Pattern Formation in Quantum Turing Machines

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Abstract

We investigate the iteration of a sequence of local and pair unitary transformations, which can be interpreted to result from a Turing-head (pseudo-spin \( S \)) rotating along a closed Turing-tape (\( M \) additional pseudo-spins). The dynamical evolution of the Bloch-vector of \( S \), which can be decomposed into \( 2^M \) primitive pure state Turing-head trajectories, gives rise to fascinating geometrical patterns reflecting the entanglement between head and tape. These machines thus provide intuitive examples for quantum parallelism and, at the same time, means for local testing of quantum network dynamics.

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Despite the lack of a clear-cut definition, the physics of *complexity* [1] has intrigued physicists for many years. For continuous classical systems with few degrees of freedom the notion of chaos has attracted much interest as a sign of uncontrollability [2]. For discrete classical systems in the form of cellular automata the notion of computational irreducibility has been introduced to account for the lack of “short-cuts”, i.e. our inability to predict the respective state evolution without following the detailed dynamics step by step [3]. The linearity of quantum dynamics appears to make the respective evolution “well-behaved” from the start. The limit of control, nevertheless, abounds even for modestly large quantum networks [4] due to the, typically, exponentially large Hilbert-space, in which the state evolves [5]. It has been shown that if this kind of “quantum-complexity” could be harnessed, new efficient modes of computation should become available [6,7]. However, one will first have to find ways to circumvent that disastrous exponential blow-up.

A quantum network (composed of $N$ subsystems) is defined by its Hamiltonian-operator $\hat{H}$. This, $\hat{H}$, is also the generator of the respective unitary (system-) dynamics, $\hat{U}(t) = \exp (-i\hat{H}t/\hbar)$, which transforms a given initial state $|\psi_0> >$ into a final state $|\psi'> >$ after some given time $\Delta t$: We thus have a one-parameter transformation $\hat{U}(\Delta t)$ operating on arbitrary initial states (requiring a number of state parameters which grows exponentially with $N$). To improve control it is therefore tempting to consider, instead, arbitrary unitary transformations acting on one given initial state: In fact, this type of scenario underlies most current quantum-computational schemes [7].

Any system-dynamics can be approximated as an iterative sequence of unitary basis operators (so-called “gates” [8]). In this letter we address a quantum Turing machine (QTM) architecture [9–13] which can be understood as a specific and formalized version of such an iterative map. Typically, one will be unable to “observe” the network in full detail; one then usually resorts to “macro-observables”. Here we focus, instead, on a single microscopic subsystem, the “Turing-head” $S$. To predict its state exactly, the full network state is required, though. However, while the evolution of arbitrary initial states by a given map seems exponentially “hard”, the evolution of some specific initial state by a whole class of maps turns out to be “easy” and is not at all limited to small $N$-networks. Furthermore we will show that the evolution of the Turing-head in its reduced space gives rise to geometrical patterns reflecting the entanglement between Turing-head and Turing-tape. These patterns can be thought to result from the superposition of exponentially many “basic” Turing-machines, an intuitive example of “quantum parallelism”.

The quantum network to be considered here is composed of $N (= M + 1)$ pseudo-spins $|j(\mu)>$, $j = 0, 1$; $\mu = S, 1, 2, \cdots, M$ (Turing-head $S$, Turing-tape spins $1, 2, \cdots, M$) so that its network-state $|\psi> >$ lives in the $2^{M+1}$-dimensional Hilbert-space spanned by the product wave-functions $|j(S)k(1)\cdots l(M)> = |jk\cdots l>$. Correspondingly, any (unitary) network-operator can be expanded as a sum of product-operators. The latter may be based on the following traceless $SU(2)$-generators

\[
\begin{align*}
\hat{\lambda}_x(\mu) &= \hat{P}_{01}(\mu) + \hat{P}_{10}(\mu) \\
\hat{\lambda}_y(\mu) &= i\hat{P}_{01}(\mu) - i\hat{P}_{10}(\mu) \\
\hat{\lambda}_z(\mu) &= \hat{P}_{11}(\mu) - \hat{P}_{00}(\mu), \\
\end{align*}
\]

where $\hat{P}_{ij}(\mu) = |i(\mu)> < j(\mu)|$ is a (local) transition operator.
We now consider the iterative map for which each full cycle \( p = 1, 2, \cdots \) consists of a sequence of \( 2M \) unitary transformations \( \hat{U}_n, n = 1, 2, \cdots, 2M \). At step \( m, m = n + 2M(p - 1) \), we thus have

\[
|\psi_m > = \hat{U}_n \cdots \hat{U}_2 \hat{U}_1 (\hat{U}_{2M} \cdots \hat{U}_2 \hat{U}_1)^{p-1} |\psi_0 > .
\] (2)

Presently we identify the \( \hat{U}_n \) with the local unitary transformation on the Turing-head \( S, \hat{U}_\alpha(S) \), and the quantum-controlled-NOT (QCNOT) on \((S, \mu), \hat{U}(S, \mu)\), respectively,

\[
\hat{U}_{2\mu - 1} = \hat{U}_\alpha(S) = \hat{I}(S) \cos (\alpha / 2) - i \lambda_x (S) \sin (\alpha / 2)
\] (3)

\[
\hat{U}_{2\mu} = \hat{U}(S, \mu) = \hat{P}_{00}(S) \hat{\lambda}_x (\mu) + \hat{P}_{11}(S) \hat{1} (\mu) = \hat{U}^+(S, \mu).
\] (4)

However, the basic results of this paper apply also to different transformations \( \hat{U}(S, \mu) \), e.g. with \( \lambda_x (\mu) \) replaced by \( i \lambda_x (\mu) \). In any case, the sequence of eq. (4) may be interpreted to emerge from a Turing-head rotation along the closed Turing-tape, thus iterating between local and QCNOT-operations. Any such QTM is specified by its tape-size \( M \), the external control-parameters \( \alpha, \mu = 1, 2, \cdots M \), and the initial state \( |\psi_0 > \). Without loss of generality we will restrict ourselves to \( \alpha_1 = \alpha_2 = \cdots = \alpha_M = \alpha \). The state \( |\psi_0 > \) will be taken to be a product of Turing-head and tape wave-functions. This initial “no-correlation” assumption is typical also for system-bath models [4]. In fact, the Turing-tape may be considered as a special (finite) bath-model for system \( S \).

We restrict ourselves to the Bloch-vector \( \vec{\lambda} \) of the Turing-head \( S \) (our “system of interest”)

\[
\vec{\lambda}_\mu = < \psi_m | \hat{\lambda}_{ij}(S) \otimes \hat{1}(1) \otimes \cdots \otimes \hat{1}(M) | \psi_m > .
\] (5)

The Bloch-vectors of the Turing-tape could be calculated along the same lines, but the Turing-head plays a specific role by construction. Due to the entanglement with the Turing-tape, the Turing-head will, in general, appear to be in a “mixed state”, \( |\vec{\lambda}_m|^2 < 1 \).

The tape spin-states

\[
| \pm (\mu) > = \frac{1}{\sqrt{2}}(|0(\mu) > \pm |1(\mu) >), \quad \mu = 1, 2, \cdots, M
\] (6)

are eigenstates of \( \hat{\lambda}_x (\mu) \) with eigenvalues \( \pm 1 \) respectively. If spin \( \mu \) is in one of these states, the QCNOT-operation \( \hat{U}(S, \mu) \) cannot create any entanglement, irrespective of the head state \( |\varphi(S) > :\)

\[
\hat{U}(S, \mu) |\varphi(S) > \otimes | + (\mu) > = |\varphi(S) > \otimes | + (\mu) >
\] (7)

\[
\hat{U}(S, \mu) |\varphi(S) > \otimes | - (\mu) > = \hat{\lambda}_x (S) |\varphi(S) > \otimes | - (\mu) > .
\] (8)

For the \( 2^M \) orthonormalized initial tape-states

\[
|\mathcal{P}_0^j > \in \{|\mathcal{P}_0^\pm \cdots \pm > = | \pm (1) > \otimes | \pm (2) > \otimes \cdots \otimes | \pm (M) > \}
\] (9)

and with \( |\varphi_0(S) > = \cos (\varphi_0 / 2) |0(S) > - i \sin (\varphi_0 / 2) |1(S) > , \) the network-state \( |\psi_m > \) remains a product-state,

\[
|\psi_m > = |\varphi_m(S) > \otimes |\mathcal{P}_0^j > ,
\] (10)
and the Turing-head described by

\[ \lambda_i^m(\mathcal{P}_j^0) = \langle \mathcal{P}_j^0 | < \varphi_m(S) | \hat{\lambda}_i(S) \otimes \hat{1}(1) \otimes \cdots \otimes \hat{1}(M) | \varphi_m(S) > | \mathcal{P}_j^0 > \]  (11)

performs a pure state trajectory on the Bloch-circle

\[ (\lambda_y^m(\mathcal{P}_j^0))^2 + (\lambda_z^m(\mathcal{P}_j^0))^2 = 1. \]  (12)

We show examples for \( \varphi_0 = \pi/6 \) and \( M = 1, 2 \) (Fig 1). The step number \( m \) is marked to specify the apparent “jumping”. The explicit machine rules for the Turing-head are given in Table I (\( M = 1 \)). The orbits for tape \( M \) are contained in those for \( kM \) (\( k = 2, 3, \cdots \)).

For given Turing-tape size \( M \) the initial state \( | \mathcal{P}_j^0 > \) gives rise to a periodic orbit whose period does not depend on \( \alpha \), if

\[ | \varphi_m(S) > | \mathcal{P}_j^0 > \]

with \( n_i = 0, 1, 2, \cdots \), contains \( q | - > \)-states, where \( \sum_{i=0}^{q} n_i + q = M; q = \text{odd for } M \text{ odd or even, while } q \text{ can be even only for } M \text{ even, satisfying} \]

\[ \sum_{i=0}^{q/2} n_{2i} = \frac{M - q}{2}. \]  (14)

Otherwise \( | \varphi_m(S) > | \mathcal{P}_j^0 > \) generates an aperiodic orbit (i.e. an effective rotation controlled by \( \alpha \)). The aperiodic (“quasi-periodic”) primitives also become strictly periodic, if \( \alpha \) is a rational multiple of \( \pi \).

Any initial state with \( S \) in the pure state \( | \varphi_0(S) > \) can thus be written as

\[ | \psi_0 > = \sum_{j=1}^{2^M} a_j | \varphi_0(S) > | \mathcal{P}_j^0 > , \]  (15)

i.e. \( | \psi_0 > \) can be specified by the coefficients \( \{ a_j \} \). With eq. (15) and using the orthogonality of the \( | \mathcal{P}_j^0 > \), the resulting motion of the Turing-head depends only on the modulus of \( a_j \) and is given by (cf. eqs. (5), (10))

\[ \lambda_k^m(\psi_0) = \sum_{j=1}^{2^M} |a_j|^2 \lambda_k^m(\mathcal{P}_j^0). \]  (16)

This decomposition can be seen as an intuitive example for quantum parallelism: The individual Turing-head performs exponentially many primitive trajectories “in parallel”. We may restrict the sum in (14), (16) to the periodic (aperiodic) primitives only. Equal weight superpositions of the 4 periodic (4 aperiodic) orbits lead to the isolated point (quasi-1-dimensional) patterns as shown in Fig 2 (\( M = 3, \varphi_0 = 0 \)). The special equal-weight superposition with \( a_j = (1/2^M)^{1/2} \) corresponds to the initial state \( | \psi_0 > = | \varphi_0(S) > \otimes |00\cdots0 > \), which is a complete product-state. There are other non-product states though, leading to the same equal-weight result for the Turing-head, i.e. to the same pattern.

For \( | \psi_0 > = |00\cdots0 > \), the typical initial state also for quantum computation [6], and for large \( M \) the construction of the Turing-head motion based on the decomposition approach
(2^M primitives with equal weight) becomes impractical. Surprisingly, the Bloch-vector of \( S \) can easily be found for any \( M \) and any step-number \( m = n + 2M(p - 1) \) from

\[
\begin{align*}
\lambda_x^m &= 0 \\
\lambda_y^m &= Y_{m,M}(\alpha) \\
\lambda_z^m &= Z_{m,M}(\alpha)
\end{align*}
\]  

(17)

using the recursion relations (Table I). Alternatively, the Bloch-vector \( \vec{\lambda}_m \) can be calculated directly from the initial state \( |1\cdots0\rangle \). The resulting geometrical patterns for \( M = 1, 2, 3, 10 \) are shown in Fig 3 including all steps up to \( m = 3000 \). These patterns, reminiscent of Poincaré sections in classical phase spaces (for open quantum systems compare [13]), decompose into various sub-manifolds (which reflect higher-order invariants). In the process of their built-up the Bloch-vector \( \vec{\lambda}_m \) jumps between these sub-manifolds, just as between the discrete points of the corresponding superposition of all the periodic orbits (a one-to-one correspondence, compare Fig 2); the latter thus play an important role reminiscent of Gutzwiller’s periodic-orbit-theory [16]. For \( M = 1, 2 \) the sub-manifolds are circles with radius \( r \), center \( \vec{c}_j \), defining the invariants \( I(\vec{\lambda}) = \prod_{j=1}^{2M+1} (|\vec{\lambda} - \vec{c}_j| - r) = 0 \) (for \( \phi_0 = 0 \) two of the circles coincide). We note in passing that the initial state \( |10\cdots0\rangle \) generates a Turing-head trajectory with \( \vec{\lambda}_m \) of Fig 3 replaced by \(-\vec{\lambda}_m\) (The individual tape spin may be in any state \( |0\rangle, |1\rangle \)). The unitary evolution of a mixed state can thus be constructed as weighted combinations of these trajectories, at each step \( m \). They lead to “shrunk” patterns.

The unitary transformations \( \hat{U}_\alpha(S) \) and \( \hat{U}(S, \mu) \) do not commute for \( \alpha \neq 0 \): Even without introducing any time-parameters, the sequence of transformation thus defines a specific order. This ordering can be made explicit by associating a time \( \Delta t \) with each step \( m \). The Fourier transform of this discrete dynamics (underlying the built-up of the Turing-head pattern) will thus give complementary information, accessible to spectroscopy. This would amount to testing the “non-classicality” of the respective trajectory rather than testing the non-classicality of states. Absolute time-scales become relevant as we compare \( \Delta t \) with the decoherence time \( \tau_c \). Even short times \( \tau_c \) might be overcome by running the Turing machine fast enough , i.e. by choosing \( 2M\Delta t \ll \tau_c \). Note that the Turing-head dynamics is robust with respect to phase changes of the Turing-tape states.

In conclusion, we have shown that the QTM architecture allows for a discrete dynamical evolution which, when viewed from the reduced subspace of the Turing-head, appears as some highly ordered geometric pattern. For specific initial states (“input”), these patterns (“output”) can be easily calculated for any tape size. They constitute a sensitive local test for the functioning of the total network in its exponentially large Hilbert-space. The “output” becomes available for any large enough observation period and does not suffer from the notorious “halting problem” [17]. These findings, we believe, are the first concrete results pertaining to QTM’s, a field which up to now has not shown much potential for future applications.

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FIGURES

FIG. 1. The primitives $\mathcal{P}_0^+$ (aperiodic) and $\mathcal{P}_0^-$ (periodic) for $M = 1$, and $\mathcal{P}_0^{++}$ (aperiodic), $\mathcal{P}_0^{--}$, $\mathcal{P}_0^{+-}$, $\mathcal{P}_0^{-+}$ (periodic) for $M = 2$; $\alpha = \pi/\sqrt{3}$, $\varphi_0 = \pi/6$.

FIG. 2. Equal-weight superpositions ($a_j = 1/2$) of 4 periodic (4 aperiodic) orbits for $|\psi_0 \rangle = |0000 \rangle$, $M = 3$, and total step number $m = 3000$. The equal-weight superposition $(1/\sqrt{2})$ of these two, in turn, generates the pattern for initial state $|0000 \rangle$ (see Fig 3 for $M = 3$); $\alpha = \pi/\sqrt{3}$.

FIG. 3. Turing-head-patterns for $|\psi_0 \rangle = |00 \cdots 0 \rangle$, $M = 1, 2, 3, 10$, and total step number $m = 3000$; $\alpha = \pi/\sqrt{3}$.
TABLE I. State-evolution of Turing-head for $M = 1$ and initial states $|P_0^j> = \lambda_m^m(P_0^j) = Y_m^{(j)}$, $\lambda_m^m(P_0^j) = Z_m^{(j)}$, $j = 1$ (aperiodic), 2 (periodic).

| $Y_0^{(1)}$ | $Y_1^{(1)}$ | $Y_2^{(1)}$ | $Y_3^{(1)}$ | $Y_4^{(1)}$ | $Z_0^{(1)}$ | $Z_1^{(1)}$ | $Z_2^{(1)}$ | $Z_3^{(1)}$ |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| $Y_0^{(1)} = \sin \phi_0$ | $Y_1^{(1)} = \sin (\phi_0 + \alpha)$ | $Y_2^{(1)} = \sin (\phi_0 + 2\alpha)$ | $Y_3^{(1)} = \sin (\phi_0 + 3\alpha)$ | $Y_4^{(1)} = \sin (\phi_0 + 4\alpha)$ | $Z_0^{(1)} = -\cos \phi_0$ | $Z_1^{(1)} = -\cos (\phi_0 + \alpha)$ | $Z_2^{(1)} = -\cos (\phi_0 + 2\alpha)$ | $Z_3^{(1)} = -\cos (\phi_0 + 3\alpha)$ |

TABLE II. Recursion relations for the reduced state evolution of $S$ in the case of $|\psi_0> = |00\cdots0>$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_M$. Let $Y_m = Y_{m,M}$, $Z_m = Z_{m,M}$, $Z_{m,0} := -1$, and $m' := m - 4p + 2$, where $p$ is the cycle number for step $m$; $m = n + 2M(p - 1)$, $n = 1, 2, \cdots, 2M$. $Y_0 = 0$, $Y_1 = \sin \alpha$, $Z_0 = -1$, $Z_1 = -\cos \alpha$.

| $Y_m$ | $Y_{m,M}$ | $Y_{m-1,M}$ | $Z_m$ | $Z_{m-1,M}$ |
|-------|-----------|--------------|-------|------------|
| $Y_m = -Y_1 Z_{m-1} - Z_1 Y_{m-1}$ | $Y_{m,M} = Y_{m-1,M} + Y_1 Z_{m',M-2}$ | $Y_{m,M} = Y_{m-1,M} - Y_1 (-Z_1)^{M-1}$ | $Z_m = -Z_1 Z_{m-1} + Y_1 Y_{m-1}$ | $Z_m = -Z_1 Z_{m-2} + Y_1 Y_{m-2}$ |

| $n$ | $n$ | $n$ | $n$ |
|------|------|------|------|
| $n = \text{odd}$ | $n = \text{even} \neq 2M$ | $n = 2M$, $p = \text{odd}$ | $n = \text{odd}$ |
| $n = 2M, p = \text{even}$ | $n = \text{even}$ | $n = \text{odd}$ | $n = \text{even}$ |
