Time-reversal symmetry and random polynomials

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We analyze the density of roots of random polynomials where each complex coefficient is constructed of a random modulus and a fixed, deterministic phase. The density of roots is shown to possess a singular component only in the case for which the phases increase linearly with the index of coefficients. This means that, contrary to earlier belief, eigenvectors of a typical quantum chaotic system with some antiunitary symmetry will not display a clustering curve in the stellar representation. Moreover, a class of time-reverse invariant quantum systems is shown, for which spectra display fluctuations characteristic of orthogonal ensemble, while eigenvectors conferr to predictions of unitary ensemble.

The distribution of roots of polynomials of high degree with random coefficients was investigated recently in connection with properties of quantum chaotic systems [1–4]. In particular the authors of the cited references considered the coherent state representation of eigenstates of a quantum mechanical spin system with the total spin $S$. The polynomials in question have the form

$$P(z) = \sum_{k=0}^{N} C_{k} a_{k} z^{k}, \quad N = 2S \quad (1)$$

where $C_{k}$ stand for binomial coefficients and $a_{k}$ are components of an eigenvector. The complex variable $z$ is connected to the Bloch sphere angular variables $\theta, \phi$ via $z = \tan(\theta/2) \exp(i\phi)$. It was shown by Lebœuf and Voros [1], that for large values of $S$ when the quantum system in question is chaotic the distribution of the roots is given by

$$\rho(z) = \frac{N}{\pi} \frac{1}{1 + |z|^{2}} \quad (2)$$

corresponding to the uniform distribution of the roots over the Bloch sphere. This is the consequence of the fact, that in the semiclassical limit $N \to \infty$ the components with respect to a "generic basis" of the eigenvectors of a chaotic systems are independently normally distributed (see [3] and references therein).

The details of the distribution of the components $a_{k}$ depend on symmetries of the system in question. For systems which are not time-reversal invariant the eigenvector components are complex, with independently, normally distributed real and imaginary parts, whereas for time-reversal invariant systems the eigenvectors can be made real (also with normally distributed components). In the latter situation the uniform distribution (2) is modified. In particular, the roots tend to concentrate on the real line $\text{Im}z = 0$, which is a symmetry line for the roots (if $z_{0}$ is a root then it’s complex conjugate $z_{0}^{*}$ is also a root) [3], see also below). When projected back on the sphere the symmetry line is the great circle $\phi = 0$.

This simplest situation corresponds to the case when the time-reversal operator is represented by the complex conjugation operator. On the other hand it is known that generalized time-reversal symmetries, represented by the complex conjugation supplemented by a unitary transformation, influence statistical properties of eigenvector components in the same way as the conventional time-reversal symmetry [3]. As an illustration the authors of Refs. [3] considered various models of the so called kicked top system [1], which is described by the one-step evolution operator of the form $U = \exp(-i\hat{f}_{1}) \exp(-i\hat{f}_{2}) \exp(-i\hat{f}_{3})$ with $f_{i} = f_{i}(S_{x}, S_{y}, S_{z}), i = 1, 2, 3$ polynomial functions of the components of the spin operator $S = (S_{x}, S_{y}, S_{z})$. The simplest case displaying chaotic dynamics in the classical limit is obtained by choosing $U_{0} = \exp(-i\mu S_{x}) \exp(-i\mu S_{y})$ with appropriate values of the parameters $\mu$ and $p$. It has two generalized time-reversal symmetries $T_{1} = \exp(-i\mu S_{x})K$ and $T_{2} = \exp(-i\mu S_{y}) \exp(i\pi S_{y}) \exp(i\pi S_{z})K$, $T_{i}U_{0}^{-1} = U_{i}$ both being compositions of linear rotations with the complex conjugation operator $K$. The rotations shift the symmetry line from the great circle $\phi = 0$ to other ones, the phenomenon exhibited by the numerical investigations performed by the authors of Refs. [3].

A non-homogeneous distribution of zeros of Husimi functions is linked to statistical properties of coherent states expanded in the eigenbasis of the Floquet operator. In particular, the number of relevant eigenstates [1] and the entropy of coherent states [1] was found for this model to be smaller than average along the symmetry lines $T_{i}$. A smaller number of significantly occupied eigenstates denotes a larger number of weakly occupied states, in consistency with investigated clustering of zeros of eigenstates in Husimi representation along the symmetry curves. Moreover, the distribution of expansion coefficients of a coherent state localized sufficiently far away from the symmetry lines is statistically indistinguishable from properties of a generic coherent state of a system without any antiunitary symmetry [1]. This corresponds to recent result of Prosen [1], who showed that the densities of zeros of random polynomials with real and complex coefficients are equal sufficiently far away from the real axis.
In order to break the generalized time-reversal symmetry the original model $U_0$ was supplemented by a nonlinear rotation $f_1 = qS^2 / 2S$ (in Ref. [3]) or $f'_1 = qS^2 / 2S$ (in Ref. [3]) instead of $f_1 = 0$. In their numerical investigations Bogomolny et al. observed vanishing of the concentration of the roots which they attributed to the breaking of the time-reversal symmetry. In what follows we will argue that the concentration of the roots on the symmetry lines happens in the case of generalized time invariance only exceptionally and as such cannot be treated as a criterium discriminating between the time reversal invariant and noninvariant systems. In particular the kicked tops $U_1 = \exp(-iqS^2)U_0$ and $U_2 = \exp(-iqS^2)U_0$ differ with respect to the statistical properties of the spectra for generic values of the parameter $q$. The additional rotation term brakes all generalized time-reversal symmetries for the first top and $U_1$ pertains to the circular unitary ensemble (CUE), while the second still possess such a symmetry

$$T' = \exp(-iqS^2)\exp(-i\mu S_z)\exp(-iqS^2)K,$$  \hspace{1cm} (3)

and its spectrum is typical to circular orthogonal ensemble (COE), irrespective of the value of $q$. Note that the above operator is constructed of a nonlinear unitary rotation (quadratic term $S^2_z$ in the exponent), in contrast to the operators $T_1$ and $T_2$.

Inasmuch as level statistics reveals directly the symmetry properties of quantum systems, special care has to be taken interpreting the statistical properties of eigenvectors, since their distribution depends on the basis chosen. For example, the distribution of eigenvectors of $U_0$ in the $S_z$ basis does not confer to COE predictions. The agreement with random matrices is recovered in $S_z$ basis: the geometric symmetry of the top manifests itself in the structure of operator $U_0$. It splits into two parities of size $S$ and $S + 1$, which have to be treated separately to achieve results according to random matrices. In earlier references [10–12] the variables of the top were exchanged $x \leftrightarrow z$, what gives the same effect.

The distribution of eigenvectors can be characterized by their mean entropy $H$, which for random matrices of size $N$ is equal to $H(N, \beta) = \Psi(N \beta / 2 + 1) - \Psi(\beta / 2 + 1)$, where $\Psi$ stands for the digamma function and $\beta = 1$ for COE and $\beta = 2$ for CUE [3]. Figure 1 presents the entropy of eigenvectors relative to the entropy of CUE for two tops $U_1$ and $U_2$ as a function of the control parameter $q$. Observe similar behaviour for "unitary" top $U_1$ and the "orthogonal" top $U_2$. The dips in the data for unitary top at $q = 0$ and $q = p = 6.0$ correspond to transitions to the orthogonal class, while $U_2$ pertains to COE for any value of $q$ due to the symmetry [3]. This difference is visualized in level spacing distribution $P(s)$ displayed in the inset. An explanation of this fact is simple: out of any "orthogonal" spectrum $D_1$ by a generic unitary rotation $W$ one can produce an operator $U_W = WD_1W^\dagger$ which enjoys COE-like properties of the spectrum and CUE-like properties of the eigenvectors. This is exactly the case of the top $U_2$, for which the operator $\exp(-iqS^2)$ plays the role of $W$. Observe that $U_2$ is similar to the orthogonal top $U'_2 = \exp(-i\mu S_z)\exp[-i(p + q)S^2_z / 2S]$. A similar effect is visible in the distribution of zeros of Husimi function representing eigenvectors: both tops show lack of roots concentration lines as shown for $U_1$ in [3] and for $U_2$ in [3], even though they belong to different universality classes.

In order to understand the above announced results let us derive the density of roots $\rho$ of a polynomial [1], where $a_k$ are Gaussian distributed random quantities with fixed but arbitrary phases $\varphi_k$:

$$a_k = r_k e^{i\varphi_k},$$  \hspace{1cm} (4)

the $r_k$ being distributed according to

$$P(r_k) = \frac{1}{\sqrt{2\pi}} e^{-r_k^2/2}.$$  \hspace{1cm} (5)

We will make use of the same technique employed in Ref. [3], namely representing $\rho(r, \varphi)$ by the Kac–formula,

$$\rho(z) = \delta[P(z)] \left| \frac{dP(z)}{dz} \right|,$$  \hspace{1cm} (6)

and then expressing the delta–functions for the real – and imaginary parts of $P(z)$ as Fourier integrals. We then get, in full analogy with Eq.(C6) in Ref. [3]:

$$\rho(r, \varphi) = \frac{1}{(2\pi)^2} \int d\xi_1 \int d\xi_2 \left\{ \sum_{k=0}^{N} k^2 C_N^k r_k^2 r_{l-k}^2 + \sum_{k\neq l=0}^{N} kl \sqrt{C_N^k C_N^l} r_k r_l r_k^2 r_{l-k}^2 e^{i(\varphi_k - \varphi_l) + i(k-l)\varphi} \right\} \exp \left\{ \sum_{k=0}^{N} r_k (\alpha_n \cos \varphi_n + \beta_n \sin \varphi_n) \right\},$$  \hspace{1cm} (7)

where

$$\alpha_n = ir^n \sqrt{C_N^n} (\cos(n\varphi) \xi_1 + \sin(n\varphi) \xi_2)$$  \hspace{1cm} (8)

$$\beta_n = ir^n \sqrt{C_N^n} (\cos(n\varphi) \xi_2 - \sin(n\varphi) \xi_1)$$  \hspace{1cm} (9)

and $z = re^{i\varphi}$. Averaging over the random coefficients $r_k$ amounts now to simple Gaussian integrations. The resulting average density can be cast in the following form:

$$\langle \rho(r, \varphi) \rangle = \frac{1}{(2\pi)^2} \int d\xi_1 \int d\xi_2 (A + B\xi_1 \xi_2 + C\xi_1^2 + D\xi_2^2) \exp(-a\xi_1^2 - b\xi_2^2 - 2c\xi_1 \xi_2),$$  \hspace{1cm} (10)

where
Thus, the points \((r, \varphi)\) for which the average density of roots diverges are the zeros of the polynomial in Eq.\((23)\). However, \(Q(r, \varphi)\) is positive semi–definite. The only possibility of \(Q(r\varphi) = 0\) is given by \(r = 0\) (which is always a solution and thus always a point of singular density), or by simultaneous vanishing of all coefficients: 
\[
\sin(\varphi_m + m\varphi - \varphi_n - n\varphi) = 0 \quad \text{for all} \ m, n.
\]
In the latter case \(Q(r, \varphi)\) will be zero for all \(r\), implying immediately that lines of singular density can be only straight lines in the \(z\)-plane. On the other hand, assuming that \(\varphi_k\) is a differentiable function of the index \(k\), one finds that \(\varphi_k = -k\varphi + \text{const.}\) with a \(k\)-independent constant. Since the phases \(\varphi_k\) were chosen as constants, the only way to fulfill this equation is by \(\varphi_k = k\alpha + \beta, \varphi = -\alpha\). For any other choice of the \(k\)-dependence of the \(\varphi_k\), lines with more or less pronounced maxima of \(\rho(r, \varphi)\) may still exist, but the singular character of the density is lost — with the exception of the origin.

Above reasoning proves our claim that curves of singular density are only possible if the phases \(\varphi_k\) increase linearly with the index \(k\). This is exactly the case of the top \(U_0\), for which the symmetries \(T_1\) and \(T_2\) manifest themselves as singularities along straight lines on the complex plane, which correspond to great circles on the sphere.

On the other hand, all the deviations from the above form result in a blurring of the sharp lines seen when plotting numerically obtained roots of random polynomials, irrespective of whether a particular symmetry of the possibly underlying physical system is still preserved or not. To demonstrate this effect we have analyzed random polynomials \([\tilde{1}]\) with coefficients \([\tilde{1}]\) given by \(\varphi_k = qk^2/N\). This assumption corresponds to the problem induced by the generalized time-reversal symmetry \([\tilde{3}]\) of the top \(U_2\). For \(q = 0\) (real coefficients \(a_k\)) the distribution of zeros suffers a singularity along the real axis, while for larger value of \(q\) the clustering curve twists and acquires a finite width. This is visible in figures 2 and 3 where we plotted on a complex plane zeros of 50 random polynomials with \(N = 40\) (part a) and the density of zeros obtained according to Eq.\((19)\) (part b). For \(q=0.2\) the symmetry line already deviates from the real axis. For \(q = 0.5\) a ridge in the density of zeros is still observed, at \(q \sim 1\) the distribution of zeros is almost homogeneous. Interestingly, the qualitative character of the density does not change much with \(N\).

Let us mention here that the density of zeros of random polynomials \([\tilde{1}]\) with fixed phases can be obtained using slightly different techniques proposed by Edelman and Kostlan \([\tilde{2}]\), Shepp and Vanderbei \([\tilde{4}]\) or Prosen \([\tilde{5}]\). Moreover, the density of roots of some generalized random polynomials was recently discussed in \([\tilde{16}]\).

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FIG. 1. Mean entropy of eigenvectors compared with the entropy $H_{\text{CUE}}$ of the unitary ensemble drawn as a function of the perturbation parameter $q$ for two models: "unitary" top $U_1(\Delta)$ and "orthogonal" top $U_2(\phi)$ with $\mu = 1.7, p = 6.0$ and spin length $S = 40$. The dashed line represents the value $H_{\text{COE}}/H_{\text{CUE}} \approx 0.91$. The inset shows the cumulative level spacing distribution $P(s)$ obtained for both models out of 100 operators $U$ with fixed $q = 2.0$ and $p$ varying from 6.0 to 12.0 and compared to the Wigner surmises for both universality classes.

FIG. 2. The distribution of roots of 50 random polynomials with quadratically increasing phases ($q = 0.2$) shown in part a) follows the analytically obtained density shown in the contour plot in part b). The concentration line of the zeros deviates from the real axis and is no longer a line of singular density.

FIG. 3. As in Fig 2 for $q = 0.5$. The concentration line of the zeros is even more blurred than for $q = 0.2$. 
\[ N = 40, \alpha = 0.2 \]
