The Effect of Spatial Curvature on the Classical and Quantum Strings

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Abstract

We study the effects of the spatial curvature on the classical and quantum string dynamics. We find the general solution of the circular string motion in static Robertson-Walker spacetimes with closed or open sections. This is given closely and completely in terms of elliptic functions. The physical properties, string length, energy and pressure are computed and analyzed. We find the back-reaction effect of these strings on the spacetime: the self-consistent solution to the Einstein equations is a spatially closed ($K > 0$) spacetime with a selected value of the curvature index $K$ (the scale factor is normalized to unity). No self-consistent solutions with $K \leq 0$ exist. We semi-classically quantize the circular strings and find the mass $m$ in each case. For $K > 0$, the very massive strings, oscillating on the full hypersphere, have $m^2 \sim Kn^2$ ($n \in N_0$) independent of $\alpha'$ and the level spacing grows with $n$, while the strings oscillating on one hemisphere (without crossing the equator) have $m^2\alpha' \sim n$ and a finite number of states $N \sim 1/(K\alpha')$. For $K < 0$, there are infinitely many string states with masses $m \log m \sim n$, that is, the level spacing grows slower than $n$.

The stationary string solutions as well as the generic string fluctuations around the center of mass are also found and analyzed in closed form.
1 Introduction and Results

The propagation of strings in Friedmann-Robertson-Walker (FRW) cosmologies has been investigated using both exact and approximative methods, see for example Refs.[1-8] (as well as numerical methods, which shall not be discussed here). Except for anti de Sitter spacetime, which has negative spatial curvature, the cosmologies that have been considered until now, have been spatially flat. In this paper we will consider the physical effects of a non-zero (positive or negative) curvature index on the classical and quantum strings. The non-vanishing components of the Riemann tensor for the generic D-dimensional FRW line element, in comoving coordinates:

\[ ds^2 = -dt^2 + a^2(t) \frac{d\vec{x}d\vec{x}}{(1 + \frac{K}{4}\vec{x}\vec{x})^2}, \]  

are given by:

\[ R_{tit} = \frac{-a a_{tt}}{(1 + \frac{K}{4}\vec{x}\vec{x})^2}; \quad R_{ijij} = \frac{a^2(K + a_t^2)}{(1 + \frac{K}{4}\vec{x}\vec{x})^4}; \quad i \neq j \]  

where \( a = a(t) \) is the scale factor and \( K \) is the curvature index. Clearly, a non-zero curvature index introduces a non-zero spacetime curvature; the exceptional case provided by \( K = -a_t^2 = \text{const.} \), corresponds to the Milne-Universe. From Eqs.(1.2), it is also seen that the curvature index has to compete with the first derivative of the scale factor. The effects of the curvature index are therefore conveniently discussed in the family of FRW-universes with constant scale factor, the so-called static Robertson-Walker spacetimes. This is the point of view we take in the present paper.

We consider both the closed \((K > 0)\) and the hyperbolic \((K < 0)\) static Robertson-Walker spacetimes, and all our results are compared with the already known results in the flat \((K = 0)\) Minkowski spacetime. We determine the evolution of circular strings, derive the corresponding equations of state, discuss the question of strings as self-consistent solutions to the Einstein equations [6], and we perform a semi-classical quantization. We find all the stationary string configurations in these spacetimes and we perform a canonical quantization, using the string perturbation series approach [1], for a static string center of mass.
The radius of a classical circular string in the spacetime (1.1), for $a = 1$, is determined by:

$$\dot{r}^2 + V(r) = 0; \quad V(r) = (1 - K r^2)(r^2 - b \alpha'^2),$$

(1.3)

where $b$ is an integration constant. This equation is solved in terms of elliptic functions and all solutions describe oscillating strings (Fig.1 shows the potential $V(r)$ for $K > 0$, $K = 0$, $K < 0$).

For $K > 0$, when the spatial section is a hypersphere, the string either oscillates on one hemisphere or on the full hypersphere. The energy is positive while the average pressure can be positive, negative or zero; the equation of state is given by Eqs.(3.19), (3.25). Interestingly enough, we find that the circular strings provide a self-consistent solution to the Einstein equations with a selected value of the curvature index, Eq.(3.28). Self-consistent solutions to the Einstein equations with string sources have been found previously in the form of power law inflationary universes [6]. We semi-classically quantize the circular strings using the stationary phase approximation method of Ref.[9].

The strings oscillating on one hemisphere give rise to a finite number $N_-$ of states with the following mass-formula:

$$m_-^2 \alpha' \approx \pi n, \quad N_- \approx \frac{4}{\pi K \alpha'}.\quad (1.4)$$

As in flat Minkowski spacetime, the scale of these string states is set by $\alpha'$. The strings oscillating on the full hypersphere give rise to an infinity of more and more massive states with the asymptotic mass-formula:

$$m_+^2 \approx K n^2.\quad (1.5)$$

The masses of these states are independent of $\alpha'$, the scale is set by the curvature index $K$. Notice also that the level spacing grows with $n$. A similar result was found recently for strings in anti de Sitter spacetime [7, 8].

For $K < 0$, when the spatial section is a hyperboloid, both the energy and the average pressure of the oscillating strings are positive. The equation of state is given by Eq.(3.35). In this case, the strings can not provide a self-consistent solution to the Einstein equations. After semi-classical quantization, we find an infinity of more and more massive states. The mass-formula is given by Eq.(4.26):

$$\sqrt{-K m^2 \alpha'^2} \log \sqrt{-K m^2 \alpha'^2} \approx -\frac{\pi}{2} K \alpha' n$$

(1.6)
Notice that the level spacing grows faster than in Minkowski spacetime but slower than in the closed static Robertson-Walker spacetime. A summary of the classical and semi-classical features of the circular strings is presented in Tables I and II. Figs. 2-4 depict the mass quantization conditions for the $K > 0$ and $K < 0$ cases.

On the other hand, the stationary strings are determined by:

$$\phi' = \frac{L}{r^2}, \quad r^2 + U(r) = 0; \quad U(r) = (1 - Kr^2)(\frac{L^2}{r^2} - 1),$$

(1.7)

where $L$ is an integration constant (Fig. 5 shows the potential $U(r)$ for $K > 0$, $K = 0$, $K < 0$). For $K > 0$, all the stationary string solutions describe circular strings winding around the hypersphere. The equation of state is of the extremely unstable string type [1]. For $K < 0$, the stationary strings are represented by infinitely long open configurations with an angle between the two "arms" given by:

$$\Delta \phi = \pi - 2 \arctan(\sqrt{-KL}).$$

(1.8)

The energy density is positive while the pressure densities are negative. No simple equation of state is found for these solutions. A summary of the results for the stationary strings is presented in Table III.

Finally we compute the first and second order fluctuations around a static string center of mass, using the string perturbation series approach [1] and its covariant versions [3, 10]. Up to second order, the mass-formula for arbitrary values of the curvature index (positive or negative) is identical to the well-known flat spacetime mass-formula; all dependence on $K$ cancels out.

2 The Static Robertson-Walker Spacetimes

To clarify our notation we start by reviewing a few fundamental aspects of the Robertson-Walker spacetimes with curved spatial sections. The general line element is:

$$ds^2 = -dt^2 + a^2(t)[d\xi^2 + f^2(\xi)d\Omega^2_{D-2}],$$

(2.1)
where the function \( f(\xi) \) is given by:
\[
\begin{align*}
  f(\xi) &= \sin \xi, \quad 0 \leq \xi \leq \pi, \quad K > 0 \\
  f(\xi) &= \xi, \quad 0 \leq \xi < \infty, \quad K = 0 \\
  f(\xi) &= \sinh \xi, \quad 0 \leq \xi < \infty, \quad K < 0
\end{align*}
\] (2.2)

The spatial sections are closed, flat or hyperbolic depending on whether \( K \) is positive, zero or negative. Usually a non-zero curvature index \( K \) is scaled to either plus or minus 1 by a redefinition of the scale factor \( a(t) \), but for our purposes of considering the so-called static Robertson-Walker spacetimes, it is convenient to set the constant scale-factor equal to unity and keep \( K \) arbitrary. We shall also use the coordinates defined by setting:
\[
r = f(\xi),
\] (2.3)
in which case the line element takes the form (after a rescaling):
\[
ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2_{D-2}],
\] (2.4)

Notice that in the case of closed spatial sections, the latter coordinates cover only half of the spatial hypersurface (for \( r \in [0, 1/\sqrt{K}] \)), which is in that case a hypersphere of radius \( 1/\sqrt{K} \). Finally it is also useful to have the comoving coordinates defined by:
\[
r = \frac{R}{1 + \frac{K}{4}R^2},
\] (2.5)
with the corresponding line element:
\[
ds^2 = -dt^2 + a^2(t)[\frac{dR^2 + R^2 d\Omega^2_{D-2}}{(1 + \frac{K}{4}R^2)^2}].
\] (2.6)

For a general \( D \)-dimensional curved spacetime with curvature index \( K \), cosmological constant \( \Lambda \) and an energy-momentum tensor of the fluid form, the Einstein equations read:
\[
(D - 1)(D - 2)(K + a_t^2) = 2a^2(G\rho + \Lambda),
\] (2.7)
\[
2(D - 2)aa + (D - 3)(D - 2)(K + a_t^2) = 2a^2(\Lambda - GP),
\] (2.8)
where $\rho$ is the energy density, $P$ is the pressure, $G$ is a positive constant (essentially the gravitational constant in $D$ dimensions) and $a_t \equiv da(t)/dt$. In both equations the curvature index has to compete directly with the derivative of the scale factor, thus for instance in inflationary models (like de Sitter or power law universes), the effect of the curvature index will soon be negligible. In this paper we are interested precisely in the effects of the curvature index on the classical and quantum string propagation, and it follows that this investigation is most suitably performed in spacetimes with vanishing $a_t$, and not only because of simplicity. These spacetimes of constant scale factor are denoted the static Robertson-Walker spacetimes. By scaling we obtain $a = 1$, and the Einstein equations take the form:

\begin{align}
(D - 1)(D - 2)K &= 2(G\rho + \Lambda), \\
(D - 3)(D - 2)K &= 2(\Lambda - GP).
\end{align}

(2.9) \hspace{1cm} (2.10)

Usually energy and pressure are supposed to be non-negative. However, considering string sources in the Einstein equations, this is not necessarily true \[\text{[4]},\] not even in flat Minkowski spacetime \[\text{[7]}\]. In fact, in the next section we shall return to the question of self-consistent string solutions to the Einstein equations in the case of vanishing cosmological constant. We will consider bosonic strings with equations of motion and constraints given by:

\begin{align}
\ddot{x}^\mu - x''^\mu + \Gamma^\mu_{\rho\sigma}(\dot{x}^\rho \dot{x}^\sigma - x'^\rho x'^\sigma) &= 0, \\
g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu &= g_{\mu\nu} (\dot{x}^\mu \dot{x}^\nu + x'^\mu x'^\nu) = 0,
\end{align}

(2.11)

where dot and prime stand for derivative with respect to $\tau$ and $\sigma$, respectively. For the $2 + 1$ dimensional ($d\Omega^2_{D-2} = d\phi^2$) metric defined by the line element (2.4), for $a = 1$, they take the form:

\begin{align}
\ddot{t} - t'' &= 0, \\
\ddot{r} - r'' + \frac{Kr}{1 - Kr^2} (\dot{r}^2 - r'^2) - r(1 - Kr^2)(\dot{\phi}^2 - \phi'^2) &= 0, \\
\ddot{\phi} - \phi'' + \frac{2}{r}(\dot{r}\dot{\phi} - \dot{\phi} r') &= 0, \\
- \frac{\dot{t}r'}{1 - Kr^2} + r^2 \dot{\phi}' &= 0, \\
- (\dot{t}^2 + t'^2) + \frac{1}{1 - Kr^2} (t^2 + r'^2) + r^2(\dot{\phi}^2 + \phi'^2) &= 0.
\end{align}

(2.12)
3 Circular Strings, Physical Interpretation, Self-Consistency

In this section we shall give a complete description of the evolution and physical interpretation of circular string configurations in the static Robertson-Walker spacetimes. A plane circular string effectively lives in 2 + 1 dimensions so we will drop the dimensions perpendicular to the string plane. The line element, in the coordinates (2.4), is then:

\[ ds^2 = -dt^2 + \frac{dr^2}{1-Kr^2} + r^2d\phi^2. \]  
(3.1)

Circular strings have been intensively studied in static spacetimes [2, 3, 5, 4, 4, 4, 4, 4, 4], but until now not in the static Robertson-Walker spacetimes. The general equations determining the evolution and dynamics of the circular strings [3, 5] can however be used directly here too. The ansatz \( (t = t(\tau), r = r(\tau), \phi = \sigma) \), corresponding to a circular string, leads to [3]:

\[ \dot{t} = \sqrt{b}\alpha', \quad \dot{r}^2 + V(r) = 0, \]  
(3.2)

where \( \alpha' \) is the string tension, \( b \) is a positive integration constant with the dimension of \((\text{mass})^2\) and the potential \( V(r) \) is given by:

\[ V(r) = (1-Kr^2)(r^2 - b\alpha'^2). \]  
(3.3)

By insertion of Eqs.(3.2)-(3.3) into Eq.(3.1), we obtain the induced line element on the world-sheet:

\[ ds^2 = r^2(\tau)(-d\tau^2 + d\sigma^2), \]  
(3.4)

and the string length is given by:

\[ l(\tau) = 2\pi|r(\tau)|. \]  
(3.5)

Energy and pressure of the circular strings can be obtained from the 2 + 1 dimensional spacetime energy-momentum tensor:

\[ \sqrt{-g}T^{\mu\nu} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu)\delta^{(3)}(X - X(\tau, \sigma)). \]  
(3.6)
After integration over a spatial volume that completely encloses the string, the energy-momentum tensor for a circular string takes the form of a fluid:

\[ T^\mu_\nu = \text{diag.}(-\rho, P, P), \quad (3.7) \]

where in the comoving coordinates:

\[ \rho = \frac{1}{\alpha' t} = \sqrt{b}, \quad (3.8) \]
\[ P = \frac{1}{2\alpha' t} \frac{\dot{R}^2 - R^2}{(1 + \frac{K}{4} R^2)^2} = \frac{1}{2\sqrt{b}\alpha'^2} \left[ \frac{\dot{r}^2}{1 - K r^2} - r^2 \right] = \frac{b\alpha'^2 - 2r^2}{2\sqrt{b}\alpha'^2}. \quad (3.9) \]

Let us now consider separately the three different cases of vanishing, positive and negative curvature index. For \( K = 0 \) we have flat Minkowski spacetime, the circular string configurations there and their physical interpretation were already discussed in Ref.[7]. The circular string potential (3.3) is shown in Fig.1a and the equations of motion (3.2) determining the string radius are solved by:

\[ t(\tau) = \sqrt{b}\alpha' \tau, \quad r(\tau) = \sqrt{b}\alpha' \cos \tau, \quad (K = 0) \quad (3.10) \]

i.e. the string motion follows a pure harmonic motion with period \( T_\tau = 2\pi \) in the world-sheet time. The energy and pressure, Eqs.(3.8)-(3.9), are given by:

\[ \rho = \sqrt{b}, \quad (3.11) \]
\[ P = -\frac{\sqrt{b}}{2} \cos 2\tau. \quad (3.12) \]

Notice that during an oscillation of the string, the equation of state "oscillates" between \( P = \rho/2 \), corresponding to ultra-relativistic matter in 2+1 dimensions, and \( P = -\rho/2 \), corresponding to extremely unstable strings [4]. For further discussion on this point, see Ref.[7]. Using the exact time-dependent pressure (3.12), it is clear from Eqs.(2.9)-(2.10) that the circular strings do not provide a self-consistent solution in Minkowski spacetime. If using instead the average values over one oscillation:

\[ <\rho> = \sqrt{b}, \quad <P> = 0, \quad (3.13) \]

corresponding to cold matter, we see that Eq.(2.10) with vanishing cosmological constant is fulfilled, while Eq.(2.9) leads to:

\[ K = G\sqrt{b}. \quad (3.14) \]
The circular strings thus generate a positive curvature index and we conclude that even after averaging over an oscillation, they do not provide a self-consistent solution in Minkowski spacetime.

For positive curvature index, the spatial hypersurface is a sphere and we have to distinguish between the two cases $\sqrt{bK\alpha'} \leq 1$ and $\sqrt{bK\alpha'} > 1$, as is clear from the location of the zeros of the potential, see Figs.1b,1c. For $\sqrt{bK\alpha'} \leq 1$, the solution of Eqs.(3.2)-(3.3) is:

$$t_-(\tau) = \sqrt{b\alpha'}\tau, \quad r_-(\tau) = \sqrt{b\alpha'}\text{sn}[\tau, k_-] \quad (K > 0),$$

(3.15)

where $\text{sn}[\tau, k_-]$ is the Jacobi elliptic function and the elliptic modulus is given by:

$$k_- = \sqrt{bK\alpha'} \in [0, 1].$$

(3.16)

The solution describes a string oscillating between zero radius and maximal radius $r_{\text{max}} = \sqrt{b\alpha'}$ with period $T_\tau = 4K(k_-)$ in the world-sheet time, where $K(k_-)$ is the complete elliptic integral of the first kind. Since $r_{\text{max}} \leq 1/\sqrt{K}$, the string oscillates on one hemisphere; it does not cross the equator. The energy and pressure are obtained from Eqs.(3.8)-(3.9):

$$\rho_- = \sqrt{b} = \frac{k_-}{\sqrt{K\alpha'}},$$

(3.17)

$$P_- = \frac{k_-}{2\sqrt{K\alpha'}}(1 - 2\text{sn}^2[\tau, k_-]).$$

(3.18)

During an oscillation of the string, the equation of state "oscillates" between $P_- = \rho_-/2$ and $P_- = -\rho_-/2$. This is similar to the situation in Minkowski spacetime. The average values are given by:

$$<\rho_-> = \frac{k_-}{\sqrt{K\alpha'}}, \quad <P_-> = \frac{1}{2\sqrt{K\alpha'}}\left[\frac{k_-^2 - 2}{k_-} + 2 \frac{E(k_-)}{K(k_-)}\right],$$

(3.19)

so that $2\sqrt{K\alpha'} < P_- > \in [-1, 0]$ where the limit $-1$ corresponds to $k_- \to 1$ and the limit $0$ corresponds to $k_- \to 0$. Thus in average the pressure is negative (in the limit $k_- = 0$, there are no strings at all) and the equation of state is written:

$$<P_-> = (\gamma(k_-) - 1) <\rho_->; \quad \gamma(k_-) = \frac{3}{2} - \frac{1}{k_-^2}[1 - \frac{E(k_-)}{K(k_-)}].$$

(3.20)
When \( k_- \) increases from \( k_-=0 \) to \( k_-=1 \), the function \( \gamma(k_-) \) decreases from \( \gamma(0)=1 \) to \( \gamma(1)=1/2 \), that is, from cold matter type (\( \gamma=1 \)) to extremely unstable string type (\( \gamma=1/2 \)).

Returning now to the Einstein equations (2.9)-(2.10) without cosmological constant, we see that Eq.(2.9) can be fulfilled using the average values (3.19), but Eq.(2.10) cannot. We conclude that the string solutions, Eq.(3.15), for \( \sqrt{bK}\alpha' \leq 1 \), do not provide a self-consistent solution to the Einstein equations.

We now consider the case where \( \sqrt{bK}\alpha' > 1 \). The solution of Eqs.(3.2)-(3.3) is (see Fig.1c):

\[
\begin{align*}
t_+(\tau) &= \sqrt{b}\alpha'\tau, \\
r_+(\tau) &= \frac{1}{\sqrt{K}} \text{sn}[\tau/k_+, k_+], \quad (K > 0) \\
\end{align*}
\]  

(3.21)

where the elliptic modulus is now given by:

\[
k_+ = \frac{1}{\sqrt{bK}\alpha'} \in ]0,1[.
\]  

(3.22)

This string solution is oscillating between zero radius and maximal radius \( r_{\text{max}} = 1/\sqrt{K} \), corresponding to the radius of the hypersphere, with period \( T_\tau = 4k_+K(k_+) \) in the world-sheet time. The physical interpretation of this solution is a string oscillating on the full hypersphere. For \( \tau = 0 \) it starts with zero radius on one of the hemispheres. It expands and reaches the equator for \( \tau = k_+K(k_+) \). It then crosses the equator and contracts on the other hemisphere until it collapses to a point for \( \tau = 2k_+K(k_+) \). It now expands again, crosses the equator and eventually collapses to its initial configuration of zero radius for \( \tau = 4k_+K(k_+) \). The energy and pressure of this solution are given by:

\[
\begin{align*}
\rho_+ &= \sqrt{b} = \frac{1}{\sqrt{K}\alpha'k_+}, \\
P_+ &= \frac{1 - 2k_+^2 \text{sn}^2[\tau/k_+, k_+]}{2k_+\sqrt{K}\alpha'}. \\
\end{align*}
\]  

(3.23)

(3.24)

During an oscillation, the equation of state "oscillates" between \( P_+ = \rho_+/2 \) and \( P_+ = (1 - 2k_+^2)\rho_+/2 \). The average values are given by:

\[
\begin{align*}
< \rho_+ > &= \frac{1}{\sqrt{K}\alpha'k_+}, \\
< P_+ > &= \frac{1}{2k_+\sqrt{K}\alpha'}\left[ \frac{2E(k_+)}{K(k_+)} - 1 \right],
\end{align*}
\]  

(3.25)
so that $2\sqrt{K}\alpha' < P_+ \in ] - 1, \infty]$ where the limit $-1$ corresponds to $k_+ \to 1$ and the limit $\infty$ corresponds to $k_+ \to 0$. Thus the average pressure can be negative, zero or positive for these solutions. The equation of state is:

$$< P_+ > = (\gamma(k_+) - 1) < \rho_+ >; \quad \gamma(k_+) = \frac{1}{2} + \frac{E(k_+)}{K(k_+)}.$$  (3.26)

When $k_+$ increases from $k_+ = 0$ to $k_+ = 1$, the function $\gamma(k_+)$ decreases from $\gamma(0) = 3/2$ to $\gamma(1) = 1/2$, that is, from ultra-relativistic matter type ($\gamma = 3/2$) to extremely unstable string type ($\gamma = 1/2$). Let us consider also the question of self-consistency in this case. Using the average values (3.26) in the Einstein equations (2.9)-(2.10), without cosmological constant, we find that the self-consistency conditions are satisfied with:

$$K = \frac{G}{k_+ \sqrt{K}\alpha'}, \quad 2E(k_+) = K(k_+),$$  (3.27)

which yield the (numerical) solution:

$$k_+ = 0.9089..., \quad K = \left(\frac{G}{\alpha'}\right)^{2/3} \times 1.0658...$$  (3.28)

It follows that a gas of oscillating circular strings described by Eq.(3.21) for $k_+ = 0.9089...$, provides a self-consistent solution to the Einstein equations. The solution is a spatially closed static Robertson-Walker spacetime with scale factor normalized to $a = 1$ and curvature index $K = \left(\frac{G}{\alpha'}\right)^{2/3} \times 1.0658...$

Finally we consider circular strings in the spatially hyperbolic case, corresponding to negative curvature index. The potential is shown in Fig.1d. and Eqs.(3.2)-(3.3) are solved by:

$$t(\tau) = \sqrt{b_0} \alpha' \tau, \quad r(\tau) = \frac{k}{\sqrt{-K}} s d[\mu \tau, k], \quad (K < 0)$$  (3.29)

where we introduced the notation:

$$\mu = \sqrt{1 - Kb_0\alpha'^2}; \quad k = \sqrt{\frac{-Kb_0\alpha'^2}{1 - Kb_0\alpha'^2}} \in [0, 1]$$  (3.30)

The solution describes a string oscillating between zero radius and maximal radius $r_{\text{max}}$ with period $T_\tau$ in the world-sheet time:

$$r_{\text{max}} = \frac{1}{\sqrt{-K}} \frac{k}{\sqrt{1 - k^2}}, \quad T_\tau = 4K(k)/\mu.$$  (3.31)
The energy and pressure are given by:

\[
\rho = \sqrt{b} = \frac{k}{\sqrt{-K\alpha'\sqrt{1-k^2}}},
\]

\[
P = \frac{k - 2k(1-k^2)sd^2[\mu\tau,k]}{2\sqrt{1-k^2}\sqrt{-K\alpha'}}.
\]

During an oscillation, the equation of state "oscillates" between \( P = \rho/2 \) and \( P = -\rho/2 \). This is like the situation in flat Minkowski spacetime. The average values are given by:

\[
<\rho> = \frac{k}{\sqrt{-K\alpha'\sqrt{1-k^2}}}, \quad <P> = \frac{1}{2\sqrt{1-k^2}\sqrt{-K\alpha'}}[-k + \frac{2}{k}(1 - \frac{E(k)}{K(k)})],
\]

so that \( 2\sqrt{-K\alpha'} < P > \in [0, \infty[ \) where the limit 0 corresponds to \( k \to 0 \) and the limit \( \infty \) corresponds to \( k \to 1 \). The average pressure is always positive (for \( k = 0 \), there are no strings at all). The equation of state takes the form:

\[
<P> = (\gamma(k) - 1) <\rho>; \quad \gamma(k) = \frac{1}{2} + \frac{1}{k^2}(1 - \frac{E(k)}{K(k)}).
\]

When \( k \) increases from \( k = 0 \) to \( k = 1 \), the function \( \gamma(k) \) increases from \( \gamma(0) = 1 \) to \( \gamma(1) = 3/2 \), that is, from cold matter type \( (\gamma = 1) \) to ultra-relativistic matter type \( (\gamma = 3/2) \). Clearly, these solutions can not provide a self-consistent solution to the Einstein equations (2.9)-(2.10).

This concludes our analysis of classical circular string configurations in the static Robertson-Walker spacetimes. A summary of the results is presented in Table I.

### 4 Semi-Classical Quantization

In this section we perform a semi-classical quantization of the circular string configurations discussed in the previous section. We use an approach developed in field theory by Dashen et. al. [9], based on the stationary phase approximation of the partition function. The method can be only used for time-periodic solutions of the classical equations of motion. In our string problem, these solutions include all the circular string solutions in the static
Robertson-Walker spacetimes, discussed in the previous section. The method has recently been used to quantize circular strings in de Sitter and anti de Sitter spacetimes \[7\] also, and we shall follow the analysis of Ref.\[7\] closely.

The result of the stationary phase integration is expressed in terms of the function:

\[
W(m) \equiv S^{\text{cl}}(T(m)) + m T(m),
\]

where \(S^{\text{cl}}\) is the action of the classical solution, \(m\) is the mass and the period \(T(m)\) is implicitly given by:

\[
\frac{dS^{\text{cl}}}{dT} = -m.
\]

In string theory we must choose \(T\) to be the period in a physical time variable. In the static Robertson-Walker spacetimes, it is convenient to take \(T\) to be the period in the comoving time \(t\). From Eq.(3.2), it follows that:

\[
T = \sqrt{b} \alpha' T_\tau.
\]

The bound state quantization condition is \[9\]:

\[
W(m) = 2\pi n, \quad n \in \mathbb{N}_0,
\]

\(n\) being ”large”. We will consider the two cases of positive and negative curvature index. The case of zero curvature index was considered in Ref.\[7\] and the results in that case will come out anyway in the limit \(K \to 0\) (from above or below). The classical action is given by:

\[
S^{\text{cl}} = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \int_0^{T_\tau} d\tau \, g_{\mu\nu} \left[ \dot{X}^{\mu} \dot{X}^{\nu} - X'^{\mu} X'^{\nu} \right]
= -\frac{2}{\alpha'} \int_0^{T_\tau} d\tau \, r^2(\tau),
\]

where we used Eqs.(3.2)-(3.3). We first consider positive \(K\) in the case \(\sqrt{bK\alpha'} \leq 1\), i.e. the solution (3.15), corresponding to strings oscillating on one hemisphere. The period in comoving time is given by:

\[
T_- = \frac{4k_- K(k_-)}{\sqrt{K}}.
\]
The classical action over one period becomes:

\[ S_{\text{cl}} = \frac{8}{K\alpha'}[E(k_-) - K(k_-)]. \] (4.7)

A straightforward calculation yields:

\[ \frac{dT_-}{dk_-} = \frac{4}{\sqrt{K}} \frac{E(k_-)}{1-k_-^2}, \quad \frac{dS_{\text{cl}}}{dk_-} = -\frac{8}{K\alpha'} \frac{k_- E(k_-)}{1-k_-^2}. \] (4.8)

Now Eq.(4.2) leads to:

\[ m_- = \frac{2k_-}{\sqrt{K\alpha'}}, \] (4.9)

and the quantization condition (4.4) becomes:

\[ W_- = \frac{8}{K\alpha'}[E(k_-) - (1-k_-^2)K(k_-)] = 2\pi n \] (4.10)

This equation determines a quantization of the parameter \( k_- \), which through Eq.(4.9) yields a quantization of the mass. A full parametric plot of \( K\alpha'W_- \) as a function of \( K\alpha'^2m_-^2 \) for \( k_- \in [0,1] \) is shown in Fig.2. A fair approximation is provided by the straight line connecting the two endpoints:

\[ W_- \approx 2m_-^2\alpha', \] (4.11)

so that the mass quantization condition becomes:

\[ m_-^2\alpha' \approx \pi n \] (4.12)

The total number of states is then estimated to be:

\[ N_- \approx \frac{4}{\pi K\alpha'}. \] (4.13)

This is the number of quantized circular string states oscillating on one hemisphere without crossing the equator. The circular strings oscillating on the full hypersphere are obtained for \( \sqrt{bK}\alpha' > 1 \), and are given by the solution (3.21). Their period in comoving time is:

\[ T_+ = \frac{4K(k_+)}{\sqrt{K}}. \] (4.14)
The classical action over one period becomes:

$$S_{\text{cl}} = \frac{8}{K\alpha'} \frac{E(k_+)}{k_+} - \frac{K(k_+)}{k_+}. \quad (4.15)$$

A straightforward calculation yields:

$$\frac{dT_+}{dk_+} = \frac{4}{\sqrt{K}} \left[ E(k_+) - \frac{K(k_+)}{k_+} \right] - \frac{8}{K\alpha'} \left[ \frac{E(k_+)}{k_+^2(1 - k_+^2)} - \frac{K(k_+)}{k_+^2} \right].$$

Now Eq.(4.2) leads to:

$$m_+ = \frac{2}{\sqrt{K\alpha'k_+}}, \quad (4.17)$$

and the quantization condition (4.4) becomes:

$$W_+ = \frac{8}{K\alpha'} \frac{E(k_+)}{k_+} = 2\pi n \quad (4.18)$$

This equation determines a quantization of the parameter $k_+$, which through Eq.(4.17) yields a quantization of the mass. A parametric plot of $K\alpha'W_+$ as a function of $K\alpha'^2m_+^2$ for $k_+ \in [0, 1]$ is shown in Fig.3. Asymptotically, when the mass grows indefinitely, corresponding to $k_+ \to 0$ (see Eq.(4.17)), we find:

$$K\alpha'^2m_+^2 \approx \frac{4}{k_+^2}, \quad K\alpha'W_+ \approx \frac{4\pi}{k_+} \quad (4.19)$$

and the mass quantization condition becomes:

$$m_+^2 \approx Kn^2. \quad (4.20)$$

Several interesting remarks are now in order: First, notice that the mass is independent of $\alpha'$. The scale of these very massive states is set by the curvature index $K$. Secondly, since the mass is proportional to $n$, the level spacing ($\Delta(m^2\alpha')$ as a function of $n$) grows proportionally to $n$. This is completely different from flat Minkowski spacetime where the level spacing is constant. Finally, it should be mentioned that the same behaviour for very massive strings was found recently in anti de Sitter spacetime [7, 8]. At this point it is tempting to consider the partition function for a gas of strings at finite temperature, but it must be stressed that a discussion of the thermodynamic
properties (for instance existence or non-existence of a Hagedorn temperature) must be based on exact quantization of generic strings, and not only on a semi-classical quantization of special circular string configurations. It must be noticed, however, that it has been shown recently \[7, 8\], that for Minkowski, de Sitter and anti de Sitter spacetimes, the spectrum of generic strings and the semi-classical spectrum of circular strings are in complete agreement.

Let us now consider the semi-classical quantization of the circular strings in the spatially hyperbolic spacetime, i.e. we return to the solutions (3.29). The period in comoving time is:

$$T = \frac{4kK(k)}{\sqrt{-K}}. \quad (4.21)$$

The classical action over one period becomes:

$$S^{\text{cl}} = \frac{8}{K\alpha'} \frac{E(k) - (1 - k^2)K(k)}{\sqrt{1 - k^2}}. \quad (4.22)$$

A straightforward calculation yields:

$$\frac{dT}{dk} = \frac{4}{\sqrt{-K}} \frac{E(k)}{1 - k^2}, \quad \frac{dS^{\text{cl}}}{dk} = \frac{8}{K\alpha'} \frac{kE(k)}{(1 - k^2)^{3/2}}. \quad (4.23)$$

Now Eq.(4.2) leads to:

$$m = \frac{2}{\sqrt{-K\alpha'}} \frac{k}{\sqrt{1 - k^2}}, \quad (4.24)$$

and the quantization condition (4.4) becomes:

$$W = \frac{8}{K\alpha'} \frac{E(k) - K(k)}{\sqrt{1 - k^2}} = 2\pi n \quad (4.25)$$

This equation determines a quantization of the parameter $k$, which through Eq.(4.24) yields a quantization of the mass. A parametric plot of $-K\alpha'W$ as a function of $-K\alpha'^2m^2$ for $k \in [0, 1]$ is shown in Fig.4. Asymptotically, when the mass grows indefinitely, corresponding to $k \to 1$ (see Eq.(4.24)), we find the quantization condition:

$$\sqrt{-Km^2\alpha'^2} \log \sqrt{-Km^2\alpha'^2} \approx -\frac{\pi}{2} K\alpha' n \quad (4.26)$$
Formally, it corresponds to a mass-formula in the form \( m \log m \propto n \), i.e., the level spacing grows faster than in Minkowski spacetime (where it is constant) but slower than in the closed static Robertson-Walker spacetime (where it grows proportional to \( n \)).

We close this section by ensuring that our results reproduce correctly the spectrum in flat Minkowski spacetime by taking the limit \( K \rightarrow 0 \). Starting for instance from Eqs.(4.9)-(4.10) in the spatially closed universe, we find in the limit \( K \rightarrow 0 \):

\[
m = 2\sqrt{b}, \quad W = 2\pi \sqrt{b} \alpha' \quad (4.27)
\]

which by Eq.(4.4) gives:

\[
m^2 \alpha' = 4n \quad (4.28)
\]

If we subtract the intercept \(-4\), this is the well-known (exact) mass formula for closed bosonic strings in flat Minkowski spacetime. Same result is obtained by starting from Eqs.(4.17)-(4.18) or from Eqs.(4.24)-(4.25).

The main conclusions of this section are presented in Table II.

5 Stationary Strings

In this section we will supplement our results on exact classical string solutions by considering the family of stationary strings. Stationary strings certainly do not tell much about string propagation in curved spacetimes, which is our main aim here, but since they are equilibrium configurations, existing only when there is an exact balance between the local gravity and the string tension, their actual shape reflects the geometry and topology of the underlying spacetime and they provide information about the interaction of gravity on strings. We shall follow a recent approach [14] where the stationary strings are described by a potential in the (stationary) radial coordinate. The stationary string ansatz \((t = \tau, \ r = r(\sigma), \ \phi = \phi(\sigma))\) in Eq.(2.12) leads to:

\[
\phi' = \frac{L}{r^2}, \quad r^2 + U(r) = 0, \quad (5.1)
\]

where \( L \) is an integration constant and the potential \( U(r) \) is given by:

\[
U(r) = (1 - Kr^2)\left(\frac{L^2}{r^2} - 1\right). \quad (5.2)
\]
Notice that we consider plane stationary strings in the backgrounds (3.1), for $K$ positive, negative or zero. Possible extra transverse dimensions have been dropped for simplicity. Beside energy and pressure, it is also interesting to consider the string length. By insertion of the stationary string ansatz and Eqs.(5.1)-(5.2) in the line element (3.1), we find that the string length element is:

$$dl = d\sigma,$$  

(5.3)

thus $\sigma$ measures directly the length of stationary strings in the static Robertson-Walker spacetimes.

For vanishing curvature index, corresponding to flat Minkowski spacetime, it is well-known that the only stationary strings are the straight ones. Indeed, the potential $U(r)$ is given by (Fig.5a.):

$$U(r) = \frac{L^2}{r^2} - 1,$$  

(5.4)

and Eqs.(5.1) are solved by:

$$r(\sigma) = \sqrt{\sigma^2 + L^2},$$

(5.5)

$$\phi(\sigma) = \arctan(\sigma/L),$$

(5.6)

which for $\sigma \in ]-\infty, +\infty[$ describes an infinitely long straight string parallel to the $y$-axis with "impact-parameter" $L$. The string energy and pressure densities are obtained from Eq.(3.6):

$$\frac{d\rho}{dl} = \frac{d\rho}{d\sigma} = \frac{d}{d\sigma} \int d^3X \sqrt{-g} T^{00} = \frac{1}{2\pi \alpha'},$$

(5.7)

$$\frac{dP_y}{dl} = \frac{dP_y}{d\sigma} = \frac{d}{d\sigma} \int d^3X \sqrt{-g} T^{y\, y} = -\frac{1}{2\pi \alpha'},$$

(5.8)

while $P_x = 0$. Obviously the integrated energy and pressure are infinite. Eqs.(5.7)-(5.8) represent the well-known string equation of state in $1 + 1$ effective dimensions.

Let us now turn to the more interesting cases of non-vanishing curvature index. For positive $K$, the potential is given by (Fig.5b.):

$$U(r) = (1 - |K| r^2)(\frac{L^2}{r^2} - 1),$$

(5.9)
For $L > 1/\sqrt{K}$ there are no solutions since the "impact-parameter" $L$ is larger than the radius of the hypersphere, thus we need only consider the case $L \leq 1/\sqrt{K}$. The solution of Eqs.(5.1) is:

$$r^2(\sigma) = \frac{1}{K} \frac{KL^2 + \tan^2(\sqrt{K}\sigma)}{1 + \tan^2(\sqrt{K}\sigma)},$$

(5.10)

$$\phi(\sigma) = \arctan[\tan(\sqrt{K}\sigma)]/\sqrt{KL}].$$

(5.11)

Consider first the "degenerate" cases $L = 0$ and $L = 1/\sqrt{K}$:

$$\phi = \pi/2, \quad r(\sigma) = \pm \frac{1}{\sqrt{K}} \cos(\sqrt{K}\sigma); \quad \text{for} \quad L = 0 \quad (5.12)$$

$$\phi = \sqrt{K}\sigma, \quad r(\sigma) = 1/\sqrt{K}; \quad \text{for} \quad L = 1/\sqrt{K} \quad (5.13)$$

Eqs.(5.12) describe a stationary circular string winding around the hypersphere from pole to pole, while Eqs.(5.13) describe a stationary circular string winding around the hypersphere along the equator. More generally, the solution (5.10)-(5.11) describes a stationary circular string of radius $1/\sqrt{K}$ winding around the hypersphere for arbitrary values of $L$, as can be seen as follows: The hypersphere is parametrized by:

$$x = \frac{1}{\sqrt{K}} \sin \xi \sin \phi$$

$$y = \frac{1}{\sqrt{K}} \cos \xi$$

$$z = \frac{1}{\sqrt{K}} \sin \xi \cos \phi$$

(5.14)

where $\sin \xi = \sqrt{K}r$. A rotation by the angle $\theta$ in the $y - z$ plane leads to:

$$\tilde{x} = x$$

$$\tilde{y} = y \cos \theta - z \sin \theta$$

$$\tilde{z} = y \sin \theta + z \cos \theta$$

(5.15)
Consider now the general solution (5.10)-(5.11) and take \( \theta = \arccos(\sqrt{KL}) \)

\[
\begin{align*}
\tilde{x} &= \frac{1}{\sqrt{K}} \sin(\sqrt{K} \sigma) \\
\tilde{y} &= 0 \\
\tilde{z} &= \frac{1}{\sqrt{K}} \cos(\sqrt{K} \sigma)
\end{align*}
\]

(5.16)
i.e. the solution (5.13). All stationary strings are identical up to rotations, so we need only consider (say) the solution (5.13). By integrating the components of the energy-momentum tensor, Eq.(3.6), we find:

\[
\rho = \frac{1}{\sqrt{K} \alpha'},
\]

(5.17)

\[
P_x = P_z = -\frac{1}{2\sqrt{K} \alpha'},
\]

(5.18)
corresponding to the extremely unstable string type equation of state \( P = -\rho/2 \) in 2 + 1 effective dimensions.

For negative curvature index, the potential (5.2) is:

\[
U(r) = (1 + |K| r^2) \left( \frac{L^2}{r^2} - 1 \right),
\]

(5.19)

and stationary strings can only exist for \( r \geq L \), see Fig.5c. Eqs.(5.1) are solved by:

\[
r^2(\sigma) = \frac{1}{K} KL^2 - \frac{\tanh^2(\sqrt{-K} \sigma)}{1 - \tanh^2(\sqrt{-K} \sigma)},
\]

(5.20)

\[
\phi(\sigma) = \pm \left\{ \frac{\pi}{2} - \arctan(\sqrt{-KL} \tan^{-1}(\sqrt{-K} \sigma)) \right\}.
\]

(5.21)

For \( \sigma \in ] - \infty, +\infty[ \), the solution describes a string stretching from spatial infinity towards \( r = L \) and back towards spatial infinity. The angle between the two "arms" is given by:

\[
\Delta \phi = \pi - 2 \arctan(\sqrt{-KL}) \in [0, \pi].
\]

(5.22)

The total string length and energy are infinite, while the energy density is given by:

\[
\frac{d\rho}{dl} = \frac{1}{2\pi \alpha'}.
\]

(5.23)
The pressures in the two directions are generally different due to lack of symmetry. In the comoving coordinates (2.6) we find the following integral expressions, using also Eq.(3.6):

\[ P_x = -\frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} d\sigma \left[ R' \cos \phi - R\phi' \sin \phi \right]^2, \]  
(5.24)

\[ P_y = -\frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} d\sigma \left[ R' \sin \phi - R\phi' \cos \phi \right]^2, \]  
(5.25)

where \( R \) and \( r \) are related by Eq.(2.5). Using the explicit solutions, Eqs.(5.20)-(5.21), the pressure densities are:

\[ \frac{dP_x}{dl} = \frac{dP_x}{d\sigma} = \frac{8KL^2}{\pi\alpha'(1-KL^2)^2} e^{-\sqrt{-K}|\sigma|}, \]  
(5.26)

\[ \frac{dP_y}{dl} = \frac{dP_y}{d\sigma} = \frac{-8}{\pi\alpha'(1-KL^2)^2} e^{-\sqrt{-K}|\sigma|}. \]  
(5.27)

The pressure densities are negative (\( K \) is negative) but asymptotically go to zero exponentially. The integrated pressures are thus finite. In fact, Eqs.(5.24)-(5.25) can be integrated explicitly:

\[ P_x = \frac{2}{\pi\alpha'\sqrt{-K}(1-KL^2)} \left\{ \frac{-2}{3} + \frac{1}{KL^2} - \frac{1}{KL^2} \sqrt{-K} \left[ \frac{\pi}{2} \arcsin \frac{1}{\sqrt{1-KL^2}} \right] \right\}, \]  
(5.28)

\[ P_y - P_x = \frac{-4}{3\pi\alpha'\sqrt{-K}(1-KL^2)} < 0 \]  
(5.29)

In the two extreme limits \( L \to \infty, \ L \to 0 \), we have:

\[ P_x \to 0, \ \ P_y \to 0; \ \ \text{for} \ L \to \infty \]  
(5.30)

\[ P_x \to 0, \ \ P_y \to -\frac{4}{3\pi\alpha'\sqrt{-K}}; \ \ \text{for} \ L \to 0 \]  
(5.31)

where the latter case describes a straight string through \( r = 0 \). This concludes our discussion of the stationary strings and their physical interpretation in the static Robertson-Walker spacetimes. A summary of the results is presented in Table III.
6 String Perturbation Series Approach

Until now, our analysis has been based on exact solutions to the string equations of motion and constraints (2.11). However, it is interesting to consider also approximative methods. In this section we use the string perturbation series approach, originally developed by de Vega and Sánchez [1], to describe string fluctuations around the string center of mass. The target space coordinates are expanded as:

\[ x^\mu(\tau, \sigma) = q^\mu(\tau) + \eta^\mu(\tau, \sigma) + \xi^\mu(\tau, \sigma) + ... \]  

(6.1)

and after insertion of this series into Eqs.(2.12), one solves the string equations of motion and constraints order by order in the expansion. For a massive string, the zeroth order equations determining the string center of mass read:

\[ \ddot{q}^\mu + \Gamma^\mu_{\rho\sigma} \dot{q}^\rho \dot{q}^\sigma = 0, \quad g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = -m^2 \alpha'^2 \]  

(6.2)

For a string with center of mass in the (say) \( x-y \) plane of the \( D \)-dimensional static Robertson-Walker spacetime (2.4), Eqs.(6.2) lead to:

\[ t = \sqrt{p^2 + m^2 \alpha'^2}, \quad \dot{\phi} = \frac{L}{r^2}, \quad \dot{r}^2 = (1 - K r^2)(p^2 - \frac{L^2}{r^2}) \alpha'^2, \]  

(6.3)

where \( p \) and \( L \) are integration constants with the physical interpretation of momentum and angular momentum, respectively. Eqs.(6.3) can be easily solved in terms of elementary functions, but we shall not need the general solutions here. The equations for the first and second order string fluctuations will turn out to be quite complicated in the general case, so we shall consider here only a static string center of mass, \( p = L = 0 \):

\[ t = m \alpha' \tau, \quad r = \text{const.} \equiv r_0, \quad \text{all angular coordinates constant.} \]  

(6.4)

It is convenient to consider from the beginning only first order string fluctuations in the directions perpendicular to the geodesic of the center of mass. We thus introduce normal vectors \( n_R^\mu \), \( R = 1, 2, ..., (D - 1) \):

\[ \eta^\mu = \delta x^R n_R^\mu, \]  

(6.5)
where $\delta x^R$ are the comoving fluctuations, i.e. the fluctuations as seen by an observer travelling with the center of mass of the string. It can be shown that the first order fluctuations fulfill the equations \[\tag{6.6}\]

$$\ddot{C}^R_n + (n^2 \delta RS - R_{\mu\sigma\nu} n^\nu \dot{n}_R^\sigma \dot{q}^\mu) C^S_n = 0,$$

where $R_{\mu\rho\sigma\nu}$ is the Riemann tensor of the background and $C^S_n$ are the modes of the fluctuations:

$$\delta x^R(\tau, \sigma) = \sum_n C^R_n(\tau) e^{-in\sigma} \quad \tag{6.7}$$

In the present case of static Robertson-Walker spacetimes, the Riemann tensor is non-zero but the projections appearing in Eq.(6.6) actually vanish, as can be easily verified. It follows from the explicit expressions of the normal vectors:

$$n^r = (0, \sqrt{1-Kr_0^2}, 0), \quad n^i = (0, 0, ..., 0, \frac{1}{r_0}, 0, ..., 0), \quad \tag{6.8}$$

that the first order string fluctuations are ordinary plane waves:

$$\eta^t(\tau, \sigma) = 0, \quad \eta^r(\tau, \sigma) = \sqrt{1-Kr_0^2} \sum_n [a_n e^{-in(\sigma+\tau)} + \tilde{a}_n e^{-in(\sigma-\tau)}], \quad \tag{6.9}$$

$$\eta^i(\tau, \sigma) = \frac{1}{r_0} \sum_n [a^i_n e^{-in(\sigma+\tau)} + \tilde{a}^i_n e^{-in(\sigma-\tau)}]. \quad \tag{6.10}$$

Here $\eta^t$, $\eta^r$ and $\eta^i$ denote the fluctuations in the temporal, radial and angular directions, respectively. The second order fluctuations are determined by:

$$\ddot{\xi}^t - \xi''' = 0, \quad \tag{6.12}$$

$$\ddot{\xi}^r - \xi''' = -\frac{Kr_0}{1-Kr_0^2} [\dot{\eta}^r]^2 - (\eta^r)^2] + r_0(1-Kr_0^2) \sum_i [(\dot{\eta}^i)^2 - (\eta^i)^2], \quad \tag{6.13}$$

$$\ddot{\xi}^i - \xi''' = -\frac{2}{r_0} \sum_i [\dot{\eta}^r \dot{\eta}^i - \eta^r \eta^i]. \quad \tag{6.14}$$

These are just ordinary wave equations with source terms, and can be easily solved. Thereafter, we have to expand also the constraint equations. In the present case we find up to second order:

$$m^2 \alpha^2 = \frac{1}{1-Kr_0^2} [\dot{\eta}^r]^2 + (\eta^r)^2] + r_0^2 \sum_i [(\dot{\eta}^i)^2 + (\eta^i)^2], \quad \tag{6.15}$$
\[ \frac{1}{1 - K r_0^2} \dot{\eta}^r \dot{\eta}^r + r_0^2 \sum_i \dot{\eta}^i \dot{\eta}^i = 0. \]  

(6.16)

Introducing the notation \( A_n = A_n^0, \quad A_n^\alpha = (A_n^0, A_n^i) \), these equations read explicitly:

\[ m^2 \alpha' \alpha'^2 = -2 \sum_\alpha \sum_{n,l} l(n - l) [A_{n-l}^{\alpha} A_l^0 e^{-in(\sigma+\tau)} + \tilde{A}_{n-l}^{\alpha} \tilde{A}_l^0 e^{-in(\sigma-\tau)}], \]  

(6.17)

\[ \sum_\alpha \sum_{n,l} [A_{n-l}^{\alpha} A_l^0 e^{-in(\sigma+\tau)} - \tilde{A}_{n-l}^{\alpha} \tilde{A}_l^0 e^{-in(\sigma-\tau)}] = 0, \]  

(6.18)

which are just the usual flat spacetime constraints. All dependence of the curvature index \( K \) (positive or negative) has canceled out in these formulae.

For \( n = 0 \), in particular, we get the usual flat spacetime mass-formula:

\[ m^2 \alpha'^2 = 2 \sum_\alpha \sum_l l^2 [A_l^\alpha A_{-l}^{\alpha} + \tilde{A}_l^\alpha \tilde{A}_{-l}^{\alpha}], \]  

(6.19)

with the constraint that there must be an equal amount of left and right movers:

\[ \sum_\alpha \sum_l l^2 [A_l^\alpha A_{-l}^{\alpha} - \tilde{A}_l^\alpha \tilde{A}_{-l}^{\alpha}] = 0. \]  

(6.20)

Notice that the spectrum found here (the flat spacetime spectrum) is very different (when \( K \neq 0 \)) from the spectrum of the circular strings, as discussed in Section 4. This is however in no ways contradictory. The circular string ansatz merely picks out particular states in the complete spectrum, so there is no apriori reason to believe that the circular string spectrum should be similar to the generic spectrum. Furthermore, the perturbation approach used in this section is also semi-classical in nature, in the sense that it is based on fluctuations around a special solution, namely the static string center of mass, Eq.(6.4).

### 7 Concluding Remarks

We have solved the equations of motion and constraints for circular strings in static Robertson-Walker spacetimes. We computed the equations of state and found a self-consistent solution to the Einstein equations. The solutions have been quantized semi-classically using the stationary phase approximation.
method and the resulting spectra were analyzed and discussed. We also found all stationary string configurations in these spacetimes and we computed the corresponding physical quantities, string length, energy and pressure. Finally we calculated the first and second order string fluctuations around a static center of mass, using the string perturbation series approach.

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Figure Captions

Fig.1. The potential $V(r)$ introduced in Eqs.(3.2)-(3.3) for a circular string in the static Robertson-Walker spacetimes: (a) flat ($K = 0$), (b) closed ($K > 0$ and $\sqrt{bK} \leq 1$), (c) closed ($K > 0$ and $\sqrt{bK} > 1$), (d) hyperbolic ($K < 0$). In the cases of closed spatial sections, the radial coordinate is only defined up to the equator $r_{\text{max}} = 1/\sqrt{K}$ (=3, in the cases shown).

Fig.2. Parametric plot of $K\alpha'W_-$ as a function of $K\alpha'^2m_-^2$, Eqs.(4.9)-(4.10), for $k \in [0,1]$ in the closed static Robertson-Walker spacetime for $\sqrt{bK} \leq 1$. Notice that $K\alpha'W_- \in [0,8]$ and $K\alpha'^2m_-^2 \in [0,4]$. For $W_- = 2\pi n$ ($n \geq 0$) there can only be a finite number of states.

Fig.3. Parametric plot of $K\alpha'W_+$ as a function of $K\alpha'^2m_+^2$, Eqs.(4.17)-(4.18), for $k \in ]0,1[$ in the closed static Robertson-Walker spacetime for $\sqrt{bK} \leq 1$. Notice that $K\alpha'W_+ \in ]8,\infty[$ and $K\alpha'^2m_+^2 \in ]4,\infty[$. For $W_+ = 2\pi n$ ($n \geq 0$) there will be infinitely many states.

Fig.4. Parametric plot of $-K\alpha'W$ as a function of $-K\alpha'^2m^2$, Eqs.(4.24)-(4.25), for $k \in [0,1]$ in the hyperbolic static Robertson-Walker spacetime. Notice that $-K\alpha'W \in [0,\infty[$ and $-K\alpha'^2m^2 \in [0,\infty[$. For $W = 2\pi n$ ($n \geq 0$) there will be infinitely many states.

Fig.5. The potential $U(r)$ introduced in Eqs.(5.1)-(5.2) for a stationary string in the static Robertson-Walker spacetimes: (a) flat ($K = 0$), (b) closed ($K > 0$ and $\sqrt{KL} \leq 1$), (c) hyperbolic ($K < 0$). In the case of closed spatial sections, the radial coordinate is only defined up to the equator $r_{\text{max}} = 1/\sqrt{K}$ (=4, in the case shown).
Table Captions

Table I. Classical circular strings in the static Robertson-Walker spacetimes. Notice that a self-consistent solution to the Einstein equations, with the string back-reaction included, can be obtained only for $K > 0$.

Table II. Semi-classical quantization of the circular strings in the static Robertson-Walker spacetimes. Notice in particular the different behaviour of the high mass spectrum of strings in the three cases.

Table III. Stationary strings in the static Robertson-Walker spacetimes. Notice that the pressure densities are always negative in all three cases.
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