CATEGORIFICATION OF TENSOR POWERS OF THE VECTOR
REPRESENTATION OF $U_q(\mathfrak{gl}(1|1))$

ANTONIO SARTORI

Abstract. We consider the monoidal subcategory of finite dimensional representations of $U_q(\mathfrak{gl}(1|1))$ generated by the vector representation, and we provide a diagram calculus for the intertwining operators. We construct a categorification using subquotient categories of the BGG category $O(\mathfrak{gl}_n)$. Using Soergel’s functor $V$ we compute the Soergel modules corresponding to the indecomposable projective modules in these subcategories and the homomorphism spaces between them; hence we describe the regular blocks of these categories as modules over some explicit diagram algebras. We construct diagrammatically standard and proper standard modules for the proper stratified structure of these algebras. Finally, we prove that they have an interesting connection with the geometry of Springer fibres associated with hook partitions.

Contents

1. Introduction

2. Categorification of $\mathfrak{gl}(m|n)$-representations
   2.1. Super Howe duality
   2.2. Categorification of $\mathfrak{gl}(m|n)$

Part I: Representation theory of $U_q(\mathfrak{gl}(1|1))$

3. The Hecke algebra and Hecke modules
   3.1. Hecke algebra
   3.2. Induced Hecke modules

4. Representations of $U_q(\mathfrak{gl}(1|1))$
   4.1. The quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$
   4.2. Representations of $U_q(\mathfrak{gl}(1|1))$

5. A semisimple subcategory of $U_q(\mathfrak{gl}(1|1))$-representations
   5.1. Representations $V(\alpha)$
   5.2. Super Schur-Weyl duality for $V^{\otimes n}$
   5.3. Diagrams for the intertwining operators

Part II: Categorification of $U_q(\mathfrak{gl}(1|1))$-representations

6. Subquotient categories of $O$
   6.1. Serre quotients and projectively presented modules
   6.2. Subquotient categories of $O$
   6.3. The parabolic categories of $p$-presentable modules
   6.4. Functors between categories $O^p_{\lambda,q}$-pres

Date: May 28, 2013.
Key words and phrases. Categorification, Lie superalgebras, Category $O$, induced Hecke modules, Soergel modules, Springer fibre, hook partition.
This work has been supported by the Graduiertenkolleg 1150, funded by the Deutsche Forschungsgemeinschaft.
7. The categorification
   7.1. Combinatorics
   7.2. The Grothendieck group
   7.3. Categorification of the intertwiners
   7.4. Canonical basis
   7.5. The bilinear form
   7.6. Categorification of the action of \( U_q(\mathfrak{gl}(1|1)) \)

Part III: A diagram algebra from Soergel modules

8. Symmetric polynomials
   8.1. Complete symmetric polynomials
   8.2. Ideals generated by complete symmetric polynomials
   8.3. Morphisms between quotient rings
   8.4. Duality
   8.5. Schubert polynomials

9. Some canonical bases elements
   9.1. Combinatorics
   9.2. Canonical basis elements

10. Soergel modules
    10.1. Soergel’s theorems
    10.2. Some Soergel modules
    10.3. Morphisms between Soergel modules
    10.4. Grading

11. A diagram algebra for \( \Omega_k(\mathfrak{g}) \)
    11.1. Diagrams
    11.2. The algebra structure
    11.3. Graded cellular structure
    11.4. The properly stratified structure
    11.5. A bilinear form and self-dual projective modules
    11.6. Diagrammatical versions of the functors \( \mathcal{E}_k \) and \( \mathcal{F}_k \)

12. Cohomology of the Springer fibre
    12.1. Fixed points and stable manifolds
    12.2. The cohomology rings

References

1. Introduction

  The Jones polynomial is a classical invariant of links in the three-dimensional space defined using the vector representation of the Lie algebra \( \mathfrak{sl}_2 \) (or more precisely of the quantum algebra \( U_q(\mathfrak{sl}_2) \)). In his fundamental paper [Kho00], Khovanov constructed a graded homology theory for links whose graded Euler characteristic is the Jones polynomial. Khovanov homology has two main advantages over the Jones polynomial: first, it has been proven to be a finer invariant and second, it has values in a category of complexes and it also assigns to cobordisms between links chain maps between chain complexes. This categorical approach to classical invariants is often called categorification. Khovanov’s work raised great interest in categorification, and since then a categorification program for general representations of more general semisimple Lie algebras and even Kac-Moody algebras has been developed.
by several authors and motivated various generalizations (see for example [FKS06, MS09, Web10, KL09, Ron08]). The main tools in all these works come from representation theory and geometry related to it.

Another very important invariant of knots is the Alexander polynomial, which is much older than the Jones polynomial. Originally defined using the topology of the knot complement, the Alexander polynomial is not the quantum invariant corresponding to some semisimple Lie algebra, like the Jones polynomial. Instead, it can be defined using the representation theory of the general Lie superalgebra corresponding to some semisimple Lie algebra, like the Jones polynomial. (or, more precisely, its quantum enveloping superalgebra [Vir06, Sar13]; alternatively, one can use the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ where $q$ is a root of unity, but we will not consider this approach). A categorification of the Alexander polynomial exists, but comes from a very different area of mathematics: a homology theory, known as Heegard-Floer homology, whose Euler characteristic gives the Alexander polynomial, has been developed using symplectic geometry ([OS05], [MOST07]). This homology theory, however, does not have an interpretation or a counterpart in representation theory yet.

Our work is motivated by the attempt to construct/understand categorifications of Lie superalgebras (and hopefully a categorification of the Alexander polynomial) using tools from representation theory. In fact, there are only a few other recent papers studying representation theoretical categorifications of Lie superalgebras and related structures ([Kho10], [FL13]). In particular, we hope to start a categorification program for $\mathfrak{gl}(1|1)$ beginning with a categorification of tensor powers of the vector representation and of their subrepresentations. We point out that a counterpart of our construction in the symplectic geometry setting has been developed by Tian [Tia12, Tia13]. Our main result can be summarized as follows:

**Theorem.** Let $V$ be the vector representation of $U_q(\mathfrak{gl}(1|1))$, fix $n > 0$ and consider the commuting actions of $U_q(\mathfrak{gl}(1|1))$ and of the Hecke algebra $H_n = H(S_n)$ on $V^{\otimes n}$:

$$U_q(\mathfrak{gl}(1|1)) \otimes V^{\otimes n} \otimes H_n.$$  

For each $n > 0$ there exists a triangulated category $\mathcal{D}^{\mathbb{C}}(\mathcal{O}(n))$ and two families of endofunctors $\{E, F\}$ and $\{E_i, i = 1, \ldots, n-1\}$ which commute with each other and which on the Grothendieck group level give the actions $\mathfrak{gl}(1|1)$ and of the Hecke algebra $H_n$ on $V^{\otimes n}$ respectively:

$$[E_i], [F] \otimes K^C(\mathcal{D}^{\mathbb{C}}(\mathcal{O}(n))) \otimes [E_i].$$

A remarkable property of the finite dimensional representations of $\mathfrak{gl}(1|1)$ (and more in general of $\mathfrak{gl}(m|n)$) is that they need not be semisimple. For example, if $V$ is the vector representation of $\mathfrak{gl}(1|1)$, then $V \otimes V^*$ is a four-dimensional indecomposable non irreducible representation. It is not clear how the lack of semisimplicity should affect the categorification, but it is plausible that this provides additional difficulties. What we can categorify in the present work is indeed only a semisimple monoidal subcategory of the representations of $\mathfrak{gl}(1|1)$, that contains the vector representation $V$, but not its dual $V^*$. We remark that we will develop all the details for the quantum version, but in order to keep this introduction clean we avoid to introduce the quantum enveloping algebra now.

Our categorification relies on a very careful analysis of the representation theory of $\mathfrak{gl}(1|1)$ and its canonical basis (see also [Zha09a]). In the categorification, indecomposable projective modules correspond to canonical basis elements, that we can compute explicitly via a diagram calculus, analogous to the diagram calculus developed in [FK07] for $\mathfrak{sl}_2$. The key-tool for our construction is the so called super Schur-Weyl duality $\mathfrak{S}^2$: the symmetric group algebra $\mathbb{C}[S_n]$ acts on the tensor power $V^{\otimes n}$, and this action commutes with the action of $\mathfrak{gl}(1|1)$. Considered as
\[ C[S_n] \text{-modules, the weight spaces of } V^\otimes n \text{ are induced modules of type } (\dagger) \]
\[
(\text{tr}_S \otimes \text{sgn}_{S_{n-k}}) \otimes C[S_k \times S_{n-k}] C[S_n].
\]

An important point is the following observation:

**Theorem** (See Proposition 5.7). Lusztig’s canonical basis of \( V^\otimes n \), defined using the action of \( \mathfrak{gl}(1|1) \), agrees with the canonical basis defined in term of the symmetric group action.

This Schur-Weyl duality is strictly related to a version of super Howe duality that connects representations of \( \mathfrak{gl}(1|1) \), or more generally \( \mathfrak{gl}(m|n) \), with representations of \( \mathfrak{gl}_N \) [CW01]. In fact, the whole categorification process we develop works more generally for tensor powers of the vector representation of \( \mathfrak{gl}(m|n) \). We will sketch the main ideas for the general case in Section 2 while in the rest of the paper we will restrict to \( \mathfrak{gl}(1|1) \) and work out the details in this case; this turns out to be already very laborious. To develop the \( \mathfrak{gl}(1|1) \)-categorification theory we will use super Schur-Weyl duality instead of Howe duality, and hence reduce the problem to symmetric group categorification. The two approaches are equivalent, but we personally prefer to work out the detail based on the first one.

The fundamental tool used in our construction is the BGG category \( \mathcal{O} \) (cf. [Hum08]), which plays already an important role in many other representation theoretical categorifications. In particular, we will construct a categorification of tensor powers of \( V \) and of their subrepresentations using some subquotient categories of \( \mathcal{O}(\mathfrak{gl}_n) \). These categories are build in two steps: first one takes a parabolic subcategory and then a “\( q \)-presentable” quotient; the two steps can be reversed, and one gets the same result. The process is sketched by the following picture, which is also helpful to remember how we index our categories:

We will give the precise definitions and discuss the technical details in Section 6.

The construction of these subquotient categories is motivated by the following. In general, a semisimple module \( M \) is usually categorified via some abelian category \( \mathcal{C} \), but this category \( \mathcal{C} \) does not decompose into blocks according to the decomposition of \( M \) into summands; this is indeed one of the main points of the categorification: \( \mathcal{C} \) is supposed to have more structure than \( M \). Usually the submodules generated by canonical basis elements in \( M \) give a filtration of \( M \) (but not a decomposition!), that corresponds to the filtration of \( \mathcal{C} \). This principle has been applied in [MS08] to categorify induced modules for the symmetric group: the category \( \mathcal{O}_0(\mathfrak{gl}_n) \) is well-known to be a categorification of the regular representation of the symmetric group \( S_n \); these induced modules for the symmetric group are direct summands of the regular representation of \( C[S_n] \); hence they can be categorified via subquotient categories of \( \mathcal{O}_0(\mathfrak{gl}_n) \).

In particular, [MS08] provide some categories, which we denote by \( \mathcal{Q}_k(n) \), categorifying the induced modules \( (\dagger) \), and define on them a categorical action of \( C[S_n] \)
CATEGORIZATION OF REPRESENTATIONS OF $U_q(\mathfrak{gl}(1|1))$

using translation functors. To categorify $V^\otimes n$ we sum up these categories $\mathcal{Q}_k(n)$ for $k = 0, \ldots, n$. In addition, we consider also the corresponding singular blocks $\mathcal{Q}_k(a)$ of the same subquotients categories; note that singular blocks do not appear in [MS08] since they do not provide categorifications of $\mathbb{C}[S_n]$-modules; in our picture, they categorify subrepresentations of $V^\otimes n$. Translation functors of category $O(gl_n)$ restrict to all these categories and categorify the action of the intertwining operators of the $gl(1|1)$-action.

We remark that the categories $\mathcal{Q}_k(a)$ have a natural grading (inherited from the Koszul grading on $O(gl_n)$) and all the functors we consider are actually graded functors between these categories. As a result, the categorification lifts to a categorification of representations of the quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$.

We will work out all the details in the graded setting.

What is left to complete the picture is to define functors that categorify the action of $gl(1|1)$ itself. There is a natural way to define adjoint functors $E$ and $F$ between $\mathcal{Q}_k(a)$ and $\mathcal{Q}_{k+1}(a)$, which portend to categorify the action of the generators $E$ and $F$ of $U(gl(1|1))$. Although $E$ is exact, $F$ is only right exact in general, and we need to derive our categories and functors in order to have an action on the Grothendieck groups. However, the following problem arises. The categories we consider are equivalent to categories of modules over some finite dimensional algebras. Unfortunately, these algebras are not always quasi-hereditary; in general they are only properly stratified (the definition of standardly and properly stratified algebras has been modeled to describe the properties of some generalized parabolic subcategories of $O$, introduced by [FKM02], that include as particular cases the categories that we consider). A properly stratified algebra does not have in general finite global dimension (this happens if and only if the algebra is quasi-hereditary). As a consequence, finite projective resolutions do not always exist, and we are forced to consider unbounded derived categories. But the Grothendieck groups of these unbounded derived categories vanish, [Miy06]. A workaround to this problem has been developed in [AS13], using the additional structure of a mixed Hodge structure, which in our case is given by the grading. Given a graded abelian category, [AS13] define a proper subcategory of the left unbounded derived category of graded modules; this subcategory is big enough to contain projective resolutions, but small enough to prevent the Grothendieck group to vanish. We describe in detail how the categories we consider and the functors $E$ and $F$ can be derived using these techniques.

Of course at this point one would like to understand and describe these categories $\mathcal{Q}_k(n)$ explicitly. Very surprisingly (at least for us), this is indeed possible. To give an idea, let us present the categorification of $V^\otimes 2$. We let $R = \mathbb{C}[x]/(x^2)$ and $A = \text{End}_\mathbb{C}(\mathbb{C} \oplus R)$. The algebra $A$ can be identified with the path algebra of the quiver

$$
\begin{align*}
\begin{array}{c}
1 \quad 2
\end{array}
\end{align*}
\begin{array}{c}
\overset{a}{\rightarrow}
\overset{b}{\rightarrow}
\end{array}
$$

with the relation $ba = 0$.

We indicate by $e_1$ and $e_2$ the two idempotents corresponding to the vertices of the quiver. Let us identify $\mathbb{C}$ with $A/e_1 A$ and notice that $\mathbb{C}$ becomes then naturally an $(A, \mathbb{C})$-bimodule. Moreover, notice that $R$ is naturally isomorphic to the endomorphism ring of the projective module $A e_2$, so that we can consider $A e_2$ as an $(A, R)$-bimodule. The categorification of $V^\otimes 2$ is then given by the following picture:
where \( p = \mathfrak{gl}_2 \) is the trivial parabolic subalgebra. This should be compared with the standard categorification of \( W^\otimes 2 \) (see [FKS06]), where \( W \) is the vector representation of \( \mathfrak{sl}_2 \):

\[
\begin{array}{ccc}
\mathcal{O}_0^p(\mathfrak{gl}_2) & \mathcal{O}_0(\mathfrak{gl}_2) & \mathcal{O}_0^p(\mathfrak{gl}_2) \\
\mathbb{C}\text{-mod} & \mathbb{C} \otimes - & \mathbb{C} \otimes - \\
\text{Hom}_\lambda(\mathfrak{A}_\lambda, -) & \text{Hom}_\lambda(\mathfrak{A}_\lambda, -) & \text{Hom}_\lambda(\mathfrak{A}_\lambda, -) \\
\mathfrak{A}_\lambda \otimes R - & \mathfrak{A}_\lambda \otimes R - & \mathfrak{A}_\lambda \otimes R - \\
\mathcal{C}\text{-mod} & \mathcal{C}\text{-mod} & \mathcal{C}\text{-mod}
\end{array}
\]

In particular, note that the first and the second leftmost weight spaces are categorized in the same way. This will hold for all tensor powers \( V^\otimes n \) and \( W^\otimes n \) and is due to the fact that these weight spaces for \( \mathfrak{gl}(1|1) \) and for \( \mathfrak{sl}_2 \) are the same as modules for the symmetric group. The second leftmost weight space, in particular, is categorized using the well-known category of modules over the path algebra of the Khovanov-Seidel quiver [KS02].

One should however note the remarkable difference in the rightmost weight space of our example. Here our categorification differs from the \( \mathfrak{sl}_2 \) picture and leaves the world of highest weight categories.

In general, the description of our categories is slightly more involved, but still explicit. We will develop the instruments for that in Part [III] where we will compute the endomorphism algebras of the projective generators using Soergel’s functor \( \mathcal{V} \) and Soergel modules ([Soe90]; for the importance of Soergel bimodules see also [Kho07] and [EW12]). For this, we restrict for simplicity to consider only the regular blocks \( \mathcal{Q}_k(n) \), although we believe the same process can be applied more generally to singular blocks. We determine the Soergel modules \( \mathcal{V}P(w \cdot 0) \) corresponding to indecomposable projective modules in category \( \mathcal{O}(\mathfrak{gl}_n) \), where \( w \) is in some subset \( D \) of \( S_n \) consisting of shortest/longest coset representatives. We compute then the homomorphisms spaces \( \text{Hom}(\mathcal{V}P(w \cdot 0), \mathcal{V}P(w' \cdot 0)) \) and the subspaces of morphisms that factor through some \( \mathcal{V}P(z \cdot 0) \) for \( z \notin D \); the quotient of the former by the latter gives the homomorphism space between the corresponding parabolic projective modules in the parabolic category \( \mathcal{O}(\mathfrak{gl}_n) \). By construction, we get in this way the endomorphism algebra \( \mathcal{A}_k \) of a projective generator of \( \mathcal{Q}_k(n) \). As far as we know, this is the first work in which the Soergel functor is used to compute explicit endomorphism algebras corresponding to blocks of subquotient categories of \( \mathcal{O} \). The crucial point that makes our computation work is the fact that the Soergel modules we consider are cyclic. This is equivalent to the corresponding Schubert varieties being rationally smooth (cf. [Str09]). In some sense, what we consider is the maximal subset of the symmetric group such that the corresponding Schubert varieties are all rationally smooth (cf. [GR02]).

We provide then a diagrammatical description of this algebra \( \mathcal{A}_k \) and we reprove in purely elementary terms the fact, known from Lie theory, that \( \mathcal{A}_k \) is cellular and properly stratified, by explicitly constructing standard and proper standard.
modules. As a byproduct, we can describe the functors $E$ and $F$ as bimodules and compute their endomorphism rings, proving that they are indecomposable. We remark that one could expect an action of a KLR algebra on powers of $E$ and $F$. However, notice that since $E^2 = 0$ and $F^2 = 0$ it does not make sense to investigate the endomorphism spaces $\text{End}(E^k)$ and $\text{End}(F^k)$ for $k > 1$. At the moment it is not clear to us how one could get a 2-categorification for $gl(1|1)$-representations.

The Soergel functor and Soergel modules interplay the category $O(gl_n)$ with the cohomology of the flag variety. In our case, since the category $Q_k(n)$ is a quotient of the parabolic category $O_p(gl_n)$, where $p$ corresponds to a composition of $n$ of type $(n-k,1,\ldots,1)$, one expects a connection with the cohomology of the Springer fibre of hook type sitting inside the full flag variety. Mimicking [SW12], we compute the cohomology rings of the closed attracting cells of this Springer fibre for an action of the torus and we prove that they are isomorphic to the endomorphism rings of the indecomposable projective modules of our categories $Q_k(n)$. It should be possible to construct a convolution product on these cohomology rings as in [SW12] so that we recover the full algebra $A_k$. We believe that this interpretation could be used to establish a connection with the approach of Tian ([Tia12], [Tia13]).

Outline of the paper. After this introduction, the paper contains a general section on super Howe duality and applications to categorification. In this section we recall the statement of super Howe duality and we show how it can be used to deduce a categorification of $gl(m|n)$-representations from a categorification of $gl_k$-representations.

The rest of the paper is concerned with $gl(1|1)$-categorification and is divided into three parts. In Part I we describe in detail properties of the quantum superalgebra $U_q(gl(1|1))$ and of its representation theory. We restrict then to a semisimple subcategory $\text{Rep}$ of representations. The main achievement of this part is the construction in §5.3 of a diagram calculus for the intertwining operators using webs, similar to the $sl_2$-diagram calculus of [FK97]. In particular, we define a diagrammatical category $\text{Web}$ and a full functor $T : \text{Web} \rightarrow \text{Rep}$.

This allows to compute explicitly the canonical bases and the action of $U_q(gl(1|1))$. In Part II we introduce the interesting subquotient categories $Q_k(\alpha)$ of $O(gl_n)$. We study in detail their properties and the action of the functors between them. We then show how they can be used to construct a categorification of the representations in $\text{Rep}$, defining a functor $\mathcal{F} : \text{Web} \rightarrow O\text{Cat}$, where $O\text{Cat}$ is a category containing all our categories $Q_k(\alpha)$. The main result of this part is the following:

Theorem (See Theorem 7.8 and Theorem 7.10). We have a commuting diagram:

$$
\begin{array}{ccc}
\text{Web} & \xrightarrow{\mathcal{F}} & O\text{Cat} \\
\downarrow{\mathcal{F}} & & \downarrow{K_0} \\
\text{Rep} & \xrightarrow{T} & \\
\end{array}
$$

At least on the level of derived categories, the $U_q(gl(1|1))$-action on representations in $\text{Rep}$ can be lifted to an action of functors on the corresponding categories $Q(\alpha)$.

We refer to Section 7 for all the details.

Part III can be read almost independently from the first two parts. After some commutative algebra preliminaries (Section 8) we compute the Soergel modules corresponding to projective modules $P(w_k w \cdot 0)$ in $O(gl_n)$, where $w$ is a shortest coset representative for $(S_k \times S_{n-k}) \backslash S_n$ and $w_k \in S_k$ is the longest element. We describe
the space of homomorphisms between them, and we are able to identify which morphisms get annihilated when restricted to the parabolic subcategory $\mathcal{O}^p(\mathfrak{g}_n)$, where $p$ is a parabolic subalgebra with only one non-trivial block (cf. Theorem 10.17).

Using these homomorphism spaces, we construct diagram algebras $A_{n,k}$; the main result of this part is the following:

**Theorem** (See Theorem 11.12). The category of finitely generated modules over $A_{n,k}$ is equivalent to $\mathcal{Q}_k(\mathfrak{g})$.

Moreover, we construct diagrammatically indecomposable projective, standard and proper standard modules, describing explicitly the properly stratified structure of $A_{n,k}$. Finally, in Section 12 we consider the relation with the cohomology of the Springer fibre, proving that the endomorphism rings of indecomposable projective $A_{n,k}$-modules are isomorphic to the cohomology rings of some generalized Schubert varieties in the Springer fibre of hook type.

**Acknowledgements.** The present paper is part of the author’s PhD thesis. The author would like to thank his advisor Catharina Stroppel for her help and support.

2. Categorification of $\mathfrak{gl}(m|\ell)$-representations

In this introductory section we present a categorification result for tensor products of the vector representation of the general Lie superalgebra $\mathfrak{gl}(m|\ell)$, which can be obtained from a categorification of $\mathfrak{gl}_n$-representations using a version of Howe duality. The details will be worked out in the rest of the paper in the particular case of $\mathfrak{gl}(1|1)$. Although in the rest of the paper we compute explicitly in the quantized case, we prefer to avoid to introduce the quantum enveloping algebra of $\mathfrak{gl}(m|\ell)$ in this section. Hence we will treat the classical (non-quantized) case, although the whole section can be translated for quantum enveloping algebras.

2.1. Super Howe duality. Let $I_{m|\ell} = \{1, \ldots, m + \ell\}$ with a parity function $|\cdot| : I_{m|\ell} \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$
|i| = \begin{cases} 
0 & \text{if } i \leq m \\
1 & \text{if } i > m 
\end{cases}
$$

for each $i \in I_{m|\ell}$. Let also $\mathbb{C}^{m|\ell}$ be a $m + \ell$ dimensional super vector space with basis $\{e_i \mid i \in I_{m|\ell}\}$ such that $|e_i| = |i|$, where as usual $|v|$ denotes the degree of an homogeneous element $v \in \mathbb{C}^{m|\ell}$. Then the Lie superalgebra $\mathfrak{gl}(m|\ell)$ is the super vector space of matrices $\text{End}(\mathbb{C}^{m|\ell})$ equipped with the Lie super bracket

$$
[x, y] = xy - (-1)^{|x||y|}yx.
$$

In particular note that $\mathfrak{gl}(m|0) = \mathfrak{gl}(0|m) = \mathfrak{gl}_m$. The Lie superalgebra $\mathfrak{gl}(m|\ell)$ acts by matrix multiplication on $\mathbb{C}^{m|\ell}$; this is the vector representation of $\mathfrak{gl}(m|\ell)$.

If $V$ is a super vector space, we define an action of the symmetric group $S_N$ on the tensor power $\otimes^N V$ by setting

$$
\sigma V(x_1 \otimes \cdots \otimes x_N) = (-1)^{|x_{\ell+1}|x_1 \otimes \cdots \otimes x_{\ell+1} \otimes x_1 \otimes \cdots \otimes x_N}
$$

for every simple reflection $\sigma \in S_N$. Let $\pi^S, \pi^\Lambda \in \mathbb{C}[S_n]$ be the idempotents projecting onto the trivial and sign representations respectively. We set then

$$
\pi^S V = \pi^S \cdot (\otimes^N V) \quad \text{and} \quad \pi^\Lambda V = \pi^\Lambda \cdot (\otimes^N V).
$$

In particular, notice that if $V$ is a vector space (i.e. it is concentrated in zero degree) then these definitions coincide with the usual symmetric and exterior powers of $V$. 

If \( v_1, \ldots, v_r \) is a basis of \( V \), then a basis of \( \Lambda^N V \) is given by
\[
(2.5) \quad v_1 \wedge \cdots \wedge v_N = \pi^N \cdot (v_1 \otimes \cdots \otimes v_N)
\]
for all sequences \( (i_1, \ldots, i_N) \) of indices \( i \in \{1, \ldots, r\} \) such that \( i_1 \leq i_2 \leq \cdots \leq i_N \) and if \( i_\ell = i_{\ell+1} \) then \( |v_{i_\ell}| = 1 \). Moreover a basis of \( S^N V \) is given by
\[
(2.6) \quad v_1 \otimes \cdots \otimes v_N = \pi^S \cdot (v_1 \otimes \cdots \otimes v_N)
\]
for all sequences \( (i_1, \ldots, i_N) \) of indices \( i \in \{1, \ldots, r\} \) such that \( i_1 \leq i_2 \leq \cdots \leq i_N \) and if \( i_\ell = i_{\ell+1} \) then \( |v_{i_\ell}| = 0 \).

We have the following result (cf. [CW01, CW10]):

**Proposition 2.1** (Super Howe duality). Let \( p, m, N \in \mathbb{Z}_{>0} \) be positive integers and \( q, n \in \mathbb{Z}_{\geq 0} \). The natural actions of \( \mathfrak{gl}(p|q) \) and \( \mathfrak{gl}(m|n) \) on \( \Lambda^N (\mathbb{C}^p \otimes \mathbb{C}^m) \) commute with each other and generate each other’s centralizer. As a \( \mathfrak{gl}(m|n) \)-module, \( \Lambda^N (\mathbb{C}^p \otimes \mathbb{C}^m) \) decomposes as the direct sum
\[
(2.7) \quad \bigoplus_{i_1 + \cdots + i_{p+q} = N} \Lambda^{i_1} \mathbb{C}^m \otimes \cdots \otimes \Lambda^{i_{p+q}} \mathbb{C}^m \otimes S^{i_{p+1}} \mathbb{C}^m \otimes \cdots \otimes S^{i_{p+q}} \mathbb{C}^m.
\]

Note that inverting the roles of \( p|q \) and \( m|n \) we have a similar decomposition \( (2.7) \) as a \( \mathfrak{gl}(p|q) \)-module.

**Proof.** The first part is [CW01] Theorem 3.3 and Corollary 3.2. We check the decomposition \( (2.7) \).

Let \( \{e_1, \ldots, e_{p+q}\} \) and \( \{f_1, \ldots, f_{m+n}\} \) be the standard bases of \( \mathbb{C}^p \) and \( \mathbb{C}^m \) respectively. We fix the following ordered basis of \( \mathbb{C}^p \otimes \mathbb{C}^m \):
\[
(2.8) \quad e_1 \otimes f_1, \ldots, e_1 \otimes f_{m+n}, \ldots, e_{p+q} \otimes f_1, \ldots, e_{p+q} \otimes f_{m+n}.
\]

We then get a basis of \( \Lambda^N (\mathbb{C}^p \otimes \mathbb{C}^m) \) as in \( (2.5) \). Let \( M \) be equal to \( (2.7) \). We define an isomorphism \( \Psi \) from \( \Lambda^N (\mathbb{C}^p \otimes \mathbb{C}^m) \) to \( \mathfrak{gl}(p|q) \) in the following way. Given a basis vector \( w = (e_{i_1} \otimes f_{j_1}) \wedge \cdots \wedge (e_{i_N} \otimes f_{j_N}) \) of \( \Lambda^N (\mathbb{C}^p \otimes \mathbb{C}^m) \), define functions \( a, b : \{1, \ldots, p+q\} \to \{1, \ldots, N\} \) by \( a(h) = \min \{\ell \mid i_\ell = h\} \) and \( b(h) = \max \{\ell \mid i_\ell = h\} \) or \( a(h) = b(h) = \bullet \) if this set is empty. Set also \( c(h) = b(h) - a(h) + 1 \), with the convention \( \bullet - \bullet = -1 \). Then we define
\[
(2.9) \quad \Psi(w) = \Lambda^{c(1)} \mathbb{C}^m \otimes \cdots \otimes \Lambda^{c(p+1)} \mathbb{C}^m \otimes S^{c(q)} \mathbb{C}^m
\]
to be the element
\[
(2.10) \quad (f_{j_{a(1)}} \wedge \cdots \wedge f_{j_{a(m)}}) \otimes \cdots \otimes (f_{j_{a(m+1)}} \wedge \cdots \wedge f_{j_{a(m+n)}}) \otimes (f_{j_{b(m+1)}} \wedge \cdots \wedge f_{j_{b(m+n)}}) \otimes \cdots \otimes (f_{j_{b(m+1)}} \wedge \cdots \wedge f_{j_{b(m+n)}}).
\]
It is straightforward to check that this is indeed an element of the basis, and that \( \Psi \) is bijective and \( \mathfrak{gl}(m|n) \)-equivariant. \( \square \)

### 2.2. Categorification of \( \mathfrak{gl}(m|n) \)

Set now \( V = \mathbb{C}^m \). Our goal is to construct a categorification of \( V^\otimes N \) for \( N > 0 \).

Set \( p = N \) and \( q = 0 \) in Proposition 2.1. We have then that \( \Lambda^N (\mathbb{C}^N \otimes V) \) decomposes as a \( \mathfrak{gl}(m|n) \)-module as
\[
(2.11) \quad \bigoplus_{i_1 + \cdots + i_N = N} \Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_N} V
\]
and as a \( \mathfrak{gl}_N \)-module as
\[
(2.12) \quad \bigoplus_{j_1 + \cdots + j_{m+n} = N} \Lambda^{j_1} \mathbb{C}^N \otimes \cdots \otimes \Lambda^{j_m} \mathbb{C}^N \otimes S^{j_{m+1}} \mathbb{C}^N \otimes \cdots \otimes S^{j_{m+n}} \mathbb{C}^N.
\]
Notice that one summand of (2.11) is in particular $V^\otimes N$. A categorification of the $\mathfrak{gl}_N$-module (2.12), although not written in the literature, is in principle known to experts, and is what we are going to use to categorify the $\mathfrak{gl}(m|n)$-module (2.11).

Remark 2.2. Another kind of duality, called super Schur-Weyl duality, relates $\mathfrak{gl}(m|n)$ and the symmetric group $S_N$: the natural action of $\mathbb{C}[S_N]$ on $V^\otimes N$ is $\mathfrak{gl}(m|n)$-equivariant; moreover, the map $\mathbb{C}[S_N] \to \text{End}_{\mathfrak{gl}(m|n)}(V^\otimes N)$ is always surjective, and it is injective if and only if $N \leq (m+1)(n+1)$ (see [BR87, Ser84]).

In order to state the categorification theorem, we need some notation. Let us fix the standard basis $\{v_1, \ldots, v_{m+n}\}$ of $V = \mathbb{C}^{m+n}$, with

$$|v_i| = \begin{cases} 0 & \text{for } i = 1, \ldots, m, \\ 1 & \text{for } i = m+1, \ldots, n. \end{cases}$$

Let $\mathfrak{h} \subset \mathfrak{gl}(m|n)$ be the subalgebra of diagonal matrices. Then $V^\otimes N$ decomposes as direct sum of weight spaces for the action of $\mathfrak{h}$. Let $\Lambda$ be the set of compositions $\lambda = (\lambda_1, \ldots, \lambda_{m+n})$ of $N$ with at most $m+n$ parts (that is, we allow $\lambda_i = 0$ for some indices $i$). Then the weight spaces of $V^\otimes N$ are indexed by $\Lambda$, and the correspondence is given by

$$(V^\otimes N)_{\lambda} = \text{span}\{v_{\sigma(a_1)} \otimes \cdots \otimes v_{\sigma(a_\lambda)} | \sigma \in S_N\},$$

where

$$a^\lambda = (1, \ldots, 1, 2, \ldots, 2, \ldots, m+n, \ldots, m+n).$$

We can now state our main result:

**Theorem 2.3.** Given $\lambda \in \Lambda$, let $\mathfrak{q}_{\lambda} \subset \mathfrak{gl}_N$ be the standard parabolic subalgebra corresponding to the composition $(\lambda_1, \ldots, \lambda_m, 1, \ldots, 1)$ and $\mathfrak{p}_{\lambda} \subset \mathfrak{gl}_N$ be the standard parabolic subalgebra corresponding to the composition $(1, \ldots, 1, \lambda_{m+1}, \ldots, \lambda_{m+n})$. Then there is an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}_{0}^{\mathfrak{p}_{\lambda}, \mathfrak{q}_{\lambda}\text{-pres}}(\mathfrak{gl}_N)) \longrightarrow (V^\otimes N)_{\lambda}$$

sending equivalence classes of standard modules to standard basis vectors.

For each index $i = 1, \ldots, N-1$ choose a singular weight $\lambda_i$ for $\mathfrak{gl}_N$ whose stabilizer under the dot action is generated by the simple reflection $s_i$. Then defining $\theta_i = T_{\lambda_i}^0 \circ T_{\lambda_i}^N$ we get a categorical action of the generators $s_i + 1$ of $\mathbb{C}[S_N]$, which intertwines the $\mathfrak{gl}(m|n)$-action at the level of the Grothendieck group.

We refer to Section 5 for the definitions of the categories appearing in (2.16) and of the translation functors $T_{\lambda}^0$ and $T_{\lambda}^N$.

**Proof.** The first claim follows from the definition of the categories $\mathcal{O}_{0}^{\mathfrak{p}_{\lambda}, \mathfrak{q}_{\lambda}\text{-pres}}(\mathfrak{gl}_N)$ (cf. Section 5). The second claim can be proven generalizing the proof of Theorem 7.3. \qed

Combining Zuckermann’s/coapproximation functors and their adjoints (see $\S 6.4$ for the definitions) one can define functors $\mathcal{E}_j$, $\mathcal{F}_j$ for $j = 1, \ldots, m+n-1$ between some opportune unbounded derived categories, as in $\S 7.6$. These functors commute with the functors $\theta_i$ and give an action of $\mathfrak{gl}(m|n)$ at the level of the Grothendieck groups.

We remark that for $n = 0$ Theorem 2.3 gives exactly the categorification of $(\mathbb{C}^m)^\otimes N$ developed in [MS09].

In principle it is possible to work out the combinatorics of Theorem 2.3 in detail. However, a careful analysis would be quite long; moreover, a categorification of
(3.12), although known in principle, is not written in the literature. Hence, we will restrict ourselves in the rest of the paper to the case of $\mathfrak{gl}(1|1)$.

## Part I. Representation theory of $U_q(\mathfrak{gl}(1|1))$

This part is divided into three sections. In Section 3, which can be skipped at a first reading and used as a reference, we collect some results on the Hecke algebra and on induced sign/trivial modules to which we will refer in the rest of the paper. In Section 4 we will define the quantum enveloping algebra $U_q(\mathfrak{gl}(1|1))$ and its representation theory. In Section 5 we will study in detail the semisimple subcategory of $U_q(\mathfrak{gl}(1|1))$-representations that we will categorify in the rest of the paper.

### 3. The Hecke algebra and Hecke modules

In this section we recall the definition of the bar involution and the canonical basis of the Hecke algebra for the symmetric group $S_n$. We then study in detail induced sign/trivial modules.

#### 3.1. Hecke algebra

Let $S_n$ denote the symmetric group of permutations of $n$ elements, generated by the simple reflections $s_i$ for $i = 1, \ldots, n - 1$. For $w \in S_n$ we denote by $\ell(w)$ the length of $w$. Moreover, we denote by $<_\text{Bruhat}$ the Bruhat order on $S_n$.

The Hecke algebra of the symmetric group $W = S_n$ is the unital associative $\mathbb{C}(q)$-algebra $H_n$ generated by $\{H_i \mid i = 1, \ldots, n - 1\}$ with relations

\begin{align}
(3.1a) & \quad H_iH_j = H_jH_i \quad \text{if } |i - j| > 2, \\
(3.1b) & \quad H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}, \\
(3.1c) & \quad H_i^2 = (q^{-1} - q)H_i + 1.
\end{align}

It follows from (3.1a) that the elements $H_i$ are invertible with $H_i^{-1} = H_i + q - q^{-1}$. For $w \in S_n$ such that $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression, we define

\[ H_w = H_{i_1} \cdots H_{i_r}. \]

Thanks to (3.1a), this does not depend on the chosen reduced expression. The elements $H_w$ for $w \in W$ form a basis of $H_n$, called standard basis, and we have

\begin{equation}
H_wH_i = \begin{cases} 
H_{ws_i} & \text{if } \ell(ws_i) > \ell(w), \\
H_{ws_i} + (q^{-1} - q)H_w & \text{otherwise}.
\end{cases}
\end{equation}

We can define on $H_n$ a bar involution by $H_w^\dagger = H_w^{-1}$, and $\overline{H} = q^{-1}H - Hq^{-1}$. We also have a bilinear form $\langle - , - \rangle$ on $H_n$ such that the standard basis elements are orthonormal:

\begin{equation}
\langle H_w, H_{w'} \rangle = \delta_{w,w'} \quad \text{for all } w, w' \in W.
\end{equation}

By standard arguments one can prove the following:

**Proposition 3.1** ([KL79], in the normalization of [Soe97]). There exists a unique basis $\{H_w \mid w \in W\}$ of $H_n$ consisting of bar-invariant elements such that

\begin{equation}
H_w = H_w + \sum_{w' < w} P_{w',w}(q)H_{w'}
\end{equation}

with $P_{w',w} \in q\mathbb{Z}[q]$ for every $w' < w$.

The basis $H_w$ is called Kazhdan-Lusztig basis. We will also call it canonical basis of $H_n$. 

Remark 3.2. There is an inductive way to construct the canonical basis elements. First, note that $H_{w} = H_{\epsilon}$. Then set $C_{i} = H_{i} + q$: since $C_{i}$ is bar invariant, we must have $H_{w_{i}} = C_{i}$. Now suppose $w = w's_{i} \succ w'$: then $H_{w_{i}}C_{i}$ is bar invariant and is equal to $H_{w}$ plus a $\mathbb{Z}[q, q^{-1}]$-linear combination of some $H_{w''}$ for $w'' \prec w$. It follows that
\[
H_{w_{i}}C_{i} = H_{w} + p \quad \text{for some } p \in \bigoplus_{w'' \prec w} \mathbb{Z}H_{w''}.
\]

3.2. Induced Hecke modules. We will consider induced Hecke modules which are a mixed version of induced sign and induced trivial modules (see also \cite{Soe97}).

In the following, all modules over the Hecke algebra will be right modules.

Let $W_{p}, W_{q}$ be two parabolic subgroups of $W$ (that is, they are generated by simple reflections) such that the elements of $W_{p}$ commute with the elements of $W_{q}$. Note that $W_{p+q} = W_{p} \times W_{q}$ is also a parabolic subgroup of $W$. Let also $H_{p}, H_{q}$ be the corresponding Hecke algebras, and let $H_{p+q} = H_{p} \times H_{q}$. Let $sgn_{p}$ be the one dimensional sign module for $H_{p}$ (on which each generator $H_{i} \in H_{p}$ acts as $-q$), and let $trv_{q}$ be the one dimensional trivial module for $H_{q}$ (on which each generator $H_{i} \in H_{q}$ acts as $q^{-1}$). Consider the induced module
\[
\mathcal{M}_{q}^{p} = \text{Ind}_{H_{p+q}}^{H_{p}}(sgn_{p} \boxtimes trv_{q}) = (sgn_{p} \boxtimes trv_{q}) \otimes H_{p+q}H_{n}.
\]

If $W_{p}$ is trivial, we omit $p$ from the notation and we write $\mathcal{M}_{q}$. Analogously, if $W_{q}$ is trivial we omit $q$ and we write $\mathcal{M}_{p}$.

Let $W_{p}, W_{q}$ and $W_{p+q}$ be the set of shortest coset representatives for $W_{p} \setminus W$, $W_{q} \setminus W$ and $W_{p+q} \setminus W$ respectively. Then a basis of $\mathcal{M}_{q}^{p}$ is given by
\[
\{N_{w} = 1 \otimes H_{w} | w \in W_{p+q}\}
\]
(where 1 is some chosen generator of the $\mathbb{C}(q)$-vector space $sgn_{p} \boxtimes trv_{q}$).

The action of $H_{n}$ on $\mathcal{M}_{q}^{p}$ is given explicitly by the following lemma:

Lemma 3.3. For all $w \in W_{p+q}$ we have
\[
N_{w} \cdot H_{i} = \begin{cases} 
N_{ws_{i}} & \text{if } ws_{i} \in W_{p+q} \text{ and } \ell(ws_{i}) > \ell(w), \\
N_{ws_{i}} + (q^{-1} - q)H_{w} & \text{if } ws_{i} \in W_{p+q} \text{ and } \ell(ws_{i}) < \ell(w), \\
-qN_{w} & \text{if } ws_{i} = s_{j}w \text{ for } s_{j} \in W_{p}, \\
q^{-1}N_{w} & \text{if } ws_{i} = s_{j}w \text{ for } s_{j} \in W_{q}.
\end{cases}
\]

The module $\mathcal{M}_{q}^{p}$ inherits a bar involution by setting $\overline{N_{w}} = 1 \otimes \overline{H_{w}}$. Moreover, the bilinear form \[(3.3)\] induces a bilinear form on $\mathcal{M}_{q}^{p}$.

A canonical basis can be defined on $\mathcal{M}_{q}^{p}$ by the following generalization of Proposition 3.1.

Proposition 3.4. There exists a unique basis \[\{\overline{N}_{w} | w \in W_{p+q}\}\] of $\mathcal{M}_{q}^{p}$ consisting of bar-invariant elements satisfying
\[
\overline{N}_{w} = N_{w} + \sum_{w' \prec w} \mathcal{P}_{w',w}(q)N_{w'}
\]
with $\mathcal{P}_{w',w} \in q\mathbb{Z}[q]$ for every $w' \prec w$.

As described in Remark 3.2 one can construct inductively the canonical basis of $\mathcal{M}_{q}^{p}$. In particular, for $W_{p+q} \ni ws_{i} \succ w$ one always has
\[
\overline{N}_{w_{i}}C_{i} = \overline{N}_{ws_{i}} + p
\]
where $p$ is a $\mathbb{Z}$-linear combination of $\overline{N}_{w'}$ for $w' \prec ws_{i}$.

\footnote{We use this notation because they will correspond later to two parabolic subalgebras $p, q \subseteq gl_{n}$.}
Maps between Hecke modules (I). We will now construct maps between induced modules $\mathcal{M}_p^q$ corresponding to different pairs of parabolic subgroups $W_p, W_q$. First, we consider the case in which we change the subgroup $W_q$.

Let $W_q' \subseteq W_q$ be also a parabolic subgroup of $W$. Let us define a map $i = \iota_q' : \mathcal{M}_q^p \to \mathcal{M}_q'^p$ by

\[
i : N_w \mapsto \sum_{x \in W' \cap W_q} q^{l(w_w) - l(x)} N_{xw}
\]

where $w_w$ is the longest element of $W_q' \cap W_q = (W_q' \backslash W_q)^{\text{short}}$. Note that for $w \in W^{p+q}$ and $x \in W' \cap W_q$ the product $xw$ is an element of $W^{p+q}$.

The map (3.11) is natural, in the sense that if $W_q' \subseteq W_q$ is another subgroup of $W$ generated by simple reflections then $\iota_q' = \iota_q'' \circ \iota_q'$: this follows because each element of $(W_q' \backslash W_q)^{\text{short}}$ factors in a unique way as the product of an element of $(W_q' \backslash W_q)^{\text{short}}$ and an element of $(W_q' \backslash W_q)^{\text{short}}$.

Lemma 3.5. The map $i$ just defined is an injective homomorphism of $\mathcal{H}_n$-modules that commutes with the bar involution. Moreover it sends the canonical basis element $\sum_{w} N_w$ to the canonical basis element $\sum_{w'} N_{w'}$.

Proof. The injectivity is clear, because $i(N_w)$ is a linear combination of $N_{w'}$ for $w' \prec w$ and the coefficient of $N_{w'}$ is $1$. To prove that $i$ is a homomorphism of $\mathcal{H}_n$-modules, it is sufficient to consider the case $W_q' = \{e\}$. In fact, we have a commutative diagram of injective maps

\[
\begin{array}{ccc}
\mathcal{M}_q^p & \xrightarrow{i_q} & \mathcal{M}_q'^p \\
\downarrow & & \downarrow \\
\mathcal{M}_q^p & \xrightarrow{i_q'} & \mathcal{M}_q'^p
\end{array}
\]

and if $i_q$ and $i_q'$ are both $\mathcal{H}_n$-equivariant then so is $i_q'$.

Hence let $i = i_q$ and let us show using (3.8) that $i(N_w)H_i = i(N_w)H_i$ for all $i = 1, \ldots, n - 1$ and for each basis element $N_w \in \mathcal{M}_q^p$. Note first that $W^{p+q} \subseteq W^p$; moreover, if $w_s \in W^{p+q}$ then $xw_s \in W^p$ for every $x \in W_q$, so that the first two cases of (3.8) are clear. Suppose then that we are in the fourth case, that is $w_s = s_j w$ for some $s_j \in W_p$; then $xw_s = x s_j xw$ for every $x \in W_q$, because elements of $W_p$ commute with elements of $W_q$. We are left with the third case of (3.8), that we will now examine.

Pick an index $i$ such that $w_s = s_j w$ for some $s_j \in W_q$, and let $A^\perp = \{ x \in W_q \mid \ell(xw_s) \geq \ell(xw) \}$; note that for $x \in A^\perp$ we have $\ell(xs_j) > \ell(x)$ and that the right multiplication by $s_j$ is a bijection between $A^\perp$ and $A^\perp$ (unless $W_q = \{e\}$, but this case is trivial since $i$ is just the identity). Compute:

\[
i(N_w)H_i = \sum_{x \in A^\perp} q^{\ell(w_s) - \ell(x)} N_{xw_s} + \sum_{x \in A^\perp} q^{\ell(w) - \ell(x)} (N_{xw_s} + (q^{-1} - q)N_{xw})
\]

\[
= \sum_{x \in A^\perp} q^{\ell(w) - \ell(x)} N_{xw_s} + \sum_{x \in A^\perp} q^{\ell(w) - \ell(x)} (N_{xw_s} + (q^{-1} - q)N_{xw})
\]

\[
= \sum_{x' \in A^\perp} q^{\ell(w) - \ell(x') + 1} N_{x'w} + \sum_{x \in A^\perp} q^{\ell(w) - \ell(x)} (N_{xw_s} + (q^{-1} - q)N_{xw})
\]

\[
= \sum_{x \in A^\perp} q^{\ell(w) - \ell(x)} N_{xw_s} + q^{-1} \sum_{x \in A^\perp} q^{\ell(w) - \ell(x)} N_{xw}
\]
\[ q^{-1} \sum_{x^i \in A^+} q^\ell(w_i) - \ell(x) N_{x^i w} + q^{-1} \sum_{x \in A^-} q^\ell(w_i) - \ell(x) N_{x w} \]

\[ = q^{-1} i(N_w) = i(q^{-1} N_w) = i(N_w H_i). \]

It remains to show the bar invariance. Again, by the same argument as before it is sufficient to consider the case \( W' = \{ e \} \). It is enough to check it for a basis; in fact we will prove by induction that \( i(N_w) \) is bar invariant for every \( w \in W^{p+q} \).

For \( w = e \), we have \( i(N_e) = i(N_e) = \sum_{x \in W_q} q^\ell(w_i) - \ell(x) N_x \), that is well-known to be the canonical basis element for \( H_q \) corresponding to the longest element of \( W_q \); hence it is bar invariant. For the inductive step, suppose \( w s_i = w \) and use (3.10):

\[ \begin{align*}
    i(N_{w s_i}) &= i(N_w C_i - p) = i(N_w) C_i - i(p) \\
    &= i(N_{w s_i}) C_i - i(p) = i(N_{w s_i}).
\end{align*} \]

(3.13)

The last claim follows by the unicity of the canonical basis elements, because \( i(N_w) \) is bar invariant and the coefficient of \( N_{w'} \) in its standard basis expression is

- 1 if \( w' = w_q^t w \),
- a multiple of \( q \) if \( w' = x w'' \) for some \( x \in W' \cap W_q \) and \( w'' \in W_q \) with \( w'' \not\succeq w \) (but \( w' \neq w_q^t w \)),
- 0 otherwise.

Now we define a left inverse \( Q : M^p_q \to M^q_p \) of \( i \) by setting

\[ Q(N_e) = \frac{1}{c_q} N_e, \quad \text{where} \quad c_q = \sum_{x \in W' \cap W_q} q^{\ell(w_q^t)} - 2\ell(x). \]

(3.14)

It is easy to show that \( Q \) is indeed well-defined (since \( M^q_p \) is a quotient of \( M^p_q \), and \( Q \) is, up to a multiple, the quotient map). Moreover

\[ Q \circ i(N_w) = Q \left( \sum_{x \in W' \cap W_q} q^{\ell(w_q^t)} - \ell(x) N_{x w} \right) = \frac{1}{c_q} \sum_{x \in W' \cap W_q} q^{\ell(w_q^t)} - 2\ell(x) N_w = N_w \]

for all basis elements \( N_w \in M^q_p \).

Maps between Hecke modules (II). Now let us examine the case in which we change the subgroup \( W_p \). Namely let \( W_p \subset W_p \) be a parabolic subgroup of \( W \), and define a linear map \( j : M^p_q \to M^p_q \) by

\[ j : N_w \mapsto \sum_{x \in W' \cap W_p} (-q)^\ell(x) N_{x w} \]

(3.16)

As for Lemma 3.3 it is easy to prove that \( j \) is an injective homomorphism of \( H \)-modules, anyway it does not commute with the bar involution and it does not send canonical basis elements to canonical basis elements. Instead, \( j \) sends the dual canonical basis (defined to be the basis that is dual to the canonical basis with respect to the bilinear form) to the dual canonical basis.

Define also a \( H \)-modules homomorphism \( z : M^p_q \to M^p_q \) by setting \( z(N_e) = N_e \). This gives a well-defined homomorphism because \( M^p_q \) is a quotient of \( M^p_q \).

Lemma 3.6. The map \( z \) is bar invariant and sends the canonical basis element \( N_w \in M^p_q \) to \( \bar{N_w} \in M^p_q \) if \( w \in W^{p+q} \) and to 0 otherwise. Moreover \( z \circ j = \sum_{x \in W' \cap W_p} q^{2\ell(x)} \id \).
Proof. The map $z$ is bar invariant by definition: in fact obviously $z(N_c) = \overline{z(N_c)}$, and then by multiplying with the $C_i$’s one can see that $z$ is bar invariant on a set of generators.

If $w \in W^{P_{q}}$ then it is easily seen that $z(N_w) \in N_w + \sum_{w \prec w} qZ[q]N_w'$. By uniqueness of the canonical basis elements it has to be $z(N_w) = N_w \in \mathcal{M}_w^q$. If $w \notin W^{P_{q}}$ then by the same reasoning $z(N_w) = 0$.

Moreover

\[(3.17) \quad z \circ j(N_w) = z \left( \sum_{x \in W' \cap W_p} (-q)^{f(x)} N_{xw} \right) = \sum_{x \in W' \cap W_p} q^{2f(x)} N_{xw},\]

hence the last assertion follows as well. □

4. Representations of $U_q(\mathfrak{gl}(1|1))$

We recall the definition of the quantum enveloping algebra $U_q(\mathfrak{gl}(1|1))$ and of its braided structure. We describe then the representations of $U_q(\mathfrak{gl}(1|1))$ and the decomposition of their tensor products.

4.1. The quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$. We now study in detail the quantum enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$, that is a $q$-deformed version of the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(1|1)$. We will collect some known results which are hard to find in the literature.

Let $P = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be the weight lattice of $\mathfrak{gl}(1|1)$ (see also [2.1]) and $P^* = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$ its dual with the natural bilinear pairing $(h_i, \varepsilon_j) = \delta_{i,j}$, and set $\alpha = \varepsilon_1 - \varepsilon_2$ to be the simple root of $\mathfrak{gl}(1|1)$. Then $U_q(\mathfrak{gl}(1|1))$ is defined to be the unital superalgebra over $\mathbb{C}(q)$ with generators $E, F, q^h (h \in P^*)$ and relations

\[(4.1) \quad q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for} \, \, h, h' \in P^*, \]

\[q^h E = q^{(h,\alpha)} E q^h, \quad q^h F = q^{-\langle h, \alpha \rangle} F q^h \quad \text{for} \, \, h \in P^*, \]

\[EF + FE = \frac{K - K^{-1}}{q - q^{-1}} \quad \text{where} \, \, K = q^{h_1 + h_2}; \]

\[E^2 = F^2 = 0.\]

Note that $K$ is a central element of $U_q(\mathfrak{gl}(1|1))$.

By a superalgebra we mean a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. The elements $q^h$ are in degree 0, while $E$ and $F$ are in degree 1. As usual, we will use the notation $|.|$ to indicate the degree. Elements of degree 0 are called even, while elements of degree 1 are called odd.

The Hopf superalgebra structure. The superalgebra $U_q(\mathfrak{gl}(1|1))$ is made into a Hopf superalgebra via the comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$, that are defined by

\[(4.2) \quad \Delta(E) = E \otimes K^{-1} + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K \otimes F, \]

\[S(E) = -EK, \quad S(F) = -K^{-1}F, \]

\[\Delta(q^h) = q^h \otimes q^h, \quad S(q^h) = q^{-h}; \]

\[\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(q^h) = 1.\]

Notice that from the centrality of $K$ it follows that $S^2 = \text{id}$; this is a particular property of $U_q(\mathfrak{gl}(1|1))$, that for instance does not hold in $U_q(\mathfrak{gl}(m|n))$ for general $m, n$ (see [BKK00] for a definition of the general linear quantum supergroup).
Remark 4.1. One should always pay attention to signs when working with superobjects. We recall that if \( A \) is a superalgebra then \( A \otimes A \) can be given a superalgebra structure by declaring \( (a \otimes b) (c \otimes d) = (-1)^{|b||c|}ac \otimes bd \). Analogously, if \( M \) and \( N \) are \( A \)-supermodules, than \( M \otimes N \) becomes an \( A \)-supermodule with action \( (a \otimes b) \cdot (m \otimes n) = (-1)^{|b||m|}am \otimes bn \) for \( a, b \in A, m \in M, n \in N \).

If \( H \) is a Hopf superalgebra and \( M, N \) are (finite dimensional) \( H \)-supermodules then the comultiplication \( \Delta \) defines a map \( H \to H \otimes H \) and hence makes it possible to give \( M \otimes N \) an \( H \)-module structure. But note that signs appear in the action. The antipode \( S \), moreover, allows to turn \( M^* \) into an \( H \)-module: but again, a sign appears, because \( x \varphi(v) = (-1)^{|\varphi||x|} \varphi(S(x)v) \) for \( x \in H, \varphi \in M^* \). A good rule to keep in mind is that signs appear whenever an odd element steps over some other odd element.

We define a bar involution on \( U_q(\mathfrak{gl}(1\,|\,1)) \) by setting:

\[
E = E, \quad F = F, \quad \bar{q}^h = q^{-h}, \quad \bar{q} = q^{-1}.
\]

Note that \( \bar{\Delta} = (\otimes) \circ \Delta \circ (\otimes) \) defines another comultiplication on \( U_q(\mathfrak{gl}(1\,|\,1)) \), and by definition \( \bar{\Delta}(\bar{x}) = \Delta(x) \).

We also define the Cartan anti-involution \( \omega \):

\[
\omega(E) = F, \quad \omega(F) = E, \quad \omega(q^h) = q^{-h}, \quad \omega(q) = q^{-1},
\]

with \( \omega(xy) = \omega(y)\omega(x) \) for \( x, y \in U_q(\mathfrak{gl}(1\,|\,1)) \).

The braided structure. The quantum group \( U_q(\mathfrak{gl}(1\,|\,1)) \) possesses a braided Hopf superalgebra structure (cf. [Zha02], [Oht02]), thanks to the existence of a universal \( R \)-matrix. This \( R \)-matrix has been explicitly computed by Khoroshkin and Tolstoy (cf. [KT91]). We adapt their definition to our notation\(^2\).

The universal \( R \)-matrix has the form \( R = \Theta \Upsilon \), where

\[
\Upsilon = q^{\frac{b_1}{2} + b_2 (b_1 - b_2) + \frac{a_1}{2} + a_2 (a_1 - a_2)},
\]

\[
\Theta = 1 + (q - q^{-1}) F \otimes E.
\]

Actually the expression for \( \Upsilon \) does not make sense in \( U_q(\mathfrak{gl}(1\,|\,1)) \otimes U_q(\mathfrak{gl}(1\,|\,1)) \). We need to consider a bigger algebra \( \hat{U}_q(\mathfrak{gl}(1\,|\,1)) \) over \( \mathbb{C}[[q]] \), that is the topological completion of the algebra generated by \( E, F, h (h \in \mathbb{P}^* \) with the same relations of \( U_q(\mathfrak{gl}(1\,|\,1)) \), where we put \( q = e^h \in \mathbb{C}[[h]] \) and \( q^h = q^h = e^{bh} \). Then \( R \) exists in the completed tensor product \( \hat{U}_q(\mathfrak{gl}(1\,|\,1)) \otimes \hat{U}_q(\mathfrak{gl}(1\,|\,1)) \) (cf. [Kas95] Chapters XVI and XVIII). Note that \( \Upsilon \) is characterized by the property that it acts on a weight vector \( w_1 \otimes w_2 \) as \( q^{\mu_1 - \mu_2} \), where \( w_1 \) and \( w_2 \) have weights \( \mu_1 \) and \( \mu_2 \) respectively.

A remarkable property of \( \hat{U}_q(\mathfrak{gl}(1\,|\,1)) \) is that only the definition of \( \Upsilon \) requires the completion, while \( \Theta \) is defined in \( U_q(\mathfrak{gl}(1\,|\,1)) \); this depends on the fact that \( E^2 = F^2 = 0 \).

The following properties hold for every \( x \in \hat{U}_q(\mathfrak{gl}(1\,|\,1)) \):

\[
\Theta \bar{\Delta}(x) = \Delta'(x) \Theta,
\]

\[
\Upsilon \Delta(x) = \bar{\Delta}(x) \Upsilon,
\]

where \( \Delta' \) is the opposite comultiplication \( \Delta' = \sigma \Delta \) with \( \sigma(a \otimes b) = (-1)^{|a||b|} (b \otimes a) \).

These properties imply

\[
R \Delta(x) = \Delta'(x) R.
\]

\(^2\)Our comultiplication is the opposite of [KT91], hence we have to take the opposite \( R \)-matrix, cf. also [Kas95] Chapter 8.
Moreover we have the quasi-triangularity identities
\[(\Delta \otimes 1)(R) = R_{13}R_{23} \quad \text{and} \quad (1 \otimes \Delta)(R) = R_{13}R_{12}\]
and the Yang-Baxter relation
\[(R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}).\]

The element \(\Theta\) is called \textit{quasi} \(R\)-\textit{matrix}; it satisfies:
\[
\Theta \Theta = \Theta = 1 \otimes 1,
\]
We will also need the element
\[
\Theta' = \sigma(\Theta) = 1 + (q^{-1} - q)E \otimes F.
\]
From (4.17) it follows that for every \(x \in \hat{U}_q(\mathfrak{gl}(1,1))\)
\[
\Theta \Delta(x) = \Delta(x) \Theta'.
\]
One can check that
\[
(\Delta \otimes 1)(\Theta') \Theta'_{12} = (1 \otimes \Delta)(\Theta') \Theta'_{23},
\]
hence we can define \(\Theta'^{(3)}\) as the expression (4.14). More generally, one can define \(\Theta'^{(n)}\) for every \(n\).

4.2. 
\textbf{Representations of} \(U_q(\mathfrak{gl}(1|1))\). We define a parity function \(P : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}\) by setting \(|\epsilon_1| = 0, |\epsilon_2| = 1\) and extending additively. By a representation of \(U_q(\mathfrak{gl}(1|1))\) we mean from now on a finite-dimensional \(U_q(\mathfrak{gl}(1|1))\)-supermodule with a decomposition into weight spaces \(M = \bigoplus_{\lambda \in \mathbb{P}} M_{\lambda}\), such that \(q^h\) acts as \(q^{(h,\lambda)}\) on \(M_{\lambda}\). We suppose further that \(M\) is \(\mathbb{Z}/2\mathbb{Z}\)-graded, and the grading is uniquely determined by the requirement that \(M_{\lambda}\) is in degree \(|\lambda|\).

It is not difficult to find all simple representations: they are indexed by their highest weight \(\lambda \in \mathbb{P}\). We denote by \(V(\lambda)\) the simple representation with highest weight \(\lambda\). If \(\lambda \in \text{Ann}(h_1 + h_2)\), then \(V(\lambda)\) is one-dimensional: we have \(V(\lambda) = \mathbb{C}(q)(v^{\lambda})\) and
\[
E v^{\lambda} = 0, \quad F v^{\lambda} = 0, \quad q^h v^{\lambda} = q^{(h,\lambda)} v^{\lambda}, \quad K v^{\lambda} = 0.
\]
In particular for \(\lambda = 0\) we have the trivial representation, that we will simply denote by \(\mathbb{C}(q)\) in the following.

If \(\lambda \notin \text{Ann}(h_1 + h_2)\) then \(V(\lambda)\) is two-dimensional; let us denote by \(v^{\lambda}_1\) its highest vector. Let us also introduce the following notation that will be very useful:
\[
q^{(\lambda)} = q^{(h_1 + h_2, \lambda)}, \quad |\lambda| = |\langle h_1 + h_2, \lambda \rangle|,
\]
where, as usual, \([k]\) is the quantum number defined by
\[
[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = q^{-k+1} + q^{-k+3} + \cdots + q^{-k-3} + q^{-k-1}.
\]
Even if the second equality holds only for \(k > 0\), we define \([k]\) for all integers \(k\) using the first equality; in particular we have \([-k] = -[k]\).

Then \(V(\lambda) = \mathbb{C}(q)\langle v^{\lambda}_1 \rangle \oplus \mathbb{C}(q)\langle v^{\lambda}_0 \rangle\) with \(|v^{\lambda}_1| = |\lambda|, |v^{\lambda}_0| = |\lambda| + 1\) and
\[
E v^{\lambda}_1 = 0, \quad F v^{\lambda}_1 = [\lambda] v^{\lambda}_0, \quad q^h v^{\lambda}_1 = q^{(h,\lambda)} v^{\lambda}_1, \quad K v^{\lambda}_1 = q^{\lambda} v^{\lambda}_1, \quad E v^{\lambda}_0 = v^{\lambda}_1, \quad F v^{\lambda}_0 = 0, \quad q^h v^{\lambda}_0 = q^{(h,\lambda-\alpha)} v^{\lambda}_0, \quad K v^{\lambda}_0 = q^{\lambda} v^{\lambda}_0.
\]

\textbf{Remark 4.2.} As a remarkable property of \(U_q(\mathfrak{gl}(1|1))\), we notice that since \(E^2 = F^2 = 0\) all simple \(U_q(\mathfrak{gl}(1|1))\)-modules (even the ones with non-integral weights) are finite dimensional. In fact, formulas (4.17) define two-dimensional simple representations of \(U_q(\mathfrak{gl}(1|1))\) for all \(\lambda \in \mathbb{C}\epsilon_1 \oplus \mathbb{C}\epsilon_2\).
The following lemma is the first step to decompose a tensor product of $U_q(\mathfrak{gl}(1|1))$-representations:

**Lemma 4.3.** Let $\lambda, \mu \in P$ with $\lambda, \mu, \lambda + \mu \not\in \text{Ann}(h_1 + h_2)$. Then we have

$$V(\lambda) \otimes V(\mu) = V(\lambda + \mu) \oplus V(\lambda + \mu - \alpha).$$

**Proof.** We have

$$E(v_0^\lambda \otimes v_0^\mu) = v_1^\lambda \otimes q^{-\mu}v_0^\mu + (-1)^{|\lambda|+1}v_0^\lambda \otimes v_1^\mu,$$

$$F(v_1^\lambda \otimes v_1^\mu) = [\lambda]v_0^\lambda \otimes v_1^\mu + (-1)^{\lambda}[\mu]q^\lambda v_1^\lambda \otimes [\mu]v_0^\mu.$$ 

Under our assumptions, these two vectors are linearly independent. One can verify easily that $v_0^\lambda \otimes v_0^\mu$ and $E(v_0^\lambda \otimes v_0^\mu)$ span a module isomorphic to $V(\lambda + \mu - \alpha)$, while $v_1^\lambda \otimes v_1^\mu$ and $F(v_1^\lambda \otimes v_1^\mu)$ span a module isomorphic to $V(\lambda + \mu)$. 

In the following, we set

$$P' = \{ \lambda \in P \mid \lambda \not\in \text{Ann}(h_1 + h_2) \}.$$ 

Also, $P^\pm = \{ \lambda \in P \mid (h_1 + h_2, \lambda) \geq 0 \}$ will be the set of positive/negative weights and $P' = P^+ \cup P^-$. Note that in analogy with the classical Lie situation, we could set $\alpha^V = h_1 + h_2$.

**The vector representation.** The vector representation of $U_q(\mathfrak{gl}(1|1))$ is $V = V(\varepsilon_1)$. This is a 2-dimensional $\mathbb{C}(q)$-module; the basis vectors are $x = v_0^\varepsilon_1$, $y = v_0^{-\varepsilon_1}$, with $|x| = 0, |y| = 1$ and the action of $U_q(\mathfrak{gl}(1|1))$ is given by

$$Ex = 0, \quad Fx = y, \quad q^h x = q^{(h, \varepsilon_1)} x, \quad Kx = qx,$$

$$Ey = x, \quad Fy = 0, \quad q^h y = q^{(h, -\varepsilon_1)} y, \quad Ky = qy.$$

For $V^\otimes n$ we obtain from Lemma 4.3 the following decomposition:

**Proposition 4.4** ([BM12 Theorem 6.4]). The tensor powers of $V$ decompose as

$$V^\otimes m = \bigoplus_{\ell=0}^{m-1} \binom{m}{\ell} V(m\varepsilon_1 - \ell\alpha).$$

Let $V^*$ be the dual of $V$. In $V^*$ we have the basis $\{x^*, y^*\}$, dual to the standard basis of $V$. Explicitly, the $U_q(\mathfrak{gl}(1|1))$-module structure with respect to this basis is given by

$$Ex^* = -qy^*, \quad Fx^* = 0, \quad q^h x^* = q^{-(h, \varepsilon_1)} x^*, \quad Kx^* = q^{-1} x^*,$$

$$Ey^* = 0, \quad Fy^* = -q^{-1} x^*, \quad q^h y^* = q^{-(h, -\varepsilon_1)} y^*, \quad Ky^* = q y^*.$$

If we identify $v_1^{-\varepsilon_2} = -qy^*$ and $v_0^{-\varepsilon_2} = x^*$ we see that $V^*$ is isomorphic to $V(-\varepsilon_2)$.

The following generalizes Proposition 4.4 and follows from Lemma 4.3 by induction:

**Theorem 4.5.** Suppose $m \neq n$. Then we have the following decomposition:

$$V^\otimes m \otimes V^{\ast \otimes n} = \bigoplus_{\ell=0}^{m+n-1} \binom{m+n}{\ell} V(m\varepsilon_1 - n\varepsilon_2 - \ell\alpha).$$

For $m = n$ the tensor product is not semisimple. It decomposes into a sum of indecomposable reducible representations isomorphic to $V \otimes V^*$. 
Intertwining operators. Thanks to the quasi-triangular structure of $U_q(\mathfrak{gl}(1|1))$, we can define braiding operators on tensor products of representations. First, let $\lambda, \mu \in P'$ with $\lambda + \mu \in P'$.

Let $P : V(\lambda) \otimes V(\mu) \rightarrow V(\mu) \otimes V(\lambda)$ be given by $v \otimes w \mapsto (-1)^{|v||w|}w \otimes v$. We define a map $R^{-1} = R^{-1}P : V(\lambda) \otimes V(\mu) \rightarrow V(\mu) \otimes V(\lambda)$. By (4.23), this operator intertwines the action of $U_q(\mathfrak{gl}(1|1))$. Explicitly:

$$
R^{-1}(v_0^\lambda \otimes v_0^\mu) = (-1)^{(l|\lambda|+1)(l|\mu|+1)} q^{-l(\lambda,\lambda-\alpha)\lambda} v_0^\lambda \otimes v_0^\mu,
$$

$$
R^{-1}(v_0^\lambda \otimes v_1^\mu) = (-1)^{(l|\lambda|+1)|\mu|} (q^{-l(\mu,\lambda-\alpha)\lambda} v_1^\mu \otimes v_0^\lambda + (-1)^{|\mu|} q^{-l(\mu-\alpha,\lambda)\lambda}(q^{-1}-q)|\mu| v_0^\mu \otimes v_1^\mu)
$$

$$
R^{-1}(v_1^\lambda \otimes v_0^\mu) = (-1)^{(l|\lambda|+1)|\mu|} q^{-l(\mu-\alpha,\lambda)\lambda} v_0^\mu \otimes v_1^\lambda
$$

$$
R^{-1}(v_1^\lambda \otimes v_1^\mu) = (-1)^{(l|\lambda|+1)|\mu|} v_1^\mu \otimes v_1^\lambda.
$$

When we want to stress the dependency on $\lambda$ and $\mu$, we will write $\tilde{R}^{-1}_{\lambda,\mu}$.

Lusztig’s bar involution and canonical basis. We briefly recall from [Lus10] some facts about the bar involution and based modules. For a brief but more detailed introduction see also [FK97] §1.5.

**Definition 4.6.** A bar involution on a $U_q(\mathfrak{gl}(1|1))$-module $W$ is an involution $\overline{\bullet}$ such that $\overline{\overline{x}} = x$ for all $x \in U_q(\mathfrak{gl}(1|1))$, $v \in W$.

Note that $v_1^\lambda = v_1^\lambda$, $v_0^\lambda = v_0^\lambda$ define a bar involution on every simple representation $V(\lambda)$, $\lambda \in P'$.

Assume we have bar involutions on $U_q(\mathfrak{gl}(1|1))$-modules $W, W'$. Then define on $W \otimes W'$

$$
\overline{w \otimes w'} = \Theta'(w \otimes w').
$$

It follows from (1.13) that this defines a bar involution on $W \otimes W'$. Moreover, (1.11) allows us to repeat the construction for bigger tensor products, and the result is independent of the bracketing.

We call $B_\lambda = \{v_1^\lambda, v_0^\lambda\}$ the standard basis of $V(\lambda)$. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a sequence of weights $\lambda_i \in P'$. On the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_\ell)$ we have the standard basis

$$
B_\lambda = B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_\ell} = \{v_{\eta_1}^{\lambda_1} \otimes \cdots \otimes v_{\eta_\ell}^{\lambda_\ell} | \eta_i \in \{0, 1\} \text{ for all } i\}
$$

obtained tensoring the elements of the standard basis of the factors.

On the elements of (4.29) we fix a partial ordering induced from the Bruhat ordering on permutations, as follows. The symmetric group $S_\ell$ acts from the right on the set of sequences $\{0, 1\}^\ell$, hence on $B_\lambda$. The weight space of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_\ell)$ of weight $\lambda_1 + \cdots + \lambda_\ell - (\ell - k)\alpha$ is spanned by the subset $\{B_\lambda\}_k$ of the standard basis (4.29) consisting of elements such that $\sum_i \eta_i = k$. The action of $S_\ell$ on each subset $\{B_\lambda\}_k$ is transitive; mapping the identity $e \in S_\ell$ to the minimal element

$$
v_{\lambda_1}^1 \otimes \cdots \otimes v_{\lambda_k}^1 \otimes v_{0_k^\lambda} \otimes \cdots \otimes v_{0_{\ell-k}^\lambda}
$$

determines a bijection

$$
(S_k \times S_{\ell-k}) \setminus S_\ell \rightarrow \{B_\lambda\}_k,
$$

where $\{S_k \times S_{\ell-k}\} \setminus S_\ell$ is the set of shortest coset representatives for $S_k \times S_{\ell-k} \setminus S_\ell$. The Bruhat order of the latter induces a partial order on $\{B_\lambda\}_k$ and hence on $B_\lambda$. Notice that the minimal element (4.30) is bar invariant.

---

3We work with $\tilde{R}^{-1}$ instead of $\tilde{R}$ because it will fit better with our diagrammatic.
We have then the following analogue of [Lus10 Theorem 27.3.2]:

**Theorem 4.7.** In $V(\lambda_1) \otimes \cdots \otimes V(\lambda_\ell)$, for each standard basis element $v^\lambda_1 \otimes \cdots \otimes v^\lambda_\ell$ in $B_\lambda$ there is a unique bar-invariant element

$$v^\lambda_1 \otimes \cdots \otimes v^\lambda_\ell$$

such that $v^\lambda_1 \otimes \cdots \otimes v^\lambda_\ell = v^\lambda_1 \otimes \cdots \otimes v^\lambda_\ell$ is a $q\mathbb{Z}[q]$-linear combination of elements of the standard basis that are smaller than $v^\lambda_1 \otimes \cdots \otimes v^\lambda_\ell$.

**Proof.** The proof is completely analogous to [Lus10 Theorem 27.3.2].

**Definition 4.8.** The elements \((4.32)\) constitute the canonical basis of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_\ell)$.

**Example 4.9.** On the two-dimensional weight space of $V \otimes V$, the bar involution is given by

$$\overline{x \otimes y} = x \otimes y$$

$$\overline{y \otimes x} = y \otimes x + (q - q^{-1})x \otimes y.$$ 

The canonical basis is then

$$x \triangleleft y = x \otimes y$$

$$y \triangleleft x = y \otimes x + qx \otimes y.$$ 

**Example 4.10.** In $V^*$ let $X = v_1^{-\varepsilon_2}$ and $Y = v_0^{-\varepsilon_2}$. On the two-dimensional weight space of $V^* \otimes V^*$, the bar involution is given by

$$\overline{X \otimes Y} = X \otimes Y$$

$$\overline{Y \otimes X} = Y \otimes X + (q - q^{-1})X \otimes Y$$

and its canonical basis is

$$X \triangleleft Y = X \otimes Y$$

$$Y \triangleleft X = Y \otimes X + qX \otimes Y.$$ 

**5. A semisimple subcategory of $U_q(\mathfrak{gl}(1|1))$-representations**

In the last section we have seen explicitly the well-known fact that the category of representations of $U_q(\mathfrak{gl}(1|1))$ is not semisimple. From now on, we will restrict ourselves to consider only a subset of semisimple representations of $U_q(\mathfrak{gl}(1|1))$, that contains the tensor powers of the vector representation $V$. In particular, we will study in detail the intertwining operator, using Schur-Weyl duality for the tensor powers of the vector representation \((\S 5.2)\) and developing a diagram calculus similar to the one in [FK97] \((\S 5.3)\).

**5.1. Representations $V(a)$**. Given a natural number $a$ let $V(a) = V(a\varepsilon_1)$. For a sequence $a = (a_1, \ldots, a_\ell)$ of natural numbers let us denote $V(a) = V(a_1) \otimes \cdots \otimes V(a_\ell)$. Let $\text{Rep}$ be the monoidal subcategory of finite dimensional $U_q(\mathfrak{gl}(1|1))$-representations generated by $V(a)$ for $a \in \mathbb{N}$: the objects of $\text{Rep}$ are exactly \{$V(a) \mid a \in \bigcup_{\ell \geq 0} \mathbb{N}^\ell$\}. Note that this category is not abelian (it is not even additive). Anyway, by adding all direct sums and kernels we would get a monoidal abelian semisimple category, with simple modules $V(m_1\varepsilon_1 + m_2\varepsilon_2)$ for $m_1, m_2 \in \mathbb{N}$.

Since $V(0)$ is the trivial one-dimensional representation and hence the unit of the monoidal structure, it is enough to consider sequences $a$ not containing $0$; from now on, we will always suppose that our sequences consist of strictly positive integer numbers. If $a_1 + \cdots + a_\ell = n$, we will often call the sequence $a$ a composition of $n$. The sequence

$$a = (1, \ldots, 1)$$

where
will be called the regular composition of $n$. Any other composition of $n$ will be called singular. Notice that $V(n) = V(\varepsilon_1)^{\otimes n} = V^{\otimes n}$.

We repeat formulas from the previous section for the special case of the representations $V(a)$. Let $v_i^a = v_i^{a+1}$ for $i = 0, 1$. Then $V(a)$ is a 2-dimensional vector space with basis vectors $v_0^a$ in degree 0 and $v_1^a$ in degree 1; the action of $U_q(\mathfrak{gl}(1|1))$ is given by

\begin{equation}
\begin{aligned}
Ev_1^a &= 0, \\
Fv_1^a &= v_0^a, \\
\eta^h v_1^a &= q^{h, a+1} v_1^a, \\
K v_1^a &= q^a v_1^a,
\end{aligned}
\end{equation}

(5.2)

\begin{equation}
\begin{aligned}
Ev_0^a &= [a] v_1^a, \\
Fv_0^a &= 0, \\
\eta^h v_0^a &= q^{h, a+1} v_0^a, \\
K v_0^a &= q^a v_0^a.
\end{aligned}
\end{equation}

(5.3)

**Projections and embeddings.** Let $a, b \geq 1$. By Lemma 4.3, $V(a + b)$ is a subrepresentation of $V(a) \otimes V(b)$. Let us define explicit maps $\Phi_{a, b} : V(a) \otimes V(b) \rightarrow V(a + b)$ and $\Phi^{a, b} : V(a + b) \rightarrow V(a) \otimes V(b)$. We set

\[ \Phi_{a, b} : V(a) \otimes V(b) \rightarrow V(a + b) \]

\[ v_0^a \otimes v_0^b \mapsto 0 \]

\[ v_0^a \otimes v_1^b \mapsto q^{-b} \begin{bmatrix} a + b - 1 \\ b \end{bmatrix} v_1^{a+b} \]

\[ v_1^a \otimes v_0^b \mapsto \begin{bmatrix} a + b - 1 \\ a \end{bmatrix} v_0^{a+b} \]

\[ v_1^a \otimes v_1^b \mapsto \begin{bmatrix} a + b \\ a \end{bmatrix} v_1^{a+b} \]

(5.3)

and

\[ \Phi^{a, b} : V(a + b) \rightarrow V(a) \otimes V(b) \]

\[ v_0^{a+b} \mapsto v_0^a \otimes v_b^b + q^a v_1^a \otimes v_0^b \]

\[ v_1^{a+b} \mapsto v_1^a \otimes v_1^b, \]

(5.4)

where we recall that

\begin{equation}
[k] = [k][k-1] \cdots [1] \quad \text{for all } k \geq 1,
\end{equation}

(5.5)

\[ \frac{n}{k} = \frac{[n]!}{[k]![n-k]!} \quad \text{for all } n \geq 1, \ 1 \leq k \leq n. \]

(5.6)

One can check that the two maps $\Phi_{a, b}$ and $\Phi^{a, b}$ are indeed $U_q(\mathfrak{gl}(1|1))$-equivariant. Moreover, we have

\begin{equation}
\Phi_{a, b} \Phi^{a, b} = \begin{bmatrix} a + b \\ a \end{bmatrix} \text{id}.
\end{equation}

(5.7)

One can also check that $\Phi_{a, b}$ and $\Phi^{a, b}$ commute with the bar involution.

**Standard and canonical basis.** Let $a \in \mathbb{Z}_{>0}$. The elements $v_0^a$, $v_1^a$ give the standard basis of $V(a)$. Let now $\alpha = (a_1, \ldots, a_t)$ be a sequence of (strictly) positive numbers. For any sequence $\eta = (\eta_1, \ldots, \eta_t) \in \{0, 1\}^t$ we let $v_\eta^a = v_{\eta_1}^{a_1} \otimes \cdots \otimes v_{\eta_t}^{a_t}$. The elements \( \{v_\eta^a | \eta \in \{0, 1\}^t\} \) are called the standard basis vectors of $V(a)$.

According to Definition 4.8, for each standard basis vector $v_\eta^a$ there exists a corresponding canonical basis vector

\[ v_\eta^{\wedge a} = v_{\eta_1}^{a_1} \cdots \hat{v}_{\eta_\ell}^{a_\ell} \cdots v_{\eta_t}^{a_t}. \]

(5.8)
The bilinear form. Fix a sequence of positive numbers $a = (a_1, \ldots, a_\ell)$. We define a symmetric bilinear form on $V(a)$ by setting
\begin{equation}
(v_{\eta}^a, v_{\gamma}^a)_a = q^{\sum x_i ^ a b_i ^ a} 
\left[ \begin{array}{c}
\beta_1^\eta \\
\beta_1^\gamma \\
\vdots \\
\beta_\ell^\eta \\
\beta_\ell^\gamma
\end{array} \right] 
\delta_{\eta_1}^\gamma \cdots \delta_{\eta_\ell}^\gamma,
\end{equation}
where $\delta_i^j$ is the Kronecker delta,
\begin{equation}
\beta_j^\eta = a_j - 1 + \eta_j = \begin{cases}
a_j - 1 & \text{if } \eta_j = 0, \\
a_j & \text{otherwise}
\end{cases}
\end{equation}
and
\begin{equation}
\left[ h_1 + \cdots + h_\ell \right]_{h_1, \ldots, h_\ell} = \frac{[h_1 + \cdots + h_\ell]!}{[h_1]! \cdots [h_\ell]!}.
\end{equation}

Note that $q^{\sum x_i ^ a b_i ^ a}$ in (5.9) is exactly the factor needed so that the value of (5.9) is a polynomial in $q$ with constant term 1. We introduce the following non-standard notation:
\begin{equation}
[h]_0 = q^{h-1} [h] \\
[h]_0! = q^{h(h-1)/2} [h]!
\end{equation}
\begin{equation}
\left[ a + b \right]_0 = q^{ab} \left[ a + b \right],
\end{equation}
\begin{equation}
\left[ h_1 + \cdots + h_\ell \right]_0 = q^{\sum x_i h_i} \left[ h_1 + \cdots + h_\ell \right].
\end{equation}

Hence we can rewrite (5.9) as
\begin{equation}
(v_{\eta}^a, v_{\gamma}^a)_a = \left[ \begin{array}{c}
\beta_1^\eta \\
\beta_1^\gamma \\
\vdots \\
\beta_\ell^\eta \\
\beta_\ell^\gamma
\end{array} \right]_0 
\delta_{\eta_1}^\gamma \cdots \delta_{\eta_\ell}^\gamma.
\end{equation}

Notice that we have
\begin{equation}
[h]_0^\ell = [h]_0 [h-1]_0 \cdots [2]_0,
\end{equation}
\begin{equation}
\left[ h_1 + \cdots + h_\ell \right]_0 = \frac{[h_1 + \cdots + h_\ell]_0 !}{[h_1]_0 ! \cdots [h_\ell]_0 !}.
\end{equation}

Lemma 5.1. For all $v \in V(a) \otimes V(b), v' \in V(a+b)$ we have
\begin{equation}
(\Phi_{a,b}(v), v')_a = (v, q^{-ab}\Phi_{a,b}(v')).
\end{equation}

Proof. This is a straightforward calculation on the basis vectors:
\begin{align*}
(\Phi_{a,b}(v_0^a \otimes v_1^b), y_{a+b}) &= q^{-b} \left[ a + b - 1 \right]_b 
= (v_0^a \otimes v_1^b, q^{-ab}\Phi_{a,b}(y_{a+b})) \\
(\Phi_{a,b}(v_1^a \otimes v_0^b), y_{a+b}) &= \left[ a + b - 1 \right]_a 
= (v_1^a \otimes v_0^b, q^{-ab}\Phi_{a,b}(y_{a+b})) \\
(\Phi_{a,b}(v_1^a \otimes v_1^b), x_{a+b}) &= \left[ a + b \right]_a 
= (v_1^a \otimes v_1^b, q^{-ab}\Phi_{a,b}(x_{a+b})). \quad \square
\end{align*}

Lemma 5.2. For all standard basis vectors $v_{\eta}^a, v_{\gamma}^a \in V(a)$ we have
\begin{equation}
(F v_{\eta}^a, v_{\gamma}^a)_a = \frac{\beta_1^{\eta+a} + \cdots + \beta_\ell^{\eta+a}}{[\beta_1^{\eta+a} + \cdots + \beta_\ell^{\eta+a}]_0} (v_{\eta}^a, E v_{\gamma}^a)_a.
\end{equation}
Proof. Suppose that there exists an index \( r \) such that \( \eta_i = \gamma_i \) for all \( i \neq r \) and \( \eta_r = 0 \), \( \gamma_r = 1 \) (otherwise both sides of (5.2.10) are zero). Up to a sign (that we ignore, because it is the same in both formulas), we have

\[
(Fv^a_\eta, v^a_\gamma)_a = (q^{a_1+\cdots+a_r-1}v^a_\gamma, v^a_\gamma)_a = q^{a_1+\cdots+a_r-1} \left[ \frac{\beta_{1}^\eta + \cdots + \beta_{r}^\gamma}{\beta_{1}^\gamma, \ldots, \beta_{r}^\gamma} \right]_0
\]

and

\[
(v^a_\eta, Ev^a_\gamma)_a = (v^a_\eta, [a_r]q^{-a_{r+1}+\cdots-a_r}v^a_\gamma)_a = [a_r]q^{-a_{r+1}+\cdots-a_r} \left[ \frac{\beta_{1}^\eta + \cdots + \beta_{r}^\gamma}{\beta_{1}^\gamma, \ldots, \beta_{r}^\gamma} \right]_0.
\]

Since \( \beta_i^\eta = \beta_i^\gamma \) for \( i \neq r \) while \( \beta_r^\eta = \beta_r^\gamma + 1 = a_r \), we have

\[
(Fv^a_\eta, v^a_\gamma)_a = [a_r] \left[ \frac{\beta_1^\eta + \cdots + \beta_r^\gamma}{\beta_1^\gamma} \right] q^{a_1+\cdots+a_r-1-a_r} q^{1-a_1-\cdots-a_r},
\]

that proves the claim. \( \square \)

**Remark 5.3.** If we enlarge \( U_q(\mathfrak{gl}(1\mid1)) \) with a new generator \( E' \) such that

\[
E = q^{2h} - 1
\]

then we get an adjsution between \( F \) and \( E' \).

**Dual standard and dual canonical basis.** We define the dual standard basis \( \{v^\bullet_\eta \mid \eta \in \{0,1\}^r \} \) of \( V(\alpha) \) to be the basis dual to the standard basis with respect to the bilinear form \( (\cdot, \cdot)_a \):

\[
(v^\bullet_\eta, v^\bullet_\gamma)_a = \begin{cases} 1 & \text{if } \eta = \gamma, \\ 0 & \text{otherwise.} \end{cases}
\]

Of course, since the standard basis is already orthogonal, each \( v^\bullet_\eta \) is a multiple of \( v^a_\eta \). In particular, one has

\[
\left[ \frac{\beta_1^\eta + \cdots + \beta_r^\gamma}{\beta_1^\gamma, \ldots, \beta_r^\gamma} \right]_0 v^\bullet_\eta = v^a_\eta.
\]

Moreover, we define the dual canonical basis to be the basis dual to the canonical basis with respect to the bilinear form \( (\cdot, \cdot)_a \):

\[
(v^\bigcirc_\eta, v^\bigcirc_\gamma)_a = \begin{cases} 1 & \text{if } \eta = \gamma, \\ 0 & \text{otherwise.} \end{cases}
\]

### 5.2. Super Schur-Weyl duality for \( V^\otimes n \)

Let us consider \( \tilde{R}^{-1} \) on \( V \otimes V \); explicitly, we have

\[
\tilde{R}^{-1}(y \otimes y) = -qy \otimes y, \quad \tilde{R}^{-1}(y \otimes x) = x \otimes y + (q^1 - q)y \otimes x \\
\tilde{R}^{-1}(x \otimes y) = y \otimes x, \quad \tilde{R}^{-1}(x \otimes x) = q^{-1}x \otimes x.
\]

On can easily check that

\[
(\tilde{R}^{-1})^2 = (q^1 - q)\tilde{R}^{-1} + \text{Id}.
\]

Notice that the action of \( \tilde{R}^{-1} \) can be expressed in terms of a projection (5.3) and an embedding (5.4):

\[
\Phi^{1,1} \Phi_{1,1} = \tilde{R}^{-1} + q.
\]

We can consider on \( V^\otimes n \) the operators

\[
\tilde{R}_{i,i+1}^{-1} = \text{id}^\otimes_{i-1} \otimes \tilde{R}^{-1} \otimes \text{id}^\otimes_{n-i-1}.
\]
It follows from (5.18) that they are intertwiners for the action of $U_q(\mathfrak{gl}(1|1))$. By (5.27) and (4.10), we have then an $U_q(\mathfrak{gl}(1|1))$-equivariant right action of the Hecke algebra $H_n$ on $V^{\otimes n}$.

The following result is also known as super Schur-Weyl duality. The non-quantized version was originally proved by Berele and Regev ([BR87]) and independently by Sergeev ([Ser84]).

**Proposition 5.4** ([Mit06]). The map

$$
H_n \longrightarrow \text{End}_{U_q(\mathfrak{gl}(1|1))}(V^{\otimes n})
$$

$$
H_i \longrightarrow \hat{R}^{-1}_{i,i+1}
$$

is surjective. As a module for $H_n$, we have

$$
V^{\otimes n} = \bigoplus_{k=1}^{n} (S(\mu_{n,k}) \otimes S(\mu_{n,k})),
$$

where $\mu_{n,k}$ is the hook partition $(k, 1^{n-k})$ and $S(\mu_{n,k})$ is the $q$-version of the corresponding Specht module.

It follows in particular that the kernel of (5.30) is the two-sided ideal $I_n$ generated by the idempotents projecting onto simple representations of $H_n$ corresponding to Young shapes with $n$ boxes that are not hooks (a Young shape is said to be of hook type if the corresponding partition is $(k, 1, \ldots, 1)$). For $n \leq 3$ there are no such Young shapes. For $n = 4$, the only Young shape that is not a hook is $\boxplus$, and the corresponding idempotent is, up to a multiple,

$$
(H_1 + q)(H_3 + q)H_2(H_1 - q^{-1})(H_3 - q^{-1}).
$$

For $n \geq 4$ every Young shape that is not a hook contains some $\boxplus$ and it is easy to prove that the ideal $I_n$ is generated by

$$
(H_{i-1} + q)(H_{i+1} + q)H_i(H_{i-1} - q^{-1})(H_{i+1} - q^{-1})
$$

for $i = 2, \ldots, n - 2$.

As often occurs with the Hecke algebra, it is more convenient to choose generators $C_i = H_i + q$. We introduce the Super Temperley Lieb Algebra as follows:

**Definition 5.5.** For $n \geq 1$, the Super Temperley Lieb Algebra $\text{STL}_n$ is the unital associative $\mathbb{C}(q)$-algebra generated by $\{C_i \mid i = 1, \ldots, n-1\}$ subject to the relations

$$
C_i^2 = (q + q^{-1})C_i,
$$

$$
C_iC_j = C_jC_i,
$$

$$
C_iC_{i+1}C_i - C_i = C_{i+1}C_iC_{i+1} - C_{i+1},
$$

for $|i - j| > 1$, and

$$
C_{i-1}C_{i+1}C_i((q + q^{-1}) - C_{i-1})((q + q^{-1}) - C_{i+1}) = 0,
$$

$$
((q + q^{-1}) - C_{i-1})((q + q^{-1}) - C_{i+1})C_iC_{i-1}C_{i+1} = 0.
$$

Since the first three relations are just the relations that the generators $C_i = H_i + q$ satisfy in the Hecke algebra, it follows that $\text{STL}_n$ is a quotient of $H_n$. Moreover, by the discussion above, we have

$$
\text{STL}_n = \text{End}_{U_q(\mathfrak{gl}(1|1))}(V^{\otimes n}).
$$

Consider the weight space decomposition

$$
V^{\otimes n} = \bigoplus_{k=0}^{n}(V^{\otimes n})_k
$$
where
\[(V^\otimes n)_k = \{ v \in V^\otimes n \mid q^h v = q^{(h,kx_1+(n-k)z)} v \}.\]

Clearly, every weight space is a module for the Hecke algebra. We have:

**Proposition 5.6.** Let \( W_q = \langle s_1, \ldots, s_{k-1} \rangle \) and \( W_p = \langle s_{k+1}, \ldots, s_{n-1} \rangle \) as subgroups of \( S_n \). With the notation of §3 we have
\[(V^\otimes n)_k \cong M^p_q\]
as right \( H_n \)-modules. The isomorphism is given explicitly by
\[\Psi : M^p_q \rightarrow (V^\otimes n)_k\]
\[N_w \mapsto v^a_{\eta_{\text{min}}, w},\]
where
\[
\eta_{\text{min}} = (1, \ldots, 1, 0, \ldots, 0),
\]
and \( S_n \) acts on sequences of \( \{0,1\}^n \) from the right by permutations.

**Proof.** It is straightforward to check that, by the definition of the action of \( H_n \) on \( V^\otimes n \) (5.26), we have \( v_{\eta_{\text{min}}} \cdot H_w = v_{\eta_{\text{min}}, w} \) whenever \( w \in W^{p+q} \). In particular, (5.39) is a bijection. We need to show that the action of the Hecke algebra is the same on both sides. This follows comparing (3.8) and (5.26).

As a consequence, there is a second notion of canonical basis on \( (V^\otimes n)_k \), defined using the Hecke algebra action from Section 3. Not surprisingly, this coincides with Lusztig canonical basis (compare with [FKK98, Theorem 2.5]):

**Proposition 5.7.** Under the isomorphism \( \Psi \) (5.39), the canonical basis element \( N_w \) of \( M^p_q \) is mapped to the canonical basis element \( v^a_{\eta_{\text{min}}, w} \).

**Proof.** By the uniqueness results (Proposition 5.4 and Theorem 4.7), it is enough to show that the bar involution of \( M^p_q \) is mapped to the bar involution of \( (V^\otimes n)_k \) under (5.39). On \( M^p_q \) the bar involution is uniquely determined by \( \bar{N}_c = N_c \) and \( \bar{XH_i} = \bar{X}H_i = \bar{X}H_i^{-1} \) for all \( X \in M^p_q \). It is enough to show that the same holds for Lusztig bar involution on \( (V^\otimes n)_k \). Of course \( v_{\eta_{\text{min}}} = v_{\eta_{\text{min}}, w} \), and one can show by standard methods (cf. [Zha09a, Lemma 2.3]) that
\[(5.41) \quad R^{-1}_{i,i+1} v_{\eta} = R_{i,i+1} \bar{v}_{\eta},\]
for all standard basis elements \( v_{\eta} \).

Moreover, the form \( \langle \cdot, \cdot \rangle \) on \( M^p_q \) and the form \( \langle \cdot, \cdot \rangle_a \) on \( (V^\otimes n)_k \) are proportional under \( \Psi \):

**Lemma 5.8.** Let \( \Psi \) be the isomorphism (5.39). Then
\[(5.42) \quad \langle \Psi(X), \Psi(Y) \rangle_a = [k]! \langle X, Y \rangle \quad \text{for all } X, Y \in M^p_q.
\]

**Proof.** It is enough to check (5.42) on the standard basis \( \{ N_w \mid w \in W^{p+q} \} \) of \( M^p_q \).

We have
\[\langle \Psi(N_w), \Psi(N_z) \rangle_a = (v_{\eta_{\text{min}}, w}, v_{\eta_{\text{min}}, z})_a = \begin{cases} [k]! & \text{if } w = z, \\ 0 & \text{otherwise.} \end{cases}\]

By definition, this is the same as \( [k]! \langle N_w, N_z \rangle \).

**5.3. Diagrams for the intertwining operators.** In this section we will provide a diagram calculus for the intertwining operators in the category \( \text{Rep} \).
The category Web. We start by defining a diagrammatical category Web. First, a web diagram is an oriented plane graph with edges labeled by positive integers. Only single and triple vertices are allowed. Single vertices must lie on the bottom (resp. top) line if they are sources (resp. targets) for the corresponding edge. Around a triple vertex, the sum of the labels of the ingoing edges must agree with the sum of the labels of the outgoing vertices; this means that only the following labeling are allowed for any strictly positive numbers $a$, $b$:

\[
\begin{align*}
\sum_{i=1}^{k} a_i &= \sum_{i=1}^{k} b_i \\
&= \sum_{i=1}^{k} a_i + b_i 
\end{align*}
\]

Note that we do not draw the orientation of the edges, because we suppose all edges to be oriented upwards. The source of a web is the sequence $a = (a_1, \ldots, a_\ell)$ of labels on the bottom line. The target is the sequence $a' = (a'_1, \ldots, a'_{s})$ on the top line.

If we have two webs $\psi, \varphi$ and the target of $\varphi$ is the same as the source of $\psi$, then we can compose $\psi$ and $\varphi$ by concatenating vertically:

\[
\psi \circ \varphi = \begin{array}{c}
\psi \\
\varphi
\end{array}
\]

Additionally, we can always concatenate two webs $\varphi, \psi$ horizontally, putting the second on the right of the first; in this case we use a tensor product symbol:

\[
\varphi \otimes \psi = \begin{array}{c}
\varphi \\
\psi
\end{array}
\]

The category $\text{Web}'$ is the monoidal category whose objects are sequences $a = (a_1, \ldots, a_\ell)$ of strictly positive integers; a morphism from $a$ to $a'$ is a $\mathbb{C}(q)$-linear combination of web diagrams with source $a$ and target $a'$. Composition of morphisms corresponds to vertical concatenation of web diagrams. Horizontal concatenation of web diagrams gives, on the other side, a monoidal structure on $\text{Web}'$.

We define the two elementary webs $\Upsilon_{a,i}$ and $\Upsilon^{a,i}$ by the diagrams:

\[
\Upsilon_{a,i} = \begin{array}{c}
\cdots \\
\text{a_1} & \text{a_{i-1}} & \text{a_i} & \text{a_{i+1}} & \text{a_{i+2}} & \text{a_\ell}
\end{array}
\]

\[
\Upsilon^{a,i} = \begin{array}{c}
\cdots \\
\text{a_1} & \text{a_{i-1}} & \text{a_i} & \text{a_{i+1}} & \text{a_{i+2}} & \text{a_\ell}
\end{array}
\]

and notice that each web diagram can be obtained as composition of elementary webs diagrams. We let also

\[
\tilde{a}_i = (a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_\ell)
\]

be the target of $\Upsilon_{a,i}$. 
We define the category $\mathbf{Web}$ to be the quotient of $\mathbf{Web}'$ by the following relations:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \quad = \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \\
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \\
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array}
\end{align*}
\]

(5.48)

(5.49)

Thanks to these relations, we can also consider multiple vertices with only one outgoing (resp. ingoing) edge: we define them to be equal to concatenations of elementary graphs like (5.49) (resp. (5.44)). For example:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \quad = \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \\
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \\
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array}
\end{align*}
\]

(5.50)

Webs as intertwiners. Now we are going to define a monoidal functor $\mathcal{T} : \mathbf{Web} \to \mathbf{Rep}$. On objects we define it to be $\mathcal{T}(a) = V(a)$. To define $\mathcal{T}$ on morphisms, it suffices to consider elementary pieces of webs. An oriented edge is an identity morphism from the source $a$ to the target $a$ in $\mathbf{Web}$, hence the functor $\mathcal{T}$ assigns to it the identity morphism of $V(a)$:

\[
\mathcal{T} \left( \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \right) = \begin{array}{c}
V(a) \\
\text{id} \\
V(a)
\end{array}
\]

(5.51)

To triple vertices we assign projections and inclusions of subrepresentations, as follows:

\[
\mathcal{T} \left( \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \right) = \begin{array}{c}
V(a + b) \\
\Phi_{a,b} \\
V(a) \otimes V(b)
\end{array}
\]

(5.52)

\[
\mathcal{T} \left( \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\end{tikzpicture}
\end{array} \right) = \begin{array}{c}
V(a) \otimes V(b) \\
\Phi_{a,b} \\
V(a + b)
\end{array}
\]

(5.53)

It is straightforward to verify that $\mathcal{T}$ assigns the same morphism to both sides of (5.48) and (5.49), hence $\mathcal{T}$ is well-defined. In what follows, we are going to omit to write the functor $\mathcal{T}$ and consider a web just as a homomorphism of the corresponding representations.
Remark 5.9. From the web calculus we introduced we cannot see explicitly the action of the Hecke algebra. Anyway, we could enhance our web calculus allowing edges labeled by 1 to cross (with over- or undercrossings). Relation (5.28) becomes graphically

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram1}
\end{array}
\end{array}
\end{align*}
\]

Matrix coefficients. Let \( \varphi \) be a web from \( a = (a_1, \ldots, a_\ell) \) to \( a' = (a'_1, \ldots, a'_{\ell'}) \). Given \( \eta \in \{0,1\}^\ell, \gamma \in \{0,1\}^\ell' \), the matrix coefficient

\[
\langle \varphi(v^a_\eta), v^{a'}_\gamma \rangle,
\]

that is the coefficient of \( v^{a'}_\gamma \) in \( \varphi(v^a_\eta) \) when expressed in the standard basis, is represented by the diagram

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram2}
\end{array}
\end{array}
\end{align*}
\]

where the \( i \)-th line below is labelled by \( \land \) if \( \eta_i = 1 \) and by \( \lor \) if \( \eta_i = 0 \) (and analogously above).

Diagrams provide a convenient way to compute matrix coefficients, as we are going to explain. Let us fix a diagram \( \varphi \) and suppose that we want to compute the coefficient (5.55). We start with the picture (5.56). Then we label every edge of the graph with \( \land \) and \( \lor \), in all possible ways. Such a “completely labeled” graph is evaluated according to the local rules in Figure 1 (the missing label possibilities are evaluated to zero, and the total evaluation is obtained via multiplication). To evaluate the initial picture, sum the evaluations over all possible “complete labeling”.

Canonical basis. Fix a sequence \( a = (a_1, \ldots, a_\ell) \) and consider a standard basis element \( v^a_\eta \) of \( V(a) \). This standard basis element is represented by a (trivial) diagram, obtained as follows: take the identity web \( a \rightarrow a \) and label the edges from the left to the right with an \( \land \) if \( \eta_i = 1 \) and a \( \lor \) if \( \eta_i = 0 \), (as in Figure 2). We call it the standard basis diagram corresponding to \( v^a_\eta \).

Starting from this standard basis diagram, one can obtain the corresponding canonical basis element as follows. For every consecutive \( \lor \land \) (in this order), join the corresponding two edges as follows:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram3}
\end{array}
\end{array}
\end{align*}
\]

Repeat this process using at each step also the \( \lor \)'s created in the previous steps, until no more \( \lor \land \) is left. At the end, we will obtain some diagram \( C(v^a_\eta) \) that we call the canonical basis diagram corresponding to \( v^a_\eta \) (see Figure 3 and Example 5.12 below).
CATEGORIZATION OF REPRESENTATIONS OF $U_q(\mathfrak{gl}(1|1))$

Figure 1. Evaluation of elementary diagrams.

Figure 2. The standard basis diagram for $v^{(3,1,4,4,2,1,1)}_{(1,0,1,1,0,1,0)}$.

Figure 3. The canonical basis diagram for $v^{(3,1,4,4,2,1,1)}_{(1,0,1,1,0,1,0)}$.

Remark 5.10. Note that this canonical basis diagram is obtained joining recursively each edge labeled by $a \lor$ with all immediately following edges labeled by $\land$'s. If we use multiple vertices (as defined by (5.50)), we can construct the canonical basis diagram in just one step. In particular, the construction is independent of the order in which we consider the pairs $\lor \land$.

We claim that canonical basis diagrams correspond to canonical basis element via $\mathcal{F}$. In fact, the diagram $C(v^a_\eta)$ has an underlying web that represents some embedding $V(a') \to V(a)$, where $a'$ is some composition that is refined by $a$; this web carries on the bottom the labels of a basis element of $V(a')$, that is at the same time a standard basis element and a canonical basis element. Hence the diagram $C(v^a_\eta)$ is an “evaluated web”, that gives a bar-invariant element of $V(a)$ (since $\mathcal{F}(\varphi)$ sends bar-invariant elements to bar-invariant elements for all webs $\varphi$). Examining the evaluation rules (Figure 1), one sees that the matrix coefficients of $C(v^a_\eta)$ are all in $q\mathbb{Z}[q]$ except for the coefficient of $v^a_\eta$, that is 1. Summarizing, we have:
Proposition 5.11. The diagram \( C(v_\alpha^\eta) \) gives the canonical basis element \( v_\alpha^\eta \) of \( V(a) \).

Example 5.12. Let \( a = (3, 1, 4, 2, 1, 1) \) and consider the element \( v_\alpha^\eta \) \((1,0,1,1,0,1,0) \in V(a) \). The corresponding standard and canonical basis diagrams are pictured in Figures 2 and 3. In particular, evaluating the canonical basis diagram according to the rules in Figure 1, we get the corresponding canonical basis element

\[
(5.58) \quad v_{(1,0,1,1,0,1,0)}^\alpha = v_{(1,0,1,1,1,0,0)}^\alpha + qv_{(1,1,1,0,1,0,0)}^\alpha + q^2v_{(1,0,1,1,1,0,0,0)}^\alpha + q^3v_{(1,1,0,1,1,0,0,0)}^\alpha + q^7v_{(1,1,1,0,1,0,0,0)}^\alpha.
\]

Action of \( E \) and \( F \). Using our diagram calculus we can easily compute the action of \( F \) (in an analogous way as [FK97] for \( sl_2 \)).

Proposition 5.13. Fix some representation \( V(a) \) and consider a canonical basis element \( v_\alpha^\eta \). We have

\[
(5.59) \quad F(v_\alpha^\eta) = \begin{cases} v_{0}^\eta_1 \otimes v_{0}^\eta_2 \otimes \cdots \otimes v_{0}^\eta_r & \text{if } \eta_1 = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Proof. Suppose that \( \eta_i = 1 \) for \( i = 1, \ldots, h \), while \( \eta_{h+1} = 0 \) (possibly \( h = 0 \)). The canonical basis diagram \( C(v_\alpha^\eta) \) is

\[
(5.60)
\]

where there are some vertices in the box. We can also represent it as

\[
(5.61)
\]

because this is the same element according to our diagrammatic calculus. On the bottom we read the labels of \( v_\gamma^\alpha \) for some composition \( a' \) refining \( a \), where \( \gamma = (1, 0, \ldots, 0) \). We can easily compute

\[
(5.62) \quad F(v_1^\alpha \otimes v_0^\alpha_i \otimes \cdots \otimes v_0^\alpha_i') = v_0^\alpha_1 \otimes v_0^\alpha_2 \otimes \cdots \otimes v_0^\alpha_r'.
\]

Hence

\[
(5.63) \quad F = \begin{pmatrix}
\end{pmatrix} = \begin{pmatrix}
\end{pmatrix}
\]

that is exactly our assertion. \( \square \)
By (5.20) it follows that $E$ sends the dual canonical basis to the dual canonical basis (up to a multiple):

**Proposition 5.14.** Fix some representation $V(a)$ and consider a dual canonical basis element $v_\eta^\otimes a$. We have

$$E(v_\eta^\otimes a) = \begin{cases} \frac{[\eta_1^n + \cdots + \eta_\ell^n]}{q^{a_1 + \cdots + a_\ell - 1}} v_1^{a_1} \otimes v_2^{a_2} \otimes \cdots \otimes v_\ell^{a_\ell} & \text{if } \eta_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Part II. Categorification of $U_q(gl(1|1))$-representations**

This part is the technical core of the paper, in which we construct the categorification using subquotient categories of $O$. We have tried to separate the categorification itself (in Section 7) from the Lie theoretical tools, which are collected in Section 6. The reader who is interested in the categorification but not in all the Lie theoretical details may want to skim quickly Section 6 and pass then to Section 7.

6. Subquotient categories of $O$

We are going to define the subquotient categories which we will use for the categorification. This section is purely Lie theoretical. We will start with a quick reminder about Serre quotient categories (§6.1). We will then give two equivalent definitions of the categories $O^p, q \text{-pres}_\lambda$ (§6.2 and §6.3) and describe their properly stratified structure. Finally, in §6.4 we introduce the functors between these categories that we will use for categorifying the action of $U_q(gl(1|1))$ and of the intertwining operators in the next section.

6.1. Serre quotients and projectively presented modules. We recall some standard facts about Serre quotient categories.

**Serre quotients.** Let $A$ be an abelian category. A non-empty full subcategory $S \subset A$ is called a **Serre subcategory** if it is closed under extensions. Given a Serre subcategory $S \subset A$ one defines the **quotient category** $A/S$ to be the category with the same objects of $A$ and with morphisms

$$\text{Hom}_{A/S}(M, N) = \lim_{\longrightarrow} \text{Hom}_A(M', N')$$

where the direct limit is taken over all pairs $M' \subseteq M$, $N' \subseteq N$ such that $M/M' \in S$ and $N' \in S$. The quotient category comes with an exact quotient functor $Q: A \to A/S$.

Now suppose that $A = A\text{-mod}$ for some finite-dimensional algebra $A$, and suppose that the simple $A$-modules are up to isomorphism $L(\lambda)$ for $\lambda \in \Lambda$. For each subset $\Gamma \subseteq \Lambda$ define $S_\Gamma$ to be the Serre subcategory of $A\text{-mod}$ consisting of the modules with all composition factors of type $L(\lambda)$ for $\lambda \in \Gamma$. For all $\lambda \in \Lambda$ let $P(\lambda)$ be the projective cover of $L(\lambda)$. Then we have the following (see for example [AM11, Proposition 33]):

**Proposition 6.1.** Set $P = P_\Gamma = \bigoplus_{\lambda \in \Lambda - \Gamma} P(\lambda)$. Then there is an equivalence of categories

$$A/S_\Gamma \cong \text{mod} - \text{End}_A(P)$$

The quotient functor is $Q = \text{Hom}_A(P, -)$.

In particular, thanks to (6.2), the category $A/S_\Gamma$ becomes an abelian category. Notice that the quotient functor has as left adjoint the functor $\bigotimes_{\text{End}(P)^{op}} -$. 
Presentable modules. Let $\mathcal{C}$ be an additive subcategory of the abelian category $A$. We define the category of $\mathcal{C}$-presentable objects to be the full subcategories of $A$ consisting of all objects $M \in A$ having a presentation
\[
Q_1 \longrightarrow Q_2 \longrightarrow M
\]
with $Q_1, Q_2 \in \mathcal{C}$. Now suppose as before $A = A-mod$ for a finite-dimensional algebra $A$. Given a projective module $P \in A-mod$ we let $\text{Add}(P)$ be the full subcategory of $A-mod$ consisting of all modules which admit a direct sum decomposition with summands being direct summands of $P$, and we consider the category $\text{Add}(P) = \text{mod}$-$A$-$\text{mod}$ of $A$-presentable objects to be the full subcategories of $A$.

6.2. Subquotient categories of $\mathcal{O}$. Let us fix a positive integer $n$. Let $\mathfrak{gl}_n$ be the general Lie algebra of $n \times n$ matrices with the standard Cartan decomposition $\mathfrak{gl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be the standard Borel subalgebra. Consider the BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n) = \mathcal{O}(\mathfrak{gl}_n, \mathfrak{b})$: this is the full subcategory of finitely generated $U(\mathfrak{gl}_n)$-modules that are weight modules for the action of $\mathfrak{b}$ with integral weights and that are locally $\mathfrak{n}^+$-finite. We stress that we consider only modules with integral weights. We recall some standard facts on the category $\mathcal{O}$; for more details we refer to [Hum92].

The category $\mathcal{O}$ is a highest weight category ([CPSSS]). For a weight $\lambda$ of $\mathfrak{gl}_n$ we let $M(\lambda)$ be the Verma module with highest weight $\lambda$. We let $L(\lambda)$ be the unique simple quotient of $M(\lambda)$ and $P(\lambda)$ be its projective cover. The modules $L(\lambda)$ for $\lambda$ running over the integral weights of $\mathfrak{gl}_n$ give a full set of pairwise non-isomorphic simple objects in $\mathcal{O}$.

We consider the dot action of the Weyl group $S_n$ on $\mathfrak{h}^*$, given by $w \cdot \lambda = w(\lambda + \rho) - \rho$. Two simple objects $L(\lambda)$, $L(\mu)$ are in the same block of $\mathcal{O}$ if and only if $\lambda$ and $\mu$ are in the same $S_n$-orbit under the dot action. For an integral dominant weight $\lambda$ we let $\Omega^\lambda$ be the block of $\mathcal{O}$ containing $L(\lambda)$. We have then a block decomposition $\mathcal{O} = \bigoplus_{\lambda} \mathcal{O}^\lambda$.

Given a standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{gl}_n$ with Levi factor $\mathfrak{l}$, let $\mathcal{O}^\mathfrak{p}$ be the full subcategory of $\mathcal{O}$ consisting of modules that, viewed as $U(\mathfrak{l})$-modules, are direct sums of finite dimensional simple $\mathfrak{l}$-modules. Let $W_\mathfrak{p} \subset S_n$ be the standard parabolic subgroup corresponding to $\mathfrak{p}$, and let $W^\mathfrak{p}$ be the set of shortest coset representatives for $W_\mathfrak{p}/S_n$. Then $\mathcal{O}^\mathfrak{p}$ is also the full Serre subcategory of $\mathcal{O}$ containing the simple objects $L(w \cdot \lambda)$ for $\lambda$ dominant and $w \in W^\mathfrak{p}$. Let $P^\mathfrak{p}(w \cdot \lambda)$ be the projective cover of $L(w \cdot \lambda)$ in $\mathcal{O}^\mathfrak{p}$ and let $M^\mathfrak{p}(\lambda)$ be the corresponding parabolic Verma module. The block decomposition of $\mathcal{O}$ induces a block decomposition $\mathcal{O}^\mathfrak{p} = \bigoplus_{\lambda} \mathcal{O}^\mathfrak{p}_\lambda$.

Let now $\mathfrak{p}, \mathfrak{q}$ be two orthogonal standard parabolic subalgebras of $\mathfrak{gl}_n$ (by orthogonal we mean that the corresponding subsets $P_\mathfrak{p}, P_\mathfrak{q}$ of the simple roots $\Pi$ of $\mathfrak{gl}_n$ are orthogonal; this is equivalent to imposing that $\mathfrak{p} + \mathfrak{q}$ is also a parabolic subalgebra of $\mathfrak{gl}_n$ and $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{b}$). Let $W_\mathfrak{p}, W_\mathfrak{q}$ be the corresponding parabolic subgroups of the Weyl group $S_n$. Note that, since $\mathfrak{p}$ and $\mathfrak{q}$ are orthogonal, $W_\mathfrak{p} \times W_\mathfrak{q}$ is also a subgroup of $S_n$. Consider the general Lie algebras $\mathfrak{gl}_\mathfrak{p}, \mathfrak{gl}_\mathfrak{q} \subset \mathfrak{gl}_n$ with Weyl groups $W_\mathfrak{p}$ and $W_\mathfrak{q}$ respectively, so that $\mathfrak{p} = \mathfrak{gl}_\mathfrak{p} + \mathfrak{b}$ and $\mathfrak{q} = \mathfrak{gl}_\mathfrak{q} + \mathfrak{b}$.

Following [AM92], we let:

- $\mathcal{P}^\mathfrak{p}$ be the additive semisimple subcategory of $\mathcal{O}(\mathfrak{gl}_\mathfrak{p})$ generated by the dominant simple module $L(0)$. 


\( \mathcal{P}_q = \text{Add}(P(w_q, 0)) \) be the additive subcategory of \( \mathcal{O}(\mathfrak{gl}_q) \) generated by the anti-dominant indecomposable projective module \( P(w_q, 0) \), where \( w_q \in W_q \) is the longest element.

Let also \( \overline{\mathcal{P}}_q \) be the category of \( \mathcal{P}_q \)-presentable modules (cf. \( \S 6.1 \)).

**Remark 6.2.** The category \( \overline{\mathcal{P}}_q \) is equivalent to the category of finitely generated modules over the endomorphism algebra of a projective generator of \( \mathcal{P}_q \) (see \( \S 6.1 \)), and therefore is an abelian category. In particular, if \( W_q \cong S_k \) then by Soergel’s endomorphism and structure theorems (cf. Section \( \S 10 \)), \( \overline{\mathcal{P}}_q \) is equivalent to the category of finitely generated modules over \( \mathbb{C}[x_1, \ldots, x_k]/(e_1, \ldots, e_k) \), where \( e_1, \ldots, e_k \) are the non-constant elementary symmetric polynomials. On the other side, the category \( \mathcal{P}_q \) is equivalent to the category of \( \mathbb{C} \)-mod.

Let \( \mathfrak{a} = \mathfrak{a}_p + q = (\mathfrak{gl}_p \oplus \mathfrak{gl}_q) + \mathfrak{h} \) and define \( \mathfrak{n}_{p+q} \) by \( \mathfrak{p} + q = \mathfrak{a} \oplus \mathfrak{n}_{p+q} \). Taking the external tensor product we obtain a subcategory \( \mathcal{P}^p \boxtimes \overline{\mathcal{P}}_q \) of \( (\mathfrak{gl}_p \oplus \mathfrak{gl}_q) \)-modules.

Extending the action by zero, we can consider this as a category of \( \mathfrak{a} \)-modules. Let \( \mathcal{P}^p \) be the additive closure of the full subcategory of \( \mathfrak{a} \)-modules which have the form \( E \otimes P \), where \( E \) is a simple finite dimensional \( \mathfrak{a} \)-module and \( P \in \mathcal{P}^p \boxtimes \overline{\mathcal{P}}_q \) is a projective object. Finally, let \( \mathcal{A}^p = \overline{\mathcal{P}}_q \) be the category of \( \mathcal{P}^p \)-presentable \( \mathfrak{a} \)-modules.

In other words, \( \mathcal{A}^p = \overline{\mathcal{P}}_q \) is the additive closure

\[
(E \otimes (L(0) \boxtimes P(w_q, 0))) | E \text{ is a simple finite dimensional } \mathfrak{a} \text{-module}
\]

and \( \mathcal{A}^p = \overline{\mathcal{P}}_q \).

**Definition 6.3.** We define \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) to be the full subcategory of \( \mathfrak{gl}_q \)-modules that are:

1. finitely generated;
2. locally \( \mathfrak{n}_{p+q} \)-finite;
3. direct sum of objects of \( \mathcal{A}^p_q \) as \( \mathfrak{a} \)-modules.

The categories \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) fall into a more general family of categories that were first introduced in [FKM02] (called generalized parabolic subcategories of \( \mathcal{O} \)) and then generalized in [Maz04]. Our definition follows [MS08], and in particular is a special case of [MS08] Definition 32. Notice that the category \( \mathcal{A}^p_q \) is admissible (in the sense of [MS08] \( \S 6.3 \)) by [MS08] Lemma 33. However, in [MS08] only the trivial block is studied, while we will be interested also in singular blocks.

**Lemma 6.4.** The category \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) is a subcategory of \( \mathcal{O}^p \).

**Proof.** Conditions (2) and (3) together imply that modules of \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) are locally \( \mathfrak{n}^\mathfrak{p} \)-finite; condition (3) also implies that modules of \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) are weight modules for \( \mathfrak{h} \); hence \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) is a subcategory of \( \mathcal{O} \). By condition (3), moreover, objects of \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) are direct sums of finite dimensional simple \( \mathfrak{gl}_p \)-modules.

Hence \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \) is a subcategory of \( \mathcal{O}^p \).

It follows in particular that the block decomposition \( \mathcal{O}^p = \bigoplus_{\mathfrak{A}} \mathcal{O}^p_\mathfrak{A} \) induces a direct sum decomposition

\[
\mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) = \bigoplus_{\mathfrak{A}} \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_{\mathfrak{A}})_{\mathfrak{A}}.
\]

**Lemma 6.5.** We have the following inclusions of full subcategories:

1. if \( \mathfrak{p}' \subset \mathfrak{p} \) then \( \mathcal{O}(\mathfrak{p} + q, \mathcal{A}^p_q) \subset \mathcal{O}(\mathfrak{p}' + q, \mathcal{A}^{p'}_q) \);
2. if \( \mathfrak{q}' \subset \mathfrak{q} \) then \( \mathcal{O}(\mathfrak{p} + \mathfrak{q}, \mathcal{A}^p_q) \subset \mathcal{O}(\mathfrak{p} + q', \mathcal{A}^{p+q'}_q) \).
We warn the reader, however, that the second inclusion will not be an exact inclusion of abelian categories (once we will have defined the abelian structure on the categories \( \mathcal{O}\{p+q, \mathcal{A}^p_q\} \), see \( \S 6.3 \).

**Proof.** Let \( M \in \mathcal{O}\{p+q, \mathcal{A}^p_q\} \). By definition, \( M \) is finitely generated and locally \( \mathfrak{n}^+ \)-finite. Write \( M = \bigoplus \alpha M_{\alpha} \) as an \( \mathfrak{a}_{p+q} \)-module, with \( M_{\alpha} \in \mathcal{A}^p_q \). Let \( P_{\alpha} \to Q_{\alpha} \to M_{\alpha} \) be a \( \mathcal{P}^p_q \)-presentation of \( M_{\alpha} \). Considering this as a sequence of \( \mathfrak{a}_{p+q} \)-modules (resp. \( \mathfrak{a}_{p+q} \)-modules), we see that it is enough to show that

(i) every object of \( \mathcal{P}^p_q \) decomposes, as an \( \mathfrak{a}_{p+q} \)-module, into a direct sum of objects of \( \mathcal{P}^p_q \);

(ii) every object of \( \mathcal{P}^p_q \) decomposes, as an \( \mathfrak{a}_{p+q} \)-module, into a direct sum of objects of \( \mathcal{P}^p_q \).

Since \( \S 6.3 \) is straightforward (every object of \( \mathcal{P}^p_q \) is, as an \( \mathfrak{a}_{p+q} \)-module, an object of \( \mathcal{P}^p_q \)), let us verify \( \S 6.3 \). For this it is enough to check that, for every dominant integral weight \( \lambda \) of \( \mathfrak{gl}_q \), the anti-dominant projective module \( P(w_{\lambda} \cdot \lambda) \in \mathcal{O}(\mathfrak{gl}_q) \) decomposes, as a \( \mathfrak{gl}_q \)-module, as direct sum of objects of type \( E \otimes P(w_{\mu} \cdot \mu) \) for some weight \( \mu \) of \( \mathfrak{gl}_q \) and some finite dimensional \( \mathfrak{gl}_q \)-module \( E \). This follows because \( \mathcal{O}(\mathfrak{gl}_q) \supset P(w_{\lambda} \cdot \lambda) = U(\mathfrak{gl}_q) \otimes_{\mathcal{O}(\mathfrak{gl}_q)} P(w_{\lambda} \cdot \lambda|_{\mathfrak{gl}_q}) \), and \( P(w_{\lambda} \cdot \lambda) \) can be obtained from \( P(w_{\lambda} \cdot \lambda) \) in \( \mathcal{O}(\mathfrak{gl}_q) \) by tensoring with finite dimensional modules. \( \square \)

6.3. The parabolic categories of \( p \)-presentable modules. We will give now another definition of the blocks of \( \mathcal{O}\{p+q, \mathcal{A}^p_q\} \). Let \( \lambda \) be a dominant integral weight for \( \mathfrak{gl}_n \) with stabilizer \( S_{\lambda} \) under the dot action. Define

\[
\Lambda^p_{\mathfrak{gl}}(\lambda) = \left\{ w \in (S_n/S_{\lambda})_{\text{short}} \mid wS_{\lambda} \subset W^p \right\}.
\]

Notice that \( w^q W^g \) is simply the set of longest coset representatives for \( W^q \backslash S_n \). When we omit \( p \) or \( q \) from the notation, we assume them to be the borel \( \mathfrak{b} \). If \( \lambda \) is regular then in particular \( \Lambda^p_{\mathfrak{gl}}(\lambda) = \{ w_{\lambda} w \mid w \in W^p \} \) is the set of elements of \( S_n \) that are shortest coset representatives for \( W^p \backslash S_n \) and longest coset representatives for \( W^q \backslash S_n \). Let

\[
\mathcal{P}^p_{\mathfrak{gl}}(\lambda) = \bigoplus_{x \in \Lambda^p_{\mathfrak{gl}}(\lambda)} P^p(x \cdot \lambda)
\]

and as in \( \S 6.3 \) let \( \text{Add}(\mathcal{P}^p_{\mathfrak{gl}}(\lambda)) \) be the full subcategory of \( \mathcal{O}^p_{\mathfrak{gl}} \) consisting of all modules which admit a direct sum decomposition with summands being direct summands of \( \mathcal{P}^p_{\mathfrak{gl}}(\lambda) \).

**Definition 6.6.** We define the category \( \mathcal{O}_{\lambda}^{p,q}\text{-\text{pres}} \) to be the full subcategory of \( \mathcal{O}_{\lambda}^p \) which consists of all \( \text{Add}(\mathcal{P}^p_{\mathfrak{gl}}(\lambda)) \)-presentable modules.

**Proposition 6.7.** For all integral dominant weights \( \lambda \), the categories \( \mathcal{O}\{p+q, \mathcal{A}^p_q\}_{\lambda} \) and \( \mathcal{O}_{\lambda}^{p,q}\text{-\text{pres}} \) coincide.

**Proof.** First we show the inclusion \( \mathcal{O}_{\lambda}^{p,q}\text{-\text{pres}} \subset \mathcal{O}\{p+q, \mathcal{A}^p_q\}_{\lambda} \). Consider \( P^p(w_{\lambda} \cdot \lambda) \) in \( \mathcal{O}_{\lambda}^p \). Let \( L(\lambda|_{\mathfrak{gl}_q}) \otimes \mathbf{P}(w_{\lambda} \cdot \lambda|_{\mathfrak{gl}_q}) \in \mathcal{O}(\mathfrak{gl}_q) \) denote the \( (\mathfrak{gl}_q \oplus \mathfrak{gl}_q) \)-module obtained as external tensor product of the finite-dimensional simple \( \mathfrak{gl}_q \)-module \( L(\lambda|_{\mathfrak{gl}_q}) \in \mathcal{O}(\mathfrak{gl}_q) \) and of the antidominant indecomposable projective module \( P(w_{\lambda} \cdot \lambda|_{\mathfrak{gl}_q}) \in \mathcal{O}(\mathfrak{gl}_q) \). Consider it as an \( \mathfrak{a} \)-module by extending the action to \( \mathfrak{h} \) with the weight \( \lambda \), and then as a \( (p+q) \)-module by letting \( n_{p+q} \) act by zero. By the analogue of the BGG construction of projective modules in \( \mathcal{O} \) [BGG76], we have

\[
P^p(w_{\lambda} \cdot \lambda) = U(\mathfrak{gl}_n) \otimes_{p+q} (L(\lambda|_{\mathfrak{gl}_q}) \otimes \mathbf{P}(w_{\lambda} \cdot \lambda|_{\mathfrak{gl}_q})).
\]
Since $U(\mathfrak{gl}_n)$ decomposes as direct sum of finite dimensional modules for the adjoint action of $\mathfrak{gl}_p \oplus \mathfrak{gl}_q$, it follows that \((\ref{eq:1})\), as an $\mathfrak{a}$-module, decomposes as direct sum of objects of $\mathcal{P}_q$. Since by tensoring \((\ref{eq:2})\) with finite dimensional $\mathfrak{gl}_m$-modules we can obtain all projective modules $P^\mathfrak{g}(x \cdot \lambda)$ for $x \in \Lambda^p_q(\lambda)$, and $\mathcal{P}_q$ is closed under tensor product with finite dimensional modules, the same holds for them. Now, if $M \in \mathcal{O}_q^{\mathfrak{g}, \text{pres}}$ then we have a presentation $Q_1 \to Q_2 \to M$ with $Q_1, Q_2 \in \text{Add}(\mathcal{P}_q^\mathfrak{g}(\lambda))$. Considering this as a sequence of $\mathfrak{a}$-modules, it follows that $M$ decomposes as a direct sum of objects of $\mathcal{A}_q^\mathfrak{g} = \mathcal{P}_q^\mathfrak{g}$.

Now let us show the other inclusion $\mathcal{O}\{\mathfrak{p} + q, \mathcal{A}_q^\mathfrak{g} \}_{\lambda} \subseteq \mathcal{O}_q^{\mathfrak{g}, \text{pres}}$. Let $M \in \mathcal{O}\{\mathfrak{p} + q, \mathcal{A}_q^\mathfrak{g} \}_{\lambda}$. By Lemma \ref{lem:1} we have $M \in \mathcal{O}_q^\mathfrak{g}$. As an $\mathfrak{a}$-module, $M$ is generated by elements of weight $x \cdot \lambda$ with $sx < x$ for any simple reflection $s \in W_q$ (i.e. $x \cdot \lambda$ is an anti-dominant weight for $\mathfrak{gl}_q$). Of course then this is also true as a $\mathfrak{gl}_p$-module. Hence the projective cover $Q$ of $M$ in $\mathcal{P}_q^\mathfrak{g}$ is an element of $\text{Add}(\mathcal{P}_q^\mathfrak{g}(\lambda))$. Let $K = \ker(Q \to M)$ in $\mathcal{P}_q^\mathfrak{g}$, and consider the short exact sequence $K \to Q \to M$ as a sequence of $\mathfrak{a}$-modules. Since all objects of $\mathcal{A}_q^\mathfrak{g}$ are finitely generated, we may assume (taking direct summands) that $K \subseteq Q$ is a short exact sequence of finitely generated $\mathfrak{a}$-modules, that is, we can suppose $M \in \mathcal{A}_q^\mathfrak{g}$ and, by the first paragraph, $Q \in \mathcal{P}_q^\mathfrak{g}$. We can write $Q = Q_M \oplus Q'$ where $Q_M$ is the projective cover of $M$, and $K = Q' \cap \ker(Q_M \to M)$. Since $M \in \mathcal{A}_q^\mathfrak{g}$, we have a presentation $P_M \to Q_M \to M$ with $P_M, Q_M \in \mathcal{P}_q^\mathfrak{g}$, hence we have a surjective map $P_M \to \ker(Q_M \to M)$ and therefore a surjective map $P' \to K$ for some $P' \in \mathcal{P}_q^\mathfrak{g}$. Since as an $\mathfrak{a}$-module $P'$ is generated by elements of weight $x \cdot \lambda$ with $sx < x$ for any simple reflection $s \in W_q$, the same holds for $K$. Hence we can apply the same construction we did for $M$ to $K$ and get a presentation $P \to Q \to M$ with $P, Q \in \text{Add}(\mathcal{P}_q^\mathfrak{g}(\lambda))$.

In particular, for $\mathfrak{p} = \mathfrak{b}$ and $\lambda = 0$ we get the category $\mathcal{O}_0^{\mathfrak{g}, \text{pres}}$ of \cite{MS05}. The results of \cite{MS05} Section 2 carry over to the case of an arbitrary integral weight $\lambda$. In particular, we have:

**Lemma 6.8.** The category $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$ is an abelian category with a simple preserving duality and is equivalent to $\text{End}(\mathcal{P}_\lambda^\mathfrak{g}(\lambda)) - \text{mod}$. For $x \in \Lambda_\lambda^\mathfrak{g}(\lambda)$ the modules $P(x \cdot \lambda)$ are obviously objects of $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$. Each $P(x \cdot \lambda)$ has a unique simple quotient $S(x \cdot \lambda)$ in $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$, and the set $\{S(x \cdot \lambda) \mid x \in \Lambda_\lambda^\mathfrak{g}(\lambda)\}$ gives a full set of pairwise non-isomorphic simple objects of $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$.

We want to extend these results to the general case $\mathfrak{p} \neq \mathfrak{b}$. First, let us recall the definition of the Zuckerman’s functor $\mathfrak{z} : \mathcal{O} \to \mathcal{O}_\mathfrak{p}$. Given $M \in \mathcal{O}$, the object $\mathfrak{z}M$ is the biggest quotient of $M$ that lies in $\mathcal{O}_\mathfrak{p}$. The functor $\mathfrak{z}$ is right exact and $\mathfrak{z}P(x \cdot \lambda) = P^\mathfrak{g}(x \cdot \lambda)$ for each $\lambda \in \Lambda^\mathfrak{g}(\lambda)$.

**Lemma 6.9.** The category $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$ coincides with the full subcategory of objects of $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$ that are in $\mathcal{O}_\mathfrak{p}$.

**Proof.** Since both are full subcategories of $\mathcal{O}(\mathfrak{gl}_n)$, we need only to prove that they have the same objects. Let $M \in \mathcal{O}_\mathfrak{p}^{\mathfrak{g}, \text{pres}} \cap \mathcal{O}_\mathfrak{p}$ and consider a presentation $P \to Q \to M \to 0$ with $P, Q \in \text{Add}(\mathcal{P}_\mathfrak{g}(\lambda))$. Applying $\mathfrak{z}$ yields a presentation $\mathfrak{z}P \to \mathfrak{z}Q \to M \to 0$ with $\mathfrak{z}P, \mathfrak{z}Q \in \text{Add}(\mathcal{P}_\mathfrak{g}(\lambda))$.

The other inclusion follows by Proposition \ref{prop:5} and Lemma \ref{lem:6}.

**Lemma 6.10.** The category $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$ is the Serre subcategory of $\mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$ generated by the simple objects $S(x \cdot \lambda)$ for $x \in \Lambda_\lambda^\mathfrak{g}(\lambda)$.

**Proof.** First let us prove that $S(x \cdot \lambda) \in \mathcal{O}_\lambda^{\mathfrak{g}, \text{pres}}$ is in $\mathcal{O}_\mathfrak{p}$ if $x \in \Lambda^\mathfrak{g}_\mathfrak{p}(\lambda)$. Let $P \to Q \to S(x \cdot \lambda)$ be a presentation of $S(x \cdot \lambda)$ with $P, Q \in \text{Add}(\mathcal{P}_\mathfrak{g})$. Applying the Zuckerman’s functor $\mathfrak{z}$ yields a presentation of $\mathfrak{z}S(x \cdot \lambda)$ with $\mathfrak{z}P, \mathfrak{z}Q \in \text{Add}(\mathcal{P}_\mathfrak{g})$.
Since \( \mathfrak{g}P(x \cdot \lambda) \neq 0 \) (because \( L(x \cdot \lambda) \) is a quotient of \( P(x \cdot \lambda) \) in \( \mathcal{O} \)) and \( S(x \cdot \lambda) \) is a quotient of \( P(x \cdot \lambda) \), it follows that \( \mathfrak{g}S(x \cdot \lambda) \neq 0 \). On the other side, \( \mathfrak{g}S(x \cdot \lambda) \in \mathcal{O}_\lambda^{\text{q-pres}} \) by Lemma [6.10]. But \( \mathfrak{g}S(x \cdot \lambda) \) is a non-zero quotient in \( \mathcal{O}_\lambda^{\text{q-pres}} \) of the simple module \( S(x \cdot \lambda) \), hence \( \mathfrak{g}S(x \cdot \lambda) = S(x \cdot \lambda) \). It follows that \( S(x \cdot \lambda) \in \mathcal{O}_\lambda^{\text{q-pres}} \).

On the other side, if \( x \not\in \Lambda^p_\lambda(\lambda) \) but \( x \not\in \Lambda^p_\lambda(\lambda) \), then clearly \( S(x \cdot \lambda) \notin \mathcal{O}_\lambda^{\text{q-pres}} \). Since \( \mathcal{O}_\lambda^{\text{q-pres}} \) is closed under extensions, it follows that the objects of \( \mathcal{O}_\lambda^{\text{q-pres}} \) that are also in \( \mathcal{O}_\lambda^{\text{p-pres}} \) are exactly the objects whose composition factors are of type \( S(x \cdot \lambda) \) for \( x \in \Lambda^p_\lambda(\lambda) \).

It follows that the modules \( S(x \cdot \lambda) \) for \( x \in \Lambda^p_\lambda(\lambda) \) give a full set of pairwise non-isomorphic simple objects of \( \mathcal{O}_\lambda^{\text{p,q-pres}} \). Moreover, the projective cover of \( S(x \cdot \lambda) \) is \( P^\mathfrak{g}(x \cdot \lambda) \).

The graded abelian structure. The category \( \mathcal{O}_\lambda^{\text{p,q-pres}} \) is equivalent to the category of finitely generated (right) modules over \( \text{End}(\mathcal{P}_\lambda(\lambda)) \); via this equivalence we can define on \( \mathcal{O}_\lambda^{\text{p,q-pres}} \) a natural abelian structure. However, as we already pointed out, this abelian structure is not induced by the abelian structure of \( \mathcal{O}_\lambda \).

Of course, \( \mathcal{O}_\lambda \) is equivalent to \( \text{End}(\mathcal{P}(\lambda)) \)-mod where \( \mathcal{P}(\lambda) \) is a minimal projective generator of \( \mathcal{O}_\lambda \). The algebra \( \text{End}(\mathcal{P}_\lambda(\lambda)) \) can be obtained from \( A_\lambda = \text{End}(\mathcal{P}(\lambda)) \) in two steps. First, let \( e^\mathfrak{g} \in \text{End}(\mathcal{P}(\lambda)) \) be the idempotent projecting onto the direct sum of the projective modules \( P(x \cdot \lambda) \) for \( x \notin \Lambda^\mathfrak{g}(\lambda) \). Then \( \text{End}(\mathcal{P}_\lambda(\lambda)) = A_\lambda/A_\lambda e^\mathfrak{g}A_\lambda \). Moreover, let \( \overline{f}_q \in A_\lambda/A_\lambda e^\mathfrak{g}A_\lambda \) be the idempotent projecting onto the direct sum of the projective modules \( P^\mathfrak{g}(x \cdot \lambda) \) for \( x \in \Lambda^p_\lambda(\lambda) \).

By Lemma [6.10] the two steps can be done also in the inverse order: let \( f_q \in A_\lambda \) be the idempotent projecting onto the direct sum of the projective modules \( P(x \cdot \lambda) \) for \( x \in \Lambda^p_\lambda(\lambda) \). Then \( \text{End}(\mathcal{P}_\lambda(\lambda)) = f_q A_\lambda f_q \). Moreover, let \( \overline{f}_q \in f_q A_\lambda f_q \) be the idempotent projecting onto the direct sum of the projective modules \( P(x \cdot \lambda) \) for \( x \notin \Lambda^p_\lambda(\lambda) \). Then \( \text{End}(\mathcal{P}_\lambda(\lambda)) = (f_q A_\lambda f_q)/(f_q A_\lambda f_q e^\mathfrak{g} A_\lambda f_q) \). It follows that

\[
(f_q A_\lambda f_q)/(f_q A_\lambda f_q e^\mathfrak{g} A_\lambda f_q) = \overline{f}_q (A_\lambda/A_\lambda e^\mathfrak{g} A_\lambda) f_q.
\]

As far as we understand, this is not a trivial result, but instead a consequence of Lemma [6.10].

By [BCS06], the algebra \( A_\lambda \) has a natural Koszul grading. This induces a grading on the algebra \( \mathcal{O}_\lambda^{\text{p,q-pres}} \). Summarizing, we have:

**Proposition 6.11.** The category \( \mathcal{O}_\lambda^{\text{p,q-pres}} \) is equivalent to the category of finitely generated right modules over a finite dimensional positively graded algebra.

We will denote by \( \mathcal{Z}\mathcal{O}_\lambda^{\text{p,q-pres}} \), the graded version of \( \mathcal{O}_\lambda^{\text{p,q-pres}} \), that is the category of finitely generated graded modules over the algebra \( \mathcal{O}_\lambda^{\text{p,q-pres}} \). There is an obvious forgetful functor \( \mathfrak{f} : \mathcal{Z}\mathcal{O}_\lambda^{\text{p,q-pres}} \to \mathcal{O}_\lambda^{\text{p,q-pres}} \), which forgets the grading. By a graded lift of an object \( M \in \mathcal{O}_\lambda^{\text{p,q-pres}} \) we mean an object \( \hat{M} \in \mathcal{Z}\mathcal{O}_\lambda^{\text{p,q-pres}} \) such that \( \mathfrak{f}\hat{M} = M \). The object \( M \) is then called gradable. Of course, not all modules are gradable (cf. [Str03a]), but all interesting ones will be. In particular, the techniques of [Str03a] ensure that simple and indecomposable projective modules are gradable, both as objects of \( \mathcal{O}_\lambda \) and of \( \mathcal{O}_\lambda^{\text{p,q-pres}} \) (although the grading is different). We take their standard graded lifts to be determined by requiring that the simple head is concentrated in degree 0. We will use the notation \( qM = M(1) \) for the graded shift of a module.

**The properly stratified structure.** The results of [MS05, Section 2] extend to the categories \( \mathcal{O}_\lambda^{\text{p,q-pres}} \). Let us briefly sketch them.
As in [MS05 Proposition 2.6], one can define a simple-preserving duality on \( \mathcal{O}^{p,q}_{\lambda} \). Alternatively, one can observe that \( A_\lambda \) is a symmetric algebra (and this induces the duality on \( \mathcal{O} \)) and then prove the following:

**Lemma 6.12.** The algebra \( \mathcal{O}^{p,q}_{\lambda} \) is a symmetric algebra; this induces a simple-preserving duality on \( \mathcal{O}^{p,q}_{\lambda} \).

The module \( P^p(x \cdot \lambda) \) is the projective cover of \( S(x \cdot \lambda) \) in \( \mathcal{O}^{p,q}_{\lambda} \). Hence \( S(x \cdot \lambda) = P^p(x \cdot \lambda)/\text{Tr}_{q_{\mathfrak{g}}}(\text{rad} P^p(x \cdot \lambda)) \), where given two modules \( M, N \) the trace of \( M \) in \( N \) is defined to be \( \text{Tr}_M N = \bigcup_{f:M \rightarrow N} \text{Im} f \). Let \( P^p(x) = \bigoplus_{w \in \Sigma} P^p(w \cdot \lambda) \) and set \( \Delta(x \cdot \lambda) = P^p(x \cdot \lambda)/\text{Tr}_{P^p(x \cdot \lambda)} P^p(x \cdot \lambda) \). As in [MS05 Lemma 2.8], one can show that the modules \( \Delta(x \cdot \lambda) \) satisfy a universal property, and as in [MS05 Proposition 2.9] this can be used to show that

\[
\Delta(x \cdot \lambda) \cong U(\mathfrak{gl}_n) \otimes_{p+q} P^{(s)}(x \cdot \lambda),
\]

where \( P^{(s)}(x \cdot \lambda) \) is the projective cover in \( \mathcal{O}^{p,q}_{\lambda} \) of the highest weight module with highest weight \( x \cdot \lambda \). Moreover, one can define

\[
\Delta(x \cdot \lambda) \cong U(\mathfrak{gl}_n) \otimes_{p+q} S^{(s)}(x \cdot \lambda),
\]

where \( S^{(s)}(x \cdot \lambda) \) is the simple module in \( \mathcal{A}^{p,q}_{\lambda} \) with highest weight \( x \cdot \lambda \). The same argument of [MS05 Theorem 2.16] gives:

**Theorem 6.13.** The algebra \( \mathcal{O}^{p,q}_{\lambda} \) with the order induced by the Bruhat order on \( \Lambda^q_{\lambda} \) is a graded properly stratified algebra (see Definition 11.20). The modules \( \Delta(x \cdot \lambda) \) and \( \Delta(x \cdot \lambda) \) are the standard and proper standard modules respectively.

It is easy to show that also the modules \( \Delta(x \cdot \lambda) \) and \( \Delta(x \cdot \lambda) \) are gradable. Again, we choose their standard lifts by requiring the simple heads to be concentrated in degree 0.

### 6.4. Functors between categories \( \mathcal{O}^{p,q}_{\lambda} \)

We conclude this section examining the natural functors that can be defined between the categories we have introduced. In particular, for \( p' \supseteq p \) and \( q' \supseteq q \) we will define functors

\[
\begin{array}{ccc}
\mathcal{O}^{p,q}_{\lambda} & \xrightarrow{j} & \mathcal{O}^{p',q'}_{\lambda} \\
\mathcal{O}^{p,q}_{\lambda} & \xleftarrow{i} & \mathcal{O}^{p',q'}_{\lambda} \\
\end{array}
\]

**Zuckernann’s functors.** Suppose \( p' \) is also a standard parabolic subalgebra of \( \mathfrak{gl}_n \) with \( p' \subset p \). Let us fix an integral dominant weight \( \lambda \). We have then an inclusion functor \( j : \mathcal{O}^{p}_{\lambda} \rightarrow \mathcal{O}^{p'}_{\lambda} \). Since the abelian structure of \( \mathcal{O}^{p'}_{\lambda} \) is the restriction of the abelian structure of \( \mathcal{O}^{p,q}_{\lambda} \), this is an exact functor. Using Lemma 6.5, we see that this restricts to an exact functor \( j : \mathcal{O}^{p,q}_{\lambda} \rightarrow \mathcal{O}^{p',q'}_{\lambda} \).

The left adjoint of \( j : \mathcal{O}^{p,q}_{\lambda} \rightarrow \mathcal{O}^{p,q}_{\lambda} \) is the **Zuckernann’s functor** \( \tilde{j} : \mathcal{O}^{p,q}_{\lambda} \rightarrow \mathcal{O}^{p,q}_{\lambda} \), defined on \( M \in \mathcal{O}^{p,q}_{\lambda} \) by taking the maximal quotient that lies in \( \mathcal{O}^{p,q}_{\lambda} \). The functor \( \tilde{j} \) is right exact, but not exact in general. Being right exact, \( \tilde{j} \) sends a presentation \( P \rightarrow Q \rightarrow M \) with \( P, Q \in \text{Add}(\mathcal{O}^{p,q}_{\lambda}(\lambda)) \) to a presentation \( \tilde{j}P \rightarrow \tilde{j}Q \rightarrow \tilde{j}M \) of \( \tilde{j}M \) with \( \tilde{j}P, \tilde{j}Q \in \text{Add}(\mathcal{O}^{p,q}_{\lambda}(\lambda)) \), hence it restricts to a functor \( \tilde{j} : \mathcal{O}^{p,q}_{\lambda} \rightarrow \mathcal{O}^{p,q}_{\lambda} \).

Notice that the definitions of \( j \) and \( \tilde{j} \) make sense in the graded setting too, hence we have also adjoint functors

\[
\begin{array}{ccc}
\mathcal{O}^{p,q}_{\lambda} & \xrightarrow{j} & \mathcal{O}^{p',q'}_{\lambda} \\
\mathcal{O}^{p,q}_{\lambda} & \xleftarrow{\tilde{j}} & \mathcal{O}^{p,q}_{\lambda} \\
\end{array}
\]
Coapproximation functors. Suppose \( q' \) is also a standard parabolic subalgebra of \( \mathfrak{gl}_n \) with \( q' \subset q \) and let us fix an integral dominant weight \( \lambda \). According to Lemma 6.3, we have an inclusion functor \( i : \mathcal{O}^p_{\lambda^q} \rightarrow \mathcal{O}^p_{\lambda^q} \). This is right exact but not left exact in general (cf. [MS05, Example 2.3] for an example).

Its right adjoint \( \Omega : \mathcal{O}^p_{\lambda^q} \rightarrow \mathcal{O}^p_{\lambda^q} \) is called coapproximation, and can be described Lie theoretically as follows. Take \( M \in \mathcal{O}^p_{\lambda^q} \), and let \( p : Q \rightarrow \text{Tr} \mathfrak{p}(\lambda)(M) \) be a projective cover in \( \mathcal{O}^p_{\lambda^q} \) (notice that \( \mathfrak{p}(\lambda) \) is a direct summand of \( \mathfrak{p}(\lambda) \) and in particular an object of \( \mathcal{O}^p_{\lambda^q} \)). Then define \( \Omega(M) = Q/\text{Tr} \mathfrak{p}(\lambda)(\text{ker} p) \). It is not difficult to show that \( \Omega \) is an exact functor. Moreover, it induces also a functor \( \Omega : \mathcal{O}^p_{\lambda^q} \rightarrow \mathcal{O}^p_{\lambda^q} \).

To compute the action of \( \Omega \) on proper standard modules, we will need the following easy fact:

**Lemma 6.14.** Let \( q' \subset q \) and let \( w \in \Lambda^p_q(\lambda) \). Then there exists a unique \( x \in W_q \) such that \( xw \in \Lambda^p_q(\lambda) \) and \( \ell(xw) = \ell(x) + \ell(w) \).

**Proof.** Let \( S_\lambda \) be the stabilizer of the weight \( \lambda \). Since \( p \) is orthogonal to \( q \), we may assume \( p = \mathfrak{b} \). Moreover, since \( \Lambda^p_q(\lambda) \subset (S_n/S_\lambda)_{\text{short}} \), it is clearly sufficient to prove the result for \( w \in (S_n/S_\lambda)_{\text{short}} \). Then the lemma is simply a statement about double cosets. Let \( z \) be the shortest element in the double coset \( W_q w S_\lambda \). Then all shortest coset representatives for \( S_n/S_\lambda \) contained in \( W_q w S_\lambda \) can be obtained as \( yz \) for \( y \in W_q \) (and in particular \( w = y_1 z \) for \( y_1 \in W_q \)). Let \( y_0 \in W_q \) be the shortest element such that \( y_0 z \in S_\lambda \cap (W_q S_\lambda)_{\text{long}} \neq \varnothing \) (this exists, since this is the unique element such that \( y_0 z w_\lambda \) is the longest element of the double coset \( W_q w S_\lambda \), where \( w_\lambda \) is the longest element of \( S_\lambda \)). Setting \( x = y_0 y_1^{-1} \) we get the claim. \( \square \)

First, we suppose \( q' \) to be the trivial parabolic subalgebra \( \mathfrak{b} \), and we compute the action of \( \Omega \) on Verma modules.

**Proposition 6.15.** Consider the coapproximation functor \( \Omega : \mathcal{O}^p_q \rightarrow \mathcal{O}^p_{\lambda^q} \). Let \( w \in \Lambda^p_q(\lambda) \), and let \( x \in W_q \) be the element given by Lemma 6.14 such that \( xw \in \Lambda^p_q(\lambda) \). Then we have \( \Omega M^p(w \cdot \lambda) = q^{\ell(x)} x \Omega M^p(w \cdot \lambda) \).

We will need some preliminary results in the ungraded setting.

**Lemma 6.16.** Suppose \( w \in \Lambda^p_q(\lambda) \), let \( M(w \cdot \lambda) \) be a Verma module in \( \mathcal{O}_\lambda \) and \( M^p(w \cdot \lambda) \) be its parabolic quotient in \( \mathcal{O}^p_q \). Then for every simple reflection \( s \in W_q \) such that \( \ell(sw) > \ell(w) \) the map \( M^p(sw \cdot \lambda) \rightarrow M^p(w \cdot \lambda) \) induced at the quotient by the inclusion \( M(sw \cdot \lambda) \hookrightarrow M(w \cdot \lambda) \) is injective.

**Example 6.17.** Notice that in the statement of the lemma it is essential to assume that the simple reflection \( s \) is orthogonal to the parabolic subalgebra \( p \). As a counterexample when this is not true, consider the regular block \( \mathcal{O}^p_q(\mathfrak{gl}_n) \), where \( p \subset \mathfrak{gl}_n \) is the standard parabolic subalgebra corresponding to the composition \((2, 1)\). Then the inclusion \( M(s_2 \cdot 0) \hookrightarrow M(0) \) of Verma modules in \( \mathcal{O}(\mathfrak{gl}_n) \) induces a map \( M^p(s_2 \cdot 0) \rightarrow M^p(0) \) which is not injective (the kernel is isomorphic to the simple module \( L(s_2 s_1 \cdot 0) \)).

**Proof of Lemma 6.17.** Let \( v_{sw}, v_w \) be the highest weight vectors of \( M(sw \cdot \lambda) \) and \( M(w \cdot \lambda) \) respectively. Then (cf. [Hum08 §1.4]) the inclusion \( M(sw \cdot \alpha) \hookrightarrow M(w \cdot \lambda) \) is determined by \( v_{sw} \rightarrow \ell^{\text{char}} f_{sw} v_{sw} \) for some \( k \in \mathbb{N} \), where \( f_{sw} \in \mathbb{N}^+ \) is the standard generator of \( U(\mathfrak{gl}_n) \) corresponding to the simple root \( \alpha_s \). This indeed defines an injective map because the Verma modules are free as \( U(\mathfrak{n}^-) \)-modules and \( U(\mathfrak{n}^-) \) has no zero divisors.
Let $\mathfrak{gl}_n = \mathfrak{p} \oplus \mathfrak{u}^\mathfrak{p}_n$. The parabolic Verma modules can be defined through parabolic induction, hence they are free as $U(\mathfrak{u}^\mathfrak{p}_n)$-modules (although in general not of rank one). Since the simple reflection $s$ is orthogonal to the set of reflections $W_p$, the element $f_{α_α}$ lies in $U(\mathfrak{u}^\mathfrak{p}_n)$ and the map on the quotients is again given by multiplication by it. By the same argument as before, this map has to be injective. □

**Lemma 6.18.** With the same notation as before, $\mathrm{coker} \left( M_P(sw \cdot λ) \to M_P(w \cdot λ) \right)$ has only composition factors of type $L(y \cdot λ)$ with $sy > y$.

**Proof.** The inclusion is given by multiplication by $f_{α_α}$. By the PBW Theorem, it follows immediately that the cokernel is locally $(f_{α_α})_{k \in \mathbb{N}}$-finite, hence all its composition factors are indexed by elements of $S_n$ that are shortest coset representatives for $\langle s \rangle \backslash S_n$. □

**Lemma 6.19.** For every $w \in \Lambda^p(λ)$ and $x \in W_q$ such that $xw \in \Lambda^p(λ)$ we have $q^{(x)} \Omega M_P(xw \cdot λ) = \Omega M_P(w \cdot λ)$.

**Proof.** Of course, it is sufficient to prove it for a simple reflection $s \in W_q$. Then the result follows from Lemma 6.18 if we apply the exact functor $\Omega$ to the short exact sequence

\[(6.14) \quad qM_P(sw \cdot λ) \hookrightarrow M_P(w \cdot λ) \twoheadrightarrow Q.\]

**Lemma 6.20.** Let $w \in \Lambda^p_q(λ)$. Then $\Omega M_P(w \cdot λ) = \overline{Δ}(w \cdot λ)$.

**Proof.** The projective module $P_P(w \cdot λ)$ has a filtration by parabolic Verma modules in $O^p_λ$. Hence the projective module $P_P(w \cdot λ) = \Omega P_P(w \cdot λ)$ in $O^p_λ$-pres has a filtration by modules $\Omega M_P(y \cdot λ)$ for $y \in \Lambda^p(λ)$, $y \preceq w$.

Now the proper standard module $\overline{Δ}(w \cdot λ)$ is defined to be the maximal quotient $Q$ of $P_P(w \cdot λ)$ in $O^p_λ$-pres satisfying

\[(6.15) \quad \langle \mathrm{rad} \ Q : S(z \cdot λ) \rangle = 0 \quad \text{for all } z \preceq w.\]

Obviously the quotient $\Omega M_P(w \cdot λ)$ at the top of $P_P(w \cdot λ)$ satisfies (6.15). Any bigger quotient contains the simple head of some $\Omega M_P(y \cdot λ)$ for $y \prec w$. Consider such a $y$ and let $x' \in W_q$ be the element given by Lemma 6.13 for $y$. By Lemma 6.13 the simple head in $O^p_λ$-pres of $\Omega M_P(y \cdot λ)$ is the simple head of $\Omega M_P(x'y \cdot λ)$, where $x' \in W_q$ is the element given by Lemma 6.13 for $y \in \Lambda^p(λ)$; but this is the simple head of $\Omega M_P(x'y \cdot λ)$, that is $S(x'y \cdot λ)$. Notice that $x'y \preceq w$ (this follows because $y \prec w$ and both $x'y, w \in \Lambda^p_q(λ)$). Hence $\Omega M_P(w \cdot λ)$ is indeed the maximal quotient satisfying (6.15). □

The proof of the proposition follows now easily:

**Proof of Proposition 6.15.** By Lemma 6.19 we have $\Omega M_P(w \cdot λ) = q^{(x)} \Omega M_P(xw \cdot λ)$ and by Lemma 6.20 this is $q^{(x)} \overline{Δ}(xw \cdot λ)$. □

Using Proposition 6.15 it is easy to prove a general result for any standard parabolic subalgebra $q'$ with $q' \subset q$:

**Corollary 6.21.** Let $q'$ be a standard parabolic subalgebra of $\mathfrak{gl}_n$ with $q' \subset q$ and consider the coapproximation functor $Ω : ZO_λ^{q',q'-\text{pres}} \to ZO_λ^{p,q'-\text{pres}}$. Let $w \in \Lambda^p_q(λ)$ and let $x \in W_q$ be the element given by Lemma 6.13. Then we have $ΩΔ(w \cdot λ) = p^{(x)}ΩΔ(xw \cdot λ)$.

**Proof.** Let $Ω_q : ZO_λ^{q',q'-\text{pres}}$ and $Ω_q : ZO_λ^{p,q'-\text{pres}}$ be the coapproximation functors. It follows from the definition that $Ω_q Ω_q' = Ω_q$. By Proposition 6.15 we have $Ω_q M_P(w \cdot λ) = Δ(w \cdot λ)$ and $Ω_q M_P(w \cdot λ) = p^{(x)}Δ(xw \cdot λ)$, and the claim follows. □
Using the coapproximation functor $\Omega$ we can compute proper standard filtration of standard modules:

**Proposition 6.22.** Suppose that $q$ has only one block (that is, $W_q \cong S_k$ for some integer $k$) and let $\lambda$ be a dominant regular weight. Then for all $w \in \Lambda^0_q(\lambda)$ the proper standard filtration of the standard module $\Delta(w \cdot \lambda)\Omega^\mu_q$-pres has length $k!$. In particular, in the Grothendieck group of $\Omega^\mu_q$-pres we have

\[ [\Delta(w \cdot \lambda)] = q^{k(k-1)/2} [\Delta(w \cdot \lambda)]. \]

**Proof.** Since $\lambda$ is regular, $w$ is a longest coset representative for $W_q \setminus S_n$, hence $w = w_0 w'$. It is well-known that in a Verma flag of the projective module $P^\mu(w \cdot \lambda)$ all Verma modules $P^\mu(xw' \cdot \lambda)$ for $x \in W_q$ appear exactly once. Applying $\Omega$, by Proposition 6.14 we get a filtration of $P^\mu(w \cdot \lambda)$ in $\Omega^\mu_q$-pres with $\Delta(w \cdot \lambda)$ appearing exactly $k!$ times. Of course, this is the part of the filtration that builds the standard module $\Delta(w \cdot \lambda)$. By the Kazhdan-Lusztig conjecture, in the Grothendieck group of $\Omega^\mu_q$ we have

\[ [P^\mu(w \cdot \lambda)] = \sum_{x \in W_q} q^{\ell(wu) - \ell(x)}[P^\mu(xw' \cdot \lambda)] + \sum_{x \in W_q, x < w'} q^2[\Delta(w \cdot \lambda)]. \]

Applying $\Omega$ and considering only the part of the filtration that builds $\Delta(w \cdot \lambda)$ we get

\[ [\Delta(w \cdot \lambda)] = \sum_{x \in W_q} q^{2\ell(wu) - \ell(x)}[\Delta(w \cdot \lambda)], \]

which is the same as (6.16). \qed

**Graded lifts of translation functors.** Remember that we have translation functors $T^\mu_\lambda : \mathcal{O}_\lambda(gl_n) \to \mathcal{O}_\mu(gl_n)$, defined classically in the ungraded settings. In [Str03a], graded lifts are introduced if $\lambda$ is regular and $\mu$ semi-regular, or the opposite. We need to work in a more general case.

Let $\lambda, \mu$ be weights with stabilizers $S_\lambda, S_\mu$ respectively, and suppose $S_\lambda \subset S_\mu$. As in [Str03a, Section 8], it follows that the translation functors $T^\mu_\lambda$ and $T^\mu_\mu$ have graded lifts. We need to specify the graded shifts. Recall that translation functors are uniquely determined by their action on the dominant Verma module. Hence it is sufficient to specify the shift for it. We fix therefore $T^\mu_\lambda M(\lambda) = P(\mu)$ and $T^\mu_\mu M(\mu) = P(x_0 \cdot \lambda)$, where $x_0$ is the longest element in $(S_\mu/S_\lambda)_{\text{short}}$.

**Lemma 6.23.** We have graded adjunctions

\[ (T^\mu_\mu, q^{\ell(x_0)}T^\mu_\lambda) \quad \text{and} \quad (T^\mu_\lambda, q^{-\ell(x_0)}T^\mu_\mu). \]

**Proof.** It suffices to check the shifts, hence to verify (6.19) on the dominant Verma modules. We have

\[ q^{\ell(x_0)}C = \text{Hom}(P(x_0 \cdot \lambda), M(\lambda)) = \text{Hom}(T^\mu_\mu P(\mu), M(\lambda)) \]

\[ = \text{Hom}(P(\mu), q^{\ell(x_0)}T^\mu_\lambda M(\lambda)) = \text{Hom}(P(\mu), q^{\ell(x_0)}P(\mu)) = q^{\ell(x_0)}C \]

and

\[ C = \text{Hom}(P(\mu), M(\mu)) = \text{Hom}(T^\mu_\mu M(\lambda), M(\mu)) \]

\[ = \text{Hom}(M(\lambda), q^{-\ell(x_0)}T^\mu_\mu M(\mu)) = \text{Hom}(M(\lambda), q^{-\ell(x_0)}P(x_0 \cdot \lambda)) = C. \]

For the first calculation, we used that the well-known fact that the composition factor $L(x_0 \cdot \lambda)$ appears in $M(\lambda)$ only once in degree $\ell(x_0)$; for the second one, we used the also well-known fact that the shifted Verma module $q^{\ell(x_0)}M(\lambda)$ appears at the bottom of the projective module $P(x_0 \cdot \lambda)$. \qed
Translation functors in $Z\mathcal{O}^{P,q\text{-pres}}$. Translation functors preserve the subcategories we have introduced:

**Lemma 6.24.** Given two dominant weights $\lambda, \mu$, the translation functor $T^\mu_\lambda$ restricts to a functor $\mathcal{O}^{P,q\text{-pres}}_{\lambda} : \mathcal{O}^{P,q\text{-pres}}_{\lambda} \to \mathcal{O}^{P,q\text{-pres}}_{\mu}$. Moreover, translation functors commute with the functors $\mathfrak{z}, \mathfrak{i}, \Omega$.

**Proof.** It follows directly from the definition that tensoring with a finite dimensional $\mathfrak{gl}_n$-module defines an exact endofunctor of the category $\mathcal{O}(p+q,A^2)$. In particular, the translation functor $T^\mu_\lambda$ preserves the category $\mathcal{O}(p+q,A^2)$.

Since $\mathfrak{z}, \mathfrak{i}$ are inclusions, it follows that $T^\mu_\lambda$ commutes with them. By adjunction, it commutes also with $\mathfrak{z}, \Omega$. \qed

Of course we have also the graded version

\begin{equation}
T^\mu_\lambda : Z\mathcal{O}^{P,q\text{-pres}}_\lambda \to Z\mathcal{O}^{P,q\text{-pres}}_\mu.
\end{equation}

If $\lambda$ and $\mu$ are weights with stabilizers $S_\lambda, S_\mu$ with $S_\lambda \subset S_\mu$, we will use the expressions translation onto the wall and translation out of the wall to indicate the translation functors $T^\mu_\lambda$ and $T^\lambda_\mu$ respectively (notice that in the literature these expressions are often used only when $\lambda$ is regular). We will need the following easy result to compute the action of translation functors \((6.22)\) in the category $Z\mathcal{O}^{P,q\text{-pres}}$.

**Lemma 6.25.** Let $S_\lambda, S_\mu$ be standard parabolic subgroups of $S_n$ with $S_\lambda \subset S_\mu$. Then for every $w \in (S_n/S_\lambda)^{\text{short}}$ there exist unique elements $w' \in (S_n/S_\mu)^{\text{short}}$, $x \in (S_n/S_\lambda)^{\text{short}}$ such that $w = w'x$. Moreover $\ell(w) = \ell(w') + \ell(x)$.

**Proof.** The element $w$ determines some coset $wS_\mu$, in which there is a unique shortest coset representative $w'$. Hence $w = w'x$ for some $x \in S_\mu$ with $\ell(w) = \ell(w') + \ell(x)$. Since $w \in (S_n/S_\lambda)^{\text{short}}$ we have $\ell(wt) > \ell(w)$ for all $t \in S_\lambda$; but then also $\ell(xt) > \ell(x)$ for all $t \in S_\lambda$, hence $x \in (S_n/S_\lambda)^{\text{short}}$. \qed

Now we compute how translation functors act on proper standard modules. First, we consider translation onto the wall:

**Proposition 6.26.** Let $\lambda, \mu$ be dominant weights with stabilizers $S_\lambda, S_\mu$ respectively, and suppose $S_\lambda \subset S_\mu$. Let $w \in \Lambda^2_\lambda(\mu)$, and write $w = w'x$ as given by Lemma 6.25. Then we have

\begin{equation}
T^\mu_\lambda \overline{\Delta}(w \cdot \lambda) = \begin{cases} q^{-\ell(x)} \overline{\Delta}(w' \cdot \mu), & \text{if } w' \in \Lambda^2_\mu(\mu), \\ 0, & \text{otherwise.} \end{cases}
\end{equation}

**Proof.** First, we compute in the usual category $\mathcal{O}(\mathfrak{gl}_n)$. It is well-known that translating a Verma module to the wall gives a Verma module. In fact if we forget the grading then $T^\mu_\lambda M(w \cdot \lambda) = M(w' \cdot \mu)$ (cf. [Hum08, Theorem 7.6]). The graded version can be computed generalizing [Str03a, Theorem 8.1], and is $T^\mu_\lambda M(w \cdot \lambda) = q^{-\ell(x)} M(w' \cdot \mu)$.

Now since the functors $\mathfrak{z}$ and $\Omega$ commute with $T^\mu_\lambda$, using Proposition 6.15 we have

\begin{equation}
T^\mu_\lambda \overline{\Delta}(w \cdot \lambda) = T^\mu_\lambda \overline{\mathfrak{z}} M(w \cdot \lambda) = \overline{\mathfrak{z}} T^\mu_\lambda M(w \cdot \lambda) = q^{-\ell(x)} \overline{\mathfrak{z}} M(w' \cdot \mu).
\end{equation}

If $w' \notin \Lambda^2_\mu(\mu)$ then $\mathfrak{z} M(w' \cdot \mu) = 0$. Otherwise we get $q^{-\ell(x)} \overline{\Delta}(w' \cdot \mu)$. \qed

Now let us compute translation of proper standard modules out of the wall:
Proposition 6.27. Let $\lambda, \mu$ be dominant weights with stabilizers $S_\lambda, S_\mu$ respectively, and suppose $S_\lambda \subset S_\mu$. Then for every $w \in \Lambda^2_\mu(\mu)$ we have
\begin{equation}
[T^\lambda_m \Delta(w \cdot \mu)] = \sum_{y \in (S_\mu/S_\lambda)^{\text{short}}} q^{\ell(y_0) - \ell(y) + \ell(x_y)} [\Delta(x_y w y \cdot \lambda)],
\end{equation}
where $y_0$ is the longest element of $(S_\mu/S_\lambda)^{\text{short}}$, and for every $y \in (S_\mu/S_\lambda)^{\text{short}}$ the element $x_y$ is the element given by Lemma 6.17 for $w y \in \Lambda^\mu(\lambda)$.

Note that $w \in \Lambda^2_\mu(\mu)$ implies that $w S_\mu \subseteq W^p$; but as $S_\lambda \subseteq S_\mu$, we have then $w y S_\lambda \subseteq W^p$, and in particular $w y \in \Lambda^\mu(\lambda)$ for all $y \in (S_\mu/S_\lambda)^{\text{short}}$.

Proof. Consider $M(w \cdot \mu)$ in $Z^\mathcal{O}$. Then we have
\begin{equation}
[T^\lambda_m M(w \cdot \mu)] = \sum_{y \in (S_\mu/S_\lambda)^{\text{short}}} q^{\ell(y_0) - \ell(y)} [M(w y \cdot \lambda)].
\end{equation}
This is well-known in the ungraded setting (see for example [Hum08 Theorem 7.12]); the graded version follows as in [Str05]. Since the Zuckermann’s functor $\mathfrak{z}$ is exact on modules that admit a Verma flag, we can apply $\mathfrak{z}$ to both sides of (6.26). Hence we get in $Z^\mathcal{O}^p$:
\begin{equation}
[T^\lambda_m P(w \cdot \mu)] = \sum_{y \in (S_\mu/S_\lambda)^{\text{short}}} q^{\ell(y_0) - \ell(y)} [P^\mu(w y \cdot \lambda)].
\end{equation}
Now we can apply the exact functor $\mathcal{O}$ to both sides. Using Proposition 6.15 and the commutativity of $\mathcal{O}$ with $T^\lambda_m$ we obtain the claim. \hfill \square

Now we compute translations of projective modules out of the wall:

Proposition 6.28. Let $\lambda, \mu$ be dominant weights with stabilizers $S_\lambda, S_\mu$ respectively, and suppose that $S_\lambda \subset S_\mu$. Then for every $w \in \Lambda^2_\mu(\mu)$ we have in $Z^\mathcal{O}^{p,q,\text{pres}}$,
\begin{equation}
T^\lambda_m P^\mu(w \cdot \mu) = P^\mu(w y_0 \cdot \lambda)
\end{equation}
where $y_0$ is the longest element of $(S_\mu/S_\lambda)^{\text{short}}$.

Proof. Let $P(w \cdot \lambda) \in Z^\mathcal{O}$. By [Hum08 Theorem 7.11] we have $T^\lambda_m P(w \cdot \mu) = P(w y_0 \cdot \lambda)$ as ungraded modules. By (6.27), the top Verma module is not shifted under translation, hence this also holds as graded modules. Applying the Zuckermann’s functor $\mathfrak{z}$ we get (6.28) in $Z^\mathcal{O}^p$, hence also in $Z^\mathcal{O}^{p,q,\text{pres}}$. Notice that we get for free that $w y_0 \in \Lambda^2_\mu(\lambda)$ (although it would be easy to check it directly). \hfill \square

Using the adjunctions (6.11), we can then compute translations of simple modules onto the wall:

Proposition 6.29. Let $\lambda, \mu$ be dominant weights with stabilizers $S_\lambda, S_\mu$ respectively, and suppose that $S_\lambda \subset S_\mu$. Let $y_0$ be the longest element of $(S_\mu/S_\lambda)^{\text{short}}$. Then for every $w \in \Lambda^2_\mu(\lambda)$ we have in $Z^\mathcal{O}^{p,q,\text{pres}}$
\begin{equation}
T^\lambda_m S(w \cdot \lambda) = \begin{cases} q^{-\ell(y_0)} S(z \cdot \mu) & \text{if } w = z y_0 \text{ for some } z \in \Lambda^2_\mu(\mu) \subseteq S_\mu \\ 0 & \text{otherwise}. \end{cases}
\end{equation}
Proof. We use the previous result together with the adjunction $(T^\lambda_m, q^{\ell(y_0)} T^\mu_m)$. For every projective module $P^\mu(z \cdot \mu) \in Z^\mathcal{O}^{p,q,\text{pres}}$ we have
\begin{equation}
\text{Hom}(T^\lambda_m P^\mu(z \cdot \mu), S(w \cdot \lambda)) \cong \text{Hom}(P^\mu(z \cdot \mu), q^{\ell(y_0)} T^\mu_m S(w \cdot \lambda)).
\end{equation}
The left hand side is 0 unless $w = z y_0$, in which case it is $\mathcal{O}$, and the claim follows. \hfill \square
7. The Categorification

In this section, which contains the main theorems of the paper, we will construct explicitly the categorification of the representations studied in Section 5. We will define the categorification itself in §7.2 and construct the action of the intertwining operators and of \( U_q(\mathfrak{gl}(1\mid 1)) \) in §7.3 and §7.6 respectively. In §7.5 we will categorify the bilinear form (5.9) and in §7.4 we will prove that the indecomposable projective modules categorify the canonical basis.

In the following, we fix a positive integer \( n \) and we let \( k \in \{0, \ldots, n\} \). If \( \Pi = \{\alpha_1, \ldots, \alpha_{n-1}\} \) is the set of the simple roots of \( \mathfrak{gl}_n \), we let \( p \) and \( q \) be the standard parabolic subalgebras of \( \mathfrak{gl}_n \) with corresponding sets of simple roots \( \Pi_p = \{\alpha_1, \ldots, \alpha_{k-1}\} \) and \( \Pi_q = \{\alpha_k, \ldots, \alpha_{n-1}\} \), so that \( W_{p+q} = S_k \times S_{n-k} \subset S_n \).

For every composition \( a \) of \( n \) we fix, once and forever, a dominant integral weight \( \lambda_a \) for \( \mathfrak{gl}_n \) with stabilizer \( S_a \) under the dot action. We suppose for future notational convenience that if \( n \) is the regular composition of \( n \) (5.1) then \( \lambda_n = 0 \). We set

\[
(7.1) \quad \Lambda_k(a) \overset{\text{def}}{=} \Lambda_p^q(\lambda_a) \quad \text{and} \quad Q_k(a) \overset{\text{def}}{=} Z_{\mathcal{O}_p,q-\text{pres}}^a.
\]

From now on, for \( w \in \Lambda_k(a) \) we denote by \( S(w) \in Q_k(a) \) the simple module \( S(w \cdot \lambda_a) \) and by \( Q(w) \) its projective cover \( P(w \cdot \lambda_a) \). We let also \( \Delta(w) \) and \( \overline{\Delta}(w) \) be the corresponding standard and proper standard module.

7.1. Combinatorics. A hook partition of shape \((n-k, k)\) is given by a row of length \( n-k \) and a column of length \( k \) (cf. Figure 4). Notice that for us the box in the corner belongs to the row, but not to the column. If \( a = (a_1, \ldots, a_\ell) \) is a composition of \( n \), a \((n-k, k)\)-tableau of type \( a \) is a tableau filled with the integers (7.2)

\[
1, \ldots, 1, 2, \ldots, 2, \ldots, \ell, \ldots, \ell.
\]

If we number the boxes of the hook partition of shape \((n-k, k)\) from 1 to \( n \) starting with the column from the bottom to the top and ending with the row from the left to the right, then the permutation group \( S_n \) acts from the left on the set of \((n-k, k)\)-tableaux of type \( a \) permuting the boxes. The stabilizer of this action is \( S_a \).

Define the minimal \((n-k, k)\)-tableau \( T^\text{min}_a \) of type \( a \) to be the tableau obtained putting the numbers (7.2) in order first in the column, from the bottom to the top, then in the row, from the left to the right (see Figure 5). Set also

\[
(7.3) \quad T_a(w) = T^\text{min}_a \cdot w
\]

for each \( w \in S_a \). Then we can define a bijection \( w \mapsto T_a(w) \) between \((S_n/S_a)^\text{short} \) and \((n-k, k)\)-tableaux of type \( a \).
Figure 5. These are \((1,2,2,2)\)-tableaux of type \((3,4)\). The leftmost tableau is the minimal one. Notice that only the last one is admissible.

We say that a tableau is \textit{admissible} if:

(a) the numbers in the row are strictly increasing (from left to right);
(b) the numbers in the column are non-increasing (from the bottom to the top).

For an example see Figure 5.

\textbf{Proposition 7.1.} The bijection

\begin{equation}
(S_n/S_\alpha)^{\text{short}} \xrightarrow{1-1} \{(n-k,k)\text{-tableaux of type } \alpha\}
\end{equation}

\begin{equation}
w \mapsto T_\alpha(w)
\end{equation}

restricts to a bijection

\begin{equation}
\Lambda_k(\alpha) \xrightarrow{1-1} \{\text{admissible } (n-k,k)\text{-tableaux of type } \alpha\}.
\end{equation}

\textbf{Proof.} Given \(w \in (S_n/S_\alpha)^{\text{short}}\), it is enough to observe that the condition \([a]\) is equivalent to \(w \in W^p\) and the condition \([b]\) is equivalent to \(wS_\alpha \cap wqW^q \neq \emptyset\). \(\square\)

\section{The Grothendieck group.}

Let \(A\) be an abelian category. We recall that the Grothendieck group \(K(A)\) is the quotient of the free \(\mathbb{Z}\)-module on generators \([M]\) for \(M \in A\) modulo the relation \([B] = [A] + [C]\) for each short exact sequence \(A \hookrightarrow B \rightarrow C\). If the category \(A\) is graded then \(K(A)\) becomes a \(\mathbb{Z}[q,q^{-1}]\)-module under \(q[M] = [qM] = [M(1)]\). For an abelian graded category \(A\) we let moreover

\begin{equation}
K^C(q)(A) = C(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K(A).
\end{equation}

Fix a composition \(\alpha\) of \(n\) and an integer \(0 \leq k \leq n\). A basis of the Grothendieck group \(K(Q_k(\alpha))\) as a \(\mathbb{Z}[q,q^{-1}]\)-module is given by the simple modules \(S(w)\) for \(w \in \Lambda_k(\alpha)\). Equivalence classes of the proper standard modules give also a basis, whereas the standard modules do not give a basis over \(\mathbb{Z}[q,q^{-1}]\) in general (although they always give a basis of \(K^C(q)(Q_k(\alpha))\) over \(C(q)\)).

According to Proposition 7.1 the set \(\Lambda_k(\alpha)\) is in bijection with the set of admissible \((n-k,k)\)-tableaux of type \(\alpha\). For \(w \in \Lambda_k(\alpha)\) let \(v(w) = v_{\eta}^\alpha \in V(\alpha)\), where

\begin{equation}
\eta_i = \begin{cases} 0 & \text{if the number } i \text{ appears in the row of } T_\alpha(w), \\ 1 & \text{otherwise}. \end{cases}
\end{equation}

We write also \(v_{(T_\alpha(w))} = v(w)\). We can then define an isomorphism

\begin{equation}
K^{C(q)}(Q_k(\alpha)) \rightarrow V(\alpha)_k
\end{equation}

\begin{equation}
[\Delta(w)] \mapsto \frac{1}{(v(w),v(w))} v(w).
\end{equation}
Notice that if \( a = (a_1, \ldots, a_\ell) \) then for \( k < n - \ell \) the category \( \Omega_k(a) \) is empty. We set
\[
\Omega(a) = \bigoplus_{k=n-\ell} \Omega_k(a)
\]
and we get an isomorphism
\[
K^{C(q)}(\Omega(a)) \cong V(a).
\]

7.3. Categorification of the intertwiners. Let \( \mathcal{O}\text{Cat} \) be the category whose objects are finite direct sums of the categories \( \Omega_k(a) \) for all \( n \geq 0, 0 \leq k \leq n \) and for all compositions \( a \) of \( n \), and whose morphisms are all functors between these categories. We define a functor \( \mathcal{F} : \text{Web} \to \mathcal{O}\text{Cat} \) as follows. If \( a = (a_1, \ldots, a_\ell) \) is an object of \( \text{Web} \) with \( n = \sum a_i \), then we set
\[
\mathcal{F}(a) = \bigoplus_{k=n-\ell} \Omega(a).
\]
We define \( \mathcal{F} \) on the elementary webs (5.45) and (5.46) by
\[
\mathcal{F}(\Lambda_{a,i}) = T^{\bar{a}_i}_a, \quad \mathcal{F}(\Gamma^{\mu,i}) = T^{\bar{a}_i}_a
\]
where \( \bar{a}_i \) was defined in (5.44) and if \( \lambda_b, \lambda' \) are the fixed dominant weights of \( \mathfrak{gl}_n \) with stabilizers \( S_b S_{b'} \) then \( T^{\lambda_b}_b = T^{\lambda'_{b'}}_{b'} \). Since the elementary web diagrams generate all web diagrams, we get a functor \( \mathcal{F} : \text{Web}' \to \mathcal{O}\text{Cat} \). By the following lemma, the functor \( \mathcal{F} \) respects the relations (5.48) and (5.49) and hence descends to a functor \( \mathcal{F} : \text{Web} \to \mathcal{O}\text{Cat} \).

Lemma 7.2. If \( a, a', a'' \) are three compositions of \( n \) with \( S_a \supseteq S_{a'} \supseteq S_{a''} \) then
\[
T^{a''}_{a'} = T^{a'}_{a''} T^{a}_{a'} \quad \text{and} \quad T^{a''}_{a} = T^{a''}_{a} T^{a}_{a''}.
\]

Proof. We prove that this holds in the whole category \( \mathcal{O} \). By the classification theorem of projective functors [Hum08, Theorem 10.8] it is enough to show that the equalities (7.13) hold when the functors are applied to a dominant Verma module. This follows from [Hum08, Theorem 10.8]. \( \square \)

The functor \( \mathcal{F} \) categorifies the functor \( \mathcal{F} \) (cf. 5.3):

Theorem 7.3. The following diagram commutes:
\[
\begin{array}{ccc}
\text{Web} & \xrightarrow{\mathcal{F}} & \mathcal{O}\text{Cat} \\
\mathcal{F} & \downarrow & \\
\text{Rep} & \xrightarrow{K^{C(q)}} & \\
\end{array}
\]

Proof. Let \( a = (a_1, \ldots, a_\ell) \) and \( a' = (a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_\ell) \). We need to show that if \( \varphi : a \to a' \), \( \psi : a' \to a \) are the webs with only one triple vertex then \( K^{C(q)}(T^{a}_{a'}) = \mathcal{F}(\varphi) \) and \( K^{C(q)}(T^{a'}_{a}) = \mathcal{F}(\psi) \). Of course it is sufficient to check this on the basis of proper standard modules. Hence it suffices to check that
\[
[T^{a}_{a'}(w \cdot \lambda)] = \Phi_{a_1,a_{i+1}}[\Delta(w \cdot \lambda)],
\]
\[
[T^{a}_{a'}(w' \cdot \mu)] = \Phi_{a_1,a_{i+1}}[\Delta(w' \cdot \mu)]
\]
for all \( w \in \Lambda_k(a) \) and \( w' \in \Lambda_k(a') \) (for all possible values of \( k \)), where \( \Phi_{a_1,a_{i+1}} \) and \( \Phi_{a_1,a_{i+1}} \) are the maps (5.3) and (5.4) (tenored with identities).
Let us fix $k$ and start with (7.15). Fix $w \in \Lambda_k(\mathbf{a})$ and write $w = w'x$ with 
$w' \in (S_n/S_{a'})_{\text{short}}, \ x \in (S_{\mathbf{a}}/S_{\mathbf{a}})_{\text{short}}$ as given by Lemma 6.25. By Proposition 6.26 we have

$$T_a \Delta(w \cdot \lambda) = \begin{cases} q^{-\ell(x)} \Delta(w', \mu) & \text{if } w' \in \Lambda_k(\mathbf{a}'), \\ 0 & \text{otherwise.} \end{cases}$$

(7.17)

In what follows, we only write the $i$-th and $(i + 1)$-th tensor factors of $v_{(w')}$ and the $i$-th tensor factor of $v_{(w')}$, since the other ones are clearly the same. Let $T_a(w)$ be the $(n - k, k)$-tableau of type $a$ corresponding to $w$, and notice that the tableau $T_a(w)$ can be obtained from $T_a(w)$ by decreasing by one all entries greater or equal to $i + 1$.

We have four cases (see Figure 6):

(a) If $v_{(w')} = v_{0}^{a_i} \otimes v_{0}^{a_{i+1}}$ then $T_a(w)$ has both an entry $i$ and an entry $i + 1$ in the row. Then $T_a(w)$ has two entries $i$ in the row, and is not admissible; of course this also holds for $T_a'(w')$ since $w' = w^{-1}$. Hence $w' \notin \Lambda_k(\mathbf{a}')$ and $T_a \Delta(w \cdot \lambda) = 0$.

(b) If $v_{(w')} = v_{0}^{a_i} \otimes v_{1}^{a_{i+1}}$ then $T_a(w)$ has an entry $i$ but no entry $i + 1$ in the row. It is easy to see that in this case $x$ is a permutation of length $a_{i+1}$ composed with the longest element of $(S_{a_1+a_{i+1}+1}/(S_{a_1-1} \times S_{a_{i+1}}))_{\text{short}}$ and therefore $T_a \Delta(w \cdot \lambda) = q^{-a_{i+1}} \Delta(w', \mu)$.

(c) If $v_{(w')} = v_{1}^{a_i} \otimes v_{0}^{a_{i+1}}$ then $T_a(w)$ has an entry $i + 1$ but no entry $i$ in the row. Then $x$ is the longest element of $(S_{a_1+a_{i+1}+1}/(S_{a_1} \times S_{a_{i+1}}))_{\text{short}}$ and therefore $T_a \Delta(w \cdot \lambda) = q^{-a_{i+1}} \Delta(w', \mu)$.

(d) If $v_{(w')} = v_{0}^{a_i} \otimes v_{1}^{a_{i+1}}$ then all entries $i$ and $i + 1$ of $T_a(w)$ are in the column. Then $x$ is the longest element of $(S_{a_1+a_{i+1}+1}/(S_{a_1} \times S_{a_{i+1}}))_{\text{short}}$ and hence $T_a \Delta(w \cdot \lambda) = q^{-a_{i+1}} \Delta(w', \mu)$.

In cases (b) and (c) the tableau $T_a'(w')$ has one entry $i$ in the row, hence $v_{(w')} = v_{0}^{a_i} \otimes v_{0}^{a_{i+1}}$, while in case (d) the tableau $T_a'(w')$ has all entries $i$ in the column and hence $v_{(w')} = v_{1}^{a_i+a_{i+1}}$. Hence in all four cases we have $[T_a \Delta(w \cdot \lambda)]$ is equal to $\varphi(\Delta(w \cdot \lambda))$ up to a multiple, and we are left to check that the coefficients fit. For example in case (b) comparing with (5.3) we must check that

$$q^{-a_{i+1}} q^{-a_1} \Delta(v_{(w')} x_{(w)})_{\mathbf{a}} = q^{-a_{i+1}} \left[ a_i + a_{i+1} - 1 \right]_{a_{i+1}}.$$  

(7.18)

Using the formula (5.16) for the bilinear form and the notation as in (5.10), we compute the l.h.s. of (7.18):

$$q^{-a_{i+1}} q^{-a_1} \Delta(v_{(w')} x_{(w)})_{\mathbf{a}} = [\beta_1 + \cdots + \beta_i]_0! \Delta(\beta_j)_0! \[\Delta(\beta_{i+1})_0! \]_{a_1} \cdots \[\Delta(\beta_{i+1})_0! \]_{a_{i+1}},$$

(7.19)

where if $v_{(w')} = v_{0}^{a_i}$ and $v_{(w')} = v_{0}^{a_i}$ we set $\beta_j = \beta_j^0$ and $\beta_j' = \beta_j^0$. Substituting $\beta_j' = \beta_j$ for $j < i$, $\beta_j' = \beta_{j+1}$ for $j > i$, $\beta_i' = a_i + a_{i+1} - 1$, $\beta_i = a_i - 1$, $\beta_i = a_i - 1$, we get exactly the r.h.s. of (7.18). Similarly we can handle cases (c) and (d).

Now let us consider (7.16). Let $w' \in \Lambda_k(\mathbf{a}')$, and consider the corresponding tableau $T = T_a'(w')$. Suppose first that $v_{(w')} = v_{(w)} = v_{0}^{a_i} \otimes v_{0}^{a_{i+1}}$; then $T$ has exactly one entry $i$ in the row, and we can apply Lemma 7.4 below. Note that the tableaux $T'''$ and $T'$ of Lemma 7.4 correspond to $v_{0}^{a_i} \otimes v_{1}^{a_{i+1}}$ and $v_{1}^{a_i} \otimes v_{0}^{a_{i+1}}$, respectively. Hence we just need to check that the coefficients are the right ones. Let us start with the first term of the r.h.s. of (7.20): comparing (7.20) with (5.4),
where as before if $v$ compute the r.h.s. of (7.21):

Using the formula (5.16) for the bilinear form and the notation as in (5.10), we have

$$\begin{align*}
\cdots & i \quad i+1 \quad \cdots \\
\vdots & \vdots \\
i & \vdots \quad \vdots \\
i & \vdots \quad \vdots \\
i+1 & \vdots \\
i+1 & \vdots \\
T_a(w) & \vdots \\
\end{align*}$$

Case (a) $v(w) = v_i^a \otimes v_i^{a+1}$

$$\begin{align*}
\cdots & i+1 \quad \cdots \\
\vdots & \vdots \\
i & \vdots \\
i & \vdots \\
i+1 & \vdots \\
i+1 & \vdots \\
T_a(w) & \vdots \\
\end{align*}$$

Case (b) $v(w) = v_i^a \otimes v_i^{a+1}$

$$\begin{align*}
\cdots & i+1 \quad \cdots \\
\vdots & \vdots \\
i & \vdots \\
i & \vdots \\
i+1 & \vdots \\
i+1 & \vdots \\
T_a(w) & \vdots \\
\end{align*}$$

Case (c) $v(w) = v_i^a \otimes v_i^{a+1}$

$$\begin{align*}
\cdots & \cdots \\
\vdots & \vdots \\
i & \vdots \\
i & \vdots \\
i+1 & \vdots \\
i+1 & \vdots \\
T_a(w) & \vdots \\
\end{align*}$$

Case (d) $v(w) = v_i^a \otimes v_i^{a+1}$

$\text{FIGURE 6.}$ Here are depicted the tableaux $T_a(w)$ and $T_a'(w)$ in each of the four cases of the proof of Theorem 7.3 using the isomorphism defined by (7.8), we must show that

$$(7.20) \quad \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix} \begin{bmatrix} (v(T), v(T')) a' \\ 0 \end{bmatrix} = 1$$

or equivalently

$$(7.21) \quad \begin{bmatrix} a_i + a_{i+1} - 1 \\ a_{i+1} \end{bmatrix} (v(T), v(T')) a' = (v(T'), v(T')) a.$$
Let $a, a'$ as in the proof of Theorem 7.23. Let $T$ be an admissible tableau of type $a'$ with exactly one entry $i$ in the row. Construct admissible tableaux $T', T''$ of type $a$ as follows: first increase of 1 all entries of $T$ greater than $i$; then substitute the first $a_i+1$ entries $i$ with $i+1$ (here first means, as always for our hook diagrams, that we first go through the column from the bottom to the top and then through the row from the left to the right). Call the result $T'$. Moreover, let $T'' = T' \cdot x_0$ where $x_0$ is the longest element of $(S_{a'}/S_a)^{\short}$. Then we have

$$[\tau_a^w \overline{\Delta}(T)] = \left[\begin{array}{c} a_i + a_{i+1} - 1 \\ a_i \end{array}\right] \frac{(v(T'), v(T))_{a'}}{(v(T'), v(T))_a} = 1,$$

that follows as before. □

**Lemma 7.4.** Let $a, a'$ as in the proof of Theorem 7.23. Let $T$ be an admissible tableau of type $a'$ with exactly one entry $i$ in the row. Construct admissible tableaux $T', T''$ of type $a$ as follows: first increase of 1 all entries of $T$ greater than $i$; then substitute the first $a_i+1$ entries $i$ with $i+1$ (here first means, as always for our hook diagrams, that we first go through the column from the bottom to the top and then through the row from the left to the right). Call the result $T'$. Moreover, let $T'' = T' \cdot x_0$ where $x_0$ is the longest element of $(S_{a'}/S_a)^{\short}$. Then we have

$$[\tau_a^w \overline{\Delta}(T)] = \left[\begin{array}{c} a_i + a_{i+1} - 1 \\ a_i \end{array}\right] \frac{(v(T'), v(T))_{a'}}{(v(T'), v(T))_a} = 1,$$

that follows as before. □

**Proof.** What we need is just to translate Proposition 7.21. Let $T = T_a(w)$ for $w \in \Lambda_k(a')$. Consider the sum on the r.h.s. of (7.25). First consider the set $\{T_a(wy) \mid y \in (S_{a'}/S_a)^{\short}\}$: this consists of all tableaux obtained by permuting the entries $i$ and $i+1$ of $T$. Notice now that for all $y \in (S_{a'}/S_a)^{\short}$ the tableau $T_a(xywy)$ is obtained from $T_a(wx)$ permuting the entries $i$ and $i+1$ in the column so that it becomes admissible; in particular $\ell(x_y) + \ell(w) + \ell(y) = \ell(xywy)$ and the set $\{T_a(xywy) \mid y \in (S_{a'}/S_a)^{\short}\}$ consists of the two tableaux $T'$ and $T''$. Notice also that for each $y \in (S_{a'}/S_a)^{\short}$ we have $x_ywy = wx_y'$ where $y'$ is the longest element of $(S_{a'}/S_a)^{\short}$ with $\ell(x_y') = \ell(x_y)$; in particular also $\ell(x_y') + \ell(y) = \ell(x_y')$. Let

$$b' = (a_1, \ldots, a_i + a_{i+1} - 1, 1, a_{i+2}, \ldots, a_k),$$
$$b = (a_1, \ldots, a_i + a_{i+1} - 1, 1, a_{i+2}, \ldots, a_k).$$

Then we have $T' = T_a(yw_0'y)$ and $T'' = T_a(yw_0y)$ where $y_0$ is the longest element of $(S_{a'}/S_a)^{\short}$ and $y_0$ is the longest element of $(S_{a'}/S_a)^{\short}$. Now we can compute the two coefficients of (7.25):

$$\sum_{y \in (S_{a'}/S_a)^{\short}} q^{\ell(y_0)} q^{\ell(y) + \ell(y')} = \sum_{y \in (S_{a'}/S_a)^{\short}} q^{2\ell(y) + \ell(y')} = q^{\ell(y_0) + \ell(y')} \sum_{y \in (S_{a'}/S_a)^{\short}} q^{2\ell(y)} = q^{\ell(y_0) - \ell(y')} \sum_{y \in (S_{a'}/S_a)^{\short}} q^{2\ell(y)}.$$
while
\begin{equation}
\sum_{y \in (S_{a'} \cap S_a)} q^{\ell(y)} = \sum_{y \in (S_{a'} \cap S_a)} q^{2\ell(y)} = \sum_{z \in (S_{a_i+a_{i+1}}/(S_{a_i-1} \times S_{a_{i+1}}))^{\text{short}}} q^{2\ell(z)} = \left[ \frac{a_i + a_{i+1} - 1}{a_i + a_{i+1}} \right]_0
\end{equation}
where we restricted \( S_{a_i+a_{i+1}} \) (since the permutations act trivially elsewhere) and we substituted \( y = zz' \) for \( z' = s_{a_i+a_{i+1}-1} \cdots s_{a_{i+1}} s_{a_i} \); the element \( z_0 \) is the longest element of \( (S_{a_i+a_{i+1}}/(S_{a_i-1} \times S_{a_{i+1}}))^{\text{short}} \). □

Lemma 7.5. Let \( a, a' \) as in the proof of Theorem 7.3. Let \( T \) be an admissible tableau of type \( a' \) with all entries \( i \) in the column. Construct an admissible tableaux \( T' \) of type \( a \) as follows: first increase of \( 1 \) all entries of \( T \) greater than \( i \); then substitute the first \( a_{i+1} \) entries \( i \) with \( i+1 \) (here first means, as always for our hook diagrams, that we first go through the column from the bottom to the top and then through the row from the left to the right). Then we have
\begin{equation}
[T_a^{\mathbf{\Delta}}(T)] = \left[ \frac{a_i + a_{i+1}}{a_i} \right]_0 [\mathbf{\Delta}(T')],
\end{equation}
where for an admissible tableau \( T_a(w) \) we wrote \( \mathbf{\Delta}(T_a(w)) \) for \( \mathbf{\Delta}(w) \).

Proof. The proof is similar to the previous one, but easier. We just need to compute
\begin{equation}
\sum_{y \in (S_{a'} \cap S_a)} q^{\ell(y)} = \sum_{x \in (S_{a'} \cap S_a)^{\text{short}}} q^{2\ell(y)} = \left[ \frac{a_i + a_{i+1} + 1}{a_i} \right]_0.
\end{equation}
Let us consider in particular the regular composition \( n \) of \( n \). For every \( i = 1, \ldots, n-1 \) let \( a^{(i)} = (a_1^{(i)}, \ldots, a_{i-1}^{(i)}) \) where \( a_j^{(i)} = 2 \) if \( j = i \) or \( a_j^{(i)} = 1 \) otherwise. For \( i = 1, \ldots, n-1 \) define \( C_i = T_a^{\mathbf{\Delta}^{(i)}} \circ T_a^{\mathbf{\Delta}^{(i)}} \) as a functor \( C_i : \mathcal{Q}(n) \rightarrow \mathcal{Q}(n) \). As a consequence of Theorem 7.3 we have

Corollary 7.6. The endofunctors \( C_i \) on \( \mathcal{Q}(n) \) categorify the action of the Super Temperley Lieb Algebra \( \text{STL}_n \) (see Definition 5.5).

It is known that the functors \( C_i \), as endofunctors of \( \mathcal{Z} \mathcal{O} \), satisfy the relations
\begin{align}
(7.32a) \quad & C_i^2 = C_i(1) \oplus C_i(-1), \\
(7.32b) \quad & C_i C_j = C_j C_i, \\
(7.32c) \quad & C_i C_{i+1} C_i \oplus C_{i+1} = C_{i+1} C_i C_{i+1} \oplus C_i,
\end{align}
for all \( i, j = 1, \ldots, n-1 \) with \( |i-j| > 1 \). In fact, these are the categorical versions of the relations of the Hecke Algebra \( \mathcal{H}_n \). By Corollary 5.6 the relations \( 5.3.4 \) and \( 5.3.4 \) are satisfied in the Grothendieck group. We conjecture that their categorical version is satisfied by the functors \( C_i \):

Conjecture 7.7. The functors \( C_i \) on \( \mathcal{Q}(n) \) satisfy the relations
\begin{align}
(7.32d) \quad & C_{i-1} C_{i+1} C_i C_{i-1} C_{i+1} \oplus [2] C_{i-1} C_{i+1} C_i \\
& = [2] (C_{i-1} C_{i+1} C_i C_{i-1} C_{i+1}) + C_{i-1} C_{i+1} C_i C_{i-1} C_{i+1}, \\
(7.32e) \quad & C_{i-1} C_{i+1} C_i C_{i-1} C_{i+1} \oplus [2] C_{i-1} C_{i+1} C_i \\
& = [2] (C_{i-1} C_{i+1} C_i C_{i-1} C_{i+1}) + C_{i-1} C_{i+1} C_i C_{i-1} C_{i+1})
\end{align}
for all \( i = 2, \ldots, n-2 \).
Notice that for a functor $\mathcal{F}$ we denoted $[2]\mathcal{F} = \mathcal{F}(1) \oplus \mathcal{F}(-1)$ and $[2]^2 \mathcal{F} = \mathcal{F}(2) \oplus \mathcal{F} \oplus \mathcal{F}(\mathcal{F}(-2))$.

Although apparently harmful, we believe Conjecture 7.7 to be quite hard. The difficulty is due to the lack of a classification of projective functors on the parabolic category $\mathcal{O}^p$ if $p$ is not the Borel subalgebra $\mathfrak{b}$.

### 7.4. Canonical basis

Now we give a categorical interpretation of the canonical basis of $V(\mathfrak{a})$. First we restrict to consider the regular composition $n$. Recall that by Proposition 5.7 the canonical basis of $(V^\otimes \mathfrak{a})_k$ can be interpreted as a canonical basis for the Hecke algebra action. In this section we will use the Hecke module structure of the Grothendieck groups of our categories.

Let $p, q \subset \mathfrak{g}_n$ be the parabolic subalgebras defined at the beginning of the section, such that $\mathcal{Q}_p(n) = \mathcal{Z}\mathcal{O}_\Lambda^{p,q}\text{-pre}\mathcal{S}$. Using the notation introduced in Section 3 we fix isomorphisms

\begin{equation}
K^{(\Lambda, q)}(\mathcal{O}_\Lambda) \to \mathcal{H}_n \\
[M(w \cdot \lambda)] \mapsto H_w \\
K^{(\Lambda, q)}(\mathcal{O}_\Lambda^p) \to \mathcal{M}_p \\
[M^p(w \cdot \lambda)] \mapsto N_w.
\end{equation}

As well-known, by the Kazhdan-Lusztig conjecture projective modules are sent to the canonical basis elements of $\mathcal{H}_n$ and $\mathcal{M}_p$ by the two isomorphisms.

Composing the isomorphism (7.3) with the isomorphism (5.38) we get an isomorphism

\begin{equation}
K_0(\mathcal{Z}\mathcal{O}_\Lambda^{p,q}\text{-pre}\mathcal{S}) \to \mathcal{M}_p^n \\
\Delta(w_q w \cdot \lambda) \mapsto N_w
\end{equation}

for $w \in ((S_k \times S_n) \setminus S_n)_{\text{short}}$, where $w_q \in S_k$ is the longest element.

**Lemma 7.8.** The coapproximation functor $\mathcal{Q} : \mathcal{Z}\mathcal{O}_\Lambda^{p,q}\text{-pre}\mathcal{S} \to \mathcal{Z}\mathcal{O}_\Lambda^{p,q}\text{-pre}\mathcal{S}$ categorifies the map $Q : \mathcal{M}_p \to \mathcal{M}_p^n$ (defined in (7.3)).

**Proof.** Let $w \in \Lambda^P(n)$. By Proposition 6.13 we have $\mathcal{Q}\mathcal{M}_p(w \cdot 0) = q^\ell(x)\mathcal{L}(xw \cdot 0)$ where $x \in W_q$ is given by Lemma 6.14. Now $[\mathcal{M}_p(w \cdot 0)] = N_w \in \mathcal{M}_p$ and $[\Delta(xw \cdot 0)] = \frac{1}{|\mathcal{L}(xw)|}N_w \in \mathcal{M}_p^n$. On the other side, by definition $\mathcal{Q}N_w = e_q^{-1}q^{-\ell(w_q)+\ell(x)}N_{w_q x w}$. The claim follows since

\begin{equation}
e_q^{-1}q^{-\ell(w_q)+\ell(x)} = \frac{1}{|\mathcal{L}(w_q)|}q^\ell(x).
\end{equation}

\[ \square \]

**Lemma 7.9.** Under the isomorphism (7.8) we have $[Q(w_q w)] \mapsto N_w$ for all $w \in \mathcal{W}^{p+q}$.

**Proof.** By Lemma 3.5 and the discussion after it, it follows that $Q$ sends the canonical basis element $\mathcal{S}_{w_q w}$ to $N_w$. By Lemma 7.8 we have

\begin{equation}
[Q(w_q w)] = [\mathcal{Q}\mathcal{M}_p(w_q w \cdot 0)] = Q[\mathcal{M}_p(w_q w \cdot 0)] = Q\mathcal{S}_{w_q w} = N_w.
\end{equation}

Now let us consider a general composition $\mathfrak{a}$.

**Proposition 7.10.** Under the isomorphism (7.8) the class of the indecomposable projective module $Q(w)$ maps to the canonical basis element $v_{\ell(\mathfrak{a})}^{(\mathfrak{a})} \in V(\mathfrak{a})$ corresponding to the standard basis element $v_{\ell(\mathfrak{a})}$.

**Proof.** By Lemma 7.9 we know the result for the regular composition $\mathfrak{n}$. Consider an embedding $V(\mathfrak{a}) \to V^{\otimes n}$ given by a web diagram $\varphi$. We know that $\mathcal{F}(\varphi) : \mathcal{Q}_k(\mathfrak{a}) \to \mathcal{Q}_n(\mathfrak{e})$, that categorifies $\varphi$, sends indecomposable projectives to indecomposable projectives (Proposition 5.24). On the other side, it follows immediately from our diagrammatic calculus that what $\varphi$ sends to a canonical basis element is a canonical basis element.

$\square$
7.5. The bilinear form. We give now a categorical interpretation of the bilinear form \((5.9)\). Given a \(\mathbb{Z}\)-graded complex vector space \(M = \bigoplus_{i \in \mathbb{Z}} M^i\), let \(h(M) = \sum_{i \in \mathbb{Z}} (\dim \, M^i) q^i \in \mathbb{Z}[q, q^{-1}]\) be its graded dimension. Now let \(M, N\) be objects of \(\mathcal{Q}_k(\mathfrak{a})\). Set
\[
(7.37) \quad h(\operatorname{Ext}(M, N)) = \sum_{i \in \mathbb{Z}} (-1)^i h(\operatorname{Ext}^i(M, N)).
\]
Let also \(\overline{\cdot}\) be the involution of \(\mathbb{Z}[q, q^{-1}]\) given by \(\overline{q} = q^{-1}\).

**Proposition 7.11.** For \(M, N \in \mathcal{Q}_k(\mathfrak{a})\) we have
\[
(7.38) \quad h(\operatorname{Ext}(M, N^*)) = ([M], [N])_a.
\]

**Proof.** First, note that the l.h.s. of \((7.38)\) defines a bilinear form on the Grothendieck group. Hence we only need to prove that the two sides coincide on a basis.

By the properties of properly stratified algebras (cf. [Fri07, Lemma 4]) we have
\[
(7.39) \quad \operatorname{Ext}^i(\Delta(z), (\Delta(w))^*) = \begin{cases} 
\mathbb{C} & \text{if } z = w \text{ and } i = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence we are left to prove that
\[
(7.40) \quad \frac{([\Delta(z)], v_{(w)})_a}{(v_{(w)}, v_{(w)})_a} = \delta_{z, w} \quad \text{for all } w, z \in \Lambda_k(\mathfrak{a})
\]

or equivalently that
\[
(7.41) \quad [\Delta(z)] = v_{(z)} = (v_{(z)}, v_{(z)})_a [\Delta(z)] \quad \text{for all } z \in \Lambda_k(\mathfrak{a}).
\]

By the properties of a properly stratified algebra, it suffices for that to prove that the proper standard module \(\Delta(z)\) appears \((v_{(z)}, v_{(z)})\)-times in some proper standard filtration of the indecomposable projective \(P(z)\). Since we know which basis the proper standard and the indecomposable projective modules categorify, this follows. \(\square\)

By Proposition 7.11 and since
\[
(7.42) \quad \operatorname{Ext}^i(\Delta(z), (\Delta(w))^*) = \operatorname{Ext}^i(P(z), (L(w))^*) = \begin{cases} 
\mathbb{C} & \text{if } z = w \text{ and } i = 0, \\
0 & \text{otherwise},
\end{cases}
\]

we have:

**Theorem 7.12.** Under the isomorphism \((7.8)\) we have the following correspondences:

\[
\begin{align*}
\{ \text{standard modules} \} & \longleftrightarrow \text{standard basis}, \\
\{ \text{proper standard modules} \} & \longleftrightarrow \text{dual standard basis}, \\
\{ \text{indecomposable projective modules} \} & \longleftrightarrow \text{canonical basis}, \\
\{ \text{simple modules} \} & \longleftrightarrow \text{dual canonical basis}.
\end{align*}
\]

7.6. Categorification of the action of \(U_q(\mathfrak{gl}(1|1))\). We want now to define functors that categorify the actions of \(U_q(\mathfrak{gl}(1|1))\).
Functors $\mathcal{E}$ and $\mathcal{F}$. Fix an integer $n$, a composition $\mathbf{a} = (a_1, \ldots, a_\ell)$ of $n$ and an integer $n - \ell \leq k < n$. Let $\lambda = \lambda_\mathbf{a}$, and let $\mathfrak{p}, \mathfrak{q}, \mathfrak{p}', \mathfrak{q}'$ be the parabolic subalgebras of $\mathfrak{g}_n$ such that $\Omega_k(\mathbf{a}) = \mathcal{Z}\mathcal{O}_{\mathfrak{p}'-\text{pres}}^\lambda$ and $\Omega_{k+1}(\mathbf{a}) = \mathcal{Z}\mathcal{O}_{\mathfrak{q}'-\text{pres}}^\lambda$. Notice that $\mathfrak{p}' \subset \mathfrak{p}$ and $\mathfrak{q} \subset \mathfrak{q}'$. We have a diagram

\[
\begin{diagram}
\node{Z\mathcal{O}_{\mathfrak{p}'-\text{pres}}^\lambda}
\arrow{s,l}{j}
\node{\Omega_k(\mathbf{a})}
\arrow{s,l}{i}
\node{\mathcal{Z}\mathcal{O}_{\mathfrak{q}'-\text{pres}}^\lambda}
\arrow{s,l}{\mathcal{E}_k}
\node{\Omega_{k+1}(\mathbf{a})}
\end{diagram}
\]

Let us define $\mathcal{E}_k = \Omega \circ j$ and $\mathcal{F}_k = j \circ i$. We get then a pair of adjoint functors $(\mathcal{F}_k, \mathcal{E}_k)$:

\[
(\mathcal{E}_k, \mathcal{F}_k)
\]

We can compute explicitly the action of $\mathcal{F}_k$ on projective modules and of $\mathcal{E}_k$ on simple modules:

**Proposition 7.13.** For $w \in \Lambda_{k+1}(\mathbf{a})$ we have

\[
\mathcal{F}_k Q(w) = \begin{cases} Q(w), & \text{if } w \in \Lambda_k(\mathbf{a}), \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** Consider the diagram (7.43). Of course $\Lambda_k(\mathbf{a}) = \Lambda_k^\lambda(\lambda) \subset \Lambda_k^\mathfrak{p}'(\lambda)$, and we have $iQ(w) = P\mathfrak{p}'(w \cdot \lambda) \in \mathcal{Z}\mathcal{O}_{\mathfrak{p}'-\text{pres}}^\lambda$. By the definition of the Zuckermann’s functor we have then $jP\mathfrak{p}'(w \cdot \lambda) = P\mathfrak{p}'(w \cdot \lambda) = Q(w) \in \Omega_k(\mathbf{a})$ if $w \in \Lambda_k^\lambda(\lambda)$, or 0 otherwise. \qed

**Proposition 7.14.** For $w \in \Lambda_k(\mathbf{a})$ we have

\[
\mathcal{E}_k S(w) = \begin{cases} S(w), & \text{if } w \in \Lambda_{k+1}(\mathbf{a}), \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** Consider the diagram (7.43). By Lemma 6.10 the simple objects of $\Omega_k(\mathbf{a})$ are the simple objects $S(w \cdot \lambda)$ of $\mathcal{Z}\mathcal{O}_{\mathfrak{p}'-\text{pres}}^\lambda$ such that $w \in \Lambda_k(\mathbf{a})$. In particular, $jS(w) = S(w \cdot \lambda)$ for each $w \in \Lambda_k(\mathbf{a})$. Let $\Omega_{\mathfrak{p}'} : \mathcal{Z}\mathcal{O}_{\mathfrak{p}'}^\lambda \to \mathcal{Z}\mathcal{O}_{\mathfrak{p}'-\text{pres}}^\lambda$ and $\Omega_{\mathfrak{q}} : \mathcal{Z}\mathcal{O}_{\mathfrak{q}'}^\lambda \to \mathcal{Z}\mathcal{O}_{\mathfrak{q}'-\text{pres}}^\lambda$ be the corresponding coapproximation functors. As we already noticed, it follows from the definition that $\Omega_{\mathfrak{p}'} = \Omega \circ \Omega_{\mathfrak{q}}$. Since $S(w \cdot \lambda) = \Omega_k L(w \cdot \lambda)$, we have $\Omega S(w \cdot \lambda) = \Omega_{\mathfrak{p}'} L(w \cdot \lambda)$. This is $S(w) \in \Omega_{k+1}(\mathbf{a})$ if $w \in \Lambda_{k+1}(\mathbf{a})$, or 0 otherwise. \qed

**Unbounded derived categories.** Being the composition of exact functors, the functor $\mathcal{E}_k$ is exact. On the other side, being the composition of right-exact functors, $\mathcal{F}_k$ is right exact, but not exact in general. Therefore, $\mathcal{F}_k$ does not induce a map between the Grothendieck groups, unless we pass to derived category. Unfortunately, properly stratified algebras do not have, in general, finite global dimension (this happens if and only if they are quasi-hereditary). Hence, we shall consider unbounded derived categories. The main problem with unbounded derived categories is that their Grothendieck group is trivial (see [My86]). A workaround to this problem has been developed by Achar and Stroppel in [AS13]. We recall briefly their main definitions and results, adapted to our setting.

Consider a finitely dimensional positively graded $\mathbb{C}$-algebra $A = \bigoplus_{i \leq 0} A_i$ with semisimple $A_0$, and let $\mathcal{A} = A\text{--gmod}$. Each simple object of $\mathcal{A}$ is concentrated in
one degree. Achar and Stroppel define a full subcategory $D^\vee A$ of the unbounded derived category $D^-A$ by

$$D^\vee A = \left\{ X \in D^-A \ \middle| \ \text{for each } m \in \mathbb{Z} \text{ only finitely many of the } H^i(X) \right\}.$$  

Recall that the Grothendieck group $K(T)$ of a small triangulated category $T$ is defined to be the free abelian group on isomorphism classes $[X]$ for $X \in T$ modulo the relation $[B] = [A] + [C]$ whenever there is a distinguished triangle of the form $A \to B \to C \to A[1]$. As for abelian categories, if $T$ is graded then $K(T)$ is naturally a $\mathbb{Z}[q, q^{-1}]$-module. Let

$$I = \{ x \in D^\vee A \ | \ \beta \leq m \} \in K(D^\vee A)$$

for all $m \in \mathbb{Z}$, where $\beta : D^\vee A \to D^\vee A$ is induced by the exact functor $\beta : A \to A$ defined on the graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ by $\beta_i M = \bigoplus_{i \leq m} M_i$. Then $K(D^\vee A) = K(D^\vee A)/I$ is the topological Grothendieck group of $D^\vee A$. The names is motivated by the fact that one can define on $K(D^\vee A)$ a $(q)$-adic topology with respect to which $K(D^\vee A)$ is complete. It follows that $K(D^\vee A)$ is a $\mathbb{Z}[q][q^{-1}]$-module.

On the other side, let $K(A)$ be the completion of the $\mathbb{Z}[q, q^{-1}]$-module $K(A)$ with respect to the $(q)$-adic topology (AS13 §2.3). Then the natural map $K(A) \to K(D^\vee A)$ is injective and induces an isomorphism of $\mathbb{Z}[q][q^{-1}]$-modules

$$K(A) \cong K(D^\vee A).$$

Moreover, if $\{ L_i \ | \ i \in I \}$, with $I$ finite, is a full set of pairwise non-isomorphic simple objects of $A$ concentrated in degree 0 and $P_i$ is the projective cover of $L_i$, then both $\{ L_i \ | \ i \in I \}$ and $\{ P_i \ | \ i \in I \}$ give a $\mathbb{Z}[q][q^{-1}]$-basis for $K(A)$. In particular $\hat{K}(A) \cong \mathbb{Z}[q][q^{-1}] \otimes_{\mathbb{Z}[q, q^{-1}]} K(A)$.

In our setting, we have for each category $\mathcal{O}_A$ naturally

$$K^C(q)(\mathcal{O}_A) \cong \mathbb{C}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} K(\mathcal{O}_A)$$

$$\cong \mathbb{C}(q) \otimes_{\mathbb{Z}[q][q^{-1}]} \hat{K}(\mathcal{O}_A).$$

The same holds in particular for $\mathfrak{Q_k}(\mathfrak{a})$. We define also

$$K^C(q)(D^\vee A) = \mathbb{C}(q) \otimes_{\mathbb{Z}[q][q^{-1}]} K(D^\vee A).$$

Let $A_{\geq m}$ be the full subcategory of $A$ consisting of objects $M = \bigoplus_{i \geq m} M_i$. An additive functor $F : A \to A'$ is said to be of finite degree amplitude if there exists some $\alpha > 0$ such that $F(A_{\geq m}) \subset A'_{\geq m+\alpha}$ for all $m \in \mathbb{Z}$. Let $F : A \to A'$ be a right-adjoint functor that commutes with the degree shift. If $F$ has finite degree amplitude, then the left-derived functor $LF$ induces a continuous homomorphism of $\mathbb{Z}[q][q^{-1}]$-modules $[LF] : \hat{K}(A) \to \hat{K}(A')$.

**Derived functors $\mathcal{E}$ and $\mathcal{F}$.** Let us now go back to our functors $\mathcal{E}_k$ and $\mathcal{F}_k$. Being exact, $\mathcal{E}_k$ induces a functor $\mathcal{E}_k : D^\vee (\mathfrak{Q_k}(\mathfrak{a})) \to D^\vee (\mathfrak{Q_{k+1}}(\mathfrak{a}))$. On the other side, it is immediate to check that the functors $i$ and $j$ and therefore also $\mathcal{F}_k$ have finite degree amplitude. Hence $LF_k$ restricts to a functor $LF_k : D^\vee (\mathfrak{Q_{k+1}}(\mathfrak{a})) \to D^\vee (\mathfrak{Q_k}(\mathfrak{a}))$. Since $\mathcal{E}_k$ is exact, it follows by standard arguments that we have a pair of adjoint functors $(LF_k, \mathcal{E}_k)$:

$$\mathcal{E}_k : D^\vee (\mathfrak{Q_k}(\mathfrak{a})) \leftrightarrow D^\vee (\mathfrak{Q_{k+1}}(\mathfrak{a})) : LF_k$$

**Remark 7.15.** Since $i$ sends projective modules to projective modules, it follows from [Wei94] Comparison Theorem 10.8.2 that $LF_k = L \circ Li$. 


Theorem 7.16. The functors $L\mathcal{F}_k$ and $E_k$ categorify $F$ and $F'$ respectively, that is, the following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{D}^\nabla \Omega_k(a) & \xrightarrow{L\mathcal{F}_k} & \mathcal{D}^\nabla \Omega_{k+1}(a) \\
\downarrow{K^{\mathcal{C}(q)}} & & \downarrow{K^{\mathcal{C}(q)}} \\
V(a)_k & \xrightarrow{F} & V(a)_{k+1}
\end{array}
\quad \begin{array}{ccc}
\mathcal{D}^\nabla \Omega_k(a) & \xrightarrow{E_k} & \mathcal{D}^\nabla \Omega_{k+1}(a) \\
\downarrow{K^{\mathcal{C}(q)}} & & \downarrow{K^{\mathcal{C}(q)}} \\
V(a)_k & \xrightarrow{F'} & V(a)_{k+1}
\end{array}
\]

Proof. We use Proposition 7.13 to check that the first diagram commutes on the basis given by indecomposable projective modules. Let $w \in \Lambda_{k+1}(a)$ and write $v(w) = v_0^a$. Then in $K^{\mathcal{C}(q)}(\mathcal{D}^\nabla \Omega_k(a))$ we have $[Q(w \cdot \lambda)] = v_0^a$. Now $w$ is in $\Lambda_k(a)$ if and only if it is a shortest coset representative for $W_p \setminus S_n$. Let $T^{k+1}_a(w)$ (resp. $T^k_a(w)$) be the $(n-k+1, k+1)$-tableau (resp. $(n-k, k)$-tableau) of type $a$ corresponding to $w$. Obviously $T^k_a(w)$ can be obtained from $T^{k+1}_a(w)$ by removing the upper box $b$ of the column and adding it to the row on the left. Clearly $T^k_a(w)$ is admissible if and only if the entry of this box $b$ is 1. Hence $w \in \Lambda_k(a)$ if and only if $\eta_l = 1$, and in this case we have $[Q(w)] = v_0^a \otimes v_{r_1}^{a_2} \cdots \otimes v_{r_{\ell-1}}^{a_{\ell}}$ in $K^{\mathcal{C}(q)}(\mathcal{D}^\nabla \Omega_k(a))$. By Proposition 7.13 this is the action of $F$.

Since $E_k$ is the adjoint functor of $L\mathcal{F}_k$, the commutativity of the second diagram follows from the adjunction (5.20) and Proposition 7.11 (of course we could also argue as for $L\mathcal{F}_k$ and check directly the commutativity of the second diagram above using Proposition 7.14). \qed

We define $E = \bigoplus_{k=n-\ell}^{n-1} E_k$ and $\mathcal{F} = \bigoplus_{k=n-\ell}^{n-1} \mathcal{F}_k$ as endofunctors of $\Omega(a)$. We have:

Lemma 7.17. The functors $E$ and $\mathcal{F}$ satisfy $E \circ E = \mathcal{F} \circ \mathcal{F} = 0$.

Proof. Let $S \in \Omega(a)$ be a simple module. It follows from Proposition 7.14 that $E^2 S = 0$. Since $E$ is exact, this implies that $E = 0$.

On the other side, it follows from 7.13 that $F^2$ is zero on projective modules. Since $\mathcal{F}$ is right exact and any object of $\Omega(a)$ has a projective presentation, it follows that $\mathcal{F}^2$ is the zero functor. \qed

Since $\mathcal{L}$ sends projective modules to projective modules, it follows (cf. [We91, Corollary 10.8.3]) that $L\mathcal{F} \circ L\mathcal{F} = L(\mathcal{F} \circ \mathcal{F}) = 0$.

We summarize the results of this section in the following:

Theorem 7.18. Let $\varphi$ be a web defining a morphism $V(a) \to V(a')$. Then the diagram

\[
\begin{array}{ccc}
\mathcal{D}^\nabla \Omega(a') & \xrightarrow{E, L\mathcal{F}} & \mathcal{D}^\nabla \Omega(a') \\
\downarrow{\mathcal{G}(\varphi)} & & \downarrow{\mathcal{G}(\varphi)} \\
\mathcal{D}^\nabla \Omega(a) & \xrightarrow{E, L\mathcal{F}} & \mathcal{D}^\nabla \Omega(a)
\end{array}
\]

(7.53)
commutes and categorifies (i.e. gives, after applying the completed Grothendieck group $K^C(q)$) the diagram

\[
\begin{array}{c}
V(a') \\
\mathcal{F}(\varphi) \\
V(a)
\end{array}
\begin{array}{c}
\xrightarrow{E', F} \\
\xrightarrow{E', F}
\end{array}
\begin{array}{c}
V(a') \\
\mathcal{F}(\varphi) \\
V(a)
\end{array}
\]

(7.54)

In particular, for $a = n$ we have two families of endofunctors $\{E, L\}$ and $\{C_i | i = 1, \ldots, n-1\}$ of $\mathcal{D}^{Q}(\pi)$ which commute with each other and which on the Grothendieck group level give the actions of $U_q(gl(1|1))$ and of the Hecke algebra $H_n$ on $V^\otimes n$ respectively.

PART III. A diagram algebra from Soergel modules

This part is almost independent of the previous ones. We develop some instruments (using symmetric polynomials and Soergel modules) with which we will construct a diagram algebra, similar to the generalized Khovanov algebras [BS11]. By the construction, this algebra is isomorphic to the endomorphism algebra of a projective generator of the category $Q_k(\pi)$ (see Theorem 11.12). We will conclude with a section on the cohomology of the Springer fibre of hook type.

8. Symmetric polynomials

In this section we are going to study some rings obtained as quotients of a polynomial ring modulo an ideal generated by complete symmetric functions in some subsets of variables. We start recalling some easy standard facts about symmetric polynomials.

We let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring. We consider it as a graded ring with $\deg x_i = 2$ for every $i$.

8.1. Complete symmetric polynomials. The complete symmetric polynomials are defined as

\[
h_j(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n} x_{i_1} \cdots x_{i_j}
\]

for every $j \geq 1$ so that for example $h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$. We set also $h_0(x_1, \ldots, x_n) = 1$, while if $n = 0$ (i.e., we have zero variables), we let $h_i() = 0$ for every $i \geq 1$. The polynomials $h_i(x_1, \ldots, x_n)$ are invariant under the action of $S_n$ permuting the variables, and in fact generate the algebra $R^{S_n}$ of invariant polynomials.

We will consider complete symmetric polynomials in some subset of the variables of $R$. The following formula helps us to decompose a complete symmetric polynomial in $k$ variables as complete symmetric polynomials in $\ell$ and $k-\ell$ variables, for every $\ell = 1, \ldots, k-1$:

\[
h_j(x_1, \ldots, x_k) = \sum_{n=0}^j h_n(x_1, \ldots, x_\ell)h_{j-n}(x_{\ell+1}, \ldots, x_k).
\]

Another formula allows us to express a complete symmetric polynomial in $k-1$ variables in terms of complete symmetric polynomials in $k$ variables:

\[
h_j(x_1, \ldots, x_{k-1}) = h_j(x_1, \ldots, x_k) - x_kh_{j-1}(x_1, \ldots, x_k).
\]
For $1 \leq i \leq n-1$ let $R^{s_i}$ be the subring of $R$ consisting of polynomials invariant under the simple transposition $s_i$. We recall from \cite{Dem73} the definition of the classical Demazure operator $\partial_i : R \to R^{s_i}(2)$, given by

\begin{equation}
\partial_i : f \mapsto \frac{f - s_i(f)}{x_i - x_{i+1}}.
\end{equation}

The operator $\partial_i$ is linear, vanishes on $R^{s_i}$ and satisfies

\begin{equation}
\partial_i(fg) = f\partial_i(g) \quad \text{whenever } f \in R^{s_i}.
\end{equation}

Let also $P_i : R \to R$ be defined by $P_i(f) = f - x_i\partial_i(f)$. It is easy to show that $P_i$ has also values in $R^{s_i}$. The following commutation rules hold:

\begin{equation}
[P_i, x_{i+1}] = -x_is_i, \quad [\partial_i, x_i] = s_i.
\end{equation}

The operators $\partial_i$ and $P_i$ can be used to define the decomposition $R \cong R^{s_1} \oplus x_1R^{s_1}$ as a $R^{s_1}$-module, by

\begin{equation}
f \mapsto P_1f \oplus x_1\partial_1f.
\end{equation}

Demazure operators have the nice property of sending complete symmetric polynomials to other complete symmetric polynomials:

**Lemma 8.1.** For all $j \geq 1$ we have

\begin{equation}
\partial_kh_j(x_1, \ldots, x_k) = h_{j-1}(x_1, x_2, \ldots, x_{k+1}).
\end{equation}

**Proof.** We have

\begin{align}
\partial_kh_j(x_1, \ldots, x_k) &= \partial_k \left( \sum_{\ell=0}^{j} h_{j-\ell}(x_1, \ldots, x_{k-1})x_k^\ell \right) \\
&= \sum_{\ell=0}^{j} h_{j-\ell}(x_1, \ldots, x_{k-1})\partial_k(x_k^\ell) \\
&= \sum_{\ell=1}^{j} h_{j-\ell}(x_1, \ldots, x_{k-1}) \frac{x_k^\ell - x_{k+1}^\ell}{x_k - x_{k+1}} \\
&= \sum_{\ell=1}^{j} h_{j-\ell}(x_1, \ldots, x_{k-1})h_{\ell-1}(x_k, x_{k+1}) \\
&= h_{j-1}(x_1, x_2, \ldots, x_{k+1}).
\end{align}

Notice that in the first and last equalities we used (8.2). \qed

### 8.2. Ideals generated by complete symmetric polynomials

We are going to study quotients rings of $R$ generated by some of the $h_i$’s. Let

\begin{equation}
\mathcal{B}' = \{ b = (b_1, \ldots, b_n) \in \mathbb{N}^n \mid b_i \geq b_{i+1} \geq b_1 - 1 \}.
\end{equation}

In other words, $\mathcal{B}'$ is the set of weakly decreasing sequences of positive numbers such that the difference between two consecutive items is at most one. For every sequence $b \in \mathcal{B}'$ let $I_b \subset R$ be the ideal generated by

\begin{equation}
h_{b_1}(x_1), h_{b_2}(x_1, x_2), \ldots, h_{b_n}(x_1, \ldots, x_n).
\end{equation}

Set also $R_b = R/I_b$.

Let us fix a lexicographic monomial order on $R$ with

\begin{equation}
x_n > x_{n-1} > \cdots > x_1.
\end{equation}

Then we have:

**Lemma 8.2.** The polynomials (8.11) are a Groebner basis for $I_b$. 

Lemma 8.7. Let \( R \). Morphisms between quotient rings.

\[
\text{Lemma 8.8. Let } \] C

Proposition 8.3.

\[
\text{Example 8.4. Let } b = (1, \ldots, 1). \text{ Then } x_i = h_1(x_1, \ldots, x_i) - h_1(x_1, \ldots, x_{i-1}) \text{ lies in } I_b \text{ for each } i, \text{ hence } I_b = (x_1, \ldots, x_n) \text{ and } R_b \cong \mathbb{C} \text{ is one dimensional.}
\]

Example 8.5. Let \( b = (n, n-1, \ldots, 1). \) Then it is easy to show that the ideal \( I_b \) is the ideal generated by the symmetric polynomials in \( n \) variables with zero constant term, and \( R_b \) is the ring of the coinvariants \( R/(R^n) \), isomorphic to the cohomology of the full flag variety of \( \mathbb{C}^n \). As given by Proposition 8.3 it has dimension \( n! \) and it is well-known that a monomial basis is given by

\[
\text{(8.13)} \quad \{ x_1^{j_1} \cdots x_n^{j_n} | 0 \leq j_i \leq n - i \}. 
\]

8.3. Morphisms between quotient rings. Next, we are going to determine all \( R \)-module homomorphisms between rings \( R_b \).

Proposition 8.6. Let \( b, b' \in \mathcal{B}' \), and let \( c_i = \max\{ b'_i - b_i, 0 \} \). Then a \( \mathbb{C} \)-basis of \( \text{Hom}_R(R_b, R_{b'}) \) is given by

\[
\text{(8.15)} \quad \{ 1 \mapsto x_1^{j_1} \cdots x_n^{j_n} | c_i \leq j_i < b'_i \}. 
\]

Remark 8.4. ...that \( \mathfrak{C}_1 \) is well-known that a monomial basis is given by

\[
\text{(8.13)} \quad \{ x_1^{j_1} \cdots x_n^{j_n} | 0 \leq j_i \leq n - i \}. 
\]

Lemma 8.7. Let \( b \in \mathcal{B}' \). Then \( h_a(x_1, \ldots, x_i) \in I_b \) for every \( a \geq b_i \).

Proof. We prove by induction on \( \ell \geq 0 \) that \( h_{b_i + \ell}(x_1, \ldots, x_i) \in I_b \) for every \( i = 1, \ldots, n \). For \( \ell = 0 \) the statement follows from the definition. For the inductive step, choose an index \( i \) and pick \( j < i \) maximal such that \( b_j = b_i + 1 \) (or let \( j = 0 \) if such an index does not exist) and write using iteratively (8.3):

\[
h_{b_i + \ell}(x_1, \ldots, x_i) = h_{b_i + \ell}(x_1, \ldots, x_j) + x_{j+1}h_{b_i + \ell - 1}(x_1, \ldots, x_{j+1}) + \cdots + x_i h_{b_i + \ell - 1}(x_1, \ldots, x_{i-1}) + x_i h_{b_i + \ell - 1}(x_1, \ldots, x_i).
\]

Since \( b_i + \ell = b_j + \ell - 1 \), the terms on the right all lie in \( I_b \) by the inductive hypothesis.

Lemma 8.8. Let \( b = (b_1, \ldots, b_n) \in \mathcal{B}' \) and

\[
(8.16) \quad b' = (b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1}, \ldots, b_n)
\]

for some \( i \). Suppose that also \( b' \in \mathcal{B}' \). Then \( I_{b'} \subset I_b \) while \( x_i I_{b'} \subset I_{b'} \).

Proof. It follows directly from Lemma 8.7 that \( I_{b'} \subset I_b \). For the other assertion, since \( h_{b_j}(x_1, \ldots, x_j) \in I_{b'} \) for every \( j \neq i \), we only need to prove that \( x_i h_{b_i}(x_1, \ldots, x_i) \in I_{b'} \). By (8.8) we have

\[
(8.17) \quad x_i h_{b_i}(x_1, \ldots, x_i) = h_{b_i+1}(x_1, \ldots, x_i) - h_{b_i+1}(x_1, \ldots, x_{i-1}).
\]

Since we suppose \( b' \in \mathcal{B}' \), it follows that \( b_{i-1} = b_i + 1 \), hence the r.h.s. of (8.17) lies in \( I_{b'} \).
We will call two sequences \( \mathbf{b}, \mathbf{b}' \in \mathcal{B} \) that satisfy the hypothesis of Lemma 8.8 (without regarding the order) near each other.

**Lemma 8.9.** Let \( \mathbf{b}, \mathbf{b}' \in \mathcal{B} \) and set \( c_i = \max\{b'_i - b_i, 0\} \). Then \( x_1^{c_1} \cdots x_n^{c_n} I_\mathbf{b} \subseteq I_{\mathbf{b}'} \).

**Proof.** We can find a sequence \( \mathbf{b} = b^{(0)}, b^{(1)}, \ldots, b^{(N)} = \mathbf{b}' \) with \( b^{(k)} \in \mathcal{B} \) for each \( k \) and \( N = \sum \left| b_i - b'_i \right| \) such that \( b^{(i)} \) and \( b^{(i+1)} \) are near each other. Then the claim follows applying iteratively Lemma 11.4. \( \square \)

**Lemma 8.10.** Let \( \mathbf{b}, \mathbf{b}' \in \mathcal{B} \). Let \( c_i = \max\{b'_i - b_i, 0\} \). Suppose \( p \in R \) be such that \( pI_\mathbf{b} \subseteq I_{\mathbf{b}'} \). Then \( x_1^{c_1} \cdots x_n^{c_n} | p \).

**Proof.** We prove the claim by induction on the leading term of \( p \), using the lexicographic order \( \{8.12\} \). Let \( p' \) be the leading term of \( p \) and pick an index \( 1 \leq i \leq n \). By assumption, \( ph_i (x_1, \ldots, x_i, \ldots) \in I_{\mathbf{b}'} \). By the theory of Groebner basis, the leading term of \( p\psi_h (x_1, \ldots, x_i) \) is divisible by \( x_1^{b'_i} \cdots x_n^{b'_n} \), and this leading term is just \( p' x_i^{b'_i} \). It follows immediately that \( x_1^{c_1} \cdots x_n^{c_n} | p' \). By Lemma 8.9 we then know that \( p' I_\mathbf{b} \subseteq I_{\mathbf{b}'} \), hence also \( (p - p') I_\mathbf{b} \subseteq I_{\mathbf{b}'} \). By induction, we may assume that \( x_1^{c_1} \cdots x_n^{c_n} | (p - p') \), and we are done. \( \square \)

**Proof of Proposition 8.6.** It follows from Lemma 8.9 that the elements of \( \{8.15\} \) indeed define morphisms \( R_\mathbf{b} \to R_{\mathbf{b}'} \). By Proposition 8.3 they are linearly independent, and by Lemma 8.10 they are a set of generators. \( \square \)

### 8.4. Duality

The category of finite dimensional \( R \)-modules has a duality, given by

\[
M^* = \text{Hom}_\mathbb{C}(M, \mathbb{C}).
\]

In fact, the vector space \( M^* \) is endowed with an \( R \)-action by setting \((r \cdot f)(m) = f(r \cdot m)\) for all \( f \in M^*, m \in M, r \in R \) (since \( R \) is commutative). If \( M \) is graded, the dual inherits a grading declaring \((M^*)_j = (M^{-j})^*\).

Now consider some \( \mathbf{b} \in \mathcal{B} \). The monomial basis \( \{8.14\} \) of \( R_\mathbf{b} \) has a unique element of maximal degree \( \mathbf{b} = 2(b_1 + \cdots + b_n - n) \), namely \( x^{b-1} \) where \( \mathbf{b} - 1 \) is the sequence \((b_1 - 1, \ldots, b_n - 1)\). We define a symmetric bilinear form \( (\cdot, \cdot) \) on \( R_\mathbf{b} \) by letting \((p, q)\) be the coefficient of \( x^{b-1} \) in the expression of \( pq \in R_\mathbf{b} \) as a linear combination of elements of the basis \( \{8.13\} \).

**Lemma 8.11.** The form \((\cdot, \cdot)\) on \( R_\mathbf{b} \) is non-degenerate, hence we get an isomorphism of graded \( R \)-modules

\[
(8.19) \quad R_\mathbf{b} \cong R_\mathbf{b}^*(-\mathbf{b}) \quad \text{for any } \mathbf{b} \in \mathcal{B}. \]

The degree shift comes out because the bilinear form pairs the degree \( i \) component of \( R_\mathbf{b} \) with its component of degree \( b - i \).

By the properties of a duality, we have

\[
(8.20) \quad \text{Hom}_R(R_\mathbf{b}, R_{\mathbf{b}'}) \cong \text{Hom}_R(R_{\mathbf{b}'}, R_\mathbf{b}) \cong \text{Hom}_R(R_{\mathbf{b}'}, R_\mathbf{b})(\mathbf{b}' - \mathbf{b})
\]

for any \( \mathbf{b}, \mathbf{b}' \in \mathcal{B} \). It is not difficult to see that the composite isomorphism is given explicitly by

\[
(8.21) \quad \Theta : (1 \mapsto p) \mapsto \left( 1 \mapsto \frac{x^{b-1}}{x^{\mathbf{b}' - \mathbf{b}} - p} \right).
\]
8.5. **Schubert polynomials.** We recall some basic facts about Schubert polynomials, referring to [Mac91] for more details. Let $w \in S_n$ be a permutation; then the operator $\partial_w = \partial_{s_1} \cdots \partial_{s_{\ell(w)}}$, where $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ is some reduced expression, does not depend on the particular chosen reduced expression and is hence well-defined. Let $w_n \in S_n$ be the longest element. Then one defines the **Schubert polynomial**

$$
\mathcal{S}_w(x_1, \ldots, x_n) = \partial_{w_1}^{-1} \cdots \partial_{w_n}^{-1} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}
$$

for each $w \in S_n$. The Schubert polynomials give a basis of $R/(R^S_n)$. It follows from the definition that $\deg \mathcal{S}_w(x_1, \ldots, x_n) = 2\ell(w)$.

For our purposes, it will be more convenient to have a monomial basis of $R/(R^S_n)$, indexed by permutations $w \in S_n$.

**Definition 8.12.** For each $w \in S_n$ we define $\mathcal{S}'_w(x_1, \ldots, x_n)$ to be the leading term of $\mathcal{S}_w(x_1, \ldots, x_n)$ in the lexicographic order \[8.12\].

Being the leading terms of a basis of $R/(R^S_n)$, it follows by the theory of Groebner bases that also the monomials $\mathcal{S}'_w(x_1, \ldots, x_n)$ give a basis.

**Remark 8.13.** We already noticed in Example 8.5 that $R/(R^S_n) \cong R_b$ for $b = (n, n-1, \ldots, 1)$. Hence we already have a monomial basis of $R/(R^S_n)$ given by Proposition 8.3. In fact, this basis coincides with the basis $\{\mathcal{S}'_w(x_1, \ldots, x_n) \mid w \in S_n\}$; the advantage of using Schubert polynomials is that they give us a way to index these basis elements through permutations.

There is an easy way to construct the monomials $\mathcal{S}'_w(x_1, \ldots, x_n)$ (cf. [HLS93]): let $c_w(i) = \# \{ j < i \mid w(j) > w(i) \}$; then $\mathcal{S}'_w(x_1, \ldots, x_n) = x_1^{c_w(1)} \cdots x_{n-1}^{c_w(n-1)}$.

**Example 8.14.** The following table contains the Schubert polynomials and the polynomials $\mathcal{S}'_w$ in the case $n = 3$.

| $w \in S_3$ | $\mathcal{S}_w$ | $\mathcal{S}'_w$ |
|------------|-----------------|-----------------|
| $e$         | $x_1$           | $1$             |
| $s$         | $x_1$           | $x_1$           |
| $t$         | $x_1 + x_2$     | $x_2$           |
| $st$        | $x_1 x_2$       | $x_1 x_2$       |
| $ts$        | $x_1^2$         | $x_1^2$         |
| $w_3$       | $x_1^2 x_2$     | $x_1^2 x_2$     |

9. **Some canonical bases elements**

In the following, we will compute some canonical basis elements in the Hecke algebra for some special permutations of the symmetric group $S_n$. We will need these expressions to compute the dimension of Soergel modules in the next section.

For the definition of $H_n$ we refer to \[3.1\]. We remark that all actions of $S_n$ will be from the right.

9.1. **Combinatorics.** Let us fix an integer $0 \leq k \leq n$. If $s_1, \ldots, s_{n-1}$ are the simple reflections in $S_n$, let $W_k$ be the subgroup generated by $s_1, \ldots, s_{k-1}$ and $W_k^\perp$ be the subgroup generated by $s_{k+1}, \ldots, s_{n-1}$. Notice that $W_k \times W_k^\perp = S_k \times S_{n-k} \subset S_n$. Let $w_k$ be the longest element of $W_k$, and let $D = D_{n,k}$ be the set of shortest coset representatives $((W_k \times W_k^\perp)S_n)^\text{short}$. The set $D$ is in natural bijection with $\land \lor$-sequences consisting of $k$ $\land$’s and $n - k$ $\lor$’s, by mapping the identity $e \in S_n$ to the sequence

$$
e = \land \cdots \land \lor \cdots \lor_{n-k}
$$

(9.1)
and letting $S_n$ act by permutations on the right. From now on, we identify an element $z \in D$ with the corresponding $\land \lor$-sequence.

**Remark 9.1.** To keep the connection with the previous sections, let $\mathfrak q, \mathfrak p \subset \mathfrak g_{n}$ be the standard parabolic subalgebras corresponding to $W_k$ and $W_k^+$ respectively. Then $W_k = W_{\mathfrak q}$, $W_k^+ = W_{\mathfrak p}$ and $D = W^{\mathfrak p+\mathfrak q}$. Moreover, multiplication on the left by $w_k$ defines a bijection between $D$ and $\Lambda_k(n) = \Lambda^\mathfrak p_\mathfrak q(0)$.

There are a few ways to encode an element $z \in D$, that we are now going to explain.

**The position sequences:** In an $\land \lor$-sequence $z \in D$, we number the $\land$'s (resp. the $\lor$'s) from 1 to $k$ (resp. from 1 to $n - k$) from the left to the right. Moreover, we number the positions of an $\land \lor$-sequence from 1 to $n$ from the left to the right. We let $\land^i$ be the position of the $i$-th $\land$ and $\lor^j$ be the position of the $j$-th $\lor$ in $z$. For example, in the sequence $z = \land \lor \land \land \lor \land \land$, we have $\land^2 = 4$ and $\lor^2 = 2$. Notice that both the sequences $(\land^1, \ldots, \land^k)$ and $(\lor^1, \ldots, \lor_{n-k})$ uniquely determine $z$.

**The $\land$-distance sequence:** We set
\begin{equation}
\land_i = \land^i - i \quad \text{for } i = 1, \ldots, k,
\end{equation}
so that
\begin{equation}
(\land^1, \ldots, \land^k) = (1 + \land_1, \ldots, k + \land_k).
\end{equation}
In other words, $\land_i$ measures how many steps the $i$-th $\land$ of the initial sequence $e$ has been moved to the left by the permutation $z$. This defines a bijection $z \mapsto z^\land$ between $D$ and the set
\begin{equation}
\{ z^\land = (z^1, \ldots, z^n) \mid 0 \leq z^1 \leq \cdots \leq z^n \leq n - k \}.
\end{equation}
Define the permutation
\begin{equation}
t_{i,\ell}^\land = s_i s_{i+1} \cdots s_{i+\ell - 1}
\end{equation}
for all $i = 1, \ldots, n - 1$ and $\ell = 1, \ldots, n - i$ (and set $t_{i,0}^\land = e$). Then we have a reduced expression for $z$:
\begin{equation}
z = t_{n-1}^\land \cdots t_{1}^\land z^\land.
\end{equation}

**The $\lor$-distance sequence:** Analogously, set
\begin{equation}
\lor_i = i - \lor^i_{k-i} \quad \text{for } i = k + 1, \ldots, n
\end{equation}
so that
\begin{equation}
(\lor^1, \ldots, \lor^n_{n-k}) = (k + 1 - \lor^1_{k+1}, \ldots, n - \lor^n_{n-k}).
\end{equation}
In other words, $\lor_i$ measures how many steps the $(i - k)$-th $\lor$ of $e$ has been moved to the left by the permutation $z$. This defines a bijection $z \mapsto z^\lor$ between $D$ and the set
\begin{equation}
\{ z^\lor = (z^1_{k+1}, \ldots, z^n) \mid k \geq z^1_{k+1} \geq \cdots \geq z^n \geq 0 \}.
\end{equation}
Define
\begin{equation}
t_{k+i,\ell}^\lor = s_{k+i-1} s_{k+i-2} \cdots s_{k+i-\ell}
\end{equation}
for $i = 1, \ldots, n - k$ and $\ell = 1, \ldots, k$ (and set $t_{k+i,0}^\lor = e$). Then we have another reduced expression for $z$:
\begin{equation}
z = t_{n}^\lor \cdots t_{n-k+1}^\lor z^\lor.
\end{equation}
The $b$-sequence: Finally we want to assign to the element $z \in D$ its $b$-sequence $b^z$. Let

$$B = \{ b = (b_1, \ldots, b_n) \in \mathbb{N}^n \mid k + 1 \geq b_1 \geq \cdots \geq b_n = 1, \quad b_i \leq b_{i+1} + 1 \quad \forall i = 1, \ldots, n - 1 \}$$

and define $b^z \in B$ by

$$b^z_i = \# \{ j \mid \wedge^z_j > i \} + 1.$$  

In other words, $b^z_i - 1$ is the number of $\wedge$‘s on the right of position $i$. It is clear that $b^z$ uniquely determines the element $z \in D$. In fact, this defines a bijection between $D$ and $B$.

Example 9.2. Let $n = 8, k = 4$ and consider the element $z = s_4s_5s_6 \in D$. The corresponding $\wedge\vee$-sequence and the $b$-sequences are:

$$\wedge \wedge \vee \wedge \vee \wedge \vee$$

$$b^z = 4 \ 3 \ 3 \ 2 \ 2 \ 2 \ 1 \ 1$$

We also have $z^\wedge = (0, 0, 1, 3)$ and $z^\vee = (2, 1, 1, 0)$.

9.2. Canonical basis elements. One of the rare examples of explicitly known canonical basis elements is the following (cf. [Soe97, Prop. 2.9]):

Lemma 9.3. Let $w_k$ be the longest element of $S_k$. Then the canonical basis element $C_{w_k}$ is given by

$$C_{w_k} = \sum_{w' \in S_k} q^{\ell(w_k) - \ell(w')} H_{w'}. \quad (9.14)$$

This expression holds both in $H_k$ and in $H_n \supset H_k$ for any $n > k$.

In the next proposition we will generalize (9.14) and give explicit formulas for the canonical basis elements $C_{w_k,z}$ for $z \in D$. But first we introduce the following notation: we set

$$\sum_{w' \in S_k} \binom{q}{w'} f(w') = \sum_{w' \in S_k} q^{-\ell(w')} f(w') \quad \text{and} \quad \sum_{i=0}^{h} \binom{q}{i} g(i) = \sum_{i=0}^{h} q^{-i} g(i)$$

for whatever functions $f$ defined on $S_n$ and $g$ defined on $\{0, \ldots, h\}$.

Proposition 9.4. Let $z \in D$, with $z = \ell_{k+1}^\vee \ell_{k+1}^\wedge \cdots \ell_{n+1}^\vee$. Then

$$C_{w_k,z} = \sum_{w' \in S_k} \binom{q}{w'_{k+1}} \sum_{i_{k+1}=0}^{s'_{k+1}} \cdots \sum_{i_n=0}^{s'_{n}} q^{\ell(w_k,z)} H_{w'_{k+1} \cdots w'_{n+1}}. \quad (9.16)$$

Proof. First, we note that all words $w't_{k+1}^\wedge \cdots t_{n+1}^\vee$ that appear in the expression on the right are actually reduced words. This is clear if we look at the action of this permutation on the string

$$\wedge \wedge \cdots \wedge \vee \wedge \cdots \vee$$

on the right: the length of the permutation is the cardinality of the set $X$ of the couples of symbols of this string that have been inverted. To $X$ belong $\ell(w')$ couples consisting of two $\wedge$‘s; moreover, every $\vee_{k+\alpha}$ appears in $X$ exactly $i_\alpha$ times coupled with some $\wedge$ or some $\vee_{k+\beta}$ for $\beta < \alpha$. Hence the length of the permutation $w't_{k+1}^\wedge \cdots t_{n+1}^\vee$ is exactly $\ell(w') + i_{k+1} + \cdots + i_n$, and therefore this is a reduced expression.

Now, in the r.h.s. of (9.16) the coefficient of $H_{w_k,z}$ is one, while the coefficient of every other basis element $H_{w't_{k+1}^\wedge \cdots t_{n+1}^\vee}$ is divisible by $q$. Hence the only thing we have to prove is that the r.h.s of (9.16) is bar invariant.
We proceed by induction on the length of $z$, the case $z = 0$ being given by Lemma [7.3]. Let $h$ be the greatest index such that $z_h^\vee \neq 0$. Hence we have $z = t_{k+1}^\vee z_{k+1} \cdots t_1^\vee z_1$. First suppose that $z_h^\vee \geq 2$. Set $z' = t_{k+1}^\vee z_{k+1} \cdots t_{h}^\vee z_{h-1}$ and $j = h - z_h^\vee$ so that $z = z's_j$. We compute:

$$
(9.18) \quad C_{w_k z} C_j = \left( \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_h=0}^{z_h^\vee -1} q^\ell(w_k z') H_w t_{k+1}^\vee t_{k+1} \cdots t_{h}^\vee t_{h} \right) (H_j + q)
$$

$$
(9.19) \quad = \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_h=0}^{z_h^\vee -1} q^\ell(w_k z') - z_h^\vee + 1 H_w t_{k+1}^\vee t_{k+1} \cdots t_{h}^\vee t_{h} + t_{h+1}^\vee t_{h+1}^\vee
$$

$$
(9.20) \quad + \left( \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_h=0}^{z_h^\vee -1} q^\ell(w_k z') H_w t_{k+1}^\vee t_{k+1} \cdots t_{h}^\vee t_{h} \right) H_j
$$

$$
(9.21) \quad + \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_h=0}^{z_h^\vee -1} q^\ell(w_k z') + 1 H_w t_{k+1}^\vee t_{k+1} \cdots t_{h}^\vee t_{h}.
$$

The element $C_{w_k z} C_j$ is obviously bar-invariant. Moreover, the sum of (9.19) and (9.21) gives exactly the r.h.s. of (9.16) for $C_{w_k z}$; hence we only need to prove that (9.20) is bar invariant. But it is easy to check that in (9.20) the term $H_j$ on the right acts as $q^{-1}$; hence (9.20) is equal to the r.h.s. of (9.16) for $C_{w_k z'}$, where $z' = t_{k+1}^\vee z_{k+1} \cdots t_{h}^\vee z_{h-1}$, and this is bar-invariant by induction.

Now suppose instead that $z_h^\vee = 1$. Set $z' = t_{k+1}^\vee z_{k+1} \cdots t_{h-1}^\vee z_{h-1}$ so that $z = z's_{h-1}$, and compute:

$$
(9.22) \quad C_{w_k z} C_{h-1} = \left( \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_{h-1}=0}^{z_{h-1}^\vee -1} q^\ell(w_k z') H_w t_{k+1}^\vee t_{k+1} \cdots t_{h-1}^\vee t_{h-1} \right) C_{h-1}
$$

$$
(9.23) \quad = \left( \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_{h-1}=0}^{z_{h-1}^\vee -1} q^\ell(w_k z') H_w t_{k+1}^\vee t_{k+1} \cdots t_{h-1}^\vee t_{h-1} \right) C_{h-1}
$$

$$
(9.24) \quad + \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_{h-1}=0}^{z_{h-1}^\vee -1} q^\ell(w_k z') + 1 H_w t_{k+1}^\vee t_{k+1} \cdots t_{h-1}^\vee t_{h-1}.
$$

This is exactly the r.h.s. of (9.16) for $C_{w_k z}$; hence this is also bar invariant. □

We will need some other canonical basis elements that we now compute.

**Proposition 9.5.** Let $z \in D$, with $z = t_{k+1}^\vee z_{k+1} \cdots t_{h}^\vee z_{h}$. Suppose that for some index $j$ we have $z_j^\vee = z_j^\vee$. Then $C_{s_j w_k z}$ is equal to

$$
(9.25) \quad \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_j=0}^{z_j^\vee -1} q^\ell(w_k z') H_{s_j w_k t_{k+1}^\vee t_{k+1} \cdots t_{j}^\vee t_{j+1} \cdots t_{h}^\vee t_{h}}
$$

$$
+ \sum_{w' \in S_k} \sum_{i_k+1=0}^{z_{k+1}} \cdots \sum_{i_{j-1}=0}^{z_{j-1}^\vee -1} q^\ell(w_k z') + 1 H_{s_j w_k t_{k+1}^\vee t_{k+1} \cdots t_{j}^\vee t_{j+1} \cdots t_{h}^\vee t_{h}}.
$$
Proof. We prove the claim by induction on the length of \( z \), using Proposition \[\ref{prop:C_wz}\] for the expression of \( C_{wz} \). In the following computation, we do not write the sums over \( w' \in S_k \) and over the indices \( i_h \) for \( h \neq j, j + 1 \).

\[
(9.26) \quad C_j C_{wz} = C_j \sum_{i_j=\ell} z_{j+1}^\gamma(q)_{i_j=0} \sum_{i_{j+1}=0} q^f(\gamma z) H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij+1}^{\gamma} \cdots}
\]

\[
(9.27) \quad = C_j \sum_{i_j=\ell} z_{j+1}^\gamma(q)_{i_j=0} \sum_{i_{j+1}=0} q^f(\gamma z) H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij+1}^{\gamma} \cdots} + C_j \sum_{i_j=\ell} q^f(\gamma z) \sum_{i_{j+1}=0} H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij+1}^{\gamma} \cdots}
\]

Permutations \( w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij+1}^{\gamma} \cdots \) occurring in \((9.27)\) become longer when multiplied on the left with \( s_j \). Hence \((9.27)\) becomes

\[
(9.29) \quad = C_j \sum_{i_j=\ell} q^f(\gamma z) \sum_{i_{j+1}=0} H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij+1}^{\gamma} \cdots}
\]

This is exactly \((9.28)\). Hence we are left to show that \((9.28)\) is bar invariant.

For the permutations occurring in \((9.28)\) we have

\[
(9.30) \quad w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij}^{\gamma} \cdots = s_j w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij+1}^{\gamma} \cdots.
\]

Hence \((9.28)\) is equal to

\[
(9.31) \quad = \sum_{i_j=\ell} q^f(\gamma z) - 1 H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij}^{\gamma} \cdots} + \sum_{i_j=\ell} q^f(\gamma z) H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij}^{\gamma} \cdots}
\]

Note that for \( i_j = z_{j+1}^\gamma \) the second sum runs over an empty set of indices. We can rewrite \((9.31)\) as

\[
(9.32) \quad \sum_{i_j=\ell} q^f(\gamma z) - 2 H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij}^{\gamma} \cdots} + \sum_{i_j=\ell} q^f(\gamma z) - 1 H_{w' \cdots t_{j,ij+1}^{\gamma} t_{j+1,ij}^{\gamma} \cdots}
\]
or, renaming the indices and swapping the sums,

\begin{align}
\sum_{i_j=0}^{(q)} \sum_{i_{j+1}=0}^{(q)} q^{(w_k z) - 2} H_{s_j w^r \cdots t^r_{j+1} i_{j+1} \cdots} + \\
\sum_{i_j=0}^{(q)} \sum_{i_{j+1}=0}^{(q)} q^{(w_k z) - 1} H_{w^r \cdots t^r_{j+1} i_{j+1} \cdots}.
\end{align}

Let \( z' \in D \) be determined by \( z'^{\nu}_j = z^{\nu}_k \) for \( h \neq j, j + 1 \) while \( z'^{\nu}_j = z^{\nu}_j - 1 \). By induction (9.33) is \( C_{s_j w_k z'} \), hence it is bar invariant. \( \square \)

We will not need the explicit expression (9.25), but only the following

**Corollary 9.6.** Let \( z \in D \), with \( z = t^\nu_{k+1} z^\nu_{k+1} \cdots t^\nu_{n-1} z^\nu_{n-1} \). Suppose that for some index \( j \) we have \( z^\nu_j = z^\nu_{j+1} \). Then the canonical basis element \( C_{s_j w_k z} \) is a sum of

\begin{align}
kl((z^\nu_{k+1} + 1) \cdots (z^\nu_j + 1)(z^\nu_{j+1} + 2)(z^\nu_{j+2} + 1) \cdots (z^\nu_n + 1))
\end{align}

standard basis elements with monomial coefficients in \( q \).

10. **Soergel modules**

In this section we will describe some Soergel modules as quotient rings \( R_n \) (defined in Section \( \S \)). The strategy is the following: we prove that the ideal \( I_n \) is contained in the annihilator and we then use a dimension argument comparing the dimension of \( R_n \) (Proposition \( \S 3 \)) and of the Soergel module (given by the corresponding canonical basis element computed in Section \( \S 4 \)). We compute then the homomorphism spaces between these Soergel modules (10.3).

Fix a positive integer \( n \) and let \( R = \mathbb{C}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables. For \( 0 \leq \ell \leq n \) let \( J_\ell \) be the ideal generated by the non-constant symmetric polynomials in \( \ell \) variables \( x_1, \ldots, x_\ell \). Let moreover \( B = \mathbb{C}[x_1, \ldots, x_n]/J_n \). For a simple reflection \( s_i \in S_n \), let \( B^{s_i} \) denote the invariants under \( s_i \). In the following, we will abbreviate \( \otimes_{B^{s_i}} \) by \( \otimes_i \) while \( \otimes \) will be simply \( \otimes_B \). We let also \( B = B_1 \).

10.1. **Soergel’s theorems.** We first recall the main results from [Soe90]. We refer to Section \( \S \) for definitions and notation regarding the BGG category \( \mathfrak{O} \).

**Theorem 10.1 ([Soe90] Endomorphismensatz 3).** Let \( w_n \) be the longest element of \( S_n \). We have a canonical isomorphism \( \text{End}_\mathfrak{O}(P(w_n \cdot 0)) \cong B \).

Let \( \mathcal{V} = \text{Hom}_\mathcal{O}(P(w_n \cdot 0), \cdot) : \mathcal{O}_0 \to B \) be the Soergel’s functor. We have then:

**Theorem 10.2 ([Soe90] Struktursatz 2).** Let \( P, Q \in \mathcal{O}_0(\mathfrak{gl}_n) \) be two projective modules. Then \( \mathcal{V} \) induces an isomorphism

\begin{align}
\text{Hom}_\mathcal{O}(P, Q) \to \text{Hom}_B(\mathcal{V}P, \mathcal{V}Q).
\end{align}

For an indecomposable projective \( P(z \cdot 0) \) with \( z \in S_n \), we call \( C_z = \mathcal{V}P(z \cdot 0) \) a Soergel module.

**Theorem 10.3 ([Soe90] Zerlegungssatz 1 and Theorem 4]).** The \( B \)-module \( C_z \) is indecomposable. Let \( z = s_{i_1} \cdots s_{i_r} \) be some reduced expression; then \( C_z \) can be identified with the unique summand of \( B_{i_r} \otimes \cdots \otimes B_{i_1} \otimes \mathbb{C} \) that is not isomorphic to any \( C_{w'} \) for \( w' \prec w \). This is also the unique summand containing \( 1 \otimes \cdots \otimes 1 \).
Consider for example a simple reflection $s_i \in S_n$. According to the theorem, the indecomposable object $C_i = C_{x_i}$ is a summand of $B_i \otimes C$. But it is immediate to check that the two dimensional $R$-module $B_i \otimes C$ is indecomposable, hence $C_i = B_i \otimes C$. This module is in fact isomorphic to $R/(x_1, \ldots, x_i, x_i + 1, x_i + 2, \ldots, x_n)$.

Notice that since $B$ is a quotient of $R$ we have
\begin{equation}
\text{Hom}_B(M, N) \cong \text{Hom}_R(M, N)
\end{equation}
for all $M, N \in B\text{-mod}$. Hence, it is harmful to consider $B$-modules as $R$-modules.

10.2. Some Soergel modules. We will now determine explicitly some modules $C_w$. In the following, we will use the notation introduced in \[9.1\]. First we recall an easy well-known fact:

**Lemma 10.4.** As a $\mathbb{C}$-vector space, a basis of $B \otimes_{i_1} B \otimes \cdots \otimes_{i_m} B \otimes_{i_r} C$ is given by
\begin{equation}
\{x_{i_1}^{r_1} \otimes \cdots \otimes x_{i_r}^{r_r} \otimes 1 \mid r_j \in \{0, 1\}\}.
\end{equation}

**Proposition 10.5.** For every $z \in D$, the module $C_wz$ is cyclic, that is $C_wz \cong R/\text{Ann}_R C_wz \cong B/\text{Ann}_B C_wz$.

**Proof.** By Proposition \[9.3\], $H_e$ appears exactly once with coefficient $q^{\ell(wz)}$ in the canonical basis element $C_wz$. By the Kazhdan-Lusztig Conjecture, this implies that in $0_0(g\mathfrak{l}_n)$ the dominant Verma module $M(0)$ appears exactly once in some Verma flag of the indecomposable projective $P(wz \cdot 0)$. By \[Str03b, Lemma 7.3\], this is equivalent to $C_wz$ being cyclic. \[□\]

**Lemma 10.6.** For every $z \in D$ the dimension of $C_wz$ over $\mathbb{C}$ is given by
\begin{equation}
\dim_\mathbb{C} C_wz = k! (z_{k+1}^\vee + 1) \cdots (z_n^\vee + 1) = b_1^z \cdots b_n^z.
\end{equation}

**Proof.** The first equality follows from Proposition \[9.4\] and the Kazhdan-Lusztig conjecture. We want to show the second equality. As in Example \[9.2\] we imagine the $b$-sequence written on top of the $\land \lor$-sequence for $z$. Over the $\land$’s we have the numbers between 1 and $k$, each appearing once: hence their contribute is $k!$. Over the $\lor$’s we have a number measuring how many $\land$’s are on its right, plus one: this coincides with how many times this $\lor$ has been moved to the left plus one, that is, $z_{k+1}^\vee + 1$. The claim follows immediately. \[□\]

**Lemma 10.7.** The module $C_wz$ is isomorphic to $R/(J_k, x_{k+1}, \ldots, x_n)$.

**Proof.** Let $J' = (J_k, x_{k+1}, \ldots, x_n)$. By Proposition \[10.5\] the module $C_wz$ is cyclic over $B$. Choose any reduced expression $s_{i_1} \cdots s_{i_N}$ for $w_k$ and build the corresponding module $B_{w_k} = B_{i_N} \otimes \cdots \otimes B_{i_1} \otimes C$. Since all polynomials of $J'$ are symmetric in the first $k$ variables, we have $J' \subseteq \text{Ann}_R B_{w_k} \subseteq \text{Ann}_R C_{w_k}$, hence $C_{w_k}$ is a quotient of $R/J'$. Notice that $J' = I_{e'}$ for $e \in S_n$ the identity element. By Lemma \[10.6\] and Proposition \[8.3\] $\dim_\mathbb{C} C_{w_k} = \dim_\mathbb{C} R/J'$, hence $C_{w_k} = R/J'$. \[□\]

As we said, we will use the same notation introduced in \[9.4\]. For $t_{i, l}^\wedge$ let
\begin{equation}
B_{t_{i, l}^\wedge} = B_{i+\ell-1} \otimes B_{i+\ell-2} \otimes \cdots \otimes B_i
\end{equation}
and for $z = t_{k, x_k}^\wedge \cdots t_{1, x_1}^\wedge$ let
\begin{equation}
B_z^\wedge = B_{t_{k, x_k}^\wedge} \otimes \cdots \otimes B_{t_{1, x_1}^\wedge}.
\end{equation}
Moreover, for $t_{i, l}^\vee$ let
\begin{equation}
B_{t_{i, l}^\vee} = B_{i-\ell} \otimes B_{i-\ell+1} \otimes \cdots \otimes B_{i-1}
and for \( z = t_{k+1, z_{k+1}}^\vee \cdots t_{k, z_k}^\vee \) let
\[
B^\vee_{\mu} = B_{\mu, z_{k+1}}^\vee \otimes \cdots \otimes B_{\mu, z_1}^\vee.
\]

From Soergel Theorem \[14.3\] and Proposition \[10.3\], it follows that \( C_{w_k} \) is isomorphic both to the \( B \)-submodule of \( B_{\nu}^\vee \otimes C_{w_k} \) generated by \( 1 = 1 \otimes \cdots \otimes 1 \) and to the \( B \)-submodule of \( B_{\nu}^\vee \otimes C_{w_k} \) generated by \( 1 = 1 \otimes \cdots \otimes 1 \).

The following lemma is the crucial step to determine the annihilator of \( C_{w_k} \).

**Lemma 10.8.** Let \( z \in D \), and let \( m \) be the number of non-zero \( z_i^\wedge \)'s. Then \( h_t(x_1, \ldots, x_{k-m+z_{k-m+1}^\wedge}) \in \text{Ann} \ C_{w_k} \) for all \( \ell > m \).

**Proof.** Let us prove the assertion by induction on the sum \( N \) of the \( z_i^\wedge \)'s (that is, up to a shift, the length of \( z \)). If \( N = 0 \) then also \( m = 0 \), and \( h_t(x_1, \ldots, x_k) \in \text{Ann} \ C_{w_k} = (J_k, x_{k+1}, \ldots, x_n) \) for every \( \ell > 1 \) by Lemma \[ 10.7 \].

For the inductive step, let \( i = k - m + z_{k-m+1}^\wedge \), write \( z = z_i^\wedge s_i \) and compute in \( B \otimes_i (B_{\nu}^\vee \otimes C_{w_k}) \):

\[
h_{t+1}(x_1, \ldots, x_i) \cdot (1 \otimes 1)
\]

\[
= (P_i(h_{t+1}(x_1, \ldots, x_i)) + x_i \partial_i(h_{t+1}(x_1, \ldots, x_i))) \cdot 1 \otimes 1
\]

\[
= 1 \otimes (P_i(h_{t+1}(x_1, \ldots, x_i)) + x_i \partial_i(h_{t+1}(x_1, \ldots, x_i))) \cdot 1.
\]

Since \( C_{w_k} \) is a summand of \( B \otimes_i (B_{\nu}^\vee \otimes C_{w_k}) \), it is sufficient to show that \[10.9\] is zero. In fact, we prove that both terms \( P_i(h_{t+1}(x_1, \ldots, x_i)) \) and \( \partial_i(h_{t+1}(x_1, \ldots, x_i)) \) act as zero on \( B^\vee_{\nu} \).

Let us start considering the second term. Let \( y \in D \) be determined by \( y_i^\wedge = z_i^\wedge \) for \( i \neq k - m + 1, k - m + 2 \), while \( y_{k-m+1}^\wedge = 0 \) and \( y_{k-m+2}^\wedge = z_{k-m+1}^\wedge \). Notice that our chosen reduced expression for \( z \) splits as \( z' = yz'' \), so that

\[
B^\vee_y = B_{t-1}^\vee B_{t-2}^\vee \cdots B_{k-m-1}^\vee B_1 B_{j-1} \cdots B_{i+1} = B_{z''}^\vee B_{z'}^\vee.
\]

for \( j = k - m + 1 + z_{k-m+2}^\wedge \), where we omitted the tensor product signs. By \[ 8.8 \], \( \partial_i(h_{t+1}(x_1, \ldots, x_i)) = h_t(x_1, \ldots, x_{i+1}) \); being symmetric in the variables \( x_{i+1} \) for \( a \neq i, i+1 \), this steps over \( B_{z''}^\vee \) and acts on \( B_{z'}^\vee \otimes C_{w_k} \). By induction, this action is zero.

Now let us consider the action of the term \( P_i(h_{t+1}(x_1, \ldots, x_i)) \). Write

\[
P_i(h_{t+1}(x_1, \ldots, x_i)) = h_{t+1}(x_1, \ldots, x) - x_i \partial_i h_{t+1}(x_1, \ldots, x_i)
\]

\[
= h_{t+1}(x_1, \ldots, x_i) - x_i h_{t+1}(x_1, \ldots, x_{i+1}).
\]

Of these two summands, the second acts as zero exactly as before. For the first one, write \( y's_{i+1} = y \) so that \( B_{y}^\vee = B \otimes_{i+1} B_{y}^\vee \). Then \( h_{t+1}(x_1, \ldots, x_i) \) steps over the first tensor product, and by induction acts as zero on \( B_{y}^\vee \).

**Proposition 10.9.** Let \( z \in D \) with \( b \)-sequence \( b^z \). Then the complete symmetric polynomial \( h_{t\downarrow}^\vee(x_1, \ldots, x_i) \) lies in \( \text{Ann} \ C_{w_k} \) for all \( i = 1, \ldots, n \).

**Proof.** We divide the indices \( i \in \{1, \ldots, n\} \) corresponding to the positions in the \( \forall \vee \)-sequence of \( z \) in three subsets: we call an index \( i \) such that \( z_i^\wedge = 0 \) initial, we call an index \( i \) for which \( b_i^\wedge = 1 \) final, and we call all other indices in between in the middle:

\[
\wedge \cdots \wedge \forall \vee \cdots \vee \forall \vee \cdots \vee
\]

where \( \wedge \times \) stands for a \( \wedge \) or a \( \vee \). Notice that it can happen that an index \( i \) is both initial and final if and only if there are no \( \forall \vee \)'s, that is \( k = n \). Since in this case we already know the claim, we can exclude it.

If \( i \) is final, then \( w_{k+z} \in S_i \subset S_n \) (where \( S_i \) is the subgroup generated by the first \( i - 1 \) simple transpositions) and obviously \( h_1(x_1, \ldots, x_i) \) annihilates \( C_{w_k} \).
If $z$ is not the identity (in which case there are no indexes in the middle), then $i = k - h + z_{k-1}^{h+1}$ is in the middle, and the statement of Lemma 10.8 is that $h_{i\gamma}(x_1, \ldots, x_i) \in \text{Ann} \mathcal{C}_{w_{a\gamma}}$. For the other indexes in the middle, we can use Lemma 10.8 after letting $h_{i\gamma}(x_1, \ldots, x_i)$ step over some initial tensor symbols of $B^\wedge_{i\gamma}$.

If $i$ is initial, then $z$ is a permutation in the subgroup of $S_n$ generated by $s_{i+1}, \ldots, s_{n-1}$, hence $h_{i\gamma}(x_1, \ldots, x_i)$, when acting on $B^\wedge_{i\gamma} \otimes \mathcal{C}_{w_{a\gamma}}$, can step over $B^\wedge_{i\gamma}$, and we only need to prove that $h_{i\gamma}(x_1, \ldots, x_i) \in \text{Ann} \mathcal{C}_{w_{a\gamma}}$. In fact, renaming the indexes this follows from the following general statement: $h_a(x_1, \ldots, x_i) \in J_k$ for every $1 \leq \ell \leq k$ and $a > k - \ell$. This well-known fact can easily be proved by (reversed) induction on $h$: if $h = k$ the claim is obvious; for the inductive step, just use (8.3). □

We can now identify the Soergel modules with the rings $R_b = R/I_b$ defined in §8.2.

**Theorem 10.10.** Let $z \in D$ with $b$-sequence $b^\gamma$. Then $\text{Ann} \mathcal{C}_{w_{a\gamma}} = I_b$ and $\mathcal{C}_{w_{a\gamma}} \cong R_b$. A basis of $R_b$ is given by

\[(10.12) \quad \{x_1^{c_1} \cdots x_{n-1}^{c_{n-1}} \cdot 1 \mid 0 \leq c_i < b_i^\gamma \}.
\]

**Proof.** For simplicity let $b = b^\gamma$. By Proposition 8.3, $I_b \subseteq \text{Ann} \mathcal{C}_{w_{a\gamma}}$, so we have a surjective map $R/I_b \twoheadrightarrow R/(\text{Ann} \mathcal{C}_{w_{a\gamma}})$. By Proposition 8.3 and Lemma 10.8 their dimension agree, hence $I_b = \text{Ann} \mathcal{C}_{w_{a\gamma}}$. The basis of $R_b$ is given by Proposition 8.3. □

Translating Proposition 8.3, we can determine the homomorphism spaces between the Soergel modules $\mathcal{C}_{w_{a\gamma}}$:

**Corollary 10.11.** Let $z, z' \in D$ with $b$-sequences $b^\gamma, b'^\gamma$. Let $c_i = \max\{b_i'^\gamma - b_i^\gamma, 0\}$ for $i = 1, \ldots, n - 1$. Then a $C$-basis of $\text{Hom}_R(\mathcal{C}_{w_{a\gamma}}, \mathcal{C}_{w_{a\gamma}'})$ is given by

\[(10.13) \quad \{1 \mapsto x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} \mid c_i \leq j_i < b_i'^\gamma \}.
\]

In particular, by Theorem 10.2, we have determined all homomorphism spaces $\text{Hom}_R(P(\mathcal{C}_{w_{a\gamma}}, \mathcal{C}_{w_{a\gamma}'}, 0))$ for $z, z' \in D$. Let $p \subseteq \mathfrak{gl}_n$ be the standard parabolic subalgebra with Weyl group $W_p = W_{k_\gamma} \subseteq S_n$. In what follows, we want to compute the homomorphism spaces between the projective modules $P^p(\mathcal{C}_{w_{a\gamma}}, 0)$ for $z \in D$ in the parabolic category $\mathcal{O}_p^p(\mathfrak{gl}_n)$. Since $\mathcal{O}_p^p(\mathfrak{gl}_n)$ is a Serre subcategory of $\mathcal{O}_0(\mathfrak{gl}_n)$, these are obtained as quotients of the corresponding homomorphism spaces in the non-parabolic category $\mathcal{O}_0(\mathfrak{gl}_n)$ by morphisms that factor through projective modules $P^p(w' \cdot o) \in \mathcal{O}_0(\mathfrak{gl}_n)$ for some $w' \notin W^p$ (cf. §6.2). Hence we need to compute other Soergel modules, corresponding to these elements $w'$. Surprisingly, it will suffice to consider elements $w'$ that differ from some $w_{a\gamma}$ only by a simple reflection, as in the following proposition:

**Proposition 10.12.** Let $z \in D$ with $b$-sequence $b^\gamma$. Suppose $z'^\gamma = z_j^\gamma + 1$ for some index $j$. Let $\ell = j - z_j^\gamma$, so that $s_{\ell z} = z_{z\ell}$ Then $\mathcal{C}_{s_{\ell z}\gamma} = \mathcal{C}_{s_{z\ell}\gamma}$ is the quotient of $R$ modulo the ideal generated by the complete symmetric polynomials

\[(10.14) \quad h_{a_i}(x_1, \ldots, x_i) \quad \text{for } i = 1, \ldots, n,
\]

where $a_i = b_i^\gamma$ for $i \neq \ell$ while $a_\ell = b_\ell^\gamma + 1$.

Notice that the sequence $a = (a_1, \ldots, a_n)$ is not an element of $\mathcal{B}^\gamma$, since $a_\ell = a_{\ell - 1} + 1$. 

\[\text{CATEGORIZATION OF REPRESENTATIONS OF } U_q(\mathfrak{sl}_2(1)) \]
Proof. The proof is analogous to the proof of Theorem \[10.10\] By Corollary \[9.8\] the module \( C_{s_j w_k z} \) is cyclic. In particular, it is the submodule generated by 1 inside \( B \otimes_z C_{w_k z} \). First, let us prove that the polynomials \(10.14\) lie in \( \text{Ann} C_{s_j w_k z} \), or equivalently that they vanish on \( B \otimes_z C_{w_k z} \). This is clear for \( i \neq \ell \); in this case, these polynomials can step over the first tensor product, and then they vanish because they lie in \( \text{Ann} C_{w_k z} \) by Theorem \[10.10\]. For the remaining case, we have

\[
(10.15) \quad h_{\alpha}(x_1, \ldots, x_\ell) \cdot (1 \otimes 1) = 1 \otimes (P_\ell(h_{\alpha}(x_1, \ldots, x_\ell)) \cdot 1) + x_\ell \otimes (\partial_\ell(h_{\alpha}(x_1, \ldots, x_\ell)) \cdot 1).
\]

By \(8.3\) all terms contain \( h_{\alpha}(x_1, \ldots, x_\ell) \) or \( h_{\alpha-1}(x_1, \ldots, x_{\ell+1}) \), both which both lie in \( \text{Ann} C_{w_k z} \), and we are done.

It remains to prove that the polynomials \(10.14\) are a set of generators. Let \( I \) be the ideal generated by them. We know that \( C_{s_j w_k z} \) is a quotient of \( R/I \). As for Lemma \[8.2\] the polynomials \(10.14\) are a basis of \( I \). As for Proposition \[5.3\] the quotient \( R/I \) has dimension \( a_1 \cdots a_n \). By Corollary \[9.8\] and an argument similar to the proof of Lemma \[10.6\] this coincides with the dimension of \( C_{s_j w_k z} \), and we are done. \( \square \)

### 10.3. Morphisms between Soergel modules

In each basis set \(10.13\) there is exactly one morphism of minimal degree, which we call the minimal degree morphism \( C_{w_k z} \rightarrow C_{w_k z'} \). For each \( z \in D \), the vector space \( \text{Hom}_R(C_{w_k z}, C_{w_k z'}) \) is a ring that is naturally isomorphic to \( C_{w_k z} \). Moreover, for \( z, z' \in D \) the vector space \( \text{Hom}_R(C_{w_k z}, C_{w_k z'}) \) is naturally a \((C_{w_k z}, C_{w_k z'})\)-bimodule. This bimodule is cyclic, generated by the minimal degree morphism. In what follows, we will often refer to this fact saying that the minimal degree morphisms divides all other morphisms.

We let \( D' \) be the set of shortest coset representatives for \( W_{\ell} \setminus S_n \). In particular, for every \( z \in D \) we have \( z, w_k z \in D' \).

**Definition 10.13.** For \( z, z' \in D \) we say that a morphism \( C_{w_k z} \rightarrow C_{w_k z'} \) is illicit if it factors through some \( C_y \) for \( y \notin D' \).

We let \( W_{z, z'} \) be the vector subspace of \( \text{Hom}_R(C_{w_k z}, C_{w_k z'}) \) consisting of all illicit morphisms. Since it is a \((C_{w_k z}, C_{w_k z'})\)-submodule, we can define the quotient bimodule

\[
(10.16) \quad Z_{z, z'} = \text{Hom}_R(C_{w_k z}, C_{w_k z'})/W_{z, z'}.
\]

In particular, we have that

\[
(10.17) \quad \text{Hom}_{O'}(P^p(w_k z \cdot 0), P^p(w_k z' \cdot 0)) \cong Z_{z, z'}.
\]

We are going to determine all the subspaces \( W_{z, z'} \), and consequently the quotients \( Z_{z, z'} \).

**Lemma 10.14.** Let \( z, z' \in D \), and suppose that for some index \( j \) we have

\[
(10.18) \quad z^\nu_j = \begin{cases} z^\nu_j + 1 & \text{for } i = j, j + 1, \\ z^\nu_j & \text{otherwise.} \end{cases}
\]

In particular \( z' = z s_{k \ell+1} \) for \( \ell = j - z^\nu_j - 1 \), and the corresponding \( \land \lor \)-sequence in positions \( \ell, \ell + 1, \ell + 2 \) are

\[
(10.19) \quad z = \cdots \land \lor \cdots \quad \text{and} \quad z' = \cdots \lor \land \cdots.
\]

Then

\[
(10.20) \quad W_{z, z'} = \text{Hom}_R(C_{w_k z}, C_{s_k z'}) \quad \text{and} \quad W_{z', z} = \text{Hom}_R(C_{w_k z'}, C_{s_k z}).
\]
Proof. It is enough to show that \( \varphi \in \text{Hom}_R(C_{w_{k+1}}, C_{w_{k+1}'}), \) \( \varphi : 1 \mapsto x_j x_{j+1} \) and \( \psi \in \text{Hom}_R(C_{w_{k+1}'}, C_{w_{k+1}}), \) \( \psi : 1 \mapsto 1 \) are illicit, since they divide all other morphisms. First of all, note that by construction

\[
(10.21) \quad b^i_j = \begin{cases} b^i_j + 1 & \text{for } i = \ell, \ell + 1, \\
 b^i_j & \text{otherwise.}
\end{cases}
\]

Let \( y = s_j z = z s_{\ell+1} \), and note that \( y \notin D' \). We know \( C_{w_{k+1}'} \) by Proposition 10.12. Since \( \text{Ann}(C_{w_{k+1}'}) \subseteq \text{Ann}(C_{w_{k+1}}) \subseteq \text{Ann}(C_{w_{k+1}'}) \), the morphism \( \psi \) can be written as the composition of the natural quotient maps

\[
(10.22) \quad C_{w_{k+1}'} \xrightarrow{1} C_{w_{k+1}} \xrightarrow{1} C_{w_{k+1}},
\]

hence it is illicit.

On the other side, \( x_{\ell+1} \text{Ann}(C_{w_{k+1}}) \subseteq \text{Ann}(C_{w_{k+1}}) \) because by (8.3)

\[
(10.23) \quad x_{\ell+1} h_{b_{\ell+1}^j} (x_1, \ldots, x_{\ell+1}) = h_{b_{\ell+1}^j+1} (x_1, \ldots, x_{\ell+1}) - h_{b_{\ell+1}^j+1} (x_1, \ldots, x_\ell)
\]

and \( h_{b_{\ell+1}^j+1} (x_1, \ldots, x_\ell) \in \text{Ann}(C_{w_{k+1}}) \) by the arguments of the proof of Lemma 8.7. Moreover, \( x_{\ell} \text{Ann}(C_{w_{k+1}'}) \subseteq \text{Ann}(C_{w_{k+1}'}) \) because by (8.3) we can write

\[
(10.24) \quad x_{\ell} h_{b_{\ell}^j} (x_1, \ldots, x_\ell) = h_{b_{\ell}^j+1} (x_1, \ldots, x_\ell) - h_{b_{\ell}^j+1} (x_1, \ldots, x_{\ell-1})
\]

and the r.h.s. is in \( \text{Ann}(C_{w_{k+1}'}) \) by Lemma 8.7. Hence \( \varphi \) can be written as the composition

\[
(10.25) \quad C_{w_{k+1}} \xrightarrow{x_{\ell+1}} C_{w_{k+1}'}, \quad C_{w_{k+1}'} \xrightarrow{x_{\ell}} C_{w_{k+1}'},
\]

and therefore is illicit.

\[\square\]

Lemma 10.15. Let \( z \in D \) and suppose \( z_j^j = z_{j+1}^j \) for some index \( j \). Let \( \ell = j - z_j^j \), so that \( s_j z = z s_{\ell} \). Then the endomorphism \( 1 \mapsto x_\ell \) of \( C_{w_{k+1}} \) is illicit.

Proof. Let \( y = s_j w_{k+1} z \notin D' \). First, we claim that \( x_\ell \text{Ann}(C_{w_{k+1}}) \subseteq \text{Ann}(C_y) \) and hence that \( 1 \mapsto x_\ell \) defines a morphism \( C_{w_{k+1}} \to C_y \). By Theorem 10.10 and Proposition 10.12, the only thing to check is that \( x_\ell h_{b_\ell^j} (x_1, \ldots, x_\ell) \in \text{Ann}(C_y) \). By (8.3) we have

\[
(10.26) \quad x_\ell h_{b_\ell^j} (x_1, \ldots, x_\ell) = h_{b_\ell^j+1} (x_1, \ldots, x_\ell) - h_{b_\ell^j+1} (x_1, \ldots, x_{\ell-1}) \in \text{Ann}(C_y).
\]

On the other side, again by Theorem 10.10 and Proposition 10.12, it is clear that \( 1 \mapsto 1 \) defines a morphism \( C_y \to C_{w_{k+1}} \). Hence the endomorphism \( 1 \mapsto x_\ell \) of \( C_{w_{k+1}} \) factors through \( C_y \) and is therefore illicit.

\[\square\]

More generally we have:

Lemma 10.16. Let \( z \in D \). For every \( j \) between \( k+1 \) and \( n-1 \) the morphism

\[
(10.27) \quad 1 \mapsto x_\ell x_{\ell+1} \cdots x_{\ell'},
\]

where \( \ell = j - z_j^j \) and \( \ell' = (j+1) - z_{j+1}^j - 1 \), is illicit.

Proof. Let \( y \in D \) be defined by \( y_i^j = z_i^j \) for \( i \neq j \), while \( y_j^j = z_{j+1}^j \). From Corollary 10.11 we have that \( 1 \mapsto 1 \) and \( 1 \mapsto x_\ell x_{\ell+1} \cdots x_{\ell' - 1} \) define morphisms \( C_{w_{k+1}} \to C_{w_{k+1}'} \) and \( C_{w_{k+1}'} \to C_{w_{k+1}} \) respectively. By Lemma 10.15 the endomorphism \( 1 \mapsto x_{\ell'} \) of \( C_{w_{k+1}}' \) is illicit, and so is (10.27), since it can be expressed as composition of these three morphisms.

\[\square\]

Theorem 10.17. For all \( z, z' \in D \) define a subbimodule \( \tilde{W}_{z,z'} \) of the homomorphism space \( \text{Hom}_R(C_{w_{k+1}}, C_{w_{k+1}'}) \) as follows:

(i) if for some index \( 1 \leq j \leq n-k-1 \) we have \( \forall_j^z \geq \forall_{j+1}^{z'} \) or \( \forall_j^z \geq \forall_{j+1}^{z'} \), then we set \( \tilde{W}_{z,z'} = \text{Hom}(C_{w_{k+1}}, C_{w_{k+1}'}) \);
(ii) otherwise we define \( \tilde{W}_{z,z'} \) to be the subbimodule generated by the morphisms

\[
1 \mapsto (x_{v_j} x_{v_{j+1}} \cdots x_{v_{n-k}})(x_1^{z_1} \cdots x_n^{z_n}) \quad \text{for } 1 \leq j \leq n - k,
\]

where \( c_i = \max\{b_i^z - b_i^{z'}, 0\} \) and

\[
\beta(j) = \begin{cases} 
\min\{\nu_j^{z+1}, \nu_j^{z'}\} - 1 & \text{if } j < n - k, \\
if j = n - k.
\end{cases}
\]

Then we have \( \tilde{W}_{z,z'} = W_{z,z'} \).

**Example 10.18.** Let us consider the following example:

\[
\begin{array}{cccccccccccc}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} \\
\text{10} & \text{10} & \text{9} & \text{8} & \text{7} & \text{6} & \text{5} & \text{5} & \text{5} & \text{4} & \text{3} & \text{2} & \text{1} & \text{1} \\
\text{\(b\)} & \text{\(z\)} & \text{\(z'\)} & \text{\(b'\)} & \text{\(x_2 x_3 x_4\)} & \text{\(x_9 x_{10} x_{11} x_{12}\)}
\end{array}
\]

For convenience, we have written the subscripts of the \( \nu \)'s, indicating their progressive number. We are in case (ii) and the generating morphisms (10.28) of \( \tilde{W}_{z,z'} \) are

\[
\begin{array}{c|c}
\text{j} & \text{morphism} \\
\hline
1 & 1 \mapsto (x_2 x_3 x_4)(x_1 x_5 x_6 x_7) \\
2 & 1 \mapsto (x_9)(x_1 x_5 x_6 x_7) \\
3 & 1 \mapsto (x_9 x_{10} x_{11} x_{12})(x_1 x_5 x_6 x_7) \\
4 & 1 \mapsto (x_{13})(x_1 x_5 x_6 x_7)
\end{array}
\]

In the picture, the case \( j = 1 \) is highlighted in solid blue and the case \( j = 3 \) is highlighted in dashed purple.

**Proof of Theorem 10.17.** First, we assume that \( \nu_j^z \geq \nu_{j+1}^z \) for some index \( 1 \leq j < n - k \). Pick \( j \) minimal with this property. Notice that by the minimality of \( j \) we have \( \nu_{j-1}^z < \nu_j^z \) (if \( j > 1 \)), and hence on the left of the \( j \)-th \( \vee \) of \( z \) there is an \( \wedge \) (this remains true also if \( j = 1 \), since in this case \( \nu_1^z \geq \nu_2^z > \nu_1^z \geq 1 \)). Let \( \alpha = \nu_j^z \) and \( \ell = \nu_{j+1}^z - \nu_j^z \), and let

\[
\begin{align*}
\text{(10.30)} & \quad \nu_j^z \wedge \nu_{j+1}^z \\
\text{(10.31)} & \quad z^{(1)} = z s_{\alpha + \ell - 1} s_{\alpha + \ell - 2} \cdots s_{\alpha + 1} \\
\text{(10.32)} & \quad z^{(2)} = z^{(1)} s_{\alpha - 1} s_{\beta}
\end{align*}
\]

where on the right we pictured the \( \wedge \vee \)-sequences between positions \( \alpha - 1 \) and \( \alpha + \ell \). The composition

\[
C_{w_k z} \xrightarrow{1} C_{w_k z}^{(1)} \xrightarrow{\ell - 1} C_{w_k z}^{(2)}
\]

is illicit by Lemma 10.14. Composing with the minimal degree morphism \( C_{w_k z}^{(2)} \to C_{w_k z'} \) we obtain the minimal degree morphism \( C_{w_k z} \to C_{w_k z'} \), which is therefore illicit. It follows that \( \text{Hom}_R(C_{w_k z}, C_{w_k z'}) = W_{z,z'} \).

A straightforward dual argument (cf. §8.3) proves that \( \text{Hom}_R(C_{w_k z'}, C_{w_k z}) = W_{z',z} \). Swapping \( z \) and \( z' \) it follows that \( \text{Hom}_R(C_{w_k z}, C_{w_k z'}) = W_{z,z'} \) if \( \nu_j^z \geq \nu_{j+1}^z \).
Now assume we are in case (ii) and fix an index \( j \). First, let us consider the case \( \vee_{j+1}^\prime < \vee_j^\prime \), so that \( \beta(j) = \vee_j^\prime \). Let \( \gamma = \vee_j^\prime \), \( \delta = \vee_{j+1}^\prime \), \( \epsilon = \vee_{j+1}^\prime \). Let

\[
\begin{align*}
\text{(10.34)} & \quad z \\
\text{(10.35)} & \quad z^{(1)} = z_s s_{\gamma+1} s_{\delta-1} \wedge \cdots \wedge \wedge \\
\text{(10.36)} & \quad z^{(2)} = z^{(1)} s_{\epsilon-2} s_{\delta+1} \wedge \cdots \wedge \wedge \\
\text{(10.37)} & \quad z^{(3)} = z^{(2)} s_{\delta-1} s_{\delta} \wedge \cdots \wedge \wedge \\
\end{align*}
\]

where on the right we pictured the \( \wedge \vee \)-sequences between positions \( \gamma \) and \( \epsilon \). The composition

\[
\text{(10.38)} \quad C_{w,k} \xrightarrow{1} C_{w,k}(z^{(1)}) \xrightarrow{x_{\gamma+1} x_{\delta+1} \cdots x_{\epsilon-1}} C_{w,k}(z^{(2)}) \xrightarrow{x_{\delta-1} x_{\delta}} C_{w,k}(z^{(3)})
\]

is illicit by Lemma 10.14. By construction the composition of (10.38) with the minimal degree morphism \( C_{w,k}(z^{(3)}) \to C_{w,k} \) equals the morphism (10.28) from \( C_{w,k} \) to \( C_{w,k} \), that is therefore illicit.

Let us now consider the other case \( \vee_{j+1}^\prime \leq \vee_j^\prime \). By Lemma 10.16 the endomorphism of \( C_{w,k} \) defined by

\[
\text{(10.39)} \quad 1 \mapsto x_{\vee_{j+1}^\prime} \cdots x_{\vee_j^\prime} \quad \text{is illicit. This morphism divides the morphism (10.28), which is therefore illicit.}
\]

To conclude the proof we are left to check that in case (ii) \( \mathcal{W}_{z,z'} \subseteq \mathcal{W}_{z,z'} \). Unfortunately we cannot check this directly. Instead, by Lemma 11.7 in the next section we have that the dimensions of the quotients of \( \text{Hom}_R(C_{w,k}, C_{z,k}) \) by \( \mathcal{W}_{z,z'} \) and \( \mathcal{W}_{z,z'} \) agree. This implies that \( \mathcal{W}_{z,z'} = \mathcal{W}_{z,z'} \). \( \Box \)

10.4. Grading. In order to keep the computations more transparent, we decided to postpone the introduction of the grading until now. The ring \( R \) is graded with \( \deg x_i = 2 \). Since the ideal \( J_n \) is homogeneous, \( B \) is also graded, and the graded definition of the module \( B_i \) is \( B_i = B \otimes_i B(-1) \). By Soergel theorems all \( C_m \) for \( w \in S_n \) are graded. In the graded version of the module \( C_{w,k} \) the cyclic generator is in degree \( -\ell(w,k) \).

The spaces \( \mathcal{W}_{z,z'} \) are homogeneous submodules, and the quotients \( \mathcal{Z}_{z,z'} \) are then graded modules.

By our discussion in §5 and with the opportune degree shifting we put on the modules \( C_{w,k} \), it follows that all modules \( C_{w,k} \) are graded self-dual. In particular

\[
\text{(10.40)} \quad \text{Hom}_R(C_{w,k}, C_{w,k}) \cong \text{Hom}_R(C_{w,k}, C_{w,k})
\]

as graded vector spaces for every \( z, z' \in D \). An explicit isomorphism was described in (5.21).

11. A diagram algebra for \( \Omega_k(\mathfrak{n}) \)

We want now to define diagram algebras over \( \mathbb{C} \) which are isomorphic to the endomorphism algebras of projective generators of \( \Omega_k(\mathfrak{n}) \). These algebras are analogous to the generalized Khovanov algebras defined in [BS11]. In particular, as the diagrams used in [BS11] are essentially the same diagrams from [FK97] describing the intertwining operators of \( \mathfrak{gl}_2 \) representations, our diagrams are essentially the same web diagrams that we used in (5.20) for the intertwining operators of the representations of \( \mathfrak{gl}(1|1) \).

We point out that the major difficulty is the definition of the multiplication of two diagrams, which is not simply stacking one on the top of the other (as in many other diagram algebras), but instead a quite involved process. Brundan and Stroppel use Khovanov’s TQFT to define this multiplication. Since there is not an analogous of such a TQFT for \( \mathfrak{gl}(1|1) \), we construct the multiplication in an indirect way using
composition of morphisms between Soergel modules. A drawback of our definition of the multiplication is that it is not clear how one can define diagrammatically bimodules for the diagram algebra, as in [BS10].

Using the same arguments and techniques of [BS11] we will describe explicitly the graded cellular structure (§11.3) and the graded properly stratifies structure (11.4). In §11.6 we will use the diagram algebra and Soergel modules to compute the endomorphism rings of the functors $E_k$ and $F_k$ defined in §7.6 proving that they are indecomposable.

11.1. Diagrams. We start introducing the diagrams on which our algebras will be build. We will redefine some keywords that are commonly used in Lie theory (such as weight and block) in a diagrammatical sense. When using these keywords, it will always be clear if we refer to diagrams or to Lie theory.

Weights. A number line $L$ is a horizontal line containing a finite number of vertices indexed by a set of consecutive integers in increasing order from left to right. Given a number line, a weight is obtained by labeling each of the vertices by $\wedge$ or $\vee$.

On the set of weights there is the partial order called Bruhat order, generated by $\wedge\vee \succ \vee\wedge$. For weights $\lambda, \mu$ declare $\lambda \sim \mu$ if $\mu$ can be obtained from $\lambda$ by permuting $\wedge$’s and $\vee$’s.

Blocks. A block $\Gamma$ is a $\sim$-equivalence class of weights. From now on, let us fix a block $\Gamma$. Let also $k$ be the number of $\wedge$’s and $n-k$ be the number of $\vee$’s of any weight of $\Gamma$. The weights of $\Gamma$ can be identified with $\wedge\vee$-sequences in the sense of §9.1, and hence with elements of $D_{n,k}$. For a weight $\lambda$, we can then define as in §9.1 the position sequences $(\wedge^1, \ldots, \wedge^k)$ and $(\vee^1, \ldots, \vee^{n-k})$ and the $b$-sequence $b^\lambda$.

Enhanced weights. An enhanced weight $\lambda^\sigma$ is a weight $\lambda$ together with a bijection $\sigma$ between the vertices labelled $\wedge$ in $\lambda$ and the set $\{1, \ldots, k\}$. By numbering the $\wedge$’s from the left to the right we may view $\sigma$ as en element in $S_k$ and call it the underlying permutation. We call $\lambda$ the underlying weight. We will also say that we obtain the enhanced weight $\lambda^\sigma$ by enhancing the weight $\lambda$ with the permutation $\sigma$.

Notice that there are exactly $k!$ enhanced weights with the same underlying weight.

We define a partial order on the set of enhanced weights by the following rule:

\[(11.1) \quad \lambda^\sigma \preceq \mu^\tau \iff \lambda \prec \mu \text{ or } (\lambda = \mu \text{ and } \ell(\sigma) \leq \ell(\tau)).\]

Fork diagrams. An $m$-fork is a tree with a unique branching point (the root) of valency $m$; the other $m$ vertices of the tree are called the leaves. An 1-fork will be also called a ray. This is an example of a 5-fork:

```
 leaves
   / \
 /   \ 
 /     \\ 
\root
```

Let $V$ be the set of vertices of the number line $L$, and let $H_-$ (resp. $H_+$) be the half-plane below (resp. above) $L$. A lower fork diagram is a diagram made by the number line $L$ together with some forks contained in $H_-$, such that the leaves of each $m$-fork are $m$ distinct consecutive vertices in $V$; we require each vertex of $V$ to be a leaf of some fork. The forks and rays of a lower fork diagram will be also called lower forks and lower rays.

Upper rays, upper forks and upper fork diagrams are defined in an analogous way. If $c$ is a lower fork diagram, the mirror image $c^*$ through the horizontal number line
CATEGORIFICATION OF REPRESENTATIONS OF $U_q(\mathfrak{gl}(1|1))$

is an upper fork diagram, and vice versa. The following are examples of a lower fork diagram $c$ and its mirror image $c^*$:

![Diagram](image)

Oriented diagrams. If $c$ is a lower fork diagram and $\lambda$ is a weight with the same underlying number line, we can glue them to obtain a diagram $c\lambda$. We call $c\lambda$ an unenhanced oriented lower fork diagram if:

- each $m$-fork for $m \geq 1$ is labeled with exactly one $\lor$ and $m-1$ $\land$'s;
- the diagram begins at the left with a (possibly empty) sequence of rays labeled $\land$, and there are no other rays labeled $\land$ in $c$.

Notice that by definition each $\land$ and $\lor$ of $\lambda$ labels some fork of $c$. We define analogously an unenhanced oriented upper fork diagram. The orientation of an unenhanced oriented lower (or upper) fork diagram is the corresponding weight.

An (enhanced) oriented lower fork diagram $c\lambda^\sigma$ is an unenhanced oriented lower fork diagram $c\lambda$ together with a permutation $\sigma \in S_k$ such that $\lambda^\sigma$ is an enhanced weight. Similarly we define an (enhanced) oriented upper fork diagram.

For $m \geq 1$ and $1 \leq i \leq m$ we define $\lambda(m, i)$ to be the weight formed by one $\lor$ and $m-1$ $\land$'s, where the $\lor$ is at the $i$-th place. Note that a lower fork diagram $c$ consisting of only a lower $m$-fork admits exactly $m!$ orientations, and they are exactly the $\lambda(m, i)^\sigma$ for $i \in \{1, \ldots, m\}$, $\sigma \in S_{m-1}$.

By a fork diagram we mean a diagram of the form $ab$ obtained by gluing a lower fork diagram $a$ underneath an upper fork diagram $b$, assuming that they have the same underlying number lines. An unenhanced oriented fork diagram is a fork diagram $a\lambda b$ obtained by gluing an oriented lower fork diagram $a\lambda$ and an oriented upper fork diagram $\lambda b$, as in the picture:

![Diagram](image)

An (enhanced) oriented fork diagram is obtained by additionally enhancing the corresponding weight.

Degrees. Define the degree of an unenhanced oriented lower (or upper) $m$-fork by setting $\deg(c\lambda(m, i)) = \deg(\lambda(m, i)c^*) = (i - 1)$. Define then the degree of an unenhanced oriented lower (resp. upper) fork diagram to be the sum of the degrees of all the lower (resp. upper) forks. Finally, the degree of an unenhanced oriented fork diagram $a\lambda b$ is

$$\deg(a\lambda b) = \deg(a\lambda) + \deg(\lambda b).$$
Moreover, define the degree of a permutation $\sigma$ as $\deg(\sigma) = 2\ell(\sigma)$. Then we define the degree of enhanced oriented diagrams by
\begin{align}
(11.3) \quad & \deg(a\lambda^\sigma) = \deg(a\lambda) + \deg(\sigma), \\
(11.4) \quad & \deg(\lambda^\sigma b) = \deg(\lambda b) + \deg(\sigma), \\
(11.5) \quad & \deg(a\lambda^\sigma b) = \deg(a\lambda b) + \deg(\sigma) = \deg(a\lambda) + \deg(\lambda b) + \deg(\sigma)
\end{align}

In particular, enhancing with the neutral element $e \in S_k$ preserves the degree.

**Example 11.1.** Consider the fork diagram $a\lambda b$ given by:

We have $\deg(a\lambda) = 1$ and $\deg(\lambda b) = 2 + 3 = 5$, so that $\deg(a\lambda b) = 6$. We can enhance the diagram with any permutation $\sigma \in S_5$, and then $\deg(a\lambda^\sigma b) = 6 + 2\ell(\sigma)$.

*The lower fork diagram associated to a weight.* For each weight $\lambda$ there is a unique lower fork diagram, denoted $\lambda^\downarrow$, such that $\lambda^\downarrow e$ is an oriented lower fork diagram of degree 0. One can construct it easily in the following way: examine the weight $\lambda$ from the left to the right and find all maximal subsequences consisting of a $\vee$ followed by some $\wedge$'s; draw a lower fork under each of these subsequences, and then draw lower rays under the remaining $\wedge$'s and $\vee$'s. Analogously $\lambda^\downarrow = (\lambda^\uparrow)^*$ is the unique upper fork diagram such that $\lambda^\downarrow \lambda^\uparrow$ is an oriented upper fork diagram of degree 0.

For weights $\mu$ and $\lambda$, we use the notation $\mu \subset \lambda$ to indicate that $\mu \sim \lambda$ and $\mu^\wedge \lambda^\vee$ is an oriented lower fork diagram.

**Lemma 11.2.** Let $\lambda, \mu$ be two weights in the same block $\Gamma$. If $\lambda = \mu$ then $\lambda = \mu$. If $\mu \lambda$ is oriented then $\mu \leq \lambda$ in the Bruhat order.

*Proof.* Being in the same block, the weights $\lambda$ and $\mu$ have the same number of $\wedge$'s and $\vee$'s. The claims follow then easily from the construction.

In particular, given our fixed block $\Gamma$, it follows that every lower fork diagram $a$ (such that $a\mu$ is oriented for some $\mu \in \Gamma$) determines a unique weight $\lambda$ with $\lambda = a$. In what follows, we will sometime interchange $a$ and $\lambda$ in the notation: for example, we will write $\vee^a_i$ for $\vee^\lambda_i$ or $b^a$ for $b^\lambda$ and so on.

We collect now some lemmas that we will need later.

**Lemma 11.3.** Let $\lambda, \mu$ be two weights in the same block $\Gamma$.

(i) The lower fork diagram $\lambda^\mu$ is oriented if and only if
\begin{equation}
(11.6) \quad \forall^i \lambda \leq \forall^i \mu < \forall^i \lambda + 1 \quad \text{for all } i \in 1, \ldots, n-k-1.
\end{equation}

(ii) There exists an oriented fork diagram $\lambda^\mu \eta$ for some $\eta \in \Gamma$ if and only if
\begin{equation}
(11.7) \quad \forall^i \lambda < \forall^i \mu + 1 \quad \text{and} \quad \forall^i \mu < \forall^i \lambda + 1 \quad \text{for all } i \in 1, \ldots, n-k-1.
\end{equation}

*Proof.* It is clear that (ii) follows from (i) so let us prove (i). It is easy to see that the lower fork diagram $\lambda^\mu$ is oriented if and only if each lower fork of $\lambda$ is labeled by exactly one $\vee$; this is exactly the same as (11.6).

**Lemma 11.4.** Consider weights $\lambda, \mu \in \Gamma$ with corresponding $b$-sequences $b^\lambda, b^\mu$.

(a) If $\mu \geq \lambda$ then $b^\mu_i \leq b^\lambda_i$ for all $i = 1, \ldots, n$.

(b) If $\lambda^\mu$ is oriented then $b^\lambda_i - b^\mu_i \leq 1$ for all $i = 1, \ldots, n$. 

(c) If \( \lambda \eta^\tau \mu \) is oriented (for some weight \( \eta \in \Gamma \)), then \( |b_i^\lambda - b_i^\mu| \leq 1 \) for all \( i = 1, \ldots, n \).

**Proof.** If \( \mu \geq \lambda \) then the \( i \)-th \( \lor \) of \( \mu \) is not on the right of the \( i \)-th \( \lor \) of \( \lambda \), and the first claim follows.

Let \( \lambda \eta^\mu \) be oriented. By Lemma \( \ref{lem:oriented_weights} \) we have \( \lor_i^\lambda \leq \lor_i^\mu < \lor_{i+1}^\mu \). This means that for every vertex \( v \in V \) there is at most one \( \lambda \) more to the right of \( v \) in \( \lambda \) than in \( \mu \). This is exactly \( \ref{lem:oriented_weights} \).

The last claim follows from the second: if \( \lambda \eta^\mu \) is oriented (for some weight \( \eta \) with \( \mathbf{b} \)-sequence \( \mathbf{b}^\prime \)), then \( b_i^\lambda - b_i^\mu = b_i^\lambda - b_i^\eta + b_i^\eta - b_i^\mu \in \{1, -1, 0\} \). \( \square \)

Since we have identified \( \Gamma \) with \( D_{n,k} \), we can define the length \( \ell(\lambda) \) of any weight \( \lambda \in \Gamma \) to be the length of the corresponding permutation in \( D_{n,k} \). This has not be confused with the length of the permutation \( \sigma \) of an enhanced weight \( \lambda^\tau \).

**Lemma 11.5.** Consider weights \( \lambda, \eta \) in the same block \( \Gamma \). Then

\[
\deg(\lambda \eta^\tau) = \ell(\lambda) - \ell(\eta) + 2\ell(\sigma).
\]

**Proof.** Since \( \lambda \eta \) is oriented, the weight \( \eta \) is obtained from \( \lambda \) permuting the \( \land \)'s and \( \lor \)'s on each lower fork of \( \lambda \). The degree of \( \lambda \eta \) is the sum of how much each \( \lor \) of \( \lambda \) has been moved to the right to reach the corresponding \( \lor \) of \( \eta \); hence it is just the length of this permutation. In other words, if we let \( z, z' \in D_{n,k} \) be the permutations corresponding to \( \lambda, \eta \) respectively, then we have \( z = z'y \) for some \( y \in S_n \) with \( \ell(z') = \ell(z) + \ell(y) \), and \( \deg(\lambda \eta) = \ell(y) \). \( \square \)

**11.2. The algebra structure.** We connect now our diagrams with the commutative algebra from Section \( \ref{sect:algebra_structure} \). Let us always fix a block \( \Gamma \) with \( k \land \)'s and \( n-k \lor \)'s.

**Relations with polynomial rings.** We associate to the weight \( \lambda \) the ring \( R_\lambda = R/I_\lambda \) (that has been defined in \( \ref{sect:relations} \)), and we want to describe \( Z_{z,z'} \) from \( \ref{sect:relations} \) diagrammatically.

Given an oriented lower fork diagram \( \lambda \eta^\tau \), we define the polynomial

\[
p_{\lambda \eta^\tau} = \mathcal{S}_\tau(x_{\lambda^\tau}, \ldots, x_{\lambda^\tau}) \cdot \prod_{j=1}^{n-k} x_{\lor_j^\lambda} x_{\lor_{j+1}^\lambda} \cdots x_{\lor_{j-1}^\lambda} \in R.
\]

with \( \mathcal{S}_\tau(x_{\lambda^\tau}, \ldots, x_{\lambda^\tau}) \) as defined in \( \ref{sect:relations} \). Notice that the terms on the right always make sense because, since \( \lambda \eta^\tau \) is oriented, \( \lor_j^\tau \geq \lor_j^\lambda \) for all indices \( j \) (cf. Lemma \( \ref{lem:oriented_weights} \)). Often we will consider \( p_{\lambda \eta^\tau} \) as a polynomial in the quotient \( R_\lambda \), but it will be convenient to have a chosen lift in \( R \). Notice that we have

\[
\deg(p_{\lambda \eta^\tau}) = 2(\ell(\sigma) + \ell(\lambda) - \ell(\eta)).
\]

**Proposition 11.6.** Let \( \lambda, \mu \in \Gamma \) be weights, and let \( z, z' \) be the corresponding elements of \( D \). Let \( W_{\mu,\lambda} \) be the graded vector space with homogeneous basis

\[
\{\mu \eta^\tau \lambda | \mu \eta^\tau \lambda \text{ is an oriented fork diagram}\}.
\]

With \( \tilde{W}_{z,z'} \) as defined in Theorem \( \ref{thm:isomorphism} \) we have an isomorphism of graded vector spaces

\[
\Psi : W_{\mu,\lambda} \longrightarrow \text{Hom}_R(C_{w_k z}, C_{w_k z'})/\tilde{W}_{z,z'}
\]

\[
\mu \eta^\tau \lambda \mapsto (1 \mapsto p_{\mu \eta^\tau} \lambda).
\]
Proof. First, note that \( p_{\mu\nu} = p_{\mu\nu} \mathcal{S}'(x_{\lambda_1}, \ldots, x_{\lambda_N}) \). By definition we have \( p_{\mu\nu} = x_j^{i_1} \cdots x_k^{i_m} \) with \( \varepsilon_j = b_j^\nu - b_j^\nu \). By Lemma 11.3, \( b_j^\nu \geq b_j^\nu \) for every \( j \), hence \( \varepsilon_j \geq b_j^\nu - b_j^\nu \). By Corollary 10.11, \( 1 \mapsto p_{\mu\nu} \) induces a well-defined morphism in \( \text{Hom}_R(C_{\mu\nu}, C_{\lambda_1}) \), hence also in the quotient.

Let us show that (11.12) is homogeneous of degree 0. The degree of the morphism \( 1 \mapsto p_{\mu\nu} \) in \( \text{Hom}_R(C_{\mu\nu}, C_{\lambda_1}) \) is \( \deg(p_{\mu\nu}) = \ell(wkz') + \ell(wkz) \), that is the same as \( \deg(p_{\mu\nu}) - \ell(z') + \ell(z) = \deg(p_{\mu\nu}) - \ell(\mu) + \ell(\lambda) \). By (10.10) this is \( \ell(\lambda) + \ell(\mu) - 2\ell(\eta) + 2\ell(\sigma) \). By Lemma 11.3, this is the same as \( \deg(\mu\nu \chi) \).

Next, we want to see that \( p_{\mu\nu} \) is always a monomial of the basis (10.13). For that, note that by definition \( \varepsilon_j = 1 \) exactly when \( b_j^\mu = b_j^\nu + 1 \). Moreover, the monomial \( \mathcal{S}'(x_{\lambda_1}, \ldots, x_{\lambda_N}) = x_j^{i_1} \cdots x_k^{i_m} \) is by construction in the basis of \( R_\eta \), that means that \( i_j \leq b_j^\nu \) for every \( j \). It follows that \( i_j + \varepsilon_j < b_j^\nu \), hence \( p_{\mu\nu} \) is a monomial of the basis (10.13).

We claim now that none of the \( p_{\mu\nu} \) is \( \tilde{W}_{z,z'} \). Note that by the construction of the monomial \( \mathcal{S}'(x_{\lambda_1}, \ldots, x_{\lambda_N}) = x_{j_1}^{i_1} \cdots x_{j_m}^{i_m} \) does not appear in \( p_{\mu\nu} \). By Lemma 11.3, we have \( \nu_j < \nu_{j+1} \) and \( \nu_j < \nu_{j+1} \). This means that both \( x_j^{i_1} \cdots x_j^{i_m} \) and \( x_j^{i_1} \cdots x_j^{i_m} \) do not divide \( p_{\mu\nu} \).

To conclude the proof, we need to construct an inverse of \( \Psi \). Take a basis monomial \( \mathfrak{m} = x_j^{i_1} \cdots x_j^{i_m} \in \text{Hom}_R(C_{\mu\nu}, C_{\lambda_1}) \) that does not lie in \( \tilde{W}_{z,z'} \). For every \( j \), let \( \ell_j \) be the maximum such that \( x_j^{i_1} x_j^{i_2} \cdots x_j^{i_m} \) divide \( \mathfrak{m} \). As \( \mathfrak{m} \) does not lie in \( \tilde{W}_{z,z'} \), it should be \( \ell_j < \nu_{j+1} \) and \( \ell_j < \nu_{j+1} \). Form a weight \( \nu \) in the same block of \( \lambda \) and \( \mu \) with the \( \nu \)'s in positions \( \ell_1, \ell_2, \ldots, \ell_{\kappa-1} \) by Lemma 11.3, the diagram \( \mu\nu \chi \) is oriented. Let \( \mathfrak{m}' \) be the quotient of \( \mathfrak{m} \) by \( p_{\mu\nu} \). By construction, \( b_j^\nu = b_j^\nu \) if \( x_j \) does not appear in \( p_{\mu\nu} \), and \( b_j^\nu = b_j^\nu + 1 \) if \( x_j \) appears (with coefficient 1) in \( p_{\mu\nu} \). Hence, it is clear that \( \mathfrak{m}' \) is a monomial \( \mathcal{S}'(x_{\lambda_1}, \ldots, x_{\lambda_N}) \). By construction, it follows that in this way we get an inverse of the map (11.12), that is hence an isomorphism.

As a consequence, we obtain the following result, that completes the proof of Theorem 10.14.

**Lemma 11.7.** For all \( z, z' \in D \) we have
\[
\text{dim}_{\mathbb{C}} \text{Hom}_{R}(C_{\mu\nu}, C_{\lambda_1})/\tilde{W}_{z,z'} = \text{dim}_{\mathbb{C}} Z_{z,z'}.\tag{11.13}
\]

*Proof.* By Proposition 11.6, the dimension of \( \text{Hom}_R(C_{\mu\nu}, C_{\lambda_1})/\tilde{W}_{z,z'} \) is the same as \( \text{dim}_{\mathbb{C}} W_{\lambda,\mu} \), where \( \lambda, \mu \in \Gamma \) are the weights corresponding to \( z, z' \). This dimension is simply \( k! \) times the number of unenhanced weights \( \eta \) such that \( \mu\eta \chi \) is oriented.

On the other side, by (10.17) we have
\[
Z_{z,z'} \cong \text{Hom}_{Q}(P^n(wkz), P^n(wkz')) \cong \text{Hom}_{Q_k}(Q(wkz), Q(wkz')), \tag{11.14}
\]

where \( n \) is the regular composition of \( n \). Since the modules \( Q(wkz) \), \( Q(wkz') \) are projective, we can compute the dimension of (11.14) using Proposition 7.11 and we get
\[
\text{dim}_{\mathbb{C}} Z_{z,z'} = ([Q(wkz)], [Q(wkz')])_{k} = 1, \tag{11.15}
\]
where \( (\cdot,\cdot)_{k} = 1 \) is the form \( (\cdot,\cdot)_{k} \) evaluated at \( q = 1 \). If \( v_{(wkz)}^\circ \) and \( v_{(wkz')}^\circ \) are the canonical basis elements corresponding to \( v_{(wkz)} \) and \( v_{(wkz')} \), we get by Proposition 7.10
\[
\text{dim}_{\mathbb{C}} Z_{z,z'} = (v_{(wkz)}^\circ, v_{(wkz')}^\circ)_{k} = 1. \tag{11.16}
\]
By the orthogonality of the standard basis elements \( v_w \) for \( w \in \Lambda_k(n) \) we can write
\[
(v_w^\triangledown, v_w^\triangledown)_{\eta} = \frac{1}{|k|!} \sum_{w \in \Lambda_k(n)} (v_{(w,k)}^\triangledown, v_w^\triangledown)_{\eta}.
\]

Let \( C(v_{(w,k)}) \), \( C(v_{w,k}) \) be the canonical basis diagrams corresponding to \( v_{(w,k)} \) and \( v_{w,k} \). Notice that the underlying trees correspond exactly to the lower fork diagrams \( \Delta \) and \( \Delta \).

By the definition of the bilinear form, \( (v_{(w,k)}^\triangledown, v_w^\triangledown)_{\eta} \) is equal to \( |k|! \) times the evaluation of the diagram \( \mathcal{D} \) obtained by labeling the canonical basis diagram \( C(v_{(w,k)}) \) with \( \wedge \)'s and \( \vee \)'s according to the standard basis diagram of \( v_w \). This labeled diagram corresponds in a natural way to a lower fork diagram \( \Delta \eta \). If one analyses the evaluation rules (Figure 1), one sees immediately that the evaluation of \( \mathcal{D} \) is a monomial in \( q \) if the corresponding diagram \( \Delta \eta \) is oriented, or zero otherwise. Hence the r.h.s. of (11.17), evaluated for \( q = 1 \), is equal to \( k! \) times the number of unenhanced weights \( \eta \) such that the lower fork diagrams \( \Delta \eta \) and \( \mu \eta \) are both oriented.

The algebra structure. Thanks to Proposition 11.6, we can define a graded algebra \( A = Ap \) over \( \mathbb{C} \). As a graded vector space, a homogeneous basis is given by
\[
\{a\lambda \sigma \beta\} \mid \text{for all } \alpha, \lambda, \beta \in \Gamma, \sigma \in S_k \text{ such that } \alpha \supset \lambda \subset \beta
\]
that is the same as
\[
\{(a\lambda \beta \sigma)\} \mid \text{for all oriented fork diagrams } a\lambda \beta \sigma \text{ with } \lambda \in \Gamma.
\]
The degree on this basis is given by the degrees on fork diagrams. For \( \lambda \in \Gamma \) we write \( e_\lambda \) for \( (\lambda \lambda \lambda) \). Note that the vectors \( e_\lambda \) give a basis for the degree 0 component of \( A \).

From Proposition 11.6 we get the following:

**Corollary 11.8.** There is an isomorphism of graded vector spaces
\[
A \cong \bigoplus_{z, z' \in D} \text{Hom}(C_{w_k z}, C_{w_k z'})/\mathcal{W}_{z, z'}.
\]
This defines a graded algebra structure on \( A \).

**Remark 11.9.** Explicitly, the multiplication of the basis vectors \( (a\lambda \sigma b) \) and \( (c\mu \tau d) \) can be computed in the following way. First, if \( b^* \neq c \) then set it to be zero. Now suppose \( b = c^* \). Then take \( p_{a\lambda \tau} \) and \( p_{a\lambda \sigma} \) in \( R \) and multiply them. By construction, the result gives a well defined morphism of the corresponding Soergel modules: write it as a linear combination of the basis (11.13) and translate it in the diagrammatic algebra \( A \) using the isomorphism of Proposition 11.6.

**Example 11.10.**

\[
a\lambda b = \ldots \quad \text{and} \quad c\mu d = \ldots.
\]

Let also \( \sigma = s_1 \in S_3, \tau = e \in S_3 \). We want to compute the product \( (a\lambda \sigma b)(c\mu \tau d) \).

First notice that \( b^* = c \). By (11.9) we have
\[
p_{a\lambda \sigma} = x_1 \cdot x_1 x_4
\]
\[
p_{c\mu \tau} = 1 \cdot x_1
\]
(for the computation of the polynomials $\mathcal{E}_\sigma$ and $\mathcal{E}_\tau$, we refer to Example 8.14). The product is $p_{\lambda\nu}p_{\mu\sigma} = x_1^\lambda x_4$. The $b$-sequence of $a$ is $(4, 3, 2, 1, 1)$, hence $x_1^3 x_4$ is not an element of the monomial basis $\{11.19\}$ of $R_n$. We need to do some computations in the ring $R_n$: using the relations $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1^4 = 0$ we have

$$x_1^4 x_4 \equiv -x_1^4 - x_1^3 x_2 - x_1^3 x_3 \equiv -x_1^4 x_2 - x_1^4 x_3.$$  

This is now a linear combination of monomials of the basis $\{11.13\}$. The monomial $-x_1^3 x_2$, although not zero in $R_n$, is of type $\{10.28\}$, hence defines an illicit morphism and is zero in the quotient. We are left only with the monomial $x_1 x_2$, and the biggest index $i$ such that $x_1 x_2 \cdots x_i | m$ is 1. Hence the monomial $m$ corresponds to a diagram $aq\eta d$ where $\eta$ has $\vee$’s in positions 2, 5. Moreover, the permutation $\pi$ is determined by $\mathcal{E}_\lambda(x_1, x_3, x_4) = x_1^2 x_3$. By Example 8.14 $\pi$ is the longest element of $S_4$. Hence $(a\lambda\sigma)b(c\mu\tau d) = -(a\eta\tau d)$, where

\[
\begin{tikzpicture}
  \node (A) at (0,0) {a};
  \node (B) at (1,0) {b};
  \node (C) at (2,0) {c};
  \node (D) at (3,0) {d};
  \node (E) at (0,-1) {\lambda};
  \node (F) at (1,-1) {\mu};
  \node (G) at (2,-1) {\sigma};
  \node (H) at (3,-1) {\tau};
  \draw (A) -- (B) -- (C) -- (D);
  \draw (E) -- (B);
  \draw (F) -- (C);
  \draw (G) -- (D);
  \draw (H) -- (A);
\end{tikzpicture}
\]

By construction, $p_{\lambda\lambda} = 1$ for any $\lambda \in \Gamma$. Under the isomorphism of Proposition 11.6, the element $e_\lambda$ is sent to $id_{W_{w_k}} \in \text{End}_R(C_{w_k})$, where $z \in D$ corresponds to $\lambda$; hence the elements $e_\lambda$ satisfy

$$e_\lambda(a\mu\sigma b) = \begin{cases} a\mu\sigma b & \text{if } a = \lambda, \\
0 & \text{otherwise,} \end{cases} \quad (a\mu\sigma b) e_\lambda = \begin{cases} a\mu\sigma b & \text{if } b = \lambda, \\
0 & \text{otherwise} \end{cases}$$

for any basis element $a\mu\sigma b \in A$. That is, the vectors $\{e_\lambda | \lambda \in \Gamma\}$ are mutually orthogonal idempotents whose sum is the identity $1 \in A$. The decomposition $\{11.20\}$ can be written as

$$A = \bigoplus_{\lambda, \mu \in \Gamma} e_\lambda Ae_\mu.$$  

A basis of the summand $e_\lambda Ae_\mu$ is

$$\{\lambda \eta \sigma \tau | \text{ for all } \eta \in \Gamma, \sigma \in S_k \text{ such that } \lambda \supset \eta \subset \mu\}.$$  

Duality. Recall from Example 8.21 that for every $z, z' \in D$ we have an isomorphism

$$\Theta : \text{Hom}_R(C_{w_k}, C_{w_{z'}}) \rightarrow \text{Hom}_R(C_{w_k}, C_{w_{z'}}),$$

\[
(1 \mapsto p) \mapsto (1 \mapsto x^{b-b'} p),
\]

where $b, b'$ are the $b$-sequences of $z, z'$ respectively and $b - b' = (b_1 - b'_1, \ldots, b_n - b'_n)$ and the notation is as in (8.21).

**Lemma 11.11.** Let $\lambda, \mu \in \Gamma$ and let $z, z'$ be the corresponding elements of $D_{n,k}$. We have $\Theta(W_{z',z}) = W_{z',z}$. Therefore the isomorphism $\Theta$ descends to an isomorphism $\Theta : Z_{z',z} \rightarrow Z_{z',z}$ that fits with the duality on diagrams:

$$\Theta(\Psi(\mu\eta\sigma)) = \Psi(\lambda \eta \sigma \tau)$$

for every enhanced weight $\eta \sigma$ such that $\lambda \eta \sigma \tau$ is oriented.
Proof. Let \( b, b' \) be the \( b \)-sequences of \( \lambda \) and \( \mu \) respectively. Note that
\[
(11.29) \quad \frac{x^{b-1}}{x^{b'}-1} = x^{b-b'} = \prod_{\nabla < \nabla'} (x_{\nabla} \cdots x_{\nabla'-1}) = \prod_{\nabla < \nabla'} (x_{\nabla}^{-1} \cdots x_{\nabla'-1}^{-1})
\]
as an element in \( \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). If \( (1 \mapsto m) \) is a monomial morphism of the basis \([10.13]\) of \( \text{Hom}_R(C_{w_1z}, C_{w_2z}) \), it follows immediately that \( (1 \mapsto m) \in W_{z',z} \) if and only if \( (1 \mapsto a^{b'}-b) \in \mathcal{R}_{z',z} \), hence \( \Theta(W_{z',z}) = W_{z',z} \).

Moreover, it follows from equation \([11.29]\) for the polynomials \( p_{\lambda\mu} \) and \( p_{\mu\nu} \) that
\[
p_{\lambda\mu} = x^{b-b'}p_{\lambda\mu}, \quad \text{hence } \Theta(\Psi(\mu_{\lambda\mu}^\lambda)) = \Psi(\mu_{\lambda\mu}^\lambda).
\]

As a corollary, we have that the linear map \( * : A \rightarrow A \) defined by
\[
(11.30) \quad (a\lambda b)^* = (b^*\lambda a^*)
\]
is an algebra anti-isomorphism.

**Endomorphism algebra of a projective generator of \( \mathcal{Q}_k(\pi) \).** As it follows from the definition, the algebra \( A \) only depends on the number of \( \land \)'s and \( \lor \)'s in the block \( \Gamma \). We define hence \( A_{n,k} \) to be \( A\Gamma \) for some block \( \Gamma \) with \( k \land \)'s and \( n-k \lor \)'s.

We can now state the main result of this section, that follows from the construction of the algebras \( A_{n,k} \):

**Theorem 11.12.** Let \( \mathcal{Q}(\pi) = \bigoplus_{w \in \Lambda_k(\pi)} Q(w) \) be the sum of all indecomposable projective modules up to isomorphism in \( \mathcal{Q}_k(\pi) \). Then we have an isomorphism of graded categories
\[
A_{n,k}_{\text{g}} \cong \mathcal{Q}_k(\pi).
\]

Proof. Let \( p,q \subseteq \mathfrak{gl}_n \) be the standard parabolic subalgebras such that \( \mathcal{Q}_k(\pi) = Z_0 \mathfrak{sl}_n \)-pres. Recall that \( Q(w) = p^p(w \cdot 0) \), hence \( \mathcal{Q}(\pi) \) is the sum of all parabolic projective modules \( p^p(w_qz \cdot 0) \) for \( z \in D_{n,k} \), where \( w_q \in W_q = W_k \) is the longest element. Hence
\[
(11.32) \quad \text{End}_{\mathcal{O}_k}(\mathcal{Q}(\pi)) = \bigoplus_{z,z' \in D_{n,k}} \text{Hom}_{\mathcal{O}_k}(p^p(w_qz \cdot 0), p^p(w_{q}z' \cdot 0)).
\]

By \([10.17]\), we have
\[
(11.33) \quad \text{Hom}_{\mathcal{O}_k}(p^p(w_qz \cdot 0), p^p(w_qz' \cdot 0)) \cong Z_{z,z'}
\]
as \( (Z_{z'}z', Z_{z}z) \)-bimodules. By Corollary \([11.3]\) and by the definition of the algebra structure on \( A_{n,k} \) it follows then that \( A_{n,k} \cong \text{End}_{\mathcal{O}_k}(\mathcal{Q}(\pi)) \) as graded \( \mathbb{C} \)-algebras.

**Remark 11.13.** The grading of \( \mathcal{Q}_k(\pi) \) is inherited from the Koszul grading on the category \( \mathcal{O}_0 \) in \([RGS96]\).

### 11.3. Graded cellular structure.

From Theorem \([11.12]\) and the graded quasi-hereditary structure on \( Z_0 \mathfrak{sl}_n \) one could deduce a graded cellular structure on \( \mathcal{Q}_k(\pi) \). Instead we explicitly construct it here on the diagram side. The construction follows the one in \([BS11]\).
Proposition 11.14. Let \((a\lambda b)\) and \((c\mu d)\) be basis vectors of \(A\). Then

\[
(a\lambda^\tau b)(c\mu^\tau d) = \begin{cases} 
0 & \text{if } b \neq c^*, \\
\ell^\tau_{(a\lambda\sigma^c)}(\mu^\tau) \cdot (a\mu^\tau d) \quad & \text{if } b = c^* \text{ and } (a\mu d) \text{ is oriented}, \\
+ \sum_{\ell(\tau) > \ell(\sigma)} \ell^\tau_{(a\lambda\sigma^c)}(\mu^\tau) \cdot (a\mu^\tau d) + (\dagger) & \text{otherwise},
\end{cases}
\]

where:

(i) the scalars \(\ell^\tau_{(a\lambda\sigma^c)}(\mu^\tau)\) are independent of \(d\);

(ii) \((\dagger)\) denotes a linear combination of basis vectors of \(A\) of the form \((a\nu^\lambda d)\) with \(\nu \succeq \mu\);

(iii) \(\ell^\tau_{(a\lambda\sigma^c)}(\mu^\tau)\) is equal to 1 if \(b = c^* = \lambda\) and \(\sigma = e\), otherwise it is zero.

Proof. If \(b \neq c^*\) the claim is obvious, so let us suppose \(b = c^*\). Of course we have

\[
(\nu^\lambda b)(c\mu^\tau d) = \sum_{\nu \in \Gamma, \chi \in S_k} C(\nu^\lambda)(a\nu^\lambda d).
\]

for some coefficients \(C(\nu^\lambda) \in \mathbb{C}\). Let us first prove that only terms with \(\nu^\lambda \succeq \mu^\tau\) occur in the sum, i.e. if \(C(\nu^\lambda) \neq 0\) then \(\nu^\lambda \succeq \mu^\tau\).

Before going on, let us remark where is the subtlety in the following argument. We want to understand which element of \(A\) corresponds to the morphism \(1 \mapsto p_{a\lambda^\tau} p_{c\mu^\tau}^*: \) in general this morphism is not a monomial morphism of the basis (10.13), and we have to use the relations defining \(R_a\) to rewrite it as a linear combination of the monomial morphisms (10.13).

Let us fix some \(\nu^\lambda\) such that \(C(\nu^\lambda) \neq 0\). First, let us prove that \(\nu \succeq \mu\). By definition, \(\nu \succeq \mu\) is equivalent to \(\nu^\lambda_j \geq \mu^\tau_j\) for all \(j = 1, \ldots, n - k\). Fix an index \(j\). If \(\nu^\lambda_j \geq \mu^\tau_j\) then also \(\nu^\mu_j \geq \nu^\mu_j\) by Lemma 11.3(i). Hence suppose \(\nu^\mu_j < \nu^\mu_j\). By construction the monomial

\[
(x_j \nu^\mu_j x_{j+1} \cdots x_{j-1})(x_j \nu^\mu_j x_{j+1} \cdots x_{j-1})
\]

divides \(p_{a\lambda^\tau} p_{c\mu^\tau}\). In particular, since \(\nu^\lambda_j \geq \nu^\mu_j\) also \(x_j \nu^\mu_j x_{j+1} \cdots x_{j-1}\) divides \(p_{a\lambda^\tau} p_{c\mu^\tau}\). Hence, if \(p_{a\lambda^\tau} p_{c\mu^\tau}\) is a monomial of the basis (10.13), we can conclude that \(\nu^\mu_j \geq \nu^\mu_j\). Otherwise, we get the same conclusion using the technical Lemma 11.15 below.

Now to check that \(\nu^\lambda \succeq \mu^\tau\) we have to show that in the case \(\nu = \mu\) we have \(\ell(\chi) \geq \ell(\tau)\). So let us suppose \(\nu = \mu\). Since the multiplication is graded, we must have

\[
\deg(a\lambda^\tau b) + \deg(c\mu^\tau d) = \deg(a\mu^\lambda d).
\]

If \(a = \mu\) we write \(\ell(a)\) for \(\ell(\rho)\), and similarly for \(b, c, d\). Then using Lemma 11.3 we get from (11.36)

\[
\ell(\chi) = 2\ell(\tau) + 2\ell(\sigma) + 2\ell(b) - 2\ell(\lambda).
\]

Since \(\lambda^\tau b\) is oriented, by Lemma 11.2 the diagram \(b\) corresponds to some weight that is smaller or equal than \(\lambda\) in the Bruhat order. This implies that \(\ell(\lambda) \leq \ell(\lambda)\) (notice that under the identification of \(\Gamma\) with \(D_{n,k}\), the Bruhat order on weights corresponds to the opposite of the usual Bruhat order on permutations). It follows that \(\ell(\lambda) \geq \ell(\tau)\). This concludes the first part of the proof.

Now suppose that \(C(\mu^\tau) \neq 0\); if we substitute in (11.37) \(\chi = \tau\) we get \(2\ell(\sigma) + 2\ell(b) - 2\ell(\lambda) = 0\). Since \(\ell(b) \geq \ell(\lambda)\) we must have \(\ell(\sigma) = 0\) and \(\ell(b) = \ell(\lambda)\). This implies \(\sigma = e\) and \(b = \lambda\). It is easy to see that in this case the morphism
1 \mapsto p_{\alpha \lambda \gamma} p_{\mu \delta} \) is an element of the monomial basis (10.13), and hence we have exactly \((a \lambda^b)(c \mu^d) = (a \mu^d)\). \qed

Lemma 11.15. Fix some \(b \in \mathcal{B}\) and let \(m\) be an index such that \(b_{m-1} = b_m\). Suppose that \(x_m x_{m+1} \cdots x_{m+\ell}\) divides some polynomial \(p \in R\). Write \(p = \sum c_i x^i\) in \(R_b\), where \(x^i\) are monomials of the basis (10.13). Then \(x_m x_{m+1} \cdots x_{m+\ell}\) divides all monomials \(x^i\) for which \(c_i \neq 0\).

Proof. We will use the relations defining the ideal \(I_b\) to write the expression of \(p\) as a linear combination of basis monomials. Of course, it is sufficient to examine the case in which \(p = x_j^j\) is a monomial.

Consider the maximum \(r\) for which \(j_r \geq b_r\): if there is no such \(r\), then \(p\) is a monomial of the basis (10.13) and we are done. If \(r < m\) or \(r > m + \ell\) then using the relation \(h_b(\lambda, x_1, \ldots, x_r)\) we can rewrite \(p\) as a linear combination of monomials \(x_j^j\) with \(j_r < j_r \) and \(x_m x_{m+1} \cdots x_{m+\ell}\) divides \(x_j^j\): so by an induction argument we may suppose \(m < r < m + \ell\). If \(\ell \geq 1\) we can write

\[
(11.38) \quad x_{r-1} x_j^j = x_{r-1} h_j, (x_1, \ldots, x_r) - \sum_{s=0}^{j_r-1} x_{r-s} h_{j_{r-s}} (x_1, \ldots, x_{r-1}).
\]

Since \(h_j, (x_1, \ldots, x_r) \in I_b\) because \(j_r \geq b_r\), and also \(x_{r-1} h_j, (x_1, \ldots, x_{r-1}) \in I_b\) by (10.13), the expression (11.38) gives in \(R_b\)

\[
(11.39) \quad x_{r-1} x_j^j = \sum_{s=0}^{j_r-1} x_{r-s} h_{j_{r-s}} (x_1, \ldots, x_{r-1}).
\]

In the special case \(\ell = 0, r = m\), we write instead

\[
(11.40) \quad x_m^m = h_j, (x_1, \ldots, x_m) - \sum_{s=0}^{j_m-1} x_m^s h_{j_m-s} (x_1, \ldots, x_{m-1}).
\]

that in \(R_b\) is

\[
(11.41) \quad x_m^m = - \sum_{s=1}^{j_m-1} x_m^s h_{j_m-s} (x_1, \ldots, x_{m-1}),
\]

since \(j_m \geq b_{m-1}, b_m\). Both in (11.39) and (11.41), on the r.h.s. we have a sum of monomials \(x_j^j\) with \(1 \leq j_r < j_r\): by an induction argument on \(j_r\), the claim follows. \qed

The main result of this section is the graded cellular algebra structure of \(A\) in the sense of [GL90], [HM10]. A graded cellular algebra is an associative unital algebra \(H\) together with a cell datum \((X, I, C, \deg)\) such that:

- (GC1) \(X\) is a finite partially ordered set;
- (GC2) \(I(\lambda)\) is a finite set for each \(\lambda \in X\);
- (GC3) \(C: \bigcup_{\lambda \in X} I(\lambda) \times I(\lambda) \to H, (i, j) \mapsto C_{i,j}^\lambda\) is an injective map whose image is a basis of \(H\);
- (GC4) the map \(H \to H, C_{i,j}^\lambda \mapsto C_{i,j}^{\lambda}\) is an algebra anti-automorphism;
- (GC5) if \(\lambda \in X\) and \(i, j \in I(\lambda)\) then for any \(x \in H\) we have that

\[
(11.42) \quad x C_{i,j}^\lambda = \sum_{i' \in I(\lambda)} r_x(i', i) C_{i',j}^\lambda \pmod{H_{>\lambda}},
\]

where the scalar \(r_x(i', i)\) is independent of \(j\) and \(H_{>\lambda}\) is the subspace of \(H\) spanned by \(\{C_{s, l}^\mu | \mu > \lambda\) and \(k, l \in I(\mu)\});
- (GC6) \(\deg: \bigcup_{\lambda \in X} I(\lambda) \to \mathbb{Z}, i \mapsto \deg_i^\lambda\) is a function such that the \(\mathbb{Z}\)-grading on \(H\) defined by declaring \(\deg C_{i,j}^\lambda = \deg_i^\lambda + \deg_j^\lambda\) makes \(H\) into a graded algebra.
Proposition 11.16. The algebra $A$ is a graded cellular algebra with cell datum $((\Gamma \times S_k, \trianglelefteq), I, C, \deg)$ where:

(a) $I(\lambda^\sigma) = \{ \alpha \in \Gamma | \alpha \subset \lambda \}$;
(b) $C$ is defined by setting $C^\lambda_{\alpha, \beta} = (\lambda^\sigma \alpha \beta)$;
(c) $\deg^\lambda_{\alpha} = \deg(\lambda^\sigma) - \ell(\sigma)$.

Proof. Conditions (GC1-3) and (GC6) are direct consequences of the definitions. Condition (GC4) follows from Lemma 11.11. Condition (GC5) follows from Proposition 11.14. □

11.4. The properly stratified structure. As before, let us fix a block $\Gamma$ and let $A = A_\Gamma$. We construct now explicitly the properly stratified structure on $A$ (that we already now exists by Theorem 6.13 and Corollary 11.8). The construction is similar to the one of [BS11].

An $A$-module will always be a finite dimensional graded left $A$-module. Let $A^{-gmod}$ be the category of such modules. If $M = \bigoplus M_i$ is a graded $A$-module then we will write $M(\langle j \rangle)$ for the same module structure but with new grading defined by $(M(\langle j \rangle))_i = M_i - j$. If $M, N$ are graded $A$-modules then $\text{Hom}_A(M,N)$ is a graded vector space.

Irreducible and projective $A$-modules. As we already noticed, the algebra $A$ is unital with $1 = \sum_{\lambda \in \Gamma} e_{\lambda}$. Let $A_{>0}$ be the sum of all components of $A$ of strictly positive degree. Then

$$(11.43) \quad A/A_{>0} = \bigoplus_{\lambda \in \Gamma} e_{\lambda} C e_{\lambda} \cong \bigoplus_{\lambda \in \Gamma} C$$

is a semisimple algebra, with a basis given by the images of the idempotents $e_{\lambda}$. The image of $e_{\lambda}$ spans a one dimensional $A/A_{>0}$-modules, and hence also a one dimensional $A$-module which we denote $L(\lambda)$. Thus $L(\lambda)$ is a copy of the field concentrated in degree 0, and $(a\mu^\sigma b) \in A$ acts on it as 1 if $(a\mu^\sigma b) = (\lambda^\sigma \lambda \mu)$ and as 0 otherwise. The modules

$$(11.44) \quad \{ L(\lambda)(j) | \lambda \in \Gamma, j \in \mathbb{Z} \}$$

give a complete set of isomorphism classes of irreducible graded $A$-modules.

For any finite dimensional graded $A$-module $M$, let $M^*$ denote its graded dual. That is, $(M^*)_j = \text{Hom}_C(M_{-j}, C)$ and $x \in A$ acts on $f \in M^*$ by $xf(m) = f(x^*m)$. Clearly we have that

$$(11.45) \quad L(\lambda)^* \cong L(\lambda)$$

for each $\lambda \in \Gamma$.

For each $\lambda \in \Gamma$ let also $P(\lambda) = Ae_{\lambda}$. This is a graded $A$-module with basis

$$(11.46) \quad \{(a\mu^\sigma \mu) | \text{for all } \nu, \mu \in \Gamma \text{ and } \sigma \in S_k \text{ with } \nu \subset \mu \supset \lambda \}.$$  

The module $P(\lambda)$ is a projective module; in fact, it is the projective cover of $L(\lambda)$ in $A^{-gmod}$. The modules

$$(11.47) \quad \{ P(\lambda)(j) | \lambda \in \Gamma, j \in \mathbb{Z} \}$$

give a complete set of isomorphism classes of indecomposable projective $A$-modules.
Cell modules and standard modules. We introduce now standard modules. The terminology will be motivated at the end of the section. For \( \mu \in \Gamma \), define \( \Delta(\mu) \) to be the vector space with basis
\[
\{(\mu^\tau) \mid \text{for all } \lambda \in \Gamma, \tau \in S_k \text{ such that } \lambda \subset \mu\}
\]
or equivalently
\[
\{(c\mu^\tau) \mid \text{for all oriented lower fork diagrams } c\mu^\tau\}.
\]
We put a grading on \( \Delta(\mu) \) by defining the degree of \( (c\mu^\tau) \) to be \( \deg(c\mu^\tau) \), and we make it into an \( A \)-module through
\[
(a\lambda^\tau b)(c\mu^\tau) = \begin{cases} 
\sum_{\tau' \in S_k} t_{(a\lambda^\tau b)}^{\tau'}(\mu^\tau')(a\mu^\tau') & \text{if } b = c^* \text{ and } (a\mu) \text{ is oriented,} \\
0 & \text{otherwise,}
\end{cases}
\]
where \( t_{(a\lambda^\tau b)}^{\tau'}(\mu^\tau) \) is the scalar defined by Proposition 11.14. Note that \( t_{(a\lambda^\tau b)}^{\tau'}(\mu^\tau) \) was defined only for \( \tau' = \tau \) or for \( \ell(\tau') > \ell(\tau) \); otherwise we set \( t_{(a\lambda^\tau b)}^{\tau'}(\mu^\tau) = 0 \).

**Theorem 11.17.** For \( \lambda \in \Gamma \) enumerate the distinct elements of the set \( \{ \mu \in \Gamma \mid \mu \supset \lambda \} \) as \( \mu_1, \mu_2, \ldots, \mu_m = \lambda \) so that if \( \mu_i \prec \mu_j \) then \( i > j \). Set \( M(0) = \{0\} \) and for \( i = 1, \ldots, m \) define \( M(i) \) to be the subspace of \( P(\lambda) \) generated by \( M(i-1) \) and the vectors
\[
\{(c\mu_i^\tau) \mid \text{for all oriented lower fork diagrams } c\mu_i^\tau\}.
\]
Then
\[
\{0\} = M(0) \subset M(1) \subset \cdots \subset M(m) = P(\lambda)
\]
is a filtration of \( P(\lambda) \) as an \( A \)-module such that
\[
M(i)/M(i-1) \cong \Delta(\mu_i)(\deg \mu_i \lambda)
\]
for each \( i = 1, \ldots, m \).

**Proof.** It follows from Proposition 11.14 that \( M(i) \) is indeed a submodule of \( P(\lambda) \). The map
\[
f_i : \Delta(\mu_i)(\deg \mu_i \lambda) \longrightarrow M(i)/M(i-1)
\]
\[
(c\mu_i^\tau) \longmapsto (c\mu_i^\tau) + M(i-1)
\]
gives an isomorphism of graded vector spaces. This map is of degree zero because
\[
\deg(c\mu_i^\tau) = \deg(c\mu_i^\tau) + \deg(\mu_i \lambda).
\]
Through this vector space isomorphism we can transport the \( A \)-module structure of \( M(i)/M(i-1) \) to \( \Delta(\mu_i) \). Using Proposition 11.14 we see that the module structure we get on \( \Delta(\mu_i) \) is given by (11.50). Hence (11.50) defines indeed an \( A \)-module structure on \( \Delta(\mu_i) \) and (11.54) is an isomorphism of \( A \)-modules. Since any weight \( \mu \) arises as \( \mu_i \) for some \( \lambda \) as in the statement of the theorem (take for example \( \lambda = \mu_i, i = m \)), we conclude also that (11.50) defines an \( A \)-module structure for every \( \mu \). \qed

Let us now define cell modules. Let \( \mu^\tau \in \Gamma \times S_k \) be an enhanced weight and define \( V(\mu^\tau) \) to be the vector space with basis
\[
\{(\mu^\tau) \mid \text{for all } \lambda \in \Gamma \text{ such that } \lambda \subset \mu\}
\]
or equivalently
\[
\{(c\mu^\tau) \mid \text{for all oriented lower fork diagrams } c\mu^\tau\}.
\]
We remark that the difference with \((11.48)\) and \((11.49)\) is that now the permutation \(\tau\) is fixed. As before, we put a grading on \(V(\mu^\tau)\) by defining the degree of \((a \lambda^\tau)\) to be \(\text{deg}(c_{\mu^\tau})\), and we make it into an \(\mathcal{A}\)-module through
\[
(11.58) \quad (a \lambda^\tau b)(c_{\mu^\tau}) = \begin{cases} t_{(a \lambda^\tau b)}(\mu^\tau) \cdot (a \mu^\tau) & \text{if } b = c^* \text{ and } (a \mu) \text{ is oriented}, \\ 0 & \text{otherwise}. \end{cases}
\]

From Proposition \((11.44)\) we have that \(t_{(a \lambda^\tau b)}(\mu^\tau)\) does not depend on \(\tau\). Hence \((11.58)\) is the same as
\[
(11.59) \quad (a \lambda^\tau b)(c_{\mu^\tau}) = \begin{cases} (a \mu^\tau) & \text{if } b = c^* = \overline{\lambda}, \sigma = e \text{ and } (a \mu) \text{ is oriented}, \\ 0 & \text{otherwise}. \end{cases}
\]

It will follow from Theorem \((11.18)\) that this indeed defines an \(\mathcal{A}\)-module structure. It is clear from \((11.59)\) that all cell modules \(V(\mu^\tau)\) for a fixed \(\mu\) are isomorphic (up to a degree shift). Explicitly we have \(V(\mu^\tau) \cong V(\mu^e)\langle\text{deg}(\tau)\rangle\). We recall that \(\text{deg}(\tau) = 2\ell(\tau)\). Therefore for a weight \(\mu \in \Gamma\) we define the proper standard module \(\overline{\Delta}(\mu)\) to be the vector space with basis \((11.60)\)
\[
\{c_{\mu^\lambda} \mid \text{for all } \lambda \in \Gamma \text{ such that } \lambda \subset \mu\}
\]
or equivalently
\[
(11.61) \quad \{c_{\mu^\lambda} \mid \text{for all unenhanced oriented lower fork diagrams } c_{\mu^\lambda}\}.
\]

We put a grading on \(\overline{\Delta}(\mu)\) by defining the degree of \((c_{\mu^\lambda})\) to be \(\text{deg}(c_{\mu^\lambda})\), and we make it into an \(\mathcal{A}\)-module through
\[
(11.62) \quad (a \lambda^\tau b)(c_{\mu^\lambda}) = \begin{cases} (a \mu^\tau) & \text{if } b = c^* = \overline{\lambda}, \sigma = e \text{ and } (a \mu) \text{ is oriented}, \\ 0 & \text{otherwise}. \end{cases}
\]

Of course we have an isomorphism \(\overline{\Delta}(\mu) \cong V(\mu^e)\).

**Theorem 11.18.** Let \(\mu \in \Gamma\). Enumerate the elements of \(S_k\) as \(\sigma_1, \sigma_2, \ldots, \sigma_k = e\) in such a way that if \(\ell(\sigma_i) > \ell(\sigma_j)\) then \(i < j\). Let \(N(0) = \{0\}\) and for \(i = 1, \ldots, k!\) define \(N(i)\) to be the subspace of \(\Delta(\mu)\) generated by \(N(i-1)\) and the vectors
\[
(11.63) \quad \{c_{\mu^{\lambda^\sigma^i}} \mid \text{for all oriented lower fork diagrams } c_{\mu^{\lambda^\sigma^i}}\}.
\]

Then
\[
(11.64) \quad \{0\} = N(0) \subset N(1) \subset \cdots \subset N(k!) = \Delta(\mu)
\]
is a filtration of \(\Delta(\mu)\) as an \(\mathcal{A}\)-module such that
\[
N(i)/N(i-1) \cong \overline{\Delta}(\mu)/\langle2\ell(\sigma_i)\rangle.
\]

**Proof.** It follows from Proposition \((11.44)\) that \(N(i)\) is indeed a submodule of \(\Delta(\mu)\). The map
\[
(11.66) \quad f_i : \overline{\Delta}(\mu)/\langle2\ell(\sigma_i)\rangle \longrightarrow N(i)/N(i-1)
\]
\[
(c_{\mu^\lambda}) \mapsto (c_{\mu^{\lambda^\sigma^i}} + N(i-1))
\]
gives an isomorphism of graded vector spaces. The degree shift comes from
\[
(11.67) \quad \text{deg}(c_{\mu^{\lambda^\sigma^i}}) = \text{deg}(c_{\mu^\lambda}) + 2\ell(\sigma_i).
\]

Through \(f_i\) we can transport the module structure of \(N(i)/N(i-1)\) to \(\overline{\Delta}(\mu)\). The module structure on \(N(i)/N(i-1)\) is described by \((11.50)\). It follows that \(\overline{\Delta}(\mu)/\langle2\ell(\sigma_i)\rangle\) is endowed with the module structure of \(V(\mu^{\sigma^i})\) described by \((11.58)\); this shows in particular that \((11.58)\) defines indeed an \(\mathcal{A}\)-module structure. We have already argued that this is the same as the module structure described by \((11.62)\) on \(\overline{\Delta}(\mu)\).

\(\Box\)
The Grothendieck group. The Grothendieck group $K(A-\text{gmod})$ of $A-\text{gmod}$ is a free $\mathbb{Z}$-module with basis given by equivalence classes of simple modules. The group $K(A-\text{gmod})$ becomes a $\mathbb{Z}[q, q^{-1}]$-module if we set $q[M] = [M(1)]$ for all graded $A$-modules $M$. It is also free as a $\mathbb{Z}[q, q^{-1}]$-module, with basis $\{[L(\lambda)] | \lambda \in \Gamma\}$.

For $\lambda, \mu \in \Gamma$, define

\begin{equation}
\label{eq:11.72}
d_{\lambda, \mu} = \begin{cases}
q^{\deg(\Delta(\mu))} & \text{if } \lambda \subset \mu, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

By Theorem \ref{thm:11.17} we have that

\begin{equation}
\label{eq:11.73}
[P(\lambda)] = \sum_{\mu \in \Gamma} d_{\lambda, \mu} [\Delta(\mu)]
\end{equation}

and by Theorem \ref{thm:11.19} we have that

\begin{equation}
\label{eq:11.74}
[\Sigma(\mu)] = \sum_{\lambda \in \Gamma} d_{\lambda, \mu} [L(\lambda)].
\end{equation}

Moreover, by Theorem \ref{thm:11.15} we have that

\begin{equation}
\label{eq:11.75}
[\Delta(\mu)] = [k]_0! \cdot [\Sigma(\mu)]
\end{equation}

where we recall that

\begin{equation}
\label{eq:11.76}
[k]_0 = \frac{q^{2k} - 1}{q^k - 1} \quad \text{and} \quad [k]_0! = [k][k - 1] \cdots [1].
\end{equation}

Since $d_{\lambda, \lambda} = 1$, the matrix $(d_{\lambda, \mu})$ is upper triangular with determinant 1, hence it is invertible over $\mathbb{Z}[q, q^{-1}]$. In particular, the proper standard modules give also a $\mathbb{Z}[q, q^{-1}]$-basis of $[A-\text{gmod}]$. On the other side, notice that the matrix $[k]_0! \text{Id}$ is not invertible over $\mathbb{Z}[q, q^{-1}]$ unless $k = 0, 1$. In particular, standard and projective modules do not give a basis of the Grothendieck group in general.

We recall the definition of a graded properly stratified algebra in the sense of [Maz04] (see also [FKM02, Fri07]).
Definition 11.20. Let $B$ be a finite dimensional associative graded algebra over a field $\mathbb{K}$ with a simple preserving duality and with equivalence classes of simple modules $\{L(\lambda)j) | \lambda \in \Lambda, j \in \mathbb{Z}\}$ where $\langle \lambda, \cdot \rangle$ is a partially ordered finite set. For each $\lambda \in \Lambda$ let

(i) $P(\lambda)$ denote the projective cover of $L(\lambda)$,

(ii) $\Delta(\lambda)$ be the maximal quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = 0$ for all $\mu \succ \lambda$,

(iii) $\overline{\Delta}(\lambda)$ be the maximal quotient of $\Delta(\lambda)$ such that $[\text{rad} \overline{\Delta}(\lambda) : L(\mu)] = 0$ for all $\mu \succeq \lambda$.

Then $B$ is properly stratified if the following conditions hold for every $\lambda \in \Lambda$:

(PS1) the kernel of the canonical epimorphism $P(\lambda) \to \Delta(\lambda)$ has a filtration with subquotients isomorphic to graded shifts of $\Delta(\mu)$, $\mu \succ \lambda$;

(PS2) the kernel of the canonical epimorphism $\Delta(\lambda) \to \overline{\Delta}(\lambda)$ has a filtration with subquotients isomorphic to graded shifts of $\overline{\Delta}(\lambda)$;

(PS3) the kernel of the canonical epimorphism $\overline{\Delta}(\lambda) \to L(\lambda)$ has a filtration with subquotient isomorphic to graded shifts of $L(\mu)$, $\mu \prec \lambda$.

The modules $\Delta(i)$ and $\overline{\Delta}(i)$ are called standard and proper standard modules respectively.

Theorem 11.21. For every block $\Gamma$ the algebra $A_\Gamma$ is a graded properly stratified algebra. The partially ordered set indexing the simple modules is $(\Gamma, \prec)$. The modules $\Delta(\mu)$ and $\overline{\Delta}(\mu)$ are the standard and proper standard modules respectively. Moreover, the diagonal matrix of the multiplicity numbers of the proper standard modules in the filtrations of the standard modules is a multiple of the identity.

Proof. We already noticed that $A = A_\Gamma$ is a finite dimensional associative unital graded algebra over $\mathbb{C}$ with a duality with respect to which the simple modules are self-dual. For $\lambda \in \Gamma$ let $L(\lambda) = L(\lambda)$ and define $P(\lambda)$, $\Delta(\lambda)$ and $\overline{\Delta}(\lambda)$ as in (i), (ii) above.

By the unicity of the projective cover we have $P(\lambda) \cong \overline{\Delta}(\lambda)$. From (11.73) and (11.74) we have that $\Delta(\lambda)$ is a quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = 0$ for every $\mu > \lambda$; from Theorem 11.17 it follows that it is maximal with this property, hence $\Delta(\lambda) \cong \overline{\Delta}(\lambda)$. By the same argument using (11.74) and Theorem 11.18 we get that $\overline{\Delta}(\lambda) \cong \overline{\Delta}(\lambda)$. Hence we need to show that properties (PS1), (PS2), (PS3) are satisfied. But this follows immediately from Theorems 11.17, 11.18 and 11.19. □

11.5. A bilinear form and self-dual projective modules. We define a bilinear form on $A$ and we determine which projective modules are self-dual.

Defect. Let $\lambda$ be a weight in some block $\Gamma$. We say that an $\wedge$ of $\lambda$ is initial if it has no $\vee$’s on its left. Let us define the defect of $\lambda$ to be

$$\text{def}(\lambda) = \#\{\text{non initial } \wedge \text{’s of } \lambda\}. \tag{11.77}$$

We have the following elementary result:

Lemma 11.22. The maximal degree of $e_\lambda A e_\lambda$ is $k(k-1) + 2\text{def}(\lambda)$ and the homogeneous subspace of maximal degree of $e_\lambda A e_\lambda$ is one dimensional.

Proof. It is straightforward to notice that the homogeneous subspace of maximal degree of $e_\lambda A e_\lambda$ is one dimensional: the diagram of maximal degree is $\Delta(\lambda)$, where $\eta$ orients every fork of $\lambda$ with maximal degree (that is, each $\vee$ is at the rightmost position) and $\sigma$ is the longest element of $S_k$. By definition, the degree of this diagram is obtained by adding $2\ell(\sigma)$ to the sum of $2(m-1)$ for every $m$-fork of $\lambda$. Hence, this degree is $2\ell(\sigma)$ plus twice the number of non-initial $\wedge$’s of $\lambda$. □
Lemma 11.23. Consider \( \lambda, \mu \in \Gamma \) and suppose that \( e_\lambda A e_\mu \) is not trivial. Then the homogeneous subspaces of minimal and maximal degree of \( e_\lambda A e_\mu \) are one dimensional. The minimal degree is

\[
(11.78) \quad \sum_{i=1}^{n-k} |\nu_i^\lambda - \nu_i^\mu|
\]
and the maximal degree is

\[
(11.79) \quad k(k-1) + \sum_{i=1}^{n-k} |\nu_i^{\min}-1-\nu_i^\lambda| + |\nu_i^{\min}-1-\nu_i^\mu|
\]

where we set \( \nu_i^{\min} = \min\{\nu_i^\lambda, \nu_i^\mu\} \) and \( \nu_i^{\min} = n+1 \).

If \( \text{def}(\lambda) \geq \text{def}(\mu) \) then the sum of (11.78) and (11.79) is equal to the maximal degree of \( e_\lambda A e_\mu \).

Proof. We use the condition (11.6) to determine if a diagram is oriented. The minimal degree diagram is \( \lambda \eta \mu \) where \( \nu_i^\eta = \max\{\nu_i^\lambda, \nu_i^\mu\} \) and its degree is given by (11.78). The maximal degree diagram is \( \lambda \eta \mu \) where \( \nu_i^\eta = \min\{\nu_i^\lambda, \nu_i^\mu\} \) - 1, and its degree is given by (11.79).

Let us now check the last assertion. The sum of (11.78) and (11.79) is

\[
(11.80) \quad k(k-1) + \sum_{i=1}^{n-k} 2(\nu_i^{\min}-1-\nu_i^{\min}).
\]

This is the maximal degree of \( e_\eta A e_\eta \) where \( \eta \in \Gamma \) is the weight with \( \nu_i^\eta = \nu_i^{\min} \). Of course \( \text{def}(\eta) = \max\{\text{def}(\lambda), \text{def}(\mu)\} \) and by Lemma 11.22 the maximal degrees of \( e_\lambda A e_\lambda \) and \( e_\eta A e_\eta \) are the same.

Notice that a weight \( \lambda \) is of maximal defect if and only if it starts with a \( \vee \). If \( \lambda \) is not of maximal defect, let \( \hat{\lambda} \) be obtained from \( \lambda \) by swapping the first \( \vee \) and the first \( \wedge \). Otherwise, let \( \hat{\lambda} = \lambda \). In particular, \( \hat{\lambda} \) is always of maximal defect.

Lemma 11.24. For every \( \lambda \in \Gamma \) the socle of \( P(\lambda) \) contains a degree shift of \( L(\hat{\lambda}) \).

In facts, the socle of \( P(\lambda) \) is simple, hence it is isomorphic to a degree shift of \( L(\hat{\lambda}) \), but we will not need this in what follows.

Proof. It is straightforward to check that the diagram of maximal degree in \( A e_\lambda \) is of type \( \lambda \eta \mu \). The claim follows.

A bilinear form. For every \( \lambda \in \Gamma \) of maximal defect, let us choose a non-zero element \( \xi_\lambda^{\max} \in e_\lambda A e_\lambda \) of maximal degree (for example, we can choose it to be the diagram \( \lambda \eta \mu \) of the previous proof). For every element \( z \in A \) write \( e_\lambda z e_\lambda = t\xi_\lambda^{\max} + \text{terms of lower degree} \), and set \( \Theta(\lambda)(z) = t \).

Define

\[
\Theta(z) = \sum_{\text{def}(\lambda) \text{ max}} \Theta(\lambda)(z).
\]

Finally, define a bilinear form \( \Theta : A \times A \rightarrow \mathbb{C} \) by setting \( \Theta(y,z) = \Theta(yz) \). Obviously, this form is associative in the sense that \( \Theta(y,zw) = \Theta(yz,w) \) for every \( y, z, w \in A \).

Lemma 11.25. For every \( \lambda \), the form \( \Theta \) restricted to \( e_\lambda A e_\lambda \) is symmetric and non degenerate.

Proof. Let \( \lambda \) correspond to \( z \in D \). Up to a degree shift, \( e_\lambda A e_\lambda \cong Z_{z,z} \). Since \( Z_{z,z} \) is commutative, note that \( \Theta \) is symmetric on \( e_\lambda A e_\lambda \). Consider the monomial basis \( \{ 1 \mapsto x^T \} \) that consists of the elements of (10.13) that are not divided by (10.28). It is clear that for every element \( \varphi \) in that basis there exists exactly one element \( \varphi^T \) with \( \Theta(\varphi, \varphi^T) \neq 0 \). This proves that the form is non degenerate.
Let $e_{\text{def}} = \sum_{\text{def}(\lambda)} \max e_{\lambda}$.

**Lemma 11.26.** The form $\theta$ restricted to $e_{\text{def}} A \times A e_{\text{def}}$ is non degenerate; that is, if $\theta(y, t) = 0$ for every $y \in e_{\text{def}} A$, then $t = 0$ and viceversa.

**Proof.** We may take $t \in e_y A e_{\lambda}$ for some $\lambda$ of maximal defect and suppose $\theta(y, t) = 0$ for every $y \in e_{\lambda} A e_{\mu}$. Let $y_0$ be a generator of the minimal-degree subspace of $e_{\lambda} A e_{\mu}$ (which by Lemma 11.23 is one dimensional). In particular, $\theta(y', y_0 t) = 0$ for every $y' \in e_{\lambda} A e_{\lambda}$. By Lemma 11.26 this implies that $y_0 t = 0$. From the following Lemma 11.27 it follows then that $t = 0$.

The viceversa follows because $\theta(y, t) = \theta(t^*, y^*)$. \hfill \Box

**Lemma 11.27.** Suppose $\lambda$ is of maximal defect and let $0 \neq t \in e_{\mu} A e_{\lambda}$. Let also $0 \neq y_0 \in e_{\lambda} A e_{\mu}$ be of minimal degree. Then $y_0 t \neq 0$.

**Proof.** First, let $0 \neq t_0 \in e_{\mu} A e_{\lambda}$ be of minimal degree, and let us prove that $y_0 t_0 \neq 0$. By definition, $y_0 t_0 = (h_i \in [b_{\lambda}^t - b_{\mu}^t] \in \{0, 1\}$. First let us suppose that $1 \mapsto x^h$ is an element of the basis (10.13), that is $h_i < b_{\lambda}^t$ for every $i$. It is quite easy to argue that for every $i$ there exist an index $j$ with $\lambda^t \leq j < \lambda^t_{i+1}$ and $b_{\lambda}^t = b_{\mu}^t$; in fact it is sufficient to choose $j = \lambda^t$ if $\lambda^t \geq \lambda^t_i$ or $j = \lambda^t_i$ otherwise. This means that $1 \mapsto x^h$ is not illicit (cf. Theorem 10.17), hence it is not zero.

We should now consider the case in which $1 \mapsto x^h$ is not an element of the basis (10.13). This happens if $h_i = 1$ for some $i$ with $b_{\lambda}^t = 1$ and $b_{\mu}^t = 2$. Let $j$ be such that $\lambda^t_j$ is the rightmost $\lambda$ in a position $\lambda^t_i \leq i$. It is easy to argue that for $e_{\mu} A e_{\lambda}$ to be non trivial we must actually have $\lambda^t_i < i$. Let also $i' = \lambda^t_{\max} = \max\{\lambda^t_1, \lambda^t_i\} < i$. Then we have $b_{\lambda}^t = b_{\mu}^t \geq 2$. Using the relation $h_1(x_1, \ldots, x_i)$ to write $x^h$ in our fixed monomial basis we get in particular a term divided by $x_{i'}$. Applying the technique of the previous paragraph to this term we get that $y_0 t_0 \neq 0$: the only thing to notice is that $x_{i'} x_{i^t} \cdots x_{i^t_{i+1}}$ never divides a monomial basis element, since $b_{\lambda}^t_{i^t_{i+1}-1} = 1$.

Now, it follows from the proof of Lemma 11.26 that there is some element $u \in R$ such that $y_0 t_0 u$ generates the maximal degree subspace of $e_{\lambda} A e_{\lambda}$. In particular $y_0 t_0 u \neq 0$. By Lemma 11.23 $y_0 u$ is of maximal degree in $e_{\mu} A e_{\lambda}$. It is then clear by our characterization of $e_{\mu} A e_{\lambda}$ that there exists an element $u' \in R$ such that $u't = t_0 u$. Now $y_0 t u' = y_0 u t = y_0 t_0 u \neq 0$ implies that $y_0 t \neq 0$. \hfill \Box

**Self-dual projective modules.** Finally, we can determine which indecomposable projective modules are self-dual.

**Lemma 11.28.** Let $\lambda$ be of maximal defect. Then $P(\lambda)$ is self-dual up to a degree shift. In particular, it is an injective module.

**Proof.** By Lemma 11.26 the map
\[
(11.82) \quad y \mapsto \theta(y^*, ;)
\]
defines an isomorphism between $P(\lambda)$ and its dual up to a degree shift. \hfill \Box

**Theorem 11.29.** Let $\lambda \in \Gamma$. Then $P(\lambda)$ is an injective module if and only if $\lambda$ is of maximal defect.

**Proof.** By Lemma 11.28 if $\lambda$ is of maximal defect then $P(\lambda)$ is injective. On the other side, suppose $P(\lambda)$ is injective. Then $P(\lambda)$ is a tilting module, and by standard theory it is self dual (as an ungraded module). In particular, the socle of $P(\lambda)$ is $L(\lambda)$. By Lemma 11.23 $\lambda$ has to be of maximal defect. \hfill \Box
11.6. Diagrammatical versions of the functors $\mathcal{E}_k$ and $\mathcal{F}_k$. To conclude the section, we show how the functors $\mathcal{E}_k$ and $\mathcal{F}_k$ of Section 6.3 can be translated in this setting. This will allow us to compute their endomorphism algebras and to prove that they are indecomposable.

Let us fix an integer $n$. For all $k = 0, \ldots, n$ let us set in this section $A_k = A_{n,k}$.

Let $\Gamma_k$ be the subset of weights of $\Gamma_k$ of maximal defect, and let $\Gamma'_k = \Gamma_k - \Gamma_k$. Notice that $\lambda \in \Gamma'_k$ if and only if $\lambda$ starts with a $\vee$. Let also

$$e^\vee_k = \sum_{\lambda \in \Gamma'_k} e_\lambda, \quad e_k = \sum_{\lambda \in \Gamma_k} e_\lambda.$$  \hfill (11.83)

In the notation of the previous section, $e^\vee_k = e_{\text{def}}$.

Consider now $P^\vee_k = A_k e^\vee_k$, that is the sum of all indecomposable projective-injective $A_k$-modules. We want to describe a right $A_{k+1}$-action on it.

For any $\lambda \in \Gamma'_k$ let $\lambda^{(\lambda)} \in \Gamma'_{k+1}$ be the weight obtained from $\lambda$ after substituting the first $\vee$ with an $\wedge$. Conversely, given $\mu \in \Gamma'_{k+1}$ let $\mu^{(\vee)} \in \Gamma'_k$ be the weight obtained from $\mu$ after substituting the first $\wedge$ with a $\vee$. Of course, the map $\lambda \mapsto \lambda^{(\vee)}$ defines a bijection $\Gamma'_k \to \Gamma'_{k+1}$ with inverse $\mu \mapsto \mu^{(\vee)}$.

**Lemma 11.30.** Let $\lambda, \mu \in A_{k+1}^\vee$. Then we have a natural $R$-modules isomorphism

$$\text{Hom}_R(C_\lambda, C_\mu) \cong \text{Hom}_R(C_{\lambda^{(\vee)}}, C_{\mu^{(\vee)}})$$

that induces a surjective map

$$e_\mu A_{k+1} e_\lambda \longrightarrow e_{\mu^{(\vee)}} A_k e_{\lambda^{(\vee)}}.$$  \hfill (11.85)

**Proof.** Since the $b$-sequences of $\lambda$ and $\lambda^{(\vee)}$ are the same, the first claim follows. By Theorem 10.17 $W_{\lambda^{(\vee)}, \mu^{(\vee)}}$ is generated by $W_{\lambda, \mu}$ together with the morphism $1 \mapsto x_1, \ldots, x_j$ where $j = \min \{\lambda^{(\vee)}_1, \mu^{(\vee)}_1\}$. Hence $e_{\mu^{(\vee)}} A_k e_{\lambda^{(\vee)}}$ is a quotient of $e_\mu A_{k+1} e_\lambda$. \hfill \Box

**Corollary 11.31.** We have a surjective algebra homomorphism

$$e_{\lambda^{(\vee)}} e_{\mu^{(\vee)}} A_{k+1} \longrightarrow e_{\lambda^{(\vee)}} e_{\mu^{(\vee)}}.$$  \hfill (11.86)

Let us now define $F_k$ to be the $(A_k, A_{k+1})$-bimodule $P^\vee_k$, where the right $A_{k+1}$-structure is induced by the idempotent truncation $A_{k+1} \rightarrow e_{\lambda^{(\vee)}} e_{\mu^{(\vee)}} A_{k+1}$ composed with $e_{\lambda^{(\vee)}} e_{\mu^{(\vee)}}$. The bimodule $F_k$ defines a right-exact functor

$$A_{k+1} \longrightarrow \text{gmod} \xrightarrow{F_k \otimes A_{k+1}} A_k \longrightarrow \text{gmod}. $$  \hfill (11.87)

For each indecomposable projective module $P(\mu) = A_{k+1} e_\mu \in A_{k+1} \longrightarrow \text{gmod}$ we have

$$F_k \otimes A_{k+1} (A_{k+1} e_\mu) = \begin{cases} A_k e_\lambda & \text{if } \lambda \in \Gamma_k, \\
0 & \text{otherwise.} \end{cases} $$  \hfill (11.88)

**Proposition 11.32.** Under the equivalence of categories $A_k \longrightarrow \text{gmod} \cong \Omega_k(n)$ the functor $F_k \otimes A_{k+1}$ just defined corresponds to the functor $\mathcal{F}_k$.

**Proof.** Let $p, q, p', q' \subset \mathfrak{gl}_n$ be the parabolic subalgebras such that $\Omega_k(n) = Z_0^p q$-pres and $\Omega_{k+1} = Z_0^{p'} q'$-pres. Recall that the functor $\mathcal{F}_k$ is defined as the composition of the inclusion $i : \Omega_{k+1} \rightarrow Z_0^{p'} q'$-pres and the Zuckermann’s functor $j : Z_0^{p'} q'$-pres $\rightarrow \Omega_k$. Let $B = \text{End}_Q(\mathcal{P}_q^p(0))$, where $\mathcal{P}_q^p(0)$ is the projective generator (6.8) of $Z_0^{p'} q'$-pres. Let also $f_q \in B$ be the idempotent projecting onto the direct sum of the projective modules $P^p(x \cdot 0)$ for $x \in A_q^p(0)$ and $e^p \in B$ be the idempotent
projecting onto the direct sum of the projective modules $P^p(x \cdot 0)$ for $x \notin \Lambda^p_0(0)$. Then we have (cf. [5.3])

$$A_k = B/B e e B \quad \text{and} \quad A_{k+1} = f_{q'} B f_{q'}.$$  

Moreover, the inclusion functor $i$ corresponds to $B \otimes f_{q'} B f_{q'}$ while the Zuckermann’s functor corresponds to $(B/B e e B) \otimes_B$. Hence the functor $\mathcal{F}_k$ corresponds to

$$M \mapsto (B/B e e B) \otimes_B f_{q'} B f_{q'},$$

that is the same as

$$M \mapsto (B/B e e B) \mathcal{T}_{q'} \otimes_B f_{q'} B f_{q'},$$

where $\mathcal{T}_{q'}$ is the image of $f_{q'}$ in $B/B e e B$. Obviously $(B/B e e B) \mathcal{T}_{q'} = P_{q'}$ as a left $A_k$-module. It is easy to notice that also the right $A_{k+1}$-module structure is the same, since in both cases it is the natural structure induced by the bigger algebra $\text{End}_B(\mathcal{P}(0))$, where $\mathcal{P}(0)$ is the projective generator [6.8] of $O_0$.

The usual hom-tensor adjunction gives a natural isomorphism

$$\text{Hom}_{A_k}(F_k \otimes A_{k+1}, M, N) \cong \text{Hom}_{A_{k+1}}(M, \text{Hom}_{A_k}(F_k, N))$$

for all $M \in A_k$-gmod, $N \in A_{k+1}$-gmod. As a corollary of Proposition 11.32 and by the unicity of the adjoint functor we get:

Proposition 11.33. Under the equivalence of categories $A_k$-gmod $\cong \mathcal{Q}_k(r)$ the functor $\text{Hom}_{A_k}(F_k, -)$ corresponds to the functor $\mathcal{E}_k : \mathcal{Q}_k \to \mathcal{Q}_{k+1}$.

Notice that, since $F_k$ is a projective $A_k$-module, the functor $\text{Hom}_{A_k}(F_k, -)$ is exact (as we already knew for $\mathcal{E}_k$).

Proposition 11.34. We have $\text{End}(\mathcal{E}_k) \cong \text{End}(\mathcal{F}_k) \cong \mathbb{C}[x_1, \ldots, x_n]/I_k$ where $I_k$ is the ideal generated by the complete symmetric functions

$$h_k+1(x_1, \ldots, x_m) \quad \text{for all} \quad 1 \leq m \leq n-k,$$

$$h_n-m+1(x_1, \ldots, x_m) \quad \text{for all} \quad n-k+1 \leq m \leq n.\quad (11.93)$$

In particular, $\mathcal{E}_k$ and $\mathcal{F}_k$ are indecomposable functors.

Proof. Let us first compute $\text{End}(\mathcal{F}_k)$. By Proposition 11.32 we have $\text{End}(\mathcal{F}_k) \cong \text{End}_{A_k - A_{k+1}}(F_k)$. Since the structure as right $A_{k+1}$-module is induced by the map (11.30), this is the same as $\text{End}_{A_k - A_k}(F_k)$, that is the centre of $c_k A_k c_k'$. This algebra is the endomorphism algebra of the indecomposable projective-injective modules of $A_k$-gmod. But, by the construction, it is also the endomorphism algebra of the indecomposable projective injective modules of $O_0$ by a standard argument using the parabolic version of Soergel’s functor $V$ (see [Str03], Section 10) it follows that the centre of $c_k A_k c_k'$ coincides with the centre of $O_0$. Brundan [Bru08, Main Theorem] showed that this centre is canonically isomorphic to $\mathbb{C}[x_1, \ldots, x_n]/I_k$, where $I_k$ is the ideal generated by

$$h_r(x_1, \ldots, x_m) \quad \text{for all} \quad 1 \leq m \leq n - k, \quad r \geq 0 \quad (11.94)$$

Notice that this result builds on a conjecture of Khovanov [Kho04, Conjecture 3] (proved in [Bru08, Main Theorem], [Str09, Theorem 1]), that the centre of $O_0$ agrees with the cohomology ring of a Springer fibre. Under this identification, the presentation (11.94) can be deduced from Tanisaki presentation [Tan82] of the cohomology of the Springer fibre.

Using (8.30) one can easily prove that the polynomials (11.94) generate the same ideal as (11.93).
For $E_k$, by Proposition 11.33 we have $\text{End}(E_k) \cong \text{End}(\text{Hom}(F_k, \cdots))$. Notice that for all $N \in A_k \text{-} \text{gmod}$ we have a natural isomorphism

\begin{equation}
\text{End}_{A_k}(F_k, N) \cong \text{End}_{A_k}(F_k, A_k) \otimes_{A_k} N,
\end{equation}

where $\text{Hom}_{A_k}(F_k, A_k)$ is regarded as a $(A_{k+1}, A_k)$-bimodule, hence $\text{End}(E_k) \cong \text{End}_{A_{k+1}-A_k}(\text{Hom}_{A_k}(F_k, A_k))$. Since $F_k = A_k e_k\nu$ as a left $A_k$-module, we have $\text{Hom}_{A_k}(F_k, A_k) \cong e_k\nu A_k$ as a right $A_k$-module. If we let $E_k$ be the $(A_{k+1}, A_k)$-bimodule obtained from $F_k$ by swapping the actions of $A_k$ and $A_{k+1}$ (using their dualities), we have then $\text{Hom}_{A_k}(F_k, A_k) \cong E_k$ as $(A_{k+1}, A_k)$-bimodules. It follows that

\begin{equation}
\text{End}(E_k) \cong \text{End}_{A_{k+1}-A_k}(E_k) \cong \text{End}_{A_k-A_{k+1}}(F_k) \cong \text{End}(F_k).
\end{equation}

The fact that the functors $E_k$ and $F_k$ are indecomposable follows since $\text{End}(E_k) \cong \text{End}(F_k)$ is a graded local ring. □

12. Cohomology of the Springer fibre

In this section we will prove that the endomorphism rings of the indecomposable projective modules $A e_\lambda$ are isomorphic to the cohomology rings of some subvarieties of the Springer fibre. Conjecturally, it should be possible to describe the whole algebra $A_n$ using a convolution product on the direct sum of cohomologies. We warn the reader that in this section we will use an ad hoc notation, which differs sometimes from the one used in the rest of the paper.

Let $G = GL(n)$ be the general linear group of invertible $n \times n$ matrices, $B$ the Borel subgroup of upper triangular matrices, $T$ the torus of invertible diagonal matrices. Fix an integer $\ell \leq n$ and let $N$ be the standard nilpotent matrix of Jordan type $(\ell, 1^{n-\ell})$. If $\{e_1, \ldots, e_{\ell}, f_1, \ldots, f_{n-\ell}\}$ is the standard basis of $\mathbb{C}^n$, then $N e_i = e_{i-1}$ for $i = 2, \ldots, \ell$, and $N f_i = N f_{i+1} = 0$. Let $\mathcal{B}_N = (G/B)^N$ be the Springer fibre consisting of all flags fixed by $\text{Id} + N$.

To keep the connection with the previous sections, we think $\ell = n - k$. In Section 11 we described the Soergel modules for the parabolic category $\mathcal{O}_p^\ell$, where $p$ was of type $(1, \ldots, 1, n - k)$. But dealing with the Springer fibre, we prefer to follow the standard convention and to “reorder variables, indices and positions” so that the composition $(1, \ldots, 1, n - k)$ becomes a partition $(n - k, 1, \ldots, 1)$. This is the reason why in this section we are using a somehow ‘dual’ notation.

We consider a Young hook shape of type $(\ell, 1^{n-\ell})$. This shape is formed by a row with $\ell$ boxes and a column with $n - \ell$ boxes; according to our convention, the box in the corner belongs to the row and not to the column: note that this makes a difference between the hook shape of type $(1, 1^{n-1})$ and the one of type $(0, n)$. A tableau of shape $(\ell, 1^{n-\ell})$ is obtained by filling the row with numbers $r_\ell, \ldots, r_1$ from the left to the right and the column with numbers $c_1, \ldots, c_{n-\ell}$ from the top to the bottom, such that $\{r_1, c_j\} = \{1, \ldots, n\}$.

**Definition 12.1.** We say that a tableau is

- row-strict if $r_\ell > r_{\ell-1} > \cdots > r_1$,
- row-strict-column-strict if moreover $c_1 < c_2 < \cdots < c_{n-\ell}$,
- standard if moreover $r_\ell = n$.

We denote by $\text{Rs}(\ell, n)$, $\text{RsCs}(\ell, n)$, $\text{St}(\ell, n)$ respectively the sets of row-strict, row-strict-column-strict and standard tableaux of shape $(\ell, 1^{n-\ell})$.

Note that this is not the usual definition (although there is a straightforward correspondence with the usual definition).
Let \( \tau \in \text{St}(n, \ell) \). Define \( Y_\tau \) to be the subset of \( \mathcal{B}_N \) consisting of all flags \( F_\bullet \) such that
\[
\text{Im } N^{\ell-1} \subseteq F_{r_1} \subseteq \ker N,
\]
\[
\text{Im } N^{\ell-2} \subseteq F_{r_2} \subseteq \ker N^2,
\]
\[
\vdots
\]
\[
\text{Im } N \subseteq F_{r_{\ell-1}} \subseteq \ker N^{\ell-1}.
\]
Then (cf. \cite[Theorem 2.1]{Ful03}) \( Y_\tau \) is a locally closed subset of \( \mathcal{B}_N \) whose closure is an irreducible component.

For future convenience, we rewrite the conditions (12.1) in the following equivalent way:
\[
\langle e_1 \rangle \subseteq F_{r_1} \subseteq \langle e_1 \rangle + Q,
\]
\[
\langle e_1, e_2 \rangle \subseteq F_{r_2} \subseteq \langle e_1, e_2 \rangle + Q,
\]
\[
\vdots
\]
\[
\langle e_1, \ldots, e_{\ell-1} \rangle \subseteq F_{r_{\ell-1}} \subseteq \langle e_1, \ldots, e_{\ell-1} \rangle + Q,
\]
where \( Q = \langle f_1, \ldots, f_{n-\ell} \rangle \). Of course, the last condition is unnecessary since for a standard tableau we have \( F_{r_\ell} = F_n = \mathbb{C}^n \).

12.1. Fixed points and stable manifolds. Let \( S \subset T \subset GL(n) \) be the centralizer of \( N \) in \( T \). One can easily see that \( S \) is a \((n-\ell+1)\)-dimensional torus and consists of all invertible diagonal matrices whose first \( \ell \)-elements are all equal. The action of \( T \) on \( G/B \) induces an action of \( S \) on \( \mathcal{B}_N \).

**Lemma 12.2.** We have a bijection \( \tau \mapsto F_\bullet(\tau) \) between \( \text{Rs}(n, \ell) \) and the set of fixed points for the action of \( S \) on \( \mathcal{B}_N \), given by
\[
F_\bullet(\tau) = \langle e_p \mid p \leq R_i \rangle + \langle f_q \mid c_q \leq i \rangle
\]
where \( R_i \) is the number of elements \( r_j \) in the first row of \( \tau \) that are smaller than or equal to \( i \).

**Proof.** It is clear that if \( F_\bullet \in \mathcal{B}_N \) is fixed by \( S \) then each \( F_i \) is generated by some of the standard basis vectors. Conversely, every flag generated by basis vectors is obviously fixed by \( S \). Such a flag is in \( \mathcal{B}_N \) if and only if whenever \( e_j \in F_i \) then also \( e_{j-1} \in F_i \).

Fix the cocharacter
\[
\mathbb{C}^\times \rightarrow S
\]
\[
t \mapsto \text{diag}(t^{-1}, \ldots, t_{\ell}^{-1}, t, t^2, \ldots, t^{n-\ell}).
\]
This determines an action of the one-dimensional torus \( \mathbb{C}^\times \) on \( \mathcal{B}_N \). For \( \tau \in \text{Rs}(n, \ell) \) let us define the attracting variety
\[
\mathcal{Y}_\tau^0 = \{ F_\bullet \in \mathcal{B}_N \mid \lim_{t \to \infty} t \cdot F_\bullet = F_\bullet(\tau) \}
\]
and let \( \overline{\mathcal{Y}_w} = \overline{\mathcal{Y}_r} \) be its closure.

To state the next result we need some additional notation. We number the first \( n \) vertices on the number line from \( n \) to \( 1 \) from the left to the right. Then we have obviously:
Lemma 12.3. There is a bijection between RsCs(n, ℓ) and a block \( \Gamma_{n-\ell} \) consisting of weights with \( n-\ell \) \&’s and \( \ell \) \lor’s, given by putting the \( \lor \)’s in positions \( r_1, r_2, \ldots, r_\ell \) and the \( \& \)’s in positions \( e_1, e_2, \ldots, e_{n-\ell} \).

Recall that in §11.5 we defined for every weight \( \lambda \) a weight \( \tilde{\lambda} \) of maximal defect. This assignment together with the lemma above gives for every row-strict-column-strict tableau \( \tau \) a standard tableau \( \tilde{\tau} \).

Proposition 12.4. Let \( \tau \in \text{RsCs}(n, \ell) \). The set \( \mathcal{Y}_\tau \) is the set of flags \( F_\bullet \in \mathcal{B}_N \) such that

\[
\langle e_1 \rangle \subseteq F_{r_1} \subseteq \langle e_1 \rangle + Q,
\langle e_1, e_2 \rangle \subseteq F_{r_2} \subseteq \langle e_1, e_2 \rangle + Q, \\
\vdots \\
\langle e_1, \ldots, e_{\ell-1} \rangle \subseteq F_{r_{\ell-1}} \subseteq \langle e_1, \ldots, e_{\ell-1} \rangle + Q,
\langle e_1, \ldots, e_\ell \rangle \subseteq F_{r_\ell} \subseteq \langle e_1, \ldots, e_\ell \rangle + Q
\]

where \( Q = \langle f_1, \ldots, f_{n-\ell} \rangle \). In particular \( \mathcal{Y}_\tau \subseteq \mathcal{Y}_{\tilde{\tau}} \) and if \( \tau \) is a standard tableau then \( \mathcal{Y}_\tau = \mathcal{Y}_{\tilde{\tau}} \).

Proof. First observe that since \( P = \langle e_1, \ldots, e_\ell \rangle \) has minimal weight for \( C^\times \), no vector not in \( P \) is attracted to \( P \) as \( t \to \infty \). Hence we have

\[
\dim \lim_{t \to \infty} (t \cdot F_i) \cap P \leq \dim F_i \cap P
\]

for all \( i \).

Now let \( F_\bullet \in \mathcal{B}_N \) and suppose \( \langle e_1, \ldots, e_i \rangle \nsubseteq F_{r_i} \) for some \( i \). Then \( \dim F_{r_i} \cap P < i \). By (12.7) this also holds for the limit. Hence \( t \cdot F_\bullet \not\to F_\bullet(\tau) \). On the other side, it is clear that if (12.6) holds then generically \( t \cdot F_\bullet \to F_\bullet(\tau) \).

Lemma 12.5. For every \( \tau \in \text{RsCs}(n, \ell) \) we have a fibration

\[
\mathcal{Y}_\sigma \longrightarrow \mathcal{Y}_\tau \longrightarrow \mathcal{Y}(n - \ell, (r_\ell, 1^{n-\ell-r_\ell}))
\]

where \( \sigma \) is the standard tableau obtained from \( \tau \) after removing all boxes containing entries \( i > r_\ell \), while \( \mathcal{Y}(n - \ell, (r_\ell, 1^{n-\ell-r_\ell})) \) is the partial flag variety of \( C^{n-\ell} \) consisting of flags

\[
F_\bullet : \quad \{0\} = F_0 \subset F_{r_\ell-1} \subset F_{r_\ell-2} \subset \cdots \subset F_{n-\ell} = C^{n-\ell}.
\]

Proof. The fibration \( \mathcal{Y}_\tau \to \mathcal{Y}(n - \ell, (r_\ell, 1^{n-\ell-r_\ell})) \) is defined by

\[
F_\bullet \mapsto \{0\} = F_0 \subset F_{r_\ell}/P \subset F_{r_\ell+1}/P \subset \cdots \subset F_{n-\ell} = C^n/P = C^{n-\ell}
\]

where as before \( P = \langle e_1, \ldots, e_\ell \rangle \).

12.2. The Cohomology Rings. In the following we will denote by \( H^* \) the cohomology with complex coefficients. Our next goal is to compute the cohomology rings \( H^*(\mathcal{Y}_\tau) \) for all \( \tau \in \text{RsCs}(n, \ell) \).

Proposition 12.6. We have

\[
\dim H^*(\mathcal{Y}_\tau) = (n - \ell)! \cdot r_1(r_2 - r_1) \cdots (r_\ell - r_{\ell-1}).
\]

Proof. For \( \tau \in \text{St}(n, \ell) \), since \( \mathcal{Y}_\tau = \mathcal{Y}_{\tilde{\tau}} \), this is just [Fund3, Theorem 3.2]. If \( \tau \) is not standard, we use the fibration (12.8). Since we are dealing with complex varieties, the dimension of the cohomology of the total space is just the product of the dimensions of the fibre and of the base space. Notice that the tableau \( \sigma \) is standard (\( \sigma \in \text{St}(r_\ell, \ell) \)), hence we know already that

\[
\dim H^*(\mathcal{Y}_\sigma) = (r_\ell - \ell)! \cdot r_1(r_2 - r_1) \cdots (r_\ell - r_{\ell-1}).
\]
Since the dimension of the cohomology of $\mathfrak{g}(n-\ell, (r_\ell, 1^{n-\ell-\ell_\ell}))$ is
\begin{equation}
(n-\ell)(n-\ell-1)\cdots(r_\ell-\ell+1),
\end{equation}
the claim follows.

\textbf{Surjectivity.} We want now to find a set of generators. The following argument is inspired by [DCPS1].

Let $\tau \in \mathrm{RsCs}(n, \ell)$. Let $p : Y_\tau \to \mathbb{P}(\ker N)$ be the projection $F_1 \mapsto F_1$. We fix the following complete flag of $\ker N$:
\begin{align}
W_0 &= \{0\} \\
W_1 &= \langle e_1 \rangle \\
W_2 &= \langle e_1, f_1 \rangle \\
\vdots \\
W_{n-\ell} &= \langle e_1, f_1, \ldots, f_{n-\ell-1} \rangle \\
W_{n-\ell+1} &= \ker N.
\end{align}

We let $\Delta^j = \mathbb{P}(W_j) - \mathbb{P}(W_{j-1})$; this is of course an open affine cell of $\mathbb{P}(\ker N)$, isomorphic to $\mathbb{C}^{n-1}$. Let moreover $V^j = p^{-1}(\mathbb{P}(W_j))$.

Given a tableau $\tau$ and an entry $a$ of $\tau$, we define $\tau^a$ to be the tableau obtained from $\tau$ by removing the box containing $a$ and then subtracting 1 to all entries bigger than $a$. Note that if $\tau \in \mathrm{RsCs}(n, \ell)$ then $\tau^a$ is also a row-strict-column-strict tableau.

\textbf{Lemma 12.7.} The set $V^j - V^{j-1}$ is either empty or isomorphic to $\Delta^j \times Y_{\tau^j}$, for $j > 1$ and to $\Delta^j \times Y_{\tau^1}$ for $j = 1$.

\textbf{Proof.} Let $U = \mathbb{P}(\ker N) - \mathbb{P}(W_1)$ and $U' = \mathbb{P}(W_1)$, so that $U \cup U' = \mathbb{P}(\ker N)$. Notice that $p$ is surjective onto $\mathbb{P}(\ker N)$ if and only if 1 is in the row of $\tau$, that is $r_1 \neq 1$; otherwise $p$ is onto $\mathbb{P}(W_1)$. Now $p|_{U'}$ is a locally trivial fibration with fibre isomorphic to $Y_{\tau^1}$, (in this specific case, the base space is even a point), while $p|_{U'}$ is a locally trivial fibration with fibre isomorphic to $Y_{\tau^1}$ (if non empty). In particular, for every $j$ the projection $p$ restricted to $V^j - V^{j-1}$ is a locally trivial fibration; since the base space is isomorphic to $\mathbb{C}^{n-1}$, the fibration has to be trivial, hence isomorphic (if non empty) to the product of $\Delta^j$ and the fibre.

Thanks to Lemma 12.7 we have a recursive construction of a cell decomposition of $Y_\tau$ with even dimensional cells.

\textbf{Proposition 12.8.} For every $\tau \in \mathrm{RsCs}(n, \ell)$ the inclusion $Y_\tau \hookrightarrow G/B$ of $Y_\tau$ in the full flag variety induces a surjective homomorphism $H^*(G/B) \to H^*(Y_\tau)$ in cohomology.

\textbf{Proof.} We prove by induction on $n$ that it is possible to construct a cell decomposition of $G/B$ with even dimensional cells such that $Y_\tau$, with the CW-structure that we have defined, is a subcomplex of it. For $n = 0$ there is nothing to prove, so let us consider $n > 0$. Notice that $G/B = Y_{\sigma}$ where $\sigma$ is the unique element of $\mathrm{RsCs}(n, 0)$. Complete the flag $W_\bullet$ of (12.13) for $\tau$ to a full flag of $\mathbb{C}^n$. Then by Lemma 12.7 $V^j$ - $V^{j-1}$ isomorphic to $\Delta^j \times Y_{\tau^j}$, where $\tau^j \in \mathrm{RsCs}(n-1, 0)$, while $V^j - V^{j-1}$ is either isomorphic to $\Delta^j \times Y_{\tau^j}$ for some $a_j$ or empty. By induction, we can suppose that $Y_{\tau^j}$ is a subcomplex of $Y_{\sigma^j}$; then the claim for $Y_\tau$ follows.

Since the cells are even dimensional, they give a basis of the cohomology as a vector space. It follows that the homomorphism $H^*(G/B) \to H^*(Y_\tau)$ in cohomology is surjective.
The isomorphism with $\mathbb{Z}_{z,z}$. For $\tau \in \text{RsCs}(n, \ell)$ let us define an ideal $I_\tau$ of $R = \mathbb{C}[x_1, \ldots, x_n]$ as follows. Let $b$ be the $b$-sequence of the weight corresponding to $\tau$. Let
\begin{equation}
I'_\tau = (h_b(x_n, \ldots, x_i))_{i=n, \ldots, 1}
\end{equation}
and
\begin{equation}
I''_\tau = (x_{r_1} x_{r_1-1} \cdots x_{r_1-1+1})_{i=h, \ldots, 1}
\end{equation}
where $r_0 = 0$. Set
\begin{equation}
I_\tau = I'_\tau + I''_\tau.
\end{equation}
Finally, set
\begin{equation}
R_\tau = \mathbb{C}[x_1, \ldots, x_n]/I_\tau.
\end{equation}
Note that according to Theorems 10.10 and 10.17 and Corollary 11.8 we have
\begin{equation}
R_\tau \cong \mathbb{Z}_{z,z} = e_\lambda A_n, n-\ell e_\lambda
\end{equation}
where $z \in D_{n,n-\ell}$ and $\lambda \in \Gamma_{n-\ell}$ are the permutation and the weight corresponding to the tableau $\tau$. Since we work in the dual pictures (with reordered indices), the isomorphism is given by $x_i \mapsto x_{n-i}$.

We recall that the elementary symmetric polynomials are defined as
\begin{equation}
e_j(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} x_{i_1} \cdots x_{i_j}
\end{equation}
for $0 \leq j \leq n$. We are now ready to state the main theorem of this section.

**Theorem 12.9.** For every $\tau \in \text{RsCs}(n, \ell)$ the cohomology ring of $Y_\tau$ is isomorphic to $R_\tau$. The Chern class of the canonical bundle $F_1/F_{1-1}$ over $Y_\tau$ is sent to the class of $x_1$ under this isomorphism.

The proof will consist of several reduction steps. Let us remark that by Proposition 12.8 we know that the cohomology ring is generated by the Chern classes of its canonical line bundles $F_1/F_{1-1}$ (since this holds for the full flag variety). Moreover, by Proposition 12.4 we already know that the dimensions agree. Hence it suffices to prove that for every $\tau \in \text{RsCs}(n, \ell)$ the Chern classes of the canonical bundles $F_1/F_{1-1}$ on $Y_\tau$ satisfy the relations of the ideal $I_\tau$.

**Lemma 12.10.** Let $\tau$ be the row-strict-column-strict tableau of shape $(1^n)$. Then Theorem 12.9 holds for $Y_\tau$.

**Proof.** In this case, $R_\tau$ is the cohomology ring of the full flag variety. But $Y_\tau$ is the full flag variety, since conditions 12.4 are void for it (the row of $\tau$ is empty). \qed

**Lemma 12.11.** Suppose $\lambda \in \Gamma_{n-\ell}^\vee$ is a weight starting with a $\vee$, and let $\lambda^{(\wedge)}$ as defined in 11.7. Let $\tau, \sigma$ be the tableaux corresponding to $\lambda$ and $\lambda^{(\wedge)}$ respectively. If Theorem 12.9 holds for $\sigma$, then it holds for $\tau$.

**Proof.** For notation convenience, let $a = r_{1-1}$. We have $I_\tau = I_\sigma + (x_n \cdots x_{a+1})$. A flag $F_\bullet \in Y_\tau$ obviously satisfies the relations 12.10 also for $\sigma$. Moreover, if it is invariant for the nilpotent $N_\tau$ of shape $(h, 1^{n-\ell})$, it is a fortiori invariant for the nilpotent $N_\sigma$ of shape $(\ell - 1, 1^{n-\ell+1})$. Hence we have an inclusion map $Y_\tau \hookrightarrow Y_\sigma$, and the relations that $x_1, \ldots, x_n$ satisfy in $H^\ast(Y_\sigma)$ are also satisfied in $H^\ast(Y_\tau)$.

We are left to prove that the relation $x_n \cdots x_{a+1}$ holds on $H^\ast(Y_\tau)$. By 12.10 for $\tau$, we know that $F_\sigma \subset K = (e_1, \ldots, e_{\ell-1}) + Q$. Let us work in $K$-theory for bundles over $Y_\tau$ and write $[\mathbb{C}^n/F_\sigma] = [\mathbb{C}^n/K] + [K/F_\sigma]$. Since the bundle $\mathbb{C}^n/K$ is a one dimensional trivial bundle, the $(n-a)$-th Chern class of $\mathbb{C}^n/F_\sigma$ is trivial.
But this class is equal to the elementary symmetric function $e_{n-a}(x_n, \ldots, x_{a+1}) = x_n \cdots x_{a+1}$ by the Whitney sum formula, and we are done. \hfill \Box

**Lemma 12.12.** Suppose $\tau$ is a row-strict-column-strict tableau that is not standard. If Theorem 12.9 holds for the standard tableau $\tilde{\tau}$ (defined in (11.5)), then it also holds for $\tau$.

**Proof.** Remember that $\tilde{\tau}$ is obtained permuting the leftmost $\land$ with the leftmost $\lor$ of the $\land\lor$-sequence corresponding to $\tau$. As before, since $\mathcal{Y}_{\tilde{\tau}} \subseteq \mathcal{Y}_{\tau}$, all relations of $H^*(\mathcal{Y}_{\tilde{\tau}})$ also hold in $H^*(\mathcal{Y}_{\tau})$. Hence we need to prove that in $H^*(\mathcal{Y}_{\tau})$ the relations $h_b(x_n, \ldots, x_i)$ for $i > r_\ell$ hold.

The variety $\mathcal{Y}_{\tau}$ consists of all flags $F_\tau$ in $\mathcal{Y}_{\tilde{\tau}}$ that satisfy also $P \subseteq F_\tau$. Let $a \geq r_\ell$.

We argue as in the previous proof: we have $[F_\tau] = [F_a/P] + [P]$; since $P$ is a trivial bundle, by the Whitney sum formula we have

$$e_i(x_a, \ldots, x_1) = 0 \quad \text{for all } i > a - \ell. \tag{12.21}$$

Note that $a - \ell$ is equal to the number of $\land$’s that are on the right of position $a$, that is $b_\ell - 1$. Let us consider the following identity of symmetric functions:

$$h_{a-\ell+1}(x_n, \ldots, x_{a+1}) \quad \tag{12.22}$$

$$= (-1)^{a-\ell+1} \left( \sum_{i=0}^{a-\ell} (-1)^i h_i(x_n, \ldots, x_{a+1}) e_{a-\ell+i+1}(x_n, \ldots, x_1) \right).$$

It follows that $h_{a-\ell+1}(x_n, \ldots, x_{a+1}) = 0$. \hfill \Box

**Proof of Theorem 12.9.** Let $\tau \in RsCs$. Applying repeatedly Lemmas 12.11 and 12.12 we can restrict to the case in which $\tau$ is a sequence with $\land$’s only. Then the theorem holds by Lemma 12.10. \hfill \Box

We conjecture that, as in [SW12], it is possible to define a convolution product on the direct sum

$$\bigoplus_{\tau, \tau' \in RsCs(n, \ell)} H^*(\mathcal{Y}_{\tau} \cap \mathcal{Y}_{\tau'}) \tag{12.23}$$

such that the resulting algebra is isomorphic to the algebra $A_{n,n-\ell}$ from [11.2]. Hence this would give a geometric realization of the endomorphism algebras $A_{n,n}$ from Lie theory (for $\lambda$ a regular weight) and of their diagrammatical realizations (11.20).

**References**

[AM11] T. Agerholm and V. Mazorchuk, *On selfadjoint functors satisfying polynomial relations*, J. Algebra 330 (2011), 448–467.

[AS13] P. N. Achar and C. Stroppel, *Completions of Grothendieck groups*, Bull. Lond. Math. Soc. 45 (2013), no. 1, 200–212.

[Aus74] M. Auslander, *Representation theory of Artin algebras. I, II*, Comm. Algebra 1 (1974), 177–268; ibid. 2 (1974), 269–310.

[BGG76] I. N. Bernstein, I. M. Gel’fand, and S. I. Gel’fand, *A certain category of $\mathfrak{g}$-modules*, Funkcional. Anal. i Priložen. 10 (1976), no. 2, 1–8.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.

[BJS93] S. C. Billey, W. Jockusch, and R. P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. 2 (1993), no. 4, 345–376.

[BK00] G. Benkart, S.-J. Kang, and M. Kashiwara, *Crystal bases for the quantum superalgebra $U_q(gl(m,n))$*, J. Amer. Math. Soc. 13 (2000), no. 2, 295–331.

[BM12] G. Benkart and D. Moon, *Planar Rook Algebras and Tensor Representations of $gl(1|1)$*, ArXiv e-prints (2012), 1201.2482.
[Zha09a] H. Zhang, The quantum general linear supergroup, canonical bases and Kazhdan-Lusztig polynomials, Sci. China Ser. A 52 (2009), no. 3, 401–416.

[Zha09b] H. Zhang, The quantum general linear supergroup, canonical bases and Kazhdan-Lusztig polynomials., Sci. China, Ser. A 52 (2009), no. 3, 401–416.

Mathematisches Institut, Endenicher Allee 60, Universität Bonn, 53115 Bonn, Germany

E-mail address: sartori@math.uni-bonn.de

URL: http://www.math.uni-bonn.de/people/sartori