CONGRUENCES RELATED TO BELL POLYNOMIALS VIA A DIFFERENTIAL OPERATOR

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Abstract. In this paper, by the differential operator we give some congruences and new proof for some known congruence concerning the Bell polynomials, derangement polynomials and \( r \)-Lah polynomials.

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1. INTRODUCTION

Recall that the \( n \)-th Bell polynomial (named also Touchard polynomial) \( B_n(x) \) and the \( n \)-th \( r \)-Bell polynomial \( B_{n,r}(x) \) are defined by

\[
B_0(x) = 1 \quad \text{and} \quad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,
\]

\[
B_{0,r}(x) = 1 \quad \text{and} \quad B_{n,r}(x) = \sum_{k=0}^{n} \binom{n+r}{k+r} x^k,
\]

where \( \binom{n}{k} \) is \( (n,k) \)-th Stirling number of the second kind which counts the number of partition of the set \([n]\) into \( k \) non-empty subsets, and \( \binom{n}{k}_r \) is the \( r \)-Stirling numbers of the second kind which counts the number of partition of the set \([n]\) into \( k \) non-empty subsets such that the numbers \( 1, \ldots, r \) are in distinct subsets see [2].

The numbers \( B_n := B_n(1) \) is the \( n \)-th Bell number which counts the number of all partitions of the set \( [n] := \{1, \ldots, n\} \) and \( B_{n,r} := B_{n,r}(1) \) is the \( n \)-th \( r \)-Bell number which counts the number of all partitions of set \([n+r]\) into \( k + r \) non-empty subsets such that the first \( r \) elements are in distinct subsets, see [6]. These polynomials are also given by the Dobinski formula

\[
B_n(x) = \exp(-x) \sum_{j=0}^{n} \frac{x^j}{j!} \quad \text{and} \quad B_{n,r}(x) = \exp(-x) \sum_{j=0}^{n} (j+r) \frac{x^j}{j!}.
\]

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In 1933 Touchard [11] established for any prime number $p$ the following congruence
\[ B_{n+p} \equiv B_{n+1} + B_n \pmod{p}, \quad n \in \mathbb{N}. \]
Later Gertsch and Robert [3], by the umbral calculus method proved that
\[ B_{n+p}(x) \equiv B_{n+1}(x) + x^p B_n(x) \pmod{p \mathbb{Z}_p[x]}, \quad n \in \mathbb{N}. \]
Benyattou and Mihoubi [1] proved that
\[ B_{n+p,r}(x) \equiv B_{n+1,r}(x) + x^p B_{n,r}(x) \pmod{p \mathbb{Z}_p[x]}, \quad n, r \in \mathbb{N}, \]
where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers.

Several congruences involved the Bell polynomials are given in [1,3,9,10] and are linked to other polynomials such as the Lah and derangement polynomials. Motivated by the congruences studied in the above references we use in this paper the differential operator and its properties to establish some congruences and to give a new proof for some known congruence concerning the Bell polynomials, $r$-Bell polynomials, derangement polynomials and $r$ Lah polynomials. This paper is organized as follows:

In the next section by the differential operator we define the Bell polynomials also we give a new proof of some known congruences on the Bell and $r$-Bell polynomials.

Lemma 1. Let $D = \frac{d}{dx}$ be the differential operator and $P = x + xD$ and let $f$ be a polynomial. Then for any non-negative integers $n, r$ there hold
\[ (P)^r 1 = P^n x^r = x^r B_{n,r}(x), \quad (2.1) \]
\[ (P)^r f(P) 1 = x^r f(P + r) 1. \quad (2.2) \]
In particular, we get
\[ B_n(x) = P^{n-m} B_m(x), \quad 0 \leq m \leq n, \quad (2.3) \]
\[ x^r = (P)^r 1, \quad (2.4) \]
where \((x)_r\) is the Pochhammer symbol defined above.

Proof. To prove the identity \(P^n x^r = x^r B_{n,r}(x)\) we proceed by induction \(n \geq 0\). For \(n = 0\) or \(n = 1\), we have

\[ P^0 x^r = x^r \quad \text{and} \quad P^1 x^r = (x + xD)x^r = x^{r+1} + rx^r = x^r B_{1,r}(x). \]

Assume that \(P^n x^r = x^r B_{n,r}(x)\) for all \(k \in \{0, 1, \ldots, n\}\). Then

\[ P^{n+1} x^r = P(P^n x^r) = (x + xD)x^r B_{n,r}(x) = (r + x)x^r B_{n+1,r}(x) + x^{r+1} D B_{n+1,r}(x), \]

and from (1.1) we obtain

\[ xD B_{n,r}(x) = B_{n+1,r}(x) - (r + x) B_{n,r}(x), \]

hence

\[ P^{n+1} x^r = (r + x)x^r B_{n+1,r}(x) + (r + x) B_{n+1,r}(x) - (r + x) B_{n,r}(x) = x^r B_{n+1,r}(x), \]

which completes the step induction. We also have

\[
\begin{align*}
\langle P \rangle_r P^n 1 &= \sum_{k=0}^{r} (-1)^{r-k} \left[ \begin{array}{c} r \\ k \end{array} \right] P^{n+k} 1 = P^r \sum_{k=0}^{r} (-1)^{r-k} \left[ \begin{array}{c} r \\ k \end{array} \right] B_k(x) = P^r x^r = x^r B_{n,r}(x).
\end{align*}
\]

The identity (2.2) follows because

\[
\begin{align*}
\langle P \rangle_r P^n 1 &= x^r B_{n,r}(x) = x^r \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) r^{n-k} B_k(x) = x^r \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) r^{n-k} P^k 1 = x^r (P + r)^n 1.
\end{align*}
\]

For the particular cases, by taking \(r = 0\) in (2.1) we get \(B_{h}(x) = P^0 1\). This means that

\[ B_{h}(x) = P(P^{n-1}) = P B_{n-1}(x) = \cdots = P^n B_{n-m}(x), \]

by taking \(n = 0\) in (2.1) we get \(\langle P \rangle_r 1 = x^r\).

Remark 1. Since \(B_{h+m}(x) = P^n B_m(x) = P^m B_h(x)\), the following symmetric identity follows

\[ B_{h+m}(x) = \sum_{r=0}^{m} \left( \begin{array}{c} m \\ r \end{array} \right) x^r B_{h,r}(x) = \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) x^r B_{h,r}(x). \]

Let \(r_1, \ldots, r_q, n \geq 0\) be integers such that \(r_1 \leq \cdots \leq r_q\) and let

\[ r_q = (r_1, \ldots, r_q), \quad |r_q| = r_1 + \cdots + r_q. \]

Recall the \(r_q\)-Stirling numbers and the \(r_q\)-Bell polynomials introduced and studied by Maamra and Mihoubi [4, 5, 7], can be defined by

\[
\sum_{j=0}^{n+|r_q|-1} \left\{ \begin{array}{c} n+|r_q| \\ j+q \end{array} \right\} r_q \big( (x + rq) r_1 \cdots (x + rq) r_{q-1} (x + rq)^n, \big),
\]

\[ B_{h}(x; r_q) = \sum_{j=0}^{n+|r_q|-1} \left\{ \begin{array}{c} n+|r_q| \\ j+q \end{array} \right\} x^j. \]
Lemma 2. There holds
\((P)_{r_1} \cdots (P)_{r_{q-1}} P^s x^q = (P)_{r_1} \cdots (P)_{r_q} P^q 1 = x^q B_n(x; r_q)\).

Proof. Setting \((u)_{r_1} \cdots (u)_{r_q} = \sum_{k=0}^{\lfloor r_q \rfloor} a_k(r_q) u^k\).

From Theorem 7 given in [4] and Lemma 1 we have
\[\sum_{k=0}^{\lfloor r_q \rfloor} a_k(r_q) B_{n+k}(x) = P^{s+1} - P^{s+1} = B_{n+p}(x) - B_{n+1}(x) = x^q B_n(x) \pmod{p Z_p[x]}\]

Lemma 3. Let \( f \) be a polynomial in \( \mathbb{Z}[x] \). For any non negative integers \( n, s \geq 1 \), and for any prime \( p \) there holds
\[ f(P)(P^{s'} - P)1 \equiv (x^p + x^{p^2} + \cdots + x^{p^s}) f(P)1 \pmod{p \mathbb{Z}_p[x]}\]

Proof. It suffices to take \( f(x) = x^n \) and proceed by induction on \( s \). Indeed, for \( s = 1 \), use the Touchard’s congruence for polynomials
\[ B_{n+p}(x) \equiv B_{n+1}(x) + x^p B_n(x) \pmod{p \mathbb{Z}_p[x]}\]

to get
\[ P^n(P^p - P)1 = P^{n+p} 1 - P^{n+1} 1 = B_{n+p}(x) - B_{n+1}(x) \equiv x^p B_n(x) = x^p P^n 1 \pmod{p \mathbb{Z}_p[x]}\]

Assume it is true for \( s \). Then
\[ P^n(P^{s+1} - P)1 = ((P^{s'} - P + P^p - P)P^s 1 \equiv ((P^{s'} - P)P^p + P - P)P^{s} 1 = (P^{s'} - P)^p P^{s+1} + (P^p - P)P^s 1 \equiv (x^p + x^{p^2} + \cdots + x^{p^s})(P^{s'} - P)^{p-1} P^{s+1} 1 + x^p P^n 1 \]

\[ \vdots \]

\[ \equiv (x^p + x^{p^2} + \cdots + x^{p^s})^p P^n 1 + x^p P^n 1 \equiv (x^{p^2} + x^{p^3} + \cdots + x^{p^{s+1}})P^n 1 + x^p P^n 1 \equiv (x^p + x^{p^2} + \cdots + x^{p^{s+1}})P^n 1 \pmod{p \mathbb{Z}_p[x]}\]

hence, the proof is completed. \( \square \)

Now we give new proof for a congruence concerning \( B_n(x; r_q) \).
Proposition 1. Let \( n, r, r_q \geq \cdots \geq r_1 \geq 0 \) and \( s \geq 1 \) be non-negative integers. For any prime \( p \) there holds
\[
\mathcal{B}_{n+1} (x; r_q) \equiv (x^p + \cdots + x^{p^r}) \mathcal{B}_n (x; r_q) + \mathcal{B}_{n+1} (x; r_q) \pmod{p\mathbb{Z}_p [x]}.
\]
(2.5)

In particular, for \( s = q = 1 \), \( r_q = r \) and or \( r = 0 \) we get
\[
\mathcal{B}_{n+p,r} (x) \equiv x^p \mathcal{B}_{n,r} (x) + \mathcal{B}_{n+1,r} (x) \pmod{p\mathbb{Z}_p [x]},
\]
\[
\mathcal{B}_{n+p} (x) \equiv x^p \mathcal{B}_n (x) + \mathcal{B}_{n+1} (x) \pmod{p\mathbb{Z}_p [x]}.
\]

Proof. For (2.5), by Lemma 2 and Lemma 3 we have
\[
x^{s} \mathcal{B}_{n+p,r} (x; r_q) = x^{s}(\mathcal{B}_{n+p,r} (x; r_q) - \mathcal{B}_{n+1,r} (x; r_q)) + x^{s} \mathcal{B}_{n+1} (x; r_q)
\]
\[
= (P)_{r_1} \cdots (P)_{r_q} P^s (P^r - P) 1 + x^{s}\mathcal{B}_{n+1} (x; r_q)
\]
\[
\equiv (x^{p} + x^{p^2} + \cdots + x^{p^r})(P)_{r_1} \cdots (P)_{r_q} P^s 1 + x^{s}\mathcal{B}_{n+1} (x; r_q)
\]
\[
= x^{s}(x^{p} + \cdots + x^{p^r}) \mathcal{B}_n (x; r_q) + x^{s} \mathcal{B}_{n+1} (x; r_q) \pmod{p\mathbb{Z}_p [x]}.
\]
This completes the proof. \( \square \)

3. Congruences Related to Bell Polynomials Via a Differential Operator

In this section, we give some general congruences on the \( r \)-Lah polynomials and the derangement polynomials. To start, let us give a short introduction to these polynomials. Recall that the \((n,k)\)-th \( r \)-Lah number \( L_{r} (n,k) \) counts the number of partitions of the set \( \{n + r\} \) into \( k + r \) ordered lists, such that the numbers of set \( \{r\} \) are in distinct lists, see also\([8]\), and the \( r \)-Lah polynomials associated to \( r \)-Lah number are defined by
\[
L_{n,r} (x) = \sum_{k=0}^{n} L_{r} (n,k) x^k,
\]
with exponential generating function
\[
\sum_{i=0}^{n} L_{n,r} (x) \frac{t^n}{n!} = \frac{1}{(1-t)^2} \exp \left( \frac{t}{1-t} x \right).
\]
The derangement polynomials are defined by
\[
D_n (x) = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} k! (x-1)^{n-k},
\]
where \( D_n (0) := D_n \) is the number of derangements of \( n \) elements, see \([10]\).

Lemma 4. For any non-negative integer \( n, r \) there hold
\[
L_{n,r} (x) = (P + 2r)_n 1,
\]
(3.1)
\[
D_n (1-x) = (-1)^n (P - 1)_n 1.
\]
(3.2)
Proof. Since \( \langle x \rangle^n := (x + n - 1)_n \) and the property
\[
\langle x \rangle^m = \langle x \rangle_m \quad \text{(2.4)},
\]
then, from the known identity [8] \((x + 2r)_n = \sum_{k=0}^{n} L_r(n,k) x^k\), and by the relation (2.4) the \(r\)-Lah polynomials can be written as
\[
L_{n,r}(x) = \sum_{k=0}^{n} L_r(n,k) x^k = \sum_{k=0}^{n} L_r(n,k) (P)_k 1 = (P+2r)_n 1.
\]
We also have by the relation (2.4)
\[
(P-1)_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (P)_{n-k} 1 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^n-x = (-1)^n D_n(1-x).
\]

Proposition 2. For any prime number \(p\) and any integers \(n,m,r \geq 0\), there holds
\[
L_{n+mp,r}(x) \equiv x^{mp} L_{n,r}(x) \pmod{p \mathbb{Z}_p[x]}.
\]
Prove. From (3.1) we have
\[
L_{n+mp,r}(x) = \langle P+2r \rangle_{n+mp} 1 = \langle P+2r \rangle_p \langle P+2r+p \rangle_{n+(m-1)p} 1
\]
\[
\equiv \langle P+2r+p-1 \rangle_p \langle P+2r \rangle_{n+(m-1)p} 1
\]
\[
= \langle P+2r \rangle_{n+(m-1)p} \sum_{k=0}^{p-1} \binom{p}{k} (2r+p-1)_{p-k} (P)_{p-k} 1
\]
\[
\equiv \langle P+2r \rangle_{n+(m-1)p} \left( \frac{2r+p-1}{p} \right) (P+2r)_{n+(m-1)p} 1 + (P)_{p} (P+2r)_{n+(m-1)p} 1
\]
\[
\equiv x^p \langle P+2r+p \rangle_{n+(m-1)p} 1 \equiv x^p L_{n+(m-1)p,r}(x) \pmod{p \mathbb{Z}_p[x]}.
\]
So, we get successively
\[
L_{n+mp,r}(x) \equiv x^{mp} L_{n+(m-1)p,r}(x) \equiv \ldots \equiv x^{(m-1)p} L_{n+p,r}(x) \equiv x^{mp} L_{n,r}(x) \pmod{p \mathbb{Z}_p[x]}.
\]

Proposition 3. Let \(p\) be a prime number and \(m,n\) be non-negative numbers. There holds
\[
D_{n+mp}(1-x) \equiv (-x)^{mp} D_n(1-x) \pmod{p \mathbb{Z}_p[x]}.
\]
For \(x = 1\), we obtain
\[
D_{n+mp} \equiv (-1)^{mp} D_n \pmod{p}.
\]
Proof. By (3.2), the identity \((x)_{n+m} = (x)_n (x-n)_m\) and the congruence
\[
\binom{p}{j} \equiv 0 \pmod{p}, \quad 1 \leq j \leq p-1,
\]
we obtain
\[
D_{n+p}(1-x) = \sum_{j=0}^{p-1} (\binom{p}{j} (-1)^{p-j} (P-1)^{p-j} \cdot (P-1)^{n-j+1} \cdot (x)_j (x-n)_m)
\]
and one can proceed easily by induction on \(m \geq 0\) to complete the proof. \(\square\)

Proposition 4. For any non-negative integers \(n, r\) and for any prime number \(p\), there holds
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (2r+n)_{n-k} D_k(1-x) = L_{n,r}(x). \quad (3.4)
\]
In particular for \(n = p - 1\) we get
\[
\sum_{k=0}^{p-1} (2r+p-1)_{p-1-k} D_k(1-x) \equiv L_{p-1,r}(x) \pmod{p\mathbb{Z}[x]}, \quad (3.5)
\]
for \(n = p - 1, r = 0\) we get
\[
\sum_{k=0}^{p-1} \frac{D_k(1-x)}{k!} \equiv -L_{p-1}(x) \pmod{p\mathbb{Z}[x].} \quad (3.6)
\]

Proof. For (3.4) we have
\[
L_{n,r}(x) = (P+2r)_n = (P+2r+n-1)_n
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} (2r+n)_{n-k} (P-1)_k \cdot 1
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (2r+n)_{n-k} D_k(1-x).
\]
For (3.5) and (3.6) take \(n = p - 1\) or \(n = p - 1, r = 0\) and use the congruences
\[
\binom{p-1}{k} \equiv (-1)^k \pmod{p}, \quad k!(p-1)_{p-1-k} \equiv -1 \pmod{p}, \quad 0 \leq k \leq p - 1.
\]
\(\square\)
Proposition 5. For any non-negative integers \( n \) and for any prime number \( p \), there holds
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k B_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_k (1-x).
\] (3.7)

In particular for \( n = p-1 \) or \( n = p \) we obtain
\[
\sum_{k=0}^{p-1} B_{p-1-k}(x) \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^k D_k (1-x) \pmod{\mathbb{Z}_p[x]},
\] (3.8)
\[
D_p (1-x) \equiv -x^p \pmod{\mathbb{Z}_p[x]}.
\] (3.9)

Proof. For (3.7) we have
\[
(P-1)^n 1 = \sum_{k=0}^{n} \binom{n}{k} (P-1)_k 1.
\]

Then
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k P^{n-k} 1 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k B_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_k (1-x).
\]

For (3.8) and (3.9) take \( n = p-1 \) or \( n = p \) and use the congruences
\[
\binom{p-1}{k} \equiv (-1)^k \pmod{p}, \quad \binom{p}{k} \equiv 0 \pmod{p} \quad 1 < k < p.
\]

\( \square \)

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