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Moduli spaces of semistable pairs in Donaldson–Thomas theory

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Abstract. Let \((X, O_X(1))\) be a polarized smooth projective variety over the complex numbers. Fix \(D \in \text{coh}(X)\) and a nonnegative rational polynomial \(\delta\). Using GIT we construct a coarse moduli space for \(\delta\)-semistable pairs \((E, \varphi)\) consisting of a coherent sheaf \(E\) and a homomorphism \(\varphi: D \rightarrow E\). We prove a chamber structure result and establish a connection to the moduli space of coherent systems constructed by Le Potier in (Faisceaux semi-stable et systèmes cohérents. Cambridge University Press, Cambridge, 1995; Systèmes cohérents et structures de niveau. Astérisque 214, 1993).

0. Introduction

String theorists are highly interested in counting curves on Calabi–Yau threefolds. This can be done by integrating over a virtual cycle on the moduli space of these curves (cf. [1]). The arising moduli problems are not compact and there have been different approaches to their compactification (cf. [2]). Following Pandharipande and Thomas we consider pairs \((\mathcal{E}, s)\) consisting of a coherent sheaf \(\mathcal{E}\) with one dimensional support and a section \(s \in H^0(\mathcal{E})\). Such a pair is called stable if firstly the sheaf is pure and secondly \(s\) considered as a homomorphism \(O_X \rightarrow \mathcal{E}\) is generically surjective. Thus a stable pair \((\mathcal{E}, s)\) provides us with a Cohen–Macaulay curve \(C_{\mathcal{E}} = \text{supp}\mathcal{E}\) and a finite number of points on this curve, namely the cokernel of \(s\).

In [3] the moduli space for such stable pairs is contructed. More generally Le Potier considered so-called coherent systems \((\Gamma, \mathcal{E})\) consisting of a sheaf \(\mathcal{E}\) together with a subspace \(\Gamma \subseteq H^0(\mathcal{E})\). Since the pairs introduced above can only cover the case of irreducible curves there is a need for generalizations of the notion of stable pairs. We want a section \(s\) for every irreducible component of \(C_{\mathcal{E}}\). Thus we should consider pairs \((\mathcal{E}, \varphi)\) with a homomorphism \(\varphi: O_X' \rightarrow \mathcal{E}\) or even more generally with a homomorphism \(\varphi: D \rightarrow \mathcal{E}\) for an arbitrary but fixed coherent sheaf \(D\). There is a generalized notion of stability for such pairs.

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In this article we will construct a coarse moduli space for semistable pairs on
an arbitrary polarized smooth projective variety and relate these moduli spaces to
the moduli spaces of coherent systems in the case $D = \mathcal{O}_X^r$.

We will now give a more detailed overview of the content of this article. In
Sect. 1 we will define the generalized notion of stability depending on a parameter
$\delta \in \mathbb{Q}[x]$ and discuss some basic properties of semistable pairs. Next in Sect. 2 we
will prove the boundedness of the family of $\delta$-semistable pairs which will enable
us to define the parameter space for our moduli problem in Sect. 3. The core of this
article is contained in Sect. 4 where we perform the GIT construction of the moduli
space. In Sect. 5 we prove the usual chamber structure result summarizing how
the moduli spaces change if we vary the parameter $\delta$. Last but not least we show
in Sect. 6 that the moduli space of coherent systems by Le Potier can be obtained
from the moduli space of stable pairs as a quotient by a group action.

**Notations** The main guidelines for the notations used in this article are [4] and
[5]. By a scheme one should always think of a scheme of finite type over some
fixed algebraically closed field $k$ of characteristic zero. If $V$ is a finite dimensional
$k$-vector space we let $\mathbb{P}(V) := (V \setminus \{0\})/\sim$ be the set of lines passing through the
origin. For any vector $v \in V$ we denote its equivalence class in $\mathbb{P}(V)$ by $[v]$. Finally
by $\text{PGL}_n$ we denote the quotient $\text{GL}_{n+1}/k^* \text{Id}$. Thus $\text{PGL}_n$ is the automorphism
group of $\mathbb{P}^n$.

1. Semistable pairs

In order to define pairs we first fix some notation. Let $(X, \mathcal{O}_X(1))$ be a polarized
smooth projective variety, $D$ a coherent $\mathcal{O}_X$-module and $\delta$ a rational polynomial
$\geq 0$, i.e., $\delta(m) \geq 0 \forall m \gg 0$.

**Definition 1.1.** A pair $(\mathcal{E}, \varphi)$ consists of a coherent $\mathcal{O}_X$-module $\mathcal{E}$ and a homo-
morphism $\varphi: D \to \mathcal{E}$. By $P = P_\mathcal{E}$ we denote the Hilbert polynomial of $\mathcal{E}$. This
is a polynomial of degree $d = \dim \mathcal{E}$ and its leading coefficient is just the rank
of the sheaf $\mathcal{E}$ which we denote by $r$ or $r_\mathcal{E}$. We call a pair pure if $\mathcal{E}$ is pure. A
homomorphism of pairs $\alpha: (\mathcal{E}, \varphi) \to (\mathcal{E}', \psi)$ is a homomorphism of $\mathcal{O}_X$-modules
$\alpha: \mathcal{E} \to \mathcal{E}'$ such that there is a scalar $\lambda \in k$ making the following diagramm
commute

\[
\begin{array}{ccc}
D & \xrightarrow{\lambda \cdot \text{id}} & D \\
\downarrow{\varphi} & & \downarrow{\psi} \\
\mathcal{E} & \xrightarrow{\alpha} & \mathcal{E}'
\end{array}
\]

In the obvious way we define the notion of an isomorphism of pairs.

**Lemma 1.2.** Let $(\mathcal{E}, \varphi)$ be a pair. Then for any $\lambda \in k^* (\mathcal{E}, \lambda \varphi)$ is isomorphic to
$(\mathcal{E}, \varphi)$.

**Proof.** Obvious. \qed