NOTES ON THE CONSTRUCTION OF THE MODULI SPACE OF CURVES

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The purpose of these notes is to discuss the problem of moduli for curves of genus \( g \geq 3 \) \(^1\) and outline the construction of the (coarse) moduli scheme of stable curves due to Gieseker. We present few complete proofs. Instead we try and explain the subtleties and give precise references to the literature. The notes are broken into 4 parts.

In Section 1 we discuss the general problem of constructing a moduli “space” of curves. We will also state results about its properties, some of which will be discussed in the sequel.

We begin Section 2 by recalling from [DM] (and also [Vi]) the definition of a groupoid, and define the moduli groupoid of curves, as well as the quotient groupoid of a scheme by a group. We then give the conditions required for a groupoid to be a stack, and prove that the quotient groupoid of a scheme by a group is a stack. After discussing properties of morphisms of stacks, we define a Deligne-Mumford stack and prove that if a group acts on a scheme so that the stabilizers of geometric points are finite and reduced then the quotient stack is Deligne-Mumford. We conclude Section 2 with the definition of the moduli space of a Deligne-Mumford stack.

In Section 3 the notion of a stable curve is introduced, and we define the groupoid of stable curves. The groupoid of smooth curves is a sub-groupoid. We then prove that the groupoid of stable curves of genus \( g \geq 3 \) is equivalent to the quotient groupoid of a Hilbert scheme by the action of the projective linear group with finite, reduced stabilizers at geometric points. By the results of the previous section we can conclude the the groupoid of stable curves is a Deligne-Mumford stack defined over \( Spec \mathbb{Z} \) (as is the groupoid of smooth curves). We also discuss the results of [DM] on the irreducibility of the moduli stack.

In Section 4 we prove that a geometric quotient of a scheme by a group is the moduli space for the quotient stack. We then discuss the method of geometric invariant theory for constructing geometric quotients for the actions of reductive groups. Finally, we briefly outline

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\(^1\)Because every curve of genus 1 and 2 has non-trivial automorphisms, the problem of moduli is more subtle in this case than for curves of higher genus.
Gieseker’s approach to constructing the coarse moduli space over an algebraically closed field as the quotient of the aforementioned Hilbert scheme.

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1. Basics

Definition 1.1. Let $S$ be a scheme. By a smooth curve of genus $g$ over $S$ we mean a proper, flat, family $C \to S$ whose geometric fibers are smooth, connected 1-dimensional schemes of genus $g$.

Note: By the genus of a smooth, connected curve $C$ over an algebraically closed field, we mean $\dim H^0(C, \omega_C) = \dim H^1(C, \mathcal{O}_C) = g$, where $\omega_C$ is the sheaf of regular 1-forms on $C$. If the ground field is $\mathbb{C}$, then $C$ is a smooth, compact Riemann surface, and the algebraic definition of genus is the same as the topological one.

The basic problem of moduli is to classify curves of genus $g$. As a start, it is desirable to construct a space $\mathcal{M}_g$ whose geometric points represent all possible isomorphism classes of smooth curves. In the language of complex varieties, we are looking for a space that parametrizes all possible complex structures we can put on a fixed surface of genus $g$.

However, as modern (post-Grothendieck) algebraic geometers we would like $\mathcal{M}_g$ to have further functorial properties. In particular given a scheme $S$, a curve $C \to S$ should correspond to a morphism of $S$ to $\mathcal{M}_g$ (when $S$ is the spectrum of an algebraically closed field this is exactly the condition of the previous paragraph).

In the language of functors, we are trying to find a scheme $\mathcal{M}_g$ which represents the functor $\mathcal{F}_{\mathcal{M}_g} : \text{Schemes} \to \text{Sets}$ which assigns to a scheme $S$ the set of isomorphism classes of smooth curves of genus $g$ over $S$.

Unfortunately, such a moduli space cannot exist because some curves have non-trivial automorphisms. As a result, it is possible to construct non-trivial families $C \to B$ where each fiber has the same isomorphism class. Since the image of $B$ under the corresponding map to the moduli space is a point, if the moduli space represented the functor $\mathcal{F}_{\mathcal{M}_g}$ then $C \to B$ would be isomorphic to the trivial product family.

Given a curve $X$ and a non-trivial (finite) group $G$ of automorphisms of $X$ we construct a non-constant family $C \to B$ where each fiber is
isomorphic to $X$ as follows. Let $B'$ be a scheme with a free $G$ action, and let $B = B'/G$ be the quotient. Let $C' = B' \times X$. Then $G$ acts on $C'$ by acting as it does on $B'$ on the first factor, and by automorphism on the second factor. The quotient $C'/G$ is a family of curves over $B$. Each fiber is still isomorphic to $X$, but $C$ will not in general be isomorphic to $X \times B$.

There is however, a coarse moduli scheme of smooth curves. By this we mean:

**Definition 1.2.** ([GIT, Definition 5.6, p.99]) There is a scheme $M_g$ and a natural transformation of functors $\phi : F_{M_g} \to \text{Hom}(\underline{\cdot}, M_g)$ such that

1. For any algebraically closed field $k$, the map $\phi : F_{M_g}(\Omega) \to \text{Hom}(\Omega, M_g)$ is a bijection, where $\Omega = \text{Spec} k$.
2. Given any scheme $M$ and a transformation $\psi : F_{M_g} \to \text{Hom}(\underline{\cdot}, M)$ there is a unique transformation $\chi : \text{Hom}(\underline{\cdot}, M_g) \to \text{Hom}(\underline{\cdot}, M)$ such that $\psi = \chi \circ \phi$.

The existence of $\phi$ means that given a family of curves $C \to B$ there is an induced map to $M_g$. We do not require however, that a map to moduli gives a family of curves (as we have already seen a non-constant family with iso-trivial fibers). However, condition 1 says that giving a curve over an algebraically closed field is equivalent to giving a map of that field into $M_g$.

Condition 2 is imposed so that the moduli space is a universal object.

In his book on geometric invariant theory Mumford proved the following theorem.

**Theorem 1.1.** ([GIT, Chapter 5]) Given an algebraically closed field $k$ there is a coarse moduli scheme $M_g$ of dimension $3g - 3$ defined over $\text{Spec} k$, which is quasi-projective and irreducible.

The proof of this theorem will be subsumed in our general discussion of the construction of $\overline{M}_g$, the moduli space of stable curves.

A natural question to ask at this point, is whether $M_g$ is complete (and thus projective). The answer is no. It is quite easy to construct curves $C \to \text{Spec} \mathcal{O}$ with $\mathcal{O}$ a D.V.R. which has function field $K$, where $C$ is smooth, but the fiber over the residue field is singular. Since $C$ is smooth, the restriction $C_K \to \text{Spec} K$ is a smooth curve over $\text{Spec} K$, so there is a map $\text{Spec} K \to M_g$. The existence of such a family does not prove anything, since we must show that we can not replace the special fiber by a smooth curve. The total space $\tilde{C}$ of a family with modified special fiber is birational to $C$. Since $C$ and $\tilde{C}$ are surfaces (being curves over 1-dimensional rings) there must be a sequence of
birational transformations (centered in the special fiber) taking one to the other. It appears therefore, that it suffices to construct a family $C \to \text{Spec } \mathcal{O}$ such that no birational modification of $C$ centered in the special fiber will make it smooth. Unfortunately, because $\mathcal{M}_g$ is only a coarse moduli scheme, the existence of such a family does not prove that $\mathcal{M}_g$ is incomplete. The reason is that there may be a map $\text{Spec } \mathcal{O} \to \mathcal{M}_g$ extending the original map $\text{Spec } K \to \mathcal{M}_g$ without there being a family of smooth curves $\hat{C} \to \text{Spec } \mathcal{O}$ extending $C_K \to \text{Spec } K$. However, we will see when we discuss the valuative criterion of properness for Deligne-Mumford stacks that it suffices to show that for every finite extension $K \subset K'$ we can not complete the induced family of smooth curves $C_{K'} \to \text{Spec } K'$ to a family of smooth curves $C' \to \text{Spec } \mathcal{O}'$, where $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in $K'$.

**Example:** The following family shows that $\mathcal{M}_3$ is not complete. It can be easily generalized to higher genera. Consider the family $x^4 + xyz^2 + y^4 + t(z^4 + z^3x + z^3y + z^2y^2)$ of quartics in $\mathbb{P}^2 \times \text{Spec } \mathcal{O}$ where $\mathcal{O}$ is a D.V.R. with uniformizing parameter $t$. The total space of this family is smooth, but over the closed point the fiber is a quartic with a node at the point $(0 : 0 : 1) \in \mathbb{P}^2$. Moreover, even after base change, any blow-up centered at the singular point of the special fiber contains a $(-2)$ curve, so there is no modification that gives us a family of smooth curves.

Since $\mathcal{M}_g$ is not complete, a natural question is to ask whether it is affine. The answer again is no. This follows from the fact that $\mathcal{M}_g$ has a projective compactification in the moduli of abelian varieties, such that boundary has codimension 2. In particular, this means that there are complete curves in $\mathcal{M}_g$. On the other hand, Diaz proved the following theorem ([Di]).

**Theorem 1.2.** Any complete subvariety of $\mathcal{M}_g$ has dimension less than $g - 1$.

It is not known how close this bound is to being sharp.

Finally we state a spectacular theorem due largely to Harris and Mumford.

**Theorem 1.3.** For $g > 23$, $\mathcal{M}_g$ is of general type.

The importance of this theorem is that until its proof, people believed that $\mathcal{M}_g$ was rational, or at least unirational. The reason for this belief was that for $g \leq 10$, the (uni)rationality of $\mathcal{M}_g$ had been affirmed by the Italian school.
2. **Stacks**

Let $S$ be a scheme, and let $\mathcal{S} = (\text{Sch}/S)$ be the category of schemes over $S$.

2.1. **Groupoids.**

**Definition 2.1.** A category over $S$ is a category $F$ together with a functor $p_F : F \rightarrow S$. If $B \in \text{Obj}(\mathcal{S})$ we say $X$ lies over $B$ if $p_F(X) = B$.

**Definition 2.2.** (see also [Vi, Definition 7.1]) If $(F, p_F)$ is a category over $S$, then it is a groupoid over $S$ if the following conditions hold:

1. If $f : B' \rightarrow B$ is a morphism in $S$, and $X$ is an object of $F$ lying over $B$, then there is an object $X'$ over $B'$ and a morphism $\phi : X' \rightarrow X$ such that $p_F(\phi) = f$.
2. Let $X, X', X''$ be objects of $F$ lying over $B, B', B''$ respectively. If $\phi : X' \rightarrow X$ and $\psi : X'' \rightarrow X$ are morphisms in $F$, and $h : B' \rightarrow B''$ is a morphism such that $p_F(\psi) \cdot h = p_F(\phi)$ then there is a unique morphism $\chi : X' \rightarrow X''$ such that $\psi \cdot \chi = \phi$ and $p_F(\chi) = h$.

Note that condition (2) implies that a morphism $\phi : X' \rightarrow X$ of objects over $B'$ and $B$ respectively is an isomorphism if and only if $p_F(\phi) : B' \rightarrow B$ is an isomorphism. (To see that $p_F(\phi)$ being an isomorphism is sufficient to ensure that $\phi$ is an isomorphism, apply condition (2) where one of the maps is $p_F(\phi)$ and the other the identity, and lift $p_F(\phi)^{-1} : B \rightarrow B'$ to $\phi^{-1} : X \rightarrow X'$. The other direction is trivial.) Define $F(B)$ to be the subcategory consisting of all objects $X$ such that $p_F(X) = B$ and morphisms $f$ such $p_F(f) = id_B$. Then $F(B)$ is a groupoid; i.e. a category where all morphisms are isomorphisms. This is the reason we say $F$ is a groupoid over $S$. Also note that condition (2) implies that the object $X'$ over $B'$ in condition (1) is unique up to canonical isomorphism. This object will be called the pull-back of $X$ via $f$ and denoted $f^*X$.

**Example:** If $F : S \rightarrow \text{Sets}$ is a contravariant functor, then we can associate a groupoid (also called $F$) whose objects are pairs $(B, \beta)$ where $B$ is an object of $S$ and $\beta \in F(B)$. A morphism $(B', \beta') \rightarrow (B, \beta)$ is an $S$-morphism $f : B' \rightarrow B$ such that $F(f)(\beta) = \beta'$. In this case $F(B)$ in the groupoid sense is just the set $F(B)$ in the functor sense; i.e. all morphisms in the groupoid $F(B)$ are the identity. In particular, any $S$-scheme $B$ is a groupoid via its functor of points $\text{Hom}(\underline{\cdot}, B)$.

**Example:** If $X/S$ is a scheme and $G/S$ is a group scheme acting on $X$ then we define the quotient groupoid $[X/G]$ as follows. The sections
of \([X/G]\) over \(B\) are \(G\)-principal bundles \(E \to B\) together with a \(G\)-equivariant map \(E \to X\). A morphism from \(E' \to B'\) to \(E \to B\) is a commutative diagram

\[
\begin{array}{ccc}
E' & \to & E \\
\downarrow & & \downarrow \\
B' & \to & B
\end{array}
\]

such that \(E' \simeq E \times_B B'\). If the action is free and a quotient scheme \(X/G\) exists, then there is an equivalence of categories between \([X/G]\) and the groupoid associated to the scheme \(X/G\).

**Example:** Of particular importance to us is the groupoid \(F_{M_g}\) defined over \(\text{Spec } \mathbb{Z}\). The objects of \(F_{M_g}\) are smooth curves as defined in part 1. A morphism from \(X' \to B'\) to \(X \to B\) is a commutative diagram

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
B' & \to & B
\end{array}
\]

which induces an isomorphism \(X' \simeq X \times_B B'\). The functor \(\mathcal{M}_g \to \text{Sch}/\mathbb{Z}\) sends \(X \to B\) to \(B\). We will eventually prove that \(F_{M_g}\) is a quotient groupoid as in the previous example.

**Warning:** The groupoid we have just defined is not the groupoid associated to the moduli functor we defined in Part 1. The groupoid here is not a functor, since if \(X/B\) is a curve with non-trivial automorphisms \(F(B)\) will not be a set because there are morphisms which are not the identity. (A set is a groupoid where all the morphisms are the identity.)

### 2.2. Definition of a stack

Let \((F,p_F)\) be a groupoid. Let \(B\) be an \(S\)-scheme and let \(X\) and \(Y\) be any objects in \(F(B)\). Define a functor \(\text{Iso}_B(X,Y) : \text{Sch}/B \to \text{Sets}\) by associating to any morphism \(f : B' \to B\), the set of isomorphisms in \(F(B')\) between \(f^*X\) and \(f^*Y\).

If \(X = Y\) then \(\text{Iso}_B(X,X)\) is the functor whose sections over \(B'\) mapping to \(B\) are the automorphisms of the pull-back of \(X\) to \(B'\).

In the case of curves Deligne and Mumford proved that \(\text{Iso}_B(X,X)\) is represented by a scheme \(\text{Iso}_B(X,Y)\), because \(X/B\) and \(Y/B\) have canonical polarizations ([DM, p.84]). When \(X = Y\) then Deligne and Mumford prove directly that the \(\text{Iso}_B(X,Y)\) is finite and unramified over \(B\) ([DM, Theorem 1.11]). Applying the theorem to \(B = \text{Spec } k\), where \(k\) is an algebraically closed field, this theorem proves that every curve has a finite automorphism group.

Note that the scheme \(\text{Iso}_B(X,X)\) need not be flat over \(B\). For example, if \(X/B\) is a family of curves, the number of points in the
fibers of $\text{Iso}_B(X, X)$ over $B$ will jump over the points $b \in B$ where the fiber $X_b$ has non-trivial automorphisms.

**Definition 2.3.** A groupoid $(F, p_F)$ over $S$ is a stack if

1. $\text{Iso}_B(X, Y)$ is a sheaf in the étale topology for all $B$, $X$ and $Y$.
2. If $\{B_i \to B\}$ is a covering of $B$ in the étale topology, and $X_i$ is a collection of objects in $F(B_i)$ with isomorphisms
   \[ \phi_{ij} : X_{j|B_i \times_B B_j} \to X_{i|B_i \times_B B_j} \]
   in $F(B_i \times_B B_j)$ satisfying the cocycle condition. Then there is an object $X \in F(X)$ with isomorphisms $X|_{B_i} \sim X_i$ inducing the isomorphisms $\phi_{ij}$ above.

Note: If $F$ is a functor, then condition (1) is satisfied since $\text{Iso}_B(X, Y)$ will always be either the empty sheaf or the constant sheaf. In this case condition (2) just asserts that the functor is a sheaf in the étale topology. Condition (2) is not immediate and may easily fail if $F$ is not representable (A representable functor will be a stack, since condition (2) is equivalent to saying that the functor of points is a sheaf in the étale topology). For example, the moduli functor we defined in Part 1 is not a stack, since it doesn’t satisfy condition (2). In particular, as noted above, given a curve $C$ with automorphism group $G$ and $B' \to B$ Galois with group $G$, there are two ways to descend the family $C \times B'/B'$ to a family over $B$, so a section of $F$ over $B$ is not determined by its pull-back to an étale cover.

However, the moduli groupoid defined above is a stack. We will not prove this here, but instead we will prove that the moduli groupoid is the quotient groupoid of a scheme by $\text{PGL}(N + 1)$.

**Proposition 2.1.** (cf. [Vi, Example 7.17]) The groupoid $[X/G]$ defined above is a stack.

**Proof:** Let $e, e'$ be sections of $[X/G](B)$ corresponding to principal bundles $E \to B$ and $E' \to B$ and $G$-maps $f : E \to X$ and $f' : E' \to X$. Then $\text{Iso}_B(e, e')$ is empty unless $E = E'$ and $f = f'$. If $e = e'$, then the isomorphisms correspond to elements of $g$ which preserve $f$. In other words, $\text{Iso}_B(e, e)$ is just the $G$-subgroup which is the stabilizer of the $G$-map $f : E \to X$ (see [GIT, Definition 0.4] for the definition of stabilizer).

The functor which associates to $G$-map $f : E \to X$ its stabilizer is represented by the scheme $\text{Stab}_X(G)$; i.e., the stabilizer of the identity map $X \to X$. Since $\text{Iso}_B(e, e')$ is represented by a scheme it is a sheaf in the étale topology.

Furthermore, a principal $E \to B$ is determined by étale descent, so condition (2) is satisfied. □
2.3. **Representable morphisms.** Most of the material in this section is taken from [DM, Section 4].

Let $F$ and $G$ be stacks over $S$. A morphism of stacks is just a functor of groupoids which commutes with the projection functor to $S$. If $f : F \to G$ and $h : H \to G$ are morphisms of stacks, then we can define the fiber product $F \times_G H$ as the groupoid whose sections over a base $B$ are pairs $(x, y) \in F(B) \times H(B)$ such that $f(x)$ is isomorphic to $h(y)$. It can be easily checked that this groupoid is a stack.

**Definition 2.4.** A morphism $f : F \to G$ of stacks is said to be representable if for any map of a scheme $B \to G$ the fiber product $F \times_G X$ is represented by a scheme.

**Remark:** When we say that a stack is a scheme, we mean that the stack is the stack associated to the functor of points of the scheme.

**Example:** There is a projection map $X \to [X/G]$ corresponding to the trivial $G$-bundle $X \times_S G$ on $X$. This morphism is representable because giving a map $B \to [X/G]$ is equivalent to giving a principal bundle $E \to B$. The fiber product $B \times_{[X/G]} X$ is just the the scheme $E$.

Let $P$ be a property of morphisms of schemes which is stable under base change and of a local nature on the target.

**Definition 2.5.** ([DM, Definition 4.3]) A representable morphism of stacks $f : F \to G$ has property $P$, if for all maps of scheme $B \to G$ the corresponding morphism of schemes $F \times_G B \to B$ has property $P$.

**Example:** The projection morphism $X \to [X/G]$ is smooth since for any $B \to [X/G]$ the corresponding map $E \to B$ is smooth because $E$ is a principal bundle over $E$.

2.4. **Definition of a Deligne-Mumford stack.**

**Definition 2.6.** A stack is Deligne-Mumford if

1. The diagonal $\Delta_X : F \to F \times_S F$ is representable, quasi-compact and separated.

2. There is a scheme $U$ and an étale surjective morphism $U \to F$. Such a morphism $U \to F$ is called an (étale) atlas.

**Remark.** In [DM], such a stack is called an algebraic stack. To conform to current terminology we use the term Deligne-Mumford stack. A more general class of stacks was studied by Artin, and they are now called Artin stacks. The basic difference is that an Artin stack need only have a smooth atlas. Condition (1) above is equivalent to the following condition:
(1’) Every morphism $B \to F$ from a scheme is representable, so condition (2) makes sense.

(This fact is stated in [DM] and proved in [Vi, Prop 7.13]).

**Remark:** Condition (2) asserts the existence of a universal deformation space for deformations over Artin rings.

Vistoli also proves the following proposition:

**Proposition 2.2.** [Vi, Prop 7.15] The diagonal of a Deligne-Mumford stack is unramified

As a consequence of this proposition we can prove [Vi, p. 666]

**Corollary 2.1.** If $F$ is a Deligne-Mumford stack, $B$ quasi-compact, and $X \in F(B)$ then $X$ has only finitely many automorphisms.

**Remark.** The are Artin stacks which are not Deligne-Mumford where each object has a finite automorphism group. In this case the diagonal is quasi-finite but ramified. Objects in the groupoid have *infinitesimal automorphisms*. This phenomenon only occurs in characteristic $p$, because all groups are smooth in characteristic 0.

Proof: Let $B \to F$ be map corresponding to $X$, and let $B \to F \times_S F$ be the composition with diagonal. The pullback $B \times_{F \times_S F} F$ can be identified with scheme $Iso_B(X, X)$. Since $F$ is a Deligne-Mumford stack the map $Iso_B(X, X)$ is unramified over $X$. Furthermore, since $B$ is quasi-compact, the map $Iso_B(X, X) \to X$ can have only finitely many sections. Therefore, $X$ has only finitely many automorphisms over $B$. □

**Note:** The above proof shows that diagonal $F \to F \times_S F$ is not in general an embedding, since $Iso_B(X, X)$ need not be isomorphic to $B$. It is however, a local embedding. This is the main technical difficulty in doing intersection theory directly on Deligne-Mumford stacks ([Vi]). However, using the equivariant intersection theory developed in [EG], one can avoid these difficulties for quotient stacks.

The following theorem is stated (but not proved) in [DM, Theorem 4.21]. We give the proof below with a slight additional assumption. This is the only proof in these notes which does not appear in the literature.

**Theorem 2.1.** Let $F$ be a quasi-separated stack over a Noetherian scheme $S$. Assume that

1. The diagonal is representable and unramified,
2. There exists a scheme $U$ of finite type over $S$ and a smooth surjective $S$-morphism $U \to F$.

Then $F$ is a Deligne-Mumford stack.
Remark: This theorem says that condition (1) and the existence of a versal deformation space (condition (2)) is actually equivalent to the existence of a universal deformation space.

Remark: We give the proof below under the additional assumption that the residue fields of the closed points of $S$ are perfect. In particular we prove the theorem for stacks of finite type over $\text{Spec} \ Z$. Using the theorem we will prove that the stack of stable curves is a Deligne-Mumford stack of finite type over $\text{Spec} \ Z$.

Proof. The only thing to prove is that $F$ has an étale atlas of finite type over $S$. Let $u \in U$ be any closed point in $f^{-1}(u)$. Set $I_{u} = \delta^{-1}(u \times_{S} u)$. Let $z \in U_{u}$ be a closed point which is separable (i.e. étale) over $u$. (The set of such closed points is dense in a smooth variety). Since $U_{u}$ is smooth, the point $z$ is cut out by a regular sequence in the local ring of $U_{u}$ at $z$.

The diagonal $\delta : F \to F \times_{S} F$ is unramified. Thus, the map $U_{u} \to u \times_{S} U$ obtained by pulling back the morphisms $u \times_{S} U \to F \times_{S} F$ along the diagonal is unramified. We assume $U$ is of finite type and that the residue fields of $S$ are perfect. Thus, $k(u)$ is a finite, hence separable, extension of the residue field of its image in $S$. Hence the morphism $u \times_{S} U \to U$ is unramified and so is the composition $U_{u} \to u \times_{S} U \to U$.

Let $x$ be the image of $z$ in $U$. By [EGA4 18.4.8] there are étale neighborhoods $W'$ and $U'$ of $x$ and $z$ respectively and a closed immersion $W' \hookrightarrow U'$ such that the diagram commutes

\[
\begin{array}{ccc}
W' & \hookrightarrow & U \\
\text{étale} & \downarrow & \text{étale} \\
U_{0} & \to & U
\end{array}
\]

Let $z'$ be any point lying over $y$. Let $Z_{u}$ be the closed subscheme of $U'$ defined by lifts to $\mathcal{O}_{U'}$ of the local equations for $z' \in W'$. By construction, $Z_{u}$ intersects $U'$ transversally at $z'$. We will show that the induced morphism $Z \to F$ is étale in a neighborhood of $Z$.

By definition, this means that for every map of a scheme $B \to F$, the induced map of schemes $B \times_{F} Z_{u} \to B$ is étale in a neighborhood of $z' \times_{F} Z_{u}$. Since $U \to F$ is smooth and surjective, it suffices to check that the morphism is étale after base change to $U$.

By construction, $Z_{u} \subset U'$ is cut out by a regular sequence in a neighborhood of $z' \in U'$ (since $z'$ is a smooth point of $W'$). Thus $Z_{u} \times_{F} U \to U' \times_{F} U$ is a regular embedding in a neighborhood of $z' \times_{F} U$. Since $U' \times_{F} U \to U'$ is smooth, we can apply [EGA4,Theorem 17.12.1], and conclude that $Z_{u} \times_{F} U' \to U'$ is smooth in a neighborhood of $z'$. Moreover, the relative dimension of this morphism is 0. Therefore, $Z_{u} \to F$ is étale in a neighborhood of $z'$.
Since \( U \) is of finite type over \( S \), the \( Z_u \)'s are as well. The union of the \( Z_u \)'s cover \( F \) (since their pullbacks via the morphism \( U \to F \) cover \( U \)). Also, \( U \) is Noetherian because it is of finite type over a Noetherian scheme. Thus a finite number of the \( Z_u \)'s will cover the \( F \). (To see this, we can pullback via the map \( U \to F \). The pullback of the \( Z_u \)'s form an étale cover of \( U \) which is Noetherian.) \( \square \)

**Remark:** Without the assumption that the residue fields of \( S \) are perfect we do not know that any point \( u \in U \) is actually unramified over \( F \). In this case we need to analyze the image of \( u \) in \( F \), which is not a point, but rather a gerbe over a point.

The theorem has a useful corollary.

**Corollary 2.2.** Let \( X/S \) be a Noetherian scheme of finite type and let \( G/S \) be a smooth group scheme acting on \( X \) with finite, reduced stabilizers, then \( [X/G] \) is a Deligne-Mumford stack.

Proof: The condition on the action ensures that \( \text{Iso}_B(E,E) \) is unramified over \( E \) for any map \( B \to [X/G] \) corresponding to the principal bundle \( E \to B \). This in turn implies that the diagonal is also unramified, so condition (1) is satisfied. Furthermore, condition (2) is satisfied by the smooth map \( X \to [X/G] \). \( \square \)

2.5. **Further properties of Deligne-Mumford stacks.** Not all morphisms of stacks are representable, so we can not define algebro-geometric properties of morphisms as we did for representable morphisms. However, if we consider morphisms of Deligne-Mumford stacks then we can define properties of morphisms as follows (see [DM, p. 100])

Let \( P \) be a property of morphisms of schemes which at source and target is of a local nature for the étale topology. This means that for any family of commutative squares

\[
\begin{array}{ccc}
X_i & \xrightarrow{g_i} & X \\
\downarrow{f_i} & & \downarrow{f} \\
Y_i & \xrightarrow{h_i} & Y
\end{array}
\]

where the \( g_i \) (resp \( h_i \)) are étale and cover \( X \) (resp. \( Y \)), then \( f \) has property \( P \) if and only if \( f_i \) has property \( P \) for all \( i \).

Examples of such properties are \( f \) flat, smooth, étale, unramified, locally of finite type, locally of finite presentation, etc.

Then if \( f : F \to G \) is any morphism of Deligne-Mumford stacks we say that \( f \) has property \( P \) if there are étale atlases \( U \to F, U' \to G \) and a compatible morphism \( U \to U' \) with property \( P \).

Likewise, if \( P \) is property of schemes which is local in the étale topology (for example regular, normal, locally Noetherian, of characteristic
then a Deligne-Mumford stack $F$ has property $P$ if for one (and hence every) étale atlas $U \to F$, the scheme $U$ has property $P$.

An open substack $F \subset G$ is a full subcategory of $G$ such that for any $x \in \text{Obj}(F)$, all objects in $G$ isomorphic to $x$ are also in $F$. Furthermore, the inclusion morphism $F \to G$ is represented by an open immersions In a similar way we can talk about closed (or locally closed) substacks.

Using these notions, we say that a map of Deligne-Mumford stacks $F \to G$ is separated if for any map of a scheme $B \to G$ the fiber product $F \times_G B$ is separated as a stack over $B$. It is proper if it is separated, of finite type and locally over $F$ there is a Deligne-Mumford stack $H \to F$ and a representable proper map $H \to G$ commuting with the projection to $F$ and the original map $F \to G$.

$$
\begin{array}{c}
H \\
\downarrow \\
F \to G
\end{array}
$$

**Remark:** By a theorem of Vistoli [Vi, Prop. 2.6] and Laumon–Moret-Baily [L-MB, Theorem 10.1] every Noetherian stack has a finite cover by a scheme. Using this fact we can say that a morphism $F \to G$ is proper if there is a finite cover $X \to F$ by a scheme such that the composition $X \to F \to G$ is a proper representable morphism. (Recall that any morphism from a scheme to a stack is representable). Similarly we say that a morphism $f : F \to G$ of Noetherian stacks is (quasi)-finite if for any finite cover $X \to F$, the composition $X \to F \to G$ is representable and (quasi)-finite.

As is the case with schemes, there are valuative criteria for separation and properness ([DM, Theorem 4.18-4.19]). The valuative criterion for separation is equivalent to the criterion for schemes, but we only construct an isomorphism between two extensions.

**Theorem 2.2.** A morphism $f : F \to G$ is separated iff the following condition holds:

For any complete discrete valuation ring $V$ and fraction field $K$ and any morphism $f : \text{Spec } V \to G$ with lifts $g_1, g_2 : \text{Spec } V \to F$ which are isomorphic when restricted to $\text{Spec } K$, then the isomorphism can be extended to an isomorphism between $g_1$ and $g_2$.

**Theorem 2.3.** A separated morphism $f : F \to G$ is proper if and only if for any complete discrete valuation ring $V$ with field of fractions $K$ and any map $\text{Spec } V \to G$ which lifts over $\text{Spec } K$ to a map to $F$,
there is a finite separable extension $K'$ of $K$ such that the lift extends to all of $\text{Spec } V'$ where $V'$ is the integral closure of $V$ in $K'$.

**Remark:** When applied to schemes, Theorem 2.3 appears to be stronger than the usual valuative criterion for properness. However, this is not the case, as it is easy to show that if there is a lift $\text{Spec } V' \to F$, then there is in fact a lift $\text{Spec } V \to F$, as long as the image of $V'$ is contained in an affine subscheme of $F$ - which is always the case if $F$ is a scheme.

Finally we conclude this section with the definition of the moduli space of a Deligne-Mumford stack. This definition is completely analogous to Mumford’s definition ([GIT, p. 99]) of a coarse moduli scheme mentioned above.

**Definition 2.7.** The moduli space of a Deligne-Mumford stack $F$ is a scheme $M$ together with a proper morphism $\pi : F \to M$, such that

(*) for any algebraically closed field $k$ there is a bijection between the connected components of the groupoid $F(\Omega)$ and $M(\Omega)$, where $\Omega = \text{Spec } k$.

Furthermore, $M$ is universal in the sense that if $F \to N$ is a proper map satisfying (*) there is a morphism $M \to N$ such that the map $F \to N$ factors through $\pi$.

**Remark:** The reader at this point may wonder why we need the valuative criterion for stacks as stated in Theorem 2.3. The difference is explained as follows. Let $F$ be a complete Deligne-Mumford stack whose sections are schemes. If $M$ be a moduli space $F$, then $M$ is also complete. Let $B = \text{Spec } \mathcal{O}$ where $\mathcal{O}$ is a DVR with function field $K$ and residue field $k$, and suppose there is a map $\text{Spec } K \to F$ corresponding to a section $X_K \to \text{Spec } K$ of $F$ over $\text{Spec } K$. Since $M$ is complete, the induced map $\text{Spec } K \to M$ can be extended to a map $B \to M$. However, there need not be a section $X \to B$ which restricts to $X_K \to \text{Spec } K$. We can only assert that there is a finite cover $B' \to B$ and a section $X' \to B'$ which restricts over the generic fiber to $X_K \times_B B'$.

3. **Stable curves**

In this section we discuss stable curves and the compactification of the moduli of curves to the moduli of stable curves.

**Definition 3.1.** [DM, Definition 1.1] A Deligne-Mumford stable (resp. semi-stable) curve of genus $g$ over a scheme $S$ is a proper flat family $C \to S$ whose geometric fibers are reduced, connected, 1-dimensional schemes $C_s$ such that:
(1) \( C_s \) has only ordinary double points as singularities.

(2) If \( E \) is a non-singular rational component of \( C \), then \( E \) meets the other components of \( C_s \) in more than 2 points (resp. in at least 2 points).

(3) \( C_s \) has arithmetic genus \( g \); i.e. \( \dim H^1(\mathcal{O}_{C_s}) = g \).

Remark: Clearly, a smooth curve of genus \( g \) is stable. Condition (2) ensures that stable curves have finite automorphism groups, so that we will be able to form a Deligne-Mumford stack out of the category of stable curves. We will not use the notion of semi-stable curves until we discuss geometric invariant theory in Section 4.

Denote by \( \mathcal{M}_g \) the groupoid over \( \text{Spec } \mathbb{Z} \) whose sections over a scheme \( B \) are families of stable curves \( X \to B \). As is the case with smooth curves, we define a morphism from \( X' \to B' \) to \( X \to B \) as a commutative diagram

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
B' & \to & B
\end{array}
\]

which induces an isomorphism \( X' \simeq X \times_B B' \).

3.1. **The stack of stable curves is a Deligne-Mumford stack.**

Let \( \pi : C \to S \) be a stable curve. Since \( \pi \) is flat and its geometric fibers are local complete intersections, the morphism is a local complete intersection morphism. It follows from the theory of duality that there is a canonical invertible dualizing sheaf \( \omega_{C/S} \) on \( C \). If \( C/S \) is smooth, then this sheaf is the relative cotangent bundle. The key fact we need about this sheaf is a theorem of Deligne and Mumford [DM, p. 78].

**Theorem 3.1.** Let \( C \to S \) be a stable curve of genus \( g \geq 2 \). Then \( \omega_{C/S}^{(n)} \) is relatively very ample for \( n \geq 3 \), and \( \pi_*(\omega_{C/S}^{(n)}) \) is locally free of rank \((2n-1)(g-1)\).

Remark: When \( \pi \) is smooth, the theorem follows from the classical Riemann-Roch theorem for curves. The general case is proved by analyzing the locally free sheaf obtained by restricting \( \omega_{C/S} \) to the geometric fibers of \( C/S \). In particular, if \( S = \text{Spec } k \), with \( k \) algebraically closed, then \( \omega_{C/S} \) can be described as follows. Let \( f : C' \to C \) be the normalization of \( C \) (note \( C' \) need not be connected). Let \( x_1, \ldots, x_n, y_1, \ldots, y_n \) be the points of \( C' \) such that the \( z_i = f(x_i) = f(y_i) \) are the double points of \( C \). Then \( \omega_{C/S} \) can be identified with the sheaf of 1-forms \( \eta \) on \( C' \) regular except for simple poles at the \( x \)'s and \( y \)'s and with \( \text{Res}_{x_i}(\eta) + \text{Res}_{y_i}(\eta) = 0 \).

As a result, every stable curve can be realized as a curve in \( \mathbb{P}^{N=(2n-1)(g-1)-1} \) with Hilbert polynomial \( P_{g,n}(t) = (2nt-1)(g-1) \). There is a subscheme...
Theorem 3.2. $F_{\mathcal{M}_g} = \left[ H_{g,n} / \text{PGL}(N+1) \right]$ and $F_{\overline{\mathcal{M}}_g} = \left[ \overline{H}_{g,n} / \text{PGL}(N+1) \right]$.

Note that the theorem asserts that the quotient is independent of $n$.

Proof: Given a family of stable curves $C \rightarrow B$, let $E \rightarrow B$ be the principal $\text{PGL}(N + 1)$ bundle associated to the projective bundle $\mathcal{P}^n(\pi_* (\omega_{C/B}^{\otimes n}))$. Let $\pi' : C \times B \rightarrow E$ be the pullback family. The pullback of this projective bundle to $E$ is trivial and is isomorphic to $\mathcal{P}^n(\pi'_* (\omega_{C \times B/E}^{\otimes n}))$, so there is a map $E \rightarrow \overline{H}_{g,n}$ which is clearly $\text{PGL}(N + 1)$ invariant. Thus there is a functor $F_{\mathcal{M}_g} \rightarrow \left[ \overline{H}_{g,n} / \text{PGL}(N+1) \right]$ which takes $F_{\mathcal{M}_g}$ to $\left[ H_{g,n} / \text{PGL}(N+1) \right]$.

The next step is to show that if $C$ is a stable curve then any automorphism of $C/B$ is induced by an automorphism of the projective bundle $\mathcal{P}^n(\omega_{C/B}^{\otimes n})$. This is proved for smooth curves in [GIT, Proposition 5.2], and is easily generalized to stable curves because $\pi_* (\omega_{C/B}^{\otimes n})$ has the same properties as in the smooth case. It then follows that $F_{\mathcal{M}_g}$ is a full subcategory of the quotient $\left[ H_{g,n} / \text{PGL}(N+1) \right]$.

Now if $E \rightarrow B$ is a section of $\left[ \overline{H}_{g,n} / \text{PGL}(N+1) \right]$ then there is a family $C_E \rightarrow E$ of curves of genus $g$ together with an isomorphism $\mathcal{P}^n(\pi_* (\omega_{C_E/B}^{\otimes n})) \simeq \mathcal{P}^n_E$. Now $\text{PGL}(N+1)$ acts by changing the isomorphism. In particular, if $g \in \text{PGL}(N+1)$ and $e \in E$ then the fiber of $\pi_E$ is the same over $e$ as it is over $ge$. We can therefore form a quotient $C/B$ such that $C_E \simeq C \times B E$. Hence we have defined a section of $F_{\overline{\mathcal{M}}_g}$, so there is an equivalence of categories between $F_{\mathcal{M}_g}$ and $\left[ \overline{H}_{g,n} / \text{PGL}(N+1) \right]$ as desired. □

Corollary 3.1. $F_{\mathcal{M}_g}$ and $F_{\overline{\mathcal{M}}_g}$ are Deligne-Mumford stacks.

Proof: We have just shown that $F_{\mathcal{M}_g}$ and $F_{\overline{\mathcal{M}}_g}$ are quotients of a scheme by a smooth group, so they have smooth atlases. Every stable curve defined over an algebraically closed field has a finite and reduced automorphism group, so the diagonal is unramified. Therefore, they are Deligne-Mumford stacks by theorem 2.1. □
3.2. Properness of $\overline{F}_{\mathcal{M}_g}$. Given that $\overline{F}_{\mathcal{M}_g}$ is a Deligne-Mumford stack, the valuative criterion of properness and the following stable reduction theorem show that it is proper over $Spec \, \mathbb{Z}$.

**Theorem 3.3.** Let $B$ be the the spectrum of a DVR with function field $K$, and let $X \to B$ be a family of curves such that its restriction $X_K \to Spec \, K$ is a stable curve. Then there is a finite extension $K'/K$ and a unique stable family $X' \to B \times_K K'$ such that the restriction $X' \to Spec \, K'$ is isomorphic to $X_K \times_K K'$.

Remarks on the proof of Theorem 3.3. This theorem was originally proved (but not published) in characteristic zero by Mumford and Mayer ([GIT, Appendix D]). There is a relatively straightforward algorithmic version of this theorem in characteristic 0 which I learned from Joe Harris. Blow up the singular points of the special fiber of $X/B$ until the total space of the family is smooth and the special fiber has only nodes as singularities. The modified special fiber will have a number of components with positive multiplicity coming from the exceptional divisors in the blowups. Next, do a base change of degree equal to the g.c.d. of the multiple components. After base change all components of the special fiber will have multiplicity 1. Then contract all (-1) and (-2) rational components in the total space. The special fiber is now stable. Furthermore, the total space of the new family is a minimal model for the surface. Since minimal models of surfaces are unique, the stable limit curve is unique. □

This algorithmic proof fails in characteristic $p > 0$, because after blowing up some components of the special fiber may have multiplicity divisible by $p$. In this case, it will not be possible to make the component become reduced after base change.

Deligne and Mumford proved the stable reduction theorem in arbitrary characteristic using Neron models of the Jacobians of the curves ([DM]). Later Artin and Winters gave a direct geometric proof using the theory of curves on surfaces.

3.3. Irreducibility of $F_{\mathcal{M}_g}$ and $\overline{F}_{\mathcal{M}_g}$. Using the description of the moduli stacks as quotients of $H_g$ and $\overline{H}_g$ we can deduce properties of the stacks from the corresponding properties of the Hilbert scheme. In particular, deformation theory shows that $H_g$ and $\overline{H}_g$ are smooth over $Spec \, \mathbb{Z}$ ([DM, Cor 1.7]). Since the map $H_g \to F_{\mathcal{M}_g}$ (resp. $\overline{H}_g \to F_{\mathcal{M}_g}$) is smooth we see that $\overline{F}_{\mathcal{M}_g}$ is smooth.

Further analysis ([DM, Cor 1.9]) shows that the scheme $\overline{H}_g - H_g$ representing polarized, singular, stable curves is a divisor with normal crossings in $\overline{H}$. This property descends to the moduli stacks.
Theorem 3.4. [DM, Thm 5.2] \( F_{\mathcal{M}_g} \) is smooth and proper over \( \text{Spec} \ Z \). The complement \( F_{\mathcal{M}_g} - F_{\mathcal{M}_g} \) is a divisor with normal crossings in \( F_{\mathcal{M}_g} \).

The main result of [DM] is the following theorem:

Theorem 3.5. [DM] \( F_{\mathcal{M}_g} \) is irreducible over \( \text{Spec} \ Z \).

Remark: Deligne and Mumford gave two proofs of this theorem. In both cases they deduce the result from the classical characteristic 0 result stated below. We outline below their second proof, which uses Deligne-Mumford stacks.

Proposition 3.1. \( F_{\mathcal{M}_g} \times \text{Spec} \ Z \text{Spec} \ C \) is irreducible.

Proof: It was shown classically that there is a space \( H_{k,b} \) parametrizing degree \( k \) covers of \( \mathbb{P}^1 \) simply branched over \( b \) points defined over the complex numbers. In [Fut], Fulton showed that the functor \( F_{H_{k,b}} \) whose sections over a base \( B \) are families of smooth curves \( C \to B \) together with a degree \( k \) map \( C \to \mathbb{P}^1_B \) expressing each geometric fiber as a cover of \( \mathbb{P}^1 \) simply branched over \( b \) points is represented by a scheme which we also call \( H_{k,b} \). In characteristic greater than \( k \) it is a finite étale cover of \( \mathbb{P}^b_b = (\mathbb{P}^1)^b - \Delta \), where \( \Delta \) is the union of all diagonals (This fact was known classically over \( \mathbb{C} \). In low characteristic the map may fail to be finite). Since \( P_b \) is obviously irreducible, it can be proved that \( H_{k,b} \) is irreducible in high characteristic by showing that the monodromy of the covering \( H_{k,b} \to P_b \) acts transitively on the fiber over a base point in \( P_b \) for all \( k, b \). Since there is a universal family of branched covers \( C_{k,b} \to H_{k,b} \) there is a map \( H_{k,b} \to F_{\mathcal{M}_g} \) (where \( g = b/2 - k + 1 \)). By the Riemann-Roch theorem for smooth curves, every curve of genus \( g \) can be expressed as a degree \( k \) cover of \( \mathbb{P}^1 \) with \( b \) simple branch points, as long as \( k > g + 1 \). Thus for \( k \) (and thus \( b \)) sufficiently large, the map is surjective. Therefore \( F_{\mathcal{M}_g} \) is irreducible in characteristic greater than \( k \), and thus \( F_{\mathcal{M}_g} \times \mathbb{C} \) is irreducible. \( \square \)

Proof of Theorem 3.5(outline): Since \( F_{\mathcal{M}_g} - F_{\mathcal{M}_g} \) is a divisor \( F_{\mathcal{M}_g} \) is irreducible if and only if \( F_{\mathcal{M}_g} \). The stack \( F_{\mathcal{M}_g} \) is smooth, so it suffices to show that it is connected.

In [DM, Section 5], Deligne and Mumford construct a stack \( n F_{\mathcal{M}_g} \) whose sections are curves with a “Jacobi structure of level \( n \)” ([DM, Paragraph 5.14]). The extra structure eliminates all non-trivial automorphisms when \( n \geq 3 \), so this stack is in fact represented by a scheme. Now, \( n F_{\mathcal{M}_g} \) is a finite cover of \( F_{\mathcal{M}_g} \), and they deduce the connectedness of \( n F_{\mathcal{M}_g} \times \mathbb{C} \) from the connectedness of \( F_{\mathcal{M}_g} \times \mathbb{C} \) ([DM, Theorem 5.13, Lemma 5.16]). They also prove that the fibers of
\[ nF_{M_g} \rightarrow \text{Spec } \mathbb{Z}[e^{2\pi i/n},1/n] \] have the same number of connected components \[\text{[DM], Cor 5.11}\]. Since \(n \geq 3\) was arbitrary, the irreducibility of \(F_{M_g}\) over \(\text{Spec } \mathbb{Z}\) follows. \(\square\)

Remark: In \[\text{[HM]}\] Harris and Mumford constructed a compactification of \(H_{k,b}\) where the boundary represents stable curves expressed as branched covers of chains of \(P^1\)’s. The existence of this compactification implies that every smooth curve admits degenerations to singular stable curves. Fulton \[\text{[Fu82]}\] used this fact to resurrect an argument of Severi to give a purely algebraic proof that \(F_{M_g}\) is irreducible in characteristic 0. This combined with the results of \[\text{[DM]}\] give a purely algebraic proof that \(F_{M_g}\) is irreducible over \(\text{Spec } \mathbb{Z}\).

4. Construction of the moduli scheme

As we have previously seen, the moduli stack is a quotient stack of a smooth scheme \(H_g\) by \(PGL(N+1)\). In this, the final section, we discuss the construction of a quotient scheme \(H_g/PGL(N+1)\) over an algebraically closed field \(k\). We first prove that such a scheme is unique and is the coarse moduli space for the quotient stack. We then briefly discuss Gieseker’s GIT construction of a quotient scheme.

Throughout this section, we will assume that all schemes are defined over an algebraically closed field \(k\).

4.1. Geometric quotients.

**Definition 4.1.** [\[GIT\], Definitions 0.5, 0.6] Let \(X\) be a scheme defined over a field \(k\), and let \(G/k\) be an algebraic group acting on \(X\). A \(k\)-scheme \(Y\) is a geometric quotient of \(X\) by \(G\) if there is a morphism \(X \rightarrow Y\) such that

1. \(f\) is surjective, affine and \(G\) invariant.
2. \(f^*(O_X)^G = O_Y\).
3. If \(W \subset X\) is closed and \(G\) invariant, then \(f(W)\) is closed in \(Y\). Furthermore, if \(W_1\) and \(W_2\) are disjoint \(G\) invariant subsets of \(X\), then \(f(W_1)\) and \(f(W_2)\) are disjoint.
4. The geometric fibers of \(f\) are orbits.

**Remark:** The purpose of the geometric invariant theory developed by Mumford is to construct geometric quotients for the action of reductive\(^2\) groups on projective varieties.

The following is a restatement of \[\text{[GIT], Prop 0.1}\].

\(^2\)The definition of a reductive group is given in \[\text{[GIT]}\] Appendix A]. For the purpose of these notes, it suffices to know that \(SL(N+1,k)\) is reductive for a field \(k\).
Proposition 4.1. If $X \xrightarrow{f} Y$ is a geometric quotient and if $X \xrightarrow{g} Z$ is a $G$ invariant morphism, then there is a unique morphism $\phi : Y \rightarrow Z$ such that $g = \phi \circ f$.

Note that the proposition implies that if a geometric quotient exists then it is unique.

Now let $X$ be a scheme with a $G$ action such that the stabilizers of geometric points are finite and reduced. We have seen that the groupoid $[X/G]$ is a Deligne-Mumford stack. Assume the action of $G$ on $X$ is locally proper; i.e., $X$ can be covered by open invariant subschemes $U$ such that $G$ acts properly on $U$.

Proposition 4.2. [Vi, Proposition 2.11] If $f : X \rightarrow Y$ is a geometric quotient of $X$ by $G$, and $f$ is universally submersive, then $Y$ is the (coarse) moduli space for the stack $[X/G]$.

Proof: Let $\eta \in [X/G](B)$ be a section corresponding to a principal $G$-bundle $E \xrightarrow{s} B$ together with a $G$ invariant map $\phi : E \rightarrow X$. Since the map $f : X \rightarrow Y$ is $G$ invariant, there is a unique map $\psi : B \rightarrow Y$ making the obvious square commute. Hence we can associate to $\eta$ a unique section of of $\text{Hom}(B,Y)$. Thus there is a morphism of stacks $[X/G] \rightarrow Y$.

Now if $\Omega = \text{Spec} \ K$ where $K$ is algebraically closed, then $\text{Hom}(\Omega,Y)$ is, by Condition (4) of the definition, the set of orbits of $K$-valued points of $X$. This is exactly $[X/G](\Omega)$. Therefore, the map induces a bijection $[X/G](\Omega) \rightarrow Y(\Omega)$ as required in the definition of a coarse moduli space.

Using the local properness of the $G$ action together with the universal submersiveness of $f$ we can prove that the morphism $[X/G] \rightarrow Y$ is proper ([Vi, Proposition 2.11].

Finally, note that the universal property of $Y$ implied by Prop 4.1 also shows that $Y$ satisfies the universal property necessary for a (coarse) moduli space. □

Remark: If $G$ is reductive and $f : X \rightarrow Y$ is a geometric quotient, so that $f$ is affine, then [GIT, Proposition 0.8] the action of $G$ is actually proper. This condition is satisfied in all geometric invariant theory quotients of quasi-projective varieties.

4.2. Construction of quotients by geometric invariant theory.
In this paragraph we discuss the geometric invariant theory necessary to construct $\mathcal{M}_g$ and and $\overline{\mathcal{M}}_g$ as quotients of Hilbert schemes of $n$-canonically embedded (stable) curves. Our source is [Gi, Chapter 0].

---

$^3f : X \rightarrow Y$ is submersive if $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in $X$. If this property remains after base change, then we say $f$ is universally submersive.
For a full treatment of geometric invariant the classic reference is Mumford’s [GIT].

Let \( X \subset \mathbb{P}^N \) be a projective scheme, and let \( G \) be a reductive group acting on \( X \) via a representation \( G \to GL(N+1) \).

**Definition 4.2.** (1) A closed point \( x \in X \) is called semi-stable if there exists a non-constant \( G \)-invariant homogeneous polynomial \( F \) such that \( F(x) \neq 0 \).

(2) \( x \in X \) is called stable if: \( \dim o(x) = \dim G \) (where \( o(x) \) denotes the orbit of \( x \)) and there exists a non-constant \( G \)-invariant polynomial such that \( F(x) \neq 0 \) and for every \( y_0 \) in \( X_F = \{ y \in X | F(y) \neq 0 \} \), \( o(y_0) \) is closed in \( X_F \).

Let \( X^{ss} \) denote the semi-stable points of \( X \), and \( X^s \) denote the stable points. Then \( X^s \subset X^{ss} \) are both open in \( X \). However, they may be empty.

The following is the first main theorem of geometric invariant theory.

**Theorem 4.1.** There exists a projective scheme \( Y \) and a universally submersive morphism \( f_{ss} : X^{ss} \to Y \) such that \( f_{ss} \) satisfies properties (1)-(3) of the definition of a geometric quotient (such a morphism is often called a good quotient in the literature). Furthermore, there exists \( U \subset Y \) open such that \( f^{-1}(U) = X^s \) and \( f_s : X^s \to U \) is a geometric quotient of \( X^s \) by \( G \).

**4.3. Criteria for stability.** Let \( X \subset \mathbb{P}^N \) be a projective scheme, and let \( \tilde{X} \subset \mathbb{A}^{N+1} \) be the affine cone over \( X \). Assume as above, that a reductive group \( G \) acts on \( X \) via a representation \( G \to GL(N+1) \). Then \( G \) acts on \( \tilde{X} \) as well. The stability of \( x \in X \) can be rephrased in terms of the stability of the points \( \tilde{x} \in \tilde{X} \) lying over \( x \).

**Proposition 4.3.** [GIT, Chapter 1, Proposition 2.2 and Appendix B] A geometric point \( x \in X \) is semi-stable if for one (and thus for all) \( \tilde{x} \in \tilde{X} \) lying over \( X \), \( 0 \notin o(\tilde{x}) \). The point \( x \) is stable if \( o(\tilde{x}) \) is closed in \( \mathbb{A}^{N+1} \) and has dimension equal to the dimension of \( G \).

The second main theorem of geometric invariant theory is Mumford’s numerical criterion for stability which we now discuss.

**Definition 4.3.** A 1-parameter subgroup of \( G \) is a homomorphism \( \lambda : G_m \to G \). This will be abbreviated to \( \lambda \) is a 1-PS of \( G \).

Now if \( \lambda \) is a 1-PS of \( G \), then the since \( \lambda \) is 1-dimensional, there is a basis \( \{ e_0, \ldots, e_N \} \) of \( \mathbb{A}^{N+1} \) such that the action of \( \lambda \) is diagonalizable with respect to this basis; i.e. \( \lambda(t)e_i = t^{r_i}e_i \) where \( t \in G_m \) and \( r_i \in \mathbb{Z} \).

If \( \tilde{x} = \sum x_i e_i \in \tilde{X} \), then the set of \( r_i \) such that \( x_i \) is non-zero is called
the \( \lambda \)-weights of \( \tilde{x} \). Note that if \( x \in \mathbb{P}^N \) then the \( \lambda \)-weights are the same for all points in \( \mathbb{A}^{N+1} - 0 \) lying over \( x \).

**Definition 4.4.** \( x \in X \) is \( \lambda \)-semi-stable if for one (and thus for all) \( \tilde{x} \in \tilde{X} \) lying over \( x \), \( \tilde{x} \) has a non-positive \( \lambda \) weight. A point \( x \) is \( \lambda \)-stable if \( \tilde{x} \) has a negative \( \lambda \) weight.

**Theorem 4.2.** \([\text{GIT}]\) A point \( x \in X \) is (semi)stable if and only if \( x \) is \( \lambda \)-(semi)stable for all 1-PS \( \lambda : \mathbb{G}_m \to G \).

Remark on the Proof: It is easy to see that if \( x \) is unstable (i.e. not semi-stable) with respect to \( \lambda : \mathbb{G}_m \to G \) then \( x \) is unstable. The reason is that if all the weights of \( \lambda \) are positive then 0 will be in the closure of the \( G \)-orbit of \( \tilde{x} \) in \( \mathbb{A}^{N+1} - 0 \). The converse is more difficult. \( \square \)

Example (cf. \([\text{GIT}, \text{Proposition 4.1}]\)). The set of homogeneous forms of degree 4 in two variables forms a 5-dimensional vector space \( V \). We will view \( \mathbb{P}(V) \) as the space parametrizing 4-tuples of (not necessarily) distinct points in \( \mathbb{P}^1 \). There is a natural action of \( SL(2) \) on \( V \) inducing an action on \( \mathbb{P}(V) \). Let us use the numerical criterion to determine the stable and semi-stable locus in \( \mathbb{P}(V) \).

If \( v \in V \) is a form of degree 4 and \( \lambda \) is a 1-PS subgroup of \( SL(2) \), then we can write \( v = a_4X_0^4 + a_3X_0^3X_1 + a_2X_0^2X_1^2 + a_1X_0X_1^3 + a_0X_1^4 \), and \( \lambda \) acts by \( \lambda(t)(X_0) = t^rX_0 \), \( \lambda(t)(X_1) = t^{-r}X_0 \) and \( r > 0 \) (the weight on \( X_1 \) must be the negative of the weight on \( X_0 \), since \( \lambda \) maps to \( SL(2) \)). The possible weights of \( v \) are \( \{4r, 2r, 0, -2r, -4r\} \). In order for \( v \) to be \( \lambda \)-stable one of \( a_1 \) or \( a_0 \) must be non-zero. It is \( \lambda \)-semi-stable if one of \( a_2, a_1 \) or \( a_0 \) is non-zero. On the other hand, we can consider the 1-PS, \( \tau \) which acts by \( \tau(t)X_0 = t^{-r}X_0 \) and \( \tau(t)X_1 = t^rX_1 \). In order for \( v \) to be \( \tau \)-stable one of \( a_4 \) and \( a_3 \) must be non-zero, while it is \( \tau \)-semi-stable if \( a_2 \) is non-zero. Combining the conditions imposed by \( \lambda \) and \( \tau \) we see that if \( v \) is stable, then one of \( a_0 \) or \( a_1 \) is non-zero and one of \( a_3 \) or \( a_4 \) is non-zero. This condition is equivalent to the condition that \( (1 : 0) \) and \( (0 : 1) \) are not multiple points of the subscheme of \( \mathbb{P}^1 \) cut out by the form \( v \). Likewise, \( v \) is semi-stable if \( (1 : 0) \) or \( (0 : 1) \) is cut out with multiplicity no more than 2. Finally \( v \) is unstable if \( (1 : 0) \) or \( (0 : 1) \) is cut out with multiplicity more than 2.

From this analysis it is clear that if \( v \in V \) cuts out 4 distinct points then it will be stable for every 1-PS. Likewise if \( v \) cuts out a subscheme of \( \mathbb{P}^1 \) with each point having multiplicity 2 or less then it is semi-stable for every 1-PS. Conversely, if \( v \) cuts a point of multiplicity 3 or more then \( v = X_0^3(a_0X_0 + a_1X_1) \) for some choice of coordinates on \( \mathbb{P}^1 \). Then
\( v \) will have strictly positive weights for a 1-PS \( \lambda \) acting diagonally by 
\( \lambda(t)X_0 = t^rX_0 \) for \( r > 0 \).

4.4. **Gieseker’s construction of \( \overline{M}_g \).** Let \( \text{Hilb}^{N+1}_{P(t)} \) be the Hilbert scheme of curves in \( \mathbb{P}^N \) with Hilbert Polynomial \( P(t) \). Now if \( X \subseteq \mathbb{P}^N \) is a curve with Hilbert polynomial \( P(t) \), then there exists \( m > 0 \) (independent of \( X \)) such that the restriction map \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m)) \) is surjective and \( \dim H^0(X, \mathcal{O}_X(m)) = P(m) \). Taking the \( P(m) \)-th exterior power of \( \phi_m \) we obtain a linear map \( V^m = \bigwedge^{P(m)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to \bigwedge^{P(m)} H^0(X, \mathcal{O}_X(m)) = k \) unique up to scalars; i.e., an element of \( \mathbb{P}(V^m) \). The corresponding point in \( \mathbb{P}(V^m) \) is called the \( m \)-th Hilbert point of \( X \) and is denote \( H_m(X) \). In this way we obtain a map \( \text{Hilb}^{N+1}_{P(t)} \to \mathbb{P}(V^m) \). For \( m \) sufficiently large this map is an embedding.

Both \( SL(N+1) \) and \( PGL(N+1) \) act on \( \mathbb{P}(V^m) \) via the \( m \)-th exterior power representation of \( SL(N+1) \to GL(m) \). Now the action of \( SL(N+1) \) factors through the action of \( PGL(N+1) \) (the stabilizer of \( SL(N+1) \) at a geometric point is the group of \( N+1 \) roots of unity) so we have the following proposition.

**Proposition 4.4.** If \( X \subseteq \mathbb{P}(V^m) \) then \( X \to Y \) is a geometric quotient by \( SL(N+1) \) if and only if it is a geometric quotient by \( PGL(N+1) \).

**Proof:** If \( X \to Y \) is a geometric quotient by \( SL(N+1) \) then the geometric fibers are \( SL(N+1) \) orbits. These orbits are the same as the \( PGL(N+1) \) orbits. Likewise, \( \mathcal{O}_X^{SL(N+1)} = \mathcal{O}_X^{PGL(N+1)} \). Thus, \( \mathcal{O}_Y \simeq f_*\mathcal{O}_X^{PGL(N+1)} \). Finally if \( W \) and \( V \) are \( PGL(N+1) \) invariant, they are also \( SL(N+1) \) invariant. Thus if they are disjoint, then since \( X \to Y \) is an \( SL(N+1) \) quotient, their images will be disjoint as well. Hence \( X \to Y \) is a \( PGL(N+1) \) quotient. The converse is similar. \( \square \)

Let \( g \geq 3 \) and \( d \geq 20(g-1) \) be integers. Consider the Hilbert scheme \( \text{Hilb}^{N+1}_{P(t)} \) of curves in \( \mathbb{P}^N \) with Hilbert polynomial \( P(t) = dt - g + 1 \) (the curves parametrized necessarily have arithmetic genus \( g \)). The first step in Gieseker’s construction is to prove the following theorem. The proof is 10 pages long and uses the numerical criterion.

**Theorem 4.3.** [Gi] Theorem 1.0.0 | There exists an integer \( m_0 >> 0 \) such that if \( X \) is smooth then \( H_{m_0}(X) \) is \( SL(N+1) \) stable.

**Remark:** The theorem is not necessarily true for arbitrary \( m_0 >> 0 \). However there are infinitely many \( m_0 \) for which the theorem is true ([Gi, Remark after Theorem 1.0.0]).

The next, and technically most difficult step is to prove the following theorem. The proof takes 50 pages!
Theorem 4.4. [Gi, Theorem 1.0.1] For the same integer \( m_0 \), every point in \( \text{Hilb}^{N+1}_{P(t)} \cap \mathbf{P}(V^{m_0})^{ss} \) parametrizes a Deligne-Mumford semi-stable curve.

Let \( U \subset \text{Hilb}^{N+1}_{P(t)} \) be the subscheme of semi-stable curves with respect to the \( m_0 \)-th Hilbert embedding. Let \( Z_U \subset \mathbf{P}^N_U \) be the restriction of the universal family of projective curves. As before, view a point \( h \in U \) as parametrizing a curve \( X_h \) and a very ample line bundle \( L_h \) of degree \( d \) on \( X_h \). Set \( U_c = \{ h \in U | L_h \simeq \omega_{X_h}^n \} \). This is a constructible subscheme of \( U \) which is empty unless \( 2g - 2 \) divides \( d \). Gieseker then proves that \( U_c \) is in fact closed in \( U \). He also proves that \( U_c \) is smooth ([Gi, Theorem 2.0.1]) and parametrizes only all Deligne-Mumford stable curves; thus, \( U_c \simeq \overline{M}_{g,n} \). Since \( U_c \) is closed in \( U \) there is a projective quotient \( U_c/\mathbf{PGL}(N+1) \). Finally note that \( \mathbf{PGL}(N+1) \) (and thus \( \mathbf{SL}(N+1) \)) acts with finite stabilizers on points of \( U_c \) because the curves parametrized have finite automorphism groups. Hence the points of \( U_c \) are in fact \( \mathbf{SL}(N+1) \) stable. Thus a universally submersive geometric quotient \( U_c/\mathbf{SL}(N+1) \) exists. Since this is isomorphic to a geometric quotient \( U_c/\mathbf{PGL}(N+1) \simeq \overline{M}_{g,n}/\mathbf{PGL}(N+1) \) we have succeeded in constructing a coarse moduli scheme for the stack of stable curves. \( \square \)

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