Solving A Class of Nonsmooth Resource Allocation Problems with Directed Graphs though Distributed Smooth Multi-Proximal Algorithms

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Abstract

In this paper, two distributed multi-proximal primal-dual algorithms are proposed to deal with a class of distributed nonsmooth resource allocation problems. In these problems, the global cost function is the summation of local convex and nonsmooth cost functions, each of which consists of one twice differentiable function and multiple nonsmooth functions. Communication graphs of underlying multi-agent systems are directed and strongly connected but not necessarily weighted-balanced. The multi-proximal splitting is designed to deal with the difficulty caused by the unproximable property of the summation of those nonsmooth functions. Moreover, it can also guarantee the smoothness of proposed algorithms. Auxiliary variables in the multi-proximal splitting are introduced to estimate subgradients of nonsmooth functions. Theoretically, the convergence analysis is conducted by employing Lyapunov stability theory and integral input-to-state stability (iISS) theory with respect to set. It shows that proposed algorithms can make states converge to the optimal point that satisfies resource allocation conditions.

Key words: Distributed resource allocation, nonsmooth cost function, directed graph, splitting method.

1 Introduction

In this paper, we consider a class of distributed nonsmooth convex resource allocation problems with directed graphs. A wide range of problems in the field of coordination of multi-agent systems [1]-[3], economic dispatch of power systems [4] and machine learning belong to this class of problems. As examples, in distributed constrained coordination of multi-agent systems with directed graphs, the local cost function of agent \( i \) usually consists of a smooth function and multiple nonsmooth functions standing for different constraints and tasks. Moreover, multi-agent systems are required to maintain some configurations described by resource allocation conditions. When considering a classical machine learning problem - the fused LASSO problem [5] - with constraints and directed graphs, the least squares loss is smooth. The \( l_1 \) penalty and indicator functions of local constraints in this problem are usually nonsmooth. Then resource allocation conditions are employed here as global constraints. As common features, each global cost function in these problems is summed up by local cost functions, and each local cost function consists of a smooth convex function and multiple nonsmooth convex...
functions. Even though nonsmooth functions are proximable, their summation might not be, where a function being proximal means that the proximal operator of this function has a closed or semi-closed form solution and is computationally easy to evaluate [6]. Besides, connected graphs of these problems are directed and maybe weight-unbalanced, where a directed graph being weight-unbalanced means that the in-degree and out-degree of some nodes in this graph are unequal. The difficulty of these problems is to tackle nonsmooth cost functions and directed connecting graphs simultaneously. Due to important applications and challenges mentioned above, these problems have attracted increasing attentions.

**Literature review**

Communication between agents in multi-agent systems has attracted much attention due to the importance of information exchange. Recently, continuous-time distributed algorithms for resource allocation problems have been widely investigated with different kinds of connected graphs [7]-[13]. For undirected graphs, [7] designed an initialization-free distributed algorithm for distributed resource allocation problems. [8] proposed a new distributed private-guaranteed algorithm to solve economic dispatch problems with undirected graphs. As to directed graphs, [9] proposed a continuous-time algorithm via singular perturbation for distributed resource allocation problems. While [9] did not consider local constraints. In [10], a distributed projection-based algorithm was designed to deal with distributed resource allocation problems with weight-balanced graphs. [11] investigated constrained nonsmooth resource allocation problems via a distributed algorithm, which can solve resource allocation problems with strongly convex cost functions and weight-balanced digraphs, as well as resource allocation problems with strictly convex cost functions and connected undirected graphs. For distributed resource allocation problems with weight-unbalanced graphs, [12] proposed a distributed adaptive algorithm to achieve the optimal solution. While this algorithm fails to solve resource allocation problems with local constraints and weight-unbalanced graphs simultaneously.

Nonsmoothness is a natural property of many resource allocation problems in real-world science and engineering areas. Two important categories of existing algorithms for solving distributed nonsmooth optimization and resource allocation problems are shown here. The first category is subgradient-based algorithms proposed in [14]-[18], whose convergence was proven based on nonsmooth analysis [19]. [15] designed a distributed continuous-time projected algorithm to deal with distributed constrained nonsmooth optimization problems. [17] investigated the distributed nonsmooth constrained optimization problem with distributed

**Contribution**

In this paper, two smooth primal-dual algorithms are proposed for a class of distributed nonsmooth convex resource allocation problems with directed graphs. A distributed estimator of the left eigenvector associated with zero eigenvalue of Laplacian matrix of the directed graph is considered in the second algorithm. The global cost function in these problems is a summation of local cost functions, and each of them consists of a smooth convex function and multiple nonsmooth convex functions. Although each nonsmooth function is proximable, their summation might not be. Contributions of this paper are summarized as follows.

(i) This paper explores a class of nonsmooth resource allocation problems with directed graphs. Compared with [7]-[12], this paper considers resource allocation problems with weight-unbalanced graphs. In contrast to [13], smooth algorithms are designed for nonsmooth resource allocation problems with local constraints.

(ii) Distributed smooth primal-dual algorithms employing multi-proximal splitting are proposed in this paper. The multi-proximal splitting is used to deal with the unproximable property of the summation of nonsmooth functions and ensure smoothness of proposed algorithms.

(iii) A Lyapunov function and an iISS-Lyapunov function with respect to the set of equilibria are designed. Then the convergence and correctness of proposed algorithms are proved by using Lyapunov stability theory and iISS theory, which provides novel insights into analysis of the asymptotically convergent system with inputs by employing iISS theory with respect to set.
The rest of this paper is organized as follows. In Section II, some basic definitions of graph theory, proximal operator and iISS theory are presented. Section III shows the nonsmooth resource allocation problem with directed graph. In Section IV, we propose two distributed multi-proximal splitting based smooth continuous-time primal-dual algorithms with and without left eigenvector estimator, respectively. Then proofs for the convergence and correctness of these algorithms are also presented. In Section V, simulations show the effectiveness of our proposed algorithm. Finally, Section VI concludes this paper.

2 Mathematical Preliminaries

In this section, we introduce necessary notations, definitions and preliminaries about graph theory, proximal operator and integral input-to-state stability (iISS).

2.1 Graph Theory

A weighted directed graph $G$ is denoted by $G(V, E, A)$, where $V = \{1, \ldots, n\}$ is a set of nodes, $E$ is a set of edges, and $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a weighted adjacency matrix. An edge $e_{ij} \in E$ indicates that agent $i$ can receive information from agent $j$. If $e_{ij} \in E$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. Moreover, $a_{ii} = 0, i \in I$. Agent $j \in N_i$ denotes agent $j$ is a neighbour of agent $i$. The in-degree and out-degree of agent $i$ are $d_i^n = \sum_{j=1}^n a_{ij}$ and $d_i^o = \sum_{j=1}^n a_{ji}$, respectively. The Laplacian matrix is $L_n = D^n - A$, where $D^n \in \mathbb{R}^{n \times n}$ is diagonal with $D^n_{ii} = \sum_{j=1}^n a_{ij}, i \in \{1, \ldots, n\}$. We use $\| \cdot \|$ to indicate Euclidean norm. Let $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^+$ denotes the set of positive real numbers. $\text{diag}\{b_1, \ldots, b_n\} \in \mathbb{R}^{n \times n}$ is denoted as the diagonal matrix, whose $i$-th diagonal element is $b_i \in \mathbb{R}$ for $i \in \{1, \ldots, n\}$. $I_n$ is the $n$-dimensional identity matrix. Let $0_n \in \mathbb{R}^n$ denote the vector of all zeros. $O_n$ is the $n$-dimensional null matrix, which means that every element in $O_n$ is zero. $(\cdot)^T$ denotes transpose of matrix.

Lemma 1 ([1]) Assume that graph $G$ is strongly connected with the Laplacian matrix $L_n$. Then:

1. There is a positive left eigenvector $h = (h_1, h_2, \ldots, h_n)^T$ associated with the zero eigenvalue such that $h^T L = 0_n$ and $\sum_{i=1}^n h_i = 1$.

2. $\min_{x \in \mathbb{R}^n} x^T Lx \geq \lambda_2(L) \|x\|^2$, where $L = (HL + L^T H)/2$ with $H = \text{diag}(h_1, h_2, \ldots, h_n)$ and $\lambda_2(L)$ being its second smallest eigenvalue.

2.2 Proximal Operator

Let $f(\delta)$ be a lower semi-continuous convex function for $\delta \in \mathbb{R}^r$. Then the proximal operator $\text{prox}_f[\theta]$ of $f(\delta)$ at $\theta \in \mathbb{R}^r$ is

$$\text{prox}_f[\theta] = \arg \min_{\delta} \{ f(\delta) + \frac{1}{2}\|\delta - \theta\|^2 \}. \quad (1)$$

Let $\partial f(\delta)$ denote the subdifferential of $f(\delta)$. If $f(\delta)$ is convex, then $\partial f(\delta)$ is monotone, that is, $(\zeta_1 - \zeta_2)^T (\delta_1 - \delta_2) \geq 0$ for all $\delta_1 \in \mathbb{R}^r, \delta_2 \in \mathbb{R}^r$, $\zeta_1 \in \partial f(\delta_1)$, and $\zeta_2 \in \partial f(\delta_2)$. $\delta = \text{prox}_f[\theta]$ is equivalent to

$$\theta - \delta \in \partial f(\delta). \quad (2)$$

2.3 Integral Input-to-State Stability with respect to set

Consider the system

$$\dot{x} = f(x, u), x(0) = x_0, t \geq 0, \quad (3)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Inputs are measurable and locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz continuous. Equilibria of system (3) consist a closed set $\mathcal{M}$. For each $\xi \in \mathbb{R}^n$, the point-to-set distance from $\xi$ to $\mathcal{M}$ is denoted by

$$\|\xi\|_{\mathcal{M}} = d(\xi, \mathcal{M}) = \inf\{\|\xi - \psi\|, \psi \in \mathcal{M}\}. \quad (4)$$

In particular, $\|\xi\|_{(0)} = \|\xi\|$. Let $\mathcal{K}$ denote the class of functions $a(x) : [0, \infty) \to [0, \infty)$ which are strictly increasing, continuous and $a(0) = 0$; $\mathcal{K}_\infty$ denotes the class of functions $a(x) : [0, \infty) \to [0, \infty)$ which are strictly increasing, continuous, and $\lim_{x \to +\infty} a(x) = +\infty$. $\mathcal{L}$ is the class of functions $a(x, y) : [0, +\infty) \to [0, \infty)$ which are continuous, decreasing and $\lim_{x \to +\infty} a(x) = 0$; $\mathcal{K}_\mathcal{L}$ is the class of functions $a(x, y) : [0, \infty) \to [0, \infty)$ where $a(x, y)$ belongs to class $\mathcal{K}$ with respect to $x : [0, \infty)$ and to class $\mathcal{L}$ with respect to $y : [0, \infty)$ [27]. A positive definite function $a(x) : [0, \infty) \to [0, \infty)$ is one that $a(0) = 0$ and $a(x) > 0$ when $x > 0$. A function $V(x) \in \mathbb{R}$ is semipositive if and only if for each $r$ in the range of $V(x)$, the sublevel set $\{x|V(x) \leq r\}$ is compact. A positive definite function with respect to $\mathcal{M}$ is one that is zero at $\mathcal{M}$ and positive otherwise [28,29]. A nonempty set $\mathcal{M}$ is 0-invariant for system (3) if the solution starting from $\mathcal{M}$ is defined for all $t \geq 0$ and stays in $\mathcal{M}$ when $u \equiv 0_m$. System (3) is said to be forward complete if the solution $x(t, x_0, u)$ is defined for all $t > 0$ [30].

Define $DV(x) = [DV(x)]^T$. Then definitions of integral input-to-state stability (iISS) and iISS-Lyapunov function with respect to a closed and 0-invariant set $\mathcal{M}$ are given below.
Definition 1 System (3) is Integral Input-to-State Stability (iISS) with respect to a closed and 0-invariant set $\mathcal{M}$ if system (3) is forward complete and there exist functions $a_1 \in \mathcal{K}_\infty$, $a_2 \in \mathcal{KL}$ and $a_3 \in \mathcal{K}$, such that

$$a_4(\|x\|_\mathcal{M}) \leq V(x) \leq a_5(\|x\|_\mathcal{M}),$$

(6)

and

$$DV(x)f(x,u) \leq -a_6(\|x\|_\mathcal{M}) + a_7(\|u\|)$$

(7)

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Note that $V$ in Definition 2 is positive definite and proper (i.e., radially unbounded) with respect to $\mathcal{M}$. If the 0-input system $\dot{x} = f(x,0_m)$ is globally asymptotically stable (GAS) with respect to $\mathcal{M}$, the system (3) is to be said 0-GAS with respect to $\mathcal{M}$.

Similar to definitions of dissipation and zero-output dissipation in [31], here we introduce concepts of dissipation and zero-output dissipation with respect to $\mathcal{M}$.

Definition 3 The system (3) with output $p : \mathbb{R}^n \to \mathbb{R}^r$ is dissipative with respect to a closed and 0-invariant set $\mathcal{M}$ if system (3) is forward complete and there exists a continuously differentiable, proper, and positive definite function $V$ with respect to $\mathcal{M}$, together with a continuous positive definite function $a_8$ and a function $a_9 \in \mathcal{K}$, such that

$$DV(x)f(x,u) \leq -a_8(\|p(x)\|) + a_9(\|u\|)$$

(8)

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Moreover, if (8) holds with $p = 0$, i.e., if there exist a proper and positive definite function $V$ with respect to $\mathcal{M}$, and an $a_9 \in \mathcal{K}$, such that

$$DV(x)f(x,u) \leq a_9(\|u\|)$$

(9)

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, we say that the system (3) is zero-output dissipative (ZOD) with respect to $\mathcal{M}$.

Consider a system

$$\dot{x}(t) = J(x(t)), x(0) = x_0, t \geq 0$$

(10)

where $J : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous. The following result is a special case of Theorem 3.1 in [32].

Lemma 2 Let $\mathcal{D}$ be a compact, positive invariant set with respect to system (10), $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and $x(\cdot) \in \mathbb{R}^q$ be a solution of (10) with $x(0) = x_0 \in \mathcal{D}$. Assume $V(x) \leq 0 \forall x \in \mathcal{D}$, and define $\mathcal{Z} = \{x \in \mathcal{D} : \dot{V}(x) = 0\}$. If every point in the largest invariant subset $\bar{\mathcal{M}}$ of $\mathcal{Z} \cap \mathcal{D}$ is Lyapunov stable, where $\mathcal{Z}$ is the closure of $\mathcal{Z} \subset \mathbb{R}^n$, then system (10) converges to one of its equilibria.

3 Problem Description

In this section, the resource allocation problem with a directed graph is formulated. We consider a network of $n$ agents with first-order dynamics, interacting over a graph $\mathcal{G}$. The nonsmooth resource allocation problem is given as

$$\min_{x \in \mathbb{R}^{nq}} F(x), \text{ s.t. } \sum_{i=1}^n x_i = \sum_{i=1}^n d_i,$$  

(11)

where $F(x) = \sum_{j=0}^m F_j(x) = \sum_{i=1}^n f_i(x_i), f_i(x_i) = \sum_{j=0}^m f_{ij}(x_i)$, and $F_j(x) = \sum_{i=1}^m f_{ij}(x_i), j \in \{0, 1, \ldots, m\}, m \geq 2$. Note that $x_i \in \mathbb{R}^q$ is the state of $i$-th agent and $x = [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbb{R}^{nq}$.

For each agent $i \in \{1, \ldots, n\}$, there are $m + 1$ function $f_i^0, \ldots, f_i^m : \mathbb{R}^q \to \mathbb{R}$, contained in the local cost function $f_i(x_i) : \mathbb{R}^q \to \mathbb{R}$, where $f_i^0$ is a smooth convex function, $f_i^j$ is a nonsmooth convex function for $j \in \{1, \ldots, m\}$. Each agent $i$ only has the information about $f_i^j$ for $j \in \{0, 1, \ldots, m\}$. The constraint presented in (11) indicates that all solutions must achieve resource allocation conditions $\sum_{i=1}^n x_i = \sum_{i=1}^n d_i$. Each agent only exchanges information with its neighbours in a fully distributed manner.

Assumptions below are made for the wellposedness of the problem (11) in this section.

Assumption 1 $f_i^0$ is twice continuously differentiable and strongly convex for all $i \in \{1, \ldots, n\}$, which means that there exists a constant $c > 0$ such that for agent $i$,

$$\left(\nabla f_i^0(\vartheta_1) - \nabla f_i^0(\vartheta_2)\right)^T(\vartheta_1 - \vartheta_2) \geq c\|\vartheta_1 - \vartheta_2\|^2,$$  

(12)

where $\vartheta_1 \in \mathbb{R}^q, \vartheta_2 \in \mathbb{R}^q, \vartheta_1 \neq \vartheta_2$. Without loss of generality, we assume $c > m - 1$.

Assumption 2 Each $f_i^j$ is (nonsmooth) lower semi-continuous closed proper convex functions for all $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, and it is proximable.
Assumption 3 The weighted graph \( G \) is directed and strongly connected.

Assumption 4 There exists at least one feasible point to problem (11).

Remark 1 The condition \( c > m - 1 \) in Assumption 1 is mild. When \( 0 < c \leq m - 1 \), there always exists a function \( f^0_i(x) = K_i f^0_i(x) \) for agent \( i \) with \( K > \frac{m - 1}{c} \) such that
\[
(\nabla f^0_i(\theta_1) - \nabla f^0_i(\theta_2))^T(\theta_1 - \theta_2) \geq Kc\|\theta_1 - \theta_2\|^2 > (m - 1)\|\theta_1 - \theta_2\|^2.
\]
Then, we arrive at the following lemma by the Karush-Kuhn-Tucker (KKT) condition of convex optimization problems.

Lemma 3 Under Assumptions 1-4, a feasible point \( x^* \in \mathbb{R}^q \) is a solution of problem (11) if and only if there exist \( x^* \in \mathbb{R}^q \), a constant \( v_0 \in \mathbb{R}^q \), and \( v^* \in \mathbb{R}^q \) such that
\[
0_{nq} \in \nabla F^0(x^*) + \sum_{j=1}^m \partial F^j(x^*) - v^*, \quad (13a)
\]
\[
\sum_{i=1}^n x^*_i = \sum_{i=1}^n d_i, v^*_i = v_0 \text{ for } i \in \{1, \ldots, n\}, \quad (13b)
\]
where \( v = [v^T_1, v^T_2, \ldots, v^T_n]^T \) is the Lagrange multiplier, \( \nabla F^0(x) = [(\nabla f^0_1(x_1))^T, (\nabla f^0_2(x_2))^T, \ldots, (\nabla f^0_n(x_n))^T]^T \), and \( \partial F^j(x) = [(\partial f^j_1(x_1))^T, (\partial f^j_2(x_2))^T, \ldots, (\partial f^j_n(x_n))^T]^T \) for \( j \in \{1, \ldots, m\} \).

The proof of Lemma 3 is omitted since it is a trivial extension of the proof for Theorem 3.25 in [33].

4 Distributed Algorithms with Multi-Proximal Operator

The purpose of this section is to design two continuous-time distributed algorithms based on multi-proximal splitting to solve the nonsmooth resource allocation problem (11) for two cases that with known left eigenvector \( h \) and with a distributed estimator of left eigenvector \( h \), respectively.

In order to tackle the difficulty caused by the unproximable property of \( \sum_{j=1}^m f^j_i(x_i) \) for each agent \( i \), here we introduce a class of auxiliary variables \( z^j(t) \in \mathbb{R}^q \) for \( j \in \{1, \ldots, m-1\} \) combined with a constant parameter \( \gamma \in \mathbb{R}^+ \) such that there exist feasible points \( z^j \) splitting (13a) as
\[
-\nabla F^0(x^*) + v^* + \gamma \sum_{j=1}^{m-1} z^j \in \partial F^m(x^*), \quad (14a)
\]
\[
-\gamma z^j \in \partial F^j(x^*), \quad j \in \{1, \ldots, m-1\}. \quad (14b)
\]
According to the property (2) of proximal operator, we can transfer (14) as
\[
x^* = \text{Prox}_{F^0}(x^* - \nabla F^0(x^*) + v^* + \gamma \sum_{j=1}^{m-1} z^j), \quad (15)
\]
\[
x^* = \text{Prox}_{F_j}(x^* - \gamma z^j), \quad j \in \{1, \ldots, m-1\},
\]
where for any \( \xi = [\xi_1^T, \xi_2^T, \ldots, \xi_n^T]^T \in \mathbb{R}^{nq}, \xi_i \in \mathbb{R}^q, i \in \{1, \ldots, n\}, \text{Prox}_{F_j}[\xi] = [(\text{Prox}_{f^j_1}(\xi_1))^T, (\text{Prox}_{f^j_2}(\xi_2))^T, \ldots, (\text{Prox}_{f^j_n}(\xi_n))^T]^T \). \( x^* \) and \( v^* \) are defined in (13). From (14b), it is clear that \( -\gamma z^j \) is presented to estimate a subgradient in \( \partial F^j(x^*) \) for \( j \in \{1, \ldots, m-1\} \).

4.1 Algorithm Design with Known Left Eigenvector \( h \)

In this subsection, we present a distributed smooth multi-proximal primal-dual algorithm for solving problem (11) with the information of left eigenvector \( h \).

According to (13) and (14a), we propose a smooth algorithm as
\[
\dot{x}_i(t) = \text{Prox}_{f^i}[x_i(t) - \nabla f^0_i(x_i(t)) + v_i(t) + \gamma \sum_{j=1}^{m-1} z^j_i(t)] - x_i(t),
\]
\[
\dot{z}_i^j(t) = \text{Prox}_{f_j}[x_i(t) - \gamma z^j_i(t)] - x_i(t),
\]
\[
\dot{v}_i(t) = -h_i^{-1}(x_i(t) - d_i) - \alpha \sum_{k \in N_i} a_{ik}(v_i(t) - v_k(t)) - w_i(t),
\]
\[
\dot{w}_i(t) = \alpha \sum_{k \in N_i} a_{ik}(v_i(t) - v_k(t)), \quad w_i(0) = 0_q, \quad (16)
\]
where \( t \geq 0, 0 < \gamma < \frac{1}{m-1}, i \in \{1, \ldots, n\}, \) and \( j \in \{1, \ldots, m-1\} \).

Remark 2 Because all proximal operators \( \text{Prox}_{f^j_i}(\cdot) \) for \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m-1\} \) are continuous and nonexpansive, the proposed algorithm (16) is locally Lipschitz continuous even though each \( f^j_i(x_i) \) in problem (11) is nonsmooth, which means that the smoothness of algorithm (16) is guaranteed.

Algorithm (16) can be written in a compact form as
\[
\dot{x}(t) = \text{Prox}_{F^0}[x(t) - \nabla F^0(x(t)) + v(t)]
\]
\[
+ \gamma \sum_{j=1}^{m-1} z^j(t) - x(t), \quad (17a)
\]
\[
\dot{z}^j(t) = \text{Prox}_{F_j}[x(t) - \gamma z^j(t)] - x(t), \quad (17b)
\]
\[
\dot{v}(t) = -H_q(x(t) - d) - \alpha L_q v(t) - w(t), \quad (17c)
\]
\[
\dot{w}(t) = \alpha L_q v(t), \quad w(0) = 0_{nq}, \quad (17d)
\]
where \( j \in \{1, \cdots, m-1\} \), \( H_{nq} = \text{diag}\{h_1, \cdots, h_n\} \otimes I_q \), 
\( d = [d^T_1, \cdots, d^T_n]^T \in \mathbb{R}^{rq} \), and \( L_{nq} = L_n \otimes I_q \). The matrix \( L_n \otimes I_q \) is the Kronecker product of matrices \( L_n \) and \( I_q \).

**Remark 3** From (17b), it is shown that \(-\gamma z^j\) is the proximal-based estimator of a subgradient in \(\partial F^j(x)\) for \( j \in \{1, \cdots, m-1\} \). With the help of estimator \(-\gamma z^j\), the corresponding proximal operator (17a), which employs the information of \(-\gamma z^j\) instead of \(\partial F^j(x)\) for \( j \in \{1, \cdots, m-1\} \), is presented to tackle the difficulty caused by the unproximable property of \(\sum_{j=1}^{m-1} F^j(x)\). The scheme combined by (17a) and (17b) is called the **multi-proximal splitting**, which may be viewed as an extension of three operator splitting shown in [23].

**Lemma 4** Under Assumptions 1-4, if \((x^*, z^*, v^*, w^*) \in (\mathbb{R}^{rq}, \mathbb{R}^{(m-1)rq}, \mathbb{R}^{rq}, \mathbb{R}^{rq})\) is an equilibrium of algorithm (17) and \((1_n \otimes I_q)^T H_{nq} w^* = 0_q\), then \(x^*\) is a solution of problem (11), where \( z = \{z^1, \cdots, z^q\}^T \).

**PROOF.** If \((x^*, z^*, v^*, w^*)\) is an equilibrium of algorithm (17), then according to the property (2) of proximal operator and algorithm (17), it yields that for \( j \in \{1, \cdots, m-1\}, \)

\[
-\nabla F^0(x^*) + v^* + \gamma \sum_{j=1}^{m-1} z^j \in \partial F^m(x^*),
\]

\[
-\gamma z^j \in \partial F^j(x^*),
\]

\[
-H_{nq}^{-1}(x^* - d) - \alpha L_{nq} v^* - w^* = 0_{nq},
\]

\[
\alpha L_{nq} v^* = 0_{nq}.
\]

From (18a), (18b) and (18d), there exists a \( v_0^0 \in \mathbb{R}^q \) such that

\[
0_{nq} \in -\nabla F^0(x^*) - \sum_{j=1}^{m-1} \partial F^j(x^*) + v^* + z^j, \]

\[
v^* = 1_n \otimes v^0.
\]

Summing (18c) and (18d) yields that \( - (x^* - d) - H_{nq} w^* = 0_{nq} \), which means that

\[
\sum_{i=1}^{n} (x_i^0 - d_i) = - \sum_{i=1}^{n} h_i I_q w_i^0 = - (1_n \otimes I_q)^T H_{nq} w^* = 0_q. \]

Considering (19) together with (20) and according to Lemma 3, \( x^* \) is a solution of problem (11).

Then we state the convergence result of the proposed distributed algorithm (17). Let \((x^*, z^*, v^*, w^*)\) be an equilibrium of algorithm (17). Define a Lyapunov candidate \( V(x, z, v, w) = V_1(x, z) + V_2(x) + V_3(v, w) \), where

\[
V_1(x, z) = (\eta + 1) \frac{1}{2} \|\bar{x}^*\|^2 + \frac{1}{2} \gamma \sum_{j=1}^{m-1} (\|z^j\|^2 - 2(\bar{x}^*)^T \bar{z}^j),
\]

\[
V_2(x) = (\eta + 1) \|F^0(x) - F^0(x^*) - (\bar{x}^*)^T \nabla F^0(x^*)\|,
\]

\[
V_3(v, w) = \frac{\eta}{2} (\bar{v}^*)^T H_{nq} \bar{v}^* + \frac{1}{2} (\bar{v}^* + \bar{w}^*)^T H_{nq} (\bar{v}^* + \bar{w}^*),
\]

and \( \eta > 0, \bar{x}^* \triangleq x - x^*, \bar{z}^j \triangleq z^j - z^j, \bar{v}^* \triangleq v - v^*, \bar{w}^* \triangleq w - w^* \).

By analysing the convergence of (17), the main theorem of this subsection is obtained as below.

**Theorem 1** Consider algorithm (17). Suppose Assumptions 1-4 hold. If following inequalities

\[
\alpha > \frac{(\eta + 1)^2}{\eta \lambda_2(L_{nq})}, \quad \eta > \max\{\frac{1}{b_2 h^*} - 1, 0\}
\]

hold, where \( b_2 = c - \frac{1}{2}(1 + \gamma)(m - 1) \frac{1}{\beta} \), \( \gamma = \frac{1 + (\gamma - 1) \frac{m - 1}{2}}{2} < \frac{2}{1 + \gamma}, h^* = \min_{i \in I} \{h_1, \cdots, h_n\} \), then the trajectory of \( x(t) \) converges, and \( \lim_{t \to \infty} x(t) \) is the solution of problem (11).

**PROOF.** It can be easily verified that \( V(x^*, z^*, v^*, w^*) = 0 \). Next, we will show that \( V(x, z, v, w) > 0 \) for all \((x, z, v, w) \neq (x^*, z^*, v^*, w^*)\).

Since all \( F^0(x) \) for \( i \in \{1, \cdots, n\} \) are convex, then \( F^0(x) - F^0(x^*) - (\bar{x}^*)^T \nabla F^0(x^*) \geq 0 \). Hence \( V_2(x) \geq 0 \).

Since \( 0 < \gamma < \frac{1}{m - 1} \),

\[
V_1(x, z) = \frac{\eta}{2} \sum_{j=1}^{m-1} \|\frac{1}{m - 1} \bar{x}^* - \gamma (m - 1) \bar{z}^j\|^2 + \gamma (1 - \gamma (m - 1)) \|\bar{z}^j\|^2 \geq 0.
\]

Since \( V_2(x) \geq 0, V(x, z, v, w) \geq V_1(x, z) + V_2(x, v, w) \geq 0 \). Clearly \( V(x, z, v, w) \) is positive definite, radically unbounded, \( V(x, z, v, w) \geq 0 \) and is zero if and only if \((x, z, v, w) = (x^*, z^*, v^*, w^*)\).
It follows from algorithm (17) that

\[ x + \dot{x} = \text{Prox}_{F_0}[x - \nabla F_0(x) + v + \sum_{j=1}^{m-1} \gamma z^j], \]

\[ x^* = \text{Prox}_{F_0}[x^* - \nabla F_0(x^*) + v + \sum_{j=1}^{m-1} \gamma z^j], \] (24)

\[ x + \dot{z}^j = \text{Prox}_{F_j}[x - \gamma z^j], j \in \{1, \cdots, m-1\}, \]

\[ x^* = \text{Prox}_{F_j}[x^* - \gamma z^j], j \in \{1, \cdots, m-1\}. \]

Since \( f_j^1(\cdot) \) is convex, \( \partial f_j^1(\cdot) \) is monotone for agent \( i \in \{1, \cdots, n\} \), where \( j \in \{1, \cdots, m-1\} \). According to the property (2) of proximal operator, it follows from (24) that for \( j \in \{1, \cdots, m-1\} \),

\[ (\sum_{j=1}^{m-1} \tilde{z}^j - \nabla F_0(\tilde{x}^*) + \tilde{v}^* - \dot{x})^T(\tilde{x}^* + \dot{x}) \geq 0, \]

\[ (-\gamma \tilde{z}^j - \tilde{z}^j)^T(\tilde{x}^* + \dot{z}^j) \geq 0, \] (25)

where \( \nabla F_0(\tilde{x}^*) = \nabla F_0(x) - \nabla F_0(x^*) \).

From (25), it can be shown that for \( j \in \{1, \cdots, m-1\} \),

\[ \gamma \sum_{j=1}^{m-1} ((\tilde{z}^j)^T \tilde{x}^*) - (\nabla F_0(\tilde{x}^*))^T \tilde{x}^* + (\tilde{v}^*)^T \tilde{x}^* + (\tilde{v}^*)^T \tilde{x}^* \]

\[ + \sum_{j=1}^{m-1} ((\tilde{z}^j)^T \tilde{x}) - (\nabla F_0(\tilde{x}^*))^T \tilde{x} - (\tilde{v}^*)^T \dot{x} \geq 0, \]

and

\[ -\gamma (\tilde{z}^j)^T \tilde{x}^* - (\tilde{v}^*)^T \dot{z}^j - \gamma (\tilde{z}^j)^T \dot{z}^j - \|\dot{z}^j\|^2 \geq 0. \] (27)

The derivative of Lyapunov candidate \( \dot{V}(x, z, v, w) \) along the trajectory of algorithm (17) satisfies

\[ \dot{V}(x, z, v, w) = (\eta + 1)(\tilde{x}^*)^T \dot{x} + \gamma (\eta + 1) \sum_{j=1}^{m-1} (\tilde{z}^j)^T \dot{z}^j - \gamma (\eta + 1) \sum_{j=1}^{m-1} ((\tilde{z}^j)^T \tilde{x}) + (\eta + 1)(\nabla F_0(\tilde{x}^*))^T \dot{x} + \dot{V}_3(v, w), \] (28)

where

\[ \dot{V}_3(v, w) \leq - (\eta + 1)(\tilde{v}^*)^T \tilde{x}^* - (\tilde{w}^*)^T \tilde{x}^* - (\tilde{w}^*)^T H_{nq} \tilde{w}^* \]

\[ - (\eta + 1)(\tilde{v}^*)^T H_{nq} \tilde{w}^* - \alpha \eta (\tilde{v}^*)^T L_{nq} \tilde{v}^*, \] (29)

and \( L_{nq} = (H_{nq} L_{nq} + L_{nq}^T H_{nq})/2 \).

According to (26)-(29), it follows that

\[ \dot{V}(x, z, v, w) \]

\[ \leq - (\eta + 1)\|\dot{x}\|^2 - (\eta + 1) \sum_{j=1}^{m-1} \|\dot{z}^j\|^2 \]

\[ - (\eta + 1)(1 + \gamma) \sum_{j=1}^{m-1} (\tilde{z}^j)^T \dot{z}^j - (\tilde{w}^*)^T H_{nq} \tilde{w}^* \]

\[ - (\eta + 1)(\nabla F_0(\tilde{x}^*))^T \dot{x} + (\eta + 1)(\tilde{v}^*)^T \tilde{x} - (\tilde{w}^*)^T \tilde{x}^* - \alpha \eta (\tilde{v}^*)^T L_{nq} \tilde{v}^* - (\eta + 1)(\tilde{v}^*)^T \tilde{w}^*. \] (30)

Then according to Assumption 1, there exists a parameter \( \beta > 0 \) such that

\[ (1 + \gamma) \sum_{j=1}^{m-1} (\tilde{z}^j)^T \dot{z}^j \geq - \frac{1}{2}(1 + \gamma) \beta \sum_{j=1}^{m-1} \|\dot{z}^j\|^2 - \frac{(1 + \gamma)(m - 1)}{2\beta} \|\tilde{x}^*\|^2. \] (31)

Hence we have the conclusion that

\[ \dot{V}(x, z, v, w) \]

\[ \leq - (\eta + 1)\|\dot{x}\|^2 - (\eta + 1)b_1 \sum_{j=1}^{m-1} \|\dot{z}^j\|^2 - (\tilde{w}^*)^T \tilde{x}^* \]

\[ - \alpha \eta (\tilde{v}^*)^T L_{nq} \tilde{v}^* - (\eta + 1)b_2 \|\tilde{x}^*\|^2 - (\tilde{w}^*)^T H_{nq} \tilde{w}^* \]

\[ + (\eta + 1)(\tilde{v}^*)^T \tilde{x} - (\eta + 1)(\tilde{v}^*)^T H_{nq} \tilde{w}^*, \]

where \( b_1 = 1 - \frac{1}{2}(1 + \gamma) \beta \) and \( b_2 = c - \frac{1}{2}(1 + \gamma)(m - 1) \frac{1}{2}. \)

In order to illustrate that there always exists a \( \beta > 0 \) such that \( b_1 > 0 \) and \( b_2 > 0 \), here we define a function \( B(\gamma) \) of \( \gamma \) that \( B(\gamma) = \frac{2}{(\gamma + 1)^2} - \frac{(\gamma + 1)(m - 1)}{2c} \). The derivative of \( B(\gamma) \) is shown as

\[ \frac{dB(\gamma)}{d\gamma} = - \frac{2}{(\gamma + 1)^2} - \frac{m - 1}{2c} < 0. \] (33)

Note that \( 0 < \gamma < \frac{1}{m-1} \leq 1 \) and \( c > m - 1. \) According to (33), we have \( B_{\min}(\gamma) > B(1) = 1 - \frac{m-1}{c} > 0. \)

As the result, there exists a \( \beta \) such that

\[ \frac{(1 + \gamma)(m - 1)}{2c} < \beta < \frac{2}{1 + \gamma}. \] (34)
which means that $b_1 = 1 - \frac{1}{2}(1 + \gamma)\beta > 0$, and $b_2 = c - \frac{1}{2}(1 + \gamma)(m - 1)\frac{1}{\beta} > 0$.

In light of the above analysis and using the inequality $x^T y \leq \frac{1}{2\gamma}||x||^2 + \frac{1}{2}||y||^2$, equation (32) can be written as

$$\dot{V}(x, z, v, w) \leq -\epsilon_1 \|\dot{x}\|^2 - \epsilon_2 \sum_{j=1}^{m-1} \|\dot{z}_j\|^2 - \epsilon_3 \|\bar{x}\|^2$$

$$- \epsilon_4 \|\bar{v}\|^2 - \epsilon_5 (\bar{w}^*)^T H_{nq\bar{w}}^\gamma,$$

where $\epsilon_1 = \gamma + \frac{1}{2}$, $\epsilon_2 = (\gamma + 1)b_1$, $\epsilon_3 = (\gamma + 1)b_2 - \frac{1}{\pi}$, $\epsilon_4 = \alpha\eta\lambda_2(L_{nq}) - (\gamma + 1)^2$, and $\epsilon_5 = \frac{1}{4}$.

According to (22), it follows that $\epsilon_k > 0$ for $k \in \{1, 2, 3, 4, 5\}$. Additionally, since $V(x, z, v, w)$ is positive-definite, radially unbounded, lower bounded, $(x^*, z^*, v^*, w^*)$ is Lyapunov stable. It follows from the LaSalle invariant principle and Lemma 2 that $(x(t), z(t), v(t), w(t))$ converges to an equilibrium of algorithm (17) in the largest invariant set $\mathcal{M}$ in $E = \{x, z, v, w\}|x = x^*, v = v^*, w = w^*, \bar{v} = \bar{v}^*, \gamma \sigma_j^T \in \partial F_j^T(x^*)\}$. According to Lemma 4, $x^*$ is a solution of problem (11).

**4.2 Algorithm Design with Distributed Estimator of Left Eigenvector $h$**

However, the left eigenvector $h$ corresponding to $\lambda_1(L_{nq}) = 0$ may not be known by any single agent, since $h$ is a global variable for multi-agent systems. In this subsection, we present a distributed smooth multi-proximal primal-dual algorithm for solving the problem (11) with a distributed estimator of left eigenvector $h$.

Similar to algorithm (17), according to (13) and (14a), we propose a smooth algorithm as

$$\dot{x}(t) = \text{Prox}_{F_n}[x(t) - \nabla F^0(x(t)) + v(t)] + \gamma \sum_{i=1}^{m-1} z_i^j(t) - x(t),$$

$$\dot{z}_i^j(t) = \text{Prox}_{F_i^j}[z_i^j(t) - \gamma z_i^j(t)] - x(t),$$

$$\dot{v}(t) = -Y^{-1}(x(t) - d) - \alpha L_{nq} v(t) - w(t),$$

$$\dot{w}(t) = \alpha L_{nq} v(t), \quad w(0) = 0_{nq},$$

$$\dot{y}(t) = -L_{nn} y(t), \quad y(0) = [I_{n1}, \ldots, I_{nn}]^T \in \mathbb{R}^{nn},$$

where $j \in \{1, \ldots, m - 1\}$, $Y = \text{diag}(y_1^1, \ldots, y_n^1) \otimes I_q$, $L_{nn} = L_n \otimes I_n$, and $I_n^i$ is the $i$-th row of $I_n$.

**Remark 4** When the directed graph $\mathcal{G}$ of problem (11) is weight-balanced, it follows that $h_i = h_j$, $i, j \in \mathcal{V}$. While $\mathcal{G}$ is usually weight-unbalanced, hence a distributed estimator of $h$ is required for problem (11). Variable $y$ in algorithm (36) is designed to obtain the estimated value of $h$. Lemma 5 combined with Theorem 2 will show that $y_i^* = h_i$, where $y_i^* = \lim_{t \to \infty} y_i(t)$ for $i \in \{1, \ldots, n\}$.

**Lemma 5** Under Assumptions 1-4, if $(x^*, z^*, v^*, w^*, y^*) \in (\mathbb{R}^{nq}, \mathbb{R}^{(m-1)nq}, \mathbb{R}^{nq}, \mathbb{R}^{nq}, \mathbb{R}^{nn})$ is an equilibrium of algorithm (36), $(I_n \otimes I_q)^T H_{nq} w^* = 0_q$ and $y^* = I_n \otimes h$, then $x^*$ is a solution of problem (11).

**PROOF.** If $(x^*, z^*, v^*, w^*, y^*)$ is an equilibrium of algorithm (36), similar to the proof of Lemma 4, it can be shown that there exists a $v^i \in \mathbb{R}^q$ such that

$$0_{nq} \in -\nabla F^0(x^*) + \sum_{j=1}^{m-1} \partial F_j^0(x^*) + v^*, \quad (37)$$

$$v^* = 1_n \otimes v^0,$$

and

$$-Y^{-1}(x^* - d) - \alpha L_{nq} v^* - w^* = 0_{nq}, \quad (38a)$$

$$\alpha L_{nq} v^* = 0_{nq}, L_{nn} y^* = 0_{nq}. \quad (38b)$$

Adding (38a) and (38b) yields that $-(x^* - d) - Y w^* = 0_{nq}$, which means that

$$\sum_{i=1}^{n} (x_i^* - d_i) = -\sum_{i=1}^{n} y_i^* I_q w_i(t) = -\sum_{i=1}^{n} h_i I_q^* \quad (39)$$

$$= - (1_n \otimes I_q)^T H_{nq} w^* = 0_q,$$

where $y_i^*$ is the $(i-1)q + i$-th element of $y^*$. According to Lemma 3, $x^*$ is a solution of problem (11). □

Next, we will state the convergence result of the proposed distributed algorithm (36).

Firstly, some lemmas should be given to obtain the final result.

**Lemma 6** Assume system (3) can be written as

$$\dot{x} = f(x, u) = g(x) + u. \quad (40)$$

If system (40) is forward complete, 0-GAS with respect to a closed and 0-invariant set $\mathcal{M}$, ZOD with respect to $\mathcal{M}$ with a positive definite function $W_1$ that

$$a_{10}(\|x\|) \leq W_1(x) \leq a_{11}(\|x\|)$$

$$DW_1(x)f(x, u) \leq a_{12}(\|u(t)\|), \quad (41)$$

where $a_{10}$, $a_{11}$, and $a_{12}$ are positive constants. Then

$$\lim_{t \to \infty} x(t) = h \in \mathcal{R}, \quad \lim_{t \to \infty} u(t) = 0,$$

where $h$ is an element of $\mathcal{M}$.
for $a_{10}, a_{11} \in K_\infty$ and $a_{12} \in K$, then system (40) is iISS with respect to $M$ with $a_{13} \in KL$ such that

$$a_{10}(\|x(t, x_0, u)\|_M) \leq a_{13}(\|x_0\|_M) + \int_0^t 2(a_{12}(\|u(s)\|) + \|u(s)\|)ds.$$  \hspace{1cm} (42)

Moreover, if $a_{12}(\|u(t)\|) = k\|u(t)\|^2$, where $k \in \mathbb{R}^+$, $u(t)$ is exponentially convergent to zero, then system (40) converges to $M$.

**Proof.** If system (40) is forward complete and 0-GAS with respect to $M$, then by Theorem 2.8 and Remark 4.1 in [30], there exists a smooth function $W_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ and functions $a_{14}, a_{15}, a_{16} \in K_\infty$ such that

$$a_{14}(\|x\|_M) \leq W_2(x) \leq a_{15}(\|x\|_M)$$

and

$$DW_2(x)f(x, 0) \leq -a_{16}(\|x\|_M).$$  \hspace{1cm} (43)

Then according to (43), proof of Lemma IV.10 and Proposition II.5 in [31], there exists an iISS Lyapunov function $W_3$ with respect to $M$ such that

$$W_3(x) = W_1(x) + \pi(W_2(x)),$$  \hspace{1cm} (44)

where $\pi(r) \triangleq \int_0^r \frac{ds}{1+\kappa(x_1^2(s))}$, and $\kappa(r) \triangleq r + \max\{\|x\|_M \leq r\} \{\|DW_2(x)\|\}$. From (40), (41) and (44), we have the conclusion that

$$DW_3(x)f(x, u) \leq -\rho(W_3(x)) + \|u(t)\| + a_{12}(\|u(t)\|),$$  \hspace{1cm} (45)

where $\rho$ is a positive definite function. Then according to (45) and Corollary IV.3 in [31], there exist an $a_{17} \in KL$ such that

$$W_3(x(t)) \leq a_{17}(W_3(x_0), t) + \int_0^t 2(\|u(\tau)\| + a_{12}(\|u(\tau)\|))d\tau.$$  \hspace{1cm} (46)

Since $a_{10}(\|x\|_M) \leq W_1(x) \leq W_3(x) \leq W_1(x) + W_2(x)$, equation (42) holds.

Let $U(t) = \int_t^\infty 2(k\|u(\tau)\|^2 + \|u(\tau)\|)d\tau$ for $t \geq 0$. Since $u(t)$ is exponentially convergent to zero, $U(t) \leq M_U$ for a $M_U \in \mathbb{R}^+$. $U(t)$ is decreasing, and $\lim_{t \rightarrow \infty} U(t) = 0$. From (42), for $t \geq 0$, it follows that

$$\|x(t)\|_M \leq a_{10}^{-1}(a_{13}((\|x(0)\|_M, 0) + M_U) \leq M_X.$$  \hspace{1cm} (47)

For any $\varepsilon > 0$, choose $T_U \geq 0$ and $T_X \geq 0$ such that $U(T_U) \leq a_{10}(\varepsilon)/2$ and $a_{13}(M_X, T_X) \leq a_{10}(\varepsilon)/2$. Let $T \triangleq T_X + T_U$. Then from (42), for any $t \geq T$,

$$a_{10}(\|x(t)\|_M) \leq a_{13}(\|x(T_U)\|_M, t-T_U) + \int_{T_U}^t 2(k\|u(\tau)\|^2 + \|u(\tau)\|)d\tau \leq a_{13}(M_X, T_X + (t-T)) + U(T_U) \leq a_{13}(M_X, T_X) + U(T_U) \leq a_{10}(\varepsilon),$$  \hspace{1cm} (48)

which means that $\|x(t)\|_M \leq \varepsilon$ for all $t \geq T$. Hence system (40) converges to $M$. □

Let $M_{Y_j} = \{(\tau x^*-\phi_2 z^*)^T, (\phi_3 z^*)^T\} | (x^*, z^*, v^*, w^*, y^*) \in M_Y \}$ for $j \in \{1, \ldots, m-1\}$ and $\phi_k \in K$ for $k \in \{1, 2, 3\}$, where $M_Y$ is the largest invariant set in $E = \{(x, z, t, u, y)| x = x^*, v = v^*, w = w^*, y = y^*, \gamma \in \partial F^j(x^*) \}$ for $j \in \{1, \ldots, m-1\}$. Then it follows that for any $\xi \in \mathbb{R}^{2\eta}$, $\xi \in M_{Y_j}$ if and only if $\xi = [(\phi_1 x^* - \phi_2 z^*)^T, (\phi_3 z^*)^T] \in M_Y$.

**Lemma 7 Consider algorithm (36). For $\xi$ and $M_{Y_j}$ with $j \in \{1, \ldots, m-1\}$, it follows that**

(i) For each $\xi \in \mathbb{R}^{2\eta}$, there exists a unique $x \in \mathbb{R}^{n\eta}$ and $z_j \in \mathbb{R}^{n\eta}$ such that $\xi = [(\phi_1 x^* - \phi_2 z_j^*)^T, (\phi_3 z_j^*)^T]^T$.

(ii) Let $P_{M_{Y_j}}(\xi) \triangleq \arg\min_{x, z} M_{Y_j}(\xi, \|x - v\|^2)$. Then $P_{M_{Y_j}}(\xi) = [(\phi_1 x^* - \phi_2 z_j^*)^T, (\phi_3 z_j^*)^T]$ for some $(x^*, z^*, v^*, w^*, y^*) \in M_Y$.

(iii) For each $\xi \in \mathbb{R}^{2\eta}$, there exists an $(x^*, v^*, w^*, y^*) \in M_Y$ such that $\|\xi\|^2_{M_{Y_j}} = [(\phi_1 x^* - \phi_2 z_j^*)^T, (\phi_3 z_j^*)^T]^T$.

**Proof.** Obviously (i) is true. It follows from (i) that there exists a unique $(x^*, z^*, v^*, w^*, y^*)$ such that $P_{M_{Y_j}}(\xi) = [(\phi_1 x^* - \phi_2 z_j^*)^T, (\phi_3 z_j^*)^T]^T$. By definition of $P_{M_{Y_j}}(\xi), (\phi_1 x^* - \phi_2 z_j^*)^T, (\phi_3 z_j^*)^T \in \mathbb{R}^{2\eta}$, which means that $(x^*, z^*, v^*, w^*, y^*) \in M_Y$. Thus (ii) is proved.

Note that $\|\xi\|^2_{M_{Y_j}} = \|\xi - P_{M_{Y_j}}(\xi)\|^2$. Then according to (i) and (ii), it shows that $\|\xi\|^2_{M_{Y_j}} = \|[(\phi_1 x^* - \phi_2 z_j^*) - (\phi_1 x^* - \phi_2 z_j^*)]^T, (\phi_3 z_j^*)^T\|^2 = \|[(\phi_1 x^* - \phi_2 z_j^*) - (\phi_2 z_j^* - \phi_2 z_j^*)] + (\phi_3 z_j^*)^T\|^2$. Hence (iii) is proved.

Similarly to the analysis of (iii), there holds that $\nabla V(\xi) = -\frac{1}{2} \nabla \|\xi - P_{M_{Y_j}}(\xi)\|^2 = \xi - P_{M_{Y_j}}(\xi)$. Then according to (ii), it is shown that $\xi - P_{M_{Y_j}}(\xi) = [(\phi_1 x^* - \phi_2 z_j^*)^T, (\phi_3 z_j^*)^T]^T$. This completes the proof of (iv). □
Then, the main theorem of this subsection is given below.

**Theorem 2** Consider algorithm (36). Suppose Assumptions 1-4 hold. If inequalities (22) hold, then the trajectory of $x(t)$ converges, and \( \lim_{t \to \infty} x(t) \) is the solution of problem (11).

**PROOF.** Define $\phi = \text{col}(x, z, v, w)$. The first-order system controlled by (36) can be considered as

\[
\dot{\phi} = g_1(\phi) + g_2(\phi, y) + g_3(y),
\]

where $g_1(\phi) = \text{col}(\dot{x}, \dot{z}, G_1 \dot{w}), G_1 = -H_{nq}^{-1}(x - d) - \alpha L_{nq}v - w, g_2(\phi, y) = \text{col}(0_{nq}, 0_{(m-1)nq}, G_2, 0_{nq}), G_2 = (H_{nq}^{-1} - Y^*)\dot{x}^*, g_3(y) = \text{col}(0_{nq}, 0_{(m-1)nq}, u, 0_{nq}),$ and $u(t) = (H_{nq}^{-1} - Y^*)((x^*) - d)$.

i) Firstly, with only the first part in (49), we consider the system

\[
\dot{\phi} = g_1(\phi).
\]

From Theorem 1, it is clear that under system (50), \( (x(t), z(t), v(t), w(t)) \) converges to the largest invariant set \( \mathcal{M} \) in \( E = \{(x, z, v, w) | x = x^*, v = v^*, w = w^*, -\gamma x^j \in \partial F^j(x^*) \text{ for } j \in \{1, \cdots, m-1\}\} \).

ii) Consider the system

\[
\dot{\phi} = g_1(\phi) + g_2(\phi, y),
\]

where \( [(\phi^*)^T, (y^*)^T]^T \) is an equilibrium of algorithm (36), and $g_2(\phi, y)$ satisfies that $g_2(\phi^*, y^*) = 0$.

From (51) and the Lyapunov candidate $V_Y(x, z, v, w, y) = V(x, z, v, w) + V_3(y)$, where $V_3(y) = \frac{1}{2}\|\dot{y}^*\|^2$ and $\dot{y}^* \triangleq y - y^*$, it yields that

\[
\dot{V}_Y(x, z, v, w, y) \leq -\epsilon_1 \|\dot{x}\|^2 - \epsilon_2 \sum_{j=1}^{m-1} \|\dot{x}^j\|^2 - \epsilon_3 \|\ddot{x}^*\|^2 - \epsilon_4 \|\ddot{y}^*\|^2 \leq -\epsilon_5 (\ddot{y}^*)^T H_{nq} \ddot{y}^* - \frac{1}{2} (\ddot{y}^*)^T (L_{nn} + L_{nn}^*) \ddot{y}^* + D V_Y,
\]

where

\[
D V_Y = \frac{\partial V_3(v, w)}{\partial v} G_2 \leq (n + 1)(\ddot{y}^*)^T Q x^* + (\ddot{y}^*)^T Q x^* \leq \zeta_1 (\ddot{y}^*)^T Q v^* + \zeta_2 (\ddot{y}^*)^T Q x^* + \zeta_3 (\ddot{y}^*)^T Q w^* \leq \rho(t) \left[ \zeta_1 \|v^*\|^2 + \zeta_2 \|x^*\|^2 + \zeta_3 \|w^*\|^2 \right]
\]

and $Q = I_{nq} - H_{nq}Y_{nq}^{-1}, \zeta_1 = \frac{\eta + 2}{2}, \zeta_2 = \frac{\eta}{2} + 1, \zeta_3 = \frac{1}{2}$, $\rho(t) = \max_{i \in I} |1 - h_i(y_i(t))^{-1}|$, since $y(t) = e^{-L_m x} y(0)$ and $y(0) = [P_1^y, \cdots, P_n^y]^T$ from (36), it is shown that $\lim_{t \to \infty} y(t) = 1(h^T \otimes I_q)y(0) = 1_n \otimes h$. Therefore, $y^* = 1_n \otimes h$. Then according to Lemma 2.6 in [35], $y(t)$ is exponentially convergent to $1_n \otimes h$, and $y_i(t) > 0$ for all $i \in \{1, \cdots, n \}$ and $t > 0$. As the result, $\rho(t)$ and $w(t)$ are both exponentially convergent to zero.

With (52) and (53), it is followed that

\[
\dot{V}_Y(x, z, v, w, y) \leq -\epsilon_1 \|\dot{x}\|^2 - \epsilon_2 \sum_{j=1}^{m-1} \|\dot{x}^j\|^2 - \epsilon_3 \|\ddot{x}^*\|^2 - \epsilon_4 \|\ddot{y}^*\|^2 - \epsilon_5 (\ddot{y}^*)^T H_{nq} \ddot{y}^* - \frac{1}{2} (\ddot{y}^*)^T (L_{nn} + L_{nn}^*) \ddot{y}^* + D V_Y.
\]

Since $\rho(t) \rightarrow 0$ when $t \rightarrow \infty$, there exists $T_0 > 0$ such that when $t > T_0$, $l_1 \geq \frac{1}{2} \zeta_3, l_2 \geq \frac{1}{2} \epsilon_4, l_3 \geq \frac{1}{2} \epsilon_5$. Therefore $\dot{V}_Y(x, z, v, w, y) \leq 0$ when $t > T_0$.

When $t \leq T_0$, since $0 < \rho(t) < 1$,

\[
\dot{V}_Y(x, z, v, w, y) \leq \zeta_2 (\ddot{y}^*)^T H_{nq} \ddot{y}^* + \frac{1}{2} (\ddot{y}^*)^T (L_{nn} + L_{nn}^*) \ddot{y}^* + D V_Y \leq \zeta_2 (\ddot{y}^*)^T H_{nq} \ddot{y}^* + \frac{1}{2} (\ddot{y}^*)^T (L_{nn} + L_{nn}^*) \ddot{y}^* + (\epsilon_5 + \epsilon_2 l_2) \|\ddot{y}^*\|^2 \leq \zeta_2 (\ddot{y}^*)^T H_{nq} \ddot{y}^* + \frac{1}{2} (\ddot{y}^*)^T (L_{nn} + L_{nn}^*) \ddot{y}^* \leq \frac{1}{2} (\ddot{y}^*)^T (L_{nn} + L_{nn}^*) \ddot{y}^* + \epsilon_1 V_3(x, z, v, w, y).
\]

Note that

\[
\frac{(n + 1)\|\ddot{y}^*\|^2 + \frac{1}{2} \|\ddot{y}^*\|^2 + \frac{1}{2} \|\ddot{y}^*\|^2}{2h_{\text{max}}} \geq \frac{1}{2h_{\text{max}}} \|\ddot{y}^*\|^2 + \frac{1}{h_{\text{max}}} (\ddot{y}^*)^T \ddot{y}^* \geq \epsilon_2 \|\ddot{y}^*\|^2 + \epsilon_3 \|\ddot{y}^*\|^2 = \min \{\epsilon_2, \epsilon_3\} (\|\ddot{y}^*\|^2 + \|\ddot{y}^*\|^2)
\]

From (55) and (56), it is shown that

\[
\dot{V}_Y(x, z, v, w, y) \leq \frac{\zeta_2}{\min \{\epsilon_2, \epsilon_3\}} V_3(x, z, v, w) + \epsilon_1 V_3(x, z, v, w, y) \leq \kappa_1 V_Y(x, z, v, w, y),
\]

where $\kappa_1 = \max \{\frac{\zeta_2}{\min \{\epsilon_2, \epsilon_3\}}, \epsilon_1\}$.

According to (57), when $t = T_0$,

\[
V_Y(T_0) \leq e^{\kappa T_0} V_Y(0).
\]
Lemma 6, (49) is 0-GAS with respect to $M$ from (61) and Definition 3, it is clear that system (49) converges to one of its equilibria in $\mathcal{M}_Y$. Similar to the analysis in proof of Theorem 1, it is clear that $(1_n \otimes I_q)H_{pq}v^* = 0_q$. Then according to Lemma 5, $x^*$ is the solution of problem (11). This completes the proof.

**Remark 5** For $\xi = [(\varphi_1 x - \varphi_2 z^*)^T, (\varphi_3 z^*)^T]^T$ with $j \in \{1, \ldots, m - 1\}$, let $P_{\mathcal{M}_Y, j}(\xi) = [(\varphi_1 x^* - \varphi_2 z^*)^T, (\varphi_3 z^*)^T]^T$, $P_{\mathcal{M}_Y, j}(x) = \tilde{x}_j^*$, and $P_{\mathcal{M}_Y, j}(z^*) = \tilde{z}_j^*$, where $\mathcal{M}_{Y_j} \triangleq \{x^*[z^*], z^*, w^*, y^* \in \mathcal{M}_Y\}$ and $\mathcal{M}_{Y_j} \triangleq \{z^*[x^*, z^*, w^*, y^*] \in \mathcal{M}_Y\}$. Hence $\tilde{x}_j^* = x^*$ and usually $\tilde{z}_j^* \neq z_j^*$. While it is true that $\tilde{z}_j^* \in \mathcal{M}_{Y_j}$, hence $\tilde{V}_{\mathcal{M}_Y}(x, z, v, w, y)$ can be deduced based on the analysis of $V_{\mathcal{M}_Y}(x, z, v, w, y)$ in proof of Theorem 1.

**Remark 6** In proof of Theorem 2, the first-order system controlled by (36) had been separated to three parts. Since the existence of estimation error between $y$ and $h$, the Lyapunov function $V_{\mathcal{M}_Y}(x, z, v, w, y)$ of system (51) may increase before $T_0$. Then we proved that the Lyapunov function $V_{\mathcal{M}_Y}(x, z, v, w, y)$ of system (51) is bounded when $t \leq T_0$ and $\tilde{V}_{\mathcal{M}_Y}(x, z, v, w, y) \leq 0$ when $t > T_0$. Finally, with the help of iISS theory with respect to set, it is proved that system (49) is asymptotically convergent to its equilibria in $\mathcal{M}_Y$, which provides new ideas about stability analysis of asymptotically convergent system with exponentially convergent inputs.

### 5 Simulations

In this section, simulations are performed to validate the proposed algorithm (36). Consider the fused LASSO problem with four agents moving in a 2-D space with first-order dynamics (3) as

$$\min_{x \in \mathbb{R}^n} F(x), \quad \text{s.t.} \quad \sum_{i=1}^4 x_i = \sum_{i=1}^4 d_i,$$  

where $x_i = [x_i^1, x_i^2]^T \in \mathbb{R}^2, i \in \{1, 2, 3, 4\}, F(x) = \sum_{j=0}^3 f_j(x) = 2||x - s||^2 + \iota(x) + ||x - p||_1 + ||Dx||_1$, $\iota(x) = \begin{cases} 0, & \text{if } x \in \Omega \\ \infty, & \text{if } x \notin \Omega \end{cases}$ and

$$D = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{8 \times 8}.$$  

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The local cost function $f_i(x_i)$ for agent $i$ is consisted by

\[
\begin{align*}
   f_1^0(x_i) &= 2\|x_i - s_i\|^2, \\
   f_1^1(x_i) &= \|x_i - p_i\|_1, \\
   f_1^2(x_i) &= \|x_1^2 - x_i^2\|_1, \\
   f_1^3(x_i) &= \begin{cases} 0, & \text{if } x_i \in \Omega_i \\
   \infty, & \text{if } x_i \notin \Omega_i, \end{cases}
\end{align*}
\]

where $s_i = [s_1^T, s_2^T]^T = [i - 2.5, 0]^T$, $p_i = [p_1^T, p_2^T]^T = [0, i - 2.5]^T$ and $\Omega_i = \{\delta \in \mathbb{R}^2 \mid \|\delta - x_i(0)\|^2 \leq 64\}$. Then $f_1^0(x_i)$, $f_1^1(x_i)$, $f_1^2(x_i)$ and $f_1^3(x_i)$ represent respectively the quadratic objective, the $l_1$ penalty with an anchor $p_i$, another $l_1$ penalty associated with the matrix $D$, and the indicator function of the constraint set $x_i \in \Omega_i$ for each agent $i$. Resource allocation conditions are described as $d_1 = [2, -1]^T$, $d_2 = [-1, 1]^T$, $d_3 = [-1, -1]^T$ and $d_4 = [2, 2]^T$.

Based on (64), the gradient of $f_1^0$ and proximal operators of $f_1^1$, $f_1^2$ and $f_1^3$ for agent $i$ are shown as

\[
\nabla f_1^0(x_i) = [4(x_1^1 - s_1^1), 4(x_2^1 - s_2^1)]^T,
\]

\[
\text{prox}_{f_1^1}[\eta_1] = (\phi(\eta_1^1, p_1^1), \phi(\eta_1^2, p_1^2))
\]

\[
\text{prox}_{f_1^2}[\eta_2] = (\phi(\eta_2^1, \delta_2^1), \phi(\eta_2^2, \delta_2^2))
\]

\[
\text{prox}_{f_1^3}[\eta_3] = \arg \min_{\delta \in \Omega} \|\delta - \eta_3\|^2
\]

where $\eta_j \in \mathbb{R}^2$, $j \in \{1, 2, 3\}$. For $\xi_1 \in \mathbb{R}$ and $\xi_2 \in \mathbb{R}$, the function $\phi(\xi_1, \xi_2)$ is defined as follows

\[
\phi(\xi_1, \xi_2) = \begin{cases}
   \xi_1 - 1, & \text{if } \xi_1 > \xi_2 + 1 \\
   \xi_2, & \text{if } \xi_2 - 1 \leq \xi_1 \leq \xi_2 + 1 \\
   \xi_1 + 1, & \text{if } \xi_1 < \xi_2 - 1.
\end{cases}
\]

Note that the proximal operator of $f_1^1 + f_1^2 + f_1^3$, e.g., \(\text{prox}_{f_1^1 + f_1^2 + f_1^3}[\eta_1] = \arg \min_{\delta \in \Omega} \{\|\delta - p_1\|_1 + \|\delta - \delta_1\|_2 + \frac{1}{2}\|\delta - \eta_1\|^2\}\) is not proximable, where $\eta_1 \in \mathbb{R}^2$. Hence proximal algorithms [20]-[23] may not fit for this problem.

The Laplacian matrix of weight-unbalanced directed graph $G$ is given as

\[
L_4 = \begin{bmatrix}
1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

We set $\alpha = 5$ and $\gamma = 0.2$ as coefficients in algorithm (36). Initial positions of agents 1, 2, 3, and 4 are set as $x_1(0) = [-4, 5.5]^T$, $x_2(0) = [6, 5]^T$, $x_3(0) = [5, -3.5]^T$, and $x_4(0) = [-5, -5]^T$. We set initial values for Lagrange multipliers $v_i$ and auxiliary variables $z_1^i, z_2^i, w_i$ for $i \in \{1, 2, 3, 4\}$ as zeros.

Motions of system (3) versus time and trajectories of $\sum_{i=1}^{4} x_i^j$ with algorithm (36) are shown in Fig.1, which show that resource allocation conditions $\sum_{i=1}^{4} x_i^1 = \sum_{i=1}^{4} d_i^1 = 5$ and $\sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} d_i^2 = 1$ are satisfied. Fig.2 gives trajectories of $x_i(t)$ for $i \in \{1, 2, 3, 4\}$. Fig.3 shows the trajectory of $F(x)$, which proves that the global cost function is minimized. It can be seen from Fig.1-3 that all agents converge to the optimal solution which minimizes the global cost function and satisfies resource allocation conditions. Fig.4 - Fig.7 show trajectories of Lagrange multipliers $v_i(t)$ and auxiliary variables $z_1^i, z_2^i, w_i$ for $i \in \{1, 2, 3, 4\}$ respectively, which also verify the boundedness of system (3) steered by algorithm (36).

Fig. 1. Motions of system (3) in a 2-D space and trajectories of $\sum_{i=1}^{4} x_i^j$ for $j \in \{1, 2\}$ with algorithm (36)

Fig. 2. Trajectories of $x_i(t)$ for $i \in \{1, 2, 3, 4\}$ with algorithm (36)

Fig. 3. The trajectory of $F(x)$ with algorithm (36)

6 Conclusion

In this paper, a class of nonsmooth resource allocation problems with directed graphs was solved via
two distributed multi-proximal operator based primal-dual algorithms. The second algorithm considered a distributed estimator of the left eigenvector $h$ corresponding to $\lambda_1(L_{nq}) = 0$. These two algorithms were smoothed thanks to the multi-proximal splitting. Moreover, the design of the second proposed algorithm can also give a new viewpoint to tackle many widely studied distributed constrained resource allocation problems. Future extensions will involve considering nonsmooth resource allocation problems with switching topologies and more complex communication situations such as time delay and packet losses.

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