. Yau, arXiv:1704.05490 [math.DG]
11. L. Andersson, M. Eichmair and J. Metzger, arXiv:1006.4601 [gr-qc]
12. M. Eichmair, Commun. Math. Phys. 319, 575–593 (2013)
13. G.J. Galloway, R. Schoen, Commun. Math. Phys. 266, 571 (2006)

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.