Hadronic contributions to the anomalous magnetic moment of the electron and the hyperfine splitting of muonium

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Abstract
Motivated by recent progress of theory and experiment on the anomalous magnetic moment of the electron, $a_e$, we update the hadronic contributions to $a_e$. Using our up-to-date compilation of $e^+e^- \to \text{hadrons}$ data, we find the leading order hadronic contribution $a_e^{\text{had,LO,VP}} = (1.866 \pm 0.010_{\text{exp}} \pm 0.005_{\text{rad}}) \cdot 10^{-12}$ and the next-to-leading order hadronic contribution $a_e^{\text{had,NLO,VP}} = (-0.2234 \pm 0.0012_{\text{exp}} \pm 0.0007_{\text{rad}}) \cdot 10^{-12}$, where the first and second errors are from the error of the experimental data and the uncertainty in the treatment of radiative corrections, respectively. These values are compatible with earlier evaluations by other groups, but have significantly improved uncertainties due to the more precise input data used. We also update the leading order hadronic contribution to the ground state hyperfine splitting of muonium, obtaining $\Delta \nu_{\text{Mu}}^{\text{had,VP}} = (232.68 \pm 1.25_{\text{exp}} \pm 0.72_{\text{rad}}) \text{Hz}$. This value is consistent with the most precise evaluation in the literature and reduces its error by a factor of two.

1 Introduction
With the start of the LHC, the hunt for physics beyond the Standard Model (SM) at the energy frontier has entered a new era, though so far no direct signal for ‘new physics’ has been observed. At the same time, experiments at lower energies are measuring SM parameters with unprecedented accuracy and are becoming ever more sensitive to quantum effects, possibly from
physics beyond the SM. A prime example for this is the anomalous magnetic moment of leptons [1], caused by loop effects mainly from Quantum Electro-Dynamics (QED) but also influenced by the strong and weak force and possibly by effects from new physics. For the muon, the anomaly $a_\mu$ is sensitive to all sectors of the SM and has been measured with 0.5 ppm accuracy, using a muon storage ring at Brookhaven [2]. This measurement, when compared to the SM prediction of $a_\mu$, shows a discrepancy of $3.3 \sigma$ [3, 4, 5] which may be a sign for new physics.

The electron’s anomalous magnetic moment, $a_e$, has been measured at Harvard via atomic spectroscopy [7],

$$a_e = 1159\,652\,180.73\,(28) \cdot 10^{-12}.$$  

This corresponds to a 0.24 ppb accuracy, hence $a_e$ is known even much more precisely than $a_\mu$. However, due to the small mass of the electron, $a_e$ is less sensitive to quantum effects from higher mass scales and is mainly generated by QED effects, though it also receives small contributions from the strong and weak interaction. It is therefore an ideal place to determine the fine structure constant $\alpha$.

Another precision observable is the ground state hyperfine splitting (HFS) of muonium. Similar to $a_e$, it is sensitive to quantum effects at low mass scales and mainly QED dominated. It is very useful to determine the electron-to-muon mass ratio and hence the muon mass.

To determine SM parameters or to become sensitive to new physics, it is crucial that the theoretical predictions of such precision observables have accuracies similar to or smaller than the experimental uncertainties. For the lepton anomalies and also the HFS of muonium this has been achieved by higher-order calculations in all sectors of the SM (and various extensions of it) [3, 8, 9]. For $a_e$, the group of Kinoshita and collaborators have, after a many-year effort, completed calculations including full five-loop effects [10, 11, 12, 13, 14, 15, 16]. These results have reduced the error of the theoretical prediction of $a_e$ by more than four-fold. It is therefore timely to provide an up-to-date evaluation of the hadronic contributions. In this short letter we calculate the hadronic contributions due to vacuum polarisation effects for the anomalous magnetic moment of the electron and also for the hyperfine splitting of the ground state of muonium.

2 Anomalous magnetic moment of the electron

At present, the most precise determination of the fine structure constant is through the measurement of the anomalous magnetic moment of the electron, $a_e$, using the Harvard measurement [7] quoted in Eq. (1) above. With their new complete five-loop and updated and improved four-loop QED contributions, and using results from [17, 18, 19, 20] for the hadronic and electroweak (EW) contributions, Kinoshita et al. arrive at [10]

$$\alpha^{-1}(a_e) = 137.035\,999\,1736\,(68)(46)(26)(331),$$

1The tau lepton’s anomaly, $a_\tau$, is even more sensitive to physics at higher energy scales than $a_\mu$, but is very difficult to measure with high precision due to the short lifetime of the tau. See [6] for a recent review and SM prediction of $a_\tau$. 

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corresponding to a 0.25 ppb accuracy. This value is slightly different but compatible (at the level of 1.3 σ) with the second best determination of α which is using Rubidium atoms \cite{21, 22} and reads \( \alpha^{-1}(\text{Rb}) = 137.035 \pm 999 \pm 049 \) (90) [0.66 ppb]². The four errors quoted in (2) stem from the four- and five-loop QED contributions, from the hadronic and electroweak effects, and from the uncertainty of the experimental value for \( a_e \). The group of Gabrielse is working on further improvements of the measurement of \( a_e \) and also a similar determination of the positron’s anomaly which, assuming CPT invariance, can be combined with the electron’s anomaly similar to the case of the muon. However, currently the experimental error of \( a_e \) is dominating the theoretical uncertainties, which have been reduced dramatically by the new QED calculations of Kinoshita and collaborators. Their four- and five-loop errors, i.e. the first and second error in (2), are mainly coming from the statistical uncertainties of the numerical Monte Carlo integrations and can be improved with increased computational efforts when needed. To understand the composition of the third error, let us list the hadronic and electroweak contributions as used in (10):

\[
\begin{align*}
\alpha_{\text{had}}^\text{LO, VP} &= 1.875 \pm 18 \cdot 10^{-12}, \\
\alpha_{\text{had}}^\text{NLO, VP} &= -0.225 \pm 5 \cdot 10^{-12}, \\
\alpha_{\text{had}}^\text{1-by-1} &= 0.035 \pm 10 \cdot 10^{-12}, \\
\alpha_{\text{EW}} &= 0.0297 \pm 5 \cdot 10^{-12}.
\end{align*}
\]

Here, similar to the case of the anomalous magnetic moment of the muon, the error from the electroweak effects is negligible compared to the uncertainties from the hadronic contributions. The latter are dominated by the leading order (LO) and next-to-leading order (NLO) hadronic vacuum polarisation (VP) corrections. For \( a_e \), the contributions from light-by-light (l-by-l) scattering diagrams are rather small, but they still add significantly to the uncertainty of the \( a_{\text{had}} \). A further scrutiny of the model calculations, and possibly also ‘first-principle’ determinations based on lattice calculations, are under way for the anomalous magnetic moment of the muon. This in turn will also allow to improve the estimates for \( a_{\text{had}}^\text{1-by-1} \). In the following, we will discuss our up-to-date determination of \( a_{\text{had}}^\text{LO, VP} \) and \( a_{\text{had}}^\text{NLO, VP} \). Note that while the hadronic cross section data used as input in these calculations have improved significantly in recent years due to both direct scan measurements and via the method of radiative return (see e.g. \cite{24} for a recent review of the field), no up-to-date determination of the VP induced contributions is available so far.

### 2.1 Leading order hadronic VP contributions

The calculation of the leading order hadronic VP contributions to \( a_e \) is very similar to the case of the muon and uses a dispersion integral over the hadronic cross section times a well-known kernel function \( K_\ell(s) \), see e.g. the detailed discussion in \cite{25}:

\[
\alpha_{\text{had}}^\text{LO, VP} = \frac{1}{4\pi^3} \int_{m_e^2}^{\infty} ds \ K_\ell(s)a_{\text{had}}^0(s),
\]

²Note that this value is a big improvement from the previous measurement by the same group \cite{23}.
Table 1: Contributions to $a_e^{\text{had,LO,VP}}$ (in units of $10^{-12}$) and to $\Delta \nu_{\text{Mu}}^{\text{had,VP}}$ (in Hz) from different energy regions, obtained with the data compilation as used in [4]. The first four lines give our predictions close to threshold where no data are available and are based on chiral perturbation theory (ChPT), see [25] for details. For $2.6 < \sqrt{s} < 3.73$ GeV perturbative QCD (pQCD) with errors comparable to those of the latest BES data [26] in this energy region is used. In the region below 2 GeV the sum of exclusive channels, supplemented by isospin relations for channels where no data are available, is used, see [4] for details.

where $\ell = e, \mu$, and $\sigma_0^{\text{had}}(s)$ is the undressed (i.e. excluding VP corrections) total hadronic cross section. If we define the function $\hat{K}_\ell(s)$ by

$$\hat{K}_\ell(s) = \frac{3s}{m_\ell^2} K_\ell(s),$$

then $\hat{K}_\ell(s)$ is a monotonically increasing function of order one with $\hat{K}_\ell(s) \to 1$ for $s \to \infty$. In the case of $a_e$, the kernel $\hat{K}_e(s)$ is very close to one throughout. In fact, $\hat{K}_e(s)$ can be expanded in terms of $m_\ell^2/s$ as

$$\hat{K}_e(s) = 1 + \left(3 \ln \frac{m_\ell^2}{s} + \frac{25}{4}\right) \frac{m_\ell^2}{s} + \mathcal{O}\left(\frac{m_\ell^4}{s^2 \ln \frac{m_\ell^2}{s}}\right),$$

and the deviation of $\hat{K}_e(s)$ from one is almost always negligible. (In the case of $a_\mu$, $\hat{K}_\mu(s = m_\mu^2) = 0.40$, $\hat{K}_\mu(s = 4m_{s\pm}^2) = 0.63$, and $\hat{K}_\mu(s) \to 1$ for $s \to \infty$.) Hence the low energy contributions to the dispersion integral are even more important than in the case of the muon. Using our up-to-date comprehensive compilation of hadronic cross section data [4] we obtain contributions to $a_e$ from different energy regions as displayed in Tab. [4]. The errors given in Tab. [4] contain the statistical and systematic uncertainties from the input data including correlations over different energies for the various hadronic final states which are added incoherently.
In addition, we have to take into account the uncertainties in the treatment of radiative corrections from final state radiation and VP effects, see [4, 25] for a detailed discussion. These are conservatively estimated to be
\[
\delta a_{\text{FSR}} = 0.005, \quad \delta a_{\text{VP}} = 0.002
\]
and have to be added in quadrature to the error of the total \(a_{\text{had,LO,VP}}\) which hence reads
\[
a_{\text{had,LO,VP}} = (1.866 \pm 0.011) \cdot 10^{-12}.
\]
This number is to be compared with the result from \([17]\) given in Eq. (3), which has been used in \([10]\) and \([22]\). As expected, the mean value has changed only slightly, but the total error is significantly improved by about 40\% due to the improved hadronic cross section data. Compared with the more recent evaluation given in \([8]\), which is \(a_{\text{had,LO,VP}} = (1.860 \pm 0.015) \cdot 10^{-12}\), the agreement is very good with a further but less significant improvement in the error.

### 2.2 Next-to-leading order hadronic VP contributions and total \(a_{\text{had}}\)

Contrary to the next-to-leading order (NLO) contributions induced by so-called hadronic light-by-light scattering diagrams, the NLO corrections including hadronic VP diagrams can be calculated with help of dispersion integrals, see e.g. [4, 18, 25]. Due to the small mass of the electron, diagrams with two hadronic VP insertions or one hadronic and one VP insertion from a heavy lepton (muon or tau) are strongly suppressed, and the diagrams with one hadronic VP insertion and one additional photon or electron loop are practically the only relevant contributions. With our latest compilation of hadronic cross section data we obtain the value
\[
a_{\text{had,LO,VP}} = (−0.2234 \pm 0.0012_{\text{exp}} \pm 0.0007_{\text{rad}}) \cdot 10^{-12},
\]
where the uncertainty due to the statistical and systematic errors of the hadronic cross section data used as input and the additional error due to radiative corrections applied to the data are given separately. This result is close to the result quoted in Eq. (4) \([18]\), but has a much smaller error. These NLO corrections lead to a reduction of the hadronic LO VP corrections by about 12\%, so should not be neglected.

Combining the results from Eqs. (11) and (12) with \(a_{\text{had,1-by-1}}\) from \([5]\) \([19]\)\(^4\), our estimate of the hadronic contributions to the electron’s anomaly is
\[
a_{\text{had}} = (1.678 \pm 0.014) \cdot 10^{-12}.
\]

Equations (11) and (12) are the main result of this letter. Given the current uncertainties of the QED result for \(a_e\) and the uncertainty of the experimental measurement, the improvements in the hadronic contributions may not look dramatic and do not affect the determination of \(\alpha\) significantly. However, anticipating future improvements for \(a_{\text{QED}}\) and \(a_{\text{exp}}\), it is important that the hadronic contributions are now much better under control and will not limit the indirect determination of \(\alpha\) for the foreseeable future.

\(^3\)The LO and NLO VP contributions are, to a very good approximation, totally correlated, which is taken into account accordingly in the error combination.

\(^4\)This value for the light-by-light contributions is compatible with the evaluation by Jegerlehner and Nyffeler \([8, 27]\), who quote \(a_{\text{had,1-by-1}} = 0.039 (13) \cdot 10^{-12}\).
3 Hyperfine splitting of muonium

Another high-precision observable is the HFS of the ground state of muonium. Comparing its measured and predicted values is currently the best method to determine the electron-to-muon mass ratio and hence the muon mass [22]. The HFS is mainly QED dominated, but also receives higher-order contributions due to hadronic and EW interactions, see [9, 22, 28] for detailed reviews and further references.

The experimental value of the muonium HFS is [29, 30]:

\[ \nu_{\text{Mu}}(\text{exp}) = 4,463,302,776(51) \text{ Hz}, \]  

(14)

while the theoretical value quoted in the 2010 version of CODATA [22] is

\[ \nu_{\text{Mu}}(\text{theory}) = 4,463,302,891(272) \text{ Hz}, \]  

(15)

where the uncertainty is dominated by that of the mass ratio \( m_e/m_\mu \). According to Ref. [22], among the theoretical uncertainty of 272 Hz, the uncertainty of the QED contributions is 98 Hz and that of the hadronic contribution 4 Hz. The hadronic contribution quoted in Ref. [22] is

\[ \Delta \nu_{\text{had}}^{\text{Mu}} = 236(4) \text{ Hz}, \]  

(16)

which is the sum of the hadronic VP contribution \( \Delta \nu_{\text{had,VP}}^{\text{Mu}} = 231.2(2.9) \text{ Hz} \) [31] and the hadronic higher-order contribution \( \Delta \nu_{\text{had,HO}}^{\text{Mu}} = 5(2) \text{ Hz} \) [31]. Contrary to the case of the lepton anomalies, for the muonium HFS the hadronic light-by-light contributions are completely negligible at the current level of accuracy [32]. While the error in the theoretical prediction of the muonium HFS is dominated by the estimates of unknown higher-order QED contributions, the hadronic contribution is not negligible and the error quoted in (16) is just about one order of magnitude below the current experimental error. In fact, there is a planned experiment to measure the muonium HFS at J-PARC [33], which aims at reducing the experimental uncertainty by a factor of two or more compared to Eq. (14). \footnote{Also note that in their most recent work on QED corrections to the HFS of muonium, Eides and Shelyuto quote an uncertainty of about 10 Hz as current goal for the theoretical uncertainty of the HFS [34].}

We therefore take the opportunity to improve the hadronic contributions. The hadronic VP contributions to the muonium HFS, \( \Delta \nu_{\text{had,VP}}^{\text{Mu}} \), have previously been evaluated by a number of groups [31, 35, 36, 37, 38, 39, 40]. They can be written as a dispersion integral [35],

\[ \Delta \nu_{\text{had,VP}}^{\text{Mu}} = \frac{1}{2\pi^3} \frac{m_e}{m_\mu} \nu_F \int_0^{\infty} ds \frac{K_{\text{Mu}}(s) \sigma_{\text{had}}^0(s)}{m_s^2}, \]  

where \( \nu_F \) is the so-called Fermi energy,

\[ \nu_F = \frac{16}{3} R_\infty \alpha^2 \frac{m_e}{m_\mu} \left[ 1 + \frac{m_e}{m_\mu} \right]^{-3}, \]  

(18)
where \( R_\infty \) is the Rydberg constant, \( R_\infty = 3 \, 289 \, 841 \, 960 \, 364(17) \text{ kHz} \) \(^{22}\). The explicit form of the kernel function \( K_{\mu}(s) \) is given e.g. in Ref. \([39]\). After correcting typos in Ref. \([39]\) it reads

\[
K_{\mu}(s) = \begin{cases} 
-2 \left( \frac{s}{4m_\mu^2} + 2 \right) L \left( \sqrt{s} \frac{1}{4m_\mu^2} \right) + \left( \frac{s}{4m_\mu^2} + \frac{3}{2} \right) \ln \frac{s}{m_\mu^2} - \frac{1}{2}, & \text{(for } s < 4m_\mu^2 \text{)} , \\
- \left( \frac{s}{4m_\mu^2} + 2 \right) \beta \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \left( \frac{s}{4m_\mu^2} + \frac{3}{2} \right) \ln \frac{s}{m_\mu^2} - \frac{1}{2}, & \text{(for } s \geq 4m_\mu^2 \text{)} ,
\end{cases}
\]  

(19)

where

\[
L(\tau) \equiv -\frac{\sqrt{1 - \tau^2}}{\tau} \tan^{-1} \frac{\sqrt{1 - \tau^2}}{\tau}, \quad \beta \equiv \sqrt{1 - \frac{4m_\mu^2}{s}} .
\]  

(20)

The kernel \( K_{\mu}(s) \) is a monotonically decreasing function with \( K_{\mu}(s = m_\mu^2) = 5.53 \) and \( K_{\mu}(s = 4m_{\pi^\pm}^2) = 1.98 \), approaching zero as \( s \to \infty \). For large \( s \), an expansion in terms of \( m_\mu^2/s \) is useful:

\[
K_{\mu}(s) = \left( -\frac{9}{2} \ln \frac{m_\mu^2}{s} + \frac{15}{4} \right) \frac{m_\mu^2}{s} + \mathcal{O} \left( \frac{m_\mu^4}{s^2} \ln \frac{m_\mu^2}{s} \right).
\]  

(21)

Basically the dependence of \( K_{\mu}(s) \) on \( s \) is similar to that of \( K_\mu(s) \). Since the leading term in the above expansion is \( (m_\mu^2/s) \ln(s/m_\mu^2) \) rather than \( (m_\mu^2/s) \), it puts only slightly more emphasis on higher energies compared to the hadronic VP corrections to \( a_\mu \).

Using the same input as above, we obtain the hadronic contributions from different energy regions as listed in Tab. \([1]\). As in the case of \( a_e \) above, the error displayed in the last line of Tab. \([1]\) contains only the uncertainties of the experimental data. Adding an error from the conservatively estimated uncertainty due to radiative corrections, \( \pm 0.72 \text{ rad Hz} \), our final result for the hadronic VP contributions to the muonium HFS reads

\[
\Delta \nu_{\mu, \text{VP}}^{\text{had}} = (232.68 \pm 1.44) \text{ Hz} .
\]  

(22)

Our result is compatible with and, as expected, considerably more accurate than the previous result quoted in Eq. \((16)\). Note however, that so far no attempt has been made at a complete calculation of higher-order hadronic VP corrections which have been estimated to be of the same order or bigger than the error of the leading order ones \([31, 39]\).

### 4 Conclusions

We have used our comprehensive compilation of hadronic cross section data to determine the hadronic vacuum polarisation contributions to the anomalous magnetic moment of the electron, \( a_e \), to leading and next-to-leading order. We have also evaluated the hadronic VP contributions to the hyperfine splitting of the ground state of muonium. Our main results are given in Eqs. \((11)\), \((12)\) and \((22)\). While the changes of the central values compared to earlier determinations are modest, the corresponding uncertainties have been significantly improved and will, for the foreseeable future, not affect precision determinations of physical constants from these observables.
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