The Voter Basis and the Admissibility of Tree Characters

Andrew Beveridge¹ · Ian Calaway²

Received: 29 August 2018 / Accepted: 20 January 2021 / Published online: 12 February 2021
© The Author(s), under exclusive licence to Springer Nature B.V. part of Springer Nature 2021

Abstract
Let ⪰ be a total order on the power set of a finite set [n]. A subset S ⊆ [n] is separable when for any X, Y ⊆ S and any Z ⊆ [n] − S, the ordering of X and Y is the same as the ordering of X ∪ Z and Y ∪ Z. The character of a preference order is the collection of all separable subsets. Motivated by questions in the theories of voting, marketing and social choice, the admissibility problem asks which collections C ⊆ ℘([n]) can arise as characters of preference orders. We introduce a linear algebraic technique to construct preference orders. Each vector in our 2ⁿ-dimensional voter basis induces a simple preference preorder (where ties are allowed) with nice separability properties. Given any collection C ⊆ ℘([n]) that contains both ∅ and [n], and such that all pairs of subsets are either nested or disjoint, we use the voter basis to construct a preference order with character C.

Keywords Preference order · Separability · Admissibility · Poset · Referendum elections

1 Introduction
Ranking sets of alternatives has received widespread attention in the social sciences [1]. For economists, interdependent consumer preferences provide insight into which goods are complements or substitutes. Such information helps vendors to choose inventory, or to design marketing materials and store layouts that encourage cluster purchasing of interrelated items [17]. Meanwhile, understanding the implications of preference interdependencies is critical in social choice theory [9, 14]. Indeed, interrelated preferences can result in problematic outcomes for referendum elections. A voter must cast their votes for multiple simultaneous proposals, so they are forced to guess the overall outcome when expressing their preferences. This encourages strategic voting, rather than expressing true
preferences. We contribute to the study of the admissibility problem \cite{12}, which seeks to characterize the achievable patterns of ranking interdependencies.

**Definition 1** A preference preorder \( \succeq \) is a total preorder (reflexive, transitive, connected) on \( \mathcal{P}([n]) \). A preference order \( \succeq \) is a total order (antisymmetric, reflexive, transitive, connected) on \( \mathcal{P}([n]) \).

A preference preorder allows for ties between distinct subsets of \([n]\). A preference order is a preference preorder with no ties. We view a preference preorder as a ranking of all possible outcomes of a decision processes. The relation \( A \succ B \) means that we prefer outcome \( A \) to outcome \( B \). In a preference preorder, \( A \sim B \) means that we are indifferent between the two (that is, \( A \succeq B \) and \( B \succeq A \)). In economics, the ground set \([n] = \{1, 2, \ldots, n\}\) could be a set of goods available in a store, where the outcome \( A \) corresponds to a consumer’s purchases on a particular shopping excursion. In social choice theory, the ground set could be a set of proposals in a referendum election, where the outcome \( A \) corresponds to the “yes” votes of a given voter.

Researchers with interests in the social sciences commonly replace the power set \( \mathcal{P}([n]) \) with \( X[n] = \mathbb{Z}_2^n \) in the canonical way: \( A \subset [n] \) corresponds to the binary word \( x = x_1 x_2 \cdots x_n \) where \( x_i = 1 \) when \( i \in A \), and \( x_i = 0 \) otherwise. For example, the preference order

\[
\{1, 2\} \succ \{1\} \succ \emptyset \succ \{2\},
\]

can be written in bit string notation as

\[
11 \succ 10 \succ 00 \succ 01.
\]  (1)

This formulation is particularly suited for studying multiple-criteria binary decision processes. There are a few advantages to using bit strings in the study of separability. First, we can distinguish between a collection \( S \) of criteria and an outcome \( x_S \) for the criteria in \( S \). For example, in an election, the set \( S \) represents a referendum slate, while the bit string \( x_S \) represents a voter’s cast ballot. Second, this notation naturally extends to criteria with more than two choices by replacing bit strings with words over the appropriate alphabet.

**Definition 2** The bit string \( x = x_1 x_2 \cdots x_n \in X[n] \) is an outcome for \([n]\). For \( S \subset [n] \), a partial outcome \( x_S \) is a bit string whose entries are indexed by \( S \). We define \( X_S \) to be the set of all partial outcomes on \( S \), and we use \( 0_S \) and \( 1_S \) to denote the all-zero and all-one partial outcome on set \( S \), respectively. Finally, we employ concatenation \( x = x_S x_{\sim S} \) to partition a bit string, allowing ourselves to reorder the indices as convenient.

Here are two situations where the preference order of Eq. 1 corresponds to a reasonable individual’s preferences. First, consider a two-item shopping trip for burgers and buns. The ranking corresponds to the preference order

burgers and buns \( \succ \) only burgers \( \succ \) neither \( \succ \) only buns.

Second, this preference order could reflect a voter’s preference for the outcome of a city referendum election, where the first proposal is whether to sponsor a new professional sports team and the second proposal is whether to build a new stadium. This voter’s least preferred outcome would be to build a new stadium without bringing a team to play there.

We now formulate our notion of dependence and independence among outcomes. For sets \( S, T \subset [n] \), let \( T \setminus S = \{i : i \in T \text{ and } i \notin S\} \) denote the relative complement of \( S \) in \( T \). When \( T = [n] \), we use \( -S \) to denote the complement of \( S \).
**Definition 3** Let \( \succeq \) be a preference preorder. The set \( S \subset [n] \) is \( \succeq \)-separable when for every \( X, Y \subset S \) and every \( Z \subset -S \),

\[
X \succeq Y \iff X \cup Z \succeq Y \cup Z.
\]

(2)

Otherwise, the set \( S \) is \( \succeq \)-nonseparable. When the preference order is clear, we use the terms separable and nonseparable for brevity.

Another advantage to bit string notation emerges when we reformulate this separability condition. The set \( S \) is \( \succeq \)-separable when for every \( x_S, y_S \) and \( z_{-S} \),

\[
x_S z_{-S} \succeq y_S z_{-S} \iff x_S 0_{-S} \succeq y_S 0_{-S}.
\]

(3)

In other words, a subset \( S \subset [n] \) is separable with respect to preference order \( \succeq \) when the ranking of partial outcomes on \( S \) is independent of the partial outcomes on \(-S\).

The benefit of condition (3) over condition (2) is the explicit appearance of the set \( S \): we have replaced the subsets \( X \subset S \), \( Y \subset S \) and \( Z \subset -S \) of equation with the partial outcomes \( x_S, y_S \) and \( z_{-S} \). This will be helpful because the technical arguments in Section 3 will partition outcomes into multiple blocks. Using a notation that explicitly captures that partition will be essential. From here forward, we embrace the differentiation between sets and outcomes by exclusively using “outcome” to refer to a bit string.

Note that \( \emptyset \) and \([n]\) are vacuously separable for any preference preorder. Returning to our burgers-and-buns preference order (1), we observe that burgers are separable for the shopper’s preference. Indeed, conditioning on the two possible outcomes for the second item (buns), we have \( 11 \succ 01 \) and \( 10 \succ 00 \), which means that regardless of whether the store is out of buns, the shopper prefers buying burgers over not buying burgers. Meanwhile, buns are not separable for this preference order. Conditioning on the outcome for first item (burgers), we have \( 11 \succ 10 \) and \( 00 \succ 01 \). If burgers are in stock, then she prefers to buy buns. However, if she cannot buy burgers, then her bun preference flips: she would prefer buying nothing over buying buns alone.

In practice, nonseparable preferences can be problematic. Brams et al. [4] showed that nonseparable preferences can lead to an election paradox where no voter’s ballot matches the final outcome. Lacy and Niou [14] went further to show that the final outcome could be every voter’s least favored result. Given their potential consequences, a mathematical understanding of the complexities of separable preferences is in order.

### 1.1 The Admissibility Problem

We turn to the study of separable and nonseparable sets.

**Definition 4** Let \( \succeq \) be a preference preorder. The collection

\[
\text{char}(\succeq) = \{ S \subset [n] : S \text{ is } \succeq \text{-separable} \}
\]

(4)

is called the character of \( \succeq \). When \( \text{char}(\succeq) = P([n]) \), we say that \( \succeq \) is completely separable, and when \( \text{char}(\succeq) = \{ \emptyset, [n] \} \), we say that \( \succeq \) is completely nonseparable.

For example, the character for the preference order in Eq. 1 is

\[
\text{char}(\succeq) = \{ \emptyset, \{1\}, \{1, 2\} \}.
\]

Both completely separable and completely nonseparable preference orders of \( P([n]) \) have been constructed for arbitrary \( n \). In particular, Hodge and TerHaar [12] showed that as \( n \rightarrow \)
the probability that a randomly chosen preference order is completely nonseparable tends to 1.

Completely separable preference orders appear in the literature under various names [19]. Indeed, when every subset of \([n]\) is separable, we have a preference order that satisfies de Finetti’s axiom [7], namely that

\[ A \preceq B \iff A \cup C \preceq B \cup C \text{ when } (A \cup B) \cap C = \emptyset. \]

Maclagan referred to orders satisfying de Finetti’s axiom as boolean term orders and studied their combinatorial and geometric properties [15]. In probability theory (where they enjoy applications in economics), they are known as comparative probability orders and linear qualitative probabilities [8, 13, 18]. For more on the structure and enumeration of completely separable preferences, see [3, 5, 15].

Generating a wide range of separability patterns is valuable for both social choice theory and economics. Simulation of electorates with diverse separabilities is essential for measuring the impact of nonseparability, and to test possible mitigation strategies. Simulation of trading economies with nuanced dependency patterns within and between sectors is essential for understanding potential cascade effects between markets. Generalizing the study of complete separability, Hodge and TerHaar [12] introduced the following concept.

**Definition 5** The collection \(C \subset \mathcal{P}([n])\) is admissible when there exists a preference order \(\succeq\) such that \(\text{char}(\succeq) = C\).

Note that this definition requires that \(\succeq\) is a preference order (rather than merely a preference preorder).

Hodge and TerHaar posed the admissibility problem: which collections \(C \subset \mathcal{P}([n])\) are admissible? An admissible collection must contain both \(\emptyset\) and \([n]\), since each of these sets trivially satisfies (3). Bradley, Hodge and Kilgour [3] proved that admissible collections are closed under intersections. Hodge and TerHaar [12] proved that this closure condition is sufficient for \(n \leq 3\), but not for larger \(n\). When \(n = 4\) there is exactly one inadmissible collection (up to permutations of [4]) satisfying this intersection closure condition:

\[\{\emptyset, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{3, 4\}, \{1, 2, 3, 4\}\}.\]

Hodge, Krines and Lahr [11] used ‘preseparable extensions’ to construct certain classes of admissible collections by recursively stitching together total orders on disjoint ground sets. For such an admissible collection \(C\), there is at least one proper, nonempty \(S\) such that both \(S\) and \(-S\) are in \(C\). Recently, Bjorkman, Gravelle and Hodge [2] used Hamilton paths on the hypercube to generate orders called cubic preferences. The admissible collections that they construct consist of nested subsets \(\emptyset \subset S_1 \subset S_2 \cdots \subset S_k \subset [n]\).

Herein, we extend the landscape of preference characters by introducing a linear algebraic framework and then using it to generate preferences with a tree structure of separable sets. Our admissible families are sublattices of the boolean lattice \(B_n\) of subsets of \([n]\) ordered by inclusion, so we first recall some helpful poset terminology. Set containment induces a partial order on any collection \(C \subset \mathcal{P}([n])\). For \(A, B \in C\), we define \(A \prec B\) when \(A \subset B\). We say that \(B\) covers \(A\) when \(A \prec B\) and there is no \(C \in C\) such that \(A \prec C \prec B\). The Hasse diagram of a poset is an acyclic directed graph that has an edge from vertex \(A\) to \(B\) whenever \(B\) covers \(A\). The graph layout is drawn so that \(B\) appears above all sets that it covers, so that all edges are oriented upwards. The unique maximal element is \([n] \in C\) and unique minimal element is \(\emptyset \in C\).

This brings us to our main result.
Definition 6 A tree collection $C \subset \mathcal{P}([n])$ is a collection of subsets of $[n]$ such that $\emptyset, [n] \in C$, and for every $A, B \in C$, one of the following is true:

$$A = B, \ A \subset B, \ A \supset B, \text{ or } A \cap B = \emptyset.$$ 

So every pair of subsets is either nested or disjoint. A tree character is an admissible tree collection.

Note that a tree collection is closed under intersection and its Hasse diagram of $C \setminus \{\emptyset\}$ is a tree rooted at $[n]$. Also, the admissible collections of $[2]$ are an example of a tree collection, so our result gives another proof of admissibility of that family.

Theorem 1 Every tree collection is admissible.

Our proof of this theorem occurs in two parts. Given a tree collection $C$, we first construct a preorder $\succeq$ with character $C$. We then make a minor alteration to create a total order $\succeq'$ with the same tree character.

Our method for constructing preference orders with tree characters is particularly valuable for economic trade applications. Goods are frequently organized hierarchically, either by sector categorization or via clustering methods. Running trade simulations with preference dependencies are drawn from these hierarchies would help to measure the robustness of models where dependencies between goods cause cascading effects of economics shocks.

1.2 The Preference Space $\mathbb{P}^n$

Our proof of Theorem 1 uses linear algebraic methods to construct a preference order with the desired character. We introduce the $2^n$-dimensional voter basis, whose vectors induce preference orders with nice separability properties. We believe that this flexible voter basis has the potential to significantly expand the set of known preference characters, and perhaps more importantly, to provide insight into the structure of completely separable preferences (aka comparative probability orderings).

We adopt the linear algebraic viewpoint by converting a preference order into a $2^n$-dimensional vector. Consider the preference space $\mathbb{P}^n \cong \mathbb{R}^{2^n}$ whose vector entries are indexed by the bit strings from $X_{[n]} = \mathbb{Z}_2^n$. We view a preference vector in $\mathbb{P}^n$ as a utility function on outcomes, where a higher valued entry corresponds to a more preferred outcome. Starting with a preference preorder $\succeq$, we construct the preference vector $v_{\succeq}$ by setting $v(x)$ to be the length of the shortest chain from outcome $x$ to the least preferred outcome. For example, the preference order in Eq. 1 corresponds to the preference vector

$$v_{\succeq} = \begin{bmatrix} v(11) \\ v(10) \\ v(01) \\ v(00) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$ 

Note that we have listed the entries of $v_{\succeq}$ in reverse lexicographical order. This aligns with two standard conventions for describing completely separable preferences: (a) the election outcome $11 \cdots 1$ is typically most preferred, and (b) the singleton outcomes satisfy $100 \cdots 0 > 010 \cdots 0 > \cdots > 000 \cdots 1$.

If not, we can remedy this situation. Let $m = m_1 m_2 \cdots m_n$ be the most preferred outcome and let $I = \{i : m_i = 0\}$. To achieve (a), we change every $x \in X_{[n]}$ by inverting the
value of $x_i$ for $i \in I$. We then reorder the positions to achieve (b). In terms of a referendum election, this corresponds to (a) negating the language of some proposals, and (b) reordering the proposals.

Conversely, any vector $p \in P^n$ induces a preference preorder $\succeq_p$ where we rank the outcomes $x \succeq y$ whenever $p(x) \geq p(y)$. Of course, this mapping from preference vectors to preference preorders is many-to-one: all that matters is the order of the relative magnitudes of the entries of vector $p$.

For convenience, we adapt our separability definitions for a preference vector $p$.

**Definition 7** Let $\succeq$ be the preference preorder induced by $p \in P^n$. The set $S \subseteq [n]$ is $p$-separable (resp. $p$-nonseparable) when $S$ is $\succeq$-separable (resp. $\succeq$-nonseparable). We also define char($p$) = char($\succeq$).

Naturally, our preference construction hinges upon picking a useful basis for the preference space $P^n$. We use hatted notation to denote the reverse bijection from $\mathbb{Z}_2^n$ to $P([n])$:

$$\hat{x} = \{i \in [n] \mid x_i = 1\}. \quad (5)$$

For example, $\hat{10110} = \{1, 3, 4\}$. We also define the parity indicator function on $P([n])$

$$\delta_{\text{even}}(S) = \begin{cases} 1 & \text{if } |S| \text{ is even}, \\ 0 & \text{if } |S| \text{ is odd}. \end{cases} \quad (6)$$

**Definition 8** The voter basis $V_n = \{v_A \mid A \subseteq [n]\}$ is the collection of vectors whose entries $v_A(x)$ indexed by the outcomes $x \in \mathbb{Z}_2^n$ are given by

$$v_A(x) = \delta_{\text{even}}(\hat{x} \cap A). \quad (7)$$

The voter basis $V_3$ is shown in Table 1. For each $A \subseteq [n]$, the entries of $v_A$ only take on two values: 0 and 1. Therefore, the preference preorder $\succeq_{v_A}$ partitions $P([n])$ into two equal parts: the preferred subsets and unpreferred subsets of $[n]$. Along with their simple structure, the voter basis vectors have nice separability properties.

**Theorem 2** The voter basis $V_n$ has the following properties:

(a) $V_n$ is a basis for $P^n$.

### Table 1 The voter basis for $P^3$

| subset | bit string | $v_{\{1,2,3\}}$ | $v_{\{1,2\}}$ | $v_{\{1,3\}}$ | $v_{\{1\}}$ | $v_{\{2,3\}}$ | $v_{\{2\}}$ | $v_{\{3\}}$ | $v_{\emptyset}$ |
|--------|------------|------------------|----------------|----------------|-------------|----------------|-------------|-------------|------------|
| $\{1,2,3\}$ | 111 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $\{1,2\}$ | 110 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $\{1,3\}$ | 101 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| $\{1\}$ | 100 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| $\{2,3\}$ | 011 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\{2\}$ | 010 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| $\{3\}$ | 001 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| $\emptyset$ | 000 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
(b) Let $\preceq$ be the preference preorder induced by basis vector $v_A$. Then $S \subset [n]$ is separable if and only if $A \subset S$ or $A \cap S = \emptyset$. Equivalently,

$$\text{char}(v_A) = \{ S \mid A \subset S \text{ or } A \cap S = \emptyset \}.$$ 

While we do not take up the question of completely separable preferences in this current work, we are hopeful that the voter basis will be useful for illuminating this important (and difficult) family of preference orders. The voter basis also has deep connections to representation theory: we originally developed $V_n$ using representation theory for the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$. To maintain the focus of this exposition, we defer those connections to future work, and instead provide an elementary proof that $V_n$ is a basis for $P^n$.

We conclude this section by drawing connections to previous research that employs vector representations in the study of election preferences. Hodge and Klima [10] represent a preference order as a column vector of bit strings, with the voter’s $i$th preference appearing in the $i$th row. Treating each row as a vector in $\mathbb{Z}_2^n$, we obtain a $2^n \times n$ binary preference matrix. For example, the preference order of Eq. 1 corresponds to the $4 \times 2$ binary preference matrix

$$\begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}.$$ 

This representation has proven quite useful in many of the constructions mentioned above. As a side note, the absence of an algebraic structure for these matrices was part of the motivation for our definition of the preference space $P^n$. Looking at election outcomes more globally, Daughtery et al. [6] introduced the profile space $M^n \cong \mathbb{Q}^n$ to decompose an election according to the actual ballots cast. For example, a ballot for a ranked choice election with $n$ candidates corresponds to a permutation of $[n]$. Using a basis $\{v_\sigma \mid \sigma \in S_n\}$, where we view $\sigma \in S_n$ a total order of $[n]$, the collection of voter ballots corresponds to the linear combination $\sum_{\sigma \in S_n} a_\sigma v_\sigma$ where $a_\sigma$ is the number of ballots cast with candidate ranking $\sigma$. To capture such aggregate behavior of the electorate in the preference space $P^n$, we would create a linear combination of the preference vectors across the electorate. Simplifying would give a single preference vector that captures the overall utility score for each election outcome. Finally, we note that a preference vector $v \in P^n$ is equivalent to the value function as defined in Bradley et al. [3], though our vector space viewpoint is crucial to the methods herein.

### 1.3 Roadmap

The remainder of this paper is organized as follows. Section 2 develops the voter basis. We prove Theorem 2, consider the separability properties of voter basis vectors, and introduce some helpful notation for describing and combining partial outcomes. We then introduce a rank function on $P([n])$ and construct a vector $w$ that induces a completely nonseparable preference order. The coefficients of $w$ are useful in our subsequent constructions. In Section 3, we prove Theorem 1: tree collections are admissible. We conclude in Section 4, suggesting some directions for future research.
2 The Voter Basis

We formulate some elementary results using bit string notation, and prove Theorem 2. Recall from Definition 2 that $X_S$ denotes the set of all partial outcomes on the set $S \subset [n]$, and that $0_S$ denotes the all-zero partial outcome on $S$. The simplest preference preorder on $S$ arises when a voter is indifferent between all the outcomes.

**Definition 9** A set $S$ is **trivially separable** with respect to $\succeq$ when for all $x_S \in X_S$ and all $u_{-S} \in X_{-S}$, we have

$$x_S u_{-S} \sim 0_S u_{-S}.$$  

**Lemma 1** If $S$ is trivially separable then $-S$ is separable.

**Proof** Suppose $S$ is trivially separable, i.e., for all $x_S \in X_S$ and $u_{-S} \in X_{-S}$, $x_S u_{-S} \sim 0_S u_{-S}$, then $-S$ is trivially separable.

Bradley, Hodge and Kilgour [3] showed that set intersections preserve separability.

**Lemma 2** [3] If $S$ and $T$ are $\succeq$-separable then so is $S \cap T$.

We include a proof to acquaint the reader with the general flow of the bit string proofs that follow. It is intuitive that the intersection of separable sets should also be separable. However, a rigorous proof requires partitioning outcomes according to the four sets $S \cap T$, $S - T$, $T - S$ and $-S \cap T$. The bit string notation is well-suited to this task.

**Proof** We partition each outcome $z \in X_{[n]}$ as

$$z = z_{S \cap T} z_{S \cap \neg T} z_{S - T} z_{T - S} z_{\neg S \cup T}.$$  

Suppose that $x_{S \cap T} 0_{S \cap T} \succeq y_{S \cap T} 0_{S \cap T}$, and let $v_{S \cap T}$ be any other bit string on $0_S \cap T$. Then

$$x_{S \cap T} 0_{S \cap T} \succeq y_{S \cap T} 0_{S \cap T}$$  

$$\Rightarrow (x_{S \cap T} 0_{S - T}) 0_{T - S} 0_{S \cup T} \succeq (y_{S \cap T} 0_{S - T}) 0_{T - S} 0_{S \cup T}$$

$$\Rightarrow (x_{S \cap T} v_{S - T}) v_{T - S} v_{S \cup T} \succeq (y_{S \cap T} v_{S - T}) v_{T - S} v_{S \cup T}$$

$$\Rightarrow x_{S \cap T} v_{S \cup T} \succeq y_{S \cap T} v_{S \cup T}.$$  

Therefore $S \cap T$ is $\succeq$-separable. 

We now prove that $\mathcal{V}_n$ is a basis for the preference space $\mathbb{P}^n$, and then investigate the separability properties of a voter basis vector $v_S$. The following elementary lemma, suggested to us by Jeremy Martin [16], leads to a quick proof that $\mathcal{V}_n$ is a basis.

**Lemma 3** Let $W_n$ be the $2^n \times 2^n$ matrix whose entries are indexed by subsets of $[n]$ and whose $(S, T)$-th entry is

$$W_n(S, T) = (-1)^{|S \cap T|}.$$  

*Springer*
Then \( \det(W_1) = -2 \) and \( \det(W_n) = 2^n2^n-1 \) for \( n \geq 2 \).

**Proof** We have

\[
W_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} W_1 & W_1 \\ W_1 & -W_1 \end{bmatrix},
\]

where we have used the ordering \((\emptyset, \{1\})\) and \((\emptyset, \{1\}, \{2\}, \{1, 2\})\) for the rows and columns of \( W_1 \) and \( W_2 \), respectively. Clearly, \( \det(W_1) = -2 \). Elementary row operations on \( W_2 \) yield

\[
W_2 = \begin{bmatrix} W_1 & W_1 \\ W_1 & -W_1 \end{bmatrix} \sim \begin{bmatrix} W_1 & W_1 \\ W_1 & -2W_1 \end{bmatrix} \sim \begin{bmatrix} W_1 & 0 \\ 0 & -2W_1 \end{bmatrix}
\]

and these row addition operations do not change the determinant. Therefore \( \det(W_2) = (-2)^2 \det(W_1)^2 = 2^4 \). The same block matrix structure holds for \( W_n \) in terms of \( W_{n-1} \), where we order by subsets of \([n-1]\) followed by subsets containing element \( n \). Induction gives

\[
\det(W_n) = (-2)^{2^n-1} \det(W_{n-1})^2 = 2^{2^n-1} \left(2^{n-1}2^{2^{n-2}}\right)^2 = 2^{n2^n-1}.
\]

\( \square \)

**Proof of Theorem 2(a)** Let \( w_S \) denote the column of \( W_n \) indexed by \( S \subseteq [n] \). Observe that \( w_S = 2v_S - v_\emptyset \) where \( v_\emptyset = 1 \) is the all-ones vector. Therefore \( V_n \) is a basis for \( \mathbb{P}^n \).

It is important to note that the row/column order in this proof is different from the order displayed in Table 1, which adheres to preference order conventions. The recursive ordering is essential for the inductive proof. Also, we could have used the \( w_S \) vectors as our basis, but the plentiful zeros of the \( v_S \) vectors simplify the arguments below. Next, we introduce some terminology and notation for outcomes.

**Definition 10** The outcome \( x \) is even (odd) when the size of the corresponding set \( |\hat{x}| \) is even (odd). The outcome \( x \) is even in \( A \) (odd in \( A \)) when \( |\hat{x} \cap A| \) is even (odd).

For example, consider the set \( X_3 = \mathbb{Z}_3^2 \). The outcomes 011, 010, 001, and 000 are even in \( \{1\} \) whereas the remaining four outcomes are odd in \( \{1\} \). The outcomes 000, 001, 110 and 111 are even in \( \{1, 2\} \), while the other four outcomes are odd in \( \{1, 2\} \). As a second example, Definition 8 can now be restated as: \( x \) is even in \( A \) if and only if \( v_A(x) = 1 \).

Each voter basis vector \( v_A \) induces a preference order in which outcomes that are even in \( A \) are preferred to outcomes that are odd in \( A \). The vector \( v_\emptyset \) induces the trivial preference order (complete indifference). The vector \( v_{\{n\}} \) prefers the even subsets of \([n]\) over the odd subsets. More generally, for nonempty \( A \subseteq [n] \), the vector \( v_A \) partitions the outcomes into a set of \( 2^{n-1} \) preferred outcomes and a set of \( 2^{n-1} \) undesirable outcomes. For example, when \( n = 3 \), the voter basis vector \( v_{\{1\}} = v_{100} \) induces the ordering

\[
\{011, 010, 001, 000\} > \{111, 110, 101, 100\}.
\]

When \( n = 4 \), the vector \( v_{\{1, 2\}} = v_{1100} \) induces the ordering

\[
\{1111, 1110, 1101, 1100, 0011, 0010, 0001, 0000\} > \{1011, 1010, 1001, 1000, 0111, 0110, 0101, 0100\}.
\]

The following notation streamlines our nonseparability proofs.
Definition 11 Let $S \subset [n]$ and let $x_S \in X_S$ be a partial outcome. For $i \in S$, we let $1_i$ and $0_i$ denote that the partial outcome $x_i$ is fixed as 1 or 0, respectively. We use $0_\emptyset$ to denote the partial outcome that is all-zero on elements in $S$ that have not already been specified.

For example, if $S = \{1, 2, 3, 4, 5\}$ then $x_S = 1_2 0_\emptyset$ denotes the outcome 01000. In the proofs below, we will often use this notation to construct sparse partial outcomes $x_S, y_S$ and $u_S, v_S$ so that $x_S u_S > y_S u_S$ while $x_S v_S < y_S v_S$. As an example of the four resulting outcomes, suppose that $n = 6$ and let $S = \{1, 2, 3\}$. Consider the partial outcomes $x_S = 0_\emptyset$ and $y_S = 1_2 0_\emptyset$ on $S$ and the partial outcomes $u_S = 0_\emptyset$ and $v_S = 1_5 0_\emptyset$ on $-S = \{4, 5, 6\}$. Concatenating each pairing gives

$$x_S u_S = 000000,$$
$$y_S u_S = 010000,$$
$$x_S v_S = 000010,$$
$$y_S v_S = 010010.$$

We are now ready to prove Theorem 2(b): the set $S$ is separable for the preference order induced by the vector $v_A$ if and only if $A \subset S$ or $A \cap S = \emptyset$. Applying this theorem for $n = 4$, the preference order induced by $v_{\{1\}}$ (or any other singleton set) induces a completely separable ordering. The preference order induced by $v_{\{1,2\}}$ has character

$$\{\emptyset, \{3\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}.$$

Finally, $v_{\{1,2,3,4\}}$ induces a completely nonseparable preorder on $X_{\{4\}}$. As these examples show, the voter basis vectors have very useful separability properties. Theorem 2 (b) reveals the potential of these basis vectors as building blocks for constructing preference orders. In particular, the nonseparable properties of $v_{\{n\}}$ will be essential for removing unwanted separabilities.

Proof of Theorem 2(b) Given $S \subset [n]$, let $x$ and $y$ be outcomes that are identical on $-S$. There are three cases to consider; we handle the two separable cases first. We fix $A \subset [n]$ and let $\geq$ be the preference order induced by $v_A$.

Case 1: $A \subset S$. We decompose $x = x_S u_S = x_A x_S \cap A u_S$ and $y = y_S u_S = y_A y_S \cap A u_S$. We claim that the preference relation between these outcomes is independent of the shared binary digits $u_S$. Indeed, if $x_A$ and $y_A$ are the same parity, then both or neither are even in $A$, so that $v_A(x_S u_S) = v_A(y_S u_S)$ for all $u_S \in X_S$. If $x_A$ and $y_A$ are not the same parity, then we may assume that $x_A$ is even and $y_A$ is odd, so that $v_A(x_S u_S) = 1 > 0 = v_A(y_S u_S)$ for all $u_S \in X_S$. Either way, the preference between outcomes $x$ and $y$ depends only on the parities of $x$ and $y$ in $A$, which is independent of $u_S$. Therefore, $S$ is separable whenever $A \subset S$.

Case 2: $A \cap S = \emptyset$. We decompose $x$ and $y$ as $x = x_S u_A u_{S \cup A}$ and $y = y_S u_A u_{S \cup A}$. The outcomes are identical on $A$, so their parity in $A$ is the same. Therefore $v_A(x_S u_A u_{S \cup A}) = v_A(y_S u_A u_{S \cup A})$ for all $u_S \in X_S$ which means that $S$ is trivially separable.

Case 3: $S \cap A \neq \emptyset$ and $A - S \neq \emptyset$. Note that this includes the case where $\emptyset \subset S \subset A$. We construct a pair of outcomes on $S$ to certify that $S$ is nonseparable. Let $s \in S \cap A$ and let $a \in A - S$. Let $x_S = 0_\emptyset$ be the all-zero outcome and let $y_S = 1_s 0_\emptyset$ be the singleton outcome on $s$. Now let $u_S = 0_\emptyset$ be the all-zero outcome and $v_S = 1_a 0_\emptyset$ be the singleton outcome on $a$. We have

$$v_A(x_S u_S) = 1 > 0 = v_A(y_S u_S)$$
$$v_A(x_S v_S) = 0 < 1 = v_A(y_S v_S),$$
so that our preference between \( x_S \) and \( y_S \) depends on the outcome on \(-S\). Therefore the set \( S \) is nonseparable.

We conclude this subsection with a trio of elementary results concerning the entries of \( v_T \). The corollaries will be used frequently in the next section to construct preference vectors with desired properties.

**Lemma 4**  Let \( S, T \subset [n] \). Consider outcomes \( x = x_S u_{-S} \) and \( y = y_S u_{-S} \) that agree on \(-S\). We have \( v_T(x) = v_T(y) \) if and only if the partial outcomes \( x_{S \cap T} \) and \( y_{S \cap T} \) have the same parity.

**Proof**  The values of the entries \( v_T(x) \) and \( v_T(y) \) depend solely on the respective parity of the partial outcomes \( x_{S \cap T} u_{T-S} \) and \( y_{S \cap T} u_{T-S} \). These parities agree if and only if the parities of \( x_{S \cap T} \) and \( y_{S \cap T} \) agree.

**Corollary 1**  If \( T \subset -S \) and the outcomes \( x = x_S u_{-S} \) and \( y = y_S u_{-S} \) agree on \(-S\), then \( v_T(x) = v_T(y) \).

**Corollary 2**  If \( S \subset T \) and the outcomes \( x = x_S u_{-S} \) and \( y = y_S u_{-S} \) agree on \(-S\), then \( v_T(x) = v_T(y) \) if and only if \( x_S \) and \( y_S \) have the same parity.

Lemma 4 highlights that fact that when outcomes agree on some subset \( U \), then the voter basis vectors indexed by subsets of \( U \) do not contribute to preference differences between the two outcomes. The two corollaries are analogous to observations we made in the proof of Theorem 2(b). Considering the preference order induced by the voter basis vector \( v_T \), sets disjoint from \( T \) are trivially separable, and supersets of \( T \) are separable.

### 2.1 A Completely Nonseparable Preference Order

We introduce a rank function on \( \mathcal{P}([n]) \) and then use it to construct a preference vector \( w \) that induces a preference order that is completely nonseparable. The proof is elementary, but is valuable in that it provides an opportunity to evaluate the separability of a preference vector, prior to grappling with the more intricate arguments below. Moreover, vectors derived from \( w \) will be used in the proof of Theorem 1.

**Definition 12**  The rank function \( \rho : \mathcal{P}([n]) \to [2^n] \) maps a subset \( A \subset [n] \) to its position in the ordering of subsets of \([n]\) that lists the sets by increasing set size, and then within a fixed size, lists the sets lexicographically.

For example, when \( n = 3 \), our ordering is

\[
\emptyset < \{1\} < \{2\} < \{1, 2\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\},
\]

so that \( \rho(\emptyset) = 1 \), and \( \rho(\{1, 3\}) = 6 \), and so on. Note that, in general, if \( A \subsetneq B \subset [n] \) then \( \rho(A) < \rho(B) \). In other words, the rank function \( \rho \) induces a preference order on \( \mathcal{P}([n]) \) that is monotone with respect to set containment.

We define

\[
w = \sum_{A \in \mathcal{P}([n])} 2^{\rho(A)} v_A
\]

and consider the properties of the preference preorder induced by \( w \).
Lemma 5 If $x \neq y$ are distinct outcomes on $[n]$, then $w(x) \neq w(y)$, so $w$ induces a total order on $\mathcal{P}([n])$.

Proof Let $a \in [n]$ be an element where $x$ and $y$ disagree. Without loss of generality, $x = 1_a x_{[n]−a}$ and $y = 0_a y_{[n]−a}$. The entry $w(x)$ includes the summand $2^\rho([a])$ while $w(y)$ does not. Since every positive integer has a unique binary representation, $w(x) \neq w(y)$.

Lemma 6 The preference order induced by $w$ is completely nonseparable, meaning $\text{char}(w) = \text{char}(\preceq_w) = \{\emptyset, [n]\}$.

Proof Let $A$ be a nontrivial proper subset of $[n]$, and let $a \in A$ and $b \in \complement A$. We have

$$w(0_a 0_a) - w(1_a 0_a) = \sum_{S \in [n]} 2^\rho(S) > 0.$$ 

Meanwhile

$$w(0_a 1_b 0_a) - w(1_a 1_b 0_a) = \sum_{S \in [n] \setminus \{a, b\}} \left(2^\rho(S \cup \{a\}) - 2^\rho(S \cup \{a, b\})\right) < 0$$

because the ranking function satisfies $\rho(S \cup \{a\}) < \rho(S \cup \{a, b\})$ for all $S \in [n] \setminus \{a, b\}$. Our preference between the outcomes $0_A u_{\complement A}$ and $1_s 0_{A\setminus a} u_{\complement A}$ depends on the value of $u_{\complement A}$. Therefore the set $A$ is nonseparable.

We close this subsection with one final observation. The sum of the coefficients of $w$ is

$$\sum_{A \in \mathcal{P}([n])} 2^\rho(A) = \sum_{k=1}^{2^n} 2^k = 2^{2^n+1} - 2 < 2^{2^n+1}.$$ 

Later on, in the proof of Theorem 1, we will break unwanted symmetries by adding a small vector whose nonzero coefficients are drawn from $2^{−2^n−1}w$. With that in mind, for every $A \in \mathcal{P}([n])$, we introduce the constant

$$d_A = 2^\rho(A)−2^n−1.$$ 

We choose this scaling because

$$\sum_{A \in \mathcal{P}([n])} d_A = 1 − 2^{−2^n} < 1,$$ 

which will be smaller than all the other nonzero coefficients. Crucially, the impact of this small vector will be inconsequential, except when comparing entries that are otherwise equal. We can now generalize Lemma 5.

Lemma 7 Let $\emptyset \neq S \subset \mathcal{P}([n])$ be a family of subsets of $[n]$, and let $x \neq y$ be distinct outcomes on $[n]$ such that there is at least one $S \in S$ where $x_S$ and $y_S$ have different parities. Let

$$u = \sum_{A \in S} d_A v_A.$$ 

Then $u(x) \neq u(y)$ and $−1 < u(x) − u(y) < 1$.

Proof Without loss of generality, $x_S$ is odd and $y_S$ is even. The entry $u(x)$ includes the summand $d_S$ while $u(y)$ does not. As in the proof of Lemma 5, $u(x) \neq u(y)$, since the
summands are scaled powers of 2. We have \(-1 < u(x) - u(y) < 1\) since \(\sum_{A \in \mathcal{P}([n])} d_A < 1\).

3 Tree Characters

We turn to the proof of Theorem 1: given a tree collection \(C\), there is a preference order \(\succeq\) with \(\text{char}(\succeq) = C\). In Section 3.1, we develop some terminology and then state Theorem 3, the preorder variant of our main theorem. In particular, we specify the linear combination of \(V_n\) that produces a preference preorder with the desired tree character. We offer two examples of preference vectors constructed using the tree character theorem. In Section 3.2, we prove Theorem 3, deferring the details of four technical lemmas to Section 3.3. Finally, we prove our main Theorem 1 in Section 3.4.

3.1 The Tree Character Theorem

We start by building on Definition 6, which defines a tree collection \(C\) to be a collection of subsets of \([n]\) which contains \(\emptyset\) and \([n]\), and such that every pair of sets is either nested or disjoint.

**Definition 13** Let \(C\) be a tree collection and let \(A, B \in C\). If \(B\) covers \(A\), then \(B\) is the parent of \(A\), and \(A\) is a child of \(B\). The children of \(B\) are called siblings. The \(k\)th generation consists of all sets that are at distance \(k\) from the root \([n]\). For \(A \neq \emptyset\), we use \(g(A)\) to denote the generation of \(A\). Ancestors and descendants are defined in the natural way.

In a tree collection, every proper nonempty set has a unique set that covers it. For example, the collection of sets

\[C_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\]

is a tree collection. Figure 1a represents \(C - \{\emptyset\}\) as a rooted tree. We have \(g(\{1, 2\}) = 1\) and \(g(\{4, 5\}) = 2\) and \(g(\{4\}) = 3\). On the other hand, the collection

\[C_2 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}\]

is not a tree collection, since both \(\{1, 2\}\) and \(\{1, 3\}\) cover \(\{1\}\).

As we construct our preference vector for a tree collection \(C\), we will also need to keep track of the elements of \([n]\) that appear in generation \(k\), but do not appear in generation \(k + 1\). For example, in \(C_1\), the set \(\{4, 5\}\) has one child \(\{4\}\) but the element 5 is “missing” from the next generation. For convenience, we collect these missing elements into sets of ghost children.

**Definition 14** Let \(C\) be a tree collection and consider \(A \in C\) with children \(A_1, A_2, \ldots, A_\ell\) where \(\ell \geq 1\). If \(\bigcup_i A_i \neq A\), then the ghost child of \(A\) is \(A - \bigcup_i A_i\). The Hasse diagram that includes the ghost children is called a haunted Hasse diagram.

Note that if \(A\) does not have any children, then it does not have a ghost child either. Figure 1b shows the haunted Hasse diagram for the tree collection \(C_1\). With the addition of ghost children, every element in the set \([n]\) appears in exactly one leaf of the haunted Hasse diagram.
During our construction, we will use ghost children to prevent (unwanted) unions of siblings from becoming separable. For example, in Fig. 1b, the children of the set \([8]\) are \([1, 2]\) and \([3, 4, 5]\). We will use the ghost child \([6, 7, 8]\) to prevent the set \([1, 2, 3, 4, 5]\) from also being separable. More precisely, to break unwanted separability on unions of siblings \(A_i \cup A_j\), we include a tiny vector in the direction of \(v_{A_i \cup A_j}\), as described in Theorem 1 below.

**Definition 15** Let \(C\) be a tree collection. Let \(A \in C\) whose children and ghost child (when present) are \(A_1, A_2, \ldots, A_k\). The sibling linkage \(\mathcal{L}(A)\) of \(A\) is

\[
\mathcal{L}(A) = \{A_i \cup A_j : 1 \leq i < j \leq k\}
\]

Elements \(A_i \cup A_j\) of the sibling linkage \(\mathcal{L}(A)\) are called siblinks. The set

\[
\mathcal{L} = \bigcup_{A \in C} \mathcal{L}(A)
\]

is the sibling linkage of the tree collection \(C\).

We now state the preference preorder variant of our main theorem: for every tree collection \(C\), there exists a preference preorder \(\succeq\) with \(\text{char}(\succeq) = C\). Our main Theorem 1 will follow quickly from this one; see Section 3.4.

**Theorem 3** Consider a tree collection \(C \subset \mathcal{P}([n])\). Let \(\alpha = 2 - 2^{-(n-1)}\). For nonempty \(A \in C\), let \(c_A = \alpha^{\rho(A)}\) and let \(d_B = 2^{\rho(B)} - 2^{\rho(B) - 1}\) where \(\rho(B)\) is the rank of set \(B\). Define

\[
v_C = \sum_{A \in C \setminus \{\emptyset\}} c_A v_A + \sum_{B \in \mathcal{L}} d_B v_B.
\]

Then \(C\) is the collection of separable sets in the preference preorder induced by the preference vector \(v_C\). In other words, \(\text{char}(v_C) = C\).

We illuminate the form and function of the coefficients \(c_A\) and \(d_B\) in Section 3.2 below. For now, it is enough to mention that \(c_A\) creates the separability of \(A \in C\), and \(d_B\) breaks unwanted separabilities of some sets outside of \(C\). Finally, we note that \(\alpha\) is essentially equal to 2, but choosing \(\alpha = 2\) would invalidate Lemma 12 below.
We conclude this section with two examples which use Theorem 1 to construct preference orders for give tree collections. First, consider the simple example for \( n = 3 \) with tree collection

\[ \mathcal{C}_3 = \{ \emptyset, \{1\}, \{2\}, \{1, 2, 3\} \}. \]

The root \( \{1, 2, 3\} \) has children \( \{1\}, \{2\} \) and ghost child \( \{3\} \). Therefore, the siblinks are

\[ \mathcal{L}_3 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}. \]

We have \( \alpha = 2 - 2^2 = 7/4 \) and the siblink coefficients are of the form \( 2^{\rho(B) - 9} \) for the rank function \( \rho \) corresponding to Eq. 8. Our desired vector is

\[ v_{\mathcal{C}_3} = v_{\{1,2,3\}} + \frac{7}{4} v_{\{1\}} + \frac{7}{4} v_{\{2\}} + \frac{1}{16} v_{\{1,2\}} + \frac{1}{8} v_{\{1,3\}} + \frac{1}{4} v_{\{2,3\}}. \]

Using the voter basis vectors listed in Table 1, the entries of \( 16 v_{\mathcal{C}_3} \) are

\[
\begin{array}{cccccccc}
111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\
7 & 17 & 46 & 32 & 48 & 30 & 57 & 79
\end{array}
\]

which corresponds to preference order

\[ 111 < 110 < 010 < 100 < 011 < 001 < 000. \]

It is easy to check that the separable sets are precisely those in \( \mathcal{C}_3 \). We also note that this particular example produced an order since there are no ties.

Finally, we construct a preference vector in \( \mathbb{P}^9 \cong \mathbb{R}^{2^9} \) for the tree collection

\[ \mathcal{C}_4 = \{ \emptyset, \{1\}, \{2\}, \{7\}, \{8\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \].

(11)

This example will produce a preorder, as noted below. The haunted Hasse diagram of \( \mathcal{C}_4 \) is shown in Fig. 2. There are four nonempty sibling linkages for \( \mathcal{C}_4 \):

\[
\begin{align*}
\mathcal{L}([9]) &= \{\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\}, \\
\mathcal{L}([1, 2, 3, 4]) &= \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \\
\mathcal{L}([5, 6, 7, 8, 9]) &= \{\{5, 6, 7, 8, 9\}\}, \\
\mathcal{L}([7, 8, 9]) &= \{\{7, 8\}, \{7, 9\}, \{8, 9\}\}.
\end{align*}
\]

So the set of all siblinks is

\[ \mathcal{L}_4 = \{\{1, 2\}, \{7, 8\}, \{7, 9\}, \{8, 9\}, \{1, 3, 4\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\}. \]

As described in Eq. 10, the preference vector \( v_{\mathcal{C}_4} \) consists of two summations. First, we create a linear combination of basis vectors indexed by elements of \( \mathcal{C}_4 \). The coefficient of \( v_A \) is determined by the generation of \( A \) in the haunted Hasse diagram. This gives us the first part of our vector \( v_{\mathcal{C}_4} \):

\[
\begin{align*}
v_{\mathcal{C}_4} &= \alpha v_{\{1,2,3,4\}} + \alpha v_{\{5,6,7,8,9\}} + \alpha^2 v_{\{1\}} + \alpha^2 v_{\{2\}} + \alpha^2 v_{\{5,6\}} \\
&\quad + \alpha^2 v_{\{7,8,9\}} + \alpha^3 v_{\{7\}} + \alpha^3 v_{\{8\}},
\end{align*}
\]

(12)

where \( \alpha = 2 - 2^{-8} \). Next, we create a linear combination of basis vectors indexed by the siblinks

\[
\frac{1}{2^{2^9+1}} \sum_{B \in \mathcal{L}_4} 2^{\rho(B)} v_B
\]

(13)
where $\rho(B)$ is the rank of siblink $B \in \mathcal{L}_4$ in the ordering of $\mathcal{P}([4])$. We obtain $\mathbf{v}_{\mathcal{C}_4} \in \mathcal{P}_9$ by adding expressions Eqs. 12 and 13. We can routinely check that $\text{char}(\mathbf{v}_{\mathcal{C}_4}) = \mathcal{C}_4$, though this is best done via mathematical software. Finally, we note that the vector $\mathbf{v}_{\mathcal{C}_4}$ induces a preorder: the elements 3 and 4 are treated identically, as are the elements 5 and 6.

### 3.2 Tree Characters for Preference Preorders

In this subsection, we prove Theorem 3. We start with a few observations about the coefficients in Theorem 1. First, if $A \in \mathcal{C} \cap \mathcal{L}$ then the coefficient of $\mathbf{v}_{A}$ is $c_A + d_A$. Second, we have $c_A \geq 1$ for every $A \in \mathcal{C}$, while $\sum_{A \in \mathcal{L}} d_A < 1$. A third property is described by the following lemma, which illuminates the choice of the constant $\alpha = 2 - 2^{-(n-1)}$.

#### Lemma 8

Suppose that $n \geq 3$. Let $\alpha = 2 - 2^{-(n-1)}$. For $1 \leq m \leq n - 1$, we have

$$0 < \alpha^m - \sum_{i=0}^{m-1} \alpha^i = \alpha^m - \frac{\alpha^m - 1}{\alpha - 1} < 1,$$

and consequently, for $1 \leq r < s \leq n - 1$, we have

$$\sum_{i=r}^{s-1} \alpha^i < \alpha^s < \alpha^r + \sum_{i=r}^{s-1} \alpha^i.$$

While we are tempted to take $\alpha = 2$, this would replace the final inequality with the equality $2^s = 2^r + \sum_{i=r}^{s-1} 2^i$, which breaks the proof of Lemma 12 below. The choice of $\alpha = 2 - 2^{-(n-1)}$ gives the behavior we need for tree characters of $[n]$.

#### Proof

The first inequality chain is equivalent to $-1 < \alpha^m (\alpha - 2) < \alpha - 2$. Dividing by the negative quantity $\alpha - 2$ gives

$$1 < \frac{\alpha^m}{2 - \alpha},$$

which clearly holds for $\alpha = 2 - 2^{-(n-1)}$ and $1 \leq m \leq n - 1$. The second inequality chain follows directly from the first. \qed

---

Fig. 2 The haunted Hasse diagram for the tree collection $\mathcal{C}_4$, which has two ghost children: (3, 4) and (9)
The proof of Theorem 3 is quite technical. We defer the details to a series of four lemmas whose proofs appear in the next subsection. Consider the preference order induced by $v_C$ of Eq. 10. We must show that the sets in $C$ are separable and that all other sets are nonseparable.

**Lemma 9** Every set $A \in C$ is $v_C$-separable.

We prove this lemma in the next subsection. Turning to sets that are not members of $C$, we introduce some additional definitions to partition these sets into three categories.

**Definition 16** Let $C$ be a tree collection. A nonempty set $B \in C$ is unbreakable when $B$ cannot be written as the union of sets in $C - \{B\}$. The set of all unbreakable sets is denoted $U(C)$.

The collection of unbreakable sets is a superset of the join-irreducibles of the lattice $C$ (which are elements that cover exactly one other element). For example, in the tree collection $C_4$ from Eq. 11 and Fig. 2, the join-irreducibles are

$$\{1\} , \{2\} , \{5, 6\} , \{7\} , \{8\}$$

while $U(C_4)$ also includes the sets

$$\{1, 2, 3, 4\} , \{7, 8, 9\} .$$

**Definition 17** Let $C$ be a tree collection and let $B \subset [n]$. Let

$$\Upsilon'_C(B) = \{ A \in U(C) \mid A \subset B \}$$

be the collection of $C$-unbreakable sets contained in $B$ and let

$$\Upsilon_C(B) = \{ A \in \Upsilon'_C(B) \mid \forall A' \in \Upsilon'_C(B), A \not\subset A' \} .$$

be the collection of maximal sets in $\Upsilon'_C(B)$. The construct of $B$ is

$$K_C(B) = \bigcup_{A \in \Upsilon_C(B)} A .$$

When $K_C(B) = B$, we say that $B$ is constructible.

We make some elementary observations about constructs and unbreakable sets. First, if $A$ is unbreakable, then $A - K_C(A) \neq \emptyset$. Second, for any $B \subset [n]$, the tree structure of $C$ ensures that the collection $\Upsilon_C(B)$ of maximal $C$-unbreakable subsets of $B$ is pairwise disjoint. Third, every set $A \in C$ is $C$-constructible: either $A$ is unbreakable, or it can be written as the disjoint union of unbreakable sets. And finally, if $B \notin C$ can be written as the union of sets in the tree collection $C$, then $B$ is constructible using maximal $C$-unbreakable subsets of $B$.

For an example, let us return to tree collection $C_4$. The following four sets are not $C_4$-constructible:

$$\{5\} , \{1, 2, 4\} , \{2, 6\} , \{1, 2, 7, 9\} ,$$
while these five sets are $C_4$-constructible:

$$\{1, 2\} = \{1\} \cup \{2\},$$
$$\{2, 5, 6, 8\} = \{2\} \cup \{5, 6\} \cup \{8\},$$
$$\{1, 2, 3, 4, 7\} = \{1, 2, 3, 4\} \cup \{7\},$$
$$\{1, 2, 7, 8\} = \{1\} \cup \{2\} \cup \{7\} \cup \{8\},$$
$$\{5, 6, 7, 8, 9\} = \{5, 6\} \cup \{7, 8, 9\}.$$ 

Note that $\{1, 2\}$ and $\{5, 6, 7, 8, 9\}$ are $C_4$-constructed from siblings in $C_4$. Meanwhile, the other three sets are $C_4$-constructed using elements that are not siblings.

Here are the lemmas that handle sets that are not contained in $C$. Their proofs are deferred to the subsection that follows.

**Lemma 10** Consider a set $B \notin C$ that is not $C$-constructible. Then the set $B$ is not $v_C$-separable.

**Lemma 11** Consider a constructible set $B \notin C$ where the elements of $\Upsilon_C(B)$ are siblings in $C$. The set $B$ is not $v_C$-separable.

**Lemma 12** Consider a constructible set $B \notin C$ where at least two elements of $\Upsilon_C(B)$ are not siblings in $C$. The set $B$ is not $v_C$-separable.

**Proof of Theorem 3** If $B \in C$, then $B$ is separable by Lemma 9. Consider $B \notin C$. If $B$ is not $C$-constructible, then $B$ is nonseparable by Lemma 10. If $B$ is constructible, then either this construction uses a set of siblings or does not. Lemmas 11 and 12 show that $B$ is nonseparable in either case. In summary, only elements of $C$ are separable. Therefore, $\text{char}(v_C) = C$.

All that remains is to justify these four lemmas. The proofs become more intricate as we progress. In particular, Lemma 11 requires ghost children and siblinks to force nonseparability among unions of siblings in the tree collection. Let us begin.

### 3.3 Proofs of the Tree Character Lemmas

We start with a few observations and some helpful notation. Let $\mathcal{C}$ be a tree collection of $[n]$ and let $\mathcal{L} = \bigcup_{A \in \mathcal{C}} \mathcal{L}(A)$ be the union of all siblinks of $\mathcal{C}$. The vector $v_C$ constructed in Theorem 1 is

$$v_C = \sum_{A \in \mathcal{C}} c_A v_A + \sum_{A \in \mathcal{L}} d_A v_A.$$ 

The coefficients are given by $c_A = \alpha^{g(A)}$ where $\alpha = 2 - 2^{-(n-1)}$ and $g(A)$ is the generation of $A \in \mathcal{C}$, and $d_A = 2^{\rho(A)-2^{n-1}}$ where $\rho(A)$ is the rank of the set $A$ in our ordering of $\mathcal{P}([n])$ from Definition 12. For convenience, we define $c_S = 0$ when $S \notin \mathcal{C}$ and $d_S = 0$ when $S \notin \mathcal{L}$, so that we denote the coefficient of $S$ as $c_S + d_S$, regardless of whether $S$ is a member of the tree collection or the sibling linkage.

Suppose that $A_r \in \mathcal{C}$ is in the $r$th generation and that $A_r \subset A_{r-1} \subset \cdots \subset A_0 = [n]$ is its complete ancestral chain of supersets in $\mathcal{C}$. Then Lemma 8 ensures that

$$c_{A_r} = \alpha^r > \sum_{j=0}^{r-1} \alpha^j = \sum_{j=0}^{r-1} c_{A_j}. \quad (14)$$
Next, we observe that $|\mathcal{L}| < 2^n$ because each siblink is the union of two disjoint nonempty sets. Equation 9 yields

$$\sum_{A \in \mathcal{L}} d_A \leq \sum_{A \in \mathcal{P}[n]} d_A < 1.$$  \hspace{1cm} (15)

We employ the following notation for partial sums of the coefficients of $v_C$ that are even with respect to a given set $A$. For $S \subset \mathcal{C}$ and $T \subset \mathcal{L}$, we define

$$C(A, S) = \sum_{S \in S} \delta_{\text{even}}(A \cap S) \llbracket S \rrbracket,$$

$$D(A, T) = \sum_{T \in T} \delta_{\text{even}}(A \cap T) d_T,$$

where the parity indicator function $\delta_{\text{even}}(\cdot)$ is defined in Eq. 6. It will be convenient to use this same notation with a partial outcome $x_A$, in which case we define

$$C(x_A, S) = \sum_{S \in S} \delta_{\text{even}}(A \cap S) \llbracket S \rrbracket,$$

$$D(x_A, T) = \sum_{T \in T} \delta_{\text{even}}(A \cap T) d_T,$$

Given two sets $A$ and $B$, we will often need to compare $v_C(A)$ with $v_C(B)$. Equation 15 leads to the handy observation

$$C(A, \mathcal{C}) - C(B, \mathcal{C}) - 1 < v_C(A) - v_C(B) < C(A, \mathcal{C}) - C(B, \mathcal{C}) + 1.$$ \hspace{1cm} (16)

In particular, $C(A, \mathcal{C}) - C(B, \mathcal{C}) \in \mathbb{Z}$, so Eq. 15 allows us to ignore the fractional contribution from the siblink coefficients when this difference is nonzero.

For a given set $B \subset [n]$, and a collection of subsets $\mathcal{T} \subset \mathcal{P}[n]$, we will also be interested in members of $\mathcal{T}$ that are subsets of $B$, supersets of $B$ or disjoint from $B$. We define

$$\text{des}_\mathcal{T}(B) = \{ T \in \mathcal{T} : T \subsetneq B \},$$

$$\overline{\text{des}}_\mathcal{T}(B) = \text{des}_\mathcal{T}(B) \cup \{ B \},$$

$$\text{anc}_\mathcal{T}(B) = \{ T \in \mathcal{T} : B \subsetneq T \},$$

$$\text{disj}_\mathcal{T}(B) = \{ T \in \mathcal{T} : B \cap T = \emptyset \}.$$

For $A \in \mathcal{C}$, the tree structure of $\mathcal{C}$ leads to the partitions $\mathcal{C} = \overline{\text{des}}_\mathcal{C}(A) \cup \text{anc}_\mathcal{C}(A) \cup \text{disj}_\mathcal{C}(A)$ and $\mathcal{L} = \overline{\text{des}}_\mathcal{L}(A) \cup \text{anc}_\mathcal{L}(A) \cup \text{disj}_\mathcal{L}(A)$. Consequently, we decompose $v_C(A)$ as

$$v_C(A) = C(A, \overline{\text{des}}_\mathcal{C}(A)) + C(A, \text{anc}_\mathcal{C}(A)) + C(A, \text{disj}_\mathcal{C}(A))$$

$$+ D(A, \overline{\text{des}}_\mathcal{L}(A)) + D(A, \text{anc}_\mathcal{L}(A)) + D(A, \text{disj}_\mathcal{L}(A)),$$

in the separability proofs that follow. But first, we prove a quick but useful lemma.

**Lemma 13** Let $A \in \mathcal{C}$. Consider distinct outcomes $x = x_A u_{\circ A}$ and $y = y_A u_{\circ A}$ that are identical on $\circ A$. If $C(x_A, \overline{\text{des}}_\mathcal{C}(A)) \neq C(y_A, \overline{\text{des}}_\mathcal{C}(A))$ then

$$|C(x_A, \overline{\text{des}}_\mathcal{C}(A)) - C(y_A, \overline{\text{des}}_\mathcal{C}(A))| \geq c_A \geq 1.$$
Proof If \( c_B \) is a summand in either \( C(x_A, \overline{\text{des}}_C(A)) \) or \( C(y_A, \overline{\text{des}}_C(A)) \), then \( B \subset A \), so that \( c_B = 2g(B) \geq 2g(A) = c_A \). Since \( c_A \) divides every term in both \( C(x_A, \overline{\text{des}}_C(A)) \) and \( C(y_A, \overline{\text{des}}_C(A)) \), it also divides their difference.

We are now prepared to prove our four lemmas. The nonseparability proofs use the notation \( x_{[1,2,3]} = 130_n = 001 \) from Definition 11 for constructing sparse outcomes on \([n]\).

3.3.1 Proof of Lemma 9

Let \( \succeq \) be the ordering induced by \( v_C \). We prove that if \( A \in C \) then \( A \) is separable. Consider distinct partial outcomes \( x_A \) and \( y_A \). Let \( u_{\delta A} \) be any partial outcome on \( -A \) and define \( x = x_A u_{\delta A} \) and \( y = y_A u_{\delta A} \). We claim that the sign of the difference

\[
v_C(x) - v_C(y) = (C(x, C) + D(x, L)) - (C(y, C) + D(y, L))
\]

(a) only depends upon \( x_A \) and \( y_A \), and (b) is independent of the particular choice of \( u_{\delta A} \).

Recall that in the ordering induced by \( v_C \), the inequality \( v_C(x) > v_C(y) \) corresponds to the preference \( x \succ y \). So this claim is equivalent to the separability of set \( A \).

Case 1: \( C(x, \overline{\text{des}}_C(A)) > C(y, \overline{\text{des}}_C(A)) \). We claim that \( x_A u_{\delta A} > y_A u_{\delta A} \). We have

\[
v_C(x) - v_C(y) > C(x, C) - C(y, C) - 1
\]

by Eq. 16. Corollary 1 shows that when \( B \in \circ A \), we have \( v_B(x_A u_{\delta A}) = v_B(y_A u_{\delta A}) \). Therefore, we can ignore the sets in \( \text{disj}_C(A) \) when calculating \( v_C(x) - v_C(y) \).

\[
v_C(x) - v_C(y) > C(x_A, \overline{\text{des}}_C(A)) + C(y_A, \text{anc}_C(A))
\]

\[
- C(y_A, \overline{\text{des}}_C(A)) - C(y_A, \text{anc}_C(A)) - 1
\]

\[
\geq 2g(A) - C(y_A, \text{anc}_C(A)) - 1
\]

\[
\geq 2g(A) - (2g(A) - 1) - 1 = 0
\]

by Lemma 13 and Eq. 14. We conclude that \( x_A u_{\delta A} > y_A u_{\delta A} \) for every choice of \( u_{\delta A} \).

Case 2: \( C(x, \overline{\text{des}}_C(A)) < C(y, \overline{\text{des}}_C(A)) \). We have \( x_A u_{\delta A} < y_A u_{\delta A} \) for every choice of \( u_{\delta A} \) by reversing the roles of \( x \) and \( y \) in Case 1.

Case 3: \( C(x, \overline{\text{des}}_C(A)) = C(y, \overline{\text{des}}_C(A)) \). We claim that

\[
v_C(x) - v_C(y) = D(x, L) - D(y, L).
\]

(17)

As in Case 1, we can ignore the sets in \( \text{disj}_C(A) \). Next, observe that \( x_A \) and \( y_A \) have the same parity. If this were not true, then exactly one of \( C(x, \overline{\text{des}}_C(A)) \) and \( C(y, \overline{\text{des}}_C(A)) \) would include \( c_A \), which would guarantee \( C(x, \overline{\text{des}}_C(A)) \neq C(y, \overline{\text{des}}_C(A)) \). Indeed, \( c_A \) is the unique smallest summand, so no combination of other terms could properly compensate for the small difference. By Corollary 2, the matching parity of \( x_A \) and \( y_A \) means that \( C(x_A, \text{anc}_C(A)) = C(y_A, \text{anc}_C(A)) \). Since \( \text{anc}_C(A) \) is a nested chain of subsets, we conclude that \( C(x, \text{anc}_C(A)) = C(y, \text{anc}_C(A)) \) as well. Our assumption that \( C(x, \overline{\text{des}}_C(A)) = C(y, \overline{\text{des}}_C(A)) \) means that \( C(x, C) = C(y, C) \), so Eq. 17 holds.

We now calculate \( D(x, L) - D(y, L) \). We can ignore the sets in \( \text{disj}_C(A) \) since their contributions only depend on the shared partial outcome \( u_{\delta A} \). Since \( x_A \) and \( y_A \) have the same parity, Corollary 2 guarantees that \( D(x_A, \text{anc}_C(L(A)) = D(y_A, \text{anc}_C(L(A)) \). Therefore,

\[
v_C(x) - v_C(y) = D(x_A, \overline{\text{des}}_L(A)) - D(y_A, \overline{\text{des}}_L(A))
\]
and this value is independent of the choice of partial outcome \( u \circ A \). This proves that every \( A \in C \) is separable in the ordering induced by \( v_C \).

### 3.3.2 Proof of Lemma 10

Let \( \succeq \) be the ordering induced by \( v_C \). Let \( B \not\in C \) be a set that is not \( C \)-constructible. We show that \( B \) is nonseparable by assembling two partial outcomes \( x_B \not\succeq y_B \) so that the preference between \( x = x_B u \prec_B u \) and \( y = y_B u \prec_B u \) depends on the choice of \( u \prec_B u \).

Let \( B \subseteq [n] \) be a nonempty set that is not \( C \)-constructible. Let \( K = K_C(B) \subseteq B \) be the \( C \)-construct of \( B \). (Note that we might have \( K = \emptyset \); the set \( B = \{1\} \) in the Hasse diagram of Fig. 2 is one such example.) Consider the set \( F = \{ A \in C \mid A \cap (B - K) \neq \emptyset \} \). Note that \( F \neq \emptyset \) since \( [n] \in F \). Pick a minimal set \( A' \in F \), meaning that \( A' \) does not contain any other member of \( F \). Observe that \( A' - B \neq \emptyset \) by our choice of \( K \). Let \( a_1 \in A' \cap (B - K) \) and let \( a_2 \in A' - B \). By the structure of our tree collection and the minimality of \( A \), if \( a_1 \in A \) for some \( A \in C \), then \( A' \subseteq A \).

We have the freedom to construct \( x = x_B w \prec_B u \) and \( y = y_B w \prec_B u \) any way we like. Using the notation from Definition 11 and recalling that \( a \in A \), \( B \subseteq [n] \), \( v \in B \) we take

\[
x_B = 0_s, \quad u \prec_B u = 0_s, \quad y_B = 1_{a_1} 0_s, \quad v \prec_B u = 1_{a_2} 0_s.
\]

Let \( S \in C \cup L \) such that \( a_1 \notin S \). By Corollary 1, we have the equality \( v_S(x_B w \prec_B u) = v_S(y_B w \prec_B u) \). This means that for any partial outcome \( w \prec_B u \), the difference between \( v_C(x_B w \prec_B u) \) and \( v_C(y_B w \prec_B u) \) must be caused by coefficients of sets \( S \in C \cup L \) with \( a_1 \in S \). As noted above, \( A' \subseteq S \) because \( A' \) is the minimal set in \( F \) that contains \( a_1 \). Therefore \( a_2 \in S \) as well.

The preference order of \( x_B w \prec_B u \) and \( y_B w \prec_B u \) is determined by the sign of \( v_C(x_B w \prec_B u) - v_C(y_B w \prec_B u) \). We have

\[
C(x_B w \prec_B u, C) - C(y_B w \prec_B u, C) = c_{A'}(v_{A'}(x_B w \prec_B u) - v_{A'}(y_B w \prec_B u)) + \sum_{A \in \text{ance}(A')} c_A (v_A(x_B w \prec_B u) - v_A(y_B w \prec_B u)).
\]

We use Eq. 16 to bound the impact of the sublink coefficients. Observe that \( x_B u \prec_B u \) is even in \( A' \) while \( y_B u \prec_B u \) is odd in \( A' \). Therefore,

\[
v_C(x_B u \prec_B u) - v_C(y_B u \prec_B u) > c_{A'} - \sum_{A \in \text{ance}(A')} c_A - 1 \geq 0
\]

by Eq. 14. Similarly, \( x_B v \prec_B u \) is odd in \( A' \) while \( y_B v \prec_B u \) is even in \( A' \), so

\[
v_C(x_B v \prec_B u) - v_C(y_B v \prec_B u) < -c_{A'} + \sum_{A \in \text{ance}(A')} c_A + 1 \leq 0.
\]

We have shown that \( x_B u \prec_B u \succ y_B u \prec_B u \), while \( x_B v \prec_B u \prec y_B v \prec_B u \). Therefore, \( B \) is nonseparable.

### 3.3.3 Proof of Lemma 11

Let \( \succeq \) be the ordering induced by \( v_C \). Consider a constructible set \( B \notin C \) where \( C_B(P) = \{B_1, B_2, \ldots, B_k\} \subseteq C \) and all of the \( B_i \) are children of \( P \in C \). Note that \( k \geq 2 \) since \( B = K_C(B) = \bigcup_{i=1}^k B_i \) while \( B \notin C \). We prove that \( B \) is nonseparable.
Let the remaining children of \( P \) be \( A_1, \ldots, A_\ell \) where \( \ell \geq 1 \) (because \( B \not\subseteq P \)) and perhaps one \( A_j \) is a ghost child of \( P \). For \( i = 1, 2, \) let \( b_i \in B_i - K_C(B_i) \), and for \( 1 \leq j \leq \ell \), pick any \( a_j \in A_j \). Consider the partial outcomes \( x_B = 1_{b_2} 0_a \) and \( y_B = 1_{b_1} 0_u \) on \( B \) and the partial outcomes \( u_B = 0_a \) and \( v_B = 1_{a_1} 1_{a_2} \cdots 1_{a_\ell} 0_s \). We will show that our preference between \( x_B u_B \) and \( y_B u_B \) is the opposite of our preference between \( x_B v_B \) and \( y_B v_B \).

Because \( P \) is the parent of both \( B_1, B_2 \), the partial outcomes \( x \) and \( y \) have the same parity in every set in \( C - \{ B_1, B_2 \} \). In addition, \( B_1 \) and \( B_2 \) are in the same generation, so for any partial outcome \( w_B \) on \( -B \), we have
\[
C(x_B w_B, C) = C(y_B w_B, C),
\]
which means that
\[
\nu_C(x_B w_B) - \nu_C(y_B w_B) = D(x_B w_B, L) - D(y_B w_B, L).
\]
The parities of these outcomes agree on all sublinks, except for those of the form \( B_1 \cup A_j \) and \( B_2 \cup A_j \). Our choice of partial outcomes \( u \) and \( v \) flips the parities of the outcomes in these sets so that
\[
D(x_B u_B, L) - D(y_B u_B, L) = D(y_B v_B, L) - D(x_B v_B, L)
\]
and this value is nonzero by Lemma 7. Therefore, \( x_B u_B > y_B u_B \) while \( x_B v_B < y_B v_B \), so \( B \) is nonseparable.

### 3.3.4 Proof of Lemma 12

Let \( \preceq \) be the ordering induced by \( \nu_C \). In this section, we prove the nonseparability of a \( C \)-constructible set \( B \not\subseteq C \) where at least two sets in \( \Upsilon_C(B) \) have different parents.

For a nonempty set \( S \), note that there is at least one outcome that is even on \( S \) (the all-zero outcome \( 0_s \)) and at least one outcome that is odd on \( S \) (the indicator outcome \( 1_s \) for \( s \in S \)). We start with a lemma that constructs an outcome with a specified behavior on a given chain of subsets in \( C \). We anticipate that the construction technique of this lemma will be useful beyond its application in proving Lemma 12.

**Lemma 14** Consider a chain of nested sets \( \emptyset \neq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r \). For any \( T \subseteq [r] \), there is an outcome \( w \) such that for \( 1 \leq i \leq r \), we have \( \nu_{A_i}(w) = 1 \) if \( i \in T \) and \( \nu_{A_i}(w) = 0 \) if \( i \in [r] - T \).

**Proof** For \( 1 \leq i \leq r \), let \( a_i \in A_i - A_{i-1} \), where we take \( A_0 = \emptyset \). We recursively construct an outcome of the form
\[
w = w_{a_1} w_{a_2} \cdots w_{a_r} 0_s.
\]
If \( 1 \in T \) then take \( w_{a_1} = 0 \). If \( 1 \notin T \) then take \( w_{a_1} = 1 \). For \( 2 \leq i \leq r \), take \( w_{a_i} \) to be 0 or 1 depending on whether \( w_{a_1} \cdots w_{a_{i-1}} \) is even or odd and whether \( i \in T \).

As an example, consider the nested chain \( A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \) given by
\[
\{1, 2\} \subseteq \{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}
\]
and let \( T = \{1, 3\} \). We choose \((a_1, a_2, a_3, a_4) = (1, 3, 4, 7)\) so that \( a_i \in A_i - A_{i-1} \), where we set \( A_0 = \emptyset \). The outcome \( 0_1 1_3 0_4 1_7 0_5 = 00100010 \) is even on \( A_1, A_3 \) and odd on \( A_2, A_4 \).
We now begin the proof of Lemma 12 in earnest. Consider a constructible set $B \notin C$ with $\anc_C(B) = \{B_1, B_2, \ldots, B_k\} \subset C$ where $B_1$ and $B_2$ are not siblings in $C$. For $i = 1, 2$ let $K_i = K_C(B_i) \subseteq B_i$ and $b_i \in B_i - K_i$.

Consider the partial outcomes $x_B = 1_{b_2} 0_s$ and $y_B = 1_{b_1} 0_s$ on $B$. We must track the parity of the partial outcomes $x_B$ and $y_B$ with respect to $S \subset C$. Observe that $x_B$ is even in $B_1$ and odd in $B_2$, while $y_B$ is odd in $B_1$ and even in $B_2$. More generally, if $B_1 \subset S$ but $S \cap B_2 = \emptyset$ then $x_B$ is even in $S$ while $y_B$ is odd in $S$, and if $B_2 \subset S$ but $S \cap B_1 = \emptyset$ then $x_B$ is odd in $S$ while $y_B$ is even in $S$. Next, we note that if $S \cap B_1 = \emptyset$ and $S \cap B_2 = \emptyset$, then both $x_B$ and $y_B$ are zero (hence even) in $S$. Finally, if $B_1 \cup B_2 \subset S$, then $x_B$ and $y_B$ have the same parity in $S$ by Corollary 2.

Let $g_1 = g(B_1)$ and $g_2 = g(B_2)$ denote the generations of $B_1$ and $B_2$, respectively. Let $P$ be the first shared ancestor of $B_1$ and $B_2$. We may assume that $g_1 \geq g_2 > g$ and that $\anc_C(B_1) - \anc_C(B_2) \neq \emptyset$ because $B_1$ and $B_2$ are not siblings. We are ready to use Lemma 14 to construct $u_{-B}$ and $v_{-B}$ that change the preference between the partial outcomes $x_B$ and $y_B$. There are two cases, depending on shared ancestry of $B_1$ and $B_2$.

We may assume that either $g_1 > g_2 = g + 1$ or $g_1 \geq g_2 > g + 1$. For $i = 1, 2$, let $Q_i \in C$ be the child of $P$ that contains $B_i$. (If $B_2$ is a child of $P$ then $B_2 = Q_2$.) Let $R_1$ be the parent of $B_1$ (so that when $g_1 = g + 2$, we have $R_1 = Q_1$).

Let $r_1 \in R_1 - B_1$. Observe that that $r_1 \notin B_2$, while $r_1, b_1, b_2 \in P$. We take $u_{-B} = 0_s$ to be the all-zero outcome and $v_{-B} = 1_{r_1} 0_s$ to be the indicator outcome on $r_1$. Let $S \in \anc_C(B_1) \cap \desc_C(P)$ and $T \in \anc_C(B_2) \cap \desc_C(P)$. Note that the ancestry $\anc_C(B_2) \cap \desc_C(P) = \emptyset$ when $B_2$ is a child of $P$; in this case, statements below concerning $T$ hold vacuously.

First, we consider outcomes that end in $u_{-B}$. Observe that $x_B u_{-B} = 1_{b_2} 0_s$ is even in $B_1$, but odd in $B_2$. Meanwhile, $y_B u_{-B} = 1_{b_1} 0_s$ is odd in $B_1$, but even in $B_2$. The outcome $x_B u_{-B}$ is even in $S$ (odd in $T$) while $y_B u_{-B}$ is odd in $S$ (even in $T$). Therefore

$$C(x_B u_{-B}, C) - C(y_B u_{-B}, C) = \alpha^{g_1} + \sum_{i=g+1}^{g_1-1} \alpha^i - \left(\alpha^{g_2} + \sum_{i=g+1}^{g_2-1} 2^i\right) = \sum_{i=g+2}^{g_1} \alpha^i > 1.$$

Next, we consider outcomes that end in $v_{-B}$. Observe that $x_B v_{-B} = 1_{b_2} 1_{r_1} 0_s$ is even in $B_1$ but odd in $B_2$, while $y_B u_{-B} = 1_{b_1} 1_{r_1} 0_s$ is odd in $B_1$ but even in $B_2$. Turning to the ancestry to $P$, we see that $x_B v_{-B}$ is odd in $S$ (even in $T$) while $y_B v_{-B}$ is even in $S$ (odd in $T$). Therefore

$$C(x_B v_{-B}, C) - C(y_B v_{-B}, C) = \alpha^{g_1} + \sum_{i=g+1}^{g_2-1} \alpha^i - \left(2^{g_1} + \sum_{i=g+1}^{g_1-1} \alpha^i\right) = \alpha^{g_1} - \left(2^{g_2} + \sum_{i=g_2}^{g_1-1} \alpha^i\right) < 0$$

by Lemma 8.

We have shown that $x_B u_{-B} > y_B u_{-B}$ and $x_B v_{-B} < y_B v_{-B}$, so the set $B$ is nonseparable.
3.4 Tree Characters for Preference Orders

The hard work is now behind us, and we can finally prove Theorem 1. Given a tree collection $\mathcal{C} \subseteq \mathcal{P}([n])$, Theorem 3 constructs a preorder $\succeq$ with character $\mathcal{C}$. All that remains is to adjust this preorder, breaking ties while also preserving its separability properties.

We formulate the main argument as a general lemma which will be useful in future voter basis constructions. This lemma breaks preorder ties by adding a small vector in $\text{span}(v_1, v_2, \ldots, v_n)$.

**Lemma 15** Let the preference vector $v \in \mathcal{P}^n$ induce preference preorder $\succeq$ with character $\mathcal{C}$. Suppose that for every nonseparable set $B \notin \mathcal{C}$, there exist partial outcomes $x_B, y_B, u_B, v_B$ such that

$$x_B u_B > y_B u_B \quad \text{and} \quad x_B v_B < y_B v_B.$$ \hspace{1cm} (18)

Let

$$\gamma = \min\{|v(x) - v(y)| : x, y \in X_n \text{ such that } v(x) \neq v(y)\}.$$  

Then the preference vector

$$v' = v + \sum_{i=1}^n \frac{\gamma}{2^i}v_{[i]}$$

Induces a preference order $\succeq'$ with character $\mathcal{C}$.

The technical condition (18) on each nonseparable set $B$ means that at least one preference between partial outcomes on $B$ truly flips, rather than merely changing to indifference.

**Proof** First, we observe that entries of $v'$ are pairwise distinct, so $v'$ induces a total order. Consider distinct outcomes $x \neq y$. If $v(x) - v(y) = 0$ then

$$|v'(x) - v'(y)| = \left| \sum_{i=1}^n \frac{\gamma}{2^i}(v_{[i]}(x) - v_{[i]}(y)) \right| \geq \gamma/2^n > 0.$$

On the other hand, if $|v(x) - v(y)| \geq \gamma$, then $|v'(x) - v'(y)| \geq \gamma/2^{n+1} > 0$.

Consider any $\succeq$-nonseparable set $B \notin \mathcal{C}$. Let $x_B, y_B, u_B, v_B$ be the partial outcomes guaranteed by the lemma. We have $v(x_B u_B) - v(y_B u_B) \geq \gamma$, and therefore

$$v'(x_B u_B) - v'(y_B u_B) \geq \gamma/2^{n+1} > 0.$$  

Similarly, 

$$v'(x_B v_B) - v'(y_B v_B) \leq -\gamma/2^{n+1} < 0.$$  

Since our $\succeq'$-preference for partial outcomes $x_B, y_B$ flips, the set $B$ is $\succeq'$-nonseparable.

Now consider any $\succeq$-separable set $A \in \mathcal{C}$. First, suppose that $x_A > y_A$, meaning that $v(x_A u_A) - v(y_A u_A) > \gamma > 0$ for every partial outcome $u_A$. It follows directly that $v'(x_A u_A) - v'(y_A u_A) > \gamma/2^{n+1} > 0$. So $x_A > y_A$ and $A$ is $\succeq'$-separable.

Second, suppose that we have $\succeq$-indifference $x_A \sim y_A$, so that $v(x_A u_A) = v(y_A u_A)$ for any partial outcome $u_A$. We now have

$$v'(x_A u_A) - v'(y_A u_A) = \sum_{i \in x_A} \frac{\gamma}{2^i}v_{[i]} - \sum_{i \in y_A} \frac{\gamma}{2^i}v_{[i]} \neq 0$$

because $x_A \neq y_A$. We are no longer indifferent, but our $\succeq'$-preference is independent of partial outcome $u_A$. Therefore $A$ is $\succeq'$-separable.  

\[ \square \]
We conclude with our proof of Theorem 1

(Theorem 1) Every tree collection is admissible.

Proof Let $\mathcal{C} \subseteq \mathcal{P}(\{1, \ldots, n\})$ be a tree collection, and let $v \in \mathcal{P}^n$ be the preference vector constructed in Theorem 3. (If $v$ induces a total order, then we are done.) We claim that $v$ adheres to the conditions of Lemma 15.

There are three types of nonseparable sets in the construction. Their nonseparability is proven in Lemmas 10, 11 and 12. In each of these lemmas, we prove that $B$ is nonseparable by constructing partial outcomes $x_B, y_B, u_B, v_B$ such that $x_B u_B \succ y_B u_B$ and $x_B v_B \prec y_B v_B$, as required by Lemma 15. Therefore we can construct a preference vector $v'$ that induces a preference order with character $\mathcal{C}$.

4 Conclusions and Future Work

The admissibility problem asks which collections of sets correspond to characters of preference orders. We have used the voter basis to create preference orders with desired separability properties. In particular, we have shown that every tree collection is admissible. Our tree construction taps into the potential of the voter basis for character construction: this flexible method has strong potential to further expand the known families of preference characters. We also wonder whether the voter basis can provide insight into the class of completely separable preferences.

Another open question is whether admissibility for preorders is equivalent to admissibility for totals orders. Our Lemma 15 provides a way to alter a preference preorder vector $v$ to obtain a preference order vector $v'$ without changing the collection of separable sets. This lemma imposes a technical condition on the nonseparable sets. In particular, our construction does not apply to a preorder with a nonseparable set that only alternates between indifference and preference (rather than reversing the preference ranking).

Finally, we return to the voter basis itself. Our proof that $V_n$ forms a basis for the preference space $\mathcal{P}^n$ (Theorem 2(a)) is short and effective. However, its simplicity hides the deep connection between constructing voter preferences and the symmetries of the hypercube. We are hopeful that the representation theory underlying the voter basis can provide further insight into the admissibility problem and the structure of completely separable preferences.

Acknowledgements We thank Tom Halverson for many insightful conversations and his help in developing the voter basis; Jeremy Martin for suggesting Lemma 3; and Trung Nguyen and Tuyet-Anh Tran for their careful readings of earlier drafts. We also thank the anonymous referees for their feedback, which has significantly improved the exposition.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Barberá, S., Bossert, W., Pattanaik, P.K.: Ordering sets of objects. In: Barberá, S., Hammond, P.J., Seidl, C. (eds.) Handbook of Utility Theory, vol. 2, chapter 17, pp. 893–977. Springer (2004)
2. Bjorkman, B., Gravelle, S., Hodge, J.K.: Cubic preferences and the character admissibility problem. Math. Soc. Sci. 99, 5–17 (2019)
3. Bradley, W.J., Hodge, J.K., Kilgour, D.M.: Separable discrete preferences. Math. Soc. Sci. 49(1), 335–353 (2005)
4. Brams, S.J., Kilgour, D.M., Zwicker, W.S.: Voting on referenda: the separability problem and possible solutions. Elect. Stud. 16, 359–377 (1997)
5. Christian, R., Conder, M., Slinko, A.: Flippable pairs and subset comparisons in comparative probability orderings. Order 24, 193–213 (2007)
6. Daugherty, Z., Eustis, A.K., Minton, G., Orrison, M.E.: Voting, the symmetric group, and representation theory. Am. Math. Mon. 116(8), 667–687 (2009)
7. de Finetti, B.: Sul significato soggettivo della probabilità. Fundam. Math. 17, 298–329 (1931)
8. Fishburn, P.C.: Finite linear qualitative probability. J. Math. Psychol. 40, 64–77 (1996)
9. Hodge, J.K.: The mathematics of referendums and separable preferences. Math. Mag. 84(4), 268–277 (2011)
10. Hodge, J.K., Klima, R.E.: The Mathematics of Voting and Elections: a Hands-on Approach. Volume 22 of Mathematical World. American Mathematical Society, Providence (2005)
11. Hodge, J.K., Krines, M., Lahr, J.: Preseparable extensions of multidimensional preferences. Order 26(2), 125–147 (2009)
12. Hodge, J.K., TerHaar, M.: Classifying interdependence in multidimensional binary preferences. Math. Soc. Sci. 55(2), 190–204 (2008)
13. Kraft, C.H., Pratt, J.W., Seidenberg, A.: Intuitive probability on finite sets. Ann. Math. Stat. 30, 408–419 (1959)
14. Lacy, D., Niou, E.M.S.: A problem with referendums. Journal of Theoretical Politics 12(1), 5–31 (2000)
15. Maclagan, D.: Boolean term orders and the root system $b_n$. Order 15, 279–295 (1999)
16. Martin, J.: Personal communication (2018)
17. Shocker, A.D., Bayus, B.L., Kim, N.amwoon.: Product complements and substitutes in the real world: the presence of other products. J. Mark. 68, 28–40 (2004)
18. Slinko, A.: Additive representability of finite measurement structures. In: Brams, S.J., Gehrlein, W.V., Roberts, F.S. (eds.) The Mathematics of Preference, Choice and Order, pp. 113–133. Springer (2009)
19. Sloane, N.J.A.: The On-line Encyclopedia of Integer Sequences. https://oeis.org/A005806 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.