Affine projective Osserman structures

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Abstract

By considering the projectivized spectrum of the Jacobi operator, we introduce the concept of projective Osserman manifold in both the affine and in the pseudo-Riemannian settings. If \( M \) is an affine projective Osserman manifold, then the deformed Riemannian extension metric on the cotangent bundle is both spacelike and timelike projective Osserman. Since any rank-1-symmetric space is affine projective Osserman, this provides additional information concerning the cotangent bundle of a rank-1 Riemannian symmetric space with the deformed Riemannian extension metric. We construct other examples of affine projective Osserman manifolds where the Ricci tensor is not symmetric and thus the connection in question is not the Levi-Civita connection of any metric. If the dimension is odd, we use methods of algebraic topology to show the Jacobi operator of an affine projective Osserman manifold has only one non-zero eigenvalue and that eigenvalue is real.

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1. Introduction

1.1. Osserman geometry in the Riemannian setting

Let \( \mathcal{R} \) be the curvature operator of a Riemannian manifold \( \mathcal{M} := (M, g) \) of dimension \( m \). The Jacobi operator \( \mathcal{J}(x) : y \rightarrow \mathcal{R}(y, x)x \) is a self-adjoint endomorphism of the tangent bundle. Following the seminal work of Osserman [23], one says that \( \mathcal{M} \) is Osserman if the eigenvalues of \( \mathcal{J} \) are constant on the unit sphere bundle

\[ S(M, g) := \{ \xi \in TM : g(\xi, \xi) = 1 \}. \]

Work of Chi [10], of Gilkey \textit{et al} [18], and of Nikolayevsky [19, 20] shows that any complete and simply connected Osserman manifold of dimension \( m \neq 16 \) is a rank-1-symmetric space; the 16-dimensional setting is exceptional and the situation is still not clear in that setting although there are some partial results due, again, to Nikolayevsky [21].

There has been much activity recently in Osserman Geometry. Brozos-Vázquez and Merino [1] showed that in dimension 4, the Osserman condition and the Rakić duality principle...
are equivalent. Nikolayevsky [22] showed that a conformally Osserman manifold (here one uses the Weyl conformal tensor to define the Jacobi operator) is locally isometric to a rank-1-symmetric space in dimension 16 modulo a certain assumption on algebraic curvature tensors in dimension 16. Brozanos-Vázquez et al [3] have examined conformally Osserman manifolds using warped product structures.

1.2. Osserman geometry in the pseudo-Riemannian geometry

Suppose that \( M = (M, g) \) is a pseudo-Riemannian manifold of signature \((p, q)\) for \(p > 0\) and \(q > 0\). The pseudo-sphere bundles are defined by setting:

\[
S^\pm(M, g) = \{ \xi \in TM : g(\xi, \xi) = \pm 1 \}.
\]

One says that \((M, g)\) is spacelike (resp. timelike) Osserman if the eigenvalues of \( J \) are constant on \( S^+(M, g) \) (resp. \( S^-(M, g) \)). The situation is rather different here as the Jacobi operator is no longer diagonalizable and can have nontrivial Jordan normal form as shown by García-Río et al [13]; in the algebraic context, the Jordan normal form can be arbitrarily complicated [17]. One says \((M, g)\) is nilpotent if \( J(x) \) is nilpotent for any tangent vector \( x \); this does not imply \((M, g)\) is flat in the pseudo-Riemannian setting.

Even in signature \((2, 2)\), the classification is incomplete. We refer to recent work by Díaz-Ramos et al [14] examining non-diagonalizable Jacobi operators and to recent work of Derdzinski [11] studying questions concerning type III Jordan–Osserman metrics raised by Díaz-Ramos et al [12]. We also refer to related work of Calviño-Louzao et al [6] treating similar questions arising in Ivanov–Petrova geometry related to the skew-symmetric curvature operator \( R(x, y) \).

Walker geometry is intimately related with many questions in mathematical physics. Chaichi et al [7] have studied conditions for a Walker metric to be Einstein, Osserman, or locally conformally flat and obtained thereby exact solutions to the Einstein equations for a restricted Walker manifold. Chudecki and Prazanowski [8, 9] examined Osserman metrics in terms of 2-spinors and provided some new results in HH-geometry using the close relation between weak HH-spaces and Walker and Osserman spaces using results of [12].

1.3. Affine Osserman manifolds

Let \( \nabla \) be a torsion free connection on a smooth manifold \( M \); the pair \((M, \nabla)\) is said to be an affine manifold. The first work on Osserman geometry in the affine setting is due to García-Río et al [16]. One has \( J(\lambda x) = \lambda^2 J(x) \) for \( \lambda \in \mathbb{R} \). This rescaling must be taken into effect. If \( T \) is a linear map of a finite-dimensional real vector space, let \( \text{Spec}(T) \subset \mathbb{C} \) be the spectrum of \( T \); this is the set of roots of the characteristic polynomial \( P_T(\lambda) := \det(T - \lambda \text{Id}) \). One says that an affine manifold \((M, \nabla)\) is affine Osserman if \( \text{Spec}(J(x)) = \{0\} \) for any tangent vector \( x \); i.e. \( J(x) \) is nilpotent. This notion clearly is invariant under rescaling and there are many examples. One has, for example, the following result of García-Río et al [13].

**Theorem 1.1.** Define a torsion free connection on \( \mathbb{R}^m \) by setting

\[
\nabla_{\partial_i} \partial_j = \sum_{k=\max(i,j)}^{m} \Gamma^k_{ij}(\partial_1, \ldots, \partial_{k-1}) \partial_k \quad \text{for } \Gamma^k_{ij} = \Gamma^k_{ji}.
\]

Then \((M, \nabla)\) is affine Osserman.

Such examples are important in neutral signature Osserman geometry. Let \((M, \nabla)\) be an affine manifold. Let \((x^1, \ldots, x^m)\) be local coordinates on \( M \). If \( \omega \in T^*M \), expand \( \omega = \sum_i \gamma_i \, dx^i \) to define the dual fiber coordinates \((y_1, \ldots, y_m)\) and thereby obtain canonical local coordinates.
Lemma 1.5. Suppose there exists a positive and once this is done, \( J \) is smooth. We will establish the following result in section 2.

\[ g_{\nu,\nu}(\partial_{\nu_i}, \partial_{\nu_j}) = -2\gamma_{ij} \Gamma_{ij}^k(x) + \Phi_{ij}(x), \]
\[ g_{\nu,\nu}(\partial_{\nu_i}, \partial_{\nu_j}) = \delta_{ij}, \quad g_{\nu,\mu}(\partial_{\nu_i}, \partial_{\mu_j}) = 0. \]  

We have chosen to work in coordinates; a coordinate free discussion is given in the appendix following the discussion in Calvino-Louzao et al. [4]. One has:

**Theorem 1.2.** Let \((M, \nabla)\) be an affine Osserman manifold and let \( \Phi \) be a smooth symmetric 2-tensor on \( M \). Then the deformed Riemannian extension \((T^*M, g_{\nu,\nu})\) is a pseudo-Riemannian nilpotent Osserman manifold of neutral signature.

**Theorem 1.3.** Let \((M, \nabla)\) be an affine Osserman manifold. Then the modified Riemannian extension \((T^*M, g_{\nu,1})\) is a pseudo-Riemannian Osserman manifold of neutral signature so that if \( \xi_{\pm} \in S^\pm(T^*M, g_{\nu,1}) \), then \( \text{Spec} \{J(\xi_{\pm})\} = \pm(0, 1, \frac{1}{2}) \) with multiplicities \((1, 1, 2m-2)\), respectively.

Note that the structures can be chosen so that Jacobi operators for the metrics in theorem 1.2 and in theorem 1.3 have non-trivial Jordan normal form.

1.4. Projectivizing the spectrum

Since \( \mathcal{J}(\lambda \xi) = \lambda^2 \mathcal{J}(\xi) \), it is necessary to take this rescaling into account. This played no role, of course, if we assume that \( \text{Spec} \{J(\xi)\} = \{0\} \) for all \( \xi \). But it motivates the following.

**Definition 1.4.**

1. Let \((M, \nabla)\) be an affine manifold. We say \((M, \nabla)\) is an affine projective Osserman manifold if given any pair of non-zero tangent vectors \( x, y \), there is a real scaling factor \( s(x, y) \neq 0 \) so

\[ \text{Spec} \{J(x)\} = s(x, y) \cdot \text{Spec} \{J(y)\} \neq \{0\}. \]

2. Let \((M, g)\) be a pseudo-Riemannian manifold. We say \((M, g)\) is spacelike projective Osserman (resp. timelike projective Osserman) if given any pair of vectors \( x, y \in S^\pm(M, g) \) (resp. \( S^{-}(M, g) \)), there is a real scaling factor \( s(x, y) \neq 0 \) so

\[ \text{Spec} \{J(x)\} = s(x, y) \cdot \text{Spec} \{J(y)\} \neq \{0\}. \]

Although in principle, we allowed \( s(x, y) \) to be negative, in fact \( s(x, y) \) can be chosen to be positive and once this is done, \( s \) is smooth. We will establish the following result in section 2.

**Lemma 1.5.** Let \((M, \nabla)\) be an affine manifold. Let \( \mathcal{O} \) be a connected open subset of \( TM \). Suppose there exists \( s(x, y) \) so that \( \text{Spec} \{J(x)\} = s(x, y) \text{Spec} \{J(y)\} \neq \{0\} \) for all \( x, y \in \mathcal{O} \). Then:
Theorem 1.6. \[ \text{result is an immediate consequence of lemma 1.5.} \]

Theorem 1.7. Let \( p \) \text{et al timelike projective Osserman but not spacelike projective Osserman.} \]

Similarly, there exists a pseudo-Riemannian manifold \((M, g)\) of signature \( p, q \) which is spacelike projective Osserman and not timelike projective Osserman. Similarly, there exists a pseudo-Riemannian manifold \((M, \tilde{g})\) of signature \( (p, q) \) which is timelike projective Osserman but not spacelike projective Osserman.

In section 4, we will generalize theorem 1.2 to the projective setting:

Theorem 1.8. Let \( \Phi \) be a symmetric 2-tensor on an affine manifold \((M, \nabla)\). The following assertions are equivalent:

1. \((M, \nabla)\) is an affine projective Osserman manifold.
2. \((T^*M, g_{\nabla}, \Phi)\) is a spacelike projective Osserman manifold.
3. \((T^*M, g_{\nabla}, \Phi)\) is timelike projective Osserman manifold.

Let \( \rho(x, y) := \text{Tr}[z \rightarrow R(z, x)y] \) be the Ricci tensor. This tensor need no longer be symmetric so we let \( \rho_s(x, y) := \frac{1}{2} [\rho(x, y) + \rho(y, x)] \) be the symmetric part of this tensor. Any Riemannian Osserman manifold is necessarily an affine projective Osserman manifold. The fact that \((M, g)\) is Riemannian is crucial here. If \((M, g)\) is a pseudo-Riemannian Osserman manifold of higher signature, then \( \mathcal{J}(\xi) \) is nilpotent for any null vector \( \xi \) and thus \( \text{Spec}(\mathcal{J}(\xi)) = \{0\} \).

If \((M, g)\) is a rank-1-symmetric space, then \((M, g)\) is Osserman and hence \((M, g)\) is an affine projective Osserman manifold. If \( m = 2 \) and if \( 0 \neq x \), let \( \{0, \lambda(x)\} \) be the eigenvalues of \( \mathcal{J}(x) \) where each eigenvalue is repeated according to its multiplicity. Then \( \rho(x, x) = \rho_s(x, x) = \text{Tr}[\mathcal{J}(x)] = \lambda(x) \). The following result is now immediate and provides examples to which theorem 1.8 applies.

Theorem 1.9.

1. Any rank-1-symmetric space is an affine projective Osserman manifold where we let \( \nabla \) be the Levi-Civita connection.
2. If \( m = 2 \) and if \((M, \nabla)\) is an affine manifold, then \((M, \nabla)\) is an affine projective Osserman manifold if and only if \( \rho_s(x, x) \neq 0 \) for all \( x \neq 0 \), i.e. \( \rho_s \) is definite.
1.5. The algebraic context

Let $V$ be a real vector space of dimension $m$ and let $A \in \text{End}(V) \otimes V^*$. We say that $(V, A)$ is an \textit{affine curvature model} if $A$ has the symmetries of the curvature operator of an affine connection for all $x, y, z \in V$:

\[
A(x, y)z = -A(y, x)z, \\
A(x, y)z + A(y, z)x + A(z, x)y = 0.
\]

The first symmetry is the $\mathbb{Z}_2$ anti-symmetry and the second symmetry is the first Bianchi identity. If $(M, \nabla)$ is an affine manifold, then $(\text{TP}M, \text{RP})$ is an affine curvature model for any $P \in M$. Conversely, given an affine curvature model $(V, A)$, then there exists a complete affine manifold $(M, \nabla)$ and a point $P$ of $M$ so that $(V, A)$ is isomorphic to $(\text{TP}M, \text{RP})$, i.e. every affine curvature model can be geometrically realized by a complete affine manifold (see Euh \textit{et al} [2]).

Let $(V, A)$ be an affine curvature model. The associated \textit{Jacobi operator} is given by setting $J(v)w := A(w, v)v$. One says that $(V, A)$ is an \textit{affine projective Osserman curvature model} if $\text{Spec}\{J(v)\} = \{0, \lambda(v)\} \neq \{0\}$ for $0 \neq v, w \in V$. In section 5, we will prove the following result which has an immediate application to the geometric setting.

**Theorem 1.10.** Let $(V, A)$ be a an affine projective Osserman curvature model of odd dimension $m$. If $0 \neq v \in V$, then $\text{Spec}\{J(v)\} = \{0, \lambda(v)\}$ where $\lambda(v)$ is a smooth real valued function on $V - \{0\}$ which never vanishes. The eigenvalue $0$ appears with multiplicity $1$ and the eigenvalue $\lambda(v)$ appears with multiplicity $m - 1$. In this situation $\rho(v, v) = (m - 1)\lambda(v)$ so the symmetric Ricci tensor $\rho$, defines a non-degenerate definite inner product on $V$.

In section 6, we will prove the following result.

**Theorem 1.11.** Let $\mathcal{M}_\epsilon := (\mathbb{R}^m, A)$ where the non-zero components of $A$ are determined by:

\[
A_{ij} = 1 \text{ for } 1 \leq i \neq j \leq m \text{ and } A_{122} = A_{121} = -\epsilon.
\]

(1) $\mathcal{M}_\epsilon$ is an affine projective Osserman model for any $\epsilon$.

(2) $\mathcal{M}_\epsilon$ is geometrically realizable by an affine projective Osserman manifold.

**Remark 1.12.** The Ricci tensor of the model in theorem 1.11 is given by

\[
\rho(\epsilon_i, \epsilon_j) = \begin{cases} 
\epsilon & \text{if } i = 1, \; j = 2 \\
-\epsilon & \text{if } i = 2, \; j = 1 \\
0 & \text{if } i = j \\
m - 1 & \text{otherwise}
\end{cases}.
\]

If $\epsilon \neq 0$, then $\rho_\epsilon$ is not symmetric and $A$ is not a Riemannian algebraic curvature operator and, in particular, is not the curvature operator of constant sectional curvature $+1$.

The tensor of theorem 1.11 is a perturbation of the curvature tensor of constant sectional curvature $+1$. In section 7, we present two algebraic examples which are perturbations of the Fubini–Study metric on complex projective space and on quaternionic projective space, respectively, and which are affine projective Osserman models.
2. The proof of lemma 1.5

Let \((M, \nabla)\) be an affine manifold and let \(\mathcal{O}\) be an open connected subset of \(TM\). Assume \(\text{Spec}[\mathcal{J}(x)] = s(x, y)\text{Spec}[\mathcal{J}(y)] \neq \{0\}\) for all \(x, y \in \mathcal{O}\). Let \(\sigma(t)\) be a path in \(\mathcal{O}\). Since the number of eigenvalues in \(\text{Spec}[\mathcal{J}(\sigma(t))]\) is independent of \(t\), eigenvalues do not coalesce or bifurcate and consequently the eigenvalue multiplicities are constant as well along \(\sigma\). Thus

\[
\text{Tr}(\mathcal{J}(t)\mathcal{J}(t)') = s(\sigma(0), \sigma(t))^2\text{Tr}(\mathcal{J}(0)\mathcal{J}(0)')\text{ for any } k.
\]

(2a)

Since \(\mathcal{J}(\sigma(0))\) is not nilpotent, \(\text{Tr}(\mathcal{J}(\sigma(0))^k) \neq 0\) for some \(k\). Fix such a \(k\). Since \(\mathcal{O}\) is connected, equation (2a) implies that \(\text{Tr}(\mathcal{J}(x)^k) \neq 0\) for any \(x \in \mathcal{O}\) and that \(s(\sigma(0), \sigma(t))^k\) is smooth. If \(k\) is odd, since \(s(\sigma(0), \sigma(0)) = 1\) and \(s(\sigma(0), \sigma(t)) \neq 0\), we have \(s(\sigma(0), \sigma(t)) > 0\). Since the endpoints were arbitrary, \(s(x, y) > 0\) for all \((x, y)\) and the Lemma follows.

On the other hand, if \(\text{Tr}(\mathcal{J}(\sigma(0))^k) = 0\) for all odd \(k\), then \(\text{Spec}[\mathcal{J}(\sigma(0))]\) is symmetric about the origin and we may assume \(s(\sigma(0), \sigma(t))\) is positive. Again, we can take the \(k\)th root to establish lemma 1.5.

3. The proof of theorem 1.7

Let \(p > 0\) and let \(q > 0\) be given. Let \((S^q, g_p)\) denote the sphere in \(\mathbb{R}^{q+1}\) with the standard metric of constant sectional curvature +1. Let \((\mathbb{R}^q, g_p)\) denote \(\mathbb{R}^q\) with a flat negative definite metric. Let \(M = (\mathbb{R}^p \times S^q, g_p \oplus \Bar{g}_q)\); this metric has signature \((p, q)\). If \(\xi = (\xi_p, \xi_q) \in TM\), then \(\mathcal{J}(\xi) = 0 \oplus \mathcal{J}(\xi_q)\). If \(\xi\) is spacelike, then \(\xi_q \neq 0\) and \(\text{Spec}[\mathcal{J}(\xi)] = \text{Spec}[\mathcal{J}(\xi_q)]\) and thus \((M, g)\) is a spacelike projective Osserman manifold; 0 is an eigenvalue of multiplicity \(p\). On the other hand, if \(\xi_q = 0\) and \(\xi_p \neq 0\), then \(\xi\) is timelike and \(\text{Spec}[\mathcal{J}(\xi)] = \{0\}\) so \((M, g)\) is not a timelike projective Osserman manifold. This proves the first assertion of theorem 1.7; the second follows similarly.

4. The proof of theorem 1.8

Let \(\sigma\) be the canonical projection from \(T^*M\) to \(M\). Let \(\xi \in T(T^*M)\) and let \(a = \alpha \xi \in TM\). Relative to the canonical frame \((\partial_1, \ldots, \partial_m, \partial_1, \ldots, \partial_m)\) for \(T(T^*M)\), one has (see García-Río et al [16]) that

\[
\mathcal{J}_{\xi v} (\xi) = \begin{pmatrix} \mathcal{J}_{\mathcal{V}} (a) & 0 \\ * & \mathcal{J}_{\mathcal{V}} (a) \end{pmatrix}
\]

where * is some linear map from \(\text{Span}[\partial_1] \to \text{Span}[\partial_2]\). Consequently

\[
\text{Spec}[\mathcal{J}_{\xi v} (\xi)] = \text{Spec}[\mathcal{J}_{\mathcal{V}} (a)].
\]

If \(\xi_k \in S^\perp(T^*M, g_{\mathcal{V}}, \Phi)\), then \(a := \alpha_s \xi_k \neq 0\). The implication (1) \(\Rightarrow\) (2) and the implication (1) \(\Rightarrow\) (3) of theorem 1.8 now follow. Conversely, suppose that Assertion (2) holds or that Assertion (3) holds. Let \(a \neq 0\). Choose \(\xi \in S^\perp(T^*M, g_{\mathcal{V}}, \Phi)\) so that \(\sigma_s(\xi) = ta\) for some \(t \neq 0\); The implications (2) \(\Rightarrow\) (1) and (3) \(\Rightarrow\) (1) now follow.

5. The proof of theorem 1.10

Let \((V, A)\) be an affine projective Osserman curvature model. Fix a basepoint \(0 \neq x \in V\) and let \(\text{Spec}[\mathcal{J}(x)] = \{0, \lambda_1, \ldots\}\); by hypothesis \(\text{Spec}[\mathcal{J}(x)] \neq \{0\}\). If \(0 \neq y \in V\), \(\text{Spec}[\mathcal{J}(y)] = \{0, s(x, y)\lambda_1, \ldots\}\). Let

\[
V_1(y) = \begin{cases} \ker((\mathcal{J}(y) - s(x, y)\lambda_1)^m) & \text{if } \lambda_1 \in \mathbb{R} \\ \ker((\mathcal{J}(y) - s(x, y)\lambda_1)^m(\mathcal{J}(y) - s(x, y)\lambda_1)^m) & \text{otherwise} \end{cases}
\]

6
be the generalized eigenspace corresponding to $\lambda_1$ if $\lambda_1$ is real and to $\{\lambda_1, \bar{\lambda}_1\}$ otherwise. As noted previously, the eigenvalue multiplicities are constant. Thus these generalized eigenspaces have constant dimension and vary smoothly with $y$. Put an auxiliary inner product $\langle \cdot, \cdot \rangle$ on $V$ and let $S^{m-1}$ be the unit sphere of $(V, \langle \cdot, \cdot \rangle)$. Let $y \in S^{m-1}$. Since $J(y)y = 0$, $J(y)$ induces an endomorphism of the quotient space $V/\mathbb{R}$ which we may identify with $T_yS^{m-1}$. Since $m - 1$ is even, $S^{m-1}$ has no non-trivial sub-bundles. Since $\{0\} \neq V_1$ is a sub-bundle of $T_yS^{m-1}$, we conclude $V_1 = T_yS^{m-1}$ for $y \in S^{m-1}$. This implies that $0$ is an eigenvalue of multiplicity $1$ and that

$$\text{Spec}(J(y)) = \begin{cases} \{0, s(y, x)\lambda_1\} & \text{if } \lambda \in \mathbb{R} \\ \{0, s(y, x)\lambda_1, s(y, x)\bar{\lambda}_1\} & \text{otherwise} \end{cases}.$$

This completes the proof if $\lambda_1$ is real. Thus we suppose $\lambda_1$ is complex and argue for a contradiction. We complexify and decompose

$$T_y(S^{m-1}) \otimes_{\mathbb{R}} \mathbb{C} = W_{\lambda(y, x)\lambda_1} \oplus W_{\lambda(y, x)\bar{\lambda}_1}$$

into the generalized eigenbundles corresponding to $\lambda$ and $\bar{\lambda}$ where

$$W_\mu(y) := \{x \in T_yS^{m-1} \otimes_{\mathbb{R}} \mathbb{C} : (J(y) - \mu)\xi = 0\}.$$

Since $J(-y) = J(y)$, we obtain a corresponding decomposition of the tangent bundle of projective space

$$T(\mathbb{R}P^{m-1}) \otimes_{\mathbb{R}} \mathbb{C} = W_\lambda \oplus W_{\bar{\lambda}}.$$

Since $W_\lambda = \bar{W}_{\bar{\lambda}}$, the first Chern class vanishes:

$$0 = c_1(T(\mathbb{R}P^{m-1}) \otimes_{\mathbb{R}} \mathbb{C}) \in H^2(\mathbb{R}P^{m-1}; \mathbb{Z}_2) = \mathbb{Z}_2.$$

On the other hand, $\mathbb{R}P^{m-1}$ is not orientable since $m - 1$ is even. Thus $w_1(T(\mathbb{R}P^{m-1}))$ generates $H^1(\mathbb{R}P^{m-1}; \mathbb{Z}_2) = \mathbb{Z}_2$. Since the generator of the first cohomology group $H^1(\mathbb{R}P^{m-1}; \mathbb{Z}_2)$ squares to the generator of the second cohomology group $H^2(\mathbb{R}P^{m-1}; \mathbb{Z}_2)$, this implies

$$0 \neq w_1^2(T(\mathbb{R}P^{m-1})) \in H^2(\mathbb{R}P^{m-1}; \mathbb{Z}_2) = \mathbb{Z}_2.$$

This is a contradiction since

$$w_1^2(T(\mathbb{R}P^{m-1})) = c_1(T(\mathbb{R}P^{m-1}) \otimes_{\mathbb{R}} \mathbb{C}).$$

This contradiction completes the proof.

6. The proof of theorem 1.11

If $m = 2$, then theorem 1.11 follows from theorem 1.9 (2) so we shall assume that $m \geq 3$. We have defined $\mathfrak{m}_e := (\mathbb{R}^m, A)$, where the non-zero components of $A$ are determined by

$$A_{ij}^j = 1 \text{ for } 1 \leq i \neq j \leq m \quad \text{and} \quad A_{122} = A_{121} = -\varepsilon.$$

If $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on $\mathbb{R}^m$, then $\rho_1 = (m - 1) \langle \cdot, \cdot \rangle$. We let $G := SO(2) \times SO(m - 2)$ act on $\mathbb{R}^m$. We lower indices and regard $A \in \otimes^2 V^*$:

$$A = -\varepsilon(e^1 \otimes e^3) \otimes (e^1 \otimes e^1 + e^2 \otimes e^2) + \sum_{i<j} (e^i \otimes e^j) \otimes (e^i \otimes e^j).$$

Consequently $A$ is invariant under the action of $G$ so $\text{Spec}(J(x)) = \text{Spec}(J(gx))$ for all $g \in G$. Let $x = a_1 e_1 + \cdots + a_m e_m$ belong to $S^{m-1}$. We may use the action of $SO(2)$ to ensure
that $a_2 = 0$ and we may use the the action of $SO(m - 2)$ to ensure that $a_i = 0$ for $i > 3$ in examining Spec($\mathcal{J}(x)$). Thus we may assume that $x = \cos \theta e_1 + \sin \theta e_3$ so

$\mathcal{J}(x)e_i = e_i$ for $i \geq 4$,

$\mathcal{J}(x)(\cos \theta e_1 + \sin \theta e_3) = 0$.

Thus 0 is an eigenvalue of multiplicity at least 1 and +1 is an eigenvalue of multiplicity at least $m - 2$. Since Tr($\mathcal{J}(x)$) = $\rho(x, x)$ = $(m - 1)$, we conclude that +1 is an eigenvalue of multiplicity $m - 1$. Consequently, $\mathcal{M}$ is an affine projective Osserman curvature model for any $\varepsilon$.

Define a torsion free connection $\nabla$ on $\mathbb{R}^m$ by setting:

$\Gamma_{mm} = 2$, $\Gamma_{im} = \Gamma_{mi} = \Gamma_{ii} = 1$ for $i < m$; $\Gamma_{11} = -\Gamma_{22} = (x_1 + x_2)$.

We have $R_{ijkl} = \partial_k \Gamma_{ij} - \partial_j \Gamma_{ik} + \Gamma_{jm} \Gamma_{ik} - \Gamma_{im} \Gamma_{jk}$. There are no terms in $\varepsilon^2$ and the only terms in $\varepsilon$ which are quadratic in the Christoffel symbols are

$0 = \Gamma_{ml} \Gamma_{ij} - \Gamma_{il} \Gamma_{mj}$

Thus $\Span{\Gamma_{ij}}$ is invariant under the action of $\mathcal{M}$ by affine isomorphisms; thus

$0 = \Gamma_{m2} \Gamma_{22} - \Gamma_{22} \Gamma_{m2}$.

Consequently, the quadratic terms give rise to:

$R_{im} = \Gamma_{im} \Gamma_{mm} - \Gamma_{ii} \Gamma_{im} = 2 - 1$ for $i < m$.

$R_{ml} = \Gamma_{ml} \Gamma_{mm} - \Gamma_{ii} \Gamma_{ml} = 2 - 1$ for $i < m$.

$R_{ij} = \Gamma_{ij} \Gamma_{ii} = 1$ for $i \neq j < m$.

We complete the proof by examining the terms involving the derivatives of $\Gamma$ and verifying:

$R_{ij} = \partial_3 \Gamma_{ij} = -\varepsilon$ and $R_{221} = \partial_3 \Gamma_{11} = \varepsilon$.

**Remark 6.1.** Suppose $\varepsilon \neq 0$. If $x = e_3$, then $\mathcal{J}(x)$ is diagonal. If $x = \frac{1}{\sqrt{2}}(e_1 + e_3)$, then

$\mathcal{J}(x)(e_1 + e_3) = 0$, $\mathcal{J}(x)(e_1 - e_3) = e_1 - e_3$, $\mathcal{J}(x)e_2 = \frac{1}{2} \varepsilon e_1 + e_2$.

Thus $\Span{v_1 := e_2 + \frac{1}{2} \varepsilon (e_1 + e_3), v_2 := (e_1 - e_3)}$ is invariant under the action of $\mathcal{J}(x)$. As $\mathcal{J}(x)v_2 = v_2$ and $\mathcal{J}(x)v_1 = v_1 + v_2$, we have non-trivial Jordan normal form in this instance.

**Remark 6.2.** Suppose $\varepsilon = 0$. Since the Christoffel symbols are constant, the group of translations acts transitively on $\mathcal{M}$ by affine isomorphisms; thus $\mathcal{M}$ is affine homogeneous. However, if we set $\sigma(t) = (0, \ldots, 0, x(t))$, then the geodesic equation becomes $\ddot{x} + 2\dot{x} = 0$ which blows up in finite time for suitable initial conditions. Thus $(\mathbb{R}^m, \nabla)$ is geodesically incomplete. Finally, we compute

$\nabla R(\partial_m, \partial_1, \partial_1; \partial_1) = \nabla_{\partial_m} R(\partial_m, \partial_1) \partial_1 - R(\nabla_{\partial_m} \partial_1, \partial_1) \partial_1 - R(\partial_m, \nabla_{\partial_1} \partial_1) \partial_1 - R(\partial_m, \partial_1) \nabla_{\partial_1} \partial_1$

$= (2 - 2 - 2) \partial_0 \neq 0$.

Consequently, $\nabla R \neq 0$. Thus this manifold is not locally symmetric. This shows that the affine manifold $\mathcal{M}$ is not affine equivalent to the standard affine structure on the sphere $S^m$.

If $\varepsilon \neq 0$, then there is a translation group of rank $m - 1$ which acts on $(\mathcal{M}, \nabla)$ preserving the structures. Furthermore, this manifold is affine curvature homogeneous. However, we have additional entries in $\nabla R$:

$\nabla R(\partial_2, \partial_1, \partial_1; \partial_1) = -2 \Gamma_{11} \partial_2$, and $\nabla R(\partial_1, \partial_2, \partial_2; \partial_2) = -2 \Gamma_{22} \partial_1$.

Since $\Gamma_{11}$ and $\Gamma_{22}$ vanish if and only if $x_1 + x_2 = 0$, $(\mathcal{M}, \nabla)$ is not 1-affine curvature homogeneous and has affine cohomogeneity 1.
7. Two algebraic examples

In section 6, we considered a model based on the tensor of constant sectional curvature 1. In this section, we examine examples which are related to the curvature operators of complex projective space. These examples have non-symmetric Ricci tensors and non-trivial Jordan normal form. We do not know if any of the examples in this section can be realized geometrically.

7.1. A complex example

Let \( m = 2\hat{m} \) be even, let \((\cdot, \cdot)\) be the usual positive definite inner product on \( \mathbb{R}^m \) for \( m \) even, and let \( J \) be a Hermitian complex structure; this means that

\[ J(\cdot, \cdot) = (\cdot, \cdot) \text{ and } J^2 = -\text{Id}. \]

We can choose an orthonormal basis \( \{e_1, \ldots, e_m\} \) for \( \mathbb{R}^m \) so that if \( 1 \leq j \leq \hat{m} \), then

\[ J e_j = \begin{cases} e_{2j} & \text{if } i = 2j - 1 \\ -e_{2j-1} & \text{if } i = 2j \end{cases}. \]

Define algebraic affine curvature operators by setting

\[ A_0(x, y)z := (y, z)x - (x, z)y, \]
\[ A_J(x, y)z := (Jy, z)Jx - (Jx, z)Jy - 2(Jx, y)Jz, \]
\[ \mathcal{E}(e_1, e_2)e_1 = -e_1 \text{ and } \mathcal{E}(e_2, e_1)e_2 = e_2. \]

The tensor \( A_0 + A_J \) is the curvature operator of the Fubini–Study metric on complex projective space \( \mathbb{C}P^m \). If \( (x, x) = 1 \), then

\[ J_{\lambda_0 + \lambda_1}(x) \cdot y = \begin{cases} 0 & \text{if } y = x \\ (\lambda_0 + 3\lambda_1)y & \text{if } y = Jx \\ \lambda_0 y & \text{if } y \perp \{x, Jx\} \end{cases}. \]

Thus \( \lambda_0 A_0 + \lambda_1 A_J \) is an affine projective Osserman curvature model.

**Lemma 7.1.** Let \( \mathcal{M}_c := (\mathbb{R}^m, \lambda_0 A_0 + \lambda_1 A_J + \varepsilon \mathcal{E}) \). The eigenvalues of \( J(x) \) for \( (x, x) = 1 \) are \( \{0, \lambda_0 + 3\lambda_1, \lambda_0, \ldots, \lambda_0\} \) where each eigenvalue is repeated according to multiplicity. Thus \( \mathcal{M}_c \) is an affine projective Osserman curvature model.

**Proof.** The tensor \( A_0 \) is invariant under the action of the full orthogonal group \( O(m) \), the tensor \( A_J \) is invariant under the action of the unitary group \( U(\hat{m}) \), and the tensor \( \mathcal{E} \) is invariant under the action of the group

\[ G := U(1) \times U(\hat{m} - 1) \subset U(\hat{m}) \subset O(m). \]

We suppose \( x_1 \in \mathbb{R}^m \) satisfies \( (x_1, x_1) = 1 \). We use the action of \( G \) to assume \( x_1 = \cos \theta e_1 + \sin \theta e_3 \) when studying \( J(x) \). Then \( J(x) = \lambda_0 \text{Id} \) on \( \text{Span}\{e_1, e_2\} \) and these vectors are eigenvectors. The central question is what happens to \( \text{Span}\{e_I\}_{I \leq 2} \) so we may assume \( m = 4 \). Let

\[ x_1 := \cos \theta e_1 + \sin \theta e_3, \quad x_2 := -\sin \theta e_1 + \cos \theta e_3, \quad x_3 := Jx_1 = \cos \theta e_2 + \sin \theta e_4, \quad x_4 := Jx_2 = -\sin \theta e_2 + \cos \theta e_4. \]

Let \( * \) be a coefficient which we do not need to specify. We then have

\[ J(x_1)x_1 = 0, \]
\[ J(x_1)x_2 = \lambda_0 x_2, \]
\[ J(x_1)x_3 = (\lambda_0 + 3\lambda_1)x_3 + *e_1 = *x_1 + *x_2 + (\lambda_0 + 3\lambda_1)x_3, \]
\[ J(x_1)x_4 = \lambda_0 x_4 + *e_1 = *x_1 + *x_2 + \lambda_0 x_4. \]
The matrix of $\mathcal{J}(x_1)$ on this four-dimensional subspace is therefore given by

$$\mathcal{J}(x_1) = \begin{pmatrix} 0 & 0 & \ast & \ast \\ 0 & \lambda_0 & \ast & \ast \\ 0 & 0 & \lambda_0 + 3\lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}.$$ 

The lemma now follows.

**Remark 7.2.** If we take $\theta = \frac{\pi}{2}$, then $x_1 = e_3$ and $\mathcal{J}(x_1)$ is diagonal. If we take $\theta = \frac{\pi}{4}$, then $x_1 = (e_1 + e_3)/\sqrt{2}$ and the same argument given in remark 6.1 shows

$$\mathcal{J}(x)(e_1 + e_3) = 0, \quad \mathcal{J}(x)(e_1 - e_3) = e_1 - e_3,$$

$$\mathcal{J}(x)(e_2 - e_4) = \frac{1}{2}e_1 + (e_2 - e_4),$$

$$\mathcal{J}(x)(e_2 - e_4 + \frac{1}{2}e_3) = \frac{1}{2}e_1 + e_2 - e_4$$

and, again, the Jordan normal form is non-trivial if $\varepsilon \neq 0$.

### 7.2. A quaternion example

Let $m = 4k$ and let $\{J_1, J_2, J_3\}$ give $\mathbb{R}^{4k}$ an orthogonal quaternion structure, i.e.

$$(J_1x, J_2x) = (x, x), \quad J_1J_2 + J_2J_1 = -2\delta_{ij} \text{Id}, \quad \text{and } J_1J_2 = J_3.$$ 

Let

$$\mathcal{E} := -(e^1 \wedge e^2) \otimes (e^3 \otimes e^4 + e^2 \otimes e^3) + (e^3 \wedge e^4)(e^3 \otimes e^3 + e^4 \otimes e^4).$$

**Lemma 7.3.** Let $\mathcal{M}_k : (\mathbb{R}^m, \lambda_0A_0 + \lambda_1A_1 + \lambda_2A_2 + \lambda_3A_3 + \varepsilon \mathcal{E})$. The eigenvalues of $\mathcal{J}(x)$ for $(x, x) = 1$ are $\{0, \lambda_0 + 3\lambda_1, \lambda_0 + 3\lambda_2, \lambda_0 + 3\lambda_3, \lambda_0, \ldots, \lambda_0\}$ where each eigenvalue is repeated according to multiplicity. Thus $\mathcal{M}_k$ is an affine projective Osserman curvature model.

**Proof.** Let $\mathbb{H} := \text{Span}_\mathbb{R}\{1, i, j, k\}$ denote the quaternions. We take an orthonormal basis $\{e^1_v, e^2_v, e^3_v, e^4_v\}$ for $\mathbb{R}^{4k}$ where $1 \leq v \leq k$ so that $e^1_v = J_v e^1_v$, $e^2_v = J_v e^2_v$, and $e^3_v = J_v e^3_v$. This permits us to identify $\mathbb{R}^m = \mathbb{H}^k$ with the quaternions where $\{J_1 = i, J_2 = j, J_3 = k\}$ are the quaternions acting from the left. Let $\text{Sp}(k)$ be the group of isometries of $\mathbb{R}^m$ which commute $\{J_1, J_2, J_3\}$; this is the set of $k \times k$ orthogonal quaternion matrices acting from the right. The affine algebraic curvature tensor in question is invariant under the action of $\text{Sp}(1) \times \text{Sp}(k - 1)$. Consequently, in considering Spec($\mathcal{J}(x)$), it suffices to consider the special case $x = \cos \theta e^1_v + \sin \theta e^2_v$. The remaining variables $e^2_v$ for $v \geq 3$ play no role and may be ignored. We compute

$$\mathcal{J}(x)(\cos \theta e^1_v + \sin \theta e^2_v) = 0,$$

$$\mathcal{J}(x)(-\sin \theta e^1_v + \cos \theta e^2_v) = \lambda_0( -\sin \theta e^1_v + \cos \theta e^2_v),$$

$$\mathcal{J}(x)(\cos \theta e^1_v + \sin \theta e^2_v) = (\lambda_0 + 3\lambda_1)( \cos \theta e^1_v + \sin \theta e^2_v) + \ast e^1_v,$$

$$\mathcal{J}(x)(-\sin \theta e^1_v + \cos \theta e^2_v) = \lambda_0( -\sin \theta e^1_v + \cos \theta e^2_v + \ast e^1_v),$$

$$\mathcal{J}(x)(\cos \theta e^1_v + \sin \theta e^2_v) = (\lambda_0 + 3\lambda_2)( \cos \theta e^1_v + \sin \theta e^2_v),$$

$$\mathcal{J}(x)(-\sin \theta e^1_v + \cos \theta e^2_v) = \lambda_0( -\sin \theta e^1_v + \cos \theta e^2_v),$$

$$\mathcal{J}(x)(\cos \theta e^1_v + \sin \theta e^2_v) = (\lambda_0 + 3\lambda_1)( \cos \theta e^1_v + \sin \theta e^2_v),$$

$$\mathcal{J}(x)(-\sin \theta e^1_v + \cos \theta e^2_v) = \lambda_0( -\sin \theta e^1_v + \cos \theta e^2_v).$$
The last four vectors are eigenvectors. The matrix of $J(x)$ with respect to the first four vectors takes the form:

$$J(x) = \begin{pmatrix} 0 & 0 & * & * \\ 0 & \lambda_0 & * & * \\ 0 & 0 & \lambda_0 + 3\lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}.$$ 

We have indicated by ‘*’ the upper diagonal terms which play no role in determining the eigenvalue structure. The desired result now follows.

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Appendix. The deformed Riemannian extension

We have worked in local coordinates. It is, however, possible to define the deformed Riemannian extension invariantly. Let $(M, \nabla)$ be an affine manifold. Let $\pi: T^*M \to M$ be the natural projection from the cotangent bundle to $M$. Let $X$ be a smooth vector field on $M$. Let $\langle \cdot, \cdot \rangle$ denote the natural pairing between a tangent vector and a cotangent vector. Let $P \in M$ and let $\omega \in T^*_P M$. The evaluation $\iota$ and the complete lift $X^C$ are characterized invariantly by the relations:

$$\iota X(P, \omega) = \langle X(P), \omega \rangle \text{ and } X^C(\iota Z) = \iota [X, Z].$$

In a system of local coordinates $\{x^i, y^j\}$, expand $X = X^j \partial_{x^j}$. We have

$$\iota X = X^i y_i \text{ and } X^C = X^j \partial_{y^j} - y_i (\partial_{x^i} X^j) \partial_{y^j}.$$ 

The Riemannian extension $g_\nabla$ is characterized by the identity:

$$g_\nabla(X^C, Y^C) = -\iota (\nabla_X Y + \nabla_Y X).$$

Let $\Phi$ be a symmetric $(0, 2)$-tensor field on $M$. The deformed Riemannian extension $g_{\nabla, \Phi}$ is the metric of neutral signature $(\tilde{m}, \tilde{m})$ on the cotangent bundle given by

$$g_{\nabla, \Phi} = g_{\nabla} + \pi^* \Phi.$$ 

It is given in a coordinate formalism in equation (1a). We remark that the modified Riemannian extension is the metric on $T^*M$ given invariantly by $\omega \circ \omega + g_{\nabla, \Phi}$; a coordinate formalism is given in equation (1b).

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