Quasihole wavefunctions for the Calogero model

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ABSTRACT

The one-quasihole wavefunctions and their norms are derived for the system of particles on the line with inverse-square interactions and harmonic confining potential.
The Calogero-Sutherland-Moser class of models of interacting particles in one dimension [1-3] has received a great deal of attention, mostly due to its interesting mathematical properties and connection to fractional statistics [4-7].

It is of interest to have explicit energy wavefunctions for these systems, since they are needed to calculate correlation functions. So far, it is mostly the periodic (Sutherland) model that has been used for the purpose of such calculations [8-10], and its wavefunctions (termed Jack polynomials in the mathematical literature [11]) have been extensively studied [12]. The original harmonic (Calogero) system, on the other hand, has been rather neglected in this respect, due, mainly, to its translation non-invariance. Its wavefunctions are, likewise, rather obscure. Although in principle they can be obtained either with the original diagonalization method [1] or with the operator method [13,14], their general explicit form is unknown. The original wavefunctions found by Calogero are, to this day, the only explicitly known ones. Nevertheless, this system has the advantage of possessing ladder operators, and would thus be more suited to an algebraic approach.

In this note, we present a set of wavefunctions of interest in the many-body properties of this system, namely the one-hole wavefunctions, along with their norms. Generically, hole states are simpler than particle states in these systems. For instance, the one-hole wavefunctions of the Sutherland model are identical to the free fermion ones, upon division by the Vandermonde determinant in the relevant power. (Cf. also the Laughlin states.) As we will see, the Calogero holes are not so simple, but they are still amenable to a complete analysis.

The model of consideration is described by the hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i>j} \frac{l(l-1)}{(x_i - x_j)^2} + \frac{1}{2} \sum_{i=1}^{N} x_i^2 \]  

where we chose units such that the particle masses and oscillator frequency be one. Particle statistics are irrelevant, due to the impenetrability of the mutual potential, and we shall consider symmetric wavefunctions. The dynamics are determined by
the coupling \( l \), with \( l = 0, 1 \) corresponding to noninteracting bosons or fermions respectively. For \( l = 1 \) we have the usual picture of the Fermi sea with particle and hole excitations. By following these states as they evolve for \( l \neq 1 \), we are led to the notion of (quasi)particle and (quasi)holes.

For noninteracting particles, the wavefunction of a particle at energy \( n \) in the coherent state representation is simply \( z_i^n \), and the action of the oscillator creation operator \( a_i^\dagger \) is multiplication by \( z_i \). The wavefunctions are therefore identical in form to the momentum eigenstates of free particles on the circle (for which \( z_i = \exp(ix_i) \)). In the case of fermions, thus, the known connection of free fermions on the circle to representations and characters of \( U(N) \) [15-18] carries over to the \( l = 1 \) harmonic system as well. Each excited state of the fermion system can be mapped to a \( U(N) \) Young tableau, and thus to an irrep \( R \) of \( U(N) \). This is done by mapping the particle excitations to successive rows of the tableau. Equivalently, each hole excitation can be mapped to a column of the tableau. Since a hole excitation at energy \( n \) is the same as \( n \) particles excited by one unit, the two pictures are dual descriptions of the same quantum state. A general excited state in the fermion case can be obtained as

\[
|R, f > = \chi_R(a_i^\dagger)|0, f >
\]

where \( |0, f > \) is the fermionic \( N \)-body ground state and \( \chi_R \) is the character of the representation \( R \) expressed in terms of the operator matrix \( \text{diag}(a_1^\dagger, \cdots a_N^\dagger) \). The one-hole state, in particular, corresponds to a tableau with a single column, that is, the \( n \)-fold fully antisymmetric representation of \( U(N) \). It is then expressed as

\[
|n, f > = \chi_n(a_i^\dagger)|0, f > = \sum_{\text{distinct}} a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0, f >
\]

where the sum is over all combinations of distinct indices.

The corresponding states in the bosonic case, obtained by “collapsing” each fermionic state in a way that successive particle distances in the energy spectrum
are reduced by one unit, can similarly be expressed as

\[ |R, b > = \sum_{\text{distinct}} (a_{i1}^\dagger)^{n_1} \cdots (a_{iN}^\dagger)^{n_N} |0, b > \]

(4)

where \( n_1, \ldots n_N \) are the lengths of the rows of the Young tableau. The “one-hole” state thus becomes simply a state with \( n \) particles in the first level above the ground state level and is expressed as

\[ |n, b > = \sum_{\text{distinct}} a_{i1}^\dagger \cdots a_{in}^\dagger |0, b > \]

(5)

a form identical to the fermionic one. This is a property specific to the fully antisymmetric (one-hole) state.

In the interacting system we can define raising and lowering operators

\[ a_i = \frac{1}{\sqrt{2}} (x_i + ip_i - \sum_{j \neq i} \frac{l}{x_i - x_j} M_{ij}) , \quad a_i^\dagger = \frac{1}{\sqrt{2}} (x_i - ip_i + \sum_{j \neq i} \frac{l}{x_i - x_j} M_{ij}) \]

(6)

where \( M_{ij} \) is the operator exchanging particles \( i \) and \( j \). These satisfy

\[ [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 , \quad [a_i, a_j^\dagger] = \delta_{ij} (1 + l \sum_{k \neq i} M_{ik}) + (1 - \delta_{ij}) l M_{ij} \]

(7)

The hamiltonian can be written as

\[ H = \sum_{i=1}^{N} a_i^\dagger a_i \]

(8)

and upon acting on symmetric states, on which \( M_{ij} = 1 \), it coincides with (1) up to a constant equal to the ground state energy. The ground state

\[ \psi_o = \prod_{i>j} |x_i - x_j|^{l} e^{-\frac{1}{2} \sum_i x_i^2} \]

(9)

is annihilated by all lowering operators, and all excited states can be obtained by acting with symmetric combinations of raising operators. The corresponding
(quasi)particle and (quasi)hole states, thus, can be obtained by acting with the corresponding bosonic combination. In particular, the fermionic states can be obtained by acting with the bosonic combinations of the above operators with \( l = 1 \). This achieves, therefore, a ‘bosonization’ of the system. (Note that the usual lowering operators do not individually annihilate the vacuum in the fermionic case, while the above ones still do.)

The one-hole states thus are given by the expression (5) in terms of the generalized operators. The resulting wavefunction is a polynomial in the coordinates \( h(x_i) \) multiplying the ground state. To streamline its derivation, we introduce the commuting operators

\[
A_i = e^{-\frac{i}{2}a^2} a_i^\dagger \ e^{\frac{i}{2}a^2} = a_i^\dagger - la , \quad \text{where} \quad a = \sum_{i=1}^{N} a_i
\]

it terms of which the hole state can be written

\[
|n, l> = \chi_n(a_i^\dagger)|0, l> = e^{\frac{i}{2}a^2} \chi_n(A_i)|0, l>
\]  

Each \( A_i (=-a_i + \sqrt{2}x_i - la) \) acting on the vacuum gives \( \sqrt{2}x_i \), and satisfies

\[
[x_i, A_j] = \frac{l}{\sqrt{2}}(1 - M_{ij}) , \quad i \neq j
\]

Since in \( \chi_n(A_i) \) each index \( i \) appears at most once in each term, the commutator (12) which arises upon reordering such terms acting on the vacuum commutes with all remaining operators. Therefore, it can be pulled through to act on the vacuum, giving zero. The net result is that \( \chi_n(A_i)|0, l> = 2^{n/2} \chi_n(x_i)|0, l> \). The operator \( a \), on the other hand, acts as a derivative on each \( x_i \), that is

\[
[a, \chi_n(x_i)] = \frac{1}{\sqrt{2}} \sum_j \frac{\partial}{\partial x_j} \chi_n(x_i) = \frac{1}{\sqrt{2}}(N - n + 1) \chi_{n-1}(x_i)
\]
The final result for the polynomial part of the wavefunction is

\[
    h_n(x_i) = 2^n \sum_{k=0}^{\infty} \left( \frac{l}{4} \right)^k \frac{(N - n + 2k)!}{k!(N - n)!} \sum \text{distinct } x_{i_1} \cdots x_{i_{n-2k}}
\]  

(we assume that for negative index the characters vanish).

The use of raising and lowering operators in the derivation of the wavefunctions was of mainly conceptual advantage. Indeed, these states could have been derived from Schrödinger’s equation, starting with \( \chi_n(x_i) \) and recursively generating the other terms. The advantage of this formalism becomes much more substantial, however, when calculating the norms of these states. To do that, consider the operators

\[
    a_i(s) = a_i + s
\]

satisfying the same commutation relations as the \( a_i \). All hole states are generated from the function of \( s \)

\[
    |Z(s)\rangle = \prod_{i=1}^{N} a_i^\dagger(s)|0, l >
\]

and it suffices to calculate the norm of \( |Z(s)\rangle \). To this end, define the matrix elements

\[
    Z_n = < 0, l | a_i (s) \cdots a_i a_{i_1}^\dagger (s) \cdots a_{i_n}^\dagger (s) |0, l >
\]

\[
    Y_n = < 0, l | a_i (s) \cdots a_i a_{i_1}^\dagger (s) \cdots a_{i_n}^\dagger (s) |0, l >
\]

Clearly the above elements are independent of the specific choice of (distinct) indices and depend only on \( s \). By commuting through the operators \( a_i(s), a_{i_1}^\dagger(s) \) in \( Z_n, Y_n \) and using the vacuum condition \( a_i(s)|0, l > = s|0, l > \), we obtain the
recursion relations

\[ Z_n = [1 + l(N - n)]Z_{n-1} + sY_n \]

\[ Y_n = s^*Z_{n-1} - l(n - 1)Y_{n-1} \] (18)

Solving the above relations with initial conditions \( Z_0 = 1, Y_0 = 0 \), we obtain

\[ < Z(s) | Z(s) > = Z_N = \sum_{n=0}^{N} (ss^*)^n {N \choose n} \prod_{k=0}^{N-n-1} (1 + lk) \] (19)

From (19) we can simply read off the norms of the hole states

\[ < h_n | h_n > = \frac{N!}{n!(N-n)!} \prod_{k=0}^{n-1} (1 + lk) \] (20)

In the large-\( N \) limit where \( N - n \gg 1 \) (\( n \) need not be small), the limiting form of the above norms is

\[ < h_n | h_n > = \frac{N!}{\Gamma(\frac{1}{l})(N-n)!} l^n n^{\frac{1}{l}} \] (21)

It is convenient to express the above set of states in terms of a generating function. Define the differently normalized characters

\[ \omega_n(x_i) = (N - n)! \sum_{\text{distinct}} x_{i_1} \cdots x_{i_n} \] (22)

Then the generating function for the hole states \( h_n \) is

\[ h(s) = \sum_{n=0}^{N} s^n 2^{-\frac{2}{l}} (N - n)!h_n = e^{\frac{4s^2}{l}} \omega(s) = e^{\frac{4s^2}{l}} \sum_{n=0}^{N} s^n \omega_n \] (23)

(It is understood that only the first \( N \) powers of \( h(s) \) are actually energy eigenfunctions.)
Since each $\omega_n$ is homogeneous in $x_i$ with degree $n$, a rescaling of the $x_i$ amounts to a rescaling of $s$ and thus to a rescaling of $l$. We conclude from (23)

$$h_n(x_i; l) = l^n h_n\left(\frac{x_i}{\sqrt{l}}; 1\right)$$

Thus the polynomial part of the hole wavefunction is simply a rescaling of the fermion hole wavefunction.

The Hamiltonian of the Calogero model can be separated into center of mass and relative coordinates. The above hole states are, in general, not eigenstates of the center of mass motion, but rather superpositions of center of mass oscillations of energies from 0 to $n$. To isolate the center of mass coordinate $x$, consider the center-of-mass frame coordinates $y_i$

$$y_i = x_i - x = x_i - \frac{1}{N} \sum_j x_j$$

Using (13) we can expand

$$\omega_n(x_i) = \omega_n(y_i + x) = \sum_k \frac{x^k}{k!} \omega_{n-k}(y_i)$$

The generating function $h(s)$ then becomes

$$h(s) = e^{\frac{1}{4} s^2 + sx} \sum_{n=0}^{\infty} s^n \omega_n(y_i)$$

Finally, using the generating function for the Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{s^n}{2^n n!} H_n(x) = e^{sx - \frac{s^2}{4}}$$

we obtain

$$h(s) = e^{\frac{1}{4} (l+\frac{1}{N}) s^2 + s \omega(s, y_i)} \sum_n \left(\frac{s}{\sqrt{2N}}\right)^n H_n(\sqrt{N}x)$$

$H_n(\sqrt{N}x)$ is an eigenstate of the center of mass oscillation (the frequency being the same, but the mass being $\sqrt{N}$). The remaining part, being a function only
of relative coordinates, has no center of mass excitations, and therefore the above is a generating function for the energy eigenstates separately for each $H_n$. So we obtain

$$\bar{h}(s) = \sum_n s^n \bar{h}_n(x_i) = e^{\frac{1}{2}(l+\frac{1}{N})s^2} \sum_n s^n \omega_n(x_i - x)$$

where $\bar{h}_n(x_i)$ are the ‘bare’ hole wavefunctions, stripped of all center of mass excitations. We stress that the original hole states $h_n$ are superpositions of states of the form $\bar{h}_k H_{n-k}$. Note also that $\bar{h}_1 = 0$, since the relative coordinates sum to zero. It is interesting that moving to the center of mass coordinate essentially amounts to a shift in the coupling constant $l$ by $1/N$.

Concluding, we remark that the above techniques could be generalized further to obtain more general classes of states for the model. Such results for the Calogero model are encouraging and suggest that a treatment of the properties of the inverse-square system in the thermodynamic limit in the operator formalism may be feasible. Other open questions, such as the existence of a duality symmetry between the Sutherland models with couplings $l$ and $l^{-1}$ [19,8,9], of which we have no realization yet in the Calogero model, are issues for further investigation.

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