GLOBAL WELL-POSEDNESS AND SCATTERING OF THE ENERGY-CRITICAL MAXWELL-KLEIN-GORDON SYSTEM IN THE LORENZ GAUGE

SEOKCHANG HONG

Abstract. We study initial value problem of the (1 + 4)-dimensional Maxwell-Klein-Gordon system (MKG) in the Lorenz gauge. Since (MKG) in the Lorenz gauge does not possess an obvious null structure, it is not easy to handle the nonlinearity. To overcome this obstacle, we impose an additional angular regularity. In this paper, we prove global well-posedness and scattering of (MKG) for small data in a scale-invariant space which has extra weighted regularity in the angular variables. Our main improvement is to attain the scaling critical regularity exponent and prove global existence of solutions to (MKG) in the Lorenz gauge.

1. Introduction

In this paper, we investigate global well-posedness and scattering of the (1 + 4)-dimensional Maxwell-Klein-Gordon system in the Lorenz gauge. The Maxwell-Klein-Gordon (MKG) system describes a physical phenomena of a spin-zero particle in an electromagnetic field. The (MKG) system is obtained by coupling the Klein-Gordon scalar field $\phi : \mathbb{R}^{1+4} \rightarrow \mathbb{C}$ with an electromagnetic field $F$. To be precise, we consider the covariant form of the (MKG) system:

$$
\partial^\nu F_{\mu\nu} = \text{Im}(\phi \overline{D_\mu} \phi),
$$

$$
\overline{D_\mu} D_\mu A = m^2 \phi,
$$

where $F = dA$ is the associated curvature 2-form given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A = (A_\mu)$, $\mu = 0, 1, \cdots, 4$ is a real-valued 1-form on $\mathbb{R}^{1+4}$. We write $\overline{D_\mu} = \partial_\mu - iA_\mu$ for the connection. Here the constant $m \geq 0$ is a mass. Since we study the initial value problem of (MKG) in a scale-invariant Besov space, we put $m = 0$ in the sequel. In other words, we consider the mass-less (MKG) system.

The (MKG) in $\mathbb{R}^{1+4}$ obeys the law of conservation of energy. The conserved energy of a solution $(A, \phi)$ at time $t$ is defined as

$$
E_{\text{MKG}}(\phi, A) := \frac{1}{2} \int_{\mathbb{R}^4} |F_{\mu\nu}(t)|^2 + |D_\mu \phi(t)|^2 \, dx.
$$

We also note that the (MKG) system is invariant under the scaling $(A, \phi) \mapsto (\lambda^{-1} A, \lambda^{-1} \phi) (\lambda^{-1} t, \lambda^{-1} x)$, and hence the critical Sobolev space for the system is $\dot{H}^1$. Consequently, the Cauchy problem of the (1 + 4)-dimensional (MKG) system can be regarded as energy-critical problem.

We remark that the gauge potential $A_\mu$ is not necessarily unique. Indeed, for a sufficiently smooth real-valued function $\Lambda$ on $\mathbb{R}^{1+4}$, the (MKG) system is invariant under the transform $(A, \phi) \mapsto (A + \Lambda, \phi)$.
(A − dΛ, e^{iΛ} \phi). Hence we have gauge freedom and it is reasonable to choose a representative, which is suitable for our purpose. (See also [19].)

1.1. Maxwell-Klein-Gordon equations in the Lorenz gauge. Imposing the Lorenz gauge \( \partial^\mu A_\mu = 0 \) we obtain the following nonlinear wave equations after some computation

\[
\Box \phi = 2iA^\mu \partial_\mu \phi + A^\mu A_\mu \phi,
\]

\[
\Box A = -\text{Im}(\phi \partial \overline{\phi}) - A|\phi|^2,
\]

where \( \Box = \partial^2 - \Delta \) is the d’Alembertian. The Maxwell-Klein-Gordon system has been extensively studied [8, 19]. Typical choices of gauge are the Lorenz gauge \( \partial^\mu A_\mu = 0 \) and the Coulomb gauge \( \partial^j A_j = 0 \). Keel-Roy-Tao [6] proved global existence below the energy space and almost optimal well-posedness is proven by Machedon-Sterbenz [12]. In the Lorenz gauge, Pecher [17] showed almost critical well-posedness in the Fourier-Lebesgue spaces.

From now on we exclusively consider (1 + 4)-dimensional setting. The (MKG) system is relatively well-understood in the Coulomb gauge. Selberg [18] proved well-posedness in \( H^{1+} \), which is almost optimal and Krieger-Sterbenz-Tataru [10] showed global well-posedness with small \( H^1 \) norm. Then Oh-Tataru [13, 14, 15] obtained global well-posedness with arbitrarily large data. Concerning the Lorenz gauge, Pecher [16] proved local well-posedness for \( H^{7\over 6+} \) data. Thus well-posedness for critical regularity data in the Lorenz gauge is still open.

We briefly discuss the technical difficulty in proving low regularity well-posedness. When we are concerned with quadratic nonlinearity, the most difficult type of interaction is when the two inputs give rise to an output which is close to the light cone in the space-time Fourier space. However, if nonlinearity has a certain cancellation property, it would be possible to obtain better regularity properties. Such cancellation typically given by null structure plays a crucial role to attain the critical regularity. We refer the readers to [7, 4, 9, 11] for the study on the null form estimates and their applications.

In the Lorenz gauge, however, the main obstacle is that the null structure inside the equations [18] is not enough to attain the critical regularity as we are concerned with a low dimensional setting. By not enough, we mean that the cancellation property of parallel interaction given by the null structure is only the angle \( \angle(\xi_1, \xi_2) \) between the two input frequencies. If the null structure is enough, for instance, it yields more than \( \angle(\xi_1, \xi_2) \), such as the type of null form \( Q_0 \), defined by 

\[
Q_0(u, v) = \partial_1 u \partial_1 v - \partial_2 u \partial_2 v
\]

which gives \( \angle(\xi_1, \xi_2)^2 \), we could overcome this problem. Unfortunately, we cannot expect such an enough null structure in this case and hence it is not easy to handle the parallel interactions.

Inspired by the work of Sterbenz [23], we expect that the rotation generators \( \Omega_{ij} = x_i \partial_j - x_j \partial_i \) plays a crucial role since they eliminate such delicate interactions. Furthermore, we will get a significant gain over the classical Strichartz estimates for the wave equation [23, 3], and hence obtain a crucial improvement at the level of multilinear estimates. See also [11, 24, 27] and references therein for the study on the Dirac-Klein-Gordon and Yang-Mills systems via angular regularity. In this paper, we study global well-posedness and scattering of the mass-less (MKG) in a scale-invariant Besov
Our main improvement is to attain the scaling critical regularity exponent and prove global existence of solutions to (MKG) using a scale-invariance of function space. Then scattering is followed by Theorem 1.1.

**Theorem 1.2** (Scattering). For any given initial data \((\phi_0, \phi_1, a, \dot{a})\) satisfying the conditions (1.3), (1.6), and (1.7), there exist unique functions \((\phi_0^+, \phi_1^+, a^+, \dot{a}^+)\) and \((\phi_0^-, \phi_1^-, a^-, \dot{a}^-)\) such that

\[
\lim_{t \to \pm \infty} \left( \| \phi(t) - \phi^+(t) \|_{\dot{B}_{11}^{1,1} \times \dot{B}_{11}^{0,1}} + \| \partial_t \phi(t) - \partial_t \phi^+(t) \|_{\dot{B}_{11}^{0,1}}
\right.
\]

\[
+ \| A(t) - A^+(t) \|_{\dot{B}_{11}^{1,1} \times \dot{B}_{11}^{1,1}} + \| \partial_t A(t) - \partial_t A^+(t) \|_{\dot{B}_{11}^{0,1}} = 0.
\]

Moreover, the scattering operator which maps \((\phi_0, \phi_1, a, \dot{a})\) to \((\phi_0^+, \phi_1^+, a^+, \dot{a}^+)\) is a local diffeomorphism in \(\dot{B}_{11}^{1,1} \times \dot{B}_{11}^{0,1} \times \dot{B}_{11}^{1,1} \times \dot{B}_{11}^{0,1}\).
1.2. Strategy of proof. We follow the approach due to the works of Sterbenz and Wang [24, 27]. That is, we shall define the appropriate function spaces and use bilinear decomposition for angles and estimate the nonlinearity in the function spaces. As the author of [24] mentioned, the estimate of cubic terms is very straightforward. Indeed, we will only use Hölder’s inequality and Strichartz estimates, and hence estimates of bilinear forms in (1.3) will be the crucial part of this paper. To do this, we consider all possible frequency interactions such as High × High and Low × High interactions. By the mercy of an extra weighted regularity in angular variables, we enjoy the improved space-time Strichartz estimates and hence the High × High interaction can be treated rather easier than the Low × High case. (See Section 5.4.)

We remark that in the Low × High interaction, the situation that the low frequency controls the angular regularity becomes more difficult case, since we cannot exploit the angular concentration estimate. (See Remark 5.1.)

Organisation. The rest of this paper is organised as follows. In Section 2, we recall the Strichartz estimates and reveal null structure of $A^\mu \partial_\mu \phi$. We construct the function spaces via the Littlewood-Paley projections in Section 3. We introduce the angular decompositions of bilinear form in Section 4. Then Section 5 will be the main part of this paper, devoted to the proof of our main result.

Notations. As usual different positive constants, which are independent of dyadic numbers $\mu, \lambda$, and $d$ are denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$, i.e., $\frac{1}{C}B \leq A \leq CB$ for some absolute constant $C$. We also use the notation $A \ll B$ if $A \leq \frac{1}{C}B$ for some large constant $C$. Thus for quantities $A$ and $B$, we can consider three cases: $A \approx B$, $A \ll B$ and $A \gg B$. In fact, $A \lesssim B$ means that $A \approx B$ or $A \ll B$. We shall use the notation $A \pm$ which means that for small positive $\epsilon > 0$, we may replace $A \pm$ by $A \pm \epsilon$. For example, we shall write the improved Strichartz estimates with additional angular regularity as

$$\|e^{\mp i|\tau|}f_1\|_{L_t^\infty L_x^2} \lesssim \|\Omega^\frac{1}{2} f_1\|_{L_x^2}.$$  

The spatial and space-time Fourier transforms are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^4} e^{-ix \cdot \xi} f(x) \, dx,$$

$$\hat{u}(\tau, \xi) = \int_{\mathbb{R}^{1+4}} e^{-i(t \tau + x \cdot \xi)} u(t, x) \, dt \, dx.$$  

We also write $F_x(f) = \hat{f}$ and $F_{t,x}(u) = \hat{u}$. We denote the backward and forward wave propagation of a function $f$ on $\mathbb{R}^4$ by

$$e^{\mp i|\tau|} f = \int_{\mathbb{R}^4} e^{ix \cdot \xi} e^{\mp i|\tau|} \hat{f}(\xi) \, d\xi.$$  

The notation $V$ denotes the parametrix for the inhomogeneous wave equation with zero initial data, that is, $u = Vf$ if and only if

$$\Box u = f, \quad u(0, \cdot) = 0, \quad \partial_t u(0, \cdot) = 0.$$  

Let $E$ be any fundamental solution to the homogeneous wave equation, that is, $\Box E = \delta$, where $\delta$ is the Dirac delta distribution. Then we can represent the parametrix operator $V$ via the following
formula:

$$V(f) = E * f - W(E * f),$$

where, for any smooth well-defined function $g = g(t, x)$, $W(g)$ denotes the solution of linear homogeneous wave equation with initial data $(g, \partial_t g)$.

Finally, we recall some basic facts from harmonic analysis on the sphere. The most of the ingredients here can be found in [24]. We also refer the readers to [21] for the systematic introduction to the analysis on the sphere. We denote the standard infinitesimal generators of the rotations on $\mathbb{R}^4$ by $\Omega_{ij} = x_i \partial_j - x_j \partial_i$. Then the Laplace-Beltrami operator $\Delta_{S^3}$ can be written as

$$\Delta_{S^3} = \sum_{i<j} \Omega_{ij}^2.$$  

We define the fractional order of the spherical Laplacian

$$|\Omega|^\sigma = (-\Delta_{S^3})^{\frac{\sigma}{2}}.$$  

We also use the inhomogeneous form of $\Delta_{S^3}$:

$$\langle \Omega \rangle^\sigma f = f_0 + |\Omega|^\sigma f,$$

where $f_0$ is the radial part of $f$, given by

$$f_0(r) = \frac{1}{|S^3|} \int_{S^3} f(r\omega) \, d\omega.$$  

An important fact of the operators $\langle \Omega \rangle^\sigma$ is that they are commutative with the Fourier transform:

$$\mathcal{F}[\langle \Omega \rangle^\sigma f] = \langle \Omega \rangle^\sigma \mathcal{F}(f).$$  

Then the homogeneous Besov space with additional angular regularity is defined as

$$\|f\|_{\dot{B}^s_{\infty,1}} := \|\langle \Omega \rangle^\sigma f\|_{B^s_{\infty,1}},$$

where $\|f\|_{B^s_{\infty,1}} = \sum_{\lambda \in \mathbb{Z}^4} \lambda^s \|P_\lambda f\|_{L^2}$, and $P_\lambda$ is the Littlewood-Paley projection onto the set $\{\xi \in \mathbb{R}^4 : |\xi| \approx \lambda\}$.

**Remark 1.3.** Strictly speaking, the following Leibniz rule is not true:

$$\langle \Omega \rangle (fg) = (\langle \Omega \rangle f)g + f(\langle \Omega \rangle g).$$

This is because obviously the operator $|\Omega|$ is a non-local operator. On the other hand, $\Omega_{ij}$ is clearly a local operator and the Leibniz rule holds. However, for convenience, by abuse of notation we replace any instance of an single $\Omega_{ij}$ with the operator $\langle \Omega \rangle$ and assume the above Leibniz rule for $\langle \Omega \rangle$ is true. See also [24, 27].

2. **Strichartz estimates and Null structure**

2.1. **Strichartz estimates.** We first introduce the classical Strichartz estimates for the homogeneous wave equations, based solely on translation invariant derivatives of the initial data. (See also [24, 14].)
Proposition 2.1. Let \( n = 4 \) be the number of spatial dimensions, and let \( \sigma = \frac{3}{2} \) be the corresponding Strichartz admissibility exponent. If \( f \) is any function of the spatial variable only, denote by \( f_1 = P_1 f \) its unit frequency projection. Then one has the following family estimates for \( 2 \leq q \):

\[
\|e^{\pm it|\nabla|} f_1\|_{L_t^q L_x^r} \lesssim \|f_1\|_{L_x^2},
\]

where \( \frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2} \).

In sequel, we only use a few selected subset of admissible pair \((q, r)\), namely,

\[(2.2) \quad \|e^{\pm it|\nabla|} f_1\|_{L_t^\infty L_x^2} \lesssim \|f_1\|_{L_x^2},
\]

\[(2.3) \quad \|e^{\pm it|\nabla|} f_1\|_{L_t^2 L_x^\infty} \lesssim \|f_1\|_{L_x^2},
\]

\[(2.4) \quad \|e^{\pm it|\nabla|} f_1\|_{L_t^2 L_x^6} \lesssim \|f_1\|_{L_x^2}.
\]

The sharpness of the above space-time estimates is proven by Knapp type counterexample, which does not have radial symmetry. Hence it is natural to expect that the Strichartz estimates would be improved if one impose spherical symmetry. We refer to [23, 3] for the details and other differential operators.

Proposition 2.2. Let \( n = 4 \) be the number of spatial dimensions, and let \( \sigma_\Omega = 3 \) denote the four dimensional angular Strichartz admissible exponent. Let \( f_1 \) be a unit frequency function of the spatial variable only. Then for indices \((q, r)\) such that \( \frac{1}{q} + \frac{\sigma}{r} \geq \frac{\sigma}{2} \) and \( \frac{1}{q} + \frac{\sigma_\Omega}{r} < \frac{\sigma_\Omega}{2} \), and for every \( 0 < \epsilon \), there is a \( C_\epsilon \) which depends only on \( \epsilon \) such that the following estimates hold:

\[
\|e^{\pm it|\nabla|} f_1\|_{L_t^{\mathcal{Q}} L_x^s} \lesssim C_\epsilon \|\langle \Omega \rangle^s f_1\|_{L_x^2},
\]

where \( s = (1 + \epsilon)(\frac{3}{r} + \frac{2}{q} - \frac{3}{2}) \).

In particular, we shall use often the following space-time estimate:

\[
\|e^{\pm it|\nabla|} f_1\|_{L_t^2 L_x^{3+}} \lesssim \|(\Omega)^{\frac{s}{2}} f_1\|_{L_x^2}.
\]

2.2. Null form of \( A^\mu \partial_\mu \phi \). In this section, we derive the null form in \( A^\mu \partial_\mu \phi \). See also [16]. We begin with the standard \( Q \)-type null form introduced in [7]:

\[
Q_{0j}(u, v) = \partial_t u \partial_j v - \partial_j u \partial_t v,
Q_{jk}(u, v) = \partial_j u \partial_k v - \partial_k u \partial_j v.
\]

We decompose the spatial part \( A \) of the gauge potential \( A \) into the divergence-free and curl-free parts

\[
A = A^{df} + A^{cf},
\]

where

\[
A^{df}_j := R^k(R_j A_k - R_k A_j), \quad A^{cf}_j := -R^k R_k A^k.
\]
Here \( R_j = |\nabla|^{-1} \partial_j \) is the Riesz transform. Then we see that
\[
A^\mu \partial_{\mu} \phi = A^0 \partial_t \phi + A \cdot \nabla \phi \\
= -A_0 \partial_t \phi + A^c_1 \cdot \nabla \phi + A^{df} \cdot \nabla \phi \\
=: N_1 + N_2.
\]
By the Lorenz gauge condition \( \partial_t A_0 = \partial^i A_j \), we get
\[
N_1 = -A_0 \partial_t \phi - R_j R_k A^k \partial^i \phi \\
= -A_0 \partial_t \phi - (|\nabla|^{-1} \partial_j A_0) \cdot \nabla \phi \\
= \partial_j (|\nabla|^{-1} R^j A_0) \partial_t \phi - \partial_t (|\nabla|^{-1} R_j A_0) \partial^i \phi \\
= -Q_{0j} (|\nabla|^{-1} R^j A_0, \phi).
\]
We also have
\[
N_2 = R^k (R_j A_k - R_k A_j) \partial^i \phi \\
= (|\nabla|^{-2} \partial^k \partial_j A_k \partial^i \phi + A_j \partial^i \phi \\
= -\frac{1}{2} (|\nabla|^{-2} (\partial_j \partial^i A_k - \partial_j \partial_k A^i)) \partial^i \phi - |\nabla|^{-2} (\partial^k \partial_j A_k - \partial^k \partial_k A_j) \partial^i \phi \\
= -\frac{1}{2} (\partial_j (|\nabla|^{-1} R^j A_k - R_k A^i) \partial^i \phi - \partial^k (|\nabla|^{-1} (R_j A_k - R_k A_j) \partial^j \phi \\
= -\frac{1}{2} Q_{jk} (|\nabla|^{-1} (R^j A^k - R^k A^j), \phi).
\]
We shall denote the Fourier symbols of \( Q_{0j} \) and \( Q_{jk} \) by \( q_{0j}(\xi, \eta) \) and \( q_{jk}(\xi, \eta) \), respectively. Then the symbols satisfy the following estimates:
\[
|q_{0j}(\xi, \eta)|, |q_{jk}(\xi, \eta)| \lesssim |\xi| |\eta| \angle(\xi, \eta),
\]
where \( \angle(\xi, \eta) = \arccos(\frac{\xi \cdot \eta}{|\xi||\eta|}) \) is the angle between \( \xi \) and \( \eta \). (See [20].)
The above notations seem too complicated. However, an important point here is that the bilinear form \( A^\mu \partial_{\mu} \phi \) contributes an additional angle between two input frequencies. In other words, it is not necessary to distinguish the null forms \( Q_{0j} \) and \( Q_{jk} \). In the sequel, we will by abuse of notation simply write \( Q(\varphi, \phi) \) for the bilinear form \( A^\mu \partial_{\mu} \phi \). That is, we even ignore the vector components of \( A^\mu \) and denote it by \( \varphi \) shortly.

3. Function spaces

This section is devoted to the introduction of preliminary setup to be used in the proof of Theorem 1.1. We shall establish the function spaces in this section. The readers can find the following notations in [24, 27].

3.1. Multipliers. Let \( \beta \) be a smooth bump function given by \( \beta(s) = 1 \) for \( |s| \leq 1 \) and \( \beta(s) = 0 \) for \( |s| \geq 2 \). We define the dyadic scaling of \( \beta \) by \( \beta_\lambda(s) = \beta(\lambda^{-1} s) \) for \( \lambda \in 2^\mathbb{Z} \). Now we define the Fourier multipliers which give localisation with respect to the spatial frequency, space-time frequency, distance to the cone (modulation), distance to the lower cone and the upper cone as
Then we have
\[ p_{\lambda}(\xi) = \beta_{2\lambda}(|\xi|) - \beta_{\lambda/2}(|\xi|), \quad s_{\lambda}(\tau, \xi) = \beta_{2\lambda}((\tau, \xi)) - \beta_{\lambda/2}((\tau, \xi)), \]
(3.2) \[ c_{d}(\tau, \xi) = \beta_{2d}(|\tau| - |\xi|) - \beta_{d/2}(|\tau| - |\xi|), \]
(3.3) \[ c_{d}^{+}(\tau, \xi) = \beta_{2d}(\tau + |\xi|) - \beta_{d/2}(\tau + |\xi|), \quad c_{d}^{-}(\tau, \xi) = \beta_{2d}(\tau - |\xi|) - \beta_{d/2}(\tau - |\xi|). \]

Then we define the corresponding Fourier projection operators. For example, \( F_{x}(P_{\lambda} f) = p_{\lambda} F_{x}(f) \) and \( F_{t,x}(\lambda_{d}, u) = s_{\lambda} F_{t,x}(u) \). We also write \( S_{\lambda,d} = S_{\lambda} C_{d} \), \( S_{\lambda,d}^{\pm} = S_{\lambda}^{\pm} C_{d}^{\pm} \), and denote
\[ S_{\lambda,\leq d} = \sum_{d \leq d} S_{\lambda,d}, \quad S_{\lambda,> d}^{\pm} = \sum_{d > d} S_{\lambda,> d}^{\pm}, \]
and the projection \( S_{\lambda,d} \) is defined in the obvious way.

For any given \( 0 < \eta \leq 1 \), we define \( C_{\eta} \) to be a collection of finitely overlapping caps of size \( \eta \) in the unit sphere \( S^{3} \) in \( \mathbb{R}^{4} \). Then we denote a smooth partition of unity subordinate to the angular sectors \( \{ \xi \neq 0 : \frac{1}{|\xi|} \in \omega \} \) by \( \{ b_{\eta}^{\omega} \}_{\omega \subset C_{\eta}} \). The corresponding angular localisation operator is denoted by \( B_{\eta}^{\omega} \). For simplicity, we allow the abuse of notation and identify the angular sectors \( \omega \subset C_{\eta} \) and their centre. We define
\[ S_{\lambda,d}^{\pm} = B_{\eta}^{\omega} \frac{1}{(4\pi)^{\frac{d}{2}}} P_{\lambda} S_{\lambda,d}, \quad S_{\lambda,> d}^{\pm} = B_{\eta}^{\omega} \frac{1}{(4\pi)^{\frac{d}{2}}} P_{\lambda} S_{\lambda,> d}. \]

We have the following lemma concerning boundedness of the above localisation operators. (See also \([24, 27]\).)

**Lemma 3.1.**

(1) The following multipliers are given by \( L^{1}_{\Lambda} L^{1}_{x} \) kernels and are uniformly bounded in \( L^{1}_{\Lambda} L^{1}_{x} \), \( \lambda^{-1} \nabla S_{\lambda}, B_{\frac{1}{(4\pi)^{\frac{d}{2}}} P_{\lambda}, S_{\lambda,> d}^{\pm}} \), \( (\lambda d) V S_{\lambda,d}^{\omega} \), and also those operators are bounded in mixed Lebesgue spaces \( L^{q}_{\Lambda} L^{1}_{x} \).

(2) The following multipliers are uniformly bounded in the \( L^{q}_{\Lambda} L^{2}_{x} \) spaces for \( 1 \leq q \leq +\infty \):
\[ S_{\lambda,d}, S_{\lambda,> d}. \]

### 3.2. Function spaces.

Now we establish the function spaces with space-time frequency localised in the support of \( s_{\lambda}(\tau, \xi) \). We let
\[ \| u \|_{\lambda,\leq d}^{p} := \sum_{d \leq d} d^{\frac{p}{2}} \| S_{\lambda,d} u \|_{L^{p}_{\lambda} L^{2}_{x}}, \]
and
\[ \| u \|_{Y_{\lambda}} := \lambda^{-1} \| S_{\lambda} u \|_{L^{1}_{\Lambda} L^{2}_{x}}. \]

We also define the space \( Z_{\Omega} \) by
\[ \| f \|_{Z_{\Omega}^{{\Lambda}}} := \lambda^{-1} \sum_{d \leq \lambda} \int_{\Omega} \| S_{\lambda,d} f \|_{L^{\infty}_{x}} dt. \]

Then we have
\[ \| f \|_{Z_{\Omega}^{{\Lambda}}} \leq \lambda^{-1} \| S_{\lambda}(\Omega) f \|_{L^{1}_{\Lambda} L^{2}_{x}} \leq \| S_{\lambda}(\Omega) f \|_{Y_{\lambda}}, \]
and hence \( \langle \Omega \rangle^{-1} Y_\lambda \subset Z_{\Omega, \lambda} \). We also have

\[
\text{(3.5)} \quad \sup_{d \leq \lambda} \lambda^{-1} \left\| \sup_{\omega} \left\| S_{X, d, f} \right\|_{L_1^2} \right\|_{L_1^\infty} \lesssim \| f \|_{F_{\Omega, \lambda}}.
\]

Finally, we define the function space

\[
F_{\Omega, \lambda} = \langle \Omega \rangle^{-1} (X_{\lambda}^{1, 1} + Y_\lambda) \cap S_\lambda (L_1^\infty L_2^2) \cap Z_{\Omega, \lambda}.
\]

As \( X_{\lambda}^{1, 1} \subset S_\lambda (L_1^\infty L_2^2) \), naturally we have \( \Box X_{\lambda}^{1, 1} \subset \Box S_\lambda (L_1^\infty L_2^2) \), and by duality, we obtain

\[
\text{(3.6)} \quad \lambda, d S_{\lambda}(L_1^1 L_2^2) \subset (\Box X_{\lambda}^{1, 1})' = (\lambda X_{\lambda}^{-1, 1})' = \lambda^{-1} X_{\lambda}^{1, \infty},
\]

and recall an obvious relation \( X_{\lambda}^{1, 1} \subset X_{\lambda}^{1, \infty} \), hence we have

\[
\text{(3.7)} \quad d^{\frac{1}{2}} \| S_{X, d, f} \|_{L_1^2} \lesssim \| (\Omega)^{-1} f \|_{F_{\Omega, \lambda}}, \quad \text{for } d \in 2^{\mathbb{Z}}, \quad 0 < d \leq \lambda,
\]

\[
\text{(3.8)} \quad d^{\frac{1}{2}} \| S_{X, d \leq f} \|_{L_1^2} \lesssim \| (\Omega)^{-1} f \|_{F_{\Omega, \lambda}}.
\]

Now we define the Besov type function space, which we will iterate to prove our main theorem:

\[
\text{(3.9)} \quad \| u \|_{F_{\Omega}} := \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda \| S_{\lambda} u \|_{F_{\Omega, \lambda}}.
\]

We list several space-time estimates to be used throughout this paper.

\[
\text{(3.10)} \quad \| S_{\mu} (\Omega) u \|_{L_1^p L_2^2} \lesssim \| u \|_{F_{\Omega, \mu}},
\]

\[
\text{(3.11)} \quad \| S_{\mu} (\Omega) u \|_{L_1^2 L_2^p} \lesssim \| u \|_{F_{\Omega, \mu}},
\]

\[
\text{(3.12)} \quad \| S_{\mu} (\Omega) u \|_{L_1^2 L_2^{\infty}} \lesssim \mu^\frac{1}{2} \| u \|_{F_{\Omega, \mu}},
\]

\[
\text{(3.13)} \quad \| S_{\mu} (\Omega) u \|_{L_1^1 L_2^3} \lesssim \mu^{\frac{1}{2} +} \| u \|_{F_{\Omega, \mu}}.
\]

We end this section with angular concentration estimates and Sobolev embedding estimates.

**Lemma 3.2** (Lemma 5.2 of [24]). Let \( 2 \leq n \) be a given integer. Then for any test function \( u \) on \( \mathbb{R}^n \), and any \( 2 \leq p < \infty \), one has the following estimate:

\[
\sup_{\omega} \| B_{\eta}^{\omega} u \|_{L_p} \lesssim \eta^s \| (\Omega)^s u \|_{L_p},
\]

where \( 0 \leq s < \frac{n - 1}{p} \).

**Lemma 3.3.** Let \( f \) be a test function on \( \mathbb{R}^4 \). Then one has the following frequency localised estimate:

\[
\text{(3.14)} \quad \| B_{\eta}^{\omega} P_{\lambda} f \|_{L_p^\infty} \lesssim \eta^{\lambda (\frac{1}{2} - \frac{1}{p})} \| f \|_{L_p^\infty},
\]

and also by scaling argument, one has

\[
\text{(3.15)} \quad \| B_{\eta}^{\omega} P_{\lambda} f \|_{L_p^\infty} \lesssim \eta^{\lambda (\frac{1}{2} - \frac{1}{p})} \lambda^{\frac{1}{2} (\frac{1}{2} - \frac{1}{p})} \| f \|_{L_p^\infty}.
\]
We observe that in Lemma 3.3, the function $f$ is not required to be localised in an angular sector. Thus we can use Lemma 3.3 first and then apply Lemma 3.2. For example, we write

\[
\sup_{\omega} \| S_{\mu,d} u \|_{L^\infty} \lesssim \mu \left( \frac{d}{\mu} \right)^{1-} \sup_{\omega} \| S_{\mu,d} u \|_{L^2} \lesssim \mu \left( \frac{d}{\mu} \right)^{1-} \| S_{\mu,d} u \|_{L^2} \lesssim \mu \left( \frac{d}{\mu} \right)^{1-} \| S_{\mu,d} u \|_{L^2}.
\]

4. Bilinear decompositions for angles

We shall discuss various bilinear decompositions for frequency localised products of the form:

\[
(4.1) \quad S_{\lambda_0}(S_{\lambda_1} u S_{\lambda_2} v).
\]

By the standard Littlewood-Paley trichotomy, the localised product (4.1) vanishes unless

\[
\min\{\lambda_0, \lambda_1, \lambda_2\} \lesssim \text{med}\{\lambda_0, \lambda_1, \lambda_2\} \approx \max\{\lambda_0, \lambda_1, \lambda_2\}.
\]

We focus on two important interactions in (4.1), namely, the High×High and Low×High frequency interactions:

\[
(4.2) \quad \lambda_0 \lesssim \lambda_1 \approx \lambda_2,
\]

\[
(4.3) \quad \lambda_1 \lesssim \lambda_0 \approx \lambda_2.
\]

In what follows, we give the bilinear decompositions for angles for the type (4.2), (4.3). We refer the reader to [22, 24, 27] for more details.

Lemma 4.1 (Lemma 6.1 of [24]). For the following localised products of the form:

\[
S_{\mu,d}(S_{\lambda_1} u S_{\lambda_2} v),
\]

one has the following angular decomposition:

\[
S_{\mu,d}(S_{\lambda_1} u S_{\lambda_2} v) = \sum_{\omega_1, \omega_2, \omega_3} S_{\mu,d}^\pm(S_{\lambda_1} u S_{\lambda_2} v) * b_{\omega_1}^\pm b_{\omega_2}^\pm b_{\omega_3}^\pm,
\]

for the convolution of the associated cutoff functions in Fourier side. There is a similar decomposition for the terms $S_{\mu,d}(S_{\lambda_1} u S_{\lambda_2} v)$ and $S_{\mu,d}(S_{\lambda_1} u S_{\lambda_2} v)$, where $d$ is in the range $d < c\mu$ and $c \ll 1$ is the small number fixed above.

We also have the angular decomposition for the interaction of type (4.3).

Lemma 4.2 (Lemma 6.2 of [24]). For the following localised products of the form:

\[
S_{\lambda_1} u S_{\lambda_2} v,(S_{\mu,d} u S_{\lambda_1} u S_{\lambda_2} v),
\]
one has the following angular decomposition:
\[
s_{\lambda, \leq \min \{d, c\mu\}}^{\pm}(s_{\mu,d}^{\pm} * s_{\lambda, \leq \min \{d, c\mu\}}^{\pm}) = \sum_{\omega_1, \omega_2, \omega_3} b_{\omega_1}^{\omega_2} B_{\omega_3}^{\omega_2} s_{\lambda, \leq \min \{d, c\mu\}}^{\pm}(s_{\mu,d}^{\pm} * b_{\omega_3}^{\omega_2} s_{\lambda, \leq \min \{d, c\mu\}}^{\pm}),
\]
for the convolution of the associated cutoff functions in Fourier side. There is a similar decomposition for the terms \(S_{\lambda, \leq d}(S_{\mu, \leq d} u S_{\lambda, \leq d} v)\) and \(S_{\lambda, d}(S_{\mu, \leq d} u S_{\lambda, \leq d} v)\) in the range \(d < c\mu\), where \(c \ll 1\) is a fixed small number.

We note that the angular sectors involved in the summation on the right-hand side of the decompositions in Lemma 4.1 and Lemma 4.2 are essentially diagonal. To avoid verbatim, we write angular decompositions throughout this paper as follows:
\[
S_{\mu,d}(S_{\lambda, \leq \min \{d, c\mu\}} \varphi S_{\lambda, \leq \min \{d, c\mu\}} \phi)
= \sum_{\omega} S_{\mu,d}^{\omega}(B_{\omega}^{\omega} S_{\lambda, \leq \min \{d, c\mu\}} \varphi S_{\lambda, \leq \min \{d, c\mu\}} \phi),
\]
(4.4)
\[
S_{\lambda, \leq \min \{d, c\mu\}}(S_{\mu,d} \varphi S_{\lambda, \leq \min \{d, c\mu\}} \phi)
= \sum_{\omega} B_{\omega}^{\omega} S_{\lambda, \leq \min \{d, c\mu\}}(S_{\mu,d} \varphi S_{\lambda, \leq \min \{d, c\mu\}} \phi).
\]
(4.5)
We also need the following angular decomposition, which will be only used in Section 5.3.

Lemma 4.3 (Lemma 6.4 of [24]). For the following expression:
\[
S_{\lambda,d}^{\omega}(S_{\mu} u S_{\lambda, \leq c\mu} v),
\]
one has the following angular restriction:
\[
s_{\lambda,d}^{\omega, \pm}(s_{\mu}^{\pm} * s_{\lambda, \leq c\mu}^{\pm}) = s_{\lambda,d}^{\omega, \pm}(s_{\mu}^{\pm} * b_{\omega}^{\omega} s_{\lambda, \leq c\mu}^{\pm})
\]
for the convolution of the associated cutoff functions in Fourier space. Here the angles are restricted to the range \(|\omega_1 - \omega_3| \approx \left(\frac{d}{c\lambda}\right)\).

5. PROOF OF MAIN THEOREM

This section is devoted to the proof of our main result. The proof of scattering is followed by Theorem 1.1. See [22][24][27] for details. Now we focus on the proof of Theorem 1.1. In view of Duhame’s principle and Picard’s iteration, we need to show the following nonlinear estimates:
\[
\|VQ(\varphi, \phi)\|_{F_t} \lesssim \|\varphi\|_{F_t} \|\phi\|_{F_t},
\]
(5.1)
\[
\|V(\phi_1 \partial \phi_2)\|_{F_t} \lesssim \|\phi_1\|_{F_t} \|\phi_2\|_{F_t},
\]
(5.2)
\[
\|V(\phi \varphi \psi)\|_{F_t} \lesssim \|\phi\|_{F_t} \|\varphi\|_{F_t} \|\psi\|_{F_t}.
\]
(5.3)
The bilinear estimates [22] is already known. (See [24,]) The treatment of cubic terms is very straightforward. It suffices to show that
\[
\|S_{\mu}(S_1 u S_1 v S_1 w)\|_{L_t^1 L_{x}^2} \lesssim \mu \|S_{\mu} u\|_{F_t} \|S_{\mu} v\|_{F_t} \|S_{\mu} w\|_{F_t},
\]
(5.4)
\[
\|S_1(S_{\mu}(S_1 u S_1 v S_1 w))\|_{L_t^1 L_{x}^2} \lesssim \mu \|S_{\mu} u\|_{F_t} \|S_{\mu} v\|_{F_t} \|S_{\mu} w\|_{F_t}.
\]
(5.5)
We simply use the Bernstein’s inequality, Hölder’s inequality and the Strichartz estimates. Indeed, for the proof of (5.4), we see that
\[ \|S_\mu(S_1(\Omega)uS_1vS_1w)\|_{L^1_tL^2_x} \lesssim \mu^{-2} \|S_\mu(S_1(\Omega)uS_1vS_1w)\|_{L^1_tL^2_x}^+ \]
\[ \lesssim \mu^{-2} \|S_1(\Omega)u\|_{L^\infty_tL^2_x} \|S_1v\|_{L^2_tL^\infty_x} \|S_1w\|_{L^2_tL^\infty_x} \]
\[ \lesssim \|S_1u\|_{F_{0,1}} \|S_1v\|_{F_{0,1}} \|S_1w\|_{F_{0,1}}. \]

To prove (5.5), we write
\[ |S_1(S_\mu(\Omega)uS_1vS_1w)|_{L^1_tL^2_x} \lesssim \|S_\mu(\Omega)u\|_{L^2_tL^\infty_x} \|S_1v\|_{L^2_tL^\infty_x} \|S_1w\|_{L^2_tL^\infty_x} \]
\[ \lesssim \mu^{\frac{3}{2}} \|S_\mu(\Omega)u\|_{L^2_tL^\infty_x} \|S_1v\|_{L^2_tL^\infty_x} \|S_1w\|_{L^2_tL^\infty_x} \]
\[ \lesssim \mu^{\frac{3}{2}} \|S_\mu u\|_{F_{0,\mu}} \|S_1v\|_{F_{0,1}} \|S_1w\|_{F_{0,1}}, \]
where we used \( \|S_\mu(\Omega)u\|_{L^2_tL^\infty_x} \lesssim \mu^\frac{3}{2} \|u\|_{F_{0,\mu}}. \)

From now on we exclusively consider the bilinear estimates (5.1). We first apply the dyadic decomposition on the space-time frequency of the bilinear form \( Q(\varphi, \phi). \) It suffices to treat the High×High and Low×High interactions as follows:
\[ \sum_{\mu \leq \max\{\lambda_1, \lambda_2\}} \mu \|VQ(S_{\lambda_1}(\varphi, S_{\lambda_2}(\phi))\|_{F_{0,\mu}} \lesssim \lambda_1 \lambda_2 \|\varphi\|_{F_{0,\lambda_1}} \|\phi\|_{F_{0,\lambda_2}}, \]
\[ \|VQ(S_{\mu}(\varphi, S_{\lambda}(\phi))\|_{F_{0,\lambda}} \lesssim \mu \|\varphi\|_{F_{0,\mu}} \|\phi\|_{F_{0,\lambda}}. \]

As we are concerned with a scale-invariant function space, it is reasonable to assume that the high frequency \( \lambda = \lambda_1 = \lambda_2 = 1 \) and the low frequency \( \mu \lesssim 1. \) In consequence, our aim is to prove the following bilinear estimates:
\[ \mu \|VQ(S_1(\varphi, S_1(\phi))\|_{F_{0,\mu}} \lesssim \mu^0 \|S_1(\phi)\|_{F_{0,1}} \|S_1(\phi)\|_{F_{0,1}}, \]
\[ \|VQ(S_{\mu}(\varphi, S_1(\phi))\|_{F_{0,1}} \lesssim \mu \|S_{\mu}(\phi)\|_{F_{0,\mu}} \|S_1(\phi)\|_{F_{0,1}}. \]

We present the main scheme of the proof of (5.6) and (5.7). We first note that the Fourier projection operator \( S_\mu \) and the parametrix \( V \) do not commute when \( \mu \lesssim 1. \) Indeed, we have (see (24) for derivation)
\[ S_\mu VQ(S_1(\varphi, S_1(\phi)) = S_\mu V S_\mu Q(S_1(\varphi, S_1(\phi)) - \sum_{\mu \lesssim \sigma \lesssim 1} W(P_{\mu}S_{\sigma}VQ(S_1(\varphi, S_1(\phi)). \]

For the second term of the right-hand side of (5.8), we shall prove the following:
\[ \sum_{\mu} \sum_{\mu \lesssim \sigma \lesssim 1} \|P_{\mu}S_{\sigma}VQ(\Omega)(S_1(\varphi, S_1(\phi)||_{L^\infty_tL^2_x} \lesssim \|S_1(\phi)\|_{F_{0,1}} \|S_1(\phi)\|_{F_{0,1}}, \]
which is very straightforward. Then we further decompose \( S_{\mu}Q(S_1(\varphi, S_1(\phi)) \) into the space-time frequency which is away from the light cone and near the cone, respectively. For this purpose, we write
\[ S_{\mu}Q(S_1(\varphi, S_1(\phi)) = S_{\mu}Q(S_1(\varphi, S_{1,\leq \phi} + S_{\mu}Q(S_{1,\leq \phi}, S_{1,\leq \phi}) \]
\[ + S_{\mu}Q(S_{1,\leq \phi} + S_{1,\leq \phi} + S_{\mu}Q(S_{1,\leq \phi} + S_{1,\leq \phi}) \]
\[ =: \mathcal{H}^1 + \mathcal{H}^2 + \mathcal{H}^3. \]
The first and second terms $\mathcal{H}^1, \mathcal{H}^2$ are rather easier than the third term $\mathcal{H}^3$, which is near the cone. We will use the angular decomposition \cite{4.4} to $\mathcal{H}^3$. Then we apply the Hölder inequality, Sobolev embedding, and angular concentration estimates and the Strichartz estimates to obtain the required estimates.

In the proof of \cite{5.1}, we simply get
\[
S_1 V Q(S_1 \varphi, S_1 \phi) = S_1 V S_1 Q(S_1 \varphi, S_1 \phi)
\]
and we write
\[
S_1 Q(S_\mu \varphi, S_1 \phi) = S_1 Q(S_\mu \varphi, S_{1, \mu \leq} \phi) + S_{1, \mu \leq} Q(S_\mu \varphi, S_{1, \mu \leq} \phi)
\]
(5.11)
\[
= : \mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3.
\]
The Low×High interaction is more difficult than the High×High interaction. This is because of the regularity $B^1$ of the scalar field $\varphi$. We will lose $\mu$ from $\varphi$, and hence we must gain more. To overcome this problem, we first observe that the angular decomposition as Lemma \cite{4.2} will be very large when the modulation $d$ is close to the low frequency $\mu$. Thus we shall divide the Low×High interaction into two cases: $(\frac{d}{\mu})^\frac{1}{2} \ll \mu$ and $\mu \lesssim (\frac{d}{\mu})^\frac{1}{2}$. We still use Lemma \cite{4.2} for the first case. In the second case, instead of Lemma \cite{4.2} we use a smaller sector with size $\mu$. In this case, the range of $d$ is given by $\mu^3 \lesssim d \lesssim \mu$ and hence the summation on $d$ makes no problem. We refer the readers to \cite{27} for the change of weight between $\mu$ and $(\frac{d}{\mu})^\frac{1}{2}$.

We introduce the outline of the remainder of this section. We first treat the proof of \cite{5.1} in Section \ref{5.1}. Then the estimates of the High×High and Low×High interaction away from the light cone is given in Section \ref{5.2} and \ref{5.3} which are rather easier than the frequency near the cone. Section \ref{5.4} and \ref{5.5} are devoted to the estimates of the High×High and Low×High interaction near the cone.

5.1. **High×High interaction including Commutator term.** We simply use the Bernstein’s inequality, the boundedness of multipliers (Lemma \cite{3.1}), and Hölder’s inequality.

\[
\sum_{\mu \lesssim 1} \mu \left\| \sum_{\mu \lesssim \sigma \lesssim 1} P_\mu S_{\sigma, \sigma} V Q(S_1 \varphi, S_1 \phi) \right\|_{L^\infty_t L^2_x} \lesssim \sum_{\mu \lesssim 1} \mu \sum_{\mu \lesssim \sigma \lesssim 1} \left\| P_\mu S_{\sigma, \sigma} V Q(S_1 \varphi, S_1 \phi) \right\|_{L^\infty_t L^2_x} \lesssim \sum_{\mu \lesssim 1} \mu \sum_{\mu \lesssim \sigma \lesssim 1} \mu^2 \left\| S_{\sigma, \sigma} V Q(S_1 \varphi, S_1 \phi) \right\|_{L^\infty_t L^1_x} \lesssim \sum_{\mu \lesssim 1} \mu \sum_{\mu \lesssim \sigma \lesssim 1} \left( \frac{\mu}{\sigma} \right)^2 \left\| S_1 \varphi S_1 \phi \right\|_{L^\infty_t L^1_x} \lesssim \sum_{\mu \lesssim 1} \mu \left\| S_1 \varphi \right\|_{L^\infty_t L^2_x} \left\| S_1 \phi \right\|_{L^\infty_t L^2_x} \lesssim \left\| S_1 \varphi \right\|_{F_1} \left\| S_1 \phi \right\|_{F_1}.
\]

5.2. **High×High interaction away from cone.** We recall the definition of the $Y$ space. We first observe that
\[
\sum_{\mu} \left\| S_\mu (S_1 \varphi S_1 \phi) \right\|_{L^1_t L^2_x} \lesssim \sum_{\mu \lesssim 1} \sum_{\sigma \lesssim \mu} \left\| P_\sigma S_\mu (S_1 \varphi S_1 \phi) \right\|_{L^1_t L^2_x}.
\]
We use the Bernstein's inequality to gain $\sigma^{\frac{2}{3}}$. The Hölder inequality and Strichartz estimates give the desired estimates as follows.

\[
\begin{align*}
\mu \| V \mathcal{H}^1 \|_{F_{1.\mu}} & \lesssim \mu \| V \mathcal{H}^1 \|_{(\Omega)^{-1} Y_{\mu}} \\
& \lesssim \sum_{\mu \leq 1} \sum_{\sigma \leq \mu} \sigma^{\frac{2}{3}} \| \langle \Omega \rangle (S_1 \varphi S_{1.c \mu \leq} \phi) \|_{L^1 L^{\frac{2}{3}}} \\
& \lesssim \sum_{\mu \leq 1} \sum_{\sigma \leq \mu} \sigma^{\frac{2}{3}} \| S_1 \langle \Omega \rangle \varphi \|_{L^1 L^2} \| S_{1,c \mu \leq} \langle \Omega \rangle \phi \|_{L^2 L^2} \\
& \lesssim \sum_{\mu \leq 1} \sum_{\sigma \leq \mu} \sigma^{\frac{2}{3}} (c \mu)^{-\frac{1}{2}} \| S_1 \varphi \|_{F_{1.t}} \| S_1 \phi \|_{F_{1.t}} \\
& \lesssim e^{-\frac{1}{3}} \mu^{\frac{1}{3}} \| S_1 \varphi \|_{F_{1.t}} \| S_1 \phi \|_{F_{1.t}},
\end{align*}
\]

where we used $\mu^{\frac{1}{3}} \| S_{1.c \mu \leq} \langle \Omega \rangle u \|_{L^2 L^2} \lesssim \| S_1 u \|_{F_{1.t}}$. The estimate of $\mathcal{H}^2$ is very similar. Indeed,

\[
\mu \| V \mathcal{H}^2 \|_{F_{1.\mu}} \lesssim \sum_{\mu \leq 1} \sum_{\sigma \leq \mu} \| P_0 S_\mu Q \langle \Omega \rangle (S_{1.c \mu \leq} \varphi, S_{1.c \mu \leq} \phi) \|_{L^1 L^2} \\
\lesssim \sum_{\mu \leq 1} \sum_{\sigma \leq \mu} \sigma^{\frac{2}{3}} \| \langle \Omega \rangle (S_{1.c \mu \leq} \varphi S_{1.c \mu \leq} \phi) \|_{L^1 L^2} \\
\lesssim \sum_{\mu \leq 1} \sum_{\sigma \leq \mu} \sigma^{\frac{2}{3}} (c \mu)^{-\frac{1}{2}} \| S_1 \varphi \|_{F_{1.t}} \| S_1 \phi \|_{F_{1.t}} \\
\lesssim e^{-\frac{1}{3}} \mu^{\frac{1}{3}} \| S_1 \varphi \|_{F_{1.t}} \| S_1 \phi \|_{F_{1.t}},
\]

5.3. **Low $\times$ High interaction away from cone.** We first use the Hölder’s inequality with respect to $t$ and then $x$ to get

\[
\| V \mathcal{L}^1 \|_{F_{1.\mu}} \lesssim \| V \mathcal{L}^1 \|_{(\Omega)^{-1} Y_{\mu}} = \| S_1 Q \langle \Omega \rangle (S_\mu \varphi, S_{1.c \mu \leq} \phi) \|_{L^1 L^2} \\
\lesssim \| S_\mu \langle \Omega \rangle \varphi \|_{L^1 L^2} \| S_{1.c \mu \leq} \langle \Omega \rangle \phi \|_{L^2 L^2} \\
\lesssim \mu^{\frac{1}{3}} \| S_\mu \varphi \|_{L^2 L^2} \| S_{1.c \mu \leq} \langle \Omega \rangle \phi \|_{L^2 L^2} \\
\lesssim \mu^{\frac{1}{3}} \mu^{\frac{1}{2}} \| S_\mu \varphi \|_{F_{1.\mu}} \| S_{1.c \mu \leq} \langle \Omega \rangle \phi \|_{F_{1.t}} \\
\lesssim e^{-\frac{1}{3}} \mu^{\frac{1}{3}} \| S_\mu \varphi \|_{F_{1.\mu}} \| S_{1.c \mu \leq} \langle \Omega \rangle \phi \|_{F_{1.t}}.
\]

Here we also used $\| S_\mu \langle \Omega \rangle \varphi \|_{L^2 L^2} \lesssim \mu^{\frac{1}{3}} \| S_\mu \varphi \|_{F_{1.\mu}}$. The estimate of $\mathcal{L}^2$ in the $X$ space is quite similar.

\[
\| V \mathcal{L}^2 \|_{F_{1.\mu}} \lesssim \| V \mathcal{L}^2 \|_{(\Omega)^{-1} X^*_{\mu}} \lesssim \sum_{c \mu \leq d} d^{-\frac{1}{2}} \| S_{1.d} Q \langle \Omega \rangle (S_\mu \varphi, S_{1.c \mu \leq} \phi) \|_{L^1 L^2} \\
\lesssim (c \mu)^{-\frac{1}{2}} \| S_\mu \langle \Omega \rangle \varphi \|_{L^2 L^2} \| S_{1.c \mu \leq} \langle \Omega \rangle \phi \|_{L^2 L^2} \\
\lesssim e^{-\frac{1}{3}} \mu^{\frac{1}{3}} \| S_\mu \varphi \|_{F_{1.\mu}} \| S_{1.c \mu \leq} \langle \Omega \rangle \phi \|_{F_{1.t}}.
\]
We need to estimate $\mathcal{L}^2$ in the $Z$ space. To do this, we apply Lemma 3.1 Hölder inequality with respect to $t$ and the Bernstein’s inequality and then Lemma 3.2.

$$
\|V \mathcal{L}^2\|_{Z_{\alpha,1}} = \sum_{\epsilon \mu < d} d^{-1} \int \sup_{\omega} \|B_{d}^{\epsilon} S_{1,d} Q(S_{\mu}, S_{1, \leq \epsilon \mu})\|_{L^\infty_t} dt
$$

$$
\lesssim \sum_{\epsilon \mu < d} d^{-1} \int \sup_{\omega_1, \omega_2} \|B_{d}^{\epsilon} S_{1,d} Q(S_{\mu}, B_{(d/2)}^{\epsilon} S_{1, \leq \epsilon \mu})\|_{L^\infty_t} dt
$$

$$
\lesssim \sum_{\epsilon \mu < d} d^{-1} \|S_{\mu} \phi\|_{L^6_t L^\infty_x} \left( \sup_{\omega} \|B_{(d/2)}^{\epsilon} S_{1, \leq \epsilon \mu} \phi\|_{L^2_t} \right)
$$

$$
\lesssim \sum_{\epsilon \mu < d} d^{-1} \|S_{\mu} \phi\|_{L^6_t L^\infty_x} \|S_{1}(\Omega) \phi\|_{L^2_t L^2_x}
$$

$$
\lesssim \mu^{\frac{1}{12}} \|S_{\mu} \phi\|_{F_{\mu,1}} \|S_{1} \phi\|_{F_{1}}.
$$

5.4. **High×High interaction near cone.** We further decompose $\mathcal{H}^3$ as follows:

$$
S_{\mu} Q(S_{1, \leq \epsilon \mu} \varphi, S_{1, \leq \epsilon \mu} \phi) = \sum_{d < \mu} S_{\mu, \leq d} Q(S_{1, \leq d} \varphi, S_{1} d \phi) + \sum_{d < \epsilon \mu} S_{\mu, \leq d} Q(S_{1, d} \varphi, S_{1, \leq \epsilon \mu} \phi)
$$

$$
+ \sum_{\epsilon \mu \leq d} S_{\mu, d} Q(S_{1, \leq \min(d, \epsilon \mu)} \varphi, S_{1, \leq \min(d, \epsilon \mu)} \phi)
$$

$$
=: \mathcal{H}^3_{1} + \mathcal{H}^3_{2} + \mathcal{H}^3_{3}.
$$

We estimate the term $\mathcal{H}^3_{1}$ in the $Y$ space. We apply in order the angular decomposition (Lemma 1.1), Sobolev embedding estimate (Lemma 3.3), Hölder inequality, and then Strichartz estimates.

$$
\mu \|V \mathcal{H}^3_{1}\|_{Y_{\alpha,1} Y_{\mu}} \lesssim \sum_{d < \epsilon \mu} \left( \sum_{\omega} \|S_{\mu, \leq d} Q(B_{(d/2)}^{\epsilon} S_{1, \leq d} \varphi, B_{(d/2)}^{\epsilon} S_{1} d \phi)\|_{L^2_t}^2 \right)^{\frac{1}{2}}
$$

$$
\lesssim \sum_{d < \epsilon \mu} \mu^{\frac{1}{12}} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \left( \sum_{\omega} \|Q(B_{(d/2)}^{\epsilon} S_{1, \leq d} \varphi, B_{(d/2)}^{\epsilon} S_{1} d \phi)\|_{L^2_t}^2 \right)^{\frac{1}{2}}
$$

$$
\lesssim \sum_{d < \epsilon \mu} \mu^{-\frac{1}{12}} \|S_{1, \leq d} \varphi\|_{L^6_t L^\infty_x} \|S_{1, \leq d} \phi\|_{L^2_t L^2_x}
$$

$$
\lesssim \mu^{\frac{1}{12}} \|S_{1} \phi\|_{F_{1}} \|S_{1} \phi\|_{F_{1}}.
$$
The estimate of $\mathcal{H}_3^3$ is very similar. We omit the details. The treatment of $\mathcal{H}_3^3$ is also similar. Indeed, we apply the angular decomposition, Sobolev estimates and Hölder inequality.

\[
\mu \| V \mathcal{H}_3^3 \|_{(\Omega)^{-1} X_{d,T}^2} = \sum_{d \leq \mu} d^{\frac{1}{2}} \| S_{\mu,d}(\Omega) Q(S_{1, \leq \min(d, c \mu)} \varphi, S_{1, \leq \min(d, c \mu)} \phi) \|_{L_x^2 L_t^2}
\]

\[
\lesssim \sum_{d \leq \mu} d^{\frac{1}{2}} \mu^\frac{\delta}{2} d^\frac{\delta}{2} \left( \frac{d}{\mu} \right) \tag{\frac{1}{2}}
\]

\[
\times \left\| \left( \sum_{\omega} (\Omega) (B^{-\omega}_{\frac{d}{\mu}} S_{1, \leq \min(d, c \mu)} \varphi S_{1, \leq \min(d, c \mu)} \phi)^2 \right)^{\frac{1}{2}} \right\|_{L_x^2}
\]

\[
\lesssim \sum_{d \leq \mu} \mu^{\frac{\delta}{2}} d^\frac{\delta}{2} \| S_{1, \Omega} \varphi \|_{L_x^2 L_t^2} \| S_{1, \Omega} \phi \|_{L_x^2 L_t^2}
\]

\[
\lesssim \mu^{\frac{\delta}{2}} \| S_{1} \varphi \|_{F_{L_x^2}} \| S_{1} \phi \|_{F_{L_x^2}}
\]

We need to estimate the term $\mathcal{H}_3^3$ in the $Z$ space also. We first use Lemma 3.1 and then the following step is quite similar.

\[
\mu \| V \mathcal{H}_3^3 \|_{Z_{L_x^2}} \lesssim \sum_{d \leq \mu} (d \mu)^{-1} \left\| \sum_{\omega} S_{\mu,d}(\Omega) Q(S_{1, \leq \min(d, c \mu)} \varphi, S_{1, \leq \min(d, c \mu)} \phi) \right\|_{L_x^2} dt
\]

\[
\lesssim \sum_{d \leq \mu} (d \mu)^{-1} \mu^{\frac{\delta}{2}} - \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \int \| S_{1, \leq \min(d, c \mu)} \varphi, S_{1, \leq \min(d, c \mu)} \phi \|_{L_x^2} dt
\]

\[
\lesssim \sum_{d \leq \mu} \mu^{\frac{\delta}{2}} - d^\frac{\delta}{2} \int \| S_{1, \leq \min(d, c \mu)} \varphi \|_{L_x^{2^*}} \| S_{1, \leq \min(d, c \mu)} \phi \|_{L_x^{2^*}} dt
\]

\[
\lesssim \mu^{\frac{\delta}{2}} \| S_{1} \varphi \|_{L_x^2 L_x^{2^*}} \| S_{1} \phi \|_{L_x^2 L_x^{2^*}}
\]

\[
\lesssim \mu^{\frac{\delta}{2}} \| S_{1} \varphi \|_{F_{L_x^2}} \| S_{1} \phi \|_{F_{L_x^2}}
\]

5.5. \textbf{Low × High interaction near cone.} As the previous section, we further decompose $\mathcal{L}_3$ as follows:

\[
S_{1, \leq c \mu} Q(S_{\mu} \varphi, S_{1, \leq c \mu} \phi) = \sum_{d \leq c \mu} S_{1, \leq d} Q(S_{\mu, \leq d} \varphi, S_{1, \leq d} \phi) + \sum_{d \leq \mu} S_{1, d} Q(S_{\mu, \leq d} \varphi, S_{1, \leq d} \phi)
\]

\[
+ \sum_{d \leq \mu} S_{1, \leq \min(d, c \mu)} Q(S_{\mu, d} \varphi, S_{1, \leq \min(d, c \mu)} \phi)
\]

\[
=: \mathcal{L}_3^1 + \mathcal{L}_3^2 + \mathcal{L}_3^3.
\]

To deal with the term $\mathcal{L}_3^1$ and $\mathcal{L}_3^2$, we first apply Lemma 4.2. Then we use in order the Hölder inequality with respect to $t$, and then $x$. Then we use Lemma 3.3 and Strichartz estimates to obtain
the required estimates. The explicit treatment of $\mathcal{L}H^3_1$, $\mathcal{L}H^3_2$ is as follows.

\[
\|V\mathcal{L}H^3_1\|_{(\Omega)^{-1}Y_1} \lesssim \sum_{d < c\mu} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \left\| \sup_{\omega} \| S^{\omega}_{\mu, \leq d}(\Omega)\varphi \|_{L^\infty_{\omega}} \right\| \left( \sum_{\omega} \| B^{\omega}_{(\frac{d}{\mu})^{\frac{1}{2}}} S_{1, d}(\Omega)\phi \|_{L^2_{\omega}}^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \sum_{d < c\mu} \mu^{-\frac{1}{2}} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \| S_{\mu, \leq d}(\Omega)\varphi \|_{L^2_{\omega}L^6_\omega} d^{-\frac{2}{3}} \| S_1\phi \|_{F_0}
\]

\[
\lesssim \sum_{d < c\mu} \mu^{-\frac{1}{2}} d^{\frac{3}{2}} \| S_{\mu, \leq d}(\Omega)\varphi \|_{F_{\alpha}, \mu} \| S_1\phi \|_{F_0}
\]

\[
\lesssim \mu \| S_{\mu, \leq d}(\Omega)\varphi \|_{F_{\alpha}, \mu} \| S_1\phi \|_{F_0}.
\]

We also need to estimate the term $\mathcal{L}H^3_2$ in the $Z$ space. From now on, the proof is quite different.

\[
\|V\mathcal{L}H^3_2\|_{z_{\alpha, 1}} = \sum_{d \leq \mu} \int \sup_{\omega} \| B^{\omega}_{(\frac{d}{\mu})^{\frac{1}{2}}} S_{1, d} V\mathcal{L}H^3_2 \|_{L^\infty_{\omega}} dt
\]

\[
\lesssim \sum_{d \leq \mu} \left( \frac{d}{\mu} \right)^{-\frac{1}{2}} \int \sup_{\omega} \| S^{\omega}_{\mu, \leq d}(\Omega)Q(S_{\mu, \leq d}\varphi, S_{1, \leq d}\phi) \|_{L^\infty_{\omega}} dt
\]

\[
\lesssim \sum_{d \leq \mu} (d\mu)^{-\frac{1}{2}} \int \sup_{\omega} \| S^{\omega}_{\mu, \leq d}(\Omega)Q(S_{\mu, \leq d}\varphi B^{\omega}_{(\frac{d}{\mu})^{\frac{1}{2}}} S_{1, \leq d}) \|_{L^\infty_{\omega}} dt
\]

\[
\lesssim \sum_{d \leq \mu} (d\mu)^{-\frac{1}{2}} \int \sup_{\omega} \| S^{\omega}_{\mu, \leq d}(\Omega)Q(S_{\mu, \leq d}\varphi S_{1, \leq d}) \|_{L^\infty_{\omega}} dt
\]

\[
\lesssim \sum_{d \leq \mu} \left( \frac{d}{\mu} \right)^{1-} \mu^{\frac{3}{2}} \int \| S_{\mu, \leq d}\varphi \|_{L^2_{\omega}}^2 \| S_1\phi \|_{L^2_{\omega}}^2 dt
\]

\[
\lesssim \sum_{d \leq \mu} \left( \frac{d}{\mu} \right)^{\frac{3}{2}} \| S_{\mu, \leq d}\varphi \|_{L^2_{\omega}L^6_\omega} \| S_1\phi \|_{L^2_{\omega}L^6_\omega}.
\]

In this manner, we can obtain the required estimate only if $d \ll \mu^3$. On the other hand, for $d \ll \mu^3$, we have $\mu \lesssim (\frac{d}{\mu})^{\frac{3}{2}}$, and hence instead of Lemma 4.2, we make the use of a smaller
angular decomposition with size $\mu$ for output frequency and high input frequency.

$$
\sum_{\mu^3 \leq d < c \mu} (d\mu)^{-\frac{1}{2}} \int \sup_{\omega} \| B_{\frac{d}{\mu}}^{\omega} S_{1:d}(S_{\mu^3 \leq d} \varphi S_{1:d} \phi) \|_{L^\infty_x} dt \\
\lesssim \sum_{\mu^3 \leq d < c \mu} (d\mu)^{-\frac{1}{2}} \int \sup_{\omega} \| B_{\frac{d}{\mu}}^d S_{1:d}(S_{\mu^3 \leq d} \varphi S_{1:d} \phi) \|_{L^\infty_x} dt \\
\lesssim \sum_{\mu^3 \leq d < c \mu} (d\mu)^{-\frac{1}{2}} \sum_{\omega_1, \omega_2} \| B_{\frac{d}{\mu}}^\omega S_{1:d}(S_{\mu^3 \leq d} B_{\frac{d}{\mu}}^\omega S_{1:d} \phi) \|_{L^\infty_x} dt \\
\lesssim \sum_{\mu^3 \leq d < c \mu} (d\mu)^{-\frac{1}{2}} \int \| S_{\mu^3 \leq d} \varphi \|_{L^\infty_x} \| B_{\frac{d}{\mu}} S_{1:d} \phi \|_{L^\infty_x} dt \\
\lesssim \sum_{\mu^3 \leq d < c \mu} (d\mu)^{-\frac{1}{2}} \mu^{\frac{1}{2}} \| S_{\mu^3 \leq d} \varphi \|_{L^\infty_x} \| B_{\frac{d}{\mu}}^\omega S_{1:d} \phi \|_{L^\infty_x} \| B_{\frac{d}{\mu}}^\omega S_{1:d} \phi \|_{L^\infty_x} \| B_{\frac{d}{\mu}}^\omega S_{1:d} \phi \|_{L^\infty_x} \\
\lesssim \mu^{\frac{1}{2}} \| S_{\mu^3 \varphi} \|_{F_{\mu^3, \mu}} \| S_{1:d} \phi \|_{F_{\mu^3, \mu}}.
$$

Here, the summation on $d$ makes only the loss of log $\mu$ and hence we get the desired estimates. The term $\mathcal{L}_3^3$ is the most crucial part of our proof. We write

$$
\| V \mathcal{L}_3^3 \|_{F_{\mu^3, \mu}} \lesssim \sum_{d \leq \mu} \| S_{1 : \leq \min(d, c \mu)} Q(S_{\mu^3 \varphi}, S_{1 : \leq \min(d, c \mu)} \langle \Omega \rangle \phi) \|_{L^1_x L^2_t} \\
+ \sum_{d \leq \mu} \| S_{1 : \leq \min(d, c \mu)} Q(S_{\mu^3 \varphi \langle \Omega \rangle \phi}, S_{1 : \leq \min(d, c \mu)} \phi) \|_{L^1_x L^2_t} \\
=: \mathcal{J}_1 + \mathcal{J}_2.
$$

The term $\mathcal{J}_1$ is rather easier than $\mathcal{J}_2$. We simply recall the property of the $Z$ space $[50]$. 

$$
\mathcal{J}_1 \lesssim \sum_{d \leq \mu} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \| \sup_{\omega} \| S_{\mu^3 \varphi \langle \Omega \rangle \phi} \|_{L^\infty_x} \| S_{1 : \leq \min(d, c \mu)} \langle \Omega \rangle \phi \|_{L^\infty_x L^2_t} \\
\lesssim \mu \| S_{\mu^3 \varphi} \|_{F_{\mu^3, \mu}} \| S_{1 \phi} \|_{F_{\mu^3, \mu}}.
$$

Now we are left to consider the $\mathcal{J}_2$. We further decompose the range of $d$ into $d \ll \mu^3$ and $\mu^3 \leq d \leq \mu$. If $d \ll \mu^3$, then we apply Lemma $[42]$ Hölder inequality and then Sobolev estimates.

$$
\mathcal{J}_2^{d \ll \mu^3} \lesssim \sum_{d \ll \mu^3} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \left( \sum_{\omega} \| S_{\mu^3 \varphi \langle \Omega \rangle \phi} \|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}} \| \sup_{\omega} \| B_{\frac{d}{\mu}} \langle \Omega \rangle \phi \|_{L^1_t L^2_x} \| S_{1 : \leq \min(d, c \mu)} \phi \|_{L^2_t L^\infty_x} \\
\lesssim \sum_{d \ll \mu^3} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \mu \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \| S_{\mu^3 \varphi} \|_{F_{\mu^3, \mu}} \| S_{1 \phi} \|_{F_{\mu^3, \mu}} \\
\lesssim \sum_{d \ll \mu^3} \mu^{\frac{1}{2}} \left( \frac{d}{\mu} \right)^{\frac{1}{2}} \| S_{\mu^3 \varphi} \|_{F_{\mu^3, \mu}} \| S_{1 \phi} \|_{L^2_t L^\infty_x}.
$$
Thus we can get the desired estimate for $d \ll \mu^3$. On the other hand, if $\mu^3 \lesssim d \lesssim \mu$, we write
\[
\mathcal{J}_2^{\mu^3 \lesssim d \lesssim \mu} = \sum_{d \lesssim \mu} \left( \sum_{|\omega_1 + \omega_3| \approx \mu} B_{\mu^2}^{\omega_1 \omega_3} S_{1, \Omega} \cdot \sum_{|\omega_2 + \omega_3| \approx \mu} Q(S_{\mu, \omega_2}^{\omega_3 \omega_3} (\Omega) \varphi, B_{\mu^2}^{\omega_3} S_{1, \Omega} \cdot \sum_{|\omega_3| \approx \mu} \phi) \right)_{L_1 L_2^2}.
\]
We rewrite the $L_2^2$ norm via duality and then use H"older inequality.
\[
\| \cdots \|_{L_2^2} \lesssim \sup_{\|h\|_{L_2^2}} \left( \frac{d}{\mu} \right)^{\frac{3}{2}} \sum_{\omega_2} \| S_{\mu, \omega_2}^{\omega_2 \omega_2} (\Omega) \varphi \|_{L_2^2} \sum_{\omega_1, \omega_2, \omega_3} B_{\mu^2}^{\omega_2 \omega_3 \omega_3} \| P_{\mu} B_{\mu^2}^{\omega_3} S_{1, \Omega} \cdot \sum_{\|\omega_3\| \approx \mu} \phi \|_{L_1 L_2^2}.
\]
Then
\[
\mathcal{J}_2^{\mu^3 \lesssim d \lesssim \mu} \lesssim \sum_{d \lesssim \mu} \left( \frac{d}{\mu} \right)^{\frac{3}{2}} d^{-\frac{1}{2}} \mu \| S_{\mu} (\Omega) \varphi \|_{L_2^2 L_2^2} \left( \frac{d}{\mu} \right)^{-\frac{1}{2}} \left\| \sup_{\omega} \| B_{\mu}^{\omega_3} S_{1, \Omega} \cdot \sum_{\|\omega_3\| \approx \mu} \phi \|_{L_1 L_2^2} \right\|_{L_2^2}
\]
\[
\lesssim \sum_{d \lesssim \mu} \left( \frac{d}{\mu} \right)^{-\frac{1}{2}} \mu \| S_{\mu} (\Omega) \varphi \|_{L_2^2 L_2^2} \| S_{\mu} \varphi \|_{F_{0, \mu}} \| S_1 \phi \|_{F_{0, \mu}}.
\]

**Remark 5.1.** In the Low×High regime, the most difficult interaction is when the low frequency controls the angular regularity. In this case, we cannot exploit the angular concentration estimates to gain some positive power of $\mu$. Hence we only use the Sobolev embedding estimates.

**Acknowledgements**

The author is supported by NRF-2018R1D1A3B07047782 and NRF-2016K2A9A2A13003815.

**References**

[1] T. Candy, S. Herr, *Transference of bilinear restriction estimates to quadratic variation norms and the Dirac-Klein-Gordon system*, Analysis and PDE 11, no. 5, (2018): 1171–1240.

[2] T. Candy, S. Herr, *Conditional large initial data scattering results for the Dirac-Klein-Gordon system*, Forum of Mathematics, Sigma 6, (2018), 55pp.

[3] Y. Cho, S. Lee, *Strichartz estimates in spherical coordinates*, Indiana University Mathematics Journal 62, no. 3, (2013): 991–1020.

[4] D. Foschi, S. Klainerman, *Bilinear space-time estimates for homogeneous wave equations*, Annales Scientifiques de l’ Ecole Normale Superieure 33, no. 2, (2000): 211–274.

[5] M. Keel, T. Tao, *Endpoint Strichartz estimates*, American Journal of Mathematics 120, no. 5, (1998): 955–980.

[6] M. Keel, T. Roy, T. Tao, *Global well-posedness of the Maxwell-Klein-Gordon equation below the energy norm*, Discrete and Continuous Dynamical systems - A 30, (2011): 573–621.

[7] S. Klainerman, M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Communications on Pure and Applied Mathematics 46, no. 9, (1993): 1221–1268.

[8] S. Klainerman, M. Machedon, *On the Maxwell-Klein-Gordon equation with finite energy*, Duke Mathematical Journal 74, (1994): 19–44.

[9] S. Klainerman, D. Tataru, *On the optimal local regularity for Yang-Mills equations in $\mathbb{R}^{4+1}$*, Journal of American Mathematical Society 12, (1999): 93–116.

[10] J. Krieger, J. Sterbenz, D. Tataru, *Global well-posedness for the Maxwell-Klein-Gordon equation in $4 + 1$ dimensions: small energy*, Duke Mathematical Journal 164, no. 6, (2015): 973–1040.
[11] S. Lee, A. Vargas, *Sharp null form estimates for the wave equation*, American Journal of Mathematics 130, no. 5, (2008): 1279–1326.

[12] M. Machedon, J. Sterbenz, *Almost optimal local well-posedness for the (3 + 1)−dimensional Maxwell-Klein-Gordon equations*, Journal of American Mathematical Society 17, (2004): 297–359.

[13] S.-J. Oh, D. Tataru, *Local well-posedness of the (4+1)−dimensional Maxwell-Klein-Gordon equation at energy regularity*, Annals of PDE 2, no.1, (2016): 70pp.

[14] S.-J. Oh, D. Tataru, *Global well-posedness and scattering of the (4 + 1)−dimensional Maxwell-Klein-Gordon equation*, Inventiones mathematicae 205, no.3, (2016): 781–877.

[15] S.-J. Oh, D. Tataru, *Energy dispersed solutions for the (4 + 1)−dimensional Maxwell-Klein-Gordon equation*, Annals of PDE 2, no.1, (2016): 70pp.

[16] H. Pecher, *Local well-posedness for low regularity data for the higher dimensional Maxwell-Klein-Gordon system in Lorenz gauge*, Journal of Mathematical Physics 59, no.10, (2018): 101503, 20pp.

[17] H. Pecher, *Almost optimal local well-posedness for the Maxwell-Klein-Gordon system with data in Fourier-Lebesgue spaces*, Communications on Pure and Applied Analysis 19, no.6, (2019): 3303–3321.

[18] S. Selberg, *Almost optimal local well-posedness of the Maxwell-Klein-Gordon equations in 1 + 4 dimensions*, Communications in partial differential equations 27, (2002): 1183–1227.

[19] S. Selberg, A. Tesfahun, *Finite-energy global well-posedness of the Maxwell-Klein-Gordon equations in Lorenz gauge*, Communications in partial differential equations 35, (2010): 1029–1105.

[20] S. Selberg, A. Tesfahun, *Null structure and local well-posedness in the energy class for the Yang-Mills equations in Lorenz gauge*, American Journal of Mathematics 129, no.3, (2007): 611–664.

[21] R.S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equation*, Duke Mathematical Journal 44, no.3, (1977): 705–714.

[22] X. Wang, *On global existence of 3D charge critical Dirac-Klein-Gordon system*, Nonlinear Differential Equations and Applications 22, no. 4, (2015): 849–875.

[23] S. Hong, *Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea*

Email address: seokchangible@snu.ac.kr