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1. Editor’s note

In addition to the interesting research announcements in Section 3, I am very pleased to announce the solution of a problem about γ-sets, implicit since Gerlits and Nagy’s 1982 paper Some properties of C(X), I, and explicit in the 1996 paper of Just, Miller, Scheepers, and Szeptycki The combinatorics of open covers II and in several later papers by these and by other authors. Details are available in the paper announced in Section 3.11 below, and are reproduced in Section 2 below.

From a personal perspective, I am interested in problems of this sort since my Master’s thesis. In general, the question is: Assume that we take infinite sets of natural numbers, which are rapidly thinning out in some combinatorial sense (a scale, a tower, etc.), and then add all finite sets of naturals. As a subspace of the Cantor...
space $P(\mathbb{N})$, which selection hypotheses does our set satisfy? This approach differs from the classical one, in that we do not consider the topology during the construction. E.g., we do not take into account potential open covers in a transfinite-inductive construction. Results of this form were obtained by Fremlin and Miller; Just, Miller, Scheepers, and Schetycki; Scheepers; Bartoszynski; Bartoszynski and Tsaban; and Tsaban and Zdomskyy.

The present solution, which is joint with my Master’s Student Tal Orenshtein, grew out of this series of intermediate advances, and in addition relies on the method from Galvin and Miller’s $\gamma$-sets and other singular sets of real numbers (1984), and on Francis Jordan’s method from There are no hereditary productive $\gamma$-spaces (2008), with one additional twist which makes everything fit together. The intermediate advances which were motivated by related (but other) questions in the field of selection principles. This is a beautiful demonstration of the importance of treating questions in wider contexts than the ones in which they were initially posed.

Readers not interested in generalizations or in new proof methods or in weakenings of Martin’s Axiom (but which are interested in something), may still be happy with the fact that the new result gives apparently the first proof that there are uncountable $\gamma$-sets in all Random reals models, obtained by adding any number of Random reals to a model of the Continuum Hypothesis.

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2. $\gamma$-SETS FROM A WEAK HYPOTHESIS

In the paper 3.11, we construct sets of reals satisfying $S_1(\Omega, \Gamma)$, traditionally called $\gamma$-sets, from a weak set theoretic hypothesis. The problem thus settled has some history, which we now survey briefly. This involves combinatorial cardinal characteristics of the continuum [1]. We give the necessary the definitions as we proceed.

$\gamma$-sets were introduced by Gerlits and Nagy in [5], their most influential paper, as the third property in a list numbered $\alpha$ through $\epsilon$. This turned out to be the most important property in the list, and obtained its item number as it name. One of the main results in [5] is that for Tychonoff spaces $X$, $C(X)$ with the topology of pointwise convergence is Fréchet-Urysohn if, and only if, $X$ is a $\gamma$-set.

While uncountable $\gamma$-sets exist in ZFC,¹ Borel’s Conjecture (which is consistent with, but not provable within, ZFC) implies that all metrizable $\gamma$-sets are countable.

Since we are dealing with constructions rather than general results, we restrict attention to subsets of $\mathbb{R}$ (or, since the property is preserved by continuous images, subsets of any topological space which can be embedded in $\mathbb{R}$).

¹The axioms of Zermelo and Fraenkel, together with the axiom of Choice, the ordinary axioms of mathematics.
Gerlits and Nagy proved in [5] that Martin’s Axiom implies that all spaces of cardinality less than \( c \) are \( \gamma \)-sets. There is a simple reason for that: The critical cardinality of a property \( P \), denoted \( \text{non}(P) \), is the minimal cardinality of a set not satisfying \( P \). Let \( \binom{\mathcal{P}}{\mathcal{U}} \) be the property: Each \( \mathcal{U} \in \Omega(X) \) contains a set \( \mathcal{V} \in \Gamma(X) \).

Gerlits and Nagy proved that \( S_1(\Omega, \Gamma) = \binom{\mathcal{P}}{\mathcal{U}} \) [5]. Let \( A \subseteq^* B \) mean that \( A \setminus B \) is finite. \( A \) is a pseudointersection of \( \mathcal{F} \) if \( A \subseteq^* B \) for all \( B \in \mathcal{F} \). Let \( p \) be the minimal cardinality of a family \( \mathcal{F} \) of infinite subsets of \( \mathbb{N} \) which is closed under finite intersections, and has no pseudointersection. Then \( \text{non}(\binom{\mathcal{P}}{\mathcal{U}}) = p \) [4], and Martin’s Axiom implies \( p = c \) [4].

By definition, for each property \( P \), every space of cardinality smaller than \( \text{non}(P) \) satisfies \( P \). Thus, the real question is whether there is \( X \) of cardinality at least \( \text{non}(P) \), which satisfies \( P \). Galvin and Miller [4] proved a result of this type: \( p = c \) implies that there is a \( \gamma \)-set of reals, of cardinality \( p \). Just, Miller, Scheepers and Szeptycki [7] have improved the construction of [4]. We introduce their construction in a slightly more general form, that will be useful later.

Cantor’s space \( \{0,1\}^\mathbb{N} \) is equipped with the Tychonoff product topology, and \( P(\mathbb{N}) \) is identified with \( \{0,1\}^\mathbb{N} \) using characteristic functions. This defines the topology of \( P(\mathbb{N}) \). The partition \( P(\mathbb{N}) = [\mathbb{N}]^{\mathcal{R}_0} \cup [\mathbb{N}]^{<\mathcal{R}_0} \), into the infinite and the finite sets, respectively, is useful here.

For \( f, g \in \mathbb{N}^\mathbb{N} \), let \( f \leq^* g \) if \( f(n) \leq g(n) \) for all but finitely many \( n \). \( b \) is the minimal cardinality of a \( \leq^* \)-unbounded subset of \( \mathbb{N}^\mathbb{N} \). A set \( B \subseteq [\mathbb{N}]^{\mathcal{R}_0} \) is unbounded if the set of all increasing enumerations of elements of \( B \) is unbounded in \( \mathbb{N}^\mathbb{N} \), with respect to \( \leq^* \).

**Definition 2.1.** A tower of cardinality \( \kappa \) is a set \( T \subseteq [\mathbb{N}]^{\mathcal{R}_0} \) which can be enumerated bijectively as \( \{x_\alpha : \alpha < \kappa\} \), such that for all \( \alpha < \beta < \kappa \), \( x_\beta \subseteq^* x_\alpha \).

An unbounded tower of cardinality \( \kappa \) is an unbounded set \( T \subseteq [\mathbb{N}]^{\mathcal{R}_0} \) which is a tower of cardinality \( \kappa \). (Necessarily, \( \kappa \geq b \).)

Let \( t \) be the minimal cardinality of a tower which has no pseudointersection. Rothberger proved that \( t \leq b \) [1]. \( t = b \) if, and only if, there is an unbounded tower of cardinality \( t \).

Just, Miller, Scheepers and Szeptycki [7] proved that if \( T \) is an unbounded tower of cardinality \( \mathcal{R}_1 \), then \( T \cup [\mathbb{N}]^{<\mathcal{R}_0} \) satisfies \( S_1(\Omega, \Omega) \), as well as a property, which was later proved by Scheepers [10] to be equivalent to \( S_1(\Gamma, \Gamma) \). In Problem 7 of [7], we are asked the following.

**Problem 2.2** (Just-Miller-Scheepers-Szeptycki [7]). Assume that \( T \subseteq [\mathbb{N}]^{\mathcal{R}_0} \) is an unbounded tower of cardinality \( \mathcal{R}_1 \) (so that \( \mathcal{R}_1 = b \)). Is \( T \cup [\mathbb{N}]^{<\mathcal{R}_0} \) a \( \gamma \)-set, i.e., satisfies \( S_1(\Omega, \Gamma) \)?

Scheepers proves in [9] that for each unbounded tower \( T \) of cardinality \( t = b \), \( T \subseteq [\mathbb{N}]^{\mathcal{R}_0} \) satisfies \( S_1(\Gamma, \Gamma) \).
Miller [8] proves that in the Hechler model, there are no uncountable $\gamma$-sets. In this model, $\aleph_1 = p = t < b$, and thus $\aleph_1 = t$ does not suffice to have an uncountable $\gamma$-set of reals. At the end of [8] and in its appendix, Miller proves that $\diamondsuit(b)$, a property strictly stronger than $\aleph_1 = b$, implies that there is an uncountable $\gamma$-set of reals. He concludes that it is still open whether $b = \aleph_1$ is enough to construct an uncountable $\gamma$-set.

We show that the answer is positive, and indeed also answer a question of Gruenhage and Szeptycki [6]: A classical problem of Malykhin asks whether there is a countable Fréchet-Urysohn topological group which is not metrizable. Gruenhage and Szeptycki prove that $F \subseteq \mathbb{N}^\mathbb{N}$ is a $\gamma$-set if, and only if, a certain construction associated to $F$ provides a positive answer to Malykhin’s Problem [6]. They define a generalization of $\gamma$-set, called weak $\gamma$-set, and combine their results with results of Nyikos to prove that $p = b$ implies that there is a weak $\gamma$-set in $\mathbb{N}^\mathbb{N}$ [6, Corollary 10]. They write: “The relationship between $\gamma$-sets and weak $\gamma$-sets is not known. Perhaps $b = p$ implies the existence of a $\gamma$-set.” Our solution confirms their conjecture.

Theorem 2.3. For each unbounded tower $T$ of cardinality $p$ in $[\mathbb{N}]^{\aleph_0}$, $T \cup [\mathbb{N}]^{<\aleph_0}$ satisfies $S_1(\Omega, \Gamma)$.

Zdomskyy points out that our proof actually shows that a wider family of sets are $\gamma$-sets. For example, if we start with $T$ an unbounded tower of cardinality $p$, and thin out its elements arbitrarily, $T \cup [\mathbb{N}]^{<\aleph_0}$ remains a $\gamma$-set. This may be useful for constructions of examples with additional properties, since this way, each element of $T$ may be chosen arbitrarily from a certain perfect set.

In particular, we have that in each model of ZFC where $p = b$, there are $\gamma$-sets of cardinality $p$.

Corollary 2.4. In each of the Cohen, Random, Sacks, and Miller models of ZFC, there are $\gamma$-sets of reals with cardinality $p$.

As discussed above, there are no uncountable $\gamma$-sets in the Hechler model [8]. Since the Laver and Mathias models satisfy Borel’s Conjecture, there are no uncountable $\gamma$-sets in these models, too.

Earlier, Corollary 2.4 was shown for the Sacks model by Ciesielski, Millán, and Pawlikowski in [2], and for the Cohen and Miller models by Miller [8], using specialized arguments. It seems that the result, that there are uncountable $\gamma$-sets in the Random reals model (constructed by extending a model of the Continuum Hypothesis), is new.

$\diamondsuit(b)$ is defined in Dzamonja-Hrusak-Moore [3].
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3. RESEARCH ANNOUNCEMENTS

3.1. Ultrafilters with property (s).
   http://www.ams.org/journal-getitem?pii=S0002-9939-09-09919-5
   Arnold W. Miller

3.2. On a converse to Banach’s Fixed Point Theorem.
   http://www.ams.org/journal-getitem?pii=S0002-9939-09-09904-3
   Marton Elekes

3.3. Analytic groups and pushing small sets apart.
   http://www.ams.org/journal-getitem?pii=S0002-9947-09-04665-0
   Jan van Mill

3.4. Club-guessing, stationary reflection, and coloring theorems. We obtain strong coloring theorems at successors of singular cardinals from failures of certain instances of simultaneous reflection of stationary sets. Along the way, we establish new results in club-guessing and in the general theory of ideals.
   http://arxiv.org/abs/0905.3754
   Todd Eisworth
3.5. More on the pressing down game. We investigate the pressing down game and its relation to the Banach Mazur game. In particular we show: Consistently relative to a supercompact, there is a nowhere precipitous normal ideal $I$ on $\aleph_2$ such that player nonempty wins the pressing down game of length $\aleph_1$ on $I$ even if player empty starts. For the proof, we construct a forcing notion to force the following: There is normal, nowhere precipitous ideal $I$ on a supercompact $\kappa$ such that for every $I$-positive $A$ there is a normal ultrafilter containing $A$ and extending the dual of $I$.

http://arxiv.org/abs/0905.3913
Jakob Kellner, Saharon Shelah

3.6. A note on discrete sets. We give several partial positive answers to a question of Juhasz and Szentmiklossy regarding the minimum number of discrete sets required to cover a compact space. We study the relationship between the size of discrete sets, free sequences and their closures with the cardinality of a Hausdorff space, improving known results in the literature.

http://arxiv.org/abs/0905.3588
Santi Spadaro

3.7. Antidiamond principles and topological applications.

http://www.ams.org/journal-getitem?pii=S0002-9947-09-04705-9
Todd Eisworth and Peter Nyikos

3.8. Partitions and indivisibility properties of countable dimensional vector spaces. We investigate infinite versions of vector and affine space partition results, and thus obtain examples and a counterexample for a partition problem for relational structures. In particular we provide two (related) examples of an age indivisible relational structure which is not weakly indivisible.

http://arxiv.org/abs/0907.3771
C. Laflamme, L. Nguyen Van The, M. Pouzet, N. Sauer

3.9. Group-valued continuous functions with the topology of pointwise convergence. Let $G$ be a topological group with the identity element $e$. Given a space $X$, we denote by $C_pXG$ the group of all continuous functions from $X$ to $G$ endowed with the topology of pointwise convergence, and we say that $X$ is: (a) $G$-regular if, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, there exist $f \in C_pXG$ and $g \in G \setminus \{e\}$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$; (b) $G^*$-regular provided that there exists $g \in G \setminus \{e\}$ such that, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, one can find $f \in C_pXG$ with $f(x) = g$ and $f(F) \subseteq \{e\}$. Spaces $X$ and $Y$ are $G$-equivalent provided that the topological groups $C_pXG$ and $C_pYG$ are topologically isomorphic.

We investigate which topological properties are preserved by $G$-equivalence, with a special emphasis being placed on characterizing topological properties of $X$ in terms of those of $C_pXG$. Since $\mathbb{R}$-equivalence coincides with $l$-equivalence, this line of research
“includes” major topics of the classical $C_p$-theory of Arhangel’skiï as a particular case (when $G = \mathbb{R}$).

We introduce a new class of TAP groups that contains all groups having no small subgroups (NSS groups). We prove that: (i) for a given NSS group $G$, a $G$-regular space $X$ is pseudocompact if and only if $C_pXG$ is TAP, and (ii) for a metrizable NSS group $G$, a $G^*$-regular space $X$ is compact if and only if $C_pXG$ is a TAP group of countable tightness. In particular, a Tychonoff space $X$ is pseudocompact (compact) if and only if $C_pX\mathbb{R}$ is a TAP group (of countable tightness). Demonstrating the limits of the result in (i), we give an example of a precompact TAP group $G$ and a $G$-regular countably compact space $X$ such that $C_pXG$ is not TAP.

We show that Tychonoff spaces $X$ and $Y$ are $\mathbb{T}$-equivalent if and only if their free precompact Abelian groups are topologically isomorphic, where $\mathbb{T}$ stays for the quotient group $\mathbb{R}/\mathbb{Z}$. As a corollary, we obtain that $\mathbb{T}$-equivalence implies $G$-equivalence for every Abelian precompact group $G$. We establish that $\mathbb{T}$-equivalence preserves the following topological properties: compactness, pseudocompactness, $\sigma$-compactness, the property of being a Lindelöf $\Sigma$-space, the property of being a compact metrizable space, the (finite) number of connected components, connectedness, total disconnectedness. An example of $\mathbb{R}$-equivalent (that is, $l$-equivalent) spaces that are not $\mathbb{T}$-equivalent is constructed.

http://arxiv.org/abs/0907.4941

Dmitri Shakhmatov, Jan Spevak

3.10. Stationary and convergent strategies in Choquet games. If POINT has a winning strategy against EMPTY in the Choquet game on a space, the space is said to be a Choquet space. Such a winning strategy allows POINT to consider the entire finite history of previous moves before making each new move; a stationary strategy only permits POINT to consider the previous move by EMPTY. We show that POINT has a stationary winning strategy for every second countable $\mathbb{T}_1$ Choquet space. More generally, POINT has a stationary winning strategy for any $\mathbb{T}_1$ Choquet space with an open-finite basis.

We also study convergent strategies for the Choquet game, proving the following results.

A $\mathbb{T}_1$ space $X$ is the open image of a complete metric space if and only if POINT has a convergent winning strategy in the Choquet game on $X$.

A $\mathbb{T}_1$ space $X$ is the compact open image of a metric space if and only if $X$ is metacompact and POINT has a stationary convergent strategy in the Choquet game on $X$.

A $\mathbb{T}_1$ space $X$ is the compact open image of a complete metric space if and only if $X$ is metacompact and POINT has a stationary convergent winning strategy in the Choquet game on $X$.

http://arxiv.org/abs/0907.4126

François G. Dorais and Carl Mummert
3.11. **Linear $\sigma$-additivity and some applications.** We show that countable increasing unions preserve a large family of well-studied covering properties, which are not necessarily $\sigma$-additive. Using this, together with infinite-combinatorial methods and simple forcing theoretic methods, we explain several phenomena, settle problems of Just, Miller, Scheepers and Szeptycki; Gruenhage and Szeptycki; Tsaban and Zdomskyy; and Tsaban, and construct topological groups with very strong combinatorial properties. (See also Sections 1 and 2 above.)

http://arxiv.org/abs/0906.5136

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4. UNSOLVED PROBLEMS FROM EARLIER ISSUES

Issue 1. Is \( \binom{\Omega}{\Gamma} = \binom{\Omega}{\Gamma} \)?

Issue 2. Is \( \mathcal{U}_{\text{fin}}(\mathcal{O}, \Omega) = \mathcal{S}_{\text{fin}}(\Gamma, \Omega) \)? And if not, does \( \mathcal{U}_{\text{fin}}(\mathcal{O}, \Gamma) \) imply \( \mathcal{S}_{\text{fin}}(\Gamma, \Omega) \)?

Issue 4. Does \( \mathcal{S}_1(\Omega, T) \) imply \( \mathcal{U}_{\text{fin}}(\Gamma, \Gamma) \)?

Issue 5. Is \( p = p^* \)? (See the definition of \( p^* \) in that issue.)

Issue 6. Does there exist (in ZFC) an uncountable set satisfying \( \mathcal{S}_{\text{fin}}(\mathcal{B}, \mathcal{B}) \)?

Issue 8. Does \( X \not\in \text{NON}(\mathcal{M}) \) and \( Y \not\in \text{D} \) imply that \( X \cup Y \not\in \text{COF}(\mathcal{M}) \)?

Issue 9 (CH). Is \( \text{Split}(\Lambda, \Lambda) \) preserved under finite unions?

Issue 10. Is \( \text{cov}(\mathcal{M}) = \text{o}\mathfrak{d} \)? (See the definition of \( \text{o}\mathfrak{d} \) in that issue.)

Issue 11. Does \( \mathcal{S}_1(\Gamma, \Gamma) \) always contain an element of cardinality \( \mathfrak{b} \)?

Issue 12. Could there be a Baire metric space \( M \) of weight \( \aleph_1 \) and a partition \( \mathcal{U} \) of \( M \) into \( \aleph_1 \) meager sets where for each \( \mathcal{U}' \subset \mathcal{U}, \bigcup \mathcal{U}' \) has the Baire property in \( M \)?

Issue 14. Does there exist (in ZFC) a set of reals \( X \) of cardinality \( \mathfrak{d} \) such that all finite powers of \( X \) have Menger’s property \( \mathcal{S}_{\text{fin}}(\mathcal{O}, \mathcal{O}) \)?

Issue 15. Can a Borel non-\( \sigma \)-compact group be generated by a Hurewicz subspace?

Issue 16 (MA). Is there \( X \subseteq \mathbb{R} \) of cardinality continuum, satisfying \( \mathcal{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \)?

Issue 17 (CH). Is there a totally imperfect \( X \) satisfying \( \mathcal{U}_{\text{fin}}(\mathcal{O}, \Gamma) \) that can be mapped continuously onto \( \{0, 1\}^\mathbb{N} \)?

Issue 18 (CH). Is there a Hurewicz \( X \) such that \( X^2 \) is Menger but not Hurewicz?

Issue 19. Does the Pytkeev property of \( C_p(X) \) imply that \( X \) has Menger’s property?

Issue 20. Does every hereditarily Hurewicz space satisfy \( \mathcal{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) \)?

Issue 21 (CH). Is there a Rothberger-bounded \( G \leq \mathcal{Z}^\mathbb{N} \) such that \( G^2 \) is not Menger-bounded?

Issue 22. Let \( \mathcal{W} \) be the van der Waerden ideal. Are \( \mathcal{W} \)-ultrafilters closed under products?

Issue 23. Is the \( \delta \)-property equivalent to the \( \gamma \)-property \( \left( \begin{array}{c} \Omega \\ \Gamma \end{array} \right) \)?

Previous issues. The previous issues of this bulletin are available online at http://front.math.ucdavis.edu/search\&t=%22SPM+Bulletin%22

Contributions. Announcements, discussions, and open problems should be emailed to tsaban@math.biu.ac.il

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