METRICALLY THIN SINGULARITIES OF INTEGRABLE CR
FUNCTIONS

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Abstract. In this article, we consider metrically thin singularities $A$ of
the tangential Cauchy-Riemann operator on smoothly embedded Cauchy-
Riemann manifolds $M$. The main result states removability within the space
of locally integrable functions on $M$ under the hypothesis that the $(\dim M -
2)$-dimensional Hausdorff volume of $A$ is zero and that the CR-orbits of $M$
and $M \setminus A$ are comparable.

1. Introduction

A smooth real submanifold $M$ of a complex manifold $X$ is called (embedded)
CR-manifold if the dimension of the maximal complex subspace $T^c_p M$ of
the tangent space $T_p M$ does not depend on $p \in M$. In this case the complex
tangent spaces are the fibers of a smooth vector bundle, the complex tangent
bundle $T^c M$, whose (complex) rank $\text{CR-dim} M = \dim_c T^c_p M$ is called the CR-
dimension of $M$. Then the CR-vectorfields, i.e. the sections of $T^c M$, form
a system of first order differential operators, also denoted as the system of
the tangential Cauchy-Riemann equations. Functions annihilated by all CR-
vectorfields will be called CR-functions.

This paper is devoted to the study of the singularities of integrable CR-
functions. A closed set $A \subset M$ is called $L^p$-removable if

$$L^p_{\text{loc,CR}} (M \setminus A) \cap L^p_{\text{loc}} (M) = L^p_{\text{loc,CR}} (M),$$

i.e. if any $f \in L^p_{\text{loc}} (M)$ which is CR outside of $A$ is automatically CR on all of
$M$. This notion appeared probably for the first time in the classical Riemann-
Removability Theorem (stating the $L^\infty_{\text{loc}}$ removability of isolated points for the
Cauchy-Riemann operator on the complex plane). Later on, the question was	reated by Bochner, Carleson e.a. also in the other $L^p$-spaces for the ordinary
Cauchy-Riemann and the Laplace operator, and by Harvey and Polking for gen-
eral linear partial differential operators. Their main theorem (Theorem 4.1. of
[HIPO]), which is best possible in the general setting, implies for tangential
Cauchy-Riemann operators that $E$ is $(L^p, \overline{\partial}_b)$-removable if the Hausdorff mea-
sure $H^{\dim M - p'} (E) < \infty$, $p' = p/(p - 1)$. We stress that on the general level
considered by Harvey and Polking no results in $L^1$ are possible, as is already

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shown by the meromorphic function of one variable $1/z$, which is locally integrable near the singularity at the origin.

The main theorem of this article shows that essentially stronger removability phenomena hold true for tangential Cauchy-Riemann operators on embedded CR-manifolds which are not Riemann surfaces. Our approach relies on the close interplay between CR-functions and complex analysis by means of analytic extension. The rough idea tells that singularities of CR-functions should behave in a way comparable to holomorphic functions. For boundaries of complex domains, this point of view was illustrated by contributions of many authors (we refer to the excellent survey [CS]). In the present paper, we focus in particular on CR-manifolds of arbitrary codimension.

As background for analytic extension we need to recall the notion of CR-orbits. For a point $p$ of a CR-manifold $M$ its CR-orbit $O(p, M)$ is defined as the union of all points which are connected with $p$ by a piecewise differentiable path $\gamma \subset M$ whose (one-sided) derivatives are always non-vanishing and contained in $T_c M$. By general results of Sussmann it is known that CR-orbits are injectively immersed submanifolds of $M$. For information on their tremendous bearing on analytic extension and further references, we refer the reader to contributions of Treves, Trépreau, Tumanov, Jöricke and Merker.

We shall constantly employ the following terminology: A property $P$ is said to be true on almost every CR-orbit if there is a union $N$ of CR-orbits such that $P$ holds true on any orbit $O$ out of $N$. Furthermore a CR-manifold $M$ will be called globally minimal if it contains only one orbit.

The following theorem contains the essence of the article.

**Theorem 1.1.** Let $M$ be an embedded CR-manifold of class $C^3$, dimension $d \geq 3$, and CR-dimension $\dim_{CR} M = m \geq 1$. Then every closed subset $E$ of $M$ with $H^{d-2}(E) = 0$ such that for almost all CR orbits, $O_{CR} \setminus E$ is globally minimal is $L^p_{loc}$-removable for $1 \leq p \leq \infty$.

The above theorem is a refinement of results the authors proved in [MP], where either the hypothesis on Hausdorff measure was too restrictive (we supposed $H^{d-3}(E)$ locally finite), or $M$ was assumed to be real analytic. This research was as a whole inspired by corresponding results of Lupacciolu, Stout and Chirka on analytic extension of continuous CR-functions defined on parts of boundaries of complex domains.

The metrical hypothesis is optimal for trivial reasons: Taking for $E$ the transverse intersection of $M$ with a complex hypersurface $X$ of the ambient space we plainly get $H^{d-2}(E) > 0$. If $X$ has a holomorphic defining function $f$, then $1/f|_M$ is around $E$ locally integrable but not CR.

The organization of the proof is as follows: In a first reduction (section 2) we shall see that the problem restricts to solving the corresponding question on almost all CR-orbits. More concretely we shall be left with the hypothesis that both $M$ and $M \setminus (M - E)$ are globally minimal.

Next we employ known techniques using Bishop-discs to extend our CR-function $f$ from $M \setminus E$ to a large portion of a wedge $W$ attached to $M$ (section 3). Then the final step should be to remove the singularity in the wedge with good $L^1$-estimates and to recover thereafter $f$ globally as an $L^1$-limit (similary
as in the theory of Hardy spaces) thereby proving that $f$ is CR everywhere. At the removal of the singularity in $W$ we encounter a special problem which is overcome by the following theorem which seems to be of independent interest.

**Theorem 1.2.** Let $D \subseteq \mathbb{C}^N$ be a domain equipped with a foliation $F$ by holomorphic curves of class $C^2$. Further let $E \subset D$ be a closed union of leaves with $H_{2N}(E) = 0$. Then a function $F \in L^1(D) \cap \mathcal{H}(D \setminus E)$ extends holomorphically to $D$ as soon as $F$ extends to an open subset $D'$ of $D$ which intersects each leaf contained in $E$.

The proof is an extension of techniques used by Henkin and Tumanov to prove a related statement on continuous CR-functions on manifolds foliated by holomorphic curves.

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### 2. Reduction to CR-orbits

In this section we shall reduce the proof of theorem 1.1 to the case that both $M$ and $M \setminus E$ are globally minimal. This is a consequence of the following two statements.

**Lemma 2.1.** Under the assumptions of Theorem 1.1, for almost every orbit $O$ the intersection $E \cap O$ satisfies $H^{d_O - 2}_E(E \cap O) = 0$, where $d_O = \dim_R O$.

Here the index $E$ in $H^{d_O - 2}_E(E \cap O)$ indicates that the Hausdorff content is computed with respect to the manifold topology of $O$ and the pullback of the euclidean metric to $O$.

**Proof.** First we recall the following local property of CR-orbits: Near every point $p \in M$ there exist a (topologically trivial) $C^2$-foliation of $M$ into leaves of the dimension $k = \dim O(p, M)$ such that for any $q$ near $p$ its orbit $O(q, M)$ contains the leaf passing through $q$. This implies that the dimension of the CR-orbits passing near $p$ cannot be less than $\dim O(p, M)$ and that in case of equality the leaf is an open subset of the corresponding orbit. A well-known property of Hausdorff-measures assures that for almost every leaf $L$ the intersection $E \cap L$ satisfies $H^{\dim L - 2}_E(E \cap L) = 0$.

To exploit this local information we need the following simple covering lemma.

**Lemma 2.2.** There is a countable covering of $M$ by foliated open sets $U_j \subset M$ as above such that every point $p \in M$ appears in at least one $U_j$ whose leaves are of dimension $\dim O(p, M)$.

**Proof.** We argue by induction on the dimension of orbits. By paracompactness of $M$ we easily find a countable covering $\{U_j^{(0)}\}$ as required for the open submanifold

$$M^{(0)} = \{p \in M : \dim O(p, M) = \dim M\}.$$

In this case the $U_j^{(0)}$ are just equipped with the trivial codimension-zero-foliation. In the next we look at

$$M^{(1)} = \{p \in M : \dim O(p, M) \geq \dim M - 1\},$$
which is by the above local property of CR-orbits also an open subset of $M$. We choose for any point $p \in M \setminus M_0$ an open set $U_p(1)$ equipped with a convenient codimension-one-foliation and obtain thereby a covering $\{U_j^{(0)}, U_p(1)\}$. Again by paracompactness we extract a countable covering $\{U_j^{(1)}\} \supset \{U_j^{(0)}\}$.

By induction we treat the open submanifolds

$$M^{(k)} = \{p \in M : \dim O(p, M) \geq \dim M - k\}$$

in analogous fashion. Hence we get an increasing chain of systems $\{U_j^{(k)}\}$ and the desired covering after at most $\dim M - 2\text{CR-dim} M$ steps. \hfill $\square$

From Lemma 2.2 and the preceding remark we deduce that the set of all CR-orbits $O$ of a given dimension $k$ for which the intersection $O \cap E$ with some leaf of with some fixed $U_j$ (with leaf-dimension $k$) has too large Hausdorff content is of zero measure. As countable unions of sets of measure zero are still of measure zero, Lemma 2.1 is proved. \hfill $\square$

Next we need the following result from [PO].

**Theorem 2.3.** If $M$ is $C^3$, a function $f \in L^1_{\text{loc}}(M)$ is CR if and only if $f|_O$ belongs to $L^1_{\text{loc,CR}}(O)$ for almost every CR-orbit $O$.

Lemma 2.1 obviously implies that the set-theoretical assumptions of Theorem 1.1 are inherited by almost all orbits. Theorem 2.3 shows, first, that the restriction of an $f \in L^1_{\text{loc,CR}}(M \setminus E)$ is for almost every orbit a CR-function on $O \setminus E$ and, secondly, that it is enough to remove the singularity orbit-wise. Hence the desired reduction to globally minimal manifolds is complete.

### 3. Reduction to singularities of holomorphic functions

Now we shall use the technique of analytic discs to reduce the proof to a special problem concerning removable singularities of integrable holomorphic functions. By the preceding section and the fact that the CR-orbits of a manifold $M$ of class $C^{2,\alpha}, 0 < \alpha < 1$, are of class $C^{2,\beta}$, for any $\beta \in (0, 1)$ (s. [MP]), it is enough to prove the following theorem.

**Theorem 3.1.** Let $M$ be $C^{2,\alpha}, 0 < \alpha < 1$, $\text{dim}_{\text{CR}} M = m \geq 1$, $\dim M \geq 3$.

Then every closed subset $E$ of $M$ such that $M$ and $M \setminus E$ are globally minimal and such that $H^{d-2}_{\text{loc}}(E) = 0$, is $L^1_{\text{loc}}$-removable.

**Proof.** As the proof is quite long, it will subdivided into several steps:

**Step one: general setup.** Fix a function $f \in L^1_{\text{loc,CR}}(M \setminus E)$. Define

$$A = \{\Psi \subset E \text{ closed; } M \setminus \Psi \text{ globally minimal and } f \in L^1_{\text{loc,CR}}(M \setminus \Psi) \cap L^1_{\text{loc}}(M)\}$$

and define $E_{\text{nr}} = \cap_{\Psi \in A} \Psi$. Then $M \setminus E_{\text{nr}}$ is globally minimal too. If $E_{\text{nr}} = \emptyset$, we are done.

To reach a contradiction, we assume $E_{\text{nr}} \neq \emptyset$ and denote $E_{\text{nr}}$ from now on again by $E$. As $M \setminus E$ is globally minimal, we may, according to Proposition 1.16 of [MP], (after a slight deformation of $M$ fixing $E$) assume that $f \in L^1_{\text{loc}}(M) \cap \mathcal{H}(\Psi(M \setminus E))$.

First we look for a small analytic disc $A$ attached to $M$ whose boundary has non-trivial intersection with $E$ without being contained in $E$. Further we shall
achieve that $A$ is round in the following sense: There is a point $p \in bA$ and
holomorphic coordinates $z, w$ around $p$ exhibiting such that $\{z = 0\}$ corresponds
to $T^c_p M$ such that the projection of $A$ to $\{z = 0\}$ is a translated coordinate disc.
Indeed, let $q \in E$ first be an arbitrary point and $\gamma$ a piece-wise differentiable
CR-curve linking $q$ with a point $q' \in M \setminus E$. After shortening $\gamma$ we may suppose
$\{q\} = E \cap \gamma$ and $\gamma$ to be a smoothly embedded segment. Therefore $\gamma$ can be
described as the integral curve of some non-vanishing CR vectorfield $X$ defined
in a neighborhood of $\gamma$.

It suffices to show that $f$ is CR near the endpoint $q$. Indeed, a standard
argument using the dynamical flow of $X$ proves the existence of arbitrarily
small neighborhoods $V$ for which $(M \setminus E) \cup V$ is globally minimal. If $f$ is
CR near $q$ and $V$ small enough, this contradicts the minimality of $E$.

**Step two: construction of analytic discs.** Let us briefly recall some basic
notions concerning analytic discs. An analytic disc $A$ is a holomorphic mapping
$A : \Delta \to \Omega$ extending continuously to $\overline{\Delta}$. Sometimes, we will by $A, \partial A$ also
denote the images $A(\Delta), A(\partial \Delta)$. The disc $A$ is said to be attached to $M$ if
$\partial A \subset M$.

Locally, we will usually work in holomorphic euclidean coordinates $(z = x + iy, w)$
transferring a given point $p \in M$ to the origin such that $M$ is given
near $p$ as a graph of a function
\begin{equation}
    x = h(y, w),
\end{equation}

satisfying $h(0,0) = 0$ and $\nabla h(0,0) = 0$. A disc $A$ will be called round if in
appropriated coordinates $(z, w)$ its $w$-components appear as a round disc in some
complex line in $\mathbb{C}^m$.

By an observation from [TU] we may approximate $\gamma$ by a finite chain of round
analytic discs $A_1, \ldots, A_k, A_1(1) = q, A_j(-1) = A_{j+1}(1), A_k(-1)$ very close to
$q'$, where all $A_j$ are of small diameter.

In what follows we shall prove for a round disc $A$ of small size that $f$ is
CR near all of $\partial A$ as soon as $\partial A \not\subset E$. If $A_j$ is the first of the above discs
whose boundary is not contained in $E$ this will imply that $f$ is CR near $\partial A_j$.
Careful examination of the constructions below yields that we can successively
apply the argument to $A_{j+1}, \ldots, A_k$. In particular $f$ is CR near $q$ leading to a
contradiction.

**Step three: partial analytic extension.** For a small round disc $A$ with $\partial A \not\subset E$
we have to prove that $f$ is CR near $\partial A$. By symmetry it is enough to argue near
$p = A(1)$. We shall embed $A$ into family of analytic discs sweeping out a wedge
$W$ glued to $M$. Using the continuity principle we shall extend $f$ holomorphically
to a subset of $W$ of full measure and prove that the extension $F$ is integrable
on $W$.

We shall first develope $A$ in a preliminary family $A_{\rho,s'}$ of analytic discs,
$0 \leq \rho < \rho_2, \rho_2 > \rho_1$, $I_{\rho_2} = (0, \rho_2), s' = (a_2, \ldots, a_m, y_1^0, \ldots, y_m^0)$ running in a
neighborhood of 0 in $\mathbb{C}^{m-1} \times \mathbb{R}^m$, as follows: Set for the $w$-component
\begin{equation}
    W_{\rho,a}(\zeta) = (\rho(\zeta - \rho_1), a_2, \ldots, a_{m-1})
\end{equation}

and take then $A_{\rho,s'}$ of the form
\begin{equation}
    A_{\rho,s'}(\zeta) = (X_{\rho,s'}(\zeta) + iY_{\rho,s'}(\zeta), \rho(\zeta - \rho_1), a_2, \ldots, a_{m-1}),
\end{equation}
where $Y_{\rho,s'}$ is the solution of Bishop’s equation

$$Y_{\rho,s'} = T_I h(Y_{\rho,s'}, W_{\rho,s'}) + y^0.$$  

Here $T_I$ denotes the renormalized Hilbert-transform which associates to a (vector valued) real function on $\partial \Delta$ its harmonic conjugate vanishing at the point $1 \in \partial \Delta$. Since the discs $W_{\rho,s'}$ are of small $C^{2,\alpha}$ norm, general properties of Bishop’s equation give the existence of solutions depending smoothly on the data.

Differentiating Bishop’s equation we see that there exists $\mathcal{V}$, a neighborhood of $0$ in $\mathbb{C}^{n-1}$, $\mathcal{V}$, a neighborhood of $0$ in $\mathbb{R}^n$, such that the mapping

$$I_{\rho_2} \times \mathcal{A} \times \mathcal{Y} \times b\Delta \ni (\rho, a, y^0, \zeta) \mapsto A_{\rho,a,y^0}(\zeta) \in M$$

is an embedding. This shows that a neighborhood in $M$ of $A_{0,0,0}(\Delta) = p_1$ is foliated by $C^2$-smooth real discs $D_{a,y^0} = D_{s'} = \{A_{\rho,a,y^0}(\zeta) : M : 0 \leq \rho < \rho_2, \zeta \in b\Delta\}$. Moreover, since $H_{d-2}(E) = 0$, the set $S_E = \{s' \in S' : D_{s'} \cap E \neq \emptyset\}$ is a closed subset of $S'$ foliated by $\mathcal{A} \times \mathcal{Y}$ of Lebesgue measure zero. By construction, each disc $A_{\rho,s'}$ with $s' \not\in S_E'$ is therefore analytically isotopic to a point in $M \setminus E$.

Furthermore, by means of normal deformations of the family near $A(-1)$ as in [MP1], Proposition 2.6, we can develop $A$ in a regular family $A_{\rho,s',v}$ which has the property that, for each $v \in \mathcal{V}$, the set $S_{E,v}' = \{s' \in S' : D_{s',v} \cap E \neq \emptyset\}$ is a closed subset of $S'$ of Lebesgue measure zero. Therefore, each disc $A_{\rho,s',v}$ with $A_{\rho,s',v}(b\Delta) \cap E = \emptyset$ is analytically isotopic to a point in $M \setminus E$, since $S' \setminus S_{E,v}'$ is dense and open in $S'$.

Then the isotopy property and a version of the continuity principle (s. [ME]) imply that $\mathcal{H}(\mathcal{V}(M \setminus E))$ extends holomorphically into

$$\mathcal{W} = \{A_{\rho,a,y^0,v}(\zeta) \in \mathbb{C}^n : \rho \in I_{\rho_1}, a \in \mathcal{A}_1, y^0 \in \mathcal{Y}_1, v \in \mathcal{V}_1, \zeta \in \Delta_1\}$$

minus the set

$$E_{\mathcal{W}} = \{A_{\rho,s',v}(\zeta) \in \mathbb{C}^n : A_{\rho,s',v}(b\Delta) \cap E \neq \emptyset\}$$

for which $H_{2m+2n-1}(E_{\mathcal{W}}) = 0$. Let $f \in \mathcal{H}(\omega)$ and let $F$ denote its extension to $\mathcal{W} \setminus E_{\mathcal{W}}$.

Let us finally show $F \in L^1(\mathcal{W})$. For each fixed $v$ the boundaries of the discs $A_{\rho,s',v}$ foliate some open subset of $M$. By applying a standard estimate to almost every of these discs and integrating over $\rho, s'$ we get an $L^1$ estimate for the restriction of $F$ to the $(d+1)$-dimensional submanifold of $W$ swept out by these discs. Integrating over $v$ we finally get $F \in L^1(\mathcal{W})$.

**Step four: extension of $F$.** The singular set $E_{\mathcal{W}}$ contains the closed subset

$$E'_{\mathcal{W}} = \{A_{\rho,s',v}(\zeta) \in \mathbb{C}^n : A_{\rho,s',v}(b\Delta) \subset E\}.$$  

The main problem now consists in removing $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ from $W \setminus E'_{\mathcal{W}}$. This is a consequence of Theorem 1.2. Indeed, the foliation of $W \setminus E'_{\mathcal{W}}$ being given by the holomorphic discs $A_{\rho,s',v}$ we observe that $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ satisfies $H^{2d-2}(E_{\mathcal{W}} \setminus E'_{\mathcal{W}}) = 0$ because of $H^{d-2}(E) = 0$. Finally we have to verify that $F$ extends holomorphically through some point of each leaf of $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$. But that is clear by definition of $E_{\mathcal{W}} \setminus E'_{\mathcal{W}}$ and our initial assumption $f \in L^1_{loc}(M) \cap \mathcal{H}(\mathcal{V}(M \setminus E))$.  


Assuming Theorem 1.2 we are left with $E'_{W}$. Its definition and $H^{d-2}(E) = 0$ yield $H^{2N-2}(E'_{W}) = 0$. Hence almost every complex line is disjoint from $E'_{W}$ and we can extend $F$ through $E'_{W}$ by the ordinary continuity principle.

**Step five: $L^1$ boundary values.** Finally, we wish to recover $f$ near $p$ as the $L^1$-limit of $F$. For $0 < \delta << 1$, $0 < r << 1$, define the approach manifolds

$$M_{e,r} = \{A_{\rho,s',v}(re^{i\theta}) : |1 - \rho| < \delta, s' \in S, |\theta| < \delta\}.$$ 

In the last but one step we proved $F \in L^1(W)$ by estimating along almost every disc. Similarly, we prove now a uniform $L^1$ bound for the restrictions of $F$ to the approach manifolds.

Namely the Embedding-Theorem of Carleson (s. [JO]) yields a uniform bound

$$\int_{-\delta}^{\delta} |F(A_{\rho,s',v}(re^{i\theta}))| d\theta < C \|f \circ A_{\rho,s',v}\|_{H^1},$$

valid for almost all discs $A_{\rho,s',v}$ with $\partial A_{\rho,s',v} \cap E = \emptyset$ (as usual $\| \cdot \|_{H^1}$ denotes the Hardy-space norm). Integrating over $s'$ gives the desired estimate

$$\int_{M_{e,r}} |F| d\theta ds' < C \|f\|_{L^1(M')},$$

where $M'$ is a sufficiently large neighborhood of $p$ in $M$.

As in the usual theory of Hardy spaces (s. for the theory in several variables the standard reference [ST]) $F$ attains near $p$ a weak boundary $F^*$, which is an integrable CR-function. Obviously, $F^*$ coincides (almost everywhere) with $f$, and the proof is ready.

4. $L^1$-Removability for holomorphic functions

We shall now complete the proof of Theorem 1.1 by showing Theorem 1.2. But before a remark concerning the relation of the two theorems is in order.

**Remark 4.1.** The reader may have noticed that we need in the proof of Theorem 1.1 only a weaker version of Theorem 1.2 where even $H^{2N-1}(L) = 0$ holds true. Under this additional hypothesis some special cases get much simpler:

If $F$ in Theorem 1.2 is even a holomorphic foliation, there is an elementary proof by the continuity principle, which moreover does not use the integrability of $F$. For dimensional reasons the same is true for $N = 2$ (this argument generalizes readily to differential foliations by analytic hypersurfaces). For the general situation and $N \geq 3$ we do not know if a proof working only with holomorphic hulls exists.

**Proof of Theorem 1.2.** We take inspiration from an argument used by Henkin and Tumanov (s. [HT], Lemma 6) to treat a related question for continuous CR-functions on CR-manifolds foliated by complex curves.

Let $E_{nr}$ be the complement of the maximal open subset of $D$ to which $F$ extends holomorphically. We assume $E_{nr} \neq \emptyset$ and have to deduce a contradiction. Our arguments are local and work near any point of $p_0 \in E_{nr}$ which is not an inner point of the set $L_{p_0} \cap E_{nr}$ with respect of the leaf-topology of the leaf $L_{p_0}$ through $p_0$. 

Around \( p_0 \) we may choose a neighborhood \( Q \) with the following properties: Near \( \overline{Q} \) there are coordinates \( w = u + iv \) and \( r = (r_1, \ldots, r_{N-2}) \), where \( w \) is holomorphic and the \( r_j \) of class \( C^2 \), such that \( Q \) is given as \( \{ 0 < u < 1, 0 < v < 1, 0 < r_j < 1, j = 1, \ldots, N - 2 \} \) and the foliation \( \mathcal{F} \) corresponds to the level sets of the mapping \( r \). Contracting \( U \) around \( p_0 \) we may further assume that \( F \) is holomorphic near the bottom \( \{ v = 0, 0 \leq u \leq 1, 0 \leq r_j \leq 1, j = 1, \ldots, N - 2 \} \). Further we may assume the existence of holomorphic coordinates \( z = (z_1, \ldots, z_N) \) near \( \overline{Q} \). It shall be convenient to work with slightly smaller product domain \( Q' = \{ 0 < v < 1 \} \times B' \) where we get \( B' \) by smoothing the edges of the bottom. In the following we will tacitly suppose appropriate contractions of \( Q \) and \( Q' \) around \( p_0 \) which does not destroy the precedingly achieved properties.

We choose near \( \overline{Q} \) a basis \( \overline{l_1}, \ldots, \overline{l_N} \) of complex anti-linear vectorfields whose coefficients are \( C^2 \) with respect to \( (w, r) \) such that \( L_1 \) is tangent to \( \mathcal{F} \). As observed in [HT], the \( \overline{l_1}, \ldots, \overline{l_N} \) may be corrected such that all brackets of the form \([\overline{l_1}, \overline{l_i}]\) are tangent to \( \mathcal{F} \).

Next we take a subdomain \( G \subset Q' \) containing \( p_0 \) which is the region squeezed between the bottom \( B' \) and a smooth hypersurface \( M \) which cuts \( \{ v = 0 \} \) transversely along \( bB' \) and is transverse to the leaves of \( \mathcal{F} \). By Fubini’s Theorem, after a slight deformation of \( M \) the restriction \( F|_M \) may be supposed to be integrable with respect to \( (2N - 1) \)-dimensional volume. We shall show that (after some additional modifications) \( F|_M \) is an integrable CR-function. Afterwards, the usual Hartogs-Bochner-Theorem gives a holomorphic extension of \( F \) to \( G \), in contradiction to the choice of \( p_0 \). Let us rename \( B := B', Q := Q' \).

For technical reasons we have to fix in advance a special approximation of \( F \). Let \( \chi \) be a smooth non-negative, compactly supported, rotation-invariant function of the holomorphic coordinates \( z \) with \( \int \chi dm(z) = 1 \). Take a smooth function \( \eta \geq 0 \) whose support is contained in a small neighborhood of \( \overline{Q} \) and which equals 1 near \( \overline{Q} \). Setting \( \chi_\epsilon(z) = (1/\epsilon)^{2N} \chi(z/\epsilon) \), we define for sufficiently small \( \epsilon > 0 \)

\[
F_\epsilon = (\eta F) * \chi_\epsilon,
\]

where \( * \) denotes convolution with respect to Lebesgue-measure in \( z \). It is standard that \( F_\epsilon \) approximates \( F \) in \( L^1(Q) \). From the mean-value property of holomorphic functions and the rotation-invariance of \( \chi \) we further deduce that, for near any given point \( z \in Q \setminus E_{nr} \), \( F_\epsilon \) will coincide with \( F \) for \( \epsilon \) sufficiently small. In particular, this is true near the bottom \( B \).

We extract a subsequence \( \epsilon_k \searrow 0 \) and claim that after a slight deformation of \( M \) we can assume that the restrictions of \( F_k = F_{\epsilon_k} \) to \( M \) tend in \( L^1(M) \) to \( M \). Indeed, this is a consequence of the following variant of Fubini’s theorem: If we have a series of functions converging in \( L^1 \) on a product set, then their restriction to almost all slices will converge to the restriction of the limit.

Now we can return to the main part of the proof. Fix a point \( q \in M \cap E_{nr} \). We have to show

\[
(2) \quad \int_M F \wedge \overline{\partial} \phi = 0,
\]
for any smooth \((N, N - 2)\)-form \(\phi\) such that \(\text{supp } \phi \cap M\) is contained in some small neighborhood of \(q\) in \(M\). By the very definition of the tangential CR complex, the right side of (2) is not changed if we add to \(\phi\) an \((N, N - 2)\)-form contained in the differential ideal generated by a defining function \(\rho\) of \(M\) and its derivative \(\overline{\partial} \rho\) (s. [BOGG], 8.1). As \(M\) is transverse to \(\mathcal{F}\), we may restrict our attention to more special \(\phi\): Let \(\omega_j\) be a complex anti-linear dual base of \(\mathcal{T}_j\). Then the coefficient of \(\omega_j^1\) in \(\overline{\partial} \rho\) is non-vanishing, and it is enough to consider \(\phi = \sum J \phi J d\omega_j\) such that all coefficients \(\phi_J\) where \(J\) contains 1 are zero (here \(\sum J\) means summation over increasing indices, \(\phi_J\) are \((N, 0)\)-forms).

Fixing such a \(\phi\), we wish to approximate its coefficients (with respect to the base \(\omega_j, \overline{\omega_j}\)) by functions which are holomorphic along the leaves. In order to apply known techniques for approximation in the complex plane we use the holomorphic \(w\)-coordinate and argue fiberwise. As the intersection of \(\text{supp } \phi\) with a leaf \(L_w\) is contained in a short segment \(I_w\), we may approximate the coefficient functions on \(L_w\) by taking convolution integrals over \(I_w\) with holomorphic kernels (similar as for instance in the proof of the Approximation-Theorem of Baouendi and Treves). As the integrals depend smoothly on \(r\) we get that this sequence of approximating functions \(\phi_j\) tends in \(C^2\) to \(\phi\). In particular, \(\phi_j \rightarrow 0\) on \(\partial M\).

The fact that \(\text{supp } \phi_j\) can no longer be assumed to be of compact support seems to cause complications. Nevertheless, it shall be possible to establish

\[
\int_M F \wedge \overline{\partial} \phi_j = \int_{\partial M} F \wedge \phi_j,
\]

which implies (2) by going to the limit, since \(\phi_j \rightarrow 0\) on \(\partial M\).

Now we verify as in [HT] that \(\overline{\partial} F_k \wedge \overline{\partial} \phi_j\) is of the form \(\mathcal{T}_1 F_k \wedge \alpha\) where \(\alpha\) is a \(C^2\) form independent of \(k\) (but of course depending on \(j\)). Indeed, for a selection \(\mathcal{T}_1, \mathcal{T}_{s_1}, \ldots, \mathcal{T}_{s_{N-2}}\) we apply Cartan’s formula

\[
(\mathcal{T}_1, \mathcal{T}_{s_1}, \ldots, \mathcal{T}_{s_{N-2}}) \vdash \overline{\partial} \phi_j = \\
\mathcal{T}_1(\mathcal{T}_{s_1}, \ldots, \mathcal{T}_{s_{N-2}}) \vdash \phi_j + \sum (-1)^{k+1} \mathcal{T}_{s_k}(\mathcal{T}_1, \ldots, \mathcal{T}_{s_{k-1}}, \ldots) \vdash \phi_j \\
+ \sum (-1)^{k+l+1}(\ldots, \mathcal{T}_{s_k}, \mathcal{T}_{s_l}, \ldots) \vdash \phi_j + \sum (-1)^{k+1}(\mathcal{T}_1, \mathcal{T}_{s_k}, \ldots) \vdash \phi_j,
\]

where \(\vdash\) denotes interior multiplication of vectors and forms. By the choice of \(\phi\) and of the \(L_s\) such that \(\mathcal{T}_1, \mathcal{T}_s = a_s \mathcal{T}_1\), all the right-hand terms vanish. Consequently, \((\mathcal{T}_1, \mathcal{T}_{s_1}, \ldots, \mathcal{T}_{s_{N-2}}) \vdash \overline{\partial} \phi_j = 0\), that is to say, if we write \(\overline{\partial} \phi_j = \sum J \beta_j d\omega_j\), all \(\beta_j\) with \(J\) containing 1 vanish identically. Therefore in the wedge product with \(\overline{\partial} F_k = \sum \beta_j \mathcal{T}_1 F_k d\omega_j\) only the term \(\mathcal{T}_1 F_k d\omega_1\) survives.

Consequently we get

\[
\int_M F_k \wedge \overline{\partial} \phi_j = \int_B F_k \wedge \overline{\partial} \phi_j + \int_G \mathcal{T}_1 F_k \wedge \alpha.
\]

So it remains to prove the following.

**Claim.** For fixed \(j\) and for \(k \rightarrow \infty\), the integral \(\int_G \mathcal{T}_1 F_k \wedge \alpha\) tends to zero.
Since by applying Stokes theorem afterwards, we will obtain (3):

$$\int_M F \wedge \overline{\partial} \phi_j = \int_B F \wedge \overline{\partial} \phi_j \frac{\partial M}{\partial B} \int_{\partial M} F \wedge \phi_j \longrightarrow_{j \to \infty} 0.$$  

Proof. Let $R$ be the closed set of all $r^0 = (r^0_1, \ldots, r^0_{N-2})$ such that the holomorphic curve $\{ r = r^0 \}$ has non-void intersection with $E_{nr}$. By a result of Whitney, there is a so called regularized distance function $\delta_R(r)$ which is comparable with the euclidean distance function $\text{dist}(r, R)$ (s. [ST]) and satisfies:

(i) $1/C \text{dist}(r, R) \leq \delta_R(r) \leq C \text{dist}(r, R), \forall r \in B, C \geq 1$;

(ii) $\delta_R(r) \in C^0(B) \cap C^2(B\setminus F)$;

(iii) $|\nabla \delta_R(r)| \leq C$ on $B\setminus F$.

Fix an arbitrary $\epsilon > 0$. Further we take a small $\delta > 0$ whose precise choice shall be explained later. For the moment we only require $\delta$ to be a regular value of $\delta_R$, i.e. to be generic in the sense of the theorem of Sard. Hence $Q \cap \{ \delta_R = \delta \}$ a $C^2$-smooth hypersurface (fibered by leaves of $F$ and with possibly, a number of connected components tending to $\infty$ as $\delta \to 0$.

Depending on $\delta$ we choose a large $k$ such that, for any $p \in Q$ with $\delta_R(z) \geq \delta$, the distance of $p$ to the singularity $E_{nr}$, measured with respect to the holomorphic coordinate $z$, is greater than $\epsilon_k$ (the scaling parameter appearing in the definition of $F_k$).

In the decomposition

$$\int_G \overline{T}_1 F_k \wedge \alpha = \int_{G \cap \{ \delta_R > \delta \}} \overline{T}_1 F_k \wedge \alpha + \int_{G \cap \{ \delta_R < \delta \}} \overline{T}_1 F_k \wedge \alpha$$

the first term on the right is obviously zero since $F$ is holomorphic on $G \cap \{ \delta_R > \delta \}$. For the second term we integrate by parts

$$\int_{G \cap \{ \delta_R < \delta \}} \overline{T}_1 F_k \wedge \alpha = \int_{G \cap \{ \delta_R < \delta \}} F_k \wedge \overline{T}_1 \alpha + \int_{\partial (G \cap \{ \delta_R < \delta \})} \sigma(\overline{T}_1) F_k \wedge \alpha,$$

where $\overline{T}_1$ denotes the formally adjoint operator of $T_1$ and $\sigma(\overline{T}_1)$ the factor from the symbol of $T_1$ appearing in the boundary integral.

We claim that the first summand on the right gets small if $\delta$ is small and $k$ large. Indeed, as $R$ is of $(2N - 2)$-dimensional volume zero and $F$ integrable, there is a $\delta$ such that

$$\int_{Q \cap \{ \delta_R < \delta \}} |F| < \epsilon.$$  

By the theorem of Sard, we may suppose that $\delta$ is a regular value of $\delta_R$, and therefore $Q \cap \{ \delta_R = \delta \}$ a smooth hypersurface. Next we choose $k$ so large that

$$\int_{Q \cap \{ \delta_R < \delta \}} |F_k| < 2\epsilon.$$  

As $|\overline{T}_1 \alpha| < C$ on $\overline{Q}$, we have $|\int_{G \cap \{ \delta_R < \delta \}} F_k \wedge \overline{T}_1 \alpha| \leq 2C \epsilon$. The second term decomposes in three boundary integrals

$$\int_{\partial (G \cap \{ \delta_R < \delta \})} = \int_{G \cap \{ \delta_R = r \}} + \int_{M \cap \{ \delta_R = r \}} + \int_{B \cap \{ \delta_R = r \}}$$
For the first boundary term we remark that
\[ \int_{G \cap \{ \delta_R = \delta \}} \sigma(\mathcal{L}_1) F_k \wedge \alpha = 0, \]
as \( \mathcal{L}_1 \) is tangential to \( G \cap \{ \delta_R = \delta \} \), whence \( \sigma(\mathcal{L}_1) \) vanishes on \( G \cap \{ \delta_R = r \} \). The rest of the boundary integral is estimated in analogous manner as the interior integral. We have only to use that \( F_k |_{G} \to F |_{G} \) in \( L^1 \) and that \( F_k \) coincides with \( F \) near \( B \). After summing up the proof of the claim is finished.

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