Behavior of a Chiral Condensate in Schwarzschild and Reissner-Nordström Space-Times

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Abstract

We consider the behavior of a chiral condensate around and inside a spherically-symmetric black hole. We use a zeroth-order heat kernel expansion to compute the effective action, and then we minimize the effective action at every point in space to obtain the profile of the chiral condensate. We find that any black hole horizon is associated with a second-order phase transition between confined and deconfined phases. This holds at the single horizon of a Schwarzschild black hole and at the two horizons of a Reissner-Nordström black hole. Additionally, between the event horizon and the central singularity, there is always at least one first-order phase transition between a confined phase and a deconfined phase. Depending on the charge-to-mass ratio of the black hole, there may be either one or three first-order phase transitions between the event horizon and the central singularity.

1 Introduction

The QCD phase diagram is a matter of great interest to contemporary particle physics. At high energies and/or high temperatures, it is known that quarks cease to be confined inside hadrons and instead enter a deconfined phase. If gravity is included in the theory, strong space-time curvature can also cause QCD matter to transition into a deconfined phase \cite{1, 2}. Such strong curvature can be found in the vicinity of black holes.

In Ref. \cite{3}, the authors found that the chiral condensate approaches zero near the event horizon of a Schwarzschild black hole. Thus, the event horizon is surrounded by a “bubble” of approximately restored chiral symmetry (or in other words, a deconfined phase). As the mass of the black hole increases, the radius of this bubble (relative to the radius of the black hole) decreases. At the boundary of the bubble, a second-order phase transition takes place, in which the deconfined phase inside the bubble meets the confined phase in the wider universe.

To attain these results, the authors used a second-order heat kernel expansion \cite{3}. We show that, for large Schwarzschild black holes, similar results can be attained with a zeroth-order heat kernel expansion. Because the zeroth-order heat kernel expansion is much simpler than the second-order expansion, it can be easily generalized to more exotic scenarios.

First, we generalize the results of Ref. \cite{3} to include the region inside the event horizon. We show that, inside the event horizon, a first-order phase transition takes place between the confined and deconfined phases. At this transition, the value of the chiral condensate changes abruptly.

The inclusion of charge causes an inner horizon to form inside the event horizon. Around this new horizon, another second-order phase transition between confined and deconfined phases takes place. Regardless of the charge and mass of the black hole, there is always a first-order phase transition between the inner horizon and the central singularity. However, depending on the charge-to-mass ratio of the black hole, there may be either zero or two first order phase transitions between the event horizon and the inner horizon.

Throughout this article, we use Planck units, so that $c = G = \hbar = 4\pi\epsilon_0 = 1$. 

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Effective Action for General Black Hole Space-Time

In a Minkowski background, a general, spherically-symmetric black hole metric takes the form

$$ds^2 = -f_{\text{Mink}}(r) \, dt^2 + f_{\text{Mink}}^{-1}(r) \, dr^2 + r^2 \, d\Omega^2. \quad (1)$$

Every black hole has a temperature $T_{\text{BH}}$. To account for thermal effects, we work in Euclidean time. The Euclidean time direction is periodic, and its period is equal to the inverse temperature $\beta$. In Euclidean time, we may write the metric as

$$ds^2 = f_{\text{Eucl}}(r) \, dt^2 + f_{\text{Eucl}}^{-1}(r) \, dr^2 + r^2 \, d\Omega^2, \quad (2)$$

where

$$f_{\text{Eucl}}(r) = |f_{\text{Mink}}(r)|. \quad (3)$$

From here on, we will simply write $f_{\text{Eucl}}(r)$ as $f(r)$. We model baryonic matter by a four-fermion interaction with interaction strength $\lambda$. Let $N$ be the total number of different types of quarks, counting both flavors and colors. We may write the classical action as

$$S = \int d^4x \left\{ \bar{\psi} i\gamma^\mu \nabla_\mu \psi + \frac{\lambda}{2N} (\bar{\psi}\psi)^2 \right\}. \quad (4)$$

(The astute reader may notice that there is no factor of $\sqrt{g}$ beside the volume element in Eqn. 4. This is because the determinant of the metric in Schwarzschild space-time is identical to that in Minkowski space-time.) We define the condensate $\sigma$ as

$$\sigma = -\frac{\lambda}{N} \langle \bar{\psi}\psi \rangle. \quad (5)$$

We define the related quantity $\sigma_\epsilon$ by

$$\sigma_\epsilon^2 = \sigma^2 + \epsilon \sqrt{\int d\sigma \frac{d\sigma}{dr}}, \quad (6)$$

where $\epsilon = \pm 1$ is the spin eigenvalue of the fermion field and $f$ is shorthand for $f(r)$. We assume that the underlying field $\psi$ is spherically symmetric and time-independent, so $\sigma$ and $\sigma_\epsilon$ must both be spherically symmetric and time-independent as well.

We define $\omega_\nu(u)$ by

$$\omega_\nu(u) = \sum_{n=1}^\infty (-1)^n n^{-\nu} K_\nu(n\beta u). \quad (7)$$

To approximate the effective action, we use a resummed heat kernel expansion. To zeroth-order in the resummed heat kernel expansion, we may write the effective action $\Gamma[\sigma]$ as

$$\Gamma[\sigma] = \hat{\Gamma}_0[\sigma] + \hat{\Gamma}_1[\sigma] + \hat{\Gamma}_2[\sigma] + \hat{\Gamma}_3[\sigma], \quad (8)$$

where

$$\hat{\Gamma}_0[\sigma] = \int d^4x \left( \frac{\sigma^2}{2\lambda} \right), \quad (9)$$

$$\hat{\Gamma}_1[\sigma] = \frac{\beta}{32\pi^2} \sum_{\epsilon=\pm 1} \int d^3x \, f^{-2} \left\{ \frac{3\sigma^2}{4} - \frac{\sigma^4}{2} \ln \left( \frac{f\sigma^2}{\ell^2} \right) \right\}, \quad (10)$$

and

$$\hat{\Gamma}_2[\sigma] = \frac{\beta}{32\pi^2} \sum_{\epsilon=\pm 1} \int d^3x \, f^{-2} \left\{ \frac{16\sigma^2}{f\beta^2} \omega_2 \left( \sqrt{f} \sigma_\epsilon \right) \right\}. \quad (11)$$

In Eqn. 10 the quantity $\ell$ is a renormalization length scale. Because the Euclidean time is periodic with period $\beta$, we may rewrite $\hat{\Gamma}_0[\sigma]$ as

$$\hat{\Gamma}_0[\sigma] = -\beta \int d^3x \left( \frac{\sigma^2}{2\lambda} \right). \quad (12)$$
Realistic astrophysical black holes have a very low temperature, which implies a large value of $\beta$. The Bessel function $K_2(x)$ is approximately zero for large $x$. Therefore, with $\beta$ large, $\omega_2(u)$ is approximately zero. Hence, we may write the effective action as

$$\Gamma [\sigma] = -\beta \int d^3x \left( \frac{\sigma^2}{2\lambda} \right) + \frac{16\pi^2}{\beta} \sum_{\epsilon = \pm 1} \int d^3x f^{-2} \left\{ \frac{3\sigma^4}{4} - \frac{\sigma^4}{2} \ln \left( \frac{f \sigma^2}{\ell^2} \right) \right\}. \quad (13)$$

### 3 Effective Potential and Phase Transitions

For a black hole much larger than the Planck mass (an astrophysically realistic assumption), the metric function $f(r)$ varies slowly with respect to $r$, even at the event horizon. Therefore, we would expect that the condensate $\sigma$ would also vary slowly with respect to $r$. With these approximations, we may simplify $\Gamma [\sigma]$ as

$$\Gamma [\sigma^2] = -\beta \int d^3x \left( \frac{\sigma^2}{2\lambda} \right) + \frac{16\pi^2}{\beta} \int d^3x f^{-2} \left\{ \frac{3\sigma^4}{4} - \frac{\sigma^4}{2} \ln \left( \frac{f \sigma^2}{\ell^2} \right) \right\}. \quad (14)$$

(For future convenience, we have rewritten $\Gamma [\sigma]$ as $\Gamma [\sigma^2]$ in Eqn. 14.) The Lagrangian $L (\sigma^2)$ corresponding to this action is

$$L (\sigma^2) = -\beta \left( \frac{\sigma^2}{2\lambda} \right) + \frac{16\pi^2}{\beta} f^{-2} \left\{ \frac{3\sigma^4}{4} - \frac{\sigma^4}{2} \ln \left( \frac{f \sigma^2}{\ell^2} \right) \right\}. \quad (15)$$

The effective potential is given by the negative of the Lagrangian:

$$V (\sigma^2) = -L (\sigma^2) = \frac{\beta \sigma^2}{2\lambda} - \frac{16\pi^2}{\beta} f^{-2} \left\{ \frac{3\sigma^4}{4} - \frac{\sigma^4}{2} \ln \left( \frac{f \sigma^2}{\ell^2} \right) \right\}. \quad (16)$$

At every point in space, the condensate will tend to minimize the effective potential. There is a critical value of $f$, which is given by

$$f_{\text{crit}} = \sqrt{\frac{\lambda \ell^2}{8\pi^2}}. \quad (17)$$

When $f > f_{\text{crit}}$, the potential is minimized at $\sigma^2 = 0$. When $f < f_{\text{crit}}$, the potential is minimized at

$$\sigma^2_{\text{min}} = -\frac{16\pi^2 f^2}{2\lambda} \left[ W_{-1} \left( -\frac{16\pi^2 f^3}{2\epsilon \lambda \ell^2} \right) \right]^{-1}, \quad (18)$$

where $W_{-1}(x)$ is the lower branch of the Lambert $w$ function (see Appendix A). Thus, there is a phase transition between the confined and deconfined phases, which takes place at $f = f_{\text{crit}}$. This phase transition is accompanied by a discontinuous “jump” in $\sigma^2_{\text{min}}$, which means that the phase transition is first-order.

There is also a second-order phase transition, which takes place at $f = 0$. From Eqn. 18, we see that, as $f \to 0$, the condensate value $\sigma^2_{\text{min}}$ also approaches zero. However, unlike at $f = f_{\text{crit}}$, there is no sudden jump in $\sigma^2_{\text{min}}$. Hence, this phase transition is second-order.

### 4 Chiral Condensate Inside a Schwarzschild Black Hole

First, let us consider a Schwarzschild black hole, for which the metric function (with a Minkowski background) $f_{\text{Mink}}(r)$ takes the form

$$f_{\text{Mink}}(r) = 1 - \frac{2M}{r}, \quad (19)$$

where $M$ is the mass of the black hole. The event horizon of this black hole is located at $r_s = 2M$. Thus, in Euclideanized space-time, the metric function $f(r)$ takes the form

$$f(r) = |1 - \frac{2M}{r}|. \quad (20)$$
In Figures 3 and 4 (see Appendix B), we graph the condensate $\sigma^2(r)$ as a function of $r/r_s$ for several values of $\lambda$, focusing on the region outside the event horizon. (Regardless of the mass of the black hole, the condensate profile as a function of $r/r_s$ does not change.)

We confirm that Schwarzschild black holes are surrounded by a bubble of deconfined phase. (By assumption, the black holes in Figures 3 and 4 are very large, so the bubble only extends a short distance from the event horizon.) These results indicate that our zeroth-order heat kernel expansion was able to replicate the qualitative features identified in Ref. 3, which used a second-order heat kernel expansion.

Now, we apply the zeroth-order formalism developed so far to the region inside the event horizon. In Figure 5, we see that there is a confined phase inside the event horizon as well as outside. Just inside the event horizon, there is a second-order phase transition between the confined and deconfined phases.

As one moves further into the black hole, the condensate increases rapidly (see Figure 6). Eventually, there is a first-order phase transition from a confined to a deconfined phase, at which the condensate $\sigma^2(r)$ suddenly drops to zero. To find the location of this phase transition, we must solve the equation

$$f(r_{pt}) = f_{crit},$$

where $r_{pt}$ is the $r$ coordinate of the phase transition. To solve Eqn. 21, we plug in Eqn. 20 for $f(r)$. In symbols,

$$|1 - \frac{2M}{r_{pt}}| = f_{crit}.$$ (22)

To guarantee sensible asymptotic behavior, we must impose the condition $f_{crit} > 1$ (see Appendix A). Thus, there is no solution to Eqn. 22 outside the event horizon of the black hole. Inside the event horizon, we use simple algebra to obtain the following solution:

$$r_{pt} = \frac{2M}{1 + f_{crit}}.$$ (23)

As $\lambda$ or $\ell$ increases, the first-order phase transition takes place closer to the central singularity. Below, we have plotted $r_{pt}/r_s$ as a function of $\lambda$, for several values of $\ell$. (Recall that $r_s = 2M$.)

![Figure 1: $r_{pt}/r_s$ vs. $\lambda$, for several values of $\ell$](image)
Next, let us consider a Reissner-Nordström (RN) black hole. The metric function $f_{\text{Mink}}(r)$ is
\begin{equation}
  f_{\text{Mink}}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},
\end{equation}
where $M$ is the mass of the black hole and $Q$ is its charge. Any horizons associated with this black hole must satisfy the equation $f_{\text{Mink}}(r_h) = 0$, where $r_h$ is the $r$ coordinate of the horizon\(^7\). If $Q < M$, there are two horizons, located at
\begin{equation}
  r_{\pm} = M \pm \sqrt{M^2 - Q^2}.
\end{equation}
The outer horizon ($r_+$) is the event horizon of the black hole, while the inner horizon ($r_-$) is a Cauchy horizon\(^8\),\(^9\).

We may write the Euclideanized metric function $f(r)$ as
\begin{equation}
  f(r) = |1 - \frac{2M}{r} + \frac{Q^2}{r^2}|.
\end{equation}
In terms of $r_{\pm}$, we may write $f(r)$ as
\begin{equation}
  f(r) = |1 - \frac{r_+}{r}| \times |1 - \frac{r_-}{r}|,
\end{equation}
where $\times$ is just scalar multiplication. At both horizons, $f(r) = 0$; therefore, each horizon is surrounded by a bubble of deconfined phase. The deconfinement transition that takes place at the horizons is a second-order phase transition (see Figures 8, 10, and 11).

Now, we seek to understand how many first-order phase transitions occur for a given black hole. Outside the event horizon ($r > r_+$), it is easy to see that $f(r) < 1$. Because $f_{\text{crit}} > 1$, there can be no first-order phase transition outside the event horizon. Inside the inner horizon, $f(r)$ is given by
\begin{equation}
  f(r) = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right).
\end{equation}
It is easy to see that, as $r$ decreases, $f(r)$ increases monotonically; at $r = 0$, $f(r)$ becomes singular. Regardless of the value of $f_{\text{crit}}$, $f(r)$ will eventually surpass it. Hence, a single first-order phase transition takes place inside the inner horizon, and the central singularity is surrounded by a bubble of deconfined phase. Between the horizons, $f(r)$ is given by
\begin{equation}
  f(r) = \left(\frac{r_+}{r} - 1\right) \left(1 - \frac{r_-}{r}\right).
\end{equation}
In order for a first-order phase transition to occur between the horizons, there must be a point where $f(r) = f_{\text{crit}}$. (If such a point exists, there will be two first-order phase transitions. As $r$ decreases, $f(r)$ increases past $f_{\text{crit}}$ and then decreases below $f_{\text{crit}}$ again.) To find the maximum value of $f(r)$ between the horizons, we must solve the equation
\begin{equation}
  f'(r_{\text{max}}) = 0,
\end{equation}
where $r_{\text{max}}$ is the $r$ coordinate at which the maximum occurs. Plugging Eqn. 29 into Eqn. 30 we find that
\begin{equation}
  \frac{r_+}{r_{\text{max}}} \left(\frac{r_-}{r_{\text{max}}} - 1\right) + \frac{r_-}{r_{\text{max}}} \left(\frac{r_+}{r_{\text{max}}} - 1\right) = 0.
\end{equation}
Rearranging Eqn. 31 we find that
\begin{equation}
  (r_+ + r_-) r_{\text{max}} = 2r_+ r_-,
\end{equation}
or equivalently,
\begin{equation}
  r_{\text{max}} = \frac{Q^2}{M}.
\end{equation}
Plugging Eqn. 33 into Eqn. 26 we find that the maximum value of $f(r)$ between the horizons is given by
\begin{equation}
  f_{\text{max}} = \frac{M^2}{Q^2} - 1.
\end{equation}
In order for there to be two first-order phase transitions (as opposed to no first-order phase transitions) between the horizons, the following equation must be satisfied:

\[ f_{\text{max}} > f_{\text{crit}}, \quad (35) \]

or equivalently,

\[ Q^2 < \frac{M^2}{1 + f_{\text{crit}}}. \quad (36) \]

Below, we have displayed a phase diagram in the variables \( Q \) and \( M \), which shows the transition between black holes that exhibit two first-order phase transitions between the horizons and black holes that exhibit no first-order phase transitions between the horizons. (We set \( \lambda = 10 \) and \( \ell = 10^3 \). The charge \( Q \) and the mass \( M \) are both in multiples of \( 10^{40} \) Planck units.)

![Phase Diagram for Reissner-Nordström Black Holes](image)

**Figure 2:** Phase Diagram for Reissner-Nordström Black Holes. The red region corresponds to black holes with two phase transitions between the horizons, the green region corresponds to black holes with no phase transitions between the horizons, and the blue region corresponds to naked singularities with no horizons.

In Figures 9 and 10, we considered a black hole with \( Q/M = 0.5 \) and a chiral condensate with the parameters \( \lambda = 1 \) and \( \ell = 10 \). This black hole satisfies the bound given in Eqn. 36. As expected, this black hole exhibits three first-order phase transitions.

In Figures 11 we considered a black hole with \( Q/M = 0.75 \), \( \lambda = 1 \), and \( \ell = 10 \). This black hole does not satisfy the bound given in Eqn. 36. As expected, this black hole exhibits just one first-order phase transition, which takes place inside the inner horizon.
6 Conclusion

In this article, we have extended the treatment of chiral condensates in black hole space-times to include charged black holes. We have analyzed the behavior of the chiral condensate for all values of \( r \), both outside and inside the event horizon. While the charge-to-mass ratio of a black hole has relatively little effect on the behavior of the condensate outside the event horizon, it significantly affects the behavior of the condensate inside the event horizon. We constructed a phase diagram describing how the qualitative behavior of the chiral condensate inside the event horizon changes with the mass and charge of the black hole.

In Ref. [3], the authors performed a second-order resummed heat kernel expansion on the exterior of a Schwarzschild black hole, and our zeroth-order expansion approximately matched their results. To further check the accuracy of the zeroth-order expansion, we would like to perform a second-order resummed heat kernel expansion in the interior of a Schwarzschild black hole. Additionally, we would like to perform a second-order expansion on a Reissner-Nordström space-time. After extracting the behavior of the chiral condensate from these expansions (which must be done numerically), we intend to compare this behavior to that predicted by our zeroth-order expansion. If the second-order numerical results approximately match the zeroth-order analytic results for these particular black holes, it would strongly suggest that our zeroth-order analytic results are correct for general Reissner-Nordström black holes, both inside and outside the event horizon.

Due to its simplicity and ease of use, we believe that the zeroth-order resummed heat kernel expansion may be suitable for more complicated space-times, where a second-order expansion may not be possible. We would like to study chiral condensates in non-spherically symmetric space-times, time-dependent space-times, or (if possible) space-times that break both spherical symmetry and time-translation symmetry.

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7
Appendix A: Computation of the Minimum of the Effective Potential

At every point in space, the condensate will tend to minimize the effective potential. Thus, we must solve the equation

\[ V'(\sigma^2_{\text{min}}) = 0. \]  (37)

Explicitly, Eqn. 37 takes the form

\[ V'(\sigma^2_{\text{min}}) = \frac{\beta}{2\lambda} - \frac{\beta}{16\pi^2} f^{-2} \left\{ \frac{3}{2} \sigma^2_{\text{min}} - \sigma^2_{\text{min}} \ln \left( \frac{f\sigma^2_{\text{min}}}{\ell^2} \right) - \frac{\sigma^2_{\text{min}}}{2} \right\} \]

(38)

Rearranging Eqn. 38, we find that

\[ -\frac{16\pi^2 f^2}{2\lambda} \left( \frac{1}{\sigma^2_{\text{min}}} \right) + \ln \left( \frac{1}{\sigma^2_{\text{min}}} \right) = \ln \left( \frac{f}{\ell^2} \right) - 1. \]  (39)

Next, we take the exponential of both sides of Eqn. 39 (and multiply by a constant) to obtain

\[ -\frac{16\pi^2 f^2}{2\lambda} \left( \frac{1}{\sigma^2_{\text{min}}} \right) \exp \left( -\frac{16\pi^2 f^2}{2\lambda} \left( \frac{1}{\sigma^2_{\text{min}}} \right) \right) = -\frac{16\pi^2 f^2}{2\lambda} \frac{f}{e\ell^2}. \]  (40)

The Lambert W function is defined by \([10]\)

\[ w(x) \exp(w(x)) = x. \]  (41)

Thus, we may solve Eqn. 40 as

\[ -\frac{16\pi^2 f^2}{2\lambda} \left( \frac{1}{\sigma^2_{\text{min}}} \right) = w \left( -\frac{16\pi^2 f^3}{2e\lambda \ell^2} \right), \]  (42)

or equivalently,

\[ \sigma^2_{\text{min}} = -\frac{16\pi^2 f^2}{2\lambda} \left[ w \left( -\frac{16\pi^2 f^3}{2e\lambda \ell^2} \right) \right]^{-1}. \]  (43)

The Lambert W function \(w(x)\) is only defined for \(x \geq -1/e\) \([10]\). This constraint implies that

\[ -\frac{16\pi^2 f^3}{2e\lambda \ell^2} \geq -\frac{1}{e}, \]  (44)

or equivalently,

\[ \frac{16\pi^2 f^3}{2\lambda \ell^2} \leq 1. \]  (45)

This defines a critical value for \(f\), which we may write as

\[ f_{\text{crit}} = \sqrt[3]{\frac{\lambda \ell^2}{8\pi^2}}. \]  (46)
When \( f > f_{\text{crit}} \), there is no solution to Eqn. 38. Thus, there is no local minimum (and, by extension, no global minimum) for the potential \( V(\sigma^2) \). For any choice of the parameters \( \beta, \lambda, f, \) and \( \ell, V'(\sigma^2) \) is positive for sufficiently large \( \sigma^2 \). Thus, when \( f > f_{\text{crit}} \), \( V(\sigma^2) \) is a monotonically increasing function for all values of \( \sigma^2 \).

To minimize \( V(\sigma^2) \) for \( f > f_{\text{crit}} \), we must make \( \sigma^2 \) as small as possible. Since \( \sigma^2 \) must be non-negative, we find that \( \sigma^2_{\text{min}} = 0 \). Thus, \( f \) acts as an order parameter for the transition from a confined to a deconfined phase, with its critical value given by \( f = f_{\text{crit}} \). This is consistent with the results obtained in Ref. [5].

Asymptotically far from the black hole, we expect that \( f(r) \to 1 \) and that the condensate exists in a confined phase. This imposes the condition \( f_{\text{crit}} > 1 \), which restricts the possible values of \( \lambda \) and \( \ell \).

Let us return to the case where \( f < f_{\text{crit}} \). In order to minimize the effective potential (as opposed to simply extremizing it), the value \( \sigma^2_{\text{min}} \) must satisfy the equation \( V''(\sigma^2_{\text{min}}) > 0 \). After differentiating Eqn. 38 and performing some algebra, we obtain

\[
\ln \left( \frac{f \sigma^2_{\text{min}}}{\ell^2} \right) < 0,
\]

or equivalently,

\[
\sigma^2_{\text{min}} < \frac{\ell^2}{f}.
\]

When \( x < 0 \), the \( w \) function is multi-valued with two branches: \( W_0(x) \) and \( W_{-1}(x) \), with \( W_{-1}(x) \leq W_0(x) \) \[10\]. Plugging Eqn. 43 into Eqn. 48, we find that

\[
w \left( -\frac{16\pi^2 f^3}{2e\lambda \ell^2} \right) < -\frac{16\pi^2 f^3}{2\lambda \ell^2}.
\]

For \( x \in (-1, 0) \), the branches \( W_0(x) \) and \( W_{-1}(x) \) satisfy the inequality \[6\]

\[
W_{-1} \left( \frac{x}{e} \right) < x < W_0 \left( \frac{x}{e} \right).
\]

From Eqn. 50, we see that only the branch \( W_{-1}(x) \) can satisfy Eqn. 49. Thus, we may write \( \sigma^2_{\text{min}} \) as

\[
\sigma^2_{\text{min}} = -\frac{16\pi^2 f^2}{2\lambda} \left[ W_{-1} \left( -\frac{16\pi^2 f^3}{2e\lambda \ell^2} \right) \right]^{-1}.
\]
Appendix B: Additional Figures and Plots for Schwarzschild Black Hole

Figure 3: Profile of the condensate $\sigma^2(r)$ outside the event horizon, with $\ell = 10^3$

Figure 4: Same as Figure 3 but enlarged to display the region just outside the event horizon
Figure 5: Profile of the condensate $\sigma^2(r)$ across the event horizon, with $\ell = 10$

Figure 6: Profile of the condensate $\sigma^2(r)$ deep inside the black hole, with $\ell = 10$
Appendix C: Additional Figures and Plots for Reissner-Nordström Black Hole

Figure 7: Profile of the condensate $\sigma^2(r)$ outside the event horizon, with $\lambda = 1$ and $\ell = 10^3$

Figure 8: Profile of the condensate $\sigma^2(r)$ close to the event horizon, with $\lambda = 1$ and $\ell = 10^3$
Figure 9: Profile of the condensate inside the event horizon, with $Q/M = 0.5$, $\lambda = 1$, and $\ell = 10$, showing the first-order phase transition farthest from the inner horizon.

Figure 10: Profile of the condensate inside the event horizon, with $Q/M = 0.5$, $\lambda = 1$, and $\ell = 10$, showing the first-order phase transitions close to the inner horizon.
Figure 11: Profile of the condensate inside the event horizon, with $Q/M = 0.75$, $\lambda = 1$, $\ell = 10$