CHARACTERISATION OF THE AFFINE PLANE USING $\mathbb{A}^1$-HOMOTOPY THEORY

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Abstract. In this article we prove that any $\mathbb{A}^1$-contractible smooth complex surface is isomorphic as a variety to $\mathbb{C}^2$. We show that the $\mathbb{A}^1$-connected component of a variety $X$ contains the information about $\mathbb{A}^1$-s in $X$.

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1. Introduction

A natural problem in affine algebraic geometry is to determine whether a given affine variety $X$ is isomorphic to the affine $n$-space $\mathbb{A}^n_k$ over a field $k$. Several invariants are used to answer this question. For instance, a complex affine variety isomorphic to $\mathbb{A}^3_\mathbb{C}$ is topologically contractible, has trivial Picard group and has trivial group of units. Ramanujam, in his fundamental work [35], discovered that a non-singular complex algebraic surface is isomorphic to $\mathbb{A}^2_\mathbb{C}$ if and only if it is topologically contractible and simply connected at infinity. On the other hand, Miyanishi gave characterisation of $\mathbb{A}^2_\mathbb{C}$ using $\mathbb{G}_a$- action [29, Theorem 1]. Thus, the invariants coming from the $\mathbb{G}_a$-actions, such as Makar-Limanov invariant as well as topological contractibility play an important role in algebraic characterisation of the affine spaces. This article arose from an attempt to relate these invariants.

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We will use the bridge of $k^1$-homotopy theory to establish a relation between algebra and topology. Let $k$ be a field and $Sm/k$ be the category of smooth finite type $k$-schemes. Denote the category of presheaves of sets on $Sm/k$ by $PSh(Sm/k)$ and the category of presheaves of simplicial sets on $Sm/k$ by $\Delta^{op}PSh(Sm/k)$. We will call it the category of spaces. The left Bousfield localisation of the global projective model structure [17, Section 2] on $\Delta^{op}PSh(Sm/k)$ with respect to the Nisnevich hypercovers gives Nisnevich local projective model structure [17, Section 6]. Following [32, Section 3.2], we consider the left Bousfield localisation of the Nisnevich local projective model structure on $\Delta^{op}PSh(Sm/k)$ with respect to the class of maps $X \times A^1 \to X$ for all smooth schemes $X$. The resulting model structure is called the unstable $A^1$- model structure and the resulting homotopy category is denoted by $H(k)$. A space $X$ is called $A^1$-contractible if the natural map $X \to \text{Spec} k$ is an isomorphism in $H(k)$. For example, $A^n_k$s are $A^1$-contractible. Note that over $\mathbb{C}$, $A^1$-contractibility implies topological contractibility [2, Lemma 2.5].

In dimension 1 and 2, the real line and the real plane are the only topologically contractible open manifolds. Whitehead first constructed topologically contractible open manifold of dimension 3 which is not homeomorphic to $\mathbb{R}^3$ [38]. In fact for every $n \geq 3$, there are infinitely many pairwise topologically contractible open manifolds of dimension $n$ ([28], [27], [34], [9], [20]). In dimension 1, $A^1$ is the only $A^1$-contractible smooth scheme [2, Claim 5.7]. On the other hand, for every $n \geq 4$, Asok and Doran constructed infinitely many pairwise non-isomorphic $A^1$-contractible smooth schemes of dimension $n$, which are quotients of the affine spaces by a $G_a$-action [2, Theorem 5.1]. In dimension $n \geq 6$, there are arbitrary dimensional moduli of pairwise non-isomorphic $A^1$-contractible smooth schemes of dimension $n$ arising from the quotient of an action of a unipotent group on an affine space [2, Theorem 5.3]. However, they proved their method always produces affine space in dimension 1 and 2 [2, Claim 5.7 and Claim 5.8]. In dimension three, Koras-Russell threefolds are $A^1$-contractible [13, Theorem 1.1] [23, Theorem 4.2] but not isomorphic to $A^3_C$ [18, Corollary 9.7]. There are topologically contractible surfaces not isomorphic to $A^2_C$ [35, §3] [11, Theorem A]. It is therefore natural to ask whether $A^2$ is the only $A^1$-contractible variety of dimension 2 ([4, Conjecture 5.2.3]). In this article we will prove the following result:

**Theorem 1.1.** A smooth complex surface $X$ is isomorphic to $A^2_C$ if and only if $X$ is $A^1$-contractible.

The idea of the proof is to use algebraic characterisation of the affine plane ([36, Theorem 2]). For that we prove if $X$ is $A^1$-connected affine complex surface then it has negative logarithmic Kodaira dimension. We prove this in Theorem 4.9, Section 4. This is the main result in this paper. As a corollary of Theorem 1.1 we get:
Corollary 1.2. Koras-Russell threefold is not a product of two proper subvarieties.

Also we get a generalised Zariski’s cancellation about affine surface:

Corollary 1.3. Let $X$ be a smooth complex surface. Suppose $X \times Y$ is $\mathbb{A}^1$-contractible. Then $X \cong \mathbb{A}^2_C$. In particular, if $X \times Y \cong \mathbb{A}^3_C$, then $X \cong \mathbb{A}^2_C$.

The above characterisation also establishes that $\mathbb{A}^1$-contractibility is indeed a stronger notion than topological contractibility as Ramanujam surface $[35, \S 3]$ is not $\mathbb{A}^1$-contractible but topologically contractible. In particular, it answers [2, Question 6.4]. It is shown that $M(X) \cong M(Spec \ C)$ in $DM_{gm}(\mathbb{C})$, where $X$ is the Ramanujam surface [1, Theorem 1]. This shows that motivic contractibility is weaker than $\mathbb{A}^1$-contractibility.

To prove our main result, we had to analyse $\pi_{10}$ of a variety. Section 2, Section 3 and Section 6 grew out of this analysis. In Section 2, we first recall the definition of $\mathbb{A}^1$-connected component sheaf and its related $\mathbb{A}^1$-invariant sheaf $L(X)$ [5, Definition 2.9]. The main result here is Corollary 2.14, which says that for a space $X$, the sheaves $\pi_{01}(X)$ and $L(\pi_{01}(X))$ have same sections over any finitely generated separable field extension $F/k$ (see also [6, Theorem 2.2]).

Asok and Morel constructed the birational $\mathbb{A}^1$-invariant sheaf, denoted by $\pi_{01}^b(X)$, for a smooth proper scheme $X$ [3, Section 6]. Over any finitely generated separable field extension, its sections agree with the sections of the $\mathbb{A}^1$-chain connected component sheaf $S(X)$ [3, Theorem 6.2.1]. In Section 3, we prove that the sheaf $\pi_{01}^b(X)$ is the connected component sheaf of $X$ in the birational model structure on the category of spaces (see Theorem 3.4) which answers the question raised in [24, Theorem 4]. In Section 5, we prove Theorem 1.1 together with the characterisations of $\mathbb{A}^2_C$ and $\mathbb{A}^4_C$.

In Section 6, we introduce a new invariant $O_{ch}(X)$ for an affine variety $X$ which is a subring of the Makar-Limanov invariant of $X$. Unlike Makar-Limanov invariant, $O_{ch}(-)$ is functorial and homotopy invariant but it is not representable in $H(k)$ (see Lemma 6.17). We prove that it is the ring of regular functions on the $\mathbb{A}^1$-chain connected component sheaf of $X$ (see Proposition 6.12) i.e.

$$O_{ch}(X) \cong Hom_{Shv(Sm/k)}(S(X), \mathbb{A}^1).$$

Theorem 6.9 provides evidence that the ring $O_{ch}(X)$ detects $\mathbb{A}^1$-s in an affine variety $X$. From Section 4 onwards, we assume $k$ to be an algebraically closed field unless otherwise mentioned.

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2. First properties of $\pi_0^{A^1}$

The goal of this section is to show that the field valued points of the $A^1$-connected component of a space can be computed using an $A^1$-invariant sheaf associated with the space (see Definition 2.1, Definition 2.6, Corollary 2.14). For any space $\mathcal{X}$, define $\pi_0^{A^1}(\mathcal{X})$ to be the Nisnevich sheaf associated to the presheaf

$$U \in Sm/k \mapsto Hom_{H(k)}(U, \mathcal{X}).$$

This is called the $A^1$-connected component sheaf of $\mathcal{X}$. A space $\mathcal{X}$ is called $A^1$-connected if $\pi_0^{A^1}(\mathcal{X}) \cong \text{Spec } k$ as sheaves.

**Definition 2.1.** ([32, Example 2.4], [31, Definition 7]) A presheaf of sets $\mathcal{F}$ on $Sm/k$ (resp. a $k$-scheme $X$) is said to be $A^1$-invariant (resp. $A^1$-rigid) if for each smooth $k$-scheme $U$, the natural map $\mathcal{F}(U) \to \mathcal{F}(A^1_U)$ (resp. $X(U) \to X(A^1_U)$) induced by the projection map $A^1_U \to U$ is a bijection.

**Remark 2.2.**

1. The $A^1$-rigid $k$-schemes are examples of $A^1$-fibrant spaces. These $A^1$-fibrant objects have trivial $A^1$-homotopy sheaf of groups [32, Example 2.4].
2. Two $A^1$-rigid $k$-schemes are isomorphic in $H(k)$ if and only if they are isomorphic as $k$-schemes [3, Lemma 2.1.9].
3. Abelian varieties, $G_m$, any smooth projective curve of positive genus are the examples of $A^1$-rigid $k$-schemes [3, Example 2.1.10].
4. For an $A^1$-rigid scheme $X$, $\pi_0^{A^1}(X)$ is isomorphic to $X$ [3, Lemma 2.1.9].
5. Any open or closed subscheme of an $A^1$-rigid $k$-scheme is $A^1$-rigid.
6. Finite product of $A^1$-rigid $k$-schemes is $A^1$-rigid.

The next lemma shows that $A^1$-homotopy theory of smooth schemes has as building blocks $A^1$-rigid smooth $k$-schemes. These building blocks have no higher homotopies by Remark 2.2. This is different from the local nature of étale homotopy theory and also different from the usual homotopy theory of manifolds.

**Lemma 2.3** (Local nature). $X$ is a locally finite type $k$-scheme, then $X$ has a local base of $A^1$-rigid $k$-schemes at each of its points.

**Proof.** Since $X$ is of locally finite type, $X$ has an open covering by closed subschemes of $A^1_k$. So it is enough to prove the theorem for $A^1_k$ by Remark 2.2. For any point $P \in A^1_k$, $P$ is in some basic open set $D((x_1 - \alpha_1)(x_2 - \alpha_2)..(x_n - \alpha_n))$. This basic open set is a finite product of $G_m$-s, so it is $A^1$-rigid by Remark 2.2. Thus all open subsets of this basic open set form a local base at $P$ of $A^1$-rigid $k$-schemes by Remark 2.2. $\Box$

Inspired by the discreteness of topological connected components, Morel conjectured the following about $\pi_0^{A^1}(\mathcal{X})$ [31, Conjecture 12].

**Conjecture 2.4** (Morel). $\pi_0^{A^1}(\mathcal{X})$ is $A^1$-invariant for any space $\mathcal{X}$. 

Remark 2.5. $\pi_0^A(X)$ is $A^1$-invariant for the following $X$:

1. $X$ is an $A^1$-connected space.
2. $X$ is a motivic $H$-group or a homogeneous spaces for motivic $H$-groups [8, Theorem 4.18].
3. $X$ is a smooth projective surface [5, Corollary 3.15] [7, Theorem 1.2] or a smooth toric variety [37, Lemma 4.2, Lemma 4.4].

Other than these cases, the conjecture remains open.

Definition 2.6. [5, Definition 2.9] Let $F$ be a presheaf of sets on $Sm/k$, $S(F)$ is defined as the Nisnevich sheaf associated to the presheaf $S^{pre}(F)$ given by

$$S^{pre}(F)(U) := F(U)/\sim$$

for $U \in Sm/k$, where $F(U)/\sim$ is the quotient of $F(U)$ by the equivalence relation generated by $\sigma_0(z) \sim \sigma_1(z), \forall z \in F(A^1_U)$ and $\sigma_0, \sigma_1 : F(A^1_U) \to F(U)$ are induced by the 0-section and the 1-section $U \to A^1_U$ respectively. For any $n > 1$, $S^n(F)$ is defined inductively as the sheaves

$$S^n(F) := S(S^{n-1}(F)).$$

For any sheaf $F$, there is a canonical epimorphism $F \to S(F)$. The sheaf $L(F)$ is defined as

$$L(F) := \lim_{\to} S^n(F).$$

Therefore there is an induced epimorphism $F \to L(F)$.

Remark 2.7. (1) For $X \in Sm/k$, $S(X)$ is the $A^1$-chain connected component sheaf $\pi_0^{ch}(X)$ of $X$ [5, Remark 2.11].
(2) $L(F)$ is a homotopy invariant sheaf [5, Theorem 2.13].
(3) The canonical epimorphism $F \to L(F)$ uniquely factors through $F \to \pi_0^{h1}(F)$ [5, Remark 2.15]. The morphism $\pi_0^{h1}(F) \to L(F)$ is an isomorphism if and only if $\pi_0^{h1}(F)$ is homotopy invariant [5, Corollary 2.18].

Definition 2.8. Suppose, $G \in PSh(Sm/k)$. $G$ is called homotopy invariant in one variable if for each finitely generated separable field extension $F$ of $k$ the map $G(F) \to G(A^1_F)$ induced by projection is a bijection.

Example 2.9. (1) Any $A^1$-invariant presheaf is homotopy invariant in one variable.
(2) $\pi_0^{h1}(X)$ is homotopy invariant in one variable for any space $X$ [8, Corollary 3.2].

The following results give a method of comparing $\pi_0^{h1}(F)$ and $L(F)$. Corollary 2.14 was already proved in [6, Theorem 2.2]. However our proof works in a more general setting.
Lemma 2.10. Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on $\text{Sm}/k$ which is homotopy invariant in one variable and $X = \text{Spec } R$, spectrum of an essentially smooth discrete valuation ring. Then the map $\mathcal{G}(X) \to S(\mathcal{G})(X)$ is surjective.

Proof. First note that, for every finitely generated separable field extension $F$ of $k$, the map $\mathcal{G}(F) \to S(\mathcal{G})(F)$ is a bijection. Surjectivity follows because of the epimorphism $\mathcal{G} \to S(\mathcal{G})$. For injectivity, suppose $a, b \in \mathcal{G}(F)$ such that $a$ and $b$ map to the same element of $S(\mathcal{G})(F)$. Then there are chains of $\mathbb{A}_1$-s in $\mathcal{G}$ joining $a$ and $b$. But any $H \in \mathcal{G}(\mathbb{A}_1)$ factors through $\mathcal{G}(F)$. Therefore $a = b$ in $\mathcal{G}(F)$.

Now suppose, $X = \text{Spec } R$ where $R$ is an essentially smooth discrete valuation ring. Let $\alpha$ be an element of $S(\mathcal{G})(X)$. The element $\alpha$ gives an element of $S(\mathcal{G})(\text{Spec } R^h)$ ($R^h$ is the Henselization of $R$). The map $\mathcal{G}(\text{Spec } R^h) \to S(\mathcal{G})(\text{Spec } R^h)$ is surjective. So there is a Nisnevich neighbourhood $W \to X$ of the closed point of $X$ and $\alpha' \in \mathcal{G}(W)$ such that $\alpha'$ maps to $\alpha|_W$. Suppose, $F = \text{Frac}(R)$ and $L = K(W)$. Since over $F$ we have bijection $\mathcal{G}(F) \to S(\mathcal{G})(F)$, there is $\beta \in \mathcal{G}(F)$ such that $\beta$ maps to $\alpha|_F$. The following square is an elementary distinguished square in the Nisnevich topology (Definition 7.1):

$$
\begin{array}{ccc}
\text{Spec } L & \longrightarrow & W \\
\downarrow & & \downarrow \\
\text{Spec } F & \longrightarrow & X
\end{array}
$$

Since the morphism $\mathcal{G} \to S(\mathcal{G})$ is bijection for sections over fields, we have $\beta|_L = \alpha'|_L$. As $\mathcal{G}$ is a sheaf, $\beta$ and $\alpha'$ lift to an element $\bar{\alpha} \in \mathcal{G}(X)$. This $\bar{\alpha}$ maps to $\alpha$. \qed

Theorem 2.11. Let $\mathcal{G}$ be a Nisnevich sheaf of sets on $\text{Sm}/k$ which is homotopy invariant in one variable. Then for each $X \in \text{Sm}/k$ with $\text{dim}(X) \leq 1$, the map $\mathcal{G}(X) \to S(\mathcal{G})(X)$ is surjective.

Proof. The proof of the theorem follows from Lemma 2.10 and the proof in [8, Theorem 3.1]. \qed

Corollary 2.12. Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on $\text{Sm}/k$ which is homotopy invariant in one variable. Then $S(\mathcal{G})$ is also homotopy invariant in one variable.

Proof. Using Lemma 2.10 and Theorem 2.11, we get the proof. \qed

Corollary 2.13. Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on $\text{Sm}/k$ which is homotopy invariant in one variable and $F/k$ is a finitely generated separable field extension. Then the maps $\mathcal{G}(F) \to \mathcal{L}(\mathcal{G})(F)$ and $\mathcal{G}(\mathbb{A}_1^1_F) \to \mathcal{L}(\mathcal{G})(\mathbb{A}_1^1_F)$ are bijections.

Proof. The map $\mathcal{G}(F) \to S(\mathcal{G})(F)$ is a bijection by Lemma 2.10. Since $S(\mathcal{G})$ is homotopy invariant in one variable by Corollary 2.12, the map $S(\mathcal{G})(F) \to \mathcal{L}(\mathcal{G})(F)$ is also a bijection. \qed
\( S^2(\mathcal{G})(F) \) is a bijection. By induction we get, the map
\[
S^n(\mathcal{G})(F) \to S^{n+1}(\mathcal{G})(F)
\]
is a bijection \( \forall \, n \). Since \( \mathcal{L}(\mathcal{G}) \) is the colimit of \( S^n(\mathcal{G}) \), the map \( \mathcal{G}(F) \to \mathcal{L}(\mathcal{G})(F) \) is a bijection. As \( \mathcal{L}(\mathcal{G}) \) is an \( \mathbb{A}^1 \)-invariant sheaf ([5, Theorem 2.13]) and \( \mathcal{G} \) satisfies \( \mathcal{G}(F) \cong \mathcal{G}(\mathbb{A}^1_F) \), the map \( \mathcal{G}(\mathbb{A}^1_F) \to \mathcal{L}(\mathcal{G})(\mathbb{A}^1_F) \) is a bijection. □

Corollary 2.14. Suppose \( X \in \Delta^\text{op} PSh(Sm/k) \). The maps \( \pi_{0}^{\mathbb{A}^1}(X)(F) \to \mathcal{L}(\pi_{0}^{\mathbb{A}^1}(X))(F) \) and \( \pi_{0}^{\mathbb{A}^1}(X)(\mathbb{A}^1_F) \to \mathcal{L}(\pi_{0}^{\mathbb{A}^1}(X))(\mathbb{A}^1_F) \) are bijections for any finitely generated separable field extension \( F/k \).

Proof. It follows from Corollary 2.13 and Corollary 3.2 in [8]. □

Question 2.15. For a proper scheme \( X \in Sm/k \), we have \( S(X)(F) \cong S^2(X)(F) \) for each finitely generated separable extension \( F/k \) [5, Theorem 3.9]. Thus \( S(X) \) is homotopy invariant in one variable. On the other hand, the real sphere \( T \) in \( \mathbb{A}^3\mathbb{R} \) contains no non-constant \( \mathbb{A}^1\mathbb{R} \) but \( S^2(T)(\mathbb{R}) \) is point. Therefore it is natural to ask whether \( S(X) \) is homotopy invariant in one variable for any scheme \( X \in Sm/k \) with \( k = \bar{k} \).

3. Birational Connected Component

The first example of an \( \mathbb{A}^1 \)-invariant sheaf associated to the \( \mathbb{A}^1 \)-connected component of a scheme was constructed by Asok and Morel. Let \( X \in Sm/k \) be a proper scheme. There is a birational (thus homotopy invariant) sheaf \( \pi_{0}^{\mathbb{A}^1}(X) \) ([3, Section 6.2] such that its sections over any finitely generated separable field extension \( L \) of \( k \) is the \( \mathbb{A}^1 \)-chain connected component of \( L \)-rational points, i.e. \( S(X)(L) = \pi_{0}^{\mathbb{A}^1}(X)(L) \) ([3, Theorem 6.2.1]).

Remark 3.1. (1) For a proper scheme \( X \in Sm/k \), there is a canonical morphism \( \pi_{0}^{\mathbb{A}^1}(X) \to \pi_{0}^{\mathbb{A}^1}(X) \) which is a bijection on sections over any finitely generated separable field extensions of \( k \) [3, Proposition 6.2.6], [5, Corollary 3.10].

(2) \( \pi_{0}^{\mathbb{A}^1} \) is birational invariant of smooth proper schemes [24, Theorem 1]. However \( \pi_{0}^{\mathbb{A}^1} \) is not birational invariant sheaf of smooth proper schemes [5, Example 4.8].

In this section in Theorem 3.4, we will prove that \( \pi_{0}^{\mathbb{A}^1}(X) \) is isomorphic to the connected component sheaf of \( X \) in the birational model structure (Proposition 3.2). This answers the question raised in [24, Theorem 4]. In [33, Definition 2.6] Pablo constructed birational unstable motivic homotopy category (equivalent construction by Theorem 3.8).

Proposition 3.2. The left Bousfield localisation of the projective model structure on \( \Delta^\text{op} PSh(Sm/k) \) with respect to the following class of maps
\[
\{ U \overset{i}{\to} X \in Sm/k \mid i \text{ is an open immersion with dense image} \}.
\]
exists. It gives a model structure on \( \Delta^{op} PSh(Sm/k) \) called the unstable birational model structure.

**Proof.** Existence of the left Bousfield localisation is proved in [22, Theorem 4.1.1]. \( \Box \)

The resulting homotopy category associated to the birational model structure will be denoted by \( H_b(k) \).

**Definition 3.3.** For any space \( X \), the connected component presheaf associated to the birational model structure is defined as \( U \mapsto \text{Hom}_{H_b(k)}(U, X) \), for \( U \in Sm/k \). It will be denoted by \( \pi^b_0(X) \).

The aim of this section is to prove the following result:

**Theorem 3.4.** There is an isomorphism of the presheaves: \( \pi^{hA_1}_0(X) \cong \pi^b_0(X) \), for \( X \in Sm/k \) a proper scheme.

Let \( f : U \to X \in Sm/k \) be a Nisnevich covering and \( U_* \) be the corresponding Čech simplicial scheme. Here \( U_n \) is the smooth scheme given by \( U \times_X U \times_X \cdots \times_X U \) (the product is taken \( n + 1 \) times). Let \( f : U_* \to X \) be the corresponding map of simplicial schemes. We show that inverting the birational morphisms in \( A_1 \)-homotopy category is equivalent to only inverting the birational morphisms in the global projective model structure (Theorem 3.8).

**Lemma 3.5.** The map \( f : U_* \to X \) is a birational weak equivalence (i.e it is an isomorphism in \( H_b(k) \)).

**Proof.** Any Nisnevich covering has a section over a dense open set. Therefore there is an open dense set \( V \subset X \) such that the restriction \( f^{-1}(V) \to V \) has a section. We have the following commutative diagram in \( \Delta^{op} PSh(Sm/k) \):

\[
\begin{array}{ccc}
  f^{-1}(V)_* & \rightarrow & U_* \\
  \downarrow & & \downarrow f \\
  V & \rightarrow & X
\end{array}
\]

where the left vertical map is induced by the restriction and the upper horizontal map is induced by the inclusion. The left vertical map is a sectionwise weak equivalence, since there is a section. The map \( V \to X \) is an inclusion of dense open set, so it is a birational weak equivalence. As the map \( f : U \to X \) is an étale map, for each \( n \), the \( (n + 1) \)-fold product \( f^{-1}(V) \times_X f^{-1}(V) \cdots \times_X f^{-1}(V) \) is open and dense in \( U \times_X U \cdots \times_X U \) fitting in the pullback square:

\[
\begin{array}{ccc}
  f^{-1}(V) \times_V \cdots \times_V f^{-1}(V) & \rightarrow & U \times_X U \cdots \times_X U \\
  \downarrow & & \downarrow \\
  V & \rightarrow & X
\end{array}
\]
Therefore the morphism $f^{-1}(V) \rightarrow U$ is a birational weak equivalence [32, Proposition 2.14]. Hence $f : U \rightarrow X$ is a birational weak equivalence. □

Corollary 3.6. Any Nisnevich weak equivalence is a birational weak equivalence.

Proof. The local projective model structure on $\Delta^{op}PSh(Sm/k)$ is the left Bousfield localisation of the projective model structure at the class of Čech hypercovers ([17, Theorem 6.2, Example A.11]),

$$\{ U \rightarrow X \mid U \rightarrow X \text{ is a Nisnevich covering} \}$$

Since the map $U \rightarrow X$ is a birational weak equivalence by Lemma 3.5, the result follows. □

Lemma 3.7. $X \in Sm/k$, the projection map $X \times A^1 \rightarrow X$ is a birational weak equivalence.

Proof. Suppose $X \in \Delta^{op}PSh(Sm/k)$ and $X \in Sm/k$. Consider a presheaf of sets $F_{X,X}$ on $Sm/k$ as $Y \mapsto \text{Hom}_{H^b(k)}(Y \times X, X)$. $F_{X,X}$ is a birational sheaf on $Sm/k$. Then we have a bijection $F_{X,X}(\mathbb{P}^1_k) \rightarrow F_{X,X}(\text{Spec } k)$ [24, Appendix A]. This implies the projection map $\mathbb{P}^1_k \times X \rightarrow X$ is an isomorphism in $H^b(k)$. Composing it with birational map $A^1_k \times X \rightarrow \mathbb{P}^1_k \times X$, we get that the projection map $X \times A^1 \rightarrow X$ is an isomorphism in $H^b(k)$. □

Theorem 3.8. Any $A^1$-weak equivalence is a birational weak equivalence. Therefore the unstable birational model structure is equivalent to the motivic unstable birational model structure in [33, Definition 2.6].

Proof. The left Bousfield localisation of the projective model structure (universal model structure) on $\Delta^{op}PSh(Sm/k)$ at the class of the Čech hypercovers and the projection maps $A^1_X \rightarrow X \in Sm/k$ gives the $A^1$-model structure [16, Proposition 8.1]. Therefore, the $A^1$-weak equivalences are generated by the Čech hypercovers and the projection maps $A^1_X \rightarrow X$. Both are birational weak equivalences [Lemma 3.5 and Lemma 3.7]. Hence the result follows. □

3.1. Proof of the Theorem 3.4.

Proof. Suppose $U \in Sm/k$ is irreducible. Then, $\pi_0^{bh}(X)(U) = S(X)(k(U))$ by [3, Definition 6.2.5]. By [24, Theorem 6.6.3], we have the following natural bijection

$$\text{Hom}_{S_b^{-1}Sm}(U, X) \cong \pi_0^{bh}(X)(U),$$

for each $U \in Sm/k$. The Yoneda embedding of $Sm/k$ in $\Delta^{op}PSh(Sm/k)$ as representable constant simplicial presheaf induces a functor $\eta : S_b^{-1}Sm \rightarrow H_b(k)$ because of the universal property of localisation [19, Section 1]). The functor $\eta$ is universal and it factors the functor $\pi : Sm/k \rightarrow H_b(k)$. This gives the map

$$\text{Hom}_{S_b^{-1}Sm}(U, X) \rightarrow \text{Hom}_{H_b(k)}(U, X).$$
Thus we have a morphism $\eta : \pi_{bA}^{b1}(X) \to \pi_0^b(X)$.

Consider the following commutative diagram of presheaves on $Sm/k$,

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & \pi_0^{bA}(X) \\
\pi \downarrow & & \downarrow\cong \\
\pi_0^b(X) & \xrightarrow{\gamma} & \pi_0^b(\pi_{bA}^{b1}(X))
\end{array}
\]

induced by the natural transformation $Id \to \pi_0^b()$. The top horizontal morphism of presheaves $\alpha : X \to \pi_0^{bA}(X)$ is induced by the canonical functor $\alpha : Sm/k \to S^{-1}_{b}Sm$. The right most vertical map is an isomorphism, since $\pi_0^{bA}(X)$ is a fibrant object in the birational model structure. This gives a morphism $\theta : \pi_0^b(X) \to \pi_0^{bA}(X)$. The morphism $\eta \circ \theta \circ \pi$ is same as $\eta \circ \alpha$ and $\eta \circ \alpha$ is the natural morphism $\pi$. The morphism $\pi : X \to \pi_0^b(X)$ induces a bijection

$\text{Hom}_{PSh(Sm/k)}(\pi_0^b(X), \pi_0^b(X)) \cong \text{Hom}_{PSh(Sm/k)}(X, \pi_0^b(X))$,

since $\pi_0^b(X)$ is birational local. This gives $\eta \circ \theta$ is the identity morphism. So $\theta$ is a monomorphism. On the other hand, the morphism $\alpha$ factors through $\theta$ and the morphism $\alpha$ is sectionwise surjective, since $\pi_0^{bA}(X)$ is a birational sheaf and its section over $U$ is the $\mathbb{A}^1$-equivalence classes of $k(U)$-rational points of $X$. Hence $\theta$ is an epimorphism and consequently it is an isomorphism.

\[\square\]

4. Existence of $\mathbb{A}^1$ and $\mathbb{A}^1$-connectedness

The aim of this section is to relate $\mathbb{A}^1$-homotopy theory with the existence of affine lines in a variety. The main result in this section is Theorem 4.9 where we establish this. By the phrase “there is an $\mathbb{A}^1$ in $X$”, we mean the existence of a non-constant map from $\mathbb{A}^1_k$ to $X$.

**Definition 4.1.** (1) A $k$-variety $X$ is said to be dominated by images of $\mathbb{A}^1$, if there is an open dense subset $U$ of $X$ such that for every $p \in U(k)$, there is an $\mathbb{A}^1$ in $X$ through $p$ [26, §1]. In this case, the variety is called $\mathbb{A}^1$-uniruled or log-uniruled [14, Definition 3].

(2) A $k$-variety $X$ is called $\mathbb{A}^1$-ruled if there is an open dense subset $V$ of $X$ such that $V$ is isomorphic to $\mathbb{A}^1 \times U$ for some $k$-variety $U$ [14, Definition 1].

**Remark 4.2.** By definition, the $\mathbb{A}^1$-ruled varieties are dominated by images of $\mathbb{A}^1$. Suppose $k$ is an uncountable field of characteristic 0. A smooth affine $k$-surface is $\mathbb{A}^1$-ruled if and only if it has negative logarithmic Kodaira dimension [30, §4, §5]. Moreover in this case, the notions log-uniruled, $\mathbb{A}^1$-ruled and the negativity of logarithmic Kodaira dimension coincide [26, Theorem 1.1]. However in higher dimensions, being $\mathbb{A}^1$-ruled is a stronger notion than $\mathbb{A}^1$-uniruled [14, Proposition 9].
Suppose $\mathcal{F}$ is a Nisnevich sheaf of sets on $Sm/k$ and $W \in Sm/k$, $f \in \mathcal{F}(W)$.

**Definition 4.3.** An element $\alpha \in \mathcal{F}(\text{Spec } k)$ is in the image of $f$ if $\exists \gamma \in W(\text{Spec } k)$ such that the composition $\text{Spec } k \rightarrow W \overset{f}{\rightarrow} \mathcal{F}$ is $\alpha$.

**Definition 4.4.** A homotopy $H \in \mathcal{F}(\mathbb{A}^1_k)$ is said to be non-constant if $H(0) \neq H(1) \in \mathcal{F}(W)$, where $H(0)$ and $H(1)$ are induced by the 0-section and the 1-section from $W$ to $\mathbb{A}^1_k$ respectively.

**Remark 4.5.** Let $\mathcal{F}$ be a sheaf and $X \in Sm/k$. A section $\alpha \in \mathcal{S}(\mathcal{F})(X)$ is given by a Nisnevich covering $W \rightarrow X$, a section $\gamma \in \mathcal{F}(W)$ and a Nisnevich covering $W' \rightarrow W \times_X W$ such that $p_1^*(\gamma)|_{W'}$ and $p_2^*(\gamma)|_{W'}$ in $\mathcal{F}(W')$ are joined by a chain of $\mathbb{A}^1$-homotopies (where $p_1, p_2 : W \times_X W \rightarrow W$ are the projection maps). If $p_1^*(\gamma)|_{W'} = p_2^*(\gamma)|_{W'}$, then $\gamma$ can be lifted to some element $\alpha' \in \mathcal{F}(X)$ and in this case $\alpha'$ maps to $\alpha$ via the canonical morphism $\mathcal{F} \rightarrow \mathcal{S}(\mathcal{F})$. Otherwise, we will get an element $\mathbf{H} \in \mathcal{S}(\mathbb{A}^1_k)$ such that $p_1^*(\gamma)|_{W'} = H(0) \neq H(1)$ as sections. This is essentially the data of ghost homotopy mentioned in [5, Definition 3.2].

**Condition 4.6.** Suppose $X, W \in Sm/k$, $\alpha \in X(k)$ and $n \geq 0$. A homotopy $H \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$ is said to satisfy the condition $^\ast(W, \alpha)$, if $H$ satisfies the following properties:

1. $H$ is a non-constant homotopy.
2. $H(0)$ factors through $X$ i.e. there is a morphism $\psi : W \rightarrow X$ such that the following diagram commutes:

   $\begin{array}{ccc}
   W & \xrightarrow{\psi} & X \\
   \downarrow{\iota_0} \quad & & \downarrow \\
   \mathbb{A}^1_k & \xrightarrow{H} & \mathcal{S}^n(X)
   \end{array}$

   Here $\iota_0 : W \rightarrow \mathbb{A}^1_k$ is the 0-section and the right vertical map is the canonical epimorphism $X \rightarrow \mathcal{S}^n(X)$.

3. $\alpha \in \text{Im}(H(0))$ (By (2), $H(0) : W \rightarrow X$).

**Proposition 4.7.** Suppose $X, W \in Sm/k$, $\alpha \in X(k)$ and $n \geq 1$. Let $H \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$ be a homotopy, where $W$ is irreducible and $H$ satisfies $^\ast(W, \alpha)$. Then there is $W' \in Sm/k$ irreducible and a homotopy $H' \in \mathcal{S}^m(X)(\mathbb{A}^1_k)$ for some $m < n$ such that $H'$ satisfies $^\ast(W', \alpha)$.

**Proof.** The morphism $X \rightarrow \mathcal{S}^n(X)$ is an epimorphism and $H \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$. Thus,

1. There is a Nisnevich covering $f : V \rightarrow \mathbb{A}^1_k$,
4.5

(2) There is a morphism \( \phi : V \to X \) such that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & X \\
\downarrow{f} & & \downarrow \\
\mathbb{A}^1_W & \xrightarrow{H} & S^n(X)
\end{array}
\]

The morphism \( \phi \) gives an element of \( S^{n-1}(X)(V) \) via the epimorphism \( X \to S^{n-1}(X) \). The elements \( p_1^*(\phi) \) and \( p_2^*(\phi) \) are same in \( S^n(X)(V \times_{\mathbb{A}^1_W} V) \) (where \( p_1, p_2 : V \times_{\mathbb{A}^1_W} V \to V \) are the projection maps). Therefore, there is a Nisnevich covering \( V' \to V \times_{\mathbb{A}^1_W} V \) and there is a chain of non-constant homotopy (since \( H \) is a non-constant homotopy, so \( p_1^*(\phi)|_{V'} \neq p_2^*(\phi)|_{V'} \in S^{n-1}(X)(V') \) by Remark 4.5) \( \mathbb{A}^1 \)-homotopies \( G_1, G_2, \ldots, G_k \in S^{n-1}(X)(\mathbb{A}^1_{V'}) \) such that

\[
G_1(0) = p_1^*(\phi)|_{V'} \text{ and } G_k(1) = p_2^*(\phi)|_{V'}.
\]

Suppose \( V = \bigsqcup_{i=1}^n V_i \), \( V_i \)'s are the irreducible components of \( V \). Then \( V \times_{\mathbb{A}^1_W} V \) is the union of \( V_i \times_{\mathbb{A}^1_W} V_j \) varying \( i \) and \( j \) (note that, each \( V_i \times_{\mathbb{A}^1_W} V_j \) is non-empty since \( W \) is irreducible) and for each irreducible component \( V_0 \) of \( V' \) which is also a connected component, there are dominant maps (étale maps) from \( V_0 \) to \( V_i \) (for some \( i \)) induced by the projection maps \( p_1 \) and \( p_2 \). We have the following cases.

**Case 1:** Suppose \( \alpha \notin \overline{\text{Im}(\phi)} \). Consider the following diagram:

\[
\begin{array}{ccc}
W' & \xrightarrow{f} & V \\
\downarrow{H(0)} & & \downarrow{\phi} \\
W & \xrightarrow{i_0} & \mathbb{A}^1_W \\
\downarrow{H} & & \downarrow \\
& & S^n(X)
\end{array}
\]

where \( i_0 : W \to \mathbb{A}^1_W \) is the 0-section. Here the left square is cartesian and the lower triangle is commutative, since \( H(0) \) factors through \( X \). We have \( \phi|_{W'} \neq H(0)|_{W'} \) as morphisms to \( X \), since \( \alpha \notin \overline{\text{Im}(\phi)} \). But they are same in \( S^n(X)(W') \). Suppose \( m \geq 0 \) is the least such that these two maps are same in \( S^{n+1}(X)(W',V') \). Thus there is a Nisnevich covering \( W'' \to W' \) and there is a non-constant homotopy (by Remark 4.5) \( H' \in S^n(X)(\mathbb{A}^1_{W'}) \), such that \( H'(0) = H(0)|_{W''} \). There is an irreducible component (say \( W_0 \)) of \( W'' \) such that \( H'|_{\mathbb{A}^1_{W_0}} \) is non-constant. Since the map \( W_0 \to W \) is dominant and \( \alpha \in \overline{\text{Im}(H(0))} \), \( \alpha \in \overline{\text{Im}(H'|_{\mathbb{A}^1_{W_0}}(0))} \).

**Case 2:** Suppose \( \alpha \in \overline{\text{Im}(\phi)} \). Moreover assume that there is an irreducible component (say \( V_0 \)) of \( V' \) that maps to \( V_i \times_{\mathbb{A}^1_W} V_j \) (for some \( i \) and \( j \)) with \( \alpha \in \overline{\phi(V_i)} \) and \( p_1^*(\phi)|_{V_0} \neq p_2^*(\phi)|_{V_0} \). Then there is some \( t \) such that
$G_{t|\mathbb{A}^1_{V_0}}$ is the required non-constant homotopy (if for each $t$, $G_{t|\mathbb{A}^1_{V_0}}$ is constant, then $p_1^*(\phi)$ and $p_2^*(\phi)$ agree in $V_0$). Since the map $V_0 \to V_i$ is dominant, $\alpha \in \text{Im}(G_{t|\mathbb{A}^1_{V_0}}(0))$. In particular, if $\alpha \in \phi(V_i)$ for every $i$, then we can take any irreducible component $V_0$ of $V'$ such that $G_{1|\mathbb{A}^1_{V_0}}$ is the non-constant homotopy.

**Case 3:** Suppose $\alpha \in \text{Im}(\phi)$ and there is a $j$ such that $\alpha \notin \phi(V_j)$. If needed, renumbering $V_i$-s, we can assume that $\alpha \in \phi(V_1), \phi(V_2), \ldots, \phi(V_t)$ and $\alpha \notin \phi(V_{t+1}), \ldots, \phi(V_n)$. Moreover we can assume that for each irreducible component $V_0$ of $V'$ that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ with $m \leq i$ we have, $p_1^*(\phi)|_{V_0} = p_2^*(\phi)|_{V_0}$ in $S^{n-1}(X)(V_0)$. Otherwise the conclusion follows from Case 2. Thus we have for every $m \leq i$,

$$p_1^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} = p_2^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} \in S^{n-1}(X)(V_m \times_{\mathbb{A}^1_W} V_l).$$

Suppose there is a $t < n - 1$ and there is an irreducible component $W_0$ of $V'$ that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ for some $m, l$ with $m \leq i$ such that

$$p_1^*(\phi)|_{W_0} \neq p_2^*(\phi)|_{W_0} \in S^t(X)(W_0).$$

Since $p_1^*(\phi)|_{W_0}$ and $p_2^*(\phi)|_{W_0}$ are same in $S^{n-1}(X)(W_0)$, we choose $t$ such that $p_1^*(\phi)|_{W_0}$ is same with $p_2^*(\phi)|_{W_0}$ in $S^{t+1}(X)(W_0)$. Then there is a Nisnevich covering $V'' \to W_0$ and a non-constant homotopy (by Remark 4.5) $H' \in S^t(X)(\mathbb{A}^1_{W'})$ such that $H'(0) = p_1^*(\phi)|_{V''}$. So there is an irreducible component $W'_0$ of $V''$ such that $H'|_{\mathbb{A}^1_{W'_0}}$ is non-constant. Since the map $W'_0 \to V_m$ is dominant, we have $\alpha \in \text{Im}(H'|_{\mathbb{A}^1_{W'_0}})$.

On the other hand, if there is no such $t$ then for every irreducible component $V_0$ of $V'$ that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ for some $m \leq i$, we have $p_1^*(\phi)|_{V_0} = p_2^*(\phi)|_{V_0}$ as morphisms to $X$. Therefore we have,

$$p_1^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} = p_2^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l}, \quad \forall m \leq i \forall l$$

as morphisms to $X$. But then all $\phi(V_l)$ are same for every $l$, since $p_1 : V_m \times_{\mathbb{A}^1_W} V_l \to V_m$ and $p_2 : V_m \times_{\mathbb{A}^1_W} V_l \to V_l$ are dominant maps. It is a contradiction, since we have assumed there is some $j$ such that $\alpha \notin \phi(V_j)$.

Therefore, the proposition is proved. \qed

**Remark 4.8.** In case for any Nisnevich sheaf of sets $\mathcal{F}$, using the same argument as in the proof of Proposition 4.7, we have the following: Suppose there is a non-constant homotopy $H \in S(\mathcal{F})(\mathbb{A}^1_W)$ for some $W \in Sm/k$ such that the image of $H(0)$ contains some $\alpha \in \mathcal{F}($Spec $k)$. Then there is a non-constant homotopy $H' \in \mathcal{F}(\mathbb{A}^1_{W'})$ for some $W' \in Sm/k$ such that image of $H'(0)$ contains $\alpha$.

The next theorem is the main theorem of this section. It shows that being $\mathbb{A}^1$-connected gives $\mathbb{A}^1$-s in a variety.
**Theorem 4.9.** Suppose \(X \in Sm/k\) is \(\mathbb{A}^1\)-connected with \(\dim(X) \geq 2\). Then one of the following holds:

1. \(\forall \, x \in X(k)\), there is a non-constant \(\mathbb{A}^1\) through \(x\).
2. There is a non-constant homotopy \(H : \mathbb{A}^1_Y \to X\), for some irreducible \(Y \in Sm/k\), such that the dimension of the closure of the image of \(H\) is at least 2.

In particular for a surface \(X\), if \(X\) is \(\mathbb{A}^1\)-connected, then \(X\) is dominated by images of \(\mathbb{A}^1\).

**Proof.** As \(X\) is \(\mathbb{A}^1\)-connected, the sheaf \(\mathcal{L}(X)\) is trivial [5, Corollary 2.18]. Suppose that \(\exists \, \alpha \in X(k)\) such that there is no non-constant \(\mathbb{A}^1\) through \(\alpha\). Choose \(\beta \in X(k)\) with \(\beta \neq \alpha\). Also \(\alpha \neq \beta \in \mathcal{S}(X)(\text{Spec } k)\), but \(\alpha = \beta \in \mathcal{L}(X)(\text{Spec } k)\). Therefore, there is an \(n \geq 1\) such that \(\alpha = \beta \in \mathcal{S}^n(X)(\text{Spec } k)\) and \(\alpha \neq \beta \in \mathcal{S}^n(X)(\text{Spec } k)\). So there is a non-constant \(\mathbb{A}^1\)-homotopy \(H \in \mathcal{S}^n(X)(\mathbb{A}^1_k)\) such that \(H(0) = \alpha\). Hence by applying Proposition 4.7 repeatedly, there exists some \(Y \in Sm/k\) irreducible, along with a non-constant homotopy \(H' : \mathbb{A}^1_Y \to X\), such that \(\alpha \in \overline{\text{Im}(H'(0))}\). Since \(k\) is algebraically closed, the \(k\)-rational points are dense, so \(H'(0) \neq H'(1)\) at some \(k\)-rational point. Therefore the image of \(H'\) contains a non-constant \(\mathbb{A}^1\) and we have \(\overline{\text{Im}(H')}\) contains \(\alpha\). Therefore \(\overline{\text{Im}(H')}\) is of dimension at least 2, as we have assumed that there is no non-constant \(\mathbb{A}^1\) through \(\alpha\).

Since \(H'\) is a non-constant homotopy, by shrinking \(Y\) we can assume that \(H'(0, y) \neq H'(1, y), \forall \, y \in Y(k)\) and the dimension of the closure of image is at least 2. Thus if \(X\) is a surface, the map \(H'\) is dominant. So there is a non-empty open subset \(U\) of \(X\) such that \(U\) is contained in the image of \(H'\). Each \(u \in U(k)\) has the preimage \((t, y) \in \mathbb{A}^1_Y\) for some \(k\)-point \(y\) in \(Y\). Therefore, \(u\) is in the image of \(H''|_{\mathbb{A}^1_Y \times \{y\}}\). Thus \(X\) is dominated by images of \(\mathbb{A}^1\).

**Corollary 4.10.** Suppose \(X \in Sm/k\) is \(\mathbb{A}^1\)-connected. Then there is a non-constant \(\mathbb{A}^1\) in \(X\).

**Remark 4.11.** In Corollary 4.10, the assumption that \(k\) is an algebraically closed field, is necessary. The unit sphere \(T\) in \(\mathbb{A}^2_\mathbb{R}\) given by the equation \(x^2 + y^2 + z^2 = 1\) is \(\mathbb{A}^1\)-connected, however there is no non-constant \(\mathbb{A}^1_\mathbb{R}\) in \(T\).

5. **Characterisation of Affine Space**

In this section we give some characterisations of \(\mathbb{C}^n\) for the cases \(n = 2, 3\) and 4 using \(\mathbb{A}^1\)-homotopy theory. We give here the proof of the main theorem (Theorem 1.1). A variant of the main theorem (Theorem 5.1) and its consequences are also given in this section.

**Proof of the Theorem 1.1.** Suppose \(X\) is \(\mathbb{A}^1\)-contractible. Then the Picard group of \(X\) is trivial and the group of units of \(X\) is \(\mathbb{C}^*\). Moreover by Theorem 4.9, \(X\) is dominated by images of \(\mathbb{A}^1\). Thus the logarithmic
Kodaira dimension of $X$ is $-\infty$ by [26, Theorem 1.1]. Therefore, using [36, Theorem 2], we get $X \cong \mathbb{A}^2_C$. □

Since the topologically contractible complex surfaces have trivial Picard group and trivial group of units, the same proof gives a mixed characterisation of the affine plane.

**Theorem 5.1.** A smooth complex surface is isomorphic to $\mathbb{A}^2_C$ if and only if it is topologically contractible and $\mathbb{A}^1$-connected.

**Corollary 5.2.** Suppose $X$ is a smooth complex surface which is topologically contractible and of non-negative logarithmic Kodaira dimension. Then $X$ is not $\mathbb{A}^1$-connected. However $X$ is motivically contractible i.e. $M(X) \cong M(\text{Spec } C)$ in $DM_{gm}(C)$ [1, Theorem 1]. For example, the Ramanujam surface [35, §3] and the tom Dieck-Petrie surfaces [11, Theorem A] are not $\mathbb{A}^1$-connected but motivically contractible.

**Remark 5.3.** Affine modification does not always preserve $\mathbb{A}^1$-contractibility. The Ramanujam surface and the tom Dieck-Petrie surfaces are affine modifications of $\mathbb{A}^2_C$ and the Koras-Russell threefolds are affine modifications of $\mathbb{A}^3_C$. These are topologically contractible [25, Example 3.1 and 3.2] and the Koras-Russell threefolds are $\mathbb{A}^1$-contractible [13, Theorem 1.1]. However the Ramanujam surface and the tom Dieck-Petrie surfaces are not even $\mathbb{A}^1$-connected (Corollary 5.2).

**Proof of Corollary 1.3.** $X \times Y$ is $\mathbb{A}^1$-contractible and $X$ is a retract of $X \times Y$. Therefore $X$ is also $\mathbb{A}^1$-contractible. Hence, $X \cong \mathbb{A}^2_C$ by Theorem 1.1. □

**Corollary 5.4.** An $\mathbb{A}^1$-contractible smooth complex affine threefold is isomorphic to $\mathbb{A}^2_C$ if and only if it is isomorphic to a product of two proper sub-varieties of lower dimension. Similarly an $\mathbb{A}^1$-contractible smooth variety is isomorphic to $\mathbb{A}^4_C$ if and only if it is isomorphic to a product of two proper sub-varieties each of dimension two.

Let $k$ be a field of characteristic zero and $R$ is a $k$-algebra. The following definition is related to the property of being $\mathbb{A}^1$-ruled.

**Definition 5.5.** [18, Definition 1.1.6] A locally nilpotent $k$-derivation $D : R \to R$ is a $k$-linear derivation such that for each $a \in R$ $\exists n \in \mathbb{N}$ such that $D^n(a) = 0$. The derivation $D$ has a slice if $\exists s \in R$ with $D(s) = 1$. We denote the kernel of $D$ by $R^D$ which is a $k$-algebra and the set of all locally nilpotent $k$-derivations on $R$ will be denoted by $LND_k(R)$.

Locally nilpotent derivation is an essential tool in affine algebraic geometry to characterise the polynomial rings. Miyanishi showed a two-dimensional affine U.F.D. over an algebraically closed field $k$ with no non-trivial units is isomorphic to $k[x,y]$ if it admits a non-trivial locally nilpotent $k$-derivation. [29, Theorem 1].
Remark 5.6. (1) Suppose $D \in \text{LND}_k(R)$ has a slice $s \in R$. Then $R = R^D[s]$ i.e. $R$ is a polynomial ring over $R^D$ of one variable and $D = \frac{d}{ds}$, derivative with respect to $s$ [18, Corollary 1.22].
(2) The locally nilpotent $k$-derivations on an affine $k$-domain $B$ correspond to the algebraic $\mathbb{G}_a$-actions on $\text{Spec } B$ [18, Section 1.5].
(3) Let $X$ be an affine variety such that $\mathcal{O}(X)$ is a U.F.D. Then $X$ is $\mathbb{A}^1$-ruled if and only if there is a non-trivial locally nilpotent derivation on $\mathcal{O}(X)$ [14, Proposition 2].

The fact that $\mathbb{A}^2_C$ is the only $\mathbb{A}^1$-contractible smooth complex surface has the following consequences.

Corollary 5.7. A smooth affine complex threefold $X$ is isomorphic to $\mathbb{A}^3_C$ if and only if $X$ is $\mathbb{A}^1$-contractible and there exists a locally nilpotent derivation with a slice.

Proof. Suppose $X$ is $\mathbb{A}^1$-contractible and there exists a locally nilpotent derivation on $\mathcal{O}(X)$ with a slice. Then by Remark 5.6, $X \cong U \times \mathbb{A}^1$, where $U$ is a smooth affine surface. The surface $U$ is $\mathbb{A}^1$-contractible, being a retract of $X$. Therefore by Theorem 1.1, $U \cong \mathbb{A}^2_C$ and hence $X \cong \mathbb{A}^3_C$.

In this context, there is an algebraic characterisation of the polynomial ring $k[x, y, z]$. A three dimensional finitely generated $k$-algebra, which is also a U.F.D., is isomorphic to $k[x, y, z]$ if and only if its Makar-Limanov invariant is trivial [18, Theorem 9.9]. In this section, we construct a new functorial invariant $\mathcal{O}_{ch}(X)$ (Definition 6.1) which is a subobject of $\mathcal{ML}(X)$. We show that $\mathcal{O}_{ch}(X) = k$ implies the existence of $\mathbb{A}^1$-s in $X$ (Theorem 6.9).
**Definition 6.1.** Let $X$ be an affine $k$-variety and $\mathcal{O}(X)$ be the ring of regular functions on $X$. For a fixed $g : \mathbb{A}^1_k \rightarrow X$, define

$$\mathcal{O}_{ch,g}(X) := \{ f \in \mathcal{O}(X) \mid f \circ g \text{ is constant} \}$$

We define, $\mathcal{O}_{ch}(X) = \bigcap_{g \in Hom_{Sch/k}(\mathbb{A}^1_k,X)} \mathcal{O}_{ch,g}(X)$.

We get the following immediate properties of $\mathcal{O}_{ch,g}(X)$.

**Lemma 6.2.** Suppose $X$ is an affine $k$-variety.

1. A morphism $\phi : \mathbb{A}^1_k \rightarrow \mathbb{A}^1_k$ is constant if and only if the induced $k$-algebra homomorphism $\bar{\phi} : k[T] \rightarrow k[T]$ takes $T$ to an element of $k$.
2. $\mathcal{O}_{ch,g}(X)$ is a $k$-subalgebra of $\mathcal{O}(X)$. In particular, $\mathcal{O}_{ch}(X)$ is a $k$-subalgebra of $\mathcal{O}(X)$.
3. Suppose $f_1, f_2 \in \mathcal{O}(X)$. If the product $f_1 f_2 \in \mathcal{O}_{ch,g}(X)$, then $f_1 \in \mathcal{O}_{ch,g}(X)$ or $f_2 \in \mathcal{O}_{ch,g}(X)$.
4. Suppose, $X$ is of dimension at least two. Then the morphism $\tilde{i} : X \rightarrow Spec(\mathcal{O}_{ch,g}(X))$ induced by the inclusion $i : \mathcal{O}_{ch,g}(X) \rightarrow \mathcal{O}(X)$ is birational.

**Proof.** (1), (2) and (3): The proofs are quite straightforward. So we leave it to the reader.

(4): Since $X$ is of dimension at least 2, so $g : \mathbb{A}^1_k \rightarrow X$ is not dominant. The image of $g$ is closed in $X$ and it is given by some ideal $I$ of $\mathcal{O}(X)$. So its complement is the union of basic open set $D(f)$’s, $f \in I$. Choose $f \in I$ with $D(f)$ is non-empty. Then $\bar{g}(f) = 0$ (where $\bar{g} : \mathcal{O}(X) \rightarrow k[T]$ is induced by $g$). So $f h + \mu \in \mathcal{O}_{ch,g}(X)$ $\forall h \in \mathcal{O}(X), \forall \mu \in k$ by Part (1). We have an injective homomorphism $i_* : \mathcal{O}_{ch,g}(X) \rightarrow \mathcal{O}(X)$ induced by the inclusion $i$. The map $i_*$ is also surjective. Indeed for $\frac{fh}{f^k} \in \mathcal{O}(X)_f$, $fh \in \mathcal{O}_{ch,g}(X)$ and the element $\frac{fh}{f^k}$ is mapped to $\frac{h}{f}$. Hence $\mathcal{O}_{ch,g}(X)_f$ and $\mathcal{O}(X)_f$ are isomorphic and therefore $X$ and $Spec(\mathcal{O}_{ch,g}(X))$ are birational. □

**Remark 6.3.** In the above Lemma 6.2, the assumption about the dimension of $X$ is necessary. If $X = \mathbb{A}^1_k$ and $g$ be the identity map on $\mathbb{A}^1_k$, then $\mathcal{O}_{ch,g}(X)$ is trivial. Also observe that, $\mathcal{O}_{ch,g}(X)$ may not always be finitely generated $k$-subalgebra of $\mathcal{O}(X)$. For instance, suppose $X = \mathbb{A}^2_k$ and $g$ is the $y$-axis. Then $\mathcal{O}_{ch,g}(X) = k + xk[x,y]$. This subring of $k[x,y]$ is not Noetherian. For this, consider the chain of ideals $\{I_n\}_n$ in $\mathcal{O}_{ch,g}(X)$: $I_n$ is the ideal generated by $\{x, xy, xy^2, ..., xy^{n-1}\}$. This chain of ideals does not stabilize.

**Remark 6.4.** We can describe $\mathcal{O}_{ch,g}(X)$ explicitly. The image of the affine line $g : \mathbb{A}^1_k \rightarrow X$ is closed in $X$. Let $\bar{g} : \mathcal{O}(X) \rightarrow k[T]$ be the $k$-algebra homomorphism induced by $g$. A regular function on $X$ is in $\mathcal{O}_{ch,g}(X)$ if and only if its image is in $k$ under $\bar{g}$. Thus for $\phi \in \mathcal{O}_{ch,g}(X)$, $\phi - \bar{g}(\phi) \in Ker(\bar{g})$. Therefore, $\mathcal{O}_{ch,g}(X) = k + Ker(\bar{g})$. If $\phi = \lambda + \theta$ for some constant $\lambda$ and
\( \theta \in \text{Ker}(g) \), then \( \phi \) takes value \( \lambda \) along \( \text{Im}(g) \).

**Suppose \( g_1 \) and \( g_2 \) are two intersecting \( \mathbb{A}^1 \)-s in \( X \):**

If \( \phi \in \mathcal{O}_{\text{ch},g_1}(X) \cap \mathcal{O}_{\text{ch},g_2}(X) \), then \( \phi = \lambda + \theta = \lambda' + \theta' \) for some constants \( \lambda, \lambda', \theta, \theta' \) are in kernel of \( g_1 \) and \( g_2 \) respectively. Since \( g_1 \) and \( g_2 \) intersect, \( \lambda = \lambda' \). Therefore \( \mathcal{O}_{\text{ch},g_1}(X) \cap \mathcal{O}_{\text{ch},g_2}(X) = k + (\text{Ker}(\tilde{g}_1) \cap \text{Ker}(\tilde{g}_2)) \), if \( g_1 \) and \( g_2 \) intersect.

**Suppose \( g_1 \) and \( g_2 \) are parallel:**

If the images of \( g_1 \) and \( g_2 \) are disjoint, then \( \mathcal{O}_{\text{ch},g_1}(X) \cap \mathcal{O}_{\text{ch},g_2}(X) \) properly contains \( k + (\text{Ker}(\tilde{g}_1) \cap \text{Ker}(\tilde{g}_2)) \) from the following lemma (Lemma 6.6).

**Definition 6.5.** A line \( g : \mathbb{A}^1 \rightarrow X \) is called isolated if it does not intersect any other lines in \( X \). for any line \( h : \mathbb{A}^1 \rightarrow X \), if \( \text{Im}(h) \neq \text{Im}(g) \) then \( \text{Im}(h) \cap \text{Im}(g) = \emptyset \).

**Lemma 6.6.** Suppose \( X \) is an affine \( k \)-variety.

1. Suppose \( g_1, g_2, \ldots, g_n \) are pairwise parallel lines in \( X \) (i.e. \( \text{Im}(g_i) \cap \text{Im}(g_j) = \emptyset, \forall i \neq j \)) and \( c_1, c_2, \ldots, c_n \) are \( n \) many constants. Then there is \( f \in \mathcal{O}(X) \) such that \( f = c_i \) along \( \text{Im}(g_i) \).

2. Let \( X \) be a smooth affine surface such that \( \mathcal{O}(X) \) is a U.F.D. Suppose there is a line \( g : \mathbb{A}^1 \rightarrow X \) which is isolated. Then \( \mathcal{O}_{\text{ch}}(X) \) is non-trivial.

**Proof.**

1. Since \( g_i \) and \( g_j \) are parallel, \( \text{Ker}(\tilde{g}_i) + \text{Ker}(\tilde{g}_j) = \mathcal{O}(X) \). Indeed, if \( \text{Ker}(\tilde{g}_i) + \text{Ker}(\tilde{g}_j) \) is contained in some maximal ideal of \( \mathcal{O}(X) \), then there is a common point of \( g_1 \) and \( g_2 \). So the ideals \( \text{Ker}(\tilde{g}_i) \) and \( \text{Ker}(\tilde{g}_j) \) are pairwise comaximal. Thus by Chinese remainder theorem, there exists \( f \in \mathcal{O}(X) \) such that \( f = c_i \) along \( \text{Im}(g_i) \).

2. Since \( \mathcal{O}(X) \) is a U.F.D., there is a \( f \in \mathcal{O}(X) \) irreducible such that the zero set of \( f \) is the closed set \( \text{Im}(g) \). Then \( f \) is non-zero in the complement of \( \text{Im}(g) \). So for any other line \( h : \mathbb{A}^1 \rightarrow X \) with \( \text{Im}(h) \neq \text{Im}(g) \), \( f \) is everywhere non-zero along \( \text{Im}(h) \) as \( g \) is an isolated line. Hence \( f \) must be constant along \( \text{Im}(h) \), since \( \mathbb{G}_m \) is \( \mathbb{A}^1 \)-rigid (Remark 2.2). Therefore \( f \) is a non-trivial element in \( \mathcal{O}_{\text{ch}}(X) \).

Thus in particular, for a smooth affine surface with trivial Picard group if \( \mathcal{O}_{\text{ch}}(X) \) is trivial, then there is no parallel family of lines in \( X \). Note that \( \mathbb{A}^1 \times \mathbb{G}_m \) has lines parallel to \( x \)-axis and any polynomial of \( y \) is in \( \mathcal{O}_{\text{ch}}(\mathbb{A}^1 \times \mathbb{G}_m) \).

**Definition 6.7.** A chain connected component of \( X \) is defined to be the largest subset of \( X(k) \) such that any two points in it can be joined by a chain of \( \mathbb{A}^1 \)-s.

**Remark 6.8.** Let \( T \subset X(k) \) be a chain component. Then \( T \) is the union of lines in \( X \) such that the points in the images of the lines are in \( T \).
Theorem 6.9. Let $X$ be a smooth affine surface such that $\mathcal{O}(X)$ is a U.F.D. Then $\mathcal{O}_{ch}(X)$ is trivial if and only if there is some dense chain connected component of $X$.

Proof. Suppose there is some chain connected component of $X$ which is dense in $X$. Let $T$ be the union of all lines in that chain connected component and $f$ is in $\mathcal{O}_{ch}(X)$. The function $f$ is constant along $T$, since any two points of $T$ can be joined by chain of lines. But $T$ is dense in $X$. Therefore $f$ is constant.

On the other hand, assume that $\mathcal{O}_{ch}(X)$ is trivial. If possible, there is a chain component (say $T$) which is the union of finitely many lines (i.e. finitely many distinct images). Then it is closed. There is some $f \in \mathcal{O}(X)$ such that its zero set is $T$. Then $f$ is non-zero along every other line outside $T$. Therefore it is constant along each line outside $T$. But $f$ is non-constant. This gives a contradiction. Therefore every chain connected component is a union of infinitely many lines (i.e. infinitely many distinct images). Choose any such chain connected component, say $S$. Its closure cannot be of dimension 1, since it contains infinitely many lines, hence $S$ is dense in $X$.

Remark 6.10. Suppose $\mathcal{O}_{ch}(X)$ is trivial. Then $\mathcal{O}(X)^* = k^*$. Indeed, if $f \in \mathcal{O}(X)^*$, then $f$ is a morphism from $X$ to $\mathbb{G}_m$. Therefore $f$ is constant along each line in $X$, since $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid (Remark 2.2). Hence $f \in \mathcal{O}_{ch}(X)$. So $f$ is constant.

Suppose, $\alpha : Y \to X$ is a morphism of affine $k$-varieties. For an affine line $g$ in $Y$, $\alpha \circ g$ is an affine line in $X$. So $f \circ (\alpha \circ g)$ is constant if $f \in \mathcal{O}_{ch}(X)$. Thus the morphism $\alpha$ induces $\alpha^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ that restricts to a $k$-algebra homomorphism $\mathcal{O}_{ch}(X) \to \mathcal{O}_{ch}(Y)$. Therefore, $\mathcal{O}_{ch}(X)$ is functorial in $X$.

Proposition 6.11. Let $k$ be a field of characteristic 0. Suppose $X$ is an affine $k$-variety. Then $\mathcal{O}_{ch}(X) \subset ML(X)$. Therefore, if $ML(X)$ is trivial then $\mathcal{O}_{ch}(X)$ is trivial.

Proof. Suppose $f \in \mathcal{O}_{ch}(X)$ and $D$ is a locally nilpotent $k$-derivation on $\mathcal{O}(X)$. We need to show that $f \in Ker(D)$. We have a $k$-algebra homomorphism $\exp(D) : \mathcal{O}(X) \to \mathcal{O}(X)[T]$ defined as $\exp(D)(g) = \sum_{n=0}^{\infty} \frac{D^n(g)}{n!} T^n$. Fix $x \in X(k)$. Consider the following composition of $k$-algebra homomorphisms:

$$k[T] \xrightarrow{T \mapsto f} \mathcal{O}(X) \xrightarrow{\exp(D)} \mathcal{O}(X)[T] \xrightarrow{\text{evaluation at } x} k[T]$$

This composition takes $T$ to an element of $k$ by Lemma 6.2. Suppose, $\exp(D)(f) = \sum_{n=0}^{\infty} f_n T^n \in \mathcal{O}(X)[T]$, where $f_n \in \mathcal{O}(X)$. Then for every $n \geq 1$, $f_n(x) = 0$, $\forall x \in X(k)$. Since $X(k)$ is dense in $X$, $f_n$-s are zero for every $n \geq 1$. Thus $\exp(D)(f) = f$. Hence $f \in Ker(D)$.
For an affine variety $X \in Sm/k$, the sheaf $S(X)$ of $A^1$-chain connected components of $X$ is the Nisnevich sheaf on $Sm/k$ associated to the presheaf $\pi_0(Sing^A_*(X))$ [5, Remark 2.10]. The natural map

$$S(X) \to \pi^A_0(X)$$

is an epimorphism. We have two morphisms $\theta_0$ and $\theta_1$ from $\text{Hom}(A^1_k, X)$ to $X$ induced by the 0-section and the 1-section respectively. So $S(X)$ is the coequaliser of $\theta_0$ and $\theta_1$ in $\text{Shv}(Sm/k)$.

There is a natural epimorphism from $X$ to $S(X)$. Thus a morphism from $S(X)$ to $A^1_k$ gives an element in $O(X)$.

**Proposition 6.12.** As $k$-subalgebras of $O(X)$, we have

$$\mathcal{O}_{ch}(X) = \text{Hom}_{\text{Shv}(Sm/k)}(S(X), A^1_k).$$

**Proof.** Suppose $\phi \in \mathcal{O}_{ch}(X)$, $\phi$ gives a morphism $X$ to $A^1_k$. We will show that $\phi \circ \theta_0 = \phi \circ \theta_1$ as morphisms from $\text{Hom}(A^1_k, X)$ to $A^1_k$ i.e. to show that $\forall U \in Sm/k$ and $f : A^1_1 \to X$, $\phi \circ f \circ \sigma_0 = \phi \circ f \circ \sigma_1$ where $\sigma_0, \sigma_1 : U \to A^1_k$ are the 0-section and the 1-section respectively. For any $x \in U(k)$, we get a morphism $i_x : A^1_k \to A^1_1$. Composing $f$ with $i_x$, we get a morphism from $A^1_k$ to $X$. Since $\phi$ is constant along this line $f \circ i_x$, $\phi(f(\sigma_0(x))) = \phi(f(\sigma_1(x)))$. The $k$-points are dense in $U$, so $\phi \circ f \circ \sigma_0 = \phi \circ f \circ \sigma_1$. Therefore, we have a unique morphism from $S(X)$ to $A^1_k$. So $\phi \in \text{Hom}_{\text{Shv}(Sm/k)}(S(X), A^1_k)$.

Conversely suppose, $\eta \in \text{Hom}_{\text{Shv}(Sm/k)}(S(X), A^1_k)$ and $\theta : A^1_k \to X$ is a line in $X$. Then $\eta$ gives a morphism from $X$ to $A^1_k$. The line $\theta$ is homotopic to the constant map. Define $H : A^1_k \times A^1_k \to X$ as composition of $\theta$ with the multiplication map $A^1_k \times A^1_k \to A^1_k ((s, t) \mapsto st)$. Then $H \sigma_0 = \theta$ and $H \sigma_1$ is the constant map, where $\sigma_0, \sigma_1 : A^1_k \to A^1_k \times A^1_k$ are the 0-section and the 1-section respectively. Since $\eta$ is a regular function on $S(X)$, $\eta$ is same along the 0-section and the 1-section of $H$. Therefore $\eta$ is constant along $\text{Im}(\theta)$. Thus, $\eta \in \mathcal{O}_{ch}(X)$. \qed

**Corollary 6.13.** If $X \in Sm/k$ is $A^1$-chain connected, then $\mathcal{O}_{ch}(X)$ is trivial.

**Remark 6.14.** Koras-Russell threefold of the first kind is $A^1$-chain connected [15, Example 2.21]. Therefore if $X$ is Koras-Russell threefold of the first kind, then $\mathcal{O}_{ch}(X)$ is trivial.

The relationship of $\mathcal{O}_{ch}(X)$ with $S(X)$ has a useful consequence.

**Corollary 6.15.** $\mathcal{O}_{ch}(-)$ is homotopy invariant i.e. $\mathcal{O}_{ch}(X) = \mathcal{O}_{ch}(X \times A^1_k)$ (both are $k$-subalgebras of $O(X \times A^1_k)$), $\forall X \in Sm/k$.

**Proof.** The morphism

$$\text{Id} \times i_0 : Sing^A_*(X) \to Sing^A_*(X \times A^1_k)$$

is a simplicial homotopy equivalence [32, Corollary 3.5]. Therefore the corollary follows from the Proposition 6.12. \qed
Remark 6.16. Makar-Limanov invariant is not homotopy invariant. If $X$ is Koras-Russel threefold over $\mathbb{C}$, then $ML(X) = \mathbb{C}[T]$ (18, Corollary 9.7) and $ML(X \times \mathbb{A}^1_k) = \mathbb{C}$ (12, Section 1). From Proposition 6.11 and homotopy invariance of $O_{ch}(X)$, we have $O_{ch}(X) = O_{ch}(X \times \mathbb{A}^1_k) = \mathbb{C}$.

So far we have observed that $O_{ch}$, the presheaf of $k$-algebras on the category of affine varieties over $k$ is homotopy invariant and it is directly related to $\mathbb{A}^1$-chain-connected component sheaf (Proposition 6.12). However $O_{ch}$ is not representable in $H(k)$.

Lemma 6.17. $O_{ch}$ is not representable in $H(k)$.

Proof. If possible, $O_{ch}$ is given by $\mathbb{A}^1$-connected component presheaf of some $\mathbb{A}^1$-fibrant object $X$. Then $O_{ch}$ satisfies the gluing property by Lemma 7.4, for any elementary distinguished square as in 7.1. Suppose, $X$ is the affine line $\mathbb{A}^1_k$ and the elementary distinguished square [Definition 7.1] is given by Zariski open covering $U = \mathbb{A}^1_k - \{0\}$ and $V = \mathbb{A}^1_k - \{1\}$. Then both $U$ and $V$ contain no affine lines so, $O_{ch}(U) = O(U)$ and $O_{ch}(V) = O(V)$. So $O(U \times O(U \cap V) O(V) = O(A^1_1)$. But $O_{ch}(A^1_k)$ is trivial. So the map $O_{ch}(A^1_1)$ to $O_{ch}(U) \times O_{ch}(U \cap V) O_{ch}(V)$ is not surjective. Hence $O_{ch}$ does not satisfy the gluing property.

Remark 6.18. For an affine variety $X \in Sm/k$, let $X_{Nis}$ be the small Nisnevich site. Then $O_{ch}|_{X_{Nis}}$ is not a sheaf whenever $O_{ch}(X) \neq O(X)$ by Lemma 2.3. If $O_{ch}(X) = O(X)$, then $O_{ch}|_{X_{Nis}}$ is a sheaf.

Question 6.19. Let $X \in Sm/k$ be an affine surface such that $O(X)$ is a U.F.D. Suppose $O_{ch}(X)$ is trivial. Is $X \cong \mathbb{A}^2_k$?

Remark 6.20. We have seen $O_{ch}(X)$ as the regular functions on $\mathbb{A}^1$-chain connected components of $X$ [Proposition 6.12]. We define,

$$O_{ch}^{(n)}(X) := Hom_{Shv(Sm/k)}(S^n(X), \mathbb{A}^1_k).$$

Then $O_{ch}^{(n)}(X)$ is a $k$-subalgebra of $O(X)$ and $O_{ch}^{(n+1)}(X) \subset O_{ch}^{(n)}(X)$ (since there is an epimorphism from $S^n(X)$ to $S^{n+1}(X)$). Thus inside $O(X)$, we have a decreasing chain of $k$-subalgebras, not necessarily Noetherian. Does the above chain of $k$-subalgebras stabilize?

Remark 6.21. Suppose $X \in Sm/k$ is $\mathbb{A}^1$-connected. Then there is an $n$ such that $S^n(X)$ is trivial. Indeed since $L(X)$ is trivial, the identity map on $X$ and the constant map given by a $k$-rational point are same in $S^n(X)$ for some $n$. Thus any map from $Spec O$ ($O$ is a Henselian local ring) to $X$ is same with the constant map in $S^n(X)$. Therefore the chain $\{O_{ch}^{(n)}(X)\}_n$ in Remark 6.20 stabilizes for some $n$.

Question 6.22. Given an affine variety $X$, define $X_i$ inductively as follows: $X_0 = X$ and $X_i = Spec(O_{ch}(X_{i-1}))$. What is the relation between $X_i$ and the spectrum of $O_{ch}^{(i)}(X)$?
There is a canonical map from $X_i$ to $X_{i+1}$ for every $i$. Note that if $X$ does not have any non-constant $\mathbb{A}^1_k$, then $\mathcal{O}_{ch}(X) = \mathcal{O}(X)$. Does there exist some $n$ such that $X_n$ has no non-constant $\mathbb{A}^1$?

### 7. Appendix

**Definition 7.1.** An elementary distinguished square (in the Nisnevich topology) is a cartesian square in $Sm/k$ of the form

$$
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow p \\
U & \stackrel{j}{\longrightarrow} & X
\end{array}
$$

such that $p$ is an étale morphism, $j$ is an open embedding and $p^{-1}(X - U) \rightarrow (X - U)$ is an isomorphism (we put the reduced induced structure on the corresponding closed sets) [32, Definition 3.1.3].

See [32, Chapter 3] for more details.

**Definition 7.2.** A simplicial presheaf $\mathcal{X}$ on $Sm/k$ is said to satisfy the Nisnevich Brown–Gersten property if for any elementary distinguished square in the Nisnevich topology as in Definition 7.1, the induced square of simplicial sets

$$
\begin{array}{ccc}
\mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\
\downarrow & & \downarrow \\
\mathcal{X}(U) & \longrightarrow & \mathcal{X}(U \times_X V)
\end{array}
$$

is homotopy cartesian [32, Definition 3.1.13].

**Remark 7.3.** Any fibrant space for the Nisnevich local model structure satisfies the Nisnevich Brown–Gersten property [32, Remark 3.1.15]. Any $\mathbb{A}^1$-fibrant space satisfies Nisnevich Brown-Gersten property.

The next lemma shows that the connected component presheaf of a Nisnevich fibrant object satisfies the gluing property for any elementary distinguished square.

**Lemma 7.4.** Suppose $\mathcal{X}$ is a Nisnevich fibrant object. Then for any elementary distinguished square as in 7.1, the induced map

$$
\pi_0(\mathcal{X}(X)) \rightarrow \pi_0(\mathcal{X}(U)) \times_{\pi_0(\mathcal{X}(U \times_X V))} \pi_0(\mathcal{X}(V))
$$

is surjective.

**Proof.** [8, Lemma 2.2].
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