Non-trivial Linear Systems on Smooth Plane Curves

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0 Introduction

Let $C$ be a smooth plane curve of degree $d$ defined over an algebraically closed field $k$. In [10], while studying space curves, Max Noether considered the following question. For $n \in \mathbb{Z}_{\geq 1}$ find $\ell(n) \in \mathbb{Z}_{\geq 0}$ such that there exists a linear system $g_n^{\ell(n)}$ on $C$ but no linear system $g_n^{\ell(n)+1}$ and classify those linear systems $g_n^{\ell(n)}$ on $C$. The arguments given by Noether in the answer to this question contained a gap. In [1] C. Ciliberto gave a complete proof for Noether’s claim using different arguments. In [6] R. Hartshorne completed Noether’s original arguments by solving the problem also for integral (not necessarily smooth) plane curves (see Remark 1.3).

The linear systems $g_n^{\ell(n)}$ are either non-special, or special but not very special, or very special but trivial. By a very special (resp. trivial) linear system on a smooth plane curve $C$ we mean:

**Definition.** A linear system $g_n^r$ on $C$ is very special if $r \geq 1$ and $\dim |K_C - g_n^r| \geq 1$. (Here $K_C$ is a canonical divisor on $C$). A base point free complete very special linear system $g_n^r$ on $C$ is trivial if there exists an integer $m \geq 0$ and an effective divisor $E$ on $C$

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of degree \(md - n\) such that \(g^r_n = |mg^2_d - E|\) and \(r = \frac{m^2 + 3m}{2} - (md - n)\). A complete very special linear system \(g^r_n\) on \(C\) is trivial if its associated base point free linear system is trivial.

In this paper, we consider the following question. For \(n \in \mathbb{Z}_{\geq 1}\) find \(r(n)\) such that there exists a very special non-trivial complete linear system \(g^{r(n)}_n\) on \(C\) but no such linear system \(g^{r(n)+1}_n\). Our main result is the following:

**Theorem.** Let \(g^r_n\) be a base point free very special non-trivial complete linear system on \(C\). Write \(r = \frac{(x + 1)(x + 2)}{2} - \beta\) with \(x, \beta\) integers satisfying \(x \geq 1, 0 \leq \beta \leq x\). Then

\[
n \geq n(r) := (d - 3)(x + 3) - \beta.
\]

This theorem only concerns linear systems of dimension \(r \geq 2\). But 1-dimensional linear systems are studied in [3]. From these results one finds that \(C\) has no non-trivial very special linear system of dimension 1 if \(d \leq 5\) and for \(d \geq 6\), \(C\) has non-trivial very special complete linear systems \(g^1_{3d - 9}\) but no such linear system \(g^1_n\) with \(n < 3d - 9\). The proof of this theorem is also effective for case \(r = 1\) if one modifies it little bit.

Concerning the original problem one can make the following observation. For \(x \geq d - 2\) one has \(r > g(C)\) and of course \(C\) has no non-trivial very special linear systems \(g^r_n\). For \(x \leq d - 3\) one has \((d - 3)(x + 2) \leq (d - 3)(x + 3) - x\). So, if the bound \(n(r)\) is sharp, then also the bound \(r(n)\) can be found. Concerning the sharpness of the bound \(n(r)\), we prove it in case \(\text{char}(k) = 0\) for \(x \leq d - 6\). In case \(\text{char}(k) \neq 0\) we prove that there exists smooth plane curves of degree \(d\) with a very special non-trivial \(g^{r(n)}_{n(r)}\) in case \(x \leq d - 6\). Finally for the case \(x > d - 6\) we prove that there exist no base point free complete very special non-trivial linear systems of dimension \(r\) on \(C\). Hence, at least in case \(\text{char}(k) = 0\) the numbers \(r(n)\) are determined.

**Some Notations**

We write \(P_a\) to denote the space of effective divisors of degree \(a\) on \(P^2\). If \(P\) is a linear subspace of some \(P_a\) then we write \(P.C\) for the linear system on \(C\) obtained by intersection with divisors \(\Gamma \in P\) not containing \(C\). We write \(F(P.C)\) for the fixed point divisor of \(P.C\) and \(f(P.C)\) for the associated base point free linear system on \(C\),
so $f(P.C) = \{D - F(P.C) : D \in P.C\}$. If $Z$ is a 0-dimensional subscheme of $P^2$ then $P_a(-Z)$ is the subspace of divisors $D \in P_a$ with $Z \subset D$.

1 Bound on the degree of non-trivial linear systems

A complete linear system $g_r^n$ on a smooth curve $C$ is called very special if $r \geq 1$ and $\dim |K_C - g_r^n| \geq 1$. From now on, $C$ is a smooth plane curve of degree $d$ and $g_r^n$ is a very special base point free linear system on $C$ with $r \geq 2$.

Definition 1.1 $g_r^n$ is called a trivial linear system on $C$ if there exists an integer $m \geq 0$ and an effective divisor $E$ on $C$ of degree $md - n$ such that $g_r^n = |mg_d^2 - E|$ and $r = \frac{m^2 + 3m}{2} - (md - n)$.

Theorem 1.2 Write $r = \frac{(x+1)(x+2)}{2} - \beta$ with $x, \beta$ integers satisfying $x \geq 1, 0 \leq \beta \leq x$. If $g_r^n$ is not trivial, then $n \geq n(r) := (d - 3)(x + 3) - \beta$.

Remark 1.3 In the proof of this theorem we are going to make use of the main result of Hartshorne [6] which describes the linear systems on $C$ of maximal dimension with respect to their degrees. The result is as follows:

Let $g_r^n$ be a linear system on $C$ (not necessarily very special). Write $g(C) = \frac{(d-1)(d-2)}{2}$.

i) If $n > d(d - 3)$ then $r = n - g$ (the non-special case)

ii) If $n \leq d(d - 3)$ then write $n = kd - e$ with $0 \leq k \leq d - 3, 0 \leq e < d$, one has

$$
\begin{cases}
    r \leq \frac{(k-1)(k+2)}{2} & \text{if } e > k + 1 \\
    r \leq \frac{k(k+3)}{2} - e & \text{if } e \leq k + 1.
\end{cases}
$$
Hartshorne also gives a description for the case one has equality. This theorem (a claim originally stated by M. Noether with an incomplete proof) is also proved in [6]. However, Hartshorne also proves the theorem for integral plane curves using the concept of generalized linear systems on Gorenstein curves. We need this more general result in the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume \( g^r_n = rg^{1}_{n/r} \) and \( n < (x + 3)(d - 3) - \beta \). Noting \( 2r = (x + 1)(x + 2) - 2\beta \geq x^2 + x + 2 \geq x + 3, \) we have \( \frac{(x + 3)(d - 3) - \beta}{r} < 2(d - 2). \) Hence, \( g^{1}_{n/r} = |g^2_n - P| \) for some \( P \in C. \) Since \( \dim |rg^{1}_{n/r}| = r, \) certainly \( \dim |2g^{1}_{n/r}| = 2. \) But \( \dim |2g^2_n - 2P| = 3. \) A contradiction.

Since \( g^r_n \) is special, there exist an integer \( 1 \leq m \leq d - 3 \) and a linear system \( P \subset P_m \) such that \( g^r_n = f(P,C) \) and \( P \) has no fixed components. In Lemma 1.4 we are going to prove that, because \( g^r_n \) is not a multiple of a pencil, a general element \( \Gamma \) of \( P \) is irreducible.

Now, for each element \( \Gamma' \) of \( P \) we have \( F(P,C) \subset \Gamma' \) (inclusion of subschemes of \( P^2 \)). In particular \( F(P,C) \subset \Gamma \cap \Gamma' \). This remark is known in the literature as Namba’s lemma. As a subscheme of \( \Gamma \), \( F(P,C) \) is an effective generalized divisor on \( \Gamma \) (terminology of [3]). We find that for each \( \Gamma' \in P \) with \( \Gamma' \neq \Gamma \) the residual of \( F(P,C) \) in \( \Gamma \cap \Gamma' \) (we denote it by \( \Gamma \cap \Gamma' - F(P,C) \)) is an element of the generalized complete linear system on \( \Gamma \) associated to \( O_{\Gamma}(m - F(P,C)) \). Hence, we obtain a generalized linear system \( g^1_{m^2 - (md - n)} \) on \( \Gamma \).

Now we are going to apply Hartshorne’s theorem (Remark 1.3) to this \( g^{r-1}_{m^2 - (md - n)} \) on \( \Gamma \).

Since \( g^r_n \) is non-trivial on \( C, \) we know that \( r > \frac{m^2 + 3m}{2} - (md - n). \) If \( m^2 - (md - n) > m(m - 3), \) then i) in Remark 1.3 implies \( r - 1 \leq m^2 + n - md - \frac{(m - 1)(m - 2)}{2} \) so \( r < \frac{m^2 + 3m}{2} - (md - n), \) a contradiction.

So \( m^2 - (md - n) \leq m(m - 3) \) and we apply ii) in Remark 1.3. We find \( x \leq m - 3 \) and \( m^2 + n - md \geq mx - \beta, \) so \( n \geq \varphi(m) := -m^2 + m(d + x) - \beta. \) Since \( x + 3 \leq m \leq d - 3, \) we find \( n \geq \varphi(x + 3) = \varphi(d - 3) = (d - 3)(x + 3) - \beta = n(r) \). This completes the proof of the theorem except for the proof of Lemma 1.4.

Lemma 1.4 Let \( C \) be a smooth plane curve of degree \( d \) and let \( g^r_n \) be a base point free complete linear system on \( C \) which is not a multiple of a one-dimensional linear system. Suppose there exists a linear system \( P \subset P_e \) without fixed component for some \( e \leq d - 1 \) such that \( g^r_n = f(P,C) \). Then the general element of \( P \) is irreducible.
Remark 1.6

Theorem 1.2: paper one uses arguments like in [1]. That classification is completely contained in our result for which

\[ g \]

In [9] one makes a classification of linear systems on smooth plane curves that is surjective, while \( \dim H^i \leq 1 \) for \( i \neq 1 \). For \( t \in \mathbb{Z}_{\geq 1}, e = (e_1, \ldots, e_t) \in (\mathbb{Z}_{\geq 1})^t \) with \( \sum_{i=1}^t e_i = e \) and \( m = \{ m_{ij} | 1 \leq i \leq t, 1 \leq j \leq s \} \), let

\[ V(t, e, m) = \{ \Gamma_1 + \cdots + \Gamma_t : \Gamma_i \in \mathbb{P}_e, \text{is irreducible and } i(\Gamma_i, C; P_j) = m_{ij} \}. \]

It is not so difficult to prove that this defines a stratification of \( \mathbb{P}_e \) by means of locally closed subspaces.

Since \( \mathbb{P} \) is irreducible there is a unique triple \((t_0, e_0, m_0)\) such that \( \mathbb{P} \cap V(t_0, e_0, m_0) \) is an open non-empty subset of \( \mathbb{P} \). In particular, \( \mathbb{P} \subset \{ \Gamma_1 + \cdots + \Gamma_{t_0} : \Gamma_i \in \mathbb{P}_{e_0}, \text{and } i(\Gamma_i, C; P_j) \geq m_{0ij} \} \). We need to prove that \( t_0 = 1 \), so assume that \( t_0 > 1 \). Let forget the subscript 0 from now on.

Let \( F_i = \sum_{j=1}^s m_{ij} P_j \subset C \). For each \( D \in g_n^* \) there exists \( \Gamma = \Gamma_1 + \cdots + \Gamma_t \) with \( \Gamma_i \in \mathbb{P}_e(-F_i) \) and \( D = \Gamma.C - (F_1 + \cdots + F_s) = \sum_{i=1}^t (\Gamma_i.C - F_i) \). Writing \( D_i = \Gamma_i.C - F_i \in |e_i g_d^2 - F_i| \) we find \( D = \sum_{i=1}^t D_i \). Suppose for some \( 1 \leq i \leq t \) we have \( \dim |e_i g_d^2 - F_i| = 0 \). If \( \Gamma' \) and \( \Gamma'' \) are in \( \mathbb{P}_e(-F_i) \) then \( \Gamma'.C = \Gamma''.C \), but \( e_i < d \) so \( \Gamma' = \Gamma'' \). This implies that \( \mathbb{P}_e(-F_i) = \{ \Gamma_0 \} \), but then \( \Gamma_0 \) is a fixed component of \( \mathbb{P}_e \), a contradiction. Hence, for \( 1 \leq i \leq t \), we have \( \dim |e_i g_d^2 - F_i| \geq 1 \). Now, let \( L_i \) be the irreducible sheaf on \( C \) associated to \( |e_i g_d^2 - F_i| \) and let \( L \) be the irreducible sheaf on \( C \) associated to \( g_n^* \). Then \( L = L_1 \otimes \cdots \otimes L_t \) and we find that the natural map

\[ H^0(C, L_1) \otimes \cdots \otimes H^0(C, L_t) \rightarrow H^0(C, L) \]

is surjective, while \( \dim H^0(C, L_i) \geq 2 \) for \( 1 \leq i \leq t \). From [1, Corollary 5.2], it follows that \( g_n^* \) is a multiple of a pencil. But this is a contradiction.

Remark 1.5: In [3] one makes a classification of linear systems on smooth plane curves for which \( r \) is one less than the maximal dimension with respect to the degree. In that paper one uses arguments like in [6]. That classification is completely contained in our Theorem 1.2.

Remark 1.6: If \( r \geq n - \frac{(d-1)(d-2)}{2} + d - 1 \) then \( \dim |K_C - g_n^*| \geq d - 2 \). Hence, \( |K_C - g_d^2 - g_n^*| = |(d-4)g_d^2 - g_n^*| \neq 0 \). So in this case we can assume \( m \leq d - 4 \) in the proof of Theorem 1.2. Then, in the proof of Theorem 1.2 using Bertini’s theorem, we can...
prove that, for $D \in g_n^r$ general there exists an irreducible curve $\Gamma$ of degree $d - 3$ with $\Gamma.C \geq D$ (see argument in [3, p.384]). So we don’t need the proof of Lemma 1.4 under that assumption on $r$.

**Remark 1.7** In that case $n = n(r)$, we find $m = x + 3$ and $m^2 + n - md = mx - \beta$. So the generalized linear system $g_{m^2+n-md}^{r-1} = g_{mx-\beta}^{r-1}$ on $\Gamma$ is of maximal dimension with respect to its degree. The description of those linear systems in [3] implies that it is induced by a family of plane curves of degree $x$ containing some subspace $E \subset \Gamma$ of length $\beta$. Writing $Z = F(P.C) \subset \Gamma$ we find $|P_m, \Gamma - Z| = |P_x, \Gamma - E|$ and so $Z \in |P_3, \Gamma + E|$. In order to find non-trivial $g_{n(r)}^r$’s it is interesting to find for a smooth curve $C$ of degree $d$, a curve $\Gamma$ of degree $m$ and a curve $\Gamma'$ of degree 3 such that $\Gamma \cap \Gamma' \subset C$. We discuss this in §3.

First we solve the following postulation problem: let $\Gamma'$ be the union of 3 distinct lines $L_1, L_2, L_3$ and let $D_i$ be an effective divisor of degree $a$ on $L_i$ with $D_i \cap (L_j \cup L_k) = \emptyset$ for $\{i, j, k\} = \{1, 2, 3\}$. Give necessary and sufficient conditions for the existence of a smooth curve $\Gamma$ of degree $a$ such that $\Gamma.L_i = D_i$ for $i = 1, 2, 3$.

## 2 Carnot’s theorem

We begin with pointing out the following elementary fact:

**Lemma 2.1** Let $m \geq 4$ and $m_j$ ($j = 1, \ldots, \ell$) be positive integers satisfying $\sum_{j=1}^{\ell} m_j = m$

and let $\Phi(X) = \sum_{j=1}^{m} a_j X^j$ be a non-zero polynomial of degree at most $m$. If $\Phi(X)$ is divisible by $(X - X_j)^{m_j}$ for $\ell$ distinct values $X_1, \ldots, X_\ell$, then the ratio $a_0 : \cdots : a_m$ is uniquely determined. In particular, $a_m \neq 0, a_0 = (-1)^m a_m \prod_{j=1}^{\ell} X_j^{m_j}$ and $a_{m-1} = -a_m \sum_{j=1}^{\ell} m_j X_j$.

Using this, we have the following Carnot’s theorem and infinitesimal Carnot’s theorems. Another generalization of this theorem is given by Thas et al. [11] (see also [12]). They call this B. Segre’s generalization of Menelaus’ theorem.
Lemma 2.2 (Carnot, cf. [4, Proposition 1.8], [8, p.219]) Let \( L_1, L_2, L_3 \) be lines in \( \mathbb{P}^2 \) so that \( L_1 \cap L_2 \cap L_3 = \emptyset \) and let \( D_i = \sum_{j=1}^{\ell_i} m_{ij} P_{ij}, \) \((\sum_{j=1}^{\ell_i} m_{ij} = m, i = 1, 2, 3)\) be an effective divisor on \( L_i \) such that \( D_{i_1} \cap L_{i_2} = \emptyset \) if \( i_1 \neq i_2. \) Assume \((x : y : z)\) is a coordinate system on \( \mathbb{P}^2 \) such that \( L_1, L_2, L_3 \) correspond to the coordinate axes \( x = 0, y = 0, z = 0, \) respectively. Let the coordinate of \( P_{ij} \) be given by \((x_{ij}, y_{ij}, z_{ij})\) \( (\text{of course} \ x_{ij} = y_{ij} = z_{ij} = 0) \). Then, there exists a curve \( \Gamma \) not containing any one of the lines \( L_1, L_2, L_3 \) of degree \( m \) satisfying \((\Gamma, L_i) = D_i \) \((i = 1, 2, 3)\) if and only if

\[
(2.1) \quad \prod_{j=1}^{\ell_1} \left( \frac{y_{1j}}{x_{1j}} \right)^{m_{1j}} \prod_{j=1}^{\ell_2} \left( \frac{z_{2j}}{x_{2j}} \right)^{m_{2j}} \prod_{j=1}^{\ell_3} \left( \frac{x_{3j}}{y_{3j}} \right)^{m_{3j}} = (-1)^m.
\]

**Proof.** Assume there exists a curve \( \Gamma \) of degree \( m \) not containing any one of the lines \( L_1, L_2, L_3 \) satisfying \((\Gamma, L_i) = D_i \) \((i = 1, 2, 3)\). Such a curve is given by \( \Phi(x, y, z) = \sum_{i+j+k=m} a_{ijk} x^i y^j z^k = 0. \) In this description, if \( i(\Gamma, L_1; P_{ij}) = m_{ij} \), then \( \Phi(0, y, z) \) is divisible by \((y_{1j} z - z_{1j} y)^{m_{1j}}. \) This implies \( a_{00m} = (-1)^m a_{0m0} \prod_{j=1}^{\ell_1} \left( \frac{y_{1j}}{x_{1j}} \right)^{m_{1j}}. \) Similarly, we have \( a_{m00} = (-1)^m a_{00m} \prod_{j=1}^{\ell_2} \left( \frac{z_{2j}}{x_{2j}} \right)^{m_{2j}} \) and \( a_{0m0} = (-1)^m a_{m00} \prod_{j=1}^{\ell_3} \left( \frac{x_{3j}}{y_{3j}} \right)^{m_{3j}}. \) Since \( \Gamma \) does not contain an intersection point \( L_{i_1} \cap L_{i_2} \) for \( i_1 \neq i_2, \) we have \( a_{m00} a_{0m0} = 0. \) Hence, we have \((2.1)\).

For the converse, take \( a_{000} = 1. \) Then, by \((2.1)\) we can determine \( a_{ijk} \) so that \( \Gamma \) has the desired property. This completes the proof.

Next, we see two infinitesimal cases i. e. case \( D_{i_1} \cap L_{i_2} = \emptyset \) and case \( L_1 \cap L_2 \cap L_3 = \emptyset \).

**Lemma 2.3** Let \( L_1, L_2, L_3 \) be lines in \( \mathbb{P}^2 \) so that \( L_1 \cap L_2 \cap L_3 = \emptyset \) and let \( D_i = \sum_{j=1}^{\ell_i} m_{ij} P_{ij}, \) \((\sum_{j=1}^{\ell_i} m_{ij} = m)\) be an effective divisor on \( L_i \) such that \( m_{11} = m_{21} = 1, P_{11} = \)
Lemma 2.4 Let $L_1, L_2, L_3$ be lines in $\mathbb{P}^2$ so that $L_1 \cap L_2 \cap L_3 \neq \emptyset$ and let $D_i = \sum_{j=1}^{\ell_i} m_{ij} P_{ij}, (\sum_{j=1}^{\ell_i} m_{ij} = m)$ be an effective divisor on $L_i$ such that $D_i \cap L_j = \emptyset$ if $i \neq j$.

Let $(x : y : z)$ be a coordinate system on $\mathbb{P}^2$ such that $L_1, L_2, L_3$ correspond to the line $y - z = 0, y = 0, z = 0$, respectively. Let the coordinate of $P_{ij}$ be given by $(x_{ij} : y_{ij} : z_{ij})$ (of course $y_{ij} = z_{ij}, y_{i2} = 0, z_{i3} = 0$). Then, there exists a curve $\Gamma$ of degree $d$ not containing any one of the lines $L_1, L_2, L_3$ such that $(\Gamma.L_i) = D_i$ ($i = 1, 2, 3$) if and only if

\[
(2.3) \quad \sum_{j=1}^{\ell_1} m_{1j} \frac{x_{1j}}{y_{1j}} - \sum_{j=1}^{\ell_2} m_{2j} \frac{x_{2j}}{y_{2j}} - \sum_{j=1}^{\ell_3} m_{3j} \frac{x_{3j}}{y_{3j}} = 0.
\]
Proof. Again, we use the same notation as in the proof of Lemma 2.3. Assume there exists a curve $\Gamma$ having the desired property. Since $\Gamma$ does not contain $L_2 \cap L_3$, we have $a_{m00} \neq 0$. By Lemma 2.1,

$$a_{m-1,10} = - \sum_{j=1}^{\ell_3} m_{3j} \frac{x_{3j}}{y_3} a_{m00} \quad \text{and} \quad a_{m-1,01} = - \sum_{j=1}^{\ell_2} m_{2j} \frac{x_{2j}}{x_2} a_{m00}.$$  

To consider the condition on $L_1$, we take the coordinate system $(\xi : \eta : \zeta)$ with $\xi = x, \eta = y, \zeta = z - y$. Put

$$\Psi(\xi, \eta, \zeta) = \Phi(\xi, \eta, \eta + \zeta) = \sum_{i+j+k=m} a_{ijk} \xi^{i} \eta^{j} (\zeta + \eta)^{k} = \sum_{i+j+k=m} b_{ijk} \xi^{i} \eta^{j} \zeta^{k}.$$  

Then, $b_{m00} = a_{m00}$ and $b_{m-1,10} = a_{m-1,10} + a_{m-1,01}$. In this description, if $i(\Gamma, L_i; P_{ij}) = m_{ij}$, then $\Psi(\xi, \eta, 0)$ is divisible by $(\xi_{ij} \eta - \eta_{ij} \zeta)^{m_{ij}}$. This implies that $b_{m-1,10} = - \sum_{j=1}^{\ell_{ij}} m_{ij} \frac{x_{1j}}{y_{1j}} b_{m00}$. Then, we have (2.3).

For the converse, noting that if $a_{m00} \neq 0$ then $\Gamma$ does not contain $L_i$ $(i = 1, 2, 3)$ as a component, we can find a $\Gamma$ having the desired property.

Remark 2.5 In each of the lemmas 2.2, 2.3 and 2.4, if (2.1) (resp. (2.2), (2.3)) holds, we can find a smooth curve $\Gamma$ of degree $m$ with $\Gamma \cap L_i = D_i$ for $i = 1, 2, 3$. Indeed, let $P \subset P_m$ be the linear system of divisors $\Gamma$ of degree $m$ on $P^2$ satisfying, as schemes, $D_i \subset \Gamma \cap L_i$. Clearly $L_1 + L_2 + L_3 + P_{m-3} \subset P$ and we proved that $U = \{ \Gamma \in P : \Gamma \text{ does not contain any of the lines } L_1, L_2, L_3 \}$ is a non-empty open subset of $P$. Because of Bertini’s theorem, for $\Gamma \in U$ we have $L_i \cap \Gamma = \{ P_{ij}, \ldots, P_{ii} \}$. Consider the linear system $P' = \{ \Gamma \cap P^2 \setminus (L_1 \cup L_2 \cup L_3) : \Gamma \in P \}$ on $M = P^2 \setminus (L_1 \cup L_2 \cup L_3)$. Since $\Gamma \cap M \in P'$ for any $\Gamma \in P_{m-3}$, $P'$ separates tangent directions and points on $M$. Because of Bertini’s theorem in arbitrary characteristics (see [3]), we find that a general element $\Gamma$ of $P'$ is smooth. So, a general element $\Gamma$ of $P$ satisfies $\text{Sing}(\Gamma) \subset \{ P_{ij} : i = 1, 2, 3 \text{ and } 1 \leq j \leq \ell \}$. But, using $\Gamma' \in P_{m-3}$ suited we find $\Gamma = \Gamma' + L_1 + L_2 + L_3$ is smooth at $P_{ij}$, except for the case $P_{11} = P_{21}$ in Lemma 2.3. In that case, however, if $\Gamma \in U$ then $\Gamma$ is smooth at $P_{11}$ because of Bezout’s theorem. This completes the proof of the remark. (For Bertini’s theorem in arbitrary characteristics, one can also use [5].)
3  Sharpness of the bound

The next proposition implies that it is enough to solve the postulation problem mentioned in Remark 1.7 in order to prove sharpness for the bound \( n(r) \) in Theorem 1.2.

**Proposition 3.1.** Let \( C \) be a smooth plane curve of degree \( d \) and let \( r = \frac{(x + 1)(x + 2)}{2} - \beta \) with \( x, \beta \in \mathbb{Z} \) satisfying \( 4 \leq x + 3 \leq d - 3, 0 \leq \beta \leq x \). Let \( n = n(r) = (d - 3)(x + 3) - \beta \). Suppose there exists a smooth curve \( \Gamma \) of degree \( m = x + 3 \), a curve \( \Gamma' \) of degree 3 and an effective divisor \( E \) of degree \( \beta \) on \( \Gamma \) such that \( Z \subset C \), where the divisor \( Z = (\Gamma \cap \Gamma') + E \) on \( \Gamma \) is considered as a closed subscheme of \( \mathbb{P}^2 \). Then \( |mg_d^2 - Z| \) is a non-trivial \( g^r_n \) on \( C \).

**Proof.** We write \( E = P_1 + \cdots + P_\beta \). Let \( L_1, \ldots, L_\beta \) be general lines through \( P_1, \ldots, P_\beta \), resp., and let \( L_{\beta+1}, \ldots, L_x \) be general lines in \( \mathbb{P}^2 \). If \( P \in C \), then we write \( \mu_P(Z) \) for the multiplicity of \( Z \) at \( P \).

i) \( |mg_d^2 - Z| \) is base point free.

Suppose \( P \) is a base point for \( |mg_d^2 - Z| \). Since \( \Gamma.C - Z \in |mg_d^2 - Z| \) one finds \( P + Z \leq \Gamma.C \), hence \( i(\Gamma, C; P) > \mu_P(Z) \geq i(\Gamma, \Gamma'; P) \). Also \( (\Gamma' + \sum_{i=1}^r L_i).C - Z \in |mg_d^2 - Z| \), hence \( P \in (\Gamma' + \sum_{i=1}^r L_i).C - Z = (\Gamma'.C - \Gamma' \cap \Gamma) + (\sum_{i=1}^r L_i.C - E) \) (sum of two effective divisors). Since \( P \notin \sum_{i=1}^r L_i.C - E \), we find \( P \in \Gamma'.C - \Gamma' \cap \Gamma \). This implies \( i(\Gamma', C; P) > i(\Gamma, \Gamma'; P) \). But \( i(\Gamma, \Gamma'; P) \geq \min(i(\Gamma, C; P), i(\Gamma', C; P)) \) (so called Namba’s lemma), hence we have a contradiction.

ii) \( \dim(|mg_d^2 - Z|) \geq r \).

Indeed, \( (\Gamma' + \mathbf{P}_x(-E)).C - Z \subset |mg_d^2 - Z| \) and \( \dim(\Gamma' + \mathbf{P}_x(-E)) = \frac{(x + 1)(x + 2)}{2} - \beta - 1 \). But also \( \Gamma.C - Z \subset |mg_d^2 - Z| \) while \( \Gamma.C \notin (\Gamma' + \mathbf{P}_x(-E)).C \). This proves the claim.

iii) \( \dim(|mg_d^2 - Z|) = r \).

If \( \dim(|mg_d^2 - Z|) > r \) then on \( \Gamma \) it induces a linear system \( g^{r'}_m \) with \( r' \geq r \). But Hartshorne’s theorem (see 1.3) implies that this is impossible.

iv) \( |mg_d^2 - Z| \) is not trivial.

First of all, \( |mg_d^2 - Z| \) is very special. Indeed \( (d - 3 - m)g_d^2 + Z \subset |K_C - (mg_d^2 - Z)| \). If \( d - 3 = m \) then from the Riemann-Roch theorem, one finds \( \dim |Z| = 1 \).
Suppose \(|mg^2 - Z|\) would be trivial, i.e. \(|mg^2 - Z| = |kg^2 - F|\) with \(r = \frac{k^2 + 3k}{2} - (dk - n)\). Since \(g^r\) is very special, one has \(k < d - 3\). Consider \(D = (\Gamma'.C - \Gamma \cap \Gamma') + (\sum_{i=1}^n L_i, C - E)\) as in step i). There exists \(\gamma \in \mathbb{P}_k(-F)\) with \(\gamma.C = D + F\). Because of Bezout’s theorem one has \(\gamma = \gamma' + L_1 + \cdots + L_x\). If \(P \in E\) then \(P \notin \Gamma'.C - \Gamma' \cap \Gamma\), otherwise both \(i(\Gamma', C; P) > i(\Gamma', \Gamma; P)\) and \(i(\Gamma, C; P) > i(\Gamma', \Gamma; P)\), a contradiction to Namba’s lemma. This implies \(\gamma'.C \geq \Gamma'.C - \Gamma' \cap \Gamma\). Once more from Namba’s lemma, we obtain \(\Gamma'.\gamma' \geq \Gamma'.C - \Gamma' \cap \Gamma\) and so
\[
\text{deg}(\Gamma'.\gamma') = 3(k - x) \geq \text{deg}(\Gamma'.C - \Gamma' \cap \Gamma) = 3(d - x - 3),
\]
a contradiction to \(k < d - 3\).

**Corollary 3.2** Let \(C\) be a smooth plane curve of degree \(d\). Assume there exists a plane curve \(\Gamma'\) of degree 3 and a smooth plane curve \(\Gamma\) of degree \(a\) \((4 \leq a \leq d - 6)\) such that \(\Gamma \cap \Gamma' \subset C\) (as schemes). Then \(C\) possesses a non-trivial linear system \(g^r\) for \(r = \frac{(a - 2)(a - 1)}{2} - \beta, 0 \leq \beta \leq a - 3\) and \(n = n(r) = a(d - 3) - \beta\).

**Proof.** \(\Gamma.C = \Gamma \cap \Gamma' + D\) for some effective divisor \(D\) of degree \(a(d - 3)\) on \(C\). But \(a(d - 3) \geq d - 3 \geq a + 3\), so we can choose an effective divisor \(E \subset D\) of degree \(\beta\) and then one has to apply Proposition 3.1 to \(|mg^2 - Z|\) for \(Z = \Gamma \cap \Gamma' + E\).

**Construction 3.3** Fix \(\Gamma' \in \mathbb{P}_a, \Gamma \in \mathbb{P}_a\) \((a \geq 4)\) general and look at \(\mathbb{P}_{a+\varepsilon}(-\Gamma \cap \Gamma')\) \((\varepsilon \geq 1)\). Clearly \(\Gamma' + \mathbb{P}_{a+\varepsilon - 3} \subset \mathbb{P}_{a+\varepsilon}(-\Gamma \cap \Gamma'), \Gamma + \mathbb{P}_{\varepsilon} \subset \mathbb{P}_{a+\varepsilon}(-\Gamma \cap \Gamma')\). Take \(P \in \mathbb{P}_2(\Gamma \cap \Gamma')\). If \(P \notin \Gamma'\) then using \(\Gamma' + \mathbb{P}_{a+\varepsilon - 3}\) one can separate tangent vectors at \(P\). If \(P \notin \Gamma\) then one uses \(\Gamma + \mathbb{P}_{\varepsilon}\). Kleiman’s Bertini theorem \([\text{1}]\) in arbitrary characteristics implies that a general element \(C \in \mathbb{P}_{a+\varepsilon}(-\Gamma \cap \Gamma')\) is smooth outside \(\Gamma \cap \Gamma'\). But if \(\Gamma'' \in \mathbb{P}_{\varepsilon}\) is general then \(\Gamma + \Gamma''\) is smooth on \(\Gamma \cap \Gamma'\). This implies that a general element \(C \in \mathbb{P}_{a+\varepsilon}(-\Gamma \cap \Gamma')\) is smooth. This proves that, for each \(d, 1 \leq x \leq d - 6, 0 \leq \beta \leq x\), there exists a smooth plane curve \(C\) of degree \(d\) possessing a non-trivial \(g^r\) with \(r = \frac{(x + 1)(x + 2)}{2} - \beta\) and \(n(r) = (d - 3)(x + 3) - \beta\).

In trying to prove this statement for all smooth plane curves of degree \(d\), we only succeeded in case \(\text{char}(k) = 0\). This is the following theorem.
Theorem 3.4 Let $C$ be a smooth plane curve of degree $d$ over an algebraically closed field of characteristic zero. Let $d > m \geq 4$. There exists $\Gamma' \in \mathbb{P}_3$ and a smooth $\Gamma \in \mathbb{P}_m$ such that, as schemes, $\Gamma \cap \Gamma' \subset C$.

**Proof.** Fix two general lines $L_1, L_2$ in $\mathbb{P}^2$, let $S = L_1 \cap L_2$. We may assume neither $L_1$ nor $L_2$ is a tangent line of $C$ and $S \not\subset C$. Choose points $P_{11}, \ldots, P_{1m}$ on $C \cap L_1$ and $P_{21}, \ldots, P_{2m}$ on $C \cap L_2$. Choose a general point $S'$ in $\mathbb{P}^2 \setminus C \cup L_1 \cup L_2$. The pencil of lines in $\mathbb{P}^2$ through $S'$ induces a base point free $\psi$ corresponding to the line $L$. Fix two general lines $0 \in \mathbb{P}^2$ such that the associated divisor looks like $2Q + E$ with $Q \notin E$ and $E$ consists of $d - 2$ different points (here we use characteristic zero).

- If $Q$ is a ramification point of $g_q$ then the associated divisor looks like $2Q + E$ with $Q \notin E$ and $E$ consists of $d - 2$ different points (here we use characteristic zero).
- If $Q \in L_i \cap C$ $(i = 1, 2)$ then $Q$ is not a ramification of $g_q$. The associated divisor is $Q + E$ with $E \cap (L_1 \cup L_2) = \emptyset$.
- The line $SS'$ is not a tangent line of $C$.

On the symmetric product $C^{(m)}$ we consider $V = \{E \in C^{(m)} : \text{there exists } D \in g_q \text{ with } E \leq D\}$. In terminology of $\Psi_d^1$ it is the set $V^1_d(g_q)$ and we consider $V$ with its natural scheme structure. From Chapter 2 in loc. cit., it follows that $V$ is a smooth curve.

Let $D_0 \in g_q$ corresponding to the line $SS'$ and let $V_0 = \{E \in V : E \leq D_0\}$. We define a map $\psi : V \setminus V_0 \to \mathbb{P}^1$ as follows. Associated to $E \in V \setminus V_0$ there is a line $L$ through $S'$. Write $E = P_{31} + \cdots + P_{3m}$. We distinguish 3 possibilities:

i) $E \cap (L_1 \cup L_2) = \emptyset$. Choose coordinates $x, y, z$ on $\mathbb{P}^2$ such that $L_1, L_2, L$ corresponds to the coordinate axes $x = 0, y = 0, z = 0$, respectively. Let $(x_{ij} : y_{ij} : z_{ij})$ be the coordinates of $P_{ij}$ $(i = 1, 2, 3; 1 \leq j \leq m)$. Then

$$
\psi(E) = \prod_{j=1}^{m} \left( \frac{y_{1j}}{z_{1j}} \right) \prod_{j=1}^{m} \left( \frac{y_{2j}}{x_{2j}} \right) \prod_{j=1}^{m} \left( \frac{x_{3j}}{y_{3j}} \right).
$$

As long as we take $L_1, L_2, L$ as axes, this value is independent of the coordinates.

ii) $E \cap (L_1 \cup L_2) \neq \emptyset$. Say $P_{11} = P_{31} \in E \cap (L_1 \cup L_2)$. Choose coordinates as before and let $\alpha x + z = 0$ be the equation of the tangent line to $C$ at $P_{11}$ ($\alpha \neq 0$). Then

$$
\psi(E) = \alpha \prod_{j=2}^{m} \left( \frac{y_{1j}}{z_{1j}} \right) \prod_{j=1}^{m} \left( \frac{x_{2j}}{x_{2j}} \right) \prod_{j=2}^{m} \left( \frac{x_{3j}}{y_{3j}} \right).
$$

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Again taking $L_1, L_2, L$ as axes, this value is independent of the coordinates. (Of course this is a function to $C$ and we consider $P^1 = C \cup \{\infty\}$.)

iii) If $\psi(E)$ is not defined in $C$ then $\psi(E) = \infty$.

For $E \in V_0$, we define $\psi(E) = (-1)^m$

This map is a holomorphic map. Indeed, fixing coordinates $(x : y : z)$ such that $L_1, L_2$ corresponds to $x = 0, y = 0$, resp. and $S' = (1 : 1 : 0)$, we can write $z - \gamma(x - y) = 0$ for the pencil of lines through $S'$ (except for $SS'$). If $E \in V \setminus V_0$ and $E$ is a part of a divisor of $g_d^1$ consisting of $d$ different points, then $\gamma$ is a local coordinate of $V$ at $E$. In case i) we write down $\psi$ locally as a holomorphic function in $\gamma$. It is easy to check that $\psi$ is continuous at $E$ in case ii).

For $E \in V_0$, write $\beta z + (x - y) = 0$ for the pencil of lines through $S'$ close to $SS'$. Let $(x_{3j} : x_{3j} : z_{3j})$ be the coordinates at the points $P_{3j}$ of $E$. For $E' \in V$ close to $E$ we have $E' = \sum_{j=1}^m P'_{3j}$ and coordinates $(x'_{3j} : \beta z'_{3j} + x'_{3j} : z'_{3j})$ at $P'_{3j}$. Here we can assume that $x'_{3j} = x_{3j}(\beta), z'_{3j} = z_{3j}(\beta)$ are holomorphic functions in $\beta$ (local coordinate at $V$ in $E$) and $x_{3j} = x_{3j}(0), z_{3j} = z_{3j}(0)$.

Choose new coordinates $\xi = \beta z + x - y, \eta = y, \zeta = x$. The coordinates of $P_{1j}$ are $(0 : y_{1j} : \beta z_{1j} + y_{1j}),$ of $P_{2j}$ are $(x_{2j} : 0 : \beta z_{2j} + x_{2j})$, of $P'_{3j}$ are $(x'_{3j} : \beta z'_{3j} + x'_{3j} : 0)$. We find

\[
(3.1) \quad \psi(E') = \prod_{j=1}^m \frac{y_{1j}}{\beta z_{1j} - y_{1j}} \prod_{j=1}^m \frac{\beta z_{2j} + x_{2j}}{x_{2j}} \prod_{j=1}^m \frac{x'_{3j}}{\beta z'_{3j} + x'_{3j}}
\]

\[
= (-1)^m - \left( \sum_{j=1}^m \frac{z_{3j}}{x_{3j}} - \sum_{j=1}^m \frac{z_{1j}}{y_{1j}} - \sum_{j=1}^m \frac{z_{2j}}{x_{2j}} \right) \beta + o(\beta).
\]

Hence, $\psi$ is continuous at $E$.

Since $V$ is smooth and $\psi$ is continuous on $V$ and holomorphic except for a finite number of points, $\psi$ is a holomorphic map $V \to P^1$.

At some component of $V$, $\psi$ is not constant. Indeed, look at a fibre $2Q + E \in g_d^1$ with $E \in C^{(d-2)}$. Take a close fibre $P_1 + P_2 + E'$ with $P_1, P_2$ close to $Q$. Choose $F \leq E'$ with $\deg(F) = m - 1$ and consider $P_1 + F \in V$. Let $W$ be the irreducible component of $V$ containing $P_1 + F$. Using monodromy one finds $P_2 + F \in W$. But clearly $\psi(P_1 + F) \neq \psi(P_2 + F)$, hence $\psi : W \to P^1$ is a covering. In particular $\psi^{-1}((-1)^m) \neq 0$.

If for some $E \in W \setminus V_0$ we have $\psi(E) = (-1)^m$ then the theorem follows from Lemmas 2.2, 2.3 and Remark 2.5. So, we have to take a closer look to $\psi$ at $V_0$. By the equation
More concretely, is the subscheme of $W$ irreducible? What are the dimension of those irreducible components? And so on.

Since $\dim n$ for any values of $n$ on a (general) smooth plane curve, classify those linear systems and study $C$ to prove that $g$ exactly $\geq r$. Suppose that each zero of $\psi - (-1)^m$ belonging to $V_0$ is simple. Then $\psi - (-1)^m$ has exactly $\binom{d}{m}$ zeros at those points. Now we look at zeros of $\psi$ on $V \setminus V_0$. The number of zeros is finite. For case i) there is none. For case ii) we have two possibilities. If $E \in V$ corresponds to a line $L$ through $S'$ containing one of the points $P_1, \ldots, P_m$ but $E \cap (L_1 \cup L_2) = \emptyset$. There are $m \binom{d-1}{m-1}$ such possibilities. If $E \in V$ corresponds to a line $L$ through $S'$ not containing any of the points $P_1, \ldots, P_m$ but $E \cap L_1 \neq \emptyset$. There are $(d - m) \binom{d-1}{m-1}$ such possibilities. So, on the components of $V$ where $\psi$ is not constant, $\psi$ has at least $m \binom{d-1}{m} + (d - m) \binom{d-1}{m-1}$ zeros. But this number is greater than $\binom{d}{m}$, so $\psi - (-1)^m$ has a zero on $V \setminus V_0$. This completes the proof of the theorem.

Remark 3.5 In order to obtain the bound $r(n)$ mentioned in the introduction, we have to prove that $C$ possesses no base point free very special non-trivial linear systems $g^n_C$ with $r \geq \frac{(d-4)(d-3)}{2} - (d - 5)$ (i.e. $x \geq d - 5$). In the introduction we already noticed that $x \leq d^2 - 3$. Assume $g^n_C$ is a very special non-trivial linear system. From Theorem 1.2 we find $n \geq n((d-4)(d-5)/2 - (d - 5)) = d^2 - 6d + 11$. But then $\deg(K_C - g^n_C) \leq (d-1)(d-2) - 2(d^2 - 6d + 11) = 3d - 11$. However, very special linear systems $g^n_C$ of degree $m \leq 3d - 11$ are trivial. So, the associated base point free linear system $g^n_C$ of $|K_C - g^n_C|$ is of type $|ag^3_C - E|$ with $a \leq d - 4$, $E$ effective and $\dim |K_C - g^n_C| = -2 \frac{3a}{2} - \deg E$. If $E \neq \emptyset$, then for $P \in E$ one has $\dim |ag^3_C - E + P| > \dim |ag^3_C - E|$, so $\dim |g^n_C - P| = r$. This implies that $g^n_C$ has a base point, hence $E = \emptyset$. But then, $g^n_C = |(d - 3 - a)g^3_C - F|$ and since $\dim |ag^3_C + F| = \dim |ag^3_C|$, we have $r = \frac{(d - 3 - a)^2 + 3d - 3 - a}{2} - \deg F$. This is a contradiction to the fact that $g^n_C$ would be non-trivial.

Remark 3.6 It would be interesting to find an answer to the following questions: For which values of $n$ do there exist non-trivial base point free very special linear systems $g^n_C$ on a (general) smooth plane curve. Classify those linear systems and study $W_n$ on $J(C)$. More concretely, is the subscheme of $W_n(r)$ corresponding to non-trivial linear systems irreducible? What are the dimension of those irreducible components? And so on.
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