Blowing-up solutions for time-fractional equations on a bounded domain

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Abstract
This paper proposes initial-boundary value problems for time-fractional analogs of Kuramoto-Sivashinsky, Korpusov-Pletner-Sveshnikov, Cahn-Allen, and Hoff equations due to a bounded domain. Adequate conditions for the blowing-up of solutions in limited time of previously mentioned conditions are displayed. The Pohozhaev nonlinear capacity strategy is considered. Illustrative examples are given for each of the investigated equations.

Keywords
Caputo derivative, time-fractional, generalized Kuramoto-Sivashinsky equation, Korpusov-Pletner-Sveshnikov equation, Cahn-Allen equation, Hoff equation, blow-up

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Introduction
Fractional calculus works on the powers of the differential equations that are not numbers such that most wonders in science and designing are communicated by partial differential equations. Fractional calculus was initiated as a pure mathematical aspect in the middle of the 19th century.\textsuperscript{1,2} The concept of fractional or non-integer order derivation and integration can be followed back to the beginning of numbers order calculus itself.\textsuperscript{3}

For the most part, physical marvel might depend on its current state and on its chronicled states, which can be displayed effectively by applying the hypothesis of derivatives and integrals of fractional order.\textsuperscript{4} Due to this, several analytical techniques are used to derive exact, explicit, and numerical solutions of nonlinear fractional partial differential equations, where the modeling of physical phenomena is very interest to many scientists and researchers up to now.\textsuperscript{5}

With these achievements, we study time-fractional equations for funding blowing-up solutions by using

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the Pohozhaev nonlinear capacity method; more absolutely, on the choice of test functions agreeing to initial and boundary conditions beneath thought for the time-fractional equations. In 2021, Alsaedi et al. gave a simple case of the analysis of a rough blow-up, that is, the case where the solution tends to infinity as \( s \to \tau \) on \([0, \infty)\), more exactly, when the integral

\[
\int_{0}^{\infty} z(v, s) \Phi(v) \, dv,
\]

tends to infinity as \( s \to \tau \) for the given function \( \Phi \). The aim of this paper is to study the blowing-up solutions of the following the time-fractional equations:

TF1. Time-fractional generalized Kuramoto-Sivashinsky equation

\[
\frac{\partial}{\partial 0, s} z + z z_{v} + \xi_{1} z_{vv} + \xi_{2} z_{vvv} + \xi_{3} z_{vvvv} = 0;
\]

TF2. Time-fractional Korpusov-Pletner-Sveshnikov equation

\[
\frac{\partial}{\partial 0, s} (z + z_{vv}) + \xi_{2} z_{vv} + \frac{\xi_{3}}{2} (z^{2})_{vv} = 0;
\]

TF3. Time-fractional Cahn-Allen equation

\[
\frac{\partial}{\partial 0, s} (z + z_{vv}) - z_{vv} + z^{3} - z = 0;
\]

TF4. Time-fractional Hoff equation

\[
\frac{\partial}{\partial 0, s} (z + z_{vv}) = \xi_{2} z + \xi_{3} z^{3};
\]

for \( v \in J = (0, \infty) \), \( s > 0 \), here \( \xi_{i} \) (\( i = 1, 2, 3 \)), non-zero constants with initial conditions

\[
z(v, 0) = z_{0}(v), \quad v \in [0, \infty),
\]

where \( z_{0} \) is a given function. This work is devoted to blowing-up solutions of time-fractional analogues of the above equations. The approach to the problem is based on the Pohozhaev nonlinear capacity method; more precisely, on the choice of test functions according to initial and boundary conditions under consideration. The solutions for nonlinear partial differential equations has attracted a large number of researchers, as many papers have emerged around this study (see Refs. 12–15). We donate a straightforward case of the investigation of a harsh blow-up, that is, the case where the solution tends to infinity as \( s \to \tau \) on \( J \) more precisely, for the given function \( \Phi \), the integral

\[
\int_{0}^{\infty} z(v, s) \Phi(v) \, dv,
\]

which tends to infinity as \( s \to \tau \). The rest of the paper can be outlined concisely as follows: “Preliminaries” section contains some definitions and properties of fractional order integral and differential operators that will be used later. In the “Blowing-up solutions of the time-fractional equations” section the Pohozhaev nonlinear capacity method has been applied to above equation and illustrates the obtained results by some examples at the end of each section.

**Preliminaries**

For real-valued function \( v \in L^{1}(J) \), the fractional integral of Riemann–Liouville is defined by

\[
\mathcal{I}^{\xi}_{\tau_{1}, \tau_{2}} v(s) = (v * K_{\xi})(s) = \int_{\tau_{1}}^{s} \frac{(s - \eta)^{\xi - 1}}{\Gamma(\xi)} v(\eta) \, d\eta, \quad (\xi > \tau_{2}),
\]

where \( J = \tau_{1}, \tau_{2} \),

\[
K_{\xi}(s) = \frac{\xi^{\xi - 1}}{\Gamma(\xi)},
\]

\( \Gamma \) is the Euler gamma function and “*” is the convolution operation. The Sobolev space is defined for the function \( v \) by (Kilbas et al. 16)

\[
W^{m, 1}(J) = \left\{ v \in L^{1}(J) : \frac{d^{m}v}{ds^{m}} \in L^{1}(J) \right\}.
\]

Let \( v * K_{\xi} \in W^{m, 1}(J) \) and \( m = [\xi] + 1, \xi > 0 \). The Riemann-Liouville fractional derivative \( D^{\xi}_{\tau_{1}} \) of order \( \xi > 0(m - 1 < \xi < m, m \in \mathbb{N}) \) is defined as

\[
D^{\xi}_{\tau_{1}} v(s) = \frac{d^{m}v}{ds^{m}} \mathcal{I}^{m - \xi}_{\tau_{1}} v(s) = \frac{d^{m}}{ds^{m}} \int_{\tau_{1}}^{s} \frac{(s - \eta)^{m - 1 - \xi}}{\Gamma(m - \xi)} v(\eta) \, d\eta.
\]

The Caputo fractional derivative \( D^{\xi}_{\tau_{1}} \) of order \( \xi \in \mathbb{R}(m - 1 < \xi < m, m \in \mathbb{N}) \) is defined as

\[
D^{\xi}_{\tau_{1}} v(s) = \mathcal{I}^{m - \xi}_{\tau_{1}} v(\eta) \frac{d^{m}}{d\eta^{m}} \left[ v(s - \tau_{1}) - v'(\tau_{1}) - \frac{v(\tau_{1})}{1!} - \ldots - \frac{v^{(m - 1)}(\tau_{1})}{(m - 1)!} \right].
\]

If \( v \in C^{m}(J) \), then the Caputo fractional derivative \( D^{\xi}_{\tau_{1}} \) of order \( \xi \in \mathbb{R} \) is defined as

\[
D^{\xi}_{\tau_{1}} v(s) = \mathcal{I}^{m - \xi}_{\tau_{1}} v^{(m)}(s) = \int_{\tau_{1}}^{s} \frac{(\xi - \eta)^{m - 1 - \xi}}{\Gamma(m - \xi)} v^{(m)}(\eta) \, d\eta.
\]

**Proposition 2.1.** Let \( 0 < \xi \leq 1 \) and \( v \in C([0, \tau]), v' \in L^{1}([0, \tau]) \). Then

\[
\int_{0}^{\infty} z(v, s) \Phi(v) \, dv,
\]
\[ 2s(s)\partial^\alpha_{\theta, 0} v(s) \geq \partial^\alpha_{\theta, 0} v^2(s), \quad \forall s \in (0, \tau]. \quad (8) \]

Let a given function \( z \) be monotone. We consider the FDE

\[
\begin{cases}
\partial^\alpha_{\theta, 0} z(s) = z^2(s), & 0 < \xi < 1, s > 0, \\
z(0) = z_0 \in \mathbb{R}.
\end{cases}
\quad (9)
\]

Theorem 2.2 assure the blow-up of solutions to (9).

**Theorem 2.2.** The solution of problem (9) blows-up in a finite time\(^{18}\)

\[
\left( \frac{\Gamma(\xi + 1)}{4z_0} \right)^{\frac{1}{\xi}} \leq \tau \leq \left( \frac{\Gamma(\xi + 1)}{z_0} \right)^{\frac{1}{\xi}},
\quad (10)
\]

is \( \lim_{\tau \to +\infty} z(\tau) = +\infty \), whenever \( z_0 > 0 \).

**Blowing-up solutions of the time-fractional equations**

**The time-fractional generalized Kuramoto-Sivashinsky equation**

In this section we consider the time-fractional generalized Kuramoto-Sivashinsky equation (1) where \( \xi_1, \xi_2, \xi_3 \) are the parameters and \( 0 < \xi \leq 1 \) is the fractional order. The problem (1) is also called KdV-Burgers-Kuramoto-Sivashinsky equation with time-fractional derivative as it may be a generalization of the taking after well-known equations. The equation (1) coincides with

- The classical fractional Burgers equation take the form

\[
z_s + \xi_1 z_{ss} = \xi_2 z_{sss},
\quad (11)
\]

whenever \( \xi_1 > 0 \) and \( \xi_2 = \xi_3 = 0 \) in (1);

- The classical fractional KdV equation take the form

\[
z_s + \xi_1 z_{ss} + z_{vvv} = 0,
\quad (12)
\]

whenever \( \xi_1 = \xi_3 = 0 \) and \( \xi_2 = 1 \) in (1);

- The classical fractional Kuramoto-Sivashinsky equation take the form

\[
z_s + \xi_2 z_{ss} + z_{ss} + z_{vvv} = 0,
\quad (13)
\]

whenever \( \xi_2 = 0, \xi_1 = \xi_3 = 1 \) in (1), and \( \zeta = 1 \). We investigate the question of the blow-up of a classical solution of problem (1). Assume that \( z \in C^1((0, \tau]), \) Using integrating by parts from multiplication the time-fractional generalized Kuramoto-Sivashinsky equation (1) and \( \varphi \), we get

\[
\frac{\partial z}{\partial x} + \frac{z}{s(s)} \int_0^\infty z(v, s)\varphi(v) dv
\]

\[
= \int_0^\infty z(v, s)(\xi_2\varphi''(v) - \xi_1\varphi''(v) - \xi_3\varphi''(v)) dv,
\quad (14)
\]

where

\[
\Phi(s) = \mathcal{H}(z(x), \varphi(x)) - \mathcal{H}(z(0, s), \varphi(0)),
\]

and \( \forall \eta \in \mathbb{R} \), \( \eta \in (0, \tau] \).

\[
\mathcal{H}(z(v, s), \varphi(v)) = -0.5z^2(v, s)\varphi(v) - \xi_1 z_s(v, s)\varphi(v) + \xi_1 z(v, s)\varphi'(v)
\]

\[
- \xi_2 z_v(v, s)\varphi(v) + \xi_2 z_v(v, s)\varphi'(v) - \xi_2 z_s(v, s)\varphi''(v)
\]

\[
- \xi_3 z_{vv}(v, s)\varphi(v) + \xi_3 z_v(v, s)\varphi'(v) - \xi_3 z_s(v, s)\varphi''(v).
\quad (15)
\]

Consider monotonically nondecreasing function \( \varphi(v) \), and satisfy the following properties

\[
\left\{ \begin{array}{l}
\vartheta_1 : = 0.5 \int_0^\infty \left( \frac{\xi_2\varphi''(v) - \xi_1\varphi'(v)}{\varphi(v)} \right)^2 dv < \infty \\
\vartheta_2 : = 2 \int_0^\infty \frac{\varphi(v)^2}{\varphi(v)} dv < \infty.
\end{array} \quad (16) \right.
\]

Then we have

\[
2 \int_0^\infty z(v, s)(\xi_2\varphi''(v) - \xi_1\varphi''(v) - \xi_3\varphi''(v)) dv
\]

\[
+ \int_0^\infty z^2(v, s)\varphi'(v) dv = \int_0^\infty \left( \gamma\varphi''(v) - \xi_1\varphi''(v) - \xi_3\varphi''(v) \right)^2 \varphi(v) dv,
\]

where

\[
z(v, s) = z(v, s) + \frac{(\xi_2\varphi''(v) - \xi_1\varphi''(v) - \xi_3\varphi''(v))}{\varphi(v)}.
\]

The Hölder inequality implies that

\[
\left( \int_0^\infty z(v, s) dv \right)^2 \leq \int_0^\infty z^2(v, s) dv \int_0^\infty \left( \frac{\varphi(v)}{\varphi(v)} \right)^2 dv.
\]

Then, expression (14) takes the form

\[
\frac{\partial z}{\partial x} + \frac{z}{s(s)} \int_0^\infty z(v, s)\varphi(v) dv
\]

\[
= \int_0^\infty z(v, s)(\xi_2\varphi''(v) - \xi_1\varphi''(v) - \xi_3\varphi''(v)) dv + \Phi(s) - \vartheta_1,
\]

where \( \Phi(s) = \int_0^\infty z(v, s)\varphi(v) dv \) and
\[ \Phi(s) = H(z(x, s), \varphi(x)) - H(z(0, s), \varphi(0)). \]

**Theorem 3.1.** Let \( z_0(v) \in L^1(\mathcal{J}) \), the function \( \psi \) satisfy conditions (16) and the solution \( z \) of the equation (1) belongs to \( C_{1,2}^{1,4}(\mathcal{J}) \times (0, \tau) \). If \( \Phi(s) - 1 \rightarrow 0 \), for all \( s > 0 \), and \( \mathcal{W}(0) > 0 \), then \( \mathcal{W}(s) \rightarrow +\infty, \forall s \rightarrow \tau \), where \( \tau \) satisfies estimate (10).

**Proof.** Obviously

\[ \partial_{+}^{\delta_1,0,s} \mathcal{W}(s) \geq \mathcal{W}^2(s), \]

where \( \mathcal{W}(s) = \partial_2 \mathcal{W}(s) \). Since the function \( \mathcal{W}(s) \) is an upper solution of equation (9), so \( \mathcal{W}(s) \rightarrow +\infty \) for \( s \rightarrow \tau \), where inequality (10) holds for \( \tau \). Whereupon \( \mathcal{W}(s) \rightarrow +\infty \) for \( s \rightarrow \tau \).

**Example 3.2.** (Fractional Burgers Equation) Let \( \xi_2 = \xi_1 = 0 \) in problem (1) on \([0, 1]\) and the solution of problem (1) satisfy the boundary conditions

\[
\begin{align*}
\{ & z(0, s) = 0, \quad s \geq 0 \\
& z(1, s) = z(1, s) - z(1, s) - 0.5 v z'(1, s) = 0, \quad s \geq 0
\end{align*}
\]

Then, if \( \psi(v) = 2v - 1 \), we obtain \( \delta_1 := 0, \delta_2 := \frac{1}{2} \) and \( \Phi(s) = \delta_1 = 0, \forall s \geq 0 \). Hence it follows from Theorem 3.1 that the solution of problem (1) blows up in finite time under the condition

\[
\int_{0}^{1} z_0(v) v^2 dv > 0.
\]

**Example 3.3.** (Fractional KdV Equation) Consider the problem (1) with \( \xi_1 = \xi_3 = 0, \xi_2 = 1 \) on \([0, 1]\) equipped via boundary conditions

\[
\begin{align*}
\{ & z(0, s) = 0, \quad s \geq 0 \\
& z(1, s) = 0, \quad s \geq 0 \\
& z_r(1, s) = z_r(0, s) + z_r(1, s), \quad s \geq 0
\end{align*}
\]

Then, if \( \psi(v) = \frac{1}{2} v^2 \), we obtain \( \delta_1 := 0, \delta_2 := \frac{1}{2} \) and \( \Phi(s) = \delta_1 = 0, \forall s \geq 0 \). Indeed, Theorem 3.1 implies that the solution of problem (1) blows up in finite time under the condition \( \int_{0}^{1} z_0(v) v^2 dv > 0 \).

**Example 3.4.** (Fractional Kuramoto-Sivashinsky Equation) Consider the problem (1) with \( \xi_2 = 0 \) and \( \xi_1 = \xi_3 = 1 \) on \([0, 1]\) via boundary conditions:

\[
\begin{align*}
\{ & z(0, s) = 0, \quad s \geq 0 \\
& z(1, s) = 0, \quad s \geq 0 \\
& z_r(1, s) = z_r(0, s) + z_r(1, s), \quad s \geq 0
\end{align*}
\]

Letting \( \psi(v) = v \), we obtain \( \delta_1 := 0, \delta_2 := \frac{1}{2} \) and \( \Phi(s) = \delta_1 = 0, \forall s \geq 0 \). So, Theorem 3.1 holds and the solution of problem (1) blows up in finite time under the condition

\[
\int_{0}^{1} z_0(v) v^2 dv > 0.
\]

The time-fractional Korpusov-Pletner-Sveshnikov equation

In this section we consider the time-fractional Korpusov-Pletner-Sveshnikov equation (2), here, \( \xi_1, \xi_2 \) are the parameters and \( \xi \) is the fractional order with \( 0 < \xi < 1 \). The equation (2) is called the Korpusov-Pletner-Sveshnikov equation with time-fractional derivative as it is a generalization of the following well-known equations. We study the question of the blow-up of a classical solution \( z \in C_{1,2}^{1,2}(\mathcal{J}) \) and the solution \( z \in C_{1,\xi}^{1,2}(\mathcal{J}) \times (0, \tau) \) of the problem (2) exists. Multiplying the time-fractional Korpusov-Pletner-Sveshnikov equation (2) by \( \psi \) and integrating by parts, we obtain

\[
\begin{align*}
\partial_{+}^{\delta_1,0,s} z(v, s)(\varphi(v) + \varphi'(v)) &= - \int_{0}^{\infty} z(v, z)(\xi_1 \varphi''(v)) dv \\
&- 0.5 \int_{0}^{\infty} z^2(v, s)(\xi_2 \varphi'(v)) dv + \Phi(s),
\end{align*}
\]

where

\[
\Phi(s) = H(z(x, s), \varphi(x)) - H(z(0, s), \varphi(0)),
\]

and

\[
H(z(v, s), \varphi(v)) = - \partial_{+}^{\delta_1,0,s} z_r(0, s) \varphi(v) + \partial_{+}^{\delta_1,0,s} z_r(0, s) \varphi'(v) \\
- \beta z_r(0, s) \varphi(v) + \xi_1 z_r(0, s) \varphi'(v) \\
- 0.5 \xi_2 (z_r')^2 + 0.5 \xi_2 z_r^2(v, s) \varphi'(v).
\]

Consider monotonically nondecreasing function \( \varphi(v) \) with

\[
\xi_2 \varphi'(v) \geq 0, \quad \forall v \in \mathcal{J},
\]

and following properties

\[
\begin{align*}
\partial_1 := 0.5 \int_{0}^{\infty} \frac{(\xi_2 \varphi'(v))^2}{(\xi_2 \varphi'(v))} dv < \infty \\
\partial_2 := 2 \int_{0}^{\infty} \frac{(\xi_2 \varphi'(v))^2}{(\xi_2 \varphi'(v))} dv < \infty.
\end{align*}
\]

Then we have
\[ 2 \int_{0}^{\infty} z(v, s)(\xi_1 \varphi''(v)) \, dv + \int_{0}^{\infty} \tilde{z}^2(v, s)(\xi_2 \varphi'(v)) \, dv = \int_{0}^{\infty} \tilde{z}^2(v, s)(\xi_2 \varphi'(v)) \, dv - \int_{0}^{\infty} \left( \frac{\xi_2 \varphi''(v)}{\xi_2} \right) \, dv, \]

where \( z(v, s) = z(v, s) + \frac{\xi_1}{\xi_2} \). Form the Hölder inequality, we get
\[
\left( \int_{0}^{\infty} z(v, s)(\varphi(v) + \varphi''(v)) \, dv \right)^2 \leq \int_{0}^{\infty} \tilde{z}^2(v, s)(\xi_2 \varphi'(v)) \, dv \times \int_{0}^{\infty} (\varphi(v) + \varphi''(v))^2 \, dv. \]

Then, expression (17) takes the form
\[
\vartheta_{+0,v} W(s) \geq \vartheta_{2}^{-1} W^2(s) + \Phi(s) - \vartheta_1, \quad (19) \]

where
\[ W(s) = \int_{0}^{\infty} \tilde{z}^2(v, s)(\varphi(v) + \varphi''(v)) \, dv \]
and
\[ \Phi(s) = H(z(v, s), \varphi(v)) - H(z(0, s), \varphi(0)). \]

**Theorem 3.5.** Let \( z_0(v) \in L^1_1(\mathbb{R}^+) \), the function \( \varphi \) satisfy conditions (18) and the solution \( z \) of the equation (2) belongs to \( C^{1,2}_r(\mathbb{R}^+ \times [0, \tau]) \). If \( \Phi(s) \geq 0, \quad \forall s > 0 \), and \( W(0) > 0 \), then \( W(s) \to + \infty \) as \( s \to \tau \), where \( \tau \) satisfies estimate (10).

**Proof.** Obviously \( \vartheta_{+0,v} W(s) \geq \vartheta_{2}^{-1} W^2(s) \), where \( W(s) = \vartheta_{2} W(s) \). Since the function \( W(t) \) is an upper solution of equation (9), therefore \( W(s) \to + \infty \) as \( s \to \tau \), where estimate (10) holds for \( \tau \). Whereupon \( W(s) \to + \infty \) as \( s \to \tau \).

**Example 3.6.** (Fractional Korpusov-Pletner-Sveshnikov Equation) Let \( \xi_1 = \sqrt{2} \), \( \xi_2 = 1 \) and consider problem (2) on \( [0, 1] \) with Dirichlet type boundary conditions
\[
\begin{align*}
\begin{cases}
  z(0, s) = 0, & s \geq 0, \\
  z(1, s) = 0, & s \geq 0, \\
  \vartheta_{+0,s} z(0, s) - \vartheta_{+0,s} z(0, s) = v(s), & s \geq 0.
\end{cases}
\end{align*}
\]
Suppose that \( v(s) \geq \frac{1}{s} \), for all \( s > 0 \). Then, if \( \varphi(v) = \frac{1}{12} v^2 \), we obtain \( \vartheta_1 : = \frac{1}{2}, \quad \vartheta_2 : = \frac{1}{10} \) and
\[ \Phi(s) - \vartheta_1 = v(s) - \frac{1}{6} \geq 0, \quad \forall s > 0. \]
So, from Theorem (3.5), we conclude that the solution of the problem (2) blows up in finite time via
\[
\int_{0}^{1} z_0(v)(v - 1) \, dv > 0.5.
\]

**The time-fractional Cahn-Allen equation**

In this section we consider the time-fractional Cahn-Allen equation (3), where \( \zeta \) is the fractional order with \( 0 < \zeta \leq 1 \). The equation (3) is called the Cahn-Allen equation with time-fractional derivative as it is a generalization of the following well-known equations. We investigate the question of the blow-up of a classical solution \( z \in C^{1,2}_r(\mathbb{R}^+ \times [0, \tau]) \) of problem (3). Assume that \( \varphi \in C^2(\mathbb{R}) \) and the solution \( z \in C^{1,2}_r(\mathbb{R}^+ \times [0, \tau]) \) of problem (3) exists. Multiplying the time-fractional Cahn-Allen equation (3) by \( z_0 \), we have
\[ \vartheta_{+0,v} W(s) = \vartheta_{2}^{-1} W^2(s) + \Phi(s) - \vartheta_1, \quad (19) \]

where
\[ \Phi(s) = H(z(v, s), \varphi(v)) - H(z(0, s), \varphi(0)). \]

Then we have
\[ -2 \int_{0}^{\infty} z^2(v, s) \varphi'(v) \, dv + \int_{0}^{\infty} z^2(v, s) \varphi'(v) \, dv = \int_{0}^{\infty} \tilde{z}^2(v, s) \varphi'(v) \, dv - \int_{0}^{\infty} \varphi'(v) \, dv, \]

where \( H(z(v, s), \varphi(v)) = -0.5 \vartheta_{+0,v} z^2(v, s) \varphi(v) + 0.5 z^2(v, s) \varphi(v) -0.25 U^2(v, s) \varphi(v) + 0.5 z^2(v, s) \varphi(v) \).

Consider the monotonically nondecreasing function \( v \), such that \( \varphi'(v) \equiv 0, \quad \forall v \in \mathbb{R} \), and satisfy the following properties
\[
\begin{align*}
\begin{cases}
  \vartheta_1 : = 0.5 \int_{0}^{1} \varphi'(v) \, dv < \infty, \\
  \vartheta_2 : = 0.5 \int_{0}^{1} \varphi'(v) \, dv < \infty.
\end{cases}
\end{align*}
\]

Then we have
\[
-2 \int_{0}^{\infty} z^2(v, s) \varphi'(v) \, dv + \int_{0}^{\infty} z^2(v, s) \varphi'(v) \, dv = \int_{0}^{\infty} \tilde{z}^2(v, s) \varphi'(v) \, dv - \int_{0}^{\infty} \varphi'(v) \, dv,
\]

where \( \bar{z}(v, s) = z(v, s) - 1 \). By employing the Hölder inequality, we have
\[
\left( \int_{0}^{1} z^2(v, s) (0.5 \varphi'(v)) \, dv \right)^2 \leq \int_{0}^{1} z^2(v, s) \varphi'(v) \, dv \int_{0}^{1} (0.5 \varphi'(v))^2 \, dv,
\]

and
\[
\vartheta_{+0,v} W(s) = \vartheta_{2}^{-1} W^2(s) + \varphi'(v) \equiv 0.5 \vartheta_{+0,v} \left( \tilde{z}^2(v, s) \varphi(v) \right), \quad (22)
\]
we also get
Then, expression (20) takes the form
\[
\dot{\mathcal{W}}_{t} + \mathcal{W}(s) \geq \dot{\mathcal{W}}_{s}^{-1}\mathcal{W}^{2}(s) + \Phi(s) - \tilde{\theta}_{1},
\]
where
\[
\mathcal{W}(s) = \int_{0}^{s} \tilde{z}(v, s)(\varphi(v) + \varphi''(v)) \, dv,
\]
and
\[
\Phi(s) = \mathcal{H}(s(x, s), \varphi(s)) - \mathcal{H}(z(0, s), \varphi(0)).
\]

**Theorem 3.7.** Let \(z_{0}(v) \in L^{1}(\tilde{\gamma})\), the function \(\varphi\) satisfy condition (21) and the solution \(z\) of the equation (3) belongs to \(C_{1,v}^{1,2}(\tilde{\gamma} \times [0, \tau])\). If
\[
\Phi(s) - \dot{\theta}_{1} = 0, \quad \forall s > 0,
\]
and \(\mathcal{W}(0) > 0\), then \(\mathcal{W}(s) \to + \infty\) as \(s \to \tau\), where \(\tau\) satisfies estimate (10).

**Proof.** Obviously \(\dot{\mathcal{W}}_{t} + \mathcal{W}(s) \geq \dot{\mathcal{W}}_{s}^{-1}\mathcal{W}^{2}(s)\) where \(\dot{\mathcal{W}}(s) = \dot{\theta}_{1}\mathcal{W}(s)\). In the other hand the function \(\mathcal{W}(s)\) is an upper solution of equation (9), therefore \(\mathcal{W}(s) \to + \infty\), \(\forall s \to \tau\), where \(\tau\) satisfies estimate (10). Whereupon \(\mathcal{W}(s) \to + \infty\) as \(s \to \tau\).

**Example 3.8.** (Fractional Cahn-Allen Equation)
Consider the problem (3) on \([0, 1]\) under nonlocal dynamical boundary conditions
\[
\begin{align*}
\tilde{z}(1, s) &= 0, \\
-0.5\dot{\mathcal{W}}_{t} + \mathcal{W}(s) + 0.5\dot{\mathcal{W}}_{s}(v, s)(v) &= 0.5, \quad \forall s \geq 0.
\end{align*}
\]
Letting \((v) = \nu\), we obtain \(\dot{\theta}_{1} = \frac{1}{2}, \quad \dot{\theta}_{2} = \frac{1}{2}\) and
\[
\Phi(s) - \dot{\theta}_{1} = 0, \quad (\forall s > 0).
\]
Therefore Theorem 3.7 implies that the solution of problem (3) blows up in finite time via the condition
\[
\int_{0}^{1} z_{0}(v) \nu \, dv > 0.5.
\]

**The time-fractional Hoff equation**
In this section we consider the time-fractional Hoff equation where \(\xi\) is the fractional order with \(0 < \xi \leq 1\).
We investigate the question of the blow-up of a classical solution \(z \in C_{1,v}^{1,2}(\tilde{\gamma} \times [0, \tau])\) of the problem (4). Assume that \(\varphi \in C^{2}(\tilde{\gamma})\) and the solution \(z \in C_{1,v}^{1,2}(\tilde{\gamma} \times [0, \tau])\) of the problem (4) exists. Multiplying the time-fractional Hoff equation (4) by \(\varphi\) and integrating by parts, we have
\[
-0.5 \int_{0}^{\tilde{z}} \tilde{z}(v, s) \varphi'(v) \, dv.
\]

Then, expression (20) takes the form
\[
\dot{\mathcal{W}}_{t} + \mathcal{W}(s) \geq \dot{\mathcal{W}}_{s}^{-1}\mathcal{W}^{2}(s) + \Phi(s) - \dot{\theta}_{1},
\]
where
\[
\mathcal{W}(s) = \int_{0}^{s} \tilde{z}(v, s)(\varphi(v) + \varphi''(v)) \, dv,
\]
and
\[
\Phi(s) = \mathcal{H}(s(x, s), \varphi(s)) - \mathcal{H}(z(0, s), \varphi(0)).
\]

Consider monotonically nondecreasing function \(\varphi(v)\), such that \(\varphi(v) > 0, \forall v \in \tilde{\gamma}\) and satisfy the following properties
\[
\left\{ \begin{array}{l}
\dot{\theta}_{1} := 0.5 \int_{0}^{\tau} E_{\xi}(v) \, dv < \infty \\
\dot{\theta}_{2} := -2 \int_{0}^{\tau} \left( \frac{\varphi(v) + \varphi''(v)}{E_{\xi}(v)} \right) \, dv < \infty.
\end{array} \right.
\]

Then we have
\[
0.5 \int_{0}^{\tau} \tilde{z}^{2}(v, s)(\xi_{1}\varphi(v)) \, dv + 0.25 \int_{0}^{\tau} \tilde{z}^{4}(v, s)(\xi_{2}\varphi(v)) \, dv
\]
\[
= 0.25 \int_{0}^{\tau} \tilde{z}^{4}(v, s)(\xi_{2}\varphi(v)) \, dv
\]
\[
-0.25 \int_{0}^{\tau} \left( \xi_{1}\varphi(v) \right)^{2} \, dv,
\]
where \(\tilde{z}(v, s) = z(v, s) + \frac{\xi_{1}}{E_{\xi}(v)}\). The Hölder inequality implies that the estimate
\[
\left( \int_{0}^{\tau} \tilde{z}^{2}(v, s)(\varphi(v) + \varphi''(v)) \, dv \right)^{2} \leq \int_{0}^{\tau} \tilde{z}^{2}(v, s)(\xi_{2}\varphi(v)) \, dv
\]
\[
\times \int_{0}^{\tau} \left( \frac{\varphi(v) + \varphi''(v)}{E_{\xi}(v)} \right) \, dv.
\]

Then, expression (25) takes the form
\[
\dot{\mathcal{W}}_{t} + \mathcal{W}(s) \geq \dot{\mathcal{W}}_{s}^{-1}\mathcal{W}^{2}(s) + \Phi(s) - \dot{\theta}_{1},
\]
where
\[
\mathcal{W}(s) = \int_{0}^{s} \tilde{z}(v, s)(\varphi(v) + \varphi''(v)) \, dv,
\]
and
\[
\Phi(s) = \mathcal{H}(z(x, s), \varphi(s)) - \mathcal{H}(z(0, s), \varphi(0)).
\]

**Theorem 3.9.** Let \(z_{0}(v) \in L^{1}(\tilde{\gamma})\) and the solution \(z\) of the equation (4) is such that
\[
z \in C_{1,v}^{1,2}(\tilde{\gamma} \times [0, \tau]),
\]
and let the function $\phi$ satisfy conditions (26). If $\Phi(s) - \vartheta_1 > 0, (\forall s > 0)$, and $W(0) > 0$, then $W(s) \to + \infty$ for each $s \to \tau$, where $\tau$ satisfies estimate (10).

**Proof.** Obviously $\bar{W}_t(s) \geq W_2(s)$ where $\bar{W}(s) = \vartheta_1 W(s)$. Since the function $\bar{W}(s)$ is an upper solution of equation (9), therefore $\bar{W}(s) \to + \infty$ for $W \to \tau$, where $\tau$ satisfies estimate (10). Whereupon $W(s) \to + \infty$ for $s \to \tau$.

**Example 3.10.** (Fractional Hoff Equation) Let consider the problem (4) on $[0, 1]$ under nonlocal dynamical boundary conditions

$$
\begin{align*}
&z(1, s) = 0, \\
&\vartheta_1 z_0(v, s) \phi(v) - \vartheta_2 z_0(v, s) \phi'(v), \quad s \geq 0,
\end{align*}
$$

Letting $\phi(v) = \frac{1}{2} v$, we obtain

$$
\vartheta_1 : = - \frac{1}{16}, \quad \vartheta_2 : = \frac{1}{6},
$$

and $\Phi(s) - 1 = 0, (\forall s > 0)$. Thus, Theorem 3.9 implies that the solution of the problem (4) blows up in finite time under the condition

$$
\int_0^1 z_0(v) v dv > \frac{1}{16}.
$$

**Conclusion**

We donate a straightforward case of the investigation of a harsh blow-up, that is, the case where the solution tends to infinity as $s \to \tau$ on $\bar{\mathcal{J}}$ more precisely, when for the given function $\Phi$, the integral (6) tends to infinity as $s \to \tau$.

**Author contributions**

AB: Actualization, formal analysis, methodology, initial draft, validation, and investigation. MKAK: Methodology, actualization, validation, investigation, formal analysis, and initial draft. MB: Methodology, actualization, validation, investigation, formal analysis, and initial draft. MES: Validation, actualization, formal analysis, methodology, investigation, simulation, initial draft, software, and was a major contributor in writing the manuscript. XGY: Methodology, actualization, validation, investigation, formal analysis, and initial draft. All authors read and approved the final manuscript.

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