Strategic Network Formation with Attack and Immunization

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November 18, 2015

Abstract

Strategic network formation arises in settings where agents receive some benefit from their connectedness to other agents, but also incur costs for forming these links. We consider a new network formation game that incorporates an adversarial attack, as well as immunization or protection against attack. An agent’s network benefit is the expected size of her connected component post-attack, and agents may also choose to immunize themselves from attack at some additional cost. Our framework can be viewed as a stylized model of settings where reachability rather than centrality is the primary interest (as in many technological networks such as the Internet), and vertices may be vulnerable to attacks (such as viruses), but may also reduce risk via potentially costly measures (such as anti-virus software).

The reachability network benefit model has been studied in the setting without attack or immunization [4], where it is known that the set of equilibrium networks is the empty graph as well as any tree. We show that the introduction of attack and immunization changes the game in dramatic ways; in particular, many new equilibrium topologies emerge, some more sparse and some more dense than trees. Our interests include the characterization of equilibrium graphs, and the social welfare costs of attack and immunization.

Our main theoretical contributions include a strong bound on the edge density at equilibrium. In particular, we show that every equilibrium network contains at most only $2n - 4$ edges for $n \geq 4$, where $n$ denotes the number of agents (and this upper bound is tight). This demonstrates that despite permitting topologies denser than trees, the amount of “overbuilding” introduced by attack and immunization is sharply limited. We further show that attack and immunization do not significantly erode social welfare: every non-trivial equilibrium in our model asymptotically has social welfare at least as that of any equilibrium in the original attack-free model.

We complement and illustrate our sharp theoretical results with simulation results demonstrating fast convergence of a new bounded rationality dynamic, swapstable best response, which generalizes linkstable best response but is considerably more powerful in our model. These simulations further elucidate the wide variety of asymmetric equilibria possible and demonstrate topological consequences of the dynamics, including heavy-tailed degree distributions arising from immunization decisions.
1 Introduction

Networks built by distributed, strategic individuals arise in contexts where agents receive some benefit from their connectedness to others, but also incur some cost for forming these links. The area of strategic network formation \[4\] studies the structure of equilibrium networks formed as the result of various choices for the network benefit function, as well as the social welfare of these equilibria. In many network formation games, the costs incurred from forming links have been direct: each edge costs \( C \geq 0 \) for an agent to purchase. More recently, motivated by scenarios as diverse as financial crises, terrorism and technological vulnerability, games with indirect connectivity costs have been considered: an agent’s connections expose her to negative, contagious shocks the network might endure.

In this work, we begin with the simple and well-studied reachability network formation game \[9\] in which players purchase links to each other, and enjoy a network benefit equal to the size of their connected component in the collectively formed graph. We modify this model by introducing an adversary who is allowed to examine the network, and choose a single vertex or player to attack. This attack then spreads throughout the entire connected component of the originally attacked vertex, destroying all of these vertices. However, players also have the option of purchasing immunization against attack. Thus the attack spreads only to those non-immunized (or vulnerable) vertices reachable from the originally attacked vertex. We assume the adversary seeks to maximize destruction, and thus will attack the largest component of vulnerable vertices, breaking ties randomly. A player’s overall payoff is thus the expected size of their post-attack component, minus their edge and immunization expenditures.

We note that in this model, the spread of the initial or “seed” attack to reachable non-immunized vertices is certain, and the protection of immunized vertices is absolute. Clearly it would be natural to consider relaxations such as probabilistic attack spreading and probabilistic immunization, as well as other generalizations such as multiple initial attack vertices. However, we shall see that there is already quite a bit to say about the basic model we examine here.

Our game can be viewed as a stylized model of settings where reachability rather than centrality is the primary interest in joining a network which is vulnerable to adversarial attack. Examples include technological networks such as the Internet, where packet transmission times are sufficiently low that being “central” \[9\] or a “hub” \[6\] are less of a concern, but in the presence of attacks such as viruses or DDoS, mere reachability may be compromised. Parties may reduce such risks via potentially costly measures such as anti-virus software or other security mechanisms. While our simplified model is obviously not directly applicable to this example, we do believe our theoretical results provide some high-level insights about such scenarios. In Section 8 we shall discuss possible variants of our model.

Immunization against attack has recently been studied in games played on a network where risk of contagious shocks are present \[7\], but only in the setting in which the network is first designed by a centralized party, after which agents or vertices make individual immunization decisions. Here we endogenize both the immunization decisions and the network formation, which leads to a model incomparable to this earlier work.

The original reachability network formation game \[4\] permitted a sharp and very simple characterization of all equilibrium networks: any tree as well as the empty graph. We begin by demonstrating that once attack and immunization are introduced, the set of possible equilibria becomes considerably more complex, including networks that contain multiple cycles, as well as others which are disconnected but nonempty. This diversity of equilibrium topologies leads us to our primary questions of interest: How dense can equilibria become — in particular, does the presence of the attacker encourage the creation of massive redundancy of connectivity? And on a related note, does the introduction of attack and immunization result in dramatically lower social welfare compared to the original reachability game?

**Our Results and Techniques:** The main theoretical contributions of this work are to show that our game still exhibits strong edge sparsity at equilibrium, and enjoys favorable social welfare properties despite the presence of the attacker. In particular, we show that equilibrium networks in our game with \( n \geq 4 \) players have at most \( 2n - 4 \) edges, fewer than twice as many edges as can appear in any nonempty equilibria of the original reachability game without attack. The starting point for this result is construction of a useful meta-graph representation that captures the structure of equilibrium graphs. We then use some tools from extremal graph theory to upper bound the resources globally invested by the players to mitigate connectivity disruptions due to any attack, obtaining our sparsity result.

We further show that if at least one edge is purchased and at least one vertex immunizes in an equilibrium, the resulting network is always connected. These results imply that any new equilibrium network (one which was not an equilibrium of the original reachability game) is either a sparse but connected graph, or is a...
forest on vertices who chose not to immunize. The latter case occurs only when the immunization cost is quite high relative to the connectivity cost, and in the former case we further show the social welfare is at least \( n^2 - O(n^{5/3}) \), which is asymptotically the maximum possible with a polynomial rate of convergence. Together, these results provide us with a complete picture of social welfare in our model. We show this welfare lower bound by first proving any equilibrium network with both immunization and an edge is connected, then showing that there cannot be many targeted vertices who are “critical” for global connectivity, where “critical” is defined formally in terms of both the vertex’s probability of attack and the size of the components remaining after her attack. Thus, players myopically optimizing their own utility create networks which are highly resilient in presence of attack.

We complement our sharp theoretical results with simulation results demonstrating fast and general convergence of a new notion of best response dynamics, called \textit{swapstable} best response, which generalizes linkstable best response but is much more powerful for our game. (The computational complexity of full best response dynamics remains an interesting open problem.) These simulations provide a dynamic counterpart to our static equilibrium characterizations, and illustrate a number of interesting further features of equilibria, such as heavy-tailed degree distributions resulting from early immunization decisions.

\section*{Organization}

The remainder of this paper is organized as follows. We formally present our model and review some related work in Section\ref{sec:Model}. We present in Section\ref{sec:NetworkTopologies} some interesting network topologies that arise as equilibria in our model and illustrate the richness of the solution space. Section\ref{sec:MetaGraphConstruction} describes a meta-graph construction that allows us to establish key structural properties of the equilibrium networks. Building on the properties established in Section\ref{sec:MetaGraphConstruction}, we present our sparsity result in Section\ref{sec:Sparsity}. We present our lower bound on social welfare in Section\ref{sec:LowerBound}. Section\ref{sec:EmpiricalResults} is dedicated to our empirical results. Finally, we conclude with some directions for future work in Section\ref{sec:Conclusion}.

\section{Model} \label{sec:Model}

We assume the \( n \) vertices of a graph correspond to individual players. Each player has the choice to purchase edges to other players, at a cost of \( C_E > 0 \) per edge. Each player additionally decides whether to immunize herself at a cost of \( C_I > 0 \), or remain vulnerable.

A (pure) \textit{strategy} for player \( i \) (denoted by \( s_i \)) is a pair consisting of the subset of players \( i \) purchased an edge to and her immunization choice. More formally, we will denote the subset of edges which \( i \) buys an edge to as \( x_i \subseteq \{1, \ldots, n\} \), and the binary variable \( y_i \in \{0,1\} \) as her immunization choice (\( y_i = 1 \) when player \( i \) immunizes). Then \( s_i = (x_i, y_i) \). We assume that edge purchases are unilateral — players do not need approval or reciprocation in order to purchase an edge to another — but that the connectivity benefits and risks are bilateral. We shall restrict our attention to pure strategy equilibria (and indeed our results show they both exist and are structurally diverse).

Let \( s = (s_1, \ldots, s_n) \) denote the strategy profile for all the players. Fixing \( s \), the set of edges purchased by all the players induces an undirected graph, and the set of immunization decisions forms a bipartition of the vertices. We will denote a game \textit{state} as a pair \((G, I)\), where \( G = (V, E) \) is the undirected graph induced by the edges purchased by all the players, and \( I \subseteq V \) is the set of players who decide to immunize. We use the notation \( U = V \setminus I \) to denote the vulnerable vertices (i.e., the players who decide not to immunize). We will refer to a subset of vertices in \( U \) as a \textit{vulnerable region} if they form a maximal connected component.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Blue and red vertices denote \( I \) and \( U \), respectively. In this example there are two targeted (maximum size vulnerable) regions, indicated by the rectangles. The adversary will randomly select and destroy one of these regions.}
\end{figure}

Fixing a game state \((G, I)\), an adversary inspects the formed network and the immunization pattern and
chooses to attack some vertex in the network. If the adversary attacks a vulnerable vertex \( v \in U \), then the attack starts at \( v \) and spreads, killing \( v \) and any other vulnerable vertices reachable from \( v \). Immunized vertices act as “firewalls” through which the attack cannot spread. We assume the adversary will choose to attack and destroy the vulnerable vertex which maximizes the total number of vertices killed. Equivalently, the adversary chooses to attack and destroy the largest vulnerable region. We refer to any maximum-sized vulnerable region as a targeted region, and vertices within a targeted region as targeted vertices.\(^1\) We denote the set of all targeted vertices by \( T \subseteq U \). If there are multiple targeted vertices i.e., \( |T| > 1 \), the adversary will choose one of these vertices uniformly at random to attack which, again, is equivalent to choosing randomly among all maximum-sized vulnerable regions (see Figure[1]).

Player \( i \)’s utility (or payoff) after an attack is equal to the size of her connected component (defined to be zero in the event she is killed) minus any expenses (edge purchases and immunization) incurred. If \( |T| > 1 \) then player \( i \)’s expected utility (fixing a game state) is equal to the expected size of her connected component less her expenditures, where the expectation is taken over the adversary’s choice of attack. More formally, let \( CC_i(v) \) denote the size of the connected component of player \( i \) when vertex \( v \in T \) is attacked by the adversary. Then the expected utility of \( i \) in strategy profile \( s \) denoted by \( u_i(s) \) is computed as follows.

\[
    u_i(s) = \frac{1}{|T|} \left( \sum_{v \in T} CC_i(v) \right) - |x_i| \cdot C_k - y_i \cdot C_t.
\]

Finally, we refer to the sum of expected utilities of all the players when playing strategy profile \( s \) as the (social) welfare of \( s \).

**Equilibrium Concepts** In this work, we will analyze the networks formed in our game under two types of equilibria. We model each of the \( n \) players as strategic agents who choose deterministically which edges to purchase and whether or not to immunize, knowing the exogenous behavior of the adversary defined as above. We say a strategy profile \( s \) is a pure-strategy Nash equilibrium (Nash equilibrium for short) if, for any player \( i \), fixing the behavior of the other players to be \( s_{-i} \), the expected utility for \( i \) cannot strictly increase playing any action \( s'_i \) over \( s_i \).

In addition to Nash, we study another equilibrium concept that is closely related to linkstable equilibrium (e.g. [5]), which is a bounded-rationality generalization of Nash. We consider a generalization of linkstable equilibrium that we refer to as swapstable equilibrium. A strategy profile is said to be a swapstable equilibrium if no individual agent’s expected utility (fixing other agent’s strategies) can strictly improve under any of the following deviations, which we call swap deviations:

1. Dropping any single purchased edge.
2. Purchasing any single unpurchased edge.
3. Dropping any single purchased edge and purchasing any single unpurchased edge.
4. Dropping any single purchased edge, and changing her immunization status.
5. Purchasing any single unpurchased edge, and changing her immunization status.
6. Dropping any single purchased edge and purchasing any single unpurchased edge, and changing her immunization status.

The first two of these deviations correspond to standard link stability. The third permits the more powerful swapping of one purchased edge for another. The last three simply repeat the first three, but allow the player to also flip her immunization status, which is particular to our game. Our interest in swapstable networks derives from the fact that while they only consider “simple” or “local” deviation rules, they share several properties with Nash networks that linkstable networks do not. In that sense, swap stability is a bounded rationality concept that moves us closer to full Nash. Intuitively, in our game (and in many of our proofs), we exploit the fact that if a player is connected to some other set of vertices via an edge to a vulnerable vertex, and that set also contains an immune vertex, the player would prefer to connect to the immune vertex instead. This deviation involves a swap of edges, not just a single addition or deletion. It is worth mentioning explicitly that, by definition, every Nash equilibrium is a swapstable equilibrium and every swapstable equilibrium is a linkstable equilibrium. It can be shown that the reverse of none of these statements are true in our game (see Appendix[5]).

\(^1\)To be more precise, we will refer to vertices who choose not to immunize as vulnerable, whether or not they are targeted vertices. We will only use targeted vertices to refer to those vertices which are part of maximum-sized vulnerable regions.
2.1 Related Work

Our paper is a contribution to the study of network design and defense. This problem has been extensively studied in economics, electrical engineering, and computer science (see e.g., [1, 2, 11, 17]). Most of the existing work takes the network as given and examines optimal security choices (see e.g., [3, 8, 12, 14, 15]). To the best of our knowledge, our paper offers the first model in which links and immunization are both chosen by players.

Combining these two dimensions — linking and immunization — within a common framework yields new insights. We start with a discussion of the network formation literature. In a setting with no attack, our model reduces to the original model of one-sided reachability network formation, due to Bala and Goyal [4]. They showed that a Nash equilibrium network is either a tree network or an empty network. By contrast, our work shows that in the presence of a security threat, Nash networks exhibit very different properties: both networks containing cycles and partially connected networks can emerge in equilibrium. Moreover, we show that while networks may contain cycles, they are sparse (we provide a tight upper bound on the number of links in any equilibrium network of our game).

Turning to the work on security, it is instructive to relate our work to a recent paper by Cerdeiro et al. [7] who study optimal design of networks in a setting where players make immunization choices but the network design is given. They show that, for different costs of immunization, an optimal network is either a hub-spoke network or a network containing $k$-critical vertices or a partially connected network (observe that a $k$-critical vertex can secure $n - k$ vertices by immunization). Our analysis extends this work by showing that there is a pressure toward the emergence of $k$-critical vertices even when linking is decentralized. Our work also contributes to the study of welfare costs of decentralization. Cerdeiro et al. [7] show that the Price of Anarchy is bounded, when the network is centrally designed while immunization is decentralized. By contrast, our work shows that the Price of Anarchy is unbounded when network and immunization are both decentralized. However, we also show that non-trivial equilibrium networks have a Price of Anarchy very near one. This highlights the key role of linking and resonates with the original results on the Price of Anarchy in the context of pure network formation games (see e.g., the discussion in [10]).

Finally, in a recent paper, Blume et al. [6] study network formation in a context where new links generate direct benefits but infection can flow through paths of connections and immunization is not a choice for agents. They demonstrate a fundamental tension between socially optimal and stable networks: the former lie just below a linking threshold that keeps contagion under check, while the latter admit linking just above this threshold leading to extensive contagion and very low payoffs. In contrast, we show that in a setting with linking and immunization, non-trivial equilibrium networks attain high levels of social welfare (a Price of Anarchy approaching 1).

3 Diversity of Equilibrium Networks

In this section, we provide several examples of Nash equilibria (thus also swapstable and linkstable equilibria) of our game. In contrast to the reachability network formation game [4], our game exhibits equilibrium networks which contain cycles, as well as non-empty graphs which are not connected. These examples prove that the tight characterization of the reachability game, where equilibrium networks are either trees or the empty graph, fails to hold for our more general game. However, in the following sections, we show that an approximate version of this characterization continues to hold.

In figures throughout the paper we represent immunized and vulnerable vertices as blue and red, respectively. Furthermore, although we treat the networks as undirected graphs throughout this paper (since the connectivity benefits and risks are bilateral), we use directed edges in the figures in this section to denote which player purchased the edge — i.e., a directed edge $i \rightarrow j$ will denote that player $i$ has purchased an edge to player $j$. The equilibrium networks discussed in this section remain equilibria for a wide range of $C_E$ and $C_I$; see Appendix C for details.

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2 A vertex is $k$-critical in a connected network if the size of the largest connected component after removing that vertex is $k$.

3 The empty graph and trees can also form at equilibrium in our game (see Appendix C).
3.1 Forests

We first assert that non-empty networks with multiple connected components can form in the equilibrium of our game (see Figure 2).

![Figure 2: An Example of a forest equilibrium. \(C_E = 1\) and \(C_I = 9\).]

3.2 Simple Cycles

We next assert that unlike the reachability game, our game admits cycles at equilibrium. In particular, we show that a cycle of alternating immunized and vulnerable vertices can form in the equilibria (see Figure 3).

![Figure 3: An example of a cycle equilibrium. \(C_E = 1.5\) and \(C_I = 3\).]

3.3 Flowers

Thus far, our examples contained at most \(n\) edges. We now assert that there are equilibrium networks which contain \(\Omega(n)\) more edges than the cycle.

![Figure 4: An example of a 4-petal flower equilibrium. \(C_E = 0.1\) and \(C_I = 3\).]

In particular, consider the flower equilibrium (see Figure 4) with \(F\) petals where each petal has a similar immunization pattern to the cycle in Figure 3. The number of edges in this equilibrium is \(n + F - 1\). So the larger the number of petals \(F\) is, the denser the equilibrium is; we point out that the densest flower equilibrium construction has \(4n/3 - O(1)\) edges.

Although the densest flower equilibrium requires \(C_E\) to be inversely proportional to the number of players (so \(C_E \to 0\) as \(n \to \infty\)), the flower network remains an equilibrium even even when \(C_E > 1\). In this regime, however, the number of petals \(F\) would be \(\Omega(\sqrt{n})\) in the densest flower; so the densest flower equilibrium when \(C_E > 1\) has \(n + \Omega(\sqrt{n})\) edges.

\[4\] We later show that this disconnectedness in non-empty equilibria can only happen if all the vertices in equilibria are vulnerable (see Proposition 3).
3.4 Complete Bipartite Graph

We finish our examples by asserting that the complete bipartite graph can also form in equilibrium when the cost of purchasing an edge is small (and independent of \( n \)) but immunization is expensive (see Figure 5).

Figure 5: An example of complete bipartite equilibrium. \( C_E = 0.1 \) and \( C_I = 4 \).

On the one hand, the examples above show that equilibrium networks can be substantially denser in our game compared to the non-attack reachability game. It is thus natural to ask just how dense they can be. In Section 5, we prove that the equilibria of our game cannot contain more than \( 2n - 4 \) edges when \( n \geq 4 \). So while these networks can be denser than trees, they remain quite sparse, and thus the threat of attack does not result in too much “overbuilding” or redundancy of connectivity at equilibrium. Our density upper bound is tight, as the complete bipartite graph in Figure 5 has exactly \( 2n - 4 \) edges.

On the other hand, the examples also show that equilibrium networks can be disconnected (even before the attack), and this might raise concerns regarding the social welfare at equilibrium compared to the reachability game. In Section 6, we show that all equilibria in our game which contain at least one edge and at least one immunized vertex (and are thus non-trivial, in the sense that are different than any equilibrium of the reachability game without attack) are connected, and have immunization patterns such that, even after the attack, the network remains highly connected. This allows us to prove that such equilibria in fact enjoy very good social welfare.

4 Structural Properties of Equilibrium Networks

In this section, we prove several structural properties of the equilibrium networks of our game. These properties give us a precise understanding of the types of behavior can exist in the equilibria, and also will be useful in proving our main results in Sections 5 and 6. All the results in this section hold for Nash, swapstable and linkstable equilibria.

Consider any network \( G = (V, E) \) (not necessarily an equilibrium). Recall that we use \( I \) and \( U \) to denote the set of immunized and vulnerable vertices, respectively. We now describe a useful partition of the vertices of \( G \). Let \( G_{ul} \) denote the subgraph of \( G \) on \( U \). Consider the connected components of \( G_{ul} \) and let \( T_1, \ldots, T_k \) denote the components of maximum size; these are exactly the targeted regions of \( G \). Each \( T_i \) will be an element of the partition; we now define the remaining elements inductively. Let \( X \) denote the union of all partition elements defined so far (thus initially, \( X = \bigcup_{i=1}^k T_i \)). If \( X \not\subseteq V \), let \( v \) be any vertex in \( V - X \), and let \( \mathcal{N}_v \) be the set of all vertices reachable from \( v \) without visiting any vertices in \( X \) (i.e. the set of vertices reachable avoiding the partition elements so far). Then we let \( \mathcal{N}_v \) be a new partition element, and repeat this procedure until \( X = V \). See Figure 6a.

Note that by construction, in this partition no immunized vertex lies in any of the original targeted elements \( T_i \); and each non-targeted element \( \mathcal{N}_v \) either contains at least one immunized vertex, or consists of a set of vulnerable vertices that are a connected component of \( G \) (because any immunized vertex that was reachable would have been included in such an element). Again see Figure 6a.

By virtue of our construction, the partitioning has several other properties summarized below in Lemma 1; these properties hold regardless of whether or not \( G \) is an equilibrium.

**Lemma 1.** Let \( G = (V, E) \) be a network, and consider the partition of \( V \) into targeted and non-targeted regions described above. Then:

1. There is no edge in \( G \) between vertices in two different targeted regions \( T_i \) and \( T_j \).
2. There is no edge in \( G \) between vertices in two different non-targeted regions \( \mathcal{N}_i \) and \( \mathcal{N}_j \).
3. Any edge in $G$ between a targeted vertex $u \in T_i$ and a non-targeted vertex $v \in N_j$ implies that $v$ is immunized i.e., $v \in I$.

Proof. First, if there were an edge in $G$ between two vertices in $T_i$ and $T_j$, then neither $T_i$ nor $T_j$ would be maximal connected components in $G_t$, contradicting their definition. For the second part, assume without loss of generality that $N_i$ was created in the partition construction before $N_j$. If there were some edge between a vertex in $N_i$ and a vertex in $N_j$, then $N_i$ would have included all of $N_j$, again contradicting the definition. Finally, if there were an edge between some $u \in T_i$ and $v \in N_j$ and $v$ were vulnerable, $T_i$ would not be maximal connected component in $G_t$.

The following lemma gives additional properties obeyed by the partition at equilibrium.

Lemma 2. Suppose $G = (V, E)$ is an equilibrium network (Nash, swapstable or linkstable), and $T_i$ and $N_j$ denote targeted and non-targeted regions, respectively. Then:

1. The subgraph of $G$ induced by any $T_i$ or any $N_j$ is a tree.
2. There is at most one edge $(u, v) \in E$ such that $u \in T_i$ and $v \in N_j$ for all $T_i$ and $N_j$.
3. The number of targeted regions is strictly bigger than one when $|V| > 1$.

Proof. First, consider a targeted region $T_i$. In case of an attack, either all vertices in this region survive or all of them die. If all vertices survive the attack, then the connectivity of each vertex in the region receives is the size of her connected component in the remaining graph, which includes this targeted region and anything it is connected to in the remaining graph (since the targeted region is connected by definition). Thus if $T_i$ is not a tree, any edge on a cycle in $T_i$ can be discarded without altering the connectivity of any vertex in $T_i$ after any attack (improving the utility of some player since $C_k > 0$). An identical argument holds for non-targeted regions $N_j$, without needing to consider the case where the region is attacked.

Second, suppose there were more than one edge between vertices in a targeted region $T_i$ and vertices in a non-targeted region $N_j$. Suppose $(u, v)$ for $u \in T_i, v \in N_j$ is one such edge. If $u$ bought the edge, she pays $C_k$ for the edge whether or not her targeted region dies, and this payment does not yield her any more connectivity (if her region survives, she is connected to $N_j$ with or without this edge). Thus, $u$ would not buy such an edge in any equilibrium. Similarly, $v$ would not buy this edge either: if $T_i$ survives, $v$ is connected to $T_i$ without the edge $(u, v)$; if $T_i$ does not survive, the edge is useless and costs $C_k > 0$. So, $v$ would not buy $(u, v)$ in equilibrium. Thus, there can be at most one edge $(u, v)$ such that $u \in T_i$ and $v \in N_j$.

We now prove the last case by contradiction. Suppose there is only one targeted region $T_i$. Then $T_i$ must be a singleton vertex; otherwise, some player in $T_i$ must have purchased an edge and the utility of this player is negative as every attack uniquely targets $T_i$ (killing it with probability 1). Let $u$ denote the singleton vertex in $T_i$. Then the expected utility of $u$ is 0, and neither $u$ nor any other vertex will purchase an edge incident to $u$. 

![Diagram](image.png)

Figure 6: An example of a network $G$ and its meta-graph $G'$. 

(a) A sample network $G$. $T$ and $N$ denote targeted and non-targeted regions, respectively. Red and purple denote the targeted and non-targeted regions, respectively. 

(b) The meta-graph $G'$ corresponding to $G$ in Figure 6A. Red and purple denote targeted and non-targeted regions, respectively.
Lemma 2 since there is at most one edge between any pair of non-targeted regions $N_i$ and $N_j$. Let $N_i$ be the partition that $v, v'$ belong to. Since $v$ is best responding, it must be that $|N_i| + C_1 + C_2 \geq 1/2$, since $v$ could choose to not buy $(v, v')$ and not to immunize for expected utility of at least $1/2$. This implies that $u$ cannot be best responding in this case, since buying an edge to $v$ and immunizing would give $u$ an expected utility of $|N_i| + C_1 + C_2 + 1 \geq 3/2 > 0$, a contradiction to $G$ being an equilibrium.

Lemma 2 allows us to represent the graph $G$ in a very particular and helpful way. We create a meta-graph $G' = (V', E')$ as follows. The vertices $V'$ are labeled by the elements of the partition discussed above (each representing either a targeted region or a non-targeted region of $G$). Each edge in $E'$ corresponds to an edge $(u, v) \in E$ such that $u \in T, v \in N_j$ (see Figure 6b).

We now show that $G'$ is bipartite. Furthermore, when $G$ is an equilibrium network, $G'$ is a simple graph (as opposed to a multigraph).

**Observation 1.** Let $G = (V, E)$ be any network and $G'$ the meta-graph described above. Then $G'$ is a bipartite graph between the targeted regions $T_i$ and the non-targeted regions $N_j$, and if $G$ is an equilibrium, there is at most one edge between any $T_i$ and $N_j$.

**Proof.** The fact that the graph is bipartite with this partitioning follows from Lemma 1 which states that there are no edges in $G$ between vertices in any pair of non-targeted regions $N_i$ and $N_j$ or targeted regions $T_i$ and $T_j$. The fact that $G'$ is a simple graph (not a multigraph) when $G$ is in an equilibrium follows from Lemma 2 since there is at most one edge $(u, v) \in E$ between any $u \in T_i$ and $v \in N_j$ in equilibrium.

## 5 Sparsity of Equilibrium Networks

In this section, we show that despite the existence of equilibrium networks containing $\Omega(n^4)$ cycles (and $2n - 4$ edges) shown in Section 4, any network formed in either a Nash, swapstable or linkstable equilibrium has at most $2n - 4$ edges and is thus quite sparse. This result also implies that the lower bound on sparsity shown in the previous section is tight. We begin by showing that $G'$, the meta-graph described in Section 4, is sparse by adapting techniques and results from graph theory. We then show that there is a simple connection between the edge density of $G'$ and the edge density of the original equilibrium graph $G$.

More specifically, we show that $G'$ is chord-free. A graph is chord-free graph if no cycle of length at least 4 has a chord, where a chord is an edge that is not part of the cycle but connects two vertices on the cycle. We then use Mader’s Theorem 16 to prove that chord-free graphs (and hence $G'$) is sparse. Finally, since the elements in the partitioning of $G$ (corresponding to vertices of $G'$) are sparse themselves and $G'$ is a simple graph (not a multigraph), we conclude that equilibrium network $G$ itself must be sparse.

First, let us state Mader’s theorem.

**Theorem 1.** (Mader’s Theorem 16) Let $G = (V, E)$ be any undirected graph with minimum degree of $d$. Then, there exists an edge $(u, v) \in V$ such that there are $d$ internally vertex-disjoint paths between $u$ and $v$.

We now use Mader’s Theorem to show that any chord-free graph has at most $2n - 4$ edges when $n \geq 4$.

**Theorem 2.** Let $G = (V, E)$ be a chord-free graph on $n \geq 4$ vertices. Then $G$ contains at most $2n - 4$ edges.

**Proof.** While $G$ contains a vertex of degree at most 2, we remove this vertex from $G$ and repeat this process until either the number of remaining vertices falls to 4 or the minimum degree in the residual graph is at least 3. Let $G(V, E)$ be the resulting graph upon termination of this process, and let $\tilde{n} \geq 4$ denote the number of vertices in $G$.

If $\tilde{n} = 4$, then the assertion of the theorem follows from the following two observations: (i) we removed at most $2(\tilde{n} - 4)$ edges in the process, and (ii) any chord-free graph on 4 vertices contains at most 4 edges. Combining these observations together, we can conclude that the total number of edges in $G$ is at most $2(\tilde{n} - 4) + 4 = 2n - 4$. 

9
Otherwise, \( \tilde{G} \) is a graph with minimum degree of at least 3. Moreover \( \tilde{G} \) is chord-free (since \( G \) is chord-free and vertex deletion maintains the chord-free property). Now by Theorem 1, \( \tilde{G} \) contains an edge \((u, v)\) such that there are at least 3 vertex-disjoint paths connecting \( u \) and \( v \). This implies that there are at least two vertex disjoint paths connecting \( u \) and \( v \), other than the edge \((u, v)\). So, there exists some cycle that contains \( u \) and \( v \) (but not the edge \((u, v)\)) with length at least 4. However, the edge between \( u \) and \( v \) would be a chord for such a cycle. This is a contradiction since \( \tilde{G} \) is chord-free. So, \( \tilde{G} \) must be a graph with 4 vertices, and hence there must be at most \( 2n - 4 \) edges in \( G \).

We next prove that \( G' \), the meta-graph described in Section 4 is chord-free.

**Lemma 3.** Let \( G = (V, E) \) be an equilibrium (Nash, swapstable or linkstable) network and \( G' = (V', E') \) the construction described in Section 4. Then \( G' \) is chord-free.

**Proof.** Suppose not. Then there exists a cycle of size at least 4 in \( G' \) that has a chord. Consider any such cycle. By definition there exist vertices \( u, v, y, z \in V' \) such that (i) there are at least two vertex disjoint paths between \( u \) and \( v \), (ii) \( y \) is one path from \( u \) to \( v \), and (iii) \( y, z \in E' \). We show that dropping the edge between \( y \) and \( z \) would be a linkstable deviation (and hence a swapstable and Nash deviation) that increases the expected payoff of the vertex that purchased this edge. This would contradict our assumption that \( G \) was an equilibrium network.

Since there is an edge between \( y \) and \( z \), one must represent a targeted region and the other a non-targeted region, by Observation 1. Let \( z \) denote the non-targeted region. First, note that the edge cannot be purchased by a vertex in \( z \). Any vertex in \( z \) will always be connected to the entirety of this cycle, and its neighborhood, other than the vertices in the attacked targeted region, even without this edge; thus, this edge is not linkstable if a vertex in \( z \) has bought it. Second, we show that no vertex in \( y \) would also buy this edge in an equilibrium: if \( y \) survives, every vertex in \( y \) has connectivity to the entire surviving part of the cycle, and if not, the vertex in \( y \) who purchased the edge gets no benefit from that edge; so the edge is also not linkstable even when it is purchased by a vertex in \( y \). Thus, no vertex in either of \( z \) and \( y \) would have purchased such an edge; contradicting the assumption that \( G \) was an equilibrium network.

Finally, we use Theorem 2 and Lemma 3 to show that the equilibrium networks in our game are all sparse.

**Theorem 3.** Suppose \( G = (V, E) \) is an equilibrium network (Nash, swapstable or linkstable) on \( n \) vertices. Then \( G \) contains at most \( 2n - 4 \) edges when \( n \geq 4 \).

**Proof.** Let \( G' = (V', E') \) be the meta-graph described in Section 4. Let \( n' \) the number of vertices in \( G' \). We consider two cases: (1) \( n' \geq 4 \) or (2) \( n' \leq 3 \).

In case (1), since \( G' \) is chord-free by Lemma 3 and not a multigraph by Lemma 2, Theorem 2 implies \( G' \) has at most \( 2n' - 4 \) edges (since \( n' \geq 4 \)). Furthermore, each vertex \( v' \in V' \) actually represents a tree in \( G \) by Lemma 2. For every \( v' \in V' \), if \( v' \) represents \( k_{v'} \) vertices in \( G \), this implies that \( n' = n - \sum_{v' \in V'} (k_{v'} - 1) \). Thus, \( G \) can only have at most

\[
2n' - 4 + \sum_{v' \in V'} (k_{v'} - 1) = 2\left(n - \sum_{v' \in V'} (k_{v'} - 1)\right) - 4 + \sum_{v' \in V'} (k_{v'} - 1) = 2n - 4 - \sum_{v' \in V'} (k_{v'} - 1) \leq 2n - 4
\]

edges, as desired.

In case (2), since the vertices in \( G' \) represent trees in \( G \) by Lemma 2, \( G \) can have at most \( n \) edges which is at most \( 2n - 4 \) when \( n \geq 4 \).

We conclude this section by remarking that the upper bound provided in Theorem 3 is indeed tight as the complete bipartite graph in Figure 5 exhibits the same edge density.

### 6 Social Welfare in Equilibrium Networks

The sparsity results of Section 5 show that despite the potential presence of cycles at equilibrium, there are still sharp limits on collective expenditure on edges in our game. However, these results do not directly lower bound the social welfare, due to connectivity concerns: if the graph could become highly fragmented after the attack at equilibrium, or is sufficiently fragmented prior to the attack, the reachability benefits to players...
could be sharply lower than in the attack-free reachability game. In this section we show that when $C_1$ and $C_k > 1$ are both constants with respect to $n$, none of these concerns are actually realized in any “interesting” equilibrium network, described precisely below.

In the original reachability network formation game \[1], the maximum welfare achievable in any equilibrium is \( n^2 - O(n) \) (see Appendix A). Here we will show that under a mild assumption, the welfare achievable in any “non-trivial” equilibrium of our game is \( n^2 - O(n^{5/3}) \). Obviously, with no restrictions, this cannot be true: just as in the original reachability game, for edge cost $C_k > 1$, the empty graph remains an equilibrium in our game and has social welfare only \( O(n) \) (each vertex has an expected payoff of $1 - 1/n$). We thus assume the equilibrium network contains at least one edge and at least one immunized vertex. We refer to all equilibrium networks that satisfy the above assumption as non-trivial equilibria. They capture the equilibria that are new to our game compared to the original attack-free setting — the network is not empty, and at least one player has chosen immunization.

Limiting attention to non-trivial equilibria is necessary if we hope to guarantee that the welfare at equilibrium is \( \Omega(n^2) \) when $C_k > 1$; as already noted, without the edge assumption, the empty graph is an equilibrium; and without the immunization assumption, $n/3$ disjoint components where each component consists of 3 vulnerable vertices is an equilibrium (for carefully chosen $C_k$ and $C_1$). In both cases, the social welfare is only \( O(n) \).

First in Lemma 4 we show that all non-trivial (Nash or swapstable) equilibrium networks have some nice properties when $C_k > 1$ i.e. we show that they are actually connected, and that no two vulnerable vertices share an edge.

**Lemma 4.** Let $C_k > 1$, and consider a (Nash or swapstable) equilibrium network $G$. If $G$ is non-trivial then $G$ is connected, and the size of all vulnerable (and, thus, targeted) regions, if there are any, is exactly 1.

The proof of Lemma 4 depends on several structural properties of equilibrium networks that we explore in Appendix B. So we defer the proof of Lemma 4 to Appendix B.3. We next state our main result of this section regarding the social welfare in non-trivial equilibria.

**Theorem 4.** Let $C_k > 1$, and consider a (Nash or swapstable) equilibrium network $G = (V, E)$ over $n$ vertices. If $G$ is non-trivial and $C_k$ and $C_1$ are constants (independent of $n$), then the welfare of $G$ is $n^2 - O(n^{5/3})$.

**Block-Cut Tree Decomposition** Before proving Theorem 4, we describe the notion of block-cut tree decomposition of a graph which will be useful in our proof. The block-cut tree decomposition (see e.g., [18]) of an undirected graph $G = (V, E)$, denoted by $T = (B \cup C, E')$, is defined as follows. A vertex $b \in B$ (called a block) corresponds to some subset $V_b$ of $V$ which is a maximal two-connected component in $G$. A vertex $v \in C$ (called a cut vertex) corresponds to some vertex $v \in V$, the removal of which would increase the number of connected components in $G$; an edge $e = (b, v) \in E'$ means that $v \in V_b$, and that the removal of $v$ from $G$ would disconnect $V_b \setminus \{v\}$ from some other part of $G$. In contrast to the standard convention that cut vertices are also part of the blocks their removal would disconnect, we will assume throughout that cut vertices are not part of the blocks. This is simply to avoid over-counting the number of vertices in a subtree of $T$. Also, note that all the leaves in $T$ must be blocks since any cut vertex has degree at least 2. We remark that the block-cut tree decomposition of any undirected graph $G$ can be efficiently computed in $O(|E| + |V|)$ time.

We define the size of a block $b$ (denoted by $|b|$) to be number of vertices in $V_b$ (which is $|V_b|$). Also we define the size of a subtree $T_v$, rooted at $v \in B \cup C$ (denoted by $|T_v|$) to be the number of vertices contained in the union of all blocks and cut vertices in $T_v$.

In our problem, the block-cut tree decomposition of an equilibrium network $G$ has the nice property that if a targeted vertex inside of a block is attacked, $G$ remains connected after the attack. However, if a targeted cut vertex is attacked, then $G$ becomes disconnected. This property becomes handy when we compute the welfare in equilibrium networks. We now present the proof of Theorem 4.

\[\text{Footnotes:}\]

5We view this condition as the natural or most interesting regime of our model, since in most circumstances we do not imagine the cost of edge formation or immunization to grow with the population size.

6 In Appendix E we examine some properties of equilibrium networks of our game under which the welfare matches the $n^2 - O(n)$ welfare of the reachability game.

7 These nice properties might not hold when $C_k \leq 1$. For example See Figure 18 where the non-trivial equilibrium network is not connected.
Proof of Theorem 4. First of all, Lemma 4 implies that $G$ has a single connected component, all the vulnerable vertices are targeted and the size of any targeted region, if there are any, is 1. Also since there are at most $2n - 4$ edges in $G$ by Theorem 3 and the number of immunized vertices is at most $n$, the collective expenditure of vertices in $G$ is at most $C_{\text{max}} = (2n - 4)C_E + nC_I$.

Let $T = (B \cup C, E')$ be the block-cut tree decomposition of $G$. The decomposition has the nice property that an attack to a targeted vertex in any block of $T$ leaves $G$ with a single connected component after attack. However, an attack to a targeted cut vertex of $T$ can disconnect $G$. So to analyze the welfare, we only consider the targeted cut vertices in $T$ and in particular we only focus on targeted cut vertices of $T$ with the property that the attack on such a vertex sufficiently reduces the size of the largest connected component in the resulting graph. More precisely, let $\epsilon = 2\sqrt{C_E/n^{1/3}}$. We refer to a targeted cut vertex $v$ as a heavy cut vertex if after an attack to $v$, the size of the largest connected component in $G \setminus \{v\}$ is strictly less than $(1 - \epsilon)n$. We then show that the total probability of attack to heavy cut vertices is small if $G$ is a non-trivial equilibrium. This implies that with high probability (which we specify shortly) the network retain a large connected component after the attack and, hence, the welfare is high.

Root $T$ arbitrarily on some targeted cut vertex $r \in C$. If there is no such cut vertex, then the size of largest connected component in $G$ after any attack is at least $n - 1$. So the social welfare in that case is at least $(n - 1)^2 - C_{\text{max}}$ and we are done. So assume $r$ exists and consider the set of cut vertices $H_r \subseteq C$ such that for all $v \in H_r$, (a) $v$ is targeted, (b) $|T_v| \geq \epsilon n$, and (c) no targeted cut vertex $v' \in T_v \setminus \{v\}$ has the property that $|T_{v'}| \geq \epsilon n$ (i.e., $v$ is the deepest node in the tree $T_v$ that satisfies property (b)). Observe that each $v \in H_r$ is a heavy cut vertex. We consider two cases based on the size of $H_r$: (1) $|H_r| = 1$ and (2) $|H_r| > 1$.

Consider case (1) where $|H_r| = 1$. Let $H_r = \{v\}$. Consider the following two cases: (a) $v = r$ and (b) $v \neq r$ where $r$ is the root of the tree. In case (a), let $p$ be the probability of attack to $v$. Now consider

![Figure 7: Case 1(a): $v$ is the only heavy cut vertex and is the root of $T$. The triangles denote the subtrees rooted at the child blocks of $v$.](image)

the deviation in which $v$ immunizes but maintains the same edge purchases as in her current strategy in $G$. Since $G$ is an equilibrium, $v$ (weakly) prefers her current strategy to this deviation. The connectivity of $v$ after an attack to any targeted vertex other than $v$ is at least $n - \epsilon n$. So for $v$ to not prefer immunizing: $p(n - \epsilon n) \leq C_E$. Furthermore, if any vertex other than $v$ is attacked, the size of the largest connected component after the attack is at least $(1 - \epsilon)n$ (See Figure 7). This implies the welfare is at least

$$
(1 - p)((1 - \epsilon)n)^2 - C_{\text{max}} > \left(1 - \frac{C_I}{(1 - \epsilon)n}\right)(1 - 2\epsilon)n^2 - C_{\text{max}}
$$

$$
> n^2 - 4\sqrt{C_E/n^{1/3}} - \frac{n^{4/3}}{n^{1/3} - 2\sqrt{C_E}} - C_{\text{max}} = n^2 - O(n^{5/3})
$$

since $C_{\text{max}}$ is $O(n)$.

For case (b), observe that the targeted cut vertices on the path from $v$ to $r$ (the root of $T$) are the only possible heavy cut vertices in the network. So let $p_v$ denote the probability that some cut vertex on the path from $v$ to $r$ (root of $T$) is attacked. We consider two cases: (b1) $p_v \leq \sqrt{C_E n^{-1/3}}$, and (b2) $p_v > \sqrt{C_E n^{-1/3}}$. We show that in case (b1) the welfare is as claimed in the statement of Theorem 4 and case (b2) cannot happen if $G$ is a non-trivial equilibrium network.

---

8Recall that for any $v \in B \cup C$, we denote $T_v$ to be the subtree rooted at $v$. Furthermore, we define the size of $T_v$ (denoted by $|T_v|$) to be the cardinality of the union of all the blocks and cut vertices in $T_v$.

9We count both $v$ and $r$ to be on the path.
In case (b1), with probability $1 - p_v$, the size of the largest connected component after the attack is at least $(1 - \epsilon)n$. Hence the welfare in case (b1) is at least

$$(1 - p_v) ((1 - \epsilon) n)^2 - C_{\text{max}} \geq \left(1 - \frac{\sqrt{C_E n}}{n^{1/3}}\right) (1 - 2\epsilon) n^2 - C_{\text{max}}$$

$$= \left(1 - \frac{\sqrt{C_E n}}{n^{1/3}}\right) (1 - \frac{4\sqrt{C_E}}{n^{1/3}}) n^2 - C_{\text{max}}$$

$$> n^2 - 5\sqrt{C_E n^{5/3}} - C_{\text{max}} = n^2 - O(n^{5/3})$$

since $C_{\text{max}}$ is $O(n)$.

In case (b2), with probability $1 - p_v$, the size of the largest connected component after the attack is at least $(1 - \epsilon)n$. Hence the welfare in case (b2) is at least

$$(1 - p_v) ((1 - \epsilon) n)^2 - C_{\text{max}} \geq \left(1 - \frac{\sqrt{C_E n}}{n^{1/3}}\right) (1 - 2\epsilon) n^2 - C_{\text{max}}$$

$$= \left(1 - \frac{\sqrt{C_E n}}{n^{1/3}}\right) (1 - \frac{4\sqrt{C_E}}{n^{1/3}}) n^2 - C_{\text{max}}$$

$$> n^2 - 5\sqrt{C_E n^{5/3}} - C_{\text{max}} = n^2 - O(n^{5/3})$$

since $C_{\text{max}}$ is $O(n)$.

In case (b2), since $r$ is cut vertex, $r$ has at least two child blocks. Consider any child block of $r$ that is not in the same subtree of $r$ as $v$ (e.g., $b$ in Figure 8) and call this child block $b$. Since all targeted regions are singletons, $r$’s connects only to immunized vertices in $b$—let $w$ be one such immunized vertex. Now consider the deviation that $w$ purchases an edge to an immunized vertex $w'$ in $T_v$ ($w'$ exists because by the choice of $\epsilon$, $|T_v| \geq 2$ and $v$ is targeted). Then with probability $p_v$, the vertex $w$ will get an additional benefit of at least $|T_v| - 1 \geq \epsilon n - 1$ from thus purchase minus the connection cost $C_E$. However,

$$p_v(\epsilon n - 1) \geq \left(\sqrt{C_E n}^{-1/3}\right) \left(2\sqrt{C_E n^{2/3}} - 1\right) = 2C_E n^{1/3} - \sqrt{C_E n}^{-1/3} > C_E n^{1/3} > C_E,$$

which shows that the $w$ can increase her expected utility strictly in this deviation; a contradiction to the assumption that network $G$ is an equilibrium network.

In case (2), let $r'$ be a cut vertex that is the lowest common ancestor of vertices in $H_r$. If $r' \neq r$, we root the tree on $r'$ and repeat the process of finding heavy cut vertices. Note that $H_r \subseteq H_{r'}$ but we might increase the set of heavy cut vertices when we root $T$ in $r'$. See e.g. Figure 9.

Observe that the vertices in $H_{r'}$ and the targeted cut vertices on the path from some $v \in H_{r'}$ to $r'$ (new root) are the only possible heavy cut vertices in the network. Let $p_v$ denote the probability that some targeted cut vertex on the path from $v$ to $r'$ is attacked. We consider two cases: (a) $\Sigma_{v \in H_{r'}} p_v \leq n^{-1/3}$, and (b) $\Sigma_{v \in H_{r'}} p_v > n^{-1/3}$. We show that in case (a) the welfare is as claimed in the statement of Theorem 4, and that case (b) cannot happen if $G$ is a non-trivial equilibrium network.

First consider case (a). In this case, with probability of at least $1 - \Sigma_{v \in H_{r'}} p_v$, the size of the largest
Figure 9: An example of re-rooting the tree in case 2. Heavy cut vertices in $H$ are in red. The small rectangles and circles denote blocks and cut vertices, respectively. The triangles denote the subtrees rooted at critical cut vertices.

connected component after an attack is at least $(1 - \epsilon)n$. Hence the welfare in this case is at least

$$
\left(1 - \sum_{v \in H_{r'}} p_v\right) \left((1 - \epsilon) n\right)^2 - C_{\text{max}} \geq \left(1 - n^{-1/3}\right) \left(1 - 2\epsilon\right) n^2 - C_{\text{max}}
$$

$$
= \left(1 - n^{-1/3}\right) \left(1 - \frac{4\sqrt{C_E}}{n^{1/3}}\right) n^2 - C_{\text{max}}
$$

$$
> n^2 - \left(1 + 4\sqrt{C_E}\right) n^{5/3} - C_{\text{max}} = n^2 - O\left(n^{5/3}\right)
$$

since $C_{\text{max}}$ is $O(n)$.

Next, we consider case (b), namely $\sum_{v \in H_{r'}} p_v > n^{-1/3}$. First note that since $|H_{r'}| \leq 1/\epsilon$ there exists a $v^* \in H_{r'}$ such that

$$
p_{v^*} > \frac{n^{-1/3}}{|H_{r'}|} \geq n^{-1/3} \epsilon = 2\sqrt{C_E} n^{-2/3}.
$$

by the pigeonhole principle. Also $|T_{v'}| \geq 2$ for all $v' \in H_{r'}$ because $v'$ is a cut vertex. This implies that there is at least one immunized vertex in each $T_{v'}$ for all $v' \in H_{r'}$ because there can be no edges between targeted vertices in $G$. By the choice of the root, there exists a $v' \in H_{r'}$ such that every time a heavy cut vertex on the path from $v^*$ to the root is attacked then $T_{v^*}$ and $T_{v'}$ end up in different connected components. Now, consider the deviation that an immunized vertex $w$ in $T_{v^*}$ purchases an edge to an immunized vertex in $T_{v'}$.

In this deviation, $w$ would get an additional connectivity benefit of at least $p_{v^*} (|T_{v'}| - 1)$ – this benefit occurs whenever there is an attack to a cut vertex on the path from $v^*$ to the root (which happens with probability of $p_{v^*}$) and the connectivity benefit in this case is at least $|T_{v'}| - 1 \geq (\epsilon n - 1)$. Also, the extra expenditure of $w$ in this deviation is $C_E$. However,

$$
p_{v^*} (\epsilon n - 1) \geq \left(2\sqrt{C_E} n^{-2/3}\right) \left(2\sqrt{C_E} n^{-1/3} - 1\right) n = 4C_E - 2\sqrt{C_E} n^{-2/3} \geq 2C_E > C_E,
$$

which shows that $w$ can increase her expected utility strictly in this deviation; a contradiction to the assumption that network $G$ is an equilibrium network.

We conclude this section by noting that while linkstable equilibria are also connected when $C_E > 1$, the size of targeted regions in such equilibria can indeed be bigger than 1 (see Figure 15). The welfare result of Theorem 4 can thus be extended to linkstable equilibria if the size of targeted regions in such equilibria is bounded by some constants (independent of $n$) when $C_E > 1$. We leave the study of welfare in non-trivial linkstable equilibria as an open question.
7 Simulations

In this section we complement and illustrate our theoretical results with extensive simulations investigating various properties of swapstable best response dynamics (defined below). More precisely, we have implemented a simulation that allows the specification of the following parameters: number of players or vertices \( n \); edge cost \( C_E \); immunization cost \( C_I \); and initial edge density. The first three of these parameters are as discussed in the theoretical results, but the last is new and specific to the simulations. Note that for any \( C_E \geq 1 \), it is already a Nash equilibrium for players to purchase no edges (empty graph) and no immunizations. Thus to sensibly study any type of best response dynamics, it is necessary to “seed” the process with at least some initial connectivity. With regard to motivation, one could view these initial edge purchases as having occurred prior to the introduction of attack and immunization. Below we will examine simulations starting both from very sparse initial connectivity (multiple connected components) and rather dense initial connectivity, for varying combinations of the other parameters. In all cases the initial connectivity was chosen randomly (via the Erdős-Rényi model).

Our simulations proceed in rounds, where each round consists of a swapstable best response update for all \( n \) players in some fixed order. More precisely, in the update for player \( i \) we fix the edge and immunization purchases of all other players, and compute the expected payoff of \( i \) if she were to alter her current action according to swap deviations stated in Section 2.

Swapstable dynamics is a rich but “local” best response process, and thus more realistic than full Nash best response dynamics (whose computational complexity is unknown for our game) from a bounded rationality perspective. We also note that the phenomena we report on here appear to be qualitatively robust to a variety of natural modifications of the dynamics, such as restriction linkstable best response instead of swapstable, changes to the ordering of updates, and so on. Recall as well that all of our formal results hold for swapstable as well as Nash equilibria, so the theory remains relevant for the simulations.

The first question that arises in the consideration of any kind of best response dynamic is whether and when it will converge to the corresponding equilibrium notion. Interestingly, empirically it appears that swapstable best response dynamics always converges, and does so rather rapidly. In Figure 10 we show the average number of rounds to convergence over many trials, starting from dense initial connectivity (average degree 5), for varying values of the population size \( n \). The growth in rounds appears to be strongly sublinear in \( n \) (recall that each round updates all \( n \) players, so the overall amount of computation is still superlinear in \( n \)). We are thus led to conjecture the general convergence of swapstable dynamics.

![Figure 10: Average number of rounds for swapstable convergence vs. \( n \), for \( C_E = C_I = 2 \).](image.png)

In Section 3 we gave a number of formal examples of Nash and swapstable equilibria. These examples tended to exhibit a large amount of symmetry, especially those containing cycles, due to the large number of cases that need to be considered in the proofs. Figure 11 shows a sampling of “typical” equilibria found via

---

10This conjecture needs some specification, since it is possible to give a particular initial graph and a particular ordering of players that causes the dynamics to cycle. However, this example also relies on a worst-case rule for breaking best response ties, and thus we suspect the more natural variant with randomized ordering and randomized tie-breaking converges generally. To our knowledge standard potential game arguments do not seem to apply here. See Appendix F for more details.
Figure 11: Sample equilibria reached by swapstable best response dynamics for \( n = 50 \). Blue vertices have chosen to immunize, red vertices have not. Left panel: \( C_E = 0.5, C_I = 2 \). Middle panel: \( C_E = 2, C_I = 2 \). Right panel: \( C_E = 0.5, C_I = 20 \). See text for discussion.

Figure 12: Number of edges (left panel), number of immunizations (middle panel), and average welfare (right panel) for \( N = 50 \) and varying values for \( C_I \) and \( C_E \).

The sample equilibria above provide snapshots only at the conclusion of swapstable dynamics. In Figure 12 we examine entire paths, again at \( n = 50 \). In each simulation we started from denser initial graphs (average degree 5), and each panel visualizes a different quantity as a function of the number of rounds, and for three cost regimes: \( C_E = C_I = 2 \) (blue curves); \( C_E = C_I = 6 \) (red curves); and \( C_E = C_I = 10 \) (green curves). In each panel there are 10 simulations shown for each cost regime.

In the left panel, we show the evolution of the total number of edges in the graph over successive rounds. Here we see that for all cost regimes, there is initially a precipitous decline in connectivity, as the overly dense initial graph cannot be supported at equilibrium (as predicted by the theory). Thus in these early rounds all players are choosing to drop edge purchases. The ultimate connectivity, however, depends on the cost regime. In the inexpensive regime (blue), connectivity falls monotonically until it levels out very near the threshold for global connectivity at \( n − 1 \) (horizontal black line), resulting in trees or perhaps just one cycle. In the moderate cost regime (red), we see a bifurcation — in some of the trials, connectivity continues to fall all

\[ \text{In these simulations the initial edge density was only } 1/(2n), \text{ so the initial graph was very sparse and fragmented.} \]
the way to the empty graph at equilibrium, while other fall below, while others fall well below the $n - 1$ tree threshold, but then “recover” back to that threshold (which we shall discuss shortly), resulting in a non-monotonic pattern of connectivity. In the expensive cost regime, all trials again result in a monotonic fall of connectivity, but this time all the way to the empty graph equilibrium.

For the same cost regimes and trials, the middle panel shows the number of immunizations purchased over successive rounds. In the inexpensive regime (blue), we see that immunizations, sometimes many, are purchased in very early rounds. As suggested by the theory of Section 6, these immunizations act as a “safety net” that prevents connectivity from falling below the tree threshold, as players can obtain the benefits of global connectivity but keep attack vulnerability low by connecting to immune players. In one trial we see that all players end up purchasing immunity. More typically, immunizations grow initially and then decline at equilibrium. In the moderate cost regime (red), we see that the explanation for the connectivity bifurcation discussed above can be traced to immunization decisions. In those trials where connectivity is recovered, some players eventually choose to immunize and thus provide the focal points for edge repurchasing. In the many trials that resulting in the empty graph, immunizations never occurred (these trials remain at $y = 0$ and thus do not appear in the plot). In the expensive regime, no trials are visible because immunizations are never purchased.

Finally, the right panel shows the evolution of the average social welfare per player over successive round. In the inexpensive regime (blue), welfare increase slowly and modestly from negative values in the dense initial graph, then increase dramatically as the benefits of immunization are realized. In the moderate cost regime (red), we see a bifurcation of social welfare corresponding directly to the bifurcation of connectivity. In the expensive regime, all trials simply converge from below to the minimum $(1-1/n)$ welfare of the empty graph. Again in light of the results of Section 6, the relationship between $C_E$, $C_I$ and $n$ is determining whether convergence is to a non-trivial equilibrium with immunizations and thus high social welfare, or to a highly fragmented network with no immunizations and low social welfare.

![Histogram over 200 trials of the ratio of maximum degree to average degree at swap-stable equilibrium for $n = 100$, $C_E = 0.5$, $C_I = 2$.](image_url)

We conclude our experimental discussion by noting that for many parameter regimes, the dynamics described above seem to result in heavy-tailed degree distributions — a property commonly observed in large-scale social and other networks which is easy to capture in stochastic generative models such as preferential attachment, but more rare in strategic network formation. This can be seen anecdotaly in Figure 11, where we see sample equilibria consisting of “hubs” of varying size connected by sparse strands of vertices. More systematically, in Figure 13 we show the results of 200 simulations of swapstable dynamics for $n = 100$, $C_E = 0.5$ and $C_I = 2$. The figure shows a histogram of the ratio of the maximum degree to the average degree across all 200 equilibria found. The lowest ratio observed is 6, the average is 15.8, and the maximum is 41 — so the highest degree is consistently an order of magnitude greater than the average or more. Furthermore, in all 200 trials the highest-degree vertex chose immunization, despite the average rate of immunization across the population being only 23%. Consistent with the discussion of the dynamics above, there thus seems to be an amplification process at work, where vertices that immunize early become the recipients of many edge purchases, since they provide other vertices connectivity benefits that are relatively secure against attack without the cost of immunization. More generally, at equilibrium there is a strong positive correlation (0.43, 0.43,
with a negligible P-value) between a vertex’s degree and its immunization decision. Interestingly, however, it appears that while essential to overall social welfare, immunizers are not themselves the relative beneficiaries: there is a strongly negative (-0.75) correlation between payoff and the decision to immunize.

8 Future Directions

We conclude by mentioning some areas and problems for further study. The adversarial attack and the connectivity benefit to players considered in this work are perhaps the simplest choices for these elements of our model. It would be interesting to consider both more complicated attack models (e.g., when the adversary attacks in a way that minimizes the social welfare rather than the number of vertices killed) or different benefit functions (e.g., centrality notions of network benefit such as average distance to other players). One could also consider richer dynamics for the spread of the attack, such as standard diffusion models like independent cascades or threshold contagion. It is also of interest to understand the complexity of computing Nash best responses in our model, and whether swapstable best response provably always converges (as we have seen empirically). Finally, the immunization we considered in this work was perfect (i.e., a vertex will deterministically survive if she immunizes). A more realistic model might assume immunization improves an agent’s defenses, but does not guarantee her survival. One interesting model might allow agents to invest in immunization to any degree $p$ between 0 and 1, where 0 gives no security, 1 gives perfect security, and intermediate values $p$ provide immunity with probability $p$.

9 Acknowledgments

We would like to thank Chandra Chekuri, Yang Li and Aaron Roth for useful suggestions.

References

[1] ALPCAN, T., AND BAAR, T. Network Security: A Decision and Game-Theoretic Approach, 1st ed. Cambridge University Press, 2010.

[2] ANDERSON, R. Security Engineering: A Guide to Building Dependable Distributed Systems, 2nd ed. Wiley Publishing, 2008.

[3] ASPNES, J., CHANG, K., AND YAMPOLSKIY, A. Inoculation strategies for victims of viruses and the sum-of-squares partition problem. Journal of Computer and System Sciences 72, 6 (2006), 1077–1093.

[4] BALA, V., AND GOYAL, S. A noncooperative model of network formation. Econometrica 68, 5 (2000), 1181–1230.

[5] BLOCH, F., AND JACKSON, M. Definitions of equilibrium in network formation games. International Journal of Game Theory 34, 3 (2006), 305–318.

[6] BLUME, L., EASLEY, D., KLEINBERG, J., KLEINBERG, R., AND TARDOS, É. Network formation in the presence of contagious risk. In Proceedings of the 12th ACM Conference on Electronic Commerce (2011), pp. 1–10.

[7] CERDEIRO, D., DZIUBINSKI, M., AND GOYAL, S. Contagion risk and network design. Working Paper (2014).

[8] CUNNINGHAM, W. Optimal attack and reinforcement of a network. Journal of ACM 32, 3 (1985), 549–561.

[9] FABRIKANT, A., LUTHRA, A., MANEVA, E., PAPADIMITRIOU, C., AND SHENKER, S. On a network creation game. In Proceedings of the 22nd ACM Symposium on Principles of Distributed Computing (2003), pp. 347–351.

[10] GOYAL, S. Connections: An introduction to the economics of networks. Princeton University Press, 2007.

[11] GOYAL, S. Conflicts and networks. The Oxford Handbook on the Economics of Networks. (2015).
A Reachability Network Formation Game

In this section, for completeness, we prove properties of the equilibrium network in the original reachability network formation game [4]. This case coincides with setting $C_I$ equal to zero in our formulation. Obviously, when $C_I = 0$, it is a dominant strategy for any player to immunize.

Bala and Goyal [4] state that for for a wide range of edge purchasing cost $C_E$, any equilibrium network is either a tree or the empty network.

**Proposition 1** (Proposition 4.1 of Bala and Goyal [4]). When $C_E \in (0, n - 1)$, every network that forms in an equilibrium of the reachability game is either a tree or the empty network.

**Proof.** Note that it is easy to see when $C_E \in (1, n - 1)$, the empty network is an equilibrium. Now consider any equilibrium that is not the empty network when $C_E \in (0, n - 1)$. We claim such network

- cannot have any cycles.
- cannot have more than one connected component.

These two properties imply that the equilibrium is indeed a tree as claimed.

So, first, suppose there is an equilibrium network that has a cycle. Pick any edge on this cycle. Clearly, the player who purchased this edge would remain connected to any other player that she was connected to if she drops her edge. Since $C_E > 0$ she would strictly increase her utility by dropping this edge, contradicting the assumption that the network was an equilibrium.

Second, assume the equilibrium has more than one connected component. Now, note that the size of each connected component that has an edge should be at least $C_E + 1$. Otherwise, any vertex that is in a component with size strictly less than $C_E + 1$ who purchased an edge would strictly increase her utility by dropping all her purchased edges. Since the graph is non-empty there exists a connected component in that network. It is easy to see that any vertex in any other connected component (which can be a singleton vertex) would strictly increase her utility by purchasing an edge to the mentioned connected component (since the size of the mentioned connected component is at least $C_E + 1$). This contradicts the assumption that the network was an equilibrium.

Proposition 1 implies the following about the welfare of the reachability game at equilibrium.

**Corollary 1.** In the reachability game when $C_E \in (0, n - 1)$, the maximum social welfare achieved in any equilibrium is $n^2 - C_E(n - 1)$.

**Proof.** By proposition 1 there are only two types of equilibrium. The empty graph equilibrium has a social welfare of $n$. Any tree equilibrium has a social welfare of $n^2 - C_E(n - 1)$ which is straitly bigger than the social welfare of the empty graph equilibrium when $C_E \in (0, n - 1)$. 

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[12] Gueye, A., Walrand, J., and Anantharam, V. A network topology design game: How to choose communication links in an adversarial environment. In Proceedings of the 2nd International ICTS Conference on Game Theory for Networks (2011).

[13] Jordan, C. Sur les assemblages de lignes. Journal für die reine und angewandte Mathematik 70 (1869), 185–190.

[14] Kearns, M., and Ortiz, L. Algorithms for interdependent security games. In Neural Information Processing Systems (2003).

[15] Laszka, A., Szeszler, D., and Buttyán, L. Linear loss function for the network blocking game: An efficient model for measuring network robustness and link criticality. In Proceedings of the 3rd International Conference on Decision and Game Theory for Security (2012), pp. 152–170.

[16] Mader, W. Über die maximalzahl kreuzungsfreier h-wege. Archiv der Mathematik (Basel) 31 (1978), 387–402.

[17] Roy, S., Ellis, C., Shiva, S., Dasgupta, D., Shandilya, V., and Wu, Q. A survey of game theory as applied to network security. In Proceedings of the 43rd Hawaii International Conference on System Sciences (2010), pp. 1–10.

[18] West, D. Introduction to Graph Theory, 2 ed. Prentice Hall, 2000.
We complement Proposition 1 by showing that, indeed for any tree, there exist a wide range of edge purchasing cost $C_E$ and a specific edge purchasing pattern that make that tree an equilibrium in the reachability game.

**Proposition 2.** For any tree on $n$ vertices and $c \in (0, n/2)$, there exists an edge purchasing pattern which makes that tree an equilibrium of the reachability game.

**Proof.** Given any tree, pick any vertex which satisfies the following property as the root of the tree: no subtree of this root has size bigger than $n/2$ (see Lemma 5 for a proof that such vertex exists). We claim that the pattern that every vertex buys an edge to its parent in this tree is an equilibrium. First of all it is easy to see that in the construction, the root does not purchase any edges and she is also connected to any other vertex in the network, so she does not want to purchase any edges. Now consider any other vertex that is not the root. This vertex purchases exactly one edge in the construction. If she drops that edge, her expenditure decreases by $C_E$. However, she loses connectivity to at least $n/2$ vertices. Since $C_E < n/2$, this deviation will strictly decrease her payoff. Finally, it is easy to see that such vertex does not benefit by purchasing more edges or changing her edge purchasing decision (i.e., dropping her currently purchased edge and buying one or more edges to other vertices in the network) because she is already connected to every other vertex in the network using a single edge. This completes the proof.

Note that the range of edge purchasing cost in Proposition 2 is a strict subset of the range in Proposition 1. This is because for higher edge purchasing costs, only specific (and not all) trees can form in an equilibrium.

We wrap up this section by presenting Lemma 5, which we used in the proof of Proposition 2.

**Lemma 5** (Jordan [13]). Consider a graph $G = (V, E)$ where $|V| = n$. If $G$ is a tree, then there exists a vertex $v \in V$ such that rooting the tree on $v$, no sub-tree has size more than $n/2$.

### B Difference Between Solution Concepts

As we mentioned in Section 2, linkstable equilibria and swapstable equilibria are both generalizations of Nash equilibria. In particular, this implies that any Nash equilibrium is also a swapstable and linkstable equilibrium. Furthermore, since linkstable equilibria is also a generalization of swapstable equilibria, then any swapstable equilibrium is also a linkstable equilibrium.

In this section we show that these solutions concepts are in fact different in our game. To do so, we first show an example of a swapstable equilibrium which is not a Nash equilibrium (Example 1). We then show an example of a linkstable equilibrium which is neither a swapstable equilibrium nor a Nash equilibrium (Example 2).

We start with the former and show that the set of swapstable equilibria is indeed larger than the set of Nash equilibria in our game (see Figure 14).

![Figure 14](image_url)

Figure 14: An example of swapstable equilibrium which is not a Nash equilibrium. $C_{E1} = 1$ and $C_{I1} = 4$.

**Example 1.** Let $n = 3k$ and consider $k$ disjoint trees of size 3. In each tree there is a vertex that both other vertices purchase an edge to that vertex. When $c \in (0, 3/2)$, $C_I \in [4, 6]$ and $k \geq 9$, the mentioned network is a swapstable equilibrium but is not a Nash equilibrium.

**Proof.** Due to the symmetry in this network, we only need to consider the deviations for two types of vertices: the vertex that does not purchase an edge and the vertex that does.

Let’s consider the vertex that does not purchase an edge first. Her utility is $3(1 - 1/k)$ and her swapstable deviations are as follows:

1. adding one edge.
2. adding one edge and immunizing.
3. immunizing.

We show that none of these deviations are beneficial by showing that the utility before the deviation is always (weakly) bigger than the utility after deviation.

Case 1 trivially does not happen, because if the vertex adds an edge to a different tree, she would be a part of the unique targeted region which cannot happen in any equilibrium by Lemma 2. She also does not want to purchase an edge to her tree, since she is connected to every vertex in the tree already.

In case 2, she will survive with probability 1 after immunization. As far as the edge purchasing decision, she would get the maximum utility if she purchases an edge to a different tree.

\[ C_I \ge 4 \quad \text{and} \quad C_E \ge 0 \implies C_I + C_E \ge 4 \quad \text{and} \quad k \ge 9 \implies 3 \left( 1 - \frac{1}{k} \right) \ge \left( 3 + 3 \left( 1 - \frac{1}{k-1} \right) \right) - C_E - C_I. \]

In case 3, she will survive with probability one but she has to pay for immunization.

\[ C_I \ge 4 \quad \text{and} \quad k \ge 9 \implies 3 \left( 1 - \frac{1}{k} \right) \ge 3 - C_I. \]

The only other type of vertex is the one who purchased an edge. Such vertex has a utility of \( 3 \left( 1 - \frac{1}{k} \right) - C_E \) and her swapstable deviations are as follows:
1. dropping her purchased edge.
2. dropping her purchased edge and immunizing.
3. adding one more edge.
4. adding one more edge and immunizing.
5. immunizing.
6. swapping her edge.
7. swapping her edge and immunizing.

We again show that none of these deviations are beneficial.

In case 1, after dropping her edge but the size of her connected component will also decrease.

\[ C_E \in (0, \frac{3}{2}) \quad \text{and} \quad k \ge 9 \implies 3 \left( 1 - \frac{1}{k} \right) \ge 1. \]

In case 2, once she drops her purchased edge, she is no longer a targeted vertex. So immunization has no benefits in this case and as long as case 1 is not beneficial, case 2 cannot be beneficial either.

Case 3's analysis is exactly the same as the analysis of case 1 of the vertex who does not purchase an edge.

In case 4, she will survive with probability 1 after immunization. As far as the edge purchasing decision, she would get the maximum utility if she purchases an edge to a different tree.

\[ C_I \ge 4 \quad \text{and} \quad C_E \ge 0 \implies C_I + C_E \ge 4 \quad \text{and} \quad k \ge 9 \implies 3 \left( 1 - \frac{1}{k} \right) - C_E \ge \left( 3 + 3 \left( 1 - \frac{1}{k-1} \right) \right) - 2 \cdot C_E - C_I. \]

In case 5, she will survive with probability one but she has to pay for immunization which is costly.

\[ C_I \ge 4 \quad \text{and} \quad k \ge 9 \implies 3 \left( 1 - \frac{1}{k} \right) - C_E \ge 3 - C_E - C_I. \]

In case 6, swapping her purchased edge to a vertex in another tree will cause her to be a part of the unique targeted region which cannot happen in any equilibrium by Lemma 2. Also, obviously, swapping to another vertex in her tree will leave her utility unchanged.

In case 7, swapping her purchased edge to a vertex in the same tree and immunizing is not beneficial as long case 5 is not beneficial (it has the same expenditure and benefit as in case 5). So we only need to consider the case when she swaps her edge to a vertex in a different tree and immunizes.

\[ C_I \ge 4 \implies 3 \left( 1 - \frac{1}{k} \right) \ge \left( 1 + 3 \left( 1 - \frac{1}{k} \right) \right) - C_I. \]
Finally, it is easy to come up with a strictly beneficial Nash deviation which implies that the above network is not a Nash equilibrium. Consider any vertex that did not purchase an edge. She has a utility of \(3(1 - 1/k)\). Consider her utility when she buys one edge to all the other trees and immunizes herself.

\[
C_t \leq 5 \quad \text{and} \quad k \geq 9 \quad \text{and} \quad C_E \leq \frac{3}{2} < 2 \quad \Rightarrow \quad (3k - 3) - C_E \cdot (k - 1) - C_t \geq (3k - 3) - 2(k - 1) - 5 = k - 6 \geq 3.
\]

So her utility strictly increases by this deviation, implying that such network cannot be a Nash equilibrium.

In our second example (see Figure 15), we show that the set of linkstable equilibria is indeed larger than the set of swapstable equilibria (which itself is larger than the set of Nash equilibria).

\[C_E = 2 \quad \text{and} \quad C_I = 4.\]

**Example 2.** Consider a cycle consisting of \(n = 3k\) vertices. If (i) every player buys the edge in her counter clockwise direction on the cycle, (ii) every third vertex in the cycle immunizes (so there are \(k\) immunized vertices in the cycle) and (ii) \(C_E \in (0, n/2 - 5), C_I \in (3, n/2 + 3)\) and \(k \geq 7\), then the cycle is a linkstable equilibrium but not a swapstable equilibrium.

**Proof.** First, consider any immunized vertex. This vertex clearly cannot change her immunization, regardless of how she changes her edge purchases. Since she is always connected to the vulnerable vertex to her counter clockwise direction, changing the immunization, will result in forming the unique largest targeted region which by Lemma 2 cannot happen in any equilibrium. So as long as the payoff of the immunized vertex is greater than zero before the deviation, she will not change her immunization decision.

\[
C_E < \frac{n}{2} - 5 \quad \text{and} \quad C_I < \frac{n}{2} + 3 \quad \Rightarrow \quad C_E + C_I < n - 2 \quad \Rightarrow \quad (n - 2) - C_E - C_I > 0.
\]

Furthermore, fixing the immunization decision, the linkstable edge purchasing deviations for any of the immunized vertices are as follows.

1. adding one more edge.
2. dropping her purchased edge.

In each case we consider the utilities after and before the deviation and show that given the conditions in the statement of the example, the deviation in not beneficial.

In case 1, before the deviation the immunized vertex remains connected to any vertex that survives. So adding more edges will only increase the expenditure while the connectivity benefit is the same. Since \(C_E > 0\), this deviation is not beneficial.

In case 2, dropping her edge might cause the network to become disconnected after the attack and, hence, it decreases the connectivity benefit of the vertex.

\[
C_E < \frac{n}{2} - 5 < \frac{n}{2} - \frac{3}{2} = \frac{3}{2}(k - 1) \quad \Rightarrow \quad (n - 2) - C_E - C_I > \frac{1}{k} \left(1 + 4 + \ldots + (3k - 2)\right) - C_I.
\]

Now, we consider the vulnerable vertices. Any such vertex (if survives) will remain connected to any other survived vertex. So no vulnerable vertex wants to add more edges (while keeping her purchased edge). So the possible deviations of such vertices that we need to consider are as follows.

1. immunizing.
2. dropping her purchased edge.
3. dropping her purchased edge and immunizing.

Similar to the case of the immunized vertex, we compare the utilities after and before the deviation and show that given the conditions in the statement of the example, the deviation is not beneficial.

We divide the vulnerable vertices into two disjoint categories:

(i) one that purchases an edge to an immunized vertex (type i) and
(ii) one that purchases an edge to a vulnerable vertex (type ii). For the 2nd and 3rd deviation, we need to distinguish between these two types.

In case 1, the vertex who immunizes survives. Also her vulnerable neighbor is not targeted anymore.

\[ C_I > 3 \implies C_I > 3 - \frac{6}{n} = 3 - \frac{2}{k} \implies (1 - \frac{1}{k})(n - 2) - C_E > (n - 2) - C_E - C_I. \]

In case 2, dropping her edge might cause the network to become disconnected after the attack and, hence, it decreases the connectivity benefit of the vertex. For a type i vertex,

\[ C_E < \frac{n}{2} - 5 < \frac{n}{2} - \frac{7}{2} + \frac{6}{n} \implies (1 - \frac{1}{k})(n - 2) - C_E > \frac{1}{k} \left(3 + 6 + \ldots + (3k - 3)\right). \]

For a type ii vertex, the analysis is slightly different since after dropping her purchased edge, the vertex will not be a part of any targeted region anymore.

\[ C_E < \frac{n}{2} - 5 < \frac{n}{2} - 4 + \frac{6}{n} \implies (1 - \frac{1}{k})(n - 2) - C_E > \frac{2}{k-1} \left(0 + 3 + 6 + \ldots + (3k - 6)\right). \]

In case 3, a type i vertex always survives if she immunizes but dropping the edge will decrease her connectivity benefit.

\[ C_E < \frac{n}{2} - 5 \text{ and } C_I > 3 \implies C_E - C_I < \frac{n}{2} - 8 < \frac{n}{2} - 5 + \frac{6}{n} \]
\[ \implies (1 - \frac{1}{k})(n - 2) - C_E > \frac{1}{k-1} \left(3 + 6 + \ldots + (3k - 3)\right) - C_I. \]

Note that a type ii vertex survives if she drops her edge. So immunization will only increase her expenditure. So a type ii vertex always prefers the deviation in case 2 to the deviation in case 3. And since we showed that the deviation in case 2 is not beneficial for a type ii vertex, the deviation in case 3 cannot be beneficial either.

Now it is easy to see that this network cannot form in any swapstable equilibrium. Consider two vulnerable vertices (denoted by \( u \) and \( v \)) that are connected to each other with an edge. Suppose without loss of generality that \( u \) has purchased the edge between \( u \) and \( v \). Now, it is to see that \( u \) can get a strictly higher payoff by dropping the edge purchased to \( v \) and instead buying an edge to the immunized vertex that \( v \) is connected to.

This way, \( u \) is not a part of any targeted region anymore (so she survives with probability 1 instead of \( 1 - 1/k \)) but she is still connected to every vertex she was connected to before the deviation. Since her expenditure is exactly the same this deviation will increase her utility and show that such network cannot form in any swapstable equilibrium.

\[ \square \]

C Diversity in Equilibrium

We showed in Section 3 that the equilibria of our network formation game can be quite diverse. In this section we present the examples of this diversity more formally. All the results of this section hold for Nash, swapstable and linkstable equilibria. Also similar to Section 3 blue and red vertices denote immunized and targeted vertices in the figures, respectively. Moreover, directed edges are used in the figures to determine the players who purchased the edges in the network. We proceed to present some showcases of the equilibria of our game in the remainder of this section.

C.1 Empty Graphs

We first show that an empty graph with all immunized vertices or an empty graph with all targeted vertices can form in equilibria of our game. Furthermore, we show that these are the only empty equilibrium networks of our game.
Lemma 6. There exists a range of values for parameters $C_E$ and $C_I$ such that the empty graph is a (Nash, swapstable or linkstable) equilibrium network.

Proof. First, it is easy to check that the empty network with all targeted vertices is an equilibrium when $C_E \geq 1$ and $C_I \geq 1/n$. No player would strictly prefer to purchase an edge, immunize or do them both. Also, when $C_E \geq 1$ and $C_I \leq 1/n$, the empty network with all immunized vertices is an equilibrium. This shows that regardless of value of $C_I$ when $C_E \geq 1$, the empty network is an equilibrium.

Lemma 7. Let $G$ be a (Nash, swapstable or linkstable) equilibrium network. If $G$ is the empty network, then the vertices in $G$ are either all immunized or all targeted.

Proof. For contradiction, assume an empty equilibrium network with both targeted and immunized vertices. Let $k > 0$ denote the number of players that are targeted. Since we are in an equilibrium, any immunized player (weakly) prefers immunization to remaining targeted:

$$1 - C_I \geq \left(1 - \frac{1}{k+1}\right) \implies C_I \leq \frac{1}{k+1}.$$  

Similarly, any targeted player (weakly) prefers to remain targeted compared to immunizing herself:

$$\left(1 - \frac{1}{k}\right) \geq 1 - C_I \implies C_I \geq \frac{1}{k},$$

which contradicts with the range of $C_I$ in the previous equation.

C.2 Trees

As we pointed out earlier, all the nonempty equilibria of the reachability game are trees. In this section we show that trees can also form in the equilibria of our game. In particular, we show two specific tree constructions: one in which all the vertices are immunized and the other in which all the leaves are targeted. Note that there are indeed tree equilibria that fall outside of these two categories (as some of the leaves can be targeted and some can be immunized). However, as we will see in Lemma 22 when $C_E > 1$, no non-leaf vertex can be targeted in any tree equilibrium.

We start by showing that a tree with all immunized vertices can form in equilibria.

Lemma 8. Consider any tree on $n$ vertices. Suppose $C_E \in (0, n/2)$ and $C_I \in (0, n/2)$. Then, there exists an edge purchasing pattern which makes that tree an equilibrium when all the vertices are immunized.

Proof. Since all the vertices are immunized and have a payoff of $n - C_E - C_I > 0$, no player would change her immunization decision (regardless of how she changes her edge purchases), because if so, she would form the unique largest targeted region and will be killed by the adversary.

Now that immunization decision are fixed, it is easy to see that rooting the tree as in Proposition 2 and the pattern of purchasing an edge towards the root will result in an equilibrium network.

We then show that the equilibria proposed in Lemma 8 can be used as a black-box to prove new tree equilibria in our game.

Figure 16: An example of tree equilibrium with $C_E = 2$ and $C_I = 1.9$.

Lemma 9. Consider any tree on $k \leq n/2$ immunized vertices. Add $n - k$ targeted leaf vertices to this tree such that every immunized vertex has at least one targeted neighbor (see Figure 16). Then, for $C_E \in (0, k/2)$, $C_I \in ((n-1)/(n-k), k/2 - 1)$ and $k \geq 7$, there exists an edge purchasing pattern the makes this network an equilibrium.
Proof. Root the tree of immunized vertices as described in Proposition 2 and let any immunized vertex to purchase an edge towards the root. Also let all the targeted vertices to buy the edge that connects them to the immunized tree.

Consider any immunized vertex. She is connected to all the surviving vertices after any attack and she has done so with (at most) one edge purchase. She would not change her immunization decision because she would form the unique targeted vertex. She would not add any more edges either and it is easy to see that her current edge purchase is the best in case she can only purchase one edge.

Consider any targeted vertex. She is connected to all the surviving vertices after any attack and she has done so with only edge purchase. She would not change her immunization decision either because the cost of immunization dominate the benefit she would get for surviving with probability one. She would not add any more edge either and it is easy to see that her current edge purchase is the best in case she can only purchase one edge.

Hub-Spoke
We now prove that a hub-spoke network can form in the equilibrium of our game. Although a hub-spoke network is just a special case of the tree networks described in Lemma 9, we provide a separate proof for this network which applies to a wider range of parameters in comparison to Lemma 9.

Hub-spoke network is an interesting equilibrium because it satisfies all of the following properties at the same:
(i) it is also an equilibrium in the classic reachability game,
(ii) the network is very efficient because there are no more linkage than a tree (a minimum required to connect all the players) and the number of immunized vertices is only one and (iii) the welfare at equilibrium is high, because all the surviving vertices remain connected after each attack.

Figure 17: An example of hub-spoke equilibrium with $C_E = 1$ and $C_I = 1$

Lemma 10. If $C_E \in (0, n - 3]$ and $C_I \in [1, n - 1]$ then a hub-and-spoke network is an equilibrium when the hub immunizes and the spokes buy the edges to the hub.

Proof. First observe that the expected size of the connected component of immunized and targeted vertices are $n - 1$ and $(1 - 1/(n - 1))(n - 1) = n - 2$, respectively. Also the expenditure of immunized and targeted vertices are $C_I$ and $C_E$, respectively.

Let us first consider the hub. The hub does not want to buy another edge because she already has an edge to any other vertex in the graph. So her only possible deviation is to change her immunization. If she changes her immunization decision, she will be the part of the unique targeted region. So her utility becomes zero while her expenditure becomes zero as well. So to prefer to not change her immunization decision her utility should be at least zero in the current strategy.

$$C_I \leq n - 1 \implies (n - 1) - C_I \geq 0.$$

Now consider any spoke vertex. Since the network is symmetric, it only suffices to consider the possible deviations of one spoke vertex. The spoke has the following deviations which we analyze one by one.

1. immunizing.
2. dropping her purchased edge.
3. dropping her purchased edge and immunizing.
4. dropping her purchased edge and adding new edge(s).
5. dropping her purchased edge, adding new edge(s) and immunizing.
6. adding more edges.
7. adding more edges and immunizing.

In each case we consider the utilities before and after the deviation and show that given the conditions in the
statement of Lemma 10, the deviation in not beneficial.

In case 1, the spoke survives after immunizing but has to pay the price of immunization.

\[ C_I \geq 1 = \Rightarrow (n-2) - C_E \geq (n-1) - C_I - C_E. \]

In case 2, the spoke becomes disconnected from the rest of the network.

\[ C_E \leq n-3 < n-3 + \frac{C_I}{n-1} = \Rightarrow (n-2) - C_E \geq 1 - \frac{C_I}{n-1}. \]

In case 3, the spoke survives after immunization but similar to case 2 she becomes disconnected from the
rest of the network.

\[ C_E \leq n-3 \text{ and } C_I \geq 1 = \Rightarrow C_E - C_I \leq n-4 < n-3 = \Rightarrow (n-2) - C_E \geq 1 - C_I. \]

In case 4, adding more edges will cause the vertex to form the unique targeted region so as long as the her
current utility before deviation is non-negative, she would not drop her edge and add any new edges.

\[ C_E \leq n-3 < n-2 = \Rightarrow (n-2) - C_E > 0. \]

In case 5, the spoke definitely survives after immunizing but her payoff in case of survival is capped at
\( n-1 \) (which will happen if she purchases at least 2 edges to 2 other spokes). Suppose she adds \( i \geq 1 \) edges.

\[ C_E \geq 0 \text{ and } C_I \geq 1 = \Rightarrow (n-2) - C_E \geq n - 1 - iC_E - C_3. \]

In case 6, adding more edges will only cause the vertex to form the largest targeted region which cannot
happen in any equilibrium by Lemma 2. So similar to case 4, she does not add more edges if her current
payoff is strictly greater than zero.

In case 7, after the vertex immunizes, she becomes connected to every vertex that survives. So adding
more edges strictly decrease her payoff since \( C_E > 0 \). So as long as the condition for case 1 holds, the spoke
does not deviate in case 7 either.

C.3 Forest

Nonempty networks with multiple connected components (with no immunized vertex) can form in the equi-
libria of our game. We start by showing that a forest consisting of targeted trees of equal size can form in equilibria.

**Lemma 11.** Let \( n = kF \). Then \( k \) disjoint targeted trees of size \( F \) can form in an equilibrium when \( C_E \in (0, F/4] \), \( C_I \geq (k - 7/4)F, k \geq 4 \) and \( F \geq 2 \).

**Proof.** Similar to the construction proposed in Proposition 3 in each tree, we can fix a root and guarantee that
each player only purchases one edge (towards the root). Hence, the possible deviations of any player who
purchased an edge are as follows:

1. dropping her purchased edge.
2. dropping her purchased edge and immunizing.
3. dropping her purchased edge and adding more edge(s).
4. dropping her purchased edge, adding more edge(s) and immunizing.
5. immunizing.
6. adding more edges.
7. adding more edge(s) and immunizing.
By construction, the root of the tree is the only vertex that does not purchase any edge. Hence, the deviations of
the root is only limited to cases 5-7. So in the first four cases we only consider a vertex who purchased an
edge.

In case 1, dropping the edge will cause the vertex to survive but her connectivity benefit decreases but it is at most $F/2$ due to Lemma[5]

$$C_E \leq \frac{F}{4} \text{ and } k \geq 4 \implies (1 - \frac{1}{k})F - C_E \geq \frac{F}{2}. $$

In case 2, when the vertex drops her purchased edges, she is not part of any targeted region anymore, so she survives with probability 1 even without immunization. So as long as the deviation in case 1 is not beneficial, the deviation in case 2 is not beneficial either.

In case 3, if the vertex drops her purchased edge and buys any edge(s) to any other connected component
she will form the unique targeted region which cannot happen in any equilibrium by Lemma[2]. So the only other case to consider is when she drops her edge and buys edge(s) to the same connected component she was a part of. Since she only requires one edge to connect to the component, she does not have a better deviation in this case either.

In case 4, note that the payoff of a vertex who does not purchase an edge and deviates according to case
6 is strictly better than the deviation in this case. So showing that the deviation in case 6 is not beneficial (as
we do shortly) is sufficient to show that the deviation in case 4 is not beneficial either.

For cases 5-7, first consider a vertex who purchased one edge. In case 5, when she immunizes, she
survives with probability one.

$$C_t \geq (k - \frac{7}{4})F \text{ and } k \geq 4 \implies C_t > \frac{3F}{4} \implies (1 - \frac{1}{k})F - C_E > F - C_E - C_t.$$ 

In case 6, adding more edges to other components will only result in forming the unique targeted region which cannot happen in any equilibrium by Lemma[2]. Also, the vertex is already connected to the vertices in her component and any edge beyond a tree is redundant.

Finally, in case 7, when the vertex immunizes, she survives with probability of 1. Furthermore, since
$C_E < F/4$ she would benefit the most by buying an edge to any connected component she is not connected to.

$$C_E > 0 \text{ and } C_t \geq (k - \frac{7}{4})F \implies (1 - \frac{1}{k})F - C_E > (k - 1)F - kC_E - C_t.$$ 

Now we consider cases 5-7 for the vertex that did not purchase any edge. It’s easy to verify that the argument in case 6 still holds. The argument in cases 5 and 7 also hold with the only difference that now we have to subtract a $C_t$ from both sides of the final inequalities in both cases, since the vertex initially has not purchased any edges.

The forest network in Lemma[11] was symmetric (all the trees have the same size). We now assert an
equilibrium.

**Lemma 12.** Let $n = kF + n'$. Then $k$ disjoint targeted trees of size $F$ along with $n'$ vulnerable singleton vertices can form in the equilibrium if $C_E \in (1, F/4)$, $C_t \geq (k - 1)(F - 1)$, $k \geq 4$, $F \geq 5$ and $n' \geq 0$.

**Proof.** Similar to the construction proposed in Proposition[2] for each of the trees, we can fix a root and
guarantee that each player only purchases one edge (towards the root).

First, observe that no vertex would like to add or drop an edge without changing her immunization decision. The singleton vertices, would not like to buy an edge to another singleton vertex (since $C_E > 1$) nor buy an edge to a vertex in the trees (since they would form the unique targeted region). Vertices in a tree would not want to add an edge to another vertex either, they are already connected to other vertices in their tree and would form the unique targeted region if they buy an edge to a vertex outside of their tree. Furthermore, they would not drop their only purchased edge because

$$C_E \leq \frac{F}{4} \text{ and } k \geq 4 \implies (1 - \frac{1}{k})F - C_E \geq \frac{F}{2}, $$

where again $F/2$ is the maximum connectivity benefit they can receive according to Lemma[5].

Now we show that no vertex in any of the trees could immunize and add more edges to strictly increase her
expected payoff. We only consider the root vertices, since they are the only vertices who has not purchased
an edge yet (so given the same deviation, the payoff of no vertex who purchased an edge can be higher than the root). We also point out that the most beneficial deviation is when the root buys an edge to every other tree (since she pays $C_E$ for an edge but gets a benefit of $(1 - 1/(k - 1))F > C_E$). So we have,

$$C_E \geq 1 \text{ and } C_I \geq (k - 1)(F - 1) \implies (1 - \frac{1}{k})F \geq (k - 1)F - (k - 1)C_E - C_I.$$ 

Similarly, a singleton vulnerable vertex would like to buy an edge to every tree after immunization in her best deviation which is still not beneficial.

$$C_E \geq 1 \text{ and } C_I \geq (k - 1)(F - 1) \implies 1 \geq (k - 1)F - (k - 1)C_E - C_I.$$ 

We conclude this section by pointing out that other non-symmetric forest equilibria can form in the equilibria of our game e.g., when there are some targeted trees along with vulnerable trees of smaller size and singleton vertices.

### C.4 Cycles

We now assert that unlike the reachability game, cycles can form in equilibria of our game. Indeed, we show that an alternating cycle of immunized and targeted vertices can form in equilibria.

**Lemma 13.** A cycle of $n = 2k$ alternating immunized and targeted vertices can form in equilibria when (i) every vertex buys an edge to the vertex in her clockwise direction in the cycle, and (ii) $C_E \in (1, n/2 - 2)$, $C_I \in (2, n/2 + 1)$ and $k \geq 4$.

**Proof.** The proof is by case analysis for immunized and targeted vertices, respectively. First observe that the expected size of the connected component of immunized and vulnerable vertices are $n - 1$ and $(1 - 1/k)(n - 1)$, respectively. Also the expenditure of immunized and vulnerable vertices are $C_E + C_I$ and $C_E$, respectively.

Let’s start with an immunized vertex. Note that if an immunized vertex changes her immunization decision, she will deterministically get killed by the adversary regardless of her edge purchasing decision because the immunized vertex is already connected to a targeted vertex. So as long as the payoff of an immunized vertex is greater than zero, she cannot change her immunization decision.

In case 1, after purchasing $i \geq 1$ edge(s) the expected size of the connected component of the immunized vertex is (trivially) at most $n - 1$. So the deviation is not beneficial because she is currently achieving the same expected size using only edge purchase.

In case 2,

$$C_E < \frac{n}{2} - 2 < \frac{n}{2} - 1 \implies (n - 1) - C_I - C_E > 0.$$ 

So for an immunized vertex, given that she never benefits by changing her immunization decision, the possible deviations are as follows.

1. dropping her purchased edge and adding new edge(s).
2. dropping her purchased edge.
3. adding more edge(s).

We compare the utilities after and before any of these deviation and show that given the conditions in the statement of Lemma 13 none of these deviations are beneficial.

In case 1, after purchasing $i \geq 1$ edge(s) the expected size of the connected component of the immunized vertex is (trivially) at most $n - 1$. So the deviation is not beneficial because she is currently achieving the same expected size using only edge purchase.

In case 2,

$$C_E < \frac{n}{2} - 2 < \frac{n}{2} - 1 \implies (n - 1) - C_I - C_E > \frac{1}{k} \left(1 + 3 + \ldots + (2k - 1)\right) - C_I.$$ 

In case 3, no matter which new edge(s) the immunized vertex purchases, her expected connected component has size at most $n - 1$. So adding more edges will only increase the expenditure.

Now consider a targeted vertex. Since the network is symmetric, it suffices to consider one such vertex. Her deviations are as follows.

1. dropping her purchased edge and adding new edge(s).
2. dropping her purchased edge.
3. adding more edge(s).
4. immunizing.
5. dropping her purchased edge, adding new edge(s) and immunizing.
6. dropping her purchased edge and immunizing.
7. adding more edge(s) and immunizing.

Again, we compare the utilities after and before each deviation and show that given the conditions in the statement of Lemma 13, the deviations are not beneficial.

Case 1 is similar to the analysis of case 1 for the immunized vertex with the only difference that 
\[ n - 1 \]
should be replaced by 
\[ (1 - 1/k)(n - 1) \].

In case 2,
\[ C_E < \frac{n}{2} - 2 < \frac{n}{2} - 2 + \frac{2}{n} \implies (1 - \frac{1}{k})(n - 1) - C_E > \frac{1}{k} \left( 2 + \ldots + (2k - 2) \right) \).

Similar to the case 3 for the immunized vertex, in case 3 no matter which new edge(s) the targeted vertex purchases, her expected connected component is at most 
\[ (1 - 1/k)(n - 1) \]. So adding more edges will only increase her expenditure.

In case 4, the vertex will survive with probability 1, but she has to pay for immunization.

\[ C_I > 2 > 2 - \frac{1}{k} = \frac{1}{k} (n - 1) \implies (1 - \frac{1}{k})(n - 1) - C_E > (n - 1) - C_E - C_I. \]

In case 5, when the targeted vertex immunizes, she survives with probability one. Suppose she adds \( i \geq 1 \) edges, then the size of her connected component after the attack is at most 
\( n - 1 \).

\[ C_E > 1 \text{ and } C_I > 2 \implies C_I + iC_E > 3 > \frac{1}{k}(n - 1) \implies (1 - \frac{1}{k})(n - 1) - C_E > (n - 1) - iC_E - C_I. \]

In case 6,
\[ \frac{n}{2} - 2 < 2 - \frac{1}{k} < \frac{n}{2} - 2 + \frac{2}{n} \implies (1 - \frac{1}{k})(n - 1) - C_E > \frac{1}{k} \left( 2 + \ldots + (2k - 2) \right) - C_I. \]

In case 7, the targeted vertex survives after immunization. Suppose she adds \( i \geq 1 \) edges, then the size of her connected components is at most 
\( n - 1 \).

\[ C_E > 1 \text{ and } C_I > 2 \implies C_I + iC_E > 3 > \frac{1}{k}(n - 1) \]
\[ \implies (1 - \frac{1}{k})(n - 1) - C_E > (n - 1) - (i + 1)C_E - C_I. \]

\[ \Box \]

C.5 Flowers

We next show that multiple cycles can form in equilibrium. The flower equilibrium in this section is an illustration of such phenomenon. In the flower equilibrium each petal has the same pattern of immunization as the cycle in Lemma 13.

Lemma 14. Let \( n = F(2k - 1) + 1 \). Consider a flower network containing \( F \) petals (cycles) of size \( 2k \) where all the cycles share exactly one vertex. Assume each petal is composed of alternating immunized and targeted vertices, and the shared vertex is immunized. Then the flower network can form in an equilibrium when (i) in each petal, the targeted vertices buy both of the edges to their immunized neighbors, and (ii) \( C_I \in (2, (2k - 1)F) \), \( C_E \in (0, \min \{(k - 1)F - 2, (\left( k - 1 \right)^2 + 5) / (2kF)\}) \), \( k \geq 2 \) and \( F \geq 3 \).
Proof. First note that the expected size of the connected component for immunized and targeted vertices are
\((2k - 1)F\) and \((1 - 1/(kF))(2k - 1)F\), respectively. Also the expenditure of immunized and targeted
vertices are \(C_I\) and \(2C_E\), respectively.

First, consider any immunized vertex. Any such vertex is connected to every survived vertex in the
network after any attack. So no such vertex wants to add any edges. Furthermore, she does not want to
change her immunization decision because (regardless of her edge purchases) she will form the unique largest
targeted region. So as long as her current payoff is bigger than zero, she would not change her action. This
means
\[ C_I < (2k - 1)F \iff (2k - 1)F - C_I > 0. \]

Now, consider any targeted vertex. Such vertex (when survives) is also connected to to every survived
vertex in the network after any attack. Since she managed to do so with only two edge purchases, it suffices
for us to only consider deviations such that the number of edges purchased by the targeted vertex is at most
2. So her possible deviations are as follows.
1. buying two edges and immunizing.
2. buying one edge.
3. buying one edge and immunizing.
4. buying no edges.
5. buying no edges and immunizing.

We compare the utilities before and after each deviation and show that given that the conditions in
Lemma [4] that none of the deviations are beneficial.

Remind that the current edge purchases of any targeted vertex connect her to any survived vertex. So if an
targeted vertex is buying two edges, she can do no better than her current purchases. So in case 1, it suffices
to check only the deviation in the immunization decision.

In case 2, first observe that if a targeted vertex is going to buy a single edge, she will buy it to the central
immunized vertex if she wants to maximize her expected size of the connected component after attack\(^{12}\). Second, among all the targeted vertices in a petal, the targeted vertex with the maximum expected size of
the connected component is the vertex who is \(k - 1\) hops away from the central immunized vertex.\(^ {13} \) So it
suffices to consider the deviation of such vertex.

\[ C_E < \min \left\{ (k - 1)F - 2, \frac{(k - 1)^2 + 5}{2kF} \right\} \implies \frac{(k - 1)^2 + 5}{2kF} \implies (1 - \frac{1}{kF})(2k - 1)F - 2C_E > (2k - 1)F - 2C_E - C_I. \]

The same argument holds for case 3, with the only difference than the vertex will survive with probability
of 1.

\[ C_E < \min \left\{ (k - 1)F - 2, \frac{(k - 1)^2 + 5}{2kF} \right\} < \frac{(k - 1)^2 + 5}{2kF} \text{ and } C_I > 2 - \frac{1}{kF} \implies \frac{(1 + 3 + \ldots + (k - 3) + 1 + 3 + \ldots + (k - 1)) - C_E}{kF} \]

\[^{12}\] She would remain connected to at least \((F - 1)\) of the petals if she survives.
\[^{13}\] Fix a petal and consider a targeted vertex who purchased an edge to the central immunized vertex. If the attack happens outside of
this petal, then the size of connected is the same for all the targeted vertices in the petal. If the attack happens in the petal and the vertex
survives, her expected utility is at least half of her petal size (and sometimes more) regardless of the attack.
In case 4, she still survives with the same probability but her size of connected component is only 1 anytime she survives.

\[
C_E \leq \min\{(k - 1)F - 2, \frac{(k - 1)^2 + 5}{2kF}\} \leq (k - 1)F - 2 \leq kF - \frac{F}{2} - \frac{3}{2} + \frac{1}{2k} + \frac{1}{2kF} \\
\implies (1 - \frac{1}{kF})(2k - 1)F - 2C_E \geq (1 - \frac{1}{kF}).
\]

In case 5, she survives with probability 1, but the size of her connected component is 1.

\[
C_B \leq \min\{(k - 1)F - 2, \frac{(k - 1)^2 + 5}{2kF}\} \leq (k - 1)F - 2 \leq kF - \frac{F}{2} - \frac{1}{2} + \frac{1}{2k} \quad \text{and} \quad C_I > 2 \\
\implies (1 - \frac{1}{kF})(2k - 1)F - 2C_B > 1 - C_I.
\]

The number of edges in the flower equilibrium is \(n + F - 1\). So to get the densest flower equilibrium, it suffices to set \(F\) as large as possible or \(k\) as small as possible. Setting \(k = 2\) will result in a flower equilibrium with \(4n/3 - O(1)\) edges. We finally point out that among all the examples of equilibrium in this section with strictly more than \(n\) edges, the flower is the only example that remains an equilibrium even when \(C_B > 1\).

### C.6 Complete Bipartite Graph

We finally show that a specific form of complete bipartite graph can form in equilibria. The equilibria presented in Lemma 15 has \(2n - 4\) edges which shows that our upper bound on the density of equilibria (Theorem 3) is tight.

**Lemma 15.** Consider a complete bipartite graph \(G = (U \cup V, E)\) with \(|U| = 2\) and \(|V| \geq 1\). \(G\) can form in equilibria if all the vertices in \(U\) are targeted, all the vertices in \(V\) are immunized, the vertices in \(U\) purchase all the edges in \(E\), \(C_E \in [0, 1/2]\) and \(C_I \in [(n - 1)/2, n - 1]\).

**Proof.** The proof is by case analysis for immunized and targeted vertices, respectively. First observe that the expected size of the connected component of immunized and targeted vertices are \(n - 1\) and \((n - 1)/2\), respectively. Also the expenditure of immunized and targeted vertices are \(C_I\) and \((n - 2)C_E\), respectively.

Consider an immunized vertex first. If she changes her immunization decision, she will deterministically get killed by the adversary regardless of her edge purchasing decision because the immunized vertex is already connected to a targeted vertex and hence she will form the unique largest targeted region. So as long as her payoff is greater than zero, she would not change her immunization decision.

\[
C_I \leq n - 1 \implies (n - 1) - C_I \geq 0.
\]

Also the immunized vertex remains connected to every vertex that survives, regardless of the attack. So she would not want to by any edges.

Now consider a targeted vertex. First, note that since \(C_B \leq 1/2\), the current utility of a targeted vertex is at least \(1/2\). Next, it is easy to observe that no deviation of a targeted vertex can be beneficial if she purchases an edge to the other targeted vertex, regardless of her choice of immunization or her other edge purchases. If she does not immunize, buying an edge to the other targeted vertex will result in forming the largest unique targeted region which cannot happen in any equilibrium by Lemma 2. If she immunizes, the other targeted vertex becomes the unique largest targeted region, so she would not benefit by purchasing an edge in this case either.

This, together with the symmetry of the network with respect to immunized vertices imply that the deviations of a targeted vertex that we need to consider are as follows.

1. purchasing \(k \in \{0, \ldots, n - 3\}\) edges to immunized vertices.
2. purchasing \(k \in \{0, \ldots, n - 3\}\) edges to immunized vertices and immunizing.

We compare the utilities of the targeted vertex before and after the deviation and show that none of the above deviations are beneficial.
In case 1,
\[ k \leq n - 3 < n - 2 \text{ and } C_E \leq \frac{1}{2} \implies (n - k - 2)C_E \leq \frac{n - k - 2}{2} \]
\[ \implies \frac{n - 1}{2} - (n - 2)C_E \geq \frac{k + 1}{2} - kC_E. \]

In case 2,
\[ C_E \leq \frac{1}{2} \text{ and } C_I \geq \frac{n - 1}{2} \geq \frac{k + 1}{2} \implies (n - k - 2)C_E - C_I \leq \frac{n - k - 2}{2} - \frac{k + 1}{2} \]
\[ \implies \frac{n - 1}{2} - (n - 2)C_E \geq (k + 1) - kC_E - C_I. \]

\[ \square \]

D Properties of Non-trivial Equilibria When \( C_E > 1 \)

In this section we examine several structural properties of non-empty equilibrium networks in our game. In particular, in section D.1 we show that all non-trivial (Nash, swapstable or linkstable) equilibria are connected when \( C_E > 1 \). Furthermore, we show all non-empty equilibria with no immunized vertex have multiple connected components. Next, in Section D.2 we show that targeted regions are singletons in any non-trivial (Nash or swapstable) equilibria. We wrap up this section by presenting the proof of Lemma 4 in Section D.3.

D.1 Connectivity in Equilibria

In this section, we examine the connectivity of the networks formed in equilibria. In particular, in Proposition 3 we show that any non-trivial (Nash, swapstable or linkstable) equilibrium network is connected when \( C_E > 1 \). Furthermore, we show in Proposition 4 that in the absence of any immunized vertex in the (Nash, swapstable or linkstable) equilibrium, only forests can form in equilibria when the equilibrium has at least one edge.

We start by analyzing the connectivity of non-trivial equilibrium networks. More specifically, in Lemma 16 we show that every immunized vertex in non-trivial equilibria has an adjacent edge. Then, in Lemma 17 we use this observation to show that all the immunized vertices in a non-trivial equilibrium network are in the same connected component. Next, in Lemma 18 we show that all non-targeted vertices are in the same connected component as the immunized vertices. Finally, in Lemma 19 we show that no isolated targeted regions can exist in non-trivial equilibria when \( C_E > 1 \).

Lemma 16. Suppose \( G = (V, E) \) is a non-trivial (Nash, swapstable or linkstable) equilibrium network. Then, for all \( u \in I \), there is an edge \((u, v) \in E\).

Proof. Suppose not. Then there exists an immunized vertex \( u \) with no adjacent edge. By assumption of the Lemma, there exists an edge \((x, y) \in E\). Let \( N_x \) be the connected component of \( x \) (and \( y \)) before the attack.

Let \( k \) denote the number of targeted regions in \( G \). Since \( G \) is an equilibrium network, any vertex in \( N_x \) that is buying an edge does not (strictly) prefer to drop any of her edges. Let \( \mu \) denote the size of the connected component of \( x \) after she drops the edge \((x, y)\). Trivially, \( \mu \geq 1 \). So, either \( N_x \) is targeted and
\[ (1 - \frac{1}{k})|N_x| - C_E \geq \mu \geq 1 \implies (1 - \frac{1}{k})|N_x| \geq C_E + 1, \]
or \( N_x \) is not targeted and
\[ |N_x| - C_E \geq \mu \geq 1 \implies |N_x| \geq C_E + 1 \]

If \( u \in I \) were to buy an edge to \( x \), in the former case she would get an expected utility of
\[ (1 - \frac{1}{k})|N_x| + 1 - C_E - C_I \geq 2 - C_I > 1 - C_I, \]

14Remind that non-trivial equilibria have at least one edge and one immunized vertex.
and in the latter case she would get an expected utility of
\[ |N_u| + 1 - C_E - C_t \geq 2 - C_t > 1 - C_t. \]
Thus, she strictly prefers to buy an edge rather than remaining isolated; a contradiction. Therefore, there must be an edge adjacent to \( u \in I \).

Lemma 17. Suppose \( G = (V, E) \) is a non-trivial (Nash, swapstable or linkstable) equilibrium network. Then, all the immunized vertices of \( G \) are in the same connected component.

Proof. Suppose not. Then the immunized vertices are in multiple connected components. Let \( G_1 \) and \( G_2 \) be two such components and let us denote the immunized vertices in them by \( u_1 \) and \( u_2 \), respectively (if there are more than one immunized vertex in each component, pick one arbitrarily). We show that the deviation that \( u_1 \) purchases an edge to \( u_2 \) is strictly beneficial for \( u_1 \). Consider any edge adjacent to \( u_2 \) that is purchased by either \( u_2 \) or some other vertex in \( G_2 \). This edge exists due to Lemma 16. Call the vertex who purchased the edge \( v \). Let \( \mu \) denote the connectivity benefit of \( v \). Since we are in an equilibrium, then \( v \) (weakly) prefers to maintain this edge. So
\[ \mu - C_E \geq R \implies \mu \geq C_E + R, \]
where \( R \) denote the connectivity benefit of \( v \) after dropping the edge. \( R \geq (1 - 1/k) \) if \( v \) is targeted and otherwise \( R \geq 1 \). Now the expected utility of \( u_1 \) after buying an edge to \( u_2 \) will increase by at least \( \mu - C_E \geq R > 0 \); a deviation that strictly improves \( u_1 \)’s expected payoff. So the immunized vertices cannot be in different connected component.

Lemma 18. Suppose \( G = (V, E) \) is a non-trivial (Nash, swapstable or linkstable) equilibrium network. Then, all the non-targeted vulnerable vertices of \( G \) are in the same connected component as any of the immunized vertices of \( G \).

Proof. First note that by Lemma 17 all the immunized vertices are in the same connected component of \( G \). Call this component \( G \). Assume by contradiction that there exists a non-targeted vulnerable vertex \( w \) that is not in \( G \). Pick any immunized vertex in \( G \) and call it \( u \). We show that the deviation that \( w \) purchases an edge to \( u \), strictly increases \( w \)’s expected payoff. \( u \) has an adjacent edge by Lemma 16. Let \( v \) be the vertex who purchased the adjacent edge to \( u \). Let \( \mu \) denote the connectivity benefit of \( v \). Since we are in an equilibrium, then \( v \) (weakly) prefers to maintain this edge. So
\[ \mu - C_E \geq R \implies \mu \geq C_E + R, \]
where \( R \) denote the connectivity benefit of \( v \) after dropping the edge. \( R \geq (1 - 1/k) \) if \( v \) is targeted and otherwise \( R \geq 1 \). Now the expected utility of \( w \) after buying an edge to \( u \) will increase by at least \( \mu - C_E \geq R > 0 \); a deviation that strictly improves \( w \)’s expected payoff. So all the non-targeted vulnerable vertices in \( G \) as well.

Lemma 19. Suppose \( G = (V, E) \) is a non-trivial (Nash, swapstable or linkstable) equilibrium network. Then, no isolated targeted region can exist in \( G \).

Proof. We consider two cases: (1) targeted regions are singletons or (2) targeted regions have size strictly bigger than 1.\(^{15}\)

In case (1), suppose by contradiction there exists some targeted vertex \( u \in U \) which has no adjacent edge. We will show that some vertex in \( G \) has a beneficial deviation in this case. Let \( k \) denote the number of targeted regions, and \( R = 1 - 1/k \) the expected payoff of any targeted vertex in \( G \) with no adjacent edge; in particular, the expected payoff of \( u \) is also \( R \). Note that by Lemma 17 all the immunized vertices are in the same connected component of \( G \). Let \( v \in I \) be an immunized vertex in \( G \). Let \( N_v \) denote the connected component (before attack) of \( v \). We will consider 2 cases in turn: (a) there exists an immunized vertex in \( N_v \) who is purchasing an edge, and (b) all the edges in \( N_v \) are purchased by targeted vertices.

In case (a), let \( v' \in N_v \) and \( v' \in I \) be an immunized vertex who purchased at least one edge in \( N_v \). Since we are in an equilibrium, \( v' \) weakly prefers her current strategy to dropping any of her edges along with changing her immunization. Suppose she purchased \( i \) edges. This implies
\[ \mu - iC_E - C_t \geq \mu' - (i - 1)C_E, \]

\(^{15}\) The targeted regions for Nash and swapstable equilibria are singletons when \( C_E > 1 \) as we will show in Lemma 20. So case (2) only applies to linkstable equilibria.
where \( \mu \) and \( \mu' \) are the connectivity benefit of \( v' \) before and after the deviation. Since \( \mu' \geq 0 \), then \( \mu \geq C_E + C_I \). Now consider the deviation that \( u \) purchases an edge to \( v' \) and immunizes. In this deviation, her connectivity benefit is \( 1 + \mu \). So, her expected utility after her deviation is
\[
1 + \mu - C_E - C_I \geq 1 > R.
\]

So \( u \) can strictly increase her expected utility; a contradiction to \( G \) being an equilibrium.

In case (b), if there exists a targeted vertex \( u' \) in \( N_v \) who purchased an edge \((u', v')\) to an immunized vertex \( v' \) such that the marginal benefit of the edge is strictly bigger than \( C_E \), \( u \) can also purchase an edge to \( v' \) and increase her utility strictly; a contradiction. So then assume the marginal benefit from all the edge purchases in \( N_v \) is exactly zero. Consider an edge \((u', v')\) in \( N_v \) and consider the deviation that \( u \) purchases an edge to \( v' \) (which is immunized). Since \( C_E > 1 \), there is at least another targeted vertex \( u'' \) in \( N_v \) who purchase an edge to \( v' \). This means the marginal benefit of \( u \) for the edge \((u, v')\) is strictly bigger than zero; because \( u \) would get at least the same benefit as \( u' \) for the edge and will also get strictly higher benefit in case of the attack to \( u'' \) (or any other targeted vertex besides \( u' \) who purchased an edge to \( v' \)). Again a strictly beneficial deviation for \( u \), which is a contradiction.

Finally, consider case (2) where the targeted regions have size strictly bigger than 1. Suppose the size of targeted regions are \( T \) and there are \( k \) targeted regions in \( G \). Consider a targeted vertex \( u \) in an isolated targeted region who purchased an edge to another targeted vertex \( u' \). Since, we are in an equilibrium, \( u \) (weakly) prefers to keep her edge to \( u' \) which implies
\[
(1 - \frac{1}{k})T - C_E \geq \mu \geq 1 \implies (1 - \frac{1}{k})T \geq C_E + 1,
\]
where \( \mu \geq 1 \) is the size of connected component of \( u \) if she drops the edge \((u, u')\). Now consider an immunized vertex \( v \) in \( G \) and a deviation that \( v \) purchases an edge to \( u \). Then \( v \)’s expected utility after this deviation is increased by
\[
(1 - \frac{1}{k})T - C_E \geq 1,
\]
which is a contradiction. \( \square \)

Figure 18: An example of non-trivial Nash equilibrium with more than one connected component. \( C_E = 0.5 \) and \( C_I = 4 \).

Note that the condition that \( C_E > 1 \) is necessary in Lemma \[19\] as illustrated in the example in Figure \[18\] where a non-trivial (Nash, swapstable and linkstable) equilibrium has an isolated targeted vertex. We now use Lemmas \[17,19\] to analyze the connectivity of non-trivial equilibrium networks.

**Proposition 3.** Let \( G \) be a non-trivial (Nash, swapstable or linkstable) equilibrium network. Then, \( G \) is connected when \( C_E > 1 \).

**Proof.** Suppose by contradiction that \( G \) has multiple connected components. Lemma \[17\] shows that all the immunized vertices in \( G \) are in the same connected component. Call that component \( G_v \). Next, consider any vulnerable vertex \( v \) that is not targeted and is not in \( G_v \). Lemma \[18\] shows that such isolated \( v \) cannot exist. Next, consider any targeted vertex which is not part of \( G_v \). Lemma \[19\] shows that such isolated targeted vertices cannot exist when \( C_E > 1 \). So when \( C_E > 1 \), \( G \) is connected when it has an immunized vertex and an edge. \( \square \)

\[^{16}\]This means the expected utility of \( u' \) would drop by an amount bigger than zero when she drops the edge \((u', v')\).
We wrap up this section by analyzing the connectivity in equilibrium networks with an edge but no immunized vertices. We highlight that Proposition 4 hold for all ranges of $C_E$.

**Proposition 4.** Let $G$ be a (Nash, swapstable or linkstable) equilibrium network with at least one edge but no immunized vertex. Then $G$ is a forest.

**Proof.** We show that any nonempty equilibrium with no immunized vertex should have more than one con-

cnected component and cannot have any cycles. Since there is no immunized vertex in the network, the equi-
lum cannot have a single connected component since all the vertices will die in the attack and whoever purchased an edge in such network would strictly prefer to drop it. Finally, none of the connected com-
ponents can have a cycle because any edge beyond the tree in that component is redundant. These imply that the nonempty equilibria with no immunization are all forests.

**D.2 Singleton Targeted Regions**

In this section we show that when $C_E > 1$, in any non-trivial Nash or swapstable equilibrium, targeted regions are singletons; hence, no two vulnerable vertices will share an edge.

**Lemma 20.** Suppose $C_E > 1$ and let $G$ be a non-trivial (Nash or swapstable) equilibrium network. Then the size of targeted regions are at most 1.

**Proof.** Suppose not. Then there exists some targeted region $T_i$ with $|T_i| > 1$. By Lemma 2, the subgraph of $G$ on $T_i$ forms a tree. Then, this tree must have at least two leaves $x, y \in T_i$. We claim that there is some vertex in $T_i$ who would strictly prefer to swap her edge to some immunized vertex in $G$ rather than an edge which connects her to the remainder of $T_i$.

Proposition 4 shows that any non-trivial Nash or swapstable network is connected, so $G$ is connected. Also since $G$ contains some immunized vertex (since $G$ is non-trivial), any connection between $T_i$ and the rest of $G$ is through immunized vertices. We consider two cases and show that none of them is possible.

1. **One of $x$ or $y$ buys her edge in the tree.** Suppose without loss of generality $x$ buys an edge in the tree.

   - Since $G$ is connected, there exists an immunized vertex $z$ which is connected to some vertex in $T_i$. If $x$ is not connected to $z$, then $x$ would strictly prefer to buy an edge to $z$ over buying her tree edge. So suppose $x$ is connected to $z$. Then if $y$ also bought her tree edge, she would also strictly prefer an edge to $z$. If $y$ did not buy her tree edge, then her parent bought it; since $C_E > 1$, this implies $y$ must be connected to some immunized vertex $z'$ (or it would not be worth connecting to $y$); then, $y$'s parent would strictly prefer to buy an edge to $z'$ over an edge to $y$. Thus, $x$ cannot have bought her tree edge (either $y$ or her parent would like to re-wire if this were the case).

2. **Neither $x$ nor $y$ buys her connecting edge in the tree.** Since $C_E > 1$, both $x$ and $y$ must have immunized neighbors (or their edges being purchased by $x$'s parent and $y$'s parent would not be best responses by those vertices). But then, both $x$'s parent and $y$'s parent would strictly prefer to buy an edge to $x$ and $y$'s immunized neighbor rather than to $x$ and $y$, respectively.

Thus, since these cases are exhaustive and none of them is possible, no two vulnerable vertices in the same $T_i$ can be connected by an edge in $G$. By construction, $T_i$ are the maximum-sized connected components of vulnerable vertices; thus, if the $T_i$s are singletons, no non-targeted region $N_i$ can contain any vulnerable vertex. Finally, since any vertex in $T_i$ is not connected to a vertex in another targeted region $T_j$ (by Lemma 1), no pair of vulnerable vertices is connected by an edge in $G$.

Lemma 20 does not hold for non-trivial linkstable equilibria. For example Figure 15 illustrates a linkstable equilibrium that has targeted regions of size bigger than 1 when $C_E > 1$. This supports our previous claim that several structural properties of Nash networks also hold for swapstable networks but might not hold for linkstable networks.

**D.3 Proof of Lemma 4**

**Proof of Lemma 2** By Proposition 4 $G$ is connected. Also Lemma 20 shows that the size of targeted regions in $G$ is at most 1.
E  Higher Welfare in Special Cases

In this section, we show that the welfare achievable in the equilibria of our game can be $n^2 - O(n)$ (matching the welfare of the reachability game) in special cases. In particular, in Lemmas 21 and 22, we show conditions under which the network remains connected after the attack. This paired with the fact that the targeted regions are singletons when $C_E > 1$ (see Lemma 20) imply that the connectivity benefit in such networks is $(n-1)^2$. Finally, the assumption that $C_E$ and $C_I$ are $O(1)$ as in Section 6 immediately results in a lower bound of $n^2 - O(n)$ on the welfare.

So in the remainder of this section, we examine properties under which the network remains connected after attack. Again consider the block-cut tree decomposition of the equilibrium network. We first in Lemma 21 show that in order for any cut vertex in the block-cut tree decomposition to be targeted in an equilibrium, the probability of attack to such cut vertex should be low. So Lemma 21 implies that when the probability of attack to a cut vertex is high, all the cut vertices are immunized.

**Lemma 21.** Let $C_E > 1$ and $G$ be a non-trivial (Nash or swapstable) equilibrium on $n$ vertices. Let $v$ be a cut vertex in the block-cut tree decomposition of $G$. Then $v$ is immunized if the probability of attack to a targeted vertex is strictly bigger than $2 \cdot \max\{C_E, C_I\}/n$.

**Proof.** Suppose by contradiction that $v$ is targeted. Root the block cut tree decomposition of $G$ at $v$. Then $v$ has at least two child blocks (because $v$ is cut vertex). Let $n^*$ denote the size of the largest subtree of $v$ and let $b^*$ denote the child block of $v$ which is part of the largest subtree (so $n^* = |T_{b^*}|$).

Since $C_E > 1$, all vulnerable vertices are targeted and all targeted regions are singletons by Lemma 4. This implies that the edge $(v, u_b)$ in each child block $b$ of $v$ corresponds to $u_b$ being an immunized vertex. Finally, we let $p = 1/|T|$ to denote the probability of attack to each targeted vertex where $T$ is the set of targeted vertices.

In case of attack to $v$, the children of $v$ becomes disconnected. Since $G$ is in equilibrium, for any $u_b \neq u_{b^*}$ to not purchase an edge to $u_{b^*}$ we need:

$$p \cdot n^* \leq C_E.$$  

Furthermore, one possible deviation for $v$ is to immunize. In that case, her connectivity payoff increases by at least $p \cdot (n - n^*)$ because she will remain connected to all her children except one of them regardless of the attack. In this deviation, however, she pays an extra $C_I$ for immunization. Since we are in an equilibrium, this implies:

$$p \cdot (n - n^*) \leq C_I.$$  

Finally, either $n^* \geq n/2$ or $n - n^* \geq n/2$. So if $p > 2 \cdot \max\{C_E, C_I\}/n$ one of Equations 2 or 3 would be violated. So

$$p \leq \frac{2 \cdot \max\{C_E, C_I\}}{n}.$$

We next show that in any tree equilibrium, all the non-leaf vertices are immunized.

**Lemma 22.** Let $C_E > 1$ and $G = (V, E)$ be a non-trivial (Nash or swapstable) equilibrium on $n$ vertices. Then any non-leaf vertex in $G$ is immunized.

**Proof.** Lemma 4 implies that $G$ is connected, vulnerable regions are singletons and all vulnerable vertices are targeted. We prove $G$ has no non-targeted vertex by contradiction. So suppose $G$ has some non-targeted vertices. We consider two cases and reach a contradiction in each case: (1) $G$ only has one targeted non-leaf vertex or (2) $G$ has more than one targeted non-leaf vertices.

In case (1), let $v$ be the targeted non-leaf vertex. If we root the tree on $v$, then the tree has at least two subtrees (because $v$ is a non-leaf). So assume $v$ has $d \geq 2$ children and let $T_1, \ldots, T_d$ denote the subtrees of $v$. Let $T^*$ denote the largest subtree of $v$ ($T^*$ might not be unique) and let $u^*$ be the immunized vertex in $T^*$ such that $(v, u^*) \in E$ (remind that since $v$ is targeted and targeted regions are singletons by Lemma 4, when $C_E > 1$ then $u^*$ should be immunized). Now consider any other subtree $T_i \neq T^*$ of $v$. Pick any leaf $u_i$ in that subtree. Since $c > 1$, then $u_i$ has purchased her edge to her parent. Now consider the deviation that $u_i$ swaps her edge to $u^*$. In this deviation, she would get a strictly higher payoff in case of an attack to $v$ and would get the same payoff in case of an attack to any other targeted vertex. This contradicts the assumption that $G$ is an equilibrium.

In case (2), there exists a vertex $u$ such that if we root the tree on $u$, then there exists a targeted non-leaf vertex $v$ that satisfies the following two properties: (a) all the non-leaf vertices in $T_u$ (except $v$) are immunized
and (b) \(|T_v| < n/2\). We first show that \(u\) exists. Suppose we root the tree arbitrarily. Then there exists a non-leaf targeted vertex \(v'\) such that all the non-leaf vertices in \(T_{v'}\) (except \(v'\)) are immunized. If \(|T_{v'}| < n/2\), then we are done and we can pick the current root as \(u\). Otherwise, we can root the tree at \(v'\). So when we pick \(v'\) as root, all the vertices that were not in \(T_{v'}\) in the initial rooting will now become another subtree of \(v'\) (call this subtree \(T'\) and \(|T'| < n/2\)). Since there is at least another targeted non-leaf vertex in \(T'\), we know there exists a targeted non-leaf vertex \(v\) (in \(T'\)) that satisfies properties (a) and (b) as well.

Finally, consider vertex \(v\) that satisfies (a) and (b). Since all the non-leaf vertices in \(T_v\) are immunized and there are at least two non-leaf targeted vertices in the tree, then \(v\) should have an immunized parent \(w\) (again remind that since \(v\) is targeted and targeted regions are singletons by Lemma 4 when \(\Gamma > 1\) then \(w\) should be immunized). Now consider the deviation that a leaf in \(T_v\) swaps her edge from her parent to \(w\). In this deviation, she would get a strictly higher payoff in case of an attack to \(v\) and would get the same payoff in case of an attack to any other targeted vertex. This again contradicts the assumption that \(G\) is an equilibrium.

As we mentioned earlier, if the conditions of Lemma 21 or 22 were satisfied for a non-trivial equilibrium \(G\), then the network \(G\) remains connected after each attack. And this in turn will result in a very high social welfare in equilibria.

### F Best Response Cycles

In this section, we show that Nash (and also swapstable and linkstable) best response dynamics can cycle if we start from an arbitrary initial graph, the players best respond in a fix order but when a player has several best responses, ties are broken adversarially.

![Figure 19: Nash best response cycles. \(C_E = 7/6\) and \(C_I = 20\).](image)

**Example 3.** Consider the network in Figure 19 (with all vulnerable vertices) with \(n = 20\), \(C_E = 7/6\) and \(C_I = 20\) to be the initial configuration in running the Nash best response dynamics. If the vertices Nash best respond in the increasing order of their labels, then there exists a tie breaking rule which causes the best response dynamics to cycle.

**Proof.** Since the components are symmetric, we only analyze one of the components. Vertices 1 and 2 are currently best responding (although each has a deviation with the same payoff but we break ties in favor of their current action). Vertex 3’s best response is to drop her edge. Vertex 4’s best response is to connect back to the same component she was a part of before vertex 3’s best response. We break ties by forcing vertex 4 to purchase an edge to vertex 1.

After the first round, we are in the same pattern as before but the labels of the vertices are different. So in the next round vertex 2 would drop her edge and vertex 3 would buy an edge to vertex 4. In the third round, vertex 1 would drop her edge. In the fourth round, and vertex 2 would buy an edge to vertex 3. In the fourth round, vertex 4 would drop her edge. In the fifth round, vertex 1 would buy an edge to 2, vertex 3 would drop her edge and vertex 4 would buy an edge to vertex 1. So we are back in the same configuration that we were at the beginning or round 2 (see Figure 19).

Since we considered Nash best responses, but all the best responses chosen by the adversary were linkstable deviations, Example 3 also shows that swapstable and linkstable best response dynamics can cycle.
if the order of vertices who best respond are fixed but the ties in the best responses of a vertex are broken adversarially.

We suspect that this phenomenon is the result of adversarial tie-breaking and/or the ordering on the vertices. So we conjecture that starting from any initial graph, the swapstable (and linkstable) best response dynamics will converge to an equilibrium network when the ordering of the vertices who best respond in each round is random and the ties in the best responses of a vertex are broken randomly.

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17 A similar observation has been made for the convergence of Nash best responses in the original reachability game.

18 In our experimental results in Section 7, we used a fixed tie-breaking rule and yet the simulations always converged to an equilibrium.