We present a dynamical description and analysis of non-equilibrium transitions in the noisy Ginzburg-Landau equation based on a canonical phase space formulation. The transition pathways are characterized by nucleation and subsequent propagation of domain walls or solitons. We also evaluate the Arrhenius factor in terms of an associated action and find good agreement with recent numerical optimization studies.

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Phenomena far from equilibrium are widespread including turbulence in fluids, interface and growth problems, chemical and biological systems, and problems in material science and nanophysics. Here the dynamics of complex systems driven by weak noise, corresponding to rare events, is of particular interest in the context of e.g., nucleation during phase transitions, chemical reactions, and conformational changes in macromolecules. The weak noise limit is associated with a long time scale corresponding to the separation in energy scales of the thermal energy and the energy barriers between metastable states; the transition takes place by sudden jumps between metastable states followed by long waiting times in the vicinity of the states. The fundamental issue is thus the determination of the transition pathways and the associated transition rates.

A particularly interesting non-equilibrium problem of relevance in the nanophysics of switches is the influence of thermal noise on two-level systems with spatial degrees of freedom, see [1, 2, 3]. In a recent paper by E, Ren, and Vanden-Eijden [4] this problem has been addressed using the Ginzburg-Landau equation driven by thermal noise. These authors implement a powerful numerical optimization technique for the determination of the space time configuration minimizing the Freidlin-Wentzell action and in this way determine the orbits and their associated action yielding the switching probabilities in the long time-low temperature limit. The picture that emerges is that the transition probabilities in the weak noise limit are associated with soliton propagation and nucleation resulting from soliton collisions.

In recent work we have addressed a related problem in nonequilibrium physics, namely the Kardar-Parisi-Zhang equation or the equivalent noisy Burgers equation describing for example a growing interface in a random environment. Using a canonical phase space method derived from the weak noise limit of the Martin-Siggia-Rose functional [6, 7] or directly from the Fokker-Planck equation [8, 9], we have in the one dimensional case analyzed the resulting coupled field equations minimizing the action.

In this letter we apply a soliton approach in the canonical phase space formulation to the noisy Ginzburg-Landau equation and attempt to account for some of the numerical results of E et al. We thus give analytical arguments for the propagation of noise induced domain walls or solitons, the nucleation events associated with domain wall creation and annihilation, and the associated time dependent action. Details of our analysis will be given elsewhere.

The noisy Ginzburg-Landau equation for a field \( u(x, t) \) driven by white noise has the form

\[
\frac{\partial u}{\partial t} = -\Gamma \frac{\delta F}{\delta u} + \eta \quad \text{and} \quad \langle \eta(x,t)\eta(0,0) \rangle = \Delta \delta(x)\delta(t),
\]

with free energy

\[
F = \frac{1}{2} \int dx \left( \left( \frac{\partial u}{\partial x} \right)^2 + V(u) \right).
\]
In the switching problem considered by E et al., \( V(u) \) is given by the “Mexican Hat” double well potential
\[
V(u) = k_0^2(1 - u(x)^2)^2.
\]
with strength parameter \( k_0 \). \( \Gamma \) is a kinetic transport coefficient setting the time scale.

The Ginzburg-Landau equation in its deterministic form has been used both in the context of phase ordering kinetics and in its complex form in the study of pattern formation. In the noisy case for a finite system the equation has been studied in [4]; see also an analysis of the related \( \phi^4 \) theory in [5]. In the present problem the noisy equation provides a generalization of the classical Kramers problem [6] to spatially extended systems.

The equation admits a fluctuation-dissipation theorem yielding the stationary distribution \( P_{\text{stat}} \propto \exp[-2F/F] \). The equilibrium states follow from
\[
\delta F/\delta u = -d^2u/dx^2 - 2k_0^2u(1 - u^2) = 0,
\]
giving the two degenerate uniform ground states \( u = \pm 1 \) with \( F = 0 \), as well as nonuniform domain wall solutions
\[
u_{dw}(x) = \pm \tanh k_0(x - x_0), \quad (4)
\]
centered at \( x_0 \), connecting the two ground states. The associated free energy is
\[
F_{dw} = 4k_0^2/3. \quad (5)
\]
Since the spectrum of \( d^2F/du^2 = -d^2/dx^2 - 2k_0^2(1 - 3u^2) \) is positive for the ground states and has zero-eigenvalue Goldstone modes (translation modes) for the domain wall solutions, see e.g. [6], we infer that the free energy landscape possesses two global minima at \( u = \pm 1 \) and a series of local metastable saddle points of free energy \( 4nk_0^2/3 \), corresponding to \( n \) connected domain wall states at positions \( x_i, i = 1, \cdots, n \).

In order to address the issue of noise-driven transitions in the Ginzburg-Landau equation [7] we apply the phase space approach developed for the Kardar-Parisi-Zhang equation [8,9]. This method is based on a weak noise WKB-like approximation, \( P(u(x), t) \propto \exp[-S(u(x), t)/\Delta] \), applied to the Fokker-Planck equation \( \Delta \rho/\Delta t = HP \) for the transition probability \( P(u(x), t), \) driven by the Hamiltonian or Liouvillian \( H = \{ H(u(x)), \{ \partial/\partial u(x) \} \} \). To leading order in \( \Delta \) the action or weight function \( S \) then satisfies the Hamilton-Jacobi equation \( \delta S/\delta u(x) = \delta H/\delta u(x) \). Moreover, the underlying principle of least action \( \delta S = 0 \) yields Hamiltonian equations of motion \( \partial u/\partial t = \delta H/\delta p(x) \) and \( \partial p/\partial t = -\delta H/\delta u(x) \). We note that the equations of motion are identical to the saddle point equations in the Martin-Siggia-Rose functional formulation [10,11].

For the Ginzburg-Landau equation we then obtain explicitly the coupled deterministic field equations
\[
\frac{\partial u}{\partial t} = \Gamma \frac{\partial^2 u}{\partial x^2} + 2\Gamma k_0^2u(1 - u^2) + p, \quad (6)
\]
derived from the Hamiltonian
\[
H = \frac{1}{2} \int dx [p(1 - u^2) + 4k_0^2u(1 - u^2)]. \quad (7)
\]

In the weak noise limit the transition pathways are determined by the orbits in the \((u, p)\) phase space from an initial configuration \( u_1(x) \) at time \( t = 0 \) to a final configuration \( u_2(x) \) at time \( t = T \). The conjugate field \( p \) is a slaved variable representing the noise driving the system. Finally, the transition rate \( P(u_1 \rightarrow u_2, T) \propto \exp[-S/\Delta] \) is determined by the action
\[
S(u_1 \rightarrow u_2, T) = \int_{u_1,0}^{u_2, T} dx dt \left[ p \frac{\partial u}{\partial t} - H \right]. \quad (9)
\]
associated with the orbit.

To linear order, the effect of the noise field \( p \) is to impart a velocity to the static domain wall [12]. This can be seen by expanding \( u \) and \( p \) on the translation mode associated with the static domain wall, \( u_{\text{tm}} = -(1/m)\partial u_{\text{dw}}/\partial x \), where the mass \( m \) is to be determined. Setting \( u = u_{\text{dw}} + u_{\text{tm}} \) and \( p = p_0 u_{\text{tm}} \) and using \( \partial^2 F/\partial u^2 \) \( u_{\text{tm}} = 0 \) we obtain from (8) and (9) \( du_{\text{tm}}/dt = 0 \) and \( dp_0/\partial t = 0 \) with solutions \( u_0 = p_0 t \) and \( p_0 = \text{const.} \). Consequently,
\[
u(x, t) \sim u_{\text{dw}} - \frac{p_0 t}{m} u_{\text{dw}} \sim u_{\text{dw}} \left( x - \frac{p_0 t}{m} \right), \quad (10)
\]
describing a domain wall of mass \( m \) propagating with velocity \( v = p_0/m \). With the above normalization of the translation mode the noise field \( p_0 \) is a momentum contributing to the total momentum \( \Pi = \int dx u \partial p/\partial x \) (generator of translation) and thus canonically conjugate to the position \( x \) of the domain wall. From \( \int dx \cosh^{-4} k_0 x = 4/3k_0 \) we infer the mass \( m \). For the energy and action associated with the propagation of a single domain wall in time \( T \) we then have
\[
E_0 = \frac{p_0^2}{2m}, \quad S_0 = \frac{T p_0^2}{2m}, \quad m = \frac{4k_0}{3}. \quad (11)
\]
This analysis generalizes directly to a dilute gas of connected non-overlapping domain walls. The noise \( p \) then gives rise to individual velocities imparted to each domain wall and the energies, momenta, and actions simply add up.

In addition to the translation modes, time-dependent diffusive modes are also excited, corresponding to small Gaussian fluctuations about the local minima. In the neighborhood of \( u = \pm 1 \) it follows from (8) and (7) that \( u \) and \( p \) in a plane wave decomposition, \( \exp(\pm ik x) \) evolve with time-dependence \( \exp(\pm ik(2 + 4k_0^2)t) \). About the local saddle points corresponding to domain wall propagation, the extended modes are phase shifted corresponding to the trapping of a localized deformation mode with
time dependence $\exp(\pm \Gamma 3k_0^2 t)$ and the translation mode discussed above, see e.g. [3].

The dynamical interpretation of the noise-induced switching in the Ginzburg-Landau equation is now clear. The transition from e.g. $u = +1$ to $u = -1$ starts by nucleating a small region of size $\sim k_0^{-1}$, and then propagating a domain wall or walls, with superposed linear modes subject to energy and momentum conservation and topological constraints, until the whole system is in the $u = -1$ state. The energy of the initial state is given by $E = (1/2) \int dx u^2$, and the noise field thus has to be assigned initially in order to reach the switched state $u = -1$ in a prescribed time $T$. For topological reasons the domain walls must nucleate and annihilate in pairs in general accompanied by absorption or emission of linear modes. Since the linear modes also carry positive action the dynamical modes with lowest action correspond to nucleation or annihilation of domain wall pairs with equal and opposite momenta, i.e., equal speeds.

In the case of periodic boundary conditions the momentum $\Pi = \int dx u \partial p / \partial x$ of the initial and final states is zero. The system is translational invariant and the formation and annihilation of one or several domain wall pairs moving with the same speed take place at equidistant positions along the axis. For fixed boundary conditions the translational invariance is broken and the momentum $\Pi$ is nonvanishing corresponding to nucleation and annihilation of domain walls at the boundaries. This general scenario of switching is completely consistent with the numerical analysis in [4].

Switching a system of size $L$ in time $T$ by means of a single domain wall, corresponding to the pathway via the lowest local saddle point of the free energy at $F_{dw} = m$, the propagation velocity $v = p_0/m = L/T$ and we obtain the action $S_1(T) = mL^2/2T$ and associated transition probability $P \propto \exp(-mL^2/2\Delta T)$. In the thermodynamic limit $L \to \infty$, $P \to 0$ as a result of the broken symmetry in the double well potential. At long times the action falls off as $1/T$. At intermediate times $t$ and positions $x$ we have $P \propto \exp(-mx^2/2\Delta t)$ and we infer that the domain walls in the stochastic interpretation perform a random walk with mean square displacement $2\Delta t/m$, corresponding to diffusive behavior.

In Fig. 1a we show a domain wall nucleating at the left boundary and propagating with constant velocity $v = 1/T$ to the right boundary, where it annihilates. We have used the same parameter values as in [4], i.e., $\delta = \Gamma = 0.3$, $2\Gamma k_0^2 = \delta^{-1}, T = 7$, and a system size $L = 1$. In Fig. 1b we have plotted the trajectory of the domain wall in space and time. The switching can also take place by nucleating two domain walls at the boundaries. These then move at half the velocity $v/2$ and subsequently annihilate at the center. This process corresponds to the pathway via the local saddle point of the free energy at $F_{dw} = 2m$, and the action is given by $S_2(T) = 2S_1(4T)$. Snapshots of this process are shown in Fig. 1c and the corresponding space-time plot in Fig. 1d. Combining the contributions from nucleation and the subsequent domain wall propagation, we can write a heuristic expression for the total action:

$$S_n(T) = nS_{\text{nuc}} + mL^2/2nT,$$

where $S_{\text{nuc}}$ is the action for nucleating a single domain wall and $n$ is the number of walls. The action of nucleation is easily estimated from the Arrhenius factor associated with the Kramers escape from the ground state $u = +1$ to the saddle point in the free energy in [2], i.e., $P \propto \exp(-\Gamma F_{dw}/\Delta)$. We thus obtain a nucleation action $S_{\text{nuc}}$ of order $\Gamma k_0$. A more detailed argument, to be presented elsewhere, based on estimating $S_{\text{nuc}}$ from the equations of motion in [2] and [4] and the action in [4], yields for domain wall nucleation and annihilation

$$S_{\text{nuc}} \sim 6.5\Gamma k_0.$$

In Fig. 2a we have plotted $S$ versus $T$ for $n = 1-6$ domain walls using the parameter values in [4]. Choosing $S_{\text{nuc}}$ according to (13) we find excellent agreement with the numerical results. As also discussed in [4] we note that the switching scenario depends on $T$. At shorter switching times it becomes more favorable to nucleate more domain walls. In the present formulation this feature is associated with the finite nucleation or annihilation action $S_{\text{nuc}}$. This is evidently a finite size effect in the

![Figure 1: In a) the switching from $u = +1$ to $u = -1$ in time $T$ is effectuated by means of a domain wall propagating with velocity $v = 1/T$. The domain wall is nucleated at $x = 0$ and annihilated at $x = 1$. In b) the process is depicted in an $(x, t)$ plot.](image)
FIG. 2: In a) the switching from $u = +1$ to $u = -1$ in time $T$ takes place by means of two domain walls propagating in opposite directions with velocity $v = 1/2T$. The domain walls nucleate at the boundaries and annihilate at the center. In b) the switching process is depicted in an $(x, t)$ plot.

FIG. 3: The action $S(T)$ given by (12) is plotted as a function of $T$ for transition pathways involving up to $n = 6$ domain walls. The lowest action and thus the most probable transition is associated with an increasing number of domain walls at shorter times, indicated by the heavy limiting curve. The curves correspond to choosing $S_{\text{nuc}} = 5\Gamma k_0$.

sense that the action at a fixed $T$ diverges in the thermodynamic limit $L \to \infty$, corresponding to the broken symmetry.

In this letter we have presented a dynamical description and analysis of a specific non equilibrium transition in the noisy Ginzburg-Landau equation based on a canonical phase space formulation. We find good agreement both qualitatively and quantitatively with the numerical finding of E. et al. [4] based on an optimization of the Freidlin-Wentzel action. The dynamical approach offers in the nonperturbative weak noise or low temperature limit an alternative way of determining dynamical pathways and the Arrhenius part of the associated transition rates.

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