Canonical Structure of 2D Black Holes.†

José Navarro-Salas¹,², Miguel Navarro²,³,⁴ and César F. Talavera¹,²,³

1.- Departamento de Física Teórica, Burjassot-46100, Valencia, Spain.
2.- IFIC, Centro Mixto Universidad de Valencia-CSIC, Burjassot-46100, Valencia, Spain.
3.- The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom.
4.- Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, 18002, Granada, Spain.

Abstract

We determine the canonical structure of two-dimensional black-hole solutions arising in 2D dilaton gravity. By choosing the Cauchy surface appropriately we find that the canonically conjugate variable to the black hole mass is given by the difference of local (Schwarzschild) time translations at right and left spatial infinities. This can be regarded as a generalization of Birkhoff’s theorem.

†Work partially supported by the C.I.C.Y.T. and the D.G.I.C.Y.T.
Two-dimensional dilaton gravity models have attracted much attention in the last few years because they are useful toy models for quantum gravity. The string inspired model proposed by Callan, Giddings, Harvey and Strominger (CGHS-model) \cite{1} (see the review \cite{2}) provides an excellent scenario to study black-hole physics (the black hole solutions were first found in \cite{3, 4}). An exact, non-perturbative solution to the theory is still lacking, although some progress has been achieved recently \cite{5, 6, 7} using canonical quantization methods.

In any canonical quantization approach the issue of the reduced phase-space of the theory is of great importance. For the CGHS-model this question was addressed in ref. \cite{8} (see also \cite{9}) following the strategy of the covariant phase-space formalism \cite{10, 11}. However only the case of compact spatial section was analysed in \cite{8} owing to the subtleties of the non-compact case. A similar study was carried out in \cite{12} although their results are valid for the case of compact spatial section only. The aim of this paper is to perform a rigorous analysis of the canonical structure of the model for an open spatial section. We shall restrict ourselves to the case of pure gravity because the relevant questions we want to address already emerge in the pure gravity situation. We shall also consider spherically symmetric Einstein gravity, which is equivalent to a 2D dilaton gravity model. This model has been analysed in a rather involved way through Ashtekar’s hamiltonian approach \cite{13}.

In the covariant formulation of the canonical formalism the phase space is defined as the space of all classical solutions. This space is endowed with a presymplectic two-form

$$\omega = \int_\Sigma d\sigma_\mu (-\delta j^\mu)$$

(1)

where $\Sigma$ is a Cauchy hypersurface and $-\delta j^\mu = \omega^\mu$ (the symplectic current) can be obtained from the variation of the action $S = S[\Psi^\alpha]$

$$\delta S = \int \partial_\mu j^\mu + \frac{\delta S}{\delta \Psi^\alpha} \delta \Psi^\alpha.$$  

(2)

Now two remarks are in order. The presymplectic form (1) is constructed in the same way as a Noether charge. Therefore, without suitable boundary conditions (1) is not a well-defined quantity (finite and independent of $\Sigma$). Moreover, the two-form (1) is not necessarily non-degenerate. In general, it could have a non-trivial kernel. We can define the infinitesimal gauge-type symmetries of the theory as those generated by the kernel of the presymplectic form $\omega$ in (1). The physical (non-degenerate) symplectic form is obtained by pushing down (1) on the quotient manifold with respect to the degenerate directions (i.e., the reduced phase-space). These two crucial points will be discussed throughout the paper.

Let us start our analysis of the pure gravity CGHS model

$$S_{\text{CGHS}} = \frac{1}{2} \int_M d^2x \sqrt{-g} \left[ e^{-2\phi}(R + 4(\nabla \phi)^2 + 4\lambda^2) \right],$$

(3)
by writing down the potential one-form $j^\mu$ for the symplectic current. Following the scheme outlined above we obtain

$$j^{\alpha} = \frac{1}{2} \sqrt{-g} e^{-2\Phi} \left( 8 \partial^\alpha \Phi \delta \Phi - 2 \partial^\alpha \Phi (g^{\mu\nu} \delta g_{\mu\nu}) - 2 \partial_\beta \Phi \delta g^{\alpha\beta} + g^{\mu\nu} \delta \Gamma^\alpha_{\mu
u} - g^{\mu\alpha} \delta \Gamma^\nu_{\mu
u} \right). \quad (4)$$

The next point is to insert the general solution of the classical equations of motion into (4). It is well known that, up to space-time diffeomorphisms, the general solution of the model is given by the black hole solutions. In the conformal gauge, $ds^2 = -e^{2\rho} dx^+ dx^-$, it can be written as

$$e^{-2\Phi} = e^{-2\rho} = \frac{m}{\lambda} - \lambda^2 x^+ x^- , \quad (5)$$

where $x^+, x^-$ play the role of the null Kruskal coordinates of the Schwarzschild black hole and the parameter $m$ turns out to be the ADM mass of the black hole \[3, 14\]. To recover standard static black hole solutions one can introduce the local coordinates

$$\tau = \frac{1}{2}(\sigma^+ + \sigma^-), \quad \sigma = \frac{1}{2}(\sigma^+ - \sigma^-),$$

with

$$\lambda x^+ = e^{\lambda \sigma^+}, \quad \lambda x^- = -e^{-\lambda \sigma^-} , \quad (6)$$

arriving at

$$ds^2 = -(1 + \overline{m} e^{-2\lambda \sigma})^{-1} \left( d\sigma^+ d\sigma^- \right), \quad (8)$$

$$e^{-2\Phi} = \frac{\overline{m}}{m} + e^{2\lambda \sigma} , \quad (9)$$

where $\overline{m} = m/\lambda$. In the asymptotic region, $\sigma \to +\infty$, the solutions approach to the linear dilaton vacuum (LDV)

$$\rho \sim 0 , \quad \Phi \sim -\lambda \sigma . \quad (10)$$

We should remark that, in contrast to the standard Poincaré invariance of the flat space-time vacuum of general relativity, the only symmetry of the LDV is the time translation. In other words, the time translation is the only asymptotic symmetry of the black hole solution \[3, 14\]. Closely related to this is the issue of the gauge character of the space-time diffeomorphisms. At an intuitive level one could expect that only those diffeomorphisms leading to trivial Noether charges should be regarded as gauge symmetries \[15\]. This is in accord with our covariant definition of gauge-type symmetries. If the transformation generated by a vector field $X_H$ leaves $\omega$ unchanged and $X_H$ belongs to the kernel of $\omega$, then the Noether charge $H (i_{X_H} \omega = \delta H)$ vanishes on the covariant phase space. In general relativity the unique diffeomorphisms leading to non-trivial charges are the asymptotic Poincaré transformations \[16\] (see also \[17\]). In our case, and due to the dilaton field, the only transformations that preserve the asymptotic behaviour of the fields \[10\] are the time translations. Therefore, the most general admissible solution is given by \(\overline{m}\) and

$$ds^2 = (1 + \overline{m} e^{-2\lambda \sigma})^{-1} \left(- (d(\tau + f(\tau, \sigma)))^2 + d\sigma^2 \right) , \quad (11)$$

2
where \( f(\tau, \sigma) \) is an arbitrary function with vanishing derivatives at spatial infinity.

To evaluate \( j^\mu \) on-shell we first need to evaluate the Christoffel symbols:

\[
\begin{align*}
\Gamma^0_{00} &= -\frac{m f'(1 + \dot{f})}{\overline{m} + e^{2\lambda \sigma}} + \frac{\ddot{f}}{1 + f}, \\
\Gamma^0_{01} &= \frac{m(1 - f'^2)}{\overline{m} + e^{2\lambda \sigma}} + \frac{\dot{f}'}{1 + f}, \\
\Gamma^0_{11} &= \frac{m f'(1 - f'^2)}{(\overline{m} + e^{2\lambda \sigma})(1 + \dot{f})} + \frac{f''}{1 + f}, \\
\Gamma^1_{00} &= m(1 + \dot{f})^2 \frac{1}{\overline{m} + e^{2\lambda \sigma}}, \\
\Gamma^1_{01} &= \frac{m f'(1 + \dot{f})}{\overline{m} + e^{2\lambda \sigma}}, \\
\Gamma^1_{11} &= m(1 + f'^2) \frac{1}{\overline{m} + e^{2\lambda \sigma}}.
\end{align*}
\]

Pushing down the current (4) on the space of solutions (9), (11), and after a long computation, the expression for \( j^\mu \) is calculated to be

\[
j^\mu = \frac{1}{2} \epsilon^{\mu\nu} \partial_\nu \left( 2m \delta f + (\overline{m} + e^{2\lambda \sigma}) \delta f' - (\overline{m} + e^{2\lambda \sigma}) \frac{f'}{1 + f} \delta \dot{f} \right) - 4\lambda e^{2\lambda \sigma} \partial_\nu \delta f.
\]

In the static region \(-\infty < \sigma^+, \sigma^- <+\infty\) the symplectic current turns out to be then

\[
\omega^\mu = -\delta j^\mu = -\frac{1}{2} \epsilon^{\mu\nu} \partial_\nu \left( 2\delta m \wedge \delta f + \delta \overline{m} \wedge \delta f' - \frac{f'}{1 + f} \delta \overline{m} \wedge \delta \dot{f} \right) - \frac{1}{2} (\overline{m} + e^{2\lambda \sigma}) \frac{1}{1 + f} \delta f' \wedge \delta \dot{f}.
\]

Therefore the contribution of this asymptotically flat domain to the two-form \( \omega \) is given by

\[
\omega_I = \frac{1}{2} \left( 2\delta f \wedge \delta m + \delta f' \wedge \delta \overline{m} - \frac{f'}{1 + f} \delta \dot{f} \wedge \delta m - \frac{m + e^{2\lambda \sigma}}{1 + f} \delta \dot{f} \wedge \delta f' \right) \bigg|_{\partial \Sigma_I},
\]

where \( \partial \Sigma_I \) represents the two spatial ends (\(+\infty\) and \(-\infty\)). To find a finite resulting expression for (20), we have to require appropriate boundary conditions. The minimal requirement is to assume

\[
e^{\lambda \sigma} f', e^{\lambda \sigma} \dot{f} \sim_{+\infty} 0,
\]

and

\[
f' \sim_{-\infty} 0.
\]
Figure 1: Kruskal diagram for black-hole spacetime. $\Sigma = \Sigma_I \cup \Sigma_{II}$ is the Cauchy surface.

These conditions ensure the finiteness of (20) as well as the independence from the particular surface $\Sigma_I$:

$$\omega_I = \delta(f(+\infty) - f(-\infty)) \wedge \delta m.$$  \hspace{1cm} (23)

The above result suggests to choose the Cauchy surface $\Sigma$ in such a way that it connects the right and left infinities through the static regions I and II of Kruskal diagram (see Fig. I). Adding to (23) the contribution of the asymptotically flat region II we have

$$\omega = \omega_I + \omega_{II} = \delta \left(f(i_R^0) - f(i_L^0)\right) \wedge \delta m.$$  \hspace{1cm} (24)

where $i_R^0, i_L^0$ stands for the right and left spatial infinities. This means that the reduced phase-space of the model is two-dimensional, where the canonically conjugate variable to the mass is the global variable $(f(i_R^0) - f(i_L^0)) \equiv \mathcal{F}$ of the local time translations. Observe that the parameter $m$ clearly emerges now as the Noether charge of the asymptotic time translation $(\mathcal{F} \to \mathcal{F} + a^0)$:

$$i(\frac{\partial}{\partial f}) \omega = \delta m.$$  \hspace{1cm} (25)

Now we want to extend our previous discussion to the case of spherically symmetric Einstein gravity. Although this theory can be seen as a special case of a general class of two-dimensional dilaton gravity models, we shall treat it as a four-dimensional theory.

The symplectic current potential of general relativity is

$$j^\alpha = \frac{1}{16\pi} \sqrt{g} \left(g^{\mu\nu} \delta \Gamma^\alpha_{\mu\nu} - g^{\mu\alpha} \delta \Gamma^\nu_{\mu\nu}\right).$$  \hspace{1cm} (26)
According with Birkhoff’s theorem [18] the Schwarzschild solution represents the general solution, up to space-time diffeomorphisms, of spherically symmetric gravity in vacuum. To work out the canonical structure of the model one can proceed as in the previous case. Taking into account that the only surviving asymptotic Poincaré symmetry of the reduced theory is the time translation, the most general admissible solution in region I and II has to be of the form

\[ ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]

(27)

Inserting (27) into the time component of the current (26) we obtain

\[ j^0 = \frac{1}{16\pi} \sin \theta \frac{d}{dr} \left[ -\frac{r(r - 2m)}{1 + f} f \delta f + r(r - 2m)\delta f' - 2f \delta m + 2m \delta f \right]. \]

(28)

Integrating over \(2m < r < +\infty, 0 < \varphi < 2\pi, 0 < \theta < \pi,\) it is now easy to arrive at the following expression for \(\omega_I\)

\[ \omega_I = \frac{1}{4} \left[ -2 \frac{rf'}{1 + f} \delta m \delta f + \frac{r(r - 2m)}{1 + f} \delta f' \delta f + 2r \delta m \delta f' + 4\delta f \delta m \right] \bigg|_{r=2m}. \]

(29)

For \(\omega_I\) to be well defined the asymptotic behaviour has to be adjusted appropriately. We can require the following fall-off behaviour

\[ rf', rf \sim_{\infty} 0, \]

(30)

and

\[ f' \sim_{2m} 0. \]

(31)

Assuming (30), (31), we arrive at

\[ \omega_I = \delta(f(+\infty) - f(2m)) \wedge \delta m, \]

(32)

and the final symplectic form is

\[ \omega = \delta(f(t_H^0) - f(t_L^0)) \wedge \delta m. \]

(33)

We can observe the parallelism between the results (33) and (24). It seems to be a general property of black hole solutions and constitutes a kind of generalization of Birkhoff’s theorem in the sense that identifies the true dynamical degrees of freedom. As a byproduct, this implies that the conformal gauge does not capture the full canonical content of the two-dimensional metric in a non-compact spatial world. In going to the conformal gauge one makes use of some particular diffeomorphisms that cannot be regarded as gauge transformations.

The explicit knowledge of the reduced phase space of lagrangian models opens an avenue for their quantization [19, 11] and, at the same time, could suggest new variables
for quantizing the system at a presymplectic level. We plan to address these issues in a future publication extending the present work to the case when matter fields are present.

After completing this work we received a preprint [20] where the canonical structure of Schwarzschild black holes is analyzed with different methods. Our result for spherically symmetric gravity agrees with that of Ref. [20].

Acknowledgements

M. Navarro acknowledges to the spanish MEC for a Postdoctoral fellowship. C.F. Talavera is grateful to the Generalitat Valenciana for a FPI grant.

References

[1] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, Phys. Rev. D 45 R(1992)1005.
[2] J.A. Harvey and A. Strominger, preprint EFI-92-41, hep-th/9209055 (1992).
[3] E. Witten, Phys. Rev. D 44 (1991) 314.
[4] G. Mandal, A. Sengupta and S. Wadia, Mod. Phys. Lett. A 6 (1991) 1685.
[5] S. Hirano, Y. Kazama and Y. Satoh, Phys. Rev. D 48 (1993) 1687.
[6] K. Schoutens, E. Verlinde and H. Verlinde, Phys. Rev. D 48 (1993) 2670.
[7] A. Mikovic, Black Holes and Non-Perturbative Canonical 2D Dilaton Gravity, Imperial-TP/93-94/16.
[8] A. Mikovic and M. Navarro, Phys. Lett. B 315 (1993) 267-276.
[9] J. Navarro-Salas and C.F. Talavera, Quantum Cosmological Approach to 2d Dilaton Gravity, preprint FTUV/93-34, IFIC/93-34 (to appear in Nucl. Phys. B).
[10] C. Crnković and E. Witten, in Three Hundred Years of Gravitation, eds. S.W. Hawking and W. Israel (Cambridge, 1987) p. 676.
[11] J. Navarro-Salas, M. Navarro and V. Aldaya, Phys. Lett. B 287 (1992)109; Nucl. Phys. B 403 (1993) 291.
[12] K.S. Soh, Phys. Rev. D 49 (1994) 1906.
[13] T. Thieman and H.A. Kastrup, Nucl. Phys. B 399 (1993) 211.
[14] A. Bilal and I.I. Kogan, Phys. Rev. D 47 (1993) 5408.
[15] S. Coleman, *Aspects of Symmetry* (Cambridge, 1985).

[16] A. Ashtekar, L. Bombelli and O. Reula, in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, ed. M. Francavigilia (ESP, 1991) p. 417.

[17] C. Crnković, *Nucl. Phys.* B 288 (1987) 419.

[18] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge (1973).

[19] E. Witten, *Nucl. Phys.* B 311 (1988) 46.

[20] K. Kuchar, *Geometrodynamics of Schwarzschild Black Holes*, preprint UU-REL-94-3-1.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9405015v2