TREES, QUATERNION ALGEBRAS AND MODULAR CURVES

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Abstract. We study the action on the Bruhat-Tits tree of unit groups of maximal orders in certain quaternion algebras over \( \mathbb{F}_q(T) \) and discuss applications to arithmetic geometry and group theory.

1. Introduction

Let \( B \) be an indefinite division quaternion algebra over \( \mathbb{Q} \). Let \( \mathcal{O} \subset B \) be a maximal order. Let \( d \) be the discriminant of \( B \); recall that \( d > 1 \) is the product of primes where \( B \) ramifies. Let \( \Gamma^d = \{ \gamma \in \mathcal{O} \mid \text{Nr}(\gamma) = 1 \} \), where \( \text{Nr} \) is the reduced norm of \( B \). Upon fixing an identification of \( B \otimes \mathbb{Q} \mathbb{R} \) with \( \mathbb{M}_2(\mathbb{R}) \), we can view the group \( \Gamma^d \) as a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \). It is well-known that \( \Gamma^d \) acts discontinuously on the upper half-plane \( \mathbb{H} \) and \( X^d := \Gamma^d \backslash \mathbb{H} \) is compact; cf. [11, Ch. 5]. Shimura observed that \( X^d \) parametrizes abelian surfaces with multiplication by \( \mathcal{O} \). From this he was able to show that the algebraic curve \( X^d \) admits a canonical model over \( \mathbb{Q} \). It is hardly surprising that the properties of the algebraic curve \( X^d \) are reflected in the properties of the group \( \Gamma^d \). For example, one can compute the genus of \( X^d \) by computing the hyperbolic volume of a fundamental domain for the action of \( \Gamma^d \) on \( \mathbb{H} \) and the number of elliptic points; cf. [18, Ch. 4]. The transfer of information also goes in the opposite direction, e.g., knowing the genus of \( X^d \) and the relative position of elliptic points on this curve, one deduces a presentation for \( \Gamma^d \). An interesting special case is when \( \Gamma^d \) is generated by torsion elements; this happens only for \( d = 6, 10, 22 \), and one knows a presentation of \( \Gamma^d \) in these cases, cf. [1].

The purpose of the present paper is to study the function field analogues of the groups \( \Gamma^d \). First, we introduce some notation.

Let \( C := \mathbb{P}^1_{\mathbb{F}_q} \) be the projective line over the finite field \( \mathbb{F}_q \). Denote by \( F = \mathbb{F}_q(T) \) the field of rational functions on \( C \). The set of closed points on \( C \) (equivalently, places of \( F \)) is denoted by \( |C| \). For each \( x \in |C| \), we denote by \( \mathcal{O}_x \) and \( F_x \) the completions of \( \mathcal{O}_{C,x} \) and \( F \) at \( x \), respectively. The residue field of \( \mathcal{O}_x \) is denoted by \( \mathbb{F}_x \), the cardinality of \( \mathbb{F}_x \) is denoted by \( q_x \), and \( \deg(x) := [\mathbb{F}_x : \mathbb{F}_q] \). Let \( A := \mathbb{F}_q[T] \) be the ring of polynomials in \( T \) with \( \mathbb{F}_q \) coefficients; this is the subring of \( F \) consisting of functions which are regular away from \( \infty := 1/T \).

Let \( D \) be a quaternion division algebra over \( F \). Let \( R \) be the set of places where \( D \) ramifies (\( R \) is finite and has even cardinality, and conversely, for any choice of a finite set \( R \subset |C| \) of even cardinality there is a unique, up to an isomorphism, quaternion algebra ramified exactly at the places in \( R \); see [18, p. 74]). Denote by \( D^\times \) the

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multiplicative group of $D$. Assume $D$ is split at $\infty$, i.e., $D \otimes F \cong M_2(F)$. Fix a maximal $A$-order $\Lambda$ in $D$. Since $D$ is split at $\infty$, it satisfies the so-called Eichler condition relative to $A$. Since Pic($A$) = 1, this implies that, up to conjugation, $\Lambda$ is the unique maximal $A$-order in $D$, i.e., any other maximal $A$-order in $D$ is of the form $\alpha \Lambda \alpha^{-1}$ for some $\alpha \in D^\times(F)$, cf. [18, Cor. III. 5.7]. We are interested in the group of units of $\Lambda$:

$$\Gamma := \Lambda^\times = \{ \lambda \in \Lambda \mid \text{Nr}(\lambda) \in F^\times \}.$$ 

Via an isomorphism $D^\times(F) \cong GL_2(F)$, the group $\Gamma$ can be considered as a discrete subgroup of $GL_2(F)$. There are two analogues of the Poincaré upper half-plane in this setting. One is the Bruhat-Tits tree $T$ of $PGL_2(F)$, the other is Drinfeld upper half-plane $\Omega := \mathbb{P}^1,_{F} - \mathbb{P}^1,_{\infty}$, an $F_{\infty}$-P($\mathbb{P}^1$), where $\mathbb{P}^1,_{\infty}$ is the rigid-analytic space associated to the projective line over $F_{\infty}$. These two versions of the upper half-plane are related to each other: $\Omega$ has a natural structure of a smooth geometrically connected rigid-analytic space and $T$ is the dual graph of an analytic reduction of $\Omega$, cf. [17]. As a subgroup of $GL_2(F)$, $\Gamma$ acts naturally on both $T$ and $\Omega$ (these actions are compatible with respect to the reduction map). The quotient $\Gamma \setminus \Omega$ is a one-dimensional, connected, smooth analytic space over $F_{\infty}$, which in fact is the rigid-analytic space associated to a smooth projective curve $X_{R}$ over $F_{\infty}$, cf. [17, Thm. 3.3]. The quotient $\Gamma \setminus T$ is a finite graph (Proposition 3.1).

The properties of $\Gamma$, $\Gamma \setminus T$ and $X_{R}$ are intimately related. We explore these relationships to obtain a description of the graph $\Gamma \setminus T$, which includes a formula for the number of vertices of a given degree and a formula for the first Betti number (Theorem 3.4). Next, the description of $\Gamma \setminus T$ translates into a statement about the structure of the group $\Gamma$ (Theorem 3.5). In particular, we determine the cases when $\Gamma$ is generated by torsion elements and find a presentation for $\Gamma$ in those cases (Theorem 3.6); in §5 we find explicitly the torsion units which generate $\Gamma$ in terms of a basis of $D$. Finally, we have an application to the arithmetic of $X_{R}$. From the structure of $\Gamma \setminus T$ and a geometric version of Hensel’s lemma we deduce that $X^R(F_{\infty}) = \emptyset$ if and only if $R$ contains a place of even degree (Theorem 4.4). This last theorem is the function field analogue of a well-known result of Shimura which says that $X^d(R) = \emptyset$, i.e., $X^d$ does not have points which are rational over $R$. [16].

The curve $X_{R}$ parametrizes $D$-elliptic sheaves over $F$ with pole $\infty$, where $D$ is a maximal $O_{\Sigma}$-order in $D$ (the notion $D$-elliptic sheaf generalizes the notion of Drinfeld module). This can be used to show that $X_{R}$ has a model over $F$. As a consequence of Theorem 4.4, if $R$ contains a place of even degree, then $X_{R}$ has no $L$-rational points for any extension $L/F$ which embeds into $F_{\infty}$. In particular, $X_{R}(F) = \emptyset$.

2. Preliminaries

2.1. Graphs. We recall the terminology related to graphs, as presented in [15].

Definition 2.1. An (oriented) graph $\mathcal{G}$ consists of a non-empty set $X = \text{Ver}(\mathcal{G})$, a set $Y = \text{Ed}(\mathcal{G})$ and two maps

$$Y \to X \times X, \quad y \mapsto (o(y), t(y))$$

and

$$Y \to Y, \quad y \mapsto \bar{y}$$
which satisfy the following condition: for each \( y \in Y \) we have \( \bar{y} = y, \bar{y} \neq y \) and \( o(y) = t(\bar{y}) \).

An element \( v \in X \) is called a vertex of \( G \); an element \( y \in Y \) is called an (oriented) edge, and \( \bar{y} \) is called the inverse of \( y \). The vertices \( o(y) \) and \( t(y) \) are the origin and the terminus of \( y \), respectively. These two vertices are called the extremities of \( y \). Note that it is allowed for distinct edges \( y \neq z \) to have \( o(y) = o(z) \) and \( t(y) = t(z) \):

and it is also allowed to have \( y \in Y \) with \( o(y) = t(y) \), in which case \( y \) is called a loop:

We say that two vertices are adjacent if they are the extremities of some edge. We will assume that for any \( v \in X \) the number of edges \( y \in Y \) with \( o(y) = v \) is finite; this number is the degree of \( v \). A vertex \( v \in X \) is called terminal if it has degree 1. A circuit in \( G \) is a collection of edges \( y_{1}, y_{2}, \ldots, y_{m} \in Y \) such that \( y_{i} \neq y_{i+1} \), \( t(y_{i}) = o(y_{i+1}) \) for \( 1 \leq i \leq m-1 \), and \( t(y_{m}) = o(y_{1}) \). A graph \( G \) is connected if for any two distinct vertices \( v, w \in X \) there is a collection of edges \( y_{1}, y_{2}, \ldots, y_{m} \in Y \) such that \( t(y_{1}) = o(y_{m+1}) \) for \( 1 \leq i \leq m-1 \), \( v = o(y_{1}) \) and \( w = t(y_{m}) \).

A connected graph without circuits is called a tree. A geodesic between two vertices \( v, w \) in a tree \( T \) is a collection of edges \( y_{1}, y_{2}, \ldots, y_{m} \in Ed(T) \) such that \( y_{i} \neq y_{i+1} \), \( t(y_{i}) = o(y_{i+1}) \) for \( 1 \leq i \leq m-1 \), and \( v = o(y_{1}) \), \( w = t(y_{m}) \). Since \( T \) is connected and has no circuits, a geodesic between \( v, w \in Ver(T) \) always exists and is unique. The distance \( d(v, w) \) between \( v \) and \( w \) is the number of edges in the geodesic joining \( v \) and \( w \). In particular, \( d(v, w) = 1 \) if and only if \( v \) and \( w \) are adjacent.

A graph is finite if it has finitely many vertices and edges. A finite graph \( G \) can be interpreted as a 1-dimensional \( \Delta \)-complex [5 p. 102]. The first Betti number \( h_{1}(G) \) of \( G \) is the dimension of the homology group \( dim_{\mathbb{Q}} H_{1}(G, \mathbb{Q}) \). (One can show that \( G \) is homotopic to a bouquet of circles; \( h_{1}(G) \) is equal to the number of those circles.)

An automorphism of \( G \) is a pair \( \phi = (\phi_{1}, \phi_{2}) \) of bijections \( \phi_{1} : X \to X \) and \( \phi_{2} : Y \to Y \) such that \( \phi_{1}(o(y)) = o(\phi_{2}(y)) \) and \( \phi_{2}(y) = \phi_{2}(y) \). Let \( \Gamma \) be a group acting on a graph \( G \) (i.e., \( \Gamma \) acts via automorphisms). We say that \( v, w \in X \) are \( \Gamma \)-equivalent if there is \( \gamma \in \Gamma \) such that \( \gamma v = w \); similarly, \( y, z \in Y \) are \( \Gamma \)-equivalent if there is \( \gamma \in \Gamma \) such that \( \gamma y = z \). For \( v \in X \), denote

\[
\Gamma_{v} = \{ \gamma \in \Gamma \mid \gamma v = v \}
\]

the stabilizer of \( v \). Similarly, let \( \Gamma_{y} = \Gamma_{\bar{y}} \) be the stabilizer of \( y \in Y \). \( \Gamma \) acts with inversion if there is \( \gamma \in \Gamma \) and \( y \in Y \) such that \( \gamma y = \bar{y} \). If \( \Gamma \) acts without inversion, then we have a natural quotient graph \( \Gamma \backslash G \) such that \( Ver(\Gamma \backslash G) = \Gamma \backslash Ver(G) \) and \( Ed(\Gamma \backslash G) = \Gamma \backslash Ed(G) \).

**Definition 2.2.** ([13 p.70]) Let \( K \) be a field complete with respect to a discrete valuation \( ord_{K} \). Let \( \mathcal{O} := \{ x \in K \mid ord_{K}(x) \geq 0 \} \) be the ring of integers of \( \mathcal{O} \). Fix
a uniformizer \( \pi \), i.e., an element with \( \text{ord}_K(\pi) = 1 \). Let \( k := \mathcal{O}/\pi\mathcal{O} \) be the residue field. We assume \( k \cong \mathbb{F}_q \) is finite. Let \( V \) be a two-dimensional vector space over \( K \). A \textit{lattice} of \( V \) is any finitely generated \( \mathcal{O} \)-submodule of \( V \) which generates the \( K \)-vector space \( V \); such a module is free of rank 2. If \( \Lambda \) is a lattice and \( x \in K^\times \), then \( x\Lambda \) is also a lattice. We call \( \Lambda \) and \( x\Lambda \) \textit{equivalent lattices}. The action of \( K^\times \) on the set of lattices in \( V \) subdivides this set into disjoint equivalence classes. We denote the class of \( \Lambda \) by \([\Lambda]\).

Let \( T \) be the graph whose vertices \( \text{Ver}(T) = \{[\Lambda]\} \) are the equivalence classes of lattices in \( V \), and two vertices \([\Lambda]\) and \([\Lambda']\) are adjacent if we can choose representatives \( L \in [\Lambda] \) and \( L' \in [\Lambda'] \) such that \( L' \subset L \) and \( L/L' \cong k \). One shows that \( T \) is an infinite tree in which every vertex has degree \((q + 1)\). This is the \textit{Bruhat-Tits tree} of \( \text{PGL}_2(K) \).

Let \( \text{GL}(V) \) denote the group of \( K \)-automorphisms of \( V \); it is isomorphic to \( \text{GL}_2(K) \). One easily verifies that \( \text{GL}(V) \) acts on the Bruhat-Tits tree \( T \) and preserves the distance between any two vertices (although \( \text{GL}(V) \) acts with inversion).

Let \( \text{GL}(V)^0 := \ker \left( \text{ord}_K \circ \det : \text{GL}(V) \to \mathbb{Z} \right) \).

**Lemma 2.3.** Let \( g \in \text{GL}(V)^0 \) and \( v \in \text{Ver}(T) \). Then \( d(v, gv) = 2n \), \( n \geq 0 \).

**Proof.** See [15] p. 75. \( \square \)

Given two linearly independent vectors \( f_1 \) and \( f_2 \) in \( V \), we denote by \([f_1, f_2]\) the similarity class of the lattice \( \mathcal{O}f_1 \oplus \mathcal{O}f_2 \). Fix the standard basis \( e_1 = (1, 0) \), \( e_2 = (0, 1) \) of \( K^2 \). Note that the vertices adjacent to \([e_1, e_2]\) are

\[ [\pi^{-1}e_1, e_2], \quad [\pi e_1, e_2 + ce_1], c \in k. \]

If we express every vector in \( K^2 \) in terms of \( e_1 \) and \( e_2 \), then the action of \( g \in \text{GL}_2(K) \) on \( T \) is explicitly given by

\[ g[ae_1 + be_2, ce_1 + de_2] = [ae_1g + be_2g, ce_1g + de_2g], \]

where \( g \) acts on row vectors \( e_1 \) in the usual manner.

### 2.2. Quaternion algebras.

Given a quaternion algebra \( D \) over \( F \), we denote by \( \alpha \mapsto \alpha' \) the canonical involution of \( D \) [15] p. 1]; thus \( \alpha'' = \alpha \) and \((\alpha\beta)' = \beta'\alpha' \).

The reduced trace of \( \alpha \) is \( \text{Tr}(\alpha) = \alpha + \alpha' \); the reduced norm of \( \alpha \) is \( \text{Nr}(\alpha) = \alpha\alpha' \); the reduced characteristic polynomial of \( \alpha \) is

\[ f(x) = (x - \alpha)(x - \alpha') = x^2 - \text{Tr}(\alpha)x + \text{Nr}(\alpha). \]

**Notation 2.4.** For \( a, b \in F^\times \), let \( H(a, b) \) be the \( F \)-algebra with basis \( 1, i, j, ij \) (as an \( F \)-vector space), where \( i, j \in H(a, b) \) satisfy

- If \( q \) is odd,
  \[ i^2 = a, \quad j^2 = b, \quad ij = -ji; \]
- If \( q \) is even,
  \[ i^2 + i = a, \quad j^2 = b, \quad ij = j(i + 1). \]

**Proposition 2.5.** \( H(a, b) \) is a quaternion algebra. Moreover, any quaternion algebra \( D \) is isomorphic to \( H(a, b) \) for some \( a, b \in F^\times \) (although \( a \) and \( b \) are not uniquely determined by \( D \)).

**Proof.** See pp. 1-5 in [18]. \( \square \)
For $\alpha \in H(a, b)$, there is a unique expression $\alpha = x + yi + zj +wij$. The quadratic form corresponding to the reduced norm on $H(a, b)$ is $Q(x, y, z, w) = \alpha \cdot \alpha'$. In terms of $a$ and $b$, $Q$ is explicitly given by

- If $q$ is odd,
  
  $$Q(x, y, z, w) = x^2 - ay^2 - bz^2 + abw^2;$$

- If $q$ is even,
  
  $$Q(x, y, z, w) = x^2 + xy + ay^2 + b(z^2 + zw + aw^2).$$

One has the following useful criterion for determining whether $H(a, b)$ is split or ramified at a given place.

**Lemma 2.6.** $H(a, b)$ is split at $v \in |C|$ if and only if $Q(x, y, z, w) = 0$ has a non-trivial solution over $F_v$, i.e., there exist $(c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$ such that $Q(c_1, c_2, c_3, c_4) = 0$.

**Proof.** Let $0 \neq \alpha \in H(a, b)_v := H(a, b) \otimes_F F_v$. If $Q$ does not have non-trivial solutions over $F_v$, then $\alpha \cdot \alpha' \in F_v^\times$ and $\alpha'/(\alpha \cdot \alpha')$ is the inverse of $\alpha$. On the other hand, if $Q$ does have non-trivial solutions then there is $\alpha \neq 0$ such that $\alpha \cdot \alpha' = 0$. In this last case $H(a, b)_v$ obviously has zero divisors, so cannot be a division algebra. □

**Corollary 2.7.** If $a, b \in \mathbb{F}_q^\times$, then $H(a, b)$ is split, i.e., is isomorphic to $M_2(F)$.

**Proof.** Using Lemma 2.6 and Hensel’s lemma, $H(a, b)$ is split at $v$ if and only if $Q$ has a non-trivial solution over $F_v$. An easy consequence of Warning’s Theorem [10, Thm. 6.5] is that a quadratic form in $n$ variables over a finite field has a non-trivial solution if $n \geq 3$. Since $Q$ over $F_v$ has 4 variables, it has a non-trivial solution. Hence $H(a, b)$ is split at every $v \in |C|$; this implies $H(a, b) \cong M_2(F)$ [18, p. 74]. □

**Proposition 2.8.** Let $L$ be a finite field extension of $F$. There is an $F$-isomorphism of $L$ onto an $F$-subalgebra of $D$ if and only if $[L : F]$ divides 2 and no place in $R$ splits in $L$. Moreover, any two such $F$-isomorphisms are conjugate in $D$.

**Proof.** See [14, (32.15)]. □

### 3. The Group of Units

We return to the notation and assumptions in the introduction. Thus, $D$ is a quaternion division algebra which is split at $\infty$, $\Lambda$ is a maximal $A$-order in $D$, and $\Gamma := \Lambda^\times = \{\lambda \in \Lambda \mid \text{Nr}(\lambda) \in \mathbb{F}_q^\times\}$.

From now on we denote $K := F_\infty$, $\mathcal{O} := \mathcal{O}_\infty$, $k := F_\infty \cong \mathbb{F}_q$. The group $\Gamma$ can be considered as a discrete subgroup of $\text{GL}_2(K)$ via an embedding

$$\iota : \Gamma \hookrightarrow D^\times(F) \hookrightarrow D^\times(K) \cong \text{GL}_2(K).$$

Note that for $\gamma \in \Gamma$, $\det(\iota(\gamma)) = \text{Nr}(\gamma)$, so $\iota(\Gamma) \subset \text{GL}_2(K)^0$. We fix some embedding $\iota$, and omit it from notation. Being a subgroup of $\text{GL}_2(K)^0$, $\Gamma$ naturally acts on the Bruhat-Tits tree $\mathcal{T}$ of $\text{PGL}_2(K)$, and moreover, by Lemma 2.3, $\Gamma$ acts without inversion.

**Proposition 3.1.** The quotient graph $\Gamma \backslash \mathcal{T}$ is finite.
Proof. It is enough to show that $\Gamma \setminus \mathcal{T}$ has finitely many vertices. The group $\text{GL}_2(K)$ acts transitively on the set of lattices in $K^2$. The stabilizer of $[e_1, e_2]$ is $Z(K) \cdot \text{GL}_2(O)$, where $Z$ denotes the center of $\text{GL}(2)$. This yields a natural bijection $\text{Ver}(T) \cong \text{GL}_2(K)/Z(K) \cdot \text{GL}_2(O)$, and also

$$\text{Ver}(\Gamma \setminus \mathcal{T}) \cong \Gamma \setminus \text{GL}_2(K)/Z(K) \cdot \text{GL}_2(O).$$

We will show that the above double coset space is finite. Denote by $\Lambda$ the adele ring of $F$. Consider the group $D^\times(F)$ embedded diagonally into $D^\times(\Lambda)$. Since $D$ is a division algebra, $D^\times(F) \setminus D^\times(\Lambda)/Z(K)$ is compact (cf. [18 Ch. III.1]). Denote $D^\times_y := \prod_{x \in |\mathcal{T}|} D^\times(\mathcal{T}_x)$. Since $A$ is a principal ideal domain, the strong approximation theorem for $D^\times$ yields (cf. [18 Ch. III.4])

$$D^\times(\Lambda) \cong D^\times(F) \cdot \text{GL}_2(K) \cdot D^\times_y.$$ 

Note that $\Gamma = D^\times(F) \cap D^\times_y$. Thus, $\Gamma \setminus \text{GL}_2(K)/Z(K)$ is compact since it is the image of $D^\times(F) \setminus D^\times(\Lambda)/Z(K)$ under the natural quotient map $D^\times(\Lambda) \to D^\times(\Lambda)/D^\times_y$. Finally, since $\text{GL}_2(O)$ is open in $\text{GL}_2(K)$, $\Gamma \setminus \text{GL}_2(K)/Z(K) \cdot \text{GL}_2(O)$ is finite. □

Proposition 3.2. Let $v \in \text{Ver}(T)$ and $y \in \text{Ed}(T)$. Then $\Gamma_v \cong F_q^\times$ or $\Gamma_v \cong F_{q^n}^\times$, and $\Gamma_y \cong F_q^\times$.

Proof. By choosing an appropriate basis of $K^2$, we can assume $v = [e_1, e_2]$. The stabilizer of $v$ in $\text{GL}_2(K)^0$ is $\text{GL}_2(O)$. Since this last group is compact in $\text{GL}_2(K)$, whereas $\Gamma$ is discrete, $\Gamma_v = \Gamma \cap \text{GL}_2(O)$ is finite. In particular, if $\gamma \in \Gamma_v$, then $\gamma^n = 1$ for some $n \geq 1$. We claim that the order $n$ of $\gamma$ is coprime to the characteristic $p$ of $F$. Indeed, if $p|n$ then $(\gamma^n/p - 1) \in D$ is non-zero but $(\gamma^n/p - 1)^p = 0$. This is not possible since $D$ is a division algebra. Consider the subfield $F(\gamma)$ of $D$ generated by $\gamma$ over $F$. By Proposition 2.3, $[F(\gamma) : F] = 1$ or $2$. Since $\gamma \in D$ is algebraic over $F_q$, we conclude that $[F_q(\gamma) : F_q] = 1$ or 2.

It is obvious that $F_q^\times \subset \Gamma_v$. Assume there is $\gamma \in \Gamma_v$ which is not in $F_q^\times$. From the previous paragraph, $\gamma$ generates $F_q^\times$ over $F_q$. Considering $\gamma$ as an element of $\text{GL}_2(O)$, we clearly have $a + b\gamma \in M_2(O)$ for $a, b \in F_q$ (embedded diagonally into $\text{GL}_2(K)$). But if $a$ and $b$ are not both zero, then $a + b\gamma \in \Lambda$ is invertible, hence belongs to $\Gamma$ and $\text{GL}_2(O)$. We conclude that $F_q(\gamma)^\times \cong F_q^\times \subset \Gamma_v$, and moreover, every element of $\Gamma_v$ is of order dividing $q^2 - 1$. Suppose there is $\delta \in \Gamma_v$ which is not in $F_q(\gamma)^\times$. Since $\delta$ is algebraic over $F_q$, $\delta$ and $\gamma$ do not commute in $D$ (otherwise $F(\gamma, \delta)$ is a subfield of $D$ of degree $q > 2$ over $F$). Then $\Gamma_v / F_q^\times$ is a finite subgroup of $\text{PGL}_2(K)$ whose elements have orders dividing $q + 1$ and which contains two non-commuting elements of order $(q + 1)$. This contradicts Dickson’s classification of finite subgroups of $\text{PGL}_2(K)$ [21 II.8.27].

Now consider $\Gamma_y$. Clearly $F_q^\times \subset \Gamma_y$. Let $v$ and $w$ be the extremities of $y$. Note that there are natural inclusions $\Gamma_v \subset \Gamma_y$, $\Gamma_y \subset \Gamma_w$ and $\Gamma_y = \Gamma_v \cap \Gamma_w$. If $\Gamma_y$ is strictly larger than $F_q^\times$, then from the discussion on the stabilizers of vertices, we have $\Gamma_y = \Gamma_w \cong F_{q^2}^\times$ (an equality of subgroups of $\Gamma$). Therefore, $\Gamma_y \cong F_{q^2}^\times$. On the other hand, the stabilizer of $y$ in $\text{GL}_2(K)^0$ is isomorphic to the Iwahori subgroup $I$ of $\text{GL}_2(O)$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{\pi}$. Since $I$ does not contain a subgroup isomorphic to $F_{q^2}^\times$, we get a contradiction. □
Corollary 3.3. Let $v \in \text{Ver}(\mathcal{T})$ be such that $\Gamma_v \cong \mathbb{F}_q^\times$. Then $\Gamma_v$ acts transitively on the vertices adjacent to $v$.

Proof. By Proposition 3.2, a subgroup of $\mathbb{F}_q^\times$ which stabilizes an edge with origin $v$ is $\mathbb{F}_q^\times$. Hence $\Gamma_v/\mathbb{F}_q^\times$ acts freely on the set of vertices adjacent to $v$. Since this quotient group has $q+1$ elements, which is also the number of vertices adjacent to $v$, it has to act transitively.

We introduce a function which will simplify the notation in our later discussions:

$$\varphi(R) = \begin{cases} 
0, & \text{if some place in } R \text{ has even degree;} \\
1, & \text{otherwise.}
\end{cases}$$

Let

$$g(R) = 1 + \frac{1}{q^2 - 1} \prod_{x \in R} (q_x - 1) - \frac{q}{q + 1} \cdot 2^{#R-1} \cdot \varphi(R).$$

Theorem 3.4.

1. The graph $\Gamma \setminus \mathcal{T}$ has no loops;
2. $h_1(\Gamma \setminus \mathcal{T}) = g(R)$;
3. Every vertex of $\Gamma \setminus \mathcal{T}$ is either terminal or has degree $q+1$.
4. Let $V_1$ and $V_{q+1}$ be the number of terminal and degree $q+1$ vertices of $\Gamma \setminus \mathcal{T}$, respectively. Then

$$V_1 = 2^{#R-1} \varphi(R) \quad \text{and} \quad V_{q+1} = \frac{1}{q-1}(2g(R) - 2 + V_1).$$

Proof. The graph $\Gamma \setminus \mathcal{T}$ has no loops since adjacent vertices of $\mathcal{T}$ are not $\Gamma$-equivalent, as follows from Lemma 2.3.

By Theorem 4.2, $\Gamma \setminus \mathcal{T}$ is the dual graph of $X^R$ (see 4.3 for notation). Using 4.6, one concludes that the arithmetic genus of $X^R$ is equal to $h_1(\Gamma \setminus \mathcal{T})$. Now by the flatness of $X^R$, the arithmetic genus of $X^R$ is equal to the genus of $X^R$. Finally, the genus of $X^R$ is equal to $g(R)$ by [12, Thm. 5.4]. Overall, we get $h_1(\Gamma \setminus \mathcal{T}) = g(R)$.

Let $v \in \text{Ver}(\mathcal{T})$. By Proposition 3.2, $\Gamma_v \cong \mathbb{F}_q^\times$ or $\mathbb{F}_q^\times$. In the second case, by Corollary 3.3, the image of $v$ in $\Gamma \setminus \mathcal{T}$ is a terminal vertex. Now assume $\Gamma_v \cong \mathbb{F}_q^\times$. We claim that the image of $v$ in $\Gamma \setminus \mathcal{T}$ has degree $q+1$. Let $e, y \in \text{Ed}(\mathcal{T})$ be two distinct edges with origin $v$. It is enough to show that $e$ is not $\Gamma$-equivalent to $y$ or $y$. On the one hand, $e$ cannot be $\Gamma$-equivalent to $y$ since $\Gamma_v \cong \mathbb{F}_q^\times$ stabilizes every edge with origin $v$. On the other hand, if $e$ is $\Gamma$-equivalent to $y$ then $v$ is $\Gamma$-equivalent to an adjacent vertex, and that cannot happen.

Let $S$ be the set of terminal vertices of $\Gamma \setminus \mathcal{T}$. Let $G$ be the set of $\Gamma$ classes of subgroups of $\Gamma$ isomorphic to $\mathbb{F}_q^\times$. We claim that there is a bijection $\varphi : S \to G$ given by $\tilde{v} \mapsto \Gamma_v$, where $v$ is a preimage of the terminal vertex $\tilde{v} \in S$. The map is well-defined since if $w$ is another preimage of $\tilde{v}$ then $v = \gamma w$ for some $\gamma \in \Gamma$, and so $\Gamma_w = \gamma^{-1} \Gamma_v \gamma$ is a conjugate of $\Gamma_v$. If $\varphi$ is not injective, then there are two vertices $v, w \in \text{Ver}(\mathcal{T})$ such that $\Gamma_v \cong \mathbb{F}_q^\times$, $\Gamma_w \cong \gamma^{-1} \Gamma_v \gamma$ for some $\gamma \in \Gamma$, but $v$ and $w$ are not in the same $\Gamma$-orbit. Then $\Gamma_{\gamma v} = \Gamma_w \gamma^{-1} = \Gamma_v$, but $\gamma v \neq v$. The geodesic connecting $v$ to $\gamma w$ is fixed by $\Gamma_v$, so every edge on this geodesic has stabilizer equal to $\Gamma_v$. This contradicts Proposition 3.2. Finally, to see that $\varphi$ is surjective it is enough to show that every torsion element in $\Gamma$ fixes some vertex. Suppose $\gamma \in \Gamma$ does not fix any vertices in $\mathcal{T}$. Since $\gamma$ acts without inversion, a
result of Tits [15] Prop. 24, p. 63 implies that there is a straight path \( P \) in \( T \) on which \( \gamma \) induces a translation of amplitude \( m \geq 1 \). If \( \gamma \) is torsion, then for some \( n \geq 2 \), \( \gamma^n = 1 \) induces a translation of amplitude \( mn \) on \( P \), which is absurd.

Let \( A := \mathbb{F}_q[T] \) and \( L := \mathbb{F}_q^\infty F \) (note that \( A \) is the integral closure of \( A \) in \( L \)). If \( q \) is odd, let \( \xi \) be a fixed non-square in \( \mathbb{F}_q \). If \( q \) is even, let \( \xi \) be a fixed element of \( \mathbb{F}_q \) such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\xi) = 1 \) (such \( \xi \) always exists, cf. [10] Thm. 2.24). Consider the polynomial \( f(x) = x^2 - \xi \) if \( q \) is odd, and \( f(x) = x^2 + x + \xi \) if \( q \) is even. Note that \( f(x) \) is irreducible over \( \mathbb{F}_q \); this is obvious for \( q \) odd, and follows from [10] Cor. 3.79 for \( q \) even. Thus, a solution of \( f(x) = 0 \) generates \( \mathbb{F}_q^\infty \) over \( \mathbb{F}_q \). Denote by \( P \) the set of \( \Gamma \)-conjugacy classes of elements of \( \Lambda \) whose reduced characteristic polynomial is \( f(x) \). By a theorem of Eichler (Cor. 5.12, 5.13, 5.14 on pp. 94-96 of [14]),

\[
\#P = h(A) \prod_{x \in R} \left( 1 - \left( \frac{L}{x} \right) \right),
\]

where \( h(A) \) is the class number of \( A \) and \( \left( \frac{L}{x} \right) \) is the Artin-Legendre symbol:

\[
\left( \frac{L}{x} \right) = \begin{cases} 
1, & \text{if } x \text{ splits in } L; \\
-1, & \text{if } x \text{ is inert in } L; \\
0, & \text{if } x \text{ ramifies in } L.
\end{cases}
\]

Note that a place of even degree splits in \( L \), and a place of odd degree remains inert, and since \( h(A) = 1 \), this implies \( \#P = 2^{\# R^\times} P(R) \).

If \( \lambda \in \Lambda \) is an element with reduced characteristic polynomial \( f(x) \), then it is clear that \( \lambda \in \Gamma \) and \( \mathbb{F}_q(\lambda)^\times \subset \Gamma \) is isomorphic to \( \mathbb{F}_q^\times \). Hence \( \lambda \mapsto \mathbb{F}_q(\lambda)^\times \) defines a map \( \chi : P \to G \). As before, it is easy to check that \( \chi \) is well-defined and surjective. We will show that \( \chi \) is 2-to-1, which implies the formula for \( \chi \). Obviously \( \lambda \neq \lambda' \) and these elements generate the same subgroup in \( \Gamma \), as the canonical involution on \( D \) restricted to \( F(\lambda) \) is equal to the Galois conjugation on \( F(\lambda)/F \). Since \( \lambda \) and \( \lambda' \) are the only elements in \( \mathbb{F}_q(\lambda) \) with the given characteristic polynomial, it is enough to show that \( \lambda \) and \( \lambda' \) are not \( \Gamma \)-conjugate. Suppose there is \( \gamma \in \Gamma \) such that \( \lambda' = \gamma \lambda \gamma^{-1} \). One easily checks that \( 1, \lambda, \gamma, \gamma \lambda \) are linearly independent over \( F \), hence generate \( D \). If \( q \) is odd, then \( \lambda' = -\lambda \). If \( q \) is even, then \( \lambda' = \lambda + 1 \). Using this, one easily checks that \( \gamma^2 \) commutes with \( \lambda \), e.g., for \( q \) odd:

\[
\gamma^2 \lambda \gamma^{-2} = \gamma \lambda' \gamma^{-1} = -\gamma \lambda \gamma^{-1} = -\lambda' = \lambda.
\]

Hence \( \gamma^2 \) lies in the center of \( D \), and therefore, \( \gamma^2 = b \in \mathbb{F}_q^\times \). Looking at the relations between \( \lambda \) and \( \gamma \), we see that \( D \) is isomorphic to \( H(\xi, b) \). This last quaternion algebra is split according to Corollary [2.7] which leads to a contradiction.

The number of edges \( E = \# \text{Ed}(\Gamma \setminus T) \), ignoring the orientation, is equal to

\[
E = (V_1 + (q + 1)V_{q+1})/2.
\]

By Euler’s formula, \( E + 1 = q(R) + V_1 + V_{q+1} \). This implies the expression for \( V_{q+1} \), and finishes the proof of the theorem.

The next theorem is the group-theoretic incarnation of Theorem 3.5.

**Theorem 3.5.** Let \( \Gamma_{\text{tor}} \) be the normal subgroup of \( \Gamma \) generated by torsion elements.

1. \( \Gamma / \Gamma_{\text{tor}} \) is a free group on \( g(R) \) generators.
2. If \( \phi(R) = 0 \), then \( \Gamma_{\text{tor}} = \mathbb{F}_q^\times \).
(3) If \( \wp(R) = 1 \), then the maximal finite order subgroups of \( \Gamma \) are isomorphic to \( \mathbb{F}_{q^2}^\times \), and, up to conjugation, \( \Gamma \) has \( 2^{#R-1} \) such subgroups.

**Proof.** By \[15\] Cor. 1, p.55, \( \Gamma/\Gamma_{\text{tor}} \) is the fundamental group of the graph \( \Gamma \setminus T \). The topological fundamental group of any finite graph is a free group. Hence \( \Gamma/\Gamma_{\text{tor}} \) is a free group. The number of generators of this group is equal to \( \dim_\mathbb{Q} H_1(\Gamma \setminus T, \mathbb{Q}) \). This proves (1). Parts (2) and (3) follow from the proofs of Proposition 3.2 and Theorem 3.4. \( \square \)

**Theorem 3.6.** \( \Gamma = \Gamma_{\text{tor}} \) if and only if one of the following holds:

1. \( R = \{ x, y \} \) and \( \{ \deg(x), \deg(y) \} = \{ 1, 2 \} \). In this case, \( \Gamma \) has a presentation
   \[
   \Gamma \cong \langle \gamma_1, \gamma_2 \mid \gamma_1^{q^2-1} = \gamma_2^{q^2-1} = 1, \gamma_1^{q+1} = \gamma_2^{q+1} \rangle.
   \]
2. \( q = 4 \) and \( R \) consists of the four degree-1 places in \( |C| - \infty \). In this case, \( \Gamma \) has a presentation
   \[
   \Gamma \cong \langle \gamma_1, \ldots, \gamma_8 \mid \gamma_1^{q^2-1} = \cdots = \gamma_8^{q^2-1} = 1, \gamma_1^{q+1} = \cdots = \gamma_8^{q+1} \rangle.
   \]

**Proof.** By Theorem 3.5, \( \Gamma = \Gamma_{\text{tor}} \) if and only if \( g(R) = 0 \). From the formula for \( g(R) \) one easily concludes that \( g(R) = 0 \) exactly in the two cases listed in the theorem. In Case (1), according to Theorem 3.4, \( V_1 = 2 \) and \( V_{q+1} = 0 \), so \( \Gamma \setminus T \) is an edge:

In Case (2), \( V_1 = 8 \) and \( V_{q+1} = 2 \), so \( \Gamma \setminus T \) is the tree:

\[ \text{The knowledge of the quotient tree } \Gamma \setminus T \text{ allows to reconstruct } \Gamma \text{ (15, I.4.4)}: \Gamma \text{ is the graph of groups } \Gamma \setminus T, \text{ where each terminal vertex of } \Gamma \setminus T \text{ is labeled by } \mathbb{F}_{q^2}^\times, \text{ each non-terminal vertex is labeled by } \mathbb{F}_q^\times, \text{ and each edge is labeled by } \mathbb{F}_q^\times (\text{the monomorphisms } \Gamma_y \to \Gamma_{\ell(y)} \text{ are the natural inclusions } \mathbb{F}_q^\times \to \mathbb{F}_q^\times); \text{ see [15] p. 37]. In other words, } \Gamma \text{ is the amalgam of the groups labeling the vertices of } \Gamma \setminus T \text{ along the subgroups labeling the edges. The presentation for } \Gamma \text{ follows from the definition of amalgam; see [15, I.1].} \square \]

Theorem 3.4 allows to determine \( \Gamma \setminus T \) in some other cases, besides the case when \( \Gamma \setminus T \) is a tree treated in Theorem 3.6.

**Theorem 3.7.** Suppose \( R = \{ x, y \} \) and \( \{ \deg(x), \deg(y) \} = \{ 1, 2 \} \). Then \( \Gamma \setminus T \) is the graph which has 2 vertices and \( q + 1 \) edges connecting them:
Proof. From Theorem 3.4 $V_1 = 0$ and $V_{q+1} = 2$. This implies the claim.

The example in Theorem 3.7 is significant for arithmetic reasons. Assume $q$ is odd. As is shown in [13], the curve $X_R$ is hyperelliptic if and only if $R = \{x, y\}$ and $\{\deg(x), \deg(y)\} = \{1, 2\}$. Thus, Theorems 3.7 and 4.1 imply that the closed fibre over $\infty$ of the minimal regular model over Spec$(\mathcal{O})$ of a hyperelliptic $X_R$ consists of two projective lines $\mathbb{P}^1_{\mathbb{F}_q}$ intersecting transversally at their $q + 1 \mathbb{F}_q$-rational points. This can be used to determine the group of connected components $\Phi_\infty$ of the closed fibre of the Néron model of the Jacobian of $X_R$ over Spec$(\mathcal{O})$. Using a result of Raynaud (cf. [5] p. 283), one obtains $\Phi_\infty \cong \mathbb{Z}/(q + 1)\mathbb{Z}$.

4. MODULAR CURVES

GL$_2(K)$ acts on Drinfeld’s upper half-plane $\Omega$ by linear fractional transformations. As we discussed in the introduction, the quotient $\Gamma \setminus \Omega$ is the underlying rigid-analytic space of a smooth, projective curve $X_R$ over $K$. The genus of this curve is computed in [12] using arithmetic methods, and it turns out to be equal to $g(R)$. The theory of Mumford curves allows to construct a model of $X_R$ over Spec$(\mathcal{O})$.

Theorem 4.1. There is a scheme $\mathcal{X}^R$ over Spec$(\mathcal{O})$ which is proper, flat and regular, and whose generic fibre $\mathcal{X}^R_k := \mathcal{X}^R \times_\mathcal{O} \text{Spec}(K)$ is isomorphic to $X^R$. The geometric special fibre $\mathcal{X}^R \bar{k} := \mathcal{X}^R \times_\mathcal{O} \text{Spec}(\bar{k})$ is reduced, connected and has at most ordinary double points, where $\bar{k}$ denotes the algebraic closure of $k$. The normalization of components of $\mathcal{X}^R_k$ are $k$-rational curves, and the double points of $\mathcal{X}^R_k$ are $k$-rational with two $k$-rational branches. Moreover, $\Gamma \setminus \mathcal{T}$ is the dual graph of $\mathcal{X}^R_k$: this means that the irreducible components $E$ of $\mathcal{X}^R_k$ and the double points $x$ of $\mathcal{X}^R_k$ are naturally in bijection with the vertices $v$ and the edges $\{y, \bar{y}\}$ of $\Gamma \setminus \mathcal{T}$ respectively, such that $x$ is contained in $E$ if and only if $v = o(y)$ or $t(y)$.

Proof. This follows from [9] Prop. 3.2 after making the following two observations: (1) since $\Gamma$ acts without inversion of $\mathcal{T}$, the graph $(\Gamma \setminus \mathcal{T})^*$ in [9] is $(\Gamma \setminus \mathcal{T})$ itself; (2) for any $y \in \text{Ed}(\Gamma \setminus \mathcal{T})$ the image of the stabilizer $\Gamma_y$ is trivial in PGL$_2(F_\infty)$ by Proposition 3.2, so the length of $y$ (in Kurihara’s terminology) is 1.

Remark 4.2. $X^R$ is not necessarily the minimal regular model of $X^R$ over Spec$(\mathcal{O})$; one obtains the minimal model by removing the terminal vertices from $\Gamma \setminus \mathcal{T}$.

Next, we recall Jordan-Livně’s geometric version of Hensel’s lemma:

Theorem 4.3. Let $\mathcal{X}$ be a proper, flat and regular scheme over $\mathcal{O}$. Let $\mathcal{X}_K := \mathcal{X} \times_\mathcal{O} \text{Spec}(K)$ be the generic fibre and $\mathcal{X}_k := \mathcal{X} \times_\mathcal{O} \text{Spec}(k)$ be the special fibre. Then $\mathcal{X}_K$ has a $K$-rational point if and only if $\mathcal{X}_k$ has a smooth $k$-rational point.

Proof. See [7] Lem. 1.1.

Theorem 4.4. $X^R(K) \neq \emptyset$ if and only if $\phi(R) = 1$.

Proof. By Theorem 4.1 $X^R$ is the generic fibre of $\mathcal{X}^R$ which is proper, flat and regular over Spec$(\mathcal{O})$. Hence by Theorem 4.3 $X^R$ has a $K$-rational point if and only if $\mathcal{X}^R_k$ has a smooth $k$-rational point. The irreducible components of $\mathcal{X}^R_k$ are $\mathbb{P}^1_k$’s intersecting transversally at $k$-rational points, and exactly two components
pass through a singularity (note that the irreducible components do not have self-intersections, as \( \Gamma \setminus \mathcal{T} \) has no loops.) Since \( \# \mathcal{P}_1(k) = q + 1 \), \( \mathcal{X}_q^R \) has a smooth \( k \)-rational point if and only if there is an irreducible component whose corresponding vertex in \( \Gamma \setminus \mathcal{T} \) has degree less than \( q + 1 \). By Theorem 3.4 \( \Gamma \setminus \mathcal{T} \) has a vertex of degree less than \( q + 1 \) if and only if \( \varphi(R) = 1 \).

5. Explicit generators

Over \( \mathbb{Q} \) one knows not only the cases when \( \Gamma^d \) is generated by torsion elements, but also the explicit description of the generators of \( \Gamma^d \) in terms of a basis of \( D \); cf. [1] p. 92 or [8]. For example, \( \Gamma^6 \) is isomorphic to the subgroup of \( \text{SL}_2(\mathbb{R}) \) generated by

\[
\gamma_1 = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 2 - \sqrt{2} \\ -6 - 3\sqrt{2} & -\sqrt{2} \end{pmatrix}, \quad \gamma_2 = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -2 + \sqrt{2} \\ 6 + 3\sqrt{2} & -\sqrt{2} \end{pmatrix}, \\
\gamma_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}, \quad \gamma_4 = \frac{1}{2} \begin{pmatrix} 1 & 3 - 2\sqrt{2} \\ -9 - 6\sqrt{2} & 1 \end{pmatrix}
\]

which have orders 4, 4, 6, 6, respectively.

In this section we will find explicit generators for \( \Gamma \) in Case (1) of Theorem 3.6. As a consequence of our calculations, we will also obtain in this case a direct proof of Theorem 3.4 for odd \( q \).

First, we explicitly describe \( D \) in terms of generators and relations when \( \varphi(R) = 1 \), and then describe a maximal \( A \)-order in \( D \). For each \( x \in |C| - \infty \), denote by \( \mathfrak{p}_x \prec A \) the corresponding prime ideal of \( A \). Let \( \mathfrak{r} \) be the monic generator of the ideal \( \prod_{x \in R} \mathfrak{p}_x \) (this is the discriminant of \( D \)).

**Lemma 5.1.** Assume \( \varphi(R) = 1 \). If \( q \) is odd, let \( \xi \in \mathbb{F}_q \) be a fixed non-square. If \( q \) is even, let \( \xi \in \mathbb{F}_q \) be a fixed element such that \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\xi) \neq 0 \). Then \( H(\xi, \mathfrak{r}) \cong D \).

**Proof.** It is enough to check that \( H \) is ramified exactly at the places in \( R \). For this one can use the same argument as in the proof of Corollary 2.7. Reducing the quadratic form \( Q \mod \mathfrak{p}_v \) for every \( v \in |C| - \infty \), one easily checks that \( H(\xi, \mathfrak{r}) \) is ramified at every place in \( R \) and is split at every place in \( |C| - R - \infty \): Note that \( x^2 - \xi y^2 = 0 \) (resp. \( x^2 + xy + \xi y^2 = 0 \)) has no non-trivial solutions over \( \mathbb{F}_v \), \( v \in R \), when \( q \) is odd (resp. even), since the corresponding quadratic has no roots over \( \mathbb{F}_q \) due to the choice of \( \xi \) and \( [\mathbb{F}_v : \mathbb{F}_q] \) is odd by assumption. Finally, \( H \) is automatically split at \( \infty \) since the number of places where a quaternion algebra ramifies must be even.

From now on we assume that \( \varphi(R) = 1 \) and identify \( D \) with \( H(\xi, \mathfrak{r}) \).

**Lemma 5.2.** The free \( A \)-module \( \Lambda \) in \( D \) generated by

\[
x_1 = 1, \quad x_2 = i, \quad x_3 = j, \quad x_4 = ij
\]

is a maximal order.

**Proof.** It is obvious that \( \Lambda \) is an order. To show that it is maximal we compute its discriminant, i.e., the ideal of \( A \) generated by \( \text{det}(\text{Tr}(x_i x_j))_{ij} \). When \( q \) is odd

\[
\text{det}(\text{Tr}(x_i x_j))_{ij} = \text{det} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\xi & 0 & 0 \\ 0 & 0 & 2\mathfrak{r} & 0 \\ 0 & 0 & 0 & -2\xi \end{pmatrix} = -16\xi^2 \mathfrak{r}^2,
\]
when \( q \) is even

\[
\det(\text{Tr}(x_ix_j))_{ij} = \det \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & r \\
0 & 0 & r & 0
\end{pmatrix} = r^2.
\]

In both cases, the discriminant of \( \Lambda \) is \( \prod_{x \in R} \mathbb{P}_2^2 \), and this implies that \( \Lambda \) is maximal, cf. [18, pp. 84-85]. \( \Box \)

From Lemma 5.2 we see that finding the elements \( \lambda = a + bi + cj + dj \in D \) which lie in \( \Lambda \) is equivalent to finding \( a, b, c, d, \in A \) such that

\[
(a^2 - \xi b^2) - r(c^2 - \xi d^2) \in \mathbb{F}_q^\times \quad \text{if } q \text{ is odd}
\]

\[
a^2 + ab + \xi b^2 + r(c^2 + cd + \xi d^2) \in \mathbb{F}_q^\times \quad \text{if } q \text{ is even},
\]

(this is the condition \( \text{Nr}(\lambda) \in \mathbb{F}_q^\times \) written out explicitly). We are particularly interested in torsion elements of \( \Gamma \), hence instead of looking for general units, we will try to find elements in \( \Lambda \) which are algebraic over \( \mathbb{F}_q^\times \). Since the actual calculations differ for "\( q \) even" and "\( q \) odd", we have to treat these cases separately, but first we make the following simplifying observation.

If \( R \) consists of two rational places, then without loss of generality we can assume \( r = T(T - 1) \). Indeed, \( F \) is the function field of \( \mathbb{P}_1^1 \), so \( \text{Aut}(F/\mathbb{F}_q) \cong \text{PGL}_2(\mathbb{F}_q) \), where the matrix \( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \) acts by \( T \mapsto \frac{xT + y}{zT + w} \). It is well-known that there is a linear fractional transformation which fixes \( \infty \) and maps any two given \( \mathbb{F}_q \)-rational points of \( \mathbb{P}_1^1(\mathbb{F}_q) \) to 0 and 1. The action of \( \text{Aut}(F/\mathbb{F}_q) \) does not affect the structure of \( \Gamma \setminus T \) (but of course, quaternion algebras ramified at different sets of places are not isomorphic as \( A \)-algebras).

**\( q \) is odd.** Consider the equation \( \gamma^2 = \xi \) in \( \Lambda \). If we write \( \gamma = a + bi + cj + dj \), then

\[
\gamma^2 = a^2 + b^2\xi + c^2r - d^2\xi r + 2(abi + acj + adij).
\]

Therefore, \( \gamma^2 = \xi \) is equivalent to

\[
(5.1) \quad a = 0 \quad \text{and} \quad b^2\xi + c^2r - d^2\xi r = \xi.
\]

A possible solution is \( b = 1, d = c = 0 \). This gives the obvious \( \theta_1 = i \) as a torsion unit.

Now let \( r = T(T - 1) \). One easily checks that

\[
b = 2T - 1, \quad c = 0, \quad d = 2
\]

satisfies (5.1). Thus, \( \theta_2 = (2T - 1)i + 2ij \) is a torsion unit.

Next, we study the action of \( \theta_1, \theta_2 \) on \( T \). Since \( r = T(T - 1) \) has even degree and is monic, \( \sqrt{r} \in K \). The map

\[
i \mapsto \begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{r} & 0 \\ 0 & -\sqrt{r} \end{pmatrix},
\]

defines an embedding of \( D \) into \( \mathbb{M}_2(K) \).

Consider \( L := F(\sqrt{r}) \). The place \( \infty \) splits in \( L \). Denote by \( \infty_1 \) and \( \infty_2 \) the two places of \( L \) over \( \infty \). The integral closure of \( A \) in \( L \) is \( \mathcal{A} := A[\sqrt{r}] \). Let \( \pi := ((2T - 1) + 2\sqrt{r}) \in \mathcal{A} \). Since \( \pi^{-1} = (2T - 1) - 2\sqrt{r} \), \( \pi \) is a unit in \( \mathcal{A} \). Since
\( \pi \) is not a constant, its valuations are non-zero at \( \infty_1 \) and \( \infty_2 \). If \( n := \text{ord}_{\infty_1}(\pi) \), then
\[
\min(n, -n) = \text{ord}_{\infty_1}(\pi + \pi^{-1}) = \text{ord}_{\infty_1}(4T - 2) = -1.
\]
This implies that we can extend the valuation at \( \infty \) to \( L \) so that \( \pi \) is a uniformizer of \( F_\infty \). We conclude that \( \theta_2 \) acts on \( T \) as the matrix
\[
\theta_2 = i((2T - 1) + 2j) = \begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \pi^{-1} \\ \xi & 0 \end{pmatrix}.
\]
Let \( v, w \in \text{Ver}(T) \) be \( v = [e_1, e_2] \) and \( w = [\pi e_1, e_2] \). Now
\[
\theta_1 \cdot v = [e_2, \xi e_1] = [\xi e_1, e_2] = [e_1, e_2] = v
\]
and
\[
\theta_2 \cdot w = [e_2, \xi \pi e_1] = [\xi \pi e_1, e_2] = [\pi e_1, e_2] = w.
\]
Thus, we found two adjacent vertices in \( T \) and elements in their stabilizers which are not in \( F_q^* \). Proposition 3.2 implies that \( \Gamma_v \cong F_q^* \) and \( \Gamma_w \cong F_q^* \). By Corollary 3.3, the images of \( v \) and \( w \) in \( \Gamma \setminus T \) are adjacent terminal vertices. This implies that \( \Gamma \setminus T \) is an edge, and proves Theorem 3.4 in the case when \( \xi = -1 \) and \( 1 - \theta_i \) generates \( F_q(\theta_i)^* \). Hence \( \Gamma \) is isomorphic to the subgroup of \( \text{GL}_2(K) \) generated by the matrices
\[
\gamma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 1 \\ 2(T + 1) + 2\sqrt{T(T - 1)} \\ T + 1 \end{pmatrix}
\]
both of which have order 8 and satisfy \( \gamma_1^4 = \gamma_2^4 = -1 \).

**q is even.** Here we will look for solutions of the equation \( \gamma^2 + \gamma + \xi = 0 \) in \( \Lambda \).

Again writing \( \gamma = a + bi + cj + dij \),
\[
\gamma^2 + \gamma + \xi = (a + a^2 + b^2\xi + c^2 \tau + cd\tau + d^2\xi \tau + \xi) + b(b + 1)i + c(b + 1)j + d(b + 1)i_j.
\]
Hence \( \gamma^2 + \gamma + \xi = 0 \) if and only if
\[
(5.2) \quad b = 1 \quad \text{and} \quad a + a^2 + \tau(c^2 + cd + d^2\xi) = 0.
\]
An obvious solution is \( a = c = d = 0 \), which gives \( \theta_1 = i \) as a torsion unit.

Now let \( \tau = T(T + 1) \). Then
\[
a = T, \quad c = 1, \quad d = 0
\]
satisfy (5.2), so \( \theta_2 = T + i + j \) is a torsion unit.

When \( q \) is even, a technical complication arises in the study of the action of \( \Gamma \) on \( T \). This is due to the fact that neither \( F(i) \) nor \( F(j) \) embed into \( K \) (in the extension \( F(i)/F \) the place \( \infty \) remains inert, and \( \infty \) ramifies in the extension \( F(j)/F \), since \( j \) is non-separable over \( F \)). In particular, our choice of generators of \( D \) does not provide an easy explicit embedding of \( \Gamma \) into \( \text{GL}_2(K) \) (although, we can embed \( \Gamma \) into \( \text{GL}_2(F_q^*K) \) via
\[
i \mapsto \begin{pmatrix} \kappa & 0 \\ 0 & \kappa + 1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix},
\]
where $\kappa \in \mathbb{F}_{q^2}$ is a solution of $x^2 + x + \xi = 0$). Instead, we appeal to Theorem 3.5. A tedious calculation shows that $\mathbb{F}_q(\theta_1)^\times$ and $\mathbb{F}_q(\theta_2)^\times$ are not conjugate in $\Gamma$ (in fact, it is enough to check that $\theta_1$ is not a $\Gamma$-conjugate of $\theta_2$ or $\theta_2 + 1$). Hence by Theorem 3.5 these subgroups generate $\Gamma$. Again, choosing generators $\gamma_1$ and $\gamma_2$ of these cyclic groups, one obtains two torsion elements which generate $\Gamma$.

We finish with a remark on Case (2) of Theorem 3.6: \(q = 4\) and \(r = T^4 + T\). We can write down some solutions of (5.2) in this case. Let $c, d \in \mathbb{F}_4$ be not both zero. Then

\[\alpha := c^2 + cd + \xi d^2 \in \mathbb{F}_4^\times.\]

Let $s \in \mathbb{F}_4^\times$ be such that $s^2 = \alpha$ (always exists and is unique). Let $a = sT^2 + s^2T + m$, where $m = 0, 1$. Then, since $s^4 = s$ and $m^2 = m$,

\[a^2 + a = (\alpha T^4 + sT^2 + m) + (sT^2 + \alpha T + m) = \alpha(T^4 + T).\]

Now it is clear that $\theta = a + i + cj + dij$, with $a, c, d$ as above, satisfies (5.2). Nevertheless, it seems rather challenging to find explicitly enough torsion units which will generate the 8 non-conjugate subgroups of $\Gamma$ isomorphic to $\mathbb{F}_16^\times$.

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