Alternative Canonical Formalism for the Wess-Zumino-Witten Model

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Abstract

We study a canonical quantization of the Wess–Zumino–Witten (WZW) model which depends on two integer parameters rather than one. The usual theory can be obtained as a contraction, in which our two parameters go to infinity keeping the difference fixed. The quantum theory is equivalent to a generalized Thirring model, with left and right handed fermions transforming under different representations of the symmetry group. We also point out that the classical WZW model with a compact target space has a canonical formalism in which the current algebra is an affine Lie algebra of non–compact type. Also, there are some non–unitary quantizations of the WZW model in which there is invariance only under half the conformal algebra (one copy of the Virasoro algebra).

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1 Introduction

In [1] one of us discussed a new canonical formulation of the two dimensional nonlinear model. In this paper we will generalize the formalism to include the Wess-Zumino-Witten term. The well known [2],[3] quantization of this model depends on an integer parameter, k; for a particular value of the coupling constant the theory is conformally invariant. We will find that our quantization will depend on two integer parameters, the conventional one being the limiting case as they go to infinity keeping the difference fixed. This more general quantization scheme does not lead to any new unitary conformal field theories; the fixed point for the $\beta$ function occurs only in the limiting case. However if we drop the requirement of unitarity, there are field theories parameterized by a pair of integers, which have half-conformal invariance (invariance under one copy of the Virasoro algebra).

The unitary quantum theory is equivalent to a fermionic field theory with four-Fermi interactions; the two integers may be thought of as the numbers of the left and right moving fermions. The fact that the conventional quantization of the nonlinear model is the large $k$ limit of a fermionic system was shown in [4]. Our work can be viewed as a generalization of this result. Also, we work within canonical quantization rather than path integral quantization as in Ref.[4]. Some work in the same direction has been done in [5].

Such an equivalence of the WZW model to a fermionic theory is another example of the Bose–Fermi correspondence. Our example is also of interest as a two–dimensional analogue of Quantum Chromodynamics (QCD). The nonlinear model in two dimensions is classically scale invariant; this is broken at the quantum level. The coupling constant is scale dependent and has an ultraviolet stable zero at the origin: the theory is asymptotically free. This is analogous to the scaling behaviour of QCD.

In more detail, the field equation of the model we study is

$$\partial^\mu (\partial_\mu g^{-1}) - \rho \epsilon_{\mu \nu} \partial_\mu gg^{-1} \partial_\nu gg^{-1} = 0.$$  (1)

Here, $g : R^{1,1} \rightarrow G$ is a map from two–dimensional Minkowski space to a simple compact Lie group $G$. In ref. [1] the case $\rho = 0$ was studied. A new canonical formalism, in which the current algebra is the direct sum of two affine Lie algebras, was found. In this paper, we will show that this approach can be extended to the case with a WZW term. The main difference is that the central charges of the two affine Lie algebras are not equal.
2 The Standard Canonical Formalism

Let us review the classical formulation of the nonlinear model\[2,3\]. For a related discussion see Ref.\[1\].

The crucial property of the nonlinear model is that it can be formulated entirely in terms of the current $\partial_\mu g g^{-1}$. The equation of motion $\eqref{1}$ is equivalent to the following pair of first order equations,

$$\begin{align*}
\frac{\partial I}{\partial t} &= \frac{\partial J}{\partial x} + \rho [I,J] \\
\frac{\partial J}{\partial t} &= \frac{\partial I}{\partial x} - [I,J]
\end{align*}$$

The second equation, which is a sort of Bianchi identity, is the integrability condition for the existence of a $g : R^{1,1} \rightarrow G$ satisfying,

$$\begin{align*}
I &= \frac{\partial g}{\partial t} g^{-1} \\
J &= \frac{\partial g}{\partial x} g^{-1}
\end{align*}$$

If we also impose the boundary condition

$$\lim_{x \to -\infty} g(x) = 1$$

the solution for $g$ is unique. Then equation $\eqref{2}$ guarantees that $g$ satisfies the nonlinear model equation of motion. Note that if we had chosen space to be a circle, $\eqref{3}$ would not imply $\eqref{4}$ and $\eqref{5}$. The solution to these equations will not be periodic in general. If $(I,J)$ is viewed as a connection, $\eqref{3}$ says that it is flat. But in order for a flat connection to be ‘pure gauge’ as in $\eqref{4}$ and $\eqref{5}$, it is necessary also for the parallel transport operator around a homotopically nontrivial curve (holonomy) to be equal to the identity. It might be interesting to study the theory on the circle as well. This first order formulation in terms of currents is particularly convenient in two dimensions. In higher dimensions, the spatial components of the currents must satisfy some first class constraints.

In our case, an initial data is given by any pair of functions $I,J : R \rightarrow G$ which are square integrable ($G$ is the Lie algebra associated to $G$). The square integrability is a condition on how quickly they must decay to zero at infinity. It is needed for finiteness of energy (see $\eqref{18}$ below).

From the above discussion, it is clear that $I$ and $J$ provide co–ordinates on the phase space (space of initial data) of the nonlinear model. This is exactly what the $p$ and $q$
variables do in classical mechanics. Observables are functions on the phase space, so they can always be written as functions of $I$ and $J$ (the same way observables of a classical mechanical system can be written as functions of $p$ and $q$). Just as the Canonical Commutation Relations (CCR) determine the Poisson brackets of any two observables in classical mechanics, the commutation relations of $I$ and $J$ determine completely the algebra of observables. Like the CCR, the Poisson brackets of $I$ and $J$ also define a Lie algebra (the current algebra).

The standard action for the WZW model is

$$S = \frac{1}{4\lambda^2} \int \text{tr} \partial_\mu g \partial^\mu g^{-1} d^2x + n\Gamma.$$  

(7)

Here the WZW term $\Gamma$,

$$\Gamma = \frac{1}{24\pi} \int_B d^3y \epsilon^{ijk} \text{tr} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g g^{-1}$$  

(8)

is an integral over a three manifold $B$ with space–time as boundary. We recover the above equation of motion with

$$\rho = \frac{n\lambda^2}{4\pi}.$$  

(9)

Note that the action as well as the canonical formalism depends on the extra parameter $\lambda$, which does not affect the classical dynamics as it cancels out. In the quantum theory $n$ must be an integer so that $e^{iS}$ be independent of the extension to the third dimension.

It is possible to derive a set of Poisson Brackets for $I$ and $J$ from the action principle. But for our purposes, it is better to regard the Poisson brackets and a Hamiltonian as the basic postulates, justified by reproducing the equation of motion. This will allow us to generalize to a situation where the action is not obvious.

There is an extensive literature on the philosophy of current algebras, see [3] for more references. This approach had mixed success in theory of strong interactions. For the two dimensional nonlinear model on the other hand, this seems to be the most natural point of view.

In our case the Poisson brackets of $I$ and $J$ are,

$$\frac{1}{2\lambda^2} \{I_\alpha(x), I_\beta(y)\}_1 = f_{\alpha\gamma \beta} f_\gamma(x) \delta(x - y) + \rho f_{\alpha\beta \gamma} J_\gamma \delta(x - y)$$  

(10)

$$\frac{1}{2\lambda^2} \{I_\alpha(x), J_\beta(y)\}_1 = f_{\alpha\beta \gamma} J_\gamma(x) \delta(x - y) - \delta_{\alpha\beta} \delta'(x - y)$$  

(11)

$$\frac{1}{2\lambda^2} \{J_\alpha(x), J_\beta(y)\}_1 = 0.$$  

(12)
The subscript on the brackets distinguishes these ones from other Poisson brackets we will introduce later.

These brackets can be written more elegantly if we introduce some mathematical notation. Let us define the infinite dimensional Lie group \( \mathcal{G} \) to be the set of functions \( g \) satisfying

\[
\mathcal{G} = \{ g : R \rightarrow G \mid g(-\infty) = 1 \}
\]

The multiplication in \( \mathcal{G} \) is just pointwise:

\[
g_1 g_2(x) = g_1(x) g_2(x)
\]

As always, the Lie algebra of \( \mathcal{G} \) will be denoted by an underline:

\[
\mathcal{G}_\mathcal{G} = \{ \xi : R \rightarrow G \mid \int tr \xi^2 \, dx < \infty \}
\]

The Lie bracket in \( \mathcal{G}_\mathcal{G} \) is also pointwise. Occasionally, we will need another Lie algebra with the same vector space as above, but for which all brackets are zero. This abelian Lie algebra will be called \( \mathcal{A} \). It is important to distinguish \( \mathcal{G}_\mathcal{G} \) from \( \mathcal{A} \) although as vector spaces, they are the same function space on \( R \).

Since the Lie algebra \( \mathcal{G} \) has a natural invariant inner product, we will identify it with its dual as a vector space.

Define now the Lie algebra \( \mathcal{C}_1 \) to be the set of triples \( (\xi, \eta, b) \), where \( \xi \) and \( \eta \) are square integrable functions from \( R \) to \( G \), \( b \) is a real number. The Lie bracket is defined to be

\[
[(\xi, \eta, b), (\xi', \eta', b')] = 
(\{\xi, \xi'\}, \{\xi, \eta'\} - \{\xi', \eta\} + \rho [\xi, \xi'], - \int tr (\xi \frac{\partial}{\partial x} \eta' - \xi' \frac{\partial}{\partial x} \eta) \, dx)
\]

This algebra has an abelian subalgebra isomorphic to \( \mathcal{A} \oplus R \) when \( \xi = 0 \). If \( \rho = 0 \), \( \mathcal{C}_1 \) is the semi-direct sum of two subalgebras.

\[
\mathcal{C}_1 = \mathcal{G}_\mathcal{G} \dot{\oplus} (\mathcal{A} \oplus R)
\]

where the dot denotes a semi-direct sum of Lie algebras. By a change of basis, we can see that if \( \rho \neq 0 \), this is a semi-direct sum of an affine Lie algebra with an abelian algebra. Then \( \mathcal{C}_1 \) describes the algebra of currents given by \( \mathcal{G}_1 \), \( \mathcal{G}_2 \), \( \mathcal{G}_3 \), for any value of \( \rho \).

The hamiltonian describing the non linear model is

\[
H_1 = \frac{1}{4 \lambda^2} \int tr (I^2 + J^2) \, dx.
\]
This can of course be derived from the usual action principle; but we prefer to justify it directly by showing that, together with (12), it leads to the required equations of motion (2) and (3). This is a straightforward calculation:

\[
\frac{\partial I_\beta(y)}{\partial t} = \{H_1, I_\beta(y)\}_1 = \frac{\partial I_\beta(y)}{\partial x} + \rho f_{\alpha\beta\gamma} I_\alpha(y) J_\gamma(y)
\]

(19)

\[
\frac{\partial J_\beta(y)}{\partial t} = \{H_1, J_\beta(y)\}_1 = \frac{\partial J_\beta(y)}{\partial x} + f_{\alpha\beta\gamma} I_\alpha(y) J_\gamma(y).
\]

(20)

Note that the parameter \( \lambda \) (the ‘coupling constant’) drops out of the equations of motion.

For the record, let us note the stress tensor \( \Theta_{\mu\nu} \) of the theory as a function of the currents.

\[
\Theta_{00} = \Theta_{11} = \frac{1}{4\lambda^2} tr(I^2 + J^2)
\]

(21)

\[
\Theta_{01} = \Theta_{10} = \frac{1}{2\lambda^2} tr(IJ)
\]

(22)

This is traceless and conserved, so this formalism is conformally and Poincaré invariant. We won’t discuss the Poisson brackets of \( \Theta_{\mu\nu} \), since they will be derived in a more general context later.

### 3 The contraction of SU(2) to E(2)

There is a certain analogy between our current algebra and the Euclidean group. The spatial currents \( J \) commute, as translations do, while the time components \( I \) are analogous to rotations. One way to construct a representation of the Euclidean group is to diagonalize the commuting generators, and then representing rotations as differential operators in terms of them (the method of induced representations).

The analogous procedure in our case would be to diagonalize \( J \), which is essentially the Schrödinger representation of the field theory. Unfortunately this is very complicated to do in practice, since an appropriate measure of integration in the configuration space does not exist. The only known measure is an analogue of the Wiener measure \([7]\) which unfortunately is not the appropriate one. Even in the case of the abelian group (where the required measure is known and is a Gaussian), the inner product which makes the Hamiltonian self-adjoint is not obtained from the Wiener measure. So we look at another way of constructing representations of algebras with abelian ideals, the method of contractions.
We will describe here the construction of the representation of $E(2)$, using the method of ‘contraction’ or ‘deformation’ of Lie algebras. The discussion follows an article of Inonu [8]. This will suggest another way of dealing with the nonlinear model.

Recall that $E(2)$ is the Lie algebra of isometries of $R^2$. It is a non-compact Lie algebra, this implying that its unitary representations are infinite dimensional.

Now geometrically, we can regard $R^2$ to be the limit as the radius goes to infinity of a two dimensional sphere, $S^2$. There must therefore be some way of thinking of $E(2)$ as a limit of the isometry algebra of $S^2$. This is what the method of contractions (or, deformations) of Lie algebras does. Another example is the Galilei group which is the limit as the velocity of light goes to infinity of the Lorentz group. The Poincaré group is a contraction of the group $O(2,3)$.

In all these examples a non-simple group is obtained as the contraction of a simple group. This is useful in representation theory because the representations of these simple groups are in many ways easier to understand. We will work out the case of $E(2)$ in detail here.

The isometries of $S^2$ form the Lie algebra $SU(2)$. The generators of $SU(2)$ are the three rotations $S_i$, satisfying

$$[S_i, S_j] = \epsilon_{ijk} S_k$$

Let us now introduce a real parameter $\tau$ and define

$$S = S_3$$
$$P_a = \tau S_a ; a = 1, 2$$

Then, the commutation relations become

$$[S, P_a]_2 = \epsilon_{ab} P_b$$
$$[P_a, P_b]_2 = \tau^2 \epsilon_{ab} S$$

the subscript being present to distinguish it from the relations of $E(2)$. For any finite value of $\tau$ the above relations describe $SU(2)$ in a peculiar choice of basis. But in the limit as $\tau \to 0$ they become just $E(2)$.

The irreducible representations of $SU(2)$ are labelled by spin $j$, which is either integer or half integer. The dimension of the spin $j$ representation is $2j + 1$. If a representation of
SU(2) is to tend to one for E(2) in the limit as \( \tau \to 0 \), \( j \) must go to infinity at the same time. For, all unitary representations of E(2) are infinite dimensional.

Let us now do this explicitly. An orthonormal basis \( |m\rangle \) for the spin \( j \) representation of SU(2) is labelled by \( m = -j, -j + 1, \ldots, j - 1, j \). The matrix elements are

\[
<m|\hat{S}|n> = m \delta_{m,n}
\]

\[
<m|\hat{S}_1|n> = \frac{1}{2} \sqrt{(j-m)(j+n)} \delta_{n,m+1}
\]

\[
+ \frac{1}{2} \sqrt{(j+m)(j-n)} \delta_{n,m-1}
\]

\[
<m|\hat{S}_2|n> = -\frac{i}{2} \sqrt{(j-m)(j+n)} \delta_{n,m+1}
\]

\[
+ \frac{i}{2} \sqrt{(j+m)(j-n)} \delta_{n,m-1}
\]

As we already mentioned, to have a meaningful limit as \( \tau \to 0 \), we must also let \( j \to \infty \) such that

\[
\lim \tau j = a
\]

is finite. In this limit the matrix elements of \( S \) and \( P \) are,

\[
<m|\hat{S}|n> = m \delta_{m,n}
\]

\[
<m|P_1|n> = \frac{a}{2} (\delta_{n,m-1} + \delta_{n,m+1})
\]

\[
<m|P_2|n> = \frac{ia}{2} (\delta_{n,m-1} - \delta_{n,m+1})
\]

which is what we want.

4 New Canonical Formalism

We will now show that the current algebra \( C_1 \) is the contraction of the direct sum of two affine Lie algebras. Thus we can construct representations of it as limiting cases of representations of affine Lie algebras. However, it turns out that there is a more general canonical formalism for the WZW model where the current algebra is just a pair of affine Lie algebras. This does not follow from the usual action principle (an action principle which leads to this general canonical formalism can be found). It is also found that this new formalism preserves conformal (and hence Poincaré) invariance at the classical level.

Furthermore, the new current algebra is isomorphic to that of fermion currents. So a representation can be found in terms of fermion bilinears. Then the Hamiltonian of the
WZW model becomes identical to that of a Thirring model. This is a bosonization of the Thirring model. The special case without WZW term was discussed in [1].

By analogy to (25) (and to $E(3)$ as discussed in [1]) we are lead to a deformation of the Lie algebra $C_1$. We will attempt to deform the algebra of $I$ and $J$ such that the equations of motion remain unchanged. Let us make the ansatz ($a, \epsilon, \tau, \mu$ are parameters which we assume to be real for now):

\[
\frac{1}{2\lambda^2} \{I_\alpha(x), I_\beta(y)\}_{\frac{1}{2^\pm}} = f_{\alpha\beta\gamma} I_\gamma(x) \delta(x-y) + af_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y)
\]

\[
\frac{1}{2\lambda^2} \{I_\alpha(x), J_\beta(y)\}_{\frac{1}{2^\pm}} = f_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y) - \delta_{\alpha\beta} \delta'(x-y) + \epsilon f_{\alpha\beta\gamma} I_\gamma(x) \delta(x-y)
\]

\[
\frac{1}{2\lambda^2} \{J_\alpha(x), J_\beta(y)\}_{\frac{1}{2^\pm}} = \tau^2 f_{\alpha\beta\gamma} I_\gamma(x) \delta(x-y) + \mu f_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y)
\]

together with the hamiltonian (18) these Poisson brackets lead to the equations,

\[
\frac{\partial I}{\partial t} = \frac{\partial J}{\partial x} + (a - \epsilon)[I, J]
\]

\[
\frac{\partial J}{\partial t} = \frac{\partial I}{\partial x} + (1 - \tau^2)[I, J]
\]

There is an unwanted factor of $(1 - \tau^2)$, so we don’t quite get the equations (2), (3) we want. But if we rescale $I$ and $J$ by $(1 - \tau^2)$ this problem disappears. The equations become

\[
\frac{\partial I}{\partial t} = \frac{\partial J}{\partial x} + \frac{(a - \epsilon)}{(1 - \tau^2)}[I, J]
\]

\[
\frac{\partial J}{\partial t} = \frac{\partial I}{\partial x} + [I, J]
\]

which agree with (2), (3) if

\[
\rho = \frac{a - \epsilon}{1 - \tau^2}.
\]

Thus we arrive at the algebra
\[
\frac{1}{2\lambda^2} \{ I_\alpha(x), I_\beta(y) \}_{1/2} = (1 - \tau^2)f_{\alpha\beta\gamma}I_\gamma(x)\delta(x-y) + a(1 - \tau^2)f_{\alpha\beta\gamma}J_\gamma(x)\delta(x-y)
\]

(35)

\[
\frac{1}{2\lambda^2} \{ I_\alpha(x), J_\beta(y) \}_{1/2} = (1 - \tau^2)f_{\alpha\beta\gamma}J_\gamma(x)\delta(x-y) - (1 - \tau^2)^2\delta_{\alpha\beta}\delta'(x-y) + (1 - \tau^2)\epsilon f_{\alpha\beta\gamma}I_\gamma(x)\delta(x-y)
\]

(36)

\[
\frac{1}{2\lambda^2} \{ J_\alpha(x), J_\beta(y) \}_{1/2} = \tau^2(1 - \tau^2)^2f_{\alpha\beta\gamma}J_\gamma(x)\delta(x-y) + (1 - \tau^2)\mu f_{\alpha\beta\gamma}I_\gamma(x)\delta(x-y).
\]

(37)

Let us call this algebra \( C_2 \). This is supplemented with the rescaled hamiltonian,

\[
H_2 = \frac{1}{4\lambda^2(1 - \tau^2)^2} \int tr(I^2 + J^2)dx.
\]

(38)

Clearly in the limit \( \tau \to 0 \) we recover the standard formalism. We have just shown that this formalism leads to our equations of motion, with the above identification of \( \rho \).

The Poisson bracket relations above are an obscure way of writing \( C_2 \). Without a tedious calculation it is not even clear that they obey the Jacobi identities. But there is a change of variables which simplifies them substantially. It is suggested by the analogy of \( C_1 \) to \( E(3) \) and \( C_2 \) to \( SU(2) \oplus SU(2) \) \[1\].

Let us define \( L \) and \( R \) by

\[
I = 2\lambda^2(1 - \tau^2)(\alpha L + \beta R)
\]

\[
J = 2\tau\lambda^2(1 - \tau^2)(\gamma L + \delta R).
\]

(39)

with \( L \) and \( R \) generators of two commuting affine Lie algebras,

\[
\{ L_\alpha(x), L_\beta(y) \}_2 = f_{\alpha\beta\gamma}L_\gamma(x)\delta(x-y) + \frac{k}{2\pi}\delta_{\alpha\beta}\delta'(x-y)
\]

(40)

\[
\{ R_\alpha(x), R_\beta(y) \}_2 = f_{\alpha\beta\gamma}R_\gamma(x)\delta(x-y) - \frac{k}{2\pi}\delta_{\alpha\beta}\delta'(x-y)
\]

(41)

\[
\{ L_\alpha(x), R_\beta(y) \}_2 = 0
\]

(42)

and \( k, \tilde{k} \) are a pair of constants.

It is now straightforward (although quite tedious) to show that this algebra goes over to \( C_2 \) under the above change of variables, if we choose

\[
\alpha = 1 - \rho \tau \quad \beta = 1 + \rho \tau
\]

(43)
\[ \gamma = \tau (\rho \tau - 1) \quad \delta = \tau (\rho \tau + 1) \quad (44) \]

\[ \epsilon = \mu = \rho \tau^2 \quad a = \rho \quad (45) \]

\[ k = \frac{\pi}{2\lambda^2 \tau (1 - \rho \tau)^2} \quad \bar{k} = \frac{\pi}{2\lambda^2 \tau (1 + \rho \tau)^2} \quad (46) \]

Thus our current algebra \( C_2 \) is isomorphic to a direct sum of two affine Lie algebras. These Poisson brackets can be derived from an action principle as in ref.\[1\], but we will postpone that discussion.

Note that the change of variables to \( L \) and \( R \) is singular if \( \tau = 0 \). The equations \( (35)-(37) \) are still well defined in this limit, it is just that at \( \tau = 0 \) \( C_2 \) is no longer isomorphic to the sum of two affine Lie algebras since \( L \) and \( R \) don’t exist. On the other hand at \( \tau = \pm 1 \), \( L \) and \( R \) Poisson brackets exist, but not those of \( I \) and \( J \).

It is more convenient to use \( L \) and \( R \) as the basic dynamical variables, since they have simple Poisson brackets. The hamiltonian in this language is

\[ H_2 = \lambda^2 (1 + \tau^2) \int \text{tr} \left[ (1 - \rho \tau)^2 L^2 + 2 \frac{1 - \tau^2}{1 + \tau^2} (1 - \rho^2 \tau^2) LR + (1 + \rho \tau)^2 R^2 \right] dx \quad (47) \]

Also, a choice of independent parameters is \( \rho, k, \bar{k} \). We find for instance that

\[ \frac{2\pi \rho}{\lambda^2} = (k - \bar{k}) \frac{16kk}{(\sqrt{k} + \sqrt{\bar{k}})^4}. \quad (48) \]

Thus we see that \( k \) and \( \bar{k} \) must have the same sign for \( \rho \) to be real. Without loss of generality we can choose them to be positive. The limit \( \tau \to 0 \) which should be the conventional theory corresponds to letting \( k, \bar{k} \to \infty \) keeping \( k - \bar{k} \) fixed and equal to an even integer. In this limit,

\[ \frac{2\pi \rho}{\lambda^2} = k - \bar{k} \quad (49) \]

so that we can identify \( n = \frac{k - \bar{k}}{2} \). In general it is not necessary that this quantity be an integer, it can take any value determined as above by a pair of positive integers.

Clearly this general canonical quantization cannot be obtained by the standard action principle. We can now find (following \[1\]) an action principle which gives this canonical formalism. We define variables \( l, r : R^{1,1} \to G \) such that

\[ L = \frac{k}{2\pi} \frac{\partial l}{\partial x} l^{-1}, \quad R = \frac{\bar{k}}{2\pi} \frac{\partial r}{\partial x} r^{-1} \quad (50) \]

Now define the action

\[ S_2 = k \Gamma(l) - \bar{k} \Gamma(r) - \lambda^2 (1 + \tau^2) \int \text{tr} \left[ (1 - \rho \tau)^2 L^2 + 2 \frac{1 - \tau^2}{1 + \tau^2} (1 - \rho^2 \tau^2) LR + (1 + \rho \tau)^2 R^2 \right] dx dt \quad (51) \]
where $\Gamma$ is the WZW term defined in (8). Arguments exactly analogous to those in Ref.[1] show that this leads to the new canonical formalism. For $e^{iS}$ to be single valued, $k$ and $\bar{k}$ must be integers. We will obtain the same requirement from the representation theory of the affine Lie algebra. This form of the action does not look Lorentz invariant, but we will show that the theory is in fact classically conformally invariant. This implies in particular Lorentz invariance.

The components of the stress tensor are,

$$\Theta_{00} = \Theta_{11} = \frac{1}{4\lambda^2(1-\tau^2)^2} \text{tr} [I^2 + J^2] \quad (52)$$

$$\Theta_{01} = \Theta_{10} = \frac{1}{2\lambda^2(1-\tau^2)^2} \text{tr} [IJ] \quad (53)$$

It is often more convenient to use instead the quantities

$$\Theta = -\frac{1}{2}(\Theta_{00} + \Theta_{01}) \quad \tilde{\Theta} = \frac{1}{2}(\Theta_{00} - \Theta_{01}). \quad (54)$$

Classical conformal invariance amounts to the statements

$$(\partial_t - \partial_x)\Theta = 0 \quad (\partial_t + \partial_x)\tilde{\Theta} = 0 \quad (55)$$

together with the Poisson brackets

$$\{\Theta(u), \Theta(v)\} = \Theta([u,v]) \quad (56)$$

$$\{\tilde{\Theta}(u), \tilde{\Theta}(v)\} = \tilde{\Theta}([u,v]) \quad (57)$$

$$\{\Theta(u), \tilde{\Theta}(v)\} = 0. \quad (58)$$

Here $\Theta(u) = \int dxu(x)\Theta(x)$ etc.

In general, if $M_A$ are currents satisfying an affine Lie algebra (in the sense of Poisson brackets),

$$\{M_A(x), M_B(y)\} = f^{C}_{AB} M_C(x)\delta(x-y) + \frac{\Omega_{AB}}{2\pi} \delta'(x-y) \quad (59)$$

the conditions for

$$\Theta(x) = G^{AB} M_A(x)M_B(x) \quad (60)$$

$$\tilde{\Theta}(x) = \tilde{G}^{AB} M_A(x)M_B(x) \quad (61)$$

to satisfy the previous Poisson brackets are

$$G^{AB} = 2G^{AC}\frac{\Omega_{CD}}{2\pi}G^{DB} \quad (62)$$
\[ G^{AB} = 2G^{AC} \frac{\Omega_{CD}}{2\pi} G^{DB} \]  
(63)

\[ 0 = 2\tilde{G}^{AC} \frac{\Omega_{CD}}{2\pi} G^{DB}. \]  
(64)

These relations may be regarded as the classical analogue of the Master Virasoro equation \[9\]. In our case, we can choose \( M = (L, R) \), and write \( \Theta, \tilde{\Theta} \) as quadratic expressions in \( L \) and \( R \),

\[
\Theta = -\frac{1}{8\lambda^2(1-\tau^2)^2} (I+J)^2 = G^{AB} M_A M_B \]
(65)

\[
\tilde{\Theta} = -\frac{1}{8\lambda^2(1-\tau^2)^2} (I-J)^2 = \tilde{G}^{AB} M_A M_B. \]
(66)

The matrices \( G^{AB} \) and \( \tilde{G}^{AB} \) are

\[
G^{AB} = -\frac{\lambda^2}{2} \begin{pmatrix} (1-\rho^2)(1-\tau^2)1_3 & (1-\rho^2\tau^2)(1-\tau^2)1_3 \\ (1-\rho^2\tau^2)(1-\tau^2)1_3 & (1+\rho\tau)^2(1+\tau^2)1_3 \end{pmatrix} \]
(67)

and

\[
\tilde{G}^{AB} = \frac{\lambda^2}{2} \begin{pmatrix} (1-\rho\tau)^2(1+\tau^2)1_3 & (1-\rho^2\tau^2)(1-\tau^2)1_3 \\ (1-\rho^2\tau^2)(1-\tau^2)1_3 & (1+\rho\tau)^2(1-\tau^2)1_3 \end{pmatrix} \]
(68)

and the central term is given by

\[
\Omega^{AB} = \begin{pmatrix} k1_3 & 0 \\ 0 & -\bar{k}1_3 \end{pmatrix} \]
(69)

where by \( 1_3 \) we mean the \( 3 \times 3 \) identity matrix. It is now straightforward to check that the conditions above for classical conformal invariance are in fact satisfied. The conditions for conformal invariance at the quantum level are more stringent; we will not find any new unitary conformal field theories in our approach.

5 Quantization

So far we have mainly talked about an alternative classical formulation of the nonlinear model. Now let us begin quantizing the new formalism.

Recall that \( C_2 \) is the analogue of the CCR in the nonlinear model. Therefore finding a representation of \( C_2 \) is the first step in quantizing the nonlinear model. We need a unitary
representation to construct a quantum theory. It is desirable also that the representation be irreducible. For, each subrepresentation would otherwise form an independent dynamical system (this is why one always picks an irreducible representation for the CCR in ordinary quantization). Once operators representing $L$ and $R$ are found, we can try to construct the Hamiltonian and other observables as their functions. But this last step is going to involve a renormalization.

Representations of our current algebra can be classified by purely algebraic methods. The earlier considerations require that $k$ and $\bar{k}$ be both positive. Since $k, \bar{k}$ appear with opposite signs in the central terms of $L$ and $R$, we find that the representations of $L$ must be highest weight and that of $R$ lowest weight, if the representation is to be unitary.

For $k = 1$, there is a unique representation for the current algebra, which can be written in terms of fermions [3]. For higher level numbers, fermionic representations are a direct sum of several irreducible representations. The multiplicity of a given irreducible representation is finite if a condition, known as the Goddard–Nahm–Olive condition (GNO), [3, 14], is satisfied. Then, at least when the multiplicity is finite, the nonlinear model is equivalent to one sector of the fermionic theory. Conversely, the fermionic theory breaks up into a set of nonlinear models which don’t interact with each other.

It is convenient to use a countable basis for the Lie algebra when discussing representations. It is possible to introduce such a basis on the space of functions on the real line, if we consider only functions which vanish at infinity (to have finite energy, our variables $L, R$ must in fact vanish at infinity). A convenient basis on the space of functions on $R$ vanishing at infinity is,

$$e_r^m(x) = \left(\frac{1}{\pi(1 + x^2)}\right)^r \left(\frac{i - x}{i + x}\right)^m. \tag{70}$$

We should use $r = 1$ for currents and $r = \frac{1}{2}$ for spinor fields. This is because $e_m^{1/2}$ are orthonormal (so that Canonical Anti-commutation Relations simplify), while $e_m^1$ satisfy

$$\sum_{m \in \mathbb{Z}} e_m^1(x)e_{p-m}^1(y) = e_p^1(x-y)\delta(x-y) \tag{71}$$

$$\sum_{m \in \mathbb{Z}} i me_m^1(x)e_{-m}^1(y) = \frac{1}{2\pi}\delta'(x-y). \tag{72}$$

With

$$L_\alpha(x) = \sum_m L_{m\alpha} e_m^1(x) \quad R_\alpha(x) = \sum_m R_{m\alpha} e_m^1(x) \tag{73}$$
we have the relations

$$\{L_{m\alpha}, L_{n\beta}\} = f_{\alpha\beta\gamma} L_{m+n\gamma} - i k m \delta_{\alpha\beta} \delta_{m,-n}$$ (74)

$$\{R_{m\alpha}, R_{n\beta}\} = f_{\alpha\beta\gamma} R_{m+n\gamma} + i \bar{k} m \delta_{\alpha\beta} \delta_{m,-n}.$$ (75)

A unitary representation of this algebra will be given by a set of operators satisfying

$$L^\dagger_{m\alpha} = L_{-m\alpha} \quad R^\dagger_{m\alpha} = R_{-m\alpha} \quad \text{(unitarity)}$$ (76)

$$[\hat{L}_{m\alpha}, \hat{L}_{n\beta}] = i f_{\alpha\beta\gamma} \hat{L}_{m+n\gamma} + km \delta_{\alpha\beta} \delta_{m,-n}$$ (77)

$$[\hat{R}_{m\alpha}, \hat{R}_{n\beta}] = i f_{\alpha\beta\gamma} \hat{R}_{m+n\gamma} - \bar{k} m \delta_{\alpha\beta} \delta_{m,-n}.$$ (78)

The representation of $L$ can be chosen to be highest weight (i.e., there exists a vector with $L_m|0 >= 0$ for $m > 0$) and unitary. In this case $k$ is required to be a positive integer. Then the representation for $R$ has to be lowest weight if we want it to be unitary as well, since $k, \bar{k}$ appear with opposite signs in the central terms (recall that $k, \bar{k}$ must have the same sign from our earlier arguments). A lowest weight representation is just the complex conjugate of a highest weight representation. Hence, to every pair of irreducible highest weight representations of the affine Lie algebra of $G$, there is a quantization of the WZW model.

Thus we have a state $|0 >$ from which all other states in the Hilbert space can be constructed, satisfying

$$\hat{L}_{m\alpha}|0 >= 0, \quad m > 0$$

$$\hat{L}_{0\alpha}|0 >= 0, \quad \alpha > 0$$

$$\hat{R}_{m\alpha}|0 >= 0, \quad m < 0$$

$$\hat{R}_{0\alpha}|0 >= 0, \quad \alpha < 0$$

$$\hat{L}_{0\alpha}|0 >= \alpha.\mu|0 >, \quad \alpha \in \mathcal{H}$$

$$\hat{R}_{0\alpha}|0 >= -\alpha.\bar{\mu}|0 >, \quad \alpha \in \mathcal{H}$$

This is the statement that $|0 >$ is a highest weight vector for $\hat{L}$ and a lowest weight vector for $\hat{R}$. Here, $\alpha > 0$ means that $\alpha$ is a positive root of $G$. Also, $\mathcal{H}$ is the Cartan subalgebra of $G$ and $\mu, \bar{\mu}$ the highest weights of some representations. The necessary and sufficient condition for such a representation to exist is that $k, \bar{k}$ be integers with

$$k \geq \psi.\mu \geq 0, \quad \bar{k} \geq \psi.\bar{\mu} \geq 0.$$ (79)
where $\psi$ is the highest root of $G$ (in general $\psi$ is normalized to have length $\sqrt{2}$).

The vector space for the representation labelled by $(k, \mu)$ is constructed by acting on $|0\rangle$ by the ‘raising operators’ $\{\hat{L}_m, m < 0 \text{ and } m = 0, \alpha < 0\}$ and $\{\hat{R}_m, m > 0 \text{ and } m = 0, \alpha > 0\}$. This is, however, a standard construction described for example in [11, 6]. It should be kept in mind that the state $|0\rangle$ is not in general the ground state. The true ground state of the theory is the one that minimizes $H$ and is in general quite different from $|0\rangle$.

Let us define the normal ordering (suspending the summation convention on $\alpha$),

\[
\hat{L}_m \hat{L}_n = \hat{L}_n \hat{L}_m, \quad n < 0
\]

\[
\hat{R}_m \hat{R}_n = \hat{R}_n \hat{R}_m, \quad n > 0
\]

(80)

Then a representation of the direct sum of two Virasoro algebras, say $V_1 \oplus V_2$, may be constructed as

\[
\hat{T}(x) = \frac{\pi}{k + \tilde{h}(G)} \hat{L}_\alpha(x) \hat{L}_\alpha(x)
\]

(82)

\[
\hat{T}(x) = -\frac{\pi}{k + \tilde{h}(G)} \hat{R}_\alpha(x) \hat{R}_\alpha(x).
\]

(83)

Note that the factor in front of $L^2$ and $R^2$ has changed from the classical value. $k, \tilde{k}$ have been shifted by an integer $\tilde{h}(G)$ characteristic of the algebra, called the ‘dual Coxeter number’ [8]:

\[
\tilde{h}(G) \delta_{\alpha \epsilon} = \frac{1}{2} f_{\alpha \gamma} f_{\epsilon \gamma}.
\]

(84)

These will then satisfy

\[
\frac{1}{i} \int u(y) \hat{T}(y) \, dy, \frac{1}{i} \hat{L}_\alpha(x) = u(x) \frac{1}{i} \frac{\partial \hat{L}_\alpha}{\partial x}
\]

(85)

and

\[
\frac{1}{i} \hat{T}(u), \frac{1}{i} \hat{T}(v) = \frac{1}{i} \hat{T}([u, v]) + c \omega(u, v)
\]

(87)

\[
\frac{1}{i} \hat{T}(u), \frac{1}{i} \hat{T}(v) = \frac{1}{i} \hat{T}([u, v]) + \bar{c} \omega(u, v).
\]

(88)
It is the presence of the extra central term on the right hand side of these relations which makes \( \hat{T}, \tilde{T} \) satisfying \( V_1 \oplus V_2 \) rather than the algebra of vector fields on the real line, as in the classical theory. The cocycle \( \omega \) is

\[
\omega(u,v) = \frac{\pi i}{12} \int [u \left( \frac{d^3}{dx^3} + \frac{d}{dx} \right) v - u \leftrightarrow v] \, dx
\]

The central charges are

\[
c = \frac{k \dim G}{k + \hbar(G)}, \quad \tilde{c} = \frac{\bar{k} \dim \bar{G}}{k + \hbar(G)}.
\]

The quantities \( T, \tilde{T} \) do not however define components of the true (physical) stress tensor; these are given by \( \Theta \) and \( \tilde{\Theta} \). In general the physical stress tensor does not satisfy \( V_1 \oplus V_2 \), so that the theory is not conformally invariant. \( \hat{T}, \tilde{T} \) are still useful to construct the momentum density,

\[
\hat{\Theta}_{01}(x) = \hat{T}(x) + \tilde{T}(x).
\]

The hamiltonian is more involved:

\[
H = \int \Theta_{00}(x) dx = \int \left[ \frac{1 + \tau^2}{2\tau} (T - \tilde{T}) + gL_\alpha R_\alpha \right] dx
\]

where \( g \) is a ‘coupling constant’,

\[
g = \frac{\pi}{2\sqrt{(kk)}} \frac{(1 - \tau^2)}{\tau}.
\]

(We can trade \( \tau \) for \( g, k, \bar{k} \)). Although the first two terms in the hamiltonian density are finite for states constructed from \( |0> \), the last term is divergent. This is because, the operator \( \int \hat{L} \cdot \hat{R} dx = \sum_m \hat{L}_m \hat{R}_{-m} \) consists of two creation operators when \( m < 0 \); the correction to the energy levels in perturbation theory is divergent at order \( g^2 \). One must then introduce a cut–off \( \Lambda \) and define a regularized hamiltonian

\[
H_\Lambda = \int \left[ Z(g(\Lambda), \Lambda)(T - \tilde{T}) + g(\Lambda)\hat{L}_\alpha \hat{R}_\alpha \right] dx.
\]

The coupling constant and the ‘wave–function’ renormalization \( Z \) must depend on \( \Lambda \) in such a way that the energy levels remain finite as \( \Lambda \rightarrow \infty \). This procedure will break scale invariance.

A particular way to construct representations of the current algebra is to use fermions. Let us introduce fermions transforming under the representations \( r, \tilde{r} \) of \( G \):

\[
[b_m^i, b_n^j]_+ = \delta^{ij} \delta_{m,-n} \quad [\tilde{b}_m^i, \tilde{b}_n^j]_+ = \tilde{\delta}^{ij} \delta_{m,-n}.
\]
The corresponding field operators $\psi, \tilde{\psi}$ are defined by
\[
\psi_i(x) = \sum_m b^i_m e^m(x), \quad \tilde{\psi}_i(x) = \sum_m \tilde{b}^i_m e^m(x)
\]
and satisfy the Canonical Anti-commutation Relations (CAR). We will assume that the representations satisfy the GNO condition [3]. That is, $r, \tilde{r}$ are isomorphic to the representation on the tangent space of a symmetric space of $G$. The most obvious choice is that $r (\tilde{r})$ is the direct sum of $k(\tilde{k})$ copies of the fundamental representation.

We find a representation of the CAR satisfying
\[
b^i_m |0\rangle = 0, \quad \tilde{b}^i_m |0\rangle = 0
\]
Then we define normal ordering by
\[
\circ b^i_m b^j_n \circ = -b^i_m b^j_n \quad \text{if} \quad m > 0 \quad \text{and} \quad n > 0
\]
\[
= \frac{1}{2} [b^i_0, b^j_0] \quad \text{if} \quad m = n = 0
\]
\[
= b^i_m b^j_n \quad \text{otherwise}
\]
and
\[
\circ \tilde{b}^i_m \tilde{b}^j_n \circ = -\tilde{b}^i_m \tilde{b}^j_n \quad \text{if} \quad m < 0 \quad \text{and} \quad n > 0
\]
\[
= \frac{1}{2} [\tilde{b}^i_0, \tilde{b}^j_0] \quad \text{if} \quad m = n = 0
\]
\[
= \tilde{b}^i_n \tilde{b}^j_n \quad \text{otherwise}
\]
Then, we can construct a representation of the current algebra:
\[
\hat{L}_\alpha = \frac{i}{2} \circ \tilde{\psi}^i r_{ij\alpha} \psi^j \circ \quad \hat{R}_\alpha = \frac{i}{2} \circ \tilde{\psi}^i \tilde{r}_{ij\alpha} \psi^j \circ
\]
This representation is finitely reducible if $r, \tilde{r}$ are isomorphic to the representations of $G$ on the tangent space of symmetric spaces (the GNO condition). The constants $k, \tilde{k}$ are given by the Dynkin index (quadratic Casimirs) of the representations $r, \tilde{r}$. They are, of course, positive integers. When the GNO condition is satisfied, moreover, the quantities $\hat{T}$ and $\hat{\tilde{T}}$ form the stress tensor of a free fermion field [3].

\[
\hat{T} = \frac{1}{2} \frac{\partial \psi}{\partial x} \psi \circ \quad \hat{\tilde{T}} = \frac{1}{2} \frac{\partial \tilde{\psi}}{\partial x} \tilde{\psi} \circ
\]
Thus the hamiltonian becomes
\[
H = Z(g) \int \left\{ \circ \frac{1}{2} \frac{\partial \psi}{\partial x} \psi \circ - \circ \frac{1}{2} \frac{\partial \tilde{\psi}}{\partial x} \tilde{\psi} \circ \right\} dx
\]
\[
- \frac{g}{4} \int \circ \psi_i \rho_{ij\alpha} \psi^j(x) \circ \circ \tilde{\psi}_k \rho_{kl\alpha} \tilde{\psi}^l(x) \circ \circ dx.
\]
This is the hamiltonian of a non–abelian Thirring model. The first term is the free hamiltonian except for a ‘wave–function’ renormalization $Z(g)$. The second is the current–current coupling of the Thirring model. With this identification, $g$ is the Thirring model coupling constant. We have to use regularization and renormalization to get a well–defined theory. The Thirring model is usually defined with left and right sectors in the same representation. Our model with $k \neq \bar{k}$ will have different representations and hence will violate parity in general.

Since the theory is not expected to be finite in general, we expect conformal invariance to be broken in the quantum theory. The usual Thirring model has an ultraviolet stable fixed point, so we should expect to have such a fixed point in this theory as well.

We briefly comment on scale invariance of the quantum theory we are describing. We don’t find new fixed points for the $\beta$ function, but we expect to recover the infrared (IR) and the ultraviolet (UV) fixed points found in the conventional formalism, respectively in the limits

$$\tau \to 0, \quad \rho = \pm 1 \quad (IR)$$

$$\tau \to \infty, \quad \rho = \pm 1 \quad (UV)$$

In these limits $\lambda^2$ tends to

$$\lambda^2 \to \frac{2\pi}{k - \bar{k}} \quad \tau \to 0$$

$$0 \quad \tau \to \infty. \quad \text{(103)}$$

However for $\tau \to 0, \infty$, our transformations are singular, so that our argument is not rigorously proven.

We mention also a quantization based on a non–unitary representation of the current algebra. If we pick highest weight representations for both $L$ and $R$, we will have a pseudo–unitary representation, which is unitary with respect to a norm that is not positive. Although such a representation will have negative norm states, we mention this possibility because of a curious realization of conformal invariance (which we will call "half–conformal" invariance). Recall that the values $\rho = \pm 1$ correspond to conformally invariant theories in the conventional formalism. In the more general unitary quantization we described earlier, for a generic value of $\tau$, even these points do not describe conformal field theories. If $\rho = -1,$
\[ \Theta = -\frac{\lambda^2}{2}(1 - \tau^2)^2(L + R)^2, \quad \tilde{\Theta} = -\frac{\lambda^2}{2}(1 + \tau)^2L + (1 - \tau)^2R \]  

(104)

but, unlike the case of highest and lowest weight representations, if \( L \) and \( R \) both form a highest weight representation, the operator \( \Theta \) can now be defined without any divergence, by normal ordering. It will also satisfy the Virasoro algebra, with the usual rescaling of the constant factor. However, even when \( \rho = -1 \), \( \tilde{\Theta} \) does not satisfy the Virasoro algebra, so we still do not have full conformal invariance. If \( \rho = +1 \), instead, \( \tilde{\Theta} \) will satisfy the Virasoro algebra, while \( \Theta \) will not. Thus in this version of the theory, we get half conformal invariance when \( \lambda \) satisfies

\[ \frac{2\pi}{\lambda^2} = \frac{|k - \bar{k}|}{(\sqrt{k} + \sqrt{k})^4}. \]  

(105)

By simple arguments it is conjectured that, if the theory is also Lorentz invariant, this is all what we need for scale invariance. Namely ‘half–conformal invariance’ and Lorentz invariance guarantee the theory to be scale invariant. Usually, unitary scale invariance theories are also fully conformal invariant [12]. Our example does not contradict that expectation, since the half–conformal invariant theory is not unitary.

6 Remarks

We wish to make a comment about the algebra \( (10) - (12) \). We have so far assumed that the parameter \( \tau \) is real. It is an interesting fact that even when the field variable \( g \) is valued in a compact Lie group, as in our case, this is in fact not necessary. In the current algebra above, \( \tau \) can be purely imaginary; only \( \tau^2 \) appearing in the current algebra. The parameters \( a, \epsilon, \mu \) also remain real when we allow \( \tau \) to be purely imaginary. In this case the algebra \( C_2 \) is not isomorphic to a sum of two affine Lie algebras based on \( G \); instead it becomes isomorphic to the affine Lie algebra of the complexification of \( G \) (but still viewed as a real Lie algebra). Thus if \( G = SU(2) \), we get the affine Lie algebra of \( SL_2(C) \) or \( O(1, 3) \). The parameter \( k \) becomes complex and \( \bar{k} = -k^* \), where \( k^* \) is the complex conjugate. Thus we find the remarkable result that a WZW model with compact target space can have a non–compact current algebra. Representations of such a current algebra are typically non–unitary, [13, 14] so that quantization along these lines may not be interesting (there are also unitary representations of zero central charge [15], but they are not useful for our purpose).
On the other hand, such current algebras seem to be a good starting point for quantization in the sense of Drinfeld \[ \text{[16]} \]; this possibility is being studied.

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