Analysis on the effect of technical fluctuations on laser lineshape

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We analyze theoretically the effect of technical fluctuations on laser linewidth in terms of statistics of amplitude and phase noise and their respective bandwidths. While the phase noise tends to broaden the linewidth as the magnitude of the noise increases, the amplitude noise brings out an additional structure with its spectral density reflecting the magnitude of the noise. The effect of possible coupling between those two noises is also discussed.

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I. INTRODUCTION

Lineshape is one of the fundamental properties which tells about how monochromatic a laser is. It also provides a way to investigate the light emission mechanisms for various types of lasers [1]. However the technical fluctuations inherent in a laser and its environment prevent one from observing spectral structure determined by intrinsic dynamics only. In fact, the intrinsic limit of linewidth set by quantum noise [2] is much smaller than we usually observe in laboratory. The sources of such noise is usually thermal or mechanical and they appear as, for example, cavity drift, fluctuation of population inversion and instability pump field. We may classify the aspects of those fluctuations into phase and amplitude noise with various bandwidth. We analyze the effect of technical fluctuations on lineshape in terms of statistics of amplitude and phase noise and their respective bandwidths. While the phase noise tends to broaden the linewidth as the magnitude of the noise increases, the amplitude noise brings out an additional structure with its spectral density reflecting the magnitude of the noise. The effect of possible coupling between those two noises is also discussed.

II. PHASE NOISE

Let us write the time variation of the electric field, the spectrum of which we want to measure, as

\[ E(t) = E_0(t)e^{i[\omega_0 t + \phi(t)]}, \]

where \( E_0(t) \) is a slowly varying envelope and \( \phi(t) \) is a randomly fluctuating phase.

Firstly consider the case in which the amplitude \( E_0(t) \) is constant \( E_0 \) and \( \phi(t) \) undergoes a random walk process which imposes Gaussian statistics on \( \phi(t) \). The effective frequency is

\[ \omega_{eff}(t) = \omega_0 + \dot{\phi}(t) \]

which represents that the time derivative of \( \phi(t) \) makes jitter around the carrier frequency. According to Wiener-Khinchin theorem, the spectral lineshape is given by

\[ g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau) \rangle}{\langle E(t) \rangle^2}, \]

where \( \langle \cdot \rangle \) denotes a time average. Using Eq. (1),

\[ \langle E^*(t)E(t+\tau) \rangle = |E_0|^2 e^{i\omega_0 \tau} \langle e^{i\int_t^{t+\tau} \phi(t)dt'} \rangle. \]

For a normally distributed random variable \( x \) with its probability density function (PDF)

\[ P[x] = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}, \]

where \( \bar{x} \) is the mean of \( x \) and \( \sigma^2 \) is the variance, the characteristic function \( \langle e^{i\omega x} \rangle \) is calculated to be \( e^{i\bar{x}\omega} e^{-\frac{\sigma^2\omega^2}{2}} \). Taking \( x = \int_t^{t+\tau} \phi(t')dt' \) and \( \omega = 1, \)

\[ \langle E^*(t)E(t+\tau) \rangle = |E_0|^2 e^{i\omega_0 \tau} e^{-\frac{\sigma^2\tau^2}{2}}, \]

where \( \bar{x} = 0 \) and \( \sigma^2_x \) is given by

\[ \sigma^2_x = \langle \int_t^{t+\tau} dt' \int_t^{t+\tau} dt'' \phi(t')\phi(t'') \rangle. \]

The calculation of \( \sigma^2_x \) depends on the bandwidth of the frequency jittering.

It is readily calculated if the spectrum of \( \phi(t) \) is flat, that is, \( \langle \phi(t)\phi(t+\tau) \rangle \) is proportional to \( \delta(\tau) \). The constant of proportionality is found from

\[ \langle \phi^2(t) \rangle = \int \phi^2(t) P[\phi(t)]d\phi(t) \]

\[ = \frac{1}{\sqrt{2\pi}\Delta R} \sqrt{\frac{\pi}{2}} (2\Delta R^2)^{\frac{1}{2}} \]

\[ = (\Delta R)^2, \]

where \( \Delta R \) denotes the magnitude of jittering and the PDF of \( \phi(t) \)

\[ P[\phi] = \frac{1}{\sqrt{2\pi}\Delta R} e^{-\phi(t)^2/(2\Delta R)^2}. \]
The correlation \( \langle \dot{\phi}(t)\dot{\phi}(t+\tau) \rangle \) as a function of \( \tau \) as a model of finite bandwidth Gaussian noise. (b) The value of integration of Eq. (10) is the area surrounded by \( t'' = t' \pm \sqrt{2} \tau' = \tau \) and \( t''' = \tau \)

was used. Then

\[
\sigma_{\tau}^2 = \int_0^\tau dt''\int_0^\tau dt''\langle \dot{\phi}(t')\dot{\phi}(t'') \rangle
= (\Delta R)^2 \int_0^\tau dt'\int_0^\tau dt''\delta(t' - t'')
= \sqrt{2}(\Delta R)^2|\tau|, \tag{10}
\]

where, in the first line, \( t \) is replaced with zero regarding the noise being stationary. Finally we came to obtain

\[
g^{(1)}(\tau) = e^{i\omega_0\tau}e^{-\frac{1}{\sqrt{2}}(\Delta R)^2|\tau|} \tag{11}
\]

and its Fourier transform gives a Lorentzian line shape with FWHM of \((\Delta R)^2\).

If the spectrum of \( \dot{\phi}(t) \) has a finite bandwidth, \( \sigma_{\tau}^2 \) depends on the correlation time \( t_c \) of \( \dot{\phi}(t) \). We can model it by a square-shaped temporal correlation as in Fig. 1(a). In terms of frequency this noise is the white noise filtered by a sinc-function low-pass filter of width \( \sim t_c^{-1} \). The integration in Eq. (10) then takes the value depending on \( \tau \) (Fig 1(b))

\[
\frac{\sigma_{\tau}^2}{(\Delta R)^2} = \left\{ \begin{array}{ll}
\tau^2 & \text{if } \tau < \sqrt{2}t_c \\
2\sqrt{2}\tau t_c - 2t_c^2 & \text{if } \tau > \sqrt{2}t_c \end{array} \right. \tag{12}
\]

Thus \( g^{(1)}(\tau) \) is Gaussian up to \( \tau = \sqrt{2}t_c \) and thereafter exponentially decaying function. Of course it is continuous at \( \tau = \sqrt{2}t_c \) having the common value \( e^{-(2\Delta R)^2t_c^2} \). If \( t_c \) is longer than \((\Delta R)^{-1}\), \( g^{(1)}(\tau) \) is practically Gaussian. We have Gaussian lineshape with its linewidth \( \sim \Delta R \) in that case. In the opposite limit where \( t_c \) goes to zero, the Lorentzian lineshape due to the white noise is recovered.

Another possible case of frequency jittering is slow modulation of carrier frequency as it usually happens in a laser with its cavity slowly drifting around the resonance frequency. PDF of Eq. (9) is no more valid in such cases. Rather we start from

\[
\dot{\phi}(t) = \Delta F \cos(\Omega t) \tag{13}
\]

and accordingly

\[
\phi(t) = \frac{\Delta F}{\Omega} \sin \Omega t \equiv \phi_0 \sin \Omega t, \tag{14}
\]

where \( \Delta F \) is the amplitude of modulation and \( \Omega \) is the slow frequency. For this simple harmonic oscillation, PDF of \( \phi(t) \) is given by

\[
P[\phi(t)] = \frac{1}{\pi \sqrt{\phi_0^2 - \phi^2(t)}}. \tag{15}
\]

The expectation value of \( e^{ix} \) will be calculated using this PDF where

\[
x = \int_{-\infty}^{\infty} \phi(t)dt = \int_{-\infty}^{\infty} \Delta F \cos \Omega t dt'
= \phi_0[\sin \Omega(t + \tau) - \sin \Omega t]
= \phi_0(\sin \Omega t \cos \Omega \tau + \cos \Omega t \sin \Omega \tau - \sin \Omega t)
= \phi \times (-2 \cos^2 \frac{\Omega \tau}{2} \pm \sin \Omega \tau \sqrt{\phi_0^2 - \phi^2}). \tag{16}
\]

By inspecting the PDF in Eq. (15), we can recognize that \( \phi(t) \) spends most of its time near \( \phi(t) \simeq \phi_0 \) thereby neglect the second term in Eq. (16). Then

\[
\langle e^{ix} \rangle \simeq \int_{-\phi_0}^{\phi_0} P[\phi]e^{-2i\phi \cos^2 \frac{\Omega \tau}{2}} d\phi
= \int_{-\phi_0}^{\phi_0} \frac{1}{\pi} e^{-2i\phi \cos^2 \frac{\Omega \tau}{2}} \frac{\phi}{\sqrt{\phi_0^2 - \phi^2}} d\phi
= J_0 \left(2\phi_0 \cos^2 \frac{\Omega \tau}{2}\right), \tag{17}
\]

where the integration in the last line involving \( J_0 \), Bessel function of the first kind is performed in Ref. 3. Therefore

\[
g^{(1)}(\tau) = e^{i\omega_0\tau} J_0 \left(2\phi_0 \cos^2 \frac{\Omega \tau}{2}\right). \tag{18}
\]

The graph of \( J_0(2\phi_0 \cos^2 \Omega \tau/2) \) is given in Fig. 2(a) and (c) for different value of magnitude of modulation \( \Delta F \). It is infinite pulse train with the repetition rate \((\Omega/2\pi)^{-1}\). Its Fourier transform constitutes frequency comb whose width of envelope is determined by the inverse time duration of the pulse, \( T \). (Fig.1 (b),(d)) \( T \) gets shorter as we increase \( \Delta F \) because the argument of Bessel function changes by larger amount for the same change of \( \tau \). This is reasonable result since the more harsh we swing the carrier frequency, the wider the spectrum should be. \( \Omega \) determines the repetition rate, i.e. density (degree of fine-tooth) of the comb.

If another frequency components other than \( \Omega \) is added in Eq. (12), the fine spectral structure is easily destroyed by the complexity of argument in Bessel function in Eq. (17). The additional modulation is linearly added to \( x \) and finally to the argument of the Bessel function. This complicated argument brings about reduction of the pulse height of \( J_0 \) by only a small number of such superposition. Thus, in the spectrum, the spectral density of each tooth in the comb gets smaller and interval between adjacent teeth gets narrower.
III. INTENSITY NOISE

Next let us include the effect of the amplitude fluctuation. The criterion between amplitude and phase, in this analysis, rests with their direct appearance in intensity. Consider the intensity modulated like

\[ |E_0(t)|^2 = I_0 (1 + M \cos \Omega t) \],

where the modulation depth \( M \) is usually much less than 1. The corresponding field amplitude can be written as

\[ E_0(t) = \sqrt{I_0} \left( 1 + a_1 e^{i\Omega t} + a_{-1} e^{-i\Omega t} + a_2 e^{i2\Omega t} \cdots \right) \],

(20)

where \( 1 \gg a_1, a_{-1} \), and \( a_1, a_{-1} \gg a_2, \cdots \) for \( M \ll 1 \). This implies that the spectrum is given with sidebands, which are symmetrically apart from the carrier frequency by integer multiples of \( \Omega \). (Fig. 3 (a)) Extending this idea to the noise with finite band we can imagine superposition of many sidebands comprised of all frequency component within the band. The resultant spectrum contains the low-lying wing structure near the carrier frequency. One example where the Gaussian noise is applied is depicted in Fig. 3 (b). Note that the Lorentzian is smoothed out by Gaussian profile which brings about the considerable deviation from a Lorentzian lineshape.

The explicit calculation starts with

\[ \langle E^* (t) E(t + \tau) \rangle = e^{i\omega_0 \tau} \langle E_0^* (t) E_0(t + \tau) e^{i\int_{t}^{t+\tau} \dot{\phi}(t') dt'} \rangle \].

(21)

Since

\[ E_0^* (t) E_0(t + \tau) = I_1 (1 + |a_1|^2 e^{i\Omega \tau} + |a_{-1}|^2 e^{-i\Omega \tau} + a_1^* a_{-1} e^{i(\Omega - \omega) \tau} + \cdots) \],

(22)

each sideband make \( \bar{x} \) to shift by \( 0, \Omega, 2\Omega \cdots \) respectively while \( \sigma_x^2 \) remains the same. Thus

\[ g^{(1)} (\tau) = e^{i\omega_0 \tau} e^{-\frac{1}{2} (\Delta R)^2 } \]

\[ \times \left( 1 + |a_1|^2 e^{i\Omega \tau} + |a_{-1}|^2 e^{-i\Omega \tau} + \cdots \right) \],

(23)

so that

\[ |\mathcal{E}(\omega)|^2 \sim \frac{1}{(\omega - \omega_0)^2 + (\Delta R^2/\sqrt{2})^2}

\[ + \frac{|a_{-1}|^2}{(\omega - \omega_0 - \Omega)^2 + (\Delta R^2/\sqrt{2})^2} \cdots \].

(24)
The same result can be understood in a different way. The spectral amplitude $\mathcal{E}(\omega)$ is the Fourier transform of the product of two functions: $E_0(t)$ and $e^{i[\omega t + \phi(t)]}$. Thus the overall spectrum is given by convolution of each spectrum of the functions. If the bandwidth of the amplitude noise is Gaussian,

$$\mathcal{E}(\omega) \sim \int \frac{\delta(\omega' - \omega_0) + a_G e^{(\omega' - \omega_0)^2/\Delta_G^2}}{(\omega' - \omega - \omega_0)^2 + \Delta_L^2} d\omega', \quad (25)$$

where $\Delta_L$ and $\Delta_G$ are spectral widths of the Lorentzian and Gaussian respectively and $a_G$ is relative spectral amplitude of Gaussian noise.

If the bandwidth of amplitude fluctuation is too broad, the low-lying structure may not be easily recognized. The manifestation of amplitude noise can then be confirmed in the experiment from the intensity correlation function $g^{(2)}(\tau)$ since

$$g^{(2)}(0) = \langle 1 + \int m(\omega) \cos \Omega t \, d\Omega \rangle(1 + \int m(\Omega') \cos \Omega' t d\Omega')$$

$$= 1 + \frac{1}{2} \int d\Omega \int d\Omega' m(\Omega)m(\Omega') \delta(\Omega - \Omega')$$

$$= 1 + \frac{1}{2} \int d\Omega m^2(\Omega), \quad (26)$$

where $m(\omega)$ is modulation density. The peak near $\tau = 0$ definitely reveal the effect of amplitude fluctuation.

There might be possible coupling between amplitude and phase as is well known in semiconductor lasers. If $\phi(t)$ is affected by the intensity modulation as a quadrature $\phi(t) = 2\delta \sin \Omega t$ where $\delta$ designates the magnitude of coupling,

$$E(t) \simeq (1 + 2a_1 \cos \Omega t) \cos (\omega t + 2\delta \sin \Omega t)$$

$$= \frac{1}{2} e^{i\omega t} + (a_1 + \delta)e^{i(\omega + \Omega)} + (a_1 - \delta)e^{-i(\omega - \Omega)} + c.c. \quad (27)$$

Hence this type of coupling brings about the asymmetry in the sidebands.

**IV. CONCLUSION**

We investigated the effect of technical fluctuations on laser lineshape. The Gaussian noise in frequency make the lineshape a Lorentzian or a Gaussian depending on the correlation time of the noise. Slow swing of frequency results in a lineshape with many sidebands by the periodicity involved. The amplitude noise imposes the low-lying wing structure in the spectrum. The bandwidth of the noise only determines the width of the additional structure while the coupling between the amplitude and phase might lead to an asymmetry in the structure.

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