Abstract Similarity, Fractals and Chaos

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

David Hilbert

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To prove presence of chaos for fractals, a new mathematical concept of abstract similarity is introduced. As an example, the space of symbolic strings on a finite number of symbols is proved to possess the property. Moreover, Sierpinski fractals, Koch curve as well as Cantor set satisfy the definition. A similarity map is introduced and the problem of chaos presence for the sets is solved by considering the dynamics of the map. This is true for Poincaré, Li-Yorke and Devaney chaos, even in multi-dimensional cases. Original numerical simulations which illustrate the results are delivered.

1 Introduction

Self-similarity is one of the most important concepts in modern science. From the geometrical point of view, it is defined as the property of objects whose parts, at all scales, are similar to the whole. Dealing with self-similarity goes back to the 17th Century when Gottfried Leibniz introduced the notions of recursive self-similarity [1]. Since then, history has not recorded any thing about self-similarity until the late 19th century when Karl Weierstrass introduced in 1872 a function that being everywhere continuous but nowhere differentiable. The graph of the Weierstrass function became an example of a self-similar curve. The set constructed by Georg Cantor in 1883 is considered as the most essential and influential self similar set since it is a simple and perfect example for theory and applications of this field. Space-filling curves are substantial epitomai of continuous self-similar curves which were described by Giuseppe Peano and David Hilbert in 1890-91. Other examples of self-similar sets are Koch curve discovered by Helge von Koch in 1904 and Sierpinski gasket and carpet which are introduced by Waclaw Sierpinski in 1916. Julia sets gained significance in being generated using the dynamics of iterative function. They are discovered by Gaston Julia and Pierre Fatou in 1917-19, where they studied independently the iteration of rational functions in the complex plane. The term “fractal” was coined by Benoit Mandelbrot in 1975 [2] to describe certain geometrical structures that exhibit self-similarity. Since then, this word has been employed to denote all the above mentioned sets, and the field became known as fractal geometry. Consequently, the fractal concept is axiomatically linked with the notion of self-similarity which is considered to be one of the acceptable definitions of fractals. That is, a fractal can be defined as a set that display self-similarity at all scales. However, Mandelbrot define a fractal as a set whose Hausdorff dimension strictly larger than its topological dimension [3]. To sum up, self-similarity and fractional dimension are the most two important features of fractals. The connection between them is that self-similarity is the easiest way to construct a set that has fractional dimension [4].

Chaos, in general, can be defined as aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions [5]. The first recognition of chaos phenomenon was indicated in the work of Henri Poincaré in 1890 when he studied the problem of the stability of the solar system. In the 1950s, Edward Lorenz discovered sensitivity to initial conditions in a weather forecasting
properties like unimodality, hyperbolicity, period three and topological conjugacy to a standard chaotic chaos. More precisely, the usual construction of chaos starts with description of a map with certain

The development does not rely on any special functions, and the similarity map used in our paper is the logistic map. It was shown that a chaos equivalent to Li-Yorke type can be extended to higher-dimensional discrete systems [17–19]. This requires employing special theorems like Marotto Theorem [18]. Applying chaotic abstract similarity developed in our paper, we have shown that Devaney, Li-Yorke and Poincaré chaos can take place in the dynamics of n connected perturbed logistic maps. Examples with numerical simulations are provided in Section 5.

In this paper we concern with self-similarity. This concept is reflected in many problems that arise in various fields such as wavelets, fractals, and graph systems [20]. Interesting definitions of self-similarity and related problems of dimension and measure are discussed in papers [21–30]. The manuscripts consider sets in Euclidean space \( \mathbb{R}^n \) and define self-similar set as a union of its images under similarity transfunctions [28]. Our research develops self-similarity for metric spaces and this why we call it abstract self-similarity. The development does not rely on any special functions, and the similarity map used in our paper is applied for chaos formation. The map is different than any similarity map essentially considered in literature before. In the present paper, we are not concerned in analysis of dimension and measure but mostly rely on the distance, since our main goal is to discuss chaos problem for fractals. Nevertheless, we suppose that our suggestion may be useful for the next extension of the results obtained in [22–30] for the abstract self-similarity case. First of all those which consider dimension and measure and corresponding problems of fractals.

We define a self-similar set as a collection of points in a metric space, where it can be considered as a union of infinite shrinking sets with notation that allows to introduce dynamics in the set and then prove chaos which is the most important result of the present research. For the purpose, a specific map over the invariant self-similar set is defined. The map acting as the identity if the whole set taken as an argument for the map. This feature is equivalent to the property of self-similarity in the ordinary fractals, and this is the reason for calling this map the similarity map. We expect that the concept of abstract self-similarity will create new frontiers for chaos and fractals investigations, and we hope it will be a helpful tool in other fields such as harmonic analysis, discrete mathematics, probability, and operator algebras [20].

Our approach with respect to chaos is characterized by the priority of the domain over the map of chaos. More precisely, the usual construction of chaos starts with description of a map with certain properties like unimodality, hyperbolicity, period three and topological conjugacy to a standard chaotic
map. Next, chaotic behavior is discovered, and finally, the structure of the chaotic attractor is analyzed. For example, in Li-Yorke chaos, a map is firstly defined such that it has period three property, and then, the domain (scrambled set) for the chaos is characterize by specific properties. The same can be said for the chaos of unimodal maps, when the domain of chaos is a Cantor set. In the Smale horseshoe case, we can conclude that the map and the domain are simultaneously determined so that the construction of the domain started with an initial set and it is developed step by step using a particular map. Therefore, the structure of the domain and the nature of the map are mutually dependent on each other. For the chaos in symbolic dynamics, the domain is primarily described as infinite sequences of symbols then the map is introduced as a shift on the space. However, the map still has priority since the properties of the sequences are described with respect to the map. In the proposed approach, we first construct a domain in a metric space with specific conditions to be a suitable venue for manifestations of chaos. Thereafter, the similarity map is built on the basis of the invariance and self-similarity properties of the domain to define an abstract motion. This is why the map is appropriate for abstract self-similar set as well as for any fractal constructed through self-similarity, and thence proving chaos for these classes of fractals becomes possible. Moreover, we drop the continuity requirement for the motion since the chaotic map need not be continuous [31,32]. In paper [31], for instance, the authors ignore the continuity of some chaotic maps during the discussion of chaos conditions on the product of semi-flows. We regard the continuity of a chaotic map as an important property only from the analytical side, that is to say, it is very useful for handling the map to prove presence of chaos [15,33], however, it is not rigorously correct to consider it as an intrinsic property for chaos. Despite the discontinuity of the similarity map, opposite to our desire, one can recognize that presence of sensitivity and the irregular behavior of simulations make the discussion possible. That is, in general we have
\[
F = \bigcup_{i=1}^{m} F_i.
\]
where \( F \) is a compact set and \( d \) is a metric. We assume that \( F \) is divided into \( m \) disjoint nonempty subsets, \( F_i, i = 1, 2, \ldots, m \), such that \( F = \bigcup_{i=1}^{m} F_i \). In their own turn, the sets \( F_i \), \( i = 1, 2, \ldots, m \), are divided into \( m \) disjoint nonempty subsets \( F_{ij} \), \( j = 1, 2, \ldots, m \), such that \( F_i = \bigcup_{j=1}^{m} F_{ij} \). That is, in general we have \( F_{1i} \cup \cdots \cup F_{ni} = \bigcup_{j=1}^{m} F_{ij} \), for each natural number \( n \), where all sets \( F_{1i} \cup \cdots \cup F_{ni} \), \( j = 1, 2, \ldots, m \), are nonempty and disjoint. We assume that for the sets \( F_{1i} \cup \cdots \cup F_{ni} \), the diameter condition is valid. That is
\[
\max_{i_k=1,2,\ldots,m} \text{diam}(F_{1i_1} \cup \cdots \cup F_{ni_n}) \to 0 \text{ as } n \to \infty,
\]
where \( \text{diam}(A) = \sup\{d(x, y) : x, y \in A\} \), for a set \( A \) in \( F \).

Let us construct a sequence, \( p_n \), of points in \( F \) such that \( p_0 \in F \), \( p_1 \in F_{11} \), \( p_2 \in F_{112} \), \ldots, \( p_n \in F_{1i_1} \cup \cdots \cup F_{ni_n} \), \( n = 1, 2, \ldots \). It is clear that,
\[
F \supset F_{1i_1} \supset F_{1i_12} \supset \cdots \supset F_{1i_1i_2} \supset \cdots, i_k = 1, 2, \ldots, m, k = 1, 2, \ldots.
\]
That is, the sets form a nested sequence. Therefore, due to the compactness of \( F \) and the diameter condition, there exists a unique limit point for the sequence \( p_n \). Denote the point as \( F_{1i_1i_2} \cup \cdots \cup F_{ni_n} \in F \), accordingly to the indexes of the nested subsets. Conversely, it is easy to verify that each point \( p \in F \) admits a corresponding \( p_n \) and it can be written as \( p = F_{1i_1i_2} \cup \cdots \cup F_{ni_n} \), and this representation is a unique one due to the diameter condition. Finally, we have that
\[
F = \{ F_{1i_1i_2} \cup \cdots \cup F_{ni_n} : i_k = 1, 2, \ldots, m, k = 1, 2, \ldots \},
\]
and
\[
F_{1i_1i_2} \cup \cdots \cup F_{ni_n} = \bigcup_{j_k=1,2,\ldots,m} F_{1i_1i_2} \cup \cdots \cup F_{ni_nj_1j_2} \cdots
\]
for fixed indexes \( i_1, i_2, \ldots, i_n \).

The set \( F \) satisfies (2) and (3) is said to be the abstract self-similar set as well as the triple \( (F, d, \varphi) \) the self-similar space.

2 Abstract Self-Similarity

Let us consider the metric space \((F, d)\), where \( F \) is a compact set and \( d \) is a metric. We assume that \( F \) is divided into \( m \) disjoint nonempty subsets, \( F_i, i = 1, 2, \ldots, m \), such that \( F = \bigcup_{i=1}^{m} F_i \). In their own turn, the sets \( F_i \), \( i = 1, 2, \ldots, m \), are divided into \( m \) disjoint nonempty subsets \( F_{ij} \), \( j = 1, 2, \ldots, m \), such that \( F_i = \bigcup_{j=1}^{m} F_{ij} \). That is, in general we have \( F_{1i} \cup \cdots \cup F_{ni} = \bigcup_{j=1}^{m} F_{ij} \), for each natural number \( n \), where all sets \( F_{1i} \cup \cdots \cup F_{ni} \), \( j = 1, 2, \ldots, m \), are nonempty and disjoint. We assume that for the sets \( F_{1i} \cup \cdots \cup F_{ni} \), the diameter condition is valid. That is
\[
\max_{i_k=1,2,\ldots,m} \text{diam}(F_{1i_1} \cup \cdots \cup F_{ni_n}) \to 0 \text{ as } n \to \infty,
\]
where \( \text{diam}(A) = \sup\{d(x, y) : x, y \in A\} \), for a set \( A \) in \( F \).

Let us construct a sequence, \( p_n \), of points in \( F \) such that \( p_0 \in F \), \( p_1 \in F_{11} \), \( p_2 \in F_{112} \), \ldots, \( p_n \in F_{1i_1} \cup \cdots \cup F_{ni_n} \), \( n = 1, 2, \ldots \). It is clear that,
\[
F \supset F_{1i_1} \supset F_{1i_12} \supset \cdots \supset F_{1i_1i_2} \supset \cdots, i_k = 1, 2, \ldots, m, k = 1, 2, \ldots.
\]
That is, the sets form a nested sequence. Therefore, due to the compactness of \( F \) and the diameter condition, there exists a unique limit point for the sequence \( p_n \). Denote the point as \( F_{1i_1i_2} \cup \cdots \cup F_{ni_n} \in F \), accordingly to the indexes of the nested subsets. Conversely, it is easy to verify that each point \( p \in F \) admits a corresponding \( p_n \) and it can be written as \( p = F_{1i_1i_2} \cup \cdots \cup F_{ni_n} \), and this representation is a unique one due to the diameter condition. Finally, we have that
\[
F = \{ F_{1i_1i_2} \cup \cdots \cup F_{ni_n} : i_k = 1, 2, \ldots, m, k = 1, 2, \ldots \},
\]
and
\[
F_{1i_1i_2} \cup \cdots \cup F_{ni_n} = \bigcup_{j_k=1,2,\ldots,m} F_{1i_1i_2} \cup \cdots \cup F_{ni_nj_1j_2} \cdots
\]
for fixed indexes \( i_1, i_2, \ldots, i_n \).

The set \( F \) satisfies (2) and (3) is said to be the abstract self-similar set as well as the triple \( (F, d, \varphi) \) the self-similar space.
Let us introduce the map \( \varphi : \mathcal{F} \to \mathcal{F} \) such that
\[
\varphi(F_{i_1i_2...i_n}) = F_{i_2i_3...i_n}.
\]
(4)

Considering iterations of the map, one can verify that
\[
\varphi^n(F_{i_1i_2...i_n}) = \mathcal{F},
\]
(5)
for arbitrary natural number \( n \) and \( i_k = 1, 2, ..., m, k = 1, 2, ... \). The relations (4) and (5) give us a reason to call \( \varphi \) a similarity map and the number \( n \) the order of similarity.

In the next example of our paper and in the future studies, it is important to find the structure of abstract self-similar space for a given mathematical object.

Example 1. Let us consider the space of symbolic strings of 0 and 1, which is defined by
\[
\Sigma = \{s_1s_2s_3... : s_k = 0 \text{ or } 1\}.
\]
The distance in \( \Sigma \) is defined by
\[
d(s, t) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k - 1},
\]
(6)
where \( s = s_1s_2... \) and \( t = t_1t_2... \) be two elements in \( \Sigma \).

Considering the pattern of the self-similar set, we denote the elements of the set by \( \Sigma_{s_1s_2...} = s_1s_2..., \) and describe the \( n \)th order subsets of strings in \( \Sigma \) by
\[
\Sigma_{s_1s_2...s_n} = \{s_1s_2...s_n s_{n+1}s_{n+2}... : s_k = 0 \text{ or } 1\},
\]
where \( s_1, s_2,..., s_n \) are fixed symbols. One can show that \( d(s, t) \leq \frac{1}{2^n} \) for any two elements \( s, t \in \Sigma_{s_1s_2...s_n} \). Moreover, \( d(s_1s_2...s_n000..., s_1s_2...s_n111...) = \frac{1}{2^n} \). Therefore, \( \text{diam}(\Sigma_{s_1s_2...s_n}) = \frac{1}{2^n} \).

Consequently,
\[
\lim_{n \to \infty} \text{diam}(\Sigma_{s_1s_2...s_n}) = \lim_{n \to \infty} \frac{1}{2^n} - \frac{1}{2^{n+1}} = 0,
\]
and the diameter condition holds.

The similarity map for the space is the Bernoulli shift, \( \sigma(s_1s_2s_3...) = s_2s_3s_4,... \). That is,
\[
\varphi(\Sigma_{s_1s_2s_3...}) = \sigma(\Sigma_{s_1s_2s_3...}).
\]

On the basis of the above discussion, one can conclude that the triple \((\Sigma, d, \varphi)\) is a self-similar space. This is a purely illustrative example since it makes us perceive how self-similarity can be defined for abstract objects which are not necessarily geometrical ones. The space of symbolic strings on two symbols has been considered, since it is the most basic example that frequently used to describe the dynamics on symbolic spaces. However, more generally, the space on \( m \) symbols can also be considered.

3 Similarity and Chaos

To prove chaos for the self-similar space, we assumed in this section the separation condition. Define the distance between two nonempty bounded sets \( A \) and \( B \) in \( \mathcal{F} \) by \( d(A, B) = \inf\{d(x, y) : x \in A, y \in B\} \).

The set \( \mathcal{F} \) satisfies the separation condition of degree \( n \) if there exist a positive number \( \varepsilon_0 \) and a natural number \( n \) such that for arbitrary \( i_1i_2...i_n \) one can find \( j_1, j_2...j_n \) so that
\[
d(\mathcal{F}_{i_1i_2...i_n}, \mathcal{F}_{j_1j_2...j_n}) \geq \varepsilon_0.
\]
(7)

We call \( \varepsilon_0 \) the separation constant.

In the following theorem, we prove that the similarity map \( \varphi \) possesses the three ingredients of Devaney chaos, namely density of periodic points, transitivity and sensitivity. A point \( F_{i_1i_2i_3...} \in \mathcal{F} \) is periodic with period \( n \) if its index consists of endless repetitions of a block of \( n \) terms.
Theorem 1. If the separation condition holds, then the similarity map is chaotic in the sense of Devaney.

Proof. Fix a member $F_{i_1i_2...i_n}$ of $F$ and a positive number $\varepsilon$. Find a natural number $k$ such that $\text{diam}(F_{i_1i_2...i_k}) < \varepsilon$ and choose a $k$-periodic element $F_{i_1i_2...i_k}$ of $F_{i_1i_2...i_k}$. It is clear that the periodic point is an $\varepsilon$-approximation for the considered member. The density of periodic points is thus proved.

Next, utilizing the diameter condition, the transitivity will be proved if we show the existence of an element $F_{i_1i_2...i_n}$ of $F$ such that for any subset $F_{i_1i_2...i_k}$ there exists a sufficiently large integer $p$ so that $\varphi^p(F_{i_1i_2...i_n}) \in F_{i_1i_2...i_k}$. This is true since we can construct the sequence $i_1i_2...i_n$ such that it contains all sequences of the type $i_1i_2...i_k$ as blocks.

For sensitivity, fix a point $F_{i_1i_2...i_n} \in F$ and an arbitrary positive number $\varepsilon$. Due to the diameter condition, there exist an integer $k$ and element $F_{i_1i_2...i_k} \neq F_{i_1i_2...i_ki_k+1i_k+2}$ such that $d(F_{i_1i_2...i_ki_k+1i_k+2}, F_{i_1i_2...i_ki_k+1i_k+2}) < \varepsilon$. We precise $j_k+1,j_k+2,...$ such that

$$d(F_{i_1i_2...i_ki_k+1i_k+2...i_k+n}, F_{j_k+1j_k+2...j_k+n}) > \varepsilon_0,$$

by the separation condition. This proves the sensitivity.

For Poincaré chaos, Poisson stable motion is utilized to distinguish the chaotic behavior instead of the periodic motions in Devaney and Li-Yorke types. Existence of infinitely many unpredictable Poisson stable trajectories that lie in a compact set meet all requirements of chaos. Based on this, chaos can be appeared in the dynamics on the quasi-minimal set which is the closure of a Poisson stable trajectory. Therefore, the Poincaré chaos is referred to as the dynamics on the quasi-minimal set of trajectory initiated from unpredictable point. For more details we refer the reader to [11,12].

Next theorem shows that the Poincaré chaos is valid for the similarity dynamics.

Theorem 2. If the separation condition is valid, Then the similarity map possesses Poincaré chaos.

The proof of the last theorem is based on the verification of Lemma 3.1 in [12] adopted to the similarity map.

In addition to the Devaney and Poincaré chaos, it can be shown that the Li-Yorke chaos also takes place in the dynamics of the map $\varphi$. The proof of the following theorem is similar to that of Theorem 6.35 in [15] for the shift map defined on the space of symbolic sequences.

Theorem 3. The similarity map is Li–Yorke chaotic if the separation condition holds.

Example 2. We have shown that the space of symbolic strings is a self-similar set in Example 7. One can see that $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_0 = \{0s_2s_3...\}$ and $\Sigma_1 = \{1s_2s_3...\}$, hence,

$$d(\Sigma_0, \Sigma_1) = \inf\{d(s,t) : s \in \Sigma_0, t \in \Sigma_1\}$$

$$= d(000..., 1000...)$$

$$= d(0111..., 1111...) = 1.\$$

Therefore, the separation condition of degree 1 holds with the separation constant $\varepsilon_0$ equal to unity.

According to the results of this section, the Bernoulli shift is chaotic in the sense of Poincaré, Li-Yorke and Devaney. That is, we confirm one more time the presence of chaos which have been proven for the dynamics in [6,12,33,34].

4 Chaos in Fractals

As implementations of abstract self-similarity, we consider several examples of fractals namely Sierpinski carpet, Sierpinski gasket, Koch curve and Cantor set. This consists of two main tasks. The first one is to indicate abstract self-similarity for the fractals, and the second one is to ascertain chaos according to the results of the last section.
4.1 Chaos for Sierpinski carpet

Let $S$ be the Sierpinski carpet constructed in a unit square. In what follows, we are going to find the structure of the abstract self-similar space for the Sierpinski carpet. We shall denote the abstract set by the italic $S$. Let us start by dividing the carpet into eight subsets and denote them as $S_1, S_2, ..., S_8$ (see Fig. 1 (a)). The subsets will be determined such that any couple of adjacent subsets have common horizontal or vertical boundary line. For this reason, we shall use the boundary agreement such that: (i) The points of the common boundary of two horizontally adjacent subsets belong to the left one. (ii) The points of the common boundary of two vertically adjacent subsets belong to the lower one. Figure 1 (b) illustrates the boundary agreement, (i) and (ii), for the subsets, $S_5, S_7$ and $S_8$. For clarification the boundaries are shown by black lines and we see that the common boundary points of $S_5$ and $S_8$ belong to $S_5$ not to $S_8$ and the common boundary points of $S_7$ and $S_8$ belong to $S_7$ not to $S_8$. In the second step, each subset $S_i$, $i = 1, 2, ..., 8$ is again subdivided into eight smaller subsets denoted as $S_{ij}$, $j = 1, 2, ..., 8$.

![Figure 1](image1.png)

Figure 1: (a) The first step of abstract self-similar set construction. (b) The illustration of the boundary agreement.

Continuing in the same manner, the subsets of higher order can inductively be determined such that at each $n^{th}$ step we have $8^n$ subsets notated as $S_{i_1i_2...i_n}$, $i_k = 1, 2, ..., 8$. Figure 2 (a) and (b) show, for example, the subsets of $S_1$ and subsets of $S_{11}$ respectively.

![Figure 2](image2.png)

Figure 2: Examples of the 2$^{nd}$ and the 3$^{rd}$ order subsets of the Sierpinski carpet

To determine the distance between the points of $S$, we will apply the corresponding Euclidean distance for the set $S$ such that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two points in $S$, then, $d(x, y) =$
\( \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \). The diameter of a subset at an \( n \)th step is equal to, \( \text{diam}(S_{i_1i_2i_3...i_n}) = \frac{\sqrt{2}}{3^n} \), and therefore, it diminishes to zero as \( n \) tends to infinity, and the diameter condition holds. It is easy to check that each point in \( S \) has a unique presentation \( S_{i_1i_2i_3...i_n} \). Hence, the set \( S \) can be written as

\[
S = \{S_{i_1i_2i_3...i_n} : i_k = 1, 2, 3...8, \ k \in \mathbb{N}\}.
\]

The separation condition of degree 1 is satisfied and the separation constant is

\[
\varepsilon_0 = \left\{ \min\{d(S_i, S_j) : S_i, S_j \text{ are disjoint, } i, j = 1, 2, 3...8\} \right\} = \frac{1}{3}.
\]

Let us now define the similarity map by

\[
\varphi(S_{i_1i_2i_3...}) = S_{i_2i_3...}.
\]

Thus, we have shown that the triple \((S, d, \varphi)\) is a self similar space with the separation condition. In view of Theorems 1, 2 and 3, the similarity map \( S \) is chaotic in the sense of Poincaré, Li-Yorke and Devaney.

### 4.2 A chaotic trajectory in the Sierpinski carpet

In this section, we provide a geometric realization of the similarity map on the Sierpinski carpet and see how the map can be useful for visualizing the trajectories of the points of a self-similar set and indexing its subsets. A chaotic trajectory is seen as expected in the last section. In the paper [35], we adopt the idea of Fatou-Julia iteration (also called Escape Time Algorithm (ETA) [14]) and develop a scheme for constructing the Sierpinski carpet. The scheme is based on the iterations of the modified planar tent map

\[
T(x) = \begin{cases} 
3 \left\lfloor x \mod 1 \right\rfloor & x \leq \frac{1}{2} \text{ or } x > 1, \\ 
3(1 - x) & \frac{1}{2} < x \leq 1, 
\end{cases} \\
T(y) = \begin{cases} 
3 \left\lfloor y \mod 1 \right\rfloor & y \leq \frac{1}{2} \text{ or } y > 1, \\ 
3(1 - y) & \frac{1}{2} < y \leq 1. 
\end{cases}
\]

Depend on this, one can construct a map \( \bar{T} = (\bar{T}_1, \bar{T}_2) : S \to S \) such that the set \( S \) is invariant,

\[
\bar{T}_1(x) = \begin{cases} 
3x & 0 \leq x \leq \frac{1}{3}, \\ 
3x - 1 & \frac{1}{3} \leq x \leq \frac{1}{2}, \\
2 - 3(1 - x) & \frac{1}{2} < x < \frac{2}{3}, \\ 
3(1 - x) & \frac{2}{3} \leq x \leq 1, 
\end{cases} \\
\bar{T}_2(y) = \begin{cases} 
3y & 0 \leq y \leq \frac{1}{3}, \\ 
3y - 1 & \frac{1}{3} \leq y \leq \frac{1}{2}, \\
2 - 3(1 - y) & \frac{1}{2} < y < \frac{2}{3}, \\ 
3(1 - y) & \frac{2}{3} \leq y \leq 1.
\end{cases}
\tag{8}
\]

This map is equivalent to the similarity map \( \varphi \) defined above, therefore, the trajectory of a point \( x \in S \) can be visualized using the map (8). Figure 3 shows an example of the trajectory for the center point \( x \) of the subset

\[
S_{2773113731327718243151582246178476485265635846545627125423317216244}.
\]

The points of the trajectory are considered as the centers of the subsets which are determined by \( \bar{T}^k(x), \ k = 0, 1, 2, ..., 68 \). The idea of indexing of the subsets is illustrated in Example 3.
4.3 Sierpinski gasket as chaos domain

To construct an abstract self-similar set on the basis of the Sierpinski gasket, let us consider the Sierpinski gasket generated in a unit equilateral triangle. We firstly divided the gasket into three smaller parts to be the first order subsets and denoted them by $G_1$, $G_2$ and $G_3$ as shown in Fig. 4(a). By glancing at the figure, one can see that every two subsets share only a single point as a common boundary. For this reason we consider the following boundary agreement: The common boundary point of every couple of adjacent subsets belongs either to the left one or to the lower one. Applying the agreement, the subsets $G_1$, $G_2$ and $G_3$, become disjoint subsets of the desired abstract self-similar set $G$ such that $G = \bigcup_{i=1}^{3} G_i$.

Secondly, each subset, $G_i$, $i = 1, 2, 3$, is again subdivided into three subsets, and we notate them as $G_{ij}$, $i,j = 1, 2, 3$, (see Fig. 4(b)). Taking into account the boundary agreement, we repeat the same procedure such that at each $n^{th}$ step, we denote the resultant subsets by $G_{i_1i_2...i_n}$, $i_k = 1, 2, 3$.

The subsets of the Sierpinski gasket described above have an inverse relationship with the construction-step variable, $n$, $\text{diam}(G_{i_1i_2...i_n}) = \frac{1}{2^n}$, from which one can verify the validity of the diameter condition.

Following the arguments of the abstract self-similarity, one can deduce that a point in $G$ can be uniquely represented by $G_{i_1i_2...i_n...}$, and the abstract self-similar set $G$ can be expressed as

$$G = \{G_{i_1i_2...i_n...} : i_k = 1, 2, 3, \ k \in \mathbb{N}\}.$$
distance between any two disjoint subsets of the second order, that is

$$\varepsilon_0 = \min\{d(G_{i_1j_2}, G_{j_1j_2}) : G_{i_1j_2}, G_{j_1j_2} \text{ are disjoint}, \ i_n = 1, 2, 3\} = \frac{\sqrt{3}}{8},$$

where $d$ is the usual Euclidean distance. Thus, one can see that separation condition is valid.

The similarity map acting on the Sierpinski gasket, $G$, can be defined by $\varphi(G_{i_1j_2i_3ij_3...}) = G_{i_2j_2i_3j_3...}$. Consequently, the triple $(G, d, \varphi)$ is a self-similar space and $\varphi$ is chaotic in the sense of Poincaré, Li-Yorke and Devaney.

The same idea can be extended to the fractals associated with Pascal’s triangles. It is well known that Pascal’s triangle in mod 2 creates the classical Sierpinski gasket. Different fractals associated with Pascal’s triangles in different moduli can be considered as abstract similar sets and it can also be proved that the similarity map defined on these sets possesses chaos.

### 4.4 Koch curve and chaos

Let us consider the Koch curve, $K$, constructed from an initial unit line segment. To identify an abstract self-similar set corresponding to the Koch curve, we start by dividing $K$ into four equal parts (subsets) and denoting them as $K_1, K_2, K_3$ and $K_4$ as shown in Fig. 5(a). Since the Koch curve is a connected set, each two adjacent subsets share a single point as a common boundary. Let us denote the end points of $K$ and denoting them as $a_1$ and $a_{i+1}$. Figure 5(a) illustrates these points and for clarification the points are shown by black thick dots. It is seen in the figure that $K_1$ and $K_2$ share the point $a_2$, $K_2$ and $K_3$ share the point $a_3$ and $K_3$ and $K_4$ share the point $a_4$. In the second step, each subset $K_i$, $i = 1, 2, 3, 4$ is again subdivided into four subsets $K_{ij}$, $j = 1, 2, 3, 4$. Figure 5(b) and (c) illustrate the second step for the subsets $K_1$ and $K_2$ respectively. Again here we see that each two adjacent subsets share a single boundary point.

As in the previous cases, to determine the abstract self-similar set, we need to consider the following boundary agreement: For each adjacent subsets $K_{i_1j_2...i_{n-1}j}$ and $K_{i_1j_2...i_{n-1}j+1}$, the common boundary point $a_{i_1j_2...i_{n-1}j+1}$ belongs to $K_{i_1j_2...i_{n-1}j+1}$. This condition means that the common boundary point $a_2$ shown in Fig. 5(a), for instance, belongs to $K_2$ not to $K_1$ and the common boundary point $a_{23}$ shown in Fig. 5(c) belongs to $K_{23}$ not to $K_22$.

By applying the boundary agreement to all subsets at each step, we have fully described the disjoint subsets of the proposed abstract self-similar set for the Koch curve. From the construction of the Koch curve and by using the usual Euclidean distance, one can deduce that the distance between the end points of each subset $K_{i_1j_2...i_n}$ is $\frac{1}{3^n}$ which clearly represents the diameter of the subset. Therefore, the diameter condition holds. A point in $K$ can be represent by $K_{i_1j_2...i_{n...}}$, so that,

$$K = \left\{K_{i_1j_2...i_n...} : i_k = 1, 2, 3, 4, \ k \in \mathbb{N}\right\}.$$
The separation condition is also valid with degree 1, since for any $K_i, i = 1, 2, 3, 4$ one can find $K_j, j = 1, 2, 3, 4, j \neq i$ such that they are separated from each other by a distance of not less than $\varepsilon_0$. The separation constant, $\varepsilon_0$, can be defined by

$$\varepsilon_0 = \min\{d(K_1, K_3), d(K_1, K_4), d(K_2, K_4)\} = \frac{\sqrt{7}}{9}.$$

The similarity map for the abstract fractals of Koch curve is given by $\varphi(K_{i_1i_2i_3...}) = K_{i_2i_3...}$, and thus, we have shown that the triple $(K, d, \varphi)$ defines a chaotic self-similar space.

### 4.5 Chaos for Cantor set

A perfect example of chaos in fractals is the Cantor set. As we previously mentioned, the chaoticity in the Cantor set is determined by finding a topological conjugacy with the symbolic dynamics. To show that the Cantor set is not an exception to our approach for chaos, we shall establish an abstract self-similar set corresponding to the Cantor set. Let us consider the middle third Cantor set, $C$, initiated from a unit line segment. The first step consists of dividing $C$ into two subsets and denoted them by $C_1$ and $C_2$ (see Fig. 6 (a)). In the second step each of $C_1$ and $C_2$ is subdivided into two subsets as shown in Fig. 6 (b). These subsets are denoted by $C_{11}$, $C_{12}$, $C_{21}$ and $C_{22}$. In every next step, we repeat the same procedure for each subsets resulting from the preceding step. We denote the resultant subsets at each $n^{th}$ step by $C_{i_1i_2...i_n}, i_k = 1, 2$.

![Figure 6: The 1st and the 2nd order subsets of the abstract self-similar set for the Cantor set](image)

In the Cantor set case, we do not need any boundary agreement since all subsets are disjoint. Considering the usual Euclidean distance, the diameter of a subset, $C_{i_1i_2...i_n}$, is $\frac{1}{3^n}$. Therefore, the diameter condition holds. The points in $C$ are represented by $C_{i_1i_2...i_n}$. Hence the abstract self-similar set is defined by

$$C = \{C_{i_1i_2...i_n} : i_k = 1, 2, k \in \mathbb{N}\}.$$

From the construction, the separation condition is clearly valid and the constant $\varepsilon_0$ is defined by the distance between the subsets $C_1$ and $C_2$, so that $\varepsilon_0 = \frac{1}{3}$.

The similarity map is defined by $\varphi(C_{i_1i_2i_3...}) = C_{i_2i_3...}$, and the triple $(C, d, \varphi)$ defines a self-similar space. Theorems 1, 2 and 3 are also applicable for this case.

In connection with the above examples of chaos, we remark that the Sierpinski carpet and Koch curve indicate that a non-continuous map can have a domain which is a connected set while for continuous maps, the domains of chaos are usually disconnected. Examples of disconnected chaotic domains are the Cantor set for the logistic map [6], the modified Sierpinski triangle with exceptions in [14], the set associated with Smale’s horseshoe map [36], and the Poincaré section of the Lorenz attractor [37].

### 5 Dynamical Abstract Self-Similar Sets and Chaos

In this part of the paper, we describe a dynamical determination of abstract self-similar set by utilizing the roles of the domain and the map simultaneously. In other words, a map is used to describe the structure
of $\mathcal{F}$ and the relationships between its subsets. The set constructed by this way, we call it Dynamical Abstract Similarity Set (DASS). We start by considering the triple $(X, d, \varphi)$ and a compact set $F \subset X$, where $d$ is a metric, and $\varphi : X \to X$ is a map.

Let $m$ be a fixed natural number and consider the set $F_0 \subset X$. Denote by $F^{(1)}$ the preimage of the set $\varphi(F) \cap F_0$ under the function $\varphi$ in $F$ and assume that there exist disjoint nonempty subsets $F_i \subset F$, $i = 1, 2, \ldots, m$, such that $\bigcup_{i=1}^{m} F_i = F^{(1)}$.

Denote by $F^{(2)}$ the preimage of the set $\varphi(F) \cap F_0$ under $\varphi^2$ in $F^{(1)}$ and assume that there exist disjoint nonempty subsets $F_{ij} \subset F_i$, $j = 1, 2, \ldots, m$, such that $\bigcup_{j=1}^{m} F_{ij} = F^{(2)}$.

Once more, denote by $F^{(3)}$ the preimage of the set $\varphi(F) \cap F_0$ under $\varphi^3$ in $F^{(2)}$ and assume that there exist disjoint nonempty subsets $F_{ijk} \subset F_{ij}$, $k = 1, 2, \ldots, m$, such that $\bigcup_{k=1}^{m} F_{ijk} = F^{(3)}$.

In general, if the sets $F^{(n-1)}$ are determined, we denote by $F^{(n)}$ the preimage of the set $\varphi(F) \cap F_0$ under $\varphi^n$ in $F^{(n-1)}$ and assume that there exist disjoint nonempty subsets $F_{i_1i_2\ldots i_n} \subset F_{i_2i_3\ldots i_{n-1}}$, $i_n = 1, 2, \ldots, m$, such that $\bigcup_{i_n=1}^{m} F_{i_1i_2\ldots i_n} = F^{(n)}$.

We continue in this procedure, and assume that the following condition is satisfied

$$\max_{i_k=1,2,\ldots,m} \text{diam}(F_{i_1i_2\ldots i_n}) \to 0 \text{ as } n \to \infty.$$  

(9)

Let us construct a sequence, $p_n$, of points in $F$ such that $p_0 \in F$, $p_1 \in F_{i_1}$, $p_2 \in F_{i_1i_2}$, ... , $p_n \in F_{i_1i_2\ldots i_n}$, $n = 1, 2, \ldots$. It is clear that,

$$F \supset F_{i_1} \supset F_{i_1i_2} \supset \cdots \supset F_{i_1i_2\ldots i_n} \supset F_{i_2i_3\ldots i_{n+1}} \ldots, \quad i_k = 1, 2, \ldots, m, \quad k = 1, 2, \ldots.$$  

That is, the sets form a nested sequence. Therefore, due to the compactness of $F$ and condition (9), there exists a unique limit point for the sequence $p_n$. According to the indexes of the nested subsets, we denote the point as $F_{i_1i_2\ldots i_n} \in F$. Conversely, it can be verified that each point $p = F_{i_1i_2\ldots i_n}$ admits a corresponding $p_n$. Based on this, one can justify that the representation of each such point is a unique one. The collection of all such points constitutes the set $\mathcal{F}$, i.e.,

$$\mathcal{F} = \{F_{i_1i_2\ldots i_n} : i_k = 1, 2, \ldots, m\},$$

and for fixed indexes $i_1, i_2, \ldots, i_n$ the subsets of $\mathcal{F}$ can be represented by

$$F_{i_1i_2\ldots i_n} = \bigcup_{j_k=1,2,\ldots,m} F_{i_1i_2\ldots i_nj_1j_2\ldots}.$$  

Since $F_{i_1i_2\ldots i_n} \subset F_{i_1i_2\ldots i_{n+1}}$, the condition (9) implies that the diameter condition (1) is valid for the set $\mathcal{F}$. Thus, the set $\mathcal{F}$ is a DASS. Moreover, from the above construction, we see that the map $\varphi$ satisfies the relations (4) and (5). Therefore, $\varphi$ is a similarity map and triple $(\mathcal{F}, d, \varphi)$ is a self similar space.

Now, let us formulate the following condition: For arbitrary $i_1i_2\ldots i_n$ one can find $j_1j_2\ldots j_n$ and a positive number $\varepsilon$ such that

$$d(F_{i_1i_2\ldots i_n}, F_{j_1j_2\ldots j_n}) \geq \varepsilon.$$  

(10)

From the construction, it is clear that if the condition (10) holds for the sets $F_{i_1i_2\ldots i_n}$, then the separation condition (7) is valid for the set $\mathcal{F}$ with a separation constant $\varepsilon_0 \geq \varepsilon$. If this is the case, then in view of Theorem (1) and (3) the similarity map $\varphi$ is chaotic in the sense of Poincaré, Li-Yorke and Devaney.

The approach described above is not only an alternative way of the abstract similarity construction but it can be an essential part of the subject. For instance, it gives us a method for indexing. That is, the similarity map can be simultaneously used to number the subsets of each order depending on the enumeration of their images. The following examples illustrate the idea of DASS, indexing and chaos.

**Example 3.** Let $X = \mathbb{R}^n$ and $F_0 = F = [0, 1]^n$ is the $n$-dimensional unit cube. Consider the $n$-dimensional logistic map $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n): \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$x_{k+1}^1 = \varphi_1(x_k^1) = r_1x_k^1(1-x_k^1),$$

$$x_{k+1}^2 = \varphi_2(x_k^2) = r_2x_k^2(1-x_k^2),$$

$$\vdots$$

$$x_{k+1}^n = \varphi_n(x_k^n) = r_nx_k^n(1-x_k^n),$$

(11)
where \( r_i > 4, \ i = 1, 2, \ldots, n \) are parameters. Proceeding upon the properties of the logistic map, ETA is applied for the map \((11)\). We iterate the points in \( F \) under the map \((11)\) such that in each iteration, we keep only the points whose images do not escape the domain \( F_0 \). The resulting points from the first iteration are belong to the subsets \( F_i, i = 1, 2, \ldots, 2^n \). In the second iteration, the non-escaped points are belong to \( 2^{2^n} \) subsets and each subset is indexed as \( F_{ij}, j = 1, 2, \ldots, 2^n \) such that \( F_{ij} \subseteq F_i \) and \( \varphi(F_{ij}) = F_j \). Similarly, a subset resulting at the \( k \)th iteration is indexed as \( F_{i_1i_2\ldots i_k} \) such that \( F_{i_1i_2\ldots i_{k-1}} \subseteq F_{i_1i_2\ldots i_{k}} \) and \( \varphi(F_{i_1i_2\ldots i_k}) = F_{i_2i_3\ldots i_k} \).

Based on the algorithm, it is clear that the condition \((9)\) holds. Thus, we describe the points \( F_{i_1i_2i_3\ldots} = \lim_{k \to \infty} F_{i_1i_2\ldots i_k} \), and then the DASS, the self-similar set \( F \) corresponding to the above algorithm, is defined as the collection of the points \( F_{i_1i_2i_3\ldots} \).

For \( r_i, i = 1, 2, \ldots, n \) larger than 4, the separation condition is guaranteed to be valid for the set \( F \), and therefore, Theorem 1, 2 and 3 imply that the similarity map \( \varphi \) is chaotic in the sense of Poincaré, Li-Yorke and Devaney.

For numerical simulation, let us consider the 2-dimensional system

\[
\begin{align*}
x_{n+1} &= \varphi_1(x_n) = r_1 x_n (1 - x_n), \\
y_{n+1} &= \varphi_2(y_n) = r_2 y_n (1 - y_n),
\end{align*}
\]

with \( r_1 = 4.2 \) and \( r_2 = 4.3 \). We fix \( F_0 = F = [0, 1] \times [0, 1] \) and apply ETA for \((13)\). The first iteration will generate the sets \( F_i, i = 1, 2, 3, 4 \). In the second iteration, we get the sets \( F_{ij}, j = 1, 2, 3, 4 \), and so on. Figure 7 shows the subsets constructed in the first three iterations. The DASS corresponding to the system \((13)\) is the set resulting from an infinite iteration of this procedure. The set is a sort of Cantor dust which is the Cartesian product of two Cantor sets \([38]\).

**Example 4.** Let \( F_0 = F \) denote the initial set \([0, 1] \times [0, 1]\) and consider the 2-dimensional perturbed logistic map \( \varphi = (\varphi_1, \varphi_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
\begin{align*}
x_{n+1} &= \varphi_1(x_n, y_n; \mu_1) = r_1 x_n (1 - x_n) + \mu_1 y_n, \\
y_{n+1} &= \varphi_2(x_n, y_n; \mu_2) = r_1 y_n (1 - y_n) + \mu_2 x_n,
\end{align*}
\]

where \( r_1, r_2, \mu_1 \) and \( \mu_2 \) are parameters. The last term in the left-hand side of the both equations in system \((13)\) can be considered as a perturbation of a unimodal map. It is known that, if a unimodal map is regular \([36, 39]\), then it is structurally stable, and therefore, any small perturbation does not affect the topological properties of the map \([40]\). Such an inference can be extended for high-dimensional unimodal maps. Numerical simulations can provide an adequate verification of the unimodal properties of the perturbed map.

Similar to Example 3, we apply ETA to the map \((13)\). The points that do not escape \( F_0 \) in the first iteration are belong to the subsets \( F_i, i = 1, 2, 3, 4 \). In the second iteration, the resulting points are belong to the subsets indexed by \( F_{ij}, j = 1, 2, 3, 4 \) such that \( F_{ij} \subseteq F_i \) and \( \varphi(F_{ij}) = F_j \). Similarly, a subset resulting at the \( n \)th iteration is indexed as \( F_{i_1i_2\ldots i_n} \) such that \( F_{i_1i_2\ldots i_n} \subseteq F_{i_1i_2\ldots i_{n-1}} \) and \( \varphi(F_{i_1i_2\ldots i_n}) = F_{i_2i_3\ldots i_n} \). Figure
Figure 8: The first three iterations of DASS construction using the map (13). Depending on the choice of the coefficients $r_i$ and relying on the smallness of the coefficients $\mu_i$, we have that the diameter and separation conditions for abstract similarity and chaos are fulfilled. Moreover, the simulation results confirm that both conditions hold. Therefore, we could say that the similarity map (13) is chaotic on the self similar-set $F$. Figure 9 depicts the trajectories of some points that approximately belong to the set $F$. The irregular behavior of the trajectories reveals the presence of chaos in (13).

Example 5. Consider the space $X = \mathbb{R}^n$ and let $F$ be a compact set in $X$ such that it contains an open neighborhood of the $n$-dimensional unit cube and the set $F_0$ to be sufficiently near to the cube. Consider the $n$-dimensional perturbed logistic map $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) : \mathbb{R}^n \to \mathbb{R}^n$ which is defined by

\begin{align*}
x_{1k+1} &= \varphi_1(x^1_k, x^2_k, ..., x^n_k; \mu_1) = r_1x^1_k(1 - x^1_k) + \mu_1\chi_1(x^1_k, x^2_k, ..., x^n_k), \\
x_{2k+1} &= \varphi_2(x^1_k, x^2_k, ..., x^n_k; \mu_2) = r_2x^2_k(1 - x^2_k) + \mu_2\chi_2(x^1_k, x^2_k, ..., x^n_k), \\
&\vdots \\
x_{nk+1} &= \varphi_n(x^1_k, x^2_k, ..., x^n_k; \mu_n) = r_nx^n_k(1 - x^n_k) + \mu_n\chi_n(x^1_k, x^2_k, ..., x^n_k),
\end{align*}

(14)

where $\mu_i, \ i = 1, 2, ..., n$ are parameters and $\chi = (\chi_1, \chi_2, ..., \chi_n; \mu_i)$ is a continuous function. Due to the continuity of $\chi$ and for $r_i, \ i = 1, 2, ..., n$ larger than 4 and sufficiently small $\mu_i, \ i = 1, 2, ..., n$, one can show that a DASS can constructed using (14), and thus, chaotic behavior, in the since of Poincaré, Li-Yorke and Devaney, for $n$-dimensional perturbed logistic map would be expected to appear.
6 Conclusion

The abstraction of the self-similarity concept is now accomplished. Furthermore, we have shown that the set of symbolic strings satisfies the definition of abstract self-similarity. This example illustrates how the abstraction of a mathematical concept can be significant not only to extract its essence but also to explore more fields where it can be manifested. In addition to equipping the self-similar set with a metric, the similarity map is introduced to define abstract self-similar space. The map is proven to be chaotic in the sense of Poincaré, Li-Yorke and Devaney. The building of chaos usually begins with a map defined over its domain and then saying about a chaotic attractor that appears as a part of the domain. In our research, we start by describing a chaotic set, and only then introduce a similarity map which admits chaotic dynamics. We utilize infinite sequences to index the points of the domain. The action of the map is not just a shifting in the string space as much as a transforming of the domain points.

Self-similarity is widely spread in nature, but it is usually associated with fractal geometry. Proceeding from this point, we have shown that the Sierpinski fractals, Koch curve and Cantor set can be associated with abstract self-similarity, and consequently possess chaos. This covers already known fractals constructed through self-similarity and possibly other fractals that generated by escape-time algorithm such as Julia and Mandelbrot sets. The suggested abstract similarity definition can be elaborated through fractal sets defined by fractal dimension, chaotic dynamics development, topological spaces, physics, chemistry, and neural network theories development.

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