Engineering non-Gaussian entangled states with vortices by photon subtraction

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Abstract. We show how a quantum mechanical vortex state of a two-mode system can be generated from a squeezed vacuum by subtracting a photon. The vortex state has nonclassical properties; for example, its Wigner function can be negative, in contrast to the Wigner function of the two-mode squeezed vacuum. We show, by calculating the logarithmic negativity parameter, that the vortex state has stronger entanglement than the two-mode squeezed vacuum.

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1. Introduction

Photons, being massless, usually have their quantum states described in Fock space. Each Fock state consists of a precise number of photons. All other interesting states of the field are constructed as a combination of Fock states (see, e.g., [1]), and different combinations have different quantum properties. The most commonly studied physical properties of the radiation field are (a) photon statistics and (b) squeezing, which contains information about the phase of the field. While photon statistics is directly described in terms of the Fock states, squeezing
requires the use of a different space called quadrature space. Many of the commonly studied states such as coherent states and squeezed states are Gaussian in quadrature space.

In this paper, we present the quantum engineering of states that (i) are non-Gaussian in quadrature space and more importantly (ii) have a vortex structure. We will show the strong nonclassical and entanglement properties of such states with vortex structures. The states that we present are different from the states of quantized electromagnetic fields carrying orbital angular momentum [2–4]. In this case, the mean electric field in the coordinate space has a vortex structure; that is, the field has a mode function, which has a vortex structure. Note that the mode functions are obtained from the solutions of the classical Maxwell equations and have no quantum features by themselves.

The structure of this paper is as follows. In section 2, we demonstrate the production of a vortex state by subtracting a photon from the two-mode squeezed vacuum state. In section 3, we show the nonclassicality of the vortex state. We also establish the entanglement character of the vortex state.

2. Engineering a vortex state

Consider the well-known two-mode squeezed field defined by

\[ |\xi\rangle = \exp(\xi a^\dagger b^\dagger - \xi^* ab)|0,0\rangle, \quad \xi = r e^{i\phi}, \tag{1} \]

where \( \xi \) is a complex parameter, \( a \) and \( b \) represent two modes of the field with commutation relations \([a, a^\dagger] = [b, b^\dagger] = 1, [a, b^\dagger] = 0, \) etc. The quadrature operators \( x_a, x_b, y_a \), and \( y_b \) are defined by

\[ a = \frac{x_a + iy_a}{\sqrt{2}}, \quad b = \frac{x_b + iy_b}{\sqrt{2}}. \tag{2} \]

These obey the commutation relations \([x_a, y_a] = i, [x_b, y_b] = i\). The quadrature distribution associated with (1) is

\[ \Psi(x_a, x_b) = \frac{1}{\sqrt{(1 - \eta^2)\pi \cosh^2 r}} \exp\left[ \frac{2x_a x_b \eta - (x_a^2 + x_b^2) \eta^2}{1 - \eta^2} - \frac{1}{2}(x_a^2 + x_b^2) \right], \tag{3} \]

\( \eta = e^{i\phi} \tanh r \).

The quadrature distribution for the two-mode squeezed vacuum is a two-dimensional (2D) Gaussian. Further, \( \Psi(x_a, x_b) \) does not factorize into a product of functions with each depending on either \( x_a \) or \( x_b \). This, as is well known, reflects the entanglement character of the two-mode squeezed vacuum.

We next consider the engineering of a vortex state in quadrature space. One method that has been extensively used in engineering quantum states of radiation fields consists of adding [5–9] or subtracting [10–12] the photon from the well-known states of the quantum field. Consider the arrangement shown in figure 1. Here, a downconverter produces a two-mode squeezed state \( |\xi\rangle \) of the signal (s) and the idler (i) modes \( a \) and \( b \). We then subtract a photon from mode \( b \) via a beam splitter with low reflectivity and detection of one photon by the avalanche photo diode (APD), or even better by a single-photon detector. The beam splitter would have a reflectivity of the order of a few per cent so that the probability of finding two photons would be of the order of \( 10^{-4} \). The subtraction procedure has been demonstrated to nearly 100% accuracy in several
experiments. The use of a single-photon detector is much better [11, 13, 14]. The resulting state of the output field would be

$$|\xi\rangle^{(s)} = N b \exp(\xi a^\dagger b^\dagger - \xi^* ab)|0, 0\rangle,$$

$$N = \langle \xi | b^\dagger b | \xi \rangle^{-1/2} = 1 / \sinh r.$$  (4)

We can rewrite equation (4) as

$$|\xi\rangle^{(s)} = N \exp(\xi a^\dagger b^\dagger - \xi^* ab) \exp(-\xi a^\dagger b^\dagger + \xi^* ab) b \exp(\xi a^\dagger b^\dagger - \xi^* ab)|0, 0\rangle.$$  (5)

This can be simplified using the Bogoliubov transformation to

$$|\xi\rangle^{(s)} = N \exp(\xi a^\dagger b^\dagger - \xi^* ab)(b \cosh r + a^\dagger e^{i\phi})|0, 0\rangle.$$

$$= \exp(\xi a^\dagger b^\dagger - \xi^* ab)|1, 0\rangle e^{i\phi}.$$  (6)

Thus the state $|\xi\rangle^{(s)}$ is obtained by the application of squeezing transformation to a state with a single photon in mode $a$. The state $|\xi\rangle^{(s)}$ has many nonclassical properties which arise from the presence of both the Fock state $|1\rangle$ and the squeezing operator. We would show that the Wigner function for $|\xi\rangle^{(s)}$ can be negative. The state (6) can be written in terms of the Fock states by using the well-known decomposition of the squeezing operator [15, 16]

$$S(\xi) = \exp(e^{i\phi} \tanh r a^\dagger b^\dagger) \exp[-(\ln \cosh r) (a^\dagger a + b^\dagger b + 1)] \exp(-e^{i\phi} \tanh r a b)$$  (7)

and then

$$|\xi\rangle^{(s)} = \frac{e^{i\phi}}{\cosh r} \sum_{n=0}^{\infty} e^{in\phi} (\tanh r)^n \sqrt{n+1} |n+1, n\rangle.$$  (8)

$$= e^{i\phi} a^\dagger |\xi\rangle / (\cosh r).$$  (9)

Thus the state (8) can also be obtained by adding a photon to the state $|\xi\rangle$ in the mode $a$. The state (8) has the property that the difference between the numbers of photons in the signal and idler modes is unity. The state (8) has an unusual property. It can have a vortex structure. The quadrature distribution $\Psi_{1}(x_a, x_b)$ can be calculated using (9) and the representation of $a^\dagger$ in

Figure 1. Scheme for the production of single-photon-subtracted two-mode states using a beam splitter with low reflectivity.
terms of $x_a$ and the derivative $\partial/\partial x_a$:

$$a^\dagger = \frac{1}{\sqrt{2}} \left( x_a - \frac{\partial}{\partial x_a} \right),$$  

(10)

$$\Psi^{(s)}(x_a, x_b) = \frac{e^{i\varphi}}{\sqrt{2} \cosh r} \left( x_a - \frac{\partial}{\partial x_a} \right) \Psi(x_a, x_b),$$  

(11)

where $\Psi(x_a, x_b)$ is the quadrature distribution (3) for the two-mode squeezed vacuum. On simplification, (11) becomes

$$\Psi^{(s)}(x_a, x_b) = \frac{\sqrt{2} e^{i\varphi} (x_a - \eta x_b)}{(1 - \eta^2)^{3/2} \pi^{1/2} \cosh^2 r} \exp \left[ \frac{2x_a x_b \eta - (x_a^2 + x_b^2) \eta^2}{1 - \eta^2} - \frac{1}{2} (x_a^2 + x_b^2) \right].$$  

(12)

For $\eta = i|\eta|$ this has the structure of a vortex. The $|\Psi^{(s)}(x_a, x_b)|^2$ and contours of the constant $|\Psi^{(s)}(x_a, x_b)|^2$ are shown in figure 2, which also shows the phase of the wavefunction. The vortex structure is clear, and the ellipticity of the vortex structure is evident as $|\eta| \neq 1$. The details of a variety of states with vortex structure can be found in [17].

3. Nonclassicality and entanglement in the vortex state

Next we discuss the negativity of the Wigner function for the state $|\xi^{(s)}\rangle$. This can be obtained by first noting that the Wigner function $W_{\xi}(\alpha, \beta)$ for the Fock state $|1, 0\rangle$ is

$$W_{\xi}(\alpha, \beta) = \frac{4}{\pi^2} (4|\alpha|^2 - 1) \exp[-2(|\alpha|^2 + |\beta|^2)].$$  

(13)

It can be shown that if two density matrices $\rho$ and $\tilde{\rho}$ are related by the squeezing transformation

$$\rho = \exp(\xi a^\dagger b^\dagger - \xi^* a b) \tilde{\rho} \exp(\xi^* a b - \xi a^\dagger b^\dagger),$$  

(14)

then the Wigner functions are related by

$$W_{\rho}(\alpha, \beta) = W_{\tilde{\rho}}(\tilde{\alpha}, \tilde{\beta}),$$  

(15)

where

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta}^* \end{pmatrix} = \begin{pmatrix} \cosh r & -\sinh r e^{i\varphi} \\ -\sinh r e^{-i\varphi} & \cosh r \end{pmatrix} \begin{pmatrix} \alpha \\ \beta^* \end{pmatrix}. $$  

(16)

Hence, the Wigner function for the state $|\xi^{(s)}\rangle$ will be

$$W^{(s)}(\alpha, \beta) = \frac{4}{\pi^2} (4|\tilde{\alpha}|^2 - 1) \exp[-2(|\tilde{\alpha}|^2 + |\tilde{\beta}|^2)],$$  

(17)

where $\tilde{\alpha}$ and $\tilde{\beta}$ are given by (16). The negative regions of the Wigner function are given by

$$4|\alpha \cosh r - \beta^* \sinh r e^{i\varphi}|^2 < 1.$$  

(18)

Clearly, in 4D space there are many values of $\alpha$ and $\beta$ for which equation (18) can be easily satisfied. We next consider quantitatively the entanglement in the vortex state by computing the log negativity parameter defined by [18, 19]

$$E = \log_2(1 + 2N),$$  

(19)
Figure 2. (a) The intensity $|\Psi(x_a, x_b)|^2$, (b) the phase of $\Psi(x_a, x_b)$ and (c) the contour of $|\Psi(x_a, x_b)|^2$ as a function of $x_a$ and $x_b$ for $r = 1$ and $\eta = i \tanh r \approx 0.76i$. 

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The ratio \( \tilde{E} \), which gives the entanglement in the vortex state relative to the entanglement in the two-mode squeezed vacuum, as a function of \( r \).

where \( N \) is the modulus of the sum of all the negative eigenvalues associated with the partial transpose \([20, 21]\) \( \rho^{PT} \) of the density matrix \( |\xi\rangle\langle\xi| \). From equation (8), the partial transpose is

\[
\rho^{PT} = \sum_{n,m} c_n c_m |n + 1, m\rangle \langle m + 1, n| e^{i(n-m)\varphi},
\]

(20)

\[
c_n = (\tanh r)^n \sqrt{1 + n/cosh^2 r}.
\]

(21)

The diagonal elements in (20) are all positive. The terms \( n \neq m \) in (20) have the form

\[
|n + 1, m\rangle \langle m + 1, n| e^{i(n-m)\varphi} + |m + 1, n\rangle \langle n + 1, m| e^{i(m-n)\varphi},
\]

which can be written in the diagonal form

\[
\frac{1}{2} (e^{i\varphi} |n + 1, m\rangle + e^{i\varphi} |m + 1, n\rangle)(e^{-i\varphi} \langle n + 1, m| + e^{-i\varphi} \langle m + 1, n|)
\]

\[\quad - \frac{1}{2} (e^{i\varphi} |n + 1, m\rangle - e^{i\varphi} |m + 1, n\rangle)(e^{-i\varphi} \langle n + 1, m| - e^{-i\varphi} \langle m + 1, n|).
\]

(22)

Thus all the negative eigenvalues are \(-c_n c_m\) and hence the log negativity parameter (19) becomes

\[
E = \log_2 \left( 1 + \sum_{n \neq m} c_n c_m \right)
\]

\[= \log_2 \left( \sum_n c_n \right)^2, \text{ since } \sum_n c_n^2 = 1.
\]

(23)

The sum in (23) can be evaluated numerically. A similar calculation for the two-mode squeezed vacuum (where the corresponding \( c_n' \)s are given by \((\tanh r)^n/cosh r\)) gives the result \( \log_2(e^{2r}) \). Thus entanglement in the vortex state can be compared with that in the state \( |\xi\rangle \) by studying the ratio

\[
\tilde{E} = \left( \sum_n c_n e^{-r} \right)^2.
\]

(24)
This ratio is shown in figure 3 over a range of values of the parameter $r$. It is to be noted that $\tilde{E} > 1$ and hence in a sense the vortex state $|\xi\rangle(s)$ is more entangled than the two-mode squeezed vacuum $|\xi\rangle$.

4. Conclusions

In conclusion, we have shown how quantum states with vortices in quadrature space can be engineered by photon subtraction on the two-mode squeezed vacuum. We show that the Wigner function for the vortex state can be negative. Further, we show that entanglement in the vortex state is stronger than in the Gaussian state, i.e. the two-mode squeezed vacuum. Clearly, our method can be extended to generate higher-order vortex states in quadrature space.

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