Talk about pivots

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We consider two different methods of parameterizing the dark energy equation of state in order to assess possible “figures of merit” for evaluating dark energy experiments. The two models are $w(a) = w_0 + (1 - a)w_a$ and $w(a) = w_p + (a_p - a)w_a$. This brief note shows that the size of the error contours cannot change under such a reparameterization. This makes the figures of merit associated with these parameterizations mathematically identical. This also means that any “bias” exhibited by one model is equally present in the other.

I. INTRODUCTION

Dark energy poses challenges to physics because we know so little about it. While there are many competing models, none of them are overwhelmingly convincing. This is of course why the field is so exciting. We expect to learn a great deal by further study of the dark energy. This note addresses a specific technical issue related to modeling dark energy and forecasting the impact of future experiments.

As discussed in [1] it is standard practice to model the dark energy as a perfect fluid. Its dynamical properties can then be expressed in terms of the dark energy equation of state parameter $w$ as a function of cosmic scalefactor $a$. The general case of this parameterization implies an infinite number of degrees of freedom (in order to describe the continuous function $w(a)$), but in most implementations a finite parameter ansatz is used. Any simple ansatz will exclude possible forms for $w(a)$ and thus could distort the subsequent analysis, so the relative advantages and risks associated with particular choices of ansatz are a topic of ongoing discussion and debate.

Currently, the ansatz that is probably most commonly used is the linear expression

$$w(a) = w_0 + (1 - a)w_a$$

which we will call “normal form”. This is a special ($a_p = 1$) case of the linear expression

$$w(a) = w_p + (a_p - a)w_a$$

where $a_p$ is the “pivot scalefactor”. Equation (2) is usually used as in 2, where $a_p$ is chosen to be the value of the scalefactor where $w(a)$ is most tightly constrained by a particular data set 3. By construction, $w(a_p) = w_p$ in this ansatz, which we will call “pivot form”.

Attention has been recently drawn to the pivot ansatz by the Dark Energy Task Force (DETF) [4]. The DETF defines a figure of merit for a given data set based on the area inside a constant probability contour in the two dimensional $w_p$--$w_a$ plane. In [1] Linder looked at the effect of moving from a description based on normal form to a description based on pivot form (§III of [1]). Linder’s paper suggests that the normal form suffers less from bias than the pivot form. He considers a non-linear generalization of (2) where $a$ has an exponent $b$ differing from unity, and demonstrates a bias using this generalized pivot form in his figures 2 and 3.

This brief note focuses on the linear ($b = 1$) form of the pivot ansatz. This is the only form that is relevant to the DETF work. We show why normal form and pivot form are mathematically identical and interchangeable when it comes to defining “area” figures of merit like the one used by the DETF (this fact has already been stated in the Fisher matrix approximation in [4].) Linder’s discussion emphasizes the bias he exhibits using his non-linear ansatz and singles out the behavior of $w_p$. Linder’s discussion favors normal form over pivot form, stating that normal form is “more robust”.

While it may seem that our conclusion is in conflict with [1], we do not believe there are any concrete technical points of disagreement. For example, Linder’s figures 2 and 3 shows that the bias disappears for the linear ($b = 1$) case. Also, we agree with Linder that the choice of $a_p$ is data-dependent. So the differences between this comment and [1] are apparently only ones of emphasis. Our message is that [1] does not demonstrate any weakness of the DETF figure of merit relative to the equivalent (in fact equal) one based on normal form. Our motivation is to make this message transparent even in the non-Gaussian case.

II. A MECHANICAL ANALOGUE

As the scalefactor plays the role of the evolution parameter, it is useful to think of it as time. Identifying the equation of state $w(a)$ as $x(t)$ we transform (1) and (2) into the equation of a particle traveling at a constant velocity:

$$x(t) = x_0 - tx_a$$

It is well known that this mechanical system comes from an almost trivial Hamiltonian

$$H = \frac{p^2}{2}$$

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where the “mass” has been scaled to unity. As this is a
time-independent evolution of the system we can apply
Liouville’s theorem: the evolution of a region of phase
space is volume preserving. That is, time evolution pre-
jects the volume in the $x_0\text{-}x_a$ plane.

III. COMMENTS

The difference between Eqn. (1) and Eqn. (2) is that
they are looking at the same system at different “times”
or scalefactors. As these equations have a Hamiltonian
evolution (as is easy seen by constructing a mechanical
alogy) the phase space volume cannot change. As a
result, the area inside the error contours cannot depend
on whether you are using parameterization (1) or (2),
although the shape and orientation of the contours in
$w_0\text{-}w_a$ space can certainly change. This result is inter-
esting because we have not made any assumption about
the underlying distribution in the $w_0\text{-}w_a$ plane. (The
case of a Gaussian distribution is considered explicitly
in the Appendix.) Both pivot and normal form consider
the exact same linear family of functions $w(a)$, and just
label them by different linear combinations of their pa-
rameters. This particular relabeling does not change area
in phase space, and does not change the likelihood of a
specific function $w(a)$ given a specific data set.

More generally, it is interesting to ask what happens
when the class of models is generalized so that the coeffi-
cient of $a$ is no longer unity. In this case, it is no longer
true that the area in the $w_0\text{-}w_a$ plane is preserved. The
main reason why the above explanation fails is that the
“canonical momentum” changes in time, and while the
$q\text{-}p$ area is preserved in time, the $w_0\text{-}w_a$ area is not.
However, this generalization is not relevant to normal and
pivot forms, and thus does not impact our conclusions.

So we emphasize again our main point: The area fig-
ures of merit based on pivot and normal form are math-
ematically identical.

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APPENDIX A: THE FISHER MATRIX CASE

We have shown generally the equivalence of area fig-
ures of merit based on the pivot and normal forms. Here
we show explicitly how this result comes out of the Fisher
matrix analysis of a Gaussian distribution probability
distribution $P(x)$ as a function of parameter vector $x$.

Here we have

$$P(x) = \frac{1}{\sqrt{2\pi \det F}} \exp(-x^T F x). \quad (A1)$$

where $F$ is the covariance matrix, meaning that the eigenvectors
give the directions of the principal axes of the er-
or ellipse and the eigenvalues give the inverse variances
squared ($1/\sigma_1^2$ and $1/\sigma_2^2$) in these directions, as is easily
seen by diagonalizing $F$. The figure of merit $\sigma_1 \times \sigma_2$ is,
up to a constant factor, the area of the 1-\sigma ellipse.

When we change basis to $x' = T x$ then the quadratic
form transforms in the following way:

$$(x')^T (T^T F T) x' \equiv (x')^T F' x' \quad (A2)$$

where $F'$ is the transformed covariance matrix. In our

case, we have

$$x' = \begin{pmatrix} w_p \\ w_a \end{pmatrix} = \begin{pmatrix} 1 & 1-a_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_a \end{pmatrix}, \quad x = \begin{pmatrix} w_0 \\ w_a \end{pmatrix} \quad (A3)$$

as can easily be verified. The transformation matrix has
unit determinant, and so

$$\frac{1}{(\sigma_1')^2} \times \frac{1}{(\sigma_2')^2} = \det T^T F T = \det F = \frac{1}{\sigma_1^2} \times \frac{1}{\sigma_2^2} \quad (A4)$$

which shows that the figure of merit could not have
changed under this transformation.

It should be emphasized that this is not a separate
result, but rather a special case of the result in the body
of this note presented in a language that may be more
familiar to some readers.

[1] E. V. Linder, “Biased Cosmology: Pivots, Parameters,
and Figures of Merit”, [arXiv:astro-ph/0604280]

[2] W. Hu and B. Jain, “Joint Galaxy-Lensing Observables
and the Dark Energy”, Phys. Rev. D 70 (2004) 043009
[arXiv:astro-ph/0312395].

[3] D. Huterer and M. S. Turner, “Probing the dark energy:
Methods and strategies”, Phys. Rev. D 64 (2001) 123527
[arXiv:astro-ph/0012510].

[4] The Dark Energy Task Force is a subpanel of the AAAC
and HEPAP, charged by DoE, NASA and NSF to examine
future probes of Dark Energy. The most recent public pre-
sentation of the ongoing work of the DETF can be found
at http://www.hep.net/p5pub/AlbrechtP5March2006.pdf

[5] V. I. Arnold, “Mathematical methods of classical mechan-
ic”, Springer-Verlag (1989)