HAANTJES ALGEBRAS OF CLASSICAL INTEGRABLE SYSTEMS

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ABSTRACT. A tensorial approach to the theory of classical Hamiltonian integrable systems is proposed, based on the geometry of Haantjes tensors. We introduce the class of symplectic-Haantjes manifolds (or $\omega\mathcal{H}$ manifolds), as a natural setting where the notion of integrability can be formulated. We prove that the existence of suitable Haantjes algebras of (1,1) tensor fields with vanishing Haantjes torsion is a necessary and sufficient condition for a Hamiltonian system to be integrable in the Liouville-Arnold sense. We also show that new integrable models arise from the Haantjes geometry. Finally, we present an application of our approach to the study of the Post-Winternitz system and of a stationary flow of the KdV hierarchy.

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1. Introduction

Integrable systems are ubiquitous in many branches of modern mathematics and theoretical physics. Due to their relevance, in the last decades the search for intrinsic mathematical structures underlying the notion of integrability has been actively pursued. In particular, the investigation of the properties of exact solvability of integrable systems led to the discovery of new important analytic and geometric techniques. It is interesting to observe that finite-dimensional integrable models coming from classical or quantum mechanics share many geometric and algebraic properties with the infinite-dimensional ones described in terms of soliton equations.

The study of the geometry of classical integrable systems has a long history, dating back to the works by Liouville, Jacobi, Stäckel, Eisenhart, Arnold, etc.

In this context, the bi-

Hamiltonian approach has shown to be crucial for many respects. A bi-Hamiltonian manifold is a differentiable manifold endowed with a pencil of Poisson structures \[ \omega_N \]. In particular, the special class of \( \omega_N \) manifolds, introduced in [33, 12], is characterized by a non-degenerate Poisson bivector (whose inverse provides a symplectic structure \( \omega \)), and a compatible \( (1, 1) \) tensor field \( N \), also called recursion or hereditary operator. Such a tensor has a vanishing Nijenhuis torsion as a consequence of the existence of an underlying bi-Hamiltonian structure. The class of \( \omega_N \) manifolds offers a coherent approach to the construction of separation variables; it has been successfully applied, for instance, to the study of Gelfand-Zakharevich systems [16, 22, 12].

The purpose of this paper is to present a new formulation of classical integrability based on Haantjes operators, namely operator fields with vanishing Haantjes torsion. The latter concept was introduced in 1955 by Haantjes in [18] as a natural generalization of the Nijenhuis torsion [17]. However, the relevance of the Haantjes differential-geometric work in the realm of integrable systems quite surprisingly has not been recognized for a long time, with the exception of some interesting applications to Hamiltonian systems of hydrodynamic type [13, 6, 14].

The central notion underlying our formulation of integrability is that of Haantjes algebra, introduced in [49]. Essentially, a Haantjes algebra is a pair \( (M, \mathscr{H}) \) where \( M \) is a differentiable manifold and \( \mathscr{H} \) is a set of operator fields over \( M \), with vanishing Haantjes torsion, which satisfy suitable compatibility conditions among each others.

The study of these algebras was initiated by us for two reasons. Indeed, they play a crucial role in the theory of diagonalization of operators on differentiable manifolds: Whenever the operators of a Haantjes algebra are semisimple and commute, a set of local coordinates exists where all operators can be diagonalized simultaneously. Moreover, in the non-semisimple case, they acquire simultaneously a block-diagonal form. At the same time, Haantjes algebras naturally generalize several known interesting geometric structures arising in Riemannian geometry [3, 55, 49]. A generalization of both the Haantjes torsion and the Haantjes theorem has been recently proposed in [50].

In this work, we will show the prominent role of Haantjes algebras in the theory of integrable systems. Indeed, we shall define a new family of manifolds, called symplectic-Haantjes (or \( \omega \mathscr{H} \)) manifolds. They are symplectic manifolds endowed with a Haantjes algebra of operators that are compatible with the symplectic structure.
We shall prove that the integrability of a Hamiltonian finite-dimensional system can be characterized in terms of an Abelian algebra of Haantjes operators, whose spectral and geometric properties turn out to be particularly rich. The notion of Haantjes chain, defined in this framework, is a natural extension in the context of Haantjes geometry of previous similar notions known in the literature, as that of Lenard-Magri chain [26] and generalized Lenard chain [52, 11, 28], relevant for quasi-bi-Hamiltonian systems and their generalizations.

A comparison between our notion of $\omega\mathcal{H}$ manifolds and the recent definition of “Haantjes manifolds” due to Magri [29, 30, 31, 32] is in order. Our theory mainly differs from the fact that we assume the existence of an algebra of independent Haantjes operators which are supposed to be compatible with the symplectic form $\omega$. Besides, the Haantjes chains in our context are “shorter” than the ones defined in the recent Magri’s theory [30]. This is due to a weaker assumption that allows us to deal with both integrable and separable systems. This fact is an important novelty of the present work.

Our main result concerning integrability is a theorem establishing that the existence of a $2n$-dimensional $\omega\mathcal{H}$ manifold is a necessary and sufficient condition for a non-degenerate Hamiltonian system to be integrable in the Liouville-Arnold sense. Precisely, we shall prove the existence of $n$ Haantjes operators

$$K_\alpha = \sum_{i=1}^{n} \frac{\nu_i^{(\alpha)}(J)}{\nu_i(J)} \left( \frac{\partial}{\partial J_i} \otimes dJ_i + \frac{\partial}{\partial \phi_i} \otimes d\phi_i \right) \quad \alpha = 1, \ldots, n,$$

where $\nu_i$ and $\nu_i^{(\alpha)}$ are the frequencies of the Hamiltonian $H$ and of the $(\alpha) - nth$ linear flow associated with the given system, respectively and $(J, \phi)$ are a set of action-angle variables. Formula (1) therefore intimately relates the Haantjes algebraic-geometric structure of an integrable system with its intrinsic dynamical properties.

As a by-product of the main theorem, we will be able to define new general classes of integrable models possessing an assigned Haantjes geometry. Quite interestingly, the systems so obtained are related to the wave equation.

An advantage of the present formulation à la Haantjes (which also represents the main motivation for our study) is its generality: Haantjes tensors are indeed a larger class of tensors than those of Nijenhuis.

The proposed theory incorporates essentially all the known results on integrability of finite-dimensional systems that have been developed in a bi-Hamiltonian framework up to date, i.e. all the approaches based on Lenard chains and their generalizations (as quasi-bi-Hamiltonian systems [4], etc).

There is a neat relation between the Haantjes geometry developed in the present work and the well known Nijenhuis geometry. In fact, a subfamily of symplectic-Haantjes manifolds is provided by the class of symplectic-Nijenhuis $(\omega N)$ manifolds. Precisely, we shall show that given an $\omega N$ manifold, under mild assumptions one can construct an $\omega\mathcal{H}$ structure by taking $n$ independent powers of the recursion operator $N$. In this case, $N$ will play the role of a generator for the $\omega\mathcal{H}$ structure.

However, it is important to notice that there exist $\omega\mathcal{H}$ structures not arising from a subjacent $\omega N$ structure. Indeed, whenever the Haantjes algebra $\mathcal{H}$ is non-Abelian, obviously it cannot be cyclically generated by a single Nijenhuis operator.
This is the case, for instance, of the $\omega \mathcal{H}$ manifold associated with the superintegrable Post-Winternitz system.

Another noteworthy feature of our approach is that a priori the class of Haantjes algebras considered is not necessarily diagonalizable. This aspect represents a generalization of the $\omega N$ approach, where indeed the operator $N$ is diagonalizable by hypothesis. Nevertheless, our theory keeps the intrinsic simplicity enjoyed by the standard approach to the Lenard-Magri chains for soliton hierarchies.

At the same time, the theory of $\omega \mathcal{H}$ manifolds is motivated by the crucial problem of the construction of coordinate systems allowing the additive separation of the associated Hamilton-Jacobi (HJ) equation (the separation variables). This is one of the most important problems in the theory of classical integrable systems, which historically has motivated a large amount of research work and inspired the formulation of several fundamental geometric developments.

The problem of the construction of separation of variables (SoV) can be recast in our approach and, in many cases, solved explicitly. Our main result in this direction is Theorem 28 ensuring the existence, under mild hypotheses, of a set of distinguished coordinates provided by the Haantjes structure associated with an integrable system, that we shall call the Darboux-Haantjes coordinates. They represent separation coordinates for the Hamilton-Jacobi equation associated with the Hamiltonian functions of the given system.

The study of the general problem of separation of variables (including partial separation [47], [10]) in the Haantjes geometry is in progress; the case of multiseparable systems has been addressed in [46].

Finally, we mention that a generalization of $\omega \mathcal{H}$ manifolds that parallels the theory of Poisson-Nijenhuis manifolds [21] has been introduced in [56]. The structures arising in this perspective, of Poisson-Haantjes type, will be suitable for studying Gelfand-Zakharevich systems [16].

The structure of the paper is as follows. In Section 2, we review the main algebraic properties of Nijenhuis and Haantjes tensors, and the notion of Haantjes algebras introduced in [49]. In Section 3, we introduce the main geometric structures needed for the discussion of integrability, i.e. the $\omega \mathcal{H}$ manifolds; also, we clarify their relation with $\omega N$ manifolds. In particular, a theorem guaranteeing the existence of the DH coordinates is proved. Besides, the notion of generator of a Haantjes structure is defined. Section 4 contains the theorem that characterizes complete integrability via the Haantjes geometry. In Section 5, new integrable models related to the wave equation are deduced from suitable Haantjes structures.

In Section 6, a procedure for the construction of Haantjes structures for a given integrable system with two degrees of freedom is proposed. Also, the relevant example of the superintegrable Post-Winternitz system, whose separation coordinates are still not known, is worked out. An application of our theory to a stationary reduction of the seventh-order equation of the KdV hierarchy is discussed in Section 7. Some open problems are discussed in the final Section 8.

2. HAANJTES ALGEBRAS OF OPERATORS

Given a dynamical system defined over a finite-dimensional manifold $M$, a fundamental issue is to find suitable sets of coordinates which allow us to decouple the equations of motion. The natural frames of such coordinates, being obviously integrable, can be characterized in a tensorial manner as eigendistributions of a
suitable class of (1,1) tensor fields, i.e. the ones with a vanishing Nijenhuis or Haantjes tensor. In this section, we review some basic algebraic results concerning the theory of such tensors. For a more complete treatment, see the original papers [18, 41], the related ones [42, 15] and the recent review [20].

2.1. Preliminaries. We shall denote by $M$ a differentiable manifold and by $L : TM \to TM$ a (1,1) smooth tensor field, i.e. a smooth field of linear operators on the tangent space at each point of $M$. In the following, all tensors will be considered to be smooth.

**Definition 1.** The Nijenhuis torsion of $L$ is the skew-symmetric (1, 2) tensor field defined by

$$ \mathcal{T}_L(X, Y) := L^2[X, Y] + [LX, LY] - L \left( [X, LY] + [LX, Y] \right), $$

where $X, Y \in TM$ and $[ \cdot , \cdot ]$ denotes the commutator of two vector fields.

In local coordinates $x = (x^1, \ldots, x^n)$, the Nijenhuis torsion can be written in the form

$$ (\mathcal{T}_L)^{ij}_{jk} = \sum_{\alpha=1}^{n} \left( \frac{\partial L^i_{\alpha}}{\partial x^j} L^j_{\alpha} - \frac{\partial L^j_{\alpha}}{\partial x^i} L^i_{\alpha} + \left( \frac{\partial L^i_{\alpha}}{\partial x^k} - \frac{\partial L^k_{\alpha}}{\partial x^i} \right) L^j_{\alpha} \right), $$(3)

for the components of $\mathcal{T}_L$; among them, $n^2(n-1)/2$ are independent. Here for the sake of brevity we have used the notation $\partial_j := \frac{\partial}{\partial x^j}$ and the indices between square brackets are to be skew-symmetrized, except those in $| \cdot |$.

**Definition 2.** The Haantjes tensor associated with $L$ is the (1, 2) tensor field defined by

$$ \mathcal{H}_L(X, Y) := L^2 \mathcal{T}_L(X, Y) + \mathcal{T}_L(LX, LY) - L \left( \mathcal{T}_L(X, LY) + \mathcal{T}_L(LX, Y) \right). $$

The skew-symmetry of the Nijenhuis torsion implies that the Haantjes tensor is also skew-symmetric. Its explicit intrinsic expression is

$$ \mathcal{H}_L(X, Y) = L^4[X, Y] + [L^2 X, L^2 Y] - 2L^3 \left( [X, LY] + [LX, Y] \right) + L^2 \left( [X, L^2 Y] + 4 [LX, LY] + [L^2 X, Y] \right) - 2L \left( [LX, L^2 Y] + [L^2 X, LY] \right). $$

Its local expression in recursive form is

$$ (\mathcal{H}_L)^{ij}_{jk} = \sum_{\alpha, \beta=1}^{n} \left( L^i_{\alpha} L^j_{\beta} (\mathcal{T}_L)^{\alpha}_{\beta} + (\mathcal{T}_L)^{\alpha}_{\beta} L^j_{\alpha} L^i_{\beta} - L^i_{\alpha} \left( (\mathcal{T}_L)^{\alpha}_{\beta} L^j_{\beta} + (\mathcal{T}_L)^{\alpha}_{\beta} L^i_{\beta} \right) \right). $$

More explicitly, we have

$$ (\mathcal{H}_L)^{ij}_{jk} = \sum_{\alpha=1}^{n} \left( - (L^i_{\alpha})^j L^j_{\alpha} + \sum_{\beta=1}^{n} \left( (L^2)^{\alpha}_{\beta} + 2L^i_{\alpha} (L^j_{\beta} \partial_{j} L^i_{\alpha} - L^j_{\beta} \partial_{i} L^i_{\alpha}) \right) \right) + \sum_{\gamma=1}^{n} \left( (L^2)^{i}_{\gamma} L^j_{\gamma} \partial_{i} L^i_{\gamma} - L^i_{\gamma} (L^2)^{j}_{\gamma} \partial_{j} L^i_{\gamma} - 2L^i_{\gamma} L^j_{\gamma} L^i_{\gamma} \partial_{j} L^i_{\gamma} \right). $$

The following notion is at the heart of the theory we are going to develop.
Definition 3. A Haantjes (Nijenhuis) operator is an operator field whose Haantjes (Nijenhuis) torsion identically vanishes.

Remark 4. Any operator field on a two dimensional manifold is a Haantjes operator. Furthermore, on an n-dimensional manifold, any operator field that admits local charts where it takes a diagonal form is also a Haantjes operator.

2.2. Haantjes algebras. The concept of Haantjes algebra was defined in [49] as an abstract setting for developing the theory of Haantjes operators. Indeed, many important properties of this class of operators can be discussed in a general, basis-independent context, in which the algebraic structure is kept to a minimum. As we shall see, additional structures (as symplectic or Poisson structures) are needed if one wishes to discuss integrable models and the construction of suitable separation variables. For sake of completeness, here a very brief review of the theory of Haantjes algebras is presented.

Definition 5. A Haantjes algebra of rank m is a pair \((M, \mathcal{H})\) which satisfies the following conditions:
- \(M\) is a differentiable manifold of dimension \(n\);
- \(\mathcal{H}\) is a set of Haantjes operators \(K : TM \to TM\) that generates
  - a free module of rank \(m\) over the ring of smooth functions on \(M\)
  
\[ \mathcal{H}(fK_1 + gK_2)(X,Y) = 0, \quad \forall X,Y \in TM, \quad \forall f,g \in C^\infty(M), \quad \forall K_1, K_2 \in \mathcal{H}, \]
  - a ring w.r.t. the composition operation

\[ \mathcal{H}(K_1 K_2)(X,Y) = 0, \quad \forall K_1, K_2 \in \mathcal{H}, \quad \forall X,Y \in TM, \]

If

\[ K_1 K_2 = K_2 K_1 \quad \forall K_1, K_2 \in \mathcal{H} \]

the algebra \(\mathcal{H}\) will be said to be an Abelian Haantjes algebra. Moreover, if the identity operator \(I \in \mathcal{H}\), then \((M, \mathcal{H})\) will be said to be a Haantjes algebra with identity.

The assumptions (8), (9) ensure that the set \(\mathcal{H}\) is an associative algebra of Haantjes operators; moreover, the Hamilton-Cayley theorem implies that its rank \(m\) is not greater than \(n\).

The conditions of Definition 5 might seem difficult to realize. However, a natural class of Haantjes algebras is given, in a local chart \(\{U, x = (x^1, \ldots, x^n)\}\), by operators of the form

\[ K = \sum_{i=1}^{n} l_i(x) \frac{\partial}{\partial x^i} \otimes dx^i, \]

where \(l_i(x) := l^i_j(x)\) are arbitrary smooth functions playing the role of the eigenvalues of \(K\). The diagonal operators (11) have vanishing Haantjes tensor and satisfy the differential compatibility condition (8), by virtue of Remark 4. Furthermore, they form a commutative ring; thus they also satisfy Eqs. (9).

Definition 6. The algebra generated by operators of the form (11) will be said to be a diagonal Haantjes algebra.
2.2.1. Cyclic Haantjes algebras. A particularly relevant class of Abelian Haantjes algebras is given by those generated by a single Haantjes operator $L : TM \to TM$ [49]. One can construct directly a Haantjes algebra $\mathcal{L}$ of rank $m \leq n = \dim(M)$ by choosing as a set of generators the first $(n-1)$ powers of $L$ together with $L^0 := I$

$$\mathcal{L}(L) := \text{Span}\{I, L, \ldots, L^{n-1}, \ldots\} = \text{Span}\{I, L, \ldots, L^n\}.$$ 

The fact that $\mathcal{L}(L)$ is a Haantjes algebra is a consequence of the following result.

**Proposition 7.** [5] Let $L$ be an operator with vanishing Haantjes tensor on $M$. Then, for any polynomial $p(x, L) = \sum_{j=0}^{n-1} a_j(x) L^j$ with coefficients $a_j \in C^\infty(M)$, the associated Haantjes tensor also vanishes, i.e.

$$H_L(X, Y) = 0 \implies H_{p(x, L)}(X, Y) = 0.$$

**Proof.** See Corollary 3.3, p. 1136 of Ref. [5]. □

According to the previous discussion, one can introduce the notion of cyclic Haantjes algebras.

**Definition 8.** Let $(M, \mathcal{H})$ be an Abelian Haantjes algebra of rank $m$. An operator $L$, with minimal polynomial of degree $h \geq m$, is a generator of $\mathcal{H}$ if

$$\mathcal{H} \subseteq \mathcal{L}(L).$$

The corresponding algebra will be said to be a cyclic Haantjes algebra.

Let

$$B_{\text{cyc}} = \{I, L, L^2, \ldots, L^{m-1}\}$$

be a cyclic basis of $\mathcal{L}(L)$. A basis $B$ of $\mathcal{H}$ such that $B \subseteq B_{\text{cyc}}$ will be said to be a cyclic basis of $\mathcal{H}$.

A generator of $\mathcal{H}$ allows us to represent each Haantjes operator $K \in \mathcal{H}$ as a polynomial field in $L$ of degree at most $(h-1)$, i.e.

$$K = p_K(x, L) = \sum_{i=0}^{h-1} a_i(x) L^i,$$

where $a_i(x)$ are smooth functions on $M$. A natural problem is to establish the conditions ensuring that a given Haantjes algebra is cyclic. This problem has been solved in Proposition 47 of [49], where it was proven that each semisimple Abelian algebra (see Definition 9) is cyclic. In the next subsection, we will present an extension of that result.

2.3. Haantjes coordinates. We shall always assume that the eigenvalues of any operator field considered in this article are real functions.

Let us recall the main result of [49] about the existence of Haantjes charts for Haantjes algebras. The set of proper eigenvector fields of a generic operator $K$, corresponding to an eigenvalue $l_i(x)$, is given by $\text{Ker}(K - l_i I)$, whereas the set of its generalized eigenvector fields is $\text{Ker}(K - l_i I)^{\rho_i}$. We have denoted by $\rho_i$ the Riesz index of $l_i$ (assumed to be independent of $x$), that is, the minimum integer such that $\text{Ker}(K - l_i I)^{\rho_i} = \text{Ker}(K - l_i I)^{\rho_i+1}$.

**Definition 9.** A Haantjes algebra is said to be semisimple if each $K \in \mathcal{H}$ is semisimple (diagonalizable), that is, if each $K$ admits a local reference frame formed by proper eigenvector fields of $K$. 
If a semisimple Haantjes algebra is Abelian, then there exists a local reference eigenframe in which all $K \in \mathcal{H}$ take simultaneously a diagonal form. A crucial question is to ascertain whether an integrable common eigenframe exists, or equivalently, whether there exists a local coordinate chart where all operators can be simultaneously written in a diagonal form. The answer to this problem is offered by Theorem 11. Preliminarily, we state the following

**Definition 10.** Let $\{D_i, D_j, \ldots, D_k\}$ be a set of distributions of vector fields. We shall say that these distributions are mutually integrable if

i) each of them is integrable;

ii) any sum $D_i + D_j + \ldots + D_k$ (where all indices $i, j, \ldots, k$ are different) is also integrable.

**Theorem 11.** [49] Let $(M, \mathcal{H})$ be an Abelian Haantjes algebra of rank $m$ and $\{K_1, \ldots, K_m\}$ a basis of it. Let us consider the spectral decomposition

$$T_x M = \bigoplus_{a=1}^{\nu} V_a(x)$$

where

$$V_a(x) := D^{(1)}_{i_1}(x) \cap \ldots \cap D^{(m)}_{i_m}(x) \quad a := (i_1, \ldots, i_m)$$

and

$$D^{(\alpha)}_{i_\alpha}(x) := \text{Ker}(K_\alpha - l^{(\alpha)}_{i_\alpha}I)^{s_\alpha}(x), \quad \alpha = 1, \ldots, m, \quad i_\alpha = 1, \ldots, s_\alpha,$$

where $s_\alpha$ is the number of the distinct eigenvalues of $K_\alpha$. The distributions $V_a$ are mutually integrable; therefore, there exists a set of coordinates (that we shall call Haantjes coordinates) adapted to the decomposition (15), such that all $K \in \mathcal{H}$ can be simultaneously written in a block-diagonal form. Furthermore, if $\mathcal{H}$ is semisimple, in each set of Haantjes coordinates all $K \in \mathcal{H}$ can be simultaneously written in a diagonal form.

For the sake of clarity, we also quote Proposition 45 of Ref. [49], which is relevant in the forthcoming discussion.

**Proposition 12.** Let $L$ be a semisimple operator with $h$ pointwise distinct eigenvalues $\{\lambda_1(x), \ldots, \lambda_h(x)\}$, and $K$ be another semisimple operator field possessing $s$ pointwise distinct eigenvalues, with $s \leq h$. The following conditions are equivalent:

- $K$ belongs to the cyclic algebra of rank $h$ generated by $L$, i.e.

$$K \in \mathcal{L}(L);$$

- there exists a polynomial field $p_K(x, \lambda)$ in $\lambda$ of degree at most $(h - 1)$ such that

$$K = p_K(x, L);$$

- each eigendistribution of $L$ is included in a single eigendistribution of $K$:

$$C_{\lambda_i} := \text{ker}(L - \lambda_i I) \subseteq D_i := \text{ker}(K - l_i I),$$

where it is understood that the eigenvalues $(l_1(x), \ldots, l_h(x))$ of $K$ may not be all distinct.
We present now an extension of the result obtained in Proposition 47 of Ref. [49], since it will be relevant in the formulation of the theory of $\omega \mathcal{H}$ manifolds.

**Proposition 13.** Every semisimple Abelian Haantjes algebra $(M, \mathcal{H})$ of rank $m$ is cyclic and admits a family of Haantjes generators; among them, there exists a Nijenhuis operator.

Moreover, an operator $K \in \mathcal{H}$ is a generator of $\mathcal{H}$ if and only if it possesses $h$ distinct eigenvalues; precisely, $h = m$ if $I \in \mathcal{H}$, or $h = (m + 1)$ otherwise.

**Proof.** Let us consider the Haantjes chart
\begin{equation}
\{U, x = (y^1, \ldots, y^v)\},
\end{equation}
adapted to the spectral decomposition (15). Then, if $v \geq m$, each operator of the form
\begin{equation}
L = \sum_{a=1}^{v} \lambda_a(x) \sum_{j_a=1}^{r_a} \frac{\partial}{\partial y^{a_j_a}} \otimes dy^{a_j_a},
\end{equation}
where $r_a = \text{rank } V_a$ is a generator of $\mathcal{H}$, provided that its eigenvalue fields $\{\lambda_1(x), \ldots, \lambda_v(x)\}$ are arbitrary but distinct smooth functions at any point of $U$.

In fact, the eigendistributions of the operator (22) are given by the distributions $V_a$ defined in Eq. (16); consequently, by construction they satisfy condition (20).

Besides, as the eigenvalues of $L$ are distinct, this operator also satisfies the assumptions of Proposition 12. In particular, if the eigenvalues of $L$ are chosen to be
\begin{equation}
\lambda_a(x) = \lambda_a(y^{a_1}, \ldots, y^{a_{r_a}}) \quad a = 1, \ldots, v,
\end{equation}
then $L$ is a Nijenhuis generator, that is, its Nijenhuis torsion identically vanishes. If $v < m$, a generator can still be constructed, because we can further decompose some of the distributions $V_a$ into a direct sum of mutually integrable sub-distributions. Precisely, we have
\begin{equation}
V_a = \left\langle \frac{\partial}{\partial y^{a_1}}, \ldots, \frac{\partial}{\partial y^{a_{r_a}}} \right\rangle = \bigoplus_{i_a=1}^{\tilde{r}_a} \left\langle \frac{\partial}{\partial y^{a_1}}, \ldots, \frac{\partial}{\partial y^{a_{i_a}}} \right\rangle = \bigoplus_{i_a=1}^{\tilde{r}_a} C_{a, i_a},
\end{equation}
with $\sum \tilde{r}_a = r_a$; the previous decomposition of $V_a$ can be realized in such a way that the number of addends appearing into the direct sum
\begin{equation}
T_x M = \bigoplus_{a=1, i_a=1}^{v, \tilde{r}_a} C_{a, i_a}
\end{equation}
is not less than $m$.

Let us recall that since $\mathcal{H}$ is a semisimple algebra by assumption, then the degree of the minimal polynomial of any $K \in \mathcal{H}$ coincides with the number of its distinct eigenvalues.

Assume now that $K$ belongs to $\mathcal{H}$ and has $h$ distinct eigenvalue fields, with $h = m$ if $I \in \mathcal{H}$, or $h = (m + 1)$ otherwise. If $I \in \mathcal{H}$, then $\mathcal{L}(K) \subseteq \mathcal{H}$ and having rank $m$, it coincides with $\mathcal{H}$. Otherwise, $\mathcal{H} = \text{Span}(K, \ldots, K^m) \subset \mathcal{L}(K)$.

Conversely, assume that a generator $L$ belongs to $\mathcal{H}$. Then, by Lemma 37 of [49], we have that $h \leq m$ if $I \in \mathcal{H}$, or $h \leq m + 1$ otherwise. Furthermore, we observe that if $I \notin \mathcal{H}$ and $h = m$, then the cyclic Haantjes algebra $\mathcal{L}(L)$ would
coincide with \( \mathcal{H} \), which is absurd. Taking into account that \( h \geq m \) by Definition 8, the statement is proven. \( \square \)

As in the case \( v \geq m \) the number \( v \) coincides with the number \( h \) of the pointwise distinct eigenvalues of \( L \), we deduce the following

**Corollary 14.** Let \((M, \mathcal{H})\) be a semisimple Abelian Haantjes algebra of rank \( m \). Assume that the number of the addends of the decomposition (15) is \( v \geq m \). Then, generators of \( \mathcal{H} \) belonging to \( \mathcal{H} \) exist if and only if \( v = m \), when \( I \in \mathcal{H} \), or \( v = m + 1 \), otherwise.

In Section 3.4, we shall specialize the results of Proposition 13 and Corollary 14 to the case of \( \omega.\mathcal{H} \) manifolds.

### 2.4. Haantjes chains.

The theory of Lenard–Magri chains is a fundamental piece of the geometric approach to soliton hierarchies. Lenard–Magri chains have been introduced in order to construct integrals of motion in involution for infinite-dimensional Hamiltonian systems [25, 26] (see also [45], for a brief history about the origin of the name “Lenard chains”). Besides, some non trivial generalizations of Lenard–Magri chains have proved to be useful in the study of separation of variables for finite-dimensional Hamiltonian systems (see [37, 38, 54, 11, 12] and reference therein).

Hereafter, we propose a further generalization of the standard notions of the theory, which has the advantage to be both simple and directly connected to the theory of classical integrable systems.

**Definition 15.** Let \((M, \mathcal{H})\) be a Haantjes algebra of rank \( m \). We shall say that a smooth function \( H \) generates a Haantjes chain of closed 1-forms of length \( m \) if there exist a distinguished basis \( \{\tilde{K}_1, \ldots, \tilde{K}_m\} \) of \( \mathcal{H} \) such that

\[
\tag{25} d(\tilde{K}_\alpha dH) = 0, \quad \alpha = 1, \ldots, m,
\]

where \( \tilde{K}_\alpha : T^*M \to T^*M \) is the transposed operator of \( \tilde{K}_\alpha \). The (locally) exact 1-forms

\[
\tag{26} dH_\alpha = \tilde{K}_\alpha dH
\]

(which are supposed to be linearly independent), are called the elements of the Haantjes chain of length \( m \) generated by \( H \); the functions \( H_\alpha \) are their potential functions.

In order to inquire about the existence of Haantjes chains for an assigned Haantjes algebra, we have to consider the codistribution, of rank \( r \leq m \), generated by a given function \( H \) through an arbitrary basis \( \{K_1, K_2, \ldots, K_m\} \) of \( \mathcal{H} \), i.e.

\[
\tag{26} \mathcal{D}_H := \text{Span}\{K_1^* dH, K_2^* dH, \ldots, K_m^* dH\},
\]

and the distribution \( \mathcal{D}_H \) of the vector fields annihilated by them, which has rank \( n - r \). Note that such distributions do not depend on the particular basis chosen in \( \mathcal{H} \).

The following theorem offers a geometric characterization of the existence of a Haantjes chain generated by a smooth function \( H \) in terms of the Frobenius integrability of its associated codistribution.
Theorem 16. Let $(M, \mathcal{H})$ be a Haantjes algebra of rank $m$, and $H$ be a smooth function on $M$. Let $\mathcal{D}_H^\circ$ be the codistribution (26), assumed to be of rank $m$ (independent on $x$), and $\mathcal{D}_H$ be the distribution of the vector fields annihilated by the 1-forms of $\mathcal{D}_H^\circ$. Then, the function $H$ generates a Haantjes chain (25) if and only if $\mathcal{D}_H^\circ$ (or equivalently $\mathcal{D}_H$) is Frobenius-integrable.

Proof. By definition, the Haantjes chain (25) contains $m$ independent exact 1-forms. Therefore, they generate the integrable distribution

$$\mathcal{D}^\circ = \text{Span}\{dH_1 = K_1^T dH, \ldots, dH_m = K_m^T dH\},$$

which coincides with $\mathcal{D}_H^\circ$. Vice versa, let $\mathcal{D}_H^\circ$ be integrable and $\mathcal{D}_H$ its foliation. Then, there exist $m$ independent functions $(H_1, H_2, \ldots, H_m)$ which are constant on the leaves of $\mathcal{D}_H$. Their differentials belong to $\mathcal{D}_H^\circ$, hence they can be written as

$$dH_\alpha = \sum_{\beta=1}^{m} a_{\alpha\beta}(x) K_\beta^T dH =: K_\alpha^T dH \quad \alpha = 1, \ldots, m.$$ 

$\square$

3. The theory of symplectic-Haantjes manifolds

In this section we introduce the new class of symplectic-Haantjes manifolds; we shall call them $\omega\mathcal{H}$ manifolds, by analogy with the well-known $\omega\mathcal{N}$ manifolds [33, 12]. These new manifolds are essentially Haantjes algebras endowed with a symplectic structure. The main reason to define these manifolds is that, apart from their interesting mathematical properties, they provide a simple but sufficiently general setting in which the theory of Hamiltonian integrable systems can be naturally formulated.

3.1. Haantjes algebras and $\omega\mathcal{H}$ manifolds.

Definition 17. A symplectic–Haantjes (or $\omega\mathcal{H}$) manifold of class $m$ is a triple $(M, \omega, \mathcal{H})$ which satisfies the following properties:

i) $(M, \omega)$ is a symplectic manifold of dimension $2n$;

ii) $\mathcal{H}$ is a Haantjes algebra of rank $m$;

iii) $(\omega, \mathcal{H})$ are algebraically compatible, that is

$$\omega(X, K Y) = \omega(K X, Y) \quad \forall K \in \mathcal{H},$$

or equivalently

$$\Omega K = K^T \Omega, \quad \forall K \in \mathcal{H}. $$

Hereafter $\Omega := \omega^\flat : TM \rightarrow T^* M$ denotes the fiber bundles isomorphism defined by

$$\omega(X, Y) = \langle \Omega X, Y \rangle \quad \forall X, Y \in TM,$$

and the map $P := \Omega^{-1} : T^* M \rightarrow TM$ is the Poisson bivector induced by the symplectic structure $\omega$.

If the identity operator $I$ belongs to $\mathcal{H}$, then we shall say that $(M, \omega, \mathcal{H})$ is a $\omega\mathcal{H}$ manifold with identity. If $\mathcal{H}$ is an Abelian Haantjes algebra, the resulting $\omega\mathcal{H}$ manifold will be said to be Abelian.
Remark 18. The set of conditions of Definition 17 admits a natural, simple realization. Indeed, note that in a coordinate system \( x = (x^1, \ldots, x^{2n}) \) the operators

\[
K_\alpha = \sum_{j=1}^{2n} \lambda_j^{(\alpha)}(x) \frac{\partial}{\partial x^j} \otimes dx^j, \quad \alpha = 1, \ldots, m,
\]

according to Definition 6, generate a diagonal Haantjes algebra for any smooth function \( \lambda_j^{(\alpha)}(x) \). Besides, by imposing the algebraic compatibility conditions \( (29) \) we get in a Darboux chart the solutions \( \{x = (q^1, \ldots, q^n, p_1, \ldots, p_n)\} \) given by

\[
K_\alpha = \sum_{i=1}^{n} l_i^{(\alpha)}(x) \left( \frac{\partial}{\partial q^i} \otimes dq^i + \frac{\partial}{\partial p_i} \otimes dp_i \right), \quad \alpha = 1, \ldots, m.
\]

Here \( l_i^{(\alpha)}(x) = \lambda_i^{(\alpha)}(x) = \lambda_{n+i}^{(\alpha)}(x), \ i = 1, \ldots, n \).

As a consequence of conditions \( (29) \), one can immediately deduce the following proposition, which turns out to be crucial for many results of the present theory. For instance, it has important consequences on the spectrum and the eigenvector fields of the Haantjes operators belonging to a \( \omega \mathcal{H} \) manifold.

Proposition 19. Let \((M, \omega, \mathcal{H})\) be an \( \omega \mathcal{H} \) manifold. Then, any composed operator \( \Omega p(x, K), P q(x, K^T) \) (where \( p(x, K) \) and \( q(x, K) \) are polynomial fields in \( K \) and \( K^T \) respectively) is skew-symmetric \( \forall K \in \mathcal{H} \). Moreover, if \( \omega \mathcal{H} \) is Abelian, then \( K^T \Omega K, K_\alpha PK^T_\beta \) are also skew-symmetric \( \forall K_\alpha, K_\beta \in \mathcal{H} \).

Corollary 20. Given a \( 2n \)-dimensional \( \omega \mathcal{H} \) manifold \( M \), every generalized eigen–distribution \( \text{Ker}(K - l_i I)^{r_i}, r_i \in \mathbb{N} \), is of even rank. Therefore, the geometric and algebraic multiplicities of each eigenvalue \( l_i(x) \) are even.

Proof. Given an \( \omega \mathcal{H} \) manifold, every generalized eigen–distribution \( \text{Ker}(K - l_i I)^{r_i} \) has the same rank of the kernel of the operator \( \Omega(K - l_i I)^{r_i} \), which is skew-symmetric by virtue of Proposition 19. Thus, the second statement in the Corollary is a consequence of the fact that the geometric multiplicity of an eigenvalue \( l_i(x) \) is equal to the rank of \( \text{Ker}(K - l_i I) \), and its algebraic multiplicity to the rank of \( \text{Ker}(K - l_i I)^{r_i} \).

Due to the above corollary, and the spectral decomposition of the tangent spaces of \( M \) given by

\[
T_x M = \bigoplus_{i=1}^{s} D_i(x), \quad D_i(x) := \text{Ker}(K - l_i I)^{r_i}(x),
\]

as \( \text{rank } D_i \geq 2, \ i = 1, \ldots, s \) we conclude that the number of the distinct eigenvalues of any Haantjes operator \( K \) of an \( \omega \mathcal{H} \) structure is not greater than \( n \).

Definition 21. Given a \( 2n \)-dimensional \( \omega \mathcal{H} \) manifold, if the number of pointwise distinct eigenvalues of a Haantjes operator \( K \in \mathcal{H} \) is \( n \), we shall say that such operator is maximal.

Observe that the minimal polynomial of a maximal operator \( K \in \mathcal{H} \) has the form

\[
m_K(x, \lambda) = \prod_{i=1}^{n} \left( \lambda - l_i(x) \right)^{r_i}.
\]
Lemma 22. A Haantjes operator $K$ of a 2n-dimensional $\mathcal{H}$ manifold of class $m$ is maximal if and only if its minimal polynomial has degree $m = n$. Therefore, its minimal polynomial is the product of $n$ linear factors; thus, $K$ is pointwise semisimple.

Proof. As a consequence of Corollary 20, if $K$ admits a Jordan chain of length $\rho_i$ associated with a given eigenvalue $l_i(x)$, there must exist, for the same eigenvalue, a “twin” Jordan chain of the same length. Consequently, the number of the Jordan chains of length $\rho_i$ associated to a given eigenvalue is even, and therefore $\rho_i \leq n$.

Observe that $K$ can be maximal if and only if $\rho_i = 1$, $i = 1, \ldots, n$, due to Eq. (33). In fact, in this case every eigendistribution of $K$ has rank 2 and it is formed by proper eigenvector fields. □

In the following, we shall present some properties of both algebraic and differential-geometric nature for the Haantjes operators $K \in \mathcal{H}$. Denote by

$$E_i := \bigoplus_{j=1, j \neq i}^s D_j = \text{Im}(K - l_i I)^{\rho_i}, \quad i = 1, \ldots, s,$$

the distribution of rank $(2n - r_i)$ spanned by all of the generalized eigenvectors of a Haantjes operator $K$, except those associated with the eigenvalue $l_i$. Such a distribution will be called a characteristic distribution of $K$. Let $E_i^\circ$ denote the module of one-forms that annihilate all vector fields of the distribution $E_i$.

Proposition 23. Given an $\omega \mathcal{H}$ manifold, the relations

$$\Omega(D_j) = E_j^\perp \iff D_j = P(E_j^\circ) = D_j^\perp,$$

$$\Omega(E_j) = D_j^\circ \iff E_j = P(D_j) = E_j^\perp,$$

hold. Here $E_j^\perp$ and $D_j^\perp$ are the symplectic orthogonal distributions of $E_j$ and $D_j$, respectively.

Proof. Property (35) follows from the compatibility condition (29), taking into account that the symplectic operator $\Omega$ is invertible. In fact, for each generalized eigenvector field $Y_j \in D_j$, the one-form $\Omega Y_j$ is a generalized eigenform of $K^T$, as one infers from the relation

$$(K^T - l_j I)^{\rho_i} \Omega Y_j = 0.$$

Then, since

$$\text{Ker}(K^T - l_j I)^{\rho_i} = \text{Im}(K - l_i I)^{\rho_i},$$

we deduce that $\Omega Y_j(x)$ belongs to $E_i^\circ(x)$. Since this subspace has the same dimension of $D_j(x)$, we obtain Eq. (35). The companion relation (36) follows from Eq. (35) jointly with the observation that, by construction, $E_j(x)$ is a complementary subspace of $D_j(x)$ in $T_xM$. □

Proposition 24. Given an $\omega \mathcal{H}$ manifold, the distributions $D_j$ of each $K \in \mathcal{H}$ are integrable and of even rank. Their integral leaves are symplectic submanifolds of $M$ and are symplectically orthogonal to each other, namely

$$\omega(D_j, D_j) = \text{symplectic},$$

$$\omega(D_j, D_k) = 0 \quad j \neq k.$$
Proof. The distributions $D_j$ are integrable due to the Hantjes Theorem \[18\] and are of even rank by virtue of Corollary \[20\]. Besides, they are symplectic as
\[
D_j \cap (D_j)^\perp \equiv D_j \cap E_j = \{0\}.
\]
Finally, property \[36\] follows from the fact that $D_k \subseteq E_j \equiv D_j^\perp$, if $j \neq k$. □

**Corollary 25.** Given an Abelian $\omega \mathcal{H}$ manifold, each integral leaf $D_j$ of the eigen-distribution $D_j$, $j = 1, \ldots, s$, is an Abelian $\omega \mathcal{H}$ submanifold.

**Proof.** As the Haantjes algebra $\mathcal{H}$ is Abelian by assumption, the submanifold $D_j$ is $\mathcal{H}$-invariant. Therefore, each operator $K \in \mathcal{H}$ can be restricted to $D_j$, giving rise to a Haantjes algebra of rank $\leq m$. Finally, the compatibility condition \[29\] holds since
\[
K^T|_{D_j} \Omega|_{D_j} = (K^T \Omega)|_{D_j} = (\Omega K)|_{D_j} = \Omega|_{D_j} K|_{D_j} \quad \forall K \in \mathcal{H}.
\]
□

The $\omega \mathcal{H}$ manifold associated with the integral leaf $D_j$ will be denoted by $(D_j, \omega|_{D_j}, \mathcal{H}|_{D_j})$.

Below, we shall prove that Proposition \[24\] and Corollary \[25\] hold for the intersections of the distributions $D_{i_1}^{(\alpha)}$ defined in Eq. \[17\]. Precisely, let us consider the distributions
\[
W_{I(\alpha)} := W_{i_1, \ldots, i_\alpha} := D_{i_1}^{(1)} \cap \cdots \cap D_{i_\alpha}^{(\alpha)} \quad \alpha = 1, \ldots, m,
\]
which are integrable, being intersections of integrable distributions. Interestingly, the integral leaves of such distributions are themselves $\omega \mathcal{H}$ manifolds.

**Proposition 26.** Given an Abelian $\omega \mathcal{H}$ manifold, the distributions $W_{I(\alpha)}$ \[40\] that have constant rank are integrable and of even rank. Their integral leaves are symplectic submanifolds of $M$ and are symplectically orthogonal to each other, namely
\[
\omega (W_{I(\alpha)}, W_{I(\alpha)}) = \text{symplectic}\]
\[
\omega (W_{I(\alpha)}, W_{I(\beta)}) = 0 \quad \alpha \neq \beta.
\]
Moreover, they are $\omega \mathcal{H}$ submanifolds of $(M, \omega, \mathcal{H})$.

**Proof.** We prove the result by induction on the number $\alpha$ of the factors in the intersection \[40\]. First, observe that if $\alpha = 1$, properties \[41\] and \[42\] are fulfilled according to Proposition \[24\]. Let us suppose that these properties are satisfied by the distributions
\[
W_{I(\alpha-1)} = W_{i_1, \ldots, i_{\alpha-1}} = D_{i_1}^{(1)} \cap \cdots \cap D_{i_{\alpha-1}}^{(\alpha-1)} \quad \alpha > 1;
\]
then we will prove that they hold true for the distributions $W_{I(\alpha)} = W_{I(\alpha-1)} \cap D_{i_\alpha}^{(\alpha)}$. To this aim, we consider the foliation $\mathcal{H}_{I(\alpha-1)}$ and a generic leaf $W_{I(\alpha-1)}$ of it. Obviously, the symplectic form $\omega$ can be restricted to $W_{I(\alpha-1)}$, where it is still non degenerate, by the induction assumption \[41\]. Furthermore, as $W_{I(\alpha-1)}$ is invariant for the Haantjes algebra $\mathcal{H}$, each element of $\mathcal{H}$ can be restricted to $W_{I(\alpha-1)}$. 

Now, let us denote by \( \mathcal{H}_{W_I^{(a-1)}} \) the restriction of the Haantjes algebra to \( W_I^{(a-1)} \). Over the leaf \( W_I^{(a-1)} \) the restricted compatibility condition holds true, as \( \forall K \in \mathcal{H} \) we have

\[
(K^T)_{W_I^{(a-1)}} \Omega |_{W_I^{(a-1)}} = (K^T \Omega) |_{W_I^{(a-1)}} = (\Omega K) |_{W_I^{(a-1)}} = \Omega |_{W_I^{(a-1)}} K |_{W_I^{(a-1)}} .
\]

Therefore, the triple \( \langle W_I^{(a-1)}, \omega |_{W_I^{(a-1)}}, \mathcal{H} |_{W_I^{(a-1)}} \rangle \) is a \( \omega, \mathcal{H} \) manifold.

Notice that the generalized eigendistribution \( W_I^{(a)} \) can be characterized at any point by \( W_I^{(a)}(x) = \text{Ker}(K_\alpha - l_i^{(a)j}) |_{W_I^{(a-1)}}(x) \). Consequently, the rank of \( W_I^{(a)} \)

must be even due to Corollary 20. Moreover,

\[
W_I^{(a)}(x) = \left( \text{Ker}(K_\alpha - l_i^{(a)j}) |_{W_I^{(a-1)}} \right)^\perp(x) = P \mid_{W_I^{(a-1)}} \left( \left( \text{Ker}(K_\alpha - l_i^{(a)j}) |_{W_I^{(a-1)}} \right)^2 \right) \mid_{W_I^{(a-1)}}(x) = (E_{i_0}^{(1)}) |_{W_I^{(a-1)}}(x),
\]

where \( P \mid_{W_I^{(a-1)}} = (\Omega |_{W_I^{(a-1)}})^{-1} \). Therefore,

\[
W_I^{(a)} \cap W_I^{(a)} = (D_i^{(m)}) \mid_{W_I^{(a-1)}} \cap (E_i^{(m)}) \mid_{W_I^{(a-1)}} = \{0\}
\]

so that \( W_I^{(a)} \) is a symplectic foliation of \( M \). Observe that condition (44) also holds over any leaf \( W_I^{(a)} \). Therefore, we conclude that each integral leaf \( W_I^{(a)} \) of the distributions \( W_I^{(a)} \) is again a \( \omega, \mathcal{H} \) manifold; it will be denoted by \( \langle W_I^{(a)}, \omega |_{W_I^{(a)}}, \mathcal{H} |_{W_I^{(a)}} \rangle \).

To prove relation (42), let us compare \( W_I^{(a)} = D_i^{(1)} \cap \ldots \cap D_j^{(a)} \) with \( W_J^{(a)} = D_{i_1}^{(1)} \cap \ldots \cap D_{j_0}^{(a)} \). We distinguish two possibilities: If, \( i_1 \neq j_1, \ldots, i_a \neq j_a \), it follows that \( W_I^{(a)} \subseteq W_J^{(a)} \) as \( D_i^{(1)} \subseteq E_i^{(1)} = D_{j_1}^{(1)}, \ldots, D_i^{(a)} \subseteq E_i^{(a)} = (D_{j_0}^{(a)})^\perp \). Consequently, Eq. (42) holds true. Instead, if some indices coincide, say \( i_1 = j_1, \ldots, i_s = j_s, \) but \( i_{s+1} \neq j_{s+1}, \ldots, i_a \neq j_a \), we have:

\[
W_I^{(a)} = (D_{i_{s+1}}^{(1)} \cap \ldots \cap D_{i_{a}}^{(a)}) |_{D_{i_1}^{(1)} \cap \ldots \cap D_{i_s}^{(a)}}, \quad W_J^{(a)} = (D_{j_{s+1}}^{(1)} \cap \ldots \cap D_{j_{a}}^{(a)}) |_{D_{i_1}^{(1)} \cap \ldots \cap D_{i_s}^{(a)}},
\]

therefore we go back to the previous case.

\[\square\]

3.2. Darboux–Haantjes coordinates for \( \omega, \mathcal{H} \) manifolds. We wish to show that among the families of Haantjes coordinates defined in Section 2.3, there exist sets of coordinates in which the symplectic form \( \omega \) takes specifically the Darboux form. In order to construct such Darboux-Haantjes coordinates, we first state the following

**Lemma 27.** Let \( (M, \omega) \) be a symplectic manifold and let

\[
\{U_i, (x^{1,1}, \ldots, x^{1,2\sigma_1}; \ldots; x^{a,1}, \ldots, x^{a,2\sigma_a}; \ldots; x^{v,1}, \ldots, x^{v,2\sigma_v}) \}
\]

be a local chart in \( M \) in which \( \omega \) takes locally the block-form expression

\[
\omega = \sum_{i,j=1}^{2\sigma_1} \omega_{ij}^{(1)} dx^{1,i} \wedge dx^{1,j} + \ldots + \sum_{i,j=1}^{2\sigma_a} \omega_{ij}^{(a)} dx^{a,i} \wedge dx^{a,j} + \ldots + \sum_{i,j=1}^{2\sigma_v} \omega_{ij}^{(v)} dx^{v,i} \wedge dx^{v,j} .
\]

Then, its components satisfy the equations

\[
\frac{\partial \omega_{ij}^{(a)}}{\partial x^{b,k}} = 0 \quad a \neq b, k = 1, \ldots, 2\sigma_b .
\]
Proof. It is an immediate consequence of the fact that $\omega$ is a closed form. \hfill \square

**Theorem 28.** Let $(M, \omega, \mathcal{H})$ be an Abelian $\omega\mathcal{H}$ manifold of class $m$. Among the sets of Haantjes coordinates for $\mathcal{H}$, there exist local charts in $U \subset M$ with Darboux coordinates

\begin{equation}
(q^1, p_1, \ldots, q^v, p_v),
\end{equation}

where

\begin{equation}
(q^1, p_1) = (q^{1,1}, q^{1,\sigma_1}, p_{1,1}, \ldots, p_{1,\sigma_1}); \ldots; (q^v, p_v) = (q^{v,1}, q^{v,\sigma_v}, p_{v,1}, \ldots, p_{v,\sigma_v}),
\end{equation}

such that $\omega$ takes the Darboux form

\begin{equation}
\omega = \sum_{a=1}^{v} \sum_{i=1}^{\sigma_a} dp_{a,i} \wedge dq^{a,i}.
\end{equation}

Here $\sigma_a = 1/2 \text{ rank}(V_a)$, and $V_a$ ($a = 1, \ldots, v$) are the distributions defined in Eq. (16).

Proof. Let us consider a Haantjes chart for $\mathcal{H}$. It is adapted to the decomposition (15) and has the form (45). In such a chart, each element of the Haantjes algebra takes a block-diagonal form, due to Theorem 11. Besides, the distributions $V_a$ satisfy the properties stated in Proposition 26 as they correspond to the distributions $\mathcal{W}_\alpha$ when $\alpha = m$. Consequently, in each Haantjes chart, the symplectic form $\omega$ takes the block form (46) as

\begin{equation}
\omega \left( \frac{\partial}{\partial x^{a,i}}, \frac{\partial}{\partial x^{b,j}} \right) = 0 \quad a \neq b, \ i = 1, \ldots, 2\sigma_a, \ j = 1, \ldots, 2\sigma_b,
\end{equation}

thanks to Eq. (42). Thus, its components satisfy property (47). Then, over the leaves of every distribution $V_a$ one can find Darboux coordinates for the restriction of the symplectic form, which is still symplectic, due to Eq. (41). Therefore, one can collect such coordinates to obtain a local chart in $M$ like (48), adapted to the decomposition (15). In this chart, the symplectic form $\omega$ takes the Darboux form (49) and each Haantjes operator $K \in \mathcal{H}$ still possesses a block-diagonal form. \hfill \square

**Definition 29.** Given an $\omega\mathcal{H}$ manifold, the local coordinates where all Haantjes operators take simultaneously a block-diagonal form and, at the same time, the symplectic form takes the Darboux form (49) are called Darboux–Haantjes (DH) coordinates.

**Definition 30.** An $\omega\mathcal{H}$ manifold $(M, \omega, \mathcal{H})$ will be said to be semisimple if $\mathcal{H}$ is a semisimple Haantjes algebra.

**Corollary 31.** In a semisimple Abelian $\omega\mathcal{H}$ manifold $(M, \omega, \mathcal{H})$, each $K \in \mathcal{H}$ takes the diagonal form (31) in every set of Darboux–Haantjes coordinates.

Proof. Given a semisimple Abelian Haantjes algebra $\mathcal{H}$, in each set of Haantjes coordinates the operators $K \in \mathcal{H}$ take a diagonal form. Furthermore, due to Theorem 28, among Haantjes coordinates there exist local charts in which the symplectic form takes the Darboux form (49). In such DH coordinates, every $K \in \mathcal{H}$ takes the form (31) as a consequence of the compatibility condition (29). \hfill \square
3.3. Haantjes chains for $\omega \mathcal{H}$ manifolds. The relevance of Haantjes chains in the theory of $\omega \mathcal{H}$ manifolds is due to the following

**Lemma 32.** Let $(M, \omega, \mathcal{H})$ be an Abelian $\omega \mathcal{H}$ manifold. Then the potential functions $H_\alpha$, whose differentials belong to all Haantjes chains generated by a single function $H$, are in involution among each others and with $H$, w.r.t. the Poisson bracket defined by the Poisson operator $P = \Omega^{-1}$.

**Proof.** In fact, we have
\begin{equation}
\{H_\alpha, H_\beta\} = < dH_\alpha, P dH_\beta > = < K_\alpha^T dH, PK_\beta^T dH > = < dH, K_\alpha PK_\beta^T dH >^\text{Prop. 19} = 0,
\end{equation}
for $\alpha, \beta = 1, \ldots, m$. The involution of $H_\alpha$ with $H$ can be proved analogously. $\square$

An interesting problem is to study both Lagrangian eigendistributions and their associated Lagrangian foliations in an $\omega \mathcal{H}$ manifold. To this aim, let us compare the distribution $D_H$, spanned by the vector fields annihilated by the codistribution $D_H^\perp$ defined by Eq. (26), with the distribution, denoted by $D_H^\perp$, of the vector fields symplectically orthogonal to those of $D_H$. It is known that $D_H^\perp = P(D_H^\perp)$. Taking into account Eq. (26) and Proposition 19, it turns out that
\begin{equation}
D_H^\perp = \text{Span}\{K_1 X_H, K_2 X_H, \ldots, K_m X_H\},
\end{equation}
where $X_H = P dH$ is the Hamiltonian vector field with Hamiltonian function $H$. Thus, we deduce the following result.

**Proposition 33.** Let $(M, \omega, \mathcal{H})$ be a $2n$-dimensional Abelian $\omega \mathcal{H}$ manifold, and $H$ be a smooth function on $M$. The relation
\begin{equation}
D_H^\perp \subseteq D_H
\end{equation}
holds, namely, $D_H$ is a coisotropic distribution and $D_H^\perp$ is an isotropic one. Moreover, if
\begin{equation}
\text{rank}(D_H) = n,
\end{equation}
they coincide and form a Lagrangian distribution.

**Proof.** Each vector field belonging to $D_H^\perp$ is annihilated by any 1-form belonging to $D_H^\perp$ as
\begin{equation}
<K_\alpha^T dH, K_\beta X_H> = <dH, K_\alpha K_\beta P dH >^\text{Prop. 19} = 0, \quad \forall K_\alpha, K_\beta \in \mathcal{H}.
\end{equation}
If $\text{rank}(D_H) = n$, we also have $\text{rank}(D_H^\perp) = n = \text{rank}(P(D_H^\perp))$. Therefore, $D_H^\perp = D_H$. $\square$

**Proposition 34.** Let $(M, \omega, \mathcal{H})$ be a $2n$-dimensional Abelian $\omega \mathcal{H}$ manifold of class $m$ and $H$ be a smooth function that generates a Haantjes chain of length $m$. Then, the distribution $D_H$ (rs. $D_H^\perp$) are integrable distributions and have a coisotropic (rs. isotropic) foliation that we denote by $\mathcal{D}_H$ (rs. $\mathcal{D}_H^\perp$). In particular, if $m = n$, $\mathcal{D}_H = \mathcal{D}_H^\perp$ is a Lagrangian foliation.

**Proof.** Let $\{H_1, H_2, \ldots, H_m\}$ be the potential functions of the Haantjes chain generated by $H$. They are in involution, due to Lemma 32; then, they are integral functions of the coisotropic foliation $\mathcal{D}_H$. Consequently, the isotropic distribution $D_H^\perp$ is also integrable. In particular, if $m = n$, then $\text{rank}(D_H) = \text{rank}(D_H^\perp) = n$. Thus, from Eq. (52) it follows that $D_H = D_H^\perp$. $\square$
The following theorem clarifies the compatibility between a \(\omega H\) manifold and a set of functions in involution.

**Theorem 35.** Let \((M, \omega, H)\) be a \(2n\)-dimensional \(\omega H\) Abelian manifold of class \(m\). Let \(\{H_1, H_2, \ldots, H_m\}\) be a set of independent functions in involution and \(D^\circ\) denote the codistribution spanned by their differentials. The functions \(\{H_1, H_2, \ldots, H_m\}\) form a Haantjes chain, generated by a smooth function \(H\) in involution with them, if and only if \(H\) satisfies the condition
\[
D_H^\circ = D^\circ .
\]

**Proof.** Condition (53) is equivalent to require that \(D_H^\circ \subseteq D^\circ\) as they both have, by assumption, the same rank \(m\). Thus, if such an inclusion is satisfied, \(D_H^\circ\) is integrable and its foliation is a coisotropic foliation due to Proposition 33. Moreover, by virtue of Theorem 16, it follows that the function \(H\) generates the Haantjes chain given by \(\{dH_1, dH_2, \ldots, dH_m\}\).

Conversely, if \(\{H_1, H_2, \ldots, H_m\}\) are the potential functions of a Haantjes chain generated by \(H\), as a consequence of Eq. (25) it follows that \(\bar{K}_T^\alpha dH = dH_\alpha \in D^\circ\), for \(\alpha = 1, \ldots, m\). Then, condition (53) is satisfied. \(\square\)

3.4. **Cyclic \(\omega H\) manifolds.** A particular, especially relevant family of Abelian \(\omega H\) manifolds is represented by the class of symplectic manifolds endowed with a cyclic Haantjes algebra of rank \(m\) (see Sec. 2.2.1). This algebra is generated by a single Haantjes operator \(L\), assumed to satisfy the compatibility condition (29).

Taking into account Eq. (14), it is easy to prove that such condition holds true also for each \(K \in \mathcal{L}(L)\). In fact, representing \(K\) as a polynomial field in \(L\), we have
\[
\Omega K = \Omega p_K(x, L) = p_K(x, L^T)\Omega = K^T\Omega .
\]

We shall say that these manifolds are cyclic \(\omega H\) manifolds.

For a cyclic \(\omega H\) manifold one can construct a special class of Haantjes chains. Indeed, in this context, Theorem 16 amounts to say that a function \(H\) generates the Haantjes chain
\[
dH_\alpha = K_T^\alpha dH = p_\alpha(L^T)dH \quad \alpha = 1, \ldots, m
\]
if and only if the codistribution
\[
D_H^\circ = \text{Span}\{dH, L^T dH, \ldots, (L^T)^{m-1} dH\}
\]
is integrable. The chain (54) will be said to be a cyclic Haantjes chain.

An important class of cyclic \(\omega H\) manifolds is represented by \(\omega N\) manifolds [33, 27]. Let \((M, \omega, N)\) be a symplectic-Nijenhuis (or \(\omega N\)) manifold, that is, a manifold endowed with a symplectic form \(\omega\) and a Nijenhuis operator \(N\) that satisfy the following compatibility conditions
\[
\Omega N - N^T\Omega = 0 ,
\]
\[
d(\Omega N) = 0 .
\]
Here
\[
d\Omega(X, Y) = \mathcal{L}_X(\Omega)Y - \mathcal{L}_Y(\Omega)X + d < \Omega X, Y > + \Omega [X, Y] , \quad \forall X, Y \in TM
\]
and \(\mathcal{L}_X\) denotes the Lie derivative of a tensor field with respect to the vector field \(X\).
Example 36. Let us suppose that the Nijenhuis operator $N$ has its minimal polynomial of degree $m$. Then, the $\omega N$ manifold $M$ has a standard $\omega \mathcal{H}$ structure, given by

$$(M, \omega, K_1 = I, K_2 = N, \ldots, K_m = N^{m-1}),$$

with a Haantjes algebra of rank $m \leq \dim(M)$. In fact, each Nijenhuis operator $N$ is also a Haantjes operator; therefore, it generates the cyclic Haantjes algebra $L(N)$. Besides, the algebraic compatibility condition (56) assures that for all Haantjes operators

$$(59) \quad K = p_K(x, N) = \sum_{i=0}^{m-1} a_i(x) N^i,$$

condition iii) of Definition 17 is fulfilled.

Furthermore, the differential condition (57) implies that for all $K$, the following relation

$$(60) \quad d(\Omega, K)(X, Y) = \sum_{i=0}^{m-1} da_i \wedge (\Omega N^i)(X, Y) \quad \forall X, Y \in TM$$

holds.

Note that the cyclic Haantjes chains on $\omega N$ manifolds coincide with the notion of Nijenhuis chains [11] and of generalized Lenard chains, defined in [48, 53]. In particular, if $K_n = N^{n-1}$, the cyclic Haantjes chains coincide with the classical Lenard-Magri chains [27] (see also the analysis of the Benenti systems in terms of Killing-Stäckel in algebras [3]).

We shall now deepen into the relationship between the notion of cyclic Haantjes algebras and the theory of $\omega \mathcal{H}$ manifolds.

**Proposition 37.** Every semisimple $2n$-dimensional Abelian $\omega \mathcal{H}$ manifold $(M, \omega, \mathcal{H})$ of class $m$ is a cyclic $\omega \mathcal{H}$ manifold. Moreover, each generator of $\mathcal{H}$ that belongs to $\mathcal{H}$ must have $h$ distinct eigenvalues, with $h = m$ if $I \in \mathcal{H}$, or $h = m + 1$ otherwise. In particular, if $\mathcal{H}$ has rank $m = n$, an operator $K \in \mathcal{H}$ is a generator of $\mathcal{H}$ if and only if it is maximal. In this case, $I \in \mathcal{H}$.

Finally, if a $2n$-dimensional $\omega \mathcal{H}$ manifold of class $n$ is non-semisimple, then none of its generators can be maximal.

**Proof.** The first and the second statement are a direct consequence of Proposition 13. In particular, if $h = n$, the cyclic algebra $L(K)$ generated by a maximal operator $K \in \mathcal{H}$ has rank $n$; therefore, it coincides with $\mathcal{H}$; thus, $I \in \mathcal{H}$. Conversely, every generator of $\mathcal{H}$, being semisimple and having its minimum polynomial of degree $n$ (by virtue of the second statement), is maximal.

Finally, we observe that generators of non-semisimple Haantjes algebras cannot be maximal, since maximal operators, due to Lemma 22, are semisimple. \[\square\]

The following Proposition, which specializes the results of Proposition 13 and Corollary 14 to the case of a semisimple $2n$-dimensional Abelian $\omega \mathcal{H}$ manifold of class $m$, presents an explicit construction of a generator of $\mathcal{H}$; in particular, this generator can be chosen to be a Nijenhuis operator.

**Proposition 38.** Let $(M, \omega, \mathcal{H})$ be an Abelian $2n$-dimensional semisimple $\omega \mathcal{H}$ manifold of class $m$. Let us consider the spectral decomposition (15) and a Darboux-Haantjes chart $\{U, (q^{a,j_a}, p_{a,j_a})\}, a = 1, \ldots, v, j_a = 1, \ldots, \sigma_a = \frac{1}{2} \text{rank}(V_a)$, adapted
to the decomposition (15), namely

$$\mathcal{V}_a = \text{Span} \left\{ \frac{\partial}{\partial q^{a,j}_a}, \frac{\partial}{\partial p_{a,j}_a} \right\}.$$  

Then, if \( m \leq v \leq n \), each operator defined by

$$L = \sum_{a=1}^{v} \lambda_a(q, p) \sum_{j_a=1}^{\sigma_a} \left( \frac{\partial}{\partial q^{a,j}_a} \otimes dq^{a,j}_a + \frac{\partial}{\partial p_{a,j}_a} \otimes dp_{a,j}_a \right)$$

is a generator of \( \mathcal{H} \), provided that \( \{ \lambda_1(q, p), \ldots, \lambda_v(q, p) \} \) are arbitrary, pointwise distinct smooth functions. Therefore, every operator \( K \in \mathcal{H} \) can be written in the form

$$K = \sum_{i=1}^{m} l_i \frac{\Pi_{j \neq i} (L - \lambda_j I)}{\Pi_{j \neq i} (\lambda_i - \lambda_j)},$$

where \( l_i = l_i(q, p) \) are the eigenvalue fields of \( K \). In particular, if

$$\lambda_a(q, p) = \lambda_a(q^{a,1}_a, p_{a,1}, \ldots, q^{a,\sigma_a}_a, p_{a,\sigma_a}) \quad a = 1, \ldots, v,$$

the generator \( L \) is a Nijenhuis operator. This operator endows the manifold \( M \) with a standard \( N \) structure, since it satisfies both conditions (56) and (57), as it can be proved by a direct calculation.

If \( v < m \leq n \), by means of a further decomposition of \( \mathcal{V}_a \) we can re-obtain the case \( m \leq v \).

Finally, a generator \( L \) is maximal if and only if \( m = v = n \). In this case, \( I \in \mathcal{H} \).

**Proof.** The inequality \( v > n \) cannot hold in an \( \omega \mathcal{H} \) manifold of class \( n \) because, due to Proposition 26, the rank of each distribution \( \mathcal{V}_a \) cannot be less than 2. Therefore, we have that \( v \leq n \). Let us consider a Darboux-Haantjes chart of the form (61), adapted to the decomposition (15) (whose existence is guaranteed by Theorem 28). In this chart, if \( v \geq m \), the generator (22) takes the form (62) and it can be a Nijenhuis operator, providing that its eigenvalues fields are chosen of the form (64).

When \( v < m \), a generator can still be constructed since, as in Proposition 13, we can further decompose each distribution \( \mathcal{V}_a \) into a direct sum of 2-dimensional sub-distributions

$$\mathcal{V}_a = \bigoplus_{i=1}^{\sigma_a} \text{Span} \left\{ \frac{\partial}{\partial q^{a,i}_a}, \frac{\partial}{\partial p_a,i} \right\} \quad a = 1, \ldots, v.$$

\( \square \)

### 4. Complete Integrability and Haantjes Structures

The aim of this Section is to prove the main result of this paper, namely the equivalence between the existence of an \( \omega \mathcal{H} \) structure associated with a Hamiltonian system and its complete integrability in the sense of Liouville and Arnold. In particular, we shall prove formula (1), which relates the Haantjes geometry of a given integrable system with its dynamics. Also, we will show in a specific example how the Haantjes formulation overcomes, for the vector fields under scrutiny, an obstruction to the existence of a classical Lenard chain pointed out by R. Brouzet.

From now on, we will work with the distinguished basis \( \{ \tilde{K}_1, \ldots, \tilde{K}_m \} \), and we will drop off the tilde over \( \tilde{K}_a \) for the sake of simplicity.
4.1. **Haantjes theorem for integrable systems.** We propose a characterization of the notion of integrability in the sense of Liouville–Arnold in terms of $\omega \mathcal{H}$ manifolds.

**Theorem 39** (Liouville-Haantjes). Let $M$ be a $2n$-dimensional Abelian $\omega \mathcal{H}$ manifold of class $n$ and $\{H_1, H_2, \ldots, H_n\}$ be smooth potential functions of a Haantjes chain generated by a function $H$. Then, the foliation generated by these functions is Lagrangian. Consequently, each Hamiltonian system, with Hamiltonian functions $H$ and $H_\alpha$, $1 \leq \alpha \leq n$, is integrable by quadratures.

Conversely, let us consider a completely integrable system with $n$ degrees of freedom, defined by a Hamiltonian $H$ and a set of $n$ integrals of motion $\{H_1, \ldots, H_n\}$, in involution and independent among each other. Let $\{(J_k, \phi_k)\}$, $k = 1, \ldots, n$, denote a set of action-angle variables, with associated frequencies $\nu_k(J) := \frac{\partial H}{\partial J_k}$. If $H$ is non degenerate, that is

\[ \det \left( \frac{\partial \nu_k}{\partial J_i} \right) = \det \left( \frac{\partial^2 H}{\partial J_i \partial J_k} \right) \neq 0, \]

then $M$ admits, in any tubular neighbourhood of an Arnold torus, a semisimple $\omega \mathcal{H}$ structure whose Haantjes algebra is generated by the operators

\[ K_\alpha = \sum_{i=1}^{n} \frac{\nu^{(\alpha)}_i(J)}{\nu_i(J)} \left( \frac{\partial}{\partial J_i} \otimes dJ_i + \frac{\partial}{\partial \phi_i} \otimes d\phi_i \right) \quad \alpha = 1, \ldots, n, \]

where $\nu^{(\alpha)}_i(J)$ are the frequencies of the $(\alpha)$–$n$th linear flow.

**Proof.** To prove the first statement, by virtue of the classical Liouville-Arnold theorem, it is sufficient to note that the functions $H_\alpha$ belonging to a Haantjes chain are in involution w.r.t. the Poisson bracket defined by the symplectic form $\omega$, thanks to Lemma 32.

Let us prove the converse statement. The integrals of motion $\{H_1, \ldots, H_n\}$ are all assumed to be independent smooth functions on an open dense subset of the phase space, in involution among each others and with $H$. Due to the celebrated Arnold theorem [2], the $2n$-dimensional phase space is foliated by leaves whose compact connected components are invariant tori. Also, at least in any tubular neighbourhood of each torus, there exists a set of action-angle (AA) variables $\{(J_i, \phi_i)\}$, such that the symplectic 2-form reads

\[ \omega = \sum_{i=1}^{n} dJ_i \wedge d\phi_i. \]

Owing to condition (65), the set $\{H_1, \ldots, H_n\}$ depends on the action variables only [2]. Then, the functions $H_\alpha$ take the generic form

\[ H_\alpha = H_\alpha(J), \quad \alpha = 1, \ldots, n. \]

With these data, we shall construct a semisimple and Abelian $\omega \mathcal{H}$ structure associated with the given integrable system.

As a basis of the Haantjes algebra we wish to construct, we can take the following diagonal operators in the action-angle coordinates

\[ K_\alpha = \sum_{i=1}^{n} l^{(\alpha)}_i(J) \left( \frac{\partial}{\partial J_i} \otimes dJ_i + \frac{\partial}{\partial \phi_i} \otimes d\phi_i \right) \quad \alpha = 1, \ldots, n, \]
where \( l_1^{(\alpha)} \) are arbitrary smooth functions. They comply with Definition 5 and fulfill the compatibility condition (29).

Moreover, we impose that the integrals of motion \( \{ H_1, H_2, \ldots, H_n \} \) form a Haantjes chain generated by \( H \), i.e.

\[
K_\alpha^T dH = dH_\alpha, \quad \alpha = 1, \ldots, n.
\]

Being \( K_\alpha \) diagonal in the AA variables, such conditions are equivalent to the following system of \( 2n \) algebraic equations in the \( n \) indeterminate functions:

\[
l_1^{(\alpha)} \frac{\partial H}{\partial J_i} = \frac{\partial H_\alpha}{\partial J_i}, \quad i = 1, \ldots, n.
\]

\[
l_1^{(\alpha)} \frac{\partial H}{\partial \phi_i} = \frac{\partial H_\alpha}{\partial \phi_i}, \quad i = 1, \ldots, n.
\]

Obviously, Eqs. (72) are trivially satisfied, so that only Eqs. (71) have to be taken into account. Without loss of generality, we assume that \( \nu_i, i = 1, \ldots, n \), are non-vanishing functions. Then, equations (71) imply that the eigenvalues of the \( \alpha \)-th operator must be the ratio between the frequencies associated to the \( \alpha \)-th integral of motion and to the Hamiltonian, respectively. It is easy to prove that the Haantjes operators so obtained are linearly independent, due to the independence of the integrals of motion. Consequently, the Haantjes algebra involved in the Haantjes chain (70) is generated by the distinguished basis of operators (66).

There is a natural relation between AA variables and DH coordinates in the Haantjes geometry, as clarified by

**Proposition 40.** Any set of AA variables for a completely integrable system is a set of DH coordinates for the \( \omega \mathcal{H} \) manifold given by the symplectic form \( \omega \) and the Haantjes diagonal algebra generated by the operators (66).

**Remark 41.** The Haantjes operators (66) exist without any restriction on the form of the Hamiltonian function \( H \), except for the non-degeneracy condition (65). However, if one wishes to construct a Nijenhuis recursion operator \( N \) for \( H \), i.e. a Nijenhuis operator that, at the same time, provides a classical Lenard chain

\[
dH_\alpha = (N^T)^\alpha dH \quad \alpha = 1, \ldots, n,
\]

and has the natural vector fields \( \left( \frac{\partial}{\partial J_i}, \frac{\partial}{\partial \phi_i} \right) \) as eigenvectors, then the Hamiltonian function must take necessarily the separated form

\[
H(J) = \sum_{k=1}^n H_k(J_k),
\]

where \( H_k(J_k) \) is a smooth function of the single action variable \( J_i \) (see [33], [35]).

**Remark 42.** The non constant eigenvalues \( l_1^{(\alpha)}(x) \) of the Haantjes operators (66) \( K_\alpha \), \( \alpha = 1, 2, \ldots, n \), depending only on action variables, are integrals of motion for the Hamiltonian vector field \( X_H \), i.e. their Lie derivatives along the flow of \( X_H \) vanishes:

\[
\mathcal{L}_{X_H} l_1^{(\alpha)} = 0.
\]
However, this property does not imply that the Haantjes operators are invariant along the flow of $X_H$, as

$$\mathcal{L}_{X_H} K_n = \sum_{i,k=1}^{n} (t^{(\alpha)}_i - t^{(\alpha)}_k) \frac{\partial}{\partial J_k} \frac{\partial}{\partial \phi_i} \otimes dJ_k.$$  

4.2. The analysis of Brouzet. In [7], R. Brouzet studied the existence of a Nijenhuis recursion operator for a completely integrable system, that is a Nijenhuis operator compatible with $\omega$ and fulfilling the requirement

$$\mathcal{L}_{X_H}(N) = 0.$$  

Notice that this requirement is not satisfied by the Haantjes operators (66), according to Eq. (75). Brouzet proved that the existence of a Nijenhuis recursion operator for $X_H$ in a tubular neighbourhood of a Liouville torus implies very strong restrictions on the form of its Hamiltonian function. Accordingly, he presented an example of an integrable system with two degrees of freedom that does not admit a recursion operator compatible with the original symplectic structure. Here we show that such example does admit a simple formulation in the context of the $\omega_\mathcal{H}$ geometry.

In his analysis, Brouzet considered the symplectic manifold $M = \mathbb{R}^2 \times \mathbb{T}^2$, with the action variables $(J_1, J_2) \in \mathbb{R}^2$, the angles $(\phi_1, \phi_2)$ on the bi-dimensional torus $\mathbb{T}^2$, and the Hamiltonian function

$$H = J_1 (1 + J_2^2),$$  

which is not of the form (74) and is non degenerate in the dense open submanifold $M' := \{m \in M : J_2 \neq 0\}$. The corresponding Hamiltonian vector field

$$X_H = (1 + J_2^2) \frac{\partial}{\partial \phi_1} + 2J_1 J_2 \frac{\partial}{\partial \phi_2}$$  

is completely integrable, since any smooth function depending only on the action variables is an integral of motion for it. For instance, let us take

$$H_1 = J_1, \quad H_2 = J_2,$$

which on $M'$ are functionally independent among each others. One can easily verify that the two Hamiltonian functions in involution $\{H_1, H_2\}$ are the potential functions of a Haantjes chain w.r.t. the $\omega_\mathcal{H}$ structure given by the standard symplectic form

$$\omega = dJ_1 \wedge d\phi_1 + dJ_2 \wedge d\phi_2$$

and by the Haantjes operators

$$K_1 = \frac{1}{(1 + J_2^2)} \left( \frac{\partial}{\partial J_1} \otimes dJ_1 + \frac{\partial}{\partial \phi_1} \otimes d\phi_1 \right), \quad K_2 = \frac{1}{2J_1 J_2} \left( \frac{\partial}{\partial J_2} \otimes dJ_2 + \frac{\partial}{\partial \phi_2} \otimes d\phi_2 \right),$$

which are constructed in the open submanifold of $M'$ where $J_1 \neq 0$ according to the formulae (66).

It is interesting to observe that the authors of [24] have by-passed the Brouzet obstruction to the existence of a Nijenhuis recursion operator for the Hamiltonian (76) (and for other examples presented in [8]) by using a different strategy. Their approach consists in looking for a Nijenhuis recursion operator compatible with a
symplectic structure alternative to the original one. Instead, in our theory, the Haantjes operators are compatible with the original symplectic structure.

5. Integrable models, wave equation and Haantjes geometry

Given the equivalence between complete integrability of a Hamiltonian system and the existence of an associated $\omega H$ structure, we can use this equivalence in two ways: to construct integrable models from a given Haantjes geometry (the direct problem) or, conversely, to determine the Haantjes geometry of a given integrable system (the inverse problem). In this section, we will adopt the first point of view, in order to show the flexibility of the Haantjes approach in applicative contexts.

We consider the simplest case of a manifold $M$ of dimension 4 and of a Haantjes algebra of class 2, whose basis is $\{I, K\}$ for a suitable $K$. Indeed, by searching for Haantjes chains w.r.t. such a distinguished basis, we are able to define families of associated integrable models.

Precisely, we will show that by means of the Haantjes geometry, solutions of the two-dimensional wave equation can be used to define new integrable systems.

**Theorem 43.** Let $\xi = \frac{x+y}{\sqrt{2}}, \eta = \frac{x-y}{\sqrt{2}}, p_\xi = \frac{p_x+p_y}{\sqrt{2}}, p_\eta = \frac{p_x-p_y}{\sqrt{2}}$ be characteristic coordinates and momenta in an open set of $M$. The Hamiltonian

$$H = H_1(\xi, \eta, p_\xi, p_\eta) = f(\eta) + g(\xi) + F(p_\eta) + G(p_\xi)$$

where $f, g, F, G$ are arbitrary functions of their arguments, is integrable and admits the first integral of motion

$$H_2(\xi, \eta, p_\xi, p_\eta) = -f(\eta) + g(\xi) - F(p_\eta) + G(p_\xi).$$

**Proof.** Consider the uniform Haantjes operator in Cartesian coordinates and momenta

$$K = \frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial p_x} \otimes dp_y + \frac{\partial}{\partial p_y} \otimes dp_x.$$

We construct the Haantjes chain

$$K^T dH = dH_2.$$

This chain is defined by the differential relations

$$\begin{align*}
\frac{\partial H_1}{\partial p_y} &= \frac{\partial H_2}{\partial p_x}, \\
\frac{\partial H_1}{\partial p_x} &= \frac{\partial H_2}{\partial p_y}, \\
\frac{\partial H_1}{\partial x} &= \frac{\partial H_2}{\partial y}.
\end{align*}$$

These equations can be combined to define the wave equations

$$H_{i,p_xp_x} - H_{i,p_yp_y} = 0, \quad H_{i,xx} - H_{i,yy} = 0, \quad i = 1, 2.$$

Therefore the Hamiltonian functions

$$H_1(x, y, p_x, p_y) = F(p_x-p_y) + G(p_x+p_y) + f(x-y) + g(x+y)$$

and

$$H_2(x, y, p_x, p_y) = -F(p_x-p_y) + G(p_x+p_y) - f(x-y) + g(x+y),$$

where $F, G, f, g$ are arbitrary smooth functions of their arguments, define a completely integrable system, separable in the coordinates $(\xi, \eta, p_\xi, p_\eta).$
Example 44. Choosing the functions $F, G, f, g$ as a power of their arguments, we get the interesting family of models

\begin{align}
H_1 &= (p_x - p_y)^n + (p_x + p_y)^n + (x - y)^m + (x + y)^m, \quad n, m \in \mathbb{N} \quad (89) \\
H_2 &= -(p_x - p_y)^n + (p_x + p_y)^n - (x - y)^m + (x + y)^m. \quad (90)
\end{align}

For $n = 2$, the Hamiltonian function $H_1$ is quadratic in the momenta and corresponds to a class of separable systems that have been discussed in [43] (page 81). In particular, for $n = 2, m = 3$ one obtains the Sawada-Kotera system [1]. For $n > 2$ we have, to the best of our knowledge, a new family of integrable systems.

The direct method outlined in this section can be widely adopted to generate new models from known Haantjes operators. However, an exhaustive analysis of this approach is out of the scopes of the present work.

6. The inverse problem for systems with two degrees of freedom

In this Section, we deal with the inverse problem. More precisely, given a set of independent functions in involution, we will construct by means of them a Haantjes algebra compatible with the symplectic form. In other words, we shall determine a Haantjes algebra for an assigned integrable system, represented in physical variables.

6.1. A general procedure. Let us consider the simplest case of Hamiltonian systems with two degrees of freedom. We propose a general procedure allowing us to determine a Haantjes algebra with identity and of class 2, associated to the Haantjes chain formed by the differentials of the two given integrals of motion. We search for a generator of such Haantjes algebra, that is to say a Haantjes operator $L$ whose minimal polynomial is of degree two (namely, the maximum degree allowed by our assumptions):

\begin{equation}
ml(x, \lambda) := \lambda^2 - c_1(x)\lambda - c_2(x) ,
\end{equation}

where $c_1(x) = \frac{1}{2}\text{Trace}(L)$, $c_2(x) = -\sqrt{\det(L)}$. Let us note that such a requirement does not imply the semisimplicity of $L$ (unless the existence of two real, distinct roots of $ml(x, \lambda)$ is also assumed).

Remark 45. In the case $n = 2$, any non-isotropic (that is $L_1 \neq l(x)I$) Haantjes operator, compatible with the symplectic form, is a generator of the cyclic Haantjes algebra $L(L_1) = \text{Span}\{I, L_1\}$. Besides, any other generator of $L(L_1)$ has the form

\begin{equation}
L_2 = fI + gL_1 ,
\end{equation}

where $f$ and $g$ are arbitrary smooth functions, with $g$ nowhere vanishing. In fact,

\begin{equation}
\det(L_2 - \lambda I) = \det(fI + gL_1 - \lambda I) = g^n \det \left( L_1 - \frac{\lambda - f}{g} I \right) .
\end{equation}

Therefore, the eigenvalues $\lambda_1^{(1)}$ of $L_1$ and $\lambda_1^{(2)}$ of $L_2$ are related by the affine equations

\begin{equation}
\lambda_1^{(2)} = f + g\lambda_1^{(1)} \quad i = 1, 2 ;
\end{equation}

consequently, $\lambda_1^{(1)} = \lambda_1^{(1)} \Leftrightarrow \lambda_1^{(2)} = \lambda_2^{(2)}$. Then, we can conclude that the Haantjes algebra (even if non-semisimple) contains a maximal generator if and only if all of its generators are maximal.
The procedure can be sketched as follows. Given two independent integrals of motion in involution \( \{ H_1 = H, H_2 \} \), we look for an operator \( L \) with the following properties:

i) it is compatible with the symplectic form

\[ L^T \Omega = \Omega L \quad (95) \]

ii) it provides us with a Haantjes chain for the integrals

\[ L^T dH = dH_2 \quad (96) \]

iii) it is a Haantjes operator

\[ \mathcal{H}_L(X, Y) = 0 \quad \forall X, Y \in TM \quad (97) \]

The algebraic compatibility condition \((95)\) is very strong: chosen a set of Darboux coordinates, it allows us to reduce the number of unknown components of the operator \( L \) from 16 to 6. We obtain that it must have the form

\[ L = \begin{bmatrix} l_1^1 & l_1^2 & 0 & l_4^1 \\ l_2^1 & -l_1^4 & l_4^2 & 0 \\ 0 & l_2^2 & l_4^1 & l_4^2 \\ -l_3^2 & 0 & l_2^2 & l_2^2 \end{bmatrix}, \quad (98) \]

where \( l_j^i \) are unknown arbitrary functions on \( M \). Note that the form \((98)\) for \( L \) guarantees that condition \((91)\) is satisfied. The relations \((96)\), still algebraic, provide us with a system of 4 algebraic equations in the 6 unknown functions; it turns out that only 3 equations are independent; thus we are left with 3 unknown functions. The vanishing of the Haantjes torsion \((97)\) of \( L \) provides us with an over-determined system of 24 PDEs of first order, which can be managed with some suitable ansätze. For instance, some homogeneity properties for the components of \( L \) can be assumed.

6.2. On the superintegrable Post-Winternitz system. By means of the procedure described above, we can discuss now the inverse problem for a system which recently has attracted much attention: the Post-Winternitz (PW) system \([44]\). Indeed, it is a maximally superintegrable system \([36]\) with integrals of motion cubic and quartic in the momenta. As a consequence of maximal superintegrability, its bounded orbits are closed and periodic. Thus, as any superintegrable system, it does not fulfill the non degeneracy condition \((65)\), so that Theorem \(39\) cannot be applied. Despite its regularity properties, separation variables for the PW system are not known. Since it does not belong to the classical Stäckel family of Hamiltonian functions quadratic in the momenta, the PW system is certainly not separable by an extended point transformation.

Let us consider a set of canonical coordinates \((x, y, p_x, p_y)\); the Hamiltonian system with Hamiltonian function

\[ H = H_1 = \frac{1}{2}(p_x^2 + p_y^2) + a \frac{x}{y^2}, \quad a \in \mathbb{R}, \quad y \neq 0, \quad (99) \]

admits the two independent integrals of motion

\[ H_2 = 2p_x^3 + 3p_y^2p_x + a \left( 9y^2p_y + 6 \frac{x}{y^3}p_x \right), \quad (100) \]
and
\[
H_3 = p_y^3 - 12ay^2p_xp_y + 4a\frac{x}{y^3}p_y^2 - 2a^2\left(9y^{2/3} - \frac{2x^2}{y^2}\right).
\]

We shall prove that these integrals form the two different Haantjes chains \(\{dH_1, dH_2\}\) and \(\{dH_1, dH_3\}\); each of them is sufficient to guarantee the complete integrability of the PW system.

By performing the extended-point canonical transformation
\[
q_1 = y^\frac{1}{3}, \quad q_2 = \frac{x}{y^\frac{1}{3}}, \quad p_1 = 2\frac{x}{y^3}p_x + 3y^\frac{2}{3}p_y, \quad p_2 = y^\frac{2}{3}p_x,
\]
we reduce the Hamiltonian functions to a rational form; from it we infer the weights of the three components of (96) (still unknown after having imposed the conditions (98)) as a result of the previous approach, we get the \(\mathcal{H}\) manifold \((\omega, I, K_{2}^{(PW)})\) associated with the Haantjes chain \(\{dH, dH_2\}\), where
\[
K_{2}^{(PW)} = 3 \begin{bmatrix}
2p_x & p_y & 0 & 3y
0 & 2p_x & -3y & 0
0 & 0 & 2p_x & 0
0 & 0 & p_y & 2p_x
\end{bmatrix},
\]
which is semisimple or in a block-diagonal form (if it is not). This fact is crucial in order to find separation variables [46], whenever they exist, or, more generally, to study partial separability [47], [9].
Remark 47. The procedure described above can also be applied to the case when an Hamiltonian function is given, but no integrals of motion are known. In this situation, Eq. (96) for the Haantjes chain can be used to construct, in principle, both the integrals of motion (if they exist) and the corresponding Haantjes operators.

7. The stationary reduction of the seventh-order KdV flow revisited

In this section, in order to show the large range of applicability of the theory previously developed, we shall discuss an important example of Hamiltonian integrable system, defined on a six-dimensional symplectic manifold, which is obtained as a stationary reduction of the seventh-order equation of the Korteweg de Vries (KdV) hierarchy.

In [52], a method to obtain the Poisson pencil $P_1 - \lambda P_0$ of the stationary flows of the KdV hierarchy was presented. In [37], this method was applied to construct the stationary reduction of the seventh-order equation of the hierarchy. The restricted Poisson pencil turns out to be a degenerate pencil of co-rank one in a seven-dimensional manifold $M(7)$, being therefore a Gelfand-Zakarevich system [16]. It possesses a polynomial Casimir function of length four, starting with a Casimir of $P_0$ and ending with a Casimir of $P_1$. Then, a Marsden-Ratiu reduction procedure [34], similar to the one used in other cases [52, 39, 40], was performed to each six-dimensional symplectic leaf $S_0$ of the Poisson tensor $P_0$, in order to get rid of the Casimir of $P_0$.

Furthermore, by restricting the polynomial Casimir function to $S_0$, one of the present authors was able to obtain in [37] three Hamiltonian functions in involution which in the Darboux chart $(q_1, q_2, q_3, p_1, p_2, p_3)$ read

\begin{align}
H_1 &= p_1 p_2 + \frac{1}{2} p_3^2 - \frac{5}{8} q_1^4 + \frac{5}{2} q_1^2 q_2 + \frac{1}{2} q_1 q_3^2 - \frac{1}{2} q_2^2, \\
H_2 &= \frac{1}{2} p_1^2 + p_1 p_2 q_1 + p_3 q_1 - p_2 q_2 - p_2 p_3 q_3 - \frac{1}{2} q_1^5 - \frac{1}{4} q_1^3 q_3^2 + \frac{1}{2} q_2 q_3^2 + 2 q_1 q_2^2, \\
H_3 &= \frac{1}{2} p_3 q_1^2 + p_3 q_2 - p_1 p_3 q_3 - p_2 p_3 q_1 q_3 + \frac{1}{2} p_2 q_3^2 + \frac{1}{2} q_1 q_3^2 - q_1 q_2 q_3^2 - \frac{1}{8} q_3^4.
\end{align}

However, as typically happens in the case of Gelfand-Zakarevich systems, the reduced integrable Hamiltonian systems on $S_0$ do not possess a bi-Hamiltonian description but a $\omega N$ one [12]. Nevertheless, they can also be described in the context of our new theory. In fact, we search for a generator $L$ of a cyclic Haantjes algebra $\mathcal{H}$ of rank 3, therefore with the minimal polynomial of degree 3:

\begin{equation}
(106) \quad m_L(x, \lambda) := \lambda^3 - c_1(x) \lambda^2 - c_2(x) \lambda - c_3(x).
\end{equation}

To this aim, we follow a procedure whose first step is analogous to the one performed in two degrees of freedom. We look for an operator $K_2$ that satisfies

\begin{align}
(107) & \quad K_2^T \Omega = \Omega K_2, \\
(108) & \quad K_2^T dH = dH_2, \\
(109) & \quad \mathcal{H} K_2(X, Y) = 0 \quad \forall X, Y \in TM.
\end{align}
Condition (107) allows us to reduce the unknown components of the operator $K_2$ from 36 to 15. Under the simplest ansatz that the remaining elements of $K_2$ are linear in the Darboux coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$, we find the following, unique solution of Eqs. (107), (108) and (109):

\[
K_2 = \begin{bmatrix}
q_1 & 1 & 0 & 0 & 0 & 0 \\
-2q_2 & q_1 & -q_3 & 0 & 0 & 0 \\
-q_3 & 0 & 2q_1 & 0 & 0 & 0 \\
0 & -p_2 & -p_3 & q_1 & -2q_2 & -q_3 \\
p_2 & 0 & 0 & 1 & q_1 & 0 \\
p_3 & 0 & 0 & 0 & -q_3 & 2q_1
\end{bmatrix}.
\]

Since $K_2$ is a maximal semisimple Haantjes operator in the points where the discriminant $\Delta$ of its minimal polynomial

\[
\Delta = -(8q_1^4q_2 + 4q_1^3q_3 + 32q_1^2q_2^2 + 72q_1q_2q_3^2 + 27q_3^4 + 32q_2^2)
\]

is positive, then, by virtue of Proposition 37, it generates a cyclic Haantjes algebra $\mathcal{H}$. Thus, we search for another Haantjes operator $K_3$ such that

\[
K_3 = fI + gL + hL^2, \quad L = K_2,
\]

\[
K_3^T dH = dH_3,
\]

where $f, g, h$ are suitable smooth functions on $M$ to be determined. The unique solution is $K_3 = (q_1^2 + 2q_2)I - 2q_1L + L^2$, therefore

\[
K_3 = \begin{bmatrix}
0 & 0 & -q_3 & 0 & 0 & 0 \\
q_2^2 & 0 & -q_1q_3 & 0 & 0 & 0 \\
-q_3 & -q_3 & q_1^2 + 2q_2 & 0 & 0 & 0 \\
0 & 0 & p_2q_3 - p_3q_1 & 0 & q_3^2 & -q_1q_3 \\
0 & 0 & -p_3 & 0 & 0 & -q_3 \\
-(p_2q_3 - p_3q_1) & p_3 & 0 & -q_3 & -q_1q_3 & q_1^2 + 2q_2
\end{bmatrix}.
\]

Thus, $\{K_1, K_2, K_3\}$ is a distinguished basis of the cyclic Abelian Haantjes algebra of rank 3 generated by $L = K_2$, which provides us with the Haantjes chain $\{dH_1, dH_2, dH_3\}$.

8. Future Perspectives

The extension of the present theory to the case of quantum integrable systems is a nontrivial task. This research line would pave the way to an algebraic interpretation of the notion of Haantjes integrability developed here, in terms of infinite-dimensional commuting operators on a Hilbert separable space of quantum states.

Also, it would be interesting to compare the geometric structures underlying the vision offered here with the intrinsic, purely algebraic structures developed in [23], in the context of nilpotent integrability.

We mention that new $\omega\mathcal{H}$ structures have been recently found in Ref. [19], which is based on an earlier version of this article.

An in-depth analysis of the case of superintegrable systems [51], especially maximally superintegrable ones, has been performed [46]. Along these lines, we also wish to construct a generalization of our approach to the study of the geometry of certain classical systems, as the abovementioned Post-Winternitz model of Section
that do not possess any known system of separation coordinates. We believe that our theory can offer a proper language in which the study of separability can be formulated and carried out.

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