Dark Energy and Dark Matter in General Relativity with local scale invariance

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Abstract: We consider a generalization of Einstein’s general theory of relativity such that it respects local scale invariance. This requires the introduction of a scalar and a vector field in the action. We show that the theory naturally displays both dark energy and dark matter. We solve the resulting equations of motion assuming an FRW metric. The solutions are found to be almost identical to those corresponding to the standard ΛCDM model.

1 Introduction

In recent papers [1, 2] we have studied a scale invariant extension of the general theory of relativity. The theory postulates a scalar field Φ and a modification of the gravitational action such that [3, 4],

\[ \frac{1}{2\pi G} R \rightarrow \beta \Phi^* \Phi R \]  

where \( G \) is the Newton’s gravitational constant, \( R \) is the Ricci scalar and \( \beta \) is a dimensionless constant. The modified action involves no mass parameter and is invariant under the scale transformation. It is also convenient to introduce the concept of pseudo-scale invariance. In four dimensions the pseudo-scale transformation can be written as [3, 4],

\[ x \rightarrow x, \]
\[ \Phi \rightarrow \Phi/\Lambda, \]
\[ g^{\mu\nu} \rightarrow g^{\mu\nu}/\Lambda^2, \]
\[ A_\mu \rightarrow A_\mu, \]
\[ \Psi \rightarrow \Psi/\Lambda^{3/2}. \]  

where \( x \) is the space-time coordinate, \( g^{\mu\nu} \) the metric, \( A_\mu \) a vector field and \( \Psi \) a spin half field. The scale transformation can be expressed as a combination of the pseudo-scale and the general coordinate transformations. Hence as long as general coordinate invariance is respected, pseudo-scale invariance is equivalent to scale invariance.
We also considered a theory with local scale or pseudo-scale invariance [3–12]. The transformations in this case are given by Eq. (2) with the parameter \( \Lambda \rightarrow \Lambda(x) \). In this case we need to introduce the Weyl vector meson, \( S_\mu \), besides the scalar field. Under pseudo-scale transformation the vector field transforms as [3, 4]

\[
S_\mu \rightarrow S_\mu - \frac{1}{f} \partial_\mu \ln(\Lambda(x)).
\]

The Lagrangian in this case, ignoring all other fields besides the \( \Phi \) and \( S_\mu \), may be written as

\[
\mathcal{L} = -\frac{\beta}{8} \Phi^2 \tilde{R} + \mathcal{L}_{\text{matter}}
\]

where

\[
\mathcal{L}_{\text{matter}} = \frac{1}{2} g^{\mu\nu} (\mathcal{D}_\mu \Phi)(\mathcal{D}_\nu \Phi) - \frac{\lambda}{4} \Phi^4 - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} E_{\mu\nu} E_{\rho\sigma},
\]

\( f \) is the gauge coupling constant, \( E_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu \), \( \tilde{R} \) is the modified curvature scalar [3,4], invariant under local pseudo-scale transformation, and

\[
\mathcal{D}_\mu \equiv \partial_\mu - f S_\mu
\]

is the gauge covariant derivative. The scalar \( \tilde{R} \) is found to be

\[
\tilde{R} = R - 6f S_\kappa^{\kappa} - 6f^2 S_\mu S^\mu
\]

where \( R \) is the standard curvature scalar. The mass of the vector boson \( S_\mu \) has been constrained by cosmological observations to be very light, less than 400 eV, or very heavy, greater than Planck Mass [13].

A theory with local scale invariance is aesthetically pleasing since it does not contain any mass parameter in the action. This also implies that we cannot add the cosmological constant term in the action. Hence as long as scale invariance is unbroken, cosmological constant is identically zero. In Ref. [1,2] we argued that scale invariance is broken by the cosmological time evolution or equivalently by initial conditions. This phenomenon is called cosmological symmetry breaking in Ref. [1,2]. The authors argue that the universe is a time dependent solution of the equations of motion. All phenomena are described by making a quantum expansion around this time dependent solution. Hence if the symmetries of the action are not respected by this solution then these symmetries will be hidden, in analogy
to the phenomenon of spontaneous symmetry breaking. However as shown in Ref. [1, 2], cosmological symmetry breaking is not the same as spontaneous symmetry breaking.

Once scale invariance is broken cosmologically, the theory generates the dimensionful parameters such as the Newton’s constant and the vacuum (or dark) energy [14–23]. These essentially get related to the initial conditions imposed on the scalar field. The fact that this theory leads to dark energy has also been noticed in Ref. [24]. However the precise nature of identification is different from that in Ref. [1, 2]. In Ref. [1, 2] the authors speculated that if scale invariance is broken cosmologically, then this symmetry may not be anomalous. This symmetry may also tame the quantum corrections to the vacuum energy, hence avoiding the fine-tuning problems in cosmological constant [25–31]. Alternate proposals to solve the cosmological constant problem are discussed in Ref. [25, 32–39].

In the case of global scale invariance, the authors in Ref. [2] found a solution with $\Phi$ equal to a constant. In this case our theory reduces to a scalar-tensor model. The cosmological implications of such models have been studied extensively in recent literature [40–42]. The value of the constant field $\Phi$ can be chosen to fit the experimental value of the gravitational constant. For local scale invariance, $\Phi$ is constant in the gauge $S_0 = 0$. In Ref. [2], the authors set $S_i = 0$, where the index $i = 1, 2, 3$. However in general $S_i$ is not zero. In the present paper we give a general solution to the equations of motion of this model. We find that the solutions naturally lead to both dark energy and dark matter.

In Ref. [1, 2], the authors had suggested that the scalar field $\Phi$ may be the Higgs multiplet. They argued that cosmological symmetry breaking also leads to a breakdown of the electroweak symmetry. In this theory the Higgs particle is absent from the particle spectrum and acts as the longitudinal mode of the vector field $S_\mu$ [3, 4]. Although this is a consistent picture, the field $\Phi$ may also be associated with a scalar field in a grand unified theory (GUT). In the present paper, we assume $\Phi$ to be a real scalar field. The generalization to the case where $\Phi$ may be a standard model or a GUT scalar field multiplet is straightforward.

## 2 Equations of motion

The Einstein’s equations and the equations of motion for $\Phi$ and $S_\mu$ following from Eq. (4) are given in Ref. [2]. Here we assume that, at leading order, all the fields are independent of space coordinate and depend only on time. In this case the equations simplify considerably. We display these equations below, correcting some typographical errors in Ref. [2]. The
time-time and space-space components of the Einstein’s equation are

\[ \Phi^2 \left( R_{00} - \frac{R}{2} \right) + 3 \Phi^2 f^2 (S^i S_i - S_0 S_0) + 3 f S_0 \partial_0 (\Phi^2) - g^{ij} \partial_0 (\Phi^2) \Gamma^0_{ij} = \frac{4}{\beta} T_{00}, \]  

(8)

and

\[ \Phi^2 \left( R_{ij} - \frac{1}{2} g_{ij} R \right) + 3 f^2 \Phi^2 g_{ij} S^\mu S^\mu - 6 f^2 \Phi^2 S_i S_j - 3 f g_{ij} S^0 \partial_0 \Phi^2 + g_{ij} \partial_0 \partial_0 (\Phi^2) \Gamma^0_{ij} = \frac{4}{\beta} T_{ij}, \]  

(9)

respectively. Here dots represent derivatives with respect to time. The equation of motion for the scalar field is

\[ \ddot{\Phi} - f \Phi \dot{S}_0 + 3 \Phi \frac{\dot{\Phi}}{a} - f^2 S_0 S^\mu \Phi + \lambda \Phi^3 - 3 f \Phi S_0 \frac{\dot{a}}{a} + \frac{\beta}{4} \dot{\Phi} \ddot{R} = 0 \]  

(10)

The corresponding equations for \( S_0 \) and \( S_i \) are

\[ f S_0 = \frac{\dot{\Phi}}{\Phi}, \]  

(11)

\[ \dddot{S}_i + \frac{\dot{\Phi}}{a} \dot{S}_i + f^2 \Phi^2 S_i + \frac{3}{2} \beta f^2 \dot{\Phi}^2 S_i = 0 \]  

(12)

respectively. Since the theory has local scale invariance, we need to fix the gauge in order to obtain a unique solution. We choose the gauge \( S_0 = 0 \). In this case Eq. (11) implies that the scalar field \( \Phi \) is a constant, independent of time.

Thus, we set \( \Phi(t) = \eta = \text{constant} \) with the above mentioned gauge choice. The nonzero constant value of \( \eta \) essentially acts as dark energy in the present model. The existence of dark energy is naturally implied by cosmologically broken scale invariance. The resulting 0-0 and i-j components of the Einstein’s equations can then be written as,

\[ 3 \eta^2 \frac{\dot{a}^2}{a^2} = \frac{4}{\beta} \left[ \frac{\lambda}{4} \eta^4 + \frac{\dot{S}_i^2}{2a^2} + \left( 1 + \frac{3\beta}{2} \right) \frac{f^2 \eta^2 S_i^2}{2a^2} \right] \]  

(13)

and

\[ 3 \eta^2 \left( \frac{2 \ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \frac{4}{\beta} \left[ \frac{3\lambda}{4} \eta^4 - \frac{\dot{S}_i^2}{2a^2} + \left( 1 + \frac{3\beta}{2} \right) \frac{f^2 \eta^2 S_i^2}{2a^2} \right] \]  

(14)
respectively. In Eqs. 13 and 14, sum over the subscript $i$ is implied in terms containing $S_i^2$ or $\dot{S}_i^2$. The equations for $\eta$ and $S_i$ become,

$$3\eta^2 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \frac{4}{\beta} \left[ \frac{\lambda}{2} \eta^2 + \left( 1 + \frac{3\beta}{2} \right) \frac{f^2 S_i^2 \eta^2}{2a^2} \right] ,$$  

(15)

$$\ddot{S}_i + \frac{\dot{a}}{a} \dot{S}_i + \left( 1 + \frac{3\beta}{2} \right) f^2 \eta^2 S_i = 0$$  

(16)

respectively.

The background constant value of the field $\Phi$ can be related to the Planck mass, $M_p$, by the relation [1–4],

$$\beta \eta^2 = \frac{M_p^2}{2\pi}$$  

(17)

The mass of the vector field, $S_\mu$, is found to be,

$$M_S^2 = \left( 1 + \frac{3\beta}{2} \right) f^2 \eta^2 .$$  

(18)

If the vector field $S_\mu = 0$, then the Hubble constant, $H_0$, is given by,

$$H_0 = \sqrt{\frac{\lambda}{3\beta}} \eta = \sqrt{\frac{\lambda}{6\pi}} \frac{M_p}{\beta} ,$$  

(19)

Finally the vacuum energy density is given by,

$$\rho_\Lambda = \frac{1}{4} \lambda \eta^4 .$$  

(20)

From the relationship between the Hubble constant and the Planck mass, it is clear that either $\lambda$ is extremely small or $\beta$ is extremely large. If we assume $\beta \sim O(1)$, then $\lambda$ is found to be of order $10^{-60}$. In Ref. [1, 2] the authors argued that this small value by itself may not lead to fine tuning problems. These would arise if the quantum corrections at each order require very delicate cancellations to maintain the small value of this parameter. The quantum corrections in this model have so far not been computed. It is of course also important to explain the origin of such a small parameter. This issue is beyond the scope of the present paper. However we speculate that such small values of scalar coupling might arise due to the well known triviality of scalar field theories [43, 44]. The continuum scalar field self coupling is driven to zero by renormalization group analysis. However at length scales smaller than the Planck length, it may not be appropriate to treat space-time as a continuum. The discrete nature of space time might generate a small value for the scalar self coupling.
3 Cosmological Solution

The equation of motion for $S_i$, Eq. (16), is similar to that of a damped harmonic oscillator, with weakly time-dependent frequency and decay terms. We seek a solution of the form

$$S_i = n_i S$$

where $n_i$ is a constant unit vector. The solution for $S$ can be expressed as,

$$S = Re \left\{ A e^{-\int \frac{\dot{a}}{a} dt - i \int \omega_1 dt} + B e^{-\int \frac{\dot{a}}{a} dt + i \int \omega_1 dt} \right\},$$

where, $A$ and $B$ are assumed to be slowly varying functions of time and $\omega_1^2 = \omega^2 - \frac{H^2}{4}$, $\omega^2 = \left(1 + \frac{3\beta}{2}\right) f^2 \eta^2 = M_S^2$ and $H = \dot{a}/a$. By substituting this solution in Eq. (16) and neglecting second derivatives of $A$ and $B$, we get,

$$\frac{\dot{A}}{A} + \frac{\dot{\omega}_1}{2\omega_1} = i \frac{\dot{H}}{4\omega_1} \Rightarrow A = \frac{k_1}{\sqrt{\omega_1}} e^{\frac{i}{2} \sin^{-1} \frac{H}{2\omega}}$$

$$\frac{\dot{B}}{B} + \frac{\dot{\omega}_1}{2\omega_1} = -i \frac{\dot{H}}{4\omega_1} \Rightarrow B = \frac{k_2}{\sqrt{\omega_1}} e^{-\frac{i}{2} \sin^{-1} \frac{H}{2\omega}}$$

where, $k_1$ and $k_2$ are constants of integration and are, in general, complex. Since $\omega >> H$ we see that $A$ and $B$ vary very slowly with time compared to other terms in $S_i$. The most rapidly varying terms are those containing $\int \omega_1 dt$ in the exponent. Due to these terms, $S_i$ fluctuates rapidly with time.

Due to the presence of a vector field $S_i$ in our theory, our cosmological solution naturally contains a constant three dimensional unit vector $n_i$ defined in Eq. 21. This vector defines a direction in space and hence breaks rotational invariance. However the background metric is still isotropic since the vector $n_i$ does not contribute to Einstein’s equations. Furthermore it is unlikely to lead to very large observable consequences of the breakdown of isotropy. This is because the field $S_\mu$ does not directly interact with visible matter [3, 4]. Nevertheless, it is extremely interesting to determine the cosmological predictions of this breakdown of isotropy in view of several observations which indicate a preferred direction in the universe [45–52]. Models in which vector fields acquire nonzero vacuum or background values have also been considered by many authors [53–68]. It has been argued that many of these models, which lead to prolonged anisotropic accelerated expansion, are unstable [69]. In our model the vector field does not directly lead to anisotropic expansion, even though it acquires a non-zero background value.
3.1 Leading Order Solution

At leading order we can assume that $A$ and $B$ are time independent. The leading order solution for $S$ can then be written as

$$S = \frac{1}{\sqrt{a}}(A' \cos \theta + B' \sin \theta)$$  \hspace{1cm} (24)$$

where $\theta = \int \omega_1 dt$ and $A'$ & $B'$ are some real constants. We also find

$$\dot{S} = \frac{1}{\sqrt{a}} \left(-\frac{H}{2} p + \omega_1 q\right),$$  \hspace{1cm} (25)$$

where, $p = A' \cos \theta + B' \sin \theta$, $q = -A' \sin \theta + B' \cos \theta$. We next substitute these into the 0-0 component of the Einstein’s equation. We define,

$$\rho_{S_i} = \frac{1}{2a^2} \dot{S}_i^2 + \frac{1}{2a^2} \omega^2 S_i^2$$  \hspace{1cm} (26)$$

This essentially acts as the contribution to the energy density provided by the field $S_i$. We find,

$$\rho_{S_i} = \frac{1}{2a^3} \left[\omega^2 (p^2 + q^2) + \frac{H^2}{4} (p^2 - q^2) - \omega_1 H p q\right]$$  \hspace{1cm} (27)$$

where, $p^2 + q^2 = A'^2 + B'^2$, which is a constant. Since $\omega_1$ is very large, $S_i$ is a rapidly oscillating function of time. Hence it is reasonable to replace the oscillatory functions with their time averages. After averaging over time, $(p^2 - q^2) \to 0$ and $pq \to 0$. Hence,

$$\rho_{S_i} = \frac{1}{2a^3} \left(A'^2 + B'^2\right) \omega^2$$  \hspace{1cm} (28)$$

We next consider the i-j component of the Einstein’s equation. We define,

$$-3 P_{S_i} = -\frac{1}{2a^2} \dot{S}_i^2 + \frac{1}{2a^2} \omega^2 S_i^2$$  \hspace{1cm} (29)$$

which effectively acts as the contribution of the $S_i$ field to pressure. After substituting the time averaged values for the oscillatory functions, we get

$$P_{S_i} = 0$$  \hspace{1cm} (30)$$

Hence, we find that the field $S_i$ essentially acts as the cold dark matter. Its energy density $\rho_{S_i}$ varies as $1/a^3$ and its pressure $P_{S_i}$ is zero at leading order. A similar phenomenon is seen in the case of coherent axion oscillations [70–74].
The modified Einstein’s equations, at leading order, can now be written as

\[
H^2 = \frac{\dot{a}^2}{a^2} = \frac{\lambda}{3\beta} \eta^2 + \frac{2}{3\beta \eta^2 a^3} (A^2 + B^2) \omega^2
\]

(31)

\[
2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{\lambda}{\beta} \eta^2 .
\]

(32)

Eq. (31) generalizes the expression for the Hubble constant, Eq. (19), for the case when the vector field is non-zero.

### 3.2 Corrections to the leading order

We next calculate the corrections to the leading order result by taking into account the time dependence of the coefficients \(A\) and \(B\) in Eq. (22). Substituting for \(A\) and \(B\), from Eq. (23), in Eq. (22), we find,

\[
S = \frac{1}{\sqrt{\omega_1 a}} \left[ Q \cos(\theta - x) + P \sin(\theta - x) \right] = \frac{1}{\sqrt{\omega_1 a}} U
\]

(33)

\[
\dot{S} = \frac{1}{\sqrt{\omega_1 a}} \left[ \frac{H}{2} \left( \frac{\dot{x}}{\omega_1} - 1 \right) U + (\omega_1 - \dot{x}) V \right]
\]

(34)

where,

\[
U = N \cos \theta + M \sin \theta , \quad V = -N \sin \theta + M \cos \theta
\]

and \(M = P \cos x + Q \sin x\), \(N = -P \sin x + Q \cos x\).

Here, \(x = \frac{1}{2} \sin^{-1} \frac{H}{2\omega} = \frac{1}{2} \cos^{-1} \frac{\omega}{2} = \frac{1}{2} \sin^{-1} \omega_1\), \(\dot{x} = \dot{H}/4\omega_1 = -\dot{\omega}_1/H\) and \(P \& Q\) are some real constants.

Substituting these in Eq. (26), we get,

\[
\rho_{Si} = \frac{1}{2\omega_1 a^3} \left[ (U^2 + V^2) \omega^2 + \frac{H^2}{4} (U^2 - V^2) - \omega_1 H \left( 1 - \frac{\dot{x}}{\omega_1} \right)^2 UV 
\]

\[
\quad + \ (\dot{x}^2 - 2\omega_1 \dot{x}) \left( \frac{H^2}{4\omega_1^2} U^2 + V^2 \right) \right] .
\]

(35)

The third term on the right hand side reduces to \(\omega_1 HUV\), since \(\dot{x}/\omega_1 << 1\). The fourth term simplifies to \(-\dot{H}V^2/2\), if we neglect terms suppressed by factors of \(H/2\omega_1\). Hence we find

\[
\rho_{Si} = \frac{1}{2\omega_1 a^3} \left[ (U^2 + V^2) \omega^2 + \frac{H^2}{4} (U^2 - V^2) - \omega_1 HUV - \frac{\dot{H}}{2} V^2 \right] .
\]

(36)
We again substitute time averaged values for rapidly oscillating functions. This sets \((U^2 - V^2) \rightarrow 0\), \(UV \rightarrow 0\) and \((U^2 + V^2) = (M^2 + N^2) = P^2 + Q^2\), which is a constant. A leading order expression for \(\dot{H}\) can be computed using Eq. (31). We find

\[
\dot{H} = -\frac{1}{\beta \eta^2 a^3} (A^2 + B^2) \omega^2 .
\]  

(37)

Thus, we get,

\[
\rho_{Si} = \frac{(P^2 + Q^2) M_S}{2a^3} + \frac{(P^2 + Q^2) \lambda \eta^2}{48 \beta a^3 M_S} + \frac{(P^2 + Q^2) (A'^2 + B'^2) M_S}{6 \beta \eta^2 a^6}.
\]  

(38)

The leading term varies as \(a^{-3}\) as already found in the previous section. Here, we also find two subleading terms. One of these falls as \(1/a^3\) and the second falls much faster, as \(a^{-6}\), as the universe expands. We similarly find the corrections to the pressure term \(P_s\). We find that, using Eq. (29),

\[
-3P_s = \frac{1}{2a^3 \omega_1} \left[ (U^2 - V^2) \omega_1^2 + \omega_1 HUV + \frac{\dot{H}}{2} V^2 \right].
\]  

(39)

Again, substituting time averaged values for the rapidly oscillating functions, we get

\[
P_{Si} = -\frac{1}{6 \omega_1 a^3} \frac{\dot{H}}{4} (P^2 + Q^2) = \frac{(P^2 + Q^2) (A'^2 + B'^2) M_S}{24 \beta \eta^2 a^6}.
\]  

(40)

Hence, we get a small correction term to \(P_{Si}\), which also decays rapidly as \(a^{-6}\) as the universe expands.

The 0-0 and i-j component of the Einstein’s equations can, now, be written as,

\[
\frac{3 \beta}{4} \eta^2 H^2 = \frac{\lambda}{4} \eta^4 + \frac{(P^2 + Q^2)}{2 \omega_1 a^3} \left( \omega^2 - \frac{\dot{H}}{4} \right)
\]  

(41)

and

\[
\frac{3 \beta}{4} \eta^2 \left( 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \frac{3 \lambda}{4} \eta^4 + \frac{(P^2 + Q^2) \dot{H}}{2 \omega_1 a^3} \frac{\dot{H}}{4}
\]  

(42)

respectively. The first of the above two equations can be cast in the form,

\[
1 = \Omega_\Lambda + \Omega_{Si}
\]  

(43)

where

\[
\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}}, \quad \Omega_{Si} = \frac{\rho_{Si}}{\rho_{cr}}.
\]
\[ \rho_\Lambda = \frac{\lambda}{4} \eta^4, \rho_{S_i} = \frac{(P^2 + Q^2)}{2\omega_1 a^3} \left( \omega^2 - \frac{\dot{H}}{4} \right) \text{ and } \rho_{cr} = \frac{3\beta}{4} \eta^2 H^2. \]

Eq. (43) looks like the ΛCDM model with \( \Omega_M = \Omega_{S_i} \).

Thus, the energy density \( \rho_{S_i} \) and the corresponding pressure \( P_{S_i} \) of the vector field \( S_i \), including the correction terms, are obtained as

\[ \rho_{S_i} = \frac{c_1}{2\omega_1 a^3} \left( \omega^2 + \frac{c_2}{a^3} \right) \]

and

\[ P_{S_i} = \frac{c_1 c_2}{6\omega_1 a^6} \]

where,

\[ c_1 = P^2 + Q^2, \quad c_2 = \frac{A^2 + B^2}{4\beta} \left( 1 + \frac{3\beta}{2} \right) \]

In the limit \( x \to 0 \) and \( \omega_1 \to \omega \), we find \( A' = Q/\sqrt{\omega} \) and \( B' = P/\sqrt{\omega} \).

We can make an estimate of the term \( (P^2 + Q^2) \). Since the recent cosmological observations support a flat ΛCDM model, we can equate the second term on right hand side of Eq. (41), evaluated at present time, to \( \rho_{M,0} \). The contribution due to \( \dot{H} \) is negligible. Thus, we get,

\[ P^2 + Q^2 \approx \frac{3M_p^2 H_0^2 \Omega_M}{4\pi M_s} \]

where \( \Omega_M \) is computed at the current time.

### 4 Including the contribution due to radiation

In this section, we obtain a set of dynamical equations to study the evolution of different components of the universe since the beginning of the radiation dominated era. For this purpose we introduce, by hand, the contribution due to radiation. We expect to reproduce the usual Big Bang evolution where radiation dominates at early times, followed by dark matter and dark energy dominated eras, respectively, at late times. We introduce a radiation term with energy density \( \rho_R \) and it’s corresponding pressure term \( P_R \) in the energy-momentum tensor \( T_{\mu\nu} \). The resulting equations are solved numerically.

It is convenient to introduce the following variables,

\[ X^2 = \frac{\lambda}{3\beta} \frac{\eta^2}{H^2} = \Omega_\Lambda, \quad \text{(44)} \]
\[ Y'^2 = \frac{2S^2}{3\beta a^2 \eta^2} = \Omega_1, \]
\[ Z'^2 = \frac{2}{3\beta} \left( 1 + \frac{3\beta}{2} \right) \frac{f^2S^2}{a^2H^2} = \Omega_2, \]
\[ R = \frac{4}{3\beta a^4 \eta^2 H^2} = \Omega_R. \]

Here, \( \Omega_{S_i} = \Omega_1 + \Omega_2 \), \( \rho_{R,0} \) is the radiation energy density in the current era and the prime denotes derivative with respect to \( \ln a \). Hence for any function \( f \),
\[ f' \equiv \frac{df}{d\ln a} = \frac{1}{H} \frac{df}{dt}. \]

With these variables, we can cast the equations (13), (15) and (16), along with \( \rho_R \) and \( P_R \), in a dimensionless form, to obtain the following set of equations:
\[ X' = X(2 - 2X^2 - Z^2), \quad \text{(45)} \]
\[ Y' = -Y(2X^2 + Z^2) - \frac{\kappa}{2} XZ, \]
\[ Z' = Z(1 - 2X^2 - Z^2) + \frac{\kappa}{2} XY, \]
\[ R' = -2R(2X^2 + Z^2), \]

where, \( \kappa = \sqrt{\frac{12\beta}{\lambda}} = \sqrt{3}M_P M_S / \sqrt{2\pi \rho_V}. \)

We studied the dynamical equations numerically from the beginning of radiation dominated era (\( \ln a = -29 \)) till today (\( \ln a = 0 \)). The results are presented in Fig. [1]. In the graphs we only show results for the range \( \ln a = [-14, 0] \), as radiation is the only dominant component in the omitted regions. The plots show the results for three values of \( \kappa = 50, 200, 500 \). The initial conditions for these three cases have been chosen so as to match the final observed values of \( \Omega_M \) and \( \Omega_{\Lambda} \) [75–78].

As is evident from the plots, varying \( \kappa \) varies the frequency of oscillations. Besides that the results are almost identical, as long as \( \kappa >> 1 \). This can be understood from the expression of \( \kappa \). For fixed values of \( M_P \) and \( \rho_V \), increasing \( \kappa \) increases \( \omega \) or \( M_S \), which implies more rapid oscillations. Furthermore as seen from our analytic results, applicable when radiation energy density is negligible, we reproduce the standard \( \Lambda CDM \) model in the large \( \kappa \) limit.
Figure 1: The ratio of energy density to the critical energy density, $\Omega_i$, for different components as a function of $\ln(a)$ for $\kappa = 50, 200, 500$. 
5 Conclusions

We have analyzed a locally scale invariant generalization of Einstein’s gravity. The theory requires introduction of a scalar and a vector field. The scale invariance in the theory is broken by a recently introduced mechanism called the cosmological symmetry breaking. We have shown that this theory naturally leads to both dark energy and dark matter. Due to scale invariance the cosmological constant term is absent in the action. The solutions to the equations of motion admit a constant, non-zero value of the scalar field, which leads to a small cosmological constant or dark energy. The cold dark matter arises in the form of vacuum oscillations of the vector field. We have shown that the theory behaves very similar to the $\Lambda CDM$ model with negligible corrections. The precise values of the energy densities of different components are fixed by the initial conditions. Some of the parameters in the model take very small values and it is necessary to find an explanation for such small values. Furthermore it is important to compute quantum corrections in this model since that will determine whether the model suffers from fine tuning problems. The model can be generalized to include the standard model fields. The scalar field may then be identified with the Higgs multiplet. In this case the Higgs particle is absent from the particle spectrum and hence provides a very interesting test of the model. Alternatively the scalar field might be identified with a GUT scalar field multiplet. This possibility has so far not been studied in the literature.

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