REMARKS ON DERIVED EQUIVALENCES OF RICCI-FLAT MANIFOLDS

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Abstract. After a finite étale cover, any Ricci-flat Kähler manifold decomposes into a product of complex tori, irreducible holomorphic symplectic manifolds, and Calabi–Yau manifolds. We present results indicating that this decomposition is an invariant of the derived category. The main idea to distinguish the derived category of an irreducible holomorphic symplectic manifold from that of a Calabi–Yau manifold is that point sheaves do not deform in certain (non-commutative) deformations of the former, whereas they do for the latter. On the way, we prove a conjecture of Călăăraru on the module structure of the Hochschild–Kostant–Rosenberg isomorphism for manifolds with trivial canonical bundle as a direct consequence of recent work by Calaque, van den Bergh, and Ramadoss.

A Calabi–Yau manifold $Z$ is a simply-connected compact Kähler manifold with trivial canonical bundle and $H^0(Z,\Omega_Z^i) = 0$ for all $0 < i < \dim Z$. Note that any Calabi–Yau manifold of dimension at least three is projective.

A holomorphic symplectic manifold $Y$ is a compact Kähler manifold which admits an everywhere non-degenerate holomorphic two-form. The canonical bundle of such a $Y$ is again trivial. We call $Y$ irreducible if it is in addition simply-connected and the holomorphic two-form is unique up to scaling. A projective holomorphic symplectic manifold will simply be called projective symplectic.

We recall the following classical decomposition theorem for Ricci-flat manifolds (see [1, 2]).

Theorem 0.1. Let $X$ be a compact Kähler manifold with $0 = c_1(X) \in H^2(X,\mathbb{R})$. Then there exists a finite étale cover $\tilde{X} \to X$ such that $\tilde{X}$ itself decomposes as

$$\tilde{X} \cong \prod_i A_i \times \prod_j Y_j \times \prod_k Z_k,$$

where the $A_i$ are simple complex tori, the $Y_j$ are irreducible holomorphic symplectic, and the $Z_k$ are Calabi–Yau varieties of dimension at least three.

We would like to answer the following:

Question 0.2. Suppose $X$ and $X'$ are two smooth complex projective varieties with $c_1(X) = 0$ and $c_1(X') = 0$ such that there exists an exact equivalence

$$\mathcal{D}^b(X) \cong \mathcal{D}^b(X')$$

between their bounded derived categories of coherent sheaves. Are then the symplectic factors and the Calabi–Yau factors of $X$ and $X'$ also derived.
equivalent, i.e.

\[ D^\flat(Y_j) \cong D^\flat(Y'_{\sigma(j)}) \quad \text{and} \quad D^\flat(Z_k) \cong D^\flat(Z'_{\tau(k)}) \]

for suitable permutations \( \sigma \) and \( \tau \)?

As only for projective manifolds the (derived) category of coherent sheaves encodes enough relevant geometric information, we restrict ourselves to this case. Clearly, in the theorem the manifold \( X \) is projective if and only if all the factors \( A_i, Y_j, \) and \( Z_k \) are projective.

To be more specific, one could first consider the following case: Suppose \( Y, Y' \) and \( Z, Z' \) are irreducible projective symplectic manifolds and Calabi–Yau manifolds, respectively. Suppose further that there exists an exact equivalence \( D^\flat(Y \times Z) \cong D^\flat(Y' \times Z') \). Does this imply \( D^\flat(Y) \cong D^\flat(Y') \) and \( D^\flat(Z) \cong D^\flat(Z') \)?

**Remark 0.3.** Allowing étale covers makes things more subtle. E.g. for \( A \) an abelian surface, one knows that the Hilbert scheme \( \text{Hilb}^3(A) \) of three points on \( A \) is derived equivalent to the Hilbert scheme \( \text{Hilb}^3(\hat{A}) \), where \( \hat{A} \) denotes the dual abelian surface (see [15]).

After an étale cover \( \text{Hilb}^3(A) \) is isomorphic to \( A \times K^2(A) \) and similarly for \( \hat{A} \). Here, \( K^2(A) \) denotes the 4-dimensional generalised Kummer variety of \( A \). However, it is unknown whether \( K^2(A) \) and \( K^2(\hat{A}) \) are derived equivalent (see [15]).

So far, we are only able to show that derived equivalence does distinguish between the three types in the decomposition theorem. More precisely, we prove the following:

**Theorem 0.4.** Let \( X \) and \( X' \) be smooth projective varieties with equivalent derived categories, i.e. there exists an exact equivalence \( D^\flat(X) \cong D^\flat(X') \). If \( X \) is either an abelian variety or an irreducible projective symplectic manifold, then \( X' \) is of the same type.

The result is proved by using particular algebraic properties of Hochschild (co)homology which are invariant under derived equivalence. Following general principles, abelian varieties and symplectic varieties will be studied by means of Hochschild cohomology of degree one (see Proposition 3.1) respectively degree two (see Theorem 4.9).

On the way, we observe that a conjecture of Căldăraru [6] can in the case of manifolds with trivial canonical bundle rather easily be proved by combining the recent articles of van den Bergh and Calaque [7] and Ramadoss [19]. Căldăraru’s conjecture asserts that Hochschild homology and Dolbeault cohomology are isomorphic as modules over Hochschild cohomology via the modified Hochschild–Kostant–Rosenberg isomorphism \( I^K : \text{HH}^*(X) \cong \text{HT}^*(X) \), which is proved to be a ring isomorphism in [7] (confirming a claim of Kontsevich in [14]). Putting things together one obtains (see Corollary 2.5):

**Theorem 0.5.** Suppose \( X \) is a manifold with trivial canonical bundle. Then the modified Hochschild–Kostant–Rosenberg isomorphism defines an isomorphism

\[ (I^K, I_K) : (\text{HH}^*(X), \text{HH}_s(X)) \cong (\text{HT}^*(X), \Omega^*_s(X)) \]
compatible with the ring structure of \( \text{HH}^*(X) \) and \( \text{HT}^*(X) \) and the module structure of \( \text{HH}_*(X) \) and \( \text{HR}_*(X) \).

Working more locally, the proof should also lead to a proof of Căldăraru’s conjecture for manifolds with non-trivial canonical bundle, but this shall be pursued elsewhere.

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1. Hochschild (co-)homology

This section contains only well-known material presented in a form ready for later use. The general references are [5, 7, 14]. In Section 2 we discuss an open conjecture of Căldăraru on the module structure of Hochschild homology under the Hochschild–Kostant–Rosenberg isomorphism and prove special cases of it which are crucial for the proof of our main result.

1.1. Hochschild (co-)homology. Let \( X \) be a smooth projective variety of dimension \( n \). By \( \iota: X \xrightarrow{\sim} \Delta \subset X \times X \), we denote the diagonal embedding. One defines the Hochschild cohomology of \( X \) by

\[
\text{HH}^*(X) := \bigoplus_i \text{HH}^i(X)[-i] \quad \text{with} \quad \text{HH}^i(X) := \text{Ext}^i_{X \times X}(\iota_* \mathcal{O}_X, \iota_* \mathcal{O}_X)
\]

and the Hochschild homology by

\[
\text{HH}_*(X) := \bigoplus_i \text{HH}_i(X)[i] \quad \text{with} \quad \text{HH}_i(X) := \text{Ext}^{-i}_{X \times X}(\iota_* \omega_X^\vee[-n], \iota_* \mathcal{O}_X).
\]

**Remark 1.1.** There are other conventions on the grading of Hochschild homology in the literature. The grading used in this paper is analogous to the homological and the cohomological grading in algebraic topology. Moreover, the Mukai pairing becomes homogeneous of degree zero in this setting.

The Yoneda product, i.e. composition in the derived category, endows \( \text{HH}^*(X) \) with the structure of a graded \( \mathbb{C} \)-algebra and \( \text{HH}_*(X) \) becomes a module over it. The product in Hochschild cohomology shall be denoted by

\[
\cup: \text{HH}^p(X) \otimes \text{HH}^q(X) \to \text{HH}^{p+q}(X)
\]

and the module structure by

\[
\cap: \text{HH}^p(X) \otimes \text{HH}_q(X) \to \text{HH}_{q-p}(X).
\]

1.2. Polyvector fields. A similar structure can be defined by using polyvector fields and forms. Let \( \Omega_{X,*} \) denote the complex with trivial differentials and \( \Omega^i_X \) in degree \(-i\), i.e.

\[
\Omega_{X,*} := \bigoplus_q \Omega^q_X[q],
\]

and let \( \wedge^* T_X \) be its dual:

\[
\wedge^* T_X := \bigoplus_q \wedge^q T_X[-q] \cong \Omega^\vee_{X,*}.
\]
The cup-product defines a ring structure on the graded \(\mathbb{C}\)-algebra

\[
\mathrm{HT}^*(X) := \bigoplus \mathrm{HT}^i(X)[-i] \quad \text{with} \quad \mathrm{HT}^i(X) := \bigoplus_{p+q=i} H^p(X, \wedge^q T_X)
\]

and contraction with polyvector fields endows

\[
\mathrm{H}\Omega_*(X) := \bigoplus \mathrm{H}\Omega_i(X)[i] \quad \text{with} \quad \mathrm{H}\Omega_i(X) := \bigoplus_{q-p=i} H^p(X, \Omega_X^q)
\]

with the structure of a \(\mathrm{HT}^*(X)\)-module. We shall write these two operations as

\[
\wedge : \mathrm{HT}^p(X) \otimes \mathrm{HT}^q(X) \to \mathrm{HT}^{p+q}(X)
\]

and

\[
\iota : \mathrm{HT}^p(X) \otimes \mathrm{H}\Omega_q(X) \to \mathrm{H}\Omega_{q-p}(X).
\]

1.3. The Hochschild–Kostant–Rosenberg isomorphism. There is a unique natural isomorphism

\[
I : \iota^* \iota_* \mathcal{O}_X \to \Omega_{\ast, \ast}
\]

in \(\mathcal{D}^\mathbb{P}(X)\) such that its adjoint morphism \(\iota_* \mathcal{O}_X \to \iota_* \Omega_{\ast, \ast}\) coincides with the exponential of the universal Atiyah class \(\alpha : \iota_* \mathcal{O}_X \to \iota_* \Omega_{\ast}[1]\) of \(X\) (see \([\mathbb{2}]\) or \([\mathbb{4}]\)).

Consider the adjunctions \(\iota^* \dashv \iota_*\) and \(\iota_! \dashv \iota^*\), where \(\iota_! (F) = \iota_* (F \otimes \omega_X^\wedge [-n])\). Using those, the isomorphism \(I\) yields the two Hochschild–Kostant–Rosenberg isomorphisms

\[
I_{\text{HKR}}^\mathbb{H} : \mathbb{H}\mathbb{H}(X) = \text{Ext}_X^\mathbb{H} (\iota_* \mathcal{O}_X, \iota_* \mathcal{O}_X) \cong \text{Ext}^\mathbb{H}_X (\iota^* \iota_* \mathcal{O}_X, \mathcal{O}_X)
\]

and

\[
I_{\text{HKR}} : \mathbb{H}\Omega_i(X) = \text{Ext}_X^{-i} (\iota_* \omega_X^\wedge [-n], \iota_* \mathcal{O}_X) \cong \text{Ext}^{-i}_X (\mathcal{O}_X, \iota^* \iota_* \mathcal{O}_X)
\]

\[
\cong \text{Ext}^{-i}_X (\mathcal{O}_X, \Omega_{\ast, \ast}) = \bigoplus H^{-i+j}(X, \Omega_X^j) = \mathrm{H}\Omega_j(X).
\]

In general, these two morphisms respect neither the algebra nor the module structure. In order to compare \((\cup, \cap)\) with \((\wedge, \iota)\), the original HKR-isomorphisms have to be modified as follows. Let \(a : \hat{A}^\mathbb{H}_i \in \mathbb{H}\Omega_0(X)\), where \(\hat{A} = \hat{A}(X)\) is the characteristic class on \(X\) corresponding to the \(\hat{A}\)-genus. Then consider the induced graded isomorphisms

\[
a^{-1} \iota : \mathrm{HT}^*(X) \to \mathrm{HT}^*(X), \quad v \mapsto a^{-1} \iota v
\]

and

\[
a \wedge : \mathbb{H}\Omega_*(X) \to \mathbb{H}\Omega_*(X), \quad \alpha \mapsto a \wedge \alpha.
\]

Here, \(\iota\) denotes the contraction of a polyvector field by a form

\[
\iota : \mathbb{H}\Omega_{-p}(X) \otimes \mathrm{HT}^q(X) \to \mathrm{HT}^{p+q}(X), \quad (\alpha, v) \mapsto \alpha \iota v
\]

and \(\wedge\) denotes the map

\[
\wedge : \mathbb{H}\Omega_{-p}(X) \otimes \mathbb{H}\Omega_q(X) \to \mathbb{H}\Omega_{p+q}(X), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta
\]

induced by the exterior product of forms. Earlier we used the contraction of a form \(\alpha\) by a polyvector field \(v\), which was denoted by \(v \iota \alpha\). The order
of the arguments should avoid any confusion. We have to stress, however, that we adopt the following sign convention for the inner product: For an odd vector space $V$ and vectors $v, w \in V^\vee$ one has $v \cdot w = -w \cdot v$, that is $\cdot$ shall be symmetric in the graded sense.

Following Kontsevich (14), one then defines the following modified HKR-isomorphisms

$$I^K := (a^{-1} \cdot ) \circ I^{\text{HKR}}: HH^*(X) \xrightarrow{\sim} HT^*(X)$$

and

$$I_K := (a \wedge \cdot ) \circ I^{\text{HKR}}: HH_*(X) \xrightarrow{\sim} H\Omega_*(X).$$

By 14 and 7 one knows that $I^K: HH^*(X) \sim HT^*(X)$ is an isomorphism of $\mathbb{C}$-algebras, i.e. $I^K(v_1 \cup v_2) = I^K(v_1) \wedge I^K(v_2)$ for all $v_1, v_2 \in HT^*(X)$.

In Section 2 we will comment on the multiplicativity of $I_K$.

**Remark 1.2.** Recall that $\hat{A}(X)$ is obtained from the power series $x/(e^{x}/2 - e^{-x})/2)$, whereas one uses $x/(1 - e^{-x})$ for the Todd-genus $\text{td}(X)$. Hence $\text{td}(X) = \exp(c_1(X)/2) \wedge \hat{A}(X)$. Since inner product with $c_1(X)$ acts as a derivation on the algebra $HT^*(X)$, the modified HKR-isomorphism $I^K$ is multiplicative if and only if $(\exp(t \cdot c_1(X)) \wedge a) \cdot ) \circ I^{\text{HKR}}$ is multiplicative, where $t \in \mathbb{C}$. Thus, one could as well use $a = \sqrt{\text{td}(X)}$ for the definition of $I^K$.

Similar arguments apply to the definition of $I_K$ and its conjectured multiplicativity (see Section 2).

1.4. Hochschild (co-)homology and Chern characters under derived equivalence. Suppose

$$\Phi: \mathcal{D}^b(X_1) \sim \mathcal{D}^b(X_2)$$

is an exact equivalence of two smooth projective varieties $X_1$ and $X_2$. Due to results of Căldăraru [5] and Orlov [17] (cf. [11, Ch. 6]), one knows that $\Phi$ induces an isomorphism of $\mathbb{C}$-algebras

$$\Phi^{\text{HH}^*}: HH^*(X_1) \sim HH^*(X_2)$$

and an isomorphism of $HH^*(X_1)$-modules (via $\Phi^{\text{HH}^*}$)

$$\Phi^{\text{HH}_*}: HH_*(X_1) \sim HH_*(X_2).$$

Under the same assumptions, the following diagram commutes (see [5]):

$$\begin{array}{ccc}
\mathcal{D}^b(X_1) & \xrightarrow{\Phi} & \mathcal{D}^b(X_2) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
H\Omega_0(X_1) & \xrightarrow{I_{\text{HKR}}^{-1}} & H\Omega_0(X_2) \\
I_{\text{HKR}}^{-1} \downarrow & & \downarrow I_{\text{HKR}}^{-1} \\
HH_0(X_1) & \xrightarrow{\phi_{\text{HH}_*}} & HH_0(X_2).
\end{array}$$

In fact, the composition $I_{\text{HKR}}^{-1} \circ \text{ch}$, i.e. the Hochschild–Chern character, admits a different description in terms of Fourier–Mukai transforms, but we shall not need this (see [5]).
2. The multiplicative structure of Hochschild homology

In [6] Căldăraru conjectured that the modified HKR-isomorphism $I_K$ is an isomorphism of $\text{HH}^*(X)$-modules, where the $\text{HT}^*(X)$-module $\Omega_*(X)$ is viewed as an $\text{HH}^*(X)$-module via $I_K$. In the proof of the main result (Theorem 4.9) a special case of this conjecture is used. So, as the general conjecture is still open, we will have to provide an alternative argument for this point and in fact we will give two.

Firstly, we will show how the recent paper of Ramadoss [19] can be used to establish Căldăraru’s conjecture for varieties with trivial canonical bundle. Secondly, as [19] is technically involved and for the general benefit of having an alternative proof, we shall also explain a more direct approach showing multiplicativity of $I_K$ for classes $v \in \text{HH}^2(X)$ and algebraic classes $\beta \in \text{HH}^0(X)$ (under additional assumptions on the manifold). Fortunately, this special case is sufficient for the proof of Theorem 4.9. Moreover, the argument does apply also to other manifolds whose canonical bundle is not necessarily trivial. This justifies further the inclusion of the second argument, which relies on graph homology and the wheeling theorem.

2.1. $I_K$ for varieties with trivial canonical bundle. Let us first prove that [19] does imply Căldăraru’s conjecture for varieties with trivial canonical bundles or, more general, for certain submodules $\text{HH}'_s \subset \text{HH}_s$ respectively $\Omega'_s \subset \Omega_s$. Let

$$\Omega'_s(X) \subset \Omega_s(X)$$

be the $\text{HT}^*(X)$-submodule generated by the subspace $\bigoplus_i H^i(X, \Omega^n_X)$. Similarly, we define

$$\text{HH}'_s(X) \subset \text{HH}_s(X)$$

to be the $\text{HH}^*(X)$-submodule generated by the subspace

$$I^{-1}_{\text{HKR}} \left( \bigoplus_i H^i(X, \Omega^n_X) \right) = I^{-1}_K \left( \bigoplus_i H^i(X, \Omega^n_X) \right).$$

**Proposition 2.1.** The map $I_K$ defines an isomorphism of $\text{HH}'_s(X)$-modules

$$I_K : \text{HH}'_s(X) \cong \Omega'_s(X),$$

where the $\text{HT}^*(X)$-module $\Omega'_s(X)$ is viewed as an $\text{HH}^*(X)$-module via $I_K$.

**Proof.** We have to show that for any $\alpha \in \text{HH}'_s(X)$ the following diagram commutes

$$\begin{array}{ccc}
\text{HH}^p(X) & \xrightarrow{\cap \alpha} & \text{HH}'_{q-p}(X) \\
I_K \downarrow & & \downarrow I_K \\
\text{HT}^p(X) & \xrightarrow{J_K(\alpha)} & \Omega'_q(X).
\end{array}$$

Suppose we have shown this already for all $\alpha_0$ with $I_K(\alpha_0) \in H^i(X, \Omega^n_X)$. Then the general result follows from the following computation for a class
of the form $\alpha = v_0 \cap \alpha_0$ with $I_K(\alpha_0) \in H^i(X, \Omega^n_X)$ and any $v \in \text{HH}^p(X)$:

$$I_K(v \cap \alpha) = I_K(v \cap (v_0 \cap \alpha_0)) = I_K((v \cup v_0) \cap \alpha_0)$$

$$= I_K^t(v \cup v_0) \cup I_K(\alpha_0) \supseteq (I_K(v) \cup I_K^t(v_0)) \cup I_K(\alpha_0)$$

$$= I_K(v) \cup (I_K(v_0) \cup I_K(\alpha_0)) \supseteq I_K(v) \cup I_K(v_0 \cap \alpha_0)$$

$$= I_K(v) \cup I_K(\alpha).$$

Here ($\ast$) is due to the multiplicativity of $I_K^t$ (see [14] and [7]) and ($\supseteq$) follows from the assumed commutativity of (3) for $\alpha_0$.

Thus, it remains to prove the commutativity of (3) for all $\alpha \in \text{HH}^*(X)$ with $I_K(\alpha) \in H^i(X, \Omega^n_X)$.

But before, let us revisit the definition of the HKR-isomorphism for Hochschild cohomology. By construction, $I_{\text{HKR}}$ is given by composing the adjunction morphism $\text{Ext}_{X \times X}^{-i}(t_*\omega_X^v[-n], t_*\Omega_X) = \text{Ext}_{X \times X}^{-i}(t_*\Omega_X, t_*\omega_X) \cong \text{Ext}_{X}(\omega_X^v, t_*\omega_X) \cong \text{Ext}_{X}(\omega_X^v[-n], \Omega_X)$ in the second variable with $H^* \colon t_*\Omega_X \cong \Omega_{X,*}$. A priori, one could also apply the adjunction $\iota^* \cong \iota_*$ and composite with $I$ in the first factor (analogously to the definition of the HKR-isomorphism for Hochschild cohomology). More precisely, one could compose the adjunction morphism $\text{Ext}_{X \times X}^{-i}(t_*\omega_X^v[-n], t_*\Omega_X) \cong \text{Ext}_{X}(\omega_X^v, t_*\omega_X) \cong \text{Ext}_{X}(\omega_X^v[-n], \Omega_X)$ with $I$ to obtain $\overline{I}_{\text{HKR}} : \text{HH}_i(X) \cong H^{-i}(X, \Omega^n_X) \cong \text{HH}_i(X)$ differ by the Todd genus. This is the main result of [19]. More precisely, following Ramadoss one has

$$\text{td}(X) \cap I_{\text{HKR}} = \overline{I}_{\text{HKR}}.$$ 

Note that there is an extra sign operator in [19] which is caused by a different sign convention for the isomorphism $R$ and thus $\overline{I}_{\text{HKR}}$.

In particular

$$\text{pr}_n \circ I_{\text{HKR}} = \text{pr}_n \circ \overline{I}_{\text{HKR}},$$

where $\text{pr}_n : \text{HH}_i(X) = \bigoplus_{q-p=i} H^p(X, \Omega^n_X) \to H^{n-i}(X, \Omega^n_X)$ denotes the projection.

Back to the commutativity of (3) for classes in $H^i(X, \Omega^n_X)$. First observe that $H^i(X, \Omega^n_X) \cong \text{Ext}_{X}^i(\omega^n_X, \Omega_X) = \text{Ext}_{X}^{-i}(\omega^n_X[-n], \Omega_X)$ embeds via the diagonal into $\text{HH}_i(X)$, i.e. there is a natural map

$$t_* : H^i(X, \Omega^n_X) \hookrightarrow \text{HH}_i(X).$$

Moreover, it is easy to check that the composition of $\overline{I}_{\text{HKR}} \circ t_* : H^i(X, \Omega^n_X) \to H\Omega_{n-i}(X)$ with the projection $H\Omega_{n-i}(X) \to H^i(X, \Omega^n_X)$ is the identity. Hence, $\alpha = t_*(I_K(\alpha))$ for $I_K(\alpha) \in H^i(X, \Omega^n_X)$ and $I_K(t_*\beta) = \beta$ for any $\beta \in H^i(X, \Omega^n_X)$.

The HKR-isomorphism $I : t_*\Omega_X \cong \Omega_{X,*}$ yields the isomorphisms

$$I(\mathcal{F}) : \text{Ext}_{X}(t_*\mathcal{F}, t_*\Omega_X) \cong \text{Ext}_{X}(\omega_X^v[-n], \mathcal{F}, \Omega_X) \cong \text{Ext}_{X}(\Omega_{X,*} \otimes \mathcal{F}, \Omega_X)$$
which are functorial in $\mathcal{F}$ (as object on $X$). For $\mathcal{F} = \mathcal{O}_X$ this is the isomorphism $j^{\HKR}$ and for $\mathcal{F} = \omega_X^\vee[-n]$ it is $\tilde{I}^{\HKR}$. We continue to use the natural isomorphism $R: \Ext^X_\mathcal{O}_X(\Omega_X, \omega_X^\vee[-n]) \cong H^*(X, \wedge^* T_X \otimes \omega_X[n]) \cong \Omega_{\mathcal{O}_X}(X)$ as in the definition of $\tilde{I}^{\HKR}$ above.)

By functoriality of $I(\mathcal{F})$ applied to a morphism $\beta: \omega_X^\vee[-n] \to \mathcal{O}_X[i - n]$ induced by a class $\beta \in H^*(X, \Omega_X^\vee)$, we obtain

$$I^{\HKR}(v \cap \iota_\ast \beta) = I^{\HKR}(v \circ \iota_\ast \beta) = I^{\HKR}(v) \cup \beta$$

for all $v \in \HH^*(X)$ (our definition of $R$ is chosen exactly in a way such that this formula holds without any additional signs).

Write a class $\beta$ of maximal holomorphic degree as $\beta = I^{\HKR}(\alpha) = \tilde{I}^{\HKR}(\alpha)$. Then apply Lemma 2.2 below to $w = I^{\HKR}(v) \in H^*(X, \wedge^j T_X)$ to obtain:

$$I_K(v \cap \iota_\ast \beta) = a \wedge I^{\HKR}(v \cap \iota_\ast \beta)$$

$$= \exp(-c_1(X)/2) \wedge a^{-1} \wedge \td(X) \wedge I^{\HKR}(v \cap \iota_\ast \beta)$$

$$= \exp(-c_1(X)/2) \wedge (a^{-1} \wedge \tilde{I}^{\HKR}(v \cap \iota_\ast \beta))$$

$$= \exp(-c_1(X)/2) \wedge (a^{-1} \curlywedge (I^{\HKR}(v) \cup \beta))$$

$$= \exp(-c_1(X)/2) \wedge ((a^{-1} \cup I^{\HKR}(v)) \cup \beta)$$

$$= \exp(-c_1(X)/2) \wedge (I_K(v) \cup I_K(\iota_\ast \beta))$$

$$= \exp(-c_1(X)/2) \wedge (I_K(v) \cup I_K(\iota_\ast \beta)).$$

This proves the assertion up to the additional factor $\exp(-c_1(X)/2)$. To conclude, it suffices to show that

$$c_1(X) \wedge (w \cup \gamma) = 0$$

whenever $\gamma$ has maximal holomorphic degree, i.e. $\gamma \in H^*(X, \Omega_X^\vee)$. Using the compactness of $X$, (4) is equivalent to

$$\int_X (c_1(X) \wedge (w \cup \gamma)) \wedge \delta = 0$$

for all $\delta \in \Omega_\ast(X)$. Applying Lemma 2.2 again, one finds

$$(c_1(X) \wedge (w \cup \gamma)) \wedge \delta = \pm (c_1(X) \wedge \delta) \wedge (w \cup \gamma)$$

$$= \pm ((c_1(X) \wedge \delta) \cup w) \cup \gamma$$

$$= \pm (c_1(X) \cup (\delta \cup w)) \cup \gamma.$$

Due to the assumption that $\gamma \in H^*(X, \Omega_X^\vee)$, only those $\delta$ with $\delta \cup w \in H^*(X, T_X)$ contribute to (5). Then apply Lemma 2.3 below to conclude. \qed

**Lemma 2.2.** Let $\beta \in H^*(X, \omega_X) \subseteq \HH^*(X)$ be a form of maximal holomorphic degree. Then

$$(-1)^{\ell}(\beta' \cup w) \cup \beta = \beta' \wedge (w \cup \beta)$$

for all $w \in \HT^*(X)$ and all $\beta' \in H^*(X, \Omega_X^\vee)$. 
The lemma clearly follows from the following easy fact in linear algebra: Let $V$ be an $n$-dimensional vector space and let $\beta \in \wedge^n V^\vee$, $\beta' \in \wedge^{\ell} V^\vee$, and $w \in \wedge^i V$. Then

$$(-1)^\ell (\beta ' \lrcorner w) \lrcorner \beta = \beta ' \lrcorner (w \lrcorner \beta).$$

As the equation is linear in $\beta$, $\beta'$, and $w$, we may assume that $\beta' = e^{k-\ell+1} \wedge \ldots \wedge e^k$, $\beta = e^1 \wedge \ldots \wedge e^n$, and $w = e_k \wedge \ldots \wedge e_2 \wedge e_1$. Here $(e_j)$ is a basis of $V$ and $(e^j)$ is the dual basis. The verification of (6) for this choice is straightforward and left to the reader, but remember the sign convention for the inner product on page 15.

**Lemma 2.3.** For any complex manifold $X$ the contraction with the first Chern class of $X$

$$c_1(X) \lrcorner : H^*(X, T_X) \to H^{*+1}(X, \mathcal{O}_X)$$

is trivial.

**Proof.** The assertion can be proved by using the curvature as the Dolbeault representative of the Atiyah class $A(T_X) \in H^1(X, \mathcal{E}nd(T_X) \otimes \Omega_X)$ or, more algebraically, by using the naturality of the Atiyah class as follows: Any morphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ of complexes yields a commutative diagram

$$\begin{array}{ccc}
\mathcal{E}_1 & \xrightarrow{A(E_1)} & \mathcal{E}_1 \otimes \Omega_X[1] \\
\varphi \downarrow & & \downarrow \varphi \otimes \text{id} \\
\mathcal{E}_2 & \xrightarrow{A(E_2)} & \mathcal{E}_2 \otimes \Omega_X[1].
\end{array}$$

(7)

Applied to a class $v \in H^i(X, T_X)$ viewed as a morphism $\varphi_v : \mathcal{O}_X[-i] \to T_X$, it shows that the composition

$$\mathcal{O}_X[-i] \xrightarrow{\varphi_v} T_X \xrightarrow{A(T_X)} T_X \otimes \Omega_X[1]$$

is trivial, for $A(\mathcal{O}_X) = 0$.

As an element in $H^{i+1}(X, T_X \otimes \Omega_X)$ this trivial composition can also be computed as the contraction of $A(T_X) \in H^1(X, \mathcal{E}nd(T_X) \otimes \Omega_X) = H^1(X, (T_X \otimes \Omega_X) \otimes \Omega_X)$ by $v \in H^i(X, T_X)$ in the second factor of the product $(T_X \otimes \Omega_X) \otimes \Omega_X$. However, the Atiyah class $A(T_X)$ is symmetric in $\Omega_X$, i.e. $A(T_X) \in H^1(X, T_X \otimes S^2(\Omega_X))$ (see e.g. [13]), and hence also the contraction of $A(T_X)$ by $v$ in the third factor of $T_X \otimes \Omega_X \otimes \Omega_X$ is trivial. Taking the trace yields

$$0 = \text{tr}(A(T_X) \circ \varphi_v) = \text{tr}(v \lrcorner A(T_X)) = v \lrcorner c_1(X),$$

which is equivalent to the assertion.

**Remark 2.4.** That the contraction of $H^i(X, T_X)$ with $c_1(X)$ vanishes for $i = 0, 1$ is obvious, as it simply says that $c_1(X)$ stays of type $(1, 1)$ under infinitesimal automorphisms and deformations. The general case is geometrically less obvious. Note that the contraction by $c_1(X)$ of elements in $H^*(X, \wedge^i T_X)$ for $i > 1$ will in general be non-trivial.
Although in general different, there are interesting cases when \( H\Omega_2^f(X) = H\Omega_e(X) \) and \( HH_2^f(X) = HH_e(X) \). E.g., if the canonical bundle \( \omega_X = \Omega_X^\natural \) is trivial, then \( H^0(X, \Omega_X^\natural) \) generates the \( HT^*(X) \)-module \( H\Omega_e(X) \) and the image of
\[
\iota_* : H^0(X, \Omega_X^\natural) = \text{Hom}(\omega_X^\vee, \mathcal{O}_X) \hookrightarrow \text{Ext}^{-n}(\iota_* \omega_X^\vee[-n], \iota_* \mathcal{O}_X) = HH_n(X)
\]
generates the \( HH^*(X) \)-module \( H\Omega_e(X) \). In this case, the proposition yields

**Corollary 2.5.** Suppose \( \omega_X \cong \mathcal{O}_X \). Then the isomorphism
\[
I_K : HH_e(X) \xrightarrow{\sim} H\Omega_e(X)
\]
is an isomorphism of \( HH^*(X) \)-modules, where the \( HT^*(X) \)-module \( H\Omega_e(X) \) is viewed as an \( HH^*(X) \)-module via \( I^K \).

### 2.2. Multiplicativity of \( I_K \) for algebraic classes.
As before, we denote by
\[
\alpha = \alpha_X : \iota_* \mathcal{O}_X \to \iota_* \Omega_X[1]
\]
the universal Atiyah class of \( X \). Composition and projection onto the symmetric algebra of \( \Omega[1] \) yield morphisms \( \alpha^n : \iota_* \mathcal{O}_X \to \iota_* \Omega_X^n[n] \) and thus the exponential of the Atiyah class
\[
\exp(\alpha) : \iota_* \mathcal{O}_X \to \iota_* \Omega_X^n.
\]
See as a morphism between Fourier–Mukai kernels, \( \exp(\alpha) \) defines a natural transformation from the identity functor to the functor given by tensoring with \( \Omega_X^n \). Applied to \( \mathcal{F} \) it yields
\[
\exp(\alpha)(\mathcal{F}) = q_*(p^* \mathcal{F} \otimes \exp(\alpha)) : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^n,
\]
whose trace is the Chern character of \( \mathcal{F} \)
\[
\text{ch}(\mathcal{F}) = \text{tr}_\mathcal{F}[q_*(p^* \mathcal{F} \otimes \exp(\alpha))]: \mathcal{O}_X \to \Omega_X^n.
\]
Here \( \text{tr}_\mathcal{F} \) denotes taking the trace with respect to \( \mathcal{F} \).

**Proposition 2.6.** Suppose that either \( H^0(X, \bigwedge^2 \mathcal{T}_X) = 0 \) or that \( X \) is irreducible holomorphic symplectic. Let \( \beta \in HH_0(X) \) be an algebraic class (that is a class that is mapped to \( \text{ch}(\mathcal{F}) \) under \( I_{\text{HKR}} : HH_e(X) \to HH_e(X) \) for some \( \mathcal{F} \in \mathcal{D}^b(X) \)). Then
\[
I_K(u \cap \beta) = I^K(u) \cup I_K(\beta)
\]
for all \( u \in HH^2(X) \).

**Proof.** (The following proof has been inspired by the proof of [2, Thm. 4.5].)

Set
\[
v := I_{\text{HKR}}(u) \in HT^2(X) = H^2(X, \mathcal{O}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^0(X, \bigwedge^2 \mathcal{T}_X).
\]
As before we let \( I : \iota^* \iota_* \mathcal{O}_X \to \Omega_X^n \) denote the quasi-isomorphism [2]. Then \( v \circ I \in \text{Ext}^*_X(\iota^* \iota_* \mathcal{O}_X, \mathcal{O}_X) \) is the morphism corresponding to \( u \) under the adjunction \( \iota^* \dashv \iota_* \), i.e.
\[
\iota_* (v \circ I) \circ \eta = u,
\]
where \( \eta : \iota_* \mathcal{O}_X \to \iota_* \iota^* (\iota_* \mathcal{O}_X) \) is the natural adjunction morphism.

With these notations, the class \( \exp(\alpha)(\mathcal{F}) \) is given by
\[
I \circ q_*(p^* \mathcal{F} \otimes \eta) \in \text{Ext}^*_X(\mathcal{F}, \mathcal{F} \otimes \Omega_X^n).
\]
Let us denote by \((u \circ \beta)' \in \mathrm{Ext}^2_X(\mathcal{O}_X, \tau^*\tau_*\mathcal{O}_X)\) the element that corresponds to \(u \cap \beta = u \circ \beta \in \mathrm{HH}_{-2}(X) = \mathrm{Ext}^2_X(\mathcal{O}_X, \tau_*\mathcal{O}_X)\) under the adjunction \(\tau^* \dashv \tau_*\). (Recall, \(u_!(\mathcal{F}) = \tau_!(\mathcal{F} \otimes \omega_X^[-n])\) or, alternatively, \(u_! = S_{X \times X}^{-1} \circ \tau_* \circ S_X\), where \(S_Y\) denotes the Serre functor on \(Y\).)

Let \(\mu' \in \mathrm{Ext}^*_X(\tau^*\tau_*\mathcal{O}_X, \omega_X[1])\) be arbitrary and

\[
\mu = \tau_*\mu' \circ \eta \in \mathrm{Ext}^*_X(\tau_*\mathcal{O}_X, \tau_*\omega_X[1])
\]

the corresponding element under the adjunction \(\tau_* \dashv \tau_*\).

By definition of \(\tau_!\) via Serre duality on \(X\) and \(X \times X\), one has

\[
\int_X \mu' \circ (u \circ \beta)' = \int_{X \times X} \mu \circ u \circ \beta.
\]

It is \(\beta = I_{\text{HKR}}^1(\mathrm{ch}(\mathcal{F}))\) the Hochschild–Chern character of \(\mathcal{F}\) (see [6]). By definition,

\[
\int_{X \times X} \mu \circ u \circ \beta = \int_X \mathrm{tr}_F[q_*(p^*\mathcal{F} \otimes (\mu \circ u))].
\]

One has

\[
\int_X \mu' \circ (u \circ \beta)' = \int_{X \times X} \mu \circ u \circ \beta
= \int_X \mathrm{tr}_F[q_*(p^*\mathcal{F} \otimes (\mu \circ u))]
= \int_X \mathrm{tr}_F[q_*(p^*\mathcal{F} \otimes (\tau_*\mu' \circ \eta \circ \tau_* v \circ \tau_* I \circ \eta))]
= \int_X \mu' \circ \mathrm{tr}_F[q_*(p^*\mathcal{F} \otimes (\eta \circ \tau_* v \circ \tau_* I \circ \eta))].
\]

As \(\mu'\) was arbitrary, it follows that

\[
(u \circ \beta)' = \mathrm{tr}_F[q_*(p^*\mathcal{F} \otimes (\eta \circ \tau_* v \circ \tau_* I \circ \eta))].
\]

Applying \(I\) to both sides yields:

\[
I_{\text{HKR}}(u \cap \beta) = I \circ (u \circ \beta)'
= \mathrm{tr}_F[q_*(p^*\mathcal{F} \otimes (\tau_* I \circ \eta \circ \tau_* v \circ \tau_* I \circ \eta))]
= \mathrm{tr}_F[\exp(\alpha)(\mathcal{F}) \circ v \circ \exp(\alpha)(\mathcal{F})].
\]

(8)

The rest of the proof is done by a case-by-case analysis:

**The first case** \(v \in H^2(X, \mathcal{O}_X)\): One has

\[
I_{\text{HKR}}(u \cap \beta) = \mathrm{tr}_F[\exp(\alpha)(\mathcal{F}) \circ v \circ \exp(\alpha)(\mathcal{F})]
= \mathrm{tr}_F[\exp(\alpha)(\mathcal{F}) \circ v]
= v \land \mathrm{tr}_F[\exp(\alpha)(\mathcal{F})]
= v \land \mathrm{ch}(\mathcal{F})
= v \land \mathrm{ch}(\mathcal{F}).
\]
Furthermore, \( v = a^{-1} \otimes v \). Thus, we have
\[
I_K(u \cap \beta) = a \wedge I_{HKR}(u \cap \beta)
= a \wedge (v \otimes \text{ch}(\mathcal{F}))
= v \otimes (a \wedge \text{ch}(\mathcal{F}))
= (a^{-1} \otimes v) \otimes (a \wedge \text{ch}(\mathcal{F}))
= I_K(u) \otimes I_K(\beta).
\]

The second case \( v \in H^1(X, T_X) \): In this case, \( v \otimes \) acts as a derivation (with respect to the wedge product). Thus, one has
\[
I_{HKR}(u \cap \beta) = \text{tr}_F[\exp(\alpha)(\mathcal{F}) \circ v \circ \exp(\alpha)(\mathcal{F})]
= \text{tr}_F[\exp(\alpha)(\mathcal{F}) \circ v \circ \alpha(\mathcal{F})]
= \text{tr}_F[v \otimes \exp(\alpha)(\mathcal{F})]
= v \otimes \text{ch}(\mathcal{F}).
\]

Since the \((1,1)\)-part of \( a^{-1} \) is trivial, one clearly has \( v = a^{-1} \otimes v \). (In fact, an argument similar to the following shows that \( v = b \otimes v \) for any \( b = 1 + \lambda c_1(X) + \ldots \).) Furthermore, since \( a = \hat{\mathcal{A}}^\mathbb{R} \) stays of pure type under any deformation of \( X \), also \( v \otimes a = 0 \) (this follows also from Lemma 2.3).

Hence
\[
I_K(u \cap \beta) = a \wedge I_{HKR}(u \cap \beta) = a \wedge (v \otimes \text{ch}(\mathcal{F}))
= v \otimes (a \wedge \text{ch}(\mathcal{F})) - (v \otimes a) \wedge \text{ch}(\mathcal{F})
= (a^{-1} \otimes v) \otimes (a \wedge \text{ch}(\mathcal{F}))
= I_K(u) \otimes I_K(\beta).
\]

The final case \( v \in H^0(X, \bigwedge^2 T_X) \): By Lemma 2.7 below, the right hand side of (8) is \( a^{-1} \wedge ((a^{-1} \otimes v) \otimes (a \wedge \text{ch}(\mathcal{F}))) \). Then we have
\[
I_K(u \cap \beta) = a \wedge I_{HKR}(u \cap \beta)
= (a^{-1} \otimes v) \otimes (a \wedge \text{ch}(\mathcal{F}))
= I_K(u) \otimes I_K(\beta).
\]

Lemma 2.7. Suppose \( X \) is irreducible holomorphic symplectic. Let \( v \in H^0(X, \bigwedge^2 T_X) = \text{Hom}(\Omega^2_X, \mathcal{O}_X) \) be a holomorphic bivector field and \( \mathcal{F} \in D^!(X) \). Then
\[
\text{tr}_F[\exp(\alpha)(\mathcal{F}) \circ v \circ \exp(\alpha)(\mathcal{F})] = a^{-1} \wedge ((a^{-1} \otimes v) \otimes (a \wedge \text{ch}(\mathcal{F}))).
\]

This lemma will be proven by means of graph homology in Appendix B

3. Using \( \text{HH}^1 \)

The proof of the following result is a straightforward application of the invariance of the first Hochschild cohomology under derived equivalence.
Proposition 3.1. Suppose $A$ is an abelian variety. If for a smooth projective variety $X$ there exists an exact equivalence $\mathcal{D}^b(A) \cong \mathcal{D}^b(X)$, then $X$ is as well an abelian variety.

Proof. The order of the canonical bundle and the dimension of a smooth projective variety are derived invariants (see e.g. [11, Prop. 4.1]). Hence, $\omega_X$ is trivial and $\dim(X) = \dim(A)$.

By Theorem 0.1, there exists a finite étale cover $\pi: \hat{X} \to X$ with $\hat{X} \cong B \times \prod_i Y_i \times \prod_j Z_j$ such that $B$ is an abelian variety, the $Y_i$ are irreducible holomorphic symplectic, and the $Z_j$ are Calabi–Yau manifolds of dimension at least three. For the latter two types, the first Hochschild cohomology $\text{HH}^1 \cong H^0(T) \oplus H^1(O)$ is trivial. Thus the pull-back yields an injection

$$\pi^*: \text{HH}^1(X) \hookrightarrow \text{HH}^1(B).$$

On the other hand, the equivalence $\mathcal{D}^b(A) \simeq \mathcal{D}^b(X)$ induces an isomorphism $\text{HH}^1(A) \simeq \text{HH}^1(X)$ of vector spaces of dimension $2\dim(A) = 2\dim(X)$. Hence $2\dim(X) \leq 2\dim(B)$. Therefore, $\hat{X} = B$ and $\pi^*: \text{HH}^*(X) \to \text{HH}^*(B)$ must be bijective.

Moreover, since the pull-back $\pi^*: \text{HH}^1(X) \to \text{HH}^1(B)$ respects the direct sum decomposition, both spaces $H^0(X, T_X)$ and $H^1(X, O_X)$ are of dimension $\dim(X)$. By functoriality of the Albanese morphism, the composition $B \to X \to \text{Alb}(X)$ is étale. Hence $X \to \text{Alb}(X)$ is étale as well and, therefore, $X$ is an abelian variety. $\square$

Remark 3.2. Note that there exist indeed étale quotients of abelian varieties with trivial canonical bundle which are not abelian varieties themselves (see [22, 16.16]). Those are however, due to the proposition, never derived equivalent to an abelian variety. This also explains why we had to use the Albanese map in the proof.

Remark 3.3. If in the proposition, one furthermore assumes that $X$ is an irreducible projective symplectic or a Calabi–Yau manifold of dimension at least three, then the contradiction is obtained more easily by using $\text{HH}^1(A) \cong \text{HH}^1(X)$ and the fact that $\text{HH}^1(X) = H^0(X, T_X) \oplus H^1(X, O_X)$ is trivial for any such an $X$.

Remark 3.4. Alternatively to Remark 3.3 one could use the existence of spherical and projective objects (see [12]) on Calabi–Yau respectively irreducible symplectic manifolds (provided e.g. by any line bundle), which do not exist on abelian varieties of dimension at least two.

4. Using $\text{HH}^2$

The idea to prove that an irreducible holomorphic symplectic manifold $Y$ can never be derived equivalent to a Calabi–Yau manifold $Z$ of dimension at least three goes as follows: The infinitesimal deformations (commutative and non-commutative) parametrised by the second Hochschild cohomology are identified under any derived equivalence $\mathcal{D}^b(Y) \cong \mathcal{D}^b(Z)$, i.e. $\text{HH}^2(Y) \cong \text{HH}^2(Z)$. Algebraic classes $\alpha \in H\Omega_0(Y)$ that stay algebraic under all these deformations are of a very special form (see Proposition 4.5). In particular, they have non-trivial square with respect to the Mukai pairing. On the other
hand, a skyscraper sheaf \( k(z) \in \mathcal{D}^p(Z) \) does deform infinitesimally in all deformation directions parametrised by \( \mathbb{H}H^2(Z) \) (which are all commutative for \( Z \) a Calabi–Yau manifold), but its Mukai square is trivial.

We shall use the second Hochschild cohomology and \( \cap : \mathbb{H}H^2(X) \otimes \mathbb{H}H_0(X) \to \mathbb{H}H_2(X) \) instead of the first Hochschild cohomology as in the case of abelian varieties. Consider the two subspaces

\[
A(X) := \{ \alpha \in \mathbb{H}H_0(X) \mid \forall v \in \mathbb{H}H^2(X) : v \cap \alpha = 0 \}
\]

and

\[
R(X) := \{ \alpha \in \Omega_0(X) \mid \forall v \in \mathbb{H}T^2(X) : v \cup \alpha = 0 \}.
\]

**Remark 4.1.** The contraction with \( v \in H^1(X, T_X) \) of a cohomology class \( \alpha \in H^{p,p}(X) \) measures the change of the bidegree of \( \alpha \) under the infinitesimal deformation of \( X \) corresponding to \( v \) (Griffiths transversality). In analogy, one should think of \( R(X) \) as the set of classes of pure type which stay of pure type under all infinitesimal deformations parametrised by the second Hochschild cohomology \( \mathbb{H}H^2(X) \), which include the classical ones in \( H^1(X, T_X) \) as well as non-commutative and twisted ones in \( H^0(X, \Lambda^2 T_X) \) and \( H^2(X, \mathcal{O}_X) \), respectively.

**Lemma 4.2.** Suppose \( X \) has trivial canonical bundle. Then the isomorphism \( \mathcal{I}_K : \mathbb{H}H_0(X) \xrightarrow{\sim} \Omega_0(X) \) induces an isomorphism

\[
\mathcal{I}_K : A(X) \xrightarrow{\sim} R(X).
\]

Moreover, if \( \Phi : \mathcal{D}^p(X) \xrightarrow{\sim} \mathcal{D}^p(X') \) is an exact equivalence, then \( \Phi^{\mathbb{H}H_*} \) defines an isomorphism \( A(X) \cong A(X') \).

**Proof.** This follows from the compatibility of the modified HKR-isomorphism with the module structures, see Corollary 2.5.

The second assertion is a consequence of the general fact that any exact equivalence induces an isomorphism of the Hochschild structure. \( \square \)

The above proof relies on Corollary 2.5, which in turn uses 7 and 19. If instead one prefers to use Proposition 2.6, one would get the following result, which on the one hand works only for algebraic classes, but, on the other hand, covers manifolds with non-trivial canonical bundle.

**Lemma 4.3.** Suppose that either \( H^0(X, \Lambda^2 T_X) = 0 \) or that \( X \) is irreducible holomorphic symplectic. Let \( \alpha \in \mathbb{H}H_0(X) \) be a class such that \( \mathcal{I}_K(\alpha) \in \Omega_0(X) \) is algebraic. Then \( \alpha \in A(X) \) if and only if \( \mathcal{I}_K(\alpha) \in R(X) \).

**Proof.** By Proposition 2.6 we have \( \mathcal{I}_K(v \cap \alpha) = \mathcal{I}_K(v) \cup \mathcal{I}_K(\alpha) \) for all \( v \in \mathbb{H}T^2(X) \), which suffices to conclude. \( \square \)

**Proposition 4.4.** Let \( Z \) be a projective manifold of dimension \( m \) with \( H^0(Z, \Lambda^2 T_Z) = 0 \). Then \( H^{m,m}(Z) \subset R(Z) \).

If, moreover, \( Z \) is a Calabi–Yau manifold of dimension \( m \geq 3 \), then the following holds:

\[
(H^{0,0} \oplus H^{1,1} \oplus H^{m-1,m-1} \oplus H^{m,m})(Z) \subset R(Z).
\]
Proof. The action of $HT^2(Z) \cong H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X)$ on $H^{m,m}(Z)$ is trivial for degree reasons. Thus, $H^{m,m}(Z) \subset R(Z)$.

Let now in addition $Z$ be Calabi–Yau of dimension at least three. Thus, not only $H^0(Z, \Lambda^2 T_Z) = 0$ but also $H^2(Z, \mathcal{O}_Z) = 0$. Hence it suffices to show that the contraction with classes in $H^1(Z, T_Z)$ annihilates $H^{p,q}(Z)$ for $q = 0, 1, m - 1, m$. This is trivial for $q = 0, m$. For $q = 1$ and $q = m - 1$ it follows from $H^{0,2}(Z) = 0$ respectively $H^{m-2,m}(Z) = (H^{2,0}(Z))^\vee = 0$ (use Serre duality).

Proposition 4.5. Suppose $Y$ is a simply-connected, projective symplectic manifold of dimension $2n$. Then $R(Y)$ is the set of all classes $\alpha \in H^n(Y, \Omega^n_Y)$ satisfying

1. $\bar{\sigma} \wedge \alpha = 0$,
2. $\Lambda_\sigma \alpha = 0$ and
3. $\alpha \in H^n(Y', \Omega^n_Y)$ for all deformations $Y'$ of $Y$

for all holomorphic symplectic forms $\sigma \in H^{2,0}(Z)$. Here, $\Lambda_\sigma$ denotes the dual Lefschetz operator to $L_\sigma := \sigma \wedge \cdot$ (cf. [8]).

Proof. As we later only use that any class in $R(Y)$ satisfies (1)–(3), we only prove this inclusion. The other relies on the same arguments and is left to the reader. For the general theory of hyper-Kähler manifolds we refer to [8] and [9].

By the Decomposition Theorem [1.1] we have $Y = \bigsqcup_i Y_i$, where each factor $Y_i$ is an irreducible projective symplectic manifold. Via the isomorphism $I^K$, one has

$$H^2(Y) \cong \bigoplus_i \left( H^2(Y_i, \mathcal{O}_{Y_i}) \oplus H^1(Y_i, T_{Y_i}) \oplus H^0(Y_i, \Lambda^2 T_{Y_i}) \right)$$

with $H^2(X, \mathcal{O}_{Y_i})$ spanned by the anti-holomorphic symplectic form $\bar{\sigma}_i$ on $Y_i$, and $H^0(Y_i, \Lambda^2 T_{Y_i})$ identified with $H^0(Y_i, \mathcal{O}_{Y_i})$ by contraction with $\sigma_i$.

Consider $\alpha \in R(Y) \subset H^n(\Omega^n_Y)$. As the direct sum decomposition [9] respects the decomposition $H^n(Y) \cong \bigotimes_i H^n(Y_i)$, it follows that $\alpha \in \bigotimes_i R(Y_i)$. Thus we may assume that $Y = Y_i$.

Since $\alpha \in R(Y)$, we have $\bar{\sigma} \wedge \alpha = 0$. Standard Lefschetz theory of the anti-holomorphic form $\bar{\sigma}$ on $Y$ then shows $\alpha \in \bigoplus_{p \geq n} H^p(Y, \Omega^n_Y)$.

Contraction with the canonical generator $v \in H^0(Y, \Lambda^2 T_Y)$ is given by the dual Lefschetz operator to $\bar{\sigma}$, i.e. $v \wedge \alpha = \Lambda_\sigma \alpha = 0$. Again by Lefschetz theory, this means that $\alpha \in \bigoplus_{p \leq n} H^p(Y, \Omega^n_Y)$ and together with the above result this yields $\alpha \in H^n(Y, \Omega^n_Y)$.

By definition of $R(Y)$, we know that $v \wedge \alpha = 0$ for all $v \in H^1(Y, T_Y)$. Thus infinitesimally, $\alpha$ stays of type $(n, n)$. By the general theory of hyper-Kähler manifolds, this also shows that $\alpha$ stays of type $(n, n)$ on any hyper-Kähler rotation of $Y$. As any two deformations are connected by twistor lines, it follows that $\alpha$ is of type $(n, n)$ on any deformation of $Y$. \qed

Remark 4.6. In other words, the only classes $\alpha$ of pure type on a simply-connected holomorphic symplectic manifold $Y$ that stay of pure type under any infinitesimal deformation in $H^2(Y)$ are the classes in $H^n(Y, \Omega^n_Y)$ that
are primitive with respect to all holomorphic symplectic forms and their anti-
holomorphic conjugates, and which stay of type \((n, n)\) under any classical
deformation.

**Remark 4.7.** Note that for rational cohomology classes in \(H^n(Y, \Omega^k_Y)\), so in
particular for algebraic ones, conditions (1) and (2) are equivalent. Also
note that a class satisfying (3) need not necessarily be primitive, e.g. \(c_2(Y)\)
on a four-dimensional \(Y\) is not.

**Lemma 4.8.** Let \(X\) be a smooth projective variety such that there exists
an exact equivalence \(\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)\), where \(Y\) is an irreducible projective
symplectic manifold of dimension \(2n\). Then there exists a non-degenerate quadratic form
\[ q: \text{HH}^2(X) \rightarrow \sqrt{\text{HH}^{2n}(X)} \]
such that \(q(\alpha)^n = \alpha^{2n}\) for all \(\alpha \in \text{HH}^2(X)\) and such that the subring of
\(\text{HH}^1(X)\) generated by \(\text{HH}^2(X)\) is given by
\[ \text{SHH}^2(X) = S^*\text{HH}^2(X)/\langle \alpha^{n+1} | q(\alpha) = 0 \rangle. \]

**Proof.** One has the following isomorphisms of graded rings:
\[ \text{HH}^*(X) \cong \text{HH}^*(Y) \cong \text{HT}^*(Y) = H^*(Y, \bigwedge^* T_Y) \cong H^*(Y, \Omega^*_Y), \]
where the first isomorphism is induced by the exact equivalence \(\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)\), the second isomorphism is the modified HKR-isomorphism \(I^K\), and the third isomorphism comes from the isomorphism \(T_Y \cong \Omega_Y\) induced by the symplectic form on \(Y\).

By \([2, 8]\), the map \(\alpha \mapsto \alpha^{2n}\) on \(H^2(Y, \mathbb{C})\) possesses an \(n\)-th root, namely
the Beauville–Bogomolov quadratic form \(q_Y\) of \(Y\). By a result of Verbitsky (see \([3]\)), one has
\[ \text{SH}^2(Y, \mathbb{C}) = S^*H^2(Y, \mathbb{C})/\langle \alpha^{n+1} | q(\alpha) = 0 \rangle. \]
Since \(\text{HH}^*(X) \cong H^*(Y, \Omega^*_Y)\), the claim of the Lemma follows. \(\square\)

**Theorem 4.9.** Let \(Y\) be an irreducible projective symplectic manifold. If for
a smooth projective variety \(X\) there exists an exact equivalence \(\Phi: \mathcal{D}^b(Y) \cong \mathcal{D}^b(X)\), then \(X\) is also an irreducible projective symplectic manifold.

**Proof.** As the order of the canonical bundle and the dimension are derived invariants (see e.g. \([11\), Prop. 4.1]), we have \(\omega_X \cong \mathcal{O}_X\) and \(\dim X = \dim Y =: 2n\).

Assume that \(X\) does not possess a symplectic form. Then for any \(v \in H^0(X, \bigwedge^2 T_X)\) one would have \(v^n = 0\). By the description of \(\text{SHH}^2(X)\) given in Lemma \([4, 8]\) this would show that \(v = 0\). Thus, \(H^0(X, \bigwedge^2 T_X) = 0\).

Consider a closed point \(x \in X\) and its structure sheaf \(k(x)\). Its Mukai vector \(\text{ch}(k(x))\) is the generator of \(H^{2n}(X, \Omega^*_X)\), which by Proposition \([13]\) is contained in \(R(X)\). As \(\Phi\) is an equivalence, there exists a complex \(\mathcal{F}_x \in \mathcal{D}^b(Y)\) with \(\Phi(\mathcal{F}_x) \cong k(x)\). It follows that \(\gamma := \text{ch}(\mathcal{F}_x) \wedge \sqrt{\text{td}(Y)} = \text{ch}(\mathcal{F}_x) \wedge a \in R(Y)\). Indeed, by commutativity of \([2]\) and by Lemma \([1, 2]\) or \([4, 3]\) one has
\[ \Phi_{\text{HH}^1}(I_K^{-1}(\gamma)) = \Phi_{\text{HH}^1}((I_{\text{HKR}}^{-1}(\text{ch}(\mathcal{F}_x)))) = I_{\text{HKR}}^{-1}(\text{ch}(k(x))) = I_K^{-1}(\text{ch}(k(x))) \in A(X) \]
and hence $I_K^{-1}(\gamma) \in A(Y)$, which is equivalent to $\gamma \in R(Y)$ (again by Lemma 4.2 or [13]). Due to Proposition 4.5, one thus has $\gamma \in H^n(Y, \Omega^n_Y)$.

Since $\Phi$ is an equivalence, $\chi(F^x, F^x) = \chi(k(x), k(x)) = 0$. This leads to a contradiction as follows: On the one hand,

$$0 = \chi(F^x, F^x) = (-1)^n \int_Y \gamma \wedge \gamma.$$  

(For the last equality use that $\gamma \in H^n(Y, \Omega^n_Y)$.) On the other hand, again due to Proposition 4.5 one knows that $\gamma \in H^n(Y, \Omega^n_Y)$ is also of type $(n, n)$ on any hyper-Kähler rotation $Y'$ defined by a complex structure say $J$. Now write $\sigma = \omega_J + i\omega_K$, where $\omega_J$ is the Kähler form on $Y'$. Using $\bar{\sigma} \wedge \gamma = 0$ of Proposition 4.5 and the fact that $\gamma$ as an algebraic Chern class is real, we find that $\omega_J \wedge \gamma = 0$, i.e. $\gamma$ is an $\omega_J$-primitive $(n, n)$-class on $Y'$. Thus by the Hodge–Riemann bilinear relations

$$\int_Y \gamma \wedge \gamma \neq 0,$$

a contradiction. Thus our assumption has to be wrong and therefore $X$ admits a symplectic form.

It remains to show that $X$ is irreducible symplectic. As

$$\text{SHH}^2(X) = S^*\text{HH}^2(X)/\langle \alpha^{n+1} \mid q(\alpha) = 0 \rangle,$$

it follows that the assumptions of Proposition A.1 in the appendix about the spaces $H^{k,0}(X)$ are fulfilled, which finally proves the theorem.  

**Corollary 4.10.** An irreducible projective symplectic manifold can never be derived equivalent to a Calabi–Yau manifold of dimension at least three.  

As another step towards answering our general Question 0.2 we prove the following partial result:

**Proposition 4.11.** Let $Y_1, \ldots, Y_n$ and $Y_1', \ldots, Y_m'$ be irreducible projective symplectic varieties. Set $Y := \prod_i Y_i$ and $Y' := \prod_j Y'_j$. Assume that there exists an exact equivalence $\mathcal{D}^b(Y) \cong \mathcal{D}^b(Y')$. Then there exists a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ with $\dim(Y_i) = \dim(Y'_{\sigma(i)})$ and $b_2(Y_i) = b_2(Y'_{\sigma(i)})$.

**Proof.** Consider the variety

$$Q := \left\{ [\alpha] \mid \alpha^{\dim Y} = 0 \right\} \subset \mathbb{P}(\text{HH}^2(Y)).$$

By Lemma 4.8 the irreducible components of $Q$ are in bijection to the irreducible factors $Y_i$ of $Y$. Furthermore, each irreducible component $Q_i$ is a quadric whose rank is the second Betti number of the corresponding irreducible factor. As $Q$ is a derived invariant, the claim of the Proposition follows.  

**Remark 4.12.** For the time being we cannot exclude that a Calabi–Yau manifold $Z$ is derived equivalent to a smooth projective variety $X$ which itself is not Calabi–Yau (in our restrictive sense). E.g. we can neither exclude that $X$ is a finite étale quotient of a Calabi–Yau manifold nor that it is a product of Calabi–Yau manifolds with possibly an irreducible holomorphic
symplectic factor. An abelian factor can easily be excluded by studying the first Hochschild cohomology.

**Appendix A. Irreducible holomorphic symplectic manifolds**

The following result might be known to the experts, but we were unable to find a reference for it.

**Proposition A.1.** Let $X$ be a holomorphic symplectic manifold of complex dimension $2n$ such that $H^{k,0}(X) \cong \mathbb{C}$ for $k = 0, \ldots, 2n$ and $H^{k,0}(X) = 0$ otherwise.

Then $X$ is simply-connected, i.e. $X$ is an irreducible holomorphic symplectic manifold.

**Proof.** Let $\pi: \hat{X} \to X$ be a cover of $X$ as in the Decomposition Theorem [0.1] and let $d$ be its degree.

Assume that $\hat{X}$ contains an abelian factor. In this case, the fundamental group of $X$ would contain an infinite free abelian group, which contradicts the assumption that $H^{1,0}(X) = 0$. Thus, $\hat{X}$ is simply-connected.

Endow $X$ with a hyper-Kähler metric $g$, which induces a hyper-Kähler metric also on $\hat{X}$. By the Decomposition Theorem [0.1] $\hat{X}$ splits into a product $Y_1 \times \cdots \times Y_k$ of irreducible hyper-Kähler manifolds. Set $n_i := \frac{1}{2} \dim Y_i$.

The holomorphic Euler characteristic of $\hat{X}$ is given by

$$\chi(\hat{X}, \mathcal{O}_{\hat{X}}) = \chi(Y_1) \cdots \chi(Y_k) = (1 + n_1) \cdots (1 + n_k),$$

while the holomorphic Euler characteristic of $X$ is by assumption

$$\chi(X, \mathcal{O}_X) = 1 + n.$$

As $\chi(\hat{X}, \mathcal{O}_{\hat{X}}) = d \cdot \chi(X, \mathcal{O}_X)$, one has the following equality:

$$d \cdot (1 + n) = (1 + n_1) \cdots (1 + n_k). \tag{10}$$

Fix a point $p \in \hat{X}$ and let $q := \pi(p)$. The splitting $\hat{X} = Y_1 \times \cdots \times Y_k$ induces a splitting $T_X(p) \cong \mathbb{C}^{2n_1} \oplus \cdots \oplus \mathbb{C}^{2n_k}$ of the holomorphic tangent space at $p$ as a unitary vector space. We can choose the isomorphism in such a way that under it the holonomy group of $\hat{X}$ at $p$ is identified with $H^0 := \text{Sp}(n_1) \times \cdots \times \text{Sp}(n_k)$ with its natural action.

Let $H$ be the holonomy group of $X$ at $q$ acting unitarily on $\mathbb{C}^{2n} = \mathbb{C}^{2n_1} \oplus \cdots \oplus \mathbb{C}^{2n_k}$. Then $H/H^0$ is a certain quotient group of $\pi_1(X, q)$.

By the holonomy principle, the space $H^{2,0}(X) = H^0(X, \Omega_X^2)$ is given by the invariants of $\wedge^2 \mathbb{C}^{2n}$ under the action of $H$. By assumption, $H^{2,0}(X)$ is spanned by a unique (up to scaling) holomorphic symplectic form $\sigma$ and its pull-back can be written as $\pi^*\sigma = \sigma_1 + \cdots + \sigma_k$ with $\sigma_i$ a symplectic form on $Y_i$. We can choose the isomorphism $T_X(p) \cong \mathbb{C}^{2n}$ from above such that $\pi^*\sigma(p)$ becomes the standard symplectic form on $\mathbb{C}^{2n}$. In particular, $\sigma_i(p)$ becomes the standard symplectic form on the summand $\mathbb{C}^{2n_i}$, which we denote again by $\sigma_i$. In this way, we realize $H$ as a subgroup of the standard $\text{Sp}(n)$.

As $H^0$ is a normal subgroup of $H$, it follows that $H$ is a subgroup of the normaliser $N$ of $H^0$ in $\text{Sp}(n)$. By Lemma [A.2] below, there is a group
homomorphism $\rho: N \to \mathfrak{S}_k$ such that $A\sigma_i = \sigma_{\rho(A)(i)}$ for all $i = 1, \ldots, k$ for $A \in N$. Furthermore, one clearly has $n_i = n_{\rho(A)(i)}$. Denote the image of $H$ under $\rho$ by $\mathfrak{S}$.

Since $H^{2,0}(X)$ is one-dimensional, the holonomy principle yields that the space of invariants of $\bigwedge^2 \mathbb{C}^{2n}$ under the action of $H$ is one-dimensional as well. On the other hand, the space of invariants under the action of $H^0$ is given by elements of the form $\lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k$ with $\lambda_i \in \mathbb{C}$ arbitrary.

Thus $\mathfrak{S}$ has to act transitively on the $\sigma_1, \ldots, \sigma_k$. It follows that $m := n_1 = \ldots = n_k$, i.e. all factors are of the same dimension. Furthermore, the order of $\mathfrak{S}$ has to be a multiple of $k$.

Assume that $m > 1$. We may further assume that $k > 1$, as otherwise $d = 1$ by (10) and $X$ would already be simply-connected. Then the space of invariants of $\bigwedge^2 \mathbb{C}^{2n}$ under the action of $N$ (and thus $H$) includes the two-dimensional subspace spanned by $\sum_i \sigma_i^2$ and $\sum_{i < j} \sigma_i\sigma_j$. Hence $\dim(H^{4,0}(X)) > 1$ by the holonomy principle, contradicting the assumption. Thus only the case $m = 1$ remains.

By construction, $\mathfrak{S}$ is a subquotient of $\pi_1(X, q)$ and, therefore, its order divides $d = |\pi_1(X, q)|$. In particular, $d$ has to be a multiple of $k$, say $d = e \cdot k$ for a natural number $e$. Together with (10) this yields the equation (with $n_i = m = 1$)

$$e \cdot k \cdot (1 + k) = 2^k$$

having $e = k = 1$ as its unique solution. Thus $X$ is already simply-connected.

\[\Box\]

**Lemma A.2.** Let $n = n_1 + \ldots + n_k$ be a partition of a natural number. Consider $\mathbb{C}^{2n} = \mathbb{C}^{2n_1} \oplus \cdots \oplus \mathbb{C}^{2n_k}$ with its standard hermitian complex-symplectic structure. Denote the standard symplectic form on the summand $\mathbb{C}^{2n_i}$ by $\sigma_i$.

Let $N$ be the normaliser of $G := \text{Sp}(n_1) \times \cdots \times \text{Sp}(n_k)$ in $\text{Sp}(n)$. The natural action of $\text{Sp}(n)$ induces a natural action of $N$ on $\mathbb{C}^{2n}$ and $\bigwedge^2 \mathbb{C}^{2n}$.

Then there is a group homomorphism $\rho: N \to \mathfrak{S}_k$ such that $A\sigma_i = \sigma_{\rho(A)(i)}$ for all $A \in N$ and $i = 1, \ldots, k$.

Furthermore, the image of $\rho$ is a subgroup containing only permutations $\tau$ with $n_{\tau(i)} = n_i$ for all $i = 1, \ldots, k$.

**Proof.** As $N$ is the normaliser of $G$, it maps the space of $G$-invariants into itself. The space of invariants of $G$ is spanned by $\sigma_1, \ldots, \sigma_k$.

Let $A \in N$ and pick $i$ such that $\sigma_i$ is of minimal rank (i.e. $n_i$ is minimal among the $n_1, \ldots, n_k$). By the same reasoning as in the proof of the Proposition, we have $A\sigma_i = \sum_j \lambda^i_j\sigma_j$ for certain $\lambda^i_j \in \mathbb{C}$. The rank of a two-form does not change under a general linear transformation. Thus the sum in $A\sigma_i$ can only consist of one term, i.e. we have to have $A\sigma_i = \lambda_i\sigma_{\rho(A)(i)}$ with $n_i = n_{\rho(A)(i)}$ for a certain $\rho(A)(i)$ and $\lambda_i \in \mathbb{C}^*$.

In this way, we define $\rho(A)(i)$ for all $i$ with minimal $n_i$. As $A$ is invertible, $\rho(A)$ must be a permutation of those $i$ for which $n_i$ is minimal.

Suppose we have shown that all $A \in N$ permute (up to scaling) the forms $\sigma_\ell$ of rank $2n_\ell < r$ and let $\sigma_i$ be of rank $r$. Then $A\sigma_i = \sum_j \lambda^i_j\sigma_j$ for certain $\lambda^i_j \in \mathbb{C}^*$. Clearly, $r = \text{rk}(\sigma_i) = \text{rk}(A\sigma_i) = \sum_{\lambda^i_j \neq 0} \text{rk}(\sigma_j)$. Thus, either
$A\sigma_i = \lambda_i^j \sigma_j$ for some $j$ or $A\sigma_i$ is a linear combination of symplectic forms $\sigma_j$ of strictly smaller rank. As $A$ is invertible, the latter would contradict the assumption that $A^{-1}$ permutes all $\sigma_i$ of rank $< r$. Hence $A\sigma_i = \lambda_i^j \sigma_j = \lambda_i \sigma_{\rho(A)(i)}$ and $\text{rk}(\sigma_i) = \text{rk}(\sigma_{\rho(A)(i)})$, i.e. $n_i = n_{\rho(A)(i)}$. Hence, $A$ permutes forms $\sigma_i$ of rank $\leq r$ up to scaling by $\lambda_i \in \mathbb{C}^*$.

Continuing in this way, we obtain a permutation $\rho(A)$ of $\{1, \ldots, k\}$ with $n_{\rho(A)(i)} = n_i$ and such that $A\sigma_i = \lambda_i \sigma_{\rho(A)(i)}$ for certain $\lambda_i \in \mathbb{C}^*$. It is clear that $A \mapsto \rho(A)$ is a group homomorphism.

Now $\sigma = \sigma_1 + \cdots + \sigma_k$ is invariant under the action of the group $\text{Sp}(n)$, thus also under the normaliser $N$, and hence $\lambda_1 = \ldots = \lambda_k = 1$, which proves the Lemma.

Appendix B. Proof of Lemma 2.7

B.1. Graph homology. We need to introduce a bit of notation from graph homology. We follow mostly [21] and refer the reader it for details:

A Jacobi diagram is a uni-trivalent graph with a chosen cyclic orientation of the set of half-edges at each trivalent vertex. By

$$\mathcal{A}(*_{x_1} \cdots *_{x_k} \uparrow_{y_1} \cdots \uparrow_{y_l} \cap z_1 \cdots \cap z_m)$$

we denote the (suitably completed, see [21] Sect. 2.5, 3.3]) $\mathbb{Q}$-vector space spanned by all Jacobi diagrams whose set of univalent vertices is partitioned into sets labelled by $x_1, \ldots, x_k$, linearly ordered sets labelled by $y_1, \ldots, y_l$ and cyclicly ordered sets labelled by $z_1, \ldots, z_m$ modulo the AS, IHX and STU relations (for the definition of the AS, IHX and STU relations, see [21]). Elements of these spaces are called graph homology classes. In what follows, let $X$ and $X'$ denote disjoint collections of disjantly labelled stars, oriented intervals, and oriented circles.

Example B.1. By $x \prec y \in \mathcal{A}(*_x *_y)$ we denote the graph homology class defined by the Jacobi diagram consisting of two univalent vertices labelled with $x$ and $y$, respectively. We call this graph homology class a strut.

There are commutative, associative products

$$\mathcal{A}(*_{x_1} \cdots *_{x_k} X) \otimes \mathcal{A}(*_{x_1} \cdots *_{x_k} X') \to \mathcal{A}(*_{x_1} \cdots *_{x_k} X X'),$$

$$C \otimes D \mapsto C \cup_{x_1 \cdots x_k} D,$$

which are given by the disjoint union of the underlying Jacobi diagrams. There are further associative products

$$\mathcal{A}(\uparrow_x X) \otimes \mathcal{A}(\uparrow_x X') \to \mathcal{A}(\uparrow_x X X'),$$

$$C \otimes D \mapsto C \# x D,$$

which are given by juxtaposition of the orders of the univalent vertices labelled by $x$. There is a natural linear map

$$\pi_x: \mathcal{A}(*_x X) \to \mathcal{A}(\uparrow_x X),$$

which is given by averaging over all possibilities to linearly order the univalent vertices labelled by $x$. This map is an isomorphism of vector spaces (see, e.g. [21] Sect. 3.3]). Thus we can identify the two spaces.

There is another natural linear map

$$\text{tr}_x: \mathcal{A}(\uparrow_x X) \to \mathcal{A}(\cap_x X),$$

$Aσ_i = \lambda_i^j σ_j$ for some $j$ or $Aσ_i$ is a linear combination of symplectic forms $σ_j$ of strictly smaller rank. As $A$ is invertible, the latter would contradict the assumption that $A^{-1}$ permutes all $σ_i$ of rank $< r$. Hence $Aσ_i = \lambda_i^j σ_j = \lambda_i σ_{ρ(A)(i)}$ and $\text{rk}(σ_i) = \text{rk}(σ_{ρ(A)(i)})$, i.e. $n_i = n_{ρ(A)(i)}$. Hence, $A$ permutes forms $σ_i$ of rank $≤ r$ up to scaling by $λ_i ∈ \mathbb{C}^*$.

Continuing in this way, we obtain a permutation $ρ(A)$ of $\{1, \ldots, k\}$ with $n_{ρ(A)(i)} = n_i$ and such that $Aσ_i = λ_i σ_{ρ(A)(i)}$ for certain $λ_i ∈ \mathbb{C}^*$. It is clear that $A → ρ(A)$ is a group homomorphism.

Now $σ = σ_1 + · · · + σ_k$ is invariant under the action of the group $\text{Sp}(n)$, thus also under the normaliser $N$, and hence $λ_1 = · · · = λ_k = 1$, which proves the Lemma.

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we denote the (suitably completed, see [21] Sect. 2.5, 3.3]) $\mathbb{Q}$-vector space spanned by all Jacobi diagrams whose set of univalent vertices is partitioned into sets labelled by $x_1, \ldots, x_k$, linearly ordered sets labelled by $y_1, \ldots, y_l$ and cyclicly ordered sets labelled by $z_1, \ldots, z_m$ modulo the AS, IHX and STU relations (for the definition of the AS, IHX and STU relations, see [21]). Elements of these spaces are called graph homology classes. In what follows, let $X$ and $X'$ denote disjoint collections of disjantly labelled stars, oriented intervals, and oriented circles.

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There are commutative, associative products

$$\mathcal{A}(*_x_1 \cdots *_x_k X) \otimes \mathcal{A}(*_x_1 \cdots *_x_k X') \to \mathcal{A}(*_x_1 \cdots *_x_k X X'),$$

$$C \otimes D \mapsto C \cup_{x_1 \cdots x_k} D,$$

which are given by the disjoint union of the underlying Jacobi diagrams. There are further associative products

$$\mathcal{A}(↑_x X) \otimes \mathcal{A}(↑_x X') \to \mathcal{A}(↑_x X X'),$$

$$C \otimes D \mapsto C \# x D,$$

which are given by juxtaposition of the orders of the univalent vertices labelled by $x$. There is a natural linear map

$$\pi_x: \mathcal{A}(*_x X) \to \mathcal{A}(↑_x X),$$

which is given by averaging over all possibilities to linearly order the univalent vertices labelled by $x$. This map is an isomorphism of vector spaces (see, e.g. [21] Sect. 3.3]). Thus we can identify the two spaces.

There is another natural linear map

$$\text{tr}_x: \mathcal{A}(↑_x X) \to \mathcal{A}(\cap_x X),$$
which is given by applying the forgetful map that turns the linear order of the univalent vertices labelled by \(x\) into a cyclic order. By [21], this map is an isomorphism whenever \(X\) consists only of oriented circles, i.e. does not include stars or oriented intervals.

We introduce bilinear forms

\[
\mathcal{A}(*_{x_1} \cdots *_{x_k} X) \otimes \mathcal{A}_0(*_{x_1} \cdots *_{x_k} X') \to \mathcal{A}_0(X X'), \quad C \otimes D \mapsto \langle C, D \rangle_{x_1 \cdots x_k}
\]

which are given by summing over all ways of gluing all \(x_i\)-labelled univalent vertices of \(C\) pairwise to all \(x_i\)-labelled univalent vertices of \(D\), \(i = 1 \ldots k\). Here \(\mathcal{A}_0(\cdots)\) denotes the subspace of \(\mathcal{A}(\cdots)\) generated by those Jacobi diagrams that do not include any component \(x_i \xrightarrow{x} x_i\).

We shall also need an operator that allows us to relabel graph homology classes: Let \(\Delta^{y}_{x_1 x_2}\) be the linear operator that is given by replacing the labels of \(y\) by \(x_1\) and \(x_2\). For vertices labelled with stars, this operation can be extended: Let

\[
\Delta^{y}_{x_1 x_2} : \mathcal{A}(\ast_{y} X) \to \mathcal{A}(\ast_{x_1} \ast_{x_2} X)
\]

be the linear operator whose value is the sum over all ways of replacing the labels \(y\) by one of \(x_1\) and \(x_2\).

Example B.2. One has

\[
\langle C, D \cup_{x} D \rangle_{x} = \langle \Delta^{x}_{y_1 y_2} C, (\Delta^{x}_{y_1} D_1) \otimes (\Delta^{x}_{y_2} D_2) \rangle_{y_1 y_2}
\]

when either \(C\) or \(D_1\) and \(D_2\) having no struts \(x \xrightarrow{x} x\).

In [21], D. Thurston also introduces diagrammatic differential operators \(\partial_{C}\). Here they are denoted slightly differently: Let

\[
\mathcal{A}_0(\ast_{x} X) \otimes \mathcal{A}(\ast_{x} X') \to \mathcal{A}(\ast_{x} X X'), \quad C \otimes D \mapsto C \xrightarrow{\cup_{x}} D,
\]

be the differential operator (with respect to \(\cup_{x}\)) that is given by summing over all ways of gluing all \(x\)-labelled univalent vertices of \(C\) to some \(x\)-labelled univalent vertices of \(D\). (We can also define the inner product if the right hand side has no struts, but the left hand side (possibly) has.)

Example B.3. This differential operator can be expressed in terms of the inner product \(\langle \cdot, \cdot \rangle\), namely

\[
\Delta^{x}_{y} (C \xrightarrow{x} D) = \langle C \cup_{x} \exp(x \xrightarrow{y} D), D \rangle_{x}.
\]

Here, \(\exp\) means the exponential series with respect to the product \(\cup_{xy}\).

B.2. Weight systems. The reason why we have recalled these facts about graph homology is that graph homology classes define certain cohomology classes on holomorphic symplectic manifolds, the so-called Rozansky–Witten classes (for a detailed treatment see, e.g., [16] or [20]): Let \(C \in \mathcal{A}(\ast_{x} \uparrow_{y})\) be a graph homology class which is represented by a Jacobi diagram \(\Gamma\), whose univalent are labelled by \(x\) and \(y\), and which has a linear order of the univalent vertices labelled by \(y\) chosen. Let the number of trivalent vertices be \(k\), the number of univalent vertices labelled \(x\) be \(\ell\) and the number of univalent vertices labelled \(y\) be \(m\).

Let \(g\) be a finite-dimensional metric Lie algebra over \(\mathbb{C}\), i.e. a Lie algebra endowed with a scalar product \(\langle \cdot, \cdot \rangle\) with respect to which the adjoint maps
are skew-symmetric. Let \( E \) be a finite-dimensional \( g \)-module. Then we can construct an element
\[
\Gamma(g, \langle \cdot, \cdot \rangle, E) \in S^g \otimes \text{End}(E)
\]
as follows: Use the scalar product on \( g \) to identify \( g \) and \( g^\vee \). In particular, we can view the Lie bracket as an element in \( g^{\otimes 3} \). Take one tensor copy of this element for each of the \( k \) trivalent vertices in \( \Gamma \) and one copy of the scalar product as an element in \( g^{\otimes 2} \) for each edge connecting two univalent vertices. Use the scalar product to contract the resulting tensor tensor along the edges of the graph \( \Gamma \) connecting trivalent vertices. This process yields an element in \( g^{\otimes 2} \otimes g^{\otimes m} \). The action of \( g \) on \( E \) can be viewed as a map \( g \to \text{End}(E) \). Applying this map to the last \( m \) tensor factors in our element in \( g^{\otimes 2} \otimes g^{\otimes m} \) and then composing the endomorphisms in the order given by the linear order of the univalent vertices labelled \( y \) yields an element in \( g^{\otimes 2} \otimes \text{End}(E) \). Finally, project down to the symmetric tensors to get a well-defined element \( \Gamma(g, \langle \cdot, \cdot \rangle, E) \in S^g \otimes \text{End}(E) \). Using the properties of the Lie bracket, the ad-invariance of the scalar product and the fact that the map \( g \to \text{End}(E) \) is a morphism of Lie algebras allows one to show that \( \Gamma(g, \langle \cdot, \cdot \rangle, E) \) does only depend on the graph homology class \( C \) of \( \Gamma \) and not on \( \Gamma \) itself. Thus, we have defined an element
\[
C(g, \langle \cdot, \cdot \rangle, E) \in S^g \otimes \text{End}(E).
\]
Note that for \( m = 0 \) we obtain an element in \( S^g \equiv S^g \otimes \text{id}_E \). Given a graph homology class \( D \in \mathcal{A}(\ast_x \circ_y) \), we set
\[
D(g, \langle \cdot, \cdot \rangle, E) := \text{tr}_E C(g, \langle \cdot, \cdot \rangle, E) \in S^g,
\]
where \( C \) is any lift of \( D \) under the natural map \( \text{tr}_y: \mathcal{A}(\ast_x \circ_y) \to \mathcal{A}(\ast_x \circ_y) \). This is a well-defined assignment due to the cyclic invariance of the trace map \( \text{tr}_E \) on \( \text{End}(E) \).

This construction applies to holomorphic symplectic manifolds as follows: The (shifted) tangent sheaf \( T_X[-1] \) of a holomorphic symplectic manifold \( X \) can be viewed as a Lie algebra in the bounded derived category of \( X \): The Lie bracket is given by the Atiyah class
\[
A(T_X): T_X[-1] \otimes T_X[-1] \to T_X[-1].
\]
(In fact, this is true on any complex manifold.) A holomorphic symplectic form \( \sigma \) endows \( T_X[-1] \) with an invariant metric (of degree 2)
\[
\sigma: T_X[-1] \otimes T_X[-1] \to \mathcal{O}_X[-2].
\]
Finally, every bounded complex \( \mathcal{F} \) of coherent sheaves becomes a \( T_X[-1] \)-module, where the module action is given by the Atiyah class
\[
A(\mathcal{F}): T_X[-1] \otimes \mathcal{F} \to \mathcal{F}.
\]
As the above construction of \( C(g, \langle \cdot, \cdot \rangle, E) \) also works in the more general context of Lie algebra objects in suitable categories, one can associate to each such triple \((X, \sigma, \mathcal{F})\) classes
\[
C(X, \sigma, \mathcal{F}) \in H^k(X, \Omega^\ell_X \otimes \text{End}(\mathcal{F}))
\]
(resp. \( C(X, \sigma) = C(X, \sigma, \mathcal{F}) \in H^k(X, \Omega^\ell_X) \) if \( m = 0 \)) and
\[
D(X, \sigma, \mathcal{F}) \in H^k(X, \Omega^\ell_X).
\]
(Here, we have identified $\Omega_X$ with $T_X$ by means of the symplectic form.)

The maps $C \mapsto C(X, \sigma, F)$ and $D \mapsto D(X, \sigma, F)$ are called the (Rozansky–Witten) weight system given by $(X, \sigma, F)$.

The following examples all follow from the definitions (see also [16]).

**Example B.4.** Let $\Omega_x \in \mathcal{A}(\ast_x)$ be the so-called Wheeling element (see [21 Sect. 2.7]). Then

$$\Omega_x(X, \sigma) = a(X) \in H\Omega_0(X)$$

(see [10]).

**Example B.5.** Consider $x \overset{x}{\sim} x \in \mathcal{A}(\ast_x)$. One has

$$x \overset{x}{\sim} x(X, \sigma, F) = 2\sigma.$$

**Example B.6.** Let $C \in \mathcal{A}(\ast_x X)$ and $D \in \mathcal{A}(\ast_x)$. Then

$$(C \cup_x D)(X, \sigma, F) = C(X, \sigma, F) \wedge D(X, \sigma, F).$$

**Example B.7.** Let $C \in \mathcal{A}(\ast_x Y)$ and $D \in \mathcal{A}(\uparrow Y)$. Then

$$(C \#_Y D)(X, \sigma, F) = C(X, \sigma, F) \circ D(X, \sigma, F).$$

**Example B.8.** Consider $\exp(x \overset{y}{\sim} y) \in \mathcal{A}(\ast_x Y)$. Then

$$\exp(x \overset{y}{\sim} y)(X, \sigma, F) = \exp(\alpha)(F).$$

**Example B.9.** Let $C \in \mathcal{A}(\ast_x)$ and $D \in \mathcal{A}(\ast_x)$. Then

$$(C \downarrow_x D)(X, \sigma, F) = C(X, \sigma, F) \downarrow_x D(X, \sigma, F),$$

where we view $C(X, \sigma, F)$ as an element in $H\mathcal{T}^*(X)$ by means of the isomorphism $H^p(X, \wedge^q T_X) \cong H^p(X, \Omega^q_X)$ induced by the symplectic form $\sigma$.

**Example B.10.** Let $\pi: \mathcal{A}(\ast_x X) \to \mathcal{A}(\uparrow_x X)$ be the natural map. Then

$$\pi(C)(X, \sigma, F) = C(X, \sigma, F) \circ \exp(\alpha)(F),$$

where we view $C(X, \sigma, F)$ as an element in $H\mathcal{T}^*(X)$ by means of the isomorphism $H^p(X, \wedge^q T_X) \cong H^p(X, \Omega^q_X)$ as above.

B.3. The proof.

**Proof of Lemma 2.7.** Only the case $v \neq 0$ needs a proof, but then we may assume that $v$ is dual to the holomorphic symplectic form $\sigma$.

By Lemma 2.11 below, we have

$$(12) \quad \text{tr}_x[(\Omega_x \downarrow_x \exp(x \overset{y}{\sim} y)) \#_x (\Omega_x \downarrow_x \frac{1}{2} x \overset{x}{\sim} x)] = \text{tr}_x[\Omega_x \downarrow_x (\exp(x \overset{y}{\sim} y) \cup_x \frac{1}{2} x \overset{x}{\sim} x)].$$

It is easy to check that

$$\Omega_x \downarrow_x (\exp(x \overset{y}{\sim} y) \cup_x \frac{1}{2} x \overset{x}{\sim} x) = \frac{1}{2} y \overset{y}{\sim} y \cup_y (\Omega_y \cup_y \exp(x \overset{y}{\sim} y)).$$

Furthermore, one has

$$\Omega_x \downarrow_x \exp(x \overset{y}{\sim} y) = \Omega_y \cup_y \exp(x \overset{y}{\sim} y),$$
so (12) becomes

\[
\Omega_y \cup_y \operatorname{tr}_x[\exp(x \cup_y) \#_x (\Omega_x \cup_x \frac{1}{2}x \cup_x)]
\]

\[
= \operatorname{tr}_x[(\Omega_x \cup_x \exp(x \cup_y)) \#_x (\Omega_x \cup_x \frac{1}{2}x \cup_x)]
\]

\[
= \operatorname{tr}_x[\frac{1}{2}y \cup_y \exp(x \cup_y)]
\]

\[
= \frac{1}{2}y \cup_y \exp(x \cup_y).
\]

(13)

Applying the Rozansky–Witten weight system given by \((X, \sigma, \mathcal{F})\) to (13) then yields

\[
a \land \operatorname{tr}_x[\exp(\alpha)(\mathcal{F})] \circ (a \land v) \circ \exp(\alpha)(\mathcal{F}) = v \land (a \land \operatorname{tr}_x[\exp(\alpha)(\mathcal{F})])
\]

\[
= v \land (a \land \operatorname{ch}(\mathcal{F})),
\]

from which the claim of the Lemma easily follows. \(\square\)

**Lemma B.11.** The following relation holds in the graph homology space \(\mathcal{A}(\cup_x * y)\):

\[
\operatorname{tr}_x[(\Omega_x \cup_x X) \#_x (\Omega_x \cup_x Y)] = \operatorname{tr}_x[\Omega_x \cup_x (C \cup_x D)]
\]

for all \(C \in \mathcal{A}(\cup_x * y)\) and \(D \in \mathcal{A}(\cup_x).\)

**Proof.** Let \(H_{z,x} \in \mathcal{A}(\cup_x * y)\) be the Kontsevich integral of a bead \(x\) on a wire \(z\) (see [21]). It has the property

\[
\Delta_{x_1,x_2}^z H_{z,x} = H_{z,x_1} \#_z H_{z,x_2} \in \mathcal{A}(\star_z \cup_{x_1} \cup_{x_2}).
\]

Define an operator

\[
\Phi: \mathcal{A}(\cup_x * y) \to \mathcal{A}(\cup_x * y), \ X \mapsto (H_{z,x}, X)_{x} := (\tilde{H}_{z,x}, X)_{x},
\]

where \(\tilde{H}_{z,x} \in \mathcal{A}(\star_z \cup_z)\) is any lift of \(H_{z,x}\) under the map \(\operatorname{tr}_x \circ \pi_x: \mathcal{A}(\star_z \cup_z) \to \mathcal{A}(\star_z \cup_z).\) (That the operator is independent of the chosen lift follows from the same considerations as in the proof of [21] Lemma 5.4.) By [21] Thm. 4, we moreover have

\[
H_{z,x} = \operatorname{tr}_x[\exp(x \cup z \cup_x \Omega_x)].
\]

It follows that

\[
\Phi(C) = \langle (\exp(x \cup z) \cup_x \Omega_x, C)_{x} = \Delta_{x}^z(\Omega_x \cup_x C)\rangle.
\]

Thus one has

\[
\operatorname{tr}_y \operatorname{tr}_x[(\Omega_x \cup C) \#_x (\Omega_x \cup_x D)]
\]

\[
= \operatorname{tr}_x[(\Omega_x \cup \operatorname{tr}_y(C)) \#_x (\Omega_x \cup_x D)]
\]

\[
= \operatorname{tr}_x[(H_{x,x_1} \#_x H_{x,x_2}, \Delta_{x_1}^z \operatorname{tr}_y(C) \otimes \Delta_{x_2}^z(D))_{x_1,x_2}]
\]

\[
= \operatorname{tr}_x[\langle \Delta_{x_1,x_2}^z H_{x,z}, \Delta_{x_1}^z \operatorname{tr}_y(C) \otimes \Delta_{x_2}^z(D)\rangle_{x_1,x_2}]
\]

\[
= \operatorname{tr}_x[\Delta_{x_1,x_2}^z(H_{x,z}, \operatorname{tr}_y(C) \cup_x D)_{x}]
\]

\[
= \operatorname{tr}_x[\Omega_x \cup_z (\operatorname{tr}_y(C) \cup_x D)]
\]

\[
= \operatorname{tr}_y \operatorname{tr}_x[\Omega_x \cup_x (C \cup_x D)].
\]

Finally recall from [21] that \(\operatorname{tr}_y: \mathcal{A}(\cup_x * y) \to \mathcal{A}(\cup_x \cup_y)\) is an isomorphism. \(\square\)
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