Fermion masses, neutrino mixing and CP violation from the anti-grand unification model

C.D. Froggatt\textsuperscript{a}\textsuperscript{*} M. Gibson\textsuperscript{a}\textsuperscript{†} H.B. Nielsen\textsuperscript{b}\textsuperscript{‡}
D.J. Smith\textsuperscript{b}\textsuperscript{§}

December 25, 2021

\textsuperscript{a} Department of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK
\textsuperscript{b} Niels Bohr Institute, Blegdamsvej 17-21, DK 2100 Copenhagen, Denmark

Abstract

The fermion masses and mixing angles are fitted using only 3 free parameters in a non-supersymmetric extension of the Standard Model, with new approximately conserved chiral gauge quantum numbers broken by a set of Higgs fields. The fundamental mass scale of this anti-grand unification model is given by the Planck mass. We also calculate neutrino mixing angles and masses, as well as CP violation from the CKM matrix. A good fit is obtained to the observed fermion masses but our predictions of the neutrino masses are too small to lead to any observable neutrino oscillation effects claimed today, without introducing another mass scale. We also give some arguments in support of this type of model based on the observed fermion masses.

PACS: 12.15.Ff, 12.15.Mh

\textsuperscript{*}c.froggatt@physics.gla.ac.uk
\textsuperscript{†}m.gibson@physics.gla.ac.uk
\textsuperscript{‡}hbech@nbi.dk
\textsuperscript{§}D.Smith@nbi.dk
1 Introduction

In a previous paper [1] we presented a model to explain the origin of the Standard Model (SM) fermion mass and mixing hierarchy, based on the so-called anti-grand unified extension of the SM gauge group (SMG). This model gives rise to a characteristic structure for the quark-lepton mass matrices; in this paper we shall try to argue that some features of this model are implied by the experimentally determined fermion masses. We shall also extend the analysis in [1] to include CP violation due to a complex phase in the CKM quark mixing matrix and to calculate the neutrino mass matrix to get predictions for neutrino masses and mixing angles.

In section 2 we will describe how to calculate fermion masses and mixing angles when viewing the SM as an effective theory. The observed fermion masses and mixing angles will be discussed in section 3. An approximate parameterisation of the masses will then be described. In section 4 we will give a simple method by which a model could naturally explain such a parameterisation. Our anti-grand unified model will be described in section 5, where we show that it fulfils the requirements of section 4. The gauge quantum numbers of the quarks and leptons in our model are fixed by the requirement of anomaly cancellation.

In section 6 we will describe the construction of our model for the second and third generation fermion masses and mixing angle, in terms of two Higgs fields in addition to the usual Weinberg-Salam (WS) Higgs field. We emphasise in section 7 the general features, suggested by phenomenology and the anti-grand unified model, which underlie the structure of our mass matrices for the second and third generation fermions. The choice of the Higgs field quantum numbers and the extension of the mass matrices to include the first generation particles are discussed in section 8. This leads to definite predictions for the order of magnitude of the fermion masses and mixing angles in terms of the vacuum expectation values (VEVs) of the Higgs fields. In section 9 we present our best fit to the conventional experimental masses and mixing angles, in terms of just three VEVs of the same order of magnitude. We also make a fit using preliminary values for the light quark masses extracted from lattice QCD. These smaller “lattice” values for the light quark masses motivated us to consider, in section 10, the possibility of an alternative mass matrix structure.

Then we discuss other predictions of our model. In section 11 we consider how much CP violation would be expected from the CKM matrix in our model. Finally we turn to the neutrino masses and mixing angles in section 12. The effective Majorana mass matrices for the three light neutrinos in models with a hierarchical mass matrix can naturally have quasi-degenerate mass eigenstates with maximal mixing [8]. Maximal neutrino mixing provides a candidate explanation for the atmospheric muon neutrino deficit or the solar neutrino problem with vacuum oscillations. Our model does have such a structure but, with a see-saw mass scale equal to the Planck scale, the predicted neutrino masses are too small to give observable effects. As in most other models, it is necessary
to introduce an intermediate mass scale in order to obtain observable neutrino mixing. This is not so attractive in the anti-grand unification model. However we can obtain identical structure for the fermion mass matrices in an anomaly free SMG $\otimes U(1)^3$ model, in which the fundamental scale is unconstrained and could be taken as intermediate between the electroweak and Planck scales. We present our conclusions in section 13.

2 Modelling fermion masses

In the SM the fermions (apart from the neutrinos) all get a mass via the Higgs mechanism. Before electroweak symmetry breaking there are interactions of the form:

$$\mathcal{L}_{\text{mass}} = \overline{Q_L}H_U \Phi_{WS} U_R + \overline{Q_L}H_D \Phi_{WS} D_R + \overline{L_L}H_E \Phi_{WS} E_R + \text{h.c.}$$  \hspace{1cm} (1)

where $\Phi_{WS}$ is the WS Higgs field, $Q_L$ is the 3 SU(2) doublets of left-handed quarks, $H_U$ is the $3 \times 3$ Yukawa matrix for the up-type quarks, etc. If we represent the SU(2) doublets $\Phi_{WS}$ and $Q_L$ as 2 component column vectors, we define:

$$\tilde{\Phi}_{WS} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi_{WS}^\dagger$$  \hspace{1cm} (2)

and

$$Q_L = \begin{pmatrix} U_L \\ D_L \end{pmatrix} = (U_L \ D_L)$$  \hspace{1cm} (3)

where $\overline{U_L}$ are the CP conjugates of the 3 left-handed up-type quarks.

After electroweak symmetry breaking the WS Higgs field gets a vacuum expectation value (VEV) and we would write:

$$\mathcal{L}_{\text{mass}} = \overline{U_L}M_U U_R + \overline{D_L}M_D D_R + \overline{E_L}M_E E_R + \text{h.c.}$$  \hspace{1cm} (4)

where the mass matrices are related to the Yukawa matrices and WS Higgs VEV by:

$$M = H \langle \phi_{WS} \rangle \sqrt{2}$$  \hspace{1cm} (5)

and we have chosen the normalisation so that:

$$\langle \phi_{WS} \rangle = 246 \text{ GeV}$$  \hspace{1cm} (6)

The masses of e.g. the 3 up-type fermions are obtained from $M_U$ by diagonalising the matrix to find the 3 eigenvalues. In particular we can find unitary matrices $V_U$ and $V_D$ so that:

$$V_U^\dagger M_U V_U = \text{diag}\{m_u^2, m_c^2, m_t^2\}$$  \hspace{1cm} (7)

$$V_D^\dagger M_D V_D = \text{diag}\{m_d^2, m_s^2, m_b^2\}$$  \hspace{1cm} (8)
The quark mixing matrix is then defined as:

$$V_{\text{CKM}} = V_U^\dagger V_D$$  \hspace{1cm} (9)$$

This is where we find aesthetic problems with the SM. First, we have no way of calculating the mass matrices since there are no constraints due to e.g. gauge symmetries. The elements of the 3 Yukawa matrices are allowed to be arbitrary complex numbers. This means that there is no understanding of the origin of the masses or mixing angles within the SM. The second problem is that we would expect that, since there is no distinction between the 3 generations, the order of magnitude of the mass of each fermion would not depend on which generation it was in. Further, since, as far as the Higgs sector is concerned, there is no distinction between the different types of fermions, we would generally expect that all fermions in the SM (except for the neutrinos) would have the same order of magnitude masses. Indeed naturality would suggest that all the Yukawa matrix elements be of order unity and the fermion masses of order $\langle \phi_{WS} \rangle / \sqrt{2} = 174$ GeV. This is clearly not the case.

So in order to understand the masses of the fermions we must postulate some model beyond the SM, which can give different SM fermions very different masses without the need for arbitrarily small Yukawa couplings. One method to do this is to extend the SM gauge group so that the 3 generations are not equivalent under the more fundamental gauge group. Then we would consider eq. (9) to be the effective Lagrangian for the Yukawa-Higgs sector. In the full gauge group such terms would not generally be gauge invariant since the SM fermions would have extra “charges” (generally Abelian and non-Abelian). Of course such terms would still be expected in the effective theory at energies much below the symmetry breaking scale of the full gauge group to the SM. The difference is that now we have no reason to expect that all the Yukawa interactions in the effective theory should be of the same order of magnitude.

As an example consider the bottom quark. If we suppose that the dominant transition between the left- and right-handed components in the fundamental theory involves not only the WS Higgs field, but e.g. 2 other Higgs fields $W$ and $T$ as shown in fig. 1, we would get the following relation \cite{3} for the effective Yukawa coupling of the bottom quark in the SM:

$$h_b \simeq \frac{\langle W \rangle \langle T \rangle}{M_F M_F}$$ \hspace{1cm} (10)$$

where $\langle W \rangle$ and $\langle T \rangle$ are simply the VEVs of the new Higgs fields $W$ and $T$, and $M_F$ is the (fundamental) mass scale of the intermediate fermions. Here we are assuming that all fundamental Yukawa couplings (the $\lambda_i$) are of order 1. Thus the order of magnitude of the effective SM Yukawa coupling constants are given by the product of small symmetry breaking factors, like $\frac{\langle W \rangle}{M_F}$ and $\frac{\langle T \rangle}{M_F}$. So now we can explain why the fermions in the SM have different masses and we can construct explicit models, by choosing extended gauge quantum numbers.
for the SM fermions and making specific choices of Higgs fields $W$, $T$, etc. to allow the transitions in the fundamental theory which give masses to the SM fermions.

It is important to note that this type of model cannot give exact predictions, since we are still unable to calculate fundamental Yukawa couplings. Essentially we introduce a lot more fundamental Yukawa couplings but then make the naturality assumption that they are all of order 1. However, the problem we are addressing is the huge range of fermion masses and so we can get some understanding of this hierarchy without any knowledge of why the masses are exactly what they are. Furthermore we assume that there exists a spectrum of vector-like fermion states, all having a mass of order $M_F$, which can mediate all of the required symmetry breaking transitions.

Clearly there are many different models we could propose to model the fermion masses. In order to make progress it is therefore necessary to examine the experimentally measured masses and look for relations such as order of magnitude degeneracy. One point to note is that the effective Yukawa couplings we predict will be the values at the fundamental scale $M_F$ (assumed to be of order the Planck scale $M_{Planck} \approx 10^{19}$ GeV) rather than experimental scales such as 1 GeV.

3 Measured fermion masses

The masses of the fermions in the SM are usually quoted as running masses, at some scale such as 1 GeV, except for the top mass which is generally quoted as a pole mass. These masses can be evolved to other scales by using the renormalisation group equations (RGEs). One of the assumptions we make is that the SM is valid up to the Planck scale, but assuming a lower fundamental scale (say of order $10^{15}$ GeV) makes no essential difference. So we evolve the
masses to the Planck scale to compare them. The main effect is to change the ratios of the quark to lepton masses and also the ratio of the top quark to other quark masses. At a scale $\mu$ we can express the running masses in terms of the Yukawa couplings by:

$$m(\mu) = h(\mu)\langle \phi_{WS} \rangle \sqrt{2}$$  \hspace{1cm} (11)

The pole mass of a quark is given to leading order in the strong coupling by:

$$M = m(M) \left(1 + \frac{4}{3} \frac{\alpha_s(M)}{\pi} \right)$$  \hspace{1cm} (12)

In table 1 we show typically quoted values for the SM fermion Yukawa couplings at 1 GeV and the corresponding Yukawa couplings at the Planck scale evolved using the 1-loop RGEs for the SM (see e.g. [5]). There is some ambiguity, particularly for the light quarks, in extracting these Yukawa couplings from experiment. We shall consider alternative smaller values for the $u$, $d$ and $s$ quark masses, suggested by recent lattice calculations [6], in section 9.

In table 2 we show the magnitudes of the 3 mixing angles (without the CP phase) at 1 GeV and the Planck scale. We can see that $V_{us}$ does not change significantly, but the other 2 mixing angles are different at the two scales. We find that the running is independent of the CP phase chosen. We also find, in agreement with [5], that the ratio of $V_{cb}$ to $V_{ub}$ is approximately constant. The difference at the two scales is only sensitive to the top quark mass.

If we now look for order of magnitude equalities at the Planck scale we can see that, with the exception of the top and probably also the charm quarks, the Yukawa couplings of the fermions within each generation are order of magnitudewise degenerate. We could explain the down-type quarks and the leptons
Table 2: Experimental values of the mixing angles at 1 GeV and the Planck scale.

| Mixing angle | V(1 GeV) | V(M_{Planck}) | ln(V(M_{Planck})) |
|--------------|---------|---------------|-------------------|
| V_{us}       | 0.22    | 0.22          | -1.5              |
| V_{cb}       | 0.041   | 0.049         | -3.0              |
| V_{ub}       | 0.0035  | 0.0042        | -5.5              |

having degenerate masses in a grand-unified model such as the well known SU(5) model. However, in reality, the degeneracy is only true in an order of magnitude approximation, whereas we would expect exact equality at some scale in such a model.\(^1\) Another feature, which indicates that the degeneracies are not due to unification, is that the up quark is approximately degenerate with the down quark and the electron but the charm and, most obviously, the top quarks do not fall into the same pattern. Obviously we could say that the up quark is the exception and it is simply chance that it has the same order of magnitude mass as the down quark and the electron, but this is not entirely satisfactory. For example, if the up-type quarks get masses by a different mechanism (e.g. supersymmetry with a large tan\(\beta\) which could produce the observed top and bottom quark masses even if they actually had similar Yukawa couplings), then it would seem to be natural to expect that the up quark would be heavier than the down quark and the electron, just as the charm and top quarks are heavier than the other fermions in their generations. Therefore we take the view that there is an approximate mass degeneracy between charged fermions within each generation, which is certainly true for the lightest generation. However, in the other two generations, the charm and top quarks are exceptions to this rule. This means that the charm and top quarks must get their masses by a different mechanism from the other fermions.

4 A Natural Explanation

If we consider the hypothetical situation where the charged fermions within each generation have the same masses, then it would be natural to conclude that the masses were generated by some mechanism which didn’t distinguish between each type of fermion, but did distinguish between fermions in different generations. This would indicate that, in some model more fundamental than the SM, there should be a distinction between the 3 generations. We shall refer to these distinct “generations” in the fundamental theory as “proto-generations”. We cannot really claim that the SM fermion masses could be described by an

\(^1\)It is of course possible to avoid the predicted degeneracy by introducing a non-minimal Higgs structure.\(^2\)
approximation to such a model, since the charm and top quarks are clearly not even order of magnitudewise degenerate with the other fermions in their generations. However, we shall show that the SM fermion masses can be described by a very similar type of model.

An important point to consider is that the fermion masses actually tell us only what the eigenvalues of the mass matrices are, they do not specify the complete matrices. It seems reasonable to assume that in a model where each generation is distinguished (e.g. by having different charges under some new gauge interaction), the dominant elements in the mass matrices which would lead to the fermion masses could easily be elements on the diagonal, i.e. the transitions between left- and right-handed fermions within the same generation. Now we could explain approximate degeneracy of fermions within each generation by a model which produced mass matrices with the same order of magnitude diagonal elements. The important point is that we need not require the complete matrices to be similar, as long as the off-diagonal elements make no significant contributions to the eigenvalues.

Now we can explain why we have the exceptions of the top and charm quarks. To first approximation the largest eigenvalue of a matrix is simply the largest element. So in the down-type and charged lepton mass matrices, $M_D$ and $M_E$, the largest element is the one corresponding to the interaction between the left- and right-handed fermions in the 3rd proto-generation. In the up-type mass matrix, $M_U$, it happens that our assumption about diagonal elements being dominant was wrong. This is not totally surprising, since we don’t really have any good reason to make such an assumption without knowledge of the fundamental mass-generating mechanism. Since we know that the left-handed top and bottom quarks are in the same SU(2) doublet in the SM, this means that the right-handed top quark is not really the fermion in the 3rd proto-generation of this model. The right-handed top quark must therefore belong to the 1st or 2nd proto-generation. This makes no difference in the SM since there is no distinction between the generations, but in such extended models a distinction does exist. By identifying the right-handed top quark with the 2nd proto-generation and the right-handed charm quark with the 3rd proto-generation, we have a simple mechanism to explain why their masses differ from the other fermions within their generations.

So now we have some important criteria for a model of the fermion masses. We have made the assumption that the SM is valid up to some high scale, such as the Planck scale. We have shown that there are several order of magnitude degeneracies when the SM fermion Yukawa couplings are compared at this scale. This has led to the requirement that the generations should be distinguished in the fundamental model, and that the fermions within each of these proto-generations should have order of magnitude degenerate masses. However, the interactions between generations must distinguish between the charged leptons and the 2 different types of quarks, so that the top and charm quark masses can be explained as originating from interactions between left- and right-handed
fermions from different proto-generations (most simply being due to the fact that the right-handed top quark of the SM actually belongs to the 2nd proto-generation and the right-handed charm quark to the 3rd proto-generation). Note that this mechanism would not work if, for example, only the top quark mass deviated from the “rule” of degeneracy within generations.

5 The anti-grand unification model

We will now propose a model which has all the features proposed in the previous section. We will first describe the model and then show that it does in fact satisfy all our requirements. The model we are considering is called the anti-grand unification (AGUT) model. It has previously been considered as a candidate for explaining the fermion masses [9]. The gauge group for the model is:

\[ G = \text{SMG}_1 \otimes \text{SMG}_2 \otimes \text{SMG}_3 \otimes U(1)_f \]  

where we have defined:

\[ \text{SMG}_i = \text{SU}(3)_i \otimes \text{SU}(2)_i \otimes U(1)_i \]  

The three \( \text{SMG}_i \) groups will be broken down to their diagonal subgroup, which is the gauge group of the SM. The \( U(1)_f \) group will be totally broken. So rather than unifying the simple factors of the SMG, the AGUT model “splits” them into 3 copies of each (hence the name of the model). This gauge group (strictly speaking without the \( U(1)_f \) group) has been used to successfully predict the values of the running gauge coupling constants in the SM at the Planck scale [10], as critical couplings estimated using lattice gauge theory.

We put the SM fermions into this group \( G \) in an obvious way. We have one generation of fermions coupling to each \( \text{SMG}_i \) in exactly the same way as they would couple to the SMG in the SM. The broken chiral gauge quantum numbers of the quarks and leptons under the symmetry groups \( \text{SMG}_i \) can readily explain the mass differences between fermion generations but cannot explain all the mass splittings within each generation, such as the ratio of the top and bottom quark masses. It is for this reason that the Abelian flavour group \( U(1)_f \) is introduced. We then choose \( U(1)_f \) charges with the constraint that there should be no anomalies and no new mass-protected fermions. This leads us almost uniquely to the set of charges \( y_f \) shown in table 3. We have labelled the fermions coupling to \( \text{SMG}_i \) by the names of the ‘i’th generation of SM fermions. However, this is just a method of labelling the representations of the full gauge group and, as we have discussed in section 4, we expect that, for example, the fermion we have labelled \( c_R \) will in fact turn out to be the right-handed top quark in the SM.

Although the quantum numbers of the fermion fields are determined by the theoretical structure of the model (in particular the requirement of anomaly
Table 3: \(U(1)_f\) charges of the SM fermions.

| Fermion | \(y_f\) | Fermion | \(y_f\) | Fermion | \(y_f\) |
|---------|---------|---------|---------|---------|---------|
| \(u_L\) | 0       | \(c_L\) | 0       | \(t_L\) | 0       |
| \(u_R\) | 0       | \(c_R\) | 1       | \(t_R\) | -1      |
| \(d_R\) | 0       | \(s_R\) | -1      | \(b_R\) | 1       |
| \(e_L\) | 0       | \(\mu_L\) | 0     | \(\tau_L\) | 0   |
| \(e_R\) | 0       | \(\mu_R\) | -1    | \(\tau_R\) | 1   |

cancellation), we do have some freedom in the choice of the quantum numbers of the Higgs fields. So, to model the fermion masses in the SM, we must choose how to break \(G\) down to \(\text{SMG}\).

A crucial simplification can be made by considering only the Abelian charges, when formulating a model for symmetry breaking. We can justify this by noting that all fermions are in singlet or fundamental representations of the non-Abelian groups. In the SM the fermions obey a charge quantisation rule:

\[
y \frac{1}{2} + \frac{1}{2} \text{"duality"} + \frac{1}{3} \text{"triality"} \equiv 0 \pmod{1} \tag{15}
\]

where \(y\) is the conventional weak hypercharge. Here “duality” is 1 for the fundamental representation of \(\text{SU}(2)\) (the doublet) and 0 for the singlet. Similarly “triality” is 1 for the \(\text{SU}(3)\) triplet, -1 for the anti-triplet and 0 for the singlet. Therefore we see that, if these are the only allowed non-Abelian representations, we can calculate the non-Abelian representations from the weak hypercharge. Similarly in the AGUT model we have such a charge quantisation for the weak hypercharge \(y_i\) associated with each \(\text{SMG}_i\) separately, so that the 4 Abelian charges completely specify the representation under the full AGUT gauge group. We assume that the Higgs scalar fields also satisfy the charge quantisation rule for each \(\text{SMG}_i\) and belong to singlet or fundamental representations of the non-Abelian groups.

As we discussed in section 4, we wish to have a model where the diagonal elements in the different mass matrices \(M_U\), \(M_D\) and \(M_E\) are the same, but off-diagonal elements should, in general, differ in different matrices. We can now show that this will always be the case in the AGUT model.

Let us define the \(U(1)\) charge vector:

\[
\vec{Q} = \left( \frac{y_1}{2}, \frac{y_2}{2}, \frac{y_3}{2}, y_f \right) \tag{16}
\]

Then the net charges of the Higgs fields, other than the WS Higgs field, in a transition from one left-handed fermion \(f_L\) to a right-handed fermion \(g_R\) of the same type, are given by:

\[
\Delta \vec{Q}_{fg} = \vec{Q}_{fL} - \vec{Q}_{gR} \pm \vec{Q}_{WS} \tag{17}
\]
where we have a plus sign for up-type quarks and a minus sign for down-type quarks and charged leptons. Mass matrix elements with the same value of $\Delta \vec{Q}_{fg}$ will be mediated by the same set of Higgs fields and suppressed by the same product of symmetry breaking parameters. To compare elements in the same place in different mass matrices we will define:

$$\Delta \vec{Q}_{Tij} = \Delta \vec{Q}_{T,T_j}$$

(18)

and to consider the diagonal elements we will simplify the notation further by defining:

$$\Delta \vec{Q}_{Ti} = \Delta \vec{Q}_{Tii}$$

(19)

where $T_i$ refers to the fermion in the 'i'th proto-generation of type $T$; e.g. $D_2$ is equivalent to $s$, the strange quark.

Since we can use the Higgs fields $W$ and $W^\dagger$ (which have opposite charges) equivalently in non-supersymmetric models, we will have the same order of magnitude diagonal elements if:

$$\Delta \vec{Q}_{Ui} = \pm \Delta \vec{Q}_{Di} = \pm \Delta \vec{Q}_{Ei}$$

(20)

for any combination of plus and minus signs (possibly dependent on $i$). From eq. (17) we can see that, if we choose both signs to be minus, the charges of the WS Higgs field $\vec{Q}_{WS}$ cancel out in the relations of eq. (20). We can then easily see from table 3 that the U(1)$_f$ charges allow:

$$\Delta \vec{Q}_{Ui} = -\Delta \vec{Q}_{Di} = -\Delta \vec{Q}_{Ei}$$

(21)

The charges $y_j$ for these diagonal elements are all 0 for $j \neq i$, so the only requirement left is that the charges $y_i$ satisfy eq. (21). This is true for the simple reason that these are the usual weak hypercharges of the fermions in the SM and the relation is just the condition that the fermions have the correct charges to couple to the same WS Higgs field. Therefore we have proved that the diagonal elements in the 3 mass matrices $M_U$, $M_D$ and $M_E$ are the same for each matrix, no matter what Higgs fields we choose or how we extend the quantum numbers of the WS Higgs field. Furthermore it is fairly obvious that, in general, the off-diagonal elements will not be the same in the different matrices. So the AGUT model has satisfied the requirements discussed in section 4.

6 Constructing a model of the Higgs fields for the 2nd and 3rd generations

There are many ways to approach the construction of a realistic model of the fermion masses. In section 4 we argued that there should be approximate degeneracy of masses between fermions within each generation, with the exception of
the top and charm quarks. We explained how this could be achieved naturally in a model where the diagonal elements of each mass matrix were of the same order of magnitude but, in general, other elements were different in each matrix. As we noted in section 5, this is precisely the case in the AGUT model. Of course there are other possible models where the mass matrices are of this form. However, one important point about the AGUT model is that we have not enforced such a condition; rather it was forced upon us by the requirement of anomaly cancellation, which determined the charges of the fermions. So, however we approach the construction of a definite model (by making a definite choice of the Higgs fields to break the AGUT gauge group down to the SM), there is always the possibility of natural order of magnitude degeneracies between the masses of different fermions within the same generation. Whether or not this actually occurs depends on whether or not the appropriate diagonal elements give the dominant contribution to the eigenvalue of the relevant matrices.

We have previously presented one method for constructing a realistic model of the fermion masses and mixing angles [1]. Here we will give an alternative method which leads to the same model. First we shall choose the quantum numbers of the WS Higgs field so that this is the only Higgs field required to produce the top quark mass and, thereby, the top quark Yukawa coupling is naturally of order one. As we have explained in section 4, in this type of model the dominant element in the up-type mass matrix cannot be a diagonal element, since then we could not predict that the top quark was much heavier than the bottom quark and tau lepton. So we choose the element corresponding to a transition between $t_L$ and $c_R$, as defined in section 5, to be dominant. Correspondingly the element which determines the charm quark mass corresponds to the transition between $c_L$ and $t_R$. So all we have really done is relabel the right-handed top and charm quarks which is simply a matter of definition (identification of fermions in our model with those of the SM), since they have the same gauge representations in the SM.

We could in fact choose the dominant element to be any of the 6 off-diagonal elements in the up-type mass matrix. However, interchanging $SMG_2$ and $SMG_3$ along with a change of sign of the $U(1)_f$ charges simply corresponds to relabelling the second and third proto-generations. This relates the 6 elements in pairs so that there are only really 3 distinct choices to be made. It turns out that, with our assumption that the first generation fermions are all order of magnitudewise degenerate in mass and that their masses are due to the same diagonal element in the three mass matrices, the 3 choices lead to equivalent models for the second and third generation masses. This is because, after making the appropriate choice of WS Higgs quantum numbers, the quantum numbers of the transitions among the second and third generations are the same in all 3 cases up to permutations of the $SMG_i$ and rescaling of the $U(1)_f$ charges. However, the quantum numbers in transitions involving the first generation are dependent on which of the 3 choices is made. These charges cannot be related to a different case by such a linear transformation of the
U(1) charges. Clearly the 3 choices correspond to the arbitrary labelling of the 3 proto-generations according to the U(1)$_f$ charges. There are 3 rather than 6 choices since changing the sign of the U(1)$_f$ charges simply corresponds to interchanging the 2nd and 3rd proto-generations. So, although we will only consider the choice made above (a dominant $t_L - c_R$ mass matrix element) in this paper, our conclusions about modelling the second and third generations will be independent of this choice.

To begin with we shall concentrate on the mass matrix elements responsible for the second and third generation masses. We have chosen the elements in the up-type mass matrix which will give the top and charm quark masses. In the down-type and lepton mass matrices we must choose the diagonal elements, if we want to obtain the order of magnitude degeneracies between the bottom quark and tau lepton masses, and between the strange quark and muon masses. Therefore, ignoring the elements involving the first generation, we have the following order of magnitude elements in the mass matrices:

\[
M_U \simeq \begin{pmatrix} m_s & m_c & m_t \\ m_t & m_b & m_b \end{pmatrix}
\]  
(22)

\[
M_D \simeq \begin{pmatrix} m_s & \text{?} & \text{?} \\ \text{?} & m_b V_{23} & \text{?} \\ \text{?} & \text{?} & m_b \end{pmatrix}
\]  
(23)

\[
M_E \simeq \begin{pmatrix} m_s & \text{?} \\ \text{?} & m_b \end{pmatrix}
\]  
(24)

where $m_{\mu} \simeq m_s$ and $m_{\tau} \simeq m_b$ and we have indicated the likely position of the element which leads to the mixing $V_{23}$ between second and third generation quarks.

Now we are in a position to choose specific Higgs fields which could lead to these types of mass matrices. We can do this by noting that there are relations between the quantum numbers of different elements. In particular we define:

\[
\vec{b} = \vec{Q}_{b_L} - \vec{Q}_{b_R} - \vec{Q}_{WS}
\]  
(25)

which are the total charges carried by the Higgs fields, other than the WS Higgs field, in the transition which leads to the element in the down-type mass matrix corresponding to the bottom quark mass. The U(1) charges of the WS Higgs field are:

\[
\vec{Q}_{WS} = \vec{Q}_{c_R} - \vec{Q}_{t_L} = \left( 0, \frac{2}{3}, 0, 1 \right) - \left( 0, \frac{1}{6}, 0, 0 \right) = \left( 0, \frac{2}{3}, -\frac{1}{6}, 1 \right)
\]  
(26)

We define similar quantities for other elements in the down-type and lepton mass matrices. In the up-type mass matrix we have a different interaction with the WS Higgs field, and so we define the total charges carried by the other Higgs fields in, for example, the transition responsible for the charm quark mass by:

\[
\vec{c} = \vec{Q}_{c_L} - \vec{Q}_{t_R} + \vec{Q}_{WS}
\]  
(27)
Here we have used the requirement that the charm quark mass should be due to the off-diagonal element corresponding to the transition between \(c_L\) and \(t_R\), rather than the diagonal element corresponding to the transition between \(c_L\) and \(c_R\). It is easy to verify the following relations between these charge vectors:

\[
\vec{b} + \vec{c} + \vec{s} = \vec{0} \tag{28}
\]
\[
\vec{Q}_{b23} + \vec{s} - \vec{b} = \vec{0} \tag{29}
\]

where \(\vec{Q}_{b23}\) is defined to be the charges corresponding to the element we expect to have magnitude \(m_bV_{23}\) in eq. (23). This means that these vectors form triangles in charge space. Thus we predict that (in units of the supposed unsuppressed top quark mass) the mass of the bottom, charm or strange quark is no less than the product of the other two masses. This is in fact true in nature and also for the other case, eq. (29), with \(m_b, m_s\) and \(m_bV_{23}\).

We now try to fit the effective SM Yukawa couplings in terms of as few Higgs fields as possible. In the case of only 1 Higgs field \(W\) the prediction from eq. (28) for \(h_s\), the smallest Yukawa coupling of the three, is:

\[
h_s \simeq h_bh_c \tag{30}
\]

To see this note that each Yukawa coupling is of the form

\[
h_X \sim \langle W \rangle^{|n_X|} \tag{31}
\]

where from eq. (28)

\[
n_b + n_c + n_s = 0 \tag{32}
\]

and so, since \(h_s < h_c < h_b\) implies \(|n_s| > |n_c| > |n_b|\):

\[
|n_s| = |n_b| + |n_c| \tag{33}
\]

Eq. (31) shows that eqs. (33) and (30) are equivalent. The VEV \(\langle W \rangle\) (in units of \(M_F\)) is the suppression factor generated each time that the Higgs field \(W\) has to be applied. Now eq. (30) is not true in nature (see table 1) and so we must use at least 2 Higgs fields.

Whenever we have relations such as eqs. (28) and (29) between three matrix element charge vectors, we imagine that we can re-use chains of Higgs fields (see fig. 1) able to exchange the quantum numbers necessary to give masses to, say, the \(b\) and \(c\) quarks to also give mass to the \(s\) quark. Naturally, there will be a piece of chain common, up to charge conjugation, for \(b\) and \(c\) not used by \(s\). We seek to symbolise this chain structure in figure 2: the connected single lines represent logarithms of the suppression factors for the masses of the \(b, c\) and \(s\) quarks. The lengths of the double lines represent the logarithms of the bunches of Higgs field suppression factors \(\frac{\langle H \rangle}{M_F}\) common for those two quark masses represented by the lines forming the double lines in question. A priori we
Figure 2: The magnitudes of the logarithms of the Planck scale Yukawa couplings for \( b \) (really \( \tau \)), \( c \) and \( s \) (really \( \mu \)) are represented by the total lengths of the corresponding single lines. The lengths of the double lines labelled by \( \Pi_i \) represent \( - \ln \Pi_i \).

We now assume that just 2 Higgs fields, with linearly independent quantum numbers, are needed to give the masses of the \( b \), \( c \) and \( s \) quarks and denote them by \( W \) and \( T \). Denoting the quantum number vectors of these fields \( W \) and \( T \) respectively, we must be able to write

\[
\vec{b} = n_b \vec{Q}_W + p_b \vec{Q}_T
\]

and similar expressions for \( \vec{c} \) and \( \vec{s} \). Then eq. (32) is satisfied as above and similarly

\[
p_b + p_c + p_s = 0
\]

So, for each Higgs field, we must have a relation like:

\[
|p_c| = |p_b| + |p_s|
\]

or a similar relation with the quark names permuted. It follows that the suppression factors associated with each Higgs field can be factorised in a similar
way to eq. (30), and the 3 Yukawa couplings can be split order of magnitudewise in the following way:

\[ h_b \simeq \Pi_1 \Pi_2 \]  
\[ h_c \simeq \Pi_1 \Pi_3 \]  
\[ h_s \simeq \Pi_2 \Pi_3 \]  

The \( \Pi_i \)s are products of Higgs suppression factors, i.e. they are of the form:

\[ \Pi_i = \langle W \rangle^{n_i} \langle T \rangle^{p_i} \]  

Here, and henceforth, we express the Higgs field VEVs (apart from \( \langle \phi_{WS} \rangle \)) in units of \( M_F \).

Similarly eq. (29) requires the splittings:

\[ h_b \simeq \Pi_4 \Pi_5 \]  
\[ h_s \simeq \Pi_4 \Pi_6 \]  
\[ h_b V_{23} \simeq \Pi_5 \Pi_6 \]  

In order to motivate our final model from the numerology, we choose to replace the quark Yukawa couplings \( h_b \) and \( h_s \) by the lepton Yukawa couplings \( h_\tau \) and \( h_\mu \), because the latter fit our model better. It is anyway our prediction that \( h_b \simeq h_\tau \) and \( h_s \simeq h_\mu \), since the quantum numbers satisfy:

\[ \vec{b} = \vec{\tau} \]  
\[ \vec{s} = \vec{\mu} \]  

From eqs. (37)-(39) or their illustration in figure 2, we see, by insertion of the Planck scale Yukawa couplings from table 1, that:

\[ \ln \Pi_1 \simeq \frac{1}{2}(\ln h_b + \ln h_c - \ln h_s) \]  
\[ \simeq \frac{1}{2}(\ln h_\tau + \ln h_c - \ln h_\mu) \]  
\[ \simeq -1.8 \]  

\[ \ln \Pi_2 \simeq \frac{1}{2}(\ln h_s + \ln h_c - \ln h_b) \]  
\[ \simeq \frac{1}{2}(\ln h_\mu + \ln h_c - \ln h_\tau) \]  
\[ \simeq -2.8 \]  

\[ \ln \Pi_3 \simeq \frac{1}{2}(\ln h_b + \ln h_c - \ln h_s) \]  
\[ \simeq \frac{1}{2}(\ln h_\mu + \ln h_c - \ln h_\tau) \]  
\[ \simeq -4.5 \]  

16
Figure 3: The magnitudes of the logarithms of the Planck scale Yukawa couplings for $b$, $s$ (really $\mu$) and $b \times V_{23}$ are represented by the total lengths of the corresponding single lines. The lengths of the double lines labelled by $\Pi_i$ represent $-\ln \Pi_i$.

Similarly, from eqs. (41)-(43) or figure 3, we get:

\[
\begin{align*}
\ln \Pi_4 & \simeq -2.1 \\
\ln \Pi_5 & \simeq -2.5 \\
\ln \Pi_6 & \simeq -5.3
\end{align*}
\] (55-57)

Interestingly we notice that we have a very good order of magnitude relation:

\[
\Pi_3 \simeq \Pi_1 \Pi_2
\] (58)

and so we could try to consider $\Pi_1$ and $\Pi_2$ as single Higgs field VEVs (in units of $M_F$):

\[
\begin{align*}
\langle W \rangle &= \Pi_1 \\
\langle T \rangle &= \Pi_2
\end{align*}
\] (59-60)

We also observe that $h_b \simeq h_\tau$ is, to a good numerical approximation, split in the same way in these 2 cases eqs. (57) and (58). So we make the identifications:

\[
\begin{align*}
\Pi_4 &= \Pi_1 \\
\Pi_5 &= \Pi_2
\end{align*}
\] (61-62)
This gives us the approximate values of:

\[ \langle W \rangle \simeq e^{-1.95} \simeq 0.14 \] (63)
\[ \langle T \rangle \simeq e^{-2.65} \simeq 0.07 \] (64)

It is also numerically suggested to express \( \Pi_6 \) in terms of the 2 Higgs field VEVs as:

\[ \Pi_6 = \langle T \rangle^2 \] (65)

We can now see that it is consistent to split \( h_s \) in 2 different ways, since we have:

\[ h_s = \langle W \rangle \langle T \rangle^2 = \left( \frac{\langle T \rangle}{\Pi_2} \right) \left( \frac{\langle W \rangle}{\Pi_3} \right) = \left( \frac{\langle T \rangle}{\Pi_4} \right) \left( \frac{\langle W \rangle}{\Pi_6} \right) \] (66)

So we have shown that it may be possible to fit the masses and mixing angle of the second and third generations using the two Higgs fields \( W \) and \( T \). In particular we make the order of magnitude predictions:

\[ h_\mu \simeq h_s \simeq \langle W \rangle \langle T \rangle^2 \] (67)
\[ h_c \simeq \langle W \rangle^2 \langle T \rangle \] (68)
\[ h_t \simeq h_b \simeq \langle W \rangle \langle T \rangle \] (69)
\[ V_{cb} \simeq V_{ts} \simeq V_{23} \simeq \frac{\langle T \rangle^3}{h_b} \simeq \frac{\langle T \rangle^2}{\langle W \rangle} \] (70)

Now we have determined all the \( \Pi_i \) and so we know the values of \( |n_i| \) and \( |p_i| \) defined in eq. (40). However, we don’t know the signs of \( n_i \) or \( p_i \). Essentially, if e.g. \( n_i < 0 \) then \( |n_i| \) hermitean conjugated Higgs fields \( W^\dagger \) are used for the corresponding element in the mass matrix rather than the Higgs fields \( W \) themselves. By definition we can choose the sign of the charges of \( W \) and \( T \) so that:

\[ \vec{b} = \vec{Q}_W + \vec{Q}_T \] (71)

Then we can use eq. (28), with our reasonable assumption that \( \vec{Q}_W \) and \( \vec{Q}_T \) are linearly independent, to see that the only consistent choice is that

\[ \vec{c} = -2\vec{Q}_W + \vec{Q}_T \] (72)
\[ \vec{s} = \vec{Q}_W - 2\vec{Q}_T \] (73)

7 Predictions from a generalised model

Clearly we can now calculate the charges of the Higgs fields \( W \) and \( T \) in the AGUT model. However, we shall first highlight the predictions that only depend on the following general features suggested by the AGUT model and the data as discussed in section 4:
(a) Left-handed up-type and down-type quarks have the same charges.

(b) The “diagonal” matrix elements in the up-type mass matrix involve the opposite charges to those in the corresponding elements of the down-type and lepton matrices. The “diagonal” elements are just meant to be diagonal after some appropriate permutation of the imagined left and right handed proto-quarks and leptons.

(c) The top quark mass has zero quantum numbers suppressing it and is “off-diagonal”.

Now the linear relations, eqs. (28) and (29), are in fact consequences of these assumptions (a), (b) and (c). Using the phenomenological arguments of the previous section, we take as our final assumption:

(d) The quantum numbers of the Higgs fields $W$ and $T$ should be related to those for the mass matrix elements by eqs. (71)-(73).

With the above 4 assumptions extracted from the AGUT model and phenomenology, we obtain the Yukawa coupling matrices in the form:

$$H_U \sim \left( \begin{array}{cc} W^\dagger T^2 & (W^\dagger)^2 T \\ 1 & W^\dagger T^1 \end{array} \right)$$

(74)

$$H_D \sim \left( \begin{array}{cc} W(T^\dagger)^2 & T^3 \\ W^2(T^\dagger)^4 & WT \end{array} \right)$$

(75)

$$H_E \sim \left( \begin{array}{cc} W(T^\dagger)^2 & ? \\ ? & WT \end{array} \right)$$

(76)

where the order of magnitudes of the elements are obtained by replacing the Higgs fields by their expectation values (in units of $M_F$). So only the 2 off-diagonal elements in the charged lepton matrix are model dependent. Since we now have all the matrix elements for the quarks we can check that we obtain the required masses and mixing angle, i.e. that no elements dominate the ones we expected to be relevant for calculating the eigenvalues.

The largest eigenvalue in a $2 \times 2$ matrix is approximately the largest element and the other eigenvalue is approximately the determinant divided by the largest element. So we can see that the up-type matrix has the form we expected, since no element is larger than 1 and $\langle W \rangle^2 \langle T \rangle \gg (\langle W \rangle \langle T \rangle)^2 (\langle W \rangle \langle T \rangle)$. In the down-type matrix we see that the element we wanted to be $h_3 V_{23}$ has the correct magnitude $\langle T \rangle^3$. We also see that the other off-diagonal element in $H_D$ is greatly suppressed and does not give any significant contribution to the eigenvalues.

As already discussed in section 4, the first generation $u$, $d$ and $e$ masses could be dominated by the extra diagonal elements in the 3 generation Yukawa coupling matrices $H_U$, $H_D$ and $H_E$.

It should be noted that when we consider a specific model such as the AGUT model, problems can arise. Most obviously the off-diagonal elements in $H_E$ could
dominate one or both of the required eigenvalues. In this case the model would not be suitable. Secondly, it might turn out that the charges of the Higgs fields $W$ and $T$ were not independent and then the elements would be expressible as different combinations of these fields. In this case the matrix element would correspond to the least suppressed combination and this may be different from the elements suggested above. Finally, a similar situation arises when elements involving the first generation are considered. Then it may be necessary (as it is in fact in the AGUT model) to introduce more Higgs fields, since all the elements may not be expressible in terms of $W$ and $T$ alone. If all the Higgs fields are not linearly independent then we again have the possibility that there are other less suppressed combinations. These last two situations do not necessarily spoil the model, but a careful check is required.

8 Specific choice of Higgs fields within the AGUT model

We are now ready to calculate the charges of the Higgs fields $W$ and $T$ in the AGUT model. This is quite simple using eqs. (71) and (72). We obtain:

$$\vec{Q}_W = \frac{1}{3}(\vec{b} - \vec{c}) = \left(0, -\frac{1}{2}, \frac{1}{2}, -\frac{4}{3}\right)$$

$$\vec{Q}_T = \vec{b} - \vec{Q}_W = \left(0, -\frac{1}{6}, \frac{1}{6}, -\frac{2}{3}\right)$$

(77) (78)

Obviously these charges are linearly independent and so the second potential problem, discussed at the end of the previous section, does not arise in this model. We can now calculate all the elements involving only the second and third generations. The relevant elements of the three Yukawa matrices are given by:

$$H_U \sim \begin{pmatrix} W^\dagger T^2 & (W^\dagger)^2 T \\ W^\dagger T^\dagger & 1 \end{pmatrix}$$

$$H_D \sim \begin{pmatrix} W(T^\dagger)^2 & T^3 \\ W^2(T^\dagger)^4 & WT \end{pmatrix}$$

$$H_E \sim \begin{pmatrix} W(T^\dagger)^2 & W^3(T^\dagger)^5 \\ (W^\dagger)^2 T^4 & WT \end{pmatrix}$$

(79) (80) (81)

Now it can clearly be seen that the off-diagonal elements in the charged lepton matrix are sufficiently suppressed, so that they do not make significant contributions to the eigenvalues. Thus, as required, the first potential problem discussed at the end of the previous section does not arise in this model. So we have shown that this model could give realistic fermion masses for the second
and third generations. We must now complete the model by determining the matrix elements involving the first generation.

We notice that the charges of $W$ and $T$ do not cover the 2 dimensional space of charges $\frac{y_f}{2}$ and $y_f$, since only even $y_f$ charges can be constructed with integer numbers of these Higgs fields. Therefore, since both $W$ and $T$ have $\frac{y_f}{2} = 0$, we will need at least 2 more Higgs fields to fully cover the 3 dimensional charge space required to break $G$ down to the SMG. We will now choose 2 more Higgs fields which, together with $W$ and $T$, will fully cover this space.

As a simple proposal, we may introduce a third Higgs field $\xi$ provided with charges corresponding to the quantum number differences between the left-handed quarks in the first and second generations. Then the $H_U$ and $H_D$ matrix elements in the row corresponding to the left-handed first generation quark become the same as those in the second generation row augmented by an extra factor $\xi$. The $V_{us}$ mixing matrix element is then expected to be dominated by the $H_U$ and $H_D$ elements in the first generation row, corresponding to those in the second row which dominate the $c$ and $s$ masses. In fact $V_{us}$ is readily seen to be given, in our type of model, by the ratio of these first to second row matrix elements and should thus be equal order of magnitudewise to the $\xi$ Higgs field suppression factor:

$$V_{us} \simeq \langle \xi \rangle \approx 0.22$$  \hspace{1cm} (82)

Similarly we expect:

$$V_{ub} \simeq \langle \xi \rangle V_{23} \simeq V_{us} V_{cb}$$  \hspace{1cm} (83)

The quantum numbers for such a $\xi$ field must of course be:

$$\vec{Q}_\xi = \vec{Q}_{d_L} - \vec{Q}_{s_L} = \left( \frac{1}{6}, 0, 0, 0 \right) - \left( 0, \frac{1}{6}, 0, 0 \right) = \left( \frac{1}{6}, -\frac{1}{6}, 0, 0 \right)$$  \hspace{1cm} (84)

We must now choose one more Higgs field to fully span the 3 dimensional space of charges. Otherwise the first generation would remain massless. In order to be consistent with the well-known phenomenological relation:

$$V_{us} \simeq \sqrt{\frac{m_d}{m_s}}$$  \hspace{1cm} (85)

which motivated the Fritzsch and many subsequent quark mass ansätze, we clearly want the first generation “diagonal” matrix elements to be suppressed, by a factor $\langle \xi \rangle^2$ relative to the Yukawa matrix element dominating the $s$ quark mass. With our proposal that the first generation row has a factor $\xi$ more than the second generation row, we must thus arrange for the transition matrix element from $s_L$ to $d_R$ to have the same factor $W(T^\dagger)^2\xi$ as that from $d_L$ to $s_R$. This can be achieved—numerically—by proposing a Higgs field $S$, having a suppression factor that is actually equal to unity and that can compensate for the difference in quantum numbers between these two hoped for equally suppressed matrix
elements. This will lead to 2 different but comparable mechanisms for the down quark mass. So we take:

\[ \langle S \rangle = 1 \quad (86) \]

and the charges of \( S \) are given by:

\[
\bar{Q}_S = [\bar{Q}_{s_L} - \bar{Q}_{d_R}] - [\bar{Q}_{d_L} - \bar{Q}_{s_R}]
= \left[ \left( \frac{1}{6}, 0, 0, 0 \right) - \left( -\frac{1}{3}, 0, 0, 0 \right) \right] - \left[ \left( \frac{1}{6}, 0, 0, 0 \right) - \left( 0, -\frac{1}{3}, 0, -1 \right) \right]
= \left( \frac{1}{6}, -\frac{1}{6}, 0, -1 \right) \quad (87)
\]

We note that the SM weak hypercharge vanishes

\[ y = y_1 + y_2 + y_3 = 0 \quad (88) \]

for the Higgs fields \( W, T, \xi \) and \( S \). This guarantees that the SMG is recovered as the diagonal subgroup of the \( SMG_i \) groups.

We can now calculate the suppression of all elements in the Yukawa matrices. However, we must first note that, since we have used 4 Higgs fields, we cannot uniquely resolve the charge differences between left-handed and right-handed fermions. There will be some combination of the 4 Higgs field charges which will result in vanishing charge differences. We must find the smallest combination of the 4 Higgs fields which results in a vanishing set of charges \( \bar{Q} = 0 \). To do this we note that all fermion U(1)\(_f\) charge differences are quantised as integers. However, the 3 Higgs fields \( W, T \) and \( \xi \) can only give integer U(1)\(_f\) charge differences which are even. Therefore we must have at least two \( S \) Higgs fields involved in the combination. Then we can find the simplest combination of the other 3 Higgs fields which, together with the two \( S \) fields, give net vanishing charge differences \( \bar{Q} = 0 \). This combination is:

\[ 2\bar{Q}_S - 2\bar{Q}_\xi - 9\bar{Q}_T + 3\bar{Q}_W = 0 \quad (89) \]

Since this involves such large powers of \( T \), there is usually no ambiguity in selecting the combination of Higgs fields which suppresses the transition the least. So finally we can write out the full Yukawa matrices for each type of fermion:

\[
H_U \sim \begin{pmatrix}
S^\dagger W^\dagger T^2 (\xi^\dagger)^2 & W^\dagger T^2 \xi & (W^\dagger)^2 T \xi \\
S^\dagger W^\dagger T^2 (\xi^\dagger)^3 & W^\dagger T^2 & (W^\dagger)^2 T \\
S^\dagger (\xi^\dagger)^3 & 1 & W^\dagger T^\dagger
\end{pmatrix} \quad (90)
\]

\[
H_D \sim \begin{pmatrix}
SW(T^\dagger)^2 \xi^2 & W(T^\dagger)^2 \xi & T^3 \xi \\
SW(T^\dagger)^2 \xi & W(T^\dagger)^2 & T^3 \\
SW^2(T^\dagger)^4 \xi & W^2(T^\dagger)^4 & WT
\end{pmatrix} \quad (91)
\]

\[
H_E \sim \begin{pmatrix}
SW(T^\dagger)^2 \xi^2 & W(T^\dagger)^2 (\xi^\dagger)^3 & (S^\dagger)^2 WT^4 \xi^\dagger \\
SW(T^\dagger)^2 \xi^5 & W(T^\dagger)^2 & (S^\dagger)^2 WT^4 \xi^2 \\
S^3 W(T^\dagger)^5 \xi^3 & (W^\dagger)^2 T^4 & WT
\end{pmatrix} \quad (92)
\]
Note that one of the elements in $H_E$ has been changed after the introduction of the Higgs fields $\xi$ and $S$, but this element is still so suppressed that it has practically no relevance for any of the charged lepton masses.

\section{Results}

Now we are able to choose specific values for the 3 VEVs $\langle W \rangle$, $\langle T \rangle$ and $\langle \xi \rangle$ and calculate the resulting masses and mixing angles. The overall mass scale for the fit is set by eqs. (5) and (6). In order to find the best possible fit we must use some function which measures how good a fit is. Since we are expecting an order of magnitude fit, this function should depend only on the ratios of the fitted masses to the experimentally determined masses. The obvious choice for such a function is:

$$\chi^2 = \sum \left[ \ln \left( \frac{m}{m_{\text{exp}}} \right) \right]^2$$

where $m$ are the fitted masses and mixing angles and $m_{\text{exp}}$ are the corresponding experimental values. The Yukawa matrices are calculated at the fundamental scale which we take to be the Planck scale. We use the first order renormalisation group equations (RGEs) for the SM to calculate the matrices at lower scales. Running masses are calculated in terms of the Yukawa couplings at 1 GeV. The only exception is the top quark, where the experimentally measured mass is the pole mass and this is what we quote. We present here the result of an updated fit, using the following values \cite{4} of the SM gauge coupling constants at the electroweak scale and their values extrapolated to the Planck scale via the RGEs:

\begin{align*}
U(1) : & \quad g_1(M_Z) = 0.462 \quad g_1(M_{\text{Planck}}) = 0.614 \\
SU(2) : & \quad g_2(M_Z) = 0.651 \quad g_2(M_{\text{Planck}}) = 0.504 \\
SU(3) : & \quad g_3(M_Z) = 1.22 \quad g_1(M_{\text{Planck}}) = 0.491
\end{align*}

We cannot simply use the 3 matrices given by eqs. (90)–(92) to calculate the masses and mixing angles, since only the order of magnitude of the elements is defined. This could result in accidental cancellations if we calculated the eigenvalues and eigenvectors using these values. Therefore we calculate statistically, by giving each element a random complex phase and then finding the masses and mixing angle. We repeat this several times and calculate the geometrical mean for each mass and mixing angle. In fact we also vary the magnitude of each element randomly, by multiplying by a factor chosen to be the exponential of a number picked from a Gaussian distribution with mean value 0 and standard deviation 1.

We then vary the 3 free parameters to find the best fit given by the $\chi^2$ function. We get the lowest value of $\chi^2$ for the VEVs:

$$\langle W \rangle = 0.179$$

23
Table 4: Best fit to conventional experimental data. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

|        | Fitted     | Experimental |
|--------|------------|--------------|
| $m_u$  | 3.6 MeV    | 4 MeV        |
| $m_d$  | 7.0 MeV    | 9 MeV        |
| $m_c$  | 0.87 MeV   | 0.5 MeV      |
| $m_s$  | 1.02 GeV   | 1.4 GeV      |
| $m_n$  | 400 MeV    | 200 MeV      |
| $m_p$  | 88 MeV     | 105 MeV      |
| $M_t$  | 192 GeV    | 180 GeV      |
| $m_b$  | 8.3 GeV    | 6.3 GeV      |
| $m_t$  | 1.27 GeV   | 1.78 GeV     |
| $V_{us}$ | 0.18      | 0.22         |
| $V_{cb}$ | 0.018     | 0.041        |
| $V_{ub}$ | 0.0039    | 0.0035       |

\[ \langle T \rangle = 0.071 \]
\[ \langle \xi \rangle = 0.099 \]

The fitted value of $\langle \xi \rangle$ is approximately a factor of two smaller than the estimate given in eq. (82). This is mainly because there are contributions to $V_{us}$ of the same order of magnitude from both $H_U$ and $H_D$. The result of the fit is shown in table 4. The experimental values are those given in table 1. This fit has a value of:

\[ \chi^2 = 1.87 \]

This is equivalent to fitting 9 degrees of freedom (9 masses + 3 mixing angles - 3 Higgs VEVs) to within a factor of $\exp(\sqrt{1.87/9}) \approx 1.58$ of the experimental value. This is better than would have been expected from an order of magnitude fit and should be compared with $\chi^2 = 3.7$ for the fit with only 7 degrees of freedom in 8.

We can also fit to different experimental values of the 3 light quark masses by using recent results from lattice QCD 8. Light quark masses extracted from lattice QCD seem to be consistently lower than the conventional phenomenological values 8 given in table 1. We take the following light quark masses as typical lattice values, extrapolated to 1 GeV using the RGEs:

\[ m_u \simeq 1.3 \text{ MeV} \]
\[ m_d \simeq 4.2 \text{ MeV} \]
\[ m_s \simeq 85 \text{ MeV} \]
Table 5: Best fit using alternative light quark masses extracted from lattice QCD. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

|       | Fitted     | Experimental |
|-------|------------|--------------|
| $m_u$ | 1.9 MeV   | 1.3 MeV      |
| $m_d$ | 3.7 MeV   | 4.2 MeV      |
| $m_e$ | 0.45 MeV  | 0.5 MeV      |
| $m_c$ | 0.53 GeV  | 1.4 GeV      |
| $m_s$ | 327 MeV   | 85 MeV       |
| $m_t$ | 75 MeV    | 105 MeV      |
| $M_t$ | 192 GeV   | 180 GeV      |
| $m_b$ | 6.4 GeV   | 6.3 GeV      |
| $m_{\tau}$ | 0.98 GeV | 1.78 GeV    |
| $V_{us}$ | 0.15     | 0.22        |
| $V_{cb}$ | 0.033    | 0.041       |
| $V_{ub}$ | 0.0054   | 0.0035      |

We can now vary the Higgs VEVs to give the best fit to this alternative data. The best fit is shown in table 5. The values of the Higgs VEVs are:

$$\langle W \rangle = 0.123$$
$$\langle T \rangle = 0.079$$
$$\langle \xi \rangle = 0.077$$

and this fit has a larger value of:

$$\chi^2 = 3.81$$

Comparing this fit to the one above, using conventional light quark masses, we can see that we have an improvement in the first generation masses, since the up and down quark masses were lowered towards the electron mass. The fit to $V_{cb}$ has also improved. However, the strange quark mass, which we always predicted too large, is even worse because it’s experimental mass has also been lowered. It may seem that now our assumption, that $h_s \simeq h_\mu$ at the fundamental scale, is not correct. Of course we must remember that everything should be taken order of magnitudewise and we can always ignore one borderline case such as this. However, it would be interesting to see if we could have produced a model without the order of magnitude degeneracy between the strange quark and and muon masses.
10 Suggestion for a better model of the alternative masses

At first it would appear to be impossible to produce a model without the order of magnitude degeneracy $h_s \simeq h_\mu$, while retaining the other order of magnitude degeneracies between the leptons and down-type quarks. Indeed, we would certainly not want to spoil the natural prediction of $h_b \simeq h_\tau$, which is well-known to be quite accurate. The relation between the down and electron masses may not appear to be so accurate. However, when we consider the lower value of the down quark mass extracted from lattice QCD, along with the hindsight realisation that we actually predict $h_d \simeq 2h_e$ due to the two competing combinations of elements giving the lowest eigenvalue, we actually have quite an accurate prediction which we would not want to spoil. As we noted in section 4 if one eigenvalue involves an off-diagonal term, then so must at least one other. Thus, it would appear that we cannot spoil the unwanted relation, $h_s \simeq h_\mu$, without spoiling another relation.

However, we can in fact do precisely this. Examining the Yukawa matrices given by eqs. (91) and (92), we see that there is one off-diagonal element which has the same order of magnitude in the down-type and charged lepton matrices. It turns out that this is not by chance and is actually a consequence of the element being in the same position as the unsuppressed element in the up-type matrix, which leads to the top quark mass. To see this in general, we use the notation of section 5 for the net charge differences supplied by the Higgs fields suppressing each element in the Yukawa matrices.

If the top mass is to be unsuppressed then by definition we have, for some $i$ and $j$:

$$\Delta \vec{Q}_{Uij} = \vec{Q}_{U_{iL}} - \vec{Q}_{U_{jR}} = \vec{Q}_{D_{iL}} - \vec{Q}_{U_{jR}} = \vec{0} \quad (108)$$

We now wish to show that, for the same fixed $i$ and $j$, this implies:

$$\Delta \vec{Q}_{Dij} = -\Delta \vec{Q}_{Eij} \quad (109)$$

where we know the sign is minus (either sign would lead to the order of magnitude equality) because of the specific example of eqs. (11) and (12). So now we can see that, using eq. (108):

$$\Delta \vec{Q}_{Dij} = 2\vec{Q}_{D_{iL}} - \vec{Q}_{D_{jR}} - \vec{Q}_{U_{jR}} \quad (110)$$

$$-\Delta \vec{Q}_{Eij} = -\vec{Q}_{D_{iL}} - \vec{Q}_{E_{iL}} + \vec{Q}_{U_{jR}} + \vec{Q}_{E_{jR}} \quad (111)$$

We can see from table 3 that all left-handed fields carry zero U(1) charge and so:

$$2\vec{Q}_{D_{iL}} = -\vec{Q}_{D_{iL}} - \vec{Q}_{E_{iL}} \quad (112)$$

since the only non-zero charges are the U(1)$_i$ charges, which are equal to the weak hypercharges in the SM. An alternative, perhaps more fundamental, reason
for this equality is that it corresponds to the absence of anomalies. In particular the fields $D_{iL}$ (which is the same as $U_{iL}$) and $E_{iL}$ are the only fields coupling to SU(2)$_i$. So eq. (112) is simply the condition for cancellation of anomalies associated with the triangle Feynman diagrams with two external SU(2)$_i$ gauge bosons and one U(1) gauge boson (with 4 independent choices of the U(1) gauge group).

Similarly the relation:

$$-\vec{Q}_{D_{jR}} - \vec{Q}_{U_{jR}} = \vec{Q}_{U_{jR}} + \vec{Q}_{E_{jR}}$$

(113)
can be seen to be true. Again this is due to anomaly cancellation, though not in such a simple manner as the cancellation for the left-handed fields. So we now see that eq. (109) is true. Therefore it may be possible to produce a model, where the bottom quark and tau lepton masses come from the element in the same off-diagonal position as the unsuppressed element in the up-type matrix. This would retain the good relation, $h_b \simeq h_\tau$, as well as the order of magnitude degeneracy of the 1st generation masses, but would not enforce the less desirable prediction of $h_s \simeq h_\mu$. This interesting scenario is currently being investigated and will not be further commented on in this paper.

11 CP violation

Another prediction, which can be made from a model of the mass matrices, is the amount of CP violation due to the CKM matrix. This depends on the complex phases in the matrix usually parameterised by a phase $\delta \[4\]$. However, there are different ways of parameterising the unitary CKM matrix, and so it is better to define a parameterisation-independent quantity which is a measure of the amount of CP violation. Such a definition is possible and corresponds to the areas of the "unitarity triangles" \[12\]. The 3 sides of the triangles are defined in the complex plane as $s_i = V_{ij}V_{ik}^*$, where $j \neq k$ are fixed and $i$ labels the 3 sides. The condition that the CKM matrix $V$ is unitary determines that:

$$s_1 + s_2 + s_3 = 0$$

(114)

and so these 3 lines in the complex plane form a triangle. Also, the areas of all the different triangles are the same. Therefore we can define the amount of CP violation, $J$, in terms of the area, $A$, of any of these triangles, e.g.:

$$J = 2A = |x_1y_2 - x_2y_1|$$

(115)

where $x_i$ ($y_i$) are the real (imaginary) components of $s_i$. In our model with the conventional values of the light quark masses, we find that the model predicts:

$$J \simeq 5.8 \times 10^{-6}$$

(116)
In the case where we use the lower lattice values for the light quark masses, we have the prediction:

\[ J \simeq 1.2 \times 10^{-5} \]  

(117)

These predicted values are both lower than the experimental determination [13]:

\[ J \simeq 2.0 \times 10^{-5} - 3.5 \times 10^{-5} \]  

(118)

The model predictions agree in order of magnitude though, for the case where we take the conventional values of the light quark masses, the prediction is clearly worse than any of the mass and mixing angle predictions. However, we should really even consider this worst case as sufficiently good agreement with experiment, since our prediction of the CP violation parameter \( J \) is expected to be less accurate than that for a typical mass or mixing angle. This is because our prediction for \( J \sim \frac{T^4 \xi^2 W^2}{m^2} \) involves 8 Higgs field VEVs, whereas a typical mass prediction involves rather of the order of 3 VEVs such as \( m_s \sim WT^2 \). As noted after eq. (100), with \( \chi^2 = 1.87 \), we expect to fit a typical mass within a (one “standard deviation”) factor of \( \exp(\sqrt{1.87/9}) \simeq 1.58 = \exp(0.46) \). If now the logarithm of the uncertainty factor for \( J \) is taken to be \( \frac{1}{2} \) times as big, then \( J \) is expected to be uncertain by a factor of \( \exp(\pm 0.46) = 1.58 \). So even the worst case, eq. (116), only deviates from the experimental value, eq. (118), by a factor of \( \log_{10} \frac{2.7 \times 10^{-5}}{1.2} = 1.3 \) “standard deviations”. Thus we conclude that our CP violation predictions agree with experiment, within the accuracy we can expect.

We will now show that the amount of CP violation in our model can be well estimated in terms of the fitted mixing angles. We will first consider the simplified case, where the up-type Yukawa matrix is approximately diagonal, and so all the quark mixing is due to off-diagonal elements in the down-type Yukawa matrix. Then the down-type matrix should be of the form:

\[ H_d \sim \begin{pmatrix} h_d & h_s V_{12} & z \\ x & h_s & h_b V_{23} \\ x & x & h_b \end{pmatrix} \]  

(119)

where \( x \) denotes element which are considered to be “sufficiently suppressed”. This means that the lower off-diagonal components should not be so large as to be relevant when calculating masses and mixing angles. Since the matrix diagonalised is \( H_d H_d^\dagger \), it can easily be seen that these elements will not be relevant unless they are considerably larger than the corresponding upper off-diagonal elements. This is because they get multiplied by smaller diagonal elements, since \( h_d \ll h_s \ll h_b \). As we will show:

\[ V_{us} \simeq V_{cd} \simeq V_{12} \]  

(120)

\[ V_{cb} \simeq V_{ts} \simeq V_{23} \]  

(121)

However, the element \( z \) determines \( V_{ub} \) but only contributes to \( V_{td} \). If \( z \) is bigger than or of the same order of magnitude as the product \( h_b V_{us} V_{cb} \), we
would expect that we should have:

\[ z \simeq h_b V_{ub} \simeq h_b V_{td} \]  \hspace{1cm} (122)

but if \( z \) were smaller than this value, we would still predict the correct order of magnitude of \( V_{td} \) through the ‘indirect’ mixing between all 3 generations as opposed to the ‘direct’ mixing due to \( z \), as we shall now explain.

Consider how to approximately diagonalise the matrix \( H_d H_d^\dagger \). If \( z \) is so small that there is essentially no ‘direct’ contribution to \( V_{td} \), then we can use a 2 stage process using unitary matrices of the form:

\[
U = \begin{pmatrix} U_{2\times2} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & V_{23} & 0 \\ 0 & 0 & V_{2\times2} \end{pmatrix}.
\]  \hspace{1cm} (123)

where, in the approximation of small mixing, we have the order of magnitude unitary matrices (suppressing all phases for convenience):

\[
U_{2\times2} \simeq \begin{pmatrix} 1 & V_{12} \\ V_{12} & 1 \end{pmatrix} \quad \text{and} \quad V_{2\times2} \simeq \begin{pmatrix} 1 & V_{23} \\ V_{23} & 1 \end{pmatrix}.
\]  \hspace{1cm} (124)

The order these matrices are applied is important. It can be seen that when \( z \) is small:

\[ U^\dagger V^\dagger Y_d Y_d^\dagger VU \simeq \text{diag}(y_d^2, y_s^2, y_b^2). \]  \hspace{1cm} (125)

If we had applied the matrices in the opposite order we would not have approximately diagonalised \( Y_d Y_d^\dagger \). Therefore the CKM matrix is given by the product \( VU \). With the approximations we have made above, this is:

\[ V_{CKM} \simeq VU \simeq \begin{pmatrix} 1 & V_{12} & 0 \\ V_{12} & 1 & V_{23} \\ V_{12} V_{23} & V_{23} & 1 \end{pmatrix}. \]  \hspace{1cm} (126)

So we see that we have the symmetrical relations:

\[
V_{cd} \simeq V_{us} \simeq V_{12} \hspace{1cm} (127)
\]

\[
V_{ts} \simeq V_{cb} \simeq V_{23} \hspace{1cm} (128)
\]

but the mixing between 1st and 3rd generations is given by:

\[
V_{ub} \simeq 0 \hspace{1cm} (129)
\]

\[
V_{td} \simeq V_{12} V_{23} \equiv V_{indirect}^{13} \hspace{1cm} (130)
\]

This is what we refer to as ‘indirect’ mixing between the 1st and 3rd generations.

Now we shall consider the case when there is also ‘direct’ mixing between the 1st and 3rd generations. This is the case where:

\[ z \simeq h_b V_{13}^\text{direct} \]  \hspace{1cm} (131)
In this case we can approximately diagonalise $H_d H_d^\dagger$ by using the unitary matrix:

$$W = \begin{pmatrix} 1 & 0 & V_{13}^{\text{direct}} \\ 0 & 1 & 0 \\ V_{13}^{\text{direct}} & 0 & 1 \end{pmatrix}$$

(132)

and then applying the matrices $V$ and $U$ as above. In the approximation of small mixing angles, this will only affect the mixing between 1st and 3rd generations. We will get:

$$V_{ub} \simeq V_{13}^{\text{direct}}$$

(133)

$$V_{td} \simeq V_{13}^{\text{direct}} + V_{13}^{\text{indirect}}$$

(134)

So we have found the relation:

$$V_{td} \simeq V_{ub} + V_{us} V_{cb}$$

(135)

In fact these 3 terms correspond to the 3 sides of one of the “unitarity triangles”. The ‘direct’ and ‘indirect’ terms arise from different matrix elements of $H_d$, which have been assigned independent random phases that are averaged over. This means that, for given values for the magnitudes of $V_{ub}$ and $V_{us} V_{cb}$, the angle $\theta$ between these 2 sides should be random. We can then estimate the amount of CP violation (which is two times the area of this triangle) to be:

$$J \simeq \frac{1}{2} V_{us} V_{cb} V_{ub}$$

(136)

where the $\frac{1}{2}$ is the geometric average of sin $\theta$.

Actually there is a contribution to the CKM matrix from the up-type Yukawa matrix in our model and thus the above considerations are not quite true, in the sense that we do not really have then that the phases of two of the sides in the unitarity triangle are independent. In fact the combination of the up-type 12-transition matrix element and the 23-element in the down-type matrix leads to a contribution to the CKM matrix, which in turn gives rise to contributions to the $V_{ub}$ and the $V_{12} V_{23}$ sides of the unitarity triangle. By a similar calculation as above, and with obvious notation, we have:

$$V_{td} \simeq V_{13}^{\text{direct}} + V_{12}^{\text{down}} V_{23}$$

(137)

$$V_{ub} \simeq V_{13}^{\text{direct}} + V_{12}^{\text{up}} V_{23}$$

(138)

$$V_{cb} \simeq V_{23}$$

(139)

$$V_{us} \simeq V_{12}^{\text{down}} + V_{12}^{\text{up}}$$

(140)

The phases are such that $V_{td}$, $V_{ub}$ and $V_{us} V_{cb}$ are the three sides of a unitarity triangle. It is now clear that there are three independent quantities relevant to the triangle: $V_{13}^{\text{direct}}$, $V_{12}^{\text{down}} V_{23}$ and $V_{12}^{\text{up}} V_{23}$. Furthermore, in our model
these three quantities are of the same order of magnitude. Therefore there is a permutation symmetry among the three sides of the unitarity triangle. With such a symmetry it is impossible for the angles to be flatly distributed, since each angle must have an expectation value of $\pi/3$ in order that their sum be $\pi$. However, we expect the distributions to be given by rather smooth functions which can naturally be expanded on $\cos \theta$, $\cos 2\theta$ etc. (where the angle is called $\theta$). A contribution of the form of $\cos(\text{"odd"}\theta)$ will not influence the average of a function of $\sin \theta$, such as e.g. $\log \sin \theta$. This is because sine and, thus, functions of sine take the same value for an angle and its supplementary angle, while $\cos(\text{"odd"}\theta)$ changes sign between angle and supplementary angle. So only from even cosines such as $\cos 2\theta$, which presumably already comes with a rather small coefficient, will we expect any modification of the average of the sine of the angle. To the accuracy to which this $\cos 2\theta$ and higher even cosines can be ignored we could, for the purpose of evaluating the geometric average for $\sin \theta$, equally well assume the totally flat distribution for $\theta$ used above. Thus we do not expect much deviation from the above estimate, eq. (136), of the CP-violation strength by the inclusion of the up-type mass matrix contribution. So we can still expect a rather good agreement of this estimate with the computer calculation.

We can now calculate our theoretical prediction of CP violation, given the fitted values of the mixing angles for the 2 different fits. For the fit to the conventional light quark masses we expect:

$$J \simeq \frac{1}{2} \times 0.18 \times 0.018 \times 0.0039 \simeq 6.3 \times 10^{-6}$$

and for the fit to the alternative light quark masses:

$$J \simeq \frac{1}{2} \times 0.15 \times 0.033 \times 0.0054 \simeq 1.3 \times 10^{-5}$$

We can see that the above predictions only deviate by about ten percent from the calculated values eqs. (116) and (117). This deviation is due to the fact that the up-type Yukawa matrix contributes to the mixing between 1st and 2nd generations. However, the theoretical prediction still agrees very well and is a useful estimate in models where most of the mixing is due to the down-type Yukawa matrix and the matrix elements have uncorrelated phases.

12 Neutrino masses and mixing angles

We have constructed a successful model of the observed fermion masses and mixing angles. It is interesting to investigate what predictions this model would give for neutrino masses and mixing angles. We expect the neutrinos to get a mass in this model by the usual see-saw mechanism \[2, 14\]. This occurs when
we treat the SM as a low energy effective theory. Then we can include the non-renormalisable interaction:

\[ \mathcal{L}_\nu = \frac{1}{M_F} \bar{LL}\Phi WS H_\nu C(LL\Phi WS)^T + \text{h.c.} \quad (143) \]

with notation as in eq. (4), where \( C \) is the antisymmetric charge conjugation matrix and \( H_\nu \) is the effective light neutrino Majorana coupling matrix. After electroweak symmetry breaking we obtain mass terms for the neutrinos:

\[ \mathcal{L}_{\nu \text{ mass}} = \bar{\nu}_L M_\nu C \nu_L^T + \text{h.c.} \quad (144) \]

where the Majorana mass matrix:

\[ M_\nu = H_\nu \frac{\langle \phi WS \rangle^2}{2M_F} \quad (145) \]

is necessarily symmetric. Such terms arise in our model in the same way as the Yukawa terms for the other fermions. The only difference is that the Feynman diagrams corresponding to fig. 1 involve 2 WS Higgs fields and the transitions are between \( \nu_i L \) and \( \nu_j^c R \).

The Majorana mass matrix \( M_\nu \) is, of course, a symmetric matrix but otherwise, in models of our type with approximately conserved chiral \( U(1) \) charges, the matrix elements are generally of different orders of magnitude. As pointed out in [3], there are two qualitatively different types of eigenstate that can arise when diagonalising a symmetric mass matrix with such a hierarchical structure. In the first case, a neutrino can dominantly combine with its own antineutrino to form a Majorana particle with small mixing angles. The second case occurs when a neutrino combines dominantly with an antineutrino, which is not the CP conjugate state, to form a 2-component massive neutrino. For example the bare \( \tau \)-neutrino might combine with the bare \( \mu \)-antineutrino. Such states automatically occur in pairs, with order of magnitudewise degenerate masses and maximal mixing. In the example given, the other member of the pair would be formed by combining the bare \( \mu \)-neutrino with the bare \( \tau \)-antineutrino state. In particular, this happens when the matrix element with the largest order of magnitude is off-diagonal.

We define the lepton mixing matrix analogously to the CKM matrix. So we find unitary matrices \( U_E \) and \( U_\nu \) such that:

\[ U_E^\dagger M_E M_E^\dagger U_E = \text{diag}\{m_e^2, m_\mu^2, m_\tau^2\} \quad (146) \]
\[ U_\nu^\dagger M_\nu M_\nu^\dagger U_\nu = \text{diag}\{m_e^2, m_\mu^2, m_\tau^2\} \quad (147) \]

Then the lepton mixing matrix is defined by:

\[ U = U_\nu^\dagger U_E \quad (148) \]
Table 6: Constraints on neutrino mass squared differences $\Delta m^2$ and mixing angles $\sin^2 2\theta$ from fits [18] to the solar and atmospheric neutrino data.

| Experiment       | Mixing         | $\Delta m^2$(eV$^2$)          | $\sin^2 2\theta$ |
|------------------|----------------|-------------------------------|------------------|
| Solar (Vacuum)   | $\nu_e - \nu_{\mu,\tau}$ | $10^{-11} - 10^{-10}$        | $> 0.7$          |
| Solar (MSW)      | $\nu_e - \nu_{\mu,\tau}$ | $4 \times 10^{-6} - 10^{-5}$ | $3 \times 10^{-3} - 10^{-2}$ |
| Atmospheric      | $\nu_{\mu} - \nu_{\tau}$  | $5 \times 10^{-3} - 3 \times 10^{-2}$ | $> 0.65$          |

By this mechanism we expect small neutrino masses, which may just be large enough to be observable (if the electron neutrino is one member of an almost degenerate mass neutrino pair and hence has a large mixing angle) as solar neutrino vacuum oscillations. Our choice of the Planck scale as the fundamental scale, $M_P = M_{Planck}$, would require the element responsible for the electron neutrino mass in the Majorana coupling matrix $H_\nu$ to be essentially unsuppressed (i.e. of order unity). In this case $m_{\nu_e} \sim \langle \phi_{WS} \rangle^2 \sim 3 \times 10^{-6}$ eV, as appropriate for “just-so” (energy-dependent) vacuum oscillations [15]. There is also some experimental evidence for an oscillation between muon and tau neutrinos. This would explain the reduction in the number of muon neutrinos compared to electron neutrinos, observed on the ground after cosmic ray interactions in the atmosphere.

There are 2 alternative mechanisms for solar neutrino oscillations. There is the obvious possibility of vacuum oscillations, i.e. oscillations between the sun and the earth [16]. In order to obtain an oscillation probability which depends on the energy of the solar neutrino, as appears necessary when comparing the data to the conventional solar model calculations [17, 18], the neutrinos reaching the earth must have undergone approximately one oscillation. The corresponding “just-so” neutrino mass squared difference is $\Delta m^2 \simeq 10^{-11} - 10^{-10}$ eV$^2$. The well-known alternative to the large mixing vacuum oscillation solution is the MSW effect [19]. This is essentially an enhancement of small mixing oscillations, due to electron neutrinos interacting with electrons within the sun. Current solar and atmospheric neutrino experiments [18] constrain the difference of neutrino masses squared and mixing angles as shown in table 6.

However, we note that results from recent solar model calculations [20] give a predicted $^8B$ neutrino flux which varies by more than a factor of two. If the $^8B$ flux is taken to be an adjustable parameter within this range, it seems possible to get an acceptable energy independent fit to all the solar neutrino data [21]. By an energy independent solution we mean that $\Delta m^2$ is sufficiently large that the $\nu_e \rightarrow \nu_\alpha$ oscillation occurs many times between the sun and the earth, and what is observed is the average probability of oscillation, which does not depend on the energy of the neutrino. Such an energy independent large mixing vacuum oscillation solution to the solar neutrino problem corresponds
to a mass squared difference $10^{-2} \text{eV}^2 > \Delta m^2 > 10^{-10} \text{eV}^2$. The upper limit is provided by reactor oscillation experiments [18].

We can calculate the neutrino mass matrix in our model, if we assume that there are no new Higgs fields which were not involved in the other fermion mass matrices. In terms of the Higgs fields already introduced we find:

$$H_\nu \sim \begin{pmatrix}
(S^\dagger)^2(W^\dagger)^2T^4(\xi^\dagger)^4 & (S^\dagger)^2(W^\dagger)^2T^4\xi^\dagger & (W^\dagger)^2T(\xi^\dagger)^3 \\
(S^\dagger)^2(W^\dagger)^2T^4\xi^\dagger & W(T^\dagger)^5 & (W^\dagger)^2T \\
(W^\dagger)^2T(\xi^\dagger)^3 & (W^\dagger)^2T & S^2(W^\dagger)^2(T^\dagger)^2(\xi^\dagger)^2
\end{pmatrix}$$

(149)

where, as usual, we assume that all fundamental Yukawa couplings are of order 1. Clearly all the elements of $H_\nu$ are suppressed. The largest element is off-diagonal and of order $\langle W \rangle^2 \langle T \rangle$. As emphasised above, the matrix is symmetric and so we obtain a pair of almost degenerate eigenvalues. We find, using the fitted values eqs. (97-99), the following eigenvalues for $H_\nu$:

$$h_{\nu_\mu} \simeq h_{\nu_\tau} \simeq \langle W \rangle^2 \langle T \rangle \simeq 2.3 \times 10^{-3}$$

(150)

$$h_{\nu_e} \simeq \langle W \rangle^2 \langle T \rangle^4 \langle \xi \rangle^4 \simeq 7.8 \times 10^{-11}$$

(151)

Since a pair of off-diagonal elements dominate the matrix, there is almost maximal mixing ($\sin^2 2\theta = 1$) between $\nu_\mu$ and $\nu_\tau$ with very small mixing ($\sin^2 2\theta \simeq \langle \xi \rangle^6 \simeq 10^{-6}$) of $\nu_e$. This is not suitable for observable solar neutrino oscillations. However, we do have the correct mixing structure for atmospheric neutrino oscillations. The problem here is that this would require the difference in masses squared of $\nu_\mu$ and $\nu_\tau$ to be:

$$\Delta m^2_{\mu\tau} = |m_{\nu_\tau}^2 - m_{\nu_\mu}^2| = \Delta m^2_{\text{exp}} \simeq 10^{-2} \text{eV}^2$$

(152)

In our model the main contribution to the mass difference, between the quasi-degenerate mass eigenstates $\nu_\tau$ and $\nu_\mu$, comes from the largest diagonal element in the $2 \times 2$ submatrix of $H_\nu$ containing the dominant off-diagonal elements. From eq. (149), we see that this element is $(H_\nu)_{33}$ and hence:

$$\frac{\Delta m^2_{\mu\tau}}{m_{\nu_\tau}} \simeq \frac{(H_\nu)_{33}}{h_{\nu_\tau}} \simeq \langle T \rangle \langle \xi \rangle^2 \simeq 7 \times 10^{-4}$$

(153)

Unfortunately our scale is totally wrong, since with the natural choice of the Planck scale as the fundamental scale we predict:

$$m_{\nu_\mu} \simeq m_{\nu_\tau} \simeq h_{\nu_\tau} \frac{\langle \phi W S \rangle^2}{2M_F} \simeq 7 \times 10^{-9} \text{eV}, \quad m_{\nu_e} \simeq 2 \times 10^{-16} \text{eV}$$

(154)

and

$$\Delta m^2_{\mu\tau} = 2\frac{\Delta m^2_{\mu\tau}}{m_{\nu_\tau}} m_{\nu_\tau} \simeq 7 \times 10^{-20} \text{eV}^2$$

(155)
Hence if this model is to generate observable mixing it will be necessary to introduce a new mass scale ($M_F \sim 10^{11}$ GeV), which although both arbitrary and aesthetically unappealing is usually necessary in models of neutrino mass. Altering our interpretation of $M_F$ as the Planck scale would spoil the AGUT predictions of the gauge coupling constants in [10], and we are reluctant to do so. However, it should be noted that an $SMG \otimes U(1)^3$ model with the same U(1) charges as our AGUT model would be anomaly free. Such a model would have the same fermion mass matrix structure but would not suffer from the same objection to a lower $M_F$, as it would have no prediction for the values of the gauge coupling constants. This $SMG \otimes U(1)^3$ model does not appeal so strongly to us, precisely because it does not predict values for the gauge coupling constants and the choice of the U(1) charges seems rather arbitrary [1].

We may introduce an ad hoc new mass scale into the AGUT model and keep $M_F = M_{Planck}$, by adding another Higgs particle ($\Delta$ say) with an arbitrarily chosen VEV, which is a triplet under SU(2) in the SM. Then including the interaction:

$$L_{\nu} = \overline{T_L} \tau \cdot \Delta H_\nu C \mathcal{T}^T_L + \text{h.c.}$$

where $\tau$ are the Pauli spin matrices, leads to the neutrino mass matrix:

$$M_\nu = H_\nu \langle \Delta^0 \rangle$$

Here $\Delta^0$ is the neutral component of the Higgs triplet. The model can now give suitable masses and mixing for the atmospheric neutrino problem, by the assignment of the following quantum numbers, corresponding to the exchange of 2 WS Higgs fields, to $\Delta$:

$$\vec{Q}_\Delta = -2 \vec{Q}_{WS} = \left(0, -\frac{4}{3}, \frac{1}{3}, -2 \right)$$

It is taken to be a triplet under SU(2) in the $SMG_3$ group, but otherwise to belong to singlet or fundamental non-Abelian representations of the three $SMG_i$ groups. With this $\Delta$ we have the same prediction, eq. (149), for $H_\nu$, but with the choice of a new mass scale $\langle \Delta^0 \rangle \approx 1300$ eV we now have:

$$\Delta m^2_{\mu\tau} \approx 10^{-2} \text{ eV}^2$$

$$m_\nu_\mu \approx m_\nu_\tau \approx 3 \text{ eV}, \quad m_\nu_\tau \approx 10^{-7} \text{ eV}$$

So, as well as explaining the atmospheric neutrino problem, the neutrinos $\nu_\mu$ and $\nu_\tau$ would also be hot dark matter candidates. There is also some evidence for $\overline{\nu}_\mu \rightarrow \overline{\nu}_e$ neutrino oscillations from the LSND collaboration [22]; however, as we shall see, we would not predict any observable mixing there. The lepton mixing matrix U for our model is given by:

$$U \sim \begin{pmatrix} 1 & \langle \xi \rangle^3 & \langle T \rangle^3 \langle \xi \rangle \\ -\frac{\langle \xi \rangle^3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\langle T \rangle^3 \langle \xi \rangle}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \sim \begin{pmatrix} 1 & 10^{-3} & 4 \times 10^{-5} \\ -7 \times 10^{-4} & 0.71 & -0.71 \\ 7 \times 10^{-4} & 0.71 & 0.71 \end{pmatrix}$$
where we have used the fitted VEVs of eqs. (97-99) and ignored CP violating phases. Here $U_{l\alpha}$ denotes the mixing between flavour eigenstate $l$ and mass eigenstate $\alpha$. This leads to the bound:

$$P_{\nu_{\mu} \nu_e} \leq \sum_{\alpha,\beta} |U_{\nu_{\mu},\alpha} U_{\nu_e,\beta} U_{\nu_{\mu},\alpha} U_{\nu_e,\beta}|$$

$$\simeq 2 \times 10^{-6}$$

(162)

on the probability for $\nu_\mu \to \nu_e$ oscillations. This probability is too small to be observed by current experiments and, in particular, is incompatible with the LSND observation for which:

$$P_{\nu_{\mu} \nu_e} \simeq 10^{-3}, \quad \Delta m_{\nu_{\mu}}^2 \simeq 1 \text{ eV}^2$$

(163)

The above choice of quantum numbers for $\Delta$ is not the only possibility and, by choosing appropriate charges, it is in fact possible to explain the solar neutrino problem. For example the choice:

$$\bar{Q}_\Delta = \left(-\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{3}\right)$$

(164)

for the Abelian charges, and singlet or fundamental non-Abelian representations under the $SMG_i$ groups, leads to the mass matrix:

$$M_{\nu} \sim \langle \Delta^0 \rangle \begin{pmatrix} S(T_1)^3 W(\xi_1)^4 & S(T_1)^3 W_\xi & ST^3(W_1)^2 \xi_1^4 \\ S(T_1)^3 W_\xi & S(T_1)^3 W^2 \xi_1 & S^3T^3(W_1)^2 \xi^2 \\ ST^3(W_1)^2 \xi_1^4 & S^3T^3(W_1)^2 \xi^2 & S^3(W_1)^2 \end{pmatrix}$$

(165)

In this case the largest element $(M_{\nu})_{33}$ is diagonal and corresponds to the mass of $\nu_\tau$, while the masses of the lighter $\nu_\mu$ and $\nu_e$ states are quasi-degenerate corresponding to the off-diagonal elements $(M_{\nu})_{12} = (M_{\nu})_{21}$. For $(\Delta^0) \simeq 3 \text{ eV}$ we find that

$$\Delta m_{\nu_{\mu}}^2 \simeq 7 \times 10^{-11} \text{ eV}^2$$

$$m_{\nu_e} \simeq m_{\nu_\mu} \simeq 2 \times 10^{-5} \text{ eV} \quad m_{\nu_\tau} \simeq 10^{-1} \text{ eV}$$

(166)

(167)

which, since we have almost maximal mixing between $\nu_e$ and $\nu_\mu$, is suitable for the “just-so” vacuum oscillation solution to the solar neutrino problem given in table II.

As we noted earlier, the allowed range of values for $\sin^2 2\theta$ and $\Delta m^2$ from table II is for energy dependent solutions of the solar neutrino problem. However it is also possible to get energy independent solutions with maximal mixing and $\Delta m_{\nu_{\mu}}^2 > 10^{-10} \text{ eV}^2$. In particular this means that we can increase $(\Delta^0)$ in the above mass matrix, eq. (165) to give:

$$\langle \Delta^0 \rangle \simeq 100 \text{ eV}$$

$$\Delta m_{\nu_{\mu}}^2 \simeq 8 \times 10^{-8} \text{ eV}^2$$

$$m_{\nu_e} \simeq m_{\nu_\mu} \simeq 6 \times 10^{-4} \text{ eV} \quad m_{\nu_\tau} \simeq 3 \text{ eV}$$

(168)

(169)

(170)
Then we have an energy independent solution to the solar neutrino problem, with a sufficiently heavy $\nu_\tau$ to provide a candidate for hot dark matter. We conclude that it is possible to obtain observable neutrino oscillations in our model, but only at the expense of introducing a new mass scale. It is quite natural to obtain a pair of neutrinos with quasi-degenerate masses and a maximum mixing angle of $\theta = \pi/4$. Otherwise all mixing angles are expected to be small. This is true for a rather general class of models which generate the hierarchical structure of the effective low energy Majorana neutrino mass matrix by a similar mechanism to ours, using new approximately conserved chiral (gauge) charges. In particular we do not expect to obtain significant mixing between all three neutrinos.

13 Conclusions

We have described the development of our proposed system of Higgs fields, which generates a realistic quark-lepton mass spectrum and breaks the anti-grand unification (AGUT) gauge group $\text{SMG}^3 \otimes U(1)_f$ down to the SM gauge group SMG. The most important feature used in this development is that, in our AGUT model, the diagonal elements are suppressed to the same degree in each of the three charged fermion mass matrices $M_U$, $M_D$ and $M_E$. By using this property and consideration of possible values for the Abelian gauge quantum numbers, we were led to a scheme in which the Planck scale Yukawa couplings for the two heaviest generations were fitted in terms of two suppression factors $W$ and $T$. A couple of numerical coincidences supported this picture and we ended up with the suggested formulae, eqs. (67-70).

These expressions could arise from a general quantum number system, sharing a few properties with our AGUT model. The most important property is the rule that the same quantum number differences suppress the diagonal elements for a given generation in each of the three mass matrices. Also the top quark mass is assumed to be unsuppressed by the quantum numbers. Finally it has to be checked that the other matrix elements, assumed to be small in the phenomenologically suggested Yukawa coupling matrices in terms of $W$ and $T$, do not come out to be too large. It turned out that our AGUT model could indeed realize the general scheme, in terms of two Higgs fields vacuum expectation values $\langle W \rangle$ and $\langle T \rangle$ expressed in units of the fundamental (Planck) mass scale.

Then, more trivially, the model was extended to include the first generation; with all three first generation particles predicted to have order of magnitudewise equal masses. This was done by introducing two more Higgs fields, $\xi$ and $S$, the latter of which however gives no suppression, i.e. we took $\langle S \rangle = 1$. That really means we could have left out the Higgs field $S$ and replaced the AGUT gauge group $\text{SMG}^3 \otimes U(1)_f$ by the subgroup $\text{SMG}_{12} \otimes \text{SMG}_3 \otimes U(1)$, which survives the spontaneous breakdown due to $S$. We managed to naturally include
the well-known order of magnitude relation $V_{us} \simeq \sqrt{m_s/m_e}$. It also followed that the mass expected for the $d$ quark should be larger, by a factor of order 2, than its first order approximation of being degenerate with the $u$ quark and the electron at the Planck scale.

We were thereby led to the choice of quantum numbers given in eqs. (77, 78, 84, 87) for the Higgs fields suppressing the masses, and in eq. (26) for the Weinberg-Salam Higgs field of our model. So finally the charged fermion mass matrices are given, order of magnitudewise, by the Yukawa matrices $H_U, H_D$ and $H_E$ of eqs. (90-92).

A fit to the quark-lepton masses and mixing angles was made, using a computer program to insert and average over random complex numbers of order unity independently multiplying each of the matrix elements of $H_U, H_D$ and $H_E$. The worst deviations from experiment are the strange quark mass, predicted to be around 400 MeV or twice its conventional value, and a value of the mixing matrix element $V_{cb} \simeq 0.018$ around half its experimental value. Since our model only pretends to make order of magnitude predictions, even these worst cases should be considered as agreement with experiment. We also made a fit using a preliminary lattice QCD determination of the light quark masses. Since these lattice values are significantly smaller than the conventional values, and in particular we take the $s$ quark mass to be 85 MeV, the second fit is markedly worse; it doubled the average square logarithmic deviation between our predictions and the data. This result is a rather direct consequence of the simple prediction of our model that the order of magnitude of the muon mass and the strange quark mass should be the same at the Planck scale.

We have also estimated the amount of CP violation in our model. Since CP violation arises as a product of rather many of our suppression factors, we expect our prediction to be more uncertain than that for a typical mass or mixing angle. In fact we predict the Jarlskog invariant $J$ measuring CP violation to be a factor of 3 or 4 below the experimental value. With our expected uncertainty in the exponent, this corresponds to approximately one “standard deviation”.

Finally we considered the extension of our model to the neutrino sector. The natural see-saw mass scale is the Planck mass in our model, for which the only possible observable effect would be “just-so” solar neutrino vacuum oscillations provided the electron neutrino $\nu_e$ mixes strongly with $\nu_\mu$ or $\nu_\tau$. The symmetrical Majorana neutrino mass matrix, being hierarchical, can naturally give rise to pairs of quasi-degenerate neutrino mass eigenstates with maximum mixing. In our model, the effective light neutrino Majorana coupling matrix eq. (147) has this property, but with maximal mixing between $\nu_\mu$ and $\nu_\tau$, while $\nu_e$ becomes much lighter than and mixes very weakly with the other neutrinos. So we predict no observable neutrino oscillations, unless we modify our model and introduce another mass scale into the theory. We have discussed some ways of introducing a new mass scale into our model and given examples with observable phenomenological implications. These all give neutrino oscillation phenomenol-
ogy corresponding to a pair of neutrinos with quasi-degenerate masses and a mixing angle \( \theta = \frac{\pi}{4} \).

Apart from the neutrino sector, the AGUT model gives a good overall order of magnitude fit to the fermion masses and mixing angles.

Acknowledgements
DJS wishes to acknowledge The Royal Society for funding. HBN acknowledges funding from INTAS 93-3316, EF contract SC1 0340 (TSTS) and Cernfølge-forskning. CF acknowledges funding from INTAS 93-3316 and PPARC GR/J21231.

References
[1] C.D. Froggatt, H.B. Nielsen and D.J. Smith, Phys. Lett. B385 (1996) 150
[2] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B164 (1979) 114;
   C.D. Froggatt, Perspectives in Particle Physics ’94 eds. D. Klubučar, I.
   Picek and D. Tadić (World Scientific, 1995) p240
[3] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B147 (1979) 277
[4] Particle Data Group, Phys. Rev. D54 (1996) 1
[5] H. Arason, D.J. Castaño, B. Kesthelyi, S. Mikaelian, E.J. Piard, P.Ramond
   and B.D. Wright, Phys. Rev. D46 (1992) 3945
[6] R. Gupta and T. Bhattacharya, hep-lat/9605039;
   B.J. Gough et al, hep-ph /9610223
[7] M. Olechowski and S. Pokorski, Phys. Lett. B257 (1991) 388;
   P. Ramond, R.G. Roberts and G.G. Ross, Nucl. Phys. B406 (1993) 19
[8] H. Georgi and C. Jarlskog, Phys. Lett. B86 (1979) 297;
   J.A. Harvey, P. Ramond and D.B. Reiss, Phys. Lett. B92 (1980) 309
[9] C.D. Froggatt, G. Lowe and H.B. Nielsen, Nucl. Phys. B414 (1994) 579.
[10] D.L. Bennett, H.B. Nielsen and I. Picek, Phys. Lett. B208 (1988) 275;
    D.L. Bennett and H.B. Nielsen, Int. J. Mod. Phys. A9 (1994) 5155;
    L.V. Laperashvili, Yad. Fiz. 57 (1994) 501
[11] H. Fritzsch, Phys. Lett. B70 (1977) 436
[12] C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039
[13] S. Herrlich and U. Nierste, Phys. Rev. D52 (1995) 6505

39
[14] M. Gell-Mann, P. Ramond and R. Slansky, in: *Supergravity*, eds. P. van Nieuwenhuizen and D.Z. Freedman, (North Holland, New York, 1980) p. 315;
T. Yanagida, in: *Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe*, eds. A. Sawada and H. Sugawara, (KEK, Tsukuba-Gu, Ibaraki-ken, Japan, 1979) p. 95.

[15] S.L. Glashow and L.M. Krauss, Phys. Lett. **B190** (1987) 199.

[16] S.M. Bilenky and B.M. Pontecorvo, Phys. Rep. **41** (1978) 225;
V. Barger, K. Whisnant and R.J.N. Phillips, Phys. Rev. **D24** (1981) 538.

[17] J.N. Bahcall and M.H. Pinsonneault, Rev. Mod. Phys. **67** (1995) 781.

[18] D.L. Wark, *Proceedings of the International Europhysics Conference on High Energy Physics* (Brussels, 1995), eds. J. Lemonne, C. Vander Velde and F. Verbeure, (World Scientific, Singapore, 1996) p. 799.

[19] L. Wolfenstein, Phys. Rev. **D17** (1978) 2369;
S.P. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. **42** (1985) 913.

[20] J.N. Bahcall and M.H. Pinsonneault, **hep-ph/9610592** (1996);
A. Dar and G. Shaviv, Nucl. Phys. **B** (Proc. Suppl.) **38** (1995) 12.

[21] A. Acker and S. Pakvasa, **hep-ph/9611423** (1996);
P.I. Krastev and S.T. Petcov, **hep-ph/9612243** (1996);
P.F. Harrison, D.H. Perkins and W.G. Scott, **hep-ph/9702243**.

[22] C. Athanassopoulos et al., Phys. Rev. Lett. **75** (1995) 2650; ibid. **77** (1996) 3082;
J.E. Hill, Phys. Rev. Lett. **75** (1995) 2694.