A Low-rank Method for Parameter-dependent Fluid-structure Interaction Discretizations with Hyperelasticity

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Abstract

Fluid-structure interaction models are used to study how a material interacts with different fluids at different Reynolds numbers. Examining the same model not only for different fluids but also for different solids allows to optimize the choice of materials for construction even better. A possible answer to this demand is parameter-dependent discretization. Furthermore, low-rank techniques can reduce the complexity needed to compute approximations to parameter-dependent fluid-structure interaction discretizations.

Low-rank methods have been applied to parameter-dependent linear fluid-structure interaction discretizations. The linearity of the operators involved allows to translate the resulting equations to a single matrix equation. The solution is approximated by a low-rank method. In this paper, we propose a new method that extends this framework to nonlinear parameter-dependent fluid-structure interaction problems by means of the Newton iteration. The parameter set is split into disjoint subsets. On each subset, the Newton approximation of the problem related to the upper median parameter is computed and serves as initial guess for one Newton step on the whole subset. This Newton step yields a matrix equation whose solution can be approximated by a low-rank method.

The resulting method requires a smaller number of Newton steps if compared with a direct approach that applies the Newton iteration to the separate problems consecutively. In the experiments considered, the proposed method allows to compute a low-rank approximation up to twenty times faster than by the direct approach.

Key words. Parameter-dependent fluid-structure interaction, low-rank, ChebyshevT, tensor, Newton iteration

AMS subject classification. 15A69, 49M15, 65M22, 74F10

1 Introduction

Fluid-structure interactions (FSI) play a crucial role in various applications, ranging from classical problems of aeroelasticity to naval engineering, biology and medicine, compare [2, 29] for examples. Simulations of fluid-structure interactions serve to supplement or replace experiments. The two characteristic difficulties of fluid-structure interactions are the very stiff coupling arising from the interplay of a parabolic (fluid) and a hyperbolic equation (structure), and the moving and a priori unknown solution domain resulting from the motion and deformation of the structure. Finally, a nonlinear, poorly structured system of partial differential equations must be solved.

Two approaches are generally considered to approximate the coupled system. First, partitioned methods, which are based on an external coupling of a flow model and a structural model. These methods are highly flexible and efficient discretizations and solvers can be considered for the particular subproblems. However, partitioned methods suffer from stability problems, especially when problems with pronounced added-mass effect are considered [7]. This effect emerges when the densities of fluid and

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structure are similar, for example in shipbuilding applications or in biomedical problems. The alternative monolithic approach considers the entire system as a single entity, generally discretized using implicit methods. It results in a highly robust description, which, however, leads to algebraic systems of equations of great complexity, whose solution (in the nonlinear as well as in the linear case) is extremely costly, compare [20] [13]. For a comparison of both approaches we refer to Heil et al. [17]. Here we follow the monolithic approach, also because it allows us to formulate the parameter-dependent problem in a single unit.

The other problem, i.e., the motion of the domain, is usually addressed either by a transformation into a common coordinate system, or, especially in partitioned approaches, the coupling condition is realized by interpolation between the two subproblems. However, if a common coordinate system is chosen as the basis of the description, on the other hand an Eulerian description is suitable [11] [3] [27] [24], i.e., both the flow and structure problem are described on the changing Eulerian domain, or on the other hand a fixed reference domain is chosen for both subproblems. This second framework is called the Arbitrary Lagrangian Eulerian (ALE) coordinate system [10] [19] [30], which is based on a transformation of the Navier-Stokes equations into a fixed reference system. We take a simple route here and assume that the deformation of the fluid domain is small, and thus the geometric nonlinearity does not play a crucial role, so the ALE transformation can be considered as an identity. In the elasticity model, on the other hand, we will consider the usual nonlinear strain tensor.

Here we target parameter-dependent fluid-structure interaction problems in order to use the solution later in an optimization or design process. Alternatively, an optimization problem with fluid-structure interactions as constraints could be considered directly. This approach has already been described in the literature [6] [31] [12], but it remains very complex due to the great complexity of the coupled system. For technically realistic scenarios, no efficient and at the same time accurate solution exists so far. Therefore, in this work we develop a novel low-rank technique that solves the parametric problem using a low-rank format in an all-at-once approach. For this purpose, we follow the ideas laid out for the linear case in [32] and embed this in a Newton iteration for the nonlinear problem considered here.

We state the weak formulation of the nonlinear FSI problem in Section 2. In Section 3 the parameter-dependent discretization is linearized by means of the Newton iteration, and a Newton step is, for a subset of problems, translated to a matrix equation. We then discuss the approximability of the unknown, a matrix, by low-rank methods in Section 4, before we derive and code the novel algorithm in Algorithm 1. The 3d FSI test case from [29] Section 8.3.2.2 is considered for a numerical comparison of Algorithm 1 with standard Newton iterations in Section 5. Variants of Algorithm 1 are discussed in Section 5.3.

2 The stationary nonlinear fluid-structure interaction problem

Let Ω, F, S ⊂ ℝd be open for d ∈ {2, 3} such that F ∪ S = ¯Ω and S ∩ F = ∅. We are interested in an FSI problem that uses the stationary Navier-Stokes equations [29] Section 2.4.5.3 to model the fluid part F. On the solid part S, we use the stationary model of a Saint Venant-Kirchhoff material [29] Definition 2.18. By Γint = ∂F ∩ ∂S, we denote the interface. The outer boundary ∂Ω is split into the Dirichlet part ΓF ⊂ ∂Ω and the fluid outflow part ΓD ∩ ∂S ⊂ ∂F ∩ ∂Ω, where a Neumann outflow conditions of do-nothing type holds [13]. We denote the L2 scalar product on F and S by ⟨·, ·⟩F and ⟨·, ·⟩S, respectively. The weak formulation of the coupled nonlinear FSI problem with a vanishing right hand side is given by

\[ \langle \nabla \cdot v, \xi \rangle_F = 0, \]
\[ \mu_s (\nabla u + \nabla u^T + \nabla u^T \nabla u, \nabla \varphi)_S + \lambda_s (\text{tr}(\nabla u + \frac{1}{2} \nabla u^T \nabla u) I, \nabla \varphi)_S \]
\[ + \rho_f (\langle v \cdot \nabla \rangle v, \varphi)_F + \nu_f \rho_f \langle \nabla v + \nabla v^T, \nabla \varphi \rangle_F - \langle p, \nabla \cdot \varphi \rangle_F = 0 \quad \text{and} \]
\[ \langle \nabla u, \nabla \psi \rangle_F = 0, \]

where p ∈ L2(F) denotes the pressure, v ∈ vint + H1(F; ΓD ∩ Γint) the velocity, u ∈ H1,0(Ω) the deformation. By vint ∈ H1(F) we introduce an extension of the Dirichlet data on ΓD into the fluid domain. For the test functions of the divergence equation, momentum and deformation equation we assume ξ ∈ L2(F), ϕ ∈ H1,0(Ω) and ψ ∈ H1,0(F), respectively. H1,0(Ω, ΓD) denotes the L2(Ω) functions that are weakly differentiable and have a trace equal to zero on ΓD. The weakly differentiable functions H1,0(Ω) have a trace equal to zero on ∂Ω. The fluid parameters involved are the fluid density
\( \rho_f \in \mathbb{R} \) and the kinematic fluid density \( \nu_f \in \mathbb{R} \). The Saint Venant-Kirchhoff model equations involve the first Lamé parameter \( \lambda_r \in \mathbb{R} \) and the shear modulus \( \mu_s \in \mathbb{R} \). The interface coupling conditions at \( \Gamma_{\text{int}} = \partial S \cap \partial F \) are implicitly included in (1). The geometric condition \( u_f = u|_S \) on \( \Gamma_{\text{int}} \) follows from the global approach \( u \in H^1_0(\Omega)^d \) and likewise, the kinematic condition \( v = 0 \) on \( \Gamma_{\text{int}} \) is embedded into the trial space. The dynamic condition

\[
(\mu_s \left( \nabla u + \nabla u^T + \nabla u^T \nabla u \right) + \lambda_s \text{tr}(\nabla u) + \frac{1}{2} \nabla u^T \nabla u) n_S + \left( \nu_f \rho_f (\nabla v + \nabla v^T) - p I \right) n_F = 0
\]

follows with integration by parts as the common test function \( \phi \) is globally defined. We refer to [29, Sec. 3.4] for details on the derivation of the variational formulation. By \( n_S \) and \( n_F \) we denote outward facing unit normal vectors on \( \partial S \) and \( \partial F \), respectively.

### 2.1 Finite element formulation and Newton iteration

We will consider monolithic finite element discretizations of the coupled fluid-structure interaction problem (1). To keep the notation simple we assume that velocity \( u_h \in V_h \subset H^1_0(\Omega, \Gamma^D) \) and deformation \( u_h \in W_h \subset H^1_0(\Omega)^d \) and pressure \( p_h \in L_h \subset L^2(F) \) are conforming finite element spaces, where the pair \( V_h \times L_h \) satisfies the inf-sup condition on \( F \). We refer to the literature for details and peculiarities on finite element discretizations of FSI problems [10, 30, 1, 29]. In the following we will just skip the index \( h \) referring to the spatial discretization. The arising nonlinear algebraic problems will be solved by the Newton iteration, which usually shows optimal convergence for fluid-structure interactions [19, 13].

The solution components are combined to one single variable \( x = (v_h, u_h, p_h) \in X_h := V_h \times W_h \times L_h \) with \( M := \dim(X_h) \) and the Newton iteration \( j - 1 \mapsto j \) is formally given by

\[
\begin{align*}
(A_0 + \mu_s A_1 + \rho_f \nu_f A_2 + \rho_f J_\mu(x_{j-1}) + \mu_s J_\mu(x_{j-1}) + \lambda_s J_\lambda(x_{j-1})) \delta x_j &= b_D - g(x_{j-1}), \\
x_j &= x_{j-1} + \delta x_j.
\end{align*}
\]

The matrices \( A_0, A_1, A_2 \in \mathbb{R}^{M \times M} \) are discrete linear operators, \( J_\rho(\cdot), J_\mu(\cdot), J_\lambda(\cdot) \in \mathbb{R}^{M \times M} \) are discrete Jacobian matrices of nonlinear operators evaluated at the argument given. \( g(\cdot) \) evaluates the variational residual at the argument. The Jacobian matrices \( J_\rho(\cdot), J_\mu(\cdot) \) and \( J_\lambda(\cdot) \) depend on \( x_{j-1} \), the approximation of the previous linearization step. Solving the linear systems is challenging due to the bad conditioning of the Jacobian and the missing structure, the matrix is neither symmetric nor positive definite. The coupled system includes the Navier-Stokes equation and is hence of a saddle-point type. Linear solvers are well investigated [14, 28, 9, 1, 13] but in two dimensions, direct solvers are also applicable.

### 2.2 Parameter dependent formulation

Since the behavior of an FSI model varies if parameters such as the shear modulus or the kinematic fluid viscosity changes, we are interested in a parameter-dependent discretization with respect to \( m \in \mathbb{N} \) different parameter combinations \( (\mu_{i}^s, \nu_{i}^f) \) for \( i \in \{1, \ldots, m\} \). Here we will consider variations of the shear moduli \( \mu_i^s \) and the kinematic fluid viscosity \( \nu_i^f \). Hence, the nonlinear problem (2) is to be solved for \( m \) solutions \( x^1, x^2, \ldots, x^m \in X_h \) each of them being \( M \)-dimensional. For a given parameter choice \( i \in \{1, \ldots, m\} \), at the Newton step \( j \in \mathbb{N} \), the Jacobians \( J_\rho(\cdot), J_\mu(\cdot) \) and \( J_\lambda(\cdot) \) depend on \( x^i_{j-1} \), the approximation of the previous linearization step, and, therefore, on the parameter index \( i \).

Due to this nonlinear dependency, the parameter dependent equation can not be translated to a matrix equation similar to the linear case discussed in [32]. Instead, what we propose in this paper is to split the parameter set into \( K \in \mathbb{N} \) disjoint subsets:

\[
S^{\mu, \nu} := \{ (\mu_i^s, \nu_i^f) \}_{i \in \{1, \ldots, m\}} = \bigcup_{k=1}^K \mathcal{I}_k.
\]

The subsets have to be chosen such that the problems that are clustered within one subset \( \mathcal{I}_k \) do not differ too much from each other. Details regarding the clustering will be discussed later. For each subset \( \mathcal{I}_k, \bar{x}_c^k \in \mathbb{R}^M \), the Newton approximation of the problem related to the upper median index, is computed
up to a given accuracy \( \epsilon_N > 0 \). \( \tilde{x}_{c,N}^k \) is then used as initial guess for one Newton step on the whole subset \( \mathcal{I}_k \). This translates to a Newton step that can be written as the following matrix equation. For all \( k \in \{1, \ldots, K\} \), find \( S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|} \) such that

\[
A_0 S_k + A_1 S_k D^k_{\mu} + \rho_f A_2 S_k D^k_{\rho_f} + \rho_f J_f S_k D^k_{\rho_f} + \lambda_s J_s \tilde{x}_{c,N}^k S_k + J_{\mu} \tilde{x}_{c,N}^k S_k D^k_{\mu} + \lambda_s J_s \tilde{x}_{c,N}^k S_k = (bD - g(\tilde{x}_{c,N}^k, 0, 0) \otimes (1, \ldots, 1) - (A_1 \tilde{x}_{c,N}^k + g_{\mu}(\tilde{x}_{c,N}^k)) \otimes \text{diag}(D^k_{\mu})^T - \rho_f A_2 \tilde{x}_{c,N}^k \otimes \text{diag}(D^k_{\rho_f})^T),
\]

(3)

where

\[
D^k_{\mu} := \text{diag} (\mu_1^k), \quad D^k_{\rho_f} := \text{diag} (\rho_f^k) \in \mathbb{R}^{|\mathcal{I}_k| \times |\mathcal{I}_k|}
\]

and

\[
g_{\mu}(\cdot) \text{ evaluates the residual of the operator that is, in the FSI problem, multiplied with the shear modulus.}.
\]

\[
S_k \text{ in (3) is the Newton update for all problems related to the parameter set } \mathcal{I}_k \text{ and can be approximated by a low-rank method such as the GMRESTR or the ChebyshevT method from [32, Algorithm 2, Algorithm 3]. The low-rank approximation } \hat{S}_k \text{ to (3) is then added to the initial guess on } \mathcal{I}_k.
\]

We obtain the Newton approximation

\[
\hat{X}_k := \tilde{x}_{c,N}^k \otimes (1, \ldots, 1) + \hat{S}_k \quad \text{for all } \quad k \in \{1, \ldots, K\}.
\]

The global approximation to the parameter-dependent problem is

\[
\hat{X} := [\hat{X}_1| \ldots |\hat{X}_K].
\]

If \( \hat{S}_k \) has rank \( R_k \) for all \( k \in \{1, \ldots, K\} \), the rank of \( \hat{X} \) is at most \( K \sum_{k=1}^{K} R_k \).

To obtain \( \tilde{x}_{c,N}^k \), multiple Newton steps of the standard FSI problem (2) are needed. All \( K \) of them can be computed in parallel. On the subset itself, we perform only one Newton step for the matrix equation.

### 3 Discretization and solution of the matrix-Newton iteration

For the sake of readability, we restrict to the case of a parameter-dependent discretization with respect to two parameters. In Section 4.4, we explain how to extend the resulting method to further parameters. The parameters of interest are the \( m_1 \in \mathbb{N} \) shear moduli and the \( m_2 \in \mathbb{N} \) kinematic fluid viscosities

\[
\{\mu_i^k\}_{i \in \{1, \ldots, m_1\}} \subset \mathbb{R}, \quad \{\nu_j^k\}_{j \in \{1, \ldots, m_2\}} \subset \mathbb{R}.
\]

(5)

With this choice we face, in total, \( m = m_1 m_2 \) different FSI problems.

Consider a finite element discretization of (1) on a triangulation \( \Omega_h \), a matching mesh [29, Definition 5.9] of the domain \( \Omega \). If \( N \in \mathbb{N} \) denotes the number of mesh nodes in \( \Omega_h \), the total number of degrees of freedom \( M \in \mathbb{N} \) that results after discretization is equal to \( M = 5N \) if \( d = 2 \) and \( M = 7N \) if \( d = 3 \) (one pressure and \( d \) velocities and deformations, see [32, Chapter 3]). Similar to the linear case, we first need the discrete differential operator (discretization matrix) \( A_0 \in \mathbb{R}^{M \times M} \). \( A_0 \) discretizes all the linear operators involved in (1) with a fixed shear modulus \( \mu_\in \mathbb{R} \) and a fixed kinematic fluid viscosity \( \nu_f \in \mathbb{R} \). To be clear, restricted to the momentum equation,

\[
A_0 \text{ discretizes } \mu_s (\nabla u + \nabla u^T, \nabla \varphi)_S + \lambda_s (\text{tr}(\nabla u) I, \nabla \varphi)_S + \nu_f \rho_f (\nabla v + \nabla v^T, \nabla \varphi)_F - (p, \nabla \cdot \varphi)_F.
\]

Furthermore, we need the discrete operators \( A_1, A_2 \in \mathbb{R}^{M \times M} \) such that

\[
A_1 \text{ discretizes } (\nabla u + \nabla u^T, \nabla \varphi)_S \quad \text{and} \quad A_2 \text{ discretizes } (\nabla v + \nabla v^T, \nabla \varphi)_F.
\]

In our discrete finite element space, every unknown

\[
x_h = \begin{pmatrix}
  ph \\
  vh \\
  uh
\end{pmatrix} \in X_h = V_h \times W_h \times L_h
\]

(6)
consists of a discrete pressure \( p_h \in L_h \), a discrete velocity and deformation that are \( v_h \in V_h, u_h \in W_h \).

Since all the operators discretized by the matrices \( A_0, A_1 \) and \( A_2 \) are linear, the corresponding operators can be discretized by choosing appropriate trial functions. The application of linear operators, if the argument is contained in the finite element space of dimension \( M \), is possible via matrix-vector products. For the nonlinear operators in (1),

\[
\langle \nabla u^T \nabla u, \nabla \varphi \rangle_S, \quad \langle \text{tr}(\nabla u^T \nabla u)I, \nabla \varphi \rangle_S \quad \text{and} \quad \langle (v \cdot \nabla)v, \varphi \rangle_F,
\]

this is not possible.

### 3.1 The Jacobian matrices

To linearize the nonlinear operators in (1), we apply the Newton iteration [25, Section 7.1.1] due to its relevance and fast convergence rate when it comes to finite elements for the Navier-Stokes equations (compare [29, Section 4.4] and [21, Remark 6.44]). As an alternative, the Picard (fixed-point) iteration will be discussed in Section 5.3. For linearization via the Newton iteration, we introduce the discrete Jacobian matrices for the nonlinear operators in (1). \( J_h(x_h) \in \mathbb{R}^{N \times N} \) is the discrete Jacobian matrix of

\[
J((v \cdot \nabla)v, \varphi)_F(x_h) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\partial((v \cdot \nabla)v, \varphi)_F}{\partial v} \bigg|_{v = v_h} & 0 \\
0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{N \times N}
\]

\( J_h(x_h) \in \mathbb{R}^{N \times N} \) the one of

\[
J_{(\nabla u + \nabla u^T)I, \nabla \varphi}_S(x_h) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{N \times N}
\]

and \( J_A(x_h) \in \mathbb{R}^{N \times N} \) the one of

\[
J_{\frac{1}{2}(\text{tr}(\nabla u^T \nabla u)I, \nabla \varphi)_S}(x_h) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{N \times N}.
\]

### 3.2 The Newton iteration

At first, we fix \((i_1, i_2) \in \Xi^{m_1, m_2} := \{1, ..., m_1\} \times \{1, ..., m_2\}\) and take a look at the Newton iteration for the problem related to a shear modulus with value \( \mu_s^{i_1} \) and a kinematic fluid viscosity of \( \nu_f^{i_2} \). Even though the right hand side in (1) vanishes, we incorporate the Dirichlet boundary conditions into the right hand side vector \( b_D \in \mathbb{R}^M \). As initial guess, we choose \( x_h^{i_1, i_2} := b_D \). Let \( g(x_h^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}) \) denote the residual of the problem (1) with a shear modulus \( \mu_s^{i_1} \) and a kinematic fluid viscosity \( \nu_f^{i_2} \), evaluated at \( x_h^{i_1, i_2} \in \mathbb{R}^M \) and let

\[
A(x_h^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}) := A_0 + (\mu_s^{i_1} - \mu_s) A_1 + (\nu_f^{i_2} - \nu_f) A_2 + \mu_s^{i_1} J_\mu(x_h^{i_1, i_2}) + \lambda_s J_\lambda(x_h^{i_1, i_2}) + \mu_f J_\mu(x_h^{i_1, i_2}).
\]

At the Newton step \( j \in \mathbb{N} \), we approximate the Newton update \( \delta x_h^{i_1, i_2} \in X_h \) such that

\[
A(x_h^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}) \delta x_h^{i_1, i_2} = b_D - g(x_h^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}).
\]

The Newton approximation of the Newton step \( j \) is given as

\[
x_h^{i_1, i_2} := x_h^{i_1, i_2} + \delta x_h^{i_1, i_2}.
\]
Similar to the low-rank framework discussed in [32], we are interested in translating a number of Newton steps of the form (8) to a single matrix equation. But since the matrix $A(x_{i_1,i_2},\mu_{s_1}^{i_1},\nu_{s_2}^{i_2})$ from (7) depends, in addition to the parameters $\mu_{s_1}^{i_1}$ and $\nu_{s_2}^{i_2}$, on the Newton approximation of the previous linearization step, a direct translation like in [32] is not possible.

### 3.3 Clustering of the parameter set

We first split the parameter set

$$S_{\mu,\nu} := \{(\mu_{s_1}^{i_1},\nu_{s_2}^{i_2})\}_{i_1 \in \{1,...,m_1\}, \ i_2 \in \{1,...,m_2\}} \subset \mathbb{R} \times \mathbb{R}$$

into the disjoint subsets $I_k$, namely

$$S_{\mu,\nu} = \bigcup_{k=1}^{K} I_k.$$

As mentioned in the introduction, the problems that are clustered within one subset $I_k$ should not differ too much from each other. Grouping the right parameters to a cluster can be a challenge and depend on the parameter choice. The elements in the parameter set $S_{\mu,\nu}$ have to be ordered first. We define the most intuitive way to order the parameter set.

**Definition 1** (Little Endian Order). If for $(i_1, i_2), (l_1, l_2) \in \mathbb{N}^{m_1,m_2},$

$$(\mu_{s_1}^{i_1},\nu_{s_2}^{i_2}) < (\mu_{s_1}^{l_1},\nu_{s_2}^{l_2}) \iff i_1 + (i_2 - 1)m_1 < l_1 + (l_2 - 1)m_1,$$

the grouping operation of the indexes of the elements in $S_{\mu,\nu}$ is little endian.

There are other ways to order the elements in $S_{\mu,\nu}$. In a parameter-dependent discretization with respect to different shear moduli and first Lamé parameters, we might prefer to order the elements in the parameter set by its Poisson ratios. Even the sizes of the subsets $\{I_k\}_{k \in \{1,...,K\}}$ may be chosen problem-dependently. In this paper, any clustering results in subsets $\{I_k\}_{k \in \{1,...,K\}}$ with

$$|I_k| = \begin{cases} \lfloor \frac{m}{K} \rfloor & \text{if } k \in \{1,...,K-1\} \\ m - (K-1)\lfloor \frac{m}{K} \rfloor & \text{else.} \end{cases} \quad (9)$$

**Example 1.** For $m_1 = 4, m_2 = 5$ and $K = 3$, the subsets would be

- $I_1 = \{(\mu_{s_1}^{1},\nu_{s_2}^{1}), (\mu_{s_1}^{2},\nu_{s_2}^{2}), (\mu_{s_1}^{3},\nu_{s_2}^{3}), (\mu_{s_1}^{4},\nu_{s_2}^{4}), (\mu_{s_1}^{1},\nu_{s_2}^{5}), (\mu_{s_1}^{2},\nu_{s_2}^{5}), (\mu_{s_1}^{3},\nu_{s_2}^{5}), (\mu_{s_1}^{4},\nu_{s_2}^{5})\}$
- $I_2 = \{(\mu_{s_1}^{5},\nu_{s_2}^{1}), (\mu_{s_1}^{6},\nu_{s_2}^{2}), (\mu_{s_1}^{7},\nu_{s_2}^{3}), (\mu_{s_1}^{8},\nu_{s_2}^{4}), (\mu_{s_1}^{5},\nu_{s_2}^{5}), (\mu_{s_1}^{6},\nu_{s_2}^{5}), (\mu_{s_1}^{7},\nu_{s_2}^{5}), (\mu_{s_1}^{8},\nu_{s_2}^{5})\}$
- $I_3 = \{(\mu_{s_1}^{1},\nu_{s_2}^{1}), (\mu_{s_1}^{2},\nu_{s_2}^{2}), (\mu_{s_1}^{3},\nu_{s_2}^{3}), (\mu_{s_1}^{4},\nu_{s_2}^{4}), (\mu_{s_1}^{1},\nu_{s_2}^{5}), (\mu_{s_1}^{2},\nu_{s_2}^{5}), (\mu_{s_1}^{3},\nu_{s_2}^{5}), (\mu_{s_1}^{4},\nu_{s_2}^{5})\}$

Details regarding clustering of the parameter set will be discussed in Section 5.4.

**Definition 2** (Upper Median Index). We call the index or multi-index that corresponds to the upper median of a set when ordered as defined in Definition 1 the upper median index.

**Example 2.** For the sets in Example 1, the upper median index of $I_1$ is $\tilde{m} = (4,1)$, the one of $I_2$ is $\tilde{m} = (2,3)$ and the upper median index of $I_3$ is $\tilde{m} = (1,5)$.

### 3.4 A single Newton step formulated as matrix equation

Let $\tilde{m} = (\tilde{m}_{s_1}^{i_1},\tilde{m}_{s_2}^{i_2}) \in \mathbb{N} \times \mathbb{N}$ be the upper median index of the parameter set $I_k$ for $k \in \{1,...,K\}$. Furthermore, let $x_{\epsilon_N}^{\tilde{m}}$ be the Newton approximation of the discretized FSI problem with a shear modulus $\mu_{\tilde{m}_{s_1}^{i_1}}$ and a kinematic fluid viscosity $\nu_{\tilde{m}_{s_2}^{i_2}}$. $\epsilon_N > 0$ defines the accuracy of the Newton approximation $x_{\epsilon_N}^{\tilde{m}}$. To be more precise, the stopping criterion for the Newton iteration is fulfilled whenever the residual norm is smaller than $\epsilon_N$. 

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Definition 3 (Operator diag). We extend the definition [15, Section 1.2.6] to

\[
\text{diag} \ A_i = \begin{pmatrix} A_1 & & \\
& \ddots & \\
& & A_m \end{pmatrix} \in \mathbb{R}^{M \times M}
\]

if the argument of the operator diag(·) is a set of matrices

\[A_i \in \mathbb{R}^{M \times M} \text{ for all } i \in \{1, \ldots, m\}.\]

(11) is block diagonal. If the argument of the operator diag(·) is a single square matrix, we have

\[
\text{diag}(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{MM} \end{pmatrix} \in \mathbb{R}^{M} \text{ for } A = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} \in \mathbb{R}^{M \times M}.
\]

Definition 4 (Diagonal Matrices). Similar to the diagonal matrices from [4],

\[
D_k = \text{diag} \ (\mu_i^1, \nu_j^2)_{\mathcal{I}_k} \text{ and } \quad D_k = \text{diag} \ (\mu_i^2, \nu_j^2)_{\mathcal{I}_k}
\]

for \(k \in \{1, \ldots, K\}\), we define

\[
D_k = D_k - \mu I_{\mathcal{I}_k} \quad \text{and} \quad D_k = D_k - \nu I_{\mathcal{I}_k}
\]

where \(I_{\mathcal{I}_k}\) denotes the \(|\mathcal{I}_k| \times |\mathcal{I}_k|\) identity matrix.

With this notation, we formulate a Newton step on the whole subset \(\mathcal{I}_k\) that uses \(x^{n_k}_{\epsilon_N}\) as initial guess.

We first formulate the Newton step on \(\mathcal{I}_k\) in the vector notation, similar to [32, Definition 2]. To do so, we need the vectorization operator.

Definition 5 (Vectorization restricted to \(\mathbb{R}^{M \times m}\)). For a matrix

\[
(v_1 | \ldots | v_m) \in \mathbb{R}^{M \times m} \text{ with columns } v_i \in \mathbb{R}^M \text{ for } i \in \{1, \ldots, m\},
\]

the vectorization operator [15, Section 5.1] is defined as

\[
\text{vec} : \mathbb{R}^{M \times m} \to \mathbb{R}^{Mm}, \quad \text{vec}(v_1 | \ldots | v_m) \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{Mm}.
\]

The inverse vectorization operator is defined as

\[
\text{vec}^{-1} : \mathbb{R}^{Mm} \to \mathbb{R}^{M \times m}, \quad \text{vec}^{-1}\left(\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}\right) \mapsto (v_1 | \ldots | v_m) \in \mathbb{R}^{M \times m}.
\]

Once the initial guess for one Newton step is fixed to \(x^{n_k}_{\epsilon_N}\) for \(k \in \{1, \ldots, K\}\), a Newton step for all problems related to the parameters in \(\mathcal{I}_k\) can be formulated as follows: Find \(s_k \in \mathbb{R}^{M|\mathcal{I}_k|}\) such that

\[
\text{diag} \ A(x^{n_k}_{\epsilon_N}, \mu_i^1, \nu_j^2)_{\mathcal{I}_k} s_k = b_D \otimes (1, \ldots, 1)^T - \left(g(x^{n_k}_{\epsilon_N}, \mu_i^1, \nu_j^2)_{\mathcal{I}_k}\right)_{(\mu_i^1, \nu_j^2)_{\mathcal{I}_k}}.
\]

(12)
\( A_k \in \mathbb{R}^{M|I_k| \times M|I_k|} \) is a block diagonal matrix. The unknown, the Newton update \( s_k \), is a vector. Therefore, we will call the notation used in (12) the **vector notation**. In this notation, the approximation for the next Newton step is

\[
x^k := \begin{pmatrix} x_{eN}^{\tilde{u}_k} \\ \vdots \\ x_{eN}^{\tilde{u}_k} \end{pmatrix} + s_k \in \mathbb{R}^{M|I_k|}.
\]

We now translate (12) to a matrix equation. Let

\[
B^\mu_\nu(x_{eN}^{\tilde{u}_k}) := (b_D - g(x_{eN}^{\tilde{u}_k}, 0, 0)) \oplus (1, \ldots, 1) - (A_1 x_{eN}^{\tilde{u}_k} + g_\mu(x_{eN}^{\tilde{u}_k})) \oplus \text{diag}(D^k_{\mu})^T
\]

\[
- \rho_f A_2 x_{eN}^{\tilde{u}_k} \otimes \text{diag}(D^k)^T,
\]

where \( g_\mu(x_k) \) evaluates the operator

\[
\langle \nabla u^T \nabla v, \nabla \varphi \rangle_S
\]

at the given argument \( u = u_k \). On the subset \( I_k \), for \( k \in \{1, \ldots, K\} \), the Newton step (12) is equivalent to the following matrix equation: Find \( S_k \in \mathbb{R}^{M \times |I_k|} \) such that

\[
A_0 S_k + A_1 S_k D_{\mu}^{\nu} + \rho_f A_2 S_k D_{\nu}^{\mu} = \lambda_0 J_{\lambda}(x_{eN}^{\tilde{u}_k}) S_k + \rho_f J_{\lambda}(x_{eN}^{\tilde{u}_k}) S_k = B^\mu_\nu(x_{eN}^{\tilde{u}_k}).
\]

The approximation for the next Newton step is given by

\[
X^k := x_{eN}^{\tilde{u}_k} \oplus (1, \ldots, 1) + S_k \in \mathbb{R}^{M \times |I_k|}.
\]

**Remark 1.** Note that

\[
\text{vec}(S_k) = s_k \quad \text{as well as} \quad \text{vec}(X^k) = x^k.
\]

The global approximation to the parameter-dependent problem is

\[
\tilde{X} = [X^1| \ldots |X^K].
\]

If the order of the parameters is chosen as defined in Definition 1, the column \( p \in \mathbb{N} \) of \( \tilde{X} \) is an approximation of the finite element solution to the FSI problem related to the parameters

\[
A_{\mu}^{p-1}(\mu_{p+1}^{m_1+1}, \nu_1^{m_2+1})
\]

We now explain how to approximate the matrix \( S_k \) in (14) and why we do not apply more than one Newton step on \( I_k \).

### 4 Low-rank methods and generalization

We first examine, from a theoretical point of view, the low-rank approximability of \( S_k \) in (14). Consider the parameter-dependent matrix \( A(x_{eN}^{\tilde{u}_k}, \mu^{l_1}_s, \nu^{l_2}_f) \) and the right hand side

\[
b(\mu^{l_1}_s, \nu^{l_2}_f) := b_D - g(x_{eN}^{\tilde{u}_k}, \mu^{l_1}_s, \nu^{l_2}_f)
\]

of the equations involved in the block diagonal problem (12). Once \( x_{eN}^{\tilde{u}_k} \) is computed for \( k \in \{1, \ldots, K\} \), we consider \( A(x_{eN}^{\tilde{u}_k}, \mu^{l_1}_s, \nu^{l_2}_f) \) as a matrix-valued function \( A(\mu^{l_1}_s, \nu^{l_2}_f) \) that depends on the parameters \( \mu^{l_2}_s, \nu^{l_2}_f \) only.
Algorithm 1 Two-parameter Discretization of Nonlinear FSI Problem

**Input:** Accuracy $\epsilon_N > 0$, ranks $R_k \in \mathbb{N}$ for $k \in \{1, ..., K\}$

**Output:** $\hat{X}$, a rank $K \sum_{k=1}^{K} R_k$ approximation of the parameter-dependent FSI discretization

Split the parameter set $S^{\mu, \nu}$ into the disjoint subsets $\bigcup_{k=1}^{K} I_k$

for $k = 1, ..., K$ do

Compute the Newton approximation $x_{\epsilon_N}^{\tilde{n}_k}$ of the problem related to the upper median parameters in $I_k$ with index $\tilde{n}_k$. Stopping criterion $\|\text{residual}\|_2 \leq \epsilon_N$

Use $x_{\epsilon_N}^{\tilde{n}_k}$ as initial guess for one Newton step on the subset $I_k$. By means of a low-rank method such as GMREST or the ChebyshevT method from [32, Algorithm 1 and 3], find a rank-$R_k$ approximation

$$\hat{X}_k = x_{\epsilon_N}^{\tilde{n}_k} \otimes (1, ..., 1) + S_k \in \mathbb{R}^{M \times |I_k|},$$

where $S_k$ is a solution to

$$F(S_k, x_{\epsilon_N}^{\tilde{n}_k}) = B_k^{\mu, \nu}(x_{\epsilon_N}^{\tilde{n}_k}),$$

with the notation from (14).

end for

The global approximation is then given by

$$\hat{X} := [\hat{X}_1 | \cdots | \hat{X}_K].$$

### 4.1 Approximability of $S_k$ by a low-rank matrix

Without loss of generality, we transform, for Section 4.1, the parameter set $S^{\mu, \nu}$ such that

$$S^{\mu, \nu} \subset [-1, 1] \times [-1, 1].$$

Let $\mathcal{E}_{\rho_0} \subset \mathbb{C}$ be the open elliptic disc with foci $\pm 1$ and sum of half axes of $\rho_0 > 0$. If, in a one-parameter discretization, $A(\cdot)$ and $b(\cdot)$ are assumed to have analytic extensions on $\mathcal{E}_{\rho_0}$ and $A(\cdot)$ is assumed to be invertible on $\mathcal{E}_{\rho_0}$, the singular value decay of the matrix $S_k$ in [14] is exponential as proven in [22, Theorem 2.4]. We define the open elliptic polydisc

$$\mathcal{E}_{\rho_0}^\times := \mathcal{E}_{\rho_0} \times \mathcal{E}_{\rho_0}.$$

**Corollary 1** (Case $p = 2$ of Theorem 3.6 in [22]). Assume that

$$b : [-1, 1] \times [-1, 1] \to \mathbb{R}^M \quad \text{and} \quad A : [-1, 1] \times [-1, 1] \to \mathbb{R}^{M \times M}$$

have analytic extensions on $\mathcal{E}_{\rho_0}^\times$ and $A(\mu, \nu)$ is invertible for all $(\mu, \nu) \in \mathcal{E}_{\rho_0}^\times$. There exists $\hat{S}_k \in \mathbb{R}^{M \times M}$ of rank $R \in \mathbb{N}$ for any $t = \frac{1}{q} - 1$ with $0 < q \leq 1$ such that

$$\|S_k - \hat{S}_k\|_F \leq \sqrt{Mm_1m_2CR^{-t}},$$

with $C$ as defined in [22, Theorem 3.6].

**Proof.** Corollary 1 follows directly from [22, Theorem 3.6]. \qed

**Remark 2.** Even though the term $R^{-t}$ suggests a polynomial decay of the error for an increasing rank $R$, one has to be careful about the constant $C$. The smaller $q$ is chosen, the bigger $C$ becomes. Choosing the right $q$ is difficult. However, we do not go into detail here and refer to [22, Section 3.1] for further reading.

The matrix $S_k$ in [14] can now be approximated by a low-rank method such as the GMREST or the ChebyshevT method from [32, Algorithm 1 and 3]. Once $\hat{S}_k$, a low-rank approximation of $S_k$, is computed, a low-rank approximation of $X_k$ in [15] can be achieved by a simple rank 1 update.

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4.2 Preconditioner

The mean-based preconditioner corresponding to the formulation [12] in vector notation is

\[ P^k_T := I_{\mathcal{I}_k} \otimes A(x^t_{\alpha_k}, \hat{\mu}_k, \nu_j^k), \]

where

\[ \hat{\mu}_k := \min_{(\mu^{i_1, \nu^{j_2}) \in \mathcal{I}_k} \frac{(\mu^{i_1} - \mu_s) + \max_{(\mu^{i_2}, \nu^{j_2}) \in \mathcal{I}_k} (\mu^{i_2} - \mu_s)}{2} \]

\[ \hat{\nu}_j^k := \min_{(\mu^{i_1, \nu^{j_2}) \in \mathcal{I}_k} \frac{(\nu^{j_2} - \nu_f) + \max_{(\mu^{i_2}, \nu^{j_2}) \in \mathcal{I}_k} (\nu^{j_2} - \nu_f)}{2} \]

for \( k \in \{1, \ldots, K\} \), similar to \( P_T \) in [32] Section 3.2].

Remark 3 (Multiple Newton Steps). We point out two difficulties that would come up if multiple Newton steps in the form [12] were performed. In a second Newton step, \( \hat{X}^k \), a low-rank approximation of \( X^k \) in [13], serves as initial guess.

In the first Newton step, the right hand side matrix \( B^k_v(x_{\alpha_k}^t) \) in [14] has rank not more than 3. Even if the initial guess \( \hat{X}^k \) for a second Newton step is a low-rank matrix, the columns of \( \hat{X}^k \) differ from each other. Therefore, the residual of the problem, the right hand side, to be more precise, \( g(\cdot, \cdot) \) has to be evaluated column by column \( |\mathcal{I}_k| \) times. This is too expensive and the low-rank structure of the right hand side matrix is not assured anymore in a second Newton step.

The first Newton step is indeed a Newton step. Due to the fact that all problems use the same initial guess \( x_{\alpha_N}^t \), the Jacobian matrix assembled is correct for all problems related to the respective subset \( \mathcal{I}_k \). In a second Newton step, this would not be the case. Even for different initial guesses, one single Jacobian matrix would be used for all problems related to the respective subset \( \mathcal{I}_k \).

The resulting method, a low-rank method for a two-parameter discretization of [11], is coded in Algorithm [1].

4.3 Time dependent fluid-structure interactions

Similar to [32] Section 5.1, we consider the time-dependent nonlinear fluid-structure interaction problem that couples the Navier-Stokes with the St. Venant Kirchhoff model equations. Let \( t \in [0, T] \subseteq \mathbb{R} \) be the time variable for \( T \in \mathbb{R} \) and \( \rho_s \in \mathbb{R} \) denote the solid density. The weak formulation of the non-stationary nonlinear fluid-structure interaction problem is

\[ \langle \nabla \cdot v, \xi \rangle_F = 0, \]

\[ \mu_s (\nabla u + \nabla u^T + \nabla u^T \nabla u, \nabla \varphi) + \lambda_s (\text{tr}(\nabla u + \frac{1}{2} \nabla u^T \nabla u) I, \nabla \varphi) + \rho_s (\partial_t v, \varphi)_S \]

\[ + \rho_f (\partial_t v, \varphi)_F + \mu_f ((v \cdot \nabla) v, \varphi)_F + \nu_f (\mu_f (v + \nabla v^T, \nabla \varphi)_F - \langle p, \nabla \cdot \varphi \rangle_F = 0 \]

(17)

and

\[ \langle \nabla u, \nabla \psi \rangle_F = 0, \]

with \( v \in L^2([0, T]; v_{0N} + H^1_0(\Omega, \Gamma^D \cup \Gamma_{\text{int}}^D)), \partial_t v \in L^2([0, T]; H^{-1}(\Omega))^d \) for all \((t, x) \in [0, T] \times \Omega \). We use the notation from [11]. For time discretization, we apply the \( \theta \)-scheme from [29] Section 4.1.4.

4.3.1 Time discretization with the \( \theta \)-scheme

Let the discretization matrices \( A^t_0, A^t_\theta \in \mathbb{R}^{M \times M} \) be such that

\[ A^t_0 \] discretizes \( \langle v, \varphi \rangle_F \) and \( A^t_\theta \) discretizes \( \langle v, \varphi \rangle_S \).
We consider the discrete time interval $[0, T]$ that is split into $w + 1 \in \mathbb{N}$ equidistant time steps. The start time is $t_0 = 0$, the time steps are given by $t_i := i \Delta t$ for $i \in \{0, \ldots, w\}$, $\Delta t = t_{i+1} - t_i$ for all $i \in \{0, \ldots, w - 1\}$ and $w \Delta t = T$. Let $b^i_D \in \mathbb{R}^M$ be the right hand side at time $t_i$ and $B^i := b^i_D \otimes (1, \ldots, 1)$ for $i \in \{0, \ldots, w\}$.

Since both time-dependent operators in \([17]\), $(\ast)$ on the solid and $(\ast\ast)$ on the fluid, depend linearly on the unknown, a parameter-dependent discretization is straightforward.

We start to discuss the problem with fixed parameters $(\mu^i_s, \nu^i_f)$ in $S^{\mu, \nu}$. Let $x^{i+1, i+2, t_0}$ at time $t_0 = 0$ be the initial. We can compute $x^{i+1, i+2, t_0}$ as an approximation to the stationary problem or simply set $x^{i+1, i+2, t_0} = b^i_D$. Let $\theta \in (0, 1]$. $X^{i+1, i+2, t_{i-1}} \in \mathbb{R}^\Delta$ denotes the approximation of the previous time step. At the Newton step $j \in \mathbb{N}$, find $s \in \mathbb{R}^\Delta$ such that

$$(1 - \theta)g(x^{i+1, i+2, t_{i-1}}, \mu^i_s, \nu^i_f) - \theta g(x^{i+1, i+2, t_{i-1}}, \mu^i_s, \nu^i_f),$$

with the notation from \([8]\). The approximation at the next Newton step is given by

$$x^{i+1, i+2, t_i} = x^{i+1, i+2, t_{i-1}} + \delta x^{i+1, i+2, t_i}.$$  

After $l \in \mathbb{N}$ Newton steps, the approximation of problem \([17]\) at time $t_i$, related to the parameter combination $(\mu^i_s, \nu^i_f)$, is given by $x^{i+1, i+2, t_i} \in \mathbb{R}^\Delta$. \([18]\) can be, for a set of problems, translated to one single matrix equation similar to \([14]\). Now, the Newton update at time step $t_i$, $S^t_k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$, is time-dependent.

We consider the problems related to the parameters in the subset $\mathcal{I}_k$. Let $X^{k, t_i} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$ be the approximation at time step $t_i$ and $x^{n^k, t_i} \in \mathbb{R}^\Delta$ the Newton approximation of the upper median parameter problem for $i \in \{1, \ldots, w\}$. $x^{n^k, t_i}$ is computed via \([18]\) for $(i_l, i_r) = m^k$.

At time $t_0$, $X^{k, t_0} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$ is given as initial value as well as $x^{n^k, t_0} \in \mathbb{R}^\Delta$. On the subset $\mathcal{I}_k$ at time step $t_i$ for $i > 0$, we apply the $\theta$-scheme for $\theta \in (0, 1]$. In the Newton step on $\mathcal{I}_k$, we find $S^t_k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$ such that

$$1 - \theta)g(x^{k, t_{i-1}}, \mu^i_s, \nu^i_f) - \theta g(x^{k, t_{i-1}}, \mu^i_s, \nu^i_f),$$

where $X^{k, t_{i-1}} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$ denotes the approximation of the last time step related to the parameters $(\mu^i_s, \nu^i_f)$.

**Remark 4** (Right Hand Side). Evaluation of the right hand side in \([14]\), especially $(\ast\ast\ast)$, is expensive. $(\ast\ast)$ could be approximated by

$$(1 - \theta)B^\mu, \nu(x^{n^k, t_{i-1}}),$$

which has at most rank 3 and, therefore, is cheaper to evaluate.

**Remark 5** (Implicit Treatment of the Pressure). As discussed in \([22\text{, Section 4.1.4]}\), the pressure operator

$$-\langle p, \nabla \cdot \varphi \rangle_F$$

is always treated implicit.

Equation \([19]\) is, in matrix notation, a Newton step of the discretization of \([17]\) on $\mathcal{I}_k$ at time step $t_i$ with initial guess $x^{n^k, t_i}$. Hence, parameter-dependent discretizations of \([17]\) can be approached by low-rank methods that approximate the Newton update $S^t_k$ at time step $t_i$, the solution to \([19]\).
4.4 Extension to further parameters

Consider a parameter-dependent discretization of problem (1) with respect to \(m_1\) shear moduli, \(m_2\) kinematic fluid viscosities, \(m_3\) first Lamé parameters and \(m_4\) fluid densities. The parameter set is now given by

\[
S^{u,v,\lambda,\rho} = \{(\mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}) \}_{i_1=1,...,m_1, i_2=1,...,m_2, i_3=1,...,m_3, i_4=1,...,m_4} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.
\]

We fix the parameters \((\mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}) \in S^{u,v,\lambda,\rho}\) and consider the Newton iteration for the problem related to this parameter choice. Let the discrete operator \(A_3 \in \mathbb{R}^{M \times M}\) be such that

\[
A_3 \text{ discretizes } (\text{tr}(\nabla u) I, \nabla \varphi)_S.
\]

With \(x_j^{i_1,i_2,i_3,i_4} = b_D\), the parameter-dependent Jacobian matrix of the Newton step \(j \in \mathbb{N}\) is

\[
A(x_j^{i_1,i_2,i_3,i_4}, \mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}) := A_0 + (\mu_i^{i_1} - \mu_s) A_1 + (\nu_i^{i_2} \rho_i^{i_4} - \nu_f \rho_f) A_2 + (\lambda_i^{i_3} - \lambda_s) A_3 + \mu_i^{i_1} J_\mu(x_j^{i_1,i_2,i_3,i_4}) + \lambda_i^{i_3} J_\lambda(x_j^{i_1,i_2,i_3,i_4}) + \rho_i^{i_4} J_\rho(x_j^{i_1,i_2,i_3,i_4}),
\]

Find the Newton update \(\delta x_j^{i_1,i_2,i_3,i_4} \in X_h\) such that

\[
A(x_j^{i_1,i_2,i_3,i_4}, \mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}) \delta x_j^{i_1,i_2,i_3,i_4} = b_D - g(x_j^{i_1,i_2,i_3,i_4}, \mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}),
\]

where \(g(x_j^{i_1,i_2,i_3,i_4}, \mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4})\) denotes the residual of the problem (1) evaluated at \(x_j^{i_1,i_2,i_3,i_4}\) and parameters \((\mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4})\). The approximation for the next linearization step is given by

\[
x_j^{i_1,i_2,i_3,i_4} = x_j^{i_1,i_2,i_3,i_4} + \delta x_j^{i_1,i_2,i_3,i_4}.
\]

4.4.1 Clustering

Again, the grouping operation of the indexes of the elements in \(S^{u,v,\lambda,\rho}\) is little endian. For

\[
(i_1, i_2, i_3, i_4) \in \{1, ..., m_1\} \times \{1, ..., m_2\} \times \{1, ..., m_3\} \times \{1, ..., m_4\},
\]

it holds

\[
(\mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}) < (\mu_j^{j_1}, \nu_j^{j_2}, \lambda_j^{j_3}, \rho_j^{j_4}) \quad \text{if} \quad i_1 + (i_2 - 1)m_1 + (i_3 - 1)m_1m_2 + (i_4 - 1)m_1m_2m_3 < l_1 + (l_2 - 1)m_1 + (l_3 - 1)m_1m_2 + (l_4 - 1)m_1m_2m_3.
\]

Let the clustering of the parameter set

\[
S^{u,v,\lambda,\rho} = \bigcup_{k=1}^K \mathcal{I}_k
\]

be as defined in (13). For the sake of readability, we define

\[
\Xi^k := \{(i_1, i_2, i_3, i_4) : (\mu_i^{i_1}, \nu_i^{i_2}, \lambda_i^{i_3}, \rho_i^{i_4}) \in \mathcal{I}_k\}
\]

and simply write

\[
(i_2, i_4) \in \Xi^k \quad \text{for the set} \quad \{(i_2, i_4) : (i_1, i_2, i_3, i_4) \in \Xi^k\}.
\]

4.4.2 Diagonal matrices

To formulate the Newton step on \(\mathcal{I}_k\) as a matrix equation, we need the matrices

\[
D_{\nu_\nu}^k := \text{diag}(\nu_j^{j_2} \rho_j^{j_4}), \quad D_{\nu_\rho}^k := \text{diag}(\nu_j^{j_2} \rho_j^{j_4} I_{|\mathcal{I}_k|}), \quad D_\lambda^k := \text{diag}(\lambda_j^{j_3}), \quad D_\rho^k := \text{diag}(\rho_j^{j_4})
\]

and

\[
D_{\nu_\rho}^k := \text{diag}(\nu_j^{j_2} \rho_j^{j_4} I_{|\mathcal{I}_k|})
\]

in addition to the diagonal matrices from Definition 3.
4.4.3 The matrix equation for four parameters

Let \( \hat{m}^k = (\hat{m}_1^k, \hat{m}_2^k, \hat{m}_3^k, \hat{m}_4^k) \in \Xi^k \) be the index corresponding to the upper median parameter of the subset \( \mathcal{I}_k \) and \( x_{\hat{m}^k}^k \in \mathbb{R}^M \) the Navier approximation of the related problem. In addition to the evaluation function \( g_\mu(x_h) \) and \( g_\rho(x_h) \) such that they evaluate the operators

\[
\frac{1}{2} \langle \nabla u^T \nabla u \rangle_S \quad \text{and} \quad \langle (v \cdot \nabla) v, \varphi \rangle_F
\]

at the given argument \( u = u_h \) and \( v = v_h \), respectively. The right hand side in the matrix equation is

\[
B_{\mu,\nu,\lambda,\rho}^k(x_{\hat{m}^k}^k) := \left( b_D - g(x_{\hat{m}^k}^k, 0, 0, 0, 0) \right) \otimes (1, \ldots, 1) - \left( A_1 x_{\hat{m}^k}^k + g_\mu(x_{\hat{m}^k}^k) \right) \otimes \text{diag}(D_\mu^k)^T
- A_2 x_{\hat{m}^k}^k \otimes \text{diag}(D_\nu^k)^T - \left( A_3 x_{\hat{m}^k}^k + g_\lambda(x_{\hat{m}^k}^k) \right) \otimes \text{diag}(D_\lambda^k)^T - g_\rho(x_{\hat{m}^k}^k) \otimes \text{diag}(D_\rho^k)^T.
\]

At the Newton step on the subset \( \mathcal{I}_k \), we find \( S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|} \) such that

\[
A_0 S_k + A_1 S_k D_{\mu^k}^k + A_2 S_k D_{\nu^k}^k + A_3 S_k D_\lambda^k + J_\mu(x_{\hat{m}^k}^k) S_k D_\mu^k + J_\lambda(x_{\hat{m}^k}^k) S_k D_\lambda^k + J_\rho(x_{\hat{m}^k}^k) S_k D_\rho^k = B_{\mu,\nu,\lambda,\rho}^k(x_{\hat{m}^k}^k).
\]

For a four-parameter discretization, the matrix equation \([16]\) in Algorithm \([1]\) is replaced by \([20]\).

5 Numerical study

We consider the 3d test case from \([29]\) Section 8.3.2.2 with the domain \( \Omega \), split into solid \( S \) and fluid \( F \)

\[
\Omega := (0,1.5) \times (0,0.4) \times (-0.4,0.4), \quad S := (0.4,0.5) \times (0,0.2) \times (-0.2,0.2) \quad \text{and} \quad F := \Omega \setminus S.
\]

A parabolic inflow with an average velocity \( v_{in} = 0.15 \) is given at \( x = 0 \). At \( x = 1.5 \), the do-nothing condition \( v \cdot n_F = 0 \) holds for the velocity and the pressure. The problem is plane symmetric with respect to \( z = 0 \). This is why we compute approximations in one half of the domain only (compare Figure \([1]\)). On the symmetry plane we demand

\[
v_f \cdot n = 0 \quad \text{and} \quad u_s \cdot n = 0 \quad \text{for} \quad z = 0.
\]

On the right boundary at \( x = 1.5 \) the do-nothing outflow condition holds and on all boundaries, velocity and deformation fulfill zero Dirichlet boundary conditions.

In this paper, we use tri-quadratic finite elements on a hexahedral mesh for the pressure, the velocity and the deformation. The Navier-Stokes equations are stabilized by local projection stabilization (LPS) \([3]\) and \([29]\) Lemma 4.49, by projecting the pressure onto the space of tri-linear finite elements on the same mesh, see \([29]\) Section 4.3.2. Hereby the equal-order approach is applicable to pressure,
deformation and velocity, which is relevant for formulating the matrix equation, compare \( (6) \). Our finite element mesh \( \Omega_h \) and discretization is such that the total number of unknowns is \( M = 66759 \).

The first Lamé parameter \( \lambda_s \) to is taken as \( 2000000 \) and the fluid density \( \rho_f \) as \( \rho_f = 1000 \). The stationary fluid-structure interaction problem \( (1) \) is discretized with respect to the parameter set \( m_1 = 500 \) shear moduli \( \mu_i^1 \) and \( \mu_i^2 \) \( \in [400000, 600000] \) and \( m_2 = 10 \) kinematic fluid viscosities \( \nu^1_f \) \( \in [0.001, 0.003] \).

With this choice we cover solids with Poisson ratios between 0.38 and 0.41 and Reynolds numbers between \( \approx 20 \) and \( \approx 60 \) if the characteristic length is assumed to be \( L = 0.4 \).

All computations were performed with MATLAB\textsuperscript{®} 2018a in combination with the finite element toolkit Gascoigne 3d \cite{gascoigne} on a CentOS Linux release 7.5.1804 with a dual-socket Intel Xeon Silver 4110 and 192 GB RAM. \cite[Algorithm 6]{brandt} was used to realize the truncation operator \( T(\cdot) \) from \cite[Definition 4]{brandt} that maps, for \( R \in \mathbb{N} \), into \( T_R \), the space of Tucker tensors from \cite[Definition 3]{brandt}. Using the MATLAB builtin command \( \text{lu()} \), the preconditioners were decomposed into a permuted LU decomposition.

### 5.1 Low-rank approximation versus separate Newton iterations

As a first test we compare the performance of Algorithm \ref{alg:alg0} with the individual solution of all problems by a Newton iteration. In Algorithm \ref{alg:alg0} we set \( \epsilon_N = 10^{-6} \), \( K = 50 \) and \( R_k = 10 \) for all \( k \in \{1, ..., K\} \). The indices of the parameter set \( S^{\mu, \nu} \) were ordered little endian. The ChebyshevT and the GMREST methods both were restarted once after 6 iterations. To compute \( \tilde{x}_{m_k}^{\epsilon N} \), for \( k > 1 \), \( x_{m_k}^{\epsilon N - 1} \) served as initial guess for the Newton iteration such that mostly a single Newton step is sufficient.

#### 5.1.1 ChebyshevT parameter estimation

To estimate the parameters \( c \) and \( d \) for the ChebyshevT method (compare \cite[Section 4.3]{brandt}), a downgraded variant of Algorithm \ref{alg:alg0} was ran for a small number of \( M_c = 1575 \) degrees of freedom. Let \( \bar{I}_k \subset I_k \) such that

\[
\bar{I}_k = \{ (\mu_i^1, \nu^2_f) \in I_k : (\mu_i^1 \geq \mu_0^1 \lor \mu_i^1 \leq \mu_{\bar{0}}^1) \land (\nu^2_f \geq \nu^2_f \lor \nu^2_f \leq \nu^2_f) \ \text{for all} \ (\mu_i^1, \nu^2_f) \in I_k \}
\]

and define

\[
Bl^k(\mu^1_i, \nu^2_f) := (P^k_f)^{-1} A(x_{m_k}^{\epsilon N}, \mu_i^1, \nu_i^2_f),
\]
Table 1: Algorithm 1 Compared With the Standard Newton Method

| Method                          | Approx. Storage | Newton Steps | Computation Times (Minutes) |
|---------------------------------|-----------------|--------------|----------------------------|
| Alg. 1, $R = 500$ with GMREST (green) | $O((M + m + R)R)$ $≈ 275.65$ MB | 50+106 -156 | - 1815.5 426.4 $≈ 21$ hours |
| Alg. 1, $R = 500$ with ChebyshevT (blue) | $O((M + m + R)R)$ $≈ 275.65$ MB | 50+106 -156 | 19.8 815.4 426 $≈ 21$ hours |
| Standard Newton, 5000 times, $\epsilon_N = 10^{-6}$ (red) | $O[Nm]$ $≈ 2546.65$ MB | 5016 - - - | - - - $≈ 38757$ |
| Standard Newton, 5000 times, $\epsilon_N = 10^{-12}$ (cyan) | $O[Nm]$ $≈ 2546.65$ MB | 10421 - - - | - - - $≈ 80108$ |

50 Newton steps were required by Algorithm 1 on the subsets $T_k$, 106 to compute the approximations $x_i^{n_i^k}$. The column "Est." shows the time needed to estimate the ChebyshevT parameters $c$ and $d$, "Indiv." the time needed to compute the Newton approximations $x_i^{n_i}$ and "Comp." the time needed to approximate $S_k$ in the matrix equations and assemble $X$.

with the preconditioner $P_T^k$ from Section 4.2 The quantities

$$\Lambda_{\max}^k := \max\{\{\lambda : \lambda \in A((P_T^k)^{-1}A_k)\}\}$$

and

$$\Lambda_{\min}^k := \min\{\{\lambda : \lambda \in A((P_T^k)^{-1}A_k)\}\}$$

were approximated by

$$\hat{\Lambda}_{\max}^k := \max_{(\mu^1,\nu^2) \in \mathcal{T}_k} \{\{\lambda : \lambda \in A(B_{\mathcal{L}}(\mu^1,\nu^2))\}\}$$

and

$$\hat{\Lambda}_{\min}^k := \min_{(\mu^1,\nu^2) \in \mathcal{T}_k} \{\{\lambda : \lambda \in A(B_{\mathcal{L}}(\mu^1,\nu^2))\}\}.$$
approximate values of the discretization errors. We estimate the relative error $P$ where $m = m_1 m_2 = 4501$ (right dashed vertical line in Figure 2b) is $\approx 5.1 \cdot 10^{-4}$. Therefore, we would like to examine the quality of the approximation of the problem related to the parameter pair $(\mu_1^3, \nu_1^3)$ at $m = 4501$ a bit more. In addition, we also consider the approximation of the problem related to $m = 2001$, the parameters $(\mu_1^4, \nu_1^4)$ (left dashed vertical line in Figure 2b). There, the relative residual norm is $\approx 3.6 \cdot 10^{-4}$. For these two parameter combinations, each of which has a large error, we try to split the error into the discretization error and the error by applying the low-rank method.

Table 2: Error Analysis for Algorithm 1

| Parameters $(\mu_1^2, \nu_1^2)$ | $\epsilon_N$ | $err_{x-def}^{P_1}$ | $err_{disc}^{P_1}_{x-def}$ | $err_{drag}$ | $err_{disc}^{drag}$ |
|-------------------------------|--------------|---------------------|-----------------------------|--------------|---------------------|
| $(\mu_1^4, \nu_1^4)$         | $10^{-6}$    | $2.626 \cdot 10^{-7}$ | $4.566 \cdot 10^{-3}$      | $1.043 \cdot 10^{-7}$ | $1.375 \cdot 10^{-2}$ |
|                              | $10^{-12}$   | $2.655 \cdot 10^{-7}$ | $4.566 \cdot 10^{-3}$      | $1.046 \cdot 10^{-7}$ | $1.375 \cdot 10^{-2}$ |
| $(\mu_1^2, \nu_1^2)$         | $10^{-6}$    | $4.145 \cdot 10^{-7}$ | $1.048 \cdot 10^{-2}$      | $1.200 \cdot 10^{-7}$ | $1.129 \cdot 10^{-2}$ |
|                              | $10^{-12}$   | $4.432 \cdot 10^{-7}$ | $1.048 \cdot 10^{-2}$      | $1.177 \cdot 10^{-7}$ | $1.129 \cdot 10^{-2}$ |

5.2 Error analysis

We next explore the influence of the algebraic error arising from the approximation of the discrete equations. For this purpose, we consider two cases in detail. The relative residual norm of the approximation of the parameter combination $m = m_1 m_2 = 4501$ (right dashed vertical line in Figure 2b) is $\approx 5.1 \cdot 10^{-4}$. Therefore, we would like to examine the quality of the approximation of the problem related to the parameter pair $(\mu_1^3, \nu_1^3)$ at $m = 4501$ a bit more. In addition, we also consider the approximation of the problem related to $m = 2001$, the parameters $(\mu_1^4, \nu_1^4)$ (left dashed vertical line in Figure 2b). There, the relative residual norm is $\approx 3.6 \cdot 10^{-4}$. For these two parameter combinations, each of which has a large error, we try to split the error into the discretization error and the error by applying the low-rank method.

To measure the quality of the approximation we consider the two error functionals that have been proposed in [20] and [29] Section 8.3.2.2. These are the $x$-deflection in the point $P_1 = (0.4, 0.2, 0.2)$ from Figure 1 and the force of the fluid onto the structure in main flow direction

$$J_{x-def}(x_h) := \epsilon_1 \cdot u_h(P_1), \quad J_{drag}(x_h) := \int_{\Gamma_{int}} \sigma F \cdot n \cdot \epsilon_1 ds.$$ 

Here, $\epsilon_1 \in \mathbb{R}^3$ denotes the first unit vector and $u_h(P_1) \in \mathbb{R}^3$ is the deformation in the point $P_1$.

Let $(i_1, i_2) \in \Xi^{m_1, m_2}$. $\hat{x}^{i_1, i_2} \in \mathbb{R}^{M_2}$ denotes the approximation of the FSI problem related to the parameter combination $(\mu_1^3, \nu_1^3)$ provided by Algorithm with the GMREST method. The respective approximation provided by the standard Newton method with $\epsilon_N = 10^{-6}$ and $\epsilon_N = 10^{-12}$ is denoted by $x^{i_1, i_2} \in \mathbb{R}^M$ and $x^{i_1, i_2} \in \mathbb{R}^M$, respectively. To compute a reference solution, we go to a finer grid, where the number of degrees of freedom is $M_f = 495495$. The Newton stopping criterion is set to $\|\text{residual}\|_2 \leq 10^{-12}$ and we denote the approximation obtained by $x^{i_1, i_2} \in \mathbb{R}^{M_f}$.

We list the relative quantities

$$err_{x-def}^{P_1} := \frac{|J_{x-def}(\hat{x}^{i_1, i_2}) - J_{x-def}(x^{i_1, i_2}^{\epsilon_N})|}{|J_{x-def}(x^{i_1, i_2}^{\epsilon_N})|}$$

$$err_{drag} := \frac{|J_{drag}(\hat{x}^{i_1, i_2}) - J_{drag}(x^{i_1, i_2}^{\epsilon_N})|}{|J_{drag}(x^{i_1, i_2}^{\epsilon_N})|}$$

$$err_{disc}^{P_1}_{x-def} := \frac{|J_{x-def}(\hat{x}^{i_1, i_2}) - J_{x-def}(x^{i_1, i_2}^{\epsilon_N})|}{|J_{x-def}(x^{i_1, i_2}^{\epsilon_N})|}$$

$$err_{disc}^{drag} := \frac{|J_{drag}(\hat{x}^{i_1, i_2}) - J_{drag}(x^{i_1, i_2}^{\epsilon_N})|}{|J_{drag}(x^{i_1, i_2}^{\epsilon_N})|},$$

where $(i_1, i_2) \in \Xi^{m_1, m_2}$ and $\epsilon_N \in \{10^{-6}, 10^{-12}\}$ in Table 2. We assume that $x^{i_1, i_2}$ approximates the solution sufficiently well. Therefore, the quantities $err_{x-def}^{P_1}$ and $err_{disc}^{drag}$ can be considered as approximate values of the discretization errors. We estimate the relative error

$$\frac{|J_{drag}(\hat{x}^{i_1, i_2}) - J_{drag}(x^{i_1, i_2})|}{J_{drag}(x^{i_1, i_2})} \approx \frac{|J_{drag}(\hat{x}^{i_1, i_2}) - J_{drag}(x^{i_1, i_2})|}{J_{drag}(x^{i_1, i_2})} \leq err_{disc}^{drag} + err_{drag}.$$
where $J_{\text{drag}}^{\text{1,2}}$ denotes the force of the fluid on the structure in $x$-direction of the solution. Furthermore, the error $\text{err}_{\text{drag}}$ is the error of Algorithm 1.

As listed in Table 2, we see that the discretization errors, $\text{err}_{\text{disc, x-defl}}$ and $\text{err}_{\text{disc, drag}}$, are always bigger than $\text{err}_{\text{P1, x-defl}}$ and $\text{err}_{\text{drag}}$, the errors made by the GMREST method. In this sense, the GMREST method provided approximations with accuracies that are at least of order $10^{-4}$ more accurate than required.

5.3 Fixed-point iteration and alternative clustering

Algorithm 1 is based on the Newton iteration. We now investigate why this Newton approach is more advantageous than basing Algorithm 1 on the Picard (fixed-point) iteration in the first place.

An advantage of the fixed-point over the Newton iteration is that the right hand side does not need to be evaluated newly at every iteration. As in Section 3.1, we define the matrices $F_\rho(\cdot)$, $F_\mu(\cdot)$ and $F_\lambda(\cdot) \in \mathbb{R}^{M \times M}$ such that

\[
\begin{align*}
F_\rho(x_h) & \text{ discretizes } \langle (v_h \cdot \nabla)v, \varphi \rangle_F, \\
F_\mu(x_h) & \text{ discretizes } \langle \nabla u_h^T \nabla u, \nabla \varphi \rangle_S \quad \text{ and} \\
F_\lambda(x_h) & \text{ discretizes } \frac{1}{2} \langle \text{tr}((\nabla u_h^T \nabla u)I), \nabla \varphi \rangle_S.
\end{align*}
\]

$v_h$ is the velocity of the unknown $x_h$, the approximation of the previous linearization step. Since we solve the actual equation for $v$, replacing $(v \cdot \nabla)v$ by $(v_h \cdot \nabla)v$ coincides with the Oseen fixed point linearization of the convection term mentioned in [29, Section 4.4.1].

With the notation from (14), a fixed-point step on the subset $\mathcal{I}_k$ translates to the problem of finding $X^k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$ such that

\[
A_0 X^k + A_1 X^k D_{\mu^k} + \rho_f A_2 X^k D_{\nu^k} + F_\rho(x_{\epsilon_n}^k) X^k D_{\rho}^k + \lambda_\epsilon F_\lambda(x_{\epsilon_n}^k) X^k + \rho_f F_\rho(x_{\epsilon_n}^k) X^k = b_D \otimes (1, ..., 1). 
\]

In Algorithm 1, we replace (16) by (21). Once $x_{\epsilon_n}^k$ is computed on the respective subset, we directly approximate $X^k$ in (16) with a low-rank method. No update is needed here. The variant of Algorithm 1 that uses fixed-point iteration on the subset level (Alg. 1 with GMREST (fixed-point it. on $\mathcal{I}_k$)) is compared with Algorithm 1 in Figure 3. The relative residual norms (in green)
denoted by Alg. 1 with GMREST (Newton it. on $I_k$) in Figure 3 coincide with the ones in Figure 2b. For Algorithm 1 based on fixed-point iteration (in red), we used not only the identical setup as the one in Section 5 but also exactly the same initial guesses $x_{\tilde{m}}^{i_k}$ for $k \in \{1, \ldots, K\}$. In addition, we plot the relative residual norms of $x_{\tilde{m}}^{i_k}$ on the respective subsets $I_k$ (in blue) for visualization.

**Remark 6.** At the parameter combination corresponding to $m = 2000$, another initial guess is used than at the parameter combination corresponding to $m = 2001$. This is why there is a kink at $m = 2000$.

A Newton iteration on the subset level provides faster convergence and therefore, more accurate approximations than a fixed-point iteration on the subset level (compare Figure 3a). This is illustrated in Figure 3b, where the approximations computed by the algorithm based on the Newton iteration has a relative residual norm that is at least 10 times smaller than the ones computed by the algorithm based on the fixed-point iteration.

### 5.4 Alternative ways to cluster the parameter set

![Figure 4: The relative residual norms of the approximations obtained by Algorithm 1 with different numbers of subsets.](image)

In [5], a one-parameter discretization of a nonlinear FSI problem was performed. The solid equations used there are linear and the residual norms in the respective numerical example obtained by [5, Algorithm 1] are all smaller than $10^{-9}$. This is remarkably smaller than the ones obtained in Section 5. The reason for this is, on the one hand, that the kinematic fluid viscosity in the numerical example in [5] is, $\nu_f = 0.04$, chosen much bigger than the kinematic fluid viscosities considered in this paper. On the other hand, the equations on the solid of the problem considered in this paper are nonlinear. The problems that are grouped by a subset $I_k$ differ such that a Newton step in the form (14) converges for problems that do not lie central in $I_k$, in the worst case, to a relative residual norm not smaller than $\approx 5.1 \cdot 10^{-4}$. The question whether choosing the subsets adaptively would bring any benefits arises in this context.

In Figure 4, we compare the relative residual norms of the approximations obtained by Algorithm 1 with GMREST for different numbers of subsets. The green line corresponds to the ones of the first 200 parameter combinations in Figure 2b. For $K = 100$ (corresponding line in blue), the maximal relative residual norm of the approximations to the problems related to the first 200 parameter combinations could be reduced from $\approx 2.44 \cdot 10^{-4}$ to $\approx 6.34 \cdot 10^{-5}$. A further refinement to $K = 200$ (line in red) resulted in a maximal relative residual norm of $\approx 1.51 \cdot 10^{-5}$. Reducing the rank $R_k$ to 5 (in case of $K = 100$ and $K = 200$) did not lead to an increasing maximum in terms of relative residual norms.

However, in multi-parameter discretizations like the one considered in Section 5, smaller subsets could help to get higher accuracies for some problems. But it is not clear how to group the problems if fluid and solid parameters are varied at the same time and the cardinalities of the subsets $I_k$ are not similar.
for all \( k \in \{1, \ldots, K\} \). It is still open how the clusters Algorithm 1 uses can be chosen more sophisticatedly or adaptively refined in a multi-parameter setup that requires more accurate approximations.

6 Conclusions

Algorithm 1 allows to compute low-rank approximations for parameter-dependent nonlinear FSI problems with lower complexity than standard approaches. We reduce the complexity of the resulting algorithm by applying low-rank approaches. Even though we only perform one Newton step on the subset level, the errors of the approximations we obtain with Algorithm 1 if compared with the ones provided by full approaches on the respective mesh, are smaller than the discretization errors. Essentially, Algorithm 1 is applicable to other coupled nonlinear problems such as FSI problems in ALE formulation and can be used, in principle, for finite difference, finite volume or any finite element discretization. The approximations we obtained in Section 5 provide accuracies that are more exact than required (compare Table 2). Moreover, we have demonstrated, why, on the subset level, the Newton linearization is to be preferred to fixed-point iteration. For multi-parameter discretizations, yet, it is not clear how to improve the choice of the subsets. The problems grouped by one subset should converge to a small residual norm in one Newton step with the same initial guess.

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