Perturbative Derivation of Mirror Symmetry

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Abstract

We provide a purely perturbative (one loop) derivation of mirror symmetry for supersymmetric sigma models in two dimensions.
1. Introduction

In [1] a proof of mirror symmetry was provided in terms of equivalence of linear sigma models and certain Landau-Ginsburg theories. In the proof non-perturbative physics, i.e. generation of superpotential by vortices, played a crucial role. In this brief note we provide a perturbative derivation of mirror symmetry, which can be viewed as a purely perturbative reinterpretation of the derivation of [1].

The main motivation to revisit the proof of mirror symmetry is based on the recent results in [2], where it was shown that instanton effects in massive $\mathcal{N} = 1$ supersymmetric theories in four dimensions can be evaluated in perturbation theory (see also the followup work [3,4]). Moreover it was shown that this allows one to derive non-perturbative S-dualities for 4d supersymmetric field theories from a perturbative perspective. This leads naturally to the question of whether one can also derive mirror symmetry for two dimensional supersymmetric sigma models, which involves non-perturbative physics, using only perturbative techniques. We will show that this is indeed the case.

2. Perturbative Derivation of Mirror Symmetry

Consider, for example, a $d = 2$ supersymmetric sigma model for a non-compact toric manifold which can be realized as an $\mathcal{N} = (2,2)$ linear sigma model with one abelian vector multiplet and $n$ chiral fields charged under it [5]. The arguments below easily generalize to the case with more than one $U(1)$ gauge group. The local geometry is specified by the charges of chiral fields $Q_i$, $i = 1, \ldots n$ and the FI term $t$. Provided that charges sum up to zero $\sum_i Q_i = 0$ the theory is expected to flow to a conformal theory corresponding to a Calabi-Yau target geometry. In such a case, the Higgs-branch theory is a non-compact Calabi-Yau manifold $X$ with $h_{1,1}(X) = 1$, and Kahler class proportional to $t$. We will consider, as in [1], mirror symmetry in a more generalized sense which includes non-Ricci flat target geometries as well.

It is well known that the quantum cohomology ring for supersymmetric sigma models on Kahler manifolds with $c_1 > 0$ can be understood from purely perturbative perspective. For example for $\mathbb{C} P^n$, which can be realized as a $U(1)$ linear sigma model with $n + 1$ charged fields, integrating out the charged fields leads to a one loop generation of the superpotential [6]

$$W = \Sigma (\log \Sigma^{n+1} - (n + 1)) + t\Sigma$$
where $\Sigma$ is the twisted chiral superfield containing the gauge field. This also leads to the quantum cohomology ring

$$dW = 0 \rightarrow \Sigma^{n+1} = e^{-t},$$

as is well known, where $\Sigma$ plays the role of the Kahler class. Note that as long as $\Sigma \neq 0$ the charged fields pick up a mass $M_i \sim Q_i \Sigma$ and can be integrated out. The fact that extremization of the superpotential leads to a non-vanishing value of $\Sigma$ makes integrating out the charged fields a self-consistent scheme. The generalization of the above superpotential and quantum cohomology to other $c_1 > 0$ Kahler manifolds has been studied in [7].

For the conformal case, where $\sum_i Q_i = 0$ integrating out the charged fields would not be completely justified, as $\Sigma = 0$ is not dynamically ruled out and thus this does not give a complete self-consistent description of the theory. However, as discussed in [11] it is also natural to consider twisted mass deformations of this theory given by weakly gauging the global symmetries while freezing the corresponding vector multiplet scalars to fixed non-zero values (for earlier work on these deformations see [8]). Now, integrating out the massive charged fields is justified and the exact twisted superpotential is generated at one loop for the dynamical vector multiplet $\Sigma$. For simplicity of notation define

$$\Sigma_i = Q_i \Sigma + m_i, \quad (2.1)$$

where $m_i$ is the twisted mass of the charged fields. The superpotential is, up to a constant, [6],

$$W(\Sigma) = \sum_i \Sigma_i (\log \Sigma_i - 1) + t_i \Sigma_i, \quad (2.2)$$

where $t_i = t/nQ_i$. The path integral (which we present only schematically)

$$Z(m, t) = \int D\Sigma \ e^{W(\Sigma)} \quad (2.3)$$

can be rewritten in terms of integral over $\Sigma_i$ with a delta function that freezes the non-dynamical part: this should constrain $\Sigma_i$ to satisfy $n - 1$ linear relations that imply (2.1)

$$\sum_i R^A_i \Sigma_i = m^A, \quad (2.4)$$

That is, $R$’s are orthogonal to $Q$’s

$$\sum_i R^A_i Q_i = 0,$$
for \( A = 1, \ldots, n - 1 \) and the \( n - 1 \) physical mass terms are given by \( m^A \). The constraints (2.4) can be imposed by introducing new twisted chiral fields \( Y_A \) in the theory, playing the role of Lagrange multipliers for the constraints. These fields are such that if we integrate them out we get back the original theory with the same superpotential. In particular we consider the action

\[
Z(m, t) = \int \prod_{i,A} D\Sigma_i DY_A \ e^{\Sigma_i (\log \Sigma_i - 1) + t_i \Sigma_i - \sum_{A,i} Y_A (R^A_i \Sigma_i - m_A)},
\]

which as far as the F-terms are concerned imposes the above constraints. We are not interested in the potential deformations of the D-terms, and only keep track of the F-terms. In particular there would also be D-terms involving the \( Y_A \) fields which do not affect the F-terms and we have suppressed them in the above expression. It is important to notice that \( Y_A \) are \( \mathbb{C}^* \) valued, i.e.

\[
Y_A \sim Y_A + 2i\pi.
\]

This follows from the observation made in [3,9] about the nature of the chiral field \( \Sigma \). Recall that the top component of \( \Sigma \) contains the field strength of the \( U(1) \) vector field, \( \Sigma = \ldots + \theta^+ \bar{\theta}^{-} F \). If we consider the theory on a compact Riemann surface, then \( F \) is quantized, and consequently Lagrange multiplier enforcing (2.4) on \( F \) must be periodic with period \( 2\pi \). From the coupling \( \int d\theta^+ d\bar{\theta}^{-} Y\Sigma = \text{Im}(Y)F \) in the superpotential we conclude that it is the imaginary part of \( Y \) that is periodic, and integrating over \( Y \)’s above, we recover the original formulation of the theory, (2.3).

The path integral over \( \Sigma_i \) can be done and localizes on \( \partial \Sigma_i W = 0 \)

\[
\log(\Sigma_i) + t_i + \sum_A R^A_i Y_A = 0.
\]

This gives

\[
Z(m, t) = \int \prod_A D Y_A \ e^{-\sum_i e^{-Y_i} - \sum_A Y_A m_A},
\]

where we defined

\[
Y_i = t_i + \sum_A R^A_i Y_A.
\]

This Landau-Ginsburg theory is the known result for the mirror of a massive deformation of the local A-model theory [1].
This generalizes to the derivation of mirrors of other local and compact models. As explained in [1] mirror symmetry in the case of compact manifolds is closely related to the non-compact case (see also [7]). As there are no new ingredients, we refer the reader to [1] for detailed discussion of mirror symmetry in the compact case.

In fact in [1], by T-dualizing the charge fields to $Y_A$, it was noticed that in massive cases, the theory formulated in terms of $(Y_A, \Sigma)$ fields, gives rise to the expected super-potential in terms of $\Sigma$ fields by integrating out the $Y_A$ fields. In the derivation we have presented here the $Y_A$ play the role of Lagrange multiplier fields. However it is not too difficult to show that the insertion of the square of the charge fields $|\Phi_i|^2$ is equivalent to the insertion of $ReY_i$ which is compatible with the derivation of [1]. In this sense, the derivation above is not new, but it provides a novel perturbative perspective.

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