Stochastic Response, Cascading and Control of Colored Noise in Dynamical Systems

Rue-Ron Hsu, Jyh-Long Chern, Wei-Fu Lin, and Chia-Chu Chen

Nonlinear Science Group, Department of Physics,
National Cheng Kung University,
Tainan, Taiwan 70101, Republic of China

ABSTRACT

We analytically determine the correlation functions of the stochastic response of a generic mapping system driven by colored noise. We also address the issue of noise cascading in coupled-element systems, particularly in a uni-directionally coupled system with a dc-filtered coupling. A noise-feedback method and an opposite-pairing method are introduced to control the stochastic response. Excellent agreements are obtained between analytical results and numerical simulations.

PACS number(s): 05.45.+b, 05.40.+j, 02.50.Fz.
1 Introduction

The stochastic response of dynamical systems to noise has attracted much attention recently and is a recurrent theme in the study of nonlinear systems. Some classic works were accomplished on the persistence of dynamical system under random perturbation [1]. Recently, there are many novel and intriguing phenomena induced by noise, for instance, noise-induced order [2], synchronization by external noise [3], stochastic resonance [4], noise-driven coherence resonance [5], and noise-assisted stabilization [6]. Some experimental verifications of these phenomena have been found in extremely complicated real systems, such as the brain [7] and plasma [8], and the physical picture can be captured by dynamical models with simple additive noise. On the other hand, noise-induced multistability has been addressed in exploring the influence of multiplicative and additive noise [9], and the reentrance phenomenon was introduced in studying the dynamical system with colored noise [10]. The colored noise is the noise source with finite correlation time. In opposite to colored noise, white noise have not any correlation between time lapse. Whenever the noise is dynamical involved, the correlation functions of the stochastic response of dynamical system to colored noise is usually very difficult to obtain exactly. Several theories, for examples, the small correlation time theory [11], the functional-calculus theory [12], the decoupling theory [13], the unified colored-noise theory [14] and the interpolation
procedure [15], were proposed to treat the colored noise problem in stochastic processes. However, analytical results can only be obtained under the limit of small or infinite correlation time. Therefore, how to determine the stochastic response of the colored noise exactly, is still an important issue.

In another front, it is known that collective behaviour which arises from the coupling among the elements could be dramatically different from the behaviour of individual element, and is also a current interest in nonlinear dynamics. In the deterministic systems, some novel features, such as self-induced spatial disorder and chaotic itinerancy, have been numerically found in distributed nonlinear element systems [16]. Universal critical behaviours have also been found in coupled map lattices [17] and the first order phase transition was studied in coupled oscillator systems [18]. Nevertheless, toward a more realistic model, one should include the noisy elements. Indeed, the stochastic resonance has been explored by numerical simulation and experiment in coupled-element systems with additive noise [19]. However, it is of fundamental interest to provide more illustrative example to clarify the exact role played by noise in the coupled system.

In this paper, we analytically determine the correlation functions of the stochastic response of the steady state when the colored noise is dynamically involved in a generic mapping system. We also investigate the issue of noise cascading in coupled dynamical systems. As an analytical illustration, we adopt a special model in which all the elements are uni-directionally coupled
to each other, specifically, we consider a dc-filtered coupling. The coupling is fundamental different to the global or nearest-neighbour coupling in spatially extended system[16,17], which is receiving much attention nowadays. But, this simple model thus serves as an analytically solvable model to illustrate the nature of noise during propagation in spatial extent under the influence of dynamical structures. Furthermore, we found two elegant ways to control the stochastic response.

2 Stochastic response of colored noise

To begin with, we consider a generic one-dimension nonlinear map in which the noise is involved dynamically,

$$x_{n+1} = f_r(x_n) + \zeta_n.$$  \hspace{1cm} (1)

where $\zeta_n$ can be colored or white noise with zero mean fluctuation. For simplicity, we only treat the stochastic response of the fixed point, one can extend our results to higher periods by following the similar procedure in Reference [20]. The dispersive output can also be expressed as

$$x_n = \overline{x} + \xi_n,$$  \hspace{1cm} (2)

where $\overline{x}$ is the noise-free fixed point, $\overline{x} = f_r(\overline{x})$, and $\xi_n$ is the stochastic response. From now, the amplitude of the noise and stochastic response are assumed to be small in comparison with the fixed point, such that the
perturbation scheme is applicable. By substituting Eq.(2) into Eq.(1), a recurrence relation is obtained:

$$\xi_{n+1} = M \xi_n + \zeta_n,$$

(3)

where $M = \frac{\partial f_r (x)}{\partial x} |_x$. With $|M| < 1$, we will, after the transient, have a small response fluctuation around the stable noise-free fixed point. By iterating Eq.(3), we have

$$\xi_k = M^k \xi_0 + \sum_{j=1}^{k} (M)^{j-1} \zeta_{k-j}$$

(4)

After passing the transient state, i.e. $k >> 1$, the first term of Eq.(4) can be neglected. As a result, the mean fluctuation of stochastic response is zero, if $< \zeta_k > = 0$. Here, $< A_k >\equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} A_{k+i}$ denotes the long time average of $A_{k+i}$. The index $k$ just reminds us that we are taking average for those states after the transient $k >> 1$, and it is irrelevant to the average. The zero mean of stochastic response imply that the average of dispersive output $x_n$ is same as the fixed point $\bar{x}$. The general relations between the correlations of stochastic response and those of input noise can be obtained in compact forms:

$$< \xi_k \xi_k > = \frac{1}{1 - M^2} \left\{ < \zeta_k \zeta_k > + 2 \sum_{i=1}^{\infty} M^i < \zeta_k \zeta_{k+i} > \right\}$$

(5)

$$< \xi_k \xi_{k+\ell} > = \frac{M^\ell}{1 - M^2} \left\{ < \zeta_k \zeta_k > + 2 \sum_{i=1}^{\infty} M^i < \zeta_k \zeta_{k+i} > \right\}$$

$$+ \sum_{j=1}^{\ell} \sum_{i=0}^{\infty} M^{\ell-j+i} < \zeta_k \zeta_{k+j+i} >, \quad \ell \geq 1.$$  

(6)
If the correlations of the input noise are given, one can obtain the correlation of stochastic response after the summations are accomplished. For example, if the input noise $\zeta_k$ is a white noise and is denoted by $\eta_k$ which have zero mean and derivation $D = <\eta^2>$, then the mean fluctuation of stochastic response is zero and the correlation function of the stochastic response is

$$<\xi_k \xi_{k+\ell}> = \frac{M^\ell D}{1 - M^2}, \quad \ell \geq 0. \quad (7)$$

Eq.(7) shows that the stochastic responses are colored when the white noise is dynamically convoluted by a generic map unless the system is critical i.e. $M = 0$. The correlation time $\tau_c$ of the colored response can be identified as $\tau_c = -(|\ln |M||)^{-1}$.

Next, we use the result of Eq.(7) as the input colored noise to drive another system, $\tilde{x}_{n+1} = \tilde{f}_r(\tilde{x}_n) + \xi_n$. With Eqs.(5) and (6), the mean fluctuation and the correlation of the stochastic responses, $\tilde{\xi}_n = \tilde{x}_n - \bar{x}$, of the new system could be deduced. They are $<\tilde{\xi}_k> = 0$, and

$$<\tilde{\xi}_k \tilde{\xi}_k> = \frac{D}{(1 - M^2)^2} \left[ \frac{1}{(1 - M^2)(1 - MM)} \right], \quad (8)$$

$$<\tilde{\xi}_k \tilde{\xi}_{k+\ell}> = \frac{D}{(1 - M^2)} \left\{ \frac{1}{(1 - MM)} \right. \times \left[ \frac{M(\tilde{M}^\ell - M^\ell)}{\tilde{M} - M} + (1 + \tilde{M}M)^\ell \right] \}, \quad \ell \geq 1, \quad (9)$$

where $|\tilde{M}| = |\frac{\partial \tilde{f}_r}{\partial \tilde{x}}| < 1$ and $\bar{x} = \tilde{f}_r(\bar{x})$. The response and the correlation time are related to the dynamical structures $M$ and $\tilde{M}$ explicitly. Eqs.(8) and (9) also suggest that the correlation of the response could be greatly
enhanced by the variation of dynamical structure. Referring to Fig.1, when
the first stage is at the critical point \((M = 0)\), the stochastic response of
the second stage exhibits a symmetry at the critical point \((\tilde{M} = 0)\). The
symmetry is broken as the dynamics of the first stage moves away from its
critical point. The minimum of stochastic response will shift to \(M\tilde{M} < 0\)
region. This fact will provide us a way to control the stochastic response and
we will return to this point later.

Moreover, the extension of this result to higher periods case straightforward but tedious. For example, to explore the stochastic response of each
state of period-2, we shall divide the output data into two sets, called even and
odd parts, and estimate the correlation for each set individually. Then, the
correlations of stochastic response on period-2 states can be simply carried
out by converting \(M\) and \(\zeta_n\) to \(M_2 = \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\) and \(b_n = \frac{\partial f}{\partial x_1}\zeta_n + \zeta_{n+1}\) in
Eqs.(5) and (6) respectively. Where \(\bar{x}_1\) (\(\bar{x}_2\)) is the mean value of odd (even)
part of dispersive output, i.e. the period-2 orbit for noise-free case, and \(\bar{x}'\) is
\(\bar{x}_2\) (\(\bar{x}_1\)) when the index \(n\) of \(b_n\) is even (odd).

### 3 Noise cascading

As mentioned above, the noise cascading problem in coupled-element
systems, especially for spatially extend coupled system, is remained as a
challenge issue to be addressed. To gain more illustrative features, we take a
dynamical system, called the first stage, which is driven by a white noise. The
stochastic response is extracted by a dc-filter and used as the noise to drive another system, called the second stage. This procedure could be repeated successively. Physically, all the dynamics are described by the following mapping,

\[ x^{(1)}_{n+1} = f_r^{(1)}(x^{(1)}_n) + \eta_n \]
\[ x^{(2)}_{n+1} = f_r^{(2)}(x^{(2)}_n) + \xi^{(1)}_n \]
\[ \ldots \]
\[ x^{(m)}_{n+1} = f_r^{(m)}(x^{(m)}_n) + \xi^{(m-1)}_n, \]

where \( \xi^{(i)}_n = x^{(i)}_n - \mathbf{x}^{(i)} \), and can be extracted by a dc-filter. The essential feature of noise cascading can be addressed by treating identical systems (all \( f_r^{(i)} \) are the same). To obtain an explicit form of analytical result we will assume \(|M| << 1\). Using Eqs. (5) and (6), and keeping the lowest order of \( M \), it can be proven that the correlation functions for the \( m \)-th (\( m \geq 2 \)) stage are

\[ <\xi^{(m)}_k \xi^{(m)}_k> \approx (1 + m^2 M^2)D \]
\[ <\xi^{(m)}_k \xi^{(m)}_{k+\ell}> \approx C^\ell_{m-1} M^\ell D, \; \ell \geq 1, \]

where \( C^\ell_{m-1} \) is the binomial coefficient. The proof is shown in appendix. These results indicate that the standard deviation of the noise is amplified when more stages are included. As shown in Fig.2, the analytical result is also supported by numerical simulation. We note that the correlation
functions monotonously decrease with $\ell$ for $M > 0$ and oscillating decay for $M < 0$. Both damping factors have $\tau_c = -(|\ln |M||)^{-1}$ for $\ell >> 1$.

4 Control of noise response

The noise amplification found above eventually leads to the breakdown of the coupled-element system as long as there are too many elements. This amplification phenomenon raises an important issue, namely, whether or not a controllable stochastic response can be obtained in noise cascading. If the answer is negative then it is hopeless to study noise cascading in dynamical systems. Fortunately, we found two way to surmount the above difficulty.

The first solution to this problem can be achieved by means of feedback. Let us modify the generic one-dimension map, Eq.(1), by adding feedback response $a\xi_n$,

$$y_{n+1} = f_r(y_n) + \zeta_n + a\xi_n,$$

where $a$ is the feedback strength and $\xi_n = y_n - \bar{y}$. By using the same approach as before, we find the modified general relation between the correlations of stochastic response and those of input noise,

$$<\xi_k\xi_k> = \frac{1}{1 - (M + a)^2} \left\{ <\zeta_k^2> + 2 \sum_{i=1}^{\infty} (M + a)^i <\zeta_k\zeta_{k+i}> \right\}$$

$$<\xi_k\xi_{k+\ell}> = \frac{(M + a)^\ell}{1 - (M + a)^2} \left\{ <\zeta_k^2> + 2 \sum_{i=1}^{\infty} (M + a)^i <\zeta_k\zeta_{k+i}> \right\}$$

$$+ \sum_{j=1}^{\ell} \sum_{i=0}^{\infty} (M + a)^{\ell-j+i} <\zeta_k\zeta_{k+j+i}>, \quad \ell \geq 1.$$
This result indicates that the output response at best could be reduced to the input noise if we adjust $a$ to be $a = -M$. Fig.3 is the simulation result and shows that the feedback scheme can reduce the stochastic response efficiently. The standard derivation of output response is minimized at $a = -M$, and it is indeed smaller than the case without feedback ($a = 0$). On the other hand, as implied by the results of Eqs.(8) and (9), some suitable variations of dynamical structures may be employed to reduce the stochastic response, and thus postpone the breakdown. For a simple illustration, let us consider a two-stage system with opposite pairs. It means that we choose the stability quantity of second stage to be $\tilde{M} = -M$, which is opposite to the previous one. By using Eq.(8) and (9), the correlation of the output noise is

$$< \xi_k \xi_{k+2\ell} > = \frac{M^{2\ell} D}{1 - M^4}, \quad \ell \geq 0,$$

and $< \xi_k \xi_{k+2\ell+1} > = 0$. Since $|M| < 1$, it is clear that the output noise is suppressed in comparison with Eq.(7) and the correlation time is still $\tau_c = -(\ln |M|)^{-1}$ in this case. That offers us an opportunity to reduce the colored noise by choosing the dynamical structure with opposite pairs. Extending this opposite-pairing method to the cascading model with $p$-pair, $M^{(2p-1)} = M$ and $M^{(2p)} = -M$, $p \geq 1$, one finds that the correlation functions for the $p$-th pair are

$$< \xi_k^{(2p-1)} \xi_k^{(2p-1)} > \approx (1 + M^2 + p^2 M^4) D$$

and

$$< \xi_k^{(2p)} \xi_k^{(2p)} > \approx (1 + p^2 M^4) D.$$
These results can be proven by following the same procedure shown in the appendix. As shown in Fig.4, the output noise is indeed greatly reduced and the breakdown is postponed.

5 Concluding remarks

In summary, we analytically derive the relation between the correlation of stochastic response of a steady state in a mapping system and that of input noise. We also obtain the exact form of the correlations of the noise cascading in an uni-directionally dc-filtered coupled system. Our results suggest that noise amplification might lead to catastrophic disaster in coupled-element systems. However, in certain cases, such as the one discussed above, the amplification can be controlled either by a feedback scheme or by an opposite-pairing arrangement of dynamical structures. Let us address the implication on this point. Assume that the noise-cascading system will breakdown when the derivation of stochastic response is large than a threshold $D_{th}$, thus the noise amplification set the limit on the cascading length. For examples, the maximum lengths of the identical-coupled and opposite-pairing cascading models are $m_{max} = \left[ \sqrt{\frac{(D_{th} - D)}{D}} \right] |M|$, and $2p_{max} = 2\left[ \sqrt{\frac{(D_{th} - D)}{D}} \right] |M^2|$ respectively, where $[ \ ]$ is the Gaussian notation. Since $|M| < 1$, we will have a shorter chain for identical elements and a longer chain for the opposite-pairing one. On the other hand, the noise-feedback coupling will give an un-limited chain. Thus, it is worthwhile noting that our models, even they are not so realistic
now, may provide a scheme in which the noise could influence the size of the pattern formation [21].

Finally, there are some remarks should be mentioned. All the analytical results agree with the numerical simulations very well as long as the ratio between the amplitude of input noise and the value of fixed point is less than 5%, and the dynamical systems are not too close to the bifurcation point. The same conclusion can also be drawn for the Ikeda map. It is important to note that the dc-filtered coupling is essential throughout this investigation. Without the filter, the dynamical structures of each stage will be greatly altered by the dc-signal and the coupled system will quickly breakdown.

Acknowledgements

We thank Prof. H.-T. Su for useful discussions and reading the manuscript. The work is partially supported by the National Science Council, Taiwan, R.O.C. under the contract numbers. NSC 86-2112-M006-003 and NSC 86-2112-M006-008.
Appendix

Here, we will use mathematical induction to prove Eqs.(11) and (12). After applying the approximation $|M| << 1$ and keeping those leading terms, Eqs. (5)-(6) end up with two general relations between the $(m+1)$-th stage correlation functions and $m$-th stage correlation functions, i.e.,

$$< \xi_{k}^{(m)} \xi_{k}^{(m)} > \approx (1 + M^2) < \xi_{k}^{(m-1)} \xi_{k}^{(m-1)} > + 2M < \xi_{k}^{(m-1)} \xi_{k+1}^{(m-1)} >,$$  \hspace{0.5cm} (19)

$$< \xi_{k}^{(m)} \xi_{k+\ell}^{(m)} > \approx M^\ell < \xi_{k}^{(m-1)} \xi_{k}^{(m-1)} > + \sum_{i=1}^{\ell} M^{\ell-i} < \xi_{k}^{(m-1)} \xi_{k+i}^{(m-1)} >, \hspace{0.5cm} \ell \geq 1. \hspace{1cm} (20)$$

We note that the leading order of $< \xi_{k}^{(m-1)} \xi_{k+i}^{(m-1)} >$ is $M^i$.

Let us first consider the $m = 2$ case. We read out $< \xi_{k}^{(1)} \xi_{k}^{(1)} > \approx (1 + M^2)M^\ell D$ from Eq.(7), and put it into Eqs.(19) and (20). It is easy to check that

$$< \xi_{k}^{(2)} \xi_{k}^{(2)} > \approx (1 + 4M^2)D \hspace{1cm} (21)$$

$$< \xi_{k}^{(2)} \xi_{k+\ell}^{(2)} > \approx \ell + 1M^\ell D = C^{\ell+1}M^\ell D., \hspace{0.5cm} \ell \geq 1. \hspace{1cm} (22)$$

Next, for the cases of $m > 2$, we will show that if Eqs. (11) and (12) are true for $m$-th stage then they are also true for the $m + 1$ case. Plugging Eqs. (11)-(12) into Eqs. (19)-(20) and expanding them to those leading order terms, we obtain

$$< \xi_{k}^{(m+1)} \xi_{k}^{(m+1)} > \approx (1 + M^2)(1 + m^2M^2)D + 2C_{m-1}^m M^2 D$$
\[ <\xi_k^{(m+1)}\xi_{k+\ell}^{(m+1)}> \approx (1 + (m + 1)^2 M^2) D \]

\[ <\xi_k^{(m+1)}\xi_{k+\ell}^{(m+1)}> \approx M^{\ell}(1 + m^2 M^2) D + \sum_{i=1}^{\ell} C_{m-1}^{i+m-1} M^{\ell} D \]

\[ \approx \sum_{i=0}^{\ell} C_{m-1}^{i+m-1} M^{\ell} D = C_{(m+1)-1}^{\ell + (m+1)-1} M^{\ell} D, \quad \ell \geq 1, \quad (24) \]

where the summation identity of binomial coefficients is used. By mathematical induction, Eqs. (11) and (12) are true for any \( m \)-th stage. Q.E.D.
References

[1] D. Ludwig, SIAM Rev. 17, No 4, 605, (1975). J.P. Crutchfield and Huberman, Phys. Lett A77, 407 (1980). J.P. Crutchfield, M. Nauenberg and J. Rudnick Phys. Rev. Lett. 46, 933 (1981). G. Mayer-Kress and H. Haken, J. Stat. Phys. 26, 149 (1981). B. Shraiman, C. E. Wayne and P. Martin, Phys. Rev. Lett. 46, 935 (1981). M.J. Feigenbaum and B. Hasslacher, Phys. Rev. Lett. 49, 605 (1982). J.E. Hirsch, B. A. Huberman, and D.J. Scalapino, Phys. Rev. A25, 519 (1982). J.P. Crutchfield, J.D.Farmer and B.A. Huberman, Phys. Rep. 92, 45 (1982). W. Horsthemke and R. Lefever, ”Noise-Induced Transitions: Theory and Applications in Physics, Chemistry and Biology”, (Springer-Verlag, Berlin, 1984). S. Parkash, C. K. Peng, and P. Alstrøm, Phys. Rev. A43, 6564 (1991).

[2] K. Matsumoto, and I. Tsuda, J. Stat. Phys. 31, 87 (1983); G. Paladin, M. Serva, and A. Vulpiani, Phys. Rev. Lett. 74, 66 (1995).

[3] A.S. Pikovsky, Phys. Lett A165, 33 (1992); Y-Y. Chen, Phys. Rev. Lett. 77, 4318 (1996).

[4] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A14, L453 (1981); P. Jung, Phys. Rep. 234, 175 (1993); F. Moss, D. Pierson, and D. O’Gorman, Int. J. Bifurcation and Chaos 4, 1383 (1994); M.E.
Inchiosa, and A.R. Bulsara, *Phys. Rev. E* **52**, 327 (1995); K. Wiesenfeld, and F. Moss, *Nature* **373**, 33 (1995).

[5] A.S. Pikovsky, and J. Kurths, *Phys. Rev. Lett.* **78**, 775 (1997).

[6] V. Hakim, and W.J. Rappel, *Europhys. Lett.* **27**, 637 (1994); R. N. Mantegna, and B. Spagnolo, *Phys. Rev. Lett.* **76**, 563 (1996).

[7] E. Simonotto, M. Riani, C. Seife, M. Roberts, J. Twitty, and F. Moss, *Phys. Rev. Lett.* **78**, 1186 (1997); B. J. Gluckman, T. I. Netoff, E. J. Neel, W. L. Ditto, M. L. Spano, and S.J. Schiff, *Phys. Rev. Lett.* **77**, 4098 (1996).

[8] L. I, and J.-M. Liu, *Phys. Rev. Lett.* **74**, 3161 (1995).

[9] S. Kim, S.H. Park, and C.S. Ryu, *Phys. Rev. Lett.* **78**, 1616 (1997) and references therein.

[10] F. Castro, A.D. Sánchez, and H.S. Wio, *Phys. Rev. Lett.* **75**, 1691 (1995); Y. Jia, and J.-r. Li, *Phys. Rev. Lett.* **78**, 994 (1997).

[11] J.M. Sancho, M. San Miguel, S.L. Katz, and J.P. Gunton. *Phys. Rev. A* **26**, 1589 (1982).

[12] R.F. Fox, *Phys. Rev. A* **33**, 467 (1986); 34,4525 (1986); *J. Stat. Phys.* **46**, 1589 (1982).
[13] P. Hänggi, T.J. Mrocakowski, F. Moss, and P.V.E. McClintock, Phys. Rev. A32, 695 (1985).

[14] P. Jung, and P. Hänggi, Phys. Rev. A35, 4464 (1987); J. Opt. Soc. Am. B5, 979 (1988).

[15] F. Castro, H.S. Wio, and G. Abramson, Phys. Rev. E52, 159 (1995).

[16] K. Otsuka, and K. Ikeda, Phys. Rev. Lett. 59, 194 (1987); Phys. Rev. A39, 5209 (1989); K. Otsuka, Phys. Rev. Lett. 65, 329 (1990).

[17] W. Yang, E.-J. Ding, and M. Ding, Phys. Rev. Lett. 76, 1808 (1996); P. Marcq, H. Chaté, and P. Manneville, Phys. Rev. Lett. 77, 4003 (1996).

[18] H.-A. Tanaka, A. J. Lichtenberg, and S. Oishi, Phys. Rev. Lett. 78, 2104 (1997).

[19] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. 75, 3 (1995); M. Löcher, G. A. Johnson, and E. R. Hunt, Phys. Rev. Lett. 77, 4698 (1996).

[20] R.-R. Hsu, H.-T. Su, J.-L. Chern, and C.-C. Chen, Phys. Rev. Lett. 78, 2936 (1997).

[21] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993); A. J. Koch and H. Meinhardt Rev. Mod. Phys. 66, 1481 (1994); and references therein.
Figure Captions:

Figure 1.

Up: The block diagram for the stochastic response $\tilde{\xi}_n$ of the dynamical system $\tilde{M}$ where the correlation function of the input noise is Eq.(7). That is equivalent to a uni-directionally coupled system in which two elements are coupled by a dc-filter and with white noise input.

Bottom: $\sqrt{\frac{\langle \xi^2 \rangle}{D}}$ via $\tilde{M}$ for several fixed $M$. The solid lines denote the analytical estimations. The circle (bullet, box) denote the simulation results for the dynamical structures $M = 0.5$ (0.0, −0.5) of first stage. Here, the dynamical systems are logistic maps $f_r(x) = rx(1 - x)$ with $M = 2 - r$, and $\tilde{f}_{\tilde{r}}(\tilde{x}) = \tilde{r}\tilde{x}(1 - \tilde{x})$ with $\tilde{M} = 2 - \tilde{r}$. The input uniform white noise $\eta_n$ have amplitude 0.01.

Figure 2.

Up: The block diagram for an identical coupled-element system.

Bottom: $\sqrt{\frac{\langle \xi^2 \rangle^{(m)}}{D}}$ via stage number $m$, for two identical coupled-element systems which have different elements $f_r(x) = 1.9 \times x(1 - x)$ and $f_r(x) = 1.99 \times x(1 - x)$, respectively. The solid lines denote the analytical results Eq.(11). The bullet (box) denote the simulation results for the input noises with amplitudes 0.01 (0.001).

Figure 3.

Up: The block diagram for a feedback scheme to reduce the stochastic response.
Bottom: \( \sqrt{\frac{\langle \xi^2 \rangle}{D}} \) via the feedback strength \( a \). The input uniform white noise has amplitude 0.01. Two logistic maps \( f_r(x) = 1.8 \times x(1 - x) \) with \( M = 0.2 \) and \( f_r(x) = 2.3 \times x(1 - x) \) with \( M = -0.3 \) are demonstrated. Both simulations and analytical results show that the responses are minimized at \( a = -0.2 \) and \( a = 0.3 \) respectively.

**Figure 4.**

**Up:** The block diagram for a opposite-pairing scheme to reduce the stochastic response.

**Bottom:** \( \sqrt{\frac{\langle \xi^{(m)}^2 \rangle}{D}} \) via stage number \( m \), for the opposite-pairing coupled system which have a pair of logistic maps \( f_r(x) = 1.9 \times x(1 - x) \) and \( f_r(x) = 2.1 \times x(1 - x) \). The solid lines denote the analytical results Eqs.(17) and (18). The bullet denote the simulation result for the input noise with amplitudes 0.01.
Fig. 3
