The Nuclear Pairing Gap - How Low Can It Go?

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Abstract. The pairing gap for $^{53}$Ca obtained from new experimental data on the masses of $^{52-54}$Ca has the smallest value yet observed. This is explained in the framework of the nuclear shell model with schematic and realistic Hamiltonians as being due to shell gaps around the low-$j$ orbital $1p_{1/2}$. Minima in the pairing gaps for all nuclei are shown and discussed.

1. Introduction

One of the most robust signatures of pairing in nuclei is the odd-even oscillation in the one-neutron separation energies as a function of neutron or proton number. This is illustrated in figure 1 which shows the binding energies and one-neutron separation energies for the calcium isotopes. The figure also shows the results of a shell-model calculation in the $(0f_{7/2}, 0f_{5/2}, 1p_{3/2}, 1p_{1/2})(fp)$ model space with the GX1A Hamiltonian (also referred to as GXPF1A in the literature [1]) compared with experiment. The oscillation in the one-neutron separation energies can be quantified in terms of the energy differences

$$D_n(N) = (-1)^{N+1}[S_n(Z, N + 1) - S_n(Z, N)]$$

$$= (-1)^{N+1}[BE(Z, N - 1) + BE(Z, N + 1) - 2BE(Z, N)],$$

where $S_n(N) = BE(Z, N) - BE(Z, N - 1)$ is the one-neutron separation energy. $N$ is the number of neutrons and $Z$ is the number of protons. This quantity turns out to be always positive and reflects the fact that the even nuclei are always more bound on the average than the neighboring odd nuclei. I will distinguish the results for even and odd $N$ values denoted by, $D_{ne}$ and $D_{no}$, respectively. In the literature one commonly finds the related quantity known as the odd-even mass parameter or pairing gap $\Delta_n(N) = \frac{D_n(N)}{2}$ (see Figures 2.5 in [2]). I use $D$ rather than $\Delta$ because its values are directly connected to simple underlying quantities associated with pairing and shell gaps. Equivalent equations for $D_p$ as a function of proton number are obtained by fixing $N$ and varying $Z$. This contribution is based on material I have published in [3] and [4].

2. The Calcium Isotopes

Figure 2 shows values of $D_n$ for the calcium isotopes $(N > 20)$ obtained from experiment and from two commonly used effective Hamiltonians in the $fp$ model space, KB3G [5] and GX1A [1] and with a new ab-initio Hamiltonian that includes three-nucleon interactions [6]. I also show the excitation energies for the lowest $2^+$ states of the even nuclei. The experimental data is from the 2012 mass table [7] together with recent data for $^{51-54}$Ca [8], [9]. The experimental energy
Figure 1. The bottom panel shows the ground-state energies for the calcium isotopes obtained from the pf shell-model calculation with the GX1A Hamiltonian relative to \(^{40}\text{Ca}\) with filled circles even \(N\) and open circles for odd \(N\), all connected by a line. The crosses are the experimental data. The top panel shows the one-neutron separation energies for GX1A and experiment.

Figure 2. \(D_n\) and \(E_x(2^+\text{)}\) for the calcium isotopes as a function of neutron number. \(D_n\) values are shown by the filled circles for even \(N\) and open circles for odd \(N\), all connected by a line. The \(E_x(2^+\text{)}\) values are shown by the crosses.

shown for the \(2^+\) state in \(^{54}\text{Ca}\) was also recently measured [10]. When the GX1A Hamiltonian was developed, the experimental \(D_n\) value for \(^{51}\text{Ca}\) was almost half of the value predicted. But the new mass measurements for \(^{51,52}\text{Ca}\) [8] showed that the previous experimental mass for \(^{52}\text{Ca}\) was incorrect, and gave a \(D_n\) value for \(^{51}\text{Ca}\) that is now in good agreement with the predictions (as shown in figure 2). The new data point for \(^{53}\text{Ca}\) shown in the upper panel of figure 2 obtained from the new \(^{53,54}\text{Ca}\) mass measurements [9] turns out to be the lowest value for \(D_n\) observed in all nuclei with even \(Z\). The results are in good agreement with the KB3G and GX1A shell model predictions.

I will use the shell model with schematic and realistic Hamiltonians to understand the trends observed for \(D\), and in particular the low value for \(^{53}\text{Ca}\). This will be used to qualitatively understand the trends for minima in the \(D\) values for all nuclei.

To obtain insight into the reasons for the patterns observed in figure 2, I start with the simple “surface-delta-function” (SDI) model for the interaction [11]. The SDI differs from the delta interaction by the replacement of the radial integrals by a constant. The results for \(D\) obtained with SDI when the single-particle energies are degenerate are shown in the bottom panel of figure 3. The SDI \(J = 0\), \(T = 1\) two-body matrix elements for orbitals \(a\) and \(b\) with spins \(j_a\) and \(j_b\) are

\[
\langle aa | \text{SDI} | bb \rangle = C\sqrt{(2j_a + 1)(2j_b + 1)},
\]

where \(C\) is a constant. The interaction strength \(C\) is chosen to give a value for \(D\) that will turn out to be similar to that obtained with
the effective $pf$ shell Hamiltonians for calcium. The excitation energies of the $2^+$ states are also constant with the SDI and degenerate single-particle energies. The results for $D$ would be the same if the $pf$ orbitals were replaced by a single orbital with $j = 19/2$, but the constant $2^+$ energy would be higher (3 MeV). The $D$ value is determined by the number of $m$ states that participate in the pairing. The interaction energies obtained with SDI are $E(n) = nV_o \frac{\nu_2}{2}$ for even $n$, and $E(n) = (n-1)\nu_2$ for odd $n$, and thus $D = -\nu_o$. $\nu_o$ is the paired interaction strength for two particles with $J = 0$. All of the odd $N$ nuclei have four degenerate states with $J^\pi = 1/2^-, 3/2^-, 5/2^-$ and $7/2^-$. 

Figure 3. $D_n$ and $E_x(2^+)$ obtained with the SDI interaction as a function of the shell gap between the $0f_{7/2}$ orbital and the degenerate ($0f_{5/2}, 1p_{3/2}, 1p_{1/2}$) set of orbitals. The second panel from the top also shows the results for $D_n$ (thin lines) obtained when the shell gap is infinite. The results for the MSDI interaction are shown in the top panel.

Figure 4. $D_n$ and $E_x(2^+)$ obtained with the SDI interaction as a function of the shell gaps below and above $1p_{1/2}$ orbital. The dip is at $N = 33$. 

Figure 3 shows the numerical results obtained as the shell gap between the $0f_{7/2}$ orbital and a degenerate group of $0f_{5/2}, 1p_{3/2}, 1p_{1/2}$ orbitals is increased from zero to four MeV. The value of $D_n$ at $N = 28$ begins to rise when the value of the shell gap becomes greater than $D_o$. In the infinite gap limit the total original value of 4 MeV is divided between the lower group (1.6 MeV for the $0f_{7/2}$) and the upper group (2.4 MeV for $0f_{5/2}, 1p_{3/2}, 1p_{1/2}$). When the shell gap becomes large at $N = 28$ the value of $D$ is nearly equal to that of the $2^+$ excitation energy. Experiment and theory in figure 2 show clear signatures of shell gaps at $N = 28$ and $N = 32$. The GX1A calculation also shows a shell gap at $N = 34$ in contrast to the KB3G and NN+3N models that do not show a gap at $N = 34$. Mass measurements for the more neutron-rich calcium isotopes are required for the experimental value at $N = 34$. The experimental energy of the $2^+$ state in
Figure 5. Experimental values of $D_o$ for the Sn isotopes. The crosses show the energies of the $2^+$ states.

$^{54}$Ca [10] and indirect evidence from production cross section data [12] indicate that the shell gap at $N = 34$ is about 0.5 MeV smaller than that given by GX1A.

In order to obtain a dip in the $D_o$ value one needs to make shell gaps below and above a low-$j$ orbital. This is shown in figure 4 where there are three groups of orbitals with $1p_{1/2}$ in the middle split by one and two MeV from the other orbitals. In the limit of a completely isolated $1p_{1/2}$ orbital $D_o = -C \sqrt{(2j_a + 1)(2j_b + 1)} = -2C = 0.4$ MeV.

The main defect of the SDI model is the lack of oscillations in the $D$ values that are observed in experiment. In the 1960's this was recognized as a basic failure of the delta and SDI Hamiltonians. It was empirically fixed by adding a constant to the interaction to make the so-called modified-delta interaction [13] of the form $V(| \vec{r}_1 - \vec{r}_2 |) = A \delta(| \vec{r}_1 - \vec{r}_2 |) + B$. With the SDI form of the radial integral this becomes the so-called modified-surface-delta interaction (MSDI). The modern interpretation of this constant is that it comes from core-polarization corrections and three-body interactions. It is essential to obtain a good saturation property for the binding energies. The $D$ values obtained with $B = 0.2$ MeV and with a shell gap of four MeV are shown at the top of figure 3. This constant simply adds a term $n(n-1)B/2$ to the all energies. Thus, $D_o = -V_o - B$ for odd $n$ and $D_e = -V_o + B$ for even $n$. Half the sum of neighboring even and odd $D$ gives the pairing contribution: $D_a = \frac{1}{2}[D_e(N) + D_o(N-1)] = -V_o$, and half of the difference gives quadratic dependence: $D_b = \frac{1}{2}[D_e(N) - D_o(N-1)] = B$. Figure 2-5 in Bohr and Mottelson [2] is based upon the $D_a$ combination ($\Delta = D_a/2$).

Thus, the MSDI model with shell gaps gives a semi-quantitative understanding of all trends observed in $D_{e/o}$. The low $D_o$ value for $^{55}$Ca observed in experiment and theory in figure 2 is due to occupation of the $1p_{1/2}$ orbital at $N = 33$. The experimental value for $^{55}$Ca is $D_o = 0.65(10)$ MeV compared to the calculated values (in MeV) of 1.170 (MBPT), 0.425 (GX1A) and 0.489 (KB3G). The key quantity for the effective Hamiltonians is the $\langle (1p_{1/2})^2 | V | (1p_{1/2})^2 \rangle$ effective two-body matrix element. It is 0.151 MeV for KB3G and 0.053 MeV for GX1A compared to -0.20 MeV with the MSDI model. If the $1p_{1/2}$ orbital was completely isolated, its $D_o$ value would be negative for GX1A and KB3G. The small positive $D_o$ values obtained with GX1A and KB3G are due to mixing with the other orbitals.
Experimental values of $D_{no}$ for all nuclei with even $Z$. Values are plotted as a function of the number of neutrons ($N > Z$) and connected by lines for a given $Z$ value. The value for $^{53}$Ca is shown by the red dot. The vertical lines show the location of the magic numbers 20, 28, 50, 82 and 126.

Figure 6. Experimental values of $D_{no}$ for all nuclei with even $Z$. Values are plotted as a function of the number of neutrons ($N > Z$) and connected by lines for a given $Z$ value. The value for $^{53}$Ca is shown by the red dot. The vertical lines show the location of the magic numbers 20, 28, 50, 82 and 126.

3. The Tin Isotopes

The experimental results for the Sn isotopes are shown in figure 5. One observes a small dip at $N = 65$ due to small shell gaps around the $2s_{1/2}$ orbital with the $1d_{5/2}, 0g_{7/2}$ orbitals just below and the $1d_{3/2}, 0h_{11/2}$ orbitals just above. The well-known neutron shell gap appears at $N = 82$. The reduction in the pairing from $N = 81$ to $N = 83$ is due to the high number of $m$ states associated with the nearly degenerate $2s_{1/2}, 1d_{3/2}, 0h_{11/2}$ set of orbitals just below $N = 82$ (18 $m$ states) compared to the relatively isolated $1f_{7/2}$ orbital just above $N = 82$ (with 8 $m$ states). As shown in [14], for $Z = 54$ ($\text{Xe}$) $D$ is about the same at $N = 81$ and $N = 83$. This change of slope between $Z = 50$ and $Z = 54$ is described by the shell-model calculations carried out by the Naples group [15]. Part of the is due to the spreading out of the effective single-particle energies for the $2s_{1/2}, 1d_{3/2}, 0h_{11/2}$ orbitals as $Z$ increases. The energy of the $2s_{1/2}$ orbital moves above those of the other two orbitals, resulting in some isolation of the low-$j$ $2s_{1/2}$ state for $Z = 54$. This change was termed an “anomaly” in [14] because it could not be understood on the basis of energy-density functional (EDF) calculations. What the “anomaly” indicates is that the single-particle energies are incorrect in those EDF models.

4. Global Properties

The experimental $D_{no}$ values for all nuclei obtained from the 2012 mass table [7] and the new calcium experiment [9] are shown in figure 6. Plots of the experimental $D_{o}$ values for each isotope are shown on my website [16]. The value of $D_{o}$ for $^{53}$Ca at $N = 33$ is the lowest value obtained for all nuclei and ties with that of $^{207}$Pb that has a value of $D_{o} = 0.63$ MeV. Due to the change in nuclear size, the SDI pairing interaction strength decreases roughly as $(A)^{-1/2}$. With this scaling the minimum is 50% lower in $^{53}$Ca compared to $^{207}$Pb.

The minimum at $N = 33$ is only observed for calcium ($Z = 20$). With increasing $Z$, the $1p_{1/2}$ and $0f_{5/2}$ orbitals cross over, and at $N = 39$ there is a small minimum in $D_{o}$ coming from gaps around the $1p_{3/2}$ orbital in $^{68}$Ni ($Z = 28$) and $^{70}$Zn ($Z = 30$). Robust minima are observed for $N = 15$ due to the $1s_{1/2}$ orbital and $N = 127$ due to the $2p_{1/2}$ orbital. All of these nuclei have a $J = 1/2$ spin. The $0d_{5/2} - 1s_{1/2}$ gap disappears for carbon ($Z = 6$) [17], and the $N = 15$ dip should be gone for $^{21}$C. Mass measurements of $^{21,22}$C are required.

Another relatively low value of $D_{o}$ occurs for $N = 57$ at $^{97}$Zr, and is associated with a
relatively isolated neutron $2s_{1/2}$ orbital at that point. The dip at $N = 57$ only occurs for $Z = 40$. As $Z$ increases, the energy of the $2s_{1/2}$ orbital crosses above the energy of the $0g_{7/2}$ orbital and creating a small dip at $N = 65$ ($57+8$) in $^{115}$Sn (figure5). As $Z$ decreases, the energy of the $2s_{1/2}$ orbital should cross under the energy of the $1d_{5/2}$ orbital, creating a minimum at $N = 51$ ($57-6$) for $Z = 28$. Mass measurements of $^{78-80}$Ni are required. It is remarkable that the energy of the neutron $2s_{1/2}$ orbital migrates from being the level just above the Fermi surface at $Z = 28$ to the level just below the Fermi surface at $Z = 54$. This is due to the $\ell$-dependence of the single-particle energies in a Woods-Saxon type potential.

The experimental $D_{po}$ values for all nuclei obtained from the 2012 mass table [7] are shown in figure7. There are robust minima at $Z = 7$ (nitrogen) due to the $0p_{1/2}$ orbital, at $Z = 15$ (phosphorus) due to the $1s_{1/2}$ orbital, and at $Z = 81$ (thallium) due to the $2s_{1/2}$ orbital. The robust minimum at $Z = 29$ (copper) is due to the isolated $1p_{3/2}$ orbital. The minimum at $Z = 39$ (yttrium) starts at $N = 38$ and is due to the $2p_{1/2}$ orbital. $^{87-97}$Y all have $1/2^-$ ground state spins. Generally the shell gaps are smoothed out by the energy splitting of the Nilsson orbitals in deformed nuclei. However the $Z = 39$ dip remains in $^{99-101}$Y which are presumably deformed and have uncertain ground-state spins.

5. Summary
In summary, I have discussed how the oscillations in the neutron separation energy depend on shell gaps and effective interactions. The $D_{no}$ value obtained for $^{53}$Ca is the smallest value yet measured and is in good agreement with predictions based on previously obtained effective Hamiltonians for the $pf$ shell. Its small value is due to shell gaps on both sides of the low-$J$ $0p_{1/2}$ orbital. Other mass regions where small $D_o$ values are observed due to isolated low-$J$ orbitals were discussed.

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