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A Filippov system describing media effects on the spread of infectious diseases

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**A R T I C L E  I N F O**

Article history:
Received 3 December 2012
Accepted 27 June 2013

Key words:
Filippov system
Media coverage
Threshold policy
Sliding bifurcation
Global asymptotic stability

**A B S T R A C T**

A Filippov epidemic model with media coverage is proposed to describe the real characteristics of media/psychological impact in the spread of an infectious disease. We extend the existing models by incorporating a piecewise continuous transmission rate to describe that the media coverage exhibits its effect once the number of infected individuals exceeds a certain critical level. Mathematical and bifurcation analyses with regard to the local, global stability of equilibria and local sliding bifurcations are performed. Our main results show that the system stabilizes at either the equilibrium points of the two subsystems or the new endemic state induced by the on–off media effect, depending on the threshold levels. The finding suggests that a previously chosen level of the desired number of infected individuals can be reached when the threshold policy and other parameters are chosen properly.

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1. Introduction

When a type of contagious disease appears and starts to spread, people’s response to the threat of disease is dependent on their perception of risk, which is affected by public and private information disseminated widely by the media. Massive news coverage and fast information flow can generate a profound psychological impact on the public. Media communications have played an important role in affecting the outcome of infectious disease outbreaks, such as the 2003 severe acute respiratory syndrome (SARS) and the recent 2009 H1N1 influenza epidemic [1–6]. In particular, the propagation of information and education by media have great influence not only on individual behavior but also on the implementation of public intervention and control measures. If susceptible individuals are alert enough to avoid unnecessary contact with infected individuals or all infected individuals stay in the region being treated and cancel their trips, new transmission will apparently decrease due to reduced contact rates [7,8]. The role of media coverage on the transmission of disease is thus crucial and should be given prominence in the study of disease dynamics.

There has been an increasing interest in models in which the impact of media coverage on disease spread in a given population is considered. Liu et al., in [1], described the impact of media coverage on the transmission coefficient by a decreasing factor, resulting in the transmission coefficient $\beta \exp(-a_1E - a_2I - a_3H)$, where $E$, $I$, and $H$ are the numbers of reported exposed, infectious, and hospitalized individuals, respectively. Cui et al., in [3], proposed a compartment model with incidence rate $\mu \exp(-mI)SI$ with $m > 0$ to reflect the impact of media coverage on the transmission. They have shown that Hopf bifurcation can occur when $m > 0$ is sufficiently small. Li et al., in [6], used $\beta_2I/(m + I)$ to reflect the reduced amount of contact rate through media coverage and formulated an SIS (susceptible–infective–susceptible) epidemic model with incidence rate $(\beta_1 - \beta_2I/(m + I))SI/N$. Moreover, other forms, such as $(\mu_1 - \mu_2f(I))SI/(S + I)$, have been formulated to approximate the media-induced incidence rate (see details in [2,4,5]).
A common assumption for the models in [1–6] is that the impact of media coverage on the transmission of the infectious disease occurs as soon as the disease emerges and remains during the whole process of the disease’s spreading. However, this is not really the case. Mostly, at the initial stages of an emerging infectious disease, both the general individuals and public mass media are unaware of the disease. Media reports, information processing, and individuals’ alerted responses to the information can only arise as the number of infected individuals reaches and exceeds a certain level. For example, the A/H1N1 virus was first reported in two US children in Mexico in March 2009, but health officials have stated that it apparently infected people as early as January 2009 in Mexico [9]. Hence, at the initial stage, the number of A/H1N1 cases and information about early precautionary measures against the disease were not available. With a diseases spreading further and information about infectious disease outbreaks being reported, media coverage about an epidemic disease can encourage the public to take precautionary measures against the disease such as wearing masks, avoiding public places, avoiding travel when sick, and frequent hand washing.

Therefore, it is necessary to refine the existing mathematical models with a media-induced effect to reflect this feature. Our main purpose is then to modify the known models in order to describe the impact of media coverage on disease transmission by introducing a piecewise continuous transmission term. We shall examine the effect of media coverage on disease transmission and to address whether the improved function representing a media-induced effect influences the global dynamics of disease transmission.

The rest of the paper is organized as follows. In Section 2, we propose a Filippov epidemic model to describe the media-induced effect on the transmission of infectious diseases. In Section 3, sliding mode dynamics and the existence of a pseudo-equilibrium are investigated. In Section 4, the bifurcation set and the sliding bifurcation of the proposed system are discussed. The global dynamics under variable threshold levels is examined in Section 5, and a discussion and biological conclusion are given in Section 6. Finally, for readers’ interests, we present a brief introduction to threshold policy in the Appendix.

2. Filippov epidemic model and preliminaries

When the number of infected individuals increases and reaches a certain level \( I_\epsilon \), mass media often start to report information about the disease, including ways of transmission, number of infected individuals, and then the public try their best to avoid being infected. This consequently lowers the effective contact, resulting in a reduction in transmission rate which is usually represented by \( \exp(-\alpha I) \). We consider a population that is divided into three types: susceptible, infective, and recovered. Let \( S \), \( I \), and \( R \) denote the proportions of susceptible, infective, and recovered individuals with respect to the total population of individuals, respectively. Then the model equations are

\[
\begin{align*}
\frac{dS(t)}{dt} &= \mu - \beta \exp(-\alpha I)SL - \mu S, \\
\frac{dI(t)}{dt} &= \beta \exp(-\alpha I)SL - (\mu + \gamma)I, \\
\frac{dR(t)}{dt} &= \gamma I - \mu R, \\
\end{align*}
\]

with

\[
\epsilon = \begin{cases} 
0, & \sigma(S, I) < 0, \\
1, & \sigma(S, I) > 0,
\end{cases}
\]

where \( \sigma(S, I) \) is a given function which may depend on the number of susceptible and infected individuals, and \( \epsilon \) is a discontinuous control function. But here we choose the special function \( \sigma(S, I) = \sigma(I) = I - I_\epsilon \), which means that only the number of infected individuals is the index for mass media to exhibit its effect. We note that model (1) with \( \epsilon = 0 \) was introduced and examined in [10]. All other parameters are positive constants, where \( \mu \) is the natural birth (death) rate, \( \beta \) denotes the basic transmission rate, and \( \gamma \) represents the removed/recovered rate. Model (1) with (2) reveals a dynamical system subject to a threshold policy (TP), which is referred to as an on-off control or as a special and simple case of variable structure control in the control literature. A more detailed introduction on the TP is presented in the Appendix.

Since the recovered class \( R \) does not influence the dynamics of the first two equations of model (1), we only need to focus on the first two equations in the rest of this work. It is easy to show that

\[
\{(S, I) \in R^2_+ | 0 < S(t) + I(t) \leq 1 \} \ni \Omega
\]

is an attraction region of system (1). In fact, it follows from model (1) that \( S(t) + I(t) + R(t) = 1 \) and

\[
\left. \frac{dS(t)}{dt} \right|_{S=0} = \mu > 0, \quad \left. \frac{dI(t)}{dt} \right|_{I=0} = 0, \quad \left. \frac{d(S + I)}{dt} \right|_{S+I=1} = \mu - \mu(S + I) - \gamma I < 0,
\]

so all solutions will enter into the region \( \Omega \) and remain in it ultimately. Hence, \( \Omega \) is an attraction region. The \((S, I)\) phase plane is split into three parts: \( G^1 = \{(S, I) \in R^2_+ | \sigma(I) < 0\} \), \( G^2 = \{(S, I) \in R^2_+ | \sigma(I) > 0\} \), and the discontinuous boundary

\[
\Sigma = \{(S, I) \in R^2_+ | \sigma(I) = 0\}.
\]
Thus, in region $G^1$, there is no effect from media coverage, and the transmission rate $\beta$ remains unchanged, while in region $G^2$, media coverage exhibits its effect, and hence the transmission rate declines to $\beta \exp(-\alpha I)$.

For convenience, let the vector $Z = (S, I)^T$, and denote
\[
F_G(Z) = (\mu - \beta I - \mu S, \beta S - (\mu + \gamma)I)^T, \\
F_G(Z) = (\mu - \beta \exp(-\alpha I)I - \mu S, \beta \exp(-\alpha I)I - (\mu + \gamma)I)^T.
\]

Then system (1) with (2) can be rewritten as the following Filippov system [12]:
\[
\dot{Z} = \begin{cases} 
F_G(Z), & Z \in G^1, \\
F_G(Z), & Z \in G^2.
\end{cases}
\] (4)

The main characteristic of a Filippov system is that control is suppressed when the value of the threshold function is below a previously chosen threshold policy; above the threshold, control is applied, while in classical systems, control has been implemented all along [13–15]. In the following, we call Filippov system (4) defined in region $G^1$ system $S^1$ and that defined in region $G^2$ system $S^2$.

Since the right-hand side of system (4) is piecewise continuous, we consider its solutions in Filippov’s sense throughout the paper, which refers to solutions of the differential inclusions $\dot{Z}(t) \in F(Z(t))$, where
\[
F(Z(t)) = \bigcap_{\delta > 0} \bigcap_{\mu N=0} F_G(B(Z; \delta) \setminus N) \cup F_G(B(Z; \delta) \setminus N),
\]
$\overline{\text{co}}$ stands for the convex hull, $B(Z; \delta)$ is the open $\delta$-neighborhood of $Z$, and $\mu$ denotes the Lebesgue measure. It follows from the definition of $F_G$, $i = 1, 2$, that $F(Z(t))$ is nonempty, bounded and closed, convex, and upper semicontinuous in $Z$. Then, by [12], one gets that there exists a solution for system (4) with initial value $S(t_0) = S_0, I(t_0) = I_0$.

2.1. Dynamics for system $S^1$

We note that system $S^1$ has been studied in [10,11], but for clarity we present the main result and main method here. It is easy to obtain the basic reproduction number $R_0 = \beta / (\mu + \gamma)$.

**Proposition 1.** The disease-free equilibrium $E_0 = (1, 0)$ is globally asymptotically stable if $R_0 < 1$, while the unique endemic equilibrium $E^1 = (S^1, I^1)$ is feasible and globally asymptotically stable if $R_0 > 1$, where $S^1 = 1/R_0, I^1 = \mu / (\beta (R_0 - 1))$.

**Proof.** If $R_0 < 1$, taking a Lyapunov function $V(t) = I(t)$ and applying LaSalle’s invariance principle, one gets that $E_0$ is globally asymptotically stable. If $R_0 > 1$, we choose a Lyapunov function
\[
V(t) = \frac{1}{2} \omega_1 (S - S^1)^2 + \omega_1 S^1 \left( I - I^1 - I^1 \ln \frac{I}{I^1} \right).
\]

Then, by using LaSalle’s invariance principle, we conclude that the endemic equilibrium $E^1$ is globally asymptotically stable.

Moreover, the endemic equilibrium $E^1$ is a stable focus if $\Delta_1 < 0$, while it is a stable node if $\Delta_1 \geq 0$, where $\Delta_1 = \beta^2 \mu - 4(\mu + \gamma)^2 [\beta - (\mu + \gamma)]$.

2.2. Dynamics for system $S^2$

For system $S^2$, the basic reproduction number is also $R_0$, and the disease-free equilibrium is $E_0$. The possible endemic state $E^2 = (S^2, I^2)$ satisfies
\[
S^2 = \frac{\mu + \gamma}{\beta} \exp(\alpha I^2), \\
\mu - (\mu + \gamma)I^2 - \frac{\mu (\mu + \gamma)}{\beta} \exp(\alpha I^2) = 0.
\]

Denote
\[
g(I) = \mu - (\mu + \gamma)I - \frac{\mu (\mu + \gamma)}{\beta} \exp(\alpha I);
\]
then we have
\[
g(0) = \mu - \frac{\mu (\mu + \gamma)}{\beta} > 0
\]
provided that $R_0 > 1$ and
\[
g \left( \frac{\mu}{\mu + \gamma} \right) = -\frac{\mu (\mu + \gamma)}{\beta} \exp \left( \frac{\alpha \mu}{\mu + \gamma} \right) < 0.
\]
Further, simple calculation yields 
\[ g'(l) = - (\mu + \gamma) - \frac{\alpha \mu (\mu + \gamma)}{\beta} \exp(\alpha l) < 0 \]

for all \( l \geq 0 \). All those indicate that there exists a unique \( l \) such that \( g(l) = 0 \), denoted by \( l_s \) and such that \( 0 < l_s < \mu / (\mu + \gamma) \), which can be determined analytically. In fact, \( g(l_s) = 0 \) is equivalent to

\[ \frac{\alpha \mu}{\beta} \exp\left( \frac{\alpha \mu}{\mu + \gamma}ight) = \alpha \left( \frac{\mu}{\mu + \gamma} - l_s^2 \right) \exp\left( \alpha \left( \frac{\mu}{\mu + \gamma} - l_s^2 \right) \right). \]

According to the definition of the Lambert \( W \) function and solving the above equation with respect to \( l_s^2 \), one obtains

\[ l_s^2 = \frac{\mu}{\mu + \gamma} - \frac{1}{\alpha} \text{Lambert W}\left( \frac{\alpha \mu}{\beta} \exp\left( \frac{\alpha \mu}{\mu + \gamma} \right) \right). \]

Consequently, \( S^2 \) can be determined. The definition and properties of the Lambert \( W \) function can be found in the literature [16].

**Proposition 2.** The disease-free equilibrium \( E_0 \) is globally asymptotically stable if \( R_0 < 1 \), while the endemic state \( E^2 \) is globally asymptotically stable if \( R_0 > 1 \).

**Proof.** If \( R_0 < 1 \), we take a Lyapunov function \( V(t) = I(t) \), and the same discussion as that in the proof of Proposition 1 yields that \( E_0 \) is globally asymptotically stable. If \( R_0 > 1 \), it is easy to get that the Jacobian matrix of \( S^2 \) at \( E^2 \) possesses two eigenvalues with negative real parts, so it is locally asymptotically stable. Further, letting a Dulac function \( B = 1/(SI) \) and using Bendixon–Dulac criteria, we can exclude the existence of limit cycles, and hence \( E^2 \) is globally asymptotically stable.

Moreover, it is a stable focus if \( \Delta_2 < 0 \), where

\[ \Delta_2 = \left( \frac{\mu}{S^2} + \alpha l_s^2 (\mu + \gamma) \right)^2 - 4 \frac{l_s^2 (\mu + \gamma)}{S^2} (\alpha \mu S^2 + \mu + \gamma). \]

Note that the equilibria \( E^1 \) and \( E^2 \) are endemic equilibria for subsystems \( S^1 \) and \( S^2 \), respectively. However, they could be either in region \( G^1 \) or in \( G^2 \). If the equilibrium points are located in their opposite regions, they are named virtual equilibrium points. Otherwise, they are called real equilibrium points. Both real equilibrium and virtual equilibrium are called standard equilibrium or regular equilibrium (see the definitions in [17,18] or the Appendix).

3. Sliding domain and its dynamics

There may exist sliding segments on the boundary (3) if there are some subregions of the line \( \Sigma \) such that both vectors of the two systems \( S^1 \) and \( S^2 \) are directed towards each other. Mathematically, there are two methods to determine sufficient conditions for a sliding mode to occur on the discontinuous surface: the well-known Filippov convex method [12] and Utkin’s equivalent control method [18]. In the following, we employ the equivalent control method to determine the sliding domain and sliding mode dynamics of Filippov system (4).

Let 
\[ h(Z) = \langle \sigma_Z(Z), F_{G^1}(Z) \rangle = \langle \sigma_Z(Z), F_{G^2}(Z) \rangle = F_{G^1} \sigma(Z) \cdot F_{G^2} \sigma(Z), \]

where \( \langle \cdot \rangle \) denotes the standard scalar product and \( F \sigma(Z) = \langle \sigma_Z(Z), F(Z) \rangle \) is the Lie derivative of \( \sigma \) with respect to the vector field \( F \) at \( Z \). Then the sliding domain can be defined as

\[ \Sigma_s = \{ Z \in \Sigma | F_{G^1} \sigma(Z) \geq 0, F_{G^2} \sigma(Z) \leq 0 \}. \]

By the definition of the switching surface \( \sigma(Z) \) (i.e., \( \sigma(Z) = I - I_c \)), and solving the two inequalities given in the above equation, one obtains the sliding domain of the Filippov system (4) as follows:

\[ \Sigma_s = \left\{ (S, l) \in R^2_+ \Bigg| \frac{\mu + \gamma}{\beta} \leq S \leq \frac{\mu + \gamma}{\beta} \exp(\alpha l_c), l = l_c \right\}. \tag{5} \]

Next, we address the dynamics of Filippov system (4) on the boundary \( \Sigma_s \) [18]. Note that \( \sigma' = l' = \beta \exp(-\alpha \epsilon l) S - (\mu + \gamma) l \). Let \( \sigma' = 0 \); solving with respect to \( \epsilon \), we have

\[ \epsilon = -\frac{1}{\alpha l} \ln\left( \frac{\mu + \gamma}{\beta S} \right). \]
Substituting this and $\sigma = I - l_c = 0$ into the first equation of model (1), we obtain the sliding mode dynamics, which can be determined by the following differential equation:

$$S' = \mu - \mu S - (\mu + \gamma)I_c$$

(6)

with

$$\frac{\mu + \gamma}{\beta} \leq S \leq \frac{\mu + \gamma}{\beta} \exp(\alpha l_c).$$

It follows that there is an equilibrium $S_\ast = (\mu - (\mu + \gamma)I_c)/\mu$ for system (6) provided that

$$\frac{\mu + \gamma}{\beta} < S_\ast = \frac{\mu - (\mu + \gamma)I_c}{\mu} < \frac{\mu + \gamma}{\beta} \exp(\alpha l_c),$$

which is equivalent to

$$l_1^2 < l_c < l_1^1,$$

(7)

where $l_1^1$ and $l_1^2$ are defined in Section 2.

Since the components of both $E^1$ and $E^2$ satisfy $\mu - \mu S - (\mu + \gamma)I = 0$ and $S_1^1 < S_\ast < S_2^2$, we know that $l_1^1 > l_1^2$, which implies that the inequalities (7) are well defined. Note that the equilibrium $E_\ast = (S_\ast, I_\ast) = ((\mu - (\mu + \gamma)I_c)/\mu, I_c)$ is called a pseudo-equilibrium of Filippov system (4) according to Definition A.2 in the Appendix.

In order to analyze the relation between the sliding segment $\Sigma_\ast$ and the attraction region $\Omega$, we need to solve the following equations:

$$\frac{\mu + \gamma}{\beta} = 1 - I \quad \text{and} \quad \frac{\mu + \gamma}{\beta} \exp(\alpha I) = 1 - I.$$

It is easy to derive

$$l_1^c = \frac{\beta - (\mu + \gamma)}{\beta}$$

from the first equation. For the second equation, denote $h(I) = (\mu + \gamma)/\beta \exp(\alpha I) + I - 1$, and by implementing the similar discussion for the function $g(I)$, one can get that there exists a unique $I$ satisfying $h(I) = 0$ and $0 < I < 1$, denoted by $l_1^c$, which can be solved explicitly. In fact, $h(l_1^c) = 0$ can be written as

$$\frac{\alpha(\mu + \gamma)}{\beta} \exp(\alpha) = \alpha(1 - l_1^c) \exp(\alpha(1 - l_1^c)),$$

solving which yields

$$l_1^c = 1 - \frac{1}{\alpha} \text{Lambert W} \left( \frac{\alpha(\mu + \gamma)}{\beta} \exp(\alpha) \right).$$

Then we can get that the sliding segment $\Sigma_\ast$ is totally out of the attraction region $\Omega$ if $l_c > l_1^1$, while it lies totally in $\Omega$ for $l_c < l_1^2$. If $l_1^2 < l_c < l_1^1$, part of $\Sigma_\ast$ is out of the attraction region $\Omega$ and part of it is in $\Omega$.

4. Sliding bifurcation analysis

Bifurcation sets of equilibria and sliding modes are related to the sliding bifurcations, and all these are useful for the forthcoming global dynamics analysis.

4.1. Bifurcation sets of equilibria and sliding modes

In this subsection, we will address the richness of the possible equilibria and sliding modes that the system can exhibit. Parameters $\alpha$ and $l_c$ are fixed to build the bifurcation diagram, and all other parameters are fixed as those in Fig. 1. We define the four curves for parameters $\alpha$ and $l_c$ as follows: $L_1 = \{(\alpha, l_c)|l_c = l_1^1\}, L_2 = \{(\alpha, l_c)|l_c = l_1^2\}, L_3 = \{(\alpha, l_c)|l_c = l_1^1\}, \text{and } L_4 = \{(\alpha, l_c)|l_c = l_1^2\}$, where $l_1^i, l_2^i, i = 1, 2$, are defined as in previous sections.

The two solid lines $L_1$ and $L_2$ divide the $\alpha - l_c$ parameter space into three parts, and the coexistence of regular, virtual, or pseudo equilibria are indicated in each part. When $l_c > l_1^1$ (i.e., the region $I_1 \cup I_2 \cup I_3$), the equilibrium $E^1$ is real, while $E^2$ is virtual (denoted by $E^1_v$ and $E^2_v$, respectively), and $E_5$ does not exist. When $l_1^2 < l_c < l_1^1$ (i.e., the region $I_4 \cup I_5$), both the equilibria $E^1$ and $E^2$ are virtual (denoted by $E^1_v$ and $E^2_v$, respectively), and $E_5$ does exist. When $l_c < l_1^2$ (i.e., the region $I_6$), the equilibrium $E^2$ is real, while $E^1$ is virtual (denoted by $E^1_v$ and $E^2_v$, respectively), and $E_5$ does not exist, either. It should be
Fig. 1. Bifurcation set for Filippov system (4) with respect to $\alpha$ and $I_c$. Here we fixed all other parameters as follows: $\mu = 0.2$; $\beta = 1$; $\gamma = 0.1$.

noted that the equilibria of system (4) actually include the equilibria of $S^1$ or $S^2$, which may be real or virtual equilibria, and pseudo-equilibria, if feasible.

The two dashed lines $L_3$ and $L_4$ also divide the parameter space into three parts. In region $\Gamma_1$, the sliding segment $\Sigma_5$ is totally out of the attraction region $\Omega$. In region $\Gamma_2 \cup \Gamma_4$, $\Sigma_5$ is partly out of the attraction region $\Omega$ and it lies totally in $\Omega$ in region $\Gamma_3 \cup \Gamma_5 \cup \Gamma_6$.

This implies that, as the threshold level $I_c$ increases, the sliding segment $\Sigma_5$ will enlarge and may intersect with the boundary of the attraction region $\Omega$ such that part of it is in the attraction region or even lies out of the attraction region ultimately.

It is clear that, if one or more parameters changes, some bifurcations occur in Fig. 1. For example, boundary focus bifurcation as well as boundary node bifurcation can occur when one of the parameters varies such as $I_c$, which will be addressed in detail in the following subsection.

4.2. Boundary focus bifurcation

In this subsection, we fix all other parameters and choose the threshold value $I_c$ as the bifurcation parameter. By Section 2, we know that $E^i$ is a focus (node) if $\Delta_i < 0$ ($\Delta_i \geq 0$), where $i = 1, 2$. In the following, we only investigate the boundary focus bifurcation (i.e., $\Delta_i < 0$); the boundary node bifurcation (i.e., $\Delta_i > 0$) can be discussed similarly. To clearly show the boundary focus bifurcation of Filippov system (4), we first discuss the following two types of special point named the tangent point (denoted as $T$) and the boundary equilibrium (denoted as $E_b$). See [19,20] and the Appendix for detailed definitions of the tangent point and boundary equilibrium.

Tangent points of Filippov system (4) satisfy

$$\beta \exp(-\alpha \epsilon s_l) - (\mu + \gamma) = 0, \quad l = l_c,$$

solving which with respect to $S$ yields the following two tangent points:

$$T_1 = \left( \frac{\mu + \gamma}{\beta}, l_c \right) \quad \text{and} \quad T_2 = \left( \frac{\mu + \gamma}{\beta} \exp(\alpha l_c), l_c \right).$$

It is clear that points $T_1$ and $T_2$ are the endpoints of the sliding segment $\Sigma_5$.

The boundary equilibrium of Filippov system (4) satisfies

$$\mu - \beta \exp(-\alpha \epsilon s_l) - S = 0, \quad \beta \exp(-\alpha \epsilon s_l) - (\mu + \gamma) = 0, \quad l = l_c.$$

We require the equalities

$$\frac{\mu + \gamma}{\beta \exp(-\alpha \epsilon l_c)} = \frac{\mu}{\mu + \beta \exp(-\alpha \epsilon l_c) l_c}$$

to hold true to ensure the existence of boundary equilibria, which indicates that $l_c = l^*_1$ for $\epsilon = 0$, while, for $\epsilon = 1$, it is equivalent to

$$\frac{\alpha \mu}{\beta} \exp\left(\frac{\alpha \mu}{\mu + \gamma} - \alpha l_c\right) = \left(\frac{\alpha \mu}{\mu + \gamma} - \alpha l_c\right) \exp\left(\frac{\alpha \mu}{\mu + \gamma} - \alpha l_c\right).$$
solving which gives \( l_c = l_c^2 \). Hence, there exists boundary equilibrium

\[
E_b^1 = \left( \frac{\mu}{\mu + \beta l_c}, l_c \right) \quad \text{or} \quad E_b^2 = \left( \frac{\mu}{\mu + \beta \exp(-\alpha l_c)} l_c, l_c \right)
\]

provided that \( l_c = l_c^1 \) or \( l_c = l_c^2 \).

If the pseudo-equilibrium, tangent point, and real equilibrium (or tangent point and real equilibrium) collide simultaneously when the parameter \( l_c \) passes through a critical value, boundary focus bifurcation may occur. For example, when parameter \( l_c \) passes through the first critical value \( l_c = l_c^1 \), the equilibrium \( E_b^1 \) and tangent point \( T_1 \) can collide, as shown in Fig. 2(b), where \( l_c^1 = 0.4667 \). In this case, the boundary equilibrium \( E_b^1 \) is an attractor (see Fig. 2(b)). A stable focus \( E_b^1 \) and a tangent point \( T_1 \) coexist for \( l_c > l_c^1 \), as shown in Fig. 2(a) with \( l_c = 0.5 \). They collide at \( l_c = 0.4667 \) and are replaced by a pseudo-equilibrium \( E_b^2 \) and a tangent point \( T_1 \) for \( l_c < l_c^1 \), as shown in Fig. 2(c) with \( l_c = 0.43 \). This bifurcation shows how a stable focus can become a stable pseudo-equilibrium.

Another boundary focus bifurcation of Filippov system (4) occurs when parameter \( l_c \) passes through the second critical value \( l_c = l_c^2 \). In this case, a stable focus \( E_b^2 \) and a tangent point \( T_2 \) coexist for \( l_c < l_c^2 \). They collide at \( l_c = l_c^2 \) and are replaced by a pseudo-equilibrium \( E_b^2 \) and a tangent point \( T_2 \) for \( l_c > l_c^2 \).

5. Global behavior of the system

We shall investigate the global stability of all the possible equilibria, including the standard equilibrium and pseudo-equilibrium, under different threshold levels and \( R_0 \). If \( R_0 \) is less than 1, the disease is eradicated, and it is not necessary to consider any media-induced effect; thus we only focus on the case when \( R_0 > 1 \).

5.1. Nonexistence of limit cycles

Indeed, to prove the global stability of the equilibrium of system (4), we need to rule out the existence of limit cycles. First, we shall preclude the existence of limit cycles totally in region \( G^1 \) or \( G^2 \). Denote the right-hand sides of system \( S^1 \) by \( f^{(i)}(x) \), where \( f^{(i)}(x) = (f_1^{(i)}(x), f_2^{(i)}(x)) \), \( i = 1, 2 \). Let the Dulac function be \( B = 1/(S\bar{S}) \). For system \( S^1 \), we have

\[
\frac{\partial(Bf_1^{(1)})}{\partial S} + \frac{\partial(Bf_2^{(1)})}{\partial \bar{S}} = -\frac{\mu}{S^2} \beta < 0,
\]

so there is no limit cycle totally in \( G^1 \). Similarly, for system \( S^2 \), we get

\[
\frac{\partial(Bf_1^{(2)})}{\partial S} + \frac{\partial(Bf_2^{(2)})}{\partial \bar{S}} = -\frac{\mu}{S^2} \alpha \exp(-\alpha l) < 0,
\]

so no limit cycle is totally in \( G^2 \).

By the above discussion, we easily obtain the following result.

Lemma 1. There is no limit cycle for Filippov system (4) which is totally in region \( G^i \) (\( i = 1, 2 \)).

Next, we shall preclude the existence of limit cycles, pieces of which are in the sliding region \( \Sigma_3 \).

Lemma 2. There is no closed orbit for Filippov system (4) which contains part of the sliding segment \( T_1T_2 \).

Proof. By Section 4.1, there does exist a pseudo-equilibrium \( E_b^3 \) for \( l_c^2 < l_c < l_c^1 \) which is asymptotically stable in the sliding domain \( \Sigma_3 \). The local stability of the pseudo-equilibrium \( E_b^3 \) on the sliding domain indicates that no limit cycle containing part of the sliding segment exists in such a case. Therefore, we only need to show the result for \( l_c > l_c^1 \) or for \( l_c < l_c^2 \). In the following, we shall do it for \( l_c > l_c^1 \); for \( l_c < l_c^2 \), we can use a similar process.

In fact, since \( l_c > l_c^1 \), by the expression of the sliding domain given in (5), the sliding mode dynamics (6) satisfies \( S' < 0 \), which shows that the trajectory moves from the right to the left on the sliding segment \( T_1T_2 \) (as shown in Fig. 3).

We only need to show that the orbit starting from the tangent point \( T_1 \), denoted by \( l_1 \), cannot hit the manifold \( \Sigma \) again. Since \( E_b^1 \) is a stable focus for \( \Delta_1 < 0 \), the orbit \( l_1 \) intersects with the horizontal isoline \( g_1 \), where \( g_1 \) refers to the isoline \( \{S, l\} = (\mu + \gamma)/\beta, l < l_c \), at point \( M_1 \) first, and at point \( N_1 \) second. Hence we conclude that point \( N_1 \) lies between \( T_1 \) and \( E_b^1 \). Otherwise, the trajectory \( l_1 \) may either coincide with point \( T_1 \) or hit the manifold \( \Sigma \) at some point to the right of point \( T_1 \). If the former is true, \( l_1 \) is a closed orbit which is tangent to the sliding segment \( T_1T_2 \). Then any trajectory starting from a point out of the closed orbit \( l_1 \) cannot cross the cycle \( l_1 \), and certainly cannot tend to the endemic equilibrium \( E_b^1 \), which contradicts the stability of equilibrium \( E_b^1 \) in \( G^1 \). If the latter holds, it indicates that the endemic state \( E_b^1 \) is an unstable focus, which also contradicts that \( E_b^2 \) is stable in \( G^1 \). Hence the orbit initiating from point \( T_1 \) cannot hit the manifold \( \Sigma \) but spirally approaches the stable equilibrium \( E_b^1 \). Therefore, no closed orbit containing part of the switching segment \( T_1T_2 \) exists. This completes the proof.
Fig. 2. Boundary focus bifurcation for Filippov system (4). Here we choose $I_c$ as a bifurcation parameter and fix all other parameters as follows: $\mu = 0.2, \beta = 1, \gamma = 0.1, \alpha = 0.8, I_c = 0.5$ (a), $I_c = 0.4667$ (b), $I_c = 0.43$ (c).

Fig. 3. Phase plane $S$-$I$ for Filippov system (4), showing the switching line ($I = I_c$), sliding domain ($T_1T_2$), real equilibrium (small circle: $E_1^r$), virtual equilibrium (small circle: $E_1^v$), and the attraction region $\Omega$ bounded by the line $S+I = 1$. The horizontal and vertical isoclinic curves for the free system $S^1$ are $\{(S, I) \in R_2^+ | S = (\mu + \gamma)/\beta, I < I_c \}$ and $\{(S, I) \in R_2^+ | I = -\mu/\beta + \mu/(\beta S), I < I_c \}$, denoted by $g_S^1$ and $g_I^1$, respectively. The horizontal and vertical isoclinic curves for the controlled system $S^2$ are $\{(S, I) \in R_2^+ | I = 1/\alpha \ln(\beta S/(\mu + \gamma)), I > I_c \}$ and $\{(S, I) \in R_2^+ | I = -1/\alpha \text{Lamb} \mu/(\beta \mu - \alpha \mu/(\beta S)), I > I_c \}$, denoted by $g_S^2$ and $g_I^2$, respectively. The typical orbit $l_t$ is plotted simply to show the asymptotical stability of the focus $E_1^r$ for Case 1 without great accuracy.

We now rule out the existence of the limit cycle induced by the nonsmooth nature of the trajectories on the complement to $\Sigma_3$ in $\Sigma'$; that is, we will prove that there is no limit cycle surrounding the sliding segment $T_1T_2$ which is partly in region $G^1$ and partly in region $G^2$. So, we give the following result when the sliding segment is totally within the attraction region.

**Lemma 3.** There is no limit cycle surrounding the sliding segment $T_1T_2$ in the attraction region $\Omega$.

**Proof.** The method of proving this lemma is similar to that in [21,22], but for readers interested in the technical aspects we give the details in the following. Suppose that $X$ is a limit cycle in the interior of the attraction region $\Omega$, and that it surrounds the sliding segment $T_1T_2$ with a period $T$, as shown in Fig. 4. Denote its part below the line $I = I_c$ by $X_1$ and its part above the line $I = I_c$ by $X_2$. Denote the intersection points of the limit cycle $X$ and the line $I = I_c$ by $H_1$ and $H_2$, the intersection points of $X$ and the auxiliary line $I = I_c - \epsilon$ by $A_1$ and $A_2$, and the intersection points of $X$ and another auxiliary
line $I = I_c + \epsilon$ by $A_4$ and $A_3$, where $\epsilon$ ($\epsilon > 0$) is any sufficiently small number. Let $D_1$ be the region delimited by $X_1$ and the segment $A_2A_1$, and let $D_2$ be the region delimited by $X_2$ and the segment $A_4A_3$. We denote the boundary of $D_1$ and $D_2$ by $C_1$ and $C_2$, respectively, with the directions indicated in the Fig. 4. Let the Dulac function be $B = 1/(SI)$ defined as before. By the discussion given above and Green’s theorem, we get

$$
\int_{D_1} \left[ \frac{\partial (B^{(1)}_1)}{\partial S} + \frac{\partial (B^{(1)}_2)}{\partial I} \right] dSdl = \oint_{C_1} B \left[ f_1^{(1)} dl - f_2^{(1)} ds \right] = -\int_{A_2A_1} B^{(1)}_2 ds,
$$

$$
\int_{D_2} \left[ \frac{\partial (B^{(2)}_1)}{\partial S} + \frac{\partial (B^{(2)}_2)}{\partial I} \right] dSdl = \oint_{C_2} B \left[ f_1^{(2)} dl - f_2^{(2)} ds \right] = -\int_{A_4A_3} B^{(2)}_2 ds.
$$

Let $D_0 \subset D_1$ and

$$
\xi = \int_{D_0} \left[ \frac{\partial (B^{(1)}_1)}{\partial S} + \frac{\partial (B^{(1)}_2)}{\partial I} \right] dSdl;
$$

then, since we know that $\xi < 0$, and, furthermore,

$$
0 > \xi > \int_{D_1} \left[ \frac{\partial (B^{(1)}_1)}{\partial S} + \frac{\partial (B^{(1)}_2)}{\partial I} \right] dSdl + \int_{D_2} \left[ \frac{\partial (B^{(2)}_1)}{\partial S} + \frac{\partial (B^{(2)}_2)}{\partial I} \right] dSdl,
$$

we obtain

$$
0 > \xi > -\int_{A_2A_1} B^{(1)}_2 ds - \int_{A_4A_3} B^{(2)}_2 ds.
$$

(8)

Suppose that the abscissas of the points $H_1$, $H_2$, $A_1$, $A_2$, $A_4$, $A_3$ are $\bar{S}$, $\bar{S}$, $\bar{S} + h_1(\epsilon)$, $\bar{S} - h_2(\epsilon)$, $\bar{S} + h_4(\epsilon)$, $\bar{S} - h_3(\epsilon)$, respectively, where $h_i(\epsilon)$ ($i = 1, 2, 3, 4$) is continuous and satisfies $\lim_{\epsilon \to 0} h_i(\epsilon) = 0$ and $h_i(0) = 0$. Thus, we get

$$
\lim_{\epsilon \to 0} \left[ -\int_{A_2A_1} B^{(1)}_2 ds \right] = \lim_{\epsilon \to 0} \int_{\bar{S} - h_2(\epsilon)}^{\bar{S} + h_1(\epsilon)} \left( \beta - \frac{\mu + \gamma}{\bar{S}} \right) dS
$$

$$
= \lim_{\epsilon \to 0} \left[ \beta(\bar{S} - \bar{S} - h_2(\epsilon) + h_1(\epsilon)) - (\mu + \gamma) \ln \frac{\bar{S} - h_2(\epsilon)}{\bar{S} + h_1(\epsilon)} \right]
$$

$$
= \beta(\bar{S} - \bar{S}) - (\mu + \gamma) \ln \frac{\bar{S}}{\bar{S}}.
$$
By the same process, we get
\[
\lim_{\epsilon \to 0} \left[ - \int_{\partial_{\delta} A_3} B f(2) \, ds \right] = - \lim_{\epsilon \to 0} \int_{\Sigma + h_4} \left( \beta \exp(-\alpha l_c + \epsilon) - \frac{\mu + \gamma}{S} \right) \, dS
\]
\[
= - \lim_{\epsilon \to 0} \left[ \beta \exp(-\alpha l_c + \epsilon) \left( \frac{S - h_3(\epsilon) - h_4(\epsilon)}{S + h_4(\epsilon)} \right) - (\mu + \gamma) \ln \left( \frac{S - h_3(\epsilon)}{S + h_4(\epsilon)} \right) \right]
\]
\[
= - \beta \exp(-\alpha l_c)(S - S) + (\mu + \gamma) \ln \left( \frac{S}{S} \right).
\]
Therefore,
\[
\lim_{\epsilon \to 0} \left[ - \int_{\partial_{\delta} A_3} B f(1) \, ds - \int_{\partial_{\delta} A_3} B f(2) \, ds \right] = \beta (1 - \exp(-\alpha l_c))(S - S) > 0,
\]
which contradicts (8). This rules out the existence of the limit cycle \( X \). This completes the proof.

### 5.2. Global stability of the equilibria

To establish all the possible behaviors that the system can exhibit, we choose variable sets of values for the parameters such that the dynamics in all regions are presented. In this subsection, we consider the following three cases.

**Case 1.** Suppose that \( l_c > l^1_c \). Then equilibrium \( E^1 \) is real, while \( E^2 \) is virtual. By calculation, we know that the sliding mode does exist, but there is not a pseudo-equilibrium. In the following, we shall show that \( E^1 \) is globally asymptotically stable.

**Theorem 1.** For Filippov system (4), the endemic equilibrium \( E^1 \) is globally asymptotically stable if \( R_0 > 1 \).

**Proof.** Suppose that \( \Delta_1 < 0 \). Then the endemic equilibrium \( E^1 \) is a focus, which is locally asymptotically stable, by the above discussion. The procedure of the proof for Lemma 2 shows that the trajectory moves from the right to the left on the sliding segment \( T_1T_2 \) in this case.

Note that it is easy to get \( l^1_c > l^1_r, l^2_r > l^2_c \) and \( l^1_c^* \geq l^1_c \), where \( l^1_r, l^2_r \) are defined as those in Section 3. Then, either \( l^2_r > l^1_r \) or \( l^2_r < l^1_r \) may hold true. For different sets of values of the parameters, the sliding segment may exclusively, partly, or totally be in the attraction region \( \Omega \), so we then consider the following three possibilities.

(i) If \( l_c \geq l^1_c \), then the point \((\alpha, l_c) \in \Gamma_1 \) in the \( \alpha \) - \( l_c \) plane (shown in Fig. 1). The sliding segment \( T_1T_2 \) lies out of the attraction region \( \Omega \) entirely, as shown in Fig. 5(a).

Then the regular equilibrium \( E^1 \) and the virtual equilibrium \( E^2 \) are in region \( G^1 \cap \Omega \), and the direction of the vector field in region \( G^2 \cap \Omega \) points downward. Hence, no limit cycle which is partly in region \( G^2 \) and partly in \( G^2 \) exists in the attraction region \( \Omega \). In addition, by Lemma 1, there is no limit cycle totally in \( G^1 \) (\( i = 1, 2 \)). Therefore the local stability of the endemic equilibrium \( E^1 \) implies that \( E^1 \) is globally asymptotically stable.

(ii) If \( \max \{ l^1_r, l^2_r \} < l_c < l^1_c \), then we get \((\alpha, l_c) \in \Gamma_2 \) in the \( \alpha \) - \( l_c \) plane (see Fig. 1). In such a case the sliding segment \( T_1T_2 \) partly lies out of the attraction region \( \Omega \), as shown in Fig. 5(b). Note that again the direction of the vector field in region \( G^2 \cap \Omega \) points downward. Moreover, according to Lemma 2, we know that no closed orbit containing part of the sliding segment \( T_1T_2 \) exists. Hence, the endemic equilibrium \( E^1 \) is globally asymptotically stable.

(iii) If \( l_c \leq \max \{ l^1_r, l^2_r \} \), then we can get \((\alpha, l_c) \in \Gamma_3 \) in the \( \alpha \) - \( l_c \) plane (see Fig. 1), and the sliding segment is totally in the attraction region \( \Omega \), as shown in Fig. 5(c).

Note that a limit cycle totally in the region \( G^1 \) or \( G^2 \) does not exist. In addition, Lemma 2 precludes the possibility of a limit cycle induced by the sliding nature of the trajectories on the sliding domain \( \Sigma_5 \). It follows from Lemma 3 that there is no limit cycle in the attraction region \( \Omega \) which surrounds the sliding segment \( T_1T_2 \). Hence, the endemic equilibrium \( E^1 \) is globally asymptotically stable.

If \( \Delta_1 \geq 0 \), the endemic equilibrium \( E^1 \) is a node, which is also asymptotically stable in \( G^1 \). Note that both the regular equilibrium \( E^1 \) and virtual equilibrium \( E^2 \) are in region \( G^1 \). This indicates that all trajectories will attain the subregion of the attraction region \( \Omega \) below the manifold \( \Sigma \). Moreover, they will remain in it and approach the endemic equilibrium \( E^1 \). Hence \( E^1 \) is globally asymptotically stable. This completes the proof of Theorem 1.

**Case 2.** If \( l^2_r < l_c < l^1_r \), both \( E^1 \) and \( E^2 \) are virtual. The sliding mode does exist, and there does exist a pseudo-equilibrium \( E_S \) which is asymptotically stable in the sliding domain \( T_1T_2 \). We shall show that the pseudo-equilibrium \( E_S \) is globally asymptotically stable in the SI phase plane in the following.

**Theorem 2.** For Filippov system (4), the pseudo-equilibrium \( E_S \) is globally asymptotically stable if \( R_0 > 1 \).
Fig. 5. Phase plane S-I for Filippov epidemic model (4), showing the sliding domain \((T_1T_2)\) and asymptotical equilibrium (small circles: \(E^1_i, E^2_i\) or \(E^1_s, E^2_s\)) for different parameter sets. The isoclinic lines \(g^1_S, g^2_S\) and \(g^1_I, g^2_I\) are plotted for the free (controlled) system \(S^1 (S^2)\). The curves represent the orbits in the phase plane indicating the asymptotical equilibrium. Parameters values are chosen such that dynamics in all region \(\Gamma_i, i = 1, 2, \ldots, 6\) as shown in Fig. 1 are exhibited. (a) \(\mu = 0.2, \beta = 1, \gamma = 0.3, \alpha = 0.6, l = 0.6\) for region \(\Gamma_1\); (b) \(\mu = 0.2, \beta = 1, \gamma = 0.1, \alpha = 0.8, l = 0.6\) for region \(\Gamma_2\); (c) \(\mu = 0.2, \beta = 1, \gamma = 0.1, \alpha = 0.8, l = 0.52\) for region \(\Gamma_3\); (d) \(\mu = 0.2, \beta = 0.8, \gamma = 0.05, \alpha = 0.95, l = 0.52\) for region \(\Gamma_4\); (e) \(\mu = 0.1, \beta = 1, \gamma = 0.1, \alpha = 1.2, l = 0.4\) for region \(\Gamma_5\); (f) \(\mu = 0.2, \beta = 1, \gamma = 0.1, \alpha = 0.8, l = 0.35\) for region \(\Gamma_6\).

Proof. By Lemmas 1 and 2, we know that there is no limit cycle totally in the region \(G_i^2 (i = 1, 2)\), or containing part of the sliding segment \(T_1T_2\). Note that the sliding segment may be partly or totally in the interior of the attraction region. Then there are two possibilities to consider.
(i) If \( I_c \geq \min\{I_2^0, I_1^1\} \), it is easy to get \((\alpha, I_c) \in I_1^4 \) in the \( \alpha-I_c \) plane (see Fig. 1), and part of the sliding segment \( T_1T_2 \) is in the attraction region, as shown in Fig. 5(d). Note that again the direction of the vector field in region \( G^2 \cap \Omega \) points downward. By using similar arguments as the second possibility of Case 1, we obtain that no limit cycle exists in the attraction region.

(ii) If \( I_c < \min\{I_2^0, I_1^1\} \), then \((\alpha, I_c) \in I_2^3 \) in the \( \alpha-I_c \) plane (see Fig. 1), and the sliding segment is totally in the attraction region \( \Omega \), as shown in Fig. 5(e). Based on a similar argument to Lemma 3, we can exclude the existence of limit cycles in the attraction region \( \Omega \) which surround the sliding segment \( T_1T_2 \).

Note that both locally stable equilibria \( E^1 \) and \( E^2 \) are virtual and then cannot be attained. Consequently, any orbit initiating from region \( G^1 \) (\( G^2 \)) intersects with the switching segment in order to approach the equilibrium point \( E^1_0 \) (\( E^2_0 \)). Further, according to the above discussion, no limit cycle exists in the attraction region. We then conclude the pseudo-equilibrium \( E_s \) is globally asymptotically stable. This completes the proof of Theorem 2.

**Case 3.** If \( I_c < I_2^0 \), then we have \((\alpha, I_c) \in I_0^6 \) in the \( \alpha-I_c \) plane (see Fig. 1). The endemic equilibrium \( E^1 \) is virtual and \( E^2 \) is a real equilibrium, and there is no pseudo-equilibrium. The sliding mode does exist, and the trajectory moves from the left to the right on the sliding segment in such a case, as shown in Fig. 5(f). By a similar discussion to that in the third possibility of Case 1, we know that there is no limit cycle totally in the attraction region \( \Omega \). If the endemic equilibrium \( E^2_1 \) is locally stable, \( E^2_1 \) is globally asymptotically stable, and we get the following conclusion.

**Theorem 3.** For Filippov system (4), the endemic equilibrium \( E^2 \) is globally asymptotically stable if \( R_0 > 1 \).

**Remark.** If we consider the continuous transmission rate (i.e., \( I_c = 0 \)), then system (1)–(2) becomes system \( S^2 \). This indicates that the media coverage exhibits its effect once the epidemic disease emerges and begins to spread. By Proposition 2, we know that system \( S^2 \) will stabilize at the disease-free equilibrium \( E_0 \) or endemic equilibrium \( E^2 \) depending on the basic reproduction number \( R_0 \). In fact, when \( R_0 < 1 \), the disease-free equilibrium \( E_0 \) is globally asymptotically stable, suggesting that the disease is eradicated; when \( R_0 > 1 \), the endemic equilibrium \( E^2 \) turns out to be a global attractor, suggesting that the disease persists and the number of infected individuals approaches the level of \( I^2 \), which is determined uniquely by system \( S^2 \) itself.

However, Filippov system (4) is an epidemic system with piece-wise continuous transmission rate. It follows from Theorems 1 and 3 that it may stabilize at one of the endemic equilibria (i.e., \( E^1 \) or \( E^2 \)) of the variable structures when \( R_0 > 1 \), depending on the choice of the critical level \( I_c \). The most interesting and crucial issue is that the switched system (4) can stabilize at the pseudo-equilibrium \( E_s \) (Theorem 2), which is new and qualitatively different from the equilibrium for system \( S^2 \). This indicates that a previously chosen level of the desired number of the infected persons can be reached when the threshold level and other parameters are chosen properly, which may suggest to us an idea for disease control when the emergent infectious diseases cannot be eradicated immediately.

**6. Discussion and biological conclusion**

A few mathematical models have described the impact of media coverage on the transmission dynamics of infectious diseases (see [1,3,5,6]). The formulated models implicitly assume that the media-induced effect occurs throughout all the epidemic time. However, public mass media usually cannot be aware of or report infectious diseases when the number of infected individuals is generally small at the initial stages of disease spreading. To model this vital fact, we extend the existing models by incorporating a piecewise continuous transmission rate to describe that media reports and education can exhibit their effects once the number of infected individuals exceeds a certain level. To our best knowledge, system (1) with (2) is the first epidemic system that incorporates a piecewise continuous transmission rate to describe the real impact of media coverage on infectious diseases.

The proposed epidemic model in this paper, which is described by a piecewise smooth ordinary differential equations, describes the media-induced effect subject to variable structure control. The system is actually called a Filippov system (or switching system) in control language [12]. A commonly used and implementable control is the simple on–off control, which can stabilize the models [21,23–28]. The overall objective of this paper is to examine the effect of media coverage on disease transmission and to address whether the improved function representing the media-induced effect influences the global dynamics of disease transmission.

The analysis performed in this work establishes all possible dynamic behaviors that the present model can exhibit, as can be seen in Figs. 2 and 5. We employ the properties of the Lambert W function [16] and bifurcation analysis of piecewise smooth systems [19,20] to investigate the long-term dynamical behavior of the proposed Filippov epidemic model. In particular, the sliding mode dynamics and the local sliding bifurcations have been addressed in this paper. The crucial role of the variable structure control in determining the dynamical behavior of the model is shown. We have also obtained the global stability for the Filippov epidemic model by using qualitative techniques. Our main results suggest that the system stabilizes at the equilibrium point \( E^1_0 \), \( E^2_0 \), or \( E_s \), depending on the threshold level \( I_c \). Six different types of threshold level are considered (see Fig. 5). For each one, the sliding segment is exclusively, partly, or totally in the attraction region. Fig. 5(a)–(c) show that the endemic state \( E^1_0 \) is globally asymptotically stable, i.e., \( E^1_0 \) is a global attractor. Trajectories may approach the endemic equilibrium \( E^2_1 \) without sliding, or enter the sliding region and then slide and approach the equilibrium \( E^2_1 \), depending on the initial data. Similarly, Fig. 5(f) shows that the endemic equilibrium \( E^2_1 \) is an attractor.
The most interesting case is represented in Fig. 5(d) and (e), where the threshold level \( l_e \) is in between \( l_1^* \) and \( l_2^* \) (i.e., \( l_1^* < l_e < l_2^* \)) and both the equilibria \( E^1 \) and \( E^2 \) are virtual, from which we know that the pseudo-equilibrium \( E_S \) is globally asymptotically stable. Comparing with the former cases, trajectories will enter the sliding domain and slide to the pseudo-equilibrium \( E_S \). This implies that the infected population can stabilize at a previously chosen level (i.e., \( l_e \)) when the threshold policy and other parameters are chosen properly. Therefore, the threshold policy combines the characteristics of the free and controlled systems, and, in particular, induces a new equilibrium \( E_S \) at which system (4) stabilizes compared to the dynamic behavior for system \( S^2 \) with continuous transmission rate incorporating media coverage (see [1,3,6]).

Although the threshold policy can significantly modify the dynamical behavior of the models studied, it is important to note that the number of infected individuals in the steady state triggered by the threshold policy is confined to a set of values determined previously by the parameters of the model, if some conditions on the parameters hold true. Hence, the policy is robust [17] to bounded uncertainty in the parameters, although it may cause displacement of the endemic equilibrium.

In this paper, we have conducted a qualitative analysis of the impact of media coverage on the transmission of infectious diseases. We have provided a more realistic and more natural description for the epidemic model with media coverage by considering the real procedure of media propagation and education in disease transmission. The results obtained here could be beneficial for accurately assessing the effect of the media coverage in the control and treatment of infective diseases.

Acknowledgments

The authors are supported by the National Mega-project of Science Research No. 2012ZX10001-001, by the National Natural Science Foundation of China (NSFC, 11171268), by the Fundamental Research Funds for the Central Universities (08143042), and by the International Development Research Center, Ottawa, Canada (104519-010).

Appendix. Brief introduction to the threshold policy

A threshold policy (TP) implies that, if the value of the threshold function is below the specified threshold level, control is suppressed, while if it is above the specified threshold level, control is implemented. As a result, a system subject to a TP leads to a variable structure system. The TP is also referred to as an on–off control in the control literature (see [17,18]). In the following, we shall establish a mathematical form for a TP. To illustrate our ideas, we choose a planar system as an example. A two-dimensional system submitted to a TP can be rewritten as

\[
\frac{dx(t)}{dt} = f(x, u_t),
\]

where \( x (x \in \mathbb{R}^2) \) is the state vector, \( f \in C(\mathbb{R}^2, \mathbb{R}^2) \) is continuous, and the control \( u_t (u_t \in \mathbb{R}) \) is defined as

\[
u_t = \begin{cases} u_1(x), & \tau(x) < 0, \\ u_2(x), & \tau(x) > 0, \end{cases}
\]

where \( u_1(x) = 0, \tau(x) \) is a threshold dependent on the state vector \( x \), and \( u_2 \) is not continuous. For the case when \( \tau(x) = 0, u_1(x) \) is not defined, i.e., \( u_2(x) \) is discontinuous on the boundary \( \tau(x) = 0 \). A ‘controlled system’ refers to one in which the control \( u_t = u_2 \) is implemented, and a ‘free system’ is one in which the control is not implemented (i.e., \( u_t = 0 \)). Let

\[
G^1 = \{ x \in \mathbb{R}^2 | \tau(x) < 0 \}, \quad G^2 = \{ x \in \mathbb{R}^2 | \tau(x) > 0 \},
\]

and

\[
\Sigma = \{ x \in \mathbb{R}^2 | \tau(x) = 0 \}.
\]

Denote the right-hand side of (A.1) (i.e., \( f(x, u_t) \)) in the region \( G^i \) by \( f^{(i)}(x, u_t) \).

**Definition A.1.** Let \( x^* \) be such that \( f^{(i)}(x^*, u_i) = 0 \) for some \( u_i \) in (A.2). Then \( x^* \) is called a real equilibrium of system (A.1) with (A.2) if it belongs to \( G^i \), and a virtual equilibrium if it belongs to \( G^j, j \neq i \).

A ‘sliding mode’ may occur if there are subsets \( \Sigma_S \) of the manifold \( \Sigma \) such that the vectors \( f^{(1)}(x, 0) \) and \( f^{(2)}(x, u_2) \) are directed towards each other on them.

**Definition A.2.** A point \( x^* \) is said to be a pseudo-equilibrium of system (A.1) with (A.2) if it is an equilibrium of the sliding mode of system (A.1) with (A.2), i.e., \( \lambda f(x^*, u_1(x^*)) + (1 - \lambda) f(x^*, u_2(x^*)) = 0, \tau(x^*) = 0 \), where

\[
\lambda = \frac{(\tau(x), f_2(x, u_2(x)))}{(\tau(x), f_2(x, u_2(x))) - f_1(x, u_1(x))}.
\]

**Definition A.3.** A point \( x^* \) is called a tangent point of system (A.1) with (A.2) if \( x^* \in \Sigma_S \) and \( (\tau(x), f^{(1)}(x^*, 0)) = 0 \) or \( (\tau(x), f^{(2)}(x^*, u_2)) = 0 \).

**Definition A.4.** A point \( x^* \) is called a boundary equilibrium point of system (A.1) with (A.2) if \( f^{(1)}(x^*, 0) = 0, \tau(x^*) = 0 \) or \( f^{(2)}(x^*, u_2) = 0, \tau(x^*) = 0 \).
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