On the spectral radius of nonregular uniform hypergraphs

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Abstract

Let $G$ be a connected uniform hypergraphs with maximum degree $\Delta$, spectral radius $\lambda$ and minimum H-eigenvalue $\mu$. In this paper, we give some lower bounds for $\Delta - \lambda$, which extend the result of [S.M. Cioab˘ a, D.A. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, J. Combin. Theory, Ser. B 97 (2007) 483-486] to hypergraphs. Applying these bounds, we also obtain a lower bound for $\Delta + \mu$.

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1. Introduction

Spectral graph theory has been widely studied, and has many important applications in combinatorics, computer science, physics and so on. It is natural to generalize spectral theory to hypergraphs. Recently, there have been many attempts to develop spectral hypergraph theory based on eigenvalues of tensors [2,5-7,10-13,18,20].

An order $m$ dimension $n$ tensor $\mathcal{A} = (a_{i_1i_2...i_m})$ is a multidimensional array with $n^m$ entries ($i_j \in \{1, ..., n\}, j = 1, ..., m$). $\mathcal{A}$ is called symmetric if $a_{i_1i_2...i_k} = a_{i_{\sigma(1)}i_{\sigma(2)}...i_{\sigma(k)}}$ for any permutation $\sigma$ on $\{1, ..., k\}$. In 2005, the concept of eigenvalues of tensors was defined by Qi [14] and Lim [8]. For $\mathcal{A} = (a_{i_1i_2...i_m}) \in \mathbb{C}^{n \times n \times ... \times n}$.
and $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$, $A x^{m-1}$ is a vector in $\mathbb{C}^n$ whose $i$-th component is

$$(A x^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$  

A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $A$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $A x^{m-1} = \lambda x^{[m-1]}$, where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. In this case, $x$ is an eigenvector of $A$ associated with $\lambda$. If $\lambda$ is a real eigenvalue with a real eigenvector, then $\lambda$ is called an $H$-eigenvalue of $A$. The spectral radius of $A$ is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the set of all eigenvalues of $A$.

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a hypergraph $G$, respectively. If each edge of $G$ contains exactly $k$ distinct vertices, then $G$ is called $k$-uniform. We will use the term $k$-graph in place of $k$-uniform hypergraph. Clearly, a 2-graph is a simple undirected graph. The degree of a vertex $u$ of $G$ is the number of edges containing $u$. If all vertices of $G$ have the same degree, then $G$ is called regular. In a $k$-graph $G$, a path of length $l$ is defined to be an alternating sequence of vertices and edges $u_1, e_1, u_2, \ldots, u_l, e_l, u_{l+1}$, where $u_1, \ldots, u_{l+1}$ are distinct vertices of $G$, $e_1, \ldots, e_l$ are distinct edges of $G$ and $u_i, u_{i+1} \in e_i$ for $i = 1, \ldots, l$. If there exists a path between any two vertices of $G$, then $G$ is called connected. The distance between two vertices is the length of the shortest path connecting them. The diameter of a connected $k$-graph $G$ is the maximum distance among all vertices of $G$.

The adjacency tensor $[3]$ of a $k$-graph $G$ with $n$ vertices, denoted by $A_G$, is an order $k$ dimension $n$ symmetric tensor with entries

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } i_1 i_2 \cdots i_k \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvalues (H-eigenvalues) of $A_G$ are called eigenvalues (H-eigenvalues) of $G$. Let $\mu(G)$ denote the minimum H-eigenvalue of $G$. The spectral radius of $A_G$ is called the spectral radius of $G$, denoted by $\rho(G)$. When $k = 2$, $A_G$ is the adjacency matrix of 2-graph $G$.

For a connected $k$-graph $G$ with maximum degree $\Delta$, Cooper and Dutle \cite{5} proved that $\rho(G) \leq \Delta$, with equality if and only if $G$ is regular. It is natural to consider how small $\Delta - \rho(G)$ can be for nonregular $k$-graphs. When $G$ is nonregular 2-graph, some lower bounds on $\Delta - \rho(G)$ are given in \cite{3, 4, 9, 17, 19}. These bounds are also
lower bounds of $\Delta + \mu(G)$ because $\Delta + \mu(G) \geq \Delta - \rho(G)$.

Alon and Sudakov \[1\] proved that if $G$ is a connected nonbipartite (possibly regular) 2-graph with $n$ vertices, then

$$\Delta + \mu(G) \geq \frac{1}{n(D + 1)},$$

where $D$ is the diameter of $G$. For a connected nonregular 2-graph $G$ with $n$ vertices and $m$ edges, Cioabă, Gregory and Nikiforov \[4\] obtain the following bound

$$\Delta - \rho(G) > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \quad (1.1)$$

In this paper, we give some lower bounds on $\Delta - \rho(G)$ for a connected nonregular $k$-graph $G$, which extend the inequality (1.1) to hypergraphs. For a connected non-odd-bipartite $k$-graph $G$, we show that $\Delta + \mu(G) > \frac{1}{3n}$ when $k \geq 4$ is even.

2. Preliminaries

In this section, we collect some helpful lemmas.

**Lemma 2.1.** \[2\] Let $a_1, \ldots, a_n$ be nonnegative real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} - (a_1 \cdots a_n)^{\frac{1}{n}} \geq \frac{1}{n(n - 1)} \sum_{1 \leq i < j \leq n} (\sqrt{a_i} - \sqrt{a_j})^2.$$

**Lemma 2.2.** Let $a, b, y_1, y_2$ be positive numbers. Then $a(y_1 - y_2)^2 + by_2^2 \geq \frac{ab}{a+b}y_1^2$.

**Proof.** By computation, we have

$$a(y_1 - y_2)^2 + by_2^2 = (a + b)(y_2 - \frac{ay_1}{a + b})^2 + \frac{ab}{a + b}y_1^2 \geq \frac{ab}{a + b}y_1^2.$$ 

A hypergraph $G$ is called $f$-edge-connected if $G - U$ is connected for any edge subset $U \subseteq E(G)$ satisfying $|U| < f$. Two paths are called edge-disjoint if they does not have a common edge. The following is the Menger’s theorem for hypergraphs.

**Lemma 2.3.** \[21\] A hypergraph $G$ is $f$-edge-connected if and only if there are $f$ mutual edge-disjoint paths between each pair of vertices.
Let $\lambda$ be an eigenvalue of a $k$-graph $G$ with eigenvector $x$. Since $A_Gx^{k-1} = \lambda x^{k-1}$, we know that $cx$ is also an eigenvector of $\lambda$ for any nonzero constant $c$. So we can choose $x$ such that $\sum_{i=1}^n x_i^k = 1$. In this case, we have

$$\lambda = x^\top (A_Gx^{k-1}) = k \sum_{e \in E(G)} x^e,$$

where $x^e = x_{i_1}x_{i_2}\cdots x_{i_k}$, $i_1, i_2, \ldots, i_k$ are $k$ vertices in the edge $e$. From the proof of [14, Theorem 5], we obtain the following lemma.

**Lemma 2.4.** Let $G$ be a connected $k$-graph with $n$ vertices and $k$ is even. Then

$$\mu(G) = \min_{x \in \mathbb{R}^n, \sum_{i=1}^n x_i^k = 1} x^\top (A_Gx^{k-1}).$$

3. Main results

Let $G$ be a connected $k$-graph with $n$ vertices and $m$ edges, and let $\Delta$ and $D$ be the maximum degree and the diameter of $G$, respectively. We give some lower bounds for $\Delta - \rho(G)$ as follows.

**Theorem 3.1.** Let $G$ be a nonregular connected $k$-graph. Then

$$\Delta - \rho(G) > \frac{k(n\Delta - km)}{n(2(k-1)D(n\Delta - km) + k)}.$$

Moreover, the following statements hold:

1. If $k \geq 5$ and $G$ is $f$-edge-connected, then

$$\Delta - \rho(G) > \frac{f k(n\Delta - km)}{n[2(k-1)(n\Delta - km) + fk]}.$$

2. If $k = 4$ and $G$ is $f$-edge-connected, then

$$\Delta - \rho(G) > \frac{f(n\Delta - 4m)}{n[2(n\Delta - 4m) + f]}.$$

**Proof.** It is known that $\rho(G)$ is an eigenvalue of $G$ with a positive eigenvector $x$. We choose $x$ such that $\sum_{i=1}^n x_i^k = 1$. Let $u, v$ be two vertices such that $x_u = \max_{i \in V(G)} x_i$ and $x_v = \min_{i \in V(G)} x_i$. By $\sum_{i=1}^n x_i^k = 1$, we get $x_u > \frac{1}{n} > x_v$ (G is
regular when \( x_u^k = x_v^k = \frac{1}{n} \). By (2.1), we have

\[
\Delta - \rho(G) = \Delta \sum_{i=1}^{n} x_i^k - k \sum_{e \in E(G)} x^e = \sum_{i=1}^{n} (\Delta - d_i)x_i^k + \sum_{e \in E(G)} d_i x_i^k - k \sum_{e \in E(G)} x^e
\]

\[
= \sum_{i=1}^{n} (\Delta - d_i)x_i^k + \sum_{e=i_1 \cdots i_k \in E(G)} (x_{i_1}^k + \cdots + x_{i_k}^k - kx^e),
\]

where \( d_i \) is the degree of the vertex \( i \). Since \( x_u^k > x_v^k > 0 \), by Lemma 2.1 we have

\[
\Delta - \rho(G) > (n\Delta - km)x_v^k + \frac{1}{k-1} \sum_{i,j \in E(G)} (x_i^k - x_j^k)^2. \tag{3.1}
\]

Let \( P : u = u_0, e_1, u_1, \ldots, u_{r-1}, e_r, u_r = v \) be a path from \( u \) to \( v \), where \( e_i \) is an edge containing vertices \( u_{i-1} \) and \( u_i \). Then

\[
\sum_{i,j \in E(P)} (x_i^k - x_j^k)^2 \geq \sum_{i=0}^{r-1} (x_{u_i}^k - x_{u_{i+1}}^k)^2 + \sum_{v_i \in v_i \setminus \{u_{i-1}, u_i\}} \sum_{i=0}^{r-1} (x_{u_i}^k - x_{v_{i+1}}^k)^2 + (x_{v_{i+1}}^k - x_{u_{i+1}}^k)^2.
\]

It follows from the Cauchy-Schwarz inequality that

\[
\sum_{i,j \in E(P)} (x_i^k - x_j^k)^2 \geq \frac{1}{r} \left( \sum_{i=0}^{r-1} (x_{u_i}^k - x_{u_{i+1}}^k) \right)^2 + \sum_{v_i \in v_i \setminus \{u_{i-1}, u_i\}} \frac{1}{2r} \left( \sum_{i=0}^{r-1} (x_{u_i}^k - x_{u_{i+1}}^k) \right)^2
\]

\[
= \frac{1}{r} (x_u^k - x_v^k)^2 + \frac{(k-2)^r}{2r} (x_u^k - x_v^k)^2 = \frac{2 + (k-2)^r}{2r} (x_u^k - x_v^k)^2.
\]

So

\[
\sum_{i,j \in E(P)} (x_i^k - x_j^k)^2 \geq \frac{2 + (k-2)^r}{2r} (x_u^k - x_v^k)^2 \geq \frac{k}{2r} (x_u^k - x_v^k)^2. \tag{3.2}
\]

There is a shortest path from \( u \) to \( v \) whose length does not exceed the diameter \( D \).

By (3.1) and (3.2), we have

\[
\Delta - \rho(G) > (n\Delta - km)x_v^k + \frac{k}{2(k-1)D} (x_u^k - x_v^k)^2.
\]

The right side of the above inequality is a quadratic function of \( x_v^k \). By Lemma 2.2.
we get
\[ \Delta - \rho(G) > \frac{k(n\Delta - km)}{2(k-1)D(n\Delta - km) + k} x_u^k. \]
Since \( x_u^k \geq \frac{1}{n} \), we get
\[ \Delta - \rho(G) > \frac{k(n\Delta - km)}{n(2(k-1)D(n\Delta - km) + k)}. \]

Next we consider the cases of \( k \geq 5 \) and \( k = 4 \).

**Case 1.** \( k \geq 5 \) and \( G \) is \( f \)-edge-connected. In this case, \( \frac{2+(k-2)r}{2r} \geq \frac{k}{2} \). By (3.2), we have
\[ \sum_{i,j \in e \in E(P)} (x_i^k - x_j^k)^2 \geq \frac{k}{2}(x_u^k - x_v^k)^2. \]
Since \( G \) is \( f \)-edge-connected, by Lemma 2.3, there are \( f \) mutual edge-disjoint paths between \( u \) and \( v \). By (3.1) and (3.3), we have
\[ \Delta - \rho(G) > (n\Delta - km)x_u^k + \frac{f k}{2(k-1)}(x_u^k - x_v^k)^2. \]
By Lemma 2.2, we get
\[ \Delta - \rho(G) > \frac{f k(n\Delta - km)}{2(k-1)(n\Delta - km) + f k} x_u^k > \frac{f k(n\Delta - km)}{n[2(k-1)(n\Delta - km) + f k]} x_u^k. \]

**Case 2.** \( k = 4 \) and \( G \) is \( f \)-edge-connected. In this case, \( \frac{2+(k-2)r}{2r} \geq \frac{3}{2} \). By (3.2), we have
\[ \sum_{i,j \in e \in E(P)} (x_i^2 - x_j^2)^2 \geq \frac{3}{2}(x_u^2 - x_v^2)^2. \]
Since \( G \) is \( f \)-edge-connected, by Lemma 2.3, there are \( f \) mutual edge-disjoint paths between \( u \) and \( v \). By (3.1) and (3.4), we have
\[ \Delta - \rho(G) > (n\Delta - 4m)x_v^4 + \frac{f}{2}(x_u^2 - x_v^2)^2. \]
By Lemma 2.2, we get
\[ \Delta - \rho(G) > \frac{f(n\Delta - 4m)}{2(n\Delta - 4m) + f} > \frac{f(n\Delta - 4m)}{n[2(n\Delta - 4m) + f]}. \]

**Remark 1.** Take \( k = 2 \) in Theorem 3.1, we can obtain the inequality (1.1).

**Corollary 3.2.** Let \( G \) be a nonregular connected \( k \)-graph. If \( k \geq 4 \), then
\[ \Delta - \rho(G) > \frac{1}{3n}. \]

**Proof.** All connected \( k \)-graphs are 1-edge-connected. If \( k = 4 \), then by Theorem 3.1, we have
\[ \Delta - \rho(G) > \frac{n\Delta - 4m}{n[2(n\Delta - 4m) + 1]} \geq \frac{1}{3n}. \]

If \( k \geq 5 \), then by Theorem 3.1, we have
\[ \Delta - \rho(G) > \frac{k(n\Delta - km)}{n[2(k - 1)(n\Delta - km) + k]} > \frac{1}{3n}. \]

A \( k \)-graph \( G \) is called *odd-bipartite*, if there exists a proper subset \( V_1 \) of \( V(G) \) such that each edge of \( G \) contains exactly odd number of vertices in \( V_1 \). Clearly, odd-bipartite 2-graphs are ordinary bipartite graphs. For a connected nonbipartite graph \( G \) with \( n \) vertices, Alon and Sudakov [1] proved that \( \Delta + \mu(G) \geq \frac{1}{n(D + 1)} \), where \( \Delta \) is the maximum degree of \( G \), \( D \) is the diameter of \( G \). We consider similar problem for connected non-odd-bipartite \( k \)-graphs as follows.

**Theorem 3.3.** Let \( G \) be a connected non-odd-bipartite \( k \)-graph and \( k \geq 4 \) is even. Then
\[ \Delta + \mu(G) > \frac{1}{3n}. \]

**Proof.** It is known that 0 is an H-eigenvalue of \( G \) when \( k \geq 3 \) [15]. So \( \mu(G) \leq 0 \). If \( \mu(G) = 0 \), then \( \Delta + \mu(G) > \frac{1}{3n} \). If \( G \) is nonregular, then by Corollary 3.2, we have
\[ \Delta + \mu(G) \geq \Delta - \rho(G) > \frac{1}{3n}. \] So we suppose that \( G \) is regular and \( \mu(G) < 0 \).
Let $x$ be a real eigenvector of $\mu(G)$ satisfying $\sum_{i=1}^{n} x_i = 1$, and let $V_1 = \{i : x_i < 0\}$ and $V_2 = \{i : x_i \geq 0\}$. Since $\mu(G) < 0$, by (2.1), $V_1$ is nonempty. If $V_2$ is empty, then $-x$ is a positive eigenvector of $\mu(G)$, a contradiction to the fact that $\rho(G)$ is the unique H-eigenvalue with a positive eigenvector (see [15]). Hence $V_1$ and $V_2$ are nonempty. Since $G$ is not odd-bipartite, it has an edge $f$ such that $|f \cap V_1|$ is even. Let $H = G - f$ be the $k$-graph obtained from $G$ by deleting the edge $f$. Since $G$ is regular and non-odd-bipartite, $H$ is nonregular and $|E(H)| \geq 2$. By (2.1) and Lemma 2.4 we have

$$\mu(G) = k \sum_{e \in E(G)} x^e \geq k \sum_{e \in E(H)} x^e \geq \mu(H).$$

Hence

$$\Delta + \mu(G) \geq \Delta + \mu(H) \geq \Delta - \rho(H).$$

If $H$ is connected, then by Corollary 3.2, we have $\Delta + \mu(G) \geq \Delta - \rho(H) > \frac{1}{3n}$. If $H$ is disconnected, then there exists a component $H_1$ of $H$ such that $\rho(H) = \rho(H_1)$. Since $|E(H)| \geq 2$ and $G$ is a connected regular $k$-graph, $H_1$ is nonregular. By Corollary 3.2, we have $\Delta + \mu(G) \geq \Delta - \rho(H) = \Delta - \rho(H_1) > \frac{1}{3n}$.

**Remark 2.** If $G$ is a connected odd-bipartite $k$-graph and $k$ is even, then $\mu(G) = -\rho(G)$ [16]. In this case, $\Delta + \mu(G) = \Delta - \rho(G)$.

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