Existence Theory for Stochastic Power Law Fluids

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Abstract. We consider the equations of motion for an incompressible non-Newtonian fluid in a bounded Lipschitz domain $G \subset \mathbb{R}^d$ during the time interval $(0, T)$ together with a stochastic perturbation driven by a Brownian motion $W$. The balance of momentum reads as

$$\text{div} \, v = \text{div} \, S \, dt - (\nabla v) \cdot v \, dt + \nabla \pi \, dt + f \, dt + \Phi(v) \, dW_t,$$

where $v$ is the velocity, $\pi$ the pressure and $f$ an external volume force. We assume the common power law model $S(\varepsilon(v)) = (1 + |\varepsilon(v)|)^{p-2} \varepsilon(v)$ and show the existence of martingale weak solution provided $p > \frac{2d+2}{d+2}$. Our approach is based on the $L^\infty$-truncation and a harmonic pressure decomposition which are adapted to the stochastic setting.

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1. Introduction

The flow of a homogeneous incompressible fluid in a bounded Lipschitz body $G \subset \mathbb{R}^d$, $(d = 2, 3)$, during the time interval $(0, T)$ is described by the following set of equations on $Q := (0, T) \times G$ (see for instance [4])

$$\begin{aligned}
\rho \partial_t v + \rho (\nabla v) \cdot v &= \text{div} \, S - \nabla \pi + \rho f \quad \text{in } Q, \\
\text{div} \, v &= 0 \quad \text{in } Q, \\
v &= 0 \quad \text{on } \partial G, \\
v(0, \cdot) &= v_0 \quad \text{in } G.
\end{aligned} \tag{1.1}
$$

Here the unknown quantities are the velocity field $v : Q \rightarrow \mathbb{R}^d$ and the pressure $\pi : Q \rightarrow \mathbb{R}$. The functions $f : Q \rightarrow \mathbb{R}^d$ represent a system of volume forces, $v_0 : G \rightarrow \mathbb{R}^d$ the initial datum, $S : Q \rightarrow S^d$ is the stress deviator and $\rho > 0$ the density of the fluid. Equation (1.1)$_1$ and (1.1)$_2$ describe the conservation of mass and the conservation of balance of mass respectively. Both are valid for all homogeneous liquids and gases. In order to describe a specific fluid one needs a constitutive law which relates the stress deviator $S$ to the symmetric gradient $\varepsilon(v) := \frac{1}{2} (\nabla v + \nabla v^T)$ of the velocity $v$. In the easiest case this relation is linear, i.e.,

$$S = S(\varepsilon(v)) = \nu \varepsilon(v), \tag{1.2}$$

where $\nu > 0$ is the viscosity of the fluid. In this case we have $\text{div} \, S = \frac{\nu}{2} \Delta v$ and (1.1) are the classical Navier–Stokes equations. Its mathematical observation started with the work of Leray and Ladyshenskaya (see [27] and for a more recent approach [22, 23]). The existence of a weak solution (where derivatives are

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1 For a better understanding of the problem we start with a survey about the deterministic problem but the topic of the paper is the corresponding SPDE.
2 $S^d :=$ space of all symmetric $d \times d$ matrices; the full stress tensor is $\sigma = S - \pi I$. 

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to be understood in a distributional sense) can be shown by nowadays standard arguments. However the regularity (i.e. the existence of a strong solution) is still open.

Only fluids with simple molecular structure e.g. water, oil and certain gases fulfil a linear relation such as (1.2). Those who does not are called non-Newtonian fluids (see [2]). A special class among these are generalized Newtonian fluids. Here the viscosity is assumed to be a function of the shear rate \(|\varepsilon(\mathbf{v})|\) and the constitutive relations reads as

\[
S(\varepsilon(\mathbf{v})) = \nu(|\varepsilon(\mathbf{v})|)\varepsilon(\mathbf{v}).
\] (1.3)

An external force can produce two different reactions:

- The fluid becomes thicker (for example batter): the viscosity of a shear thickening fluid is an increasing function of the shear rate;
- The fluid becomes thinner (for example ketchup): the viscosity of a shear thinning fluid is a decreasing function of the shear rate.

The power-law model for non-Newtonian respectively generalized Newtonian fluids

\[
S(\varepsilon(\mathbf{v})) = \nu_0 (1 + |\varepsilon(\mathbf{v})|)^{p-2} \varepsilon(\mathbf{v})
\] (1.4)

is very popular among rheologists. Here \(\nu_0 > 0\) and \(p \in (1, \infty)\) are specified by physical experiments. An extensive list for specific \(p\)-values for different fluids can be found in [4]. Apparently many interesting \(p\)-values lie in the interval \([\frac{3}{2}, 2]\).

The mathematical discussion of power-law models started in the late sixties with the work of Lions and Ladyshenskaya (see [27–30]). Due to the appearance of the convective term the equations for power law fluids [the constitutive law is given by (1.4)] significantly depend on the value of \(p\). The first results were achieved by Ladyshenskaya and Lions for \(p \geq \frac{3d+2}{d+2}\) (see [27] and [30]). They show the existence of a weak solution in the space

\[
L^p \left(0, T; W^{1,p}_{0,\text{div}}(G) \right) \cap L^\infty(0, T; L^2(G)).
\]

In this case it follows from parabolic interpolation that \(\mathbf{v} \otimes \mathbf{v} : \varepsilon(\mathbf{v}) \in L^1(Q)\). So the solution is also a test-function and the existence proof is based on monotone operator theory and compactness arguments.

This results were improved by Wolf [37] to the case \(p > \frac{2d+2}{d+2}\) via \(L^\infty\)-truncation. In this situation we have that \((\nabla \mathbf{v})\mathbf{v} \in L^1(Q)\) and therefore we can test with functions from \(L^\infty(Q)\). The basic idea (which was already used in the stationary case in [20] together with the bound \(p \geq \frac{2d}{d+1}\)) is to approximate \(\mathbf{v}\) by a bounded function \(\mathbf{v}_\lambda\) which is equal to \(\mathbf{v}\) on a large set and its \(L^\infty\)-norm can be controlled by \(\lambda\).

Wolf’s result was improved to \(p > \frac{2d}{d+2}\) in [16] and [7] by the Lipschitz truncation method. Under this restriction to \(p\) we have \(\mathbf{v} \otimes \mathbf{v} \in L^1(Q)\) which means we can test by Lipschitz continuous functions. So one has to approximate \(\mathbf{v}\) by a Lipschitz continuous function \(\mathbf{v}_\lambda\) which is quite challenging in the parabolic situation.

From several points of view it is reasonable to add a stochastic part to the equation of motion.

- It can be understood as a turbulence in the fluid motion (see [31]).
- It can be interpreted as a perturbation from the physical model.
- Apart from the force \(\mathbf{f}\) we are observing there might be further quantities with a (usually small) influence on the motion.

We are therefore interested in the set of equations:

\[
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \text{div} \ S \ dt - (\nabla \varepsilon) \mathbf{v} \ dt - \nabla \pi \ dt + \mathbf{f} \ dt + \Phi(\mathbf{v}) d\mathbf{W}_t & \text{in } Q, \\
\text{div} \mathbf{v} &= 0 & \text{in } Q, \\
\mathbf{v} &= 0 & \text{on } \partial G, \\
\mathbf{v}(0) &= \mathbf{v}_0 & \text{in } G,
\end{aligned}
\] (1.5)

\(^3\) The easier steady case was observed in [21] and [15].

\(^4\) We neglect physical constants for simplicity.