ABC-formula and R-operation for decay processes

Dmitrii V. Prokhorenko

Steklov Mathematical Institute
Russian Academy of Sciences
Gubkin St. 8, 119991, Moscow, Russia
email:prokhor@mi.ras.ru

Abstract

The vacuum expectation value of the evolution operator for a general class of Hamiltonians used in quantum field theory and statistical physics and which include unstable particles is considered. An exact formula which describes the large time behavior of the evolution operator is proved.
1 Introduction

The basis object in quantum field theory is scattering matrix, which describe behavior of the system at infinite intervals of times \([1, 2]\). But there exists a lot of interesting problems where one is interested for behavior of the system at large but finite intervals of times. The general method for studying such problems is the method of stochastic limit, developed by L. Accardy, Y.G. Lu and I.V. Volovich \([3]\). The main idea of this method is the derivation of quantum stochastic differential equations which describe the evolution operator for small coupling constant \(\lambda\) and large time \(t\). Higher order corrections to answers obtained by method of stochastic limit are found in \([4]\).

We consider the following Hamiltonian in the Fock space:

\[ H = H_0 + \lambda V, \]

where \(H_0\) is a free hamiltonian, and \(V\) is an interaction, which is equal to the sum of Wick monomials with kernels from the Schwartz space. Using perturbation theory we obtain the following exact formula for the vacuum expectation value of the evolution operator:

\[ \langle 0 | U(t) | 0 \rangle = e^{A t + B + C(t)}, \]

where \(A\) and \(B\) are constants and \(C(t) \to 0\) as \(t \to \infty\).

This formula was obtained in \([5]\) by using wave operators and perturbation theory. It is called the ABC formula. The wave operators method can be used to establish this formula only for non decay Hamiltonians. The method of perturbation theory admits to consider even the decay case. But in \([5]\) the expression for \(B\) and \(C\) obtained by using perturbation theory requires a modification for the decay case, which is obtained in the present paper.

Note that to establish that the integrals, which represent the coefficients \(A\) and \(B\) are exist we use the technique similar to the technique used in the renormalization theory \([1, 6, 7, 8, 9]\).

The asymptotics of the evolution operator was studied in \([10, 11, 6]\).

In the kinetic theory of gases for the determination of the transport coefficients is used the density expansion. It was discovered have that this expansion contains divergences \([12, 13]\). These divergences appear when one studies dynamics of finite set of particles at large intervals of time. It is possible that the technique of renormalization theory can be applied for studying these divergences.

The scattering problem for elementary particles is an important problem, because most of elementary particles are unstable \([14]\). The stable particles form an irreducible unitary representation of the Poincaré group. The representation of the Poincaré group corresponding to the unstable particles was considered in \([15]\). A new wave-function renormalization prescription for an unstable particle based on the complex-pole mass renormalization was suggested in \([16]\). There was also proved that conventional wave-function renormalization prescription leads to gauge dependent results.

The Fermi golden rule and large time exponential behavior for Lee models was studied in \([17]\). About scattering problem for Lee models see also \([18]\).

In \([19]\) it was suggested that one can consider unstable particles as open quantum systems.

It is possible that the ABC-formula could be used to prove the existence of wave operators for the cases when one has decay.
The paper organized as follows. In Section 2 we state the assumptions on the Hamiltonian $H$ and formulate the main theorem. In Section 3 we find sufficient conditions for existence of the coefficient $A$. In Section 4 we find the sufficient conditions for the existence of coefficient $B$.

## 2 The main theorem

Let $\mathcal{F} = L^2(\mathbb{R}^d)$ be the Boson Fock space over one particle Hilbert space $L^2(\mathbb{R}^d)$. Let $a^+(k)$ and $a(k)$ are creation and annihilation operator in $\mathcal{F}$

$$[a(k), a(k)] = [a^+(k), a^+(k)] = 0,$$

$$[a(k), a^+(k)] = 0. \tag{1}$$

Vacuum state $|0\rangle$ in $\mathcal{F}$ is defined by the relation $a(k)|0\rangle = 0$. Consider the following Hamiltonian in the Fock space $\mathcal{F}$

$$H = \int \omega(k)a^+(k)a(k)dk + \lambda \sum_{0 \leq m,n \leq N} \int v_{m,n}(p_1, \ldots, p_m|p_1, \ldots, p_n) \prod_{i=1}^{m} a^+(p_i)dp_i \prod_{j=1}^{n} a(q_j)dq_j \tag{2}$$

$k, p_i, q_j \in \mathbb{R}^d$, $N = 1, 2, 3...$

$v_{m,n}$ — are the functions from the Schwartz space of test functions.

$v_{0,0} = 0$,

We consider the following dispersion law

$$\omega(k) = \frac{k^2}{2} - \omega_0,$$

$$\omega_0 \in \mathbb{R}, \omega_0 \neq 0.$$

but main results could be generalized to the relativistic case. We will study the evolution operator

$$U(t) := e^{itH_0}e^{-itH} \tag{3}$$

$t \in \mathbb{R}$.

Friedrichs graph [6] by definition is a quadruple $(V', R, f^+, f^-)$ where $V'$ is a finite ordered set, called the set of vertices,

$R$ is a finite set, called the set of lines.

$f^+$ and $f^-$ are the maps

$$f^\pm : R \rightarrow V' \tag{4}$$

such that $\forall r \in R f^+(r) > f^-(r)$.

We will denote by $V$ the set of all vertices except the minimal one.
The Friedrichs graph is called a connected graph if for all two vertices \( v \neq v' \) there exists a sequence of vertices and lines \( v = v_0, r_1, v_1, ... r_n, v_n = v' \) such that for all \( i = 1, ..., n \) \( v_i = f^-(r), v_i = f^+(r) \) or \( v_i = f^+(r), v_i = f^-(r) \).

According to the "linked cluster theorem" \[6\] we have

\[
U(t) =: e^{U_c(t)}:
\]  

(5)

Here the index \( c \) in \( U_c(t) \) indicate that one takes only the connected graphs, and \( : : \) means normal order \[6\].

Now we state our main result.

**Theorem 1.** If \( d \geq 3 \), then

\[
\langle 0 | e^{-itH} | 0 \rangle = e^{At + B + C(t)}
\]  

(6)

in the sense of formal power series in \( \lambda \). Here \( A, B, C(t) \) — formal power series in \( \lambda \)

\[
A = \sum_{n=2}^{\infty} A_n \lambda^n,
\]

\[
B = \sum_{n=2}^{\infty} B_n \lambda^n,
\]

\[
C(t) = \sum_{n=2}^{\infty} C_n(t) \lambda^n
\]  

(7)

and \( C_n(t) \to 0 \) as \( t \to \infty \).

**Beginning of the proof.** We have

\[
\langle 0 | e^{-itH} | 0 \rangle = \langle 0 | e^{itH_0} e^{-itH} | 0 \rangle = \langle 0 | U_c(t) | 0 \rangle
\]

Here

\[
\langle 0 | U_c(t) | 0 \rangle = \sum_{n=1}^{\infty} (-i\lambda)^n F_n(t),
\]  

(8)

where \( F_n(t) \) — is the sum over all connected Friedrichs graphs of the expressions of the form

\[
F_{n}^{\Phi}(t) = \int_{0}^{t} dt_n \ldots \int_{0}^{t_1} dt_0 \int e^{i(E_n t_n + \ldots + E_0 t_0)} f(p) dp.
\]  

(9)

Here \( f(p) \) is a product of the kernels of monomials and \( E_i = \sum_{f^+(r)=v} \omega(p_r) - \sum_{f^-(r)=v} \omega(q_r) \), and \( \{p_r\} \) are the moments assigned to the lines \( \{r \mid f^+(r) = v\} \), and \( \{p_r\} \) are moments assigned to the lines \( \{r \mid f^-(r) = v\} \).
Let us transform $F_n^\Phi$. We have
\[
F_n^\Phi(t) = \int_0^t dt_n \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_n} e^{i(E_n t_n + \cdots + E_0 t_0)} f(p) dp
\]
\[
= \int_0^t dt_0 \int_0^{t-t_0} \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_0} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp
\]
\[
= \int_0^t dt_0 \int_0^{t-t_0} \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_2} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp. \tag{10}
\]

Suppose that $T > t$. Then
\[
F_n^\Phi(t) = \int_0^t dt_0 \int_0^{t-t_0} \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_0} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp
\]
\[
- \int_0^t dt_0 \int_0^{t-t_0} \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_2} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp. \tag{11}
\]

The function
\[
\int e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp \tag{12}
\]
is a continuous function. Therefore it is locally integrable. Let us suppose that the following limits exist
\[
\lim_{T \to \infty} \int_0^T dt_n \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_0} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp \tag{13}
\]
\[
\lim_{t \to \infty} \lim_{T \to \infty} \int_0^T dt_n \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_0} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp \tag{14}
\]
Denote by $A_n$ the sum of expressions (14) over all connected graphs of degree $n$, and by $B_n$ the sum of expressions (15). We have
\[
F_n(t) = A_n t + B_n + C_n(t) \tag{15}
\]
Here $C_n(t) \to 0$ as $t \to \infty$. We will prove below that the limits (14), (15) exist if $d \geq 3$.

3 Existence of coefficients $A_n$

Let us fix some Friedrichs graph $\Phi$. We have
\[
I := A_n^\Phi = \int_0^T dt_n \int_0^{t_n} \int_0^{t_n-1} \cdots \int_0^{t_1} \int_0^{t_0} e^{i(E_n t_n + \cdots + E_1 t_1)} f(p) dp \tag{16}
\]
The graph $\Phi = (V', R, f^+, f^-)$ in (17) is a connected graph.

We will denote by $V$ the set of all vertices except the minimal one. An arbitrary element of $V$ we will denote by $v, v \in V$. So the set $V$ we can identify with the set $\{1, \ldots, n\}$.

For every $r \in R$ let us define the following set

$$V_r := \{v | f^+(r) \geq v > v - 1 \geq f^-(r)\}. \quad (17)$$

For every $v \in V$ let us define the following set $R_v$

$$R_v := \{r | f^+ (r) \geq v > v - 1 \geq f^- (r)\}. \quad (18)$$

For every $A \subset V$ put

$$R_A = \{r | \exists v \in A : r \in R_v\} \quad (19)$$

It is clear that $v \in V_r \Leftrightarrow r \in R_v$

Let us introduce new variables $\tau_i := t_i - t_{i-1}$ $i = 1, \ldots, n$. We have

$$I = \int_0^\infty d\tau_n \ldots \int_0^\infty d\tau_1 e^{i\omega_0 \sum_{r \in R\forall v \in V_r} \tau_v} \times \int \prod_r dp_r e^{-\frac{1}{2} \sum_r p_r^2 \sum_{v \in V_r} \tau_v} f(...p_r...). \quad (20)$$

At first we suppose that

$$f(...p_r...) = e^{-\frac{1}{2} \sum_r p_r^2} \quad (21)$$

Then, we have the following expression for $I$

$$I = \int_0^\infty d\tau_n \ldots \int_0^\infty d\tau_1 e^{i\omega_0 \sum_{r \in R\forall v \in V_r} \tau_v} \times \prod_r \{(2\pi)^{\frac{d}{2}}(1 + i \sum_{v \in V_r} \tau_v)^{-\frac{d}{2}}\} \quad (22)$$

We want to know when the integral $I$ exists.

Suppose $A \subset V$, and $O_A$ is a subset of $\mathbb{R}^n_+$ defined as follows

$$O_A = \{(\tau_1, \ldots, \tau_n) | \tau_i < 1 \text{ if } i \in A; \tau_i \geq 1 \text{ if } i \notin A\} \quad (23)$$

The integral $I$ exists if and only if $I$ exists over each domain $O_A$. The integral $I$ over domain $O_\emptyset$ exists if the following integral exists

$$J = \int_1^\infty d\tau_n \ldots \int_1^\infty d\tau_1 \prod_r \left(\frac{1}{(\sum_{v \in V_r} \tau_v)^{\frac{d}{2}}}\right) \quad (24)$$
Let us use the Holder inequalities to estimate this integral.

\[ |\int d\tau \prod_{k=1}^{n} f_k(\tau)| \leq \prod_{k=1}^{n} \left( \int d\tau |f_k(\tau)|^{q_k} \right)^{\frac{1}{q_k}} \]

\[ 0 < q_k \leq \infty, \sum_{k=1}^{n} \frac{1}{q_k} = 1 \]  

(25)

The existence of left hand side follows from the existence of right hand side. After integrating over \( \tau_1 \) we have

\[ J \leq C_1 \int_{1}^{\infty} d\tau_1 \cdots \int_{1}^{\infty} d\tau_2 \prod_{r} \left( \frac{1}{\sum_{v \in V_r; v \neq 1} \tau_v} \right)^{\frac{1}{2}-(q_r^1)^{-1}} \]  

(26)

By definition \( q_r^1 = \infty \) if \( r \notin R_1 \), \( \sum_{R} \frac{1}{q_r} = 1 \).

The integral over \( \tau_1 \) exists if \( \exists q_r^1 : \frac{d}{2} - (q_r^1)^{-1} > 0 \).

Let us now integrate over \( \tau_2 \). We have

\[ J \leq C_2 \int_{1}^{\infty} d\tau_1 \cdots \int_{1}^{\infty} d\tau_3 \prod_{r} \left( \frac{1}{\sum_{v \in V_r; v \neq 1} \tau_v} \right)^{\frac{1}{2}-(q_r^1)^{-1}-(q_r^2)^{-1}} \]  

(27)

The integral over \( \tau_2 \) exists if \( \frac{d}{2} - (q_r^1)^{-1} - (q_r^2)^{-1} > 0 \). Note that this condition implies the following condition: \( \frac{d}{2} - (q_r^1)^{-1} > 0 \).

So by induction we see that \( J \) exists if the following condition A) is satisfied.

A) There exist real numbers \( 0 \leq s_r^v < \infty \) such that

\[ \sum_{r \in R_v} s_r^v = 1; \sum_{v \in V_r} s_r^v < \frac{d}{2}. \]  

(28)

This condition is equal to the following condition.

B) There exists real numbers \( 0 \leq s_r^v < \infty \) such that

\[ \sum_{r \in R_v} s_r^v > 1; \]

\[ \sum_{v \in V_r} s_r^v = \frac{d}{2}. \]  

(29)

(30)

We only prove that condition B) implies A). Suppose that B) holds. Diminish a bit all nonzero \( s_{rv} \). We will have strong inequality in (31), and inequality (30) will be satisfied. Then taking all vertices from \( v = 1 \) to \( v = n \) and diminishing \( s_r^v \) we will have equality in (31).
Let us introduce the notion of quotient-graph. We can receive quotient-graph $\Phi_A \ A \subset V$ from $\Phi$ by action of operation $T_i$. $T_i\Phi$ by definition is graph, with the set of vertices $T_iV = V - \{i\}$, and the set of lines $T_iR = \{r|r_+ \neq i \vee r_- \neq i - 1\}$

Put by definition

$$\Phi_A = \{ \prod_{i \in V \setminus A} T_i \} \Phi$$

(31)

By definition, the power of divergence of $\Phi$ is the number

$$C_\Phi = |V| - \frac{d}{2} |R|.$$  

(32)

Our aim is to find the conditions necessary and sufficient for validity the condition B). If condition B) is satisfied the power of divergence of graph and all its quotient-graph is negative. Confirm this statement only for graph.

$$\frac{d}{2} |R| = \sum_{r \in R} \sum_{v \in V} s_r^v = \sum_{v \in V} \sum_{r \in R_v} s_r^v > \sum_{v \in V} 1 = |V|$$

(33)

So

$$|V| - \frac{d}{2} |R| < 0.$$  

(34)

Conversely, if power of divergence of graph and all its quotient graph is negative then condition B) is satisfied.

Let us denote by $C$ the set of coefficient $s_r^v$ satisfying the condition (31) and condition: $s_r^v = 0$ if $r \notin R_v$. Denote by $C$ the set of all elements $C$. We can identify $C$ with the subset in $\mathbb{R}^N$, where $N = \sum_{v \in V} \sum_{r \in R_v} 1$. This set is convex set i.e. if $C_1 \in C$ and $C_2 \in C$ then $m_1 C_1 + m_2 C_2 \in C$; $m_1 \geq 0$, $m_2 \geq 0$, $m_1 + m_2 = 1$

Let us introduce the following notations

$$C_v := \sum_{r \in R_v} s_r^v$$

(35)

$$|C| := \sum_{v \in V} C_v = \frac{d}{2} |R|$$

(36)

$$C_A := \sum_{v \in A} C_v$$

(37)

Suppose that there exists $C^k$ $k = 1,...,n$ such that $C^k_v > 1$ if $k \neq v$

Let us see at $C^1$. If $C^1 > 1$ the statement is proved. If $C^1 \leq 1$, consider the following convex linear combination $m_1 C^1 + m_2 C^2$, such that

$$|m_1 C^1 + m_2 C^2|_{(1)} > 1$$

(38)
\[ |m_1 C^1 + m_2 C^2|_{V\setminus\{1\}} > \frac{d}{2} |R| - 1 - \varepsilon \] (39)

In the same way we can find \( \tilde{C}^k \ k = 2, \ldots, n \) such that

\[ |\tilde{C}^k|_{V\setminus\{1\}} > \frac{d}{2} |R| - 1 - \varepsilon \]

\( \tilde{C}^k \) if \( k \neq l. \) (40)

If there exists \( k \) such, that \( \tilde{C}^k > 1 \) the statement is proved. Conversely, we can construct the sets \( \tilde{C}^k, k = 2, 3, \ldots, n \) such, that \( \tilde{C}^k > 1 \) if \( k \neq l \) and

\[ |\tilde{C}^k|_{V\setminus\{1, 2\}} > \frac{d}{2} |R| - 2 - \varepsilon \] (41)

This inductive procedure will break or, we will construct the set \( \hat{C} \) such that

\[ \hat{C}_k > 1; k \neq n \] (42)

and

\[ \hat{C}_n > \frac{d}{2} |R| - V + 1 - \varepsilon > 1 \] (43)

if \( \varepsilon \) enough small.

So it is necessary to construct \( C^k. \) In the same way as above we find, what it is enough to construct the sets \( C^{k,l} \) such that \( C^{k,l} > 1 \) if \( v \neq k, l, \ |C^{k,l}| \geq \frac{d}{2} |R| \) (automatically satisfied) and \( |C^{k,l}|_{V\setminus\{k\}} > \frac{d}{2} |R|_{V\setminus\{k\}}. \) To construct \( C^{k,l} \) as above it is sufficient to construct the sets \( C^{k,l,m} \) such that \( C^{k,l,m} > 1 \) if \( v \neq k, l, m \) and \( |C^{k,l,m}|_{V\setminus\{k\}} > \frac{d}{2} |R|_{V\setminus\{k\}} \) and \( |C^{k,l,m}|_{V\setminus\{k,l\}} > \frac{d}{2} |R|_{V\setminus\{k,l\}} \)

So by induction we see, that it is sufficient to each permutation \( P \) to construct the set \( C^P \in C \) such that

\[ |C^P|_{\{P(1), \ldots, P(k)\}} = \frac{d}{2} |R|_{\{P(1), \ldots, P(k)\}} \] (44)

It is easy to see that we can do it by induction go from quotient-graph \( \Phi_{\{P(1), \ldots, P(k)\}} \) to quotient-graph \( \Phi_{\{P(1), \ldots, P(k+1)\}}. \) Indeed \( \Phi_{\{P(1), \ldots, P(k)\}} \) is the quotient-graph of graph \( \Phi_{\{P(1), \ldots, P(k+1)\}}. \) Suppose that for \( \Phi_{\{P(1), \ldots, P(k)\}} \) such set of coefficients is constructed. Let us construct \( s^P_{r^{P(k+1)}}. \)

If \( r \notin R_{P(k+1)} \) we put \( s^P_{r^{P(k+1)}} = 0. \)

If \( r \notin R_{P(k+1)} \) and if \( r \in R_{\{P(1), \ldots, P(k)\}} \) we put \( s^P_{r^{P(k+1)}} = 0. \)

If \( r \in R_{P(k+1)} \) and if \( r \notin R_{\{P(1), \ldots, P(k)\}} \) we put \( s^P_{r^{P(k+1)}} = \frac{d}{2}. \) Then

\[
\sum_{v \in \{P(1), \ldots, P(k+1)\}} \sum_{r \in R_v} s^v_r = \\
= \sum_{v \in \{P(1), \ldots, P(k)\}} \sum_{r \in R_v} s^v_r + \sum_{r \in R_{P(k+1)}} s^P_{r^{P(k+1)}} = \\
= \frac{d}{2} |R_{\{P(1), \ldots, P(k)\}}| + \left( \frac{d}{2} |R_{\{P(1), \ldots, P(k+1)\}}| - \frac{d}{2} |R_{\{P(1), \ldots, P(k)\}}| \right) = \\
= \frac{d}{2} |R_{\{P(1), \ldots, P(k+1)\}}|. \] (45)
So we have prove that $I$ over $O_∅$ exists if the power of divergence of $Φ$ and of all its quotient-graphs is negative. Analysis of convergence of $I$ over others regions $O_A A ∉ ∅$ is similar to above. It reduces to the analysis of convergence of $J$ for $Φ_{V\setminus A}$. So we have proved

**Theorem II.** If power of divergence of graph and all its quotient-graph is negative then $I$ exists.

**Theorem III.** If $d ≥ 3$ then all conditions of theorem II are satisfied.

**Proof.** It is enough to prove that $| R | ≥ | V |$ for each connected Friedrichs graph. To the minimal vertex (vertex number 0) there exists the vertex number $i_1$ and the line which connect these vertices, in another case the vertex number 0 is a non-trivial connected component of $Φ$. To the vertices with numbers 0, $i_1$ there exists a vertexes number $i_2$ and the line which connect this vertex with 0 or $i_1$, in other case the vertexes 0, $i_1$ and the lines which ends in 0 and $i_1$ consists non-trivial connected component, e.c.t. Therefore the number of lines is greater or equal then number of vertices.

**Theorem IV.** Theorem II is valid for an arbitrary function from the Schwartz space .

**Proof.**

$$I = \int_0^∞ dτ_1... \int_0^∞ dτ_n e^{iωn} \sum_{r∈ R} \sum_{v∈ V} τ_v \times$$

$$\int \prod_r dp_r e^{-\frac{i}{2} \sum_{r∈ R} p_r^2 \sum_{v∈ V} τ_v} f(...p_r...)$$

This integral can be represented as a sum $I = \sum_{V⊂ A} I_A$, here $I_A$ has the same integrand as $I$ but this integral is taken over $O_A$. Let us estimate integrand in $O_A$. Let $F$ means Fourier transform on variables $p_r r∈ R, B = V \setminus A$. So we have:

$$|\int \prod_r dp_r e^{-\frac{i}{2} \sum_{r∈ R} p_r^2 \sum_{v∈ V} τ_v} F(F^{-1}(f))(...p_r...)| =$$

$$|\int \prod_r dp_r F(e^{-\frac{i}{2} \sum_{r∈ R_B} p_r^2 \sum_{v∈ V} τ_v}) e^{-\frac{i}{2} \sum_{r∈ R_B} p_r^2 \sum_{v∈ V} τ_v} (F^{-1}(f)...p_r...))| =$$

$$= |\prod_{r∈ R_B} (2π)\frac{1}{i(∑_{v∈ V} τ_v)} ×$$

$$\times |\int \prod_r dp_r (\exp(+i \frac{1}{2} \sum_{r∈ R_B} p_r^2 \sum_{v∈ V} τ_v)) e^{-\frac{i}{2} \sum_{r∈ R_B} p_r^2 \sum_{v∈ V} τ_v} (F^{-1}(f)...p_r...))| ≤$$

$$\text{const} \prod_{r∈ R_B} (2π)\frac{d}{2} \frac{1}{i(∑_{v∈ V_B} τ_v)}$$

(47)

Here $V_r^B := V_r \cap B$. This estimate is enough to prove theorem II.
4 Existence of expression for $B$

Let us fix some Friedrichs graph $\Phi$. Then

$$L := B^\Phi_n = \lim_{t \to \infty} \lim_{T \to \infty} \int_0^t dt_0 \int_{t_0}^T dt_n \ldots \int_0^{t_2} dt_1 \times$$

$$\times \int e^{(E_{n}t_n + \ldots + E_1t_1)} f(p) dp =$$

$$\lim_{t \to \infty} \lim_{T \to \infty} \int_{\tau_1 + \ldots + \tau_n < T; \tau_i > 0} \prod_{i=1}^n d\tau_i \min(t, \tau_1 + \ldots + \tau_n) \times$$

$$e^{i\omega_0 \sum_{r \in R} \sum_{v \in V} \tau_v} \times \int \prod_{r} dp_r e^{-\frac{i}{2} \sum_{r \in R} p_r^2 \sum_{v \in V} \tau_v} f(...) p_\ldots. \quad (48)$$

Our aim is to prove that this integral converges if conditions of theorem II are satisfied. We can represent this integral as a sum

$$L = \sum_{A \subset V} L_A,$$

here $L_A$ has the same integrand as $L$, but taken over the region $O_A$. For the sake of simplicity we consider only $L_\emptyset$. In the first time we transpose limits in $L_\emptyset$, prove the convergence of obtained integral, and prove that the transposition of the limits does not change the value of integral. If we transpose the limits we will have

$$K_\emptyset(T) = \int_{\tau_1 + \ldots + \tau_n < T; \tau_i > 1} \prod_{i=1}^n d\tau_i (\tau_1 + \ldots + \tau_n) \times$$

$$e^{i\omega_0 \sum_{r \in R} \sum_{v \in V} \tau_v} \times \int \prod_{r} dp_r e^{-\frac{i}{2} \sum_{r \in R} p_r^2 \sum_{v \in V} \tau_v} f(...) p_\ldots \quad (49)$$

or

$$K_\emptyset(T) = \int_{\tau_1 + \ldots + \tau_n < T; \tau_i > 1} \prod_{i=1}^n d\tau_i (\tau_1 + \ldots + \tau_n) e^{i\omega_0 \sum_{r \in R} \sum_{v \in V} \tau_v} \times$$

$$\prod_{r} \frac{1}{(2\pi)^{\frac{n}{2}} \left( \sum_{v \in V} \tau_v \right)} f(\tau_1, \ldots, \tau_n) \quad (50)$$

where

$$f(\tau_1, ..., \tau_n) = (-i) \int \prod_{r \in R} dx_r F^{-1}(x_r) \ e^{-\frac{i}{2} \sum_{r \in R} x_r^2 \sum_{v \in V_r} \tau_v} \quad (51)$$

Lemma 1.

$$\left| \frac{d}{d\tau} f(\tau, ..., \tau, \tau_i, ..., \tau_n) \right| \leq \frac{\text{const}}{\tau} \quad (52)$$
Proof.

\[
\frac{d}{d\tau} f(\tau, \ldots, \tau, \tau_{k+1}, \ldots, \tau_n) = \int_{r \in R} dx_r F^{-1}(x_r) e^{\frac{-i}{2} \sum_{r \in R} x^2_r \frac{1}{\tau_r}} \times \prod_{v \in \{1, \ldots, k\}} \sum_{r \in R_v} \left( \frac{x^2_r}{\tau_v} \right) \left|_{\tau_1 = \ldots = \tau_k = \tau} \right.
\]

But \( \forall v \in \{1, \ldots, k\} \)

\[
\sum_{r \in R_v} \frac{1}{\sum_{v \in V_r} \tau_v} \left|_{\tau_1 = \ldots = \tau_k = \tau} \right. \leq \frac{\text{const}}{\tau}
\]

We can estimate the integral by constant. The lemma is proved.

**Lemma 2.** If conditions of theorem II are satisfied the following limit exists

\[
\lim_{T \to \infty} K_0(T)^k = \lim_{T \to \infty} \prod_{i=1}^{n} \int_{k\tau_1 + \ldots + \tau_n \leq T; \tau_{i+1} > \ldots > \tau_n} d\tau_i \times \left( \tau_1 + \ldots + \tau_n \right) e^{i\omega_0 \sum_{r \in R} \sum_{v \in V_r} \tau_v} \times \prod_{r \in R} \frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{1}{\sum_{v \in V_r} \tau_v} \right) \left|_{\tau_1 = \ldots = \tau_k = \tau} \right. f(\tau_1 + \ldots + \tau_n) \left. \right|_{\tau_1 = \ldots = \tau_k = T}
\]

**Proof.** In the first time suppose that \( k = n \). The integral is equal

\[
K_0^k(T) = n \int_{1}^{T} d\tau \tau e^{i\omega_0 \sum_{r} |V_r| \tau} \prod_{r} \frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{1}{\sum_{v \in V_r} \tau_v} \right) f(\ldots \tau) = \frac{1}{\omega_0} \pi n \sum_{V_r} \left| V_r \right|^{\frac{d}{2}} \frac{1}{|V_r|} f(\ldots \tau) - e^{i\omega_0 \sum_{r} |V_r| \tau} \prod_{r} \frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{1}{|V_r|} \right) f(1 \ldots 1) + (d \frac{1}{2} |R| - 1) \int_{1}^{T} d\tau \tau^{\frac{d}{2} |R|} \prod_{r} \frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{1}{|V_r|} \right) e^{i\omega_0 \sum_{r} |V_r| \tau} f(\ldots \tau) - \prod_{r} \frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{1}{|V_r|} \right) \int_{1}^{T} d\tau \frac{d}{d\tau} \frac{\tau}{\tau^{\frac{d}{2} |R|}} d\tau f(\ldots \tau) e^{i\omega_0 \sum_{r} |V_r| \tau}
\]

12
But \( \frac{d}{2} |R| > |V| \geq 1 \) Therefore, the first term tends to zero, the second term is constant, the third integral converges and fourth converges because

\[
|\frac{d}{d\tau} f(\tau, \ldots, \tau)| \leq \frac{\text{const}}{\tau}.
\] (57)

Let us suppose that the statement of lemma has proved for \( k + 1 \). Let us prove the statement for \( k \). Using integration by parts on \( d\tau_k \) we have

\[
K_\theta(T)^k = \int_{k\tau_k + \ldots + \tau_n < T; \tau_i > 1; \tau_k > \ldots > \tau_n} (\tau_1 + \ldots + \tau_n) e^{i\omega_0 \sum_{r \in R} \sum_{v \in V} \tau_v} \times \\
\prod_{r} \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{d}{2}} f(\tau_1 + \ldots + \tau_n) \Big|_{\tau_1 = \ldots = \tau_k} \prod_{i=k}^n d\tau_i = \\
= \frac{1}{i\omega_0 N(T)} \int_{(k+1)\tau_{k+1} + \ldots + \tau_n < T; \tau_{i+1} > 1; \tau_{k+1} > \ldots > \tau_n} \\
\times \prod_{r \in R} \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{d}{2}} e^{i\omega_0 \sum_{r \in R} \sum_{v \in V} \tau_v} (\tau_1 + \ldots + \tau_n) f(\tau_1 + \ldots + \tau_n) \Big|_{\tau_1 = \ldots = \tau_k = \tau_{k+1}} \\
- \int_{(k+1)\tau_{k+1} + \ldots + \tau_n < T; \tau_{i+1} > 1; \tau_{k+1} > \ldots > \tau_n} \\
\prod_{r \in R} \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{d}{2}} \frac{d}{d\tau_k} (\sum_{v \in V} \tau_v) \big|_{\tau_1 = \ldots = \tau_k = \tau_{k+1}} \times \\
\prod_{r} \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{d}{2}} e^{i\omega_0 \sum_{r \in R} \sum_{v \in V} \tau_v} 
\] (58)

Here \( N = \sum_{l=1}^k \sum_{r \in R_l} 1 \). The second term is equal to \( K_\theta^{k+1}(T) \) hence it has a limit as \( T \to \infty \).

To estimate the first term we will use Holder inequality. We will take coefficients \( q_r^k \) as in theorem II. Note that \( \tau_1 = \ldots = \tau_k > \frac{T}{n} \). Let us denote the first term by \( Q \). Then

\[
|Q| \leq T \text{const} \prod_{r \in R_{(1, \ldots, k)}} \frac{1}{T^{\left(\frac{d}{2} - \sum_{v=k+1}^n (q_r^k)^{-1}\right)}}
\] (59)

Because the coefficients \((q_r^k)^{-1}\) are the same as in theorem II we have

\[
\frac{d}{2} - \sum_{v=k+1}^n (q_r^k)^{-1} \geq (q_r^1)^{-1} + \varepsilon
\] (60)
For some $\varepsilon$. Hence

$$|Q| \leq \frac{1}{T^\varepsilon} \frac{T}{\sum_{q_{i}}^1} \cdot \varepsilon = \frac{1}{T^\varepsilon} \to 0$$

(61)

So the first term tends to zero. Let us use the Leibniz rule for the third term. We get the sum of three terms.

$$k \int_{k\tau_k + \ldots + \tau_n \leq T; \tau_i > 1; \tau_k \ldots \tau_n} \prod_{i=k}^n \sum_{\tau_v} \frac{1}{(2\pi)^{\frac{n}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{n}{2}} f(\tau_1 + \ldots + \tau_n) \left| \tau_1 = \ldots = \tau_k - 1 \right.$$  

$$- \int_{k\tau_k + \ldots + \tau_n \leq T; \tau_i > 1; \tau_k \ldots \tau_n} \prod_{i=k}^n \sum_{\tau_v} \frac{1}{(2\pi)^{\frac{n}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{n}{2}} f(\tau_1 + \ldots + \tau_n) \left| \tau_1 = \ldots = \tau_k \right.$$  

$$+ \int_{k\tau_k + \ldots + \tau_n \leq T; \tau_i > 1; \tau_k \ldots \tau_n} \prod_{i=k}^n \sum_{\tau_v} \frac{1}{(2\pi)^{\frac{n}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{n}{2}} \frac{1}{\sum_{v \in V_r}} f(\tau_k, \ldots, \tau_k; \ldots, \tau_n) \left| \tau_1 = \ldots = \tau_k \right.$$  

(62)

Because $k\tau_k + \ldots + \tau_n \leq \tau_k n$ and

$$\prod_{r \in V_r} \frac{1}{\tau_r^{\frac{n}{2}}} \leq \prod_{r \in V_r} \frac{1}{\tau_r^{\frac{n}{2}}}$$

(63)

the Cauchy difference for the last term can be estimated by the Cauchy difference for $J$ constructed by factor graph $\Phi_{\{k, \ldots, n\}}$. So the last term converges. So if the statement of the lemma holds for $k + 1$ then it is holds for $k$. The lemma is proved.

Note that $K_0$ is the sum of the terms which can be estimated as $K_0(T)^k$. So we have proved the following lemma.

**Lemma 3.** The expression $K_0(T)$ have the limit as $T \to \infty$.

**Lemma 4.** We can transpose the limits in (49).

**Proof.** Let us consider the difference between the limit of (49) as $T \to \infty$ and $K_0(t)$ we have

$$L_0(t) - K_0(t) =$$

$$t \int_{\tau_1 + \ldots + \tau_n \leq t; \tau_i > 1} \prod_{i=1}^n \sum_{\tau_v} \frac{1}{(2\pi)^{\frac{n}{2}}} \left( \sum_{v \in V_r} \tau_v \right)^{\frac{n}{2}} f(\tau_1, \ldots, \tau_n)$$

(64)
Divide the region of integration into the following sectors

\[ \mathcal{O}_\sigma = \{(\tau_1, ..., \tau_n) | \tau_1 > 1; \tau_{\sigma(1)} > ... > \tau_{\sigma(n)}\} \]  \hspace{1cm} (65)

Here \( \sigma \) — is a permutation of the set \( \{1, ..., n\} \). Consider only the integral over \( \mathcal{O}_{\sigma d} \). We can consider other integrals in the same way. Now we will integrate by parts on \( \tau_1 \). We have

\[
\frac{T}{i\omega_0} \sum_{r \in \mathcal{R}_1} \int_{T < \tau_1 + ... + \tau_n < T; \tau_1 > 1; \tau_2 > ... > \tau_n} \prod_{r=1}^{n} d\tau_r e^{-i\omega_0 \sum_{r \in V} \sum_{v \in \mathcal{V}} \tau_v} \times

\prod_{r} \left( \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in \mathcal{V}_r} \tau_v \right)^{\frac{d}{2}} \right) f(\tau_1, ..., \tau_n) \bigg|_{\tau_1 = T_1} - \bigg|_{\tau_1 = \tau_2} - \bigg|_{\tau_1 = T_2} - \bigg|_{\tau_1 = T} \right)

\]

\[
= \lim_{T_1 \to \infty} \left\{ \prod_{r=1}^{n} d\tau_r e^{-i\omega_0 \sum_{r \in V} \sum_{v \in \mathcal{V}} \tau_v} \times \right.

\prod_{r} \left( \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in \mathcal{V}_r} \tau_v \right)^{\frac{d}{2}} \right) f(\tau_1, ..., \tau_n) \bigg|_{\tau_1 = \tau_2} - \bigg|_{\tau_1 = T_2} - \bigg|_{\tau_1 = T} \right)

\]

\[
= \lim_{T_1 \to \infty} \left\{ \prod_{r=1}^{n} d\tau_r e^{-i\omega_0 \sum_{r \in V} \sum_{v \in \mathcal{V}} \tau_v} \times \right.

\prod_{r} \left( \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \sum_{v \in \mathcal{V}_r} \tau_v \right)^{\frac{d}{2}} \right) f(\tau_1, ..., \tau_n) \bigg|_{\tau_1 = T} \right)

Here \( N = \sum_{r \in \mathcal{R}_1} 1 \). The first term is equal to zero. The third term tends to zero as \( T \to \infty \).

The proof of this fact is the same as the proof for the first term in (59). The fourth term tends to zero as \( T \to \infty \). The proof is similar to the proof for the last term in (59). We can estimate the second term as \( L_\theta(t) - K_\theta(t) \) for a diagram with \( V - 1 \) vertexes. The lemma IV is proved.

These four lemmas implies the following.

**Theorem V.** The coefficient \( B_n \) is well defined and is equal to the sum of the terms of the form (if conditions of theorem V are satisfied)

\[
\lim_{T \to \infty} \int_{\tau_1 + ... + \tau_n < T; \tau_1 > 0} (\tau_1 + ... + \tau_n) \times

e^{-i\omega_0 \sum_{r \in V} \sum_{v \in \mathcal{V}} \tau_v} \times \int \prod_{r} dp_r e^{-\frac{i}{2} \sum_{r \in V} p_r^2 \sum_{v \in \mathcal{V}} \tau_v} f(...p_r...) \hspace{1cm} (67)
\]

15
where \( f(...p,...) \) is the product of monomials of kernels.
Therefore the coefficients \( A_n \) and \( B_n \) exist and Theorem I completely proved.

Note that theorem II is analogous to the theorem about convergence Feynman integrals in quantum field theory.

## 5 Conclusion

We have derived the general formula which describes the large time behavior of the vacuum expectation value of the evolution operator for the general class of Hamiltonians describing the decay processes.

## 6 Acknowledgements

I would like to thank my supervisor I.V. Volovich for very useful discussions.

This work was partially supported by the Russian Foundation of Basis Research (project 05-01-008884), the grant of the president of the Russian Federation (project NSh-6705.2006.1) and the program “Modern problems of theoretical mathematics” of the Mathematical sciences department of the Russian Academy of Sciences.

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