THE COARSE QUOTIENT FOR AFFINE WEYL GROUPS AND
PSEUDO-REFLECTION GROUPS

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Abstract. We study the coarse quotient $\mathfrak{t}^* \sslash W^{\text{aff}}$ of the affine Weyl group $W^{\text{aff}}$ acting on a dual Cartan $\mathfrak{t}^*$ for some semisimple Lie algebra. Specifically, we classify sheaves on this space via a ‘pointwise’ criterion for descent, which says that a $W^{\text{aff}}$-equivariant sheaf on $\mathfrak{t}^*$ descends to the coarse quotient if and only if the fiber at each field-valued point descends to the associated GIT quotient.

We also prove the analogous pointwise criterion for descent for an arbitrary finite group acting on a vector space. Using this, we show that an equivariant sheaf for the action of a finite pseudo-reflection group descends to the GIT quotient if and only if it descends to the associated GIT quotient for every pseudo-reflection, generalizing a recent result of Lonergan.

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1. Introduction

The goal of this paper is to study the coarse quotient $\mathfrak{t}^* \sslash W^{\text{aff}}$ for use in Gan22. While this quotient appears naturally in many representation theoretic contexts (see Section 1.3), its global geometry makes direct analysis of this space difficult—for example, it is not a scheme or algebraic space, and its diagonal map is not quasicompact. However, in this paper we argue that many questions about $\mathfrak{t}^* \sslash W^{\text{aff}}$ and its category of sheaves can be understood through the use of field-valued points. Specifically, we show the following, which we also use to re-derive the main results of Lon17.
Theorem 1.1. (Theorem 4.18 Theorem 4.23) The pullback map induced by $t^*/W^{\text{aff}} \to t^*/W^{\text{aff}}$ is fully faithful. Moreover, a $W^{\text{aff}}$-equivariant sheaf $\mathcal{F}$ on $t^*$ lies in the essential image of this pullback if and only if at every field-valued point $x$ of $t^*$, the $W_x$-representation on the fiber of $\mathcal{F}$ at $x$ is trivial.

Field-valued points can also be used to better understand the coarse quotient $V//H := \text{Spec}(\text{Sym}(V^*)^H)$ in the classical setting of a finite group $H$ acting on a finite dimensional $k$-vector space $V$. For example, using field-valued points, we generalize a result of Lonergan [Lon17] on descent to the coarse quotient for finite Coxeter groups to the setting of pseudo-reflection groups, see Theorem 1.6. After giving the definition of the coarse quotient in Section 1.1, we discuss these applications in Section 1.2, and then give some motivation for the study of the coarse quotient in Section 1.3.

1.1. The Coarse Quotient $t^*/W^{\text{aff}}$. Fix a reductive group $G$ with choice of maximal torus $T$ and Borel subgroup $B \supset T$ and let $t$ denote the Lie algebra of $T$. As discussed above, this paper studies the coarse quotient $t^*/W^{\text{aff}}$, where $W^{\text{aff}} := Z\Phi \rtimes W$ denotes the affine Weyl group for $G$, $W := N_G(T)/T$ is the (finite) Weyl group and $Z\Phi$ denotes the root lattice of $G$. Here, we discuss desired properties of the coarse quotient $t^*/W^{\text{aff}}$ and give its formal definition.

1.1.1. Introduction to the Coarse Quotient $t^*/W^{\text{aff}}$. The coarse quotient $t^*/W^{\text{aff}}$ is defined to serve as an analogue of the GIT quotient for the (in general infinite) group $W^{\text{aff}}$. However, one of the first obstructions which appears when working with the coarse quotient $t^*/W^{\text{aff}}$ is defining it so as to satisfy similar properties to the usual GIT quotients of finite groups. We survey some of these properties of the usual GIT quotient $V//H$ of a Weyl group $H$ acting on some finite dimensional $k$-vector space $V$. Notice that, by definition of the GIT quotient for affine schemes,

$$\mathcal{O}(V//H) = \mathcal{O}(V)^H \simeq \lim \mathcal{O}(V) = \mathcal{O}(V \otimes_{\mathcal{O}(V)^H} \mathcal{O}(V))$$

where rightmost maps are induced by the projections and the limit is taken in the category of ordinary $k$-algebras. When $H$ is a Weyl group, the $k$-algebra $\mathcal{O}(V \otimes_{\mathcal{O}(V)^H} \mathcal{O}(V))$ admits a description as the union of graphs $\Gamma_H$ of $H$ as a closed subscheme of $V \times V$, which we review in Proposition A.2. In particular, we see that

(1) the map $\Gamma_H \to V \times_{V//H} V$ is an equivalence

and that, by definition, the affine scheme

(2) $V//H$ is a quotient of $V$ by the groupoid $\Gamma_H$.

This description shows moreover shows that

(3) the map $V(k)/H \to (V//H)(k)$ is a bijection.

Another useful feature of the quotient $V//H$ which, for example, distinguishes it from the stack quotient $V//H$, is that the fiber of the quotient map depends on the choice of point of the codomain. Specifically, we have that

(4) for any $\lambda \in V(k)$, we have an isomorphism $\{\lambda\} \times_{V//H} V \cong H_\lambda \times \text{Spec}(C_\lambda)$

where $C_\lambda$ is a certain Artinian local ring determined by the action of the action of the stabilizer $H_\lambda$ of $\lambda$ on $V$. For example, by definition one has an equivalence

$$\{0\} \times_{V//H} V \cong \text{Spec}(C_0)$$

where $C_0 = \text{Sym}(V^*)/\text{Sym}(V^*)_H^H$ is the coinvariant algebra. On the other hand, if $H$ acts faithfully, then the action of $H$ on $V$ is generically free. In particular, for a generic $k$-point $\lambda$ of $V$, the orbit map gives an equivalence

$$\{\lambda\} \times_{V//H} V \cong \coprod_{h \in H} \text{Spec}(k).$$

Finally, in representation theory, it is often of interest to study coherent sheaves on spaces such that $V//H$ and relate them to equivariant coherent sheaves $\text{Coh}(V)^H$ on $V$. Specifically, if $q : V \to V//H$ denotes the quotient map, then we have that

(5) $q^*$ lifts to a fully faithful functor $\text{Coh}(V//H) \hookrightarrow \text{Coh}(V)^H$ with an easily described essential image as we review in much more detail in Section 2.
1.1.2. Other Potential Definitions of $t^* // W^{aff}$. Using the above desiderata, we may immediately disqualify two potential definitions of $t^* // W^{aff}$:

1. When $G = SL_2$, we may identify $t^* \cong \mathbb{A}^1 = \text{Spec}(k[x])$. In this case, since $\text{Spec}(k[x]^{\mathbb{Z}}) = \text{Spec}(k)$, the na"{i}ve definition for the coarse quotient $\text{Spec}(k[x]^{W^{aff}}) = \text{Spec}(k)$ does not satisfy (1), (3), or (4).

2. If the coarse quotient is replaced with the stack quotient $t^*/W^{aff}$, (4) is not satisfied.

1.1.3. The Coarse Quotient $t^* // W^{aff}$ Via Groupoids. We will define $t^* // W^{aff}$ as a quotient of $t^*$ by the union of graphs $\Gamma$ of $W^{aff}$. However, $\Gamma$ is not a scheme if $G$ is not a torus, and so a choice of ambient category is required to define this quotient analogous to (2). We choose what is essentially the most general setting and define $t^* // W^{aff}$ as a certain colimit in the category of prestacks in the sense of GR17a Chapter 2.1]. Specifically, let $\Gamma_{[n]}$ denote the $n$-fold product $\Gamma \times_{\Gamma} \Gamma \times_{\Gamma} \ldots \times_{\Gamma} \Gamma$. Through the various source and target maps, the $\Gamma_{[n]}$ naturally form a groupoid object $\Gamma_*$, see Section 4.1. We now give a special case of our definition of the coarse quotient:

**Definition 1.2.** The coarse quotient $t^* // W^{aff}$ is the geometric realization of the simplicial object $\Gamma_*$, in other words,

$$t^* // W^{aff} := \text{colim}(\ldots \Gamma \longrightarrow t^* \longrightarrow t^*)$$

in the category of prestacks.

As we review in Remark 4.8 replacing $\Gamma$ with $\Gamma_W$ in Definition 1.2 gives $t^* // W$. By this definition, (2) is vacuously satisfied. One advantage of this definition is that analogues of (1) and (3) hold essentially immediately, as we will see below.

1.1.4. Field-Valued Points. One disadvantage, however, of Definition 1.2 is that the class of prestacks is ‘so general that it is, of course, impossible to prove anything non-trivial about’ GR17b other than the formal properties discussed above. Therefore, much of the work in studying the coarse quotient will focus on showing the analogues of point (1) and (5) from the quotient map $\pi : t^* \rightarrow t^* // W^{aff}$.

One issue, however, which immediately arises when attempting to study $t^* // W^{aff}$ in this way is that Zariski open subsets of $t^*$ are ‘too large.’ For example, when $G = SL_2$, we have

$$W^{aff} = 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and, when considering the action of this group on $\mathbb{A}^1_k$, the $k$-points fixed by some order two subgroup of $W^{aff}$ behave differently than all other $k$-points, on which the action is free. Since the $k$-points fixed by some order two subgroup of $W^{aff}$ are given by a copy of the integers, we see that these points do not form a closed subscheme; they only form an \textit{ind-closed subscheme}. In particular, the subset of points for which the $W^{aff}$-action is free is not a Zariski open subset of $\mathbb{A}^1_k$.

The main way we propose to work around this is to use \textit{field-valued points}, which can serve as a useful substitute when dealing with quotients such as $t^* // W^{aff}$. While in some algebro-geometric applications, generic points can often be more difficult to work with than general closed points, a general feature of representation theory is that $K$-points which do not factor through some $k$-point behave, informally speaking, as if they were generic $k$-points. For example, if $k$ is algebraically closed and $*$ denotes either a non-integer $k$-point of $\mathbb{A}^1_k$ or the generic point of $\mathbb{A}^1_k$, then there is a canonical isomorphism (see Proposition 4.6):

$$\mathbb{A}^1/(2\mathbb{Z}) \times_{\mathbb{A}^1} W^{aff} * \simeq * \prod *$$

whereas if $*$ denotes an integral $k$-point, one has a canonical isomorphism

$$\mathbb{A}^1/(2\mathbb{Z}) \times_{\mathbb{A}^1} W^{aff} * \simeq \text{Spec}(C_0)$$

where in this case $C_0 \cong k[\epsilon]/\epsilon^2$. A generalization of this plays an important role in representation theory, see Section 1.3.1.

1.2. Descent to the Coarse Quotient for Finite Groups. The tools used to study sheaves on the coarse quotient $t^* // W^{aff}$ also can be applied to obtain new results for quotients by reflection groups, and more generally \textit{pseudo-reflection or complex reflection groups}. We survey these results here.
1.2.1. Generalities on Descent to the Coarse Quotient. When studying sheaves on \( V \parallel H \), a key first insight is that any sheaf on \( V \) which is pulled back from \( V \parallel H \) canonically acquires an \( H \)-equivariant structure, or, equivalently, can be viewed as a sheaf on the stack quotient \( V/H \). In the setting of quasicoherent sheaves, it is in fact not too difficult to show that the pullback

\[
\phi^*: \text{QCoh}(V \parallel H) \to \text{QCoh}(V/H)
\]

is fully faithful, as we review in Proposition 2.2.

**Definition 1.3.** We say that \( \mathcal{F} \in \text{QCoh}(V/H) \) *descends to the coarse quotient* \( V \parallel H \) if \( \mathcal{F} \) is in the essential image of the pullback \( \phi^* \).

In the course of studying sheaves on \( \mathfrak{t}^* \parallel \mathcal{W}^\text{aff} \), we prove the following lemma, which says that the condition for a given \( H \)-equivariant sheaf to descend to the coarse quotient can be checked at each field-valued point of \( V \).

**Lemma 1.4.** A given \( \mathcal{F} \in \text{QCoh}(V)^H \) descends to the coarse quotient \( V \parallel H \) if and only if for every field-valued point \( x \in V(K) \), the induced \( H_x \)-representation on \( x^*(\mathcal{F}) \) is trivial, where \( H_x \) denotes the stabilizer of \( x \).

1.2.2. Descent to the Coarse Quotient for Pseudo-Reflection Groups. A case where one can profitably apply Lemma 1.4 is the case where \( H \) acts as a pseudo-reflection group:

**Definition 1.5.** We say that a finite group \( H \) is a *pseudo-reflection group* acting on some vector space \( V \) if \( H \) is generated elements which act by pseudo-reflections on \( V \), which are in turn defined as non-identity endomorphisms\(^1\) of \( V \) which fix a hyperplane of \( V \) pointwise.

The groups acting by pseudo-reflections play a distinguished role in the theory of finite groups acting on vector spaces. For example, the Chevalley-Shephard-Todd theorem (which we review in Theorem 2.3) says that the coarse quotient \( V \parallel H \) is affine space if and only if \( H \) acts on \( V \) as a pseudo-reflection group. Furthermore, a theorem of Steinberg [Ste64] says that if \( H \) is a pseudo-reflection group acting on a vector space, the stabilizer of any point is a pseudo-reflection group as well. Using this and Lemma 1.4, one can derive our first main result:

**Theorem 1.6.** Assume \( H \) is a pseudo-reflection group acting on some vector space \( V \). A sheaf \( \mathcal{F} \in \text{QCoh}(V)^H \) descends to the coarse quotient \( V \parallel H \) if and only if for every pseudo-reflection \( r \in H \), the sheaf \( \text{oblv}^H_{\langle r \rangle}(\mathcal{F}) \in \text{QCoh}(V)^{\langle r \rangle} \) descends to the coarse quotient \( V \parallel \langle r \rangle \).

Using Theorem 1.6 and the fact that any reflection of a Coxeter group is conjugate to a simple reflection, one can derive an alternate proof of the main result of [Lon17]::

**Theorem 1.7.** [Lon17] Assume \( W \) is a Coxeter group with reflection representation \( V \). A given \( \mathcal{F} \in \text{QCoh}(V)^W \) descends to the coarse quotient \( V \parallel W \) if and only if for all simple reflections \( s \in W \), \( \text{oblv}^W_{\langle s \rangle}(\mathcal{F}) \in \text{QCoh}(V)^{\langle s \rangle} \) descends to the coarse quotient \( V \parallel \langle s \rangle \).

1.2.3. Local Descent to the Coarse Quotient. Lemma 1.4 is proved via study of the *coinvariant algebra* \( C \) for the action of \( H \) on \( V \), defined as the ring \( \text{Sym}(V^\vee)/\text{Sym}(V^\vee)^H \), i.e. the quotient of \( \text{Sym}(V^\vee) \) by the ideal generated by homogeneous polynomials of positive degree fixed by \( H \). Since \( H \) acts on \( C \), we similarly obtain a stack quotient \( \text{Spec}(C)/H \) and a coarse quotient \( \text{Spec}(C) \parallel H \); however, the coarse quotient simplifies since

\[
\text{Spec}(C) \parallel H := \text{Spec}(C^H) \cong \text{Spec}(k).
\]

Similarly, we will recall that the pullback functor is fully faithful in Proposition 2.2 and therefore we can analogously define the notion of an object of \( \text{QCoh}^C(\text{Spec}(C))^H \) *descending to the coarse quotient*. A key technical tool in the proof of Lemma 1.4 is the following proposition:

**Proposition 1.8.** A given \( \mathcal{F} \in \text{QCoh}(\text{Spec}(C))^H \) descends to the coarse quotient \( \text{Spec}(C) \parallel H = \text{Spec}(k) \) if and only if the canonical \( H \)-representation on \( i^*(\mathcal{F}) \) is trivial, where \( i : \text{Spec}(k) \to \text{Spec}(C) \) is the inclusion of the unique closed point.

**Remark 1.9.** Our notation is inherently derived—see Section 1.4 for our exact conventions.

\(^1\)All finite order pseudo-reflections are diagonalizable over our characteristic zero field—see, for example, [Kan01, Section 14.3(a)]. In particular, our definition agrees with others found in the literature.
1.3. Motivation. The coarse quotient $t^* \sslash W^{\text{aff}}$ can also be generalized to the coarse quotient of $t^*$ by the action of the extended affine Weyl group $\tilde{W}^{\text{aff}} := X^\bullet(T) \rtimes W$, where $X^\bullet(T)$ is the character lattice for $T$. This coarse quotient is expected to play an important role in geometric representation theory. In this section, we highlight some roles it plays and discuss some future applications.

1.3.1. Representation Theoretic Motivation. Assume, for the sake of exposition, that $G$ is semisimple and simply connected, and recall the BGG Category $O_\lambda$, a certain abelian category of representations with generalized central character $\chi_\lambda$. These categories and categories such as $D(G)^{B \times B}$ (the category of bi-$B$-equivariant $D$-modules on $G$) can be understood through the use of Soergel modules and Soergel bimodules. We also recall that (ungraded) Soergel bimodules can be understood as certain quasicoherent sheaves on the union of graphs of the Weyl group $\Gamma_W$. As we review in Proposition 1.2, we can identify $\Gamma_W \simeq t^* \times_{t^* \mathcal{W}} t^*$.

We can use the coarse quotient $t^* \sslash W^{\text{aff}}$ to define one analogue of $\Gamma_W$ whose fibers, informally speaking, describe a subcategory of the BGG category $O$ of the given central character. One can make this remark precise as follows. First, define:

$$\Gamma' := t^*/X^\bullet(T) \times_{t^* \mathcal{W}^{\text{aff}}} t^*/X^\bullet(T)$$

and let $\lambda \in t^*(k)$ denote some $k$-point. We assume for the sake of exposition that $\lambda$ is an antidominant and regular with respect to the $(W,\cdot)$-action, and denote the associated image under the quotient map by $[\lambda] : \text{Spec}(k) \to t^*/X^\bullet(T)$. Then it is standard (see [Hum08, Chapter 4.9]) that the blocks of $O_\lambda$ are in bijection with the cosets $W/W_{[\lambda]}$, where $W_{[\lambda]}$ denotes the integral Weyl group, see Definition 3.2. This group admits some remarkable properties—e.g., $W_{[\lambda]}$ need not be a parabolic subgroup of the usual Weyl group and, moreover, as the notation suggests, this group only depends on $[\lambda]$. With this setup, we note that from Proposition 4.6 and the Endomorphismensatz [Soe90] one can derive the following:

**Proposition 1.10.** We have $W$-equivariant isomorphisms

$$\Gamma' \times_{t^*/X^\bullet(T)} \text{Spec}(k) \simeq t^*/X^\bullet(T) \times_{t^* \mathcal{W}^{\text{aff}}} \text{Spec}(k) \simeq W^\mathbb{Z} \times \text{Spec}(E_\lambda)$$

where $E_\lambda$ is the endomorphism ring of the projective cover of the simple labelled by $\lambda$.

1.3.2. Connections to Categorical Representation Theory. The coarse quotient $t^* \sslash W^{\text{aff}}$ has appeared recently in providing a coherent description of certain categories associated to the category of $D$-modules on $G$, and more generally plays an important role in categorical representation theory. For example, the coarse quotient implicitly appears in a recent result of Ginzburg and Lonergan:

**Theorem 1.11.** [Lon18, Gin18] There is an equivalence identifying the abelian category of bi-Whittaker $D$-modules on $G$ with the full subcategory of $W^{\text{aff}}$-equivariant sheaves on $t^*$ which descend to the coarse quotient in the sense of Definition 4.22.

It is argued in [BG17] that the equivalence of Theorem 1.11 can be upgraded to a monoidal equivalence conditional on a derived, mixed variant of the geometric Satake theorem, which in particular shows that the convolution monoidal structure on the derived category of bi-Whittaker $D$-modules on $G$ is symmetric. In [Gan22], we prove this upgrade of Theorem 1.11 unconditionally.

In [BG17], an argument is also sketched that there is a central functor from the category of sheaves on the coarse quotient $t^* \sslash W^{\text{aff}}$ to the center of the monoidal category of $D$-modules on $G$, i.e. the category $D(G)^{G}$. It is also conjectured that this functor can be identified with a modified version of parabolic induction [BG17, Conjecture 2.9]. In [Gan22], we use the descent conditions outlined in this paper to construct a quotient of a candidate inverse to this functor. Moreover, this restricted parabolic induction functor can also be used to better understand character $D$-modules on $G$, and was used to prove a conjecture of Braverman and Kazhdan on the acyclicity of $\rho$-Bessel sheaves on reductive groups, see [Che22] and [Che21].

Moreover, in [Gan22], we argue that $t^* \sslash W^{\text{aff}}$ plays an analogous role in the study of $D(N \backslash G/N)$ as $t^*/W$ plays in the study of $D(B \backslash G/B)$, see Section 1.3.3.

**Theorem 1.12.** [Gan22] There is some quotient category of $D(N \backslash G/N)$, denoted $D(N \backslash G/N)_{\text{nondeg}}$, and a monoidal equivalence of categories

$$D(N \backslash G/N)_{\text{nondeg}} \simeq \text{IndCoh}(t^*/X^\bullet(T) \times_{t^* \mathcal{W}^{\text{aff}}} t^*/X^\bullet(T))$$

which is $t$-exact up to cohomological shift.
Moreover, as we will show, the category $D(N\backslash G/N)_{\text{nondeg}}$ can be described explicitly—for example, if $G = SL_2$, we can identify $D(N\backslash G/N)_{\text{nondeg}}$ as the quotient of $D(N\backslash G/N)$ by the full subcategory generated under colimits by the constant sheaf $\mathbf{k}_{N\backslash G/N}$.

1.4. Conventions. We work over $k$, a field of characteristic zero, and by scheme, we mean a $k$-scheme. If unspecified, $K/k$ denotes an arbitrary field extension. What follows is written in the language of derived algebraic geometry in the sense of [GR17b], and, in particular, all categories of sheaves are written as DG categories, or equivalently, $k$-linear presentable stable $\infty$-categories. However, in Section 2 and Section 3, this usage is inessential—we only use this language to parallel the definitions of the coarse quotient $t^* / W^{\aff}$ in Section 4. In particular, the reader only interested in results such as Theorem 1.6 can read Section 2 and Section 3, replacing our notion with the classical derived categories and derived functors between them, and lose no information. However, in Section 4, the usage of higher categories becomes more essential, as $t^* / W^{\aff}$ is most naturally defined as a prestack, see [GR17b] Chapter 2, Section 1.

In particular, the classical abelian category of quasicoherent sheaves on a scheme $X$ (respectively, equivariant sheaves on a scheme $X$ with an $H$-action) on some scheme $X$ will be denoted $\text{QCoh}(X)^\bowtie$ (respectively, $\text{QCoh}(X)^H,^\bowtie$). However, as we will review below in Theorem 2.3, the Chevalley-Shephard-Todd Theorem gives that for any pseudo-reflection group $H$, the pullback functor $\phi^*$ is $t$-exact, and so it induces an exact functor of abelian categories

$$\text{QCoh}(V / H)^\bowtie \xrightarrow{\phi^*} \text{QCoh}(V)^H,^\bowtie$$

and the natural analogue of Theorem 1.6 below holds for abelian categories. However, the abelian categorical analogues of results such as Lemma 1.4 and Proposition 1.8 do not hold, see Example 2.10. The failure of these claims essentially stems from the failure of $i^*$ to be $t$-exact, or equivalently, the failure of the morphism $\text{Spec}(k) \to \text{Spec}(C)$ to be flat.

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2. Descent to the Coarse Quotient for Pseudo-Reflection Groups

2.1. Preliminary Results. We recall that $k$ denotes a field of characteristic zero and, if unspecified, $K/k$ denotes an arbitrary field extension.

2.1.1. GIT Quotients and Stack Quotients. We temporarily assume $H$ is an arbitrary finite group acting on some arbitrary affine scheme $V$. Recall the notion of a GIT quotient as in [MFK94]; we will only use the affine version so we content ourselves to define GIT quotient or coarse quotient as $X \equiv H := \text{Spec}(A^H)$ [MFK94] Theorem 1.1]. Direct computation shows the following result, a modification of which will be used to define the coarse quotient $t^* / W^{\aff}$ in Section 4.

Proposition 2.1. If $H$ is a finite group acting on an affine scheme $X = \text{Spec}(A)$ then the canonical map

$$X \equiv H \simeq \text{colim}(X \times_{X \equiv H} X \equiv X)$$

is an equivalence, where the colimit is taken in the (1,1)-category of classical affine schemes.

Let $\phi : V/H \to V \equiv H$ denote the quotient map. We now recall the following result:

Proposition 2.2. The pullback $\phi^*$ is fully faithful.

Proof. The pullback $(V \to V \equiv H)^*$ admits a continuous right adjoint given by restriction of scalars. This implies that $\phi^*$ also admits a continuous right adjoint (explicitly given by taking the $H$-invariants of the complex of vector spaces) and to show that this functor is fully faithful it suffices to prove that the unit is an isomorphism on the compact generator $A := \text{Sym}(V^\vee)^H$. However, we have by construction that this pullback functor maps $A$ to the classical module $\text{Sym}(V^\vee)$ with its canonical $H$-equivariance, and so in particular we see that the endomorphisms of this module in $\text{QCoh}(V)^H = \text{Sym}(V^\vee)-\text{mod}^H$ is given
exacty by the $H$-fixed points of the endomorphisms of $\text{Sym}(V^\vee)$ as a $\text{Sym}(V^\vee)$-module. Thus, since the endomorphisms of $\text{Sym}(V^\vee)$ as a $\text{Sym}(V^\vee)$-module identify with $\text{Sym}(V^\vee)$ itself, we see that the $H$-fixed points are $\text{Sym}(V^\vee)^H$ as desired.

2.1.2. Results on Pseudo-Reflection Groups. We now recall a key result in the theory of finite group actions on vector spaces:

**Theorem 2.3.** (Chevalley-Shephard-Todd Theorem [Bou68, Chapter 5, Theorem 4], [Che55], [ST54]) If $H$ is any finite group acting on a vector space of dimension $d$, then $V/ H \cong \mathbb{A}^d_k$ if and only if $H$ is a pseudo-reflection group.

**Remark 2.4.** More generally, in fact, the space $V/ H$ is smooth only if $H$ is a pseudo-reflection group. Since we do not use this theorem in any substantial way, we only remark that this theorem is a consequence of Theorem 2.3 and the purity of the branch locus theorem.

We were unable to locate a proof of the following standard result for general pseudo-reflection groups.

**Proposition 2.5.** If $H$ is a pseudo-reflection group acting faithfully on some vector space $V$, then $\text{Sym}(V^\vee)$ is a free $\text{Sym}(V^\vee)^H$-module of rank $|H|$.

**Proof.** The fact that $\text{Sym}(V^\vee)$ is a free $\text{Sym}(V^\vee)^H$-module of some finite rank is standard, see, for example, [Kan01, Chapter 18-3]. To show that this rank is $|H|$, note that since $H$ acts faithfully and $V$ is irreducible, there exists some $H$-invariant nonempty open subset $U \subseteq V$ on which the action is free. Since the quotient map $q : V \to V/ H$ is a flat morphism of finite type Noetherian schemes, the set $q(U)$ is open. Note also that because $q$ is a uniform categorical quotient, Chapter 1, §2, Theorem 1.1], the map $q|_U : U \to q(U)$ is a categorical quotient.

Let $j : q(U) \to V/ H$ denote the open embedding. It now suffices to show that the rank of $j^*(\text{Sym}(V^\vee))$ is $|H|$, and we may show this, in turn, by computing the rank of the free module $q|_U^*: j^*(\text{Sym}(V^\vee))$. However, note that since the action of $H$ on $U$ is free, the leftmost of the following diagrams are Cartesian by Chapter 0, §4, Proposition 0.9:

$\begin{array}{ccc}
H \times U & \xrightarrow{\text{act}} & U \\
\downarrow \text{proj} & & \downarrow q_U \\
U & \xrightarrow{q_U} & q(U) \\
\downarrow & & \downarrow j \\
& & V/ H
\end{array}$

and rightmost box is Cartesian by construction. We then see that we may compute, by base change along the ‘large’ Cartesian diagram of (6) to see that the rank of the free module $q|_U^*: j^*(\text{Sym}(V^\vee))$ is $|H|$ as desired.

We also have the following theorem of Steinberg, stated in terms of our conventions on pseudo-reflection groups:

**Theorem 2.6.** [Ste64, Theorem 1.5] The stabilizer $H_x$ of some $x \in V$ is a a pseudo-reflection group.

2.1.3. The Coinvariant Algebra. Assume $H$ is some finite group acting on a vector space $V$, and let $C_0$ denote the coinvariant algebra of this action, i.e.

$C_0 := \text{Sym}(V^\vee) \otimes_{\text{Sym}(V^\vee)^H} k$

where the ring map $\text{Sym}(V^\vee)^H \to k$ is given by the ring map sending all elements of positive degree to zero. More generally, if $x \in V(K)$ is some field-valued point, we define the algebra $C_x$ as the $K$-algebra

$C_x := \text{Sym}(V^\vee_K) \otimes_{\text{Sym}(V^\vee_K)^H_x} K$

where the ring map $\text{Sym}(V^\vee_K)^H_x \to K$ is given by evaluation at $x$. Note that the map $\tau_x : V_K \to V_K$ given by translation by $x$ gives an isomorphism between $C_x$ and the coinvariant algebra for the action of $H_x$ on $V_K$.  

---

2For a proof in the reflection group case, see [Hum90, Section 3.5].
2.1.4. Nonzero Sheaves on Smooth Schemes Have Nonzero Fiber. We now recall the following result, which says that a given (complex of) quasicoherent sheaves is nonzero only if some fiber at some field-valued point is nonzero.

**Proposition 2.7.** Assume \( X \) is a Noetherian classical scheme and \( \mathcal{F} \in \text{QCoh}(X) \) is nonzero. Then there exists some field-valued point \( x : \text{Spec}(K) \to X \) such that \( x^*(\mathcal{F}) \) is nonzero.

The following proof is due to Arinkin [Ari01, Lemma 10]. We recall it here for the convenience of the reader, and to show that the proof does not require the assumption that the associated complex of \( \mathcal{F} \) be bounded.

**Proof.** Since locally Noetherian schemes are covered by \( \text{Spec}(A) \) for Noetherian \( A \), it suffices to prove this when \( X = \text{Spec}(A) \). Now, let \( M \) denote some \( A \)-module whose fiber vanishes at every field-valued point, and consider all closed subschemes \( i : Z \to X \) such that \( i^*(M) \) is nonzero. Since \( X \) is Noetherian, there is a minimal closed subscheme with this property. Taking the fiber of \( M \) at this closed subscheme, we may replace this minimal closed subscheme by \( X \) itself. In particular, for any \( f \in A \) which is nonzero, since we have a fiber sequence \( A \xrightarrow{f} A \to A/f \), we have that multiplication by any nonzero \( f \in A \) yields an equivalence \( f : - : M \xrightarrow{\sim} M \).

If \( A \) is not an integral domain, then there exists some nonzero \( f, g \in A \) whose product is zero. We therefore see that the zero map is an isomorphism since the composite \( M \xrightarrow{f} M \xrightarrow{g} M \) is an isomorphism. Now assume \( A \) is an integral domain, and fix some \( i \in \mathbb{Z} \). The above analysis gives any nonzero \( f \in A \) acts invertibly on \( H^i(M) \) for all \( i \). Therefore, we have that, as classical \( A \)-modules, \( H^i(M) \cong K \otimes_A H^i(M) \), where \( K \) denotes the field of fractions of \( A \). However, since localization is an exact functor of abelian categories, the functor \( K \otimes_A - \) is a t-exact functor of derived categories. Thus, we have
\[
0 = H^i(K \otimes_A M) \cong K \otimes_A H^i(M) \cong H^i(M)
\]
where the first step uses the assumption that the fiber of \( M \) at every point vanishes. Since \( H^i(M) = 0 \) for all \( i \), we have that \( M \) itself vanishes in \( A\)-mod by the left and right completeness of the t-structure on \( A\)-mod. \( \square \)

2.2. Pointwise Descent to the Coarse Quotient. Assume \( H \) is a finite group which acts on some vector space \( V \), and let \( C \) denote the coinvariant algebra for this action. The main result of this section is the proof of Proposition 1.8 which provides a ‘pointwise’ criterion for an \( H \)-equivariant sheaf in \( \text{QCoh}(\text{Spec}(C)/H) \) to descend to the coarse quotient \( \text{Spec}(C) \parallel H = \text{Spec}(k) \).

**Proposition 2.8.** Let \( \phi : \text{Spec}(C)/H \to \text{Spec}(k) \) denote the terminal map of \( k \)-schemes.

1. The functor \( \phi^* \) is t-exact and fully faithful.
2. Under the equivalence \( \text{QCoh}(\text{Spec}(C)/H) \cong \text{QCoh}(\text{Spec}(C))^H \), the essential image of \( \phi^* \) is the full subcategory generated by object \( C \in \text{QCoh}(\text{Spec}(C))^H \) with its canonical equivariance.

**Proof.** The t-exactness follows since any functor \( F : \text{Vect} \to C \) to some DG category \( C \) equipped with a t-structure which sends the one dimensional vector space \( k \in \text{Vect}^\circ \) to an object in \( C^\circ \) is t-exact, and the fully faithfulness follows from Proposition 2.2. Moreover, since the category \( \text{Vect} \) is generated under colimits by \( k \), the essential image of the fully faithful functor \( \phi^* \) is generated under colimits by the essential image of \( \phi^*(k) \). Since this object is given by \( C \in \text{QCoh}(\text{Spec}(C))^H \) with its canonical equivariance, we obtain (2). \( \square \)

Let \( i : \text{Spec}(k) \to \text{Spec}(C) \) denote the embedding of the unique closed point. To prove Proposition 1.8, we first show the following lemma:

**Lemma 2.9.** Assume \( M \in \text{C-mod}^H \) has the property that the (derived) fiber \( i^*(M) \cong k \otimes_C M \in \text{Rep}_k(H) \) lies in the full subcategory generated by the trivial \( H \)-representation. Then the unit map \( M \to k \otimes_C M \) induces an equivalence \( M \xrightarrow{\sim} (k \otimes_C M)^H = k \otimes_C M \), where we view \( M \) and \( k \otimes_C M \cong i_*i^*(M) \) as objects of \( \text{Rep}(H) \) via the composite \( \text{C-mod}^H \xrightarrow{\alpha} \text{Vect}^H \cong \text{Rep}(H) \).

**Proof.** Consider the cofiber sequence
\[
C^+ \to C \to k
\]
induced by the short exact sequence of classical $C$-modules equipped with $H$-equivariance. Upon taking the (derived) tensor product with $M$, we obtain a cofiber sequence

$$C^+ \otimes_C M \to M \to k \otimes_C M$$

of objects of $C$-mod$^H$. Therefore, it suffices to show that if $M$ has the property that the derived fiber $i^*(M)$ is a complex of trivial $H$-representations, then $(C^+ \otimes_C M)^H \simeq 0$.

Note that $C$ admits a filtration $C_0 \subseteq C_1 \subseteq ... \subseteq C_l$ induced by the degree of $\text{Sym}(V^\vee)$, and the $H$-action preserves this filtration. In particular, we may filter $C^+$ by classical $C$-modules for which $C^+$ acts trivially. Furthermore, since $C^+$ has no trivial subrepresentations (by definition of the coinvariant algebra), for each of these subquotients $S$ in the filtration of $C^+$, we see that

$$(S \otimes_C M)^H \simeq ((S \otimes_k C/C^+) \otimes_C M)^H \simeq (S \otimes_k (k \otimes_C M))^H$$

is the tensor product of some nontrivial $H$-representation $S$ over $k$ with an entirely trivial representation (by assumption on $M$). Therefore we see that $(S \otimes_C M)^H \simeq 0$, and so $(C^+ \otimes_C M)^H \simeq 0$, as required.

Proof of Proposition 1.8. Because the pullback map $(\text{Spec}(k)/H \to \text{Spec}(k))^*$ corresponds to the inclusion of the trivial $H$-representation under the equivalence $\text{QCoh}(\text{Spec}(k)/H) \simeq \text{Rep}(H)$, we see that all objects in the essential image of $\hat{\phi}^*$ have the property that the $H$-representation of the (derived) fiber at the unique closed point is trivial. Moreover, the functor $\hat{\phi}^*$ admits a right adjoint $\hat{\phi}_*$ which is, at the level of homotopy categories, explicitly given by taking the $H$-fixed points of the underlying complex of $H$-representations. Therefore, it suffices to show that $\hat{\phi}_*$ is conservative on those objects of $\text{QCoh}(\text{Spec}(C))^H$ for which the canonical $H$-representation on the fiber at the closed point is trivial. Let $\mathcal{F}$ be a nonzero object in this subcategory. Since $\text{Spec}(C)$ has a unique closed point, we see that $i^*(\mathcal{F})$ is nonzero. Therefore, by Lemma 2.9 we see that the $H$-fixed points of $\mathcal{F}$ itself is nonzero, as desired. 

Example 2.10. This example shows taking derived fiber is necessary in Proposition 1.8. Let $H$ denote the order two group, and consider its action on one dimensional vector space $V$ given by scaling by $-1$. The associated coinvariant algebra is $A := k[x]/x^2$. Consider the $A$-module $k \cong A/x$ with its canonical $H$-equivariance. This object is not in the essential image of $\hat{\phi}^*$ since any object mapping to it must lie in the heart of $\text{Vect}$ by the $t$-exactness of $\hat{\phi}^*$, but the underlying $C$-modules of all objects in the essential image of $\hat{\phi}^*|_{\text{Vect}^\circ}$ have even $k$-dimension. One can explicitly compute, however, that the $H$-representation on $H^{-1}i^*(k)$ is given by the sign representation. Alternatively, using the short exact sequence of $H$-equivariant $A$-modules (all in the ordinary abelian category)

$$0 \to k_{\text{sign}} \xrightarrow{1-x} A \to k \to 0$$

where here $k_{\text{sign}}$ is $k$ as an $A$-module and equipped with the sign equivariance, one obtains an isomorphism $H^{-1}i^*(k \otimes_A k) \xrightarrow{\sim} H^0i^*(k_{\text{sign}} \otimes_A k)$, which is manifestly nontrivial as an $H$-representation.

2.3. Global Descent to the Coarse Quotient. We can determine the essential image of the fully faithful functor of Proposition 2.2 more explicitly. Recall that $H$ denotes some arbitrary finite group acting on some vector space $V$. If $x$ denotes some $K$-point of $V \parallel H$ for some field $K$, note that we obtain canonical identifications

$$X/H \times_{X/\parallel H} \text{Spec}(K) \simeq (X \times_{X/\parallel H} \text{Spec}(K))/H \simeq (H \times \text{Spec}(C_x))/H \simeq \text{Spec}(C_x)/H_x$$

and so we have a Cartesian diagram:

$$\begin{array}{ccc}
\text{Spec}(C_x)/H_x & \xrightarrow{\hat{\phi}} & \text{Spec}(K) \\
\downarrow x & & \downarrow x \\
V/H \xrightarrow{\phi} V \parallel H
\end{array}$$

Since there is a bijection of the $K$-points of $V/H$ and the $K$-points of $V \parallel H$, we may also view a given $x$ as a $K$-point of $V/H$, and this $K$-point induces a map $x : \text{Spec}(K)/H_x \to V/H$. With this notation, we may now make the following claim, which in particular proves Lemma 1.4.
Theorem 2.11. The following are equivalent for a given $\mathcal{F} \in \text{QCoh}(V^H)$:

1. The (complexes of) sheaves $\mathcal{F}$ descends to the quotient $V \sslash H$.
2. For each $K$-point $x$ of $V \sslash H$, $x^*(\mathcal{F})$ descends to the coarse quotient $\text{Spec}(C_x) \sslash H_x = \text{Spec}(K)$.
3. For each $K$-point $x$ of $V \sslash H$, the $H_x$-representation on $x^*(\text{obl}(\mathcal{F}))$ is trivial.

Proof. The fact that diagram (5) commutes implies that any object $\mathcal{F}$ which descends to $V \sslash H$ satisfies (2). In particular, the fully faithful functor $\phi^*$ maps into the full subcategory of objects satisfying (2). Since this functor admits a right adjoint $\phi_*$ (explicitly given by taking the $H$-invariants of the associated complex of $\text{Sym}(V^V)$-modules), it remains to verify that the adjoint $\phi_*$ is conservative on the full subcategory of objects satisfying (2). Let $\mathcal{F}$ be such a nonzero object. Then, by Proposition 2.7, there exists some field-valued point of $V$ for which the fiber of $\mathcal{F}$ at this point does not vanish. Using the quotient map, we identify this field-valued point as a map $x: \text{Spec}(K) \to V \sslash H$.

We wish to show that $\phi_*(\mathcal{F})$ does not vanish, and, of course, it suffices to show that $x^*\phi_*(\mathcal{F})$ does not vanish. Since $x$ is a map of quasicompact schemes, we may apply base change along diagram (8) (we may do this, for example, by applying [BFN10, Proposition 3.10], whose hypotheses hold by [BFN10, Corollary 3.22, Corollary 3.23]) and may equivalently show that $\phi_*x^*(\mathcal{F})$ does not vanish. However, by assumption, $x^*(\mathcal{F})$ lies in the essential image of $\phi^*$. Therefore the adjoint $\phi_*$ does not vanish, since the adjoint to a fully faithful functor is conservative on the essential image and $x^*(\mathcal{F})$ is nonzero since $x^*(\mathcal{F})$ is nonzero. Combining these results, we see that $\phi_*(\mathcal{F})$ does not vanish on objects satisfying (2), and so the categories given by (1) and (2) are equivalent.

To show the equivalence of (2) and (3), first note that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Spec}(K)/H_x & \xrightarrow{i} & \text{Spec}(C_x)/H_x \\
x \downarrow & & \downarrow x \\
V/H & \xrightarrow{x} & \text{Spec}(C)/H_x \\
\end{array}
\]

so that, in particular, if $\mathcal{F}$ descends to the coarse quotient $\text{Spec}(C_x) \sslash H_x = \text{Spec}(K)$ for some fixed $x$, then we see that $i^*(\mathcal{F}) \simeq (\text{Spec}(K)/H_x \to \text{Spec}(K))^*(\mathcal{G})$ for some $\mathcal{G} \in \text{QCoh}(\text{Spec}(K)) = \text{Vect}_K$. Since, under the equivalence $\text{QCoh}(\text{Spec}(K)/H_x) \simeq \text{Rep}_K(H_x)$, this functor corresponds to the inclusion of the trivial $H_x$-representation, we see that such an $\mathcal{F}$ also satisfies (3). Conversely, if $\mathcal{F}$ satisfies (3) for some fixed $x$, then we see by Proposition 1.8 that $x^*(\mathcal{F})$ descends to the coarse quotient $\text{Spec}(C_x) \sslash H_x = \text{Spec}(K)$.

Proof of Theorem 1.6. For any pseudo-reflection $r$, the following diagram commutes

\[
\begin{array}{ccc}
V/(r) & \xrightarrow{} & V \sslash \langle r \rangle \\
\downarrow & & \downarrow \\
V/H & \xrightarrow{} & V \sslash H \\
\end{array}
\]

where all arrows are the canonical quotient maps. Therefore we see that if an object of $\text{QCoh}(V/H)$ descends to $V \sslash H$, then the sheaf $\text{obl}(\mathcal{F})$ descends to $V \sslash \langle r \rangle$.

Conversely, assume $\mathcal{F} \in \text{QCoh}(V/H)$ has the property that the sheaf $\text{obl}(\mathcal{F})$ descends to $V \sslash \langle r \rangle$ for every pseudo-reflection $r$. By Theorem 2.11, it suffices to show that for every $K$-point $x$ of $V \sslash H$, the canonical $H_x$-representation of $K$-vector spaces on the fiber $x^*(\mathcal{F})$ is trivial. The group $H_x$ is itself a pseudo-reflection group by Theorem 2.6 and therefore by definition is generated by pseudo-reflections. If $r$ is a pseudo-reflection of $H_x$, we see that because $\text{obl}(\mathcal{F})$ descends to $V \sslash \langle r \rangle$, the $\langle r \rangle$-representation induced by the $H_x$-representation on the fiber $x^*(\mathcal{F})$ is trivial, again by Theorem 2.11. One can see this explicitly since the following diagram commutes:
where the horizontal arrows are induced by $x$. Therefore, since a collection of generators acts trivially, the entire $H_x$-representation on the fiber $x^*(\mathcal{F})$ is trivial. \qedhere

2.4. From $\text{QCoh}$ to $\text{IndCoh}$. We recall the category $\text{IndCoh}(X)$, which is defined if $X$ is any laft prestack, see \cite[Chapter 3, Section 5]{GR17c}. If $X$ is any laft prestack, there is a canonical symmetric monoidal functor $\Upsilon_X : \text{QCoh}(X) \to \text{IndCoh}(X)$ whose underlying functor of DG categories can be identified with $'(-\otimes_{\mathcal{O}_X}\omega_X)'$. If $X$ is a smooth classical scheme of dimension $d$, $\omega_X$ can be identified with $\mathcal{O}[d]$, and $\Upsilon_X$ is an equivalence. We show the analogous claim for the quotient stack $X/F$ where $F$ is any finite group.

**Proposition 2.12.** If $F$ is a finite group acting on a smooth scheme $X$ of dimension $d$, then $\Upsilon_{X/F} : \text{QCoh}(X/F) \to \text{IndCoh}(X/F)$ is an equivalence, and $\Upsilon_{X/F}[-d]$ is $t$-exact.

**Proof.** The map $q : X \to X/F$ is finite flat. In particular, via flat descent for $\text{QCoh}(X/F)$ \cite[Chapter 3, Section 1.3]{GR17b}, we obtain that the pullback map induces a canonical equivalence

$$q^* : \text{QCoh}(X/F) \xrightarrow{\sim} \lim_{\Delta} \text{QCoh}(F^* \times X)$$

and since $q$ is in particular proper, proper descent \cite[Chapter 3, Proposition 3.3.3]{GR17c} gives an analogous equivalence for $\text{IndCoh}(X/F)$. Since, for each integer $i$, the categories $F^i \times X$ is again a smooth scheme, we see that $\Upsilon_{F^i \times X}$ is an equivalence, and $\Upsilon_{F^i \times X}[-\dim(F^i \times X)] = \Upsilon_{F^i \times X}[-d]$ is a $t$-exact equivalence. Therefore, since we can identify $\Upsilon_{X/F}$ as the composite, read left to right, of the functors

$$\text{QCoh}(X/F) \xrightarrow{\sim} \lim_{\Delta} \text{QCoh}(F^* \times X) \xrightarrow{T^\text{ind}} \lim_{\Delta} \text{IndCoh}(F^* \times X) \xhookrightarrow{\sim} \text{IndCoh}(X/F)$$

we see that $\Upsilon_{X/F}$ is an equivalence, as desired. \qedhere

We will also use the following analogue of Proposition 1.8 for $\text{IndCoh}$ in arguing that sheaves on $t^* \parallel W^{aff}$ can be identified with $W^{aff}$-equivariant sheaves satisfying Coxeter descent, see Section 4.3.2.

**Corollary 2.13.** Assume $H$ is a Coxeter group for some given reflection representation and let $C$ denote the coinvariant algebra.

1. The functors $\Upsilon_{\text{Spec}(C)}$ and $\Upsilon_{\text{Spec}(C)/H}$ are fully faithful.
2. A given $\mathcal{F} \in \text{IndCoh(Spec(C))/H}$ descends to the coarse quotient $\text{Spec}(C)/H = \text{Spec}(k)$ if and only if $\mathcal{F} \cong \Upsilon_{\text{Spec}(C)}(\mathcal{F}')$ for some $\mathcal{F}' \in \text{QCoh(Spec(C))}$ and the canonical $H$-representation on $i^!(\mathcal{F})$ is trivial.

**Proof.** Any continuous functor whose domain is $\text{QCoh(Spec(C))} = C$-mod is determined by the essential image of $C$, since $C$ compactly generates $C$-mod. Since $C$ is Gorenstein when $H$ is a Coxeter group (for example, by Borel’s theorem we may identify the coinvariant algebra with the cohomology of the flag variety), we obtain a canonical equivalence $\Xi_{\text{Spec}(C)} \cong \Upsilon_{\text{Spec}(C)}$, where $\Xi_{\text{Spec}(C)}$ is the ind-completion of the inclusion of perfect complexes into coherent ones, and thus we obtain the fully faithfulness of $\Upsilon_{\text{Spec}(C)}$. This same reasoning gives for any positive integer $i$ that we have a canonical isomorphism $\Xi_{F^i \times \text{Spec}(C)} \cong \Upsilon_{F^i \times \text{Spec}(C)}$. From this, we obtain a canonical identification which exhibits that the following diagram commutes:

\[
\begin{array}{ccc}
\text{IndCoh(Spec(C)/H)} & \xrightarrow{\sim} & \text{IndCoh(Spec(C))/H} \\
\Upsilon_{\text{Spec}(C)/H} & & \Upsilon_{\text{Spec}(C)/H} \\
\text{QCoh(Spec(C)/H)} & \xrightarrow{\sim} & \text{QCoh(Spec(C))/H} \\
\Xi_{\text{Spec}(C)} & & \Xi_{\text{Spec}(C)}
\end{array}
\]
where the vertical arrows are the equivalences induced by pullback. Since $\Xi^H_{\text{Spec}(C)}$ is fully faithful, claim (1) follows.

Because $\Upsilon$ is compatible with the pullback of $\text{Spec}(C)/H \to \text{Spec}(k)$ and $\Upsilon_{\text{Spec}(k)}$ is an equivalence, we see that the condition $\mathcal{F} \simeq \Upsilon_{\text{Spec}(C)}(\mathcal{F}')$ is necessary for a given $\mathcal{F}$ to lie in the essential image. If such an $\mathcal{F}'$ exists, the equivalence stated in (2) follows directly from Proposition 1.8 because $\Upsilon$ is compatible with the pullback of $\text{Spec}(k)/H \to \text{Spec}(C)/H$. □

Remark 2.14. The fully faithfulness also follows directly from [Ga13, Lemma 10.3.4] the fact that $\text{Spec}(C)$ is classical, and therefore in particular eventually coconnective. We thank the anonymous referee for this comment.

2.5. Equivalent Conditions for Descent to Coarse Quotient For Coxeter Groups. We now give alternate descriptions of those $H$-equivariant sheaves on $V$ which descend to the coarse quotient when $H$ is a finite Coxeter group, which we assume for this section. We use IndCoh rather than QCoh, as IndCoh will be the sheaf theory used in the later sections. We note that, by smoothness of $V//H$ (see Theorem 2.3) and by Proposition 2.12, all of the analogous results in this section hold when IndCoh is replaced with QCoh.

We first give an alternate description in the case where $\mathcal{F}$ is generated by a single reflection. For a fixed reflection $r \in H$, let $\phi_r : V/\langle r \rangle \to V/\langle r \rangle$ denote the quotient map for the action of the order two Weyl group $\langle r \rangle$ acting on $V$, and let $i_r : Z_r := V^{(r)} \hookrightarrow V$ denote the inclusion of the closed subscheme of fixed points. Note that $\text{IndCoh}(Z_r/\langle r \rangle) \simeq \text{IndCoh}(Z_r) \otimes \text{Rep}(\langle r \rangle)$ since the action of $r$ is trivial on the fixed point locus.

Proposition 2.15. Let $r$ denote a reflection. An object $\mathcal{F} \in \text{IndCoh}(V/\langle r \rangle)$ descends to the coarse quotient $V/\langle r \rangle$ if and only if the pullback $i^!_r(\mathcal{F}) \in \text{IndCoh}(Z_r) \otimes \text{Rep}(\langle r \rangle)$ lies entirely in the summand indexed by the trivial $\langle r \rangle$-representation.

Proof. The closed subscheme $Z_r \xrightarrow{i_r} V$ and complementary open subscheme $U_r := V \setminus Z_r \xrightarrow{j_r} V$ are both affine and induce two Cartesian squares as follows

$$
\begin{array}{ccc}
Z_r/\langle r \rangle & \xrightarrow{i_r} & V/\langle r \rangle \\
\downarrow{\phi|Z_r} & & \downarrow{\phi|U_r} \\
Z_r // \langle r \rangle & \xrightarrow{i_r} & V // \langle r \rangle \\
\end{array}
$$

where each vertical arrow is obtained from the map $\phi$. Since $\phi_r^!$ is fully faithful (Proposition 2.2), its essential image is closed under extensions. In particular, an object $\mathcal{F} \in \text{IndCoh}(V/H)$ lies in the essential image if and only if $i^!_r(\mathcal{F})$ and $j^!_r(\mathcal{F})$ lie in the essential image of the respective pullbacks. However, since the action of $\langle r \rangle$ is free, the rightmost vertical arrow is an equivalence, so $j^!_r(\mathcal{F})$ is always in the essential image of $\phi_r^!$. Therefore, $\mathcal{F}$ lies in the essential image if and only if $i^!_r(\mathcal{F})$ does.

Note that the action of $r$ on $Z_r$ is trivial, and therefore we see that $Z_r//\langle r \rangle \cong Z_r$. Furthermore, we may identify the pullback $\phi_r^!|Z_r$ with the functor

$$
\text{IndCoh}(Z_r) \xrightarrow{id \otimes \text{triv}} \text{IndCoh}(Z_r) \otimes \text{Rep}(\langle r \rangle)
$$

and so we see that an object of the form $i^!_r(\mathcal{F})$ is in the essential image of the pullback $\phi_r^!$ if and only if the restriction lies entirely in the trivial summand. Combining this with the assertion that $\mathcal{F}$ lies in the essential image if and only if $i^!_r(\mathcal{F})$ does, we obtain our desired characterization of the essential image. □

We now summarize and give various equivalent conditions for a given $\mathcal{F} \in \text{IndCoh}(V)^H$ to descend to the coarse quotient $V//H$, where again we remind that $H$ is a Coxeter group in this section:

Proposition 2.16. An object $\mathcal{F} \in \text{IndCoh}(V/H)$ descends to the coarse quotient $V//H$ if and only if one of the following equivalent conditions hold:

1. For each field-valued point $x : \text{Spec}(K) \to V$, the pullback $x^!(\text{obl}^H_{\text{Spec}(K)}(\mathcal{F}))$, which canonically acquires a $H_x$-representation in $\text{Vect}_K$, is the trivial $H_x$-representation.
2. For each reflection $r \in H$, $\text{obl}^H_{\langle r \rangle}(\mathcal{F})$ descends to the coarse quotient $V//\langle r \rangle$. 

(3) For each simple reflection \( s \in H \), \( \text{oblv}^{W}_{s}(\mathcal{F}) \) descends to the coarse quotient \( V \parallel \langle s \rangle \).
(4) Each cohomology group \( H^{i}(\mathcal{F}) \in \text{IndCoh}(V)^{H,\circ} \simeq \text{IndCoh}(V)^{\circ,H} \) lies in the essential image of \( \phi_{t}^{i}|_{\text{IndCoh}(V/H)^{\circ}} \).
(5) For every reflection \( r \in H \), the cohomology group \( \text{oblv}^{W}_{r}(H^{i}(\mathcal{F})) \in \text{IndCoh}(V^{(r)})^{\circ} \) lies in the essential image of the pullback \( \phi_{r}^{i} \) restricted to \( \text{IndCoh}(V^{(r)})^{\circ} \) for all \( i \in \mathbb{Z} \).
(6) For every simple reflection \( s \in H \), the cohomology group \( \text{oblv}^{W}_{s}(H^{i}(\mathcal{F})) \in \text{IndCoh}(V^{(s)})^{\circ} \) lies in the essential image of the pullback \( \phi_{s}^{i} \) restricted to \( \text{IndCoh}(V^{(s)})^{\circ} \) for all \( i \in \mathbb{Z} \).
(7) For each simple reflection \( s \in H \), the sheaf \( \mathcal{F}_{s}^{\circ}(\text{oblv}^{W}_{s}(\mathcal{F})) \in \text{IndCoh}(V^{(s)}) \otimes \text{Rep}(\langle s \rangle) \) lies entirely in the summand indexed by the trivial representation.
(8) For every reflection \( r \in H \), the sheaf \( \mathcal{F}_{r}^{\circ}(\text{oblv}^{W}_{r}(\mathcal{F})) \in \text{IndCoh}(Z_{r}) \otimes \text{Rep}(\langle r \rangle) \) lies entirely in the summand indexed by the trivial representation.

Proof of Proposition 2.16 Because \( \Upsilon \) is an equivalence for \( */H_{x} \) and \( V/H \) and intertwines \( * \)-pullback for \( \text{Qcoh} \) and \( ! \)-pullback for \( \text{IndCoh} \), we see that \( \mathcal{F} \) descends to the coarse quotient if and only if \( \mathcal{F} \) satisfies (1) by Theorem 2.11 Similarly, we see that \( \mathcal{F} \) descends to the coarse quotient if and only if \( \mathcal{F} \) satisfies (2) Theorem 1.6.

We now show (3) \( \Rightarrow \) (2). Fix some reflection \( r \in H \), and choose some \( w \in H \) for which \( w^{-1}rw \) is a simple reflection \( s \). Then the following diagram commutes

\[
\begin{array}{ccc}
V \parallel \langle r \rangle & \xrightarrow{\phi_{r}} & V/\langle r \rangle \\
\downarrow w & & \downarrow w \\
V \parallel \langle s \rangle & \xrightarrow{\phi_{s}} & V/\langle s \rangle \\
\end{array}
\]

where the vertical arrows are the maps induced by the action of \( w \in H \) and the unlabeled arrows are the quotient maps. We then see if \( \text{oblv}^{W}_{s}(\mathcal{F}) \simeq \phi_{s}^{i}(\mathcal{F}') \) for some \( \mathcal{F}' \) then

\[ \text{oblv}^{W}_{r}(\mathcal{F}) \simeq w^{i}\text{oblv}^{W}_{s}(\mathcal{F}) \simeq w^{i}(\phi_{s}^{i}(\mathcal{F}')) \simeq \phi_{r}^{i}(w^{i}(\mathcal{F}')) \]

showing that (2) holds. Conversely, the implication (2) \( \Rightarrow \) (3) follows since simple reflections are reflections.

The equivalence of a given \( \mathcal{F} \) descending to the coarse quotient \( V \parallel H \) and the given \( \mathcal{F} \) satisfying (4) follows from the \( t \)-exactness and fully faithfulness of \( \phi_{t}^{i} \), where the \( t \)-exactness follows from the fact that \( \text{oblv}^{W} \) reflects the \( t \)-structure and the fact that the quotient map \( V \to V \parallel H \) is finite-flat, see Proposition 2.5. Replacing the map \( \phi_{t}^{i} \) with \( \phi_{r}^{i} \) and \( \phi_{s}^{i} \), this argument also gives the equivalences (2) \( \Leftrightarrow \) (5) and (3) \( \Leftrightarrow \) (6).

Finally, the equivalences (2) \( \Leftrightarrow \) (7) and (3) \( \Leftrightarrow \) (8) follow directly from Proposition 2.13. \( \square \)

3. Preliminary Computations

In this section, we review some preliminary results on the union of graphs of the affine Weyl group \( W^{\text{aff}} \) and discuss extensions to the extended affine Weyl group.

3.1. The Integral Weyl Group. For a fixed \( x \in t^{\circ}(k) \), let \([x]\) denote the image of this \( k \)-point in the quotient \( t^{\circ}/\Lambda \). The following is essentially shown in the proof of \[\text{[Jan79, Satz 1.3]}\]; we recall some details for the convenience of the reader:

**Proposition 3.1.** Fix some \( x \in t^{\circ}(k) \). The following subgroups of \( W \) are identical:

(1) The image \( \overline{W}_{x}^{\text{aff}} \) of the stabilizer \( W_{x}^{\text{aff}} \) of \( x \) under the \( W^{\text{aff}} \)-action on \( t^{\circ} \) under the quotient map \( W^{\text{aff}} \to W^{\text{aff}}/Z\Phi \cong W \).
(2) The subgroup \( W_{[x]} : = \{ w \in W : wx - x \in Z\Phi \} \).
(3) The subgroup \( W^{\star}_{[x]} : = \{ w \in W : w \cdot x - x \in Z\Phi \} \).
(4) The subgroup \( W^{x} : = \langle s_{\alpha} : \langle x, \alpha^{\vee} \rangle \in Z \rangle \), where \( \alpha \) varies over the set of roots \( \Phi \).

Furthermore, the group \( W^{x} \) is a Weyl group of the root system whose roots are \( \Phi_{[x]} : = \{ \alpha \in \Phi : \langle x, \alpha^{\vee} \rangle \in Z \} \).
3.2. The Union of Graphs of a Finite Closed Subset of $W^\text{aff}$ is Finite Flat. We now prove the following possibly known result which we were unable to locate a reference for:

Proposition 3.4. Let $S \subseteq W^\text{aff}$ denote a finite, closed (see Definition A.1) subset of the affine Weyl group $W^\text{aff}$, and let $\pi_S : \Gamma_S \rightarrow \mathfrak{t}^* \times \mathfrak{t}^*$ denote the union of graphs of those $w \in S$. Then the projection map onto the first factor $s : \Gamma_S \rightarrow \mathfrak{t}^*$ is finite flat.

As we will see, Proposition 3.4 will follow from a known extension of Borel’s theorem on the cohomology of the flag variety, which we include an alternate proof of in the appendix. Using the results of the appendix, we prove Proposition 3.4 after proving the following lemma:

Lemma 3.5. Fix some finite subset $S \subseteq W^\text{aff}$, and fix some $\lambda \in \mathfrak{t}^*(K)$ for $K$ a (classical) field. Then, if $W^\text{aff}_\lambda \subseteq W^\text{aff}$ denotes the stabilizer of $\lambda$, the coproduct of inclusions induces a canonical isomorphism:

$$
\Gamma_S \times_\mathfrak{t} \Spec(K) \simeq \prod_{w \in W^\text{aff}_\lambda} (\Gamma_S \cap W^\text{aff}_w) \times_\mathfrak{t} \Spec(K)
$$

Proof. Enumerate the $W^\text{aff}$-orbit of $\lambda$ as $\{\lambda_i\}_{i \in \mathbb{N}}$, we can partition the set $S = \bigcup_i S_i$ where all $w_i \in S_i$ have the property that $w_i \lambda = \lambda_i$. Let $Z_S$ denote the closed subscheme of $\mathfrak{t}^*$ given by $\bigcup_{i,j \in \mathbb{N}, i \neq j} \bigcup_{w_i \in S_i, w_j \in S_j} \{w_i = w_j\}$; since $S$ is finite, $Z_S$ can be expressed as a finite union of nonempty Zariski closed subsets. Furthermore, we see that $\lambda$ factors through the open complement $U_S$ of $Z_S$. Therefore, since $\Gamma_S \times_\mathfrak{t} \Spec(K) \simeq (\Gamma_S \times_\mathfrak{t} U_S) \times_{U_S} \Spec(K)$ and, by definition of $U_S$, we have that $(\Gamma_S \times_\mathfrak{t} U_S)$ can be written as a disjoint union of subschemes indexed by each $S_i$, we see that our induced map is an isomorphism.

Proof of Proposition 3.4. The map $\pi_S$ is trivially finite, so it remains to show that $\pi_S$ is flat. A standard result in commutative algebra states that, since $\pi_S$ is finite, it is enough to show that for all points $x$ in $\mathfrak{t}^*$, the length of the fiber at $x$ as an $O_{\mathfrak{t}, x}$ module is independent of $x$. Choose some field-valued $x \in \mathfrak{t}^*(L)$. We enumerate the distinct $K$-points in the fiber of $x$, say, $(x, y_1), \ldots, (x, y_m) \in \Gamma_S$, and write $y_i = w_i x$ for the minimal such $w_i \in W^\text{aff}$. Write $W^\text{aff} = MW^\text{aff}_x$ where $M$ denotes the set of minimal elements of each coset in $W^\text{aff}/W^\text{aff}_x$. We thus have an isomorphism given by Lemma 3.5.
\[ \Gamma_S \times \iota' \{ x \} \cong \coprod_i (\Gamma_{w_i W^\text{aff}_S} \times \iota' \{ x \}) \]

where \( i \) ranges over a finite index set and \( w_i \in M \). Note that each \( \Gamma_{w_i W^\text{aff}_S} \times \iota' \{ x \} \) is isomorphic via left multiplication by \( w_i^{-1} \) to the fiber \( \Gamma_{S'} \times \iota' \{ x \} \) for some subset \( S' \subseteq W^\text{aff}_x \).

Furthermore, we claim this \( S' \) is a closed subset of \( W^\text{aff}_x \) in the sense of Definition A.13. To see this, recall the canonical isomorphism \( W^\text{aff}_x \cong W^F \) given in Remark 3.3 and Proposition 3.1. If \( u \in S' \) and \( u' \leq u \) (where \( \leq \) refers to the ordering on the Coxeter group \( W^F \)), we have that by [Lus94, Lemma 2.5], \( w_i u' \leq w_i u \), so that the fact that \( S \) is closed gives that \( u' \in S' \), and so \( S' \) is closed in \( W^F \). Therefore we can apply Corollary A.14 to see that the total length of the fiber is \( \sum_{i=1}^m |S \cap w_i W_x| = |S| \), which is independent of \( x \). 

3.3. Fiber of Map to Coarse Quotient of Affine Weyl Group at Field-Valued Point. Fix some field \( K/k \) and let \( x \in t^*(K) \). We will use the following lemma later:

**Lemma 3.6.** The multiplication map induces a left \( W^\text{aff} \)-equivariant isomorphism

\[ \eta : W^\text{aff} \times (\Gamma_{W^\text{aff}} \times \iota' \text{Spec}(K)) \cong \Gamma_{W^\text{aff}} \times \iota' \text{Spec}(K) \]

where \( \Gamma_{W^\text{aff}} \) is the union of graphs of the subgroup \( W^\text{aff} \leq W^\text{aff} \), which admits a map \( \Gamma_{W^\text{aff}} \to t^* \) given by the projection \((w x, x) \mapsto x\).

**Proof.** Using the results of Lemma 3.5 for every finite subset \( S \subseteq W^\text{aff} \), we obtain an isomorphism

\[ \Gamma_{W^\text{aff}} \times \iota' \text{Spec}(K) \cong \bigcup_i (\Gamma_{w_i W^\text{aff}} \times \iota' \text{Spec}(K)) \]

where the right hand side ranges over the cosets of \( W^\text{aff} / W^F \). Therefore we may check that the multiplication map induces an isomorphism at each open subset \( \Gamma_{w_i W^\text{aff}} \times \iota' \text{Spec}(K) \). However, we see that taking the fiber product of the above multiplication map by this open subset, we obtain the map

\[ w W^\text{aff}_x \times (\Gamma_{W^\text{aff}} \times \iota' \text{Spec}(K)) \to \Gamma_{w_i W^\text{aff}} \times \iota' \text{Spec}(K) \]

which is an isomorphism. Furthermore, \( \eta \) is \( W^\text{aff} \)-equivariant because the multiplication map is \( W^\text{aff} \)-equivariant.

4. The Coarse Quotient for the Affine Weyl Group

In this section, we define the coarse quotient \( t^* \sslash W^\text{aff} \) and determine some of its basic properties. After briefly reviewing the notion of a groupoid object in Section 4.1, we define this quotient in general in Section 4.2 and show that sheaves on \( t^* \sslash W^\text{aff} \) are equivalently \( W^\text{aff} \)-equivariant sheaves satisfying conditions analogous to those of Proposition 2.16 in Section 1.3.

In order to compute the category of sheaves on \( t^* \sslash W^\text{aff} \), we will use the fact that the map \( t^* \to t^* \sslash W^\text{aff} \) is an ind-finite flat cover. Using the computations of Section 3, we show this in Corollary 4.9.

4.1. Groupoid Objects and Higher Algebra. In this section, we briefly recall the notion of groupoid objects and higher categorical contexts. For a thorough treatment of groupoid objects in the \((1,1)\) and \((\infty, 1)\) setting, see [Lur09, Section 6.1.2].

In derived algebraic geometry, the notion of a classical groupoid is replaced with the notion of a *groupoid* or an \((\infty, 1)\)-category of spaces \( \mathcal{Spc} \).

**Definition 4.1.** A *groupoid object* of an \((\infty, 1)\) category \( \mathcal{C} \) is a simplicial object \( U \) of \( \mathcal{C} \) such that for every \( n \geq 0 \) and every partition \([n] = S \sqcup S'\) such that \( S \cap S' \) consists of a single element \( s \), the canonical map \( U([n]) \to U(S) \times_{U([1])} U(S') \) is an equivalence (and, in particular, the latter term is defined).

We now recall the basic results about groupoid objects in the \((\infty, 1)\) category of spaces \( \mathcal{Spc} \), which immediately implies the analogous fact for the category of prestacks since limits and colimits in functor categories are computed termwise:
Proposition 4.2. [Lur09 Corollary 6.1.3.20] Every groupoid object of $\text{Spc}$ is effective. In particular, if $U_\bullet$ is a groupoid object of $\text{Spc}$, then a geometric realization $U_{-1}$ of it exists and the canonical map $U_1 \to U_0 \times_{U_{-1}} U_0$ is an equivalence.

4.2. The Coarse Quotient. We now wish to apply the general framework above to our specific case of interest. Let $\Gamma_{W^\text{aff}}$ denote the union of graphs of each $w \in W^\text{aff}$, where by graph we mean the closed subschemes $t^* \xrightarrow{(w, \text{id})} t^* \times t^*$. More precisely, we view $\Gamma_{W^\text{aff}}$ as the classical ind-scheme given by the union of graphs $\Gamma_S$ given by the intersection of ideals (see Remark 4.4) for $S$ a finite closed subset of the affine Weyl group. In particular $\Gamma_{W^\text{aff}}$ is naturally an ind-closed subscheme of $t^* \times t^*$. 

4.2.1. Definition of the Coarse Quotient. Let $\Gamma_{W^\text{aff}}$ denote the balanced product $\check{W}^\text{aff} \times \Gamma_{W^\text{aff}}$. Note that $\Gamma_{W^\text{aff}}$ admits canonical maps to $s, t : \Gamma_{W^\text{aff}} \to t^*$ given by the maps $s(\sigma, (w, \lambda)) = \lambda$ and $t(\sigma, (w, \lambda)) = \sigma w \lambda$. We now record the following general fact which remains valid if the subgroup $W^\text{aff} \leq \check{W}^\text{aff}$ is replaced with any closed subgroup $H \leq \check{H}$ and $\Gamma_{W^\text{aff}}$ is replaced with any $\Gamma$ with an $H$-action:

Proposition 4.3. We have an isomorphism $\check{\Gamma}_{W^\text{aff}} \xrightarrow{\sim} \check{W}^\text{aff} / W^\text{aff} \times \Gamma_{W^\text{aff}}$ in such a way that the following diagram commutes:

$$
\begin{array}{c}
\check{\Gamma}_{W^\text{aff}} := \check{W}^\text{aff} \times \Gamma_{W^\text{aff}} \\
\downarrow \sim \\
\check{W}^\text{aff} / W^\text{aff} \times \Gamma_{W^\text{aff}}
\end{array}
$$

Proof. The isomorphism is induced by the map $(\check{w}, g) \mapsto (\check{w}, \check{w}g)$, and the inverse map is induced by $(\check{w}, g') \mapsto (\check{w}, \check{w}^{-1}g')$. \qed

We now may construct a groupoid object $\Gamma_\bullet$ over $t^*$ such that $\Gamma_1 \simeq \check{\Gamma}_{W^\text{aff}}$ specifically, set $\Gamma_\bullet := \Gamma_{W^\text{aff}} \times t^* \times t^* \times \cdots \times t^*$. However, working with this object in general would be a technical nuisance, a priori: all of our fiber products are inherently derived. However, the following proposition allows us to argue that the $\Gamma_n$ systematically remain in the classical $(1,1)$-categorical setting.

Proposition 4.4. The map $s : \check{\Gamma}_{W^\text{aff}} \to t^*$ is ind-finite flat.

Proof. By Proposition 4.3 it suffices to show that the map is ind-finite flat flat when $\check{W}^\text{aff} = W^\text{aff}$. In this case, $W^\text{aff}$ is a Coxeter group and has a length function $\ell$. For each positive integer $m$ set $S_m = \{ w \in W^\text{aff} : \ell(w) \leq m \}$, and let $G_m$ denote the union of graphs of those $w \in S_m$. Then we clearly have $\Gamma = \bigcup_m G_m$, and so it suffices to show that $s_m : G_m \to t^*$ is finite flat. However, this follows from Proposition 3.4. \qed

Corollary 4.5. Each $\Gamma_{[i]}$ is a filtered colimit of classical schemes.

We define $t^* \mathbin{\#} \check{W}^\text{aff}$ as the geometric realization of $\Gamma_\bullet$, exactly as in the case when $\check{W}^\text{aff} = W^\text{aff}$ in Definition 4.2. Then we see that Proposition 4.3 immediately implies:

Proposition 4.6. We have a canonical equivalence $\check{\Gamma}_{W^\text{aff}} \xrightarrow{\sim} t^* \mathbin{\#} \check{W}^\text{aff} t^*$.

This has the following corollary, which should be compared to the finite group case in [5].

Corollary 4.7. For a fixed $x \in t^*(K)$, we have canonical isomorphisms

$$X^\bullet(T) \setminus t^* \mathbin{\#} \check{W}^\text{aff} \text{Spec(K)} \simeq X^\bullet(T) \setminus \Gamma_{W^\text{aff}} \times t^* \text{Spec(K)} \xrightarrow{\sim} W_{[x]} \times \text{Spec(CW_{[x]})}$$

where $C_{[x]}$ denotes the coinvariant algebra for the integral Weyl group (Definition 3.2) of $x$.

Proof. The first equivalence follows from direct application of Proposition 1.6 we now show the second. Note that the inclusion $W^\text{aff} \hookrightarrow \check{W}^\text{aff}$ induces an isomorphism $Z \Phi \setminus W^\text{aff} \xrightarrow{\sim} X^\bullet(T) \setminus \check{W}^\text{aff}$. Therefore we obtain canonical isomorphisms

$$X^\bullet(T) \setminus \Gamma_{W^\text{aff}} := X^\bullet(T) \setminus \check{W}^\text{aff} \times \Gamma_{W^\text{aff}} \xrightarrow{\sim} Z \Phi \setminus W^\text{aff} \times \Gamma_{W^\text{aff}} \simeq Z \Phi \setminus \Gamma_{W^\text{aff}}$$
over $t^\ast$ with respect to the (right) projection map. Furthermore, by Lemma 3.6 we see:

$$Z\Phi \backslash W_{aff} \times t^\ast \text{Spec}(K) \xrightarrow{\sim} Z\Phi \backslash W_{aff} \times (\Gamma_{W_{aff}} \times t^\ast \text{Spec}(K))$$

so that, composing with the quotient map $Z\Phi \backslash W_{aff} \xrightarrow{\sim} W$, we see the quotient map induces an isomorphism

$$Z\Phi \backslash W_{aff} \times (\Gamma_{W_{aff}} \times t^\ast \text{Spec}(K)) \xrightarrow{\sim} W \times \Gamma_{W_{aff}} \times t^\ast \text{Spec}(K)$$

where we identify the image of the stabilizer $W^0_x$ with the integral Weyl group as in Remark 3.3. Therefore, since we may identify

$$W \times \Gamma_{W_{aff}} \times t^\ast \text{Spec}(K) \simeq W \times \text{Spec}(C_x) \cong W \times \text{Spec}(C_{[x]})$$

where the first isomorphism follows from the definition of $C_{[x]}$ and the second isomorphism is given by translation.

Remark 4.8. One may also wish to sheafify $t^\ast \parallel \tilde{W}_{aff}$ for some topology. This parallels the case of the finite coarse quotient because, as $t^\ast \parallel W$ is a classical scheme, a theorem of Grothendieck implies it is a sheaf in the fpqc topology. In particular, since $t^\ast \rightarrow t^\ast \parallel W$ is fpqc (see Proposition 2.5) we obtain the canonical map

$$\text{colim}_{\Delta}(\Gamma_{W_{aff}}) \rightarrow t^\ast \parallel W$$

is an equivalence, where we take the colimit in the category of fpqc stacks and use Proposition A.2. Sheafification will not affect our analysis below since the functor IndCoh factors through sheafification, and our critical Proposition 4.6 is not affected since sheafification commutes with finite limits: see [GR17b, Section 2.3.6] for the results of [Lur09] specific to our setup.

We once and for all let $\pi : t^\ast \rightarrow t^\ast \parallel \tilde{W}_{aff}$ denote the quotient map.

Corollary 4.9. The map $\pi : t^\ast \rightarrow t^\ast \parallel \tilde{W}_{aff}$ is ind-finite flat.

Proof. We may check whether a map is ind-finite flat after base change by $\pi : t^\ast \rightarrow t^\ast \parallel \tilde{W}_{aff}$. However, by Proposition 4.12 our based changed map is canonically $t : \Gamma \rightarrow t^\ast$, which is ind-finite flat by Proposition 4.4.

We now proceed to discuss sheaves on $t^\ast \parallel \tilde{W}_{aff}$ and other associated prestacks. We would like to use the critical result that IndCoh is defined on $t^\ast \parallel \tilde{W}_{aff}$ and satisfies ind-proper descent, since we now have that the map $\pi : t \rightarrow t \parallel \tilde{W}_{aff}$ is ind-schematic and ind-proper. To show IndCoh is defined on $t^\ast \parallel \tilde{W}_{aff}$, we prove the following:

Proposition 4.10. Any prestack which is a countable discrete set of points is 0-coconnective locally finite type (lft). Furthermore, the respective quotient prestacks $t^\ast/X^\bullet(T)$ and $t^\ast \parallel \tilde{W}_{aff}$ are 0-coconnective lft prestacks.

Proof. The $n$-coconnective prestacks (respectively, the $n$-coconnective lft prestacks) are those prestacks which are in the essential image of a certain left adjoint—namely, the left Kan extension of the inclusion of $n$-coconnective affine schemes (respectively, the left Kan extension of the inclusion of $n$-coconnective finite type affine schemes in the sense of [GR17b, Chapter 2, Section 1.5]). Therefore, the condition of being $n$-coconnective and the condition of being $n$-coconnective lft are conditions closed under colimits. Classical finite type affine schemes are 0-coconnective lft. Therefore since this condition is closed under colimits, then any colimit of classical schemes is 0-coconnective lft. In turn, since $t^\ast \parallel \tilde{W}_{aff}$ is a certain colimit of 0-coconnective lft prestacks by Corollary 4.5 we see that $t^\ast \parallel \tilde{W}_{aff}$ is a 0-coconnective lft prestack.

Thus by Proposition 4.10 we obtain that IndCoh($t^\ast \parallel \tilde{W}_{aff}$) is defined. In particular, Corollary 4.9 implies that $\pi$ is in particular ind-schematic and ind-proper surjection between lft prestacks. We now record two immediate consequences of this fact.

Corollary 4.11. [GR17c, Chapter 3, Section 0.1.2] The functor $\pi^!$ admits a left adjoint satisfying base change against l-pushbacks.

We denote this left adjoint by $\pi_{\ast}^{\text{IndCoh}}$. 

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Corollary 4.12. [GR17c, Chapter 3, Section 0.4.3] If \( \overline{s} : t^* \to t^* \sslash \mathcal{W}^{\text{aff}} \) denotes the quotient map, the pullback functor \( \overline{s}^* : \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \to \text{IndCoh}(t^*) \) induces an equivalence

\[
\overline{s}^* : \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \xrightarrow{\sim} \text{Tot}(\text{IndCoh}(t^*))
\]

where \( t^* \) is the cosimplicial prestack given by the Cech nerve of \( \overline{s} \).

Remark 4.13. The definition of the coarse quotient \( t^* \sslash \mathcal{W}^{\text{aff}} \) was inspired by the definition of the coarse quotient given when \( \mathcal{W}^{\text{aff}} \) is replaced by a Coxeter group in [BG17, Section 2.7.3].

4.2.2. \( t \)-Structure for Sheaves on Quotients. We use the following proposition to define a \( t \)-structure on the category \( \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \) and determine some of its basic properties in Proposition 4.15.

Proposition 4.14. The maps \( s^* : \text{IndCoh}(t^*) \to \text{IndCoh}(\Gamma_{\mathcal{W}^{\text{aff}}}) \) and \( s^*_{\text{IndCoh}} \) are \( t \)-exact.

Proof. Because the \( t \)-structure on \( \text{IndCoh}(\Gamma_{\mathcal{W}^{\text{aff}}}) \) is by definition compatible with filtered colimits ([GR17c, Chapter 3, Section 1.2.1]) it suffices to show this claim when \( \mathcal{W}^{\text{aff}} = \mathcal{W}^{\text{aff}} \). Note that the map \( s^*_{\text{IndCoh}} \) is \( t \)-exact because \( s \) is ind-affine ([GR17c, Chapter 3, Lemma 1.4.9]), and therefore since \( s \) is ind-proper, we have that \( s^* \) is the right adjoint to the \( t \)-exact functor \( s^*_{\text{IndCoh}} \) (see Corollary 4.11) and therefore is left \( t \)-exact.

We now show that \( s^* \) is right \( t \)-exact. We first note that, to show \( s^* \) is right \( t \)-exact, it suffices to show that \( s^*(\mathcal{O}_{t^*}) \in \text{IndCoh}(\Gamma_{\mathcal{W}^{\text{aff}}}) \leq 0 \). This follows since \( \text{IndCoh}(t^*) \leq 0 \) is equivalently smallest \( \infty \)-category of \( \text{IndCoh}(t^*) \) containing \( \mathcal{O}_{t^*} \), and closed under colimits (which can be seen, for example, by the \( t \)-exact equivalence \( \Psi_{t^*} : \text{IndCoh}(t^*) \leq 0 \to \text{QCoh}(t^*) \leq 0 \) given by the fact \( t^* \) is smooth and classical). Since \( s^* \) commutes with colimits and the subcategory \( \text{IndCoh}(\Gamma_{\mathcal{W}^{\text{aff}}}) \leq 0 \) is closed under colimits, we see that it remains to show that \( s^*(\mathcal{O}_{t^*}) \in \text{IndCoh}(\Gamma_{\mathcal{W}^{\text{aff}}}) \leq 0 \). In turn, to show this, we first note that

\[
s^*(\mathcal{O}_{t^*}) \simeq s^*(\omega_{t^*}[-d]) \simeq \omega_{t^*}[-d] \simeq \operatorname{colim}_m s^*_{\text{IndCoh}}(\omega_{\Gamma_m})[-d]
\]

where the first equivalence follows from the fact that \( t^* \) is smooth, the second follows from the definition of the dualizing complex and the functoriality of \( t \)-pullback, and the third follows since we have an equivalence \( \text{IndCoh}(\Gamma) \leftarrow \text{colim}_m \text{IndCoh}(\Gamma_m) \). We claim that each dualizing complex \( \omega_{\Gamma_m} \) is concentrated in a single cohomological degree, i.e. \( \Gamma_m \) is Cohen-Macaulay. This follows from the fact that the map \( \Gamma_m \to t^* \) is a finite flat map (Proposition 3.4) to affine space, and therefore \( \Gamma_m \) is Cohen-Macaulay. Thus each object of the above colimit is contained entirely in cohomological degree zero ([GR17c, Chapter 4, Lemma 1.2.5]) and therefore so too is \( s^*(\mathcal{O}_{t^*}) \) since the \( t \)-structure is compatible with filtered colimits.

Recall the canonical quotient map \( \overline{s} : t^* \to t^* \sslash \mathcal{W}^{\text{aff}} \). Define a \( t \)-structure on \( \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \) by declaring \( \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \leq 0 \) to be the full ordinary \( \infty \)-subcategory closed under colimits and containing \( \overline{s}^*_{\text{IndCoh}}(\mathcal{O}_{t^*}) \). Similarly, we define a \( t \)-structure on \( \text{IndCoh}(t^*/\mathcal{W}^{\text{aff}}) \) (respectively, \( \text{IndCoh}(t^*/\mathcal{X}^*(T)) \)) by declaring \( \text{IndCoh}(t^*/\mathcal{W}^{\text{aff}}) \leq 0 \) to be the full ordinary \( \infty \)-subcategory closed under colimits and containing the respective \( \text{IndCoh} \) pushforward given by the quotient map of the structure sheaf \( \mathcal{O}_{t^*} \). Note that these do indeed define \( t \)-structures since the inclusion functor preserves colimits, and therefore admits a right adjoint. We now record further properties of these \( t \)-structures:

Proposition 4.15. With the \( t \)-structure on \( \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \) defined as above, we have the following:

1. The functor \( \overline{s}^*_{\text{IndCoh}} \) is \( t \)-exact.
2. The map \( \overline{s}^*_{\text{IndCoh}} \) is \( t \)-exact and reflects the \( t \)-structure.
3. The map \( \overline{s}^*_{\text{IndCoh}} : \text{IndCoh}(t^*) \to \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \) is \( t \)-exact.
4. The \( t \)-structure on \( \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \) is compatible with filtered colimits.

Proof. The first claim follows by base change (Corollary 4.11) of the Cartesian diagram in Proposition 4.10 since we may identify this functor with the composite of \( s^* \), \( t \)-exact by Proposition 4.14 with the functor \( s^*_{\text{IndCoh}} \), which is \( t \)-exact since \( s \) is ind-affine ([GR17c, Chapter 3, Lemma 1.4.9]).

Next, we show that \( \overline{s}^* \) is right \( t \)-exact. If \( \mathcal{G} \in \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) \leq 0 \), we may write \( \mathcal{G} \) as some colimit \( \operatorname{colim}(\overline{s}^*_{\text{IndCoh}}(\mathcal{O}_{t^*})) \). Since \( \overline{s}^* \) is continuous, we see that by (1) that \( \overline{s}^*(\mathcal{G}) \) is a colimit of objects in the heart of the \( t \)-structure, and thus lies in \( \text{IndCoh}(t^*) \leq 0 \).

To see the left \( t \)-exactness of \( \overline{s}^* \), let \( \mathcal{F} \in \text{IndCoh}(t^* \sslash \mathcal{W}^{\text{aff}}) > 0 \). We wish to show that \( \overline{s}^*(\mathcal{F}) \in \text{IndCoh}(t^*) > 0 \), and to show this it suffices to show that \( \text{Hom}_{\text{IndCoh}(t^*)}(\mathcal{O}_{t^*}, \overline{s}^*(\mathcal{F})) \) vanishes. However, since \( \overline{s}^* \) is ind-proper,
we see that by adjunction (Corollary 4.11) it suffices to show $\text{Hom}_{\text{IndCoh}(t^*/\tilde{W}^{\text{aff}})}(\mathbb{S}^\flat_{t^*}(\mathcal{O}_{t^*}), \mathcal{F})$ vanishes, which follows by the definition of the $t$-structure. Thus the functor $\mathbb{S}^\flat_{t^*}$ is $t$-exact, and this along with its conservativity gives (2).

Now, to show (3), note that (2) gives that $\mathbb{S}^\flat_{t^*}$ reflects the $t$-structure, so it suffices to show that $\mathbb{S}^\flat_{t^*}\mathbb{S}^\flat_{t^*}$ is $t$-exact, which is precisely (1). Finally, (4) follows from the fact that $\mathbb{S}^\flat_{t^*}$ is continuous and reflects the $t$-structure, along with the fact that $t$-structure on $\text{IndCoh}(t^*)$ is compatible with filtered colimits. □

We can use a similar argument to construct $t$-structures in the setting of a discrete group acting on some ind-scheme. Let $X$ denote a discrete group acting on some ind-scheme $\Gamma$ and let $\phi : \Gamma \rightarrow \Gamma/X$ denote the quotient map. Define a $t$-structure on $\text{IndCoh}(\Gamma/X)$ via setting $\text{IndCoh}(\Gamma/X)^{\geq 0}$ to be the full subcategory generated under colimits by objects of the form $\phi^!_{\text{IndCoh}}(\mathcal{F})$ for $\mathcal{F} \in \text{IndCoh}(\Gamma)^{\leq 0}$ (or, equivalently by the continuity of $\phi^!_{\text{IndCoh}}$, for $\mathcal{F} \in \text{IndCoh}(\Gamma)^{\leq 0}$).

**Lemma 4.16.** The functors $\phi^!_{\text{IndCoh}}$ and $\phi^*$ are $t$-exact.

**Proof.** We first show $\phi^*$ is $t$-exact. Consider the Cartesian diagram:

\[
\begin{array}{ccc}
\Gamma \times X & \xrightarrow{\text{act}} & \Gamma \\
\downarrow \text{proj} & & \downarrow \phi \\
\Gamma & \xrightarrow{\phi} & \Gamma/X
\end{array}
\]

given by the quotient. Since act is ind-affine, it is $t$-exact. We also have by direct computation that $\text{proj}^!_{t^*}$ is $t$-exact. Therefore, by base change and continuity of $\phi^*$, we see that $\phi^*$ is right $t$-exact. For left $t$-exactness, assume $\mathcal{F} \in \text{IndCoh}(\Gamma/X)^{>0}$ and $\mathcal{G} \in \text{IndCoh}(\Gamma)^{\leq 0}$. Then since $\phi$ is ind-proper, by adjunction we see

\[
\text{Hom}_{\text{IndCoh}(\Gamma)}(\mathcal{G}, \phi^*(\mathcal{F})) \simeq \text{Hom}_{\text{IndCoh}(\Gamma/X)}(\phi^!_{\text{IndCoh}}(\mathcal{G}), \mathcal{F}) \simeq 0
\]

by construction of our $t$-structure, and so $\phi^*(\mathcal{F}) \in \text{IndCoh}(\Gamma)^{>0}$. Finally, since $\phi^*$ is $t$-exact and conservative, $\phi^!_{\text{IndCoh}}$ is $t$-exact if and only if $\phi^!_{\text{IndCoh}}\phi^*$ is $t$-exact. However, this follows from base change along diagram (10). □

**Corollary 4.17.** The $t$-structures on $\text{IndCoh}(t^*/\tilde{W}^{\text{aff}})$ and $\text{IndCoh}(t^* // \tilde{W}^{\text{aff}})$ are both left-complete and right-complete.

**Proof.** By Lemma 4.16 and Proposition 4.15 both categories admit conservative, $t$-exact functors to $\text{IndCoh}(t^*)$ which commute with limits (since they are right adjoints). Any category which admits a conservative, $t$-exact functor which commutes with limits to a left-complete category is left-complete, therefore the left-completeness holds in this case, where $\text{IndCoh}(t^*)$ admits a $t$-exact equivalence to $\text{QCoh}(t^*)$ (since $t^*$ is a smooth classical scheme) and therefore is left-complete. Similarly, each functor to $\text{IndCoh}(t^*)$ is continuous and so the right-completeness follows from the fact that $\text{IndCoh}(t^*)$ is also right-complete. □

4.3. **Descent to the Coarse Quotient for Affine Weyl Groups.** The quotient map $\pi : t^* \rightarrow t^* // \tilde{W}^{\text{aff}}$ induces a canonical map of prestacks $\phi : t^*/\tilde{W}^{\text{aff}} \rightarrow t^* // \tilde{W}^{\text{aff}}$. We now study the pullback functor $\phi_!$ and show that this functor behaves similarly to the case where $\tilde{W}^{\text{aff}}$ is replaced with a finite Weyl group. For example, we show that the functor $\phi_!$ is fully faithful in Theorem 4.18. We define those sheaves in $\text{IndCoh}(t^*/\tilde{W}^{\text{aff}})$ in the essential image of $\phi_!$ as those sheaves descending to the coarse quotient $t^* // \tilde{W}^{\text{aff}}$, and provide descriptions of those sheaves descending to the coarse quotient $t^* // \tilde{W}^{\text{aff}}$ in Section 4.3.3 which parallel the description for the finite Weyl group case in Proposition 2.16.

4.3.1. **Fully Faithfulness of Affine Pullback.**

**Theorem 4.18.** The functor $\phi_!$ is fully faithful.

This subsection will be dedicated to the proof of Theorem 4.18. For a given $x \in t^*(K)$, let $C_x$ denote the coinvariant algebra for the action of $W_x^{\text{aff}}$ on $t^*$, which is in particular a $K$-algebra. The closed subscheme $\text{Spec}(C_x) \rightarrow t^*$ induces a map $\text{Spec}(C_x)/W_x^{\text{aff}} \rightarrow t^*/\tilde{W}^{\text{aff}}$ which we denote by $q!$. Furthermore, let $[x]$ denote
the image of $x$ under the quotient map $t^* \to t^*/\mathcal{X}^*(T)$, and let $\mathcal{X}$ denote the image of $x$ under the quotient map $q : t^* \to t^*/W^\text{aff}$. Since the map $\tilde{\phi}$ induces a bijection on $K$-points, so we abuse notation in also regarding $\mathcal{X}$ as a $K$-point of $t^* \parallel W^\text{aff}$.

**Proposition 4.19.** Fix some $x \in t^*(K)$. There is a $W^\text{aff}$-equivariant isomorphism

$$\Gamma_{W^\text{aff}} \times _{t^*} \text{Spec}(K) \simeq W^\text{aff} \times _{X^*} \prod_{x \in \text{orbit}_{W^\text{aff}}(x)} \text{Spec}(C_{x^*})$$

and, moreover, the rectangles of the following diagram are (derived) Cartesian:

$$\begin{array}{ccc}
W^\text{aff} \times \prod_{x \in \text{orbit}_{W^\text{aff}}(x)} \text{Spec}(C_{x^*}) & \xrightarrow{s} & \text{Spec}(C_x)/W^\text{aff} \\
\downarrow{t} & & \downarrow{q} \\
t^*/W^\text{aff} & \xrightarrow{\tilde{\phi}} & \text{Spec}(K)
\end{array}$$

$$\begin{array}{ccc}
t^* & \xrightarrow{\text{id}} & t^* \\
\downarrow{q} & & \downarrow{\tau} \\
t^*/W^\text{aff} & \xrightarrow{\tilde{\phi}} & t^* \parallel W^\text{aff}
\end{array}$$

**Proof.** We first claim the outer rectangle is Cartesian. Applying Proposition 4.3, we may prove this first claim when $W^\text{aff} = W$. Write $\Gamma_{W^\text{aff}}$ as a union of $\Gamma_S$ where $S \subseteq W^\text{aff}$ varies over the finite subsets. Because this set is filtered, colimits over it commute with all finite limits (and, in particular, Cartesian products), and so we obtain

$$\Gamma_{W^\text{aff}} \times _{t^*} \text{Spec}(L) \simeq \prod_S (\Gamma_S \times _{t^*} \text{Spec}(L)) \simeq \prod_S \prod_{x \in \text{orbit}(x)} (\Gamma_{S/\text{stab}(x^*)} \times _{t^*} \text{Spec}(L))$$

where the second equivalence follows from Lemma 3.5. Therefore we see that the outer rectangle in Proposition 4.19 is Cartesian.

The fact that the left box is Cartesian follows from the fact that the stack quotient $t^*/W^\text{aff}$ is defined as the colimit over a groupoid $U$ such that $U_1 \simeq W^\text{aff} \times t^*$, and so in particular $t^* \times _{t^*/W^\text{aff}} t^* \simeq W^\text{aff} \times t^*$ by Proposition 4.2. Now, because the outer rectangle is Cartesian and all of the maps are $W^\text{aff}$-equivariant, we may take the quotient by $W$. This is a sifted colimit because the opposite category of the simplex category is sifted, and in particular, taking the quotient by $W^\text{aff}$ preserves the Cartesian product and shows the rightmost rectangle is Cartesian. \qed

**Lemma 4.20.** In the setup and notation of Proposition 4.19 the functor $\tilde{\phi}$ admits a (continuous) left adjoint.

**Proof.** To show that $\tilde{\phi}$ admits a left adjoint, it suffices to show that $\tilde{\phi}$ commutes with (small) limits, by the adjoint functor theorem (see Lur09, Chapter 5). To see that $\tilde{\phi}$ commutes with small limits, consider the following commutative diagram:

$$\begin{array}{ccc}
t^* & \xrightarrow{\text{id}} & t^* \\
\downarrow{q} & & \downarrow{\tau} \\
t^*/W^\text{aff} & \xrightarrow{\tilde{\phi}} & t^* \parallel W^\text{aff}
\end{array}$$

Since $q$ is ind-proper and surjective on geometric points, by ind-proper descent (say) we have that $q^!$ is conservative. Therefore we may check that a map in $\text{IndCoh}(t^*/W^\text{aff})$ is an isomorphism after applying $q^!$. However, since $q^! \tilde{\phi}^! \simeq \tilde{\phi}^!$, we see that $q^! \tilde{\phi}^!$ commutes with small limits (since $\tilde{\phi}^!$ is also a right adjoint since $\tilde{\phi}$ is ind-proper by Corollary 4.9) and so $\tilde{\phi}^!$ commutes with small limits as well, and thus admits a left adjoint by the adjoint functor theorem Lur09. \qed

Denote the left adjoint to $\tilde{\phi}^!$ by $\tilde{\phi}^!_{\text{IndCoh}}$. This light abuse of notation is justified by the following:
Corollary 4.21. The three Cartesian diagrams of Proposition 4.19 satisfy base change. In particular, the canonical map \( \hat{\alpha}^*_f \text{IndCoh}(q) \rightarrow \mathfrak{P} \hat{\alpha}^*_f \text{IndCoh} \) is an isomorphism.

Proof. The left and the ‘large’ Cartesian diagrams satisfy base change since the maps \( q \) and \( \mathfrak{P} \) are ind-schematic and so satisfy base change by [GR17c, Chapter 3, Theorem 5.4.3], see also Corollary 4.11. To show base change for the other Cartesian diagram, note that we may check that the map is an isomorphism on a compact generator of \( t^*/W_\text{aff} \). We choose the generator \( G := \hat{\alpha}^*_f \text{IndCoh}(\omega_t) \). The uniqueness of left adjoints then gives that \( \hat{\alpha}^*_f \text{IndCoh}(G) \simeq \mathfrak{P} \hat{\alpha}^*_f \text{IndCoh}(\omega_t) \). Base change by the outer Cartesian diagram of Proposition 4.19 then gives the desired claim. □

Proof of Theorem 4.18. To show that \( \hat{\alpha}^f \) is fully faithful, it suffices to show that the counit map \( \hat{\alpha}^*_f \text{IndCoh} \phi^f \rightarrow \text{id} \) is an equivalence. Because \( \mathfrak{P} \) admits a left adjoint (Corollary 4.11), we have that \( G := \hat{\alpha}^*_f \text{IndCoh}(\omega_t) \) is a compact generator of \( \text{IndCoh}(t^*/W_\text{aff}) \). Therefore to show that the counit is an equivalence, by continuity it suffices to show that the map \( c(G) : \hat{\alpha}^*_f \text{IndCoh} \phi^f(G) \rightarrow G \) is an equivalence. Since \( \mathfrak{P} \) is an ind-proper cover (see Corollary 4.9), \( \mathfrak{P} \) is conservative, and so it suffices to show \( s^i(c(G)) : s^i(\hat{\alpha}^*_f \text{IndCoh} \phi^f(G)) \rightarrow s^i(G) \) is an equivalence.

However, our map \( s^i(c(G)) \) is a map in \( \text{IndCoh}(t^*) \), which is generated by the skyscraper sheaves associated to all field-valued points, a direct consequence of Proposition 2.7 and the smoothness of \( t^* \), which gives that \( \mathfrak{P}_t \) is an equivalence. Therefore, we may show this map is an isomorphism when restricted to each field-valued point \( x \in t^* \). By Proposition 4.22 we see that \( \mathfrak{P}c(G) \simeq c_\phi(\mathfrak{P}t(G)) \), where \( c_\phi \) denotes the counit of the adjunction \((\hat{\alpha}^*_f \text{IndCoh}, \hat{\alpha}^f)\) of for the finite Coxeter group. However, we have that \( \hat{\alpha}^f \) is fully faithful by Proposition 2.2. Therefore, since this holds for every field-valued point \( x \), \( \hat{\alpha}^f \) is also fully faithful. □

4.3.2. Equivalent Characterizations of Descent to the Coarse Quotient for The Affine Weyl Group. We have seen in Theorem 4.18 that the functor \( \hat{\alpha}^f \) is fully faithful. In analogy with the case where \( W_\text{aff} \) is replaced with a finite group, we make the following definition:

Definition 4.22. We say that a sheaf \( \mathcal{F} \in \text{IndCoh}(t^*/X^*(T))W \) descends to the coarse quotient \( t^* \parallel W_\text{aff} \) if it lies in the essential image of \( \hat{\alpha}^f \). When the \( W_\text{aff} \)-action is clear from context, we will simply say the given sheaf descends to the coarse quotient.

We now provide many alternative characterizations of a \( W_\text{aff} \)-equivariant sheaf descending to the coarse quotient, noting that many of the following conditions involve the usual affine Weyl group \( W_\text{aff} \) as opposed to the extended affine Weyl group \( W_\text{aff} \).

Theorem 4.23. A sheaf \( \mathcal{F} \in \text{IndCoh}(t^*)W_\text{aff} \) descends to the coarse quotient \( t^* \parallel W_\text{aff} \) if and only if it satisfies one of the following equivalent conditions:

1. For every field-valued point \( x \in t^*(K) \), the canonical \( W_\text{aff} \)-representation on \( \mathfrak{P}^i(\text{obl} \hat{\alpha}^*_f W_\text{aff} \mathcal{F}) \) is trivial.
2. For every finite parabolic subgroup \( W' \) of \( W_\text{aff} \), the object \( \text{obl} \hat{\alpha}^*_f W_\text{aff} \mathcal{F} \in \text{IndCoh}(t^*/W') \) descends to the coarse quotient \( t^* \parallel W' \).
3. The object \( \text{obl} \hat{\alpha}^*_f (\mathcal{F}) \in \text{IndCoh}(t^*/\langle r \rangle) \) descends to the coarse quotient \( t^* \parallel \langle r \rangle \) for every reflection \( r \in W_\text{aff} \).
4. The object \( \text{obl} \hat{\alpha}^*_f (\mathcal{F}) \in \text{IndCoh}(t^*/\langle s \rangle) \) descends to the coarse quotient \( t^* \parallel \langle s \rangle \) for every simple reflection \( s \in W_\text{aff} \).
5. The object \( \text{obl} \hat{\alpha}^*_f (\mathcal{F}) \in \text{IndCoh}(t^*/W) \) descends to the coarse quotient \( t^* \parallel W \).
6. For each \( n \), each cohomology group \( \tau^r_\mathcal{F} \) given by the \( t \)-structure in Section 4.2 descends to the coarse quotient \( t^* \parallel W_\text{aff} \).

Proof. We first show that \( \mathcal{F} \) descends to the coarse quotient \( t^* \parallel W_\text{aff} \) if and only if \( \mathcal{F} \) satisfies (1). For a given \( x \in t^*(K) \), note that the fact that the right box in Proposition 4.19 commutes implies that any object in the essential image of \( \hat{\alpha}^f \) has the property that the pullback to \( \text{IndCoh}(\text{Spec}(K))W_\text{aff} \simeq \text{Rep}_K(W_\text{aff}) \) is trivial. Therefore, it remains to show that the left adjoint \( \hat{\alpha}^*_f \text{IndCoh} \) of Lemma 4.20 is conservative on this subcategory, since a functor with an adjoint is an equivalence if and only if it is fully faithful and its adjoint is conservative.
Assume we are given some nonzero $F \in \text{IndCoh}(t^*)^\text{aff}$ has the property that, for every field-valued point $x$ of $t^*$, the pullback to $\text{IndCoh}(\text{Spec}(K))^\text{aff}_x \simeq \text{Rep}_K(W^\text{aff}_x)$ is trivial. Since $F$ is nonzero, its pullback $q^!(F)$ is nonzero, and so in particular there exists a field-valued point for which $x^* q^!(F) \simeq \pi^!(F)$ is nonzero by Proposition 2.7. Furthermore, since $\Upsilon_\cdot$ is an equivalence, we see that $\pi^!(\text{obl}W^\text{aff}_x(F))$ lies in the full subcategory determined by the fully faithful (see Corollary 2.13) functor $\Xi^\text{aff}_{\text{Spec}(C)}$ since $\Upsilon$ intertwines with pullback. Therefore, by Corollary 2.13 we see that the assumption that $\pi^!(\text{obl}W^\text{aff}_x(F))$ is the trivial representation implies that, in the notation of Proposition 4.19, $q^!(\text{obl}W^\text{aff}_x(F))$ lies in the essential image of $\alpha^!$. Moreover, this sheaf is nonzero since $\pi^!(F)$ is nonzero and, since $i_x : \text{Spec}(K) \to \text{Spec}(C_x)$ is surjective on geometric points, $i_x^!$ is conservative [GR17b, Chapter 4, Proposition 6.2.2], and therefore the pullback functor $i_x^!$ is conservative. Thus we in particular see that $\alpha^! \text{IndCoh}(q^!(\text{obl}W^\text{aff}_x(F)))$ is nonzero. Applying base change (Corollary 4.21), we therefore see that $\pi^!(\text{obl}W^\text{aff}_x(F))$ is nonzero, and so in particular there exists a field-valued point for which $x^* q^!(F) \simeq \pi^!(F)$ is nonzero, and therefore neither does $\alpha^! \text{IndCoh}(F)$, as required.

Now, to show that (1) $\Rightarrow$ (2), let $F \in \text{IndCoh}(t^*)^\text{aff}$ be some sheaf satisfying (1) and assume $W'$ is some parabolic subgroup of $W^\text{aff}$. We wish to show that $G := \text{obl}W^\text{aff}_x(F)$ descends to the coarse quotient $t^*/W^\text{aff}_x$. By Proposition 2.16(3), it suffices to show that the canonical $W^\text{aff}_x$-representation on $x^!(G)$ is trivial. However, note that the following diagram commutes

$$
\begin{array}{ccc}
\text{Rep}(W^\text{aff}_x) & \xrightarrow{\text{obl}W^\text{aff}_x} & \text{Rep}(W') \\
\uparrow x^! & & \uparrow x^! \\
\text{IndCoh}(t^*)^\text{aff} & \xrightarrow{\text{obl}W^\text{aff}_x} & \text{IndCoh}(t^*)^W
\end{array}
$$

and so the fact that the associated $W^\text{aff}_x$-representation structure on $x^!(G)$ is trivial implies that the associated $W'_x$-representation is trivial, as desired.

Conversely, if we are given some $F$ satisfying (2) and some field-valued $x \in t^!(K)$, it is standard (see, for example, [Lon18, Proposition 5.3]) that the subgroup $W^\text{aff}_x$ is a finite parabolic subgroup. Therefore we see that, by assumption, $\text{obl}W^\text{aff}_x(F)$ descends to the coarse quotient for $t^*/W^\text{aff}_x$, and so that by Proposition 2.16(3), the canonical $(W^\text{aff}_x)_x = W^\text{aff}_x$-representation on $x^!(\text{obl}W^\text{aff}_x(F))$ is trivial, as required.

The equivalence (2) $\iff$ (3) follows directly from the fact that one can check if a given $G \in \text{IndCoh}(t^*)^W$ descends to the coarse quotient for $t^*/W^\text{aff}_x$ if and only if $\text{obl}W^\text{aff}_x(G) \in \text{IndCoh}(t^*)^W$ descends to the coarse quotient $t^*/(r)$ for all reflections $r \in W'$, see Proposition 2.16. Similarly, the equivalence (4) $\iff$ (5) by varying $r$ over all simple reflections of $W$, see Proposition 2.16(3).

The proof of the equivalence (3) $\iff$ (4) follows nearly identically to the proof of the claim ‘(2) $\iff$ (3)’ of Proposition 2.16. The relevant addition is the standard fact (see, for example, [BM13, Lemma 2.1.1]) that any reflection of $W^\text{aff}_x$ is conjugate in $W^\text{aff}_x$ to some simple reflection of $W$.

Because $\tilde{\phi}^!$ is fully faithful (Theorem 4.18) and $t$-exact (Proposition 4.15) we have that the essential image is closed under truncations, thus showing that if $F$ descends to the coarse quotient, then so too does its cohomology groups. Since the $t$-structure on $\text{IndCoh}(t^*)/W^\text{aff}$ is left-complete and right-complete (Corollary 4.17), given $F \in \text{IndCoh}(t^*/W^\text{aff})$ has a canonical isomorphism

$$
F \simeq \lim_{m, n} \text{colim}_\tau \tau^\geq m \tau \leq n(F)
$$

where $-m, n \in \mathbb{Z}_{\geq 0}$. Therefore, since $\tilde{\phi}^!$ is a continuous right adjoint, its essential image is closed under both limits and colimits. Thus since the essential image of $\phi^!$ is also closed under extensions (by fully faithfulness) we see that if all cohomology groups of $F$ descend to the coarse quotient $t^*/W^\text{aff}$, so too does $F$. 

\section{Appendix A. Review of Borel Isomorphism Extension}

A celebrated theorem of Borel identifies the cohomology of the flag variety $H^*(G^\vee/B^\vee)$ with the coinvariant algebra $C := \text{Sym}(t)/\text{Sym}(t)_+^W$, where $G^\vee$ denotes the Langlands dual group to $G$ and $B^\vee$ denotes...
the corresponding Borel subgroup. In this section, we review a statement of an upgrade of this theorem, Theorem A.5 when the flag variety is replaced with the (closed) Schubert variety \( X_v \to G^\vee/B^\vee \) and give an alternate proof. In fact, we will generalize this theorem to the cohomology of unions of Schubert cells \( H^*(X_S) \) determined by \emph{closed} subsets of the Weyl group:

**Definition A.1.** If \( \tilde{W} \) is a Coxeter group, we say a subset \( S \subseteq \tilde{W} \) is \emph{closed} if \( w \in S \) and \( w' \leq w \) implies \( w' \in S \).

To state our desired extension to closed subsets of the Weyl group, we first obtain the following alternate description of \( C \). Consider the scheme \( t^* \times t^* \), and let \( \text{graph}(w) \) denote the closed subscheme cut out by the ideal \( I_{\text{graph}(w)} \), defined in turn to be the ideal generated by elements of the form \( wp \otimes 1 - 1 \otimes p \) for \( p \in \text{Sym}(t) \). Set \( I_W := \bigcap_{w \in W} I_{\text{graph}(w)} \), and set \( J_W \) to be the ideal generated by \( I_W \) and \( \text{Sym}(t)^+ \otimes \text{Sym}(t) \). Similarly, assume we are given a closed subset \( S \subseteq W \). Set \( I_S := \bigcap_{w \in S} I_{\text{graph}(w)} \), and set \( J_S \) to be the ideal generated by \( I_S \) and \( \text{Sym}(t)^+ \otimes \text{Sym}(t) \). Note that we obtain a canonical map

\[
\Phi : \text{Sym}(t) \otimes \text{Sym}(t)^w \text{Sym}(t) \to \text{Sym}(t \times t)/I_W
\]

We now prove the following proposition, which says that the product \( t^* \times t^* /_W t^* \) may be identified with the union of graphs of \( W \):

**Proposition A.2.** The map \( \Phi \) is an isomorphism.

**Proof.** We note that \( \Phi \) is in particular a map of \( \text{Sym}(t) \)-modules. We see that by base changing the quotient map \( t^* \to t^* / W \) by itself, by Proposition 2.5 the \( \text{Sym}(t) \)-module \( \text{Sym}(t) \otimes \text{Sym}(t)^w \text{Sym}(t) \) is finite flat. The fact that \( \text{Sym}(t) \otimes \text{Sym}(t)^w \text{Sym}(t) \) is finite implies that \( \text{Sym}(t \times t)/I_W \) is also finite as a \( \text{Sym}(t) \)-module. We also see that \( W \) acts freely on a dense open subset of \( t^* \), so that for a dense open subset of \( t^* \), the fiber of the map \( \text{Spec}(\text{Sym}(t \times t)/I_W) \to t^* \) has degree \( W \). This in particular implies that all fibers of the \( \text{Sym}(t) \)-module \( \text{Sym}(t \times t)/I_W \) have rank no less than \( |W| \), since degree of a finite morphism is a upper semicontinuous function on the target. Hence, we also see that each fiber at some point \( x \) admits a surjection from the fiber of \( \text{Sym}(t) \otimes \text{Sym}(t)^w \text{Sym}(t) \) at the point, which has rank precisely \( |W| \) since \( \text{Sym}(t) \) is a free rank \( \text{Sym}(t)^W \)-module of rank \( W \) by the Chevalley-Shephard-Todd theorem. \( \square \)

**Remark A.3.** We temporarily assume that \( G_Z \) is an adjoint type Chevalley group scheme defined over the integers with maximal torus \( T_Z \). An identical result to Proposition A.2 with a similar proof (which will not be used below) also holds for the action of the Weyl group \( W \) on the torus \( T_Z \). The analogue of Proposition 2.5 is the Pittie-Steinberg theorem \( \text{Ste}75 \), which says that for such \( G_Z \) that \( \mathcal{O}(T^\vee_Z) \) is a free \( \mathcal{O}(T^\vee_Z)^W \)-module of rank \( |W| \).

**Remark A.4.** Note that we work with the union of graphs given by the intersection of ideals, not the product. For example, if \( g = st_s \), we may pick a simple reflection \( s \in W \) and choose coordinates on \( t^* \) so that \( t^* \simeq \text{Spec}(k[h,p]) \) where \( s(h) = -h \) and \( s(p) = p \). The intersection \( I_t \cap I_s \) contains the degree 1 polynomial \( p \otimes 1 - 1 \otimes p \), whereas the product \( I_t I_s \) is generated by degree two polynomials.

In particular, Proposition A.2 shows that one can identify \( C \cong \text{Sym}(t \times t)/J_W \). For a closed subset \( S \subseteq W \), let \( X_S \) denote the closed subvariety of \( G^\vee/B^\vee \) given by the union of the Schubert cells labelled by \( w \in S \). We now give an alternate proof of the following result of \( \text{Car}92 \):

**Theorem A.5.** Fix a closed subset \( S \subseteq W \). There is an isomorphism \( H^*(X_S) \simeq \text{Sym}(t \times t)/J_S \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Sym}(t \times t)/J_W & \xrightarrow{\sim} & H^*(G^\vee/B^\vee) \\
\downarrow \quad & & \\
\text{Sym}(t \times t)/J_S & \xrightarrow{\sim} & H^*(X_S)
\end{array}
\]

where the vertical maps are the canonical quotient maps and the top arrow is the Borel isomorphism.
Remark A.6. An alternate description of the rings \( \text{Sym}(t \times t)/J_S \) in the case \( G = \text{GL}_n \) and \( S \) is the closure of some Weyl group element is computed in \([ALP92]\). We thank Victor Reiner for making us aware of this reference.

A.0.1. Results on Demazure Operators. We first will recall some definitions and results of \([BGG73a]\) and \([BGG73b]\).

Definition A.7. For a simple reflection \( s \in W \) associated to a coroot \( \alpha \), define the vector space map
\[ D_s : \text{Sym}(t) \rightarrow \text{Sym}(t) \] via \( D_s(f) := \frac{t^s - t^{-s}}{\alpha} \). For a \( w \in W \), choose a reduced expression \( w = s_1 ... s_r \) and set \( D_w := D_{s_1} ... D_{s_r} \). These \( D_w \) are known as Demazure operators.

Let \( w_0 \) denote the longest element of the Weyl group \( W \) with respect to some ordering, and let \( \ell := \ell(w_0) \). We now recall the following theorem:

Theorem A.8. \([BGG73b]\) We have the following:

1. The Demazure operators are well defined and independent of reduced expression.
2. If \( s_1, ..., s_p \) is not a simple expression, then \( D_{s_1} ... D_{s_p} \) vanishes.
3. The Poincaré dual class to the Schubert variety \([X_1] \in H_0(G/B) \) maps to \( \rho^f/\ell! \) in the coinvariant algebra, and if \( S \subseteq W \) is closed, the vector space \( H^*(X_S) \) has a basis given by the \( D_w(\rho^f/\ell!) \) for which \( w_0^{-1} \in S \).

Remark A.9. To translate between point (3) of Theorem A.8 and Theorem 3.15 in \([BGG73b]\), we note that the vector space \( H^*(X_S) \) also, in the notation of \([BGG73b]\), has basis \( P_w \) for which \( w \in S \). The notation \( P_w \) will not be used outside this remark.

A.0.2. Proof of Theorem A.8. In this subsection, we prove Theorem A.8. To prove Theorem A.8, we will first determine a specific element of \( \text{Sym}(t \times t) \) which projects to a nonzero homogeneous element of degree \( \ell := \ell(w_0) \) under the composite \( \text{Sym}(t \times t) \rightarrow \text{Sym}(t \times t)/J_W \approx H^*(G/B) \).

Proposition A.10. There exists a polynomial \( F(x, y) \in \text{Sym}(t \times t) \) such that

1. \( F(x, vx) = 0 \) if \( v \neq w_0 \).
2. \( F(x, wx) = \prod \gamma(x) \), where \( \gamma \) varies over the positive coroots, and
3. \( F(x, y) \neq 0 \) in the coinvariant algebra \( \text{Sym}(t \times t)/J_W \).

To prove Proposition A.10, we will set the following notation, closely following the notation and proof of \([BGG73b]\) Theorem 3.15.

Proposition A.11. There exists some polynomial \( Q(x, y) \in \text{Sym}(t \times t) \) of \( y \)-degree \( \ell(w_0) \) for which \( Q(x, wx) = 0 \) for \( w \neq w_0 \) and such that \( Q(x, w_0x) \) is generically nonvanishing.

Proof. There exists some polynomial \( Q' \in \text{Sym}(t \times t) \) such that \( Q'(x, wx) = 0 \) for \( w \neq wx \) and \( Q'(x, w_0x) \) generically does not vanish. Set \( R_{1 \times w_0}(y) := (1 \otimes \rho)^f/\ell! \), and for \( w \neq w_0 \) set \( R_{1 \times w} := D_{1 \times w_0}^{-1}(R_{1 \times w_0}) \).

These give minimal degree lifts of the basis of the coinvariant algebra labeled by the Schubert cells by Theorem A.8. In particular, there exist polynomials \( g_w(x, y) \in \text{Sym}(t \times t)^{1 \times W} \) such that
\[ Q'(x, y) = \sum_{w \in W} g_w(x, y)R_{1 \times w}(y). \]

Set \( Q(x, y) := \sum_{w \in W} g_w(x, x)R_{1 \times w}(y) \). Then since
\[ Q(x, vx) := \sum_{w \in W} g_w(x, x)R_{1 \times w}(vx) = \sum_{w \in W} g_w(x, vx)R_{1 \times w}(vx) = Q'(x, vx), \]
we see that \( Q(x, wx) = 0 \) for \( w \neq w_0 \) and \( Q(x, w_0x) \) is generically nonvanishing.

Choose such a \( g_w(x, y) \), \( R_{1 \times w}(y) \), and \( Q \) as in the proof of Proposition A.11. Unfortunately, such a \( Q \) need not satisfy condition (3) of Proposition A.10 even if \( Q' \) does. Therefore, we will need to modify our choice of \( Q \). Further to this end, choose any reduced expression \( w_0 = s_{\alpha_1} ... s_{\alpha_r} \) labelled by coroots \( \alpha_i \). Given this decomposition, set \( w_i := s_{\alpha_i} ... s_{\alpha_1} \), \( v_i := s_{\alpha_{i+1}} ... s_{\alpha_r} \), \( Q_i := D_{1 \times v_i}Q \), \( \gamma_1 := \alpha_1 \), and, for \( i > 1 \), we set \( \gamma_i := w_i^{-1}(\alpha_i) \).

\(^3\text{Note that the notation of \([BGG73b]\) Theorem 3.15 contains a typographic error: in their version, } \alpha_1 \text{ should be replaced with } \alpha_i. \text{ This is reflected in their \([BGG73b]\) Lemma 2.2, which is appealed to in the proof of \([BGG73b]\) Theorem 3.15.}\)
Lemma A.12. For any $Q$ satisfying the conditions of Proposition A.11 and any reduced expression for $w_0$, in the notation above, each polynomial $Q_i$ has $y$-degree $i$, $Q_i(x, w_0x) \prod_{r \geq j > 1} \gamma_j(x) = (-1)^{(r-i)}Q(x, w_0x)$ and $Q_i(x, wx) = 0$ if $w \not\subseteq w_i$.

Proof. This proof closely follows the proof of the Lemma below [BGG73b; Theorem 3.15]; we include the details for the reader's convenience. We proceed by backward induction on $i$. Note that when $i = \ell(w_0)$, we see that by assumption $Q_i(x, wx) = Q(x, wx)$ so the claim follows trivially from Proposition A.11.

Now assume that the lemma has been proved for $Q_i$ for some $i > 0$. Then we obviously have the $y$-degree of $Q_{i-1}$ is $i - 1$. Furthermore, we compute that for any $w \in W$, we have

$$Q_{i-1}(x, wx) = \frac{Q_i(x, wx) - Q_i(x, s_{a_{i-1}w}x)}{\alpha_i(wx)}.$$

In particular, if $w = w_{i-1}$, we see that by our inductive hypothesis, $Q_i(x, wx) = 0$, and furthermore that $\alpha_i(wx) = (w_{i-1}^{-1}\alpha_i)(x) = \gamma_i(x)$. Therefore, we see that in this case, we have

$$Q_{i-1}(x, wx) = -\frac{Q_i(x, s_{a_{i-1}w}x)}{\gamma_i(x)}$$

and so by induction we have

$$Q_{i-1}(x, w_{i-1}x) \prod_{r \geq j > 1} \gamma_j(x) = (-1)^{(r-i)}Q(x, w_0x).$$

Finally, if $w \not\subseteq w_i$ [BGG73b; Corollary 2.6] implies that $w \not\subseteq w_i$ and $s_{\alpha_i}w \not\subseteq w_i$. By induction we see that both terms in the numerator of Eq. (12) vanish so our claim is proved. \hfill \square

We note the following corollary of Lemma A.12.

Corollary A.13. We have $g_{w_0}(x, x) \prod_i \gamma(x) = (-1)^yQ(x, w_0x)$, and furthermore for all $w \in W$, $Q(x, w_0x)$ divides $g_w(x, x) \prod_i \gamma(x)$.

Proof. The first statement is a direct application of the $i = 0$ claim of Lemma A.12. We also use it as the base case of the second statement, which we prove by backwards induction on $\ell(w)$ for $w \in W$. Fix $w \in W$, and set $\tilde{w} := w_0 w w_0$. Apply $D_{1 \times \tilde{w}}$ to the equality $Q(x, y) := \sum_{w \in W} g_w(x, x) R_{1 \times w}(y)$ to obtain

$$D_{1 \times \tilde{w}}(Q(x, y)) = \sum_{u : \ell(\tilde{w} w u^{-1}) = \ell(\tilde{w}) + \ell(w u^{-1})} g_u(x, x) R_{1 \times w}(y)$$

where the other terms vanish by (2) of Theorem A.8. In particular, the set $\{u : \ell(\tilde{w} w u^{-1}) = \ell(\tilde{w}) + \ell(w u^{-1})\}$ contains a unique element of minimal length, namely $u = w_0 w w_0 = w$, because the element $y \in W$ of largest length for which $\ell(\tilde{w} w u^{-1}) = \ell(\tilde{w}) + \ell(y)$ is $y = \tilde{w}^{-1} w_0$. Multiply both sides of Eq. (13) by $\prod_i \gamma(x)$. By Lemma A.12, we have that $Q(x, w_0x)$ divides the left hand side, and by induction, we have that $Q(x, w_0x)$ divides all terms in the right hand side except the term $g_u(x, x) R_{1 \times w}(y)$, so therefore it divides $g_u(x, x)$. \hfill \square

Proof of Proposition A.10. Set $F(x, y) = \frac{Q(x, y) \prod_i \gamma(x)}{Q(x, w_0x)}$, which is well defined by Corollary A.13. Then by construction, $F$ satisfies (2), and $F$ satisfies (1) because $Q$ does. Furthermore, since the coefficient on $R_{1 \times w_0}$ on $F$ is $(-1)^y$ in the coinvariant algebra, (3) is satisfied, thus completing the proof of Proposition A.10. \hfill \square

Corollary A.14. If $Z \subseteq W$ is a closed subset, the underlying vector space of $\text{Sym}(t \times t)/J_Z$ has basis given by the images of $D_{1 \times u}(F)$ for which $w u^{-1} \in Z$.

Proof. Consider the map $\text{Sym}(t \times t)/J_W \to \text{Sym}(t \times t)/J_Z$. Since, generically in $t$, the union of $|Z|$ graphs will have $|Z|$ points lying above them, we have that $\dim_k(\text{Sym}(t \times t)/J_Z) \geq |Z|$.

On the other hand, assume that $w_0 u^{-1} \notin Z$. We will show that $D_{1 \times u}(F)(t x', x') = 0$ for all $t \in Z$. In particular, by Proposition A.10, these elements are all linearly independent, and so by showing this we will obtain the opposite inequality $\dim_k(\text{Sym}(t \times t)/J_Z) \leq |Z|$. Choose a reduced word decomposition for $w_0 u^{-1}$ and of $u$ to obtain a reduced word decomposition of $w_0 = (w_0 u^{-1}) u$. Apply Lemma A.12 (where, in the notation of the lemma, $i$ denotes the length of $w_0 u^{-1}$, $w_i = w_0 u, v_i = u$, and $w = l$) to see that $D_{1 \times u}(F)(x, t x') = 0$ if $t^{-1} \not\subseteq w_0 u$, where we make the coordinate change $x := t x'$. Since inversion preserves ordering, we see that this is equivalent to the condition that $t \not\subseteq w_0 u^{-1}$. However, by assumption, $t \in Z$ and is closed, so $t \not\subseteq w_0 u^{-1}$, and so our desired vanishing holds. \hfill \square
Setting $Z = S$ in Corollary A.14 precisely matches the description of Theorem A.8, which therefore proves Theorem A.5.
REFERENCES

[GR17c] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*. Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017, pp. xxxv+436. ISBN: 978-1-4704-3570-7. DOI: 10.1090/surv/221.2 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1090/surv/221.2

[Hum08] James E. Humphreys. *Representations of semisimple Lie algebras in the BGG category O*. Vol. 94. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+289. ISBN: 978-0-8218-4678-0. DOI: 10.1090/gsm/094 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1090/gsm/094

[Hum90] James E. Humphreys. *Reflection Groups and Coxeter Groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990, pp. xii+204. ISBN: 0-521-37510-X. DOI: 10.1017/CBO9780511623646 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1017/CBO9780511623646

[Jan79] Jens Carsten Jantzen. *Moduln mit einem höchsten Gewicht*. Vol. 750. Lecture Notes in Mathematics. Springer, Berlin, 1979, pp. ii+195. ISBN: 3-540-09558-6.

[Kan01] Richard Kane. *Reflection groups and invariant theory*. Vol. 5. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer-Verlag, New York, 2001, pp. x+379. ISBN: 0-387-98979-1. DOI: 10.1007/978-1-4757-3542-0 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1007/978-1-4757-3542-0

[Lon17] Gus Lonergan. “A Remark on Descent for Coxeter Groups”. In: (2017). URL: https://arxiv.org/pdf/1707.01156.pdf

[Lon18] Gus Lonergan. “A Fourier transform for the quantum Toda lattice”. In: *Selecta Math. (N.S.)* 24.5 (2018), pp. 4577–4615. ISSN: 1022-1824. DOI: 10.1007/s00029-018-0419-x URL: https://doi-org.ezproxy.lib.utexas.edu/10.1007/s00029-018-0419-x

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xvi+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1515/9781400830558

[Lur17] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, 1994, pp. xiv+292. ISBN: 3-540-56963-4. DOI: 10.1007/978-3-642-57916-5 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1007/978-3-642-57916-5

[Soc90] Wolfgang Soergel. “Kategorie O, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe”. In: *J. Amer. Math. Soc.* 3.2 (1990), pp. 421–445. ISSN: 0894-0347. DOI: 10.2307/1990960 URL: https://doi-org.ezproxy.lib.utexas.edu/10.2307/1990960

[ST54] G. C. Shephard and J. A. Todd. “Finite unitary reflection groups”. In: *Canad. J. Math.* 6 (1954), pp. 274–304. ISSN: 0008-414X. DOI: 10.4153/cjm-1954-028-3 URL: https://doi.org/10.4153/cjm-1954-028-3

[Ste64] Robert Steinberg. “Differential equations invariant under finite reflection groups”. In: *Trans. Amer. Math. Soc.* 112 (1964), pp. 392–400. ISSN: 0002-9947. DOI: 10.2307/1994152 URL: https://doi.org/10.2307/1994152

[Ste75] Robert Steinberg. “On a theorem of Pittle”. In: *Topology* 14 (1975), pp. 173–177. ISSN: 0040-9383. DOI: 10.1016/0040-9383(75)90025-7 URL: https://doi-org.ezproxy.lib.utexas.edu/10.1016/0040-9383(75)90025-7

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