Staircase Models from Affine Toda Field Theory

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We propose a class of purely elastic scattering theories generalising the staircase model of Al. B. Zamolodchikov, based on the affine Toda field theories for simply-laced Lie algebras $g = A, D, E$ at suitable complex values of their coupling constants. Considering their Thermodynamic Bethe Ansatz equations, we give analytic arguments in support of a conjectured renormalisation group flow visiting the neighbourhood of each $W_g$ minimal model in turn.

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1. Introduction

The proposals of A. B. Zamolodchikov for the scaling region of the Ising model in a magnetic field \([1]\) have stimulated a renewal of interest in two-dimensional integrable field theory. To think of a model as a perturbation of a conformal field theory by certain of its relevant operators allows much non-perturbative information to be extracted, starting with the existence of the conserved charges which guarantee integrability, and often leading on from there to hypotheses for an exact S-matrix and mass spectrum, hypotheses which in many cases have now withstood independent checks.

One such check is to ‘complete the circle’, reconstructing information on the ultraviolet limit from the infra-red data contained in the hypothesised exact S-matrix. If this matches with the content of the initial conformal field theory, a very non-trivial confirmation of the whole approach has been achieved. Better still, non-perturbative information on the whole renormalisation group trajectory between ultraviolet and infra-red can often be extracted. The machinery employed is known as the Thermodynamic Bethe Ansatz (TBA) \([2]\), first used in this (relativistic) context by Al. B. Zamolodchikov \([3]\); subsequently these techniques have been developed by many others.

The method is straightforward if the scattering is purely elastic \([3–6]\), that is if the S-matrix is diagonal in a suitable basis. The thermodynamics of the system in a box can be reconstructed from a set of coupled non-linear integral equations (the TBA equations), and these equations follow immediately from the logarithmic derivative of the S-matrix itself. If the box is small enough, UV effects dominate and (to mention one example) the central charge of the original conformal field theory can be recovered. Interesting features have been observed in the study of some of these systems of equations, pointing to the possibility of further exact results \([7]\): the TBA equations for models described by the so-called minimal ADE affine Toda S-matrices (see below) can be written in a universal way, depending only on corresponding non-affine Dynkin diagrams.

Unfortunately, the majority of cases involve non-diagonal S-matrices. To derive TBA equations directly from such scattering data, the so-called ‘colour transfer matrix’ should be diagonalised \([8]\), a forbidding problem – requiring the higher-level Bethe ansatz (see, for example, ref. \([9]\)) – in many cases. Furthermore, many perturbations of conformal field theories lead to flows, not to massive scattering in the IR limit, but to a non-trivial fixed point so that the RG flow interpolates between two conformal field theories. The \(\phi_{13}\) flows between minimal models \([10]\) provide the first examples of this phenomenon. Despite some suggestive proposals \([11,12]\), the S-matrix approach is much more delicate in such cases, owing to the well-known difficulties of treating massless particles in 1+1 dimensions.
An alternative strategy, developed by Al. B. Zamolodchikov in a series of papers [8,11,13], is to start with an educated guess for the relevant system of TBA equations, and then to check that they correctly reproduce a reasonable number of parameters typical of the UV theory, such as the central charge and the dimension of the perturbing operator. Once it has been established to general satisfaction that the proposed TBA is indeed associated with the perturbed conformal field theory under discussion, the RG flow of the vacuum energy (and possibly also of the energies of some of the excited states [14]) can be followed numerically, or IR data can be reconstructed analytically, allowing the IR limit to be identified – perhaps with some other conformal field theory.

In this way Al. B. Zamolodchikov was able to propose TBA systems for the flows of minimal models perturbed by $\phi_{13}$, the least relevant operator. Rather remarkably, these TBA systems can also be encoded on graphs (in fact the $A_n$ Dynkin diagrams). In the diagonal TBA, energy terms in the equations, one for each particle type, were naturally attached to the nodes of the corresponding Dynkin diagram, as will be explained further below. Here, the structure is a little different, with all but one or two of the energy terms being set to zero. The correct distribution to take depends on the sign of the perturbing parameter, and has a dramatic effect on the predicted IR behaviour: for one choice the flow is to a massive theory of kinks [8], while the other gives the flow to the next-lowest minimal model [11]. Subsequent work [13], further modifying the distribution of energy terms, gave ansätze (with identical diagrammatic structure) for TBA systems describing the flows between $su(2)^k \times su(2)^l / su(2)^{k+l}$ coset models.

Proceeding further, it was natural to look for TBA systems to describe analogous perturbations of $g^{(k)} \times g^{(l)} / g^{(k+l)}$ coset models, for which $\phi_{13}$ becomes $\phi_{(id, id, adj)}$. These were given for $g = A_n$, $l = 1$ in [15], and then for the general (simply-laced) case in [16]. In this latter reference a universal form of the general TBA equations was given, allowing the generalisation of the diagrammatic structure found in [7] to be identified: the relevant graph is a ‘product’ of the Dynkin diagram of $g$ (which had already been seen to be relevant for the diagonal Toda-type S-matrices) with an $A_k$ Dynkin diagram. Otherwise stated, the graph is formed by taking a tower of $k$ copies of the $g$ Dynkin diagram, with links in the vertical direction joining the replicated nodes. Presumably this replication encodes the colour structure of the non-diagonal S-matrices for these models [17], although the details remain to be worked out.

For the minimal models ($g = su(2)$), there is another intriguing result. Al. B. Zamolodchikov recently proposed [18] a simple diagonal scattering theory (the so-called ‘staircase model’), from which a TBA system followed in the standard way. However, numerical study of the solutions of this system revealed a structure much richer than any previously
observed, with hoppings of the vacuum energy (which is proportional to the effective central charge) strongly suggestive of an RG flow for an underlying field theory that passes by a sequence of \( c < 1 \) (minimal model) fixed points.

In fact, Zamolodchikov’s S-matrix is an analytic continuation of that for the \( A_1 \) affine Toda field theory. This suggests a natural generalisation to the other affine Toda field theories, and our aim in this paper is to present arguments supporting the claim that such S-matrices yield new staircase models which hop between the \( g^{(k)} \times g^{(1)} / g^{(k+1)} (W_g) \) coset conformal theories. The above-mentioned structure of a tower of Dynkin diagrams for the TBA systems found in [16] emerges as a natural consequence of the block structure of the affine Toda S-matrices.

While we were completing the writing of this paper, independent work by Martins appeared [19], proposing the same generalisation of [18] as we give here, and backing this up with a detailed numerical study of the solutions of the TBA system. This seems to us to complement our rather more analytic approach, and we therefore refer the reader to [19] for numerical confirmations of the various approximations and hypotheses to be made below.

The paper is organised as follows: in section 2 we recall the necessary features of the affine Toda S-matrices, and their relationship with the root systems of the simply-laced Lie algebras. The connection between the ‘minimal’ and ‘full’ affine Toda S-matrices is described, and this leads to a key relationship between their logarithmic derivatives. Section 3 presents the relevant TBA equations, and discusses the ‘locality’ structure of these equations as a function of the coupling constant. For a suitable analytic continuation of the coupling constant, the system becomes highly non-local and involves two scales (rather than the more usual one), the interaction of which is responsible for the intricate hopping structure of the solutions. Effective central charges are predicted for various asymptotic limits of these two scales, providing evidence that the RG behaviour being described is indeed a roaming trajectory visiting a sequence of \( W_g \)-minimal models. At the risk of repeating material that may be well-known to experts in the field, we have decided to make this part of the discussion reasonably explicit. Finally, we give a few speculations as to the physical meaning of the structures observed.

2. The affine Toda S-matrices

For real values of the coupling constant, the simply-laced affine Toda theories seem to have a number of particularly simple properties [21–23]. There are \( r \) different types of particles, \( r \) being the rank of the associated Lie algebra. Particles of different types are
distinguished one from another by conserved charges of non-zero spin, constraining the
S-matrix to be diagonal — the scattering theory is purely elastic. Furthermore (and in
contrast to the situation for non simply-laced algebras) the mass ratios remain fixed at
their classical values, and do not pick up any coupling-constant dependence via quantum
effects. As a result, the full S-matrix elements factorise into a product of two pieces: a
minimal factor, independent of the coupling constant, which contains the physical-strip
poles and thus all information on mass ratios and bound-state structure; and a second
‘Z-factor’ incorporating the coupling-constant dependence. This piece simply introduces
some zeroes into the physical strip, ensuring that as the coupling constant tends to zero
the S-matrix tends to the identity, but changes neither the bound-state structure nor the
resulting bootstrap equations. The situation for non simply-laced theories is markedly
more complicated [24], and will not be discussed below.

Despite their apparent independence, both factors must satisfy the same set of boot-
strap equations, equations which follow from the pole structure of the minimal piece. As
a result, their functional forms are very similar, a fact which will be relevant below. This
is most clearly seen when the S-matrix elements are written as a product of the functional
building blocks \( \{ x \} _{B} \) employed in ref. [22]. First, define elementary blocks \( (x) \) by

\[
(x)(\theta) = \frac{\sinh \left( \frac{\theta}{2} + \frac{i \pi x}{2h} \right)}{\sinh \left( \frac{\theta}{2} - \frac{i \pi x}{2h} \right)},
\]

where \( h \) is the Coxeter number of the algebra involved. (Other authors use the notation
\( f_\alpha(\theta) \) for \( (h\alpha)(\theta) \); including the factor of \( h \) in the notation, as here, has the advantage that
the parameter \( x \) then turns out to take integer values.) The bigger block, incorporating
the coupling-constant dependence via the function \( B(\beta) = 2\beta^2/(\beta^2+4\pi) \), is

\[
\{ x \}_B = \frac{(x - 1)(x + 1)}{(x - 1 + B)(x + 1 - B)}.
\]

(2.2)

The factorisation of this block into minimal and coupling-constant dependent pieces is
clear, as is the relationship between the two factors. To express this precisely, define a
symmetrised shift operator [21], acting on functions \( f(\theta) \), by

\[
(R_y f)(\theta) = f(\theta - i\pi y/h) f(\theta + i\pi y/h).
\]

(2.3)

Then

\[
R_y (x) = (x - y)(x + y),
\]

(2.4)

and so

\[
\{ x \}_B = \frac{R_1 (x)}{R_{1-B} (x)}.
\]

(2.5)
Thus the only difference between the two factors in the building block is a reciprocation and a change in the shift. This property is inherited by the full S-matrix elements, though it can sometimes be hidden due to the relations $0 = - (h) = 1$.

From the point of view of the bootstrap equations, it is perfectly consistent to consider the minimal part on its own, as removing the $Z$-factor leaves the physical poles and the signs of their residues unchanged. Such S-matrices are products of the corresponding minimal blocks,

$$\{x\}_{\text{min}} = (x - 1)(x + 1) = R_1(x).$$

However, the physical interpretation is very different: TBA calculations have established beyond reasonable doubt that these S-matrices correspond to perturbations of $g^{(1)} \times g^{(1)} / g^{(2)}$ coset conformal field theories.

Although the details are not essential for the main development, it is worth recalling how these S-matrix elements can be written in a uniform way for all the ADE series. The construction uses properties of a Coxeter element $w$ taken from the Weyl group of the relevant root system $\Phi$, each particle species $a$ being associated with an orbit $\Gamma_a \subset \Phi$ of $w$. Letting $\Gamma_a^+ = \Gamma_a \cap \Phi^+$ denote the intersection of such an orbit with the positive roots,

$$S^B_{ab} = \prod_{\beta \in \Gamma_a^+} \{u(\phi_a, \beta) + 1\}^{(\lambda_a, \beta)};$$

$$S^\text{min}_{ab} = \prod_{\beta \in \Gamma_a^+} \{u(\phi_a, \beta) + 1\}^{(\lambda_a, \beta)}. \quad (2.7a)$$

Here, $\lambda_a$ is the fundamental weight associated with the $a$th spot on the Dynkin diagram, $\phi_a = (1-w^{-1})\lambda_a$ is a particular root in the orbit $\Gamma_a$, and $2\pi u(\phi_a, \beta)/h$ is the angle between the projections of $\phi_a$ and $\beta$ into the $e^{2\pi i / h}$-eigenplane of $w$. For more explanation see [25], but really it is only the general form of (2.7) that will be relevant to what follows.

At the level of the S-matrices, the factorisation seen for the blocks in (2.5) can now be made explicit by introducing

$$S^F_{ab} = \pm \prod_{\beta \in \Gamma_a^+} (u(\phi_a, \beta) + 1)^{(\lambda_a, \beta)}. \quad (2.8)$$

With this definition,

$$S^\text{min}_{ab} = R_1 S^F_{ab}, \quad S^B_{ab} = \frac{R_1 S^F_{ab}}{R_1 - B S^F_{ab}}. \quad (2.9)$$

Note, the sign in (2.8) is left undetermined by the requirement that (2.9) should hold; indeed $S^F_{ab}$ can be thought of as a skewed ‘square root’ of $S^\text{min}_{ab}$. As long as the function
is used merely as a book-keeping device, this ambiguity will not cause any problems. However, it is tempting to speculate about a physical interpretation, in which case the sign might perhaps be important and a further subtlety should be considered: namely, whatever the sign choices made in (2.8), the functions $S_{ab}^F$ will in general fail, by a sign, to satisfy some of the bootstrap equations holding for $S_{ab}^{\text{min}}$ and $S_{ab}^B$. The origin of these signs can be seen by re-examining the proof, given in [25], that the bootstrap equations hold for the functions defined in (2.7). Since the bootstrap equations often involve shifts in arguments more general than the symmetrical operation defined in (2.3), the first step must be to write (2.7) using functions which transform into themselves under the single shift $T_y$, defined by

$$(T_y f)(\theta) = f(\theta + i\pi y/h).$$

This property does not hold for $(x)$ defined in (2.1), but decomposing it as $(x) = (x)_+/(-x)_+$, where

$$(x)_+ (\theta) = \sinh \left( \frac{\theta}{2} + \frac{i\pi x}{2h} \right),$$

results in a building block which satisfies

$$T_y (x)_+ = (x + y)_+. \quad (2.12)$$

Similarly $\{x\}_+$ can be defined by replacing $(.)$ by $(.)_+$ in either (2.2) or (2.6), in terms of which (2.7) becomes

$$S_{ab} = \prod_{\beta \in \Gamma_b} \{u(\phi_a, \beta) + 1\}_+^{(\lambda_a, \beta)}. \quad (2.13)$$

(The root $\beta$ now runs round the whole orbit $\Gamma_b$, rather than just the positive half as in (2.7); to see the equivalence of (2.13) and (2.7) needs a few root system identities, which can be found in [25].) Now an intermediate step in the proof of the bootstrap equations involves re-expressing the product appearing in (2.13), and for this, the relation $\{x+2h\}_+ = \{x\}_+$ must be used. It is this property which fails by a sign when the same trick is attempted for $S_{ij}^F$, since $(x+2h)_+ = -(x)_+$. Thus if the manipulations involve shifting an odd number of blocks $(x)_+$, an extra minus sign will appear, and even the freedom of signs allowed by (2.8) is not enough to eliminate all these signs from the bootstrap equations (this can already be seen in the $A_2$ case). There may be some interesting features hidden in these signs, but the question will not be pursued here. The signs in any event disappear once the logarithmic derivative has been taken, and so do not affect the TBA equations.

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This sign was missed in [10].
To discuss these logarithmic derivatives, set $\phi_f(\theta) = -i \frac{d}{d\theta} \log f(\theta)$. Following from this definition,

$$\phi_{fg} = \phi_f + \phi_g, \quad \phi_{f^{-1}} = -\phi_f, \quad \phi_{R_y f} = (\mathcal{T}_y + \mathcal{T}_{-y})\phi_f.$$ (2.14)

With the shorthands $\phi_{ab}^F$, $\phi_{ab}^{\text{min}}$ and $\phi_{ab}^B$ defined in the obvious way, combining (2.9) with (2.14) yields

$$\phi_{ab}^B = \phi_{ab}^{\text{min}} - \mathcal{T}_{1-B}\phi_{ab}^F - \mathcal{T}_{B-1}\phi_{ab}^F.$$ (2.15)

For consistency with [16], we write $\phi_{ab}^F$ as $\psi_{ab}$. Then, putting

$$\theta_0 = \frac{i\pi}{\hbar}(1 - B),$$ (2.16)

equation (2.15) can be rewritten as

$$\phi_{ab}^B(\theta) = \phi_{ab}^{\text{min}}(\theta) - \psi_{ab}(\theta + \theta_0) - \psi_{ab}(\theta - \theta_0).$$ (2.17)

This relation will be useful in the treatment of the TBA equations.

3. The TBA equations

The TBA equations for the affine Toda models have the standard form for a purely elastic scattering theory [3]. Writing the mass ratios as $\tilde{m}_a = m_a/m_1$, a system living on a circle of finite length $R$ depends on the scale through the quantity $\frac{1}{2}m_1R \equiv e^x$, and the ultraviolet limit corresponds to sending $x \to -\infty$. To avoid the complications caused by the fact that $x$ will almost always be negative, this parameter will be swapped for $y = -x$.

The TBA equations for the pseudoenergies $\varepsilon_a(\theta)$, encoding all of the finite-size effects, are

$$\varepsilon_a(\theta) = 2\tilde{m}_a e^{-y} \cosh(\theta) - \frac{1}{2\pi} \sum_{b=1}^{r} \phi_{ab}^B \ast L_b(\theta),$$ (3.1)

where $L_a(\theta) = \log(1 + e^{-\varepsilon_a(\theta)})$ and $\ast$ denotes the rapidity convolution:

$$\phi \ast L(\theta) = \int_{-\infty}^{\infty} d\theta' \phi(\theta - \theta')L(\theta').$$ (3.2)

The energy terms mentioned earlier are the functions $2\tilde{m}_a e^{-y} \cosh(\theta)$; they can be attached to the nodes of the Dynkin diagram via the association [5,22,26] between the Toda particle
masses and the Perron-Frobenius eigenvector of the Cartan matrix. The effective central charge, defined to be 
\[ -\frac{\pi}{6} R \times \text{finite size ground state energy} \ E(R), \]
given by

\[
c(y, \theta_0) = \frac{6}{\pi^2} \sum_{a=1}^{r} \int_{-\infty}^{\infty} d\theta \hat{m}_a e^{-y \cosh(\theta) L_a(\theta)}, \tag{3.3}
\]
the \( \theta_0 \)-dependence of this function entering via the \( B \)-dependence of the kernel function \( \phi^B_{ab} \) in (3.1).

To gain a general understanding of the solutions to (3.1), the form of this kernel should
be discussed. Since

\[
\phi_{(x)}(\theta) = \frac{1}{2} \coth \left( \frac{\theta}{2} + \frac{i\pi x}{2h} \right) - \frac{1}{2} \coth \left( \frac{\theta}{2} - \frac{i\pi x}{2h} \right), \tag{3.4}
\]
it follows from the definitions (2.7b) and (2.8) (and the reality of the numbers \( u(\phi_a, \beta) \)) that for \( \text{real } \theta \) the functions \( \phi_{ab}^{\text{min}}(\theta) \) and \( \psi_{ab}(\theta) \) are strongly peaked about \( \theta = 0 \), and decay like \( e^{-|\theta|} \) outside an interval of size \( O(1) \) centred at the origin. Thus so long as \( L(\theta) \) is reasonably well-behaved on the real \( \theta \)-axis, the values of \( \phi_{ab}^{\text{min}} * L(\theta) \) and \( \psi * L(\theta) \) are governed by the values taken by \( L(\theta') \) with \( \theta' \) close to \( \theta \), and the convolution (3.2) has a ‘local’ character. (To gain an intuition for this, it is helpful to think of replacing the functions \( \phi_{ab}^{\text{min}} \) and \( \psi \) by two delta-functions, each normalised to preserve the corresponding overall integral; indeed we shall be working in asymptotic regions where this seems to be a valid approximation to make.) For real values of the coupling constant \( \beta \), the shift \( \theta_0 \) in (2.17) is purely imaginary, and the above line of reasoning goes through for \( \phi^{B} \) as well, with the conclusion that the full TBA equation also has this local character. But to incorporate Zamolodchikov’s roaming idea, and in particular to introduce another scale into the problem, it is necessary to be a little more general. This should involve relaxing the requirement that \( \beta \) be real, but on general grounds it might be expected that the S-matrix elements should continue to be real analytic – that is, \( S_{ab}(\theta) \) should still be real for \( \theta \) on the imaginary axis. From the definition (2.2) of the basic block, \( \text{any } \text{real value for } B \) satisfies this condition, corresponding to \( \beta \text{ real } \) if \( B \) is between 0 and 2, and \( \beta \text{ purely imaginary } \) otherwise. A little less obvious, but clear enough once (2.3) is considered, is that values of \( B \) such that \( (1-B)^* = -(1-B) \) are also permitted (here, \( * \) denotes complex conjugation). That is, \( 1-B \) should be imaginary, and \( \theta_0 \) real. In turn, this requires that \( \beta/(2\sqrt{\pi}) \) should lie on the unit circle in the complex plane. An appealing feature of the S-matrix elements for such values of \( \beta \) is that (unlike the situation for many imaginary values of \( \beta \)) no extra poles are introduced into the physical strip, and so the interpretations of all the physical poles, bound states and bootstrap equations remain unchanged. It also connects with an
interesting physical interpretation of the roaming behaviour, a speculative point that will be described in the concluding section. For now, the important point is that this analytic continuation changes the ‘locality structure’ of the TBA equations dramatically, as can be seen by using (2.17) to rewrite (3.1):

\[ \varepsilon_a(\theta) = 2\hat{m}_a e^{-y} \cosh(\theta) - \frac{1}{2\pi} \sum_{b=1}^{r} \left[ \phi_{ab}^{\text{min}} L_b(\theta) - \psi_{ab} \ast (L_b(\theta - \theta_0) + L_b(\theta + \theta_0)) \right]. \]  

(3.5)

The \( \phi_{ab}^{\text{min}} \) term continues to ‘see’ \( L_b \) in the region \( \theta' \approx \theta \), but for the \( \psi \) term it is now the values taken by \( L_b \) at \( \theta \pm \theta_0 \), rather than at \( \theta \) itself, that are important. Furthermore, the equation now contains two scales, namely \( y \) and \( \theta_0 \). It is the interaction of these two scales which produces the staircase behaviour, the controlling parameter being their ratio, \( y/\theta_0 \). If this ratio is kept fixed while \( y \) and \( \theta_0 \) tend to infinity, then generally the effective central charge \( c(y, \theta_0) \) will tend to the central charge of one of the \( W_g \)-minimal models – precisely which one depends on the chosen value of \( y/\theta_0 \). Different asymptotic behaviours are separated by integer or half-integer values for this ratio. If on the other hand \( y \) is increased while \( \theta_0 \) is kept fixed, to study the behaviour towards the ultraviolet of one particular theory at one particular value of the coupling constant, then the changing value of \( y/\theta_0 \) results in a series of different values for the effective central charge, with a cross-over at each integer and half-integer. The rest of this section is devoted to some analytical arguments in support of these claims, leading to the expectation that the set of solutions to (3.5) at given (sufficiently large) \( \theta_0 \) do indeed form a staircase, with the central charge of a \( g^{(k)} \times g^{(1)}/g^{(k+1)} \) coset conformal field theory at the \( k \)th step.

As a first step, it seems reasonable to suppose that for fixed \( y \) the function \( L_b(\theta) \) is bounded. As soon as \( |\theta| \) is large enough for the first (energy) term in (3.5) to contribute, the exponential growth of the hyperbolic cosine will quickly swamp any influence of the convolution terms. Thus, for \( \theta \gg y \) or \( \theta \ll -y \), \( \varepsilon_a(\theta) \) is closely approximated by the functions \( \varepsilon_a^+(\theta) \) or \( \varepsilon_a^-(\theta) \) respectively, where

\[ \varepsilon_a^\pm(\theta) = \hat{m}_a e^{-y \pm \theta}. \]

(3.6)

Correspondingly, \( L_a(\theta) \) suffers a double-exponential decay beyond \( |\theta| > y \) and is soon approximately zero. This information is already enough to see why the \( k=1 \) fixed point is expected to control the effective central charge for \( y \ll \theta_0/2 \). The reason is simply that for such a value of \( y \), the \( \psi_{ab} \) term never contributes significantly to (3.5) — for \( |\theta| \gg y \), the energy term dominates and all the convolution terms can be ignored, while for
\(|\theta| \leq y\), both \(|\theta - \theta_0|\) and \(|\theta + \theta_0|\) are larger than \(y\), and hence \(L_b(\theta - \theta_0)\) and \(L_b(\theta + \theta_0)\) are essentially zero, and the \(\psi_{ab}\) term in (3.3) can again be dropped. The resulting ‘reduced’ set of equations is just that for \(S_{ab}^{\text{min}}(\theta)\), shown in refs. [3, 4] to have the central charge of the \(g^{(1)} \times g^{(1)}/g^{(2)}\) conformal field theory as the limiting \((y \to \infty)\) value of \(c(y)\). Of course, \(y\) must be less than \(\theta_0\) for the above treatment to be valid, but if \(\theta_0\) is taken large enough then the relevant central charge is approached well before the approximations break down.

At this stage it would be possible to continue by discussing what happens as \(y\) increases beyond \(\theta_0/2\), the values of the function \(L_a(\theta)\) near \(y\) starting to influence those at \(-y\), and so on. Such a treatment would mimic that already given by Al. B. Zamolodchikov in [18]. However, at least if the aim is merely to uncover the sequence of central charges visited by the trajectory, it seems better to think of the above discussion just as a warm-up exercise, and now to proceed directly to the general situation.

An assumption is needed, generalising already-observed behaviour of ‘local’ TBA equations, on the form of \(L'_a(\theta)\). For \(\theta \ll -y\), \(L\) and hence also \(L'\) are approximately zero, due to the dominance of the energy term in (3.3). Then after a transition around \(\theta = -y\) (a kink) during which \(L' \neq 0\), \(L\) should settle down to some (not necessarily zero) constant value, and \(L'\) return to zero. In the case of a local TBA equation, this is all that happens until \(\theta = +y\) is reached, at which point there is another kink and \(L\) returns to zero. However, in the case of (3.3), the non-local terms in the convolution serve to link the values of \(L\) and hence \(L'\) separated at a distance \(\theta_0\), and the ‘seed’ kink at \(\theta = -y\) would be expected to propagate in and cause further kinks at \(-y + \theta_0, -y + 2\theta_0\), and so on. Similarly, the kink at \(\theta = +y\) results in other kinks at \(\theta = y - \theta_0, y - 2\theta_0\), propagating in from the right. This process does not form any kinks for \(|\theta| > y\) since in any event the convolutions are swamped by the energy term in this region.

We can try to make this a little more precise by taking the derivative of (3.5) with respect to \(\theta\), and substituting \(\varepsilon' = -(1+e^\varepsilon)L'\). This gives an equation which couples the values of \(L'\) at \(\theta, \theta + \theta_0\) and \(\theta - \theta_0\). Repeating this at \(\theta + \theta_0, \theta + 2\theta_0\), and so on couples \(L'(\theta)\) with \(L'(\theta + n\theta_0)\) for any \(n \in \mathbb{Z}\); furthermore so long as we remain in the region \(|\theta + n\theta_0| < y\), the inhomogeneous terms \((2\tilde{m}_a e^{-y} \sinh(\theta + n\theta_0))\) can be dropped. In addition, if \(|\theta + n\theta_0| > y\), then \(L'(\theta + n\theta_0) = 0\) is forced (up to exponentially small corrections) by (3.6). Thus for \(\theta \neq \pm(y-n\theta_0)\) \((n \in \mathbb{Z})\), we expect \(L'(\theta) \sim 0\). The picture is more complicated at the ‘transition regions’ around \(\theta = \pm y\), where the energy term and the convolutions are in equal competition. The resulting non-zero values for \(L'(\theta)\) at \(\pm y\) then propagate in to \(-y + \theta_0, y - \theta_0\) and so forth. However, for this to be any more than a plausibility argument, it would have to be shown that the set of equations coupling together the values of \(L'\) at intervals of \(\theta_0\) were non-singular in a suitable sense, and also that the \(\theta\)-dependent factors
of \(-(1 + e^\varepsilon)\) did not spoil this. Nevertheless, it provides some support for the claim that
\(L'(\theta)\) is close to zero for \(\theta\) not equal to \(\pm(y-n\theta_0)\), with \(n = 0, 1, \ldots\) and \(y-n\theta_0 > -y\),
and that \(L(\theta)\) is thus a series of plateaux, separated by kinks at these points. (Note, the
kink regions should each have a size of order one, and so taking \(y\) large while preserving
the value of \(y/\theta_0\) enables their size relative to \(y\) and \(\theta_0\) to be made as small as desired.)
In the generic situation, kinks propagating in from the left will miss those propagating in
from the right, the two sets of kinks being interlaced along the \(\theta\)-axis. This only fails to be
ture if \(y/\theta_0\) is near an integer or half-integer; this should correspond to a crossover in the
critical behaviour. Otherwise, take \(y\) to lie between \((k-1)\theta_0/2\) and \(k\theta_0/2\) for some integer
\(k\). Working in from \(\theta = -\infty\), the first kink to be encountered is the seed kink at \(\theta = -y\);
then a descendant of the rightmost kink is found at \(\theta = y - (k-1)\theta_0\); then a descendant of
the leftmost kink at \(-y + \theta_0\); and so on. Between these \(2k\) kinks, there are \(2k-1\) plateaux,
within each of which the \(L_i(\theta)\), and hence also the \(\varepsilon_a(\theta)\), are approximately constant. The
\(i^{th}\) plateau, lying between the \(i^{th}\) and \((i+1)^{th}\) kinks, is centred on \(z_i = (i-k)\theta_0/2\, and
the approximately constant value of \(\varepsilon_a(\theta)\) along this plateau will be denoted \(\varepsilon_i^a \equiv \varepsilon_a(z_i)\).
These constants are easily found, since constant terms can be pulled out of the convolutions
in (3.3), leaving just the overall integrals of \(\phi^{\min}\) and \(\psi\) as proportionality factors. From
[5] and [16], these are given by
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \phi^{\min}_{ab}(\theta) = \delta_{ab} - 2C^{-1}_{ab} ; \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \psi_{ab}(\theta) = -C_{ab},
\]
where \(C_{ab}\) is the Cartan matrix of the non-affine Dynkin diagram for \(g\). (These formulae
can also be proved directly from (2.7) and (2.8) [25].) Considering equation (3.5) for
\(\theta = z_i\) results in the following consistency condition for the numbers \(\varepsilon_i^a\):
\[
\varepsilon_i^a = \sum_{b=1}^{r} \left[ (2C^{-1}_{ab} - \delta_{ab}) \log(1+e^{-\varepsilon_b^i}) - C_{ab} \left( \log(1+e^{-\varepsilon_b^{i-2}}) + \log(1+e^{-\varepsilon_b^{i+2}}) \right) \right].
\] (3.8)
If \(z_i\) lies towards an end of the chain, that is if \(i = 1, 2, 2k-2\) or \(2k-1\), the pseudo-energy
there interacts with \(\varepsilon^{\pm}(\theta)\), already fixed by (3.6); to the approximation to which we are
working here (and increasing \(y\) while keeping \(y/\theta_0\) fixed if necessary to sharpen things
up), these functions can be taken to be infinite at the points at which they need to be
evaluated. Thus, with the boundary conditions that \(\varepsilon^{-1} = \varepsilon^0 = \varepsilon^{2k} = \varepsilon^{2k+1} = \infty\), (3.8)
should hold for all \(i = 1 \ldots 2k-1\). Note that \(\varepsilon_i^i\) interacts with \(\varepsilon_i^{i\pm 2}\) rather than with its
immediate neighbours \(\varepsilon_i^{i\pm 1}\), a consequence of (3.5) and the fact that \(z_i \pm \theta_0 = z_i\pm 1, not \)
\(z_i\pm 1\). Thus the constants \(\{\varepsilon_i^a\}\) divide into two sets, \(\{\varepsilon_i^j\}\) and \(\{\varepsilon_i^{j-1}\}\), values in each set
only interacting with others in the same set or, at the ends, with the (effectively infinite)
boundary values determined by (3.6). This can be used to put (3.8) into a more familiar
tform: setting $x^j_a = \exp(-\varepsilon^{2j-1}_a)$ and $y^j_a = \exp(-\varepsilon^{2j-2}_a)$, the consistency conditions become

$$
x^j_a = \prod_{b=1}^{r} \left( 1 + x^j_b \right)^{\delta_{ab} - 2C_{ab}^{-1}} \left( 1 + x^{j-1}_b \right) C_{ab}^{-1} \left( 1 + x^{j+1}_b \right) C_{ab}^{-1}, \quad j=1...k
$$

$$
y^j_a = \prod_{b=1}^{r} \left( 1 + y^j_b \right)^{\delta_{ab} - 2C_{ab}^{-1}} \left( 1 + y^{j-1}_b \right) C_{ab}^{-1} \left( 1 + y^{j+1}_b \right) C_{ab}^{-1}, \quad j=2...k
$$

(3.9)

with the understanding that $x^0_a = x^{k+1}_a = y^1_a = y^{k+1}_a = 0$. In the $A_1$ case, there is just
one particle type and $C_{11}^{-1} = 1/2$; the system reduces to that given in [8], with $x^i_1$ and $y^i_1$
becoming $x_i$ and $y_i$ in Zamolodchikov’s notations. On the other hand, (3.9) is also the
specialisation to $l = 1$ of the constraints for the $g^{(k)} \times g^{(l)} / g^{(k+l)}$ TBA system proposed in
[10]: to see this, $x^i_a$, $y^i_a$ above should be replaced by $Y^a_i(-\infty)$, $Y^a_{i-1}(+\infty)$ respectively. Of
course, in the two above-mentioned papers the ‘morality’ is rather different from here, in
that (for fixed particle type $a$) $x^i_a$ and $y^i_a$ are the asymptotic values at $\pm \infty$ of a number of
different functions, indexed by $j$; here, they are the intermediate values of a single
function, $\exp(-\varepsilon_a(\theta))$. This changes the spirit of the central charge calculation, although
the underlying structure is the same.

One more piece of notation is needed. The interleaving of plateaux and kinks described
above can be summarised in the following inequalities:

$$
z_0 < -y < z_1 < y-(k-1)\theta_0 < z_2 < -y+\theta_0 < z_3 < \ldots
$$

$$
\ldots < z_{2k-1} < -y+(k-1)\theta_0 < z_{2k}
$$

(3.10)

(recall, $z_i = (i-k)\theta_0/2$, and $(k-1)\theta_0/2 < y < k\theta_0/2$). Define $K_i$ to be the interval
$[z_{i-1}, z_i]$, so that the $i$th kink of $L_a(\theta)$ lies in $K_i$. (For $i \neq 1, 2k$ this is also a kink for $\varepsilon_a$,
while for $i = 1$ or $2k$ it marks the transition of $\varepsilon_a$ to the exponentially growing behaviour
described by (3.6).) For $y/\theta_0$ sufficiently far from the crossover (integer or half-integer)
values, the kinks occur well inside the regions $K_i$, and by the time the boundaries of these
regions are reached, the functions $L_a$ have settled down to constants. Were it not for the
non-local nature of (3.5), these kinks would be described by a set of independent TBA
equations, with boundary conditions, effectively at $\pm \infty$, determined by (3.9). However,
the nonlocality couples the equation for the kink in $K_i$ to those for the kinks in $K_{i+2}$;
to leading order these coupled equations are exactly those proposed in [16]. In this way,
the ‘tower’ structure of the TBA systems in that paper can be traced back to the block
structure of the affine Toda S-matrices, the key relation being (2.15).
This would be enough to verify our claims; however, for completeness the remainder of section concludes the computation of the function $c(y, \theta_0)$ directly in the context of the staircase model, continuing to work in the asymptotic region where both $y$ and $\theta_0$ are large, with $(k-1)/2 < y/\theta_0 < k/2$.

Equations (3.3) and (3.5) must be re-examined with an eye to their asymptotic behaviour. The integration range $[-\infty, \infty]$ in the expression (3.3) for the effective central charge $c(y, \theta_0)$ can be restricted to $[z_1, z_{2k}] = \bigcup_{i=1}^{2k} K_i$, since the double-exponential decay of $L_a(\theta)$ causes the integrand to be vanishingly small outside this region. Within this region,

$$2\hat{m}_a e^{-y} \cosh(\theta) \sim \begin{cases} \hat{m}_a e^{-y-\theta}, & \theta \in K_1; \\ 0, & \theta \in K_2 \ldots K_{2k-1}; \\ \hat{m}_a e^{-y+\theta}, & \theta \in K_{2k}. \end{cases}$$ (3.11)

Thus the only direct contribution to $c(y, \theta_0)$ is from the kinks in $K_1$ and $K_{2k}$. Writing $c(y, \theta_0) = c_- + c_+$, where

$$c_\pm = \frac{3}{\pi^2} \sum_{a=1}^{r} \int_{z_1}^{z_{2k}} d\theta \hat{m}_a e^{-y \pm \theta} L_a(\theta),$$ (3.12)

and noting by symmetry that $c_+ = c_-$, all that remains is to calculate

$$c_- \sim \frac{3}{\pi^2} \sum_{a=1}^{r} \int_{K_1} d\theta \hat{m}_a e^{-y-\theta} L_a(\theta).$$ (3.13)

This is essentially standard, the only modifications needed to the cases in [8,15,16] being, crudely speaking, that integration by parts should be replaced by considerations based on the ‘locality’ of $\phi^\text{min}(\theta)$ and $\psi(\theta)$. The relevant property is the following: let $f(\theta)$ be any even function of $\theta$ with its support concentrated at the origin (by this we mean that the overall integral of $f(\theta)$ receives a significant contribution only in a region near $\theta=0$, the size of which is asymptotically small in comparison with $\theta_0$ and $y$; both $\phi^\text{min}(\theta)$ and $\psi(\theta)$ satisfy this condition). If $A(\theta)$ and $B(\theta)$ vary slowly (on the scale of the support of $f$) at the ends of the interval $K_i = [z_{i-1}, z_i]$, then

$$\int_{K_i} d\theta f(\theta) A(\theta) B(\theta) = -\int_{K_i} d\theta f(\theta) A(\theta) B(\theta) + \left( \int_{-\infty}^{\infty} d\theta f(\theta) \right) \left[ A(\theta) B(\theta) \right]_{z_{i-1}}^{z_i}. \quad (3.14)$$

If $f(\theta) = \delta(\theta)$, this formula really is just integration by parts; a proof of its more general validity is relegated to appendix A.

Returning now to the calculation of the effective central charge, the necessary property of the solutions of the TBA equations, allowing for the exact evaluation of (3.12) in the
asymptotic region of interest, is found by taking the derivative of (3.5), multiplying by $L_a(\theta)$, integrating $\int_{K_i} d\theta$ and finally summing on $a$. In an abbreviated notation, this yields

$$\mathcal{J}^i = \frac{\pi^2}{3} C^i - T_0^i + T_+^i + T_-^i$$

where

$$\mathcal{J}^i = \sum_{a=1}^{r} \int_{K_i} d\theta \varepsilon_a^i(\theta) L_a(\theta) = \sum_{a=1}^{r} \int_{\varepsilon_{a-1}}^{\varepsilon_{a}} d\varepsilon \log (1 + e^{-\varepsilon}) ;$$

$$C^i = \frac{3}{\pi^2} \sum_{a=1}^{r} \int_{K_i} d\theta 2\hat{m}_a e^{-y} \sinh(\theta) L_a(\theta) ;$$

$$T_0^i = \frac{1}{2\pi} \sum_{a,b=1}^{r} \int_{K_i} d\theta \phi_{ab}^{\text{min}'} L_b(\theta) L_a(\theta) ;$$

$$T_+^i = \frac{1}{2\pi} \sum_{a,b=1}^{r} \int_{K_i} d\theta \psi_{ab}^{\text{min}'} L_b(\theta \pm \theta_0) L_a(\theta) .$$

These terms can be simplified in various ways. First, using the relations

$$2\hat{m}_a e^{-y} \sinh(\theta) \sim \begin{cases} 
-\hat{m}_a e^{-y-\theta}, & \theta \in K_1 ; \\
0, & \theta \in K_2 \ldots K_{2k-1} ; \\
\hat{m}_a e^{-y+\theta}, & \theta \in K_{2k} ,
\end{cases}$$

and referring back to (3.12), we have

$$C^i \sim -\delta_{1,i} c_- + \delta_{2k,i} c_+ .$$

(Note how the relations (3.11) and (3.17), by forcing the effective energy terms to lie at the end of the $A$-type Dynkin diagram, automatically select the $l=1$ cases among the TBA systems proposed in [16].)

Of the remaining pieces, $T_0^i$ is just a surface term: applying (3.14) to (3.16c), swapping the sums on $a$ and $b$ and recalling that $\phi_{ab}^{\text{min}'} = \phi_{ba}^{\text{min}'}$ yields

$$2T_0^i = \sum_{a,b=1}^{r} (\delta_{ab} - 2C_{ab}^{\text{min}'} - 2C_{ab}^{-1}) [L_a(\theta) L_b(\theta)]_{z_{i-1}} ,$$

where (3.7) has also been used for the overall integral of $\phi_{ab}^{\text{min}'}$. A similar manoeuver for (3.16d) gives, for $1 \leq i \leq 2k$,

$$T_+^i = -T_+^{i+2} - \sum_{a,b=1}^{r} C_{ab}^{-1} [L_a(\theta) L_b(\theta \pm \theta_0)]_{z_{i-1}} ,$$

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while $I^i_{\pm} \sim 0$ if $i < 1$ or $i > 2k$, owing to the double-exponential decay of $L_a(\theta)$. The surface piece makes this equation a little hard to use; to get rid of it, define

$$
\tilde{I}^i_{\pm} = I^i_{\pm} + \frac{1}{2} \sum_{a,b=1}^{r} C^{-1}_{ab} [L_a(\theta)L_b(\theta \pm \theta_0)]^i_{z_i-1}.
$$

(3.21)

Recalling that $z_i + \theta_0 = z_{i+2}$, (3.20) is now

$$
\tilde{I}^i_{\pm} = -\tilde{I}^i_{\mp},
$$

(3.22)

and, as before, $\tilde{I}^i_{\pm} \sim 0$ if $i < 1$ or $i > 2k$. Equation (3.15) becomes

$$
\tilde{J}^i = \frac{\pi^2}{3} C^i + \tilde{I}^-_{\pm} + \tilde{I}^+_{\pm} + \frac{1}{2} \sum_{a,b=1}^{r} \left[ L_a(\theta) \left\{ (2C^{-1}_{ab} - \delta_{ab})L_b(\theta) 
- C^{-1}_{ab} (L_b(\theta-\theta_0) + L_b(\theta+\theta_0)) \right\} \right]_{\tilde{z}_i-1}.
$$

(3.23)

equation (3.19) having been used to substitute for $I^0_i$. Now the term in curly brackets is familiar from (3.8), since it is only ever evaluated when $\theta$ is equal to one of the $z_i$: once the sum on $b$ has been performed, it is just $\varepsilon_a(\theta)$. So, if we set

$$
\tilde{\mathcal{J}}^i = \mathcal{J}^i - \frac{1}{2} \sum_{a=1}^{r} \left[ L_a(\theta)\varepsilon_a(\theta) \right]_{\tilde{z}_i-1}.
$$

(3.24)

and use (3.18) as well, the result is

$$
\tilde{\mathcal{J}}^i = \frac{\pi^2}{3} \left( -\delta_{1,i} c_- + \delta_{2k,i} c_+ \right) + \tilde{I}^-_{\pm} + \tilde{I}^+_{\pm}.
$$

(3.25)

At $i = 1$, this is

$$
c_- = -\frac{3}{\pi^2} (\tilde{\mathcal{J}}^1 - \tilde{I}^-_{\pm} - \tilde{I}^+_{\pm}).
$$

(3.26)

Now $\tilde{I}^-_{\pm} = \tilde{I}^+_{\pm} = 0$, while $\tilde{I}^1_{\pm} = -\tilde{I}^3_{\pm}$. In addition, for $1 < i < 2k$, (3.23) can be rewritten, via (3.22), as $\tilde{I}^i_{\pm} = \tilde{\mathcal{J}}^i + \tilde{I}^i_{\pm}$. Thus (3.26) falls out into a cascade of terms, one for every other kink starting with the left-most:

$$
c_- = -\frac{3}{\pi^2} (\tilde{\mathcal{J}}^1 + \tilde{I}^3_{\pm})
= -\frac{3}{\pi^2} (\tilde{\mathcal{J}}^1 + \tilde{\mathcal{J}}^3 + \tilde{I}^5_{\pm})
= \ldots
= -\frac{3}{\pi^2} \sum_{j=1}^{k} \tilde{J}^{2j-1},
$$

(3.27)
the process terminating when the term $\tilde{J}^{2k+1} = 0$ is encountered. Note that the last line only depends on $y$ and $\theta_0$ via the value of $k$ that their ratio entails. To calculate $\tilde{J}^{2j-1}$, we write it as $\sum_{a=1}^{r} \tilde{J}_a^{2j-1}$, where

$$\tilde{J}_a^{2j-1} = J_a^{2j-1} - \frac{1}{2} [L_a(\theta)\varepsilon_a(\theta)]^{2j-1}_{2j-2}$$

and

$$J_a^{2j-1} = \int_{\varepsilon_a^{2j-2}}^{\varepsilon_a^{2j-1}} d\varepsilon \log (1 + e^{-\varepsilon})$$

$$= \text{Li}_2 \left(-e^{-\varepsilon_a^{2j-1}}\right) - \text{Li}_2 \left(-e^{-\varepsilon_a^{2j-2}}\right).$$

Here, $\text{Li}_2(z) = \int_{0}^{z} dt \log(1 - t)/t$ is the ‘usual’ dilogarithm function (for more information, see ref. [27]). The surface part in (3.28) turns this into the Rogers dilogarithm, $L(z) = -\frac{1}{2} \int_{0}^{z} dt \left(\frac{\log(1-t)}{1-t} + \log t \frac{1}{1-t}\right)$, via the relation

$$\mathcal{L} \left(\frac{x}{1+x}\right) = -\text{Li}_2(-x) - \log(1+x) \log x,$$

valid for $x > -1$ [27]. With the earlier definitions $x_a^j = e^{-\varepsilon_a^{2j-1}}$, $y_a^j = e^{-\varepsilon_a^{2j-2}}$, equation (3.28) becomes

$$\tilde{J}_a^{2j-1} = -\mathcal{L} \left(\frac{x_a^j}{1 + x_a^j}\right) + \mathcal{L} \left(\frac{y_a^j}{1 + y_a^j}\right).$$

The effective central charge is $c_+ + c_- = 2c_-$. Subject to the hypotheses on the plateau-like behaviour of $L_a(\theta)$, we have now shown that

$$\lim_{y, \theta_0 \to \infty} \frac{c(y, \theta_0)}{c_k} = \frac{6}{\pi^2} \sum_{j=1}^{k} \sum_{a=1}^{r} \left[ \mathcal{L} \left(\frac{x_a^j}{1 + x_a^j}\right) - \mathcal{L} \left(\frac{y_a^j}{1 + y_a^j}\right) \right],$$

where $x_a^j$ and $y_a^j$ are the real solutions of (3.9) (or zero in the case of $y_a^1$, corresponding to the first kink). In taking this limit, the ratio $y/\theta_0$ is assumed to be kept fixed. If instead $\theta_0$ is fixed at some (suitably large) value and $y$ is increased, $c(y, \theta_0)$ will run through the values $c_k$ in turn. The plateau structure should break down, and a cross-over occur, whenever the interval between a pair of kinks is of the same order as the kink size. The intervals between the kinks are of lengths $2y - (k-1)\theta_0$ and $k\theta_0 - 2y$, while the kinks themselves have an extent of order one. Thus the cross-overs should occur while $2y$ is in a region of order one about each integer multiple of $\theta_0$. (Examining the graphs in ref. [18], the correct size would seem to be about five, a value completely consistent with the form (3.4) of the functions $\phi_j(x)$. The important point is that the sizes of these cross-over regions depend neither on $\theta_0$ nor on $y$.)

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To find numerical values for the $c_k$, we use the following sum rule \[28\], which holds whenever the constants $x^j_a$ satisfy a ‘chain’ of relations of the form (3.9), of length $k$:

$$
\sum_{j=1}^{k} \sum_{a=1}^{r} \mathcal{L} \left( \frac{x^j_a}{1 + x^j_a} \right) = \frac{\pi^2}{6} \frac{r h k}{h + k + 1}.
$$

(3.33)

(So for the constants $y^i_a$ in (3.9), $k$ should be replaced by $k-1$ here.) For the simply-laced Lie algebras, the Coxeter number $h$ is equal to the dual Coxeter number $\tilde{h}$, so the right-hand side of (3.33) is in fact the central charge for a theory of Gepner parafermions \[28\] at level $k+1$. Since we do not know of any general proof of these relations, we checked them numerically (to eight-digit final precision) for all simply-laced Lie algebras of rank $\leq 8$, for $k = 1 \ldots 8$. With the relation

$$
\mathcal{L} \left( \frac{x}{1 + x} \right) = \frac{\pi^2}{6} - \mathcal{L} \left( \frac{1}{1 + x} \right),
$$

this gives

$$
c_k = r (h + 1) \left[ \frac{1}{h + 1} + \frac{k}{h + k} - \frac{k + 1}{h + k + 1} \right],
$$

a result which reproduces the central charge of the $g^{(k)} \times g^{(1)} / g^{(k+1)}$ coset model.

4. Conclusions and speculations

The hopping nature of the solutions to equation (3.3) has now been established in a fair amount of detail, but the physical meaning of this result is much less clear. While we have no final answers to this question, we give in this concluding section some speculations on the Lagrangian form of the staircase models which might be worthy of further investigation.

The usual Lagrangian density for affine Toda field theory can be written as

$$
\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - V(\phi, \beta)
$$

(4.1a)

where

$$
V(\phi, \beta) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi},
$$

(4.1b)

the sum in the second equation running over all the simple roots $\alpha_1 \ldots \alpha_r$ of (non-affine) $g$, together with $\alpha_0 = - \sum_1^r n_i \alpha_i$, the negative of the highest root. So long as $\beta$ is real, the potential has a unique minimum at $\phi = 0$, and a purely elastic scattering theory of $r$ different types of bosonic particles results \[20\], \[23\].
However the potential (4.14) has also been discussed at purely imaginary values of $\beta$ \cite{30-32}, the main aim being to find a Lagrangian basis for the $\phi_{(id,id,adj)}$-perturbed coset models (generally thinking of the perturbation as being in the massive direction). In such cases the unitarity of (4.14) is doubtful, and some form of truncation has to be invoked to restrict the space of states to a positive-definite subspace. The physical motivation for this procedure is unclear, and so it might be preferable to look for an alternative continuation of the real coupling-constant Lagrangian for which the manifest reality of the potential was preserved.

The simplest way to create a real potential from (4.17) is just to add its complex conjugate, forming $V'(\phi,\beta) = V(\phi,\beta) + V(\phi,\beta^*)$. (We assume that the field $\phi$ remains real here.) However, the non-linearity of the Toda field equations means that at the classical level there is no reason to believe the potential $V'$ to be integrable. But at the quantum level, for suitably well-chosen values of $\beta$, things may be different. The basis of this expectation is the strong-weak coupling duality of the affine Toda field theories, a quantum effect seen for the S-matrices in the fact that (from equation (2.2)) $S_{\alpha \beta}^B = S_{\alpha \beta}^{2-B}$. Since $B(\beta) = 2\beta^2/(\beta^2 + 4\pi)$, the interchange between the two possibilities is effected by the following 'duality' transformation on the coupling constant:

$$\beta \rightarrow \tilde{\beta} = 4\pi/\beta,$$  \hspace{1cm} (4.2)

relating the strong and weak coupling regimes. At the more fundamental level of lightcone canonical quantization, it has also been argued \cite{34} that the Hamiltonians arising from $V(\phi,\beta)$ and $V(\phi,\tilde{\beta})$ actually commute – a result from which the equality of the strong- and weak- coupling S-matrices follows automatically. This motivates the idea that a linear combination of $V(\phi,\beta)$ and $V(\phi,\tilde{\beta})$ might be quantum integrable. One possibility is to form the ‘self-dual’ potential $V''(\phi,\beta) = V(\phi,\beta) + V(\phi,\tilde{\beta})$, and in fact analogous non-affine potentials have already been examined by Mansfield \cite{35}, though with rather different motivations. Here, the hope is to construct a real continuation of (4.1) which retains integrability at the quantum level, and we now see that this might be possible if $V'(\phi,\beta) = V''(\phi,\beta)$, that is if $\beta^* = \tilde{\beta}$. The allowed values of $\beta$ are thus

$$\beta = 2\sqrt{\pi} e^{i\gamma},  \hspace{1cm} (4.3)$$

and the ‘roaming’ analytic continuation has been recovered, from an alternative perspective. As a function of the (real) variable $\gamma$, the proposed potential is

$$V''(\phi,\gamma) = \frac{m^2}{2\pi} \sum_{i=0}^{r} n_i e^{2\sqrt{\pi} \cos \gamma \cos [(2\sqrt{\pi} \sin \gamma) \alpha_i + 2\gamma]}.$$  \hspace{1cm} (4.4)
Taking $\gamma = 0$ recovers the real-coupling Toda theory at its self-dual point, while (since $\theta_0 = \frac{\pi}{2} \tan \gamma$) the large-$\theta_0$ limit corresponds to $\gamma \to \pi/2$. In this limit each term in the sum (4.4) is a product of a slowly-growing exponential with a rapidly-varying cosine, and near the origin the total is well approximated by

$$-\frac{m^2}{2\pi} \sum_{i=0}^{r} n_i \cos \left[ (2\sqrt{\pi} \sin \gamma) \alpha_i \phi + 2\gamma \right].$$

(4.5)

We would like to think of this as the ‘unrestricted’ potential, a manifestly real analogue, for the higher-rank algebras, of the sine-Gordon potential. In the $A_2$ case, for example, the potential (4.3) has minima lying at the vertices of a triangular lattice. If these are the classical vacua, then (at least from a visual examination of the potential) kinks should exist, joining neighbouring vacua, one for each edge of the lattice. This motivates the idea that (4.5) might provide a potential for a Landau-Ginzburg description of an unrestricted SOS-type model associated with the algebra $g$. Further from the origin, the exponential growth of the other terms in (4.4) becomes important. At first sight this is unfortunate, as an immediate consequence is that the potential (4.4) is not bounded from below, and so it seems that the theory is not well-defined. However (at least so long as $\theta_0$ is large enough) while the minima get lower at larger values of $|\phi|$, so the saddle-points between these minima get higher. At a given energy-scale, this may serve to confine the field to a neighbourhood of the origin, the vacua further away being effectively decoupled. To take a classical analogy, a particle starting at the origin of the potential (4.4) with any finite kinetic energy would be confined to a finite domain in the $\phi$-plane, despite the fact that beyond this domain points can be found at which the potential energy is arbitrarily negative. The question is thus whether 1+1 dimensions is near enough to the classical ($d = \infty$) limit. Assuming that this is true, a rather appealing picture emerges — at any given energy-scale, only a certain number of vacua are accessible to the field $\phi$, and the SOS picture provided by (4.3) is modified to an effective Landau-Ginzburg potential appropriate for a restricted SOS-type model (the possibility of such potentials for the case of $a_2$ is mentioned in [10]). As one progresses towards higher energies in the UV limit, more and more vacua become available and the restriction is lifted in stages, in a way which could reproduce the roaming behaviour already observed at the level of TBA equations.

It has to be admitted that there are many problems with this scenario, even at the naive classical level presented above. In particular, it seems hard to ensure that the easing of the restriction happens in the correct manner, so that a full series of coset models (or rather their hoped-for Landau-Ginzburg potentials) is visited. In this regard, it may be relevant that there is no compelling reason for the choice of $V''$ given above — more
generally, pick any function \( f \), with complex conjugate \( \bar{f} \). Then \( f(\beta)V(\phi, \beta) + \bar{f}(\bar{\beta})V(\phi, \bar{\beta}) \) is a candidate potential. Clearly it would be more satisfying to have a physical principle to justify one choice over all the others. It should also be stressed again that the models described by (4.4) are perforce a long way from their classical domains, and so (just as for the Landau-Ginzburg picture of the \( c < 1 \) models [36]) classical intuitions should be treated with caution.

Despite these difficulties, the search for a physical understanding of the staircase models seems to be very worthwhile. At the level of the TBA, they already unify and clarify a whole class of systems; it is not unreasonable to hope for similar insights at the field theory level as well.

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**Appendix A.**

The aim is to show that

\[
\int_{K_i} d\theta f' A(\theta)B(\theta) = -\int_{K_i} d\theta f' B(\theta)A(\theta) + \left( \int_{-\infty}^{\infty} d\theta f(\theta) \right) [A(\theta)B(\theta)]_{z_i}^{z_i-1}, \tag{3.14}
\]

subject to the conditions already mentioned in section 3. First, expand out the left-hand side as follows:

\[
\int_{K_i} d\theta f' A(\theta)B(\theta) = \int_{K_i} d\theta \int_{-\infty}^{\infty} d\theta' f'(\theta - \theta')A(\theta')B(\theta) = \int_{I \cup III \cup V} d\theta d\theta' f'(\theta - \theta')A(\theta')B(\theta), \tag{A.1}
\]

where I, III and V are three regions in the \( \theta, \theta' \) plane:

\[
I = K_i \times [-\infty, z_{i-1}] ; \quad III = K_i \times [z_i, \infty] ; \quad V = K_i \times K_i.
\]

Similarly, the first integral on the right-hand side of (3.14) is, with a trivial change of integration variable,

\[
-\int_{K_i} d\theta' f'(\theta')A(\theta') = -\int_{K_i} d\theta' \int_{-\infty}^{\infty} d\theta f'(\theta' - \theta)B(\theta)A(\theta') = \int_{II \cup IV \cup V} d\theta d\theta' f'(\theta - \theta')A(\theta')B(\theta), \tag{A.2}
\]

subject to the conditions already mentioned in section 3. First, expand out the left-hand side as follows:

\[
\int_{K_i} d\theta f' A(\theta)B(\theta) = \int_{K_i} d\theta \int_{-\infty}^{\infty} d\theta' f'(\theta - \theta')A(\theta')B(\theta) = \int_{I \cup III \cup V} d\theta d\theta' f'(\theta - \theta')A(\theta')B(\theta), \tag{A.1}
\]

where I, III and V are three regions in the \( \theta, \theta' \) plane:

\[
I = K_i \times [-\infty, z_{i-1}] ; \quad III = K_i \times [z_i, \infty] ; \quad V = K_i \times K_i.
\]

Similarly, the first integral on the right-hand side of (3.14) is, with a trivial change of integration variable,

\[
-\int_{K_i} d\theta' f'(\theta')A(\theta') = -\int_{K_i} d\theta' \int_{-\infty}^{\infty} d\theta f'(\theta' - \theta)B(\theta)A(\theta') = \int_{II \cup IV \cup V} d\theta d\theta' f'(\theta - \theta')A(\theta')B(\theta), \tag{A.2}
\]

subject to the conditions already mentioned in section 3. First, expand out the left-hand side as follows:

\[
\int_{K_i} d\theta f' A(\theta)B(\theta) = \int_{K_i} d\theta \int_{-\infty}^{\infty} d\theta' f'(\theta - \theta')A(\theta')B(\theta) = \int_{I \cup III \cup V} d\theta d\theta' f'(\theta - \theta')A(\theta')B(\theta), \tag{A.1}
\]

where I, III and V are three regions in the \( \theta, \theta' \) plane:

\[
I = K_i \times [-\infty, z_{i-1}] ; \quad III = K_i \times [z_i, \infty] ; \quad V = K_i \times K_i.
\]

Similarly, the first integral on the right-hand side of (3.14) is, with a trivial change of integration variable,
swapping the order of the integrations (the double integral is absolutely convergent due to the finite extent of $K_i$) and using the antisymmetry of $f'$ in the last line. This time the last integral is over the three regions II, IV and V where V is as before and

$$ II = (-\infty, z_{i-1}] \times K_i; \quad IV = [z_i, \infty] \times K_i. $$

Subtracting (A.2) from (A.1), the second term on the right-hand side of (3.14) must be given by

$$ \left[ \int_I - \int_{II} + \int_{III} - \int_{IV} \right] d\theta d\theta' f'(\theta - \theta') A(\theta') B(\theta). $$

Consider first the integrals over I and II. The locality of $f'$ means that in these two regions the integrand is only non-negligible near $(z_{i-1}, z_{i-1})$, and the approximately constant nature of $A(\theta') B(\theta)$ near this point allows its replacement in the integrand by $A(z_{i-1}) B(z_{i-1})$. Asymptotically we are also free to replace I by $I' = [z_{i-1}, \infty] \times [-\infty, z_{i-1}]$ and II by $II' = [-\infty, z_{i-1}] \times [z_{i-1}, \infty]$ (it may help to draw a picture of this), after which the remaining (double) integral of $f'(\theta - \theta')$ is easily seen to reduce to $-\int_\infty^- f(\theta)$. In just the same way the difference between the integrals over III and IV is well-approximated by $(\int_\infty^- f(\theta)) A(z_i) B(z_i)$, which completes the demonstration of (3.14).

Note that in the usual TBA, the integrals all run from $-\infty$ to $\infty$, in which case a naive swap of integration variables leads to the second term on the right-hand side of (3.14) being missed. This would be wrong, the reason being that the integral over the entire $\theta, \theta'$-plane is not absolutely convergent. The correct prescription is to regulate the $\theta$-integral to an interval $K_i$, say, and then let $K_i \to [-\infty, \infty]$ at the end, thus recovering (3.14) with limits of $\pm \infty$. It is worth stressing that this result does not depend at all on the 'integrable' nature of the TBA equations, equations which only need to be used once (in the form of the relation (3.15)) at each step in the evaluation of the effective central charge.

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