ON THE HARDER-NARASIMHAN FILTRATION FOR FINITE DIMENSIONAL REPRESENTATIONS OF QUIVERS

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Abstract. We prove that the Harder-Narasimhan filtration for an unstable finite dimensional representation of a finite quiver coincides with the filtration associated to the 1-parameter subgroup of Kempf, which gives maximal instability in the sense of Geometric Invariant Theory for the corresponding point in the parameter space where these objects are parametrized in the construction of the moduli space.

Keywords: Moduli space, quivers, representations, Harder-Narasimhan, GIT, Kempf

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Introduction

Let $Q$ be a finite quiver, given by a finite set of vertices and arrows between them, and a representation of $Q$ on finite dimensional $k$-vector spaces, where $k$ is an algebraically closed field of arbitrary characteristic. There exists a notion of stability for such representations given by King ([Ki]) and, more generally by Reineke ([Re]) (both particular cases of the abstract notion of stability for an abelian category that we can find in [Ru]), and a notion of the existence of a unique Harder-Narasimhan filtration with respect to that stability condition.

We consider the construction of a moduli space for these objects by King ([Ki]) and associate to an unstable representation an unstable point, in the sense of Geometric Invariant Theory, in a parameter space where a group acts. Then, the 1-parameter subgroup given by Kempf ([Ke]), which is maximally destabilizing in the GIT sense, gives a filtration of subrepresentations and we prove that it coincides with the Harder-Narasimhan filtration for that representation.

This article makes use of the same techniques that a previous work of the author in collaboration with T. Gómez and I. Sols ([GSZ]). In that article, we considered an unstable torsion free sheaf $E$ over a smooth projective variety $X$. There, we proved that the filtration associated to the 1-parameter subgroup given by Kempf, coincides
with the Harder-Narasimhan filtration of \( E \) with the definition of stability given by Gieseker.

The definition of stability for a representation of a quiver (c.f. Definition \[1.1\]) contains two sets of parameters, the coefficients of the linear functions \( \Theta \) and \( \sigma \). In [Ke], the 1-parameter subgroup is taken to maximize certain function which depends on the choice of a linearization of the action of the group we are taking the quotient by, and a \textit{length} in the set of 1-parameter subgroups (c.f. Definition \[3.1\]). In the case of sheaves the group is \( \text{SL}(N) \), which is simple, so any such length is unique up to multiplication by a scalar, whereas for finite dimensional representations of quivers we quotient by a product of general linear groups, so we have to choose a scalar for each factor in the choice of a length. Hence, we put the positive coefficients of \( \sigma \) precisely as these scalars and consider a particular linearization depending on \( \sigma \) and \( \Theta \), in order to relate the Harder-Narasimhan filtration of a representation with the filtration given in [Ke] (c.f. Theorem \[5.3\]).

Hesselink shows in [He] that the unstable locus of a smooth complex projective variety acted by a complex reductive group can be stratified by conjugacy classes of 1-parameter subgroups. Tur shows in [Tu] that Hesselink’s stratification coincides with a stratification in Harder-Narasimhan types (meaning the numerical invariants appearing on the Harder-Narasimhan filtration), for the space of quiver representations. In [Ho], Hoskins proves that these two stratifications do coincide with a Morse stratification given by the norm square of the moment map, for reductive groups acting on affine spaces, in particular quiver representations.

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\section{Harder-Narasimhan filtration for representations of quivers}

A finite quiver \( Q \) is given by a finite set of vertices \( Q_0 \) and a finite set of arrows \( Q_1 \). The arrows will be denoted by \( (\alpha : v_i \rightarrow v_j) \in Q_1 \). We denote by \( \mathbb{Z}Q_0 \) the free abelian group generated by \( Q_0 \).

Fix \( k \), an algebraically closed field of arbitrary characteristic. Let \( \text{mod } kQ \) be the category of finite dimensional representations of \( Q \) over \( k \). Such category is an
abelian category and its objects are given by tuples

\[ M = ((M_v)_{v \in Q_0}, (M_{\alpha}: M_{v_i} \to M_{v_j})_{\alpha:v_i \to v_j}) \]

of finite dimensional \( k \)-vector spaces and \( k \)-linear maps between them. The dimension vector of a representation is given by

\[ \dim_k M_v = \sum_{v \in Q_0} \dim_k M_v \cdot v \in \mathbb{N}Q_0. \]

Let \( \Theta \) be a set of numbers \( \Theta_v \) for each \( v \in Q_0 \) and define a linear function \( \Theta : \mathbb{Z}Q_0 \to \mathbb{Z} \), by

\[ \Theta(M) := \Theta(\dim M) = \sum_{v \in Q_0} \Theta_v \dim_k M_v. \]

Let \( \sigma \) be a set of strictly positive numbers \( \sigma_v \) for each \( v \in Q_0 \), and define a (strictly positive) linear function \( \sigma : \mathbb{Z}Q_0 \to \mathbb{Z} \), by

\[ \sigma(M) := \sigma(\dim M) = \sum_{v \in Q_0} \sigma_v \dim_k M_v. \]

We call \( \sigma(M) \) the total dimension of \( M \). We will refer to \( \Theta \) and \( \sigma \) indistinctly meaning the sets of numbers or the linear functions.

For a non-zero representation \( M \) of \( Q \) over \( k \), define its slope by

\[ \mu_{(\Theta, \sigma)}(M) := \frac{\Theta(M)}{\sigma(M)}. \]

**Definition 1.1.** A representation \( M \) of \( Q \) over \( k \) is \( (\Theta, \sigma) \)-semistable if for all non-zero proper subrepresentations \( M' \subset M \), we have

\[ \mu_{(\Theta, \sigma)}(M') \leq \mu_{(\Theta, \sigma)}(M). \]

If the inequality is strict for every non-zero proper subrepresentation, we say that \( M \) is \( (\Theta, \sigma) \)-stable.

**Lemma 1.2.** If we multiply the linear function \( \Theta \) by a non-negative integer, or if we add an integer multiple of the strictly positive linear function \( \sigma \) to \( \Theta \), the semistable (resp. stable) representations remain semistable (resp. stable).

**Proof.** Let \( \Theta' = a \cdot \Theta + b \cdot \sigma, a, b \in \mathbb{Z}, a > 0 \) be another linear function and note that

\[ \frac{\Theta'(M')}{\sigma(M')} \leq \frac{\Theta'(M)}{\sigma(M)} \iff \frac{a \cdot \Theta(M') + b \cdot \sigma(M)}{\sigma(M')} \leq \frac{a \cdot \Theta(M) + b \cdot \sigma(M)}{\sigma(M)} \iff \frac{\Theta(M')}{\sigma(M')} \leq \frac{\Theta(M)}{\sigma(M)}. \]
**Remark 1.3.** In [Ki], the stability condition (c.f. [Ki, Definition 1.1]) is formulated by not considering representations with different dimension vectors. This leads to the construction of a moduli space and S-filtrations (or Jordan-Hölder filtrations) but not to define a Harder-Narasimhan filtration, for which is needed a slope condition as in Definition 1.1.

This slope stability condition, the \((\Theta, \sigma)\)-stability (c.f. Definition 1.1), can be turned out into a stability condition as in [Ki], by clearing denominators

\[ \theta(M') = \Theta(M)\sigma(M') - \sigma(M)\Theta(M') , \]

where \(\theta\) is the function in [Ki, Definition 1.1] (observe that \(\theta(M) = 0\)), \(\Theta\) and \(\sigma\) are as in Definition 1.1, and \(M' \subset M\) is a subrepresentation.

We will apply this in Proposition 2.2, to relate \((\Theta, \sigma)\)-stability with GIT stability.

**Remark 1.4.** The definition of stability which appears in [Re] considers \(\sigma_v = 1\) for each \(v \in Q_0\), although we consider a strictly positive linear function \(\sigma\) in general. The notation of \(\sigma\) agrees with [AC], [ACGP], [Sch], while \(\Theta\) agrees with [Re] but in the other references it is substituted by different notations closer to classical moduli problems where the stability notion depends on parameters (\(\tau\)-stability or \(\rho\)-stability).

**Lemma 1.5.** [Ru, Definition 1], [Re, Lemma 4.1] Let \(0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0\) be a short exact sequence of non-zero representations of \(Q\) over \(k\). Then \(\mu_{(\Theta, \sigma)}(X) < \mu_{(\Theta, \sigma)}(Y)\) if and only if \(\mu_{(\Theta, \sigma)}(X) < \mu_{(\Theta, \sigma)}(Z)\) if and only if \(\mu_{(\Theta, \sigma)}(Y) < \mu_{(\Theta, \sigma)}(Z)\).

**Proof.** Note that \(\sigma(Y) = \sigma(X) + \sigma(Z)\) and, therefore

\[ \mu_{(\Theta, \sigma)}(Y) = \frac{\Theta(Y)}{\sigma(Y)} = \frac{\Theta(X) + \Theta(Z)}{\sigma(X) + \sigma(Z)} , \]

from which the statement follows. 

**Theorem 1.6.** [Ru, Theorem 2], [Re, Lemma 4.7] Given linear functions \(\Theta\) and \(\sigma\), (being \(\sigma\) strictly positive), every representation \(M\) of \(Q\) over \(k\) has a unique filtration

\[ 0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M \]

verifying the following properties, where \(M^i := M_i/M_{i-1}\)

1. \(\mu_{(\Theta, \sigma)}(M^1) > \mu_{(\Theta, \sigma)}(M^2) > \ldots > \mu_{(\Theta, \sigma)}(M^t) > \mu_{(\Theta, \sigma)}(M^{t+1})\)
2. The quotients \(M^i\) are \((\Theta, \sigma)\)-semistable

This filtration is called the Harder-Narasimhan filtration of \(M\) (with respect to \(\Theta\) and \(\sigma\)).
Proof. Using Lemma 1.5 we can prove the existence of a unique subrepresentation $M_1$, whose slope is maximal among all the subrepresentations of $M$, and of maximal total dimension $\sigma(M_1)$ between those of maximal slope (c.f. [Ru] Proposition 1.9, [Re] Lemma 4.4). Then, proceed by recursion on the quotient $M/M_1$. 

2. Moduli space of representations of quivers

Fix $k$ an algebraically closed field of arbitrary characteristic. Fix a dimension vector $d \in \mathbb{Z}Q_0$ and fix $k$-vector spaces $M_v$ of dimension $d_v$ for all $v \in Q_0$. Fix linear functions $\Theta, \sigma : \mathbb{Z}Q_0 \to \mathbb{Z}$, being $\sigma$ strictly positive. We recall the construction by King (c.f. [Ki]) of a moduli space for representations of $Q$ over $k$ with dimension vector $d$.

Consider the affine $k$-space $R_d(Q) = \bigoplus_{\alpha : v_i \to v_j} \text{Hom}_k(M_{v_i}, M_{v_j})$ whose points parametrize representations of $Q$ on the $k$-vector spaces $M_v$. The reductive linear algebraic group $G_d = \prod_{v \in Q_0} GL(M_v)$ acts on $R_d(Q)$ by

$$(g_v)_{v_i} \cdot (M_{v})_{\alpha} = (g_v M_{\alpha} g_{v_i}^{-1})_{\alpha : v_i \to v_j}$$

and the $G_d$-orbits of $M$ in $R_d(Q)$ correspond bijectively to the isomorphism classes $[M]$ of $k$-representations of $Q$ with dimension vector $d$. We will use Geometric Invariant Theory to take the quotient of $R_d(Q)$ by $G_d$ and construct a moduli space of representations of the quiver $Q$ on the $k$-vector spaces $M_v$.

The action of $G_d$ on the affine space $R_d(Q)$ can be lifted by a character $\chi$ to the (necessarily trivial) line bundle $L$ required by the Geometric Invariant Theory. Note that the subgroup of the diagonal scalar matrices in $G_d$,

$$\Delta = \{(t_1, \ldots, t_1) : t \in k^*\},$$

acts trivially on $R_d(Q)$. Then, we have to choose $\chi$ in such a way that $\Delta$ acts trivially on the fiber, in other words, $\chi(\Delta) = 1$.

Then, using the linear functions $\Theta$ and $\sigma$, consider the character

$$\chi_{(\Theta, \sigma)}((g_v)_{v}) := \prod_{v \in Q_0} \det(g_v)^{\Theta(d)\sigma_v - \sigma(d)\Theta_v}$$

of $G_d$, and note that $\chi_{(\Theta, \sigma)}(\Delta) = 1$, because $\sum_{v \in Q_0}(\Theta(d)\sigma_v - \sigma(d)\Theta_v) \cdot d_v = 0$. 
Definition 2.1. \([\text{Ki}, \text{Definition 2.1}]\) A point \(x \in \mathcal{R}_d(Q)\) is \(\chi\)-semistable if there is a relative invariant \(f \in k[\mathcal{R}_d(Q)]^{G_d, \chi_{(\Theta, \sigma)}}\) with \(n \geq 1\), such that \(f(x) \neq 0\).

The algebraic quotient will be given by
\[
\mathcal{R}_d(Q) \sslash (G_d, \chi_{(\Theta, \sigma)}) = \text{Proj} \left( \bigoplus_{n \geq 0} k[\mathcal{R}_d(Q)]^{G_d, \chi_{n(\Theta, \sigma)}} \right).
\]

Proposition 2.2. A point \(x_M \in \mathcal{R}_d(Q)\) corresponding to a representation \(M \in \text{mod} \, kQ\) is \(\chi_{(\Theta, \sigma)}\)-semistable (resp. \(\chi_{(\Theta, \sigma)}\)-stable) for the action of \(G_d\) if and only if \(M\) is \((\Theta, \sigma)\)-semistable (resp. \((\Theta, \sigma)\)-stable).

Proof. It follows easily from \([\text{Ki}, \text{Proposition 3.1}]\) and the observation in Remark \(1.3\). In \([\text{Ki}]\), given a linear function \(\theta\), a representation \(M\) is \(\theta\)-semistable if \(\theta(M) = 0\) and, for every subrepresentation \(M' \subset M\), we have \(\theta(M') \geq 0\) (c.f. \([\text{Ki}, \text{Definition 1.1}]\)). Then, \([\text{Ki}, \text{Proposition 3.1}]\) relates the \(\theta\)-stability with the \(\chi_{\theta}\)-stability, where the character is
\[
\chi_{\theta}((g_v)_v) := \prod_{v \in Q_0} \det(g_v)^{\theta_v}.
\]
Hence, the \(\chi_{(\Theta, \sigma)}\)-stability with the character given by
\[
\chi_{(\Theta, \sigma)}((g_v)_v) := \prod_{v \in Q_0} \det(g_v)^{\Theta(d)\sigma_v - \sigma(d)\Theta_v},
\]
is equivalent to the \((\Theta, \sigma)\)-stability in Definition \(1.1\) because, given a subrepresentation \(M' \subset M\), the expression
\[
\sum_{v \in Q_0} (\Theta(M)\sigma_v - \sigma(M)\Theta_v) \cdot \dim M'_v = \Theta(M)\sigma(M') - \sigma(M)\Theta(M') \geq 0
\]
is equivalent to
\[
\frac{\Theta(M')}{\sigma(M')} \leq \frac{\Theta(M)}{\sigma(M)}.
\]

Now, denote by \(\mathcal{R}^{(\Theta, \sigma)}_{d}(Q)\) the set of \(\chi_{(\Theta, \sigma)}\)-semistable points.

Theorem 2.3. \([\text{Ki}, \text{Proposition 4.3}], \quad [\text{Re}, \text{Corollary 3.7}]\) The variety \(\mathcal{M}^{(\Theta, \sigma)}_{d}(Q) = \mathcal{R}^{(\Theta, \sigma)}_{d}(Q) \sslash (G_d, \chi_{(\Theta, \sigma)})\) is a moduli space which parametrizes \(S\)-equivalence classes of \((\Theta, \sigma)\)-semistable representations of \(Q\) of dimension vector \(d\). It is projective over the ordinary quotient \(\mathcal{R}_d(Q) \sslash G_d\).

If the quiver has no oriented cycles or it is chosen so that there is a unique semisimple representation for each dimension vector (e.g. the quiver associated to a finite
dimensional algebra over an algebraically closed field in \([Ki]\)), the ordinary quotient 
\( \mathcal{R}_d(Q) \parallel G_d \) consists on one single point, hence the moduli space is projective.

By the Hilbert-Mumford criterion we can characterize \( \chi(\Theta, \sigma) \)-semistable points by
its behavior under the action of 1-parameter subgroups. A 1-parameter subgroup of
\( G_d = \prod_{v \in Q_0} GL(M_v) \) is a non-trivial homomorphism \( \Gamma : k^* \to G_d \). There exist bases
of the vector spaces \( M_v \) such that \( \Gamma \) takes the diagonal form
\[
\begin{pmatrix}
  t^{\Gamma_{v_1,1}} & & \\
  & \ddots & \\
  & & t^{\Gamma_{v_1,t_1+1}}
\end{pmatrix}
\times \cdots \times
\begin{pmatrix}
  t^{\Gamma_{v_s,1}} & & \\
  & \ddots & \\
  & & t^{\Gamma_{v_s,t_s+1}}
\end{pmatrix}
\]
where \( v_1, \ldots, v_s \in Q_0 \) are the vertices of the quiver.

Let \( x \in \mathcal{R}_d(Q) \) and suppose that \( \lim_{t \to 0} \Gamma(t) \cdot x \) exists and is equal to \( x_0 \). Then \( x_0 \) is a
fixed point for the action of \( \Gamma \), and \( \Gamma \) acts on the fiber of the trivial line bundle over
\( x_0 \) as multiplication by \( t^a \). Define the following numerical function,
\[
\mu_{\chi(\Theta, \sigma)}(x, \Gamma) = -a .
\]
The next proposition establishes the so-called "Hilbert-Mumford numerical criterion":

**Proposition 2.4.** \([Ki]\) Proposition 2.5] A point \( x_M \in \mathcal{R}_d(Q) \) corresponding to a
representation \( M \) is \( \chi(\Theta, \sigma) \)-semistable if and only if every 1-parameter subgroup \( \Gamma \) of
\( G_d \), for which \( \lim_{t \to 0} \Gamma(t) \cdot x_M \) exists, satisfies \( \mu_{\chi(\Theta, \sigma)}(x_M, \Gamma) \leq 0 \).

**Remark 2.5.** Note that in Proposition [2.4] we change the sign of the numerical function
\( \mu_{\chi(\Theta, \sigma)}(x_M, \Gamma) \) with respect to \([Ki]\) (as we did when changing the character in the
proof of Proposition 2.2), in congruence with \([Ko]\) and \([GSZ]\).

The action of a 1-parameter subgroup \( \Gamma \) of \( G_d \) provides a decomposition of each
vector space \( M_v \) associated to each vertex \( v \in Q_0 \), in weight spaces
\[
M_v = \bigoplus_{n \in \mathbb{Z}} M^n_v
\]
where \( \Gamma(t) \) acts on the weight space \( M^n_v \) as multiplication by \( t^n \). Every 1- parameter
subgroup, for which \( \lim_{t \to 0} \Gamma(t) \cdot x \) exists, determines a weighted filtration \( M_\bullet \subset M \)
of subrepresentations (c.f. \([Ki]\))
\[
0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M
\]
where $M_i$ is the subrepresentation with vector spaces $M_{v,i} := \bigoplus_{n \leq i} M_v^n$ for each vertex $v \in Q_0$, and the weight corresponding to each quotient $M^i = M_i/M_{i-1}$ is $\Gamma_i$. Note that two 1-parameter subgroups giving the same filtration are conjugated by an element of the parabolic subgroup of $G_d$ defined by the filtration. Therefore, the numerical function $\mu_{\chi(\Theta,s)}(x,M,\Gamma)$, has a simple expression in terms of the filtration $M_\bullet \subset M$ (c.f. calculation in [Ki]):

\begin{equation}
\mu_{\chi(\Theta,s)}(x,M,\Gamma) = \sum_{v \in Q_0} \left[ (\Theta(M)\sigma_v - \sigma(M)\Theta_v) \cdot \sum_{i=1}^{t_0+1} \Gamma_{v,i} \dim M^i_v \right].
\end{equation}

Let $d_i$, $d^i$ be the dimension vectors of the subrepresentation $M_i$ and the quotient $M^i = M_i/M_{i-1}$, respectively. The action of $\Gamma$ on the point corresponding to a representation $M$ has different weights for each vertex $v \in Q_0$, but collect all different weights $\Gamma_i$ corresponding to any vertex and form the vector

$$\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_t, \Gamma_{t+1})$$

verifying $\Gamma_1 < \Gamma_2 < \ldots < \Gamma_t < \Gamma_{t+1}$. Hence, (2.1) turns out to be

\begin{equation}
\mu_{\chi(\Theta,s)}(x,M,\Gamma) = \sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M) \cdot \sigma(M^i) - \sigma(M) \cdot \Theta(M^i)],
\end{equation}

and Proposition 2.4 can be rewritten in terms of filtrations of $M$.

**Proposition 2.6.** A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation $M$ of $Q$ over $k$, is $\chi(\Theta,s)$-semistable if and only if every 1-parameter subgroup $\Gamma$ of $G_d$, defining a filtration of subrepresentations of $M$,

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M,$$

satisfies that

$$\mu_{\chi(\Theta,s)}(x,M,\Gamma) = \sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M) \cdot \sigma(M^i) - \sigma(M) \cdot \Theta(M^i)] \leq 0.$$

3. **Kempf theorem**

Given a weighted filtration of $M$,

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M$$
and \( \Gamma_1 < \Gamma_2 < \ldots < \Gamma_t < \Gamma_{t+1} \), define the following function which we call the *Kempf function*,

\[
K(x_M, \Gamma) = \frac{\sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M) \cdot \sigma(M^i) - \sigma(M) \cdot \Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1} \sigma(M^i) \cdot \Gamma_i^2}}
\]

We recall a theorem by Kempf (c.f. [Ke, Theorem 2.2]) stating that whenever there exists any \( \Gamma \) giving a positive value for the numerator of the Kempf function, there exists a unique parabolic subgroup containing a unique 1-parameter subgroup in each maximal torus, giving maximum in the Kempf function i.e., there exists a unique weighted filtration of \( M \) for which the Kempf function achieves its maximum.

The Kempf function (3.1) which appears in [Ke, Theorem 2.2] is a rational function whose numerator is equal to the numerical function \( \mu_{\chi(\Theta, \sigma)}(x_M, \Gamma) \) and the denominator is the *length* of the 1-parameter subgroup \( \Gamma \). Given a reductive algebraic linear group \( G \), there is a notion of *length* defined by Kempf (c.f. [Ke, pg. 305]) in \( \Gamma(G) \), the set of all 1-parameter subgroups.

**Definition 3.1.** A *length* is a non-negative function \( \| \| \) on \( \Gamma(G) \) with values on the real numbers, invariant by conjugation by rational points of \( G \), and such that for any maximal torus \( T \subset G \), there is a positive definite integral valued form \((\cdot, \cdot)\) in \( \Gamma(T) \) with \((\Gamma, \Gamma) = \|\Gamma\|^2\), for any \( \Gamma \in \Gamma(T) \).

If \( G \) is simple, in characteristic zero all choices of length will be multiples of the Killing form in the Lie algebra \( g \) (note that in this case \( \Gamma(G) \subset g \)). For an almost simple group in arbitrary characteristic (a group \( G \) whose center \( Z \) is finite and \( G/Z \) is simple, e.g. \( SL(N) \) in positive characteristic), all lengths differ also by a scalar.

However, in this case, the group is a product of general linear groups, which is not simple. Then, there are several simple factors in the group and we can take a different multiple of the Killing form for each factor. Hence, observe that in the Kempf function (3.1), the denominator of the expression is a function verifying the properties of the definition of a length (c.f. Definition 3.1). The different multiples we take for each factor are the integer coefficients of the strictly positive linear function \( \sigma \).

Therefore, we can rewrite [Ke, Theorem 2.2] in our case as follows:

**Theorem 3.2.** Given a \( \chi(\Theta, \sigma) \)-unstable point \( x_M \in \mathcal{R}_d(Q) \) corresponding to a representation \( M \), there exists a unique weighted filtration, i.e. \( 0 \subset M_1 \subset \cdots \subset M_{t+1} = M \) and real numbers \( \Gamma_1 < \Gamma_2 < \ldots < \Gamma_t < \Gamma_{t+1} \), called the Kempf filtration of \( M \), such
that the Kempf function $K$ achieves the maximum among all filtrations and weights verifying $\Gamma_1 < \Gamma_2 < \cdots < \Gamma_t < \Gamma_{t+1}$.

Note that the length we are considering depends on the choice of $\sigma$ and the Kempf function depends both on the length and the linearization of the group action, hence depends both on $\Theta$ and $\sigma$. In order to relate the Kempf filtration of $M$ with the Harder-Narasimhan filtration, which also depends on $\Theta$ and $\sigma$, we dispose the parameters conveniently.

4. Results on convexity

Next, we prove a result about convexity for functions which are similar to the Kempf function. The vector which maximizes such functions verifies some properties that will be strongly related to the properties of the Harder-Narasimhan filtration. In this section we recall the results of [GSZ, Section 2].

Consider $\mathbb{R}^{t+1}$ together with an inner product $(\cdot, \cdot)$ defined by the diagonal matrix

$$
\begin{pmatrix}
  b^1 & 0 \\
  \vdots & \ddots \\
  0 & b^{t+1}
\end{pmatrix}
$$

where $b^i$ are positive integers. Let

$$
\mathcal{C} = \left\{ x \in \mathbb{R}^{t+1} : x_1 < x_2 < \cdots < x_{t+1} \right\},
$$

and $v = (v_1, \cdots, v_{t+1}) \in \mathbb{R}^{t+1}$ verifying $\sum_{i=1}^{t+1} b^i v_i = 0$. Define the function

$$
\mu_v : \mathcal{C} \to \mathbb{R}
$$

$$
\Gamma \mapsto \mu_v(\Gamma) = \frac{(\Gamma, v)}{||\Gamma||}
$$

and note that $\mu_v(\Gamma) = ||v|| \cdot \cos(\Gamma, v)$.

We assume that there exists $\Gamma \in \mathcal{C}$ with $\mu_v(\Gamma) > 0$ and then, we would like to find a vector $\Gamma \in \mathcal{C}$ which maximizes the function $\mu_v$. Define $w^i = -b^i \cdot v_i$, $w_i = w^1 + \cdots + w^i$, $b_i = b^1 + \cdots + b^i$ and draw a graph joining the points with coordinates $(b_i, w_i)$, each segment having slope $-v_i$. Now draw the convex envelope of this graph (thick line in Figure 1), denoting its coordinates by $(\tilde{b}_i, \tilde{w}_i)$, and define

$$
\Gamma_i = -\frac{\tilde{w}_i - \tilde{w}_{i-1}}{b_i}.
$$

In other words, the vector $\Gamma_v = (\Gamma_1, \cdots, \Gamma_{t+1})$ defined in this way, verifies that the quantities $-\Gamma_i$ are the slopes of the convex envelope graph defined by $v$. 
Theorem 4.1. [GSZ, Theorem 2.2] The vector \( \Gamma_v \) defined in this way gives the maximum for the function \( \mu_v \) on its domain.

5. Kempf filtration is Harder-Narasimhan filtration

Finally, we study the geometrical properties of the Kempf filtration by associating to it a graph which encodes the two properties satisfied by the Harder-Narasimhan filtration.

Let \( \Theta : \mathbb{Z}Q_0 \to \mathbb{Z} \) be a linear function and let \( \sigma : \mathbb{Z}Q_0 \to \mathbb{Z} \) be a strictly positive linear function. Let \( M \) be a representation of \( Q \) over an algebraically closed field \( k \) of arbitrary characteristic, which is \( (\Theta, \sigma) \)-unstable. Consider the \( \chi(\Theta, \sigma) \)-unstable point \( x_M \in \mathcal{R}_d(Q) \) associated to \( M \), by Proposition 2.2. Let \( 0 \subset M_1 \subset \cdots \subset M_{t+1} = M \) and \( \Gamma_1 < \Gamma_2 < \cdots < \Gamma_t < \Gamma_{t+1} \) be the Kempf filtration of \( M \), by Theorem 3.2.

Let \( M^i = M_i/M_{i-1} \) be the quotients of the filtration. Consider the inner product in \( \mathbb{R}^{t+1} \) given by the matrix

\[
\begin{pmatrix}
\sigma(M^1) & 0 & \cdots \\
\cdots & \cdots & \cdots \\
0 & \cdots & \sigma(M^{t+1})
\end{pmatrix}
\]

where \( \sigma(M^i) > 0 \).
Definition 5.1. Given a filtration $0 \subset M_1 \subset \cdots \subset M_{t+1} = M$ of subrepresentations of $M$, define $v = (v_1, \ldots, v_{t+1})$, where $v_i = \Theta(M) - \frac{\sigma(M)}{\sigma(M_i)} \Theta(M_i)$, the vector associated to the filtration.

Now we can identify the Kempf function (3.1) with the function in Theorem 4.1, $K(x_{M}, \Gamma) = \sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M)\sigma(M^i) - \sigma(M)\Theta(M^i)] = \sqrt{\sum_{i=1}^{t+1} \sigma(M_i)^2 \cdot \Gamma_i^2}$

Theorem 5.3. The Kempf filtration of $M$ is the Harder-Narasimhan filtration of $M$.

Proof. The vector $v$ associated to the Kempf filtration of $M$ verifies properties (1) and (2) in Lemma 5.2, which are precisely the properties (1) and (2) in Theorem 1.6 respectively. By uniqueness of the Harder-Narasimhan filtration of $M$, both filtrations do coincide.

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