GLIMM’S METHOD FOR RELATIVISTIC HYDRODYNAMICS

J. K. CANNIZZO,1,2 N. GEHRELS,1 AND E. T. VISHNIAC3

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ABSTRACT

We present the results of standard one-dimensional test problems in relativistic hydrodynamics using Glimm’s (random choice) method and compare them to results obtained using finite differencing methods. For problems containing profiles with sharp edges, such as shocks, we find Glimm’s method yields global errors ~1–3 orders of magnitude smaller than the traditional techniques. The strongest differences are seen for problems in which a shear field is superposed. For smooth flows, Glimm’s method is inferior to standard methods. The location of specific features can be off by up to two grid points with respect to an exact solution in Glimm’s method, and furthermore, curved states are not modeled optimally, since the method idealizes solutions as being composed of piecewise constant states. Thus, although Glimm’s method is superior at correctly resolving sharp features, especially in the presence of shear, for realistic applications in which one typically finds smooth flows plus strong gradients or discontinuities, standard finite-difference methods yield smaller global errors. Glimm’s method may prove useful in certain applications such as GRB afterglow shock propagation into a uniform medium.

Subject headings: hydrodynamics — methods: numerical — relativity

1. INTRODUCTION

Interest in relativistic hydrodynamics has heightened in recent years due to the explosion in the field of gamma-ray bursts (GRBs; Costa et al. 1997; van Paradijs et al. 1997; Frail et al. 1997, 2001; MacFadyen & Woosley 1999; Aloy et al. 2000; Fox et al. 2005; Gehrels et al. 2005; Bloom et al. 2006; O’Brien et al. 2006). The current paradigm for GRBs involves the extraction of energy from a newly formed ~10 M⊙ black hole and collimation into a relativistic jet, which then propagates along the line of sight to the observer. The emission is thus strongly beamed and Doppler boosted. The interaction of the jet with the circumstellar medium produces afterglow. For Newtonian hydrodynamics the density contrast across a strong shock is given by ρshock/ρbg = (Γ + 1)/(Γ − 1), where Γ is the polytropic index; in relativistic hydrodynamics ρshock/ρbg = (γΓ + 1)/(Γ − 1), where γ is the Lorentz factor (Blandford & McKee 1976). For putative values γ ≈ 102–104 thought to be required for GRB jets, a relativistic shock can have extremely high density and be very narrow due to Lorentz contraction. This poses a severe test for standard finite-difference (FD) methods and necessitates adaptive mesh refinement (Zhang & MacFadyen 2006; Morsony et al. 2007). Adaptive refinement techniques also present challenges, as it has yet to be demonstrated that increased levels of refinement on a complex, multidimensional problem lead to convergent solutions. The desired test of showing that a standard performance metric integrated over the computational volume asymptotes to a constant value with increasing level of refinement has yet to be carried out (e.g., Zhang & MacFadyen 2006).

Traditional methods for calculating hydrodynamical evolution of a relativistic fluid have relied on finite differencing, i.e., discretizing the differential equations (Norman & Winkler 1986; Marti & Müller 2003; del Zanna & Bucciantini 2002; Lucas-Serrano et al. 2004). Figure 1 presents an example of smearing inherent in standard FD methods. It shows the evolution of the Lorentz factor γ in a spherical relativistic blast wave calculation initialized with a Blandford & McKee (1976) solution, taking γ0 = 5 initially. Each panel shows the same initial conditions, with increasing grid resolution along the +x-direction. We use a three-dimensional Cartesian grid and utilize the method described in del Zanna & Bucciantini (2002). Our implementation of their method is detailed in Cannizzo et al. (2004). Within each panel, the number of grid points along the direction of propagation is increased by a factor of 4. In the bottom panel, for which there are 64 grid points per small tick mark, one can see the clear development of a forward/reverse shock feature. The inherent smearing behavior of the technique is evident by comparing successive panels.

2. BACKGROUND

Glimm (1965) presented the theoretical basis for the random choice, or Glimm’s method. It relies on first idealizing the solution in (P, ρ, v) over N grid points as consisting of N piecewise constant states and then solving the local Riemann problem N−1 times between adjacent grid points. Second, a random location is selected within a cell, the exact solution is evaluated at that point, and then that is used as the starting solution for the next time step.

Chorin (1976, 1977) developed Glimm’s method into a numerical algorithm for problems that could be formulated in terms of nonlinear hyperbolic conservation laws. Sod (1978) reviewed several techniques for Newtonian hydrodynamics and found Glimm’s method to be superior in terms of preserving the sharpness of shock edges. In the early studies using Glimm’s method, one sees clear deficiencies in the solutions, however, in terms of both shock front localization and overall stability.

A breakthrough came from Colella (1982) who proposed using the van der Corput sequence instead of a standard random number generator for determining the solution evaluation location with cells in each time step. This sequence is generated by a simple manipulation of the digits in the binary representation of consecutive integers.

The application of Glimm’s method to relativistic hydrodynamics became possible when Balsara (1994) and Marti & Müller
adjacent grid points. The solution is evaluated in each alternating half-time step at a point \( (1/2)\Delta t \), where \( \Delta t = n_{\text{CFL}}/\Delta x \) with CFL (Courant et al. 1967) number \( n_{\text{CFL}} = 0.5 \). Thus, the pure Glimm’s Method effectively adopts a CFL number of 0.5 for the full time step. Although most of our results use a simple one-dimensional Cartesian grid, Wen et al. (1997) also present geometrical correction terms for carrying out one-dimensional calculations in cylindrical or spherical symmetry.

An important advance since Wen et al. (1997) are the studies generalizing the relativistic Riemann solution to include tangential flow (Pons et al. 2000; Rezzolla et al. 2003). This allows one to extend Glimm’s method to problems involving shear and to begin to envision a two-dimensional Glimm’s method. Pons et al. obtain a solution by solving (1) the jump conditions across shocks and (2) a differential equation that comes from a self-similarity condition along rarefaction waves. Rezzolla et al. present an integral solution to the equation derived by Pons et al. and they propose an efficient Gaussian quadrature technique for solving it. To solve the local Riemann problem between adjacent grid points we use the publicly available code RiemANN_VTF written by J.-M. Marti and E. Müller (cf. Marti & Müller 2003) which uses the formalism described in Pons et al. (2000) and Rezzolla et al. (2003).

4. TESTING

Shock tube problems used in testing hydrodynamical codes are a subset of the Riemann problems, for which \( v = 0 \) for all \( x \). One-dimensional Riemann problems are typically run on a grid such that \( 0 \leq x \leq 1 \), and the thermodynamic variables \( P, \rho \), and \( \epsilon \) are discontinuous across \( x = 0.5 \) initially. Starting the simulation is equivalent to removing a diaphragm between left (\( L \)) and right (\( R \)) states. The strong gradients across \( x = 0.5 \) result in four constant states separated by three elementary waves: rarefaction, contact discontinuity, and shock waves. Analytical solutions for the time evolution of these problems for (special) relativistic hydrodynamics are given by Marti & Müller (1994) for nonshearing problems and by Pons et al. (2000) for Riemann problems with added shear (i.e., nonzero \( v_{\parallel} \)).

The level of agreement between the exact, analytical solutions and the numerical ones is quantified by the \( L_1 \) norm error, defined for one-dimensional problems as \( L_1 = \Sigma |u_j - u(x_j)| \), where \( x_j \) is the coordinate of grid point \( j \), \( u(x_j) \) is the analytical value, and \( u_j \) is the numerical value. The grid spacing is \( \Delta x \). For consistency with previous groups, we take the solution in proper density. The analytical and numerical solutions are calculated on the same grids, and the number of grid points \( N \) in the solutions is varied between trials.

4.1. Riemann Problem 1

The values in the initial left and right states are \( (p, \rho, v)_L = (40/3, 10, 0) \) and \( (p, \rho, v)_R = (2/3) \times 10^{-6}, 1, 0 \), respectively. The adiabatic index \( \Gamma = 5/3 \). The result at \( t = 0.4 \) is compared to the analytical one. The gradient in pressure \( p \) produces in the subsequent evolution a rarefaction wave moving left and a shock wave moving right, with a contact discontinuity between. The flow is mildly relativistic, with postshock velocity \( v = 0.714 \). Figure 2 shows a comparison of the Glimm solution with the exact one, computed on a grid with \( N = 400 \). The small inset panels show details of the leading and trailing edges of the density spike associated with the shock. For the time step shown, the leading edge of the Glimm solution is off the analytical solution by one grid point, and the trailing edge is exact. Table 1 presents the \( L_1 \) errors in density between three methods, FLASH (from Morsony

![Fig. 1.—Evolution of Lorentz factor $\gamma$ for a Blandford-McKee initial state with $\gamma_0 = 5$ in a three-dimensional Cartesian calculation using the method described in del Zanna & Bucciantini (2002), using the local Lax-Friedrichs flux. The four panels show increasing grid resolution in a slice along the propagation direction. The initial step is the leftmost profile in each panel, and profiles moving to the right show the shock development at eight subsequent time steps. For ease of viewing, the solutions in the first panel are connected by solid lines. The dotted line in each panel indicates the $\gamma$ value corresponding to the local maxima in $\rho$ for the nine time steps. (For the first two panels there is a [spurious] offset between the nine time steps. (For the first two panels there is a [spurious] offset between the...
et al. 2007), weighted essentially nonoscillatory (WENO; from Zhang & MacFadyen 2006), and Glimm. (This test problem has been studied by many workers previously; e.g., Hawley et al. 1984; Schneider et al. 1993; Martí & Müller 1996; Wen et al. 1997; Martí et al. 1997; Aloy et al. 1999.) The asterisked values indicate those trials for which the leading and trailing shock edge positions of the Glimm solutions are in agreement with the analytical ones.

For a small sample of individual Glimm trials, the $L_1$ error is not always a consistent indicator of success. Although shock propagation speeds are expected to be accurate in an averaged sense, within a given time step specific features in the Glimm solution can be one or two grid points off from their correct location. For problems with sharp edges, such as shocks, the error will be large (locally) at such a position. Most of the rest of the error is introduced by idealizing the curved state (Riemann fan) to be composed of a series of piecewise constant states. Even if a shock edge location is incorrect at a given time step, at a slightly later time step, or at the same time step for a run with a different number of grid points $N$, the Glimm solution may have the correct location of the shock front edges. Therefore, a better way to measure the success of the method is to plot the $L_1$ errors for a large number of different trials, all compared at the same time step with the analytical solution for the same $N$. For problems which are typically dominated by one large density enhancement, one observes bands of solutions representing those for which the calculated edges are (1) exact, (2) off by one grid point (leading or trailing edge), (3) off by two grid points total, (4) off by three grid points total, etc. We denote the cumulative grid point error in shock front localization by $s$.

This effect is shown in Figure 3 where we plot the $L_1$ errors for the $N$ FLASH and WENO; black symbols) and G solutions (Glimm; blue symbols) in Table 1 (Riemann problem 1), as well as the results using $\sim 10^2$ additional $N$ values for G (red symbols).

| $N$  | FLASH | WENO | Glimm |
|------|-------|------|-------|
| 100  | 0.13  | 0.13 | 0.029*|
| 200  | 0.070 | 0.074| 0.034 |
| 400  | 0.036 | 0.033| 0.017 |
| 800  | 0.018 | 0.021| 0.0035*|
| 1600 | 0.0085| 0.010| 0.0033 |
| 3200 | 0.0043| 0.0051| 0.0069 |

Note.—The asterisked values indicate those trials for which the leading and trailing shock edge positions of the Glimm solutions are in agreement with the analytical ones ($s = 0$).
drives a faster and higher density shock. The values in the initial left and right states are \((p;\rho;v)_L=(10^3,1,0)\) and \((p;\rho;v)_R=(0.01,1,0)\), respectively. The adiabatic index \(\Gamma=5/3\). The result at \(t=0.4\) is compared to the analytical one. The flow is relativistic, with postshock velocity \(v=0.96\). The shock speed is 0.986. The width of the shock is \(\Delta x_t \approx 0.01\) at \(t=0.4\) and for \(N=400\) is covered by 4.2 grid points (in the analytical solution). The asterisked values indicate those trials for which \(s=0\).

Figure 4 shows the \(L_1\) error plot for Riemann problem 2, with the values given in Table 2 plus Glimm values for \(\sim 10^2\) additional \(N\) values. Because of the thinness of the shock compared to Riemann problem 1, there is now a clear banded structure to the Glimm solutions. The lowest striation, which also contains the first and sixth values from Table 2, corresponds to solutions for which both leading and trailing shock edge positions are exact, \(s=0\). The next highest striation, containing Glimm entries 3–5 from the table, corresponds to \(s=1\), and the third striation, containing the second Glimm entry from the table, corresponds to \(s=2\). The first striation lies about 2 orders of magnitude below the \(F\) errors, while the second and third are within a factor \(\sim 3-10\) of \(F\).

### 4.3. Riemann Problem 3

Riemann problem 3 starts with a strong negative pressure gradient that launches a reverse shock and a positive flow speed in the left state that initiates a forward shock. Thus, there is no Riemann fan. The values in the initial left and right states are \((p;\rho;v)_L=(1,1,0.9)\) and \((p;\rho;v)_R=(10,1,0)\), respectively. The adiabatic index \(\Gamma=4/3\). The result at \(t=0.4\) is compared to the analytical one. Table 3 compares the \(L_1\) errors for the three methods.

Figure 5 shows the \(L_1\) error plot for Riemann problem 3, with the values given in Table 3 plus Glimm values for \(\sim 10^2\) more \(N\) values. None of the values in the table lie in the band for \(s=1\). The three upper limit triangles indicate solutions for which \(s=0\),

![Fig. 4. — Same as Fig. 3, but for Table 2 (Riemann problem 2). The fact that the shock is narrower than for Riemann problem 1 leads to a more pronounced striationing; with fewer points spanning the shock, the relative error introduced by being off a given number of grid points in the shock edge location is larger.](image1)

![Fig. 5. — Same as Fig. 3, but for Table 3 (Riemann problem 3). The solutions listed in the table all lie in the band for which \(s=2\). The upper limit triangles indicate three solutions which are limited only by machine precision (\(\sim 10^{-15}\)).](image2)

| \(N\) | FLASH | WENO | Glimm |
|------|-------|------|-------|
| 100  | 0.21  | 0.21 | 0.0034* |
| 200  | 0.15  | 0.14 | 0.10 |
| 400  | 0.083 | 0.093 | 0.024 |
| 800  | 0.046 | 0.055 | 0.012 |
| 1600 | 0.025 | 0.025 | 0.0061 |
| 3200 | 0.013 | 0.015 | 0.00011* |

Table 3: \(L_1\) Error for Riemann Problem 3

| \(N\) | FLASH | WENO | Glimm |
|------|-------|------|-------|
| 100  | 0.059 | 0.10 | 0.061 |
| 200  | 0.035 | 0.063 | 0.031 |
| 400  | 0.021 | 0.030 | 0.013 |
| 800  | 0.013 | 0.017 | 0.0070 |
| 1600 | 0.085 | 0.095 | 0.0038 |
| 3200 | 0.033 | 0.052 | 0.0019 |

Note.—The asterisked values indicate those trials for which the leading and trailing shock edge positions of the Glimm solutions are in agreement with the analytical ones (\(s=0\)).
i.e., all three shock edge locations in the problem are exact at \( t = 0.4 \). The only limiting precision is the machine epsilon \( \epsilon \) (~10\(^{-15}\)). For one-dimensional problems consisting only of constant states, Glimm’s method finds the exact values; therefore, the only error is introduced by shock edge location inaccuracies. In traditional methods, this test problem produces postshock pressure oscillations in the reverse shock (e.g., Lucas-Serrano et al. 2004, see their Fig. 1; Zhang & MacFadyen 2006, see their Fig. 3). Lucas-Serrano et al. (2004) note, however, that the oscillations completely disappear when the CFL number is reduced below 0.3.

4.4. “Easy” Shear: Riemann Problem 2 with \((v_L)_R \neq 0\)

We now proceed to one-dimensional problems involving shear. The “easy” shear problem takes Riemann problem 2 and adds constant background shear in the \( R \) state, \((v_L)_R = 0.99\). The adiabatic index \( \Gamma = 5/3 \), and the result at \( t = 0.4 \) is compared to the analytical one. The highest Lorentz factor in the resulting flow \( \gamma \sim 7.1 \). Unlike purely Newtonian flows in which orthogonal components of the velocity field are decoupled from each other (aside from dissipation), with special relativity we now add the condition that \( v^2 + v^1_L < 1 \). This effectively limits the component of velocity along the direction of the flow \( v \) and also the degree of density enhancement relative to background within the shock. In addition, \( \gamma \) now includes a contribution from the shear. There is also a backreaction in terms of the evolution of \( v(x, t) \) on the initially constant \( v_L \) values. Table 4 compares the \( L_1 \) errors for the three methods.

Figure 6 shows the \( L_1 \) errors for the values given in Table 4, plus \( \sim 10^2 \) additional \( N \) values for \( G \). As with Figures 4 and 5, the banded structure associated with the precision in the shock edge localization is evident. The locus of solutions for \( s = 0 \) lies \( \sim 10^{-2} \) below the \( F \) errors, while the second striation, corresponding to \( s = 1 \), lies within a factor of 10 of the \( F \) errors.

4.5. “Hard” Shear: Riemann Problem 2 with \((v_L)_R \neq 0 \) and \((v_L)_L \neq 0\)

The “hard” shear problem starts with Riemann problem 2 and adds background shear in both the \( R \) and \( L \) states, \((v_L)_R = (v_L)_L = 0.9\). The adiabatic index \( \Gamma = 5/3 \), and the result at \( t = 0.6 \) is compared to the analytical one. The highest Lorentz factor in the resulting flow is \( \gamma \sim 35.8 \). Table 5 compares the \( L_1 \) errors for the three methods. The asterisked value indicates the trial for which \( s = 0 \). This problem poses a severe challenge for the traditional methods, but is well handled by the Glimm method. In fact, the \( L_1 \) error for \( F \) for the highest \( N \) values shown are equal to those for the lowest \( N \) values for \( G \). Zhang & MacFadyen (2006) present results of the hard shear test for up to 51,200 grid points, both uniform and the adaptive mesh equivalent (see their Table 7 and Fig. 9). Their \( L_1 \) errors for \( N = 51,200 \) of \( \sim 10^{-2} \) are comparable to those in our test for \( N = 400 \). The challenge of relativistic one-dimensional shearing problems for standard FD techniques is also evident in Morsony et al. (2007), see their Fig. 24). The profiles of \( \rho \) and \( v \) for the FD shearing experiments shown in Mignone et al. (2005), Zhang & MacFadyen (2006), and Morsony et al. (2007) all exhibit a strong displacement and skewing of the shock density spike with respect to the analytical solutions.

Figure 7 shows the \( L_1 \) errors for the values given in Table 5, plus \( \sim 10^2 \) additional \( N \) values for \( G \). The \( s = 0 \) striation lies \( \sim 10^{-2} \) below the \( F \) errors, and the higher striations are still a factor \( \sim 10 \) below \( F \).

4.6. Isentropic Smooth Flow

4.6.1. Continuous Isentropic

The previous problems contained sharp gradients produced by shocks. We now look at a problem with smooth flow, the isentropic flow problem. This consists of an initial state with smooth profiles in \(\rho, p, \) and \(v\). A pulse of moving fluid is superposed on top of a constant-density, zero-velocity state. The velocity of each individual element is constant in time. Therefore, the “exact” solution at a later time \( t > 0 \) is found by advancing each element in time at

### Table 4

| \(N\) | FLASH | WENO | Glimm |
|------|-------|------|-------|
| 100  | 0.63  | 0.76 | 0.24  |
| 200  | 0.34  | 0.39 | 0.12  |
| 400  | 0.17  | 0.23 | 0.059 |
| 800  | 0.084 | 0.12 | 0.029 |
| 1600 | 0.044 | 0.066| 0.015 |
| 3200 | 0.023 | 0.034| 0.029 |

### Table 5

| \(N\) | FLASH | WENO | Glimm |
|------|-------|------|-------|
| 100  | 0.51  | ...  | 0.038 |
| 200  | 0.46  | ...  | 0.019 |
| 400  | 0.33  | 0.52 | 0.0096|
| 800  | 0.22  | 0.36 | 0.00048|
| 1600 | 0.13  | 0.23 | 0.0030 |
| 3200 | 0.083 | 0.13 | 0.0029 |
| 6400 | 0.053 | 0.065| 0.0013 |

Note.—The asterisked value indicates the trial for which the leading and trailing shock edge positions of the Glimm solutions are in agreement with the analytical ones (\(s = 0\)).
its known velocity, which yields a grid with irregular spacing, and then interpolating the result back onto a uniform grid.

The initial structure is given by

$$\rho_0(x) = \rho^* [1 + \alpha f(x)],$$  \hspace{1cm} (1)

where $\rho^*$ is the density of the constant background state, and the function $f(x) = (x^2 - 1)^2$ for $|x| < L$ and $f(x) = 0$ for $|x| \geq L$. The width of the pulse is $L$ and the amplitude is $\alpha$. The initial velocity profile within the pulse is set by taking one of the two Riemann invariants to be constant,

$$J_- = \frac{1}{2} \ln \left( \frac{1 + v}{1 - v} \right) - \frac{1}{\sqrt{\Gamma - 1}} \ln \left( \frac{\sqrt{\Gamma - 1} + c_s}{\sqrt{\Gamma - 1} - c_s} \right),$$  \hspace{1cm} (2)

where $c_s^2 = \Gamma \rho/(\rho + (\Gamma - 1)\rho)$. The other Riemann invariant is not constant,

$$J_+ = \frac{1}{2} \ln \left( \frac{1 + v}{1 - v} \right) + \frac{1}{\sqrt{\Gamma - 1}} \ln \left( \frac{\sqrt{\Gamma - 1} + c_s}{\sqrt{\Gamma - 1} - c_s} \right).$$  \hspace{1cm} (3)

One inverts the equation for $J_-$ to find the velocity

$$v = \frac{e^{2g} - 1}{e^{2g} + 1},$$  \hspace{1cm} (4)

where

$$g = J_- + \frac{1}{\sqrt{\Gamma - 1}} \ln \left( \frac{\sqrt{\Gamma - 1} + c_s}{\sqrt{\Gamma - 1} - c_s} \right).$$  \hspace{1cm} (5)

Following previous workers (Zhang & MacFadyen 2006; Morsony et al. 2007) we use a domain $-0.35 \leq x \leq 1$ and adopt $\rho^* = 100$, $\rho^* = 1$, and $v^* = 0$. We also take $\alpha = 1$ and $L = 0.35$. The adiabatic index $\Gamma = 5/3$, and the result at $t = 0.8$ is compared to the analytical one. Figure 8 shows the evolution of $\rho$, $p$, and $v$ from the initial state. Table 6 compares the $L_1$ errors for the three methods. Figure 9 shows the $L_1$ errors for the values given in Table 6, plus $\sim 10^2$ additional $N$ values for G.

### 4.6.2. Piecewise Isentropic

The Riemann problem and isentropic flow problem span extremes of two possible initial states, one with constant states and one with smooth flow. A better metric for realistic problems, where discontinuities and smooth flows are found together, would combine these. Therefore, we investigate the evolution of a structure that is initially piecewise isentropic; between the two isentropic parts we introduce a discontinuous jump in pressure and velocity.

| TABLE 6 | $L_1$ Error for Isentropic Flow |
|---------|--------------------------------|
| $N$     | FLASH | WENO | Glimm |
| 80      | 5.5e−3 | 2.1e−3 | 0.0072 |
| 160     | 1.6e−3 | 1.1e−4 | 0.0052 |
| 320     | 4.0e−4 | 1.7e−5 | 0.0033 |
| 640     | 1.0e−4 | 1.5e−6 | 0.0024 |
| 1280    | 2.5e−5 | 1.6e−7 | 0.0019 |
| 2560    | 5.4e−6 | 1.9e−8 | 0.0014 |
| 5120    | 1.6e−6 | 2.4e−9 | 0.00053 |
Since there is now no analytical solution, we carry out one ultra-high resolution run as the reference solution.

The one change we make to the isentropic flow problem is to force a jump in $p$ at $x = 0$ such that the excess above the floor level $p = 100$ drops by a factor of 2. The sharp negative gradient in $p$ at $x = 0$ drives a strong flow to the right which is superposed on the natural flow. Figure 10 shows the evolution of $\rho$, $p$, and $v$ from the initial state, and Figure 11 shows the associated errors. The “exact” solution is obtained by computing a Glimm run for $N = 10^5$ and then interpolating to the grid spacing of each of the $10^2$ trial runs. Since this is a modification of a standard test, there are no FD model errors with which to compare.

4.7. Shear Suite of Problems from Pons et al. (2000)

In their generalization of the exact special relativistic Riemann problem to include shear, Pons et al. (2000) introduce a suite of nine tests involving shear, also based on Riemann problem 2. These have been examined by Mignone et al. (2005) using the FLASH code (see their Fig. 5). In Figure 12 we present the results of applying Glimm’s method to this test suite. As with the non-shearing test problems, constant states are reproduced exactly (i.e., to within machine precision), thereby avoiding the problems with FD methods alluded to earlier.

4.8. Ultrarelativistic Shear Problems from Aloy & Rezzolla (2006)

Rezzolla et al. (2003) study the effect of shear on the standard Riemann problems and find that the standard pattern of a contact discontinuity sandwiched between a rightward-moving forward shock and a leftward-moving reverse shock, abbreviated...
field. For sufficiently large shear, the reverse shock can be replaced by a rarefaction wave; hence, the new pattern RCS arises. Aloy & Rezzolla (2006) explore the astrophysical ramifications of the Rezzolla et al. finding as a potential mechanism for accelerating jets from active galactic nuclei, microquasars, and GRBs to very high Lorentz factors. They show that by varying the left-hand pressure $p_L$ in a Riemann problem, one can change the nature of the solution.

We present two additional shearing tests that delve deeper into the ultrarelativistic regime than the “hard” shear problem presented earlier. For the first case we take $(p, \rho, v, \gamma)_R = (10^{-3}, 10^{-4}, 0.99, 20)$ and $(p, \rho, v, \gamma)_L = (10^{-6}, 10^{-2}, 0, 1)$. The Lorentz factor $\gamma$
includes both the normal and perpendicular velocities $\gamma = (1 - v^2 - v_l^2)^{-1/2}$. The adiabatic index $\Gamma = 4/3$, corresponding to the ultrarelativistic case. For this trial the shock speed $v_s = 0.151$. The result at $t = 1.8$ is compared to the analytical one. According to Aloy & Rezzolla (see their Fig. 4), $p_L = 10^{-4}$ should lie below the transition point from $\_SCS\_\_ \rightarrow \_RCS\_\_$. Figure 13 shows a comparison between the Glimm’s method solution and the exact solution for $N = 400$, and Figure 14 shows the $L_1$ norm density errors at $t = 1.8$. Since this problem is relatively new, there are no published FD results with which to compare, but one suspects that the FD errors would be comparable to or worse than those shown above in connection with the “hard” shearing problem.

For the second Aloy & Rezzolla shear problem we increase $p_L$ by 8 orders of magnitude to $10^4$. All other initial $L$ and $R$ parameters are the same. This $p_L$ value should shift the wave pattern for the Riemann solution well into the regime $\_RCS\_\_$ and yield a flow with maximum $\gamma \approx 10^{7}$ (Aloy & Rezzolla 2006, see their Fig. 4). For this trial the shock speed $v_s = 0.200$. The result at $t = 0.8$ is compared to the analytical one. Figure 15 shows a comparison between the Glimm’s method solution and the exact solution for $N = 400$, and Figure 16 shows the $L_1$ norm density errors at $t = 0.8$. For large $N$, the Glimm solutions acquire a permanent offset error in shock edge localization, rather than deviating about a mean $s = 0$. As with Figure 14 we have only Glimm errors to present, because the test is too new to have undergone published FD testing.

4.9. Spherical Blast Wave

The evolution of a relativistic blast wave in spherical symmetry has been examined by many workers. Paniutschi et al. (1997) present a detailed study using a hybrid Glimm/FD code and taking $\gamma_0 = 10^2$. Kobayashi & Zhang (2007) utilize a spherically symmetric relativistic code which uses a second-order Godunov method with an exact Riemann solver (described in Kobayashi et al. 1999) to investigate the evolution of a relativistic blast wave. Kobayashi & Zhang investigate a thin-shell case taking $\gamma_0 = 10^2$ and a thick-shell case taking $\gamma_0 = 10^3$.

The final test shown in Wen et al. (1997) is that for a relativistic blast wave with initial Lorentz factor $\gamma_0 = 10$. For comparison in Figure 17 we show results for a run with similar starting conditions. To adapt to spherical geometry we use the geometrical correction terms given in Wen et al. (1997) with $\alpha = 2$. Within a narrow radial range $0.01r_0$ centered at $r_0$ we initialize using a Blandford-McKee profile $\rho_0(r) = 10^{4,5}r_0^3 r^{7/4} \gamma^{-1}$, where $\chi = 1 + 16(1 - r/r_0)^2 \gamma_0^2$, $\gamma = \gamma_0 r \chi^{-1/2}$, $\gamma_0 = 15$, and $r_0 = 0.4$. We take $p_0(r) = 0.2\rho_0(r)$. Inside the initial shell $\rho_0 = 10^{-4}$; outside the initial shell $\rho_0 = 1$ and $p_0 = 10^{-4}$.

The profiles shown in Kobayashi & Zhang (2007) do not display obvious oscillations in the shocked shell. In our case, using a much smaller initial Lorentz factor, we see in Figure 17 a number of small oscillations, particularly in $\gamma$. This indicates that the treatment of spherical geometry is worse than that of FD conservative methods such as the one of Kobayashi & Zhang. In addition, due to the sharpness of the density shell and the strong mass jumps accompanying grid points entering into and then
leaving the shell, mass is conserved for the run shown in Figure 17 only to within $\approx 10\%$.

5. DISCUSSION

We have presented the results of a series of tests done on standard problems in relativistic hydrodynamics using Glimm’s method. To compare to previous works we utilize the $L_1$ norm errors in density. For problems involving smooth gradients such as the isentropic flow problem, Glimm’s method fares worse than the standard FD techniques, due to the fact that solutions are typically off by $\approx 1$–2 grid points. In one dimension, however, the constant states are exact to within machine precision. This is true irrespective of the presence of shear, thereby giving the method an advantage over FD methods. If there were only constant states in a solution and if the leading and trailing shock edge locations were correct, then the entire solution would also be correct (to within machine precision). The idealization of piecewise constant states for the Riemann fan, however, is a source of error, as is the incorrect position of a shock edge. A better visualization of the Glimm errors than a simple table of $L_1$ errors versus grid point number $N$ is achieved by calculating a large number of numerical and analytical values for varying $N$ and plotting the results. In such a plot one sees several bands of solutions corresponding to the total number of grid points $s$ by which the shock edge locations are off. For a given problem, the degree to which sharp edges differ from their correct locations varies both with time within a given trial and with $N$. Therefore, one cannot choose a priori the “right” resolution for any problem such that the errors are minimized; one can only see what the errors are for being off the correct solution by a given $s$ value.

For the specific problems studied in this work, Riemann problem 1 yields similar global errors between Glimm and FD methods for the ensemble of $\approx 10^2$ solutions. For Riemann problem 2, the Glimm errors are comparable to FD for solutions for which $s \approx 3$–4. The solutions with zero localization error $s = 0$ (i.e., exact matching of the shock edges to their correct values) have $L_1$ errors $\approx 10^3$ times smaller than the FD methods. For Riemann problem 3, the $s = 0$ solutions are limited only by the machine $\epsilon$ error, solutions for which $s = 1$ lie a factor $\approx 10$ below FD, and solutions with $s \approx 2$–4 are comparable to FD. For the easy shear problem, the $s = 0$ solutions have errors $\approx 10^3$ times smaller than for FD. The errors become comparable for $s > 3$–4. For the hard shear problem, the $s = 0$ solutions have errors $\approx 10^2$–$10^3$ times smaller than for FD. The errors do not become comparable for any $s$. In fact, the Glimm errors for the lowest $N$ values studied are comparable to those for the highest $N$ values in previous FD investigations. For smooth isentropic flow, the FD errors are comparable to Glimm for the smallest $N$ values. For the largest $N$ values, the FLASH errors are a factor $\approx 10^{2.5}$ smaller than for Glimm and for WENO $\approx 10^{5.5}$ times smaller than Glimm. For the relativistic blast wave test in spherical geometry (one dimension), the profiles are similar to those of a comparable run in Wen et al. (1997, see their Fig. 5).

For the local Riemann problem, the Riemann solver RIEMANN_VT.F decomposes each solution into a left wave and a right wave. Depending on the conditions, many iterations may be required; therefore, the computation time can vary greatly. Wen et al. (1997) discuss the slowness inherent in Glimm’s method and quote run times $\approx 10$ times slower than standard FD methods. We find using a $\approx 2$ GHz machine that, for example, Riemann problem 1 for
For Glimm's method to be a useful research tool, it will probably be necessary not only to have a two-dimensional version, but also to include a provision for adaptive mesh refinement. Preliminary work on a two-dimensional version has been encouraging, but more effort is required to address the issue of numerical stability.

6. CONCLUSIONS

We present the results of relativistic hydrodynamical tests using Glimm's method, along with a comparison to results using standard methods. Glimm's method in one dimension is superior to standard finite differencing for problems containing shocks, in which a sharp gradient appears. The introduction of shear does not degrade the quality of the solutions. Indeed, the work of Pons et al. (2000) generalizing the relativistic Riemann solution to include shear now also provides impetus for making a two-dimensional relativistic Glimm's method. For problems involving smooth flow, the standard finite differencing methods are much better. Although constant states are calculated exactly (i.e., to within machine precision) in Glimm's method, curved states such as Riemann fans are somewhat imprecisely modeled as being composed of a sum of piecewise constant states. Furthermore, the fact that there is an uncertainty of 1–2 grid points in the location of a given feature means that for models with smoothly varying physical parameters, the entire profile can be shifted slightly, leading to large global errors in comparison to an exact solution. The results of the piecewise isentropic run indicate that for realistic applications containing both smooth flows and sharp gradients, standard finite-difference methods give superior global behavior. Glimm's method may prove better for applications such as GRB afterglow shock propagation into a uniform medium where one is primarily interested in the physical evolution of high-entropy material only within a restricted volume (i.e., the shocked gas) and not the global evolution of low-density, low-entropy regions far away from the shock.
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