Comparison of different coding schemes for 1-bit ADC

Fedor Ivanov
National Research University
Higher School of Economics
Moscow, Russia
Email: fivanov@hse.ru

Dmitry Osipov
National Research University
Higher School of Economics,
Institute for Information Transmission Problems
Moscow, Russia
Email: d_osipov@iitp.ru

Abstract—This paper devotes to comparison of different coding schemes (various constructions of Polar and LDPC codes, Product codes and BCH codes) for the case when information is transmitted over AWGN channel with quantization with lowest possible complexity and resolution: 1-bit. We examine performance (in terms of Frame-error-rate — FER) for schemes mentioned above and give some reasoning for results we obtained. Also we give some recommendations for choosing coding schemes for a given code rate and code length.

I. INTRODUCTION

In modern communication system each receiver antenna is generally equipped with a dedicated radio-frequency (RF) chain that includes complex, power-hungry analog-to-digital converters (ADCs). Even though it is known that the transmit power can be made inversely proportional to the number of antennas, the power dissipated by each ADC scales linearly with the sampling rate (baseband bandwidth) and exponentially with the number of quantization bits. High resolution ADCs also increase receiver complexity. Therefore as the number of antennas used by communication system grows the use of low resolution ADCs seems more and more appealing. The most extreme form of a low resolution ADC is a 1-bit ADC which has low implementation complexity since it includes a single comparator. Another important benefit of a 1-bit ADC is the fact that the analog stages of the RF chain (Automatic Gain Control (AGC), mixer, and analog filters) can be discarded or converted to digital parts. Thus 1-bit ADC is very promising for power and complexity-constrained communication systems such as sensor networks and IoT [1] and even more promising for MIMO systems especially due to the constant growing popularity of massive MIMO.

Although considerable amount of efforts has been spent recently to develop advanced signal processing techniques (e.g. channel estimation [2]–[5] and equalization [6]) for receivers with 1-bit ADCs little is known about error-correction coding for communication systems employing 1-bit ADCs. Most research papers on the topic published recently deal with MU MIMO and massive MIMO scenarios [7]–[10] and/or additional processing techniques such as oversampling [11], successive cancellation [9], channel-based precoding [7] or deep learning-based autoencoders [12], [13]. This paper concentrates on a more general and simple problem setting that is not restricted to certain class of communication systems or advanced signal processing techniques.

II. CHANNEL MODEL

Let us first describe channel model we use.

We assume that information vector \( c \in \{0, 1\}^n, n \in \mathbb{N} \) modulated by BPSK modulator (mapped to \( x = \{-1, +1\}^n \) by the rule: \( x_i = (-1)^{c_i} \)) is transmitted over channel with additive white Gaussian noise (AWGN). Thus receiver obtains:

\[
y = x + \eta,
\]

where \( \eta = (\eta_1, \eta_2, \ldots, \eta_n) \), and \( \eta_i \sim \mathcal{N}(0, \sigma^2) \), \( i = 1..n \) are pairwise independent and identically distributed (by normal distribution) random variables. Under \( \sigma^2 \) we denote noise variance. Thus each \( y \) is vector of independent random variables distributed either by \( \mathcal{N}(1, \sigma^2) \) (if \( x_i = 1 \)) or by \( \mathcal{N}(-1, \sigma^2) \) (if \( x_i = -1 \)).

We assume that received signal \( y \) is quantized with lowest possible 1-bit resolution (1-bit ADC). In this paper we deal with only symmetric case and thus quantization rule \( Q: y \mapsto q \) can be expressed as follows:

\[
g_i = \text{sign}(y_i),
\]

It is clear that composite channel AWGN + 1-bit symmetric ADC is equivalent to binary symmetric channel (BSC) with transition probability [14]:

\[
p = Q(\sqrt{SNR}),
\]

where \( Q(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt \) is the complementary Gaussian distribution function and SNR is a signal-noise ratio that can be calculated from noise variance \( \sigma^2 \) as

\[
SNR = -10 \log_{10} \sigma^2.
\]

Thus capacity of the channel we use is:

\[
C = 1 - h(p),
\]

where \( h(p) = -p \log_2 p - (1 - p) \log_2 (1 - p) \) is a binary entropy function.

Fig. 1 represents dependence between SNR in Gaussian channel and capacity of the AWGN+1-bit ADC channel.
In this section we give a brief description of different coding schemes. We further compare the performance of these codes (in terms of frame-error-rate FER versus SNR) in AWGN+1-bit ADC channel. As a codes-candidates we consider Polar codes, Low-Density Parity-Check (LDPC) codes, Bose-Chaudhuri-Hocquenghem Codes (BCH codes) and Product codes also known as Turbo Product Codes (TPC).

For different decoders of these codes we consider 2 alternative outputs of channel. If decoder operates with "hard" inputs (BCH codes, Product BCH codes) then we demodulate \(q_i\) and obtain bit values \(r_i\) as follows \(r_i = 1 - 2q_i\). If decoder (LDPC codes, Polar codes) works with soft information (log-likelihood ratios) obtained from the channel then input will be: \(r_i = q_i \ln \frac{1-p}{p}\), where \(p\) is the transition probability of BSC channel.

### A. Polar Codes

Hereinafter we use standard notation: for any vector \(v = (v_0, ..., v_{N-1})\) under \(v_i^t\) we understand \((v_i, v_{i+1}, ..., v_j)\).

Polar codes were proposed by Arikan in [19].

Consider a binary \((N, K)\) polar code specified by set \(I\) of information indexes, \(|I| = K, N = 2^n, n \in \mathbb{N}\) and the corresponding encoding procedure:

\[
x_0^{N-1} = d_0^{N-1} G_N, \tag{1}
\]

where \(d \in \{0,1\}^N\) and \(G_N\) is the generator matrix of order \(N\), defined as \(G_N = F^\otimes n\) with the Arikan’s standard polarizing kernel \(F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\) and \(\otimes n\) is \(n\)-th Kronecker product of \(F\).

Considering the information \(I\) and frozen set \(\mathcal{F} = I^c\), we may write [1] as

\[
x_0^{N-1} = d_I G_N(I) \oplus d_F G_N(\mathcal{F}).
\]

We consider the frozen bits as zeroes, \(d_F = 0\), and the information bits as the information to be encoded, \(d_I = u\).

It is clear that for fixed \(N\) and \(K\) polar code is completely defined by the set of it’s information bits \(I\). Information set plays crucial role in performance of polar code. There are different approaches to construct \(I\). In this paper we consider four of them:

- 5G NR Polar (5G Polar) code is defined for any \(1 \leq K < N \leq 1024\). This code has nested construction of information set. More details can be found in [15].
- Polarization weight (PW) Polar codes (PW Polar) [16] gives the reliability ordering as a function of their indices, i.e. this method is channel and SNR independent.
- Information set of Reed-Muller like Polar code (RM Polar) consists of \(K\) indexes that corresponds to \(K\) columns/rows of matrix \(G_N\) with \(K\) highest Hamming weights.
- Gaussian approximation (GA) technique [18] is another approach to construct set of information indexes \(I\). GA is used to evaluate the reliability of channels and then \(K\) channels with highest reliabilities are chosen as information set (GA Polar). GA uses a two-segment approximation function to cut-off the computational complexity of Density Evolution (DE) [17] when applied to binary input additive white Gaussian noise (AWGN) channels, but yielding almost the same precision.

Polar codes are usually decoded either by Successive Cancellation (SC) decoding [19] or by Successive Cancellation List Decoding (SCL) decoding [20] that maintains \(L\) concurrent decoding candidates.

### B. LDPC Codes

Low-density parity-check codes were proposed by R.G. Gallager (G-LDPC codes) in [21]. The performance of G-LDPC codes for the binary symmetric channel (BSC) were studied in [22].

It is convenient to specify LDPC codes using their Tanner graph representation [23]. The Tanner graph is a bipartite graph, where the nodes on the left side are associated with the codeword bits (variable nodes) and the nodes on the right are associated with the parity-check equations (check nodes). Any LDPC code can be described in terms of bipartite graphs that are characterized by two probability vectors

\[
\tilde{\lambda} = (\tilde{\lambda}_2, \ldots, \tilde{\lambda}_c),
\]

\[
\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_d),
\]

where \(\tilde{\lambda}_i\) is the fraction of check nodes with the degree \(i\), and \(\tilde{\rho}_l\) is the fraction of check nodes with the degree \(l\). For convenience we also define the polynomials

\[
\tilde{\lambda}(x) = \sum_{l=2}^c \tilde{\lambda}_l x^{l-1},
\]

\[
\tilde{\rho}(x) = \sum_{l=2}^d \tilde{\rho}_l x^{l-1}.
\]
There are different approaches to construct LDPC codes (i.e. to specify polynomials $\lambda(x)$ and $\tilde{\rho}(x)$). One group of methods is based on progressive edge growth algorithm (PEG). This algorithm aims to construct Tanner graphs with large girth by progressively establishing edges or connections between symbol and check nodes in an edge-by-edge manner.

Another approach starts from some small matrix $H_{core}$ of size $m_c \times n_c$ which is known as core matrix. This matrix can be found by exhaustive search or by applying Density Evolution [24] to find protograph with lowest possible decoding threshold. Then each non-zero element of core matrix is substituted by $m \times m$ matrix $P_{ij}$, $i = 1..m_c$, $j = 1..n_c$ (usually permutation matrices or circulants, zeros in $\delta D$. Product Codes and $C$ more detail. Consider two systematic linear block codes $C$ errors, where $d$ is the minimal distance of $C$.\(\forall x \in \mathbb{F}_q\)).\(\exists \tilde{\rho}(x)\) cycle spectrum (ACE spectrum) of resulted LDPC code. For instance ACE algorithm [25] is the cyclic code with generator polynomial that has $\delta$ - 1 consecutive roots $\alpha^{b}, \alpha^{b+1}, \ldots, \alpha^{b+k-2} \in \mathbb{F}_q^*$, where $\alpha$ is some element of $\mathbb{F}_q^*$. In this case $n$ is a multiplicative order of $\alpha$. Important special cases are $b = 1$ (called narrow-sense BCH codes), or $n = 2^m - 1$ (called primitive BCH codes). In this paper BCH code is assumed to be narrow-sense and primitive.

BCH codes are usually decoded by well-known Berlekamp-Massey decoder (BM) [26]. It is able to correct up to $d - \frac{1}{2}$ errors, where $d \geq \delta$ is a real minimal distance of BCH code.

D. Product Codes

The concept of product codes (or turbo product codes - TPC) is a simple and efficient method to construct powerful and long codes with a large minimum Hamming distance, $\delta$, using conventional linear block codes [27].

Let us now consider the product code construction in more detail. Consider two systematic linear block codes $C_A$ and $C_B$ having parameters $(n_A, k_A, d_A)$ and $(n_B, k_B, d_B)$ respectively. The product code $P = C_A \times C_B$ is obtained by placing $k_A k_B$ information bits in a matrix of $k_A$ rows and $k_B$ columns and encoding the $k_A$ rows and $k_B$ columns using codes $C_A$ and $C_B$ respectively. Furthermore, the parameters of the resulting product code $P$ are given by $n_P = n_A n_B$, $k_P = k_A k_B$, the code rate $R_P$ is given by $R_P = R_A R_B$ and minimal distance $\delta_P = d_A d_B$. Thus, it is possible to construct powerful product codes based on linear block codes such as BCH codes and other ones.

Turbo product codes usually decoded by iterative soft-input soft-output (SISO) decoders like As indicated by Elias [27], a TPC can be decoded by sequentially decoding it component codes in order to reduce decoding complexity.

Unfortunately in the case of 1-bit ADC it is impossible to obtain reliable soft outputs (LLRs) from the channel and thus the only way to decode TPC is to apply hard-input decoders. Moreover, it is known that such powerful codes like LDPC have excellent performance only in the case of lengths from several thousand and thus can not be used to construct TPC with moderate length. It is also known that product of two polar codes result in new polar code with not optimal frozen set (in terms of performance over channel with binary input) [28] and thus it would be better to consider long conventional polar code to obtain better performance. As a summary, in this paper we consider a BCH product codes with algebraic decoding of its components.

IV. Simulation Results

In this section we present a competitive comparison of code constructions described above. Let us first give a description of the simulation setup we provide. We also fix such code parameters as rate and length. After providing comparison we discuss these results both to explain them and choose most promising code candidates for AWGN + 1-bit ADC transmission scenario.

A. Simulation Setup $N \approx 1024$

We consider the following code parameters:

- Code length: $N \approx 1024$ (we can not fix one code length for all constructions since this length is unachievable for some codes (for instance for product BCH codes). If some code can not achieve exact code length then we choose the closest possible code length.
- Code rates: we consider a set of code rates: $R \in \{0.5, 0.625, 0.75, 0.8125, 0.875, 0.9375\}$. If some code can not achieve exact code rate (for instance BCH codes does not exist for any rate) then we choose the closest possible code rate.

We examine the following coding schemes:

- LDPC codes based on DE+ACE (density evolution serves to obtain small protograph with optimal threshold and then ACE is used to extend this protograph maximizing girth) and based on PEG (obtain optimal distribution for fixed degree of variable nodes $l = 3$ for rates $0.5, 0.625, 0.75, 0.8125$ and $l = 4$ for rates $0.875, 0.9375$). LDPC codes is decoded by Sum-Product decoding with 50 iterations.
- Polar codes constructed by different techniques: 5G NR, PW, GA, RM (see section II). These codes are decoded by SCL decoder with list size 32. CRC length is 16 for all cases.
- BCH codes with the following parameters: $(1023, 512), (1023, 638), (1023, 768), (1023, 828), (1023, 893)$ and $(1023, 953)$. These codes are decoded by Berlekamp-Massey decoder.
- Product codes constructed from extended BCH codes and primitive BCH codes. As a result we obtain the following parameters:

- $(32, 26) \times (32, 21) = (1024, 546) (R \approx 0.5)$
\[-(16, 11) \times (64, 57) = (1024, 627) \ (R \approx 0.625)\]
\[-(64, 51) \times (16, 15) = (1024, 765) \ (R \approx 0.75)\]
\[-(31, 26) \times (32, 31) = (992, 806) \ (R \approx 0.8125)\]
\[-(8, 7) \times (128, 127) = (1024, 889) \ (R \approx 0.875)\]
\[-(32, 31) \times (32, 31) = (1024, 961) \ (R \approx 0.9375)\]

These codes decoded iteratively with 10 iterations.

As it was mentioned above as a communication channel we consider BI-AWGN and symmetric 1-bit ADC in receiver side. Modulation scheme is BPSK.

**B. Comparison of Different Coding Schemes**

In this section we present the simulation results for different coding schemes described above. At each figure we compare the performance (in terms of Frame Error Rate - FER) versus SNR for different coding schemes for a fixed rate.

**Simulation results for** $N \approx 1024$, $R \approx 0.5$

Simulation results for $N \approx 1024$ and $R \approx 0.5$ are presented in Fig. 2. From this figure we can see that four coding schemes (Polar 5G NR, Polar PW, Polar GA — obtained from Density Evolution for $SNR = 2$ and LDPC codes designed from small protograph obtained by density evolution) give approximately the same simulation results. LDPC code that was optimized only in terms of girth in protograph (PEG LDPC) has worse performance due to the fact that only girth metric is adequate for pure AWGN channel without any quantization. Product codes (TPC) have even worse performance due the fact that short constituent codes are not able to correct enough fraction of errors for this transmission conditions. RM polar code has the worst performance.

**Simulation results for** $N \approx 1024$ and $R \approx 0.625$

Simulation results for $N \approx 1024$ and $R \approx 0.625$ are presented in Fig. 3. From this figure we can see that nearly all coding schemes (but TPC codes) have approximately the same performance. The slightly better performance have 5G and DE Polar codes but the difference between DE Polar and LDPC DE + ACE code is negligible. Only TPC codes have significantly worse performance than all other schemes because for this coding rate (and all higher rates) it is rather difficult to optimize inner and outer codes because these codes are algebraic (that limits space of their possible parameters)

**Simulation results for** $N \approx 1024$ and $R \approx 0.75$

Simulation results for $N \approx 1024$ and $R \approx 0.75$ are presented in Fig. 4. From this figure we can see that nearly all coding schemes (but TPC codes) have approximately the same performance. The slightly better performance have 5G and DE Polar codes but the difference between DE Polar and LDPC DE + ACE code is negligible. Only TPC codes have significantly worse performance than all other schemes because for this coding rate (and all higher rates) it is rather difficult to optimize inner and outer codes because these codes are algebraic (that limits space of their possible parameters).
and rather short. Speaking about the closeness other results to each other we can conclude that for this coding rate BCH codes significantly improve their performance and other constructions (methods of their designing) can still result in good performance.

Simulation results for \( N \approx 1024 \) and \( R \approx 0.8125 \) are presented in Fig. 5. From this figure we can see that as for \( R \approx 0.8125 \) nearly all coding schemes (but TPC codes) have approximately the same performance. But instead of the results for \( R \approx 0.75 \) now we have a new leaders: slightly better performance have 5G Polar codes, LDPC (DE + ACE extension) and BCH codes. Only TPC codes have significantly worse performance as in previous case.

Simulation results for \( N \approx 1024 \) and \( R \approx 0.9375 \) are presented in Fig. 7. From this figure we can see that codes can be now divided in four groups: best BCH code, RM polar and DE Polar, LDPC codes and other polar channel independent codes and the last and worst code is TPC. Instead of the results for \( R \in \{0.75, 0.8125, 0.875\} \) the difference between all coding schemes become significant. The best performance has BCH code — in this SNR range when number of errors is small and code rate is rather high, pure AWGN channel can be approximated by BSC with high accuracy. It is known that BCH codes are good and specially designed for pure BSC channel. And also these codes probably the best short codes with very high code rate. A bit worse results give DE Polar and DE + ACE LDPC — these two codes were also specially designed for a fixed high SNR values and thus their constructions are channel dependent. Channel independent PW Polar, RM Polar and PEG LDPC are also close to each other and have a bit worse performance than one for channel-dependent codes. Only TPC codes have significantly worse performance and can not been considered as code-candidates for 1-bit ADC AWGN channel.

C. Summary

As a summary observed by simulation results we obtained above, we can make the following conclusions:
• LDPC codes and Polar codes have comparable performance at least for short code lengths ($N \approx 1024$).
• The higher SNR the better performance of BCH codes. These codes should be used for ultra-high rate transmission scenario ($R \geq 0.8125$).
• The more unquantized AWGN channel is closer to BSC channel (when SNR tends to high values) the better performance of BCH codes and minimal distance becomes most important metric in terms of performance.
• The higher SNR the more crucial the channel-dependence of the code construction
• The best performance can be obtained by using either DE + ACE LDPC codes or GA Polar codes for moderate SNR and by BCH codes for high SNR

V. CONCLUSION

In this paper we compare the performance of different coding schemes for AWGN + 1-bit ADC channel for moderate length and different rates of codes. We gave some recommendations to choose the best code constructions for a given transmission scenario.

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