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Runliang Lin, Robert Conte

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LETTER TO THE EDITOR

On a surface isolated by Gambier

Runliang Lin
Department of mathematical sciences, Tsinghua University, Beijing 100084, P.R. China
RLin@mail.tsinghua.edu.cn

Robert Conte
1. Centre de mathématiques et de leurs applications,
École normale supérieure de Cachan, CNRS,
Université Paris-Saclay,
61, avenue du Président Wilson, F–94235 Cachan, France
2. Department of mathematics, The University of Hong Kong,
Pokfulam Road, Hong Kong
Robert.Conte@cea.fr

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We provide a Lax pair for the surfaces of Voss and Guichard, and we show that such particular surfaces considered by Gambier are characterized by a third Painlevé function.

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1. Introduction. Surfaces of Voss and Guichard

Let us first recall two equivalent definitions of these surfaces: a geometric one and an analytic one. Our notation follows the review of Gambier [4].

Geometrically, the surfaces of Voss [11] and Guichard [6] are by definition those which admit a conjugate net made of geodesics. For instance, every minimal surface is such a surface.

Analytically, they can be characterized by their three fundamental quadratic forms \(dF, dF, -dF \cdot dN, dN \cdot dN\), in which \(F(u,v)\) is the current point of the surface and \(N(u,v)\) a unit vector normal to the tangent plane. Choosing the coordinates \((u,v)\) defined by the geodesic conjugate net, these are [4, p. 362]

\[
\begin{align*}
I &= X_u^2 du^2 + 2 \cos(2\omega) X_u Y_u du dv + Y_v^2 dv^2, \\
II &= \sin(2\omega)(X_u du^2 + Y_u dv^2), \\
III &= dr^2 + 2 \cos(2\omega) dv^2 + dv^2,
\end{align*}
\]

with the notation \(X_u = \partial X(u,v)/\partial u\), \(Y_u\), \(2\omega\) is the angle between the two conjugate geodesics. It is remarkable that, among the three
Gauss-Codazzi equations [4, p. 362]

\[
\begin{aligned}
\omega_{uv} - \frac{1}{2} \sin(2\omega) &= 0, \\
X_v - \cos(2\omega)Y_v &= 0, \\
Y_u - \cos(2\omega)X_u &= 0,
\end{aligned}
\]

the first one characterizes the surfaces with a constant total (Gauss) curvature.

2. Their Lax pair

Gambier succeeded in introducing a deformation parameter \( \lambda \), thus upgrading the moving frame equations to a Lax pair, but he did not write this Lax pair explicitly, so let us do it here.

The moving frame equations (Gauss-Weingarten equations) only depend on the coefficients of the first and second fundamental forms, and the spectral parameter is introduced, as in the case of surfaces with a constant mean curvature, by noticing the invariance of the Gauss-Codazzi equations (1.2) under the scaling transformation \((u, v) \rightarrow (\lambda u, \lambda^{-1} v)\). The traceless Lax pair is

\[
\partial_u \psi = L \psi, \quad \partial_v \psi = M \psi,
\]

\[
L = \begin{pmatrix}
\frac{2X_{uu}}{3X_u} + \frac{2X_u}{3X_v} \cotg(2\omega) \omega_u & 0 & Y_u \tan(2\omega) \\
-\frac{2Y_u}{\lambda^2 X_u} \cotg(2\omega) \omega_u & -\frac{X_{uu}}{3X_u} - \frac{4X_u}{3X_v} \cotg(2\omega) \omega_u & 0 \\
-\frac{1}{\lambda^2 Y_u} \cotg(2\omega) & \frac{1}{Y_v} \cotg(2\omega) & -\frac{X_{uu}}{3X_u} + \frac{2X_u}{3X_v} \cotg(2\omega) \omega_u
\end{pmatrix}, \quad (2.2)
\]

\[
M = \begin{pmatrix}
-\frac{Y_{vv}}{3Y_v} - \frac{4Y_v}{3Y_u} \cotg(2\omega) \omega_u & -\frac{2\lambda^2 X_u}{X_v} \cotg(2\omega) \omega_u & 0 \\
0 & \frac{2Y_{vv}}{3Y_v} + \frac{2Y_v}{3Y_u} \cotg(2\omega) \omega_u & X_v \tan(2\omega) \\
\frac{1}{X_u} \cotg(2\omega) & -\frac{\lambda^2}{X_v} \cotg(2\omega) & -\frac{Y_{vv}}{3Y_v} + \frac{2Y_v}{3Y_u} \cotg(2\omega) \omega_u
\end{pmatrix}, \quad (2.3)
\]

with the zero-curvature condition,

\[
[\partial_u - L, \partial_v - M] = \begin{pmatrix}
-F E_1 & E E_1 & -(E G - F^2) E_3 \\
-GE_1 & F E_1 & -(E G - F^2) E_3 \\
GE_2 - FE_3 & -FE_2 + EE_3 & 0
\end{pmatrix} = 0, \quad (2.4)
\]

denoting \( E_j, j = 1, 2, 3 \) the lhs of (1.2), and \( E, F, G \) the coefficients of the first fundamental form,

\[
E = X_u^2, \quad F = X_u Y_v \cos(2\omega), \quad G = Y_v^2.
\]

3. Surfaces applicable on a surface of revolution

Gambier [3, p. 99] investigated surfaces whose first fundamental form I, Eq. (1.1), has coefficients \( X_u, Y_v, \omega \) only depending on the single variable \( x = u + v \). Denoting for shortness \( X_u = \xi, Y_v = \eta, \)
he first obtains

$$dX = \xi du + (\xi + 2C_1)dv, \quad dY = (\eta + 2C_2)du + \eta dv,$$

$$\xi + 2C_1 = \eta \cos(2\omega), \quad \eta + 2C_2 = \xi \cos(2\omega), \quad (3.1)$$

with $C_1, C_2$ two integration constants. After a possible conformal transformation, this defines two reductions of the Gauss-Codazzi equations, either

$$\frac{d^2\omega}{dx^2} = \frac{m^2}{2} \sin(2\omega), \quad m = \text{arbitrary constant}, \quad (3.2)$$

or

$$\frac{d^2\omega}{dx^2} = \frac{e^\epsilon}{2} \sin(2\omega). \quad (3.3)$$

The first reduction (3.2) integrates with elliptic functions and is handled in full detail by Gambier [3, pp. 100–105].

As to the second reduction (3.3), Gambier unexpectedly fails to integrate it. This ordinary differential equation (ODE) belongs to the class of second order first degree ODEs

$$\frac{d^2u}{dx^2} + \sum_{j=0}^{3} A_j(x,u) \left( \frac{du}{dx} \right)^j = 0, \quad (3.4)$$

whose property is to be form-invariant under the group of point transformations

$$(u,x) \rightarrow (U,X) : u = \varphi(X,U), \quad x = \psi(X,U), \quad U = \Phi(x,u), \quad X = \Psi(x,u). \quad (3.5)$$

Roger Liouville [7] enumerated equivalence classes of (3.4) modulo the group (3.5) but, as later pointed out by Babich and Bordag [1], he forgot the important class, to which the ODE (3.3) belongs, when the invariants which he denotes $v_3$ and $w_1$ both vanish.

When $v_3$ and $w_1$ both vanish, the coefficients $A_3, A_2, A_1$ in the class (3.4) can be set to zero by a transformation (3.5), thus defining the five remarkable four-parameter nonautonomous ODEs

$$\frac{d^2U}{dx^2} = \frac{(2\omega)^3}{\pi^2} \sum_{j=0,1,2} \theta_j \varphi'(2\omega U + \omega_j, g_2, g_3),$$

$$\frac{d^2U}{dx^2} = -2\alpha \frac{\cosh U}{\sinh^3 U} - 2\beta \frac{\sinh U}{\cosh^3 U} - 2\gamma e^{2x} \sinh(2U) - \frac{1}{2} \delta e^{4x} \sinh(4U),$$

$$\frac{d^2U}{dx^2} = \frac{1}{2} e^{2x} (\alpha e^{2U} + \beta e^{-2U}) + \frac{1}{2} e^{2x} (\gamma e^{4U} + \delta e^{-4U}),$$

$$\frac{d^2U}{dx^2} = -\alpha U + \frac{\beta}{2U^3} + \gamma \left( \frac{3}{4} U^5 + 2UX^3 + X^2 U \right) + 2\delta (U^3 + UX),$$

$$\frac{d^2U}{dx^2} = \delta (2U^3 + UX) + \gamma (6U^2 + X) + \beta U + \alpha,$$

in which the summation in the first equation runs over the four half-periods $\omega_j$ of the Weierstrass elliptic function $\wp$.

The third one is precisely, up to rescaling, the ODE (3.3) isolated by Gambier, and the main result of Ref. [1] is the existence of a point transformation mapping these five four-parameter ODEs
to the representation of the Painlevé equations chosen by Garnier [5], [2] (i.e. five equations with four parameters, the last one unifying P\(_{\text{II}}\) and P\(_{\text{I}}\),

\[
P_{\text{VI}} : u'' = \frac{1}{2} \left[ \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right] u^2 - \left[ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right] u' + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{u^2} + \gamma \frac{x-1}{(u-1)^2} + \delta \frac{x(x-1)}{(u-x)^2} \right],
\]

\[
P_{\text{V}} : u'' = \left[ \frac{1}{2u} + \frac{1}{u-1} \right] u^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2} \left[ \alpha u + \beta \right] + \gamma \frac{x}{x} + \delta \frac{u(u+1)}{u-1},
\]

\[
P_{\text{III}} : u'' = \left( \frac{1}{2u} - \frac{u'}{x} + \frac{\alpha u^2 + \gamma u^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4u} \right),
\]

\[
P_{\text{IV}} : u'' = \frac{u^2}{2u} + \gamma \left( \frac{3}{2} u^3 + 4xu^2 + 2x^2u \right) + 4\delta (u^2 + xu) - 2\alpha u + \frac{\beta}{u}.
\]

\[
P_{\text{II}}' : u'' = \delta (2u^3 + xu) + \gamma (6u^2 + x) + \beta u + \alpha.
\]

The point transformations which realize this mapping are, respectively,

\[
x = \frac{e^3 - e_1}{e^2 - e_1}, \quad u = \frac{\varphi(2\omega U, g_2, g_3) - e_1}{e^2 - e_1},
\]

\[
x = e^{2x}, \quad u = \coth^2 U,
\]

\[
x = e^{2x}, \quad u = e^x e^{2U},
\]

\[
x = X, \quad u = U^2,
\]

\[
x = X, \quad u = U.
\]

Therefore the mapping between the ODE (3.3) for \(\omega(x)\) and the third Painlevé equation (3.6) for \(u(\xi)\) is either

\[
e^{2i\omega} = 2\alpha e^{-x}u, \quad \xi = -\frac{1}{4\alpha\beta} e^{2\xi}, \quad \alpha \beta \neq 0, \quad \gamma = 0, \quad \delta = 0,
\]

or equivalently

\[
e^{2i\omega} = \gamma e^{-x}u^2, \quad \xi = \sqrt{-\frac{1}{\gamma\delta} e^{2\xi}}, \quad \alpha = 0, \quad \beta = 0, \quad \gamma\delta \neq 0.
\]

As is well known, the third Painlevé equation has three kinds of solutions:

(i) two-parameter transcendental solutions, which is the generic case, and one cannot proceed beyond the description of Gambier [3, pp. 105–106];

(ii) one-parameter Riccati-type solutions, but for our case \(\gamma\delta \neq 0\) this does not happen;

(iii) zero-parameter rational\(^8\) solutions, the only ones being, with the choice (3.7),

\[
u = \left( -\frac{\beta}{\alpha} \right)^{1/2} \xi^{1/2}, \quad \gamma = 0, \quad \delta = 0,
\]

\(^8\)Algebraic solutions of (3.6) [8] are in fact rational solutions for another representative of P\(_{\text{III}}\) in its equivalence class under \((u,x) \rightarrow (g(x)u,f(x))\), with \(f(x) = \sqrt{3}, g(x) = 1\). All algebraic solutions of P\(_n\) equations, \(n = 2, 3, 4, 5\), are similarly rational.
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or equivalently with the choice (3.8),

\[ u = \left( -\frac{\delta}{\gamma} \right)^{1/4} \xi^{1/2}, \quad \alpha = 0, \quad \beta = 0. \]  

(3.10)

However, these rational solutions correspond to \( \sin(2\omega) = 0 \), forbidden because the second fundamental form would vanish. Consequently, all solutions of (3.3) are transcendental.

4. Future developments

The equation (1.2) for constant total curvature surfaces (sine-Gordon equation) possesses many closed form solutions which obey neither (3.2) nor (3.3), for instance the factorized solution [10]

\[ \tan \frac{\omega}{2} = \frac{J_1(u+v)}{J_2(u-v)}, \]  

(4.1)

in which \( J_1 \) and \( J_2 \) are Jacobi elliptic functions, a degeneracy of which is

\[ \tan \frac{\omega}{2} = \frac{\sin k(u+v)}{\sin k(u-v)}, \]  

(4.2)

or the \( N \)-soliton solution [9], which depends on \( 2N \) arbitrary constants. The difficulty to build Voss-Guichard surfaces from such solutions is the integration of the linear system (1.2) for \( X(u,v) \) and \( Y(u,v) \).

Another useful development would be to find a Darboux transformation for the system (1.2).

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