Vertex finiteness for splittings of relatively hyperbolic groups

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Abstract

Consider a group $G$ and a family $\mathcal{A}$ of subgroups of $G$. We say that vertex finiteness holds for splittings of $G$ over $\mathcal{A}$ if, up to isomorphism, there are only finitely many possibilities for vertex stabilizers of minimal $G$-trees with edge stabilizers in $\mathcal{A}$.

We show vertex finiteness when $G$ is a toral relatively hyperbolic group and $\mathcal{A}$ is the family of abelian subgroups.

We also show vertex finiteness when $G$ is hyperbolic relative to virtually polycyclic subgroups and $\mathcal{A}$ is the family of virtually cyclic subgroups; if moreover $G$ is one-ended, there are only finitely many minimal $G$-trees with virtually cyclic edge stabilizers, up to automorphisms of $G$.

1 Introduction

There are many results bounding the complexity of simplicial group actions on trees, or equivalently of graph of groups decompositions. They go under the generic name of accessibility, and they are due mainly to Linnell, Dunwoody, Bestvina-Feighn, Sela, Weidmann [21, 9, 2, 26, 28]. They play a key role in geometric group theory, for instance in the construction of JSJ decompositions or Makanin-Razborov diagrams.

Accessibility usually provides bounds for the number of edges of graph of groups decompositions (splittings) of a given group $G$ over a certain family $\mathcal{A}$ of edge groups (hierarchical accessibility [8, 22] is different). In this paper we are concerned with controlling the isomorphism type of vertex groups.

Definition 1.1. Let $G$ be a group, and let $\mathcal{A}$ be a family of subgroups closed under conjugating and taking subgroups. We say that vertex finiteness holds for splittings of $G$ over $\mathcal{A}$ if, up to isomorphism, there are only finitely many possibilities for vertex groups of decompositions of $G$ as the fundamental group of a minimal graph of groups $\Gamma$ whose edge groups belong to $\mathcal{A}$.

A graph of groups is minimal if its Bass-Serre tree is minimal, i.e. contains no proper $G$-invariant subtree. In the case of one-edge splittings, an HNN extension is always minimal; an amalgam $A *_C B$ is minimal if and only if $C \neq A, B$. We always assume that $G$ is finitely generated, so minimal graphs of groups are finite.

Equivalently, vertex finiteness states that there are finitely many isomorphism types for vertex stabilizers of minimal $G$-trees with edge stabilizers in $\mathcal{A}$.

Here are standard examples of vertex finiteness:

- $G$ is finitely generated, and $\mathcal{A}$ only contains the trivial group. Vertex groups are free factors, there are only finitely many of them up to isomorphism.

- $G$ is a free group $F_n$, and $\mathcal{A}$ is the family of cyclic subgroups (splittings over $\mathcal{A}$ are then called cyclic splittings). Every vertex group of a cyclic splitting is free of rank at most $n$ (this may be seen by abelianizing).
• $G$ is the fundamental group of a closed orientable surface of genus $g$, and $\mathcal{A}$ is the family of cyclic subgroups. Vertex groups are fundamental groups of embedded subsurfaces; they are free of rank $\leq 2g - 1$.

More generally, if $G$ is a one-ended hyperbolic group, and $\mathcal{A}$ is the class of virtually cyclic groups, there are only finitely many possible vertex groups up to the action of $\text{Aut}(G)$ [25][7].

On the other hand, here are examples where finiteness does not hold, even if one restricts to amalgams or HNN extensions (one-edge splittings):

(i) Let $G = H \ast Z$, with $H$ containing torsion elements of arbitrarily high order $n$. Let $\mathcal{A}$ be the class of finite groups. Then $\mathbb{Z}/n\mathbb{Z} \ast \mathbb{Z}$ appears as a vertex group in the amalgam $G = H \ast_{\mathbb{Z}/n\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z} \ast \mathbb{Z})$. There are examples with $G$ finitely presented (hence accessible).

(ii) Let $G$ be the Baumslag-Solitar group $BS(2, 4) = \langle a, t \mid ta^2t^{-1} = a^4 \rangle$. For any $n \geq 1$, the group $\langle x, y \mid x^{2^n} = y^2 \rangle$ is a vertex group of a cyclic splitting of $G$ (see the introduction of [19]).

(iii) In this example, $G$ is hyperbolic relative to the solvable subgroup $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$, $\mathcal{A}$ is the class of cyclic groups, and there is no vertex finiteness even among 2-acylindrical cyclic splittings. Let $G = BS(1, 2) \ast_{a=x,y} F(x, y)$, with $F(x, y)$ the free group on $x$ and $y$. For each $n$, the element $a_n = t^{-n}at^n$ is a $2^n$-th root of $a$, and $P_n = \langle a_n, x, y \rangle \simeq \langle a_n, x, y \mid a_n^{2^n} = [x, y] \rangle$ is a vertex group of the cyclic splitting $G = BS(1, 2) \ast_{(a_n)} P_n$.

(iv) Let $H$ be the discrete Heisenberg group $H = \langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1 \rangle$. Then $G = H \ast \mathbb{Z}$ is hyperbolic relative to the nilpotent group $H$, and there is no vertex finiteness among 2-acylindrical splittings of $G$ over the class $\mathcal{A}$ of nilpotent subgroups. Indeed, $H$ has infinitely many non-isomorphic subgroups $H_n = \langle a^n, b^n, c \rangle$: they are distinguished by the index of the derived subgroup in the center (we thank Pierre Pansu for suggesting this example). Each group $H_n \ast \mathbb{Z}$ is a vertex group in the splitting $G = H \ast_{H_n} (H_n \ast \mathbb{Z})$.

Our main result is the following:

**Theorem 1.2.** Vertex finiteness holds in the following cases:

1. $G$ is finitely generated, $k$ is an integer, and $\mathcal{A} = \text{Fin}_k$ is the family of finite subgroups of order $\leq k$;
2. $G$ is hyperbolic relative to virtually polycyclic subgroups, and $\mathcal{A}$ is the family of virtually cyclic (finite or infinite) subgroups;
3. $G$ is hyperbolic relative to finitely generated abelian subgroups (possibly with torsion), and $\mathcal{A}$ is the family of virtually abelian subgroups;
4. $G$ is a finitely generated, torsion-free, CSA group, abelian subgroups of $G$ are finitely generated of bounded rank, and $\mathcal{A}$ is the family of abelian subgroups.

In Assertion 3, groups in $\mathcal{A}$ are abelian or virtually cyclic. A group is CSA if maximal abelian subgroups are malnormal.

Note that Assertion 2 (or 3) implies vertex finiteness for splittings of hyperbolic groups (with an arbitrary number of ends) over virtually cyclic subgroups. Assertion 3 applies to abelian splittings (i.e. splittings over abelian groups) of limit groups, since by [11][4] limit groups are toral relatively hyperbolic (i.e. torsion-free and hyperbolic relative to finitely generated abelian groups).

**Remark 1.3 (Optimality).** Example (i) above shows that bounding the order of edge groups is necessary in Assertion 1, even if $G$ is finitely presented. Assertion 2 does not apply to
groups which are hyperbolic relative to solvable groups, by Example (iii), and acylindricity does not help.

The example in Subsection 4.2.1 will show that Assertion 3 does not extend if nilpotent parabolic subgroups are allowed. We do not know whether virtually abelian parabolic groups may be allowed (see [17] for the case of groups having a finite index subgroup as in Assertion 3). Finally, bounding the rank of abelian subgroups is necessary in 4: if \( H \) contains \( \mathbb{Z}^n \), then \( \mathbb{Z}^n \ast \mathbb{Z} \) is a vertex group in a splitting of \( H \ast \mathbb{Z} \) over \( \mathbb{Z}^n \).

The following property, which we call \textit{tree finiteness}, is stronger than vertex finiteness: there are only finitely many minimal splittings of \( G \) over \( A \), up to the action of \( \text{Out}(G) \). For instance, it is easy to check that tree finiteness holds for splittings of a finitely generated group over the trivial group. However, Example 3.2 will show that tree finiteness does not hold for splittings over \( \mathbb{Z}/2\mathbb{Z} \) (although vertex finiteness holds by Theorem 1.2).

Tree finiteness was established by Sela and Delzant [26, Corollary 4.9], [7, Theorem 3.2] for virtually cyclic splittings of one-ended hyperbolic groups, using acylindrical super accessibility. We generalize their result as follows:

**Theorem 1.4.** Let \( G \) be one-ended, and hyperbolic relative to virtually polycyclic groups. Up to the action of \( \text{Out}(G) \), there exist only finitely many minimal splittings of \( G \) over virtually cyclic groups.

**Example 1.5.** In this example, tree finiteness does not hold for splittings of a one-ended toral relatively hyperbolic group over abelian groups, even if these groups are assumed to be closed under taking roots. Let \( G \) be the free product of \( A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle a_3 \rangle \simeq \mathbb{Z}^3 \) with three free groups \( G_i = \langle x_i, y_i \rangle \), amalgamated along \( [x_i, y_i] = a_i \). For any \( b \in \mathbb{Z}^3 \), there is a one-edge splitting of \( G \) over the abelian group \( \langle a_1, b \rangle \), with vertex groups \( \langle G_1, b \rangle \) and \( \langle G_2, G_3, A \rangle \). Since \( A \) is \( \text{Aut}(G) \)-invariant (up to conjugacy), and only finitely many automorphisms of \( A \) extend to \( G \), there is no tree finiteness. Note, however, that the isomorphism type of \( \langle G_1, b \rangle \) only depends on whether \( b \) is a power of \( a_1 \) or not.

Our motivation for Theorem 1.2 was the study of automorphisms. In [17] we use Theorem 1.2 to extend Shor's theorem [27, 20] to toral relatively hyperbolic groups: up to isomorphism, there are only finitely many fixed subgroups of automorphisms. Theorem 1.2 is also an important ingredient in our proof that the set of McCool groups of \( G \) satisfies a bounded chain condition when \( G \) is toral relatively hyperbolic [15] (a McCool group of \( G \) is the subgroup of \( \text{Out}(G) \) fixing a given finite set of conjugacy classes of \( G \)).

Assertion 1 of Theorem 1.2 is proved in Section 3. The other assertions are proved simultaneously in later sections. We successively consider one-edge splittings of one-ended groups, then one-edge splittings of arbitrary groups, and finally splittings with several edges. Tree finiteness (Theorem 1.4) is proved at the end of Section 4.

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## 2 Preliminaries

### 2.1 Trees and splittings

In this paper, \( G \) will always denote a finitely generated group.

A tree will be a simplicial tree \( T \) with an action of \( G \) without inversions. Two trees are considered to be the same if there is a \( G \)-equivariant isomorphism between them.

We usually assume that the action is \textit{minimal} (there is no proper invariant subtree) and that there is \textit{no redundant vertex} (if \( T \setminus \{x\} \) has 2 components, some \( g \in G \) interchanges them). The tree \( T \) is \textit{trivial} if there is a global fixed point (minimality then implies that \( T \) is a point). An element of \( G \), or a subgroup, is \textit{elliptic} if it fixes a point in \( T \).
Lemma 2.1. Let $H$ be virtually polycyclic.

1. $H$ only contains finitely many conjugacy classes of finite subgroups.

2. Given a subgroup $A \subset H$, there exists a finite index subgroup $A_0 \subset A$ such that, up to conjugation by an element of the normalizer $N(A_0)$, there exist only finitely many subgroups $B \subset H$ containing $A$ with finite index.

2.2 Virtually polycyclic groups

We collect a few simple algebraic facts. We write $|X|$ for the cardinality of a finite set.

We usually restrict edge groups by requiring that they belong to a family $\mathcal{A}$ as in Theorem 1.2. We then say that the splitting is over groups in $\mathcal{A}$, or over $\mathcal{A}$. The group $G$ splits over $A$ if $A$ is an edge group of a non-trivial splitting.

There is a one-to-one correspondence between vertices (resp. edges) of $\Gamma$ and $G$-orbits of vertices (resp. edges) of $T$. We say that $\Gamma$ is an edge splitting if it has exactly one edge. We denote by $G_v$ the group carried by a vertex $v$ of $\Gamma$. Similarly, we denote by $e$ an edge of $\Gamma$ or $T$, and by $G_e$ the corresponding group. The groups carried by edges of $\Gamma$ incident to a given vertex $v$ will be called the incident edge groups at $v$ (we usually view them as subgroups of $G_v$).

A tree $T'$ is a collapse of $T$ if it is obtained from $T$ by collapsing each edge in a certain $G$-invariant collection to a point; conversely, we say that $T$ refines $T'$. In terms of graphs of groups, one passes from $\Gamma = T/G$ to $\Gamma' = T'/G$ by collapsing edges; for each vertex $v'$ of $\Gamma'$, the vertex group $G_{v'}$ is the fundamental group of the graph of groups $\Gamma_{v'}$ occurring as the preimage of $v'$ in $\Gamma$.

Conversely, suppose $v'$ is a vertex of a splitting $\Gamma'$, and $\Gamma_{v'}$ is a splitting of $G_{v'}$ in which incident edge groups are elliptic. One may then refine $\Gamma'$ at $v'$ using $\Gamma_{v'}$, so as to obtain a splitting $\Gamma$ whose edges are those of $\Gamma'$ together with those of $\Gamma_{v'}$. Note that $\Gamma$ is not uniquely defined because there is flexibility in the way edges of $\Gamma'$ are attached to vertices of $\Gamma_{v'}$; this is discussed in Subsection 4.2.

All maps between trees will be $G$-equivariant. Given two trees $T$ and $T'$, we say that $T$ dominates $T'$ if there is a map $f : T \to T'$, or equivalently if every subgroup which is elliptic in $T$ is also elliptic in $T'$. In particular, $T$ dominates any collapse $T'$.

Two trees belong to the same deformation space if they dominate each other. In other words, a deformation space $\mathcal{D}$ (over $\mathcal{A}$) is the set of all trees (with edge stabilizers in $\mathcal{A}$) having a given family of subgroups as their elliptic subgroups. All trees in a given deformation space over $\mathcal{A}$ have the same set of vertex stabilizers, provided that one restricts to stabilizers not in $\mathcal{A}$. We sometimes view a deformation space as a set of splittings (rather than trees).

Groups as in Assertions 2, 3, 4 of Theorem 1.2 are accessible, so there exists a Stallings-Dunwoody deformation space: it consists of trees with finite edge stabilizers whose vertex stabilizers have at most one end. In the context of Assertion 1, we shall consider the deformation space $\mathcal{D}_k$ over $\mathcal{F} \text{in}_k$ consisting of trees whose vertex stabilizers do not split over a group in $\mathcal{F} \text{in}_k$ (recall that a group is in $\mathcal{F} \text{in}_k$ if it has order $\leq k$). These deformation spaces may (and should) be viewed as JSJ deformation spaces over the class of finite groups or over $\mathcal{F} \text{in}_k$ respectively (see [13]).

A tree is reduced if $G_e \neq G_v, G_w$ whenever an edge $e$ has its endpoints $v, w$ in different $G$-orbits (being reduced in the sense of [2] is a weaker property). Equivalently, no tree obtained from $T$ by collapsing the orbit of an edge belongs to the same deformation space as $T$. If $T$ is not reduced, one may collapse edges so as to obtain a reduced tree in the same deformation space.

2.2 Virtually polycyclic groups

We collect a few simple algebraic facts. We write $|X|$ for the cardinality of a finite set.

Lemma 2.1. Let $H$ be virtually polycyclic.

1. $H$ only contains finitely many conjugacy classes of finite subgroups.

2. Given a subgroup $A \subset H$, there exists a finite index subgroup $A_0 \subset A$ such that, up to conjugation by an element of the normalizer $N(A_0)$, there exist only finitely many subgroups $B \subset H$ containing $A$ with finite index.
3. Given a subgroup $A \subset H$, there exists a number $N$ such that, if $B \subset H$ contains $A$ with finite index $n$, then $n \leq N$.

Proof. The first assertion is contained in Theorem 8.5 of [25]. It implies 2 when $A$ is trivial.

To prove 2 in general, define $C(A)$ as the commensurator of $A$, equal to the set of $g \in G$ such that $gAg^{-1} \cap A$ has finite index in $A$ and $gAg^{-1}$. Note that any $B$ containing $A$ with finite index is contained in $C(A)$. Let $A_0 = \bigcap_{g \in C(A)} gAg^{-1}$. By [25], $A_0$ is the intersection of a finite family of conjugates of $A$, so $A_0$ has finite index in $A$. It is normal in $C(A)$, and Assertion 2 follows by applying 1 to $C(A)/A_0$. In particular, there is a bound for the index of $A_0$ in $B$, so 3 is proved.

Lemma 2.2.  
1. Given $n \in \mathbb{N}$, there are finitely many isomorphism types of virtually cyclic groups $A$ such that all finite subgroups of $A$ have order $\leq n$.

2. Given two virtually cyclic groups $A$ and $B$, and $n \in \mathbb{N}$, there are only finitely many monomorphisms $i : A \to B$ such that the index of $i(A)$ in $B$ is $\leq n$, up to precomposition by an inner automorphism of $A$.

Proof. A virtually cyclic group $A$ maps with finite kernel $N$ onto a group which is either infinite cyclic or equal to the infinite dihedral group $D_\infty$ (see [24, Theorem 5.12]). In the first case, $A$ is a semidirect product $N \rtimes \mathbb{Z}$ and there are only finitely many possibilities for $A$ up to isomorphism since $|N|$ is bounded. In the second case, $A \simeq N_1 \rtimes N_2$ with $|N_1| = |N_2| = 2|N|$, and again there are only finitely many possibilities. For the second assertion, note that there are finitely many possibilities for the image of $i$. Two injections with the same image differ by an automorphism of $A$, and $\text{Out}(A)$ is finite.

Lemma 2.3. Fix a finitely generated abelian group $P$, and a subgroup $A \subset P$. Say that two subgroups $B, B'$ with $A \subset B \subset P$ and $A \subset B' \subset P$ are equivalent if there is an isomorphism $B \to B'$ equal to the identity on $A$.

Then the number of equivalence classes is finite.

Proof. Define the root-closure $e(A, B)$ as the set of elements of $B$ having a power in $A$. It contains the torsion subgroup of $B$, and it is the smallest subgroup of $B$ containing $A$ and such that $B = e(A, B) \oplus B_0$ with $B_0 \subset B$ torsion-free. Equivalently, $e(A, B)$ is the largest subgroup of $B$ containing $A$ with finite index. Note that $A \subset e(A, B) \subset e(A, P)$, with all indices finite. As $B$ varies, there are only finitely many possibilities for $e(A, B)$, and for the isomorphism type of $B_0$. When $e(A, B) = e(A, B')$, and $B_0 \simeq B'_0$, any isomorphism $B_0 \to B'_0$ extends to an isomorphism $B \to B'$ equal to the identity on $e(A, B)$, hence on $A$.

Corollary 2.4. Fix two groups $G_0$ and $P$ with a common subgroup $A$, where $P$ is finitely generated abelian. As $B$ varies among subgroups such that $A < B < P$, the groups $G_0 *_A B$ lie in finitely many isomorphism classes.

Indeed, $G_0 *_A B \simeq G_0 *_A B'$ if $B, B'$ are equivalent.

2.3 Relatively hyperbolic groups

Suppose that $G$ is as in Assertion 2 or 3 of Theorem 1.2, i.e. $G$ is hyperbolic relative to a finite family $\{P_1, \ldots, P_k\}$ of finitely generated subgroups, which are virtually polycyclic or abelian. Subgroups of $P_i$, and their conjugates, are called parabolic. A subgroup of $G$ is virtually polycyclic if and only if it is parabolic or virtually cyclic. Any infinite virtually polycyclic subgroup is contained in a unique maximal one, which is virtually cyclic (loxodromic) or conjugate to some $P_i$. 

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Lemma 2.5. Let $G$ be hyperbolic relative to virtually polycyclic subgroups.
1. $G$ only contains finitely many conjugacy classes of finite subgroups.
2. Up to isomorphism, $G$ only has finitely many virtually cyclic subgroups.
3. Given a virtually polycyclic subgroup $A \subset G$, there are only finitely many groups $B \subset G$ containing $A$ with finite index, up to conjugacy in $G$; when $B$ varies, the index of $A$ in $B$ remains bounded.

Proof. In a relatively hyperbolic group, all finite subgroups outside of a finite number of conjugacy classes are parabolic (see for instance Lemma 3.1 of [16]), so Assertion 1 follows from Lemma 2.1. Assertion 2 follows from Lemma 2.2. Assertion 3 is clear if $A$ is finite or loxodromic, and follows from Lemma 2.1 otherwise.

Note that the lemma also holds if $G$ is a CSA group as in Assertion 4. In this case, all virtually polycyclic subgroups are abelian.

2.4 About the proofs
The next four sections are devoted to the proof of Theorems 1.2 and 1.4. All splittings will be over groups in the relevant family $\mathcal{A}$.

Note that, under all assumptions, groups in $\mathcal{A}$ are virtually abelian and fall into finitely many isomorphism classes (this follows from Lemma 2.5 in the relatively hyperbolic case). It therefore suffices to prove vertex finiteness for reduced splittings since vertex groups not in $\mathcal{A}$ remain when one collapses edges to obtain a reduced splitting in the same deformation space. Another consequence is that vertex finiteness in fact holds for non-minimal splittings.

We also note that vertex groups of splittings of $G$ over $\mathcal{A}$ satisfy the assumptions of the theorem (they are finitely generated, relatively hyperbolic, or CSA). In the relatively hyperbolic case, this follows from Theorem 1.3 of [3], as explained in the proof of Theorem 3.35 of [5] (for Assertion 3, note that nonabelian virtually cyclic groups may be removed from the list of maximal parabolic subgroups); in the CSA case, vertex groups are finitely generated (because edge groups are) and CSA. This makes inductive arguments possible.

A basic method for showing vertex finiteness is to represent any vertex group $G_v$ as the fundamental group of another graph of groups whose number of edges is bounded, and where the set of possible isomorphism types of edge and vertex groups is finite. One then has to control inclusions of edge groups into vertex groups.

When edge groups are finite, it suffices to know that vertex groups only contain finitely many conjugacy classes of finite subgroups, since postcomposing an inclusion $G_e \to G_v$ with an inner automorphism of $G_v$ does not change the fundamental group of the graph of groups.

When edge groups are infinite and $G$ is one-ended, we use a canonical JSJ decomposition $\Gamma_{\text{can}}$, and its universal compatibility with the splittings considered: given any $\Gamma$, there is a splitting $\Lambda$ such that both $\Gamma$ and $\Gamma_{\text{can}}$ may be obtained from $\Lambda$ by collapsing edges.

3 Splittings over finite groups
We prove the first assertion of Theorem 1.2. All splittings considered here will be minimal and over groups belonging to $\mathcal{F}\text{in}_k$, the family of all subgroups of order $\leq k$. Linell's accessibility [21] provides a bound (depending on $G$ and $k$) for the number of edges of such splittings, as long as the splittings have no redundant vertex.

We shall first show:

Lemma 3.1. Let $G$ be a finitely generated group, $k \geq 1$, and $\mathcal{D}$ a deformation space over $\mathcal{F}\text{in}_k$. Then $\mathcal{D}$ only contains finitely many reduced trees $T$ up to the action of $\text{Out}(G)$.
More precisely: the subgroup $\text{Out}(D)$ of $\text{Out}(G)$ consisting of automorphisms leaving $D$ invariant acts on the set of reduced trees in $D$ with finitely many orbits.

The example below shows that $\text{Out}(D)$ does not always act on the whole of $D$ with finitely many orbits, because of non-reduced trees. It also shows that the number of deformation spaces of $G$ over $\text{Fin}_2$ may be infinite modulo $\text{Out}(G)$. In particular, tree finiteness does not hold for splittings over $\text{Fin}_2$ (though it holds for splittings over the trivial group).

**Example 3.2.** Let $A$ be a one-ended group whose set of elements of order 2 is not a finite union of $\text{Aut}(A)$-orbits (one can check that the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is such a group, cf. Proposition 2.1 of [10]). Let $G = A \ast B$, with $B$ one-ended. Let $D$ be the deformation space over $\text{Fin}_2$ containing the Bass-Serre tree of the defining free product $G = A \ast B$. Here, $\text{Out}(D) = \text{Out}(G)$. For any subgroup $F < A$ of order 2, the Bass-Serre tree $T_F$ of the (non-reduced) two-edge graph of groups decomposition $G = A \ast F F \ast B$ lies in $D$, and the trees $T_F$ are not contained in a finite union of $\text{Out}(G)$-orbits as $F$ varies. Moreover, the one-edge splittings $G = A \ast F (F \ast B)$ define infinitely many $\text{Out}(G)$-orbits of deformation spaces as $F$ varies.

**Proof of Lemma 3.1.** Let $\Gamma = T/G$. By accessibility, the number of edges of $\Gamma$ is bounded. To describe $\Gamma$, we need to know edge groups, vertex groups, and inclusions of edge groups into vertex groups. There are only finitely many possibilities for edge groups (up to isomorphism). Vertex groups of $\Gamma$ with order $> k$ do not depend on $\Gamma$ for $T \in D$, so there are only finitely many possibilities for vertex groups of $\Gamma$ up to isomorphism. To prove finiteness, it therefore suffices to show that there are only finitely many possibilities for the image of an edge group in a vertex group of order $> k$, up to conjugacy.

Fix a reduced $\Gamma_0 = T_0/G$, with $T_0 \in D$. Since no edge group may be properly contained in a conjugate of itself, it follows from Proposition 4.9 of [11] that any vertex group $H$ of $\Gamma_0$, with $H \notin \text{Fin}_k$, contains finitely many subgroups $E_i(H) \in \text{Fin}_k$ with the following property: given any reduced $\Gamma = T/G$ with $T \in D$, each incident edge group of the vertex group $G_v = gHg^{-1}$ of $\Gamma$ conjugate to $H$ is contained in a $G_v$-conjugate of some $gE_i(H)g^{-1}$. The required finiteness follows since any vertex group of $\Gamma$ of order $> k$ is conjugate to some $H$.

**Remark 3.3.** In Section 7 of [11], we have defined an $\text{Out}(D)$-invariant retract $\mathcal{G} \subset D$. It consists of trees $T \in D$ all of whose edges are surviving edges: given any edge $e$, one can collapse $T$ to a reduced tree $T' \in D$ without collapsing $e$. The same argument as above shows that $\mathcal{G}$ only contains finitely many trees, up to the action of $\text{Out}(D)$. This says that $\mathcal{G}/\text{Out}(D)$ is a finite complex with missing faces, or equivalently that its spine is finite (see [11]).

If $G$ is accessible, we can consider the Stallings-Dunwoody deformation space $D$ and its retract $\mathcal{G}$. Edge stabilizers of trees in $\mathcal{G}$ all belong to some fixed $\text{Fin}_k$, so $\mathcal{G}$ coincides with the retract of a deformation space over $\text{Fin}_k$, and $\mathcal{G}/\text{Out}(G)$ is finite as above.

We now prove the first assertion of Theorem 1.2. By Linnell’s accessibility, there is a (unique) deformation space $D_k$ over $\text{Fin}_k$ such that vertex stabilizers of trees in $D_k$ do not split over a group in $\text{Fin}_k$ (this is the JSJ deformation space over $\text{Fin}_k$, see [13] subsection 6.3). Recall that all trees in $D_k$ have the same vertex stabilizers of order $> k$.

Let $H = G_v$ be a vertex group of a splitting $\Gamma$ of $G$ over $\text{Fin}_k$. As explained in Subsection 2.4, we may assume that $\Gamma$ is reduced. By Lemma 4.8 of [13], one may refine $\Gamma$ to a (JSJ) splitting $\Lambda$ in $D_k$. The refinement replaces the vertex $v$ by a subgraph of groups $\Lambda_v \subset \Lambda$ whose fundamental group is $H$. We may assume that $\Lambda_v$ is reduced (but not that $\Lambda$ is). We show that there are only finitely many possibilities for $\Lambda_v$.

The splitting $\Lambda$ is not necessarily reduced, so let $p : \Lambda \to \Lambda_0$ be a collapse map to a reduced splitting in $D_k$. Since $\Lambda_v$ is reduced, the map $p$ does not collapse any edge coming
from $\Lambda_v$. In particular, the number of edges of $\Lambda_v$ is bounded by the number of edges of $\Lambda_0$, which is bounded by Lemma 3.1.

Vertex groups of $\Lambda$ of order $> k$ are vertex groups of $\Lambda_0$, so there are finitely many possibilities for vertex and edge groups of $\Lambda_v$ up to isomorphism. There remains to control inclusions $G_e \to G_u$ from edge groups of $\Lambda_e$ to vertex groups. We may assume that $G_u$ has order $> k$. This implies that the group carried by $p(u)$ in $\Lambda_0$ is $G_u$ (but the group carried by the other endpoint of $e$ may grow). Finiteness follows from the finiteness of possible images of incident edge groups in vertex groups of graphs in $\mathcal{D}_k$ as in the previous proof.

4 One-edge splittings of one-ended groups

In this section we prove Assertions 2, 3, 4 of Theorem 1.2 for one-edge splittings of one-ended groups. We will also prove Theorem 1.4 (see Subsection 4.5).

We assume that $G$ is one-ended and we consider a one-edge splitting $\Gamma$. As explained in Subsection 2.4 we may assume that $\Gamma$ is reduced (i.e. minimal). All splittings will be over groups in $\mathcal{A}$ (necessarily infinite by one-endedness).

We first explain how to obtain $\Gamma$ from a JSJ decomposition $\Gamma_{\text{can}}$ by refining and collapsing. We then discuss refining in general (Subsection 4.2).

4.1 The canonical JSJ splitting

Let $T_{\text{can}}^*$ be the canonical JSJ tree over $\mathcal{A}$ constructed in Theorems 11.1 and 13.1 of [14] (applied with $\mathcal{H} = \emptyset$), and $\Gamma_{\text{can}}$ the associated graph of groups. In all cases considered here it is the JSJ decomposition of $G$ over $\mathcal{A}$ relative to all virtually polycyclic subgroups which are not virtually cyclic. We summarize the relevant properties of $\Gamma_{\text{can}}$.

$\Gamma_{\text{can}}$ is not necessarily reduced (and may have redundant vertices). Its vertex groups $G_{v}$ are either maximal virtually polycyclic subgroups, or rigid, or QH with finite fiber. If $G_v$ is rigid, it has no non-trivial splitting over groups in $\mathcal{A}$ in which incident edge groups are elliptic. If $G_v$ is QH, there is an exact sequence $1 \to F \to G_v \to \pi_1(\Sigma_v) \to 1$ where the fiber $F$ is finite and $\Sigma_v$ is a compact 2-dimensional orbifold. Incident edge groups are preimages of boundary subgroups of $\pi_1(\Sigma_v)$ (i.e. fundamental groups of boundary components of $\Sigma_v$), and conversely such preimages are incident edge groups (up to conjugacy).

Moreover, $\Gamma_{\text{can}}$ is universally compatible. This means that, given a non-trivial one-edge splitting $\Gamma$ as above, there is a splitting $\Lambda$ which collapses onto both $\Gamma$ and $\Gamma_{\text{can}}$. It is minimal, but not necessarily reduced. After collapsing edges in $\Lambda$, we may assume that no edge of $\Lambda$ is collapsed in both $\Gamma$ and $\Gamma_{\text{can}}$. Let $\varepsilon$ be the edge of $\Lambda$ that is not collapsed in $\Gamma_{\text{can}}$, since otherwise $\Gamma$ is a collapse of $\Gamma_{\text{can}}$, and this only produces finitely many splittings.

Denote by $v$ the vertex of $\Gamma_{\text{can}}$ to which $\varepsilon$ is collapsed, and by $G_v$ the corresponding vertex group. Let $\Lambda_v \subset \Lambda$ be the one-edge splitting of $G_v$ associated to $\varepsilon$, so that $\Lambda$ is obtained from $\Gamma_{\text{can}}$ by replacing the vertex $v \in \Gamma_{\text{can}}$ by the one-edge decomposition $\Lambda_v$ of $G_v$.
Note that $\Lambda_v$ can be a trivial decomposition (i.e. an amalgam of the form $G_v = G_v \ast_{G_v} G_v$). This occurs precisely when $\Gamma_{\text{can}}$ and $\Lambda$ belong to the same deformation space. In this case, the splitting $\Lambda$ is not reduced.

4.2 Refining a splitting

Knowing $\Gamma_{\text{can}}$ and $\Lambda_v$ is not enough to determine $\Lambda$ and $\Gamma$: one must also know how edges $e$ of $\Gamma_{\text{can}}$ incident to $v$ are attached to vertices of $\Lambda_v$ (note that refining is possible only if all groups $G_e$ are elliptic in $\Lambda_v$). When $\Lambda_v$ has two vertices, one must first decide to which vertex $u$ of $\Lambda_v$ each edge $e$ is attached. This is a combinatorial choice, with only finitely many possibilities, so we will always assume that this choice has been made. One must then know, for each $e$, the injection of $G_e$ into $G_u$, and this is a possible cause of infiniteness. We demonstrate this on an example.

4.2.1 Changing attachments

We construct a splitting $\Theta_0$ of a group $G$ such that there are infinitely many ways to refine the Bass-Serre tree of $\Theta_0$ using a fixed one-edge splitting $\Lambda_v$ of a vertex group $G_v$ (see Figure 2). This will also demonstrate that there is no vertex finiteness for abelian splittings of groups which are hyperbolic relative to nilpotent groups.

Let $H$ be the Heisenberg group, which we view as a semidirect product $\mathbb{Z} \times \mathbb{Z} = \langle a, b, t \mid ab = ba, tat^{-1} = ab, tbt^{-1} = b \rangle$. The splitting $\Theta_0$ has two vertices $v, w$, with $G_v = H$ and $G_w$ a torsion-free hyperbolic group with no non-trivial cyclic splitting. They are joined by two edges $e_1, e_2$ carrying infinite cyclic groups. The inclusions of edge groups into vertex groups map both $G_{e_1}$ and $G_{e_2}$ onto $\langle a \rangle$ in $H = G_v$, and they map $G_{e_1}, G_{e_2}$ onto non-conjugate maximal cyclic subgroups of $G_w$. The splitting $\Lambda_v$ of $G_v = H$ is the HNN extension associated to the semidirect product. The group $G = \pi_1(\Theta_0)$ is hyperbolic relative to the nilpotent group $H$ by [4], and it may be checked that $\Theta_0$ is its JSJ decomposition over abelian (or nilpotent) groups relative to $H$.

Let $\Lambda_0$ be obtained by refining $\Theta_0$ using $\Lambda_v$ in the obvious way. Collapsing $e_1, e_2$ in $\Lambda$ yields an HNN extension $\Gamma_0$ with edge group $\mathbb{Z}^2 = \langle a, b \rangle$. The base group $\Lambda_0$ is the

**Figure 2: Infinitely many refinements of a graph of groups**
fundamental group of the graph of groups $\Gamma_0$ obtained from $\Theta_0$ by making the group carried by $v$ equal to $\mathbb{Z}^2 = \langle a, b \rangle$ rather than $H$.

Now let $n \in \mathbb{N}$. Consider $\Theta_0$ and define a new graph of groups $\Theta_n$ by postcomposing the inclusion $G_{e_1} \to G_v$ with conjugation by $t^n$, an inner automorphism of $G_v$; the image of $G_{e_1}$ is now generated by $t^n a t^{-n} = ab^n$. Since we changed the edge monomorphism by an inner automorphism of the vertex group, $\Theta_n$ and $\Theta_0$ are equivalent (they are associated to the same Bass-Serre tree). Then construct $\Lambda_n, \Gamma_n$ and $\Gamma'_n$ as above.

It is still true that $\Lambda_n$ refines $\Theta_n$, and the base group $A_n$ of the HNN extension $\Gamma_n$ is the fundamental group of a graph of groups $\Gamma'_n$ with vertices carrying $\mathbb{Z}^2 = \langle a, b \rangle$ and $G_w$. But the inclusion of $G_{e_1}$ into $\mathbb{Z}^2$ now has image generated by $ab^n$. In particular, the subgroup of $\mathbb{Z}^2$ generated by the incident edge groups is $\langle a, ab^n \rangle$, it has index $n$. This shows that the splittings $\Gamma'_n$ (hence also the $\Lambda_n$’s) are distinct. Moreover, $\Gamma'_n$ is the canonical cyclic JSJ decomposition of $A_n$ relative to non-cyclic abelian groups, so the $A_n$’s are pairwise non-isomorphic.

In terms of trees, the minimal $H$-invariant subtree in the Bass-Serre tree of $\Lambda_0$ is a line $L$. There are lifts of $e_1$ and $e_2$ to vertices of $L$. In the Bass-Serre tree of $\Lambda_n$, the attachment point of a given lift of $e_1$ gets shifted by a translation of length $n$ along $L$.

### 4.2.2 Practical description of a one-edge refinement

We now explain how to describe all one-edge refinements $\Lambda$ of a given graph of groups $\Theta$ at a vertex $v$ (i.e. $\Lambda$ is obtained by refining $\Theta$ at $v$ using a one-edge splitting). In the next subsection, we will take $\Theta$ to be the canonical JSJ decomposition $\Gamma_{can}$. We view $G_v$ as a subgroup of $G$, and, for each edge $e$ incident to $v$ in $\Theta$, we view $G_e$ as a subgroup of $G_v$.

By Bass-Serre theory, a graph of groups $\Lambda$ gives an action of a group $G_\Lambda$ on a tree $T_\Lambda$. We consider $\Lambda$ and $\Lambda'$ as equivalent if there is an isomorphism $\tau : G_\Lambda \to G_{\Lambda'}$ and a $\tau$-equivariant isomorphism $T_\Lambda \to T_{\Lambda'}$.

**Lemma 4.1.** Up to equivalence, any one-edge refinement $\Lambda$ of $\Theta$ at a vertex $v$ may be obtained from the following data:

1. (marked splitting): an isomorphism $\varphi : G_v \to \pi_1(\Lambda_v)$, where $\Lambda_v$ is a one-edge splitting (which may be a trivial splitting $G_v \ast_{G_e} G_e$);
2. (combinatorial attachment): when $\Lambda_v$ is an amalgam, the choice of a vertex $u_e$ of $\Lambda_v$ for each oriented edge $e$ of $\Theta$ incident to $v$;
3. (algebraic attachment): for each oriented edge $e$ of $\Theta$ incident to $v$, a monomorphism $i_e : G_e \to \varphi^{-1}(G_{u_e})$ which is the restriction of some inner automorphism $ad_{g_e} \in \text{Inn}(G_v)$.

Different data may yield equivalent splittings. For instance, postcomposing $i_e$ with an inner automorphism of $\varphi^{-1}(G_{u_e})$ does not change $\Lambda$.

**Proof.** Starting from the data, one constructs a graph of groups $\Lambda$ as follows. The underlying graph is obtained from that of $\Lambda$ by blowing up $v$ into the one-edge graph underlying $\Lambda_v$, and attaching incident edges as prescribed by the combinatorial attachment data.

The vertex groups are those of $\Theta \setminus \{v\}$, and preimages under $\varphi$ of those of $\Lambda_v$; the edge groups are those of $\Theta$, and the preimage of the edge group of $\Lambda_v$; the monomorphisms from edge groups to vertex groups are the natural ones (those of $\Theta$ and $\Lambda_v$) and the $i_e$’s. Collapsing the edge of $\Lambda_v$ yields $\Theta$ (up to equivalence) because of the requirement that $i_e$ be the restriction of an inner automorphism.

Conversely, if $\Lambda$ is a one-edge refinement of $\Theta$, with Bass-Serre tree $T_\Lambda$, one defines $\Lambda_v$ as the one-edge splitting associated to the $G_v$-invariant subtree $T_v \subset T_\Lambda$ which is collapsed to a point $\tilde{v}$ in the Bass-Serre tree $T_\Theta$ of $\Theta$. We fix an identification $\varphi$ by choosing an edge
in $T_v$. In particular, this selects a vertex in each $G_v$-orbit of vertices of $T_v$ (there is one or two orbits, so one or two selected points $u, u'$).

If $e$ is an edge of $\Theta$ incident to $v$, we view $G_e$ as the stabilizer of an edge $\tilde{e}$ of $T_{\tilde{v}}$ incident to $\tilde{v}$. In $T_{\Lambda}$, this edge is attached to a vertex $v_e$ of $T_v$. The orbit of $v_e$ determines the combinatorial attachment, and $i_e$ is induced by $\text{ad}_{g_e}$ with $g_e$ any element of $G_v$ taking this vertex $v_e$ to the selected vertex $u$ or $u'$.

\begin{remark}
If we replace a marking $\varphi : G_v \to \pi_1(\Lambda_v)$ by $\varphi' = \varphi \circ \psi$, with $\psi$ an inner automorphism of $G_v$, the refinements of $\Theta$ by $\Lambda_v$ obtained using $\varphi'$ are (up to equivalence) the same as those obtained using $\varphi$ (one simply replaces $i_e$ by $\psi^{-1} \circ i_e$). This holds, more generally, if $\psi$ acts on each incident edge group $G_e$ as conjugation by some $g_e \in G_v$.
\end{remark}

\begin{remark}
If $\varphi^{-1}(G_{u_e})$ is malnormal in $G_v$, then the different choices for $i_e$ differ by an inner automorphism of $\varphi^{-1}(G_{u_e})$ and therefore lead to equivalent splittings $\Lambda$. More generally, the same conclusions hold if $G_e$ is contained in a unique conjugate of $\varphi^{-1}(G_{u_e})$ in $G_v$, and $\varphi^{-1}(G_{u_e})$ is its own normalizer in $G_v$.
\end{remark}

\subsection{Vertex finiteness over virtually cyclic groups}

This subsection is devoted to the proof of:

\begin{proposition}
Let $G$ be one-ended, and hyperbolic relative to virtually polycyclic groups. Then vertex finiteness holds for one-edge virtually cyclic splittings of $G$.
\end{proposition}

We will actually prove:

\begin{lemma}
Let $G$ be one-ended, and hyperbolic relative to virtually polycyclic groups. Let $\Theta$ be a virtually cyclic splitting of $G_v$, and $v$ a vertex. Up to the action of $\text{Out}(G)$, there exist only finitely many minimal virtually cyclic splittings $\Lambda$ obtained by refining $\Theta$ at $v$ using a one-edge splitting $\Lambda_v$ with the following property: if $G_v$ is not virtually polycyclic or QH with finite fiber (as defined in Subsection \[4.4\]), then $\Lambda_v$ is a trivial amalgam $G_v \ast_{G_e} G_e$.
\end{lemma}

The lemma implies the proposition because, as explained in Subsection \[4.4\], any one-edge virtually cyclic splitting of $G$ is a collapse of a one-edge refinement of $\Theta = \Gamma_{\text{can}}$; vertex groups of $\Gamma_{\text{can}}$ which are not virtually polycyclic or QH with finite fiber are rigid, so can only be refined using a trivial amalgam.

In Subsection \[4.5\] we will explain that the lemma yields tree finiteness for one-edge virtually cyclic splittings, and we will use it to prove Theorem \[1.4\] (tree finiteness for arbitrary virtually cyclic splittings).

\begin{proof}[Proof of Lemma \[4.5\]]
We use the notations of Lemma \[4.1\] and we denote by $G_e$ the edge group of $\Lambda_v$. We assume (when $\Lambda_v$ has two vertices) that the combinatorial choice (deciding to which vertex of $\Lambda_v$ edges of $\Theta$ incident to $v$ will be attached) has been made. We must prove that varying the marked splitting and the $i_e$’s does not produce infinitely many $\Lambda$’s.

We distinguish several cases, depending on the nature of $\Lambda_v$.

- First suppose that $\Lambda_v$ is a trivial amalgam. In this case we may assume that $\Lambda_v$ is $G_v = G_e \ast_{G_e} G_e$ for some virtually cyclic $G_e \subset G_v$, and $\varphi$ is the identity (the marked splitting is determined by $G_e$, and changing $\varphi$ amounts to changing $G_e$ by an automorphism of $G_v$). Call $u, u'$ the vertices of $\Lambda_v$, with vertex groups $G_u = G_e$ and $G_{u'} = G_v$.

By minimality of $\Lambda$, at least one edge $e_1$ of $\Theta$ is attached to $u$. Thus $G_{e_1}$ is contained in a conjugate of $G_u = G_e$. The groups $G_{e_1}$ and $G_e$ are both infinite and virtually cyclic, so the index is finite. Since by Remark \[4.2\] the set of refinements does not change if we replace $G_e$ by a conjugate, Assertion 3 of Lemma \[2.5\] (applied in $G_v$) lets us assume that $G_e$ is fixed (because there are only finitely many possibilities for $G_e$ up to conjugacy).

We must now vary the maps $i_e$. For edges $e$ attached to $u'$, the choice of $i_e$ is irrelevant. For edges attached to $u$, Lemma \[2.5\] provides a bound for the index $[G_u : i_e(G_e)] = \ldots$.
Lemma 7.4] Λ

Finiteness is proved as in the previous case, since vertex groups of Λ

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of the tree of cylinders $T_c$, equal to $T'_c$ in the context of Assertion 3 or 4, see Theorems 11.1 and 13.1 of [14]). By minimality, at least one edge $e_1$ has to be attached to $u$, and $G_e$ is conjugate to $G_{e_1}$ since $G_{e_1} \subset G_e \subset G_v$ and $G_e$ is abelian. This means that $G_e$ is uniquely determined (up to conjugacy), so the marked splitting is determined (up to an inner automorphism of $G_u$). The choice of the $i_v$’s (algebraic attachment) is controlled by Remark 4.3 since $G_u = G_e$ is malnormal in $G_v$.

- $\Lambda_v$ is trivial, and $G_v$ is abelian. As $G_v$ is abelian, there is no choice for the algebraic attachment. Finiteness will be deduced from Corollary 2.4.

As above we denote by $u, u'$ the vertices of $\Lambda_v$ with $G_u = G_e$ and $G_{u'} = G_v$. Let $A \subset G_e$ be the subgroup of $G_v$ generated by the groups $G_e$ carried by edges attached to $u$ (this group is meaningful because $G_v$ is abelian; otherwise, it changes if $G_e$ is replaced by a conjugate). Since the combinatorial attachment is fixed, $A$ is independent of the choice of $G_e$.

If $\varepsilon$ does not disconnect $\Lambda$ (i.e. if $\Gamma$ is an HNN-extension), let $G_0$ be the fundamental group of the graph of groups obtained from $\Lambda$ by removing the interior of $\varepsilon$ and changing the group carried by $u$ from $G_u = G_e$ to $A$. It does not depend on the choice of $G_e$. The vertex group of $\Gamma$ is $G_0 \ast_A G_e$. By Corollary 2.4 (applied with $P = G_v$), there are finitely many possibilities up to isomorphism. The argument when $\varepsilon$ separates $\Lambda$ is similar.

- $\Lambda_v$ is not trivial. Then $G_v$ cannot be rigid. The case when it is QH or virtually cyclic was studied in the previous subsection, so the only remaining possibility is when $G_v$ is abelian. The groups carried by edges incident to $v$ in $\Gamma_{can}$ generate a subgroup $A \subset G_v$. The Bass-Serre tree of $\Lambda_v$ is a line on which $G_v$ acts by translations, and $\Lambda_v$ is an HNN-extension $G_v = (G_e) \ast_{G_e}$ with $A \subset G_e \subset G_v$. Thus $\Gamma$ is an HNN-extension, and its vertex group is isomorphic to $H = G_0 \ast_A G_e$, where $G_0$ is the fundamental group of the graph of groups obtained from $\Gamma_{can}$ by changing the group carried by $v$ from $G_v$ to $A$.

The group $G_e$ is the kernel of an epimorphism $G_v \to \mathbb{Z}$ vanishing on $A$. Since there may be many such epimorphisms, there may be many possibilities for the group $G_e \subset G_v$. But Corollary 2.4 says that the isomorphism type of $H = G_0 \ast_A G_e$ only depends on the equivalence class of $G_e$ as defined in Lemma 2.3, so there are only finitely many possibilities for $H$.

### 4.5 Tree finiteness

We prove Theorem 1.4, i.e. tree finiteness for virtually cyclic splittings $\Gamma$ of $G$ when $G$ is one-ended, and hyperbolic relative to virtually polycyclic groups. We assume, of course, that $\Gamma$ has no redundant vertices.

By universal compatibility of $\Gamma_{can}$, any splitting $\Gamma$ (possibly with several edges) may be obtained by collapsing a refinement of $\Gamma_{can}$. It therefore suffices to prove finiteness up to $\text{Out}(G)$ for splittings $\Gamma$ which refine $\Gamma_{can}$.

When $\Gamma$ has just one more edge than $\Gamma_{can}$, we simply apply Lemma 4.5 to $\Gamma_{can}$, noting that any $G_v$ which is not virtually polycyclic or QH is rigid, hence elliptic in $\Gamma$.

In general, we pass from $\Gamma_{can}$ to $\Gamma$ by a finite sequence of one-edge refinements. Refining a QH vertex yields vertex groups which are virtually cyclic or QH, so Lemma 4.5 applies to each intermediate splitting. It is therefore enough to find a uniform bound (depending only on $G$) for the number of edges of $\Gamma$ (we cannot apply [2] because $\Gamma$ does not have to be reduced in the sense of [2], see below). We denote by $T, T_{can}$ the Bass-Serre trees of $\Gamma, \Gamma_{can}$ respectively.

We may factor the collapse map $\pi : \Gamma \to \Gamma_{can}$ through a splitting $\Gamma'$, belonging to the same deformation space as $\Gamma$, such that the preimage of any vertex $v$ of $\Gamma_{can}$ in $\Gamma'$ is a minimal graph of groups: we obtain $\Gamma'$ from $\Gamma$ by collapsing edges of $\pi^{-1}(v)$ associated to edges of $T$ not belonging to the minimal subtree of a conjugate of $G_v$ (if $G_v$ is elliptic in $\Gamma$, we collapse the whole of $\pi^{-1}(v)$).
Let $T'$ be the Bass-Serre tree of $\Gamma'$, and $p : T' \to T_{\text{can}}$ the induced collapse map. We first claim that the number of edges of $\Gamma'$ is uniformly bounded. To prove this, we consider a vertex $v$ of $T_{\text{can}}$ such that $T_v = p^{-1}(v)$ is not a point, and we have to bound the number of edges or $T_v/G_v$. Note that $G_v$ is not rigid, so it is virtually polycyclic or QH, and the action of $G_v$ on $T_v$ is minimal. If the action of $G_v$ on $T_v$ has no redundant vertex, the number of edges of $T_v/G_v$ is 1 if $G_v$ is polycyclic ($T_v$ is a line), bounded in terms of the orbifold $\Sigma_v$ if $G_v$ is QH. In general there may be redundant vertices, but such vertices have edges of $T_{\text{can}}$ attached to them, so the number of $G_v$-orbits of redundant vertices is bounded by the valence of the image of $v$ in $\Gamma_{\text{can}}$. This proves the claim.

Now let $q : \Gamma \to \Gamma'$ the collapse map. We consider a vertex $v' \in \Gamma'$, and we bound the number of edges of $q^{-1}(v')$. Since $\Gamma$ and $\Gamma'$ belong to the same deformation space, the group $G_{v'}$ is elliptic in $\Gamma$, so $q^{-1}(v')$ is a finite tree of groups $\Lambda_{v'}$ with a vertex $w$ carrying the same group as $v'$. By minimality of the splitting $\Gamma$, the number of terminal vertices of $\Lambda_{v'}$ is bounded by the valence of $v'$ in $\Gamma'$. It therefore suffices to bound the length of a segment $S \subset \Lambda_{v'}$ consisting of vertices of valence 2 (these vertices make $T$ non-reduced in the sense of [2]).

All edge stabilizers of $\Gamma$ are infinite and virtually cyclic. The stabilizer of any edge in $\Lambda_{v'}$ contains (with finite index) the stabilizer of an edge of $\Gamma'$. Moreover, if we orient $S$ towards $w$, the sequence of edge stabilizers is strictly increasing as one moves along $S$. Applying Assertion 3 of Lemma 2.5 to the edge stabilizers of $S$ then gives the required bound.

5 One-edge splittings of arbitrary groups

In this section we prove Assertions 2, 3, 4 of Theorem 1.2 for one-edge splittings of a group $G$ with infinitely many ends. In all cases there is a bound for the order of finite subgroups of $G$, and $G$ is accessible.

Lemma 5.1 (Compare [8 Lemma 4.22], [29 Theorem 18]). Let $G$ be an accessible group with infinitely many ends. Let $C$ be a finitely generated group with finitely many ends. If $G$ splits over $C$, there is a non-trivial splitting $\Gamma'$ of $G$ over a finite group in which $C$ is elliptic.

Proof. This is clear if $C$ has 0 or 1 end (it is elliptic in any $\Gamma'$), so assume that $C$ is virtually cyclic. Let $\Gamma$ be a non-trivial one-edge splitting of $G$ over $C$, and let $\Theta$ be a Stallings-Dunwoody decomposition of $G$ (see Subsection 2.1). There are two cases.

If $\Theta$ does not dominate $\Gamma$, some vertex group $H$ of $\Theta$ is non-elliptic in $\Gamma$, so (up to conjugacy) splits over a subgroup $C' \subset C$. The group $C'$ is infinite because $H$ is one-ended, so it has finite index in $C$. It follows that $C$ is elliptic in $\Theta$, and we define $\Gamma' = \Theta$.

If $\Theta$ dominates $\Gamma$, we may obtain the Bass-Serre tree of $\Gamma$ from that of $\Theta$ by collapsing edges and performing a finite sequence of folds $T_i \to T_{i+1}$ (see [2], p. 455). There is at least one fold because $C$ is infinite, so consider the first fold such that $T_{i+1}$ has an infinite edge stabilizer $G_e$. There are several types of folds (see [2]), but in all cases $G_e$ is elliptic in $T_i$. As above $G_e$ has finite index in (a conjugate of) $C$, so we define $\Gamma'$ as the splitting associated to $T_i$. \hfill $\Box$

Remark 5.2. The following generalization was inspired by N. Touikan. If $G$ is as in Lemma 5.1, and $\Gamma$ is a splitting of $G$ over groups with finitely many ends, there is a non-trivial splitting of $G$ over a finite group in which all edge groups $C_i$ of $\Gamma$ are elliptic. This is proved by induction: the lemma is true in a relative setting, and one applies it relative to $C_1, \ldots, C_i$ to the one-edge splitting of $G$ over $C_{i+1}$.

Let now $G$ be as in Theorem 1.2. Let $\Gamma$ be a non-trivial one-edge splitting of $G$, say an amalgam $A*_{C}B$ (the argument is the same in the case of an HNN-extension). By Section
and the first assertion of Lemma 2.5, we may assume that $C$ is infinite. It is virtually cyclic or abelian, hence finitely ended.

By Lemma 3.2 of [13], we can refine $\Gamma$ to a splitting $\Lambda$ which dominates the splitting $\Gamma'$ provided by Lemma 5.1 (but we cannot assume that $\Lambda$ collapses to $\Gamma'$). We may assume that all edge groups of $\Lambda$ are finite, except for the edge $e = vw$ coming from $\Gamma$ (it carries $C$). The number of edges of $\Lambda$ is bounded by Linnell’s accessibility.

All vertex groups of $\Lambda$ except $G_v$ and $G_w$ are vertex groups of a splitting of $G$ with finite edge groups, so only finitely many isomorphism types are possible by Section 3. Similarly, $H = G_v * C G_w$ is also such a vertex group, so there are only finitely many possibilities for $H$. The group $H$ is elliptic in $\Gamma'$, because $G_v$ and $G_w$ are and $C$ (being infinite) fixes a unique point in the Bass-Serre tree of $\Gamma'$. Since $\Gamma'$ is non-trivial, $H$ is a proper subgroup of $G$.

First suppose that $G$ is torsion-free. By Grushko’s theorem, $H$ has rank smaller than $G$, so by induction we may assume that the theorem holds for $H$: there are finitely many possibilities for $G_v$ and $G_w$ up to isomorphism. We now see that $A$ and $B$ are fundamental groups of graphs of groups such that the number of edges is bounded, edge groups are trivial, and only finitely many vertex groups are possible up to isomorphism. Finiteness follows.

There are two complications if $G$ has torsion. First, one must replace the rank by another complexity, namely $c(H)$, defined as the maximal number of edges in a minimal decomposition of $H$ over finite groups without redundant vertex (minimal means that the action on the Bass-Serre is minimal, as in Subsection 2.1). This is finite by Linnell’s accessibility, and $c(H) < c(G)$, so we can argue by induction on $c(H)$.

The groups $A$ and $B$ are now fundamental groups of graphs of groups such that the number of edges is bounded, and only finitely many vertex and edge groups are possible up to isomorphism. We have to control the inclusions of edge groups into vertex groups. We cannot argue as in the proof of Lemma 3.1 because we do not know the deformation space. Instead we use the fact that the vertex groups are hyperbolic relative to virtually polycyclic groups, and therefore only contain finitely many conjugacy classes of finite subgroups by the first assertion of Lemma 2.5.

6 Splittings with several edges

We now prove Assertions 2, 3, 4 of Theorem 1.2 in full generality, i.e. for splittings with any number of edges. Recall that we need only consider reduced splittings.

We first prove the following claim by induction on $p$: given $G$, there are only finitely many possible isomorphism types for vertex groups of reduced splittings $\Gamma$ of $G$ over $A$ with at most $p$ edges.

Given a vertex $v$ of $\Gamma$, choose an edge $e$ containing $v$, and collapse $e$. We get a reduced splitting $\Gamma'$ with fewer edges, and $G_v$ is a vertex group of a one-edge splitting of a vertex group $G_w$ of $\Gamma'$. This one-edge splitting is reduced because $\Gamma$ is reduced, and the claim follows since by induction there are finitely many possibilities for $G_w$ up to isomorphism.

If $G$ is finitely presented, Bestvina-Feighn’s accessibility [2] provides a bound for the number of edges of reduced splittings of $G$ (note that groups in $A$ are small, and reduced as defined in Subsection 2.1 implies reduced in the sense of [2]). Theorem 1.2 thus follows from the claim when $G$ is relatively hyperbolic (Assertions 2 and 3).

Bestvina-Feighn’s accessibility does not apply in the CSA case if $G$ is not finitely presented, so we use acylindrical accessibility [26, 28] instead. As usual, an abelian splitting is a splitting over abelian groups.

Lemma 6.1. Let $G$ be a finitely generated, torsion-free, CSA group. Given an abelian splitting $\Gamma$ of $G$, there exists a reduced 2-acylindrical abelian splitting $\Gamma_c$ such that any
non-abelian vertex group of \( \Gamma \) is a vertex group of \( \Gamma_c \).

Proof. First assume that no edge group of \( \Gamma \) is trivial. Let \( T \) be the Bass-Serre tree of \( \Gamma \), let \( T_c \) be its tree of cylinders (for commutation, see Example 3.5 of [15]), and let \( \Gamma_c = T_c/G \) be the corresponding graph of groups. If \( v \) is a vertex of \( T \) with \( G_v \) non-abelian, it belongs to at least two cylinders, so \( G_v \) is a vertex stabilizer of \( T_c \). The tree \( T_c \) is 2-acylindrical (Proposition 6.3 of [15]), and one can make it reduced by collapsing edges (this does not change non-abelian vertex stabilizers).

If certain edge groups of \( \Gamma \) are trivial, perform the previous construction in each maximal subgraph consisting of edges with non-trivial group.

The claim and the lemma imply the theorem since by acylindrical accessibility there is a bound for the number of edges of \( \Gamma_c \). We only have to control non-abelian vertex groups of \( \Gamma \) because we assume that abelian subgroups have bounded rank.

References

[1] Emina Alibegović. A combination theorem for relatively hyperbolic groups. Bull. London Math. Soc., 37(3):459–466, 2005.

[2] Mladen Bestvina and Mark Feighn.Bounding the complexity of simplicial group actions on trees. Invent. Math., 103(3):449–469, 1991.

[3] Brian H. Bowditch. Peripheral splittings of groups. Trans. Amer. Math. Soc., 353(10):4057–4082 (electronic), 2001.

[4] François Dahmani. Combination of convergence groups. Geom. Topol., 7:933–963 (electronic), 2003.

[5] François Dahmani and Daniel Groves. The isomorphism problem for toral relatively hyperbolic groups. Publ. Math. Inst. Hautes Études Sci., 107:211–290, 2008.

[6] François Dahmani and Vincent Guirardel. The isomorphism problem for all hyperbolic groups. Geom. Funct. Anal., 21(2):223–300, 2011.

[7] Thomas Delzant. Sur l’accessibilité acylindrique des groupes de présentation finie. Ann. Inst. Fourier (Grenoble), 49(4):1215–1224, 1999.

[8] Thomas Delzant and Leonid Potyagailo. Accessibilité hiérarchique des groupes de présentation finie. Topology, 40(3):617–629, 2001.

[9] M. J. Dunwoody. The accessibility of finitely presented groups. Invent. Math., 81(3):449–457, 1985.

[10] Daciberg Gonçalves and Peter Wong. Twisted conjugacy classes in wreath products. Internat. J. Algebra Comput., 16(5):875–886, 2006.

[11] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. Groups Geom. Dyn., 1(2):135–181, 2007.

[12] Vincent Guirardel and Gilbert Levitt. The outer space of a free product. Proc. Lond. Math. Soc. (3), 94(3):695–714, 2007.

[13] Vincent Guirardel and Gilbert Levitt. JSJ decompositions: definitions, existence and uniqueness. I: The JSJ deformation space. arXiv:0911.3173v2 [math.GR], 2009.

[14] Vincent Guirardel and Gilbert Levitt. JSJ decompositions: definitions, existence and uniqueness. II: Compatibility and acylindricity. arXiv:1002.4564v2 [math.GR], 2010.

[15] Vincent Guirardel and Gilbert Levitt. Trees of cylinders and canonical splittings. Geom. Topol., 15(2):977–1012, 2011.

[16] Vincent Guirardel and Gilbert Levitt. Splitting and automorphisms of relatively hyperbolic groups, 2012. arXiv:1212.1434 [math.GR].
[17] Vincent Guirardel and Gilbert Levitt. Extension finiteness for relatively hyperbolic groups. In preparation.

[18] Vincent Guirardel and Gilbert Levitt. McCool groups. In preparation.

[19] Gilbert Levitt. On the automorphism group of generalized Baumslag-Solitar groups. *Geom. Topol.*, 11:473–515, 2007.

[20] Gilbert Levitt and Martin Lustig. Automorphisms of free groups have asymptotically periodic dynamics. *J. Reine Angew. Math.*, 619:1–36, 2008.

[21] P. A. Linnell. On accessibility of groups. *J. Pure Appl. Algebra*, 30(1):39–46, 1983.

[22] Larsen Louder and Nicholas Touikan. Strong accessibility for finitely presented groups, 2013. arXiv:1302.5451

[23] A. H. Rhemtulla. A minimality property of polycyclic groups. *J. London Math. Soc.*, 42:456–462, 1967.

[24] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[25] Daniel Segal. *Polycyclic groups*, volume 82 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1983.

[26] Z. Sela. Acylindrical accessibility for groups. *Invent. Math.*, 129(3):527–565, 1997.

[27] J. Shor. A Scott conjecture for hyperbolic groups, 1999. preprint.

[28] Richard Weidmann. On accessibility of finitely generated groups. *Q. J. Math.*, 63(1):211–225, 2012.

[29] Henry Wilton. One-ended subgroups of graphs of free groups with cyclic edge groups. *Geom. Topol.*, 16(2):665–683, 2012.

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