REPRESENTATIONS OF $p$-ADIC GROUPS OVER COEFFICIENT RINGS

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Abstract. Motivated by the Langlands program in representation theory, number theory and geometry, the theory of representations of a reductive $p$-adic group over a coefficient ring different from the field of complex numbers has been widely developed during the last two decades. This article provides a survey of basic results obtained in the 21st century.

Contents

1. Introduction 1
2. Notation 2
3. Change of basic field 3
4. Change of coefficient ring 4
5. Parabolic induction 7
6. Admissible representations and duality 8
7. Supercuspidal support 9
8. Hecke algebras 11
9. Representations in characteristic different from $p$ 12
10. Bernstein blocks 14
11. Satake isomorphism 16
12. Pro-$p$ Iwahori Hecke ring 18
13. Modules of pro-$p$ Iwahori Hecke algebras in characteristic $p$ 20
14. Representations in characteristic $p$ 22
15. Local Langlands correspondences for $GL(n, F)$ 24
16. Gelfand-Kirillov Dimension 26
References 28

1. Introduction

The theory of representations of a $p$-adic group $G$, for instance $GL(n, \mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the $p$-adic completion of $\mathbb{Q}$ is an essential part of the Langlands program. At the beginning, it was a question of studying representations in a complex vector space. But the development of its links with number theory and geometry has required to study continuous representations in vector spaces defined over other fields than $\mathbb{C}$. There are many possibilities for such a generalization. It is easy to replace $\mathbb{C}$ by an algebraic closure $\mathbb{Q}_\ell^{ac}$ of a local field $\mathbb{Q}_\ell$, where $\ell$ is a prime different from $p$. The choice of a field isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{ac}$ identifies continuous complex representations of $G$ and continuous $\ell$-adic representations. A more difficult case is that of $\ell = p$ because the topology of a $p$-adic group and of $\mathbb{Q}_p$ are the same.
One even considers representations with values not in a vector space, but in a module over some commutative ring like \( \mathbb{Z}[1/p] \) or \( \mathbb{Z}/p^i\mathbb{Z}, i \geq 1 \). The representations over these different categories of coefficient rings are now essential in the theory of automorphic forms. Their theory has been widely developed since the beginning of the 21st century and different versions of the local Langlands correspondence have emerged.

We review the main basic results on representations over coefficient rings different from \( \mathbb{C} \). In an attempt to make this paper accessible to readers with a wide range of backgrounds, we give fairly complete definitions of all terminology. Proofs are omitted, yet we give some short indication of the key points, we cite sources and we provide examples. For the theory before 2002, the reader may consult our book and our article in the proceedings of the Beijing ICM. The subject has remained confined in research articles since these last two decades and we hope that this survey provides a navigable route to the literature.

2. Notation

We work with a triple \((F,G,R)\) where \( F \) is the basic field, \( G \) the reductive \( p \)-adic group, \( R \) the coefficient ring. We assume that \( F \) is a local non-archimedean field of ring of integers \( O_F \), uniformizer \( p_F \) and residue field \( k_F \) of characteristic \( p \) with \( q \) elements, \( G \) is the group \( \bar{\mathcal{G}}(F) \) of \( F \)-points of a connected reductive \( F \)-group \( \mathcal{G} \), endowed with the topology generated by the open pro-\( p \)-subgroup and \( R \) is a commutative ring.

An \( R \)-representation \( V \) of \( G \) will always be smooth (continuous for the discrete topology on \( R \)). It is admissible if for all open compact subgroups \( K \) of \( G \), the \( R \)-module \( V^K \) of vectors fixed by \( K \) is finitely generated.

The absolute Galois group \( \text{Gal}_E \) of a field \( E \) is the group of automorphisms of an algebraic closure \( E^{ac} \) fixing \( E \). For a prime number \( r \), \( \mathbb{F}_r \) is a field with \( r \) elements, \( \mathbb{Z}_r \) is the ring of integers in the field \( \mathbb{Q}_r \) of \( r \)-adic numbers, \( \mathbb{Z}_r^{ac} \) is the ring of integers of \( \mathbb{Q}_r^{ac} \). We always denote by \( \ell \) a prime number different from \( p \).

The parabolic and parahoric subgroups of \( G \) play an essential role in the theory of \( R \)-representations of \( G \). When we will work with them we will need more notation.

The parabolic subgroups appear for the first time at the section on parabolic induction. We fix a maximal split torus \( T \) of \( G \) of \( G \)-centralizer \( Z \) and a minimal parabolic subgroup \( B = ZU \) of unipotent radical \( U \) and opposite \( B^{op} = ZU^{op} \). We denote \( W_G \) the Weyl group quotient of the \( G \)-normalizer of \( T \) by \( Z \), \( Z^+ \subset Z \) the submonoid of elements contracting \( U \) by conjugation, \( Z^- \) those contracting \( U^{op} \), \( T^+ = T \cap Z^+, T^- = T \cap Z^- \). The group \( G \) is split if \( T = Z \) and quasi-split if \( Z \) is a torus. A standard parabolic subgroup of \( G \) is a parabolic subgroup containing \( B \), that is, \( P = MN = MB \) with unipotent radical \( N \) contained in \( U \) and Levi subgroup \( M \) containing \( Z \). The opposite parabolic subgroup \( P^{op} = MN^{op} = MB^{op} \) is not standard.

The parahoric subgroups appear for the first time at the section on Hecke algebras. We fix a special parahoric subgroup \( K \) of \( G \), equal to a parahoric subgroup of \( G \) fixing a special point \( x_0 \) of the apartment of \( T \) in the adjoint Bruhat-Tits building of \( G \), and a pro-\( p \)-Iwahori

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1. that we are aware of, without geometry or derived functors
2. Représentations \( \ell \)-modulaires d’un groupe réductif \( p \)-adique avec \( \ell \) différent de \( p \), Birkaüser 1996
3. called a connected reductive \( p \)-adic group, but beware that some authors use this terminology only when \( F \) contains \( \mathbb{Q}_p \)
4. a ring is supposed to have a unit
5. When \( G \) is not quasi-split, \( Z \) is not commutative
subgroup $\tilde{J}$ of $G$, equal to the maximal open normal pro-$p$ subgroup of the Iwahori subgroup $J$ of $G$ fixing the alcove of vertex $x_0$ associated to $B$. The unique parahoric subgroup of $T$ is the maximal compact subgroup $T^0 = T \cap K = T \cap J$ and the quotient $T/T^0$ is isomorphic to the group $X_\ast(T)$ of cocharacters of $T$ via $p_F$. The unique parahoric subgroup of the connected reductive group $Z$ is $Z^0 = Z \cap K = Z \cap J$, and the quotient $Z/Z^0$ is a commutative finitely generated group (Thomas Haines and Sean Rostami [32]). For a standard parabolic subgroup $P= MN$ of $G$, $M_0^0 = M \cap K$ is a special parahoric subgroup of $M$ and $\tilde{J}_M = \tilde{J} \cap M$ is a pro-$p$ Iwahori subgroup of $M$. Put $N^0 = K \cap N$.

The pro-$p$-Iwahori subgroups of $G$ are all $G$-conjugate, but in general there are only finitely many $G$-conjugacy classes of special parahoric subgroups of $G$.

Examples There are two conjugacy classes of special parahoric subgroups of $SL(2,F)$.

The special parahoric subgroups of $GL(n,F)$ are conjugate to $GL(n,O_F)$.

The (pro-$p$) Iwahori subgroups of $GL(n,F)$ are conjugate to the inverse image by the quotient map $GL(n,O_F) \to GL(n,k_F)$ of the upper (strictly) triangular group of $GL(n,k_F)$.

3. Change of basic field

The basic field $F$ is a finite extension of $\mathbb{Q}_p$, or of $\mathbb{F}_p((t))$, called a $p$-adic field in characteristic 0 or a local function field in characteristic $p$. Many geometric methods demand $F$ to be a local function field. For example, the proof by Bao Chau Ngô [4] of the fundamental lemma essential in the Langlands theory, which asserts an equality between certain linear combinations of integral orbitals over the Lie algebras of $G$ and of endoscopic groups. On the other hand, for $F$ of characteristic $p$, the harmonic analysis is full of traps, there are inseparable semi-simple elements, there is no exponential map to pass to the Lie algebra and $G$ does not contain a co-compact discrete subgroup (except for type A), $G$ is not a $p$-adic Lie group.

But the basic field $F$ appears only through the residual field in many constructions (endoscopy, buildings, Iwahori Hecke algebras). This is a key to transfer properties between $F$ of different characteristics. For instance, Jean-Loup Waldspurger [199] proved that the fundamental lemma for $F$ of characteristic $p$ implies the fundamental lemma for $F$ of characteristic 0. There is another proof with the general transfer principle of Cluckers and Loeser in model theory and motivic integration [31, 32]. In the other direction, the fundamental lemma for the automorphic induction for $GL(n,F)$ proved by Guy Henniart and Rebecca Herb for $F$ of characteristic 0 was transfered to $F$ of characteristic $p$ by Henniart and Bertrand Lemaire [101] using close local fields. For a positive integer $m$, two non-archimedean local fields are $m$-closed, if their rings of integers modulo the $m$-th power of their respective maximal ideals are isomorphic. The Deligne-Kazhdan philosophy can be loosely stated as: the representation theory of Galois groups or of reductive groups over $m$-close local fields are the same “up to level $m$”. For instance, Radhika Ganapathy [72] proved that for two $m$-close local fields $F, F'$ and $G$ split, the category of complex representations of $G(F)$ generated by their invariants by the $m$-filtration subgroup of an Iwahori subgroup is equivalent to the same category for representations of $G(F')$. For $G$ not split, she made sense of a natural connected reductive group $\tilde{G}$ over $F'$ associated to $G$, first when $\tilde{G}$ is quasi-split (an $F$-form of a split group) and then when $\tilde{G}$ is general (an inner form of a quasi-split group) (173 3.A and 5.A).

\footnote{Fields medal in 2020}
The local field \( \mathbb{Q}_p \) is a completion of \( \mathbb{Q} \) and \( \mathbb{Q} \) is a generalisation of \( \mathbb{Q}_p \). The local is simpler than the global. The ring \( \mathbb{Z}_p \) has only one prime ideal, namely \( p\mathbb{Z}_p \), but the ring \( \mathbb{Z} \) has infinitely many prime ideals. The absolute Galois group \( \text{Gal}_{\mathbb{Q}} \) of \( \mathbb{Q}_p \) is simple compared to \( \text{Gal}_{\mathbb{Q}} \). In the same vein, the local field \( F \) is the completion of a (non-unique) global field \( E \) and \( E \) is a generalisation of \( F \), the local group \( G \) is a localisation the group \( H \) of rational points of a connected reductive group over a global field, and \( H \) is a generalisation of \( \mathbb{C}^\times \). An automorphic irreducible \( \mathbb{C} \)-representation \( V_A \) of the adelic group \( H(A) \) gives by localisation an irreducible \( \mathbb{C} \)-representation \( V \) of \( G \) and \( V_A \) is a globalisation of \( V \). The study of automorphic representations uses the theory of representations of reductive groups over local fields. In the other direction, some theorems of representations of local groups are proved by embedding the local situation into a global one.

The classical local Langlands correspondence introduced by Langlands in 1967-1970 is a generalization of local class field theory from abelian Galois groups to non-abelian Galois groups. The absolute Galois group \( \text{Gal}_{\mathbb{Q}} \) of the finite field \( \mathbb{F}_p \) is topologically generated but the Frobenius \( \text{Frob}(x) = x^p \), the subgroup of elements in \( \text{Gal}_{\mathbb{F}} \) with image an integral power of \( \text{Frob} \) in the natural quotient map \( \text{Gal}_{\mathbb{F}} \to \text{Gal}_{\mathbb{Q}} \), is the Weil group \( W_F \) of \( F \). The reciprocity map of local class field theory \( F^* \to W_F^{ab} \) identifies the irreducible \( R \)-representations of \( GL(1, F) \) with the one-dimensional \( \mathbb{R} \)-representations of \( W_F \) when \( R \) is an algebraically closed field. Langlands proposed a parametrisation of the irreducible \( \mathbb{C} \)-representations of \( G \) in terms of \( \mathbb{C} \)-representations of \( W_F \), with a purely local characterization. It is a theorem when \( G = GL(n, F) \), generalized to representations of \( GL(n, F) \) over \( R = \mathbb{F}_l^\circ, \ell \neq p \). The first proofs of local class field theory were global. To-day the proofs of the local Langlands correspondence for \( GL(n, F) \) need global arguments except for \( n = 2 \) and \( R = \mathbb{C} \), which has a local proof (Colin Bushnell and Henniart [22]). When \( F \supset \mathbb{Q}_p \) and \( R = \mathbb{C} \), Peter Scholze [172] gave a new purely local characterization. The geometrization of a (semisimple) Langlands correspondence for all \( F, G \) and \( R = \mathbb{Z}_\ell \) for almost all \( \ell \neq p \), by Laurent Fargues and Scholze in 2021, is entirely local. A local Langlands correspondence for \( GL(n, F) \), \( F \) of characteristic 0, over \( R = \mathbb{F}_p^\circ \) is a very active research area.

4. Change of coefficient ring

Many features of complex representations of \( G \) use harmonic analysis only apparently and can generalize to representations over other coefficient rings. For instance,

a) The theory of discrete series and tempered complex representations has an algebraic and combinatorial flavour.\(^7\) It was extended by Dat [38] to an algebraically closed field \( R \) of characteristic different from \( p \) with a non-trivial valuation.

b) The proof of the classification of the irreducible complex representations of an inner form of \( GL(n, F) \) by Tadic for \( F \supset \mathbb{Q}_p \) uses harmonic analysis (the simple trace formula). Alberto

\(^7\)a global field is a finite extension of \( \mathbb{Q} \) or of \( \mathbb{F}_p(T) \)

\(^8\)for \( F \) of characteristic \( p \) Wee-Teck Gan, Luis Lomeli [70], for \( F \) of characteristic 0, Shahidi (A proof of Langland’s Conjecture on Plancherel measures; Complementary Series of p-adic groups, The Annals of Math., Series 2, Vol.132, 2 (1990), 273-330) when \( G \) is quasi-split, implying the general case as in [70]

\(^9\)the kernel \( I_F \) of the quotient map is an extension of \( \prod_{\ell \neq p} \mathbb{Z}_\ell \) by a pro-p group \( P_F \)

\(^10\)proved when \( R = \mathbb{C} \) by Gérard Laumon, Michael Rapoport, and Ulrich Stuhler in 1993 if \( F \supset \mathbb{F}_p^\circ(1) \), and if \( F \supset \mathbb{Q}_p \) by Michael Harris and Richard Taylor in 2001, (Guy Henniart gave another proof), and extended by V. in 2001 to \( R = \mathbb{F}_l^\circ, \ell \neq p \)

\(^11\)there is yet nothing for \( F \) of characteristic \( p \) to my knowledge

\(^12\)the asymptotic behaviour of coefficients may be derived from the central exponents of the Jacquet modules
Minguez and Vincent Sécherre [137] gave an algebraic proof for all $F$ and all algebraic closed fields $R$ of characteristic different from $p$.

A prime $\ell \neq p$ not dividing the order of a torsion element of $G$ is called \textit{banal} for $G$ [15]. A general principle is that the properties of complex representations of $G$ described in purely algebraic terms transfer to representations of $G$ over fields $R$ of characteristic 0 or $\ell$ banal.

\textit{Example} The banal primes for $GL(m,F)$ are those coprime with $q^i - 1$ for $1 \leq i \leq m$.

The $R$-representations of $G$ form a locally small abelian Grothendieck category $\text{Mod}_R(G)$ [198]. For a commutative ring $S$ which is an $R$-algebra, the $R$-representations of $G$ are related to the $S$-representations of $G$ by the scalar extension $^{\text{14}} S \otimes_R - : \text{Mod}_R(G) \to \text{Mod}_S(G)$ and by the restriction its right adjoint: an $S$-representation is considered as an $R$-representation. One says that an $S$-representation of $G$ in the image of the scalar extension \textit{descends} to $R$, or is defined on $R$.

When $R$ is a field, many properties on admissible irreducible $R$-representations of $G$ still assume $R$ to be algebraically closed although this is not necessary. A good tool to check if this is true is the bijection (Henniart-V. [105], 106 section 2):

$$V \mapsto BC(V)$$

- from the isomorphism classes of irreducible admissible $R$-representations $V$ of $G$,
- to the Galois orbits $^{\text{15}} BC(V)$ of the isomorphism classes of the irreducible admissible $R^{ac}$-representations of $G$ defined on a finite extension of $R$.

$BC(V)$ is the set of isomorphism classes of the irreducible subquotients $V^{ac}$ of

$$R^{ac} \otimes_R V \simeq \bigoplus_{V^{ac} \in BC(V)} W(V^{ac}),$$

where $d$ is the reduced dimension of the division $R$-algebra $\text{End}_{RG} V$ over its center $E_V$, the length of the $R^{ac}$-representation $R^{ac} \otimes_R V$ of $G$ is $d[E_V : R]$, the number of element of $BC(V)$ is $[E_V^* : R]$ where $E_V^*$ is the maximal separable subextension of $E_V/R$, and $W(V^{ac})$ is an indecomposable $R^{ac}$-representation of $G$ of irreducible subquotients isomorphic to $V^{ac}$ and of length $[E_V : E_V^*]$. Any $V^{ac} \in BC(V)$ is $V$-isotypic as a $R$-representation of $G$, and defined on a maximal subfield of $\text{End}_{RG} V$ (Justin Trias [180]).

Any irreducible admissible $R^{ac}$-representation of $G$ is absolutely irreducible and has a central character by the Schur's lemma. If the characteristic of $R$ is different from $p$, any irreducible $R^{ac}$-representation of $G$ is admissible and defined on a finite extension of $R$ [106].

As $G$ is locally a pro-$p$ group, there is no Haar measure on $G$ with values in a commutative ring $R$ where $p$ is not invertible and $R$-representations of $G$ present new phenomenona. To understand them is to-day an open question.

\textit{Examples} For a field $R$ of characteristic $p$, any irreducible $R$-representation $V$ of $G$ with $\dim_R V^K < \infty$ for some open pro-$p$ subgroup $K$ of $G$, is admissible (Vytautas Paskunas [153], a simple proof is given in (Henniart [100]). As $K$ is a pro-$p$ group, $V^K \neq 0$ when $V \neq 0$, like for a finite group $G$.

\textit{13} Lemma 5.22, Corollary 5.23 for other characterizations
\textit{14} called also base change or induction
\textit{15} an orbit under the the group $\text{Aut}_R(R^{ac})$ of $R$-automorphisms of $R^{ac}$
Irreducible implies admissible when \( G = GL(2, \mathbb{Q}_p) \) (but not in general). Indeed, one reduces to \( R = \mathbb{F}_p^{ac} \); in this case irreducible implies that the centre acts by a character (Laurent Berger \[14\]) hence is admissible by Barthel-Livne and Breuil \[16\].

There exists a non-admissible irreducible \( \mathbb{F}_p^{ac} \)-representation of \( GL(2, F) \) for an unramified extension \( F \) of \( \mathbb{Q}_p \) (Daniel Le \[135\]). One does not know if any infinite dimensional irreducible non-admissible \( \mathbb{F}_p^{ac} \)-representation of \( G \) has a central character, because its dimension is equal to the cardinal of \( \mathbb{F}_p^{ac} \) and the classical proof with the Schur’s lemma does not work.

It happens that a property of admissible irreducible representations of \( G \) over a field \( R \) transfers to representations of \( G \) over any coefficient field of the same characteristic.

**Examples** In characteristic different from \( p \), for the classification of cuspidal irreducible \( R \)-representations of \( G \) by compact induction (Henniart-V. \[106\]).

In characteristic \( p \), for the classification of non-cuspidal \[14\] admissible irreducible \( R \)-representations of \( G \), for the classification of non-supersingular simple modules of the pro-\( p \)-Iwahori Hecke \( R \)-algebra of \( G \) (Noriyuki Abe, Henniart, Florian Herzig and V. \[8\], Henniart-V. \[105\]), for the existence of a supersingular admissible irreducible \( R \)-representation of \( G \) when \( F \supset \mathbb{Q}_p \) (Herzig, Karol Kozioł and V. \[109\]).

For a prime \[17\] an \( r \)-adic representation of \( G \) is a representation of \( G \) on a \( \mathbb{Q}_p^{ac} \)-vector space which is continuous for the \( r \)-adic topology on the vector space. In this article, an \( R \)-representation of \( G \) is supposed always to be smooth. A \( p \)-adic representation of \( G \) may be not smooth, but an \( \ell \)-adic representation of \( G \) is smooth if \( \ell \neq p \). A \( \mathbb{Q}_p^{ac} \)-representation of \( G \) is a smooth \( r \)-adic representation of \( G \). The choice of an isomorphism

\[
\mathbb{C} \simeq \mathbb{Q}_p^{ac}
\]

identifies the complex representations of \( G \) and the \( \mathbb{Q}_p^{ac} \)-representations of \( G \).

A **mod \( r \) representation** of \( G \) is a \( \mathbb{F}_p^{ac} \)-representation of \( G \).

An admissible \( \mathbb{Q}_p^{ac} \)-representation \( V \) of \( G \) is called integral if \( V \) is defined on a finite extension \( E/\mathbb{Q} \), and \( V \) contains an \( G \)-stable \( \mathbb{Z}_p^{ac} \)-lattice \( L \) \[15\] admissible as a \( \mathbb{Z}_p^{ac} \)-representation of \( G \) and descending to \( O_E \) \[14\] The mod \( r \) representation \( \text{red}_r(L) = L \otimes_{\mathbb{Z}_p^{ac}} \mathbb{F}_r^{ac} \) of \( G \) is called the reduction modulo \( r \) of \( L \).

By the strong Brauer-Nesbitt theorem (V. \[187\]), an \( \ell \)-adic representation \( V \) of \( G \) of finite length containing an admissible \( G \)-stable \( \mathbb{Z}_\ell^{ac} \)-lattice \( L \) defined on some \( O_E \) as above, the \( \mathbb{Z}_\ell^{ac}[G] \)-module \( L \) is finitely generated, of reduction \( \text{red}_\ell(L) \) of finite length, and the image of \( \text{red}_\ell(L) \) in the Grothendieck group of finite length \( \mathbb{F}_\ell^{ac} \)-representations of \( G \) does not depend on the choice of \( L \); it is called the **reduction mod \( \ell \)** of \( V \). Two finite length integral \( \ell \)-adic representations of \( G \) are said to be **congruent modulo \( \ell \)** when their reductions modulo \( \ell \) are isomorphic. This does not hold true for \( \mathbb{Q}_p^{ac} \)-representations of \( G \).

**Example** An irreducible \( \mathbb{Q}_p^{ac} \)-representation \( V = \text{ind}_F^G W \) of \( G = PGL(n, F) \) compactly induced from a representation \( W \) of \( K = PGL(n, O_F) \) contains an admissible \( G \)-stable \( \mathbb{Z}_p^{ac} \)-lattice \( L \) defined on some \( O_E \) as above, of infinite length reduction, and another one \( L' \) of finite length reduction. Take \( L = \text{ind}_F^G W_{z_p^{ac}} \) for a \( K \)-stable \( \mathbb{Z}_p^{ac} \)-lattice \( W_{z_p^{ac}} \) of \( W \) and \( L' = V \cap \text{ind}_F^G 1_{z_p^{ac}} \) for a small enough discrete cocompact subgroup \( \Gamma \) of \( G \).

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\[16\] cuspidal and supersingular will be defined later

\[17\] \( \ell \) is reserved for the primes different from \( p \), think \( r = \ell \) or \( p \)

\[18\] a free \( \mathbb{Z}_p^{ac} \)-submodule of scalar extension \( V \) to \( \mathbb{Q}_p^{ac} \)

\[19\] the ring \( O_E \) is principal but not \( \mathbb{Z}_p^{ac} \). The definition bypasses this difficulty
5. Parabolic induction

For any $F,G,R$ and any parabolic subgroup $P$ of $G$ of Levi quotient $M$, the parabolic induction\(^{20}\)

$$\text{ind}^G_P : \text{Mod}_R(M) \to \text{Mod}_R(G)$$

allows to construct $R$-representations of $G$ from $R$-representations of the smaller connected reductive $p$-adic group $M$. The parabolic induction has excellent properties, it commutes with small direct sums\(^{21}\) Lemma 4.3; for $p$ nilpotent in $R$, it is fully faithful\(^{198}\); for a field $R$, the parabolic induction respects finite length representations with admissible subquotients (this depends on the classification of admissible irreducible representations if the characteristic of $R$ is $p$).

The parabolic induction is exact and has a left adjoint $L^G_P$ called the Jacquet functor, equal to the coinvariant functor $(-)_N$ with respect to the unipotent radical $N$ of $P$, and a right adjoint\(^{22}\) $R^G_P$\(^{198}\). By adjointness, $L^G_P$ is right exact and $R^G_P$ is left exact. The scalar extension commutes with the three parabolic functors\(^{105}\).

For $p$ invertible in $R$, the second adjunction

$$R^G_P = \delta_P L^G_P$$

where $\delta_P$ is the modulus of $P$\(^{23}\), is a deep property proved this year by Dat, David Helm, Robert Kurinczuk and Gilbert Moss\(^{52}\) Corollary 1.3), originally proved by Bernstein when $R = \mathbb{C}$. When $R$ is noetherian, the parabolic induction $\text{ind}^G_P$ respects admissibility, the second adjunction implies that $\text{Mod}_R(G)$ is noetherian, that the parabolic induction respects projective (resp. finitely generated) $R$-representations\(^{52}\) Corollaries 1.4, 1.5), and that $L^G_P$ respects admissibility. The functor $L^G_P$ is exact, preserves infinite direct sums\(^{41}\), and when $R$ is a field, $L^G_P$ respects finite length because $L^G_P$ respects the property of being finitely generated, and an admissible finitely generated $R$-representation of $G$ has finite length (the proof uses the Moy-Prasad unrefined types when $R$ is algebraically closed but algebraically closed is not necessary).

For a field $R$ of characteristic $p$, the adjoint functors $L^G_P$ and $R^G_P$ send an admissible irreducible $R$-representation of $G$ to 0 or to an admissible irreducible $R$-representation of $M$. Irreducible is necessary, an example of an admissible $R$-representation $V$ of $G$ with $L^G_P(V)$ not admissible is given in\(^{10}\). But contrary to the case $R = \mathbb{C}$, the functors $L^G_P$ and $R^G_P$ fail to be exact (for $R^G_P$\(^{24}\) (Emerton\(^{61}\), Koziol\(^{121}\)), $\text{ind}^G_P$ does not preserve finitely generated representations, $R^G_P$ does not preserve infinite direct sums (Abe-Henniart-V.\(^{10}\) section 4.5).

When $p$ is nilpotent in the commutative ring $R$, the right adjoint $R^G_P$ respects admissibility (Abe-Henniart-V.\(^{10}\)); it is equal to the Emerton’s functor $\text{Ord}^G_{pop}$ of ordinary parts on admissible $R$-representations\(^{23}\). If moreover $R$ is artinian, Matthew Emerton\(^{61}\) extended

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\(^{20}\) $\text{ind}^G_P(W)$ is the right translation of $G$ on the $R$-module of locally constant functions $f : G \to W$ such that $f(mng) = mf(g)$ for $m \in M, n \in N, g \in G$

\(^{21}\) when $R$ is a field of characteristic $p$, $\text{ind}^G_P$ commutes with direct products\(^{167}\)

\(^{22}\) by\(^{115}\) 8.3.27 as $\text{Mod}_R(G)$ is a locally small abelian Grothendieck category and $\text{ind}^G_P$ is right exact and commutes with small direct sums

\(^{23}\) $\delta_P(m) = |\det \text{Ad}_{\text{Lie}N}(m)| \in q^N$

\(^{24}\) there is no description of $R^G_P$ on non-admissible representations
the functor of ordinary parts to a $\delta$-functor, expected to coincide with the derived functors when the characteristic of $F$ is 0.

Example When $G = SL(2, \mathbb{Q}_p)$, Kozioł [121] showed that the derived functors of $R^G_B$ and $\text{Ord}^G_B$ are equal on any absolutely irreducible $\mathbb{F}_p^{ac}$-representations of $G$.

When the characteristic of $F$ is $p$, surprisingly $R^G_B$ is exact on admissible $\mathbb{F}_p^{ac}$-representations of $G$ (Julien Hauseux [88]).

A representation of $G$ over a field $R$ is called unramified when it is trivial on the subgroup $G^0$ of $G$ generated by its compact subgroups. The group $\Psi_R(G)$ of unramified $R$-characters $\psi : G \to R^*$ of $G$ is a torus. Generic irreducibility says that for any parabolic subgroup $P$ of $G$ of Levi $M$ and any irreducible $R$-representation $W$ of $M$, the set of $\psi \in \Psi_R(M)$ such that $\text{ind}^G_P(W \otimes \psi)$ is irreducible is Zariski-dense in $\Psi_R(M)$. Generic irreducibility is probably true for any $F, G$ and any field $R$.

Example Generic irreducibility is known for $R$ of characteristic $p$ (Abe-Henniart-V. [10]) or when $F \subset \mathbb{Q}_p$ for $R$ algebraically closed of characteristic different from $p$ (Dat [38]).

Dat (38) Theorem 3.11) extended the Langlands quotient theorem when $R = \mathbb{C}$ to any algebraically closed field $R$ of characteristic different from $p$ with a non-trivial valuation $\nu$ (for example $\mathbb{Q}_p^{ac}$):

When $P = MN$ is a standard parabolic subgroup of $G$, $W$ is a $\nu$-tempered irreducible $R$-representation of $M$, and $\psi \in \Psi_R(M)$ satisfies $-\nu(\psi) \in (\mathcal{A}_P^\times)^+$, the $R$-representation $\text{ind}^G_P(W \otimes \psi)$ has a unique irreducible quotient $J(M, W, \psi)$. Any irreducible $R$-representation $V$ of $G$ is isomorphic to $J(M, W, \psi)$ for a unique triple $(P, W, \psi)$.

An admissible $R$-representation $V$ of $G$ is $\nu$-tempered (Dat [38] Definition 3.2) if for any standard parabolic subgroup $P = MN$ such that $L^G_P(V) \neq 0$, any exponent $\chi$ in $L^G_P(V)$ satisfies $-\nu(\delta_P^{1/2} \chi) \in \mathcal{A}_P^\times$. It is called discrete if $-\nu(\delta_P^{1/2} \chi) \in \mathcal{A}_P^\times$. The exponents of $L^G_P(V)$ are the $R$-characters of the split component $A_M$ of the center of $M$ appearing in $L^G_P(V)$ seen as an $R$-representation of $A_M$.

From the Dat’s theory of $\nu$-tempered representations, one deduces (David Hansen, Tasho Kaletha and Jared Weinstein [38], C.2.2):

The Grothendieck group of finite length $\ell$-adic representations of $G$ is generated by representations of the form $\text{ind}^G_P(W \otimes \psi)$, for a standard parabolic subgroup $P \subset G$ of Levi $M$, an integral irreducible $\ell$-adic representation $W$ of $M$ and an unramified $\ell$-adic character $\psi$ of $M$.

6. Admissible representations and duality

The classification of irreducible admissible $R$-representations of $G$ is an objective of the local Langlands program. There are few finite dimensional representations when $G$ is not compact modulo the center $Z(G)$, and admissibility is a crucial finiteness property.

When $R$ is a noetherian commutative ring, a subrepresentation of an admissible $R$-representation of $G$ is admissible. A quotient of an admissible $R$-representation of $G$ is admissible [193] and

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25 This coincides with the classical definition (Henniart-Lemaire [102] 2.12 Remarque 1)

26 Let $\Delta(M)$ denote the set of simple roots of $T$ in $M$, $\Delta(P)$ the set of simple roots in $P$ of $T_M$. $\mathcal{A}^\times = X \otimes_{\mathbb{Z}} \mathbb{R}$ where $X$ is the lattice of rational characters of $T$, $(\ ,\ )$ a $W_G$-invariant scalar product on $\mathcal{A}^\times$. Then $\mathcal{A}_P^\times = \sum_{\alpha \in \Delta_P} \mathbb{R}_{\geq 0} \alpha$ and $\mathcal{A}_P^{\star}$ is the cone $\{ x \in \mathcal{A}^\times \mid (x, \alpha) = 0 \text{ for } \alpha \in \Delta(M), (x, \alpha) > 0 \text{ for } \alpha \in \Delta(P) \}$
the category $\text{Mod}_R(G)^a$ of admissible $R$-representations of $G$ is abelian if $p$ is invertible in $R$, or if $R$ is a finite field of characteristic $p$ and $F \supset \mathbb{Q}_p$.

**Example** When $F \supset \mathbb{F}_p((T))$ and $p$ is not invertible in $R$, there exists an admissible representation with a non-admissible quotient (Abe-Henniart-V. [10]).

Let $R$ be a field. The **smooth dual** $V^\vee$ of an $R$-representation $V$ of $G$ is the smooth part of the contragredient action of $G$ on the linear dual $V^* = \text{Hom}_R(V,R)$.

For $R$ of characteristic different from $p$, the smooth dual is an autoduality on $\text{Mod}_R(G)^a$. In particular, $V^\vee$ is irreducible if and only if $V$ is irreducible. The smooth dual and the parabolic induction and its left adjoint satisfy

$$(\text{ind}_P^G W)^\vee \simeq \text{ind}_P^G (W^\vee \delta_P), \quad L_P^G(V^\vee) \simeq (L_{\text{prop}}^G(V))^\vee,$$

for any $R$-representation $W$ of $M$ and any admissible $R$-representation $V$ of $G$.

For $R$ of characteristic $p$, the smooth dual of any infinite dimensional admissible irreducible $R$-representation of $G$ is zero! For $F$ of characteristic 0, Jan Kohlhaase [116] developed a higher smooth duality theory on $\text{Mod}_R(G)^a$. He studied the $i$-th smooth duality functors $S^i : \text{Mod}_R(G)^a \to \text{Mod}_R(G)^a$ for $0 \leq i \leq d$ for $d = \dim_{\mathbb{Q}_p} G$ under tensor product, inflation and induction and proved that for $V \in \text{Mod}_R(G)^a$, the integer

$$d(V) = \max \{i \mid S^i(V) \neq 0\}$$

satisfies

(i) $d(V) = 0$ if and only if $V$ is finite dimensional,

(ii) $d(\text{ind}_P^G W) = d(W) + \dim_{\mathbb{Q}_p} N$, for a parabolic subgroup $P = MN$ and $W \in \text{Mod}_R(M)^a$,

(iii) $d(V) = 1$ and $S^1(V)$ coincides with the Colmez’s contragredient introduced for the $p$-adic Langlands correspondence for $G = GL(2, \mathbb{Q}_p), R = \mathbb{F}_p^ac$ and $V$ irreducible of infinite dimensional; for the Steinberg representation $St_G$ which is irreducible, $S^1(St_G)$ is indecomposable of length 2!

For $G$ unramified, $K$ an hyperspecial subgroup of $G$, $W \in \text{Mod}_{\mathbb{F}_p^ac}(K)$ and $i > \dim_{\mathbb{Q}_p} U$, we have $S^i(\text{ind}_K^G W) = 0$ (Claus Sorensen [183]).

### 7. Supercuspidal support

An $R$-representation $V$ of $G$ is called **cuspidal** if it is killed

$$L_P^G(V) = R_P^G(V) = 0$$

by the left and right adjoints of the parabolic induction for all parabolic subgroups $P \neq G$.

When $p$ is invertible in $R$, the second adjunction implies that $V$ is cuspidal if and only if $L_P^G(V) = 0$ for any proper parabolic subgroup $P$ of $G$. Any irreducible $R$-representation $V$ of $G$ is a subrepresentation of $\text{ind}_P^G W$ for some cuspidal irreducible $R$-representation $W$. Assuming that $R$ is an algebraically closed field, the pair $(M,W)$ is unique modulo $G$-conjugation, its $G$-conjugation class of $(M,W)$ is called the **cuspidal support** of $V$. Twisting

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27 the completed group algebra of $R[K]$ is noetherian when $F \supset \mathbb{Q}_p$ but not when $F \supset \mathbb{F}_p((T))$

28 the smooth dual is the set of linear forms on $V$ fixed by some open subgroup of $G$

29 the normalized induction $\text{ind}_P^G(W \otimes \delta_P^{1/2})$ commutes with the smooth dual, the second isomorphism is equivalent to the second adjunction

30 $G$ is quasi-split and splits over some unramified extension of $F$

31 algebraically closed is probably not necessary
the cuspidal support by unramified characters we get the **inertial cuspidal support** $\Omega$ of $V$. So, $\Omega$ is the set of $(M', W')$ for $(M', W')$ $G$-conjugate to $(M, W \otimes \Psi)$ and $\psi \in \Psi_R(M)$. It is an algebraic variety with regular functions $\mathcal{O}(\Omega) = (R[M/M^0]^S)^H$ where $S$ is the (finite) group of $\psi \in \Psi_R(M)$ such that $W \otimes \psi \simeq W$. The subgroup of $w \in W_G$ fixing $M$ acts on the $R$-representations of $M$, and $H$ is the group of those $w$ such that $W^w \simeq W \otimes \psi$ for some $\psi \in \Psi_R(M)$.

When $p$ is not invertible in $R$, one needs both $L_P^G$ and $R_P^G$ to define cuspidality. The mod $p$ Steinberg representation $\text{St}_G$ and the trivial representation $1_G$ of $G$ satisfy for any parabolic subgroup $P$ of Levi $M$,

$$L_P^G(\text{St}_G) = 0, R_P^G(\text{St}_G) = \text{St}_M, \quad L_P^G(1_G) = 1_M, R_P^G(1_G) = 0.$$ 

For a field $R$ of characteristic $p$, the Steinberg representation is not a subrepresentation of $\text{ind}^G_P W$ for any cuspidal admissible irreducible $R$-representation $W$. Yet, any irreducible $R$-representation $V$ of $G$ is a subquotient of $\text{ind}^G_P W$ for some $R$-representation $W$ of $Z$ (Abe-Henniart-Herzig-V. [8] IV.1 for $R$ algebraically closed field).

An admissible irreducible $R$-representation of $G$ which is not isomorphic to a subquotient of a proper parabolically induced representation $\text{ind}^G_P W$ for all $P \neq G, W$ an admissible irreducible $R$-representation of $M$, is called **supercuspidal**.

Any admissible irreducible $R$-representation $V$ of $G$ is a subquotient of $\text{ind}^G_P W$ for some supercuspidal admissible irreducible $R$-representation $W$.

For a field $R$ of characteristic $p$, $(P, W)$ is unique modulo $G$-conjugation. This follows from the classification. Any cuspidal admissible irreducible $R$-representation of $G$ is supercuspidal.

For a field $R$ of characteristic different from $p$, a cuspidal irreducible $R$-representation $V$ of $G$ is not always supercuspidal. The $G$-conjugation class of $(M, W)$ is called a **supercuspidal support** of $V$. When $R$ is algebraically closed, its twist by unramified characters is called an **inertial supercuspidal support** of $V$; if all the irreducible $R$-representations of $G$ have a unique supercuspidal support, the **Bernstein variety** $B_R(G)$ is the disjoint union of the inertial supercuspidal supports of the irreducible $R$-representations of $G$. Contrary to the cuspidal support, the supercuspidal support is not always unique.

**Examples** When $G = GL(2, \mathbb{Q}_p), R = \mathbb{F}_q^\times, \ell$ divides $p + 1$, the unique infinite dimensional irreducible subquotient of the representation $\text{ind}^G_B 1_Z$ indecomposable of length 3 is cuspidal and non-supercuspidal.

The supercuspidal support is not always unique when $R = \mathbb{F}_q^\times, \ell$ divides $q^2 + 1$ and $G$ is the finite group $Sp_8(\mathbb{F}_q)$ (Olivier Dudas [58]) or $Sp_8(F)$ (Dat [19]).

The supercuspidal support is unique if $R$ has characteristic 0, or $G$ is an inner form of $GL(n, F)$ (Minguez-Sécherre [138]), or $G$ is the unramified unitary group $U(2, 1), p \neq 2$ (Kurinczuk [125]), when $R$ is algebraically closed (probably this is not necessary).

An irreducible $\mathbb{Q}_p^\times$-representation of $G$ is integral if and only if its supercuspidal support is integral (Dat-Helm-Kurinczuk-Moss [52] Corollary 1.6). Is any irreducible mod $\ell$ representation of $G$ a subquotient of the reduction modulo $\ell$ of an integral irreducible $\ell$-adic representation ?

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32 One does not need to suppose $W$ irreducible when $R$ is an algebraically closed field of characteristic different from $p$ (Dat [49]).
For a field $R$ of characteristic 0 or $\ell$ banal for $G$, any cuspidal irreducible $R$-representation of $G$ is supercuspidal, and projective in the category of $R$-representations of $G$ with a given central character. The reduction modulo $\ell$ of any integral cuspidal irreducible $\ell$-adic representation of $G$ is irreducible and cuspidal. The reduction modulo $\ell$ of an integral irreducible $\ell$-adic representation of $G$ may be reducible. Does any irreducible mod $\ell$ representation of $G$ lift to an irreducible $\ell$-adic representation of $G$? 

8. Hecke algebras

Hecke $\mathbb{Z}$-algebras appear everywhere in the theory of representations of $G$ to find algebraic proofs of properties of representations proved formerly with harmonic analysis. An open subgroup $K$ of $G$ which is compact or compact modulo the center of $G$, defines a Hecke ring

$$\mathcal{H}(G, K) = \text{End}_{\mathbb{Z}[G]} \mathbb{Z}[K \backslash G].$$

naturally isomorphic to the opposite of $\mathbb{Z}[K \backslash G / K]$. For any commutative ring $R$, the Hecke $R$-algebra $\mathcal{H}_R(G, K) = \text{End}_{R[G]} R[K \backslash G]$ is the scalar extension to $R$ of the Hecke ring.

A famous finiteness theorem of Deligne-Bernstein when $R = \mathbb{C}$ extended by Dat-Helm-Kurinczuk-Moss [52] is the key of the proof of the second adjunction:

When $R$ is any noetherian $\mathbb{Z}$-$\mathbb{C}$-algebra, the center $\mathcal{Z}_R(G, K)$ of $\mathcal{H}_R(G, K)$ is a finitely generated $R$-algebra and $\mathcal{H}_R(G, K)$ is a finitely generated $\mathcal{Z}_R(G, K)$-module.

One proves an equivalent statement, involving the Bernstein center $\mathcal{Z}_R(G)$, which is the endomorphism ring of the identity functor of $\text{Mod}_R(G)$:

For $R$ as above, any finitely generated $R$-representation $V$ of $G$ is $\mathcal{Z}_R(G)$-admissible and the natural image of $\mathcal{Z}_R(G) \to \text{End}_{R[G]} V$ is a finitely generated $R$-algebra.

The highly non-trivial proof uses the Fargues-Scholze local version of the Vincent Laforgue’s theory of excursion operators [65].

The $R$-representations of $G$ and the right $\mathcal{H}_R(G, K)$-modules are related by the $K$-invariant functor:

$$V \mapsto V^K \simeq \text{Hom}_{R[G]}(R[K \backslash G], V) : \text{Mod}_R(G) \to \text{Mod} \mathcal{H}_R(G, K),$$

and by its left adjoint $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{H}_R(G, K)} R[K \backslash G]$. From now on, a module of a Hecke algebra will be a right module.

When $R$ is a field of characteristic different from $p$, any simple $\mathcal{H}_R(G, K)$-module is finite dimensional if the $K$-invariant functor induces a bijection between the (isomorphism classes of) irreducible $R$-representations $V$ of $G$ with $V^K \neq 0$ and the (isomorphism classes of) $\mathcal{H}_R(G, K)$-modules, as irreducible implies admissible. This is the case when the order of any finite quotient of $K$ is invertible in $R$ ([106] Theorem 3.2), or when $K$ is an Iwahori group $J$ or a pro-$p$ Iwahori subgroup $\tilde{J}$.

When any subrepresentation of any $R$-representation of $G$ generated by its $K$-invariants has the same property, the $K$-invariant functor gives an equivalence:

$$\text{Mod}_R(G)(K) \to \text{Mod} \mathcal{H}_R(G, K)$$

By a classical result of Borel, the category of complex representations generated by their vectors invariant by an Iwahori subgroup $J$ is an indecomposable factor of $\text{Mod}_C(G)$, equivalent to $\text{Mod} \mathcal{H}_C(G, J)$ by the $J$-invariant functor. The same is true with a pro-$p$ Iwahori subgroup $\tilde{J}$ without “indecomposable”.

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33. is the reduction modulo $\ell$ of an integral cuspidal irreducible $\ell$-adic representation of $G$
There are no equivalences for $R$ of characteristic $p$ and $K = \bar{J}$ in general.

Example However, the equivalence is true for $R = \mathbb{F}_p^{ac}$ and $K = \bar{J}$ if $G = GL(2, \mathbb{Q}_p)$ or $SL(2, \mathbb{Q}_p), p \neq 2$ (Rachel Ollivier [144], Kozioł [118], Ollivier-Peter Schneider [139]).

For a prime $r$, a $\mathbb{Q}_r$-representation $V$ of $G$ is called **locally integral** if for some finite extension $E/\mathbb{Q}_r$, $V^K$ admits a $\mathcal{H}(G, K)$-stable $O_E$-lattice for all open compact subgroups $K$ of $G$.

An integral irreducible $\mathbb{Q}_p^{ac}$-representation is clearly locally integral. The two notions for irreducible representations coincide if $r = \ell \neq p$ \footnote{supposing that a uniformizer of $F$ acts trivially}.

The equivalence between integral and locally integral irreducible $\mathbb{Q}_p^{ac}$-representations of $G$ is an open question. When $G$ is split, it is the analogue of the Breuil-Schneider conjecture \footnote{for example, a field of characteristic different from $p$} restricted to smooth representations (Hu [112], Sorensen [181], [182]). A finite length $\mathbb{Q}_p^{ac}$-representation $V$ of $G$ is locally integral if and only if (Dat [40]):

$$\nu(\delta_p^{-1/2} \chi) \in \rho_p - \overline{A_p}$$

for any standard parabolic subgroup $P = MN$ of $G$ with $L_P^G(V) \neq 0$, and any exponent $\chi$ of $L_P^G(V)$ \footnote{Definition in the section on Bernstein blocks}.

9. **Representations in characteristic different from $p$**

For any commutative ring $R$ and any open subgroup $K$ of $G$, an $R$-representation $W$ of $K$ define a $R$-representation $\text{ind}_K^G W$ of $G$ by **compact induction** \footnote{the $R$-module of functions $f : G \to W$ supported on finitely many cosets $Kg$, satisfying $f(kg) = \rho(k)f(g)$ for $k \in K, g \in G$ where $G$ acts by right translation}. \footnote{if for some finite $A_M$ in $\text{Lie} P$. The formula can be simplified!} 

Example $\text{ind}_K^G 1_K = R[K \setminus G]$ for the trivial $R$-representation $1_K$ of $K$.

When $R$ is a field of characteristic different from $p$, all cuspidal irreducible $R$-representations of $G$ are conjectured to be compactly induced from open subgroups of $G$ compact modulo the center of $G$.

For $R$ algebraically closed, the conjecture has been proved for the level 0 \footnote{definition in the section on Bernstein blocks} representations of any $G$ or when

$G$ has rank 1 (Martin Weissman [201]),
$G$ is an inner form of $GL(n, F)$ (Minguez-Sécherre [138]), or of $SL(n, F)$ (Peyi Cui [36], [37]),
$G$ is a classical group and $p \neq 2$ (Shaun Stevens [185], Stevens-Kurinczuk-Daniel Skodlerak [131]) or a quaternionic form of $G$ (Skodlerak [179]),
$G$ splits on a moderately ramified extension of $F$ and $p$ does not divide the order of the absolute Weyl group (Jessica Fintzen [66]),

Algebraically closed is not necessary and there is an explicit list $\mathcal{X}$ of pairs $(K, W)$ of $G$ where $K$ is an open subgroup of $G$ compact modulo the center and $W$ an $R$-representation of $K$ such that $\text{ind}_K^G W$ is irreducible cuspidal satisfying (Henniart-V. [106]):

a) any cuspidal irreducible $R$-representation of $G$ is isomorphic to $\text{ind}_K^G W$ for some $(K, W) \in \mathcal{X}$ unique modulo $G$-conjugation,

b) $\text{ind}_K^G W$ and $W$ have the same interwinning algebra $\text{End}_{R[K]} W \to \text{End}_{R[G]} \text{ind}_K^G W$, \footnote{pons of $\mathcal{H}(G, K)$-stable $O_E$-lattices for all open compact subgroups $K$ of $G$.}
c) \( \text{ind}_K^G W \) is supercuspidal if and only if \( W \) is supercuspidal, for the “natural notion of supercuspidality” of \( W \).

d) \( \mathcal{X} \) is stable by automorphisms of \( R \).

Until the end of this section, \( R \) is an algebraically closed field of characteristic different from \( p \) and \( G = GL(m, D) \) where \( D \) is a central division algebra of dimension \( d^2 \) over \( F \), \( n = md \).

Minguez and Sécherre classified the irreducible \( R \)-representations of \( G \) with a given supercuspidal support by “supercuspidal multisegments”, and those with a given cuspidal support by “aperiodic cuspidal multisegments” [139]. This is the generalisation the Bernstein-Zelevinski classification of complex irreducible representations of \( GL(n, F) \). For \( R \) of characteristic \( \ell \), the proof uses the theory of \( \ell \)-modular types (Minguez-Sécherre [138]) and deep results on affine Hecke algebras of type \( A \) at roots of unity.

Any irreducible \( \ell \)-modular representation of \( G \) is a subquotient of the reduction modulo \( \ell \) of an integral irreducible \( \ell \)-adic representation [139]. In the other direction, any irreducible \( \ell \)-modular representation \( V \) of \( G \) lifts to an \( \ell \)-adic representation when it is supercuspidal or “banal” or unramified (Dat [38], Minguez-Sécherre [139, 137, 140]) or when \( G = GL(n, F) \). Contrary to the case \( G = GL(n, F) \), some irreducible cuspidal \( \ell \)-modular representation of \( G \) may not lift and the reduction modulo \( \ell \) of an integral cuspidal irreducible \( \ell \)-adic-representation of \( G \) may be reducible; its irreducible components are cuspidal and in the same inertial class.

Example When \( q = 8, \ell = 3, d = 2 \), any integral irreducible \( \ell \)-adic representation of \( D^* \) containing an homomorphism \( \chi : O_D^* \to (\mathbb{Q}_{ac}^\ell)^* \) trivial on \( 1 + P_D \) such that \( \chi \neq \chi^q \) has dimension 2 and a non-irreducible reduction modulo \( \ell \). When \( q = 4, \ell = 17, d = 2 \), there exists an irreducible cuspidal \( \ell \)-modular representation of \( GL(2, D) \) not lifting to \( \mathbb{Q}_{ac}^\ell \) (Minguez-Sécherre [141]).

Let \( \mathcal{D}_C(G) \) denote the set of isomorphism classes of the essentially square integrable irreducible (or discrete series) complex representations of \( G \). The classical local Jacquet-Langlands correspondence

\[
JL_C : \mathcal{D}_C(GL(m, D)) \rightarrow \mathcal{D}_C(GL(n, F))
\]

is a bijection characterized by a Harish-Chandra character relation on matching elliptic regular conjugacy classes. Fixing an isomorphism \( \mathbb{C} \simeq \mathbb{Q}_{ac}^\ell \), the local Jacquet-Langlands correspondence gives an \( \ell \)-adic local Jacquet-Langlands correspondence

\[
JL_{Q_{ac}} : \mathcal{D}_{Q_{ac}}(GL(m, D)) \rightarrow \mathcal{D}_{Q_{ac}}(GL(n, F))
\]

independent of the isomorphism \( \mathbb{C} \simeq \mathbb{Q}_{ac}^\ell \), and respecting integrality. Minguez and Sécherre [141] proved that two integral representations of \( \mathcal{D}_{Q_{ac}}(GL(m, D)) \) are congruent modulo \( \ell \) if and only if their transfers to \( GL(n, F) \) are congruent modulo \( \ell \). But there is no \( \ell \)-modular local Jacquet-Langlands correspondence compatible with the \( \ell \)-adic local Jacquet-Langlands correspondence by reduction modulo \( \ell \), as for example, when \( d = 2 \) and \( q + 1 \equiv 0 \) modulo

\[38\text{Fintzen gave another proof when } G \text{ is moderately ramified and } p \text{ does not divide the order of the absolute Weyl group.}
\[39\text{If } V \text{ irreducibly parabolically induced from an unramified character of a Levi subgroup [140].]}

ell, the trivial representation $1_{Q^\ell}$ of $D^*$ corresponds to the Steinberg $St_{Q^\ell}$ of $GL(2, F)$ of reduction modulo $\ell$ of length 2 (Dat [42]). However, the Badulescu morphism [13]

$$LJ_{Q^\ell} : Gr_{\mathbb{Q}^\ell}(GL(n, F)) \rightarrow Gr_{\mathbb{Q}^\ell}(GL(m, D))$$

where $Gr_G$ is the Grothendieck group of finite length $R$-representations of $G$, gives by reduction an $\ell$-modular Badulescu morphism

$$LJ_{\mathbb{Q}^\ell} : Gr_{\mathbb{Q}^\ell}(GL(n, F)) \rightarrow Gr_{\mathbb{Q}^\ell}(GL(m, D)).$$

Sécherre and Stevens [178] introduced the interesting notions of mod $\ell$ inertial supercuspidal support and linkage for irreducible complex representations $\pi, \pi'$ of $G$.

a) Picking an isomorphism $\mathbb{C} \simeq \mathbb{Q}_{\ell}^{ac}$ one supposes that $\pi$ is an $\ell$-adic representation of $G$. The inertial class of the cuspidal support of $\pi$ contains an integral cuspidal representation $\tau$. The mod $\ell$ inertial supercuspidal support of $\pi$ is the inertial supercuspidal support of any irreducible component of $r_{\ell}(\tau)$; it depends only on the isomorphism class of $\pi$.

b) $\pi, \pi'$ are linked if there are prime numbers $\ell_1, \ldots, \ell_r$ different from $p$, and irreducible complex representations $\pi = \pi_0, \pi_1, \ldots, \pi_r = \pi'$ such that, for each $i \in \{1, \ldots, r\}$, the representations $\pi_{i-1}, \pi_i$ have the same mod $\ell_i$ inertial supercuspidal support.

When $\pi, \pi'$ are essentially square integrable, they are linked if and only if their images by the local Jacquet-Langlands correspondence $JL_{\ell}$ are linked if and only if (Andrea Dotto [55]) they have the same semi-simple endoclass (a type invariant). When $G = GL(n, F)$ and $\pi, \pi'$ are cuspidal, they have the same endoclass if and only if the associated irreducible representations of Weil group $W_F$ by the local Langlands correspondence share an irreducible component when restricted to the wild inertia group.

10. Bernstein blocks

For a commutative ring $R$, a non trivial idempotent $e$ in the Bernstein center $Z_R(G)$ decomposes the abelian category

$$\text{Mod}_R(G) = e(\text{Mod}_R(G)) \times (1 - e)(\text{Mod}_R(G))$$

into a direct product of two abelian full subcategories. When the idempotent $e \in Z_R(G)$ is primitive, the subcategory $e(\text{Mod}_R(G))$ where $e$ acts by the identity, is indecomposable (no non trivial factors) and called a block.

Bernstein and Deligne factorized $\text{Mod}_\mathbb{C}(G)$ into blocks, and described the center of each block (a finite $\mathbb{C}$-algebra). Their arguments are valid for any algebraically closed field $R$ of characteristic 0. We have the decomposition in blocks

$$\text{Mod}_R(G) = \prod_{\Omega \in B_R(G)} \text{Mod}_R(G)_{\Omega},$$

over the connected components $\Omega$ of the Bernstein variety $B_R(G)$. The Bernstein block $\text{Mod}_R(G)_{\Omega}$ consists of the $R$-representations of $G$ all of whose irreducible subquotients have inertial supercuspidal support $\Omega$. The center of $\text{Mod}_R(G)_{\Omega}$ is the ring of regular functions on the variety $\Omega$. The decomposition is based on the unicity of the supercuspidal support.

When $G$ is an inner form of $GL(n, F)$, two complex discrete series of $G$ in the same block are inertially equivalent. The Jacquet-Langlands correspondence commutes with twisting by characters, and yields a bijection between the blocks containing discrete series. Andrea Dotto [55] parametrized these blocks by two algebraic invariants (one is the endo-class) and
obtained a complete algebraic description of the Jacquet-Langlands correspondence at the level of inertial classes.

For an algebraically closed field $R$ of characteristic different from $p$, the Deligne-Bernstein decomposition remains true (Sécherre and Stevens [177]) and Bastien Drevon and Vincent Sécherre [57] described the block decomposition of the abelian category of finite length $R$-representations of $G$. Unlike the case of all $R$-representations of $G$, several non-isomorphic supercuspidal supports may correspond to the same block. A supercuspidal block is equivalent to the principal block of the multiplicative group of a suitable division algebra.

When $R$ is an algebraically closed field of characteristic banal $\ell$ for $G$, it is expected that the Deligne-Bernstein decomposition remains true and that the reduction modulo $\ell$ gives a bijection between the blocks of $\ell$-adic representations of $G$ and the blocks of mod $\ell$ representations of $G$.

When $R = W(\mathbb{F}_\ell^\text{ac})$ is the Witt ring of $\mathbb{F}_\ell^\text{ac}$ and $G = GL(n, F)$, Helm [95], [96], [97] showed that the block decomposition of $\text{Mod}_{\mathbb{F}_\ell^\text{ac}}(G)$ lifts to a block decomposition of $\text{Mod}_{W(\mathbb{F}_\ell^\text{ac})}(G)$:

$$\text{Mod}_{W(\mathbb{F}_\ell^\text{ac})}(G) = \prod_{\Omega \in B_{\mathbb{F}_\ell^\text{ac}}(G)} \text{Mod}_{\mathbb{Z}_\ell^\text{ac}}(G)_{\Omega}. $$

The block $\text{Mod}_{W(\mathbb{F}_\ell^\text{ac})}(G)_{\Omega}$ consists of the $W(\mathbb{F}_\ell^\text{ac})$-representations of $G$ such that any irreducible subgroup $V$ has a supercuspidal support in $\Omega$ modulo isomorphism, if $\ell V = 0$

- such that the reduction modulo $\ell$ of an integral element in the inertial class of the supercuspidal support of $V$ is in $\Omega$ modulo isomorphism, if $\ell V = V$.

The center of $\text{Mod}_{W(\mathbb{F}_\ell^\text{ac})}(G)_{\Omega}$ is a finitely generated, reduced, $\ell$-torsion free $W(\mathbb{F}_\ell^\text{ac})$-algebra and the center of $\text{Mod}_{W(\mathbb{F}_\ell^\text{ac})}(G)$ is naturally isomorphic to the ring of endomorphisms of the Gelfand-Graev representation of $G$.

The blocks of $\text{Mod}_{R}(G)$ have been computed in a large number of examples [43]. The principal block of $\text{Mod}_{R}(G)$ is the block containing the trivial representation of $G$. When $R = \mathbb{C}$, the principal block is equivalent to the category of modules over the Iwahori Hecke algebra. Many blocks of $\text{Mod}_{R}(G)$ are equivalent to the principal block of another group $G'$.

When $R = \mathbb{Q}_\ell^\text{ac}, \mathbb{Z}_\ell^\text{ac}$ or $\mathbb{F}_\ell^\text{ac}$, Dat explained the known coincidences between blocks of $\text{Mod}_{R}(G)$ and predicted many more by a functoriality principle for blocks involving dual groups [41], [45].

Example For an algebraically closed field $R$ of characteristic different from $p$ and $G$ an inner form of $GL(n, F)$, each block of $\text{Mod}_{R}(G)$ is equivalent to the principal block of a product of general linear groups [177].

For a commutative ring $R$ where $p$ is invertible, there is decomposition of $\text{Mod}_{R}(G)$ by the Moy-Prasad depth ([11] Appendix A).

An $R$-representation $V$ of $G$ has depth 0 if $V = \sum_x V^{G'}_{\psi}$ is the sum of its invariants $V^{G'}_{\psi}$ by the pro-$p$ radicals of the subgroups of $G$ fixing the vertices of the adjoint Bruhat-Tits building of $G$. The possible depths form a sequence of non-negative rational numbers

$$\text{ind}_{G'}^{GL(n, F)} \psi, \text{ where } \psi \text{ is a generic } W(\mathbb{F}_\ell^\text{ac})\text{-character of the unipotent radical } U \text{ of a Borel subgroup of } GL(n, F)$$

with the theory of types
$r_0 = 0 < r_1 < \ldots$. The category $\text{Mod}_R(G)^{(r)}$ of $R$-representations of $G$ of depth $r$ is abelian with an explicit finitely generated projective generator but is generally not a block. We have

$$\text{Mod}_R(G) = \prod_{n \in \mathbb{N}} \text{Mod}_R(G)^{(r_n)}.$$ When $p = 0$ in $R$, the Bernstein center of $G$ is as small as possible, equal to the Bernstein center of the center $Z(G)$ of $G$ (Ardakov-Schneider [12] when $R$ is a field but their proofs are valid for a commutative ring, see also Dotto [55]).

$$Z_R(Z(G)) = \varprojlim_{K} R[Z(G)/K], \quad K \subset Z(G) \text{ open compact subgroup}.$$ When $E/\mathbb{Q}_p$ a finite extension of ring of integers $O_E$, the category of locally finite (equal to the union of their subrepresentations of finite length) representations of $GL(2, \mathbb{Q}_p)$ on $O_E$-torsion modules with a central character decomposes as a product of blocks with a noetherian center (Paskunas and Shen-Nin Tung [157]).

### 11. Satake isomorphism

The structure of the Hecke ring of any special parahoric subgroup $K$ of $G$ is understood via the **Satake transform**:

$$Sat : \mathcal{H}(G, K) \rightarrow \mathcal{H}(Z, Z^0) \quad Sat(f)(z) = \sum_{u \in U^0 \setminus U} f(uz) \text{ for } z \in Z.$$ It is an injective ring homomorphism, and as $\mathcal{H}(Z, Z^0) \simeq \mathbb{Z}[Z/Z^0]$ is commutative, it shows that the Hecke ring $\mathcal{H}(G, K)$ is commutative. A basis of the image of $Sat$ is

$$S_\lambda = \sum_{\lambda' \in W(\lambda)} \delta^{1/2}(\lambda/\lambda')e_{\lambda'} \text{ for } \lambda \in Z^+/Z^0,$$

where $e_{\lambda} \in \mathcal{H}(Z, Z^0)$ corresponds to $\lambda$ (Henniart-V. [104], [103] Proposition 2.3). This shows that modulo isomorphism, the commutative Hecke ring $\mathcal{H}(G, K)$ does not depend on the choice of $K$.

By scalar extension to a commutative ring $R$, the Satake transform extends to a map $Sat : \mathcal{H}_R(G, K) \rightarrow \mathcal{H}_R(Z, Z^0)$. For $R = \mathbb{C}$, it is well known that $\delta^{1/2}_R Sat$ induces an isomorphism

$$\mathcal{H}_C(G, K) \simeq \mathbb{C}[Z/Z^0]^{W_G}.$$ An all-important special case was singled out by Langlands, that is where $G$ is unramified and where $K$ is a hyperspecial maximal compact subgroup of $G$. Langlands interpreted the Satake isomorphism as giving a parametrization of the isomorphism classes of complex smooth irreducible representations of $G$ with non-zero $K$-fixed vectors, by certain semisimple conjugacy classes in a complex group $\hat{G}$ “dual” to $G$.

For a field $R$ of characteristic $p$, $Sat$ induces an isomorphism (Henniart-V. [104])

$$\mathcal{H}_R(G, K) \simeq R[Z^+/Z^0].$$

\footnote{with $Z^-/Z^0$ instead of $Z^+/Z^0$ but these monoids are isomorphic}
Instead of focusing on the trivial $R$-representation $1_K$ of $K$, we consider two finitely generated $R$-representations $W, W'$ of $K$ and the Hecke $R$-bimodule
\[ \mathcal{H}_R(G, K, W, W') \simeq \text{Hom}_R(W, W'). \]
It is realized as a set of compactly supported functions $f : G \to \text{Hom}_R(W, W')$ with a certain $K$-bi-invariance. In the case $W = W'$, it is an algebra called an \textbf{Hecke algebra with weight} $W$ that we rather write $\mathcal{H}_R(G, K, W)$; the Hecke algebra with trivial weight is the Hecke $R$-algebra $\mathcal{H}_R(G, K)$. The Satake transform generalizes for any standard parabolic subgroup $P = MN$, to an injective map
\[ Sat_M : \mathcal{H}_R(G, K, W, W') \to \mathcal{H}_R(M, M^0, W_{N^0}, W'_{N^0}), \quad Sat_M(f)(\varpi) = \sum_{n \in N^0 \setminus N} f(nm)(v) \]
for $m \in M, v \in W$, where $v \to \varpi$ is the quotient map $W \to W_{N^0}$ (similarly for $W' \to W'_{N^0}$).

Another generalization considered in (Herzig [108] when $G$ is split, Henniart-V. [103])
\[ Sat'_M : \mathcal{H}_R(G, K, W, W') \to \mathcal{H}_R(M, M^0, W'^{N^0}, W'^{N^0}), \quad Sat'_M(f)(z)(v) = \sum_{u \in U^{N^0}\setminus U} f(uz)(v) \]
for $v \in W'^{N^0}$, is related to $Sat_M$ by taking duals [103]. The functional approach of $Sat_M$ (Henniart-V. [103] Section 2) is a motivation to prefer it.

When $R$ is an algebraically closed field of characteristic $p$ and $W, W'$ are irreducible, the generalized Satake transform play a role in the modulo $p$ and $p$-adic Langlands correspondence. In this situation $W_{U^0}, W'_{U^0}$ have dimension 1, the Hecke bimodule $\mathcal{H}_R(G, K, W, W')$ is non-zero if and only if the $R$-characters of $Z^0$ on $W_{U^0}, W'_{U^0}$ are $Z$-conjugate. For $M = Z$, there are explicit bases $(S_{\lambda}^{W, W'})$ of the image of $Sat_Z$, and $(T_{\lambda}^{W, W'})$ of $\mathcal{H}_R(G, K, W, W')$ such that
\[ Sat_Z(T_{\lambda}^{W, W'}) = S_{\lambda}^{W, W'} \]
for $\lambda \in Z^+(W, W')/Z^0$ where $Z^+(W, W')$ is a certain union of cosets of $Z^0$ in $Z^+$ (Abe-Herzig-V. [11]). The proof relies on the theory the pro-$p$-Iwahori Hecke $R$-algebra. A simple consequence is the “change of weight” \[43\] which is an important step in the proof of the classification of admissible irreducible $R$-representations of $G$. There is also a change of weight the pro-$p$-Iwahori Hecke algebra giving another proof for the change of weight for $G$ (Abe[3]). For an Hecke algebra $\mathcal{H}_R(G, K, W)$ with irreducible weight $W$, one gets an explicit inverse of the Satake isomorphism (Henniart-V. [103]) \[44\]
\[ Sat_Z : \mathcal{H}_R(G, K, W) \to \mathcal{H}_R(Z^+, Z^0, W_{U^0}). \]
For $G$ quasi-split, $\mathcal{H}_R(Z^+, Z^0, W_{U^0}) \simeq R[Z^+/Z^0]$ hence $\mathcal{H}_R(G, K, W)$ does not depend on the choice of $(K, W)$ modulo isomorphism.

For $G$ general, the center of $\mathcal{H}_R(G, K, W)$ contains a finitely generated subalgebra $\mathcal{Z}_T$ isomorphic to $R[Z^+/Z^0]$, and $\mathcal{H}_R(G, K, W)$ is a finitely generated $\mathcal{Z}_T$-module. One chooses an element $s$ in the center of $M$ which strictly contracts $N$ by conjugation. There is a unique element $T_s \in \mathcal{H}_R(M, M^0, W_{N^0})$ with support $M^0$s such that $T_s(s)$ is the identity on $W_{N^0}$. The generalized Satake transform
\[ Sat_M : \mathcal{H}_R(G, K, W) \to \mathcal{H}_R(M, M^0, W_{N^0}) \]

\[43\] the change of weight theorem is an isomorphism between two compactly induced representations
\[44\] this isomorphism for $Sat'$ is proved when $G$ is split in Herzig [108], and in general in Henniart-V. [103]
is a localization at $T_a$. The natural intertwiner

$$I_V : \text{ind}^G_K W \to \text{ind}^F_M(\text{ind}^M_{V_N^0} W_{N^0})$$

is injective and its localization at $T_a$ is bijective when $W$ satisfies a regularity assumption (Herzig [107], Abe [2], Henniart-V. [103]).

For a field $R$ of characteristic $p$, the **supersingularity** of an admissible irreducible $R$-representation $V$ of $G$ is defined with the Satake homomorphism (Abe-Henniart-Herzig-V. [8]).

First, assuming $R$-algebraically closed, an homomorphism from the center of an Hecke algebra $\mathcal{H}_R(G, K, W)$ with irreducible weight is said to be supersingular if it does not extend to the center of $\mathcal{H}_R(M, M^0, W_{N^0})$ via the Satake homomorphism for any $P \neq G$. As $V$ is admissible, there exists some irreducible representation $W$ of $K$ such that $\text{Hom}_{\mathcal{H}_R(G)}(\text{ind}^G_K W, V) \neq 0$. If $\text{Hom}_{\mathcal{H}_R(G)}(\text{ind}^G_K W, V)$ as a module over the center of $\mathcal{H}_R(G, K, W)$ contains an eigenvector with a supersingular eigenvalue, $V$ is called supersingular. This does not depend on the choice of $(K, W)$. For $R$ not algebraically closed, $V$ is called supersingular if some admissible irreducible $R^{ac}$-representation $V^{ac}$ of $G$ which is $V$-isotypic as an $R$-representation, is supersingular. This does not depend on the choice of $V^{ac}$ (Henniart-V. [105]).

For $G$ unramified and $K$ hyperspecial, using the geometric Satake equivalence, Xinwein Zhu [206] identified the Hecke ring $\mathcal{H}(G, K)$ with a ring associated to the Vinberg monoid of $\hat{G}$ and formulated a canonical Satake isomorphism, and proved that the commutative $\mathcal{Z}$-algebra $\mathcal{H}(G, K)$ is finitely generated. He extended his formulation to an Hecke algebra $\mathcal{H}_{OE}(G, K, W)$ with weight a finite free $O_E$-module $W$ arising from an irreducible algebraic representation $E \otimes_{O_E} W$ of $G$, where $E/F$ is a finite extension.

For $F$ of characteristic 0 and $R$ a field of characteristic $p$, Heyer (Theorem 4.3.2) defined a derived Satake homomorphism related to $Sat_M$.

For $F$ of characteristic 0, $G$ split, $K$ hyperspecial and $R = \mathbb{Z}/p^a\mathbb{Z}$, $a \geq 1$, Niccolo Ronchetti [165] established a Satake homomorphism for the derived Hecke $\mathbb{Z}/p^a\mathbb{Z}$-algebra of $(G, K)$ (a graded associative $\mathbb{Z}/p^a\mathbb{Z}$-algebra whose degree 0 subalgebra is $\mathcal{H}_{\mathbb{Z}/p^a\mathbb{Z}}(G, K)$). The relation with the Heyer derived Satake homomorphism is unclear.

12. **Pro-$p$ Iwahori Hecke ring**

The isomorphism classes of the Iwahori Hecke ring $\mathcal{H}(G, J)$ and the pro-$p$ Iwahori Hecke ring $\mathcal{H}(G, \hat{J})$ depend only on $G$, because the Iwahori subgroups of $G$ are conjugate, as well as the pro-$p$ Iwahori subgroups.

They are both natural generalisations of affine Hecke $\mathbb{Z}$-algebras. We will focus on the pro-$p$ Iwahori Hecke ring which is more involved, that we will denote by $\mathcal{H}(G)$, but all the results apply to Iwahori Hecke rings with some simplifications.

Our motivation to study the pro-$p$ Iwahori Hecke ring instead of the Iwahori Hecke ring comes from the theory of mod $p$ representation [47]. Any non-zero mod $p$ representation of $G$ has a non-zero $\hat{J}$-fixed vector, and the pro-$p$ radical of any parahoric subgroup of $G$ is contained in some $G$-conjugate of $\hat{J}$.

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45This means that the image of $Sat_M$ contains $T_a$ and that its localisation at $T_a$ is $\mathcal{H}_R(M, M^0, W_{N^0})$.

46meaning that the map $\mathcal{H}_R(M, M^0, V_{N^0}) \otimes_{\mathcal{H}_R(G, K, V)} \text{ind}^G_K V \to \text{ind}^F_M(\text{ind}^M_{V_N^0} V_{N^0})$ is bijective, if the kernel of $V \to V_{N^0}$ contains $k^{V(N^0)^0}$ for all $k \in K \setminus P^0(P^0)^0$.

47Flicker [69] studied the pro-$p$ Iwahori Hecke complex algebra when $G$ is unramified.
For any commutative ring $R$, the pro-$p$ Iwahori Hecke $R$-algebra $\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}(G)$ is a specialization of the generic pro-$p$ Iwahori Hecke $R[q_*]$-algebra $\mathcal{H}(G)(q_*, c_*)$ of $G$, introduced by Nicolas Schmidt [168], [169], when $G$ is split, and by V. [197] in general. The $q_*$ are finitely many indeterminates and the finitely many $c_* \in R[q_*$] satisfy simple conditions. The general principle is that one proves properties of the generic pro-$p$ Iwahori Hecke $R[q_*]$-algebra by specializing all $q_*$ to 1, and then one transfers them to $\mathcal{H}_R(G)$ by specialization.

Example The affine Yokonuma-Hecke algebra defined by Maria Chlouveraki and Loïc Poulain d’Andecy is a generic pro-$p$ Iwahori Hecke algebra (Chlouveraki and Sécherre [30]).

The main features of affine Hecke $R$-algebras generalize to the generic pro-$p$ Iwahori Hecke $R$-algebra, and by specialization to $\mathcal{H}_R(G)$. The $R[q_*]$-module $\mathcal{H}(G)(q_*, c_*)$ is free with an Iwahori-Matsumoto basis of elements satisfying braid relations and quadratic relations, with “alcove walk bases” associated to the Weyl chambers satisfy product formulas involving different alcove walk bases, and with Bernstein-Lusztig relations from which one deduces an explicit canonical $R[q_*]$-basis of the center $[194]$.  

Finiteness properties of the pro-$p$ Iwahori ring $\mathcal{H}(G)$:

(i) The center $Z(G)$ of $\mathcal{H}(G)$ is a finitely generated $\mathbb{Z}$-algebra and $\mathcal{H}(G)$ is a finitely generated $Z(G)$-module.

(ii) $Z(G)$ contains a canonical subring $Z_T$ isomorphic to the affine semi-group $\mathbb{Z}$-algebra $\mathbb{Z}[T^+/T^0]$, and the $Z_T$-modules $Z$ and $\mathcal{H}$ are finitely generated.

(iii) The elements of the Iwahori-Matsumoto basis of $\mathcal{H}(G)$ are invertible in $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \mathcal{H}(G)$.

(iv) For any commutative ring $R$, the center of $\mathcal{H}_R(G)$ is $Z_R(G) = R \otimes_{\mathbb{Z}} Z(G)$.

For any field $R$, any simple $\mathcal{H}_R(G)$-module is finite dimensional by (i) and (iv) [100].

Xuhua He and Radhika Ganapathy [92] gave an Iwahori-Matsumoto presentation of the Hecke ring $\mathcal{H}(G, J_n)$ of the $n$-th congruence subgroup $J_n$ of $J$ for any $n \in \mathbb{N}_{>0}$.

For a standard parabolic subgroup $P = MN$, although $\mathcal{H}_R(M)$ is not contained in $\mathcal{H}_R(G)$, there is a parabolic induction

$$\text{ind}^{\mathcal{H}(G)}_{\mathcal{H}(M)} = - \otimes_{\mathcal{H}(M)} X_{G,P} : \text{Mod } \mathcal{H}_R(M) \to \text{Mod } \mathcal{H}_R(G), \quad X_{G,P} = \text{ind}^G_P(R[J\setminus M])$$

of right adjoint Hom$_{\mathcal{H}_R(G)}(X_{G,P}, -)$ and of left adjoint a certain localisation (hence the left adjoint is exact, a surprise when $p$ is not invertible in $R$ as the functor $(-)_N$ for representations is not exact). The parabolic induction and its right adjoint for the group and for the pro-$p$ Iwahori Hecke algebra correspond to each other via the pro-$p$ Iwahori invariant functors. The same holds true for the left adjoint functor if $R$ is a field of characteristic different from $p$, but Abe gave a counter-example for $G = GL(2, \mathbb{Q}_p)$ and $R$ of characteristic $p$ (Ollivier-V. [152]). The parabolic induction is isomorphic to:

$$\text{ind}^{\mathcal{H}(G)}_{\mathcal{H}(P)} = - \otimes_{\mathcal{H}(P)} \mathcal{H}(G) : \text{Mod } \mathcal{H}_R(M) \to \text{Mod } \mathcal{H}_R(G)$$

48 the Iwahori Matsumoto presentation, the Bernstein basis, the Bernstein-Lusztig relations, the description of the center, and the geometric proofs of Görtz [78].

49 The Iwahori-Matsumoto basis of $\mathcal{H}(G)$ is given by the characteristic functions of the double cosets of $G$ modulo $J$. 

where $\mathcal{H}(P) = \mathbb{Z}[(\tilde{J} \cap P)\backslash G/(\tilde{J} \cap P)]$ is the parabolic pro-$p$ Iwahori Hecke ring of $P$ for two ring homomorphisms $\mathcal{H}(M) \leftarrow \mathcal{H}(P) \to \mathcal{H}(G)$ (Claudius Heyer [110]).

For an algebraically closed field $R$ of characteristic $p$ and an irreducible $R$-representation $W$ of a special parahoric subgroup $K$ containing $\tilde{J}$, an inverse Satake-type isomorphism

$$f : \mathcal{H}_R(Z^-, Z^0, W^0) \rightarrow \mathcal{H}_R(G, K, W)$$

is obtained by composition of two natural algebra isomorphisms (Ollivier [147] when $G$ is split, V. [195] in general). The first isomorphism is associated to a “good” alcove walk basis

$$\mathcal{H}_R(Z^-, Z^0, W^0) \rightarrow \text{End}_{\mathcal{H}_R(G)}(W^J) \otimes_{\mathcal{H}_R(K, \tilde{J})} \mathcal{H}_R(G).$$

The dimension of $W^J$ is 1. The second isomorphism

$$\text{End}_{\mathcal{H}_R(G)}(W^J) \otimes_{\mathcal{H}_R(K, \tilde{J})} \mathcal{H}_R(G) \rightarrow \mathcal{H}_R(G, K, W)$$

is associated to a natural $\mathcal{H}_R(G)$-module isomorphism $W^J \otimes_{\mathcal{H}_R(K, \tilde{J})} \mathcal{H}_R(G) \rightarrow (\text{ind}_K^G W)^J$.

When $G$ is split, $f$ is the inverse of the variant $Sat'_Z$ of the generalized Satake isomorphism (Ollivier [147]).

13. Modules of pro-$p$ Iwahori Hecke algebras in characteristic $p$

There is a numerical mod $p$ local Langlands correspondence for the pro-$p$ Iwahori Hecke algebra of $GL(n, F)$ (V. [199]): the following two sets

a) the isomorphism classes of the $n$-dimensional irreducible $\mathbb{F}_p^{ac}$-representations of $\text{Gal}(F^{ac}/F)$ with a fixed value of the determinant of the action of a Frobenius.

b) the isomorphism classes of the supersingular simple modules $\mathcal{H}_{F^{ac}}(GL(n, F))$ with a fixed action of $p_F$ embedded diagonally, have the (finite) number of elements $[4]$. This was significantly improved by Grosse-Kloene if $F \supset \mathbb{Q}_p$ [79, 80]. He constructed an exact and fully faithful functor from the category of finite length supersingular $\mathcal{H}_{F^{ac}}(GL(n, F))$-modules to the category of $\mathbb{F}_q^{d}$-representations of $\text{Gal}(F^{ac}/F)$, if $p^d \geq q^{\lceil 3/2 \rceil}$.

We recall that the pro-$p$ Iwahori Hecke ring $\mathcal{H}(G)$ of $G$ is a finitely generated module over a central subring $\mathbb{Z}_T \simeq \mathbb{Z}[T^+/T^0]$.

A nonzero (right) $\mathcal{H}_R(G)$-module $V$ is called

ordinary if the action on $V$ of any $z \in \mathbb{Z}_T$ corresponding to a non-invertible element of the semi-group $T^+/T^0$ is invertible.

supersingular if for any $v \in V$ and any $z \in \mathbb{Z}_T$ corresponding to a non-invertible element of $T^+/T^0$, there exists $n \in \mathbb{N}$ such that $z^n v = 0$.

Let $R$ be an algebraically closed field of characteristic $p$.

Classification The supersingular simple $\mathcal{H}_R(G)$-modules are classified (V. [195]). The simple $\mathcal{H}_R(G)$-modules are classified in terms of the simple supersingular $\mathcal{H}_R(M)$-modules for the Levi subgroups $M$ of the parabolic subgroups of $G$ (Noriyuki Abe [6], algebraically closed is not necessary (Henniart-V. [105]):

50 equal to the number of irreducible unitary polynomials of degree $n$ in $k_F[X]$

51 $F^{sep} = F^{ac}$ as the characteristic of $F$ is 0
For a standard parabolic subgroup $P = MN$ of $G$ and a simple supersingular $\mathcal{H}_R(M)$-module $W$, there is a notion of extension $e_P(W)$ of $W$ to $\mathcal{H}_R(M')$ for a parabolic subgroup $P' = M'N'$ of $G$ containing $P$. There is a maximal $P'$ with this property, denoted by $P(W)$. For a parabolic subgroup $Q$ with $P < Q < P(W)$, there is a generalized Steinberg $\mathcal{H}_R(M(W))$-module

$$st^P_Q(W) = \text{ind}_{\mathcal{H}_R(Q)}^{\mathcal{H}_R(M)}(e_Q(W))/\sum_{Q \leq Q' \leq Q(W)} \text{ind}_{\mathcal{H}_R(Q')}^{\mathcal{H}_R(Q)}(e_{Q'}(W)).$$

The triple $(P, W, Q)$ is called standard. The $\mathcal{H}_R(G)$-module

$$I_{H(G)}(P, W, Q) = \text{ind}_{\mathcal{H}(P(W))}^{\mathcal{H}(G)}(st^P_Q(W))$$

is simple, and any simple $\mathcal{H}_R(G)$-module is isomorphic to $I_{H(G)}(P, W, Q)$ for some standard triple $(P, W, Q)$ unique modulo $G$-conjugation. It is ordinary if and only if $P = B$.

**Extensions** The extensions between simple $\mathcal{H}_R(G)$-modules

$$\text{Ext}^i_{\mathcal{H}_R(G)}(I_{H(G)}(P_1, W_1, Q), I_{H(G)}(P_2, W_2, Q_2)), \quad i \geq 0,$$

are either 0, or extensions between supersingular simple modules of a specialization of a generic pro-$p$ Iwahori Hecke algebra which is not of a pro-$p$ Iwahori Hecke $R$-algebra (Abe [7]). In more details, considering the central characters, the extensions are 0 if $P_1 \neq P_2$. When $P = P_1 = P_2$, following the construction of the simple modules,

$$\text{Ext}^i_{\mathcal{H}_R(G)}(I_{H(G)}(P, W_1, Q), I_{H(G)}(P, W_2, Q_2)) \simeq \text{Ext}^i_{\mathcal{H}_R(M')}\left( st^P_{Q_1}(W_1), st^P_{Q_2}(W_2) \right)$$

for some $P', Q_1, Q_2'$,

$$\text{Ext}^i_{\mathcal{H}_R(G)}(st^G_{Q_1}(W_1), st^G_{Q_2}(W_2)) \simeq \text{Ext}^{i-r}_{\mathcal{H}_R(G)}(e_G(W_1), e_G(W_2)),$$

for some explicit $r \in \mathbb{N}_{\geq 0}$, and using results of Ollivier-Schneider [158],

$$\text{Ext}^i_{\mathcal{H}_R(G)}(e_G(W_1), e_G(W_2)) \simeq \text{Ext}^i_{\mathcal{H}_R(M)/I}(W_1, W_2)$$

for some ideal $I$ of $\mathcal{H}_R(M)$ acting on $W_1, W_2$ by 0. Abe computed explicitly $\text{Ext}^1$ for two supersingular simple $\mathcal{H}_R(M)/I$-modules.

When $G = GL(2, F)$, Cédric Pépin and Tobias Schmidt proved:

(i) The 2-dimensional supersingular simple $\mathcal{H}_{\mathbb{F}_p^{ac}}(G)$-modules can be realized through the equivariant cohomology of the flag variety of the dual group $\hat{G}$ over $\mathbb{F}_p^{ac}$ [158].

(ii) There is a version in families of the Breuil’s semisimple mod $p$ Langlands correspondence for $GL_2(\mathbb{Q}_p)$ [159].

(iii) There is a Kazhdan-Lusztig theory for the generic pro-$p$ Iwahori Hecke $\mathbb{Z}[q]$-algebra of $G$, where the role of $G$ is taken by the Vinberg monoid $V_\hat{G}$ and its flag variety; the monoid comes with a fibration $V_\hat{G} \to \mathbb{A}^1$ and the dual parametrisation of $\mathcal{H}_{\mathbb{F}_p^{ac}}(G)$-modules is achieved by working over the 0-fiber. They introduce the generic pro-$p$ antispherical module and the generic pro-$p$ Satake homomorphism for a generic spherical Hecke $\mathbb{Z}[q]$-algebra [160].
14. REPRESENTATIONS IN CHARACTERISTIC $p$

In this section, $R$ is a field of characteristic $p$. The admissible irreducible $R$-representations of $G$ are classified in terms of the supersingular admissible irreducible $R$-representations of the Levi subgroups of $G$ (Abe-Henniart-Herzig-V. [8] for $R$ algebraically closed, Henniart-V. [105] for $R$ not algebraically closed).

**Classification** The representation $\text{ind}_G^P W$ parabolically induced from an irreducible admissible supersingular $R$-representation $W$ of a Levi subgroup $M$ of a parabolic subgroup $P$ of $G$, has multiplicity 1 and irreducible subquotients

$$I_G(P,W,Q) = \text{ind}_G^P W(e(W) \otimes \text{St}_Q^P W)$$

for the parabolic subgroups $Q$ of $G$ containing $P$ and contained in the maximal parabolic subgroup $P(W)$ where the inflation of $W$ to $P$ extends to a representation $e(W)$, and

$$\text{St}_Q^P W = (\text{ind}_Q^P W 1_Q)/ \sum_{Q \subseteq Q' \subset P(W)} \text{ind}_Q^P W 1_{Q'}.$$

Any irreducible admissible $R$-representation $V$ of $G$ is isomorphic to $I_G(P,W,Q)$ for a unique triple $(P,W,Q)$ modulo $G$-conjugation.

A similar classification holds true for the irreducible admissible genuine mod $p$ representations of the metaplectic double cover of $Sp_{2n}(F)$ (Koziol and Laura Peskin [123]).

There is a complete description of $\text{ind}_G^P W$ for any irreducible admissible $R$-representation $W$ of $M$ [105]. As a corollary, one obtains generic irreducibility and for any admissible irreducible $R$-representation $V$ of $G$,

$$V \text{ supersingular} \iff V \text{ cuspidal} \iff V \text{ supercuspidal}.$$

The supersingularity of $V$ can also be seen on the pro-$p$ Iwahori invariants (Ollivier-V. [152] for $R$ algebraically closed, but algebraically closed is not necessary Henniart-V. [105]):

$$V \text{ supersingular} \iff V^J \text{ is supersingular} \iff \text{some non-zero subquotient of } V^J \text{ is supersingular}.$$

The $\tilde{J}$-invariant of $I_G(P,W,Q)$ of $G$ is computed and depends only on the pro-$p$ Iwahori invariant $W^{JM}$ (Abe-Henniart-V. [9], [10]).

When $P = B$, the irreducible subquotients of $\text{ind}_G^B W$ are called ordinary. An admissible $R$-representation of $G$ with ordinary irreducible subquotients is called ordinary.

The $\tilde{J}$-invariant functor induces an equivalence between the category of finite length ordinary $R$-representations of $G$ generated by their $\tilde{J}$-invariant vectors and the category of the finite length ordinary $\mathcal{H}_R(G)$-modules, assuming $R$ algebraically closed (Abe [5]).

When $F$ has characteristic 0, the higher duals ($S^i(I_G(P,W,Q))_{i \geq 0}$ are computed in terms of ($S^i(W))_{i \geq 0}$ in a few cases (Kohlhaase [116]).

The extensions between $R$-representations $\text{ind}_G^P W$ of $G$ parabolically induced from supersingular absolutely irreducible $R$-representations $W$ of Levi subgroups, are computed in many cases when $G$ is split and $R$ finite (Hauseux [85],[86],[87],[89]).

The supersingular admissible irreducible $R$-representations $V$ of $G$ are not understood, this remains an open crucial question since two decades and a stumbling block for the search of a mod $p$ or $p$-adic local Langlands correspondence if $G \neq GL(2,\mathbb{Q}_p)$. The classification of simple
supersingular $\mathcal{H}_R(G)$-modules does not help because we do not have enough information on the pro-$p$ Iwahori invariant functor.

Breuil [16] relying on the work of Barthel-Livne classified the supersingular admissible irreducible mod $p$ representations of $GL(2, \mathbb{Q}_p)$. This was the starting point of the mod $p$ local Langlands correspondence for $GL(2, \mathbb{Q}_p)$. There are two main novel features in the mod $p$ local Langlands correspondence. It involves reducible representations of $GL_2(\mathbb{Q}_p)$, and it extends to an exact functor from finite length representations of $GL_2(\mathbb{Q}_p)$ to finite length representations of $\text{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)$.

There are scarce results on supersingular admissible irreducible mod $p$ representations of $G \neq GL(2, \mathbb{Q}_p)$. Supersingular admissible irreducible mod $p$ representations are classified only for some groups close to $GL(2, \mathbb{Q}_p)$; for $SL(2, \mathbb{Q}_p)$ (Ramla Abdellatif [1], Chuangxun Cheng [27]), and for the unramified unitary group $U(1,1)(\mathbb{Q}_p^{2}/\mathbb{Q}_p)$ in two variables (Kozioł [117]).

When $F \neq \mathbb{Q}_p$, there can be many more supersingular admissible irreducible $R$-representations of $GL(2, F)$ than 2-dimensional irreducible representations of $\text{Gal}(F^{sp}/F)$ (Breuil-Paskunas [19]); they cannot be described as quotients of a compact induction by a finite number of equations (Hu [114] if $F \supset \mathbb{F}_p((T))$, Schraen [174] if $F/\mathbb{Q}_p$ is quadratic, Wu [202] in general if $F \supset \mathbb{Q}_p$).

When $R$ is a field of characteristic $p$ and $F \supset \mathbb{Q}_p$, Herzig-Koziol-V. [109] proved that any $G$ admits a supersingular admissible irreducible $R$-representation, using a local method of Paskunas [153] if the semisimple rank $r_G$ of $G$ is 1, and a global method if $r_G > 1$. The existence is not known if $F \supset \mathbb{F}_p((T))$.

There have been recent advances which strongly suggest that the study of mod $p$ representations of $G$ is best done on the derived level. When $R$ is a field of characteristic $p$, Schneider [170] introduced the unbounded derived category $D_R(G)$ of $R$-representations of $G$. When $\tilde{J}$ is torsion free (this forces $F$ to be of characteristic 0), $D_R(G)$ is equivalent to the derived category of differential graded modules over a differential graded version $\mathcal{H}_R(G)^{\bullet}$ of the pro-$p$ Iwahori Hecke $R$-algebra of $G$, by the derived $\tilde{J}$-invariant functor.

The parabolic induction $\text{ind}_R^G : \text{Mod}_R(M) \to \text{Mod}_R(G)$ being exact extends to an exact derived parabolic induction $\text{Rind}_R^G : D_R(M) \to D_R(G)$ between the unbounded derived categories. The total derived functor of $R^G_F$ is right adjoint to $\text{Rind}_R^G$. The category $D_R(G)$ has arbitrary small direct products and $\text{Rind}_R^G$ commutes with arbitrary small direct products (Heyer [111]), hence $\text{Rind}_R^G$ has a left adjoint. When $\tilde{J}$ is torsion free, the functor $\text{Rind}_R^G$ corresponds to the derived parabolic induction functor on the dg Hecke algebra side, via the derived $\tilde{J}$-invariant functor (Sarah Scherotzke and Schneider [167]).

The Kohlhaase duality functors are related to the derived duality functor $R\text{Hom}(-, R)$ (Schneider and Claus Sorensen [171]).

The structure of the cohomology algebra $\text{Ext}_{\text{Mod}_R(G)}^{\bullet}(R[\tilde{J}\backslash G], R[\tilde{J}\backslash G])$ is simpler than of $\mathcal{H}_R(G)^{\bullet}$; there is an explicit description when $G = SL(2, \mathbb{Q}_p), p \geq 5$ (Ollivier and Schneider [150], [151]).

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[52] by Brown representability; Heyer [111] gave another proof.
15. Local Langlands Correspondences for $GL(n, F)$

The classical local Langlands correspondence for $GL(n, F)$ is a bijection between the isomorphism classes of irreducible complex representations of $GL(n, F)$ and the isomorphism classes of $n$-dimensional Weil-Deligne complex representations, given by local class field theory when $n = 1$, and characterized by the requirement that the $L$ and $\epsilon$ factors attached to corresponding pairs of complex representations coincide (Henniart [99]). A $n$-dimensional Weil-Deligne complex representation is a pair $(\sigma, N)$ where $\sigma$ is a $n$-dimensional Frobenius semi-simple complex representation of the Weil group $W_F$ and $N \in \text{End}_C \sigma$ is a nilpotent endomorphism satisfying $\sigma(w)N\sigma(w)^{-1} = q^{[w]}N$ for all $w \in W_F$. The supercuspidal irreducible $\mathbb{C}$-representations of $GL(n, F)$ correspond to the $n$-dimensional irreducible $\mathbb{C}$-representations of $W_F$.

A twist of the correspondence by an unramified complex character of $GL(n, F)$ is compatible with all the automorphisms of $\mathbb{C}$. For a prime $r$, an isomorphism $\mathbb{C} \simeq \mathbb{Q}_r^{ac}$ transfers the twisted classical local Langlands correspondence to a local Langlands correspondence for $\mathbb{Q}_r^{ac}$-representations of $GL(n, F)$. For $r = \ell \neq p$, the nilpotent part is related to the action of the tame inertia group on an $\ell$-adic representation of $W_F$. By reduction modulo $\ell$ of the $\ell$-adic local Langlands correspondence composed with the Zelevinski involution on $\ell$-adic representations of $GL(n, F)$, one obtains an $\ell$-modular local correspondence. The $\ell$-modular local correspondence, is a parametrization for $\ell$-modular irreducible representations of $GL(n, F)$ by $n$-dimensional Weil-Deligne $\ell$-modular representations, defined as above with $\mathbb{F}_\ell^{ac}$ instead of $\mathbb{C}$. The supercuspidal irreducible $\ell$-modular representations of $GL(n, F)$ correspond to the $n$-dimensional irreducible $\ell$-modular representations of $W_F$. But the nilpotent part $N$ of the Weil-Deligne $\ell$-modular representation has no obvious Galois interpretation.

Dat [42], [43], [44] obtained a geometric realization of the $\ell$-adic local Zelevinski correspondence and of the $\ell$-modular local correspondence on the unipotent irreducible $\mathbb{F}_\ell^{ac}$-representations of $GL(n, F)$ when the order of $q$ in $\mathbb{F}_\ell^*$ is at least $n$ and when $q \equiv 1 \mod \ell$ and $\ell > n$. Kurinczuk and Matringe [126], [127], [128], [129], extended to $\ell$-modular representations the Rankin-Selberg local constants of Jacquet, Piatetski-Shapiro and Shalika of pairs of complex generic representations of linear groups, and the Artin-Deligne local constants of pairs of complex Weil-Deligne representations. These local constants are preserved by the classical local Langlands correspondence, but not by the $\ell$-modular local correspondence. Enlarging the space of $\ell$-modular Weil-Deligne representations to representations with non necessarily nilpotent operators, they suggested a modified $\ell$-modular local correspondence compatible with the formation of local constants and characterized by a list of natural properties. When $R$ is a noetherian $W(\mathbb{F}_\ell^{ac})$-algebra, using the Rankin-Selberg functional equations, Matringe and Moss [136] proved that a $R$-representation of $GL(n, F)$ of Whittaker type admits a Kirillov model.

\footnote{for a fixed non trivial $\mathbb{C}$-character of $F$}
\footnote{$[w]$ is the power of $q$ to which $w$ raises the elements of the residue field $k_F$}
\footnote{$N = 0$}
\footnote{in the principal block = subquotients of some $\text{Ind}_B^G(\chi)$ for $\chi$ an unramified character of a Borel subgroup $B$, this is not the definition of Lusztig}
\footnote{the regular case}
\footnote{the limit case}
When the characteristic of $F$ is 0, Breuil and Schneider \cite{Breuil} motivated by a $p$-adic extension of the local Langlands correspondence, suggested a modified local Langlands correspondence where the complex representations of $GL(n, F)$ are no more irreducible. The Langlands quotient theorem realizes an irreducible $\mathbb{C}$-representation $V$ of $GL(n, F)$ as a quotient of a certain parabolically induced representation $\text{ind}_P^G W$. In the modified version, $V$ is replaced by a twist of $\text{ind}_P^G W$ by an unramified character of $GL(n, F)$.

When the characteristic of $F$ is 0, Emerton and Helm \cite{Emerton-Helm} motivated by a local Langlands correspondence in families and by global contexts, introduced the generic $\ell$-adic local Langlands correspondence which has useful applications to the cohomology of Shimura varieties. For any finite extension $E/\mathbb{Q}_\ell$, it is a map $\rho \mapsto \pi(\rho)$ from $n$-dimensional continuous $E$-representations of the Galois group $\text{Gal}(F_{\text{ac}}/F)$ to finite length $E$-representations of $GL(n, F)$ with an absolutely irreducible generic socle and no other generic irreducible subquotients. Each $\pi(\rho)$ contains a $GL(n, F)$-stable $O_E$-lattice $\pi(\rho)^\alpha$ of reduction modulo $\ell$ having an absolutely irreducible socle and no other generic subquotients, unique modulo homothety.

For any finite extension $R/\mathbb{Q}_p$ of ring of integers $\mathcal{O}_E$ and residue field $k_E$ containing $R$, and $\rho : \text{Gal}(F_{\text{ac}}/F) \to GL(n, \mathcal{O}_E)$ a continuous representation lifting $\overline{\rho} \otimes_R k_E$ the the reduction modulo $\ell$ of $\pi(\rho)\alpha$ embeds in $\overline{\pi(\rho)} \otimes_R k_E$.

1) $\overline{\pi(\rho)}$ has an absolutely irreducible generic socle and no other generic irreducible subquotients.

2) For all finite extensions $E/\mathbb{Q}_\ell$ of ring of integers $\mathcal{O}_E$ and residue field $k_E$ containing $R$, and $\rho : \text{Gal}(F_{\text{ac}}/F) \to GL(n, \mathcal{O}_E)$ a continuous representation lifting $\overline{\rho} \otimes_R k_E$ the the reduction modulo $\ell$ of $\pi(\rho)\alpha$ embeds in $\overline{\pi(\rho)} \otimes_R k_E$.

3) $\overline{\pi(\rho)}$ is minimal with respect to the above two conditions.

For $GL(2, F)$, the correspondence is fairly concrete and explicit when $\ell \neq 2$ (Helm \cite{Helm}).

Emerton and Helm introduced also a notion of a local Langlands correspondence in families \cite{Emerton-Helm}. For any suitable complete local noetherean algebra $R$ with finite residue field $k$, it is the unique map $\rho \mapsto \pi(\rho)$ from the continuous representations $\rho : \text{Gal}(F_{\text{ac}}/F) \to GL(n, R)$ to the admissible $R$-representations of $GL(n, F)$ that interpolates the generic local Langlands correspondence over the points of $\text{Spec} R$ and satisfies certain technical hypotheses.

The existence of the map (Helm and Moss \cite{Helm-Moss}) amounts to showing that whenever there is a congruence between two $\ell$-adic representations of $\text{Gal}(F_{\text{ac}}/F)$, there is a corresponding congruence on the other side of the $\ell$-adic generic local Langlands correspondence. The key idea of the proof is the introduction of the Bernstein center $Z$ of $\text{Mod}_{Z_{\text{ac}}}(GL(n, F))$ (Helm \cite{Helm}, \cite{Helm-Moss}, \cite{Helm-Schneider}). It encodes deep information about congruences between $Z_{\text{ac}}$-representations of $GL(n, F)$. For instance, if two integral irreducible $\mathbb{Q}_p^{\text{ac}}$-representations of $GL(n, F)$ become isomorphic modulo $\ell$, then $Z$ acts on these representations by scalars congruent modulo $\ell$.

\footnote{It is a slight modification of the Breuil and Schneider correspondence transferred to $\ell$-adic representations; the socle of $V$ is the maximal semi-simple subrepresentation of $V$.}

\footnote{\(\rho\) identifies with a representation $\text{Gal}(F_{\text{ac}}/F) \to GL(n, E)$}

\footnote{For an example of a local $p$-adic Langlands correspondence in families for $GL(2, \mathbb{Q}_p)$, see Ildar Gaisin and Joaquin Rodrigues Jacinto \cite{Gaisin-Jacinto}.}
For any $R$ and $G$ quasi-split, Dat, Helm, Kurinczuk and Moss \cite{DatHelmKurinczukMoss} studied the scheme of Langlands parameters of $G$ with coefficient the smallest possible ring $R = \mathbb{Z}[1/p]$ for a local Langlands correspondence in families. In particular, this allows to study chain of congruence of Langlands parameters modulo several different primes. The blocks in the category of $\mathbb{Z}_{2a^i}[1/p]$-representations of $G$ of depth 0 are in natural bijection with the connected components of the space of tamely ramified Langlands parameters for $G$ over $\mathbb{Z}_{ac}[1/p]$; the category is indecomposable if $G$ semi-simple and simply connected, or unramified (Dat and Thomas Lanard \cite{DatLanard}). In a work in progress, Dat, Helm, Kurinczuk and Moss extend the Emerton-Helm-Moss local Langlands correspondence in families to a conjecture which asserts the existence of isomorphisms between
  
  a) the centre of $\text{Mod}_{\mathbb{Z}[q^{-1/2}]}(G)$,

  b) the ring of functions on the moduli stack of Langlands parameters\footnote{Constructions of moduli spaces of Langlands parameters have been also proposed by Fargues and Scholze (\cite{FarguesScholze} over $\mathbb{Z}_\ell$, $\ell \neq p$ using the condensed mathematics of Clausen-Scholze, and by Xinwen Zhu \cite{Zhu}). The local Langlands correspondence is now conjectured to exist at a categorical level (Denis Gaitsgory \cite{Gaitsgory}).}

  c) the descent to $\mathbb{Z}[q^{-1/2}]$ of the endomorphisms of a Gelfand-Graev representation of $G$.

They prove this conjecture when $G$ is any classical $p$-adic group after inverting an integer. The conjecture follows from a Fargues-Scholze conjecture (\cite{FarguesScholze} I.10.2)\footnote{private communication of Dat}.

The $p$-adic local Langlands correspondence for $GL(n, F)$, $F$ of characteristic 0, is a hypothetical correspondence between continuous unitary $E$-Banach space representations of $GL(n, F)$ and $n$-dimensional continuous $E$-representations of $\text{Gal}(F^{ac}/F)$, for any finite extension $E/\mathbb{Q}_p$, given by local class field theory when $n = 1$, as all local Langlands correspondences for $GL(n, F)$. Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Paskunas and Sug Woo Shin \cite{CaraianiEmertonGeeGeraghtyPaskunasShin} constructed a candidate when $p$ does not divide $2n$ using global methods. For $F = \mathbb{Q}_p$ and $n = 2$, it coincides with the $p$-adic local correspondence envisioned by Breuil twenty years ago, constructed by Pierre Colmez \cite{Colmez}, and analyzed by (Paskunas \cite{Paskunas}, Colmez, Gabriel Dospinescu, Paskunas \cite{PaskunasKurinczukMoss}).

For $n \geq 2$ and $D_n$ the central division algebra over $F$ of invariant $1/n$, Scholze\cite{Scholze} constructed a candidate for a $p$-adic and mod $p$ Jacquet-Langlands correspondence from $GL(n, F)$ to $D_n^*$ in a purely geometric way, using the cohomology of the infinite-level Lubin-Tate space. The mod $p$ Jacquet-Langlands correspondence is a canonical map from the admissible mod $p$ representations of $GL(n, F)$ to the admissible mod $p$ representations of $D_n^*$ having a continuous action of $\text{Gal}(F^{ac}/F)$. For $F = \mathbb{Q}_p$ and $n = 2$, it is studied in (Dospinescu-Paskunas-Schraen \cite{DospinescuPaskunasSchraen}).

16. GELFAND-KIRILLOV DIMENSION

Let $R$ be a field and $V$ an irreducible admissible $R$-representation of $G$. For any decreasing sequence $(K_i)_{i \geq 1}$ of open compact subgroups of $G$ with limit the trivial group, the dimensions $\dim_R V^{K_i}$ for $i \geq 1$ are finite and form an increasing sequence. If $V$ is finite dimensional, $\dim_R V^{K_i} = \dim_R V$ when $i$ is large enough. Generally, the dimension of $V$ is infinite and $(\dim_R V^{K_i})_{i \geq 1}$ tends to infinity, but how?

For $F$ of characteristic 0, one can choose an $O_F$-lattice $L$ of the Lie algebra $\mathfrak{g}$ of $G$ on which the exponential map is defined, and consider the decreasing sequence $(K_i = \exp(p_1^i L))_{i \geq 1}$.\footnote{Private communication of Dat}
When $R = \mathbb{C}$, the Harish-Chandra local character expansion of $V$ implies that $\dim_R V^{K_i}$ becomes eventually polynomial \[ \dim_R V^{K_i} = P_{L,V}(q^i), \quad P_{L,V}(X) \in \mathbb{Q}[X] \quad \text{for } i \text{ large enough.} \]
The degree $d_V$ of the polynomial $P_{L,V}[X]$ does not depend on the choice of $L$. It is half the dimension of a unipotent conjugacy class in $G$, \[ 0 \leq d_V \leq \dim_F U, \]
and is 0 if and only if $V$ is finite dimensional. The growth of $\dim_R V^{K_i}$ is measured by $q^{d_V}$. \[ 64^{\text{Harish-Chandra, Notes by Stephen DeBacker and Paul J. Sally, Admissible invariant distributions on reductive $p$-adic group, University Lecture Series Vol.16,1999,97p.}} \]

With no restriction on $F$, for $G = GL(n,F), \ K_i = 1 + p_i M(n,O_F)$ for $i \geq 1$, and $R$ of characteristic different from $p$, the dimensions $\dim_R V^{K_i}$ satisfy the above properties \[ 65^{\text{article in preparation}} \]. For $R = \mathbb{C}$, this follows from the Roger Howe local character expansion. For $R$ of characteristic $\ell$, the key of the proof is that any cuspidal irreducible $\ell$-modular representation of $GL(n,F)$ lifts to an irreducible cuspidal $\ell$-adic representation.

For $GL(2,F)$, we have $\dim_F U = 1$, and $V$ is infinite dimensional if and only if $d_V = 1$. For $GL(n,F)$, we have $\dim_F U = (n^2 - n)/2$ and $V$ is generic (in particular if $V$ is cuspidal) if and only if $d_V = \dim_F U$.

For $R$ a finite field of characteristic $p$, $G = GL(2,\mathbb{Q}_p)$, and $V$ absolutely irreducible, the dimensions $\dim_R V^{K_i}$ for $i \geq 1$ computed by Stefano Morra \[ 66^{\text{article in preparation}} \] satisfy the above properties.

For $F$ of characteristic 0, $R$ a finite field of characteristic $p$, $K$ a uniformly powerful open pro-$p$ subgroup of $G$, $K_i$ the closed subgroup of $K$ generated by $\{ k^{p^i}, k \in K \}$ for $i \geq 1$, and $V$ an admissible $R$-representation of $G$, there is a positive integer $\delta_V$ not depending on the choice of $K$ and positive real numbers $a \leq b$, such that (Calegari-Emerton\[ 64^{\text{Harish-Chandra, Notes by Stephen DeBacker and Paul J. Sally, Admissible invariant distributions on reductive $p$-adic group, University Lecture Series Vol.16,1999,97p.}} \], Emerton-Paskunas \[ 65^{\text{article in preparation}} \], Dospinescu-Paskunas-Schraen \[ 64^{\text{article in preparation}} \]): \[ a p^{i\delta_V} \leq \dim_R V^{K_i} \leq b p^{i\delta_V}. \]
The integer $\delta_V$, which is a sort of Iwasawa dimension of the dual of $V$, is called the Gelfand-Kirillov dimension of $V$. When $F/\mathbb{Q}_p$ is unramified, the admissible $R$-representations $V$ of $GL_2(F)$ studied by Breuil-Herzig-Hu-Morra-Schraen \[ 66^{\text{article in preparation}} \] in mod $p$ cohomology satisfy $\delta_V = [F : \mathbb{Q}_p]$. If $V$ is isomorphic to $I_G(P,W,Q)$ we have \[ 66^{\text{article in preparation}} \]
\[ \delta_{I_G(P,W,Q)} = \delta_W + \dim_{\mathbb{Q}_p} N_Q \]
where $N_Q$ is the unipotent radical of the parabolic subgroup $Q$ of $G$.

These partial results indicate that perhaps a notion of Gelfand-Kirillov dimension of $V$ exists for any $F,G,R$. \[ 64^{\text{Harish-Chandra, Notes by Stephen DeBacker and Paul J. Sally, Admissible invariant distributions on reductive $p$-adic group, University Lecture Series Vol.16,1999,97p.}} \]

\[ 65^{\text{article in preparation}} \]
\[ 66^{\text{article in preparation}} \]
References

[1] R. Abdellatif – Classification des représentations modulo $p$ de $SL(2,F)$, Bull. Soc. Math. France 142:3 (2014), 537-589.

[2] N. Abe – On a classification of irreducible admissible modulo $p$ representations of a $p$-adic split reductive group, Compositio Mathematica 149 (12), p. 2139-2168 (2013).

[3] ———— Change of weight theorem for pro-$p$-Iwahori Hecke algebras, Around Langlands correspondences, Contemp. Math., vol. 691, Amer. Math. Soc., Providence, RI, 2017, pp. 1-13.

[4] ———— Parabolic inductions for pro-$p$-Iwahori Hecke algebras, Advances in Mathematics Volume 355, 2019.

[5] ———— A comparison between pro-$p$-Iwahori Hecke modules and mod $p$ representations, Algebra and Number Theory, Vol. 13 (2019), No. 8, 1959-1981.

[6] ———— Modulo $p$ parabolic induction of pro-$p$-Iwahori Hecke algebra, Journal für die reine und angewandte Mathematik. 749 (2019), 1–64.

[7] ———— Extension between simple modules of pro-$p$ Iwahori Hecke algebras, To appear in Journal of the Institute of Mathematics of Jussieu.

[8] N. Abe, G. Henniart, F. Herzig, M.-F. Vignéras – A classification of admissible irreducible modulo $p$ representations of reductive $p$-adic groups. J. Amer. Math. Soc. 30, p.495-559 (2017).

[9] N. Abe, G. Henniart, M.-F. Vignéras – On pro-$p$-Iwahori invariants of R-representations of $p$-adic groups. Representation Theory 22 (2018), 119-159.

[10] ———— Mod $p$ representations of reductive $p$-adic groups: functorial properties. Trans. Amer. Math. Soc. 371 p.8297-8337 (2019).

[11] N.Abe,F. Herzig, M.-F. Vignéras – Inverse Satake isomorphism and change of weight, Volume 26, Pages 264-324 (2022).

[12] K. Ardakov, P. Schneider – The Bernstein center in natural characteristic, arXiv: 2105.06128 .

[13] A. I. Badulescu – Jacquet-Langlands et unitarisabilité, J. Inst. Math. Jussieu 6 (2007), no3, 349-379.

[14] L. Berger – Central characters for smooth irreducible modular representations of $GL_2(Q_p)$, Rend. Semin. Mat. Univ. Padova 128 (2012), 1–6.

[15] N. Bourbaki – Éléments de mathématiques. Algèbre, Chap. 8. Modules et anneaux semi-simples. Springer, Berlin-Heidelberg (2012).

[16] C. Breuil – Sur quelques représentations modulaires et $p$-adiques de $GL_2(Q_p)$ I. Compositio Math. 138, p. 165–188 (2003).

[17] C. Breuil, F. Herzig, V. Paskunas – Towards a modulo $p$ Langlands correspondence for $GL(2)$. Memoirs of Amer. Math. Soc, 216 (2012).

[18] C. Breuil, F. Herzig, V. Paskunas – Towards a modulo $p$ Langlands correspondence for $GL(2)$. Memoirs of Amer. Math. Soc, 216 (2012).

[19] C. Breuil, P. Schneider – First steps towards $p$-adic Langlands functoriality. J. reine angew. Math. 610, 149 - 180 (2007).

[20] P. Broussous, V. Sécherre, and S. Stevens – Smooth representations of $GL_m(D)$ V: Endo-classes, Doc. Math. 17 (2012), 23-77.

[21] ———— The local Langlands conjecture for $GL(2)$, Grundlehren der mathematischen Wissenschaften 335, Springer-Verlag (2006).

[22] ———— Modular local Langlands correspondence for $GL_n$, Int.Math.Res.Not. IMRN (15):4124-4145, 2014.

[23] F. Calegari and M. Emerton – Completed cohomology, a survey, Non-abelian Fundamental Groups and Iwasawa Theory, London Math. Soc. Lecture Note Ser., 393, 239-257.

[24] A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paskunas, Sug Woo Shin – Patching and the $p$-adic local Langlands correspondence, Camb. J. Math. 4 (2016), no. 2, 197-287.

[25] ———— Patching and the $p$-adic Langlands program for $GL_2(Q_p)$, Compositio Mathematica, Volume 154, Issue 3, 2018, pp. 503 - 548

[26] Chuangxun Cheng – Mod $p$ representations of $SL_2(Q_p)$, Journal of Number Theory, 133(4):1312-1330, 2013.

[27] G. Chinello, Hecke algebra with respect to the pro-$p$-radical of a maximal compact open subgroup for $GL(n, F)$ and its inner forms, J. of Algebra 478 (2017), 296-317.
[29] ———— Blocks of the category of smooth \(\ell\)-modular representations of \(GL_n(F)\) and its inner forms: reduction to level 0, Algebra Number Theory 12 (7) (2018), 1675-1713.

[30] M. Chlouveraki, V. Sécherre – The affine Yokonuma-Hecke Algebra and the pro-\(p\) Iwahori-Hecke Algebra, Math. Res. Let. 23 (2016), no.3, 707-718.

[31] R. Cluckers, T. Hales, and F. Loeser – Transfer Principle for the Fundamental Lemma, in Stabilisation de la formule des traces, variétés de Shimura, et applications arithmétiques, Editors: L. Clozel, M. Harris, J.-P. Labesse, B.-C. Ngo. International Press of Boston (2011).

[32] R. Cluckers, J.Gordon, I.Halupczok – Integrability of oscillatory functions on local fields: transfer principles. Duke Math. J., Vol. 163, No. 8, 1549–1600 (2014).

[33] P. Colmez – Representations de \(GL_2(Q_p)\) et \((\phi,\Gamma)\)-modules. Astérisque 330 pp. 281-509 (2010).

[34] P. Colmez, G.Dospinescu, J.Hauseux, W.Niziol – \(p\)-adic étale cohomology of period domains. Mathematische Annalen, Springer Verlag, 2021, 381 (1-2), pp.105-180.

[35] P. Colmez, G.Dospinescu, V.Paskunas – The \(p\)-adic local Langlands correspondence for \(GL_2(Q_p)\), Cambridge J. Math. 2 (2014), 1-47.

[36] Peiyi Cui – Modulo \(\ell\)-representations of \(p\)-adic groups \(SL_N(F)\), arXiv:1912.13473 (31/12/2019).

[37] —- Modulo \(\ell\)-representations of \(p\)-adic groups \(SL_n(F)\): maximal simple \(k\)-types, arXiv:2012.07492 (2020).

[38] J.-F. Dat – Nu-tempered representations of \(p\)-adic groups I : \(l\)-adic case. Duke Math. J. 126 (3) p.397-469 (2005).

[39] ———— Espaces symétriques de Drinfeld et correspondance de Langlands locale. Ann. Scient. éc. Norm. Sup, 39 (1) ; 1-74 (2006).

[40] ——– Représentations lisses \(p\)-tempérées des groupes \(p\)-adiques. Amer. J. Math. 131 (1) ; 227–255 (2009).

[41] ——– Finitude pour les représentations lisses des groupes \(p\)-adiques. J. Inst. Math. Jussieu, 8 (1) p.261–333 (2009).

[42] ———— Opérateur de Lefschetz sur les tours de Drinfeld et de Lubin-Tate. Compositio Math. 148 (2), 507-530 (2012).

[43] ———— Théorie de Lubin-Tate non-abélienne \(l\)-entièr, Duke Math. J. 161 (6), 951-1010 (2012).

[44] ———— Un cas simple de correspondance de Jacquet-Langlands modulo \(\ell\). Proc. London Math Soc. 104 (2012), 690-727.

[45] ———— Lefschetz operator and local Langlands mod \(\ell\) : the regular case. Nagoya Math. J. 208, 1-38 (2012), (Hiroshi Saito memorial volume)

[46] ———— Lefschetz operator and local Langlands mod \(\ell\) : the limit case. Algebra and Number Theory 8-3, 729–766 (2014).

[47] ———— A functoriality principle for blocks of \(p\)-adic linear groups, in Around Langlands Correspondences, volume 691 of Contemp. Math., 103-131, Amer. Math. Soc., Providence, RI, 2017.

[48] ———— Simple subquotients of big parabolically induced representations of \(p\)-adic groups. J. Algebra 510 p.499-507 (2018).

[49] ———— Depth 0 representations over \(Z[1/p]\), arXiv:2202.03982v1 (Feb 2022).

[50] J.-F.Dat, D.Helm, R.Kurinczuk, G.Moss – Moduli of Langlands parameters (2020)

[51] ———— Finiteness for Hecke algebras of \(p\)-adic groups, arXiv:2203.04929v1 mar 2022.

[52] J.-F.Dat, T. Lanard – Depth zero representations over \(\Z[1/p]\), arXiv 2022.03982v1 Feb 2022

[53] G.Dospinescu, V. Paskunas, B.Schraen – Gelfand-Kirillov dimension and the \(p\)-adic correspondence, arXiv:2201.12922v1, Jan 2022.

[54] A. Dotto – The inertial Jacquet-Langlands correspondence, arXiv:1707.00635v3 (mars 2021).

[55] — Mod \(p\) Bernstein centres of \(p\)-adic groups, accepted in Math.Res. Lett. 2021.

[56] B.Drevon, V. Sécherre – Décomposition en blocs de la catégorie des représentations \(\ell\)-modulaires lisses de longueur finie de \(GL_n(D)\), arXiv:2101.05898v1, Feb 2021.

[57] O. Dudas – Non-uniqueness of supercuspidal support for finite reductive groups, J. Algebra 510 (2018), 508-512.

[58] N.Dupré, J.Kohlhaase – Model categories and pro-\(p\) Iwahori-Hecke modules, 2021. [arXiv:2112.03150]

[59] M. Emerton – Ordinary parts of admissible representations of \(p\)-adic reductive groups I. Definition and first properties. Astérisque 331, 355-402 (2010).

[60] ——– Ordinary parts of admissible representations of reductive \(p\)-adic groups II. Astérisque 331, p. 383–438 (2010).
[62] M. Emerton, D. Helm – The local Langlands correspondence for $GL_n$ in families, Annales Scientifiques de l’ENS, 47 (2014) no. 4, 655-722.

[63] M. Emerton, V. Paskunas – On the effeacibility of certain $\delta$-functors. Astérisque 331 461-469 (2010)

[64] ———- On the density of supercuspidal points of fixed weight in local deformation rings and global Hecke algebras, J. de l’Ecole Poly. Math. 7 (2020), 337-371.

[65] L.Fargues, P.Scholze – Geometrization of the local Langlands correspondence, arXiv:2102.13459v1 Feb 2021.

[66] J. Fintzen – Tame cuspidal representations in non-defining characteristics, arXiv:1905.06374 2019.

[67] —— Types for tame $p$-adic groups, Annals of Mathematics 193 no. 1 (2021), p. 303-346.

[68] —— On the construction of tame cuspidal representations, Compositio Mathematica 157:12 (2021), 2733-2746.

[69] Y.Z. Flicker – The Tame Algebra, Journal of Lie Theory, Vol.21 (2011), no. 2, 469-489.

[70] W.-T. Gan, L.Lomeli – Globalizations of supercuspidal representations over function fields and applications, Journal of the European Mathematical Society Volume 20 Issue number 11 (2018) 2813-2858

[71] R. Ganapathy – The Deligne-Kazhdan philosophy and the Langlands conjectures in positive characteristic. Pro-Quest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.).

[72] ———- The local Langlands correspondence for $GSp_4$ over local function fields, Amer. J. Math. 137 (2015), no. 6, 1441-1534.

[73] R. Ganapathy, S. V arma – On the local Langlands correspondence for split classical groups over local function fields, J. Inst. Math. Jussieu 16 (2017), no. 5, 987-1074.

[74] A. Genestier, V. Lafforgue – Champs restreints pour le groupe réductifs et paramétrisation de Langlands locale, arXiv:1709.00978v2 Aug 2018.

[75] U.Görtz – Alcove Walks and Nearby Cycle on Affine Flag Manifolds. J. of Algebraic Combinatorics, 26(4) (2007), pp. 415-430.

[76] E.Grosse-Kloenne – From pro-p-Iwahori Hecke modules to $(\varphi,\Gamma)$-modules I, Duke Mathematical Journal Vol. 165, No. 8, 1529-1595 (2016)

[77] —— From pro-p Iwahori-Hecke modules to $(\varphi,\Gamma)$-modules II, International Mathematics Research Notices, vol.2018, issue 3, 865-906 (2016).

[78] ——— Supersingular modules as Galois representations, Alg. Number Th. (2020)

[79] T.Haines, S.Rostami – On the Kottwitz conjecture for local shtuka spaces, arXiv:1709.00978v2 Mar 2017

[80] —— Parabolic induction and extensions. Algebra and Number Theory, Mathematical Sciences Publishers, 2018, 12 (4), pp.779-831.

[81] —— On the exactness of ordinary parts over a local field of characteristic $p$. Pacific Journal of Mathematics, Mathematical Sciences Publishers, 2018, 295 (1), pp.17-30.

[82] J.Hauseux– Extensions entre série principales $p$-adiques et modulo $p$ de $G(F)$. Journal of the Institute of Mathematics of Jussieu, Cambridge University Press (CUP), 2016, 15 (2), pp.225-270.

[83] —— Compléments sur les extensions entre série principales $p$-adiques et modulo $p$ de $G(F)$. Bulletin de la société mathématique de France, Société Mathématique de France, 2017, 145 (1), pp.161-192.

[84] —— Parabolic induction and extensions. Algebra and Number Theory, Mathematical Sciences Publishers 2018, 12 (4), pp.779-831.

[85] —— On the exactness of ordinary parts over a local field of characteristic $p$. Pacific Journal of Mathematics, Mathematical Sciences Publishers, 2018, 295 (1), pp.17-30.

[86] J.Hauseux, T.Schmidt, C.Sorensen – Deformation rings and parabolic induction. Journal de Théorie des Nombres de Bordeaux, 2018, 30 (2), pp.695-727.

[87] —— Functorial properties of generalised Steinberg representations. Journal of Number Theory, Elsevier, 2019, 195, pp.312-329.
[92] X.He, R.Ganapathy – Tits groups of Iwahori-Weyl groups and presentations of Hecke algebras, arXiv:2107.01768v1 (Jul 2021)

[93] D. Helm – On ℓ-adic families of representations of $GL_2(\mathbb{Q}_p)$, Mathematical Research Letters, vol.17, 805-822 (2010).

[94] ———— On the modified mod $p$ local correspondence for $GL_2(\mathbb{Q}_\ell)$, Mathematical Research Letters, vol.20, 489-500 (2013).

[95] ———— The Bernstein center of the category of smooth $W(k)[GL_n(F)]$-modules, Forum of Mathematics, Sigma, Volume 4, 2016.

[96] ———— Whittaker models and the integral Bernstein center for $GL_n$, Duke Math. Journal 165 (2016), no. 9, 1597-1628.

[97] ———— Curtis Homomorphisms and the integral Bernstein center for $GL_n$, Algebra and Number Theory, 2020.

[98] D. Helm, G. Moss, Converse theorems and the local Langlands correspondence in families, Invent. Math. 214 (2018) 999-1022.

[99] ———— Une caractérisation de la correspondance de Langlands locale pour $GL(n)$, Bull. Soc. math, 130 (4), 2002, p.587-602.

[100] —– Sur les représentations modulo $p$ de groupes réductifs $p$-adiques, Contemporary Math Vol. 489, (2009).

[101] G. Henniart, B. Lemaire – Intégrales orbitales tordues sur $GL(n,F)$ et corps locaux proches : applications, Canad. J. Math. Vol. 58 (6), 2006 pp. 1229-1267.

[102] — Représentations des espaces tordus sur un groupe réductif connexe $p$-adique, Astérisque 387 (2017).

[103] G. Henniart, M.-F. Vignéras – Comparison of compact induction with parabolic induction. Special issue to the memory of J. Rogawski. Pacific Journal of Mathematics, vol 260, No 2, p. 457–495, (2012).

[104] ———— The Satake isomorphism modulo $p$ with weight. J. für die reine und angewandte Mathematik, vol. 701, p. 33–75, (2015).

[105] ———— Representations of a $p$-adic group in characteristic $p$. For Joseph Bernstein. Proceedings of Symposia in Pure Mathematics, Volume 101, p. 171-210 (2019).

[106] ———— Representations of a reductive $p$-adic group in characteristic distinct from $p$, Tunisian Journal of Mathematics 2022.

[107] F. Herzig – The classification of admissible irreducible modulo $p$ representations of a $p$-adic $GL_n$. Invent. Math. 186 p. 373–434, (2011).

[108] ———— A Satake isomorphism in characteristic $p$, Compos. Math. 147 (2011), no. 1, 263-283.

[109] F. Herzig, K.Koziol, M.-F.Vignéras – On the existence of admissible supersingular representations of $p$-adic reductive groups (with an appendix by Sug Woo Shin). Forum of Mathematics Sigma 8 (2020).

[110] C. Heyer – Parabolic induction via the parabolic pro-$p$ Iwahori Hecke algebra, arXiv:2010.08435v2 Oct 2021.

[111] —— The left adjoint of the derived parabolic induction, arXiv:2204.11381v1 Apr 2022.

[112] Yongquan Hu – Normes invariantes et existence de filtrations admissibles, J. reine angew. Math. 634 (2009), 107-141.

[113] —— Sur quelques représentations supersingulières de $GL2(Q_p)$ J. of Algebra. 324 (2010), 1577-1615.

[114] ———— Diagrammes canoniques et représentations modulo $p$ de $GL2(F)$, J. Inst. Math. Jussieu 11 (2012), 67 - 118.

[115] M. Kashiwara, P. Shapira – Categories and Sheaves. Grundlehren der mathematischen Wissenschaften, vol. 332, Springer Berlin-Heidelberg (2006).

[116] J. Kohlhaase — Smooth duality in natural characteristic. Adv. Math. 317, 1-49 (2017)

[117] K. Koziol – A classification of the irreducible mod $p$ representations of $U(1, 1)(\mathbb{Q}_p, \mathbb{Q}_p)$, Ann.Inst.Fourier (Grenoble) 66:4 (2016), 1545-1582.

[118] ———— Pro-$p$-Iwahori invariants for $SL_2$ and $L$-packets of Hecke modules. International Mathematics Research Notices, no. 4, p. 1090–1125 (2016).

[119] ———— Hecke module structure on first and top pro-$p$-Iwahori Cohomology. Acta Arithmetica 186, 349-376 (2018)

[120] ———— Homological dimension of simple pro-$p$-Iwahori–Hecke modules Math. Res. Lett. 26 (2019), no. 3, 769-804. Journal

[121] ———— Derived right adjoints of parabolic induction:an example, arXiv: 2202.09915v1 Feb 2022.
[122] K. Koziol, Fl. Herzig, M.-F. Vignéras (appendix by Sug-Woo Shin) – On the existence of admissible supersingular representations of $p$-adic reductive groups, Forum of Mathematics, Sigma 8, e2, 73pp, (2020).
[123] K.Koziol, L.Peskin – Irreducible admissible mod-p representations of metaplectic groups, Manuscripta Math. 155 (2018), no. 3-4, 539-577.
[124] R.J. Kurinczuk – Characterization of the relation between two $\ell$-modular correspondences, Comptes Rendus Mathématique, Volume 358, Issue 2 (2020), P. 201-209.
[125] R.Kurinczuk, N.Matringe – Types modulo $\ell$ for the Lie algebras of $GL_m(D)$, Duke Math. J. 163 (2014), 795-887.
[126] R. Le – On some nonadmissible smooth irreducible representations for $GL_2$, Math. Research Letters 26 (2019), no. 6, 1747-1758.
[127] S. Morra – Correspondance de Jacquet-Langlands locale et congruences modulo $\ell$, Invent. Math. 208 (2017), no2, 553-631.
[128] S. Ngo – Les lemmes fondamentaux pour les algèbres de Lie Publications Mathématiques de l’IHÉS, Tome 111 (2010), 1-169.
[129] S. Ollivier – Parabolic induction and Hecke modules in characteristic $p$ for $p$-adic $GL_n$, ANTS 4(6) 701-742 (2010).
[130] S. Ollivier, P. Schneider – Compatibility between Satake and Bernstein isomorphisms in characteristic $p$. ANTS 8(5) 1071-1111 (2014).
[131] S. Ollivier, M.-F. Vignéras – Parabolic induction modulo $p$. Selecta Mathematica, Volume 24, Issue 5, 3973-4039 (2018).
[153] V. Paskunas – Coefficient systems and supersingular representations of $GL_2(F)$, Mém. Soc. Math. Fr. (NS) 99 (2004).
[154] ———— Extensions for supersingular representations of $GL_2(Q_p)$, Astérisque (2010), no. 331, 317-353.
[155] ———— The image of Colmez’s Montreal functor, Publ. Math. I.H.E.S 188 (2013), 1-191
[156] ———— Blocks for mod $p$ representations of $GL_2(Q_p)$. In Automorphic forms and Galois representations. Vol. 2, volume 415 of London Math. Soc. Lecture Note Ser., 231-247. Cambridge Univ. Press, Cambridge, 2014.
[157] V. Paskunas, S.N.Tung – Finiteness properties of the category of mod $p$ representations of $GL_2(Q_p)$, Forum Math. Sigma, No. e80 , 2021.
[158] C.Pépin, T.Schmidt – Mod $p$ Hecke algebras and dual equivariant cohomology I: the case of $GL_2$. Preprint 2019.
[159] ———— A semisimple mod $p$ Langlands correspondence in families for $GL_2(Q_p)$. Preprint 2020.
[160] ———— Blocks for mod $p$ representations of $GL_2(Q_p)$. In Automorphic forms and Galois representations. Vol. 2, volume 415 of London Math. Soc. Lecture Note Ser., 231-247. Cambridge Univ. Press, Cambridge, 2014.
[161] D Renard –Représentations des groupes réductifs $p$-adiques, Cours spécialisés 17, S.M.F. (2010).
[162] A Roche – Parabolic induction and the Bernstein decomposition, Compositio Math. 134(2) (2002), 113-133
[163] ———— Notes on the Bernstein decomposition (2004).
[164] ———— The Bernstein decomposition and the Bernstein centre, in Ottawa Lectures on Admissible Representations of Reductive $p$-Adic Groups, Fields Institute Monographs, Volume 26, pp. 3-52 (American Mathematical Society, Providence, RI, 2009).
[165] Ronchetti N.: A Satake homomorphism for the mod $p$ derived Hecke algebra. Preprint 2019
[166] M.M. Schein – A family of irreducible supersingular representations of $GL_2(F)$ for some ramified $p$-adic fields, arXiv:2109.15244v2 Mar 2022.
[167] S.Scherotzke, P. Schneider–Derived parabolic induction. Bulletin of the London Mathematical Society , 54(1):264-274, 2022.
[168] N.A. Schmidt – Generische pro-$p$-algebren, Diplomarbeit (2009).
[169] - ———— Generic pro-$p$-Hecke algebras, arXiv:1801.00353v1 (Dec 2017)
[170] P. Schneider – Smooth representations and Hecke modules in characteristic $p$, Pacific J. Math. 279, 447-464 (2015)
[171] P. Schneider, Cl. Sorensen – Duals in natural characteristic, arXiv:2022.01800v1 Feb 2022.
[172] P. Scholze – The Local Langlands Correspondence for $GL_n$ over $p$-adic fields, Invent. Math. 192 (2013), no. 3, 663-715.
[173] ———— On the $p$-adic cohomology of the Lubin-Tate tower, with an appendix of Michael Rapoport, Ann. Sci. Éc. Norm. Supé. (4) 51 (2018), no. 4, 811-863.
[174] B. Schraen – Sur la présentation des représentations supersingulières de $GL_2(F)$, J. Reine Angew. Math. 704 (2015), 187-208.
[175] V. Sécherre, S. Stevens – Représentations lisses de $GL(m,D)$, IV, représentations supercuspidales, J. Inst. Math. Jussieu 7 (2008), no. 3, p. 527-574.
[176] ———— Smooth representations of $GL(m,D)$, VI: semisimple types, Int. Math. Res. Not. (2011).
[177] ———— Block decomposition of the category of $\ell$-modular smooth representations of $GL_n(F)$ and its inner forms, Ann. Sci. Éc.. Norm. Sup. (4) 49 (3), 669-709 (2016).
[178] ———— Towards an explicit local Jacquet-Langlands correspondence beyond the cuspidal case, Compos. Math. 155 (2019), no. 10, 1853-1887.
[179] D. Skodlerack – Cuspidal irreducible complex or $\ell$-modular representations of quaternionic forms of $p$-adic classical groups for odd $p$, arXiv:1907.02922v2 2019.
[180] ———— Semisimple characters for inner forms II: quaternionic inner forms of classical groups, Representation Theory of the A.M.S. 24:11(2020),323-359
[181] C.M. Sorensen – A proof of the Breuil-Schneider conjecture in the indecomposable case, Annals of Mathematics 177 (2013), 1-16.
[182] ———— The Breuil-Schneider conjecture, a survey. Adv. in the Theory of Numbers. Proceedings of the CNTA XIII. Fields Institute Comm., Vol. 77. A. Alaca, S. Alaca, K. S. Williams (Eds.) 2015.
[183] ———— A vanishing result for higher smooth duals, Algebra and Number Theory 13:7 (2019), 1735-1763.
[184] S. Stevens – Semisimple characters for $p$-adic classical groups, Duke Math. J. 127(1), p. 123-173 (2005).
[185] ———— The supercuspidal representations of $p$-adic classical groups, Invent. Math. 172, p. 289-352 (2008).
[186] J-Trias – Correspondance thêta locale ℓ-modulaire I: groupe métaplectique, représentation de Weil et θ-lift, [arXiv:2009.11561v2].

[187] M.-F. Vignéras – On highest Whittaker models and integral structures. In Contributions to automorphic forms, geometry, and number theory, pages 773-816. John Hopkins Univ. Press. 2004.

[188] —— Representations modulo p of the p-adic group GL(2, F). Compositio Math. 140 pp. 333-358 (2004).

[189] —— Pro-p Iwahori Hecke ring and supersingular $\mathbb{F}_p$-representations. Math. Annalen 331, p. 523-556. Erratum vol. 333(3), p. 699-701 (2005).

[190] —— Algèbres de Hecke affines génériques, Representation Theory 10, p.1–20 (2006).

[191] —— Représentations irréductibles de $GL(2, F)$ modulo $p$, in L-functions and Galois representations, ed. Burns, Buzzard, Nekovar, LMS Lecture Notes 320 (2007).

[192] —— Série principale modulo $p$ de groupes réductifs p-adiques. GAFA Geom. funct. anal. vol. 17 p. 2090–2112 (2007).

[193] —— Représentations p-adiques de torsion admissibles (shorter version of Admissibilité des représentations $p$-adiques et lemme de Nakayama, janvier 2007 (pdf)), Number Theory, Analysis and Geometry: In Memory of Serge Lang, ED. D.Gofeld,P.Jones, D.Ramakrishnan,K.Ribet,J.Tate, Springer (2011)

[194] —— The pro-$p$ Iwahori Hecke algebra of a reductive $p$-adic group II, Muenster J. of Math. 7 p. 363–379 (2014).

[195] —— The pro-$p$-Iwahori Hecke algebra of a reductive $p$-adic group III (spherical Hecke algebras and supersingular modules), Journal of the Institute of Mathematics of Jussieu , Volume 16 , Issue 3, June 2017 , pp. 571 - 608, published online (2015).

[196] —— The pro-$p$ Iwahori Hecke algebra of a reductive $p$-adic group V (parabolic induction), Pacific Journal of Math. 279, p. 499–529 (2015).

[197] —— The pro-$p$ Iwahori Hecke algebra of a reductive $p$-adic group I, Compositio Mathematica 152, p. 693-753 (2016).

[198] —— The right adjoint of the parabolic induction. Hirzebruch Volume Proceedings Arbeitstagung 2013, Birkhauser Progress in Math. 319,p. 405-424 (2016)

[199] J.-L. Waldspurger – Endoscopie et changement de caractéristique, J. Inst. Math. Jussieu 5 Issue 3 (2006), 423-525.

[200] —— Endoscopie et changement de caractéristique : intégrales orbitales pondérées, Ann. Inst. Fourier 59(5) (2009), 1753-1818.

[201] M.H. Weissman – An induction theorem for groups acting on trees, Representation Theory 23 p. 205-212 (2019).

[202] Zhixiang Wu – A note on presentations of supersingular representations of $GL_2(F)$, Manuscripta Math. 165 (2021), 583-596.

[203] J.K.Yu – Construction of tame supercuspidal representations, J.A.M.S. 14, p. 579–622 (2001).

[204] —— Bruhat-Tits theory and buildings, pp. 53-77, On the local Langlands correspondence for tori,pp. 177-183, in Ottawa Lectures on Admissible Representations of Reductive $p$-Adic Groups, Fields Institute Monographs, Volume 26, (American Mathematical Society, Providence, RI, 2009).

[205] X. Zhu –The Geometric Satake Correspondence for Ramified Groups, Ann. Sci. Ec. Norm. Supér., 48 (2015), 409-451.

[206] —— A note on integral Satake isomorphisms, [arXiv:2005.13056v3 jan 2021].

[207] —— Coherent sheaves in the stack of Langlands parameters, [arXiv:2008.02998v2 (2021)