Global well-posedness and critical norm concentration for inhomogeneous biharmonic NLS

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Abstract
We consider the inhomogeneous biharmonic nonlinear Schrödinger (IBNLS) equation in $\mathbb{R}^N$,

$$i \partial_t u + \Delta^2 u - |x|^{-b}|u|^{2\sigma} u = 0,$$

where $\sigma > 0$ and $b > 0$. We first study the local well-posedness in $\dot{H}^{s_c} \cap \dot{H}^{2}$, for $N \geq 5$ and $0 < s_c < 2$, where $s_c = \frac{N}{2} - \frac{4-b}{2\sigma}$. Next, we established a Gagliardo-Nirenberg type inequality in order to obtain sufficient conditions for global existence of solutions in $\dot{H}^{s_c} \cap \dot{H}^{2}$ with $0 \leq s_c < 2$. Finally, we study the phenomenon of $L^{\sigma_c}$-norm concentration for finite time blow up solutions with bounded $\dot{H}^{s_c}$-norm, where $\sigma_c = \frac{2N\sigma}{4-b}$. Our main tool is the compact embedding of $\dot{L}^p \cap \dot{H}^2$ into a weighted $L^{2\sigma+2}$ space, which may be seen of independent interest.

Keywords  Biharmonic Schrödinger equation · Local well-posedness · Global well-posedness · Concentration

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1 Introduction

In this paper, we study the initial value problem (IVP) for the focusing inhomogeneous biharmonic nonlinear Schrödinger (IBNLS) equation

\[
\begin{aligned}
i \partial_t u + \Delta^2 u - |x|^{-b} |u|^{2\sigma} u &= 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(\cdot, 0) &= u_0,
\end{aligned}
\tag{1.1}
\]

where \(\sigma, b > 0\) and \(u = u(t, x)\) is a complex-valued function in space-time \(\mathbb{R}^N \times \mathbb{R}\). Here, \(\Delta^2\) stands for the biharmonic operator, that is, \(\Delta^2 u = \Delta (\Delta u)\). Equation in (1.1) may be seen as an inhomogeneous version of the fourth order NLS equation,

\[
i \partial_t u + \Delta^2 u - |u|^{2\sigma} u = 0,
\tag{1.2}
\]

in much the same way, the inhomogeneous nonlinear Schrödinger (INLS) equation

\[
i \partial_t u + \Delta u + |x|^{-b} |u|^{2\sigma} u = 0,
\tag{1.3}
\]

may be seen as an inhomogeneous version of the standard NLS equation. Equation (1.2) was introduced by Karpman [13] and Karpman-Shagalov [14] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with a Kerr nonlinearity.

Let us start by observing if \(u\) is a solution of (1.1) so is \(u_\lambda\) given by

\[
u_\lambda(x, t) = \lambda^{\frac{4-b}{2\sigma}} u(\lambda x, \lambda^4 t), \quad \lambda > 0.
\]

In addition, a straightforward computation gives

\[
\|u_\lambda(t)\|_{\dot{H}^s} = \lambda^{s - s_c} \|u(t)\|_{\dot{H}^s}.
\]

where \(s_c = \frac{N}{2} - \frac{4-b}{2\sigma}\) is the critical Sobolev index. If \(s_c = 0\) (or \(\sigma = \frac{4-b}{N}\)) the IVP (1.1) is known as mass-critical or \(L^2\)-critical; if \(s_c = 2\) (or \(\sigma = \frac{4-b}{N-4}\)) it is called energy-critical or \(\dot{H}^2\)-critical; finally, the problem is known as mass-supercritical and energy-subcritical (also called intercritical) if \(0 < s_c < 2\) (or \(\frac{4-b}{N} < \sigma < 4^*\)), where

\[
4^* = \left\{ \begin{array}{ll}
\frac{4-b}{N-4} & \text{if } N \geq 5, \\
\infty & \text{if } N = 1, 2, 3, 4.
\end{array} \right.
\]

For solutions in \(H^2\) it is not difficult to see that we have the conservation of mass \(M[u]\) and energy \(E[u]\) defined by

\[
M[u(t)] = \int |u(t)|^2 dx, \quad E[u] = \frac{1}{2} \int \Delta u(t)^2 + \frac{4-b}{2\sigma} |u(t)|^{2\sigma} |u(t)|^2 dx.
\]
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and  
\[ E[u(t)] = \frac{1}{2} \int |\Delta u(t)|^2 \, dx - \frac{1}{2\sigma + 2} \int |x|^{-b} |u(t)|^{2\sigma + 2} \, dx. \]  

(1.5)

Recently, the second and third authors in [10] studied the initial value problem (1.1). They established local well-posedness in \( H^2 \) for \( N \geq 3 \), \( 0 < b < \min\{\frac{N}{2}, 4\} \) and \( \min\{1-b\cdot N, 0\} < \sigma < 4^* \). Also, they proved global well-posedness in the mass-subcritical and mass-critical cases in \( H^2 \), that is, \( \min\{1-b\cdot N, 0\} < \sigma \leq \frac{4-b}{N} \). In the mass-supercritical and energy-subcritical cases, the authors showed the small data global existence under the same assumptions on \( b \) for dimensions \( N \geq 8 \) and \( N = 3, 4 \), where the local well-posedness results were obtained. The cases \( N = \{5, 6, 7\} \) were also studied however with extra restrictions on the parameters \( b \) and \( \sigma \). To be more precise we recall the local well-posedness result in \( H^2 \), which we will use below (see [10, Theorem 1.2]).

**Theorem A** Assume \( N \geq 3 \), \( 0 < b < \min\{\frac{N}{2}, 4\} \), and \( \max\{0, \frac{1-b}{N}\} < \sigma < 4^* \). If \( u_0 \in H^2 \), then there exists \( T = T(\| u_0 \|_{H^2}, N, \sigma, b) > 0 \) and a unique solution of (1.1) satisfying  
\[ u \in C \left( [−T, T]; H^2 \right) \cap \bigcap L^q \left( [−T, T]; H^{2, r} \right), \]

where \((q, r)\) is any \( B\)-admissible pair\(^1\).

In this paper we are interested in studying local/global well-posedness for (1.1) in \( \dot{H}^{s_c} \cap \dot{H}^2 \), with \( 0 \leq s_c < 2 \). Moreover, we study some dynamical properties of the blow-up solutions to (1.1) with initial data in \( \dot{H}^{s_c} \cap \dot{H}^2 \) for the intercritical regime, i.e., \( 0 < s_c < 2 \).

Our first result concerns the local well-posedness of (1.1) in \( \dot{H}^{s_c} \cap \dot{H}^2 \) with \( 0 \leq s_c < 2 \). We only consider \( 0 < s_c < 2 \) because the case \( \dot{H}^0 \cap \dot{H}^2 = H^2 \) corresponds to Theorem A.

**Theorem 1.1** Let \( N \geq 5 \), \( 0 < b < \min\{\frac{N}{2}, 4\} \) and \( \max\{\frac{4-b}{N}, \frac{1}{2}\} < \sigma < 4^* \). For any \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^2 \), there exists \( T = T(\| u_0 \|_{\dot{H}^{s_c} \cap \dot{H}^2}, N, \sigma, b) > 0 \) and a unique solution \( u \) of (1.1) satisfying  
\[ u \in C \left( [−T, T]; \dot{H}^{s_c} \cap \dot{H}^2 \right) \cap \bigcap L^q \left( [−T, T]; \dot{H}^{s_c, p} \cap \dot{H}^{2, r} \right) \cap \bigcap L^a \left( [−T, T]; L^r \right), \]

for any pairs \((q, p)\) \( B\)-admissible and \((a, r)\) \( \dot{H}^{s_c}\)-biharmonic admissible.

As in the \( H^2 \)-theory developed in [10], the proof of Theorem 1.1 relies on the fixed point argument combined with the Strichartz estimates related to the linear problem. In some sense our arguments extend to (1.1) the strategy presented in [4], where the authors studied the NLS equation (see also [9, Proposition 1.2]). However, here the additional restriction \( 1 < 2\sigma \) appears because in our argument we need to estimate \( \Delta(|x|^{-b}|u|^{2\sigma} u) \); moreover we use an auxiliary space \( L^a_t L^r_x \), which will be important.

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\(^1\) See Sect. 2 for the definitions of \( B\)-admissible and \( \dot{H}^{s_c}\)-biharmonic admissible pairs.
to define the metric space where we work with. It is worth mentioning that Theorem 1.1 only holds for \( N \geq 5 \); this condition appears in view of our nonlinear estimates. On the other hand, it holds in the defocusing case, that is, if we replace the sign “–” in front of the nonlinearity in (1.1) by “+”. We also point out that similar results as in Theorem 1.1 were established for (1.2) and the fractional NLS equation in [5] and [6], respectively.

In the sequel we will be concerned with global well-posedness results. To do that, we first prove a Gagliardo-Nirenberg type inequality and use it to establish sufficient conditions for global existence.

**Theorem 1.2** Let \( N \geq 1, 0 < b < \min\{N, 4\}, 0 < \sigma < 4^* \) and \( 2 \leq p < \frac{(2\sigma+2)N}{N-b} \), then for any \( u \in \dot{H}^2 \cap L^p \) we have

\[
\int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{2\sigma+2} \, dx \leq K_{opt} \|\Delta u\|_{L^2}^{\frac{2\sigma(u-1)p+2(2-s_p)}{2-s_p}} \|u\|_{L^p}^{\frac{2\sigma(2-s_p)}{2-s_p}}, \tag{1.6}
\]

where \( s_p = \frac{N}{2} - \frac{N}{p} \),

\[
K_{opt} = \left( \frac{(\sigma_c - s_p) + (2 - s_p)}{\sigma(2 - s_c)} \right)^{\frac{(4-b)(p-\sigma_c)}{2p(2-s_p)}} \left( \frac{(\sigma + 1)(2 - s_p)}{\sigma(s_c - s_p) + (2 - s_p)} \right)^{\frac{8\sigma - (p-2)(4-b)}{4-2s_p}} \|V\|_{L^p}^{-\frac{8\sigma - (p-2)(4-b)}{4-2s_p}} \tag{1.7}
\]

and \( V \) is a solution to the elliptic equation

\[
-\Delta^2 V + |x|^{-b}|V|^{2\sigma} V - |V|^{p-2} V = 0 \tag{1.8}
\]

with minimal \( L^p \)-norm.

To prove Theorem 1.2 we use a variational approach. We follow the strategy in [25], where the author established the optimal constant in the standard Gagliardo-Nirenberg inequality. In our case, the main tool used in the proof is the compact embedding of \( L^p \cap \dot{H}^2 \) into the weighted Lebesgue space \( L^{2\sigma+2}(|x|^{-b}dx) \) (see Sect. 4). In the limiting case \( b = 0 \) and \( p = 2 \) the best constant in Theorem 1.2 was already established in [8].

As an immediate consequence we obtain the following result, which correspond to the cases \( p = 2 \) and \( p = \sigma_c \) in Theorem 1.2.

**Corollary 1.3** Under the same assumptions of Theorem 1.2, we have

\[
(i) \int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{2\sigma+2} \, dx \leq K_{opt} \|\Delta u\|_{L^2}^{\frac{N\sigma+b}{4}} \|u\|_{L^2}^{\frac{4-b-\sigma(N-4)}{2}}, \text{ where}
\]

\[
K_{opt} = \left( \frac{N\sigma+b}{4-b-\sigma(N-4)} \right)^{\frac{-b-N\sigma}{4}} \|V\|_{L^2}^{\frac{2\sigma+2}{2}}
\]
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and \( V \) is a solution with minimal \( L^2 \)-norm of

\[
-\Delta^2 V + |x|^{-b} V^{2\sigma} - V = 0. \tag{1.9}
\]

(ii) \[
\int_{\mathbb{R}^N} |x|^{-b} |u(x)|^{2\sigma + 2} \, dx \leq K_{opt} \| \Delta u \|_{L^2}^2 \| u \|_{L^{2\sigma_c}}^{2\sigma_c}, \text{ where} \tag{2.10}
\]

\[
K_{opt} = \frac{\sigma + 1}{\| V \|_{L^{2\sigma_c}}^{2\sigma_c}}
\]

and \( V \) is a solution with minimal \( L^{\sigma_c} \)-norm of

\[
-\Delta^2 V + |x|^{-b} V^{2\sigma} - |V|^{\sigma_c - 2} V = 0. \tag{1.10}
\]

Next we state our results concerning global existence. The first one establishes sufficient conditions for global existence in \( H^2 \).

**Theorem 1.4** Assume \( N \geq 3 \), \( \frac{4 - b}{N} < \sigma < 4^* \) and \( 0 < b < \min\{ \frac{N}{2}, 4 \} \). Suppose \( u_0 \in H^2 \) and let \( u(t) \) be the corresponding local solution of (1.1) according to Theorem A. Let \( Q \) be a solution of (1.9) with minimal \( L^2 \)-norm. If

\[
E[u_0]^{\sigma_c} M[u_0]^{2 - \sigma_c} < E[Q]^{\sigma_c} M[Q]^{2 - \sigma_c} \tag{1.11}
\]

and

\[
\| \Delta u_0 \|_{L^2}^{\sigma_c} \| u_0 \|_{L^2}^{2 - \sigma_c} < \| \Delta Q \|_{L^2}^{\sigma_c} \| Q \|_{L^2}^{2 - \sigma_c}, \tag{1.12}
\]

then \( u(t) \) is a global solution in \( H^2 \). In addition,

\[
\| \Delta u(t) \|_{L^2}^{\sigma_c} \| u(t) \|_{L^2}^{2 - \sigma_c} < \| \Delta Q \|_{L^2}^{\sigma_c} \| Q \|_{L^2}^{2 - \sigma_c} . \tag{1.13}
\]

The strategy to prove Theorem 1.4 follows the one introduced by Holmer-Roudenko [12] in order to study global solutions for the 3D cubic NLS equation in the energy space. We also point out that in [7], Farah showed a similar result for the inhomogeneous NLS equation (1.3).

**Remark 1.5** Under assumption (1.11), if we replace (1.12) by \( \| \Delta u_0 \|_{L^2}^{\sigma_c} \| u_0 \|_{L^2}^{2 - \sigma_c} > \| \Delta Q \|_{L^2}^{\sigma_c} \| Q \|_{L^2}^{2 - \sigma_c} \), we believe that radial solutions of (1.1) must blow up in finite time. This will be issue for future investigations. For the biharmonic NLS equation (1.2) a result in this direction was established in [2].

Our second result gives a sufficient condition in order to deduce that a maximal solution in \( H^{\sigma_c} \cap \dot{H}^2 \) is indeed a global one.

\[\text{Recall that } \sigma_c = \frac{2N\sigma}{4 - b}.\]
Theorem 1.6 Assume \( N \geq 5, \max\left\{ \frac{4-b}{N}, \frac{1}{2} \right\} < \sigma < \frac{4-b}{N-4} \) and \( 0 < b < \min\left\{ \frac{N}{2}, 4 \right\} \). Suppose \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^2, 0 < s_c < 2 \), and let \( u(t) \) be the corresponding solution of (1.1), according to Theorem 1.1, with maximal time of existence, say, \( T^* > 0 \). If

\[
\sup_{t \in [0, T^*)} \| u(t) \|_{L^{\sigma_c}} < \| V \|_{L^{\sigma_c}},
\]

where \( V \) is a solution of (1.10) with minimal \( L^{\sigma_c} \)-norm, then \( u(t) \) exists globally in time.

As we will see below, the proof of Theorem 1.6 follows as an application of part (ii) in Corollary 1.3.

Remark 1.7 The restriction on the dimension \( N \geq 3 \) in Theorem 1.4 and \( N \geq 5 \) in Theorem 1.6 comes from the local well-posedness theory presented in Theorem A and Theorem 1.1, respectively. The additional restriction \( 1 < 2\sigma \) also comes from Theorem 1.1. Thus, if one are able to drop these restrictions in the local well-posedness theory then Theorems 1.4 and 1.6 also holds without these assumptions.

Finally, we consider the phenomenon of \( L^{\sigma_c} \)-norm concentration. This phenomenon, sometimes also called weak concentration to differ from the concentration in \( L^2 \) (see [22]), concerns the concentration in the Lebesgue space \( L^{\sigma_c} \) of a solution \( u(t) \in \dot{H}^{s_c} \cap \dot{H}^2 \) that blows up in finite. First of all, in view of Theorem 1.6, if we suppose that \( u(t) \) blows up in finite time \( T^* > 0 \), then

\[
\sup_{t \in [0, T^*)} \| u(t) \|_{L^{\sigma_c}} \geq \| V \|_{L^{\sigma_c}}.
\]

In addition, either \( \| u(t) \|_{\dot{H}^{s_c}} \to \infty \) or \( \| u(t) \|_{\dot{H}^2} \to \infty \), as \( t \to T^* \). Here, we will assume that the blowing up solutions satisfy

\[
\sup_{t \in [0, T^*)} \| u(t) \|_{\dot{H}^{s_c}} < \infty,
\]

so that we must have

\[
\lim_{t \to T^*} \| u(t) \|_{\dot{H}^2} = \infty.
\]

This kind of blowing up solutions are known in the literature as being of type II. For the intercritical NLS equation they were studied, for instance, in [18]. Note that in the case \( s_c = 0 \), (1.15) holds in view of the conservation of the mass. Our concentration result reads as follows.

Theorem 1.8 Assume that \( N \geq 5, \max\left\{ \frac{4-b}{N}, \frac{1}{2} \right\} < \sigma < \frac{4-b}{N-4} \) and \( 0 < b < \min\left\{ 0, \frac{N}{2} \right\} \). Let \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^2 \) be such that the corresponding solution \( u \) to (1.1) blows up in finite time \( T^* > 0 \) satisfying (1.15). Let \( \lambda(t) > 0 \) be a function satisfying

\[
\lim_{t \to T^*} \lambda(t) \| u(t) \|_{\dot{H}^{s_c}} \| \frac{1}{\dot{H}^2} = +\infty.
\]
Then
\[
\liminf_{t \to T^*} \int_{|x| \leq \lambda(t)} |u(x, t)|^{\sigma_c} \, dx \geq \|V\|_{L^{\sigma_c}}^{\sigma_c},
\]
where \(V\) is a solution to the elliptic equation (1.8) with minimal \(L^{\sigma_c}\)-norm.

The main ideas to prove Theorem 1.8 comes from [9], where the author showed a similar result for the NLS equation. However, in [9] it was used a profile decomposition theorem to overcome the loss of compactness. In our case, since equation in (1.1) is not invariant by translations we will take the advantage of the compactness embedding presented in Proposition 4.2. This strategy has already been applied to the INLS equation in [3].

The rest of the paper is organized as follows. In Sect. 2, we introduce some notations and give a review of the Strichartz estimates. In Sect. 3, we establish local well-posedness in \(\dot{H}^{s_c} \cap \dot{H}^2\). In Sect. 4, we first establish a Gagliardo-Nirenberg type inequality and use it to show the global well-posedness theory in both \(H^2\) and \(\dot{H}^{s_c} \cap \dot{H}^2\), \(0 < s_c < 2\). Section 5 is devoted to show Theorem 1.8. At last, we present an appendix where we set down some remarks concerning global well-posedness and concentration in the critical space \(\dot{H}^{s_c}\).

2 Notation and preliminaries

We begin introducing the notation used throughout the paper and list some useful inequalities. We use \(c\) to denote various constants that may vary line by line. Let \(a\) and \(b\) be positive real numbers, the notation \(a \lesssim b\) means that there exists a positive constant \(c\) such that \(a \leq cb\). Given a real number \(r\), we use \(r + \epsilon\) to denote \(r + \epsilon\) for some \(\epsilon > 0\) sufficiently small. For a subset \(A \subset \mathbb{R}^N\), \(A^c = \mathbb{R}^N \setminus A\) denotes the complement of \(A\).

For \(N \geq 1\) the number \(2^s\) is such that
\[
2^s = \begin{cases} 
\frac{2N}{N+4}, & N \geq 5 \\
+\infty, & N = 1, 2, 3, 4.
\end{cases}
\] (2.1)

The norm in the usual Sobolev spaces \(H^{s,p} = H^{s,p}(\mathbb{R}^N)\) is defined by \(\|f\|_{H^{s,p}} := \|J^s f\|_{L^p}\), where \(J^s\) stands for the Bessel potential of order \(-s\), given via Fourier transform by \(\hat{J}^s f = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}\). If \(p = 2\) we denote \(H^{s,2}\) simply by \(H^s\).

Let
\[
\dot{S}(\mathbb{R}^N) := \{ f \in S(\mathbb{R}^N); (D^\alpha \hat{f})(0) = 0, \text{ for all } \alpha \in \mathbb{N}^n \},
\]
where \(S(\mathbb{R}^N)\) denotes the Schwarz space. Endowed with the topology of \(S(\mathbb{R}^N)\), \(\dot{S}(\mathbb{R}^N)\) becomes a locally convex space. By \(\dot{S}'(\mathbb{R}^N)\) we denote the topological dual of \(\dot{S}(\mathbb{R}^N)\), which can be identified with the factor space \(S'(\mathbb{R}^N)/\mathcal{P}\) with \(\mathcal{P}\) representing the collection of all polynomials of the form \(\sum a_\alpha x^\alpha, \alpha \in \mathbb{N}^n\) (see [23], page 237).
In particular, elements in $\dot{S}'(\mathbb{R}^N)$ may be considered as elements in $S'(\mathbb{R}^N)$ modulo a polynomial. Given $s \in \mathbb{R}$, the homogeneous spaces $\dot{H}^s = \dot{H}^s(\mathbb{R}^N)$ may be defined in the following way

$$\dot{H}^s(\mathbb{R}^N) := \{ f \in \dot{S}'(\mathbb{R}^N); \| f \|_{\dot{H}^s} := \| D^s f \|_{L^2} < \infty \},$$

where $D^s = (-\Delta)^{s/2}$ is the Fourier multiplier with symbol $|\xi|^s$. Equivalently, $\dot{H}^s$ may also be defined in view of the Littlewood-Paley decomposition. Moreover, $\dot{H}^s$ are reflexive Banach spaces (see [24, Proposition 1.19] for the completeness). The reflexivity follows taking into account that the map $T : \dot{H}^s(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by $Tf = (-\Delta)^{s/2}f$ is an isometry from $\dot{H}^s(\mathbb{R}^N)$ onto a closed subspace of $L^2(\mathbb{R}^N)$.

For $m \in \mathbb{N}$, $m \geq 1$, we also define the space $\dot{W}^m(\mathbb{R}^N) := \{ f \in L^1_{\text{loc}}(\mathbb{R}^N); D^\alpha f \in L^2(\mathbb{R}^N) \text{ with } \alpha \in \mathbb{N}^N, |\alpha| = m \}$ and set

$$\| f \|_{\dot{W}^m} := \sum_{|\alpha|=m} \| D^\alpha f \|_{L^2}.$$ 

Identifying functions that differ by a polynomial the spaces $\dot{H}^m$ and $\dot{W}^m$ are identical (see [21, Theorem 3.13]). So, in what follows we will not distinguish these two spaces. In particular the space $C^\infty_0(\mathbb{R}^N)$ is dense in $\dot{H}^m \cap L^p$, $1 \leq p < \infty$ (see [11, Theorem 7]).

By $L^p_b = L^p_b(\mathbb{R}^N)$, $1 \leq p < \infty$, we denote the space of all measurable functions $u$ such that

$$\| u \|_{L^p_b} := \left( \int |x|^{-b} |u|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Thus, $L^p_b$ is nothing but the weighted Lebesgue space $L^p(|x|^{-b} \, dx)$.

If $X$ is a Banach space and $I \subset \mathbb{R}$ an interval; the mixed norms in the spaces $L^q_I X$, $1 \leq q \leq \infty$, of a function $f = f(x, t)$ is defined as

$$\| f \|_{L^q_I X} = \left( \int_I \int_I \| f(\cdot, t) \|_X^q \, dt \right)^{\frac{1}{q}},$$

with the standard modification if $q = \infty$. Moreover, if $I = \mathbb{R}$ we shall use the notation $\| f \|_{L^q_I X}$.

We now recall some Strichartz type estimates associated to the linear biharmonic Schrödinger propagator. We say the pair $(q, r)$ is biharmonic Schrödinger admissible ($B$-admissible for short) if it satisfies

$$\frac{4}{q} = \frac{N}{2} - \frac{N}{r},$$

\( \text{Springer} \)
with
\[
\begin{aligned}
2 \leq r < \frac{2N}{N-4}, & \quad \text{if } N \geq 5, \\
2 \leq r < +\infty, & \quad \text{if } 1 \leq N \leq 4.
\end{aligned}
\]
For \( s < 2 \), the pair \((q, r)\) is called \(\dot{H}^s\)-biharmonic admissible if
\[
\frac{4}{q} = \frac{N}{2} - \frac{N}{r} - s
\]
with
\[
\begin{aligned}
\frac{2N}{N-2s} \leq r < \frac{2N}{N-4}, & \quad \text{if } N \geq 5, \\
2 \leq r < +\infty, & \quad \text{if } 1 \leq N \leq 4.
\end{aligned}
\]
Given \( s \in \mathbb{R} \) we introduce the Strichartz norm
\[
\|u\|_{B(\dot{H}^s)} = \sup_{(q,r) \in \mathcal{B}_s} \|u\|_{L_t^q L_x^r},
\]
where \( \mathcal{B}_s := \{(q, r) \colon (q, r) \text{ is } \dot{H}^s\text{-biharmonic admissible}\} \). For any \((q, r) \in \mathcal{B}_s\), by \((q', r')\) we denote its dual Lebesgue pair, that is, \( q' \) and \( r' \) are such that \( \frac{1}{q} + \frac{1}{q'} = 1 \) and \( \frac{1}{r} + \frac{1}{r'} = 1 \). This allow us to introduce the dual Strichartz norm
\[
\|u\|_{B'(\dot{H}^{-s})} = \inf_{(q,r) \in \mathcal{B}_{-s}} \|u\|_{L_t^{q'} L_x^{r'}}.
\]
If \( s = 0 \) then \( \mathcal{B}_0 \) is the set of all \( B \)-admissible pairs. Thus, \( \|u\|_{B(L^2)} = \sup_{(q,r) \in \mathcal{B}_0} \|u\|_{L_t^q L_x^r} \) and \( \|u\|_{B'(L^2)} = \inf_{(q,r) \in \mathcal{B}_0} \|u\|_{L_t^{q'} L_x^{r'}} \). To indicate the restriction to a time interval \( I \subset \mathbb{R} \) we will use \( B(\dot{H}^s; I) \) and \( B'(\dot{H}^{-s}; I) \).

We also recall two important inequalities.

**Lemma 2.1** (Sobolev embedding) Let \( s \in (0, +\infty) \) and \( 1 \leq p < +\infty \).

(i) If \( s \in (0, \frac{N}{p}) \) then \( H^{s,p}(\mathbb{R}^N) \) is continuously embedded in \( L^r(\mathbb{R}^N) \) where \( s = \frac{N}{p} - \frac{N}{r} \). Moreover,
\[
\|f\|_{L^r} \lesssim \|D^s f\|_{L^p},
\]
(ii) If \( s = \frac{N}{2} \) then \( H^s(\mathbb{R}^N) \subset L^r(\mathbb{R}^N) \) for all \( r \in [2, +\infty) \). Furthermore,
\[
\|f\|_{L^r} \lesssim \|f\|_{H^s}.
\]

**Proof** See [1, Theorem 6.5.1] (see also [16, Theorem 3.3]).

In particular one has
\[
\|f\|_{L^p} \lesssim \|f\|_{\dot{H}^s}, \quad \forall f \in \dot{H}^s(\mathbb{R}^N),
\]
where \( p = \frac{2N}{N-2s} \). Note that \( \dot{H}^s_c \hookrightarrow L^{\sigma_c} \) since \( \sigma_c = \frac{2N\sigma}{4-\sigma} = \frac{2N}{N-2\sigma} \).
Lemma 2.2 (Hardy-Littlewood inequality) Let $1 < p \leq q < +\infty$, $N \geq 1$, $0 < s < N$ and $\rho \geq 0$ satisfy the conditions
\[
\rho < \frac{N}{q}, \quad s = \frac{N}{p} - \frac{N}{q} + \rho.
\]
Then, for any $u \in H^{s,p}(\mathbb{R}^N)$ we have
\[
\| |x|^{-\rho} u \|_{L^q} \lesssim \| D^s u \|_{L^p}.
\]

Proof See Theorem B* in [20]. \qed

Lemma 2.3 (Fractional Gagliardo-Nirenberg inequality) Assume $1 < p, p_0, p_1 < \infty$, $s, s_1 \in \mathbb{R}$, and $\theta \in [0, 1]$. Then the fractional Gagliardo-Nirenberg inequality
\[
\| D^s u \|_{L^p} \lesssim \| u \|_{L^{p_0}}^{1-\theta} \| D^{s_1} u \|_{L^{p_1}}^\theta.
\]
holds if and only if
\[
\frac{N}{p} - s = (1 - \theta) \frac{N}{p_0} + \theta \left( \frac{N}{p_1} - s_1 \right), \quad s \leq \theta s_1.
\]

Proof See Corollary 1.3 (page 30) in [24]. \qed

Corollary 2.4 If $\alpha > 1$ then we obtain the following estimate
\[
\| \Delta (|u|^{\alpha} u) \|_{L^p} \lesssim \| u \|_{L^{p_1}}^\alpha \| \Delta u \|_{L^{p_2}},
\]
where $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1}{p_2}$.

Proof Observe that, for $\alpha > 1$,
\[
|\Delta (|u|^{\alpha} u)| \lesssim \| |u|^{\alpha} \Delta u | + |u|^{\alpha-1} |\nabla u|^2.
\]
The Hölder inequality implies
\[
\| \Delta (|u|^{\alpha} u) \|_{L^p} \lesssim \| u \|_{L^{p_1}}^\alpha \| \Delta u \|_{L^{p_2}} + \| u \|_{L^{p_1}}^{\alpha-1} \| \nabla u \|_{L^{p_3}}^2,
\]
where $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1}{p_2}$. On the other hand, an application of Lemma 2.3 (with $s = 1, s_1 = 2$ and $\theta = \frac{1}{2}$) gives
\[
\| \nabla u \|_{L^{p_3}} \lesssim \| u \|_{L^{p_1}} \| \Delta u \|_{L^{p_2}}^\frac{1}{2},
\]
where $\frac{1}{p_3} = \frac{1}{2p_1} + \frac{1}{2p_2}$. By noting that $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1}{p_2}$ and combining the last two inequalities we obtain the desired. \qed
We now recall the Strichartz estimates related to the linear problem, which are the main tools to show the local and global well-posedness. See for instance Pausader [19] (see also [10] and the references therein).

**Lemma 2.5** Let \( I \subset \mathbb{R} \) be an interval and \( t_0 \in I \). The following statements hold.

(i) *(Linear estimates)*.

\[
\|e^{it \Delta^2} f\|_{B(L^2; I)} \leq c \|f\|_{L^2},
\]
\[
\|e^{it \Delta^2} f\|_{B(\tilde{H}^s; I)} \leq c \|f\|_{\tilde{H}^s}.
\]

(ii) *(Inhomogeneous estimates)*.

\[
\left\| \int_{t_0}^t e^{i(t'-t') \Delta^2} g(\cdot, t') dt' \right\|_{B(L^2; I)} \leq c \|g\|_{B'(L^2; I)},
\]
\[
\left\| \int_{t_0}^t e^{i(t'-t') \Delta^2} g(\cdot, t') dt' \right\|_{B(\tilde{H}^s; I)} \leq c \|g\|_{B'(\tilde{H}^{-s}; I)}.
\]

We also recall another useful Strichartz estimates for the fourth-order Schrödinger equation.

**Proposition 2.6** Assume \( N \geq 3 \). Let \( I \subset \mathbb{R} \) be an interval and \( t_0 \in I \). Suppose that \( s \geq 0 \) and \( u \in C(I, H^{-4}) \) is a solution of

\[
u(t) = e^{i(t-t_0) \Delta^2} u(t_0) + i \int_{t_0}^t e^{i(t'-t') \Delta^2} F(\cdot, t') dt',
\]

for some function \( F \in L^1_{loc}(I, H^{-4}) \). Then, for any \( B \)-admissible pair \((q, r)\), we have

\[
\|D^s u\|_{L^q_t L^r_x} \lesssim \|D^s u(t_0)\|_{L^2} + \|D^{s-1} F\|_{L^2_t L^{\frac{2N}{N+2}}_x}.
\]

In particular, when \( s = 2 \),

\[
\|\Delta u\|_{L^q_t L^r_x} \lesssim \|\Delta u(t_0)\|_{L^2} + \|\nabla F\|_{L^2_t L^{\frac{2N}{N+2}}_x}.
\]

**(2.2)**

**Proof** See Proposition 2.3 in [10]. \(\square\)

We end this section with some standard facts. Recall that

\[
\|\cdot\|_{L^\frac{N}{\gamma}(B)} < +\infty \text{ if } \frac{N}{\gamma} - b > 0 \text{ and } \|\cdot\|_{L^\frac{N}{\gamma}(B^c)} < +\infty \text{ if } \frac{N}{\gamma} - b < 0,
\]

where here, and throughout the paper, \( B \) denotes the unity ball in \( \mathbb{R}^N \), that is, \( B = \{x \in \mathbb{R}^N; |x| \leq 1\} \). By setting \( f(z) = |z|^{2\alpha} z \) and \( F(x, z) = |x|^{-b} f(z) \), the complex derivative of \( f \) is

\[
f_z(z) = \sigma |z|^{2\alpha} \quad \text{and} \quad f_{\bar{z}}(z) = \sigma |z|^{2\alpha-2} z^2.
\]
Moreover, for any \( z, w \in \mathbb{C} \), we have
\[
|F(x, z) - F(x, w)| \lesssim |x|^{-b} \left(|z|^{2\sigma} + |w|^{2\sigma}\right)|z - w|.
\]

(2.4)

### 3 Local well-posedness in \( \dot{H}^{s_c} \cap \dot{H}^2, 0 < s_c < 2 \)

In this section we show the local well-posedness in \( \dot{H}^{s_c} \cap \dot{H}^2, 0 < s_c < 2 \) (Theorem 1.1). The proof follows from a contraction mapping argument, which is based on the Strichartz estimates. To do that, we first establish suitable estimates on the nonlinearity \( F(x, u) = |x|^{-b}|u|^{2\sigma} u \) in the Strichartz norms. In view of the function \( |x|^{-b} \) in the nonlinearity, in order to obtain the nonlinear estimates, we frequently need to divide them inside and outside the unit ball.

**Lemma 3.1** Let \( N \geq 5 \) and \( 0 < b < \min\{\frac{N}{2}, 4\} \). If \( \max\{\frac{2-b}{4}, \frac{4-b}{N}\} < \sigma < \frac{4-b}{N-4} \) then the following statement holds
\[
\|\nabla F(x, u)\|_{L^2_{\gamma} L^\frac{2N}{N+2}} \lesssim T^{\theta_1} \|\Delta u\|_{B_{(\dot{L}_2^\gamma \cap \dot{H}^2)}^\sigma(I)} + T^{\theta_2} \|D^c_{\gamma} u\|_{B_{(\dot{L}_2^\gamma \cap \dot{H}^2)}^{2\sigma}} \|\Delta u\|_{B_{(\dot{L}_2^\gamma \cap \dot{H}^2)}},
\]
where \( I = [0, T] \) and \( \theta_1, \theta_2 > 0 \).

**Proof** We divide the estimate on \( B \) and \( B^c \) so that
\[
\|\nabla F(x, u)\|_{L^2_{\gamma} L^\frac{2N}{N+2}} \leq C_1 + C_2,
\]
with \( C_1 = \|\nabla F(x, u)\|_{L^2_{\gamma} L^\frac{2N}{N+2}(B)} \) and \( C_2 = \|\nabla F(x, u)\|_{L^2_{\gamma} L^\frac{2N}{N+2}(B^c)} \).

First we estimate \( C_1 \). Indeed, the Hölder and Sobolev inequalities imply
\[
C_1 \leq \|x|^{-b}\|_{L^\gamma(B)} T^{\frac{1}{2q_1}} \|x\|_{L^2_{\gamma} L^2_{L^\infty}}^{2\sigma r_1} \|\Delta u\|_{L^2_{\gamma} L^2_{L^\infty}} + T^{\frac{1}{2q_1}} \|\nabla(|x|^{-b})\|_{L^q(B)} \|u\|_{L^2_{\gamma} L^2_{L^\infty}((2\sigma+1)\epsilon)} \lesssim \|x|^{-b}\|_{L^\gamma(B)} \|\Delta u\|_{L^2_{\gamma} L^2_{L^\infty}} + \|x|^{-b-1}\|_{L^q(B)} \|\Delta u\|_{L^2_{\gamma} L^2_{L^\infty}},
\]
where
\[
\begin{aligned}
\frac{N+2}{2N} &= \frac{1}{q} + \frac{1}{r_1} + \frac{1}{r_2} = \frac{N}{q} + \frac{1}{q} \\
n &\geq \frac{N}{r} - \frac{2\sigma r_1}{r} = \frac{N}{r} - \frac{N}{(2\sigma+1)\epsilon}, \quad r < \frac{N}{2} \\
\frac{1}{q_1} &= \frac{1}{q} + \frac{2\sigma+1}{q}, \\
\frac{1}{q_2} &= \frac{1}{q} + \frac{2\sigma+1}{q}.
\end{aligned}
\]

which is equivalent to
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{N}{q} &= \frac{N}{r} - 1 = \frac{N}{r} - \frac{N(2\sigma+1)}{r} + 4\sigma + 2, \\
\frac{1}{q_1} &= \frac{1}{q} + \frac{2\sigma+1}{q}, \\
\frac{1}{q_2} &= \frac{1}{q} + \frac{2\sigma+1}{q}.
\end{array} \right.
\]

\( \square \) Springer
We now need to check that $\frac{1}{q_1} > 0$ and $\|x|^{-b}\|_{L^r(B)}$ and $\|x|^{-b-1}\|_{L^r(B)}$ are finite, i.e., $\frac{N}{\gamma} > b$ and $\frac{N}{d} > b+1$, respectively, by (2.3). To this end, we choose $(q, r)$ given by

$$r = \frac{2N(N + 4 - 2b)}{N^2 - 2bN + 16} \quad \text{and} \quad q = \frac{2(N + 4 - 2b)}{N - 4}.$$ 

It is easy to see that $(q, r)$ is $B$-admissible and since $b < \frac{N}{2}$ we deduce $r < \frac{N}{2}$. In addition, the hypothesis $\sigma < \frac{4-b}{N-4}$ yields $\frac{N}{\gamma} - b = \frac{N}{d} - b - 1 > 0$ and $\frac{1}{q_1} = \frac{4-b-\alpha(N-4)}{N+4-2b} > 0$. Therefore, $C_1 \lesssim T^{\theta_1} \|\Delta u\|^{2\sigma+1}_{B(L^2;I)}$ with $\theta_1 = \frac{1}{q_1}$.

We now estimate $C_2$. Note that

$$|\nabla F(x, u)| \lesssim |x|^{-b} \left( \|u|^{2\sigma} \nabla u| + |u|^{2\sigma} \|x|^{-1} u\right).$$

Here we need to divide the proof according to $b < 2$ and $b \geq 2$.

**Case** $b < 2$. Combining Lemma 2.2, Hölder’s inequality and Sobolev’s embedding one has

$$C_2 \lesssim \left\| \|x|^{-b}\|_{L^r(B^C)} \right\|_{L^2}^{2\sigma} \|\nabla u\|_{L^2}^{\frac{4N}{N-2}} \lesssim T^{\theta_1} \|x|^{-b}\|_{L^r(B^C)} \|D^{s_c} u\|_{L^2}^{2\sigma} \|\Delta u\|_{L^\infty} \lesssim \|\Delta u\|_{L^\infty} \lesssim \|\nabla u\|_{L^2},$$

where we have used that $\|\|x|^{-1} u\|_{L^2}^{\frac{2N}{N-2}} \lesssim \|\nabla u\|_{L^2}^{\frac{2N}{N-2}}$ (see Lemma 2.2) and

$$\frac{N + 2}{2} = \frac{N}{\gamma} - \frac{N 2\sigma}{r} + 2\sigma s_c - \frac{N - 2}{2} \quad \text{and} \quad \frac{1}{q_1} = \frac{1}{2} - \frac{2\sigma}{q}.$$ 

Hence, choosing $(q, r)$ $B$-admissible defined by (since $b < 2$)

$$q = \frac{8\sigma}{2 - b} \quad \text{and} \quad r = \frac{4\sigma N}{2\sigma N - 4 + 2b},$$

we have that $\frac{N}{\gamma} - b < 0$, that is, $\|x|^{-b}\|_{L^r(B^C)}$ is finite and $\frac{1}{q_1} > 0$. Hence, $C_2 \lesssim T^{\theta_2} \|D^{s_c} u\|_{B(L^2;I)}^{2\sigma} \|\Delta u\|_{B(L^2;I)}$ with $\theta_2 = \frac{1}{q_1}$.

**Subcase** $b \geq 2$. Arguing as in the previous case and choosing $(q, r) = (\infty, 2)$ we deduce that $\frac{1}{q_1} = \frac{1}{2} > 0$ and $\frac{N}{\gamma} - b = -2 < 0$, so $\|x|^{-b}\|_{L^r(B^C)} < \infty$. This leads to $C_2 \lesssim T^{\theta_2} \|D^{s_c} u\|_{B(L^2;I)}^{2\sigma} \|\Delta u\|_{B(L^2;I)}$ with $\theta_2 = \frac{1}{2}$. Thus, the proof of the lemma is completed. \(\square\)

---

3 Here we are using that $N \geq 5$.

4 It is easy to see that $2 < r < \frac{N}{s_c}$. Furthermore, in view of $\sigma > \frac{2-b}{4}$ one has $r < \frac{2N}{N-4}$.  

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Lemma 3.2 Let $N \geq 5$ and $0 < b < \min\{\frac{N}{2}, 4\}$. If $\max\{\frac{4-b}{N}, \frac{1}{2}\} < \sigma < \frac{4-b}{N-4}$ then the following inequality holds

$$\left\| D^c F(x, u) \right\|_{B'(L^2; I)} \lesssim (T^{\theta_1} + T^{\theta_2}) \| \Delta u \|_{B(L^2; I)} \left\| D^c u \right\|_{B(L^2; I)}^{2\sigma},$$

where $I = [0, T]$ and $\theta_1, \theta_2 > 0$.

**Proof** Here we use the $B$-admissible pair $(q_e, r_e) = (\frac{4}{2-\varepsilon}, \frac{2N}{N-4+2\varepsilon})$, where $\varepsilon > 0$ is sufficiently small. From the Sobolev embedding we have

$$\left\| D^c F(x, u) \right\|_{L^\varepsilon'} \lesssim \| \Delta F(x, u) \|_{L^\varepsilon^*}, \quad (3.1)$$

where $p^* = \frac{2N\sigma}{8\sigma + 4 - b - 2\varepsilon \sigma}$. Observe that

$$|\Delta F(x, u)| \lesssim |x|^{-b} \left( \Delta(|u|^\sigma |u|) + |x|^{-2}||u|^\sigma u| + |x|^{-b} |x|^{-1} |\nabla(|u|^\sigma u)| \right)$$

and by Lemma 2.2 one has $\| x |^{-2} (|u|^{2\sigma} u) \|_{L^\beta} \lesssim \| \Delta(|u|^{2\sigma} u) \|_{L^\beta}, \| x |^{-1} |\nabla(|u|^{2\sigma} u)| \|_{L^\beta} \lesssim \| \Delta(|u|^{2\sigma} u) \|_{L^\beta}$ for any $1 < \beta < \frac{N}{2}$. Let $A$ denotes either $B$ or $B^C$. Applying the Hölder inequality, the Sobolev embedding and Corollary 2.4 (since $2\sigma > 1$) we get

$$\| \Delta F(x, u) \|_{L^\varepsilon'_{L^\beta}} \lesssim \| |x|^{-b} \|_{L^\varepsilon'_{L^\beta}} \| \Delta (|u|^{2\sigma} u) \|_{L^\beta} \| u \|_{L^\varepsilon_{L^\beta}} \| \Delta u \|_{L_s^\varepsilon} \| \Delta u \|_{L_s^\varepsilon}$$

where (using the value of $p^*$ defined above)

$$\frac{N}{\gamma} = \frac{8\sigma + 4 - b - 2\varepsilon \sigma}{2\sigma} - \frac{N}{\beta}, \quad \frac{1}{\beta} = \frac{2\sigma}{r_1} + \frac{1}{r_1}, \quad \sigma_c = \frac{N}{r} - \frac{N}{r_1} \quad \text{and} \quad \frac{1}{q_c} = \frac{1}{q_1} + \frac{2\sigma + 1}{q}.$$  

This is equivalent to

$$\left\{ \begin{array}{l}
\frac{N}{\gamma} - b = \frac{4-b+2\sigma^2 N-2\varepsilon \sigma}{r}, \\
\frac{1}{q_1} = \frac{2\sigma+1}{q_c}.
\end{array} \right. \quad (3.3)$$

We need to find $(q, r)$ $B$-admissible such that $\| x |^{-b} \|_{L^\gamma(A)}$ is finite, $r < \frac{N}{\sigma_c}$ and $\frac{1}{q_1} > 0$. Indeed, if $A = B$ we choose $(q, r)$ $B$-admissible defined by

$$r = \frac{2\sigma N(2\sigma + 1)}{2\sigma^2 N + 4 - b - 4\varepsilon \sigma} \quad \text{and} \quad q = \frac{8\sigma(2\sigma + 1)}{\sigma N - 4 + b + 4\varepsilon \sigma},$$

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so\(^5\) by using (3.4) we obtain \(\frac{N}{\gamma} - b = \varepsilon > 0\), i.e., \(|x|^{-b} \in L^\gamma(B)\). Moreover,

\[
\theta_1 = \frac{1}{q_1} = \frac{1}{q_\varepsilon} - \frac{2\sigma + 1}{q} = \frac{4 - b - \sigma(N - 4) - 2\varepsilon\sigma}{8\sigma},
\]

which is positive since \(\sigma < \frac{4-b}{N-4}\) and \(\varepsilon > 0\) is small.

On the other hand, if \(A = B^C\) we choose \((q, r)\) defined by

\[
r = \frac{2\sigma N(2\sigma + 1)}{2\sigma^2 N + 4 - b} \quad \text{and} \quad q = \frac{8\sigma(2\sigma + 1)}{\sigma N - 4 + b},
\]

which gives, from (3.4), \(\frac{N}{\gamma} - b = -\varepsilon < 0\) and implies that \(\| |x|^{-b} \|_{L^\gamma(B^C)} \) is finite and

\[
\theta_2 = \frac{1}{q_2} = \frac{4 - b - \sigma(N - 4 - 2\varepsilon\sigma)}{8\sigma} > 0.
\]

To complete the proof we need to verify that \(r < \frac{N}{\kappa_c}\) and \(\beta < \frac{N}{2}\). Indeed, note that \(r < \frac{N}{\kappa_c}\) is equivalent to \(8 - 2b - 4\varepsilon\sigma - \sigma(N - 8 + 2b) > 0\). Clearly this is true if \(N - 8 + 2b \leq 0\), otherwise if \(N - 8 + 2b > 0\) then the facts that \(\sigma < \frac{4-b}{N-4}\) and \(b < \frac{N}{2}\) imply the desired. In addition, by using the first relation in (3.3) and the value of \(\gamma\), we obtain

\[
\beta = \begin{cases} 
\frac{2N\sigma}{8\sigma + 4 - b - 2\sigma b}, & \text{if } A = B^C \\
\frac{2N\sigma}{8\sigma + 4 - b - 2\sigma b - 4\varepsilon\sigma}, & \text{if } A = B.
\end{cases}
\]

Observe now that \(\beta < \frac{N}{2}\) is equivalent to \(4 - b + 2\sigma(2 - b) - 4\varepsilon\sigma > 0\). For one hand, if \(b \geq 2\) then this inequality clearly holds. On the other hand, if \(b < 2\) then the facts that \(b < \frac{N}{2}\) and \(\sigma < \frac{4-b}{N-4}\) yield the desired.

The proof of the lemma then follows from (3.1) and (3.2).

\[\square\]

**Lemma 3.3** Let \(N \geq 5\) and \(0 < b < 4\). If \(\frac{4-b}{N} < \sigma < \frac{4-b}{N-4}\) then the following statement holds

\[
\left\| \chi_A |x|^{-b} |u|^{2\sigma} v \right\|_{B(H^{-\kappa_c}; I)} \lesssim (T^{\theta_1} + T^{\theta_2}) \| \Delta u \|_{L^\kappa_T L^2_x} \| u \|_{B(H^{\kappa_c}; I)}^{2\sigma - \theta} \| v \|_{B(H^{\kappa_c}; I)},
\]

where \(A\) is either \(B\) or \(B^C\), \(\theta_1, \theta_2 > 0\) and \(\theta \in (0, 2\sigma)\) is small enough.

\(^5\) It is easy to check that \(\frac{4}{q} = \frac{N}{2} - \frac{N}{r}\) and if \(\frac{4-b}{N} < \sigma < \frac{4-b}{N-4}\), then \(2 < r < \frac{2N}{N-4}\).
Proof Let \((\tilde{a}, r)\) be any \(\dot{H}^{-s_c}\)-biharmonic admissible pair. The Sobolev embedding and Hölder’s inequality yield

\[
\left\| |x|^{-b} |u|^{2\sigma} v \right\|_{L^q_t L^r_x(A)} \lesssim \left\| \left( |x|^{-b} \right)_{L^q_t} \right\|_{L^q_t(A)} \left\| u \right\|_{L^q_t L^r_x(A)} \left\| v \right\|_{L^q_t L^r_x(A)} \lesssim T \left( \frac{1}{q^1} \right) \left\| \left( \left| x \right|^{-b} \right)_{L^q_t} \right\|_{L^q_t(A)} \left\| \Delta u \right\|_{L^q_t L^r_x(A)} \left\| u \right\|_{L^q_t L^r_x(A)} \left\| v \right\|_{L^q_t L^r_x(A)},
\]

where

\[
\begin{align*}
\frac{1}{q^1} &= \frac{1}{q} + \frac{1}{r} + \frac{2\sigma - \theta}{r} + \frac{1}{r}, \\
2 &= \frac{N}{2} - \frac{N\theta}{q}, \\
\frac{1}{a^1} &= \frac{1}{q^1} + \frac{2\sigma - \theta}{a} + \frac{1}{a}.
\end{align*}
\]

We need to check that \(\frac{1}{q^1} > 0\) and \(\left| x \right|^{-b} \in L^q(A)\), i.e., \(\frac{N}{\gamma} > b\) if \(A = B\) and \(\frac{N}{\gamma} < b\) if \(A = B^C\). In fact, by assuming that \((a, r)\) is \(\dot{H}^{s_c}\)-admissible, the conditions \((3.5)\) are equivalent to

\[
\left\{ \begin{array}{l}
\frac{N}{\gamma} - b = N - b - \frac{N\theta}{2} + 2\theta - \frac{N(2\sigma + 2 - \theta)}{r}, \\
\frac{2}{q^1} = 2 - s_c - \frac{2(2\sigma + 2 - \theta)}{a}.
\end{array} \right.
\]

If \(A = B\), we set

\[
a = \frac{2(2\sigma + 2 - \theta)}{2 - s_c - \varepsilon} \quad \text{and} \quad r = \frac{2N(2\sigma + 2 - \theta)}{(N - 2s_c)(2\sigma + 2 - \theta) - 4(2 - s_c - \varepsilon)}
\]

for \(0 < \varepsilon < \frac{\theta(2 - s_c)}{2}\) small enough\(^6\). Hence,

\[
\frac{2}{q^1} = 2 - s_c - \frac{2(2\sigma + 2 - \theta)}{a} = \varepsilon > 0 \\
\frac{N}{\gamma} - b = N - b - \frac{N\theta}{2} + 2\theta - \frac{N(2\sigma + 2 - \theta)}{r} = \theta(2 - s_c) - 2\varepsilon > 0.
\]

On the other hand, if \(A = B^C\), we set

\[
a = \infty \quad \text{and} \quad r = \frac{2N}{N - 2s_c} = \frac{2\sigma N}{4 - b},
\]

which gives\(^7\),

\[
\frac{2}{q^1} = 2 - s_c > 0 \quad \text{and} \quad \frac{N}{\gamma} - b = -(2 - \theta)(2 - s_c) < 0.
\]

\(^6\) The value of \(\tilde{a}\) is given by \(\tilde{a} = \frac{2(2\sigma - \theta + 2)}{s_c(2\sigma - \theta) + 2 + s_c - \varepsilon}\). Moreover, since \(s_c < 2\) and \(a > \frac{4}{2 - s_c}\) we have \(2 < r < \frac{2N}{N - 4}\).

\(^7\) Here \(\tilde{a} = \frac{2}{s_c}\). Observe that since \(0 < s_c < 2\) we obtain \(2 < r < \frac{2N}{N - 4}\).
This completes the proof of the lemma.

We now show our local well-posedness result.

**Proof of Theorem 1.1** For any \((q, p)\) \(B\)-admissible and \((a, r)\) \(\dot{H}^{s_c}\)-biharmonic admissible, we define

\[
X = C \left( [-T, T]; \dot{H}^{s_c} \cap \dot{H}^2 \right) \cap L^q \left( [-T, T]; \dot{H}^{s_c-p} \cap \dot{H}^{2-p} \right) \cap L^a \left( [-T, T]; L'^r \right),
\]

and

\[
\|u\|_T = \|\Delta u\|_{B(L^2;[-T,T])} + \|D^{s_c} u\|_{S(L^2;[-T,T])} + \|u\|_{B(\dot{H}^{s_c};[-T,T])}.
\]

We shall prove that for some \(T > 0\) the operator \(G = G_{u_0}\) defined by

\[
G(u)(t) = e^{it\Delta^2} u_0 + \int_0^t e^{i(t-t')}\Delta^2 |x|^{-b}|u(t')|^{2\sigma} u(t') dt'
\]

is a contraction on the complete metric space

\[
S(a, T) = \{ u \in X : \|u\|_T \leq a \}
\]

with the metric

\[
d_T(u, v) = \|u - v\|_{B(\dot{H}^{s_c};[-T,T])}.
\]

Indeed, from Lemma 2.5 and (2.2) we deduce

\[
\|\Delta G(u)\|_{B(L^2;[-T,T])} \leq c \|\Delta u_0\|_{L^2} + \|\nabla F(x, u)\|_{L^2_{[-T,T]}L^{\frac{2N}{N+2}}}
\]

\[
\|D^{s_c} G(u)\|_{B(L^2;[-T,T])} \leq c \|D^{s_c} u_0\|_{L^2} + c \|D^{s_c} F(x, u)\|_{B'(L^2;[-T,T])}
\]

and

\[
\|G(u)\|_{B(\dot{H}^{s_c};[-T,T])} \leq c \|u_0\|_{\dot{H}^{s_c}} + c \|\chi_B F(x, u)\|_{B'(\dot{H}^{-s_c};[-T,T])}
\]

\[
+ c \|\chi_{B^c} F(x, u)\|_{B'(\dot{H}^{-s_c};[-T,T])}
\]

where \(F(x, u) = |x|^{-b}|u|^{2\sigma} u\). Without loss of generality we consider only the case \(t > 0\). It follows from Lemmas 3.1, 3.2 and 3.3 that

\[
\|\nabla F(x, u)\|_{L^2_{[-T,T]}L^{\frac{2N}{N+2}}} \lesssim T^\theta_1 \|\Delta u\|_{B(L^2;L^1)}^{2\sigma+1} + T^\theta_2 \|D^{s_c} u\|_{B(L^2;L^1)}^{2\sigma} \|\Delta u\|_{B(L^2;L^1)}
\]
\[ \| D^{\varepsilon} F \|_{B'(\mathcal{L}^2; I)} \leq c(T^{\theta_1} + T^{\theta_2}) \| \Delta u \|_{B(\mathcal{L}^2; I)} \| D^{\varepsilon} u \|_{B(\mathcal{L}^2; I)}^{2\sigma} \]

and

\[ \| \chi B \|_{B'(\mathcal{H}^{-\infty}; I)} + \| \chi B c \|_{B'(\mathcal{H}^{-\infty}; I)} \leq c(T^{\theta_1} + T^{\theta_2}) \| \Delta u \|_{L_t^\infty L_x^2} \| u \|_{B(\mathcal{H}^{-\infty}; I)}^{2\sigma + 1 - \theta} \| \Delta u \|_{L_t^\infty L_x^2} \| u \|_{B(\mathcal{H}^{-\infty}; I)} \]

where \( I = [0, T] \) and \( \theta_1, \theta_2 > 0 \). Hence, if \( u \in S(a, T) \) then

\[ \| G(u) \|_T \leq c\| u_0 \|_{\mathcal{H}^{\infty} \cap \mathcal{H}^2} + c(T^{\theta_1} + T^{\theta_2})a^{2\sigma + 1}. \]

Choosing \( a = 2c\| u_0 \|_{\mathcal{H}^{\infty} \cap \mathcal{H}^2} \) and \( T > 0 \) such that

\[ ca^{2\sigma}(T^{\theta_1} + T^{\theta_2}) < \frac{1}{4}, \tag{3.6} \]

we get \( G(u) \in S(a, T) \), which implies that \( G \) is well defined on \( S(a, T) \). To show that \( G \) is a contraction we use (2.4) and Lemma 3.3 to deduce

\[
d_T(G(u), G(v)) \leq c\| \chi B (F(x, u) - F(x, v)) \|_{B'(\mathcal{H}^{-\infty}; I)} + c\| \chi B c (F(x, u) - F(x, v)) \|_{B'(\mathcal{H}^{-\infty}; I)}
\leq c(T^{\theta_1} + T^{\theta_2}) \left( \| \Delta u \|_{L_t^\infty L_x^2} \| u \|_{S(\mathcal{H}^{\infty}; I)}^{2\sigma - \theta} + \| \Delta v \|_{L_t^\infty L_x^2} \| v \|_{S(\mathcal{H}^{\infty}; I)}^{2\sigma - \theta} \right)
\times \| u - v \|_{S(\mathcal{H}^{\infty}; I)}
\leq c(T^{\theta_1} + T^{\theta_2}) \left( \| u \|_{T}^{2\sigma} + \| v \|_{T}^{2\sigma} \right) d_T(u, v),
\]

and thus if \( u, v \in S(a, T) \), then

\[ d_T(G(u), G(v)) \leq c(T^{\theta_1} + T^{\theta_2})a^{2\sigma} d_T(u, v). \]

Therefore, by using (3.6) we have that \( G \) is a contraction on \( S(a, T) \) and by the fixed point theorem one has a unique fixed point \( u \in S(a, T) \) of \( G \). The proof is thus completed. \( \square \)

**4 Gagliardo-nirenberg inequality and global solutions**

This section is devoted to prove Theorems 1.2 and 1.4. We start by studying the relation between the best optimal constant in (1.6) and the solutions of equation (1.8).
4.1 The ground states

First of all, we recall that inequality

\[ \int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{2\sigma+2} \, dx \leq C \| \Delta u \|_{L^2}^{\frac{2\sigma(\sigma-N)p+2(2-s_p)}{2-s_p}} \| u \|_{L^p}^{\frac{2\sigma(2-s_p)}{2-s_p}}, \]  

(4.1)

holds for some constant \( C > 0 \), provided \( 0 < b < \min\{N, 4\} \), \( 0 < \sigma < 4^* \) and \( 1 \leq p \leq \frac{(2\sigma+2)N}{N-b} \) (see [17, page 1516]. Our intention is then to prove that for \( p \geq 2 \) the optimal constant one can place in (4.1) is exactly the one given in (1.7). To do so, we follow the strategy as in [25], where the author proved a similar result for the standard Gagliardo-Nirenberg inequality.

For \( f \in \dot{H}^2 \cap L^p \), we define the “Weinstein functional” as

\[ J(f) = \frac{\| \Delta f \|_{L^2}^{\frac{2\sigma(\sigma-N)p+2(2-s_p)}{2-s_p}}}{\| f \|_{L^p}^{\frac{2\sigma+2}{2s_p+2}}} \]

and set

\[ J = \inf_{f \in \dot{H}^2 \cap L^p, f \neq 0} J(f). \]  

(4.2)

From (4.1) it is clear that \( J(f) \) is well defined for any \( f \in \dot{H}^2 \cap L^p \), \( J > 0 \) and the optimal constant in (4.1) is \( 1/J \). Our first task now is to show that the infimum of \( J \) is indeed attained. As we will see below, this infimum is directly connected to the solutions of the equation

\[ -\Delta^2 \phi + |x|^{-b}|\phi|^{2\sigma} \phi = |\phi|^{p-2} \phi. \]  

(4.3)

Here, by a solution of (4.3) we mean a critical point of the functional

\[ I(f) = \frac{1}{2} \int |\Delta f|^2 \, dx - \frac{1}{2\sigma+2} \int |x|^{-b}|f|^{2\sigma+2} \, dx + \frac{1}{p} \int |f|^p \, dx. \]

In the following lemma we obtain Pohozaev-type identities which are satisfied by any solution of (4.3).

**Lemma 4.1 (Pohozaev-type identities)** Let \( \phi \in \dot{H}^2 \cap L^p \) be a solution to (4.3). Then the following identities hold

\[ \int |x|^{-b}|\phi|^{2\sigma+2} \, dx = \int |\Delta \phi|^2 \, dx + \int |\phi|^p \, dx \]  

(4.4)

\[ \text{Recalling that } s_p = \frac{N}{2} - \frac{N}{p}. \]
and
\begin{equation}
\frac{N-b}{2\sigma+2} \int |x|^{-b} |\phi|^{2\sigma+2} \, dx = \left( \frac{N}{2} - 2 \right) \int |\Delta \phi|^2 \, dx + \frac{N}{p} \int |\phi|^p \, dx. \tag{4.5}
\end{equation}

In particular,
\begin{equation}
\int |\Delta \phi|^2 \, dx = \frac{N(2\sigma+2)-(N-b)p}{2\sigma p(2-s_c)} \int |\phi|^p \, dx \tag{4.6}
\end{equation}
and
\begin{equation}
\int |x|^{-b} |\phi|^{2\sigma+2} \, dx = \left( \frac{N(2\sigma+2)-(N-b)p}{2\sigma p(2-s_c)} + 1 \right) \int |\phi|^p \, dx. \tag{4.7}
\end{equation}

**Proof** First we note that (4.6) and (4.7) follow easily from (4.4) and (4.5). So, it suffices to establish (4.4) and (4.5). Below we proceed formally, but by using a standard argument we may see that we have the regularity and decay to justify all calculations.

To obtain (4.4) we just multiply (4.3) by $\overline{\phi}$, integrate on $\mathbb{R}^N$ and use integration by parts. On the other hand, by multiplying (4.3) by $x \cdot \nabla \overline{\phi}$ and taking the real part, we obtain
\begin{equation}
-\text{Re} \int \Delta^2 \phi \cdot x \cdot \nabla \overline{\phi} \, dx + \text{Re} \int |x|^{-b} |\phi|^{2\sigma} \phi \cdot x \cdot \nabla \overline{\phi} \, dx = \text{Re} \int |\phi|^{p-2} \phi \cdot x \nabla \overline{u} \, dx. \tag{4.8}
\end{equation}

Integrating by parts,
\begin{equation}
\int \Delta^2 \phi \cdot x \cdot \nabla \overline{\phi} \, dx = 4 \int |\Delta \phi|^2 \, dx - \int \nabla \phi \cdot x \Delta^2 \overline{\phi} \, dx - N \int |\Delta \phi|^2 \, dx,
\end{equation}
and thus
\begin{equation}
\text{Re} \int \Delta^2 \phi \cdot x \cdot \nabla \overline{\phi} \, dx = \left( 2 - \frac{N}{2} \right) \int |\Delta \phi|^2 \, dx. \tag{4.9}
\end{equation}

Integration by parts also yields
\begin{align*}
\int |x|^{-b} |\phi|^{2\sigma} \phi \cdot x \cdot \nabla \phi \, dx &= \sum_{j=1}^{N} \int |x|^{-b} |\phi|^{2\sigma} \phi_j \partial_j \overline{\phi} \, dx \\
&= \sum_{j=1}^{N} \partial_j (|x|^{-b} |\phi|^{2\sigma} \phi_j) \overline{\phi} \, dx \\
&= b \int |x|^{-b} |\phi|^{2\sigma+2} \, dx - 2\sigma \text{Re} \int |x|^{-b} |\phi|^{2\sigma} \overline{\phi} \cdot x \cdot \nabla \phi \, dx \\
&\quad - N \int |x|^{-b} |\phi|^{2\sigma+2} \, dx - \int |x|^{-b} |\phi|^{2\sigma} \phi \cdot x \cdot \nabla \phi \, dx,
\end{align*}
which implies
\[
\text{Re} \int |x|^{-b} |\phi|^{2\sigma} \phi x \cdot \nabla \bar{\phi} \, dx = -\frac{N - b}{2\sigma + 2} \int |x|^{-b} |\phi|^{2\sigma + 2} \, dx. \tag{4.10}
\]

In a similar fashion, we obtain
\[
\text{Re} \int |\phi|^{p-2} \phi x \cdot \nabla \bar{\phi} \, dx = -\frac{N}{p} \int |\phi|^p \, dx. \tag{4.11}
\]

Using (4.8), (4.9), (4.10), and (4.11) we get (4.5). \qed

Under the assumptions of Theorem 1.2, inequality (4.1) shows that the embedding $\dot{H}^2 \cap L^p \hookrightarrow L^{2\sigma+2}_b$ is continuous. Below we show that such an embedding is in fact compact.

**Proposition 4.2** Let $N \geq 1$, $0 < b < \min\{N, 4\}$, $0 < \sigma < 4^*$ and $1 < p < \frac{(2\sigma+2)N}{N-b}$. Then the embedding is compact.

**Proof** First we recall from Lemma 2.3 that
\[
\dot{H}^2 \cap L^q \hookrightarrow L^r, \tag{4.12}
\]
where $1 < q < r < 2^*$ ($2^*$ is given in (2.1)).

Now, let $\{u_n\}_{n=1}^{+\infty}$ be a bounded sequence in $\dot{H}^2 \cap L^p$. We want to show that, up to a subsequence, $\{u_n\}_{n=1}^{+\infty}$ converges in $L^{2\sigma+2}_b$. Define $r = \frac{(2\sigma+2)N}{N-b+\eta}$, with $0 < \eta \ll 1$ to be chosen later. Thus, in view of (4.12), $\{u_n\}_{n=1}^{+\infty}$ is also bounded in $L^r$. Since $L^r$ is a reflexive Banach space, there exists $u \in L^r$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $L^r$ as $n \to \infty$. Defining $w_n = u_n - u$, we will show that
\[
\int |x|^{-b} |w_n|^{2\sigma+2} \, dx \to 0,
\]
as $n \to \infty$. The main idea is to split the integral on the ball $B(0, R)$ and on $\mathbb{R}^N \setminus B(0, R)$, for appropriate $R > 0$.

Let us start by recalling that for any $R > 0$ and $\alpha > N$, we have
\[
\int_{\mathbb{R}^N \setminus B(0; R)} |x|^{-\alpha} \, dx \lesssim \frac{1}{R^{\alpha-N}}. \tag{4.13}
\]

Note that for $\eta$ small enough and $0 < \sigma < 4^*$ we have $p < r < 2^*$. \[ \square \]
By choosing \( \eta \) such that \( \eta < b \) and \( N - b + \eta > 0 \) we may define \( \gamma_1 = \frac{N}{N - b + \eta} \) and \( \gamma'_1 = \frac{N}{b - \eta} \). Thus, by Hölder’s inequality, we infer

\[
\int_{\mathbb{R}^N \setminus B(0, R)} |x|^{-b} |w_n|^{2\sigma + 2} \, dx \leq \left( \int_{\mathbb{R}^N \setminus B(0, R)} |x|^{-b \gamma'_1} \, dx \right)^{\frac{1}{\gamma'_1}} \left( \int_{\mathbb{R}^N \setminus B(0, R)} |w_n|^{(2\sigma + 2) \gamma_1} \, dx \right)^{\frac{1}{\gamma_1}}. \tag{4.14}
\]

Since \( b \gamma'_1 > N \) and \((2\sigma + 2) \gamma_1 = r\), given \( \varepsilon > 0 \) from (4.13) and (4.14) we may choose \( R > 0 \) large enough such that

\[
\int_{\mathbb{R}^N \setminus B(0, R)} |x|^{-b} |w_n|^{2\sigma + 2} \, dx < \varepsilon. \tag{4.15}
\]

Now, we estimate the integral over the ball \( B(0, R) \). Initially, let us show that \( \{w_n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( H^2(B(0, R)) \). Indeed, from the standard Gagliardo-Nirenberg inequality it suffices to show that \( \{w_n\}_{n \in \mathbb{N}} \) as well as the derivatives of order two are bounded in \( L^2(B(0, R)) \). Note if \( 1 < p < 2 \) then (4.12) implies

\[
\|u_n\|_{L^2(B(0, R))} \leq \|u_n\|_{L^2} \lesssim \|u_n\|_{\dot{H}^2(B(0, R))}. \tag{4.16}
\]

Also, if \( 2 \leq p < \frac{(2\sigma + 2)N}{N - b} \) then the embedding \( L^p(B(0, R)) \hookrightarrow L^2(B(0, R)) \) gives

\[
\|u_n\|_{L^2(B(0, R))} \lesssim \|u_n\|_{L^p(B(0, R))} \lesssim \|u_n\|_{L^p}. \tag{4.17}
\]

Moreover,

\[
\sum_{|\alpha| = 2} \|D^\alpha u_n\|_{L^2(B(0, R))} \leq \sum_{|\alpha| = 2} \|D^\alpha u_n\|_{L^2(\mathbb{R}^N)} \lesssim \|\Delta u_n\|_{L^2(\mathbb{R}^N)} \lesssim \|u_n\|_{\dot{H}^2}. \tag{4.18}
\]

From (4.16), (4.17), and (4.18) we conclude that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( H^2(B(0, R)) \).

Hence, from the Rellich-Kondrashov theorem (see, for instance, [15]) and the fact that \( w_n \to 0 \) in \( L^r(\mathbb{R}^N) \), we deduce that, up to a subsequence,

\[
w_n \to 0 \text{ in } L^q(B(0, R)) \quad \text{for all } q \in (1, 2^*). \tag{4.19}
\]

Next, for \( \eta < N - b \) the numbers \( \gamma_2 = \frac{N}{N - b - \eta} \) and \( \gamma'_2 = \frac{N}{b + \eta} \) are such \( \frac{1}{\gamma_2} + \frac{1}{\gamma'_2} = 1 \). Hence, by Hölder’s inequality we have

\[
\int_{B(0, R)} |x|^{-b} |w_n|^{2\sigma + 2} \, dx \leq \left( \int_{B(0, R)} |x|^{-b \gamma'_2} \, dx \right)^{\frac{1}{\gamma'_2}} \left( \int_{B(0, R)} |w_n|^{(2\sigma + 2) \gamma_2} \, dx \right)^{\frac{1}{\gamma_2}}.
\]
It is easy to see that \( b\gamma_2' < N \) and (for \( \eta \) small enough) \( 1 < (2\sigma + 2)\gamma_2 < 2^* \). So, in view of (4.19), given \( \varepsilon > 0 \), we obtain for \( n \) large enough,
\[
\int_{B(0,R)} |x|^{-b} |w_n|^{2\sigma+2} \, dx < \varepsilon. \tag{4.20}
\]
A combination of (4.15) and (4.20) completes the proof of Proposition 4.2. \( \square \)

With the compactness embedding in Proposition 4.2 we are able to prove Theorem 1.2.

**Proof of Theorem 1.2** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a minimizing sequence for (4.2), that is, a sequence of nontrivial functions in \( \dot{H}^2 \cap L^p \) satisfying \( J(f_n) \to J \) as \( n \to \infty \). By setting
\[
\mu_n = \frac{\|f_n\|_L^N_\frac{4}{2-\gamma_2'}}{\|f_n\|_{\dot{H}^2}} \quad \text{and} \quad \theta_n = \left( \frac{\|f_n\|_{L^p}}{\|f_n\|_{\dot{H}^2}} \right)^\frac{1}{2-\gamma_2'}
\]
and defining \( g_n(x) = \mu_n f_n(\theta_n x) \) it is not difficult to check that
\[
\|g_n\|_{L^p} = 1, \quad \|g_n\|_{\dot{H}^2} = 1, \quad \text{and} \quad J(g_n) = J(f_n) \tag{4.21}
\]
and, consequently, \( \{g_n\}_{n \in \mathbb{N}} \) is also a minimizing sequence for (4.2) and bounded in \( \dot{H}^2 \cap L^p \).

Hence, there exists \( g^* \in \dot{H}^2 \cap L^p \) such that, up to a subsequence, \( g_n \to g^* \) weakly in \( \dot{H}^2 \cap L^p \) and strongly in \( L^2_{b,2\sigma+2} \) (see Proposition 4.2). From (4.21) we deduce
\[
\|g^*\|_{L^p} \leq 1 \quad \text{and} \quad \|\Delta g^*\|_{L^2} = \|g^*\|_{\dot{H}^2} \leq 1,
\]
which implies
\[
J \leq J(g^*) \leq \lim_{n \to \infty} \frac{1}{\|g_n\|_{L^2_{b,2\sigma+2}}} = \lim_{n \to \infty} \frac{1}{\|g_n\|_{L^2_{b,2\sigma+2}}} = J. \tag{4.22}
\]
Consequently, the inequalities in (4.22) must be equalities and
\[
J = J(g^*) = \frac{1}{\|g^*\|_{L^2_{b,2\sigma+2}}} \quad \text{and} \quad \|g^*\|_{L^p} = \|\Delta g^*\|_{L^2} = 1. \tag{4.23}
\]
This shows, in particular, that \( g^* \neq 0 \) and \( g^* \) is a minimizer for the functional \( J \).

Next let us prove that, up to a scaling, \( g^* \) is a solution of (1.8). Indeed, since \( g^* \) is a minimizer of \( J \), we have
\[
\frac{d}{d\varepsilon} J(g^* + \varepsilon g)|_{\varepsilon=0} = 0, \quad \text{for all} \ g \in C_0^\infty(\mathbb{R}^N),
\]
which, using (4.23), is equivalent to

\[-\sigma (s_c - s_p) + (2 - s_p) \Delta^2 g^* + (\sigma + 1)J|x|^{-b}|g^*|^{2\sigma} g^* = \frac{\sigma (2 - s_c)}{2 - s_p} |g^*|^{p - 2} g^*.

Now, we let V be defined by \(g^*(x) = \alpha V(\beta x)\) with

\[
\alpha = \left[ \frac{A J}{B^{\frac{4 - b}{4\sigma}}} \right]^{\frac{4}{(p - 2)(4 - b) - 8\sigma}}, \quad \beta = B^{\frac{1}{4\sigma}} \left[ \frac{A J}{B^{\frac{4 - b}{4\sigma}}} \right]^{\frac{p - 2}{(p - 2)(4 - b) - 8\sigma}},
\]

and

\[
A = \frac{(\sigma + 1)(2 - s_p)}{\sigma (s_c - s_p) + (2 - s_p)}, \quad B = \frac{\sigma (2 - s_c)}{\sigma (s_c - s_p) + (2 - s_p)}.
\]

Hence V is a solution of (1.8) and

\[
\|V\|_{L^p}^p = \frac{\beta^N}{\alpha^p} \|g^*\|_{L^p}^p = \frac{\beta^N}{\alpha^p} = B^{\frac{N}{4\sigma}} \left( \frac{A J}{B^{\frac{4 - b}{4\sigma}}} \right)^{\frac{N(p - 2) - 4p}{(p - 2)(4 - b) - 8\sigma}}.
\]

Note that this implies

\[
K_{opt} = \frac{1}{J} = AB^{-\frac{(4 - b)(p - \sigma_c)}{2p(2 - s_p)}} \|V\|_{L^p} \frac{8\sigma - (p - 2)(4 - b)}{4 - 2\sigma} \left( \frac{(\sigma + 1)(2 - s_p)}{\sigma (s_c - s_p) + (2 - s_p)} \right) \|V\|_{L^p} \frac{8\sigma - (p - 2)(4 - b)}{4 - 2\sigma} ,
\]

which completes the proof of the theorem. \(\square\)

As a consequence of the Theorem 1.2, we are able to establish the global well-posedness result in Theorem 1.4.

**Proof of Theorem 1.4** As usual, the idea is to get an a priori bound of the local solution in \(H^2\). Since, by (1.4) the \(L^2\) norm is already conserved, we only need to bound \(\|\Delta u(t)\|_{L^2}\). To do so, we will use the conservation of the energy in (1.5) and Corollary 1.3 (part (i)). Indeed, first note that

\[
2E(u_0) = 2E(u(t)) = \|\Delta u(t)\|_{L^2}^2 - \frac{1}{\sigma + 1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \geq \|\Delta u(t)\|_{L^2}^2 - \frac{K_{opt}}{\sigma + 1} \|\Delta u(t)\|_{L^{2\sigma+2}}^{\sigma\beta_c+2} \|u(t)\|_{L^{(2-\beta_c)}^2}^{(2-\beta_c)} . \tag{4.24}
\]

By setting \(X(t) = \|\nabla u(t)\|_{L^2}^2\) and \(B = \frac{K_{opt}}{\sigma + 1} M[u_0] \frac{\sigma (2 - \beta_c)}{2}\), the last inequality reads as

\[
X(t) - BX(t) - \frac{\sigma\beta_c+2}{2} \leq 2E[u_0], \quad \forall t \in [-T, T],
\]

\(\square\) Springer
where $T > 0$ is the existence time provided by the local theory in Theorem A.

Next, for $x \geq 0$ we define the function $f(x) = x - Bx^{\frac{\sigma sc + 2}{2}}$. Since $\sigma > \frac{4 - b}{N}$ it is easy to see that $f$ has a maximum point at

$$x_0 = \left( \frac{2}{B(\sigma sc + 2)} \right)^{\frac{2}{\sigma sc}}$$

with maximum value

$$f(x_0) = \frac{\sigma sc}{\sigma sc + 2} \left( \frac{2}{B(\sigma sc + 2)} \right)^{\frac{2}{\sigma sc}}.$$

By requiring that

$$2E[u_0] < f(x_0) \quad \text{and} \quad X(0) < x_0, \quad (4.25)$$

the continuity of $X(t)$ yields that $X(t) < x_0$, for any $t \in [-T, T]$.

Applying Lemma 4.1 we obtain

$$E[Q] = \frac{1}{2} \| \Delta Q \|_{L^2}^2 - \frac{1}{2\sigma + 2} \| Q \|_{L_b^{2\sigma + 2}}^2 = \left( \frac{N\sigma + b}{4\sigma(2 - sc)} - \frac{1}{\sigma(2 - sc)} \right) \| Q \|_{L^2}^2$$

$$= \frac{sc}{2(2 - sc)} \| Q \|_{L^2}^2.$$

In addition, since

$$K_{opt} = \left( \frac{\sigma(2 - sc)}{\sigma sc + 2} \right)^{\frac{\sigma sc}{2}} \left( \frac{2\sigma + 2}{\sigma sc + 2} \right) \| Q \|_{L^2}^{-2\sigma}, \quad (4.26)$$

a straightforward calculation gives that (1.11) and (1.12) are equivalent to those relations in (4.25). Furthermore, (1.13) is equivalent to $X(t) < x_0$. The proof of the theorem is thus completed.

**Remark 4.3** An argument similar to that in the proof of Theorem 1.4 also holds in the case $\sigma = \frac{4 - b}{N}$. Indeed, since $sc = 0$, as in (4.24) we obtain

$$2E[u_0] \geq \| \Delta u(t) \|_{L^2}^2 \left( 1 - \frac{K_{opt}}{\sigma + 1} \| u_0 \|_{L^2}^{2\sigma} \right).$$

Moreover, because $K_{opt} = (\sigma + 1)\| Q \|_{L^2}^{-2\sigma}$ (see (4.26)) it follows that

$$2E[u_0] \geq \| \Delta u(t) \|_{L^2}^2 \left( 1 - \frac{\| u_0 \|_{L^2}^{2\sigma}}{\| Q \|_{L^2}^{2\sigma}} \right). \quad (4.27)$$

If we assume

$$\| u_0 \|_{L^2} < \| Q \|_{L^2}, \quad (4.28)$$
from (4.27) we get that \( \| \Delta u(t) \|_{L^2} \) is uniformly bounded. Thus, under assumption
(4.28) we obtain the global well-posedness in \( H^2 \) in the case \( \sigma = \frac{4-b}{N} \). This result
complements the one in [10, Proposition 1.5], where the authors proved the global
well-posedness assuming that \( \| u_0 \|_{L^2} \) is sufficiently small. In the limiting case \( b = 0 \) this
result was also established in [8, Theorem 4.2].

**Proof of Theorem 1.6** Since we are assuming (1.15) it is sufficient to show that
\( \| \Delta u(t) \|_{L^2} \) remains bounded. But, using the conservation of the energy and Corollary
1.3 (part (ii)) we deduce

\[
2 E[u_0] \geq \| \Delta u(t) \|_{L^2}^2 \left( 1 - \frac{\| u(t) \|_{L^2}^{2\sigma}}{\| V \|_{L^{2\sigma}}^{2\sigma}} \right)
\]

Thus, under assumption (1.14) we obtain the desired and the proof of the theorem is
completed. \( \square \)

**Remark 4.4** Theorem 1.6 is also true in the case \( \frac{1}{2} < \sigma = \frac{4-b}{N} \), in which case we have
\( \sigma_c = 2 \) and (1.14) reduces to (4.28).

## 5 Concentration in the critical Lebesgue norm

In this section, we prove our main result about \( L^{\sigma_c} \)-norm concentration in the inter-
critical regime for finite time blow-up solution.

**Proof of Theorem 1.8** Clearly it suffices to show that

\[
\liminf_{n \to \infty} \int_{|x| \leq \lambda(t_n)} |u(x, t_n)|^{\sigma_c} \, dx \geq \| V \|_{L^{\sigma_c}}^{\sigma_c},
\]

for any sequence \( \{ t_n \}_{n \in \mathbb{N}} \) satisfying \( t_n \uparrow T^* \), as \( n \to \infty \).

To prove this, let \( \{ t_n \}_{n \in \mathbb{N}} \) be an arbitrary sequence such that \( t_n \uparrow T^* \) and define

\[
\rho_n = \left( \frac{1}{\| u(t_n) \|_{\dot{H}^{s_c}}} \right)^{\frac{1}{2-\sigma_c}} \quad \text{and} \quad v_n(x) = \rho_n^{\frac{4-b}{2\sigma}} u(\rho_n x, t_n).
\]

By using (1.15) we may take a positive constant \( C \) such that

\[
\| v_n \|_{\dot{H}^{s_c}} = \| u(t_n) \|_{\dot{H}^{s_c}} \leq C.
\]

Thus, in view of the embedding \( \dot{H}^{s_c} \subset L^{\sigma_c} \) we deduce that \( \{ v_n \}_{n \in \mathbb{N}} \) is bounded in
\( L^{\sigma_c} \). Also, a straightforward calculation gives

\[
\| v_n \|_{\dot{H}^{s_c}}^2 = \rho_n^{\frac{2(4-b)}{2\sigma} + 4-N} \| u(t_n) \|_{\dot{H}^{s_c}}^2 = \rho_n^{2(2-s_c)} \| u(t_n) \|_{\dot{H}^{s_c}}^2 = 1.
\]

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So \( \{v_n\}_{n \in \mathbb{N}} \) is a bounded sequence in \( L^{\sigma_c} \cap \dot{H}^2 \). Consequently, there exists \( v^* \in L^{\sigma_c} \cap \dot{H}^2 \) such that, up to a subsequence, \( v_n \rightharpoonup v^* \) in \( L^{\sigma_c} \cap \dot{H}^2 \), as \( n \to \infty \). This implies

\[
\| \Delta v^* \|_{L^2} \leq \liminf_{n \to \infty} \| \Delta v_n \|_{L^2} \quad \text{and} \quad \| v^* \|_{L^{\sigma_c}} \leq \liminf_{n \to \infty} \| v_n \|_{L^{\sigma_c}}. \tag{5.2}
\]

Furthermore, by Proposition 4.2 (passing to a subsequence if necessary)

\[
\lim_{n \to \infty} \| v_n \|_{L^{2\sigma_c+2}_b} = \| v^* \|_{L^{2\sigma_c+2}_b} \tag{5.3}
\]

On the other hand, note that

\[
E(v_n) = \frac{1}{2} \| \Delta v_n \|_{L^2}^2 - \frac{1}{2\sigma + 2} \int |x|^{-b} |v_n|^{2\sigma + 2} \, dx = \rho_n^{2(2-\varepsilon_c)} E(u_0)
\]

and since \( \rho_n \to 0 \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} E(v_n) = 0. \tag{5.4}
\]

Combining Corollary 1.2 together with (5.2), (5.3) and (5.4) one has

\[
0 = \liminf_{n \to \infty} E(v_n) \geq \frac{1}{2} \| \Delta v^* \|_{L^2}^2 \left( 1 - \frac{\| v^* \|_{L^{\sigma_c}}^{2\sigma}}{\| V \|_{L^{\sigma_c}}^{2\sigma}} \right),
\]

which implies \( \| v^* \|_{L^{\sigma_c}} \geq \| V \|_{L^{\sigma_c}} \).

Next, the weak convergence \( v_n \rightharpoonup v^* \) in \( L^{\sigma_c} \), gives that for any \( R > 0 \),

\[
\liminf_{n \to \infty} \int_{|y| \leq \rho_n R} |u(y, t_n)|^{\sigma_c} \, dy = \liminf_{n \to \infty} \int_{|x| \leq R} \rho_n^{\sigma_c(4-b)} |u(\rho_n x, t_n)|^{\sigma_c} \, dx = \liminf_{n \to \infty} \int_{|x| \leq R} |v_n(x)|^{\sigma_c} \, dx \geq \int_{|x| \leq R} |v^*|^{\sigma_c} \, dx,
\]

Since assumption (1.16) implies \( \lambda(t_n)/\rho_n \to \infty \), as \( n \to \infty \), we obtain

\[
\liminf_{n \to \infty} \int_{|x| \leq \lambda(t_n)} |u(x, t_n)|^{\sigma_c} \, dx \geq \int_{|x| \leq R} |v^*|^{\sigma_c} \, dx, \tag{5.5}
\]

for any \( R > 0 \). Finally, from (5.5) and the fact that \( \| v^* \|_{L^{\sigma_c}} \geq \| V \|_{L^{\sigma_c}} \) we obtain (5.1), which completes the proof. \( \square \)

**Remark 5.1** Applying the same arguments as in the proof of Theorem 1.8, we can show the \( \dot{H}^{\sigma_c} \)-norm concentration to IBNLS for \( N \geq 5 \), \( 0 < b < \min\{ \frac{N}{2}, 4 \} \) and
max\left\{ \frac{4-b}{N}, \frac{1}{2}\right\} < \sigma < \frac{4-b}{N-4}. \text{ Indeed, we note that Proposition 4.2 also allows us to obtain the following Gagliardo-Niremberg inequality for functions in } H^s \cap H^2 \int_{\mathbb{R}^N} |x|^{-b} |f(x)|^{2\sigma+2} \, dx \leq \frac{\sigma + 1}{\|W\|_{H^s}^{2\sigma}} \|\nabla f\|_{L^2}^{2} \|f\|_{H^s}^{2\sigma},

\text{ where } W \text{ is a solution to elliptic equation}

-\Delta^2 W + |x|^{-b} |W|^{2\sigma} W - (-\Delta)^s W = 0.

\text{ This gives us, in the context of proof of Theorem 1.8, that } v_n \rightharpoonup v^* \text{ in } H^2 \cap H^s \text{ and } \|v^*\|_{H^s} \geq \|W\|_{H^s}, \text{ and thus, we obtain that}

\liminf_{t \to T^*} \int_{|x| \leq \lambda(t)} |D^s u(x,t)|^2 \, dx \geq \|W\|_{H^s}^2.

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