NON-COMMUTATIVE COREPRESENTATIONS OF QUANTUM GROUPS

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We consider a twisted version of quantum groups corepresentations. This generalization amounts to include in the theory the case where quantum space coordinates and its endomorphism matrix entries belong to a non-commutative quadratic algebra.
I. INTRODUCTION

Quantum Groups arise as the abstract structure underlying the symmetries of integrable systems in (1+1) dimensions [1]. There, the theory of quantum inverse scattering give rise to some deformed algebraic structures which were first explained by Drinfel’d as deformations of classical Lie algebras [2] [3]. An analog structure was obtained by Woronowicz in the context of non-commutative $C^*$-algebras [4]. There is a third approach, due to Manin, where Quantum Groups are interpreted as the endomorphisms of certain non commutative algebraic varieties defined by quadratic algebras, called quantum linear spaces (QLS) [5]. Faddeev et al had also interpreted the Quantum Groups from the point of view of corepresentations and quantum spaces, furnishing a connection with the quantum deformations of the universal enveloping algebras and the quantum double of Hopf algebras [6] [7].

From the algebraic point of view, quantum groups are Hopf algebras and the relation with the endomorphism algebra of QLS come from their corepresentations on tensor product spaces. The usual construction of the coaction on the tensor product space involves the flip operator interchanging factors of the tensor product of the QLS with the bialgebra. This fact implies the commutativity between the matrix elements of a representation of the endomorphism and the coordinates of the QLS. Moreover, the flip operator for the tensor product is also involved in many steps of the construction of Quantum Groups. In the braided approach to $q$-deformations, the flip operator is replaced by a braiding giving rise to the quasi-tensor category of $k$-modules, where a natural braided coaction appears [8].

In the present work, we introduce a twisted coaction over the tensor product space, thus admitting non-commutative relations between endomorphism matrix entries and quantum linear space coordinates, however this has nothing to do with the braided approach mentioned above. We find the conditions under which the general algebraic framework of multiplicative quantum groups still holds. It is also shown that the bialgebras arising from this context may be regarded as a partial twisting of usual quantum groups and the connections with integrable systems is analyzed after the introduction of the spectral parameter. This twisted coaction allow us to introduce new deformation parameters in the endomorphism bialgebra of the QLS, as it is shown in the quantum plane example where a four parameters deformation is obtained, although the Yang-Baxter condition is relaxed. Also, we find a non central object playing the role of $(q, p, r, s)$-deformed determinant. In the undeformed limit for the parameters $(r, s)$ we recover the biparametric deformation $GL_{qp}(2)$ described in ref. [12].

We present a brief description of the corepresentations of bialgebras in the second section, developing our approach to modified corepresentations in the third one. In the fourth section we present the result of the previous section as twisted bialgebras. The connection with integrable systems is discussed in the fifth section and, finally, we work out the quantum plane example in the last section.

II. QUANTUM ALGEBRAS AND COREPRESENTATIONS

Let $V$ be a vector space of dimension $n$, $\{e_i\}$ a basis for $V$ and $H_o$ the trivial bialgebra of functions over $GL(n, \mathbb{C})$. This bialgebra is freely generated by the identity and the coordinates functions $T^j_i$, in the basis $\{e_i\}$, defined by

$$T^j_i : GL(n, \mathbb{C}) \longrightarrow C$$

$$T^j_i : g \longrightarrow g^j_i$$

for $g \in GL(n, \mathbb{C})$. The $T^j_i$ are group like, the coproduct and counit are given by

$$\Delta T^j_i = T^k_i \otimes T^j_k$$

$$\varepsilon(T^j_i) = \delta^j_j$$

From now on, summation over repeated index is assumed. The comodule $(\delta, V)$, with

$$\delta : V \longrightarrow H_o \otimes V$$

$$\delta(e_i) = T^j_i \otimes e_j$$

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provides a representation of $GL(n, \mathbb{C})$ in $V$, through the $g^i_j$ in the basis $b$ of $V$. It has the coassociativity property and preserves the counit, which is expressed by the relations

$$ (I_{H_o} \otimes \delta_V) \delta_V = (\Delta \otimes I_V) \delta_V \quad (4) $$

$$ (\epsilon \otimes I_{H_o}) \delta_V = I_V \quad (5) $$

In order to extend the comodule to the tensor product algebra $V \otimes V$, one introduce the coaction on $V \otimes V$,

$$ \delta_{V \otimes V} : V \otimes V \rightarrow H_o \otimes V \otimes V $$

$$ \delta_{V \otimes V} = (m \otimes I_{V \otimes V})(I_{H_o} \otimes \tau \otimes I_V)(\delta_V \otimes \delta_V) \quad (6) $$

The extension to $V \otimes^N$ is achieved via the recursive relations

$$ \delta_{V \otimes^N} : V \otimes^N \rightarrow H_o \otimes V \otimes^N $$

$$ \delta_{V \otimes^N} = (m \otimes I_{V \otimes^N})(I_{H_o} \otimes \tau_{V \otimes^N,H_o} \otimes I_V)(\delta_{V \otimes (N-1)} \otimes \delta_V) \quad (7) $$

where $\tau_{V \otimes^N,H_o}$ is the flip operator mapping $\tau_{V \otimes^N,H_o} : V \otimes^N \otimes H_o \rightarrow H_o \otimes V \otimes^N$. This definition satisfy the coassociativity and counit properties described in relations (4) and (5). It is also worth to remark that the appearance of the flip operator in (7) leads to the commutativity between the coordinates of the quantum space and its endomorphism matrix entries, as its assumed for the quantum plane and $GL_q(2)$ [5].

Building up corepresentations for objects with more structure than $V \otimes V$, as quadratic algebras for example, requires some extra conditions that we sketch below.

Let $A$ denote the quadratic algebra generated by the ideal $I(\mathfrak{B})$, where $\mathfrak{B} : V \otimes V \rightarrow V \otimes V$, then

$$ A(\mathfrak{B}) = \frac{V \otimes \mathfrak{B}}{I(\mathfrak{B})} \quad (8) $$

$V \otimes V$ is tensor algebra on $V$. In general we consider $\mathfrak{B}$ with the form

$$ \mathfrak{B} = (I_{V \otimes V} - B) $$

$$ e_i e_j - B^k_{ij} e_k e_l \quad (9) $$

$\delta_{V \otimes V}$ must be an homomorphism of quadratic algebra, i.e.,

$$ (I_{H_o} \otimes \mathfrak{B}) \delta_{V \otimes V} = \delta_{V \otimes V} \mathfrak{B} \quad (10) $$

This is satisfied if $H$ is the bialgebra arising from the quotient of the free algebra generated by the objects $T^j_i$ and the ideal $I(\mathfrak{B}, H_o)$ generated by the quadratic relation

$$ B^k_{ij} T^l_k T^s_i - T^k_i T^l_j B^s_{kl} \quad (11) $$

i.e.,

$$ H = \frac{H_o}{I(\mathfrak{B}, H_o)} \quad (12) $$

Since $I(\mathfrak{B}, H_o)$ is a coideal with relation to $\Delta$, $H$ becomes a bialgebra, namely an FRT bialgebra. The equation (12) is a central object in the so called FRT construction [6]. In this way, $A(\mathfrak{B})$ becomes in a $H$-algebra comodule.
III. GENERALIZED COREPRESENTATIONS

The main aim of this section is to build up the mathematical framework encoding the situation in which entries of the endomorphism matrix may not commute with the coordinates of the quantum linear space defined in \( [\mathbb{S}] \). We will reach it by means of a modification in the corepresentation theory, obtained by substituting the flip map \( \tau \) in the standard definition of the coaction on \( V \otimes V \) by a non-trivial map \( \gamma \).

As described in the previous section, supplying the quantum linear space with a comodule structure requires a right definition of a coaction on the tensor product space, and the standard definition of \( \delta V \otimes V \), eq. \((6)\), provides both \( V \otimes V \) with a \( H_0 \)-comodule structure and \( \mathcal{A} = V \otimes (\mathbb{S}) \) with an \( H \)-comodule structure.

Then, let us introduce the map \( \gamma \), defined by

\[
\gamma : V \otimes H \longrightarrow H \otimes V
\]

and our proposal of generalized or non-commutative coaction on tensor product space is

\[
\delta^\gamma V \otimes V = (m \otimes I_{V \otimes V}) (I_{H_0} \otimes \gamma \otimes I_V) (\delta_V \otimes \delta_V) \tag{16}
\]

Then, we shall see that a \( H_\gamma \)-comodule structure there it is possible, for some \( H_\gamma \) to be constructed and provided \( \gamma \) satisfying some requirements. The first question is finding the condition under which \( \delta^\gamma V \otimes V \) is actually a coaction. It is addressed in the following proposition:

**Proposition 1:** The map \( \delta^\gamma V \otimes V : V \otimes V \rightarrow H_0 \otimes V \otimes V \) is a coaction turning \( V \otimes V \) into a \( H_0 \)-comodule iff \( \gamma : V \otimes H_0 \longrightarrow H_0 \otimes V \) satisfies the following conditions

\[
\gamma_{ijm} = \delta_{ijm} \tag{17}
\]

\[
\theta_{kjm} = \delta_{ijm} \tag{18}
\]

\[
\theta_{kjm} = \delta_{ijm} \tag{19}
\]

*Proof:* These properties for \( \gamma \) arise straightforward from the coassociativity and counit conditions

\[
(\Delta \otimes I_{V \otimes V}) \circ \delta^\gamma V \otimes V = (I_{H_0} \otimes \delta^\gamma V \otimes V ) \circ \delta_V \otimes \delta_V \tag{20}
\]

Proving that \( \delta^\gamma V \otimes V \) is a coaction is a straightforward exercise in algebra.

A mapping \( \gamma \) satisfying the conditions \((17) \,(19)\) leads to a comodule over \( V \otimes ^N \) as stated in the following proposition.

**Proposition 2:** Let \( \delta^\gamma V \otimes ^N : V \otimes ^N \rightarrow H_0 \otimes V \otimes ^N \) be defined by,

\[
\delta^\gamma V \otimes ^N = (m_{H_0} \otimes I_{V \otimes ^N}) (I_{H_0} \otimes \gamma_{V \otimes ^{N-1},H_0} \otimes I_V) (\delta_{V \otimes ^{N-1}} \otimes \delta_V) \tag{21}
\]

where

\[
\gamma_{V \otimes ^N, H_0} (e_{i_1} \otimes \ldots \otimes e_{i_N} \otimes T^k_j) = \gamma_{V,H_0} (e_{i_1} \otimes \gamma_{V \otimes ^{N-1},H_0} (e_{i_2} \otimes \ldots \otimes e_{i_N} \otimes T^k_j))
\]

and

\[
\gamma_{V,H_0} = \gamma
\]

then \( (V^\otimes, \{\delta^\gamma V \otimes ^N\}) \) is a left \( H_0 \)-comodule.

The proof runs as the previous one, just with a more complicated algebra.
Recalling the bijection between comodules and multiplicative matrices \([5]\), let us consider the multiplicative matrix \(M\) in \(V \otimes V\) with coefficients in \(H\), corresponding to the comodule \(\delta_{V \otimes V}\), i.e.,

\[
\delta_{V \otimes V} \equiv M \in \text{End}(V \otimes V, H)
\]

\[
\delta_{V \otimes V}(e_i \otimes e_j) = M_{ij}^s \otimes e_r \otimes e_s
\]

(22)

hence \(M\) is

\[
M_{ij}^{kl} = T_{ij}^{km} T_{mn}.
\]

(23)

Let us adopt the following convention: for \(A_{kl}^{ij}\) and \(D_{kl}^{ij}\) being any pair of four-tensors, we write \((A \times B)_{ij}^{rs} = A_{ij}^{kl} \times D_{kl}^{rs}\), where \(\times\) stands for any kind of product (tensor, algebraic, etc.), and sum over repeated index is also assumed.

With this notation, conditions (18) and (19) are

\[
\Delta M = M \otimes M
\]

\[
\epsilon(M) = I
\]

(24)

The next step is to consider a quadratic structure on \(V \otimes V\) giving rise to a QLS. Now, the bialgebra \(H\) is no longer in the endomorphism algebra of the QLS. Let us consider a QLS generated by the quotient algebra

\[
A(B) = \frac{V}{I(B)}
\]

(25)

where \(B\) means the relations defining the quadratic algebra. Associated with it we now introduce a new bialgebra structure on the free algebra generated by the \(\{T_{ik}^j\}\).

**Proposition 3:** Let \(H\) the free algebra generated by the \(\{T_{ik}^j\}\), \(\gamma\) as in the previous proposition and \(I(BM - MB)\) is the ideal generated by the quadratic relation

\[
BM - MB
\]

(26)

then, the quotient algebra \(H_\gamma\) defined as

\[
H_\gamma = \frac{H}{I(BM - MB)}
\]

(27)

is a bialgebra.

*Proof:* A necessary and sufficient condition for \(H_\gamma\) to be a bialgebra is that \(I(BM - MB)\) be a coideal, i.e.,

\[
\Delta I \subset I \otimes H + H \otimes I
\]

Then, taking into account the relation (24), one gets

\[
\Delta(BM - MB) = B \Delta M - \Delta M B
\]

\[
= B(M \otimes M) - (M \otimes M) B
\]

\[
= (BM - MB) \otimes M - M \otimes (BM - MB)
\]

and

\[
\epsilon(BM - MB) = B \epsilon(M) - \epsilon(M) B = 0
\]

hence \(H_\gamma\) is a bialgebra.\(\square\)

The main result of this section is expressed in the following proposition

**Proposition 4:** \(\{\delta_{V \otimes V}^\gamma\}\) supplies \(A(B) = \frac{V}{I(B)}\) with an left \(H_\gamma\)-comodule structure.

*Proof:* This assertion means the map \(\delta_{V \otimes V}^\gamma : A(B) \to H_\gamma \otimes A(B)\) is an homomorphism of quadratic algebras, as in eq. (11). This fact is realized by the commutation relation

\[\text{5}\]
\((I_{H\gamma} \otimes \mathcal{B}) \circ \delta_{V \otimes V}^\gamma = \delta_{V \otimes V}^\gamma \circ \mathcal{B}\)

which is immediately satisfied by virtue of the ideal defining \(H_\gamma\), i.e., the condition

\[
\mathcal{B}M = M \mathcal{B}
\]

Resuming, we can make the following assertion: given a quadratic algebra \(A(\mathcal{B})\) and a map \(\gamma\) satisfying the relations (13), then \(H_\gamma = \frac{M_H}{I(\mathcal{B} M - M \mathcal{B})}\) is a bialgebra and \(\{\delta_{V \otimes V}^\gamma\}\) renders \(A(\mathcal{B})\) into a \(H_\gamma\)-comodule (a similar quadratic algebra arise in the context of quantum braided group [9]). This may be understood because of the bijection between all the structures of left comodule on \(V = \mathbb{C}^n\) and the multiplicative matrix \(M(n, H_\gamma)\), since \(M \in H_\gamma\), satisfy \(\Delta M = M \otimes M\) and \(\varepsilon(M) = I_{V \otimes V}\), for \(M \in H_\gamma\).

The existence of an antipode is not involved in the comodule structure, so we may expect the above construction still holds when \(H_\gamma\) is a Hopf algebra, giving rise to non-commutative corepresentations of Quantum Groups.

In the last section, we describe an explicit example enjoying all these properties presented above, namely a multi-parameter deformed version of the endomorphism of the quantum plane.

IV. RELATIONS WITH TWISTED BIALGEBRAS

Let us introduce a bialgebra structure on \(Hom(H_\otimes^2, k)\) by means the convolution product \(*\) of linear forms, defined as \((f * g)(T) = (f \otimes g)(\Delta T)\) for \(f, g \in Hom(H_\otimes^2, k)\) and \(T \in H_\otimes^2\). The coproduct is \((\Delta h)(T \otimes T') = (h \circ m)(T \otimes T')\) for \(h \in Hom(H_\otimes^2, k)\) and \(T, T' \in H_\otimes\). The unit is \(\varepsilon\), the counit of the \(H_\alpha\), namely \((\varepsilon * f)(T) = f(T)\).

In this framework \(\mathfrak{H}\) \(\mathfrak{H}\), \(H\) defined in eq. (13), can be presented as the bialgebra \(H(m, \Delta, \eta, \varepsilon, R)\), with \(R : H_\otimes^2 \to k\) being an invertible linear form, related to \(\mathfrak{B}\) of the previous section by

\[
R(T_i^k \otimes T_j^l) = R_{ij}^{kl} = B_{ij}^{kl}
\]

and defined by the quadratic ideal generated by the relation

\[
m^{op} = R \ast m \ast \overline{R}
\]

Here, \(m^{op} = m \circ \tau\) and \(R \ast \overline{R} = \overline{R} \ast R = \varepsilon\). Moreover, \(H\) is said dual quasi-triangular provided \(R\) satisfies

\[
R \circ (l_H \otimes m) = R_{13} \ast R_{12}, \quad R \circ (m \otimes I_H) = R_{13} \ast R_{23}
\]

Here, \(R_{12} = R \otimes \varepsilon\), \(R_{23} = \varepsilon \otimes R\), \(R_{13} = (\varepsilon \otimes R) \circ (\tau \otimes I_H)\). This last relations implies \(R\) is a solution of the Quantum Yang-Baxter equation

\[
R_{12} \ast R_{13} \ast R_{23} = R_{23} \ast R_{13} \ast R_{12}
\]

Coming back to our problem, let us work out the bialgebra structure \(H_\gamma\), eq. (27), derived from non-commutative corepresentations of the previous section. The following characterization of the \(\gamma\) map drives to a different interpretation of the bialgebra \(H_\gamma\). Let \(\theta\) be the linear map

\[
\theta : H_\alpha \longrightarrow H_\alpha
\]

\[
\theta(T_i^j) = \theta_i^m T_m^n = \tilde{T}_i^j
\]

with the properties

\[
\theta_i^j \theta_k^l - \delta_i^j \theta_k^l = 0
\]

\[
\theta_k^l \theta_j^n = \delta_k^j
\]
Proposition 5: $\theta$ is a coalgebra homomorphism, such that
\[
\Delta \tilde{T}^i_j = \tilde{T}^k_i \otimes \tilde{T}^j_k
\]
\[
\epsilon (\tilde{T}^i_i) = \delta^i_i
\]
(34)

Proof: the properties (33) implies that $\tilde{T}$ are group like elements, then satisfying the coassociativity and counit properties.

With this notation, the twisting $\gamma$, eq. (14) can be now expressed as
\[
\gamma(e_i \otimes T^k_j) = \theta(T^k_j) \otimes e_i = \tilde{T}^k_j \otimes e_i
\]
and the quadratic relation (26) can be written more explicitly as
\[
R^{ab}_{ij} T^k_a \tilde{T}^k_b = T^k_i \tilde{T}^j_k R_{ba}^{kl}
\]
(35)
This relation generates the ideal which give rise to the quadratic algebra $H_\gamma(m, \Delta, \eta, \epsilon)$. 

Proposition 7: Let $\theta$ be an automorphism in $H_\circ$, then there is an isomorphism of the bialgebra $H_\gamma(m, \Delta, \eta, \epsilon)$ with the bialgebra $H(m, \Delta, \eta, \epsilon, R)$, where the deformed product $m_\theta : H_\circ \otimes H_\circ \to H_\circ$ is defined as
\[
m_\theta = m \circ (I_H \otimes \theta)
\]
(36)

\[
(I_H \otimes \theta)(T^k_i \otimes T^j_l) = T^k_i \otimes \tilde{T}^j_l
\]
and it fulfill the relation
\[
m_\theta^{op} = R \ast m_\theta \ast \overline{R}
\]
(37)

Proof: The ideal generated by (37) is exactly (35), and since $\theta$ is an automorphism the proof is obvious. Associativity can be reached, for example, with
\[
m_\theta(T^l_i T^m_j \otimes T^n_k) = T^l_i T^m_j \tilde{T}^n_k
\]
\[
m_\theta(T^l_i \otimes T^m_j T^n_k) = T^l_i \tilde{T}^j_k \tilde{T}^n_k
\]
(38)

This definition allows us to cast the bialgebra $H_\circ(m, \Delta, \eta, \epsilon)$ into a standard FRT bialgebra with deformed product. In this way, higher tensor corepresentation of the bialgebra $H(m, \Delta, \eta, \epsilon, R)$ are equivalent to non-commutative corepresentations of $H_\circ(m, \Delta, \eta, \epsilon)$.

The twisting by 2-cocycles of quasitriangular Hopf algebras is due to Drinfel’d [10] who has shown that starting from a quasitriangular Hopf algebra, a new quasitriangular Hopf algebra is obtained twisting by a 2-cocycle the coproduct and the quasitriangular structure $R$. In our case, we shall be interested in a partial twisting: we shall need just a twisting of the product or a twisting of the dual-quasitriangular structure, but not both together. We shall show that the braiding introduced by the non-commutative coaction boils down to a partial twisting of the usual FRT bialgebras, which in general do not preserves dual-quasitriangularity. To this end, we extract some dual results from the Drinfeld analysis.

Following ref. [8], we introduce a 2-cocycle on the bialgebra $Hom(H^{\otimes 2}, k)$ as being an invertible element of $H^{\otimes 2}$, in the sense of the product $*$, satisfying the condition $\phi_{23} * ((I_H \otimes \Delta) \circ \phi) = \phi_{12} * ((\Delta \otimes I_H) \circ \phi)$.

The maps $\phi : H^{\otimes 2} \to k$ being a 2-cocycle give rise to a new bialgebra structure on $H_\circ$, namely $H_\phi(m, \Delta, \eta, \epsilon)$, with a twisted product $m_\phi$
\[
m_\phi : H_\circ^{\otimes 2} \to H_\circ
\]
(39)
\[
m_\phi = \phi * m * \overline{\phi}
\]
Moreover, if $\phi$ is a bialgebra bicharater, i.e.,
\[
\phi(m \otimes I_H) = \phi_{13} \ast \phi_{23} \\
\phi(I_H \otimes m) = \phi_{13} \ast \phi_{12}
\]
then the 2-cocycle condition leads to the Quantum Yang-Baxter equation
\[
\phi_{12} \ast \phi_{13} \ast \phi_{23} = \phi_{23} \ast \phi_{13} \ast \phi_{12}
\]
(41)

The above proposition provides the framework to interpret the bialgebra \( H_\gamma \) as twisted one. In fact, let us assume the map \( \theta \), introduced in eq. (32), can be written as
\[
\theta(T^k_j) = \theta^{km}_{jn} T^m_n = \rho^m_j T^m_n T^k_j
\]
(42)
i.e., the map \( \theta \) admits the factorization
\[
\theta^{kl}_{ij} = \rho^l_i \rho^k_j
\]
(43)
Then, we may introduce the bialgebra bicharacter \( \phi : H_\gamma^\otimes 2 \rightarrow k \), inherited from the associativity assignment (38), defined by the relations
\[
\phi(T^k_i \otimes T^l_j) = (\varepsilon \otimes \rho) (T^k_i \otimes T^l_j)
\]
\[
\rho(T^k_i) = \rho^k_i
\]
\[
\phi(T^k_i \otimes \varepsilon) = \phi(\varepsilon \otimes T^k_i) = 1
\]
\[
\phi(m \otimes I_H) = \phi_{13} \ast \phi_{23}
\]
\[
\phi(I_H \otimes m) = \phi_{13} \ast \phi_{12}
\]
(44)
with \( \rho : H_\alpha \rightarrow k \) an invertible map, i.e., there exist \( \overline{\rho} \) such that \( \overline{\rho} \ast \rho = \rho \ast \overline{\rho} = \varepsilon \), and \( \varepsilon \) is unit of the algebra \( H_\alpha \). Then \( \phi \) is a 2-cocycle, giving rise to the twisted product on \( H_\alpha \):
\[
m^\phi_{\phi}(T^k_i \otimes T^l_j) = T^k_i \tilde{T}^l_j
\]
\[
\tilde{T}^l_j = \rho^m_i T^m_n \overline{\rho}^j_i
\]
(45)
Observe that the condition (38) is trivially fulfilled. Then, the bialgebra \( H_\gamma(m, \Delta, \eta, \varepsilon, R) \), eq. (27), is isomorphic to a partial twisting by the 2-cocycle \( \phi \), of the standard FRT bialgebra \( H(m, \Delta, \eta, \varepsilon, R) \), (39). The twisting can be performed on the product, thus obtaining the bialgebra isomorphism \( H_\gamma(m, \Delta, \eta, \varepsilon, R) = H(m^\phi_\phi, \Delta, \eta, \varepsilon, R) \), with the quadratic ideal \( \mathfrak{B} M - M \mathfrak{B} \) being expressed as
\[
m^{\phi_\phi}_{\phi}(T^k_i \otimes T^l_j) = (R \ast m^\phi_\phi \ast \overline{R}) (T^k_i \otimes T^l_j)
\]
(46)
or, alternatively, one may leaves the product untwisted, but apply the twisting onto the \( R \) map,
\[
R^\phi = \overline{\phi}_{21} \ast R \ast \phi
\]
and, again, \( H_\gamma(m, \Delta, \eta, \varepsilon, R) = H(m, \Delta, \eta, \varepsilon, R^\phi) \) with the ideal, eq. (35), expressed as
\[
m^{\phi_\phi}(T^k_i \otimes T^l_j) = (R^\phi \ast m \ast \overline{R^\phi})(T^k_i \otimes T^l_j)
\]
(47)
In general, both schemes spoil out dual-quasitriangularity. However, in order to leave open the connection with statistical systems where Boltzman weight plays the role of the \( R \) matrix, and the monodromy matrix are the \( T^k_i \), it would be relevant if \( R \) still provides a representation of the bialgebra \( H_\gamma \), as in fact it happens with dual-quasitriangular bialgebras. This is achieved if \( R^\phi \) to satisfy the relations
\[
R^\phi(m \otimes I_H) = R^\phi_{13} \ast R^\phi_{23}
\]
\[
R^\phi(I_H \otimes m) = R^\phi_{13} \ast R^\phi_{12}
\]
(48)
so that \( R^\phi \) is a solution of the Quantum Yang-Baxter equation. In this way, those quantum bialgebras arising from non-commutative corepresentations with factorizable \( \theta \)-map, can be mapped into standard FRT bialgebras by a twisting of the original \( R \).
V. INTEGRABILITY

The deep relation between Hopf Algebras and two dimensional physical systems stems from the integrability condition. As we saw in the previous section, for bialgebras arising from non-commutative corepresentations with a factorizable \( \theta \) map it is possible to arrive to dual quasitriangularity structure, then connection goes as usual. We shall see in this section in which way a general bialgebra \( H \) may be associated to some integrable systems. The main question is the introduction of the spectral parameter, which is related to the coupling constant of the physical system. In doing so, we proceed as in ref. [2] by regarding a collection of vector spaces \( V(\lambda) = \mathbb{C}^n \) for every \( \lambda \in \mathbb{C} \), with basis \( b(\lambda) = \{ e_i(\lambda), i = 1, \ldots, n \} \).

For each value of the spectral parameter \( \lambda \) the coordinate functions \( T^j_i(\lambda) \) generates a bialgebra \( H_\lambda(\lambda) \) with coproduct \( \Delta T^j_i(\lambda) = T^k_i(\lambda) \otimes T^j_k(\lambda) \) and counit \( \epsilon(T^j_i(\lambda)) = \delta^j_i \). Furthermore, the union of the \( H_\lambda(\lambda) \) of these bialgebras for all values of \( \lambda \), i.e., \( H_\lambda = \cup \lambda H_\lambda(\lambda) \), is also a bialgebra with the same coproduct and counit. Also, a \( H_\lambda \)-comodule structure on \( V(\lambda) \) is obtained by the coaction \( \delta_{V(\lambda)} e_i(\lambda) = T^k_i(\lambda) \otimes e_i(\lambda) \).

Now, let us considers the map
\[
\mathfrak{B}(\lambda, \mu) : V(\lambda) \otimes V(\mu) \rightarrow V(\mu) \otimes V(\lambda)
\]
and the QLS defined by the quadratic algebra
\[
A = \frac{\oplus \lambda V^\otimes(\lambda)}{R}
\]
where
\[
\mathfrak{R} = \cup_{\lambda, \mu}[1 \otimes 1 - B(\lambda, \mu)] \ V(\lambda) \otimes V(\mu)
\]
In order to obtain a structure of \( H_\lambda \) comodule on \( A \), we define, following the previous section, the map \( \gamma \) as
\[
\gamma : V(\lambda) \otimes H_\lambda \rightarrow H_\lambda \otimes V(\lambda)
\]
\[
\gamma(e_i(\lambda) \otimes T^k_i(\mu)) = \gamma_{ijm}^k(\lambda, \mu) T^m_i(\mu) \otimes e_m(\lambda)
\]
and the coaction on the tensor product space \( V(\lambda) \otimes V(\mu) \)
\[
\delta_{V \otimes V} = (m \otimes I_{V(\mu)}) (I_{H_\lambda} \otimes \gamma(\lambda, \mu) \otimes I_{V(\mu)}) (\delta_{V(\lambda)} \otimes \delta_{V(\mu)})
\]
In analogy with \textit{proposition} 2, this can be extended to a map,
\[
\delta^\gamma : A \rightarrow H_\lambda \otimes A
\]
supplying \( A \) with a left \( H_\lambda \)-comodule structure.

The results analogous to those ones of \textit{propositions} 1-3 of the section 2 are still valid provided the replacements
\[
\gamma_{ijm}^{kl}(\lambda, \mu) = \delta_j^m \theta_{jm}^{kl}(\lambda, \mu)
\]
\[
\theta_{ij}^{pl}(\lambda, \mu) \theta_{pk}^{rs}(\lambda, \mu) - \delta_j^r \theta_{ik}^{ps}(\lambda, \mu) = 0
\]
\[
\theta_{jm}^{kn}(\lambda, \mu) = \delta_j^k
\]
and
\[
M_{ij}^{kl}(\lambda, \mu) = T^k_i(\lambda) \theta_{jm}^{kn}(\lambda, \mu) T^m_j(\mu)
\]
So that, the ideal \( \mathfrak{B} M - M \mathfrak{B} \) of \textit{proposition} 2 becomes in
\[
B_{ijmn}^{kl}(\lambda, \mu) M_{ij}^{kl}(\lambda, \mu) - M_{ij}^{mn}(\mu, \lambda) B_{mn}^{kl}(\lambda, \mu)
\]
In order to study integrability, we analyze this equation. Eq. (53). In general, a \( \gamma \) map just satisfying the condition of the \textit{prop.} 1, does not lead to integrability making necessary to impose additional conditions on it. In the following we study some options.
Our first ansatz is to require that
\[ \gamma_{imj}^{mkl}(\lambda, \mu) \equiv \delta_i^l \theta_{mj}^{mk} = \delta_i^l \delta_j^k \] (56)

In this way, assuming \( B \) to be invertible, one can multiply (55) by the inverse of \( B \) and then make the contraction of the free index of both \( B \) and \( B^{-1} \), thus reaching the integrability condition
\[ T(\lambda)T(\mu) = T(\mu)T(\lambda) \] (57)

Here, \( T(\lambda) \) means the trace \( T_{nm}^m(\lambda) \).

There is a less obvious way to recover integrability. Condition (55) in terms of \( \theta \) is,
\[ B_{ij}^{kl}(\lambda, \mu) T^{m}_{k}(\lambda)\theta_{iv}^{nu}(\lambda, \mu)T^{n}_{v}(\mu) = T^{k}_{i}(\lambda)\theta_{jv}^{nu}(\lambda, \mu)T^{n}_{v}(\mu)B_{kl}^{mn}(\lambda, \mu) \] (58)

If the following non trivial commutation holds,
\[ B_{ij}^{kl}(\lambda, \mu)\theta_{st}^{il}(\lambda, \mu) = \theta_{ti}^{si}(\lambda, \mu)B_{rj}^{kl}(\lambda, \mu) \] (59)

we may now contract (58) with \( \theta_{st}^{ai}(\lambda, \mu) \) and, after using (55), we get
\[ B^{kl}_{rj}(\lambda, \mu)\theta_{st}^{ik}(\lambda, \mu) T^{m}_{k}(\lambda)\theta_{iv}^{nu}(\lambda, \mu)T^{n}_{v}(\mu) = \theta_{si}^{ai}(\lambda, \mu) T^{k}_{i}(\lambda)\theta_{jv}^{nu}(\lambda, \mu)T^{n}_{v}(\mu)B_{kl}^{mn}(\lambda, \mu) \] (60)

and now we proceed as in the previous case: multiplying by \((B^{-1})_{ab}^{ij}\), and then performing the contractions \((a, m)\), \((b, n)\), thus getting
\[ \theta_{si}^{kl}(\lambda, \mu) T^{m}_{k}(\lambda)\theta_{it}^{nu}(\lambda, \mu)T^{n}_{u}(\mu) = \theta_{sk}^{il}(\lambda, \mu) T^{k}_{i}(\lambda)\theta_{jv}^{nu}(\lambda, \mu)T^{n}_{v}(\mu)B_{kl}^{mn}(\lambda, \mu) \] (61)

Introducing the quantum matrix \( \tilde{T} \) of the final of the previous section, the new integrability condition is written as
\[ \tilde{T}(\lambda)\tilde{T}(\mu) = \tilde{T}(\mu)\tilde{T}(\lambda) \] (62)

This last approach to integrability may have an interpretation in the framework of statistical models through their monodromy matrices \( T^{k}_{l}(\lambda) \). In those models, periodic boundary conditions drive to the transfer matrix by taking the trace over the auxiliary space of the monodromy matrix, which means a sum over all the edge states (see for example ref. [13]). The objects \( \tilde{T}(\lambda) \) means a weighted sum over these edge states, so that \( \theta \) seems to behave as a twisting factor on the boundary conditions. The periodic ones correspond to the trivial choice \( \theta_{ij}^{kl} = \delta_i^l \delta_j^k \), and we speculate that many other kind of boundary conditions would be reached by a suitable choice of \( \theta \) [14].

In the next section we present a multiparametric example constructed from the quantum plane and fulfilling the first integrability condition.

VI. THE QUANTUM PLANE

Let us consider the quantum plane \( A_q^{2|0} \) described by
\[ e_1e_2 = q \ e_2e_1 \] (63)

In the basis \( \{ e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_1 \} \), this relation can be expressed by means the quadratic form \( B \) as
\[ B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1 - q^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (64)

which is a solution of the Yang-Baxter equation
\[ B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23} \] (65)
It is worth remarking that the following construction leads to the same structure for other choice of $B$, as the symmetric and idempotent $B'$,

$$B' = \frac{1}{q + q^{-1}} \begin{bmatrix} q + q^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & 2 & 0 \\ 0 & 2 & q^{-1} - q & 0 \\ 0 & 0 & 0 & q + q^{-1} \end{bmatrix} \tag{66}$$

This $B'$ is not a solution of Yang-Baxter equation but, in the Manin construction for pseudo-symmetric quantum space $\mathbb{H}$, it enables to characterize all the endomorphism of the quantum plane by the relation $B'M - MB'$ as the only solution to the master relation $(I - B')M(I + B')$.

The endomorphism matrix $T$ is

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{67}$$

We find a multiparametric $\theta(r,p,s)$, solution of the coassociativity, counity and the integrability condition, eqs. (68,69), that it is factorizable

$$\theta_{ij}^{kl} = \rho_i^k \rho_j^l \tag{68}$$

with

$$\rho = \begin{bmatrix} 1 & r/s \\ -s/p & (1-r)/p \end{bmatrix} \tag{69}$$

and $\rho$ is its inverse.

This means that the induced map $\phi = \varepsilon \otimes \rho$ is a 2-cocycle whenever $s \neq 0$, hence there is an obstruction in to obtain the undeformed limit $s \to 0$. In this sense, this $\phi(r,p,s)$ is not a 2-cocycle for the whole spectrum of its parameters and the twisting arising from it does not yields a continuous deformation of the algebra $H_q$.

From this matrix we obtain the rules to commute $e$ and $T$,

$$\gamma(e_i \otimes T_j^k) = \theta_{jp}^{ki} T_p^s \otimes e_i \tag{70}$$

As we see, there are now three new deformation parameters $(p,r,s)$ beside to $q$, which was introduced by the quadratic algebra of the quantum plane $\mathbb{H}$.

The relations $BM - MB$ can be written in a compact form by introducing the objects $\tilde{T}_i^j = \theta_{ij}^{rp} T_s^p$, such that

$$\tilde{T} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \tag{71}$$

so that we get

$$a \tilde{c} - q \tilde{c} a = 0$$
$$a \tilde{b} - q \tilde{b} a = 0$$
$$b \tilde{c} - c \tilde{b} = 0$$
$$c \tilde{d} - q d \tilde{c} = 0$$
$$b \tilde{d} - q d \tilde{b} = 0$$
$$a \tilde{d} - d \tilde{a} + (q^{-1} - q) c \tilde{b} = 0 \tag{72}$$

and
\[ 
\gamma(e_i \otimes \bar{a}) = \bar{a} \otimes e_i \\
\gamma(e_i \otimes \bar{d}) = \bar{d} \otimes e_i \\
\gamma(e_i \otimes \bar{b}) = p\bar{b} \otimes e_i \\
\gamma(e_i \otimes \bar{c}) = p^{-1}\bar{c} \otimes e_i 
\]

(73)

The relations (72) acquire a highly non-trivial form in terms of the \( T^k_i \). In this way, one can define a four parameter deformation of the \( M(2) \), namely \( M_{q,p,r,s}(2) \),

\[ 
M_{q,p,r,s}(2) = \frac{k[T^k_i]}{I(BM - MB)} 
\]

(74)

Also, the Grassmannian plane \( A_q^{0/2}(\xi_1, \xi_2) \), defined by the relation

\[ 
\xi_1\xi_2 = -\frac{1}{q}\xi_2\xi_1
\]

is naturally a \( M_{q,p,r,s}(2) \)-comodule \([11]\). This allows us to define a determinant for this \( M_{q,p,r,s}(2) \) from the coaction

\[ 
\delta_{V \otimes V}(\xi_1\xi_2) = D \otimes \xi_1\xi_2 
\]

(75)

Thus we get

\[ 
D = M_{12}^{12} - qM_{12}^{21} = a\bar{d} - q\bar{b}\bar{c} = a d - \frac{q}{p}(1 - r) b c - \frac{r}{s} a c - q^{-1} s b d 
\]

(76)

As was explained above, this is a non-perturbative deformation: the limit to the undeformed case can’t be taken simultaneously. However, there is a sequential limit leading to another 3 and 2 parameters deformation. In fact, if we take first the limit \( r \to 0 \) we get \( M_{q,p,s}(2) \) and now the remaining \( \phi(p, s) \) is now a genuine 2-cocycle, so that the twisting is well defined in the whole spectrum of \( p \) and \( s \). Taking now a second undeformed limit, we set \( s \to 0 \), then we recover the biparametric \( M_{q,p}(2) \) obtained by Manin et al, \([12]\), as non-standard quantum groups. In these limits \( \theta \) becomes in

\[ 
\theta(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & p & 0 \\
0 & \frac{1}{p} & 0 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix} 
\]

(77)

and now, the 2-cocycle \( \phi = \epsilon \otimes \rho \) becomes in

\[ 
\rho = \begin{bmatrix} 1 & 0 \\
0 & 1/p \end{bmatrix} 
\]

(78)

and the relations \( BM - MB \), or \( R^\phi \ast m = m^{op} \ast R^\phi \), reduce to

\[ 
\begin{align*}
ac - pqca &= 0 \\
ab - p^{-1}qba &= 0 \\
bc - p^2cb &= 0 \\
bd - pqbd &= 0 \\
ad - da + p(q^{-1} - q)cb &= 0 
\end{align*} 
\]

(79)
which define a two parametric $M_{q,p}(2)$ as the quotient algebra

$$M_{q,p}(2) = \frac{k[T^j_i]}{I(BM - MB)}$$  \hspace{1cm} (80)

for $i$ and $j$ from 1 to 2. It is worth remarking that $R^\phi(q,p) = \bar{\phi}_{21}(p) * R(q) * \phi(p)$ is a solution of the Quantum Yang Baxter equation, and assuming $R^\phi$ is a bialgebra bicharacter, it supplies $M_{q,p}(2)$ with a dual-quasitriangular structure.

This $\gamma$ give rise to the following relations between matrix entries and the coordinates of the quantum plane,

$$\gamma(e_i \otimes a) = a \otimes e_i$$
$$\gamma(e_i \otimes d) = d \otimes e_i$$
$$\gamma(e_i \otimes b) = pb \otimes e_i$$
$$\gamma(e_i \otimes c) = p^{-1}c \otimes e_i$$  \hspace{1cm} (81)

Defining $\tilde{T}^j_i = \theta^{jk}_{il} T^l_k$ we get

$$\tilde{T} = \begin{bmatrix} a & pc \\ b & d \end{bmatrix}$$  \hspace{1cm} (82)

and with the coproduct

$$\Delta \tilde{T}^j_i = \tilde{T}^k_i \otimes \tilde{T}^j_k$$  \hspace{1cm} (83)

The determinant becomes in

$$D = \det_{q,p} = ad - p^{-1}qbc$$

that satisfy the following commutation relations

$$Da - aD = 0$$
$$Db - p^{-2}bD = 0$$
$$Dc - p^2 cD = 0$$
$$Dd - dD = 0$$

With these properties, and assuming that $D$ is an invertible element of $M_{q,p}(2)$, the antipode can be defined

$$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = D^{-1} \begin{bmatrix} d & -(pq)^{-1}b \\ -pqc & a \end{bmatrix}$$  \hspace{1cm} (84)

and, consequently, the Hopf algebra $GL_{q,p}(2)$ is obtained. Thus, this biparametric deformation of $M(2)$ developed in ref. [12] can be alternatively interpreted in the framework of non-commutative corepresentations.

VII. CONCLUDING REMARKS

We have introduced a new ingredient in the theory of corepresentations of Quantum (semi)Groups admitting non-commutativity between endomorphism matrix entries and quantum space coordinates. This feature give rise to an extra deformation of all the involved structures. Our approach is not a full braiding as those obtained from the quasitensor category of $k$-modules [3]. In a less ambitious project, we have just redefined the coaction showing that it is possible to introduce a non trivial map $\gamma : V \otimes H_\gamma \rightarrow H_\gamma \otimes V$ without spoiling out the Hopf algebra and
the comodule structures provided that $\gamma$ turns $H_\gamma$ into a Quantum Matrix Group. No additional modification were introduced in the usual structure of bialgebras, preserving the product and coproduct untouched. However, provided a factorizable $\gamma$, this generalization of the coaction boils down to a twisting of the algebra structure or a twist of the $R$-matrix, and in some cases it is possible to recover a quasitriangular RFT bialgebra. Although the map $\gamma$ seems too constrained, we have found non trivial solutions introducing many new deformation parameters, still under the additional condition of integrability. This last point was also analyzed, showing that integrability can be reached at least in two independent ways. Working on the Quantum Planes $A_q^{2\mid 0}$ and $A_q^{0\mid 2}$ as examples, it was shown that the biparametric deformation of $GL(2, \mathbb{C})$, namely $GL_{pq}(2)$, can be regarded as coacting in a twisted way over the standard quantum plane.

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