Erdős-Hajnal-type results for ordered paths

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Abstract

An ordered graph is a graph with a linear ordering on its vertex set. We prove that for every positive integer $k$, there exists a constant $c_k > 0$ such that any ordered graph $G$ on $n$ vertices with the property that neither $G$ nor its complement contains an induced monotone path of size $k$, has either a clique or an independent set of size at least $n^{c_k}$. This strengthens a result of Bousquet, Lagoutte, and Thomassé, who proved the analogous result for unordered graphs.

A key idea of the above paper was to show that any unordered graph on $n$ vertices that does not contain an induced path of size $k$, and whose maximum degree is at most $c(k)n$ for some small $c(k) > 0$, contains two disjoint linear size subsets with no edge between them. This approach fails for ordered graphs, because the analogous statement is false for $k \geq 3$, by a construction of Fox. We provide further examples how this statement fails for ordered graphs avoiding other ordered trees as well.

1 Introduction

Erdős and Hajnal [10] proved that graphs avoiding some fixed induced subgraph or subgraphs have very favorable Ramsey-theoretic properties. In particular, they contain surprisingly large homogeneous (that is, complete or empty) subgraphs and bipartite subgraphs. According to the celebrated Erdős-Hajnal conjecture, every graph $G$ on $n$ vertices which does not contain some fixed graph $H$ as an induced subgraph, has a clique or an independent set of size at least $n^{c}$, where $c = c(H) > 0$ is a constant that depends only on $H$. There is a rapidly growing body of literature studying this conjecture (see, e.g., [1, 2, 5, 6, 8, 11, 13, 15, 23]).

For any graph $G$ and any disjoint subsets $A,B \subset V(G)$, we say that $A$ is complete to $B$ if $ab \in E(G)$ for every $a \in A, b \in B$. If $|A| = |B| = k$ and $A$ is complete to $B$, then $A$ and $B$ are said to form a bi-clique of size $k$. Denote the maximum degree of the vertices in $G$ by $\Delta(G)$. Following [13], a family of graphs $\mathcal{G}$ is said to have the Erdős-Hajnal property if there exists a constant $c = c(\mathcal{G}) > 0$ such that every $G \in \mathcal{G}$ has either a clique or an independent set of size at least $|V(G)|^c$. The family $\mathcal{G}$ has the strong Erdős-Hajnal property if there exists a constant $b = b(\mathcal{G}) > 0$ such that for every $G \in \mathcal{G}$, either $G$ or its complement $\overline{G}$ has a bi-clique of size $b|V(G)|$. It was proved in [1]
that if a hereditary family (that is, a family closed under taking induced subgraphs) has the strong Erdős-Hajnal property, then it also has the Erdős-Hajnal property.

The aim of this paper is to discuss Erdős-Hajnal type problems for ordered graphs. An ordered graph is a graph with a total ordering on its vertex set. With a slight abuse of notation, in every ordered graph, we denote this ordering by $\prec$. If the vertex set of $G$ is a subset of the integers, then $\prec$ stands for the natural ordering. An ordered graph $H$ is an ordered subgraph (or simply subgraph) of $G$ if there exists an order preserving embedding from $V(H)$ to $V(G)$ that maps edges to edges. If, in addition, non-edges are mapped into non-edges, then $H$ is called an induced ordered subgraph of $G$. If $G$ does not have $H$ as induced ordered subgraph, then we say that $G$ avoids $H$. The ordered path with vertices $1, \ldots, k$ and edges $\{i, i+1\}$, for $i = 1, \ldots, k - 1$, is called a monotone path of size $k$.

Our main result is the following.

**Theorem 1.** For any positive integer $k$, there exists $c = c(k) > 0$ with the following property. If $G$ is an ordered graph on $n$ vertices such that neither $G$ nor its complement contains an induced monotone path of size $k$, then $G$ has either a clique or an independent set of size at least $n^c$.

Our theorem obviously implies the analogous statement for unordered graphs, which was first established by Bousquet, Lagoutte, and Thomassé [5]. The idea of their proof was the following. We call a family of graphs, $\mathcal{H}$, lopsided if there exists a constant $c = c(\mathcal{H}) > 0$ with the following property: any graph $G$ on $n$ vertices which does not contain any element of $\mathcal{H}$ as an induced subgraph, and for which $\Delta(G) < cn$, the complement of $G$ has a bi-clique of size at least $cn$. If $\mathcal{H}$ consists of a single graph $H$, then $H$ is called lopsided. They proved that the (unordered) path of size $k$ is lopsided. It follows from the arguments of Bousquet et al. that if $\mathcal{H}$ is lopsided, then the family of all graphs which avoid every element of $\mathcal{H}$ as an induced subgraph, and whose complements also avoid them, has the strong Erdős-Hajnal and, thus, the Erdős-Hajnal property.

Since then, this idea has been exploited to prove the Erdős-Hajnal conjecture for various other families of graphs: the family of graphs avoiding a tree $T$ and its complement [8], the family of graphs avoiding all subdivisions of a graph $H$ and the complements of these subdivisions [9], the family of graphs avoiding a graph $H$ as a vertex minor [7], families of graphs avoiding a fixed cycle as a pivot minor [16], etc.

However, for ordered graphs, this method does not work even in the simplest case: for monotone paths. A construction of Fox [12] shows that, for every $n$ and $\delta > 0$, there exists an ordered graph $G$ with $|V(G)| = n$ and $\Delta(G) < n^\delta$ which avoids the monotone path of size 3, and whose complement does not contain a bi-clique of size larger than $\frac{cn}{\log n}$, for a suitable constant $c = c(\delta) > 0$. Hence, using the above terminology, the monotone path of size at least 3 is not lopsided.

Although monotone paths are not lopsided, they satisfy a somewhat weaker property, as is shown by the following theorem of the authors.

**Theorem 2.** ([19]) For any positive integer $k$, there exists a constant $c = c(k) > 0$ with the following property. If $G$ is an ordered graph on $n$ vertices that does not contain an induced monotone path of size $k$, and $\Delta(G) < cn$, then the complement of $G$ contains a bi-clique of size at least $\frac{cn}{\log n}$.
Unfortunately, Theorem 1 cannot be deduced from this weaker property. Our approach is based on a technique in [24], where it was shown that the family of string graphs has the Erdős-Hajnal property.

Recently, Seymour, Scott, and Spirkl [23] extended our Theorem 2 from monotone paths to all ordered forests $T$, albeit with a weaker bound $n^{1-o(1)}$ in place of $cn \log n$. They proved that for any $0 < c < 1$, there exists $\epsilon = \epsilon(T,c) > 0$ with the following property. If $G$ is an ordered graph on $n$ vertices that does not contain $T$ as an induced ordered subgraph and $\Delta(G) < \epsilon n^{1-c}$, then the complement of $G$ contains a bi-clique of size at least $\epsilon n^{1-c}$. Therefore, if we want to guarantee a bi-clique of size $n^{1-o(1)}$ in $G$, we need to assume that the maximum degree of $G$ is $o(n)$. This is definitely a stronger condition than the one we had for monotone paths.

Our next construction shows that this stronger condition is indeed necessary. We also provide new examples of ordered trees $T$ (that do not contain a monotone path of size 3), for which one cannot expect to find linear size bi-cliques.

**Theorem 3.** For any $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ with the following property.

For any positive integer $n \geq n_0$, there is an ordered graph $G$ with $n$ vertices and $\Delta(G) \leq \epsilon n$ such that the size of the largest bi-clique in $\overline{G}$ is at most $n^{1-\delta}$, and $G$ does not contain either of the following ordered trees as an induced ordered subgraph:

$S:\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}$

$P:\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}$

The investigation of bipartite variants of the problems considered in this paper were initiated in [17]; see also [3, 22].

Our paper is organized as follows. In Section 2, we introduce the key concept needed for the proof of Theorem 1 and reduce Theorem 1 to another statement (Theorem 6). Sections 3 and 4 are devoted to the proof of this latter statement. The construction proving Theorem 3 will be presented in Sections 5.

Throughout this paper, we use the following notation, which is mostly conventional. For any graph $G$ and any subset $U \subset V(G)$, we denote by $G[U]$ the subgraph of $G$ induced by $U$. The neighborhood of $U$ is defined as $N_G(U) = N(U) = \{v \in V(G) \setminus U : \exists u \in U, uv \in E(G)\}$. If $U = \{u\}$, instead of $N(U)$, we simply write $N(u)$. For a vertex $v \in V(G)$, let $G - v$ stand for the graph obtained from $G$ by deleting the vertex $v$. Also, if $G$ is an ordered graph, the forward neighbourhood of a vertex $v \in V(G)$, denoted by $N^+_G(v) = N^+(v)$ is the set of neighbours $y$ such that $x \prec y$.

For easier readability, we omit the use of floors and ceilings, whenever they are not crucial.

2 The quasi-Erdős-Hajnal property

After introducing some notation and terminology, we outline our proof strategy for Theorem 1.

For any $k \geq 3$, let $\mathcal{P}_k$ denote the family of all ordered graphs $G$ such that neither $G$ nor its complement contains a monotone path of size $k$ as an induced subgraph. Instead of proving that $\mathcal{P}_k$ has the Erdős-Hajnal property, we prove that it has the quasi-Erdős-Hajnal property. This concept
was introduced by the second named author in [24], in order to show that the family of string graphs has the Erdős-Hajnal property.

**Definition 4.** A family of graphs, $\mathcal{G}$, has the quasi-Erdős-Hajnal property if there is a constant $c = c(\mathcal{G}) > 0$ with the following property. For every $G \in \mathcal{G}$ with at least 2 vertices, there exist $t \geq 2$ and $t$ disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq (|V(G)|/|X_i|)^c$ for $i = 1, \ldots, t$, and

(i) either there is no edge between $X_i$ and $X_j$ for $1 \leq i < j \leq t$,

(ii) or $X_i$ is complete to $X_j$ for $1 \leq i < j \leq t$.

It was proved in [24] that in hereditary families, the quasi-Erdős-Hajnal property is equivalent to the Erdős-Hajnal property. We somewhat relax the definition of the quasi-Erdős-Hajnal property, and with a slight abuse of notation, we overwrite the previous definition as follows.

**Definition 5.** A family of graphs, $\mathcal{G}$, has the quasi-Erdős-Hajnal property if there are two constants, $\alpha, \beta > 0$, with the following property. For every $G \in \mathcal{G}$ with at least 2 vertices, there exist $t \geq 2$ and $t$ disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq \alpha(|V(G)|/|X_i|)^\beta$ for $i = 1, \ldots, t$, and

(i) either there is no edge between $X_i$ and $X_j$ for $1 \leq i < j \leq t$,

(ii) or $X_i$ is complete to $X_j$ for $1 \leq i < j \leq t$.

It is easy to verify that the two definitions are in fact equivalent. If $\mathcal{G}$ satisfies Definition 4, then, obviously, it also satisfies Definition 5. In the reverse direction, setting $c = \frac{\beta}{1 - \log_2 \alpha}$ if $\alpha \leq 1$, and $c = \beta$ if $\alpha > 1$, if the inequality $t \geq \alpha(|V(G)|/|X_i|)^\beta$ holds for some $t \geq 2$, then we also have $t \geq (|V(G)|/|X_i|)^c$.

Therefore, it is enough to show that $\mathcal{P}_k$ has the quasi-Erdős-Hajnal property. The advantage of the quasi-Erdős-Hajnal property compared to the Erdős-Hajnal property is that it allows us to establish the following lopsided statement, which will imply Theorem 1.

**Theorem 6.** For every positive integer $k$, there exist two constants $\epsilon, \alpha > 0$ with the following property.

Let $G$ be an ordered graph on $n$ vertices with maximum degree at most $en$ such that $G$ does not contain a monotone path of size $k$ as an induced subgraph. Then there exist $t \geq 2$ and $t$ disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq \alpha(n/|X_i|)^{1/2}$ and there is no edge between $X_i$ and $X_j$ for $1 \leq i < j \leq t$.

In the inequality $t \geq \alpha(n/|X_i|)^{1/2}$, the exponent $1/2$ has no significance: the statement remains true with any $0 < \beta < 1$ instead of $1/2$ (with the cost of changing $\epsilon$ and $\alpha$). However, it is not true with $\beta = 1$, as it would contradict the aforementioned construction of Fox [12].

In the rest of this section, we show how Theorem 6 implies Theorem 1. Very similar ideas were used in [5, 8, 9]. The next two sections are devoted to the proof of Theorem 6.

By a classical result of Rödl [20], any graph $G$ avoiding some fixed graph $H$ contains a linear size subset that is either very dense or very sparse. A quantitatively stronger version of this result was proved by Fox and Sudakov [14].
Lemma 7. [20] For every graph $H$ and $\epsilon_0 > 0$, there exists $\delta_0 > 0$ with the following property.

For any graph $G$ with $n$ vertices that does not contain $H$ as an induced subgraph, there is a subset $U \subset V(G)$ such that $|U| \geq \delta_0 n$, and either $|E(G[U])| \leq \epsilon_0 \binom{|U|}{2}$ or $|E(G[U])| \geq (1 - \epsilon_0) \binom{|U|}{2}$.

Lemma 7 applies to unordered graphs, but it can be easily extended to ordered graphs, using the following statement.

Lemma 8. [21] For every ordered graph $H$, there exists an unordered graph $H_0$ with the property that introducing any total ordering on $V(H_0)$, the resulting ordered graph $H_0'$ always contains $H$ as an induced ordered subgraph.

By the combination of these two lemmas, we obtain the following.

Lemma 9. For every ordered graph $H$ and $\epsilon > 0$, there exists $\delta > 0$ with the following property.

For any ordered graph $G$ with $n$ vertices that does not contain $H$ as an induced ordered subgraph, there exists a subset $U \subset V(G)$ such that $|U| \geq \delta n$, and either $\Delta(G[U]) \leq \epsilon |U|$ or $\Delta(G[U]) \leq \epsilon |U|$.

Proof. By Lemma 8, there exists a graph $H_0$ such that introducing any total ordering on $V(H_0)$, the resulting ordered graph $H_0'$ contains $H$ as an induced ordered subgraph. Let $\epsilon_0 = \frac{\epsilon}{2}$, and let $\delta_0$ be the constant given by Lemma 7 with respect to $H_0$ and $\epsilon_0$.

Let $G$ be an ordered graph with $n$ vertices that does not contain $H$ as an induced ordered subgraph. Then the underlying unordered graph of $G$ does not contain $H_0$ as an induced subgraph. Hence, there exists $U' \subset V(G)$ such that $|U'| \geq \delta_0 n$, and either $|E(G[U'])| \leq \epsilon_0 \binom{|U'|}{2}$ or $|E(G[U'])| \geq (1 - \epsilon_0) \binom{|U'|}{2}$. Suppose that $|E(G[U'])| \leq \epsilon_0 \binom{|U'|}{2}$, the other case can be handled similarly. Let $W$ be the set of vertices in $U'$ whose degree in $G[U]$ is larger than $2\epsilon_0 |U|$. Then

$$\frac{1}{2} (2\epsilon_0 |W||U'|) \leq |E(G[U'])| \leq \epsilon_0 \binom{|U'|}{2},$$

so that $|W| \leq \frac{|U'|}{2}$. Setting $U = U' \setminus W$, we have $\Delta(G[U]) \leq 2\epsilon_0 |U'| \leq \epsilon |U|$ and

$$|U| \geq \frac{|U'|}{2} \geq \frac{\delta_0}{2} n.$$

Hence, $\delta = \frac{\delta_0}{2}$ will suffice.

After this preparation, it is easy to deduce from Theorem 6 that $\mathcal{P}_k$ has the quasi-Erdős-Hajnal property and, therefore, the Erdős-Hajnal property.

Proof of Theorem 1. Let $\epsilon, \alpha > 0$ be the constants given by Theorem 6, and let $\delta > 0$ be the constant given by Lemma 9, where $H$ is the monotone path of size $k$.

Let $G$ be an ordered graph on $n$ vertices such that neither $G$ nor its complement contains a monotone path of length $k$ as an induced subgraph. Then there exists $U \subset V(G)$ such that $|U| \geq \delta n$, and either $\Delta(G[U]) < \epsilon |U|$ or $\Delta(G[U]) > \epsilon |U|$. Suppose that $\Delta(G[U]) < \epsilon |U|$, the other case can be handled similarly. Applying Theorem 6 to $G[U]$, we obtain that there exist $t \geq 2$ and $t$ disjoint sets $X_1, \ldots, X_t \subset U$ such that

$$t \geq \alpha \left( \frac{|U|}{|X_1|} \right)^{1/2} \geq \alpha \delta^{1/2} \left( \frac{n}{|X_i|} \right)^{1/2}$$

5
for \( i = 1, \ldots, t \), and there is no edge between \( X_i \) and \( X_j \) for \( 1 \leq i < j \leq t \).

Thus, the family \( \mathcal{P}_k \) has the quasi-Erdős-Hajnal property with parameters \( \alpha := \alpha \delta^{1/2} \) and \( \beta := 1/2 \). Therefore, \( \mathcal{P}_k \) also has the Erdős-Hajnal property.

In the next two sections, we present the proof of Theorem 6.

### 3 The embedding lemma

The backbone of the proof of Theorem 6 is the following technical lemma, whose proof is already contained in [24], within the proof Lemma 7. For convenience and to make this paper self-contained, it is also included here.

**Lemma 10.** There exist two constants \( \epsilon_1, \alpha_1 > 0 \) with the following property. Let \( G \) be a bipartite graph with vertex classes \( A \) and \( B \), \( |A| = |B| = n \). Then at least one of the following three conditions is satisfied.

(i) There exist \( t \geq 2 \) and \( 2t \) disjoint sets \( W_1, \ldots, W_t \subset A \) and \( X_1, \ldots, X_t \subset B \) such that \( t \geq \alpha_1 \frac{n}{\lfloor |X_i| \rfloor}^{1/2} \), and \( X_i \subset N(W_i) \) for \( i = 1, \ldots, t \), but \( X_i \cap N(W_j) = \emptyset \) for \( i \neq j \).

(ii) There exist \( X_1 \subset A \) and \( X_2 \subset B \) such that \( 2 > \alpha_1 \frac{n}{\lfloor |X_i| \rfloor}^{1/2} \) and there is no edge between \( X_1 \) and \( X_2 \).

(iii) There exists \( v \in A \) such that \( |N(v)| \geq \epsilon_1 n \).

**Proof.** We show that \( \epsilon_1 = \frac{1}{2000} \) and \( \alpha_1 = \frac{1}{100} \) meet the above requirements.

Suppose that (iii) does not hold. Then the number of edges of \( H \) is at most \( \epsilon_1 n^2 \), so the number of vertices \( w \in B \) such that \( |N(w)| > \epsilon_1 n \) is at most \( n/2 \). Deleting all such vertices, and some more, we obtain a bipartite graph \( H' \) with vertex classes \( A' \) and \( B' \) of size \( n' = n/2 \) such that the maximum degree of \( H' \) is at most \( 2 \epsilon_1 n = 4 \epsilon_1 n' \).

Let \( \epsilon = 4 \epsilon_1 = \frac{1}{500} \) and \( \alpha = \frac{1}{100} \). From now on, we shall only work with \( H' \), so with a slight abuse of notation, write \( H := H' \), \( A_0 := A' \), \( B_0 := B' \), and \( n := n' \). Therefore, we have \( \Delta(H) \leq \epsilon n \).

In what follows, we describe an algorithm, which will be referred to as the main algorithm. It will output

(i)’ either an integer \( t \geq 2 \) and \( 2t \) disjoint sets \( W_1, \ldots, W_t \subset A \) and \( X_1, \ldots, X_t \subset B \) such that \( t \geq \alpha \frac{n}{\lfloor |X_i| \rfloor}^{1/2} \), and \( X_i \subset N(W_i) \) for \( i = 1, \ldots, t \), but \( X_i \cap N(W_j) = \emptyset \) for \( i \neq j \);

(ii)’ or two subsets \( X_1 \subset A \) and \( X_2 \subset B \) such that \( 2 > \alpha \frac{n}{\lfloor |X_i| \rfloor}^{1/2} \) and there is no edge between \( X_1 \) and \( X_2 \).

We declare the following constants for the main algorithm. Let \( J_0 = \lfloor \log_2 \epsilon n \rfloor + 1 \), and for \( j = 1, \ldots, J_0 \), let \( t_j = n^{1/2} 2^{j/2} \). Then

\[
\sum_{i=1}^{J_0} t_i = \sum_{i=1}^{J_0} n^{1/2} 2^{j/2} \leq 2n \epsilon^{1/2} \frac{1}{1 - 2^{-1/2}} \leq \frac{n}{4}.
\]
Also, declare the following variables. Let \( J := J_0, A := A_0, B := B_0, A^* := \emptyset \) and \( B^* := \emptyset \).

In each step of the main algorithm, we make the following changes: we move certain elements of \( A \) into \( A^* \), move certain elements of \( B \) into \( B^* \), and decrease \( J \). We think of the elements of \( A^* \) and \( B^* \) as “leftovers”. We make sure that at the end of each step of the algorithm, the following properties are satisfied:

1. \(|A| + |A^*| = |B| + |B^*| = n\),

2. \(|A^*|, |B^*| \leq 2 \sum_{i=J+1}^{J_0} t_i\),

3. for every \( v \in B \), \(|N(v) \cap A| < 2^J\).

Note that by (1) and conditions 1 and 2, we have \( |A|, |B| \geq \frac{n}{2} \). These conditions are certainly satisfied at the beginning of the algorithm. Next, we describe a general step of our main algorithm.

**Main algorithm.** If \( J = 0 \), then stop the main algorithm, and output \( X_1 = A, X_2 = B \). In this case, there is no edge between \( A \) and \( B \), by condition 3 and \(|A|, |B| \geq \frac{n}{2}\). By the choice of \( \alpha \), this output satisfies condition (ii)'.

Suppose next that \( J \geq 1 \). For \( i = 1, \ldots, J \), let \( V_i \) be the set of vertices \( v \in B \) such that \( 2^{i-1} \leq |N(v) \cap A| < 2^i \), and let \( V_0 \) be the set of vertices \( v \in B \) such that \( N(v) \cap A = \emptyset \). Then, by condition 3, we have \( B = \bigcup_{i=0}^{J} V_i \).

Let \( k, 1 \leq k \leq J \) be the largest integer for which \( t_k < |V_k| \). First, consider the case where there is no such \( k \). Then

\[
    n - \sum_{i=J+1}^{J_0} t_i - |V_0| \leq n - |B^*| - |V_0| = |B| - |V_0| = \sum_{i=1}^{J} |V_i| \leq \sum_{i=1}^{J} t_i,
\]

where the first inequality follows from condition 2, and the first equality is the consequence of condition 1. Comparing the left-hand and right-hand sides, and using (1), we get \(|V_0| \geq n/2\). In this case, stop the algorithm and output \( X_1 = V_0 \) and \( X_2 = A \). Note that \( \alpha(n/|X_i|)^{1/2} < 2 \) is satisfied for \( i = 1, 2 \), so this output satisfies condition (ii)'.

Suppose that there exists \( k \) with the desired property. Remove the elements of \( V_i \) for \( i > k \) from \( B \), and add them to \( B^* \). Then we added at most \( \sum_{i=k+1}^{J} t_i \) elements to \( B^* \). Setting \( J := k \), properties 1-3 are still satisfied.

Now we shall run a sub-algorithm. Let \( Z_0 = V_k \). With help of the sub-algorithm, we construct a sequence \( Z_0 \supset \cdots \supset Z_r \) satisfying the following properties. During each step of the sub-algorithm, we either find an output satisfying (i)', or we will move certain elements of \( A \) to \( A^* \). At the end of the \( l \)-th step of this algorithm, \( Z_l \) will be the set of vertices in \( B \) that still have at least \( 2^{k-1} \) neighbours in \( A \). We stop the algorithm if \( Z_l \) is too small.

**Sub-algorithm.** Suppose that \( Z_l \) has already been defined. If \(|Z_l| < 2t_k\), then let \( r = l \), stop the sub-algorithm, remove the elements of \( Z_l \) from \( B \), and add them to \( B^* \). Make the update \( J := k - 1 \), and move to the next step of the main algorithm. Note that \( B^* \) satisfies condition 2. Later, we will see that all the other properties are satisfied.
On the other hand, if \(|Z_l| \geq 2t_k\), we define \(Z_{l+1}\) as follows. Let \(x_l = \frac{|Z_l|}{t_k}\). Say that a vertex \(v \in A\) is heavy if
\[
|N(v) \cap Z_l| \geq \frac{x_l 2^k}{t_k} |Z_l| = \left( \frac{|Z_l|}{t_k} \right)^2 2^k = \frac{|Z_l|^2}{n} =: \Delta_l,
\]
and let \(H_l\) be the set of heavy vertices. Counting the number of edges \(f\) between \(H_l\) and \(Z_l\) in two ways, we can write
\[
|H_l| \Delta_l \leq f < |Z_l| 2^k,
\]
which gives \(|H_l| < \frac{|Z_l|}{2}\). Remove the elements of \(H_l\) from \(A\) and add them to \(A^*\). Examine how the degrees of the vertices in \(Z_l\) changed, and consider the following two cases:

**Case 1.** At least \(\frac{|Z_l|}{2}\) vertices in \(Z_l\) have at least \(2^{k-1}\) neighbors in \(A\).

Let \(T\) be the set of vertices in \(Z_l\) that have at least \(2^{k-1}\) neighbors in \(A\), so \(|T| \geq \frac{|Z_l|}{2}\). Pick each element of \(A\) with probability \(p = 2^{-k}\), and let \(S\) be the set of selected vertices. We say that \(v \in T\) is good if \(|N(v) \cap S| = 1\), and let \(Y\) be the set of good vertices. We have
\[
\mathbb{P}(v \text{ is good}) = |N(v) \cap A| p(1 - p)^{|N(v) \cap A| - 1} \geq \frac{1}{2} (1 - 2^{-k})^{2^k} \geq \frac{1}{6},
\]
so that \(\mathbb{E}(|Y|) \geq \frac{|T|}{6} \geq \frac{|Z_l|}{12}\). Therefore, there exists a choice for \(S\) such that \(|Y| \geq \frac{|Z_l|}{12}\). Let us fix such an \(S\). For each \(v \in S\), let \(Y_v\) be the set of elements \(w \in Y\) such that \(N(w) \cap S = \{v\}\). Also, note that
\[
|Y_v| \leq |N(v) \cap Z_l| \leq \min\{\epsilon n, \Delta_l\} =: \Delta'_l.
\]
In other words, the sets \(Y_v\) for \(v \in S\) partition \(Y\) into sets of size at most \(\Delta'_l\). Here, we have
\[
\frac{|Y|}{\Delta'_l} \geq \frac{|Z_l|}{12 \Delta'_l} \geq \max\left\{ \frac{n}{12 |Z_l|}, \frac{|Z_l|}{\epsilon n} \right\}.
\]
By the choice of \(\epsilon\), the right-hand side is always at least 6. But then we can partition \(S\) into \(t \geq \frac{|Y|}{3 \Delta'_l} \geq 2\) parts \(W_1, \ldots, W_t\) such that the sets \(X_i = \bigcup_{v \in W_i} Y_v\) have size at least \(\Delta'_l\) for \(i = 1, \ldots, t\). The resulting sets \(X_1, \ldots, X_t\) satisfy that
\[
t \geq \frac{|Y|}{3 \Delta'_l} \geq \frac{n}{36 |Z_l|} \geq \frac{1}{36} \left( \frac{n}{\Delta_l} \right)^{1/2} \geq \frac{1}{36} \left( \frac{n}{|X_i|} \right)^{1/2}.
\]
Stop the main algorithm, and output \(t\) and the \(2t\) disjoint sets \(W_1, \ldots, W_t\) and \(X_1, \ldots, X_t\). By the choice of \(\alpha\), this output satisfies (i)'.

**Case 2.** At most \(\frac{|Z_l|}{2}\) vertices in \(Z_l\) have at least \(2^{k-1}\) neighbors in \(A\).

In this case, define \(Z_{l+1}\) as the set of elements of \(Z_l\) with at least \(2^{k-1}\) neighbors in \(A\) (then \(Z_{l+1}\) is the set of all elements in \(B\) with at least \(2^{k-1}\) neighbors in \(A\) as well). Also, move to the next step of the sub-algorithm.

We need to check that, if the main algorithm is not terminated, then after the sub-algorithm ends, conditions 1-3 are still satisfied. Conditions 1 and 3 are clearly true, and 2 holds for \(B^*\).
It remains to show that 2 holds for $A^*$ as well. Note that, as $|Z_{l+1}| \leq \frac{|Z_l|}{2}$ for $l = 0, \ldots, r - 1$, and $|Z_{r-1}| \geq 2t_k$, we have $|Z_l| \geq 2^{r-l}t_k$ and $x_l \geq 2^{r-l}$. Compared to the first step of the sub-algorithm, $|A^*|$ increased by

$$\sum_{l=0}^{r-1} |H_l| \leq \sum_{l=0}^{r-1} t_k \leq \sum_{l=0}^{r-1} \frac{t_k}{2^{r-l}} < t_k.$$ 

Therefore, condition 2 is also satisfied.

In every step of the main algorithm, $J$ decreases by at least one, so the main algorithm will stop in a finite number of steps. When the algorithm stops, its output will satisfy either (i)' or (ii)').

4 The proof of Theorem 6

Now we are in a position to prove Theorem 6. Let $G$ be an ordered graph. The transitive closure of $G$ is the ordered graph $G'$ on the vertex set $V(G)$ in which $x$ and $y$ are connected by an edge if and only if there exists a monotone path in $G$ with endpoints $x$ and $y$.

**Proof of Theorem 6.** Let $\epsilon_1, \alpha_1$ be the constants given by Lemma 10. Furthermore, define the following constants: $c_1 = \frac{\epsilon_1}{2}$, $c_{i+1} = \epsilon_1 c_i$ (for $i = 1, 2, \ldots$), $\epsilon = \frac{c_k}{2}$, and $\alpha = \frac{\alpha_1 c_1^{1/2}}{2}$.

Let $G$ be an ordered graph on $n$ vertices such that

1. the maximum degree of $G$ is at most $\epsilon n$,
2. there exist no $t$ and $t$ disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq \alpha \left( \frac{n}{|X_i|} \right)^{1/2}$ and there is no edge between $X_i$ and $X_j$ for $1 \leq i < j \leq t$.

Then, we show that $G$ contains a monotone path of size $k$ as an induced subgraph. In particular, we find $k$ vertices $x_1 \prec \cdots \prec x_k$ with the following properties. For $s = 1, \ldots, k$,

(a) $x_1, \ldots, x_s$ is an induced monotone path.

(b) Let

$$U_s = V(G) \setminus \left( \bigcup_{i=1}^{s-1} N(x_i) \right),$$

let $G_s = G[U_s \cup \{x_s\}]$, and let $G'_s$ be the transitive closure of $G_s$. Then the forward degree of $x_s$ in $G'_s$ is at least $c_k n$.

First, we find a vertex $x_1$ with the desired properties, that is, if $G'$ is the transitive closure of $G$, then the forward degree of $x_1$ must be at least $c_1 n$. Let $A$ be the set of the first $n/2$ elements of $V(G)$, and set $B = V(G) \setminus A$. Also, let $H$ denote the bipartite subgraph of $G'$ with parts $A$ and $B$. By Lemma 10, at least one of the following three conditions is satisfied.
(i) There exist \( t \geq 2 \) and \( 2t \) disjoint sets \( W_1, \ldots, W_t \subset A \) and \( X_1, \ldots, X_t \subset B \) such that

\[
t \geq \alpha_1 \left( \frac{|A|}{|X_i|} \right)^{1/2} = 2^{-1/2} \alpha_1 \left( \frac{n}{|X_i|} \right)^{1/2} \geq \alpha \left( \frac{n}{|X_i|} \right)^{1/2},
\]

and \( X_i \subset N_H(W_i) \) for \( i = 1, \ldots, t \), but \( X_i \cap N_H(W_j) = \emptyset \) for \( i \neq j \).

(ii) There exist \( X_1 \subset A \) and \( X_2 \subset B \) such that

\[
2 > \alpha_1 \left( \frac{|A|}{|X_i|} \right)^{1/2} = 2^{-1/2} \alpha_1 \left( \frac{n}{|X_i|} \right)^{1/2} \geq \alpha \left( \frac{n}{|X_i|} \right)^{1/2},
\]

and there is no edge between \( X_1 \) and \( X_2 \).

(iii) There exists \( v \in A \) such that \( |N_H(v)| \geq \epsilon_1 |A| = c_1 n \).

As non-edges of \( G' \) are also non-edges of \( G \), (ii) cannot hold, by property 2 of \( G \) (at the beginning of the proof). Suppose that (i) holds. Note that there is no edge between \( X_i \) and \( X_j \) in \( G \), for \( 1 \leq i < j \leq t \). Suppose for contradiction that \( x \in X_i \) and \( y \in X_j \) are joined by an edge in \( G \), for some \( x < y \). Then there exists \( w \in W_i \) such that \( wx \in E(G') \), but \( wy \notin E(G') \). This is a contradiction, as this means that there is a monotone path from \( w \) to \( x \) in \( G \), so there is a monotone path from \( w \) to \( y \) as well. Hence, there is no edge between \( X_i \) and \( X_j \) for \( 1 \leq i < j \leq t \), which contradicts 2. Therefore, (iii) must hold: there exists a vertex \( x_1 \in V(G) \) whose forward degree in \( G' = G'_1 \) is at least \( c_1 n \).

Suppose that we have already found \( x_1, \ldots, x_s \) with the desired properties, for some \( 1 \leq s \leq k-1 \). Then we define \( x_{s+1} \) as follows. Let \( X \) be the forward neighbourhood of \( x_s \) in \( G_s \), let \( Y \) be the forward neighbourhood of \( x_s \) in \( G'_s \), and let \( Z = Y \setminus X \). As \( |X| \leq \epsilon n \) and \( |Y| \geq c_s n \), we have \( |Z| \geq \frac{c}{2} n \). Let \( A \) be the set of the first \( \frac{|Z|}{2} \) elements of \( Z \) with respect to \( \prec \), and let \( B = Z \setminus A \). A monotone path in \( G_s \) is said to be \emph{good} if none of its vertices, with the possible exception of the first one, belongs to \( X \). For every \( v \in A \), there exists at least one element \( x \in X \) such that \( v \in N_{G'_s}(x) \); assign the largest (with respect to \( \prec \)) such element \( x \) to \( v \). Then there is a good monotone path from \( x \) to \( v \).

Define a bipartite graph \( H \) between \( A \) and \( B \) as follows. If \( v \in A \) and \( y \in B \), and \( x \in X \) is the vertex assigned to \( v \), then join \( v \) and \( y \) by an edge if there is a good monotone path from \( x \) to \( y \). Applying Lemma 10 to \( H \), we conclude that at least one of the following three statements is true.

(i) There exist \( t \geq 2 \) and \( 2t \) disjoint sets \( W_1, \ldots, W_t \subset A \) and \( X_1, \ldots, X_t \subset B \) such that

\[
t \geq \alpha_1 \left( \frac{|A|}{|X_i|} \right)^{1/2} > \frac{\alpha_1 c_s}{2} \left( \frac{n}{|X_i|} \right)^{1/2} \geq \alpha \left( \frac{n}{|X_i|} \right)^{1/2},
\]

and \( X_i \subset N_H(W_i) \) for \( i = 1, \ldots, t \), but \( X_i \cap N_H(W_j) = \emptyset \) for \( i \neq j \).

(ii) There exist \( X_1 \subset A \) and \( X_2 \subset B \) such that

\[
2 > \alpha_1 \left( \frac{|A|}{|X_i|} \right)^{1/2} > \frac{\alpha_1 c_s}{2} \left( \frac{n}{|X_i|} \right)^{1/2} \geq \alpha \left( \frac{n}{|X_i|} \right)^{1/2},
\]

and there is no edge between \( X_1 \) and \( X_2 \).
(iii) There exists \( v \in A \) such that \(|N_H(v)| \geq \epsilon_1|A| = \frac{\epsilon_1}{2}\cdot n = c_{s+1}n\).

Suppose first that (i) holds. Then, as before, we show that there is no edge between \( X_i \) and \( X_j \) in \( G \) for \( 1 \leq i < j \leq t \). Suppose that \( u \in X_i \) and \( w \in X_j \) are joined by an edge in \( G \), for some \( u < w \). Then there exists \( v \in W_i \) such that \( vu \in E(H) \), but \( vw \notin E(H) \). Let \( x \in X \) be the vertex assigned to \( v \). Then we can find a good monotone path from \( x \) to \( u \). Since \( uu \) is an edge of \( G \), there is a good monotone path from \( x \) to \( w \), contradicting the assumption \( vw \notin E(H) \). Therefore, there cannot be any edge between \( X_i \) and \( X_j \) in \( G \), which means that (i) contradicts 2.

Suppose next that (ii) holds. Again, we can show that there is no edge between \( X_1 \) and \( X_2 \) in \( G \), contradicting 2. Suppose that \( v \in X_1 \) and \( y \in X_2 \) are joined by an edge in \( G \), and let \( x \in X \) be the vertex assigned to \( v \). There is a good monotone path from \( x \) to \( v \) in \( G_{s+1} \), so there is a good monotone path from \( x \) to \( y \), contradicting the assumption that \( vy \) is not an edge of \( H \).

Therefore, we can assume that (iii) holds. Let \( v \in A \) be a vertex of degree at least \( c_{s+1}n \) in \( H \), and let \( x_{s+1} \in X \) be the vertex assigned to \( v \). We show that \( x_{s+1} \) satisfies the desired properties. We have \( U_{s+1} = U_s \setminus X \), and the forward degree of \( x_{s+1} \) in \( G_{s+1} \) is exactly the number of vertices \( y \) such that there is a good monotone path from \( x_{s+1} \) to \( y \). That is, the forward degree of \( x_{s+1} \) is at least \(|N_H(v)| \geq c_{s+1}n\), as required. This completes the proof. \( \square \)

5 The construction—Proof of Theorem 3

In this section, we present our construction for Theorem 3. The construction involves expander graphs, which are defined as follows.

Recall that for any graph \( H \) and any \( U \subset V(H) \), we denote by \( N(U) = N_H(U) \) the neighborhood of \( U \) in \( H \). The closed neighborhood of \( U \) is defined as \( U \cup N_H(U) \), and is denoted by \( N[U] = N_H[U] \). The graph \( H \) is called an \((n, d, \lambda)\)-expander if \( H \) is a \( d \)-regular graph on \( n \) vertices, and for every \( U \subseteq V \) satisfying \(|U| \leq |V|/2\), we have \(|N[H][U]| \geq (1+\lambda)|U|\). By a well-known result of Bollobás [4], a random 3-regular graph on \( n \) vertices is a \((n, 3, \lambda_0)\)-expander with high probability for some absolute constant \( \lambda_0 > 0 \). In the rest of this section, we fix such a constant \( \lambda_0 \). For explicit constructions of expander graphs see, e.g., [18].

For any positive integer \( r \), let \( H^r \) denote the graph with vertex set \( V(H) \) in which two vertices are joined by an edge if there exists a path of length at most \( r \) between them in \( H \). Here we allow loops, so that in \( H^r \) every vertex is joined to itself. We need the following simple property of expander graphs.

Claim 11. Let \( H \) be an \((n, d, \lambda)\)-expander graph and let \( r \geq 1 \). For any subsets \( X, Y \subseteq V(H) \) such that there is no edge between \( X \) and \( Y \) in \( H^r \), we have \(|X||Y| \leq n^2(1+\lambda)^{-r}\).

Proof. Let \( X_i = N_{H^r}[X] \) and \( Y_i = N_{H^r}[Y] \) for \( i = 0, 1, \ldots, r \). It follows from the definition of expanders that, if \(|X_i| \leq \frac{n}{2} \), then

\[ |X| \leq \frac{1}{2}n(1+\lambda)^{-i}. \]

Similarly, if \(|Y_i| \leq \frac{n}{2} \), then \(|Y| \leq \frac{1}{2}n(1+\lambda)^{-i} \). If \( X \) and \( Y \) are not connected by any edge in \( H^r \), then \( X_i \) and \( Y_{r-i} \) must be disjoint for every \( i \). Let \( \ell \) be the largest number in \{0, 1, \ldots, r\} such that \(|X_\ell| \leq n/2 \).
If \( \ell = r \), then \(|X| < n(1 + \lambda)^{-r} \), and hence \(|X||Y| \leq n^2(1 + \lambda)^{-r} \).

If \( \ell < r \), then \(|X_{\ell+1}| > n/2 \) and \(|Y_{r-\ell-1}| \leq n/2 \). Therefore, we have \(|Y| \leq n(1 + \lambda)^{-(r-\ell-1)} \). Using the inequality \( 1 + \lambda \leq 2 \), we obtain

\[
|X||Y| \leq \frac{1}{4}n^2(1 + \lambda)^{-r+1} \leq n^2(1 + \lambda)^{-r}.
\]

\[\square\]

**Claim 12.** For any \( d \)-regular graph \( H \) and \( r \geq 1 \), we have \( \Delta(H^r) \leq (d + 1)^r \).

**Proof.** Trivial, by induction on \( r \). \[\square\]

Our construction is based on the following key lemma.

**Lemma 13.** Let \( k, m, f \) be positive integers. Let \( A_1, \ldots, A_k \) be disjoint sets of size \( m \), and suppose that there exists an \( (m, 3, \lambda_0) \)-expander.

Then there is a graph \( G \) on the vertex set \( V = \bigcup_{i=1}^k A_i \) such that

1. \( \Delta(G) \leq 4^{f 2^k} \),
2. there are no three vertices \( x, y, z \in V \) such that \( x \in A_a, y \in A_b, z \in A_c \) for some \( a < b < c \), and \( xy, xz \in E(G) \), but \( yz \notin E(G) \),
3. for any \( a \neq b \) and any pair of subsets \( X \subset A_a \) and \( Y \subset A_b \) not connected by any edge of \( G \), we have \(|X||Y| \leq m^2(1 + \lambda_0)^{-f} \).

**Proof.** Let \( H \) be an \( (m, 3, \lambda_0) \)-expander. Let \( \phi : V \to V(H) \) be an arbitrary function such that \( \phi \) is a bijection when restricted to the set \( A_i \), for \( i = 1, \ldots, k \). Define the graph \( G \), as follows. Suppose that \( x \in A_a \) and \( y \in A_b \) for some \( a < b \). Join \( x \) and \( y \) by an edge if there exists a path of length at most \( f 2^{a-1} \) between \( \phi(x) \) and \( \phi(y) \) in \( H \). By Claim 12, the maximum degree of \( G \) is at most \( \sum_{i=1}^{k-1} 4^{f 2^i} \leq 4^{f 2^k} \), so that \( G \) has property 1.

To see that \( G \) also has property 2, consider \( x \in A_a, y \in A_b, z \in A_c \) such that \( a < b < c \) and \( xy, xz \in E(G) \). We have to show that \( yz \notin E(G) \). By definition, there exists a path of length at most \( f 2^{a-1} \) between \( \phi(x) \) and \( \phi(y) \) in \( H \), and there exists a path of length at most \( f 2^{a-1} \) between \( \phi(x) \) and \( \phi(z) \). But then there exists a path of length at most \( f 2^{a} \leq f 2^{b-1} \) between \( \phi(y) \) and \( \phi(z) \), so \( yz \) is also an edge of \( G \).

It remains to verify that \( G \) has property 3. If \( 1 \leq a < b \leq k \) and \( X \subset A_a \) and \( Y \subset A_b \) are not connected by any edge in \( G \), then there is no edge between \( \phi(X) \) and \( \phi(Y) \) in \( Hf 2^{a-1} \). By Claim 11, we have \(|X||Y| \leq m^2(1 + \lambda_0)^{-f 2^{a-1}} \leq m^2(1 + \lambda_0)^{-f} \). \[\square\]

Now we are in a position to prove Theorem 3.

**Proof of Theorem 3.** Let \( k = \frac{2}{\varepsilon} \), \( f = \frac{\log_2 n}{4^{2^k}} \), and \( m = \frac{n}{k} \). We show that the theorem holds with \( \delta = \frac{\log_2(1 + \lambda_0)}{2^k} \).

Let \( A_1, \ldots, A_k \) be disjoint sets of size \( m \). By Lemma 13, there exists a graph \( G_0 \) on \( V = \bigcup_{i=1}^m A_i \) satisfying conditions 1-3 with the above parameters.
Define the ordered graph $G$ on the vertex set $V$ as follows. Let $<$ be any ordering on $V$ satisfying $A_1 < \cdots < A_k$. For any $x \in A_a$ and $y \in A_b$, join $x$ and $y$ by an edge of $G$ if either $a \neq b$ and $xy \in E(G_0)$, or $a = b$. Then the maximum degree of $G$ is at most $\frac{n}{k} + \Delta(G_0) \leq cn$. Notice that the complement of $G$ does not contain a bi-clique of size $n^{1-\delta}$. Indeed, if $(X,Y)$ is a bi-clique in $G$, then there exists $a \neq b$ such that $|X \cap A_a| \geq \frac{|X|^2}{k}$ and $|Y \cap A_b| \geq \frac{|Y|^2}{k}$. Thus, $\frac{|X|^2}{k} \leq \frac{n^2}{4cn^2}$, which implies that $|X| \leq n^{1-\delta}$.

It remains to show that $G$ contains neither $S$, nor $P$ as an induced ordered subgraph. Let us start with $S$. Suppose that there are four vertices, $v_0 < v_1 < v_2 < v_3$, in $G$ such that $v_0v_1, v_0v_2, v_0v_3 \in E(G)$, but $v_1v_2, v_2v_3, v_1v_3 \not\in E(G)$. Let $v_0 \in A_a$, $v_1 \in A_b$, $v_2 \in A_c$, and $v_3 \in A_d$, then $a \leq b \leq c \leq d$. If $c = a$, then $b = a$, which implies $v_1v_2 \in E(G)$, contradiction. Therefore, $a < c \leq d$. As $v_2v_3 \not\in E(G)$, we must have $c < d$ as well. But then the three vertices $v_0, v_2, v_3$ contradict property 2, so that $G$ does not contain $S$ an induced ordered subgraph.

To show that $G$ does not contain $P$, we can proceed in a similar manner. Suppose for contradiction that there are four vertices, $v_0 < v_1 < v_2 < v_3$, in $G$ such that $v_0v_2, v_0v_3, v_1v_2 \in E(G)$, but $v_0v_1, v_1v_3, v_2v_3 \not\in E(G)$. Let $v_0 \in A_a$, $v_1 \in A_b$, $v_2 \in A_c$, and $v_3 \in A_d$, where $a \leq b \leq c \leq d$. We have $a < b$, otherwise $v_0v_1 \in E(G)$. In the same way, $c < d$, otherwise $v_2v_3 \in E(G)$. Therefore, $a < c < d$, and the vertices, $v_0, v_2, v_3$, contradict condition 2 of Lemma 13.

\[ \square \]

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