The effect of a local perturbation in a fermionic ladder

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We study the effect of a local external potential on a system of two parallel spin-polarized nanowires placed close to each other. For single channel nanowires with repulsive interaction we find that transport properties of the system are highly sensitive to the transverse gradient of the perturbation: the asymmetric part completely reflects the electrons leading to vanishing conductance at zero temperature, while the flat potential remains transparent. We envisage a possible application of this unusual property in the sensitive measurement of local potential field gradients.

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Recent advances in nanotechnology have led to an explosive growth of experimental work on low-dimensional systems. Single-channel one-dimensional (1D) conductors or nanowires can now be manufactured in a controlled fashion. At room temperature, nanowire arrays have found real-life applications as highly sensitive chemical, biological, and optical sensors. Individual nanowires have been used as mechanical force and mass sensors. Furthermore, at low temperatures both nanowires and carbon nanotubes provide a fertile ground for laboratory experiments aimed at uncovering the quantum nature of interacting many-particle systems.

The quantum physics of an individual nanowire is well understood within the paradigm of the Tomonaga-Luttinger (TL) liquid. As shown in the seminal paper of Kane and Fisher, transport properties of the TL liquid are very sensitive to the application of a local external potential. While the clean TL liquid is a perfect conductor, even a weak impurity potential in the most physical case of repulsive electron-electron interaction completely reflects the electrons leading to zero conductance at temperature $T = 0$.

The dramatic response of an interacting 1D system to a local probe is in sharp contrast with the role of such a probe in higher dimensions. A natural question to ask then is how a single impurity would affect an array of nanowires in a situation intermediate between purely one- and two-dimensional cases.

Consider two repulsive spinless TL liquids away from 1/2 filling brought together to form a ladder, as shown in Fig. a). Assume that the fermions belonging to different wires interact with each other and may also undergo inter-chain hopping. Imagine that the external potential affects a single rung of the ladder and compare the two situations shown in Fig. b): (a) a single impurity located on one of the chains and (b) a pair of identical impurities on the rung. Naively one might expect that the case (b) could be described by the Kane-Fisher scenario leading to vanishing conductance at $T \to 0$, whereas the geometry of the ladder in the case (a) would suggest ballistic conductance in the zero-temperature limit. As shown in the present Letter, this conclusion cannot be correct. Moreover, for the most physical case of repulsive interaction in the ladder the right conclusion is just the opposite, with the roles of cases (a) and (b) interchanged.

Indeed, the Kane-Fisher effect – the growth of the local potential upon decreasing the energy scale, or in other words the relevance of the local perturbation in the renormalization group (RG) sense – only occurs if the impurity potential couples to the operator whose space-time correlations are dominant in the clean system. In the case of a single repulsive $V > 0$ TL liquid these are the “charge density wave” (CDW) correlations that locally couple to the backscattering part of the impurity potential and get pinned at the impurity site driving the conductance to zero. In a ladder, the relative phase of the two incommensurate CDWs on individual chains is 0 for attractive inter-chain interaction ($U < 0$) or $\pi$ for repulsive inter-chain interaction ($U > 0$), the corresponding density-wave states being labeled by CDW$^\pm$. In case (b) the symmetric potential couples only to the CDW$^+$ operator. Thus, such an impurity is transparent (at $T = 0$) for the repulsive ($U, V > 0$) ladder but suppresses conductance for $U < 0$. On the contrary, the local impurity in case (a) couples to both CDW$^+$ and CDW$^-$ leading...
The above argument (and the calculation below) show that the repulsive two-leg ladder is extremely sensitive not only to the presence of the local probe, but also to the transverse gradient of the external potential. Since the ladder is a prototypical model of two interacting and closely located nanowires, our results suggest a possible use of double nanowire devices [1] as quantum sensors of closely located nanowires, our results suggest a possible use of double nanowire devices [1] as quantum sensors of closely located nanowires. Since the transverse gradient of the external potential. Since the repulsive two-leg ladder is extremely sensitive to two identical local impurities,\( \lambda \) and \( \lambda' \) affect the single rung of the ladder has the form \( \lambda a^\dagger a + \lambda' a'^\dagger a' \), where \( \lambda \) is the density operator, \( t_{\|} \) and \( t_{\perp} \) describe the intra- and inter-chain hopping, and \( V \) and \( U \) are the intra- and inter-chain nearest-neighbor interaction constants. Note that in the absence of \( t_{\perp} \) and \( V \), this is simply the Hubbard model where the two legs of the ladder can be mapped to spin-up and spin-down.

For \( V, U > 0 \) this model describes two closely located nanowires coupled by the Coulomb interaction \( \mu \). If the wires are sufficiently far apart (e.g. for inter-wire distance of order 50nm [5]), \( t_{\perp} \) may be set to zero (we show below this is unimportant for our purposes).

A local (at \( i = 0 \)) external potential is described by

\[
H_{\text{imp}} = \sum_i \delta \lambda_{\sigma} n_{\sigma,i=0}, \quad (2)
\]

where \( \lambda_{1,2} \) is the impurity strength on the top and bottom chain respectively. We distinguish the two limiting cases: (a) a purely local impurity, corresponding to \( \lambda_{2} = 0 \), and (b) a flat transverse potential that is equivalent to two identical local impurities, \( \lambda_{1} = \lambda_{2} \) (see Fig.1).

Note that the most general form of the external field affecting the single rung of the ladder has the form \( \lambda' a^\dagger a + \lambda'' a'^\dagger a' \) (where \( \tau^a \) are the Pauli matrices) which in addition to Eq. (2) \( [\lambda^0_{\pm}] = (\lambda_{1} \pm \lambda_{2})/2 \) contains a local variation of the inter-chain hopping (\( \lambda^x \)) and a local magnetic field (\( \lambda^z \)) – see Table I. As it turns out these terms are not important for repulsive interaction.

**Qualitative discussion.** — In the absence of electron-electron interaction the single impurity problem can be solved exactly for the model (2) or treated using the scattering matrix approach for arbitrary shape of the external potential. The impurity induces a modulation of particle density. Taking into account the interaction one finds an additional scattering potential due to the oscillating part of the modulation known as the Friedel oscillation. For a single nanowire [13], already within the Hartree-Fock approximation one can observe the divergence of the (back-)scattering amplitude providing an alternative explanation for the Kane-Fisher effect.

Friedel oscillations can also be found in the ladder model. It is clear that in the presence of inter-chain hopping \( (t_{\perp} \neq 0) \) the single impurity (\( \lambda_{2} = 0 \)) will induce Friedel oscillations in both chains. What is most important, the oscillations on the two chains are out of phase. In other words, a charged impurity placed on one chain of the ladder will induce an image charge on the other chain which will be of the opposite sign.

However unlike the single-wire Kane-Fisher problem, the argument based on Friedel oscillations cannot be extended further in order to explain our results. By design, this is a weak-coupling argument. As we discuss below, the excitation spectrum of ladder models contains gapped branches, which can not be accounted for by any weak-coupling expansion. Moreover, the existence of the Friedel oscillation is essentially tied to the presence of the inter-chain hopping [10], which it turns out plays only a minor role in the longitudinal transport of the repulsive ladder. In what follows, we therefore focus on the Kane-Fisher idea [13] of pinning the incommensurate density waves by the local impurity, qualitatively outlined above.

**Calculation.** — We now outline the derivation of our results (full details will be given elsewhere [17]). First, we diagonalize the single-particle problem (without the impurity) and obtain the two-band spectrum. Second, we linearize the spectrum about the Fermi points and apply the standard bosonization procedure [18] with the conventions [19] \( R(L) = (\kappa_{\mu}/\sqrt{2\pi\alpha}) \exp(\pm i\sqrt{4\pi\phi(R(L))}) \) where \( R(L) \) are the operators of right (left) movers in the band \( \mu = \pm \) and \( \alpha \) is the bosonic ultraviolet cutoff. The chiral bosonic fields commute as \( [\phi_{\mu}^L, \phi_{\mu'}^R] = i\delta_{\mu\mu'}/4 \) and combine in the usual way \( \phi_{\mu} = \phi_{\mu}^{L} + \phi_{\mu}^{R} \) and \( \theta_{\mu} = \phi_{\mu}^{L} - \phi_{\mu}^{R} \). The anti-commuting Klein factors, \( \{\kappa_{\mu}, \kappa_{\mu'}\} = 2\delta_{\mu\mu'} \), are not dynamic variables, so we can choose the representation \( \kappa_{1,2} = i \) and \( \kappa_{1,2} = 1 \). Finally, we arrange the bosonic fields into the charge \( \phi_{\pm} = \phi_{+} + \phi_{-})/\sqrt{2} \) and pseudo-spin \( \phi_{\mp} = (\phi_{+} - \phi_{-})/\sqrt{2} \), poles (c.f. spin-charge separation).

The precise form of the effective low-energy theory depends on the value of the inter-chain hopping \( t_{\perp} \). Having in mind experimental realizations [15] of nanowires with
and becomes relevant in the RG sense. Consequently, the corresponding bosonic field gets locked to the value defined by one of the minima of the cosine potential depending on the sign of the interaction parameter. In particular, for the repulsive ladder this scenario is realized for the last, $g_f$-proportional term, so that the field $\theta_s$ gets locked to the value $\sqrt{\pi/8} + m\sqrt{\pi/2}$ (with $m$ being arbitrary integer). This implies the non-zero expectation value $\langle \sin \sqrt{2\pi\theta_s} \rangle \neq 0$. Due to the multiplicative structure of local operators as shown in Table I which always include the massless charge field, this situation does not lead to long-range order. Despite the gap opening the conductance of the clean ladder [24] is $G = 2e^2/h$ [25], as the gapped pseudo-spin mode does not carry charge.

Having now identified the ground state of the clean system, we can consider what happens when an impurity is added at $i = 0$. Applying the bosonization rules to the local perturbation (2) we find that the external potential can be written in terms of local operators from Table I

$$H_{\text{imp}} = (\lambda_1 + \lambda_2)O_{\text{CDW}+} + (\lambda_1 - \lambda_2)O_{\text{CDW}−}/2. \quad (5)$$

Let us first consider the case of a symmetric impurity $\lambda_1 = \lambda_2$ in the repulsive ladder. Then the perturbation is proportional to $\cos \sqrt{2\pi\phi_s}$ (see Table I). Now $\theta_s$ is locked and, consequently, all correlation functions of exponents of $\phi_s$ are short-ranged (exponentially decaying). To obtain the effective theory of the charge sector for energies small compared to the pseudo-spin gap, we integrate out the pseudo-spin degree of freedom. The impurity potential does not contribute in the first order, while in the second order it generates the pair backscattering term $\cos \sqrt{2\pi\phi_c}$ (0). For $K_c > 1/2$ this operator is irrelevant in the RG sense, leaving us with a single-channel weak-coupling Kane-Fisher-type problem. Following Ref. 13 we see that

$$G = 2e^2/h - A |\text{Max}(T, V)|, \quad \gamma = 4K_c - 2 > 0, \quad (6)$$

where $V$ is the voltage bias and $A$ is a non-universal constant. Thus at $T = 0$ the symmetric impurity or, equivalently, the flat external potential remains transparent. The situation is substantially different if $\lambda_1 \neq \lambda_2$, i.e. one also has a local perturbation proportional to $O_{\text{CDW}−}$. In this case (still for repulsive interaction) the impurity contributes in the first order as one integrates out the massive (CDW− phase) pseudo-spin, which is equivalent to replacing $\sin \sqrt{2\pi\theta_s}$ with its expectation value

$$\cos \sqrt{2\pi\phi_c} \sin \sqrt{2\pi\theta_s} \to \langle \sin \sqrt{2\pi\theta_s} \rangle \cos \sqrt{2\pi\phi_c}. \quad (7)$$

We are again left with what looks like a single-channel Kane-Fisher-type problem for the charge sector, but this time we are in the strong coupling regime as the scaling dimension of the impurity operator is $d = K_c/2$, implying that the operator is relevant for any $K_c < 2$.

In the strong-coupling regime, the single-chain Kane-Fisher problem reduces to that of a weak link between

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**FIG. 2:** [Color online] Phase diagram of model (1) at an incommensurate filling. The left panel corresponds to $t_\perp = 0$ and the right panel to $t_\perp \neq 0$. The dashed line corresponds to $K_c = 1$. To the right of the dashed line the local density operators (see Table I) exhibit dominant correlations. To the left of the dashed line the dominant correlations are superconducting, however the impurity (2) does not couple to such operators which are therefore irrelevant for our purposes.

| Panel | Phase Diagram |
|-------|---------------|
| Left  | TL liquid     |
| Right | CDW−          |

**Table I:** Values of $\lambda$ and $\gamma$ for different phases. The CDW− phase is always stable.

| Phase       | $\lambda$ | $\gamma$ |
|-------------|-----------|----------|
| CDW−        | 2         | 2        |
| CDW+        | 2         | 2        |
| OAF         | 2         | 2        |
| TL liquid   | 2         | 2        |

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$$K_c = 1 - (2V + U)/\pi v_F, \quad K_s = 1 + U/\pi v_F, \quad g_\perp = U - V, \quad g_f = V, \quad \bar{U} = aU, \quad \bar{V} = aV[1 - \cos(2K_Fa)]. \quad (4)$$
two semi-infinite TL liquids. In the ladder model the impurity operator \( \lambda \) has a multiplicative structure with its pseudo-spin part frozen due to ordering of the field \( \theta_y \) in the CDW− phase. Consequently, in this case the strong-coupling regime is not equivalent to the problem of weak tunneling between two semi-infinite ladders, but is rather determined by the charge sector alone. Henceforth the calculation is analogous to the single-chain case, except that there are no single-particle processes left in the effective theory of the charge channel. One therefore considers pair hopping across the boundary described by the operator \( \cos \sqrt{\pi \gamma} \theta_y \). There is no conductance at \( T = 0 \), while at \( T > 0 \) the conductance is given by

\[
G = B \left[ \text{Max}(T, V) \right]^{\gamma'}, \quad \gamma' = 4/K_c - 2 > 0.
\]

The non-universal coefficient \( B \) is proportional to the square of the effective pair tunneling rate, which is in general unknown.

The two results \( \lambda \) and \( O \) seem to be disconnected as there appears no clear way to obtain one from another by a smooth variation of the impurity strength. This is not surprising as each of the two results was obtained by an effective weak-coupling calculation - a weak impurity in the case of \( \lambda \) and a weak pair tunneling in the case of \( O \). While it might be possible to trace the connection between the two regimes for a particularly suitable model \( \lambda \), such calculation would be strongly model-dependent. In contrast, the form of the results \( \lambda \) and \( O \) as well as the \( T = 0 \) behavior (Fig. 1) are universal.

The rest of the phase diagram shows a similar behavior – if the external potential couples to the local operator with dominant correlations in the bulk, then the impurity strength flows to strong coupling driving conductance to \( G \approx 0 \) [see Eq. (5)], otherwise the conductance remains close to its clean value as in Eq. (6). An interesting example is the OAF phase, where the ballistic conductance may only be suppressed by a local magnetic field \( \lambda \theta_y \). In any of these phases, as temperature is increased beyond the size of the pseudo-spin gap, the exponents will smoothly cross over to the spinful TL liquid values given in Ref. 12; this also describes transport in the gapless TL liquid phase.

To summarize — we have considered the effect of a local external potential on transport properties of a spinless two-leg ladder. We find that in the physical case of repulsive interaction the system is extremely sensitive to the transverse gradient of the potential which grows under renormalization and at \( T = 0 \) completely suppresses conductance in contrast with the flat potential that remains transparent. We note that although any realistic potential will have some asymmetry, if this is sufficiently small there will be a wide temperature window in which it remains weak leading to small deviations from Eq. (8).

While present technology allows for spin polarized experiments, a useful sensor of local field gradients must operate away from polarizing magnets. Therefore a spinful ladder must be considered; this case is more complex in details but conceptually contains similar physics, and will be addressed elsewhere [17].

It would be interesting to extend the present analysis to multi-wire arrays, where it is expected that the single impurity should not completely prohibit conductance as is the case in truly two-dimensional systems.

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