Power Counting in the Soft-Collinear Effective Theory

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Abstract

We describe in some detail the derivation of a power counting formula for the soft-collinear effective theory (SCET). This formula constrains which operators are required to correctly describe the infrared at any order in the $\Lambda_{\text{QCD}}/Q$ expansion ($\lambda$ expansion). The result assigns a unique $\lambda$-dimension to SCET graphs solely from vertices, is gauge independent, and can be applied independent of the process. For processes with an OPE the $\lambda$-dimension has a correspondence with dynamical twist.

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I. INTRODUCTION

The physics of processes involving very energetic particles can be described in terms of a soft-collinear effective theory (SCET) \cite{1,2,3,4,5}. This effective theory has a systematic expansion in a small parameter $\lambda$, which depends on the typical offshellness of the partons participating in the hard process. Denoting the large energy scale with $Q$ and the typical transverse momentum of constituent quarks and gluons relative to the collinear axis with $p_\perp$, the small parameter is $\lambda^2 = p_\perp^2 / Q^2$. The effective theory is constructed such that physical amplitudes are expressed as a sum of operators corresponding to a power series in $\lambda$. Typically for exclusive processes one has $p_\perp \sim \Lambda_{\text{QCD}}$, so that $\lambda = \Lambda_{\text{QCD}} / Q$. The purpose of this paper is to give a detailed derivation of the power counting formula presented in Ref. \cite{6}, which depends only on the vertices of the diagram. This is the analog of having powers of $1/m_Q$ explicit in vertices in the Heavy Quark Effective Theory \cite{7}. For a given set of SCET fields our formula can be applied independent of the process or the choice of gauge. It determines which operators are needed at any desired order in the power counting to determine the cross section or decay rate at that order. As applications we show how this formula greatly simplifies the power counting of graphs and how it is used to count powers of $Q$ in matrix elements. We also discuss how the $\lambda$-dimension of operators relates to twist for processes with an OPE such as DIS.

The degrees of freedom in the effective theory have well-defined momentum scaling. Short distance fluctuations are integrated out and appear in Wilson coefficients, while long distance fluctuations are described by effective theory fields. We refer to Refs. \cite{3,4,5,6,7,8,9,10} for a detailed description of the theory, and to Refs. \cite{1,2,3,4,5,6,7,8,9,10} for examples of how the theory can be used to prove factorization theorems, sum Sudakov double logarithms, and study power corrections. Here only the features of SCET that are relevant for explaining the power counting are reviewed.

The effective theory fields we will discuss are collected in Table I, together with their momentum and $\lambda$ scaling. The momenta are given in light-cone momentum coordinates $(p^+, p^-, p_\perp)$, defined as $p^- = \vec{n} \cdot p$, $p^+ = n \cdot p$, where $n^2 = \vec{n}^2 = 0$ and $\vec{n} \cdot n = 2$. For simplicity we restrict ourselves to processes where only a single collinear direction $n$ is relevant. The fields include collinear quarks and gluons ($\xi_{n,p}, A_{n,q}^\mu$), massless soft quarks and gluons ($q_s, A_s^\mu$), and massless ultrasoft (usoft) quarks and gluons ($q_{us}, A_{us}^\mu$). For processes with heavy quarks of mass $m_Q \sim Q$, the large perturbative mass scale can be factored out as in Heavy Quark Effective Theory. Then for power counting purposes the heavy quark fields $h_v$ are identical to either the soft or usoft light quark fields. In Table I the fields are assigned a scaling with $\lambda$ such that the action for their kinetic terms is order $\lambda^0$, and we also list the scaling of label operators and Wilson lines. Label operators $\bar{P}$ and $P^\mu$ are introduced to facilitate the power counting and separation of momentum scales \cite{3}. Integrating out offshell fluctuations with $p^2 \gg (Q\lambda)^2$ induces the collinear Wilson line $W_n[\vec{n} \cdot A_{n,q}]$ and soft Wilson line $S_n[n \cdot A_s]$, which appear in a way that preserves gauge invariance \cite{4,5}. The usoft Wilson line $Y_n[n \cdot A_{us}]$ is induced by the field redefinitions $\xi_{n,p} = Y_n \xi_{n,p}^{(0)}$ and $A_{n,q} = Y_n A_{n,q}^{(0)} Y_n^\dagger$ which decouple usoft gluons from the leading order collinear Lagrangian \cite{4}.

Interactions in SCET appear either in the effective theory action or in external operators (which are often operators or currents generated by electromagnetic or weak interactions).
It is useful to divide the full action of SCET into four pieces

\[ S = S^U + S^S + S^C + S^{SC}, \]  

where \( S^U \) has purely usoft interactions, \( S^S \) contains interactions with one or more soft fields, \( S^C \) contains interactions with one or more collinear fields, and \( S^{SC} \) contain possible mixed soft-collinear terms. Once offshell fluctuations are integrated out the mixed soft-collinear interactions typically occur in external operators, however mixed collinear-usoft interactions do appear in \( S^C \). The leading terms in \( S^U, S^S, \) and \( S^C \) are the order \( \lambda^0 \) kinetic terms for the usoft, soft, and collinear fields.

The leading order Lagrangians for (u)soft light quarks and gluons are the same as QCD. For heavy quarks we have the HQET Lagrangian, \( \mathcal{L} = \bar{h}_v i\gamma_i D h_v + \ldots \). The collinear quark Lagrangian has an expansion in \( \lambda \),

\[ \mathcal{L}_c = \mathcal{L}^{(0)}_c + \mathcal{L}^{(1)}_c + \mathcal{L}^{(2)}_c + \ldots. \]  

The first three terms are

\[
\mathcal{L}^{(0)}_c = \bar{\xi}_{n,g'} \left\{ i n \cdot D + g n \cdot A_{n,q} + \left( \mathcal{P}_\perp + g A^\perp_{n,q} \right) W \frac{1}{\mathcal{P}} W^\dagger \left( \mathcal{P}_\perp + g A^\perp_{n,q} \right) \right\} \frac{i\gamma^\mu}{2} \xi_{n,p},
\]

\[
\mathcal{L}^{(1)}_c = \bar{\xi}_{n,g'} \left\{ i \mathcal{P}_\perp \frac{1}{\mathcal{P}} W^\dagger \left( \mathcal{P}_\perp + g A^\perp_{n,q} \right) + \left( \mathcal{P}_\perp + g A^\perp_{n,q} \right) W \frac{1}{\mathcal{P}} W^\dagger i \mathcal{P}_\perp \right\} \frac{i\gamma^\mu}{2} \xi_{n,p},
\]

\[
\mathcal{L}^{(2)}_c = \bar{\xi}_{n} \left\{ i \mathcal{P}_\perp \frac{1}{\mathcal{P}} W^\dagger i \mathcal{P}_\perp - \left( \mathcal{P}_\perp + g A^\perp_{n,q} \right) W \frac{1}{\mathcal{P}} W^\dagger (\bar{n} \cdot i D) W \frac{1}{\mathcal{P}} W^\dagger \left( \mathcal{P}_\perp + g A^\perp_{n,q} \right) \right\} \frac{i\gamma^\mu}{2} \xi_{n},
\]

where \( iD^\mu = i\partial^\mu + g A^\mu_u \). Here \( \mathcal{L}^{(0)}_c \) gives the order \( \lambda^0 \) interactions [4, 5]. The expression for \( \mathcal{L}^{(1)}_c \) was first given in Ref. [3], and that for \( \mathcal{L}^{(2)}_c \) in Ref. [3].

For any operator in SCET, a scaling \( \lambda^k \) can be immediately assigned by adding up the factors of \( \lambda \) associated with the scaling of its fields, derivatives, and label operators. In time-ordered products or Feynman diagrams these operators appear as vertices. To power

| Type | Momenta \( \mathbf{p}^\mu = (+, -, \perp) \) | Fields (\( f \)) | Scaling (\( e^f \)) | Operator | Scaling |
|------|---------------------------------|----------------|----------------|----------|---------|
| collinear | \( \mathbf{p}^\mu \sim (\lambda^2, 1, \lambda) \) | \( \xi_{n,p} \) | \( \lambda \) | \( \mathcal{P}_n, W \) | \( \lambda^0 \) |
| | | (\( A_{n,p}^+, A_{n,p}^\perp, A_{n,p}^\perp \)) | (\( \lambda^2, 1, \lambda \)) | \( \mathcal{P}_\mu^\perp \) | \( \lambda \) |
| soft | \( \mathbf{p}^\mu \sim (\lambda, \lambda, \lambda) \) | \( q_{s,p} \) | \( \lambda^{3/2} \) | \( S_n \) | \( \lambda^0 \) |
| | | \( A_{s,p}^\mu \) | \( \lambda \) | \( \mathcal{P}_\mu^\perp \) | \( \lambda \) |
| usoft | \( k^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \) | \( q_{us} \) | \( \lambda^3 \) | \( Y_n \) | \( \lambda^0 \) |
| | | \( A_{us}^\mu \) | \( \lambda^2 \) | | |
count these it is useful to introduce indexes $V^i_k$ which count operators in a graph. Here $V^i_k$ counts the number of operators that scale as $\lambda^k$ and are of type $i$, where the type depends on the field content. The four vertex indexes we require are:

- $V^C_k$ for vertices involving only collinear and usoft fields,
- $V^S_k$ for vertices involving only soft and usoft fields,
- $V^{SC}_k$ for vertices with both soft and collinear fields,
- $V^U_k$ for vertices with only usoft fields.

Note the important point that the mixed soft-collinear operators require a separate index. For Non-Relativistic QCD indexes analogous to these were defined in Ref. [11]. As an example of how these indexes work consider the purely collinear DIS operator

$$O_1 = \frac{1}{Q} \bar{\xi}_{n,p'} W_n \frac{g}{2} C(\bar{P}_+, \bar{P}_-, Q, \mu) W^\dagger_n \xi_{n,p}.$$ (4)

where $\bar{P}_\pm = \bar{P}_\dagger \pm \bar{P}$ and $C$ a Wilson coefficient. When taking the proton matrix element of $O_1$, the $\bar{P}_+$ dependence of the Wilson coefficient leads to a convolution involving the quark or antiquark parton distribution functions. Since $\xi \sim \lambda$ and $W_n \sim \lambda^0$ this operator scales as $\lambda^2$. Thus $k = 2$, and since the operator only involves collinear fields a single insertion of $O_1$ makes the index $V^C_2 = 1$.

Now consider an arbitrary loop graph built out of insertions of external operators along with propagators and interactions from $S$. In the effective theory each such graph scales with a unique power of $\lambda$, say $\lambda^\delta$ for some $\delta$. Since the effective theory is constructed to include all the relevant infrared degrees of freedom, the set of all SCET graphs scaling as $\lambda^\delta$ will reproduce the infrared structure of QCD at this order. Our goal is to prove that

$$\delta = 4u + 4 + \sum_k (k - 4)(V^C_k + V^S_k + V^{SC}_k) + (k - 8)V^U_k,$$ (5)

where $u = 1$ if the graph is purely usoft and $u = 0$ otherwise. We will refer to Eq. (5) as the vertex power counting formula. This result applies to any physical process whose infrared structure can be described by the fields in Table I. In Ref. [6], this formula was given\(^1\), but the steps in its derivation were not described. Eq. (5) expresses the important result that the power of $\lambda$ associated with an arbitrary diagram can be determined entirely by the scaling of operators at its vertices. This is the analog of having all powers of $1/m_Q$ explicit in the vertices in the Heavy Quark Effective Theory.

Only external operators can have $k < 4$, such as in the DIS operator in Eq. (4). Furthermore, physical considerations always limit the number of external operator insertions (usually to just 1). For example, in DIS multiple insertions of $O_1$ would require multiple

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\(^1\) The formula in Ref. [6] has a different overall offset than Eq. (5), since for the proof there only relative scalings mattered. In general there is additional information in the normalization which is explored in this paper.
electromagnetic interactions. Thus, at leading order in $\lambda$ only graphs built out of a fixed number of external operators plus $V_4^C$, $V_4^S$, $V_4^{SC}$, and $V_8^U$ vertices need to be included. These vertices are exactly those described by the order $\lambda^0$ actions derived in Refs. [2, 3, 4]. At one higher order in $\lambda$ we only need to add a single vertex with a higher power of $k$ to the vertices included above. For instance, a single $V_5^C$. From this discussion the utility of Eq. (4) for describing higher and higher orders in $\lambda$ should be fairly evident.

**Direct Power Counting and the Derivation of Eq.(5)**

The derivation of power counting formulae such as the one in Eq. (5) has a history back to Weinberg [12] and Sterman [13, 14]. Our result in Eq. (5) provides a simple way of determining the order in $\Lambda_{QCD}/Q$ of any given diagram. The part of our proof from Eqs. (4, 3) follows the concise approach of Ref. [11] (where a power counting formula for Non-Relativistic QCD was derived).

An intuitive method of power counting diagrams involves counting powers of $\lambda$ for the loop measures, propagators, vertices, and external lines. We will refer to this as the “direct” method of power counting and use it as our starting point. In the context of collinear interactions the direct method has been employed with the threshold expansion in Ref. [15], although without counting powers of $\lambda$ for external fields. For this method it is more intuitive to begin with indexes $\tilde{V}_j^j$ which are analogous to the $V_k^j$, but do not include the scaling for the fields. Thus, $\tilde{V}_k^j$ directly count the scaling of the vertex Feynman rules. For example, an operator $\bar{\xi}_n \mathcal{P}_\perp \xi_n$ would be $\tilde{V}_2^C = 1$, whereas $\bar{\xi}_n A_{n,q}^{1/2} \xi_n$ is $\tilde{V}_0^C = 1$. Below we will show that this direct power counting method can be reduced to Eq. (5). Readers not interested in the derivation can safely skip to Application 1.

Consider an arbitrary graph containing $L^i$ loops and $I^j$ propagators of type $i$ with $i \in \{C, S, U\}$, $\tilde{V}_k^j$ vertices of type $j$, and $E^f$ external lines of type $f$, where $f$ runs over the fields in Table I. Counting the powers of $\lambda$ associated with the vertices, loop measures, propagators, and external lines we find that the graph scales as $\lambda^\delta$ with

$$\delta = \sum_{k'} k' (\tilde{V}_k^C + \tilde{V}_{k'}^S + \tilde{V}_{k'}^{SC} + \tilde{V}_{k'}^U) + 4L^C + 4L^S + 8L^U - \eta_\alpha I^C - 2I^S - 4I^U + \sum f E^f.$$  

(6)

which we refer to as the direct power counting formula. To derive Eq. (5), consider an arbitrary time-ordered product of operators $\sim \int d^4x_1 \exp(\cdots) \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$ and transform all loops and internal lines to momentum space. The scaling of momenta in the vertices give the $\tilde{V}_j^j$ terms in Eq. (5). With the momenta and fields scaling as in Table I, the integrations over loop momenta give the $L^i$ terms, and the external fields give the $E^f$ terms, where $e^f$ is the appropriate power of $\lambda$ from the table. The $E^f$ term is important if we wish to compare the size of operators with different external lines. Finally, the internal propagators give the $I^i$ terms. Since the scaling of collinear gluons is not homogeneous, $(A_n^+, A_n^-, A_n^\perp) \sim (\lambda^2, \lambda^0, \lambda)$, their propagator contribution is gauge and component dependent. This dependence is encoded in the coefficient $\eta_\alpha$. In Feynman gauge $\eta_\alpha = 2$ for all collinear particles, however in a general covariant gauge $\eta_\alpha = 2$ for fermions, but the gluon propagator

$$A_n^\mu A_n^{\nu} \rightarrow \frac{-i}{p^2} \left( g_{\mu\nu} + \alpha \frac{p_{\mu} p_{\nu}}{p^2} \right),$$

(7)
gives $\eta_\alpha = \{0, 1, 2, 2, 3, 4\}$ for the $\{++, +\perp, +-, -\perp, -\perp, -\} \}$ components. We will see that the $\eta_\alpha$ gauge dependence in Eq. (3) is cancelled by a similar dependence in $\tilde{V}_{k'}^{C}$.

To proceed we switch to the $V_k^i$ indexes. The difference in scaling of the two types of vertex indexes is the powers of $\lambda$ associated with the fields in an operator. For example, both $\xi_n P_\perp \xi_n$ and $\xi_n A_{n,q}^{-1} \xi_n$ count as $V_4^C = 1$. Since the fields in a vertex are either contracted with another field or correspond to an external line we have

$$\sum_{k,i} k V_k^i = \sum_{k',i} k' \tilde{V}_{k'}^{i} + (4 - \eta_\alpha) I^C + 2 I^S + 4 I^U + \sum_f e^f E^f.$$  

(8)

For each internal line two fields are contracted and eliminated for a momentum space propagator. The difference in scaling of the two fields and the propagator induces the $I^i$ correction terms in Eq. (8). For instance two collinear fermion fields scale as $\lambda^2$ on the LHS. If they are contracted then a $(4 - \eta_\alpha) I^C = (4 - 2) I^C = 2 I^C$ term accounts for them on the RHS. The $E^f$ terms account for them if they are not contracted. In a similar way, each soft propagator or external line is also accounted for. The most non-trivial contraction is that of two collinear gluon fields $A^\mu_n A_n^\nu$ since their scaling is inhomogeneous. However, it is straightforward to see, for example in a general covariant gauge, that these contractions are correctly encoded by the $(4 - \eta_\alpha) I^C$ term in Eq. (8). Using Eq. (8) to eliminate the $\tilde{V}_{k'}$'s from Eq. (8) leaves

$$\delta = \sum_k k (V_k^C + V_k^S + V_k^{SC} + V_k^U) + 4 L^C + 4 L^S + 8 L^U - 4 I^C - 4 I^S - 8 I^U,$$  

(9)

whose terms are now explicitly gauge independent. This is made possible by the fact that the $V_k^i$ indexes assign a homogeneous $\lambda$-dimension to gauge invariant operators.

Eq. (8) can be further simplified by using Euler topological identities, which connect the number of loops, lines and vertices in an arbitrary graph. For the complete graph we have

(E1): $\sum_k (V_k^C + V_k^S + V_k^{SC} + V_k^U) + (L^S + L^C + L^U) - (I^S + I^C + I^U) = 1.$  

(10)

Using this result to remove the $L^U - I^U$ factor from Eq. (8) leaves

$$\delta = 8 + \sum_k (k - 8)(V_k^C + V_k^S + V_k^{SC} + V_k^U) - 4 L^C - 4 L^S + 4 I^C + 4 I^S.$$

(11)

To proceed further we consider two cases, i) graphs with purely usoft fields, and ii) graphs with $\geq 1$ soft or collinear field. In case i) $L^C = L^S = I^C = I^S = V_k^C = V_k^S = V_k^{SC} = 0$ and Eq. (11) gives the final result which is $\delta = 8 + \sum_k (k - 8)V_k^U$.

In case ii) we can use the hierarchy in $p^2$ of the modes in Table I to derive additional Euler relations by removing modes one at a time, starting with those propagating over the longest distance $[\mu \nu]$. This is possible since the graph must stay connected when probed at the shorter distance scales. Thus, by removing all the usoft lines in the graph one can draw a reduced graph containing only soft and collinear modes, also of course keeping vertices that are not purely usoft. The Euler identity for this reduced graph reads

(E2): $\sum_k (V_k^C + V_k^S + V_k^{SC}) + (L^S + L^C) - (I^S + I^C) = 1.$  

(12)
FIG. 1: Graphs used for the power counting examples in the text. (a), (b), and (c) correspond to the processes DIS, $B \rightarrow X_u \ell \bar{\nu}_\ell$, and $B \rightarrow D\pi$ respectively. Dashed lines are collinear quarks, double solid lines are usoft or soft heavy quarks, and the single solid lines are soft light quarks. Gluons with a line through them are collinear, while those without a line are soft or usoft.

It is easy to see that (E2) can be used to eliminate the remaining terms in Eq. (11) that depend on the loop measures and internal lines. Because the soft and collinear terms have the same prefactor, a single Euler relation eliminates all of them simultaneously. This differs from NRQCD where removing potential and soft factors require distinct topological identities [11]. Using (E2) in Eq. (11) leaves the final result [for graphs with at least one soft and/or collinear field, ie. case ii)],

$$\delta = 4 + \sum_k (k - 4)(V^C_k + V^S_k + V^{SC}_k) + (k - 8)V^U_k. \quad (13)$$

Together cases i) and ii) reproduce Eq. (5) which is the main result of this paper. It should be noted that the derivation of Eq. (5) would not be possible if offshell degrees of freedom had been retained since the power counting for fields generating offshell fluctuations is ambiguous [4]. In the remainder of the paper we illustrate how Eq. (5) can be used on a few examples of physical interest.

**Application 1: A simplified power counting of graphs**

In the direct method of power counting graphs one counts powers of $\lambda$ for the loop measures, propagators, and vertices. As our first application we use the graphs in Fig. 1 to contrast the direct formula in Eq. (6) with the simpler vertex formula in Eq. (5). As mentioned above, in the direct approach the choice of gauge can move powers of $\lambda$ between the propagators and vertices, and for simplicity we will use Feynman gauge for our examples. On the other hand, the formula in Eq. (5) is applied the same way in any gauge.

The one-loop graph in Fig. 1(a) involves an insertion of the DIS operator in Eq. (4) and two leading order interactions from the collinear Lagrangian $\mathcal{L}_c^{(0)}$ in Eq. (3). Using the vertex power counting formula in Eq. (13), we simply count $V^C_k = 1$ and $V^S_k = 2$ to give $\delta = 4 + (2 - 4) + 0 = 2$. Alternatively, in the direct power counting approach we count powers of $\lambda$ for all the loop measures, propagators, vertices, and external lines. For Fig. 1(a) we then have

$$(\lambda)^2 \left[ \lambda^4 \times \left( \frac{1}{\lambda^2} \right)^3 \times \lambda^2 \right] = \lambda^2. \quad (14)$$
Here the factor outside the square brackets is for the external fields, the first term in the square brackets counts the collinear loop measure, and the second factor counts the three collinear propagators. The last factor in the bracket counts the powers of momentum in the quark-quark-gluon vertices in $\mathcal{L}^{(0)}$, which in Feynman gauge are either $(\perp, \perp) \sim (\lambda, \lambda)$ or $(\bar{n}, n) \sim (\lambda^2, \lambda^0)$. The result in Eq. (14) is again $\delta = 2$. Since $\delta = 2$, this graph is the same order as the operator $O_1$ itself, and non-perturbative collinear gluon exchanges such as the one in Fig. 1(a) contribute to building up the parton distribution function.

Next we consider an example involving power suppression for the inclusive decay $B \to X_u \bar{\nu}_\ell$ in the endpoint region where $p_X^2 \sim m_B \Lambda_{QCD}$ and $\lambda = \sqrt{\Lambda_{QCD}/m_b}$. At leading order the $b$ to $u$ weak current matches onto the SCET currents

$$J^\mu_i = \left[ C_i(\mathcal{P}, \mu) \xi_{n,p} W \Gamma^{C_i}_\mu h_v \right]$$

(15)

where $i = 1, 2, 3$ and $\Gamma^{C_{1,2,3}}_\mu = P_R \{ \gamma^\mu_{\perp}, n^\mu \bar{n} \cdot v, v^\mu \}$ with $P_{R,L} = (1 \pm \gamma_5)/2$. In Eq. (15) the heavy quark field $h_v$ is used and coefficients $C_i$ are dimensionless. Counting factors of $\lambda$ we find that each $J^\mu_i$ counts as $V^C_4 = 1$. Using Eq. (5) we see that $\delta = 4$ is the base $\lambda$-dimension for the tree level time-ordered product of two $J^\mu_i$ currents. With the direct counting method the $\delta = 4$ result is obtained by counting $\lambda^6$ for the external heavy quark lines and $\lambda^{-2}$ for the collinear quark propagator. We will also consider currents contributing to $B \to X_u \bar{\nu}_\ell$ that are suppressed by a single power of $\lambda$:  

$$O_{i\mu} = \left[ B_i(\mathcal{P}, \mu) \xi_{n,p} \frac{i}{2} (\mathcal{P}_\perp + g A_{n,q}^{\perp}) W \frac{1}{p_I} \Gamma^{B_i}_{\alpha\mu} h_v \right],$$

$$K_{i\mu} = \left[ E_i(\mathcal{P}, \mu) \xi_{n,p} \Gamma^{E_{\alpha\mu}}_\mu \left( \mathcal{P}^{\alpha} + g A_{n,q}^{\alpha} \right) W \frac{1}{m_b} \frac{\gamma^\mu}{2} h_v \right],$$

(16)

with dimensionless Wilson coefficients $B_i$ and $E_i$, and Dirac structures $\Gamma^{B_{1,2,3,4}}_{\alpha\mu} = P_L \{ \gamma_\alpha \gamma^\perp_{\mu}, \gamma_\alpha n^\mu \bar{n} \cdot v, \gamma_\alpha v^\mu, g_{\mu\nu} \}$ and $\Gamma^{E_{1,2,3,4}}_{\mu\alpha} = P_R \{ \gamma^\perp_{\mu} \gamma_\alpha, \gamma_\alpha n^\mu \bar{n} \cdot v, \gamma_\alpha v^\mu, g_{\mu\nu} \}$. Matching the current $\bar{\nu}_{\alpha} P_L b$ at tree level gives $B_1 = 1$, $B_2 = 1$, and $E_2 = 1/2$. Each of $O_{i\mu}$ and $K_{i\mu}$ count as $V^C_5 = 1$. The currents $O_{i\mu}$ were first introduced in Ref. 8, while the currents $K_{i\mu}$ are new.  

In Fig. 1(b) we show a graph contributing to the forward scattering amplitude for $B \to X_u \bar{\nu}_\ell$ with the current $O_{i\mu}$ on the left and the current $K_{i\mu}$ on the right. The remaining vertices in the graph are from the lowest order Lagrangians, except for the one labeled $\mathcal{L}^{(2)}_c$ which we take from the $\bar{n} \cdot A_{\alpha\mu}$ in the $(\bar{n} \cdot iD)$ term in Eq. (8). Now for the loop graph in Fig. 1(b) we have $V^C_4 = 1$, $V^C_5 = 2$, $V^U_8 = 1$, and for the $\mathcal{L}^{(2)}_c$ vertex $V^C_6 = 1$ so Eq. (5) gives $\delta = 4 + 2(5 - 4) + (6 - 4) = 8$. Thus, due to the insertion of a vertex from $\mathcal{L}^{(2)}_c$ and the subleading currents the graph is suppressed by $\Lambda^4$ relative to leading order diagrams. In contrast with the direct power counting method we have

$$\lambda^6 \left[ \lambda^8 \times \frac{1}{\lambda^2} \lambda^4 \times \lambda^2 \right] \left[ \lambda^4 \times \left( \frac{1}{\lambda^2} \right)^4 \times \lambda^0 \right] \left[ \lambda^1 \lambda^1 \right] = \lambda^8.$$

(17)

2 In Ref. 8 it was shown that reparameterization invariance (RPI) uniquely fixes the $B_i$’s in terms of $C_i$’s (this was referred to as type-I RPI in Ref. 8). It is easy to show that the $K^\mu_i$ currents are not connected to $J^\mu_i$ by type-I RPI since $\delta_i (\mathcal{P}_\perp + g A_{n,q}^{\perp}) W \propto \bar{n} \cdot D_W = 0$ so that $\delta_i K_{i\mu} = \mathcal{O}(\lambda^6)$.
Here the first term counts the dimension of the external heavy quark fields. The factors in the first square bracket are the measure, propagators, and vertices respectively. In the second square bracket we give the λ factors for the collinear loop measure, four collinear propagators, and collinear vertices (in Feynman gauge). In the final square bracket we have the powers of λ from the currents. The total result δ = 8 is the same as with the vertex formula.

The last diagram in Fig. 1(c) involve the weak flavor changing operator for the non-leptonic decay $B \to D\pi$. At leading power the external operator in SCET has the form

$$Q_{\{0,8\}} = \left( h_{v'}^c ST_h \{1, T^A\} S^\dagger h_v^b \right) \left( \xi^{(d)}_{n,p} W C_{\{0,8\}} (\not P, \not P^\dagger) \Gamma_\ell \{1, T^A\} W^\dagger \xi^{(u)}_{n,p} \right),$$

where $h_{v'}$ and $h_v$ are soft HQET fields and $\Gamma_{h,\ell}$ are spin structures. From Table I we see that $Q_{\{0,8\}} \sim \lambda^5$, and because of the presence of both soft and collinear fields it counts as $V_5^{SC} = 1$. Thus, δ = 5 is the base λ-dimension for this process. In the three loop graph in Fig. 1(c) all interactions except $Q_0$ are taken from the lowest order Lagrangians. Here the advantage of the vertex power counting is more clear. Applying Eq. (13) to this graph we see that $V_5^{SC} = 1$, $V_4^C = 3$, and $V_4^S = 4$ so that δ = 4 + 1 = 5 and the graph is leading order (since $Q_1 \sim \lambda^5$). The direct counting is more involved giving

$$\lambda^5 \left\{ \frac{\lambda^{3/2}}{\lambda^2} \right\} \left[ \lambda^4 \times \frac{1}{(\lambda)^2 (\lambda^2)^2} \times \lambda \right] \left[ (\lambda^4)^2 \times \left( \frac{1}{\lambda^2} \right)^5 \times \lambda^2 \right] = \lambda^5.$$

Here the first term counts the dimension of the external heavy quark fields and collinear quark fields. The term in curly brackets counts powers of λ from the light soft spectator quark lines and the soft gluon propagator that does not participate in a loop. The factors in the first square bracket are the measure, propagators, and vertices for the soft loop. In the final square bracket we give the λ factors for the measures, propagators, and vertices in the two collinear loops (in Feynman gauge). The final δ = 5 result is of course the same. Note that the diagram in Fig. 1(c) is also order δ = 5 without the spectator interaction. With the vertex power counting this is obvious since we simply have fewer soft vertices ($V_4^S = 2$) which does not affect the result for δ. With the direct power counting we must go back and adjust the analysis in Eq. (19) to find this result.

From Eq. (3) it should be clear that the complete set of leading order graphs can be constructed by simply adding any number of $V_4^C$, $V_4^S$, and $V_8^U$ interactions. This is in fact the true strength of this power counting formula, it makes determining the set of all graphs that contribute at a given order quite simple.

**Application 2: Counting powers of Q**

As our second application we show how Eq. (3) plus dimensional analysis can be used to determine the power of Q of matrix elements in SCET. Essentially, a power of Q is assigned by dimensional analysis and then the power of $\lambda = \Lambda_{QCD} / Q$ determines how many Q's are turned into factors of $\Lambda_{QCD}$. Thus, in general a matrix element $\langle O \rangle$ of mass dimension d is order $Q^q$, where $q = d - \delta$ and $\delta$ counts the λ-dimension of the operator and states. We use relativistic normalization for our states, $\langle p' | p \rangle = 2p^0 \delta^3 (p - p')$, and find that collinear protons and pions have $| p_n \rangle \sim | \pi_n \rangle \sim Q^{-1} \lambda^{-1}$ while soft B or D mesons have $| B_v \rangle \sim | D_v \rangle \sim Q^{-1} \lambda^{-3/2}$.  

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For DIS, the operator in Eq. (4) involves \( \xi \sim Q^{3/2} \lambda \) and \( W \sim Q^0 \lambda^0 \), plus a dimensionless Wilson coefficient, so \( O_1 \sim Q^2 \lambda^2 \). Counting powers in the collinear proton matrix element

\[
\langle p_n|O_1|p_n \rangle \sim [Q^{-1} \lambda^{-1}] [Q^2 \lambda^2] [Q^{-1} \lambda^{-1}] = Q^0 \lambda^0 ,
\]

(20)

so the result is dimensionless. This agrees with the fact that the proton matrix element of \( O_1 \) is simply a convolution of a dimensionless hard coefficient and the quark parton distribution function.

For \( B \to D\pi \) the operator \( Q_0 \) involves \( \xi \), \( W \), and \( h_v \sim Q^{3/2} \lambda^{3/2} \) fields, and scales as \( Q_0 \sim Q^6 \lambda^5 \). Taking the \( B \to D\pi \) matrix element then gives

\[
\langle D\pi_n|Q_1|B \rangle \sim [\lambda^{-5/2} Q^{-2}] [\lambda^5 Q^6] [\lambda^{-3/2} Q^{-1}] = Q^3 \lambda = Q^2 \Lambda_{QCD} .
\]

(21)

This agrees with the fact that this matrix element can be calculated to give a product of \( B \to D \) form factor and a convolution with the light cone pion wavefunction. In this result the dimensions are given by \( m_B E_{\pi} f_{\pi} \sim Q^2 \Lambda_{QCD} \) (see for example Ref. [5]).

The SCET formalism can also be used to give a simple derivation of the \( Q \) scaling of the hard scattering component of the electromagnetic form factor of an arbitrary hadron, \( \gamma^* h(p) \to h'(p') \). We assume that the form factor involves the hard interaction of \( k \) collinear partons in both \( h \) and \( h' \). In this case the electromagnetic current is matched onto an operator \( Q^{3-3k} C O_\mu^{(k)} \), which contains \( k \) \( \xi_n \) fields, \( k \) \( \xi_n' \) fields, and \( C \), a dimensionless Wilson coefficient. Note that \( Q^{3-3k} O_\mu^{(k)} \) has overall mass dimension 3. The matrix element of this SCET current is

\[
\frac{1}{Q^{3k-3}} \langle h_n'|O_\mu^{(k)}|h_n \rangle \sim \frac{1}{Q^{3k-3}} [Q^{-1} \lambda^{-1}] [Q^3 \lambda^{2k}] [Q^{-1} \lambda^{-1}] = Q \lambda^{2k-2} = \frac{(\Lambda_{QCD})^{2k-2}}{Q^{2k-3}} .
\]

(22)

This scaling for the matrix element of the hard scattering component of the electromagnetic current agrees with the scaling law derived by Brodsky and Farrar [10].

**Application 3: Relation to Twist in DIS**

In the application of the operator product expansion (OPE) to inclusive processes it is useful to classify contributions according to twist. Two common definitions are *geometric twist*, \( \tau = d - s \), equal to dimension minus spin for an operator, and *dynamic twist*, \( t \), defined by the \( Q \) scaling by which the matrix element of an operator contributes to the cross section [7]. At lowest order the definitions coincide, \( t = \tau = 2 \). However, beyond twist-2 one has \( t \geq \tau \), which corresponds to the fact that operators of a given geometric twist also contribute to higher orders in \( 1/Q \) in the cross section due to Wandzura-Wilczek contributions [8]. Since the \( \lambda \)-dimension, \( \delta \), of operators also determines their order in \( \Lambda_{QCD}/Q \), one might expect that \( \delta \) and twist are related. In fact, dynamic twist \( t = \delta \).

As a simple example consider the DIS operator in Eq. (4). Recall that collinear fermion fields have twist \( \tau = 3/2 - 1/2 = 1 \), which from Table I is equal to their \( \lambda \)-dimension. In the Breit frame the proton momentum \( \bar{n} \cdot p = Q/x \) and the matrix element [8]

\[
\langle p_n|O_1|p_n \rangle = \frac{1}{x} \int_x^1 d\xi \ C_1 \left( \frac{2Q\xi}{x}, 0, Q, \mu \right) \left[ f_{i/p}(\xi) + \bar{f}_{i/p}(\xi) \right] ,
\]

(23)
where \( f_{i/p}(\xi) \) and \( \bar{f}_{i/p}(\xi) \) are spin independent quark and antiquark distribution functions. These distribution functions are related to proton matrix elements of the local twist-2 operators

\[
S \left\{ \bar{\xi}_{n,p} \gamma^{\mu} i \bar{D}_{\mu}^{\mu_1} \cdots i \bar{D}_{\mu}^{\mu_k} \xi_{n,p} \right\},
\]

where \( S \) takes the symmetric traceless combination of the indices \( \nu, \mu_1, \ldots, \mu_k \). It is easy to see that the operator \( \mathcal{O}_1 \) in Eq. (24) corresponds to the \( t = 2 \) part of this \( \tau = 2 \) tower of completely local operators,

\[
\mathcal{O}_1 = \sum_{k=0}^{\infty} c_k \mathcal{O}_1^{(k)}, \quad \mathcal{O}_1^{(k)} = \frac{\bar{n}_\nu \bar{n}_{\mu_1} \cdots \bar{n}_{\mu_k}}{2Q} \bar{\xi}_{n,p} \gamma^{\nu} i \bar{D}_{\mu_1}^{\mu_1} \cdots i \bar{D}_{\mu_k}^{\mu_k} \xi_{n,p},
\]

where \( C_1(z,0,Q,\mu) = \sum_k c_k z^k \) and \( iD_{\mu}^{\mu} = \mathcal{P}^{\mu} + i\bar{\xi}^{\mu} n \cdot \partial/2 + gA_{n,q}^{\mu} \) are collinear covariant derivatives. The tensor product \( \bar{n}_\nu \bar{n}_{\mu_1} \cdots \bar{n}_{\mu_k} \) is symmetric and traceless and picks out the leading symmetric traceless part out of the local operator. Thus, we see that in this example that the twist \( t = \tau = 2 \) is identical to the \( \lambda \)-dimension of \( \mathcal{O}_1 \) which is \( \delta = 2 \).

Next consider the matrix elements of higher order DIS operators. In Application 2 it was shown that the \( \lambda \)-dimension of an operator has a direct correspondence with the power of \( Q \) in its matrix element. Thus, it is easy to see that a mass dimension-2 operator \( \mathcal{O}_1 \) with \( \lambda \) dimension \( \delta \) has \( \langle p_n | \mathcal{O}_1 | p_n \rangle \sim Q^{2-\delta} \), so that \( t = \delta \) even beyond twist-2. Beyond twist-2 the relationship between \( \delta \) with \( \tau \) is complicated by the same features that complicate the relationship between \( t \) and \( \tau \) (for a more detailed discussion see Refs. [17, 18, 19]). Thus, there is no simple correspondence between \( \lambda \)-dimension and geometric twist. Finally note that the \( \lambda \)-dimension also classifies operators in situations where there is no OPE such as for exclusive decays.

**Conclusion**

In this paper we have given a detailed derivation of the SCET vertex power counting formula in Eq. (5), and showed its equivalence with the more intuitive but more tedious direct power counting method. The vertex formula and \( \lambda \)-dimension of operators make counting powers of \( \Lambda_{QCD}/Q \) in interactions between soft and collinear particles as simple as counting powers of \( 1/m \) in HQET. Three examples highlighting the power and simplicity of the vertex power counting were then given using the processes of DIS, \( B \rightarrow X_u \ell \bar{\nu}_\ell \), and \( B \rightarrow D \pi \). As a further application we showed how the vertex power counting can be used to determine the powers of \( Q \) and \( \Lambda_{QCD} \) to associate with matrix elements and reproduced the Brodsky and Farrar result for the \( Q \) scaling of the hard scattering electromagnetic form factor of a hadron. For deep inelastic scattering we then compared the vertex power counting approach with twist power counting. Finally, we would like to emphasize that the most important utility of Eq. (5) is to counting powers of \( \lambda, Q, \) or \( \Lambda_{QCD} \) for power corrections to processes where a priori even the size of matrix elements may otherwise be unknown.

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