Stochastic Optimization Theory of Backward Stochastic Differential Equations Driven by G-Brownian Motion *

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Abstract: In this paper, we consider the stochastic optimal control problems under G-expectation. Based on the theory of backward stochastic differential equations driven by G-Brownian motion, which was introduced in [10, 11], we can investigate the more general stochastic optimal control problems under G-expectation than that were constructed in [28]. Then we obtain a generalized dynamic programming principle and the value function is proved to be a viscosity solution of a fully nonlinear second-order partial differential equation.

Keywords: G-expectation, G-Brownian motion, Backward SDEs, Stochastic control.

1 Introduction

Non-linear BSDEs in the framework of linear expectation were introduced by Pardoux and Peng [18] in 1990. Then a lot of researches were studied by many authors and they provided various applications of BSDEs in stochastic control, finance, stochastic differential games and second order partial differential equations theory, see [1, 8, 12, 13, 19–21, 27].

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The notion of sublinear expectation space was introduced by Peng [14–16], which is a generalization of classical probability space. The G-expectation, a type of sublinear expectation, has played an important role in the researches of sublinear expectation space recently. It can be regarded as a counterpart of the Wiener probability space in the linear case. Within this G-expectation framework, the G-Brownian motion is the canonical process. Besides, the notions of the G-martingales and the Itô integral w.r.t. G-Brownian motion were also derived. There are some new structures in these notions and some new applications in the financial models with volatility uncertainty, see Peng [16, 17].

In the G-expectation framework, thanks to a series of studies [23–26], the complete representation theorem for G-martingales has been obtained by Peng, Song and Zhang [22]. Due to this contribution, a natural formulation of BSDEs driven by G-Brownian motion was found by Hu, Ji, Peng and Song [10]. In addition, the existence and uniqueness of the solution to the BSDEs driven by G-Brownian motion has been proved. They also have given the comparison theorem, Feynman-Kac Formula and Girsanov transformation for BSDEs driven by G-Brownian motion in [11]. So the complete theory of BSDEs driven by G-Brownian motion has been established.

An important application of BSDEs is that we can define the recursive utility functions from BSDEs, which can index scaling risks in the study of economics and finance [2, 5–7]. Based on these results, a type of significant stochastic optimal control problems under linear expectation with a BSDE as cost function were studied [1, 12, 20, 21, 27]. Under G-expectation, the similar problems will be useful in the future studies of finance models with volatility uncertainty. So we arise a natural question: Can we construct the similar results in G-expectation framework. When the complete results about BSDEs driven by G-Brownian motion were established in [10, 11], we try to prove the complete results of stochastic optimization theory of BSDEs driven by G-Brownian motion in this paper.

In this paper, we investigate the stochastic optimal control problems with a BSDE driven by G-Brownian motion constructed in [10, 11] as cost function. Based on the results in [10, 11], we obtain the dynamic programming principle under G-expectation. Besides, the value function is proved to be a viscosity solution of a fully nonlinear second-order partial differential equation.

The rest of the paper is organized as follows. In Section 2, we recall the G-expectation framework and adapt it according to our objective. Besides, we give the related properties of forward and backward stochastic differential equations driven by G-Brownian motion, which will be needed in the sequel sections. In Section 3, the stochastic optimal control problems with a BSDE driven by G-
Brownian motion as cost function are investigated and a dynamic programming principle under G-expectation is obtained. In Section 4, The value function is proved to be a viscosity solution of a fully nonlinear second-order partial differential equation.

2 Preliminaries

In this section, we recall the G-expectation framework established by Peng [3, 14–16]. Besides, we give some results about forward and backward stochastic differential equations driven by G-Brownian motion, which we need in the following sections. Some details can be found in [10, 11].

2.1 G-expectation and G-martingales

Definition 2.1. Let \( \Omega \) be a given set and \( \mathcal{H} \) be a linear space of real valued functions defined on \( \Omega \), namely \( c \in \mathcal{H} \) for each constant \( c \) and \( |X| \in \mathcal{H} \) if \( X \in \mathcal{H} \). The space \( \mathcal{H} \) can be considered as the space of random variables. A sublinear expectation \( \mathbb{E} \) is a functional \( \mathbb{E} : \mathcal{H} \rightarrow \mathbb{R} \) satisfying the following properties: for all \( X, Y \in \mathcal{H} \), we have

(i) Monotonicity: \( \mathbb{E}[X] \geq \mathbb{E}[Y] \), if \( X \geq Y \);
(ii) Constant preservation: \( \mathbb{E}[c] = c \), for \( c \in \mathbb{R} \);
(iii) Sub-additivity: \( \mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y] \);
(iv) Positive homogeneity: \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \), for \( \lambda \geq 0 \).

The triple \( (\Omega, \mathcal{H}, \mathbb{E}) \) is called a sublinear expectation space.

Definition 2.2. (G-Normal Distribution) A d-dimensional random vector \( X = (X_1, \cdots, X_d) \) on a sublinear expectation space \( (\Omega, \mathcal{H}, \mathbb{E}) \) is called G-normally distributed if for each \( a, b \geq 0 \), we have

\[
X + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X,
\]

where \( \bar{X} \) is an independent copy of \( X \), i.e., \( \bar{X} \) and \( X \) is identically distributed and \( \bar{X} \) is independent from \( X \). Here the letter \( G \) denotes the function

\[
G(A) := \frac{1}{2} \mathbb{E}[(AX, X)] : \mathbb{S}_d \mapsto \mathbb{R}, \tag{2.1}
\]

where \( \mathbb{S}_d \) denotes the collection of all \( d \times d \) symmetric matrices.
Proposition 2.3. Let $X$ be $G$-normal distributed. The distribution of $X$ is characterized by

$$u(t, x) = \mathbb{E}[\varphi(x + \sqrt{t}X)], \quad \varphi \in C_{b, Lip}(\mathbb{R}^d).$$

(2.2)

In particular, $\mathbb{E}[\varphi(X)] = u(1, 0)$, where $u$ is the unique viscosity solution of the following parabolic PDE defined on $[0, \infty) \times \mathbb{R}^d$:

$$\partial_t u - G(D^2 u) = 0, \quad u|_{t=0} = \varphi,$$

(2.3)

where $G$ is defined by (2.1).

Remark 2.4. It is easy to check that $G$ is a monotonic sublinear function defined on $\mathbb{S}(d)$ and $G(A) := \frac{1}{2}\mathbb{E}|AX, X| \leq \frac{1}{2}|A|\mathbb{E}[|X|^2] = \frac{1}{2}|A|\mathbb{E}[\gamma^2]$ implies that there exists a bounded, convex and closed subset $\Gamma \subset \mathbb{S}^+_d$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} Tr(\gamma A),$$

(2.4)

where $\mathbb{S}^+_d$ denotes the collection of nonnegative elements in $\mathbb{S}_d$. If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta tr[A - B]$ for any $A \geq B$, we call the $G$-normal distribution non-degenerate, which is the case we consider throughout this paper.

Definition 2.5. Let $\Omega = C^d_0([0, T])$, i.e., the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in [0, T]}$ with $\omega_0 = 0$. The corresponding canonical process is $B_t(\omega) = \omega_t$, $t \in [0, T]$. $P_0$ is wiener measure. $\mathbb{F} = \{\mathcal{F}_t^B\}_{t \geq 0}$ is the filtration generated by $B$. We let $\mathcal{H} := L_{ip}(\Omega_T)$ to be a linear space of random variables for each fixed $T \geq 0$, where $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) : n \geq 1, t_1, \cdots, t_n \in [0, T], \varphi \in C_{b, Lip}(\mathbb{R}^{d\times n})\}$.

(i) The $G$-expectation $\hat{E}$ is a sublinear expectation defined by

$$\hat{E}[X] := \hat{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)],$$

for each $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$, where $(\xi_i)_{i=1}^n$ are identically distributed $d$-dimensional $G$-normally distributed random vectors in a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{E})$ such that $\xi_{i+1}$ is independent from $(\xi_1, \cdots, \xi_i)$ for each $i = 1, 2, \cdots, n - 1$. $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-expectation space and the canonical process $\{B_t\}_{t \in [0, T]}$ in the sublinear space $(\Omega, \mathcal{H}, \hat{E})$ is called a $G$-Brownian motion.

(ii) The conditional $G$-expectation $\hat{E}_t$ of $X \in L_{ip}(\Omega_T)$ is defined by

$$\hat{E}_t[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] := \psi(B_{t_1} - B_{t_0}, \cdots, B_{t_j} - B_{t_{j-1}}),$$

where $\psi(x_1, \cdots, x_j) = \hat{E}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)]$. 

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We denote by $L^p_G(\Omega_T)$, $p \geq 1$, the completion of G-expectation space $L^p(\Omega_T)$ under the norm $\| X \|_{p,G} := (\hat{E}[|X|^p])^{\frac{1}{p}}$. For all $t \in [0, T]$, $\hat{E}[-]$ and $\hat{E}_t[-]$ are continuous mapping on $L^p_G(\Omega_T)$ endowed with the norm $\| \cdot \|_{1,G}$. Therefore, it can be extended continuously to $L^p_G(\Omega_T)$.

**Definition 2.6.** A process $\{M_t\}_{t \geq 0}$ is called a G-martingale if for each $t \in [0, T]$, $M_t \in L^1_G(\Omega_t)$ and for each $s \in [0, t]$, we have $\hat{E}_s[M_t] = M_s$.

Now we introduce the Itô integral and quadratic variation process with respect to G-Brownian motion in G-expectation space.

**Definition 2.7.** Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, \cdots, t_N\}$ of $[0, T]$, we denote $M^p_G(0, T)$ as the collection of following type of simple processes:

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) 1_{[t_k, t_{k+1}]}(t),$$

where $\xi_k \in L^p_G(\Omega_{t_k})$, $k = 0, 1, 2, \cdots, N-1$. We denote by $M^p_G(0, T)$ the completion of $M^{p,0}_G(0, T)$ under the norm $\| \cdot \|_{M^p_G(0, T)} := (\hat{E}[\int_0^T |\cdot|^p dt])^{\frac{1}{p}}$.

**Definition 2.8.** For each $\eta \in M^2_G(0, T)$, we define

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

The mapping $I : M^{2,0}_G(0, T) \to L^2_G(\Omega_T)$ is continuous and thus can be continuously extended to $M^2_G(0, T)$.

**Definition 2.9.** The quadratic variation process of G-Brownian motion is defined by

$$\langle B \rangle_t := B_t^2 - 2 \int_0^t B_s dB_s,$$

which is a continuous, nondecreasing process.

**Definition 2.10.** We now define the integral of a process $\eta \in M^1_G(0, T)$ with respect to $\langle B \rangle$ as following:

$$Q_{0,T}(\eta) = \int_0^T \eta_t d\langle B \rangle_t := \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) : M^{1,0}_G(0, T) \to L^1_G(\Omega_T).$$

The mapping is continuous and can be extended to $M^{1}_G(0, T)$ uniquely.
Then we detail some results about the quasi-analysis theory constructed in \[3\].

**Theorem 2.11.** There exists a weakly compact family \( \mathcal{P} \subset \mathcal{M}_1(\Omega_T) \), the collection of probability measures defined on \((\Omega_T, \mathcal{B}(\Omega_T))\), such that

\[
\hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \forall X \in L_{ip}(\Omega_T),
\]

\( \mathcal{P} \) is called a set of probability measures that represents \( \hat{E} \).

**Definition 2.12.** We define the capacity associated to \( \mathcal{P} \), which is a weakly compact family of probability measure represents \( \hat{E} \), as follow:

\[
\hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T),
\]

\( \hat{c} \) is also called the capacity induced by \( \hat{E} \).

Let \((\Omega^0, \mathcal{F}^0 = \{\mathcal{F}^0_t\}, \mathcal{F}, P^0)\) be a filtered probability space, and \{\(W_t\)\} be a \(d\)-dimensional Brownian motion under \(P^0\). [3] proved that \( \mathcal{P}_M := \{P_0 \circ X^{-1} | X_t = \int_0^t h_idW_s, h \in L_{ip}^2([0, T]; \Gamma^2)\} \) represents G-expectation \( \hat{E} \), where \( \Gamma^2 := \{\gamma^2 | \gamma \in \Gamma\} \) and \( \Gamma \) is the set in the representation of \( G(\cdot) \) of the formula \(2.4\).

**Definition 2.13.** (i) Let \( \hat{c} \) be the capacity induced by \( \hat{E} \). A set \( A \subset \Omega \) is polar if \( \hat{c}(A) = 0 \). A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set.

(ii) Let \( X \) and \( Y \) be two random variables, we say that \( X \) is a version of \( Y \), if \( X = Y \) q.s.

Let \( \| \psi \|_{p,G} = [\hat{E}(|\psi|^p)]^{\frac{1}{p}} \) for \( \psi \in C_b(\Omega_T) \). The completion of \( C_b(\Omega_T) \) and \( L_{ip}(\Omega_T) \) under \( \| \cdot \|_{p,G} \) are the same and we denote them by \( L_{ip}^G(\Omega_T) \).

### 2.2 Forward and Backward Stochastic Differential Equations Driven by G-Brownian Motion

We consider the following stochastic differential equations driven by \( d \)-dimensional G-Brownian motion (G-SDE):

\[
X_t = X_0 + \int_0^t b(s, X_s)ds + \sum_{i,j=1}^d \int_0^t h_{ij}(s, X_s)d\langle B^i, B^j \rangle_s + \sum_{j=1}^d \int_0^t \sigma_j(s, X_s)dB^j_s, \quad (2.5)
\]
where $t \in [0, T]$, the initial condition $X_0 \in \mathbb{R}^n$ is a given constant, $b, h_{ij}, \sigma_j$ are given functions satisfying $b(\cdot, x), h_{ij}(\cdot, x), \sigma_j(\cdot, x) \in M^2_G(0, T; \mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ and the Lipschitz condition, i.e., $|\phi(t, x) - \phi(t, x')| \leq K|x - x'|$, for each $t \in [0, T], x, x' \in \mathbb{R}^n, \phi = b, h_{ij}$ and $\sigma_j$, respectively. The solution is a process $X \in M^2_G(0, T; \mathbb{R}^n)$ satisfying the G-SDE (2.5).

**Theorem 2.14.** ([16]) There exists a unique solution $X \in M^2_G(0, T; \mathbb{R}^n)$ of the stochastic differential equation (2.5).

Now we give the results about BSDEs driven by G-Brownian motion in the G-expectation space $(\Omega_T, L^1_G(\Omega_T), \hat{E})$ with $\Omega_T = C_0([0, T], \mathbb{R}^d)$ and $\hat{\sigma}^2 = \hat{E}[B^2_T] \geq -\hat{E}[-B^2_0] = \sigma^2 > 0$. We consider the following type of G-BSDEs (we always use Einstein convention),

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \sum_{i,j=1}^d \int_t^T g_{ij}(s, Y_s, Z_s)d\langle B^i, B^j \rangle_s - \int_t^T Z_sdB_s - (K_T - K_t),$$

where $f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfy the following properties: There exists some $\beta > 1$ such that

1. (H1) for any $y, z, f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M^\beta_G(0, T)$;
2. (H2) for some $L > 0$,

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, \omega, y, z) - g_{ij}(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|).$$

For simplicity, we denote by $6(0, T)$ the collection of processes $(Y, Z, K)$ such that $Y \in S^0_G(0, T), Z \in H^p_G(0, T), K$ is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L^p_G(\Omega_T)$. Here $S^0_G(0, T)$ is the completion of $S^0_G(0, T) = \{ h(t, B_{t_1+\cdots+t_n}) : t_1, \cdots, t_n \in [0, T], h \in C_0([0, T], \mathbb{R}^n) \}$ under $\| \cdot \|_{S^0_G} = \{ \hat{E}[\sup_{t \in [0, T]} |\eta_t|^p] \}^{1/p}$ and $H^p_G(0, T)$ is the completion of $M^0_G(0, T)$ under $\| \cdot \|_{H^p_G} = \{ \hat{E}[\int_0^T |\eta_t|^p ds]^{1/p} \}$.  

**Definition 2.15.** Let $\xi \in L^p_G(\Omega_T)$ with $\beta > 1$, $f$ and $g_{ij}$ satisfy (H1) and (H2). A triplet of processes $(Y, Z, K)$ is called a solution of equation (2.6) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y, Z, K) \in 6^\alpha_G(0, T)$;

(b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \sum_{i,j=1}^d \int_t^T g_{ij}(s, Y_s, Z_s)d\langle B^i, B^j \rangle_s - \int_t^T Z_sdB_s - (K_T - K_t)$. 

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**Theorem 2.16.** (\cite{10}) Assume that $ξ ∈ L^β(Ω_T)$ and $f, g$ satisfy (H1) and (H2) for some $β > 1$. Then equation (2.6) has a unique solution $(Y, Z, K)$. Moreover, for any $1 < α < β$ we have $Y ∈ C^1_{α}(0, T), Z ∈ H^α_{G}(0, T; \mathbb{R}^d)$ and $K_T ∈ L^α_{G}(Ω_T)$.

We have the following estimates.

**Proposition 2.17.** (\cite{10}) Let $ξ ∈ L^β_{G}(Ω_T)$ and $f, g_i$ satisfy (H1) and (H2) for some $β > 1$. Assume that $(Y, Z, K) ∈ H^α_{G}(0, T)$ for some $1 < α < β$ is a solution of equation (2.6). Then

(i) There exists a constant $C_α := C(α, T, G, L) > 0$ such that

$$|Y_t|^α ≤ C_α \hat{E}_t[|ξ|^α + \int_t^T |h^0|^α ds],$$

$$\hat{E}[\sup_{t \in [0, T]} Y_t]^α ≤ C_α \hat{E}[\sup_{t \in [0, T]} \hat{E}_t[|ξ|^α] + (\hat{E}[\sup_{t \in [0, T]} |Y_t|^α])^2(\hat{E}[\int_0^T |h^0|^α ds])^2],$$

$$\hat{E}[|K_T|^α] ≤ C_α \hat{E}[\sup_{t \in [0, T]} |Y_t|^α] + \hat{E}[\sup_{t \in [0, T]} (\int_0^T |h^0|^α ds)^α],$$

where $h^0 = |f(s, 0, 0)| + \sum_{i,j=1}^d |g_{ij}(s, 0, 0)|$.

(ii) For any given $α < α' < β$, there exists a constant $C_{α,α'}$ depending on $α, α', T, G, L$ such that

$$\hat{E}[\sup_{t \in [0, T]} Y_t] ≤ C_{α,α'} \hat{E}[\sup_{t \in [0, T]} \hat{E}_t[|ξ|^α]]$$

$$+ (\hat{E}[\sup_{t \in [0, T]} \hat{E}_t[\int_0^T |h^0|^α ds)]^α + \hat{E}[\sup_{t \in [0, T]} (\int_0^T |h^0|^α ds)]^α]).$$

**Proposition 2.18.** (\cite{11}) Let $ξ^i ∈ L^β_{G}(Ω_T)$, $i = 1, 2$, and $f^i, g^i_{ij}$ satisfy (H1) and (H2) for some $β > 1$. Assume that $(Y^i, Z^i, K^i) ∈ H^α_{G}(0, T)$ for some $1 < α < β$ are the solutions of (2.6) corresponding to $ξ^i, f^i, g^i_{ij}$. Set $\hat{Y}_t = Y^1_t - Y^2_t, \hat{Z}_t = Z^1_t - Z^2_t$ and $\hat{K}_t = K^1_t - K^2_t$. Then

(i) There exists a constant $C_α := C(α, T, G, L) > 0$ such that

$$|\hat{Y}_t|^α ≤ C_α \hat{E}_t[|\hat{ξ}|^α + \int_t^T |\hat{h}_s|^α ds],$$

where $\hat{ξ} = ξ^1 - ξ^2, \hat{h}_s = |f^1(s, Y^2_s, Z^2_s) - f^2(s, Y^2_s, Z^2_s)| + \sum_{i,j=1}^d |g^1_{ij}(s, Y^2_s, Z^2_s) - g^2_{ij}(s, Y^2_s, Z^2_s)|$. 

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(ii) For any given $\alpha'$ with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on $\alpha, \alpha', T, G, L$ such that

$$
\mathbb{E}[\sup_{t \in [0,T]} |\hat{Y}_t|^\alpha] \leq C_{\alpha,\alpha'} \mathbb{E}[\sup_{t \in [0,T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha]]
$$

$$
+ (\mathbb{E}[\sup_{t \in [0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{\gamma}_s ds)^{\alpha'}]])^\beta + \mathbb{E}[\sup_{t \in [0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{\gamma}_s ds)^{\alpha'}]]).
$$

**Theorem 2.19.** ([11]) Let $(Y^i_t, Z^i_t, K^i_t)_{t \leq T}, i = 1, 2$, be the solutions of the following G-BSDES:

$$
Y^i_t = \xi^i + \int_t^T f_i(s, Y^i_s, Z^i_s)ds + \int_t^T g_i(s, Y^i_s, Z^i_s)d<B>_s - \int_t^T Z^i_sdB_s - (K^i_T - K^i_t),
$$

where $\xi^i \in L^2_{\mathcal{G}}(\Omega_T)$, $f_i, g_i$ satisfy (H1) and (H2) with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$, then $Y^1_t \geq Y^2_t$.

**Theorem 2.20.** ([11]) Let $(Y^i_t, Z^i_t, K^i_t)_{t \leq T}, i = 1, 2$, be the solutions of the following G-BSDES:

$$
Y^i_t = \xi^i + \int_t^T f_i(s, Y^i_s, Z^i_s)ds + \int_t^T g_i(s, Y^i_s, Z^i_s)d<B>_s - \int_t^T Z^i_sdB_s - (K^i_T - K^i_t) + V^i_T - V^i_t,
$$

where $\xi^i \in L^2_{\mathcal{G}}(\Omega_T)$, $f_i, g_i$ satisfy (H1) and (H2), $(V^i_t)_{t \leq T}$ are RCLL processes such that $\mathbb{E}[\sup_{t \in [0,T]} |V^i_t|^\beta] < \infty$ with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$, $V^1_t - V^2_t$ is an increasing process, then $Y^1_t \geq Y^2_t$.

### 3 A DPP for Stochastic Optimal Control Problems under G-Expectation

Now we introduce the setting for stochastic optimal control problems under $G$-expectation. We suppose that the control state space $V$ is a compact metric space. Let the set of admissible control processes $\mathcal{U}$ for the player be a set of $V$-valued stochastic processes in $M^\beta_\mathcal{G}(\iota, T; \mathbb{R}^n)$ with $\beta > 2$ and $\iota \in [0, T]$. For a given admissible control $\nu(\cdot) \in \mathcal{U}$, the corresponding orbit which regards $\iota$ as the initial time and $\xi \in L^2_{\mathcal{G}}(\Omega_\iota; \mathbb{R}^n)$ as the initial state, is defined by the solution of the following type of G-SDE:

\[ \text{...} \]
\[
\begin{aligned}
    \left\{ 
        \begin{array}{l}
            dX_t^{\xi,\nu} = b(s, X_t^{\xi,\nu}, \nu_s)ds + \sum_{i,j=1}^d h_{ij}(s, X_t^{\xi,\nu}, \nu_s)d(B_i^s, B_j^s) + \sum_{j=1}^d \sigma_j(s, X_t^{\xi,\nu}, \nu_s)dB_j^s, \\
            s \in [t, T], \\
            X_t^{\xi,\nu} = \xi,
        \end{array}
    \right.
\end{aligned}
\]

where \( b, h_{ij}, \sigma_j : [0, T] \times \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n \) are deterministic functions and satisfy the following conditions (H3):

(A1) \( h_{ij} = h_{ji} \) for \( 1 \leq i, j \leq d \);

(A2) For every fixed \((x, \nu) \in \mathbb{R}^n \times \mathcal{U}, b(\cdot, x, \nu), h_{ij}(\cdot, x, \nu), \sigma_j(\cdot, x, \nu) \) are continuous in \( t \);

(A3) There exists a constant \( \text{L} > 0 \), for any \( t \in [0, T], x, x' \in \mathbb{R}^n, \nu, \nu' \in \mathcal{U} \) such that

\[
|b(t, x, \nu) - b(t, x', \nu')| + \sum_{i,j=1}^d |h_{ij}(t, x, \nu) - h_{ij}(t, x', \nu')| + \sum_{j=1}^d |\sigma_j(t, x, \nu) - \sigma_j(t, x', \nu')| \\
\leq \text{L}(|x - x'| + |\nu - \nu'|).
\]

From the assumption (H3), we can get global linear growth conditions for \( b, h_{ij}, \sigma_j \), i.e., there exists \( C > 0 \) such that, for \( t \in [0, T], x \in \mathbb{R}^n, \nu \in \mathcal{U} \),

\[
|b(t, x, \nu)| + \sum_{i,j=1}^d |h_{ij}(t, x, \nu)| + \sum_{j=1}^d |\sigma_j(t, x, \nu)| \leq C(1 + |x| + |\nu|).
\]

Obviously, under the above assumptions, for any \( \nu(\cdot) \in \mathcal{U} \), G-SDE (3.1) has a unique solution. Moreover, we have the following estimates:

**Proposition 3.1.** Let \( \xi, \xi' \in L_p^\infty(\Omega; \mathbb{R}^n) \) with \( p \geq 2 \), \( \nu(\cdot), \nu'(\cdot) \in \mathcal{U}, t \in [0, T], \delta \in [0, T - t] \), then we have

\[
\hat{E}_t[|X_{t+\delta}^{\xi,\nu} - X_{t+\delta}^{\xi',\nu'}|^p] \leq C(\hat{K} - \xi'\nu' + \int_t^{t+\delta} \hat{E}_r[|\nu - \nu'|^p]dr),
\]

\[
\hat{E}_t[|X_{t+\delta}^{\xi,\nu}|^p] \leq C(1 + |\xi|^p),
\]

\[
\hat{E}_t[\sup_{s \in [t, t+\delta]} |X_s^{\xi,\nu} - \xi|^p] \leq C(1 + |\xi|^p)\delta^\frac{p}{2},
\]

where \( C \) depends on \( \text{L}, G, p, n, T \).

**Proof.** The proof is similar to the proof of Proposition 4.1 in [11].
Now we give bounded functions $\Phi: \mathbb{R}^n \to \mathbb{R}$, $f: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$, $g_{ij}: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}$ satisfy the following conditions: (H4)

(i) $g_{ij} = g_{ji}$ for $1 \leq i, j \leq d$.

(ii) For every fixed $(x, y, z, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{U}$, $f(\cdot, x, y, z, v)$ and $g_{ij}(\cdot, x, y, z, v)$ are continuous in $t$, $1 \leq i, j \leq d$.

(iii) There exist a constant $C$ depending on $L$, $G$, $n$ and $T$.

We have bounded functions $\Phi$ and $g_{ij}$ also satisfy global linear growth condition in $x$, i.e., there exists $C > 0$, such that for all $0 \leq t \leq T$, $v \in \mathcal{U}$, $x \in \mathbb{R}^n$,

$$|\Phi(x) - \Phi(x')| \leq L|x - x'|,$$

$$|f(t, x, y, z, v) - f(t, x', y', z', v')| + \sum_{i,j=1}^d |g_{ij}(t, x, y, z, v) - g_{ij}(t, x', y', z', v')|$$

$$\leq L(|x - x'| + |y - y'| + |z - z'| + |v - v'|).$$

From (H4), we have that $\Phi$, $f$ and $g_{ij}$ also satisfy the conditions of Theorem 2.16 on the interval $[t, T]$. Therefore, there exists a unique solution for the following G-BSDE:

$$Y_t^{\xi, v} = \Phi(X_T^{\xi, v}) + \int_t^T f(r, X_r^{\xi, v}, Y_r^{\xi, v}, Z_r^{\xi, v}, v_r)dr - \int_t^T Z_r^{\xi, v}dB_r - (K_T^{\xi, v} - K_s^{\xi, v})$$

$$+ \sum_{i,j=1}^d \int_t^T g_{ij}(r, X_r^{\xi, v}, Y_r^{\xi, v}, Z_r^{\xi, v}, v_r)d(B_r^i, B_r^j), \quad (3.2)$$

where $X_t^{\xi, v}$ is introduced by (3.1).

**Proposition 3.2.** For each $\xi$, $\xi' \in L^p_G(\Omega; \mathbb{R}^n)$ with $p \geq 2$ and $v(\cdot)$, $v'(\cdot) \in \mathcal{U}$ we have

$$|Y_t^{\xi, v} - Y_t^{\xi', v'}| \leq C|\xi - \xi'|,$$

$$|Y_t^{\xi, v}| \leq C(1 + |\xi|),$$

$$|Y_t^{\xi, v} - Y_t^{\xi', v'}| \leq C\left(\int_t^T \mathbb{E}|v(r) - v'(r)|^2dr\right)^{\frac{1}{2}},$$

where $C$ depends on $L$, $G$, $n$ and $T$. 

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Proof. The proof is similar to the Proposition 4.2 in [11].

Given a control process $\nu(\cdot) \in \mathcal{U}$, we introduce an associated cost functional

$$J(t, x; \nu) = Y_{t}^{x, \nu}, \ (t, x) \in [0, T] \times \mathbb{R}^n,$$

where the process $Y_{t}^{x, \nu}$ is defined by G-BSDE (3.2). Similar to the proof of Theorem 4.4 in [11], we have that for $t \in [0, T], \xi \in L^2_\mathcal{G}(\Omega, \mathbb{R}^n),$

$$J(t, \xi; \nu) := Y_{t}^{x, \nu}.$$

But we are more interest in the case when $\xi = x$.

Now we define the value function as follow:

$$u(t, x) := \sup_{\nu(\cdot) \in \mathcal{U}} J(t, x; \nu). \quad (3.3)$$

Proposition 3.3. $u(t, x)$ is a deterministic function of $(t, x)$.

Proof. For a partition of $[t, s]: t = t_0 < t_1 < \cdots < t_N = s$, $p \geq 2$, $t \leq s \leq T$, we denote $L_{dp}(\Omega^t_s) := \{\varphi(B_{t_1} - B_t, \cdots, B_{t_n} - B_t) : n \geq 1, t_1, \cdots, t_n \in [t, s], \varphi \in C_b, L_{dp}(\mathbb{R}^{d \times n})\}, M_{G}^{p, t}(t, s; \mathbb{R}^n)$ by the collection of simple processes $\eta(r) = \sum_{k=0}^{N-1} \xi_k 1_{[t, t_{k+1})}(r)$, where $\xi_k \in L_{dp}(\Omega^t_s; \mathbb{R}^n)$, $k = 0, 1, 2, \cdots, N - 1$ and $M_{G}^{p, t}(t, s; \mathbb{R}^n)$ by the completion of $M_{G}^{p, t}(t, s; \mathbb{R}^n)$ under the norm $||\eta||_{M_{G}^{p, t}(t, s; \mathbb{R}^n)} := \{\hat{E}[\int_t^s |\eta(r)|^p dr]\}^{\frac{1}{p}}$.

Use the similar method in Lemma 43 of [3], we can prove for $\nu \in M_{G}^{p, t}(t, s; \mathbb{R}^n)$ is a $V$-valued process, there exists $\{u = \sum_{i=1}^{N} 1_{A_i} u^i\}_{N \in \mathbb{N}}, u^i \in M_{G}^{p, t}(t, s; \mathbb{R}^n)$ is a $V$-valued process, $A_i$ is a partition of $\mathcal{B}(\Omega)$ such that $u \rightarrow \nu$ under the norm $||\eta||_{M_{G}^{p, t}(t, s; \mathbb{R}^n)}$, $N \rightarrow \infty$. When $\nu(s) \in M_{G}^{p, t}(t, s; \mathbb{R}^n)$, we note that $J(t, x; \nu)$ is a deterministic function of $(t, x)$, because $b, h, \sigma, \Phi, \cdot$ and $g$ are deterministic functions and $\tilde{B}_s := B_{s+t} - B_t$ is a G-Brownian motion. So we need to construct a sequence of admissible controls $\{\tilde{\nu}^j(\cdot)\}$ of the form

$$\tilde{\nu}_s^j = \sum_{i=1}^{N} \nu_s^j 1_{A_{ij}},$$

satisfying $\lim_{i \rightarrow \infty} J(t, x; \tilde{\nu}(\cdot)) = u(t, x)$, where $\nu^j(\cdot) \in M_{G}^{p, t}(t, s; \mathbb{R}^n)$ is a $V$-valued processes and $\{A_{ij}\}_{j=1}^{N_i}$ is a partition of $\mathcal{B}(\Omega)$. Firstly, there exists $\{\nu^k\}_{k \geq 1} \subset \mathcal{U}$, such
that \( u(t, x) = \sup_{k \geq 1} J(t, x; \nu^k) \). Then we define \( \nu, \nu' \in \mathcal{U} \),

\[
(v \vee \nu')_s = \begin{cases} 
  0, & s \in [0, t]; \\
  \nu_s, & s \in (t, T], \text{ on } \{ J(t, x; \nu) \geq J(t, x; \nu') \}; \\
  \nu'_s, & s \in (t, T], \text{ on } \{ J(t, x; \nu) < J(t, x; \nu') \}.
\end{cases}
\]

Therefore,

\[
J(t, x; v \vee \nu') \geq J(t, x; v) \vee J(t, x; \nu').
\]

Set \( \bar{v}^1 := u^1 \vee v^1, \bar{v}^k := \bar{v}^{k-1} \vee v^i, i \geq 2 \). So \( u(t, x) = \lim_{k \to \infty} J(t, x; \bar{v}^k) \). Without loss of generality, suppose \( \hat{E}[ (u(t, x) - J(t, x; \bar{v}^k))^2 ] \leq 1/k, k \geq 1 \). We denote

\[
\bar{v}^k_s = \sum_{j,k=0}^{N_i-1} \bar{v}_{j,k}(s)1_{A^i},
\]

where \( \bar{v}_{j,k} \in M^G_{\nu}(t, s; \mathbb{R}^n) \) is a \( \mathbb{V} \)-values provess, \( \{ A^i \}_{i \in N \setminus N_i-1} \) is a partition of \( \mathcal{B}(\Omega) \).

Then we can suppose for \( k \geq 1 \), \( \hat{E}[ \int_t^T |\bar{v}^k_s - \bar{v}^i_s|^2 \, ds ] \leq \frac{1}{ck} \). From Proposition 3.2, we have

\[
\hat{E}[ |J(t, x; \bar{v}^k) - J(t, x; \bar{v}^i)|^2 ] \leq C \hat{E}[ \int_t^T |\bar{v}^k_s - \bar{v}^i_s|^2 \, ds ] \leq \frac{1}{k},
\]

Therefore, \( \hat{E}[|u(t, x) - J(t, x; \bar{v}^k)|^2 ] \leq \frac{4}{k} \). Then we have

\[
J(t, x; \bar{v}^k) = \sum_{j=0}^{N_i-1} 1_{A^j} J(t, x; \bar{v}_{j,k}) \leq u(t, x).
\]

Now we suppose that

\[
J(t, x, \bar{v}) \leq \max_{0 \leq j \leq N_i-1} J(t, x; \bar{v}_{j,k}) = J(t, x; \bar{v}_{j,k}).
\]

Because \( \hat{E}[ |J(t, x; \bar{v}^k) - u(t, x)|^2 ] \to 0 \), we have

\[
u(t, x) = \lim_{k \to \infty} J(t, x; \bar{v}_{j,k}), \text{ q.s.}
\]

Hence \( \hat{E}[u(t, x)] = u(t, x) \). We have finished the proof.

\( \square \)
Lemma 3.4. For any $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, we have

$$|u(t, x) - u(t, x')| \leq C|x - x'|,$$

(3.4)

$$|u(t, x)| \leq C(1 + |x|).$$

(3.5)

Proof. By Proposition 3.2, we have for $\nu(\cdot) \in \mathcal{U}$,

$$|\vartheta(t, x; \nu(\cdot))| \leq C(1 + |x|),$$

$$|\vartheta(t, x; \nu(\cdot)) - \vartheta(t, x'; \nu(\cdot))| \leq C|x - x'|.$$

Then, $\forall \varepsilon > 0$, there exist $\nu(\cdot), \nu'(\cdot) \in \mathcal{U}$ such that

$$\vartheta(t, x; \nu(\cdot)) \leq u(t, x) \leq \vartheta(t, x; \nu(\cdot)) + \varepsilon,$$

$$\vartheta(t, x'; \nu'(\cdot)) \leq u(t, x') \leq \vartheta(t, x'; \nu'(\cdot)) + \varepsilon.$$

Now we have

$$-C(1 + |x|) \leq \vartheta(t, x; \nu(\cdot)) \leq u(t, x) \leq \vartheta(t, x; \nu(\cdot)) + \varepsilon \leq C(1 + |x|) + \varepsilon.$$

So we get (3.5). Similarly, we obtain

$$\vartheta(t, x; \nu'(\cdot)) - \vartheta(t, x'; \nu'(\cdot)) - \varepsilon \leq u(t, x) - u(t, x') \leq \vartheta(t, x; \nu(\cdot)) - \vartheta(t, x'; \nu(\cdot)) + \varepsilon.$$

Then

$$-C|x - x'| - \varepsilon \leq |u(t, x) - u(t, x')| \leq C|x - x'| + \varepsilon.$$

Thus we have proved (3.4). □

Lemma 3.5. For any $t \in [0, T]$, $\zeta \in L^2_\mathcal{F}_t(\Omega; \mathbb{R}^n)$ and $\zeta$ is $\mathcal{F}_t^\mathcal{B}$ measurable, we have $\forall \nu(\cdot) \in \mathcal{U}$,

$$u(t, \zeta) \geq Y_{t}^{t,\zeta, \nu}.$$ 

(3.6)

Conversely, $\forall \varepsilon > 0$, there exists a $\nu(\cdot) \in \mathcal{U}$, such that

$$u(t, \zeta) \leq Y_{t}^{t,\zeta, \nu} + \varepsilon.$$ 

(3.7)

Proof. We already know that $u(t, x)$ is continuous with respect to $x$ and $Y_{t}^{t,\zeta, \nu}$ is continuous with respect to $(\zeta, \nu(\cdot))$. We want to prove (3.6), only need to discuss the simple random variables $\zeta$ of the form

$$\zeta = \sum_{i=1}^{N} 1_{A_{i}} x_{i},$$

where $A_{i}$ are mutually exclusive events and $x_{i}$ are random variables with $\mathbb{E}[x_{i}^2] < \infty$. We have

$$u(t, \zeta) = \sum_{i=1}^{N} u(t, 1_{A_{i}} x_{i}).$$

Now we can use the continuity of $u(t, x)$ to show that $u(t, \zeta)$ is continuous with respect to $\zeta$. □
and \( \nu(\cdot) \) of the form
\[
\nu(\cdot) = \sum_{i=1}^{N} 1_{A_i} \nu^i(\cdot).
\]

Here \( i = 1, 2, \ldots, N \), \( x_i \in \mathbb{R}^n \), \( \nu^i \in M_{G_t}^{p,t}(t, s; \mathbb{R}^n) \) and \( \{A_i\}_{i=1}^{N} \) is a \( \mathcal{B}(\Omega) \)-partition. Then from the same technique used in the proof of Theorem 4.4 in [11], we have
\[
Y_t^{\xi,\nu} = \sum_{i=1}^{N} 1_{A_i} Y_t^{i,\nu} \leq \sum_{i=1}^{N} 1_{A_i} u(t, x_i) = u(t, \sum_{i=1}^{N} 1_{A_i} x_i) = u(t, \xi).
\]

So we have proved (3.6). Now we prove (3.7) in a similar way. We first construct a random variable \( \eta \in L^2_G(\Omega; \mathbb{R}^n) \),
\[
\eta = \sum_{i=1}^{N} x_i 1_{A_i},
\]
where \( \{A_i\}_{i=1}^{N} \) is a \( \mathcal{B}(\Omega) \)-partition and \( x_i \in \mathbb{R}^n \), such that \( |\eta - \xi| \leq \frac{\varepsilon}{3} \). Then we have
\[
|Y_t^{\xi,\nu} - Y_t^{\eta,\nu}| \leq \frac{\varepsilon}{3},
\]
\[
|u(t, \xi) - u(t, \eta)| \leq \frac{\varepsilon}{3},
\]
for \( \nu(\cdot) \in \mathcal{U} \). Now, we chose a control \( \nu^i(\cdot) \in M_{G_t}^{p,t}(t, s; \mathbb{R}^n) \), such that \( u(t, x_i) \leq Y_t^{i,\nu} + \xi \). Set \( \nu(\cdot) := \sum_{i=1}^{N} \nu^i(\cdot) 1_{A_i} \). Finally, we get
\[
Y_t^{\xi,\nu} \geq -|Y_t^{\eta,\nu} - Y_t^{\xi,\nu}| + Y_t^{\eta,\nu}
\]
\[
\geq -\frac{\varepsilon}{3} + \sum_{i=1}^{N} Y_t^{i,\nu} 1_{A_i}
\]
\[
\geq -\frac{\varepsilon}{3} + \sum_{i=1}^{N} (u(t, x_i) - \frac{\varepsilon}{3}) 1_{A_i}
\]
\[
= -\frac{2\varepsilon}{3} + \sum_{i=1}^{N} u(t, x_i) 1_{A_i}
\]
\[
= -\frac{2\varepsilon}{3} + u(t, \eta) \geq -\varepsilon + u(t, \xi).
\]

So we have (3.7).
Now we give a type of DPP for our stochastic optimal control problems. Firstly, we define a family of backward semigroups associated with the G-BSDE (3.2). Given the initial data \((t, x)\), a positive number \(\delta \leq T - t\) and a random variable \(\eta \in L^p_G(\Omega; \mathbb{R})\) with \(p > 1\), we set

\[
G^{t, x, \eta}_{t, t+\delta} := Y^{t, x, \eta}_s,
\]

where \((Y^{t, x, \eta}_s)_{t \leq s \leq t+\delta}\) is the solution of the following G-BSDE with the time horizon \(t + \delta\):

\[
Y^{t, x, \eta}_s = \eta + \int_s^{t+\delta} f(r, X^{t, x, \eta}_r, Y^{t, x, \eta}_r, Z^{t, x, \eta}_r, \nu_r) dr - \int_s^{t+\delta} Z^{t, x, \eta}_r dB_r - (K^{t, x, \eta}_T - K^{t, x, \eta}_t)
\]

\[
+ \sum_{i,j=1}^d \int_s^{t+\delta} g_{ij}(r, X^{t, x, \eta}_r, Y^{t, x, \eta}_r, Z^{t, x, \eta}_r, \nu_r) d\langle B^i, B^j \rangle_r.
\]

Obviously, for the solution \(Y^{t, x, \eta}_s\) of G-BSDE (3.2), we have

\[
G^{t, x, \eta}_{t, T}[\Phi(X^{t, x, \eta}_T)] = G^{t, x, \eta}_{t, t+\delta}[Y^{t, x, \eta}_{t+\delta}].
\]

Then we can obtain the DPP for our stochastic optimal control problems as follow:

**Theorem 3.6.** The value function \(u(t, x)\) have the following proposition: for every \(0 \leq \delta \leq T - t\), we have

\[
u(t, x) = \sup_{\nu(\cdot) \in \mathcal{U}} G^{t, x, \eta}_{t, t+\delta}[u(t + \delta, X^{t, x, \eta}_{t+\delta})].
\] (3.8)

**Proof.** We have

\[
u(t, x) = \sup_{\nu(\cdot) \in \mathcal{U}} G^{t, x, \eta}_{t, T}[\Phi(X^{t, x, \eta}_T)] = \sup_{\nu(\cdot) \in \mathcal{U}} G^{t, x, \eta}_{t, t+\delta}[Y^{t+\delta, x, \eta}_{t+\delta}].
\]

Obviously, \(X^{t, x, \eta}_{t+\delta}\) is \(\mathcal{F}_{t+\delta}\)-measurable. So by Lemma 3.5 and Theorem 2.19, we have

\[
u(t, x) \leq \sup_{\nu(\cdot) \in \mathcal{U}} G^{t, x, \eta}_{t, t+\delta}[\nu(t + \delta, X^{t, x, \eta}_{t+\delta})].
\]

Besides, for \(\varepsilon > 0\), there exists an admissible control \(\tilde{\nu}(\cdot) \in \mathcal{U}\) such that

\[
u(t + \delta, X^{t, x, \eta}_{t+\delta}) \leq Y^{t+\delta, x, \eta}_{t+\delta} + \varepsilon.
\]
Then

\[ u(t, x) \geq \sup_{\nu(\cdot) \in U} G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, X^{t, x, \nu}_{t+\delta}) - \varepsilon] \]

\[ \geq \sup_{\nu(\cdot) \in U} G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, X^{t, x, \nu}_{t+\delta})] - C\varepsilon. \]

Because \( \varepsilon \) can be arbitrarily small, we get (3.8).

**Proposition 3.7.** \( u(t, x) \) is \( \frac{1}{2} \)-Hölder continuous in \( t \).

**Proof.** For any given \((t, x) \in [0, T] \times \mathbb{R}^n \) and \( \delta > 0(\delta + \varepsilon \leq T) \), from Theorem 3.6, we know that for \( \varepsilon > 0 \), there exists a \( \nu(\cdot) \in U \) such that

\[ G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, X^{t, x, \nu}_{t+\delta})] + \varepsilon \geq u(t, x) \geq G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, X^{t, x, \nu}_{t+\delta})]. \]

Then we need to prove

\[ u(t, x) - u(t + \delta, x) \leq C\delta^{\frac{1}{2}} \text{ (respectively, } \geq -C\delta^{\frac{1}{2}} \text{).} \] (3.9)

We only check the first inequality in (3.9). The second can be proved similarly.

We have \( \forall \varepsilon > 0, \)

\[ u(t, x) - u(t + \delta, x) \leq I_1^{\delta} + I_2^{\delta} + \varepsilon, \] (3.10)

where

\[ I_1^{\delta} = G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, X^{t, x, \nu}_{t+\delta})] - G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, x)] \]

\[ I_2^{\delta} = G^{t, x, \nu}_{t, t+\delta}[u(t + \delta, x)] - u(t + \delta, x). \]

From Proposition 3.1, we have

\[ \hat{E}_t[|X^{t, x, \nu}_{t+\delta} - x|^2] \leq C(1 + |x|^2)\delta. \]

By proposition 3.2 and Lemma 3.4, we deduce that

\[ |I_1^{\delta}| \leq [C\hat{E}_t[|u(t + \delta, X^{t, x, \nu}_{t+\delta}) - u(t + \delta, x)|^2]]^{\frac{1}{2}} \leq [C\hat{E}_t[|X^{t, x, \nu}_{t+\delta} - x|^2]]^{\frac{1}{2}} \leq C^\prime \delta^{\frac{1}{2}}. \]
Based on the definition of $G_{t,i+\delta}$, we get

$$ I_\delta^2 = \hat{E}_i[u(t + \delta, x) + \int_t^{t+\delta} f(s, X_s^{l,x,u}, Y_s^{l,x,u}, Z_s^{l,x,u}, \nu_s)ds$$
$$ + \sum_{i,j=1}^d \int_t^{t+\delta} g_{ij}(s, X_s^{l,x,u}, Y_s^{l,x,u}, Z_s^{l,x,u}, \nu_s)d\langle B^i, B^j_s \rangle$$
$$ - \int_t^{t+\delta} Z_s^{l,x,u} dB_s - (K_T^{l,x,u} - K_t^{l,x,u})] - u(t + \delta, x)$$
$$ = \hat{E}_i[\int_t^{t+\delta} f(s, X_s^{l,x,u}, Y_s^{l,x,u}, Z_s^{l,x,u}, \nu_s)ds$$
$$ + \sum_{i,j=1}^d \int_t^{t+\delta} g_{ij}(s, X_s^{l,x,u}, Y_s^{l,x,u}, Z_s^{l,x,u}, \nu_s)d\langle B^i, B^j_s \rangle]$$
$$ \leq C' \delta^2(1 + \hat{E}_i[\int_t^{t+\delta} |X_s^{l,x,u}|^2 + |Y_s^{l,x,u}|^2 + |Z_s^{l,x,u}|^2 ds]^\frac{1}{2}).$$

By Proposition 3.2, we can prove the following inequality easily by the similar method in Proposition 3.5 of [15]

$$ \hat{E}_i[\int_t^{t+\delta} |Z_s^{l,x,u}|^2 ds]^\frac{1}{2} \leq C(1 + |x|).$$

So we have $I_\delta^2 \leq C' \delta^2$. Hence, by (3.10) we have

$$ u(t, x) - u(t + \delta, x) \leq C' \delta^2 + \epsilon.$$

Let $\epsilon \rightarrow 0$, we obtain the first inequality of (3.9). The proof is completed. $\Box$

### 4 Value Function and Viscosity Solution of Fully Nonlinear Second-Order Partial Differential Equation

In this section, we consider the following fully nonlinear second-order partial differential equation

$$\left\{ \begin{array}{l}
\partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, \ (t, x) \in [0, T] \times \mathbb{R}^n, \\
u(T, x) = \Phi(x),
\end{array} \right.$$  

(4.1)
where
\[
F(D^2_x u, D_x u, u, x, t) = \sup_{v \in V}\{G(H(D^2_x u, D_x u, u, x, t, v)) + \langle b(t, x, v), D_x u \rangle + f(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \ldots, \langle \sigma_d(t, x), D_x u \rangle, v)\},
\]

\[
H_{ij}(D^2_x u, D_x u, u, x, t, v) = \langle D^2_x u \cdot \sigma_i(t, x, v), \sigma_j(t, x, v) \rangle + 2\langle D_x u, h_{ij}(t, x, v) \rangle + 2g_{ij}(t, x, u, \langle \sigma_1(t, x, v), D_x u \rangle, \ldots, \langle \sigma_d(t, x, v), D_x u \rangle, v).
\]

**Remark 4.1.** The definition and uniqueness of viscosity solution of above second-order partial differential equation can be found in Appendix C in Peng [16]. So we only need to prove that \(u(t, x)\) is a viscosity solution of equation (4.1). Besides, from the result of section 3, we can have that \(u(t, x)\) is continuous in \([0, T] \times \mathbb{R}^n\).

**Definition 4.2.** A real-valued continuous function \(u(t, x) \in C([0, T] \times \mathbb{R}^n), u(T, x) \leq \Phi(x)\), for any \(x \in \mathbb{R}^n\), is called a viscosity sub-solution (super-solution) of (4.1), if for all functions \(\varphi \in C^{2,3}([0, T] \times \mathbb{R}^n)\) satisfy \(\varphi \geq u\) and \(\varphi(t, x) = u(t, x)\) at fixed \((t, x) \in [0, T] \times \mathbb{R}^n\), we have
\[
\partial_t \varphi(t, x) + F(D^2_x \varphi(t, x), D_x \varphi(t, x), \varphi(t, x), x, t) \geq 0 (\leq 0).
\]

**Theorem 4.3.** Under the assumptions (H3) and (H4), the value function \(u(t, x)\) defined by (3.3) is a viscosity solution of equation (4.1).

In order to prove the Theorem, we need three Lemma. Firstly, we set
\[
F_1(r, x, y, z, v) = \langle b(r, x, v), D_x \varphi(r, x) \rangle + \partial_t \varphi(t, x)
+ f(r, x, y + \varphi(r, x), z + (\langle \sigma_1(t, x, v), D_x \varphi(r, x) \rangle, \ldots, \langle \sigma_d(t, x, v), D_x \varphi(r, x) \rangle, v),
\]

\[
F_{ij}^2(r, x, y, z, v)
= \langle D_x \varphi(r, x), h_{ij}(r, x, v) \rangle + \frac{1}{2}\langle D^2_x \varphi(r, x)\sigma_i(r, x, v), \sigma_j(r, x, v) \rangle
+ g_{ij}(r, x, y + \varphi(r, x), z + (\langle \sigma_1(t, x, v), D_x \varphi(r, x) \rangle, \ldots, \langle \sigma_d(t, x, v), D_x \varphi(r, x) \rangle, v).
\]

Then we consider a G-BSDE defined on the interval \([t, t + \delta] (0 < \delta \leq T - t)\):
\[
Y_{s}^{i, u} = \int_{s}^{t+\delta} F_1(r, X_r^{t, u}, Y_r^{i, u}, Z_r^{i, u}, \nu_r)dr + \int_{s}^{t+\delta} Z_r^{i, u}dB_r - (K_t^{i, u} - K_{t+\delta}^{i, u})
- \sum_{i=1}^{d} \int_{s}^{t+\delta} F_{ij}^2(r, X_r^{t, u}, Y_r^{i, u}, Z_r^{i, u}, \nu_r)d\langle B^i, B^j \rangle_r,
\]

where \(\nu(\cdot) \in \mathcal{U}\) and \(X_t^{i, u}\) defined by (3.1).
Lemma 4.4. For \( s \in [t, t + \delta] \), we have

\[
G_{s,t+\delta}^{t,s} \left[ \varphi(X_{t+\delta}^{t,s}, t + \delta) \right] = \varphi(X_{s}^{t,s}, s)
\]

is the solution of (4.2).

**Proof.** From the definition of \( G_{s,t+\delta}^{t,s} \), we know that \( G_{s,t+\delta}^{t,s} \left[ \varphi(X_{t+\delta}^{t,s}, t + \delta) \right] \) is the solution of G-BSDE (3.2) on \([t, t + \delta]\) with terminal condition \( \varphi(X_{t+\delta}^{t,s}, t + \delta) \). Applying Itô’s formula to \( \varphi(X_{s}^{t,s}, s) \), we can obtain the result. \( \square \)

Now we construct a simple G-BSDE by replacing the driving process \( X_{s}^{t,s} \) by its deterministic initial value \( x \) as follow:

\[
Y_{s}^{2,u} = \int_{s}^{t+\delta} F_{1}(r, x, Y_{r}^{2,u}, Z_{r}^{2,u}, \nu_{r})dr + \sum_{i,j=1}^{d} \int_{s}^{t+\delta} F_{2}^{ij}(r, x, Y_{r}^{2,u}, Z_{r}^{2,u}, \nu_{r})d\langle B^{i}, B^{j}\rangle_{r}
\]

\[
- \int_{s}^{t+\delta} Z_{r}^{2,u}dB_{r} - (K_{t+\delta}^{1} - K_{t}^{1}) \tag{4.3}
\]

**Lemma 4.5.** We have the following estimate, for \( \nu(\cdot) \in \mathcal{U} \),

\[
|Y_{t}^{1,u} - Y_{t}^{2,u}| \leq C\delta^{\frac{3}{2}}.
\]

Where \( C \) is independent of the control processes \( \nu(\cdot) \).

**Proof.** By proposition 3.1, we have the estimate for \( p \geq 2 \)

\[
\mathbb{E}_{s}[\sup_{x \in [t, t+\delta]} |X_{s}^{t,s} - x|^{p}] \leq C(1 + |x|^{p})\delta^{\frac{p}{2}}.
\]

By proposition 2.18, we get for fixed \( p > 2 \) and \( 2 < p < \beta \),

\[
|Y_{t}^{1,u} - Y_{t}^{2,u}|^{2} \leq \mathbb{E}_{s}[\sup_{x \in [t, t+\delta]} |Y_{t}^{1,u} - Y_{t}^{2,u}|^{2}]
\]

\[
\leq C(\mathbb{E}_{s}[\sup_{x \in [t, t+\delta]} \mathbb{E}_{s}\left( \int_{t}^{t+\delta} \hat{F}_{s}(dr)^{p} \right))]^{\frac{\beta}{p}} + \mathbb{E}_{s}[\sup_{x \in [t, t+\delta]} \mathbb{E}_{s}\left( \int_{t}^{t+\delta} \hat{F}_{s}(dr)^{p} \right)),
\]

where

\[
\hat{F}_{s} = |F_{1}(r, X_{r}^{t,s}, Y_{r}^{2,u}, Z_{r}^{2,u}, \nu_{r}) - F_{1}(r, x, Y_{r}^{2,u}, Z_{r}^{2,u}, \nu_{r})|
\]

\[
+ \sum_{i,j=1}^{d} |F_{2}^{ij}(r, X_{r}^{t,s}, Y_{r}^{2,u}, Z_{r}^{2,u}, \nu_{r}) - F_{2}^{ij}(r, x, Y_{r}^{2,u}, Z_{r}^{2,u}, \nu_{r})|.
\]

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It is easy to prove that
\[ \hat{F}_r \leq C|X_r^{x,y} - x|. \]
Then we can deduce that \( |Y_t^{1,y} - Y_t^{2,y}| \leq C\delta^2 \).

Lemma 4.6. We have
\[ \sup_{\nu \in \mathbb{U}} Y_t^{2,y} = Y^0(t), \]
where \( Y_0(\cdot) \) is the solution of the following ODE:
\[
\left\{ \begin{array}{l}
-dY^0_s = F^0(s, x, Y^0_r, 0)ds, \quad s \in [t, t + \delta], \\
Y^0_{t+\delta} = 0,
\end{array} \right.
\]
where \( F^0(r, x, y, z) = \sup_{\nu \in \mathbb{V}} \{ F^1(r, x, y, z, \nu) + 2G[(F^{ij}_2(r, x, y, z, \nu))^d_{i,j=1}] \} \).

Proof. By Theorem 2.16, we know that the G-BSDE (4.3) have a unique solution \((Y, Z, K)\). Hence there exists a process
\[
V^{2,y}_s = \sum_{i,j=1}^d \int_t^s F^1_{ij}(r, x, Y^{2,y}_r, Z^{2,y}_r, \nu)dB^i_r, \int_t^s 2G[(F^{ij}_2(r, x, Y^{2,y}_r, Z^{2,y}_r, \nu))^d_{i,j=1}]dr.
\]
Here \( V^{2,y}_s, s \in [t, t + \delta] \) is a decreasing and continuous process by [9]. Besides, it satisfies \( \hat{E}[ \sup_{s \in [t, t + \delta]} |V^{2,y}_s| ] < \infty \) obviously. So \( Y_s^{2,y} \) is the solution of the following G-BSDE:
\[
Y^{2,y}_s = \int_t^{t+\delta} \{ F^1(r, x, Y^{2,y}_r, Z^{2,y}_r, \nu_r) + 2G[(F^{ij}_2(r, x, Y^{2,y}_r, Z^{2,y}_r, \nu))^d_{i,j=1}] \} dr - \int_t^{t+\delta} Z^{2,y}_r dB_r - (K^{2,y}_{t+\delta} - K^{2,y}_t) + V^{2,y}_{t+\delta} - V^{2,y}_s,
\]
where \( \nu(\cdot) \in \mathbb{U} \). In addition, we have
\[
Y^0_t = \int_t^{t+\delta} F^0(r, x, Y^0_r, Z^0_r)dr - \int_t^{t+\delta} Z^0_r dB_r - (K^0_{t+\delta} - K^0_t) + (V^0_{t+\delta} - V^0_t),
\]
where \((Z, K, V) = 0\). By the comparison theorem 2.20 and the definition of \( F^0 \), we have for \( \nu(\cdot) \in \mathbb{U} \),
\[
Y_s^{2,y} \leq Y_s^0, s \in [t, t + \delta].
\]
On the other hand, there exists a measurable function $\nu'(r, x, y, z) : [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to V$ such that

$$F^0(r, x, y, z) = F_1(r, x, y, z, \nu') + 2G[(F_{ij}^j(r, x, y, 0, \nu'))_{i,j=1}] .$$

Then we have $\nu'(r, x, Y^0_r, Z^0_r) \in \mathcal{U}$ and $Y^0_t$ is the solution of following G-BSDE:

$$Y^0_s = \int_s^{t+\delta} F_1(r, x, Y^0_r, Z^0_r, \nu'_s)dr + \sum_{i,j=1}^d \int_s^{t+\delta} F_{ij}^j(r, x, Y^0_r, Z^0_r, \nu'_s)d\langle B^i, B^j \rangle_r - \int_s^{t+\delta} Z^0_r dB_r - (K^0_{t+\delta} - K^0_t),$$

where $Z^0_{r,v} = 0$,

$$K^0_s = \sum_{i,j=1}^d \int_t^s F_{ij}^j(r, x, Y^0_r, 0, \nu')d\langle B^i, B^j \rangle_r - \int_t^s 2G[(F_{ij}^j(r, x, Y^0_r, 0, \nu'))_{i,j=1}]dr .$$

So $Y^0_t \leq \sup_{\nu(\cdot) \in \mathcal{U}} Y^2_{t,\nu}$. Now we have proved the lemma.

Then we give the proof of Theorem 4.3:

**Proof.** We set $\varphi \in C^{2,3}([0, T] \times \mathbb{R}^n)$ and $\varphi(t, x) = u(t, x)$ for fixed $(t, x) \in [0, T] \times \mathbb{R}^n$.

From Theorem 3.6, we know

$$\varphi(t, x) = u(t, x) = \sup_{\nu(\cdot) \in \mathcal{U}} G^{t,x,\nu}_{t,t+\delta}[u(X^{t,x}_{t+\delta}, t + \delta)].$$

By $\varphi \geq u(\varphi \leq u)$ and the definition of $G$

$$\sup_{\nu(\cdot) \in \mathcal{U}} [G^{t,x,\nu}_{t,t+\delta}[u(X^{t,x}_{t+\delta}, t + \delta)] - \varphi(t, x)] \geq 0(\leq 0).$$

Then from lemma 4.4

$$\sup_{\nu(\cdot) \in \mathcal{U}} Y_{t,\nu}^1 \geq 0(\leq 0).$$

Besides from lemma 4.5

$$\sup_{\nu(\cdot) \in \mathcal{U}} Y_{t,\nu}^2 \geq C\delta^\frac{1}{2}(\leq C\delta^\frac{3}{2}).$$

Finally, lemma 4.6 implies

$$Y^0(t) \geq C\delta^\frac{1}{2}(\leq C\delta^3).$$

So $F^0(r, x, 0, 0) \geq 0(\leq 0)$ and from the definition of viscosity solution of equation (4.1), we know $u(t, x)$ is a viscosity solution of equation (4.1).
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