Droplets on Lubricant Infused Surfaces: The slow dynamics of skirt formation

Zhaohe Dai and Dominic Vella

Mathematical Institute, University of Oxford, Woodstock Rd, Oxford, OX2 6GG, UK

(Dated: December 24, 2021)

A key question in the interaction of droplets with liquid-infused surfaces is what determines the apparent contact angle of droplets. Previous work has used measured values of the skirt geometry (e.g., the radius of curvature of the skirt) to determine this. Here, we consider theoretically the equilibrium of a droplet on a smooth (inverse opal) lubricant-coated surface, and argue that the small effect of gravity within the skirt and the size of the substrate are important for determining the final equilibrium. However, we also show that the evolution towards this ultimate equilibrium is extremely slow (on the order of days for typical experimental parameter values). We therefore suggest that previous experiments on inverse opal or nano-textured surfaces may have observed only slowly-evolving transients, rather than ‘true’ equilibria, potentially explaining why a wide range of skirt sizes have been reported.

I. INTRODUCTION

Liquid-infused surfaces (LISs), also known as slippery liquid infused porous surfaces (SLIPSs), are formed by coating surface with a thin layer of oil lubricant [1, 2]. This liquid coating provides a barrier that prevents droplets of other liquids from reaching the solid surface and thereby allows deposited droplets to move with ultra-low friction [3–7]. This very low friction has found a variety of applications including the creation of surfaces that are anti-biofouling [8], anti-icing [9] and facilitate water harvesting [10], as well as allowing new routes for droplet manipulation [11–13].

In many of these applications, the deformation of the lubricant surface caused by a droplet is particularly important [14]; for example, meniscus-like deformations around the base of a droplet, often called a ‘skirt’ [15–17], give rise to interactions between droplets [11–13] that is a droplet analogue of the “Cheerios effect” [18]. However, there appears to be no clear consensus on what the size of this skirt region is. For example, Semprebon et al. presented theoretical arguments for the apparent contact angle assuming given (constant) curvatures of the interfaces in the skirt region [19, 20]. Their results were consistent with some experiments, in which the radius of curvature of the skirt was constant and, further, small compared with the droplet size [5, 7, 21, 22]. However, in other experiments, Schellenberger et al. [15] observed menisci that did not have constant curvature and were of a size comparable to the capillary length. Even without this discrepancy, current models appear unable to predict either the height of the contact line in the equilibrium state or the effective contact angle; for example, the theory of Semprebon et al. [20] takes the ratio of pressures within the droplet and layer as a control parameter, but does not determine from first principles what this ratio should be.

In this paper we consider the formation and ultimate equilibrium of the skirt region formed when a droplet is deposited on the surface of a LIS. Our aim is to present results in terms of fundamental geometrical and physical control parameters of the system, rather than emergent properties. While many implementations of LIS use a microscopic texture to retain the lubricant layer within the texture, we consider the (simpler) case of a smooth substrate, often referred to as an inverse opal [15].

For the case of an inverse opal LIS, the surface remains wetted by a thin oil layer throughout and two key questions arise: (i) what is the equilibrium state? (ii) how long after deposition is this
state observed? These two questions are the focus of this paper. We begin in §II by describing a model problem that includes the essence of the forces on a lubricating layer that are induced by the presence of a droplet: the ‘pulling’ caused by the droplet’s contact line and the ‘pushing’ caused by its internal capillary pressure. Despite the apparent simplicity of this model problem, we show in §III that many disparate length scales enter: the thickness of the oil layer, the size of the substrate and gravity all play some role in selecting the final equilibrium. We consider the asymptotic limits that appear when the substrate is relatively small and large (in senses to be defined). In §IV, we study the dynamic approach to this equilibrium state. We find that the system passes through many different phases to reach the final equilibrium and, crucially, find that the true equilibrium is only approached on extraordinarily long time scales. We suggest that this long time scale, as well as the complex role of many different length scales in determining the equilibrium, may be the reason that different experiments have reported different behaviours. We finish by summarizing our findings and discussing possible refinements and extensions of the model in §V.

II. THEORETICAL FORMULATION

![Schematic illustration](image)

FIG. 1. Schematic illustration and notation for the analysis of the ‘push-and-pull’ problem considered in this paper. Left: We mimic the effect of a liquid droplet sitting on a thin oil film by introducing a constant pressure pushing down on the film (mimicking the droplet Laplace pressure) within a constant interval $|x| < x_c$, together with line forces pulling upwards at $x = \pm x_c$ (mimicking the capillary force acting at the contact lines). Right: This combination of loads leads ultimately to the formation of an equilibrium meniscus, given by $z = h(x)$.

A. Model problem: pushing and pulling a thin film

The effect of an axisymmetric droplet with contact angle $\theta$ and contact line radius $R_c$ deposited on a thin oil layer is two-fold: firstly the capillary pressure within the droplet, $p_{\text{drop}} = 2\gamma_{\text{dv}} \sin \theta / R_c$, squeezes oil from beneath the droplet; secondly, the capillary force from the contact line, $2\pi \gamma R_c \sin \theta$, pulls the oil interface upwards, sucking liquid into the wetting skirt as it goes. The squeezing action of the droplet capillary pressure has been appreciated previously by Daniel et al. [5], who noted that this pressure is ultimately balanced by the repulsive van der Waals pressure, $p_{\text{vdW}} \sim A / h^3$ with $A$ the Hamaker constant; this leads to an equilibrium film thickness beneath the droplet $h_{\text{eqm}} \sim (A R_{\text{drop}} / \gamma)^{1/3}$. Here van der Waals forces act to prevent the oil layer from draining away completely, since the oil wets the solid.

The equilibrium of the pulling of the contact line has been considered by a variety of authors; perhaps most notably, Semprebon et al. [19, 20] considered the equilibrium of the Neumann triangle at the contact line under the assumption that external contact lines form on the planar substrate. At the same time, the early stages of this formation have been studied in a related problem by Hack et al. [13]. However, how these early stages connect to the ultimate equilibrium has not, to our knowledge, been studied.
In this paper, we seek to understand the interaction of the squeezing out of liquid beneath
the droplet and the pulling up of the contact line by surface tension. To make this problem
more tractable, we consider the model two-dimensional problem shown schematically in fig. 1: the
droplet is represented by a positive pressure \( p \) acting over \( |x| \leq x_c \) while the effect of the contact
line is modelled by two line forces, each of magnitude \( \gamma \), pulling at an angle \( \Delta \theta \) to the vertical at
\( x = \pm x_c \). (Note that this corresponds to assuming an apparent contact angle \( \theta_A = \pi/2 - \Delta \theta \)
and, further, that \( \Delta \theta \) may evolve dynamically as the skirt forms.) To make the problem more tractable
still, we assume that all interfacial tensions are equal, i.e. \( \gamma_{ov} = \gamma_{od} = \gamma_{dv} = \gamma \). Global vertical
force balance on the liquid then requires that \( p = \gamma \cos \Delta \theta / x_c \).

Our primary interest lies in understanding how the combination of the two effects of the droplet
described above affects the formation of the skirt that forms around a droplet placed on a lubricating
layer. What limits the final size of the skirt? Over what time scale does the skirt develop? How does
the apparent contact angle (or the tilting of the Neumann triangle \( \Delta \theta \)) evolve away from \( \theta_A = \pi/2 \)
with time? To address these questions about the evolution of the surface of the lubricating oil
layer, we will use lubrication theory [23].

### B. Mathematical model

The two effects of the droplet are introduced into the pressure field that is imposed on the
lubricating film using a Heaviside step function \( H(x) \) for the pushing pressure and a Dirac \( \delta \)-
function to describe the capillary force from each contact line. In particular, we have

\[
p_e(x) = \frac{\gamma \cos \Delta \theta}{x_c} H(x + x_c) \times H(x_c - x) - \gamma \cos \Delta \theta [\delta(x + x_c) + \delta(x - x_c)]
\]

for \( |x| \leq x_\infty \), with \( 2x_\infty \) the lateral extent of the thin liquid film.

The pressure within the liquid film is then determined by combining this pressure with the
pressure jump due to surface tension, the hydrostatic pressure within the liquid and a contribution
from van der Waals forces. We find that

\[
p(x, z) = p_e(x) - \gamma \frac{h_{xx}}{(1 + h_x^2)^{3/2}} - \frac{A}{h^3} + \rho g (h - z),
\]

where \( A > 0 \) is the Hamaker constant (so that the minus sign gives a repulsive vdW pressure
between the interface and the substrate) and \( z \) is the vertical position within the film. Note that
(2) includes the effects of both gravity and van der Waals forces acting on the lubricant film. While
these forces usually act at very different scales, we shall see that in this problem they both play
an important role.

The pressure field in (2) will, in general, drive the liquid within the lubricating film to flow.
Typically the thickness of the oil film is small compared to its horizontal scale \( \sim x_c \) and so the
long-wavelength approximation of the Stokes equations (cf. lubrication theory) can be used
to describe the dynamics. A standard analysis [23] shows that the evolution of the film thickness
\( h(x,t) \) is described by Reynolds’ equation:

\[
\frac{\partial h}{\partial t} = \frac{1}{3\mu} \frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right),
\]

where \( \mu \) is the viscosity of the liquid in the film.

Substituting the pressure field from (2) into (3) assuming small slopes within the film \( (|h_x| \ll 1) \)
for consistency with the derivation of (3), we obtain a nonlinear diffusion equation for the film
thickness
\[ \frac{\partial h}{\partial t} = \frac{1}{3\mu} \frac{\partial}{\partial x} \left[ h^3 \left( \frac{\partial p_e}{\partial x} - \gamma h_{xxx} + \frac{3A}{h^4} h_x + \rho g h_x \right) \right], \tag{4} \]
for \( t > 0 \). (We shall show that the assumption of small slopes in equilibrium is reasonable for moderate system sizes, \( x_\infty/x_c \lesssim 100 \).)

Equation (4) requires an initial condition and four boundary conditions. We shall use a uniform initial profile of the film:
\[ h(x, 0) = h_0 \tag{5} \]
for simplicity. We shall also assume that the problem remains symmetric about \( x = 0 \); we therefore consider only \( 0 \leq x \leq x_\infty \) and have immediately symmetry conditions at \( x = 0 \), combined with requirements that at the edge of the plate, \( x_\infty \), the film slope and fluid flux should vanish i.e.
\[ h_x(0, t) = h_x(x_\infty, t) = h_{xxx}(0, t) = h_{xxx}(x_\infty, t) = 0. \tag{6} \]

We will discuss the dynamic evolution of the film thickness that arises from the solution of (4) subject to the initial condition (5) and boundary conditions (6) in §IV. First, however, we focus on understanding the equilibrium to which this equation ultimately tends.

### III. PROPERTIES OF EQUILIBRIUM

#### A. Governing equations

The static problem is described by the steady limit of Reynolds’ equation (4), which corresponds to uniform pressure, \( \partial p/\partial x = 0 \). Far from the droplet, we expect the interface to return to a constant height, \( h_\infty \) so that the pressure there is
\[ p_\infty = p(x_\infty, 0) = -A/h_\infty^3 + \rho g h_\infty, \tag{7} \]
measured relative to the (atmospheric) pressure datum. Note, however, that this far-field film height is, in general, different from the initial liquid film height, \( h_0 \), before the droplet is deposited. In particular, note that \( h_\infty \) is not known a priori and must be determined as part of the solution. Nevertheless, we expect \( h_\infty \to h_0 \) for large systems, i.e. as \( x_\infty \to \infty \).

We will see that vertical and horizontal force balance on the contact line (the Neumann relations) for the case of all \( \gamma \) being equal considered here requires the slopes near the contact line to be order unity. For the equilibrium analysis, therefore, we reintroduce the geometrically nonlinear curvature (including terms in \( h_x^2 \)) so that the homogeneity of the liquid pressure becomes
\[ p_e(x) - \frac{\gamma h_{xx}}{(1 + h_x^2)^{3/2}} - \frac{A}{h^3} + \rho g h - p_\infty = 0. \tag{8} \]

Equation (8) is an equation for the meniscus shape \( h(x) \) in equilibrium and is to be solved with symmetry boundary conditions at \( x = 0 \) and \( x_\infty \) as well as the global conservation of volume (which encodes the pertinent information about the initial condition); these conditions read
\[ h_x(0) = h_x(x_\infty) = 0 \quad \text{and} \quad \int_0^{x_\infty} h \, dx = h_0 x_\infty. \tag{9} \]
In practice, the δ-function singularity in \( p_e(x) \) is best handled by integrating (8) across \( x = x_c \), which gives two additional boundary conditions:

\[
\begin{align*}
  h(0) &= h(x_c^+), \quad \text{and} \\
  \frac{h_x(x_c^+)}{(1+h_x^2)^{1/2}} - \frac{h_x(x_c^-)}{(1+h_x^2)^{1/2}} &= -\cos \Delta \theta.
\end{align*}
\]

Physically, the second condition in (10) ensures that the line forces at the contact line are balanced vertically — this would be approximated as \( h_x(x_c^+) - h_x(x_c^-) \approx -1 \) if \( |h_x| \ll 1 \), demonstrating that the small-slope approximation is not strictly self-consistent. It is natural also to ask how the horizontal force balance at the contact line can be satisfied within this model. In short, any asymmetry between the meniscus slopes either side of the contact line necessarily induces a rotation of the direction along which the droplet’s line force acts. We denote this rotation by \( \Delta \theta \), with the sign convention that \( \Delta \theta > 0 \) in the counter-clockwise direction (see fig. 1), so that

\[
\sin \Delta \theta = \cos \theta^+ - \cos \theta^- = \left( \frac{h_x^2}{1+h_x^2} \right)^{-1/2} \left| h_x(x_c^+)^{-1/2} - (1+h_x^2)^{-1/2} \right|_{x_c^-}.
\]

from horizontal force balance, where \( \theta^+ \) and \( \theta^- \) are meniscus angles evaluated on the right and left side of the contact line (see fig. 1). Horizontal and vertical force balance combined give that the included angle between any pair of neighbouring interfaces must be \( 2\pi/3 \) — the interfaces must be arranged as a three-pointed star, but the orientation of this star is to be determined.

**B. Some insights from scaling**

We shall shortly present a detailed study of the solution of the system (8)–(9). However, to do this relies on an appropriate non-dimensionalization of the problem, which in turn depends on some understanding of how the system is likely to behave. Given that there are disparate length scales in this problem (in addition to the size of the ‘droplet’, \( x_c \), and the initial thickness of the liquid layer) it is helpful to think about this in terms of what the system would ‘like’ to do.

Since the liquid beneath the droplet, \( 0 < x < x_c \), is subject to a positive capillary pressure, magnitude \( \gamma/x_c \), it is clear that this interface will seek to be an arc of a circle with radius of curvature \( \sim x_c \) in the absence of gravity, i.e. if \( x_c \ll \ell_c = (\gamma/\rho g)^{1/2} \). In this limiting case, therefore, the profile of the meniscus beneath the droplet \( h(x) \propto x^2/x_c \); in particular, the height of the contact line \( h_c = h(x_c) \sim x_c \). While the behaviour of the inner meniscus determines the height of the contact line in this limit, the outer meniscus (beyond the contact line) is not subject to a downward pressure, and so can only return to being flat because of the combined effect of the van der Waals attraction and hydrostatic pressure. Assuming for the moment that the film thickness is such that hydrostatic pressure dominates van der Waals force throughout, we therefore expect that the outer meniscus should decay over the capillary length \( \ell_c = (\gamma/\rho g)^{1/2} \). (Previous experimental data on inverse opal surfaces confirm this importance of gravity [15], as also discussed in Appendix A.) At a scaling level, then, the total volume of liquid lifted into the skirt in this limit is

\[
u_{\text{skirt}} \sim x_c^2 + x_c \ell_c \approx x_c \ell_c
\]

(since \( x_c \ll \ell_c \) by assumption).

The volume of liquid trapped within the skirt (12) involves the droplet size and the capillary length, \( \ell_c \), but takes no account of the total volume of lubricating liquid available to the system, which is simply \( x_\infty h_0 \). Clearly if the reservoir of liquid stored within the lubricating film is sufficiently large the system will be able, at least in principle, to reach its desired equilibrium. However, if \( x_\infty h_0 \gtrsim x_c \ell_c \) then the amount of lubricant is limited and the system must do something else. Determining this alternative is the ultimate aim of this section, but the insight already gained helps to choose appropriate scales for the non-dimensionalization of the problem.
TABLE I. A summary of the typical values of control parameters with references to experimental work (upper part of the table), and the corresponding emergent dimensionless parameters and time scales (lower part of the table).

| Value | \( x_c \) | \( \gamma \) | \( A \) | \( h_0 \) | \( \mu \) |
|-------|------|------|------|-------|-------|
| \( \sim 0.5 \text{ mm} \) | \( 10 - 60 \text{ mN/m} \) | \( \sim 10^{-21} \text{ J} \) | \( 2 - 20 \mu \text{m} \) | \( \sim 10 \text{ mPas} \) |
| Reference | \[5, 15\] | \[15, 21, 22\] | \[5, 15\] | \[5, 21, 22\] | \[5, 21\] |

| Value | \( \alpha \) | \( \text{Bo} \) | \( V_\gamma \) | \( \tau_\gamma \) | \( \tau \) |
|-------|-----|------|-----|------|------|
| \( 25 - 250 \) | \( 10^{-2} - 10^{-1} \) | \( 10^{-7} - 10^{-3} \) | \( 2 - 20 \mu \text{s} \) | \( 10^{-4} \text{s} \) |

C. Non-dimensionalization

The preceding discussion showed that the intrinsic geometrical properties of the liquid film and ‘droplet’ are extremely important. We shall also see that a number of other length scales emerge from the problem in due course. However, we shall use the length scales already seen to non-dimensionalize the equilibrium problem: we use \( h_0 \) and \( x_c \) to non-dimensionalize the vertical and horizontal directions, respectively, giving the natural dimensionless variables

\[
X = x / x_c, \quad H = h / h_0, \quad \text{and} \quad \Pi(X) = \frac{x_c^2}{\gamma h_0} p_e(x). \tag{13}
\]

(Note that the scaling analysis suggests that the natural scale for \( h_c = h(x_c) \) is \( x_c \); we use \( h_0 \) to rescale \( h(x) \) to capture the natural behaviour far from the contact line.) Substituting these relationships into (8), we find that

\[
\Pi(X) - \frac{H_{XX}}{(1 + H_X^2 / \alpha^2)^{3/2}} - \frac{V_\gamma}{H^3} + \text{Bo} H - P_\infty = 0, \tag{14}
\]

for \( 0 \leq X \leq X_\infty = x_\infty / x_c \), where

\[
\alpha = x_c / h_0 \gg 1, \quad \text{Bo} = \frac{\rho g x_c^2}{\gamma} \tag{15}
\]

are the aspect ratio of the film and the Bond number of the ‘droplet’, respectively, while

\[
V_\gamma = \frac{A x_c^2}{\gamma h_0^3} = \frac{A \alpha^4}{\gamma x_c^2} \tag{16}
\]

measures the relative importance of van der Waals forces \((A / h_0^3)\) and the typical capillary pressure in the original lubricating film \((\gamma h_0 / x_c^2)\) and \(P_\infty = p_\infty x_c^2 / (\gamma h_0)\).

We have already argued that the hydrostatic pressure term must play an important role in determining the equilibrium of the meniscus beyond the contact line (see also Appendix A). It is therefore tempting to ignore the role of van der Waals forces entirely, especially since the dimensionless parameter \( V_\gamma \ll 1 \). However, \( V_\gamma \) measures the importance of van der Waals pressures for a film thickness on the order of the initial film thickness: in equilibrium, the meniscus beneath the droplet has typical thickness \([5] \sim h_{eqm} = (Ax_c / \gamma)^{1/3} \sim 20 \text{ nm} \ll h_0 \) significantly amplifying the importance of the term \( V_\gamma / H^3 \) in (14).

With the above non-dimensionalization, the boundary conditions for the solution of (14) become

\[
H_X(0) = H_X(X_\infty) = 0 \quad \text{and} \quad \int_0^{X_\infty} H \, dX = X_\infty \tag{17}
\]
while the jump conditions across the contact line (10) become

\[ H(1^+) - H(1^-) = 0 \quad \text{and} \quad \frac{H_X}{(\alpha^2 + H_X^2)^{1/2}} \bigg|_{1^+} - \frac{H_X}{(\alpha^2 + H_X^2)^{1/2}} \bigg|_{1^-} = -\cos \Delta \theta, \quad (18) \]

with the rotation angle \( \Delta \theta \) also satisfying (11).

\[ \frac{\partial H}{\partial x} (\alpha^2 + H_X^2)^{1/2} \bigg|_{1^-} = -\cos \Delta \theta, \]

\[ \frac{\partial H}{\partial x} = \frac{\alpha^2}{(\alpha^2 + H_X^2)^{1/2}} \]

\[ H(1^+) = H(1^-) = 0 \]

D. Numerical results

We solve the problem (14) subject to (17)–(18) numerically on the interval \( 0 \leq X \leq X_\infty \) using the multi-point boundary value problem feature of \texttt{bvp5c} in MATLAB. Typical results are shown in fig. 2 and fig. 3. For comparison, we also solve the linearized problem, \(|H_X| \ll \alpha, \Delta \theta \ll 1\) obtaining qualitatively similar results — compare the dashed and solid curves in fig. 3.

![Equilibrium interface profiles calculated numerically from the solution of (14).](image1)

**FIG. 2.** Equilibrium interface profiles calculated numerically from the solution of (14). (a) The effect of system size, \( X_\infty = x_\infty/x_c \), with \( \text{Bo} = 10^{-2}, \nu_\gamma = 10^{-4}, \alpha = 10^2 \) fixed. (b) The effect of initial film thicknesses (for fixed fixed plate size \( X_\infty = 10 \) and Bond number, \( \text{Bo} = 10^{-2} \)) is obtained by changing \( \alpha^{-1} \in [10^{-3}, 10^{-1}] \) but maintaining \( \nu_\gamma = \alpha^4/10^{12} \). (a) and (b) show that equilibria are significantly affected by the size of the reservoir of oil available to be sucked into the meniscus. (c) The effect of van der Waals forces as encoded by the value of \( \nu_\gamma \) is much more limited, as shown by profiles calculated with fixed \( \text{Bo} = 10^{-2} \), initial film thicknesses (so that \( \alpha = 10^2 \)) and plate size \( X_\infty = 10 \).

In many applications it is droplets of millimetric diameter that are of most interest, so that \( x_c \sim 0.5 \text{ mm} \). The far-field thickness of the oil layer may vary significantly depending on surface preparation, but is typically on the order of \( h_0 = 5 \text{ \mu m} \). We therefore have \( \alpha \sim 10^2 \) and \( \text{Bo} \sim 10^{-2} \). What is less clear is the value of \( A \), but we take a typical value \( A = 10^{-21} \text{ J} \) \cite{24} so that \( \nu_\gamma \sim 10^{-4} \). We therefore fix the size of the droplet with \( \text{Bo} = 10^{-2} \) in our simulations and vary \( \alpha \) and \( X_\infty = x_\infty/x_c \) to study the role played by the initial film thickness and the size of the supporting plate on the final skirt shape. (Note that when the drop size is fixed through a constant \( \text{Bo} \), increasing the aspect ratio \( \alpha \) corresponds to decreasing the initial film thickness provided that \( \nu_\gamma \) is chosen such that \( \nu_\gamma \propto \alpha^4 \).) Some examples of these skirt profiles are shown in fig. 2a,b and demonstrate
FIG. 3. Dependence of (a) the equilibrium skirt height at the contact line, $H_c$, (b) the pressure in the film, $P_\infty$, (c) the skirt volume, defined by Eq. (20), and (d) the far-field equilibrium film thickness, $H_\infty$, as functions of the relative plate size, $X_\infty$. Asymptotic results for large systems (dotted black lines) and small systems (solid black lines) are also shown, given by (24) and (27), respectively. The solid line in the inset of (b) is based on $H_c = \alpha^2/(-8P_\infty)$, which is given by combining (26) and (27). All calculations use $Bo = 10^{-2}$ with $V_\gamma = \alpha^4/10^{12}$ and $\alpha$ varying from 10 to $10^3$. All solid and dashed curves of the same colour are calculated using the same parameters but with the dashed curves using the small slope approximation of the curvature, $|h_x| \ll 1$, and the no-rotation assumption for the pulling force, i.e. $\Delta \theta = 0$.

that the size of the system ($X_\infty$) and the film thickness (through $\alpha$) both play important roles in determining the final equilibrium skirt shape and volume, as might have been expected from the earlier scaling discussion. However, the role of van der Waals forces in the final interface shape at a macroscopic scale is minimal unless $V_\gamma$ approaches an infeasibly large value (see fig. 2c).

Beyond the skirt profile itself, the key features of the skirt are its height, the liquid pressure within it, the volume of liquid captured within it, and the asymmetry in the meniscus shape about the contact line. The dependence of the skirt height

$$H_c = H(X = 1)$$

on the initial film thickness and the size of the plate is shown in fig. 3a.

Similarly, the predicted pressure within the liquid skirt, $P_\infty$, is shown in fig. 3b as a function of system size, $X_\infty/X_c$, for different film thicknesses. Previously, Semprebon et al. [19, 20] obtained a relationship between the contact line height, $H_c$, and the radius of curvature of the meniscus, $\kappa \propto P_\infty$, under the assumption that the skirt remains small compared to the droplet. Such geometrical arguments are entirely compatible with our approach since the film pressure is determined as part of the solution; indeed, plotting $H_c/\alpha = h_c/x_c$ as a function of $\alpha/(-P_\infty)$ produces a collapse of our numerical data onto a master curve that is very similar to that provided by Semprebon et al. [20] (compare the inset of fig. 3b with fig. 4 of Semprebon et al. [20]). However, we emphasize that our result goes beyond that of Semprebon et al. [20] by determining the height of the contact line and the curvature/pressure of the skirt in terms of the key control parameters within the system, namely $X_\infty$ and $\alpha$. In particular, we will determine explicit relationships for $H_c$ and $P_\infty$ in the limit of large or small system sizes in sections III E and III F, respectively.
A natural definition of the skirt volume is the volume of liquid lifted above the far-field liquid level, i.e. $\int_{X^-}^{X^+} [H(X) - H(X_{\infty})] \, dX$, where the limits of integration $X_{\pm}$ are defined such that $H(X^\pm) = H(X_{\infty})$. A key question is then whether the liquid in the skirt comes predominantly from the liquid beneath the droplet being squeezed out into the skirt, or rather is sucked into the skirt from the remainder of the bath. To answer this, we introduce the dimensionless skirt volume

$$V_{\text{skirt}} = \int_{X^-}^{X^+} H(X) - H(X_{\infty}) \, dX. \quad (20)$$

With this definition, a value of $V_{\text{skirt}} = 1$ suggests that the skirt is dominated by liquid that is pushed into it from beneath the droplet (since the liquid layer beneath the droplet in equilibrium $H_{\text{eqm}} = (V_\gamma/\alpha)^{1/3} \sim 10^{-2} \ll 1$, only a negligible amount of the film is expected to remain there and a dimensionless volume of liquid $\approx 1$ is pushed out from under the droplet). Conversely, a value $V_{\text{skirt}} \gg 1$ suggests instead that the skirt is dominated by liquid that is sucked into it from the remainder of the lubricant film. The dependence of the skirt volume defined in this way on the system size, $X_{\infty}$, is shown in fig. 3c. Crucially, we see that $V_{\text{skirt}} \gg 1$ for all but the very smallest system sizes, indicating that the majority of the liquid within the skirt has been sucked into it from the reservoir, rather than being squeezed out from beneath the ‘drop’.

Figure 3d shows the dependence of the equilibrium far-field film thickness on the system size. As might be expected, this shows a significant change in thickness compared to the initial condition when the system is small, while for large systems this is essentially unchanged.

The results in fig. 3 show that, unless the system is extremely large, the size of the skirt depends sensitively on how large the system is, $X_{\infty}$. The dependence on $X_{\infty}$ arises from the global conservation of mass: the skirt requires liquid to be supplied by the coating of the remainder of the plate (since $V_{\text{skirt}} \gg 1$, as already discussed), and hence on how much liquid is available on the plate. We also see that what a ‘large’ system size means depends on the thickness of the liquid film: for smaller $\alpha$ (i.e. thicker films with drop size fixed) the skirt volume saturates at smaller system sizes, again because more liquid is available to the skirt. We will discuss this further later.

The final quantity of interest in the static problem is the asymmetry of the skirt about the contact line, which is defined as $\theta^+ - \theta^-$. This asymmetry is intimately related to the rotation of the vertical pulling force, $\Delta \theta$, since the contact line must remain in vertical and horizontal force equilibrium. (In particular, for perfect symmetry $\theta^\pm = \pi/6$ and $\Delta \theta = 0$ to ensure all included angles are $2\pi/3$.) This asymmetry can be seen in fig. 4a where the deviation of the local angle from $\pi/6$ on the left side (inner meniscus) and right side (outer meniscus) are shown as a function of system size; crucially the deviation from symmetry increases with the system size ($X_{\infty}$).

The rotation angle $\Delta \theta$ defined in (11) (and see fig. 1) is given by

$$\Delta \theta = \sin^{-1} \left[ \left( 1 + \frac{H_X^2}{\alpha^2} \right)^{-1/2} \right]_{1+} - \left( 1 + \frac{H_X^2}{\alpha^2} \right)^{-1/2} \right]_{1-}, \quad (21)$$

with $\Delta \theta \approx \|H_X(1^+) + H_X(1^-)\|/2\alpha$ for $|H_X| \ll \alpha$. The rotation of the contact line is shown in fig. 4, and demonstrates that the Neumann triangle generically rotates anti-clockwise (since $\Delta \theta > 0$); $\Delta \theta$ also increases with system size (as might be expected since meniscus symmetry and rotation are intricately related). On the face of it there are two possible sources for this rotation: the applied squeezing pressure pushes the inner and outer portions of the meniscus differently and hence may lead to rotation while the interaction between the two menisci at $\pm X_c$ also leads to an asymmetry, and hence rotation. This latter effect is somewhat analogous to the ‘Cheerios effect’ [18], and can be isolated from the former effect in our model by omitting the Heaviside terms in (1); fig. 4 therefore shows the rotation angle $\Delta \theta$ predicted from this model in which the interface is subject only to a ‘pull’ from surface tension, without the ‘push’ from the capillary pressure. Interestingly, the rotation with only the ‘pull’ effect is in the opposite sense to that obtained from
FIG. 4. (a) Deviation of the meniscus angles at the contact line (as defined in fig. 1) from $\pi/6$ as a function of the system size, $X_{\infty}$. (b) Equilibrium values of the rotation angle $\Delta \theta$ defined by Eq. (21) for the scenarios in which there are only meniscus forces (‘pull only’) and that in which the droplet’s capillary pressure also pushes the interior meniscus downward (‘push and pull’). Note that the ‘self-interaction’ of the droplet’s two menisci causes a clockwise rotation of the menisci $\Delta \theta < 0$, while the effect of the droplet’s positive Laplace pressure works in the opposite direction, causing an anti-clockwise rotation, $\Delta \theta > 0$.

the full model; we conclude that the interaction between the menisci is not the dominant source of this asymmetry — rather it is the effect of the droplet’s squeezing pressure that is crucial.

Note that in all cases our numerical results with the fully nonlinear curvature and rotation $\Delta \theta$ are only slightly different to these with linearized curvatures and no rotation — compare dashed and solid curves in figs. 3 and 4. However, this does not indicate that slopes remain small; rather there seems to be a fortuitous cancellation of the two effects. Our numerical results have revealed quite different behaviours depending on the size of the system $X_{\infty}$: for small systems, the menisci are approximately symmetrical, with the droplet’s line force acting approximately vertically; for larger systems, the outer meniscus is essentially horizontal with the droplet’s two menisci inclined at $\pi/6$ to the horizontal to balance one another. To understand these behaviors we turn now to consider the limits of large ($X_{\infty} \gg 1$) and small ($X_{\infty} \gtrsim 1$) systems in turn.

E. Large systems

When $X_{\infty}$ is sufficiently large, the system approaches a well-defined limit in which further increases in $X_{\infty}$ do not change the properties of the final equilibrium: there is enough liquid coating the substrate for the skirt to find its preferred equilibrium. In this large-system limit there is a small, but positive, pressure since $P_{\infty} = Bo H_{\infty} - V_\gamma / H_{\infty}^3$, $V_\gamma \ll Bo \ll 1$ in general and the
far-field film thickness is close to its initial value, i.e. $H_\infty \sim 1$. In this case the two menisci that meet at the contact line behave very differently: the outer meniscus is affected only by gravity (since $V_\gamma \ll 1$) and so we may write:

$$H \approx (H_c - 1) \exp \left[ -\sqrt{Bo} (X - 1) \right] + 1,$$

for $X > 1$, where we have assumed $|H_X| \ll \alpha$ (i.e. small slope deformations), which is self-consistent since we expect the constant $H_c \sim O(\alpha)$ and then (22) gives that $|H_X| \sim \alpha \sqrt{Bo} \ll \alpha$.

To solve the shape of the inner meniscus, we neglect the $V_\gamma/H^3$ term (since it decays quickly as the film height increases beyond $H_{\text{eqm}} \ll 1 \ll H_c$) and integrate (14) once to have:

$$\left( 1 + \frac{1}{\alpha^2} H_X^2 \right)^{-1/2} + \frac{1}{2\alpha} Bo \cdot H^2 + \frac{1}{\alpha} H \cos \theta - 1 = 0,$$

for $0 < X < 1$, where we have assumed $H_X = 0$ as $H \to 0$ beneath the drop to determine the constant of integration. To solve for $H_c$ and $\Delta \theta$, instead of using the vertical and horizontal force balance conditions in (18) and (21), we exploit their geometrical counterparts: $\theta^+ + \Delta \theta = \pi/6$ and $\theta^+ + \theta^- = \pi/3$, where $\theta^+ \approx |H_X(1^+)/\alpha|$ can be calculated using (22) while $\theta^- = \tan^{-1} [H_X(1^-)/\alpha]$ can be given by (23). Using $H_c = \alpha(1 + \delta H_c)$ and $\Delta \theta = \pi/6 + \delta \Delta \theta$ and $Bo \ll 1$ we obtain:

$$H_c \approx \frac{\alpha}{3\sqrt{3}} (3 - 4Bo^{1/2}), \quad H_\infty \approx 1, \quad V_{\text{skirt}} \approx \frac{H_c}{\sqrt{Bo}}, \quad \Delta \theta \approx \frac{1}{6} \pi - \frac{1}{\sqrt{3}} Bo^{1/2},$$

where we assumed $H_c \gg 1$ for the calculation of $H_c$ and neglected the volume of liquid beneath the drop when calculating $V_{\text{skirt}}$. Equation (24) is used to plot the dotted lines in fig. 3 and fig. 4b, and agrees well with our numerical results in the appropriate limit.

F. Small systems

When $X_\infty$ is not too large, our numerical results show that the skirt region is small and the far-field film thickness $H_\infty \ll 1$: the vast majority of liquid initially coating the substrate is drawn into the skirt by the skirt’s negative capillary pressure; as such, the skirt growth is limited by the amount of liquid available to it. Moreover, the pressure within the skirt is relatively large in magnitude.

In this volume-limited limit, we expect the height and radius of curvature of the skirt to be comparable and hence the meniscus height not to be large enough for a significant deviation from the ideal Neumann triangle to take place: we expect the angles at the contact line to be close to $\pi/6$, with relatively little meniscus rotation, $\Delta \theta \ll 1$. These expectations are borne out by our numerical results. To understand this limit further, we follow a similar approach to that used in [19, 21], approximating the meniscus shape on both sides of the contact line by a parabola with a radius of curvature much smaller than unity (i.e. the skirt is much smaller than the ‘drop’). This small-skirt assumption gives symmetric meniscus profiles about the contact line, since there is relatively little meniscus rotation (fig. 4b), and so we may write:

$$H \approx -\frac{1}{2} P_\infty (X_s - |X - 1|)^2$$

for $1 - X_s \leq X \leq 1 + X_s$, with $2X_s \ll 1$ the horizontal size of the skirt. The slope discontinuity condition, (18), and volume-conservation conditions, (17), can be used to solve for $X_s$ and $P_\infty$. In particular, we find $P_\infty = -\alpha/(2X_s)$ and $X_\infty = -P_\infty X_s^3/3$ so that

$$P_\infty = -\left( \frac{\alpha^3}{24X_\infty} \right)^{1/2},$$

(26)
which immediately lead to

\[ H_c = \sqrt[8]{\frac{3}{8}} \alpha^{1/2} X_\infty^{1/2}, \quad H_\infty = \left( \frac{24 \gamma^2 X_\infty}{\alpha^3} \right)^{1/6}, \quad V_{\text{skirt}} = X_\infty, \quad \text{and} \quad \Delta \theta = 0, \tag{27} \]

where the far-field thickness is determined from the stabilization of the film by van der Waals pressure, i.e. \( H_\infty \approx \left( -V_\gamma / P_\infty \right)^{1/3} \). It is also worth noting that the largest unscaled slope of the meniscus here is \( |h_x| = \alpha^{-1} H_X(1) = \alpha^{-1} P_\infty X_s = 1/2 \), which is an \( O(1) \) quantity: again, even though the deformations in this case are relatively small, we still do not formally satisfy the small slope approximation we have been making throughout.

The first two results in (27) are plotted as the solid curves in fig. 3. We find good agreement with numerical results for small skirts as these expressions are derived based on the assumption that \( X_s \ll 1 \). (Note also that the prediction \( V_{\text{skirt}} = X_\infty \) is merely a consistency check since we assumed in the above derivation that the growth of the skirt is limited by the volume of liquid available.) Finally, we note that the small-skirt assumption \( X_s \ll 1 \) requires the control parameters of the system to satisfy \( X_\infty / \alpha \ll 1 \), i.e. \( x_\infty / x_c \ll x_c / (6 h_0) \) in dimensional form. Perhaps surprisingly, this condition is not clearly satisfied in experiments; for example, taking \( x_\infty \sim 1 \text{ cm}, \ x_c \sim 0.5 \text{ mm} \) and \( h_0 \sim 5 \mu \text{m} \) [4, 21, 22] we would have \( x_\infty / x_c = 20 \), while \( x_c / (6 h_0) \sim 15 \) and the separation of scales required for a small skirt is not achieved.

G. Transition from small to large systems

In the preceding sections, we have calculated the dependence of the final skirt volume on the system size. The numerical results in fig. 3c show that, on the whole, the transition between these relations is relatively sudden, being where the asymptotic results for the skirt volume given in (27) and (24) intersect. This transition happens when

\[ x_\infty \sim \frac{x_c}{h_0} \ell_c \tag{28} \]

or

\[ X_\infty \sim \alpha \text{Bo}^{-1/2}. \tag{29} \]

Physically, this result suggests that large systems are those for which the total amount of liquid coating the substrate, \( x_\infty \times h_0 \), is significantly larger than the volume that is required to make a static meniscus of height \( \sim x_c \), making use of (24) and width \( \ell_c \). It is also worth noting that the size at which the system becomes ‘large’ (and hence no longer limited by the volume of liquid in the coating) is the capillary length multiplied by a factor \( x_c / h_0 \); for a droplet with the parameters suggested in table I, this dimensionless factor is on the order of \( 10 \sim 100 \), so that experiments may well not be in the large system size limit either. This again suggests that skirts observed experimentally may be limited by the volume of liquid available within the lubricant layer.

Our study of equilibrium has shown that the size of the coated region, in particular the reservoir of oil available to the skirt is important in determining the final equilibrium of this ‘droplet’ on a lubricant-infused surface. Some features of the equilibrium we have elucidated here are in agreement with previous experiments (see for example the role of gravity in the meniscus shape discussed in Appendix A); nevertheless we are not aware of previous discussion, or experimental study, of this sensitive dependence on system size.
Before moving on to consider the dynamics of how this equilibrium is reached, we briefly consider the effect of van der Waals forces on the equilibrium that is established. These forces are important in maintaining a very thin liquid layer on the substrate in the small system limit — see equation (27). However, the strength of van der Waals forces does not otherwise enter the asymptotic results (24) and (27). The numerical results shown in Fig. 2c and fig. 5 show that the effect of the van der Waals parameter on the macroscopic equilibrium properties of the equilibrium setup are very small unless $\mathcal{V}_\gamma \sim 1$, which corresponds to unphysically large values of the Hamaker constant $A$.

We turn now to study the dynamic process through which equilibrium is established, seeking to understand the time scale on which equilibrium is established. We shall see that though van der Waals forces play little role in the macroscopic properties of the final equilibrium, they are important in determining the time taken to setup the equilibrium.

**IV. DYNAMICS**

The dynamic model developed in §II involved making the long wavelength approximation of lubrication theory. This is, strictly speaking, inconsistent with the use of a nonlinear curvature term in the capillary pressure, which is important close to the contact line, where the Neumann conditions require slopes to be order unity. While modifications to lubrication theory to account for such slopes have been proposed previously [25, 26], the differences with standard lubrication theory are generally small and certainly quantitative, rather than qualitative. Moreover, we have already seen that for the equilibrium problem (which is ultimately reached by the dynamic system) the fully nonlinear and linear problems agree very well, modulo some small quantitative differences. We shall therefore use this linearization (i.e. we use the linearized curvature and neglect the rotation of the contact line) throughout our study of the dynamics of the problem, noting that these assumptions may mean that the final equilibria reached are slightly different to those studied in the last section. While we shall predominantly solve the resulting problem numerically, this simplification allows for...
some analytical descriptions of the evolution of the initially uniform film, \( h(x, t = 0) = h_0 \).

A. Early-time behaviour

At early times, the film beneath the ‘droplet’ does not yet ‘know’ the final equilibrium thickness that it will reach. As a result, the relevant vertical length scale with which to measure deformations of the thin liquid film is \( h_0 \). Similarly, we shall find that the early motion is dominated by the pulling force at the contact line, which does not depend on the droplet size, so that the relevant horizontal length scale is also \( h_0 \). A scaling analysis of the dimensional governing equation (4) suggests that the relevant time scale for these early dynamics is

\[
\tau_\ast = \frac{3\mu h_0}{\gamma}.
\]  

(30)

Typically, \( \tau_\ast \) is on the order of 1 \( \mu s \) (see Table I).

1. Non-dimensionalization

Based on the previous discussion, we introduce a slightly different non-dimensionalization for early times to that used for the static model, namely we let

\[
\tilde{X} = \frac{x}{h_0}, \quad \tilde{H} = \frac{h}{h_0}, \quad \text{and} \quad \tilde{T} = \frac{t}{\tau_\ast},
\]  

(31)

so that the governing equation (4) becomes

\[
\frac{\partial \tilde{H}}{\partial \tilde{T}} = \frac{\partial}{\partial \tilde{X}} \left[ \tilde{H}^3 \left( \frac{\partial \tilde{H} \partial \Pi}{\partial \tilde{X} \partial \tilde{X}} - \frac{\partial^3 \tilde{H}}{\partial \tilde{X}^3} \right) + \frac{1}{\alpha^2} \left( \frac{3V_\gamma}{\tilde{H}} + \text{Bo} \tilde{H}^3 \right) \frac{\partial \tilde{H}}{\partial \tilde{X}} \right],
\]  

(32)

for \( 0 \leq \tilde{X} \leq \alpha X_\infty \), and where \( \Pi(\tilde{X}) = \Pi(\alpha - \tilde{X})/\alpha - \delta(\tilde{X} - \alpha) \) for \( \tilde{X} > 0 \). Note that (32) involves the already familiar parameters, e.g. \( \alpha \), \( \text{Bo} \), and \( V_\gamma \) that were defined in (15) and (16).

With this rescaling, the initial and boundary conditions become

\[
\tilde{H}(\tilde{X}, 0) = 1,
\]  

(33)

and

\[
\frac{\partial \tilde{H}}{\partial \tilde{X}} \bigg|_{(0, \tilde{T})} = \frac{\partial \tilde{H}}{\partial \tilde{X}} \bigg|_{(\alpha X_\infty, \tilde{T})} = \frac{\partial^3 \tilde{H}}{\partial \tilde{X}^3} \bigg|_{(0, \tilde{T})} = \frac{\partial^3 \tilde{H}}{\partial \tilde{X}^3} \bigg|_{(\alpha X_\infty, \tilde{T})} = 0.
\]  

(34)

The problem specified in (32)–(34) is solved numerically using the method of lines implemented in MATLAB, following Shampine [27]. More details of our numerical method are given in Appendix B. However, we note here that a singularity in the driving pressure, \( \Pi(\tilde{X}) \), is introduced by the step and Dirac-\( \delta \) function at the contact line (which is now located at \( \tilde{X} = \alpha \)). This singularity is treated by considering the regions with \( \tilde{X} < \alpha \) and \( \tilde{X} > \alpha \) separately and connecting them via four matching conditions at the contact line. Specifically, the film thickness is continuous, so that

\[ [\tilde{H}]^+ = 0, \]  

(35)

with \([g]^+ = g(\alpha^+) - g(\alpha^-)\), while there are jumps in the slope and curvature that can be written

\[
\left[ \frac{\partial \tilde{H}}{\partial \tilde{X}} \right]^+_\_ = -1, \quad \text{and} \quad \left[ \frac{\partial^2 \tilde{H}}{\partial \tilde{X}^2} \right]^+_\_ = -1/\alpha.
\]  

(36)
A final jump condition is provided by the conservation of fluid flux across the contact line; this is simplified to

\[
\left[ \frac{\partial^3 \tilde{H}}{\partial \tilde{X}^3} \right]^+_- = 0,
\]

by neglecting the van der Waals and hydrostatic pressure terms, which would provide a correction of order \( \sim (3V_\gamma + Bo)/\alpha^2 \ll 1 \).

Unless stated otherwise, the parameters used in numerical simulations are \( \alpha = 100, Bo = 10^{-2} \), and \( V_\gamma = 10^{-4} \) — Table I shows that these values are typical of experiments. However, to mimic the effect of varying the film thickness, we vary \( \alpha \) but choose \( V_\gamma \propto \alpha^4 \). Numerical results for the evolution of the contact line height are shown in fig. 6b with \( \alpha \) varying to mimic this film thickness variation. On the time scale of these simulations, the results show no sign of dependence on the system size, \( x_\infty \), and only a weak dependence on the film thickness \( \alpha \) at very late times. We therefore begin by focussing on understanding these very early stages of the motion.

![Diagram](a)

**FIG. 6.** Numerical results for the early time evolution of the thin film profile. (a) Schematic of the deformation: two narrow neck regions emerge from the contact line and move away from it. The skirt volume is calculated dynamically as the volume contained in the wedge above the line \( \tilde{H} = \tilde{H}(\alpha x_\infty/x_c) \); at early times, \( \tilde{H}(\alpha x_\infty/x_c) \approx 1 \), which is the dashed line labelled ‘initial’. The zoom-in view of the skirt apex illustrates the tilting of the Neumann triangle. (b) The early-time evolution of the change in skirt height, \( \tilde{H} - 1 \). Numerical results are shown by solid curves for film aspect ratios \( \alpha = 100, 200, 400 \) (curves) with different values of the system size indicated by line style: \( x_\infty/x_c = 10 \) (solid grey) and \( x_\infty/x_c = 100 \) (dashed orange). The dashed black line indicates the prediction, (44), that comes from the early-time similarity solution. Here \( Bo = 10^2, V_\gamma = \alpha^4/(10^{12}) \) so that increasing \( \alpha \) corresponds to increasing \( h_0 \).
2. Linearized analysis

In the very early stages of the motion, the system evolves away from the flat initial condition. To study this evolution, we let $\tilde{H}(\tilde{X}, \tilde{T}) = 1 + \eta(\tilde{X}, \tilde{T})$ where $|\eta| \ll 1$ and linearize (32) to give

$$\frac{\partial \eta}{\partial \tilde{T}} = \frac{\partial^2 \tilde{\Pi}}{\partial \tilde{X}^2} - \frac{\partial^4 \eta}{\partial \tilde{X}^4}. \quad (38)$$

A scaling analysis of this equation suggests that there is an evolving length scale $\tilde{X} \sim \tilde{T}^{1/4}$ [13]. In the very early stages of the motion, therefore, the effect of the second meniscus (at $X = -\alpha$ for the meniscus shown in fig. 6) and the far-field boundary at $\tilde{X} = \alpha X_\infty$ are expected to be negligible — in agreement with the numerical results of fig. 6b. Moreover, the role of the ‘pushing’ component of $\tilde{\Pi}(\tilde{X})$ is expected to be minimal (since the Heaviside term in $\tilde{\Pi}$ that represents the pushing of the drop is less singular than the Dirac $\delta$-function that represents the pulling of the contact line).

We therefore consider briefly the simplest problem of a single meniscus pulling up on the surface, corresponding to $\tilde{\Pi}(\tilde{X}) = -\delta(\tilde{X} - \alpha)$. Using the symmetric definition of the Fourier Transform

$$\hat{\eta}(k, \tilde{T}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta(\tilde{X}, \tilde{T}) e^{-ik\tilde{X}} d\tilde{X}, \quad (39)$$

we find that (38) is transformed to

$$\frac{\partial \hat{\eta}}{\partial \tilde{T}} + k^4 \hat{\eta} = \frac{1}{\sqrt{2\pi}} k^2 e^{-ika}, \quad (40)$$

which has solution

$$\hat{\eta}(k, \tilde{T}) = \frac{e^{-ika}}{\sqrt{2\pi} k^2} \left(1 - e^{-k^4\tilde{T}}\right). \quad (41)$$

Inverting (41) gives a similarity solution for the perturbation to the flat interface shape

$$\eta(\tilde{X}, \tilde{T}) = \frac{1}{2\pi} \tilde{T}^{1/4} f(\xi), \quad (42)$$

where $\xi = (\tilde{X} - \alpha)/\tilde{T}^{1/4}$ is the similarity variable and

$$f(\xi) = 2\Gamma \left(\frac{3}{4}\right) \text{}_1F_3 \left(-\frac{1}{4}, \frac{1}{4}, \frac{3}{4}; \frac{\xi^4}{256}\right) + \Gamma \left(\frac{5}{4}\right) \xi^2 \text{}_1F_3 \left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; \frac{\xi^4}{256}\right) - \pi|\xi|, \quad (43)$$

with $\text{}_1F_3(\cdot)$ the generalized hypergeometric function [28].

From the analytical expression (43) we find that $f(0) = 2\Gamma(3/4)$, immediately giving the growth of the capillary ridge with time. The position of the necks nearest to the contact line requires the numerical determination of the position of the minima of $f(\xi)$; these are found to occur at $\xi = \xi_* \approx \pm 2.30$ with $f(\xi_*) \approx -0.610$. Altogether, this similarity solution predicts that the height of the contact line evolves according to

$$\tilde{H}_c = 1 + \frac{\Gamma(3/4)}{\pi} \tilde{T}^{1/4}. \quad (44)$$

while the minima have location and film thickness

$$\tilde{X}_{\min}^\pm \approx \alpha \pm 2.30 \tilde{T}^{1/4}, \quad \tilde{H}_{\min}^\pm - 1 \approx -\frac{0.305}{\pi} \tilde{T}^{1/4}. \quad (45)$$
FIG. 7. Early-time evolution of (a) the minimum thickness and (b) the lateral position of the neck region that forms in the early stages of the motion. The dashed black lines are the predictions of the early time similarity solution (45), while the solid curves show numerical results with the labels ‘outer’ and ‘inner’ referring to the neck regions in the regions $\tilde{X} > \alpha$ and $\tilde{X} < \alpha$, respectively. Here $\alpha = 100$.

This early time similarity solution also suggests that there are two local maxima in the film height, $\tilde{H}_{\text{max}}^\pm$, whose location and film thickness are

$$\tilde{X}_{\text{max}}^\pm \approx \alpha \pm 5.81 \tilde{T}^{1/4}, \quad \tilde{H}_{\text{max}}^\pm - 1 \approx 0.009 \tilde{T}^{1/4},$$

while stagnation points between these minima and maxima lie at

$$\tilde{X}_{\text{s.p.}}^\pm \approx \alpha \pm 4.592 \tilde{T}^{1/4}.$$  

The prefactors in (46) and (47) have been determined by consideration of the function $f(\xi)$ in (43). While many local maxima, minima and stagnation points exist in this similarity solution (and indeed throughout the evolution), we shall see that these first ones beyond the contact line are particularly important in the ensuing dynamics. We will refer to these first local maxima as the ‘bumps’ in what follows, while the minima are referred to as ‘necks’.

The above calculation gives some useful predictions for the early time evolution that can be tested by comparison with the numerical results. Figure 6 shows that the height of the contact line does indeed follow the prediction of (44). Similarly, the predictions of (45) are borne out by the numerical results shown in fig. 7.

While this similarity solution gives some insight into the early stages of the motion, its neglect of both the second meniscus (at $\tilde{X} = -\alpha$) and of the pushing pressure mean that its predictions remain purely symmetric throughout, clearly limiting its utility in describing later stages of the motion. To answer any questions about how the asymmetry at the contact line evolves dynamically requires these ingredients to be put back into the model, i.e. for the full form of $\tilde{\Pi}(\tilde{X})$ to be used. The same analysis can be followed with the complete loading pressure, i.e. $\tilde{\Pi}(\tilde{X}) = \mathcal{H}(\tilde{X} + \alpha) \times \mathcal{H}(\alpha - \tilde{X})/\alpha - [\delta(\tilde{X} + \alpha) + \delta(\tilde{X} - \alpha)]$, by taking Fourier transforms of (38).

In this case, the solution is

$$\hat{\eta}(k, \tilde{T}) = \sqrt{\frac{2}{\pi}} \left( \frac{\cos \alpha k}{k^2} - \frac{\sin \alpha k}{\alpha k^3} \right) \left( 1 - e^{-k^4 \tilde{T}} \right).$$  

(48)
FIG. 8. Early-time evolution of (a) the induced rotation angle, $\Delta \theta$, and (b) the volume, $V_{skirt}$, of the skirt. In (a) the asymptotic prediction (50) is shown by the dashed line. In (b), $V_{skirt}$ is the volume of liquid near the contact line, and is defined by $V_{skirt}(t) = \int_{x_{\min}}^{\infty} [h(x, t) - h(x_{\infty}, t)] \frac{dx}{x_c h_0}$, which would be the same as (20) as $\tilde{T} \to \infty$. $V_{drop}(t)$ is the rescaled volume of the liquid squeezed out beneath the drop, i.e., $V_{drop}(t) = \int_{0}^{x_{c}} [h_0 - h(x, t)] \frac{dx}{x_c h_0}$.

We are not able to give a closed form for the inverse of (48), and note that the introduction of a length scale (the separation between the two menisci), breaks the similarity form of the solution. However, it is possible to make progress in understanding the evolution of the quantity of most interest, which is the slope of the profile surrounding the contact line. In particular, we find that the slope of the interface deformation throughout is given by

$$\frac{\partial \tilde{H}}{\partial \tilde{X}} = \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \cos \frac{\alpha k}{k} - \frac{\sin \frac{\alpha k}{k}}{\alpha k^2} \right) \left( 1 - e^{-k^2 \tilde{T}} \right) e^{ik \tilde{X}} \, dk. \quad (49)$$

This expression can be used to calculate the asymmetry in the profile either side of the contact line and induces a tilt $\Delta \theta = \pi/2 - \theta_A$ in the vertical force applied to the film to satisfy horizontal force balance. Though our numerics neglect this tilt, its required magnitude can be calculated assuming $\Delta \theta \ll 1$; using the linearized version of (21) with (49) we find that

$$\Delta \theta \approx \frac{1}{2} \left( \frac{\partial \tilde{H}}{\partial \tilde{X}} \bigg|_{(\alpha^-, \tilde{T})} + \frac{\partial \tilde{H}}{\partial \tilde{X}} \bigg|_{(\alpha^+, \tilde{T})} \right) = \frac{\Gamma(3/4) \tilde{T}^{1/4}}{\pi \alpha}, \quad (50)$$

at early times. This result is in good agreement with our numerical simulations, as shown in fig. 8a.

**B. Beyond early times**

The preceding analysis was predicated on the assumption that the relevant time scale was that on which the initial film thickness varies, since we envisage this is the limiting factor in the first moments after a droplet is deposited. However, we also know that the ultimate equilibrium is well-described by using a non-dimensionalization based on the drop size, $x_c$, in the horizontal direction.
and the initial film thickness $h_0$ in the vertical direction. A scaling analysis of Reynolds’ equation then gives us that the relevant time scale is $\tau \sim \mu x_c^4/\gamma h_0^3$. Typically this time scale $\tau = O(10^2 \text{ s})$, which is orders of magnitude longer than the early time scale $\mu h_0/\gamma$, but comparable to that over which experiments might be expected to occur.

![Graphs showing film thickness evolution](image)

**FIG. 9.** The evolution of film thickness for an initially flat oil film subjected to the spatially non-uniform pressure profile of the ersatz droplet. Here the effect of the van der Waals parameter $\mathcal{V}_\gamma$ on the time scale of evolution may be seen as $\mathcal{V}_\gamma$ varies from $10^{-5}$ (in a, d) to $10^{-4}$ (in b, e) and $10^{-3}$ (in c, f). Here $\text{Bo}^{-1} = \alpha = 10^2$ in all calculations; in (a)-(c) $X_{\infty} = 10$ while in (d)-(f) $X_{\infty} = 10^3$. Note that the film profile ultimately reaches very close to the equilibrium solution (dashed black curves), but that the time taken to do so is extraordinarily long even for $X_{\infty} = 10$ ($T \gtrsim 10^4 \sim 10$ days with a typical $\mathcal{V}_\gamma = 10^{-4}$). Throughout, line colour is used to encode dimensionless time as in the associated colorbar. Black filled and open markers are used to indicate the local minima (necks) and maxima (bumps), respectively.
1. Non-dimensionalization

We rescale lengths in the same manner as the static problem, i.e. we let $X = x/x_c$ and $H = h/h_0$; in this way, the natural timescale discussed above arises, though we formally let

$$\tau = \frac{\sqrt[3]{\mu c^4}}{\gamma h_0^3} = \alpha^4 \tau_*$$

for notational convenience. We find that the dimensionless form of the problem reads

$$\frac{\partial H}{\partial T} = \frac{\partial}{\partial X} \left[ H^3 \frac{\partial \Pi}{\partial X} - H^3 \frac{\partial^3 H}{\partial X^3} + \left( \frac{3V_\gamma}{H} + Bo H^3 \right) \frac{\partial H}{\partial X} \right],$$

subject to

$$H(X,0) = 1,$$

$$H_X(0,T) = H_X(X_\infty,T) = H_{XXX}(0,T) = H_{XXX}(X_\infty,T) = 0,$$

for $0 \leq X \leq X_\infty$, where $\Pi = \alpha H(1 - X) - \alpha \delta(X - X_c)$.

2. Numerical results

Although the new time scale introduced to study the late time dynamics is significantly longer than that relevant at early times, allowing the system to reach very close to equilibrium still requires long dimensionless times. We use the same numerical technique as already described. However, a key difficulty in achieving this numerically is the very small horizontal scales over which the forcing pressure $\Pi(X)$ changes, compared to the very long length scales on which the interface shape is changing. To address this, we solve the problem (52)–(54) numerically with a numerical scheme that smooths the Heaviside and Dirac-$\delta$ functions present in the forcing pressure over a small length scale; this scale is chosen to be small enough to resolve the early time dynamics already discussed, but large enough that we can reach close to the ultimate equilibrium. (See Appendix B for details.)

Figure 9 shows the evolution of the film thickness profile towards equilibrium interface shape found in §III. This validates the smoothing of the pressure profile $\Pi(X)$ that is used for numerical convenience, but also illustrates another important point: the skirt region grows extremely slowly. Further validation of the numerical smoothing of the contact line discontinuity is provided by the evolution of the meniscus height and wedge rotation as functions of time, see fig. 10a,b.

The numerical results in fig. 9 show features that are distinct from the behaviour already discussed at very early times. Of particular significance is the behaviour of the local maximum, $(X_{\text{max}}, H_{\text{max}})$, beyond the neck region centred on $(X_{\text{min}}, H_{\text{min}})$ that we have already discussed: this maximum plateaus at a constant height even before the size of the system, $X_\infty$, becomes important. This change in behaviour (compared to the continued growth of the maximum observed in the early-time similarity solution) is the signature of a transition to one of several regimes that the system moves through. We now focus on these regimes, but a schematic summary is given in fig. 16.

To understand these regimes, we first focus on numerical results for $\alpha = 10^2$, $V_\gamma = 10^{-4}$ with zero gravity, $Bo = 0$, and the largest system size, $X_\infty = 10^3$. The large system size and neglect of gravity simplify the problem slightly and allow us to extend the time scale over which various phenomena are observed for as long as possible. (We find that smaller system sizes show the same behaviour but only over shorter time periods, while the effect of gravity is negligible until the very latest stages of the motion, as can be seen by comparing the solid curves in fig. 11, for which $Bo = 0$, with the dashed curves, for which $Bo = 10^{-2}$. We revisit the effects of finite Bond number
FIG. 10. Late-time evolution of (a) the skirt height and (b) rotation angle with a typical value, $V_\gamma = 10^{-4}$, and different system sizes (denoted by line colour, as described in the legend of (b)). Note that dashed black curves are the results of the early-time analysis (where the true $\delta$-function forcing is implemented). The dynamic system is not sensitive to the early-time analysis until $T \sim 10^3$, at which point the film height at the outer boundary begins to change for the smallest system, $X_\infty = 10$, see fig. 9a. (c) Late-time evolution of the skirt height with $V_\gamma$ varying: $V_\gamma = 10^{-3}$ (dotted), $V_\gamma = 10^{-4}$ (solid) and $V_\gamma = 10^{-4}$ (dashed) for various plate sizes (as indicated by line colour). In all calculations $\alpha = Bo^{-1} = 10^2$. in §IVD, but note that even with $Bo = 0$ an equilibrium is ultimately reached since van der Waals forces also play the role of a restoring force.)

An important feature of the results shown in figure 11 is the evolution of the position of the stagnation point between the external film minimum and the external maximum. The early time similarity solution shows that this stagnation point is initially located close to the ‘bump’ region, as shown by comparing eqns (46) and (47). However, fig. 11 shows that it moves towards the ‘neck’ region, before ultimately moving again towards the bump. We shall see that this movement of the stagnation point delineates different phases of the motion.

3. Early–intermediate times

The early time predictions become invalid when the vertical deformation reaches the same order of magnitude as the initial film thickness, i.e. when $\alpha T^{1/4} \sim 1$ or $T \sim \alpha^{-4}$. Beyond this point, the evolution of the film thickness beneath the neck and its position slow down significantly but, perhaps surprisingly, the local maximum (the ‘bump’) still evolves horizontally following the early-time scaling $X_{\text{max}} \sim T^{1/4}$. However, two important differences from the early-time behaviour can be seen in fig. 11: firstly the prefactor $X_{\text{max}}/T^{1/4}$ decreases from that given in (46) and, secondly, the height of the bump, $H_{\text{max}}$ does not continue increasing, but rather plateaus at a new value, $H_{\text{plat}}$; an estimate of the time $T_{\text{plat}}$ needed for this to occur may be obtained from (46) and the observation that $H_{\text{plat}} \sim 1$ giving $T_{\text{plat}} \sim \alpha^{-4}$. For $T \gtrsim T_{\text{plat}}$, the bump appears to propagate outwards with an approximately constant amplitude ($H_{\text{plat}} - 1$) that is maintained for a long period (fig. 12a). This change in behaviour is associated with the relative proximity of the neck region and stagnation point: because the neck region has reached a very small thickness, the flux of fluid through it into the skirt region is very small. The neck region therefore separates the
FIG. 11. Time evolution of the properties of the local minimum (the neck region, red curves) and the maximum (the bump region, blue curves). Each of these regions evolves through several different behaviours. A key feature is the position of the stagnation point located between them — $X_{s.p.}$ is the point at which the fluid flux vanishes, $\partial P/\partial X = 0$, and is shown by the green curve. Here, $\alpha = 100$ and $V_c = \alpha^4/10^{12} = 10^{-4}$; solid curves use $Bo = 0$ while dashed curves used $Bo = 10^{-2}$. Different background shading is used to represent early times, early–intermediate times, late–intermediate times, and late times (from left to right). Black dashed and dotted curves are plotted using the corresponding analytical expressions, which are summarized in the first two panels of fig. 16.

droplet and skirt regions from what happens beyond the neck region, in which the liquid film has to interpolate between a small thickness region to its left (the neck) and a relatively large thickness region to its right (the film). This is then reminiscent of the capillary healing/levelling problem for a thin film that has been considered recently [29, 30]; we are able to make use of these results to understand this behaviour, as we now show.

Intermediate time similarity solution for the bump. To understand the propagating, but constant height, bump regime and make the analogy with capillary healing more concrete, we note that van der Waals forces may be neglected in this region of the film (because the bump itself is well beyond the scale at which van der Waals forces play a role). The evolution of the film thickness can then be approximately described by

$$\frac{\partial H}{\partial T} = -\frac{\partial}{\partial X} \left( H^3 \frac{\partial^3 H}{\partial X^3} \right). \quad (55)$$

We seek a similarity solution of (55) of the form

$$H(X, T) = \eta(\xi), \quad \text{with} \quad \xi = (X - X_{\text{min}})/T^{1/4}. \quad (56)$$

The rescaled equation reads

$$\frac{\partial}{\partial \xi} \left( \eta^3 \frac{\partial^3 \eta}{\partial \xi^3} \right) - \left( \frac{\xi}{4} + \bar{X}_{\text{min}}T^{3/4} \right) \frac{\partial \eta}{\partial \xi} = 0, \quad (57)$$
subject to boundary conditions

\[ \eta \to 0, \quad \frac{\partial \eta}{\partial \xi} \to 0 \text{ as } \xi \to 0 \quad \text{and} \quad \eta \to 1, \quad \frac{\partial \eta}{\partial \xi} \to 0 \text{ as } \xi \to +\infty. \]  

(58)

(In this phase of the motion, the film thickness beneath the neck, \( H_{\text{min}} \), is controlled by van der Waals forces so that \( H_{\text{min}} \sim (V_\gamma/\alpha)^{1/3} \ll 1 \); we take \( H_{\text{min}} = \eta(0) \approx 0 \) for simplicity.)

Equation (57) is not of similarity form because of the time dependence of the neck position \( X_{\text{min}} \); however, in this stage of the motion the neck location evolves more slowly than at early times (where \( X_{\text{min}} - 1 \sim T^{1/4} \)), we take \( X_{\text{min}} T^{3/4} \ll 1 \), and (57) simplifies to become

\[ \frac{d}{d\xi} \left( \eta^3 \frac{d^3 \eta}{d\xi^3} \right) - \frac{\xi d\eta}{4 \xi} = 0. \]  

(59)
Equation (59) is precisely the equation describing the early-time similarity solution of an initial step of liquid relaxing under surface tension (see eqn A2 of ref. [30], for example).

The numerical solution of (59) subject to (58) is plotted as the dashed curve in fig. 12. (We use a finite domain \([0, \xi_{\infty}]\), but find that \(\xi_{\infty} = 15\) is sufficient to obtain good convergence of the numerical solution.) From this numerical solution, we can readily calculate the position and height of the bump from the rescaled profile \(\eta(\xi)\); we find that the maximum is \(H_{\text{plat}} \approx 1.25\), and occurs at \(\xi \approx 2.84\). This calculation provides direct predictions for the evolution of the bump in the early–intermediate times stage, which are summarized in fig. 16b and compare very well to the numerical results of the dynamic problem (dotted lines in the yellow domain of fig. 11). (The solution shown in fig. 12 is similar to that calculated by Zheng et al. [30], see their fig. 16. However, in our solution there is no pre-wetted layer as \(\xi \to -\infty\) and so the height of the maximum is slightly larger.)

Beneath the drop as the bump propagates. The propagation of the bump was predicated on the assumption that the liquid film is effectively split in two by the stagnation point close to the neck. This then begs the question: what is happening beneath the droplet during this time? At this point in the evolution, the region beneath the droplet is a concave dimple region: the pressure within this dimple is positive and so this dimple drains into the skirt region (through the internal neck region, see fig. 16b). This drainage in turn flattens the dimple out, ultimately reaching the uniform equilibrium thickness at which the van der Waals pressure balances the Laplace pressure of the droplet, namely \(H_{\text{eqm}} = (\nu_\gamma/\alpha)^{1/3}\). (Note, however, that (i) the equilibrium shape beneath the droplet may not ultimately have such a large flat spot, because of the finite size of the droplet and (ii) the ultimate thickness of the layer beneath the drop may be slightly less than \(H_{\text{eqm}}\) because of the additional negative pressure in the skirt.)

To understand this flattening motion and the timescale on which it occurs, we consider the linear stability of the homogeneous thickness (i.e. flat) region by substituting \(H(X,T) = H_{\text{eqm}} + \delta f(X) \exp(\sigma T)\) (with \(\delta \ll 1\) arbitrary) into (52) and linearizing. This procedure gives
\[
\sigma f \approx -H_{\text{eqm}}^3 f''' + 3\nu_\gamma f''/H_{\text{eqm}},
\]
which is to be solved subject to symmetry boundary conditions, \(f'(0) = f'''(0) = 0\), and zero displacement and pressure boundary conditions, \(f(1) = f''(1) = 0\); the solutions are of the form \(f = \cos[(n + \frac{1}{2})\pi X]\) and so the slowest decaying mode has \(n = 0\) and decay rate
\[
|\sigma| = -\sigma = \frac{\pi^4 H_{\text{eqm}}^3}{16} + \frac{3\pi^2 \nu_\gamma}{4 H_{\text{eqm}}} \approx \frac{3\pi^2}{4} \alpha^{1/3} \nu_\gamma^{2/3}
\]
which is obtained by substituting for \(H_{\text{eqm}} = (\nu_\gamma/\alpha)^{1/3}\). We therefore expect the time scale for the collapse of the dimple towards the flat spot beneath the droplet should scale as \(|\sigma|^{-1} \sim (\alpha \nu_\gamma^{-1})^{1/3}\). This scaling is confirmed by plotting the film height at \(X = 0\) as function of time in numerics (see fig. 12c) as well as the numerical measurements of the time at which the inner dimple disappears to be replaced by the flat region (see fig. 12d). (Note that the presence of the early time regime therefore requires \(\alpha^{-4} \ll (\alpha \nu_\gamma)^{-1/3}\), but this is guaranteed by the scale separation \(\nu_\gamma \ll 1 \ll \alpha\).)

Deviation from self-similarity in the bump region. Once the dimple beneath the droplet has drained to the homogeneous film thickness \(\sim H_{\text{eqm}}\), there is no longer any further fluid available to fill the skirt region. Nevertheless, the contact line has not yet reached its equilibrium height and so the skirt must seek liquid elsewhere. The only available reservoir of fluid is within the bump region and so the system must increase the flux that flows beneath the outer neck region, which in turn requires the height there to be increased: \(H_{\text{min}}\) increases as observed numerically. This increase of \(H_{\text{min}}\) in turn breaks the conditions for the constant height propagation of the bump region already discussed, and so this similarity solution must break down. In fig. 12, we see that numerical simulations do indeed start to deviate from the similarity solution given by the solution.
of (59) at sufficiently large $T$. This deviation can also be observed in fig. 11 where the predictions of the similarity solution (black dotted lines) fail to describe the behavior of the bump at later times. With $H_{\text{min}}$ increasing, the ‘valve’ that the outer neck region provided is released; fluid begins to flow into the skirt from beyond the droplet region again, and the stagnation point moves again towards the bump region. These changes all have consequences for the evolution of the film profile with increasing time, and so correspond to a new regime: late–intermediate times.

4. Late–intermediate times

The end of early–intermediate times is characterized by the complete drainage of the dimple beneath the droplet, which forces the release of the ‘valve’ in the neck region: the stagnation point moves away from the neck region towards the bump and the height at the neck region increases.

![Figure 13](image-url)

FIG. 13. Evolution of the film profile at late–intermediate times: (a) raw dimensionless profiles and (b) scaled profiles showing the location of the neck (filled black circles) and bump (open circles) in each case. The data in (b) are rescaled according to (63) and plotted on a semi-logarithmic scale to highlight the exponential growth away from the neck region.

The salient feature of this late–intermediate regime is that both the neck and the bump regions are smoothing out (fig. 13a). We now focus on the evolution of the neck region that connects the capillary-controlled region (the skirt) to the outer, reservoir region that is mostly controlled by van der Waals forces (in the absence of gravity). We proceed by first conducting a scaling analysis of (52) in the outside region (so that $d\Pi/dX = 0$). We anticipate that in this phase of the motion surface tension and van der Waals forces balance, but also that the system is out of equilibrium; at a scaling level eqn (52) immediately gives that the typical vertical and horizontal scales are

$$H \sim \gamma^{2/5} T^{1/5}, \quad X \sim \gamma^{3/10} T^{2/5}. \tag{62}$$

(These scalings are analogous to those derived for thin film rupture behaviour under attractive van der Waals interactions [31].) It is therefore natural to introduce the similarity variables

$$\bar{\eta} = H(X,T) / \left( \gamma^{2/5} T^{1/5} \right), \quad \bar{\xi} = (X-1) / \left( \gamma^{3/10} T^{2/5} \right). \tag{63}$$

With this transformation, (52) gives:

$$\frac{1}{5} \bar{\eta} - \frac{2}{5} \frac{d\bar{\eta}}{d\bar{\xi}} = \frac{d}{d\bar{\xi}} \left( -\bar{\eta}^3 \frac{d^3 \bar{\eta}}{d\bar{\xi}^3} + 3 \frac{d\bar{\eta}}{d\bar{\xi}} \right). \tag{64}$$
which is, as expected of a similarity solution, independent of time. Unlike previous work on the role of van der Waals forces in thin film rupture [31], this similarity equation appears not to have appropriate exact (or even asymptotic) solutions: in particular, the far-field condition that $H \sim 1$ as $\xi \to \infty$ is inconsistent with the similarity transformation. As such, we do not expect the numerical results of the full problem to collapse perfectly when plotted in terms of these similarity variables — as indeed is the case (see fig. 13b). Nevertheless, the rescaling (63) does give a reasonable collapse of the numerical data: while some systematic evolution of the meniscus shape remains, note that the profiles shown in fig. 13b span 3 orders of magnitude in time.) The scalings in (63) also suggest that $H_{\text{min}} \sim \gamma^2/5 T^{1/5}$ and $X_{\text{min}} \sim \gamma^3/10 T^{2/5}$, in good agreement with the numerics shown in the red domain of fig. 11.

Although we believe that no detailed solution of (64) can be found satisfying appropriate boundary conditions for the problem at hand, we note that during this phase of the motion the region between the contact line and the neck, i.e. $1 \leq X \lesssim X_{\text{min}}$, is approximately parabolic, corresponding to constant pressure in this region. More surprising perhaps is that, immediately beyond the neck region, the film thickness grows approximately exponentially (see fig. 13b). This can be rationalized as a constant flux (in the similarity variables) that is dominated by the van der Waals pressure, so that $(d\bar{\eta}/d\bar{\xi})/\bar{\eta} = \text{const}$.

Finally, we note that this behaviour continues until the bump reaches the outer limits of the system, i.e. $X_{\text{max}} \sim X_{\infty}$. We will discuss in §IV C when this happens, but first turn to describe the dynamics in this final, late-time regime.

5. Late times

![Figure 14](image)

FIG. 14. Evolution of the film profile at late times with (a) $Bo = 0$ and (b) $Bo = 10^{-2}$. The coloured, dashed lines in (a) and (b) are based on (67) and (74), respectively. The inset in (a) shows the relationship between $H_\infty(T)$ and $X_{\text{min}}(T)$, with the dashed curve showing the prediction (68). (Here $X_\infty = 10^3$, $\alpha = 10^2$ and $\gamma = 10^{-4}$ in all calculations.)

After the bump reaches the outer boundary at $X = X_\infty$, the finite size of the system plays an important role in the further evolution of the system. In particular, the film thickness at the edge, $H_\infty$, starts to decrease, with the lost volume ultimately reaching the skirt, increasing its
size (fig. 14). Since this far-field region is expected to be flat, we neglect the capillary pressure in Reynolds’ equation to give

\[ \dot{H}_\infty \approx \frac{\partial}{\partial X} \left( \frac{3\gamma}{H} \frac{\partial H}{\partial X} \right), \]  

(65)

which can immediately be integrated twice to give

\[ \frac{\dot{H}_\infty}{2} (X - X_\infty)^2 = 3\gamma \log(H/H_\infty), \]  

(66)

which may be inverted directly to give the film profile

\[ H = H_\infty \exp \left[ \frac{1}{6\gamma} \dot{H}_\infty (X - X_\infty)^2 \right]. \]  

(67)

Equation (67) leads us to expect a Gaussian-like profile in the outer portion of the film as the system edge is approached ($X \rightarrow X_\infty$). This is confirmed by the plots in fig. 14a where the dashed lines indicate the prediction (67) with the value of $\dot{H}_\infty$ determined separately.

To understand how $H_\infty$ evolves, we consider the flux from the system edge into the skirt. First, however, notice that, at this late stage of the evolution, the skirt height is very close to the final equilibrium, i.e. $H_c \sim \alpha$, according to the equilibrium result (24). Assuming a parabolic skirt (since $Bo = 0$ and there is little flow), the skirt volume may then be estimated as $\sim \alpha X_{\text{min}}/6$ which must match the change in film volume over the whole system, $\sim (1 - H_\infty)X_\infty$, since $1 \ll X_{\text{min}} \ll X_\infty$; this balance leads to

\[ 1 - H_\infty(T) \sim \frac{\alpha X_{\text{min}}(T)}{6X_\infty}. \]  

(68)

This agrees qualitatively with numerics at very late times (see fig. 14a inset).

C. Approach to final equilibrium: The role of van der Waals forces and system size

We have now seen that the film beneath the droplet passes through several different phases as it approaches equilibrium. Of these phases, it is clear that it is the late–intermediate times that dominate the slow approach to equilibrium: figs 11 and 13 show that this phase occupies dimensionless times $10^5 \lesssim T \lesssim 10^8$, corresponding to very long dimensional times. However, fig. 9 also shows that this final approach is heavily influenced by both the size of the system $X_\infty$ and the strength of van der Waals forces (through the parameter $\mathcal{V}_\gamma$), even when $\mathcal{V}_\gamma \ll 1$. The dependence on even very weak van der Waals forces is all the more surprising since we showed that the final equilibrium state is essentially insensitive to $\mathcal{V}_\gamma$ unless $\mathcal{V}_\gamma$ becomes unphysically large. To understand the joint effect of $\mathcal{V}_\gamma$ and $X_\infty$ in the slow dynamic approach to equilibrium, we continue to neglect the role of gravity, i.e. taking $Bo = 0$; we shall discuss the effect of $Bo > 0$ shortly.

As might be expected, the film cannot sense any effect of either van der Waals forces or the system’s finite size at early times: figure 10 shows numerical results with various $\mathcal{V}_\gamma$ and $X_\infty$ collapsing initially. As already discussed, early times hold for $t \ll \alpha^{-4} \tau$, independent of $\mathcal{V}_\gamma$. Even in the early-intermediate stages (for which the finite film thickness does play a role), fig. 10 emphasizes that the initial slowing down in the evolution of the skirt height is independent of $\mathcal{V}_\gamma$, rather being controlled by the finite thickness of the film and the spreading of the ‘bump’. The end of this early–intermediate phase does, however, depend on van der Waals forces, taking place when $t \sim (\alpha \mathcal{V}_\gamma^2)^{-1/3} \tau \sim \mu x_c^{7/3} / (\gamma^{1/3} A^{2/3})$. 
Beyond the early and early–intermediate times, fig. 13 suggests that the duration of the late-intermediate times is determined by the time taken for the capillary-van der Waals similarity solution to feel the effect of the system size, i.e. for \( X_\infty \sim \nu_{\gamma}^{5/10} T^{2/5} \) or \( T \sim X_\infty^{5/2} \nu_{\gamma}^{-3/4} \gg 1 \). While this is correct for systems in which the total volume of liquid is comparable to that required to make the equilibrium meniscus (small systems in the equilibrium parlance), it is not correct for large systems where the reservoir of oil available to the skirt is sufficient. In these scenarios, the lateral scale of this similarity solution merely needs to have travelled far enough to suck enough liquid to fill the equilibrium meniscus. With this logic, the effective system size, \( X_{\text{eff}}^\infty \), is determined by requiring the volume of liquid in a reservoir of this scale \( \sim 1 \times X_{\text{eff}}^\infty \) from our non-dimensionalization of film thickness equal to the ultimate meniscus volume \( \sim H_c L_{\text{men}} \) where,

\[
L_{\text{men}} = \ell_{\text{men}}/x_c \sim \nu_{\gamma}^{-1/2}.
\]

Since in this limit \( h_c \sim x_c \ (H_c \sim \alpha) \), we therefore have that the effective lateral system size is

\[
X_{\text{eff}}^\infty = \min \left\{ X_\infty, \alpha \nu_{\gamma}^{-1/2} \right\}.
\]

The time required to reach equilibrium is thus estimated to be

\[
T_{\text{eqm}} \sim \begin{cases} 
\nu_{\gamma}^{-3/4} X_\infty^{5/2}, & X_\infty \lesssim \alpha \nu_{\gamma}^{-1/2} \\
\alpha^{5/2} \nu_{\gamma}^{-2}, & X_\infty \gtrsim \alpha \nu_{\gamma}^{-1/2}
\end{cases}
\]

or in dimensional terms

\[
t_{\text{eqm}} \sim \begin{cases} 
\frac{L_{\text{men}}^{5/2}}{A^{1/4} \gamma_{\mu}^{1/4}}, & X_\infty \lesssim \alpha \nu_{\gamma}^{-1/2} \\
\frac{\mu^2 x_c^{5/2} h_0^{5/2}}{A^2}, & X_\infty \gtrsim \alpha \nu_{\gamma}^{-1/2}.
\end{cases}
\]

Interestingly, this equilibrium time only depends on the film thickness and the drop size in the large system limit and depends on the system size only in the small system case. In both cases, however, the equilibrium time is sensitively dependent on the Hamaker constant \( A \).

As a test of the scaling laws developed above, we use our numerical simulations to calculate the time taken for the contact line height to reach 90% of its equilibrium value, which we denote \( T_{90\%} \). The dependence of \( T_{90\%} \) on the system size is shown in fig. 15. We find good agreement between our numerical results and (70) (albeit with a prefactor \( \approx 1/50 \)) over a wide range of values of \( X_{\infty}, \nu_{\gamma}, \) and \( \alpha \). This analysis was predicated on an equilibrium for the droplet that neglected gravity, \( (\text{Bo} = 0) \), which is somewhat artificial; we therefore turn now to consider the effects of finite Bond number for all stages of the dynamics.

**D. Finite Bond number effects**

As a first indication of the important differences introduced by finite Bo we turn to fig. 11, which shows results for \( \alpha = 100, \nu_{\gamma} = \alpha^2/10^{12} \) for both \( \text{Bo} = 0 \) and \( \text{Bo} = 10^{-2} \). In particular, fig. 11 shows that differences in the positions and properties of the dimpled and bump regions manifest themselves only at late-intermediate and late times: at early times, the deflection of the interface is small and so the relative contribution of hydrostatic pressure to the flow is minimal. To make this more quantitative, note that in the early and early–intermediate stages of the motion the ratio of typical capillary pressure to typical hydrostatic pressure, \( \gamma h_{xx}/(\rho gh) \sim \ell_{\text{c}}^2/x_{\text{max}}^2 \). Since
FIG. 15. Time for the skirt height to reach 90% of its final equilibrium value, i.e. $T_{90\%}$ is defined such that $H(1, T_{90\%}) = 0.9 H_c$. Here, numerical results are shown by points while the scaling of (70) with a prefactor 0.02 is shown by the solid line. Marker colour is used to encode $V_\gamma$. Open and filled markers use $B_0 = 0$ and $B_0 = 10^{-2}$, respectively. Circles, squares, and diamonds denote $X_\infty = 10, 10^2$ and $10^3$, respectively. All calculations use $\alpha = 10^2$ except pentagrams (for which $\alpha = 25, X_\infty = 100$).

$X_{\max} \sim T^{1/4}$ in each case, we therefore find that capillary pressure dominates hydrostatic pressure in these regimes provided that

$$T \ll T_{\text{grav}} = B_0^{-2};$$

for the parameters in fig. 11, $T_{\text{grav}} = 10^4$, agreeing well with the time scale upon which the dashed and solid curves in fig. 11 begin to diverge.

For $T \gtrsim T_{\text{grav}}$, the bump is flattened out and the dimple propagates away from the contact line region as in the absence of gravity, but both processes take place more quickly with gravity. Similarly, plotting the numerically determined values of $T_{90\%}$ for $B_0 = 10^{-2}$ (see filled markers in fig. 15) shows that gravity also accelerates the final approach to equilibrium. To estimate the size of this effect, we first note that the effective system size in this limit must be cut-off by the capillary length $\ell_c = x_c B_0^{-1/2}$; hence (69) becomes

$$X_{\text{eff}}^\infty = \min \left\{ X_\infty, \alpha (V_\gamma + B_0)^{-1/2} \right\}. $$

(73)

Using this in place of the corresponding result with $B_0 = 0$ gives reasonable collapse of the data in fig. 15, particularly for moderate values of the combined parameter $X_{\text{eff}}^{5/2} V_\gamma^{-3/4}$. However, the difference becomes significant as the time scale over which gravity plays a role increases (i.e. as $T_{90\%}/T_{\text{grav}}$ increases). Nevertheless, we note that with $V_\gamma \sim 10^{-4}, B_0 \sim 10^{-2}$, we have $T_{90\%}/T_{\text{grav}} \sim X_{\text{eff}}^{5/2}/10$ and so might expect (70) to give a reasonable estimate of the required time to equilibrium in typical experiments (as long as the system size is not too much greater than the drop). A quick estimation using $X_{\text{eff}} \sim 10$ and $\tau \sim 1 \text{ min}$ gives $t_{\text{eqm}} \sim 4 \text{ days}$!

In the very final stages of the dynamics, gravity again plays an important role: revisiting the dynamics of the final draining into the meniscus, described by (52) with $\partial H/\partial T \approx \dot{H}_\infty$ and
analyzing the solution as $X \to X_{\infty}$ we find that

$$H \sim H_{\infty} + \frac{H_{\infty}H_{\infty} (X - X_{\infty})^2}{2 (3V_{\gamma} + Bo H_{\infty}^4)},$$

which is shown as the dashed lines in fig. 14b. Hence, if $Bo \gg 3V_{\gamma}H_{\infty}^{-4} > 3V_{\gamma}$, gravity is the relevant restoring force (not van der Waals forces) pulling the interface back towards the liquid surface. Moreover, in equilibrium, we have no flux and so the far-field meniscus decay follows:

$$H - H_{\infty} \sim \alpha \exp \left[ - \left( Bo + 3V_{\gamma}H_{\infty}^{-4} \right)^{1/2}X \right].$$

Crucially, for $Bo \gg 3V_{\gamma}/H_{\infty}^4$ this is equivalent to the meniscus decay over the capillary length $\ell_c = (\gamma/\rho g)^{1/2}$. As a single measure of the relative importance of gravity and van der Waals forces, one may introduce the parameter

$$G = Bo/V_{\gamma} = \frac{pg h_{\infty}^4}{A}.$$ 

The parameter $G$ encodes whether gravity or van der Waals forces are more important in determining how an equilibrium meniscus returns to flat. Interestingly, this number tends to be $O(1)$, and is especially sensitive to the initial thickness of the lubricating film, $h_0$. Of course, the relative importance of gravity and van der Waals forces in the final equilibrium depends on $h_{\infty}$ and hence on the size of the system, $x_{\infty}$; it is therefore possible that a small system with $G \gtrsim 1$ may yet have $H_{\infty} \ll 1$ and $Bo \ll V_{\gamma}/H_{\infty}^4$.

V. CONCLUSIONS

In this paper, we have studied a model, two-dimensional problem to understand the static and dynamic behaviour of droplets placed on lubricant infused surfaces. We have focussed on understanding the equilibrium state of the droplet, particularly the size and properties of the skirt that forms around it, and how this skirt forms. To make this problem tractable, we mimicked the effect of the droplet on the lubricating film as arising from (i) the squeezing provided by its Laplace pressure and (ii) the pulling effect of the contact lines. This ‘push-and-pull’ effect leads to the formation of a skirt qualitatively similar to that often observed experimentally; further, we showed that this model reproduces previous results for the capillary pressure within the skirt. Crucially, however, this model reveals how the final skirt size depends on the amount of lubricating liquid available in the system. We provided asymptotic relationships for the skirt properties (including height and volume) in the limit of small systems (for which the available lubricant volume is limited) and large systems (for which the available volume is sufficient); interestingly, we found that experiments are likely to lie between these two limits.

Our model accounts for attractive van der Waals interactions between the lubricating oil and the substrate — these act to stabilize the liquid film beneath the droplet, which would otherwise rupture. For physically relevant values of the strength of van der Waals forces, their effect on the final equilibrium is slight and limited to determining the thickness of the thin lubricating layer that is trapped beneath the droplet, as suggested previously [5]. However, the presence of van der Waals forces leads to a rich evolution of the dynamics of skirt formation through several regimes (see fig. 16 for a schematic summary). Crucially, this dynamic role of even moderate van der Waals forces has a significant effect on the time scale taken to reach equilibrium. In particular, we showed that the time scale for the skirt to reach 90% of its equilibrium height is sensitively dependent on the value of the Hamaker constant, see (71) for example.
A key result of our model is that the evolution of the droplet’s skirt towards the ultimate equilibrium is extremely slow: the early phases of the motion are completed on a time scale $\tau = O(10 \text{ s})$ for typical parameters. Crucially, however, the skirt then evolves through several phases, as described in fig. 16, so that on experimentally accessible time scales ($O(10^3 \text{ s})$, say), the skirt may only reach 10% of its equilibrium height. As a result, we suggest that most previous experiments with oil films lubricating smooth substrates are likely to have been in an (admittedly slowly) evolving transient state. While the slow evolution of a droplet on a SLIPS substrate towards
equilibrium has been remarked upon before (see Kreder et al. [21] for example), we believe that the analysis presented here gives new insight into just how slow this motion is and the various features of the problem that cause it. A particular bottleneck seems likely to be the evolution from the early–intermediate to late–intermediate time motions, which we have shown is limited by how quickly the initial dimple beneath the droplet is drained. More evidence that the drainage of this dimple is a limiting feature of the approach to a true equilibrium is the interference measurements of Daniel et al. [5], which show a dimple that is stable over tens of minutes and whose evolution appears to be controlled by the evaporation of the droplet itself (see fig. 1c of ref. [5]).

Our model involves several simplifications that should be revisited in future work. Central among these are our assumptions of a two-dimensional droplet sitting above a smooth substrate. In reality, the droplet is likely to be axisymmetric. At early times this will not impact the analysis presented here (since the lateral size of the skirt region remains small compared to the size of the droplet). However, later the relatively large region of substrate available to drain fluid from to fill the skirt may increase the speed with which the liquid skirt can grow. Similarly, when the lubricating oil impregnates a porous coating, the finite thickness of the porous coating is likely to mean that liquid can flow through the porous medium to avoid the ‘pinch point’ beneath the dimple that causes the slow dynamics discussed here. (Even in the static problem, the axisymmetric and two-dimensional cases can have noticeable differences, as discussed in Appendix A.)

Other simplifications made in the development of our push-and-pull model might be expected to have quantitative, rather than qualitative, effects. For example, the discussion of the dynamics presented here has been on the basis of small-slope approximations, which are not self-consistent even in the simplest case that uses the same surface tension for all interfaces. Also inconsistent is the direction of the pulling line force that needs to rotate to rigorously satisfy horizontal force balance at the contact line. We showed in the static problem that the rotation is often small and the difference between using small and large slopes is only quantitative, even when the system size is unphysically large. However, more general studies that use different surface tensions may need to take the large slope and the rotation of the apparent contact angle (and hence the moving of the contact line) into account. Similarly, we have not accounted for the role of thermal fluctuations or surface roughness in the very thin films that form during this process. In our view, such intricacies should be included in response to detailed experiments that allow for theoretical predictions to be quantitatively tested. We hope that understanding the possibly long time scale of the droplet evolution in these systems may inspire such experiments.

ACKNOWLEDGMENTS

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 886028 (Z.D.) and the Leverhulme Trust through a Philip Leverhulme Prize (D.V.). We also benefited from discussions with Ian Hewitt, Rodrigo Ledesma-Aguilar, Glen McHale and Êlégio Ruiz-Gutiérrez.

Appendix A: Importance of gravity in experiments

The model we have presented elsewhere in this paper is unusual in that it incorporates both van der Waals forces and hydrostatic pressure. The parameter $\bar{G}$ defined in (76) is typically order unity, suggesting that both parameters play a roughly equal role in the outer reaches of the meniscus. However, closer to the droplet, where $h \gg h_\infty$, we expect that hydrostatic pressure is dominant, and hence that the meniscus shape will approximately follow the solution of the linearized Laplace–Young equation, $\gamma \nabla^2 h = \rho g (h - h_\infty)$. In the two-dimensional problems considered here the relevant
FIG. 17. Experimental data for the outer meniscus profile of a droplet on a LIS, captured from Fig. 5b of Schellenberger et al. [15]. In these experiments the value of the capillary length based on the published liquid properties, \( \ell_c \approx 0.97 \) mm, is used. The position of the contact line is taken to be the highest point of the interface visible, which occurs at \( r_c = 0.43\ell_c \), with the meniscus height there, \( h_c \approx 0.23 \) mm, used to rescale the meniscus height throughout. The prediction from the linearized Laplace–Young equation for a two-dimensional scenario is a pure exponential decay (solid line), while the more detailed, axisymmetric behaviour is given by (A1), and is shown by the dashed curve.

Previous experiments [15] have shown an apparent exponential decay, but over a length scale that is slightly different to the expected capillary \( \ell_c \). This could, perhaps, suggest a stronger force pulling the meniscus down than gravity alone. However, since these experiments are axisymmetric, rather than two-dimensional, the relevant solution of the Laplace–Young equation is

\[
h(r) - h_\infty = (h_c - h_\infty) \frac{K_0(r/\ell_c)}{K_0(r_c/\ell_c)}, \quad (A1)
\]

where the contact line is situated at \( r = r_c \) and is at height \( h_c \) above the plate. (Here \( K_0(\cdot) \) is the modified Bessel function of the second kind and zeroth order.)

The experimental data of Schellenberger et al. [15] allows us to compare the prediction of (A1) to the two-dimensional solution. Figure 17 shows the data obtained by digitally capturing the data from fig. 5b of ref. [15] and rescaling this data according to the expected value of the capillary length \( \ell_c \approx 0.97 \) mm. Plotted in this way, we can see that the axisymmetric meniscus profile of (A1) gives an excellent account of experimental data without any fitting parameters; further, this decay is faster than the pure exponential decay over \( \ell_c \) expected for a two-dimensional meniscus. This agreement validates the approach adopted here of retaining the effect of gravity within our model; this agreement also serves to highlight that the effect of axisymmetry may quantitatively alter the predictions of the two-dimensional analysis presented for both the static and dynamic problems considered elsewhere in this paper.

Appendix B: Numerical scheme

We only discuss the numerical method for the problem (52)-(54) here since the results can be directly converted to the problem (32)-(34) using \( \tilde{H} = H, \tilde{X} = \alpha X, \) and \( \tilde{T} = \alpha^4 T \). We discretize
the spatial domain into \( N_1 \) cells for \([0, 1]\) and \( N_2 \) cells for \([1, X_\infty]\). The cell widths are non-uniformly distributed in a way that the meshes are highly refined near the contact line. The value of \( H \) in the \( i \)th cell of width \( \Delta X_i \), denoted by \( H_i \), is evaluated at the mid-point of the cell, while the fluxes, denoted by \( Q_{i\pm 1/2} \), are evaluated at the end-points. We discretize (52) using central finite differences and obtain a set of ordinary differential equations (ODEs) using method of lines:

\[
\frac{dH_i}{dT} = -\frac{1}{\Delta X_i} (Q_{i+1/2} - Q_{i-1/2}),
\]

where

\[
Q_{i+1/2} = -(H_{i+1/2})^3 (\Pi_X - H_{X XX} + Bo H_X)_{i+1/2} - 3\gamma H_X|_{i+1/2}/H_{i+1/2}, \quad (B2a)
\]

\[
H_{X XX}|_{i+1/2} = 2(H_{XX}|_{i+1} - H_{XX}|_{i})/(\Delta X_i + \Delta X_{i+1}), \quad (B2b)
\]

\[
H_{XX}|_{i} = (H_{X|i+1/2} - H_{X|i-1/2})/\Delta X_i, \quad (B2c)
\]

\[
H_{i+1/2} = (\Delta X_{i+1} H_i + \Delta X_i H_{i+1})/(\Delta X_{i+1} + \Delta X_i), \quad (B2d)
\]

\[
\Pi_X|_{i+1/2} = 0, \quad (B2f)
\]

for both inner \((1 \leq i \leq N_1)\) and outer domain \((1 \leq i \leq N_2)\).

For early-time calculations, the inner and outer meniscus are calculated separately so two ghost cells are added on either side of the two domains (8 in total). Besides the four boundary conditions provided by (54), four matching conditions are used to connect the two menisci at the contact line, which have been discussed in (35)-(37).

For late-time calculations, using a smoothed step and delta function is found more efficient:

\[
\Pi(X) = \frac{\alpha}{2} \left( 1 + \tanh \frac{1 - X}{\varepsilon} \right) - \frac{\alpha}{2\varepsilon} \left( \sech \frac{X - 1}{\varepsilon} \right)^2, \quad (B3)
\]

for \(0 \leq X \leq X_\infty\). In this case the evolution of the meshpoints on the two menisci still follows the scheme given by (B1) and (B2a)-(B2f) in which inner meshes are presented by \( i = 1, ..., N_1 \) but the outer meshes are now presented by \( i = N_1 + 1, ..., N_1 + N_2 \). This method could connect the two menisci automatically at the contact line after rewriting (B2f) as

\[
\Pi_X|_{i\pm 1/2} = -\frac{\alpha}{2\varepsilon^2} \left( \sech \frac{X_{i\pm 1/2} - 1}{\varepsilon} \right)^2 \left( \varepsilon + 2 \tanh \frac{1 - X_{i\pm 1/2}}{\varepsilon} \right), \quad (B4)
\]

for \(1 \leq i \leq N_1 + N_2\), where \( X_{i\pm 1/2} \) denotes the position of the right/left end-point of the \( i \)th cell.

The ODEs (B1) are solved using MATLAB’s built-in solver ode15s and exploiting the system’s sparsity as well as complex step differentiation to calculate the Jacobian [27]. The two methods presented above are identical as \( \varepsilon \to 0 \), but a finite \( \varepsilon \) is used numerically, introducing some errors in early-time calculations (fig. 10). However, it is found that using \( \varepsilon = 0.01 \) provides reasonable accuracy for \( T \gtrsim 1 \). The volume of liquid is monitored as an indicator of the numerical errors accumulated, which we find to remain within 1.5% of its initial value for all simulations.

[1] David Quéré, “Non-sticking drops,” Rep. Prog. Phys. 68, 2495 (2005).
[2] Tak-Sing Wong, Sung Hoon Kang, Sindy KY Tang, Elizabeth J Smythe, Benjamin D Hatton, Alison Grinthal, and Joanna Aizenberg, “Bioinspired self-repairing slippery surfaces with pressure-stable omniphobicity,” Nature 477, 443–447 (2011).
[3] Yaxing Li, Christian Diddens, Tim Segers, Herman Wijshoff, Michel Versluis, and Detlef Lohse, “Evaporating droplets on oil-wetted surfaces: Suppression of the coffee-stain effect,” Proc. Natl Acad. Sci. USA 117, 16756–16763 (2020).

[4] J David Smith, Rajeev Dhiman, Sushant Anand, Ernesto Reza-Garduno, Robert E Cohen, Gareth H McKinley, and Kripa K Varanasi, “Droplet mobility on lubricant-impregnated surfaces,” Soft Matter 9, 1772–1780 (2013).

[5] D. Daniel, J. V. I. Timonen, R. Li, S. J. Velling, and J. Aizenberg, “Oleoplaning droplets on lubricated surfaces,” Nat. Phys. 13, 1020–1025 (2017).

[6] Armelle Keiser, Ludovic Keiser, Christophe Clanet, and David Quéré, “Drop friction on liquid-infused materials,” Soft Matter 13, 6981–6987 (2017).

[7] Armelle Keiser, Philipp Baumli, Doris Vollmer, and David Quéré, “Universality of friction laws on liquid-infused materials,” Phys. Rev. Fluids 5, 014005 (2020).

[8] Alexander K Epstein, Tak-Sing Wong, Rebecca A Belisle, Emily Marie Boggs, and Joanna Aizenberg, “Liquid-infused structured surfaces with exceptional anti-biofouling performance,” Proc. Natl. Acad. Sci. USA 109, 13182–13187 (2012).

[9] Michael J Kreder, Jack Alvarenga, Philseok Kim, and Joanna Aizenberg, “Design of anti-icing surfaces: smooth, textured or slippery?” Nat. Rev. Mater. 1, 1–15 (2016).

[10] Zongqi Guo, Lei Zhang, Deepak Monga, Howard A Stone, and Xianming Dai, “Hydrophilic slippery surface enabled coarsening effect for rapid water harvesting,” Cell Rep. Phys. Sci. 2, 100387 (2021).

[11] Bethany V Orme, Glen McHale, Rodrigo Ledesma-Agular, and Gary G Wells, “Droplet retention and shedding on slippery substrates,” Soft Matter 35, 9146–9151 (2019).

[12] Jieke Jiang, Jun Gao, Hengdi Zhang, Wenqing He, Jianqiang Zhang, Dan Daniel, and Xi Yao, “Directional pumping of water and oil microdroplets on slippery surface,” Proc. Natl. Acad. Sci. USA 116, 2482–2487 (2019).

[13] M. A. Hack, M. Costalonga, T. Segers, S. Karpitschka, H. Wijshoff, and J. H. Snoeijer, “Printing wet-on-wet: Attraction and repulsion of drops on a viscous film,” Appl. Phys. Lett. 113, 183701 (2018).

[14] Steffen Hardt and Glen McHale, “Flow and drop transport along liquid-infused surfaces,” Annu. Rev. Fluid Mech. 54 (2021), 10.1146/annurev-fluid-030121-113156.

[15] Frank Schellenberger, Jing Xie, Noemí Encinas, Alexandre Hardy, Markus Klapper, Periklis Papadopoulos, Hans-Jürgen Butt, and Doris Vollmer, “Direct observation of drops on slippery lubricant-infused surfaces,” Soft Matter 11, 7617–7626 (2015).

[16] Martin Villegas, Yuxi Zhang, Noor Abu Jarad, Leyla Soleymani, and Tohid F Didar, “Liquid-infused surfaces: a review of theory, design, and applications,” ACS Nano 13, 8517–8536 (2019).

[17] Sam Peppou-Chapman, Jun Ki Hong, Anna Waterhouse, and Chiara Neto, “Life and death of liquid-infused surfaces: a review on the choice, analysis and fate of the infused liquid layer,” Chem. Soc. Rev. 49, 3688–3715 (2020).

[18] D. Vella and L. Mahadevan, “The cheerios effect,” Am. J. Phys. 73, 817–825 (2005).

[19] C. Semprebon, G. McHale, and H. Kusumaatmaja, “Apparent contact angle and contact angle hysteresis on liquid infused surfaces,” Soft Matter 13, 101–110 (2017).

[20] Ciro Semprebon, Muhammad Subkhi Sadullah, Glen McHale, and Halim Kusumaatmaja, “Apparent contact angle of droplets on liquid infused surfaces: Geometric interpretation,” Soft Matter 17, 95539559 (2021).

[21] Michael J Kreder, Dan Daniel, Adam Tetreault, Zhenle Cao, Baptiste Lemaire, Jaakko V. I. Timonen, and Joanna Aizenberg, “Film dynamics and lubricant depletion by droplets moving on lubricated surfaces,” Phys. Rev. X 8, 031053 (2018).

[22] Glen McHale, Bethany V Orme, Gary G Wells, and Rodrigo Ledesma-Agular, “Apparent contact angles on lubricant-impregnated surfaces/slips: From superhydrophobicity to electrowetting,” Langmuir 35, 4197–4204 (2019).

[23] L Gary Leal, Advanced transport phenomena: fluid mechanics and convective transport processes, Vol. 7 (Cambridge University Press, 2007).

[24] Jacob N Israelachvili, Intermolecular and surface forces (Academic Press, 2015).

[25] J. N. Snoeijer, “Free-surface flows with large slopes: Beyond lubrication theory,” Phys. Fluids 18, 021701 (2006).

[26] B. Tavakol, G. Froehlicher, D. P. Holmes, and H. A. Stone, “Extended lubrication theory: improved estimates of flow in channels with variable geometry,” P. Roy. Soc. A 473, 20170234 (2017).
[27] Lawrence F Shampine, “Accurate numerical derivatives in matlab,” ACM Trans. Math. Soft. 33, 26–es (2007).
[28] Milton Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (Dover, New York, 1964).
[29] Michael Benzaquen, Paul Fowler, Laetitia Jubin, Thomas Salez, Kari Dalnoki-Veress, and Elie Raphal, “Approach to universal self-similar attractor for the levelling of thin liquid films,” Soft Matter 10, 8608–8614 (2014).
[30] Zhong Zheng, Marco A Fontelos, Sangwoo Shin, Michael C Dallaston, Dmitri Tseluiko, Serafim Kalliadasis, and Howard A Stone, “Healing capillary films,” J. Fluid Mech. 838, 404–434 (2018).
[31] Wendy W Zhang and John R Lister, “Similarity solutions for van der waals rupture of a thin film on a solid substrate,” Phys. Fluids 11, 2454–2462 (1999).