Solving Skyrmions

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Abstract: We find exact solutions for Skyrmions for the Skyrme-like models. Constructing first the recursion formulae at small and large distance behavior, we proceed by implementing these constraints to a chosen parametrization of the solutions. The procedure is applied to the spherically symmetric hedgehog solution and to topological number \( N > 1 \) solutions based on rational maps.

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1. Introduction

As a non-linear theory of pions, the Skyrme model [1] provides an approximate description of hadronic physics in the low-energy limit. In this theory, the nucleon emerges as a non-perturbative solution of the field equations, or more precisely as a topological soliton. This model is also seen as a prototype which might be applicable in various physical contexts where one could expect soliton solutions to occur (e.g. condensed matter (baby Skyrmions), wrapped branes [2], ...). More recently, this picture has regained attention since it could provide an explanation for the newly discovered hadronic states [3, 4].

The original Skyrme lagrangian is a naive extension of the non-linear sigma model consisting of a fourth-order field derivatives term. This is nonetheless sufficient to stabilize the soliton against scale transformations and to reach at least a 30% accuracy with respect to physical observables. In order to incorporate effects due to higher-spin mesons and improve the fit on most observables a number of alternative Skyrme-like models which preserved the form of the original lagrangian while extending it to higher orders has been proposed and analyzed [5, 6, 7, 8]. Unfortunately, all those models are limited in that they do not admit exact analytic solutions. Indeed, very few analytic
soliton solutions are known namely the one-dimensional sine-Gordon equation, the
KdV equation, the instanton and some other special cases. This is because a necessary
condition for a soliton-like solution to exist makes the search for an analytic solution
rather complex and its discovery accidental.

In the absence of exact analytical solutions, the only alternative to numerical treat-
ment is the use of aptly chosen analytical forms which provide sometimes a reason-
able approximation but which may not reproduce the correct behavior in the limits
$r \to 0, \infty$. Apart from greatly simplifying calculations of physical quantities, a great
deal of information can be extracted from such analytic forms. For instance, symmetries
and general behavior of the solution are much easier to analyze, and the characteristics
of each Skyrme-like model also become more explicit. An analytic form would also
prove useful in the stability analysis of the soliton, both classical and quantum, and
in the calculation of multi-Skyrmion interactions. For example, one can analyze the
quantum behavior of the Skyrme model soliton based on a family of trial functions,
taking into account breathing motions and spin-isospin rotations or use these solutions
to examine the two Skyrmion interactions. For a more accurate analysis, one has to
resort to numerical computations which can be time consuming (e.g. the full numerical
solutions for lower topological (or baryon number) $N \leq 17$ of Houghton et al [3]).

The purpose of this work is to find multi-skyrmion solutions analytically assuming
rational maps. For the sake of simplicity, we consider a class of models which is at
most of order six in derivative of the fields. Floratos and Piette [10] have indeed shown
that for $N \leq 5$, the solutions have the same symmetries as the pure Skyrme model
which are well represented by the rational map ansatz. The general character of the
calculation also allows to extract an exact solution for the $N = 1$ hedgehog solution
or for convenience in some cases, simplified approximations of the solutions for any
$N$. We proceed in three steps: First, we write the series expansion of the solution
near $r = 0$ and find the recursion relation for the coefficient of the series (Section III).
Second, an analogous analysis is performed in the limit $r \to \infty$. Finally, we propose a
parametrization for the full solution and use the former recursion relations to set the
parameters (Section IV). The calculations are somewhat intricate but once the relations
are found they can be easily implemented as a computer algorithm both in numerical
or in symbolic calculations. The method described here is also versatile in the sense
that one could propose a different or more appropriate parametrization and still use the
same recursion relations for $r \to 0, \infty$ to fix the parameters. We shall conclude with
comments on the advantages and limitations of the procedure and on various ways of
improving the convergence of the series (Section VI) where the chiral angle shows an
abrupt behavior (e.g. larger $N$).
2. Skyrme model

The Skyrme model \cite{1} is defined by the lagrangian:

\[
\mathcal{L} = -\frac{F_\pi^2}{16} \text{Tr} L_\mu L^\mu + \frac{1}{32e^2} \text{Tr} f_{\mu\nu} f^{\mu\nu}
\]  
\[ (2.1) \]

where \( L_\mu = U^{\dagger} \partial_\mu U \) and \( f_{\mu\nu} \equiv [L_\mu, L_\nu] \) with \( U = U(x) \) is an element of the \( SU(2) \) group. The first term, \( \mathcal{L}_1 \), corresponds to the lagrangian of the non-linear \( \sigma \)-model. This becomes obvious when one substitutes the degrees of freedom in \( U \) by \( \sigma \)- and \( \pi \)-fields using \( U = \frac{2}{F_\pi}(\sigma + i\tau \cdot \pi) \) where \( \tau \) are the three Pauli matrices. The second term, \( \mathcal{L}_2 \), contains four derivatives of the pion field and can account for nucleon-nucleon interactions via pion exchange. It was first introduced by Skyrme in order to prevent solitonic solutions arising in the non-linear \( \sigma \)-model from shrinking to zero and thus allowing for stable topological solitons.

The coefficients \( F_\pi \) and \( e \) are respectively the pion decay constant (186 MeV) and a dimensionless coupling often called the Skyrme parameter. Here, we shall use more appropriate units and rescale the lagrangian according to

\[
\mathcal{L} = \left( -\frac{1}{2} \text{Tr} L_\mu L^\mu \right) + \frac{1}{2} \left( \frac{1}{16} \text{Tr} f_{\mu\nu} f^{\mu\nu} \right)
\]  
\[ (2.2) \]

With this normalization, lengths can be understood as units of \( \frac{2\sqrt{2}}{eF_\pi} \) while energy or mass as units of \( \frac{F_\pi^2}{2\sqrt{2}e} \).

The lowest energy soliton was found by Skyrme himself and it takes the form

\[
U(r) = \exp \left[ i\tau \cdot \hat{r} F(r) \right]
\]  
\[ (2.3) \]

where \( F(r) \) is called the chiral angle or profile function of the solution, and \( \hat{r} \) is radial unit vector. Its spherical symmetry and the hairy configuration of the spin and isospin pointing out at infinity has earned it the name of hedgehog solution but more technically, such field configuration constitutes a map from physical space \( \mathbb{R}^3 \) onto the group manifold \( SU(2) \) and is assumed to go to the trivial vacuum for asymptotically large distances. The latter constraint allows imposing that \( U(r \to \infty) \to 1 \) from which one may derive the existence of a topological invariant associated with the mapping. The originality of Skyrme's idea was to identify this invariant, i.e. the winding number \( N \),

\[
N = \text{(factor)} \int \! d^3r B_0 \quad \text{with} \quad B_\mu = \epsilon_{\mu\nu\rho\sigma} L^\nu L^\rho L^\sigma
\]

where \( B_0 \) is the topological charge density, with the baryon number.
For the static hedgehog configuration (2.3), the energy density is the sum of

\[ E_1 = -\frac{1}{2} \text{Tr} L_i L^i = [2a + b] \tag{2.4} \]

\[ E_2 = -\frac{1}{16} \text{Tr} f_{ij} f^{ij} = a[a + 2b]. \tag{2.5} \]

with \( a \equiv \frac{\sin^2 F}{r^2} \) and \( b \equiv F'^2 \). Notice that although the second term in (2.2) is quartic in the derivatives of the pion field, the corresponding energy \( E_2 \) remains only quadratic in \( F' \).

Despite its relative simplicity, the Skyrme model has deep implications and is rather successful in describing low-energy hadron physics. Yet, it cannot be considered seriously as potential candidate for the full low-energy effective theory of QCD. For example, there is no compelling reason (other than simplicity) and certainly, no physical grounds to exclude higher-order derivatives in the pion field from the effective lagrangian. On the contrary, large \( N_c \) analysis suggests that the bosonization of QCD should involve an infinite number of mesons, which implies that in the decoupling limit (or large mass limit) for higher spin mesons that it leads to an all-orders lagrangian for pions. Several attempts to construct a more realistic effective lagrangian were made either by adding vector mesons to the Skyrme picture or higher-order derivatives terms to the lagrangian [6].

In the latter approach, one adds a sixth-order term involving the baryonic density \( B_\mu \) [7].

\[ \mathcal{L}_J = c_J \text{Tr} [B^\mu B_\mu] = 3a^2 b \quad \text{with} \quad B_\mu = \epsilon_{\mu\nu\rho\sigma} L^\nu L^\rho L^\sigma \]

where \( c_J \) is a constant. The term \( \mathcal{L}_3 = \frac{1}{32} \text{Tr} f_{\mu\nu} f^{\nu\lambda} f_{\lambda}^\mu \) proposed in [3] leads to an identical contribution to the static energy density,

\[ E_3 = 3a^2 b \tag{2.6} \]

Allowing for all-orders in derivatives of the pion fields, one can demonstrate that models can be constructed in terms of \( \mathcal{L}_1, \mathcal{L}_2 \) and \( \mathcal{L}_3 \) alone and that in a rather simple class of these models [4], the static energy density is a combination of the three invariants, \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_3 \), which remains quadratic in \( F' \) and thus tractable. Moreover the full lagrangian in this case can be easily written using a generating function [11].

The purpose of the present work is to find multi-skyrmion solutions analytically and, for the sake of simplicity, we shall only consider a subclass of these models, those that can be written as a linear combination of \( \mathcal{L}_1, \mathcal{L}_2 \) and \( \mathcal{L}_3 \) i.e. the most general sixth-order lagrangian. The static energy density for the \( N = 1 \) hedgehog ansatz in...
such a generalized model is simply given by

\[ \mathcal{E} = \sum_{m=1}^{3} h_m \mathcal{E}_m = 3 \chi(a) + (b - a) \chi'(a) \]

where \( h_m \) are coefficients and \( \chi(x) = \sum_{m=1}^{3} h_m x^m \). Integrating over volume leads to the mass of the soliton

\[ M_S = 4\pi \int_0^\infty r^2 dr [3\chi(a) + (b - a) \chi'(a)] \]

Using the same notation, the chiral equation becomes:

\[ 0 = \chi'(a) \left[ F'' + 2 \frac{F'}{r} - 2 \frac{\sin F \cos F}{r^2} \right] + a \chi''(a) \left[ -2 \frac{F'}{r} + F' \frac{\cos F}{\sin F} + \frac{\sin F \cos F}{r^2} \right] \]

with \( a = \frac{\sin^2 F}{r^2} \). The Skyrme lagrangian corresponds to the case \( \chi(a) = \chi_S(a) \equiv a + \frac{1}{2} a^2 \).

3. \( N > 1 \) Skyrmions and rational maps

For \( N > 1 \) Skyrmions, we shall assume that they are conveniently described by appropriate rational conformal transformations on the hedgehog solution as suggested by Houghton et al. [9]. This assertion has been verified numerically up to \( N = 5 \) by Floratos and Piette [10] where they find that the solutions exhibit the same symmetries as the pure Skyrme model.

The rational map ansatz lies on a naive connection between rational maps which are maps from \( S^2 \mapsto S^2 \) and Skyrmions which are maps from \( \mathbb{R}^3 \mapsto S^3 \). It is then possible to identify the domain \( S^2 \) with concentric spheres in \( \mathbb{R}^3 \), and the target \( S^2 \) with spheres of latitude on \( S^3 \). Amazingly, these are accurate to a few percent with respect to lowest energy solutions obtained through lengthy numerical calculations. Ioannidou et al. [12] later generalized the ansatz in the context of \( SU(N) \) Skyrme models using harmonic maps from \( S^2 \) into \( CP^{N-1} \).

Rational maps are usually described in terms of the complex coordinate \( z = \tan(\theta/2)e^{i\varphi} \) which correspond to stereographic projections on conventional polar coordinates \( \theta \) and \( \varphi \). The point \( z \) identifies a unit vector

\[ \hat{n}_z = \frac{1}{1 + |z|^2} (z + \bar{z}, z - \bar{z}, 1 - |z|^2). \]

A rational map is a conformal map defined by

\[ R(z) = \frac{p(z)}{q(z)} \]
where one of the polynomials $p(z)$ and $q(z)$ is at least of degree $N$. One can then associate a unit vector to the rational map $R(z)$ as follows:

$$\hat{n}_R = \frac{1}{1 + |R|^2}(R + \bar{R}, R - \bar{R}, 1 - |R|^2).$$

In the context of the Skyrme model, the rational map ansatz propose a solution of the form

$$U(r, z) = \exp(i \hat{n}_R \cdot \tau F(r))$$

which reduces to the hedgehog ansatz (2.3) when $N = 1$. With boundary conditions $F(0) = \pi$ and $F(\infty) = 0$, the baryon number turns out to be $N$, the degree of $R(z)$. Using this ansatz, the static energy density for this class of models takes the form

$$\mathcal{E} = \sum_{m=1}^{\infty} h_m a_N^{m-1} [3a_N + m(b_N - a_N)]$$

where now $b_N = F'' = b$ and $a_N = a \rho(z)$ with

$$\rho(z) = \left( \frac{1 + |z|^2}{1 + |R|^2} \right)^2 \left( \frac{dR}{dz} \right).$$

A remarkable advantage of the rational maps lies in the fact that the separation of the angular and radial dependence of the solution is preserved. Integrating over to obtain the mass of the soliton, we have

$$M_S = \int_0^\infty r^2 dr \int d\Omega \sum_{m=1}^{\infty} h_m (a \rho)^{m-1} [(3 - m)a\rho + mb]$$

where the integration over solid angle reads

$$\int d\Omega \to \int_{-\infty}^{\infty} \frac{2i \, dzd\bar{z}}{(1 + |z|^2)^2}$$

and the factor $2i \, dzd\bar{z}/(1 + |z|^2)^2$ is equivalent to the usual area element on a 2-sphere $\sin \theta d\theta d\phi$.

Defining

$$I_m^N = \int \frac{d\Omega}{4\pi} \rho^m = \frac{1}{4\pi} \int \left( \frac{1 + |z|^2}{1 + |R|^2} \right)^2 \frac{2i \, dzd\bar{z}}{(1 + |z|^2)^2}$$

and

$$\chi_1(a) = \sum_{m=1}^{\infty} \alpha_m a^m \quad \chi_2(a) = \sum_{m=1}^{\infty} \beta_m a^m \quad (3.1)$$

$$6$$
where \( \alpha_m = h_m I_m^N, \beta_m = h_m I_{m-1}^N \), we can write the expression for multi-Skyrmion masses

\[
M_S = 4\pi \int_0^\infty r^2 dr \left( 3\chi_1(a) - a\chi_1'(a) + b\chi_2'(a) \right)
\]

(3.2)

Some of the angular integrations are trivial for rational maps: \( I_0^N = 1, I_1^N = N, I_m^1 = 1 \). Of course, the solutions depend on the model, i.e. the weight of each terms in \( \chi_1 \) and \( \chi_2 \), or more precisely on the value of the each coefficient \( h_m \). This in turn determines which rational maps \( R \), and values of angular integrations \( I_m^N \), minimizes the mass.

From now on, we shall restrict our analysis to models of order six in derivatives of the pion field by setting coefficients \( h_m = 0 \) for \( m > 3 \). These models correspond to the most general case where the choice of rational maps affects a single non-trivial angular integration \( I_2^N \) and the expression for the mass simplifies according to

\[
\begin{align*}
\chi_1(a) &= h_1 N a + h_2 I_2^N a^2 \\
\chi_2(a) &= h_1 a + h_2 N a^2 + h_3 I_2^N a^3.
\end{align*}
\]

Using the Derrick arguments, a stable soliton exist for \( N = 1 \) only if \( h_3 \) is positive. This term prevents the Skyrmion from shrinking to zero size against a scale transformation. In this case, to find the minimal energy configuration at fixed \( N \) (fixed degree for \( R(z) \)) one proceeds as follows: (i) minimize \( I_2^N \) as a function of the coefficients of polynomials \( p(z) \) and \( q(z) \) and (ii) find the profile function \( F(r) \) which minimizes the energy. Note that since \( I_2^N \) is positive and can go up to infinity for \( N > 1 \), negative values \( h_2 \) are physically excluded, as minimization would lead to an infinitely large negative soliton mass. One then concludes that rational map configurations and symmetries which minimizes the mass for a general sixth-order model (positive \( h_1, h_2 \) and \( h_3 \)) are the same as for the Skyrme model since one must simply minimize \( I_2^N \) in both cases.

The chiral equation becomes accordingly:

\[
0 = h_1 \left( F'' + 2 \frac{F'}{r} \right) + 2N \left( -h_1 \left( \frac{\sin F \cos F}{r^2} \right) + h_2 \left( \frac{\sin^2 F}{r^2} \right) \left[ F'' + 2 \frac{F' \cos F}{\sin F} \right] \right) \\
+ I_2^N \left( -2h_2 \left( \frac{\sin^2 F}{r^2} \right) \left[ \frac{\sin F \cos F}{r^2} \right] + 3h_3 \left( \frac{\sin^2 F}{r^2} \right)^2 \left[ F'' + 2 \frac{F'}{r} \right] \right)
\]

(3.3)

Our goal is to obtain an analytic expression for the solution of the chiral equation. For this purpose, we write the chiral equation in a form that is more convenient using \( \phi = \cos F \):
\[0 = -2r^2 \chi'_1 (1 - \phi^2)^2 \phi + \chi''_1 (1 - \phi^2)^3 \phi \]  
\[+ \chi'_2 \left(-\phi'' (1 - \phi^2) r^2 - (\phi')^2 \phi r^2 - 2r (1 - \phi^2) \phi' \right) + \chi''_2 \left(2r (1 - \phi^2) \phi' + r^2 \phi (\phi')^2 \right) (1 - \phi^2)\]  
\[(3.4)\]

where \(\chi_i \equiv \chi_i \left(\frac{(1-\phi^2)}{r^2}\right)\).

We would like to represent its solution \(\phi(r)\) in the form of a series. Its explicit form is a matter of choice as long as it reproduces the solution adequately over all values of \(r\) which requires the convergence of the series. Otherwise, what choice of series is best would have to be judged on a number of criteria which have to be met at least minimally: (1) rapid convergence, i.e. fewer terms are necessary to reach numerical precision with respect to the exact solution, (2) mathematical tractability e.g. recursion formula for the coefficients of the series can be written and (3) other aesthetical criteria, e.g. a form of series leading to an energy density and other physically quantity that can be integrated analytically.

Solutions can be constructed simply by adding terms to approximate solutions proposed in the past [13, 14, 15, 16, 17, 18, 19]. However, a quick analysis reveals that some of these candidates must be discarded since they do not behave properly at small or large distances. In this work, we propose a solution which implements these constraints and a procedure to allow for the convergence of the series. However, the first step of our analysis will be independent of a specific choice of series representation for the solution. It consists in finding the small \((r \to 0)\) and large \((r \to \infty)\) distance behavior of the solution as a series expansion in powers \(r\) and \(1/r\) respectively. This is computed using the chiral equation. Once the small and large \(r\) expansion are known explicitly, the information will serve to propose an appropriate form for the series and ultimately to calculate the coefficients of the series.

### 4. Small distance behavior

First, let us write \(\phi\) as a power expansion in \(r\) in the limit \(r \to 0\). The chiral equation \((3.3)\) is symmetric with respect to the change \(r \to -r\), \(\phi(r) \to \phi(-r) = \phi(r)\), which simplifies the power expansion since only even powers of \(r\) contribute:

\[\phi(r) = \sum_{n=0}^{\infty} a_n r^{2n}\]  
\[(4.1)\]

with boundary condition \(a_0 = -1\) since \(\phi(0) = -1\).
For convenience, let us write explicitly the functions \( \chi_1 \) and \( \chi_2 \) in the chiral equation (3.7)

\[
0 = r^2 \phi \sum_{n=0}^{\infty} n(n-3) \alpha_n a^{n+1} + \phi (\phi')^2 \sum_{n=0}^{\infty} n(n-2) \beta_n a^{n-1} + 2 r \phi' \sum_{n=0}^{\infty} n(n-2) \beta_n a^n - r^2 \phi'' \sum_{n=0}^{\infty} n \beta_n a^n.
\]

Substituting \( \phi \) by the series (4.1) and \( \phi', \phi'', \phi (\phi')^2 \) and \( a^n \) by the appropriate expressions

\[
\phi (\phi')^2 = \sum_{m=0}^{\infty} b_m r^{2m} \quad a^n = \sum_{k=0}^{\infty} c_{n,k} r^{2k}
\]
in the last equation, we get

\[
0 = \sum_{n=1}^{3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ r^2 n(n-3) \alpha_n \left( c_{n+1,k} r^{2k} a_l r^{2l} \right) + n(n-2) \beta_n \left( c_{n-1,k} r^{2k} b_l r^{2l} \right) + 2 rn(n-2) \beta_n \left( 2 c_{n,k} r^{2k} a_{l+1} r^{2l-1} \right) - r^2 n \beta_n \left( 2 c_{n,k} r^{2k} (2l-1) a_{l+1} r^{2l-2} \right) \right].
\]

The coefficients \( b_k \) and \( c_{n,k} \) are given respectively by

\[
b_1 = 4 a_0 a_1^2 \quad b_k = \bar{b}_k + \Delta b_k \quad \text{for } k > 1
\]

\[
c_{0,0} = 1 \quad c_{0,k} = 0 \quad \text{for } k > 0
\]

\[
c_{n,0} = (-2 a_0 a_1)^n \quad \text{for } n > 0
\]

\[
c_{n,k} = \bar{c}_{n,k} + \Delta c_{n,k} \quad \text{for } n, k > 0
\]

with

\[
\bar{b}_k = 4 a_0 \sum_{q=2}^{k-1} q(k+1-q) a_q a_{k+1-q} + 4 \sum_{m=1}^{k-1} a_m \left( \sum_{q=1}^{k-m} q(k+1-m-q) a_q a_{k+1-m-q} \right)
\]

\[
\Delta b_k = 8 k a_0 a_1 a_k
\]

\[
\bar{c}_{n,k} = \frac{1}{2 k a_0 a_1} \left( \sum_{q=1}^{k-1} \left( q(n+1) - k \right) 2 a_0 a_{q+1} c_{n,k-q} + \sum_{q=1}^{k} \left( q(n+1) - k \right) \sum_{r=1}^{q} a_r a_{q+1-r} c_{n,k-q-1} \right)
\]

\[
\Delta c_{n,k} = -2^n n a_0 a_1^{n-1} a_{k+1}
\]

where for later convenience, we have isolated the contribution proportional to the coefficients \( a_k \) with highest \( k \) in the terms \( \Delta b_k \) and \( \Delta c_{n,k} \) respectively.
The equation holds for arbitrary $r$, which implies that it holds for any arbitrary power of $r$. Isolating the term in $r^{2(k-1)}$ we get an expression of the form

$$0 = A_{k-1} a_k + B_{k-1}.$$  \hspace{1cm} (4.3)

where $A_{k-1}$ and $B_{k-1}$ are respectively

$$A_{k-1} = \sum_{n=1}^{3} 2n (2a_1)^n \left(-\alpha_n (n-3) (n+1) + \beta_n (n^2 - 2n - 2 - 2k^2 + k)\right),$$

$$B_{k-1} = \sum_{n=1}^{3} \sum_{m=2}^{k-1} (n-3)\alpha_n \left(a_m c_{n+1,k-m-1} + a_0 \tilde{c}_{n+1,k-1} + a_1 c_{n+1,k-2}\right)$$

$$+ \beta_n ((n-2) \left(b_m c_{n-1,k-m-1} + \tilde{b}_k c_{n-1,0} + b_1 \tilde{c}_{n-1,k-1}\right)$$

$$+ 2m \left(2n - 3 - 2m\right) a_m c_{n,k-m} + 2(2n - 5) a_1 \tilde{c}_{n,k-1}).$$

From inspection we see that both $A_{k-1}$ and $B_{k-1}$ depend on lower-index coefficients $a_m$ (i.e. $m < k$) so we can write (4.3) in the form of a recursion formula for the coefficients $a_k$ for $k > 1$,

$$a_k = -\frac{B_{k-1}}{A_{k-1}} \quad \text{with} \quad a_0 = -1.$$  

The formula requires a single input parameter $a_1$ (which must be positive), aside from the boundary condition $a_0 = -1$. $a_1$ is related to the slope of the profile function at $r = 0$ as follows

$$F(r \to 0) = \pi - \sqrt{2}a_1 r + O(r^3) \quad \text{or} \quad \phi(r \to 0) = -1 + a_1 r^2 + O(r^4).$$

It is also easy to verify that $a_k$ depend on all $\alpha_n$’s and $\beta_n$’s whose values are prescribed by both the model and the topological number of the solution (see (3.1)). This has an interesting implication: The small $r$ behavior of the solution is characterized by all the terms in the lagrangian and not only the one with highest-order derivative in the pion field as one might have suspected.

5. Large distance behavior

Now let us examine $\phi$ as a power expansion in $1/r$ in the limit $r \to \infty$. Again since the chiral equation (3.5) is symmetric with respect to the change $r \to -r$, $\phi(r) \to \phi(-r) = \phi(r)$, the power expansion only contains even powers of $1/r$. Accordingly, we write

$$\phi = \sum_{m=0}^{\infty} \hat{a}_m r^{-2m}. \quad \hspace{1cm} (5.1)$$

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Allowing for the boundary condition \( \phi(\infty) = 1 \), we set \( \hat{a}_0 = 1 \). It is also easy to verify that
\[
\hat{a}_1 = 0 \quad \text{and} \quad \hat{a}_3 = 0.
\]

According to (5.1), we write the following expression as power series
\[
\phi(\phi')^2 = \sum_{m=0}^{\infty} b_m r^{-2m} \quad a^n = \sum_{m=0}^{\infty} c_{n,m} r^{-2m}
\]
where the coefficients \( \hat{b}_k \) and \( \hat{c}_{n,k} \) are given respectively by
\[
\hat{b}_5 = 16\hat{a}_0\hat{a}_2^2
\]
\[
\hat{b}_k = \hat{b}_k + \Delta \hat{b}_k \quad \text{for} \ k > 5
\]
\[
\hat{c}_{n,0} = 0
\]
\[
\hat{c}_{n,k} = 0 \quad \text{for} \ k < 3n
\]
\[
\hat{c}_{n,3n} = (-)^n
\]
\[
\hat{c}_{n,k} = \hat{c}_{n,k} + \Delta \hat{c}_{n,k} \quad \text{for} \ k > 3n
\]
with
\[
\hat{b}_k = 4\hat{a}_0 \sum_{l=3}^{k-4} l(k-1-l)\hat{a}_l \hat{a}_{k-1-l} + 4 \sum_{m=2}^{k-1} \hat{a}_m \sum_{l=2}^{k-3-m} l(k-1-m-l)\hat{a}_l \hat{a}_{k-1-m-l},
\]
\[
\Delta \hat{b}_k = 16(k-3)\hat{a}_0 \hat{a}_2 \hat{a}_{k-3}
\]
\[
\hat{c}_{n,k} = -n(-2\hat{a}_0 \hat{a}_2)^{n-1} \sum_{q=2}^{k-3n} \hat{a}_q \hat{a}_{k-3n+2-q} - \frac{1}{2\hat{a}_0 \hat{a}_2 (k-3n)} \sum_{q=1}^{k-3n-1} (q(n+1)-k+3n)\hat{c}_{1,q+3}\hat{c}_{n,k-q}
\]
\[
\Delta \hat{c}_{n,k} = -2\hat{a}_0 n(-2\hat{a}_0 \hat{a}_2)^{n-1} \hat{a}_{k-3n+2}.
\]

Here the contributions proportional to the coefficients \( a_k \) with highest \( k \) are written explicitly in \( \Delta \hat{b}_k \) and \( \Delta \hat{c}_{n,k} \) respectively.

Substituting \( \phi \) by the series (5.1) \( \phi', \phi'', \phi(\phi')^2 \) and \( a \) by the above expressions we arrive at
\[
0 = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left[ n(n-3)\alpha_n \hat{a}_m \hat{c}_{n+1,k-m} r^{-2k+2} + n(n-2)\beta_n \hat{b}_m \hat{c}_{n-1,k-m} r^{-2k} \right] (5.2)
\]
\[
-2n(n-2)\beta_n 2m \hat{a}_m \hat{c}_{n,k-m} r^{-2k} - \sum_{n=0}^{\infty} n\beta_n 2m(2m+1)\hat{a}_m \hat{c}_{n,k-m} r^{-2k} \right]. (5.3)
\]

Again, since the equation must hold for arbitrary \( r \), it implies that it holds for an arbitrary power of \( r \). Therefore isolating the term in \( r^{-2k+3} \) leads to the relation
\[
0 = \hat{A}_{k-1} \hat{a}_k + \hat{B}_{k-1} (5.4)
\]
with
\[ \hat{A}_{k-1} = 4\hat{a}_2 \left( -4\alpha_1 + \beta_1 \left( 2k^2 - 5k + 6 \right) \right) \]
\[ \hat{B}_{k-1} = -2\alpha_1 \left( \hat{a}_0 \hat{c}_{2,k+4} + \sum_{m=1}^{k-2} (\hat{a}_m \hat{c}_{2,k-m+4}) \right) \]
\[ - \beta_1 \left( 12\hat{a}_2 \hat{c}_{1,k+1} + \sum_{m=0}^{k+2} (\hat{b}_m \hat{c}_{0,k-m+3}) + \sum_{m=3}^{k-1} 2m (2m-1) \hat{a}_m \hat{c}_{1,k-m+3} \right) \]
\[ + \sum_{n=2}^{k+3} \sum_{m=0}^{n} \left( n(n-3)\alpha_n \hat{a}_m \hat{c}_{n+1,k-m+4} + n(n-2)\beta_n \hat{b}_m \hat{c}_{n-1,k-m+3} - 2n (2n - 3 + 2m) m\beta_n \hat{a}_m \hat{c}_{n,k-m+3} \right). \]

After inspection, we find that the right hand side of equation (5.4) is trivially zero for \( k < 3 \) and but otherwise contains only terms in \( a_m \) with \( m \leq k \). Furthermore, \( \hat{A}_{k-1} \) and \( \hat{B}_{k-1} \) only depend on lower-index coefficients \( a_m \) (\( m \leq k-1 \)). This allows writing a recursion formula for the coefficients \( \hat{a}_k \) for \( k \geq 3 \),
\[ \hat{a}_k = -\frac{\hat{B}_{k-1}}{\hat{A}_{k-1}}. \]

Recalling that \( \hat{a}_0 = 1, \hat{a}_1 = 0 \) and \( \hat{a}_3 = 0 \), the only remaining unknown parameter is \( \hat{a}_2 \) which must be negative. This latter parameter fixes the dominant contribution to the profile function in the large \( r \) limit,
\[ F(r \to \infty) = \frac{\sqrt{-2\hat{a}_2}}{r^2} + O\left( \frac{1}{r^6} \right) \text{ or } \phi(r \to \infty) = 1 + \frac{\hat{a}_2}{r^4} + \frac{\hat{a}_4}{r^8} + \ldots \]

Moreover, the first coefficients \( \hat{a}_k \) in the large \( r \) expansion depend only on the lowest-\( n \) \( \alpha_n \)'s and \( \beta_n \)'s. This means that the large-\( r \) behavior is not very sensitive to the model and to the topological sector of the solution. In other words, higher-order derivative terms and the topological sector begin to contribute only when the sub-dominant terms in \( r^{-2k} \) become important.

6. Solutions for Skymions

We have found recursion relations for the coefficients of a series expansion in powers \( r \) and \( 1/r \) for small (\( r \to 0 \)) and large (\( r \to \infty \)) distances respectively. We may now proceed to construct a full solution. The explicit form of the series is to a certain extent a matter of choice. Of course, it should represent the solution adequately which
means that convergence is a prerequisite and that small \((r \to 0)\) and large \((r \to \infty)\) distance behaviors must be reproduced. Other criteria may also be suggested such as: (1) faster convergence, i.e. fewer terms are necessary to reach numerical precision with respect to the exact solution, (2) mathematical tractability e.g. recursion formula for the coefficients of the series can be written, (3) the energy density can be integrated analytically, etc...

We propose here a solution and a procedure similar to that in [17] but which applies to \(\phi = \cos F\) instead of \(F\). Taking in account the small \((r \to 0)\) and large \((r \to \infty)\) distance behavior, we can write recursion relations for the coefficients of the series. We shall see that this particular ansatz also allows in principle, to calculate the mass of the soliton analytically which was not possible in [17]. The solution has the form

\[
\phi_{r_0}(r) = \phi\left(\frac{r}{r_0}\right) = 1 + \sum_{m=0}^{\infty} \frac{(r^2)^m (r_0^2)^{m+2}}{(r^2 + r_0^2)^{2m+3}} \left(c_{2m}(r^2 + r_0^2) + c_{2m+1}r_0^2\right) \quad (6.1)
\]

The small distance expansion is given by

\[
\phi_{r_0}(r) = 1 + (c_0 + c_1) + \sum_{q=0}^{\infty} \frac{(r^2)^{2q+2}}{r_0^{2q+2}} \left((q + 3)(q + 2)(-1)^{q+1} \frac{(q + 3)(q + 2)}{2} \right) (c_0 + c_1)
\]

\[
+ \sum_{k=0}^{q} \frac{(2q - k + 2)!(-1)^k}{k!(2q - 2k + 4)!} ((2q - 2k + 4)(2q - 2k + 3)c_{2q-2k}
\]

\[
+ (2q - k + 4)(2q - k + 3) (c_{2q-2k+2} + c_{2q-2k+3}))
\]

\[
= (1 + c_0 + c_1) + (-2c_0 - 3c_1 + c_2 + c_3) \left(\frac{r^2}{r_0^2}\right)^2 + (3c_0 + 6c_1 - 4c_2 - 5c_3 + c_4 + c_5) \left(\frac{r^2}{r_0^2}\right)^4 + ...
\]

Matching the coefficients at small distance with those of (4.3), we get

\[
a_0 = 1 + c_0 + c_1 = -1
\]

and for \(k \geq 1\)

\[
(r_0^2)^k a_k = \frac{(k + 2)(k + 1)(-1)^k}{2} (a_0 - 1) + c_{2k-2} + c_{2k} + c_{2k+1} + \sum_{r=1}^{k-1} \frac{(2k - r)!(-1)^r}{r!(2k - 2r + 2)!} \cdot ((2k - 2r + 2)(2k - 2r + 1)c_{2k-2r-2} + (2k - r + 2)(2k - r + 1) (c_{2k-2r} + c_{2k-2r+1})).
\]

We invert the relation to get the odd-index coefficients and obtain

\[
c_1 = a_0 - 1 - c_0 = -2 - c_0 \quad (6.2)
\]
and for $k \geq 1$

$$c_{2k+1} = (r_0^2)^k a_k - c_{2k} - c_{2k-2} - \frac{(k+2)(k+1)(-1)^k}{2} (a_0 - 1) \quad (6.3)$$

$$- \sum_{r=1}^{k-1} \frac{(2k-r)!(-1)^r}{r!(2k-2r+2)!}((2k-2r+2)(2k-2r+1)c_{2k-2r-2} + (2k-r+2)(2k-r+1)(c_{2k-2r} + c_{2k-2r+1})). \quad (6.4)$$

On the other hand, the large distance behavior reads

$$\phi_{r0}(r \to \infty) = 1 + c_0 \left( \frac{r^2}{r_0^2} \right)^{-2} + \sum_{q=0}^{\infty} \left( \frac{r^2}{r_0^2} \right)^{-q-3} \cdot \left( c_{2q+2} + \sum_{k=0}^{q} \frac{(2q-k+2)!(-1)^k}{(k+1)!(2q-2k+2)!} (2(-q+k-1)c_{2q-2k} + (k+1)c_{2q-2k+1}) \right). \quad (6.5)$$

Again by inspection

$$-2c_0 + c_1 + c_2 = 0 \quad \text{or} \quad c_2 = 2c_0 - c_1 = 3c_0 + 2$$

and matching the coefficients at large distance, we get the even-index coefficients of the solution which then reads

$$c_{2k} = \hat{a}_{k+2} \left( \frac{r_0^2}{r^2} \right)^{-k-2} - \sum_{r=0}^{k-1} \frac{(2k-r)!(-1)^r}{(r+1)!(2k-2r)!} (2(-k+r)c_{2k-2r-2} + (r+1)c_{2k-2r-1}). \quad (6.6)$$

Inserting (6.2), (6.4) and (6.6) in (6.1), we end up with a solution $\phi_{r0}$ which is now completely determined by three parameters, $a_1$, $\hat{a}_2$ and $r_0$. The first two parameters $a_1$, $\hat{a}_2$ depend on the behavior near $r \to 0$ and $r \to \infty$ respectively whereas $r_0$ can be interpreted as an intermediate scale. The strategy adopted here to reach the solution lies upon the fact that for a given $k$, the even-index coefficients $c_{2k}$ in (6.6) depend on lower-index coefficients $c_m$ with $m < 2k$ and $\hat{a}_{k+2}$ which is fixed by the $(k+2)^{th}$ coefficient of the large $r$ expansion or ultimately $\hat{a}_2$. On the other hand, the odd-index coefficients $c_{2k+1}$ in (6.4) requires $c_m$ with $m < 2k + 1$ and $a_k$ which is determined by the behavior of solution at small $r$ or the parameter $a_1$. This has two effects: First, the procedure refines the solution by alternatively the matching of the odd- and even-index coefficients, which requires higher and higher derivatives of the solution at $r \to 0$ and $r \to \infty$ respectively (6.1). Secondly, since it relies on matching the coefficients $a_k$ and $\hat{a}_{k+2}$, the accuracy of the series and convergence must improve near the end points as $k$ increases.
It remains that the parameters $a_1$, $\hat{a}_2$ and $r_0$ are still unknown at this point of the procedure. Several approaches to find these parameters are possible. One such procedure consists in introducing three more constraints on $\phi r_0$. For example, we can use the chiral equation (3.5) at three intermediate points to set the values of $a_1$, $\hat{a}_2$ and $r_0$. We could also choose to impose a continuity condition at a given $r = r_0$ assuming that the series (4.1) and (5.1) apply to $r \leq r_0$ and $r \geq r_0$ respectively. This procedure would lead to a solution which has the virtue of being completely determined by the equation of motion, but they do not guarantee that the set of parameters would be the best to render the exact solution. Since these methods would require numerical calculations at some point anyhow, we adopt a more practical approach which consists of finding numerically the set of values for $a_1$, $\hat{a}_2$ and $r_0$ which minimize the mass of the soliton.

In principle, the series (5.1) contains an infinite number of terms which allow in the end to reach the exact solution. For computational reason the series is truncated which means that, the number of terms in the series will affect not only the precision of the series but also the values of the parameters $a_1$, $\hat{a}_2$ and $r_0$ that minimize the mass of the soliton. The larger number of terms in the series, the closer we get to the exact solution and values of the parameters $a_1$, $\hat{a}_2$ and $r_0$.

We could attempt to compute the mass of the soliton analytically. Starting from (3.2) we have

$$M_S = 4\pi \int_0^1 2u^{\frac{3}{2}} (1 - u)^{\frac{3}{2}} du \left( h_1 (2Na + b) + h_2 a (I_2^N a + 2Nb) + 3h_3 I_2^N a^2 b \right)$$

where we changed variable $r$ to $u = \frac{r^2}{r^2 + r_0^2}$ or $r = r_0 \sqrt{\frac{u}{1-u}}$. Substituting $\phi$ by our solution

$$\phi r_0 (u) = 1 + \sum_{k=0}^{K} u^k (1 - u)^{k+2} (c_{2k} + c_{2k+1} (1 - u))$$

where $K$ is finite but sufficiently large for accuracy, and assuming that we can expand the integrand in powers $u$ and $(1 - u)$, it becomes possible to integrate the expression analytically since all integrals can be cast in the form of

$$M_{ab} = \int_0^1 u^{a-1} (1 - u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \quad \text{for Re}(a), \text{Re}(b) > 0.$$

Unfortunately, even for the simplest case, the Skyrme model, the calculation becomes inefficient and quite impractical as $K$ increases. So we resort to numerical integration which in this case is much faster and proceed to minimize the mass in terms of the
parameters $a_1, \hat{a}_2$ and $r_0$. As an example, we show the result of the first few values of $K$ for the Skyrme model, for $N = 1$, which requires $h_1 = h_2, h_3 = 0$ and $I_1^1 = I_2^1 = 1$

|      | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | Num. |
|------|---------|---------|---------|---------|------|
| Mass | 1.23268 | 1.23174 | 1.23151 | 1.23151 | 1.23145 |
| $a_1$| 2.17741 | 2.05754 | 2.02667 | 1.99411 | 2.01508 |
| $\hat{a}_2$ | $-2.39428$ | $-1.62650$ | $-1.22111$ | $-3.15231$ | $-2.33204$ |
| $r_0$ | 1.05398 | 1.12095 | 1.18501 | 0.939329 | $-$ |

The exact mass is obtained within an accuracy of 0.005% for $K$ as low as 3. Note however that the convergence of $a_1$ and $\hat{a}_2$ towards their exact values is not as efficient.

7. Advantages and limitations of the approach

An extensive comparison of the approximate solution (6.7) with respect to the exact numerical solution suggests that the reliability of the approach depends largely on the model considered and topological sector. The recursion formulae for the $c_k$ coefficients ensure a perfect agreement in both limits $r \to 0$ and $r \to \infty$. The agreement is still preserved when the solution remains smooth for all values of $r$, as in the case of low $N$ soliton in the Skyrme Model. Nevertheless, discrepancies between analytical and numerical results begin to appear when we consider sixth-order models or when the topological number $N$ increases. The solutions in these cases are characterized by a sharp behavior of $F$ near $F(r) = \pi/2$ (or $\phi$ near $\phi(r) = 0$).

In order to understand the origin of these discrepancies, it is instructive to look at the quantity $a = r^{-2} \sin^2 F$ for small, large and intermediate $r$. For the intermediate region where $\sin^2 F$ reaches its maximum, the energy density is found to be the largest and $F$ is almost linear. On the other hand, $F''$ and of course $\cos F$ are relatively small. Therefore, which term dominates the chiral equation (3.3) in that region depends on the relative weight of the coefficients $h_1, 2Nh_1, 2Nh_2, I_2^N h_2$ and $I_2^N h_3$ and on the highest power of $a$. As $N$ increases, $I_2^N$ increases approximately as $N^2$ and the term proportional to $h_3 I_2^N$ is expected to dominate if $h_2$ and $h_3$ are of the order same of magnitude. This implies that most of the variation of $F$ should occur in the intermediate region since the dominant term is proportional to $a^2$ which is suppressed outside this region. This leads to a configuration of the energy density which is localized on a shell of decreasing thickness as $N$ increases. Unfortunately, the recursion formulae for the $c_k$ coefficients only apply for the end regions $r \to 0$ and $r \to \infty$ which do not contribute much to the soliton mass. So we can expect the approximate solution (6.7) to lose its accuracy when $N$ gets larger. In other word, although accuracy is expected
to increase in the $r \to 0$ and $r \to \infty$ region, it does not improve in the intermediate region where most of the energy is concentrated.

Various ways of improving the analytical solution may be considered. For instance, several other trial functions have been used in conjunction with the small and large $r$ recursion formulae, including polynomials series and Padé approximants, but we found no general form that would eliminate the aforementioned discrepancies. In fact, there is no reason a priori why any trial function should evade the conclusion of the last paragraph. A better improvement would fix the behavior of the profile angle and describe accurately the energy density in the intermediate region where its contribution is more important for any $N$. This could proceed through the construction of a recursion formula at an intermediate point and eventually its implementation into a series but, it implies rather complex intricate calculations and has not been attempted yet.

Even though imperfect, we stress again that this approach is still probably the best alternative to complete numerical treatment, and that it can prove very useful whenever an analytical form of the solution is required, as we have already discussed regarding stability analysis. Because of the general character of the calculation, it easy find the exact solution for the $N = 1$ hedgehog solution for the pure Skyrme model. It is also possible — if one chooses not use the full solution — to extract approximate solutions for any $N$ since the first few terms of the series represent fairly good approximations. Moreover, this work also provide the first construction of the recursion formulae for the series expansion in the $r \to 0$ and $r \to \infty$ limits which may be helpful in many calculations.

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