Palindromic sequences generated from marked morphisms

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Abstract

Fixed points $u = \varphi(u)$ of marked and primitive morphisms $\varphi$ over arbitrary alphabet are considered. We show that if $u$ is palindromic, i.e., its language contains infinitely many palindromes, then some power of $\varphi$ has a conjugate in class $P$. This class was introduced by Hof, Knill, Simon (1995) in order to study palindromic morphic words. Our definitions of marked and well-marked morphisms are more general than the ones previously used by A. Frid (1999) or B. Tan (2007). As any morphism with aperiodic fixed point over binary alphabet is marked, our result generalizes the result of B. Tan. Labbé (2013) demonstrated that already on a ternary alphabet the property of morphisms to be marked is important for the validity of our theorem. The main tool used in our proof is the description of bispecial factors in fixed points of morphisms provided by K. Klouda (2012).

1 Introduction

In 1995 paper, Hof, Knill, and Simon studied the spectral properties of Schroedinger operators associated to one-dimensional structure having finite local complexity. Such structure can be coded by an infinite word over a finite alphabet. Hof, Knill, and Simon showed that if the word contains infinitely many palindromes, then the operator has a purely singular continuous spectrum. In the same article, they introduced a class of morphisms $P$ (defined below) and proved that any fixed point of a morphism from $P$ is palindromic, i.e., infinitely many different palindromes occur in the fixed point. In Remark 3 of [11], the question “Are there (minimal) sequences containing arbitrarily long palindromes that arise from substitution none of which belongs to class $P$?” is formulated. In accordance with their terminology, an infinite word $u$ containing palindromes longer than $N$ for any $N \in \mathbb{N}$, is called palindromic.

In 2003, Allouche et al. demonstrated [1, Theorem 13], that any periodic palindromic sequence is a fixed point of a morphism in class $P$. Let us mention, that the property “to be palindromic”

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is the property of the language of an infinite word and not the property of the infinite word itself. In 2007, Bo Tan proved [19, Theorem 4.1], that if a fixed point of a primitive substitution $\varphi$ over binary alphabet is palindromic, then there exists a substitution $\psi$ in class $\mathcal{P}$ such that languages of both substitutions coincide. In fact, Bo Tan proved a stronger result, namely, that the substitution $\varphi$ or $\varphi^2$ is conjugate to a substitution from the class $\mathcal{P}$.

In respect to the previous results, in 2008, Blondin-Massé and Labbé [3, 13] suggested the following HKS-conjecture: (Version 1) Let $u$ be the fixed point of a primitive morphism. Then, $u$ is palindromic if and only if there exists a morphism $\varphi \neq \text{Id}$ such that $\varphi(u) = u$ and $\varphi$ has a conjugate in class $\mathcal{P}$.

But this statement turned out to be false already on ternary alphabet. In 2013, Labbé found an injective primitive substitution whose fixed points contradicts this version of HKS-conjecture, see [14]. In [10], Harju, Vesti and Zamboni pointed out, that the second shift of Labbé’s counter-example is a fixed point of a substitution from the class $\mathcal{P}$. Clearly, language of shift of a uniformly recurrent word coincides with the language of the original word. Therefore it seems that the formulation of the theorem which Bo Tan stated for binary alphabet is more suitable for the formulation of the HKS-conjecture: (Version 2) Let $u$ be the fixed point of a primitive morphism $\varphi$. If $u$ is palindromic then there exists a morphism $\psi$ in class $\mathcal{P}$ such that languages of both substitutions coincide.

The possibility of different interpretations of the phrase “arise by morphisms of class $\mathcal{P}$” in the original question of Hof, Knil and Simon, led the authors of [10] to relax the statement “to be language of a fixed point of a substitution from class $\mathcal{P}$” which both previous variants impose on the language of a palindromic word $u$. Inspired by Labbé’s counter-example, Harju, Vesti and Zamboni concentrated on infinite words with finite defect (for definition of defect consult [6]). A corollary of their results is that for any word $u$ with finite defect the following statement is true: (Version 3) If $u$ is a palindromic primitive morphic word then there exist morphisms $\varphi$ and $\psi$ with conjugates in class $\mathcal{P}$ and an infinite word $v$ such that $v = \varphi(v)$ and languages of $u$ and $\psi(v)$ coincide.

The previous discussion has only one clear conclusion: the palindromicity of infinite words attracts attention of people and it is worth to study it. In this paper, we show that Version 1 of the HKS-conjecture is still valid for a large class of morphisms. We generalize the result of Bo Tan for fixed point of marked morphisms on alphabet of arbitrary size. Our definition of marked morphism is more general then the ones previously used by A. Frid [8, 9] or B. Tan [19] (see Definition 9 below). This is a good step towards a complete characterization of the cases for which Version 1 of the HKS-conjecture holds. The main result of this paper is the following.

**Theorem 1.** Let $u$ be a fixed point of a marked and primitive morphism $\varphi : A^* \rightarrow A^*$. If $u$ is palindromic, then some power of $\varphi$ has a conjugate in class $\mathcal{P}$.

Let us stress that any primitive morphism with aperiodic fixed point over binary alphabet is marked in our generalized sense. Therefore, Theorem 1 generalizes Bo Tan’s result.

The article is organized as follows. In Section 2, we recall needed notion and results on morphisms, conjugacy, cyclic and marked morphisms. Characterization of morphisms from class $\mathcal{P}$ by the left most and right most conjugates is provided in Section 3. Section 4 is devoted to structure of bispecial factors of aperiodic palindromic words. Structure of bispecial factors of marked morphisms is described in Section 5. The crucial ingredient for our considerations in this section is the description of bispecial factors of circular D0L morphisms given by Karel Klouda in [12]. Theorem 1 is proved in Section 6. The paper ends with some comments and open questions.
2 Preliminaries

2.1 Combinatorics on words

We borrow from M. Lothaire [17] the basic terminology about words. In what follows, \( \mathcal{A} \) is a finite alphabet whose elements are called letters. A word \( w \) is a finite sequence of letters \( w = w_0w_1 \cdots w_{n-1} \) where \( n \in \mathbb{N} \). The length of \( w \) is \( |w| = n \). By convention the empty word is denoted \( \varepsilon \) and its length is 0. The set of all finite words over \( \mathcal{A} \) is denoted by \( \mathcal{A}^* \). Endowed with the concatenation, \( \mathcal{A}^* \) is the monoid generated by \( \mathcal{A} \). The set of all infinite words is \( \mathcal{A}^\mathbb{N} \) and the set of biinfinite words is \( \mathcal{A}^\mathbb{Z} \). Given a word \( w \in \mathcal{A}^* \cup \mathcal{A}^\mathbb{N} \), a factor \( f \) of \( w \) is a word \( f \in \mathcal{A}^* \) satisfying \( w = xfy \) for some \( x, y \in \mathcal{A}^* \). The set of all factors of \( w \), called the language of \( w \), is denoted by \( \mathcal{L}(w) \). A word \( w \) is periodic if it is not a power of another word, that is if \( w = u^p \) for some word \( u \) and integer \( p \) then \( w = u \) and \( p = 1 \). Let \( u \in \mathcal{A}^* \). The conjugacy class of \( u \) is the set

\[
[u] = \{ v : \text{there exists } w \text{ such that } vw = uw \}.
\]

The \( k \)-th right conjugate of \( u \) is the word \( v \in [u] \) such that \( vw = uw \) and \( |v| = k \). A period of a word \( w \) is an integer \( p < |w| \) such that \( w_i = w_{i+p} \) for all \( i < |w| - p \).

An infinite word \( u = u_0u_1 \cdots \) is periodic if there exists a positive integer \( p \) such that \( u_i = u_{i+p} \) for all \( i \in \mathbb{N} \). The smallest \( p \) satisfying previous condition is called the period of \( u \). An infinite word \( u \) is eventually periodic if there exist two positive integers \( k, p \) such that \( u_i = u_{i+p} \), for all \( i \geq k \) in which case the prefix of \( u \) of length \( k \) is called the preperiod. An infinite word is aperiodic if it is not eventually periodic. An infinite word \( u \) is recurrent if any factor occurring in \( u \) has an infinite number of occurrences. An infinite word \( u \) is uniformly recurrent if for any factor \( w \) occurring in \( u \), there is some length \( n \) such that \( w \) appears in every factor of \( u \) of length \( n \).

The reversal of \( w = w_0w_1 \cdots w_{n-1} \in \mathcal{A}^n \) is the word \( \bar{w} = w_{n-1}w_{n-2} \cdots w_0 \). A palindrome is a word \( w \) such that \( w = \bar{w} \). We say that language \( \mathcal{L}(u) \) of an infinite word \( u \) is closed under reversal if \( w \in \mathcal{L}(u) \) implies \( \bar{w} \in \mathcal{L}(u) \) as well.

A factor \( w \) of \( u \) is called right special if it has more than one right extension \( x \in \mathcal{A} \) such that \( wx \in \mathcal{L}(u) \). A factor \( w \) of \( u \) is called left special if it has more than one left extension \( x \in \mathcal{A} \) such that \( xw \in \mathcal{L}(u) \). A factor which is both left and right special is called bispecial.

2.2 Morphisms

A morphism is a function \( \varphi : \mathcal{A}^* \to \mathcal{A}^* \) compatible with concatenation, that is, such that \( \varphi(uv) = \varphi(u)\varphi(v) \) for all \( u, v \in \mathcal{A}^* \). The identity morphism on \( \mathcal{A} \) is denoted by Id\( _\mathcal{A} \) or simply Id when the context is clear. A morphism \( \varphi \) is primitive if there exists an integer \( k \) such that for all \( \alpha \in \mathcal{A} \), \( \varphi^k(\alpha) \) contains each letter of \( \mathcal{A} \). A morphism also extends in a natural way to a map over \( \mathcal{A}^\mathbb{N} \). A subset \( X \) of the free monoid \( \mathcal{A}^* \) is a code if there exists an injective morphism \( \beta : \mathcal{B}^* \to \mathcal{A}^* \) such that \( X = \beta(\mathcal{B}) \).

A morphism is erasing if the image of one of the letters is the empty word. If \( \varphi \) is a nonerasing morphism, we define \( \text{FST}(\varphi) : \mathcal{A} \to \mathcal{A} \) to be the function defined by \( \text{FST}(\varphi)(a) \) is the first letter of \( \varphi(a) \). Similarly, let \( \text{LST}(\varphi) : \mathcal{A} \to \mathcal{A} \) be the function defined by \( \text{LST}(\varphi)(a) \) is the last letter of \( \varphi(a) \).

A morphism \( \varphi \) is prolongable at \( a \) if there is a letter \( a \) such that \( \varphi(a) = aw \) where \( w \) is a nonempty word. If \( \varphi \) is prolongable at \( a \), then

\[
w = aw\varphi(w)\varphi(\varphi(w)) \cdots \varphi^n(w) \cdots
\]
is a purely morphic word. It is also a fixed point of $\varphi$, i.e. $\varphi(w) = w$. A morphic word is the image of a pure morphic word under a morphism.

The mirror-image of a morphism $\varphi$, denoted by $\tilde{\varphi}$, is the morphism such that $\tilde{\varphi}(\alpha) = \overline{\varphi(\alpha)}$ for all $\alpha \in A$. The mirror-image $\tilde{\varphi}$ of a morphism $\varphi$ in class $P$ is conjugate to $\varphi$.

**Definition 2.** A morphism $\varphi$ is in class $P$ if there exists a palindrome $p$ and for every $\alpha \in A$ there exists a palindrome $q_\alpha$ such that $\varphi(\alpha) = pq_\alpha$.

If $\varphi$ is in class $P$, with $\alpha \mapsto pq_\alpha$, then the mapping $\Psi : v \mapsto \varphi(v)p$ assigns to any palindrome $v$ a new palindrome $\varphi(v)p$. If $\varphi$ is primitive then $\Psi^n(\alpha)$ is a palindrome for any letter $\alpha \in A$ and any $n \in N$ and clearly, $\Psi^n(\alpha)$ belongs $L(\varphi)$. Therefore, the language of any fixed point of a morphism in class $P$ is palindromic.

**Remark 3.** The primitivity of $\varphi \in P$ is not necessary for palindromicity of a fixed point of $\varphi$. The following example was provided by Štepán Starosta (personal communication, spring, 2014). Consider the substitution $\varphi$ on binary alphabet defined by $0 \mapsto 000$ and $1 \mapsto 10110100$. Clearly, the substitution belongs to the class $P$. The fixed point $\varphi^\infty(1)$ is not recurrent as it contains arbitrarily long blocks of zeros. It can be shown that $\varphi^\infty(1)$ has defect 0 and thus contains infinitely many palindromes.

An antimorphism is a function $\varphi : A^* \to A^*$ such that $\varphi(uv) = \varphi(v)\varphi(u)$ for all $u,v \in A^*$. For example, the reversal $\sim : A^* \to A^*$ is an involutive antimorphism. A word $w \in A^*$ fixed by an involutive antimorphism $\varphi$ is called a $\varphi$-palindrome.

The language of a morphism $\varphi$ is denoted $L(\varphi)$. This is not ambiguous as we consider primitive morphisms. Also $\text{Pal}(\varphi) = \{w \in L(\varphi) : w = \tilde{w}\}$ denotes the set of palindromes in the language.

### 2.3 Conjugacy of acyclic morphisms

Recall from Lothaire [17] (Section 2.3.4) that $\varphi$ is right conjugate of $\psi$, or that $\psi$ is left conjugate of $\varphi$, noted $\psi \triangleright \varphi$, if there exists $w \in A^*$ such that

$$\varphi(x)w = w\psi(x), \quad \text{for all words } x \in A^*, \quad (1)$$

or equivalently that $\varphi(\alpha)w = w\psi(\alpha)$, for all letters $\alpha \in A$. We say that the word $w$ is the conjugate word of the relation $\psi \triangleright \varphi$.

A morphism $\varphi : A^* \to A^*$ is cyclic [16] if there exists a word $w \in A^*$ such that $\varphi(\alpha) \in w^*$ for all $\alpha \in A$. Otherwise, we say that $\varphi$ is acyclic. If $\varphi$ is cyclic, then the fixed point of $\varphi$ is $www\cdots$ and is periodic. Remark that the converse does not hold. For example, $a \mapsto aba, b \mapsto bab$ is acyclic but its fixed point is periodic. We have the following statement.

**Lemma 4.** A morphism is cyclic if and only if it is conjugate to itself with a nonempty conjugate word.

If a morphism is acyclic, it has a left most and a right most conjugate.

**Definition 5.** Let $\varphi$ be an acyclic morphism. The right most conjugate of $\varphi$ is a morphism $\varphi_R$ such that the following two conditions hold:

(i) $\varphi_R$ is right conjugate of $\varphi$;

(ii) if $\psi$ is right conjugate to $\varphi_R$, then $\psi = \varphi_R$. 

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The leftmost conjugate of \( \varphi \) is defined analogously and denoted \( \varphi_L \).

Equivalently, \( \varphi_L \) is the leftmost conjugate of \( \varphi \) if \( w\varphi_L(x) = \varphi(x)w \) for all words \( x \in A^* \) and if \( \text{FST}(\varphi_L) \) is not constant. Also, \( \varphi_R \) is the rightmost conjugate of \( \varphi \) if \( w\varphi(x) = \varphi_R(x)w \) for all words \( x \in A^* \) and if \( \text{LST}(\varphi_R) \) is not constant.

**Example 6.** Consider the following morphisms:

\[
\begin{align*}
\varphi_1 &: a \mapsto bab, b \mapsto bab, \\
\varphi_2 &: a \mapsto abab, b \mapsto abb, \\
\varphi_3 &: a \mapsto bbaba, b \mapsto bba, \\
\varphi_4 &: a \mapsto babab, b \mapsto bab.
\end{align*}
\]

The morphism \( \varphi_1 \) is right conjugate to \( \varphi_2 \) with conjugate word \( b \) because \( \varphi_1(a) \cdot b = babbab = b \cdot \varphi_2(a) \)
and \( \varphi_1(b) \cdot b = babb = b \cdot \varphi_2(b) \). In general, the following relations are satisfied:

\[
\varphi_L = \varphi_7 \circ \varphi_6 \circ \varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1 = \varphi_R.
\]

The morphism \( \varphi_7 \) is leftmost conjugate and \( \varphi_1 \) is rightmost conjugate with conjugate word \( babab \).

Recall some simple properties of conjugate morphisms.

**Lemma 7.** Let a morphism \( \varphi \) be right conjugate of a morphism \( \psi \) and \( w \in A^* \) be the conjugate word of the relation \( \psi \rhd \varphi \).

1. For any \( k \in \mathbb{N} \), the morphism \( \varphi^k \) is right conjugate of \( \psi^k \).
2. \( \varphi \) is injective if and only if \( \psi \) is injective.
3. \( \varphi \) is primitive if and only if \( \psi \) is primitive.
4. If \( \varphi \) is primitive, then \( \mathcal{L}(\varphi) = \mathcal{L}(\psi) \).
5. \( \tilde{\psi} \) is right conjugate of \( \tilde{\varphi} \) and the corresponding conjugate word is \( \tilde{w} \).

**Proof.**

1. Let us define recursively \( w_{(1)} = w \) and \( w_{(k+1)} = \varphi(w_{(k)})w = w\psi(w_{(k)}) \) for any \( k \in \mathbb{N} \). We show that \( \varphi^k(u)w_{(k)} = w_{(k)}\psi^k(u) \) for any \( u \in A^* \).

We proceed by induction on \( k \). Assume that \( \varphi^k(u)w_{(k)} = w_{(k)}\psi^k(u) \). As \( \varphi(v)w = w\psi(v) \) for any word \( v \), we can apply this relation to \( v = \varphi^k(u)w_{(k)} = w_{(k)}\psi^k(u) \). We get \( \varphi^{k+1}(u)\varphi(w_{(k)})w = w\psi(w_{(k)})\psi^{k+1}(u) \), or equivalently \( \varphi^{k+1}(u)w_{(k+1)} = w_{(k+1)}\psi^{k+1}(u) \), as we wanted to show.

2. Let \( u, v \in A^* \). Then

\[
\psi(u) = \psi(v) \iff \psi(u)w = \psi(v)w \iff w\varphi(u) = w\varphi(v) \iff \varphi(u) = \varphi(v).
\]

3. Let us recall that a morphism with the incidence matrix is primitive if and only if there exists an exponent \( k \) such that all elements of \( k \)th power of the incidence matrix are positive. Two mutually conjugate morphisms have the same incidence matrix.

4. Let us fix an arbitrary \( n \in \mathbb{N} \). We show that \( \mathcal{L}_n(\varphi) = \mathcal{L}_n(\psi) \). According to point 3, \( \psi \) is primitive as well, and thus there exists \( R(n) \) such that any factor of \( \mathcal{L}(\varphi) \) longer than \( R(n) \) contains any factor of \( \mathcal{L}_n(\varphi) \) and any factor of \( \mathcal{L}(\psi) \) longer than \( R(n) \) contains any factor of \( \mathcal{L}_n(\psi) \). Let us find \( k \) such that for some letter \( a \) both words \( \varphi^k(a) \) and \( \psi^k(a) \) are longer than \( 2R(n) \).
As Ψ_k and Ψ^k are conjugate, there exists a word, say y, such that Ψ^k(a)y = yΨ^k(a). Using the fact that the equation xy = yz implies x = uv and z = vu for some u, v, we can write Ψ^k(a) = uv and Ψ^k(a) = vu, in particular, u, v ∈ L(Φ) and u, v ∈ L(Ψ). Since \(|Ψ^k(a)| ≥ 2R(n)|, either u or v are longer than R(n) and thus all elements from L_n(Φ) occurs in u or v and consequently in L(Ψ) as well.

5. Applying mirror image mapping to relation Φ(a)w = wΨ(a) for any letter a ∈ A, we get \(\hat{w}\hat{(a)} = \hat{(a)}\hat{w}\) as desired.

\textbf{Lemma 8.} Let \(Φ\) be a primitive acyclic morphism and \(u\) its fixed point. Denote \(Φ_L\) and \(Φ_R\) the left most and the right most conjugate of \(Φ\), respectively. By \(w ∈ A^*\), denote the conjugate word of the relation \(Φ_L ∨ Φ_R\). If \(u ∈ L(Φ)\) then \(Φ_R(u)w = wΦ_L(u) ∈ L(Φ)\).

\textbf{Proof.} Since \(Φ\) is primitive, \(u\) occurs in \(u\) infinitely many times. Hence there exists arbitrarily long word \(v\) such that \(vu ∈ L(Φ)\). Clearly, \(Φ_L(vu) ∈ L(Φ)\). Consequently, \(w^{-1}wΦ_L(vu) = w^{-1}Φ_R(vu)w = w^{-1}Φ_R(v)Φ_R(u)w ∈ L(Φ)\). Since the factor \(v\) is long enough, \(|w^{-1}Φ_R(v)| > 0\) and thus \(Φ_R(u)w ∈ L(Φ)\).

\subsection{2.4 Marked morphisms}

A. Frid \cite{8,9} defined a morphism Φ to be marked if both \(\text{Fst}(Φ)\) and \(\text{Lst}(Φ)\) are injective. On a binary alphabet, B. Tan \cite{19} defined a morphism Φ to be marked if \(\text{Fst}(Φ)\) is injective and well-marked if \(\text{Fst}(Φ)\) is the identity. It is convenient to extend the definition of A. Frid to morphisms such that the cardinality of their conjugacy class is larger than one. Also, we don’t need that \(\text{Fst}(Φ)\) be the identity in the proof of Lemma \cite{22}. Thus, we introduce the following definitions that will be useful in the sequel.

\textbf{Definition 9.} Let \(Φ\) be an acyclic morphism. We say that \(Φ\) is marked if

\[\text{Fst}(Φ_L)\ \text{and} \ \text{Lst}(Φ_R)\ \text{are injective}\]

and that \(Φ\) is well-marked if

\[\text{it is marked and if } \text{Fst}(Φ_L) = \text{Lst}(Φ_R)\]

where \(Φ_L\) (\(Φ_R\) resp.) is the left most (right most resp.) conjugate of \(Φ\).

\textbf{Lemma 10.} A marked morphism has a well-marked power.

\textbf{Proof.} Let us realize that \(\text{Fst}(Φ ∗ Ψ) = \text{Fst}(Φ) ∗ \text{Fst}(Ψ)\) for each morphisms \(Φ, Ψ : A^* → A^*\). If \(Φ\) is marked, then \(\text{Fst}(Φ_L)\) and \(\text{Lst}(Φ_R)\) are permutations of the alphabet \(A\). Denote by the integer \(d\) the cardinality of \(A\). Then for \(k = d!\), the permutations \((\text{Fst}(Φ_L))^k\) and \((\text{Lst}(Φ_R))^k\) are the identity. Note that \(Φ_L^k\) is the left most conjugate and \(Φ_R^k\) is the right most conjugate of \(Φ^k\). Since \(\text{Fst}(Φ_L^k) = (\text{Fst}(Φ_L))^k = \text{Id} = (\text{Lst}(Φ_R))^k = \text{Lst}(Φ_R^k)\) the power \(Φ^k\) is well-marked.

The power need not be \(d!\). In fact, the least positive integer \(a(d)\) for which \(p^{a(d)} = 1\) for all permutations \(p\) in \(S_d\) is sufficient. This sequence \((a(d))_d\) is well-known and indexed by A003418 in the OEIS \cite{18}. Of course, there might be an integer \(N < a(d)\) such that \(\text{Fst}(Φ_L)^N = \text{Lst}(Φ_R)^N\).

In particular, this happens for the binary alphabet.

\textbf{Lemma 11.} A marked morphism is injective.

\textbf{Proof.} Let \(Φ_L\) be the left most conjugate of a marked morphism \(Φ\). By definition \(\text{Fst}(Φ_L)\) is injective so that \(Φ_L\) is injective. From Lemma \cite{7} injectivity is preserved by conjugacy. Therefore \(Φ\) is injective.
2.5 Periods and palindromes

From Lothaire [15, 16] we borrow the following useful result: if \( w = xy = yz \), then, for some \( u, v \), and some \( i \geq 0 \) we have

\[
x = uv, y = (uv)^i u, z = vu.
\]

(2)

The words \( u \) and \( v \) are uniquely determined. Indeed, if \( |x| = |z| \) do not divide \( |y| \), then \( |u| \) is the remainder and \( i \) is the quotient of the division of \( |y| \) by \( |x| \). If \( |x| = |z| \) divides \( |y| \), there is only one solution but two ways of writing it namely \( u = r, v = \varepsilon \) and \( u = \varepsilon, v = r \). Thus, to avoid duplicate solutions, we suppose that \( |u| \) is the remainder, \( 0 \leq |u| < |x| \), and \( i \) is the quotient of the division of \( |y| \) by \( |x| \).

We start by stating results which appeared previously as Lemma 2.10 and 2.11 of [13] and Proposition 6 of [5] (and in a less general form as Lemma 5 in [6]).

**Lemma 12.** Assume that \( w = xy = yz \). Let \( u, v \) and \( i \geq 0 \) be such that Eq. (2) holds. Let \( f \) be an involutive antimorphism on \( A^* \). The following conditions are equivalent:

(i) \( x = f(z) \);

(ii) \( u \) and \( v \) are \( f \)-palindromes;

(iii) \( w \) is a \( f \)-palindrome;

(iv) \( xyz \) is a \( f \)-palindrome.

Moreover, if one of the equivalent conditions above holds then

(v) \( y \) is a \( f \)-palindrome.

**Lemma 13.** Assume that \( w = xy = yz \) with \( |y| \geq |x| \). Then, conditions (i)-(v) in Lemma 12 are equivalent.

3 Equivalent conditions for a morphism to be in class \( \mathcal{P} \)

Let \( \varphi \) be an acyclic and primitive morphism. Let \( \varphi_R (\varphi_L \text{ resp.}) \) be the right most (left most resp.) conjugate of \( \varphi \). Their existence follows from the acyclic hypothesis. Let \( w \) be the conjugate word of the relation \( \varphi_L \triangleright \varphi_R \). Then, Equation (1) is satisfied:

\[
\varphi_R(x)w = w\varphi_L(x), \quad \text{for all words } x \in A^*.
\]

Note that we have \( |\varphi_R(x)| = |\varphi_L(x)| \) for all words \( x \in A^* \). Obviously, if a word \( x \in A^* \) is such that \( |\varphi_L(x)| \geq |w| \), then \( w \) is a suffix of \( \varphi_L(x) \) and \( w \) is a prefix of \( \varphi_R(x) \).

**Lemma 14.** Let \( v, u \) be two palindromes. If \( k \) is an integer such that

\[
k = |u| + \left\lceil \frac{|v|}{2} \right\rceil + \ell \cdot |vu| \quad \text{or} \quad k = \left\lceil \frac{|u|}{2} \right\rceil + \ell \cdot |vu|
\]

for some integer \( \ell \geq 0 \), then the \( k \)-th conjugate of \( vu \) is of the form \( \alpha \cdot p \) where \( \alpha \in A \cup \{\varepsilon\} \) and \( p \) is a palindrome. Moreover, it is a palindrome if and only if \( \alpha = \varepsilon \) if and only if the ceil function is applied on an integer, that is if \( |v| \) is even in the first case and if \( |u| \) is even in the second case.
Proof. Write $u = s \tilde{s}$ and $v = t \tilde{t}$ for some words $s, t \in \mathcal{A}$ and letters (or empty word) $\alpha, \beta \in \mathcal{A} \cup \{\varepsilon\}$. The conjugates $c_1 = \beta t s \alpha s t$ and $c_2 = \alpha s t \beta t s$ of $vu$ are of the desired form. We have

$$c_1 \cdot \beta t s \alpha s t = \beta t s \alpha s t \cdot vu \quad \text{and} \quad c_2 \cdot \alpha s t \beta t s = \alpha s t \beta t s \cdot vu.$$  

Then $c_1$ is the $k_1$-th conjugate of $vu$ for $k_1 = |\beta t s \alpha s t| = |u| + \left\lceil \frac{|v|}{2} \right\rceil$ and $c_2$ is the $k_2$-th conjugate of $vu$ for $k_2 = |\alpha s t \beta t s| = \left\lfloor \frac{|w|}{2} \right\rfloor$. Note that the $k + \ell|vu|$-th conjugate of $vu$ is equal to the $k$-th conjugate of $vu$. Finally, $c_1$ is a palindrome exactly when $|v|$ is even and $c_2$ is a palindrome exactly when $|u|$ is even. 

The next lemma is true whatever the size of the alphabet. An important consequence is that HKS conjecture is satisfied for all morphism satisfying one of the equivalent condition.

Let $\mathcal{B} \subseteq \mathcal{A}$ be the set of letters such that its image under $\varphi$ is larger than the conjugate word $w$:

$$\mathcal{B} = \{ b \in \mathcal{A} : |\varphi(b)| > |w| \}.$$  

Lemma 15. Let $\varphi$ be a morphism and $\varphi_R$ ($\varphi_L$ resp.) be the right most (left most resp.) conjugate of $\varphi$. Let $w$ be the conjugate word such that $w \varphi_L(u) = \varphi_R(u)w$ for all words $u \in \mathcal{A}^*$. For all $b \in \mathcal{B}$, let $p_b$ be the nonempty word such that $\varphi_L(b) = p_bw$. Then, the following conditions are equivalent:

1. the $\left\lceil \frac{|w|+1}{2} \right\rceil$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$;
2. $\varphi$ has a conjugate in class $\mathcal{P}$;
3. $\varphi$ and $\tilde{\varphi}$ are conjugates;
4. $\varphi_L = \tilde{\varphi}_R$;
5. $w$ is a palindrome and $p_b$ is a palindrome for all $b \in \mathcal{B}$.

Proof. (1) $\implies$ (2): This is clear.

(2) $\implies$ (3): Let $\varphi' : \alpha \mapsto p q_\alpha$ be the conjugate of $\varphi$ in class $\mathcal{P}$. Then $\tilde{\varphi}$ is conjugate to $\varphi' : \alpha \mapsto q_{\alpha} p$. But $\varphi'$ and $\tilde{\varphi}'$ are conjugates. We conclude by transitivity of the conjugacy of morphisms.

(3) $\implies$ (4): The left most conjugate of $\tilde{\varphi}$ is $\tilde{\varphi}_R$. If $\varphi$ and $\tilde{\varphi}$ are conjugates, they must share the same left most conjugate. Therefore $\varphi_L = \tilde{\varphi}_R$.

(4) $\implies$ (5): Let $u$ be a word long enough so that $|\varphi_L(u)| \geq |w|$. Then $w$ is a suffix of $\varphi_L(u)$ because $w \varphi_L(u) = \varphi_R(u)w$. But $\varphi_L(u) = \tilde{\varphi}_R(u) = \varphi_R(\tilde{u})$. Therefore, $\tilde{w}$ is a prefix of $\varphi_R(\tilde{u})$. But again $w$ is a prefix of $\varphi_R(\tilde{u})$. We conclude that $w$ is a palindrome. If $b \in \mathcal{B}$, then $\tilde{w}p_bw = \tilde{\varphi}_L(b)w = \varphi_R(b)w = w \varphi_L(b) = wp_bw$ and $p_b$ is a palindrome.

(5) $\implies$ (1): Let $k = \left\lceil \frac{|w|+1}{2} \right\rceil$ and $x_a^{(k)}$ be $k$-th conjugate of the word $\varphi_L(\alpha)$.

First, suppose $b \in \mathcal{B}$. Let $z$ be the word such that $w = \tilde{z} c z$ for some letter (or empty word) $c \in \mathcal{A} \cup \{\varepsilon\}$.

$$x_b^{(k)} = czp_b \tilde{z}$$  

is a palindrome if $|w|$ is even and the product of the letter $c$ and a palindrome if $|w|$ is odd.

Now suppose $\alpha \in \mathcal{A} \setminus \mathcal{B}$. Then $|w| \geq |\varphi_L(\alpha)|$. The equation $\varphi_R(\alpha)w = w \varphi_L(\alpha)$ is depicted in Figure 1. Hypothesis of Lemma 13 are satisfied and we conclude the existence of palindromes $u_\alpha, v_\alpha \in \mathcal{A}^*$ and an integer $i \geq 1$ such that $\varphi_L(\alpha) = v_\alpha u_\alpha$, $w = (u_\alpha v_\alpha)^i u_\alpha$, $\varphi_R(\alpha) = \varphi_L(\alpha) = u_\alpha v_\alpha$ and

$$|w| = i \cdot n_\alpha + |u_\alpha|, \quad 0 \leq |u_\alpha| < n_\alpha.$$
where $n_\alpha = |\varphi_L(\alpha)|$. We show that the $\lfloor \frac{|w|+1}{2} \rfloor$-th conjugate of $\varphi$ is in class $\mathcal{P}$. Observe that $k \leq |w|$ so that the $k$-th conjugate of $\varphi_L$ exists. We have

$$k = \left\lfloor \frac{|w|+1}{2} \right\rfloor = \left\lfloor \frac{|w|}{2} \right\rfloor = \frac{i|u_\alpha v_\alpha| + |u_\alpha|}{2} = \begin{cases} \left\lfloor \frac{|u_\alpha|}{2} \right\rfloor + \frac{i}{2}|u_\alpha v_\alpha| & \text{if } i \text{ is even}, \\ \left\lceil \frac{|v_\alpha|}{2} \right\rceil + |u_\alpha| + \frac{i-1}{2}|u_\alpha v_\alpha| & \text{if } i \text{ is odd}. \end{cases}$$

If $|w|$ is even, then $i$ is odd and $|v_\alpha|$ is even or $i$ and $|u_\alpha|$ are both even. In other words, the ceil function is applied on an integer in the above formulas for $k$. From Lemma\ref{lem:conjugate_palindrome}, the $k$-th conjugate of the word $\varphi_L(\alpha)$ is a palindrome. We conclude that the $k$-th conjugate of $\varphi$ is mapping each letter $\alpha \mapsto x_\alpha^{(k)}$ on a palindrome, that is, the $k$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$.

If $|w|$ is odd, then $i$ is odd and $|v_\alpha|$ is odd or $i$ and $|u_\alpha|$ are both odd. In other words, the ceil function is applied on a half-integer in the above formulas for $k$. From Lemma\ref{lem:conjugate_palindrome}, the $k$-th conjugate of the word $\varphi_L(\alpha)$ is of the form $c_\alpha p$ where $c_\alpha \in \mathcal{A}$ is a letter and $p$ is a palindrome. Since $k \geq 1$ and because of the definition of conjugacy, the first letter of each word $x_\alpha^{(k)}$ is the same, i.e., $c_\alpha = c$. We conclude that the $k$-th conjugate of $\varphi_L$ is in class $\mathcal{P}$.

**Corollary 16.** Let $\psi$ and $\varphi$ be conjugate morphisms such that $\psi(u)w = w\varphi(u)$ for all $u \in \mathcal{A}^*$. If $w$ is a palindrome and $|w| \geq |\varphi(\alpha)|$ for all $\alpha \in \mathcal{A}$, then the $\lfloor \frac{|w|+1}{2} \rfloor$-th conjugate of $\varphi$ is in class $\mathcal{P}$.

The morphisms $\varphi_R$ and $\varphi_L$ from Example\ref{ex:conjugate_palindrome} satisfy the assumptions of the previous corollary. Therefore a conjugate of them belongs to class $\mathcal{P}$.

## 4 Properties of palindromic words

This section contains results on properties of palindromic words. It ends with a lemma giving sufficient conditions for the presence of infinitely many palindromic bispecial factors. First we recall the relationship between two properties of language: “to be palindromic” and “to be closed under reversal”.

**Lemma 17.** Let $u$ be a uniformly recurrent palindromic word. Then its language is closed under reversal.

*Proof.* Let $v \in \mathcal{L}(u)$ and $|v| = n$. Since $u$ is uniformly recurrent, there exists an integer $R(n)$ such that any factor $u \in \mathcal{L}(u)$ with $|u| \geq R(n)$ contains all factors from $\mathcal{L}_n(u)$. In particular, any palindrome $p \in \mathcal{L}(u)$ with $|p| \geq R(n)$ contains all factors of length $n$, i.e. $v$ occurs in $p$. And clearly $\tilde{v}$ occurs in $p$ as well.

Berstel, Boasson, Carton and Fagnot provided in [2] a nice example of uniformly recurrent word whose language is closed under reversal, but the language is not palindromic. It means that the
opposite implication in the previous lemma does not hold. If we restrict ourselves to eventually periodic words, the opposite implication is valid.

The next result is well-known and already present in [1,6].

**Lemma 18.** Let $u$ be an eventually periodic word with language closed under reversal. Then $u$ is purely periodic and there exist two palindromes $p$ and $q$ such that $u = (pq)^\omega$.

**Proof.** Since $\mathcal{L}(u)$ is closed under reversal, any prefix of $u$ has infinitely many occurrences in $u$ and thus $u$ is purely periodic, i.e., $u = v^\omega$ for some factor $v$. Closedness under reversal gives that $\tilde{v}$ is a factor of $u$. Any factor with the length $|v|$, in particular $\tilde{v}$ as well, occurs in $vv$. It means that there exist factors $p,q$ of $v$ such that $p\tilde{q}v = vv$. Consequently $v = pq$ and $\tilde{v} = qp$. It implies $\tilde{p}q = qp$ and thus both factors $p$ and $q$ are palindromes.

**Lemma 19.** Let $u$ be a palindromic word. Then there exists an biinfinite word $p \coloneqq \cdots p_3p_2p_1p_0p_1p_2p_3 \cdots$, where $p_0 \in A \cup \{\varepsilon\}$ and $p_i \in A$ for $i \geq 1$, such that for any non-negative integer $m$ the string $p_mp_{m-1} \cdots p_0 \cdots p_{m-1}p_m$ is a palindrome occurring in $\mathcal{L}(u)$. The word $p$ is called infinite palindromic branch of $u$.

**Proof.** Let us construct an oriented graph $G$: vertices of $G$ are palindromes occurring in $u$. A pair of palindromes $p$ and $q$ are connected with an edge starting in $p$ and ending in $q$ if there exists and letter $a$ such that $q = apa$. It is readily seen that

- for any palindrome $q$ with length at least 2 there exists exactly one edge ending in $q$;
- for any palindrome $q$ there exists at most $\#A$ edges starting in $q$;
- no edge ends in a vertex from $A \cup \{\varepsilon\}$;
- $G$ contains no oriented cycle because any edge starts in a shorter palindrome than it ends;
- any palindromic $p$ is reachable by an oriented path from one of vertices from $A \cup \{\varepsilon\}$.

It means that $G$ is a forest with $1 + \#A$ components. Since the language of $u$ contains infinitely many palindromes, at least one of these components is an infinite tree. According to König’s lemma, this component contains an infinite oriented path. Denote its starting vertex $p_0 \in A \cup \{\varepsilon\}$. The $m^{th}$-vertex $P_m$ along the path is a palindrome $P_m \coloneqq p_mp_{m-1} \cdots p_0 \cdots p_{m-1}p_m$.

It implies that there exists an biinfinite word $p \coloneqq \cdots p_3p_2p_1p_0p_1p_2p_3 \cdots$ as desired.

**Lemma 20.** Let $u$ be a uniformly recurrent palindromic word. If $u$ is not eventually periodic, then its language contains infinitely many palindromic bispecial factors.

**Proof.** Let $w$ be a factor of $u$. Recall that $u$ is a complete return word of $w$ in $u$ if $u$ is a factor of $u$, $w$ is a prefix and a suffix of $u$ and $u$ contains no other occurrences of $w$.

First we show the existence of a bispecial word containing any factor $w$. Because of $u$ is uniformly recurrent the gaps between consecutive occurrences of $w$ are bounded, or in other words, the set of complete return words to $w$ is finite. Let $v$ be the longest common prefix of all complete return words to $w$. Clearly, it has the form $v = wV$ for some (possibly empty) factor $V$ of $u$. If $wV$ contains two occurrences of $w$, then $wV$ is the unique complete return word to $w$ and $u$ is periodic which is a contradiction. Thus $wV$ contains only one occurrence of $w$ and $wV$ must be shorter then the shortest complete return word to $w$. Therefore, $wV$ is right special because it is the longest common prefix of two longer complete return words to $w$. Moreover, $wV$ is the unique right prolongation of the factor $w$ with length $|wV|$. Analogously, the longest common suffix of all complete return words to $w$ has the form $Uw$, it is left special and $Uw$ is the unique left prolongation of the factor $w$ with length $|Uw|$. As $Uw$ is left special and $wV$ is the unique
right prolongation of the factor $w$ with length $|wV|$, the factor $UwV$ is left special as well. For the same reason, $UwV$ is right special and thus $UwV$ is a bispecial factor containing the factor $w$.

Let us show that if $w$ is a palindromic factor of $u$, then the bispecial factor $UwV$ is palindromic as well. Indeed, as $u$ is closed under reversal (Lemma 17), the set of complete return words to the palindrome $w$ is closed under reversal as well. Therefore the longest common suffix of all complete return words to $w$ is just the mirror image of the longest common prefix of all complete return words to $w$, in other words $\tilde{U} = V$, i.e. the bispecial factor $UwV$ is a palindrome.

We have shown that for any palindrome $w$ there exists a palindromic bispecial factor with length at least $|w|$. Since $u$ contains infinitely many palindromes, necessarily $u$ contains infinitely many palindromic bispecial factors. 

\[ \square \]

**Remark 21.** In the proof of the previous lemma, we have shown that any palindrome $w$ can be extended to a bispecial factor of the form $Uw\tilde{U}$, where $Uw\tilde{U}$ is the unique palindromic extension of $w$ with length $|Uw\tilde{U}|$. It implies that any palindromic branch $p := \cdots p_3p_2p_1p_0p_1p_2p_3 \cdots$ contains infinitely many bispecial palindromes $P_m := p_m p_{m-1} \cdots p_0 \cdots p_{m-1} p_m$.

## 5 Properties of well-marked morphisms

**Lemma 22.** Let $\varphi$ be a well-marked morphism. Denote $\varphi_L$ and $\varphi_R$ the left most and the right most conjugate of $\varphi$, respectively. By $w \in A^*$, denote the conjugate word of the relation $\varphi_L \triangleright \varphi_R$. If there exist $u, v \in A^*$ such that

\[ \varphi_R(u)w = \varphi_R(v)w, \]  

then $w$ is a palindrome, $\tilde{u} = v$ and $\varphi_L(a) = \tilde{\varphi}_R(a)$ for any letter $a$ occurring in $u$.

The hypothesis of this lemma can be made more general (injectivity of $\varphi_L$ and $\varphi_R$ instead of well-marked) but we use it only for well-marked morphisms.

**Proof.** Suppose $u = u_0u_1 \cdots u_n$ and $v = v_0v_1 \cdots v_m$. Due to (3) we have $\tilde{w} \varphi_R(u) = \varphi_R(v)w$. It immediately gives $\tilde{w} = w$ and moreover, $\varphi_R(u_0) \cdots \varphi_R(u_0) = \varphi_L(v_0) \cdots \varphi_L(v_m)$. Hence,

\[ \text{Fst}(\varphi_L(v_0)) = \text{Fst}(\varphi_R(u_n)) = \text{Lst}(\varphi_R(u_n)). \]

Since $\varphi$ is well-marked, then

\[ \text{Lst}(\varphi_R(u_n)) = \text{Fst}(\varphi_R(u_n)). \]

We get $\text{Fst}(\varphi_L)(v_0) = \text{Fst}(\varphi_L)(u_n)$. But $\text{Fst}(\varphi_L)$ is injective since $\varphi$ is well-marked. We conclude that $u_n = v_0$. As $|\varphi_L(a)| = |\varphi_R(a)| = |\tilde{\varphi}_R(a)|$ for any letter $a \in A$, we also deduce $\varphi_L(v_0) = \tilde{\varphi}_R(v_0)$. It implies $\varphi_L(v_1) \cdots \varphi_L(v_m) = \tilde{\varphi}_R(u_{n-1}) \cdots \tilde{\varphi}_R(u_0)$. For the same reason as above, we have $v_1 = u_{n-1}$ and $\varphi_L(v_1) = \tilde{\varphi}_R(v_1)$ and $m = n$. 

\[ \square \]

**Proposition 23.** Let $\varphi$ be a primitive marked morphism. Denote $\varphi_L$ and $\varphi_R$ the left most and the right most conjugate of $\varphi$, respectively. By $w \in A^*$, denote the conjugate word of the relation $\varphi_L \triangleright \varphi_R$. Define $\Phi : \mathcal{L}(\varphi) \mapsto \mathcal{L}(\varphi)$ by

\[ \Phi(u) = \varphi_R(u)w. \]

1. If $u \in \mathcal{L}(\varphi)$ is a left (resp. right) special factor, then $\Phi(u)$ is a left (resp. right) special factor, too.
2. There exist a finite number of bispecial factors, say $u^{(1)}, u^{(2)}, \ldots, u^{(N)} \in \mathcal{L}(\varphi)$, such that any bispecial factor of $\mathcal{L}(\varphi)$ equals $\Phi^n(u^{(j)})$ for some $j = 1, 2, \ldots, N$ and $n \in \mathbb{N}$, where $\Phi^n$ denotes the $n^{th}$ iteration of $\Phi$.

The proof of the proposition is based on results of K. Klouda from [12]. They concern a very broad class of circular non-pushy D0L-systems. Any primitive injective substitution belongs to this class. Following the notation of [12], for a morphism $\psi$ over $\mathcal{A}^+$ and an unordered pair of words $(w_1, w_2)$ from $\mathcal{A}^+$, we denote

$$f_L(w_1, w_2) = \text{the longest common suffix of } \psi(w_1) \text{ and } \psi(w_2),$$
$$f_R(w_1, w_2) = \text{the longest common prefix of } \psi(w_1) \text{ and } \psi(w_2).$$

Let us summarize the relevant consequences of Theorems 22 and 36 from [12] in the case of injective primitive morphism.

**Theorem 24** (Klouda, [12]). Let $\psi$ is an injective primitive morphism. Then,

- there exist two finite sets $\mathcal{B}_L$ and $\mathcal{B}_R$ (called $L$-forky set and $R$-forky set, resp.) of unordered pairs of nonempty factors from $\mathcal{L}(\psi)$ such that the last letters of $w_1$ and $w_2$ are distinct for any pair $(w_1, w_2)$ in $\mathcal{B}_L$ and the first letters of $v_1$ and $v_2$ are distinct for any pair $(v_1, v_2)$ in $\mathcal{B}_R$;
- there exists a finite set $\mathcal{I}$ of bispecial factors (called initial factors);
- any bispecial factor $u \in \mathcal{L}(\psi) \setminus \mathcal{I}$ is equal to $f_L(w_1, w_2)\psi(u')f_R(v_1, v_2)$, where $u'$ is a suitable bispecial factor, $(w_1, w_2) \in \mathcal{B}_L$ and $(v_1, v_2) \in \mathcal{B}_R$.\\

**Proof of Proposition** [23]. According to Lemma 8, the definition of the mapping $\Phi$ is correct. There exist $a, b \in \mathcal{A}$, $a \neq b$ such that $au, bu \in \mathcal{L}(\varphi)$. According to Lemma 8, words $\varphi_R(a)\varphi_R(u)w$ and $\varphi_R(b)\varphi_R(u)w$ belong to $\mathcal{L}(\varphi)$ too. Since $\varphi$ is marked, the last letters of $\varphi_R(a)$ and $\varphi_R(b)$ differs and thus $\varphi_R(u)w$ is left special.

Analogously, there exist $c, d \in \mathcal{A}$, $c \neq d$ such that $uc, ud \in \mathcal{L}(\varphi)$. According to Lemma 8, words $\varphi_R(u)\varphi_R(c)w = \varphi_R(u)w\varphi_L(c)$ and $\varphi_R(u)\varphi_R(d)w = \varphi_R(u)w\varphi_L(d)$ belong to $\mathcal{L}(\varphi)$ too. Since $\varphi$ is marked, the first letters of $\varphi_L(c)$ and $\varphi_L(d)$ differs and thus $\varphi_R(u)w$ is right special.

We exploit results of K. Klouda from [12]. We apply Theorem 24 to the morphism $\psi = \varphi_R$. Since $\text{LST}(\varphi_R)$ is injective, $f_L(w_1, w_2) = \varepsilon$ for any pair $(w_1, w_2)$ from the $L$-forky set $\mathcal{B}_L$. On the other hand, because of $\varphi_R(v_1)w = w\varphi_L(v_1)$ and $\varphi_R(v_2)w = w\varphi_L(v_2)$ and $\text{FST}(\varphi_L)$ is injective, we have $f_R(v_1, v_2) = w$ for any pair $(v_1, v_2)$ from the $R$-forky set $\mathcal{B}_R$. It means that any bispecial non-initial factor $u$ from $\mathcal{L}(\varphi_R)$ equals to $\varphi_R(u')w = \Phi(u')$ for some suitable bispecial factor $u'$. \hfill \square

The fixed point of a cyclic morphism is periodic. But the converse does not hold. For example, the morphism $\xi$ defined by

$$a \mapsto \xi(a) = aba \quad \text{and} \quad b \mapsto \xi(b) = bab$$

is acyclic but its fixed points $(ab)^{\omega}$ and $(ba)^{\omega}$ are periodic with length of period 2 and both letters occurs in the period. Moreover, the morphism $\xi$ is primitive and well-marked, i.e., it satisfies the assumption of the previous propositions. The following corollary describes morphisms of this type.

**Corollary 25.** Let $\varphi$ be a primitive marked morphism over an alphabet $\mathcal{A}$ and $u$ be a fixed point of $\varphi$. If $u$ is eventually periodic, then there exists a period $w \in \mathcal{A}^*$ such that $u = w^{\omega}$, $|w| = \text{Card} \mathcal{A}$, every letter of $\mathcal{A}$ occurs exactly once in the period $w$ and $\varphi(w) = w^k$ for some $k \in \mathbb{N}$. 

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Proof. According to Point 1 of Proposition 23, if \( L(\varphi) \) contains one left or right special factor of length at least 1, then it contains infinitely many left and right special factors. Since language of an eventually periodic word has only finitely many left and right special factors, necessarily there are no nonempty special factors in \( L(\varphi) \) at all. Let \( u = vw^\omega \), where \( w \) is the shortest period and \( v \) is the shortest preperiod of \( u \). Obviously, \( v \) is empty, otherwise the first letter of the period \( w \) is left special. For the same reason, any letter of the alphabet occurs in the period exactly once.

Let us denote by \( c \) the first letter of the period \( w \). As \( u \) is a fixed point of \( \varphi \) the first letter of \( \varphi(c) \) is \( c \). Since any letter occurs in \( w \) exactly once, any prefix \( u \) of \( u \) which starts and ends by the letter \( c \) has the form \( u = w^kc \) for some \( k \in \mathbb{N} \). As \( \varphi(w)c \) is a prefix of \( \varphi(w)w = \varphi(w)\varphi(w) \) which is a prefix \( u \), the word \( \varphi(w)c = w^kc \) for some \( k \in \mathbb{N} \). \( \square \)

Example 26. The morphism \( \varphi \) defined by \( a \mapsto ababcaab, b \mapsto caba, c \mapsto bc \) is a primitive marked morphism over the alphabet \( \{a,b,c\} \). Its fixed point is \( (abc)^\omega \) and \( \varphi(abc) = (abc)^5 \). But the language of the fixed point \( (abc)^\omega \) unlike the fixed point \( (ab)^\omega \) of the morphisms \( \xi \) is not palindromic.

Corollary 27. Let \( \varphi \) be a primitive marked morphism over an alphabet \( A \) and \( u \) be an eventually periodic fixed point of \( \varphi \). If \( u \) is palindromic, then \( A \) is binary alphabet and \( \varphi \) belongs to class \( \mathcal{P} \).

Proof. According to Corollary 25 \( u = w^\omega \) and the period \( w \) contains each letter of alphabet exactly once. Due to Lemma 18 the period \( w = pq \), where \( p \) and \( q \) are palindromes. The only possibility is that \( |p| = |q| = 1 \) and the period has the form \( w = ab \), where \( a, b \in A, a \neq b \). Thus cardinality of the alphabet is \( |w| = 2 \).

Corollary 25 moreover says that \( \varphi(ab) = (ab)^k \). If \( \varphi(a) = (ab)^\ell \) for some \( \ell < k \) then \( \varphi(b) = (ab)^{k-\ell} \) and the morphism \( \varphi \) is not marked. Therefore, \( \varphi(a) = (ab)^\ell a \) and \( \varphi(b) = b(ab)^{k-\ell-1} \) and obviously belongs to class \( \mathcal{P} \). \( \square \)

6 Proof of Theorem 1

Proposition 28. Let \( \varphi : A^* \to A^* \) be a primitive and well-marked morphism. If the language \( L(\varphi) \) is palindromic, then \( \varphi \) has a conjugate in class \( \mathcal{P} \).

Proof. Because of Corollary 27 we can focus on \( \varphi \) with aperiodic fixed points. Since \( \varphi \) is primitive, the language \( L(\varphi) \) is uniformly recurrent and thus there exists a constant \( K \) such that any factor longer than \( K \) contains all letters from \( A \). From Lemma 20 \( L(\varphi) \) contains an infinite number of bispecial palindromes. If a bispecial palindrome \( u \) is long enough, than according to Proposition 23 there exists \( u' \) such that \( u = \varphi_R(u')w = w\varphi_L(u') \). Without loss of generality, we can assume that \( u' \) is longer than \( K \) and thus all letters of \( A \) occur in \( u' \). Due to Lemma 22 \( \varphi_L = \overline{\varphi_R} \). This together with Lemma 15 implies the statement. \( \square \)

We may now prove the main result, namely that Version 1 of the Hof-Knill-Simon Conjecture holds for general alphabet for the case of marked morphisms.

Proof of Theorem 1. From Lemma 10 \( \varphi^k \) is well-marked for some integer \( k \). From Proposition 28 \( \varphi^k \) has a conjugate in class \( \mathcal{P} \). \( \square \)

The result of B. Tan thus becomes a corollary of this theorem.

Corollary 29. Let \( A \) be a binary alphabet and \( \varphi : A^* \to A^* \) be an acyclic and primitive morphism. If the language \( L(\varphi) \) is palindromic, then \( \varphi \) or \( \varphi^2 \) has a conjugate in class \( \mathcal{P} \).
Proof. Any acyclic binary morphism \( \varphi \) is marked. If \( \varphi \) is not well-marked then so is \( \varphi^2 \).

It turns out that the square \( \varphi^2 \) is necessary only in some particular cases for a binary alphabet \( A = \{a, b\} \), namely when \( |w| < |\varphi(a)| \) and \( |w| < |\varphi(b)| \).

As we have already mentioned, for a uniformly recurrent word \( u \), the closedness of languages under mirror image does not imply that \( u \) is palindromic. The following proposition is an analogy of Theorem 3.13 from [19] for larger alphabet, which is stated for binary alphabet.

**Proposition 30.** Let \( \varphi : A^* \to A^* \) be a primitive and marked morphism. If \( L(\varphi) \) is closed under reversal, then \( L(\varphi) \) is palindromic.

Proof. Since \( L(\varphi) = L(\varphi^k) \), we can without loss of generality assume that \( \varphi \) is well marked. We exploit Proposition 23. Consider a bispecial factor \( u \) which contains all letter from the alphabet and which is longer than any initial bispecial factor on the list \( u^{(1)}, \ldots, u^{(N)} \). The same proposition says that the factor \( \Phi(u) = \varphi_R(u)w \) is bispecial. Since \( L(\varphi) \) is closed under reversal, the factor \( \varphi_R(u)w \) is bispecial, too. By Proposition 23, there exists a bispecial factor \( v \) such that \( \varphi_R(u)w = \Phi(v) = \varphi_R(v)w \). Lemma 22 forces \( u = \tilde{v} \) and \( \varphi_R = \varphi_L \). Thus, \( \varphi \) has a conjugate in class \( P \).

7 Comments and open questions

- We believe that Version 1 of the Hof-Knill-Simon conjecture can be proved for a larger class of primitive morphisms and not only for morphisms having a well-marked power. For example, consider a word \( u \) coding a three interval exchange transformation \( T \) under permutation (321) with i.d.o.c. condition. Such word is palindromic, but the morphism fixing \( u \) is in general not marked. Are powers of these morphisms conjugate to morphisms from class \( P \)?

The counterexample to Version 1 of the HKS conjecture constructed by the first author is an infinite word over ternary alphabet. In fact this word - denote it \( u \) - is coding of a three interval exchange transformation \( T \) with permutation (321). But it does not satisfy i.d.o.c. condition. It means that the factor complexity \( C \) of \( u \) is bounded by \( C(n) \leq n + K \) for some constant \( K \). Such word is usually referred to as degenerate 3iet word and it is just a morphic image of a sturmian word.

- On the binary alphabet one may consider besides mirror image mapping also another involutive antimorphism \( E \) defined by \( E(w_1w_2 \cdots w_n) = (1 - w_n)(1 - w_{n-1}) \cdots (1 - w_1) \). Let us look at the Thue-Morse word \( u_{TM} \) which is fixed by the substitution \( \varphi_{TM} : 0 \mapsto 01, 1 \mapsto 10 \). Language \( L(u_{TM}) \) contains infinite many palindromes. The second iteration \( \varphi_{TM}^2 : 0 \mapsto 0110, 1 \mapsto 1001 \) belongs to the class \( P \). On the other hand, the language \( L(u_{TM}) \) is closed under involutive antimorphism \( E \) as well and it contains infinitely many \( E \)-palindromes, i.e., factors \( w \) satisfying \( E(w) = w \). In fact, images of letters \( \varphi_{TM}(0) = 01 \) and \( \varphi_{TM}(1) = 10 \) are both \( E \)-palindromes. The question is what is an \( E \)-analogy of the class \( P \) and the HKS conjecture.

- As we have already mentioned, Harju, Vesti and Zamboni verified Version 3 of HKS conjecture for words with finite defect. Words coding symmetric exchange transformation and also Arnoux-Rauzy words belong to most prominent examples of words with defect zero. Of the
same interest is the opposite question: which substitutions from the class $\mathcal{P}$ have a fixed point with finite defect?

In this context we have to mention the conjecture stated in the last chapter of the article [4].

**Conjecture:** Let $u$ be a fixed point of a primitive morphism $\varphi$. If the defect of $u$ is finite but non-zero, then $u$ is periodic.

In [7], Bucci and Vaslet provided a morphism $\varphi$ over ternary alphabet which contradicts this conjecture. But their morphism $\varphi$ is not injective. Therefore, the validity of Conjecture is still open for morphisms over binary alphabet or injective morphisms. It can be deduced from [4], that the conjecture is true if $\varphi$ is a uniform marked morphism.

- The main open problem remains to prove validity of Version 2 of HKS conjecture or at least validity of its relaxed Version 3.

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