Abstract

We study BFKL evolution and, in particular, the energy dependence of the saturation momentum in the presence of saturation boundaries limiting the region of linear BFKL evolution. In the case of fixed coupling evolution we confirm the previously found exponential term in $Q_s(Y)$ and determine the prefactor $Y$ and $\alpha$ dependences. In the running coupling case we find $Y^{1/6}$ corrections to the $Y^{1/2}$ exponential behavior previously known. Geometrical scaling of the scattering amplitude is valid in a wide range of momenta for fixed coupling evolution and in a more restricted region for running coupling evolution.

1 Introduction

In 1983 Gribov, Levin and Ryskin\cite{1} introduced the idea of parton saturation in high energy hard scattering as a dual description of unitarity. Since that time our understanding of saturation, and unitarity, in hard reactions has progressed considerably\cite{2}. We now have a simple model, the McLerran-Venugopalan model\cite{3, 4, 5}, which exhibits gluon saturation in a simple and, likely, fairly general manner. This model is now being used in order to understand general features of heavy ion reactions\cite{6, 7, 8, 9, 10}. In deep inelastic scattering the Golec-Biernat and Wüsthoff model incorporates the

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essential elements of saturation and gives a surprisingly good fit to much
of the HERA data for $F_2$ and for diffractive production at small values of $x$\cite{11, 12}. There remain, however, many uncertainties in our understanding
and application of saturation ideas.

The points we wish to address in this paper are the value and energy
dependence of the saturation momentum and the form of the scattering am-
pplitude on the perturbative side of the saturation line. To be more specific,
and in order to illustrate the issues, consider the scattering of a QCD dipole
of size $1/Q$ on either a hadron or on another dipole of size $1/\mu$ and with
relative rapidity $Y$. Then the saturation momentum, $Q_s(Y)$, is the momen-
tum at which determining the scattering changes from a purely perturbative
problem, for $Q > Q_s$, to a nonperturbative but weak coupling problem where
unitarity limits have been reached, for $Q < Q_s$.

The main question which naturally arises is the $Y$–dependence of $Q_s$ and
what information (dynamics) is actually necessary to control in order to
calculate the $Y$–dependence of $Q_s$. In general terms one expects BFKL\cite{13,
14} dynamics, but not necessarily the BFKL saddle-point solution, to be
the relevant dynamics since this is the evolution which leads to high density
partonic systems. Of course one cannot expect linear BFKL evolution to be
accurate when $Q \lesssim Q_s(Y)$. As a rough guess one can use the BFKL saddle-
point solution for high energy scattering at large $Q$ and then define $Q_s(Y)$
as the value at which this scattering amplitude reaches its unitarity b ound.
This was done in Ref.15 for fixed coupling BFKL evolution with the result
$\ln(Q_s^2(Y)/\Lambda^2) = \frac{2\alpha N}{\pi} \chi(\lambda_0) Y$ where $\lambda_0$ is the solution to $\chi'(\lambda_0)(1 - \lambda_0) =
-\chi(\lambda_0)$ with $\chi$ the usual BFKL eigenvalue function. The danger with this
procedure is that one cannot justify using the saddle-point method for solving
asymptotic BFKL evolution even when the scattering amplitude is small.
(We shall show in Secs.6 and 7 that this procedure gives the exponential parts
of the $Y$–dependence of $Q_s$ correctly, but misses $Y$–dependent prefactors.)
This discussion was extended in Ref.16 to the running coupling case where
it was also observed that one can expect the scattering amplitude to be a
function of $Q^2/Q_s^2$ in an extended region where $Q^2/Q_s^2 > 1$. In Ref.17 a
numerical study of the Kovchegov equation\cite{18} was carried out. This has
the advantage that BFKL dynamics is used when $Q/Q_s \gg 1$ while unitarity
is imposed in a realistic way when $Q/Q_s \lesssim 1$. The energy dependence found
numerically for $Q_s$ is close to that expected from simple saddle-point BFKL
dynamics in the region $Q/Q_s > 1$.  

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In this paper we give a new procedure to solving linear BFKL dynamics in the region $Q/Q_s > 1$ in such a way that the matching with the nonlinear dynamics present when $Q/Q_s \lesssim 1$ should be smooth. To illustrate the main problem we face here consider the scattering of a dipole of size $1/Q$ on a dipole of size $1/\mu$ in fixed (weak) coupling BFKL evolution. Suppose we are given $Q_s(Y)$ and that $Q/Q_s \gg 1$. Can we expect the scattering amplitude to be given by the saddle-point approximation to BFKL evolution? The answer is no, not in general. Because of diffusion there will be many paths, in the functional integral sense, which go from $\mu$ at $Y = 0$ to $Q$ at $Y$ and on the way pass through the saturation region. These paths should not be allowed in the true solution to the scattering problem. This, however, seems to be the only difficulty with using the saddle-point approximation. We face this difficulty by converting the usual diffusive behavior in BFKL dynamics to one with an absorbing boundary near $Q_s(Y)$, that is we throw away all paths which go into the saturation region.

In order to gain confidence that diffusion with an absorptive barrier is the right thing to do, we first study a problem whose answer is known and which has many similarities to evolution in the presence of saturation, namely non-forward scattering in BFKL evolution. It is well-known that scattering at a non-zero momentum transfer $q$ cuts off all infrared dynamics below $q$. In scattering a dipole of size $x$ on a dipole of size $x_0$, with $qx,qx_0 \ll 1$ we expect that one can get the correct, non-forward, BFKL behavior by using forward BFKL evolution but with an absorptive boundary which eliminates diffusive paths that go into the momenta region below $q$. In Secs. 4 and 5 we verify that this is the case.

In Sec.6 we evaluate BFKL evolution in the presence of saturation in the case of dipole-dipole scattering and where the coupling is fixed. Our main results are given in (47) for the scattering amplitude, and in (48) for the saturation momentum. The exponent in (48) is as previously found while the $\alpha$ and $Y$—dependence of the prefactors is new. Eq.(47) exhibits geometric scaling when $\ln(Q^2/Q_s^2) \gg 1$, and what is perhaps even more remarkable is that there are no unknown prefactors in $T$. Eq.47 is valid so long as $\ln^2(Q^2/Q_s^2) \ll \frac{2\alpha}{\pi} \chi''(\lambda_0)Y$, that is within the diffusion regime for BFKL evolution in the absence of boundaries.

In Sec.7, we deal with the running coupling case. This discussion should apply to high-energy hard scattering on protons. Our main results are contained in (83), (84) and (85). In (83) we find, for $\rho_s = \ln(Q_s^2/\Lambda^2)$,
the expected leading \( \sqrt{Y} \) behavior along with a \( Y^{1/6} \) correction while (84) and (85) exhibit geometric scaling, but now only for \( \ln(Q^2/Q_s^2) \lesssim \left[\frac{N_c \chi''(\lambda_0)}{\pi \beta(1-\lambda_0) \chi(\lambda_0)}\right]^{1/6} \). What is remarkable is that \( Q_s \) has no knowledge of the target, whether that target be a hadron or a small dipole, so long as \( Y \) is in the asymptotic regime. For a very small dipole target, of size \( 1/\mu \), the asymptotic regime only begins when \( Y \gtrsim \pi b (1-\lambda_0)^2 \). In the running coupling case we are unable to determine target-dependent prefactors in \( T \) either when the target is a hadron and even when it is a small dipole.

2 BFKL Evolution; Naive View

We consider the forward scattering amplitude for a dipole of size \( 1/Q \) on a dipole of size \( 1/\mu \). In the large \( N_c \) limit and at the leading logarithmic level, this is described by BFKL evolution. When the strong coupling \( \alpha \) is fixed, the amplitude is

\[
T(Q, \mu, Y) = \frac{2\pi \alpha^2}{\mu^2} \int \frac{d\lambda}{2\pi i} \frac{1}{\lambda^2(1-\lambda)^2} \exp \left[ \frac{2\alpha N_c}{\pi} \chi(\lambda)Y - (1-\lambda) \ln \frac{Q^2}{\mu^2} \right] ,
\]

(1)

where

\[
\chi(\lambda) = \psi(1) - \frac{1}{2} \psi(\lambda) - \frac{1}{2} \psi(1-\lambda) ,
\]

(2)

with \( \psi(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda) \) as usual and the integration contour being parallel to the imaginary axis with \( O < Re(\lambda) < 1 \). \( Y \) is the relative rapidity of the two objects. Expression (1) is symmetric under the interchange \( Q \leftrightarrow \mu \), and is normalized at \( Y = 0 \) to be the elementary cross section of two dipoles at the two gluon exchange approximation, which is

\[
T(Q, \mu, Y = 0) = \frac{2\pi \alpha^2}{Q_>^2} \left( 1 + \ln \frac{Q_>}{Q_<} \right) .
\]

(3)

In the above, \( Q_> = \max(Q, \mu) \) and \( Q_< = \min(Q, \mu) \).
Now we define a particular line in the ln($Q^2/\mu^2$)–$Y$ plane by the following two conditions

\[ \frac{2\alpha N_c}{\pi} \chi'(\lambda_0) Y + \ln \frac{Q_0^2}{\mu^2} = 0, \]  

(4)

and

\[ \frac{2\alpha N_c}{\pi} \chi(\lambda_0) Y - (1 - \lambda_0) \ln \frac{Q_0^2}{\mu^2} = 0. \]  

(5)

Eq. (4) is just the saddle-point condition, while Eq. (5) means that along the line $Q^2 = Q_0^2(Y)$ the exponent in (1) at the saddle-point vanishes. As we shall see in a while, this is a line of almost, but not exactly, constant amplitude. The solution to (4) and (5) is

\[ \frac{\chi'(\lambda_0)}{\chi(\lambda_0)} = -\frac{1}{1 - \lambda_0}, \]  

(6)

\[ Q_0^2(Y) = \mu^2 \exp \left( \frac{2\alpha N_c}{\pi} \frac{\chi(\lambda_0)}{1 - \lambda_0} Y \right). \]  

(7)

The graphical solution to (6) is shown in Fig. 1.

The value of $\lambda_0$, as determined by (6), is $\lambda_0 = 0.372$, which is not too far from $1/2$, so that $Y$--evolution is the dominant one in the process and the approach is reasonable. One can now evaluate the scattering amplitude in a region around $Q_0^2(Y)$, by the saddle point method, when $\alpha Y$ is large. Since $\lambda_0$ is chosen to satisfy the saddle-point condition, we expand $\chi(\lambda)$ around this point. Then by using (4) and (5), the amplitude becomes

\[ T = \frac{\pi \alpha^2}{\lambda_0^2(1 - \lambda_0)^2 \mu^2} \left[ \frac{Q_0^2(Y)}{Q^2} \right]^{1 - \lambda_0} \int \frac{d\lambda}{2\pi i} \exp \left[ \frac{\alpha N_c}{\pi} \chi''(\lambda_0) Y(\lambda - \lambda_0)^2 + (\lambda - \lambda_0) \ln \frac{Q^2}{Q_0^2(Y)} \right]. \]  

(8)
The Gaussian integral is easily done and we obtain

\[
T = \frac{\pi \alpha^2}{\lambda_0^2(1 - \lambda_0)^2 \mu^2} \frac{1}{\sqrt{4\alpha N_c \chi''(\lambda_0) Y}} \left[ \frac{Q_0^2(Y)}{Q^2} \right]^{1 - \lambda_0} \exp \left[ -\frac{\pi \ln^2(Q^2/Q_0^2(Y))}{4\alpha N_c \chi''(\lambda_0) Y} \right].
\]

(9)

Since higher derivatives of \(\chi(\lambda)\) have been neglected, this result is valid so long as \(|\ln(Q^2/Q_0^2)| \ll 4\alpha N_c \chi''(\lambda_0) Y/\pi\).

A couple of comments need to follow here. (i) When the dipole size is inside the diffusion region \(\ln^2(Q^2/Q_0^2) \ll 4\alpha \chi''(\lambda_0) Y/\pi\), and ignoring, for the moment, the slowly varying prefactor \(1/\sqrt{\alpha Y}\), the dominant factor of the amplitude is \((Q_0^2/Q^2)^{1-\lambda_0}\). This of course has a scaling form, with momentum scale \(Q_0^2(Y)\) as given by (7); the \(Y\)–dependence of this scale is exponential and the coefficient in the exponent is known. (ii) The amplitude along the line \(Q^2 = Q_0^2(Y)\), as claimed earlier, is close to being constant, but not quite as it behaves as \(1/\sqrt{\alpha Y}\). We are going to find lines of constant amplitude later, but an important thing to notice here is, that one can evolve the system to arbitrary large rapidity along this line, without facing the problem of making the amplitude too big and violating unitarity constraints. This is happening because as we increase \(Y\), at the same time we exponentially suppress the dipole size. This is in sharp contrast to the usual BFKL evolution, where one considers final dipole sizes comparable to the initial one, or more precisely inside its diffusion radius \(\ln^2(Q^2/\mu^2) \ll 4\alpha N_c \chi''(1/2) Y/\pi\).
3 Diffusion into the Saturation Regime

The analysis so far is naive, in the sense that we have ignored problems arising from the diffusion of the solution, as given in (9), into the saturation regime. When the forward scattering amplitude becomes of order $2\pi/\mu^2$ (or equivalently the amplitude in impact parameter space of order 1), one should not trust the solution any more, since unitarity constraints are violated. Therefore the dynamics in that region cannot be represented by the BFKL evolution. We do not intend to find the amplitude in this saturation regime, where presumably we would have to solve the (non-linear) Balitsky-Kovchegov equation\[18, 19\], but the problem is that the solution in the purely perturbative region is altered because of unitarity effects. At this level, one can see from (9) that the amplitude becomes of order $2\pi/\mu^2$, when the dipole size $1/Q$ becomes

$$\ln(Q^2/Q_0^2) \approx -\ln(\sqrt{\alpha Y}/\alpha^2)/(1 - \lambda_0).$$

This logarithmic distance of $Q^2$ from $Q_0^2(Y)$ is small compared to the diffusion radius $\sqrt{\alpha Y}$ when $\alpha Y$ is large. Thus, even though we start with an initial condition that the dipole size is in the perturbative region, as $Y$ increases the diffusion drives the dipoles to bigger sizes that enter the saturation regime and therefore it will, partly, invalidate the result as given in (9).

This problem caused by diffusion can be seen from a mathematical point of view, when one tries to reproduce (9) by doing two successive evolutions in rapidity. Imagine first that we evolve the amplitude from zero rapidity to $Y/2$. Then assume that the amplitude at this rapidity is given by (9), with $Y \to Y/2$, for dipole sizes $1/Q$ such that the amplitude is less or equal to $2\pi/\mu^2$, and given by $2\pi/\mu^2$ for larger dipoles. This serves as an initial distribution at $Y/2$ in a simple way to impose unitarity. Then one can evolve to find the amplitude at rapidity $Y$ always in the perturbative regime. This is a straightforward calculation that we don’t present here and one can find that the result agrees with (9) provided that $\sqrt{\alpha Y} \lesssim \ln(\sqrt{\alpha Y}/\alpha^2)$. This is of course a small, for our purposes, evolution in $Y$. It simply states the fact, that the solution is incorrect when the rapidity is large enough so that the dipoles start diffusing into the saturation regime.

We shall come back to resolve this issue by imposing unitarity in a more proper way in section 6. Before this, and in order to motivate the work in that section, we find it useful to consider the non-forward scattering in section 4 and how this is related to the diffusion equation in the presence of an absorptive barrier in section 5.
4 Non-Zero Momentum Transfer

Just for the purposes of this section and the following one, we adopt a slightly different notation. We consider now the scattering of a dipole of size \( x \) on a dipole of size \( x_0 \), when the momentum transfer of the scattering is \( q \). In this case the amplitude is

\[
T_{nf}(x, x_0, Y, q) = \frac{\pi \alpha^2 x x_0}{2} \int \frac{d\nu}{2\pi} \frac{1}{(\nu^2 + \frac{1}{4})^2} E_0^{0\nu^*}(x_0) E_0^{0\nu}(x) \exp \left[ \frac{2\alpha N_c}{\pi} \chi(\nu) Y \right],
\]

where

\[
\chi(\nu) = \psi(1) - \frac{1}{2} \psi\left(\frac{1}{2} + i\nu\right) - \frac{1}{2} \psi\left(\frac{1}{2} - i\nu\right),
\]

and

\[
E_0^{0\nu}(x) = \frac{2i\nu}{\pi} x^{2i\nu} \int d^2 R e^{i\vec{q} \cdot \vec{R}} \frac{e^{i\vec{q} \cdot \vec{R}}}{(|\vec{R} + \frac{x}{2}||\vec{R} - \frac{x}{2}|)^{1+2i\nu}}.
\]

Following the method presented in [21,22], we calculate the amplitude in the Appendix when \( qx, qx_0 \ll 1 \). We obtain (for large \( Y \))

\[
T_{nf} = 8\pi \alpha^2 x x_0 \frac{1}{\sqrt{\pi D Y}} e^{(\alpha_p - 1)Y} \times \left\{ \exp \left[ - \frac{\ln^2(x/x_0)^2}{D Y} \right] - \exp \left[ - \frac{\ln^2(cq^2 x x_0)^2}{D Y} \right] \right\},
\]

where \( \alpha_p - 1 = 4\alpha N_c (\ln 2)/\pi, D = 56\alpha N_c \zeta(3)/\pi \), the constant appearing is \( c = e^{2\gamma}/64 \) and \( \gamma = 0.577... \). The result is valid in the domain \( |\ln(x/x_0)^2| \ll D Y \). Here we have done the standard BFKL evolution by expanding the \( \chi \) function around its extremum at \( \nu = 0 \) (corresponding to \( \lambda = 1/2 \) in the previous notation). Eq.(13) can also be derived by looking at the scattering at a fixed impact parameter \( b \) and then taking its Fourier transformation with the integration limits for \( b \) being from \( \sim \max(x, x_0) \) to \( \sim 1/q \).
The first term in the curly bracket of (13) can be recognized as the one appearing in the forward case. Notice that the second term has the same diffusion pattern and, because of the minus sign, it will eventually cut off large dipole sizes. We will return to comment on this in the next section.

# 5 Diffusion in the Presence of an Absorptive Barrier

Going back to (13) and looking at the two terms in the bracket accompanied by the $1/\sqrt{\pi D Y}$ prefactor, we see that they obey a diffusion equation. What is not so clear, for the moment, is the physical interpretation of the second term in the diffusion mechanism, a task that we now turn into. For reasons that will be apparent soon we review the diffusion equation in the presence of an absorptive barrier, which is

$$\frac{\partial \psi(\rho, t)}{\partial t} = \frac{1}{4} \frac{\partial^2 \psi(\rho, t)}{\partial t^2},$$

with the boundary condition

$$\psi(-\rho_a, t) = 0.$$  

We convert this into a Green’s function problem,

$$\psi(\rho, t) = \int_{-\infty}^{\infty} d\rho' G(\rho, \rho', t - t') \psi(\rho', t'),$$

where the kernel should satisfy (for $\rho, \rho' \geq -\rho_a$)

$$G(\rho, \rho', 0) = \delta(\rho - \rho').$$

If we define the Laplace transformation of $G(\rho, \rho', t - t')$ with respect to time by
\[
G(\rho, \rho', t - t') = \int \frac{d\omega}{2\pi i} e^{\omega(t-t')} G_\omega(\rho, \rho'),
\]

then Eqs. (14) and (17) imply

\[
\frac{\partial^2 G_\omega(\rho, \rho')}{\partial \rho^2} - 4\omega G_\omega(\rho, \rho') = -4\delta(\rho - \rho').
\]

(19)

The solution to this equation, satisfying the condition \(G_\omega(-\rho_a, \rho') = 0\), is (for \(\rho, \rho' \geq -\rho_a\))

\[
G_\omega(\rho, \rho') = \frac{1}{\sqrt{\omega}} \left[ e^{-2\sqrt{\omega} |\rho - \rho'|} - e^{-2\sqrt{\omega}(2\rho_a + \rho + \rho')} \right].
\]

(20)

Finally, we perform the integration in (18) along the imaginary axis, to arrive at

\[
G(\rho, \rho', t - t') = \frac{1}{\sqrt{\pi t}} \left[ e^{-\frac{(\rho - \rho')^2}{2t-t'}} - e^{-\frac{(2\rho_a + \rho + \rho')^2}{2t-t'}} \right].
\]

(21)

Eq. 16 is quite general and by choosing an initial condition \(\psi(\rho', t' = 0) = \delta(\rho')\), which is the relevant one for our purposes, we obtain

\[
\psi(\rho, t) = \frac{1}{\sqrt{\pi t}} \left( e^{-\frac{\rho^2}{2t}} - e^{-\frac{(2\rho_a + \rho)^2}{2t}} \right).
\]

(22)

If we let \(t \rightarrow D\gamma, \rho \rightarrow \ln(x_0^2/x^2)\) and \(\rho_a \rightarrow -\ln(cq^2x_0^2)\), then \(\psi\) becomes

\[
\psi(x, Y) = \frac{1}{\sqrt{\pi D\gamma}} \left\{ \exp \left[ -\frac{\ln^2(x/x_0)^2}{D\gamma} \right] - \exp \left[ -\frac{\ln^2(cq^2x_0)^2}{D\gamma} \right] \right\},
\]

(23)

which is, apart from the exponential increase and the \(8\pi\alpha^2x x_0\) prefactor, identical to (13).

Here we have two equivalent descriptions. Say that we want to calculate the forward scattering amplitude with momenta (inverse dipole size) \(Q \lesssim \).
$q$ cut-off, where $q$ is a fixed momentum. An intuitive way is to look at the non-forward amplitude at momentum transfer $q$. This is the only new scale entering the problem and thus will eventually offer the infrared cut-off scale in the problem. An alternative approach is to look directly at forward scattering, but at the same time do not allow diffusion, in the dipole size, to go into the region $Q \lesssim q$. The last must happen in an absorptive way; if the dipole hits the boundary at $q$, it never comes back in the region $Q \gtrsim q$ as we show in Fig.2.

Going back to Eqs.(22), (23) one can expand the exponentials in the diffusion region $\rho^2, \rho_a^2 \ll t$. Then (22) becomes

$$\psi(\rho, t) \simeq \frac{4\rho_a}{\sqrt{\pi}} \frac{(\rho + \rho_a)}{t^{3/2}}.$$  (24)

Notice the $t^{3/2}$ power in the denominator. It has the interpretation that the probability of not diffusing into $Q \lesssim q$, by cutting out these paths, is $1/\alpha Y$ times the probability of all paths.
We apply these ideas in the next section, where we return to impose unitarity effects in the BFKL evolution considered in Section 2.

6 BFKL Evolution in the Presence of Saturation

Referring to the starting equation (1) for the forward amplitude, we define a new line $Q_c^2(Y)$, which is close to the line $Q_0^2(Y)$ in the $\ln(Q^2/\mu^2) - Y$ plane, by the following conditions

$$\frac{2\alpha N_c}{\pi} \chi'(\lambda_c)Y + \ln \frac{Q_c^2}{\mu^2} = 0$$

(25)

and

$$\frac{2\alpha N_c}{\pi} \chi(\lambda_c)Y - (1 - \lambda_c) \ln \frac{Q_c^2}{\mu^2} = \frac{3}{2} \ln \left[ \frac{4\alpha N_c \chi''(\lambda_c)Y}{\pi} \right].$$

(26)

Eq.(25) is again a saddle-point condition, as Eq.(4), while (26) has an extra $(3/2) \ln(\alpha Y)$ term compared to (5). This definition of the critical line $Q_c^2(Y)$, will result in an extra $(\alpha Y)^{3/2}$ factor in the amplitude, which will cancel the $(\alpha Y)^{3/2}$ factor that is anticipated from the discussion of the previous section, more precisely from (24). Therefore we will be able to recover lines of constant amplitude. The solution to (25) and (26) is

$$(1 - \lambda_c)\chi'(\lambda_c) + \chi(\lambda_c) = \frac{3 \ln \left[ \frac{4\alpha N_c \chi''(\lambda_c)Y}{\pi} \right]}{4\alpha N_c \pi Y},$$

(27)

and

$$Q_c^2(Y) = \frac{\mu^2 \exp\left[ \frac{2\alpha N_c}{\pi} \frac{\chi(\lambda_c)Y}{1-\lambda_c} \right]}{\left[ \frac{4\alpha N_c}{\pi} \chi''(\lambda_c)Y \right]^{3/(2(1-\lambda_c))}}.$$
Eq. (27) determines $\lambda_c$ which is a function of the rapidity $Y$, and not a pure number like $\lambda_0$. However, when $\alpha Y$ is large, $\lambda_c$ is very close to $\lambda_0$. In this case (27) reduces to

$$\lambda_c - \lambda_0 = \frac{3 \ln \left[ \frac{4\alpha N_c}{\pi} \chi''(\lambda_0) Y \right]}{(1 - \lambda_0) \frac{4\alpha N_c}{\pi} \chi''(\lambda_0) Y}. \quad (29)$$

In the exponential in (28) we notice that

$$\frac{\chi(\lambda_c)}{1 - \lambda_c} = \frac{\chi(\lambda_0)}{1 - \lambda_0} + \frac{1}{2} \frac{\chi''(\lambda_0)}{1 - \lambda_0} (\lambda_c - \lambda_0)^2 + \cdots, \quad (30)$$

as the linear term cancels when we make use of (6). This exponential can now be evaluated at $\lambda_0$, since (29) and (30) imply that the remaining term is of order $\ln^2(\alpha Y)/\alpha Y$ which is small. Then Eq. (28) becomes

$$Q_c^2(Y) = \frac{\mu^2 \exp \left[ \frac{2\alpha N_c}{\pi} \frac{\chi(\lambda_0) Y}{1 - \lambda_0} \right]}{\left[ \frac{4\alpha N_c}{\pi} \chi''(\lambda_0) Y \right]^{\frac{1}{2(1 - \lambda_0)}}} = \frac{Q_c^2(Y)}{Q_0^2(Y)}. \quad (31)$$

It is clear that the lines $Q_c^2(Y)$ and $Q_0^2(Y)$ are close in the $\ln(Q^2/\mu^2) - Y$ plane.

Let’s call $E$ the exponent in (1). Expanding $\chi(\lambda)$ around $\lambda_c$, this exponent takes the form

$$E = \frac{2\alpha N_c}{\pi} Y \left[ \chi(\lambda_c) + (\lambda - \lambda_c) \chi'(\lambda_c) + \frac{1}{2} (\lambda - \lambda_c)^2 \chi''(\lambda_c) + \cdots \right]$$

$$- [(1 - \lambda_c) - (\lambda - \lambda_c)] \ln \frac{Q_c^2}{\mu^2} - [(1 - \lambda_c) - (\lambda - \lambda_c)] \ln \frac{Q_c^2}{Q_0^2}. \quad (32)$$

Making use of the definitions of $\lambda_c$ and $Q_c^2(Y)$, which are (25) and (26), the last expression simplifies to

$$E = \frac{1}{2} \frac{2\alpha N_c}{\pi} \chi''(\lambda_c) Y (\lambda - \lambda_c)^2 + (\lambda - \lambda_c) \ln \frac{Q^2}{Q_c^2}.$$
\[ + \frac{3}{2} \ln \left[ \frac{4\alpha N_c \chi''(\lambda_c) Y}{\pi} \right] - (1 - \lambda_c) \ln \frac{Q^2}{Q_c^2}, \tag{33} \]

where the last two terms are independent of the integration variable \( \lambda \). Once again we do the Gaussian integration to obtain

\[ T = \frac{\pi \alpha^2}{\lambda_c^2 (1 - \lambda_c)^2 \mu^2} \left[ \frac{4\alpha N_c \chi''(\lambda_c) Y}{\pi} \right]^{3/2} \left( \frac{Q_c^2}{Q^2} \right)^{1-\lambda_c} \]
\[ \times \frac{1}{\sqrt{4\alpha N_c \chi''(\lambda_c) Y}} \exp \left[ -\pi \ln^2 \left( \frac{Q^2}{Q_c^2} \right) \right]. \tag{34} \]

To simplify the notation we define

\[ \rho = \ln \frac{Q^2}{\mu^2}, \rho_c = \ln \frac{Q_c^2}{\mu^2}, \tag{35} \]

and

\[ t = \frac{4\alpha N_c \chi''(\lambda_c) Y}{\pi}. \tag{36} \]

Then the amplitude in terms of \( \rho \) and \( t \) becomes

\[ T(\rho, t) = \frac{\pi \alpha^2}{\lambda_c^2 (1 - \lambda_c)^2 \mu^2} e^{-\left(1-\lambda_c\right)(\rho-\rho_c)} t^{3/2} \psi(\rho - \rho_c, t), \tag{37} \]

where we defined

\[ \psi(\rho - \rho_c, t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{(\rho-\rho_c)^2}{t}}. \tag{38} \]

It is obvious that \( \psi \) represents the diffusive part of the amplitude since it satisfies
It is at this point that unitarity of the amplitude has to be imposed. Eq. (38) is not the proper solution of (39) in the presence of saturation. When solving this diffusion equation, we require that we do not include paths which go into the saturation region; momenta $Q \lesssim Q_s$, where $Q_s$ is the saturation boundary, will be cut out. We do this, following the discussion in the two previous sections, by requiring that $\psi$ vanish very close to the saturation boundary. Of course in the problem we consider before, the boundary was $Y-$independent, since the fixed momentum transfer $q$ was the infrared cutoff. Here $Q_s$ is $Y-$dependent and it will be a line parallel to the critical one $Q_c$. However, Eq. (39) is in terms of $\rho - \rho_c$ and $t$ variables and we can put a time independent absorptive barrier in $\rho - \rho_c$, which corresponds to a $Y-$dependent barrier in momentum $Q$, thus making our approach for cutting momenta $Q \lesssim Q_s(Y)$ reasonable.

We still have to exhibit the above in detail and we start by considering

$$\psi_s(\rho, t) = \psi(\rho - \rho_c, t) - \psi(\rho - \rho_c + 2\Delta, t),$$  \hspace{1cm} (40)$$

where $\psi$ is the one given by (38) (the solution with no boundary conditions), while in $\psi_s(\rho, t)$ we require $\rho \geq \rho_c - \Delta$, as we note that $\psi_s = 0$ when $\rho = \rho_c - \Delta$. $\Delta$ is a parameter that has to be determined. Eq.37 will now become

$$T = \frac{\pi \alpha^2}{\lambda_c^2 (1 - \lambda_c)^2 \mu^2} \frac{e^{-(1-\lambda_c)(\rho-\rho_c)}}{t^{3/2}} \frac{1}{\sqrt{\pi t}} \left[ e^{-\frac{(\rho-\rho_c)^2}{t}} - e^{-\frac{(\rho-\rho_c+2\Delta)^2}{t}} \right].$$  \hspace{1cm} (41)$$

When $\rho - \rho_c$ and $\Delta$ are much smaller than the diffusion radius $\sqrt{t}$, the t-dependent prefactors will cancel with the $1/t$ factor coming from the expansion of the exponentials. In this region we can also replace $\lambda_c$ by $\lambda_0$, since $(\lambda_c - \lambda_0)(\rho - \rho_c) \ll (\ln t)/\sqrt{t} \ll 1$ as implied by (29). Then we are lead to

$$T = \frac{\pi \alpha^2}{\lambda_0^2 (1 - \lambda_0)^2 \mu^2} \frac{4}{\sqrt{\pi}} \Delta (\rho - \rho_c + \Delta) e^{-(1-\lambda_0)(\rho-\rho_c)}.$$  \hspace{1cm} (42)$$
This expression becomes maximum at the point

\[ \rho_s = \rho_c - \Delta + \frac{1}{1 - \lambda_0}, \tag{43} \]

which is a finite and \( \alpha \)-independent distance away from the point where it becomes zero, as shown in Fig.3. Considering the amplitude at \( \rho_s \), we determine \( \Delta \) by setting

\[ T(\rho_s, t) = \frac{2\pi c}{\mu^2}, \tag{44} \]

with \( c \) a constant of order 1 as required by unitarity. Eqs.(42), (43) and (44) give

\[ \alpha^2 \Delta e^{(1-\lambda_0)\Delta} = \frac{ce\sqrt{\pi}\lambda_0^2 (1 - \lambda_0)^3}{2}. \tag{45} \]
We can solve this transcendental equation by iteration, for small coupling $\alpha$, and the solution is

$$\Delta(\alpha) = \frac{2}{1-\lambda_0} \ln \frac{1}{\alpha} - \frac{1}{1-\lambda_0} \ln \ln \frac{1}{\alpha} + O(\text{const}), \quad (46)$$

where all the constants appearing in the right-hand side of (45) have been absorbed in the (irrelevant) constant term in (46).

We are finally in a position to give a result for the forward scattering amplitude, and switching back to our original notation we have

$$T = \frac{2\pi}{\mu^2} \frac{2\alpha^2\Delta(\alpha)}{\sqrt{\pi} \lambda_0^2 (1-\lambda_0)^2} \left[ \ln \frac{Q^2}{Q_c^2(Y)} + \Delta(\alpha) \right] \left[ \frac{Q_c^2(Y)}{Q^2} \right]^{1-\lambda_0}, \quad (47)$$

with $Q_c^2(Y)$ given by (31) and $\Delta(\alpha)$ by (46).

This result is valid in the diffusion region $\ln^2 (Q^2/Q_c^2) \ll 4\alpha N_c \chi''(\lambda_0) Y/\pi$ when $Q \geq Q_c$ (to the right of the critical line in Fig.4) and in the region $|\ln(Q^2/Q_c^2)| \ll \Delta(\alpha)$ when $Q \leq Q_c$ (to the left of the critical line). The validity of (47) also requires the rapidity to be large enough, so that

$$Y \gg \frac{4\pi}{(1-\lambda_0)^2 N_c \chi''(\lambda_0)} \frac{\ln^2 \alpha}{\alpha} \quad \text{and} \quad Y \gg \frac{9\pi}{16(1-\lambda_0)^3 N_c \chi''(\lambda_0)} \frac{\ln^2(\alpha Y)}{\alpha}. \quad$$

The first condition is equivalent to $2\Delta(\alpha) \ll \sqrt{t}$, while the second is a consequence of replacing $\lambda_c$ by $\lambda_0$ in (28).

Expression (47) exhibits a scaling behaviour, since the amplitude depends only on the ratio $Q^2/Q_c^2(Y)$; lines with constant $Q^2/Q_c^2(Y)$ will be lines of constant amplitude. This in fact agrees with recent numerical solutions\[17\] of the Balitsky-Kovchegov equation, where it was found that the scaling behavior is extended in a domain outside the saturation region.

It is also interesting to notice that, even without the knowledge of the exact form of the non-linear effects, one is able to determine the amplitude $T$ in (47), with no need to introduce any unknown parameters.
Finally, we come to the issue of the rapidity dependence of the saturation momentum. In our approach, $\rho_s$ is defined in (43), and by using $\rho_s = \ln(Q_s^2/\mu^2)$ we are lead to

$$Q_s^2(Y) = \mu^2 \left[ \sqrt{\ln(1/\alpha)\alpha} \right] \chi_{\lambda_0}^{2} \frac{\exp\left[\frac{2\alpha N_c}{\pi} \frac{\chi_{\lambda_0}}{1-\lambda_0} Y\right]}{(\alpha Y)^{\frac{1}{2}}(1-\chi_{\lambda_0})},$$

(48)

where of course an overall constant factor is free.

7 The Running Coupling Case

We now extend our main results, given in (47) and (48), to the case of BFKL evolution using a running coupling. Surprisingly this turns out to be not too difficult, although we shall not be able to get a result quite as complete as (47) in that the overall constant in our amplitude will be undetermined even when $\ln(Q_s^2/Q_s^2) \gg 1$.

Eq.(1) is no longer a good starting point for running coupling BFKL evolution. Rather, we write $T(Q,\mu,Y)$ in the general form
\[ T(Q, \mu, Y) = \alpha(Q) \int \frac{d\omega}{2\pi i} \int \frac{d\lambda}{2\pi i} T_{\omega\lambda} \exp[\omega Y - (1 - \lambda)(\rho - \rho_c) + \gamma \ln Y] \]  

(49)

where in this Section $\rho = \ln(Q^2/\Lambda^2)$, not $\ln(Q^2/\mu^2)$, and where $\rho_c = \rho_c(Y)$ and $\gamma$ will be specified in a moment. $T_{\omega\lambda}$ has no $Q^2$ or $Y-$dependence, but does contain the $\mu$-dependence of $T$. $\Lambda$ is the usual QCD $\Lambda$-parameter and the $\omega$-integration in (49) goes parallel to the imaginary axis and to the right of any singularities $T_{\omega\lambda}$ may have in $\omega$. We now view $T$ as a function of $\rho$ and $Y$ with the $\mu$-dependence suppressed. Our normalization is as in the fixed coupling case. The BFKL equation is, schematically,

\[ \frac{dT / \alpha(\rho)}{dY} = \frac{\alpha(\rho)N_c}{\pi} K \ast (T / \alpha) \]  

(50)

where $K$ is the usual BFKL kernel. Eq.(50) is easily applied to (49) if one uses the fact that

\[ K \ast e^{-(1-\lambda)\rho} = 2\chi(\lambda)e^{-(1-\lambda)\rho} \]  

(51)

with

\[ \chi(\lambda) = \psi(1) - \frac{1}{2} \psi(\lambda) - \frac{1}{2} \psi(1 - \lambda) \]  

(52)

the usual BFKL eigenvalue function. Using (50) and (51) on (49) gives

\[ 0 = \int \frac{d\omega d\lambda}{(2\pi i)^2} T_{\omega\lambda} \exp[\omega Y - (1 - \lambda)(\rho - \rho_c) + \gamma \ln Y] \times \left\{ \omega + \frac{d\rho_c}{dY}(1 - \lambda) + \frac{\gamma}{Y} - \frac{2\alpha(\rho)N_c}{\pi} \chi(\lambda) \right\} \]  

(53)

We now choose $\rho_c(Y)$ in such a way that when $\rho = \rho_c(Y)$ the amplitude $T$ becomes almost constant, thus following closely our fixed coupling procedure.
of Sec.6. This is done by choosing $\rho_c(Y)$ so that as much as possible of $\frac{2\alpha}{\pi}N_c \chi(\lambda)$ is cancelled by $(1 - \lambda) \frac{d\rho_c}{dY}$ in the vicinity of the $\lambda-$values that dominate (49) and (53) when $\rho$ is near $\rho_c(Y)$. Suppose $\lambda-$values near $\lambda_c$ are dominant. Then write

$$\chi(\lambda) \simeq \chi(\lambda_c) + (\lambda - \lambda_c)\chi'(\lambda_c) + \frac{1}{2}(\lambda - \lambda_c)^2\chi''(\lambda_c),$$  \hspace{1cm} (54)$$

in a “diffusion approximation” much like that introduced by Camici and Ciafaloni[23]. Furthermore, when $\rho$ is not too far from $\rho_c$ write

$$\alpha(\rho) \simeq \frac{1}{b\rho_c}(1 - \frac{\rho - \rho_c}{\rho_c}).$$  \hspace{1cm} (55)$$

By requiring

$$\frac{d\rho_c}{dY}(1 - \lambda_c) - \frac{2N_c}{\pi b\rho_c} \chi(\lambda_c) + \gamma Y = 0$$  \hspace{1cm} (56)$$

and

$$\frac{d\rho_c}{dY} + \frac{2N_c}{\pi b\rho_c} \chi'(\lambda_c) = 0$$  \hspace{1cm} (57)$$

the leading parts of $\frac{2\alpha N_c}{\pi} \chi(\lambda)$ are cancelled and the bracket in (53) becomes

$$\{ \} \simeq \left\{ \omega - \frac{N_c}{\pi b\rho_c} \chi''(\lambda_c)(\lambda - \lambda_c)^2 + \frac{2N_c(\rho - \rho_c)}{\pi b\rho^2_c} \right. \times \left[ \chi(\lambda_c) + (\lambda - \lambda_c)\chi'(\lambda_c) + \frac{1}{2}(\lambda - \lambda_c)^2\chi''(\lambda_c) \right] \right\}$$  \hspace{1cm} (58)$$

for all terms of the type given in (54) and (55). The final term in (58) is of the same form, in $(\lambda - \lambda_c)$ as the second term, but smaller by a factor of $\frac{\rho - \rho_c}{\rho_c}$, so we drop it in what follows.

If we write
\[ T = \alpha(Q) \exp\left[-(1 - \lambda_c)(\rho - \rho_c) + \gamma \ln Y\right] \psi(\rho, Y)T_0(\mu) \quad (59) \]
then the factors of \((\lambda - \lambda_c)\) in \((58)\) can be taken to be \(\frac{\partial}{\partial \rho}\) factors acting on \(\psi\). Eq.\((53)\) then reads
\[ \left\{ \frac{\partial}{\partial Y} - \frac{N_c\chi''(\lambda_c)}{\pi b \rho_c} \frac{\partial^2}{\partial \rho^2} + \frac{2N_c(\rho - \rho_c)}{\pi b \rho_c^2} [\chi(\lambda_c) + \chi'(\lambda_c) \frac{\partial}{\partial \rho}] \right\} \psi = 0 \quad (60) \]
which becomes our basic equation. Of course for this whole procedure to work the integrals over \(\lambda\) in \((49)\) and \((53)\) should be dominated by values where \(\lambda \simeq \lambda_c\). We can check this at the end by verifying that \(\frac{\partial}{\partial \rho} \psi \lesssim \frac{1}{Y^{1/6}} \psi\) so that \(\lambda - \lambda_c \simeq Y^{-1/6}\) in \((49)\). Before solving \((60)\) we first turn to a determination of \(\rho_c(Y)\) from \((56)\) and \((57)\).
Comparing \((56)\) and \((57)\) to \((25)\) and \((26)\) one sees that \((57)\) is a saddle-point condition while \((56)\) is the condition that \(T\) change slowly, depending on \(\gamma\), with \(Y\). Substituting \((57)\) into \((56)\) gives
\[ \chi'(\lambda_c)(1 - \lambda_c) + \chi(\lambda_c) = \frac{\gamma \pi b \rho_c}{2N_c Y} \quad (61) \]
analogous to \((27)\). Anticipating that \(\rho_c(Y)/Y \ll 1\) we can expand about \(\lambda_0\), satisfying \((6)\), to get
\[ \lambda_c - \lambda_0 = \frac{\gamma \pi b \rho_c}{2N_c \chi''(\lambda_0)(1 - \lambda_0) Y} \quad (62) \]
analogous to \((29)\).
Now expand \((57)\) about \(\lambda_0\) keeping only terms up to first order in \(\lambda_c - \lambda_0\). One finds
\[ \frac{d\rho_c}{dY} + \frac{2N_c}{\pi b \rho_c} \chi'(\lambda_0) + \frac{2N_c}{\pi b \rho_c} (\lambda_c - \lambda_0) \chi''(\lambda_0) = 0 \quad (63) \]
Defining \(\rho_0\) to be the solution to
\[
\frac{d\rho_0}{dY} + \frac{2N_c}{\pi b \rho_0} \chi'(\lambda_0) = 0, \tag{64}
\]

one gets

\[
\rho_0(Y) = \sqrt{\frac{4N_c \chi(\lambda_0)}{\pi b}} \frac{(Y + Y_0)}{1 - \lambda_0}
\tag{65}
\]

with \(Y_0\) integration constant. Multiplying by \(\rho_c\), Eq.(63) can be written as

\[
\frac{d}{dY} \rho_c^2 = \frac{4N_c \chi(\lambda_0)}{(1 - \lambda_0)\pi b} - \frac{2\gamma \rho_0}{(1 - \lambda_0)Y} \tag{66}
\]

where we have replaced \(\rho_c\) by \(\rho_0\) in evaluating \(\lambda_c - \lambda_0\) in (63). Eq.(66) is easily integrated to give

\[
\rho_c^2 = \rho_0^2 - \frac{4\gamma \rho_0}{1 - \lambda_0} + \text{const} \tag{67}
\]

or

\[
\rho_c(Y) = \rho_0(Y) - \frac{2\gamma}{1 - \lambda_0} \tag{68}
\]

where we have dropped terms of size \(1/\sqrt{Y}\) on the right-hand side of (68). Eq.(68) is the generalization of (31) to the running coupling situation.

Now we are in a position to solve (60), at least in an approximate way. In the denominators in (60) we can replace \(\rho_c\) by \(\rho_0\) for large \(Y\). Then dropping \(Y_0\) in \(\rho_0\) gives

\[
\left[ \frac{\partial}{\partial Y} - \frac{a}{\sqrt{Y}} \frac{\partial^2}{\partial \rho^2} + \frac{c(\rho - \rho_c)}{Y} - \frac{\rho - \rho_c}{2Y} \frac{\partial}{\partial \rho} \right] \psi = 0, \tag{69}
\]

where
\[ a = \sqrt{\frac{N_c(1 - \lambda_0)(\chi''(\lambda_0))^2}{4\pi b\chi(\lambda_0)}} \]  

(70)

and

\[ c = \frac{1 - \lambda_0}{2}. \]  

(71)

Defining new variables

\[ \xi = \left(\frac{c}{a}\right)^{1/3} \frac{\rho - \rho_c}{Y^{1/6}} \]  

(72)

and

\[ t = 6a^{1/3}c^{2/3}Y^{1/6}, \]  

(73)

Eq.(69) becomes

\[
\left[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial \xi^2} + \xi - 4 \frac{\xi}{t} \frac{\partial}{\partial \xi} \right] \psi(\xi, t) = 0.
\]  

(74)

We do not know how to solve (74) exactly. However, we can find an approximate solution of the form

\[ \psi = \frac{1}{t^2} Ai(\xi - \lambda) \exp\left[ -\frac{\xi^2}{t} - \lambda t \right] \]  

(75)

where \( \lambda \) is a constant and \( Ai \) is the Airy function. One can easily check that (75) satisfies (74) up to terms of size \( \xi^2/t^2 \psi \). Since the Airy function decreases as \( \exp\left[ -\frac{2}{3}\xi^{3/2} \right] \) for large values of \( \xi \) our ansatz satisfies (74) in the region where \( \psi \) is not exponentially small. (Since we wish to use (75) over a wide range of values of \( t \) it is important that (74) be satisfied including all terms of size \( 1/t, \xi/t \) and \( \xi^2/t \) as such terms become important after integrating.
(73) over \(t\). This is indeed the case and this requires the \(\exp[-\xi^2/t]\) term in (75) whose actual value is small in the region of interest.

Return to (59). Using (75) and expressing all variables in terms of \(\xi\) and \(t\) one gets

\[
T = \exp\left[-\frac{\xi t}{3} + (6\gamma - 5) \ln t - \lambda t\right] Ai(\xi - \lambda) \tilde{T}_0
\]  

(76)

where we have replaced \(\frac{1-\lambda}{1-\lambda_0}\) by 1 in the exponent, and where some constants have been included with \(T_0\) to give \(\tilde{T}_0\). Following our procedure in the fixed coupling case we must impose the condition that \(T\) not become large in the saturation region. Since the first term in (76) grows strongly for negative value of \(\xi\) we should choose \(\lambda\) so that the maximum of \(T\) has a value on the order of \(1/\mu^2\) (see 44) at \(\xi = \xi_s\) and then goes to zero at \(\xi = \xi_1 + \lambda\), a value slightly less than \(\xi_s\), where \(\xi_1\) is the first zero of \(Ai(\xi)\). To that end write

\[
\xi = \lambda + \xi_1 + \delta\xi.
\]  

(77)

Then

\[
Ai(\xi - \lambda) \simeq Ai'(\xi_1) \delta\xi
\]  

(78)

leading to

\[
T = \exp[-\frac{1}{3}(\xi_1 + 4\lambda) t - \frac{1}{3} t \delta\xi + \ln \delta\xi + (6\gamma - 5) \ln t] Ai'(\xi_1) \tilde{T}_0.
\]  

(79)

\(T\) has a maximum at

\[
\delta\xi = \frac{3}{t}
\]  

(80)

at which the value of \(T\) is

\[
T_{\text{max}} = \exp[-\frac{1}{3}(\xi_1 + 4\lambda) t + \ln(3/e) + 6(\gamma - 1) \ln t] Ai'(\xi_1) \tilde{T}_0.
\]  

(81)
Clearly we must take $\gamma = 1$ and $\lambda = -\xi_1/4$ in order that $T_{\text{max}}$ be independent of $Y$.

Defining

$$\rho_s = \left(\frac{a}{c}\right)^{1/3} Y^{-1/6} [\xi_1 + \lambda + 3/\ell] + \rho_c$$  \hspace{1cm} (82)

one finds

$$\rho_s = \sqrt{\frac{4N_c \chi(\lambda_0)}{\pi b}} \frac{1}{1 - \lambda_0} Y + \frac{3}{4} \left(\frac{a}{c}\right)^{1/3} \xi_1 Y^{1/6} - \frac{1}{1 - \lambda_0}. \hspace{1cm} (83)$$

Rewriting (79) in terms of the more physical variables $\rho$ and $Y$ gives

$$T = e^{-(1-\lambda_0)(\rho - \rho_s)Y^{1/6}}Ai \left( \xi_1 + \left(\frac{c}{a}\right)^{1/3} \frac{1}{Y^{1/6}} \left[\rho - \rho_s + \frac{1}{1 - \lambda_0}\right] \right)^2 \tilde{T}_0'$$  \hspace{1cm} (84)

or

$$T = \left(\frac{Q^2}{Q_s^2}\right)^{-(1-\lambda_0)} \left[ \ln \frac{Q^2}{Q_s^2} + \frac{1}{1 - \lambda_0} \right] T'_0$$  \hspace{1cm} (85)

when $\ln(Q^2/Q_s^2) + \frac{1}{1 - \lambda_0} \ll Y^{1/6}$. $T_0'$, $T'_0$ and $\tilde{T}_0'$ are all related by rather trivially calculated constant factors. Eq.(85) is very close to (47), however in the present situation we have no control over the value of $T'_0$ in contrast to the fixed coupling case.

It was the choice of $\gamma = 1$ in (76) which allowed us to cancel the factor of $\alpha(Q) \sim 1/\sqrt{Y}$ present as a prefactor in the right-hand side of (49). That is, we have complete control over the $Y$ and $Q^2$-dependence of $T$ near saturation, but we have lost our ability to deal with the $\mu-$dependence. Referring to (83) it would appear that there are no free factors left in determining $Q_s$, apart from an uninteresting possibility of a not large additive constant associated with our choice of defining $\rho_s$ at exactly the maximum of $T$. In particular it seems that $\rho_s$, or $Q_s$, is completely independent of $\mu$ in contrast to the fixed coupling case where $\mu$ set the scale $Q_s$. Indeed, we believe this to be the case, and this makes the present discussion valid also for the scattering of a dipole.
of size $1/Q$ on a proton, however, one must be careful as to the range of $Y$ in which (83)-(85) can be applied. To see the limits on $Y$ we suppose that $\mu/\Lambda \gg 1$, then for $Y$ not too large there should be a range of $Y$ for which the fixed coupling description is applicable even when running coupling effects are allowed\[24\]. Referring to (48) one easily sees that so long as

$$\ln(\mu^2/\Lambda^2) \gg \frac{2\alpha(\mu)N_c}{\pi} \chi(\lambda_0) \frac{Y}{1 - \lambda_0}$$

(86)

$\alpha(\mu) - \alpha(Q_s) \ll \alpha(\mu)$, and the fixed coupling approach should be valid. This suggests that the boundary where one must change from a fixed coupling to a running coupling description occurs at a transition value of $Y$

$$Y_{\text{trans}} \simeq \frac{\pi(1 - \lambda_0)}{2bN_c\chi(\lambda_0)} \frac{1}{\alpha^2(\mu)}.$$  

(87)

If $Y \ll Y_{\text{trans}}$ the fixed coupling decrition is applicable while the running coupling description of this section is applicable when $Y \gg Y_{\text{trans}}$. In order to determine $T'_0$ in (84), or $T''_0$ in (85) we would have to follow the $Y$-evolution of the system through the transition rapidity, a task which we have not attempted.

**Appendix A**

Here we calculate the amplitude in the non-forward case following\[21, 22\]. With the definitions and notation used in Section 4, we start by exchanging the denominators of $E^0_{\nu}(x)$ as given in (12), by introducing a Feynman parameter $\beta$. After integrating over $\tilde{R}$ we find

$$E^0_{\nu}(x) = \frac{4i\nu}{\pi} (2q)^{2\nu} \Gamma^2(1 + i\nu) e^{i\vec{q} \cdot \vec{x}/2} \times \int_0^1 \frac{d\beta}{\sqrt{\beta(1 - \beta)}} e^{-i\beta \vec{q} \cdot \vec{x}} K_{2i\nu}(\sqrt{\beta(1 - \beta)} qx)$$  

(A.1)

For small $xq$ the exponentials can be approximated by 1 and expanding the Bessel function for small argument as

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\[ K_{2\nu} \left( \sqrt{\beta(1-\beta)} \ qx \right) = \frac{1}{4i\nu} \left\{ \Gamma(1+2i\nu) \left[ \frac{2}{qx\sqrt{\beta(1-\beta)}} \right]^{2i\nu} - \text{c.c.} \right\} \]  
\text{(A.2)}

where c.c. stands for complex conjugate, one finds

\[ E_0^\nu(x) = \frac{1}{\pi (2q)^{2i\nu}} \frac{\Gamma^2(1+i\nu)}{\Gamma^2(1+2i\nu)} \times \left[ \Gamma(1+2i\nu) \left( \frac{2}{qx} \right)^{2i\nu} \int_0^1 d\beta \left( \sqrt{\beta(1-\beta)} \right)^{-1-2i\nu} - \text{c.c.} \right]. \]  
\text{(A.3)}

The integral over the Feynman parameter \( \beta \) is straightforward and we are lead to

\[ E_0^\nu(x) = (2q)^{2i\nu} \frac{\Gamma(1-2i\nu)}{\Gamma(1+2i\nu)} \left[ \frac{\Gamma^2(1+i\nu)}{\Gamma^2(1-i\nu)} \left( \frac{8}{qx} \right)^{2i\nu} - \left( \frac{qx}{8} \right)^{2i\nu} \right]. \]  
\text{(A.4)}

The factor outside the bracket in the last expression is just a phase, when \( \nu \) is real, and therefore cancels when we consider the product \( E_{q^*}^{0\nu}(x_0) E_0^\nu(x) \), for which we have

\[ E_{q^*}^{0\nu}(x_0) E_0^\nu(x) = 2 \left( \frac{x_0}{x} \right)^{2i\nu} - 2 \frac{\Gamma^2(1-i\nu)}{\Gamma^2(1+2i\nu)} \left( \frac{q^2 x_0}{64} \right)^{2i\nu}, \]  
\text{(A.5)}

plus terms odd in \( \nu \) which will vanish when the \( \nu \) integration is done. The second term may be written as

\[ -2 \left( \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)} e^{-2\gamma i\nu} \right)^2 \left( \frac{e^{2\gamma q^2 x_0}}{64} \right)^{2i\nu} = \]

\[ -2 \left( 1 - \frac{4i\zeta(3)}{3} \nu^3 + \cdots \right) \left( \frac{e^{2\gamma q^2 x_0}}{64} \right)^{2i\nu}, \]  
\text{(A.6)}

so that, neglecting terms of order \( \nu^3 \), we finally have
\[ E_q^{0\nu}(x_0)E_q^{0\nu}(x) = 2 \left( \frac{x_0}{x} \right)^{2i\nu} - 2 (cq^2x_0x)^{2i\nu}, \]  

(A.7)

with \( c = e^{2\gamma}/64 \). Now the \( \nu \) integration in (10) can be done in the standard way; by expanding \( \chi(\nu) \) around \( \nu = 0 \), keeping only up to quadratic terms and performing the Gaussian integration. This leads to Eq.(13). The second term in (A.7) gives rise to the second term in (13).

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