Speed of Quantum Evolution of Entangled Two Qubits States: Local vs. Global Evolution

S. Curilef$^1$, C. Zander$^2$, and A.R. Plastino$^2$

$^1$Departamento de Física, Universidad Católica del Norte, Antofagasta, Chile
$^2$Physics Department, University of Pretoria, Pretoria 0002, South Africa

E-mail: arplastino@maple.up.ac.za

Abstract. There is a lower bound for the “speed” of quantum evolution as measured by the time needed to reach an orthogonal state. We show that, for two-qubits systems, states saturating the quantum speed limit tend to exhibit a small amount of local evolution, as measured by the fidelity between the initial and final single qubit states after the time $\tau$ required by the composite system to reach an orthogonal state. Consequently, a trade-off between the speed of global evolution and the amount of local evolution seems to be at work.

1. Introduction
The “speed” of quantum evolution, as given by the time required by a quantum system to reach a state orthogonal to the initial state, has been the focus of considerable interest in recent years, because of its importance in connection with the physical limits that the fundamental laws of quantum mechanics impose on the rapidity of information processing and information transmission [1, 2, 3]. When processing information one expects the output state of the computing device to be reasonably different from the input state [1]. In quantum mechanics two states are distinguishable if they are orthogonal. Elementary computational operations, thus, involve going from one quantum state to an orthogonal one. As a consequence, lower bounds on the time required to reach an orthogonal state also provide estimations on how fast one can perform basic computation steps [2]. These, in turn, allow for the estimation of fundamental limits on how fast can a physical computer run [1, 2]. This procedure has even been applied to estimate the total number of basic logical operations that has been performed by the Universe (regarded as a computing device) during its complete history [3]. The minimum time necessary to reach an orthogonal state is also important in connection with the time-energy Heisenberg uncertainty principle [2], which can be formulated in terms of the minimum time that a system with a given spread in energy needs in order to evolve to a distinguishable state (that is, to an orthogonal state).

Of particular interest is the problem of the “speed” of quantum evolution in the case of composite quantum systems. Composite systems evolving under the effect of local Hamiltonians do not saturate, in most cases, the quantum speed limit [4, 5, 6, 7, 8]. However, there are two features of the states of composite systems that tend to “speed up” the quantum evolution: (i) unevenly distributed energy resources among the subsystems [4], and (ii) quantum entanglement [4, 5, 6, 7, 8]. In the present work we explore the relation between quantum speed and another
aspect of composite systems’s states: the local evolution of the subsystem’s, as measured by the fidelity between the initial subsystem’s state and the final subsystem’s state (where by “final” we mean the subsystem’s state at the time when the global state is orthogonal to the initial one). We show that, for entangled two-qubits states, states that evolve fast form the global point of view (that is, that reach an orthogonal state soon) tend to exhibit little local evolution.

2. Speed of Evolution

A natural way for measuring the “speed” of quantum evolution is provided by the time interval $\tau$ that a given initial state $|\psi(t_0)\rangle$ needs to evolve into an orthogonal state $|\psi(t_0)\rangle = 0$.

$$\langle\psi(t_0)|\psi(t_0 + \tau)\rangle = 0. \quad (1)$$

A lower bound for $\tau$ is given by $\tau_{\text{min}} = \max \left( \frac{\pi \hbar}{2E}, \frac{\pi \hbar}{2\Delta E} \right)$, where $E = \langle H \rangle$ stands for the energy’s expectation value and $\Delta E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$ for the energy’s uncertainty.

3. Two-Qubits System

We are going to consider a composite system constituted by two qubits. That is, two identical (but distinguishable) subsystems each one with an associated two-dimensional Hilbert space. The Hamiltonian governing the evolution of our system is of the form

$$H = H_a \otimes I_b + I_a \otimes H_b, \quad (3)$$

where $H_{a,b}$ and $I_{a,b}$ are, respectively, the Hamiltonian and the identity operator acting on the first and the second qubits. The time evolution operator associated with the Hamiltonian (3) is local and, consequently, the amount of entanglement of the system does not change in time. The two single qubit Hamiltonians $H_i$ have the same structure, with eigenstates, $\{|0\rangle, |1\rangle\}$, and corresponding eigenenergies $E_0 = 0, E_1 = \epsilon$. Hamiltonians like (3) are relevant for the study of some fundamental aspects of quantum entanglement (see for instance [9]) and, particularly, in connection with the problem of the speed of quantum evolution of entangled states (see [4, 8]). Our composite system can be described in terms of the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, which can be rewritten as $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. The general state of our two qubit system is then $|\psi(t_0)\rangle = \sum_{j=0}^{3} a_j |j\rangle$, where the $a_j$’s are complex coefficients satisfying the normalization requirement, $\sum_{j=0}^{3} |a_j|^2 = 1$.

4. Local vs. Global Evolution

Separable pure states of the form $\phi_1 \otimes \phi_2$ do not, in general, saturate the speed limit bound. The only separable states reaching the speed limit are highly asymmetric states such as $|0\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, where all the energy resources are concentrated in one qubit (the second one in this case) which is the only qubit really participating in the time evolution. On the contrary, symmetric separable states states of the form $|\phi\rangle \otimes |\phi\rangle$ do not saturate the speed bound. Symmetric two-qubit states, with the energy resources equally shared by both qubits, need entanglement in order to reach the speed limit. Here we are going to consider two parameterized families of energetically symmetric two qubits states. We want to investigate how does the local features of the quantum evolution change as we consider states approaching the speed limit.

The first family of states that we are going to consider (family 1) is
where $\pi/2 \leq \beta \leq \pi$ and $0 \leq \gamma_i < 2\pi$. For family 1 we have

$$\frac{\tau}{\tau_{\text{min}}} = \frac{2\beta}{\pi \sqrt{1 - \cos \beta}}$$

and the time needed to reach an orthogonal state is $\tau = \hbar \beta / \varepsilon$. The second family (family 2) is given by

$$a_0 = \sqrt{\frac{-s}{2(1-s)}}$$

$$a_1 = \exp \left( i\mu_1 \right) / 2$$

$$a_2 = \exp \left( i\mu_1 \right) / 2$$

$$a_3 = \exp \left( i\mu_2 \right) \frac{1}{\sqrt{2(1-s)}}$$

where $s < 0$ and $0 \leq \mu_i < 2\pi$. In the case of family 2 we have,

$$\frac{\tau}{\tau_{\text{min}}} = \sqrt{\frac{1 - 6s + s^2}{(-1 + s)^2}}$$

and $\tau = \pi \hbar / \varepsilon$.

These two families of states are specially important because all symmetric two-qubits states actually evolving to an orthogonal state belong to either of the alluded families. How close are those states to reaching the speed limit is measured by how close is the quotient $\tau / \tau_{\text{min}}$ to unity.

Let us now consider the initial density operator $\rho(t_0) = |\psi(t_0)\rangle \langle \psi(t_0)|$, which after a time $\tau$ evolves to the density operator $\rho_{\tau} = U \rho_0 U^\dagger$, where $U = \exp(-iH\tau / \hbar)$. The partial trace

$$\rho_0^{(a)} = \text{Tr}_b \rho_0$$

gives the (marginal) density matrix describing the initial state of subsystem $a$. After a time $\tau$ it evolves into

$$\rho_{\tau}^{(a)} = \text{Tr}_b \rho_{\tau}.$$
$F[\rho, \varrho] = \text{Tr}_b \sqrt{\rho^{1/2} \varrho \rho^{1/2}}$, constitutes a very useful measure for the distinguishability between two density matrices [10]. We have $F \in [0,1]$, with $F = 0$ when the two matrices are perfectly distinguishable and $F = 1$ when they are equal. In our case we have

$$F[\rho_0^{(a)}, \rho_\tau^{(a)}] = \text{Tr}_a \sqrt{\rho_0^{(a)} \rho_\tau^{(a)}} \left[ \rho_0^{(a)} \right]^{1/2}.$$  \hspace{1cm} (16)

In Figure 1 are depicted, for both families of states, the plots of $F[\rho_0^{(a)}, \rho_\tau^{(a)}]$ vs. $\tau/\tau_{\text{min}}$. We see that in both cases $F[\rho_0^{(a)}, \rho_\tau^{(a)}]$ is a monotonically decreasing function of $\tau/\tau_{\text{min}}$. In both cases we have that $F[\rho_0^{(a)}, \rho_\tau^{(a)}]$ adopts its maximum value when $\tau/\tau_{\text{min}} = 1$. In other words, when the (composite) state reaches the quantum speed limit, the individual subsystems experience the lowest amount of local evolution (corresponding to the largest value of $F[\rho_0^{(a)}, \rho_\tau^{(a)}]$).

5. Conclusions

The concept of the speed of quantum evolution plays an important role within the physics of information [1, 2, 3]. We have here shown that composite quantum states saturating the quantum speed limit tend to exhibit a small amount of local evolution, as measured by the fidelity between the initial and final subsystem’s states after the time $\tau$ required by the (global) state of the composite system to reach an orthogonal state. Consequently, there seems to be a trade-off between the speed of global evolution and the amount of local evolution. This trade-off may be relevant for a deeper understanding of the fundamental limits on how fast can physical computers run. A systematic exploration of the aforementioned trade-off is underway and will be addressed elsewhere.
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