Multi-fluid cosmology in Einstein gravity: analytical solutions

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Abstract
We review analytical solutions of the Einstein equations which are expressed in terms of elementary functions and describe Friedmann–Lemaître–Robertson–Walker universes sourced by multiple (real or effective) perfect fluids with constant equations of state. Effective fluids include spatial curvature, the cosmological constant, and scalar fields. We provide a description with unified notation, explicit and parametric forms of the solutions, and relations between different expressions present in the literature. Interesting solutions from a modern point of view include interacting fluids and scalar fields. Old solutions, integrability conditions, and solution methods keep being rediscovered, which motivates a review with modern eyes.

Keywords Cosmology · Analytical solutions · Multi-fluid cosmologies

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C.1 Positive curvature
1 Introduction

Analytical solutions of the Einstein–Friedmann equations describing cosmology in the context of Einstein’s theory of gravity have been known since its early days, beginning with those of de Sitter [152], Friedmann [183,184], Lemaître [251–254], and Einstein himself, including the famous and ill-fated Einstein static universe [162] (see the textbooks [94,156,165,240,243,261,301,327,404] and see Ref. [37] for a popular exposition).

The qualitative analysis of the phase spaces of solutions of the relevant differential equations provide much information about the cosmological dynamics with multiple fluids (e.g., [128,174,178,373,377,393,394,403]). However, analytical solutions are often needed for the investigation of physics beyond the cosmic dynamics. Various physical motivations to look for exact solutions include the need for toy models in theoretical research; testing and calibrating numerical codes; describing the smooth transition between different eras (most notably, between radiation- and matter-dominated epochs, between inflation and radiation, or the onset of dark energy domination); obtaining solutions of the many inflationary scenarios of the early universe; modelling the acceleration of the cosmic expansion in the present era. The latter requires the simultaneous analysis of dark energy and dark/baryonic matter, and two-component dark energy models have also been proposed [194,201,397,425] (in particular, the quintom model consisting of two interacting scalar fields, one regular and one phantom, has received much attention [14,15,77–82,114,158,170,179,180,202,247,248,255,256,299,339,351,354,355,360–366,405,406,409,416–418,426–430], see [76] for a review).

Most solutions of the Einstein–Friedmann equations found in the 1960s and 1970s and the conditions for their integrability are spread in a body of literature that spans over half a century and are being forgotten, as demonstrated by the fact that two- and three-(effective) fluid solutions found in the 1960s and 1970s have been re-investigated and rediscovered in recent years. Moreover, the theoretical motivation for studying multiple fluids or effective fluids has changed: scalar fields have assumed a primary role in cosmology with the advent of the theory of inflation and with the introduction of the dark energy concept to explain the 1998 discovery that the present expansion of the universe is accelerated [330,331,340]. This situation motivates reviewing analytical multi-fluid solutions of the Einstein–Friedmann equations from a modern point of view, including scalar fields and interacting fluids in the picture, the study of which was much less motivated in the 1970s.

2 Field equations, fluids, and scalar fields

We restrict our discussion to cosmology and analytical solutions in the context of Einstein gravity, referring the reader to [89,148,171,186,314,375] for scalar-tensor
(including $f(R)$) cosmology and to Refs. [124,207,212] for more general theories of gravity. We follow the notation of Ref. [404]: the signature of the metric tensor $g_{ab}$ is $-+++\ldots$ and units in which the speed of light $c$ and Newton’s constant $G$ are unity are used.

The Einstein equations read

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}, \quad (2.1)$$

where $R_{ab}$ and $T_{ab}$ are, respectively, the Ricci tensor and the matter stress-energy tensor, while $R \equiv g^{ab} R_{ab}$ is the Ricci scalar and $\Lambda$ is the cosmological constant. In order to discuss solution methods, one must necessarily restrict the scope. We begin with the simplest situation, and the one most common in the research literature on multi-fluid cosmology, consisting of two non-interacting perfect fluids with stress-energy tensors

$$T^{(1)}_{ab} = (P_1 + \rho_1) u_a u_b + P_1 g_{ab}, \quad (2.2)$$
$$T^{(2)}_{ab} = (P_2 + \rho_2) u_a u_b + P_2 g_{ab}, \quad (2.3)$$

where $u^c$ is the common four-velocity, $\rho_{1,2}$ are the energy densities, and $P_{1,2}$ are the isotropic pressures. A constant linear barotropic equation of state

$$P = w \rho, \quad w = \text{const.} \quad (2.4)$$

is assumed for both fluids. Indeed, when multiple gravitating fluids are present, we will assume that they all have the same four-velocity, according to the assumptions of spatial homogeneity and isotropy that require the existence of a single comoving frame associated with observers who see these symmetries in the surrounding universe.

By imposing spatial homogeneity and isotropy to satisfy the Copernican principle, the Einstein equations admit as a solution the Friedmann–Lemaître–Robertson–Walker (FLRW) geometries [94,156,240,243,261,301,404]. The FLRW line element in spherical comoving coordinates $(t, r, \vartheta, \varphi)$ is

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2_{(2)} \right), \quad (2.5)$$

where $d\Omega^2_{(2)} \equiv d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2$ is the line element on the unit 2-sphere, $K$ is the curvature index (usually normalized to 0, ±1), and $a(t)$ is the scale factor encoding the expansion history of the universe. We will also use the conformal time $\eta$ defined by $dt \equiv a \, d\eta$.

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1 Section 8 discusses interacting fluids.

2 Again, the assumption that the two fluids are collinear is not mandatory: mutually tilted fluids are the subject of a considerable literature (e.g., [127,129–132,306,372,398,403]).
Assuming the FLRW line element, the spatial symmetries reduce Eq. (2.1) to the Einstein–Friedmann equations

\[ H^2 = \frac{8\pi}{3} \rho + \frac{\Lambda}{a^2}, \quad (2.6) \]

\[ \frac{\ddot{a}}{a} = -\frac{4\pi}{3} (\rho + 3P) + \frac{\Lambda}{3}, \quad (2.7) \]

\[ \dot{\rho} + 3H (P + \rho) = 0, \quad (2.8) \]

where \( H \equiv \dot{a}/a \) is the Hubble function and an overdot denotes differentiation with respect to the comoving time \( t \). Eq. (2.8) follows from the covariant conservation of the matter energy-momentum tensor \( \nabla^b T_{ab} = 0 \). Only two of the three Einstein–Friedmann equations are independent. If two of them are given, the last one can be deduced from them. Without loss of generality, we can regard the Friedmann equation (2.6) and the energy conservation equation (2.8) as independent and the acceleration equation (2.7) as derived.

If a fluid is not coupled explicitly to other matter or fields and has equation of state (2.4), the conservation equation (2.8) implies that its energy density scales as

\[ \rho(a) = \rho^{(0)} a^{3(w+1)}, \quad (2.9) \]

with \( \rho^{(0)} \) a constant. Because fluids with different equation of state parameters \( w \) scale differently with the scale factor \( a \), in the presence of multiple fluids one of them will come to dominate the cosmic dynamics if the universe keeps expanding. For example, if there are only radiation \( (w = 1/3) \) and dust \( (w = 0) \) in an expanding universe, the radiation fluid dominates early on for small \( a \) but its energy density \( \rho_r \sim a^{-4} \) decays faster than the energy density of dust \( \rho_m \sim a^{-3} \), which comes to dominate later on if \( a(t) \) always increases. Therefore, it is customary to approximate the history of the universe with a radiation era followed by a matter era. This approximation is adequate for many problems, but there are situations in which one is interested in the detailed two-fluid solution.

In cosmology, common terminology refers to a dark energy fluid if \(-1 \leq w < -1/3\) (which, according to the acceleration equation (2.7) implies that the universe accelerates, \( \ddot{a} > 0 \)), and to phantom matter if \( w < -1 \).

If the relation (2.4) between the cosmic fluid pressure \( P \) and energy density \( \rho \) is not constant, one can still define an effective equation of state parameter \( w_{\text{eff}} \equiv P/\rho \). Non-linear barotropic equations of state \( P = P(\rho) \) have been studied in the literature, especially in relation with phantom fluids and sudden future singularities [20,21,36, 38,50,69,83,87,182,195,238,309–311,350,368,370,381] but here we restrict ourselves to linear equations of state.

The energy density of a matter component \( \rho \) can be expressed in terms of the dimensionless density parameter

\[ \Omega \equiv \frac{\rho}{\rho_c}, \quad (2.10) \]
where $\rho_c(t) \equiv \frac{3H^2}{8\pi}$ is the energy density of a spatially flat universe with $K = 0$ and $\Omega = 1$.

The standard model of cosmology, the so-called $\Lambda$–Cold Dark Matter or $\Lambda$CDM model consists of a spatially flat FLRW universe containing approximately 70% dark energy (with a measured equation of state parameter consistent with the cosmological constant signature $w = -1$) and 30% matter, modelled as a dust, adding up to $\Omega_{\text{tot}} = 1$.

Due to their non-linearity, there are significant mathematical difficulties in solving analytically the Einstein–Friedmann equations of cosmology sourced by multiple fluids or effective fluids (the latter include the cosmological constant and spatial curvature, see below). It is even more difficult to solve analytically the more general Einstein equations (without spatial homogeneity and isotropy) with two fluids as the matter source, for example in stellar models (e.g., [96,97,304]). In cosmology, there are two approaches to the analytical solution: the first method attempts to integrate the Friedmann equation in comoving or in conformal time and deals with an integral that, in general, can only be expressed in terms or elliptic or hypergeometric functions [25]. The second method looks for solutions expressed in parametric form employing the conformal time as the parameter. Also in this second approach, there is no guarantee that the integration can be performed in closed form in terms of elementary functions, which is instead the main goal of the present paper. The two approaches are complementary: one approach may succeed in situations where the other fails, and vice-versa.

A large variety of situations, which is best described by qualitative methods and phase space analysis, can present themselves for general forms of matter which may include perfect or imperfect fluids, tilted fluids [127,129–132,372,398,403], fluids with non-linear and/or non-constant equation of state [20,21,36,38,50,69,83,87,182,195,238,309–311,350,368,370,381], scalar fields, etc. One also has to distinguish between non-interacting fluids and interacting (or explicitly coupled) fluids. There is now a large amount of literature devoted to the latter, because of the hypothesis that dark energy and dark matter may be coupled directly, which would in principle explain the coincidence between the orders of magnitude of the density parameters $\Omega_{\text{DE}}$ and $\Omega_{\text{DM}}$ of these two fluids.

Even under the simplifying assumptions of spatial homogeneity and isotropy, perfect fluids, and constant barotropic equations of state, the landscape of analytical FLRW solutions and their physics is rich and is covered by a considerable amount of literature spanning five decades. The older literature is sometimes forgotten, which results in analytical solutions and solution methods being rediscovered, or in the use of numerical solutions, which necessarily commit to specific values of the parameters and initial conditions, when analytical formulas are instead available. Our goal is to provide a bird’s eye view of this area of multi-fluid cosmology.

As said, the integration of the Einstein–Friedmann equations usually leads to solutions expressed by elliptic integrals or hypergeometric functions which cannot be reduced to simpler forms [25]. These expressions are not useful in practice, making for a rather sterile catalogue of formal situations in which numerical solution of the Einstein–Friedmann equations (2.16)–(2.18) is more convenient for practical purposes. Here we focus on situations in which the field equations are integrable and
the solution can be expressed explicitly in the form of a finite number of elementary functions.

**2.1 Single fluid and effective curvature / Λ-fluids**

If \( K = 0 \) and \( \Lambda = 0 \) and there is a single fluid with constant equation of state (2.4), the well-known textbook solution for the scale factor is

\[
a(t) = a_0 (t - t_0)^{2/(3(w + 1))},
\]

where \( a_0 \) and \( t_0 \) are constants and \( \rho(a) \) is given by Eq. (2.9). Models with a single real fluid, possibly with cosmological constant and spatial curvature, were reviewed and classified early on ([208, 218–220, 293, 294, 395], see also [156]).

For a fluid mixture, one defines the total effective density by

\[
\frac{8\pi}{3} \rho_{\text{tot}} = \frac{8\pi}{3} \rho + \frac{\Lambda}{a^2} \equiv \frac{8\pi}{3} \left( \rho + \rho_{\Lambda} + \rho_K \right),
\]

where

\[
\rho_{\Lambda} = \frac{\Lambda}{8\pi}, \quad \rho_K = -\frac{3K}{8\pi a^2}
\]

can have any sign, as long as \( \rho_{\text{tot}} \geq 0 \) (otherwise the Friedmann equation (2.6) cannot be satisfied because the left hand side is always non-negative).

To be consistent in treating the curvature term \(-K/a^2\) in the Friedmann equation as an effective fluid, one must also attribute to it a pressure \( P_K \). This is done by imposing that the covariant conservation equation (2.8) be satisfied for this effective component of the cosmic fluid, which yields

\[
P_K = -\frac{\rho_K}{3} = \frac{K}{8\pi a^2}
\]

(this remark appears to have been made by Hughston & Shepley [220], McIntosh [293], and McIntosh & Foyster [294] long ago and then apparently forgotten). Similarly, it is customary to use the effective energy density \( \rho_{\Lambda} \) and pressure \( P_{\Lambda} \), with \( P_{\Lambda} = -\rho_{\Lambda} = -\Lambda/(8\pi) \), for the cosmological constant.\(^3\) The acceleration equation is automatically satisfied since the combination \( \rho_K + 3P_K \) vanishes (as it should be, since the acceleration equation (2.7) does not contain the curvature index \( K \)). By defining the total pressure

\[
P_{\text{tot}} = P + P_{\Lambda} + P_K,
\]

\(^3\) Oddly, Harrison (who found many of the solutions reported in this review) distinguished between the cosmological constant and a perfect fluid with \( w = -1 \), reporting solutions where only these two components source the Einstein–Friedmann equations simultaneously [208].
the Einstein–Friedmann equations can be rewritten as

\[
H^2 = \frac{8\pi}{3} \rho_{\text{tot}}, \\
\frac{\ddot{a}}{a} = -\frac{4\pi}{3} (\rho_{\text{tot}} + 3P_{\text{tot}}), \\
\dot{\rho}_{\text{tot}} + 3H (P_{\text{tot}} + \rho_{\text{tot}}) = 0.
\] (2.16, 2.17, 2.18)

Formally, we have reduced spatially curved \((K \neq 0)\) universes, potentially with a cosmological constant \(\Lambda\), to a fictitious spatially flat universe with vanishing \(\Lambda\). The topology of the three-dimensional spatial sections cannot change, of course, and this is only a formal trick used in the cosmology literature and textbooks to introduce the total density parameter \(\Omega_{\text{tot}} \equiv \rho_{\text{tot}}/\rho_c\). This formal reduction to a single effective fluid is normally not completed by including the pressure \(P_K\) in modern literature and textbooks. Here, this extended definition of effective curvature fluid is useful to study “simple” analytical two-fluid solutions of FLRW universes with any spatial curvature, with or without \(\Lambda\).

### 2.2 Scalar field

Scalar fields are the simplest fundamental physical fields: they abound in particle physics and are widely used in cosmology. Indeed, much of modern cosmology is based on scalar fields as the form of matter propelling the expansion of the universe. Inflation in the early universe \([240,262,279,301]\) is usually described as the effect of the dynamics of a single scalar field, dubbed inflaton. Starobinsky inflation \([379,380]\), which historically was the first inflationary scenario (although Guth’s scenario \([204]\) was much better known initially), is currently favoured by cosmological observations \([5]\). It relies on quadratic corrections to the Einstein–Hilbert Lagrangian, but those can be reduced to an effective scalar field, as in all \(f(R)\) theories of gravity \([148,314,375]\). In both standard and Starobinsky inflation, a single scalar degree of freedom acts as the source of the Einstein–Friedmann equations. What is more, the current acceleration of the cosmic expansion is commonly explained with a quintessence scalar field acting as dark energy, or with modifications of gravity, among which \(f(R)\) gravity is very popular. In both cases, we have again a scalar field dominating the dynamics of the universe. In the late universe, this scalar field acts together with the dark matter fluid modelled as a zero pressure dust. Analytical solutions of the Einstein–Friedmann equations describing a scalar field and a fluid will be discussed in Sect. 7.2: here we focus on a single scalar field.

The cosmological literature offers also scalar field models of dark matter and unified dark energy–dark matter models \([12,23,57,60–62,64,134,185,205,249,250,257,280–282,282,283,318,343–345,384,385]\). If scalar fields are truly responsible for early inflation and late-time acceleration, then their physical nature amounts to two big mysteries of cosmology and fundamental physics. Certainly, scalar fields abound in high energy theories and gravitational scalars arise naturally in modifications of general relativity. The prototypical alternatives to Einstein theory are Jordan-Brans-Dicke
gravity \[70,224,225\] and its scalar-tensor generalizations \[56,315,402\], in which the gravitational coupling strength becomes a dynamical scalar field. While, with the exception of Sect. 8.4, we do not discuss these alternative theories of gravity here, there is little doubt that the scalar fields appearing in these theories, as well as in high energy physics, have contributed to the increasing use of scalars in cosmology.

The energy-momentum tensor of a scalar field minimally coupled to the curvature is

\[
T^{(\phi)}_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla_c \phi - V(\phi) g_{ab},
\]

(2.19)

where \(V(\phi)\) is the potential energy density or self-interaction potential. The stress-energy tensor (2.19) is equivalent to that of a perfect fluid \[27,173,176,276,277,335,359\] with four-velocity

\[
u^a = \frac{\nabla^a \phi}{\sqrt{-\nabla^c \phi \nabla_c \phi}},
\]

(2.20)

provided that the gradient \(\nabla^c \phi\) is timelike \((u^c\) is correctly normalized, \(u^c u_c = -1\). This condition is always satisfied in an unperturbed FLRW universe, where \(\phi = \phi(t)\) in comoving coordinates in order to respect spatial homogeneity. In general, if the scalar field has a potential \(V(\phi)\), the effective equation of state of the equivalent fluid is dynamical. Choosing a potential \(V(\phi)\) determines a dynamical equation of state but there isn’t a one-to-one correspondence \[49,169,276\].

The covariant conservation equation \(\nabla^b T^{(\phi)}_{ab} = 0\) gives the Klein–Gordon equation obeyed by \(\phi\)

\[
\Box \phi - \frac{dV}{d \phi} = 0,
\]

(2.21)

which reduces to

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d \phi} = 0
\]

(2.22)

for a scalar \(\phi(t)\) in FLRW space, or to

\[
\phi_{,\eta \eta} + 2\mathcal{H} \phi_{,\eta} + a^2 \frac{dV}{d \phi} = 0
\]

(2.23)

in terms of conformal time, where \(\mathcal{H} \equiv a_{,\eta}/a\). In general, this equation is non-linear and relatively few analytical solutions are known.

The effective energy density and pressure of a cosmological scalar field are

\[
\rho_{\phi} = T^{(\phi)}_{ab} u^a u^b = \frac{\dot{\phi}^2}{2} + V(\phi),
\]

(2.24)
\[ P_\phi = h^{ab} T_{ab}^{(\phi)} = \frac{T_{ii}}{g_{ii}} = \frac{\dot{\phi}^2}{2} - V(\phi), \] (2.25)

where \( h^{ab} \) is the projection operator on the 3-dimensional spatial sections with Riemannian metric \( h_{ab} = g_{ab} + u_a u_b \). In general, the equation of state of the effective fluid equivalent to \( \phi \) is dynamical and time-dependent, however one can in principle impose that it remains constant and determine the potential \( V(\phi) \) that achieves this peculiar situation (which may or may not be physically motivated), as done in [32,378]. One obtains

\[ w_\phi \equiv \frac{P_\phi}{\rho_\phi} = \text{const.} \] (2.26)

if the kinetic and potential energy densities are proportional to each other, \( \dot{\phi}^2/2 = \alpha V(\phi) \) with

\[ \alpha = \frac{- (w_\phi + 1)}{w_\phi - 1} \] (2.27)

for \( w_\phi \neq 1 \). The case \( w_\phi = 1 \) corresponds to \( V = 0 \) and to a stiff fluid, for which introducing the parameter \( \alpha \) does not make sense.

Interest in analytical solutions of the Einstein–Friedmann equations (2.6)–(2.8) describing FLRW universes sourced by a single scalar field ranged from mathematical curiosity to the desire of investigating dynamics beyond the standard slow-roll approximation to inflation [279], or to showing that almost any behaviour of the scale factor \( a(t) \) can be obtained with a scalar field with a suitable potential [164]. It was hoped that, since any inflationary scenario motivated by a particle physics theory eventually amounts to specifying the scalar field potential \( V(\phi) \), by spanning a range of possibilities for \( V(\phi) \) and the resulting cosmic history \( a(t) \) and reconstructing the latter through cosmography, one could use the universe as a hot laboratory to investigate physics at energies unreachable by particle accelerators. Unfortunately, the reconstruction of the potential \( V(\phi) \) by means of the spectral indices of scalar (and maybe, in the future, of tensor) perturbations provides only the values of \( V \) and of its first two derivatives at the value of \( \phi \) when these perturbations cross outside the horizon [267]. This information is insufficient to reconstruct the functional form of the inflationary potential \( V(\phi) \).

### 2.3 Multi-fluid cosmology

Suppose that there are \( n \) non-interacting fluids with densities \( \rho_i \) and pressures \( P_i = \omega_i \rho_i \), with \( \omega_i = \text{const.} \) (\( i = 1, 2, \ldots, n \)). Then the total density is

\[ \rho_{\text{tot}} = \sum_{i=1}^{n} \rho_i \] (2.28)
and, according to Dalton’s law, the pressure is the sum of the partial pressures

\[ P_{\text{tot}} = \sum_{i=1}^{n} w_i \rho_i. \]  

(2.29)

The total effective equation of state parameter is defined as

\[ w_{\text{tot}}(a) \equiv \frac{P_{\text{tot}}}{\rho_{\text{tot}}} = \frac{\sum_{i=1}^{n} P_i}{\sum_{j=1}^{n} \rho_j} = \frac{\sum_{i=1}^{n} w_i \rho_i^{(0)} a^{-3(w_i+1)}}{\sum_{j=1}^{n} \rho_j^{(0)} a^{-3(w_j+1)}}. \]  

(2.30)

It follows that, in the presence of two or more perfect fluids, the effective equation of state of the mixture is time-dependent even if the equation of state of each component is constant. The time variation of the total effective equation of state parameter is given by [293]

\[ \dot{w}_{\text{tot}} = -\frac{3H}{\rho_{\text{tot}}^2} \sum_{i<j} (w_i - w_j)^2 \rho_i \rho_j \]  

(2.31)

(this equation is derived in Appendix A).

A consequence of Eq. (2.30) is that, if dark energy is made of two distinct components with constant equations of state

\[ P_1 = w_1 \rho_1(a) = w_1 \frac{\rho_1^{(0)}}{a^{3(w_1+1)}}, \]  

(2.32)

\[ P_2 = w_2 \rho_2(a) = w_2 \frac{\rho_2^{(0)}}{a^{3(w_2+1)}}, \]  

(2.33)

the effective equation of state parameter

\[ w_{\text{tot}} = \frac{w_1 \rho_1^{(0)} + w_2 \rho_2^{(0)}}{\rho_1^{(0)} + \rho_2^{(0)}} a^{3(w_1-w_2)} \]  

(2.34)

is not constant (even when all the \( w_i \) are) and is non-linear in the scale factor or the redshift \( z \equiv \frac{a_0}{a} - 1 \) (where \( a_0 \) is the present value of the scale factor). Therefore, it cannot be reproduced by the linear parametrizations \( w(z) = w_0 + w_1 z \) or \( w(a) = w_0 + w_1 (1 - a) \) (with \( w_{1,2} \) constants) common in observational cosmology.
2.4 Reduction to quadratures

Rewriting the Friedmann equation as (2.16), one obtains

\[ \dot{a} = \pm \sqrt{\frac{8\pi}{3}} a \sqrt{\rho_{\text{tot}}}, \]  

(2.35)

where \( \rho_{\text{tot}} = \rho_{\text{tot}}(a) \). Formal integration yields

\[ \int \frac{da}{a \sqrt{\rho_{\text{tot}}(a)}} = \pm \sqrt{\frac{8\pi}{3}} (t - t_0), \]  

(2.36)

where \( t_0 \) is an integration constant. Usually, one cannot compute the integral in the left hand side explicitly in terms of elementary functions. Even when this is possible, one will have an analytical solution \( t = t(a) \) which, in general, cannot be inverted to obtain \( a = a(t) \) explicitly.

While physically there may be a big difference between a source composed of two fluids (for example, a cosmic radiation background that is still hot, inducing significant microphysical effects in a universe already dominated by dust) and a universe filled by a single fluid with exotic non-linear equation of state, the mathematics ruling the Einstein equations does not know the difference. From the purely formal point of view, one can define the total equation of state parameter for a universe sourced by two distinct fluids as in Eq. (2.30).

2.5 Symmetries of the Einstein–Friedmann equations

In any physical theory, symmetries are important and the symmetries of the Einstein–Friedmann equations have been the subject of many works ([6,7,76,77,84,88,89,98,108–113,115,138,140,172,338,387,405] and references therein). A connection has been made between Einstein–Friedmann equations and methods of supersymmetric quantum mechanics [316,348,349]. As in other areas of physics, also in FLRW cosmology Lie, Noether, and other symmetries are often studied with the ultimate goal of generating first integrals of motion and exact solutions [41,68,88,89,91,92,108,121,122,154,190,319,320,322–324,334,392,423], sometimes in relation with Penrose’s cyclic time cosmologies or conformal time [53–55,397]. The existence of such a symmetry does not automatically make the corresponding solution a physical one.

A symmetry studied in [108,319] (see also [6,7,98,111–113,115]) and generalized in [157] applies to \( K = 0 \) universes sourced by perfect fluids. It consists of a rescaling of the comoving time \( t \), the Hubble function \( H \) and all first time derivatives, and of the fluid energy density and pressure which leaves the Einstein–Friedmann equations invariant in form. The pressure is rescaled differently than the energy density, which changes the equation of state.

The symmetry transformations are given by [157]

\[ dt \to d\tilde{t} = f(\rho) dt, \]  

(2.37)
\[ \rho \rightarrow \bar{\rho} = \frac{(1 - f^2)}{8\pi G f^2} \Lambda + \frac{\rho}{f^2} = \frac{(1 - f^2)}{f^2} \rho \Lambda + \frac{\rho}{f^2}, \quad (2.38) \]

\[ P \rightarrow \bar{P} = -\bar{\rho} + \left[ \frac{4\pi G f - (8\pi G \rho + \Lambda) f'}{4\pi G f^3} \right] (P + \rho) \]

\[ = -\bar{\rho} + \left[ \frac{f - 2(\rho + \rho \Lambda) f'}{f^3} \right] (P + \rho), \quad (2.39) \]

with \( f(\rho) > 0 \) a dimensionless regular function (the differential \( d\bar{t} = f(\rho(t))dt \) is exact because \( \rho = \rho(t) \)). The inverse is

\[ dt = \frac{d\bar{t}}{f}, \quad (2.40) \]

\[ \rho = f^2 \bar{\rho} + \left( f^2 - 1 \right) \rho \Lambda, \quad (2.41) \]

\[ P + \rho = \frac{f^3}{f - 2(\rho + \rho \Lambda) f'} (\bar{P} + \bar{\rho}). \quad (2.42) \]

Using

\[ \dot{a} \equiv \frac{da}{dt} = \frac{f}{d\bar{t}} \frac{d\bar{t}}{dt} = \frac{f}{d\bar{t}}, \quad (2.43) \]

\[ H = f \bar{H} \equiv \frac{f}{a} \frac{da}{d\bar{t}}, \quad (2.44) \]

and Eq. (2.41), the Friedmann equation (2.6) for the spatially flat universe changes into

\[ \bar{H}^2 = \frac{8\pi}{3} \bar{\rho} + \frac{\Lambda}{3}, \quad (2.45) \]

while the covariant conservation equation (2.8) becomes [157]

\[ \frac{d\bar{\rho}}{d\bar{t}} + 3\bar{H} (\bar{P} + \bar{\rho}) \]

\[ = \left[ \frac{f - 2(\rho + \rho \Lambda) f'}{f^4} \right] [\dot{\rho} + 3H (P + \rho)] = 0. \quad (2.46) \]

When \( \Lambda = 0 \), one has

\[ f(\rho) = \sqrt{\frac{\rho}{\bar{\rho}}}, \quad (2.47) \]

and the symmetry (2.37)–(2.39) reduces to the one of Ref. [108] (see also [319]) given by

\[ \rho \rightarrow \hat{\rho}(\rho), \quad (2.48) \]
\( H \rightarrow \overline{H} = \sqrt{\frac{\rho}{\bar{\rho}}} H, \) \hfill (2.49)

\( P + \rho \rightarrow \overline{P} + \overline{\rho} = (P + \rho) \sqrt{\frac{\rho}{\bar{\rho}}} \frac{d\bar{\rho}}{d\rho}, \) \hfill (2.50)

\( f(\rho) = \sqrt{\frac{\rho}{\bar{\rho}}}. \) \hfill (2.51)

de Sitter universes with \( \Lambda > 0 \) are fixed points of the transformation because, if \( P = \rho = 0 \), then \( \overline{P} = -\overline{\rho} = \text{const.} \),

\[ \overline{\rho} = \frac{(1 - f^2)}{f^2} \rho \Lambda \] \hfill (2.52)

with \( f = \text{const.} \) and, by changing units of time one can set \( f \) to unity. These symmetries map perfect fluids FLRW universes with equation of state (2.4) into universes with cosmological constant and fluids with non-linear equations of state \[ 157 \]

\[ \tilde{w} + 1 = (w + 1) \frac{\rho}{\rho + (1 - f^2) \rho \Lambda} \frac{f - 2 (\rho + \rho \Lambda) f'}{f}. \] \hfill (2.53)

One can impose that the new equation of state be constant, \( i.e. \), \( \tilde{w} \equiv \overline{P}/\overline{\rho} = \text{const.} \). Then, denoting

\[ s \equiv \frac{\tilde{w} + 1}{w + 1} = \text{const.}, \] \hfill (2.54)

the non-linear ordinary differential equation

\[ f' + \frac{(s - 1) f}{2 (\rho + \rho \Lambda)} + sf \left( 1 - f^2 \right) \frac{\rho \Lambda}{2 \rho (\rho + \rho \Lambda)} = 0 \] \hfill (2.55)

must be satisfied by the transformation function \( f \). One possible solution is \[ 157 \]

\[ f(\rho) = \sqrt{\frac{\rho + \rho \Lambda}{\gamma \rho^s + \rho \Lambda}}, \] \hfill (2.56)

where \( \gamma \) is a constant with dimensions \([\gamma] = [\rho^{1-s}]\). The energy density and pressure transform according to

\[ \overline{\rho} = \gamma \rho^s, \] \hfill (2.57)

\[ \overline{P} = \gamma [\alpha (w + 1) - 1] \rho^s, \] \hfill (2.58)

and the equation of state parameter becomes

\[ \tilde{w} = s (w + 1) - 1. \] \hfill (2.59)
For example, a de Sitter universe with \( w = -1 \) is mapped into another one for any \( \alpha \); if \( s = 1 \), a perfect fluid does not change the equation of state parameter (\( \tilde{w} = w \)). If \( s = 4/3 \), dust with \( w = 0 \) is mapped into radiation. If \( s = 3/2 \) radiation becomes a stiff fluid. This symmetry is useful to generate fluid solutions with non-linear equation of state from seed solutions with linear one, but there is always the question of how physical the result of a formal technique is. Generating new scalar field solutions for new potentials starting from a given one, which would be more interesting, proves difficult or impossible [157].

Another symmetry exists for spatially flat universes [177]: the transformation

\[
\begin{align*}
\sigma \rightarrow \tilde{\sigma} &= \sigma, \\
\sigma \rightarrow d\tilde{t} &= \sigma \frac{3(w+1)(\sigma-1)}{2\sigma} d\sigma, \\
\rho \rightarrow \tilde{\rho} &= a^{-3(w+1)(\sigma-1)} \rho,
\end{align*}
\]

leaves the form of the Einstein–Friedmann equations unchanged and the perfect fluid retains its equation of state (2.4) with the same equation of state parameter \( w \), in contrast with the previous symmetries. The symmetry with \( \sigma = 1 \), which leaves the comoving time unchanged, is the best known (e.g., [7,108,138,172]).

### 3 Two fluids with constant equations of state—comoving time

Let us consider the situation in which there are two non-interacting fluids with constant equations of state \( P_1 = w_1 \rho_1, \ P_2 = w_2 \rho_2 \) with \( w_{1,2} = \text{const.} \) We use the subscripts m, r, and s to denote dust (non-relativistic matter), radiation, and a stiff fluid, respectively. It is assumed that the two fluids are collinear and not tilted with respect to each other, i.e., that they have the same four-velocity \( u^\nu \) in their stress-energy tensors, in agreement with spatial homogeneity and isotropy. Since the spatial curvature and \( \Lambda \) are now treated as effective perfect fluids with constant equations of state, we include spatially curved FLRW universes, with or without \( \Lambda \), in the discussion.

In this section we restrict to the use of comoving time \( t \) and we look for solutions of the Einstein–Friedmann equations (2.16)–(2.18) of the form \( a = a(t) \) or \( t = t(a) \). The total energy density of the mixture is \( \rho_{\text{tot}} = \rho_1 + \rho_2 \), where the energy densities of the components scale as

\[
\rho_1(a) = \frac{\rho_1^{(0)}}{a^{3(w_1+1)}}, \quad \rho_2(a) = \frac{\rho_2^{(0)}}{a^{3(w_2+1)}},
\]

and \( \rho_i^{(0)} \) are constants. Then

\[
\dot{a} = \pm \sqrt{\frac{8\pi}{3} \frac{\sqrt{\rho_1^{(0)}}}{a^{3w_1+1}} + \sqrt{\frac{\rho_2^{(0)}}{a^{3w_2+1}}}}
\]
and

\[ I \equiv \int da a^{3w_1+1} \left[ \rho_1^{(0)} + \rho_2^{(0)} a^{3(w_1-w_2)} \right]^{-1/2} \]

\[ = \pm \sqrt{\frac{8\pi}{3}} (t-t_0), \quad (3.3) \]

where \( t_0 \) is an integration constant. This expression reduces to the single fluid equation (2.36) in the limit of a single fluid \( \rho_2^{(0)} \rightarrow 0 \).

For general values of the parameters \( w_{1,2} \), this integral can be expressed in terms of the Gauss hypergeometric function \( {}_2F_1 \) as [25]

\[ I = \frac{\alpha^q}{p+1} a^{p+1} {}_2F_1 \left( -q; \frac{p+1}{r}; \frac{p+1}{r} + 1; -\frac{\beta}{\alpha} a^r \right) \quad (3.4) \]

where \( \alpha \) and \( \beta \) are constants and

\[ p = (3w_1 + 1)/2, \quad (3.5) \]

\[ q = -1/2, \quad (3.6) \]

\[ r = 3(w_1 - w_2). \quad (3.7) \]

In general, this representation is not useful for practical purposes in cosmology. However, as several authors (Jacobs when one of the fluids is a stiff fluid [222]; McIntosh [293] and MacIntosh & Foyster [294]; and, more recently, Chen, Gibbons, Li & Yang [103]) have noticed, for special values of the parameters \( w_{1,2} \) the integral can be expressed in terms of elementary functions thanks to the truncation of the hypergeometric series or to the Chebysev theorem of integration [101,278]. This theorem states that The integral

\[ J \equiv \int dx x^p (\alpha + \beta x^r)^q \quad r \neq 0, \quad p, q, r \in \mathbb{Q} \quad (3.8) \]

admits a representation in terms of elementary functions if and only if at least one of \( \frac{p+1}{r}, q, \frac{p+1}{r} + q \) is an integer.

In our situation, we have

\[ \frac{p+1}{r} = \frac{w_1 + 1}{2(w_1 - w_2)}, \quad (3.9) \]

\[ q = -\frac{1}{2}, \quad (3.10) \]

\[ \frac{p+1}{r} + q = \frac{w_2 + 1}{2(w_1 - w_2)}. \quad (3.11) \]

The authors of [103,222,293,294] have studied systematically the simple integrability cases. To proceed we need \( w_1 \) and \( w_1 \) to be rational numbers, which always happens in cosmology, or else one can use the rational approximation to the equation
of state parameter $w$. In practice, we only need to assume that one of the parameters $w_i \in \mathbb{Q}$; since we impose one of the conditions (3.12), (3.19) below in our search for integrability, the second $w$-parameter is automatically rational if the first one is.

### 3.1 First condition: $\frac{p+1}{r} = n \in \mathbb{Z}$

Imposing $\frac{p+1}{r} = n \in \mathbb{Z}$ in order to find integrability cases in terms of elementary functions [103, 293], one obtains

\[(2n - 1) w_1 - 2nw_2 = 1. \quad (3.12)\]

We can fix the first fluid (i.e., fix $w_1 \in \mathbb{Q}$) and look for physically interesting values of the equation of state parameter $w_2$ of the second fluid, as $n \in \mathbb{Z}$ runs.

#### 3.1.1 Dust plus a second (real or effective) fluid

Suppose that the first fluid is a dust, $w_1 = 0$, and we look for a second fluid that gives “simple” integrability. Then,

\[-\frac{1}{2} \leq w_2 = -\frac{1}{2n} \leq \frac{1}{2} \quad (3.13)\]

and, as $n = -\infty, \ldots, -3, -2, -1, 1, 2, 3, \ldots, +\infty$, we obtain the pairs

\[
(w_1, w_2) = (0, 0) \quad \text{(single dust fluid)},
\]

\[
\ldots
\]

\[
(w_1, w_2) = (0, 1/6),
\]

\[
(w_1, w_2) = (0, 1/4),
\]

\[
(w_1, w_2) = (0, 1/2),
\]

\[
(w_1, w_2) = (0, -1/2),
\]

\[
(w_1, w_2) = (0, -1/4),
\]

\[
(w_1, w_2) = (0, -1/6),
\]

\[
\ldots
\]

\[
(w_1, w_2) = (0, 0) \quad \text{(again, a single dust fluid}).
\]

Both limits $n \to \pm \infty$ produce a second dust, i.e., there is a single dust fluid in the FLRW universe. No value of $w_2$ particularly interesting from the physical point of view is obtained; although quintessence (but not phantom) fluids are obtained this way, their equation of state parameter $w_2$ is not close to $-1$, as required by current observations [3].
3.1.2 Radiation plus a second (real or effective) fluid

Now suppose that the first fluid is radiation, $w_1 = 1/3$, and we look for a second fluid giving integrability in terms of elementary functions. Then,

$$-\frac{1}{3} \leq w_2 = \frac{n - 2}{3n} \leq 1 \quad (3.15)$$

and, letting $n = -\infty, \ldots, -3, -2, -1, 1, 2, 3, \ldots, +\infty$, we obtain the pairs

\[
(w_1, w_2) = \left(\frac{1}{3}, \frac{1}{3}\right) \quad \text{(a single radiation fluid)},
\]

\[
\ldots
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, \frac{5}{9}\right),
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, \frac{2}{3}\right),
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, 1\right) \quad \text{(radiation plus stiff fluid)},
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, -\frac{1}{3}\right) \quad \text{(radiation plus spatial curvature)},
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, 0\right) \quad \text{(radiation plus dust)},
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, \frac{1}{9}\right),
\]

\[
\ldots
\]

\[
(w_1, w_2) = \left(\frac{1}{3}, \frac{1}{3}\right) \quad \text{(again, a single radiation fluid)}.
\]

3.1.3 Cosmological constant plus a second (real or effective) fluid

In this case $w_1 = -1$, $n$ disappears from Eq. (3.12), which gives $w_2 = -1$, producing only a cosmological constant with no other fluids.

3.1.4 Stiff matter plus a second (real or effective) fluid

In this case $w_1 = 1$,

$$0 \leq w_2 = \frac{n - 1}{n} \leq 2, \quad (3.17)$$

and we have the pairs

\[
(w_1, w_2) = (1, 1) \quad \text{(single stiff fluid)},
\]

\[
\ldots
\]

\[
(w_1, w_2) = (1, \frac{4}{3}),
\]

\[
(w_1, w_2) = (1, \frac{3}{2}),
\]

\[
(w_1, w_2) = (1, 2),
\]

\[
(w_1, w_2) = (1, 0) \quad \text{(stiff matter plus dust)},
\]

\[
(w_1, w_2) = (1, \frac{1}{2}),
\]

\[
(w_1, w_2) = (1, \frac{2}{3}),
\]

\[
\ldots
\]

\[
(w_1, w_2) = (1, 1) \quad \text{(again, a single stiff fluid)}.
\]
3.2 Second condition: \( \frac{p+1}{r} + q = n \in \mathbb{Z} \)

Let us explore now the second possibility (3.11) of expressing the integral \( I \) in terms of elementary functions. Imposing \( \frac{p+1}{r} + q = n \in \mathbb{Z} \) [103], one obtains

\[
w_2 = \frac{2nw_1 - 1}{2n + 1}.
\]

(3.19)

Solving the Friedmann equation with an hypergeometric function, McIntosh [293] and later McIntosh and Foyster [294] noted that the latter truncates and simplifies to elementary functions when one of its arguments is an integer (which is the situation when the Chebysev theorem arises, and the route followed by the authors of [103], who apparently were unaware of Refs. [293,294]). This happens when [293,294] (cf. Eqs. (22) and (24) of Ref. [293])

\[
\frac{\gamma_1}{\gamma_2} = 1 - \frac{1}{m},
\]

(3.20)

where \( \gamma_{1,2} \equiv w_{1,2} + 1 \) and \( m \in \mathbb{Z} \). This relation is equivalent to

\[
mw_1 + (1 - m)w_2 = -1.
\]

(3.21)

Since \( m \in \mathbb{Z} \), setting \( m = 2n \in \mathbb{Z} \) gives back Eq. (3.19).

Let us fix the first fluid and look for a second fluid that guarantees “simple integrability”.

3.2.1 Dust plus a second (real or effective) fluid

If the first fluid is a dust, \( w_1 = 0 \), we have

\[-1 \leq w_2 = -\frac{1}{2n + 1} \leq 1.\]

(3.22)

and the pairs

\[
(w_1, w_2) = (0, 0) \quad \text{(a single dust fluid),}
\]

\[
(w_1, w_2) = (0, 1/5),
\]

\[
(w_1, w_2) = (0, 1/3) \quad \text{(dust plus radiation),}
\]

\[
(w_1, w_2) = (0, 1) \quad \text{(dust plus stiff fluid),}
\]

\[
(w_1, w_2) = (0, -1) \quad \text{(dust plus } \Lambda),
\]

(3.23)

\[
(w_1, w_2) = (0, -1/3) \quad \text{(dust plus spatial curvature),}
\]

\[
(w_1, w_2) = (0, -1/5),
\]

\[
(w_1, w_2) = (0, -1/7),
\]

\[
\ldots
\]

\[
(w_1, w_2) = (0, 0) \quad \text{(again, a single dust fluid).}
\]
3.2.2 Radiation plus a second (real or effective) fluid

If we start from radiation, $w_1 = 1/3$, we obtain

$$-1 \leq w_2 = \frac{2n - 3}{3(2n + 1)} \leq \frac{5}{3}$$  \hspace{1cm} (3.24)

and the pairs

$$(w_1, w_2) = (1/3, 1/3) \quad \text{(single radiation fluid)},$$

$$\ldots$$

$$(w_1, w_2) = (1/3, 3/5),$$

$$(w_1, w_2) = (1/3, 7/9),$$

$$(w_1, w_2) = (1/3, 5/3),$$

$$(w_1, w_2) = (1/3, 1) \quad \text{(radiation plus stiff fluid)},$$

$$(w_1, w_2) = (1/3, -1/9),$$

$$(w_1, w_2) = (1/3, 1/15),$$

$$(w_1, w_2) = (1/3, 1/7),$$

$$\ldots$$

$$(w_1, w_2) = (1/3, 1/3) \quad \text{(again, a single radiation fluid)}.$$

3.2.3 $\Lambda$ plus a second (real or effective) fluid

Again, setting $w_1 = -1$ makes $n$ disappear from Eq. (3.19) and produces only $w_2 = -1$: there is only a cosmological constant in a spatially flat FLRW universe.

3.2.4 Stiff matter plus a second (real or effective) fluid

Setting $w_1 = 1$ (the equation of state parameter of a stiff fluid) yields

$$-1 \leq w_2 = \frac{2n - 1}{2n + 1} \leq 3$$  \hspace{1cm} (3.26)

and the pairs

$$(w_1, w_2) = (1, 1) \quad \text{(single stiff fluid)},$$

$$\ldots$$

$$(w_1, w_2) = (1, 7/5),$$

$$(w_1, w_2) = (1, 5/3),$$

$$(w_1, w_2) = (1, 3),$$

$$(w_1, w_2) = (1, -1) \quad \text{(stiff fluid plus $\Lambda$)},$$

$$(w_1, w_2) = (1, 1/3) \quad \text{(stiff fluid plus radiation)},$$

$$(w_1, w_2) = (1, 3/5),$$

$$(n, w_2) = (1, 5/7),$$

$$\ldots$$

$$(w_1, w_2) = (1, 1) \quad \text{(again, a single stiff fluid)}.$$
3.3 Summary

In all cases, when \( n \to \pm \infty \), the solutions degenerate into a single fluid solution. If both conditions (3.12) and (3.19) hold simultaneously, one necessarily has \( w_1 = w_2 = -1 \), which is the case of a single cosmological constant with \( K = 0 \) and no real fluids.

Based on the tables of values \((w_1, w_2)\) obtained, in which we always find \( w_2 \geq -1 \), one concludes that, when one the fluids is phantom and the other is dust, radiation, or stiff matter, there are no analytical solutions in the form \( t = t(a) \) or \( a = a(t) \) expressed in terms of elementary functions (there can still be simple analytical solutions in parametric form, or solutions given by a hypergeometric function or elliptic integral, or solutions with a time-dependent equation of state). The same conclusion applies to quintessence fluids with realistic values of the equation of state parameter, other than \( \Lambda \). In the tables reported, “real” quintessence fluids (other than \( \Lambda \)) have equation of state parameter \( w_2 \) far from \(-1\), while current observations give \( w \simeq -1 \) [3].

Looking at the previous tables, the situations involving physically interesting fluids in which the two-fluid solution can be expressed in terms of elementary functions are:

- dust plus spatial curvature, \( \Lambda = 0 \);
- radiation plus spatial curvature, \( \Lambda = 0 \);
- \( K = 0, \Lambda = 0 \), dust plus radiation;
- \( K = 0 \), dust plus \( \Lambda \);
- \( K = 0, \Lambda = 0 \), dust plus stiff matter;
- \( K = 0, \Lambda = 0 \), radiation plus stiff matter.

Both single fluid solutions for \( K \neq 0, \Lambda = 0 \) (studied in [103] in \( D \) spatial dimensions) and \( K = 0 \) FLRW universes with a single fluid and \( \Lambda \neq 0 \) can be obtained by regarding them as two-fluid solutions in a fictitious spatially flat universe, provided that they fall within the list of cases above.

3.4 Exotic fluids

In addition to the classic dust, radiation, non-diluting \( \Lambda \) (with \( \dot{\rho}_\Lambda = 0 \)), and hypothetical phantom dark energy which concentrates with the cosmic expansion (i.e., \( \dot{\rho} = -3(w + 1)\rho > 0 \) for \( w < -1 \)), other exotic equations of state are of physical interest.

A stiff fluid with \( w = 1 \) is believed to be appropriate to describe matter at high (nuclear) densities and it is used in astrophysics to model dense nuclear matter in the core of neutron stars. It is reasonable that, as the universe cooled from high temperature and energy scales, it underwent a period at lower temperatures and nuclear densities described by the stiff equation of state.

A stiff fluid also corresponds to a free scalar field \( \phi \) minimally coupled to the Ricci curvature. By setting \( V(\phi) = 0 \) in Eqs. (2.24) and (2.25) it is \( P_\phi = \rho_\phi \) and the scalar field behaves as a stiff perfect fluid. This regime, called “kination” in early universe literature, is achieved exactly when the potential \( V \) is absent and approximately when \( \phi^2 \ll V(\phi) \). Then the Klein–Gordon equation (2.22) admits the first integral.
\[ \dot{\phi} = \frac{C_0}{a^3}, \quad (3.28) \]

or

\[ \phi,\eta = \frac{C_0}{a^2}, \quad (3.29) \]

where \( C_0 \) is an integration constant. Authors from the 1960s and 1970s, who were not particularly interested in scalar fields,\(^4\) did not regard a stiff fluid as the realization of a scalar field and did not integrate Eq. (3.28) or Eq. (3.29). We will report the explicit form of \( \phi \) in this kination regime, when it is given explicitly in terms of elementary functions. A stiff fluid can also mimic anisotropy in the universe \([220,222,293,298,367]\).

Finally, certain string-inspired effective gravitational field theories of the primordial universe \([42–44,285,286]\) contain a massless Kalb-Ramond axion of gravitational nature which is equivalent to an effective fluid with a stiff equation of state, and other stringy axions are possible. These fields are linked to matter-antimatter asymmetry in the universe after inflation and to axionic dark matter. It is speculated that a stiff matter era precedes inflation in these theories \([286]\).

The value \(-1/3\) of the equation of state parameter is also motivated in cosmology: a frustrated cosmic string network produces an effective fluid with this equation of state \([376,400]\).

A fluid with equation of state parameter \( w = -2/3 \) describes domain walls and is interesting in cosmology \([48,72,133]\) but, unfortunately, it does not appear in the lists of integrable cases found above. In general, a network of non-intercommuting \( n \)-dimensional topological defects produces a fluid with effective equation of state parameter \( w = -n/3 \) \([376,400]\).

The other values of the equation of state parameter \( w \) for which one has integrability in terms of elementary functions are not particularly interesting from the physical point of view, except perhaps as toy models. Among these, solutions involving a \( w = 2/3 \) fluid were given by Vajk \([395]\).

### 4 Two fluids with constant equations of state—parametric solutions with conformal time

Several exact solutions, for both single- and two-fluid solutions, can be obtained in parametric form using the conformal time \( \eta \) defined by \( dt \equiv a d\eta \) as a parameter. Two-fluid solutions in this form are perhaps not as widely known as single-fluid solutions (which are instead reported in the pedagogical literature \([94,165,168,243,269,404]\)).

In terms of conformal time, the Friedmann equation for a FLRW universe with a single matter fluid (satisfying \( P = w\rho, \ w = \text{const.} \)), cosmological constant \( \Lambda \), and spatial curvature reads

---

\(^4\) Interest in scalar field cosmology arose with inflationary scenarios in the 1980s \([240,262,267,279,301]\) and, again, with the discovery of the present acceleration of the universe in 1998 \([330,331,340]\).
\[(a, \eta)^2 = \frac{8\pi}{3} \frac{\rho^{(0)}}{a^{3w-1}} + \frac{\Lambda a^4}{3} - Ka^2, \quad (4.1)\]

which gives, along the lines of the discussion already made using comoving time,

\[
\int \frac{da}{\sqrt{\frac{8\pi}{3} \frac{\rho^{(0)}}{a^{3w-1}} + \frac{\Lambda a^4}{3} - Ka^2}} = \pm (\eta - \eta_0), \quad (4.2)
\]

where \(\eta_0\) is an integration constant. From now on, this section follows Refs. [103, 294] and we assume that the equation of state parameter \(w\) is rational.

### 4.1 \(K = 0, \Lambda \neq 0\), any \(w \in \mathbb{Q}\)

For this combination of parameters, the Einstein–Friedmann equations (2.16)–(2.18) can always be integrated in terms of simple functions [103, 208, 293, 395]. Let us assume that \(w \neq -1, -1/3\), otherwise we fall into one of the cases previously discussed. The integral on the left hand side of Eq. (4.2) has the form

\[
\sqrt{3} \int da \, a^{-2} \left[ \frac{8\pi \rho^{(0)}}{a^{3(w+1)}} + \Lambda \right]^{-1/2}, \quad (4.3)
\]

\textit{i.e., the form (3.8)} with

\[
p = -2, \quad r = -3(w + 1) \neq 0, \quad q = -\frac{1}{2}, \quad (4.4)
\]

which are all rational if \(w\) is. The Chebysev theorem applies if one of

\[
\frac{p + 1}{r} = \frac{1}{3(w + 1)}, \quad \frac{p + 1}{r} + q = -\frac{3w + 1}{6(w + 1)} \quad (4.5)
\]

is an integer. These conditions correspond, respectively, to

\[
w = -1 + \frac{1}{3n}, \quad (4.6)
\]

\[
w = -1 + \frac{2}{3(2m + 1)}, \quad (4.7)
\]

with \(n, m \in \mathbb{Z}\). These two conditions are mutually exclusive and they both allow to get arbitrarily close to the phantom divide \(w = -1\) if \(|n|\) or \(|m|\) are sufficiently large, spanning both quintessence \((-1 \leq w < -1/3)\) and phantom \((w < -1)\) equations of state near the \(w = -1\) limit.

Imposing the first condition (4.6) for integrability in terms of elementary functions, as \(n\) spans the values \(n = -\infty, \ldots, -3, -2, -1, 1, 2, 3, \ldots, +\infty\), one obtains the corresponding equation of state parameters.
\[ w = -1, \ldots, -\frac{10}{9}, -\frac{7}{6}, -\frac{4}{3}, -\frac{2}{3}, -\frac{5}{6}, -\frac{8}{9}, \ldots, -1. \quad (4.8) \]

Imposing instead the second condition \((4.7)\), for \(m = -\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, +\infty\) one obtains

\[ w = -1, \ldots, -\frac{17}{15}, -\frac{11}{9}, -\frac{5}{3}, -\frac{1}{3}, -\frac{7}{9}, -\frac{13}{15}, -\frac{19}{21}, \ldots, -1. \quad (4.9) \]

The degenerate case \(w = -1\) reproduces the empty, spatially flat, de Sitter universe, while \(w = -1/3\) gives a spatially curved universe filled with a string gas and with \(\Lambda\) already discussed. None of the other values of \(w\) correspond to dust, radiation, or stiff matter, but they cover quintessence and phantom equations of state. There is no general formula expressing the solution that is valid for all values of \(w\) corresponding to integrability in terms of elementary functions.

### 4.2 Any \(K\), \(\Lambda = 0\), any \(w \in \mathbb{Q}\)

This case gives rise to more physically interesting situations. When the fluid is radiation or dust, this is a standard solution found in all cosmology textbooks [94,165,243,261,404,411]. For \(K = 0\), the solution is elementary and given by Eq. (2.11). When \(K = \pm 1\) and there is only radiation, one can eliminate the parameter \(\eta\) to obtain \(a(t)\).

For \(K = \pm 1\) and dust, one can express both \(a\) and \(t\) as simple functions of \(\eta\), but eliminating this parameter can only be done to obtain a relation \(t(a)\) that cannot be inverted explicitly to obtain \(a(t)\).

For \(K = \pm 1\) and \(w \neq 0, 1/3\), in general, one can compute \(a(\eta)\) explicitly in terms of elementary functions, while \(t(\eta)\) is reduced to a quadrature and written as an integral that may not have an explicit expression (see Sect. 5).

Explicitly, for any \(K\), \(\Lambda = 0\), and \(w \in \mathbb{Q}\), it is possible to express \(a(\eta)\) in terms of elementary functions. Various solutions have been proposed by many authors over the years (see the next sections). The general solution is given by [103]

\[
\pm \left( \frac{3w + 1}{2} \right) (\eta - \eta_0) = \begin{cases} 
- \tan^{-1} \left[ f(a) \right] & \text{if } K = +1, \\
\frac{1}{f(a)} & \text{if } K = 0, \\
\tanh^{-1} \left| \frac{f(a) + 1}{f(a) - 1} \right| & \text{if } K = -1,
\end{cases}
\]

where

\[
f(a) = \sqrt{\frac{8\pi}{3} \rho^{(0)} a^{-3w+1} - K}. \quad (4.11)
\]
4.2.1 $K = 0, \Lambda = 0, w$-fluid

For $K = 0, \Lambda = 0$, Eq. (4.10) yields [103,208,395]

\[
\begin{align*}
  a(\eta) &= \left(\frac{2\pi}{3} \rho(0)\right)^{1 \over 3w+1} \left[(3w + 1) (\eta - \eta_0)\right]^{2 \over 3w+1},
  \\
  t(\eta) &= \left(\frac{3w+1}{3w+1-1}\right)^{1 \over 3w+1} \left(\frac{2\pi}{3} \rho(0)\right)^{1 \over 3w+1} (\eta - \eta_0) \times \left[3w+1\right]^{2 \over 3w+1} + t_0.
\end{align*}
\]  

(4.12)

The initial condition at $t = t_0$ (corresponding to $\eta = \eta_0$) is $a(0) = 0$; this universe expands forever and the fluid redshifts away if $w > -1$.

4.2.2 $K = +1, \Lambda = 0, w$-fluid

In this case Eq. (4.10) yields

\[
f(a) = \pm \tan \left[\left(\frac{3w + 1}{2}\right) (\eta - \eta_0)\right].
\]  

(4.13)

Using the expression of $f(a)$ and squaring yields

\[
\frac{8\pi}{3} \rho(0) a^{-(3w+1)} = \cos^{-2} \left[\left(\frac{3w + 1}{2}\right) (\eta - \eta_0)\right]
\]  

(4.14)

and finally [103,208,395]

\[
\begin{align*}
  a(\eta) &= \left(\frac{8\pi}{3} \rho(0)\right)^{1 \over 3w+1} \left[\cos \left[\frac{3w+1}{2} (\eta - \eta_0)\right]\right]^{2 \over 3w+1},
  \\
  t(\eta) &= \int a(\eta) d\eta.
\end{align*}
\]

If the initial condition $a(\eta = 0) = 0$ is imposed, it must be $\eta_0 = \pi/(3w + 1)$. The special cases of this solution for dust and radiation were found by Friedmann in 1922 [183] and Tolman in 1931 [390], respectively.

4.2.3 $K = -1, \Lambda = 0, w$-fluid

Two-(effective) fluid solutions for $K \Lambda \neq 0$ for dust and for radiation (separately) were reported in terms of elliptic functions by Lemaître [254], Edwards [161] and Kharbedya [232]. The solution for $K = -1, \Lambda = 0$ and any $w$-fluid was reported by Tauber [389]. In this case, the exponentiation of Eq. (4.10) gives

\[
\frac{|f(a) + 1|}{|f(a) - 1|} = e^{\pm (3w+1)(\eta - \eta_0)}
\]  

(4.15)
and it is straightforward to obtain

\[
    f(a) = -\frac{1 + e^{\pm(3w+1)(\eta-\eta_0)}}{1 - e^{\pm(3w+1)(\eta-\eta_0)}} = -\coth \left[ \pm \frac{(3w + 1)}{2} (\eta - \eta_0) \right].
\]

(4.16)

Using the expression of \( f(a) \) and squaring yields \([103,395]\)

\[
\begin{cases}
    a(\eta) = \left( \frac{8\pi}{3} \rho(0) \right)^{1/3w+1} \left\{ \sinh \left[ \frac{3w+1}{2} (\eta - \eta_0) \right] \right\}^{2/3w+1}, \\
    t(\eta) = \int a(\eta) d\eta.
\end{cases}
\]

(4.17)

If \( w > -1/3 \), the initial condition is \( a(\eta_0) = 0 \). If \( w = -1/3 \), we have an empty universe with hyperbolic 3-sections and zero cosmological constant: this is the Milne universe, which is nothing but Minkowski space sliced with a hyperbolic foliation (e.g., \([165,301,404]\)).

The most well known two-fluid solution in parametric form is the one for dust plus radiation and \( K = 0 \) (see Eq. (5.18)).

### 5 Explicit solutions

In the following, we present the known explicit analytical solutions of the Einstein–Friedmann equations (2.16)–(2.18) expressible in terms of elementary functions, for two real or effective fluids with constant, linear, barotropic equation of state \( P = w\rho, \ w = \text{const.} \) Not all the situations, listed in the previous two sections, in which integration in finite terms by means of elementary functions is possible are physically motivated. Therefore, we limit ourselves to the most physically significant values of the equation of state parameters for the real fluids (dust, radiation, or stiff fluid). We first provide the solutions using comoving time \( t(a) \) or, whenever possible, their inverse \( a(t) \). Then, we provide the solution in the parametric form

\[
\begin{cases}
    a = a(\eta), \\
    t = t(\eta),
\end{cases}
\]

(5.1)

using the conformal time \( \eta \) as the parameter, which may be useful in certain applications where conformal time is preferred for computational convenience (for example, to solve the equations for scalar or tensor perturbations in slow-roll inflation \([240,262,267,279,301]\)). We also report initial conditions, asymptotics, and single-fluid limits for these solutions.
5.1 Dust, $K = 0$, $\Lambda = 0$

This well known single fluid solution is

$$a(t) = \left(6\pi \rho_m^{(0)}\right)^{1/3} (t - t_0)^{2/3}$$

(5.2)

$$\rho(a) = \frac{\rho_m^{(0)}}{a^3} = \frac{1}{6\pi (t - t_0)^2},$$

(5.3)

or, in the less commonly encountered parametric form,

$$\begin{aligned}
  a(\eta) &= \frac{2\pi}{3} \rho_m^{(0)} (\eta - \eta_0)^2, \\
  t(\eta) &= \frac{2\pi}{9} \rho_m^{(0)} (\eta - \eta_0)^3 + t_0.
\end{aligned}$$

(5.4)

The initial condition is a Big Bang $a = 0$ at $t = t_0$, corresponding to $\eta = \eta_0$.

5.2 Dust plus spatial curvature, $\Lambda = 0$

In its parametric form using conformal time, the solution is well known (Tauber [389] and Gilman [189] gave the solution for $K = 0, \pm 1$) and is found in standard cosmology textbooks, e.g., [94,261,404]. It is rarer to find it expressed in terms of comoving time, in which case it reads [13,395]

$$t(a) = -\sqrt{\frac{8\pi}{3} \rho_m^{(0)} a - a^2 + \frac{8\pi}{3} \rho_m^{(0)} \sin^{-1}\left(\sqrt{\frac{3a}{8\pi \rho_m^{(0)}}}\right)} + t_0$$

(5.5)

for $K = +1$ and

$$t(a) = \sqrt{\frac{8\pi}{3} \rho_m^{(0)} a + a^2 - \frac{8\pi}{3} \rho_m^{(0)} \sinh^{-1}\left(\sqrt{\frac{3a}{8\pi \rho_m^{(0)}}}\right)} + t_0$$

(5.6)

for $K = -1$ [13,395]. Both solutions satisfy the Big Bang initial condition $a(t_0) = 0$.

The parametric forms of these solutions are [94,261,404,411]

$$\begin{aligned}
  a(\eta) &= \frac{4\pi}{3} \rho_m^{(0)} (1 - \cos \eta), \\
  t(\eta) &= \frac{4\pi}{3} \rho_m^{(0)} (\eta - \sin \eta) + t_0,
\end{aligned}$$

(5.7)

for $K = +1$ and

$$\begin{aligned}
  a(\eta) &= \frac{4\pi}{3} \rho_m^{(0)} (\cosh \eta - 1), \\
  t(\eta) &= \frac{4\pi}{3} \rho_m^{(0)} (\sinh \eta - \eta) + t_0,
\end{aligned}$$

(5.8)
for \( K = -1 \) where, in both cases, \( t = t_0 \) corresponds to \( \eta = 0 \) and \( a = 0 \) and

\[
\rho_{m}(a) = \frac{\rho_{m}^{(0)}}{a^3}, \quad \rho_{K}(a) = \frac{-3K}{8\pi a^2}. \tag{5.9}
\]

The parametric forms make it clear that the solution for \( K = +1 \) reaches a maximum size \( a_{\text{max}} = \frac{8\pi}{3} \rho_{m}^{(0)} \) at \( \eta = \pi \) (or \( t = \frac{4\pi^2}{3} \rho_{m}^{(0)} \)), while the \( K = -1 \) universe expands forever becoming curvature-dominated.

### 5.3 Radiation, \( K = 0, \Lambda = 0 \)

This is another classic textbook solution

\[
a(t) = \left( \frac{32\pi}{3} \rho_{r}^{(0)} \right)^{1/4} \sqrt{t - t_0}, \tag{5.10}
\]

\[
\rho_{r}(a) = \frac{\rho_{r}^{(0)}}{a^4}, \tag{5.11}
\]

or, in parametric form,

\[
\begin{cases}
    a(\eta) = \sqrt{\frac{8\pi}{3} \rho_{r}^{(0)}} (\eta - \eta_0), \\
    t(\eta) = \sqrt{\frac{2\pi}{3} \rho_{r}^{(0)}} (\eta - \eta_0)^2 + t_0,
\end{cases} \tag{5.12}
\]

where \( t_0 \) is an integration constant. The Big Bang initial condition \( a = 0 \) at \( t = t_0 \) (or \( \eta = \eta_0 \)) has been imposed and the universe expands forever.

### 5.4 Radiation plus spatial curvature, \( \Lambda = 0 \)

This solution is well-known and found in standard cosmology textbooks, e.g., [94, 261,404] and as a special case of more complicated solutions [13]. For \( K = +1 \), the scale factor is

\[
a(t) = \sqrt{\frac{8\pi}{3} \rho_{r}^{(0)}} - (t - t_0)^2 \tag{5.13}
\]

or, in parametric form,

\[
\begin{cases}
    a(\eta) = \sqrt{\frac{8\pi}{3} \rho_{r}^{(0)}} \sin \eta, \\
    t(\eta) = t_0 - \sqrt{\frac{8\pi}{3} \rho_{r}^{(0)}} \cos \eta.
\end{cases} \tag{5.14}
\]
This solution has a Big Bang at \( t = t_0 - \sqrt{8\pi \rho_r^{(0)}/3} \) (or \( \eta = 0 \)) and a Big Crunch at \( t = t_0 + \sqrt{8\pi \rho_r^{(0)}/3} \) (or \( \eta = \pi \)).

For \( K = -1 \), the scale factor is

\[
a(t) = \sqrt{(t - t_0)^2 - \frac{8\pi}{3} \rho_r^{(0)}} \tag{5.15}
\]

or, in parametric form,

\[
\begin{align*}
a(\eta) &= \sqrt{\frac{8\pi}{3} \rho_r^{(0)}} \sinh \eta, \\
t(\eta) &= \sqrt{\frac{8\pi}{3} \rho_r^{(0)}} \cosh \eta + t_0.
\end{align*}
\tag{5.16}
\]

There is a Big Bang at \( \eta = 0 \), corresponding to \( t = t_0 + \sqrt{8\pi \rho_r^{(0)}/3} \). At late times \( t \to +\infty \), the solution is approximately linear, \( a(t) \sim t \).

### 5.5 \( K = 0, \Lambda = 0 \), dust plus radiation

Solutions describing dust plus radiation are, no doubt, some of the most well-motivated because of the need to describe the simultaneous presence of non-relativistic matter and cosmic background radiation or, more in general, neutrinos or ultra-relativistic particle species. FLRW universes containing dust and radiation have been studied in the years immediately following the discovery of the cosmic microwave background [329], beginning with Alpher & Herman [16] and continuing with the works of Chernin [104], Jacobs [221], Cohen [126], Roeder [346], McIntosh [290], Vajk [395], Sapar [353], Sistero [371], May [284], Coquereaux & Grossman [125], and Dabrowski & Stelmach [139]. However this exact solution for two non-interacting fluids cannot reproduce the complicated microphysics of ultrarelativistic particle species decaying, electrons combining with protons, etc. When an accurate description is needed, for example to study imprints left in the cosmic microwave background to probe the thermal history of the universe, the physics must inserted in numerical calculations in detailed scenarios [10, 117, 118, 216, 326, 386, 424].

The solution with non-interacting dust and radiation and \( \Lambda = 0 \) was found by Tolman for \( K = +1 \) [391] and by Chernin [104] and McIntosh [291] for any \( K \).

The spatially flat, \( \Lambda = 0 \), radiation plus dust solution due to Jacobs [221] (see also [126, 395]) is probably the most well known solution with two real (as opposed to effective) fluids [103]. In spite of its importance for the transition from the radiation to the dust era, it rarely appears [165] in modern textbooks. Setting \( w_1 = 0, w_2 = 1/3 \), the relevant integral (3.3) becomes
\[ \int \frac{da}{\sqrt{\rho_m^{(0)} a + \rho_r^{(0)}}} = \frac{2}{3} \left( \rho_m^{(0)} a - 2 \rho_r^{(0)} \right) \sqrt{\rho_m^{(0)} a + \rho_r^{(0)}} \]

\[ = \sqrt{\frac{8\pi}{3}} \left( t - t_0 \right). \] (5.17)

This \( K = 0 \) solution was given by Jacobs in 1967 [221] (see also [395]) as

\[ t(a) = A \left( \rho_m^{(0)} a - 2 \rho_r^{(0)} \right) \sqrt{\rho_m^{(0)} a + \rho_r^{(0)}} + t_0, \] (5.18)

which coincides with (5.17) for \( A = \left( \sqrt{6\pi} (\rho_m^{(0)})^2 \right)^{-1} \).

In parametric form, the solution is\(^5 [13,302,395]\)

\[ \begin{cases} 
    a(\eta) = \frac{2\pi}{3} \rho_m^{(0)} \eta^2 - \frac{\rho_r^{(0)}}{\rho_m^{(0)}}, \\
    t(\eta) = \frac{2\pi}{3} \rho_m^{(0)} \eta^3 - \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \eta,
\end{cases} \] (5.19)

with the initial condition \( a(t = 0) = 0 \). This universe expands forever. The radiation fluid dominates at early times, when \( a(t) \approx \sqrt{t - t_0} \), while the dust dominates at late times, with \( a(t) \approx (t - t_0)^{2/3} \).

Let us consider the single-fluid limits. The limit to dust is obtained trivially by setting \( \rho_r^{(0)} = 0 \) in Eq. (5.17), obtaining the well known power-law \( a(t) = a_0 \left( t - t_0 \right)^{2/3}, \) i.e., Eq. (2.11) with \( w = 0 \).

The radiation-only limit is not as obvious: letting \( \rho_m^{(0)} \to 0 \) in Eq. (5.17) (or in the parametric form of this solution) does not produce a meaningful result. Instead, one has to take the limit \( \rho_m^{(0)} \to 0 \) in the integral (3.3), which reduces to

\[ \int da a \sqrt{\rho_r^{(0)}} = \sqrt{\frac{8\pi}{3}} \left( t - t_0 \right), \] (5.20)

giving \( a(t) = a_0 \sqrt{t - t_0} \) (or Eq. (2.11) with \( w = 1/3 \)). In general, the single-fluid limit of a two- or three- (real or effective) fluid solution in which radiation survives is more problematic than the analogous limit in which a non-radiative fluid remains, as will be seen in the following.

\(^5\) The authors of Ref. [13,302] were apparently unaware of Vajk’s paper [13,302,395].
The parametric form of this spatially flat universe was rediscovered by Barrow & Saich \cite{32} as a solution with radiation plus a scalar field in the potential\footnote{Barrow & Saich do not seem to have made the connection between their solution and one of the exact integrability cases in the list (3.16) of the Einstein–Friedmann equations—the results of \cite{222,293,294} discussed in our Sect. 3 are not mentioned in \cite{32}.}

\[ V(\phi) = \frac{V_0 e^{2\phi_0}}{(e^{2\phi_0} + V_1)^2} \]  

(5.21)

with \( V_0, V_1, \phi_0 \) constants, but imposing that the equation of state of the effective fluid equivalent of the scalar field be constant,

\[ \frac{\dot{\phi}^2}{2} = \alpha V(\phi), \]  

(5.22)

which gives the effective equation of state parameter \( w_\phi = \frac{\alpha - 1}{\alpha + 1} \) for the scalar field fluid. As expected from the list (3.16), the case \( w_\phi = 0 \) equivalent to \( \alpha = 1 \) is integrable and gives the exact solution (5.19) for the potential (5.21) \cite{32}. The scalar field (not reported in \cite{32}) can be found by noting that the Klein–Gordon equation admits the first integral \cite{32}

\[ \dot{\phi} = \frac{\phi_0}{a^{\alpha + 1}} \]  

(5.23)

which, in our case, yields \( \rho_\phi = \phi_0^2 a^{-3} \), \( d\phi/d\eta = \phi_0/\sqrt{a} \) and

\[ \phi(\eta) = \phi_0 \sqrt{\frac{3}{2\pi \rho_m^{(0)}}} \text{arccosh} \left( \sqrt{\frac{2\pi}{3\rho_r^{(0)}} \rho_m^{(0)}} \eta \right) + \phi_1, \]  

(5.24)

where \( \phi_0, \phi_1 \) are integration constants.

The \( K = 0 \) dust-plus-radiation universe was generalized by Jacobs \cite{222} to anisotropic Bianchi I models, but it is given in terms of elliptic integrals. The dust-plus-radiation solution describing \( K = \pm 1 \) FLRW universes is given by Eqs. (6.11) and (6.13) below.

### 5.6 \( K = 0, \text{dust plus } \Lambda \)

If \( \Lambda > 0 \), this is the standard \( \Lambda \text{CDM} \) model (with \( \Lambda \) as the simplest form of dark energy) and the analytical solution of the Einstein–Friedmann equations (2.16)–(2.18) is well-known (e.g., \cite{13,100,103}).
\[
a(t) = \frac{a_0}{2^{2/3}} \left[ \left( 1 + \sqrt{1 + \frac{3M}{a_0^3\Lambda}} \right) e^{\frac{\sqrt{3\Lambda}}{2} t} + \left( 1 - \sqrt{1 + \frac{3M}{a_0^3\Lambda}} \right) e^{-\frac{\sqrt{3\Lambda}}{2} t} \right]^{2/3},
\]
(5.25)

where \(a_0 = a(t_0)\) at the present time \(t_0\),

\[
M = \Omega_{m0}H_0^2a_0^3 = \frac{8\pi}{3} \rho_0^{(0)},
\]
(5.26)

and

\[
\Omega_m(t) \equiv \frac{\rho_m(t)}{\rho_c(t)} = \frac{8\pi \rho_m(t)}{3H^2(t)}
\]
(5.27)

is the dimensionless dust density parameter. At late times the cosmological constant with constant energy density \(\Lambda/3\) dominates over the dust that redshifts away as \(a^{-3}\) and the solution (5.25) converges to the de Sitter universe

\[
a(t \to +\infty) \simeq \frac{a_0}{2^{2/3}} \left( 1 + \sqrt{1 + \frac{3M}{a_0^3\Lambda}} \right)^{2/3} e^{H_0 t}
\]
(5.28)

(where \(H_0 = \sqrt{\Lambda/3}\)), which is a phase space attractor. Indeed, this is the solution obtained by letting \(M \to 0\) in Eq. (5.25).

The limit to a spatially flat, dust-filled universe without cosmological constant is obtained straightforwardly by a formal expansion of Eq. (5.25) as \(\Lambda \to 0\), which produces \(a(t) \sim t^{2/3}\).

The solution for dust and \(\Lambda < 0\) is obtained as the special case \(w_2 = 0\) of Eq. (5.57):

\[
a(t) = \left( \frac{\rho_m}{|\rho_\Lambda|} \right)^{1/3} \sin^{2/3} \left[ \frac{\sqrt{3|\Lambda|}}{2} (t - t_0) \right].
\]
(5.29)

There are a Big Bang at \(t = 0\) and a Big Crunch at \(t = t_0 + \frac{2\pi}{\sqrt{3|\Lambda|}}\).

### 5.7 \(K = 0\), radiation plus \(\Lambda\)

The solution for the spatially flat FLRW universe containing radiation and with \(\Lambda > 0\) is [208]

\[
a(t) = \left( \frac{8\pi \rho_r^{(0)}}{\Lambda} \right)^{1/4} \sqrt{\sinh \left( \frac{\sqrt{\Lambda}}{3} t \right)}. 
\]
(5.30)
It was rediscovered in the more complicated form [13]

\[
\begin{align*}
a(t) &= \frac{1}{\sqrt{2}} \left[ \left( a_0^2 + \sqrt{\frac{3\Gamma}{\Lambda} + a_0^4} \right) e^{2\sqrt{\frac{\Lambda}{3}} t} 
+ \left( a_0^2 - \sqrt{\frac{3\Gamma}{\Lambda} + a_0^4} \right) e^{-2\sqrt{\frac{\Lambda}{3}} t} \right]^{1/2} \\
&= \sqrt{a_0^2 \cosh \left( 2\sqrt{\frac{\Lambda}{3}} t \right) + \sqrt{\frac{\rho(0)}{\rho_\Lambda}} a_0^4 \sinh \left( 2\sqrt{\frac{\Lambda}{3}} t \right)}
\end{align*}
\]

(5.31)  

where

\[
\begin{align*}
a_0 &= a(0), \\
\Gamma &= \Omega r_0 H_0^2 a_0^4 = \frac{8\pi}{3} \rho(0),
\end{align*}
\]

(5.33) (5.34)

and a zero subscript denotes quantities evaluated at the present time. Again, this universe expands forever and the cosmological constant dominates over the radiation at late times.

As for the single-fluid limits, a Taylor expansion of Eq. (5.32) as \( \Lambda \to 0 \) reproduces the spatially flat radiative universe with zero cosmological constant and with scale factor \( a(t) \sim \sqrt{t} \). Likewise, setting \( \rho(0) = 0 \) reduces the solution to the de Sitter space with pure \( \Lambda \) and scale factor \( a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}} t} \).

The solution for radiation and cosmological constant can be presented also in the simple form [397]

\[
a(t) = a_{eq} \sqrt{\sinh \left( 2\sqrt{\frac{\Lambda}{3}} t \right)}
\]

(5.35)

where

\[
a_{eq} = \left( \frac{8\pi \rho(0)}{\Lambda} \right)^{1/4}
\]

(5.36)

is the value of the scale factor at the time of equivalence between the energy densities of radiation and of \( \Lambda \) (this is a special case of the more general solution (5.55)). The expression of the solution in terms of conformal time requires elliptic integrals.

### 5.8 \( \Lambda = 0 \), stiff fluid, any \( K \)

These solutions are again given by Vajk [395]. For \( K = 0 \), we have the usual, forever expanding, solution (2.11)

\[
a(t) = \left( 24\pi \rho_s(0) \right)^{1/6} (t - t_0)^{1/3},
\]

(5.37)
\[ \rho(a) = \frac{\rho_s^{(0)}}{a^6}, \]  

(5.38)

or, in parametric form,

\[
\begin{align*}
    a(\eta) &= \left( \frac{32\pi}{3} \rho_s^{(0)} \right)^{1/4} \sqrt{\eta}, \\
    t(\eta) &= \frac{4}{3} \left( \frac{2\pi}{3} \rho_s^{(0)} \right)^{1/4} \eta^{3/2} + t_0.
\end{align*}
\]  

(5.39)

The scalar field equivalent to the stiff fluid is

\[ \phi(t) = \phi_0 \ln (t - t_0) + \phi_1, \]  

(5.40)

or

\[ \phi(\eta) = \frac{3}{2} \phi_0 \ln \eta + \phi_1 + \phi_0 \ln \left[ \frac{4}{3} \left( \frac{2\pi}{3} \rho_s^{(0)} \right)^{1/4} \right]. \]  

(5.41)

Again, for \( K = \pm 1 \) one cannot integrate \( t(\eta) \) in finite form. For \( K = +1 \) we have

\[
\begin{align*}
    a(\eta) &= \left( \frac{8\pi}{3} \rho_s^{(0)} \right)^{1/4} \sqrt{\cos(2\eta)}, \\
    t(\eta) &= \int d\eta a(\eta),
\end{align*}
\]  

(5.42)

a universe with finite size that begins in a Big Bang and ends in a Big Crunch. For \( K = -1 \), the solution is instead

\[
\begin{align*}
    a(\eta) &= \left( \frac{8\pi}{3} \rho_s^{(0)} \right)^{1/4} \sqrt{\sinh(2\eta)}, \\
    t(\eta) &= \int d\eta a(\eta),
\end{align*}
\]  

(5.43)

describing a universe beginning in a Big Bang and eternally expanding.

### 5.9 \( K = 0 \), stiff fluid plus \( \Lambda \)

One can alternatively regard this solution as being sourced by a stiff fluid plus \( \Lambda \), or by a free scalar field plus \( \Lambda \), or by a scalar field with constant potential \( V(\phi) = \Lambda/(8\pi) \). The solution is \[ 100,169 \]

\[ a(t) = a_0 \sinh^{1/3} \left( \sqrt{3\Lambda} t \right) \]  

(5.44)
where

\[ a_0 = \left( \frac{8\pi \rho_s^{(0)}}{\Lambda} \right)^{1/6} = \left( \frac{4\pi C_0^2}{\Lambda} \right)^{1/6} \]  

(5.45)

and \( C_0 \) is the constant appearing in the first integral (3.28) of the Klein–Gordon equation. If a scalar field is the stiff fluid source, then [169]

\[ \phi(t) = \phi_0 \ln \left[ \tanh \left( \frac{\sqrt{3} \Lambda t}{2} \right) \right] + \phi_1 \]  

(5.46)

with \( \phi_0 = \pm (12\pi)^{-1/2} \) and \( \phi_1 \) is an integration constant. This solution has a Big Bang singularity at \( t = 0 \), where

\[ a(t) \simeq t^{1/3}, \]  

(5.47)

\[ \phi(t) \simeq \phi_0 \ln \left( \frac{\sqrt{3} \Lambda t}{2} \right). \]  

(5.48)

At late times \( t \to +\infty \), the solution asymptotes to the phase space de Sitter attractor with constant scalar field

\[ a(t) \simeq a_0 e^{\sqrt{\frac{\Lambda}{3}} t}, \]  

(5.49)

\[ \phi(t) \simeq \phi_1. \]  

(5.50)

The total energy density and pressure are

\[ \rho_{\text{tot}} = \frac{\dot{\phi}^2}{2} - \frac{\Lambda}{8\pi}, \]  

(5.51)

\[ P_{\text{tot}} = \frac{\dot{\phi}^2}{2} + \frac{\Lambda}{8\pi}. \]  

(5.52)

The effective equation of state of the mixture is time-dependent,

\[ w(t) \equiv \frac{P_{\text{tot}}}{\rho_{\text{tot}}} = 1 - 2 \tanh^2 \left( \sqrt{3} \Lambda t \right), \]  

(5.53)

interpolating between the stiff equation of state \( P \simeq \rho \) at early times (when the cosmological constant has not yet had the time to influence the dynamics) and \( P_{\text{tot}} \simeq -\rho_{\text{tot}} \) at late times when \( \Lambda \) dominates [169].

The limit \( \Lambda \to 0 \) in Eqs. (5.44) and (5.44) reproduces the stiff fluid (or free scalar field) solution

\[ a(t) = a_0 t^{1/3} \]  

(5.54)
(i.e., the scale factor (2.11) with \( w = 1 \)). The scalar field (5.46), however, diverges in the limit \( \Lambda \to 0 \); the correct free \( \phi(t) \) can be obtained from the first integral (3.28) which yields \( \dot{\phi} = C_0/t \) and, finally, \( \phi(t) = C_0 \ln (t/t_0) \). Similarly, setting \( \rho_0^{(0)} = 0 \) does not automatically recover the de Sitter space corresponding to the surviving \( \Lambda \): one must notice that removing the stiff fluid is equivalent to setting \( \phi = \text{const.} \) while retaining its potential \( V(\phi) = \Lambda/(8\pi) \), which generates \( a(t) \propto e^{\sqrt{\Lambda/3} t} \).

### 5.10 K = 0, any real fluid plus \( \Lambda \)

Indeed, when \( K = 0 \) and \( \Lambda \neq 0 \), not only dust but any single fluid with constant equation of state parameter \( w_2 \) can be integrated explicitly using comoving time\(^7\)\cite{100,103,208,293}. For \( \Lambda > 0 \), the solution is

\[
a(t) = \left( \frac{\rho_2^{(0)}}{\rho_\Lambda} \right)^{\frac{1}{3(w_2 + 1)}} \left\{ \sinh \left[ \frac{(w_2 + 1)}{2} \sqrt{3\Lambda} (t - t_0) \right] \right\}^{\frac{2}{3(w_2 + 1)}},
\]

where \( \rho_\Lambda = \Lambda/(8\pi) \) is the effective energy density of the cosmological constant. If \( w_2 > -1 \), there is a Big Bang at \( t = t_0 \). At late times the solution converges to the de Sitter phase space attractor

\[
a(t) \simeq \left( \frac{\rho_2^{(0)}}{\rho_\Lambda} \right)^{\frac{1}{3(w_2 + 1)}} e^{\sqrt{\Lambda/3}(t-t_0)}.
\]

The limit \( \Lambda \to 0 \) reproduces, through a straightforward Taylor expansion of the hyperbolic sine, the single-fluid, no-\( \Lambda \) solution \( a(t) = a_0 (t - t_0)^{\frac{1}{3(w_2 + 1)}} \). However, regarding the other single-fluid limit, one cannot simply take \( \rho_2^{(0)} \to 0 \) in Eq. (5.55) to obtain the empty de Sitter space generated by \( \Lambda \).

For \( \Lambda < 0 \) we have\cite{208}

\[
a(t) = \left( \frac{\rho_2^{(0)}}{|\rho_\Lambda|} \right)^{\frac{1}{3(w_2 + 1)}}
\]

\[
\times \left\{ \sin \left[ \frac{(w_2 + 1)}{2} \sqrt{3|\Lambda|} (t - t_0) \right] \right\}^{\frac{2}{3(w_2 + 1)}},
\]

and there are a Big Bang at \( t_0 \) and a Big Crunch at

\[
t = t_0 + \frac{2\pi}{\sqrt{3|\Lambda|} (w_2 + 1)}
\]

\(^7\) A similar solution with a specific value of the exponent of the hyperbolic sine for dust plus dark energy with a time-dependent equation of state is obtained in Ref.\cite{337} after an ad hoc assumption on the functional form of the deceleration parameter \( q \equiv -\ddot{a}/\dot{a}^2 \) (which essentially amounts to a choice of the scale factor).
\[ w_2 > -1. \]

Again, the limit \( \Lambda \rightarrow 0 \) reproduces the single-fluid universe with scale factor
\[ a(t) = a_0 \left( t - t_0 \right)^{\frac{1}{2(w_2 + 1)}}, \]
but the simple limit \( \rho_2^{(0)} \rightarrow 0 \) in Eq. (5.55) fails to reproduce anti-de Sitter space.

For \( \Lambda = 0 \) we have the usual, forever-expanding, solution (2.11)
\[ a(t) = \left[ (w_2 + 1) \sqrt{6 \pi \rho_2^{(0)} (t - t_0)} \right]^{\frac{2}{3(w_2 + 1)}}, \quad (5.59) \]
with \( a = 0 \) at \( t = t_0 \) if \( w_2 > -1 \).

### 5.11 \( K = 0, \ \Lambda = 0, \text{ dust plus stiff matter} \)

This possibility appears in both lists (3.18) and (3.23). As a function of comoving time, the scale factor assumes the simple form found by Vajk [395]
\[ a(t) = \left[ 6 \pi \rho_m^{(0)} (t - t_0)^2 - \frac{\rho_s^{(0)}}{\rho_m^{(0)}} \right]^{1/3}, \quad (5.60) \]
\[ \rho_m(a) = \frac{\rho_m^{(0)}}{a^3}, \quad \rho_s(a) = \frac{\rho_s^{(0)}}{a^6}. \quad (5.61) \]

This universe expands forever, with the stiff fluid redshifting away considerably faster than the dust. This solution is a special case of the one found by Chavanis [100] and Dariescu \textit{et al.} [145] for dust plus a stiff fluid plus \( \Lambda \), which is given by Eq. (6.9) with the constants \( \alpha \) and \( \beta \) as in Eqs. (6.7) and (6.8). By taking the limit \( \Lambda \rightarrow 0 \) and using the second order expansions \( \sinh x = x + \ldots, \cosh x = 1 + x^2/2 + \ldots \) as \( x \rightarrow 0 \), Eq. (6.9) yields
\[ a(t) = \left[ \frac{9 \beta t^2}{4} + 3 \sqrt{\alpha} t \right]^{1/3}, \quad (5.62) \]
\[ = a_0 \left[ 6 \pi \rho_2^{(0)} t^2 + \sqrt{24 \pi \rho_1^{(0)}} t + \ldots \right]^{1/3}, \quad (5.63) \]
which reproduces the solution (5.60) for
\[ t_0 = \sqrt{\frac{\rho_s^{(0)}}{6 \pi \rho_m^{(0)}}}. \quad (5.64) \]

Although the authors of both Refs. [145,395] do not regard this as a free scalar field solution, it is straightforward to integrate Eq. (3.28), which yields
\[ \phi(t) = \frac{\phi_0}{3 \sqrt{\alpha}} \ln \left( \frac{9 \beta}{4} + \frac{3 \sqrt{\alpha}}{t} \right) + \phi_1, \quad (5.65) \]
where $\phi_{0,1}$ are integration constants.

The single-fluid limit $\rho_s^{(0)} \to 0$ leaving only a dust, applied to the scale factor (5.60) reproduces the correct $a(t) = a_0 (t - t_0)^{2/3}$ and $\phi = \text{const.}$ disappears since $\rho_s = \rho_\phi = P_\phi = \dot{\phi}^2/2$ vanishes. However, one cannot take the limit $\rho_m^{(0)} \to 0$ in Eq. (5.60) to obtain the pure stiff fluid solution.

### 5.12 $K = 0, \Lambda = 0$, radiation plus stiff matter

This solution for $K = 0, \Lambda = 0$, and radiation plus a stiff fluid was given by Vajk [395]. We have $(w_1, w_2) = (1/3, 1)$, the Friedmann equation yields

$$I = \int \frac{da}{\sqrt{\rho_r^{(0)} a^2 + \rho_s^{(0)}}} = \pm \sqrt{\frac{8\pi}{3}} (t - t_0) \quad (5.66)$$

and, integrating,

$$t(a) = t_0 \pm \frac{3}{\sqrt{32\pi \rho_r^{(0)}}} \left\{ a \sqrt{a^2 + \frac{\rho_s^{(0)}}{\rho_r^{(0)}}} - \frac{\rho_s^{(0)}}{\rho_r^{(0)}} \ln \left( \frac{\sqrt{a^2 + \frac{\rho_s^{(0)}}{\rho_r^{(0)}}} + a}{\sqrt{a^2 + \frac{\rho_s^{(0)}}{\rho_r^{(0)}}} - \rho_s^{(0)}} \right) \right\} \quad (5.67)$$

(in this $K = 0$ case, the scale factor $a$ is dimensionless). At late times this universe is dominated by radiation and $a(t) \sim \sqrt{t}$. The parametric form of this solution is [395]

$$\begin{align*}
a(\eta) &= \sqrt{\frac{8\pi}{3}} \rho_r^{(0)} \eta^2 - \frac{\rho_s^{(0)}}{\rho_r^{(0)}}, \\
t(\eta) &= \frac{\eta}{2} \sqrt{\frac{8\pi}{3}} \rho_r^{(0)} \eta^2 - \frac{\rho_s^{(0)}}{\rho_r^{(0)}} \\
&\quad - \frac{\rho_s^{(0)}}{\rho_r^{(0)}} \sqrt{\frac{3}{32\pi \rho_r^{(0)}}} \ln \left[ \sqrt{\frac{8\pi}{3}} \rho_r^{(0)} \eta \right] \left[ \sqrt{\frac{8\pi}{3}} \rho_r^{(0)} \eta^2 - \frac{\rho_s^{(0)}}{\rho_r^{(0)}} \right] + t_0, \\
\end{align*} \quad (5.68)$$

from which one sees that this universe begins in a Big Bang.

---

8 This solution has recently been rediscovered in its parametric form, using conformal time as the parameter, in Ref. [17].
Equation (3.28) for the scalar field equivalent of the stiff fluid can be integrated giving $\phi(a)$. Using $\frac{d\phi}{dt} = \dot{a} \frac{d\phi}{da}$, one obtains

$$\frac{d\phi}{da} = \frac{C_0}{a \sqrt{\rho_f^{(0)} a^2 + \rho_s^{(0)}}}, \quad (5.69)$$

which integrates to

$$\phi(a) = \phi_0 \tanh^{-1} \left( \sqrt{1 + \frac{\rho_f^{(0)}}{\rho_s^{(0)}} a^2} \right) + \phi_1, \quad (5.70)$$

where $\phi_0 = -\frac{C_0}{\sqrt{\rho_s^{(0)}}}$ and $\phi_1$ is an integration constant.

The limit $\rho_s^{(0)} \to 0$ in which the stiff fluid disappears, leaving only radiation, transforms the relation (5.67) into the correct scale factor $a(t) = a_0 \sqrt{t - t_0}$. The other limit $\rho_f^{(0)} \to 0$ in Eq. (5.67) does not produce the corresponding stiff fluid solution, which has to be recovered directly from the limit $\rho_f^{(0)} \to 0$ of the integral (5.66).

5.13 $w = 2/3$, $\Lambda = 0$

This solution was given by Vajk [395] without apparent physical motivation and was reported by other authors [293]. For $K = 0$, it is the single-fluid universe (2.11),

$$a(t) = \left( \frac{50\pi}{3} \rho^{(0)} \right)^{1/5} (t - t_0)^{2/5}, \quad (5.71)$$

$$\rho(a) = \frac{\rho^{(0)}}{a^5}, \quad (5.72)$$

or, in parametric form,

$$\begin{cases} 
  a(\eta) = \left( \frac{6\pi}{3} \rho^{(0)} \right)^{1/3} \eta^{2/3}, \\
  t(\eta) = \frac{3}{5} \left( \frac{6\pi}{3} \rho^{(0)} \right)^{1/3} \eta^{5/3} + t_0.
\end{cases} \quad (5.73)$$

There is a Big Bang singularity $a = 0$ at $t = t_0$ (or $\eta = 0$) and the universe expands forever.

For $K = \pm 1$, the comoving time cannot be expressed explicitly in terms of simple functions. When $K = +1$ we have [395]

$$\begin{cases} 
  a(\eta) = \left( \frac{8\pi}{3} \rho^{(0)} \right)^{1/3} \cos^{2/3} \left( \frac{3\eta}{2} \right), \\
  t(\eta) = \int d\eta a(\eta),
\end{cases} \quad (5.74)$$
which has a Big Bang at $\eta = -\pi/3$ and a Big Crunch at $\eta = \pi/3$.

For $K = -1$ the solution is \[ a(\eta) = \left( \frac{8\pi}{3} \rho_r^{(0)} \right)^{1/3} \sinh^{2/3} \left( \frac{3\eta}{2} \right), \]
\[ t(\eta) = \int d\eta a(\eta). \] (5.75)

In this case there is a Big Bang at $\eta = 0$ and the universe expands forever.

### 5.14 $K = 0$, $\Lambda = 0$, $w = 2/3$ fluid plus radiation

This case, with $w_1 = 1/3$, $w_2 = 2/3$, is the third entry in the list (3.16). The comoving time form of this solution is again given by Vajk as \[ \sqrt{\frac{3}{32\pi\rho_r^{(0)}}} \left[ a - \frac{3\rho_2^{(0)}}{2\rho_r^{(0)}} \right] \sqrt{a^2 + \frac{\rho_2^{(0)}}{\rho_r^{(0)}} a}
+ \frac{3}{4} \left( \frac{\rho_2^{(0)}}{\rho_r^{(0)}} \right)^2 \ln \left( \left[ a^2 + \frac{\rho_2^{(0)}}{\rho_r^{(0)}} a + a + \frac{\rho_2^{(0)}}{2\rho_r^{(0)}} \right] \right)
+ t_0, \]
\[ \rho_r(a) = \frac{\rho_r^{(0)}}{a^4}, \quad \rho_2(a) = \frac{\rho_2^{(0)}}{a^5}. \] (5.76)

The Big Bang occurs at
\[ t = t_0 + \frac{3\sqrt{3}}{4\sqrt{\pi}} \left( \frac{\rho_2^{(0)}}{\rho_r^{(0)}} \right)^2 \ln \left( \frac{\rho_2^{(0)}}{2\rho_r^{(0)}} \right) \] (5.77)

and the universe expands forever, becoming radiation-dominated at late times. The parametric form of this solution in terms of conformal time is \[ \sqrt{a^2 + \frac{\rho_2^{(0)}}{\rho_r^{(0)}} a - \frac{\rho_2^{(0)}}{2\rho_r^{(0)}} \cosh^{-1} \left( \frac{2\rho_r^{(0)}}{\rho_2^{(0)}} + 1 \right)} = \sqrt{\frac{8\pi}{3}} \rho_r^{(0)} \eta, \]
\[ t(\eta) = \int d\eta a(\eta). \] (5.78)

9 An error in the square root $\sqrt{a + \rho_2^{(0)}/\rho_r^{(0)}}$ in the first term on the right hand side of Eq. (5.76), present in the solution of Ref. [395], was corrected in [293].
The limit \( \rho_2^{(0)} \to 0 \) leaves only radiation; taking this limit in Eq. (5.76) reproduces the correct scale factor \( a(t) = a_0 \sqrt{t - t_0} \) for the single-fluid radiative universe. The limit \( \rho_t^{(0)} \to 0 \), instead, fails to reproduce the corresponding scale factor, which must be recovered directly from the usual integral.

### 5.15 \( K = 0, \Lambda = 0, w_1 = 2/3 \) fluid plus stiff fluid

This possibility appears as the second last entry of the list (3.18). This solution was again given by Vajk [395] using comoving time\(^{10} \)

\[
\begin{align*}
t(a) &= \frac{1}{15} \sqrt{\frac{3}{2\pi \rho_1^{(0)}}} \left[ a + \frac{\rho_s^{(0)}}{\rho_1^{(0)}} \right] \\
&\quad \times \left[ 3a^2 - \frac{4\rho_s^{(0)}}{\rho_1^{(0)}} a + 8 \left( \frac{\rho_s^{(0)}}{\rho_1^{(0)}} \right)^2 \right] + t_0 \\
\rho_1(a) &= \frac{\rho_1^{(0)}}{a^{\frac{3}{5}}}, \quad \rho_s(a) = \frac{\rho_s^{(0)}}{a^{\frac{3}{5}}}, \\
\end{align*}
\]

or, in parametric form [395],

\[
\begin{align*}
\left\{ \begin{array}{l}
\sqrt{a + \frac{\rho_0^{(0)}}{\rho_1^{(0)}} \left( a - \frac{2\rho_0^{(0)}}{\rho_1^{(0)}} \right)} = \sqrt{6\pi \rho_1^{(0)}} \eta, \\
t(\eta) = \int d\eta a(\eta).
\end{array} \right.
\end{align*}
\]

The Big Bang occurs at

\[
t = t_0 + \frac{4}{5} \sqrt{\frac{2}{3\pi \rho_1^{(0)}}} \left( \frac{\rho_s^{(0)}}{\rho_2^{(0)}} \right)^{5/2},
\]

or at \( t = 0 \) if the integration constant is chosen as

\[
t_0 = -\frac{4}{5} \sqrt{\frac{2}{3\pi \rho_1^{(0)}}} \left( \frac{\rho_s^{(0)}}{\rho_2^{(0)}} \right)^{5/2}.
\]

This universe expands eternally and is dominated by the \( w_1 = 2/3 \) fluid, which decays slower than the stiff fluid, in its late history.

\(^{10}\) Vajk’s solution has a sign error in the square root, which was corrected by McIntosh [293].
We can find the scalar field $\phi$ associated with the stiff fluid as a function of the scale factor $a$. By differentiating Eq. (5.80), one obtains

$$\frac{dt}{da} = (\dot{a})^{-1} = \sqrt{\frac{3}{8\pi \rho_1^{(0)}}} \frac{a^2}{\sqrt{a + \frac{\rho_{s}^{(0)}}{\rho_1^{(0)}}}}$$

(5.85)

which, substituted into $d\phi/dt = \dot{a} d\phi/da$, gives

$$\frac{d\phi}{da} = \sqrt{\frac{3}{8\pi \rho_1^{(0)}}} \frac{d\phi}{dt} \frac{a^2}{\sqrt{a + \frac{\rho_{s}^{(0)}}{\rho_1^{(0)}}}}.$$  

(5.86)

Equation (3.28) now yields

$$\frac{d\phi}{da} = \sqrt{\frac{3}{8\pi \rho_1^{(0)}}} \frac{C_0}{a \sqrt{a + \frac{\rho_{s}^{(0)}}{\rho_1^{(0)}}}}.$$  

(5.87)

which integrates to

$$\phi(a) = \phi_0 \tanh^{-1} \left( \sqrt{1 + \frac{\rho_1^{(0)}}{\rho_{s}^{(0)}} a} \right) + \phi_1,$$

(5.88)

where $\phi_{0,1}$ are integration constants.

The single-fluid limit $\rho_{s}^{(0)} \to 0$ in Eq. (5.80) reproduces the correct scale factor $a(t) = a_0 (t - t_0)^{2/5}$, but fails on the expression (5.88) of the scalar field: this happens because the limit corresponds to the value $C_0 = 0$ of the integration constant in the first integral (3.28) leading to Eq. (5.88).

6 Three (real or effective) fluids solutions

Known three-fluid solutions include:

- $(w_1, w_2, w_3) = (1/3, -1, -1/3)$, radiation plus $\Lambda$ plus spatial curvature;
- $(w_1, w_2, w_3) = (-1/3, 0, 1/3)$, spatial curvature $K \neq 0$ plus dust plus radiation;
- $(w_1, w_2, w_3) = (-1, 0, 1)$, or $\Lambda$ plus a dust plus a stiff fluid;
- $(w_1, w_2, w_3) = (1/3, 2/3, 1)$, or radiation plus $P_2 = 2\rho_2/3$ fluid plus a stiff fluid.
6.1 Radiation plus $\Lambda$ plus spatial curvature $K \neq 0$

These solutions were found by Harrison [208] and subsequently rediscovered [13]. For $K = +1$ and $\Lambda > 0$, we have [208,383]

$$a(t) = \sqrt{\frac{3}{2\Lambda}} \left\{ 1 - \cosh \left[ 2\sqrt{\frac{\Lambda}{3}} (t - t_0) \right] \right\}^{1/2} + 2 \sqrt{\frac{\Lambda}{\Lambda_c}} \sinh \left[ 2\sqrt{\frac{\Lambda}{3}} (t - t_0) \right], \quad (6.1)$$

where

$$\Lambda_c = \frac{9}{32\pi \rho_r^{(0)}} \quad (6.2)$$

is the critical value of the cosmological constant required to have a static Einstein universe with the same amount of radiation.

For $K = -1$ and $\Lambda > 0$,

$$a(t) = \sqrt{\frac{3}{2\Lambda}} \left\{ \cosh \left[ 2\sqrt{\frac{\Lambda}{3}} (t - t_0) \right] - 1 \right\}^{1/2} + 2 \sqrt{\frac{\Lambda}{\Lambda_c}} \sinh \left[ 2\sqrt{\frac{\Lambda}{3}} (t - t_0) \right], \quad (6.3)$$

Both solutions begin with a Big Bang at $t = t_0$ and asymptote to the de Sitter space with scale factor $a(t) = a_0 \exp \left( \sqrt{\frac{\Lambda}{3}} t \right)$ at late times.

For $K = +1$ and $\Lambda < 0$, the solution is [208]

$$a(t) = \sqrt{\frac{3}{2|\Lambda|}} \left\{ 1 - \cosh \left[ 2\sqrt{\frac{|\Lambda|}{3}} (t - t_0) \right] \right\}^{1/2} + 2 \sqrt{\frac{|\Lambda|}{\Lambda_c}} \sinh \left[ 2\sqrt{\frac{|\Lambda|}{3}} (t - t_0) \right], \quad (6.4)$$

6.2 $K = 0$, $\Lambda$ plus a stiff fluid plus dust

The FLRW universe with stiff matter plus dust plus $\Lambda$ was found by Chavanis [100] and, like several others, was rediscovered recently [145]. Using the slightly different notation
\[ \rho_1 = \rho_1^{(0)} \left( \frac{a_0}{a} \right)^6, \quad (6.5) \]

\[ \rho_2 = \rho_2^{(0)} \left( \frac{a_0}{a} \right)^3, \quad (6.6) \]

\[ \alpha = \frac{8\pi}{3} \rho_1^{(0)} a_0^6, \quad (6.7) \]

\[ \beta = \frac{8\pi}{3} \rho_2^{(0)} a_0^3, \quad (6.8) \]

the solution for \( \Lambda > 0 \) is

\[
a(t) = \left\{ \frac{3\beta}{2\Lambda} \left[ \cosh \left( \sqrt{3\Lambda} t \right) - 1 \right] + \sqrt{\frac{3\alpha}{\Lambda}} \sinh \left( \sqrt{3\Lambda} t \right) \right\}^{1/3}. \quad (6.9)
\]

This universe, which begins with a Big Bang at \( t = 0 \), asymptotes to the de Sitter phase space attractor \( a(t) = a_0 e^{\sqrt{\frac{8\pi}{3}} \rho_0} \) at late times.

### 6.3 \( K \neq 0 \), dust plus radiation

The solution for dust plus radiation in a spatially flat universe is given by Eq. (5.18) using comoving time or by Eqs. (5.19) in parametric form using conformal time. The corresponding solutions for spatially curved FLRW universes were found in [104,126,221,291,294,395] (see also [165,301]). The solution for \( K = +1 \) is given by (e.g., [126])

\[
\begin{align*}
t(a) &= t_0 - \sqrt{\frac{8\pi}{3} \rho_r^{(0)}} + \frac{8\pi}{3} \rho_m^{(0)} a - a^2 \\
& \quad + \frac{4\pi}{3} \rho_m^{(0)} \sin^{-1} \left[ \frac{-\frac{4\pi}{3} \rho_m^{(0)} a}{\sqrt{\left( \frac{4\pi}{3} \rho_m^{(0)} \right)^2 + \frac{8\pi}{3} \rho_r^{(0)}}} \right]. \quad (6.10)
\end{align*}
\]

This universe begins at a Big Bang, reaches a maximum size, and ends in a Big Crunch singularity. The parametric form of this solution is [13,302,395]

\[
\begin{align*}
a(\eta) &= \frac{4\pi}{3} \rho_m^{(0)} (1 - \cos \eta) + \sqrt{\frac{8\pi}{3} \rho_r^{(0)}} \sin \eta, \\
t(\eta) &= \frac{4\pi}{3} \rho_m^{(0)} (\eta - \sin \eta) + \frac{8\pi}{3} \rho_r^{(0)} (\cos \eta - 1). \quad (6.11)
\end{align*}
\]

For \( K = -1 \) the solution is [126,294,395]
\[
t(a) = t_0 + \sqrt{\frac{8\pi}{3} \rho_r(0) + \frac{8\pi}{3} \rho_m(0)} a + \frac{4\pi}{3} \rho_m(0) \ln \left\{ C \left[ \frac{8\pi}{3} \rho_r(0) + \frac{8\pi}{3} \rho_m(0) \right] a + \frac{4\pi}{3} \rho_m(0) \right\} + \frac{4\pi}{3} \rho_m(0) \sinh^{-1} \left[ \frac{a + 4\pi \rho_m(0)/3}{\sqrt{4\pi/3 \rho_m(0) \left[ -4\pi/3 \rho_m(0) + 2\rho_r(0)/\rho_m(0) \right]}} \right] + t_0 \tag{6.12}
\]

where \( C \) is a constant (missing in [126]) needed to make the argument of the logarithm dimensionless. This universe begins at a Big Bang and expands forever, becoming dust-dominated. The expression (6.12) can be cast in the alternative form presented by Vajik [395]

\[
t(a) = \sqrt{a^2 + \frac{8\pi}{3} \rho_m(0) a + \frac{8\pi}{3} \rho_r(0)}
- \frac{4\pi}{3} \rho_m(0) \sinh^{-1} \left[ \frac{a + 4\pi \rho_m(0)/3}{\sqrt{4\pi/3 \rho_m(0) \left[ -4\pi/3 \rho_m(0) + 2\rho_r(0)/\rho_m(0) \right]}} \right] + t_0 \tag{6.13}
\]

(see Appendix B for a proof of the equivalence), or in the parametric form\(^{11}\) [302,395]

\[
\begin{align*}
ad(\eta) &= \sqrt{\frac{8\pi}{3} \rho_r(0)} \sinh \eta + \frac{4\pi}{3} \rho_m(0) (\cosh \eta - 1), \\
t(\eta) &= \sqrt{\frac{8\pi}{3} \rho_r(0)} (\cosh \eta - 1) + \frac{4\pi}{3} \rho_m(0) (\sinh \eta - \eta),
\end{align*}
\tag{6.14}
\]

with the initial condition \( a(t = 0) = 0 \).

When \( \rho_m(0) \) or \( \rho_r(0) \) are set to zero, the solutions reduce, of course, to single-fluid spatially curved universes filled only with radiation or matter, respectively. In the case of matter-only, however, it is not trivial to reproduce the single-fluid expressions (5.5) for \( K = +1 \) and (5.6) for \( K = -1 \) as \( \rho_r(0) \rightarrow 0 \). Appendix C discusses these limits.

The solutions of the Einstein–Friedmann equations for dust plus radiation with spatial curvature and \( \Lambda \neq 0 \) were classified qualitatively by Payne [325].

### 6.4 \( K \neq 0, \Lambda = 0 \), radiation plus stiff fluid

The spatially curved FLRW universes filled with radiation plus a stiff fluid were given by Vajik in parametric form [395]. For \( K = +1 \),

\(^{11}\) Ref. [13] contains an error in this parametric solution.
\[
\begin{align*}
\{a(\eta) &= \sqrt{\frac{4\pi}{3} \rho_t^{(0)} + \sqrt{\frac{4\pi}{3} \rho_t^{(0)} \left( \frac{4\pi}{3} \rho_t^{(0)} + 2 \rho_s^{(0)} / \rho_t^{(0)} \right)}} \sin(2\eta), \\
t(\eta) &= \int d\eta a(\eta) \}
\end{align*}
\] (6.15)

This cosmos begins at a Big Bang, expands to a maximum size, and collapses in a Big Crunch. The scalar field equivalent of the stiff fluid can only be expressed in terms of \( \eta \), or of \( a \), by means of elliptic integrals.

For \( K = -1 \), instead, we have the forever-expanding universe

\[
\begin{align*}
\{a(\eta) &= \sqrt{\frac{4\pi}{3} \rho_t^{(0)} + \sqrt{\frac{4\pi}{3} \rho_t^{(0)} \left( \frac{4\pi}{3} \rho_t^{(0)} + 2 \rho_s^{(0)} / \rho_t^{(0)} \right)}} \sinh(2\eta) \\
&\quad - \frac{4\pi}{3} \rho_t^{(0)} \}^{1/2}, \\
t(\eta) &= \int d\eta a(\eta) \}
\end{align*}
\] (6.16)

which, during its history, is first dominated by the stiff fluid, then becomes radiation-dominated, and ends in a curvature-dominated era.

### 6.5 \( K = 0, \Lambda = 0 \), radiation plus \( P_2 = 2 \rho_2/3 \) fluid plus stiff fluid

The solution for radiation plus a \( P_2 = 2 \rho_2/3 \) fluid plus a stiff fluid was again given by Vajk [395]. It reads

\[
\begin{align*}
t(a) &= t_0 + \sqrt{\frac{3}{32\pi \rho_t^{(0)}}} \left\{ a - \frac{3 \rho_2^{(0)}}{2 \rho_t^{(0)}} \right\} \sqrt{a^2 + \frac{\rho_2^{(0)}}{\rho_t^{(0)}} a + \frac{\rho_s^{(0)}}{\rho_t^{(0)}}} \\
&\quad + \left[ \frac{3}{4} \left( \frac{\rho_2^{(0)}}{\rho_t^{(0)}} \right)^2 - \frac{\rho_s^{(0)}}{\rho_t^{(0)}} \right] \ln \left[ \sqrt{a^2 + \frac{\rho_2^{(0)}}{\rho_t^{(0)}} a + \frac{\rho_s^{(0)}}{\rho_t^{(0)}}} \right] \\
&\quad + a + \frac{\rho_s^{(0)}}{2 \rho_t^{(0)}} \right\} \}
\end{align*}
\] (6.17)

and describes a Big Bang universe which expands forever, with the stiff fluid, then the \( w_2 = 2/3 \) fluid, and then radiation becoming dominant. In the limit \( \rho_s^{(0)} \to 0 \) in which the matter content reduces to the \( w_2 = 2/3 \) fluid plus radiation, Eq. (5.76) is reproduced.
The parametric form corresponding to the universe (6.17) using conformal time as the parameter is [395]

\[
\sqrt{a^2 + \rho_{s}^{(0)} \rho_{t}^{(0)} a} + \frac{\rho_{s}^{(0)}}{\rho_{t}^{(0)}} - \frac{\rho_{s}^{(0)}}{2\rho_{t}^{(0)}}
\times \cosh^{-1}\left[\frac{2a + \rho_{s}^{(0)}}{\rho_{t}^{(0)}} \right] = \sqrt{\frac{8\pi}{3}} \rho_{t}^{(0)} \eta, \tag{6.18}
\]

In the limit \( \rho_{s}^{(0)} \to 0 \) it reproduces Eq. (5.79) for a \( w_2 = 2/3 \) fluid plus radiation.

7 Scalar field solutions

7.1 Single scalar field

We have already seen that a minimally coupled scalar field is equivalent to a perfect fluid which, in general, has a dynamical effective equation of state and that a free scalar field is equivalent to a stiff fluid. Therefore, all stiff fluid solutions are automatically free scalar solutions, and we have provided the corresponding expression of \( \phi(t) \) when it is given explicitly in terms of elementary functions.

Originally, the study of analytical solutions of scalar field cosmology was motivated by early universe inflation. When the scalar field potential \( V(\phi) \) is part of a high energy theory, exact solutions are usually not available because of the non-linearity of the Klein–Gordon equation (2.22) and phase space analyses are most informative about the dynamics (e.g., [128,377,403]). Several formal solutions exist in the literature, although some of them are mostly of mathematical interest. Only solutions that are attractors in phase space, or that display particular physical properties (such as evading slow-roll constraints part of the time, providing exact spectral indices or desired expansion histories \( a(t) \), etc.) are usually important from the physical point of view. Many of the analytical solutions found are only valid in restricted regions of initial conditions and parameters, usually due to the fact that variables changes are needed which are restricted to regions in which the scalar field \( \phi(t) \) is monotonic and invertible. Therefore, most of these solutions should be regarded as toy models (which may still be quite useful), with the notable exception of solutions that behave as attractors in phase space.

The following solutions\(^{12}\) have been found in the context of inflationary scenarios of the early universe [240,262,279] (we do not report approximate solutions here).

\(^{12}\) These solutions have been used as seeds to generate corresponding universes in scalar-tensor gravity by means of the conformal transformation from the Einstein to the Jordan frame (e.g., [2]).
7.1.1 Exponential potentials

The power-law inflationary scenario is described by the exact solution [1,30,73,259,273,303]

\[ a(t) = a_0 t^p, \quad (7.1) \]
\[ \phi(t) = \phi_0 + \alpha \ln(t), \quad (7.2) \]
\[ V(\phi) = V_0 e^{\pm \sqrt{4\pi p} \phi/m_{pl}}, \quad (7.3) \]

where \( m_{pl} \) is the Planck mass, \( V_0, \alpha, \) and \( p \) are constants, with \( p > 1 \) in order to have cosmic acceleration \( \ddot{a} > 0 \). The de Sitter solution is obtained if \( \phi = \text{const.} \), which corresponds to a cosmological constant [30,73,303]. The Klein–Gordon equation with exponential potential has a long history in mathematics, beginning with early studies of the non-linear wave equation by Liouville in 1853 [272]. Power-law inflation has the advantage that the spectral indices of scalar and tensor perturbations are also determined exactly.

Spatially curved \((K \neq 0)\) versions of power-law inflation are possible, but these solutions cannot be put in explicit form even though the field equations can be integrated explicitly [164].

The exponential potential is a trademark of higher-dimensional compactified theories and of the subsequent transformation to the Einstein conformal frame and potentials consisting of a sum of exponential terms appear in supergravity and superstring theories. A sum of exponentials is typical of perturbation expansions in superstring theories [317]. Two-exponential potentials

\[ V(\phi) = V_0 e^{2\lambda \phi} + V_1 e^{-2\lambda \phi} - 2V_0 V_1 \quad (7.4) \]

(with \( V_{0,1} \) positive constants) were studied in relation with Noether and Hojiman symmetries and other conserved quantities [90,107,116,149–151,155].

Kruger and Norbury [242] found the spatially flat FLRW solution

\[ V(\phi) = V_0 [1 + \cosh(\lambda \phi)], \quad (7.5) \]
\[ \phi(t) = \frac{1}{\lambda} \ln \left( \frac{e^{\lambda \sqrt{V_0} t} + 1}{e^{\lambda \sqrt{V_0} t} - 1} \right), \quad (7.6) \]
\[ a(t) = \left[ \exp \left( 2\lambda \sqrt{V_0} t \right) - 1 \right]^{1/3}, \quad (7.7) \]
\[ \rho_\phi(a) = 2V_0 \left( 1 + \frac{2}{a^3} + \frac{1}{a^6} \right). \quad (7.8) \]

Ellis and Madsen [164] found a solution, for any curvature index \( K \), for the potential

\[ V(\phi) = \frac{3H_0^2}{8\pi} + \phi_1^2 \sinh^2 \left[ \frac{2H_0}{\phi_1} (\phi - \phi_0) \right] \quad (7.9) \]
(determined \textit{a posteriori}), which produces

\begin{align*}
  a(t) &= a_0 \sinh (H_0 t), \\
  \phi(t) &= \phi_0 \pm \frac{\phi_1}{H_0} \ln \left( \frac{e^{H_0 t} - 1}{e^{H_0 t} + 1} \right), 
\end{align*}

(7.10)

(7.11)

where

\begin{equation}
  \phi_1^2 = \frac{1}{4\pi} \left( H_0^2 + \frac{K}{a_0^2} \right),
\end{equation}

(7.12)

\(H_0\) and \(a_0\) are positive constants, and \(\phi_0\) is an integration constant. The dimensionless density parameter

\begin{equation}
  \Omega(t) = \left( 1 + \frac{4\pi \phi_1^2}{H_0 \sinh^2 (H_0 t)} \right) \tanh^2 (H_0 t)
\end{equation}

(7.13)

approaches unity at late times when the solution enters a slow-roll regime.

Yet another exact solution by Ellis & Madsen \cite{164} for any curvature index \(K\) exhibits the unusual linear expansion

\begin{align*}
  a(t) &= a_0 t, \\
  \phi(t) &= \phi_0 \pm \phi_1 \ln t, \\
  V(\phi) &= \phi_1^2 e^{-\frac{2(\phi - \phi_0)}{\phi_1}},
\end{align*}

(7.14)

(7.15)

(7.16)

with \(a_0\) a positive constant,

\begin{equation}
  \phi_1^2 = \frac{1}{4\pi} \left( 1 + \frac{K}{a_0^2} \right)
\end{equation}

(7.17)

and constant dimensionless density parameter \(\Omega\). This (non-slow-roll) solution can be regarded as a modern version of the Milne universe which is filled by a scalar field and allows for any spatial curvature \cite{164}. By contrast, the original Milne universe is a hyperbolic \((K = -1)\) foliation of empty Minkowski spacetime \cite{301}.

Easther \cite{159} studied a potential of the form

\begin{equation}
  V(\phi) = \sum_{j=1}^{N} \gamma_j e^{-\lambda_j \gamma \phi}
\end{equation}

(7.18)

and gave solutions for \(K = \pm 1\) for \(a, t, \phi\) in parametric form, recovering a previous solution of Ref. \cite{317} and for \(K = 0\).
7.1.2 Other exact solutions

The intermediate inflationary scenario contains another exact solution [31,32,303]:

\[
V(\phi) = \frac{m^2}{\phi^\beta} \left( 1 - \frac{\beta^2}{6\phi^2} \right),
\]

(7.19)

\[
\rho(a) = \frac{m^2}{(2\beta)^{\beta/2}} \left( \ln a \right)^{-\beta/2},
\]

(7.20)

\[
\phi = \left[ \phi_0 \pm A(t - t_0) \right]^{\frac{1}{\beta + 4}},
\]

(7.21)

\[
a(t) = a_0 \exp \left[ \alpha(t - t_0)^\gamma \right],
\]

(7.22)

where \(m\) is a mass scale, \(\beta\) is a constant, and

\[
A = \frac{\beta (\beta + 4) m}{2\sqrt{3}},
\]

(7.23)

\[
\gamma = 1 - \frac{\beta + 2}{\beta + 4},
\]

(7.24)

\[
\alpha = \frac{A^{2/3} \pi}{2\beta}.
\]

(7.25)

Various solutions were provided by Ellis & Madsen [164] with the purpose of obtaining any cosmic history \(a(t)\) that could be reconstructed from cosmological observations, or to evade slow-roll. Their method consists of imposing the form of \(a(t)\) and then solving for \(\phi(t)\) and \(V(t)\), inverting to obtain \(t(\phi)\), and deriving \(V(\phi) = V(t(\phi)) \text{ a posteriori}\). Their first solution is de Sitter expansion for \(K \geq 0\), given by

\[
a(t) = a_0 e^{H_0 t},
\]

(7.26)

\[
\phi(t) = \phi_0 \pm \frac{\phi_1}{H_0} e^{-H_0 t},
\]

(7.27)

\[
V(\phi) = \frac{3H_0^2}{8\pi} + H_0^2 (\phi - \phi_0)^2,
\]

(7.28)

where

\[
\phi_1 = \sqrt{\frac{K}{4\pi a_0^2}}
\]

(7.29)

and \(a_0, H_0\) are positive constants, while \(\phi_0\) is an integration constant. The density parameter is [164]

\[
\Omega(t) = 1 + \frac{4\pi \phi_1^2}{H_0^2} e^{-2H_0 t}.
\]

(7.30)
This solution enters a slow-roll regime only at late times, as $\Omega \to 1$ and reduces to the usual de Sitter solution with cosmological constant (contained in the potential (7.28)) and constant scalar field $\phi$ if $K = 0$.

Another solution for $K > 0$ is [164]

$$a(t) = a_0 \cosh(H_0 t),$$  \hspace{1cm} (7.31)

$$\phi(t) = \phi_0 \pm \frac{2\phi_1}{H_0} \arctan \left( e^{H_0 t} \right),$$  \hspace{1cm} (7.32)

$$V(\phi) = \frac{3H_0^2}{8\pi} + \phi_1^2 \sin^2 \left[ \frac{2H_0}{\phi_1} (\phi - \phi_0) \right],$$  \hspace{1cm} (7.33)

where $a_0, H_0$ are positive constants,

$$\phi_1^2 = \frac{1}{4K} \left( \frac{K}{a_0^2} - H_0^2 \right)$$  \hspace{1cm} (7.34)

and the density parameter

$$\Omega(t) = \left[ 1 + \frac{4\pi \phi_1^2}{H_0^2 \cosh^2(H_0 t)} \right] \coth^2(H_0 t)$$  \hspace{1cm} (7.35)

goess from unity to infinity and back to unity.

Maartens, Taylor & Roussos used the number of $e$-foldings (or $\ln(a/a_i)$, where $a_i$ is the scale factor at the beginning of inflation), or ultimately the scale factor itself, as the independent variable in order to find new analytical solutions of scalar field cosmology [275]. Their solutions interpolate between exponential or power-law inflationary expansion and the radiation era, providing an effective description of the exit from inflation. Although reheating and entropy production at the end of inflation cannot be taken into account by such a simple model, these analytical solutions offer a quick way to model a complicated transition.

Methods to solve the coupled Einstein–Friedmann–Klein–Gordon equations produced other exact inflationary solutions [22,33–35,93,99,105,163,181,206,226,242,260,263–266,321,352,358,378,422,431]. In general, all these solution-generating methods have some restrictions and limitations, and they necessarily focus on the mathematics, leading to new solutions which may not be very relevant for the physical aspects of the accelerated cosmic expansion during inflation or late quintessence domination. We refer the reader to the recent review by Martin, Ringeval & Vennin [279] for a comprehensive survey of inflationary scenarios with exact and approximate solutions of the relevant field equations.

### 7.2 Scalar field plus fluid

Since a stiff fluid is equivalent to a free scalar field, solutions describing universes sourced by a single perfect fluid decoupled from a free scalar field have already been
given as two-fluid solutions one of which obeys the stiff equation of state $P = \rho$. Whenever the scalar field can be integrated explicitly, its expression has been provided. Let us turn now to the physical motivation for such solutions.

Early universe inflation, which is believed to be driven by a scalar field in the early universe [240,262,267,279], provided the first motivation for searching solutions that describe FLRW universes sourced by a single fluid plus a scalar field. However, until 1998 the interest was more mathematical than physical. Following the 1998 discovery of the present acceleration of the universe made with type Ia supernovae [330,331,340] and the introduction of dark energy in theoretical cosmology, many models of scalar field (modelling dark energy) and a single fluid (a dust modelling dark matter) were introduced. The most popular ones (at least before the idea of modifying gravity at large scales to get rid of the ad hoc dark energy altogether [148,314,375]) were, and remain, models consisting of a scalar field (the “quintessence”) interacting with the dark matter fluid (a dust) only gravitationally. Indeed, the study of these models with inverse power-law potentials $V(\phi) = V_0/\phi^\alpha$, $\alpha > 0$, begun with the work of Peebles & Ratra [328] predating the discovery of the cosmic acceleration. We refer the reader to the excellent book by Amendola & Tsujikawa [18] for a proper account of these quintessence models.

Pre-1998 analytical solutions include those of Barrow & Saich [32] and Chimento & Jakubi [107]. Barrow & Saich solved the Einstein–Friedmann equations for a spatially flat FLRW universe sourced by a scalar field in a potential $V(\phi)$ and a perfect fluid. They imposed that the scalar field effective equation of state be constant and determined the potential a posteriori.

Attempts to solve the Einstein–Friedmann–Klein–Gordon equations with a potential usually amount to imposing a certain scale factor (usually, but not always, power-law or exponential), substituting it into the field equations, and solving them while determining a scalar field potential $V(\phi)$. The last step requires expressing $V$ as a function of time $t$ and inverting the functional relation $\phi = \phi(t)$ to find $t(\phi)$ and then $V(\phi) = V(t(\phi))$. In practice, this is not always possible to do analytically. In this approach à la Synge [388], the potential $V(\phi)$ is not determined by physical considerations but is ad hoc.

For example, with loose motivation from inflation, Méndez [295] imposed $a(t) = a_0 e^{Ht}$ while allowing for spatial curvature and a perfect fluid with constant equation of state parameter in the range $0 \leq w \leq 1$. He found no solutions for $K \leq 0$ since [295]

$$
\phi(t) = \phi_0 \pm \int_0^t \frac{dt'}{\sqrt{\frac{K e^{-2Ht'} \left(\frac{2Ht'}{4\pi a_0^2} - (w + 1) \rho_0 e^{-3(w+1)Ht'}\right)}}
$$

and found one solution for $K > 0$, but the relation $\phi(t)$ cannot be inverted to give $V(\phi)$ [295]. There are actually more analytical solutions $\phi(t)$ for $K > 0$ that can be expressed in terms of elementary functions, as can be seen by applying the Chebysev theorem to Eq. (7.36). Moreover, one can in principle find solutions for $K \leq 0$ by allowing a phantom equation of state ($w < -1$) but invertibility of $\phi(t)$ is usually not possible. Méndez turned instead to a viscous imperfect fluid with bulk viscosity pro-
portional to a power of its energy density, finding analytical solutions with scalar field that is an inverse exponential of time and quadratic potential \( V = V_0 + V_1 (\phi - \phi_0)^2 \). (This reproduces the solution (7.26)–(7.28) of Ellis & Madsen [164].) However, the chain of assumptions jeopardizes the physical significance of these solutions.

Hawkins & Lidsey discovered that the Einstein–Friedmann equations (2.16)–(2.18) describing the dynamics of a spatially flat universe sourced by a perfect fluid plus a scalar field can be reformulated in terms of the non-linear Ermakov-Pinney equation [209]. Ermakov systems consisting of two coupled, second order, non-linear ordinary differential equations arise in many fields of physics. In one dimension (the case of FLRW cosmology, in which the scale factor \( a(t) \) depends on only one variable), the Ermakov system reduces to a single (“Ermakov-Pinney” or “Milne-Pinney”) equation [167,336] of the form

\[
\frac{d^2 b}{d \tau^2} + Q(\tau) b = \frac{\lambda}{b^3}, \tag{7.37}
\]

where \( \lambda \) is a constant and \( Q(\tau) \) is an arbitrary function. In fact, considering a spatially flat FLRW universe sourced by a perfect fluid with a constant equation of state \( P = w \rho = w \rho^0 / a^{3(w+1)} \) and a minimally coupled scalar field \( \phi \) not interacting directly with it and with potential \( V(\phi) \), the Einstein–Friedmann equations (2.16)–(2.18) can be combined to give

\[
\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = -4\pi \left[ \dot{\phi}^2 + (w + 1) \frac{\rho^0(a^3 (w+1))}{a} \right]. \tag{7.38}
\]

Setting \( n \equiv 3(w + 1) \) and \( a \equiv b^{2/n} \) (with \( b > 0 \)) and changing the time coordinate to \( \tau \) defined by \( d\tau = b dt \), Eq. (7.38) becomes [209]

\[
\frac{d^2 b}{d \tau^2} + 2\pi n \left( \frac{d\phi}{d\tau} \right)^2 b = -\frac{2\pi n^2 \rho^0}{3b^3}, \tag{7.39}
\]

which does not contain \( V(\phi) \) and is of the form (7.37) with

\[
Q(\tau) = 2\pi n \left( \frac{d\phi}{d\tau} \right)^2, \tag{7.40}
\]

\[
\lambda = -\frac{2\pi n^2 \rho^0}{3}. \tag{7.41}
\]

Given two linearly independent solutions \( b_{1,2}(\tau) \) of the associated linear homogeneous equation

\[
\frac{d^2 b}{d \tau^2} + Q(\tau) b = 0, \tag{7.42}
\]
the general solution of the non-linear Ermakov-Pinney equation (7.37) is [336]

\[ b_P(\tau) = \left[ A b_1^2(\tau) + B b_2^2(\tau) + 2 C b_1(\tau) b_2(\tau) \right]^{1/2}, \]  
(7.43)

where \( A, B, \) and \( C \) are constants satisfying

\[ AB - C^2 = \frac{\lambda}{W^2}, \]  
(7.44)

and the Wronskian

\[ W = b_1 \frac{db_2}{d\tau} - b_2 \frac{db_1}{d\tau} \]  
(7.45)

is constant [336].

The Ermakov-Pinney equation is closely related to the one-dimensional Schrödinger equation. Setting now \( \psi \equiv a^{-n/2} \), changing the variable \( t \) to \( \sigma \) defined by \( d\sigma = \psi(t) d\tau \), Eq. (7.38) assumes the Schrödinger form

\[ \frac{d^2\psi}{d\sigma^2} + [E - P(\sigma)] \psi(\sigma) = 0, \]  
(7.46)

where \( E \equiv -2\pi n^2 \rho^{(0)} / 3 \) and

\[ P(\sigma) = 2\pi n \left( \frac{d\phi}{d\sigma} \right)^2 \]  
(7.47)

is the potential.

In practice, the Ermakov-Pinney equation is solved by choosing the scale factor \( a(t) \), solving for \( \phi(t) \), inverting this relation to obtain \( t(\phi) \), and then determining the potential \( V(\phi) = V(t(\phi)) [209] \). As a simple check, for \( \phi = \) const. (which implies \( V(\phi) = \) const., equivalent to a cosmological constant \( \Lambda \)), one recovers Harrison’s solution (5.55) for a single fluid with \( \Lambda [209] \). In general, the potential \( V(\phi) \) may turn out to be physically unmotivated and care must be taken to restrict to situations of physical interest.

Several authors have used the Hawkins-Lidsey reduction to an Ermakov-Pinney equation [197–200,333]. D’Ambroise studied further the equivalence between the Einstein–Friedmann–Klein–Gordon system with a perfect fluid and non-linear Schrödinger equations [143]. Seven exact solutions for a scalar field and a single barotropic fluid \( P = w \rho \) with \( w = \) const. were found [143].

The case of a scalar field and two perfect fluids with constant equation of state \( P_i = w_i \rho_i \) was studied in [241] using a non-linear Schrödinger equation representation, and the D’Ambroise single-fluid solutions were generalized to include the second fluid. Again, the potentials \( V(\phi) \) found do not appear to be physical.

The reductions to the Ermakov-Pinney equation and to linear and non-linear Schrödinger equations have been applied also to spatially curved FLRW universes.
anisotropic Bianchi models [142], braneworld models [209], and quantum cosmology [120,144,347]. There is also a connection with analogue gravity: in the $K = +1$ case, the Ermakov-Pinney equation establishes an analogy with two-dimensional Bose-Einstein condensates [268].

7.3 Multiple non-interacting scalar fields

Inflation with multiple non-interacting scalar fields was studied before the 1998 discovery of the present acceleration of the universe [330,331,340], when it seemed that the total dimensionless density parameter $\Omega_{\text{tot}}$ was approximately 0.3, i.e., before the inclusion in the standard $\Lambda$CDM model of dark energy bringing $\Omega_{\text{tot}}$ up to unity. In fact, $\Omega_{\text{tot}} = 1$ is a robust prediction of single scalar field inflation and a natural way to obtain inflation with $\Omega_{\text{tot}} < 1$ consists of using multiple inflaton fields. These models of multiple field inflation with $\Omega_{\text{tot}} < 1$ were practically abandoned soon after 1998. Modern interest in multiple field inflation originates from the fact that these two-field models [8, 9,28,29,39,40,58,59,74,75,119,166,270,271,274,296,305,332,356,357,401,407] and multiple field models [46,47,217,234,235,341,342,420,421] predict non-Gaussianity in the spectrum of density perturbations. The search for non-Gaussianity by measuring $n$-point correlation functions (with $n \geq 3$) is an active line of research in observational cosmology [8,9,28,29,39,40,46,47,58,59,74,75,119,166,217,234,235,270,271,274,296,305,332,341,342,356,357,401,407,420,421].

Since, in principle, one can mimic a perfect fluid with constant equation of state with a scalar field $\phi$ by imposing that its effective equation of state be constant and determining its potential $V(\phi)$ accordingly, as done by Barrow & Saich [32], all two-fluids exact solutions presented in Secs. 5 and 6 can in principle be reproduced in this way, although the two potentials determined in this way for the two scalar fields will, in general, not be physically motivated.

A potential is necessary because a free scalar field with $V(\phi) \equiv 0$ plus a stiff fluid is equivalent to a single stiff fluid and does not give rise to new solutions beyond what already seen in Sect. 5.

8 Two interacting fluids

The modern interest in two interacting fluids in cosmology arises in relation with the dark energy problem. Early interest in the 1960s and 1970s was motivated by the need to model the conversion of radiation into non-relativistic matter in the early universe. We proceed with some general considerations applied to interacting dark energy scenarios before moving to the early literature.

8.1 Interacting dark energy and dark matter

The standard $\Lambda$CDM model of the universe contains, in addition to a small fraction of baryonic matter, dark energy and dark matter, the two most abundant constituents the nature of which is completely unknown [4]. Dark energy and dark matter, the two
biggest mysteries of ΛCDM cosmology, are normally treated as two separate physical sectors. However, it has been speculated [66] that these two sectors could couple explicitly in the so-called “interacting dark energy” scenarios. The phenomenological implications of such an interaction would be significant: the coincidence problem of why the current densities \( \Omega_{\text{DE}} \) and \( \Omega_{\text{DM}} \) of dark energy and dark matter are of the same order of magnitude today, although they evolve differently, would be alleviated. In interacting dark energy models, the interaction is modelled by modifying arbitrarily the equations of motion for dark energy and dark matter, described as interacting perfect fluids. In this picture, these fluid stress-energy tensors are not covariantly conserved separately but the total \( T_{ab}^{(\text{tot})} = T_{ab}^{(\text{DE})} + T_{ab}^{(\text{DM})} \) is:

\[
\nabla^b T_{ab}^{(\text{DM})} = Q_a, \\
\nabla^b T_{ab}^{(\text{DE})} = -Q_a, 
\]

(8.1)

so that

\[
\nabla^b T_{ab}^{(\text{tot})} = 0. 
\]

(8.2)

The four-vector \( Q^a \) describes phenomenologically the interaction. Its detailed form is picked by hand in the literature, reflecting our ignorance about the dark sectors, while a proper description would express the unknown physics of the interaction (see Refs. [26,187,432] for attempts to derive the vector \( Q^a \) from physical considerations). What is more, it is not even clear how to provide a Lagrangian description of the interaction, as should always be done in fundamental physics. With the exception of Einstein frame scalar-tensor gravity (see Sect. 8.4), we lack even an effective field theory Lagrangian for this purpose, in spite of substantial theoretical efforts to describe dark energy (and its contender, modified gravity) via effective field theories [67,196,212].

The standard description in the large literature (e.g., [18,45,63,71,86,102,123,135,137,188,210,211,223,233,258,312,313,374,396,412,419,433], see [369,408] for reviews) assumes a spatially flat FLRW universe filled by two perfect fluids satisfying the equations

\[
\dot{\rho}_1 + 3H (P_1 + \rho_1) = Q, \\
\dot{\rho}_2 + 3H (P_2 + \rho_2) = -Q. 
\]

(8.3)

(8.4)

where the quantity \( Q(t) \) models the interaction. The explicit forms of \( Q(t) \) adopted in the literature are arbitrary: usually this quantity is assumed to be a function of one or more of \( a(t), \dot{a}(t), H(t), \rho_{1,2}(t) \) or of powers of them [45,63,71,86,102,123,135,137,188,210,211,223,233,258,312,313,374,396,412,419,433]. Attempts can be made to constrain the form of \( Q(t) \) using cosmological observations [135,210,211,419].

The total fluid with energy density \( \rho_{\text{tot}} = \rho_1 + \rho_2 \) and pressure \( P_{\text{tot}} = P_1 + P_2 \) obeys the conservation equation

\[
\dot{\rho}_{\text{tot}} + 3H (P_{\text{tot}} + \rho_{\text{tot}}) = 0. 
\]

(8.5)
The extensive literature on interacting dark energy leaves the basic questions of a covariant and Lagrangian formulation almost completely unanswered, presenting models already set up in comoving coordinates in the FLRW universe. A speculative covariant description of the possible dark energy-dark matter interaction has been proposed in Ref. [175]. There, the two fluids are described by the modified energy-momentum tensors

\[ T^{(1)}_{ab} = (P_1 + \rho_1) u_a u_b + P_1 g_{ab} + q_a u_b + q_b u_a, \]  
\[ T^{(2)}_{ab} = (P_2 + \rho_2) u_a u_b + P_2 g_{ab} - q_a u_b - q_b u_a, \]

where \( u^a \) is a common 4-velocity pointing in the time direction of the observers comoving with both fluids. The four-vector \( q^c \) describes an energy flux density between the two fluids. In a spatially isotropic FLRW universe, \( q^c \) cannot have spatial components in comoving coordinates and points in the direction of comoving time, \( i.e., \)

\[ q^c = \alpha(t) u^c, \]

where \( \alpha \) is a function of time (with \( \alpha \geq 0 \) to keep \( q^c \) future-oriented).

The two fluids are imperfect fluids, but by all means not in the usual sense of dissipative fluid used in the relativity literature (\( e.g., [160,382] \)). Usually the dissipative term \( q_a u_b + q_b u_a \) in the imperfect fluid stress-energy tensor describes a purely spatial (therefore, non-causal) heat flux density \( q^c \) with \( q^c u_c = 0 \) [160,382]. Here, instead, \( q^c \) must point parallel to \( u^c \), otherwise it violates spatial isotropy. The traces of the fluids stress-energy tensors \( T^{(i)}_{ab} \) are

\[ T^{(i)} = -\rho_i + 3 P_i \mp 2\alpha. \]

While the total stress-energy tensor \( T^{(\text{tot})}_{ab} = T^{(1)}_{ab} + T^{(2)}_{ab} \) is covariantly conserved, each component \( T^{(i)}_{ab} \) of the mixture satisfies

\[ \nabla^b T^{(i)}_{ab} = u_a u^b \nabla_b P_i + u_a u^b \nabla^b (\rho_i \mp 2\alpha) + \nabla_a P_i \\
+ (P_i + \rho_i \pm 2\alpha) u^b \nabla_b u_a \\
+ (P_i + \rho_i \pm 2\alpha) u_a \nabla^b u_b. \]  

(8.10)

Here \( i = 1, 2 \), the upper sign corresponds to fluid 1, and the lower one to fluid 2. By projecting this equation on the time direction one obtains

\[ u^a \nabla^b T^{(i)}_{ab} = (\dot{\rho}_i \pm 2\dot{\alpha}) + 3H (P_i + \rho_i \pm 2\alpha), \]

(8.11)

with the usual notation \( \dot{\rho}_i \equiv u^a \nabla_a \rho_i \). If we require that \( u^a \nabla^b T^{(i)}_{ab} = 0 \), the two fluids are covariantly conserved separately, but their perfect-fluid components \( (P_i + \rho_i) u_a u_b + P_i g_{ab} \) are not because

\[ u^a \nabla^b [(P_i + \rho_i) u_a u_b + P_i g_{ab}] = \pm 2 \left( \dot{\alpha} + \alpha \nabla_b u^b \right). \]

(8.12)
In a FLRW universe this equation assumes the form

\[ \dot{\rho}_i + 3H (P_i + \rho_i) = \mp 2 (\dot{\alpha} + 3H \alpha) \]  (8.13)

and one naturally reads the right hand side as \( \pm Q(t) \), reproducing Eqs. (8.3) and (8.4). Then \( \alpha(t) \) and \( Q(t) \) are related by

\[ \dot{\alpha} + 3H \alpha + \frac{Q(t)}{2} = 0 \]  (8.14)

or, equivalently,

\[ \frac{1}{a^3} \frac{d}{dt} \left( \alpha a^3 \right) + \frac{Q(t)}{2} = 0, \]  (8.15)

yielding

\[ \alpha(t) = -\frac{1}{2a^3(t)} \int dt \ a^3(t) Q(t). \]  (8.16)

A possible physical interpretation is proposed in [175]: fluid 1, which is is not a perfect fluid, has effective energy density and pressure

\[ T_{ab}^{(1)} u^a u^b = \rho_1 + 2\alpha \neq \rho_1, \]

\[ P_1 = \frac{1}{3} T_{ab}^{(1)} h^{ab}, \]  (8.17)

while, for fluid 2,

\[ T_{ab}^{(2)} u^a u^b = \rho_2 - 2\alpha \neq \rho_2, \]  (8.18)

\[ P_2 = \frac{1}{3} T_{ab}^{(2)} h^{ab}. \]  (8.19)

The two fluids are not perfect fluids because their stress-energy tensors \( T_{ab}^{(i)} \) contain the terms \( \pm (q_a u_b + q_b u_a) \) describing an energy transfer occurring simultaneously at all points of space without three-dimensional flux. The energy lost by fluid 1 per unit time and per unit volume is transferred to fluid 2, according to the splitting (8.3)–(8.4).

An alternative interpretation [175] is the following: when \( \alpha > 0 \), the term \( 2 (\dot{\alpha} + 3H \alpha) \) appearing in fluid 1 corresponds to a dust with zero pressure and energy density \( 2\alpha \) which transfers energy instantaneously to fluid 1 taking it from fluid 2. According to fluid 2, a dust with negative energy density \( -2\alpha \) removes energy to transfer it to fluid 1. This second dust obviously violates the weak energy condition, but this problem is not more significant than the lack of causality of usual imperfect fluids with spacelike heat currents [160,382].
The relation (8.15) between the quantities $Q(t)$ and $\alpha(t)$ becomes, in this picture,

$$Q(t) = -\frac{2}{a^3} \dot{a}^3. \quad (8.20)$$

For a three-dimensional region with unit comoving volume and physical volume $a^3$, $-2\alpha a^3$ is the energy transferred between the two fluids in this volume, $-2(2\alpha a^3)\dot{a}$ is the energy transfer rate, while $Q(t)$ is the rate of energy transferred per unit volume.

### 8.2 Interacting radiation and non-relativistic matter

Early work on interacting fluids was motivated by the need to model the conversion of the cosmic fluid from radiation (with energy density and pressure $\rho_r, P_r$) to non-relativistic matter (with $\rho_m$ and $P_m = 0$) and is mainly due to Davidson & Narlikar [141] and McIntosh [289, 291, 292] for spatially flat universes, May & McVittie [287] for $K = -1, 0$ and again to May & McVittie [288] for $K = +1$. McIntosh assumed an *ad hoc* form of the total equation of state parameter

$$w_{tot}(t) \equiv \frac{P_{tot}}{\rho_{tot}} = \frac{P_r}{\rho_r + \rho_m} \quad (8.21)$$

interpolating between $1/3$ at early times and 0 at late times [291, 292], for example [291]

$$w_{tot}(t) = \frac{1}{3 (1 + t/t_0)^\alpha}, \quad (8.22)$$

where $\alpha$ and $t_0$ are constants. May & McVittie allowed for the possibility that $P_m \neq 0$ (all these early works assumed $\Lambda = 0$). Instead of prescribing a function $w_{tot}(t)$, they introduced a non-negative function $\phi$ such that $\dot{a} = g(\phi)$, where [287]

$$g(\phi) = \begin{cases} 
\cot \phi & \text{if } K = +1, \\
1/\phi & \text{if } K = 0, \\
\coth \phi & \text{if } K = -1. 
\end{cases} \quad (8.23)$$

g(\phi) satisfies the differential equation [287]

$$\frac{dg}{d\phi} + g^2 = -K \quad (8.24)$$

and the scale factor is given by

$$a(t) = \frac{3w_{tot}(t) + 1}{2\phi}. \quad (8.25)$$
The covariant conservation equation for the fluid mixture $\dot{\rho}_{\text{tot}} + 3H (P_{\text{tot}} + \rho_{\text{tot}}) = 0$ is equivalent to

$$\frac{d}{dt} \left( \frac{3w_{\text{tot}}(t) + 1}{\phi} \right) = 2g(\phi),$$  \hspace{1cm} (8.26)

which determines $\phi(t)$ and $a(t)$ once $w_{\text{tot}}(t)$ is assigned. The (non-)conservation equations for radiation and matter are

$$\dot{\rho}_r + 3H (P_r + \rho_r) = Q_r,$$  \hspace{1cm} (8.27)

$$\dot{\rho}_m + 3H (P_m + \rho_m) = Q_m,$$  \hspace{1cm} (8.28)

where $Q_r$ is the radiation energy per unit volume converted into matter energy per unit time and

$$Q_r + Q_m = 0.$$  \hspace{1cm} (8.29)

A mixture of non-interacting radiation and dust ($P_m = 0$) is obtained as the trivial case $Q_r = Q_m = 0$, with

$$w_{\text{tot}}(t) = \frac{1}{1 + \frac{\rho_m(0)}{\rho_r} a}.$$  \hspace{1cm} (8.30)

Eq. (8.24) can be rewritten as [287]

$$\frac{d}{d\phi} \left\{ \ln \left[ (3w_{\text{tot}} + 1) \frac{dt}{d\phi} \right] \right\} = \frac{2g(\phi)}{3w_{\text{tot}} + 1}$$  \hspace{1cm} (8.31)

and integrated twice to

$$\frac{3w_{\text{tot}} + 1}{\phi} = B \exp \left( 2 \int d\phi \frac{g}{3w_{\text{tot}} + 1} \right),$$  \hspace{1cm} (8.32)

$$t - t_0 = B \int \frac{d\phi}{3w_{\text{tot}} + 1} \exp \left( 2 \int d\phi \frac{g}{3w_{\text{tot}} + 1} \right)$$  \hspace{1cm} (8.33)

with $B$ and $t_0$ integration constants [287]. Then Eq. (8.25) gives

$$a(t) = \frac{B}{2} \exp \left( 2 \int d\phi \frac{g}{3w_{\text{tot}} + 1} \right).$$  \hspace{1cm} (8.34)

Using

$$F \equiv \exp \left( -2 \int d\phi \frac{g}{3w_{\text{tot}} + 1} \right),$$  \hspace{1cm} (8.35)
one has
\[ \rho_{\text{tot}}(t) = \frac{3F^2}{2\pi B^2} \left( g^2 + K \right), \]  
(8.36)
\[ P_{\text{tot}}(t) = \frac{3F^2}{2\pi B^2} w_{\text{tot}} \left( g^2 + K \right), \]  
(8.37)
and
\[ \rho_{\text{m}} = -\rho_r + 3\rho_{\text{tot}}^{(0)} F^2 \left( g^2 + K \right), \]  
(8.38)
where
\[ \rho_{\text{tot}}^{(0)} \equiv \frac{1}{2\pi B^2}. \]  
(8.39)
By further introducing \([287]\)
\[ h(\phi) \equiv \frac{P_m}{\rho_{\text{tot}}^{(0)}}, \]  
(8.40)
the transfer rate is computed as
\[ Q_m = 3\rho_{\text{tot}}^{(0)} \phi \left\{ F^4 \frac{d}{d\phi} \left( \frac{h}{F^4} \right) \right. \]
\[ \left. - \left[ \frac{6w_{\text{tot}}(1-3w_{\text{tot}})}{3w_{\text{tot}} + 1} g + 3 \frac{dw_{\text{tot}}}{d\phi} \right] F^2 \left( g^2 + K \right) \right\}. \]  
(8.41)
The relation between \(w_{\text{tot}}(t)\) and \(g(\phi)\) in \([287]\) is not physically transparent and is dictated more by the mathematics than the physics. Other models of (possibly tilted) interacting radiation and dust fluids were studied over the years, see e.g. \([129,130,306,307]\).

### 8.3 Scalar field interacting with a perfect fluid

A scalar field can couple to a fluid or to another field. Consider a perfect fluid with energy density \(\rho_1\) and pressure \(P_1\) coupling explicitly to a scalar field \(\phi\). Their interaction is again described by the equations
\[ \dot{\rho}_1 + 3H (P_1 + \rho_1) = Q, \]  
(8.42)
\[ \dot{\rho}_\phi + 3H (P_\phi + \rho_\phi) = -Q; \]  
(8.43)
adding them leads to the conservation equation for the total fluid with energy density \(\rho_{\text{tot}} = \rho_1 + \rho_\phi\) and pressure \(P_{\text{tot}} = P_1 + P_\phi\).

Historically, interest in a scalar field directly coupled to matter can be found in the Hoyle-Narlikar steady state theory \([214,215]\) in which matter is created by the so-called...
C-field (for “creation”), a scalar field transferring its energy into matter. Although is usually referred to as a scalar-tensor theory, the energy transfer from the scalar C-field requires only a direct coupling in the context of general relativity [214,292]. As the steady-state theory was abandoned by the cosmological community, the interest in a direct coupling between a scalar and ordinary matter waned.

In the 1980s literature devoted to reheating the universe after it is cooled by the inflationary expansion, an interaction between the inflaton scalar field and radiation was introduced in a phenomenological way in order to model the decay of the inflaton into ultrarelativistic particles. This process should happen due to the inflaton’s coupling to other particles and should reheat the universe to allow for primordial nucleosynthesis and the formation of what is now the cosmic microwave background [239,240]. (Later research produced more sophisticated scenarios for ending inflation, most notably parametric amplification during the so-called preheating, see Refs. [11,19] for reviews.) The phenomenological interaction has the form

\[ Q = \Gamma \phi^2, \]

where \( \Gamma \) is a positive constant. The equation of motion (8.43) for the scalar is modified to

\[ \ddot{\phi} + 3H \dot{\phi} + \dot{\Gamma} \phi + \frac{dV}{d\phi} = 0. \tag{8.45} \]

Since\(^{13} \dot{\phi} \neq 0 \) this equation becomes a Klein–Gordon one with a potential augmented by a friction term of strength \( \Gamma \) proportional to the inflaton speed \( \dot{\phi} \). Taking this phenomenological description as paradigmatic for a scalar field-fluid interaction leads to

\[ \ddot{\phi} + 3H \dot{\phi} + \dot{\Gamma} \phi + \frac{dV}{d\phi} = 0 \tag{8.46} \]

for the scalar and to

\[ \dot{\rho}_1 + 3H (P_1 + \rho_1) = \dot{\Gamma} \phi \tag{8.47} \]

for the fluid 1 which has now a source \( \dot{\Gamma} \phi \) on the right hand side. This description can be related to the covariant description of two interacting fluids [175] described in Sect. 8.1. Then,

\[ \alpha(t) = -\frac{\Gamma}{2a^3} \int dt \, a^3 \dot{\phi}^2 \tag{8.48} \]

depends only on the kinetic energy density \( \dot{\phi}^2 / 2 \) of \( \phi \). The decay of the field \( \phi \) into fluid stops when \( \phi \) becomes static.

\(^{13} \) It is \( \dot{\phi} \neq 0 \) because, if \( \phi = \phi_0 = \text{const.} \), the scalar field effective fluid reduces to a pure cosmological constant \( \Lambda = V(\phi_0) \) and decouples from the perfect fluid.
8.4 Einstein frame formulation of scalar-tensor gravity

Although we confine ourselves to Einstein gravity in the rest of this review, here we make an exception to mention one Lagrangian and covariant formulation of interacting dark energy. In the context of alternative theories of gravity, when scalar-tensor [56,70,315,402] or $f(R)$ [85,95,148,314,375] gravity is reformulated in the Einstein conformal frame, the scalar field degree of freedom looks like an ordinary scalar field in general relativity, except for the fact that it couples explicitly to matter. If the latter is the dark matter fluid in a FLRW cosmology, one then obtains a direct interaction of this dark matter with the extra scalar degree of freedom (which could be used to model dark energy) contained in the theory in addition to the two massless spin two modes of Einstein theory.

Let us illustrate how this works using Jordan-Brans-Dicke gravity [70,224,225] and cosmology [171,186]. Brans-Dicke’s is the simplest scalar-tensor framework and the prototypical alternative to general relativity. The Jordan frame Brans-Dicke action is [70]

$$S^{(BD)} = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right] + S^{(m)}, \quad (8.49)$$

where $S^{(m)} = \int d^4 x \sqrt{-g} \mathcal{L}^{(m)}$ describes the matter sector and the Brans-Dicke scalar field $\phi > 0$ is, roughly speaking, the inverse of the effective gravitational coupling strength $G_{\text{eff}}$, which becomes a field in this theory. The dimensionless parameter $\omega$ is the “Brans-Dicke coupling” [70] and the Jordan frame is the pair of dynamical variables $(g_{ab}, \phi)$.

There is a second representation of the theory, the so-called Einstein conformal frame, i.e., the pair of variables $(\tilde{g}_{ab}, \tilde{\phi})$ where the new metric $\tilde{g}_{ab}$ is obtained from $g_{ab}$ by means of the conformal transformation

$$g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \Omega = \sqrt{\phi}, \quad (8.50)$$

while the new scalar field comes from the non-linear redefinition\(^{14}\)

$$\tilde{\phi}(\phi) = \sqrt{\frac{2\omega + 3}{16\pi G}} \ln \left( \frac{\phi}{\phi_0} \right) \quad (8.51)$$

(\(\omega > -3/2\)). Under this transformation, the Brans-Dicke action (8.49) assumes its Einstein frame form [153]

$$S = \int d^4 x \left\{ \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\phi} \tilde{\nabla}_b \tilde{\phi} - U(\tilde{\phi}) \right] + e^{-8\sqrt{\frac{\pi G}{2(\omega + 3)}} \tilde{\phi}} \tilde{\mathcal{L}}^{(m)}(\tilde{g}) \right\}, \quad (8.52)$$

\(^{14}\) This non-linear redefinition turns the kinetic energy density of $\tilde{\phi}$ into its canonical form.
where $\tilde{\nabla}_a$ is the covariant derivative operator of the rescaled metric $\tilde{g}_{ab}$ and

$$U \left( \frac{\phi}{\tilde{\phi}} \right) = V \left[ \phi \left( \frac{\phi}{\tilde{\phi}} \right) \right] \exp \left( -8\sqrt{\frac{\pi G}{2\omega + 3}} \frac{\phi}{\tilde{\phi}} \right). \quad (8.53)$$

In the Einstein conformal frame the scalar field $\tilde{\phi}$ couples directly to matter through the exponential factor multiplying the Lagrangian density $\mathcal{L}^{(m)}$ in the action (8.52). As a consequence, in this frame the matter energy-momentum tensor is not covariantly conserved. Mathematically, this happens because the equation $\nabla^b T^{(m)}_{ab} = 0$ is not conformally invariant [404]. The conformally transformed stress-energy tensor $\tilde{T}^{(m)}_{ab}$ obeys the corrected equation

$$\tilde{\nabla}^b \tilde{T}^{(m)}_{ab} = -\frac{d}{d\phi} \left[ \ln \Omega (\phi) \right] \tilde{T}^{(m)}_{ab} \tilde{\nabla}_a \phi. \quad (8.54)$$

Only conformally invariant matter with vanishing trace $T^{(m)} = 0$ remains covariantly conserved after the conformal transformation.

Consider the Einstein frame stress-energy tensor derived from the usual expression [404]

$$\tilde{T}^{(m)}_{ab} = \frac{-2}{\sqrt{-\tilde{g}}} \frac{\delta \left( \sqrt{-\tilde{g}} \mathcal{L}^{(m)} \right)}{\delta \tilde{g}^{ab}}; \quad (8.55)$$

it is easy to see that

$$\tilde{T}^{(m)}_{ab} = \Omega^{-2} T^{(m)}_{ab}, \quad (8.56)$$

$$\tilde{T}^{(m)}_{ab} = \Omega^{-4} T^{(m)}_{ab}, \quad (8.57)$$

$$\tilde{T}^{(m)}_{ab} = \Omega^{-6} T^{(m)}_{ab}, \quad (8.58)$$

and

$$\tilde{T}^{(m)} = \Omega^{-4} T^{(m)}. \quad (8.59)$$

In particular, a perfect fluid stress-energy tensor maps to

$$\tilde{T}^{(m)}_{ab} = \left( \tilde{P}^{(m)} + \tilde{\rho}^{(m)} \right) \tilde{u}_a \tilde{u}_b + \tilde{P}^{(m)} \tilde{g}_{ab} \quad (8.60)$$

under the conformal transformation, and $\tilde{g}_{ab} \tilde{u}^a \tilde{u}^b = -1$ yields

$$\tilde{u}^a = \Omega^{-1} u^a, \quad \tilde{u}_a = \Omega u_a. \quad (8.61)$$
As a consequence,

\[
\left( \tilde{P}^{(m)} + \tilde{\rho}^{(m)} \right) \tilde{u}_a \tilde{u}_b + \tilde{P}^{(m)} \tilde{g}_{ab} = \Omega^{-2} \left[ \left( P^{(m)} + \rho^{(m)} \right) u_a u_b + P^{(m)} g_{ab} \right],
\]

(8.62)
as follows from Eqs. (8.58), (8.60), and (8.61). Therefore, it must be

\[
\tilde{\rho}^{(m)} = \Omega^{-4} \rho^{(m)}, \quad \tilde{P}^{(m)} = \Omega^{-4} P^{(m)}.
\]

(8.63)

Due to Eq. (8.63), a Jordan frame fluid satisfying the barotropic equation of state \( (2.4) \) is mapped into an Einstein frame fluid with the same equation of state.

Specializing to FLRW universes, the conformal transformation maps the Jordan frame fluid conservation equation

\[
\frac{d \rho^{(m)}}{dt} + 3H \left( P^{(m)} + \rho^{(m)} \right) = 0
\]

(8.64)
to

\[
\frac{d \tilde{\rho}^{(m)}}{dt} + 3 \tilde{H} \left( \tilde{P}^{(m)} + \tilde{\rho}^{(m)} \right) = \frac{d \left( \ln \Omega \right)}{d \phi} \dot{\phi} \left( 3 \tilde{P}^{(m)} - \tilde{\rho}^{(m)} \right).
\]

(8.65)

Stepping back to general spacetimes, the stress-energy tensor \( T^{(m)}_{ab} \) scales under the conformal transformation according to [404]

\[
\tilde{T}^{ab}_{(m)} = \Omega^s T^{ab}_{(m)}, \quad \tilde{T}^{(m)}_{ab} = \Omega^{s+4} T_{ab}^{(m)},
\]

(8.66)
where \( s \) is an appropriate conformal weight. In four spacetime dimensions, the Jordan frame covariant conservation equation \( \nabla^b T^{(m)}_{ab} = 0 \) corresponds to [171,404]

\[
\nabla_a \left( \Omega^s T^{ab}_{(m)} \right) = \Omega^s \nabla_a T^{ab}_{(m)} + (s + 6) \Omega^{s-1} T^{ab}_{(m)} \nabla_a \Omega + \Omega^{s-1} \delta^{ab} T^{(m)} \nabla_a \Omega.
\]

(8.67)

The choice of conformal weight \( s = -6 \) is consistent with Eq. (8.56) and gives

\[
\tilde{T}^{(m)} = \tilde{g}^{ab} \tilde{T}^{(m)}_{ab} = \Omega^{-4} T^{(m)},
\]

(8.68)
while Eq. (8.67) has the conformal cousin

\[
\nabla_a \tilde{T}^{ab}_{(m)} = -\tilde{T}^{(m)} \tilde{g}^{ab} \nabla_a \left( \ln \Omega \right)
\]

(8.69)
in the tilded world. For Brans-Dicke theory, where \( \Omega = \sqrt{\phi} \), one has \([70,171,186,402]\)

\[
\tilde{\nabla}_a \tilde{T}^{ab}_{(m)} = -\frac{1}{2\phi} \tilde{T}^{(m)} \tilde{\nabla}_b \phi = -\sqrt{\frac{4\pi G}{2\omega + 3}} \tilde{T}^{(m)} \tilde{\nabla}_b \tilde{\phi}.
\] (8.70)

Consider now a dust fluid describing dark and baryonic matter in a FLRW universe. Setting \( P^{(m)} = 0 \) yields

\[
\tilde{u}_a \tilde{u}_b \tilde{\nabla}^b \tilde{\rho}^{(m)} + \tilde{\rho}^{(m)} \tilde{u}_a \tilde{\nabla}^b \tilde{u}_b + \tilde{\rho}^{(m)} \tilde{u}_c \tilde{\nabla}^c \tilde{u}_a - \sqrt{\frac{4\pi G}{2\omega + 3}} \tilde{\rho}^{(m)} \tilde{\nabla}_a \tilde{\phi} = 0,
\] (8.71)

which shows explicitly how the scalar field interacts with ordinary matter through its gradient. To conclude, this picture gives a completely covariant and Lagrangian formulation of a scalar field interacting with a matter fluid. In the Jordan frame this scalar has gravitational nature but this information does not matter in the Einstein frame, in which \( \tilde{\phi} \) could be regarded as standard scalar field, except for its anomalous coupling to (non-conformal) matter.

In principle, when performing the conformal transformation one should also rescale all fundamental and derived units, according to Dicke’s prescription [153], with the result that the two conformal frames would then be physically equivalent [153]. This physical equivalence has been the subject of a debate filling a fairly large literature. This equivalence issue is immaterial here since there is no doubt that the transformations (8.50) and (8.51) constitute a well-defined map between the two worlds and here we are interested in mathematical solutions for a scalar field coupled to a fluid (or for two coupled effective fluids). Therefore, Jordan frame cosmological solutions in the Jordan frame (where \( G_{\text{eff}} \simeq \phi^{-1} \) varies) translate into mathematical solutions in the Einstein frame, where formally the theory can be interpreted as general relativity with the anomalous coupling of an (otherwise canonical) scalar field \( \tilde{\phi} \) to the cosmological fluid(s).

One could think of using a tensor-multi-scalar theory to obtain multi-fluids in the Einstein conformal frame. However, in general, one cannot recast the transformed fields so that more than one has canonical kinetic energy [227]. The chiral cosmological models with non-canonical kinetic energies thus obtained still have analytical solutions. A methods to search for them has been proposed in [106]. This is a generalization of a previous method designed to obtain solutions of quintom models with potential in [24,399].

There are other methods to find analytical cosmological solutions with nonminimally coupled models scalar fields [229] and to connect integrable cosmologies in the Jordan and the Einstein frames [230,231].

### 8.5 Two interacting scalar fields

There are cosmological scenarios in which both interacting fluids are actually effective fluids coming from scalar fields \( \phi \) and \( \psi \). In the previous picture, if the first fluid is
the $\psi$-fluid with self-interaction potential $U(\psi)$, then

$$\rho_1 = \frac{\dot{\psi}^2}{2} + U(\psi), \quad (8.72)$$

$$P_1 = \frac{\dot{\psi}^2}{2} - U(\psi), \quad (8.73)$$

and the equation of motion for $\psi$ becomes

$$\ddot{\psi} + 3H\dot{\psi} - \Gamma \frac{\dot{\phi}^2}{\dot{\psi}} + \frac{dU}{d\psi} = 0 \quad (8.74)$$

(assuming $\dot{\psi} \neq 0$). When $|\dot{\psi}|$ is large and increasing and $\Gamma > 0$, the extra term $-\Gamma \frac{\dot{\phi}^2}{\dot{\psi}}$ enhances the motion of $\psi$ and plays the role of antifriction for $\psi$. If $\psi$ decreases, this term describes an effective friction opposing the motion of $\psi$. The quantity $\alpha$ given by (8.48) can still be introduced.

More generally, one can couple the two scalars by introducing a common potential $V(\phi, \psi)$ that modifies their equations of motion to

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \quad (8.75)$$

$$\ddot{\psi} + 3H\dot{\psi} + \frac{\partial V}{\partial \psi} = 0. \quad (8.76)$$

This is done in preheating scenarios designed to end inflation and raise the temperature of the universe [11, 19]. One of the two fields is the inflaton and the other denotes any field to which the inflaton couples (bosonic fields are generated in the early stages and fermionic fields near the end of preheating). The interaction potential commonly used in preheating has the form

$$V(\phi, \psi) = m_\phi^2 \phi^2 + m_\psi^2 \psi^2 + g \phi^2 \psi^2. \quad (8.77)$$

Quintom models of dark energy (see Ref. [76] for a review) are two-component dark energy models in which a phantom scalar field interacts with a second scalar field. Sometimes, for simplicity, the two scalars are modelled with two fluids [425].

9 Concluding remarks

The scientist’s view of multiple fluid FLRW cosmology has changed significantly since the discovery of the first mathematical solutions describing this physics in the 1960s and 1970s. Early interest was in radiation and dust filling a (possibly spatially curved) FLRW universe and was motivated by the recent discovery of the cosmic microwave background radiation by Penzias and Wilson [329], and expanded to consider any two fluids that could give exact solutions in terms of elementary functions. The search
for FLRW solutions quickly expanded to include effective fluids describing spatial curvature and the cosmological constant.

The modern view is quite different: the standard $\Lambda$CDM model of the universe includes dark energy and dark matter fluids in a spatially flat FLRW universe. Attempts to explain the current acceleration of the universe without the ad hoc dark energy by modifying gravity at large scales instead [148,314,375] end up being not so different from dark energy scenarios because in a FLRW universe the deviations of gravity from general relativity in the field equations can be grouped in the right hand side, producing an effective perfect fluid in a wide spectrum of theories of gravity [203].

Another ingredient that was hardly present in cosmology in the 1960s is the scalar field: since 1980 we have witnessed an enormous interest in inflationary scenarios of the early universe. Inflation solves the three problems of standard Big Bang cosmology, i.e., the horizon, flatness, and monopole problems and provides a mechanism to generate density fluctuations which constitute the seeds for structure formation in the form of quantum fluctuations of the inflaton. In most inflationary scenarios, the inflaton driving the rapid expansion of the universe is a scalar field. This is true also for $R^2$ inflation [379,380] which, although correcting Einstein theory, can be seen as the result of the dynamics of the effective extra scalar degree of freedom. It is now well known that a scalar field $\phi$ with timelike gradient $\nabla_a \phi$ is equivalent to a perfect fluid which, in general, has a dynamical equation of state. It was realized rather early that a free scalar is equivalent to a stiff fluid, which appears in several integrability cases of the Einstein–Friedmann equations, but interesting physical scenarios for the scalar field were still missing.

Having learned the lesson from inflation, the scalar field was naturally the main ingredient for modelling dark energy after the 1998 discovery of the cosmic acceleration with type Ia supernovae [330,331,340]. In all these situations dark and baryonic matter form one fluid and dark energy, or its scalar field version, another fluid. It is possible, and it would be even convenient for modellers if these dark fluids were interacting directly, which has been the subject of a considerable literature. In the meantime, scalar-tensor gravity originally proposed by Brans & Dicke [70] and later generalized [56,315,402], has been the subject of much interest and many other theories of gravity have been studied (see the reviews [89,124,207,212,413,414]). After 2004, $f(R)$ gravity (a subclass of scalar-tensor gravity) was studied intensely to explain the present acceleration of the universe [148,314,375]. Nowadays, testing gravity constitutes added value for astrophysical and cosmological observations and satellite missions at vastly different scales, as well as theoretical research programs. In the last decade or so, much interest has been devoted to “second generation” scalar-tensor theories of gravity (see [237,246] for reviews) known as Horndeski [146,147,213,236,308] and beyond-Horndeski and Degenerate Higher Order Scalar-Tensor (DHOST) theories [51,52,136,191,192,244,245,300]. They explore the vast landcape of the most general scalar-field based theories of gravity that have second order equations and are compatible with theoretical and observational constraints [65]. The motivation for studying these theories is primarily cosmological and, eventually, multi-fluid FLRW universes enter the description of these cosmologies.

While trying to connect the old with the new literature, we have reported analytical multi-(effective) fluid solutions expressed by an integral in finite form containing only
elementary functions. Either the scale factor is a function \( a(t) \) of the comoving time \( t \), or the latter is a function \( t(a) \) (this relation can sometimes be inverted to obtain \( a(t) \) explicitly). Alternatively, the solution is expressed in parametric form with the conformal time \( \eta \) as the parameter.

A key element of the discussion is the treatment of the spatial curvature of FLRW universes and of the cosmological constant \( \Lambda \) as effective fluids, reducing formally the Einstein–Friedmann equations to those for an effective \( K = 0, \Lambda = 0 \), multi-fluid situation. While this procedure is followed routinely for the cosmological constant and for the effective energy density of spatial curvature, it is not normally implemented for the effective pressure of the latter.

Following the key realization of Refs. [103,222,293,294] that the integral expressing the scale factor has an hypergeometric series representation which truncates for special values of its arguments (or, alternatively, using the Chebysev theorem of integration), the two-fluid situations in which \( t(a) \) or \( a(t) \) can be obtained analytically in terms of elementary functions can be determined. One can then compile a catalogue of two-fluid solutions expressed in terms of elementary functions and select those of physical interest.

To keep this review manageable, we restricted its scope by excluding from it certain extensions, and related topics that are of interest in modern cosmology. First, we did not discuss tilted fluids [127,129–132,306,307,372,398,403] and imperfect fluids, which are interesting possibilities to consider. FLRW cosmologies do not necessarily represent perfect fluid solutions but can also describe a viscous fluid in a non-comoving frame. Imperfect fluids are characterized by spacelike energy fluxes and anisotropic stresses [160,382,404], which are excluded by spatial isotropy in a FLRW universe, but bulk viscosity is allowed and has been studied extensively in the literature. Bulk viscosity could be due, for example, to particle production in the early universe. Likewise, we did not discuss multi-fluid or scalar field anisotropic Bianchi universes and cosmic no-hair theorems [193]. Finally, multi-fluid FLRW solutions in alternative theories of gravity have been omitted, except for the Einstein frame description of scalar-tensor gravity, which is formally equivalent to a cosmological scalar field always present in the theory that couples explicitly to the matter sector and could explain interacting dark energy in a fully covariant and Lagrangian way. The literature on modified gravity and cosmology is, however, huge and a comprehensive review of multi-fluid or multi-effective fluid cosmology in all of these theories would be a book-size undertaking which goes well beyond the purposes of this work.

We did not mention Chaplygin gas cosmology, in which a single gas interpolates between different fluids during the evolution of the universe [228]. This is an interesting alternative and the relations between its FLRW solutions and those of multi-fluids are not completely explored. At the same time, we restricted to exact FLRW universes, ignoring their perturbations. However, the latter are the main source of information in modern cosmology through the imprints that they left in the cosmic microwave background and through large-scale structure surveys. Perturbations of multi-fluid FLRW universes constitute an important part of cosmology.

Even restricting the scope of this work, however, left many interesting multi-(effective) fluids to contemplate. This richness is due to the variety of real or effective matter sources, including real fluids with barotropic equations of state, scalar fields
with a large variety of physically interesting potentials, the gravitational scalar field of scalar-tensor gravity in the Einstein frame, the cosmological constant $\Lambda$, and spatial curvature. The knowledge of analytical solutions must be supplemented by phase space analysis to determine which ones are attractors in phase space and, therefore, physically important. This type of study is fundamental in theories of inflation and dark energy. Nevertheless, the proliferation of inflationary and dark energy models, and the associated phase space studies has sometimes obscured the full covariance of a theory or its Lagrangian formulation, as we have pointed out in the context of interacting dark energy.

Finally, it is rather unfortunate that in recent years old solutions, integrability conditions, and methods have been forgotten and needed to be rediscovered, which can be partially explained by the very different points of view on this subject in the last half century. We set out with the goal of providing a starting point for the less experienced reader and a reference for the expert one, although a review cannot replace the thorough study of the relevant literature. We hope that highlighting multi-fluid cosmology will bring i) a better understanding of the relations between the various possible components of the universe; ii) more efficient research in the context of Einstein theory; iii) will facilitate the comparison with analytical solutions describing similar universes in alternative theories of gravity.

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A Derivation of Eq. (2.31)

Following Ref. [293], we have

\[
 w_{\text{tot}}(a) = \frac{\sum_{i=1}^{n} w_i \rho_i^{(0)} a^{-3(w_i+1)}}{\sum_{j=1}^{n} \rho_j a^{-3(w_j+1)}} \tag{A.1}
\]

and

\[
 \dot{w}_{\text{tot}}(a, H) = -\frac{3}{\rho^2} \left\{ \sum_i w_i \rho_i^{(0)} (w_i + 1) a^{-3 w_i - 4} \dot{a} \left[ \sum_j \rho_j^{(0)} a^{-3(w_j+1)} \right] 
\right.
\]

\[
 \left. - \sum_i w_i \rho_i^{(0)} a^{-3(w_i+1)} \sum_j \rho_j^{(0)} (w_j + 1) a^{-3 w_j - 4} \right\} 
\]

\[
 = -\frac{3}{\rho^2} \left\{ \sum_{i,j} w_i (w_i - w_j) \rho_i^{(0)} \rho_j^{(0)} a^{-3(w_i+w_j)-6} \right\} \frac{\dot{a}}{a} \tag{A.2}
\]
Remembering that \( \rho_i = \rho_i^{(0)} a^{-3(w_i+1)} \), we have

\[
\sum_{i,j} w_i (w_i - w_j) \rho_i \rho_j = \sum_{i,j} (w_i - w_j) (w_i - w_j) \rho_i \rho_j
\]

\[
= \sum_{i,j} (w_i - w_j)^2 \rho_i \rho_j + \sum_{i,j} w_j (w_i - w_j) \rho_i \rho_j,
\]

(A.3)

therefore,

\[
\sum_{i,j} w_i (w_i - w_j) \rho_i \rho_j - \sum_{i,j} w_j (w_i - w_j) \rho_i \rho_j
\]

\[
= \sum_{i,j} (w_i - w_j)^2 \rho_i \rho_j,
\]

(A.4)

or

\[
\sum_{i,j} w_i (w_i - w_j) \rho_i \rho_j = \frac{1}{2} \sum_{i,j} (w_i - w_j)^2 \rho_i \rho_j.
\]

(A.5)

Equation (A.2) then becomes

\[
\dot{w}_{\text{tot}} = -\frac{3H}{2\rho_{\text{tot}}^2} \sum_{i,j} (w_i - w_j)^2 \rho_i^{(0)} \rho_j^{(0)} a^{-3(w_i+w_j)-6}
\]

\[
= -\frac{3H}{\rho_{\text{tot}}^2} \sum_{i<j} (w_i - w_j)^2 \rho_i \rho_j,
\]

(A.6)

where we used the fact that

\[
\frac{1}{2} \sum_{i,j=1}^{n} = \sum_{i<j}^{n}.
\]

(A.7)

B Equivalence of Eqs. (6.13) and (6.12)

Set

\[
x \equiv \frac{a + 4\pi \rho_m^{(0)}}{\sqrt{4\pi \rho_m^{(0)} \left( -\frac{4\pi}{3} \rho_m^{(0)} + \frac{2\rho_m^{(0)}}{\rho_c^{(0)}} \right)}}
\]

(B.1)

and use the inverse hyperbolic function identity

\[
\sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right)
\]

(B.2)
to write

$$\sinh^{-1} x = \ln \left( \frac{a + \frac{4\pi}{3} \rho_m^{(0)}}{\sqrt{\frac{4\pi}{3} \rho_m^{(0)} \left( -\frac{4\pi}{3} \rho_m^{(0)} + \frac{2\rho_m^{(0)}}{\rho_i^{(0)}} \right)}} \right)$$

$$+ \ln \left[ \frac{(a + \frac{4\pi}{3} \rho_m^{(0)})^2}{\frac{4\pi}{3} \rho_m^{(0)} \left( -\frac{4\pi}{3} \rho_m^{(0)} + \frac{2\rho_m^{(0)}}{\rho_i^{(0)}} \right)} + 1 \right]$$

$$= \ln \left[ \frac{\sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a + \frac{8\pi}{3} \rho_r^{(0)} + a + \frac{4\pi}{3} \rho_m^{(0)}}}{\sqrt{\frac{4\pi}{3} \rho_m^{(0)} \left( -\frac{4\pi}{3} \rho_m^{(0)} + \frac{2\rho_m^{(0)}}{\rho_i^{(0)}} \right)}} \right]$$

$$= \ln \left\{ C \left[ \sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a + \frac{8\pi}{3} \rho_r^{(0)} + a + \frac{4\pi}{3} \rho_m^{(0)}} \right] \right\}$$

where

$$C \equiv \frac{1}{\sqrt{\frac{4\pi}{3} \rho_m^{(0)} \left( -\frac{4\pi}{3} \rho_m^{(0)} + \frac{2\rho_m^{(0)}}{\rho_i^{(0)}} \right)}}.$$  \hspace{1cm} (B.4)

Therefore, we have

$$t(a) = \sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a + \frac{8\pi}{3} \rho_r^{(0)}}$$

$$- \frac{4\pi}{3} \rho_m^{(0)} \sinh^{-1} \left[ \frac{a + \frac{4\pi}{3} \rho_m^{(0)}}{\sqrt{\frac{4\pi}{3} \rho_m^{(0)} \left( -\frac{4\pi}{3} \rho_m^{(0)} + \frac{2\rho_m^{(0)}}{\rho_i^{(0)}} \right)}} \right]$$

$$= \sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a + \frac{8\pi}{3} \rho_r^{(0)}}$$

$$- \frac{4\pi}{3} \rho_m^{(0)} \ln \left\{ C \left[ \sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a + \frac{8\pi}{3} \rho_r^{(0)} + a + \frac{4\pi}{3} \rho_m^{(0)}} \right] \right\}$$

\hspace{1cm} (B.5)
or, Eq. (6.13) coincides with Eq. (6.12).

C Single-fluid limit of the universe with dust plus radiation and $K = \pm 1$

Here we discuss the single-fluid limits of the solutions (6.10) and (6.12) of the Einstein–Friedmann equations (2.16)–(2.18) describing spatially curved universes sourced by dust plus radiation. As expected, these limits generate the well known solutions for radiation with curvature, however the limits to dust plus curvature are not trivial in both cases $K = \pm 1$.

C.1 Positive curvature

In the positively curved case $K = +1$, the relation between comoving time and scale factor is given by Eq. (6.10), which we reproduce here for convenience:

$$t(a) = t_0 - \sqrt{\frac{8\pi}{3} \rho_r^{(0)} + \frac{8\pi}{3} \rho_m^{(0)} a - a^2}$$

$$+ \frac{4\pi}{3} \rho_m^{(0)} \sin^{-1} \left[ \frac{-\frac{4\pi}{3} \rho_m^{(0)} + a}{\sqrt{\left(\frac{4\pi}{3} \rho_m^{(0)}\right)^2 + \frac{8\pi}{3} \rho_r^{(0)}}} \right].$$

\[\text{(C.1)}\]

 Limit to dust and $K = +1$: Setting $\rho_r^{(0)} = 0$, one obtains the limit

$$t(a) = -\sqrt{\frac{8\pi}{3} \rho_m^{(0)} a - a^2} + \frac{4\pi}{3} \rho_m^{(0)} \sin^{-1} \left( \frac{3a}{4\pi \rho_m^{(0)}} - 1 \right)$$

$$+ t_0,$$

\[\text{(C.2)}\]

that looks different from Eq. (5.5), which is instead

$$t(a) = -\sqrt{\frac{8\pi}{3} \rho_m^{(0)} a - a^2} + \frac{8\pi}{3} \rho_m^{(0)} \sin^{-1} \left( \sqrt{\frac{3a}{8\pi \rho_m^{(0)}}} \right)$$

$$+ t_0,$$

\[\text{(C.3)}\]

To proceed, use the trigonometric identities

$$\sin^{-1} x = \frac{1}{2} \cos^{-1} \left( 1 - 2x^2 \right),$$

\[\text{(C.4)}\]

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x,$$

\[\text{(C.5)}\]

$$\sin^{-1} (-x) = - \sin^{-1} x,$$

\[\text{(C.6)}\]
for $x \equiv \sqrt{\frac{3a}{8\pi \rho_0}}$ to obtain

\[
\frac{8\pi}{3} \rho_m^{(0)} \sin^{-1}\left(\sqrt{\frac{3a}{8\pi \rho_r^{(0)}}}\right) = \frac{4\pi}{3} \rho_m^{(0)} \cos^{-1}\left(1 - \frac{3a}{4\pi \rho_r^{(0)}}\right)
\]

\[
= \frac{4\pi}{3} \rho_m^{(0)} \left[\frac{\pi}{2} - \sin^{-1}\left(1 - \frac{3a}{4\pi \rho_r^{(0)}}\right)\right]
\]

\[
= \frac{2\pi^2}{3} \rho_m^{(0)} + \frac{4\pi}{3} \rho_m^{(0)} \sin^{-1}\left(\frac{3a}{4\pi \rho_r^{(0)}} - 1\right),
\]

(C.7)

which yields

\[
t(a) = -\sqrt{\frac{8\pi}{3} \rho_m^{(0)} a - a^2} + \frac{4\pi}{3} \rho_m^{(0)} \sin^{-1}\left(\frac{3a}{4\pi \rho_r^{(0)}} - 1\right)
\]

\[+ t_0'
\]

(C.8)

(where $t_0' = t_0 + \frac{2\pi^2}{3} \rho_m^{(0)}$): this coincides with the well known equation (5.5).

Limit to radiation and $K = +1$: By setting $\rho_m^{(0)} = 0$, Eq. (C.1) gives

\[
t = t_0 - \sqrt{\frac{8\pi}{3} \rho_r^{(0)} - a^2},
\]

(C.9)

which is inverted to

\[
a(t) = \sqrt{\frac{8\pi}{3} \rho_r^{(0)} - (t - t_0)^2};
\]

(C.10)

this equation coincides coincides with Eq. (5.13).

C.2 Negative curvature

In the negatively curved universe $K = -1$, the relation between comoving time and scale factor is (Eq. (6.12))

\[
t(a) = t_0 + \sqrt{\frac{8\pi}{3} \rho_r^{(0)} + \frac{8\pi}{3} \rho_m^{(0)} a + a^2}
\]

\[- \frac{4\pi}{3} \rho_m^{(0)} \ln \left\{ C \left[ \frac{8\pi}{3} \rho_r^{(0)} + \frac{8\pi}{3} \rho_m^{(0)} a + a^2 + a
\right.
\]

\[+ \frac{4\pi}{3} \rho_m^{(0)} \right\} \right\},
\]

(C.11)
Limit to dust and $K = -1$: Setting $\rho_t^{(0)} = 0$ in Eq. (C.11), one obtains

$$t(a) = t_0 + \sqrt{\frac{8\pi}{3} \rho_m^{(0)} a + a^2} - \frac{4\pi}{3} \rho_m^{(0)} \ln \left\{ C \left[ \sqrt{\frac{8\pi}{3} \rho_m^{(0)} a + a^2 + a + \frac{4\pi}{3} \rho_m^{(0)}} \right] \right\}, \quad \text{(C.12)}$$

which does not coincide with Eq. (5.6); the latter gives instead

$$t = t_0 + \sqrt{\frac{8\pi}{3} \rho_r^{(0)} a + a^2} - \frac{8\pi}{3} \rho_m^{(0)} \sinh^{-1} \left( \sqrt{\frac{3a}{8\pi \rho_m^{(0)}}} \right). \quad \text{(C.13)}$$

However, using the inverse hyperbolic function identities

$$2 \sinh^{-1} x = \cosh^{-1} \left( 2x^1 + 1 \right), \quad \text{(C.14)}$$
$$\cosh^{-1} x = \ln \left( x + \sqrt{x^2 - 1} \right), \quad \text{(C.15)}$$

one has

$$\frac{8\pi}{3} \rho_m^{(0)} \sinh^{-1} \left( \sqrt{\frac{3a}{8\pi \rho_m^{(0)}}} \right) = \frac{4\pi}{3} \rho_m^{(0)} \cosh^{-1} \left( \frac{3a}{4\pi \rho_m^{(0)}} + 1 \right)$$
$$= \frac{4\pi}{3} \rho_m^{(0)} \ln \left[ \frac{3a}{4\pi \rho_m^{(0)}} + 1 + \sqrt{\left( \frac{3a}{4\pi \rho_m^{(0)}} + 1 \right)^2 - 1} \right]$$
$$= \frac{4\pi}{3} \rho_m^{(0)} \ln \left[ \frac{3}{4\pi \rho_m^{(0)}} \left( a + \frac{4\pi}{3} \rho_m^{(0)} + \sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a} \right) \right]$$
$$= \frac{4\pi}{3} \rho_m^{(0)} \ln \left( a + \frac{4\pi}{3} \rho_m^{(0)} + \sqrt{a^2 + \frac{8\pi}{3} \rho_m^{(0)} a} \right)$$
$$+ \frac{4\pi}{3} \rho_m^{(0)} \ln \left( \frac{3}{4\pi \rho_m^{(0)}} \right). \quad \text{(C.16)}$$

The last and constant term is incorporated in the constant $t_0$ and Eq. (C.16) then coincides with Eq. (5.6).

Limit to radiation and $K = -1$: Setting $\rho_m^{(0)} = 0$, one obtains immediately the radiation-only limit

$$t(a) = t_0 + \sqrt{\frac{8\pi}{3} \rho_r^{(0)} + a^2} \quad \text{(C.17)}$$
that is inverted to give

\[
    a(t) = \sqrt{(t - t_0)^2 - \frac{8\pi}{3} \rho_r(0)},
\]

which coincides with Eq. (5.15).

References

1. Abbott, L.F., Wise, M.B.: Nucl. Phys. B 244, 541–548 (1984). https://doi.org/10.1016/0550-3213(84)90329-8
2. Abreu, J.P., Crawford, P., Mimoso, J.P.: Class. Quant. Gravit. 11, 1919–1940 (1994). https://doi.org/10.1088/0264-9381/11/8/002. arXiv:gr-qc/9401024
3. Ade, P.A.R., et al.: (Planck collaboration) Astron. Astrophys. 571, A1 (2014)
4. Ade, P.A.R., et al.: [Planck] Astron. Astrophys. 571, A16 (2014). https://doi.org/10.1051/0004-6361/201321591. arXiv:1303.5076 [astro-ph.CO]
5. Ade, P.A.R., et al.: [Planck Collaboration] Astron. Astrophys. 594, A20 (2016). https://doi.org/10.1051/0004-6361/201525898. arXiv:1502.02114 [astro-ph.CO]
6. Aguirregabiria, J.M., Chimento, L.P., Lazkoz, R.: Phys. Rev. D 70, 023509 (2004). https://doi.org/10.1103/PhysRevD.70.023509. arXiv:astro-ph/0403157 [astro-ph]
7. Aguirregabiria, J.M., Chimento, L.P., Jakubi, A.S., Lazkoz, R.: Phys. Rev. D 67, 083518 (2003). https://doi.org/10.1103/PhysRevD.67.083518. arXiv:gr-qc/0303010 [gr-qc]
8. Alabidi, L., Lyth, D.: JCAP 08, 006 (2006). https://doi.org/10.1088/1475-7516/2006/08/006. arXiv:astro-ph/0604569 [astro-ph]
9. Alabidi, L.: JCAP 10, 015 (2006). https://doi.org/10.1088/1475-7516/2006/10/015. arXiv:astro-ph/0604611 [astro-ph]
10. Ali-Haimoud, Y., Hirata, C.M.: Phys. Rev. D 83, 043513 (2011). https://doi.org/10.1103/PhysRevD.83.043513. arXiv:1011.3758 [astro-ph.CO]
11. Allahverdi, R., Brandenberger, R., Cyr-Racine, F.Y., Mazumdar, A.: Ann. Rev. Nucl. Part. Sci. 60, 27–51 (2010). https://doi.org/10.1146/annurev.nucl.012809.104511. arXiv:1001.2600 [hep-th]
12. Alcubierre, M., Guzman, F.S., Matos, T., Nunez, D., Urena-Lopez, L.A., Wiederhold, P.: Class. Quant. Gravit. 19, 5017 (2002). https://doi.org/10.1088/0264-9381/19/19/314. arXiv:gr-qc/0110102 [gr-qc]
13. Aldrovandi, R., Cuzzinatto, R.R., Medeiros, L.G.: Found. Phys. 36, 1736–1752 (2006). https://doi.org/10.1007/s10701-006-9076-6. arXiv:gr-qc/0508073 [gr-qc]
14. Alinolmohammadi, M., Sadjadi, H.M.: Phys. Lett. B 668, 113–118 (2007). https://doi.org/10.1016/j.physletb.2007.03.014. arXiv:gr-qc/0608016 [gr-qc]
15. Alinolmohammadi, M.: Gen. Relativ. Gravit. 40, 107–115 (2008). https://doi.org/10.1007/s10714-007-0514-3. arXiv:0706.1360 [gr-qc]
16. Alpher, R.A., Herman, R.C.: Phys. Rev. 75(7), 1089–1095 (1949). https://doi.org/10.1103/physrev.75.1089
17. Alvarenga, F.G., Fracalossi, R., Freitas, R.C., Gonçalves, S.V.B.: Gen. Relativ. Gravit. 49(11), 136 (2017). https://doi.org/10.1007/s10714-017-2301-0. arXiv:1607.03478 [gr-qc]
18. Amendola, L., Tsujikawa, S.: Dark Energy: Theory and Observations. Cambridge University Press, Cambridge (2010)
19. Amin, M.A., Hertzberg, M.P., Kaiser, D.I., Karouby, J.: Int. J. Mod. Phys. D 24, 1530003 (2014). https://doi.org/10.1142/S0218271815300037. arXiv:1410.3808 [hep-ph]
20. Ananda, K.N., Bruni, M.: Phys. Rev. D 74, 023524 (2006). https://doi.org/10.1103/PhysRevD.74.023524. arXiv:gr-qc/0603131 [gr-qc]
21. Ananda, K.N., Bruni, M.: Phys. Rev. D 74, 023523 (2006). https://doi.org/10.1103/PhysRevD.74.023523. arXiv:astro-ph/0512224 [astro-ph]
22. Andrianov, A.A., Cannata, F., Kamenshchik, A.Y.: JCAP 10, 004 (2011). https://doi.org/10.1088/1475-7516/2011/10/004. arXiv:1105.4515 [gr-qc]
23. Arbey, A.: Phys. Rev. D 74, 043516 (2006). https://doi.org/10.1103/PhysRevD.74.043516. arXiv:astro-ph/0601274 [astro-ph]
24. Aref’eva, I.Y., Koshelev, A.S., Vernov, S.Y.: Phys. Rev. D 72, 064017 (2005). https://doi.org/10.1103/PhysRevD.72.064017. arXiv:astro-ph/0507067 [astro-ph]

25. Assad, M.J.D., Sales de Lima, J.A.: Gen. Relativ. Gravit. 20, 527 (1988)

26. Ballesteros, G., Bellazzini, B., Mercolli, L.: JCAP 05, 007 (2014). https://doi.org/10.1088/1475-7516/2014/05/007. arXiv:1312.2957 [hep-th]

27. Ballesteros, G., Comelli, D., Pilo, L.: Phys. Rev. D 94(2), 025034 (2016). https://doi.org/10.1103/PhysRevD.94.025034. arXiv:1605.05304 [hep-th]

28. Barnaby, N., Cline, J.M.: Phys. Rev. D 73, 106012 (2006). https://doi.org/10.1103/PhysRevD.73.106012. arXiv:astro-ph/0601481 [astro-ph]

29. Barnaby, N., Cline, J.M.: Phys. Rev. D 75, 086004 (2007). https://doi.org/10.1103/PhysRevD.75.086004. arXiv:astro-ph/0611750 [astro-ph]

30. Barrow, J.D.: Phys. Lett. B 187, 12–16 (1987). https://doi.org/10.1016/0370-2693(87)90063-3

31. Barrow, J.D.: Phys. Lett. B 235, 40–43 (1990). https://doi.org/10.1016/0370-2693(90)90093-L

32. Barrow, J.D., Saich, P.: Class. Quant. Gravit. 10, 279–283 (1993). https://doi.org/10.1088/0264-9381/10/2/009

33. Barrow, J.D.: Phys. Rev. D 48, 1585–1590 (1993). https://doi.org/10.1103/PhysRevD.48.1585

34. Barrow, J.D., Liddle, A.R.: Phys. Rev. D 47(12), R5219 (1993). https://doi.org/10.1103/PhysRevD.47.R5219. arXiv:astro-ph/9303011 [astro-ph]

35. Barrow, J.D.: The Book of Universes. W.W. Norton & C, New York (2011)

36. Barrow, J.D.: Phys. Rev. D 49, 1585–1590 (1994). https://doi.org/10.1103/PhysRevD.49.1585

37. Barrow, J.D.: Class. Quant. Gravit. 21, L79 (2004)

38. Battefeld, T., Easther, R.: JCAP 03, 020 (2007). https://doi.org/10.1088/1475-7516/2007/03/020. arXiv:astro-ph/0610296 [astro-ph]

39. Battefeld, D., Battefeld, T.: JCAP 05, 012 (2007). https://doi.org/10.1088/1475-7516/2007/05/012. arXiv:hep-th/0703012 [hep-th]

40. Battye, R.A., Bucher, M., Spergel, D.: arXiv:astro-ph/9908047 [astro-ph]

41. Bayin, S.S., Cooperstock, F.I., Faraoni, V.: Astrophys. J. 428, 439–446 (1994). https://doi.org/10.1086/174256. arXiv:astro-ph/9402033 [astro-ph]

42. Beltrán Jiménez, J., Rubiera-Garcia, D., Sáez-Gómez, D., Salzano, V.: Phys. Rev. D 94, 123520 (2016)

43. Ben Achour, J., Langlois, D., Noui, K.: Phys. Rev. D 93(12), 124005 (2016). https://doi.org/10.1103/PhysRevD.93.124005. arXiv:1602.08398 [gr-qc]

44. Ben Achour, J., Crisostomi, M., Koyama, K., Langlois, D., Noui, K., Tasinato, G.: JHEP 12, 100 (2016). https://doi.org/10.1007/JHEP12(2016)100. arXiv:1608.08135 [hep-th]

45. Ben Achour, J., Livine, E.R.: JHEP 12, 031 (2019). https://doi.org/10.1007/JHEP12(2019)031. arXiv:1909.13390 [gr-qc]

46. Ben Achour, J., Livine, E.R.: Class. Quant. Gravit. 37(21), 215001 (2020). https://doi.org/10.1088/1361-6382/abbb57. arXiv:2004.05841 [gr-qc]

47. Ben’Achour, J., Livine, E.R.: JHEP 03, 067 (2020). https://doi.org/10.1007/JHEP03(2020)067. arXiv:2001.11807 [gr-qc]

48. Bergmann, P.G.: Int. J. Theor. Phys. 1, 25–36 (1968). https://doi.org/10.1007/BF00668828
57. Bernal, A., Siddhartha Guzman, F.: Phys. Rev. D 74, 103002 (2006). https://doi.org/10.1103/PhysRevD.74.103002. arXiv:astro-ph/0610682 [astro-ph]
58. Bernardeau, F., Uzan, J.-P.: Phys. Rev. D 66, 103506 (2002). https://doi.org/10.1103/PhysRevD.66.103506. arXiv:hep-ph/0207295 [hep-ph]
59. Bernardeau, F., Uzan, J.-P.: Phys. Rev. D 67, 121301 (2003). https://doi.org/10.1103/PhysRevD.67.121301. arXiv:astro-ph/0209330 [astro-ph]
60. Bertacca, D., Matarrese, S., Pietroni, M.: Mod. Phys. Lett. A 22, 2893–2907 (2007). https://doi.org/10.1142/S0217732307025893. arXiv:astro-ph/0610682 [astro-ph]
61. Bertacca, D., Bartolo, N., Matarrese, S.: JCAP 05, 005 (2008). https://doi.org/10.1088/1475-7516/2008/05/005. arXiv:0712.0486 [astro-ph]
62. Bertacca, D., Bartolo, N., Diaferio, A., Matarrese, S.: JCAP 10, 023 (2008). https://doi.org/10.1088/1475-7516/2008/10/023. arXiv:0807.1020 [astro-ph]
63. Bertolami, O., Boehmer, C.G., Harko, T., Lobo, F.S.N.: Phys. Rev. D 75, 104016 (2007). https://doi.org/10.1103/PhysRevD.75.104016. arXiv:0704.1733 [gr-qc]
64. Bertolami, O., Cosme, C., Rosa, J.G.: Phys. Rev. D 91(2), 023517 (2015). https://doi.org/10.1103/PhysRevD.91.023517. arXiv:1407.4313 [gr-qc]
65. Bettoni, D., Ezquiaga, J.M., Hinterbichler, K., Zumalacárregui, M.: Phys. Rev. D 95(8), 084029 (2017). https://doi.org/10.1103/PhysRevD.95.084029. arXiv:1608.01982 [gr-qc]
66. Billyard, A.P., Coley, A.A.: Phys. Rev. D 61, 083503 (2000). arXiv:astro-ph/9908224
67. Bloomfield, J.K., Flanagan, É.É., Park, M., Watson, S.: JCAP 1308, 010 (2013). arXiv:1211.7054 [astro-ph.CO]
68. Borowiec, A., Capozziello, S., De Laurentis, M., Lobo, F.S.N., Paliathanasis, A., Paolella, M., Wojnar, A.: Phys. Rev. D 91(2), 023517 (2015). https://doi.org/10.1103/PhysRevD.91.023517. arXiv:1407.4313 [gr-qc]
69. Bouhmadi-López, M., Errahmani, A., Martin-Moruno, P., Ouali, T., Tavakoli, Y.: Int. J. Mod. Phys. D 24, 1550078 (2015)
70. Brans, C., Dicke, R.H.: Phys. Rev. 124, 925–935 (1961). https://doi.org/10.1103/PhysRev.124.925
71. Brevik, I., Obukhov, V.V., Timoshkin, A.V.: Mod. Phys. Lett. A 29(15), 1450078 (2014). https://doi.org/10.1142/S0217732314500783. arXiv:1404.1887 [gr-qc]
72. Bucher, M., Spergel, D.N.: Phys. Rev. D 60, 043505 (1999). https://doi.org/10.1103/PhysRevD.60.043505. arXiv:astro-ph/9812022 [astro-ph]
73. Burd, A.B., Barrow, J.D.: Nucl. Phys. B 308, 929-945 (1988) [erratum: Nucl. Phys. B 324, 276 (1989)]. https://doi.org/10.1016/0550-3213(88)90135-6
74. Byrnes, C.T., Choi, K.Y., Hall, L.M.H.: JCAP 10, 008 (2008). https://doi.org/10.1088/1475-7516/2008/10/008. arXiv:0807.1101 [astro-ph]
75. Byrnes, C.T., Choi, K.Y., Hall, L.M.H.: JCAP 02, 017 (2009). https://doi.org/10.1088/1475-7516/2009/02/017. arXiv:0812.0807 [astro-ph]
76. Cai, Y.F., Saridakis, E.N., Setare, M.R., Xia, J.Q.: Phys. Rept. 493, 1–60 (2010). https://doi.org/10.1016/j.physrep.2010.04.001. arXiv:0909.2776 [hep-th]
77. Cai, Y.F., Li, H., Piao, Y.S., Zhang, X.: Phys. Lett. B 646, 141–144 (2007). https://doi.org/10.1016/j.physletb.2007.01.027. arXiv:0701.2187 [hep-th]
78. Cai, Y.F., Qiu, T., Brandenberger, R., Piao, Y.S., Zhang, X.: JCAP 03, 013 (2008). https://doi.org/10.1088/1475-7516/2008/03/013. arXiv:0711.2187 [hep-th]
79. Cai, Y.F., Li, M.Z., Lu, J.X., Piao, Y.S., Qiu, T., Zhang, X.: Phys. Lett. B 651, 1–7 (2007). https://doi.org/10.1016/j.physletb.2007.05.056. arXiv:hep-th/0701016 [hep-th]
80. Cai, Y.F., Qiu, T., Piao, Y.S., Li, M., Zhang, X.: JHEP 10, 071 (2007). https://doi.org/10.1088/1126-6708/2007/10/071. arXiv:0704.1090 [hep-th]
81. Cai, Y.F., Zhang, X.: JCAP 06, 003 (2009). https://doi.org/10.1088/1475-7516/2009/06/003. arXiv:0808.2551 [astro-ph]
82. Cai, Y.F., Wang, J.: Class. Quant. Gravit. 25, 165014 (2008). https://doi.org/10.1088/0264-9381/25/16/165014. arXiv:0806.3890 [hep-th]
83. Calcagni, G.: Phys. Rev. D 69, 103508 (2004). https://doi.org/10.1103/PhysRevD.69.103508. arXiv:hep-ph/0402126 [hep-ph]
84. Calcagni, G.: Phys. Rev. D 71, 023511 (2005). https://doi.org/10.1103/PhysRevD.71.023511. arXiv:gr-qc/0410027 [gr-qc]
85. Capozziello, S., Carloni, S., Troisi, A.: Rec. Res. Dev. Astron. Astrophys. 1, 625 (2003). arXiv:astro-ph/0303041 [astro-ph]
86. Capozziello, S., Cardone, V.F., Elizalde, E., Nojiri, S., Odintsov, S.D.: Phys. Rev. D 73, 043512 (2006). https://doi.org/10.1103/PhysRevD.73.043512. arXiv:astro-ph/0508350 [astro-ph]
87. Capozziello, S., Cardone, V.F., Elizalde, E., Nojiri, S., Odintsov, S.D.: Phys. Rev. D 73, 043512 (2006)
88. Capozziello, S., Piedipalumbo, E., Rubano, C., Scudellaro, P.: Phys. Rev. D 80, 104030 (2009). https:// doi.org/10.1103/PhysRevD.80.104030. arXiv:0908.2362 [astro-ph.CO]
89. Capozziello, S., Faraoni, V.: Beyond Einstein Gravity: A Survey of Gravitational Theories for Cos- mology and Astrophysics. Springer, New York (2010)
90. Capozziello, S., Roshan, M.: Phys. Lett. B 726, 471–480 (2013). https://doi.org/10.1016/j.physletb.2013.08.047. arXiv:1308.3910 [gr-qc]
91. Capozziello, S., DeLaurentis, M., Myrzakulov, R.: Int. J. Geom. Meth. Mod. Phys. 12(09), 1550095 (2015). https://doi.org/10.1142/S0219887815500954. arXiv:1412.1471 [gr-qc]
92. Capozziello, S., DeLaurentis, M., Dialektopoulos, K.F.: Eur. Phys. J. C 76(11), 629 (2016). https:// doi.org/10.1140/epjc/s10052-016-4491-0. arXiv:1609.09289 [gr-qc]
93. Carr, B.J., Lidsey, J.E.: Phys. Rev. D 48, 543–553 (1993). https://doi.org/10.1103/PhysRevD.48.543
94. Carroll, S.M.: Spacetime and Geometry: An Introduction to General Relativity. Addison Wesley, San Francisco (2004)
95. Carroll, S.M., Duvvuri, V., Trodden, M., Turner, M.S.: Phys. Rev. D 70, 043528 (2004). https://doi. org/10.1103/PhysRevD.70.043528. arXiv:astro-ph/0306438 [astro-ph]
96. Carter, B., Langlois, D.: Nucl. Phys. B 454, 402–424 (1995). https://doi.org/10.1016/0550-3213(95)00425-R. arXiv:hep-th/9611082 [hep-th]
97. Carter, B., Langlois, D.: Nucl. Phys. B 531, 478–504 (1998). https://doi.org/10.1016/S0550-3213(98)00430-1. arXiv:gr-qc/9806024 [gr-qc]
98. Cataldo, M., Chimento, L.P.: Int. J. Mod. Phys. D 17, 1981–1989 (2008). https://doi.org/10.1142/S0218271808013790. arXiv:0905.06090 [gr-qc]
99. Charters, T., Mimoso, J.P.: JCAP 08, 022 (2010). https://doi.org/10.1088/1475-7516/2010/08/022. arXiv:0909.2282 [hep-ph]
100. Chavanis, P.H.: Phys. Rev. D 92, 103004 (2015)
101. Chebyshev, P.L.: Sur l'integration des différentielles irrationnelles. Journal de Mathematiques (series 1) 18, 87–111 (1853)
102. Chen, Xm., Gong, Yg., Saridakis, E.N.: JCAP 04, 001 (2009). https://doi.org/10.1088/1475-7516/ 2009/04/001. arXiv:0812.1117 [gr-qc]
103. Chen, S., Gibbons, G.W., Li, Y., Yang, Y.: JCAP 12, 035 (2014). https://doi.org/10.1088/1475-7516/ 2014/12/035. arXiv:1409.3352 [astro-ph.CO]
104. Chernin, A.D.: Sov. Astron. 9, 871 (1966)
105. Chervon, S.V., Fomin, I.V., Pozdeeva, E.O., Sami, M., Vernov, S.Y.: Phys. Rev. D 79(6), 063522 (2014). https://doi.org/10.1103/PhysRevD.79.063522. arXiv:1404.1126 [gr-qc]
106. Chervon, S.V., Fomin, I.V., Beesham, A.: Eur. Phys. J. C 78(4), 301 (2018). https://doi.org/10.1140/ epjc/s10052-018-5795-z. arXiv:1704.08712 [gr-qc]
107. Chimento, L.P., Lazkoz, R.: Phys. Rev. Lett. 91, 211301 (2003). https://doi.org/10.1103/PhysRevLett. 91.211301. arXiv:gr-qc/0307111 [gr-qc]
108. Chimento, L.P., Lazkoz, R.: Class. Quant. Gravit. 23, 3195–3204 (2006). https://doi.org/10.1088/ 0264-9381/23/9/027. arXiv:astro-ph/0505254 [astro-ph]
109. Chimento, L.P., Pavon, D.: Phys. Rev. D 73, 063511 (2006). https://doi.org/10.1103/PhysRevD.73. 063511. arXiv:gr-qc/0505096 [gr-qc]
110. Chimento, L.P., Zimdahl, W.: Int. J. Mod. Phys. D 17, 2229–2254 (2008). https://doi.org/10.1142/ S0218271808013820. arXiv:gr-qc/0609104 [gr-qc]
111. Chimento, L.P., Devecchi, F.P., Forte, M.I., Kremer, G.M.: Class. Quant. Gravit. 25, 085007 (2008). https://doi.org/10.1088/0264-9381/25/8/085007. arXiv:0707.4455 [gr-qc]
112. Chimento, L.P., Forte, M.I., Lazkoz, R., Richarte, M.G.: Phys. Rev. D 79, 043502 (2009). https://doi. org/10.1103/PhysRevD.79.043502. arXiv:0811.3643 [astro-ph]
113. Chimento, L.P., Lazkoz, R., Richarte, M.G.: Phys. Rev. D 83, 063505 (2011). https://doi.org/10.1103/ PhysRevD.83.063505. arXiv:1011.2345 [astro-ph.CO]
114. Chimento, L.P., Forte, M.I., Richarte, M.G.: Mod. Phys. Lett. A 28, 1250235 (2013). https://doi.org/ 10.1142/S0217732312502355. arXiv:1106.0781 [astro-ph.CO]
117. Chluba, J., Thomas, R.M.: Mon. Not. Roy. Astron. Soc. \textbf{412}, 748 (2011). https://doi.org/10.1111/j.1365-2966.2010.17940.x. arXiv:1010.3631 [astro-ph.CO]

118. Chluba, J., Ali-Haimoud, Y.: Mon. Not. R. Astron. Soc. \textbf{456}(4), 3494–3508 (2016). https://doi.org/10.1093/mnras/stw2691. arXiv:1510.03877 [astro-ph.CO]

119. Choi, K.Y., Hall, L.M.H., van de Bruck, C.: JCAP \textbf{02}, 029 (2007). https://doi.org/10.1088/1475-7516/2007/02/029. arXiv:astro-ph/0701247 [astro-ph]

120. Christodoulakis, T., Helias, C., Kevrekidis, P.G., Kevrekidis, I.G., Papadopoulos, G.O.: arXiv:gr-qc/0302120 [gr-qc]

121. Christodoulakis, T., Dimakis, N., Terzis, P.A.: J. Phys. A \textbf{47}, 095202 (2014). https://doi.org/10.1088/1751-8113/47/9/095202. arXiv:1304.4359 [gr-qc]

122. Christodoulakis, T., Karagiorgos, A., Zampeli, A.: Symmetry \textbf{10}(3), 70 (2018). https://doi.org/10.3390/sym10030070

123. Christodoulakis, T., Helias, C., Kevrekidis, P.G., Kevrekidis, I.G., Papadopoulos, G.O.: arXiv:astro-ph/0701247 [astro-ph]

124. Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C.: Phys. Rep. \textbf{513}, 1–189 (2012). https://doi.org/10.1016/j.physrep.2012.01.001. arXiv:1106.2476 [astro-ph.CO]

125. Coquereaux, R., Grossmann, A.: Ann. Phys. USA \textbf{143}, 296 (1982)

126. Cohen, J.M.: Nature \textbf{216}, 249 (1967)

127. Coley, A.A.: Astrophys. Space Sci. \textbf{155}, 193–201 (1989)

128. Coley, A.A.: Dynamical Systems and Cosmology. Kluwer, Dordrecht (2003)

129. Coley, A.A., Tupper, B.O.J.: J. Math. Phys. \textbf{27}, 406 (1986)

130. Coley, A.A., Tupper, B.O.J.: Can. J. Phys. \textbf{64}, 204 (1986)

131. Coley, A.A., van den Hoogen, R.J.: Class. Quant. Gravit. \textbf{9}, 651–665 (1992)

132. Coley, A.A., van den Hoogen, R.J.: Class. Quant. Gravit. \textbf{12}, 1977–1994 (1995). https://doi.org/10.1088/0264-9381/12/8/015. arXiv:gr-qc/9605061 [gr-qc]

133. Conversi, L., Melchiorri, A., Mersini-Houghton, L., Silk, J.: Astropart. Phys. \textbf{21}, 443–449 (2004). https://doi.org/10.1016/j.astropartphys.2004.02.006. arXiv:astro-ph/0402529 [astro-ph]

134. Cosme, C., Rosa, J.G., Bertolami, O.: JHEP \textbf{05}, 129 (2018). https://doi.org/10.1007/JHEP05(2018)129. arXiv:1802.09434 [hep-ph]

135. Costa, A.A., Xu, X.D., Wang, B., Ferreira, E.G.M., Abdalla, E.: Phys. Rev. D \textbf{89}(10), 103531 (2014). https://doi.org/10.1103/PhysRevD.89.103531. arXiv:1311.7380 [astro-ph.CO]

136. Conversi, L., Melchiorri, A., Mersini-Houghton, L., Silk, J.: Astropart. Phys. \textbf{21}, 443–449 (2004). https://doi.org/10.1016/j.astropartphys.2004.02.006. arXiv:astro-ph/0402529 [astro-ph]

137. Cruz, N., Lepe, S., Pena, F.: Phys. Lett. B \textbf{646}, 177–182 (2007). https://doi.org/10.1016/j.physletb.2006.12.070. arXiv:gr-qc/0609013 [gr-qc]

138. Dabrowski, M.P., Stachowiak, T., Szydłowski, M.: Phys. Rev. D \textbf{68}, 103519 (2003). https://doi.org/10.1103/PhysRevD.68.103519. arXiv:hep-th/0307128 [hep-th]

139. Dabrowski, M., Stelmach, J.: Ann. Phys. USA \textbf{166}, 422 (1986)

140. Dabrowski, M.P., Kiefer, C., Sandhofer, B.: Phys. Rev. D \textbf{74}, 044022 (2006). https://doi.org/10.1103/PhysRevD.74.044022. arXiv:hep-th/0605229 [hep-th]

141. Davidson, W., Narlikar, J.V.: Progr. Phys. \textbf{29}, 539 (1966)

142. D’Ambroise, J.: arXiv:0711.3916 [hep-th]

143. D’Ambroise, J.: arXiv:1005.1410 [gr-qc]

144. D’Ambroise, J., Williams, F.L.: Int. J. Pure Appl. Maths. \textbf{34}, 117 (2007)

145. Dariescu, M.-A., Mihu, D.-A., Dariescu, C.: Roman. J. Phys. \textbf{62}, 101 (2017)

146. Deffayet, C., Esposito-Farese, G., Vikman, A.: Phys. Rev. D \textbf{79}, 084003 (2009). https://doi.org/10.1103/PhysRevD.79.084003. arXiv:0901.1314 [hep-th]

147. Deffayet, C., Gao, X., Steer, D.A., Zahariade, G.: Phys. Rev. D \textbf{84}, 064039 (2011). https://doi.org/10.1103/PhysRevD.84.064039. arXiv:1103.3260 [hep-th]

148. De Felice, A., Tsujikawa, S.: Liv. Rev. Rel. \textbf{13}, 3 (2010). https://doi.org/10.12942/lrr-2010-3. arXiv:1002.4928 [gr-qc]

149. de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P., Stornaiolo, C.: Phys. Rev. D \textbf{42}, 1091–1097 (1990). https://doi.org/10.1103/PhysRevD.42.1091

150. de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P., Stornaiolo, C.: Phys. Lett. \textbf{149}A, 79–83 (1990)

151. de Ritis, R., Platania, G., Rubano, C., Sabatino, R.: Phys. Lett. \textbf{161}A, 230 (1991)

152. de Sitter, W.: Mon. Not. R. Astron. Soc. \textbf{78}, 3–28 (1917)
153. Dicke, R.H.: Phys. Rev. 125, 2163–2167 (1962). https://doi.org/10.1103/PhysRev.125.2163
154. Dimakis, N., Christodoulakis, T., Terzis, P.A.: J. Geom. Phys. 77, 97–112 (2014). https://doi.org/10.1016/j.geomphys.2013.12.001. arXiv:1311.4358 [gr-qc]
155. Dimakis, N., Karagiorgos, A., Zampeli, A., Paliathanasis, A., Christodoulakis, T., Terzis, P.A.: Phys. Rev. D 93(12), 123518 (2016). https://doi.org/10.1103/PhysRevD.93.123518. arXiv:1604.05168 [gr-qc]
156. d’Inverno, R.: Introducing Einstein’s Relativity. Clarendon Press, Oxford (1992)
157. Dussault, S., Faraoni, V.: Eur. Phys. J. C 80(11), 1002 (2020). https://doi.org/10.1140/epjc/s10052-020-08590-8. arXiv:2009.03235 [gr-qc]
158. Dutta, S., Lakshmanan, M., Chakraborty, S.: Int. J. Mod. Phys. D 25(14), 1650110 (2016). https://doi.org/10.1142/S0218271816501108. arXiv:1607.03396 [gr-qc]
159. Easther, R.: Class. Quant. Gravit. 10, 2203–2216 (1993). https://doi.org/10.1088/0264-9381/10/11/005. arXiv:gr-qc/9308010 [gr-qc]
160. Eckart, C.: Phys. Rev. 58, 919–924 (1940). https://doi.org/10.1103/PhysRev.58.919
161. Edwards, D.: Mon. Not. R. Astron. Soc. 159, 51 (1972)
162. Einstein, A.: Sitzungsb. König. Preuss. Akad. 142–152 (1917)
163. Ellis, G.F.R.: Class. Quant. Gravit. 5, 891–901 (1988). https://doi.org/10.1088/0264-9381/5/6/010
164. Ellis, G.F.R., Madsen, M.: Class. Quant. Gravit. 8, 667 (1991)
165. Ellis, G.F.R., Maartens, R., MacCallum, M.A.H.: Relativistic Cosmology. Cambridge University Press, Cambridge (2012)
166. Enqvist, K., Vaihkonen, A.: JCAP 09, 006 (2004). https://doi.org/10.1088/1475-7516/2004/09/006. arXiv:hep-ph/0405103 [hep-ph]
167. Ermakov, V.P.: Univ. Izv. Kiev 20, 1 (1880)
168. Faraoni, V.: Am. J. Phys. 67, 732 (1999). https://doi.org/10.1119/1.19361. arXiv:physics/9901006 [physics]
169. Faraoni, V.: Am. J. Phys. 69, 372–376 (2001). https://doi.org/10.1119/1.1290250. arXiv:physics/0006030 [physics]
170. Faraoni, V.: Phys. Rev. D 69, 123520 (2004). https://doi.org/10.1103/PhysRevD.69.123520. arXiv:gr-qc/0404078 [gr-qc]
171. Faraoni, V.: Cosmology in Scalar-Tensor Gravity. Kluwer, Dordrecht (2004). https://doi.org/10.1007/978-1-4020-1989-0
172. Faraoni, V.: Phys. Rev. D 85, 024040 (2012). https://doi.org/10.1103/PhysRevD.85.024040. arXiv:1201.1448 [gr-qc]
173. Faraoni, V., Protheroe, C.S.: Gen. Relativ. Gravit. 45, 103–123 (2013). https://doi.org/10.1007/s10714-012-1462-0. arXiv:1209.3726 [gr-qc]
174. Faraoni, V., Dent, J.B., Saridakis, E.N.: Phys. Rev. D 90(6), 063510 (2014). https://doi.org/10.1103/PhysRevD.90.063510. arXiv:1405.7288 [gr-qc]
175. Faraoni, V., Coté, J.: Phys. Rev. D 98(8), 084019 (2018). https://doi.org/10.1103/PhysRevD.98.084019. arXiv:1808.02427 [gr-qc]
176. Faraoni, V.: Symmetry 12(1), 147 (2020). https://doi.org/10.3390/sym12010147. arXiv:2001.02126 [gr-qc]
177. Felten, J.E., Isaacman, R.: Rev. Mod. Phys. 58, 689–698 (1986). https://doi.org/10.1103/RevModPhys.58.689
178. Feng, B., Li, M., Piao, Y.S., Zhang, X.: Phys. Lett. B 634, 101–105 (2006). https://doi.org/10.1016/j.physletb.2006.01.066. arXiv:astro-ph/0407432 [astro-ph]
179. Feng, B.: arXiv:astro-ph/0602156 [astro-ph]
180. Fomin, I.V.: Russ. Phys. J. 61, 843–851 (2018). https://doi.org/10.1007/s11182-018-1468-5
181. Frampton, P.H., Ludwick, K.J., Scherrer, R.J.: Phys. Rev. D 84, 063003 (2011). https://doi.org/10.1103/PhysRevD.84.063003. arXiv:1106.4996 [astro-ph.CO]
182. Friedmann, A.: Zeit. Physik A 10, 377–386 (1922). https://doi.org/10.1007/BF01332580
183. Friedmann, A.: Zeit. Physik A 21, 326–322 (1924). https://doi.org/10.1007/BF01328280. Translated in Gen. Relativ. Gravit. 31, 31 (1999)
184. Fuchs, B., Mielke, E.W.: Mon. Not. R. Astron. Soc. 350, 707 (2004). https://doi.org/10.1111/j.1365-2966.2004.07679.x. arXiv:astro-ph/0401575 [astro-ph]
186. Fujii, Y., Maeda, K.: The Scalar-Tensor Theory of Gravity. Cambridge University Press, Cambridge (2003)

187. Gergely, L.A., Tsujikawa, S.: Phys. Rev. D 89, 064059 (2014). arXiv:1402.0553 [hep-th]

188. Gergely, L.A., Tsujikawa, S.: Phys. Rev. D 89, 064059 (2014). arXiv:1402.0553 [astro-ph]

189. Gilman, R.C.: Phys. Rev. D 2, 1400–1410 (1970). https://doi.org/10.1103/PhysRevD.2.1400

190. Gionti, G., S.J., Paliathanasis, A.: Mod. Phys. Lett. A 33(16), 1850093 (2018) https://doi.org/10.1142/S0217732318500931. arXiv:1711.11106 [gr-qc]

191. Gleyzes, J., Langlois, D., Piazza, F., Vernizzi, F.: Phys. Rev. D 114(21), 211101 (2015). https://doi.org/10.1103/PhysRevD.114.211101. arXiv:1404.6495 [hep-th]

192. Gleyzes, J., Langlois, D., Piazza, F., Vernizzi, F.: JCAP 02, 018 (2015). https://doi.org/10.1088/1475-7516/2015/02/018. arXiv:1408.1952 [astro-ph.CO]

193. Goldwirth, D.S., Piran, T.: Phys. Rept. 214, 223–291 (1992). https://doi.org/10.1016/0370-1573(92)90073-9

194. Gong, Y., Chen, X.: Phys. Rev. D 76, 123007 (2007). https://doi.org/10.1103/PhysRevD.76.123007. arXiv:0708.2977 [astro-ph]

195. Gorini, V., Kamenshchik, A., Moschella, U., Pasquier, V.: Phys. Rev. D 69, 123512 (2004)

196. Gubitosi, G., Piazza, F., Vernizzi, F.: JCAP 1302, 032 (2013). arXiv:1210.0201 [hep-th]

197. Gumjudpai, B.: Gen. Relativ. Gravit. 41, 249–265 (2009). https://doi.org/10.1007/s10714-008-0665-x. arXiv:0710.3598 [gr-qc]

198. Gumjudpai, B.: Astropart. Phys. 30, 186–191 (2008). https://doi.org/10.1016/j.astropartphys.2008.09.006. arXiv:0708.3674 [gr-qc]

199. Gumjudpai, B.: JCAP 09, 028 (2008). https://doi.org/10.1088/1475-7516/2008/09/028. arXiv:0805.3796 [gr-qc]

200. Gumjudpai, B.: arXiv:0904.2746 [gr-qc]

201. Guo, Z.K., Piao, Y.S., Zhang, X.M., Zhang, Y.Z.: Phys. Lett. B 608, 177–182 (2005). https://doi.org/10.1016/j.physletb.2005.01.017. arXiv:astro-ph/0410654 [astro-ph]

202. Guo, Z.K., Piao, Y.S., Zhang, X., Zhang, Y.Z.: Phys. Rev. D 74, 127304 (2006). https://doi.org/10.1103/PhysRevD.74.127304. arXiv:astro-ph/0608165 [astro-ph]

203. Gurses, M., Heydarzade, Y.: Eur. Phys. J. C 80(11), 1061 (2020). https://doi.org/10.1140/epjc/s10052-020-08641-0. arXiv:2009.12825 [gr-qc]

204. Guth, A.H.: Phys. Rev. D 23, 347–356 (1981). https://doi.org/10.1103/PhysRevD.23.347

205. Guzman, F.S., Urena-Lopez, L.A.: Phys. Rev. D 68, 024023 (2003). https://doi.org/10.1103/PhysRevD.68.024023. arXiv:astro-ph/0303440 [astro-ph]

206. Harko, T., Lobo, F.S.N., Mak, M.K.: Eur. Phys. J. C 74, 2784 (2014). https://doi.org/10.1140/epjc/s10052-014-2784-8. arXiv:1310.7167 [gr-qc]

207. Harko, T., Lobo, F.S.N.: Extensions of f(R) Gravity. Cambridge University Press, Cambridge (2018)

208. Harrison, E.R.: Mon. Not. R. Astron. Soc. 137, 69 (1967)

209. Hawkins, R.M., Lidsey, J.E.: Phys. Rev. D 66, 023523 (2002). https://doi.org/10.1103/PhysRevD.66.023523. arXiv:astro-ph/0112139 [astro-ph]

210. He, J.H., Wang, B., Jing, Y.P.: JCAP 07, 030 (2009). https://doi.org/10.1088/1475-7516/2009/07/030. arXiv:0902.0660 [gr-qc]

211. He, J.H., Wang, B., Abdalla, E.: Phys. Rev. D 83, 063515 (2011). https://doi.org/10.1103/PhysRevD.83.063515. arXiv:1012.3904 [astro-ph.CO]

212. Heisenberg, L.: Phys. Rept. 796, 1–113 (2019). https://doi.org/10.1016/j.physrep.2018.11.006. arXiv:1807.01725 [gr-qc]

213. Horndeski, G.W.: Int. J. Theor. Phys. 10, 363–384 (1974). https://doi.org/10.1007/BF01807638

214. Hoyle, F., Narlikar, J.V.: Proc. R. Soc. Lond. A273, 1 (1963)

215. Hoyle, F., Narlikar, J.V.: Proc. R. Soc. Lond. A294, 138 (1966)

216. Hu, W., Silk, J.: Phys. Rev. D 48, 485–502 (1993). https://doi.org/10.1103/PhysRevD.48.485

217. Huang, Q.G.: JCAP 05, 005 (2009). https://doi.org/10.1088/1475-7516/2009/05/005. arXiv:0903.1542 [hep-th]

218. Hughston, L.P.: Astrophys. J. 158, 98 (1969)

219. Hughston, L.P., Jacobs, K.C.: Astrophys. J. 160, 147 (1970)

220. Hughston, L.P., Shepley, L.C.: Astrophys. J. 160, 333 (1970)

221. Jacobs, K.C.: Nature 215, 1156 (1967)

222. Jacobs, K.C.: Astrophys. J. 153, 661 (1968)
223. Jamil, M., Saridakis, E.N., Setare, M.R.: Phys. Rev. D 81, 023007 (2010). https://doi.org/10.1103/PhysRevD.81.023007. arXiv:0910.0822 [hep-th]

224. Jordan, P.: Naturwiss. 26, 417 (1938)

225. Jordan, P.: Z. Phys. 157, 112–121 (1959). https://doi.org/10.1007/BF01375155

226. Joseph, A., Saha, R.: arXiv:1219.06782 [gr-qc]

227. Kaiser, D.I.: Phys. Rev. D 81, 084044 (2010). https://doi.org/10.1103/PhysRevD.81.084044. arXiv:1003.1159 [gr-qc]

228. Kamenshchik, A.Y., Moschella, U., Pasquier, V.: Phys. Lett. B 511, 265 (2001). [arXiv:gr-qc/0103004]

229. Kamenshchik, A.Y., Tronconi, A., Venturi, G., Vernov, S.Y.: Phys. Rev. D 87(6), 063503 (2013). https://doi.org/10.1103/PhysRevD.87.063503. arXiv:1211.6272 [gr-qc]

230. Kamenshchik, A.Y., Pozdeeva, E.O., Tronconi, A., Venturi, G., Vernov, S.Y.: Class. Quant. Gravit. 31, 105003 (2014). https://doi.org/10.1088/0264-9381/31/10/105003. arXiv:1312.3540 [hep-th]

231. Kamenshchik, A.Y., Pozdeeva, E.O., Tronconi, A., Venturi, G., Vernov, S.Y.: Class. Quant. Gravit. 33(1), 015004 (2016). https://doi.org/10.1088/0264-9381/33/1/015004. arXiv:1509.00590 [gr-qc]

232. Kharbediya, L.I.: Astron. Zh. (USSR) 53, 1145 (1976). (in Russian)

233. Khurshudyan, M., Chubaryan, E., Pourhassan, B.: Int. J. Theor. Phys. 53, 2370 (2014). https://doi.org/10.1007/s10773-014-2036-6. arXiv:1402.2385 [gr-qc]

234. Kim, S.A., Liddle, A.R.: Phys. Rev. D 74, 063522 (2006). https://doi.org/10.1103/PhysRevD.74.063522. arXiv:astro-ph/0608186 [astro-ph]

235. Kim, S.A., Liddle, A.R., Seery, D.: Phys. Rev. Lett. 105, 181302 (2010). https://doi.org/10.1103/PhysRevLett.105.181302. arXiv:1005.4410 [astro-ph.CO]

236. Kobayashi, T., Yamaguchi, M., Yokoyama, J.: Prog. Theor. Phys. 126, 511–529 (2011). https://doi.org/10.1143/PTP.126.511. arXiv:1105.5723 [hep-th]

237. Kobayashi, T.: Rept. Prog. Phys. 82(8), 086901 (2019). https://doi.org/10.1088/1361-6633/ab2429. arXiv:1901.07183 [gr-qc]

238. Kofinas, G., Maartens, R., Papantonopoulos, E.: J. High Energy Phys. 2003, 066 (2003)

239. Kofman, L.A., Linde, A.D., Starobinsky, A.A.: Phys. Lett. B 157, 361–367 (1985). https://doi.org/10.1016/0370-2693(85)90381-8

240. Kolb, E.W., Turner, M.S.: The Early Universe. Addison-Wesley, Redwood City (1990)

241. Kritpetch, C., Sanongkhun, J., Vanichchapongjaroen, P., Gumjudpai, B.: Mod. Phys. Lett. A 35(19), 2050157 (2020). https://doi.org/10.1142/S0217732320501576. arXiv:1908.11265 [gr-qc]

242. Kruger, A.T., Norbury, J.W.: Phys. Rev. D 61, 087303 (2000). https://doi.org/10.1103/PhysRevD.61.087303. arXiv:gr-qc/0004039 [gr-qc]

243. Kruger, A.T., Norbury, J.W.: Phys. Rev. D 61, 087303 (2000). https://doi.org/10.1103/PhysRevD.61.087303. arXiv:gr-qc/0004039 [gr-qc]

244. Lazkoz, R., Leon, G.: Phys. Lett. B 638, 303–309 (2006). https://doi.org/10.1016/j.physletb.2006.05.075. arXiv:astro-ph/0602590 [astro-ph]

245. Lazkoz, R., Leon, G., Quiros, I.: Phys. Lett. B 649, 103–110 (2007). https://doi.org/10.1016/j.physletb.2007.03.060. arXiv:astro-ph/0701353 [astro-ph]

246. Lee, J.W.: J. Korean Phys. Soc. 54, 2622 (2009). https://doi.org/10.3938/jkps.54.2622. arXiv:0801.1442 [astro-ph]

247. Lemaître, G.: Ann. Soc. Sci. Bruxelles A 47, 49 (1927)

248. Lemaître, G.: Ann. Soc. Sci. Bruxelles B 95, 483 (1931)

249. Lemaître, G.: Ann. Soc. Sci. Bruxelles IA 53, 51 (1933)

250. Leon, G., Leyva, Y., Socorro, J.: Phys. Lett. B 732, 285–297 (2014). https://doi.org/10.1016/j.physletb.2014.03.053. arXiv:1208.0061 [gr-qc]

251. Leon, G., Paliathanasis, A., Morales-Martínez, J.L.: Eur. Phys. J. C 78(9), 753 (2018). https://doi.org/10.1140/epjc/s10052-018-6225-y. arXiv:1808.05634 [gr-qc]

252. Lemaître, G.: Astron. Zh. (USSR) 53, 1145 (1976). (in Russian)

253. Lemaître, G.: Mon. Not. R. Astron. Soc. 95, 483 (1931)

254. Lemaître, G.: Ann. Soc. Sci. Bruxelles IA 53, 51 (1933)

255. Lemaître, G.: Ann. Soc. Sci. Bruxelles A 47, 49 (1927)

256. Lemaître, G.: Ann. Soc. Sci. Bruxelles B 95, 483 (1931)

257. Lemaître, G.: Ann. Soc. Sci. Bruxelles IA 53, 51 (1933)

258. Lemaître, G.: Ann. Soc. Sci. Bruxelles IA 53, 51 (1933)
257. Li, B., Rindler-Daller, T., Shapiro, P.R.: Phys. Rev. D 89(8), 083536 (2014). https://doi.org/10.1103/PhysRevD.89.083536. arXiv:1310.6061 [astro-ph.CO]
258. Li, Y.H., Zhang, J.F., Zhang, X.: Phys. Rev. D 90(6), (2014). https://doi.org/10.1103/PhysRevD.90.063505. arXiv:1404.5220 [astro-ph.CO]
259. Liddle, A.R.: Phys. Lett. B 220, 502–508 (1989). https://doi.org/10.1016/0370-2693(89)90776-4
260. Liddle, A.R., Scherrer, R.J.: Phys. Rev. D 59, 023509 (1999). https://doi.org/10.1103/PhysRevD.59.023509. arXiv:astro-ph/9809272 [astro-ph]
261. Liddle, A.: An Introduction to Modern Cosmology. Wiley, New York (2015)
262. Liddle, A.R., Lyth, D.H.: Cosmological Inflation and Large-Scale Structure. Cambridge University Press, Cambridge (2000)
263. Lidsey, J.E.: Phys. Lett. B 273, 42–46 (1991). https://doi.org/10.1016/0370-2693(91)90550-A
264. Lidsey, J.E.: Class. Quant. Gravit. 8, 923–933 (1991). https://doi.org/10.1088/0264-9381/8/5/016
265. Lidsey, J.E.: Gen. Relativ. Gravit. 25, 399–407 (1993). https://doi.org/10.1007/BF00757120
266. Lidsey, J.E., Waga, I.: Phys. Rev. D 51, 444–449 (1995). https://doi.org/10.1103/PhysRevD.51.444. arXiv:astro-ph/9408016 [astro-ph]
267. Lidsey, J.E., Liddle, A.R., Kolb, E.W., Copeland, E.J., Barreiro, T., Abney, M.: Rev. Mod. Phys. 69, 373–410 (1997). https://doi.org/10.1103/RevModPhys.69.373. arXiv:astro-ph/9508078 [astro-ph]
268. Lidsey, J.E.: Class. Quant. Gravit. 21, 777–786 (2004). https://doi.org/10.1088/0264-9381/21/4/002. arXiv:gr-qc/0307037 [gr-qc]
269. Lima, J.A.S.: Am. J. Phys. 69, 1245–1247 (2001). https://doi.org/10.1119/1.1405506. arXiv:astro-ph/0109215 [astro-ph]
270. Linde, A.D., Mukhanov, V.: Phys. Rev. D 56, R535–R539 (1997). https://doi.org/10.1103/PhysRevD.56.R535. arXiv:astro-ph/9610215 [astro-ph]
271. Linde, A.D., Mukhanov, V.: JCAP 04, 009 (2006). https://doi.org/10.1088/1475-7516/2006/04/009. arXiv:astro-ph/0511738 [astro-ph]
272. Liouville, J.J.: Math. Pures Appl. Paris 18(1), 71 (1853)
273. Lucchin, F., Matarrese, S.: Phys. Rev. D 32, 1316 (1985). https://doi.org/10.1103/PhysRevD.32.1316
274. Lyth, D.H., Ungarelli, C., Wands, D.: Phys. Rev. D 67, 023503 (2003). https://doi.org/10.1103/PhysRevD.67.023503. arXiv:astro-ph/0208055 [astro-ph]
275. Madsen, M.S.: Astrophys. Space Sci. 113, 205 (1985)
276. Madsen, M.S.: Class. Quant. Gravit. 5, 627–639 (1988). https://doi.org/10.1088/0264-9381/5/4/010
277. Marchisotto, E.A., Zakeri Coll, G.-A.: Math. J. 25, 295–308 (1994)
278. Martin, J., Ringeval, C., Vennin, V.: Phys. Dark Univ. 5–6, 75–235 (2014). https://doi.org/10.1016/j.dark.2014.01.003. arXiv:1303.3787 [astro-ph.CO]
279. Matos, T., Guzman, F.S.: Class. Quant. Gravit. 17, L9–L16 (2000). https://doi.org/10.1088/0264-9381/17/11/102. arXiv:gr-qc/01030028 [gr-qc]
280. Matos, T., Guzman, F.S., Urena-Lopez, L.A.: Class. Quant. Gravit. 17, 1707–1712 (2000). https://doi.org/10.1088/0264-9381/17/7/309. arXiv:astro-ph/9908152 [astro-ph]
281. Matos, T., Guzman, F.S., Urena-Lopez, L.A., Nunez, D.: https://doi.org/10.1007/0-306-47115-9_16. arXiv:astro-ph/0102419 [astro-ph]
282. Matos, T., Vázquez-Gonzalez, A., Magana, J.: Mon. Not. R. Astron. Soc. 393, 1359–1369 (2009). https://doi.org/10.1111/j.1365-2966.2008.13957.x. arXiv:0806.0683 [astro-ph]
283. May, T.L.: Astrophys. J. 199, 322 (1975)
284. May, T.L., McVittie, G.C.: Mon. Not. R. Astron. Soc. 148, 407 (1970)
285. May, T.L., McVittie, G.C.: Mon. Not. R. Astron. Soc. 153, 491–500 (1971)
286. McIntosh, C.B.G.: Nature 216, 1297–1298 (1967)
287. McIntosh, C.B.G.: Mon. Not. R. Astron. Soc. 138, 423 (1968)
288. McIntosh, C.B.G.: Mon. Not. R. Astron. Soc. 140, 461 (1968)
289. McIntosh, C.B.G.: J. Math. Phys. 11, 250–252 (1970)
290. McIntosh, C.B.G.: Austral. J. Phys. 25, 75–82 (1972)
291. McIntosh, C.B.G., Foyster, J.M.: Austral. J. Phys. 25, 83–89 (1972)
295. Méndez, V.: Class. Quant. Gravit. 13, 3229–3239 (1996). https://doi.org/10.1088/0264-9381/13/12/013

296. Meyers, J., Sivanandam, N.: Phys. Rev. D 83, 103517 (2011). https://doi.org/10.1103/PhysRevD.83.103517. arXiv:1011.4934 [astro-ph.CO]

297. Mishra, S., Chakraborty, S.: Eur. Phys. J. C 78(11), 917 (2018). https://doi.org/10.1140/epjc/s10052-018-6405-9. arXiv:1811.08279 [gr-qc]

298. Misner, C.W.: Astrophys. J. 151, 431–457 (1968). https://doi.org/10.1086/149448

299. Mohseni Sadjadi, H., Alimohammadi, M.: Phys. Rev. D 83, 103517 (2011). https://doi.org/10.1103/PhysRevD.83.103517. arXiv:1011.4934 [astro-ph.CO]

300. Motohashi, H., Noui, K., Suyama, T., Yamaguchi, M., Langlois, D.: JCAP 07, 033 (2016). https://doi.org/10.1088/1475-7516/2016/07/033. arXiv:1603.09355 [hep-th]

301. Mukhanov, V.: Physical Foundations of Cosmology. Cambridge University Press, Cambridge (2005)

302. Mukhanov, V.F., Feldman, H.A., Brandenberger, R.H.: Phys. Rep. 215, 203–333 (1992). https://doi.org/10.1016/0370-1573(92)90044-Z

303. Muslimov, A.G.: Class. Quant. Gravit. 7, 231–237 (1990). https://doi.org/10.1088/0264-9381/7/2/015

304. Naidu, N.F., Carloni, S., Dunsby, P.: Phys. Rev. D 104, 044014 (2021). https://doi.org/10.1103/PhysRevD.104.044014. arXiv:2102.05693 [gr-qc]

305. Naruko, A., Sasaki, M.: Prog. Theor. Phys. 121, 193–210 (2009). https://doi.org/10.1143/PTP.121.193. arXiv:0807.0180 [astro-ph]

306. Nesteruk, A.V.: Class. Quant. Gravit. 11, L15 (1994)

307. Nesteruk, A.V., Ottewill, A.C.: Class. Quant. Gravit. 12, 51–57 (1995). https://doi.org/10.1088/0264-9381/12/1/005

308. Nicolis, A., Rattazzi, R., Trincherini, E.: Phys. Rev. D 79, 064036 (2009). https://doi.org/10.1103/PhysRevD.79.064036. arXiv:0811.2197 [hep-th]

309. Nojiri, S., Odintsov, S.D.: Phys. Rev. D 70, 103522 (2004). https://doi.org/10.1103/PhysRevD.70.103522. arXiv:hep-th/0408170 [hep-th]

310. Nojiri, S., Odintsov, S.D.: Phys. Rev. D 72, 023003 (2005)

311. Nojiri, S., Odintsov, S.D.: Phys. Rev. Lett. 69, 144–150 (2006). https://doi.org/10.1016/j.physletb.2006.06.065. arXiv:hep-th/0606025 [hep-th]

312. Nojiri, S., Odintsov, S.D.: Phys. Rev. Lett. 69, 440–444 (2007). https://doi.org/10.1016/j.physletb.2007.02.042. arXiv:hep-th/0702031 [hep-th]

313. Nojiri, S., Odintsov, S.D.: Phys. Rept. 505, 59 (2011). https://doi.org/10.1016/j.physrep.2011.04.001. arXiv:1011.0544 [gr-qc]

314. Nordtvedt, K., Jr.: Astrophys. J. 161, 1059–1067 (1970). https://doi.org/10.1086/150607

315. Nowakowski, M., Rosu, H.: Phys. Rev. E 65, 047602 (2002). https://doi.org/10.1103/PhysRevE.65.047602. arXiv:physics/0111006 [physics]

316. Özer, M., Taha, M.O.: Phys. Rev. D 45, R997–R999 (1992). https://doi.org/10.1103/PhysRevD.45.8997

317. Padmanabhan, T., Choudhury, T.R.: Phys. Rev. D 66, 081301 (2002). https://doi.org/10.1103/PhysRevD.66.081301. arXiv:hep-th/0205055 [hep-th]

318. Paliathanasis, A., Tsamparlis, M., Basilakos, S.: Phys. Rev. D. 102(6), 063524 (2020). https://doi.org/10.1103/PhysRevD.102.063524. arXiv:2005.11726 [gr-qc]

319. Paliathanasis, A., Tsamparlis, M., Basilakos, S.: Phys. Rev. D 84, 123514 (2011). https://doi.org/10.1103/PhysRevD.84.123514. arXiv:1111.4547 [astro-ph.CO]

320. Paliathanasis, A., Tsamparlis, M., Basilakos, S.: Phys. Rev. D 90(10), 103524 (2014). https://doi.org/10.1103/PhysRevD.90.103524. arXiv:1410.4930 [gr-qc]

321. Paliathanasis, A., Tsamparlis, M., Basilakos, S.: Phys. Rev. D 90(4), 043529 (2014). https://doi.org/10.1103/PhysRevD.90.043529. arXiv:1408.1789 [gr-qc]

322. Paliathanasis, A., Tsamparlis, M.: Phys. Rev. D 93(4), 043528 (2016). https://doi.org/10.1103/PhysRevD.93.043528. arXiv:1511.00439 [gr-qc]

323. Paliathanasis, A., Capozziello, S.: Mod. Phys. Lett. A 31(32), 1650183 (2016). https://doi.org/10.1142/S0217732316501832. arXiv:1602.08914 [gr-qc]

324. Payne, A.D.: Austral. J. Phys. 23, 177–185 (1970)

325. Peebles, P.J.E.: Astrophys. J. 153, 1 (1968). https://doi.org/10.1086/149628

326. Peebles, P.J.E.: Principles of Physical Cosmology. Princeton University Press, Princeton (1993)
328. Peebles, P.J.E., Ratra, B.: Astrophys. J. Lett. 325, L17 (1988). https://doi.org/10.1086/185100
329. Penzias, A.A., Wilson, R.W.: Astrophys. J. 142, 419–421 (1965). https://doi.org/10.1086/148307
330. Perlmutter, S., et al.: [Supernova Cosmology Project], Nature 391, 51–54 (1998) https://doi.org/10.1038/34124. arXiv:astro-ph/9712212 [astro-ph]
331. Perlmutter, S., et al.: [Supernova Cosmology Project], Astrophys. J. 517, 565–586 (1999). https://doi.org/10.1086/307221. arXiv:astro-ph/9812133 [astro-ph]
332. Peterson, C.M., Tegmark, M.: Phys. Rev. D 84, 023520 (2011). https://doi.org/10.1103/PhysRevD.84.023520. arXiv:1011.6675 [astro-ph.CO]
333. Phetnora, T., Sooksan, R., Gumjudpai, B.: Gen. Relativ. Gravit. 42, 225–240 (2010). https://doi.org/10.1007/s10714-009-0831-9 . arXiv:0805.3794 [gr-qc]
334. Piedipalumbo, E., De Laurentis, M., Capozziello, S.: Phys. Dark Univ. 27, 100444 (2020). https://doi.org/10.1016/j.dark.2019.100444. arXiv:1912.08089 [gr-qc]
335. Pimentel, L.O.: Class. Quant. Gravit. 6, L263 (1989)
336. Pinney, E.: Proc. Am. Math. Soc. 1, 681 (1950)
337. Pradhan, A.: Indian J. Phys. 88, 215–223 (2014). arXiv:1211.1882 [physics.gen-ph]
338. Pucheu, M.L., Bellini, M.: Nuovo Cim. B 125, 851–859 (2010). https://doi.org/10.1393/ncb/i2010-10888-0. arXiv:0906.1824 [gr-qc]
339. Qiu, T.: Mod. Phys. Lett. A 25, 909–921 (2010). https://doi.org/10.1142/S021773231000006X. arXiv:1002.3971 [hep-th]
340. Riess, A.G., et al.: [Supernova Search Team] Astron. J. 116, 1009–1038 (1998). https://doi.org/10.1086/300499. arXiv:astro-ph/9805201 [astro-ph]
341. Rigopoulos, G.I., Shellard, E.P.S., van Tent, B.J.W.: Phys. Rev. D 73, 083522 (2006). https://doi.org/10.1103/PhysRevD.73.083522. arXiv:astro-ph/0506704 [astro-ph]
342. Rigopoulos, G.I., Shellard, E.P.S., van Tent, B.J.W.: Phys. Rev. D 76, 083512 (2007). https://doi.org/10.1103/PhysRevD.76.083512. arXiv:astro-ph/0511041 [astro-ph]
343. Rindler-Daller, T., Shapiro, P.R.: Mod. Phys. Lett. A 29(2), 1430002 (2014). https://doi.org/10.1142/S021773231430002X. arXiv:1312.1734 [astro-ph.CO]
344. Robles, V.H., Matos, T.: Mon. Not. Roy. Astron. Soc. 422, 282–289 (2012). https://doi.org/10.1111/j.1365-2966.2012.20603.x. arXiv:1201.3032 [astro-ph.CO]
345. Robles, V.H., Bullock, J.S., Boylan-Kolchin, M.: Mon. Not. R. Astron. Soc. 483(1), 289–298 (2019). https://doi.org/10.1093/mnras/sty3190. arXiv:1807.06018 [astro-ph.CO]
346. Roeder, R.C.: Nature 216, 774 (1967)
347. Rosu, H.C., Espinoza, P., Reyes, M.: Nuovo Cim. B 114, 1439–1444 (1999). arXiv:gr-qc/9910070 [gr-qc]
348. Rosu, H.C., Ojeda-May, P.: Int. J. Theor. Phys. 45, 1191–1196 (2006). https://doi.org/10.1007/s10773-006-9123-2. arXiv:gr-qc/0510004 [gr-qc]
349. Rosu, H.C., Khmelnytskaya, K.V.: SIGMA 7, 013 (2011). https://doi.org/10.3842/SIGMA.2011.013. arXiv:1012.1920 [gr-qc]
350. Szydlowski, M., Stachowski, A., Borowiec, A., Wojnar, A.: Eur. Phys. J. C 76, 567 (2016)
351. Sadeghi, J., Setare, M.R., Banijamali, A., Milani, F.: Phys. Lett. B 662, 92–96 (2008). https://doi.org/10.1016/j.physletb.2008.02.062. arXiv:0804.0553 [hep-th]
352. Salopek, D.S., Bond, J.R.: Phys. Rev. D 42, 3936–3962 (1990). https://doi.org/10.1103/PhysRevD.42.3936
353. Sapar, A.: Astron. Zh. (USSR) 47, 503 (1970). (in Russian)
354. Saridakis, E.N., Wellner, J.M.: Phys. Rev. D 81, 123523 (2010). https://doi.org/10.1103/PhysRevD.81.123523. arXiv:1012.5304 [hep-th]
355. Saridakis, E.N.: Nucl. Phys. B 830, 374–389 (2010). https://doi.org/10.1016/j.nuclphysb.2010.01.005. arXiv:0903.3840 [astro-ph.CO]
356. Sasaki, M., Valiviita, J., Wands, D.: Phys. Rev. D 74, 103003 (2006). https://doi.org/10.1103/PhysRevD.74.103003. arXiv:astro-ph/0607627 [astro-ph]
357. Sasaki, M.: Prog. Theor. Phys. 120, 159–174 (2008). https://doi.org/10.1143/PTP.120.159. arXiv:0805.0974 [astro-ph]
358. Schunck, F.E., Mielke, E.W.: Phys. Rev. D 50, 4794–4806 (1994). https://doi.org/10.1103/PhysRevD.50.4794. arXiv:gr-qc/9407041 [gr-qc]
359. Semiz, I.: Phys. Rev. D 85, 068501 (2012). https://doi.org/10.1103/PhysRevD.85.068501
360. Setare, M.R.: Phys. Lett. B 641, 130–133 (2006). https://doi.org/10.1016/j.physletb.2006.08.039. arXiv:hep-th/0611165 [hep-th]
428. Zhang, X.: Phys. Rev. D 79, 103509 (2009). https://doi.org/10.1103/PhysRevD.79.103509. arXiv:0901.2262 [astro-ph.CO]

429. Zhao, G.B., Xia, J.Q., Li, M., Feng, B., Zhang, X.: Phys. Rev. D 72, 123515 (2005). https://doi.org/10.1103/PhysRevD.72.123515. arXiv:astro-ph/0507482 [astro-ph]

430. Zhao, W.: Phys. Rev. D 73, 123509 (2006). https://doi.org/10.1103/PhysRevD.73.123509. arXiv:astro-ph/0604460 [astro-ph]

431. Zhuravlev, V.M., Chervon, S.V., Shchigolev, V.K.: J. Exp. Theor. Phys. 87, 223–228 (1998). https://doi.org/10.1134/1.558649

432. Ziaeepour, H.: Phys. Rev. D 86, 043503 (2012). https://doi.org/10.1103/PhysRevD.86.043503. arXiv:1112.6025 [astro-ph.CO]

433. Zimdahl, W.: Int. J. Geom. Methods Mod. Phys. 11, 1460014 (2014). https://doi.org/10.1142/S0219887814600147. arXiv:1404.7334 [astro-ph.CO]

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