Use of Physical Variables in the Chern-Simons Theories

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ABSTRACT

The use of the physical variables in the fashion of Dirac in the three-dimensional Chern-Simons theories is presented. Our previous results are reinterpreted in a new aspect.

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I. INTRODUCTION

This talk is devoted to the use of the physical variables in the fashion of Dirac [1] to the Chern-Simons (CS) theories in 2+1 dimensions with reinterpretations of our previous results [2]. In order to give motivations of our work, it would be better to introduce the CS theories [3] to this audience first.

The (pure) CS Lagrangian is defined by

\[ L_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \]  

(1)

where \( \epsilon^{012} = 1, \) \( g_{\mu\nu} = \text{diag}(1,-1,-1). \) But this theory is not so interesting in the aspect of the physical variables which are gauge invariant because there is no non-vanishing gauge invariant variable according to the equations of motion, \( F_{\mu\nu} = 0; \) only the pure-gauge variables are remained. Because of this problem, let us now consider the CS theories with matter couplings. Here, we only consider the coupling to the charged (complex) scalar fields for simplicity (the coupling with fermion fields is not different much) [4, 5, 6]:

\[ L = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi, \]  

(2)

where \( D_\mu = \partial_\mu + i A_\mu. \) Then, \( L \) is invariant up to the total divergence under the gauge transformations

\[ \phi \rightarrow e^{-i\Lambda} \phi, \quad A_\mu \rightarrow A_\mu + \partial_\mu \Lambda. \]  

(3)

The equations of motion are given by

\[ \frac{\kappa}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} = J^\mu \]  

(4)

with the conserved current \( J^\mu = i [(D^\mu \phi)^* \phi - \phi^* D^\mu \phi]; \) in the component form they are expressed by

\[ \kappa B = J^0, \]  

(5)

\[ \kappa E^k = \epsilon_{ki} J^i \]  

(6)

with the (scalar) magnetic and electric fields, \( B = \frac{1}{2} \epsilon^{ij} F_{ij}, E^k = F^{k0}, \) respectively. Here, now one can find the non-vanishing gauge invariant variables and hence this theory is more interesting in the aspect of the physical variables. Moreover, the integral form of the constraint (4), which is given by

\[ \Phi = \int B \, d^2 x = \frac{1}{\kappa} Q, \]  

(7)

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implies that when there is a particle with charge \( Q \), it carries the magnetic flux \( \Phi = \frac{1}{\kappa} Q \). In this sense, the CS theory realizes the Wilczek’s charge-flux composite \[7\] and this is a role of the CS field to the matters in the particle picture \[8\]. Its (quantum) field theoretical analysis was first given by Hagen in the Coulomb gauge, i.e., \( \nabla \cdot A = 0 \) \[4\]. The (formal) solution of gauge fields \( A^\mu \) which solves the constraints (5), (6) are given by

\[
A^i(x, t) = \frac{1}{2\pi\kappa} \int d^2\mathbf{y} \, \epsilon_{ij} \frac{(x-y)^j}{|x-y|^2} J^0(y, t),
\]

\[
A^0(x, t) = \frac{1}{2\pi\kappa} \int d^2\mathbf{y} \, \epsilon_{ij} \frac{(x-y)^j}{|x-y|^2} J^i(y, t).
\]

In this solution, we obtain first of all the following commutation relations,

\[
\left[ Q, \phi(x, t) \right] = \phi(x, t), \tag{9}
\]

\[
\left[ B(y, t), \phi(x, t) \right] = \frac{1}{\kappa} \delta^2(x-y) \phi(y, t), \tag{10}
\]

where \( Q = \int J^0 d^2\mathbf{x} = i \int d^2\mathbf{x} \, (\pi \phi - \phi^* \pi^*) \). These represent that field \( \phi(x) \) carries the unit charge and delta-function magnetic field, \( \frac{1}{\kappa} \delta^2(x-y) \) at \( x \) point; these describe the charge-flux composite in the context of quantum field theory. Moreover, if we consider the angular momentum generator which is constructed from the symmetric gauge invariant energy-momentum tensor, we can find there is unconventional term in addition to the expected orbital angular momentum part:

\[
M^{12} = \int d^2\mathbf{x} \, \epsilon_{ij} \mathbf{x}^i T^{0j} = \int d^2\mathbf{x} \, \epsilon_{ij} \mathbf{x}^i (\pi D^j \phi + c.c) = \int d^2\mathbf{x} \, (\pi \mathbf{x} \times \nabla \phi + c.c.) + \frac{1}{2\pi\kappa} \int d^2\mathbf{x} \, \int d^2\mathbf{y} \, J^0(x) \frac{\mathbf{x} \cdot (x-y)}{|x-y|^2} J^0(y) = J_{\text{orb}} + \frac{1}{4\pi\kappa} Q^2. \tag{11}
\]

The first term is the usual orbital angular momentum part. The second term is an unconventional term which can not be removed by a redefinition of the angular momentum generator because this anomalous term is uniquely governed by \([M^{01}, M^{02}] = iM^{12}\) in the Poincaré algebra \[4, 9\]. Now, it is easy to find that the first and second terms produce the following transformation for \( \phi \):

\[
[M^{12}, \phi] = -i\mathbf{x} \times \nabla \phi + \frac{1}{2\pi\kappa} Q \phi. \tag{12}
\]

Here, the first term shows the orbital angular momentum and the second one shows the (anomalous) spin for scalar fields which is a very surprising result. [This unconventional transformation
is usually called as the rotational anomaly but here the use of word ‘anomaly’ is not usual one because the anomaly exists even at the classical label in our case. This is the first non-trivial result in the theory which was given first by Hagen [4, 9]. In addition to this, several authors considered furthermore the so-called anyonic [here, ‘anyon’ means the object which interpolates between boson and fermion] commutation relation inspired by the works of Polyakov [5] and Semenoff [6]: In the Semenoff’s work, for example, he introduced the composite operators like

\[ \bar{\phi}(x) := \exp \left[ \frac{i}{2\pi\kappa} \int d^2y \Theta(x - y) J^0(y) \right] \phi(x), \]

\[ \bar{\pi}(x) := \exp \left[ -\frac{i}{2\pi\kappa} \int d^2y \Theta(x - y) J^0(y) \right] \pi(x) \]

and he found that their commutation relations are given by

\[ \bar{\phi}(x) \bar{\phi}(y) - e^{-\frac{i\Delta}{2\pi\kappa}} \phi(y) \phi(x) = 0, \]

\[ \bar{\pi}(x) \pi(y) - e^{-\frac{i\Delta}{2\pi\kappa}} \pi(y) \pi(x) = 0, \]

\[ \phi(x) \pi(y) - e^{\frac{i\Delta}{2\pi\kappa}} \pi(y) \phi(x) = i\delta^2(x - y), \]

where, \( \Theta(x - y) \) is the angle between the vector \( x - y \) and the \( x_1 \)-axis and \( \Delta := \Theta(x - y) - \Theta(y - x) = \pi (mod \ 2\pi) \). So, depending on the value of the CS coupling constant \( \kappa \), the composite operators can represent the fermions \( (\frac{\Delta}{2\pi\kappa} = \pi) \) or generally anyons \( (\frac{\Delta}{2\pi\kappa} \neq \pi) \) as well as bosons \( (\frac{\Delta}{2\pi\kappa} = 0) \). These are all stories about the CS theories in the canonical quantization with the Coulomb gauge. Now, our question is

**What is the physical (gauge independent) effect and what is the unphysical (gauge artifact) one?**

**II. GAUGE INvariant FORMULATION**

In order to shed some light to the (gauge artifact) problem, let us consider the Dirac’s physical variables instead of gauge varying base fields \( \phi, \pi, \cdots, \) etc. [4 2]

\[ \hat{\phi}(x) := \phi(x)e^{iW}, \ \hat{\pi}(x) := \pi(x)e^{-iW}, \ A_\mu(x) := A_\mu(x) - \partial_\mu W, \]

and their complex conjugates with \( W = \int d^2z \ c_k(x, z) A^k(z) \). These are gauge invariant for each Dirac dressing function \( c_k(x, z) \) under the gauge transformation [3] if the dressing function satisfies

\[ \partial_z^k c_k(x, z) = -\delta^2(x - z). \]
Then, it is easy to find that the physical variable $\hat{\phi}(x)$, which is gauge invariant\footnote{Here, one should distinguish gauge invariance and gauge independence: Our physical variables are gauge invariant but they are gauge dependent because its explicit form are gauge dependent. I’d like to thank Dr. J. Watson for providing a source of this note in his talk\cite{1}.} carries one unit charge ($[Q(y), \hat{\phi}(x)] = \hat{\phi}(x)$) and point magnetic field ($[B(y), \hat{\phi}(x)] = \frac{1}{\kappa} \delta^2(x-y)\hat{\phi}(x)$) at the matter point together with the vector potential around the matter point ($[A^i(y), \hat{\phi}(x)] = \frac{1}{\kappa} \epsilon_{ik} c_k(x,y)\hat{\phi}(x)$). This situation corresponds to the dual picture of the QED case where $\hat{\phi}(x)$ carries the radial $E$ field together with the scalar potential around the matter point $[1, 11]$.

Next, let us consider the space-time transformation properties of the physical variables. To this end, let us consider the improved Poincaré generators\cite{12} constructed from the symmetric (Belinfante) energy-momentum tensor\cite{13} which being (manifestly) gauge invariant and satisfying the Poincaré algebra:

\begin{align*}
P^0 & = \int d^2x \left[ |\pi|^2 + |D^i \phi|^2 + m^2 |\phi|^2 \right], \\
P^i & = \int d^2x \left[ \pi D^i \phi + (D^i \phi)^* \pi^* \right], \\
M^{12} & = \int d^2x \, \epsilon_{ij} x^i \left[ \pi D^j \phi + (D^j \phi)^* \pi^* \right], \\
M^{0i} & = x^0 P^i - \int d^2x \, x^i \left[ |\pi|^2 + |D^i \phi|^2 + m^2 |\phi|^2 \right]. \tag{17}
\end{align*}

First of all, we consider the spatial translation generated by

$$[\hat{\phi}(x), P^i] = \partial^j \hat{\phi}(x) - i \hat{\phi}(x) \int d^2z \left( \partial_x^2 c_k(x,z) + \partial^i c_k(x,z) \right) A^k(z). \tag{18}$$

This shows the translational anomaly in general. However, we assume that this anomaly should not appear in order that $\hat{\phi}$ responds conventionally to translations because the translation invariance is a genuine property of space-times which is independent on the matter contents. With this assumption, we obtain the condition that $c_k(x,z)$ be translationally invariant

$$\partial_x^2 c_k(x,z) = -\partial_x^i c_k(x,z), \tag{19}$$

i.e., $c_k(x,z) = c_k(x - z)$. Furthermore, this condition also guarantees the correct spatial translation law for another physical field in $(15)$: $[A_i(x), P^j] = i \partial^j A_i(x)$.

By applying similar assumption to the time translation we obtain

$$[\hat{\phi}(x), P^0] = i \partial^0 \hat{\phi}(x) + \hat{\phi}(x) \int d^2z \, \partial^0 (c_k(x-z)) A^k(z)$$

$$:= i \partial^0 \hat{\phi}(x), \tag{20}$$

$$[A_i(x), P^0] = i \partial^0 A_i(x)$$
with a necessary condition of
\[ \partial^i c_k(x - z) = 0. \]  

(21)

However, for the rotation and Lorentz boost, the anomaly is present as the spin or other properties of \( \hat{\phi}, A_i \):

\[ [\hat{\phi}(x), M^{12}] = i\epsilon_{ij}x_j\partial_i\hat{\phi}(x) + \Xi^{12}(x)\hat{\phi}(x), \]
\[ [A_i(x), M^{12}] = i\epsilon_{jk}x_j\partial_kA_i(x) - i\epsilon_{ij}A_j(x) + i\partial_i\Xi^{12}(x), \]
\[ [\hat{\phi}(x), M^{0j}] = ix^0\partial^j\hat{\phi}(x) - ix^j\partial^0\hat{\phi}(x) + \Xi^{0j}(x)\hat{\phi}(x), \]
\[ [A_i(x), M^{0j}] = ix^0\partial^jA_i(x) - ix^j\partial^0A_i(x) - i\delta_{ij}A_0 + i\partial_i\Xi^{0j}(x)\hat{\phi}(x); \]  

(22)

or in the compact form these are expressed as follows

\[ [F_{\alpha}(x), M^{\mu\nu}] = ix^\mu\partial^\nu F_{\alpha}(x) - ix^\nu\partial^\mu F_{\alpha}(x) + i\Sigma_{\alpha\beta}^{\mu\nu}F_{\beta}(x) + i\Omega_{\alpha}^{\mu\nu}(x), \]
\[ \Omega_{\hat{\phi}}^{\mu\nu}(x) = -i\Xi^{\mu\nu}(x)\hat{\phi}(x), \quad \Omega_{\hat{\phi}^*}^{\mu\nu}(x) = i\Xi^{\mu\nu}(x)\hat{\phi}^*(x), \]
\[ \Omega_{A_i}^{\mu\nu}(x) = \partial_i\Xi^{\mu\nu}(x), \]  

(23)

where

\[ \Xi^{\mu\nu} = -\Xi^{\nu\mu}, \]
\[ \Xi^{12}(x) = -x \times A(x) + \frac{1}{\kappa} \int d^2z \ z \cdot c(x - z)J_0(z), \]
\[ \Xi^{0i}(x) = -x_iA^0(x) + \frac{1}{\kappa} \int d^2z \ z_i \ c(x - z) \times J(z). \]  

(24)

with \( F_{\alpha} = (A_\mu, \hat{\phi}, \hat{\phi}^*) \) and the spin-factors \( \Sigma^{\mu\nu}_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}; \Sigma^{\mu\nu}_{\phi(\phi^*)} = 0 \) for the gauge and scalar fields, respectively. The \( \Omega_{\alpha}^{\mu\nu} \) represents the gauge invariant anomaly term. Now, I would like to note several interesting aspects of our formulation in the followings.

**A. On-shell expression of \( A_\mu \)**

The physical variable \( A_\mu \) in (15) is rather formal one. But, if we use the equations of motion (1), (3), we can obtain the following expressions:

\[ A_i(x) \approx -\frac{1}{\kappa} \int d^2z \ \epsilon_{ik}c_k(x - z)J^0(z), \]
\[ A_0(x) = \frac{1}{\kappa} \int d^2z \ c(x - z) \times J(z), \]  

(25)
respectively. Then, it is straightforward to show that the anomaly terms can be written in the simple forms:

\[ \Xi^{12}(x) \approx -\frac{1}{\kappa} \int d^2z \ (x - z) \cdot c(x - z) J^0(z), \]

\[ \Xi^{0i}(x) = \frac{1}{\kappa} \int d^2z \ (x^i - z^i) \ c(x - z) \times J(z). \]  

(26)

This can be considered as a generalization of the Coulomb gauge to the arbitrary \( c_k \) case: With a particular choice of \( c_k \) which corresponds to the Coulomb gauge, this reduces to the previous result in Coulomb gauge.

**B. Gauge fixing vs "c_k"**

Now, I’d like to note an issue, which has been controversial, about the connection of gauge fixing and “\( c_k \)” function. The problem which makes the situation complicate is that the connection is not one to one in general; in most case of (partial) gauge fixings which allow some remaining gauge transformations, \( c_k \) is not uniquely determined. But there is one exceptional case of the Coulomb gauge where the connection is one to one and so only one \( c_k \) is allowed.\(^4\) Let us consider these two cases in detail.

**a) axial gauge, \( A_1 ≈ 0 \)**

In this case, the gauge transformation function \( \Lambda \) should be \( x^1 \) independent, i.e., \( \partial_1 \Lambda = 0 \) in order to preserve the given gauge choice under the gauge transformation: \( A_1 \to A_1 + \partial_1 \Lambda := 0, \ A_2 \to A_2 + \partial_2 \Lambda. \) On the other hand, the physical matter field \( \hat{\phi} \), which is gauge invariant, becomes

\[ \hat{\phi} = e^{i \int c_k A_k} \hat{\phi} = e^{i \int c_2 A^2} \hat{\phi} \]

in this gauge. So, in order to make it gauge invariant in that gauge,

\[ \hat{\phi} \to e^{i \int c_2(A^2 - \partial_2^2 \Lambda)} e^{-i \Lambda} \hat{\phi} := \hat{\phi} \]

one should require

\[ \partial_2^2 c_2(x - z) = -\delta^2(x - z), \ \partial_1^1 c_1(x - z) = 0. \]  

(27)

The general solution of this equation can be expressed by two arbitrary functions, \( F \) and \( G \):

\[ c_1(x - z) = F(x_1 - z_1), \]

\[ c_2(x - z) = -\delta(x_1 - z_1) e(x_2 - z_2) + G(x_1 - z_1). \]  

(28)

\(^4\)However, when the gauge fixing is not introduced, the fixing \( c_k \) does not mean fixing a gauge.
The existence of these two arbitrary functions makes the analysis complicate. But for the simplest case of \( F = G = 0 \), the physical matter field \( \hat{\phi} \) becomes
\[
\hat{\phi}(x) = e^{i \int_{-\infty}^{x_{2}} dz A^{2}(x_{1}, z) \phi(x)}
\]
and it carries the vector potential tail along straight line from the matter point, \( A^{1}(y) = -\frac{1}{\kappa} \delta(x_{1} - y_{1}) \epsilon(x_{2} - y_{2}) \); this is the usual line-integral form of the physical variables.\[1\]

b) Coulomb gauge, \( \nabla \cdot A = 0 \)

In this case, the gauge transformation function \( \Lambda \) should satisfy Laplace equation \( \nabla^{2} \Lambda = 0 \) over all space in order to maintain the given gauge under the gauge transformation: \( \nabla \cdot A \rightarrow \nabla \cdot A + \nabla^{2} \Lambda := 0 \). However, since there is only the trivial solution of \( \Lambda = 0 \) for the Laplace equation over all space, we don’t need to introduce the gauge compensating factor in the physical matter field: \( \hat{\phi} = \phi \). In other words, \( W = \int d^{2}z c_{j}(x, z) A^{j}(z) := 0 \) in the general formula (15). Then, it is to not difficult to show that there is a unique solution of
\[
c_{j}^{\dagger}(x - z) = -\frac{1}{2\pi \lvert x - z \rvert^{2}}. \quad (29)
\]
Hence, this case corresponds to a complete gauge fixing.\[5\]

C. (Anyonic) Commutation relations

The next thing I would like to note is the commutation relations of the physical variables which show the anyonic commutation relations in general:
\[
\hat{\phi}(x) \hat{\phi}(y) - e^{-\frac{i}{\kappa} \Delta(x - y)} \hat{\phi}(y) \hat{\phi}(x) = 0,
\]
\[
\hat{\phi}(x) \hat{\pi}(y) - e^{\frac{i}{\kappa} \Delta(x - y)} \hat{\pi}(y) \hat{\phi}(x) = \delta^{2}(x - y),
\]
\[
[A_{i}(x), \hat{\phi}(y)] = \frac{1}{\kappa} \hat{\phi}(y) [\epsilon_{ik} c_{k}(y - x) + \partial_{i} \Delta(x - y)] ,
\]
\[
[A_{i}(x), A_{j}(y)] = \frac{i}{\kappa} \left[ \epsilon_{ij} \delta^{2}(x - y) + \xi_{ij}(x - y) + \partial_{i} \partial_{j} \Delta(x - y) \right] , \cdots , etc, \quad (30)
\]
where we have introduced two functions
\[
\Delta(x - y) = \int d^{2}z \ \epsilon^{kl} c_{k}(x - z) c_{l}(y - z),
\]
\[
\xi_{ij}(x - y) = \epsilon_{ik} \partial_{j} c_{k}(y - x) - \epsilon_{jk} \partial_{i} c_{k}(x - y), \quad (31)
\]
which are totally antisymmetric under the interchange of all the indices. If we consider the particular case of Coulomb gauge, we can find that \( \Delta(x - y) = 0 \) \[\xi_{ij}(x - y) = -\epsilon_{ij} \delta^{2}(x - y)\].

\[5\] A generalization of this case was known [14].
But in general this is not true! This implies that the physical variable $\hat{\phi}$ satisfies the anyonic commutation relation depending on $c_k$. Hence, this can be a natural definition of the anyon in the minimal form of \cite{13}!; we can introduce any arbitrary gauge invariant factors except for the minimally introduced gauge compensating factor of \cite{13} but there is no unique way to fix them\footnote{In the particular case of non-relativistic delta-function sources, there is an unique way to define the additional factor \cite{15}. Moreover, there is an interesting formulation where the additional factor is determined by the temporal non-locality \cite{16} contrast to ours where the temporal locality was assumed from the start in \cite{15}. But it is unclear that their formulation provides a new determination of the factor since there is no retardation effect; there are only the instantaneous interactions in our case.}. On the other hand, we note that the Semenoff’s construction of \cite{13}, what we have introduced in the former part of this talk, can be considered like as \cite{17}

$$\bar{\phi}(x) = \exp \left[ \frac{i}{2\pi\kappa} \int d^2y \, \Theta(x-y) J^0(y) \right] \hat{\phi}(x)$$

(32)

in our formulation; these operator has the (gauge invariant) exchanging phase factor of (14) in addition to the phase factor of (30) for the minimal part $\hat{\phi}$.

**D. Physical and unphysical effects of the rotational anomaly for $\phi$**

Now, to discuss the gauge artifact problem on the rotational anomaly let us note the following general relation

$$M_{12}^{12} \approx M_{12}^c - \frac{\kappa}{2} \int d^2z \, \partial^k (z^k \mathbf{A} \cdot \mathbf{A} - \mathbf{z} \cdot \mathbf{A} \mathbf{A}^k),$$

(33)

where $M_{12}^c$ is the canonical angular momentum

$$M_{12}^c = \int d^2z \left\{ -\mathbf{z} \times [\pi \nabla \phi + (\nabla \phi) \star \pi] + \kappa \mathbf{z} \cdot \mathbf{A} (\nabla \cdot \mathbf{A}) + \frac{\kappa}{2} \partial^k (\mathbf{z} \cdot \mathbf{A} \mathbf{A}^k) \right\}.$$  

(34)

The surface terms in $M_{12}^{12} - M_{12}^c$ and $M_{12}^c$, which are gauge invariant for the rapidly decreasing gauge transformation function $\Lambda$, give the gauge independent spin terms \"$(1/4\pi\kappa)Q^2$\" (unconventional) and \"0\" (conventional) in $M_{12}^{12}$, respectively. From which the anomalous spin $\frac{Q}{2\pi\kappa}$ of (12) for the matter field is readily seen to follow for general gauges. On the other hand, the second term of $M_{12}^c$, which vanishes only in the Coulomb gauge, gives for the general gauges the gauge restoring contribution to the rotation transformation for the matter field. This can be summarized by

$$[M_{12}^{12}, \phi] = -i\mathbf{x} \times \nabla \phi + \frac{1}{2\pi\kappa} Q \phi + \text{(gauge restoring terms for non-rotation invariant gauge)},$$

(35)

in which the second term is the gauge invariant, i.e., physical (anomalous) spin term and the last one is gauge dependent artifact term.
III. CONCLUSION

In summary, we have shown that the use of the Dirac’s physical variables in the CS-matter theories are:

a) generalization of the Coulomb gauge result to arbitrary $c_k$ cases,

b) providing a natural definition of anyons modulus the gauge invariant factors which will be determined by more fundamental theorems.

Involving the gauge artifact problems, what have been clarified in this talk are:

a) the anomalous spin of the base matter field is a physical effect and not the gauge artifact; and hence the Coulomb gauge is the best one to obtain the physical spin directly. On the other hand, since the boson commutation relation of the base matter field, $[\phi(x), \phi(y)] = 0$ is gauge independent (thought not gauge invariant), this is a counter example [4] for the quantum mechanical spin-statistics connection [4].

b) the minimal form of the physical matter fields is anyon field (i.e., satisfies the anyonic commutation relations) but depending on $c_k$; its physical contents depend on $c_k$ explicitly. The gauge artifact nature of this anyon, when a gauge fixing is introduced, is not so manifest because of complication in relation of $c_k$ and the gauge fixing. This problem has not been completely resolved yet; though the physical matter field with additional gauge invariant factor has the physical exchanging phase factor additionally, the problem for the minimal part remains.

As a further work, the non-Abelian generalization will be interesting. But I have no idea at present mainly because our result depends on the detailed expression of the physical variables [1] which will be very non-trivial ones [19].

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References

[1] P. A. M. Dirac, Can. J. Phys. 33, 650 (1955).

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7This is contrast to the formulation of Nair et.al [18] where the details of the physical variables are not important.
[2] M.-I. Park and Y.-J. Park, Phys. Rev. D58, 101702 (1998); hep-th/9903069; M.-I. Park, “Unraveling the complexity of the Chern-Simons gauge theory”, Ph. D. thesis (Sogang Univ., 1997).

[3] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. 48, 975 (1982); Ann. Phys. (N.Y.) 140, 372 (1982); 185, 406 (E) (1988).

[4] C. R. Hagen, Ann. Phys. (N.Y.) 157, 342 (1984); Phys. Rev. D31, 2135 (1985).

[5] A. M. Polyakov, Mod. Phys. Lett. A325, 325 (1988).

[6] G. W. Semenoff, Phys. Rev. Lett. 61, 517 (1988).

[7] F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982); 49, 957 (1982).

[8] R. Jackiw, Ann. Phys. 201, 83 (1990).

[9] C. R. Hagen, Phys. Rev. D31, 331 (1985); D31, 848 (1985).

[10] J. Watson, “Pinch technique approach to gauge-dependent Green’s function”, in this volume.

[11] M.-I. Park, “(2+1)-dimensional QED, anomalous surface-term contributions and superconductivity”, hep-th/9805033.

[12] C. G. Callan, S. Colman, and R. Jackiw, Ann. Phys. 59, 42 (1970).

[13] F. J. Belinfante, Physica (Utrechet) 7, 449 (1940).

[14] A. Foerster and H. O. Girotti, Nucl. Phys. B342, 680 (1990); R. Banerjee, A. Chatterjee, and V. V. Sreedhar, Ann. Phys. 222, 254 (1993).

[15] C. R. Hagen, Phys. Rev. Lett. 63, 1025 (1989); 70, 3518 (1993); R. Jackiw and S. Y. Pi, Phys. Rev. D42, 3500 (1990), (E) 3500 (1990).

[16] M. Lavelle and D. McMullan, Phys. Lett. B436, 339 (1998).

[17] K. Lee, Nucl. Phys. B373, 735 (1992); R. Banerjee, Phys. Rev. Lett. 69, 17 (1992); E. Graziano and K. D. Rothe, Phys. Rev. D49, 5512 (1994).

[18] D. Karabali and V. P. Nair, Nucl. Phys. B464, 135 (1996); Phys. Lett. B379, 141 (1996); D. Karabali, C. Kim and V. P. Nair, Nucl. Phys. B524, 661 (1998); Phys. Lett. B434, 103 (1998).
[19] M. Lavelle and D. McMullan, 329, 68 (1994); Phys. Rep. 279, 1 (1997); M. Bellon, L. Chen and K. Haller, Phys. Lett. B 373, 185 (1996); Phys. Rev. D55, 2347 (1996).