Total dominator chromatic number of specific graphs

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Abstract
Let $G$ be a simple graph. A total dominator coloring of $G$ is a proper coloring of the vertices of $G$ in which each vertex of the graph is adjacent to every vertex of some color class. The total dominator chromatic number $\chi_d^t(G)$ of $G$ is the minimum number of colors among all total dominator coloring of $G$. In this paper, we study the total dominator chromatic number of some specific graphs.

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1 Introduction

In this paper, we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G = (V,E)$ be such a graph and $\lambda \in \mathbb{N}$. A mapping $f : V(G) \rightarrow \{1, 2, ..., \lambda\}$ is called a $\lambda$-proper coloring of $G$ if $f(u) \neq f(v)$ whenever the vertices $u$ and $v$ are adjacent in $G$. A color class of this coloring is a set consisting of all those vertices assigned the same color. If $f$ is a proper coloring of $G$ with the coloring classes $V_1, V_2, ..., V_\lambda$ such that every vertex in $V_i$ has color $i$, sometimes write simply $f = (V_1, V_2, ..., V_\lambda)$. The chromatic number $\chi(G)$ of $G$ is the minimum number of colors needed in a proper coloring of a graph. The concept of a graph coloring and chromatic number is very well-studied in graph theory.

A dominator coloring of $G$ is a proper coloring of $G$ such that every vertex of $G$ dominates all vertices of at least one color class (possibly its own class), i.e., every vertex of $G$ is adjacent to all vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ of $G$ is the minimum number of color classes in a dominator coloring of $G$. The concept of dominator coloring was introduced and studied by Gera, Horton and Rasmussen [3].

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Kazemi [6,7] studied the total dominator coloring, abbreviated TD-coloring. Let $G$ be a graph with no isolated vertex, the total dominator coloring is a proper coloring of $G$ in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TD-chromatic number, $\chi^t_d(G)$ of $G$ is the minimum number of color classes in a TD-coloring of $G$. The TD-chromatic number of a graph is related to its total domination number. A total dominating set of $G$ is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A total dominating set of $G$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. The literature on the subject on total domination in graphs has been surveyed and detailed in the book [4]. It is not hard to see that for every graph $G$ with no isolated vertex, $\gamma_t(G) \leq \chi^t_d(G)$. Computation of the TD-chromatic number is NP-complete ([4]). The TD-chromatic number of some graphs, such as paths, cycles, wheels and the complement of paths and cycles has computed in [6]. Also Henning in [5] established the lower and upper bounds on the TD-chromatic number of a graph in terms of its total domination number. He has shown that, for every graph $G$ with no isolated vertex satisfies $\gamma_t(G) \leq \chi^t_d(G) \leq \gamma_t(G) + \chi(G)$. The properties of TD-colorings in trees has studied in [5,6]. Trees $T$ with $\gamma_t(T) = \chi^t_d(T)$ has characterized in [5].

The join $G = G_1 + G_2$ of two graph $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V_1$ and $V_2$. For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of $G$ with $|V|$ copies of $H$, by adding edges between the $i$th vertex of $G$ and all vertices of $i$th copy of $H$.

In this paper, we continue the study of TD-colorings in graphs. We compute the TD-chromatic number of corona and join of graphs, in Section 2. In Section 3, we compute TD-chromatic number of some specific graphs.
2 TD-chromatic number of corona and join of graphs

In this section, first we compute the TD-chromatic number of corona and join of two graphs. First we state the following results:

**Theorem 1.** (1)

(i) Let $P_n$ be a path of order $n \geq 2$. Then
\[
\chi_t^d(P_n) = \begin{cases} 
2 \left\lceil \frac{n}{3} \right\rceil - 1 & \text{if } n \equiv 1 \pmod{3}, \\
2 \left\lceil \frac{n}{3} \right\rceil & \text{otherwise}.
\end{cases}
\]

(ii) Let $C_n$ be a cycle of order $n \geq 5$. Then
\[
\chi_t^d(C_n) = \begin{cases} 
4 \left\lfloor \frac{n}{6} \right\rfloor + r & \text{if } n \equiv r \pmod{6}, \ r = 0, 1, 2, 4, \\
4 \left\lfloor \frac{n}{6} \right\rfloor + r - 1 & \text{if } n \equiv r \pmod{6}, \ r = 3, 5.
\end{cases}
\]

Here, we consider the corona of $P_n$ and $C_n$ with $K_1$. The following theorem gives the TD-chromatic number of these kind of graphs:

**Theorem 2.**

(i) For every $n \geq 2$, $\chi_t^d(P_n \circ K_1) = n + 1$.

(ii) For every $n \geq 3$, $\chi_t^d(C_n \circ K_1) = n + 1$.

**Proof.**

(i) We color the $P_n \circ K_1$ with numbers $1, 2, ..., n + 1$, as shown in the Figure 4. Observe that, we need $n + 1$ color for TD-coloring. We shall show that we are not able to have TD-coloring with less colors. Suppose that the $i$-th vertex of $P_n$ has colored with color $i - 1$. If we change the color of this vertex by color 1, and give the vertex pendant to this vertex, the color $i - 1$, then obviously this new coloring cannot be a TD-coloring. Therefore, we have the result.
Figure 1: Total dominator coloring of $P_n \circ K_1$ and $C_n \circ K_1$, respectively.

(ii) The proof is similar to Part (i). \qed

The following theorem is easy to obtain:

**Theorem 3.** For every $n \geq 2$, $\chi_d(P_n \circ K_m) = n + 1$.

In the following theorem, we consider graphs of the form $G \circ H$:

**Theorem 4.**

(i) For every connected graph $G$, $\chi_d^t(G \circ K_1) = |V(G)| + 1$,

(ii) For every two connected graphs $G$ and $H$,

$$\chi_d^t(G \circ H) \leq \chi_d^t(G) + |V(G)| \chi_d^t(H).$$

(iii) For every two connected graphs $G$ and $H$,

$$\chi_d^t(G \circ H) \leq |V(G)| + |V(H)|.$$

**Proof.**
(i) We color all vertices of graph $G$ with numbers $\{1, 2, ..., |V(G)|\}$ and all pendant vertices with another color, say, $|V(G)| + 1$. It is easy to check that we are not able to have TD-color of $G \circ K_1$ with less color. Therefore we have the result.

(ii) For TD-coloring of $G$ and $H$, we need $\chi_d^t(G)$ and $\chi_d^t(H)$ colors. We observe that if we use $\chi_d^t(G) + |V(G)|\chi_d^t(H)$ colors, then we have a TD-coloring of $G \circ H$. So $\chi_d^t(G \circ H) \leq \chi_d^t(G) + |V(G)|\chi_d^t(H)$.

(iii) We color the vertices of $G$, by $|V(G)|$ colors and for every copy of $H$, we use $|V(H)|$ another colors. We observe that this coloring gives a TD-coloring of $G \circ H$. So $\chi_d^t(G \circ H) \leq |V(G)| + |V(H)|$. □

**Remark 1.** The upper bound for $\chi_d^t(G \circ H)$ in Theorem 4(iii) is a sharp bound. As an example, for the graph $C_4 \circ K_2$ and $K_2 \circ K_3$ we have the equality (Figure 2).

![Figure 2: Total dominator coloring of $C_4 \circ K_2$ and $K_2 \circ K_3$, respectively.](image)

Here, we state and prove a formula for the TD-chromatic number of join of two graphs:

**Theorem 5.** Let $G$ and $H$ be two connected graphs, then

$$\chi_d^t(G + H) = \chi_d^t(G) + \chi_d^t(H).$$

**Proof.** For the TD-coloring of $G + H$, the colors of vertices of $G$ cannot be used for the coloring of vertices of $H$, and the colors of the vertices of $H$ cannot use for coloring of the vertices of $G$, so

$$\chi_d^t(G + H) \geq \chi_d^t(G) + \chi_d^t(H).$$
Figure 3: Friendship graphs $F_2, F_3, F_4$ and $F_n$, respectively.

Now, it suffices to consider the coloring of $G$ and the coloring of $H$ in the TD-coloring of $G + H$. Therefore, we have the result.  

3 Total dominator chromatic number of specific graphs

In this section, we consider the specific graphs and compute their TD-chromatic numbers.

The friendship (or Dutch-Windmill) graph $F_n$ is a graph that can be constructed by coalescence $n$ copies of the cycle graph $C_3$ of length 3 with a common vertex. The Friendship Theorem of Paul Erdős, Alfred Rényi and Vera T. Sós \cite{2}, states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. Figure 3 shows some examples of friendship graphs.

The generalized friendship graph $D_q^n$ is a collection of $n$ cycles (all of order $q$), meeting at a common vertex. The generalized friendship graph may also be referred to as a flower \cite{8}.

By Figure 4, we have the following result for the TD-chromatic number of these kind of graphs:

Theorem 6.

(i) For every $n \geq 2$, $\chi_d^t(F_n) = 3$.

(ii) For every $n \geq 2$, $\chi_d^t(D_q^n) = n + 2$.

(iii) For every $n \geq 2$, $\chi_d^t(D_5^n) = 2n + 2$.  

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Here, we shall consider the ladder graph. We need the definition of Cartesian product of two graphs. Given any two graphs $G$ and $H$, we define the Cartesian product, denoted $G \square H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices $(u_1, v_1)$ and $(u_2, v_2)$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$.

The $n$-ladder graph can be defined as $P_2 \square P_n$ and denoted by $L_n$. Figure 5 shows a TD-coloring of ladder graphs.

**Theorem 7.** For every $n \geq 2$,

$$\chi_{td}(L_n) = \begin{cases} 
  n + 1 & \text{if } n \text{ is odd}, \\
  n & \text{if } n \text{ is even}.
\end{cases}$$

**Proof.** It follows from a TD-coloring which has shown if Figure 5.

Here, we generalize the ladder graph $P_2 \square P_n$ to grid graphs $P_n \square P_m$. The following theorem gives the TD-chromatic number of grid graphs:

![Figure 4: Total dominator coloring of $F_n$, $D_4^n$ and $D_5^n$, respectively.](image-url)

![Figure 5: Total dominator coloring of $L_{2k+1}$ and $L_{2k+2}$, respectively.](image-url)
Theorem 8. Let $m, n \geq 2$. The TD-chromatic number of grid graphs $P_n \square P_m$ is,

$$\chi_d^t(P_n \square P_m) = \begin{cases} 
  k\chi_d^t(L_n) & \text{if } m = 2k \text{ and } n = 2s, \\
  k\chi_d^t(L_n) + \chi_d^t(P_n) & \text{if } m = 2k + 1 \text{ and } n = 2s, \\
  s\chi_d^t(L_m) + \chi_d^t(P_m) & \text{if } m = 2k \text{ and } n = 2s + 1, \\
  \chi_d^t(P_{n-1} \square P_{m-1}) + \chi_d^t(P_{m+n-1}) & \text{if } m = 2k + 1 \text{ and } n = 2s + 1.
\end{cases}$$

Proof. We prove two first cases. The proof of another cases are similar. Suppose that for some $k$ and $s$, we have $m = 2k$ and $n = 2s$. We use induction on $m$.

Case 1. If $m = 2$ and $n = 2s$, then we have a ladder and the result follows from Theorem 7. For $m = 2$, as you see in Figure 6 we have two $L_n$. Since in TD-coloring of $4 \times n$ grid graph, we cannot use the colors of vertices in the first ladder, for the second ladder, so we need $2\chi_d^t(L_n)$ colors. Since in the $P_n \square P_{2k}$, there are exactly $k$ ladder $L_n$, we have the result by induction hypothesis.

Case 2. Now suppose that $n = 2s$ and $m = 2k + 1$. First for TD-coloring of $P_n \square P_{2k}$, we need $k\chi_d^t(L_n)$ colors, by Case 1. It remains to color a path $P_n$. Therefore we need $k\chi_d^t(L_n) + \chi_d^t(P_n)$ colors to obtain a TD-coloring of $P_n \square P_m$. \qed

Now, we consider cactus graphs. A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle.
If all blocks of a cactus $G$ are cycles of the same size $i$, the cactus is $i$-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus $G$ has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that $G$ is a chain triangular cactus. We call the number of triangles in $G$, the length of the chain. An example of a chain triangular cactus is shown in Figure 7. Obviously, all chain triangular cacti of the same length are isomorphic. Hence, we denote the chain triangular cactus of length $n$ by $T_n$. See [1].

Figure 7: Total dominator coloring of $T_{2k-1}$ and $T_{2k}$, respectively.

Using the TD-coloring in Figure 7 we have the following theorem for the TD-chromatic number of $T_n$.

**Theorem 9.** For every $k \in \mathbb{N}$, $\chi_{td}(T_{2k-1}) = \chi_{td}(T_{2k}) = 2k + 1$.

By replacing triangles in the definitions of triangular cactus, by cycles of length 4 we obtain cacti whose every block is $C_4$. We call such cacti, square cacti. We see that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square. We consider an ortho-chain of length $n$, $O_n$.

Using the TD-coloring in Figure 8 we have the following theorem for the TD-chromatic number of $O_n$.

Figure 8: Total dominator coloring of $O_n$.  

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**Theorem 10.** For every $n \in \mathbb{N}$, $\chi^d_t(O_n) = 2n$.

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