Simultaneous Unknown Input And Sensor Noise Reconstruction For Nonlinear Systems With Boundary Layer Sliding Mode Observers

Ankush Chakrabarty, Gregery T. Buzzard, Stanisław H. Żak, Fanglai Zhu, Ann E. Rundell

Abstract

While sliding mode observers (SMOs) using discontinuous relays are widely analyzed, most SMOs are implemented using a continuous approximation of the discontinuous relays. This continuous approximation results in the formation of a boundary layer in a neighborhood of the sliding manifold in the observer error space. Therefore, it becomes necessary to develop methods for attenuating the effect of the boundary layer and guaranteeing performance bounds on the resulting state estimation error. In this paper, a method is proposed for designing boundary-layer based SMOs for simultaneously reconstructing system states and unknown inputs in the presence of measurement noise. The proposed scheme is attractive because (i) it generalizes existing SMO schemes to a wider class of nonlinear systems; (ii) observer gains are computed using linear matrix inequalities; and, (iii) the unknown input and measurement noise are reconstructed simultaneously. Two numerical examples are presented to illustrate the performance of the proposed scheme and verify that the pre-computed bounds on the error state are satisfied.

1 A. Chakrabarty (corresponding author: chakraa@purdue.edu) and S. H. Żak (zak@purdue.edu) are affiliated with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN, USA.
2 G. T. Buzzard (buzzard@purdue.edu) is with the Department of Mathematics at Purdue University, West Lafayette, IN, USA.
3 F. Zhu (zhufanglai@tongji.edu.cn) is affiliated with the College of Electronics and Information Engineering, Tongji University, Shanghai, P. R. China.
4 A. E. Rundell (rundell@purdue.edu) is with the Weldon School of Biomedical Engineering at Purdue University, West Lafayette, IN, USA.
Index Terms

Unknown input observers, low-pass filtering, incremental quadratic constraints, secure observer, sliding mode, descriptor systems.

I. INTRODUCTION

Simultaneous estimation of plant states and exogenous disturbances for general nonlinear systems is crucial from an application perspective and a challenging theoretical problem. Exogenous disturbances can be categorized as (i) disturbances acting on the state vector, which models attack vectors in cyberphysical systems, actuator faults in mechanical systems and modeling uncertainties; and (ii) disturbances in the measurement channels arising from sensor noise or faults in the sensor.

Applications of simultaneous unknown inputs and sensor noise estimation in fault-tolerant controller designs and fault detection and isolation are discussed in [1]. The application of sliding modes [2]–[5] to state and unknown input observer design has been widely developed in the context of linear systems. In [6] and [7], an equivalent output error injection term is proposed to recover the state and measurement disturbance signals. Linear matrix inequalities for the construction of the observer gains and the reconstruction of the state disturbances are proposed in [8]–[10] for linear systems. An extension to Lipschitz nonlinear systems has been proposed in [11]–[16], and one-sided Lipschitz nonlinear systems in [17]. In this paper, the sliding mode observer construction is extended to a general class of nonlinearities which can be characterized by a set of symmetric matrices.

Descriptor systems provide an attractive approach for simultaneous estimation of the states and exogenous disturbances. Early applications of descriptor systems in this context can be found in [18], [19]. Sliding mode observer based on descriptor systems is proposed in [20], and an approach which minimizes the $L_2$ gain from the unknown inputs to the states is presented in [21], provided the gain is allowed to be arbitrarily small. A recent paper [22], [23] also discusses reconstruction of the unknown signals using second-order sliding modes. A functional unknown input observer for descriptor systems with applications to fault diagnosis is presented in [24].

In this paper, a single sliding mode observer is proposed for simultaneously estimating the system states, unknown inputs and measurement disturbances using a descriptor system formalism.
The method extends linear matrix inequalities available in the literature to encompass a wider variety of nonlinearities that contain Lipschitz and quasi-Lipschitz continuous functions. Such nonlinearities are characterized by symmetric matrices and satisfy matrix inequalities known as incremental quadratic constraints \([25]–[27]\). Sufficient conditions are provided for the asymptotic stability of the proposed observer estimation error. The proposed observer is guaranteed to estimate the plant states and exogenous disturbance signals of a nonlinear system within a specified performance level. The nonlinearities do not have to satisfy the standard Lipschitz conditions.

In [28, p. 32] and [29], it was proposed to employ low-pass filters to reject the high-frequency components of the discontinuous relay term, in the context of sliding mode control. In this paper, we propose to use a continuous approximation of the discontinuous relay term. We then provide a method for constructing a low-pass filter for recovering the exogenous disturbances. We also demonstrate that if the disturbance is piecewise/sectionally uniformly continuous, then a low-pass filter can be constructed which recovers the disturbance signal with a prescribed accuracy.

The rest of the paper is organized as follows. In Section II, we provide our notation. In Section III, we define the class of nonlinear systems considered and state the objective of this paper. Also, the robust observer architecture is presented. In Section IV, we discuss linear matrix inequality-based sufficient conditions to guarantee attenuation of the effect of the unknown input signal. Furthermore, we provide performance bounds on the observation error. Next, we discuss conditions which ensure finite time convergence to a boundary-layer sliding mode. In Section VI, we discuss how to recover the unknown input signal, we provide sufficient conditions for the existence of a low-pass filter operating at a given unknown input reconstruction accuracy. Additionally, we formulate explicit bounds on the unknown input reconstruction error. In Section VII, we test our proposed observer scheme on two numerical examples, and offer conclusions in Section VIII. In the Appendix, we discuss how to characterize common classes of nonlinearities using symmetric multiplier matrices.

II. Notation

We denote by \( \mathbb{R} \) the set of real numbers, and \( \mathbb{N} \) denotes the set of natural numbers. Let \( p \in \mathbb{N} \). For a function \( f : \mathbb{R} \mapsto \mathbb{R} \), we denote \( C^p \) the space of \( p \)-times differentiable functions. The function \( f \in L_p \) if \( \left( \int_{-\infty}^{\infty} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} < \infty \) and \( f \in L_\infty \) if \( \sup_{\mathbb{R}} |f| < \infty \). For every
$v \in \mathbb{R}^n$, we denote $\|v\| = \sqrt{v^\top v}$, where $v^\top$ is the transpose of $v$. The sup-norm or $\infty$-norm is defined as $\|v\|_\infty \triangleq \sup_{t \in \mathbb{R}} \|v(t)\|$. We denote by $\lambda_{\min}(P)$ the smallest eigenvalue of a square matrix $P$. The symbol $\succ$ ($\prec$) indicates positive (negative) definiteness and $A \succ B$ implies $A - B \succ 0$ for $A, B$ of appropriate dimensions. Similarly, $\succeq$ ($\preceq$) implies positive (negative) semi-definiteness. The operator norm is denoted $\|P\|$ and is defined as the maximum singular value of $P$. For a symmetric matrix, we use the $\star$ notation to imply symmetric terms, that is,

$$\begin{bmatrix} a & b \\ b^\top & c \end{bmatrix} \equiv \begin{bmatrix} a & \star \\ \star & c \end{bmatrix}.$$ 

For Lebesgue integrable functions $g, h$, we use the symbol $\star$ to denote the convolution operator, that is,

$$g \star h \triangleq \int_{-\infty}^{\infty} h(t - \tau)g(t) \, d\tau = \int_{-\infty}^{\infty} h(t)g(t - \tau) \, d\tau.$$ 

### III. Problem Statement and Proposed Solution

In this section, we describe the class of systems considered and formally state our objective.

#### A. Plant model and problem statement

We consider a nonlinear plant modeled by

$$\begin{align*}
\dot{x} &= Ax + B_u u + B_f f + B_g g + G_w a, \\
y &= C x + D w_s.
\end{align*} \tag{1a} \tag{1b}$$

Here, $x \triangleq x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u \triangleq u(t) \in \mathbb{R}^{n_u}$ is the control action vector, $y \triangleq y(t) \in \mathbb{R}^{n_y}$ is the vector of measured outputs. The nonlinear function $g \triangleq g(t, u, y) : \mathbb{R} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \mapsto \mathbb{R}^{n_y}$ models nonlinearities in the system whose arguments are known at each time instant $t$.

Let the function $f \triangleq f(t, u, y, q) : \mathbb{R} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_q} \mapsto \mathbb{R}^{n_f}$ denote the system nonlinearities whose argument $q$ is not known, where

$$q \triangleq C_q x,$$

and $C_q \in \mathbb{R}^{n_q} \times \mathbb{R}^{n_x}$.

The signal $w_a \triangleq w_a(t) \in \mathbb{R}^{n_a}$ is the unknown input vector that models state disturbances, unmodeled dynamics, actuator faults or attack vectors. The signal $w_s \triangleq w_s(t) \in \mathbb{R}^{n_s}$ models the
noise in the output sensor or sensor faults. We refer to the vectors $w_a$ and $w_s$ as the \textbf{exogenous disturbances}. The matrices $A$, $B_u$, $B_g$, $B_f$, $G$, $C$ and $D$ are of appropriate dimensions.

To proceed, we make the following assumptions.

**Assumption 1.** The right-hand-side of (1a) is locally Lipschitz.

**Assumption 2.** The matrices $G$ and $D$ have full column rank, that is, $\text{rank}(G) = n_a$ and $\text{rank}(D) = n_s$. Furthermore, the matching condition is satisfied, that is, $\text{rank}(CG) = \text{rank}(G)$.

**Assumption 3.** For every complex number $s \in \mathbb{C}$ with $\Re(s) \geq 0$,

$$\text{rank} \begin{bmatrix} sI - A & G & 0 \\ C & 0 & D \end{bmatrix} = n_x + n_a + n_s.$$  

**Assumption 4.** The unknown disturbance signals $w_a$ and $w_s$ are bounded. That is, there exist scalars $\rho_a, \rho_s > 0$ such that

$$\|w_a(t)\| \leq \rho_a$$

and

$$\|w_s(t)\| \leq \rho_s$$

for all $t \in [t_0, \infty)$, where $t_0$ is the initial time. Furthermore, the unknown input $w_a(t)$ is Lebesgue integrable.

Finally, we make an assumption on the classes of nonlinearities considered in this paper. To this end, we need the following definition.

**Definition 1 (Incremental Quadratic Constraint).** Let $\mathcal{M}$ denote the set of symmetric matrices such that any matrix $M \in \mathcal{M}$ is an \textbf{incremental multiplier matrix} for $f(t, u, y, q)$. A matrix $M \in \mathbb{R}^{(n_q+n_f) \times (n_q+n_f)}$ is an incremental multiplier matrix if it satisfies the incremental quadratic constraint ($\delta QC$)

$$\begin{bmatrix} \delta q \\ \delta f \end{bmatrix}^\top M \begin{bmatrix} \delta q \\ \delta f \end{bmatrix} \geq 0. \quad (2)$$

where

$$\delta q \triangleq q_1 - q_2 \quad (3a)$$
and
\[ \delta f \triangleq f(t, u, y, q_1) - f(t, u, y, q_2) \quad (3b) \]
for all \((t, u, y, q_1, 2) \in \mathbb{R} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_q} \).

For more details regarding the construction of the incremental multiplier matrix for different categories of nonlinearities, we refer the reader to the Appendix.

**Remark 1.** The concept of characterizing nonlinearities with incremental quadratic constraints is introduced in [25] where the authors represent classes of nonlinear functions using corresponding classes of symmetric matrices. For convenience, we summarize methods for constructing incremental multiplier matrices for common nonlinearities in Section A. The class of nonlinearities considered in this paper is more general than the class of locally Lipschitz and one-sided Lipschitz nonlinearities commonly considered in the current fault estimation literature [14], [17], [23], [30], [31].

Our objective is to design a sliding mode observer for the nonlinear system modeled by (1) which simultaneously reconstructs the state vector \(x(t)\) along with the unknown inputs \(w_u(t)\) and the measurement noise \(w_s(t)\). To this end, we first construct an auxiliary descriptor system that is used as a platform for the observer design.

**B. Generalized descriptor formulation**

Let
\[ \bar{x} \triangleq \begin{bmatrix} x \\ w_s \end{bmatrix} \in \mathbb{R}^{n_x+n_s} \]
be an augmented state vector. Then we represent the nonlinear plant (1) as a generalized descriptor system
\[
\begin{align*}
\bar{E} \dot{\bar{x}}(t) &= \bar{A} \bar{x}(t) + B_u u(t) + B_f f(t, u, y, C_q E \bar{x}) \\
&\quad + B_g g(t, u, y) + G w_u(t), \\
y(t) &= \bar{C} \bar{x}(t),
\end{align*}
\]
where,
\[
\bar{E} = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_s} \end{bmatrix} \in \mathbb{R}^{n_x \times (n_x + n_s)},
\]
\[
\bar{A} = \begin{bmatrix} A & 0_{n_x \times n_s} \end{bmatrix} \in \mathbb{R}^{n_x \times (n_x + n_s)},
\]
and \( \bar{C} = \begin{bmatrix} C & D \end{bmatrix} \in \mathbb{R}^{n_y \times (n_x + n_s)} \).

**Remark 2.** We refer to (4) as a ‘generalized’ descriptor system to distinguish this class of systems from the class of singular/descriptor systems where \( \bar{E} \) and \( \bar{A} \) are square matrices with \( \bar{E} \) singular, see for example, [32].

To proceed, we require the following technical result.

**Lemma 1.** Suppose the number of measured outputs is greater than or equal to the number of sensor faults; that is, \( n_y \geq n_s \). Then there exist two matrices \( T_1 \in \mathbb{R}^{(n_x+n_s) \times n_x} \) and \( T_2 \in \mathbb{R}^{(n_x+n_s) \times n_y} \) such that
\[
T_1 \bar{E} - T_2 \bar{C} = I_{n_x+n_s}. \tag{5}
\]

**Proof:** Let \( T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \) and
\[
V = \begin{bmatrix} \bar{E} \\ -\bar{C} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -C & -D \end{bmatrix}.
\]

Computing \( T \) reduces to solving the linear equation \( TV = I \). By Assumption [2] we know that \( D \) has full column rank, which implies \( V \) has full column rank. Hence, a left inverse of \( V \) exists. We denote \( V^\ell \) as a left inverse of \( V \), that is, \( V^\ell V = I \). Clearly, \( T = V^\ell \) is a solution to \( TV = I \).

Therefore, \( T_1 \) can be computed by taking the first \( n_x \) columns of \( V^\ell \) and \( T_2 \) is the matrix constructed using the last \( n_y \) columns of \( V^\ell \). This concludes the proof.

**Remark 3.** A particular choice of such a left inverse is the Moore-Penrose pseudoinverse, that is, \( V^\dagger = (V^\top V)^{-1}V^\top \).
C. Proposed observer for simultaneous state and exogenous disturbance reconstruction

Let \( e_y = y - \bar{C}\hat{x} \). We propose the following **boundary layer sliding mode observer** architecture to estimate the plant states \( x \) and the exogeneous disturbance signals \( w_a \) and \( w_s \):

\[
\dot{z} = Qz + (L_1 - QT_2)y + T_1 B_u u + T_1 B_g g + T_1 B_f \hat{f} + T_1 G \hat{w}_a, \tag{6a}
\]

\[
\hat{x} = z - T_2 y, \tag{6b}
\]

\[
\hat{f} = f(t, u, y, C_q \bar{E}\hat{x} + L_2 e_y), \tag{6c}
\]

\[
\hat{w}_a^\eta = \begin{cases} 
\rho \frac{F e_y}{\|F e_y\|} & \text{if } \|F e_y\| \geq \eta \\
\rho \frac{F e_y}{\eta} & \text{if } \|F e_y\| < \eta,
\end{cases} \tag{6d}
\]

where \( y \) is an available (measured) output, \( \hat{w}_a^\eta \) is a continuous **injection term** for the sliding mode observer parametrized by the **smoothing coefficient** \( \eta > 0 \) and

\[
Q \triangleq T_1 \bar{A} - L_1 \bar{C}.
\]

Our aim is to design the observer gain matrices which include (i) the linear gain \( L_1 \in \mathbb{R}^{(n_x+n_s) \times n_y} \), (ii) the innovation term \( L_2 \in \mathbb{R}^{n_q \times n_y} \) which improves the estimate of the known nonlinearity \( f \) while adding a degree of freedom in the design methodology, and, (iii) the matrix \( F \in \mathbb{R}^{n_y \times n_a} \) with gain \( \rho > 0 \). The signal \( \hat{w}_a^\eta(t) \) is used to recover the unknown input signal \( w_a(t) \).

**Remark 4.** Assumption [?] implies that the observer ODEs also have unique classical solutions as \( \hat{w}_a^\eta \in C^\infty \), and hence, the functions \( \bar{x}, \hat{x} \) are absolutely continuous.

IV. OBSERVER DESIGN

In this section, we provide stability guarantees for the proposed observer \((6)\) and conditions for unknown input and measurement noise reconstruction.

A. Derivation of error dynamics

We investigate the error dynamics of the proposed observer. To this end, we require the following result.
Lemma 2. Let $Q = T_1\bar{A} - L_1\bar{C}$ and $R = L_1 - QT_2$, where $T_1, T_2$ are constructed as described in Lemma 1. Then

$$T_1\bar{A} - QT_2\bar{C} - R\bar{C} - Q = 0.$$ 

Proof: We show the matrix identity by verification,

$$T_1\bar{A} - QT_2\bar{C} - R\bar{C} - Q = T_1\bar{A} - QT_2\bar{C} - (L_1 - QT_2)\bar{C} - T_1\bar{A} + L_1\bar{C}$$

$$= T_1\bar{A} - QT_2\bar{C} - L_1\bar{C} + QT_2\bar{C} - T_1\bar{A} + L_1\bar{C} = 0.$$ 

We define the observer error to be

$$\bar{e} = \bar{x} - \hat{\bar{x}}.$$ 

Using (5), the observer error dynamics are given by

$$\dot{\bar{e}} = \dot{\hat{\bar{x}}} - \dot{\bar{x}}$$

$$= \dot{\bar{x}} - \dot{z} + T_2\bar{C}\dot{\bar{x}}$$

$$= T_1\bar{E}\dot{\bar{e}} - \dot{\bar{z}}$$

$$= Q\bar{e} + (T_1\bar{A} - QT_2\bar{C} - R\bar{C} - Q)\bar{x} + T_1B_f(f - \hat{f})$$

$$+ T_1G(w_a - \hat{w}_a^\eta).$$

Using Lemma 2 yields

$$\dot{\bar{e}} = (T_1\bar{A} - L_1\bar{C})\bar{e} + T_1B_f(f - \hat{f}) + T_1G(w_a - \hat{w}_a^\eta).$$

(7)

Our objective is to design the observer gains $L_1$, $L_2$ and $F$ to ensure that the error dynamical system (7) is ultimately bounded and the effect of the unknown input $w_a$ is attenuated.

In the next subsection we analyze the stability of the error dynamics and propose sufficient conditions for the observer design with guaranteed performance.

B. Stability of observer error dynamics

In order to investigate the stability properties of the observer error (7), we need the following technical lemma.
Lemma 3. Suppose \( M = M^\top \) is an incremental multiplier matrix (see Definition 1) for the nonlinearity \( f \) and let
\[
\xi \triangleq \begin{bmatrix} \bar{e} \\ f - \hat{f} \end{bmatrix},
\]
where \( \hat{f} \) is defined in (6c). Then the condition
\[
\xi^\top \begin{bmatrix} C_q \hat{E} - L_2 \hat{C} & 0 \\ 0 & I \end{bmatrix}^\top M \begin{bmatrix} C_q \hat{E} - L_2 \hat{C} & 0 \\ 0 & I \end{bmatrix} \xi \geq 0
\]
holds for any \( \bar{x}, \hat{x} \in \mathbb{R}^{n_x+n_s} \).

Proof: Recall that \( y = C \bar{x} \). From (4a) and (6c), we have
\[
f - \hat{f} = f(t, u, y, C_q \hat{E} \bar{x}) - f(t, u, y, C_q \hat{E} \hat{x} + L_2(y - C \hat{x})).
\]
Let \( q_1 = C_q \hat{E} \bar{x}, q_2 = C_q \hat{E} \hat{x} + L_2(y - C \hat{x}) = C_q \hat{E} \hat{x} + L_2 \hat{C}(\bar{x} - \hat{x}), \delta q \triangleq q_1 - q_2, \) and \( \delta f \triangleq f(t, u, y, q_1) - f(t, u, y, q_2) \). Hence, we obtain \( \delta q = (C_q \hat{E} - L_2 \hat{C}) \delta \bar{e} \). Now, we can write
\[
\begin{bmatrix} \delta q \\ \delta f \end{bmatrix} = \begin{bmatrix} C_q \hat{E} - L_2 \hat{C} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{e} \\ \delta f \end{bmatrix}
\]
\[
= \begin{bmatrix} C_q \hat{E} - L_2 \hat{C} & 0 \\ 0 & I \end{bmatrix} \xi. \tag{8}
\]
Recalling that the matrix \( M \) is an incremental multiplier matrix of \( f \), and substituting (8) into the incremental quadratic constraint (2), we obtain the desired matrix inequality.

Herein, we present sufficient conditions in the form of matrix inequalities for the observer design.

Theorem 1. If there exist matrices \( L_1, F, P = P^\top, M = M^\top \) and scalars \( \alpha, \rho, \mu > 0 \), where \( M \) is an incremental multiplier matrix for the nonlinearity \( f \), such that
\[
\Xi + \Phi^\top M \Phi \preceq 0, \tag{9a}
\]
\[
G^\top T_1^\top P = F \hat{C}, \tag{9b}
\]
\[
\mu P \succeq I \tag{9c}
\]
\[
\rho \geq \rho_a, \tag{9d}
\]
where
\[
\Xi = \begin{bmatrix}
(T_1 \bar{A} - L_1 \bar{C})^\top P + P(T_1 \bar{A} - L_1 \bar{C}) + \alpha P & *
\end{bmatrix}
\begin{bmatrix}
B_f^\top T_1^\top P
\end{bmatrix}
\]
and
\[
\Phi = \begin{bmatrix}
C_q \bar{E} - L_2 \bar{C} & 0
\end{bmatrix}
\begin{bmatrix}
0 & I
\end{bmatrix},
\]
then the observer estimation error (7) is ultimately bounded by
\[
\limsup_{t \to \infty} \|\bar{e}(t)\| \leq \sqrt{\frac{2\mu \rho_a}{\alpha}},
\]
where \(\eta > 0\) is the smoothing coefficient of the sliding mode injection term \(\hat{w}_a^\eta\).

**Proof:** Consider a quadratic function of the form
\[
\mathcal{V}(\bar{e}(t)) = \bar{e}(t)^\top P \bar{e}(t).
\]
Herein, for readability, we omit the argument of \(\bar{e}(t)\). Then, the time derivative of \(\mathcal{V}(\bar{e})\) evaluated on the trajectories of the error dynamical system (7) is given by
\[
\dot{\mathcal{V}}(\bar{e}) = 2\bar{e}^\top P(T_1 \bar{A} - L_1 \bar{C}) \bar{e} + 2\bar{e}^\top P T_1 B_f (f - \hat{f})
+ 2\bar{e}^\top P T_1 G (w_a - \hat{w}_a^\eta).
\]
Let \(\xi = \begin{bmatrix}\bar{e}^\top (f - \hat{f})^\top\end{bmatrix}^\top\). Then from (9a), we get
\[
0 \geq \xi^\top (\Xi + \Phi^\top M \Phi) \xi
= \begin{bmatrix}\bar{e}^\top & (f - \hat{f})^\top\end{bmatrix}
\begin{bmatrix}
(T_1 \bar{A} - L_1 \bar{C})^\top P + P(T_1 \bar{A} - L_1 \bar{C}) + \alpha P & *
\end{bmatrix}
\begin{bmatrix}
B_f^\top T_1^\top P
\end{bmatrix}
\begin{bmatrix}
\bar{e} & (f - \hat{f})
\end{bmatrix}
+ 2\bar{e}^\top P T_1 B_f (f - \hat{f})
+ 2\alpha \bar{e}^\top P \bar{e} + \xi^\top \Phi^\top M \Phi \xi
= \dot{\mathcal{V}}(\bar{e}) + \alpha \mathcal{V}(\bar{e}) + \xi^\top \Phi^\top M \Phi \xi - 2\bar{e}^\top P T_1 G (w_a - \hat{w}_a^\eta).
\]
From Lemma 3 we know that \(\xi^\top \Phi^\top M \Phi \xi \geq 0\). Then for \(\|F(y - \hat{C} \hat{x})\| \geq \eta\), we have
\[
\dot{\mathcal{V}}(\bar{e}) \leq -\alpha \mathcal{V}(\bar{e}) + 2\bar{e}^\top P T_1 G (w_a - \hat{w}_a^\eta)
\leq -\alpha \mathcal{V}(\bar{e}) + 2\|w_a\| \|\bar{e}^\top P T_1 G\| - 2\bar{e}^\top P T_1 G \hat{w}_a.
\]
Hence, recalling the definition of $\dot{\hat{w}}_n^\eta$ from (6d) and condition (9b), we get
\[
\dot{V}(\bar{e}) \leq -\alpha V(\bar{e}) + 2\rho_a \| \bar{e}^T P T_1 G \| \| w_a \| - 2 \bar{e}^T P T_1 G \frac{F(y - \hat{\hat{x}})}{\| F(y - \hat{\hat{x}}) \|} \\
= -\alpha V(\bar{e}) + 2\rho_a \| \bar{e}^T P T_1 G \| - 2 \bar{e}^T P T_1 G \frac{\hat{C} \bar{e}}{\| \hat{C} \bar{e} \|} \\
= -\alpha V(\bar{e}) + 2\rho_a \| \bar{e}^T P T_1 G \| - 2 \rho_a \| G^T T_1^T P \bar{e} \| \\
= -\alpha V(\bar{e}) + 2\| \bar{e}^T P T_1 G \| (\rho_a - \rho).
\]
By choosing $\rho \geq \rho_a$, we get
\[
\dot{V}(\bar{e}) \leq -\alpha V(\bar{e}),
\]
which implies global exponential stability of the observer error $\bar{e}$ to the set $\| F \hat{C} \bar{e} \| < \eta$; see for example, [33], for global exponential stability to a set.

Now consider error trajectories that satisfy $\| F \hat{C} \bar{e} \| < \eta$. Then, from (11), we obtain
\[
\dot{V}(\bar{e}) \leq -\alpha V(\bar{e}) + 2\| \bar{e}^T P T_1 G \| \| w_a \| - 2 \bar{e}^T P T_1 G \frac{\hat{C} \bar{e}}{\eta} \\
= -\alpha V(\bar{e}) + 2\| \bar{e}^T P T_1 G \| \| w_a \| - 2 \bar{e}^T P T_1 G \frac{G^T T_1^T P \bar{e}}{\eta} \\
= -\alpha V(\bar{e}) + 2\| \bar{e}^T P T_1 G \| \| w_a \| - 2 \frac{\| G^T T_1^T P \bar{e} \|^2}{\eta} \\
\leq -\alpha V(\bar{e}) + 2\| \bar{e}^T P T_1 G \| \| w_a \|
\]
because $\| G^T T_1^T P \bar{e} \|^2 / \eta > 0$. Next, note that $\| F \hat{C} \bar{e} \| = \| G^T T_1^T P \bar{e} \| < \eta$. Hence,
\[
\dot{V}(\bar{e}) \leq -\alpha V(\bar{e}) + 2\eta \rho_a.
\]
Summarizing, we write
\[
\dot{V}(\bar{e}) \leq \begin{cases} 
-\alpha V(\bar{e}) & \text{if } \| F e_y \| \geq \eta \\
-\alpha V(\bar{e}) + 2\eta \rho_a & \text{if } \| F e_y \| < \eta.
\end{cases}
\]
The above implies that for any $\bar{e} \in \mathbb{R}^{n_x + n_y}$, the inequality (13) holds.

Using the differential form of the Bellman-Grönwall inequality yields
\[
V(e(t)) \leq e^{-\alpha(t-t_0)} V(e(t_0)) + 2\eta \rho_a \int_{t_0}^t e^{-\alpha(t-t_0)} d\tau.
\]
Using this inequality and (9c) gives the inequality
\[
\|\bar{e}\|^2 \leq \mu \mathcal{V}(e(t))
\]
\[
\leq \mu e^{-\alpha(t-t_0)}\mathcal{V}(e(t_0)) + 2\mu \eta \rho_a \int_{t_0}^{t} e^{-\alpha(\tau-t_0)} d\tau
\]
\[
= \mu e^{-\alpha(t-t_0)}\mathcal{V}(e(t_0)) + 2\frac{\mu \eta \rho_a}{\alpha} (1 - e^{-\alpha(t-t_0)}).
\]
Hence,
\[
\limsup_{t \to \infty} \|\bar{e}(t)\|^2 \leq \frac{2\mu \eta \rho_a}{\alpha},
\]
which yields the desired bound and thereby concludes the proof.

**Remark 5.** We note that the conditions in Theorem 1 are not linear matrix inequalities (LMIs) in \( L_2, \alpha, \mu \) and \( M \). However, if \( L_2, \mu \) and \( \alpha \) are pre-fixed, then we can obtain LMI conditions by choosing \( Y_1 = PL \) and rewriting \( \Xi \) in (9) as
\[
\Xi_1 = \begin{bmatrix}
A^T T_1 P - \bar{C}^T Y_1 + PT_1 A - Y_1 \bar{C} + \alpha P & * \\
B_f^T T_1 P & 0
\end{bmatrix}.
\]
Furthermore, we should minimize \( \mu \) to compute stronger eventual bounds on the observer error. Methods for constructing LMIs without pre-fixing \( L_2 \) are provided in [27].

**Remark 6.** Suppose the conditions of Theorem 1 are satisfied. Let \( \hat{x} = \begin{bmatrix} I_{n_x} & 0_{n_x} \end{bmatrix} \hat{x} \) and \( \hat{w}_s = \begin{bmatrix} 0_{n_x} & I_{n_s} \end{bmatrix} \hat{x} \). Then the following holds for the plant state estimation error:
\[
\limsup_{t \to \infty} \|x(t) - \hat{x}(t)\| \leq \sqrt{\frac{2\mu \eta \rho_a}{\alpha}}, \tag{14a}
\]
and the measurement noise estimation error satisfies
\[
\limsup_{t \to \infty} \|w_s(t) - \hat{w}_s(t)\| \leq \sqrt{\frac{2\mu \eta \rho_a}{\alpha}}. \tag{14b}
\]
For a fixed \( \eta, \alpha \) and \( L_2 \), we can minimize \( \mu \) over the space of feasible solutions. This attenuates the effect of \( \eta \), thereby producing more accurate estimates of the state and sensor noise vectors.

**Remark 7.** Note that \( \limsup_{t \to \infty} \|\bar{e}\| \to 0 \) as \( \eta \to 0 \), which verifies the fact that under ideal sliding (\( \eta = 0 \)) the matched disturbance can be completely rejected, and exact estimates of \( x \) and \( w_s \) can be obtained.
Remark 8 (Observer Existence Conditions). Suppose $M$ is an incremental multiplier matrix for the nonlinearity $f$. From [23], we know that the triple $(L_1, P, F)$ exists if the following conditions are satisfied:

(i) $\text{rank}(\bar{C}T_1G) = \text{rank}(T_1G) = n_a$,

(ii) $\text{rank} \begin{bmatrix} sI - A & G & 0 \\ C & 0 & D \end{bmatrix} = n_x + n_a + n_s$

for all $s \in \mathbb{C}$ with $\Re(s) \geq 0$.

In certain applications such as fault detection [30] and attack detection [34]–[36], it becomes necessary to reconstruct the unknown input $w_a(t)$. We propose a method for unknown input reconstruction in the following section.

V. Unknown Input Reconstruction

In this section, we present a method to reconstruct the actuator fault/unknown input signal $w_a(t)$ for a class of nonlinear systems satisfying incremental quadratic constraints. We begin with the following assumption on the plant states.

Assumption 5. The state vector $x(t)$ of the nonlinear plant (1) is bounded for all $t \geq t_0$.

Furthermore, in this subsection, we consider $f$ to be a nonlinear function with argument $q$ only. We use the following definition from [37, p. 406].

Definition 2 (Minimal Modulus of Continuity). The minimal modulus of continuity for any nonlinearity $\varphi(q)$ is given by

$$\gamma_{\varphi}(r) = \sup \{ \| \varphi(q_1) - \varphi(q_2) \| : q_1, q_2 \in \mathbb{R}^{n_q}, \| q_1 - q_2 \| \leq r \},$$

for all $r \geq 0$.

Remark 9. An important property of the modulus of continuity is that it is a non-decreasing function, that is, if $0 < r_1 < r_2$ then $\gamma_f(r_1) \leq \gamma_f(r_2)$. This follows from the definition of the supremum.

We also pose a restriction on the class of nonlinearities considered.
Assumption 6. The nonlinearity \( f(q) \) is uniformly continuous on \( \mathbb{R}^n_q \).

We now present the following technical result on the modulus of continuity of globally uniformly continuous functions.

Lemma 4. If \( f(q) \) is uniformly continuous on \( \mathbb{R}^n_q \) then \( \gamma_f(r) \to 0 \) as \( r \to 0 \).

Proof: Let \( \varepsilon > 0 \). By uniform continuity, there exists \( \delta > 0 \) such that \( \|q_1 - q_2\| \leq \delta \) forces \( \|f(q_1) - f(q_2)\| \leq \varepsilon \). This implies \( \sup\{\|f(q_1) - f(q_2)\| : \|q_1 - q_2\| \leq \delta\} \leq \varepsilon \) which, in turn, implies that \( \gamma(\delta) \leq \varepsilon \). Since \( \gamma(\cdot) \) is non-decreasing, we have \( \gamma(r) \leq \varepsilon \) for all \( r \in [0, \delta] \), which concludes the proof.

Remark 10. Note that if a function \( f \) is continuously differentiable with bounded derivative, Hölder continuous with exponent \( \beta \in (0,1] \), or globally Lipschitz continuous, then \( f \) is also uniformly continuous.

A. Convergence to the sliding manifold

We are now ready to state and prove the following theorem which provides conditions for the observer error trajectories to converge to the boundary-layer sliding manifold

\[
S_\eta = \{ \tilde{e} \in \mathbb{R}^{n_x+n_s} : \|F\tilde{C}\tilde{e}\| < \eta \}
\]

in finite time.

Theorem 2. Let

\[
\delta f = f(C_q\tilde{E}\tilde{x}) - f(C_q\tilde{E}\hat{x} - L_2(y - \hat{C}\hat{x})),
\]

(15)

\[
S = F\tilde{C}, \ \sigma = S\tilde{e}, \text{ and let}
\]

\[
\lambda_1 \triangleq \lambda_{\min}(G^\top T_1^\top P T_1 G),
\]

(16)

and suppose Assumptions 1–6 hold. If there exists a feasible observer which satisfies the conditions in Theorem 1 and

\[
\lambda_1 \rho \geq \sup_{t \geq t_0} \|\Psi\|,
\]

(17)

where

\[
\Psi \triangleq S \left( (T_1\tilde{A} - L_1\tilde{C})\tilde{e} + T_1 Gw_a + T_1 B_f \delta f \right)
\]
then the observer error trajectory $\bar{e}(t)$ converges to $S_\eta$ in finite time.

**Proof:** If $\|S\bar{e}\| < \eta$, we are done. Hence, for the remainder of this proof, we consider error trajectories satisfying $\|S\bar{e}\| \geq \eta$. It is enough to show that

$$\sigma^T \dot{\sigma} < -\zeta \|\sigma\|$$

for some $\zeta > 0$ in order to prove finite-time convergence to $S_\eta$, as argued in [38]. To this end,

$$\sigma^T \dot{\sigma}$$

$$= \sigma^T S \bar{e}$$

$$= \sigma^T S (\bar{T}_1 \bar{A} - L_1 \bar{C}) \bar{e} + T_1 B_f \delta f + T_1 G (w_a - \hat{w}^a)$$

$$= \sigma^T S (\bar{T}_1 \bar{A} - L_1 \bar{C}) \bar{e} + T_1 B_f \delta f + T_1 G w_a$$

$$- \rho \bar{e}^T S \bar{T}_1 G \frac{S \bar{e}}{\|S \bar{e}\|}$$

$$= \sigma^T S (\bar{T}_1 \bar{A} - L_1 \bar{C}) \bar{e} + T_1 B_f \delta f + T_1 G w_a$$

$$- \rho \bar{e}^T (T_1 G)^T P(T_1 G) \frac{S \bar{e}}{\|S \bar{e}\|}$$

from (9b). From Remark 8 we know that $T_1 G$ is full column rank. Hence

$$(T_1 G)^T P(T_1 G) > 0,$$

since $P > 0$.

Recalling that $\lambda_1$ is the minimal eigenvalue of the symmetric positive definite matrix $(T_1 G)^T P(T_1 G)$, we get

$$\sigma^T \dot{\sigma}$$

$$\leq \sigma^T S ((\bar{T}_1 \bar{A} - L_1 \bar{C}) \bar{e} + T_1 B_f \delta f + T_1 G w_a) - \rho \lambda_1 \frac{\|S \bar{e}\|^2}{\|S \bar{e}\|}$$

$$\leq \|\sigma\| \left( \|S ((\bar{T}_1 \bar{A} - L_1 \bar{C}) \bar{e} + T_1 B_f \delta f + T_1 G w_a)\| - \rho \lambda_1 \right).$$

We claim that for every $\zeta > 0$, we can select $\rho$ large enough to ensure $\sigma^T \dot{\sigma} \leq -\zeta \|\sigma\|$. To prove our claim, we first demonstrate that

$$\sup_{t \geq t_0} \left\| F \bar{C} \left( ((\bar{T}_1 \bar{A} - L_1 \bar{C}) \bar{e}(t) + T_1 B_f \delta f + T_1 G w_a(t)) \right) \right\| < \infty.$$ 

(19)
Using the triangle inequality, we have

\[
\sup_{t \geq t_0} \| S \left( (T_1 \bar{A} - L_1 \bar{C}) \bar{e}(t) + T_1 G w_a(t) + T_1 B_f \delta f \right) \| \\
\leq \sup_{t \geq t_0} \left( \| S(T_1 \bar{A} - L_1 \bar{C}) \| \| \bar{e}(t) \| + \| ST_1 G \| \| w_a(t) \| \\
+ \| ST_1 B_f \| \| f(C_q \bar{E} \bar{x}) - f(C_q \bar{E} \bar{x} - L_2(y - C \bar{x})) \| \right) \\
\leq \sup_{t \geq t_0} \| S(T_1 \bar{A} - L_1 \bar{C}) \| \| \bar{e}(t) \| + \| ST_1 G \| \rho_w \\
+ \| ST_1 B_f \| \gamma_f(\| C_q \bar{E} - L_2 \bar{C} \| \| \bar{e}(t) \| ),
\]

since \( w_a(t) \) and \( \bar{e}(t) \) are bounded by Assumptions 4 and 5 and by (10).

From (12), we also know that \( \| \bar{e}(t) \| \) decays exponentially when \( \| S \bar{e} \| \geq \eta \). This implies that \( \| \bar{e}(t) \| \) is bounded and decreasing with increasing \( t \). By Remark 9 this implies that \( \gamma_f(\| C_q \bar{E} - L_2 \bar{C} \| \| \bar{e}(t) \| ) \) also decreases with increasing \( t \), since \( \bar{e}(t_0) \) is bounded. Hence

\[
\sup_{t \geq t_0} \gamma_f(\| C_q \bar{E} - L_2 \bar{C} \| \| \bar{e}(t) \| )
\]

is bounded. As all the terms are bounded, thereby finite, the condition (19) holds and the gain \( \rho \) selected using (17) is well-defined.

### VI. Low-pass filtering the unknown input signal

In this section, we discuss a filtering method to reconstruct a class of unknown input signals. To this end, we need the following definition.

**Definition 3 (Piecewise uniformly continuous).** The unknown input signal \( w_a(t) \) is **piecewise** (or sectionally) **uniformly continuous** if

(i) the signal \( w_a(t) \) exhibits finite (in magnitude) jump discontinuities at abscissae of discontinuity denoted

\[
\mathcal{T} \triangleq \{ t_i : i \in \mathbb{I} \},
\]

where \( \mathbb{I} \subseteq \mathbb{N} \) is an arbitrary (possibly infinite) index set. Specifically, \( \mathbb{I} \) is the set of integers \( i \) satisfying \( a < i < b \) for some \( a < b \), where \( a \) may be \( -\infty \) and \( b \) may be \( \infty \), and \( t_i < t_{i+1} \) whenever \( i, i + 1 \in \mathbb{I} \).

(ii) there exists a scalar \( c > 0 \) such that \( |t_{i+1} - t_i| > c \) for every \( i, i + 1 \in \mathbb{I} \);
(iii) the unknown input signal $w_a$ is uniformly continuous on the closure of each open interval in $\mathbb{R} \setminus I$ and this uniformity is independent of the interval. More formally, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\tau_1 < \tau_2$ are in $\mathbb{R} \setminus I$ satisfying $|\tau_1 - \tau_2| < \delta$ and such that there is no $i \in I$ with $\tau_1 < t_i < \tau_2$, then $\|w_a(\tau_1) - w_a(\tau_2)\| \leq \epsilon$.

**Assumption 7.** The unknown input signal $w_a(t)$ is piecewise uniformly continuous.

**Remark 11.** Assumption 7 ensures that unknown input signals do not exhibit Zeno behaviour (infinite number of jumps in finite time intervals), which is a reasonable assumption for actuator faults in practical systems or attack vectors in cyberphysical systems.

The main result of the section requires the following proposition.

**Proposition 1.** Let the smooth window function $h(t) : \mathbb{R} \mapsto \mathbb{R}$ satisfy the following conditions:

(i) $h(t) \in C^\infty$, that is, $h$ is smooth;
(ii) $h(t) \geq 0$ for $t \in [-1, 1]$ and $h(t) = 0$ elsewhere;
(iii) $\int_{-\infty}^{\infty} h(t) = 1$.

Let

$$h_\beta(t) = \frac{1}{\beta} h \left( \frac{t}{\beta} \right)$$

(20)

Let $\tau_1, \tau_2 \in \mathbb{R}$ and suppose the function $\psi : (\tau_1, \tau_2) \to \mathbb{R}$ is uniformly continuous. Then for every $\epsilon > 0$, there exists a $\beta > 0$ such that

$$\|\psi(t) - (h_\beta * \psi)(t)\| \leq \epsilon,$$

(21)

for every $t \in [\tau_1 + \beta, \tau_2 - \beta]$. Here, ‘$*$’ denotes the convolution operator.

**Proof:** We begin by noting that conditions (i)–(iii) in the proposition statement imply that the function $h_\beta$ is non-negative on the compact support $[-\beta, \beta]$, and,

$$\int_{-\infty}^{\infty} h_\beta(t) = 1.$$  

(22)

Using the definition of convolution, we have

$$\psi(t) - h_\beta(t) * \psi(t) = \psi(t) - \int_{-\infty}^{\infty} \psi(t - \tau) h_\beta(\tau) \, d\tau.$$  

Applying (22) to the above yields

$$\psi(t) - h_\beta(t) * \psi(t) = \int_{-\infty}^{\infty} (\psi(t) - \psi(t - \tau)) h_\beta(\tau) \, d\tau.$$  

July 15, 2015
Since by definition, $\psi$ is uniformly continuous on $[\tau_1, \tau_2]$, for every $\varepsilon > 0$ there exists a $\beta > 0$ such that for any $t_1, t_2 \in [\tau_1, \tau_2]$ satisfying $|t_1 - t_2| \leq \beta$, we obtain $|\psi(t_1) - \psi(t_2)| \leq \varepsilon$.

Therefore, we get the estimate

$$|\psi(t) - h_\beta(t) \ast \psi(t)| \leq \int_{-\infty}^{\infty} |(\psi(t) - \psi(t - \tau))h_\beta(\tau)| d\tau \leq \int_{-\infty}^{\infty} |\psi(t) - \psi(t - \tau)||h_\beta(\tau)| d\tau \leq \varepsilon \int_{-\infty}^{\infty} h_\beta(\tau) d\tau,$$

since $h_\beta$ is non-negative. Using (22), we get (21). This concludes the proof.

**Remark 12.** Proposition 1 implies that a uniformly continuous function can be approximated arbitrarily closely using a low-pass filter with window length $2\beta$ and impulse response $h_\beta(t)$.

We are now ready to extend this idea to piecewise uniformly continuous unknown input signals. It is intuitive that piecewise uniformly continuous signals with jump discontinuities can be recovered by low-pass filtering, but the filter performance will be degraded at small neighborhoods about the points of discontinuity. The accuracy of the unknown input reconstruction in intervals between jump discontinuities is characterized in the following proposition.

**Proposition 2.** Let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ satisfy Assumption 7 and $\varphi \in L_\infty$. Let the function $h_\beta(t)$ be defined as in (20) and let

$$\mathcal{I}_\beta \triangleq \bigcup_{i \in \mathbb{I}} [t_i - \beta, t_i + \beta]$$

(23)

denote the union of closed neighborhoods around each abscissa of discontinuity of $\varphi$. Then for every $\varepsilon > 0$, there exists a $0 < \beta < c$ such that

$$\|\varphi(t) - (h_\beta \ast \varphi)(t)\| \leq \begin{cases} 
\varepsilon, & \text{for } t \in \mathbb{R} \setminus \mathcal{I}_\beta \\
2\|\varphi\|_\infty, & \text{for } t \in \mathcal{I}_\beta.
\end{cases}$$

(24)

As before, ‘$\ast$’ denotes the convolution operator and $\| \cdot \|_\infty$ is the $L_\infty$ norm.

**Proof:** We begin by noting that the existence of the constant $c$ in Definition 3 implies that $\mathcal{T}$ is a set of Lebesgue measure zero. Hence, the convolution integrals over $\mathbb{R}$ are well-defined.

First, fix $\varepsilon > 0$ and recall the definition of $h_\beta(t)$ in (20). Since the function $h_\beta$ has compact support $[-\beta, \beta]$, the convolution integral is evaluated over the window of length $2\beta$. Thus, we
can directly apply Proposition 1 with $\psi = \varphi_{[t_i-\beta, t_i+\beta]}$ to obtain (21) for any interval in $I_\beta$.

By (iii) in Definition 3, we can select $\beta$ independent of $i \in \mathbb{I}$. From this, we conclude

$$\|\varphi(t) - (h_\beta \ast \varphi)(t)\| \leq \varepsilon,$$

for $t \in \mathbb{R} \setminus I_\beta$.

However, the same cannot be said for the points $t \in I_\beta$ because the function $\varphi$ is not uniformly continuous across the point of discontinuity. Since $\beta < c$, we know that the function jumps just once in the interval $(t_i - \beta, t_i + \beta)$. Then we write

$$\|\varphi(t') - (h_\beta \ast \varphi)(t')\|$$

$$= \left\| \int_{-\beta}^\beta (\varphi(t') - \varphi(t' - \tau)) h_\beta(\tau) \, d\tau \right\|$$

$$\leq \int_{-\beta}^\beta \|\varphi(t') - \varphi(t' - \tau)\| \|h_\beta(\tau)\| \, d\tau$$

$$\leq 2\|\varphi\|_\infty,$$

since $\int_{-\beta}^\beta h_\beta(\tau) \, d\tau = 1$ by construction. This concludes the proof.

The following theorem is the main result of this section. It is an extension of a low-pass filtering method proposed in [39], which was for linear systems with sliding manifolds of co-dimension equal to one.

Herein, we show that under mild conditions, for any given reconstruction error threshold, there exists a low-pass filter which recovers the unknown input $w_a(t)$ and the reconstruction error is guaranteed to be below the specified threshold. This is demonstrated for the case when there are multiple unknown inputs using our proposed boundary layer sliding mode observer.

**Theorem 3.** Suppose Assumptions 1–7 hold, and there exists a feasible solution $(P, L_1, L_2, F, M, \alpha, \mu)$ satisfying the conditions (9) in Theorem 1. Let $\rho$ be selected as in (17) and $I_\beta$ be defined as in (23). Then for a given $\varepsilon > 0$, there exist scalars $\beta_1, \ldots, \beta_n > 0$, a sufficiently large $T > 0$, a sufficiently small $\eta > 0$ and a low-pass filter

$$h_\beta(t) = \begin{bmatrix}
\frac{1}{\beta_1}h_{\beta_1} \left( \frac{t}{\beta_1} \right) \\
\vdots \\
\frac{1}{\beta_n}h_{\beta_n} \left( \frac{t}{\beta_n} \right)
\end{bmatrix}$$

(25)
such that
\[ \| w_a(t) - (h_\beta \ast \hat{w}_n)(t) \| \leq \begin{cases} 2\rho_a + \varepsilon/2, & \text{for } t \in I_\beta \\ \varepsilon, & \text{for } t \in [T, \infty) \setminus I_\beta \end{cases} \]
for all \( t \geq T \).

\textbf{Proof}: We begin by fixing \( \varepsilon > 0 \) and choosing \( \beta_k > 0 \) for \( 1 \leq k \leq n_w \) such that \( h_\beta(t) \) satisfies (24) for the \( j \)th component of \( w_a \). We define
\[
H_\beta \equiv \begin{bmatrix} \frac{1}{\beta_1} \| h_{\beta_1}/dt \|_1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\beta_{n_w}} \| h_{\beta_{n_w}}/dt \|_1 \end{bmatrix},
\]
where \( \| \cdot \|_1 \) denotes the \( L_1 \) norm.

Let \( \chi_1 = \| H_\beta \|_1 \), \( \chi_2 = \| F\bar{C}(T_1 A - L_1 C) \|_1 \), and \( \chi_3 = \| F\bar{C}T_1 Bf \|_1 \), with \( \| \cdot \| \) denoting the operator norm. We note that for a given \( \beta_k \)'s and \( \varepsilon \), we can choose \( \varepsilon_1 \) sufficiently small to ensure that
\[
\max\{\chi_1 \varepsilon_1, \chi_2 \varepsilon_1, \chi_3 \gamma f(\| C_q E - L_2 C \|_1 \varepsilon)\} \leq \frac{\varepsilon}{6},
\]
where \( \lambda_1 = \lambda_{\min}(G^{\top}T_1^{\top}P T_1 G) \) as defined in (16). Recall that \( \lambda_1 > 0 \), as \( G^{\top}T_1^{\top}P T_1 G \succ 0 \). Note that by construction \( F\bar{C} = G^{\top}T_1^{\top}P \), which implies
\[
\lambda_1 = \lambda_{\min}(FCT_1 G).
\]

Let \( S = F\bar{C} \), and \( t_S \) be the time at which the error trajectories enter the boundary layer sliding manifold. We know that \( t_S < \infty \) as \( \rho \) satisfies (17) in Theorem 2. Furthermore, we know that \( \bar{e}(\cdot) \) is an absolutely continuous function (see Remark 4).

Therefore, we can apply integration by parts for \( t > t_S \) and use the compact support and smoothness of \( h_\beta \) to obtain
\[
\int_{-\infty}^{\infty} h_\beta(t - \tau)S\bar{e}(\tau) d\tau = \int_{-\infty}^{\infty} \dot{h}_\beta(t - \tau)S\bar{e}(\tau) d\tau,
\]
which implies
\[
\int_{t-\beta}^{t+\beta} h_\beta(t - \tau)S\bar{e}(\tau) d\tau = \int_{t-\beta}^{t+\beta} \dot{h}_\beta(t - \tau)S\bar{e}(\tau) d\tau.
\]
Let $\delta f$ be defined as in (15). Replacing the error-derivative $\dot{\bar{e}}$ using (7) gives

$$\int_{t-\beta}^{t+\beta} \dot{h}_\beta(t-\tau) S\bar{e}(\tau) \, d\tau =$$

$$\int_{t-\beta}^{t+\beta} h_\beta(t-\tau) S(T_1\bar{A} - L_1\bar{C})\bar{e}(\tau) \, d\tau$$

$$+ \int_{t-\beta}^{t+\beta} h_\beta(t-\tau) S T_1 B_f \delta f \, d\tau$$

$$+ \int_{t-\beta}^{t+\beta} h_\beta(t-\tau) S T_1 G (w_a(\tau) - \hat{w}_a^a(\tau)) \, d\tau. \quad (29)$$

We now rewrite the last term in (29) as

$$(h_\beta \ast ST_1 Gw_a)(t) - (h_\beta \ast ST_1 G\hat{w}_a^a)(t)$$

$$= \int_{t-\beta}^{t+\beta} \dot{h}_\beta(t-\tau) S\bar{e}(\tau) \, d\tau - \int_{t-\beta}^{t+\beta} h_\beta(t-\tau) S T_1 B_f \delta f \, d\tau$$

$$- \int_{t-\beta}^{t+\beta} h_\beta(t-\tau) S(T_1\bar{A} - L_1\bar{C})\bar{e}(\tau) \, d\tau. \quad (30)$$

From Theorem 1, we know that

$$\limsup_{t \to \infty} \|\bar{e}(t)\| \leq \sqrt{2\mu_0\rho_a/\alpha},$$

and from Theorem 2 we get $\|S\bar{e}\| \leq \eta$ for $t > t_S$. Thus, for a given $\beta > 0$ and $\varepsilon_1 > 0$ chosen as in (27), there exists a sufficiently small $\eta > 0$, and sufficiently large $T > t_S$ for which

$$\sup_{\tau \in [t-\beta,t+\beta]} \|S\bar{e}(\tau)\| < \varepsilon_1$$

and

$$\sup_{\tau \in [t-\beta,t+\beta]} \|\bar{e}(\tau)\| < \varepsilon_1 \quad (31)$$

for all $t \geq T$.

The inequality (31) along with Lemma 4 implies

$$\sup_{\tau \in [t-\beta,t+\beta]} \|\delta f\|$$

$$= \sup_{\tau \in [t-\beta,t+\beta]} \|f(C_q\bar{E}\bar{x}(\tau)) - f(C_q\bar{E}\hat{x}(\tau) + L_2\bar{C}\bar{e}(\tau))\|$$

$$\leq \sup_{\tau \in [t-\beta,t+\beta]} \|\gamma_f (C_q\bar{E}\bar{e}(\tau) - L_2\bar{C}\bar{e}(\tau))\|$$

$$\leq \gamma_f (\|C_q\bar{E} - L_2\bar{C}\|\varepsilon_1).$$
Therefore, using (22) and (26), we upper bound the right hand side terms in (30). That is,

\[
\left\| \int_{-\infty}^{\infty} h_\beta(t-\tau) S \bar{e}(\tau) \, d\tau \right\| \\
\leq \|H_\beta\| \sup_{\tau \in [t-\beta, t+\beta]} \|S \bar{e}(\tau)\| \\
\leq \chi_1 \varepsilon_1, \tag{32a}
\]

\[
\left\| \int_{-\infty}^{\infty} h_\beta(t-\tau) S T_1 B_f (f(q(\tau)) - f(\hat{q}(\tau))) \, d\tau \right\| \\
\leq \sup_{\tau \in [t-\beta, t+\beta]} \|ST_1 B_f\| \|f(q(\tau)) - f(\hat{q}(\tau))\| \\
\leq \chi_3 \gamma_f \left(\|C_q \bar{E} - L_2 \bar{C}\| \varepsilon_1\right), \tag{32b}
\]

\[
\left\| \int_{-\infty}^{\infty} h_\beta(t-\tau) S (T_1 \bar{A} - L_1 \bar{C}) \bar{e}(\tau) \, d\tau \right\| \\
\leq \|S(T_1 \bar{A} - L_1 \bar{C})\| \sup_{\tau \in [t-\beta, t+\beta]} \|\bar{e}(\tau)\| \\
\leq \chi_2 \varepsilon_1, \tag{32c}
\]

for \( t \geq T \).

We know that \( ST_1 G \) is symmetric positive definite, and hence, \( \lambda_1 > 0 \). Therefore,

\[
\|w_a(t) - \hat{w}_a^\eta(t)\| \leq \frac{\|ST_1 G(w_a(t) - \hat{w}_a^\eta(t))\|}{\lambda_1}. \tag{33}
\]

Applying to the above (30) and (32) gives

\[
\|(h_\beta * w_a)(t) - (h_\beta * \hat{w}_a^\eta)(t)\| \\
\leq h_\beta(t) \|w_a(t) - \hat{w}_a^\eta(t)\| \\
\leq \frac{h_\beta(t) \|ST_1 G(w_a(t) - \hat{w}_a^\eta(t))\|}{\lambda_1} \\
\leq \frac{(\chi_1 + \chi_2) \varepsilon_1 + \chi_3 \gamma_f (\|C_q \bar{E} - L_2 \bar{C}\| \varepsilon_1)}{\lambda_1}.
\]

By construction of \( \varepsilon_1 \) in (27), we get

\[
\|(h_\beta * w_a)(t) - (h_\beta * \hat{w}_a^\eta)(t)\| \leq \varepsilon/2. \tag{34}
\]
Recall the definition of $I_\beta$ from (23). We now use (24) in Proposition 2 and (34) to obtain
\[
\|w_a(t) - (h_\beta * \hat{\eta}_a)(t)\| \\
= \|w_a(t) - (h_\beta * w_a)(t) + (h_\beta * w_a)(t) - (h_\beta * \hat{\eta}_a)(t)\| \\
\leq \|w_a(t) - (h_\beta * w_a)(t)\| + \|(h_\beta * w_a)(t) - (h_\beta * \hat{\eta}_a)(t)\| \\
\leq \begin{cases} 
2\rho_a + \varepsilon/2, & \text{for } t \in I_\beta \\
\varepsilon, & \text{for } t \in [T, \infty) \setminus I_\beta
\end{cases}
\]
for $t \geq T > t_S$ and $\eta$ sufficiently small. This concludes the proof.

**Remark 13.** Note that Theorem 3 implies that there exists a low-pass filter capable of reconstructing the unknown input $w_a(t)$. In this paper, we do not provide a method for the construction of such a low-pass filter explicitly. This is a challenging open research problem.

**VII. Examples**

In this section, the performance of the proposed observer-based state and exogenous disturbance reconstruction methodology is tested on two numerical examples. In the first example, the unknown nonlinearity is not Lipschitz continuous and there is one unknown input and one measurement noise signal. In the second example, we illustrate the effect of lowpass filtering on multiple exogenous disturbances with piecewise uniformly continuous unknown inputs.

**A. Example 1**

We modify the single joint flexible robot described in [23] with an additional non-Lipschitz nonlinearity to test our observer design methodology. The nonlinear plant is modeled as in (1)
Fig. 1. Simulation Results. (Top left) The unmeasured variable $x_2(t)$ is shown in blue, and the dashed red line is the estimated trajectory $\hat{x}_2(t)$. We note that the estimate is satisfactorily close to the actual. (Top right) The error $e(t)$ is plotted in blue with the dashed black lines showing the error bound computed to be 0.082. (Bottom Left) The estimate of the actuator fault $w_a(t)$ shown after 40 s. Note that the low pass filtered estimate is highly accurate.

with system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.75 & -0.0015 & 3.75 & 0 \\ 0 & 0 & 0 & 1 \\ 3.75 & 0 & -3.75 & -0.0013 \end{bmatrix},$$

$$B_f = B_g = \begin{bmatrix} 0 \\ -1.1104 \\ 0 \\ 1 \end{bmatrix}, \quad G = B_u = \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ 1.3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$  

Here, the nonlinearity $f = x_2|x_2|$ is not globally Lipschitz and its argument, $x_2$, is not measured directly. The function $g = 2.3 \sin(x_1)$ is known at all $t \geq t_0$ because $x_1$ is a measured output.
The control input is set to zero. Hence $n_x = 4$, $n_y = 3$, $n_a = 1$, $n_s = 1$ and $n_f = 1$ and $C_q = \begin{bmatrix} 0 & -1 & 0 & 0 \end{bmatrix}$. Thus $f(q) = q|q|$ with $q = C_q x$. From (36), we deduce that this nonlinearity has an incremental multiplier matrix

$$M = \zeta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\zeta > 0$.

We select $\alpha = 0.5$ and $L_2 = \begin{bmatrix} -8.92 & 0.62 & 5.58 \end{bmatrix}$. Using CVX [40], we obtain a feasible solution to the LMIs in (9), namely $\zeta = 0.05$ and $\mu = 16.75$, the matrix

$$P = \begin{bmatrix} 26.66 & -3.97 & -4.8 & -9.12 & 8.57 \\ -3.97 & 1.1 & 2.62 & 0.83 & 2.2 \\ -4.8 & 2.62 & 23.96 & -6.32 & 33.54 \\ -9.12 & 0.83 & -6.32 & 9.2 & -22.12 \\ 8.57 & 2.2 & 33.54 & -22.12 & 73.9 \end{bmatrix},$$

the observer gain

$$L_1 = \begin{bmatrix} 4.61 & -2.06 & -2.68 \\ 97.33 & -153.73 & -95.82 \\ 15.7 & -27.18 & -13.65 \\ -67.44 & 126.91 & 69.09 \\ -30.76 & 55.24 & 29.91 \end{bmatrix},$$

and the sliding surface matrix

$$F = \begin{bmatrix} 6.39 & 2.73 & -3.24 \end{bmatrix}.$$

We find that minimizing the norm of $Y_1$, hence $L_1$, usually enables faster runtimes using MATLAB’s ode15s or ode23s. For simulation purposes, we consider a randomly generated initial condition

$$x(t_0) = \begin{bmatrix} 2.09 & -2.17 & -0.31 & -8.58 \end{bmatrix}^T$$

and the actuator and sensor faults are chosen arbitrarily to be $w_a = \text{sawtooth}(2t + 1)$ and $w_s = \text{square}(4t)$, respectively. Hence, $\rho_a = \rho_s = 1$. The observer is initialized at $z = 0$ and the
sliding mode gain is set at $\rho = 100$. Finally, the boundary layer sliding mode injection term $\hat{w}_a^\eta$ is computed using $\eta = 10^{-4}$. From Theorem [1], we get the error state bound

$$\limsup_{t \to \infty} \|e(t)\| \leq \sqrt{\frac{2\mu \eta \rho_a}{\alpha}} \approx 0.082.$$  

A 9th-order Butterworth low-pass filter with window length $\beta = 0.24$ s is used to obtain the actuator fault signal estimate. The corresponding MATLAB implementation is `butter(9, 0.12, ’low’).` We compute the experimental mean squared error $\|w_a(t) - \hat{w}_a^\eta(t)\|^2 \approx 2.59 \times 10^{-4}$ from $t \in [40, 80]$, which verifies that our reconstruction is highly accurate. Note that because $w_a$ is a sawtooth waveform, there is a spike in error around 78 s due to the effect of the point of discontinuity, as discussed in Theorem [3]. The simulation results are shown in Figure [1]. We observe that the states and faults are reconstructed accurately by the proposed observer.
B. Example 2

We now test our method on a randomly generated system of the form (1) with multiple unknown inputs and a globally Lipschitz nonlinearity. Here,

\[
A = \begin{bmatrix}
2.44 & 5.32 & 9.29 & 8.63 \\
1.1 & -4.11 & 1.82 & 2.53 \\
-0.09 & 0.9 & -2.91 & 0.06 \\
-4.53 & -3.45 & -8.59 & -12.14
\end{bmatrix},
\]

\[B_g = 0, \quad B_u = 0,\]

\[
B_f = \begin{bmatrix}
0.04 & 1.77 \\
1.37 & 0.3 \\
-6.14 & -0.56 \\
-2.71 & 0.05
\end{bmatrix},
\]

\[
G = \begin{bmatrix}
0 \\
-1 \\
0 \\
1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix},
\]

and the nonlinearity is \(f = \sin(2x_2)\). We set

\[
C_q = \begin{bmatrix}
0 & 2 & 0 & 0
\end{bmatrix}.
\]

Since the nonlinearity is globally Lipschitz, we choose

\[
M = \zeta \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

for some \(\zeta > 0\). We fix \(\alpha = 0.5\), \(L_2 = \begin{bmatrix}
-0.04 & -0.23 & 1.42
\end{bmatrix}\) and use CVX to obtain \(\zeta = 29.52\), \(\mu = 16.08\),

\[
P = \begin{bmatrix}
8.33 & -1.42 & -1.11 & 4.51 & 5.89 \\
-1.42 & 8.99 & 4.41 & -1.15 & 0.1 \\
-1.11 & 4.41 & 20.48 & -3.84 & -1.34 \\
4.51 & -1.15 & -3.84 & 13.72 & -3.35 \\
5.89 & 0.1 & -1.34 & -3.35 & 11.08
\end{bmatrix},
\]

July 15, 2015 DRAFT
\[
L_1 = \begin{bmatrix}
17.33 & 5.44 & 17.33 \\
4.77 & 1.3 & 4.77 \\
-2.65 & 0.19 & -2.65 \\
-13.44 & -4.68 & -13.44 \\
-15.54 & -5.23 & -15.54
\end{bmatrix}
\]

and
\[
F = \begin{bmatrix}
-9.96 & -47.96 & -9.99 \\
3.93 & -3.17 & 3.96
\end{bmatrix}.
\]

We generate a random initial condition
\[
\hat{x}(t_0) = \begin{bmatrix}
0.35 & 2.44 & 1.63 & -2.09
\end{bmatrix}^T
\]

and the actuator and sensor faults are chosen to be
\[
w_a = \begin{bmatrix}
\cos(t) \\
\text{sawtooth}(4t)
\end{bmatrix}
\]

and \(w_s = \sin(3t)\), respectively. Hence, \(\rho_a \approx 1.77\) and \(\rho_s = 1\). The observer is initialized at \(z = 0\) and the sliding mode gain is chosen \(\rho = 20\). The continuous injection term \(\hat{w}_a^n\) is computed with \(\eta = 10^{-3}\). Therefore, from Theorem 1, we get the error state bound
\[
\limsup_{t \to \infty} \|\bar{e}(t)\| \leq \sqrt{\frac{2\mu \eta \rho_a}{\alpha}} \approx 0.302.
\]

Two different smoothing filters with \(\beta_1 = 0.3\) s, \(\beta_2 = 0.1\) s (MATLAB: `smooth`) are used to obtain estimates of the unknown input. The simulation results are shown in Figure 2. We note that although the smooth sinusoidal unknown input \(w_{a_1}\) is reconstructed accurately, the sawtooth input \(w_{a_2}\) exhibits overshoots at the points of discontinuities, as predicted by Theorem 3.

VIII. Conclusions

In this paper, we develop a methodology for constructing implementable boundary layer sliding mode observers for a wide class of nonlinear systems. We provide sufficient conditions in the form of linear matrix inequalities for the observer design. We formulate ultimate bounds on the reconstruction error of states, unknown inputs and measurement noise signals. The class of nonlinearities considered in the paper is much wider than the nonlinearities considered in the literature, for a wide class of exogenous disturbances. We also demonstrate the requirement of
low-pass filtering to recover the unknown input signal and provide a formal proof of unknown input reconstruction error bounds for unknown inputs exhibiting jump discontinuities, which has not been investigated previously. The proposed methodology has a variety of applications including fault detection and reconstruction for mechanical systems, high confidence control of cyber-physical systems and secure communication.

ACKNOWLEDGEMENTS

The authors would like to thank Professor Martin J. Corless of the School of Aeronautics and Astronautics, Purdue University, West Lafayette, for his useful comments and suggestions.

This research was supported by a National Science Foundation (NSF) grant DMS-0900277.

REFERENCES

[1] J. Chen and R. J. Patton, Robust Model-Based Fault Diagnosis for Dynamic Systems. Boston, MA: Kluwer Academic Publishers, 1999.
[2] V. I. Utkin, “Survey paper variable structure systems with sliding modes,” IEEE Transactions on Automatic control, vol. 22, no. 2, 1977.
[3] R. A. DeCarlo, S. H. Žak, and G. P. Matthews, “Variable structure control of nonlinear multivariable systems: a tutorial.” Proceedings of the IEEE, vol. 76, no. 3, pp. 212–232, 1988.
[4] A. E. Rundell, S. V. Drakunov, R. DeCarlo et al., “A sliding mode observer and controller for stabilization of rotational motion of a vertical shaft magnetic bearing,” IEEE Transactions on Control Systems Technology, vol. 4, no. 5, pp. 598–608, 1996.
[5] R. DeCarlo, S. Drakunov, and X. Li, “A unifying characterization of robust sliding mode control: A Lyapunov approach,” Journal of Dynamic Systems, Measurement, and Control, vol. 122, no. 4, pp. 708–718, 2000.
[6] C. Edwards, S. K. Spurgeon, and R. J. Patton, “Sliding mode observers for fault detection and isolation,” Automatica, vol. 36, no. 4, pp. 541–553, 2000.
[7] C. P. Tan and C. Edwards, “Sliding mode observers for detection and reconstruction of sensor faults,” Automatica, vol. 38, no. 10, pp. 1815–1821, 2002.
[8] S. Hui and S. H. Žak, “Observer design for systems with unknown inputs,” Int. J. Appl. Math. Comput. Sci., vol. 15, no. 4, pp. 431–446, 2005.
[9] K. Kalsi, J. Lian, S. Hui, and S. H. Žak, “Sliding-mode observers for systems with unknown inputs: A high-gain approach,” Automatica, vol. 46, no. 2, pp. 347–353, 2010.
[10] K. Kalsi, S. Hui, and S. H. Žak, “Unknown input and sensor fault estimation using sliding-mode observers,” Proceedings of the 2011 American Control Conference, pp. 1364–1369, 2011.
[11] C. P. Tan and C. Edwards, “Sliding mode observers for robust detection and reconstruction of actuator and sensor faults,” International Journal of Robust and Nonlinear Control, vol. 13, no. 5, pp. 443–463, 2003.
[12] Q. P. Ha and H. Trinh, “State and input simultaneous estimation for a class of nonlinear systems,” Automatica, vol. 40, no. 10, pp. 1779–1785, 2004.
[13] X.-G. Yan and C. Edwards, “Nonlinear robust fault reconstruction and estimation using a sliding mode observer,” Automatica, vol. 43, pp. 1605–1614, 2007.
[14] R. Raoufi and H. Marquez, “Simultaneous sensor and actuator fault reconstruction and diagnosis using generalized sliding mode observers,” American Control Conference (ACC), 2010, pp. 7016–7021, 2010.
[15] P. S. Teh and H. Trinh, “Design of unknown input functional observers for nonlinear systems with application to fault diagnosis,” Journal of Process Control, vol. 23, no. 8, pp. 1169–1184, 2013.
[16] K. C. Veluvolu, M. Defoort, and Y. C. Soh, “High-gain observer with sliding mode for nonlinear state estimation and fault reconstruction,” Journal of the Franklin Institute, vol. 351, no. 4, pp. 1995–2014, 2014.
[17] L. Li, Y. Yang, Y. Zhang, and S. X. Ding, “Fault estimation of one-sided Lipschitz and quasi-one-sided Lipschitz systems,” in Proceedings of the Chinese Control Conference, no. 2, 2014, pp. 2574–2579.
[18] M. Darouach and M. Boutayeb, “Design of observers for descriptor systems,” IEEE Transactions on Automatic Control, vol. 40, no. 7, pp. 1323–1327, 1995.
[19] M. Hou and P. Muller, “Observer design for descriptor systems,” IEEE Transactions on Automatic Control, vol. 44, no. 1, pp. 164–169, 1999.
[20] Z. Gao and H. Wang, “Descriptor observer approaches for multivariable systems with measurement noises and application in fault detection and diagnosis,” Systems and Control Letters, vol. 55, no. 4, pp. 304–313, 2006.
[21] D.-J. Lee, Y. Park, and Y.-S. Park, “Robust $H_\infty$ sliding mode descriptor observer for fault and output disturbance estimation of uncertain systems,” IEEE Transactions on Automatic Control, vol. 57, no. 11, pp. 2928–2934, 2012.
[22] F. Zhu, “State estimation and unknown input reconstruction via both reduced-order and high-order sliding mode observers,” Journal of Process Control, vol. 22, no. 1, pp. 296–302, 2012.
[23] F. Zhu, L. Xu, W. Zhang, and W. Fan, “State estimation with unknown input and measurement disturbance reconstruction based on descriptor systems,” in 53rd IEEE Conf. on Decision and Control, 2014, pp. 5524–5529.
[24] F. J. Bejarano, “Functional unknown input reconstruction of descriptor systems: Application to fault detection,” Automatica, vol. 57, pp. 145–151, 2015.
[25] L. D’Alto and M. Corless, “Incremental quadratic stability,” Numerical Algebra, Control and Optimization, vol. 3, no. 1, pp. 175–201, 2013.
[26] K. Vijayaraghavan, R. Rajamani, and J. Bokor, “Quantitative fault estimation for a class of nonlinear systems,” International Journal of Control, vol. 80, no. 1, pp. 64–74, 2007.
[27] B. Açıkme and M. Corless, “Observers for systems with nonlinearities satisfying incremental quadratic constraints,” Automatica, vol. 47, no. 7, pp. 1339–1348, 2011.
[28] V. Utkin, J. Guldner, and J. Shi, Sliding mode control in electro-mechanical systems. CRC press, 2009, vol. 34.
[29] P. Kachroo and M. Tomizuka, “Chattering reduction and error convergence in the sliding-mode control of a class of nonlinear systems,” IEEE Transactions on Automatic Control, vol. 41, no. 7, pp. 1063–1068, 1996.
[30] X.-G. Yan and C. Edwards, “Nonlinear robust fault reconstruction and estimation using a sliding mode observer,” Automatica, vol. 43, pp. 1605–1614, 2007.
[31] Y. Jinyong and L. Zhiyuan, “Fault reconstruction of descriptor nonlinear system based on sliding mode observer,” 2010 29th Chinese Control Conference (CCC), pp. 1132–1137, 2010.
[32] E. Yip and R. Sincovec, “Solvability, controllability, and observability of continuous descriptor systems,” IEEE Transactions on Automatic Control, vol. 26, no. 3, pp. 702–707, 1981.
[33] M. Corless and G. Leitmann, “Bounded controllers for robust exponential convergence,” *Journal of Optimization Theory and Applications*, vol. 76, no. 1, pp. 1–12, 1993.

[34] A. Teixeira, H. Sandberg, and K. H. Johansson, “Networked control systems under cyber attacks with applications to power networks,” in *American Control Conference (ACC), 2010*. IEEE, 2010, pp. 3690–3696.

[35] F. Pasqualetti, F. Dorfler, and F. Bullo, “Attack detection and identification in cyber-physical systems,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 11, pp. 2715–2729, 2013.

[36] Y. Mo, J. P. Hespanha, and B. Sinopoli, “Resilient detection in the presence of integrity attacks,” *Signal Processing, IEEE Transactions on*, vol. 62, no. 1, pp. 31–43, 2014.

[37] N. Aronszajn and P. Panitchpakdi, “Extension of uniformly continuous transformations and hyperconvex metric spaces,” *Pacific J. Math.*, vol. 6, no. 3, pp. 405–439, 1956.

[38] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. New Jersey: Prentice Hall Englewood Cliffs, 1991.

[39] S. Hui and S. Záková, “Stress estimation using unknown input observer,” *American Control Conference (ACC), 2013*, pp. 259–264, 2013.

[40] M. Grant, S. Boyd, and Y. Ye, “CVX: Matlab software for disciplined convex programming,” 2008. [Online]. Available: [http://cvxr.com/cvx/download/](http://cvxr.com/cvx/download/)

[41] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Transactions on Systems, Man and Cybernetics*, no. 1, pp. 116–132, 1985.

**APPENDIX**

**INCREMENTAL MULTIPLIER MATRICES FOR COMMON NONLINEARITIES**

We present systematic methods for the computation of incremental multiplier matrices for a variety of nonlinearities analyzed in this paper and encountered in practical systems. We refer the reader to [25, Section 6] for a detailed discussion of methods used to compute incremental multiplier matrices and corresponding derivations of these matrices.

We begin by recalling the definition of $\delta q$ and $\delta f$ given in (3).

**A. Incrementally sector bounded nonlinearities**

An incrementally sector bounded nonlinearity satisfies the inequality

$$(M_{11} \delta q + M_{12} \delta f)^\top X (M_{21} \delta q + M_{22} \delta f) \geq 0, \quad (35)$$

for some fixed matrices $M_{11}, M_{12}, M_{21}, M_{22}$ and for all $X \in \mathcal{X}$, where $\mathcal{X}$ is a set of matrices. After representing the nonlinearity in the form (35), the incremental quadratic constraint (IQC) in (2) is satisfied by choosing

$$M = \begin{bmatrix} M_a & M_b \\ M_b^\top & M_c \end{bmatrix},$$
where,
\[ M_a = M_{11}^T X M_{21} + M_{21}^T X M_{11}, \]
\[ M_b = M_{11}^T X M_{22} + M_{21}^T X^T M_{12}, \]
\[ M_c = M_{12}^T X M_{22} + M_{22}^T X^T M_{12}. \]

B. Incrementally positively real nonlinearities

For a class of incrementally positively real nonlinearities, that is, nonlinearities satisfying
\[ \delta f^T X \delta q \geq 0, \]
the corresponding incremental multiplier matrix is given by
\[ M = \kappa \begin{bmatrix} 0 & X^T \\ X & 0 \end{bmatrix}, \]
with \( \kappa > 0 \).

C. Globally Lipschitz nonlinearities

For a globally Lipschitz nonlinearity that satisfies \( \| \delta f \| \leq L_f \| \delta q \| \) for some \( L_f > 0 \), we write
\[ (L_f \delta q + \delta f)^T (L_f \delta q - \delta f) \geq 0 \]
and inequality (35) is satisfied by choosing
\[ M = \kappa \begin{bmatrix} L_f^2 I & 0 \\ 0 & -I \end{bmatrix} \]
with \( \kappa > 0 \).

D. Quasi-Lipschitz nonlinearities

Another class of nonlinearities considered in this paper is the so-called ‘one-sided’ or ‘quasi’ Lipschitz nonlinearities that satisfy
\[ \delta q^T Q \delta f \leq \Sigma_f \delta q^T R \delta q, \]
for some \( \Sigma_f \in \mathbb{R} \), \( Q \in \mathbb{R}^{n_q \times n_f} \) and \( R = R^T \in \mathbb{R}^{n_q \times n_q} \). An incremental multiplier matrix for this class of nonlinearities is given by
\[ M = \kappa \begin{bmatrix} 2 \Sigma_f R & -Q \\ -Q^T & 0 \end{bmatrix}, \]
with \( \kappa > 0 \).
E. Nonlinearities with derivatives residing in a polytope

Suppose we have a nonlinearity \( f \) that satisfies
\[
\frac{\partial f}{\partial q} \in \Theta,
\]
where \( \Theta \) is a polytope with vertex matrices \( \theta_1, \ldots, \theta_r \). In other words,
\[
\frac{\partial f}{\partial q} = \theta(\chi),
\]
where \( \theta(\chi) = \sum_{k=1}^{r} \chi_k \theta_k \), and \( \chi_k \) satisfies \( \chi_k \geq 0 \) for all \( k \in \{1, \ldots, r\} \) and \( \sum_{k=1}^{r} \chi_k = 1 \). Then a corresponding incremental multiplier matrix
\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}
\]
(37)
satisfies the matrix inequalities
\[
M_{22} \preceq 0
\]
\[
M_{11} + M_{12} \theta_k + \theta_k^T M_{12}^T + \theta_k^T M_{22} \theta_k \succeq 0
\]
for all \( k = 1, \ldots, r \). An example of this class of nonlinearity is \( f(q) = \begin{bmatrix} \sin(q_1) & \cos(q_2) \end{bmatrix} \)
whose derivative is
\[
\begin{bmatrix} \cos(q_1) & 0 \\ 0 & -\sin(q_1) \end{bmatrix}
\]
which lies in a polytope \( \Theta \) with vertices
\[
\theta_1 = -\theta_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \theta_3 = -\theta_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Another example that falls into this category is the Takagi-Sugeno fuzzy model, proposed in [41].

F. Nonlinearities with derivatives residing in a cone

Suppose we have a nonlinearity \( f \) that satisfies
\[
\frac{\partial f}{\partial q} \in \Omega,
\]
where \( \Omega \) is a cone with vertex matrices \( \omega_1, \ldots, \omega_r \). In other words,
\[
\frac{\partial f}{\partial q} = \omega(\chi),
\]
where \( \omega(\chi) = \sum_{k=1}^{r} \chi_k \omega_k \), and \( \chi_k \) satisfies \( \chi_k \geq 0 \) for all \( k \in \{1, \ldots, r\} \). Then a corresponding incremental multiplier matrix of the form (37) satisfies the matrix inequalities

\[
M_{22} \theta_k = 0
\]

\[
M_{12} \theta_k + \theta_k^T M_{12}^T \succeq 0
\]

for all \( k = 1, \ldots, r \). An example of this class of nonlinearity is \( f(q) = \begin{bmatrix} q_1 & q_2^5 / 5 \end{bmatrix} \), whose derivative is \( \begin{bmatrix} 1 & 0 \\ 0 & q_2^4 \end{bmatrix} \) which lies in a cone \( \Omega \) with vertices

\[
\omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]