Symmetries and Supersymmetries of the Dirac Hamiltonian with Axially-Deformed Scalar and Vector Potentials

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We consider several classes of symmetries of the Dirac Hamiltonian in 3+1 dimensions, with axially-deformed scalar and vector potentials. The symmetries include the known pseudospin and spin limits and additional symmetries which occur when the potentials depend on different variables. Supersymmetric patterns have been identified in specific limits of such spherical potentials [4, 5]. In the present Letter we further explore classes of symmetries and supersymmetries when these potentials are axially-symmetric, i.e., azimuthal angle \( \phi \). Of particular interest are the potentials enter the Dirac equation through the combinations

\[
\begin{align*}
A(\rho, z) & = E + M + V_S(\rho, z) - V_V(\rho, z) \\
B(\rho, z) & = E - M - V_S(\rho, z) - V_V(\rho, z)
\end{align*}
\]

For each solution with \( \Omega > 0 \), there is a degenerate time-reversed solution with \( -\Omega < 0 \), hence, we confine the discussion to solutions with \( \Omega > 0 \). Of particular interest are bound Dirac states with \( |E| < M \) and normalizable wave functions in potentials satisfying \( \rho V_S(\rho, z), \rho V_V(\rho, z) \to 0 \) for \( \rho \to 0 \) and \( V_S(\rho, z), V_V(\rho, z) \to 0 \) for \( \rho \to \infty \) or \( z \to \pm \infty \). The boundary conditions imply that the radial wave functions fall off exponentially for large distances and behave as a power law for \( \rho \to 0 \). Furthermore, for \( z = 0 \) and \( \rho \to \infty \), \( f^-/g^+ \propto (M - E) > 0 \) and \( g^-/f^+ \propto (M + E) > 0 \), while for \( z = 0 \) and \( \rho \to 0 \), \( f^-/g^+ \propto B(0)\rho \) and \( g^-/f^+ \propto -A(0)\rho \). These properties have important implications for the structure of radial nodes. In particular, it follows that for potentials with the indicated asymptotic behaviour and \( A(0), B(0) > 0 \), as encountered in nuclei, a necessary condition for a nodeless bound eigenstate of a Dirac Hamiltonian is

\[
A^- = 0 \quad \text{or} \quad f^+ = 0 .
\]

The Dirac equation, \( H \psi = E \psi \), leads to a set of four coupled partial differential equations involving the radial wave functions. Their solutions are greatly simplified in the presence of symmetries. We now discuss four classes of relativistic symmetries and possible supersymmetries within each class.

The symmetry of class I, referred to as pseudospin symmetry, occurs when the sum of the scalar and vector potentials is a constant, \( V_S(\rho, z) + V_V(\rho, z) = \Delta_0 \). The symmetry generators, \( \hat{S}_i \), commute with the Dirac Hamiltonian and span an \( SU(2) \) algebra \( \hat{S}_1, \hat{S}_2, \hat{S}_3 \)

\[
\hat{S}_i = \begin{pmatrix} U_p \hat{s}_i U_p & 0 \\ 0 & \hat{s}_i \end{pmatrix} \quad i = x, y, z \quad U_p = \frac{\sigma \cdot p}{p} .
\]

Here \( \hat{s}_i = \sigma_i/2 \) are the usual spin operators, defined in terms of Pauli matrices. The Dirac eigenfunctions in the pseudospin limit satisfy

\[
\hat{S}_2 \psi_{1\lambda}^{(\rho)} = \mu \psi_{1\lambda}^{(\rho)} \quad \mu = \pm 1/2
\]

and form degenerate \( SU(2) \) doublets. Their wave functions have been shown to be of the form

\[
\psi_{\Omega=\lambda-\frac{1}{2}}^{(\rho, -\mu)} \quad \psi_{\Omega=\lambda+\frac{1}{2}}^{(\rho, \mu)} ,
\]

where \( \psi_{\Omega=\lambda}^{(\rho, \pm \mu)} \)

For each solution with \( \Omega > 0 \), there is a degenerate time-reversed solution with \( -\Omega < 0 \), hence, we confine the discussion to solutions with \( \Omega > 0 \). Of particular interest are

\[
\begin{align*}
\psi_{\Omega=-\lambda-\frac{1}{2}}^{(\rho, -\mu)} & : \{ g^+, \rho \} \\
\psi_{\Omega=\lambda+\frac{1}{2}}^{(\rho, \mu)} & : \{ g, g^-, f, 0 \}
\end{align*}
\]

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where $\hat{A} = \Omega - \hat{\mu} \geq 0$ is the eigenvalue of $\hat{J}_z - \hat{S}_z$. The relativistic pseudospin symmetry has been tested in numerous realistic mean field calculations of nuclei, especially for doublets near the Fermi surface [10]. The dominant upper components of the states in Eq. (6), involving $g^+$ and $g^-$, correspond to non-relativistic pseudospin doublets with asymptotic (Nilsson) quantum numbers $|N,n_3,\Lambda\Omega = \Lambda + 1/2$ and $|N,n_3,\Lambda + 2\Omega = \Lambda + 3/2$, respectively. The doublet is expressed in terms of the pseudo-orbital angular momentum projection, $\hat{A} = \Lambda + 1$, which is added to the pseudospin projection, $\hat{\mu} = \pm 1/2$, to form doublet states with $\Omega = \hat{\Lambda} \pm 1/2$. Such doublets play a crucial role in explaining features of deformed nuclei, including superdeformation and identical bands [5, 11].

The symmetry of class $\Pi$, referred to as spin symmetry, occurs when the difference of the scalar and vector potentials is a constant, $V_S(\rho, x) - V_V(\rho, x) = \Xi_0$. The symmetry group is again $SU(2)$ and its generators [5]

$$\hat{\Sigma}_i = \begin{pmatrix} 0 & 0 & \hat{s}_i \hat{U}_p \hat{s}_i \hat{U}_p \\ \hat{U}_p & \hat{s}_i & 0 \end{pmatrix} \quad i = x, y, z$$

(6)

commute with the Dirac Hamiltonian. The Dirac eigenfunctions in the spin limit satisfy

$$\hat{\Sigma}_i \Psi^{(\mu)}_{\Omega} = \mu \Psi^{(\mu)}_{\Omega} \quad \mu = \pm 1/2$$

(7)

and form degenerate $SU(2)$ doublets. Their wave functions are of the form [5]

$$\Psi^{(1/2)}_{\Omega=\Lambda+1/2} : \{ g, 0, f, -f \}$$

(8a)

$$\Psi^{(-1/2)}_{\Omega=\Lambda-1/2} : \{ 0, g, f', -f' \}$$

(8b)

where $\Lambda = \Omega - \mu \geq 0$ is the eigenvalue of $\hat{J}_z - \hat{S}_z$. The upper components of the two states in Eq. (6) form the usual non-relativistic spin doublet with a common radial wave function, $g$, an orbital angular momentum projection, $\Lambda$, and two spin orientations $\Omega = \Lambda \pm 1/2$. The relativistic spin symmetry has been shown to be relevant to the structure of heavy-light quark mesons [3].

The Dirac Hamiltonian has additional symmetries when the scalar and vector potentials depend on different variables. The symmetry of class $\Pi$ occurs when the potentials are of the form $V_S = V_S(z)$ and $V_V = V_V(\rho)$. In this case, the Dirac Hamiltonian commutes with the following Hermitian operator

$$\hat{R}_z = [ M + V_S(z) ] \hat{\beta} \hat{\Sigma}_3 + \gamma_5 \hat{p}_z$$

(9)

where $\hat{\Sigma}_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The Dirac eigenfunctions satisfy

$$\hat{R}_z \Psi^{(\epsilon)}_{\Omega} = \epsilon \Psi^{(\epsilon)}_{\Omega}$$

(10)

A separation of variables is possible by choosing the Dirac wave function in the form

$$\Psi^{(\epsilon)}_{\Omega} : \{ u_1 h_+, u_2 h_-, u_1 h_-, -u_2 h_+ \} / \sqrt{\rho}$$

(11)

where $u_\pm \equiv u_\pm(\rho)$, $h_\pm \equiv h_\pm(z)$ and, for simplicity, we have omitted the label $\epsilon$ from these wave functions. The Dirac equation then reduces to a set of two coupled first-order ordinary differential equations in the variable $\rho$,

$$[d/d\rho - \Omega/\rho] u_1(\rho) - [E - V_V(\rho) + \epsilon] u_2(\rho) = 0$$

(12a)

$$[d/d\rho + \Omega/\rho] u_2(\rho) + [E - V_V(\rho) - \epsilon] u_1(\rho) = 0$$

(12b)

and a separate set in the variable $z$,

$$[M + V_S(z) + d/dz] h_2(z) = \epsilon h_1(z)$$

(13a)

$$[M + V_S(z) - d/dz] h_1(z) = \epsilon h_2(z)$$

(13b)

where $h_\pm(z) = h_\pm(\rho) \pm h_\pm(\rho)$. The separation constant, $\epsilon$, plays the role of a mass for the transverse motion and is determined from imposed boundary conditions. A special case when the symmetry class $\Pi$, with $V_S(z) = 0$ and $\epsilon = \pm \sqrt{M^2 + \rho^2}$, was considered for electron channeling in crystals [12]. For $V_S(z) = 0$, $\hat{R}_z$ of Eq. (9), reduces to the transverse polarization operator relevant to studies of synchrotron radiation in storage rings and QED processes in magnetic flux tubes (e.g., $e^+ e^-$ production and Bremsstrahlung) [13].

The symmetry of class $\Pi$ occurs when the potentials are of the form $V_S = V_S(\rho)$ and $V_V = V_V(\rho)$. In this case, the following Hermitian operator

$$\hat{R}_\rho = [ M + V_S(\rho) ] \hat{\beta} \hat{\Sigma}_3 - i \hat{\beta} \gamma_5 (\hat{\Sigma} \times \hat{\rho})$$

(14)

commutes with the Dirac Hamiltonian and the Dirac eigenfunctions satisfy

$$\hat{R}_\rho \Psi^{(\epsilon)}_{\Omega} = \epsilon \Psi^{(\epsilon)}_{\Omega}$$

(15)

Again, a separation of variables is possible with the choice of wave function,

$$\Psi^{(\epsilon)}_{\Omega} : \{ \xi_1 w_+, -i \xi_2 w_-, i \xi_1 w_-, -\xi_2 w_+ \} / \sqrt{\rho}$$

(16)

where $\xi_1 \equiv \xi_1(\rho)$ and $w_\pm \equiv w_\pm(z)$. The Dirac equation then reduces to a set of ordinary differential equations in the variable $\rho$,

$$[d/d\rho - \Omega/\rho] \xi_1(\rho) - [\epsilon + M + V_S(\rho)] \xi_2(\rho) = 0$$

(17a)

$$[d/d\rho + \Omega/\rho] \xi_2(\rho) + [\epsilon - M + V_S(\rho)] \xi_1(\rho) = 0$$

(17b)

and a separate set in the variable $z$,

$$[E - V_V(z) - id/dz] w_2(z) = \epsilon w_1(z)$$

(18a)

$$[E - V_V(z) + id/dz] w_1(z) = \epsilon w_2(z)$$

(18b)

where $w_\pm(z) = w_\pm(\rho) \pm w_\pm(\rho)$. The quantum number, $\epsilon$, plays the role of an energy for the transverse motion. A particular selection of potentials within the symmetry class $\Pi$ was encountered in the study of the Schwinger mechanism for particle production in a strong confined field ($V_V(\rho) = \alpha_V z$) [14, 15], $q \bar{q}$ pair-creation in a flux...
The SUSY algebra imply that if $\Psi^+$ and $\Psi^-$ are conserved, anticommuting operators for Dirac Hamiltonians ($H$) exhibiting a supersymmetric structure.

| SUSY | $\hat{R}$ | $\hat{B}$ | $\hat{B}^2 = f(H)$ |
|------|------------|------------|---------------------|
| $V_S(\rho, z) + V_V(\rho, z) = \Delta_0$ | $\hat{S}_z$ (I) | $2(M + \Delta_0 - H)\hat{S}_z$ | $(M + \Delta_0 - H)^2$ |
| $V_S(\rho, z) - V_V(\rho, z) = \Xi_0$ | $\hat{S}_z$ (I) | $2(M + \Xi_0 + H)\hat{S}_z$ | $(M + \Xi_0 + H)^2$ |
| $V_S = V_S(z)$, $V_V = \frac{\partial V}{\partial \rho}$ | $\hat{R}_\rho$ (II) | $\Sigma_3(iJ_{\gamma_\rho}[H - \Sigma_3\hat{R}_\rho] - \frac{1}{\rho} (\hat{\Sigma} \cdot \rho) \hat{R}_\rho}$ | $J^2_0(H^2 - \hat{R}^2_\rho) + \alpha_0^2 \hat{R}^2_\rho$ |
| $V_S = \frac{\partial V}{\partial \rho}$, $V_V = V_V(z)$ | $\hat{R}_\rho$ (II) | $\Sigma_3(iJ_{\gamma_\rho}[M - \Sigma_3\hat{R}_\rho] - \frac{1}{\rho} (\hat{\Sigma} \cdot \rho) \hat{R}_\rho}$ | $J^2_0(\rho^2 - M^2) + \alpha^2_0 \hat{R}^2_\rho$ |

The operator $\hat{R}$ has non-zero eigenvalues, $r$, which come in pairs of opposite signs. $\hat{B}^2 = \hat{B}\hat{B} = f(H)$, is a function of the Hamiltonian. A $Z_2$-grading operator, $P_r = R/|r|$, and Hermitian supercharges $Q_1 = \hat{B}$, $Q_2 = i\hat{Q}_1P_r$ can now be constructed. The triad of operators $Q_{\pm} = (Q_1 \pm iQ_2)/2$ and $H = Q^2_{\pm} = f(H)$ form the standard SUSY algebra. In the present analysis, $f(H)$ is a quadratic function of the Dirac Hamiltonian, $H$, and the relevant $\hat{R}$ and $\hat{B}$ operators are listed in Table I.

In the pseudospin symmetry limit, the relevant operator $\hat{B}$ connects the doublet states of Eq. (5). The spectrum, for each $\Lambda \neq 0$, consists of twin towers of pairwise degenerate pseudospin doublet states, with $\Omega_1 = \Lambda - 1/2$ and $\Omega_2 = \Lambda + 1/2$, and an additional non-degenerate nodeless state at the bottom of the $\Omega_1 = \Lambda - 1/2$ tower. Such nodeless states correspond in the non-relativistic nuclear deformed shell-model to the “intruder” states, $[N, n_3, \Lambda = N - n_3]\Omega = \Lambda + 1/2$, which, empirically, are found not to be part of a doublet [11]. The latter property follows from the fact that a nodeless bound Dirac state satisfies the criteria of Eq. (2), hence has a wave function as in Eq. (55) with $g^+, g, f \neq 0$ and $f/g^+ > 0$. Its pseudospin partner state has a wave function as in Eq. (55). The radial components satisfy $Bg^+ = |B - 2(\Lambda/\rho)|f/g^+|g^+|$, where $B$ is defined in Eq. (55). This relation is satisfied, to a good approximation, for mean-field potentials relevant to nuclei, and the r.h.s. is non-zero and, consequently, $g^- \neq 0$. If so, then the partner state [58] is also nodeless, but it cannot be a bound eigenstate since its radial components do not fulfill the condition of Eq. (2). Altogether, the ensemble of Dirac states with $\Omega_2 - \Omega_1 = 1$ exhibits a supersymmetric pattern of good SUSY, as illustrated in Fig. (2a).

In the spin symmetry limit, the relevant operator $\hat{B}$ connects the doublet states of Eq. (5). The spectrum, for each $\Lambda \neq 0$, consists of twin towers of pairwise degenerate spin-doublet states with $\Omega_1 = \Lambda - 1/2$ and $\Omega_2 = \Lambda + 1/2$. None of these towers have a single non-degenerate state. This follows from the fact that, in view of Eq. (2), a nodeless bound state has a wave function as in Eq. (88) with $g, f, f^- \neq 0$ and $g/f^- > 0$. Its spin partner has a wave function as in Eq. (89). The radial components satisfy $Af^+ = |A - 2(\Lambda/\rho)|f/f^+|f^+$, where $A$ is defined in Eq. (128). For relevant potentials the r.h.s. of this relation can vanish, hence $f^+$ has a node. Therefore, the

The essential ingredients of supersymmetric quantum mechanics [15, 16] are the supersymmetric Hamiltonian, $H$, and charges $Q_+, Q_-$, which generate the supersymmetry (SUSY) algebra $[H, Q_{\pm}] = \{Q_{\pm}, Q_{\pm}\} = 0$, $\{Q_-, Q_+\} = H$. Accompanying this set is an Hermitian $Z_2$-grading operator satisfying $[H, P] = \{P_\pm, P\} = 0$ and $P^2 = \mathbb{1}$. The +1 and −1 eigenspaces of $P$ define the “positive-parity”, $H_+$, and “negative-parity”, $H_-$, sectors of the spectrum, with eigenvectors $\Psi^+$ and $\Psi^-$, respectively. The SUSY algebra imply that if $\Psi^+$ is an eigenstate of $H$, then also $\Psi^- = Q_-\Psi^+$ is an eigenstate of $H$ with the same energy, unless $Q_+\Psi^+$ vanishes or produces an unphysical state, (e.g., non-normalizable).

The resulting spectrum consists of pairwise degenerate levels with a non-degenerate single state (the ground state) in one sector when the supersymmetry is exact. If all states are pairwise degenerate, the supersymmetry is said to be broken. Typical spectra for good and broken SUSY are shown in Fig. 1. Degenerate doublets, signaling a supersymmetric structure, can emerge in a quantum system with a Hamiltonian $H$, from the existence of two Hermitian, conserved and anticommuting operators, $\hat{R}$ and $\hat{B}$

\[ [H, \hat{R}] = [H, \hat{B}] = \{\hat{R}, \hat{B}\} = 0 \]

![FIG. 1: Typical spectra of good and broken SUSY. The operators $Q_-$ and $Q_+$ connect degenerate states in the $H_+$ and $H_-$ sectors.](image-url)
spin-partner of a nodeless state is not nodeless and can be a bound eigenstate, since the restrictions of Eq. (2) do not apply. Altogether, the ensemble of Dirac states with \( \Omega_2 - \Omega_1 = -1 \) exhibits a supersymmetric pattern of broken SUSY, as illustrated in Fig. (2b).

Within the symmetry class III, a supersymmetry is obtained for \( V_4(\rho) = \alpha \rho / \rho \) and \( V_5(z) \) arbitrary. The energy eigenvalues are \( E_{n_{\rho},\Omega}^{(c)} = \epsilon/\sqrt{1 + \alpha^2 V/(n_{\rho} + \gamma)^2} \) for \( n_{\rho} = 0, 1, 2, \ldots \), with \( \gamma = \sqrt{\Omega^2 - \alpha^2 V} \). From Eqs. (18) we see that if \( [h_1(z), h_2(z)] \) are solutions with \( \epsilon > 0 \), then \( [h_1(z), -h_2(z)] \) are solutions with \( \epsilon < 0 \). Accordingly, the doublet wave functions are as in Eq. (11), with the replacements, \( u_i \rightarrow u_{i}^{(c)}(\rho) \) for \( \Psi^{(c)}_{n_{\rho},\Omega} \), and \( u_i \rightarrow u_{i}^{(-c)}(\rho) \), \( h_{+} \rightarrow -h_{+}(z) \) for \( \Psi^{(-c)}_{n_{\rho},\Omega} \). For \( n_{\rho} \geq 1 \), the states \( \Psi^{(c)}_{n_{\rho},\Omega} \) are degenerate. For \( n_{\rho} = 0 \) only one state is an acceptable solution, which has \( \epsilon > 0 \) (assuming \( \alpha \rho > 0 \)) and is annihilated by the relevant operator \( \hat{B} \). For each \( \Omega \) and \( \epsilon \) the spectrum resembles a supersymmetric pattern of good SUSY, with the towers \( H_{+} \) (\( H_{-} \)) of Fig. 1 corresponding to states with \( \epsilon > 0 \) (\( \epsilon < 0 \)).

Within the symmetry class IV, a supersymmetry is obtained for \( V_5(\rho) = \alpha S / \rho \) (\( \alpha S < 0 \)) and \( V_6(z) \) arbitrary. The allowed values are \( \tilde{\epsilon} = \pm M \sqrt{1 - \alpha^2 S/(n_{\rho} + \tilde{\gamma})^2} \) for \( n_{\rho} = 0, 1, 2, \ldots \), where \( \tilde{\gamma} = \sqrt{\Omega^2 - \alpha^2 S} \). From Eqs. (18) we see that if \( [w_1(z), w_2(z)] \) are solutions with \( \tilde{\epsilon} > 0 \), then \( [w_1(z), -w_2(z)] \) are solutions with \( \tilde{\epsilon} < 0 \) and the same energy, \( E \). Accordingly, the doublet wave-functions are as in Eq. (10), with the replacements, \( \xi_i \rightarrow \xi_{i}^{(c)}(\rho) \) for \( \Psi_{n_{\rho},\Omega}^{(c)} \), and \( \xi_i \rightarrow \xi_{i}^{(-c)}(\rho) \), \( w_{\pm} \rightarrow -w_{\mp}(z) \) for \( \Psi_{n_{\rho},\Omega}^{(-c)} \). For \( n_{\rho} \geq 1 \) the states \( \Psi_{n_{\rho},\Omega}^{(c)} \) are degenerate. For \( n_{\rho} = 0 \) only one state, with \( \tilde{\epsilon} > 0 \), is an acceptable solution, which is annihilated by the relevant operator \( \hat{B} \). Again, for each \( \Omega \) and \( \tilde{\epsilon} \) the resulting spectrum resembles a supersymmetric pattern of good SUSY.

In summary, we have considered classes of symmetries and related supersymmetries of Dirac Hamiltonians with cylindrically-deformed scalar and vector potentials. The symmetries arise when the potentials obey a constraint on their sum or difference, or when they depend on different variables. The known pseudospin and spin symmetry limits are by themselves supersymmetric. Additional supersymmetries arise when one of the potentials has a \( 1/\rho \) dependence and the second potential depends on \( z \). It is gratifying to note that some of the indicated (super)symmetries are manifested empirically, to a good approximation, in physical dynamical systems.

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