How degenerate is the parametrization of neural networks with the ReLU activation function?

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Abstract

Neural network training is usually accomplished by solving a non-convex optimization problem using stochastic gradient descent. Although one optimizes over the networks parameters, the main loss function generally only depends on the realization of the neural network, i.e. the function it computes. Studying the optimization problem over the space of realizations opens up new ways to understand neural network training. In particular, usual loss functions like mean squared error and categorical cross entropy are convex on spaces of neural network realizations, which themselves are non-convex. Approximation capabilities of neural networks can be used to deal with the latter non-convexity, which allows us to establish that for sufficiently large networks local minima of a regularized optimization problem on the realization space are almost optimal. Note, however, that each realization has many different, possibly degenerate, parametrizations. In particular, a local minimum in the parametrization space needs not correspond to a local minimum in the realization space. To establish such a connection, inverse stability of the realization map is required, meaning that proximity of realizations must imply proximity of corresponding parametrizations. We present pathologies which prevent inverse stability in general, and, for shallow networks, proceed to establish a restricted space of parametrizations on which we have inverse stability w.r.t. a Sobolev norm. Furthermore, we show that by optimizing over such restricted sets, it is still possible to learn any function which can be learned by optimization over unrestricted sets.

1 Introduction and Motivation

In recent years much effort has been invested into explaining and understanding the overwhelming success of deep learning based methods. On the theoretical side, impressive approximation capabilities of neural networks have been established [9, 10, 16, 20, 32, 33, 37, 39]. No less important are recent results on the generalization of neural networks, which deal with the question of
how well networks, trained on limited samples, perform on unseen data [2, 3, 5-7, 17, 29]. Last but not least, the optimization error, which quantifies how well a neural network can be trained by applying stochastic gradient descent to an optimization problem, has been analyzed in different scenarios [11, 13, 22, 24, 25, 27, 38]. While there are many interesting approaches to the latter question, they tend to require very strong assumptions (e.g. (almost) linearity, convexity, or extreme over-parametrization). Thus a satisfying explanation for the success of stochastic gradient descent for a non-smooth, non-convex problem remains elusive.

In the present paper we intend to pave the way for a functional perspective on the optimization problem. This allows for new mathematical approaches towards understanding the training of neural networks, some of which are demonstrated in Section 1.2. To this end we examine degenerate parametrizations with undesirable properties in Section 2. These can be roughly classified as

C.1 unbalanced magnitudes of the parameters
C.2 weight vectors with the same direction
C.3 weight vectors with directly opposite directions.

Under conditions designed to avoid these degeneracies, Theorem [31] establishes inverse stability for shallow networks with ReLU activation function. This is accomplished by a refined analysis of the behavior of ReLU networks near a discontinuity of their derivative. Proposition [12] shows how inverse stability connects the loss surface of the parametrized minimization problem to the loss surface of the realization space problem. In Theorem [1.3] we showcase a novel result on almost optimality of local minima of the parametrized problem obtained by analyzing the realization space problem. Note that this approach of analyzing the loss surface is conceptually different from previous approaches as in [11, 13, 23, 30, 31, 36].

1.1 Inverse Stability of Neural Networks

We will focus on neural networks with the ReLU activation function \( \rho(x) := x_+ \), and adapt the mathematically convenient notation from [33], which distinguishes between the parameterization of a neural network and its realization. Let us define the set \( A_L \) of all network architectures with depth \( L \in \mathbb{N} \), input dimension \( d \in \mathbb{N} \), and output dimension \( D \in \mathbb{N} \) by

\[
A_L := \{(N_0, \ldots, N_L) \in \mathbb{N}^{L+1} : N_0 = d, N_L = D\}.
\]

The architecture \( N \in A_L \) simply specifies the number of neurons \( N_i \) in each of the \( L \) layers. We can then define the space \( P_N \) of parametrizations with architecture \( N \in A_L \) as

\[
P_N := \prod_{\ell=1}^{L} \left( \mathbb{R}^{N_\ell \times N_{\ell-1}} \times \mathbb{R}^{N_\ell} \right),
\]

the set \( \mathcal{P} = \bigcup_{N \in A_L} P_N \) of all parametrizations with architecture in \( A_L \), and the realization map

\[
\Theta : \mathcal{P} \to C(\mathbb{R}^d, \mathbb{R}^D)
\]

\[
\Theta = ((A_\ell, b_\ell))_{\ell=1}^{L} \mapsto \mathcal{R}(\Theta) := W_1 \circ \cdots \circ W_{L-1} \circ \rho \circ W_L(x) := A_{\ell} x + b_\ell
\]

where \( W_\ell(x) := A_{\ell} x + b_\ell \) and \( \rho \) is applied component-wise. We refer to \( A_{\ell} \) and \( b_\ell \) as the weights and biases in the \( \ell \)-th layer.

Note that a parametrization \( \Theta \in \Omega \subset \mathcal{P} \) uniquely induces a realization \( \mathcal{R}(\Theta) \) in the realization space \( \mathcal{R}(\Omega) \), while in general there can be multiple non-trivially different parametrizations with the same realization. To put it in mathematical terms, the realization map is not injective. Consider the basic counterexample

\[
\Theta = ((A_1, b_1), \ldots, (A_{L-1}, b_{L-1}), (0, 0)) \quad \text{and} \quad \Gamma = ((B_1, c_1), \ldots, (B_{L-1}, c_{L-1}), (0, 0))
\]

from [34] where regardless of \( A_\ell, B_\ell, b_\ell \) and \( c_\ell \) both realizations coincide with \( \mathcal{R}(\Theta) = \mathcal{R}(\Gamma) = 0 \). However, it is well-known that the realization map is locally Lipschitz continuous, meaning that close\( ^{1} \) parametrizations in \( P_N \) induce realizations which are close in the uniform norm on compact
We proceed by demonstrating how inverse stability opens up new perspectives on the optimization problem which arises in neural network training. Specifically, consider a loss function \( L: C(R^d, R^D) \to [0, \infty) \) on the space of continuous functions. For illustration, we take the commonly used mean squared error (MSE) which, for training data \( ((x^i, y^i))_{i=1}^n \subseteq (R^d \times R^D)^n \), is given by

\[
L(g) = \frac{1}{n} \sum_{i=1}^n \| g(x^i) - y^i \|_2^2, \quad \text{for } g \in C(R^d, R^D). \tag{7}
\]

Typically, the optimization problem is solved over some subspace of parametrizations \( \Omega \subseteq P_N \), i.e.

\[
\min_{\Gamma \in \Omega} L(\mathcal{R}(\Gamma)) = \min_{\Gamma \in \Omega} \frac{1}{n} \sum_{i=1}^n \| \mathcal{R}(\Gamma)(x^i) - y^i \|_2^2. \tag{8}
\]

From an abstract point of view, by writing \( g = \mathcal{R}(\Gamma) \in \mathcal{R}(\Omega) \), this is equivalent to the corresponding optimization problem over the space of realizations \( \mathcal{R}(\Omega) \), i.e.

\[
\min_{g \in \mathcal{R}(\Omega)} L(g) = \min_{g \in \mathcal{R}(\Omega)} \frac{1}{n} \sum_{i=1}^n \| g(x^i) - y^i \|_2^2. \tag{9}
\]

However, the loss landscape of the optimization problem (8) is only properly connected to the loss landscape of the optimization problem (9) if the realization map is inverse stable on \( \Omega \). Otherwise a realization \( g \in \mathcal{R}(P_N) \) can be arbitrarily close to a global minimum in the realization space but every parametrization \( \Phi \) with \( \mathcal{R}(\Phi) = g \) is far away from the corresponding global minimum in the parametrization space. Moreover, local minima of (8) in the parametrization space must correspond to local minima of (9) in the realization space if and only if we have inverse stability.
Proposition 1.2 (Parametrization minimum ⇒ realization minimum). Let \( N \in \mathcal{A}_L, \Omega \subseteq \mathcal{P}_N \) and let the realization map be \((s, \alpha)\) inverse stable on \( \Omega \) w.r.t. \( \| \cdot \| \). Let \( \Gamma^* \in \Omega \) be a local minimum of \( \mathcal{L} \circ \mathcal{R} \) on \( \Omega \) with radius \( r > 0 \), i.e. for all \( \Phi \in \Omega \) with \( \| \Phi - \Gamma^* \|_\infty \leq r \) it holds that
\[
\mathcal{L}(\mathcal{R}(\Gamma^*)) \leq \mathcal{L}(\mathcal{R}(\Phi)).
\] (10)

Then \( \mathcal{R}(\Gamma^*) \) is a local minimum of \( \mathcal{L} \) on \( \mathcal{R}(\Omega) \) with radius \( (\frac{r}{s})^{1/\alpha} \), i.e. for all \( g \in \mathcal{R}(\Omega) \) with \( \| g - \mathcal{R}(\Gamma^*) \| \leq (\frac{r}{s})^{1/\alpha} \) it holds that
\[
\mathcal{L}(\mathcal{R}(\Gamma^*)) \leq \mathcal{L}(g).
\] (11)

See Appendix A.1.2 for a proof and Example A.1 for a counterexample in the case that inverse stability is not given. Note that in [9] we consider a problem with convex loss function but non-convex feasible set, see [32] Section 3.2. This opens up new avenues of investigation using tools from functional analysis and allows utilizing recent results [19, 34] exploring the topological properties of neural network realization spaces.

As a concrete demonstration we provide with Theorem A.2 a strong result obtained on the realization space, which estimates the quality of a local minimum based on its radius and the approximation capabilities of the chosen architecture for a class of functions \( S \). Specifically let \( C > 0 \), let \( \Lambda : \mathcal{B} \to [0, \infty) \) be a quasi-convex regularizer, and define
\[
S := \{ f \in \mathcal{B} : \Lambda(f) \leq C \}.
\] (12)

We denote the sets of regularized parametrizations by
\[
\Omega_N := \{ \Phi \in \mathcal{P}_N : \Lambda(\mathcal{R}(\Phi)) \leq C \}
\] (13)
and assume that the loss function \( \mathcal{L} \) is convex and \( c \)-Lipschitz continuous on \( S \). Note that virtually all relevant loss functions are convex and locally Lipschitz continuous on \( C(\mathbb{R}^d, \mathbb{R}^D) \). Employing Proposition 1.2 inverse stability can then be used to derive the following result for the practically relevant parametrized problem, showing that for sufficiently large architectures local minima of a regularized neural network optimization problem are almost optimal.

Theorem 1.3 (Almost optimality of local parameter minima). Assume that \( S \) is compact in the \( \| \cdot \| \)-closure of \( \mathcal{R}(\mathcal{P}) \) and that for every \( N \in \mathcal{A}_L \) the realization map is \((s, \alpha)\) inverse stable on \( \Omega_N \) w.r.t. \( \| \cdot \| \). Then for all \( \varepsilon, r > 0 \) there exists \( \Omega_N \) such that for every \( N \in \mathcal{A}_L \) with \( N \geq N_1(\varepsilon, r), \ldots, N_{L-1}(\varepsilon, r) \) the following holds:
Every local minimum \( \Gamma^* \) with radius at least \( r \) of \( \min_{\Gamma \in \Omega_N} \mathcal{L}(\mathcal{R}(\Gamma)) \) satisfies
\[
\mathcal{L}(\mathcal{R}(\Gamma^*)) \leq \min_{\Gamma \in \Omega_N} \mathcal{L}(\mathcal{R}(\Gamma)) + \varepsilon.
\] (14)

See Appendix A.1.2 for a proof and note that here it is important to have an inverse stability result, where the parameters \((s, \alpha)\) do not depend on the size of the architecture, which we achieve for \( L = 2 \) and \( \mathcal{B} = \mathcal{P}_d \). Suitable \( \Lambda \) would be Besov norms which constitute a common regularizer in image and signal processing. Moreover, note that the required size of the architecture in Theorem 1.3 can be quantified, if one has approximation rates for \( S \). In particular, this approach allows the use of approximation results in order to explain the success of neural network optimization and enables a combined study of these two aspects, which, to the best of our knowledge, has not been done before. Unlike in recent literature, our result needs no assumptions on the sample set (incorporated in the loss function, see (7)), in particular we do not require “overparametrization” with respect to the sample size. Here the required size of the architecture only depends on the complexity of \( S \), i.e. the class of functions one wants to approximate, the radius of the local minima of interest, the Lipschitz constant of the loss function, and the parameters of the inverse stability.

In the following we restrict ourselves to two-layer ReLU networks without biases, where we present a proof for \((4, 1/2)\) inverse stability w.r.t. the Sobolev semi-norm on a suitably regularized space of parametrizations. Both the regularizations as well as the stronger norm (compared to the uniform norm) will shown to be necessary in Section 2. We now present, in an informal way, a collection of our main results. A short proof making the connection to the formal results can be found in Appendix A.1.2.

Corollary 1.4 (Inverse stability and implications - colloquial). Suppose we are given data \( ((x_i, y_i))_{i=1}^n \in (\mathbb{R}^d \times \mathbb{R}^D)^n \) and want to solve a typical minimization problem for ReLU networks with shallow architecture \( N = (d, N_1, D) \), i.e.
\[
\min_{\Gamma \in \mathcal{P}_N} \frac{1}{n} \sum_{i=1}^n \| \mathcal{R}(\Gamma)(x^i) - y^i \|_2^2.
\] (15)
First we augment the architecture to \( \tilde{N} = (d + 2, N_1 + 1, D) \), while omitting the biases, and augment the samples to \( \tilde{x} = (x_1', \ldots, x_d', 1, -1) \). Moreover, we assume that the parametrizations

\[
\Phi = (([a_1| \ldots |a_{N_1+1}]^T, 0), ([c_1| \ldots |c_{N_1+1}], 0)) \in \Omega \subseteq P_N
\]

are regularized such that

C.1 the network is balanced, i.e. \( \|a_i\|_\infty = \|c_i\|_\infty \).

C.2 no non-zero weight vectors in the first layer are redundant, i.e. \( a_i \nparallel a_j \).

C.3 the last two coordinates of each weight vector \( a_i \) are strictly positive.

Then for the new minimization problem

\[
\min_{\Phi \in \Omega} \frac{1}{n} \sum_{i=1}^{n} \| \mathcal{R}(\Phi)(\tilde{x}^i) - y^i \|_2^2
\]

the following holds:

1. If \( \Phi_* \) is a local minimum of (17) with radius \( r \), then \( \mathcal{R}(\Phi_*) \) is a local minimum of \( \min_{\gamma \in \Gamma(\Omega)} \frac{1}{n} \sum_{i=1}^{n} \| g(\tilde{x}^i) - y^i \|_2^2 \) with radius at least \( \frac{r}{2} \) w.r.t. \( \| \cdot \|_{W^{1,\infty}} \).

2. The global minimum of (17) is at least as good as the global minimum of (15), i.e.

\[
\min_{\Phi \in \Omega} \frac{1}{n} \sum_{i=1}^{n} \| \mathcal{R}(\Phi)(\tilde{x}^i) - y^i \|_2^2 \leq \min_{\Gamma \in \mathcal{P}_N} \frac{1}{n} \sum_{i=1}^{n} \| \mathcal{R}(\Gamma)(x^i) - y^i \|_2^2.
\]

3. By further regularizing (17) in the sense of Theorem C.3 we can estimate the quality of its local minima.

This argument is not limited to the MSE loss function but works for any loss function based on evaluating the realization. The omission of bias weights is standard in neural network optimization literature [11, 13, 22, 24]. While this severely limits the functions that can be realized with a given architecture, it is sufficient to augment the problem by one dimension in order to recover the full range of functions that can be learned [1]. Here we augment by two dimensions, so that the third regularization condition [C.3] can be fulfilled without losing range. Moreover, note that, for simplicity of presentation, the regularization assumptions stated above are stricter than necessary and possible relaxations are discussed in Section 3.

2 Obstacles to inverse stability - degeneracies of ReLU parametrizations

In the remainder of this paper we focus on shallow ReLU networks without biases and define the corresponding space of parametrizations with architecture \( N = (d, m, D) \) as \( \mathcal{N}_N := \mathbb{R}^{m \times d} \times \mathbb{R}^{D \times m} \). The realization map \( \mathcal{R} \) is, for every \( \Theta = (A, C) = ([a_1| \ldots |a_m]^T, [c_1| \ldots |c_m]) \in \mathcal{N}_N \), given by

\[
\mathbb{R}^d \ni x \mapsto \mathcal{R}(\Theta)(x) = C \rho(Ax) = \sum_{i=1}^{m} c_i \rho(a_i, x).
\]

Note that each function \( x \mapsto c_i \rho(a_i, x) \) represents a so-called ridge function which is zero on the half-space \( \{ x \in \mathbb{R}^d : \langle a_i, x \rangle \leq 0 \} \) and linear with constant derivative \( c_i a_i^T \in \mathbb{R}^D \times \mathbb{R}^d \) on the other half-space. Thus, the \( a_i \) are the normal vectors of the separating hyperplanes \( \{ x \in \mathbb{R}^d : \langle a_i, x \rangle = 0 \} \) and consequently we refer to the weight vectors \( a_i \) also as the directions of \( \Theta \). Moreover, for \( \Theta \in \mathcal{N}_N \) it holds that \( \mathcal{R}(\Theta)(0) = 0 \) and, as long as the domain of interest \( U \subseteq \mathbb{R}^d \) contains the origin, the Sobolev norm \( \| \cdot \|_{W^{1,\infty}(U)} \) is equivalent to its semi-norm, since

\[
\| \mathcal{R}(\Theta) \|_{L^\infty(U)} \leq \sqrt{d} \text{diam}(U) \mathcal{R}(\Theta) \|_{W^{1,\infty}}\]

This is a slight abuse of notation, justified by the the fact that \( \mathcal{R} \) acts the same on \( \mathcal{P}_N \) with zero biases \( b_1, b_2 \) and weights \( A_1 = A \) and \( A_2 = C \).
Then for every sequence \( (\Phi_k) \in N_{(2,2,1)} \) with \( \mathcal{R}(\Phi_k) = g_k \) it holds that
\[
\lim_{k \to \infty} \|\mathcal{R}(\Phi_k) - \mathcal{R}(\Gamma)\|_{L^\infty((-1,1)^2)} = 0 \quad \text{and} \quad \lim_{k \to \infty} \|\Phi_k - \Gamma\|_{\infty} = \infty.
\]

In particular, note that inverse stability fails here even for a non-degenerate parametrization of the zero function \( \Gamma = (0,0) \). However, for this type of counterexample the magnitude of the gradient of \( \mathcal{R}(\Phi_k) \) needs to go to infinity, which is our motivation for looking at inverse stability w.r.t. \( | \cdot |_{W^{1,\infty}} \).

### 2.2 Failure of inverse stability w.r.t. Sobolev norm

In this section we present four degenerate cases where inverse stability fails w.r.t. \( | \cdot |_{W^{1,\infty}} \). This collection of counterexamples is complete in the sense that we can establish inverse stability under assumptions which are designed to exclude these four pathologies.

**Example 2.2** (Failure due to complete unbalancedness). Let \( r > 0, \Gamma := ((r,0),0) \in N_{(2,1,1)} \) and \( g_k \in \mathcal{R}(N_{(2,1,1)}) \) be given by (see Figure 2)
\[
g_k(x) = \frac{1}{r} \rho((0,1), x), \quad k \in \mathbb{N}.
\]

Then for every \( k \in \mathbb{N} \) and \( \Phi_k \in N_{(2,1,1)} \) with \( \mathcal{R}(\Phi_k) = g_k \) it holds that
\[
|\mathcal{R}(\Phi_k) - \mathcal{R}(\Gamma)|_{W^{1,\infty}} = \frac{1}{r} \quad \text{and} \quad \|\Phi_k - \Gamma\|_{\infty} \geq r.
\]

This is a very simple example of a degenerate parametrization of the zero function, since \( \mathcal{R}(\Gamma) = 0 \) regardless of choice of \( r \). The issue here is that we can have a weight pair, i.e. \(((r,0),0)\), where the product is independent of the value of one of the parameters. Note that in Example 2.1 one can see a slightly more subtle version of this pathology by considering \( \Gamma_k := ((k,0), \frac{1}{k}) \in N_{(2,1,1)} \) instead. In that case one could still get an inverse stability estimate for each fixed \( k \); the parameters of inverse

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1For \( m \in \mathbb{N} \) we abbreviate \( [m] := \{1, \ldots, m\} \).
stability \((s, \alpha)\) would however deteriorate with increasing \(k\). In particular this demonstrates the need for some sort of balancedness of the parametrization, i.e. control over \(|c_i|_\infty\) and \(|a_i|_\infty\) individually relative to \(\|c_i\|_\infty \|a_i\|_\infty\).

Inverse stability is also prevented by redundant directions as the following example illustrates.

**Example 2.3** (Failure due to redundant directions). Let

\[
\Gamma := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, (1, 1) \in \mathcal{N}_{(2,2,1)}
\]

and \(g_k \in \mathcal{R}(\mathcal{N}_{(2,2,1)})\) be given by (see Figure 3)

\[
g_k(x) := 2\rho((1,0), x)) + \frac{1}{k}\rho((0,1), x), \quad k \in \mathbb{N}.
\]

Then for every \(k \in \mathbb{N}\) and \(\Phi_k \in \mathcal{N}_{(2,2,1)}\) with \(\mathcal{R}(\Phi_k) = g_k\) it holds that

\[
|\mathcal{R}(\Phi_k) - \mathcal{R}(\Gamma)|_{W^{1,\infty}} = \frac{1}{k} \quad \text{and} \quad \|\Phi_k - \Gamma\|_\infty \geq 1.
\]

The next example shows that not only redundant weight vectors can cause issues, but also weight vectors of opposite direction, as they would allow for a (balanced) degenerate parametrization of the zero function.

**Example 2.4** (Failure due to opposite weight vectors 1). Let \(a_i \in \mathbb{R}^d, i \in [m]\), be pairwise linearly independent with \(\|a_i\|_\infty = 1\) and \(\sum_{i=1}^m a_i = 0\). We define

\[
\Gamma := (a_1 \cdots a_m) - (a_1 \cdots a_m)^T, (1, \ldots, 1, -1, \ldots, -1) \in \mathcal{N}_{(d,2m,1)}.
\]

Now let \(v \in \mathbb{R}^d\) with \(\|v\|_\infty = 1\) be linearly independent to each \(a_i, i \in [m]\), and let \(g_k \in \mathcal{R}(\mathcal{N}_{(d,2m,1)})\) be given by (see Figure 4)

\[
g_k(x) = \frac{1}{k}\rho(v, x), \quad k \in \mathbb{N}.
\]

Then there exists a constant \(C > 0\) such that for every \(k \in \mathbb{N}\) and every \(\Phi_k \in \mathcal{N}_{(d,2m,1)}\) with \(\mathcal{R}(\Phi_k) = g_k\) it holds that

\[
|\mathcal{R}(\Phi_k) - \mathcal{R}(\Gamma)|_{W^{1,\infty}} = \frac{1}{k} \quad \text{and} \quad \|\Phi_k - \Gamma\|_\infty \geq C.
\]

Thus we will need an assumption which prevents each individual \(\Gamma\) in our restricted set from having pairwise linearly dependent weight vectors, i.e. coinciding hyperplanes of non-differentiability. This, however, does not suffice as is demonstrated by the next example, which shows that the relation between the hyperplanes of the two realizations matters.

**Example 2.5** (Failure due to opposite weight vectors 2). We define the weight vectors

\[
a_1^k = (k, k, \frac{1}{k}), \quad a_2^k = (-k, k, \frac{1}{k}), \quad a_3^k = (0, -\sqrt{2}k, \frac{1}{\sqrt{2}k}), \quad c^k = (k, k, \sqrt{2}k)
\]

and consider the parametrizations (see Figure 5)

\[
\Gamma_k := \left([- a_1^k | - a_2^k | - a_3^k] \right)^T, c^k \right) \in \mathcal{N}_{(3,3,1)}, \quad \Theta_k := \left([a_1^k \left| a_2^k \right| a_3^k] \right)^T, c^k \right) \in \mathcal{N}_{(3,3,1)}.
\]

Then for every \(k \in \mathbb{N}\) and every \(\Phi_k \in \mathcal{N}_{(3,3,1)}\) with \(\mathcal{R}(\Phi_k) = \mathcal{R}(\Theta_k)\) it holds that

\[
|\mathcal{R}(\Phi_k) - \mathcal{R}(\Gamma_k)|_{W^{1,\infty}} = 3 \quad \text{and} \quad \|\Phi_k - \Gamma_k\|_\infty \geq k.
\]

Note that \(\Gamma\) and \(\Theta\) need to have multiple exactly opposite weight vectors which add to something small (compared to the size of the individual vectors), but not zero, since otherwise reparametrization would be possible (see Lemma A.5).
3 Inverse stability for two-layer ReLU Networks

We now establish an inverse stability result using assumptions designed to exclude the pathologies from the previous section. First we present a rather technical theorem for output dimension one which considers a parametrization \( \Gamma \) in the unrestricted parametrization space \( \mathcal{N} \) and a function \( g \) in the corresponding function space \( \mathcal{R}(\mathcal{N}) \). The aim is to use assumptions which are as weak as possible, while allowing us to find a parametrization \( \Phi \) of \( g \), whose distance to \( \Gamma \) can be bounded relative to \( |g - \mathcal{R}(\Gamma)|_{W^{1,\infty}} \). We then continue by defining a restricted parametrization space \( \mathcal{N}_\ast \) for which we get uniform inverse stability (meaning that we get the same estimate for every \( \Gamma \in \mathcal{N}_\ast \)).

**Theorem 3.1** (Inverse stability at \( \Gamma \in \mathcal{N}_\ast \)). Let \( d, m \in \mathbb{N} \), \( N := (d, m, 1) \), \( \beta \in [0, \infty) \), let \( \Gamma = \left( [a_1^T \ldots a_m^T]^T, c^\Gamma \right) \in \mathcal{N}_\ast \), \( g \in \mathcal{R}(\mathcal{N}) \), and let \( I^\Gamma := \{ i \in [m] : a_i^\Gamma \neq 0 \} \).

Assume that the following conditions are satisfied:

C.1 It holds for all \( i \in [m] \) with \( \|e_i^T a_i^\Gamma\|_\infty \leq 2|g - \mathcal{R}(\Gamma)|_{W^{1,\infty}} \) that \( \|a_i^\Gamma\|_\infty \leq \beta \).

C.2 It holds for all \( i, j \in I^\Gamma \) with \( i \neq j \) that \( \frac{a_i^\Gamma}{\|a_i^\Gamma\|_\infty} \neq \frac{a_j^\Gamma}{\|a_j^\Gamma\|_\infty} \).

C.3 There exists a parametrization \( \Theta = \left( [a_1^\Theta \ldots a_m^\Theta]^T, c^\Theta \right) \in \mathcal{N}_\ast \) such that \( \mathcal{R}(\Theta) = g \) and

\[ (a) \text{ it holds for all } i, j \in I^\Gamma \text{ with } i \neq j \text{ that } \frac{a_i^\Theta}{\|a_i^\Theta\|_\infty} \neq -\frac{a_j^\Theta}{\|a_j^\Theta\|_\infty} \text{ and for all } i, j \in I^\Theta \text{ with } \]

\[ i \neq j \text{ that } \frac{a_i^\Theta}{\|a_i^\Theta\|_\infty} \neq -\frac{a_j^\Theta}{\|a_j^\Theta\|_\infty}; \]

\[ (b) \text{ it holds for all } i \in I^\Gamma, j \in I^\Theta \text{ that } \frac{a_i^\Gamma}{\|a_i^\Gamma\|_\infty} \neq \frac{a_j^\Theta}{\|a_j^\Theta\|_\infty} \]

where \( I^\Theta := \{ i \in [m] : a_i^\Theta \neq 0 \} \).

Then there exists a parametrization \( \Phi \in \mathcal{N}_\ast \) with

\[ \mathcal{R}(\Phi) = g \quad \text{and} \quad \|\Phi - \Gamma\|_\infty \leq \beta + 2|g - \mathcal{R}(\Gamma)|_{W^{1,\infty}}. \quad (35) \]

The proof can be found in Appendix A.3.2. Note that each of the conditions in the theorem above corresponds directly to one of the pathologies in Section 2.2. Condition C.1 which deals with unbalancedness, only imposes an restriction on the weight pairs whose product is small compared to the distance of \( \mathcal{R}(\Gamma) \) and \( g \). As can be guessed from Example 2.2 and seen in the proof of Theorem 3.1 such a balancedness assumption is in fact only needed to deal with degenerate cases, where \( \mathcal{R}(\Gamma) \) and \( g \) have parts with mismatching directions of negligible magnitude. Otherwise a matching reparametrization is always possible. Note that a balanced \( \Gamma \) (i.e. \( \|c_i^\Gamma\| = \|a_i^\Gamma\|_\infty \)) satisfies Condition C.1 with \( \beta = (2|g - \mathcal{R}(\Gamma)|_{W^{1,\infty}})^{1/2} \).

It is also possible to relax the balancedness assumption by only requiring \( \|c_i^\Gamma\| \) and \( \|\Gamma_i\|_\infty \) to be close to \( \|c_i^\Gamma a_i^\Gamma\|_\infty^{1/2} \) which would still give a similar estimate but with a worse exponent. In order to see that requiring balancedness does not restrict the space of realizations, observe that the ReLU is positively homogeneous (i.e. \( \rho(\lambda x) = \lambda \rho(x) \) for all \( \lambda \geq 0, x \in \mathbb{R} \)). Thus balancedness can always be achieved simply by rescaling.

Condition C.2 requires \( \Gamma \) to have no redundant directions, the necessity of which is demonstrated by Example 2.5. Note that prohibiting redundant directions does not restrict the space of realizations,
see \((87)\) in the appendix for details. From a practical point of view, enforcing this condition could be achieved by a regularization term using a barrier function. Alternatively one could employ a non-standard approach of combining such redundant neurons by changing one of them according to \((87)\) and either setting the other one to zero or removing it entirely\(^4\).

From a theoretical perspective the first two conditions are rather mild, in the sense that they only restrict the space of parametrizations and not the corresponding space of realizations. Specifically we can define the restricted parametrization space
\[
\mathcal{N}_{(d,m,D)}^*: = \{\Gamma \in \mathcal{N}_{(d,m,D)}^*: \|c_i^\Gamma\|_\infty = \|a_i^\Gamma\|_\infty \text{ for all } i \in [m] \text{ and } \Gamma \text{ satisfies C.2}\}
\]
for which we have \(R(\mathcal{N}_N^*) = R(\mathcal{N}_N)\). Note that the above definition as well as the following definition and theorem are for networks with arbitrary output dimensions, as the balancedness condition makes this extension rather straightforward.

In order to satisfy Conditions C.3d and C.3e we need to restrict the parametrization space in a way which also restricts the corresponding space of realizations. One possibility to do so is the following approach, which also incorporates the previous restrictions as well as the transition to networks without biases.

**Definition 3.2** (Restricted parametrization space). Let \(N = (d, m, D) \in \mathbb{N}^3\). We define
\[
\mathcal{N}_N^* := \{\Gamma \in \mathcal{N}_N': (a_i^\Gamma)_d > 0 \text{ for all } i \in [m]\}.
\]

While we no longer have \(R(\mathcal{N}_N^*) = R(\mathcal{N}_N)\), Lemma A.6 shows that for every \(\Theta \in \mathcal{P}_{(d,m,D)}\) there exists \(\Gamma \in \mathcal{N}_{(d+2,m+1,D)}^*\) such that for all \(x \in \mathbb{R}^d\) it holds that
\[
R(\Gamma)(x_1, \ldots, x_d, 1, -1) = R(\Theta)(x_1, \ldots, x_d).
\]

In particular, this means that for any optimization problem over an unrestricted parametrization space \(\mathcal{P}_{(d,m,D)}\), there is a corresponding optimization problem over the parametrization space \(\mathcal{N}_{(d+2,m+1,D)}^*\), whose solution is at least as good (see Corollary 1.4). Our main result now states that for such a restricted parametrization space we have uniform \((4, 1/2)\) inverse stability w.r.t. \(\| \cdot \|_{W^{1,\infty}}\), a proof of which can be found in Appendix A.3.2.

**Theorem 3.3** (Inverse stability on \(\mathcal{N}_N^*\)). Let \(N \in \mathbb{N}^3\). For all \(\Gamma \in \mathcal{N}_N^*\) and \(g \in R(\mathcal{N}_N^*)\) there exists a parametrization \(\Phi \in \mathcal{N}_N^*\) with
\[
R(\Phi) = g \quad \text{and} \quad \|\Phi - \Gamma\|_\infty \leq 4\|\Phi - R(\Gamma)\|_{W^{1,\infty}}^{1/2}.
\]

### 4 Outlook

This contribution investigates the potential insights which may be gained from studying the optimization problem over the space of realizations, as well as the difficulties encountered when trying to connect it to the parametrized problem. While Theorem 1.3 and Theorem 3.3 offer some compelling preliminary answers, there are multiple ways in which they can be extended.

To obtain our inverse stability result for shallow ReLU networks we studied sums of ridge functions. Extending this result to deep ReLU networks requires understanding their behaviour under composition. In particular, we have ridge functions which vanish on some half space, i.e. colloquially speaking each neuron may “discard half the information” it receives from the previous layer. This introduces a new type of degeneracy, which one will have to deal with.

Another interesting direction is an extension to inverse stability w.r.t. some weaker norm like \(\| \cdot \|_{L^\infty}\) or a fractional Sobolev norm under stronger restrictions on the space of parametrizations (see Lemma A.7 for a simple approach using very strong restrictions).

Lastly, note that Theorem 1.3 is not specific to the ReLU activation function and thus also incentivizes the study of inverse stability for any other activation function.

From an applied point of view, Conditions C.1, C.3 motivate the implementation of corresponding regularization (i.e. penalizing unbalancedness and redundancy in the sense of parallel weight vectors) in state-of-the-art networks, in order to explore whether preventing inverse stability leads to improved performance in practice. Note that there already are results using, e.g. cosine similarity, as regularizer to prevent parallel weight vectors [4, 35] as well as approaches, called Sobolev Training, reporting better generalization and data-efficiency by employing a Sobolev norm based loss [12].

\(^4\)This could be of interest in the design of dynamic network architectures [26, 28, 40] and is also closely related to the co-adaption of neurons, to counteract which, dropout was invented [21].
Acknowledgment

The research of JB and DE was supported by the Austrian Science Fund (FWF) under grants I3403-N32 and P 30148. The authors would like to thank Pavol Harár for helpful comments.

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A Appendix - Proofs and Additional Material

A.1 Section 1

A.1.1 Additional Material

Example A.1 (Without inverse stability: parameter minimum $\nRightarrow$ realization minimum). Consider the two domains

$$D_1 := \{(x_1, x_2) \in (-1, 1)^2 : x_2 > |x_1|\}, \quad D_2 := \{(x_1, x_2) \in (-1, 1)^2 : x_1 > |x_2|\}. \quad (40)$$

For simplicity of presentation, assume we are given two samples $x^1 \in D_1, \ x^2 \in D_2$ with labels $y^1 = 0, \ y^2 = 1$. The problem occurs, since inverse stability fails due to unbalancedness of $g$.

Let $\|R\|_{L^2} \leq 1$.

$$\sup_{\Phi \in \Omega_N} \inf_{f \in S} \|R(\Phi) - f\| < \eta \quad \text{(approximability).} \quad (46)$$

Let $g_*$ be a local minimum with radius $r \geq 2\eta$ of the optimization problem $\min_{g \in R(\Omega_N)} \mathcal{L}(g)$. Then it holds for every $g \in R(\Omega_N)$ (in particular for every global minimizer) that

$$\mathcal{L}(g_*) \leq \mathcal{L}(g) + \frac{\eta}{2} \|g_* - g\| \eta. \quad (47)$$

Proof. Define $\lambda := \frac{r}{2\|g_* - g\|}$ and $f := (1 - \lambda)g_* + \lambda g \in S$. Due to (46) there is $\Phi \in \Omega_N$ such that $\|\mathcal{R}(\Phi) - f\| \leq \eta$ and by the assumptions on $g_*$ and $\mathcal{L}$ it holds that

$$\mathcal{L}(g_*) \leq \mathcal{L}(\mathcal{R}(\Phi)) \leq \mathcal{L}(f) + c\eta \leq (1 - \lambda)\mathcal{L}(g_*) + \lambda \mathcal{L}(g) + c\eta.$$

This completes the proof. See Figure 7 for illustration.
Figure 7: The figure illustrates the proof idea of Theorem A.2. Note that decreasing $\eta$, $c$, $\|g_* - g\|$ or increasing $r'$ leads to a better local minimum due to the convexity of the loss function (red).

A.1.2 Proofs

Proof of Proposition 1.2. By Definition 1.1 we know that for every $g \in \mathcal{R}(\Omega)$ with $\|g - \mathcal{R}(\Gamma_*)\| \leq \left(\frac{r_s}{s}\right)^{1/\alpha}$ there exists $\Phi \in \Omega$ with $\mathcal{R}(\Phi) = g$ and $\|\Phi - \Gamma_*\|_{\infty} \leq s\|g - \mathcal{R}(\Gamma_*)\|^\alpha \leq r$. (48)

Therefore by assumption it holds that $\mathcal{L}(\mathcal{R}(\Gamma_*)) \leq \mathcal{L}(\mathcal{R}(\Phi)) = \mathcal{L}(g)$. (49)

which proves the claim.

Proof of Theorem 1.3. Let $\varepsilon, r > 0$, define $r' := \left(\frac{r_s}{s}\right)^{1/\alpha}$ and $\eta := \min \left\{ \left(2c r' \text{diam}(S)\right)^{-1} \varepsilon, \frac{r'}{2} \right\}$. Then compactness of $S$ implies the existence of an architecture $n(\varepsilon, r) \in \mathcal{A}_L$ such that for every $N \in \mathcal{A}_L$ with $N_1 \geq n_1(\varepsilon, r), \ldots, N_{L-1} \geq n_{L-1}(\varepsilon, r)$ the approximability assumption (46) is satisfied. Let now $\Gamma_*$ be a local minimum with radius at least $r$ of $\min_{\Gamma \in \Omega} \mathcal{L}(\mathcal{R}(\Gamma))$. As we assume uniform $(s, \alpha)$ inverse stability, Proposition 1.2 implies that $\mathcal{R}(\Gamma_*)$ is a local minimum of the optimization problem $\min_{g \in \mathcal{R}(\Omega)} \mathcal{L}(g)$ with radius at least $r' = \left(\frac{r_s}{s}\right)^{1/\alpha} \geq 2\eta$. Theorem A.2 establishes the claim.

Proof of Corollary 1.4. We simply combine the main observations from our paper. First, note that the assumptions imply that the restricted parametrization space $\Omega$, which we are optimizing over, is the space $\mathcal{N}^*_{(d+2, N_1+1, D)}$ from Definition 3.2. Secondly, Theorem 3.3 implies that the realization map is $(4, 1/2)$ inverse stable on $\Omega$. Thus, Proposition 1.2 directly proves Claim 1. For the proof of Claim 2 we make use of Lemma A.6. It implies that for every $\Theta \in \mathcal{P}(d, N_1, D)$ there exists $\Gamma \in \Omega$ such that it holds that

$$\frac{1}{n} \sum_{i=1}^{n} \|\mathcal{R}(\Gamma)(\tilde{x}^i) - y^i\|^2 = \frac{1}{n} \sum_{i=1}^{n} \|\mathcal{R}(\Theta)(x^i) - y^i\|^2;$$

which proves the claim.

A.2 Section 2

A.2.1 Additional Material

Lemma A.3 (Reparametrization in case of linearly independent weight vectors). Let

$$\Theta = (A^\Theta, C^{\Theta}) = ([a_{11}^\Theta] \ldots [a_{m1}^\Theta]^T, [c_1^\Theta] \ldots [c_m^\Theta]) \in \mathcal{N}^*_{(d, m, D)}$$

(51)
with linearly independent weight vectors \((a_i^\Theta)_{i=1}^m\) and \(\min_{i\in [m]} \|c_i^\Theta\|_\infty > 0\) and let
\[
\Phi = (A^\Phi, B^\Phi) = ([a_1^\Phi| \ldots |a_m^\Phi]^T, [c_1^\Phi| \ldots |c_m^\Phi]) \in \mathcal{N}(d,m,D)
\]
(52)
with \(\mathcal{R}(\Phi) = \mathcal{R}(\Theta)\). Then there exists a permutation \(\pi: [m] \rightarrow [m]\) such that for every \(i \in [m]\) there exist \(\lambda_i \in (0, \infty)\) with
\[
a_i^\Phi = \lambda_i a_{\pi(i)}^\Theta \quad \text{and} \quad c_i^\Phi = \frac{1}{\lambda_i} c_{\pi(i)}^\Theta.
\]
(53)

This means that, up to reordering and rebalancing, \(\Theta\) is the unique parametrization of \(\mathcal{R}(\Theta)\).

**Proof.** First we define for every \(s \in \{0,1\}^m\) the corresponding open orthant
\[
O_s := \{x \in \mathbb{R}^m : x_1(2s_1 - 1) > 0, \ldots, x_m(2s_m - 1) > 0\} \subseteq \mathbb{R}^m.
\]
(54)

By assumption \(A^\Theta\) has rank \(m\), i.e. is surjective, and therefore the preimages of the orthants
\[
H_s := \{x \in \mathbb{R}^d : A^\Theta x \in O_s\} \subseteq \mathbb{R}^d, \quad s \in \{0,1\}^m,
\]
(55)
are disjoint, non-empty open sets. Note that on each \(H_s\) the realization \(\mathcal{R}(\Theta)\) is linear with
\[
\mathcal{R}(\Theta)(x) = C^\Theta \, \text{diag}(s) A^\Theta x \quad \text{and} \quad D\mathcal{R}(\Theta)(x) = C^\Theta \, \text{diag}(s) A^\Theta.
\]
(56)

Since \(A^\Theta\) has full row rank, it has a right inverse. Thus we have for \(s, t \in \{0,1\}^m\) that
\[
C^\Theta \, \text{diag}(s) A^\Theta = C^\Theta \, \text{diag}(t) A^\Theta \quad \Rightarrow \quad C^\Theta \, \text{diag}(s) = C^\Theta \, \text{diag}(t).
\]
(57)

Note that \(C^\Theta \, \text{diag}(s) = C^\Theta \, \text{diag}(t)\) can only hold if \(s = t\) due to the assumptions that \(\|c_i^\Theta\|_\infty \neq 0\) for all \(i \in [m]\). Thus the above establishes that for \(s, t \in \{0,1\}^m\) it holds that
\[
C^\Theta \, \text{diag}(s) A^\Theta = C^\Theta \, \text{diag}(t) A^\Theta \quad \text{if and only if} \quad s = t,
\]
(58)
i.e. \(\mathcal{R}(\Theta)\) has different derivatives on its \(2^m\) linear regions. In order for \(\mathcal{R}(\Phi)\) to have matching linear regions and matching derivatives on each one of them, there must exist a permutation matrix \(P \in \{0,1\}^{m \times m}\) such that for every \(s \in \{0,1\}^m\)
\[
P A^\Phi x \in O_s \quad \text{for every} \quad x \in H_s.
\]
(59)

Thus, there exist \((\lambda_i)_{i=1}^m \in (0, \infty)^m\) such that
\[
A^\Phi = \text{diag}(\lambda_1, \ldots, \lambda_m) P^T A^\Theta.
\]
(60)
The assumption that \(D\mathcal{R}(\Theta) = D\mathcal{R}(\Phi)\), together with \ref{56} for \(s = (1, \ldots, 1)\), implies that
\[
C^\Phi = C^\Theta P \, \text{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_m}),
\]
(61)
which proves the claim.

**Example A.4** (Failure due to unbalancedness). Let
\[
\Gamma_k := ((k,0), \frac{1}{k^2}) \in \mathcal{N}(2,1,1), \quad k \in \mathbb{N},
\]
(62)
and \(g_k \in \mathcal{R}(\mathcal{N}(2,1,1))\) be given by
\[
g_k(x) = \frac{1}{k} \rho((0,1), x), \quad k \in \mathbb{N}.
\]
(63)

The only way to parametrize \(g_k\) is \(g_k(x) = \mathcal{R}(\Phi_k)(x) = c \rho((0,a), x))\) with \(a, c > 0\) (see Lemma A.3), and we have
\[
|\mathcal{R}(\Phi_k) - \mathcal{R}(\Gamma_k)|_{W^{1,\infty}} \leq \frac{1}{k} \quad \text{and} \quad \|\Phi_k - \Gamma_k\|_{\infty} \geq k.
\]
(64)

**Lemma A.5.** Let \(d, m \in \mathbb{N}\) and \(a_i \in \mathbb{R}^d\), \(i \in [m]\), such that \(\sum_{i \in [m]} a_i = 0\). Then it holds for all \(x \in \mathbb{R}^d\) that
\[
\sum_{i \in [m]} \rho(a_i, x) = \sum_{i \in [m]} \rho(-a_i, x).
\]
(65)

**Proof.** By assumption we have for all \(x \in \mathbb{R}^d\) that \(\sum_{i \in [m]} \langle a_i, x \rangle = 0\). This implies for all \(x \in \mathbb{R}^d\) that
\[
\sum_{i \in [m]} \langle a_i, x \rangle = \sum_{i \in [m]} \langle a_i, x \rangle = \sum_{i \in [m]} \langle a_i, x \rangle = 0,
\]
(66)
which proves the claim.

\[\square\]
A.2.2 Proofs

Proof of Example 2.7 We have for every \( k \in \mathbb{N} \) that
\[
\|g_k\|_{L^\infty((-1,1)^2)} \leq \frac{1}{k^4} \quad \text{and} \quad |g_k|_{W^{1,\infty}} = k^2.
\] (67)

Assume that there exists sequence of networks \( (\Phi_k)_{k\in \mathbb{N}} \subseteq \mathcal{N}_{(2,2,1)} \) with \( \mathcal{R}(\Phi_k) = g_k \) and with uniformly bounded parameters, i.e. \( \sup_{k\in \mathbb{N}} \|\Phi_k\|_\infty < \infty \). Note that there exists a constant \( C \) (depending only on the network architecture) such that the realizations \( \mathcal{R}(\Phi_k) \) are Lipschitz continuous with
\[
\text{Lip}(\mathcal{R}(\Phi_k)) \leq C\|\Phi_k\|_\infty^2
\]
(see [34, Prop. 5.1]). It follows that \( |\mathcal{R}(\Phi_k)|_{W^{1,\infty}} \leq \text{Lip}(\mathcal{R}(\Phi_k)) \) is uniformly bounded which contradicts (67).

Proof of Example 2.2 The only way to parametrize \( g_k \) is \( g_k(x) = \mathcal{R}(\Phi_k)(x) = cp((0, a), x) \) with \( a, c > 0 \) (see also Lemma A.3), which proves the claim.

Proof of Example 2.3 Any parametrization of \( g_k \) must be of the form \( \Phi_k := (A, c) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{1 \times 2} \) with
\[
A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & a_2 \\ a_1 & 0 \end{bmatrix}
\]
(68)
(see Lemma A.3). Thus it holds that \( \|\Phi_k - \Gamma\|_\infty \geq \|(1, 0) - (0, a_2)\|_\infty \geq 1 \) and the proof is completed by direct calculation.

Proof of Example 2.4 Let \( \Phi_k \) be an arbitrary parametrization of \( g_k \) given by
\[
\Phi_k = ([\tilde{a}_1 \vert \tilde{a}_2] \cdots [\tilde{a}_{2m}]^T, \tilde{c}) \in \mathcal{N}_{(d,2m,1)}
\]
(69)
As \( g_k \) has two linear regions separated by the hyperplane with normal vector \( v \), there exists \( j \in [2m] \) and \( \lambda \in \mathbb{R} \setminus \{0\} \) such that
\[
\tilde{a}_j = \lambda v.
\]
(70)
The distance of any weight vector \( \pm a_i \) of \( \Gamma \) to the line \( \{\lambda v : \lambda \in \mathbb{R}\} \) can be lower bounded by
\[
\|\pm a_i - \lambda v\|_\infty \geq \frac{1}{\lambda \|v\|_2}\|\pm a_i - \lambda v\|_2 \geq \frac{1}{\lambda \|v\|_2}\left[\|a_i\|_2^2\|v\|_2^2 - \langle a_i, v \rangle^2\right], \quad i \in [m], \lambda \in \mathbb{R}.
\]
(71)
The Cauchy-Schwarz inequality and the linear independence of \( v \) to each \( a_i, i \in [m], \) establishes that \( C := \frac{1}{\lambda \|v\|_2}\min_{i \in [m]}\left[\|a_i\|_2^2\|v\|_2^2 - \langle a_i, v \rangle^2\right] > 0 \). Together with the fact that \( \mathcal{R}(\Gamma) = 0 \), this completes the proof.

Proof of Example 2.5 Since \( x = \rho(x) - \rho(-x) \) for every \( x \in \mathbb{R} \), the difference of the realizations is linear, i.e.
\[
\mathcal{R}(\Theta_k) - \mathcal{R}(\Gamma_k) = (c_1^k a_1^k + c_2^k a_2^k + c_3^k a_3^k, x) = ((0, 0, 3), x)
\]
(72)
and thus the difference of the gradients is constant, i.e.
\[
|\mathcal{R}(\Theta_k) - \mathcal{R}(\Gamma_k)|_{W^{1,\infty}} = 3, \quad k \in \mathbb{N}.
\]
(73)
However, regardless of the balancing and reordering of the weight vectors \( a_i^k, i \in [3] \), we have that
\[
\|\Theta_k - \Gamma_k\|_\infty \geq k.
\]
(74)
By Lemma A.3 up to balancing and reordering, there does not exist any other parametrization of \( \Theta_k \) with the same realization.
A.3 Section 3

A.3.1 Additional Material

Lemma A.6. Let \( d, m, D \in \mathbb{N} \) and \( \Theta \in \mathcal{P}(d, m, D) \). Then there exists \( \Gamma \in \mathcal{N}^{*}_{(d+2, m+1, D)} \) such that for all \( x \in \mathbb{R}^d \) it holds that

\[
\mathcal{R}(\Gamma)(x_1, \ldots, x_d, 1, -1) = \mathcal{R}(\Theta)(x).
\] (75)

Proof. Since \( \Theta \in \mathcal{P}(d, m, D) \) it can be written as

\[
\Theta = \left( (A, b), (c, e) \right) = \left( ([a_1 \ldots a_m]^T, b), ([c_1 \ldots |c_m|, e] \right)
\] (76)

with

\[
\mathcal{R}(\Theta)(x) = \sum_{i=1}^{m} c_i \rho(\langle a_i, x \rangle + b_i) + e, \quad x \in \mathbb{R}^d,
\] (77)

where \( A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m, C \in \mathbb{R}^{D \times m} \), and \( e \in \mathbb{R}^D \). We define for \( i \in [m] \)

\[
b_i^+ := \begin{cases} b_i + 1 : b_i \geq 0 \\ 1 \end{cases} \quad \text{and} \quad b_i^- := \begin{cases} 1 : b_i \geq 0 \\ -b_i + 1 : b_i < 0 \end{cases}
\] (78)

and observe that \( b_i^+ > 0, b_i^- > 0 \), and \( b_i^+ - b_i^- = b_i \). For \( i \in [m] \) let

\[
c_i^* := \begin{cases} c_i : \|c_i\|_{\infty} \neq 0 \\ (1, \ldots, 1) : \|c_i\|_{\infty} = 0 \end{cases}
\] (79)

and

\[
a_i^* := \begin{cases} (a_i, \ldots, a_{i-d}, b_i^+ , b_i^-) : \|c_i\|_{\infty} \neq 0 \\ (0, \ldots, 0, 1, 1) : \|c_i\|_{\infty} = 0 \end{cases}.
\] (80)

Note that we have

\[
\mathcal{R}(\Theta)(x) = \sum_{i=1}^{m} c_i^* \rho(\langle a_i^*, (x_1, \ldots, x_d, 1, -1) \rangle) + e, \quad x \in \mathbb{R}^d.
\] (81)

To include the second bias \( e \) let

\[
c_m^* := \begin{cases} e : e \neq 0 \\ (1, \ldots, 1) : e = 0 \end{cases}, \quad \text{and} \quad a_{m+1}^* := \begin{cases} (0, \ldots, 0, 2, 1) : e \neq 0 \\ (0, \ldots, 0, 1, 1) : e = 0 \end{cases}.
\] (82)

In order to balance the network, let \( a_i^\Gamma = a_i^* (\|\varphi_i\|_\infty)^{1/2} \) and \( c_i^\Gamma = c_i^* (\|\varphi_i\|_\infty)^{1/2} \) for every \( i \in [m+1] \). Then the claim follows by direct computation. \( \square \)

A.3.2 Proofs

Proof of Theorem 3.1. Without loss of generality, we can assume for all \( i \in [m] \) that \( a_i^\Theta = 0 \) if and only if \( c_i^\Theta = 0 \). We now need to show that there always exists a way to reparametrize \( \mathcal{R}(\Theta) \) such that the architecture remains the same and \( (35) \) is satisfied. For simplicity of notation we will write \( r := |g - \mathcal{R}(\Gamma)|_{W^1, \infty} \) throughout the proof. Let \( f_i^\Gamma : \mathbb{R}^d \rightarrow \mathbb{R} \) resp. \( f_i^\Theta : \mathbb{R}^d \rightarrow \mathbb{R} \) be the part that is contributed by the \( i \)-th neuron, i.e.

\[
\mathcal{R}(\Gamma) = \sum_{i=1}^{m} f_i^\Gamma \quad \text{with} \quad f_i^\Gamma(x) := c_i^\Gamma \rho(\langle a_i^\Gamma, x \rangle),
\] (83)

\[
g = \mathcal{R}(\Theta) = \sum_{i=1}^{m} f_i^\Theta \quad \text{with} \quad f_i^\Theta(x) := c_i^\Theta \rho(\langle a_i^\Theta, x \rangle).
\] (84)

*In case one of them is zero, the other one can be set to zero without changing the realization.
Further let
\[ H^+_{\Gamma,i} := \{ x \in \mathbb{R}^d : \langle a_i^\Gamma, x \rangle > 0 \}, \]
\[ H^0_{\Gamma,i} := \{ x \in \mathbb{R}^d : \langle a_i^\Gamma, x \rangle = 0 \}, \]
\[ H^-_{\Gamma,i} := \{ x \in \mathbb{R}^d : \langle a_i^\Gamma, x \rangle < 0 \}. \]  
(85)

By conditions C.2 and C.3a we have for all \( i, j \in I^\Gamma \) that
\[ i \neq j \implies H^0_{\Gamma,i} \neq H^0_{\Gamma,j}. \]  
(86)

Further note that we can reparametrize \( \mathcal{R}(\Theta) \) such that the same holds there. To this end observe that
\[ c\rho((a, x)) + c' \rho((a', x)) = (c + c' \|a\|_\infty) \rho((a, x)), \]  
(87)
given that \( a' \) is a positive multiple of \( a \). Specifically, let \( (J_k)_{k=1}^K \) be a partition of \( I^\Theta \) (i.e. \( J_k \neq \emptyset \), \( \bigcup_{k=1}^K J_k = I^\Theta \) and \( J_k \cap J_{k'} = \emptyset \) if \( k \neq k' \)), such that for all \( k \in [K] \) it holds that
\[ i, j \in J_k \implies \frac{a_j^\Theta}{\|a_j^\Theta\|_\infty} = \frac{a_i^\Theta}{\|a_i^\Theta\|_\infty}. \]  
(88)

We denote by \( j_k \) the smallest element in \( J_k \) and make the following replacements, for all \( i \in I^\Theta \), without changing the realization of \( \Theta \):
\[ a_i^\Theta \mapsto a_{j_k}^\Theta, \quad c_i^\Theta \mapsto \sum_{j \in J_k} c_j^\Theta \frac{\|a_j^\Theta\|_\infty}{\|a_{j_k}^\Theta\|_\infty}, \quad \text{if } i \in J_k \text{ and } i = j_k, \]  
(89)
\[ a_i^\Theta \mapsto 0, \quad c_i^\Theta \mapsto 0, \quad \text{if } i \in J_k \text{ and } i \neq j_k. \]  
(90)

Note that we also update the set \( I^\Theta := \{ i \in [m] : a_i^\Theta \neq 0 \} \) accordingly. Let now
\[ H^+_{\Theta,i} := \{ x \in \mathbb{R}^d : \langle a_i^\Theta, x \rangle > 0 \}, \]
\[ H^0_{\Theta,i} := \{ x \in \mathbb{R}^d : \langle a_i^\Theta, x \rangle = 0 \}, \]
\[ H^-_{\Theta,i} := \{ x \in \mathbb{R}^d : \langle a_i^\Theta, x \rangle < 0 \}. \]  
(91)

By construction and condition C.3a we have for all \( i, j \in I^\Theta \) that
\[ i \neq j \implies H^0_{\Theta,i} \neq H^0_{\Theta,j}. \]  
(92)

Note that we now have a parametrization \( \Theta \) of \( g \), where all weight vectors \( a_i^\Theta \) are either zero (in which case the corresponding \( c_i^\Theta \) are also zero) or pairwise linearly independent to each other nonzero weight vector.

Next, for \( s \in \{0, 1\}^m \), let
\[ H^s := \bigcap_{i \in [m], s_i = 1} H^+_{\Gamma,i} \bigcap \bigcap_{i \in [m], s_i = 0} H^-_{\Gamma,i}, \]
\[ H^s := \bigcap_{i \in [m], s_i = 1} H^+_{\Theta,i} \bigcap \bigcap_{i \in [m], s_i = 0} H^-_{\Theta,i}, \]  
(93)

and
\[ S^\Gamma := \{ s \in \{0, 1\}^m : H^s \neq \emptyset \}, \quad S^\Theta := \{ s \in \{0, 1\}^m : H^s \neq \emptyset \}. \]  
(94)

The \( H^s, s \in S^\Gamma \), and \( H^s, s \in S^\Theta \), are the interiors of the different linear regions of \( \mathcal{R}(\Gamma) \) and \( \mathcal{R}(\Theta) \) respectively. Next observe that the derivatives of \( f^\Gamma, f^\Theta \) are (a.e.) given by
\[ Df^\Gamma(x) = 1_{H^+_{\Gamma,i}}(x) c_i^\Gamma a_i^\Gamma, \quad Df^\Theta(x) = 1_{H^+_{\Theta,i}}(x) c_i^\Theta a_i^\Theta. \]  
(95)

Note that for every \( x \in H^s, y \in H^0 \) we have
\[ D\mathcal{R}(\Gamma)(x) = \sum_{i \in [m]} Df^\Gamma_i(x) = \sum_{i \in [m]} s_i c_i^\Gamma a_i^\Gamma =: \Sigma^\Gamma_s, \]
\[ D\mathcal{R}(\Theta)(y) = \sum_{i \in [m]} Df^\Theta_i(y) = \sum_{i \in [m]} s_i c_i^\Theta a_i^\Theta =: \Sigma^\Theta_s. \]  
(96)
Next we use that for \( s \in S^r, t \in S^\Theta \) we have \(|\Sigma_s^r - \Sigma_t^\Theta| \leq r\) if \( H_s^r \cap H_t^\Theta \neq \emptyset\), and compare adjacent linear regions of \( \mathcal{R}(\Gamma) - \mathcal{R}(\Theta)\). Let now \( i \in I^T\) and consider the following cases:

**Case 1:** We have \( H_{1,i}^r \neq H_{\Theta,j}^\Theta\) for all \( j \in I^\Theta\). This means that the \( Df_k^\Theta\), \( k \in [m]\), and the \( Df_k^r\), \( k \in [m]\{i\}\), are the same on both sides near the hyperplane \( H_{1,i}^r\), while the value of \( Df_k^\Theta\) is 0 on one side and \( c_i^r a_i^r\) on the other. Specifically, there exist \( s^+ , s^- \in S^r\) and \( s^* \in S^\Theta\) such that \( s_i^+ = 1\), \( s_i^- = 0\), \( s_j^+ = s_j^-\) for all \( j \in [m]\{i\}\), and \( H_{1,i}^r \cap H_{j}^\Theta \neq \emptyset\), \( H_{1,i}^r \cap H_{j}^\Theta \neq \emptyset\), which implies

\[
\|c_i^r a_i^r\|_\infty = \|(\Sigma_s^r - \Sigma_{s^*}^r) - (\Sigma_s^r - \Sigma_{s^*}^r)\|_\infty \leq 2r. \tag{97}
\]

**Case 2:** There exists \( j \in I^\Theta\) such that \( H_{1,i}^r = H_{\Theta,j}^\Theta\). Note that \( \Theta\) ensures that \( H_{1,i}^r \neq H_{1,i}^r\) for \( k \in [m]\{i\}\) and \( \Theta\) ensures that \( H_{\Theta,j}^\Theta \neq H_{1,i}^r\) for \( k \in [m]\{j\}\). Moreover, Condition C.3b implies \( H_{1,i}^r = H_{\Theta,j}^\Theta\). This means that the \( Df_k^\Theta\), \( k \in [m]\{j\}\), and the \( Df_k^r\), \( k \in [m]\{i\}\), are the same on both sides near the hyperplane \( H_{1,i}^r = H_{\Theta,j}^\Theta\), while the values of \( Df_k^\Theta\) and \( Df_k^r\) change. Specifically there exist \( s^+ , s^- \in S^r\) and \( t^+ , t^- \in S^\Theta\) such that \( s_i^+ = 1\), \( s_i^- = 0\), \( s_j^+ = s_j^-\) for all \( k \in [m]\{i\}\), \( t_j^+ = 1\), \( t_j^- = 0\), \( t_k^+ = t_k^-\) for all \( k \in [m]\{j\}\), and \( H_{1,i}^r \cap H_{\Theta,j}^\Theta \neq \emptyset\), \( H_{1,i}^r \cap H_{\Theta,j}^\Theta \neq \emptyset\), which implies

\[
\|c_i^r a_i^r - c_j^\Theta a_j^\Theta\|_\infty = \|(\Sigma_s^r - \Sigma_{s^*}^r) - (\Sigma_s^r - \Sigma_{s^*}^r)\|_\infty \leq 2r. \tag{98}
\]

Analogously we get for \( i \in I^T\) that \( H_{\Theta,i}^r \neq H_{\Theta,j}^\Theta\) for all \( j \in I^\Theta\) implies \( \|c_i^r a_i^r\|_\infty \leq 2r\). Next let

\[
I_1 := \{ i \in [m] : H_{1,i}^r \neq H_{\Theta,j}^\Theta\} \cup \{ i \in [m] : a_i^r = 0\}. \tag{99}
\]

and

\[
I_2 := [m] \setminus I_1 = \{ i \in [m] : \exists j \in I^\Theta\text{ such that } H_{1,i}^r = H_{\Theta,j}^\Theta\}. \tag{100}
\]

Colloquially speaking, this shows that for every \( f_i^r\) with \( i \in I_2\) there is a \( f_j^\Theta\) with exactly matching half-spaces, i.e. \( H_{1,i}^r = H_{\Theta,j}^\Theta\), and approximately matching gradients (Case 2). Moreover, all unmatched \( f_i^r\) and \( f_j^\Theta\) must have a small gradient (Case 1).

Specifically, the above establishes that there exists a permutation \( \pi : [m] \to [m]\) such that for every \( i \in I_1\) it holds that

\[
\|c_i^r a_i^r\|_\infty, \|c_{\pi(i)}^\Theta a_{\pi(i)}^\Theta\|_\infty \leq 2r, \tag{101}
\]

and for every \( i \in I_2\) that

\[
\|c_i^r a_i^r - c_{\pi(i)}^\Theta a_{\pi(i)}^\Theta\|_\infty \leq 2r. \tag{102}
\]

We make the following replacements, for all \( i \in [m]\), without changing the realization of \( \Theta\):

\[
a_i^\Theta \rightarrow a_{\pi(i)}^\Theta, \quad c_i^\Theta \rightarrow c_{\pi(i)}^\Theta. \tag{103}
\]

In order to balance the weights of \( \Theta\) for \( I_1\), we further make the following replacements, for all \( i \in I_1\) with \( a_i^\Theta \neq 0\), without changing the realization of \( \Theta\):

\[
a_i^\Theta \rightarrow (\|a_i^\Theta\|_\infty)^{-1/2} a_i^\Theta, \quad c_i^\Theta \rightarrow (\|c_i^\Theta\|_\infty)^{-1/2} c_i^\Theta. \tag{104}
\]

This implies for every \( i \in I_1\) that

\[
|c_i^\Theta|, |a_i^\Theta| \leq (2r)^{1/2}. \tag{105}
\]

Moreover, due to Condition C.1 we get for every \( i \in I_1\) that

\[
|c_i^r|, |a_i^r| \leq \beta. \tag{106}
\]

Thus we get for every \( i \in I_1\) that

\[
|c_i^\Theta - c_i^r|, |a_i^\Theta - a_i^r| \leq \beta + (2r)^{1/2}. \tag{107}
\]
Next we (approximately) match the balancing of \((c_i^\Theta, a_i^\Theta)\) to the balancing of \((c_i^\Gamma, a_i^\Gamma)\) for \(i \in I_2\), in order to derive estimates on \(|c_i^\Theta - c_i^\Gamma|\) and \(\|a_i^\Theta - a_i^\Gamma\|_{\infty}\) from (102). Specifically, we make the following replacements, for all \(i \in I_2\), without changing the realization of \(\Theta\):

\[
a_i^\Theta \rightarrow \left(\frac{c_i^\Theta}{\|c_i^\Theta\|_{\infty}}\right)^{1/2} a_i^\Theta, \quad c_i^\Theta \rightarrow \left(\frac{\|c_i^\Theta\|_{\infty}}{|c_i^\Gamma|}\right)^{1/2} c_i^\Theta, \quad \text{if } \|c_i^\Gamma a_i^\Gamma\|_{\infty} \leq 2r, \quad (108)
\]

\[
a_i^\Theta \rightarrow \frac{c_i^\Theta}{c_i^\Gamma} a_i^\Theta, \quad c_i^\Theta \rightarrow c_i^\Gamma, \quad \text{if } \|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r, |c_i^\Gamma| > \|a_i^\Gamma\|_{\infty}, \quad (109)
\]

\[
a_i^\Theta \rightarrow a_i^\Gamma, \quad c_i^\Theta \rightarrow \|a_i^\Theta\|_{\infty} c_i^\Theta, \quad \text{if } \|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r, |c_i^\Gamma| < \|a_i^\Gamma\|_{\infty}, \quad (110)
\]

\[
a_i^\Theta \rightarrow \left(\frac{|c_i^\Gamma|}{\|c_i^\Theta\|_{\infty}}\right)^{1/2} a_i^\Theta, \quad c_i^\Theta \rightarrow \left(\frac{\|c_i^\Theta\|_{\infty}}{|c_i^\Gamma|}\right)^{1/2} c_i^\Theta, \quad \text{if } \|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r, |c_i^\Gamma| = \|a_i^\Gamma\|_{\infty}. \quad (111)
\]

Let now \(i \in I_2\) and consider the following cases:

**Case A:** We have \(|c_i^\Gamma a_i^\Gamma\|_{\infty} \leq 2r\) which, together with (102), implies \(\|c_i^\Theta a_i^\Theta\|_{\infty} \leq 4r\). Due to (108) and Condition \(C.1\), it follows that

\[
|c_i^\Theta - c_i^\Gamma|, \|a_i^\Theta - a_i^\Gamma\|_{\infty} \leq \beta + 2r^{1/2}. \quad (112)
\]

**Case B1:** We have \(|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r\) and \(|c_i^\Gamma| > \|a_i^\Gamma\|_{\infty}\) which ensures \(|c_i^\Gamma| > \|c_i^\Gamma a_i^\Gamma\|_{\infty}\). Due to (109) we get \(c_i^\Theta = c_i^\Gamma\) and it follows that

\[
\|a_i^\Theta - a_i^\Gamma\|_{\infty} = \frac{1}{|c_i^\Gamma|} \|c_i^\Theta a_i^\Theta - c_i^\Gamma a_i^\Gamma\|_{\infty} \leq \frac{2r}{\|c_i^\Gamma a_i^\Gamma\|_{\infty}^{1/2}} \leq (2r)^{1/2}. \quad (113)
\]

**Case B2:** We have \(|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r\) and \(|c_i^\Gamma| < \|a_i^\Gamma\|_{\infty}\) which ensures \(\|a_i^\Gamma\|_{\infty} > \|c_i^\Gamma a_i^\Gamma\|_{\infty}\). Due to (110) we get \(a_i^\Theta = a_i^\Gamma\) and it follows that

\[
|c_i^\Theta - c_i^\Gamma| = \frac{1}{\|a_i^\Gamma\|_{\infty}} \|c_i^\Theta a_i^\Theta - c_i^\Gamma a_i^\Gamma\|_{\infty} \leq \frac{2r}{\|c_i^\Gamma a_i^\Gamma\|_{\infty}^{1/2}} \leq (2r)^{1/2}. \quad (114)
\]

**Case B3:** We have \(|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r\) and \(|c_i^\Gamma| = \|a_i^\Gamma\|_{\infty}\). Note that \(|c_i^\Gamma a_i^\Gamma\|_{\infty} > 2r\) and (102) ensure that \(\text{sgn}(c_i^\Theta) = \text{sgn}(c_i^\Gamma)\), and that for \(x, y > 0\) it holds that \(|x - y| \leq |x^2 - y^2|^{1/2}\). Combining this with the definition of \(I_2\), the reverse triangle inequality, and (111) implies that

\[
\|a_i^\Theta - a_i^\Gamma\|_{\infty} \leq (2r)^{1/2} \quad \text{and} \quad |c_i^\Theta - c_i^\Gamma| \leq (2r)^{1/2}. \quad (115)
\]

Combining (107), (112), (113), (114), and (115) establishes that

\[
\|\Theta - \Gamma\|_{\infty} \leq \beta + 2r^{1/2}, \quad (116)
\]

which completes the proof. \(\square\)

**Proof of Theorem 3.3** Let \(\Theta \in \mathcal{N}'_N\) be a parametrization of \(g\), i.e. \(\mathcal{R}(\Theta) = g\). We write

\[
\Gamma = \left(\begin{array}{c}
a_1^\Gamma \\ \vdots \\ a_m^\Gamma \\
\end{array}\right), \quad \left[\begin{array}{c}
c_1^\Gamma \\ \vdots \\ c_m^\Gamma \\
\end{array}\right], \quad \Theta = \left(\begin{array}{c}
a_1^\Theta \\ \vdots \\ a_m^\Theta \\
\end{array}\right), \quad \Theta = \left(\begin{array}{c}
c_1^\Theta \\ \vdots \\ c_m^\Theta \\
\end{array}\right) \in \mathcal{N}^e_{(d, m, D)} \quad (117)
\]

and \(r := |g - \mathcal{R}(\Gamma)|_{\infty}\). For convenience of notation we consider the weight vectors \(a_i^\Gamma, c_i^\Theta\) here as row vectors in order to write the derivatives of the ridge functions as \(c_i^\Gamma a_i^\Gamma, c_i^\Theta a_i^\Theta\) in \(\mathbb{R}^{D \times d}\) without transposing.

We will now adjust the approach used in the proof of Theorem 3.1 to work for multi-dimensional outputs in the case of balanced networks. By definition of \(\mathcal{N}'_N\), the \((a_i^\Theta)_{i=1}^m\) are pairwise linearly independent and we can skip the first reparametrization step in (89) and (90).

The following “hyperplane-jumping” argument, which was used to get the estimates (27) and (98), works analogously since Conditions \(C.2\) and \(C.3\) are fulfilled by definition of \(\mathcal{N}'_N\). This establishes the existence of a permutation \(\pi: [m] \rightarrow [m]\) and sets \(I_1, I_2 \subseteq [m]\), as defined in (99) and (100), such that for every \(i \in I_1\) it holds that

\[
\|c_i^\Gamma a_i^\Gamma\|_{\infty}, \|c_{\pi(i)}^\Theta a_{\pi(i)}^\Theta\|_{\infty} \leq 2r, \quad (118)
\]

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and for every $i \in I_2$ that
\[
\|c_i^\Gamma a_i^\Gamma - c_i^\Theta a_i^\Theta\|_\infty \leq 2r. \tag{119}
\]

As in (103), we make the following replacements, for all $i \in [m]$, without changing the realization of $\Theta$:
\[
a_i^\Theta \rightarrow a_{\pi(i)}^\Theta, \quad c_i^\Theta \rightarrow c_{\pi(i)}^\Theta. \tag{120}
\]

Note that the weights of $\Theta$ are already balanced, i.e., we have for every $i \in [m]$ that
\[
\|c_i^\Theta\|_\infty = \|a_i^\Theta\|_\infty = \|c_i^\Theta\|_\infty^{1/2}\|a_i^\Theta\|_\infty^{1/2} = \|c_i^\Theta a_i^\Theta\|_\infty^{1/2}. \tag{121}
\]

Thus, we can skip the reparametrization step in (104) and get directly for every $i \in I_1$ that
\[
\|c_i^\Theta - c_i^\Gamma\|_\infty \leq \|c_i^\Theta\|_\infty + \|c_i^\Gamma\|_\infty = \|c_i^\Theta a_i^\Theta\|_\infty^{1/2} + \|c_i^\Gamma a_i^\Gamma\|_\infty^{1/2} \leq 2(2r)^{1/2} \tag{122}
\]

and analogously $\|a_i^\Theta - a_i^\Gamma\|_\infty \leq 2(2r)^{1/2}$.

For $i \in I_2$ we need to slightly deviate from the proof of Theorem 3.1. We can skip the reparametrization step in (108)-(111) due to balancedness and need to distinguish three cases:

Case A.1: We have $\|c_i^\Theta a_i^\Theta\|_\infty \leq 2r$ which, together with (119), implies $\|c_i^\Theta a_i^\Theta\|_\infty \leq 4r$. Due to balancedness it follows that
\[
\|c_i^\Theta - c_i^\Gamma\|_\infty, \|a_i^\Theta - a_i^\Gamma\|_\infty \leq 4r^{1/2}. \tag{123}
\]

Case A.2: We have $\|c_i^\Theta a_i^\Theta\|_\infty > 2r$ which, together with (119), implies $\|c_i^\Theta a_i^\Theta\|_\infty \leq 4r$. Again it follows that
\[
\|c_i^\Theta - c_i^\Gamma\|_\infty, \|a_i^\Theta - a_i^\Gamma\|_\infty \leq 4r^{1/2}. \tag{124}
\]

Case B: We have $\|c_i^\Theta a_i^\Theta\|_\infty > 2r$ and $\|c_i^\Gamma a_i^\Gamma\|_\infty > 2r$. Due to the definition of $I_2$ there exists $e_i \in \mathbb{R}^d$, $\lambda_i^\Gamma, \lambda_i^\Theta \in (0, \infty)$ with $e_i\|_\infty = 1, \lambda_i^\Theta = \lambda_i^\Theta e_i$, and $a_i^\Gamma = \lambda_i^\Gamma e_i$. As in (115) we obtain that
\[
\|a_i^\Theta - a_i^\Gamma\|_\infty = \|e_i\|_\infty |\lambda_i^\Theta - \lambda_i^\Gamma| \leq (|\lambda_i^\Theta|^2 - (\lambda_i^\Gamma)^2)^{1/2} \tag{125}
\]
\[
\leq \|c_i^\Theta\|_\infty a_i^\Theta\|_\infty - \|c_i^\Gamma\|_\infty a_i^\Gamma\|_\infty \leq (2r)^{1/2}.
\]

Let now w.l.o.g. $\|a_i^\Gamma\|_\infty \geq \|a_i^\Theta\|_\infty$ (otherwise we switch their roles in the following) which implies that $\lambda_i^\Gamma = \Delta_i + \lambda_i^\Theta$ with $\Delta_i = \lambda_i^\Gamma - \lambda_i^\Theta \geq 0$. Then it holds that
\[
\|c_i^\Theta - c_i^\Gamma\|_\infty = \|c_i^\Theta a_i^\Theta - c_i^\Gamma a_i^\Gamma\|_\infty \leq \|c_i^\Theta a_i^\Theta - c_i^\Theta a_i^\Theta\|_\infty + \|c_i^\Theta a_i^\Theta - c_i^\Gamma a_i^\Gamma\|_\infty \leq \|c_i^\Theta\|_\infty |\lambda_i^\Gamma - \lambda_i^\Theta| + 2r \lambda_i^\Theta \Delta_i + \lambda_i^\Theta \tag{126}
\]
\[
= \frac{(2r)^{1/2}(\Delta_i + \lambda_i^\Theta) - (\lambda_i^\Theta - (2r)^{1/2})(2r)^{1/2} - \Delta_i)}{\Delta_i + \lambda_i^\Theta} \leq (2r)^{1/2}.
\]

The last step holds due to (125) and the balancedness of $\Theta$ which ensure that
\[
\lambda_i^\Theta = \|c_i^\Theta a_i^\Theta\|_\infty^{1/2} > (2r)^{1/2} \geq |\lambda_i^\Theta - \lambda_i^\Gamma| = \Delta_i. \tag{127}
\]

This completes the proof. \(\square\)

### A.4 Section 4

#### A.4.1 Additional Material

**Lemma A.7** (Inverse stability for fixed weight vectors). Let $N = (d, m, D) \in \mathbb{N}^3$, let $A = [a_1 \ldots a_m]^T \in \mathbb{R}^{m \times d}$ with
\[
\frac{a_i}{\|a_i\|_\infty} \neq \frac{a_j}{\|a_j\|_\infty} \quad \text{and} \quad (a_i)_{d-1}, (a_i)_d > 0 \tag{128}
\]

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for all \(i \in [m], j \in [m] \setminus \{i\}\), and define
\[
N_A^N := \{ \Gamma \in N_N : a_i^\Gamma = \lambda_i a_i \text{ with } \lambda_i \in (0, \infty) \text{ and } \| c_i^\Gamma \|_\infty = \| a_i^\Gamma \|_\infty \text{ for all } i \in [m] \}. \tag{129}
\]

Then for every \(B \in (0, \infty)\) there is \(C_B \in (0, \infty)\) such that we have uniform \((C_B, 1/2)\) inverse stability w.r.t. \(\| \cdot \|_{L^\infty((-B, B)^d)}\). That is, for all \(\Gamma \in N_A^N\) and \(g \in \mathcal{R}(N_A^N)\) there exists a parametrization \(\Phi \in N_A^N\) with
\[
\mathcal{R}(\Phi) = g \quad \text{and} \quad \| \Phi - \Gamma \|_\infty \leq \frac{C_B}{2} \| g - \mathcal{R}(\Gamma) \|_{L^\infty((-B, B)^d)}^{1/2}. \tag{130}
\]

**Proof.** Note that the non-zero angle between the hyperplanes given by the weight vectors \((a_i)_{i=1}^m\) establishes that the minimal perimeter inside each linear region intersected with \((-B, B)^d\) is lower bounded. As the realization is linear on each region, this implies the existence of a constant \(C_B' \in (0, \infty)\), such that for every \(\Theta \in N_N\) it holds that
\[
|\mathcal{R}(\Theta)|_{W^{1,\infty}} \leq C_B' \| \mathcal{R}(\Theta) \|_{L^\infty((-B, B)^d)}. \tag{131}
\]

Now note that for \(N_A^N\) we can get the same uniform \((4, 1/2)\) inverse stability result w.r.t. \(\| \cdot \|_{W^{1,\infty}}\) as in Theorem 3.3 by choosing \(\pi\) to be the identity in (118). Together with (131) this implies the claim. \(\square\)