Phantom singularities and their quantum fate: general relativity and beyond
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Abstract Cosmological observations allow the possibility that dark energy is caused by phantom fields. These fields typically lead to the occurrence of singularities in the late Universe. We review here the status of phantom singularities and their possible avoidance in a quantum theory of gravity. We first introduce phantom energy and discuss its behavior in cosmology. We then list the various types of singularities that can occur from its presence. We also discuss the possibility that phantom behavior is mimicked by an alternative theory of gravity. We finally address the quantum cosmology of these models and discuss in which sense the phantom singularities can be avoided.

Keywords Phantom Energy · Alternative Theories of Gravity · Cosmological Singularities · Quantum Cosmology
1 Introduction

The origin and fate of our Universe are central issues in cosmology. Observations indicate that the expansion of the Universe is currently accelerating and that it was probably also accelerating at an early stage, the inflationary phase. The causes of those periods of accelerated expansion are unknown. It is also not known whether the early-time and the late-time accelerations have the same origin or not. It could be that the Universe was in a highly symmetric (de Sitter like) phase in the past and will return to this in the future, cf. [1]. The source of the present acceleration could be a cosmological constant, an additional dynamical field, or a modification of general relativity. The case of a dynamical field is usually referred to as dark energy [2]. Most models for dark energy employ one or more scalar fields. During the cosmic stages of acceleration the strong energy condition \( \rho + 3p > 0 \), where \( \rho \) is the energy density and \( p \) is the pressure, is violated by the fields that drive the acceleration. Depending on whether the null energy condition \( \rho + p \geq 0 \) is fulfilled or not, one talks about standard or phantom fields. Note that for both cases one must have a positive energy density, \( \rho > 0 \), since in cosmological spatially flat scenarios only this implies that the square of the Hubble rate is positive (as it must). There exist, of course, negative energy densities in Nature, such as the Casimir energy, but they are usually tiny. Phantom fields with \( \rho < 0 \) are occasionally used as toy models; see, for example, section II.B in [2].

The current observational situation is such that both standard fields and phantoms seem possible [3,4,5]. Even more, it has been shown that phantom models based on the “vacuum-metamorphosis model” help to alleviate the tension of the Hubble constant \( H_0 \) as inferred from Planck CMB data when interpreted within a \( \Lambda \)CDM cosmological model and local observational data [6]. It is also possible that the current acceleration of the cosmic expansion is a signal of the breakdown of general relativity at cosmological scales and, therefore, dark energy is just an effective description encapsulating modifications to the general relativistic cosmic predictions. This possibility will be confronted with upcoming observational data in the near future [5,8]. We remind in this regard that alternative theories of gravity are not only suitable to describe the

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1 Throughout this review dark energy denotes a fluid that can be described by a field that is minimally coupled to gravity and non-interacting with the standard model fields. Otherwise we consider that the field constitutes a genuine modification of general relativity and we have an alternative theory of gravity. Note that interactions within the dark sector are not excluded by this definition.

2 We assume here that these fields can be described by an energy–momentum tensor that has the form of a perfect fluid. For homogeneous and isotropic geometries in general relativity, this is required for consistency.

3 We use units where the speed of light \( c = 1 \). In SI units, this condition reads \( \rho + p/c^2 \geq 0 \), where both terms have the dimension \( \text{kg/m}^3 \). In these units it is seen that the second term is usually tiny compared to the first one. Taking air at sea level as an example, one has \( \rho = 1.29 \text{ kg/m}^3 \) and \( p/c^2 = 1.13 \times 10^{-12} \text{ kg/m}^3 \). This means that pressures must be immensely large (and negative) in order to compensate for densities.

4 This model arises from taking into account loop corrections in the presence of a massive scalar field.
late-time universe but also provide the model that best fit the observations of the early Universe [3,4,5,7].

This motivates the topic of our short review. Phantom fields are very exotic in many respects, their energy densities increase with the universe size, but they are a viable possibility from the observational point of view. Among their exotic aspects is the prediction of new types of singularities, notably the big rip, in which the observable Universe reaches infinite size in finite time. This raises the question about the nature of these singularities, their differences from the big bang singularity, and their fate in a quantum theory of gravity. Such a theory is not yet available in final form, but the question can be sensibly addressed in existing approaches [9]. Alternative theories of gravity can describe an accelerating universe without the need of dark energy and, therefore, they may also lead to phantom energy singularities. Thus, the formulation of a quantum alternative cosmology is a raising field of study [10,11,12,13,14].

In standard cosmology, the classical singularity theorems by Penrose, Hawking, and others, occupy a central place [15,16]. How is a singularity defined? In the words of Hawking and Penrose ([16], p. 15):

A spacetime is singular if it is timelike or null geodesically incomplete but cannot be embedded in a larger spacetime.

Typical assumptions in the proof of those theorems are an energy condition (such as the strong energy condition), a causality condition (such as the absence of closed timelike curves), and a boundary or initial condition (as the assumption that gravity is strong enough to create a trapped region) [17]. A singularity defined in the above manner may or may not involve a singularity in the curvature or matter properties (energy density, pressure). One can also have curvature singularities without geodesic incompleteness. Ellis et al. thus use the following more careful definition ([18], p. 145):

We shall define a singularity as a boundary of spacetime where either the curvature diverges . . . or geodesic incompleteness occurs. The relation between these two kinds or aspects of singularities is still not fully clear; but often they will occur together.

We shall adopt this point of view also here.

Our review is organized as follows. In section 2, we discuss energy conditions in general and their status for phantom fields in particular, emphasizing how to mimic a phantom behavior in alternative theories of gravity. Section 3 reviews and classifies cosmic singularities. Section 4 is devoted to the quantum fate of phantom singularities; we mainly discuss this question in quantum geometrodynamics (with the Wheeler–DeWitt equation as its main equation [9]), but also address loop quantum cosmology and the quantum cosmology of alternative theories of gravity. Our section 5 concludes with a brief summary and outlook.
2 Phantom energy

A satisfactory theory of gravity should be based on a relation between matter and geometry that provides us with a description of how matter moves in a given spacetime and how it affects its curvature. Although this relation is fixed by the theory, no restriction on the nature of the material content is typically imposed by it. Therefore, even in general relativity (GR), all possible four-dimensional geometries seem to be allowed solutions. Restrictions are obtained by causality conditions (e.g. avoidance of closed timelike curves) and energy conditions (ECs). Imposing ECs, one attempts to restrict the material content that should be taken into account. In particular, they are assumptions made for the form of the stress–energy tensor in agreement with our daily experience \[19,20\]. Hence, they can provide us with some hints about the characteristics of the spacetimes that are physically meaningful solutions of the theory. Nevertheless, the interpretation of the observational data currently available in the framework of GR allows the existence of a fluid that violates all the classical ECs as the most abundant substance in our Universe. As we will summarize, such unorthodox phantom fluid may lead to instabilities when quantization of gravity is not considered. Moreover, those observational data may point to the need of modifying GR at cosmological scales, where its validity has not been properly tested but it is extrapolated. In the framework of alternative theories of gravity, an effective phantom behavior can be described without introducing phantom scalar fields. But one should mention that effects of dark energy (and dark matter) can also be mimicked by infrared effects of quantum gravity as they occur, for example, in the approach of asymptotic safety \[21\].

2.1 Energy conditions

In order to understand the gravitational effects felt by a family of observers living in a given geometry, the Raychaudhuri equation can be taken into account \[15,22\]. This is a purely geometric relation in Riemannian geometry, which describes the convergence or divergence of a congruence of timelike (or lightlike) curves. For geodesic motion, it reads

\[
\frac{d\theta}{ds} = \omega_{ab}\omega^{ab} - \sigma_{ab}\sigma^{ab} - \frac{1}{3}\theta^2 - R_{ab}V^aV^b,
\]

where \(R_{ab}\) is the Ricci tensor, \(\omega_{ab}\) is the vorticity, \(\sigma_{ab}\) is the shear, \(\theta\) is the expansion, and \(V^a\) is the timelike unit vector tangent to the congruence. This equation plays a central role in the proof of the singularity theorems \[15,22\]. Restricting ourselves to congruences with vanishing vorticity (which have a tangent vector that can be expressed as a gradient of a scalar field \[23\]), the Raychaudhuri equation implies that those observers will get

\[5\] Vanishing vorticity is equivalent to the congruence being locally hypersurface orthogonal.
closer if \( V^a R^b_{\alpha \beta} V^\beta \geq 0 \). Therefore, the attractive character of gravity is guaranteed if the timelike convergence condition (TCC) is satisfied, which requires \( V^a R^b_{\alpha \beta} V^\beta \geq 0 \) for any timelike vector \( V^a \). Analogously, the focusing of null geodesics is implied by the null convergence condition (NCC), that is, \( k^a R^b_{a \beta} k^\beta \geq 0 \) for any null vector \( k^a \).

As we have discussed in the introduction, a spacetime can be called singular if a freely falling observer reaches the end of his or her path (although not only then). Therefore, as the TCC and NCC can be used to prove timelike and null geodesics incompleteness under certain circumstances \([15,18]\), they are of special interest to conclude the singular character of some geometries. But we emphasize already here that in the case of phantom energy, singularities can appear because of the violation of energy conditions. That is, if there is phantom energy, the classical singularity theorems do not apply; however, this does not imply that the geometry is geodesically complete nor that the curvature invariants are finite throughout the spacetime.

The first EC that we will discuss is intimately related with the TCC. Indeed, it can be understood as the requirement of gravity to be attractive in GR. This is the strong energy condition (SEC), which is mathematically formulated as \( V^a \left( T^b_{\alpha \beta} - \frac{1}{2} T \delta^b_{\alpha \beta} \right) V^\beta \geq 0 \), where \( T^b_{\alpha \beta} \) are the (1,1)-components of the stress–energy tensor of the matter fields and \( T \equiv T^a_{\alpha \beta} \). This condition is equivalent to substituting the Einstein equations into the TCC, so it makes sense only in the context of GR. For a perfect fluid, the SEC implies that \( \rho + p \geq 0 \) and \( \rho + 3p \geq 0 \), where \( \rho \) and \( p \) are, respectively, the energy density and pressure of the perfect fluid as measured in its rest frame. The Hawking and Hawking–Penrose singularity theorems assume the TCC and thus the SEC in the framework of GR \([15]\). Hence, if the TCC is satisfied during the cosmological evolution, we should accept a singular origin of our Universe \([22]\). Nevertheless, the SEC (and the TCC) has to be violated in cosmology when the expansion of the Universe accelerates, both during the early inflationary phase and right now. Therefore, it has been advocated that the SEC should be abandoned \([24]\).

The weak energy condition (WEC) has a clear physical meaning independent of the particular theory of gravity assumed. It requires that the energy density measured by any observer must be non-negative, that is, \( V^a T^b_{\alpha \beta} V^\beta \geq 0 \) for any timelike vector \( V^a \). A perfect fluid satisfies the WEC if \( \rho \geq 0 \) and \( \rho + p \geq 0 \). In the framework of GR, the WEC implies the NCC. However, the NCC can also be satisfied by imposing only that the WEC is fulfilled in the limit of null observers. This is the null energy condition (NEC), which requires that \( k^a T^b_{a \beta} k^\beta \geq 0 \) for any null vector \( k^a \); that is, \( \rho + p \geq 0 \) for perfect fluids.

Regarding black hole geometries, the Penrose theorem and the Second Law of black hole thermodynamics can be proven using the NCC \([22]\). Moreover, as the TCC was necessarily violated during the early inflationary phase of our Universe, a theorem pointing out the existence of an initial singularity for open universes was demonstrated under certain circumstances assuming only that
the NCC is satisfied \cite{29}. Therefore, the theorem requires just the fulfillment of the NEC in GR, whereas the SEC could be violated \cite{25}; see \cite{28} for a complete analysis of the ECs close to some cosmological events of interest in Friedmann–Lemaître–Robertson–Walker (FLRW) backgrounds.

The dominant energy condition (DEC) states that the energy density measured by any observer is non-negative and propagates locally in a causal way. Therefore, by its very definition, the DEC implies the WEC. Mathematically, the DEC requires that $V_a T^{a b} V^b \geq 0$ and $F_a F^a \leq 0$, with $F^a = -T^{a b} V^b$ being the flux four-vector and $V^a$ any timelike vector. This leads to $\rho \geq 0$ and $|p| \leq \rho$ for a perfect fluid. The zeroth law of black hole thermodynamics (i.e. that the surface gravity is constant over the horizon of a stationary black hole) requires the fulfillment of the DEC (\cite{22}, p. 334).

According to the definitions of the ECs, it can be noted that, on one hand, the DEC implies the WEC which leads to the NEC and, on the other hand, fulfillment of the SEC implies that the NEC is satisfied (but not necessarily the WEC); see, for example, Fig. 1 in \cite{29}. Therefore, violations of the NEC would lead to violations of all the other mentioned ECs. In the classical realm, it is enough to consider a non-minimally coupled scalar field\footnote{For closed universes, a similar theorem can be obtained assuming very strong requirements \cite{25}. These requirements can be violated (avoiding the conclusion of the theorem) for models of interest \cite{26} as, for example, in the emergent universe scenario discussed in \cite{27}, which describes an inflationary period without initial singularity.} to obtain such violations \cite{24}. Moreover, violations of all the ECs generically appear when considering quantum vacuum states in semiclassical physics (see the references in \cite{19} and \cite{20}).

In view of those violations, averaged energy conditions, which consist in integrating the ECs along timelike or null geodesics, were taken into account. Although they only require the ECs to be satisfied “on average”, there are also known violations of these conditions \cite{19,20}. A different approach is based on calculating quantum inequalities \cite{30}, which are bounds on the negativity of an average of the energy density. Moreover, noting that negative energies in one region of the spacetime seem to be overcompensated by positive energies in other regions, the related quantum interest conjecture was formulated \cite{32} (see also \cite{30,31} and references therein for developments). Apart from that, following a local point-wise approach, nonlinear ECs were formulated \cite{33}. The most interesting example is the flux energy condition (FEC) that requires energy of any sign to propagate in a causal way as seen by any observer. This condition is satisfied in some semiclassical situations \cite{20,33}. Furthermore, the semiclassical or quantum ECs have been recently formulated, based on noting that violations of the ECs are usually small. For example, the quantum weak energy condition (QWEC) demands that the energy density measured by any observer should not be excessively negative (so it can include, for example, the Casimir energy), introducing a specific bound to quantify this claim \cite{20,33};

\footnote{This kind of couplings can be interpreted as a modification of GR (a scalar-tensor theory of gravity). It should be noted, however, that assuming a conformal coupling is natural in some branches of physics.}
see \cite{20} and references therein for more details about these extensions. Whereas phantom energy violates the FEC, the QWEC and the QNEC have been used to minimize the violation of the ECs by phantom fields in cosmological \cite{34} or spherically symmetric solutions \cite{35}.

2.2 The phantom fluid

Soon after the discovery of the accelerated expansion that our Universe is currently undergoing, it was noted that dark energy could violate not only the SEC, but also the NEC $\rho + p \geq 0$. That is, the effective equation of state parameter of dark energy $w := p/\rho$, which needs to be smaller than $-1/3$ to describe accelerated expansion for a general relativistic universe, could be $w < -1$, cf. figure. In this case, dark energy is called phantom energy \cite{36}. Violations of the NEC would then not only be related with semiclassical effects or appear in small quantities, but would be relevant for classical fields leading to macroscopic effects. Nowadays, it is feasible that phantom energy is the most abundant cosmological ingredient; indeed, we have $w = -1.03 \pm 0.03$ according to the combination of Planck data and other astrophysical data \cite{4} and $w = -1.00_{-0.05}^{+0.04}$ according to the combination of Planck 2015 data with DES performed by the DES Collaboration \cite{5}. In addition, as we mentioned in the introduction, phantom models based on the vacuum-metamorphosis model helps to alleviate the tension of $H_0$ as inferred from Planck CMB data when interpreted within a $\Lambda$CDM cosmological model and local Hubble constant data \cite{6}. The definition of phantom fields, stated above, does not impose a restriction on the sign of $\rho$. It appears reasonable, however, to demand $\rho > 0$.

Anyway, as can be seen from one of the Friedmann–Lemaître equations,

$$\dot{a}^2 = -K + \frac{8\pi G}{3} a^2 \rho,$$

where $K$ is the curvature parameter, a negative energy density is possible only for negatively curved Friedmann universes, in which case (for $w < -1$) the universe expands from a regular null hypersurface and contracts to another regular null hypersurface \cite{37}.

The potential existence of such exotic fluids challenges our understanding of nature. The energy density of phantoms increases with the cosmic expansion and its dominance leads to a superaccelerating universe (a universe with a cosmic acceleration larger than that of a de Sitter model), whose scale factor may even blow up at a finite time (see section 3.1.1 for more details). Moreover, a fluid violating the NEC could allow the existence of exotic geometrical objects such as wormholes \cite{19}; therefore, if dark energy is of phantom nature, those objects may exist in our Universe with macroscopic size \cite{38,39}.

The most important potential shortcoming of phantom energy concerns its stability. The simplest fundamental description of a phantom fluid is in terms of
of a scalar field that is minimally coupled with gravity and has a canonical kinetic term with the “wrong” sign \[36\]. Even if such a “ghost field” does not interact classically with other fields, it will interact at least with the graviton\[9\]. This will lead to a quantum instability of the vacuum if we do not consider the theory as an effective description valid only under a given cutoff\[29,41\]. The ghost instability may be avoided when considering a non-canonical kinetic term for the scalar field. From studying perturbations of such “k-essence fields” around a FLRW background solution, one can conclude, however, that if the field violates the NEC without introducing a ghost, the speed of sound for the perturbations is imaginary \[42\], that is, the system has a gradient instability. More complicated Lagrangians for a scalar field, which are still minimally coupled to gravity but include second derivatives of the field in such a way that the field equations remain second order, may lead to violations of the NEC without instabilities or a superluminal speed of sound at least in FLRW backgrounds \[43\]. Therefore, even if the potential semiclassical instability of phantom energy could lead us to abandon the consideration of this exotic fluid, we prefer to keep an open mind until its fundamental (classical and quantum) field description is unveiled, at least as long as the observational data allow this possibility.

\[9\] See, for example, \[40\] for a review on classical and quantum ghost fields.

\[10\] It should be emphasized that the potential ghost instability appears when the quantization of gravity is not considered together with that of the field (more comments along these lines will be included in section 4).
2.3 Effective phantom behavior

The potential existence of a dark fluid with anti-gravitational properties (that is, violating the SEC) as the most abundant cosmological ingredient in our Universe is a result of interpreting the observational data in the framework of GR under the assumption of the cosmological principle. However, it is possible that the current accelerated expansion of our Universe is just a signal of the need of modifying GR at large scales. Alternative theories of gravity were already suggested as well-motivated candidates to describe gravitational phenomena at high energies or short scales. At present, the Starobinsky model of cosmic inflation is considered to be a plausible description of the early Universe (Sec. 6). Therefore, on the one hand, one could think that the potential existence of a phantom fluid (whose energy density grows with time as a result of the NEC violation) will drive the Universe again to a high-energy regime where modifications of GR might be necessary. On the other hand, one can go a step further and interpret that GR may break down in both the ultraviolet cosmic regime (during the early cosmological inflationary phase) and the infrared cosmic regime (recently in the cosmological evolution), with GR being only valid in a limited intermediate regime of the cosmic evolution. This interpretation suggests the need for considering alternative theories as the fundamental of gravitational phenomena. Thus, following this second point of view, the apparent existence of a dark fluid violating the SEC may indicate that we are entering a regime for which the appropriate gravitational theory deviates from GR significantly. This interpretation would appear more justified if the hypothetical fluid violated the WEC, explaining the counterintuitive nature of such a fluid in the classical realm.

In order to understand in a quantitative way this idea, let us note that alternative theories of gravity typically introduce higher-order curvature invariants and/or new (scalar, vector, or tensor) gravitational fields non-minimally coupled to gravity in the Lagrangian. Therefore, the modified Einstein equations of a large class of alternative theories of gravity can be written as

\[ g(\Psi^i) (G_{ab} + H_{ab}) = 8\pi G T^{(m)}_{ab}, \]

(3)

where \( G_{ab} \) is the Einstein tensor, \( T^{(m)}_{ab} \) is the stress–energy tensor associated with the material content (we assume that we have here no dark energy component), \( H_{ab} \) encapsulates additional gravitational terms of geometrical nature or involving the new fields, and \( g(\Psi^i) \) takes into account potential modifications of the gravitational coupling with \( \Psi^j \) denoting curvature invariants and gravitational fields. It is easy to rewrite (3) as

\[ G_{ab} = 8\pi G_{\text{eff}} \left[ T^{(m)}_{ab} + T^{(\text{eff})}_{ab} \right], \]

(4)

\(^{11}\) The case of a cosmological constant, understood as a fluid with \( p = -\rho \), also violates the SEC, but saturates the NEC (so it can be considered as a limiting case between phantom and dark energy).
with
\[ G_{\text{eff}} = \frac{G}{g(\Psi)} \quad \text{and} \quad T_{ab}^{(\text{eff})} = -\frac{g(\Psi)}{8\pi G} H_{ab}. \] (5)

Thus,
\[ T_{ab}^{(\text{tot})} = T_{ab}^{(m)} + T_{ab}^{(\text{eff})}. \] (6)

The SEC, which is the requirement of the TCC being satisfied in GR, makes no sense once we consider a different theory of gravity. One can formulate a generalization of this condition requiring the TCC to be satisfied in the new theory \[45\], obtaining an inequality involving both stress–energy tensors contracted with a timelike vector and their traces. Following this procedure, one can study violations of the TCC induced by a non-vanishing effective stress-energy tensor.

Whereas \(T_{ab}^{(m)}\) is associated with real matter fields, \(T_{ab}^{(\text{eff})}\) is of purely gravitational origin comprising geometric terms and/or extra gravitational fields. Therefore, one can impose that the WEC be satisfied for the matter stress–energy tensor, but there is no reason why the energy density associated with the effective stress–energy tensor should be positive, since this tensor just encapsulates the modification with respect to GR \[20,24\]. Indeed, in some theories one can easily obtain \(T_{ab}^{(\text{eff})} k^a k^b < 0\); see, for example, \[46,47\]. Following this spirit, in alternative theories of gravity one can obtain geometries where the NCC is violated supported by a material content with a positive energy density as measured by any observer. That is, one can violate the NCC without violating the energy conditions in alternative theories of gravity.\[12\] This is the ultimate reason for the possibility of describing phantom cosmologies in the framework of alternative theories of gravity without the introduction of exotic fluids; see, for example, \[49,50\].

3 Cosmic singularities

The characterization of singularities is a fascinating and at the same time a difficult task to tackle in a relativistic gravitational theory, no matter if it is GR or any extension of it. To give a metaphor: “How abrupt can the tip of a mountain appear? The answer depends on the shape of its summit or on the different ways of reaching and/or unreaching the summit.” Singularities are a familiar concept in various branches of physics, especially in fluid dynamics; see, for example, \[51\] and the references therein. Within a relativistic gravitational theory, the shape of the mountain can be characterized by its geometry and therefore by its curvature, while the ways of reaching its summit can be described through parametric curves, the timelike and lightlike geodesics being the easiest way to characterize the paths, which describe freely falling object that do not experience any acceleration.

\[\text{Notice as well that it is possible to have a phantom-like behavior, that is, to define an effective energy density, } \rho_{\text{eff}} \text{, such that } \dot{\rho}_{\text{eff}} > 0 \text{ in an expanding universe if the spatial geometry is spherical and } \rho_{\text{eff}} := A - \frac{1}{2}. \] (5)
As we have already stated in the introduction, a singularity can be defined as a boundary of spacetime where either the curvature is ill-defined (e.g. it or some of its derivatives diverge) or geodesic incompleteness occurs [18]. In the light of these criteria, we will next summarize work carried out in the context of dark energy singularities and abrupt events.

3.1 Classification of singularities

First of all, we would like to recall that within GR there is a direct and bijective (linear) relation between the energy density and pressure of the effective (total) fluid filling a FLRW universe and the (squared) Hubble rate and its cosmic time derivative:

\[
H^2 + \dot{H} = -\frac{4\pi G}{3} (\rho + 3p).
\]  

(7)

This relation does no longer need to exist in alternative theories of gravity. For this reason, we will characterize cosmological singularities by the Hubble rate and its cosmic time derivative(s), which define univocally the scalar curvature of spacetime, rather than through the total energy density and pressure of matter filling the Universe. We will as well distinguish between cosmic singularities that happen at a finite cosmic time and abrupt events that happen at an infinite cosmic time.

3.1.1 Metric classification

1. Cosmic curvature singularities (at a finite time): GR and beyond

The following classification is a generalization of the scheme proposed in [61].

**Type 0-a** A big bang singularity is a past singularity that takes place at vanishing scale factor where the Hubble rate and its cosmic time derivative diverge [18].

**Type 0-b** A big crunch singularity is a future singularity that takes place at vanishing scale factor where the Hubble rate and its cosmic time derivative diverge [18]. For a FLRW universe, this singularity is usually present in a universe which is spatially closed and filled with matter satisfying the strong energy conditions (but see [52] for a counterexample).

**Type I** A big rip singularity takes place at finite cosmic time with infinite scale factor, where the Hubble parameter and its cosmic time derivative diverge [36,53,54,29,55,56,57,65,2,59]. The occurrence of this singularity is intrinsic to phantom dark energy.

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13 We note that this relation follows from the Raychaudhuri equation [4] in the Friedmann limit if Einstein’s equation hold.
Type II A sudden singularity takes place at finite cosmic time with finite scale factor, where the Hubble parameter remains finite but its cosmic time derivative diverges \cite{52,61,62}. It is also known as a big brake if it takes place in the future with an infinite deceleration or a big démarrage, named also big boost, if it happens in the past with an infinite acceleration. The latter case is realized by a generalized Chaplygin gas \cite{65,67}, see also section 4.2.2 below.

Type III A big freeze singularity takes place at finite cosmic time with finite scale factor, where the Hubble parameter and its cosmic time derivative diverge \cite{61,65,68,70}. This singularity has also been named finite scale factor singularity in \cite{71}. A phantom generalized Chaplygin gas can induce a big freeze in the future.

Type IV This singularity takes place at finite cosmic time with finite scale factor, where the Hubble parameter and its cosmic time derivative remain finite, but higher cosmic time derivatives of the Hubble parameter diverge \cite{61,68,69,70,72,73,74}. They have been also called generalized sudden singularities \cite{75}. We emphasize that the scalar curvature is well defined not only through its finiteness but also through its differentiability. Therefore, a type IV singularity can be seen in GR as an event where the curvature is not (or not completely) differentiable, that is, it is not a $C^\infty$ function. Beyond GR this singularity can affect the equations of motion through derivatives of the scalar curvature, as, for example, in metric $f(R)$ models \cite{72}.

Type V A $w$-singularity takes place at finite cosmic time with finite scale factor, where the Hubble parameter vanishes and its cosmic time derivative is finite in such a way that the barotropic index of the fluid dominating the cosmic dynamic blows up \cite{76}. Notice that in a type IV singularity this behavior does not necessarily happen, but it is not forbidden. While within GR this behavior will not affect the boundedness of the curvature, it can affect its differentiability as for type IV. In addition, at first order in the cosmological perturbations, the unboundedness of $w$ can have an effect on the evolution equations of the gravitational potential when approaching the singular point (cf. Eq. (3.16) in \cite{77}).

Type VI A $Q$-singularity takes place at finite cosmic time in models with interacting dark energy and dark matter in which the interacting term $Q$ blows up \cite{73,74}. While this does not mean any geometrical issue at the background level, it might highlight possible instabilities at the perturbative level. More precisely, what happens is that a divergence in $Q$ may lead to a blow up of the time derivative of the equation of state and this on its own can result in a very large speed of sound of dark energy and consequently in an instability of the cosmological perturbations. Therefore, as in a $w$-singularity, a $Q$-singularity can be seen at the perturbative level.
2. Abrupt cosmic events

**Little rip** This takes place at an infinite cosmic time with infinite scale factor, where the Hubble rate and its cosmic time derivative diverge. It can be visualized as a big rip sent towards an infinite cosmic time [81, 82, 83, 84, 85, 86, 87, 88]. It is worth mentioning that the first time this behavior was found was in an alternative theory of gravity [81]. Mathematically equivalent solutions can be obtained in some specific inflationary models where the expansion is driven by a perfect fluid with bulk viscosity [89, 90].

**Little sibling of the big rip** This takes place at infinite cosmic time with infinite scale factor, where the Hubble rate diverges, but its cosmic time derivative remains finite. Such a behavior can happen precisely because it takes place at an infinite cosmic time [34, 91].

We emphasize that these abrupt events are intrinsic to phantom dark energy.

3.1.2 Affine classification

An alternative way of characterizing a spacetime singularity is by analyzing its causal curves, in particular its geodesics. We can therefore say that a spacetime has a singularity if there are causal curves that are incomplete [15, 92, 93]. We note that while the definition of cosmological singularities used in the previous subsection is independent of observers, the current one will depend on them. A timelike geodesic, for example, might be incomplete, while a lightlike geodesic might be complete. This is what happens in a big rip model with a barotropic equation of state where $-5/3 < w < -1$; that is, under these conditions a lightlike geodesic will never reach the big rip, as it would take it an infinite affine proper time, while a timelike geodesic will hit the big rip in a finite affine proper time [93, 37].

We start by recalling the (crucial) causal geodesic equation in a spatially flat FLRW universe, which at first order of a power expansion reads [93]:

$$\dot{t} = \sqrt{\delta + \frac{P^2}{a^2(t)}}, \quad P = \text{constant}, \quad \delta = 0, \pm 1. \quad (8)$$

As pointed out in [92, 93] (see also [94]), this equation is well defined for any finite non-vanishing value of the scale factor. A careful analysis of causal geodesics in dark energy dominated universes was carried out in [92, 93] under the assumption that the scale factor $a(t)$ allows a generalized Puiseux expansion, that is, a generalized Taylor expansion where the exponents are real and

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14 We will be referring to abrupt events as those cosmic curvature singularities that take place at an infinite cosmic time and where all the structure in the Universe can be destroyed in the far future (at a finite cosmic time from now.). Therefore, this definition excludes the pseudo-rip given that it corresponds to a mild event. In fact, it corresponds to a model where the Hubble parameter increases monotonically and reaches a constant value at infinite cosmic time and scale factor [80].
not necessarily natural numbers and where there is always a minimum finite exponent from which the expansion is made around the singular event. This analysis was extended to some inspired modified theories of gravity [95], $w$-singularities [96], and more recently to $Q$-singularities [78]. In all these works, it was concluded that cosmic curvature dark energy singularities would be reached by causal geodesics in a finite cosmic time with the exception of light-like geodesics in case of a barotropic equation of state with $-5/3 < w < -1$.

The next question to address is: how strong are these singularities? In the previous subsection, the strength of the singularity was characterized by the divergence of the Hubble rate and its cosmic time derivatives at the singularity, that is, the degree of divergence for quantities that characterize or define the curvature as well as its analyticity. As we are dealing with geodesics, the simplest way is to analyze scalars constructed from the contraction between the Ricci tensor and the velocity of the observer. This is precisely what the criteria by Tipler and Królak achieve, see below.

Before proceeding further, we recall that the idea of a strong singularity was first introduced by Ellis and Schmidt at the end of the seventies [97]. So far, we have assumed that the observers are point-like objects but in fact they can have a finite volume and therefore they are subject to tidal forces. With this in mind, it is understood that a singularity is strong if tidal forces imply a wrecking of the object that is heading towards the singularity. Tipler and Królak independently proposed criteria to quantify the effect of tidal forces on such an object. Let us have a closer look at them.

**Tipler criterion:** Following Tipler [98], a singularity is strong if the volume characterizing the object\(^{15}\) vanishes when the geodesics reach the singularity. This criterion was adapted to expanding FLRW universes filled with phantom dark energy [92,93] and implies there that this volume diverges. In fact, for an expanding FLRW universe filled with phantom dark energy, the spacetime contains a strong singularity at $\tau_0$ if

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' R_{ij} u^i u^j,$$

\[(9)\]

diverges when $\tau$ approaches $\tau_0$. Here, $u^i$ stands for the geodesic four-velocity with affine parameter $\tau$. This definition applies to lightlike and timelike geodesics.

**Królak criterion:** Following Królak [100], a singularity is strong if the proper-time derivative of the volume of the object heading towards the singularity is negative. Therefore, this criterion is less restrictive than the Tipler one. The Królak criterion was also applied to expanding FLRW universes filled with phantom dark energy [92,93] and implies there that the cosmic derivative of the volume must be positive. In fact, for an expanding FLRW universe filled with phantom dark energy, the spacetime contains a strong singularity at $\tau_0$.

\[^{15}\text{We will not enter into the technicality of how the volume is defined; instead we refer to [98,92,93] for further details.}\]
Phantom singularities

if \[ \int_0^\tau d\tau' R_{ij} u^i u^j \]
diverges when \( \tau \) approaches \( \tau_0 \). Again, \( u^i \) stands for the geodesic velocity with respect to an affine parameter \( \tau \). This definition applies to lightlike as well as timelike geodesics.

Applying these criteria \[92,93,95,96,78\], it was shown that all dark energy cosmic singularities are weak with the exceptions of: (i) the big rip, which is a strong singularity, and (ii) the big freeze, for which the Królak and Tipler criteria give opposite results: while for the former the singularity is strong, for the latter it is weak; therefore, no conclusions can be drawn in this case. Given that it has been shown that all bounded structure of the universe facing a big freeze in the future will be destroyed \[101\], we believe that the Królak criterion is more appropriate.

We finally like to emphasize two things. First, by analyzing the geodesics it is possible to find spacetime pathologies which are hidden in the definition of the spacetime curvature. This is the case for directional singularities where the Ricci curvature vanishes at the singularity, but where the projection of the curvature along some causal geodesics diverges \[94\]. Second, one may also apply the generalized geodesic deviation equation \[102\] when investigating the approach to singularities.

3.2 Phantom singularities and alternative theories of gravity

Among the cosmic singularities and abrupt events discussed above, only some of them require a violation of the null convergence condition (at least from a phenomenological point of view), that is, \( p^{(\text{tot})} + \rho^{(\text{tot})} < 0 \) (cf. equation \[6\]); those correspond to the big rip, pseudo-rip, little rip, and little sibling of the big rip. If, in addition, we impose that the singularity takes place in the future, the big freeze will also be included.

So far, we have assumed that a dark energy singularity arises from a perfect fluid within GR. But this is not the only possibility. Such a singularity could also arise from modified gravity, for example from an action containing a function of the Ricci scalar, \( f(R) \), within the metric formalism. As is well known, a perfect fluid with a constant equation of state (or more complicated variations) can be perfectly modeled in this setup; see, for example, \[103\] and the references therein. Indeed, if \( f(R) \) is a linear combination of the following form \[103\]:

\[
f(R) = C_+ R^{\beta_+} + C_- R^{\beta_-}, \quad C_\pm = \text{constant},
\]

where

\[
\beta_\pm = \frac{1}{2} \left\{ 1 + \frac{1 + 3w}{6(1 + w)} \pm \sqrt{\frac{2(1 - 3w)}{3(1 + w)}} + \left[ 1 + \frac{1 + 3w}{6(1 + w)} \right]^2 \right\},
\]
we will find for a FLRW universe an expansion that is equivalent to that of a FLRW universe in GR with a perfect fluid satisfying \( p = w \rho \) with constant \( w \). In particular, one can mimic a perfect fluid \( w < -1 \) at the background level. Other \( f(R) \) metric theories could induce a sudden, big freeze, or type IV singularity, given that any kind of modified generalized Chaplygin gas can (i) be mimicked by a metric theory and (ii) is known to induce the singularities listed above (for certain choices of the parameters of the model). In summary, what we want to stress is that the phantom nature of a dark energy could be rooted in some alternative theory of gravity (as already argued in section 3.1 above), and we have given here a simple example within the framework of \( f(R) \) gravity.

Before ending this subsection, we want to emphasize two things. First, even in the absence of true phantom matter in GR, we can sometimes define an effective dark energy that grows as the Universe expands. Such a description might pretend the existence of a phantom field, while it could be simply an effect of non-flat spatial curvature [48], or of extra dimensions [104,105]. Second, phantom matter does not necessarily imply the existence of a future singularity (‘doomsday’). For example, an equation of state that violates the NEC but that asymptotically approaches the equation of state corresponding to a cosmological constant might simply lead to a de Sitter universe [106].

After having classified the different types of phantom singularities, we next address the issue of how to cure them or at least how to smooth them. The obvious answer is to consider a quantum treatment, as we will do in the next section. We expect as well that GR will be modified when approaching those singularities, that is, we expect a semiclassical regime when approaching the occurrence of the singularity. We will consider this second option in the rest of this subsection.

Given that phantom singularities happen at very high energies and at very late time, ultraviolet (UV) and/or infrared (IR) corrections to GR could occur and a simple approach to describing those corrections is within the framework of alternative theories of gravity. We end with two examples that characterize this idea.

First, five-dimensional braneworld models with an induced gravity term on the brane and a Gauss-Bonnet contribution to the bulk action include IR and UV corrections to GR in a natural way [107]. Within this setup it has been shown that the big rip singularity can be alleviated and substituted by a sudden singularity [108].

Second, the Eddington-inspired-Born Infeld (EiBI) model has attracted some attention lately [109]. It corresponds to a Palatini theory, that is, to a theory where the connection that defines the curvature is not the Christoffel symbols of the physical metric. It has the virtue of removing the big bang singularity [109] and smoothing some phantom dark energy singularities [101].
4 Quantum fate of classical singularities

4.1 Criteria for singularity avoidance

In GR, we have well defined criteria for singularities (such as geodetic incompleteness), and we have rigorous singularity theorems \[15,16,18\]. None of these are available in quantum gravity. The main reason for this is, of course, that no such theory exists in final form \[9\]. But even given particular approaches, we are still far from presenting exact definitions of singularities and their potential avoidance in quantum gravity. Nevertheless, heuristic criteria for singularity avoidance exist and have been successfully applied to cosmological models.

There are various approaches to quantum gravity, but the most straightforward and conservative one is the attempt to directly quantize GR. For this purpose, one can formulate GR in canonical form to arrive at a picture of wave functions in configuration space \[9\]. In the following, we shall concentrate on quantum geometrodynamics, where the configuration space is the space of all three-geometries \((3)\mathcal{G}\) called superspace, but we also include a brief discussion of loop quantum cosmology. In quantum geometrodynamics, the central equation is the Wheeler–DeWitt equation

\[ H\Psi = 0, \]  

where \(\Psi\) is the wave functional on the configuration space of three-geometries and non-gravitational fields.\[16\]

In classical GR, singularities can occur at particular values of the three-geometry. In his pioneering paper \[110\], DeWitt focused attention on three-geometries with vanishing volume (corresponding, in cosmology, to the big bang and big crunch singularities). Calling a singular boundary of configuration space \(B_Q\), DeWitt adopts the following criterion (\[110\], p. 1129):

The fact that \(B_Q\) is not the empty set . . . is not necessarily embarrassing to the quantum physicist, for he may be able to dispose of it by simply imposing, on the state functional, the following condition:

\[ \Psi\left[(3)\mathcal{G}\right] = 0 \text{ for all } (3)\mathcal{G} \text{ on } B_Q. \]  

\(14\)

Provided it does not turn out to be ultimately inconsistent, this condition . . . yields two important results. Firstly, it makes the probability amplitude for catastrophic 3-geometries vanish, and hence gets the physicist out of his classical collapse predicament. Secondly, it may permit the Cauchy problem for the “wave equation”\[18\] . . . to be handled in a manner very similar to that of the ordinary Schrödinger equation.

\[16\] In the full theory, the Wheeler–DeWitt equation is complemented by the diffeomorphism constraints, which imply that \(\Psi\) is invariant with respect to infinitesimal three-dimensional coordinate transformations.

\[17\] For example, by allowing only the trivial solution \(\Psi \equiv 0\) (our comment).

\[18\] This equation is the Wheeler–DeWitt equation \[13\].
This “DeWitt criterion” is based on the heuristic idea that wave functionals in quantum gravity can be related, in some sense, to probability amplitudes as in ordinary quantum theory. This is not obvious, since (13) does not contain any external time parameter, so the main argument for the probability interpretation (conservation of probability in time, that is, unitarity) is not applicable.

The DeWitt criterion $\Psi \rightarrow 0$ can only serve as a sufficient, not as a necessary condition. Let us recall, for example, that the solution of the Dirac equation for the ground state of the hydrogen atom diverges at the origin,

$$
\psi_0(r) \propto (2mZ\alpha r)^{\sqrt{1-Z^2\alpha^2}-1}e^{-mZ\alpha r} \xrightarrow{r \to 0} \infty.
$$

Nevertheless, this state is not singular, since the integral $\int dr \ r^2 |\psi_0|^2$ remains finite.

Another criterion adopted in quantum cosmology is the breakdown of the classical approximation when approaching the singularity [2]. This means that wave packets necessarily disperse in this region and that the classical singularity theorems do not apply. The general situation of late-time singularities in quantum cosmology is carefully reviewed in [111].

The vanishing of the wave function as a criterion for singularity avoidance is also adopted in other contexts. The authors of [112] discuss the situation for eleven-dimensional supergravity. They employ a cosmological billiard description near a spacelike singularity and find that the wave function solution of the corresponding Wheeler–DeWitt equation approaches zero there. Such a behavior also occurs in models of gravitational collapse and can there be interpreted as the avoidance of a black-hole singularity [113,114]. In the latter case, $\Psi = 0$ at the position of the classical singularity follows from the unitary evolution with respect to a dust proper time.

4.2 Quantum phantom cosmology

In most investigations, restriction is made to FLRW universes. In this case, the Wheeler–DeWitt equation reads (with units $2G/3\pi = 1$) [9,117]

$$
\frac{1}{2} \left( \frac{h^2}{a^2} \frac{\partial}{\partial a} \left( \frac{\partial}{\partial a} \right) + \ell \frac{h^2}{a^3} \frac{\partial^2}{\partial \phi^2} - Ka + \frac{La^3}{3} + 2a^3V(\phi) \right) \psi(a, \phi) = 0,
$$

where $a$ is the scale factor, $K = 0, \pm 1$, $\Lambda$ the cosmological constant, and $\phi$ a homogeneous scalar field mimicking matter; $\ell$ can assume the values $+1$ (ordinary field) and $-1$ (phantom field). The factor ordering has been chosen in order to achieve covariance in configuration space (Laplace–Beltrami factor ordering). We note that the scale factor $a$ can also be interpreted as a kind of “phantom”. In the full Wheeler–DeWitt equation, the kinetic term related with the (local) volume of the three-geometry is negative, a feature that can be related to the attractivity of gravity [115].
4.2.1 Examples with big rip, little rip, and little sibling of the big rip

The first discussion of phantom fields in quantum cosmology was presented in [2]. The simplest case is a vanishing potential and a vanishing cosmological constant. This is not a realistic model, because it corresponds to stiff matter \((w = 1)\) in \(p = wρ\) and \(ρ < 0\). Nevertheless, it is an instructive example because it can be solved exactly. As mentioned above, this model has, in the phantom case, only solutions for negative spatial curvature \(K = -1\) [37]. Classically, this universe collapses from infinity, reaches a minimum for the scale factor (bounce) and re-expands to infinity. At infinity, it exhibits a little rip abrupt event. Solving the Wheeler–DeWitt equation (15) for this case, one has to impose the boundary condition \(ψ \to 0\), because this region is classically forbidden and we thus demand the wave function to go to zero there [19]. The solution is then a particular Bessel function for \(a\) and a plane wave for the phantom field. A more realistic model for the little rip is discussed in [88]. As shown there, it is possible to apply the DeWitt criterion and to find solutions of (15) for which the wave functions vanish in the little rip region; in this sense, the little rip can be avoided.

A real big-rip singularity is obtained for a phantom field with a Liouville potential \(V(φ) = V_0 \exp(-λκφ)\) [2]: classically, scale factor and energy density diverge at finite time. A big bang singularity is not present, as is true for all models with \(w < -1\) and \(ρ > 0\) [37]. For this model, one can find exact wave-packet solutions for the Wheeler–DeWitt equation. These solutions clearly exhibit that wave packets necessarily disperse when approaching the region of the classical big rip singularity. Following our second criterion above, this can be taken as a signal of singularity avoidance: the semiclassical approximation breaks down and the concept of an expanding universe ceases to hold before the big rip region is reached. This is an explicit example for the occurrence of quantum gravitational behavior for large-size universes. Another example is the occurrence of a quantum region when approaching the turning point of a classically recollapsing universe [118].

For negative cosmological constant and a phantom field, a model is obtained in which the universe evolves between two big rips in a finite time [2]. It contracts from the first one, reaches a minimum for the scale factor, and expands to the other one. The phantom field has a potential containing a \(\cosh^2 φ\)-term. For this model, the Wheeler–DeWitt equation can be solved in the region near the big rips, and it is found again that wave packets necessarily disperse, that is, one approaches the timeless quantum region before reaching the classical singularities.

Likewise, the quantum cosmology of models that induce a little rip and a little sibling of the big rip have been analyzed in [88,91], respectively. These

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19 Although this is a natural condition and is standard in quantum mechanics, it is not always implemented in quantum cosmology. The wave functions following from the no-boundary (Hartle–Hawking) condition, for example, typically increase in classically forbidden regions [118].
analyses have been carried by invoking a perfect fluid or a scalar field. In the case of matter described by a scalar field, the Wheeler-DeWitt equation is the same as shown in \cite{15}.

Investigations are not restricted to FLRW models. It is possible to discuss the presence of big rip and its quantum fate in the framework of anisotropic models, notably Bianchi models \cite{119}. The study of anisotropic models is an important step towards the understanding of the general case (BKL conjecture).

4.2.2 Other phantom-induced singularities

Big rip and little rip are intrinsic to the presence of phantom fields. But phantoms allow the occurrence of singularities which can also occur for ordinary fields.

Of particular interest is the situation where we have a generalized Chaplygin gas effective equation of state,

\[ p = -\frac{A}{\rho^\beta}, \tag{16} \]

with parameters \(A\) and \(\beta\). Such an equation of state is of relevance when discussing dark energy and dark matter. Phantom fields can induce big freeze (type III) and big démarrage (type II) singularities. In \cite{65}, the Wheeler–DeWitt equation was investigated for these cases\footnote{The quantum cosmology of a generalized Chaplygin gas was first carried in \cite{120}.} (and for ordinary scalar fields, too). It was shown that classes of solutions can be found that avoid the singularities in the sense of the DeWitt criterion. This is also possible for ordinary scalar fields, so phantoms do in this respect not play a different role.

The situation is somewhat different for type IV singularities \cite{74}. These singularities, which are of a rather mild nature (for example, geodesics are unaffected by it), are generically not avoided, only in particular cases.

4.2.3 Loop quantum cosmology

Singularity avoidance of phantom (and other) fields has also been discussed in loop quantum cosmology \cite{121,117}. This is the application of loop quantum gravity to cosmology, with loop quantum gravity being a variant of canonical quantum gravity, distinguished from geometrodynamics by its different use of variables. The central equation of loop quantum cosmology is still of the form \cite{13}, but it is claimed to have fundamentally the form of a difference equation. The criteria for singularity avoidance are somewhat different from the criteria used above (see e.g. the list in \cite{117}, p. 508). Much emphasis is taken on effective modifications of the Friedmann equations. It follows that the matter density \(\rho\) is replaced there by

\[ \rho \rightarrow \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right), \tag{17} \]
where $\rho_{\text{crit}}$ denotes a critical density of the order of the Planck density. This modification may introduce a bounce at small scale factors and thus to an avoidance of the big bang singularity. A similar situation can occur for phantom-induced singularities. It has, in fact, been shown that a big rip singularity is avoided and replaced by a transition from an expanding to a contracting branch [122,123]. As in geometrodynamics, type IV singularities are much less likely to be avoided. Singularity avoidance for Bianchi spacetimes has been discussed in [124].

4.3 Quantum cosmology of alternative theories of gravity

While so far the analysis of dark energy singularities has been mainly focused on GR, it should be as well analyzed within alternative theories of gravity where indeed these singularities can appear as we mentioned previously. So, we consider here that these theories are classical theories of gravity to be quantized, as they also predict singularities. As there are many ways of extending GR, we will focus on the two main stream to define a modified theory of gravity: (i) a metric $a$ approach and (ii) a Palatini approach.

4.3.1 Quantum cosmology in metric $f(R)$ gravity

Nowadays, $f(R)$ metric gravity is one of the best candidates not only to describe the early inflationary era through the Starobinsky model, see [7], but also to describe the late-time universe. The Starobinsky model was quantized back in the eighties by getting the correct Wheeler–DeWitt equation for a homogeneous and isotropic universe after introducing at the classical level a proper Lagrange multiplier which takes into account the relation between the scale factor and the scalar curvature [125]. It should be highlighted that for a homogeneous and isotropic universe, the Wheeler-DeWitt equation in this case has two degrees of freedom even if the universe is empty [125]. Quantum cosmology for a more general class of higher-derivative theories was discussed in [126,127].

The big rip singularity in the framework of $f(R)$ quantum geometrodynamics and invoking the DeWitt criterion has been recently analyzed in [12], where it was shown the existence of solutions to the Wheeler–DeWitt equation fulfilling this condition. It is worthy to note that this equation is always hyperbolic for any $f(R)$-cosmology, even if the classical model mimics a phantom expansion.

4.3.2 Quantum cosmology in Palatini EiBI gravity

We have already mentioned the EiBI theory. It has the bonus of removing the big bang singularity at the classical level and even some phantom dark energy singularities, though not the big rip. This has motivated the analysis carried in [10,11,13,14]. The quantization of this theory has to be done with great
care. In fact, a thorough analysis of the classical Hamiltonian with constraints must be carried out in order to get a self-consistent modified Wheeler–DeWitt equation. This new Wheeler–DeWitt equation is derived with the use of Dirac brackets. What should be highlighted in the quantization of this theory is that the auxiliary scale factor, that is, the one compatible with the Palatini connection, is the one that appears in the Wheeler–DeWitt equation rather than the standard scale factor. This has important consequences when imposing boundary conditions close to a singularity.

In this kind of theories, it can be shown that the big rip singularity present in the classical theory, and induced by a phantom perfect fluid or a phantom scalar field, is expected to be removed when quantum effects encoded on the modified Wheeler–DeWitt equation are taken into account \[13,14\]. A similar analysis has been carried for the little rip and little sibling of the big rip reaching similar conclusions \[14\].

5 Discussion and perspectives

So far, dark energy remains a mystery. Current observations allow the possibility that it is caused by the presence of phantom fields or by an alternative theory of gravity that mimicks their behavior. We have reviewed here the properties of these fields and have discussed in some detail the singularities caused by them in the classical theory as well as their possible quantum avoidance.

A possible modification of general relativity can arise from its quantization or from its violation already at the classical macroscopic level. Modifications from quantization are expected because it would not be natural to have a hybrid unified theory of interactions in which one part (gravity) stays classical \[9\]. Quantum modifications are expected to lead to tiny corrections in most situations and to become relevant when approaching the region of classical singularities. The situation is different for an alternative classical theory of gravity. Such a theory could directly explain the dynamics of dark energy, without invoking any additional fields, be them standard or phantom. But such a theory may also predict the occurrence of singularities and would thus point to the need of its quantization. So far, however, it is not known whether important features of general relativity such as the initial value problem or the existence of singularity theorems continue to hold in alternative theories. Much work is thus needed to study those theories in the classical and quantum realm \[12,13\]. It is hard to imagine that a decision on these issues can be reached without observational input. It is therefore important to investigate whether quantum gravitational effects such as the ones calculated in \[128,129\] can be observed and help us to reach such a decision.

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