HIGHER ORDER MOMENTS OF THE COSMIC SHEAR AND OTHER SPIN-2 FIELDS

MATIAS ZALDARRIAGA\textsuperscript{1,2} AND ROMÁN SCOCCHIMARRO\textsuperscript{1}

Received 2002 August 3; accepted 2002 October 28

ABSTRACT

We present a method for defining higher order moments of a spin-2 field on the sky using the transformation properties of these statistics under rotation and parity. For the three-point function of the cosmic shear, we show that the eight logically possible combinations of the shear in three points can be divided into two classes; four combinations are even under parity transformations, and four are odd. We compute the expected value of the even-parity ones in the nonlinear regime using the halo model and conclude that on small scales, of the four combinations there is one that is expected to carry most of the signal for triangles close to isosceles. On the other hand, for collapsed triangles, all four combinations are expected to have roughly the same level of signal, although some of the combinations are negative and others positive. We estimate that a survey of a few square degrees area is enough to detect this signal above the noise at arcminute scales.

Subject headings: cosmic microwave background — cosmology: theory — gravitational lensing — large-scale structure of universe

On-line material: color figures

1. INTRODUCTION

Gravitational lensing is quickly becoming a major tool for doing cosmology. Mass determinations of clusters of galaxies are now routine, and there are several detections of weak lensing by the large-scale structure of the universe and measurements of galaxy-galaxy lensing. Detections of weak lensing (Bacon, Refregier, & Ellis 2000; Kaiser, Wilson, & Luppino 2000; van Waerbeke et al. 2000; Wittman et al. 2000) are particularly important, since they will provide independent determination of the power spectrum of the density of the universe, while the skewness (properly normalized) is essentially a measure of the latter (Bernardeau, van Waerbeke, & Mellier 1997; Jain & Seljak 1997; Schneider et al. 1998; van Waerbeke et al. 2001). The N-body simulations have been used to create mock maps from which higher order moments have been measured. Semianalytic techniques such as the halo model (see Cooray & Sheth 2002 for a recent review) or proposals about the behavior of higher order correlations in the nonlinear regime (Scoccimarro & Frieman 1999) have also been used to make weak-lensing predictions (Hui 1999; Cooray, Hu, & Miralda-Escude 2000; Cooray & Hu 2001).

So far most of these theoretical approaches have dealt with \( \kappa \); however, the quantity that is directly measured is the shear \( \gamma \). The crucial difference between \( \kappa \) and \( \gamma \) is that the shear is a spin-2 field on the sky, while \( \kappa \) is just a scalar. Thus, the shear has two components at each position on the sky, so when designing any statistics for the shear, one has to make sure that the statistic does not depend on some arbitrary choice of coordinate system. Moreover, the transformation from shear to \( \kappa \) cannot be done exactly if one only has a shear map over a finite region on the sky and only at the position of the background galaxies. To circumvent this issue, measures of the shear at different points on the sky can be combined to form statistics that are invariant under rotation and can be expressed as weighted averages of \( \kappa \), the best example being the aperture mass (Kaiser et al. 1994; Schneider et al. 1998). We show that the amplitude of the three-point function varies with triangle configuration and can be either positive or negative. As a result, not all

The nonlinear growth of structure is very important on the length scales relevant for weak lensing, so estimates for the level of non-Gaussianity in \( \kappa \) come from two different techniques: N-body simulations (Couchman, Barber, & Thomas 1999; Jain, Seljak, & White 2000; White & Hu 2000) and semianalytic models (Jain & Seljak 1997; Schneider et al. 1998; van Waerbeke et al. 2001). The N-body simulations have been used to create mock maps from which higher order moments have been measured. Semianalytic techniques such as the halo model (see Cooray & Sheth 2002 for a recent review) or proposals about the behavior of higher order correlations in the nonlinear regime (Scoccimarro & Frieman 1999) have also been used to make weak-lensing predictions (Hui 1999; Cooray, Hu, & Miralda-Escude 2000; Cooray & Hu 2001).

What is actually measured is the reduced shear \( \gamma/(1 - \kappa) \); thus, on sufficiently small scales this may change theoretical predictions. The spin properties of the three-point function we discuss apply to both the shear and the reduced shear.

\textsuperscript{1}Center for Cosmology and Particle Physics, Physics Department, New York University, 4 Washington Place, New York, NY 10003.

\textsuperscript{2}Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540.
combinations of the three-point function are optimal from a signal-to-noise ratio perspective.

A natural way to look for non-Gaussianity is then to look at statistics of the aperture mass. In this paper we focus on another approach, directly defining higher order statistics in terms of the shear field that are independent of the coordinate system. Any statistic of the shear can ultimately be written as a linear combination of the statistics we present in this paper. We show, however, that not all the possible higher order moments are expected to have the same level of cosmological signal, and moreover, the sign of these statistics can be positive or negative depending on the configuration of the points. The advantage of our approach is that it allows us to isolate the configurations that have most cosmological signal and avoid suppression of the signal-to-noise ratio in the measurements or cancellations that would result from an arbitrary linear combination.

A first attempt has been made to define a three-point function for the shear field in Bernardeau, van Waerbeke, & Mellier (2003). Their prescription corresponds to integrating over a particular linear combination of the four different shear three-point functions that we define in this work. They have applied their statistic to a cosmic shear survey and reported a detection at an approximately $5\sigma$ level (Bernardeau, Mellier, & van Waerbeke 2002). Given these encouraging results, it is worth considering in more detail how to construct shear three-point functions and what is expected theoretically about their order of magnitude and sign depending on the particular configuration of the points.

This paper is written using weak-lensing language, but identical issues arise if one wants to define higher order moments of the CMB polarization field. The analog of the shear components are the $Q$ and $U$ Stokes parameters, and the analog of the projected mass density is usually called the $E$ field. Even if the initial conditions are Gaussian, higher order moments of the CMB polarization can be generated by secondary effects such as lensing, so the statistics presented here will be equally applicable for the CMB.

2. DEFINING HIGHER ORDER MOMENTS FOR SPIN-2 FIELDS

In this section we show how to define higher order moments of the weak-lensing shear or CMB polarization fields in a way that is geometrically meaningful. The problem is that the shear is a spin-2 field, and thus at each point on the sky it has two components. Just as in the case of a vector field, the values of these components depend on the choice of coordinate system. If at any given point one rotates the coordinate system used to define the shear components by an angle $\alpha$ (counterclockwise in our convention), the shear field has to be transformed as

$$
\begin{align*}
\gamma_1' &= (\cos 2\alpha)\gamma_1 + (\sin 2\alpha)\gamma_2 , \\
\gamma_2' &= - (\sin 2\alpha)\gamma_1 + (\cos 2\alpha)\gamma_2 .
\end{align*}
$$

Any meaningful statistic of the shear measured in a set of points has to depend only on the distances and relative orientation of these points and not on their absolute position on the sky or their orientation with respect to some fiducial origin. For example, the two-point function has to depend only on the distance between the two points, and the three-point function only on the size and shape of the triangle formed by the three points.

The way to overcome this problem of definition for the second moment is well known: one uses the separation vector to define a “natural” coordinate system. The idea is to align the coordinate system so that one of the axes lies on the great circle joining the two points and use that coordinate system to define the two components of the shear (Miralda-Escudé 1991; Kaiser 1992; Kamionkowski, Kosowsky, & Stebbins 1997).

In general, the way to define higher order moments that are invariant under rotation is by contracting the measured shear at the three points with suitable combinations of the vectors that define the sides of the triangle, are invariant under translation, and have the correct spin to compensate for the spin of the shear. Clearly, this procedure is not unique, as there are many such combinations. In this paper we propose a simple and intuitive, but at the same time geometrically meaningful, way to define higher order moments.

An $N$-point function is characterized by the set of points $X = \{\theta_i\}$ ($i = 1, \ldots, N$) at which the shear or the CMB polarization is measured. We denote two-dimensional vectors on the sky with boldface. We then define the “center of mass” of $X$,

$$
o = \frac{1}{N} \sum_{i=1}^{N} \theta_i ,
$$

and use $o$ as the origin when defining the shear at each of the points in $X$. At every point we can define a component of the shear along the direction that separates $o$ and $\theta_i$, which we can call $\gamma_+ (\theta_i)$, and a component that is measured in the coordinate system that is rotated by $45^\circ$, which we call $\gamma_x (\theta_i)$. Figure 1 illustrates how $\gamma_+$ and $\gamma_x$ are defined. This is totally analogous to the procedure for the two-point function, but now the two points used in the definition of $\gamma_+$ and $\gamma_x$ are $o$ and $\theta_i$ instead of the two points in the two-point function. The key to our procedure is that by defining an origin based only on the points in $X$, we make sure that our statistics depend only on intrinsic properties of $X$ and not on a fiducial origin.

Another important property of spin-2 fields is that they can be decomposed into two scalar potentials, one that is

![Fig. 1.—Point $o$ being used to define the $\gamma_+$ and $\gamma_x$ components of the shear at point $\theta$. The two rods indicate the ellipticity at $\theta$ that would produce a positive $\gamma_+$ and $\gamma_x$. [See the electronic edition of the Journal for a color version of this figure.]](image-url)
even under parity \( (E) \) and one that is odd \( (B) \) (Kamionkowski et al. 1997; Zaldarriaga & Seljak 1997; Crittenden et al. 2001; Schneider et al. 2002). In fact, for finite sky coverage or for maps with holes, there is a third family of modes, the ambiguous modes for which one does not have enough information to decide whether the contribution is coming from \( E \) or from \( B \) (Lewis, Challinor, & Turok 2002; Bunn et al. 2002). Weak gravitational lensing only produces \( E \) modes \( (E = \text{nothing but the projected mass density} \ k) \).

The properties under parity transformation of \( \gamma_+ \) and \( \gamma_x \) are different. As is clear from the definition, to obtain \( \gamma_x \) one has to rotate the coordinate system counterclockwise. This rotation changes direction when we do a parity transformation. This means that under parity, \( \gamma_+ \rightarrow \gamma_+ \) but \( \gamma_x \rightarrow -\gamma_x \).

As a consequence of the difference in behavior of \( \gamma_+ \) and \( \gamma_x \), any estimator that contains an odd number of \( \gamma_x \) is odd under parity. The difference in their parity behavior makes some of the three-point functions vanish. To illustrate what this means, we can consider the case of the three-point function for equilateral triangles. In principle, there are eight different combinations of the two shear components at the three points; however, only four of them, \( \langle \gamma_+ \gamma_+ \gamma_+ \rangle \), \( \langle \gamma_+ \gamma_+ \gamma_x \rangle \), \( \langle \gamma_+ \gamma_x \gamma_+ \rangle \), and \( \langle \gamma_x \gamma_+ \gamma_+ \rangle \), contain any signal from weak lensing because they are the only four that are even under parity. We show patterns that produce positive values for these correlations in Figure 2. For a discussion on how the difference in behavior under parity of the three-point functions of general configurations constrains their possible values, see Takada & Jain (2002a).

Being able to separate combinations that have signal from those that do not is very important, since one must not to dilute the signal one is trying to measure when combining or binning the measured statistics to obtain a detection. Moreover, for equilateral triangles, for example, the four odd combinations, \( \langle \gamma_+ \gamma_+ \gamma_+ \rangle \), \( \langle \gamma_+ \gamma_+ \gamma_x \rangle \), \( \langle \gamma_+ \gamma_x \gamma_x \rangle \), and \( \langle \gamma_x \gamma_+ \gamma_+ \rangle \), may prove useful for monitoring systematic problems in the data or identifying interesting physical effects such as clustering of the background sources, the effects of multiple scatterings, etc. (Bernardeau 1998; Jain et al. 2000).

Any connected \( N \)-point function of the shear can be written as a linear combination of the connected \( N \)-point functions of the two potentials \( E \) and \( B \). It is worth noting that in the case of connected \( N \)-point functions with an odd number of legs, there will be some configurations (such as those in which the lengths of all sides are equal) that can be separated into a set that only depends on the higher order correlations of \( E \) and a set that only depends on the higher moments of \( B \) (assuming \( E \) and \( B \) are independent). We can argue this only on the basis of their behavior under parity. The product of an odd number of \( E \)'s is even under parity, while a product of an odd number of \( B \)'s is odd; thus, the even-parity \( N \)-point functions of the shear receive contributions only from \( E \), and the odd ones only from \( B \). For an \( N \)-point function with an even number of legs this is not true, because the product of an odd number of \( E \)'s or \( B \)'s is even under parity. The clearest example of this is the two-point correlation function, where \( \langle \gamma_+ \gamma_+ \rangle \) and \( \langle \gamma_x \gamma_x \rangle \) receive contributions from both \( (EE) \) and \( (BB) \), while \( \langle \gamma_+ \gamma_x \rangle \) is zero if there is no \( E-B \) cross-correlation.

As we mentioned above, the way to make a three-point function (or an \( N \)-point function for that matter) that is scalar is to contract the shear at the three points with some combination of the vectors that form the sides that transform appropriately under rotations to cancel the spin of shear. That is, we need to construct spin-2 combinations of these vectors. Our proposed scheme is easy to understand. For each member of \( X \), we define \( \theta = \theta - \alpha \), and we construct two spin-2 quantities:

\[
P_+ = \frac{(\theta^2 - \theta_x^2, 2\theta_x \theta_\perp)}{\theta^2} = (\cos 2\phi, \sin 2\phi),
\]

\[
P_\times = \frac{(-2\theta_x \theta_\perp, \theta_x^2 - \theta_z^2)}{\theta^2} = (-\sin 2\phi, \cos 2\phi).
\]

The statistics we proposed are obtained by contracting the above quantities with the shear three-point function. For example,

\[
\langle \gamma_+ \gamma_+ \gamma_+ \rangle = P^\mu_+ P^\nu_+ P^\rho_+ \gamma_\mu \gamma_\nu \gamma_\rho,
\]

where the index \( \mu \) runs over the two components of the shear. The other three-point functions that we define are obtained by replacing some of the \( P_+ \) by \( P_\times \). Finally, we note that the vector \( \theta_1 \) is nothing but \( \theta_1 = (\theta_1 - \theta_2 + (\theta_1 - \theta_3))/3 \), i.e., basically the sum of the vectors that define the sides of the triangle that cross at \( \theta_1 \). The same is true for the other vertices.

3. WORKED EXAMPLES

Our objective in this section is to get some intuition into how these higher order correlations behave. To keep things simple, we just focus on the three-point function. To get some idea of how these functions behave, we work in the context of the halo model, i.e., the dark matter is assumed to be distributed in a collection of halos of different mass. Although analytic approximations and fits to numerical simulations exist for the profile of these halos and their mass function, this modeling of the shear field is clearly simplistic, since in reality halos are not spherical. This is particularly
bound to affect the configuration dependence of the three-point function. Thus, definite predictions will need more detailed modeling and comparison with direct measurements using numerical simulations.

Our aim in this paper is more modest; we only want to gain some insight into how these different three-point functions behave: whether they are positive or negative, for example. We start by considering the simple case of lensing by a singular isothermal sphere (SIS). We do this because the calculation of the three-point function can be done analytically, and it provides a useful check on the numerical code used to do the same calculations in the halo model, presented in §3.2.

### 3.1. The Singular Isothermal Sphere

In this section we calculate the three-point function that results from a single halo with a power-law density run. For definiteness, we write down formulas for an SIS, but other power laws can be calculated in an analogous way. For an SIS, the density $\rho(r)$ depends on the distance from the center $r$ as $\rho(r) \propto r^{-3}$; therefore, both the projected mass density $\kappa$ and the shear $\gamma$ scale with the projected separation as $r^{-1}$.

The shear pattern around a spherical halo is tangential, centered at the origin of the halo, which we call $\mathbf{u}$. When defining the shear components at a point $\theta$, we need to rotate the shear elements so as to define them relative to the vector $\theta - \mathbf{o}$. We can write

$$\gamma_{\perp}(\theta) = \frac{\cos(2\alpha)}{|\theta - u|}, \quad \gamma_{\times}(\theta) = \frac{\sin(2\alpha)}{|\theta - u|}. \quad (5)$$

The cosine and sine of $2\alpha$ can be calculated in terms of

$$\cos(\alpha) = \frac{(\theta - u) \cdot (\theta - o)}{|\theta - u||\theta - o|}, \quad (6)$$

$$\sin(\alpha) = \frac{\mathbf{z} \cdot [(\theta - u) \times (\theta - o)]}{|\theta - u||\theta - o|}, \quad (7)$$

where $\mathbf{z}$ is the unit vector perpendicular to the plane of the sky. The three-point function $\zeta$ is obtained by integrating over the position of the center of the halo $\mathbf{u}$:

$$\zeta^{++}_{\gamma}(l_1, l_2, l_3) = \int d^2 u \frac{\cos(2\alpha_1) \cos(2\alpha_2) \cos(2\alpha_3)}{|\theta_1 - u| |\theta_2 - u| |\theta_3 - u|}, \quad (8)$$

$$\zeta^{\times\times}_{\gamma}(l_1, l_2, l_3) = \int d^2 u \frac{\cos(2\alpha_1) \sin(2\alpha_2) \sin(2\alpha_3)}{|\theta_1 - u| |\theta_2 - u| |\theta_3 - u|}, \quad (9)$$

where $l_1$, $l_2$, and $l_3$ are the lengths of the triangle sides, $l_1^2 = |\theta_1 - \theta_2|^2$, $l_2^2 = |\theta_2 - \theta_3|^2$, and $l_3^2 = |\theta_1 - \theta_3|^2$. There are two permutations of the second equation that give the remaining three-point correlators, $\zeta^{++} \times$ and $\zeta^{\times\times}$. To calculate equations (8) and (9), we proceed as follows. First, for simplicity, we take the origin of coordinates to coincide with the center of mass, so $\mathbf{o}$ vanishes. We then write the $\mathbf{u}$ dependence inside the cosine and sine in equations (6) and (7) in terms of $d_i \equiv \theta_i - \mathbf{u}$. This can be done simply using that

$$\mathbf{u} \cdot \theta_i = \frac{1}{2} (u^2 + |\theta|^2 - d_i^2), \quad (10)$$

whereas the magnitude of $\mathbf{u}$ can be conveniently written as

$$u^2 = \frac{1}{2} \left( d_1^2 + d_2^2 + d_3^2 \right) - \frac{1}{2} \left( l_1^2 + l_2^2 + l_3^2 \right). \quad (11)$$

Similarly, we have $|\theta_1|^2 = (2l_1^2 + 2l_2^2 - l_3^2)/9$ and cyclic permutations. In this way, after appropriate translation in $\mathbf{u}$, the integrals in equations (8) and (9) are of the form

$$J(\nu_1, \nu_2, \nu_3) = \int \frac{d^2 u}{(u^2)^{\nu_1} [(l_1 - u)^2]^{\nu_2} [(l_2 - u)^2]^{\nu_3}}, \quad (12)$$

which can be evaluated by using dimensional regularization techniques (see Scoccimarro 1997 and references therein) in terms of Appell’s hypergeometric function of two variables, $F_4$, with the series expansion

$$F_4(a, b; c, d; x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j (a)_{i+j}}{i! j! (c)_{i} (d)_{j}}, \quad (13)$$

where $(a)_{i+j} \equiv \Gamma(a + i)/\Gamma(a)$. In our case, the arguments of $F_4$ have a special symmetry that allows us to write them in terms of regular hypergeometric functions:

$$F_4(\alpha, \gamma + \gamma' - \alpha - 1, \gamma, \gamma'; x(y - 1), y(y - 1)) = F(\alpha, \gamma + \gamma' - \alpha - 1, \gamma, x) F(\alpha, \gamma + \gamma' - \alpha - 1, \gamma'), \quad (14)$$

which can be easily evaluated on the computer using MATHEMATICA. Figure 3 shows the results for different triangle configurations. Since the SIS is scale free, the overall size of the triangle scales the results by a factor, so we take $l_3 = 1$. The top panel shows the four $\zeta_{\gamma}$’s for $l_2 = l_1$ as a function of the angle $\phi$ between $l_1$ and $l_2$. We see that for most of the triangles $|\phi| \leq 0.3\pi$, the signal is dominated by $\zeta^{++}_{\gamma}$, with a maximum close to equilateral triangles ($\phi = 2\pi/3$). On the other hand, the three contributions involving $\gamma_{\times}$ are generally smaller in magnitude.

\footnote{The divergence of $\zeta^{++}_{\gamma}$ as $\phi \to \pi$ is a peculiarity of the SIS when $l_3 \to 0$. The quantity $\zeta^{\times\times}_{\gamma}$ is regular in that limit, despite what Fig. 3 suggests.}
This can be understood geometrically from Figure 4. Let us consider the halo center to be inside the triangle, which minimizes the distances to the vertices and thus maximizes the signal. Since \( \cos(2\alpha) > 0 \) for \( |\alpha| < \pi/4 \), as long as the internal angles of the triangle are smaller than \( \pi/2 \), \( \zeta^{++} \) is positive. As \( \phi \to \pi \), \( \alpha_2 \to 0 \), whereas \( |\alpha_1| = |\alpha_3| \) cannot be larger than \( \pi/2 \); thus \( \zeta^{++} \) remains positive as \( \phi \to \pi \). On the other hand, as \( \phi \to 0 \), \( \alpha_1 = \alpha_3 \to 0 \), whereas \( |\alpha_2| \to \pi \) if \( u \) is off center; that explains why \( \zeta^{++} \) becomes negative as \( \phi \to 0 \).

For the three other three-point functions involving \( \gamma_x \), things are more subtle. Since \( \sin(2\alpha) \) changes sign at \( \alpha = 0 \), \( \gamma_x \) can be positive or negative depending on the location of \( u \) relative to the bisector of the vertex. As a result, moving \( u \) inside the triangle leads to cancellations and thus smaller amplitudes for \( \zeta_x \). Figure 3 (bottom) shows results for configurations in which \( l_2 = 2l_1 \). A similar pattern is seen, where for isosceles triangles (\( \phi \approx 0.6\pi \)) the \( \zeta^{++} \) is positive and maximum, whereas the remaining three contributions are smaller, becoming comparable only for collapsed triangles, which can avoid cancellations.

The physical interpretation of the different amplitudes is sketched in Figure 5. In the top row we show patterns that would produce positive values for \( \zeta^{++}, \zeta^{--}, \zeta^{+-}, \) and \( \zeta^{++} \) from left to right, respectively, and the bottom row shows patterns that would produce negative values. It is clear from the figure that the top examples correspond to patterns that would be produced by overdensities, and the bottom patterns would be produced by underdensities. In each, we indicate with a square the regions where the overdensity (underdensity) should be located to produce such a pattern.

### 3.2. Superposition of NFW Profiles

In this section we calculate the three-point functions using the halo model (Peacock & Smith 2000; Seljak 2000; Ma & Fry 2000; Scoccimarro et al. 2001). For simplicity, we restrict ourselves to the one-halo term that dominates on the small scales at which the three-point function of the shear is easiest to measure in observations. For examples of calculations of weak gravitational lensing higher order moments in the context of the halo model, see Cooray & Hu (2001) and Takada & Jain (2002b). Measurements of higher order moments of the convergence field in numerical simulations are given in Jain et al. (2000) and White & Hu (2000); van Waerbeke et al. (2001) also present results for aperture mass statistics, which are directly related to the cosmic shear.

Under our assumptions, the averaged shear can be written as an integral over the radial distance (\( d_\chi = c \, dt/a \)), mass (\( M \)), and angular location (\( u \)) of the halos:

\[
\gamma(\chi, \theta, \phi, \theta_1, \theta_2, \chi) = \int d_\chi \, d_\chi^2 (\chi) \int dM \frac{dn}{dM} \int d^2 u \times \tilde{\gamma}(\chi, \theta, \phi, \theta_1, \theta_2, \chi) \times \tilde{\gamma}(\chi, \theta, \phi, \theta_1, \theta_2, \chi),
\]

(15)

where \( a \) is the expansion factor of the universe, \( d_\chi \) is the comoving angular diameter distance, and \( dn/dM \) is the mass function of the halos. We have introduced the notation \( \tilde{\gamma}(\chi, \theta, \phi, \theta_1, \theta_2, \chi) \) to indicate the shear produced at position \( \theta \) by a halo at radial distance \( \chi \) and angular position \( u \). Note that for convenience, we are using the same symbol regardless of whether \( \gamma_+ \) or \( \gamma_\times \) is involved.

To obtain an estimate for the three-point function, we evaluate equation (15) assuming that the background sources used to measure the shear are all at redshift \( z_s = 1 \) and that the cosmological model is the so-called \( \Lambda \)-dominated cold dark matter (LCDM) model (\( \Omega_m = 0.3, \Omega_\Lambda = 0.7, h = 0.7, \sigma_8 = 0.9, \) and \( n = 1 \)). We assume that the mass function of halos is that given by Sheth & Tormen (1999) and Jenkins et al. (2001) and that dark matter halos have an NFW profile (Navarro, Frenk, & White 1997). In particular, we use

\[
\rho(r) = \frac{\rho_s}{r/r_s(1 + r/r_s)^2},
\]

(16)

where \( r \) is measured in comoving coordinates and \( r_s \) is related to the virial radius of the halo by the concentration parameter \( c \); \( r_s = r_{200}/c \). The mass of the halo is given by...
can understand the dependence of each of the curves with $\phi$ in exactly the same way.

The expected level of $\zeta_3$ on arcminute scales is $\zeta_3 \sim 10^{-6}$. We estimate the number of triangles ($N_T$) needed to detect the three-point function in a particular configuration above the noise produced by the intrinsic ellipticity of the background galaxies as follows: If the typical ellipticity of the background galaxies is $\sim 0.3$, then the typical noise added to each component of the intrinsic shear is $\sigma \sim 0.3/\sqrt{2} \approx 0.2$. The expected noise in the three-point function is then $\sigma^3/\sqrt{N_T}$. This implies that one needs roughly $N_T \sim 6.5 \times 10^7$ triangles of a particular shape to estimate the three-point function. In a survey covering a solid angle $\Omega$ with a mean density of galaxies $n$, there are roughly $N_T = n \Omega (\pi R^2 n)^{-1/2}$ triangles with sides of scale $R$. Current surveys have about 20 galaxies arcmin$^{-2}$, implying that in a few square degrees, there are enough triangles with sides of the order of 1$'$ to detect this signal.

4. SUMMARY AND DISCUSSION

In this paper we have introduced a new way of defining higher order correlation functions of a spin-2 field such as the weak-lensing shear or the CMB polarization by using the “center of mass” of the configuration as the origin from which the components of the shear are defined. In principle, for an $N$-point function there are $2^N$ different statistics of the shear. We have shown that these statistics can be divided according to their behavior under parity transformation and that one does not expect a cosmological signal in the odd ones for some configurations, such as for equilateral triangles.

In order to gain intuition about the behavior of these statistics, we calculated the four even three-point functions under some simple assumptions. We calculated analytically what would be expected to be produced by an ensemble of singular isothermal spheres. We showed that $\zeta_3^{++}$ is positive and is expected to carry the bulk of the signal for triangles that are not too elongated. For elongated triangles, we showed that all four three-point functions have similar values, but some of them are positive and others are negative. If two of the points are very close to one another, $\zeta_3^{++}$ and $\zeta_3^{++}$ carry most of the signal.

We estimated the three-point functions in the context of the halo model using the contributions from the one-halo term, which should be a reasonable approximation at small angular scales. We showed that the configuration dependence in this case is almost identical to that found for the SIS. We estimated that in order to detect a signal above the noise (due to the intrinsic ellipticity of galaxies) at scales of the order of 1$'$, a survey of a few square degrees is necessary.

Clearly, the fact that we both restricted ourselves to the one-halo term and assumed the halos to be spherical affects the behavior of the three-point function. A detailed study using numerical simulations will be needed to improve on the calculation presented here (K. Benabed, R. Scoccimarro, & M. Zaldarriaga 2003, in preparation). The potential rewards of detecting a non-Gaussian signal in the shear maps are enormous, and the tantalizing detections reported so far (Bernardeau et al. 2002) make this a very exciting time to study these issues in detail.

When this paper was under completion, a similar proposal for calculating the shear three-point function was put forward by Schneider & Lombardi (2003).

5 In this equation $c$ is the speed of light, not the concentration parameter of dark matter halos.
We thank Mashiro Takada and Bhuvnesh Jain for pointing out an error in the first version of the manuscript. M. Z. is supported by a David and Lucille Packard Foundation Fellowship for Science and Engineering and NSF grant AST 00-98506. R. S. is supported by NASA ATP grant NAG5-12100. M. Z. and R. S. are supported by NSF grant PHY-0116590.

REFERENCES

Bacon, D. J., Refregier, A. R., & Ellis, R. S. 2000, MNRAS, 318, 625
Bernardeau, F. 1998, A&A, 338, 375
Bernardeau, F., Mellier, Y., & van Waerbeke, L. 2002, A&A, 389, L28
Bernardeau, F., Van Waerbeke, L., & Mellier, Y. 1997, A&A, 322, 1 ———. 2003, A&A, 397, 405
Bullock, J. S., Kolatt, T. S., Sigad, Y., Somerville, R. S., Kravtsov, A. V., Klypin, A. A., Primack, J. R., & Dekel, A. 2001, MNRAS, 321, 559
Bunn, E. F., Zaldarriaga, M., Tegmark, M., & de Oliveira-Costa, A. 2002, Phys. Rev. D, in press
Cooray, A., & Hu, W. 2001, ApJ, 548, 7
Cooray, A., Hu, W., & Miralda-Escudé, J. 2000, ApJ, 535, L9
Cooray, A., & Sheth, R. K. 2002, Phys. Rep., 372, 1
Couchman, H. M. P., Barber, A. J., & Thomas, P. A. 1999, MNRAS, 308, 180
Crittenden, R. G., Natarajan, P., Pen, U. L., & Theuns, T. 2001, ApJ, 559, 552
Hui, L. 1999, ApJ, 519, L9
Jain, B., & Seljak, U. 1997, ApJ, 484, 560
Jain, B., Seljak, U., & White, S. D. M. 2000, ApJ, 530, 547
Jenkins, A., Frenk, C. S., White, S. D. M., Colberg, J. M., Cole, S., Evrard, A. E., Couchman, H. M. P., & Yoshida, N. 2001, MNRAS, 321, 372
Kaiser, N. 1992, ApJ, 388, 272
Kaiser, N., Squires, G., Fahlman, G., & Woods, D. 1994, in Clusters of Galaxies, ed. F. Durret, A. Mazure, & J. T. T. Van (Gif-sur-Yvette: Editions Frontières), 56
Kaiser, N., Wilson, G., & Luppino, G. 2000, ApJ, submitted (astro-ph/0003338)