Influence of qubits’ nonradiative decay into a common bath on the transport properties of microwave photons

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We consider the influence of nonradiative damping of qubits on the microwave transport of photons, propagating in an open one-dimensional microstrip line. Within the framework of the formalism of a non-Hermitian Hamiltonian we obtained the expressions for the transmission and reflection coefficients for two qubits which explicitly account for the indirect interaction between qubits due to nonradiative decay into common bath. It is shown that this interaction leads to the results that are significantly different from those already known.

I. INTRODUCTION

Coherent interaction of solid-state qubits with microwave modes of the coplanar waveguide is a field of extensive theoretical and experimental research (see the reviews [1,2] and references therein). Unlike the real atoms in an optical cavity, the artificial atoms have significant nonradiative damping channels when the energy of the excited state of a qubit radiates not into a waveguide, but transmits to other degrees of freedom, not related to the radiation. The calculation of microwave transport with due account of the nonradiative damping of qubits is of great importance from the point of manipulation and control of qubits by means of a microwave field.

Traditionally, the qubits damping is assumed to be uncorrelated: every qubit decays to its individual bath. From the formal point of view, it corresponds to the addition to the qubit excitation frequency $\omega$ of the imaginary quantity $-i\gamma$, where $\gamma$ is the decay rate of the excited state of a qubit into a nonradiative channel [3–6]. It seems to be reasonable for one qubit in a waveguide. However, for two qubits their decay through a common nonradiative channel leads to their indirect interaction, so that the damping rates of individual qubits are not independent any more [7]. In this case, the expressions for reflection and transmission coefficients are substantially modified, so that a simple substitution $\omega \rightarrow \omega - i\gamma$ is no longer correct. In particular, as was indicated in [8] such simple replacement is not sufficient for correct description of the entanglement between qubits.

More generally, the influence of a common bath on the decoherence and relaxation in qubit systems has been studied in [9,10] for superconducting qubits and in [11,12] for quantum dots. Specially prepared common bath can also be used for the creation of non-decaying entangled states in multi-qubit systems [13,14].

In the present work we study a microwave photon transport through an open waveguide with imbedded two qubits at a distance $d$ from each other. Qubits are characterized by their rates of spontaneous emission into a waveguide $\Gamma_1, \Gamma_2$, the rates of nonradiative damping into local channels $\kappa_1, \kappa_2$ and in a common bath $\gamma_1, \gamma_2$. The damping in a common bath gives rise to off-diagonal elements in Hamiltonian matrix which significantly influences the transport characteristics of microwaves photons.

The calculation of transport coefficients is carried out in the formalism of the effective non-Hermitian Hamiltonian, which has a rather wide range of applicability in the study of various kinds of open quantum systems (see the review articles [15,16] and numerous examples and references therein). This formalism in application to photon transport through one-dimensional chain of two-level systems is described in detail in [17], hence here we will only limit it to a concise summary.

II. FORMULATION OF THE PROBLEM

The Hamiltonian of two qubits interacting with a photon field in an open one dimensional waveguide, is as follows [18]:

$$ H = \sum_{i=1,2} \frac{1}{2} \Omega_i \left(1 + \sigma^{(i)}_Z\right) + \sum_k \omega_k a_k^+ a_k + J \left(\sigma^{(1)}_+ \sigma^{(2)}_- + \sigma^{(2)}_+ \sigma^{(1)}_-\right) + \sum_k \sum_{i=1,2} \lambda_i(\alpha^+_k e^{-ikx_i} a_k + a_k e^{ikx_i}) \sigma^{(i)}_Z + H_\gamma $$

where $\Omega_i$, $(i = 1, 2)$ is the qubit excitation frequency, $\omega_k$ is the waveguide modes, $J$ is the strength of the direct exchange interaction between qubits, $\lambda_i$ is the interaction strength of the $i$-th qubit located at the point $x_i$ with a photon field, $a_k^+, a_k$ are creation and annihilation photon operators, $\sigma^{(i)}_n$ are the Pauli spin matrices, where a superscript refer to the qubit number. Further, we take a coordinate origin in the middle point between two qubits, so that $x_1 = -d/2$, $x_2 = d/2$.

The fourth term in (1) is responsible for spontaneous emission of qubits into a waveguide, and the quantity $H_\gamma$, which we do not specify here, is responsible for nonradiative decay of the excited state of a qubit.

III. CALCULATION OF THE EFFECTIVE NON-HERMITIAN HAMILTONIAN

According to the method of the effective non-Hermitian Hamiltonian, the Hilbert space is divided into two mutually

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orthogonal subspaces: the internal subspace $Q$ and the external subspace $P$. The subspace $Q$ describes a closed system of the stationary states $|n\rangle$ that does not interact with the external environment. The subspace $P$ contains, in addition to the states of a closed system, also the states from the continuum, to which the system $Q$ decays due to its interaction with the system $P$. Due to this interaction the system $Q$ becomes unstable: discrete energies acquire negative imaginary components, which means in a time domain a decay of the system $Q$.

In our case the subspace $Q$ consists of two vectors, corresponding to the states in which one of the two qubits is in the excited state $|e\rangle$, and the other is in the ground state $|g\rangle$: $|1\rangle = |e_1 g_2\rangle$, $|2\rangle = |g_1 e_2\rangle$. The subspace $P$ contains vectors with both qubits being in the ground state and one photon being either in the waveguide or in the nonradiative decay channel. The dynamics of the entire system described by the Hamiltonian (1) can be projected on the evolution of the system $Q$ through an effective the non-Hermitian Hamiltonian whose matrix elements in the basis of the states of the system $Q$ can be written as follows [17]:

$$
\langle m | H_{eff} | n \rangle = \langle m | H | n \rangle - i \frac{1}{2} \sum_{c=1}^{3} A_m^c A_n^c + \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \frac{A_m(q)A_n^*(q)}{k - q + i\varepsilon} \tag{2}
$$

where the states $|m\rangle, |n\rangle$ belong to the set $|1\rangle, |2\rangle$, $k$ is the wave vector of the photon scattered in the waveguide. The summation in (2) runs over three decay channels: two local channels ($c = 1, 2$) and a common channel ($c = 3$). The imaginary quantity $i\varepsilon$ allows to avoid singularities in the integral and ensures divergent scattering waves. The amplitudes $A_m(q)$ are matrix elements of the transition between states of the subspace $Q$ and the states of subspaces $P$, where there is one photon in the waveguide. The amplitudes $A_m^c$ describe the damping of the excited state of the $m$-th qubit in a nonradiative decay channel. For local channels the amplitudes $A_1^c, A_2^c$ are different from zero, while channels $A_2^c = A_3^c = 0$. For third common channel the amplitudes $A_1^c, A_3^c$ are different from zero.

A direct calculation of the amplitudes $A_m(q)$ and the integral in (2) leads to the following result [17]:

$$
A_m(q) = \sqrt{\Gamma_m e^{i\varepsilon_m}} \tag{3}
$$

where $\Gamma_m$ is the rate of spontaneous emission of $m$-th qubit into a waveguide:

$$
\Gamma_m = \frac{L \lambda_m^2}{\hbar^2 v_g} \tag{4}
$$

$L$ - the waveguide length, $v_g$ - the group velocity of photons in a waveguide.

$$
\frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \frac{A_m(q)A_n^*(q)}{k - q + i\varepsilon} = -i \sqrt{\Gamma_m} e^{i k d_{mn}} \tag{5}
$$

where $d_{mn} = x_m - x_n$, $k$ is the wavevector which is related to the frequency of scattered photon, $k = \omega / v_g$.

We consider the amplitudes $A_m^c$ as constant quantities which do not depend on the energy of the scattered photon: $A_1^2 = \sqrt{\kappa_1}$, $A_2^2 = \sqrt{\kappa_2}$. $A_1^3 = \sqrt{\gamma_1}$, $A_2^3 = \sqrt{\gamma_2}$. In addition, we assume the amplitudes $A_1^2, A_2^2$ are independent on the inter-qubit distance $d$ which corresponds to Markov approximation: the interaction between qubits in a common bath is non-retarded. In other words, the interaction wavelength in a common channel is much larger than the inter-qubit distance.

Hence, in the basis set $|1\rangle, |2\rangle$ we obtain from (2) the matrix of effective Hamiltonian:

$$
H_{eff} = \begin{pmatrix}
\Omega_1 - i\Gamma_1 & A \\
A & \Omega_2 - i\Gamma_2
\end{pmatrix} \tag{6}
$$

where

$$
\Gamma_i = J - i \sqrt{\Gamma_1 \Gamma_2 e^{i k d}} - \frac{i}{2} \sqrt{\gamma_1 \gamma_2} \tag{7}
$$

The positions of resonances in a complex plane are determined by the equation $\det(\tilde{\omega} - H_{eff}) = 0$ which gives the following result:

$$
\tilde{\omega}_\pm = \frac{1}{2}(\Omega_1 + \Omega_2) - i \frac{1}{2}(\tilde{\Gamma}_1 + \tilde{\Gamma}_2) \\
\pm \frac{1}{2} \sqrt{(\Omega_1 - \Omega_2 + i(\tilde{\Gamma}_1 - \tilde{\Gamma}_2))}^2 + 4\Lambda^2 \tag{8}
$$

Off-diagonal elements in the matrix (6), in addition to the direct interactions $J$ describe an indirect interaction $\sqrt{\Gamma_1 \Gamma_2 e^{i k d}}$ due to spontaneous radiation of the excited qubits, and the interaction $\sqrt{\gamma_1 \gamma_2} / 2$, caused by nonradiative decay into a common channel. We note that the contribution of spontaneous emission depends on the frequency of the scattered photon $\omega$ ($k = \omega / v_g$). As a result the position of the resonances (8) in the complex plane depends on the frequency of the scattered photon. This is a manifestation of the retardation effect. Thus, in this formalism, the non-Markovian effects in photon-qubit interaction are taken into account automatically. Therefore, the position of the resonances on the real frequency axis is determined, in general, by a nonlinear equation $\omega = \Re[\tilde{\omega}_\pm(\omega)]$.

For identical qubits we obtain from (8):

$$
\Re \omega_\pm = \Omega \pm J \pm \Gamma \sin k d \tag{9}
$$

$$
\Im \omega_\pm = -\Gamma(1 \pm \cos k d) - \frac{1}{2} J - \frac{1}{2} \pm \frac{1}{2} \gamma \tag{10}
$$
As it follows from (10), the resonance width $\text{Im} \omega$ does not depend on the parameter $\gamma$ and in the limit $kd \ll 1$ is defined only by relaxation $\kappa$ in a local channel. This result is due to the interaction of non-radiative decay into a common channel (the last term on the right hand side of Eq. (7)). If it had not been taken into account, then the widths of both resonances would be proportional to $\gamma$. Thus, with the increase of $\gamma$ the width of one resonance increases linearly, while the width of other one remains unchanged. Fig. 1 shows the resonance spectrum $S(\omega)$ for two identical qubits, which is determined by the zeros of the real part of the determinant $D(\omega)$ ($S(\omega) \approx 1/D(\omega)$). The left peak corresponds to the frequency $\text{Re} \omega_-$, the right one to the frequency $\text{Re} \omega_+$. With the increase of $\gamma$, the width and the amplitude of the left peak remains unchanged while the width of the right peak increases and its amplitude decreases. Here and below the quantity $k_0 = \Omega/v_g$.

**IV. TWO QUBIT ENTANGLEMENT**

The wave function for two qubit system interacting with a photon field reads (see Eq.42 in [17]):

$$\Psi_Q = \sum_{n,m=1}^{2} |n \rangle \lambda_m R_{nm} e^{ikx_m}$$

In equation (11) the matrix $R_{nm}$ is the inverse of the matrix $(\omega - H_{\text{eff}})_{mn}$:

$$R = \frac{1}{D(\omega)} \begin{pmatrix} \omega - \Omega_2 + i\Gamma_2 & \Lambda \\ \Lambda & \omega - \Omega_1 + i\Gamma_1 \end{pmatrix}$$

(12)

where

$$D(\omega) = \text{det}(\omega - H_{\text{eff}})$$

$$= (\omega - \Omega_1 + i\Gamma_1)(\omega - \Omega_2 + i\Gamma_2) - \Lambda^2$$

(13)

Determinant (13) can also be written as follows

$$D(\omega) = (\omega - \bar{\omega}_-)(\omega - \bar{\omega}_+)$$

(14)

where the complex roots $\bar{\omega}_\pm$ are determined from (8).

From (12) we obtain the two qubit wavefunction:

$$\Psi_Q = \frac{\Lambda}{D(\omega)} (|\Omega + i\Gamma_1 \rangle (|1 \rangle e^{-ikd} + |2 \rangle e^{ikd}) + \frac{\Lambda}{D(\omega)} |\Omega + i\Gamma_1 \rangle (|1 \rangle e^{ikd} + |2 \rangle e^{-ikd})$$

(15)

This expression allows us to determine the probability amplitudes for the excitation of each of the qubits ($\langle 1 | \Psi_Q \rangle$, $\langle 2 | \Psi_Q \rangle$), and the degree of entanglement of two-qubit state. It follows from (15) that the maximally entanglement states $\Psi_Q \approx (|1 \rangle \pm |2 \rangle)$ at arbitrary frequency $\omega$ of the scattered photon take place in the limit $kd \ll 1$, as well as for discrete frequencies determined from the relation $kd = n\pi/2$, where $n = 1, 2, 3......$
FIG. 3: a) The dependence of transmission $|t|$ on the frequency of scattered photon and b) on the nonradiative decay parameter $\gamma$; solid lines (A) correspond to equation (11a) from [6], dotted lines (B) are calculated from (20).

FIG. 4: Two dimension (in the plain $kd, \omega$) pattern of transmission factor $|t|$ Near $kd = 3\pi/2$ the transparency region is well visible.

V. CALCULATION OF THE PHOTON TRANSPORT COEFFICIENTS

In accordance with the method of non Hermitian Hamiltonian the photon transition from a state with an initial momentum $k$ into a state with a final momentum $q$ is given by the amplitude [16]:

$$T^{qk} = \sum_{m,n=1}^{2} A_m(q) R_{mn} A_n^*(k)$$

(16)

Here the coefficients of photon transport $t^{qk}$ are the elements of scattering matrix $S^{qk}$ which is related to $T^{qk}$ as follows:

$$t^{qk} \equiv S^{qk} = \delta^{qk} - iT^{qk}$$

(17)

For further consideration we rewrite the determinant (13) in the following form:

$$D(\omega) = \delta_1 \delta_2 - J^2 + i\delta_1 \Gamma_2 + i\delta_2 \Gamma_1 + 2iJ \sqrt{\Gamma_1 \Gamma_2} e^{ikd}$$

$$-\Gamma_1 \Gamma_2 (1 - e^{2ikd}) + A$$

(18)

where $\delta_j = \omega - \Omega_j + \frac{i}{2} \kappa_j + \frac{i}{2} \gamma_j, j = 1, 2$.

In the last line of (13) we define the quantity $A$, which is responsible for the interaction between qubits due to nonradiative decay channels:

$$A = \frac{\sqrt{\gamma_1 \gamma_2}}{4} \left( \sqrt{\gamma_1 \gamma_2} + 4i(J - i\sqrt{\Gamma_1 \Gamma_2} e^{ikd}) \right)$$

(19)

The photon transmission corresponds to $q = k$, the reflection corresponds to $q = -k$. Then, in accordance with (17) the transmission $t$ and reflection $r$ coefficients can be expressed as: $t = t^{kk} = 1 - iT^{kk}, r = t^{-kk} = -iT^{-kk}$. By using (16) and the explicit expression for $R$ matrix (12) we write these coefficients in the following form:

$$t = \frac{1}{D(\omega)} \left( \delta_1 \delta_2 - J^2 - 2J \sqrt{\Gamma_1 \Gamma_2} \sin kd + B \right)$$

(20)

$$r = -\frac{i}{D(\omega)} \left[ \delta_2 \Gamma_1 e^{i kd} + \delta_1 \Gamma_2 e^{-i kd} + 2\Gamma_1 \Gamma_2 \sin kd + 2J \sqrt{\Gamma_1 \Gamma_2} + C \right]$$

(21)

where $D(\omega)$ was defined in (18), and the quantities $B$ and $C$ as well as $A$ (19) describes the contribution of nonradiative decay channels into the interaction between the qubits:

$$B = \frac{\sqrt{\gamma_1 \gamma_2}}{4} \left( \sqrt{\gamma_1 \gamma_2} + 4i(J + \sqrt{\Gamma_1 \Gamma_2} \sin kd) \right)$$

(22)

$$C = -\sqrt{\Gamma_1 \Gamma_2} \sqrt{\gamma_1 \gamma_2}$$

(23)

Neglecting the quantities $A$, $B$ and $C$, in the expressions (18), (20) and (21) we obtain for the coefficients $t$ and $r$ the...
expressions which exactly coincide with those already known (expressions (11a) and (11b) in the work [6]).

In Fig. 2 and Fig. 3 the transmission $|t|$ calculated from (20) is compared with the one calculated from the expression (11a) in [6]). The calculations were carried out for identical qubits: $\Omega_1 = \Omega_2, \Gamma_1 = \Gamma_2, \gamma_1 = \gamma_2$ for fixed values $\Omega_1 = 3$ GHz, $\Gamma_1 = 3 \times 10^{-4} \Omega_1$. From Fig. 2 it is clearly seen that in the region $J < 2\Gamma_1$ the expression (20) leads to the results that substantially differ from those which follow from the expression (11a) in [6]. Even more clearly this difference is evident in Fig. 3 where the presence of indirect interaction between qubits due to nonradiative decay channels results in a significant suppression of the photon transmission. Two dimensional (in the plane $k_0d, \omega$) distribution of the transmission $|t|$ calculated from (20) is shown in Fig. 4. Near $k_0d = 3\pi/2$, the region, where the transmission coefficient is close to unity in a wide frequency range, is clearly visible. The numerical simulations showed that the width of this region along the vertical axis is proportional to the magnitude $J$ and inversely proportional to the rate of nonradiative decay $\gamma$.

VI. CONCLUSION.

It was shown that the existence of nonradiative decay into a common bath results in additional interaction between qubits, which renders significant influence on the transmission and reflection of microwave photons. We studied in detail a two qubit system in an open photonic waveguide. The analytic expressions for the transmission and reflection coefficients were obtained. It is shown that there are parameter regions, where the inter-qubit correlations due to relaxation into a common bath lead to the results which are significantly different from the known ones.

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