A HOLOMORPHIC REPRESENTATION OF THE MULTIDIMENSIONAL JACOBI ALGEBRA

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Abstract. We present a holomorphic representation of the Jacobi algebra $b_n \rtimes \mathfrak{sp}(n, \mathbb{R})$ by first order differential operators with polynomial coefficients on the manifold $\mathbb{C}^n \times D_n$. We construct the Hilbert space of holomorphic functions on which these differential operators act.

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The coherent states (CS), invented by Schrödinger in the early days of quantum mechanics, offer a bridge between quantum and classical physics, see e.g. [63, 79, 1] and references therein. On the other hand, the group-theoretic generalization [62] from the standard Glauber’s field CSs (attached to the Heisenberg-Weyl (HW) group) to coherent states associated to any Lie group, attracted mathematicians, as a convenient way to representation theory and quantization [17]. Today, one speaks about CS-type groups [53, 58], a class of groups which contains all compact groups, all simple hermitian groups, certain solvable groups and also mixed groups such as the semidirect product of the HW group and the symplectic group. The latter, called after Eichler and Zagier Jacobi group [29], captured the interest of mathematicians working in representation theory (e.g. Satake [69], Berndt and Schmidt [21] and Neeb [58]), and automorphic forms (e.g. Kähler in his last three papers [43, 44, 45]). Independently, the Jacobi group has already been studied by physicists under the name of Schrödinger or Hagen group in the seventies [60, 36]. On the other hand, the Jacobi group governs the so called squeezed states in Quantum Optics. I shall point out in several places below these parallel contexts in which the Jacobi group appears.

The aim of this paper is to study holomorphic representations of the Jacobi group on a certain homogeneous Kähler manifold $M$ attached to the group. Propositions 3.11 and 3.12 express our main results. Strictly speaking, mathematicians have produced general schemes [69, 58] from which, some of the formulas presented here should be obtained as particular cases. From this point of view, the paper addresses mainly to a reader more familiar with the methods of the Theoretical or Mathematical Physics, who does not want to read hundreds of pages to know what the CS-groups really are, but who needs explicit formulas, like those presented by our Remark 3.1. Such formulas are useful when studying the equations of motion on $M$ generated by Hamiltonians which are algebraic functions in the generators of the group $[18,8,9]$, here the Jacobi group. However, while the scalar product or the reproducing kernel (3.26), related to the Kähler potential on $M$, is known [58], as far as we know, the resolution of unity (3.48) has not been written down explicitly for this case. The formulas furnished by the Propositions 3.11, 3.12 can be used for Berezin’s quantization [17]. Remark 3.11 shows the equivalence of the present approach with the similar theorems obtained in reference [29].

The starting point of the story is generally considered to be the Segal-Bargmann-Fock realization $a \mapsto \frac{\partial}{\partial z}; \ a^+ \mapsto z$ of the canonical commutation relations (CCR) $[a,a^+] = 1$ on the symmetric Fock space $\mathcal{F}_H := \Gamma^{\text{hol}}(\mathbb{C}, \frac{i}{2\pi} \exp(-|z|^2)dz \wedge d\bar{z})$ attached to the Hilbert space $\mathcal{H} := L^2(\mathbb{R}, dx)$. In fact, much earlier Sophus Lie had done the differential realization of the generators $K^{0,-,+}$ of the group $\text{SU}(1,1)$, $K^- \mapsto \frac{\partial}{\partial w}; \ K^0 \mapsto k + w \frac{\partial}{\partial w}; \ K^+ \mapsto 2kw + w^2 \frac{\partial}{\partial w}$ on the unit disk $w \in \mathbb{D}_1 = \text{SU}(1,1)/\text{U}(1)$. In [10] we have presented an explicit realization of a holomorphic representation for the the CS-group $G^1_1$, which is the semidirect product of the real three dimensional HW group $H_1$ with the group $\text{SU}(1,1)$, called.

**Introduction**
the Jacobi group $[29]$. These formulas contain both the standard Segal-Bargmann-Fock representation $[2]$ and the realization of the generators of the group SU(1,1) already mentioned.

At this point, let us mention that the present investigation is part of a larger program, we started earlier. In reference $[12]$ we advanced the conjecture that the generators of all CS-groups $[53, 58]$ admit representations by first order differential operators with holomorphic polynomials coefficients on CS-manifolds. Let us recall some of our own earlier progress in this field. Our method $[7]$ permits to get the holomorphic differential action of the generators of a continuous unitary representation $\pi$ of a Lie group $G$ with the Lie algebra $\mathfrak{g}$ on a homogeneous space $M = G/H$. Following Perelomov $[62]$, we consider homogeneous manifolds $M$ realized as Kähler CS-orbits $[11, 12, 14]$. Previously, we produced explicit representations for hermitian groups, using CS based on compact $[8]$ and noncompact $[9]$ hermitian symmetric spaces, and also on Kähler CS-orbits of semisimple Lie groups $[11, 14]$. In all such situations the differential action of the generators of the group $G$ on holomorphic functions defined on a Kähler homogeneous orbit $G/H$ can be written down as a sum of two terms, the first one as a polynomial $P$, and, the second one, as sum of partial derivatives times some polynomials $Q$-s. On hermitian symmetric spaces the degrees of the polynomials $P, Q$ are less than or equal to two $[8, 9]$. We have analyzed the simplest example of a nonsymmetric homogeneous manifold, SU(3)/SU(1)×SU(1)×SU(1), where the maximum degree of the polynomials $P, Q$ already equals three $[11, 14]$. More precisely, the present paper is devoted to a concrete realization of a differential holomorphic representation of the Lie algebra $\mathfrak{g}_n^J := \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})$ on the homogeneous space $M := \mathcal{H}_n/\mathbb{R} \times \mathfrak{sp}(n, \mathbb{R})/\mathfrak{u}(n)$, where $\mathcal{H}_n$ denotes the $(2n+1)$-dimensional HW group. The complex $n(n+3)/2$-dimensional manifold $M$ is realized as $\mathcal{D}_n^J := \mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ is the $n(n+1)/2$-dimensional Siegel ball, endowed with a Kähler structure deduced from the scalar product of two CSs.

Now we digress a little about the connection between the present paper and other fields. As it had already been mentioned, the physical object associated with the Jacobi group are the squeezed states of the Quantum Optics $[79, 72, 27, 28]$, discovered already in the early days of Quantum Mechanics $[47]$. Instead of starting from a matrix representation of the Jacobi group (see e.g. p. 182 in $[50]$), in our presentation we use methods similar to those of the squeezed states $[73]$. It is well known that for the harmonic oscillator CSs the uncertainties in momentum and position are equal to $1/\sqrt{2}$ (in units of $\hbar$). “The squeezed states” $[47, 73, 54, 22, 81, 39, 77]$ are the states for which the uncertainty in position is less than $1/\sqrt{2}$. The squeezed states are a particular class of “minimum uncertainty states” (MUS) $[56]$, i.e. states which saturate the Heisenberg uncertainty relation. In the present paper we do not insist on possible “physical” applications $[28]$ of our paper to the squeezed states. Let us just mention that “Gaussian pure states” (“Gaussons”) $[71]$ are more general MUSs. In fact, as it was shown in $[11]$, these states are CSs based on the manifold $\mathcal{X}_n^J := \mathcal{H}_n \times \mathbb{R}^{2n}$, where $\mathcal{H}_n$ is the Siegel upper half plane $\mathcal{H}_n := \{ Z \in M(n, \mathbb{C}) | Z = U + \mathbb{R}V, U, V \in M(n, \mathbb{R}), \Im(V) > 0, U^t = U; V^t = V \}$. 

...
$M(n,R)$ denotes the $n \times n$ matrices with entries in $R$, $R = \mathbb{R}$ or $\mathbb{C}$, and $X^t$ denotes the transpose of the matrix $X$. In [15] we have started the generalization of CSs attached to the Jacobi group $G^J_n$ to the (“multidimensional”) Jacobi group $G^J_n$. The connection of our construction of coherent states based on $D^J_n$ [15] with the Gaussons of [71] is a subtle one and it should be investigated separately. $D_n$ denotes the Siegel ball $D_n := \{ Z \in M(n,\mathbb{C}) | Z = Z^t, 1 - Z \bar{Z} > 0 \}$. In §4 we recall the clue of this connection in the case $n = 1$ [16], which is offered by the Kähler-Berndt’s construction [43, 44, 45, 46, 19, 21].

Furthermore we try to make a technical presentation of the content of the paper. Firstly, the notation referring to the HW group is fixed in §1. In §2 some known facts are recalled: the definition of the symplectic algebra and the symplectic group, the Gauss and Cartan decomposition, the differential action of the generators, the scalar product. §3 is devoted to the Jacobi group. We introduce coherent states associated to the Jacobi group $G^J_n$ based on the homogeneous space $D^J_n$. The explicit formulas giving the differential action of the generators of the Jacobi group are given in Remark 3.1. Remark 3.4, which can be considered as generalizing the Holstein-Primakoff-Bogoliubov type equations, implies in Remark 3.5 the action of the group $Sp(n,\mathbb{R})$ on $H_n/\mathbb{R} \equiv \mathbb{C}^n$. Lemma 3.6 is very important for our construction: it connects the normalized “squeezed” vector $\Psi_{\alpha,W} = D(\alpha)S(W)e_0, \alpha \in \mathbb{C}^n, W \in D_n$ with Perelomov’s un-normalized vector $e_{z,W}$. As a consequence, we get in Remark 3.7 the expression of the scalar product of two CS vectors associated to the Jacobi group, based on the manifold $D^J_n$. As we had already emphasized, this expression is already known (see p. 532 in [58], or the article [38] and (5.28) in [69]), but here we present a simple proof. Proposition 3.8 allows to find the action of the Jacobi group $G^J_n$ on the base manifold $D^J_n$ and the composition law in the Jacobi group. Remark 3.10 identifies our expression with the one obtained in context of Jacobi forms [29]. In §3.6 we find out the Kähler two-form $\omega$ on the manifold $D^J_n$. Applying the technique of Chapter IV from Hua’s book [40] and a lemma from Berezin’s paper [17], we determine the Liouville measure on $D^J_n$. This permits the explicit construction of the symmetric Fock space $F_K$ attached to the reproducing kernel $K$, summarized in Proposition 3.11. Proposition 3.12 gives the continuous unitary holomorphic representation $\pi_K$ of the group $G^J_n$ on $F_K$. The §4 recalls some of the further results established for the Jacobi group $G^J_1$. Finally, we discuss the connection between different contexts in which the same group appears under the names of Jacobi, Schrödinger or Hagen.

1. THE HEISENBERG-WEYL GROUP

The Heisenberg-Weyl group $H_n$ is the nilpotent group with the $2n+1$-dimensional real Lie algebra isomorphic to the algebra

$$\mathfrak{h}_n = \{ sl + \sum_{i=1}^{n} (x_i a_i^+ - \bar{x}_i a_i) \}_{a \in \mathbb{R}, x_i \in \mathbb{C}},$$

(1.1)
where $a_i^+$ ($a_i$) are the boson creation (respectively, annihilation) operators which verify the CCR
\[ [a_i, a_j^+] = \delta_{ij}; \quad [a_i, a_j] = [a_i^+, a_j^+] = 0. \] (1.2)
The vacuum verifies the relations:
\[ a_i e_\alpha = 0, \quad i = 1, \cdots, n. \] (1.3)

The displacement operator
\[ D(\alpha) := \exp(\alpha a^+ - \bar{\alpha}a) = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^+) \exp(-\bar{\alpha}a), \] (1.4)
verifies the composition rule:
\[ D(\alpha_2)D(\alpha_1) = e^{i \theta_h(\alpha_2, \alpha_1)} D(\alpha_2 + \alpha_1), \quad \theta_h(\alpha_2, \alpha_1) := \Im(\alpha_2 \bar{\alpha}_1). \] (1.5)

Here we have used the notation $\alpha \beta = \sum_i \alpha_i \beta_i$, where $\alpha = (\alpha_i)$. The composition law of the HW group $H_n$ is:
\[ (\alpha_2, t_2) \circ (\alpha_1, t_1) = (\alpha_2 + \alpha_1, t_2 + t_1 + \Im(\alpha_2 \bar{\alpha}_1)). \] (1.6)

If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, $(p, q) \mapsto \alpha$:
\[ \alpha = p + iq, \quad p, q \in \mathbb{R}^n, \] (1.7)
then
\[ \Im(\alpha_2 \bar{\alpha}_1) = (p_1^t, q_1^t) J \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (2.1)

2. The symplectic group

2.1. The symplectic algebra. The real symplectic Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ is a real form of the simple Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ of type $c_n$ and $X \in \mathfrak{sp}(n, \mathbb{R})$ if
\[ X^t J + JX = 0 \quad \text{or} \quad X = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \quad b = b^t, \quad c = c^t, \]
where $a, b, c \in M(n, \mathbb{R})$, and similarly for $\mathfrak{sp}(n, \mathbb{C})$.

In the complex realization (1.7), to $X \in \mathfrak{sp}(n, \mathbb{R})$, $X \in M(2n, \mathbb{R})$ corresponds $X_C \in \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(n, n)$
\[ X_C = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \] (2.1)
where
\[ a^* = -a, \quad b^t = b \] (2.2)
(cf. theorems in [41], [3], [33]). So, we consider the realization of the Lie algebra of the group $\text{Sp}(n, \mathbb{R})$:
\[ \mathfrak{sp}(n, \mathbb{R}) = < \sum_{i,j=1}^n (2a_{ij}K_{ij}^0 + b_{ij}K_{ij}^+ - \bar{b}_{ij}K_{ij}^-) >, \] (2.3)
where the matrices $a = (a_{ij})$, $b = (b_{ij})$ verify the conditions (2.2).
The generators $K^{0,+,-}$ verify the commutation relations

$$[K^\pm_{ij}, K^\pm_{kl}] = 0, \quad \text{(2.4a)}$$

$$2[K^{-}_{ij}, K^{+}_{kl}] = K^{0}_{ij} \delta_{li} + K^{0}_{kl} \delta_{ki} + K^{0}_{ki} \delta_{lj} + K^{0}_{li} \delta_{kj}, \quad \text{(2.4b)}$$

$$2[K^{-}_{ij}, K^{0}_{kl}] = -K^{+}_{ij} \delta_{kl} - K^{+}_{ji} \delta_{lk}, \quad \text{(2.4c)}$$

$$2[K^{+}_{ij}, K^{0}_{kl}] = -K^{0}_{ij} \delta_{kl} - K^{0}_{ji} \delta_{lk}, \quad \text{(2.4d)}$$

$$2[K^{0}_{ij}, K^{0}_{kl}] = K^{0}_{ij} \delta_{kl} - K^{0}_{ji} \delta_{lk}. \quad \text{(2.4e)}$$

With the notation: $X := d\pi(X)$, we have the correspondence: $\mathfrak{sp}(n, \mathbb{R}) \ni X \mapsto X$, where the real symplectic Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ is realized as $\mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(n,n)$

$$X = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \leftrightarrow X = \sum_{i,j=1}^{n} (2a_{ij}K^{0}_{ij} + z_{ij}K^{+}_{ij} - \bar{z}_{ij}K^{-}_{ij}), \quad b = iz, \quad \text{(2.5)}$$

where (2.2) is verified. In Table 1 we give the realization of the generators of the real symplectic group in matrices, as operators obtained via the derived representation, and a bi-fermion realization.

### 2.2. The symplectic group.

For $g \in \text{GL}(2n, \mathbb{R})$, we have

$$g \in \text{Sp}(n, \mathbb{R}) \iff g^t J g = J. \quad \text{(2.6)}$$

If in (2.6) $g \in \text{GL}(2n, \mathbb{C})$, then $g \in \text{Sp}(n, \mathbb{C})$. We remind also that $g \in \text{U}(n,n)$ iff $gKg^* = K$, where $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Under the identification (1.7) of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, we have the correspondence

$$A \in M(2n, \mathbb{R}) \rightarrow A_{\mathbb{C}} \in M(2n, \mathbb{R})_{\mathbb{C}}, \quad A_{\mathbb{C}} = W A W^{-1}, \quad W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \bar{1} \\ 1 & -\bar{1} \end{pmatrix},$$

where

$$M(2n, \mathbb{R})_{\mathbb{C}} = \left\{ \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} : P, Q \in M(n, \mathbb{C}) \right\}.$$
Remark 2.1. To every \( g \in \text{Sp}(n, \mathbb{R}) \) as in \((2.6)\), \( g \mapsto g_c \in \text{Sp}(n, \mathbb{C}) \cap \text{U}(n, n) \), or denoted just \( g = (\begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array}) \), \((2.7)\)

where
\[
\begin{align*}
aa^* - bb^* &= 1; \quad ab^t = ba^t, \quad (2.8a) \\
a^*a - b^t\bar{b} &= 1; \quad a^t\bar{b} = b^*a. \quad (2.8b)
\end{align*}
\]

Remark 2.2. The linear canonical transformations, i.e. the transformations which leaves invariant \((1.2)\), are given by elements of the group \(\text{Sp}(n, \mathbb{R})\) under the realization \(\text{Sp}(n, \mathbb{C}) \cap \text{U}(n, n)\).

See also Remark 3.4 below.

If \( g \in \text{Sp}(n, \mathbb{R}) \) is given by \((2.7)\), then
\[
g^{-1} = \left( \begin{array}{cc} a^* & -b^t \\ -b^* & a^t \end{array} \right). \quad (2.9)
\]

Gauss decomposition. Let us consider an element \( g \in \text{Sp}(n, \mathbb{R}) \). The following relations are true:
\[
\begin{align*}
g &= \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) = \left( \begin{array}{cc} 1 & Y \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \gamma & 0 \\ 0 & \delta \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ Y^\prime & 1 \end{array} \right), \quad (2.10)
\end{align*}
\]
or
\[
\begin{align*}
g &= \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ U & 1 \end{array} \right) \left( \begin{array}{cc} \gamma^\prime & 0 \\ 0 & \delta^\prime \end{array} \right) \left( \begin{array}{cc} 1 & U^\prime \\ 0 & 1 \end{array} \right), \quad (2.11)
\end{align*}
\]
where
\[
\begin{align*}
Y &= b\bar{a}^{-1}; \quad Y^\prime = \bar{a}^{-1}\bar{b}; \quad \delta = \bar{a}; \quad \gamma = a - b\bar{a}^{-1}\bar{b} = (a^*)^{-1} = (\delta^t)^{-1}; \\
1 - YY^* &= (aa^*)^{-1} > 0; \quad Y = Y^t; \quad 1 - Y^tY^\prime* = ((a^*a)^t)^{-1} > 0; \quad Y^\prime = Y^\prime t; \quad (2.12)
\end{align*}
\]

\[
U = \bar{Y} = ba^{-1}; \quad U^\prime = \bar{Y}^\prime = a^{-1}\bar{b}; \quad \gamma^\prime = \bar{a}; \quad \delta^\prime = \bar{a}; \quad (a^t)^{-1}. \quad (2.13)
\]

Cartan decomposition. Let us consider an element \( g \in \text{Sp}(n, \mathbb{R}) \). The following relations are true:
\[
\begin{align*}
g &= \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) = \left( \begin{array}{cc} m & n \\ p & q \end{array} \right) \left( \begin{array}{cc} v & 0 \\ 0 & \bar{v} \end{array} \right) = \exp \left( \begin{array}{cc} 0 & Z \\ Z & 0 \end{array} \right) \left( \begin{array}{cc} v & 0 \\ 0 & \bar{v} \end{array} \right), \quad (2.15)
\end{align*}
\]
where
\[
\begin{align*}
m &= \cosh \sqrt{ZZ}; \quad n = \sinh \sqrt{ZZ} \\
p &= \tilde{n}; \quad q = \tilde{m}; \quad (2.16)
\end{align*}
\]
or
\[
\begin{align*}
m &= (1 - YY^t)^{-1/2}; \quad n = (1 - YY^t)^{-1/2}Y^t; \quad v = (1 - YY^t)^{1/2}. \quad (2.17)
\end{align*}
\]
Z and Y above are related by formulas (2.26d) and (2.26e) below, with the correspondence $Z \leftrightarrow Z$, $Y \leftrightarrow W$, where $Y = b\bar{a}^{-1}$, i.e.

$$Z = \frac{\text{arctanh} \sqrt{YY^+}}{\sqrt{YY^+}} Y = \frac{1}{2\sqrt{YY^+}} \log \frac{1 + \sqrt{YY^+}}{1 - \sqrt{YY^+}}; \quad Y = b\bar{a}^{-1}. \quad (2.18)$$

**Hermitian symmetric spaces.** We briefly recall some well known facts about hermitian symmetric spaces [37, 78]. We use the notation:

- $X_n$: hermitian symmetric space of noncompact type, $X_n = G_0/K$;
- $X_c$: compact dual of $X_n$, $X_c = G_c/K$;
- $G_0$: real hermitian group;
- $G = G_c^0$: the complexification of $G_0$;
- $P$: a parabolic subgroup of $G$;
- $K$: maximal compact subgroup of $G_0$;
- $G_c$: compact real form of $G$.

The compact manifold $X_c$ of $2n(n+1)$-complex dimension has a complex structure inherited from the identification $X_c = G_c/K = G/P$.

The group $G_c$ acts transitively on $X_c$ with isotropy group $K = G_0 \cap P = G_c \cap P$.

$X_n = G_0/K = G_n(x_0)$ is open in $X_c$, where $x_0$ is a base point of $G$ corresponding to $K$.

$X_c$ includes $X_n$ under Borel embedding $X_n \subset X_c$: $gK \rightarrow gP, g \in G_0$.

In our case: $G_0 = \text{Sp}(n,\mathbb{R})$, $G = \text{Sp}(n,\mathbb{C})$, $G_c = \text{Sp}(n) = \text{Sp}(n,\mathbb{C}) \cap U(2n) \subset \text{SU}(2n)$, $K = U(n)$, and

$$P = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a^t c = c^t a, \ a^t d = 1 \right\}. \quad (2.19)$$

Let

$$m^+ = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b^t = b \right\}. \quad (2.19)$$

Then

$$Z \rightarrow \tilde{Z} = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \quad \xi(Z) = (\exp \tilde{Z})x_0, \quad (2.20)$$

and $\xi$ maps the symmetric $n \times n$ matrices $Z$, $1 - Z\tilde{Z} > 0$ of $m^+$ onto a dense open subset of $X_c$ that contains $X_n$. This gives the Harish-Chandra embedding: $X_n \subset \xi(m^+) \subset X_c$. The non-compact hermitian symmetric space $X_n = \text{Sp}(n,\mathbb{R})/U(n)$ admits a realization as a bounded homogeneous domain, the Siegel ball $\mathcal{D}_n$

$$\mathcal{D}_n := \{ W \in M(n,\mathbb{C}) | W = W^t, 1 - WW^t > 0 \}. \quad (2.21)$$

$X_n$ is a hermitian symmetric space of type CI (cf. Table V. p. 518 in [37]), identified with the symmetric bounded domain of type II, $\mathfrak{R}_{II}$ in Hua’s notation [40].

### 2.3. Coherent states for the symplectic group.

Coherent states associated to the real symplectic group were considered in several references as [57, 18].
see also §8 in [63]. We consider a particular case of the positive discrete series representation [51] of $\text{Sp}(n, \mathbb{R})$. The vacuum is chosen such that
\begin{align*}
K_{ij}^+ e_0 &\neq 0, \\
K_{ij}^- e_0 &= 0, \\
K_{ij}^0 e_0 &= \frac{k}{4} \delta_{ij} e_0.
\end{align*}

We have the relations:
\begin{align*}
\pi\left(\begin{array}{cc}
v & 0 \\
0 & \bar{v}
\end{array}\right) e_0 &= (\det v)^{k/2} e_0, \ v \in U(n); \\
\pi\left(\begin{array}{cc}
a & 0 \\
0 & d
\end{array}\right) e_0 &= (\det a)^{k/2} e_0, \ da = 1, \ v \in U(n); \\
\pi\left(\begin{array}{cc}
a & 0 \\
0 & d
\end{array}\right) &= \exp(\sum 2A_{ij} K_{ij}^0), \ a = \exp A.
\end{align*}

We introduce some notation, and we find out:
\begin{align*}
S(Z) &= \exp(\sum z_{ij} K_{ij}^+ - \bar{z}_{ij} K_{ij}^-), \ Z = (z_{ij}); \\
S(W) &= \exp(W K^+) \exp(\eta K^0) \exp(-W K^-); \\
&= \exp(-\bar{W} K^-) \exp(-\eta K^0) \exp(W K^+);
\end{align*}
\begin{align*}
W &= Z \tanh \frac{\sqrt{ZZ^*}}{\sqrt{Z^* Z}}; \\
Z &= \arctanh \sqrt{WW^*} W = \frac{1}{2} \sqrt{WW^*} \log \frac{1 + \sqrt{WW^*}}{1 - \sqrt{WW^*}}; \\
\eta &= \log(1 - WW^*) = -2 \log \cosh \sqrt{ZZ^*}.
\end{align*}

We have $S(Z) = S(W)$. In (2.18) $Y$ is that from the Gauss decomposition (2.10).

Perelomov’s un-normalized CS-vectors are:
\begin{align*}
e_Z := \exp(\sum z_{ij} K_{ij}^+) e_0 &= \pi\left(\begin{array}{cc}
1 & iZ \\
0 & 1
\end{array}\right) e_0, \ Z = (z_{ij}), \ Z = Z^t.
\end{align*}

Let us consider an element $g \in \text{Sp}(n, \mathbb{R})$.

**Remark 2.3.** The following relations between the normalized and un-normalized Perelomov’s CS-vectors hold:
\begin{align*}
S(Z) e_0 &= \det(1 - WW^*)^{k/4} e_W, \\
\overline{S(Z)} e_0 &= \pi(g) e_0 = \pi\left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right) e_0 = (\det\bar{a})^{-k/2} e_Z = \left(\frac{\det a}{\det\bar{a}}\right)^{\frac{k}{2}} S(Z) e_0, \ Z = -ib\bar{a}^{-1}, \\
S(g) e_{W/t} &= \det(Wb^* + a^*)^{-k/2} e_{Y/t},
\end{align*}
where $W \in \mathcal{D}_n$, and $Z \in \mathbb{C}^n$ in (2.24) are related by equations (2.24a), (2.24b), and the linear-fractional action of the group $\text{Sp}(n, \mathbb{R})$ on the unit ball $\mathcal{D}_n$ in (2.25a) is

$$Y := g \cdot W = (a \cdot W + b)(\bar{b} \cdot W + \bar{a})^{-1} = (Wb^* + a^*)^{-1}(b^t + Wa^t). \quad (2.31)$$

Using the results of [10, 13], it can be proved

**Remark 2.4.** If $S(Z)$ is defined by (2.26a), then:

$$S(Z_2)S(Z_1) = S(Z_3)e^{2AK^0}; \quad (2.32a)$$

$$S(Z_2)S(Z_1)e_0 = (\det v)^{k/2}S(Z_3)e_0; \quad (2.32b)$$

$$W_3 = (1-W_1W_1^*)^{-1/2}(1+W_2W_2^*)^{-1}(1-W_1W_1^*)^{1/2}; \quad (2.33a)$$

$$e^A = v = (MM^+)^{-1/2}M; \quad (2.33b)$$

$$M = (1-W_1W_1^*)^{-1/2}(1+W_1W_1^*)(1-W_2W_2^*)^{-1/2}; \quad (2.33c)$$

$$\det(v) = \left(\frac{\det(1+W_1W_2^*)}{\det(1+W_1W_1^*)}\right)^{1/2}, \quad (2.33d)$$

where in (2.33a) $W_i$ and $Z_i$, $i = 1, 2, 3$, are related by the relations (2.26a), (2.26b).

**2.4. The differential action for the group $\text{Sp}(n, \mathbb{R})$.** We consider CS-vectors given by (2.27)

$$e^W = e^Xe_0, \quad X = \sum w_{ij}K_{ij}^+; \quad W = (w_{ij}), \quad W = W^t.$$

It is easy to see that:

$$K_{kl}^+e^W = \frac{\partial}{\partial w_{kl}}e^W, \quad (2.34a)$$

$$K_{kl}^0e^W = \left(\frac{k}{4}\delta_{kl} + w_{il}\frac{\partial}{\partial w_{ik}}\right)e^W, \quad (2.34b)$$

$$K_{kl}^{-}e^W = \left(\frac{k}{2}w_{kl} + w_{il}w_{ik}\frac{\partial}{\partial w_{il}}\right)e^W. \quad (2.34c)$$

The proof is based on the general formula

$$\text{Ad}(\exp X) = \exp(\text{ad}_X), \quad (2.35)$$

valid for Lie algebras $\mathfrak{g}$, which here we write down explicitly as

$$Ae^X = e^X(A - [X, A] + \frac{1}{2}[X, [X, A]] + \cdots). \quad (2.36)$$

The differential action for the group $\text{Sp}(n, \mathbb{R})$ is obtained:
\(K^- = \frac{\partial}{\partial W}, \quad (2.37a)\)

\(K^+ = \frac{k}{2} W + W \frac{\partial}{\partial W} W, \quad (2.37b)\)

\(K^0 = \frac{k}{4} 1 + \frac{\partial}{\partial W} W. \quad (2.37c)\)

We have used the convention:

\[
\left[ \left( \frac{\partial}{\partial W} W \right) f(W) \right]_{kl} := \frac{\partial f(W)}{\partial w_{ki}} w_{il}, \quad W = (w_{ij}).
\]

Formulas of the type \((2.37)\) have been obtained by several authors \cite{52,68}.

2.5. The scalar product. The scalar product of two coherent state vectors indexed by the points of the Siegel ball \(D_n\) is:

\(K(Z, Z') = (e_Z, e_{Z'})_{\mathcal{H}} = \det(1 - Z'Z)^{-k/2}. \quad (2.38)\)

The Kähler two-form on \(D_n\) is:

\[
\omega = i \frac{k}{2} \text{Tr}[(1 - W\bar{W})^{-1} dW \wedge (1 - \bar{W}W)^{-1} d\bar{W}]. \quad (2.39)
\]

The Jacobian \(J = \det dW'/dW\) of the transformation \(W' = g \cdot W = f(g, W)\) such that \(W'(W_1) = 0\), modulo an irrelevant unitary transformation, is obtained (see \cite{10}) from

\[
dW' = (1 - W_1\bar{W}_1)^{-\frac{k}{2}} dW(1 - \bar{W}_1W_1)^{-\frac{k}{2}}.
\]

The density \(Q = |J|^2\) of the \(\text{Sp}(n, \mathbb{R})\)-invariant volume is

\(Q = \det(1 - W\bar{W})^{-(n+1)},\)

and the Bergman kernel, modulo the volume of the domain \(D_n\) (cf. \cite{40}) is

\(K_{D_n}(X, Y) = \det(1 - XY)^{-(n+1)}\).

The scalar product in the space of functions \(f_\psi(z) = (e_z, \psi) \in \mathcal{F}_\mathcal{H}\) is

\((\phi, \psi)_{\mathcal{F}_\mathcal{H}} = \Lambda_1 \int_{1-W\bar{W}>0} \bar{f}_\phi(W) f_\psi(W) \det(1 - W\bar{W})^q dW, \quad q = \frac{k}{2} - n - 1, \quad (2.40)\)

where \(\Lambda_1 = J_n^{-1}(q)\), with \(J_n(p) \quad (p > -1)\) defined by:

\[
J_n(p) = \frac{\pi^{\frac{n(n+1)}{2}}}{(p+1)\cdots(p+n)} \frac{\Gamma(2p+3)\Gamma(2p+5)\cdots\Gamma(2p+2n-1)}{\Gamma(2p+n+2)\Gamma(2p+n+3)\cdots\Gamma(2p+2n)}. \quad (2.41)
\]

We can write \((2.41)\) in the form

\[
J_n(p) = 2^n \pi^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} \frac{\Gamma(2p+2i)}{\Gamma(2p+n+i+1)}. \quad (2.42)
\]

With \((2.42)\), we have

\[
\Lambda_1 = 2^{-n} \pi^{-\frac{n(n+1)}{2}} \prod_{i=1}^{n} \frac{\Gamma(k-i)}{\Gamma(k-2i)}. \quad (2.43)
\]
which is formula (4.3) in [18]. The holomorphic multiplier representation of the group \( \text{Sp}(n, \mathbb{R}) \) on the space of functions defined on the manifold \( M = D_n \) has the expression

\[
\pi(g)f(W) = \mu(g,W)f(g^{-1} \cdot W),
\]

with the multiplier \( \mu \) related to the cocycle \( J \), i.e.

\[
J(g_1g_2,W) = J(g_1, g_2 \cdot W), J(g_2,W),
\]

by the relation

\[
J(g,W) = \mu(g^{-1},W)^{-1}.
\]

For \( g \in \text{Sp}(n, \mathbb{R}) \) given by (2.7), the multiplier is

\[
J(g,W) = \det(a^* + W b^*)^{\frac{k}{2}}.
\]

Note that

\[
K(g \cdot X,g \cdot Y) = J(g,Y)K(X,Y)J(g,X)^*.
\]

With (2.9), (2.31), we have

\[
g^{-1} \cdot W = (a^*W - b^t)(-b^*W + a^t)^{-1} = (-Wb + a)^{-1}(-b + Wa),
\]

and finally we have the continuous unitary holomorphic representation of \( \text{Sp}(n, \mathbb{R}) \) on the space of holomorphic functions of \( D_n \) attached to the kernel (2.38) (see [35])

\[
\pi\left( \begin{array}{cc} a & b \\ b & a \end{array} \right) f(W) = \det(a - Wb)^{-\frac{k}{2}} f((-Wb + a)^{-1}(-b + Wa)), W \in D_n. \quad (2.44)
\]

Starting from the development given by Hua [10] (see also [59]) of the determinant \( \det(1 - XY^*)^{-1} \) for the complex Grassmann manifold in Schur functions [55], Berezin [17], [18] found out that the admissible set for \( k \) for the space of functions \( \mathcal{F} \) endowed with the scalar product \( \langle \cdot, \cdot \rangle \), i.e. the set of values of \( k \) on which the integral, or its analytic continuation, converges, for a sufficiently large set of functions \( \mathcal{F} \), is the (Wallach [76]) set \( \Sigma \)

\[
\Sigma = \{0, 1, \ldots, n-1\} \cup ((n-1), \infty). \quad (2.45)
\]

The integral (2.40) deals with a non-negative scalar product if \( k \geq n - 1 \), in which the domain of convergence \( k \geq 2n \) is included, and the separate points \( k = 0, 1, ..., n - 1 \). The corresponding coherent states are a “generalized overcomplete family of states” (cf. [18]).

In this paper we are not concerned with the analytic continuation of the discrete series representations. Here are some references for the analytic continuation of the discrete series or related topics: [70, 74, 84, 67, 70, 76, 80, 51, 80, 25].
3. The Jacobi group $G_n^J$

3.1. The Jacobi algebra. The Jacobi algebra is the semi-direct sum

$$g_n^J := h_n \rtimes \mathfrak{sp}(n, \mathbb{R}),$$

where $h_n$ is an ideal in $g$, i.e. $[h_n, g] = h_n$, determined by the commutation relations:

$$[a^+_k, K^+_{ij}] = [a_k, K^-_{ij}] = 0,$$
$$[a_i, K^+_{kj}] = \frac{1}{2} \delta_{ik} a^+_j + \frac{1}{2} \delta_{ij} a^+_k,$$
$$[K^-_{kj}, a^+_i] = \frac{1}{2} \delta_{ik} a^+_j + \frac{1}{2} \delta_{ij} a^+_k,$$
$$[K^0_{ij}, a^+_k] = \frac{1}{2} \delta_{jk} a^+_i,$$
$$[a_k, K^0_{ij}] = \frac{1}{2} \delta_{ik} a^+_j.$$

3.2. Coherent states for the Jacobi group. Perelomov’s coherent state vectors associated to the group $G_n^J$ with Lie algebra the Jacobi algebra (3.1), based on the complex $N$-dimensional, $N = \frac{n(n+3)}{2}$, manifold $M$:

$$M := H_n/\mathbb{R} \times \mathfrak{sp}(n, \mathbb{R})/\mathbb{U}(n),$$
$$M = D_n := C^n \times D_n,$$

are defined as

$$e_{z,W} = \exp(X) e_0, \quad X := \sum_i z_i a^+_i + \sum_{ij} w_{ij} K^+_{ij}, \quad z \in C^n; W \in D_n.$$ (3.4)

The vector $e_0$ verify (1.3) and (2.22).

3.3. The differential action. The differential action of the generators of the Jacobi group follows from the formulas:

$$a^+_k e_{z,W} = \frac{\partial}{\partial z_k} e_{z,W},$$
$$a_k e_{z,W} = (z_k + w_{kl} \frac{\partial}{\partial z_l}) e_{z,W},$$
$$K^0_{kl} e_{z,W} = \left( \frac{k}{4} \delta_{kl} + \frac{z_l}{2} \frac{\partial}{\partial z_k} + w_{kl} \frac{\partial}{\partial w_{ik}} \right) e_{z,W},$$
$$K^+_{kl} e_{z,W} = \frac{\partial}{\partial w_{kl}} e_{z,W},$$
$$K^-_{kl} e_{z,W} = \left( \frac{k}{2} w_{kl} + \frac{z_k z_l}{2} + \frac{1}{2} (z_i w_{ik} + z_k w_{id}) \frac{\partial}{\partial z_i} + w_{im} w_{kl} \frac{\partial}{\partial w_{im}} \right) e_{z,W}.$$ (3.5e)

and we have
Remark 3.1. The differential action of the generators of Jacobi group $G_n^J$ is given by the formulas:

\[ a = \frac{\partial}{\partial z}, \quad (3.6a) \]
\[ a^+ = z + W \frac{\partial}{\partial z}, \quad (3.6b) \]
\[ K^- = \frac{\partial}{\partial W}, \quad (3.6c) \]
\[ K^0 = \frac{k}{4} + \frac{1}{2} \frac{\partial}{\partial z} \otimes z + \frac{\partial}{\partial W} W, \quad (3.6d) \]
\[ K^+ = \frac{k}{2} W + \frac{1}{2} (W \frac{\partial}{\partial z} \otimes z + z \otimes \frac{\partial}{\partial z} W) + W \frac{\partial}{\partial W} W. \quad (3.6e) \]

In (3.6) $A \otimes B$ denotes the Kronecker product of matrices, here $A \otimes B = (a_{kl} b_{ij})$, $A = (a_k) = (a_1, ..., a_n)$, $B = (b_l) = (b_1, ..., b_n)$.

3.4. Holstein-Primakoff-Bogoliubov type equations. We recall the Holstein-Primakoff-Bogoliubov equations, a consequence of the fact that the Heisenberg algebra is an ideal in the Jacobi algebra

\[ S^{-1}(Z) a_k S(Z) = (\cosh(\sqrt{ZZ}) a)_k + (\frac{\sinh(\sqrt{ZZ})}{\sqrt{ZZ}} Z a^+_k), \quad (3.7a) \]
\[ \bar{S}^{-1}(Z) a_k^+ \bar{S}(Z) = (\frac{\sinh(\sqrt{ZZ})}{\sqrt{ZZ}} \bar{Z} a) + (\cosh(\sqrt{ZZ}) a^+_k), \quad (3.7b) \]

and the CCR are still fulfilled in the new creation and annihilation operators. Above $a$ ($a^+$) denotes the column vector formed from $a_1, ..., a_n$ (respectively, $a_1^+, ..., a_n^+$).

Let us introduce the notation:

\[ \bar{A} = \begin{pmatrix} A \\ \bar{A} \end{pmatrix}; \quad D = D(Z) = \begin{pmatrix} m & n \\ p & q \end{pmatrix}, \quad (3.8) \]
\[ D(Z) = e^X, \text{ where } X := \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix}, \quad (3.9) \]

where $m$, $n$, $p$, $q$ are given by (2.15).

Remark 3.2. With the notation (3.8), (3.15), equations (3.7) become:

\[ S^{-1}(Z) a \bar{S}(Z) = \bar{D}(Z) \bar{a}. \]

Remark 3.3. If $D$ and $\bar{S}(Z)$ are defined by (1.1), respectively (2.26a), then

\[ D(\alpha) \bar{S}(Z) = \bar{S}(Z) D(\beta), \quad (3.10) \]

where

\[ \beta = m \alpha - n \bar{\alpha} = \cosh(\sqrt{ZZ}) \alpha - \frac{\sinh(\sqrt{ZZ})}{\sqrt{ZZ}} Z \bar{\alpha} \quad (3.11a) \]
\[ = (1 - WW)^{-1/2}(\alpha - W \bar{\alpha}), \quad (3.11b) \]
and

\[ \alpha = m\beta + n\bar{\beta} = \cosh(\sqrt{ZZ})\beta + \frac{\sinh(\sqrt{ZZ})}{\sqrt{ZZ}}Z\bar{\beta} \]  
(3.12a)

\[ = (1 - WW)^{-1/2}(\beta + W\bar{\alpha}). \]  
(3.12b)

With the convention (3.8), equation (3.11a) can be written down as:

\[ \tilde{\beta} = D(-Z)\bar{\alpha}; \; \tilde{\alpha} = D(Z)\tilde{\beta}. \]  
(3.13)

Let us introduce the notation

\[ S(Z, A) := \exp(\sum 2a_{ij}K^0_{ij} + z_{ij}K^+_{ij} - \bar{z}_{ij}K^-_{ij}). \]  
(3.14)

If \( g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \text{Sp}(n, \mathbb{R}) \), then

\[ S^{-1}(g) a S(g) = \alpha a + \bar{\beta} a^+, \]  
(3.15a)

\[ S^{-1}(g) a^+ S(g) = \bar{\beta} a + \bar{\alpha} a^+. \]  
(3.15b)

and we have the following (generalized Holstein-Primakoff-Bogoliubov) equations:

**Remark 3.4.** If \( S \) denotes the representation of \( \text{Sp}(n, \mathbb{R}) \), with the convention (3.8), we have

\[ S^{-1}(g) \tilde{a} S(g) = g \cdot \tilde{a}. \]  
(3.16)

**Remark 3.5.** In the matrix realization of Table 1

\[ S(g)D(\alpha)S^{-1}(g) = D(\alpha_g), \]  
(3.17)

where one has the expression of the natural action of \( \text{Sp}(n, \mathbb{R}) \times \mathbb{C}^n \to \mathbb{C}^n: g \cdot \tilde{\alpha} := \alpha_g, \)

\[ \alpha_g = a \alpha + b \bar{\alpha}. \]  
(3.18)

3.5. **The action of the Jacobi group.** Now we find a relation between the normalized (“squeezed”) vector

\[ \Psi_{\alpha,W} := D(\alpha)S(W)e_0, \; \alpha \in \mathbb{C}^n, W \in \mathcal{D}_n, \]  
(3.19)

and the un-normalized Perelomov’s CS-vector (3.3), which is important in the proof of Proposition 3.8.

**Lemma 3.6.** The vectors (3.19), (3.3), i.e.

\[ \Psi_{\alpha,W} := D(\alpha)S(W)e_0; \; e_z,W' := \exp(za^+ + W'K^+)e_0, \]

are related by the relation

\[ \Psi_{\alpha,W} = \det(1 - W\bar{W})^{k/4} \exp(-\frac{\bar{\alpha}}{2}z)e_z,W, \]  
(3.20)

where

\[ z = \alpha - W\bar{\alpha}. \]  
(3.21)
Proof. We obtain successively

\[
\Psi_{\alpha,W} = \det(1 - W\bar{W})^{k/4} D(\alpha) \exp(W\hat{K}) e_0
\]

\[
= \det(1 - W\bar{W})^{k/4} \exp\left(-\frac{1}{2} |\alpha|^2\right) \exp(\alpha a^+) \exp(-\bar{\alpha} a) \exp(W\hat{K}) e_0
\]

\[
= \det(1 - W\bar{W})^{k/4} \exp\left(-\frac{1}{2} |\alpha|^2\right) \exp(\alpha a^+) \exp(-\bar{\alpha} a) \times
\]

\[
\exp(W\hat{K}) \exp(\bar{\alpha} a) \exp(-\bar{\alpha} a)e_0
\]

\[
= \det(1 - W\bar{W})^{k/4} \exp\left(-\frac{1}{2} |\alpha|^2\right) \exp(\alpha a^+) \exp(-\bar{\alpha} a) e_0
\]

where

\[
E := \exp(-\bar{\alpha} a) \exp(W\hat{K}) \exp(\bar{\alpha} a).
\]

(3.22)

As a consequence of (2.36),

\[
\exp(Z) \exp(X) \exp(-Z) = \exp(X + [Z, X] + \frac{1}{2} [Z, [Z, X]] + \cdots),
\]

where, if we take

\[
Z = -\bar{\alpha} a; \quad X = W\hat{K},
\]

then

\[
[Z, X] = -\bar{\alpha}_k w_{kj} a^+_j = -\alpha^t Wa^+;
\]

\[
[Z, [Z, X]] = \bar{\alpha}_p w_{kj} a^+_j = \bar{\alpha}^t W\bar{\alpha},
\]

where \(\alpha^t = (\alpha_1, \ldots, \alpha_n)\). We find for \(E\) defined by (3.22) the value

\[
E = \exp(W\hat{K} - \bar{\alpha}^t Wa^+ + \frac{\bar{\alpha}^t W\bar{\alpha}}{2}),
\]

and finally

\[
\Psi_{\alpha,W} = \exp(-\frac{1}{2} \alpha^t z) \det(1 - W\bar{W})^{k/4} e_{\alpha-W\bar{\alpha},W},
\]

i.e. (3.20). \qed

Remark 3.7. Starting from (3.20), we obtain the expression (3.25) of the reproducing kernel \(K = (e_{z,W}, e_{z,W})\) and (3.26) of \(K(z, W; \bar{x}, \bar{V}) := (e_{z,V}, e_{z,W}) = (e_{\bar{z},\bar{V}}, e_{x,\bar{V})}\).

Proof. Indeed, the normalization \((\Psi_{\alpha,W}, \Psi_{\alpha,W}) = 1\) imply that

\[
(e_{z,W}, e_{z,W}) = \det(1 - W\bar{W})^{-k/2} \exp F, F := \frac{1}{2} (\bar{\alpha}^t z + c.c.).
\]

(3.23)

With the notation (3.21), we have

\[
\alpha = (1 - W\bar{W})^{-1}(z + W\bar{z}),
\]

and then \(F\) in (3.23) can be rewritten down as

\[
2F = 2z^t(1 - W\bar{W})^{-1} z + z^t W(1 - W\bar{W})^{-1} z + z^t(1 - W\bar{W})^{-1} W\bar{z},
\]

(3.24)

\[
(e_{z,W}, e_{z,W}) = \det(M)^{k/2} \exp \frac{1}{2} [2 < z, Mz> + < W\bar{z}, Mz> + < z, MW\bar{z}>],
\]

(3.25)
\[ M = (1 - W \bar{W})^{-1}. \]

Above \(< x, y > = \bar{x}^t y = \sum \bar{x}_i y_i. \] Finally, we find out
\[
(e_{x,V}, e_{y,W}) = \det(U)^{k/2} \exp \frac{1}{2} \left[ 2 < x, U y > + < V y, U y > + < x, UW \bar{x} > \right], \tag{3.26}
\]
\[ U = (1 - W \bar{V})^{-1}. \]

\[ \square \]

From the following proposition we can see the holomorphic action of the Jacobi group
\[ G^J_n := H_n \rtimes \text{Sp}(n, \mathbb{R}) \tag{3.27} \]
on the manifold \( M \).

**Proposition 3.8.** Let us consider the action \( S(g)D(\alpha)e_{z,W} \), where \( g \in \text{Sp}(n, \mathbb{R}) \) has the form \( \left[ \begin{array}{cc} I & \lambda \\ \lambda & I \end{array} \right] \), \( D(\alpha) \) is given by \( \left[ \begin{array}{cc} 1 & \alpha \\ -\alpha & 1 \end{array} \right] \), and the coherent state vector is defined in \( \text{(3.1)} \). Then we have the formula\( \text{(3.28)} \) and the relations \( \text{(3.29)-(3.33)} \): \[ S(g)D(\alpha)e_{z,W} = \lambda e_{z_1,W_1}, \quad \lambda = \lambda(g,a;z,W), \tag{3.28} \]
\[ z_1 = (Wb^* + a^*)^{-1}(z + \alpha - W\bar{\alpha}), \tag{3.29} \]
\[ W_1 = g \cdot W = (aW + b)(\bar{b}W + \bar{a})^{-1} = (Wb^* + a^*)^{-1}(b^t + Wa^t), \tag{3.30} \]
\[ \lambda = \det(Wb^* + a^*)^{-k/2} \exp\left( \frac{i}{2} z - \frac{1}{2} z_1 \right) \exp(i \theta_h(\alpha, x)), \tag{3.31} \]
\[ x = (1 - W \bar{W})^{-1}(z + W \bar{z}), \tag{3.32} \]
\[ y = a(\alpha + x) + b(\bar{\alpha} + \bar{z}). \tag{3.33} \]

**Proof.** With Lemma \( \text{3.6} \) we have \( e_{z,W} = \lambda_1 \Psi_{\alpha_0, W}, \) where \( \alpha_0 \) is given by \( \text{3.32} \) and \( \lambda_1 = \det(1 - W \bar{W})^{-k/4} \exp\left( \frac{i}{2} z \right) \). Then \( I := S(g)D(\alpha)e_{z,W} \) becomes successively
\[ I = \lambda_1 S(g)D(\alpha)\Psi_{\alpha_0, W} = \lambda_1 S(g)D(\alpha)D(\alpha_0)S(W)e_0 = \lambda_2 S(g)D(\alpha_1)S(W)e_0, \]
where \( \alpha_1 = \alpha + \alpha_0 \) and \( \lambda_2 = \lambda_1 e^{i \theta_h(\alpha_1, \alpha_0)} \). With equations \( \text{3.17}, 3.18 \), we have \( I = \lambda_2 D(\alpha_2)S(g)S(W)e_0, \) where \( \alpha_2 = a\alpha_1 + b\bar{\alpha}_1 \). But \( \text{2.28} \) implies \( I = \lambda_3 D(\alpha_2)S(g)e_{0,W} \), with \( \lambda_3 = \lambda_2 \det(1 - W \bar{W})^{-k/4} \). Now we use \( \text{2.30} \) and we find \( I = \lambda_4 D(\alpha_2)e_{0,W_1} \), where, in accord with \( \text{2.31} \), \( W_1 \) is given by \( \text{3.29} \), and \( \lambda_4 = \det(Wb^* + a^*)^{-k/2} \lambda_3 \). We rewrite the last equation as \( I = \lambda_5 D(\alpha_2)S(W_1)e_0, \) where \( \lambda_5 = (1 - W_1 \bar{W}_1)^{-k/4} \lambda_4 \). Then we apply again Lemma \( \text{3.6} \) and we find \( I = \lambda_6 e_{z_1,W_1}, \) where \( \lambda_6 = \lambda_5 (1 - W_1 \bar{W}_1)^{k} \exp\left( \frac{i}{2} z_1 \right) \), and \( z_1 = \alpha_2 - W_1 \bar{\alpha}_2 \). Proposition \( 3.8 \) is proved. \( \square \)
Corollary 3.9. The action of the Jacobi group \( D_n = \text{Sp}(n, \mathbb{R})/U(n) \) on the manifold \( (3.3b) \), where \( D_n = \text{Sp}(n, \mathbb{R})/U(n) \), is given by equations \( (3.28), (3.29) \). The composition law in \( G \) is

\[
(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)),
\]

where \( g \cdot \alpha := \alpha_g \) is given by \( (2.18) \), and if \( g \) has the form given by \( (2.7) \), then \( g^{-1} \cdot \alpha = a^* \alpha - b^* \bar{\alpha} \).

Remark 3.10. Combining the expressions \( (3.29)-(3.33) \) and taking into account the relations \( (2.8) \), the factor \( \lambda \) in \( (3.28) \) can be written down as

\[
\lambda = \det(Wb^* + a^*) - \frac{k}{2} \exp(-\lambda_1),
\]

where

\[
2\lambda_1 = z^t(\bar{a} + \bar{b}W)^{-1}\bar{b}z + (a^t + \bar{a}^t \bar{b}^{-1} \bar{a})(\bar{a} + \bar{b}W)^{-1}\bar{b}(2z + z_0); z_0 = \alpha - W\bar{\alpha} \quad \text{(3.36)}
\]

or

\[
2\lambda_1 = z^t(W + T)^{-1}z + (a^t + \bar{a}^t T)(W + T)^{-1}(2z + z_0); T = \bar{b}^{-1}\bar{a} \quad \text{(3.37)}
\]

In the case \( n = 1 \) the expression \( (3.35)-(3.36) \) is identical with the expression given in Theorem 1.4 in \([29]\) of the Jacobi forms, under the the identification of \( c, d, \tau, z, \mu, \lambda \) in \([29]\) with, respectively, \( \bar{b}, \bar{a}, w, z, \alpha, -\bar{\alpha} \) in our notation. Note also that the composition law \( (3.34) \) of the Jacobi group \( G' \) and the action of the Jacobi group on the base manifold \( (3.3b) \) is similar with that in the paper \([20]\). See also the Corollary 3.4.4 in \([21]\).

3.6. The Kähler two-form \( \omega \) and the volume form. Now we follow the general prescription \([12, 14]\). We calculate the Kähler potential as the logarithm of the reproducing kernel \( (3.25) \), \( f := \log K \), i.e.

\[
f = -\frac{k}{2} \log (1 - W\bar{W}) + \bar{z}_i(1 - W\bar{W})^{-1}z_j + \frac{1}{2} [z_i(1 - W\bar{W})^{-1}]_{ij} z_j + \frac{1}{2} [(1 - W\bar{W})^{-1}]_{ij} \bar{z}_j.
\]

The Kähler two-form \( \omega \) on the manifold \( (1.3) \) is given by the formula:

\[
-\omega = f z_i \bar{z}_j dz_i \wedge d\bar{z}_j + f \bar{z}_i \bar{w}_{\alpha \beta} d\bar{z}_i \wedge d\bar{w}_{\alpha \beta} - f z_i \bar{w}_{\alpha \beta} d\bar{z}_i \wedge d\bar{w}_{\alpha \beta} + f \bar{w}_{\alpha \beta} \bar{z}_j d\bar{w}_{\alpha \beta} \wedge d\bar{z}_j, \quad (3.39)
\]
where
\[ f_{z_i z_j} = (1 - \bar{W}W)^{-1}_{ji}, \]  
\[ 2f_{z_i w_{a \beta}} = 2(1 - \bar{W}W)^{-1}_{i\alpha} [W(1 - \bar{W}W)^{-1}z]_{\beta} \]  
\[ + (1 - \bar{W}W)^{-1}_{i\alpha} [(1 - \bar{W}W)^{-1}z]_{\beta} \]
\[ + [z'(1 - \bar{W}W)^{-1}]_{\alpha}(1 - \bar{W}W)^{-1}_{ji}, \]  
\[ 2f_{w_{a \beta} \bar{w}_{\gamma \delta}} = k(1 - \bar{W}W)^{-1}_{\beta \gamma}(1 - \bar{W}W)^{-1}_{\delta \alpha} \]  
\[ + 2[z'(1 - \bar{W}W)^{-1}W]_{\gamma}(1 - \bar{W}W)^{-1}_{\delta \alpha}[W(1 - \bar{W}W)^{-1}z]_{\beta} \]
\[ + 2[z'(1 - \bar{W}W)^{-1}]_{\alpha}(1 - \bar{W}W)^{-1}_{\beta \gamma}[(1 - \bar{W}W)^{-1}z]_{\delta} \]
\[ + [z'(1 - \bar{W}W)^{-1}]_{\gamma}(1 - \bar{W}W)^{-1}_{\delta \alpha}[W(1 - \bar{W}W)^{-1}W]_{\gamma} \]
\[ + [z'(1 - \bar{W}W)^{-1}]_{\alpha}(1 - \bar{W}W)^{-1}_{\beta \gamma}[W(1 - \bar{W}W)^{-1}z]_{\delta}, \]
i.e.
\[ -\omega = \frac{k}{2} \text{Tr}[(1 - \bar{W}W)^{-1}dW \wedge (1 - \bar{W}W)^{-1}d\bar{W}] \]  
\[ + \text{Tr}[dz^i \wedge (1 - \bar{W}W)^{-1}d\bar{z}] \]
\[ - \text{Tr}[dz^i (1 - \bar{W}W)^{-1} \wedge dW \bar{x}] + \text{c.c.} \]
\[ + \text{Tr}[z^i dW(1 - \bar{W}W)^{-1} \wedge dW \bar{x}], \]  
where \( x \) is defined in (3.32).

We can write down the two-form \( \omega \) as
\[ -\omega = \frac{k}{2} \text{Tr}(Bn\bar{B}) + \text{Tr}(A^t \bar{M} \wedge \bar{A}), \]  
\( A = dz^i \wedge dW \bar{x}, B = MdW, M = (1 - \bar{W}W)^{-1}. \) (3.42)

Now we determine the Liouville form. We apply the following technique (see Hua’s book, Ch. IV). Let \( z' = f(g, z) \) the action of the group \( G \) on the circular domain \( M \). Let us determine the element \( g \in G \) such that \( z'(z_1) = 0 \) Then the density of the volume form is \( Q = |J|^2, \) where \( J \) is the Jacobian \( J = \partial z'/\partial z. \) We apply this method to our manifold (3.3) and the Jacobi group \( \text{J}^n \). (3.27)

The transformation with the desired properties is:
\[ z' = U(1 - W_1 \bar{W}_1)^{1/2}(1 - \bar{W}W_1)^{-1} \]  
\[ z - (1 - \bar{W}W_1) \](1 - W_1 \bar{W}_1)^{-1}z_1 + (W - W_1)(1 - \bar{W}_1 W_1)^{-1}z_1, \]
\[ W' = U(1 - W_1 \bar{W}_1)^{-1/2}(W - W_1)(1 - \bar{W}_1 W_1)^{-1}(1 - W_1 \bar{W}_1)^{1/2}U^t, \]  
where \( U \) is a unitary matrix.

We find that
\[ \partial z'/\partial z = U(1 - W_1 \bar{W}_1)^{1/2}(1 - \bar{W}W_1)^{-1}, \]  
\[ dW' = AdWA^t, A = U(1 - W_1 \bar{W}_1)^{1/2}(1 - \bar{W}W_1)^{-1}. \]  
(3.45)
In order to calculate the Jacobian of the transformation (3.45), we use the following property extracted from p. 398 in Berezin’s paper [17]: Let \( A \) be a matrix and \( L_A \) the transformation of a matrix of the same order \( n \), 
\[
L_A \xi = A \xi A^t.
\]
If the matrices \( A \) and \( \xi \) are symmetric, then 
\[
\det L_A = (\det A)^{n+1}.
\]
The overall determinant of the transformation (3.43) is 
\[
J = \left| \frac{\partial z'/\partial z'}{\partial W'/\partial z} \frac{\partial z'/\partial W}{\partial W'/\partial W} \right| = \partial z'/\partial z \partial W%/\partial W,
\]
(3.46)
because \( \partial W'/\partial z = 0 \). Finally, taking \( W_1 = W \), we find out 
\[
Q = \det (1 - W \bar{W})^{-(n+2)}.
\]
(3.47)

3.7. The scalar product. If \( f_\psi(z) = (e^\bar{z}, \psi) \), then 
\[
(\phi, \psi) = \Lambda \int_{z \in \mathbb{C}^n, 1 - W \bar{W} > 0} f_\phi(z, W) f_\psi(z, W) Q K^{-1} dz dW.
\]
(3.48)

\( Q \) is the density of the volume form given by (3.47), \( K \) is the reproducing kernel (3.25), and 
\[
\int dz = n \prod_{i=1}^{n} \Re z_i \Im z_i; \quad dW = \prod_{1 \leq i \leq j \leq n} \Re w_{ij} \Im w_{ij}.
\]
(3.49)

We have \( K^{-1} = \det(1 - W \bar{W})^{k/2} \exp(-F) \) with \( F \) given by (3.24).
In order to find the value of the constant \( \Lambda \) in (3.48), we take the functions \( \phi, \psi = 1 \), we change the variable \( z = (1 - W \bar{W})^{1/2} x \) and we get 
\[
1 = \Lambda \int_{1 - W \bar{W} > 0} \det(1 - W \bar{W})^{k/2 - n} dW \int_{x \in \mathbb{C}^n} \exp(-|x|^2) \exp\left( -\frac{x^t \bar{W} \cdot x + \bar{x}^t \cdot W x}{2} \right) dx.
\]

We apply equations (A1), (A2) in Bargmann [3]: 
\[
I(B, C) = \int \exp\left( \frac{1}{2}(x.B x + \bar{x}.C \bar{x}) \right) \pi^{-n} e^{-|x|^2} \prod_{k=1}^{n} d\Re x_k d\Im x_k = [\det(1 - CB)]^{-\frac{1}{2}},
\]
where \( B, C \) are complex symmetric matrices such that \( |B| < 1, |C| < 1 \). Here \( B = \bar{W}, \ C = -W \). So, we get 
\[
1 = \Lambda \pi^n \int_{1 - W \bar{W} > 0} \det(1 - W \bar{W})^p dW, \quad p = \frac{k - 3}{2} - n.
\]
We apply Theorem 2.3.1 p. 46 in Hua’s book [40] 
\[
\int_{1 - W \bar{W} > 0, W = \bar{W}^t} \det(1 - W \bar{W})^\lambda dW = J_n(\lambda),
\]
and we find out: 
\[
\Lambda = \pi^{-n} J_n^{-1}(p),
\]
(3.50)
where, for \( p > -1 \), \( J_n(p) \) is given by formula (2.41) or by formula (2.42). So, we find out for \( \Lambda \) 
\[
\Lambda = \frac{k - 3}{2\pi} \prod_{i=1}^{n-1} \frac{(\frac{k-3}{2} - n + i)\Gamma(k + i - 2)}{\Gamma[k + 2(i - n - 1)]}.
\]
(3.51)
Proposition 3.11. Let us consider the Jacobi group $G^J_n$ with the composition rule (3.24), acting on the coherent state manifold (3.26), (3.29). The manifold $D^J_n$ has the Kähler potential (3.31) and the $G^J_n$-invariant Kähler two-form $\omega$ given by (3.42). The Hilbert space of holomorphic functions $\mathcal{H}_K$ associated to the positive definite holomorphic kernel $K \colon M \times M \to \mathbb{C}$ given by (3.20) is endowed with the scalar product (3.33) where the normalization constant $\Lambda$ is given by (3.51) and the density of volume given by (2.41).

Recalling Proposition IV.1.9. p. 104 in [15], Proposition 3.11 can be formulated as follows:

Proposition 3.12. Let $h := (g, \alpha) \in G^J_n$, where $G^J_n$ is the Jacobi group (3.27), and we consider the representation $\pi(h) := S(g)D(\alpha)$, $g \in \text{Sp}(n, \mathbb{R})$, $\alpha \in \mathbb{C}^n$, and let the notation $x := (z, W) \in D^J_n := \mathbb{C}^n \times D_n$. Then the continuous unitary representation $(\pi_K, \mathcal{H}_K)$ attached to the positive definite holomorphic kernel $K$ defined by (3.26) is

\[ (\pi_K(h).f)(x) = J(h^{-1}, x)^{-1} f(h^{-1}.x), \]

(3.52)

where the cocycle $J(h^{-1}, x)^{-1} := \lambda(h^{-1}, x)$ with $\lambda$ defined by (3.25)–(3.30) and the function $f$ belongs to the Hilbert space of holomorphic functions $\mathcal{H}_K \equiv \mathcal{H}_K$ endowed with the scalar product (3.33), where $\Lambda$ is given by (3.51).

Comment 3.13. The value of $\Lambda$ (3.33) given by (3.61) corresponds to the one given in equation (7.16) in [16]. Taking above $n = 1$, $k \to 4k$. Note that $p$ defining the normalization constant $\Lambda$ in (3.51) for the Jacobi group is related with $q$ in (2.40) defining the normalization constant for the group $\text{Sp}(n, \mathbb{R})$ by the relation $p = q - \frac{1}{2}$.

4. The Jacobi group $G^J_1$

4.1. Kähler-Berndt’s Kähler two-form $\omega$. We recall [16] that in the case $n = 1$ formula (3.24) reads:

\[ -\omega = \frac{2k}{(1 - w\bar{w})^2} dw \wedge d\bar{w} + \frac{A \wedge \bar{A}}{1 - w\bar{w}}, \quad A = dz + \bar{a}dw, \quad a_0 = \frac{z + \bar{z} w}{1 - w\bar{w}}. \]

(4.1)

Rolf Berndt -alone or in collaboration - has studied the real Jacobi group $G^J_1(\mathbb{R})$ in several references, from which I mention [19-20-21]. The Jacobi group appears (see explanation in [16]) in the context of the so called Poincaré group or The New Poincaré group investigated by Erich Kähler as the 10-dimensional group $G^K$ which invariates a hyperbolic metric [18, 15, 15]. Kähler and Berndt have investigated the Jacobi group $G^J_0(\mathbb{R}) := \text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acting on the manifold $\mathcal{H}_1 := \mathcal{H}_1 \times \mathbb{C}$, where $\mathcal{H}_1$ is the upper half plane $\mathcal{H}_1 := \{ v \in \mathbb{C} | \Im(v) > 0 \}$. We recall that the action of $G^J_0(\mathbb{R})$ on $\mathcal{H}_1$ is given by $(g, (v, z)) \to (v_1, z_1)$, $g = (M, l)$, where

\[ v_1 = \frac{av + b}{cv + d}, \quad z_1 = \frac{z + l_1 v + l_2}{cv + d}; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad (l_1, l_2) \in \mathbb{R}^2. \]

(4.2)

It can be proved [16] that
Remark 4.1. When expressed in the coordinates \((v,u) \in X_J^1\) which are related to the coordinates \((w,z) \in D_J^1\) by the map

\[
w = \frac{v-1}{v+1}, \quad z = \frac{2uv}{v+1}, \quad w \in D_1, \quad v \in H_1, \quad z \in \mathbb{C},
\]

the Kähler two-form is identical with the one

\[
-w = -\frac{2k}{(v-\bar{v})^2} dv d\bar{v} + \frac{2}{i(v-\bar{v})} B \wedge \bar{B}, \quad B = du - \bar{u} dv, \quad v, u \in \mathbb{C}, \quad \Im(v) > 0,
\]

considered by Kähler-Berndt \([43, 44, 45, 46, 19, 21]\). If we use the EZ \([21, 29]\) coordinates adapted to our notation

\[
v = x + iy; \quad u = pv + q, \quad x, p, q, y \in \mathbb{R}, \quad y > 0,
\]

the \(G_1^0(\mathbb{R})\)-invariant Kähler metric on \(X_J^1\) corresponding to the Kähler-Berndt’s Kähler two-form \(\omega\) reads

\[
ds^2 = k^2 y^2 (dx^2 + dy^2) + \frac{1}{y} [(x^2 + y^2) dp^2 + dq^2 + 2xdpdq],
\]

i.e. the equation at p. 62 in \([21]\) or the equation given at p. 30 in \([19]\).

4.2. The base of functions. We reproduce the following result proved in \([16]\)

\[
K(z,w; \bar{z},\bar{w}') := (e_{z,\bar{w}}, e_{w',\bar{z}'}) = \sum_{n,m} f_{n,m} (z,w) \bar{f}_{n,m} (z',w')
\]

\[
= (1 - w\bar{w}')^{-2k} \exp \frac{2\bar{z}'z + \bar{z}'^2 + \bar{z}^2 - \bar{z}'^2 w}{2(1 - \bar{w}w')},
\]

Here

\[
f_{n,m} (z,w) = f_{n,m} (w) \frac{P_n (z,w)}{\sqrt{n!}},
\]

\[
f_{n,m} (z) := (e_{z}, e_{k+k+n}) = \sqrt{\Gamma(n+2k)} \frac{\Gamma(n+2k)}{n! \Gamma(2k)} z^n,
\]

\[
P_n (z,w) := \left( \frac{1}{\sqrt{2}} \right)^n w^{\frac{n}{2}} H_n \left( \frac{-1z}{\sqrt{2w}} \right),
\]

i.e. the polynomials \(P_n (z,w)\) have the expression

\[
P_n (z,w) = n! \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{w^k}{2^k} \frac{z^{n-2k}}{k!(n-2k)!}.
\]

The first 6 polynomials \(P_n (z,w)\) are:

\[
\begin{align*}
P_0 &= 1; & \quad P_1 &= z; \\
P_2 &= z^2 + w; & \quad P_3 &= z^3 + 3zw; \\
P_4 &= z^4 + 6z^2w + 3w^2; & \quad P_5 &= z^5 + 10z^3w + 15zw.
\end{align*}
\]
4.3. Digression on Jacobi and Schrödinger groups. We have called the algebra \( \mathfrak{g}_1 \), the \textit{Jacobi algebra} and the group \( \mathfrak{g}_2 \), the \textit{Jacobi group}, in agreement with the name used in [21], or at p. 178 in [58], where the algebra \( \mathfrak{g}_1 := \mathfrak{h}_1 \ltimes \mathfrak{sl}(2, \mathbb{R}) \) is called “Jacobi algebra”. The denomination adopted in the present paper is of course in accord with the one used in [58] because of the isomorphism of the Lie algebras

\[ \mathfrak{su}(1, 1) \sim \mathfrak{sl}(2, \mathbb{R}) \sim \mathfrak{sp}(1, \mathbb{R}) \sim \mathfrak{so}(2, 1) \]  

Also the name “Jacobi algebra” is used in [58] p. 248 to call the semi-direct sum of the \((2n + 1)\)-dimensional Heisenberg algebra and the symplectic algebra, \( \mathfrak{h}\mathfrak{s}\mathfrak{p} := \mathfrak{h}_n \ltimes \mathfrak{sp}(n, \mathbb{R}) \). The group corresponding to this algebra is called sometimes in the Mathematical Physics literature (see e.g. §10.1 in [1], which is based on [71]) the \textit{metaplectic group}, but in reference [58] the term “metaplectic group” is reserved to the 2-fold covering group of the symplectic group, cf. p. 402 in [58] (see also [3] and [11]). Other names of the metaplectic representation are the \textit{oscillator representation}, the \textit{harmonic representation} or the \textit{Segal-Shale-Weil representation}, see references in Chapter 4 of [33] and [21].

• Apparently, the denomination of the semi-direct product of the Heisenberg and symplectic group as Jacobi group was introduced by Eichler and Zagier, cf. Chapter I in [29]. In their monograph [29], Eichler and Zagier have introduced the notion of \textit{Jacobi form} on \( \text{SL}_2(\mathbb{Z}) \) as a holomorphic function on \( \mathfrak{X}_1 \) satisfying three properties. One of this properties, generalized to other groups, was studied by Pyatetskii-Shapiro, who referred to it as the \textit{Fourier-Jacobi} expansion (see §15 in [65]), and to some coefficients as \textit{Jacobi forms}, a name adopted by Eichler and Zagier to denote also the group appearing in this context. The denomination Jacobi group was adopted also in the monograph [21] and this group is important in Kähler’s approach, see Chapter 36 in [45].

• The Jacobi algebra, denoted \( \mathfrak{s}\mathfrak{t}(n, \mathbb{R}) \) by Kirillov in §18.4 of [48] or \( \mathfrak{t}\mathfrak{s}\mathfrak{p}(2n + 2, \mathbb{R}) \) in [49], is isomorphic with the subalgebra of Weyl algebra \( \mathcal{A}_n \) of polynomials of degree maximum 2 in the variables \( p_1, \ldots, p_n, q_1, \ldots, q_n \) with the Poisson bracket, while the Heisenberg algebra \( \mathfrak{h}_n \) is the nilpotent ideal isomorphic with polynomials of degree \( \leq 1 \) and the real symplectic algebra \( \mathfrak{sp}(n, \mathbb{R}) \) is isomorphic to the subspace of symmetric homogeneous polynomials of degree 2.

• U. Niederer has introduced in [60] and developed in [61] the concept of the \textit{maximal kinematical invariance group} (MKI) of the (free) Schrödinger equation. The Schrödinger group is a 12-parameter group containing, in addition to the Galilei group \( G_5^3 \), the group of dilations and a 1-parameter group of transformations, called expansions, which is, in some respects, very similar to the special conformal transformations of the conformal group. Let us consider the wave equation

\[ \Delta(t, x)\psi(t, x) = 0, \]  

where \( \Delta(t, x) \) is a wave operator, in particular the Schrödinger operator

\[ \Delta(t, x) = i\partial_0 + \frac{1}{2m}\Delta_3, \quad (t, x) \in \mathbb{R}^4, \]  

and let

\[ (t, x) \rightarrow g(t, x) \]  

(4.13)
be any invertible coordinate transformation \( g \). Equation (4.12) is invariant under the transformation (4.13) if
\[
\psi(t, x) \rightarrow (T_g \psi)(t, x) = f_g |g^{-1}(t, x)| \psi[g^{-1}(t, x)],
\]
is again a solution of the wave equation when \( \psi \) is a wave function of (4.12).

Niederer has determined the (one-cocycle) \( f_g \) and the Schrödinger transformation
\[
g(t, x) = \left( d^2 \frac{t + b}{1 + \alpha(t + b)}, d \frac{Rx + vt + a}{1 + \alpha(t + b)} \right), \alpha, b, d \in \mathbb{R}, R \in SO(3), v, a \in \mathbb{R}^3.
\] (4.14)

In fact, \( T_g \) is a projective representation of the Schrödinger group. For the Schrödinger group see also [5]. Another names for the Schrödinger group are the Hagen group or the conformal Galilean group [30]. Other references on the same subject are [23, 66, 4, 64, 24, 6, 32].

In [26] is given the Levi-Malcev decomposition of the Schrödinger group in \( n + 1 \)-space-time dimensions:
\[
\text{Sch}(n) = \underbrace{\mathbb{R}^n \rtimes G^G_n}_{\text{radical}} \rtimes \underbrace{\text{SL}(2, \mathbb{R}) \rtimes \text{SO}(n)}_{\text{SS-Levi part}},
\]
where the Galilei group is
\[
G^G_n = (\text{SO}(n) \times \mathbb{R}^n) \rtimes \mathbb{R}^{n+1}.
\]

In the case \( n = 1, \) (4.14) with \( t \in \mathbb{C}, \Im(t) > 0, x \in \mathbb{C}, (d = 1, R = 1) \) corresponds to (4.2) with
\[
M = \left( \begin{array}{cc} 1 & b \\ \alpha & 1 + \alpha b \end{array} \right) \in \text{SL}(2, \mathbb{R}), (l_1, l_2) = (v, a) \in \mathbb{R}^2.
\]

It interesting that the paper [45] starts with the Chapter \( \text{Relativität nach Galilei}. \) The relationship between the Jacobi group \( G^J \), the New Poincaré group \( G^K \), the de Sitter group \( \text{SO}_0(1, 4) \) and the anti de Sitter group \( \text{SO}_0(2, 3) \), the (standard) Poincaré group \( G^P = \text{SO}_0(1, 3) \rtimes \mathbb{R}^4 \) and his contraction, the Galilei group \( G^{P}_G \), is discussed at p. 33 in [40].

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