An Algebra of Singular Integrals
of Non-convolution Type

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Abstract

We find a class of pseudo differential operators classified by their symbols. These symbols satisfy a differential inequality with a mixture of homogeneities in the frequency space. When taking the singular integral realization, the class of operators can be equivalently defined by their kernels. We prove that the pseudo differential operators in this class are $L^p$ continuous, admitted in a weighted norm inequality for $1 < p < \infty$. Moreover, the class itself forms an algebra.

1 Introduction

In this paper, we prove an $L^p$ theorem for a class of pseudo differential operators. By taking singular integral realization, they are singular integrals of non-convolution type. The class hereby investigated, classified either by the symbols, or equivalently by the kernels, forms an algebra under the composition of operators. We show that the pseudo differential operators in this class are $L^p$ continuous, for $1 < p < \infty$. With clarity on some more definitions, these operators are admitted in a weighted norm inequality.

Our class of operators first arises when studying some sub-elliptic problems. These include the Grushin type operators, oblique derivative problems and Cauchy-Szegö projections. We give some examples in the end.

Let $f \in \mathcal{S}$ be a Schwartz function. A pseudo differential operator $T_\sigma$ is defined by

$$
(T_\sigma f)(x) = \int \sigma(x, \xi) e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi
$$

where $\sigma$ denotes the symbol.

Let $N = N_1 + N_2 + \cdots + N_n$ be the dimension of a product space. We write

$$
x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_n} \quad (1.1)
$$
with $N_i$ the homogeneous dimension of the $i$-th subspace, for $i = 1, 2, \ldots, n$.

**Definition of $S_\rho$**: Let $0 < \rho < 1$. A symbol $\sigma$ belongs to the class $S_\rho$ if $\sigma(x, \xi) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ and satisfies the differential inequality

$$
\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi) \right| \leq A_{\alpha,\beta} \prod_{i=1}^{n} \left( \frac{1}{1 + |\xi_i| + |\xi|^\rho} \right)^{|\alpha_i|} (1 + |\xi|^\rho)^{B_i} \quad (1.2)
$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta$.

Observe that $S_\rho$ forms a sub class of the *exotic* class $S_{\rho,p}^0$. See chapter VII of [7] for an introduction. Pseudo differential operators with their symbols consisted in $S_{\rho,p}^0$ are bounded on $L^2$-space.

In the other hand, $\sigma$ in (1.2) satisfies a variant of Marcinkiewicz theorem. Namely,

$$
\left| \left( \frac{\partial}{\partial \xi} \right)^\alpha \sigma(x, \xi) \right| \leq A_\alpha \quad (1.3)
$$

uniformly in $x$. If $T_\sigma$ is translation invariant, i.e: $\sigma = \sigma(\xi)$, then it is $L^p$ continuous for $1 < p < \infty$. See [4]-[6] for references.

By taking singular integral realization, we write $T_\sigma f(x)$ in the form of

$$
\int f(x - y) \Omega(x, y) dy.
$$

Its kernel $\Omega$, is a distribution $\Omega(x, \cdot)$ that coincides with a smooth function

$$
\Omega(x, y) = \int_{\mathbb{R}^N} e^{2\pi iy \cdot \xi} \sigma(x, \xi) d\xi \quad (1.4)
$$

for $y \neq 0$. We will show that $\sigma \in S_\rho$ if and only if $\Omega$: *has a size estimate of*

$$
\left| \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) \right| \leq A_{\alpha,\beta} \prod_{i=1}^{n} \left( \frac{1}{|y_i| + |y|^\rho} \right)^{N_i + |\alpha_i| + |\beta|} \quad (1.5)
$$

for $y \neq 0$, and decays rapidly as $y \to \infty$.

Moreover, recall an normalized bump function $\varphi$ is smooth, supported on the unit ball, with all its derivatives uniformly bounded by 1 upon to a sufficiently large order. For $I \cup J = \{1, 2, \ldots, n\}$. Let $\varphi_i$ be a normalized bump function for every $i \in I$ and $y^i$ is the projection of $y$ on the subspace $\bigoplus \mathbb{R}^{N_j}$ for $j \in J$. $\Omega$ carries the cancellation property

$$
\left| \left( \prod_{j \in I} \frac{\partial^{\alpha_j}}{\partial y_j^{\alpha_j}} \right) \int \cdots \int_{\mathbb{R}^{N_i}} \left( \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) \right) \varphi_i(R_i y_i) dy_i \right| \leq A_{\alpha,\beta} \prod_{j \in I} \left( \frac{1}{|y_j| + |y^i|^\rho} \right)^{N_j + |\alpha_j| + |\beta|} (1 + \sum_{i \in I} R_i)^{B_i} \quad (1.6)
$$

for $y^i \neq 0$, and decays rapidly as $y^i \to \infty$. 

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Lastly, we introduce the following class of weight functions defined on the product space.

**Definition of** $A_p$: Let $\lambda$ be an non-negative, locally integrable function on $\mathbb{R}^N$. Then, $\lambda \in A_p$ for $1 < p < \infty$ if
\[
\frac{1}{|Q|} \int_Q \lambda(x) dx \cdot \left( \frac{1}{|Q|} \int_Q \left( \frac{1}{\lambda(x)} \right)^{\frac{q}{p}} dx \right)^{\frac{q}{p}} \leq A
\]  
with $1/p + 1/q = 1$, for all $Q = B_1 \times B_2 \times \cdots \times B_n$. Each $B_i$ is a standard ball in $\mathbb{R}^{N_i}$, for $i = 1, 2, \ldots, n$. The smallest constant $A$, denoted by $A_p(\lambda)$, is taking to be the $A_p$ norm of $\lambda$.

Given $d\mu$ to be an nonnegative Borel measure. Our main result is the following:

**Theorem 1** Suppose $\sigma \in S_\rho$. Then the operator $T_\sigma$, initially defined on \( S \), extends to a bounded operator on $L_p(\mathbb{R}^N, d\mu)$ such that
\[
\int_{\mathbb{R}^N} \left| T_\sigma f(x) \right| \, d\mu(x) \leq A_{p,\rho} \int_{\mathbb{R}^N} \left| f(x) \right|^p \, d\mu(x)
\]
if and only if $d\mu(x) = \lambda(x) dx$ is absolutely continuous, with $\lambda \in A_p$ for $1 < p < \infty$. Moreover, $S_\rho$ forms an algebra under the composition of operators.

We develop our $L^p$ estimate in the framework of Littlewood-Paley projections. Such methodology in general has been applied to several earlier works, for example in [3] and [4] for Fourier multiplier operators and translation invariant singular integrals; in [8] for pseudo differential operators. The key investigation here would be the combinatorial estimate on the Dyadic decomposition of the frequency space, where the symbol in $S_\rho$ satisfies a desired size estimate, in the sense of differential inequality (1. 2). It appears that such estimate is crucial, as it is required in each major step of our analysis. In particular, we show that every partial sum operator in the framework is bounded by the strong maximal function, which satisfies the weighted $L^p$-norm inequality with respect to the measure $\lambda(x) dx$ for $\lambda \in A_p$.

The paper is organized as follows: We derive a lemma in section 2 which plays a principal role in later estimation. As a direct consequence, $S_\rho$ forms an algebra under the composition of operators. In section 3, we give the combinatorial estimate. In section 4, we show that every partial sum operator of Littlewood-Paley projections is bounded by the strong maximal function. The $L^p$ estimate is developed in section 5. We show that $T_\sigma$ admits the desired weighted inequality in section 6, based on some earlier results proved in [3]. In section 7, we estimate the kernel of $T_\sigma$ with $\sigma \in S_\rho$. By carrying out a mollified estimation in analogue to the first section of [1], we show that our class of operators can be equivalently defined by their kernels. The last section is devoted for applications.

**Abbreviations:**

○: Except otherwise indicated, we write $\int = \int_{\mathbb{R}^N}$ and $\iint = \iint_{\mathbb{R}^N \times \mathbb{R}^N}$.

○: We write $L^p$ as $L^p(\mathbb{R}^N, dx)$, and $L^p(\lambda)$ as $L^p(\mathbb{R}^N, \lambda(x)dx)$ with $\lambda \in A_p$, for $1 < p < \infty$.

○: We always write $A$ as a positive, generic constant with a subindex indicating its dependence.
2 Principal Lemma

The main objective of this section is to prove a lemma which is principal for later estimation. The original framework of the relevant investigation was developed by Beals and C. Fefferman [10] in 1974, also by Boutet de Monvel [9] in the same year. It has later been refined by Nagel and Stein in [2] where our estimate follows.

Let \( \phi = \phi(x, y, \xi, \eta) \) to be any smooth function. Its norm is bounded uniformly by a norm function \( \delta = \delta(\xi, \eta) \) which has polynomial growth in \( |\xi| \) and \( |\eta| \). Moreover, \( \phi \) satisfies differential inequality

\[
\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial \eta^\nu} \phi(x, y, \xi, \eta) \right| \leq A_{\alpha, \beta, \mu, \nu} \delta(\xi, \eta) \prod_{i=1}^{n} \left( 1 + \frac{1}{1 + |\xi| + |\xi|^p} \right)^{\alpha_i} \prod_{j=1}^{n} \left( 1 + \frac{1}{1 + |\eta| + |\eta|^p} \right)^{\beta_j} (1 + |\xi| + |\eta|)^{p(\mu + |\nu|)}.
\]

We consider the integral of the type

\[
\int \int e^{2\pi i (x-y)(\xi-\eta)} \phi(x, y, \xi, \eta) dy d\eta.
\]

Let \( \phi \in C_c^\infty(\mathbb{R}^n) \) with \( |\phi| \leq 1 \), is constantly 1 in the unit ball and zero outside a compact support. Define \( \phi^\varepsilon(y) = \phi(\varepsilon y) \) and \( \phi^\varepsilon = \phi \phi^\varepsilon \). We have \( \phi^\varepsilon \) satisfies inequality (2.1) uniformly in \( 0 < \varepsilon \leq 1 \).

Define the differential operator

\[
D = I - \left( \frac{1}{4\pi^2} \right) \left( 1 + \frac{1}{1 + |\xi| + |\xi|^2} \right)^p \Delta_y - \left( \frac{1}{4\pi^2} \right) \left( 1 + |\xi|^2 + |\eta|^2 \right)^p \Delta_\eta
\]

and respectively

\[
Q = 1 + \left( \frac{1}{1 + |\xi|^2} + \frac{1}{1 + |\eta|^2} \right)^p |\xi| - \eta|^2 + \left( 1 + |\xi|^2 + |\eta|^2 \right)^p |x - y|^2.
\]

Denote \( L = Q^{-1}D \). We have the identity

\[
L^N \left( e^{2\pi i (x-y)(\xi-\eta)} \right) = e^{2\pi i (x-y)(\xi-\eta)}
\]

for every \( N \geq 1 \). By replacing \( \phi \) with \( \phi^\varepsilon \) inside the integral (2.2), integrating by parts with respect to \( y \) and \( \eta \) gives

\[
\int \int (tL)^N (\phi^\varepsilon(x, y, \xi, \eta)) e^{2\pi i (x-y)(\xi-\eta)} dy d\eta
\]

where \( tL = tDQ^{-1} \). One major estimate is to show that

**Lemma 2.1**

\[
\left| (tL)^N \phi^\varepsilon(x, y, \xi, \eta) \right| \leq A_N \left( \frac{1}{Q} \right)^N \left( 1 + \frac{|\xi|^2}{1 + |\xi|} + \frac{|\eta|^2}{1 + |\eta|} \right)^{pN} \delta(\xi, \eta)
\]

uniformly in \( 0 < \varepsilon \leq 1 \) and for every \( N \geq 1 \).
Proof: Let

\[ a = \left( \frac{1}{1 + |\xi|^2} + \frac{1}{1 + |\eta|^2} \right)^{\rho} \quad \text{and} \quad b = \left( 1 + |\xi|^2 + |\eta|^2 \right)^{\rho}. \]  

(2. 8)

We have

\[ (l)^N \varphi^\varepsilon (x, y, \xi, \eta) = \left( \frac{1}{Q} - \Delta_y a Q - \Delta_\varepsilon b Q \right)^N \varphi^\varepsilon (x, y, \xi, \eta). \]  

(2. 9)

If all differentiations fall on \( \varphi \), we have

\[ a \left| \Delta_y \varphi^\varepsilon (x, y, \xi, \eta) \right| \leq a \left( 1 + |\xi|^2 + |\eta|^2 \right)^{\rho} \vartheta(\xi, \eta) \]  

(2. 10)

\[ \leq \left( 1 + \frac{|\xi|^2}{1 + |\eta|^2} + \frac{|\eta|^2}{1 + |\xi|^2} \right)^{\rho} \vartheta(\xi, \eta) \]  

and

\[ b \left| \Delta_\varepsilon \varphi^\varepsilon (x, y, \xi, \eta) \right| \leq \frac{b}{\left( 1 + |\eta|^2 \right)^{\rho}} \vartheta(\xi, \eta) \]  

(2. 11)

\[ \leq \left( 1 + \frac{|\xi|^2}{1 + |\eta|^2} + \frac{|\eta|^2}{1 + |\xi|^2} \right)^{\rho} \vartheta(\xi, \eta). \]

Turn to the general settings. From (2. 4), direct computation shows

\[ \left| \frac{\partial^\alpha}{\partial \eta^\alpha} \frac{1}{Q} \right| \leq \left( \frac{1}{1 + |\eta|} \right)^{|\alpha|} \frac{1}{Q} \leq \left( \frac{1}{1 + |\eta|} \right)^{|\alpha|} \frac{1}{Q} \left( 1 + \frac{|\xi|^2}{1 + |\eta|^2} + \frac{|\eta|^2}{1 + |\xi|^2} \right)^{\rho} \]  

(2. 12)

and

\[ \left| \frac{\partial^\beta}{\partial y^\beta} \frac{1}{Q} \right| \leq \frac{1}{Q} \left( 1 + \frac{|\xi|^2}{1 + |\eta|^2} + \frac{|\eta|^2}{1 + |\xi|^2} \right)^{\rho} \]  

(2. 13)

for every multi-index \( \alpha \) and \( \beta \).

From (2. 8)-(2. 9), it is suffice to estimate the term

\[ \left( \frac{b}{Q} \right)^{N_0} \left( \frac{\partial^{\alpha_1}}{\partial \eta^{\alpha_1}} b \frac{1}{Q} \right)^{N_1} \left( \frac{\partial^{\alpha_2}}{\partial \eta^{\alpha_2}} b \frac{1}{Q} \right)^{N_2} \cdots \left( \frac{\partial^{\alpha_k}}{\partial \eta^{\alpha_k}} b \frac{1}{Q} \right)^{N_k} \varphi^\varepsilon (x, y, \xi, \eta) \]  

(2. 14)

where

\[ N_0 + N_1 + N_2 + \cdots + N_k = N, \]  

(2. 15)

\[ |\alpha_1|N_1 + |\alpha_2|N_2 + \cdots + |\alpha_k|N_k + \beta = 2N \]

for every \( 1 \leq k \leq N \).

From (2. 8), we have

\[ \left| \frac{\partial^\alpha b}{\partial \eta^\alpha} \right| \leq A_\alpha \left( \frac{1}{1 + |\eta|} \right)^{|\alpha|} b. \]  

(2. 16)
The estimates in (2.12)-(2.13), together with (2.16) imply that

\[
\left| \frac{\partial^{\alpha} b}{\partial \eta^{\alpha} Q} \right| \leq A_{\alpha} \sum_{|\beta|+|\gamma|=|\alpha|} b \left( \frac{1}{1+|\eta|} \right)^{|\beta|} \left( \frac{1}{1+|\eta|} \right)^{|\gamma|} \frac{1}{Q} \leq A_{\alpha} \left( \frac{1}{1+|\eta|} \right)^{|\alpha|} b \left( 1 + \frac{|\xi|^2}{1+|\eta|^2} + \frac{|\eta|^2}{1+|\xi|^2} \right)^{\varphi}. \tag{2.17}
\]

By bringing in the estimate (2.17) into the combinatorial settings in (2.15), we obtain differential inequality (2.7). \( \square \)

**Lemma 2.2** Suppose \( \varphi = \varphi(x, y, \xi, \eta) \) is smooth, absolutely bounded and satisfies the differential inequality (2.1). Let

\[
\Phi(x, \xi) = \iint e^{2\pi i (x \cdot y - \xi \cdot \eta)} \varphi(x, y, \xi, \eta) dy d\eta. \tag{2.18}
\]

Then, \( \Phi(x, \xi) \) is absolutely bounded and satisfies the differential inequality

\[
\left| \frac{\partial^{\alpha} \partial^{\beta}}{\partial \xi^{\alpha} \partial \eta^{\beta}} \Phi(x, \xi) \right| \leq A_{\alpha, \beta, \rho} \prod_{i=1}^{n} \left( \frac{1}{1 + |\xi_i| + |\xi_i|} \right)^{|\alpha|} (1 + |\xi|)^{\rho|\beta|}. \tag{2.19}
\]

**Proof:** 1. We momentarily assume that \( \varphi \) has compact support in \( y \) by replacing \( \varphi \) with \( \varphi^\epsilon \) as in the previous lemma. The restriction will be removed by taking approximation in the sense of distributions.

Integration by parts with respect to \( y \) in (2.18) implies

\[
\frac{\partial^{\alpha} \partial^{\beta}}{\partial \xi^{\alpha} \partial \eta^{\beta}} \Phi(x, \xi) \tag{2.20}
\]

consists of the terms

\[
\iint e^{2\pi i (x \cdot y - \xi \cdot \eta)} \prod_{i=1}^{n} \prod_{j=1}^{n} \left( \frac{\partial^{\alpha_i} \partial^{\beta_i}}{\partial \xi^{\alpha_i} \partial \eta^{\beta_i}} \frac{\partial^{\alpha_j} \partial^{\beta_j}}{\partial \eta^{\alpha_j} \partial \eta^{\beta_j}} \varphi^\epsilon(x, y, \xi, \eta) \right) dy d\eta \tag{2.21}
\]

with \(|\alpha_i| + |\alpha_j| = |\alpha|\) and \(|\beta_i| + |\beta_j| = |\beta|\) for every \( i \) and \( j \).

Since \( \varphi \) is absolutely bounded, from differential inequality (2.1), the integrant inside (2.21) has norm bounded by

\[
A_{\alpha, \beta} \prod_{i=1}^{n} \left( \frac{1}{1 + |\xi_i| + |\eta|} \right)^{|\alpha_i|} \left( \frac{1}{1 + |\xi_i| + |\eta|} \right)^{|\alpha_i|} \prod_{j=1}^{n} (1 + |\xi|)^{\rho|\beta_j|} (1 + |\eta|)^{\rho|\beta_j|} \tag{2.22}
\]

for every multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \).

2. When \(|\xi - \eta| \leq \frac{1}{2}(1 + |\xi| + |\eta|)\), indeed we have \(|\xi| \sim |\eta|\). By lemma 2.1, the integral in (2.21) has its norm bounded by

\[
A_{\alpha, \beta, N} \left( \iint Q^{-N} dy d\eta \right) \prod_{i=1}^{n} \left( \frac{1}{1 + |\xi_i| + |\eta|} \right)^{|\alpha_i|} (1 + |\xi|)^{\rho|\beta|} \tag{2.23}
\]
for every $N \geq 1$.

When $|\xi - \eta| \geq \frac{1}{2}(1 + |\xi| + |\eta|)$. An $N$-fold of integration by parts with respect to $y$ inside (2.21) gains a factor of

$$
\left( \frac{1}{1 + |\xi| + |\eta|} \right)^{(1-\rho)N}.
$$

(2.24)

It is clear that for $N$ sufficiently large, depending on $\alpha$, $\beta$ and $\rho$, the integral in (2.21) has its norm bounded by (2.23).

3. From (2.4), we have

$$
Q \geq 1 + \left( \frac{1}{1 + |\xi|^2} \right)^{\rho/2} |\xi - \eta|^2 + \left( 1 + |\xi|^2 \right)^{\rho} |x - y|^2.
$$

(2.25)

By changing variables

$$
\eta \rightarrow \left( 1 + |\xi|^2 \right)^{\rho/2} \eta \quad \text{and} \quad y \rightarrow \left( \frac{1}{1 + |\xi|^2} \right)^{\rho/2} y,
$$

(2.26)

we have

$$
\iint Q^{-N} dyd\eta \leq \iint \frac{dyd\eta}{\left( 1 + |\eta|^2 + |y|^2 \right)^N}
$$

which is absolutely convergent provided that $N$ is sufficiently large. Lastly, $\varphi^\varepsilon \rightarrow \varphi$ pointwisely as $\varepsilon \rightarrow 0$. Taking

$$
\Phi(x, \xi) = \lim_{\varepsilon \rightarrow 0} \iint e^{2\pi i (x-y)(\xi-\eta)} \varphi^\varepsilon(x, y, \xi, \eta) dyd\eta.
$$

(2.28)

Since all estimates hold uniformly in $0 < \varepsilon \leq 1$, the lemma is proved. □

As a consequence from Lemma 2.2, the symbol class $S_\rho$ forms an algebra under the composition of operators $T_{\sigma_1} \circ T_{\sigma_2} = T_{\sigma_1 \sigma_2}$.

**Theorem 2.1** Suppose $\sigma_1 \in S_\rho$ and $\sigma_2 \in S_\rho$. Then, $\sigma_1 \circ \sigma_2 \in S_\rho$.

**Proof:** From direct computations, we have

$$
\sigma_1 \circ \sigma_2(x, \xi) = \iint e^{2\pi i (x-y)(\xi-\eta)} \sigma_1(x, \eta) \sigma_2(y, \xi) dyd\eta.
$$

The function $\sigma_1(x, \eta) \sigma_2(y, \xi)$ is absolutely bounded and satisfies differential inequality (2.1). Lemma 2.2 implies that $\sigma_1 \circ \sigma_2$ satisfies differential inequality (1.2). □

3 **The Combinatorial Estimate**

Let $\phi \in C^\infty_c(\mathbb{R}^N)$, such that $\phi \equiv 1$ when $|\xi| \leq 1$ and $\phi \equiv 0$ when $|\xi| > 2$. Define

$$
\psi(\xi) = \phi(\xi) - \phi(2\xi)
$$

(3.1)
so that $\psi$ is supported on a spherical shell $\frac{1}{2} \leq |\xi| \leq 2$.

For computational purpose, we write $q = 1/\rho$ hence that $1 < q < \infty$. Let $t_i$ be integers for every $i = 1, 2, \ldots, n$. We define the $n$-tuples

$$t_i = \left(2^{-a t_i}, \ldots, 2^{-a t_i}, 2^{t_i}, 2^{-a t_i}, \ldots, 2^{-a t_i}\right) \quad (3.2)$$

where $2^{-t_i}$ is located in the $i$-th component, and the non-isotropic dilations

$$t_i \xi = \left(2^{-a t_i} \xi_1, \ldots, 2^{-a t_i} \xi_{i-1}, 2^{-t_i} \xi_i, 2^{-a t_i} \xi_{i+1}, \ldots, 2^{-a t_i} \xi_n\right) \quad (3.3)$$

$i = 1, 2, \ldots, n$.

In the other hand, we define the $n$-tuples

$$t = \left(2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_n}\right), \quad t^{-1} = \left(2^{t_1}, 2^{t_2}, \ldots, 2^{t_n}\right) \quad (3.4)$$

and the dilations

$$t \xi = \left(2^{-t_1} \xi_1, 2^{-t_2} \xi_2, \ldots, 2^{-t_n} \xi_n\right), \quad t^{-1} \xi = \left(2^{t_1} \xi_1, 2^{t_2} \xi_2, \ldots, 2^{t_n} \xi_n\right). \quad (3.5)$$

With respect to the dilations (3.3)-(3.5), we let

$$\delta_t(\xi) = \prod_{i=1}^{n} \psi(t_i \xi). \quad (3.6)$$

The partial sum operator $\Delta_t$ is defined by

$$\left(\Delta_t f\right)(\xi) = \delta_t(\xi) \hat{f}(\xi) = \prod_{i=1}^{n} \psi\left(2^{-a t_i} \xi_1, \ldots, 2^{-a t_i} \xi_{i-1}, 2^{-t_i} \xi_i, 2^{-a t_i} \xi_{i+1}, \ldots, 2^{-a t_i} \xi_n\right) \hat{f}(\xi). \quad (3.7)$$

The support of $\delta_t(\xi)$ lies inside the intersection of $n$ elliptical shells with different homogeneities of given dilations.

We aim to prove the following lemma:

**Lemma 3.1** Let $I \cup J = \{1, 2, \ldots, n\}$ with $1 \leq |I| \leq n$ such that

$$t_{i_1} + 2 + \log_2 n) / q < t_{i_2} < a t_{i_1} - \left(2 + \log_2 n\right) \quad \text{for all } i_1, i_2 \in I, \quad (3.8)$$

and

$$a t_j - \left(2 + \log_2 n\right) \leq t_i \quad \text{for some } i \in I \text{ and all } j \in J. \quad (3.9)$$

By assuming that at least one

$$t_i > \frac{1}{q-1} \left(2 + 2 \log_2 n\right) \quad \text{for some } i \in \{1, 2, \ldots, n\} \quad (3.10)$$

The support of $\delta_t(\xi)$ is non-empty if and only if

$$|\xi_i| \sim 2^{t_i} \quad \text{for every } i \in I,$$

$$|\xi_j| \lesssim 2^{t_j} \quad \text{for every } j \in J \quad \text{and} \quad |\xi_i| \sim 2^{t_i} \sim 2^{a t_i} \quad \text{for some } i \in I.$$
Remark 3.1 A direct outcome from Lemma 3.1 is that we have \( t_i \leq q t_j \) for every pair of \( i, j \in \{1, 2, \ldots, n\} \), provided that \( t \) satisfies (3.10) with the support of \( \delta_t(\xi) \) is non-empty.

Proof: First, by writing down

\[
\text{supp } \delta_t(\xi) = \text{supp} \prod_{i=1}^{n} \psi \left( 2^{-at_j} \xi_1, \ldots, 2^{-at_j} \xi_{i-1}, 2^{-at_i} \xi_i, 2^{-at_i} \xi_{i+1}, \ldots, 2^{-at_i} \xi_n \right),
\]

(3.11)

the definition of \( \psi \) in (3.1) implies that \( \xi \) satisfies

\[
|\xi_i| < 2^{t_i+1}, \quad i = 1, 2, \ldots, n,
\]

(3.12)

\[
|\xi_i| < 2^{at_j+1}, \quad \text{for } j \neq i
\]

whenever \( \xi \in \text{supp} \delta_t(\xi) \). Next, we write \( \xi = (\xi_i, \xi'_i) \in \mathbb{R}^{N_i} \times \mathbb{R}^{N-N_i} \) with \( \xi'_i \) denotes the complement of \( \xi_i \) in \( \mathbb{R}^N \). Our estimation will be obtained in several steps.

1: Suppose that for every \( i, j \in \{1, 2, \ldots, n\} \),

\[
\left( t_j + 2 + \log_2 n \right)/a < t_i < q t_j - \left( 2 + \log_2 n \right).
\]

(3.13)

We shall have

\[
|\xi_i| > \frac{2^{t_i-1}}{\sqrt{2}} \quad \text{or} \quad |\xi'_i| > \frac{2^{at_j-1}}{\sqrt{2}}
\]

(3.14)

and

\[
|\xi_i| > \frac{2^{at_j-1}}{\sqrt{2}} \frac{1}{\sqrt{n-1}}.
\]

(3.15)

In the other hand, from the first inequality in (3.12), we have

\[
|\xi_i| < 2^{t_i+1}.
\]

(3.16)

From (3.15) and (3.16), the support in (3.11) is empty provided that

\[
t_j < q t_i - 2 - \frac{1}{2} \log_2 2(n-1).
\]

(3.17)

From (3.13), we have \( t_j < q t_i - \left( 2 + \log_2 n \right) \) for every \( i \) and \( j \). Inequality (3.17) is true provided by \((1/2) \log_2 2(n-1) < \log_2 n \) for every \( n \geq 2 \). We therefore have

\[
|\xi_i| \sim 2^{t_i}, \quad i = 1, 2, \ldots, n.
\]

(3.18)

2: Suppose that there exists \( i \in \{1, 2, \ldots, n\} \) such that

\[
qt_j - \left( 2 + \log_2 n \right) \leq t_i \quad \text{for all } j \neq i.
\]

(3.19)

We shall have

\[
|\xi_i| > \frac{2^{t_i-1}}{\sqrt{2}} \quad \text{or} \quad |\xi'_i| > \frac{2^{at_j-1}}{\sqrt{2}}.
\]

(3.20)
Suppose $\xi_i$ does not satisfy the estimate in (3.14), then alternatively we must at least have one $\xi_j$ for some $l \neq i$ such that

$$|\xi_j| > \frac{2^{at_l-1}}{\sqrt{2}} \frac{1}{\sqrt{n-1}}. \quad (3.21)$$

In the other hand, from the first inequality in (3.12), we have

$$|\xi_j| < 2^{t_{j+1}}. \quad (3.22)$$

From (3.21) and (3.22), the support in (3.11) is empty provided that

$$qt_i - t_j - 2 - \frac{1}{2} \log_2 2(n-1) > 0. \quad (3.23)$$

Recall that $qt_j \leq t_i + 2 + \log_2 n$ from (3.13). We have

$$qt_i - t_j - 2 - \frac{1}{2} \log_2 2(n-1)$$

$$> \left(q - \frac{1}{q}\right) t_i - \left(1 + \frac{1}{q}\right) (2 + \log_2 n)$$

$$> 0 \quad (3.24)$$

provided that $t_i$ satisfies (3.10) and the fact that $(1/2) \log_2 2(n-1) < \log_2 n$ for every $n \geq 2$. Otherwise, suppose $t_j$ satisfies (3.10) for some $j \neq i$. Then, (3.13) implies

$$t_i > \frac{q}{q-1} \left(2 + \log_2 n\right) - \left(2 + \log_2 n\right)$$

$$= \frac{1}{q-1} \left(2 + \log_2 n\right) \quad (3.25)$$

which again implies $t_i$ satisfies (3.10). All together with the second inequality in (3.12), we must have

$$|\xi_i| \sim 2^{t_i} \sim 2^{qt_i} \quad \text{and} \quad |\xi_j| \leq 2^{t_j} \quad \text{for all } j \neq i.$$

3: The lemma will be concluded by combining the estimations in the two previous steps. Write the support in (3.11) as

$$\text{supp} \prod_{i \in I} \psi \left(2^{-t_i} \xi_i, 2^{-qt_i} \xi_i'\right) \cap \text{supp} \prod_{j \in J} \psi \left(2^{-t_j} \xi_j, 2^{-qt_j} \xi_j'\right) \quad (3.26)$$

whereas $1 \leq |I| \leq n$.

Since there exists at least one $i \in \{1, 2, \ldots, n\}$ such that $t_i$ satisfies (3.10). By (3.9), either $i \in I$ or there exists $i \in I$ such that $t_i$ satisfies (3.10). By carrying out the same estimation in step 1, we obtain

$$|\xi_i| \sim 2^{t_i} \quad \text{for every } i \in I. \quad (3.27)$$

In the other hand, suppose $t_i$ satisfies (3.10) for some $i \in I$. For $i \neq t$, we either have $t_t > t_i$ so that $t_j \geq qt_j - \left(2 + \log_2 n\right)$ for all $j \in J$, or we have $t_t < t_i$ so that $t_i$ satisfies (3.10). By carrying out the same estimation in step 2, we obtain

$$|\xi_j| \leq 2^{t_j} \quad \text{for every } j \in J \text{ and } qt_j \sim t_t \text{ for some } t_i \in I. \quad (3.28)$$

$\square$
Lemma 3.2  Let $t$ and $s$ satisfy (3.10). There exists a constant $\text{const}$, such that the intersection of

$$\text{supp}\delta_t(\xi) \cap \text{supp}\delta_s(\xi)$$

is non-empty only when

$$\sum_{i=1}^{n} |t_i - s_i| < \text{const}.$$

Proof: We set $I_1 \cup J_1 = I_2 \cup J_2 = \{1, 2, \ldots, n\}$ for which $I_1$, $I_2$, $J_1$, $J_2$ take the same meaning of $I$ and $J$ in Lemma 3.1. Consider $t_i$ for $i \in I_1 \cup J_1$ and $s_i$ for $i \in I_2 \cup J_2$ respectively. From Lemma 3.1, we have $|\xi_i| \sim 2^t$ for every $i \in I_1$ and $|\xi_j| \leq 2^t$ for every $j \in J_1$. Similarly, $|\xi_i| \sim 2^s$ for every $i \in I_2$ and $|\xi_j| \leq 2^s$ for every $j \in J_2$.

Therefore, for any $i \in I_1$, we need $s_i \geq t_i$ in order to have the intersection of supports to be non-empty. Suppose $i \in I_2$, we have $|\xi_i| \sim 2^s$ so that $|t_i - s_i|$ must be then bounded by some constant. Otherwise, suppose $i \in J_2$. Lemma 3.1 implies there exists an $s \in J_2$ such that $|\xi_i| \sim 2^s \sim 2^{q_t}$. But again, we need $t_i \geq s_i = q_s$ for every other $j \neq i$ by Remark 3.1. Thus that $|t_i - s_i|$ must be bounded by some constant for every $i \in I_1$. The same argument goes for every $i \in I_2$.

Next, from Lemma 3.1, there exists $t \in I_1$ such that

$$q_t - \left(2 + \log_2 n\right) \leq t_i \quad \text{and} \quad |\xi_i| \sim 2^t \sim 2^{q_t}$$

for all $j \in J_1$. It is necessary that $t_i \geq t$ for all other $i \in I_1$. Therefore, since $|t_i - s_i|$ is bounded for every $i \in I_1 \cup I_2$, we must have $|t_j - s_j|$ bounded for every $j \in J_1 \cap J_2$. \hfill \Box

4 Maximal Function Bound

A strong maximal function operator is defined by

$$Mf(x) = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |f(x - y)|dy$$

(4.1)

where $Q = B_1 \times B_2 \times \cdots \times B_n$. Each $B_i \subset \mathbb{R}^N$ here denotes a standard ball centered at $x_i = 0$.

We are going to show that for $\sigma \in S_{\rho}$, then

$$\left| (\Delta_{t, \sigma} f)(x) \right| \leq Mf(x)$$

(4.2)

whenever $t$ satisfies (3.10). A direct computation shows

$$(\Delta_{t, \sigma} f)(x) = \int f(x - y)\Omega_t(x, y)dy$$

(4.3)

$$= \int f(x - y) \left( \int e^{2\pi i y \cdot \xi} \delta_t(\xi) \omega(x - y, \xi) d\xi \right)dy$$

where

$$\omega(x, \xi) = \iint e^{2\pi i (x - y) \cdot (\xi - \eta)} \sigma(y, \eta) dy d\eta.$$
Since $\sigma(x, \xi)$ satisfies the differential inequality (2.1). Lemma 3.2 implies $\omega(x, \xi)$ is absolutely bounded and satisfies the differential inequality
\[
\left| \frac{\partial^\alpha}{\partial \xi^\alpha} \omega(x, \xi) \right| \leq A_{\alpha, \rho} \prod_{i=1}^{\|\alpha\|} \left( \frac{1}{1 + |\xi_i|^p + |\xi|^p} \right)^{|\alpha_i|}
\] (4.5)
for every multi-index $\alpha$.

By changing dilations with respected to $\xi = t^{-1}\eta$ and $y = tz$, (4.3) cab be rewritten as
\[
\int f(x - tz) \left\{ \int e^{2\pi i z \cdot \eta} \delta_t (t^{-1}\eta) \omega(x - tz, t^{-1}\eta) d\eta \right\} dz.
\] (4.6)

Observe that by definition (3.1),
\[
\text{supp} \delta_t (t^{-1}\eta) = \text{supp} \prod_{i=1}^{\|\alpha\|} \psi(t^{-1}t_i \xi_i)
\] (4.7)
\[
= \text{supp} \prod_{i=1}^{\|\alpha\|} \psi(2^{|\alpha_i|}t_i \xi_i, \ldots, 2^{|\alpha_{i-1}|}t_i \xi_1, \xi_i, 2^{|\alpha_{i+1}|}t_i \xi_{i+1}, \ldots, 2^{|\alpha_n|}t_i \xi_n)
\]
is inside a ball with radius 2. Moreover, from Remark 3.1, we have $t_i \leq q t_j$ for every pair of $i, j \in \{1, 2, \ldots, n\}$ provided that the support is non-empty. Therefore, the dilation $t^{-1}t_i$ has all its components bounded, for every $i$. Indeed, we have $2^{|\alpha_i|}t_i \leq 1$ for every $i$ and $j$.

Let $I \cup J = \{1, 2, \ldots, n\}$ as in Lemma 3.1. Decompose $\xi = (\xi_I, \xi_J)$ where $\xi_I$ consists all components $\xi_i$ for $i \in I$ and $\xi_J$ consists of all components $\xi_j$ for $j \in J$. We have $|\eta_i| \sim 1$ for every $i \in I$, by Lemma 3.1.

Integrating by parts with respect to $\eta$ in (4.6) gives that the integral inside the bracket equals to
\[
\left( \frac{1}{2\pi i z} \right)^{|\alpha|} \int e^{2\pi i z \cdot \eta} \left( \frac{\partial^\alpha}{\partial \eta^\alpha} \delta_t (t^{-1}\eta) \omega(x - tz, t^{-1}\eta) \right) d\eta
\] (4.8)
for every multi index $\alpha$. From differential inequality (4.5) and Lemma 3.1, we have
\[
\left| \frac{\partial^\alpha}{\partial \eta_{I_i}^\alpha \partial \eta_{J_j}^\beta} \omega(x - tz, t^{-1}\eta) \right|
\leq \prod_{i \in I} (2^{t_i})^{|\alpha_i|} \left( \frac{1}{1 + 2^{|\alpha_i|} |\eta_i| + |t^{-1}\eta|^p} \right)^{|\alpha_i|} \prod_{j \in J} (2^{t_j})^{|\beta_j|} \left( \frac{1}{1 + 2^{|\beta_j|} |\eta_j| + |t^{-1}\eta|^p} \right)^{|\beta_j|}
\leq A_{\alpha, \beta, \rho}
\] (4.9)
provided that there exists \( t \in I \) such that \( 2^t \sim 2^{|j|} \) for every \( j \in J \), from Lemma 3.1.

The estimates in (4.8)-(4.9) imply that

\[
\left|(\Delta_t T_\sigma f)(x)\right| \leq A_{N,a} \int |f(x - tz)| \left( \frac{1}{1 + |z|^N} \right) dz \tag{4.10}
\]

for every \( N \geq 1 \). The function \( \left(1 + |z|^N\right)^{-1} \) can be uniformly approximated by the absolutely convergent series \( \sum a_k \chi_{B(k)} \) where each \( a_k \) is positive constant and \( B(k) \subset \mathbb{R}^N \) is the standard ball centered at the origin with radius \( k \). We then have

\[
\int |f(x - tz)| \left( \sum_{k=1}^{\infty} a_k \chi_{B(k)} \right) dz \leq \left( \sum_{k=1}^{\infty} a_k |B(k)| \right) \sup_{Q \subset \mathbb{R}^N} \frac{1}{|Q|} \int_{Q} |f(x - y)| dy \tag{4.11}
\]

\[
\leq M f(x).
\]

We thus obtain the result in (4.2).

The \( L^p \) continuity of \( M \) follows that

\[
\left( M f \right)(x) \leq \left( M_{e_1} \left( M_{e_2} \left( \cdots M_{e_n} f \right) \right) \right)(x) \tag{4.12}
\]

as was investigated in [5] with each \( M_{e_i} \) the standard maximal function operator on \( \mathbb{R}^N \) for \( i = 1, 2, \ldots, n \).

It is clear that for all \( t \) that (3.10) does not hold, the summation of \( \delta_t(\xi) \) has a compact support. Before moving forward to the \( L^p \) estimate, we hereby give a remark on the following fact:

**Remark 4.1**

\[
\left| \sum \Delta_t T_\sigma f \right| \leq M f, \quad \left| \sum T_\sigma \Delta t f \right| \leq M f \tag{4.13}
\]

where the summations are taking over all \( t \) s such that (3.10) does not hold, or in other words,

\[
t_i \leq \frac{1}{a - 1} (2 + \log_2 n), \quad \text{for all } i = 1, 2, \ldots, n.
\]

Let \( \varphi = \varphi(\xi) \) be a smooth function supported on a ball centered at the origin. We decompose \( T_\sigma = T_{0\varphi} + T_{\sigma(1-\varphi)} \). In the other hand, define the operator \( \widetilde{T}_\sigma \) by

\[
\left( \widetilde{T}_\sigma f \right)(\xi) = (1 - \varphi(\xi)) \left( T_{\sigma(1-\varphi)} f \right)(\xi). \tag{4.14}
\]

By adjusting the size of the ball, we can assume that

\[
\widetilde{T}_\sigma f = \sum_{t,s} \Delta_t T_\sigma \Delta_s f \tag{4.15}
\]

where the summation is taking over all \( t \) and \( s \) such that (3.10) is satisfied. By Remark 4.1, it is suffice to estimate the \( L^p \) continuity of \( \widetilde{T}_\sigma \). For the sake of brevity, we will take \( T_\sigma = \widetilde{T}_\sigma \) for the next section.
5 \textbf{L}^p \textbf{Estimate}

In this section, we prove the \textbf{L}^p continuity of pseudo differential operator $T_\sigma$ with $\sigma \in S_\rho$. The estimation will be developed in the framework of Littlewood-Paley projections. If $T_\sigma$ is translation invariant where $\sigma = \sigma(\xi)$, then $\Delta_t T_\sigma \Delta_s$ vanishes when $t$ and $s$ away, in the sense of that

$$\sum_{i=1}^{n} |t_i - s_i| > \text{const}$$

by Lemma 3.2. In general, we have the following estimate of decaying:

**Lemma 5.1**

$$\left| \left( \Delta_t T_\sigma \Delta_s f \right)(x) \right| \lesssim \prod_{i=1}^{n} 2^{-\varepsilon|t_i - s_i|} \left( \mathbb{M} f \right)(x)$$

for some $\varepsilon = \varepsilon(\rho) > 0$.

**Proof:**

1. Recall the formulae in (4. 3)-(4. 4). By direct computations, we have

$$\left( \Delta_t T_\sigma \Delta_s f \right)(x) = \int \int f(x - y) \Omega_{t,s}(x, y, \xi) d\xi dy$$

where

$$\Omega_{t,s}(x, y, \xi) = \int \int e^{2\pi i y \cdot \xi} \delta_t(\xi) \omega_s(x - y, \eta) d\eta d\xi$$

and

$$\omega_s(x, \xi) = \int \int e^{2\pi i (x-y)(\xi-\eta)} \delta_s(\eta) \sigma(y, \eta) dy d\eta.$$  

Observe that $\delta_s(\eta) \sigma(y, \eta)$ satisfies differential inequality (2. 1). By Lemma 2.2, $\omega_s(x, \xi)$ is absolutely bounded and satisfies differential inequality (2. 19).

2. We momentarily assume that $\sigma(x, \xi)$ has $x$-compact support. Integration by parts in $y$ inside $\omega_s$ implies

$$\omega_s(x, \xi) = \left( \frac{1}{2\pi i} \right)^2 \int \int e^{2\pi i (x-y)(\xi-\eta)} \delta_s(\eta) \sigma(y, \eta) dy d\eta.$$  

Again, we have $\delta_s(\eta) \Delta_\eta \sigma(y, \eta)$ satisfies differential inequality (2. 1). By Lemma 2.2, $\omega_s(x, \xi)$ is bounded by

$$\left( \frac{|\eta|}{|\xi - \eta|^s} \right)^3$$

for which $\xi \in \text{supp} \delta_t(\xi)$ and $\eta \in \text{supp} \delta_s(\eta)$.

3. We set $I_1 \cup J_1 = I_2 \cup J_2 = \{1, 2, \ldots, n\}$ for which $I_1$, $I_2$ take the same meaning of $I$ and $J_1$, $J_2$ take the same meaning of $J$ in Lemma 3.1. Consider $t_i$ for $i \in I_1 \cup J_1$ and $s_i$ for $i \in I_2 \cup J_2$ respectively.

Let $\eta_i$ to be the largest component of $\eta$ so that $|\eta| \sim |\eta_i| \sim 2^i$. 

\(14\)}
Assume \( t_i \neq s_i \) for some \( i \in \{1, 2, \ldots, n\} \). We first consider the case for \( t_i < s_i \) and \( qt_i < s_i \). From Remark 3.1, we have \( t_i \leq qt_i \) for every \( i, j \in \{1, 2, \ldots, n\} \) whenever the support of \( \delta_t(\xi) \) is non-empty. By Lemma 3.1, we must have \( |\xi| \leq \eta| \|. Therefore,

\[
\frac{|\eta|}{|\xi - \eta|^q} \lesssim 2^{-(q-1)|\eta|} \quad t_i < s_i. \tag{5.7}
\]

The rest of possible cases will be estimated separately.

**Case 1:** Suppose that \( i \in I_1 \cap I_2 \). By Lemma 3.1, we have \( |\xi| \sim 2^{|t|} \) and \( |\eta| | \sim 2^q | \|. Such that

\[
|\xi - \eta| | \sim \begin{cases} 2^{|t_i|} = 2^{|t_i|} 2^{(q - s)} & t_i > s_i, \\ 2^{|s_i|} = 2^{|s_i|} 2^{(q - s)} & t_i < s_i. \end{cases} \tag{5.8}
\]

Since \( q \leq s_i \) for every \( i \) from Remark 3.1. (5.8) implies

\[
\frac{|\eta|}{|\xi - \eta|^q} \lesssim 2^{-q|\eta|}. \tag{5.9}
\]

**Case 2:** Suppose \( i \in I_1 \cap J_2 \). By Lemma 3.1, we have \( |\xi| \sim 2^{|t|} \) and \( |\eta| \mid \lesssim 2^q | \|. If \( t_i < s_i \), we are in the first case of (5.8). Let \( s_i > t_i \). By Lemma 3.1, there exists an \( j \in I_2 \) such that \( s_j \sim q s_i \) and \( |\eta| \mid \sim 2^{|s_j|} \sim 2^q | \|. In the other hand, we have \( t_j < q t_i \) since \( i \in I_1 \). Therefore, \( |\xi_i - \eta| | \sim 2^{|s_j|} \) with \( s_i > t_j \). If \( q t_j > s_i \), we go back to the estimate in (5.7). If \( q t_j > s_i \), we are in the second case of (5.8). The same argument applies to \( i \in J_1 \cap I_2 \).

Lastly, suppose \( i \in J_1 \cap J_2 \). By Lemma 3.1, there exists \( j \in I_1 \) such that \( |\xi_j| \sim 2^{|t_j|} \sim 2^q | \|. If \( i \in I_2 \), we are in Case 1. If \( i \in J_2 \), we are in Case 2.

4. By putting together all estimates (5.7)-(5.9), one easily generalizes that

\[
|\delta_t(\xi)\omega(x, \xi)| \leq \prod_{i=1}^{n} 2^{-\varepsilon|t_i - s_i|}, \quad \varepsilon = \varepsilon(q) > 0. \tag{5.10}
\]

Let \( y = t \zeta \). By carrying out the same estimations from (4.6)-(4.10), with \( \Omega_t(x, y) \) replaced by \( \Omega_{t,\xi}(x, y) \), we have

\[
(\Delta_t T_0^N \xi f) (x) \leq A_{N,\xi} \prod_{i=1}^{n} 2^{-\varepsilon|t_i - s_i|} \int \left| f(x - t \zeta) \left( \frac{1}{1 + \zeta^2} \right)^{N/2} \right| \, \zeta \, dz \tag{5.11}
\]

for \( \varepsilon = \varepsilon(q) > 0 \) and every \( N \geq 1 \).

Since all estimates are independent from the size of the \( x \)-support of \( \sigma(x, \xi) \), the compactness can be removed by taking approximation in the sense of distribution. (5.11) indeed implies the decaying of \( \Delta_t T_0^N \xi f \) in (5.1), by following the argument in the end of the previous section. \( \square \)

Let \( h_i \) be integers for every \( i = 1, 2, \ldots, n \). We define the \( n \)-tuples \( (t + h)_i \) and the non-isotropic dilations \( (t + h)_i \xi \), simply by replacing \( t_i \) with \( t_i + h_i \) respectively in (3.2) and (3.3). We then have

\[
\delta_{t + h}(\xi) = \prod_{i=1}^{n} \psi((t + h)_i \xi)
\]
as in (3. 6). The partial sum operator $\Delta_{t+h}$ is defined by $(\widehat{\Delta_{t+h} f})(\xi) = \delta_{t+h}(\xi) \hat{f}(\xi)$ as in (3. 7). Define the operator

$$\Lambda_{t,h} = \Delta T_0 \Delta_{t+h}.$$  

(5. 12)

We have the following results of almost orthogonality:

**Lemma 5.2**

$$\left| \left( \Lambda_{t,h}^* \Lambda_{s,h} f \right)(x) \right| \leq \prod_{i=1}^{n} 2^{-\epsilon |l_i| - |s_i|} \times \prod_{i=1}^{n} 2^{-2\epsilon |h_i|}(Mf)(x)$$  

(5. 13)

$$\left| \left( \Lambda_{t,h} \Lambda_{s,h}^* f \right)(x) \right| \leq \prod_{i=1}^{n} 2^{-\epsilon |l_i| - |s_i|} \times \prod_{i=1}^{n} 2^{-2\epsilon |h_i|}(Mf)(x)$$

for some $\epsilon = \epsilon(\rho) > 0$.

**Proof:** Recall from (5. 2)-(5. 4). We have $(\Delta T_0 \Delta_s)^* = \Delta_s T_0^* \Delta_t$. By symmetry, it is suffice to estimate one of the inequalities in (5. 13).

A direct computation shows

$$\left( \Lambda_{t,h}^* \Lambda_{s,h} f \right)(x) = \int f(x-y) \Omega_{t+h}(x,y) \overline{\Lambda_{s,h} f}(x,y) \, dy$$

(5. 14)

$$= \int f(x-y) \left( \int e^{2\pi i y \cdot \xi} \delta_{t+h}(\xi) \omega_{t,s}(x-y,\xi) \, d\xi \right) \, dy.$$

where

$$\omega_{t,s}(x, \xi) = \int \int e^{2\pi i (x-y) \cdot (\eta-\xi)} \delta_{s+h}(\eta) \omega_t(y, \xi) \overline{\omega_s}(y, \eta) \, dy \, d\eta.$$  

(5. 15)

From Lemma 5.1, both $\omega_t(x, \xi)$ and $\omega_s(x, \eta)$ in (5. 15) have their norm bounded by $\prod 2^{2|a_i|}$ for some $\epsilon = \epsilon(\rho) > 0$. In the other hand, we have $\omega_t(x, \xi) \overline{\omega_s}(x, \eta)$ satisfies differential inequality (2. 1). By Lemma 2.2, we have $\omega_{t,s}(x, \xi)$ satisfies differential inequality (2. 19) again, with its norm bounded by

$$\prod_{i=1}^{n} 2^{-2\epsilon |h_i|}, \quad \epsilon = \epsilon(\rho) > 0.$$  

(5. 16)

By carrying out the same estimation in the proof of Lemma 5.1, with $\omega_{s,h}(x, \xi)$ replaced by $\omega_{t,s}(x, \tilde{\xi})$, the desired result follows.  

We now start to conclude the $L^p$ continuity of $T_0$. Let $f \in L^2 \cap L^p$ and $g \in L^2 \cap L^q$ with $1/p + 1/q = 1$. Consider the identity

$$\int f(T_0 g) \, dx = \int \left( \sum_t \Delta T_0 f \right) \left( \sum_s \Delta_s g \right) \, dx.$$  

(5. 17)

By Lemma 3.2, for each $t$ fixed, the intersection of $\text{supp} \delta_t(\xi)$ and $\text{supp} \delta_s(\xi)$ is non-empty only upon to a finitely many of $s$, such that $\sum |l_i - s_i| < \text{const}$. Write $s = t + m$ and denote $|m| = \sum |l_i - s_i|$. Plancherel theorem implies that (5. 17) equals to

$$\sum_{|m| < \text{const}} \int \sum_t \Delta T_0 f \Delta_t m g \, dx.$$  

(5. 18)
In the other hand, for each \( t \) fixed, we write \( f = \sum_h \Delta_t h f \) where the summation is taking over all \( n \)-tuples \( h = (h_1, h_2, \ldots, h_n) \) of integers. We further write (5. 18) as

\[
\sum_{|m| < \text{const}} \left\{ \sum_h \left( \int \sum_t \Delta_t \sigma \Delta_t h f \Delta_t m g \, dx \right) \right\}.
\]

By applying Schwarz inequality and then H"older inequality, we have

\[
\int (T_\sigma f) g \, dx \leq \text{const} \times \sum_h \int \left( \sum_t \left| \Delta_t \sigma \Delta_t h f \right|^2 \right)^{\frac{1}{2}} \left( \sum_t \left| \Delta_t g \right|^2 \right)^{\frac{1}{2}} \, dx
\]

\[
\leq \text{const} \times \sum_h \left\| \sum_t \left| \Delta_t \sigma \Delta_t h f \right|^2 \right\|_{L^p} \left\| \left( \sum_t \left| \Delta_t g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q}.
\]

The restriction of \( L^2 \)-boundedness can be removed by taking a sequence of functions \( f_j \in L^2 \cap L^p \) converging to \( f \in L^p \) as \( j \to \infty \) in \( L^p \) space, then using the inequalities in (5. 20).

By taking the supremum of all \( g \) with \( \| g \|_{L^{q}} = 1 \), (5. 20) implies that

\[
\left\| T_\sigma f \right\|_{L^p} \leq \sum_h \left\| \sum_t \left| \Delta_t \sigma \Delta_t h f \right|^2 \right\|_{L^p}.
\]

Recall from (4. 2), we have \( \Delta_t \sigma f \) bounded by \( M f \) for every \( t \). Let \( h \) be fixed. By using the vector-valued inequality of strong maximal function in [3], and then Littlewood-Paley inequality, we have

\[
\left\| \sum_t \left| \Delta_t \sigma \Delta_t h f \right|^2 \right\|_{L^p} \leq \left\| \sum_t \left( |\Delta_t h f|^2 \right) \right\|_{L^p}
\]

\[
\leq \left\| \sum_t \left( |\Delta_t h f|^2 \right) \right\|_{L^r} \sim \| f \|_{L^r}, \quad 1 < r < \infty.
\]

In the other hand, recall from Lemma 5.2. By applying Cotlar-Stein Lemma in [7], together with Lemma 3.2 and Plancherel theorem, we have

\[
\left\| \sum_t \left| \Delta_t \sigma \Delta_t h f \right|^2 \right\|_{L^2} \leq \prod_{i=1}^{n} 2^{-\varepsilon|h_i|} \| f \|_{L^2}, \quad \varepsilon = \varepsilon(\rho) > 0.
\]

Let \( p \in (r, 2) \) and \( p \in [2, r) \). By Riesz interpolation theorem in [13], we have

\[
\left\| \sum_t \left| \Delta_t \sigma \Delta_t h f \right|^2 \right\|_{L^p} \leq \prod_{i=1}^{n} 2^{-\varepsilon'h_i} \| f \|_{L^p}
\]

with \( \varepsilon' > 0 \) depending on \( p \) and \( \rho \).

Lastly, by summing over all the \( h_i \) in the summand of (5. 21), we obtain the desired result. Namely, let \( \sigma \in S_\rho \) with \( 0 < \rho < 1 \), we have

\[
\left\| T_\sigma f \right\|_{L^p} \leq A_{p, \rho} \| f \|_{L^p}
\]

for \( 1 < p < \infty \).
6 Weighted Norm Inequality

The studies of weighted norm inequalities for maximal functions and singular integrals can be traced back to the earlier results in 1972 by Muckenhoupt [11], and then in 1974 by Coifman and C. Fefferman [12].

Write \( x = (x_i, x'_i) \in \mathbb{R}^{N_i} \times \mathbb{R}^{N-N_i} \) where \( x'_i \) denotes the complement of \( x_i \) in \( \mathbb{R}^N \). We first introduce the following class of weight functions defined in [3].

**Definition of \( A_p^* \):** Let \( \lambda \) be a non-negative, locally integrable function on \( \mathbb{R}^{N_i} \) for every \( i = 1, 2, \ldots, n \). Then, \( \lambda \in A_p^* \) for \( 1 < p < \infty \) if

\[
\frac{1}{|B_i|} \int_{B_i} \lambda(x_i, x'_i) dx_i \cdot \left( \frac{1}{|B_i|} \int_{B_i} \left( \frac{1}{\lambda(x_i, x'_i)} \right)^q dx_i \right)^{\frac{p}{q}} \leq A,
\]

\( i = 1, 2, \ldots, n \)

with \( 1/p + 1/q = 1 \), for all \( B_i \subset \mathbb{R}^{N_i} \) uniformly in \( x'_i \in \mathbb{R}^{N-N_i} \).

In below, we recall the two results proved in [3] by R. Fefferman and Stein.

(*) Given \( d\mu \) to be an nonnegative Borel measure.

\[
\int |Mf(x)|^p d\mu(x) \leq A_p \int |f(x)|^p d\mu(x) \tag{6.2}
\]

if and only if \( d\mu(x) = \lambda(x) dx \) is absolutely continuous, with \( \lambda \in A_p^* \), for \( 1 < p < \infty \).

(**)

\[
\left\| \left( \sum_i |\Delta_i f(x)|^2 \right)^{1/2} \right\|_{L^p(\lambda)} \sim \|f\|_{L^p(\lambda)} \tag{6.3}
\]

with \( \lambda \in A_p^* \) for \( 1 < p < \infty \).

Noted that the second result (**) stated hereby is proved in [3] with the language of square functions. We are going to show that the two classes \( A_p \) and \( A_p^* \) are equivalent, with a mollification on a set of measure zero. Without losing of generality, we assume \( n = 2 \) such that \( (x, y) \in \mathbb{R}^{N_i} \times \mathbb{R}^{N_2} \).

First, suppose \( \lambda \in A_p \). By Hölder inequality, we have

\[
\frac{1}{|B_2|} \int_{B_2} \left( \frac{1}{|B_1|} \int_{B_1} \lambda(x, y) dx \right)^{\frac{p}{q}} \left( \frac{1}{|B_1|} \int_{B_1} \left( \frac{1}{\lambda(x, y)} \right)^q dx \right)^{\frac{1}{q}} dy
\]

\[
\leq \left( \frac{1}{|B_1||B_2|} \int_{B_1 \times B_2} \lambda(x, y) dxdy \right)^{\frac{1}{q}} \left( \frac{1}{|B_1||B_2|} \int_{B_1 \times B_2} \left( \frac{1}{\lambda(x, y)} \right)^q dxdy \right)^{\frac{1}{q}} \leq A. \tag{6.4}
\]

Suppose the integrant on the left of inequality (6.4) is strictly greater than \( A \), on any subset of \( \mathbb{R}^{N_2} \), with non-empty interior. Then, taking \( B_2 \) to be contained in that set will reach a
contradiction. A more direct argument by applying Lesbegue differentiation theorem was given in [16].

Turn to the converse. Suppose \( \lambda \in A_p \). By \((*)\) in (6.2), choosing \( f \) to be non-negative, we have

\[
\int \int (Mf(x, y))^p \lambda(x, y) dx dy \leq A_p \int \int (f(x, y))^p \lambda(x, y) dx dy. \tag{6.5}
\]

Let \( B_1 \times B_2 \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \). It is clear that inequality (6.5) holds for

\[
\int \int (Mf(x, y))^p \lambda(x, y) \chi_{B_1 \times B_2} dx dy
\]

where \( \chi_{B_1 \times B_2} \) is the characteristic function on \( B_1 \times B_2 \). For any \( x \in B_1 \), let \( B_1(x) \) denotes the ball centered on \( x \) with twice the radius of \( B_1 \). Certainly \( B_1 \subset B_1(x) \) for every \( x \in B_1 \). The same is true for any \( y \in B_2 \). Therefore, we shall have

\[
2^{-\left(N_1+N_2\right)} \frac{1}{|B_1||B_2|} \int B_1 \times B_2 f(u, v) dudv \leq \frac{1}{|B_1(x)||B_2(y)|} \int B_1(x) \times B_1(y) f(u, v) dudv \leq Mf(x, y), \quad (x, y) \in B_1 \times B_2. \tag{6.6}
\]

Let \( N = N_1 + N_2 \), the estimates (6.5)-(6.6) imply

\[
\left( \frac{1}{|B_1||B_2|} \int B_1 \times B_2 f(x, y) dx dy \right)^p \left( \int B_1 \times B_2 \lambda(x, y) dx dy \right) \leq 2^{Np} A_p \int \int f(x, y)^p \lambda(x, y) dx dy. \tag{6.7}
\]

By choosing \( f = (1/\lambda + \varepsilon)^{\frac{q}{p}} \) for some \( \varepsilon > 0 \) and taking into account that

\[
\frac{q}{p} = q - 1 \quad \text{and} \quad \frac{p}{q} = p - 1,
\]

inequality (6.7) implies

\[
\left( \frac{1}{|B_1||B_2|} \int B_1 \times B_2 \lambda(x, y) dx dy \right) \left( \frac{1}{|B_1||B_2|} \int B_1 \times B_2 \left( \frac{1}{\lambda(x, y) + \varepsilon} \right)^{\frac{q}{p}} dx dy \right)^{\frac{p}{q}} \leq A \tag{6.8}
\]

where the constant \( A = 2^{Np} A_p \) depends only on \( p \) and \( \lambda \). By letting \( \varepsilon \rightarrow 0 \), we have \( \lambda \in A_p \).

To conclude that \( T_\sigma f \) with \( \sigma \in S_\rho \) admits the desired weighted norm inequality, we simply carry out the same estimation in section 5, with the \( L^p \) norm replaced by \( L^p(\lambda) \) norm. In particular, (5.20) holds provided by \( \lambda \in A_p \) if and only if \( (1/\lambda)^{p/q} \in A_q \) with \( 1/p + 1/q = 1 \). See chapter 3 of [3] and chapter V of [7]. In (5.22), we need apply the vector-valued weighted norm inequality for \( Mf \), proved in chapter 3 of [3], together with \((**)\), which are now valid for \( \lambda \in A_p \).
7 Estimation on Kernel

In this section, we are going to show that the class of operator $T_\sigma$ with $\sigma \in S_\rho$ can be equivalently classified by

$$\int f(x - y)\Omega(x, y)dy$$

where $\Omega$ satisfies differential inequality (1. 5) and cancellation property (1. 6). This will be showed by obtaining the following theorem.

**Theorem 7.1** The symbol $\sigma \in S_\rho$ if and only if the kernel $\Omega(x, y)$ with $y \neq 0$, satisfies differential inequality (1. 5) and the cancellation property (1. 6).

We divided the proof into two major parts.

from symbol to kernel: Recall that the kernel $\Omega$ is a distribution $\Omega(x, \cdot)$ that coincides with a smooth function

$$\Omega(x, y) = \int_{\mathbb{R}^N} e^{2\pi i y \cdot \xi} \sigma(x, \xi) d\xi$$

for $y \neq 0$. Suppose $\sigma(x, \xi)$ satisfies differential inequality (1. 2). We write

$$\frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) = \left(\frac{1}{2\pi i}\right)^{-|\alpha|} \int e^{2\pi i y \cdot \xi} \prod_{i=1}^n \xi_i^{\alpha_i} \left(\frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi)\right) d\xi$$

(7. 2)

where the summation is taking over all $t$ satisfying (3. 10).

Let $\xi \in \text{supp} \delta_t(\xi)$. For each component $\xi_i$, we have

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \sigma(x, \xi) \right| \leq \left(\frac{1}{1 + |\xi| + |\xi|^p}\right)^{|\alpha|}, \quad i = 1, 2, \ldots, n$$

(7. 3)

for every multi-index $\alpha$. From Lemma 3.1, we either have $|\xi| \sim 2^j$ or otherwise there exists $\xi$ such that $|\xi| \sim 2^{j_i} \sim 2^{j_i/\rho}$. Thus that the derivative in (7. 3) is bounded by $2^{j_i}$, for every $i$.

In the other hand, by differential inequality (1. 2), we have

$$\left| \frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi) \right| \leq (1 + |\xi|)^{\rho|\beta|}$$

(7. 4)

for every multi-index $\beta$. Clearly, from Lemma 3.1, (7. 4) is further bounded by constant multiple of $2^{j_i}|\beta|$, for some $i \in [1, 2, \ldots, n]$.

Now, consider the norm of

$$(y_j)^{\mu_i}(y_j)^{\nu_i} \int_{\mathbb{R}^N} e^{2\pi i y \cdot \xi} \delta_t(\xi)\xi_i^{\alpha_i} \left(\frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi)\right) d\xi_i$$

$$= \left(\frac{1}{2\pi i}\right)^{|\mu|+|\nu|} \int_{\mathbb{R}^N} e^{2\pi i y \cdot \xi} \frac{\partial^{\mu_i}}{\partial x_i^{\mu_i}} \frac{\partial^{\nu_j}}{\partial x_j^{\nu_j}} \delta_t(\xi)\xi_i^{\alpha_i} \left(\frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi)\right) d\xi_i, \quad j \neq i.$$
By previous estimates in (7.3) and (7.4), we have the norm of (7.5) bounded by a constant multiple of

\[ 2^{-|\mu_i|} \times 2^{-|\nu_i|} \times 2^{(N_i + |\alpha_i| + |\beta|)t_i}. \]

By Lemma 3.1, we have \( t_i \leq t_i / \rho \) for every pair of \( i, j \in \{1, 2, \ldots, n\} \). Hence that the norm of (7.5) is further bounded by a constant multiple of

\[ 2^{- \left( |\mu_i| + \rho |\nu_i| \right) t_i} \times 2^{(N_i + |\alpha_i| + |\beta|)t_i}. \] (7.6)

Let \( |\nu_i| \) fixed, we consider the summations separately for

\[ \sum_{2^i \leq |y_i|^{-1}} \text{ and } \sum_{2^i > |y_i|^{-1}} \]

For \( 2^i \leq |y_i|^{-1} \), we choose \( |\mu_i| = 0 \) so that the summation

\[ \sum_{2^i \leq |y_i|^{-1}} 2^{(N_i + |\alpha_i| + |\beta| - \rho |\nu_i|)t_i} \leq \left( \frac{1}{|y_i|} \right)^{N_i + |\alpha_i| + |\beta| - \rho |\nu_i|}. \] (7.7)

For \( 2^i > |y_i|^{-1} \), we choose \( |\mu_i| > N_i + |\alpha_i| + \rho |\beta| - \rho |\nu_i| \) so that

\[ \left( \frac{1}{|y_i|} \right)^{|\nu_i|} \sum_{2^i > |y_i|^{-1}} 2^{(N_i + |\alpha_i| + |\beta| - |\mu_i|)t_i} \leq \left( \frac{1}{|y_i|} \right)^{N_i + |\alpha_i| + |\beta| - |\mu_i|}. \] (7.8)

In the other hand, let \( |\mu_i| \) fixed, we consider

\[ \sum_{2^i \leq |y_i|^{-1}/\rho} \text{ and } \sum_{2^i > |y_i|^{-1}/\rho} \]

For \( 2^i \leq |y_i|^{-1}/\rho \), we choose \( |\nu_i| = 0 \) so that the summation

\[ \sum_{2^i \leq |y_i|^{-1}/\rho} 2^{(N_i + |\alpha_i| + |\beta| - |\mu_i|)t_i} \leq \left( \frac{1}{|y_i|} \right)^{(N_i + |\alpha_i| + |\beta| - |\mu_i|)/\rho}. \] (7.9)

For \( 2^i > |y_i|^{-1}/\rho \), we choose \( |\nu_i| > (N_i + |\alpha_i| + \rho |\beta| - |\mu_i|) / \rho \) so that

\[ \left( \frac{1}{|y_i|} \right)^{|\nu_i|} \sum_{2^i > |y_i|^{-1}/\rho} 2^{(N_i + |\alpha_i| + |\beta| - |\mu_i|)t_i} \leq \left( \frac{1}{|y_i|} \right)^{(N_i + |\alpha_i| + |\beta| - |\mu_i|)/\rho}. \] (7.10)

By summing all together, we shall have

\[ \prod_{i=1}^{n} \left| y_i |\mu_i| |y_i| |\nu_i| \right| \left| \frac{\partial^{|\mu_i|}}{\partial y_i^{\alpha_i}} \frac{\partial}{\partial x_i^\beta} Q(x, y) \right| \leq A_{\mu, \nu} \] (7.11)

for every multi-index \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) and \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \), provide that

\[ |\mu_i| + \rho |\nu_i| \geq N_i + |\alpha_i| + |\beta| \]

\[ i = 1, 2, \ldots, n \] (7.12)
We can then verify differential inequality (1.5). It is also clear that $\Omega(x, y)$ in (7.1) decays rapidly as $y \to \infty$.

To show the cancellation property, let $\varphi_i \in C^\infty(\mathbb{R}^N_i)$ be an normalized bump function, and $R_i > 0$ for every $i = 1, 2, \ldots, n$. Observe that

$$\partial_i^\alpha \varphi_i(x, \xi_i^\prime) = \int_{\mathbb{R}^{N-N_i}} e^{2\pi i y_i \xi_i^\prime} \left( \frac{\partial^\alpha}{\partial x^\alpha} \sigma(x, \xi) R_i^{-N} \varphi_i(-R_i^{-1} \xi_i) d\xi_i. \right.$$  \hfill (7.13)

We have (7.13) equals to

$$\int_{\mathbb{R}^{N-N_i}} e^{2\pi i y_i \xi_i^\prime} \left( \frac{\partial^\alpha}{\partial x^\alpha} \sigma(x, \xi) R_i^{-N} \varphi_i(-R_i^{-1} \xi_i) d\xi_i. \right.$$  \hfill (7.14)

Observe that $\partial_i^\alpha \sigma(x, \xi)$ satisfies differential inequality (1.2) with norm bounded by

$$(1 + |\xi|)^{p[\alpha]} \sim |\xi_i|^{p[\alpha]} + (1 + |\xi_i^\prime|)^{p[\alpha]} = \delta(\xi).$$

We rewrite (7.14) as

$$\int_{\mathbb{R}^{N-N_i}} |\xi_i|^{p[\alpha]} \left( \frac{\partial^\alpha}{\partial x^\alpha} \sigma(x, \xi) \delta^{-1}(\xi) R_i^{-N} \varphi_i(-R_i^{-1} \xi_i) d\xi_i. \right.$$  \hfill (7.15)

Now, $\partial_i^\alpha \sigma(x, \xi) \delta^{-1}(\xi)$ is absolutely bounded and satisfies differential inequality (1.2). The sufficient smoothness of $\varphi_i$ implies the $L^1$ norm of $\varphi_i$ is bounded. By changing dilations $\xi_i = R_i \eta_i$, so that $0 < |\eta_i| < 1$, it follows that $\delta(x, \xi_i^\prime)$ satisfies differential inequality (1.2) in $\mathbb{R}^{N-N_i}$, with its norm bounded by $(1 + R_i)^{p[\alpha]}$. An analogue of estimation also valid on $\bigoplus \mathbb{R}^N$, whenever $i \in \mathbf{I}$ as a subset of $\{1, 2, \ldots, n\}$. By carrying out the induction, we obtain the cancellation property (1.6).

**from kernel to symbol**

Suppose $\Omega(x, y)$ satisfies differential inequality (1.5) and cancellation property (1.6). We have

$$\sigma(x, \xi) = \int \Omega(x, y) e^{-2\pi i y \xi} d\eta.$$  \hfill (7.16)

Let

$$\rho_i(\xi) = 1 + |\xi_i| + |\xi|^p.$$  \hfill (7.17)

In order to prove $\sigma(x, \xi)$ satisfies differential inequality (1.2), we are going to show that

$$\prod_{i=1}^n \rho_i(\xi)^{|\alpha|} \frac{\partial^\alpha}{\partial \xi^\alpha} \int \left( \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) \right) e^{-2\pi i y \xi} d\eta.$$
is absolutely bounded by
\[ A_{\alpha,\beta}(1 + |\xi|)^{\rho[\beta]} \] (7. 18)
for every multi-index \( \alpha \) and \( \beta \).

Differentiations with respect to \( \xi \) inside (7. 17) imply that it is equal to \((-1)^{|\alpha|} (2\pi i)^{|\alpha|} \) times
\[ \int \prod_{i=1}^{n} (\rho_i(\xi)y_i)^{\alpha_i} \left( \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) \right) e^{-2\pi i y \cdot \xi} dy. \] (7. 19)

Let \( \varphi_i \) for \( i = 1, 2, \ldots, n \) be normalized bump functions. Consider
\[ \int \left( \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) \right) e^{-2\pi i y \cdot \xi} \left\{ \prod_{i=1}^{n} (\rho_i(\xi)y_i)^{\alpha_i} \varphi_i(\rho_i(\xi) |y_i|) \right\} dy. \] (7. 20)

Observe that the function
\[ \prod_{i=1}^{n} y_i^{\alpha_i} \varphi_i(|y_i|) \exp \left\{ -2\pi i \left( \frac{y_i \cdot \xi_i}{\rho_i(\xi)} \right) \right\} \] (7. 21)
is a product of \( n \) normalized bump functions of \( y_i \) for \( i = 1, 2, \ldots, n \). By cancellation property (1. 6), we have (7. 20) is bounded by
\[ A_{\alpha,\beta}(1 + \rho_1(\xi) + \rho_2(\xi) + \cdots + \rho_n(\xi))^{\rho[\beta]} \]
which is further bounded by (7. 18).

In the other hand, we have a remaining term of
\[ \int \left( \frac{\partial^\beta}{\partial x^\beta} \Omega(x, y) \right) e^{-2\pi i y \cdot \xi} \prod_{i=1}^{n} (\rho_i(\xi)y_i)^{\alpha_i} \left( 1 - \prod_{i=1}^{n} \varphi_i(\rho_i(\xi) |y_i|) \right) dy. \] (7. 22)
The integration is taking over
\[ \supp \left( 1 - \prod_{i=1}^{n} \varphi_i \right). \] (7. 23)

By definition of \( \varphi_i, i = 1, 2, \ldots, n \), we have \(|y_i| > \rho_i(\xi)|\) in the support (7. 23). Since \( \Omega(x, y) \) has rapidly decaying as \( y \to \infty \), the integral converges absolutely.

8 Applications

In this last section, we give three examples. These are the Grushin type operators, oblique derivative problems and Cauchy-Szegő projections. For further detail of regarding estimates, please see chapter VII of [2], chapter IV, XII of [7] and [14].

Example 1: Let \( x = (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1} \) with its dual variable \( \xi = (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1} \). Consider the Grushin type operator
\[ \partial_i^2 + t^2 \Delta_y. \] (8. 1)
The inverse of (8.1) can be carried out by using Hermite functions, as showed in [15]. We have the symbol of the inverse operator equals to
\[ \sigma(t, \tau, \eta) = -\frac{1}{2} \left( \frac{1}{|\eta|} \right)^{1/2} \int_0^{2\pi} \left( \frac{1}{s} \right)^{1/2} M \left( \frac{s}{2\pi |\eta| r}, \frac{\tau}{|\eta|^2} \right) ds. \] (8.2)
where \( M \) denotes the Mehler kernel which has the explicit form
\[ M(r, u, v) = \left( \pi(1 - r^2) \right)^{-1/4} \exp \left\{ 4ruv - \frac{(u^2 + v^2)(1 + r^2)}{2(1 - r^2)} \right\}. \] (8.3)
Recall from §14, chapter IV of [2]. Let \( \rho = \rho(t, \tau, \eta) \) be a distance function such that
\[ \rho(t, \tau, \eta) \sim 1 + |\tau| + |t||\eta| + |\eta|^{1/2}. \] (8.4)
The symbol \( \sigma(t, \tau, \eta) \) defined in (8.2)-(8.3) has its norm bounded by \( \rho(t, \tau, \eta)^{-2} \) and satisfies the differential inequality
\[ \left| \frac{\partial^\alpha}{\partial \tau^\alpha \partial \eta^\beta} \sigma(t, \tau, \eta) \right| \leq A_{\alpha, \beta} \left( \frac{1}{\rho(t, \tau, \eta)} \right)^{|\alpha| + |\beta|}. \] (8.5)
In the other hand, the \( t \)-derivatives of \( \sigma(t, \tau, \eta) \) satisfy a variant of differential inequalities defined inductively on the order of derivatives. Each one gains at most a factor of \( (1 + |\tau| + |\eta|)^{1/2} \).
Observe that
\[ 1 + |\tau| + |\eta|^{1/2} \leq \rho(t, \tau, \eta) \quad \text{and} \quad 1 + |\tau|^2 + |\eta| \leq \rho(t, \tau, \eta)^2 \] (8.6)
uniformly in \( t \). (8.5)-(8.6) implies that \( \sigma(t, \tau, \eta) \) satisfies differential inequality (1.2) with \( \rho = 1/2 \).
Let \( \sigma_i \) denote the symbol in (8.2) with \( t \) chosen to be \( x_i \) so that \( \tau = \xi_i \) respectively for \( i = 1, 2, \ldots, n \). We have \( \sigma_i \in S_{-1}^0 \). Moreover, let \( (\sigma_1 \circ \cdots \circ \sigma_n) \) denotes the symbol of the composition operator \( T_{\sigma_1} \circ \cdots \circ T_{\sigma_n} \). We have
\[ \left| \frac{\partial^n}{\partial \xi^\alpha \partial x^\beta} \left( \sigma_1 \circ \cdots \circ \sigma_n \right)(x, \xi) \right| \leq A_{\alpha, \beta} \prod_{i=1}^n \left( \frac{1}{1 + |\xi_i| + |\xi_i|^2} \right)^{|\alpha_i|} \left( 1 + |\xi| \right)^{\frac{|\beta|}{2}} \] (8.7)
for every multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta \). Hence that \( (\sigma_1 \circ \cdots \circ \sigma_n) \in S_{-1}^0 \).

**Example 2:** Let \( \mathcal{S} \) be a smooth bounded, open set in \( \mathbb{R}^{n+1} \), and \( X \) be a given smooth real vector field on its boundary \( \partial \mathcal{S} \). Consider the oblique derivative problem as follows:
\[
\begin{cases}
\Delta u = 0 & \text{on } \mathcal{S}, \\
Xu = f & \text{on } \partial \mathcal{S}.
\end{cases}
\] (8.8)
The question can be reduced to that of inverting a pseudo differential operator of order one on \( \partial \mathcal{S} \). An introduction of further background can be found in [17] and [18]. If \( X \) is transverse to \( \partial \mathcal{S} \), then it is the classical Neumann problem and the boundary operator has a symbol
which is elliptic. In general, we decompose $X$ as $X_T + \alpha \hat{n}$ where $X_T$ is tangential to $\partial \mathcal{X}$ and $\hat{n}$ is the outer normal field on $\partial \mathcal{X}$. Noted that $\alpha = \alpha(x)$ is considered here to be emergent type, as a necessary condition for the existence of solution. This means that $\alpha$ never changes its sign on the integral curve along $X_T$. In other words, $X$ can change its orientation only if $\partial \mathcal{X}$ is non-orientable. See [19] for references. In the other hand, we say $\alpha$ is in finite type if there exists an integer $k$ and a constant $\delta > 0$, such that

$$|\alpha| + |X_T\alpha| + \cdots + |X_T^k\alpha| \geq \delta.$$  

(8. 9)

An investigation of general oblique derivative problems with $k \geq 1$ in (8. 9) can be found in [20]. In the present example, for simplicity we only consider $k = 1$ of which the relevant estimates also hold for higher value $k$ s.

By appropriately choosing the local coordinates $(x_1, x_2, \ldots, x_n)$, we have $X$ is transverse to the manifold $M = \{x_n = 0\}$ which has co-dimension one to $\partial \mathcal{X}$. The boundary operator involved has a symbol essentially equals to

$$i\xi_n \pm q(x, \xi)x_n, \quad q(x, \xi) = \left(\sum_{i,j} a_{ij}(x)\xi_i\xi_j\right)^{\frac{1}{2}}$$

(8. 10)

with $\{a_{ij}(x)\}$ smoothly varying, real positive definite symmetric matrix. It is well known that $i\xi_n + q(x, \xi)x_n$ can only have a right parametrix and $i\xi_n - q(x, \xi)x_n$ can only have a left parametrix. The two problems are adjoint to each other. Write $x = (t, y)$ with $t = x_n$ and its dual $\xi = (\tau, \eta)$. For the sake of brevity, in what follows we seek only for the right parametrix of

$$i\tau + t|\eta|, \quad t > 0.$$  

(8. 11)

Recall the strategy given in § 15, chapter IV of [2]. By inverting the ordinary differential operator $\partial_t + t|\eta|$, where the inverse is taking to be $L^2$-orthogonal to the non-trivial null space $e^{-\frac{t}{2}|\eta|^2}$, our desired right parametrix is

$$\sigma(t, \tau, \eta) = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} e^{-|\eta|^2 z^2} \left\{ \int_z e^{-|t|^2(t^2-s^2)} e^{2t\eta(t-s)\tau} ds \right\} dz.$$  

(8. 12)

The symbol in (8. 12) has its norm bounded by $\rho(t, \tau, \eta)^{-1}$ defined in (8. 4), and satisfies differential inequality (8. 5). Its $t$-derivatives satisfy a variant of differential inequalities defined inductively on the order of derivatives. Each one gains at most a factor of $(1 + |t| + |\eta|)^4$. We thus have $\sigma \in S^4_\frac{1}{2}$. Moreover, let $\sigma_i$ denote the symbol in (8. 2) with $t$ chosen to be $x_i$ so that $\tau = \xi_i$ respectively for $i = 1, 2, \ldots, n$. The symbol $(\sigma_1 \circ \cdots \circ \sigma_n)$ belongs to $S^4_\frac{1}{2}$.

**Example 3:** Our last example is the Cauchy-Szegö projections on the Heisenberg groups. Let $x = (z, t) \in \mathbb{C}^n \times \mathbb{R}$ and $y = (w, s) \in \mathbb{C}^n \times \mathbb{R}$. Define the multiplication law $\otimes$ by

$$x \otimes y = \left( z + w, t + s + 2\text{Im}(z \cdot \overline{w}) \right).$$

(8. 13)

An inverse element is taking to be $x^{-1} = (-z, -t)$ and the identity is taking to be the origin $(0, 0)$. The multiplication law $\otimes$ defined in (8. 13) turns the space $\mathbb{C}^n \times \mathbb{R}$ into the Heisenberg group, denoted by $H^n$. 

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Essentially, a Cauchy-Szegő kernel is a distribution, represented by the function

\[ K(t, z) = \left( \frac{1}{t + i|z|^2} \right)^{n+1} \]  \hspace{1cm} (8. 14)

away from the origin. As showed in [14], its Fourier transform behaves like \( e^{-|\eta|^2/2\tau} \) for \( \tau > 0 \) and is zero for \( \tau \leq 0 \). We shall be looking at the local version of the Cauchy-Szegő projection, defined by

\[
\int_{H^n} \varphi_1(x) K(y^{-1} \otimes x) \varphi_2(y) f(y) dy
\]  \hspace{1cm} (8. 15)

where \( \varphi_1, \varphi_2 \) are some smooth cut-off functions.

Let \( \xi = (\zeta, \tau) \) be the dual variable of \( x = (z, t) \) on the space \( \mathbb{C}^n \times \mathbb{R} \), implemented by the inner product \( x \cdot \xi = \text{Re}(z \cdot \zeta) + t\tau \). The symbol corresponding to the Cauchy-Szegő projection is derived from

\[
\sigma(\zeta, \tau) = \begin{cases} 
\psi(\zeta, \tau) e^{-|\zeta|^2/2\tau} & \tau > 0, \\
0 & \tau \leq 0
\end{cases}
\]  \hspace{1cm} (8. 16)

where \( \psi \) is a smooth cut-off function that vanishes near the origin and equals 1 for large \( (\zeta, \tau) \). See chapter VII of [7].

Let \( \rho = \rho(\zeta, \tau) \) be a distance function such that

\[
\rho(\zeta, \tau) \sim 1 + |\zeta| + |\tau|^{1/2}.
\]  \hspace{1cm} (8. 17)

The symbol \( \sigma \) in (8. 16) satisfies the differential inequality

\[
\left| \frac{\partial^\alpha}{\partial \zeta^\alpha} \frac{\partial^\beta}{\partial \tau^\beta} \sigma(\zeta, \tau) \right| \leq A_{\alpha,\beta} \left( \frac{1}{\rho(\zeta, \tau)} \right)^{|\alpha|+2|\beta|} \]  \hspace{1cm} (8. 18)

for every multi-index \( \alpha \) and \( \beta \).

In the other hand, the kernel defined by \( \widetilde{\Omega}(\zeta, \tau) = \sigma(\zeta, \tau) \), is a distribution smooth away from the origin, satisfies the differential inequality

\[
\left| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial t^\beta} \Omega(z, t) \right| \leq A_{\alpha,\beta} \left( \frac{1}{|z| + |t|^2} \right)^{2n+|\alpha|} \left( \frac{1}{|t| + |z|^2} \right)^{1+|\beta|} \]  \hspace{1cm} (8. 19)

and decays rapidly as \( (z, t) \rightarrow \infty \).

We thus can verify the size of Cauchy-Szegő kernel given by (8. 14), for which

\[
|\Omega(z, t)| \lesssim \left( \frac{1}{|z| + |t|^2} \right)^{2n} \left( \frac{1}{|t| + |z|^2} \right)^{1+|\alpha|} \left( \frac{1}{|t| + |z|^2} \right)^{n+1}.
\]  \hspace{1cm} (8. 20)

In fact, from (8. 15), it is not restricted to allow the kernel has rapidly decaying at the infinity. As showed in 7.15, chapter XII of [7], Cauchy-Szegő projection given by (8. 15) can be considered as a pseudo differential operator \( T_\alpha \).
Its symbol has a form of
\[ a(x, \xi) = \varphi(x)\sigma(L_x \xi) \] (8. 21)
where \( \varphi \) is a smooth cut-off function and \( L_x \) is a linear transformation, which actually depends linearly on \( x \). To be precise, recall the multiplication law \( \otimes \) defined in (8. 13). We write \( y^{-1} \otimes x = M_x(x - y) \) where \( M_x \) is a linear transformation with determinant 1, and depends linearly on \( x \). \( L_x \) is taking to be the transpose inverse of \( M_x \).

In what follows, we take \( n = 1 \) for brevity. Accordingly, we have
\[ L_x = \begin{bmatrix} 1 & 0 & 2\tau_2 \\ 0 & 1 & -2\tau_1 \\ 0 & 0 & 1 \end{bmatrix}. \] (8. 22)

Then, the symbol \( a \) is written explicitly as
\[ a(x, \xi) = \varphi(x)\psi(L_x \xi) \exp\left( \frac{(\zeta_1 + 2\tau_2 \tau)^2 + (\zeta_2 - 2\tau_1 \tau)^2}{-2\tau} \right), \quad \tau > 0 \] (8. 23)
and is zero for \( \tau \leq 0 \). Recall from chapter VII, XII of [7], we have \( a(x, \xi) \) in (8. 23) belongs to the exotic class \( S^{0}_{\frac{1}{2}, \frac{1}{2}} \). In the following, we aim to show that \( a \in S_{\frac{1}{2}} \).

Suppose \( |\zeta| \geq |\tau|^\frac{1}{2} \). Without losing of generality, let \( |\zeta| \sim |\zeta_1| \sim |\tau|^\frac{1}{2} + \delta \) for some \( 0 \leq \delta \leq \frac{1}{2} \). By varying of \( \tau \), we can have \( |2\tau_2 \tau| \sim |\zeta_1| \) and \( |\zeta_1 + 2\tau_2 \tau| \sim |\tau|^\frac{1}{2} - \varepsilon \), for some \( \delta \leq \varepsilon \leq \frac{1}{2} \). Therefore,
\[ |\zeta_1 + 2\tau_2 \tau| \sim |\tau|^{-\left(\frac{1}{2} + \varepsilon\right)} \lesssim |\zeta_1|^{-1} \sim |\zeta|^{-1}. \] (8. 24)

When \( |\zeta| \leq |\tau|^\frac{1}{2} \), the quotient in (8. 24) is dominated by \( |\tau|^{-\frac{1}{2}} \). We thus have each \( \zeta \)-derivative gains a factor of \((1 + |\zeta| + |\tau|^\frac{1}{2})^{-1}\).

Turning to the \( \tau \)-derivatives. For \( |\zeta| \leq |\tau|^\frac{1}{2} \). Then \( |\zeta_1| \sim |2\tau_2 \tau| \sim |\tau|^\frac{1}{2} + \delta \) for some \( 0 \leq \delta \leq \frac{1}{2} \) implies \( |z| \sim |\tau|^{-\frac{1}{2} - \delta} \). For \( |\tau^\frac{1}{2} | \leq |\zeta| \leq |\tau| \). Let \( |\zeta_1| \sim |\tau|^\frac{1}{2} + \delta \) for some \( 0 \leq \delta \leq \frac{1}{2} \). In order to have \( |\zeta_1 + 2\tau_2 \tau| \sim |\tau|^\frac{1}{2} - \varepsilon \) for some \( \delta \leq \varepsilon \leq \frac{1}{2} \), we necessarily have \( |2\tau_2 \tau| \sim |\zeta_1| \) which implies \( |z_2| \sim |\tau|^{-\frac{1}{2} + \delta} \). Therefore, in all cases for \( |\zeta| \leq |\tau| \), each \( \tau \)-derivative is dominated by
\[ |z_2| \frac{|\zeta_1 + 2\tau_2 \tau|}{\tau} + \frac{|\zeta_1 + 2\tau_2 \tau|}{\tau^2} \lesssim |\tau|^{-\frac{1}{2} + \delta} |\tau|^{-\frac{1}{2} - \delta} \sim |\tau|^{-1}. \] (8. 25)

The same estimate goes to \( \zeta_2 - 2\tau_1 \tau \). Together with that \( a \in S^{0}_{\frac{1}{2}, \frac{1}{2}} \) showed in [7] and [14], we conclude that \( a \in S_{\frac{1}{2}} \).

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