Estimates for generalized Bohr radii in one and higher dimensions

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Abstract. In this article, we study a generalized Bohr radius \( R_{p,q}(X) \), \( p, q \in [1, \infty) \) defined for a complex Banach space \( X \). In particular, we determine the exact value of \( R_{p,q}(C) \) for the cases (i) \( p, q \in [1, 2] \), (ii) \( p \in (2, \infty) \), \( q \in [1, 2] \), and (iii) \( p, q \in [2, \infty) \). Moreover, we consider an \( n \)-variable version \( R_{p,q}^n(X) \) of the quantity \( R_{p,q}(X) \) and determine (i) \( R_{p,q}^n(H) \) for an infinite-dimensional complex Hilbert space \( H \) and (ii) the precise asymptotic value of \( R_{p,q}^n(X) \) as \( n \to \infty \) for finite-dimensional \( X \). We also study the multidimensional analog of a related concept called the \( p \)-Bohr radius. To be specific, we obtain the asymptotic value of the \( n \)-dimensional \( p \)-Bohr radius for bounded complex-valued functions, and in the vector-valued case, we provide a lower estimate for the same, which is independent of \( n \).

1 Introduction and the main results

The celebrated theorem of Harald Bohr [13] states (in sharp form) that for any holomorphic self-mapping \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) of the open unit disk \( \mathbb{D} \),

\[
\sum_{n=0}^{\infty} |a_n| r^n \leq 1
\]

for \( |z| = r \leq 1/3 \), and this quantity \( 1/3 \) is the best possible. Inequalities of the above type are commonly known as Bohr inequalities nowadays, and appearance of any such inequality in a result is generally termed as the occurrence of the Bohr phenomenon. This theorem was an outcome of Bohr’s investigation on the “absolute convergence problem” of ordinary Dirichlet series of the form \( \sum a_n n^{-s} \), and did not receive much attention until it was applied to answer a long-standing question in the realm of operator algebras in 1995 (cf. [19]). Starting there, the Bohr phenomenon continues to be studied from several different aspects for the last two decades, for example, in certain abstract settings (cf. [1]), for ordinary and vector-valued Dirichlet series (see, f.i., [3, 15]), for uniform algebras (see [28]), for free holomorphic functions (cf. [30]), for a Faber–Green condenser (see [26]), for vector-valued functions (cf. [17, 23, 24]), for Hardy space functions (see [5]), and for functions in several variables (see, for example, [2, 8, 12, 21, 29]). We also urge the reader to glance through the
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We will now concentrate on a variant of the Bohr inequality, introduced for the first time in [9] in order to investigate the Bohr phenomenon on Banach spaces. Let us start by defining an n-variable analog of this modified inequality. For this purpose, we need to introduce some concepts. Let \( \mathbb{D}^n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \|z\|_{\infty} = \max_{1 \leq k \leq n} |z_k| < 1\} \) be the open unit polydisk in the n-dimensional complex plane \( \mathbb{C}^n \), and let \( X \) be a complex Banach space. Any holomorphic function \( f : \mathbb{D}^n \to X \) can be expanded in the power series

\[
(1.1) \quad f(z) = x_0 + \sum_{|\alpha| \in \mathbb{N}} x_{\alpha} z^{\alpha}, \quad x_{\alpha} \in X,
\]

for \( z \in \mathbb{D}^n \). Here and hereafter, we will use the standard multi-index notation: \( \alpha \) denotes an \( n \)-tuple \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) of nonnegative integers, \( |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \), \( \alpha! := \alpha_1! \alpha_2! \cdots \alpha_n! \), \( z \) denotes an \( n \)-tuple \((z_1, z_2, \ldots, z_n)\) of complex numbers, and \( z^\alpha \) is the product \( z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \). For \( 1 \leq p, q < \infty \) and for any \( f \) as in (1.1) with \( \|f\|_{\mathcal{H}^\infty(\mathbb{D}^n, X)} \leq 1 \), we denote

\[
R^n_{p,q}(f, X) = \sup \left\{ r \geq 0 : \|x_0\|^p + \left( \sum_{k=1}^\infty \sum_{|\alpha| = k} \|x_{\alpha} z^{\alpha}\|^q \right)^{\frac{1}{q}} \leq 1 \text{ for all } z \in r\mathbb{D}^n \right\},
\]

where \( \mathcal{H}^\infty(\mathbb{D}^n, X) \) is the space of bounded holomorphic functions \( f \) from \( \mathbb{D}^n \) to \( X \) and \( \|f\|_{\mathcal{H}^\infty(\mathbb{D}^n, X)} = \sup_{z \in \mathbb{D}^n} \|f(z)\| \). We further define

\[
R^n_{p,q}(X) = \inf \{ R^n_{p,q}(f, X) : \|f\|_{\mathcal{H}^\infty(\mathbb{D}^n, X)} \leq 1 \}.
\]

Following the notations of [9], throughout this article, we will use \( R_{p,q}(f, X) \) for \( R^n_{p,q}(f, X) \) and \( R_{p,q}(X) \) for \( R^n_{p,q}(X) \). Clearly, \( R_{1,1}(\mathbb{C}) = 1/3 \). The reason for reshaping the original Bohr inequality in the above fashion becomes clear from [9, Theorem 1.2], which shows that the notion of the classical Bohr phenomenon is not very useful for \( \text{dim}(X) \geq 2 \). For a given pair of \( p \) and \( q \) in \([1, \infty)\), it is known from the results of [9] that depending on \( X \), \( R_{p,q}(X) \) may or may not be zero. A characterization theorem in this regard has further been established in [6]. However, the question of determination of the exact value of \( R_{p,q}(X) \) is challenging, and to the best of our knowledge, there is lack of progress on this problem—even for \( X = \mathbb{C} \). In fact, only known optimal result in this direction is the following:

\[
(1.2) \quad R_{p,q}(\mathbb{C}) = \frac{p}{2 + p},
\]

for \( 1 \leq p \leq 2 \) (cf. [9, Proposition 1.4]), along with rather recent generalizations of (1.2) (see, for example, [27]). This motivates us to address this problem in the first theorem of this article.

**Theorem 1.1** Given \( p, q \in [1, \infty) \), let us denote

\[
A_{p,q}(a) = \frac{(1 - a^p)^{\frac{1}{2}}}{1 - a^2 + a(1 - a^p)^{\frac{1}{2}}}, \quad a \in [0, 1)
\]

for \( 1 \leq p \leq 2 \).
and
\[ S_{p,q}(a) = \left( \frac{(1 - a^p)^{\frac{q}{2}}}{(1 - a^2 + (1 - a^p)^{\frac{q}{2}})} \right)^{\frac{1}{q}}, \quad a \in [0,1). \]

Furthermore, let \( \hat{a} \) be the unique root in \((0,1)\) of the equation
\[ x^p + x^q = 1. \]

Then
\[ R_{p,q}(\mathbb{C}) = \begin{cases} 
\inf_{a \in [\hat{a},1)} A_{p,q}(a) & \text{if } p, q \in [1,2], \\
\min \left\{ \frac{1}{\sqrt{2}}, \inf_{a \in [\hat{a},1)} A_{p,q}(a) \right\} & \text{if } p \in (2,\infty) \text{ and } q \in [1,2], \\
\frac{1}{\sqrt{2}} & \text{if } p, q \in [2,\infty). 
\end{cases} \]

For \( p \in [1,2] \) and \( q \in (2,\infty) \), \( R_{2,q}(\mathbb{C}) = 1/\sqrt{2} \), \( R_{p,q}(\mathbb{C}) = \inf_{a \in [\hat{a},1)} A_{p,q}(a) \) if \( p < 2 \) and in addition the inequality
\[ q\hat{a}^2 + p\hat{a}^{p+2} \leq p\hat{a}^p + q\hat{a}^{p+2} \]
is satisfied. In all other scenarios, we have, in general,
\[ 0 < \inf_{a \in (0,1)} S_{p,q}(a) \leq R_{p,q}(\mathbb{C}) \leq \frac{1}{\sqrt{2}}. \]

Remarks 1.2 (a) A closer look at the proof of Theorem 1.1 reveals that the conclusions of this theorem remain unchanged if the interval \([1,2]\) is replaced by \((0,2]\) everywhere in its statement. However, doing so includes cases where positive Bohr radius is nonexistent; for example, \( R_{p,q}(\mathbb{C}) = \inf_{a \in [\hat{a},1)} A_{p,q}(a) \) if \( p < 2 \) and in addition the inequality
\[ q\hat{a}^2 + p\hat{a}^{p+2} \leq p\hat{a}^p + q\hat{a}^{p+2} \]
is satisfied. In all other scenarios, we have, in general,
\[ 0 < \inf_{a \in (0,1)} S_{p,q}(a) \leq R_{p,q}(\mathbb{C}) \leq \frac{1}{\sqrt{2}}. \]

(b) Following methods similar to the proof of Theorem 1.1, it is easy to see that for any given complex Hilbert space \( \mathcal{H} \) with dimension at least 2, the following statements are true:

(i) For \( p, q \in [2,\infty) \), \( R_{p,q}(\mathcal{H}) = 1/\sqrt{2} \).

(ii) For \( p \in [1,2] \) and \( q \in [2,\infty) \), inequalities (1.5) are satisfied with \( R_{p,q}(\mathbb{C}) \) replaced by \( R_{p,q}(\mathcal{H}) \).

Note that the assumption \( q \geq 2 \) is justified by [6, Corollary 4]. Later, in Theorem 1.4, we obtain a more complete result for \( \dim(\mathcal{H}) = \infty \).

We now turn our attention to the Bohr radius \( R_{p,q}^n(X) \), where \( X \) is a complex Banach space. The first question we encounter is the identification of the Banach spaces \( X \) with \( R_{p,q}^n(X) > 0 \), which is in fact equivalent to the one-dimensional version of the same problem.

Proposition 1.3 For any given \( n \in \mathbb{N} \) and \( p, q \in [1,\infty) \), \( R_{p,q}^n(X) > 0 \) for some complex Banach space \( X \) if and only if \( R_{p,q}(X) > 0 \) for the same Banach space \( X \).
Note that from [6, Theorem 1], it is known that $R_{p,q}(X) > 0$ if and only if there exists a constant $C$ such that

$$
\Omega_X(\delta) \leq C \left( (1 + \delta)^q - (1 + \delta)^{q-p} \right)^{1/q}
$$

for all $\delta \geq 0$. We mention here that for any $\delta \geq 0$, $\Omega_X(\delta)$ is defined to be the supremum of $\|y\|$ taken over all $x, y \in X$ such that $\|x\| = 1$ and $\|x + zy\| \leq 1 + \delta$ for all $z \in \mathbb{D}$ (see [22]). Now, in view of the above discussion, it looks appropriate to consider the Bohr phenomenon, i.e., studying $R^n_{p,q}(X)$ for particular Banach spaces $X$. We resolve this problem completely for $X = \mathcal{H}$, a complex Hilbert space of infinite dimension. While this question remains open for $\text{dim} \mathcal{H} < \infty$, we succeed in determining the correct asymptotic behavior of $R^n_{p,q}(X)$ as $n \to \infty$ for any finite-dimensional complex Banach space $X$ with $R_{p,q}(X) > 0$.

**Theorem 1.4** For any given $n \in \mathbb{N}$, $p \in [1, \infty)$, $q \in [2, \infty)$ and for any infinite-dimensional complex Hilbert space $\mathcal{H}$,

$$
R^n_{p,q}(\mathcal{H}) = \inf_{a \in [0,1)} \left( 1 - (1 - (S_{p,q}(a))^2)^{\frac{1}{2}} \right) \frac{n}{\text{dim} \mathcal{H}},
$$

$S_{p,q}(a)$ as defined in the statement of Theorem 1.1. For any complex Banach space $X$ with $\text{dim}(X) < \infty$ and with $R_{p,q}(X) > 0$, we have

$$
\lim_{n \to \infty} \frac{R^n_{p,q}(X)}{\sqrt{n \log n}} = 1.
$$

At this point, we like to discuss another interesting related concept called the $p$-Bohr radius. First, we pose an $n$-variable version of the definition of $p$-Bohr radius given in [10]. For any $p \in [1, \infty)$ and for any complex Banach space $X$, we denote

$$
r^n_p(f, X) = \sup \left\{ r \geq 0 : \|x_0\|^p + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} \|x_0 z^\alpha\|^p \leq 1 \text{ for all } z \in r \mathbb{D}^n \right\},
$$

where $f$ is as given in (1.1) with $\|f\|_{L^\infty(\mathbb{D}^n, X)} \leq 1$, and then define the $n$-dimensional $p$-Bohr radius of $X$ by

$$
r^n_p(X) = \inf \{ r^n_p(f, X) : \|f\|_{L^\infty(\mathbb{D}^n, X)} \leq 1 \}.
$$

Again, following the notations of [10], we will write $r_p(f, X)$ for $r^n_1(f, X)$ and $r_p(X)$ for $r^n_1(X)$. Clearly, for $X = \mathbb{C}$, one only needs to consider $p \in [1, 2)$, as $r^n_p(\mathbb{C}) = 1$ for all $p \geq 2$ and for any $n \in \mathbb{N}$. The quantities $r_p(\mathbb{C})$ and $r^n_p(\mathbb{C})$ were first considered in [20]. Unlike $R_{p,q}(\mathbb{C})$, a precise value of $r_p(\mathbb{C})$ has already been obtained in [25]. We make further progress by determining the asymptotic behavior of $r^n_p(\mathbb{C})$ for all $p \in (1, 2)$ (the case $p = 1$ is already resolved) in the first half of Theorem 1.5.

On the other hand, to get a nonzero value of $r^n_p(X)$ where $\text{dim}(X) \geq 2$, one necessarily has to consider $p \geq 2$ and work with $p$-uniformly PL-convex complex Banach spaces $X$. A complex Banach space $X$ is said to be $p$-uniformly PL-convex...
for all $x, y \in X$. Denote by $I_p(X)$ the supremum of all $\lambda$ satisfying (1.7). Now, if we assume $r_n^p(X) > 0$ for some $n \in \mathbb{N}$, then evidently $r_n^p(X) > 0$ (as any member of $H^\infty(D, X)$ can be considered as a member of $H^\infty(D^n, X)$ as well), and therefore [10, Theorem 1.10] asserts that $X$ is $p$-uniformly $\mathbb{C}$-convex, which is equivalent to saying that $X$ is $p$-uniformly PL-convex. The second half of our upcoming theorem shows that for any $p$-uniformly PL-convex complex Banach space $X$ ($p \geq 2$) with $\dim(X) \geq 2$, the Bohr radius $r_n^p(X) > 0$ for all $n \in \mathbb{N}$ and unlike $r_p^n(\mathbb{C})$ or $R_{p,q}^n(X)$, $r_p^p(X)$ does not converge to 0 as $n \to \infty$.

**Theorem 1.5** For any $p \in (1, 2)$ and $n > 1$, we have

$$r_n^p(\mathbb{C}) \sim \left(\frac{\log n}{n}\right)^{\frac{2-n}{2p}}.$$  

For any $p$-uniformly PL-convex ($p \geq 2$) complex Banach space $X$ with $\dim(X) \geq 2$, we have

$$\left(\frac{I_p(X)}{2^p + I_p(X)}\right)^{\frac{2p}{2}} \leq r_p^p(X) \leq 1$$

for all $n \in \mathbb{N}$.

We clarify that for any two sequences $\{p_n\}$ and $\{q_n\}$ of positive real numbers, we write $p_n \sim q_n$ if there exist constants $C, D > 0$ such that $Cq_n \leq p_n \leq Dq_n$ for all $n > 1$. In Section 2, we will give the proofs of all the results stated so far.

2 Proofs of the main results

We start by recalling the following result of Bombieri (cf. [14]), which is at the heart of the proof of our Theorem 1.1.

**Theorem A** For any holomorphic self-mapping $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of the open unit disk $\mathbb{D}$,

$$\sum_{n=1}^{\infty} |a_n|r^n \leq \left\{ \begin{array}{ll}
\frac{r(1-a^2)}{1-ar^2} & \text{for } r \leq a, \\
\frac{r\sqrt{1-az^2}}{\sqrt{1-a^2}} & \text{for } r \in [0,1) \text{ in general,}
\end{array} \right.$$  

where $|z| = r$ and $|a_0| = a$.

It should be mentioned that the above result is not recorded in the present form in [14]. For a direct derivation of the first inequality in Theorem A, see the proof of Theorem 9 of [7]. The second inequality is an easy consequence of the Cauchy–Schwarz inequality combined with the fact that $\sum_{n=1}^{\infty} |a_n|^2 \leq 1 - |a_0|^2$. 

(2 ≤ $p$ < $\infty$) if there exists a constant $\lambda > 0$ such that

$$\|x\|^p + \lambda \|y\|^p \leq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta$$

(1.7)
Proof of Theorem 1.1  
Given a holomorphic function \( f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \) mapping \( \mathbb{D} \) inside \( \mathbb{D} \), a straightforward application of Theorem A yields

\[
|a_0|^p + \left( \sum_{n=1}^{\infty} |a_n| r^n \right)^q \leq \begin{cases} 
\left( a^p + (1 - a^2)^q \left( \frac{r}{1 - ar} \right)^q \right) & \text{for } r \leq a, \\
\left( a^p + (1 - a^2)^q \left( \frac{r}{\sqrt{1 - r^2}} \right)^q \right) & \text{for } r \in [0, 1). 
\end{cases}
\]

(2.1)

Now,
\[
a^p + (1 - a^2)^q \left( \frac{r}{1 - ar} \right)^q \leq 1
\]
whenever \( r \leq A_{p,q}(a) \). A little calculation reveals that \( A_{p,q}(a) \leq a \) whenever \( a^p + a^q \geq 1 \), i.e., whenever \( a \geq \tilde{a} \), \( \tilde{a} \) being the root of equation (1.3). Thus, from (2.1), it is clear that

\[
|a_0|^p + \left( \sum_{n=1}^{\infty} |a_n| r^n \right)^q \leq 1
\]
for \( r \leq \inf_{a \in [\tilde{a}, 1]} A_{p,q}(a) \), provided that \( a \geq \tilde{a} \). On the other hand,

\[
a^p + (1 - a^2)^q \left( \frac{r}{\sqrt{1 - r^2}} \right)^q \leq 1
\]
for \( r \leq S_{p,q}(a) \), i.e., inequality (2.2) remains valid for \( r \leq \inf_{a \in [0, \tilde{a}]} S_{p,q}(a) \), provided that \( a \leq \tilde{a} \). Therefore, we conclude that for any given \( p, q \in [1, \infty) \),

\[
R_{p,q}(\mathbb{C}) \geq \min \left\{ \inf_{a \in [0, \tilde{a}]} S_{p,q}(a), \inf_{a \in [\tilde{a}, 1]} A_{p,q}(a) \right\}.
\]

(2.3)

We also record some other facts which we will need to use later. Observe that for all \( p, q \in [1, \infty) \),

\[
S_{p,q}(a) = \sqrt{\frac{T(a)}{1 + T(a)}} \text{ where } T(a) = \frac{(1 - a^p)^{\frac{q}{p}}}{1 - a^2},
\]
and therefore

\[
S'_{p,q}(a) = \frac{T'(a)}{2 \sqrt{T(a)(1 + T(a))}}
\]
for \( a \in (0, 1) \), where

\[
T'(a) = \frac{2a^{p-1}T(a)}{1 - a^p} \left( \frac{a^2(1 - a^p)}{a^p(1 - a^2)} \right) - \frac{p}{q}.
\]

(2.4)

Setting \( y = 1/a \) for convenience, we write

\[
\frac{a^2(1 - a^p)}{a^p(1 - a^2)} = \frac{y^p - 1}{y^2 - 1} = P(y)
\]
defined on \((1, \infty)\). Note that

\[
\frac{d}{da} P(y) = P'(y) \frac{dy}{da} = -y^3 \frac{py^p - py^{p-2} - 2y^p + 2}{(y^2 - 1)^2},
\]

and that

\[
Q'(y) = y^{p-3}(y^2 - 1)p(p - 2),
\]

where \(Q(y) = py^p - py^{p-2} - 2y^p + 2\).

Furthermore, observe that for the disk automorphisms \(\phi_a(z) = (a - z)/(1 - az)\), \(z \in \mathbb{D}, a \in [\bar{a}, 1)\), \(R_{p,q}(\phi_a, \mathbb{C}) = A_{p,q}(a)\), and hence \(R_{p,q}(\mathbb{C}) \leq \inf_{a \in [\bar{a}, 1)} A_{p,q}(a)\). Moreover, for \(\xi(z) = z\phi_{1/\sqrt{2}}(z), z \in \mathbb{D}\), we have \(R_{p,q}(\xi, \mathbb{C}) = 1/\sqrt{2}\). Combining these two facts, we write

\[
R_{p,q}(\mathbb{C}) \leq \min \left\{ (1/\sqrt{2}), \inf_{a \in [\bar{a}, 1)} A_{p,q}(a) \right\}.
\]

We now deal with the problem case by case.

Case \(p, q \in [1, 2]\): Let us start with \(p < 2\). From (2.6), it is evident that \(Q'(y) < 0\) for \(p < 2\), and hence \(Q(y) < Q(1) = 0\) for all \(y \in (1, \infty)\). Thus, from (2.5), it is clear that \(P(y)\) is strictly increasing in \((0, 1)\) with respect to \(a\). Consequently, for all \(y \in (1, \infty)\),

\[
P(y) < \lim_{a \to 1^-} P(y) = \frac{P}{2},
\]

and using the above estimate in (2.4) gives, for all \(a \in (0, 1)\),

\[
T'(a) \leq 0,
\]

as \(q \leq 2\). Therefore, \(S_{p,q}(a)\) is strictly decreasing in \((0, 1)\), and after some calculations, we have, as a consequence,

\[
\inf_{a \in [0, \bar{a}]} S_{p,q}(a) = S_{p,q}(\bar{a}) = A_{p,q}(\bar{a}) \leq \inf_{a \in [\bar{a}, 1)} A_{p,q}(a).
\]

Hence, from (2.3), we have \(R_{p,q}(\mathbb{C}) \geq \inf_{a \in [\bar{a}, 1)} A_{p,q}(a)\). For \(p = 2\), if \(q < 2\), then \(T'(a) < 0\) for all \(a \in (0, 1)\), which (as in the case \(p < 2\)) again gives \(R_{2,q}(\mathbb{C}) \geq \inf_{a \in [\bar{a}, 1)} A_{2,q}(a)\). Otherwise, if \(p = q = 2\), then \(\bar{a} = 1/\sqrt{2}\), and for all \(a \in [0, 1)\), we get

\[
S_{2,2}(a) = 1/\sqrt{2} = \inf_{a \in [\bar{a}, 1)} A_{2,2}(a).
\]

Therefore, for all \(p, q \in [1, 2]\), we have \(R_{p,q}(\mathbb{C}) \geq \inf_{a \in [\bar{a}, 1)} A_{p,q}(a)\), and from (2.7), it is known that \(R_{p,q}(\mathbb{C}) \leq \inf_{a \in [\bar{a}, 1)} A_{p,q}(a)\). This completes the proof for this case.

Case \(p \in (2, \infty), q \in [1, 2]\): From (2.6), it is clear that \(Q'(y) > 0\) for \(p > 2\), and therefore \(Q(y) > Q(1) = 0\) for all \(y \in (1, \infty)\). It follows from (2.5) that \(P(y)\) is strictly decreasing in \((0, 1)\) with respect to \(a\). Thus, for \(q < 2\), the value of the quantity

\[
P(y) - \frac{p}{q} = \frac{a^2(1 - a^p)}{a^p(1 - a^2)} - \frac{p}{q}
\]
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decreases from
\[
\lim_{a \to 0^+} \left( P(y) - \frac{p}{q} \right) = +\infty \text{ to } \lim_{a \to 1^-} \left( P(y) - \frac{p}{q} \right) = p((1/2) - (1/q)) < 0,
\]
i.e., \( P(y) - \frac{p}{q} > 0 \) in \((0, b_1)\) and \( P(y) - \frac{p}{q} < 0 \) in \((b_1, 1)\) for some \( b_1 \in (0, 1)\), where \( P(b_1) = \frac{p}{q} \). As a consequence, \( T'(a) = 0 \) only for \( a = 0, b_1, \) and \( T'(a) > 0 \) in \((0, b_1)\), \( T'(a) < 0 \) in \((b_1, 1)\). Hence, \( S_{p,q}(a) \) strictly increases in \((0, b_1)\), and then strictly decreases in \((b_1, 1)\), which implies that
\[
\inf_{a \in [0, \tilde{a}]} S_{p,q}(a) = \min \left\{ S_{p,q}(0), S_{p,q}(\tilde{a}) \right\} = \min \left\{ (1/\sqrt{2}), A_{p,q}(\tilde{a}) \right\}.
\]
Moreover, from the proof of the case \( p, q \in [2, \infty) \), we have \( R_{p,2}(\mathbb{C}) = 1/\sqrt{2} \). These two facts combined with (2.3) readily yield
\[
R_{p,q}(\mathbb{C}) \geq \min \left\{ (1/\sqrt{2}), \inf_{a \in [\tilde{a}, 1]} A_{p,q}(a) \right\},
\]
and making use of (2.7), we arrive at our desired conclusion.

Case \( p, q \in [2, \infty) \): Applying (2.7) of this paper, (1.9) of [9], and [10, Remark 1.2] together, the proof follows immediately from the observation:
\[
(1/\sqrt{2}) \geq R_{p,q}(\mathbb{C}) \geq R_{2,2}(\mathbb{C}) \geq (1/\sqrt{2}) r_2(\mathbb{C}) = 1/\sqrt{2}.
\]

Case \( p \in [1, 2], q \in (2, \infty) \): The fact that \( R_{2,q}(\mathbb{C}) = 1/\sqrt{2} \) is evident from the proof of the case \( p, q \in [2, \infty) \). Furthermore, as we have already seen, from (2.1) it is clear that inequality (2.2) holds for \( r \leq S_{p,q}(a), a \in [0, 1] \), and therefore for
\[
r \leq \inf_{a \in [0, 1]} S_{p,q}(a).\]
From this and (2.7), we have (1.5) as an immediate consequence. The assertion \( \inf_{a \in [0, 1]} S_{p,q}(a) > 0 \) is validated from the fact that \( S_{p,q}(a) \neq 0 \) for all \( a \in [0, 1] \) and that \( \lim_{a \to 1^-} S_{p,q}(a) = 1 \). Now, we will show that the imposition of the additional condition (1.4) gives an optimal value for \( R_{p,q}(\mathbb{C}) \). We know that for \( p < 2 \), \( P(y) \) is strictly increasing in \((0, 1)\) with respect to \( a \), and as a result, \( P(y) - \frac{p}{q} \) increases from
\[
\lim_{a \to 0^+} \left( P(y) - \frac{p}{q} \right) = -p/q \text{ to } \lim_{a \to 1^-} \left( P(y) - \frac{p}{q} \right) = p((1/2) - (1/q)) > 0,
\]
i.e., \( P(y) - \frac{p}{q} < 0 \) in \((0, b_2)\) and \( P(y) - \frac{p}{q} > 0 \) in \((b_2, 1)\) for some \( b_2 \in (0, 1) \), where \( P(b_2) = \frac{p}{q} \). As a consequence, \( T'(a) = 0 \) only for \( a = 0, b_2, \) and \( T'(a) < 0 \) in \((0, b_2)\), \( T'(a) > 0 \) in \((b_2, 1)\). Hence, \( S_{p,q}(a) \) strictly decreases in \((0, b_2)\), and then strictly increases in \((b_2, 1)\). Now, if we assume the condition (1.4) in addition, it is equivalent to saying that \( T'(\tilde{a}) \leq 0 \), i.e., \( \tilde{a} \leq b_2 \). Thus, \( \inf_{a \in [0, \tilde{a}]} S_{p,q}(a) = S_{p,q}(\tilde{a}) = A_{p,q}(\tilde{a}) \). Consequently, from (2.3), we get \( R_{p,q}(\mathbb{C}) \geq \inf_{a \in [\tilde{a}, 1]} A_{p,q}(a) \), which completes our proof for this case. □

Proof of Proposition 1.3 As any holomorphic function \( f: \mathbb{D} \to X \) can also be considered as a holomorphic function from \( \mathbb{D}^n \) to \( X \), it immediately follows that \( R^n_{p,q}(X) > 0 \) for any \( n \in \mathbb{N} \) implies that \( R_{p,q}(X) > 0 \). Thus, we only need to establish
the converse. Any holomorphic \( f : \mathbb{D}^n \to X \) with an expansion (1.1) can be written as

\[
(2.9) \quad f(z) = x_0 + \sum_{k=1}^{\infty} P_k(z), \quad z \in \mathbb{D}^n,
\]

where \( P_k(z) := \sum_{|\alpha|=k} x_\alpha z^\alpha \). Thus, for any fixed \( z_0 \in \mathbb{T}^n \) (the \( n \)-dimensional torus), we have

\[
(2.10) \quad g(u) := f(uz_0) = x_0 + \sum_{k=1}^{\infty} P_k(z_0) u^k : \mathbb{D} \to X
\]

is holomorphic, and if \( \|f\|_{H^\infty(\mathbb{D}^n, X)} \leq 1 \), then \( \|g\|_{H^\infty(\mathbb{D}, X)} \leq 1 \). Hence, starting with the assumption \( R_{p,q}(X) = R > 0 \), we have \( \|P_k(z_0)\| \leq \left(1/R^k\right)(1 - \|x_0\|)^{1/q} \), and since \( z_0 \) is arbitrary, we conclude that \( \sup_{z \in \mathbb{T}^n} \|P_k(z)\| \leq \left(1/R^k\right)(1 - \|x_0\|)^{1/q} \) for any \( k \in \mathbb{N} \). Therefore, for a given \( k \in \mathbb{N} \) and for any \( \alpha \) with \( |\alpha| = k \), we have

\[
\|x_\alpha\| = \left| \frac{1}{(2\pi)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} \frac{P_k(z)}{z^\alpha} dz_1 \cdots dz_n \right| \leq \sup_{z \in \mathbb{T}^n} \|P_k(z)\| \leq \frac{1}{R^k} \left(1 - \|x_0\|\right)^{1/k}.
\]

As a result, we have, for all \( r < R \),

\[
\|x_0\|^p + \left( \sum_{k=1}^{\infty} r^k \|x_\alpha\|^q \right)^{\frac{q}{p}} \leq \|x_0\|^p + \left(1 - \|x_0\|\right)^{\left(\frac{R}{R-r}\right)^n - 1} \]

which is less than or equal to 1 whenever \( r \leq R \left(1 - (1/2)^{1/n}\right) \), thereby asserting that \( R_{p,q}^n(X) > 0 \).

\[ \text{Proof of Theorem 1.4} \]

(i) Before we start proving the first part of this theorem, note that the choice of \( q \in [2, \infty) \) is again justified due to Proposition 1.3 and [6, Corollary 4]. Now, given a holomorphic \( f : \mathbb{D}^n \to X \) with an expansion (1.1) and with \( \|f(z)\| \leq 1 \) for all \( z \in \mathbb{D}^n \), we have, for any fixed \( R \in (0, 1) \),

\[
(2\pi)^{-n} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \|f(\Re^{i\theta_1}, \Re^{i\theta_2}, \ldots, \Re^{i\theta_n})\|^2 d\theta_1 d\theta_2 \cdots d\theta_n \leq 1,
\]

which is the same as saying that

\[
\|x_0\|^2 + \sum_{|\alpha| \in \mathbb{N}} \|x_\alpha\|^2 M^{|\alpha|+|\beta|} \leq 1
\]

with \( M := \sum_{\alpha \neq \beta} \langle x_\alpha, x_\beta \rangle \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} e^{i(\theta_1(\alpha_1 - \beta_1) + \cdots + \theta_n(\alpha_n - \beta_n))} d\theta_1 d\theta_2 \cdots d\theta_n \).

Here, \( \langle \cdot, \cdot \rangle \) is the inner product of \( X \), \( \alpha \) and \( \beta \) denote as usual \( n \)-tuples \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( (\beta_1, \beta_2, \ldots, \beta_n) \) of nonnegative integers, respectively. As we know \( \int_0^{2\pi} e^{ik\theta} d\theta = 0 \) for any \( k \in \mathbb{Z}\setminus\{0\} \), \( M = 0 \). Letting \( R \to 1- \) in the above inequality, we therefore get

\[
\|x_0\|^2 + \sum_{k=1}^{\infty} \sum_{|\alpha| = k} \|x_\alpha\|^2 \leq 1.
\]

Taking \( z \in r\mathbb{D}^n \) and using this inequality, we obtain
In our context, for any given holomorphic function \(f\), from the proof of [6, Theorem 1] that for all \(k\)
\[
\|x_0\|^p + \left(\sum_{k=1}^{\infty} \sum_{|a|=k} \|x_a z^a\|^q\right) \leq \|x_0\|^p + \left(\sum_{k=1}^{\infty} \sum_{|a|=k} \|x_a\|^2\right)^{\frac{q}{2}} \left(\sum_{k=1}^{\infty} \sum_{|a|=k} |z^a|^2\right)^{\frac{q}{2}}
\]
\[
\leq \|x_0\|^p + (1 - \|x_0\|^2)^{\frac{q}{2}} \left(\sum_{k=1}^{\infty} \frac{k+n-1}{k} \chi^2\right)^{\frac{q}{2}}
\]
\[
= \|x_0\|^p + (1 - \|x_0\|^2)^{\frac{q}{2}} \left(\frac{1}{(1 - r^2)^n} - 1\right)^{\frac{q}{2}},
\]
which is less than or equal to 1 if
\[
r \leq \left(1 - (1 - (S_{p,q}(\|x_0\|))^2)^{\frac{1}{2}}\right)^{\frac{1}{q}},
\]
and therefore
\[
R^n_{p,q}(\mathcal{H}) \geq \inf_{\alpha \in (0,1)} \left(1 - (1 - (S_{p,q}(\alpha))^2)^{\frac{1}{n}}\right)^{\frac{1}{2}}.
\]
As the quantity on the right-hand side of inequality (2.11) becomes \(\sqrt{1 - (1/2)^{1/n}}\) at \(x_0 = 0\) and converges to 1 as \(\|x_0\| \rightarrow 1\), we conclude that the infimum in inequality (2.12) is attained at some \(b_3 \in [0,1)\). Since every Hilbert space \(\mathcal{H}\) has an orthonormal basis and, in our case, \(\dim(\mathcal{H}) = \infty\), we can choose a countably infinite set \(\{e_a\}_{|a| \in \mathbb{N} \cup \{0\}}\) of orthonormal vectors in \(\mathcal{H}\). Setting \(r_3 = (1 - (1 - (S_{p,q}(b_3))^2)^{\frac{1}{2}})^{\frac{1}{2}}\), we construct
\[
\chi(z) := b_3 e_0 + \frac{1 - b_3^2}{(1 - b_3^2)^{\frac{1}{2}}} \sum_{k=1}^{\infty} r_3 \left(\sum_{|a|=k} z^a e_a\right) : \mathbb{D}^n \rightarrow \mathcal{H},
\]
which satisfies \(\|\chi(z)\| \leq 1\) for all \(z \in \mathbb{D}^n\), and \(r_3 = R^n_{p,q}(\chi, \mathcal{H}) \geq R^n_{p,q}(\mathcal{H})\). This completes the proof for the first part of this theorem.

(ii) The proof for this part is rather lengthy, so we break it into a couple of steps. Prior to each step, we will provide some auxiliary information whenever needed.

Background for Step 1: If \(R_{p,q}(X) > 0\), we have
\[
\Omega_X(\delta) \leq C \left(1 + (1 + \delta)^q - (1 + \delta)^{q-p}\right)^{1/q}, \quad \delta \geq 0
\]
for some constant \(C\) (see (1.6) in the introduction). Given any such \(X\), and given any holomorphic function \(G(u) = \sum_{n=0}^{\infty} y_n u^n : \mathbb{D} \rightarrow X\) with \(\|G(u)\| \leq 1\) in \(\mathbb{D}\), it is known from the proof of [6, Theorem 1] that
\[
\|y_k\| \leq 2\Omega_X(1 - \|y_0\|) \leq 2C \left((2 - \|y_0\|)^q - (2 - \|y_0\|)^{q-p}\right)^{1/q}
\]
for all \(k \geq 1\).

Step 1: In our context, for any given holomorphic \(f : \mathbb{D}^n \rightarrow X\) with an expansion (1.1) and with \(\|f\|_{H^\infty(\mathbb{D}^n \times X)} \leq 1\), we define the holomorphic function \(g(u) = x_0 + \sum_{k=1}^{\infty} P_k(z_0) u^k : \mathbb{D} \rightarrow X\) as in (2.10), which satisfies \(\|g(u)\| \leq 1\) for all \(u \in \mathbb{D}\), \(z_0\) being any chosen point on \(\mathbb{T}^n\). Since \(R_{p,q}(X) > 0\), making use of inequality (2.13), we
conclude that for any $k \geq 1$,
\[
\|P_k(z_0)\| \leq 2C \left( (2 - \|x_0\|)^q - (2 - \|x_0\|)^{q-p} \right)^{1/q}
\]
for any $z_0 \in \mathbb{T}^n$. Therefore,
\[
(2.14) \quad \sup_{z \in \mathbb{T}^n} \|P_k(z)\| \leq 2C \left( (2 - \|x_0\|)^q - (2 - \|x_0\|)^{q-p} \right)^{1/q}
\]
for any $k \in \mathbb{N}$, $C$ being the constant for which (1.6) is satisfied.

Background for Step 2: For $1 \leq p < \infty$ and for a linear operator $U : X_0 \to Y_0$ between the complex Banach spaces $X_0$ and $Y_0$, we say that $U$ is $p$-summing if there exists a constant $c \geq 0$ such that regardless of the natural number $m$ and regardless of the choice of $f_1, f_2, \ldots, f_m$ in $X_0$, we have
\[
\left( \sum_{i=1}^m \|U(f_i)\|^p \right)^{1/p} \leq c \sup_{\phi \in B_{X_0^*}} \left( \sum_{i=1}^m |\phi(f_i)|^p \right)^{1/p},
\]
where $B_{X_0^*}$ is the open unit ball in the dual space $X_0^*$. The least $c$ for which the above inequality always holds is denoted by $\pi_p(U)$, and the set of all $p$-summing operators from $X_0$ into $Y_0$ is denoted by $\Pi_p(X_0, Y_0)$. Now, from [18, Proposition 2.3], we know that:

**Fact I.** If $U : X_0 \to Y_0$ is a bounded linear operator and $\dim(U(X_0)) < \infty$, then $U$ is $p$-summing for every $p \in [1, \infty)$.

Moreover, [18, Theorem 2.8] states that:

**Fact II.** If $1 \leq p < q < \infty$, then $\Pi_p(X_0, Y_0) \subset \Pi_q(X_0, Y_0)$. Moreover, for $U \in \Pi_p(X_0, Y_0)$, we have $\pi_q(U) \leq \pi_p(U)$.

Step 2: Coming back to our proof now, we set $X_0 = Y_0 = X$ and $U = I$—the identity operator on $X$. As $X$ is finite-dimensional, $\dim(I(X)) < \infty$ in this case and thus using Fact I, we have $I \in \Pi_p(X, X)$ for all $p \geq 1$. Therefore,
\[
\left( \sum_{|\alpha| = k} \|x_\alpha\|^k \right)^{\frac{1}{k+1}} \leq \pi_{\frac{1}{k+1}}(I) \sup_{\phi \in B_{X^*}} \left( \sum_{|\alpha| = k} |\phi(x_\alpha)|^{\frac{1}{k+1}} \right)^{\frac{k+1}{k}}
\]
for all $k \in \mathbb{N}$. Since $2k/(k+1) > 1$ for all $k \geq 2$, Fact II asserts that $\pi_{\frac{1}{k+1}}(I) \leq \pi_1(I)$. Hence, there exists a constant $D = \pi_1(I)$ (depending only on $X$) such that
\[
(2.15) \quad \left( \sum_{|\alpha| = k} \|x_\alpha\|^k \right)^{\frac{1}{k+1}} \leq D \sup_{\phi \in B_{X^*}} \left( \sum_{|\alpha| = k} |\phi(x_\alpha)|^{\frac{1}{k+1}} \right)^{\frac{k+1}{k}}
\]
for all $k \in \mathbb{N}$.

Background for Step 3: From [4, Theorem 1.1], we know that for any $\varepsilon > 0$, there exists $\mu > 0$ such that, for any complex $k$-homogeneous polynomial ($k \geq 1$) $P(z) = \sum_{|\alpha| = k} c_\alpha z^\alpha$ ($c_\alpha \in \mathbb{C}$), we have
\[
\left( \sum_{|\alpha| = k} |c_\alpha|^k \right)^{\frac{1}{k+1}} \leq \mu (1 + \varepsilon)^k \sup_{z \in \mathbb{D}^n} |P(z)|.
\]
Step 3: Recall from (2.9) now that $P_k(z) = \sum_{|a|=k} x_a z^a$, $x_a \in X$, and hence 
\[ \phi(P_k(z)) = \sum_{|a|=k} \phi(x_a) z^a \] 
for any $\phi \in B_X$. Consequently, using the above inequality, we get that for any $\varepsilon > 0$, there exists $\mu > 0$ such that

\[
\sup_{\phi \in B_X} \left( \sum_{|a|=k} |\phi(x_a)| \right)^{\frac{k+1}{2k}} \leq \mu (1 + \varepsilon)^{k} \sup_{\phi \in B_X} \sup_{z \in \mathbb{T}^n} |\phi(P_k(z))| = \mu (1 + \varepsilon)^{k} \sup_{z \in \mathbb{T}^n} \|P_k(z)\|
\]

for all $k \geq 1$. Combining this inequality with inequalities (2.14) and (2.15) appropriately, we get

\[
\left( \sum_{|a|=k} \|x_a\| \right)^{\frac{k+1}{2k}} \leq 2 \mu CD (1 + \varepsilon)^{k} ((2 - \|x_0\|)^q - (2 - \|x_0\|)^{q-p})^{1/q}.
\]

It follows that

\[
\left( \sum_{k=1}^\infty r^k \sum_{|a|=k} \|x_a\| \right)^q \leq \left( \sum_{k=1}^\infty r^k \left( \sum_{|a|=k} \|x_a\| \right)^{\frac{k+1}{2k}} \right)^q \left( \left( n + k - 1 \right) \left( \frac{k+1}{2k} \right)^q \right),
\]

where $X = \mu^q C_1^q ((2 - \|x_0\|)^q - (2 - \|x_0\|)^{q-p})$, $C_1 = 2CD$. Hence, for $z \in r\mathbb{D}$, the inequality

\[
\|x_0\|^p + \left( \sum_{k=1}^\infty \sum_{|a|=k} \|x_a z^a\| \right)^q \leq 1
\]

is satisfied if

\[
(2.16) \quad \left( \frac{X}{1 - \|x_0\|^p} \right)^{rac{1}{q}} \left( \sum_{k=1}^\infty r^k (1 + \varepsilon)^{k} \left( n + k - 1 \right) \left( \frac{k+1}{2k} \right) \right) \leq 1.
\]

Now, analyzing the function $f_1(t) = ((2 - t)^p - 1)/(1 - t)^p$, $t \in [0, 1)$, we see that $f_1(t) \leq f_1(0) = 2^p - 1$ for all $t \in [0, 1)$, and hence

\[
\frac{X}{1 - \|x_0\|^p} = \mu^q C_1^q (2 - \|x_0\|)^{q-p} f_1(\|x_0\|) \leq \begin{cases} 
\mu^q C_1^q 2^{q-p} (2^p - 1) & \text{if } q \geq p, \\
\mu^q C_1^q (2^p - 1) & \text{if } q \leq p.
\end{cases}
\]

Thus, inequality (2.16) is satisfied if

\[
C_2 \left( \sum_{k=1}^\infty r^k (1 + \varepsilon)^{k} \left( n + k - 1 \right) \left( \frac{k+1}{2k} \right) \right) \leq 1,
\]

where $C_2$ is a new constant depending on $\mu, p, q$ and the Banach space $X$. Using the estimate

\[
\binom{n+k-1}{k} \leq \frac{(n+k-1)^k}{k!} < \left( \frac{e}{k} \right)^k (n+k-1)^k < e^k \left( 1 + \frac{n}{k} \right)^k,
\]
we get, by setting \( r = (1 - 2\varepsilon)\sqrt{(\log n)/n} \),

\[
\sum_{k=1}^{\infty} r^k (1 + \varepsilon)^{k} \left( n + \frac{k - 1}{k} \right)^{\frac{k-1}{k}} \leq \sum_{k=1}^{\infty} \left( \sqrt{\frac{\log n}{n}} \sqrt{e(1 - 2\varepsilon)(1 + \varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k-1}{k}}.
\]

Hence, inequality (2.16) is satisfied if

\[
C_2 \sum_{k=1}^{\infty} \left( \sqrt{\frac{\log n}{n}} \sqrt{e(1 - 2\varepsilon)(1 + \varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k-1}{k}} \leq 1.
\]

Starting here, we will follow the similar lines of argument as in [4, pp. 743–744]. For \( n \) large enough,

\[
t_n := \frac{\sqrt{\log n}}{n^{1/4}} \sqrt{2e(1 - 2\varepsilon)(1 + \varepsilon)} < 1,
\]

and for \( k > \sqrt{n} \), observe that

\[
\left( 1 + \frac{n}{k} \right)^{\frac{k-1}{k}} < (2\sqrt{n})^{\frac{k}{2}}.
\]

Using both the above facts,

\[
\sum_{k > \sqrt{n}} \left( \sqrt{\frac{\log n}{n}} \sqrt{e(1 - 2\varepsilon)(1 + \varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k-1}{k}} \leq \sum_{k > \sqrt{n}} \left( \frac{\sqrt{\log n}}{n^{1/4}} \sqrt{e(1 - 2\varepsilon)(1 + \varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k-1}{k}} \leq \frac{t_n}{1 - t_n},
\]

which goes to 0 as \( n \to \infty \). For \( k \leq \sqrt{n} \), we start by making \( n \) sufficiently large such that \( 2 < k_0 \leq \log n \) can be chosen for which the inequalities

\[
k^{-\varepsilon} \leq 1 + \frac{\varepsilon}{2}, \quad \sum_{k_0 \leq k \leq \sqrt{n}} ((1 - 2\varepsilon)(1 + \varepsilon)^{3/2}) \leq \frac{1}{2C_2} \quad \text{and} \quad \left( \frac{1}{n} \right)^{\frac{k-2}{2(k-1)}} \leq \frac{\varepsilon}{2}
\]

are satisfied. Observing that \( x^{1/(x-1)} \) is decreasing and \((x - 2)/2(x - 1)\) is increasing in \((1, \infty)\), we obtain, for \( k \geq k_0 \),

\[
\left( \frac{k}{k+1} \left( 1 + \frac{n}{k} \right) \right)^{\frac{k-1}{k}} \leq \left( \left( \frac{1}{n} \right)^{\frac{k-2}{2(k-1)}} + k \right)^{\frac{k-1}{k}} \leq \left( \left( \frac{1}{n} \right)^{\frac{k_0-2}{2(k_0-1)}} + k_0 \right)^{\frac{k-1}{k}} \leq (1 + \varepsilon)^{\frac{k-1}{k}} \leq 1 + \varepsilon,
\]

which, after a little simplification, gives

\[
\left( 1 + \frac{n}{k} \right)^{\frac{k-1}{k}} \leq (1 + \varepsilon)^{\frac{k}{2}} \frac{n^{\frac{k}{2}}}{n^{1/k} k^{\frac{k}{2}}}.
\]
Since \( x \mapsto n^{1/x} x \) is decreasing up to \( x = \log n \) and increasing thereafter, we have \( n^{1/k}k \geq e \log n \). Therefore,

\[
\sum_{k_a \leq k \leq \sqrt{n}} \left( \sqrt{\frac{\log n}{n}} \sqrt{e(1-2\varepsilon)(1+\varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k+1}{2}} \leq \sum_{k_a \leq k \leq \sqrt{n}} \left( \sqrt{e \log n(1-2\varepsilon)(1+\varepsilon)}^{3/2} \sqrt{\frac{1}{n^{1/k}k}} \right)^k \leq \sum_{k_a \leq k \leq \sqrt{n}} \left( (1-2\varepsilon)(1+\varepsilon)^{3/2} \right)^k \leq \frac{1}{2C_2}.
\]

It remains to analyze the case \( 1 \leq k \leq k_0 \). In this case, we observe that for \( n \) large enough,

\[
\frac{k}{n} + 1 \leq \frac{k_0}{n} + 1 \leq \varepsilon + 1,
\]

and hence

\[
\left( 1 + \frac{n}{k} \right)^{\frac{k+1}{2}} \leq \left( 1 + \varepsilon \right)^{\frac{k+1}{2}} \left( \frac{n}{k} \right)^{\frac{k+1}{2}}.
\]

Making use of the above inequality and the fact that \( x \mapsto n^{1/x} x \) is decreasing in \([1, k_0]\) (i.e., \( n^{1/k}k \geq n^{1/k_0}k_0 \)), it is easily seen that

\[
\sum_{k_0 \leq k \leq \sqrt{n}} \left( \sqrt{\frac{\log n}{n}} \sqrt{e(1-2\varepsilon)(1+\varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k+1}{2}} \leq \sum_{k_0 \leq k \leq \sqrt{n}} \left( \sqrt{e \log n(1-2\varepsilon)(1+\varepsilon)}^{3/2} \frac{k^{1/(2k)}}{k_0^{1/2} n^{1/(2k_0)}} \right)^k,
\]

which tends to 0 as \( n \to \infty \). Combining all the above three estimates, we have

\[
\sum_{k=1}^{\infty} \left( \sqrt{\frac{\log n}{n}} \sqrt{e(1-2\varepsilon)(1+\varepsilon)} \right)^k \left( 1 + \frac{n}{k} \right)^{\frac{k+1}{2}} \leq \frac{1}{2C_2} + o(1)
\]

for \( n \) large enough. Therefore, inequality (2.17) is satisfied for large enough \( n \). Hence, for any given \( \varepsilon > 0 \), \( R_{p,q}^n(X) \geq (1-2\varepsilon)\sqrt{\log n/\sqrt{n}} \) for sufficiently large \( n \). This yields the following:

\[
\liminf_{n \to \infty} R_{p,q}^n(X) \sqrt{n/\sqrt{\log n}} \geq 1.
\]

**Step 4 :** In view of the above, it is only left to show that

\[
(2.18) \quad \limsup_{n \to \infty} R_{p,q}^n(X) \sqrt{n/\sqrt{\log n}} \leq 1.
\]

As \( R_{p,q}^n(X) \leq R_{p,q}^n(C) \), it is sufficient to establish this part for \( X = C \). The proof is exactly the same as the proof for the case \( p = q = 1 \) given in [12, p. 2977], but
for the sake of completeness, we reproduce the argument here. From the Kahane–Salem–Zygmund inequality, it is known that there is a constant $B$ such that for every collection of complex numbers $c_a$ and every integer $k > 1$, there is a choice of plus and minus signs for which the supremum of the modulus of $\sum_{|a|=k} c_a e^{\alpha a}$ in $\mathbb{D}$ does not exceed $B \left( n \sum_{|a|=k} |c_a|^2 \log k \right)^{1/2}$. We choose $c_a = k!/\alpha !$. Then $\sum_{|a|=k} |c_a|^2 \leq k! n^k$. By the definition of the generalized Bohr inequality in our context, we get

\[
\left( \left( R_{p,q}^n(\mathbb{C}) \right)^k n^k \right)^q = \left( \sum_{|a|=k} |c_a| \left( R_{p,q}^n(\mathbb{C}) \right)^k \right)^q 
\leq B^q \left( n \sum_{|a|=k} |c_a|^2 \log k \right)^{q/2} \leq B^q \left( n^{k+1} (k! \log k)^{1/2} \right)^q,
\]
or, equivalently,

\[
R_{p,q}^n(\mathbb{C}) \leq B^{1/k} n^{1/k} (k! \log k)^{1/k}.
\]

We use Stirling’s formula $\lim_{k \to \infty} k! (\sqrt{2\pi k} (k/e)^k)^{-1} = 1$ to conclude that

\[
R_{p,q}^n(\mathbb{C}) \leq \sqrt{\frac{k}{n}} \left( \frac{B^{1/k} n^{1/k} (k! \log k)^{1/k}}{\sqrt{e}} \right)
\]
for a new constant $B_1$. Setting $k = \lfloor \log n \rfloor$ ($\lfloor \cdot \rfloor$ is the floor function), we observe

\[
\limsup_{n \to \infty} R_{p,q}^n(\mathbb{C}) \sqrt{\frac{n}{\log n}} \leq \lim_{n \to \infty} \frac{B^{1/\lfloor \log n \rfloor} \lfloor \log n \rfloor \frac{1}{\lfloor \log n \rfloor} \lfloor \log n \rfloor \frac{1}{2 \lfloor \log n \rfloor}}{\sqrt{e}} = 1,
\]

which implies our desired inequality (2.18). This completes the proof.

**Proof of Theorem 1.5**

(i) Given a complex-valued holomorphic function $f$ with an expansion (1.1) in $\mathbb{D}$ (“$x_a$’s” are complex numbers in this case) and satisfying $\|f\|_{\mathcal{H}^\infty(\mathbb{D},\mathbb{C})} \leq 1$, an application of Hölder’s inequality yields

\[
|x_0|^p + \sum_{k=1}^{\infty} r^{kp} \sum_{|a|=k} |x_a|^p = \sum_{k=0}^{\infty} \sum_{|a|=k} |x_a|^{2-p} r^{kp} |x_a|^{2p-2} 
\leq \left( \sum_{k=0}^{\infty} r^{kp} \sum_{|a|=k} |x_a|^{2-p} \right)^{2-p} \left( \sum_{k=0}^{\infty} \sum_{|a|=k} |x_a|^2 \right)^{p-1} 
\leq \left( \sum_{k=0}^{\infty} r^{kp} \sum_{|a|=k} |x_a|^{2-p} \right)^{2-p}.
\]

Therefore, $r_p^n(\mathbb{C}) \geq (r_1^n(\mathbb{C}))^{(2-p)/p}$. Since $\lim_{n \to \infty} r_1^n(\mathbb{C}) \left( \sqrt{n}/\sqrt{\log n} \right) = 1$ (cf. [4]), we have

\[
\liminf_{n \to \infty} r_p^n(\mathbb{C}) \left( \frac{n}{\log n} \right)^{\frac{2-p}{2p}} \geq \liminf_{n \to \infty} \left( r_p^n(\mathbb{C}) \sqrt{\frac{n}{\log n}} \right)^{\frac{2-p}{2p}} = 1,
\]
and thus \( r^n_p(\mathbb{C}) \geq C((\log n)/n)^{(2-p)/2p} \) for some constant \( C > 0 \) and for all \( n > 1 \).

The upper bound \( r^n_p(\mathbb{C}) \leq D ((\log n)/n)^{(2-p)/2p} \) for some \( D > 0 \) has already been established in [20, p. 76]. This completes the proof.

(ii) To handle the second part of this theorem, we first construct \( g(u) \) as in (2.10) from a given holomorphic \( f : \mathbb{D}^n \to X \) with an expansion (1.1) and satisfying \( \|f\|_{H^\infty(\mathbb{D}^n, X)} \leq 1 \). Now, since \( X \) is \( p \)-uniformly \( PL \)-convex, from the proof of [11, Proposition 2.1(ii)], we obtain

\[
\|P_1(z_0)\| \leq \frac{2}{(I_p(X))^{\frac{1}{p}}} (1 - \|x_0\|^p)^{\frac{1}{2p}}
\]

for any arbitrary \( z_0 \in \mathbb{T}^n \). Using a standard averaging trick (see, f.i., [10, p. 94]), it can be shown that the \( P_1(z_0) \) in the above inequality could be replaced by \( P_k(z_0) \) for any \( k \geq 2 \). Thus, we conclude that

\[
(2.19) \quad \sup_{z \in \mathbb{T}^n} \|P_k(z)\| \leq \frac{2}{(I_p(X))^{\frac{1}{p}}} (1 - \|x_0\|^p)^{\frac{1}{2p}}.
\]

Now, from [16, Lemma 25.18], it is known that there exists \( R > 0 \) such that

\[
\left( \sum_{|\alpha|=k} \|x_\alpha\|^p \right)^{1/p} \leq \int_{\mathbb{T}^n} \|P_k(z)\|^p dz.
\]

Using inequality (2.19) gives

\[
\sum_{|\alpha|=k} \|x_\alpha\|^p \leq \frac{2^p}{I_p(X)}^{1/p} (1 - \|x_0\|^p) R^k p^{k-1}.
\]

Assuming \( r < R \), it is easy to see that

\[
\|x_0\|^p + \sum_{k=1}^{\infty} r^k \sum_{|\alpha|=k} \|x_\alpha\|^p \leq \|x_0\|^p + \frac{2^p}{I_p(X)} (1 - \|x_0\|^p) \sum_{k=1}^{\infty} \left( \frac{r}{R} \right)^k p^k
\]

\[
\leq \|x_0\|^p + \frac{2^p}{I_p(X)} (1 - \|x_0\|^p) \frac{r^p}{R^p - r^p},
\]

which is less than or equal to 1 if

\[
\left( \frac{I_p(X)}{r^p + I_p(X)} \right)^{\frac{1}{p}} \leq \left( \frac{I_p(X)}{2^p + I_p(X)} \right)^{\frac{1}{p}}.
\]

as from the arguments in [16, p. 627], it is clear that we can take \( R^p = I_p(X)/(I_p(X) + 2^p) \). This proves the lower estimate for \( r^n_p(X) \), and the upper estimate is trivial due to the fact that \( r^n_p(X) \leq r^n_p(\mathbb{C}) = 1 \) for \( p \geq 2 \).

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