Localisations of half-closed modules and the unbounded Kasparov product

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Abstract

In the context of the Kasparov product in unbounded $KK$-theory, a well-known theorem by Kucerovsky provides sufficient conditions for an unbounded Kasparov module to represent the (internal) Kasparov product of two other unbounded Kasparov modules. In this article, we discuss several improved and generalised variants of Kucerovsky’s theorem. First, we provide a generalisation which relaxes the positivity condition, by replacing the lower bound by a relative lower bound. Second, we also discuss Kucerovsky’s theorem in the context of half-closed modules, which generalise unbounded Kasparov modules to symmetric (rather than self-adjoint) operators. In order to deal with the positivity condition for such non-self-adjoint operators, we introduce a fairly general localisation procedure, which (using a suitable approximate unit) provides a ‘localised representative’ for the $KK$-class of a half-closed module. Using this localisation procedure, we then prove several variants of Kucerovsky’s theorem for half-closed modules. A distinct advantage of the localised approach, also in the special case of self-adjoint operators (i.e., for unbounded Kasparov modules), is that the (global) positivity condition in Kucerovsky’s original theorem is replaced by a (less restrictive) ‘local’ positivity condition, which is closer in spirit to the well-known Connes-Skandalis theorem in the bounded picture of $KK$-theory.

Keywords: Unbounded $KK$-theory; the Kasparov product; symmetric operators.

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1 Introduction

Let $A$, $B$, and $C$ be $C^*$-algebras. Kasparov’s KK-theory [Kas80] provides the abelian group $KK(A,B)$ of homotopy equivalence classes of (bounded) Kasparov $A$-$B$-modules. One of the main features of KK-theory is the existence of an associative bilinear pairing called the Kasparov product:

$$KK(A,B) \times KK(B,C) \rightarrow KK(A,C).$$

However, it is not possible to compute the Kasparov product explicitly in general. An important improvement was made by Connes and Skandalis [CS84], who provided sufficient conditions (the so-called connection condition and positivity condition) which ensure that a certain Kasparov module represents the Kasparov product of two other given Kasparov modules.

It was shown by Baaj and Julg [BJ83] that elements in KK-theory can also be represented by unbounded Kasparov modules. Many examples of elements in KK-theory are constructed from geometric situations and are most naturally represented by unbounded Kasparov modules. For example, the fundamental class in the $K$-homology of a spin manifold is naturally represented by the Dirac operator (viewed as an unbounded operator on the Hilbert space of spinors). A distinct advantage of the unbounded picture is that the Kasparov product is often easier to compute; in fact, under suitable assumptions, the unbounded Kasparov product can be explicitly constructed [Mes14, KL13, BMS16].

However, the unbounded Kasparov modules introduced by Baaj and Julg require the unbounded operators to be self-adjoint. This self-adjointness is rather natural in the case of unital $C^*$-algebras (e.g. for compact manifolds), but imposes a completeness condition in the non-unital case (e.g. for non-compact manifolds) [MR16]. Thus, unfortunately, the Baaj-Julg framework does not include typical examples such as Dirac-type operators on non-compact, incomplete, Riemannian spin manifolds (which in general are only symmetric but not self-adjoint). Nevertheless, it was shown by Baum-Douglas-Taylor [BDT89] that any elliptic symmetric first-order differential operator $D$ on a Riemannian manifold $M$ yields a class $[D] := [F_D]$ in the $K$-homology of $M$, given by the bounded transform $F_D := D(1 + D^*D)^{-\frac{1}{2}}$. Hilsum [Hil10] later provided an abstract definition of half-closed modules, which generalise unbounded Kasparov modules by replacing the self-adjointness condition by a more flexible symmetry condition. Moreover, Hilsum proved that the bounded transform of a half-closed module still yields a well-defined class in KK-theory. Hilsum’s framework encompasses the symmetric elliptic operators of Baum-Douglas-Taylor as well as the vertically elliptic operators on submersions of (possibly incomplete) smooth manifolds studied in [KS18, KS20, Dun20].

An important step in the development of the unbounded Kasparov product was provided by Kucerovsky’s theorem [Kuc97], which translated the Connes-Skandalis result to the unbounded framework of Baaj and Julg. Thus, given two unbounded Kasparov modules $(A, (E_1)_B, D_1)$ and $(B, (E_2)_C, D_2)$, Kucerovsky provided sufficient conditions for another unbounded Kasparov module $(A, E_C, D)$ to represent the (internal) Kasparov product. However, while the positivity condition of Connes-Skandalis is a local condition, the positivity condition of Kucerovsky is a global condition (and is therefore more restrictive). Moreover, Kucerovsky’s theorem only applies to the self-adjoint (complete) case.
In this article, we discuss several improved and generalised variants of Kucerovsky’s theorem. In particular, we will make the following two improvements to the so-called positivity condition:

(I) we replace the lower bound by a *relative* lower bound with respect to the operator representing the Kasparov product;

(II) we ensure that the positivity condition only needs to be checked *locally* (rather than globally), which is closer in spirit to the positivity condition of Connes-Skandalis.

Our ‘local’ approach to the positivity condition in particular also allows us to consider the non-self-adjoint (incomplete) case of half-closed modules. In fact, it was one of our main goals in this article to enhance our understanding of the unbounded Kasparov product of two half-closed modules. The main motivating examples come from vertically elliptic operators on submersions of open manifolds, for which the Kasparov product was already described in [Dun20]. We will show that similar results can be obtained in the context of noncommutative $C^*$-algebras as well. Specifically, we prove several localised versions of Kucerovsky’s theorem for half-closed modules.

We emphasise here that our localised approach can also be of advantage in the self-adjoint (complete) case of unbounded Kasparov modules, since the ‘local’ positivity condition is more flexible and therefore more broadly applicable. Indeed, consider for instance the simple example of the Riemannian submersion of Euclidean spaces $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. Then the vertical operator $D_1 := -i \partial_y + \sin(e^x)$ defines a class in $KK^1(C_0(\mathbb{R}^2), C_0(\mathbb{R}))$, and the operator $D_2 := -i \partial_x$ defines a class in $KK^1(C_0(\mathbb{R}), \mathbb{C})$. Their Kasparov product is the fundamental class in the $K$-homology of $C_0(\mathbb{R}^2)$, represented by the standard Dirac operator on $\mathbb{R}^2$, which can be checked by simply ignoring the bounded perturbation $\sin(e^x)$ in $D_1$.

However, this fact cannot be checked directly (without ignoring the perturbation) by an application of Kucerovsky’s theorem. Indeed, the commutator $[D_1, D_2] = ie^x \cos(e^x)$ is not bounded (nor relatively bounded by $D_1$ and/or $D_2$), and Kucerovsky’s positivity condition fails. This illustrates the problem with the uniformity of the estimate in Kucerovsky’s positivity condition. But, as the commutator $[D_1, D_2]$ is certainly locally bounded, our local positivity condition (see Definition 6.6) is satisfied without needing to change the operator $D_1$.

We point out that Kaad and Van Suijlekom have also proved a variant of Kucerovsky’s theorem for half-closed modules [KS19, Theorem 6.10], which partially addresses item (II). However, in their work, if the left module is essential, then the required assumptions in [KS19, Theorem 6.10] imply that $D_1$ is self-adjoint, so that the left module is in fact an unbounded Kasparov module (see [KS19, Remark 6.5]). In the context of differential operators on Riemannian submersions, to obtain the self-adjointness of $D_1$ one needs to assume for instance (cf. [KS20]) that the submersion is *proper* (which means that the fibres are compact). It was one of the main goals of this article to obtain a version of Kucerovsky’s theorem which includes the case of submersions with non-compact and incomplete fibres as well, so that in particular the results in [Dun20] are recovered as a special case.

Let us summarise the contents and main results of this article. We start Section 2 by recalling some preliminaries on $KK$-theory and the Kasparov product, and then review Hilsum’s framework of half-closed modules in Section 2.2. In Section 2.3 we collect a few useful lemmas regarding regular symmetric operators on Hilbert modules. In Section 2.4...
we discuss the connection condition, which can be dealt with in the non-self-adjoint case in the same way as in the self-adjoint case, without additional difficulty. We will show that the connection condition for half-closed modules implies the connection condition of Connes-Skandalis.

In Section 3, we first restrict our attention to the self-adjoint case of unbounded Kasparov modules, and we show that the lower bound in Kucerovsky’s positivity condition can be replaced by a relative lower bound (as mentioned in (1) above). To motivate this generalisation, let us first compare to the aforementioned Connes-Skandalis Theorem (see Theorem 2.2). Consider three (bounded) Kasparov modules $(\mathcal{A}, (E_1)_B, F_1)$, $(\mathcal{B}, (E_2)_C, F_2)$, and $(\mathcal{A}, E_B, F)$ (where $E = E_1 \hat{\otimes}_B E_2$). Roughly speaking, the positivity condition of Connes-Skandalis requires the (graded) commutator $[F, F_1 \hat{\otimes} 1]$ to be positive modulo compact operators. If we think of this in terms of pseudodifferential operators, this means there exists a positive zeroth-order pseudodifferential operator with the same principal symbol as $[F, F_1 \hat{\otimes} 1]$. On the other hand, Kucerovsky’s positivity condition (roughly speaking) requires the (graded) commutator $[D, D_1 \hat{\otimes} 1]$ to be positive modulo bounded operators (i.e., modulo ‘zeroth-order’ operators). This seems somewhat unnatural, since if we think of this in terms of differential operators, it means that the positivity condition depends not only on the (second-order) principal symbol of $[D, D_1 \hat{\otimes} 1]$, but also on the first-order part of the symbol (even though the KK-classes of $D$ and $D_1$ are determined by their principal symbols alone). A more natural condition would therefore be to require $[D, D_1 \hat{\otimes} 1]$ to be positive modulo ‘first-order’ operators, meaning that there exists a positive second-order differential operator with the same principal symbol as $[D, D_1 \hat{\otimes} 1]$. We will prove in Theorem 3.3 that Kucerovsky’s theorem can indeed be generalised to this more natural positivity condition.

The unbounded Kasparov product has been studied in [LM19] in the context of weakly anti-commuting operators. As a corollary to Theorem 3.3, we obtain here an alternative proof of [LM19, Theorem 7.4].

In the remainder of this article, we consider the symmetric (non-self-adjoint) case of half-closed modules. In Section 4 we describe our localisation procedure for half-closed modules, and provide the construction of a localised representative $\tilde{F}_D$ for a half-closed module $(\mathcal{A}, E_B, D)$. This construction was first given by Higson [Hig89] (see also [HR00, §10.8]) for the case of elliptic symmetric first-order differential operators, and was generalised by the author [Dun20] to the case of vertically elliptic symmetric first-order differential operators on submersions of open manifolds. We will show in Section 4.2 that Higson’s construction of a localised representative can be generalised further to the abstract noncommutative setting of a half-closed module $(\mathcal{A}, E_B, D)$. The construction is based on the assumption that there exists an almost idempotent approximate unit $(u_n)_{n \in \mathbb{N}}$ in the dense $\ast$-subalgebra $\mathcal{A} \subset A$. Such approximate units always exist in any $\sigma$-unital $C^\ast$-algebra $A$; the only additional assumption here is that it must lie in the subalgebra $\mathcal{A}$. Under this assumption, we show how to construct a localised representative for the $KK$-class of a half-closed module.

A detailed treatment of the local version of the positivity condition is given in Section 5. The main technical result consists of showing that our ‘local’ positivity condition for half-closed modules implies that the positivity condition of Connes-Skandalis is satisfied ‘locally’. The proof relies on the following two features of the localised representative. First, it allows us to work ‘locally’ with self-adjoint (rather than only symmetric) operators. Second, each localised term can be rescaled independently (which is crucial in order to obtain a uniform
constant in the positivity condition).

In Section 6, we then finally provide several localised versions of Kucerovsky’s theorem for half-closed modules, which provide sufficient conditions for a half-closed module \((A, E, C, D)\) to represent the (internal) Kasparov product of two half-closed modules \((A, (E_1)_E, D_1)\) and \((B, (E_2)_C, D_2)\). All these versions require the existence of a suitable approximate unit \(\{u_n\} \subset A\) as in Section 4.2. Our first main result (Theorem 6.3) requires (in addition to a domain condition) the following local version of the positivity condition: for each \(n \in \mathbb{N}\) there exists \(c_n \in [0, \infty)\) such that for all \(\psi \in \text{Dom}(Du_n) \cap \text{Ran}(u_n)\) we have

\[
\left\langle u_n(D_1 \otimes 1)\psi \middle| Du_n\psi \right\rangle + \left\langle Du_n\psi \middle| u_n(D_1 \otimes 1)\psi \right\rangle \geq -c_n\left\langle \psi \middle| (1 + (u_nDu_n)^2)\frac{1}{2}\psi \right\rangle. \tag{1.1}
\]

Using the results from Section 5, Eq. (1.1) ensures that the positivity condition of Connes-Skandalis is satisfied ‘locally’. The construction of the localised representative \(\tilde{F}_D\) using a ‘partition of unity’ then allows us to prove that the positivity condition is in fact satisfied globally.

Roughly speaking, condition (1.1) requires that the anti-commutator \([D_1 \otimes 1, u_nDu_n]\) is positive modulo ‘first-order operators’ (where ‘first-order’ is defined relative to \(u_nDu_n\)). However, to obtain a better analogy to the Connes-Skandalis positivity condition, it would be more natural to require \(u_n[D_1 \otimes 1, D]u_n\) to be positive modulo ‘first-order operators’ (where ‘first-order’ should be defined relative to \(D\)). Unless \(D_1\) commutes with \(u_n\) (as in [KS19]), this additional step is non-trivial (indeed, although we know that \([D_1, u_n]\) is bounded, this alone does not guarantee that we may consider \([D_1 \otimes 1, u_n]Du_n\) to be ‘first-order’). In Section 6.1, we will consider two possible sufficient conditions which allow us to make this additional step.

The first sufficient condition assumes, instead of (1.1), a strong local positivity condition, which requires that (for each \(n \in \mathbb{N}\) there exist \(\nu_n \in (0, \infty)\) and \(c_n \in [0, \infty)\) such that for all \(\psi \in \text{Dom}(D)\) we have

\[
\left\langle (D_1 \otimes 1)u_n\psi \middle| Du_n\psi \right\rangle + \left\langle Du_n\psi \middle| (D_1 \otimes 1)u_n\psi \right\rangle \\
\geq \nu_n\left\langle (D_1 \otimes 1)u_n\psi \middle| (D_1 \otimes 1)u_n\psi \right\rangle - c_n\left\langle u_n\psi \middle| (1 + D^*D)\frac{1}{2}u_n\psi \right\rangle.
\]

The second sufficient condition assumes, instead of (1.1), a local positivity condition, which requires simply that (for each \(n \in \mathbb{N}\) there exists \(c_n \in [0, \infty)\) such that for all \(\psi \in \text{Dom}(D)\) we have

\[
\left\langle (D_1 \otimes 1)u_n\psi \middle| Du_n\psi \right\rangle + \left\langle Du_n\psi \middle| (D_1 \otimes 1)u_n\psi \right\rangle \geq -c_n\left\langle u_n\psi \middle| (1 + D^*D)\frac{1}{2}u_n\psi \right\rangle,
\]

along with a ‘differentiability’ condition, which requires that the operator \(u_n[D, u_n]u_{n+2}\) maps \(\text{Dom}(Du_n^2)\) to \(\text{Dom}(D_1 \otimes 1)\).

We note that the latter ‘differentiability’ condition is quite naturally satisfied in the context of first-order differential operators on smooth manifolds, when \(u_n\) are compactly supported smooth functions and \(D\) is elliptic (as in [Dun20]). In Section 6.2 we will show that the strong local positivity condition is in fact fairly natural in the constructive approach to the unbounded Kasparov product.
1.1 Acknowledgements

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1.2 Notation

Let $A$ and $B$ denote $\sigma$-unital $\mathbb{Z}_2$-graded $C^*$-algebras. By an approximate unit for $A$ we will always mean an even, positive, increasing, and contractive approximate unit for the $C^*$-algebra $A$. Let $E$ be a $\mathbb{Z}_2$-graded Hilbert module over $B$ (for an introduction to Hilbert modules and further details, see for instance [Lan95, Bla98]). For $\xi \in E$, we consider the short-hand notation

$$\langle \xi \rangle := \langle \xi | \xi \rangle,$$

where $\langle \cdot | \cdot \rangle$ denotes the $B$-valued inner product on $E$. We denote the set of adjointable operators on $E$ as $\text{End}_B(E)$, and the subset of compact endomorphisms as $\text{End}_B^0(E)$. For any operator $T$ on $E$, we write $\deg T = 0$ if $T$ is even, and $\deg T = 1$ if $T$ is odd. The graded commutator $[\cdot, \cdot]$ is defined (on homogeneous operators) by $[S, T] := ST - (-1)^{\deg S \deg T} TS$.

For any $S, T \in \text{End}_B(E)$ we will write $S \sim T$ if $S - T \in \text{End}_B^0(E)$. Similarly, for self-adjoint $S, T$ we will write $S \succeq T$ if $S - T \sim P$ for some positive $P \in \text{End}_B(E)$; in this case we will say that $S - T$ is positive modulo compact operators.

Given a $*$-homomorphism $A \to \text{End}_B(E)$, an operator $T \in \text{End}_B(E)$ is called locally compact if $aT$ is compact for every $a \in A$.

Given any regular operator $D$, we define the bounded transform $F_D := D(1 + D^*D)^{-\frac{1}{2}}$. The graph inner product of $D$ is given for $\psi \in \text{Dom} D$ by

$$\langle \psi | \psi \rangle_D := \langle \psi | \psi \rangle + \langle D\psi | D\psi \rangle,$$

and the corresponding graph norm is given by $\|\psi\|^2_D := \|\langle \psi | \psi \rangle_D\|$.

2 Preliminaries on $KK$-theory

Kasparov [Kas80] defined the abelian group $KK(A, B)$ as a set of homotopy equivalence classes of Kasparov $A$-$B$-modules. We start by briefly recalling the main definitions; for more details we refer to e.g. [Bla98, §17].

**Definition 2.1.** A (bounded) *Kasparov $A$-$B$-module* $(A, \pi E_B, F)$ is given by a $\mathbb{Z}_2$-graded countably generated right Hilbert $B$-module $E$, a $(\mathbb{Z}_2$-graded) $*$-homomorphism $\pi : A \to \text{End}_B(E)$, and an odd adjointable endomorphism $F \in \text{End}_B(E)$ such that for all $a \in A$:

$$\pi(a)(F - F^*), \quad [F, \pi(a)], \quad \pi(a)(F^2 - 1) \in \text{End}_B^0(E).$$

Two Kasparov $A$-$B$-modules $(A, \pi_0 E_0 B, F_0)$ and $(A, \pi_1 E_1 B, F_1)$ are called unitarily equivalent (denoted with $\simeq$) if there exists an even unitary in $\text{Hom}_B(E_0, E_1)$ intertwining the $\pi_j$ and $F_j$ (for $j = 0, 1$).
A homotopy between \((A,\pi_0 E_0 B_2, F_0)\) and \((A,\pi_1 E_1 B_1, F_1)\) is given by a Kasparov \(A\)-\(C([0,1], B)\)-module \((A, z \widetilde{E}_{C([0,1], B)}, \tilde{F})\) such that (for \(j = 0, 1\))

\[
ev_j(A, z \widetilde{E}_{C([0,1], B)}, \tilde{F}) \simeq (A, \pi_j E_{j B}, F_j).
\]

Here \(\simeq\) denotes unitary equivalence, and \(\ev_1(A, z \widetilde{E}_{C([0,1], B)}, \tilde{F}) := (A, \bar{\pi} \otimes \rho E_B, \tilde{F} \otimes 1)\), where the \(*\)-homomorphism \(\rho_t : C([0,1], B) \to B\) is given by \(\rho_t(b) := b(t)\).

A homotopy \((A, z \widetilde{E}_{C([0,1], B)}, F)\) is called an operator-homotopy if there exists a Hilbert \(B\)-module \(E\) with a representation \(\pi : A \to \text{End}_B(E)\) such that \(\widetilde{E}\) equals the Hilbert \(C([0,1], B)\)-module \(C([0,1], E)\) with the natural representation \(\pi\) of \(A\) on \(C([0,1], E)\) induced from \(\pi\), and if \(\tilde{F}\) is given by a norm-continuous family \(\{F_t\}_{t \in [0,1]}\). A module \((\pi, E, F)\) is called degenerate if \(\pi(a)(F - F^*) = [F, \pi(a)] = \pi(a)(F^2 - 1) = 0\) for all \(a \in A\).

The \(KK\)-theory \(KK(A, B)\) of \(A\) and \(B\) is defined as the set of homotopy equivalence classes of (bounded) Kasparov \(A\)-\(B\)-modules. Since homotopy equivalence respects direct sums, the direct sum of Kasparov \(A\)-\(B\)-modules induces a (commutative and associative) binary operation (‘addition’) on the elements of \(KK(A, B)\) such that \(KK(A, B)\) is in fact an abelian group [Kas80, §4, Theorem 1].

If no confusion arises, we often simply write \((A, E_B, F)\) instead of \((A, \pi E_B, F)\), and its class in \(KK\)-theory is simply denoted by \([F] \in KK(A, B)\).

2.1 The Kasparov product

Let \(A\) be a \((\mathbb{Z}_2\text{-graded})\) separable \(C^\ast\)-algebra, and let \(B\) and \(C\) be \((\mathbb{Z}_2\text{-graded})\) \(\sigma\)-unital \(C^\ast\)-algebras. It was shown by Kasparov [Kas80, §4, Theorem 4] that there exists an associative bilinear pairing, called the (internal) Kasparov product:

\[
KK(A, B) \times KK(B, C) \to KK(A, C).
\]

Given two \(KK\)-classes \([F_1] \in KK(A, B)\) and \([F_2] \in KK(B, C)\), the Kasparov product is denoted by \([F_1] \otimes_B [F_2]\).

An important improvement was provided by Connes and Skandalis [CS84], who gave sufficient conditions which allow to check whether a given Kasparov module represents the Kasparov product. For convenience, let us first introduce some notation. Given a Hilbert \(B\)-module \(E_1\) and a Hilbert \(C\)-module \(E_2\) with a \(*\)-homomorphism \(B \to \text{End}_C(E_2)\), we consider the (graded) internal tensor product \(E := E_1 \otimes_B E_2\). For any \(\psi \in E_1\), we define the operator \(T_\psi : E_2 \to E\) as \(T_\psi \eta = \psi \otimes \eta\) for any \(\eta \in E_2\). The operator \(T_\psi\) is adjointable, and its adjoint \(T_\psi^* : E \to E_2\) is given by \(T_\psi^*(\xi \otimes \eta) = (\psi|\xi) \cdot \eta\). Furthermore, we also introduce the operator \(\tilde{T}_\psi\) on the Hilbert \(C\)-module \(E \oplus E_2\) given by

\[
\tilde{T}_\psi := \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix}.
\]

We cite here a slightly more general version of the theorem by Connes and Skandalis. First, as explained by Kucerovsky [Kuc97, Proposition 5], it suffices to check the connection condition for \(\psi \in \pi_1(A) E_1\) (rather than all \(\psi \in E_1\)). Second, as described in the comments following [Bla98, Definition 18.4.1], the positivity condition in fact only requires a lower bound greater than \(-2\).
Theorem 2.2 ([CS84, Theorem A.3]). Consider two Kasparov modules \((A, \pi_1(E_1)_B, F_1)\) and \((B, \pi_2(E_2)_C, F_2)\), and consider the Hilbert \(C\)-module \(E := E_1 \otimes_B E_2\) and the \(*\)-homomorphism \(\pi := \pi_1 \otimes 1\): \(A \to \text{End}_C(E)\). Suppose that \((A, \pi E_C, F)\) is a Kasparov module such that the following two conditions hold:

**Connection condition:** for any \(\psi \in \pi_1(A) \cdot E_1\), the graded commutator \([F \oplus F_2, \tilde{T}_\psi]\) is compact on \(E \oplus E_2\);

**Positivity condition:** there exists a \(0 \leq \kappa < 2\) such that for all \(a \in A\) we have that \(\pi(a)[F_1 \otimes 1, F]\pi(a^*) + \kappa \pi(aa^*)\) is positive modulo compact operators on \(E\).

Then \((A, \pi E_C, F)\) represents the Kasparov product of \((A, \pi_1 E_1, F_1)\) and \((B, \pi_2 E_2, F_2)\):

\[ [F] = [F_1] \otimes_B [F_2] \in KK(A, C). \]

Moreover, an operator \(F\) with the above properties always exists and is unique up to operator-homotopy.

2.2 Half-closed modules

Let \(A\) and \(B\) be \(\mathbb{Z}_2\)-graded \(\sigma\)-unital \(C^*\)-algebras, and let \(E\) be a \(\mathbb{Z}_2\)-graded countably generated Hilbert \(B\)-module. For any densely defined operator \(\mathcal{D}\) on \(E\), we consider the following subspaces of \(\text{End}_B(E)\):

\[
\text{Lip}(\mathcal{D}) := \{ T \in \text{End}_B(E) : \text{Dom}\mathcal{D}\subset \text{Dom}\mathcal{T}, \text{ and } [\mathcal{D},\mathcal{T}] \text{ is bounded on Dom } \mathcal{D}\},
\]

\[
\text{Lip}^*(\mathcal{D}) := \{ T \in \text{Lip}(\mathcal{D}) : T \cdot \text{Dom}\mathcal{D}^* \subset \text{Dom}\mathcal{D}\}. 
\]

If \(\mathcal{D}^*\) is also densely defined, we note that \(T \in \text{Lip}(\mathcal{D})\) implies \(T^* \in \text{Lip}(\mathcal{D}^*)\), and then \(-[\mathcal{D},\mathcal{T}]^*\) equals the closure of \([\mathcal{D}^*,\mathcal{T}^*]\) (see [Hil10, Lemma 2.1]). Moreover, if \(\mathcal{D}\) and \(\mathcal{T}\) are symmetric, then we have \([\mathcal{D}^*,\mathcal{T}] = [\mathcal{D},\mathcal{T}]\).

**Definition 2.3 ([Hil10, §2]).** A half-closed \(A\)-\(B\)-module \((A, E_B, \mathcal{D})\) is given by a \(\mathbb{Z}_2\)-graded countably generated Hilbert \(B\)-bimodule \(E\), an odd regular symmetric operator \(\mathcal{D}\) on \(E\), a \(*\)-homomorphism \(A \to \text{End}_B(E)\), and a dense \(*\)-subalgebra \(A \subset A\) such that

1. \(A \subset \text{Lip}^*(\mathcal{D})\);
2. \((1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}}\) is locally compact.

If furthermore \(\mathcal{D}\) is self-adjoint, then \((A, E_B, \mathcal{D})\) is called an unbounded Kasparov \(A\)-\(B\)-module.

Unbounded Kasparov modules were first introduced by Baaj and Julg [BJ83], who proved that their bounded transforms yield Kasparov modules. This statement was generalised to half-closed modules by Hilsom.

**Theorem 2.4 ([Hil10, Theorem 3.2]).** Let \((A, E_B, \mathcal{D})\) be a half-closed \(A\)-\(B\)-module, and consider a closed extension \(\mathcal{D} \subset \mathcal{\hat{D}} \subset \mathcal{D}^*\). Then the bounded transform \(F_{\mathcal{\hat{D}}} = \mathcal{\hat{D}}(1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}}\) yields a Kasparov \(A\)-\(B\)-module \((A, E_B, F_{\mathcal{\hat{D}}})\), and its class is independent of the choice of the extension \(\mathcal{\hat{D}}\).

The following theorem by Kucerovsky provides an analogous version of the Connes-Skandalis result (Theorem 2.2) for the Kasparov product of unbounded Kasparov modules. (Note that we have simplified the domain condition using [Kuc97, Lemma 10(i)].)
Theorem 2.5 ([Kuc97, Theorem 13]). Let \((A, \pi_1(E_1)_B, D_1)\) and \((B, \pi_2(E_2)_C, D_2)\) be unbounded Kasparov modules. Suppose that \((A, \pi_1(E_1 \hat{\otimes}_B E_2)_C, D)\) is an unbounded Kasparov module such that:

1. for all \(\psi\) in a dense subspace of \(A \cdot \text{Dom} D_1\), we have \(\tilde{T}_\psi \in \text{Lip}(D \oplus D_2)\).
2. we have the domain inclusion \(\text{Dom} D \subset \text{Dom} D_1 \hat{\otimes} 1\);
3. there exists \(c \in \mathbb{R}\) such that for all \(\psi \in \text{Dom}(D)\) we have

\[
\langle (D_1 \hat{\otimes} 1)\psi | D\psi \rangle + \langle D\psi | (D_1 \hat{\otimes} 1)\psi \rangle \geq c\langle \psi | \psi \rangle.
\]

Then \((A, \pi_1(E_1 \hat{\otimes}_B E_2)_C, D)\) represents the Kasparov product of \((A, \pi_1(E_1)_B, D_1)\) and \((B, \pi_2(E_2)_C, D_2)\).

In Section 3 we will show that the lower bound in the positivity condition (3) can be weakened to a relative bound. Moreover, using the localisation procedure from Section 4, we will provide several variants of the above theorem for half-closed modules in Section 6.

2.3 Regular symmetric operators

Let \(B\) be a \(\mathbb{Z}_2\)-graded \(\sigma\)-unital C*-algebra, and let \(E\) be a \(\mathbb{Z}_2\)-graded countably generated Hilbert \(B\)-module. Throughout this subsection, we consider a regular symmetric operator \(D\) on \(E\) and a positive number \(r \in (0, \infty)\). For \(\lambda \in [0, \infty)\), we introduce the notation

\[ R^r_D(\lambda) := (r^2 + \lambda^2)^{-1/2}. \]

We recall that we have the integral formula

\[ (r^2 + D^*D)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} R^r_D(\lambda) d\lambda, \quad (2.1) \]

where the integral converges in norm. Let us consider the ‘bounded transform’

\[ F^r_D := D(r^2 + D^*D)^{-\frac{1}{2}}. \]

The following lemma is exactly as in [Dun20, Lemma 2.7], except that we have replaced \(1 + D^*D\) by \(r^2 + D^*D\).

Lemma 2.6. For all \(\psi \in E\) we have

\[ \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} DR^r_D(\lambda) \psi d\lambda = F^r_D \psi. \]

Moreover, for any continuous function \(f: \mathbb{R} \to \mathbb{R}\) such that \(f(x^2)(1 + x^2)^{-\frac{1}{2}}\) is bounded, we also have

\[ \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} f(D^*D) R^r_D(\lambda) \psi d\lambda = f(D^*D)(r^2 + D^*D)^{-1/2} \psi. \]
The following lemma was proven on Hilbert spaces in [Les05, Proposition A.1], and it was shown in [LM19, Lemma 7.7] that the argument can be generalised to Hilbert modules.

**Lemma 2.7** (cf. [LM19, Lemma 7.7]). Let $P$ be an invertible regular positive self-adjoint operator on $E$, and let $T$ be a symmetric operator on $E$ with Dom $P \subset$ Dom $T$. If $TP^{-1}$ is bounded, then the densely defined operator $P^{-\frac{1}{2}}TP^{-\frac{1}{2}}$ extends to an adjointable endomorphism on $E$, and $\|P^{-\frac{1}{2}}TP^{-\frac{1}{2}}\| \leq \|TP^{-1}\|$.

**Lemma 2.8.** Let $a = a^* \in \text{Lip}(D)$, and consider $b \in \text{End}_B(E)$ such that $(1 + D^*D)^{-\frac{1}{2}}b$ is compact on $E$. Then the following statements hold:

1. The operators $[\mathcal{D}R_D^r(\lambda), a]b$ and $(1 + \lambda)^{\frac{1}{2}}[R_D^r(\lambda), a]b$ are compact and of order $O(\lambda^{-1})$.
2. The operator $[F_D^r, a]b$ is compact.

**Proof.** The proof of (1) is an abstract generalisation of [Dun20, Lemma 2.9]. We have

$$[R_D^r(\lambda), a]b = [(r^2 + \lambda + D^*D)^{-1}, a]b = -(r^2 + \lambda + D^*D)^{-1}[D^*D, a](r^2 + \lambda + D^*D)^{-1}b.$$ 

We note that, a priori, $[D^*D, a]$ may not be well-defined, since it is not clear if $a$ preserves Dom $D^*$. However, rewriting $[D^*D, a] = D^*[D, a] + (-1)^{\text{deg} a}[D^*, a]D$, we obtain the well-defined expression

$$[R_D^r(\lambda), a]b = -R_D^r(\lambda)^{\frac{1}{2}}[D^*, a](D^*R_D^r(\lambda)^{\frac{1}{2}}) + (-1)^{\text{deg} a}(R_D^r(\lambda)^{\frac{1}{2}}D^*)[D, a]R_D^r(\lambda)^{\frac{1}{2}}b.$$

Since $R_D^r(\lambda)^{\frac{1}{2}}b$ is compact, one sees that the right-hand-side of this expression is compact and of order $O(\lambda^{-\frac{1}{2}})$. Moreover, $D[R_D^r(\lambda), a]b$ is also well-defined, compact, and of order $O(\lambda^{-1})$. Finally, we see that

$$[\mathcal{D}R_D^r(\lambda), a]b = [\mathcal{D}, a]R_D^r(\lambda)b + D[R_D^r(\lambda), a]b$$

is compact and of order $O(\lambda^{-1})$. Thus we have proven (1).

Using Lemma 2.6, we have for any $\psi \in E$ that

$$[F_D^r, a]b\psi = -\frac{1}{\pi} \int_0^\infty \lambda^{-1/2}[\mathcal{D}R_D^r(\lambda), a]b\psi d\lambda.$$ 

By (1), $[\mathcal{D}R_D^r(\lambda), a]b$ is compact and of order $O(\lambda^{-1})$. Hence the above integral converges in norm to a compact operator, which proves (2).}

**Lemma 2.9.** Let $a \in \text{End}_B(E)$ be such that $a(1 + D^*D)^{-\frac{1}{2}}$ is compact, and let $b \in \text{Lip}^*(D)$. Then $ab(1 + D^*D)^{-\frac{1}{2}}$ is compact.

**Proof.** Consider the domain inclusions $\iota$: Dom $D \hookrightarrow E$ and $\tau$: Dom $D^* \hookrightarrow E$. By assumption, $a \circ \iota$ is compact. Furthermore, $b$ maps Dom $D^*$ into Dom $D$, and $b$: Dom $D^* \rightarrow$ Dom $D$ is bounded with respect to the graph norms (indeed, its norm is bounded by $\|b\| + \|D, b\|$). Moreover, by [Hil10, Lemma 2.2 & Remark 2.4], $b$: Dom $D^* \rightarrow$ Dom $D$ is adjointable. Hence $a \circ b = (a \circ \iota) \circ b$: Dom $D^* \rightarrow E$ is the composition of an adjointable and a compact operator, and therefore it is compact. Since $(1 + D^*D)^{-\frac{1}{2}}$ is a bounded map from $E$ to Dom $D^*$, the statement follows. 


2.4 The connection condition

**Definition 2.10.** Consider three half-closed modules \((A, \pi_1(E_1)_B, D_1)\), \((B, \pi_2(E_2)_C, D_2)\), and \((A, \pi E_C, D)\), where \(E := E_1 \hat{\otimes}_B E_2\) and \(\pi = \pi_1 \hat{\otimes} 1\), and suppose that \(\pi_1\) is essential. The **connection condition** requires that for all \(\psi\) in a dense subspace \(\mathcal{E}_1\) of \(\text{Dom}\ D_1\), we have

\[
\tilde{T}_\psi := \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix} \in \text{Lip}(D \oplus D_2).
\]

We can adapt the argument from [Kuc97, Proposition 14] to symmetric operators (see also [KS19, Proposition 6.11]), to obtain the following result. For simplicity, we restrict our attention to the case where \(\pi_1\) is essential (since this is our case of interest in later sections). However, we note that the case of non-essential representations \(\pi_1\) can be dealt with in the same way as in [Kuc97, Proposition 14].

**Proposition 2.11.** The connection condition of **Definition 2.10** implies the connection condition of **Theorem 2.2** for \(F_1 = F_D\) and \(F_2 = F_{D_2}\).

**Proof.** It suffices to consider elements \(a \psi b \in E_1\), for even elements \(a \in A\) and \(b \in B\), and homogeneous \(\psi \in \mathcal{E}_1 \subset \text{Dom} D_1\), where \(\mathcal{E}_1\) is the dense subset from the connection condition. Since \(T_{a \psi b} = a T_\psi b\) and \(T_{a \psi b}^* = b^* T_\psi a^*\), we have

\[
\tilde{T}_{a \psi b} = \begin{pmatrix} 0 & a T_\psi b \\ b^* T_\psi a^* & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b^* \end{pmatrix} \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix} = a \tilde{T}_\psi a^*.
\]

Hence the graded commutator with \(F_D \oplus D_2\) is given by

\[
[F_D \oplus D_2, \tilde{T}_{a \psi b}] = (F_D \oplus D_2, \tilde{a} [\tilde{T}_\psi a^*] + (-1)^{\deg_\psi} \tilde{a} \tilde{T}_\psi [F_D \oplus D_2, \tilde{a}] a^*].
\]

By **Definition 2.10**, we know that \(\tilde{T}_\psi \in \text{Lip}(D \oplus D_2)\), so it follows from Lemma 2.8(2) that the second term is compact. Since the first and third terms are also compact (by **Theorem 2.4**), this proves the statement. \(\square\)

3 Kucerovsky’s theorem revisited

In this section, we consider only the special case of unbounded Kasparov modules (i.e., the case of self-adjoint operators). Thus we consider the following setting.

**Assumption 3.1.** Let \(A\) be a \((\mathbb{Z}_2\text{-graded})\) separable \(C^*\)-algebra, let \(B\) and \(C\) be \((\mathbb{Z}_2\text{-graded})\) \(\sigma\)-unital \(C^*\)-algebras, and let \(A \subset A\) and \(B \subset B\) be dense \(*\)-subalgebras. Consider three unbounded Kasparov modules \((A, \pi_1(E_1)_B, D_1)\), \((B, \pi_2(E_2)_C, D_2)\), and \((A, \pi E_C, D)\), where \(E := E_1 \hat{\otimes}_B E_2\) and \(\pi = \pi_1 \hat{\otimes} 1\).

Our aim in this section is to improve Kucerovsky’s **Theorem 2.5** by weakening the positivity condition, as follows.

**Definition 3.2.** In the setting of **Assumption 3.1**, the **positivity condition** requires that the following assumptions hold:
(1) we have the domain inclusion \( \text{Dom}(D) \subset \text{Dom}(D_1 \otimes 1) \);
(2) there exists \( c \in [0, \infty) \) such that for all \( \psi \in \text{Dom}(D) \) we have
\[
\langle (D_1 \otimes 1)\psi \mid D\psi \rangle + \langle D\psi \mid (D_1 \otimes 1)\psi \rangle \geq -c\langle \psi \mid (1 + D^2)^{\frac{1}{2}}\psi \rangle.
\]

We then obtain the following improvement of Theorem 2.5.

**Theorem 3.3.** In the setting of Assumption 3.1, assume furthermore that the connection condition (Definition 2.10) and the positivity condition (Definition 3.2) are satisfied. Then \((A, E_C, D)\) represents the Kasparov product of \((A, \pi_1(E_1)_B, D_1)\) and \((B, \pi_2(E_2)_C, D_2)\).

Before proving the theorem, we first describe a few consequences. First of all, in the bounded picture, we know that two Kasparov modules \((A, \pi E_B, F)\) and \((A, \pi E_B, F')\) are equivalent if for each \( a \in A \), the operator \( a[F, F']\) is positive modulo compact operators (see [Ko07, Lemma 11] or [Bla98, Proposition 17.2.7]). The following statement gives an unbounded analogue (albeit a ‘global’ analogue, since we no longer ‘localise’ by elements \( a \in A \)), which says, roughly speaking, that two unbounded Kasparov modules \((A, \pi E_B, D)\) and \((A, \pi E_B, D')\) are equivalent if \([D, D']\) is positive modulo ‘first-order operators’ (this generalises [Ko07, Corollary 17]).

**Corollary 3.4.** Let \((A, \pi E_B, D)\) and \((A, \pi E_B, D')\) be unbounded Kasparov modules such that \( \text{Dom}(D) \subset \text{Dom}(D') \), and suppose there exists \( c \in [0, \infty) \) such that for all \( \psi \in \text{Dom}(D) \) we have
\[
\langle D'\psi \mid D\psi \rangle + \langle D\psi \mid D'\psi \rangle \geq -c\langle \psi \mid (1 + D^2)^{\frac{1}{2}}\psi \rangle.
\]
Then the two unbounded Kasparov modules \((A, \pi E_B, D)\) and \((A, \pi E_B, D')\) are homotopy-equivalent, i.e. \([D'] = [D] \in KK(A, B)\).

**Proof.** We claim that \([D]\) equals the internal Kasparov product (over \( B \)) of \([D']\) with \( 1_B \in KK(B, B) \). The class \( 1_B \) is represented by the unbounded Kasparov module \((B, B_2, 0)\). We can identify \( E \otimes_B B \simeq E \), and then for each \( \psi \in E \), the map \( T_{E}\psi : B \to E \) is given by \( b \mapsto \psi b \). Its adjoint \( T_{E}^*: E \to B \) is given by \( \phi \mapsto \langle \psi | \phi \rangle \). For each \( \psi \in \text{Dom} D \), we then see that the operators \( D T_{E}\psi = T_{D}\psi \) and \( T_{E}^* D = T_{D}^* \psi \) are both bounded. Hence the connection condition is satisfied. Since the positivity condition holds by hypothesis, the claim follows from Theorem 3.3.

The following lemma provides a sufficient condition for the positivity condition, given in terms of operators instead of form estimates. This sufficient condition is in particular useful for the construction of the unbounded Kasparov product from weakly anti-commuting operators (see Corollary 3.6).

**Lemma 3.5.** Let \( D \) and \( S \) be odd regular self-adjoint operators on a \( \mathbb{Z}_2 \)-graded Hilbert \( C^* \)-module \( E \), such that \( \text{Dom}(D) \subset \text{Dom}(S) \). Suppose there exists a core \( F \subset \text{Dom}(D) \) such that \( S \cdot F \subset \text{Dom}(D) \) and \( D \cdot F \subset \text{Dom}(S) \), so that \([D, S]\) is well-defined on \( F \). Assume that on \( F \) we have the equality \([D, S] = P + R\), where \( P \) is a densely defined positive symmetric operator on \( F \) (i.e. \( F \subset \text{Dom}(P) \) and \( \langle \psi | P \psi \rangle \geq 0 \) for all \( \psi \in F \)), and \( R \) is a densely defined symmetric operator which is relatively bounded by \( D \) (i.e. \( \text{Dom}(D) \subset \text{Dom}(R) \)). Then for all \( \psi \in \text{Dom}(D) \) we have
\[
\langle S\psi \mid D\psi \rangle + \langle D\psi \mid S\psi \rangle \geq -\|R(1 + D^2)^{\frac{1}{2}}\psi \| \|\psi \| (1 + D^2)^{\frac{1}{2}}\psi \|.
\]
Proof. From Lemma 2.7 we know that \( \| (1 + D^2)^{-\frac{1}{2}} R(1 + D^2)^{-\frac{1}{2}} \| \leq c := \| R(1 + D^2)^{-\frac{1}{2}} \| \), and hence we have for \( \psi \in \text{Dom} \mathcal{D} \) the inequality
\[
\pm \langle \psi | R \psi \rangle = \pm \langle (1 + D^2)^{-\frac{1}{2}} R(1 + D^2)^{-\frac{1}{2}} \psi | (1 + D^2)^{-\frac{1}{2}} R(1 + D^2)^{-\frac{1}{2}} \psi \rangle \leq c \langle \psi | (1 + D^2)^{\frac{1}{2}} \psi \rangle.
\]
Using also the positivity of \( P \), we then find for all \( \psi \in \mathcal{F} \) that
\[
\langle \mathcal{S} \psi | \mathcal{D} \psi \rangle + \langle \mathcal{D} \psi | \mathcal{S} \psi \rangle = \langle \psi | P \psi \rangle + \langle \psi | R \psi \rangle \geq -c \langle \psi | (1 + D^2)^{\frac{1}{2}} \psi \rangle.
\]
For arbitrary \( \psi \in \text{Dom} \mathcal{D} \), we choose a sequence \( \psi_n \in \mathcal{F} \) such that \( \| \psi_n - \psi \|_D \to 0 \) as \( n \to \infty \). Since \( \text{Dom} \mathcal{D} \subset \text{Dom} \mathcal{S} \), we note that we then also have the convergence \( \| \mathcal{S} \psi_n - \mathcal{S} \psi \| \to 0 \). Applying the above inequality to \( \psi_n \) we obtain
\[
\langle \mathcal{S} \psi | \mathcal{D} \psi \rangle + \langle \mathcal{D} \psi | \mathcal{S} \psi \rangle = \lim_{n \to \infty} \langle \mathcal{S} \psi_n | \mathcal{D} \psi_n \rangle + \langle \mathcal{D} \psi_n | \mathcal{S} \psi_n \rangle \geq -c \lim_{n \to \infty} \langle \psi_n | (1 + D^2)^{\frac{1}{2}} \psi_n \rangle = -c \langle \psi | (1 + D^2)^{\frac{1}{2}} \psi \rangle.
\]

Let \( \mathcal{S} \) and \( \mathcal{T} \) be odd regular self-adjoint operators on a \( \mathbb{Z}_2 \)-graded Hilbert \( C \)-module \( E \). We consider the linear subspace
\[
\mathcal{F}(\mathcal{S}, \mathcal{T}) := \text{Dom}(\mathcal{S} \mathcal{T}) \cap \text{Dom}(\mathcal{T} \mathcal{S}) = \{ \psi \in \text{Dom} \mathcal{S} \cap \text{Dom} \mathcal{T} : \mathcal{S} \psi \in \text{Dom} \mathcal{T}, \mathcal{T} \psi \in \text{Dom} \mathcal{S} \}.
\]

Then \( \mathcal{S} \) and \( \mathcal{T} \) are called weakly anti-commuting [LM19, Definition 2.1] if
\begin{itemize}
  \item there is a constant \( C > 0 \) such that for all \( \psi \in \mathcal{F}(\mathcal{S}, \mathcal{T}) \) we have
  \[
  \langle [\mathcal{S}, \mathcal{T}] \psi | [\mathcal{S}, \mathcal{T}] \psi \rangle \leq C \left( \langle \psi | \psi \rangle + \langle \mathcal{S} \psi | \mathcal{S} \psi \rangle + \langle \mathcal{T} \psi | \mathcal{T} \psi \rangle \right);
  \]
  \item there is a core \( \mathcal{E} \subset \text{Dom} \mathcal{T} \) such that \( (\mathcal{S} + \lambda)^{-1}(\mathcal{E}) \subset \mathcal{F}(\mathcal{S}, \mathcal{T}) \) for \( \lambda \in i \mathbb{R}, |\lambda| \geq \lambda_0 > 0 \).
\end{itemize}

We now obtain a different proof of the following result due to Lesch and Mesland.

**Corollary 3.6** ([LM19, Theorem 7.4]). Suppose we are given two unbounded Kasparov modules \( (\mathcal{A}, \pi_1(E_1)_B, D_1) \) and \( (\mathcal{B}, \pi_2(E_2)_C, D_2) \). We write \( E := E_1 \otimes_B E_2, \pi = \pi_1 \otimes 1 \), and \( \mathcal{S} := D_1 \otimes 1 \). Let \( \mathcal{T} \) be an odd regular self-adjoint operator on \( E \), and consider the operator \( \mathcal{D} := \mathcal{S} + \mathcal{T} \) on the domain \( \text{Dom} \mathcal{D} := \text{Dom} \mathcal{S} \cap \text{Dom} \mathcal{T} \). We assume that the following conditions hold:
\begin{enumerate}
  \item for all \( \psi \) in a dense subset of \( \text{Dom} \mathcal{D}_1 \), we have
  \[
  \tilde{T}_\psi := \begin{pmatrix} 0 & \psi \\ T_\psi & 0 \end{pmatrix} \in \text{Lip}(\mathcal{T} \oplus \mathcal{D}_2);
  \]
  \item we have \( \mathcal{A} \subset \text{Lip}(\mathcal{T}) \);
  \item \( \mathcal{S} \) and \( \mathcal{T} \) are weakly anti-commuting.
\end{enumerate}

Then \( (\mathcal{A}, \pi_1(E_1)_C, \mathcal{D}) \) is an unbounded Kasparov module that represents the Kasparov product of \( (\mathcal{A}, \pi_1(E_1)_B, D_1) \) and \( (\mathcal{B}, \pi_2(E_2)_C, D_2) \).
Proof. By (3) and [LM19, Theorem 2.6], \(D\) is regular and self-adjoint. Using (2) we clearly have \(A \subset \text{Lip}(T) \cap \text{Lip}(S) \subset \text{Lip}(D)\). As in the proof of [LM19, Theorem 7.4], we know that \(a(D \pm i)^{-1}\) is compact for any \(a \in A\). Thus \((A, \pi(E)_C, D)\) is indeed an unbounded Kasparov module. For any \(\psi \in \text{Dom}D_1\) we have bounded operators \(ST_\psi = T_{D_1}\psi\) and \(T_\psi^*S = T_{D_1}^*\psi\), which means that \(T_\psi \in \text{Lip}(S \oplus 0)\). Combined with (1) this ensures that the connection condition (Definition 2.10) is satisfied.

We note that \(F := F(S, T)\) is a core for \(D\) [LM19, Theorem 2.6.(4)], that \(S \cdot F \subset \text{Dom}S\) [LM19, Theorem 5.1], and that \([S, T]\) is relatively bounded by \(D\) [LM19, Theorem 2.6.(1)]. Thus Lemma 3.5 applies with \(F = F(S, T)\), \(P = S^2\) and \(R = [S, T]\), and we see that the positivity condition (Definition 3.2) is also satisfied. Hence follows from Theorem 3.3 that \((A, \pi(E)_C, D)\) represents the Kasparov product of \((A, \pi_1(E_1)_B, D_1)\) and \((B, \pi_2(E_2)_C, D_2)\).

\(\square\)

Remark 3.7. Under suitable assumptions [Mes14, KL13, BMS16], one can construct an operator \(T\) of the form

\[T = 1 \otimes_\nabla D_2,\]

where \(\nabla\) is a suitable ‘connection’ on \(E_1\), and prove that \(T\) satisfies the conditions of Corollary 3.6. In many geometric examples, the connection \(\nabla\) is naturally determined by the given geometry [BMS16, KS18, KS20, Dun20]. In fact, such an operator \(T\) always exists, if one is willing to allow \((A, \pi_1(E_1)_B, D_1)\) and \((B, \pi_2(E_2)_C, D_2)\) to be replaced by homotopy-equivalent modules [MR16].

### 3.1 Proof of Theorem 3.3

Before we proceed with the proof, let us first introduce some notation.

**Notation 3.8.** Let \(D\) and \(S\) be regular self-adjoint operators on a Hilbert \(C\)-module \(E\), such that \(\text{Dom}D \subset \text{Dom}S\). We consider the quadratic form \(Q\) defined for \(\psi \in \text{Dom}D\) by

\[Q(\psi) := 2 \text{Re}(D\psi|S\psi) = \langle D\psi|S\psi \rangle + \langle S\psi|D\psi \rangle.\]

For \(\lambda, \mu \in [0, \infty)\), we use the notation

\[R_D(\lambda) := (1 + \lambda + D^2)^{-1}, \quad R_S(\mu) := (1 + \mu + S^2)^{-1}.\]

We introduce the following bounded operators:

\[k_D(\lambda) := \sqrt{1 + \lambda R_D(\lambda)}, \quad k_S(\mu) := \sqrt{1 + \mu R_S(\mu)},\]

\[h_D(\lambda) := DR_D(\lambda), \quad h_S(\mu) := SR_S(\mu).\]

Furthermore, we define

\[M_1(\lambda, \mu) := h_D(\lambda)h_S(\mu), \quad M_2(\lambda, \mu) := k_D(\lambda)h_S(\mu),\]

\[M_3(\lambda, \mu) := h_D(\lambda)k_S(\mu), \quad M_4(\lambda, \mu) := k_D(\lambda)k_S(\mu).\]

**Lemma 3.9.** For \(\psi \in E\), we have the inequality

\[\sum_{m=1}^4 \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\mu\lambda)^{-\frac{1}{2}} \langle M_m(\lambda, \mu)\psi | (1 + D^2)^{\frac{1}{2}} M_m(\lambda, \mu)\psi \rangle d\lambda d\mu \leq \langle \psi | \psi \rangle.\]
Proof. Let us consider the four integrals (for \( m = 1, 2, 3, 4 \)) given by

\[
I_m := \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\mu \lambda)^{-\frac{1}{2}} \langle M_m(\lambda, \mu) \psi \mid (1 + D^2)^{\frac{1}{2}} M_m(\lambda, \mu) \psi \rangle d\lambda d\mu.
\]

We note that \( h_D(\lambda)^2 + k_D(\lambda)^2 = R_D(\lambda) \). Moreover, by Lemma 2.6 we have the strongly convergent integral

\[
\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (1 + D^2)^{\frac{1}{2}} (h_D(\lambda)^2 + k_D(\lambda)^2) d\lambda = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (1 + D^2)^{\frac{1}{2}} R_D(\lambda) d\lambda = 1.
\]

Computing the integrals over \( \lambda \), we thus obtain the equalities

\[
I_1 + I_2 = \frac{1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} \langle h_S(\mu) \psi \mid h_S(\mu) \psi \rangle d\mu,
\]

\[
I_3 + I_4 = \frac{1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} \langle k_S(\mu) \psi \mid k_S(\mu) \psi \rangle d\mu.
\]

Summing up the latter two equalities and computing the remaining norm-convergent integral over \( \mu \), we obtain

\[
\sum_{m=1}^4 I_m = \frac{1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} \langle \psi \mid R_S(\mu) \psi \rangle d\mu = \langle \psi \mid (1 + S^2)^{-\frac{1}{2}} \psi \rangle \leq \langle \psi \mid \psi \rangle.
\]

The following result relates our positivity condition to the positivity condition of Connes-Skandalis (Theorem 2.2).

**Proposition 3.10.** Let \( D \) and \( S \) be odd regular self-adjoint operators on a \( \mathbb{Z}_2 \)-graded Hilbert \( C \)-module \( E \), such that \( \text{Dom} D \subset \text{Dom} S \). Suppose there exists a constant \( c \in [0, \infty) \) such that for all \( \psi \in \text{Dom} D \) we have

\[
\langle S\psi \mid D\psi \rangle + \langle D\psi \mid S\psi \rangle \geq -c \langle \psi \mid (1 + D^2)^{-\frac{1}{2}} \psi \rangle.
\]

Then for any \( 0 < \kappa < 2 \) there exists an \( \alpha \in (0, \infty) \) such that \( [F_D, F_{\alpha S}] + \kappa \) is positive:

\[
[F_D, F_{\alpha S}] \geq -\kappa.
\]

**Proof.** Recall that for \( \psi \in E \) we have by Lemma 2.6 that \( F_D\psi = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R_D(\lambda) \psi d\lambda \), and similarly for \( F_S \). Applying Lemma 2.6 twice, we can then rewrite

\[
\langle \psi \mid [F_D, F_S] \psi \rangle = 2 \Re \langle \psi \mid F_D F_S \psi \rangle
\]

\[
= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\mu \lambda)^{-\frac{1}{2}} 2 \Re \langle \psi \mid D R_D(\lambda) S R_S(\mu) \psi \rangle d\lambda d\mu.
\]
By the same computation as in [Kuc97, Lemma 11] (or as in Lemma 5.5 below, taking the special case $v = \phi = \rho = 1$), the integrand on the right-hand-side can be rewritten as

$$2 \text{Re} \langle \psi | D \mathcal{R} \mathcal{D}(\lambda) \mathcal{S} R \mathcal{S}(\mu) \psi \rangle = \sum_{m=1}^{4} Q(M_m(\lambda, \mu) \psi).$$

By Eq. (3.1) we have $Q(\psi) \geq -c(\psi \mid (1 + D^2)^{1/2} \psi)$, and therefore we obtain

$$\langle \psi \mid [F_D, F_S] \psi \rangle \geq -c \sum_{m=1}^{4} \frac{1}{\pi^2} \int_{0}^{\infty} \int_{0}^{\infty} (\mu \lambda)^{-\frac{1}{2}} \langle M_m(\lambda, \mu) \psi \mid (1 + D^2)^{1/2} M_m(\lambda, \mu) \psi \rangle d\lambda d\mu.$$}

Applying the inequality from Lemma 3.9, we conclude that

$$\langle \psi \mid [F_D, F_S] \psi \rangle \geq -c(\psi \mid \psi).$$

Finally, if we replace $\mathcal{S}$ by $\alpha \mathcal{S}$ for some $\alpha > 0$, then we see from Eq. (3.1) that $c$ should be replaced by $\alpha c$. Thus, by choosing $\alpha$ small enough, we can ensure that $\alpha c < \kappa < 2$. \hfill \Box

**Proof of Theorem 3.3.** We can represent $[\mathcal{D}]_1$ by their bounded transforms $F_D$, $F_{\alpha \mathcal{D}_1}$, and $F_{\mathcal{D}_2}$, respectively. The statement of the theorem follows from Theorem 2.2, where the connection condition is satisfied by [Kuc97, Proposition 14] (see also Proposition 2.11 if $\pi_1$ is essential), and (choosing $\alpha$ small enough) the positivity condition is satisfied by Proposition 3.10. \hfill \Box

### 4 Localisations of half-closed modules

#### 4.1 Localisations of unbounded operators

Let $\mathcal{D}$ be a regular symmetric operator on a Hilbert $B$-module $E$. For any $b = b^* \in \text{Lip}^*(\mathcal{D})$, we will consider the **localisation** of $\mathcal{D}$ given by the operator $b \mathcal{D} b$. We recall the following lemma.

**Lemma 4.1** ([KS19, Lemma 3.2]). Let $\mathcal{D}$ be a regular symmetric operator on a Hilbert $B$-module $E$, and let $b = b^* \in \text{Lip}^*(\mathcal{D})$. Then (the closure of) $b \mathcal{D} b$ is regular and self-adjoint, and Dom $\mathcal{D}$ is a core for $b \mathcal{D} b$.

**Lemma 4.2.** Let $(\mathcal{A}, E_B, \mathcal{D})$ be a half-closed module, and consider homogeneous self-adjoint elements $a, b \in \mathcal{A}$ such that $ab = a$. Then $a(b \mathcal{D} b \pm i)^{-1}$ is a compact endomorphism.

**Proof.** Consider the regular self-adjoint operator $	ilde{\mathcal{D}} := \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix}$, and write $a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $b = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Repeatedly using $ab = a$, we see that

$$(\tilde{\mathcal{D}} \pm i) a = (\tilde{\mathcal{D}} \pm i) b = [\tilde{\mathcal{D}}, a] b + a((-1)^{\text{deg} a} \tilde{\mathcal{D}} \pm i) b = [\tilde{\mathcal{D}}, a] b + a((-1)^{\text{deg} a} b \tilde{\mathcal{D}} b \pm i).$$

We multiply from the right by $(b \mathcal{D} b \pm i)^{-1}$ and from the left by

$$(\tilde{\mathcal{D}} \pm i)^{-1} = \begin{pmatrix} \pm i & \mathcal{D}^* \\ \mathcal{D} & \pm i \end{pmatrix}^{-1} = \begin{pmatrix} \mp i (1 + \mathcal{D}^* \mathcal{D})^{-1} (1 + \mathcal{D}^* \mathcal{D})^{-1} \mathcal{D}^* \\ (1 + \mathcal{D}^* \mathcal{D})^{-1} \mathcal{D} \mp i (1 + \mathcal{D}^* \mathcal{D})^{-1} \end{pmatrix}.$$
Using that $[\mathcal{D}^*, a] = [\mathcal{D}, a]$ (more precisely, their closures are equal) and that $b\mathcal{D}^*b = b\mathcal{D}b$ on $\text{Dom} b\mathcal{D}b$, this yields

\[
\begin{pmatrix}
(a(b\mathcal{D}b \pm i)^{-1} 0 & 0 \\
0 & a(b\mathcal{D}b \pm i)^{-1}
\end{pmatrix} = b^2 \begin{pmatrix}
(a(b\mathcal{D}b \pm i)^{-1} 0 & 0 \\
0 & a(b\mathcal{D}b \pm i)^{-1}
\end{pmatrix}
\]

\[
= b^2 \begin{pmatrix}
\mp i(1 + \mathcal{D}^*\mathcal{D})^{-1} & (1 + \mathcal{D}^*\mathcal{D})^{-1} \mathcal{D}^* \\
\mp i(1 + \mathcal{D}^*\mathcal{D})^{-1} & \mp i(1 + \mathcal{D}^*\mathcal{D})^{-1}
\end{pmatrix} \times
\begin{pmatrix}
[\mathcal{D}, a]b & 0 \\
0 & 1
\end{pmatrix} + (-1)^{\deg a} a \begin{pmatrix}
\pm i & (-1)^{\deg a} b \mathcal{D}b \\
-\pm i & (-1)^{\deg a} b \mathcal{D}b
\end{pmatrix} (b\mathcal{D}b \pm i)^{-1}.
\]

We note that $b^2 (1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}}$ is compact, and by Lemma 2.9 also $b^2 (1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}}$ is compact. Since

\[
\begin{pmatrix}
\pm i & (-1)^{\deg a} b \mathcal{D}b \\
-\pm i & (-1)^{\deg a} b \mathcal{D}b
\end{pmatrix} (b\mathcal{D}b \pm i)^{-1}
\]

is bounded, it follows that $a(b\mathcal{D}b \pm i)^{-1}$ is compact. \hfill \Box

**Lemma 4.3.** Let $(A, E_B, \mathcal{D})$ be a half-closed module. Consider homogeneous self-adjoint elements $a \in A$ and $b, c \in A$ such that $b, c$ are even, $ac = a$, and $cb = c$. Let $\mathcal{D}_b := b\mathcal{D}b$. Then $a(F_D - F_{\mathcal{D}_b})$ is compact on $E$.

**Proof.** The proof closely follows the argument of [Dun20, Lemma 2.10] (which in turn was inspired by [Hil10, Lemma 3.1]). The main difference here is that we have to take care of the fact that the operators $c(\mathcal{D}_b - \mathcal{D})$ and $c(\mathcal{D}_b - \mathcal{D}^*)$ do not vanish (see below).

Since $a(F_D - F_{\mathcal{D}_b})$ is compact by Theorem 2.4, it suffices to show that $a(F_D^* - F_{\mathcal{D}_b})$ is compact. We can rewrite

\[
a(F_D^* - F_{\mathcal{D}_b}) = a \begin{pmatrix}
(1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}} & \frac{1}{2} - \frac{1}{2} \mathcal{D}_b \\
\frac{1}{2} - \frac{1}{2} \mathcal{D}_b & (1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}}
\end{pmatrix} - a \begin{pmatrix}
\frac{1}{2} - \frac{1}{2} \mathcal{D}_b & (1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}} \\
(1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}} & \frac{1}{2} - \frac{1}{2} \mathcal{D}_b
\end{pmatrix}.
\]

Using Lemma 2.6, we have for any $\psi \in E$ that

\[
a(F_D^* - F_{\mathcal{D}_b})\psi = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} T(\lambda) \psi d\lambda,
\]

where

\[
T(\lambda) := a \begin{pmatrix}
\frac{1}{2} & (1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}} \\
(1 + \mathcal{D}^*\mathcal{D})^{-\frac{1}{2}} & \mathcal{D}_b
\end{pmatrix}.
\]

We claim that $T(\lambda)$ is a compact operator on $E$, and that $\|T(\lambda)\|$ is of order $\mathcal{O}(\lambda^{-1})$ as $\lambda \to \infty$. It then follows that $\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} T(\lambda) d\lambda$ is in fact a norm-convergent integral of compact operators, which proves the statement. To prove the claim, we rewrite

\[
T(\lambda) = a(1 + \mathcal{D}_b^2)^{-1}(1 + \mathcal{D}_b^2) \mathcal{D}^*(1 + \mathcal{D} + \mathcal{D}^*)^{-1} - a(1 + \mathcal{D}_b^2)^{-1} \mathcal{D}_b (1 + \mathcal{D} + \mathcal{D}^*) (1 + \mathcal{D} + \mathcal{D}^*)^{-1}
\]

\[
+ a(1 + \mathcal{D}_b^2)^{-1} \mathcal{D}_b (1 + \mathcal{D} + \mathcal{D}^*) (1 + \mathcal{D} + \mathcal{D}^*)^{-1} - (1 + \lambda) a(1 + \mathcal{D}_b^2)^{-1}(1 + \mathcal{D} + \mathcal{D}^*) (1 + \mathcal{D} + \mathcal{D}^*)^{-1}.
\]
We note that the operators on the last line are still well-defined. For instance, we have $\text{Ran}(D^*(1 + \lambda + DD^*)^{-1}) \subset \text{Dom} D \subset \text{Dom} D_b$, so that $(\mathcal{D}_b - D)D^*(1 + \lambda + DD^*)^{-1}$ is a well-defined bounded operator. We also note that $b \cdot \text{Dom} D^* \subset \text{Dom} D$, so that $\mathcal{D}_b$ is well-defined on $\text{Dom} D^*$. Next, since $cb = c$, we note that $c(\mathcal{D}_b - D) = c(b^2 - 1)D + cb[D, b] = c[D, b]$ and similarly $c(D^* - \mathcal{D}_b) = -c[D^*, b]$. Noting that $[D^*, b] = [D, b]$ and using that $ac = a$, we find that

$$T(\lambda) = a[c, (1 + \lambda + D^2_b)^{-1}D_b](\mathcal{D}_b - D)\sigma(1 + \lambda + DD^*)^{-1}$$

$$+ a(1 + \lambda)[c, (1 + \lambda + D^2_b)^{-1}](\mathcal{D}_b - D)(1 + \lambda + DD^*)^{-1}$$

$$+ a(1 + \lambda + D^2_b)^{-1}D_b c[D, b]D^*(1 + \lambda + DD^*)^{-1}$$

$$- a(1 + \lambda)(1 + \lambda + D^2_b)^{-1}c[D, b](1 + \lambda + DD^*)^{-1}.$$ 

From Lemma 4.2 we know that $c(\mathcal{D}_b \pm i)^{-1}$ and $a(\mathcal{D}_b \pm i)^{-1}$ are compact. In particular, also $a(1 + \lambda + D^2_b)^{-1/2}$ is compact. Furthermore, we may apply Lemma 2.8.(1) to see that $a[c, (1 + \lambda + D^2_b)^{-1}D_b]$ is compact and of order $O(\lambda^{-1})$, and that $a[c, (1 + \lambda + D^2_b)^{-1}]$ is compact and of order $O(\lambda^{-2}).$ Using these facts, we see that $T(\lambda)$ is indeed compact and of order $O(\lambda^{-1}).$

### 4.2 Construction of the localised representative

We show here that the construction of a localised representative for vertical operators on submersions of open manifolds, as described in [Dun20, §2.4], generalises to the abstract (noncommutative) setting of half-closed modules. Recall that a (positive, increasing, contractive) approximate unit $\{u_n\}_{n \in \mathbb{N}}$ is called almost idempotent if $u_{n+1}u_n = u_n$ for all $n \in \mathbb{N}$ [Bla06, Definition II.4.1.1].

**Assumption 4.4.** We consider a half-closed $A$-$B$-module $(A, \pi E_B, D)$ for which the representation $\pi : A \to \text{End}_B(E)$ is essential. We assume that the $*$-subalgebra $A \subset C \subset A$ contains an (even) almost idempotent approximate unit $\{u_n\}_{n \in \mathbb{N}}$ for $A$.

**Remark 4.5.**

1. Since $\pi$ is essential, it follows that $\pi(u_n)$ converges strongly to the identity on $E$ (as $n \to \infty$).
2. We know from [Bla06, Corollary II.4.2.5] that a $\sigma$-unital $C^*$-algebra $A$ always contains an almost idempotent approximate unit $\{u_n\}$. Our main assumption is that we can find such $\{u_n\}$ inside the dense $*$-subalgebra $A$.
3. In the special case where $A$ is unital, we can of course consider $u_n = 1_A$ for all $n \in \mathbb{N}$.

**Definition 4.6.** The ‘partition of unity’ $\{\chi_k\}_{k \in \mathbb{N}}$ corresponding to the approximate unit $\{u_n\}_{n \in \mathbb{N}}$ is defined by

$$\chi_0 := u_0^{1/2}, \quad \chi_k := (u_k - u_{k-1})^{1/2} \quad (k > 0).$$

While $\chi_k^2 = u_k - u_{k-1}$ always lies in $A$ for each $k \in \mathbb{N}$, we note that we do not know if also $\chi_k$ lies in $A$ (we only know $\chi_k \in A$).

**Lemma 4.7.** The following statements hold:

1. $\chi_j \chi_k = 0$ for all $j > k + 1$;
(2) \( u_n \chi_k = \chi_k \) for all \( k < n \);
(3) \( u_n \chi_k = 0 \) for all \( k > n + 1 \).

Proof. (1) Since \( \{ u_n \} \) is almost idempotent, we know that \( u_j u_k = u_k \) for all \( j > k \). Then for \( j > k + 1 \) we have

\[
\chi_j^2 \chi_k^2 = (u_j - u_{j-1})(u_k - u_{k-1}) = (u_j - u_{j-1})u_k - (u_j - u_{j-1})u_{k-1} = u_k - u_k - u_{k-1} + u_{k-1} = 0.
\]

In particular, \( \chi_j^2 \) commutes with \( \chi_k^2 \), and therefore their square roots \( \chi_j \) and \( \chi_k \) also commute and we see that \( \chi_j \chi_k = (\chi_j^2 \chi_k^2)^{\frac{1}{2}} = 0 \).

(2) For \( n > k \) we have

\[
u_n^2 \chi_k^2 = u_n^2(u_k - u_{k-1}) = u_k - u_{k-1} = \chi_k^2.
\]

In particular, \( u_n^2 \) commutes with \( \chi_k^2 \), which implies that \( u_n \) commutes with \( \chi_k \) and \( u_n \chi_k = \chi_k \).

(3) For \( k > n + 1 \) we have

\[
u_n \chi_k = u_n(u_k - u_{k-1}) = u_n - u_n = 0.
\]

In particular, \( u_n \) commutes with \( \chi_k^2 \), which implies that \( u_n^{\frac{1}{2}} \) commutes with \( \chi_k \) and \( u_n \chi_k = u_n^{\frac{1}{2}}(u_n \chi_k)^{\frac{1}{2}} = 0 \).

We pick a sequence of elements \( v_k \in \{ u_n \}_{n \in \mathbb{N}} \) such that \( v_k u_{k+1} = u_{k+1} \) for each \( k \in \mathbb{N} \) (the simplest choice is of course to take \( v_k = u_{k+2} \), but in later sections it will be convenient to choose \( v_k = u_{k+3} \) or \( v_k = u_{k+4} \), so we allow for this additional flexibility). Pick a sequence \( \{ \alpha_k \}_{k \in \mathbb{N}} \subset (0, \infty) \) of strictly positive numbers, and consider the operators

\[
D_k := v_k D v_k, \quad F_{\alpha_k} D_k := \alpha_k D_k (1 + \alpha_k^2 D_k^2)^{-\frac{1}{2}}.
\]

Since \( \{ u_n \}_{n \in \mathbb{N}} \subset A \subset \text{Lip}^s(D) \), we know from Lemma 4.1 that (the closure of) \( D_k \) is regular and self-adjoint, and that \( \text{Dom} D \) is a core for \( D_k \). In particular, the operator \( F_{\alpha_k} D_k \) is well-defined via continuous functional calculus.

Definition 4.8. For any sequence \( \{ \alpha_k \}_{k \in \mathbb{N}} \subset (0, \infty) \) of strictly positive numbers, we define the localised representative of \( D \) as

\[
\tilde{F}_D(\alpha) := \sum_{k=0}^{\infty} \chi_k F_{\alpha_k} D_k \chi_k.
\]

Lemma 4.9. The operator \( \tilde{F}_D(\alpha) \) is well-defined as a strongly convergent series.

Proof. First, since for each \( n \) the sum \( \sum_{k=0}^{\infty} \chi_k u_n \) is finite (see Lemma 4.7), we see that \( \tilde{F}_D(\alpha) u_n \psi \) is a finite (hence convergent) sum for each \( \psi \in E \). Hence \( \tilde{F}_D(\alpha) \) converges.
strongly on the dense subset \( \{ u_n \psi \mid n \in \mathbb{N}, \psi \in E \} \). Second, using the operator inequalities
\[
\pm \sum_{k=0}^{K} \chi_k F_{\alpha_k} D_k \chi_k \leq \sum_{k=0}^{K} \chi_k^2 = u_K \leq 1.
\]
Hence the partial sums are uniformly bounded, and therefore the series converges strongly on all of \( E \).

**Lemma 4.10** ([Dun20, Lemma 2.8]). Let \( D \) be a regular self-adjoint operator on a Hilbert \( B \)-module \( E \). Let \( a \in \text{End}_B(E) \) such that \( a(D \pm i)^{-1} \) is compact. Then for any \( \alpha > 0 \), the operator \( a(F_D - F_{aD}) \) is compact.

**Theorem 4.11.** Consider the setting of Assumption 4.4. Then for any \( a \in A \), the operator \( a(\tilde{F}_D(\alpha) - F_D) \) is compact. Hence \( (A, E_B, \tilde{F}_D(\alpha)) \) is a (bounded) Kasparov \( A \)-\( B \)-module, and \( [\tilde{F}_D(\alpha)] = [F_D] \in \text{KK}(A, B) \). In particular, the class \( [\tilde{F}_D(\alpha)] \) is independent of the choices made in the construction.

**Proof.** Since we have norm-convergence \( au_n \to a \), it suffices to prove the compactness of \( u_n(\tilde{F}_D(\alpha) - F_D) \). Recall that \( \tilde{F}_D(\alpha) = \sum_{k=0}^{\infty} \chi_k F_{\alpha_k} D_k \chi_k \), where \( D_k = v_k D v_k \). By applying Lemma 4.2 (using \( \chi_k = \chi_k u_{k+1} \) from Lemma 4.7) we know that \( \chi_k(D_k \pm i)^{-1} \) is compact; hence we can apply Lemma 4.10 to see that \( \chi_k(F_{\alpha_k} D_k - F_{D_k}) \) is compact. We know from Theorem 2.4 that the commutator \( [F_D, \chi_k] \) is compact. Furthermore, applying Lemma 4.3 (with \( a = \chi_k, c = u_{k+1} \), and \( b = v_k \)) we know that \( \chi_k(F_{D_k} - D) \) is compact. From Lemma 4.7 we know that \( u_n \tilde{F}_D(\alpha) \) is given by a finite sum, and therefore
\[
u_n(\tilde{F}_D(\alpha) - F_D) = \sum_k \chi_k F_{\alpha_k} D_k \chi_k - u_n F_D \stackrel{4.10}{\sim} \sum_k u_n (\chi_k F_{D_k} \chi_k - \chi_k^2 F_D) \stackrel{4.3}{\sim} 0.
\]

## 5 Local positivity

The aim in this section is to show that the local positivity condition (see Definition 6.6 below) implies a *localised* version of the positivity condition in Theorem 2.2. This was already proven by the author for the case of first-order differential operators on smooth manifolds [Dun20, Proposition 3.1], and in fact, many of the arguments of [Dun20, §3] can be adapted to the following more abstract context.

**Assumption 5.1.** Let \( D \) be an odd regular symmetric operator on a \( \mathbb{Z}_2 \)-graded Hilbert \( C \)-module \( E \), let \( S \) be an odd regular self-adjoint operator on \( E \), and let \( \chi, \rho, \phi, v \in \text{End}_C(E) \) be even and self-adjoint. We define the operator \( D_v := vDv \). We assume that the following conditions hold:

1. \( \text{Dom}(Dv) \cap \text{Ran}(v) \subset \text{Dom}(Sv) \);
2. \( \rho \chi = \chi, \phi \rho = \rho, v \rho = \rho, v \phi = \phi v, \| \rho \| \leq 1, \) and \( \| \phi \| \leq 1 \);
3. \( \rho, \phi \in \text{Lip}(D) \cap \text{Lip}(S) \) and \( v \in \text{Lip}^*(D) \cap \text{Lip}(S) \).
(L4) \( \chi(1 + D_v^2)^{-\frac{1}{2}} \) is compact;
(L5) there exists a constant \( c \in [0, \infty) \) such that for all \( \psi \in \text{Dom}(Dv) \cap \text{Ran}(v) \) we have
\[
Q_v(\phi \psi) := \langle Dv \phi \psi \mid vS \phi \psi \rangle + \langle vS \phi \psi \mid Dv \phi \psi \rangle \geq -c \langle \phi \psi \mid (1 + D_v^2)^{\frac{1}{2}} \phi \psi \rangle.
\]

Since \( v \in \text{Lip}^*(D) \), we note that \( D_v \) is regular and self-adjoint (see Lemma 4.1). Since \( \rho, \phi \in \text{Lip} D \) and \( \rho, \phi \) commute with \( v \), we know that we also have \( \rho, \phi \in \text{Lip} D_v \). Furthermore, since \( \phi \) preserves \( \text{Dom}(Dv) \cap \text{Ran}(v) \) and we have \( \text{Dom}(Dv) \cap \text{Ran}(v) \subseteq \text{Dom}(Sv) = \text{Dom}(vS) \), we see that \( Q_v \) is well-defined.

**Notation 5.2.** For \( \lambda, \mu \in [0, \infty) \) and for \( r \in (0, \infty) \), we use the notation
\[
R^r_{D_v}(\lambda) := (r^2 + \lambda + D_v^2)^{-1}, \quad R_S(\mu) := (1 + \mu + S^2)^{-1}.
\]

We consider the following bounded operators:
\[
k^r_{D_v}(\lambda) := \sqrt{r^2 + \lambda} R^r_{D_v}(\lambda), \quad k_S(\mu) := \sqrt{1 + \mu} R_S(\mu),
\]
\[
h^r_{D_v}(\lambda) := D_v R^r_{D_v}(\lambda), \quad h_S(\mu) := S R_S(\mu).
\]

We also define the operator
\[
B(\lambda) := [S, \rho] \phi h^r_{D_v}(\lambda) \rho.
\]

We now redefine the operators \( M_m(\lambda, \mu) \), using \( D_v \) instead of \( D \), and inserting \( \rho \):
\[
M_1(\lambda, \mu) := h^r_{D_v}(\lambda) \rho k_S(\mu), \quad M_2(\lambda, \mu) := k^r_{D_v}(\lambda) \rho k_S(\mu),
\]
\[
M_3(\lambda, \mu) := h^r_{D_v}(\lambda) \rho k_S(\mu), \quad M_4(\lambda, \mu) := k^r_{D_v}(\lambda) \rho k_S(\mu).
\]

Furthermore, we define
\[
\hat{B}(\lambda, \mu) := 2 \Re \left\langle B(\lambda) k_S(\mu) \chi \psi \mid k_S(\mu) \chi \psi \right\rangle + 2 \Re \left\langle B(\lambda) h_S(\mu) \chi \psi \mid h_S(\mu) \chi \psi \right\rangle,
\]
\[
\hat{M}(\lambda, \mu) := \sum_{m=1}^4 2 \Re \left\langle [\phi, D] v M_m(\lambda, \mu) \chi \psi \mid v S \phi M_m(\lambda, \mu) \chi \psi \right\rangle,
\]
\[
\hat{Q}(\lambda, \mu) := \sum_{m=1}^4 Q_v(\phi M_m(\lambda, \mu) \chi \psi).
\]

**Lemma 5.3 ([KS19, Lemma 3.4]).** For any \( r \in \mathbb{R} \) with \( |r| > \|Dv\| \), the operator
\[
ir + Dv^2 : \text{Dom}(Dv^2) \rightarrow E
\]
is bijective, its inverse \( (ir + Dv^2)^{-1} \) is adjointable on \( E \), and we have the equality
\[
(ir + D_v)^{-1} v = v (ir + D_v^2)^{-1}.
\]

**Lemma 5.4** (cf. [KS19, Lemma 6.15]). For any \( r > \|Dv\| \), we have the following inclusions:
\[
\text{Ran} \left( k^r_{D_v}(\lambda)v \right) \subset \text{Dom}(Dv) \cap \text{Ran}(v), \quad \text{Ran} \left( h^r_{D_v}(\lambda)v \right) \subset \text{Dom}(Dv) \cap \text{Ran}(v).
\]
Proof. From Lemma 5.3 we know for any \( r > \|D_vv\| \) that
\[
\text{Ran } ((ir + D_v)^{-1}v) = v \cdot \text{Dom}(Dv^2) = \text{Dom}(Dv) \cap \text{Ran}(v).
\]
The inclusions then follow, because we can rewrite
\[
R_{D_v}^\nu(\lambda)v = (i\sqrt{r^2 + \lambda + D_v})^{-1}v = -i\sqrt{r^2 + \lambda + D_v}^{-1}v - i\sqrt{r^2 + \lambda}R_{D_v}^\nu(\lambda)v.
\]

We will study the positivity (modulo compact operators) of the operator \( \chi[F_{D_v}, F_{\rho}] \chi \). Using \( \rho \chi = \chi \), we can rewrite
\[
\chi[F_{D_v}, F_{\rho}] \chi = \chi(\rho F_D F_{\rho} + F_{\rho}F_D \rho) \chi
\]
\[
= \chi(F_{D_v} \rho F_{\rho} + F_{\rho}F_{D_v} \rho) \chi + \chi([\rho, F_{D_v}]F_{\rho} + F_{\rho} [F_{D_v}, \rho]) \chi.
\]

We note that \( [F_{D_v}, \rho] \chi \) is compact by Lemma 2.8.2 (using \( \rho \in \text{Lip}(D_v) \) and condition (1.4)), and therefore it suffices to consider instead the operator \( \chi(F_{D_v} \rho F_{\rho} + F_{\rho}F_{D_v}) \chi \). Furthermore, consider for any \( 0 \neq r \in \mathbb{R} \) the operator
\[
F_{D_v}^\nu := D_v(r^2 + D_v^2)^{-\frac{1}{2}}.
\]

Since the function \( x \mapsto x(1 + x^2)^{-\frac{1}{2}} - x(r^2 + x^2)^{-\frac{1}{2}} \) lies in \( C_0(\mathbb{R}) \), we know that also \( (F_{D_v} - F_{D_v}^\nu) \chi \) is compact. Hence we may replace \( F_{D_v} \) by \( F_{D_v}^\nu \). Applying Lemma 2.6 twice, we then rewrite
\[
\langle \psi \mid \chi(F_{D_v}^\nu \rho F_{\rho} + F_{\rho}F_{D_v}^\nu \rho) \chi \rangle = 2 \text{Re} \langle \chi \psi \mid F_{D_v}^\nu \rho F_{\rho} \chi \rangle
\]
\[
= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} \text{Re} \langle \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho S R_S(\mu) \chi \rangle d\lambda d\mu.
\]

Our first task is to study the integrand on the right-hand-side. Via a straightforward but somewhat tedious calculation, we will rewrite this integrand in terms of the operators defined above.

Lemma 5.5. For any \( \psi \in E \) we have
\[
2 \text{Re} \langle \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho S R_S(\mu) \chi \rangle = \tilde{B}(\lambda, \mu) + \tilde{M}(\lambda, \mu) + \tilde{Q}(\lambda, \mu).
\]

Proof. We compute:
\[
\langle \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho S R_S(\mu) \chi \rangle
\]
\[
= \langle \rho(1 + \mu + S^2) \rho S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho S R_S(\mu) \chi \rangle
\]
\[
= \langle \rho S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho^2 \rho S k_S(\mu) \chi \psi \rangle + \langle \rho^2 \rho S h_S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho h_S(\mu) \chi \psi \rangle
\]
\[
= \langle \rho S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \phi \rho \rho S \phi S k_S(\mu) \chi \psi \rangle + \langle \rho S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \phi S \rho h_S(\mu) \chi \psi \rangle
\]
\[
+ \langle \phi(\phi \rho, S) h_S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \rho h_S(\mu) \chi \psi \rangle + \langle \phi \rho S h_S(\mu) \chi \psi \mid D_v R_{D_v}^\nu(\lambda) \phi h_S(\mu) \chi \psi \rangle
\]
\[
= \langle B(\lambda) k_S(\mu) \chi \psi \mid k_S(\mu) \chi \psi \rangle + \langle h_S(\mu) \chi \psi \mid B(\lambda) h_S(\mu) \chi \psi \rangle
\]
where we have inserted the definition of $B(\lambda)$. To rewrite the terms on the last line, we consider either $\xi = \rho k_S(\mu)\psi$ or $\xi = \rho h_S(\mu)\chi\psi$. Since $\nu \phi = \rho$, we note that in both cases we have $\xi = \langle \psi \in \text{Ran}(\nu)\rangle$, so from Lemma 5.4 and condition (L1) we know that $k_{D_\nu}(\lambda)\xi$ and $h_{D_\nu}(\lambda)\xi$ lie in $\text{Dom}(\nu S)$. Since $\phi$ preserves $\text{Dom}(\nu S)$, we can compute
\[
\langle \phi D_\nu R_{D_\nu}(\lambda)\xi \mid S\phi \xi \rangle
= \langle \phi D_\nu R_{D_\nu}(\lambda)\xi \mid \nu S\phi (r^2 + \lambda + D_{\phi}^2) R_{D_\nu}(\lambda)\xi \rangle
= \langle \phi D_\nu v k_{D_\nu}(\lambda)\xi \mid v S\phi k_{D_\nu}(\lambda)\xi \rangle + \langle \nu S\phi h_{D_\nu}(\lambda)\xi \mid \phi D_\nu h_{D_\nu}(\lambda)\xi \rangle
= \langle \phi D_\nu v k_{D_\nu}(\lambda)\xi \mid v S\phi k_{D_\nu}(\lambda)\xi \rangle + \langle D_{\phi} v k_{D_\nu}(\lambda)\xi \mid v S\phi k_{D_\nu}(\lambda)\xi \rangle
+ \langle \nu S\phi h_{D_\nu}(\lambda)\xi \mid \phi D_\nu h_{D_\nu}(\lambda)\xi \rangle.
\]
Taking twice the real part, and inserting the definition of $Q_v$, this yields
\[
2 \text{Re} \langle \phi D_\nu R_{D_\nu}(\lambda)\xi \mid S\phi \xi \rangle = 2 \text{Re} \langle \phi D_\nu v k_{D_\nu}(\lambda)\xi \mid v S\phi k_{D_\nu}(\lambda)\xi \rangle
+ 2 \text{Re} \langle \phi D_\nu v h_{D_\nu}(\lambda)\xi \mid v S\phi h_{D_\nu}(\lambda)\xi \rangle + Q_v \langle \phi k_{D_\nu}(\lambda)\xi \rangle + Q_v \langle \phi h_{D_\nu}(\lambda)\xi \rangle.
\]
Inserting the latter into Eq. (5.2), we thus obtain
\[
2 \text{Re} \langle \chi \psi \mid D_\nu R_{D_\nu}(\lambda)\rho S R_S(\mu)\psi \rangle
= 2 \text{Re} \langle B(\lambda) k_S(\mu)\psi \mid k_S(\mu)\psi \rangle + 2 \text{Re} \langle h_S(\mu)\psi \mid B(\lambda) h_S(\mu)\psi \rangle
+ 2 \langle \langle \phi D_\nu v k_{D_\nu}(\lambda)\rho k_S(\mu)\psi \mid v S\phi k_{D_\nu}(\lambda)\rho k_S(\mu)\psi \rangle
+ 2 \langle \langle \phi D_\nu v h_{D_\nu}(\lambda)\rho k_S(\mu)\psi \mid v S\phi h_{D_\nu}(\lambda)\rho k_S(\mu)\psi \rangle
+ Q_v \langle \phi k_{D_\nu}(\lambda)\rho h_S(\mu)\psi \rangle + Q_v \langle \phi h_{D_\nu}(\lambda)\rho h_S(\mu)\psi \rangle
+ 2 \langle \langle \phi D_\nu v h_{D_\nu}(\lambda)\rho h_S(\mu)\psi \mid v S\phi h_{D_\nu}(\lambda)\rho h_S(\mu)\psi \rangle
+ Q_v \langle \phi h_{D_\nu}(\lambda)\rho h_S(\mu)\psi \rangle + Q_v \langle \phi h_{D_\nu}(\lambda)\rho h_S(\mu)\psi \rangle
= 2 \text{Re} \langle B(\lambda) k_S(\mu)\psi \mid k_S(\mu)\psi \rangle + 2 \text{Re} \langle B(\lambda) h_S(\mu)\psi \mid h_S(\mu)\psi \rangle
+ \sum_{m=1}^4 2 \text{Re} \langle \langle \phi D_\nu v M_m(\lambda,\mu)\psi \mid v S\phi M_m(\lambda,\mu)\psi \rangle + \sum_{m=1}^4 Q_v \langle \phi M_m(\lambda,\mu)\psi \rangle.
\]

**Lemma 5.6.** We have the inequality
\[
\pm \frac{1}{\pi^2} \int_0^\infty \frac{1}{(\lambda\mu)^{\frac{3}{2}}} B(\lambda,\mu)d\lambda d\mu \leq C_B \langle \chi \psi \mid \chi \psi \rangle,
\]
where $C_B := 2\|\langle S, \rho \rangle\|$. 

**Proof.** We note that the integral over $\lambda$ of $\pi^{-\frac{1}{2}} \lambda^{-\frac{3}{2}} D_\nu R_{D_\nu}(\lambda)$ converges strongly to $F_{D_\nu} = D_\nu (r^2 + D_{\phi}^2)^{-\frac{1}{2}} \leq 1$, and thus we have the operator inequality
\[
\pm \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{3}{2}} (B(\lambda) + B(\lambda^*))d\lambda = \pm \|\langle S, \rho \rangle \phi F_{D_\nu} \rho \pm \rho F_{D_\nu} \phi \| S, \rho \| \leq 2 \|\langle S, \rho \rangle\| = C_B.
\]
Inserting this into the definition of \( \hat{B}(\lambda, \mu) \), the remaining integral over \( \mu \) is norm-convergent, and using \( k_S(\mu)^2 + h_S(\mu)^2 = R_S(\mu) \) we find

\[
\pm \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} \hat{B}(\lambda, \mu) d\lambda d\mu
\]

\[
\leq \frac{1}{\pi} C_B \int_0^\infty \mu^{-\frac{1}{2}} \langle k_S(\mu) \chi \psi \rangle d\mu + \frac{1}{\pi} C_B \int_0^\infty \mu^{-\frac{1}{2}} \langle h_S(\mu) \chi \psi \rangle d\mu
\]

\[
= \frac{1}{\pi} C_B \int_0^\infty \mu^{-\frac{1}{2}} \langle \chi \psi \rangle d\mu = C_B \langle \chi \psi \rangle (1 + S^2)^{-\frac{1}{2}} \chi \psi \leq C_B \langle \chi \psi \rangle \chi \psi.
\]

The following lemma generalises Lemma 3.9.

**Lemma 5.7.** Let \( T \) be any densely defined symmetric operator on \( E \) such that \( T(r^2 + D_v^2)^{-\frac{1}{2}} \) is bounded. For \( \psi \in E \), we have the inequality

\[
\sum_{m=1}^4 \frac{1}{\pi^2} \int_0^\infty (\mu \lambda)^{-\frac{1}{2}} \langle M_m(\lambda, \mu) \psi \rangle T M_m(\lambda, \mu) \psi d\lambda d\mu \leq \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| \| \rho \|^2 \langle \psi | \psi \rangle.
\]

**Proof.** Let us consider the four integrals (for \( m = 1, 2, 3, 4 \)) given by

\[
I_m := \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\mu \lambda)^{-\frac{1}{2}} \langle M_m(\lambda, \mu) \psi \rangle T M_m(\lambda, \mu) \psi d\lambda d\mu.
\]

Since \( T(r^2 + D_v^2)^{-\frac{1}{2}} \) is bounded and \( T \) is symmetric, we know from Lemma 2.7 that also \( (r^2 + D_v^2)^{-\frac{1}{2}} T (r^2 + D_v^2)^{-\frac{1}{2}} \) is bounded, and

\[
\| (r^2 + D_v^2)^{-\frac{1}{2}} T (r^2 + D_v^2)^{-\frac{1}{2}} \| \leq \| T(r^2 + D_v^2)^{-\frac{1}{2}} \|.
\]

We obtain the operator inequality

\[
h_{D_v}(\lambda) T h_{D_v}(\lambda) = (r^2 + D_v^2)^{\frac{1}{2}} h_{D_v}(\lambda) (r^2 + D_v^2)^{-\frac{1}{2}} T (r^2 + D_v^2)^{-\frac{1}{2}} h_{D_v}(\lambda) (r^2 + D_v^2)^{\frac{1}{2}}
\]

\[
\leq (r^2 + D_v^2)^{\frac{1}{2}} h_{D_v}(\lambda) \| (r^2 + D_v^2)^{-\frac{1}{2}} T (r^2 + D_v^2)^{-\frac{1}{2}} \| h_{D_v}(\lambda) (r^2 + D_v^2)^{\frac{1}{2}}
\]

\[
\leq \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| (r^2 + D_v^2)^{\frac{1}{2}} h_{D_v}(\lambda)^2.
\]

Similarly, we also have the operator inequality

\[
k_{D_v}(\lambda) T k_{D_v}(\lambda) \leq \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| (r^2 + D_v^2)^{\frac{1}{2}} k_{D_v}(\lambda)^2.
\]

We note that \( h_{D_v}(\lambda)^2 + k_{D_v}(\lambda)^2 = R_{D_v}(\lambda) \). Moreover, by Lemma 2.6, the integral of \( \lambda^{-\frac{1}{2}} (r^2 + D_v^2)^{\frac{1}{2}} R_{D_v}(\lambda) \) converges strongly to \( \pi \), and we find

\[
\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (h_{D_v}(\lambda) T h_{D_v}(\lambda) + k_{D_v}(\lambda) T k_{D_v}(\lambda)) d\lambda \leq \| T(r^2 + D_v^2)^{-\frac{1}{2}} \|.
\]
Thus we obtain the inequalities
\[
I_1 + I_2 \leq \frac{1}{\pi} \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| \| \rho \|^2 \int_0^\infty \mu^{-\frac{1}{2}} \langle h_S(\mu) \psi \mid h_S(\mu) \psi \rangle d\mu,
\]
\[
I_3 + I_4 \leq \frac{1}{\pi} \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| \| \rho \|^2 \int_0^\infty \mu^{-\frac{1}{2}} \langle k_S(\mu) \psi \mid k_S(\mu) \psi \rangle d\mu.
\]
Summing up the latter two inequalities and computing the remaining norm-convergent integral over \( \mu \), we obtain
\[
\sum_{m=1}^4 I_m \leq \frac{1}{\pi} \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| \| \rho \|^2 \int_0^\infty \mu^{-\frac{1}{2}} \langle \psi \mid R_S(\mu) \psi \rangle d\mu
\]
\[
= \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| \| \rho \|^2 \langle \psi \mid (1 + S^2)^{-\frac{1}{2}} \psi \rangle
\]
\[
\leq \| T(r^2 + D_v^2)^{-\frac{1}{2}} \| \| \rho \|^2 \langle \psi \mid \psi \rangle. \quad \square
\]

**Lemma 5.8.** We have the inequality
\[
\frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} \hat{Q}(\lambda, \mu) d\lambda d\mu \geq -C_Q \langle \chi \psi \mid \chi \psi \rangle,
\]
where \( C_Q := c \| \phi(1 + D_v^2)^{\frac{1}{2}} \phi(r^2 + D_v^2)^{-1} \| \).

**Proof.** By condition (L5) in Assumption 5.1, we have the inequality
\[
Q_v(\phi M_m(\lambda, \mu) \chi \psi) \geq -c \langle M_m(\lambda, \mu) \chi \psi \mid \phi(1 + D_v^2)^{\frac{1}{2}} \phi M_m(\lambda, \mu) \chi \psi \rangle.
\]
The statement then follows by applying Lemma 5.7 with the operator \( T = \phi(1 + D_v^2)^{\frac{1}{2}} \phi \). \( \square \)

It remains to consider the term \( \hat{M}(\lambda, \mu) \) in Lemma 5.5. Of course, in the special case where \( \phi = \text{Id} \), we have \( \hat{M}(\lambda, \mu) = 0 \), and then we obtain the following result.

**Proposition 5.9.** Consider the setting of Assumption 5.1. Suppose furthermore that \( \phi = \text{Id} \). Then for any \( 0 < \kappa < 2 \) there exists an \( \alpha > 0 \) such that the operator \( \chi[F_D, F_{\alpha S}] \chi + \kappa \chi^2 \) is positive modulo compact operators:
\[
\chi[F_D, F_{\alpha S}] \chi \geq -\kappa \chi^2.
\]

**Proof.** As mentioned at the start of this section, we know that \( \chi[F_D, F_S] \chi \) is equal to \( \chi(F_D^* \rho F_S + F_S \rho F_D^*) \chi \) modulo compact operators. For any \( \psi \in E \), we thus need to estimate the integral in Eq. (5.1). From Lemma 5.5 we obtain the equality
\[
\langle \psi \mid \chi(F_D^* \rho F_S + F_S \rho F_D^*) \chi \psi \rangle = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} \langle \hat{B}(\lambda, \mu) + \hat{M}(\lambda, \mu) + \hat{Q}(\lambda, \mu) \rangle d\lambda d\mu.
\]
Since $\phi = \text{Id}$, we have $\hat{M}(\lambda, \mu) = 0$. Using Lemmas 5.6 and 5.8, we then obtain the estimate
$$\langle \psi \mid \chi (F_{D,}^\dagger \rho F_S + F_S \rho F_{D,}^\dagger) \chi \psi \rangle \geq -(C_B + C_Q) \langle \chi \psi \rangle.$$ Since this holds for any $\psi$, we have the operator inequality
$$\chi (F_{D,}^\dagger \rho F_S + F_S \rho F_{D,}^\dagger) \chi \geq -(C_B + C_Q) \chi^2.$$
We have therefore shown that
$$\chi[F_{D,}, F_S] \chi \gtrsim -(C_B + C_Q) \chi^2.$$
Next, we fix $0 < \kappa < 2$. If we replace $S$ by $\alpha S$ for some $\alpha > 0$, then the constants $C_B$ and $C_Q$ are replaced by $\alpha C_B$ and $\alpha C_Q$, respectively. Indeed, for $C_B = 2 \| [S, \rho] \|$ this is obvious, and we note that $C_Q$ is proportional to the constant $c$ from condition $(L5)$ in Assumption 5.1. Thus, by choosing $\alpha$ small enough, we can ensure that $\alpha (C_B + C_Q) < \kappa < 2$. \hfill \Box

In the following, we provide two different sufficient conditions which allow us to deal with the term $\hat{M}(\lambda, \mu)$ also for non-trivial $\phi$.

**Proposition 5.10.** In addition to the setting of Assumption 5.1, suppose furthermore that the following strengthening of condition $(L5)$ is satisfied:

$(L5')$ there exist constants $\nu \in (0, \infty)$ and $c \in [0, \infty)$ such that for all $\psi \in \text{Dom}(D \nu) \cap \text{Ran}(\nu)$ we have
$$Q_v(\phi \psi) \geq \nu \langle \nu S \phi \psi \rangle - c \langle \phi \psi \mid (1 + D_0^2)^{\frac{1}{2}} \phi \psi \rangle.$$ (5.3)

Then for any $0 < \kappa < 2$ there exists an $\alpha > 0$ such that the operator $\chi[F_{D,}, F_{\alpha S}] \chi + \kappa \chi^2$ is positive modulo compact operators:
$$\chi[F_{D,}, F_{\alpha S}] \chi \gtrsim -\kappa \chi^2.$$

**Proof.** First of all, for any $\beta > 0$ we can estimate
$$\pm \hat{M}(\lambda, \mu) = \pm \sum_{m=1}^4 2 \text{Re} \langle \beta^{-1}[\phi, D] v M_m(\lambda, \mu) \chi \psi \mid \beta v S \phi M_m(\lambda, \mu) \chi \psi \rangle \leq \sum_{m=1}^4 \beta^2 \langle v S \phi M_m(\lambda, \mu) \chi \psi \rangle + \beta^{-2} \langle [\phi, D] v M_m(\lambda, \mu) \chi \psi \rangle.$$ Combined with condition $(L5')$ this yields
$$\hat{M}(\lambda, \mu) + \hat{Q}(\lambda, \mu) \geq (\nu - \beta^2) \langle v S \phi M_m(\lambda, \mu) \chi \psi \rangle - \beta^{-2} \langle [\phi, D] v M_m(\lambda, \mu) \chi \psi \rangle - c \langle \phi M_m(\lambda, \mu) \chi \psi \mid (1 + D_0^2)^{\frac{1}{2}} \phi M_m(\lambda, \mu) \chi \psi \rangle.$$ Taking the double integral of this inequality, and estimating the second and third terms by applying Lemma 5.7 with $T = \beta^{-2} v [\phi, D] v + c \phi (1 + D_0^2)^{\frac{1}{2}} \phi$, we obtain
$$\frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} (\hat{M}(\lambda, \mu) + \hat{Q}(\lambda, \mu)) d\lambda d\mu \geq (\nu - \beta^2) P - C_M(\beta, c) \langle \chi \psi \rangle.$$ (5.4)
where we define
\[
P := \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} \langle v S \phi M_\mu (\lambda, \mu) \psi \rangle d\lambda d\mu \geq 0,
\]
\[
C_M(\beta, c) := \beta^{-2} \| v[D, \phi][\phi, D] v(r^2 + D_v^2)^{-\frac{1}{2}} \| + c \| (1 + D_v^2)^{\frac{1}{2}} \phi (r^2 + D_v^2)^{-\frac{1}{2}} \|.
\]
As in the proof of Proposition 5.9, we need to consider the integral
\[
\langle \psi | \chi (F^T_{D_v} \rho F_S + F_S \rho F^T_{D_v}) \psi \rangle = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} (\hat{B}(\lambda, \mu) + \hat{M}(\lambda, \mu) + \hat{Q}(\lambda, \mu)) d\lambda d\mu.
\]
Combining Eq. (5.4) with Lemma 5.6, we obtain the estimate
\[
\langle \psi | \chi (F^T_{D_v} \rho F_S + F_S \rho F^T_{D_v}) \psi \rangle \geq (\nu - \beta^2) P - (C_B + C_M(\beta, c)) \| \chi \psi \|. \tag{5.5}
\]
We now replace \( S \) by \( \alpha S \) for some \( \alpha \in (0, \infty) \). We recall that \( C_B \) is then replaced by \( \alpha C_B \). Moreover, we note that condition (L5') implies
\[
\langle D_v \phi \psi | \alpha S \phi \psi \rangle + \langle \alpha S \phi \psi | D_v \phi \psi \rangle \geq \alpha^{-1} \nu \langle \alpha \nu S \phi \psi | \alpha \nu S \phi \psi \rangle - \alpha c \langle \phi \psi | (1 + D_v^2)^{\frac{1}{2}} \phi \psi \rangle,
\]
and hence \( \nu \) and \( c \) are replaced by \( \alpha^{-1} \nu \) and \( \alpha c \). We choose \( \beta \) large enough such that
\[
\beta^{-2} \| v[D, \phi][\phi, D] v(r^2 + D_v^2)^{-\frac{1}{2}} \| < \frac{1}{3} \kappa.
\]
We then choose \( \alpha \) small enough such that
\[
\alpha^{-1} \nu - \beta^2 > 0, \quad \alpha c \| \phi (1 + D_v^2)^{\frac{1}{2}} \phi (r^2 + D_v^2)^{-\frac{1}{2}} \| < \frac{1}{3} \kappa, \quad \alpha C_B < \frac{1}{3} \kappa.
\]
These choices ensure that \( \alpha C_B + C_M(\beta, \alpha c) < \kappa \). Thus from Eq. (5.5) we obtain
\[
\langle \psi | \chi (F^T_{D_v} \rho F_a S + F_a S \rho F^T_{D_v}) \psi \rangle \geq -\kappa \| \chi \psi \|.
\]
Since this holds for any \( \psi \), we have the operator inequality
\[
\chi (F^T_{D_v} \rho F_a S + F_a S \rho F^T_{D_v}) \chi \geq -\kappa \chi^2.
\]
Since \( \chi [F_{D_v}, F_a S] \chi \) is equal to \( \chi (F^T_{D_v} \rho F_a S + F_a S \rho F^T_{D_v}) \chi \mod \text{compact operators}, \) this completes the proof. \( \square \)

**Proposition 5.11.** In addition to the setting of Assumption 5.1, suppose furthermore that the following condition is satisfied:

(L6) The operators \( \phi \) and \( \phi [D_v, \phi] \) map Dom(\( D_v \)) to Dom(\( S \)).

Then for any \( 0 < \kappa < 2 \) there exists an \( \alpha > 0 \) such that the operator \( \chi [F_{D_v}, F_a S] \chi + \kappa \chi^2 \) is positive modulo compact operators:
\[
\chi [F_{D_v}, F_a S] \chi \geq -\kappa \chi^2.
\]
Proof. First, we will derive an estimate for the double integral of \( \hat{M}(\lambda, \mu) \). Since by assumption \( \phi \) maps \( \text{Dom}(D_v) \) to \( \text{Dom}(S) \), we know that \( S\phi(r^2 + D_v^2)^{-\frac{1}{2}} \) is bounded, and therefore also \( [D_v, \phi]S\phi(r^2 + D_v^2)^{-\frac{1}{2}} \) is bounded. Furthermore, we have assumed that \( \phi[D_v, \phi] \) also maps \( \text{Dom}(D_v) \) to \( \text{Dom}(S) \), which ensures that \( \phi S\phi[D_v, \phi][r^2 + D_v^2]^{-\frac{1}{2}} = S\phi[D_v, \phi][r^2 + D_v^2]^{-\frac{1}{2}} + [\phi, S][\phi, D_v][r^2 + D_v^2]^{-\frac{1}{2}} \) is bounded as well. By applying Lemma 5.7 with the symmetric operator \( T = [D_v, \phi]S\phi + \phi S\phi[D_v, \phi] \), we obtain the inequality

\[
\pm \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} M(\lambda, \mu) d\lambda d\mu \leq C_M \langle \chi \psi | \chi \psi \rangle,
\]

where \( C_M := \|([D_v, \phi]S\phi + \phi S\phi[D_v, \phi])(r^2 + D_v^2)^{-\frac{1}{2}}\| \). As in the proof of Proposition 5.9, we need to consider the integral

\[
\langle \psi | \chi (F_{D_v} \rho F_S + F_S \rho F_{D_v}) \chi \psi \rangle = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty (\lambda \mu)^{-\frac{1}{2}} (\hat{M}(\lambda, \mu) + \hat{Q}(\lambda, \mu)) d\lambda d\mu.
\]

Combining Lemmas 5.6 and 5.8 with the above inequality for \( \hat{M}(\lambda, \mu) \), we now obtain the estimate

\[
\langle \psi | \chi (F_{D_v} \rho F_S + F_S \rho F_{D_v}) \chi \psi \rangle \geq -(C_B + C_Q + C_M) \langle \chi \psi \rangle.
\]

The proof then proceeds exactly as in Proposition 5.9.

\[\square\]

Remark 5.12. The condition (L6) is quite naturally satisfied in the context of first-order differential operators \( S \) and \( D \) on smooth manifolds, when \( \phi, v \) are compactly supported smooth functions and \( D_v \) is elliptic on \( \text{supp}(\phi) \) (this is the setting considered in [Dun20]).

6 The Kasparov product of half-closed modules

Assumption 6.1. Let \( A \) be a \((\mathbb{Z}_2\text{-graded})\) separable \( C^* \)-algebra, let \( B \) and \( C \) be \((\mathbb{Z}_2\text{-graded})\) \( \sigma \)-unital \( C^* \)-algebras, and let \( A \subset A \subset B \subset B \) be dense \(*\)-subalgebras. Consider three half-closed modules \((A, \pi_1(E_1), B, D_1), (B, \pi_2(E_2), C, D_2), \) and \((A, \pi E_C, D)\), where \( E := E_1 \otimes_B E_2 \) and \( \pi = \pi_1 \otimes 1 \), and suppose that \( \pi_1 \) is essential. We assume that the \(*\)-subalgebra \( A \subset A \) contains an (even) almost idempotent approximate unit \( \{u_n\}_{n \in \mathbb{N}} \) for \( A \).

For ease of notation, we will usually identify \( u_n \equiv u_n \otimes 1 \equiv \pi(u_n \otimes 1) \) on \( E \). We recall the ‘partition of unity’ \( \{\chi_k^2\}_{k \in \mathbb{N}} \) from Definition 4.6.

Proposition 6.2. In the setting of Assumption 6.1, assume that the connection condition (Definition 2.10) is satisfied. For each \( k \in \mathbb{N}, \) consider elements \( v_k, w_k \in \{u_n\} \) with \( v_k u_{k+1} = w_k u_{k+1} = u_{k+1} \), and write \( D_k := v_k D v_k \) and \( D_{1,k} := w_k D_1 w_k \). We assume furthermore that for some \( 0 < \kappa < 2 \) the following condition is satisfied:

- for each \( k \in \mathbb{N}, \) there exists \( \alpha_k \in (0, \infty) \) such that \( \chi_k [F_{D_k}, F_{\bar{x}_k} \otimes 1] \chi_k + \kappa \chi_k^2 \) is positive modulo compact operators.

Then \((A, \pi E_C, D)\) represents the Kasparov product of \((A, \pi_1(E_1), B, D_1)\) and \((B, \pi_2(E_2), C, D_2)\).
Proof. We can represent $\mathcal{D}$ and $\mathcal{D}_2$ by their bounded transforms $F_D$ and $F_{D_2}$, and (by Theorem 4.11) we can represent $\mathcal{D}_1$ by a localised representative $\tilde{F}_{D_1}(\alpha)$. We have seen in Proposition 2.11 that the connection condition of Theorem 2.2 is satisfied. Thus it remains to show that $F_D$ and $\tilde{F}_{D_1}(\alpha)$ (for the given sequence $\{\alpha_k\}_{k \in \mathbb{N}}$) satisfy the positivity condition of Theorem 2.2.

To prove the positivity condition, it suffices to consider $u_n[F_D, \tilde{F}_{D_1}(\alpha) \otimes 1]|u_n$, since we have norm-convergence $au_n \to a$. From Lemma 4.7 we know that $\sum_k \chi_k u_n$ is a finite sum. We know from Lemma 4.3 (applied with $a = \chi_k$, $c = u_{k+1}$, and $b = v_k$) that $\chi_k(F_D - F_{D_k}) \sim 0$. Using furthermore that $[F_D, \chi_k] \sim 0$ by Theorem 2.4, we have

$$u_n[F_D, \tilde{F}_{D_1}(\alpha) \otimes 1]|u_n = \sum_k u_n[F_D, \chi_k(F_{\alpha_k}D_{1,k} \otimes 1)]\chi_k u_n$$

$$\sim 2.4 \sum_k u_n\chi_k[F_D, F_{\alpha_k}D_{1,k} \otimes 1] \chi_k u_n$$

$$\sim 4.3 \sum_k u_n\chi_k[F_D, F_{\alpha_k}D_{1,k} \otimes 1] \chi_k u_n.$$ 

By hypothesis, we have a sequence $\{\alpha_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ such that $\chi_k[F_D, F_{\alpha_k}D_{1,k} \otimes 1] \chi_k + \kappa \chi_k^2$ is positive modulo compact operators. We then conclude that

$$u_n[F_D, \tilde{F}_{D_1}(\alpha) \otimes 1]|u_n \sim \sum_k u_n\chi_k[F_D, F_{\alpha_k}D_{1,k} \otimes 1] \chi_k u_n$$

$$\gtrsim -\sum_k \kappa u_n \chi_k^2 u_n = -\kappa u_n^2.$$ 

Thus the positivity condition of Theorem 2.2 also holds, and the statement follows. 

Theorem 6.3. In the setting of Assumption 6.1, assume that the connection condition (Definition 2.10) is satisfied. We assume furthermore that for each $n \in \mathbb{N}$ the following conditions are satisfied:

- we have the domain inclusion $\text{Dom}(D u_n) \cap \text{Ran}(u_n) \subset \text{Dom}(D_1 u_n \otimes 1)$;
- there exists $c_n \in [0, \infty)$ such that for all $\psi \in \text{Dom}(D u_n) \cap \text{Ran}(u_n)$ we have

$$\langle u_n(D_1 \otimes 1)\psi, D u_n \psi \rangle + \langle D u_n \psi, u_n(D_1 \otimes 1)\psi \rangle \geq -c_n \langle \psi, (1 + (n D u_n)^2)^{1/2} \psi \rangle.$$ (6.1)

Then $(\mathcal{A}, \pi_E, \mathcal{D})$ represents the Kasparov product of $(\mathcal{A}, \pi_1(E_1)_B, \mathcal{D}_1)$ and $(\mathcal{B}, \pi_2(E_2)_C, \mathcal{D}_2)$.

Proof. Let us write $v_k := u_{k+2}$ and $w_k := u_{k+3}$. For any $k \in \mathbb{N}$, we will check that Assumption 5.1 is satisfied by the operators $D, S = S_k = D_{1,k} \otimes 1 := w_k D_1 w_k \otimes 1, \chi = \chi_k, \rho = u_{k+1}, \phi = \text{Id}$, and $v = v_k$. We shall write $D_k := D v_k v_k$. We have by assumption the domain inclusion $\text{Dom}(D v_k) \cap \text{Ran}(v_k) \subset \text{Dom}(D_1 v_k \otimes 1)$. Noting that $\text{Dom}(D_1 v_k \otimes 1) = \text{Dom}(D_1 w_k^2 v_k \otimes 1) = \text{Dom}(S_k v_k)$, we see that condition (L1) is satisfied. Since $\phi = \text{Id}$ and $\{u_n\}$ is almost idempotent, and using Lemma 4.7, we see that condition (L2) is satisfied. By assumption, we have $u_n \in \text{Lip}^*(D) \cap \text{Lip}^*(D_1 \otimes 1)$, and since $\{u_n\}$ is commutative, it follows that we also have $u_n \in \text{Lip}(S_k)$, so condition
(L3) is satisfied. Condition (L4) follows from Lemma 4.2 (using that $\chi_k u_{k+1} = \chi_k$ and $u_{k+1} v_k = u_{k+1}$). Finally, for $\psi \in \text{Dom}(Dv_k) \cap \text{Ran}(v_k)$ we have $w_k \psi = \psi$, and then it follows from Eq. (6.1) that

$$Q_v(\psi) = \langle Dv_k \psi \mid v_k S_k \psi \rangle + \langle v_k S_k \psi \mid Dv_k \psi \rangle = \langle Dv_k \psi \mid v_k(D_1 \otimes 1)\psi \rangle + \langle v_k(D_1 \otimes 1)\psi \mid Dv_k \psi \rangle \geq -c_{k+2} \langle \psi \mid (1 + D_1^2)^{1/2} \psi \rangle,$$

which shows condition (L5). Thus Assumption 5.1 is indeed satisfied.

Hence we can apply Proposition 5.9 for each $k \in \mathbb{N}$, and for any $0 < \kappa < 2$ we obtain a sequence $\{\alpha_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ such that $\chi_k [F_{D_k}, F_{\alpha_k D_{1,k}} \otimes 1] \chi_k + \kappa \chi_k^2$ is positive modulo compact operators. The statement then follows from Proposition 6.2.

**Remark 6.4.** We note that Eq. (6.1) may be replaced by

$$\langle u_n(D_1 \otimes 1)u_n \psi \mid u_n D u_n \psi \rangle + \langle u_n D u_n \psi \mid u_n(D_1 \otimes 1)u_n \psi \rangle \geq -c_n \langle \psi \mid (1 + (u_n D u_n)^2)^{1/2} \psi \rangle.$$
Remark 6.7. For the domain condition (1) in Definition 6.6, it is of course sufficient (but not necessary!) to have the domain inclusion $\text{Dom}(D) \subset \text{Dom}(D_1 \otimes 1)$.

Lemma 6.10. Consider $D_k := u_{k+4}D_{k+4}$ and $S_k := u_{k+4}D_{1}u_{k+4} \otimes 1$, and $\psi \in \text{Dom}(D)$. Then we have
\[
\Re \langle D_k u_{k+2} \psi \mid S_k u_{k+2} \psi \rangle = 2 \Re \langle D_{k+2} \psi \mid (D_1 \otimes 1) u_{k+2} \psi \rangle + 2 \Re \langle [u_{k+4}, D] u_{k+2} \psi \mid [D_1 \otimes 1, u_{k+3}] u_{k+2} \psi \rangle \geq 2 \Re \langle D_{k+2} \psi \mid (D_1 \otimes 1) u_{k+2} \psi \rangle - 2 \| [D, u_{k+4}] [D_1 \otimes 1, u_{k+3}] \| \langle u_{k+2} \psi \mid u_{k+2} \psi \rangle.
\]

Proof. We compute
\[
\Re \langle D_k u_{k+2} \psi \mid S_k u_{k+2} \psi \rangle = \Re \langle u_{k+4} D_{k+2} \psi \mid u_{k+4}(D_1 \otimes 1) u_{k+3} u_{k+2} \psi \rangle = \Re \langle u_{k+4} D_{k+2} \psi \mid u_{k+4} [D_1 \otimes 1, u_{k+3}] u_{k+2} \psi \rangle + \Re \langle D_{k+2} \psi \mid [D_1 \otimes 1, u_{k+3}] u_{k+2} \psi \rangle + \Re \langle D_{k+2} \psi \mid [D_1 \otimes 1, u_{k+3}] u_{k+2} \psi \rangle.
\]
The inequality is then clear.

6.1.1 Strong local positivity

Definition 6.9. In the setting of Assumption 6.1, the strong local positivity condition requires that for each $n \in \mathbb{N}$ the following assumptions hold:

(1) we have the inclusion $u_n \cdot \text{Dom}(D) \subset \text{Dom}(D_1 \otimes 1)$;

(2) there exist $\nu_n \in (0, \infty)$ and $c_n \in [0, \infty)$ such that for all $\psi \in \text{Dom}(D)$ we have
\[
\Re \langle (D_1 \otimes 1) u_n \psi \mid D u_n \psi \rangle + \Re \langle D u_n \psi \mid (D_1 \otimes 1) u_n \psi \rangle \geq \nu_n \Re \langle (D_1 \otimes 1) u_n \psi \rangle - c_n \Re \langle u_n \psi \mid (1 + D^*)^2 u_n \psi \rangle.
\]

Lemma 6.10. In the setting of Assumption 6.1, assume that the strong local positivity condition is satisfied. Then writing $S_k := u_{k+4}D_{1}u_{k+4} \otimes 1$, $D_k := u_{k+4}D_{k+4}$, and $\phi_k := u_{k+2}$, there exists for each $k \in \mathbb{N}$ a constant $d_k \in (0, \infty)$ such that for all $\psi \in \text{Dom}(D)$ we have
\[
\Re \langle S_k \phi_k \psi \mid D_k \phi_k \psi \rangle + \Re \langle D_k \phi_k \psi \mid S_k \phi_k \psi \rangle \geq \nu_{k+2} \Re \langle u_{k+4} S_k \phi_k \psi \rangle - d_k \Re \langle \phi_k \psi \mid (1 + D_{k+4}^2)^{1/2} \phi_k \psi \rangle.
\]

Proof. Combining Lemma 6.8 with the strong local positivity condition, we have
\[
2 \Re \langle D_k \phi_k \psi \mid S_k \phi_k \psi \rangle \geq 2 \Re \langle D_k \phi_k \psi \mid (D_1 \otimes 1) \phi_k \psi \rangle - 2 \| [D, u_{k+4}] [D_1 \otimes 1, u_{k+3}] \| \Re \langle \phi_k \psi \rangle.
\]
where we used that \( c_{k+3} = 0 \). The second term in Eq. (6.2) can be estimated as follows. Since \( u_{k+3}^2 \cdot D_{k+4} \cdot u_{k+4}^2 \) is essentially bounded by \( D_k \). Using Lemma 2.7, we find that

\[
\langle \phi_k \psi \mid (1 + D^*D)^{1/2} \phi_k \psi \rangle = \langle \phi_k \psi \mid u_{k+4}^2(1 + D^*D)^{1/2}u_{k+4}^2\phi_k \psi \rangle \\
\leq \| T(1 + D_k^2)^{-\frac{1}{2}} \| \langle \phi_k \psi \mid (1 + D_k^2)^{1/2} \phi_k \psi \rangle.
\]

(6.3)

We then obtain the desired inequality with

\[
d_k = \nu_{k+2}\| [D_1 \otimes 1, u_{k+4}^2][D_1 \otimes 1, u_{k+3}] \| + c_{k+2}\| T(1 + D_k^2)^{-\frac{1}{2}} \| \\
+ 2\| [D, u_{k+4}^2][D_1 \otimes 1, u_{k+3}] \|.
\]

\[\square\]

**Theorem 6.11.** We consider the setting of Assumption 6.1: let \( A \) be a \((\mathbb{Z}_2,\text{-graded})\) separable \( C^*\)-algebra, let \( B \) and \( C \) be \((\mathbb{Z}_2,\text{-graded})\) \( \sigma \)-unital \( C^*\)-algebras, and let \( A \subset A \) and \( B \subset B \) be dense \(*\)-subalgebras. Consider three half-closed modules \((A, \pi_1(E_1)_B(D_1), (B, \pi_2(E_2)_C(D_2), (A, \pi(E_1, D_2), \text{where } E := E_1 \otimes_B E_2 \text{ and } \pi = \pi_1 \otimes 1, \text{ and suppose that } \pi_1 \text{ is essential.})

We assume that the \(*\)-subalgebra \( A \subset A \) contains an (even) almost idempotent approximate unit \( \{ u_n \}_{n \in \mathbb{N}} \) for \( A \). We assume furthermore that the following conditions are satisfied:

Connection condition (Definition 2.10): for all \( \psi \) in a dense subspace \( E_1 \) of \( \text{Dom}(D_1) \), we have

\[ \tilde{T}_\psi := \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix} \in \text{Lip}(D \oplus D_2); \]

Strong local positivity condition (Definition 6.9): for each \( n \in \mathbb{N} \) the following assumptions hold:

1. we have the inclusion \( u_n \cdot \text{Dom}(D) \subset \text{Dom}(D_1 \otimes 1) \);
2. there exist \( \nu_n \in (0, \infty) \) and \( c_n \in [0, \infty) \) such that for all \( \psi \in \text{Dom}(D) \) we have

\[
\langle (D_1 \otimes 1)u_n \psi \mid Du_n \psi \rangle + \langle Du_n \psi \mid (D_1 \otimes 1)u_n \psi \rangle \\
\geq \nu_n \langle (D_1 \otimes 1)u_n \psi \rangle - c_n \langle u_n \psi \mid (1 + D^*D)^{\frac{1}{2}}u_n \psi \rangle.
\]

Then \((A, \pi(E_1, C, D))\) represents the Kasparov product of \((A, \pi_1(E_1)_B(D_1), (B, \pi_2(E_2)_C(D_2)\).
Proof. Let us write \( v_k := w_k := u_{k+4} \). For any \( k \in \mathbb{N} \), we will check that Assumption 5.1 is satisfied by the operators \( D, S = S_k = D_1 \otimes 1 := w_k D_1 w_k \otimes 1 \), \( \chi = \chi_k \), \( \rho = u_{k+1} \), \( \phi = u_{k+2} \), and \( v = v_k \). We shall write \( D_v := D_v = v_k D_v v_k \).

We have by assumption the domain inclusion \( u_n.\text{Dom}(D) \subset \text{Dom}(D_1 \otimes 1) \) for any \( n \in \mathbb{N} \). For any \( \psi \in \text{Dom}(D_v) \) we note that \( \nu \psi = u_{k+4} \nu \psi \in u_{k+5} \cdot \text{Dom}(D) \subset \text{Dom}(D_1 \otimes 1) \), and therefore \( \psi \in \text{Dom}(S_k v) \), which shows condition (L1). Conditions (L2)-(L4) follow as in the proof of Theorem 6.3. Furthermore, for any \( \psi \in \text{Dom}(D_v) \) we have \( \phi \psi = \phi v \psi \), where \( v \psi \in \text{Dom}(D) \). Condition (L5)’ is then satisfied by Lemma 6.10. So by Proposition 5.10 we know that there exists a sequence \( \{\alpha_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) such that \( \chi_k[F_{D}, F_{v} D_1 \otimes 1] \chi_k + \kappa \chi_k \) is positive modulo compact operators. The statement then follows from Proposition 6.2.

6.1.2 A ‘differentiability’ condition

Lemma 6.12. In the setting of Assumption 6.1, assume that the local positivity condition is satisfied. Then writing \( S_k := u_{k+4} D_1 u_{k+4} \otimes 1 \), \( D_k := u_{k+4} D u_{k+4} \), and \( \phi_k := u_{k+2} \), there exists for each \( k \in \mathbb{N} \) a constant \( d_k \in [0, \infty) \) such that for all \( \psi \in \text{Dom}(D) \) we have

\[
\langle S_k \phi_k \psi \rangle D_k \phi_k \psi \rangle \geq -d_k \langle \phi_k \psi \rangle (1 + D_k^2)^{1/2} \phi_k \psi \rangle.
\]

Proof. Combining Lemma 6.8 with the local positivity condition, we have

\[
2 \Re \langle D_k \phi_k \psi \rangle S_k \phi_k \psi \rangle \geq 2 \Re \langle D \phi_k \psi \rangle (D_1 \otimes 1) \phi_k \psi \rangle - 2 \| [D, u_{k+4}^2] [D_1 \otimes 1, u_{k+3}] \| \langle \phi_k \psi \rangle \phi_k \psi \rangle
\]

\[
\geq -c_{k+2} \langle \phi_k \psi \rangle (1 + D^* D)^{1/2} \phi_k \psi \rangle - 2 \| [D, u_{k+4}^2] [D_1 \otimes 1, u_{k+3}] \| \langle \phi_k \psi \rangle \phi_k \psi \rangle.
\]

As in Eq. (6.3), an application of Lemma 5.7 with \( T := u_{k+4} (1 + D^* D)^{1/2} u_{k+4} \) yields

\[
\langle \phi_k \psi \rangle (1 + D^* D)^{1/2} \phi_k \psi \rangle \leq \| T(1 + D_k^2)^{-1/2} \| \langle \phi_k \psi \rangle (1 + D_k^2)^{1/2} \phi_k \psi \rangle.
\]

Thus we obtain the desired inequality with

\[
d_k = c_{k+2} \| T(1 + D_k^2)^{-1/2} \| + 2 \| [D, u_{k+4}^2] [D_1 \otimes 1, u_{k+3}] \|.
\]

Theorem 6.13. We consider the setting of Assumption 6.1: let \( A \) be a \((\mathbb{Z}_2\text{-graded})\) separable \( C^*\)-algebra, let \( B \) and \( C \) be \((\mathbb{Z}_2\text{-graded})\) \( \sigma \)-unital \( C^*\)-algebras, and let \( A \subset A \) and \( B \subset B \) be dense \( \ast \)-subalgebras. Consider three half-closed modules \((A, \pi_1(E_1)B, D_1), (B, \pi_2(E_2)C, D_2), (A, \pi E_C, D), \) where \( E := E_1 \otimes_B E_2 \) and \( \pi = \pi_1 \otimes 1 \), and suppose that \( \pi_1 \) is essential. We assume that the \( \ast \)-subalgebra \( A \subset A \) contains an (even) almost idempotent approximate unit \( \{u_n\}_{n \in \mathbb{N}} \) for \( A \). We assume furthermore that the following conditions are satisfied:

Connection condition (Definition 2.10): for all \( \psi \) in a dense subspace \( E_1 \) of \( \text{Dom}(D_1) \), we have

\[
\widetilde{T}_\psi := \begin{pmatrix}
0 & T_\psi \\
T_\psi^* & 0
\end{pmatrix} \in \text{Lip}(D \oplus D_2);
\]

Local positivity condition (Definition 6.6): for each \( n \in \mathbb{N} \) the following assumptions hold:
(1) we have the inclusion \( u_n \cdot \text{Dom}(\mathcal{D}) \subset \text{Dom}(\mathcal{D}_1 \otimes 1) \);
(2) there exists \( c_n \in [0, \infty) \) such that for all \( \psi \in \text{Dom}(\mathcal{D}) \) we have
\[
\langle (\mathcal{D}_1 \otimes 1)u_n\psi \mid D u_n \psi \rangle + \langle Du_n \psi \mid (\mathcal{D}_1 \otimes 1)u_n \psi \rangle \geq -c_n \langle u_n \psi \mid (1 + \mathcal{D}^* \mathcal{D})^{1/2} u_n \psi \rangle.
\]

Finally, assume that the following additional condition holds:

(\ast) for each \( n \in \mathbb{N} \), \( u_n[D, u_n]u_{n+2} \) maps \( \text{Dom}(\mathcal{D} u_{n+2}^2) \) to \( \text{Dom}(\mathcal{D}_1 \otimes 1) \).

Then \( (\mathcal{A}, \pi E C, \mathcal{D}) \) represents the Kasparov product of \( (\mathcal{A}, \pi_1(E_1)_B, \mathcal{D}_1) \) and \( (\mathcal{B}, \pi_2(E_2)_C, \mathcal{D}_2) \).

\textbf{Proof.} Most of the proof is similar to the proof of Theorem 6.11, so we shall be brief here. Conditions (L1)-(L4) in Assumption 5.1 are again satisfied by the operators \( \mathcal{D} \), \( S = \mathcal{D}_{1,k} \otimes 1 := w_k \mathcal{D}_1 w_k \otimes 1 \), \( \chi = \chi_k \), \( \rho = u_{k+1} \), \( \phi = u_{k+2} \), and \( v = v_k \), where \( v_k = w_k = u_{k+4} \). Condition (L5) is shown in Lemma 6.12. Thus Assumption 5.1 is indeed satisfied. Furthermore, since \( \phi_k[D, \phi_k] = u_{k+2}[\mathcal{D}, u_{k+2}]u_{k+4} \) maps \( \text{Dom}(\mathcal{D}_k) \) to \( \text{Dom}(\mathcal{D}_1 \otimes 1) \subset \text{Dom}(S_k) \) by condition (\ast), also condition (L6) is satisfied. Hence we can apply Proposition 5.11 for each \( k \in \mathbb{N} \), and we obtain a sequence \( \{\alpha_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) such that \( \chi_k[F_{\mathcal{D}_k}, \alpha_k \mathcal{D}_{1,k} \otimes 1] \chi_k + \kappa \chi_k^2 \) is positive modulo compact operators. The statement then follows from Proposition 6.2. \hfill \Box

\textbf{Remark 6.14.} If for each \( n \in \mathbb{N} \), \( u_n[D, u_n] \) maps \( \text{Dom}(\mathcal{D}) \) to \( \text{Dom}(\mathcal{D}_1 \otimes 1) \), and \( u_{n+2} \) commutes with \( [\mathcal{D}, u_n] \), then one can check that the condition (\ast) is satisfied. This situation for instance occurs if \( \mathcal{D} \) and \( \mathcal{D}_1 \) are first-order differential operators on smooth manifolds (with \( \mathcal{D} \) elliptic) and \( u_n \) are compactly supported smooth functions. This is precisely the setting studied in [Dun20], and Theorem 6.13 can be viewed as a generalisation of the statement and proof of [Dun20, Theorem 4.2].

\subsection*{6.2 The constructive approach}

In this subsection we show that the strong local positivity condition is quite naturally satisfied in (a localised version of) the constructive approach to the unbounded Kasparov product. In the following, we will in particular assume the existence of a suitable operator \( \mathcal{T} \). As explained in Remark 3.7, one should keep in mind here the case where (under suitable assumptions) \( \mathcal{T} = 1 \otimes \nabla \mathcal{D}_2 \) is constructed from a suitable ‘connection’ \( \nabla \) on \( E_1 \).

\textbf{Assumption 6.15.} Let \( A \) be a \((\mathbb{Z}_2\text{-graded})\) separable \( C^*\)-algebra, let \( B \) and \( C \) be \((\mathbb{Z}_2\text{-graded})\) \( C^*\)-algebras, and let \( A \subset A \) and \( B \subset B \) be dense \( s\)-subalgebras. Consider two half-closed modules \( (\mathcal{A}, \pi_1(E_1)_B, \mathcal{D}_1) \) and \( (\mathcal{B}, \pi_2(E_2)_C, \mathcal{D}_2) \), and suppose that \( \pi_1 \) is essential. We write \( E := E_1 \otimes_B E_2 \), \( \pi = \pi_1 \otimes 1 \), and \( S := \mathcal{D}_1 \otimes 1 \), and we consider an odd symmetric operator \( \mathcal{T} \) on \( E \). We denote by \( \mathcal{D} := \overline{S + \mathcal{T}} \) the closure of the operator \( S + \mathcal{T} \) on the initial domain \( \text{Dom}(S) \cap \text{Dom}(\mathcal{T}) \). We assume that the following conditions are satisfied:

(A1) the intersection \( \text{Dom}(S) \cap \text{Dom}(\mathcal{T}) \) is dense in \( E \), and the triple \( (\mathcal{A}, \pi E C, \mathcal{D}) \) is a half-closed module;
(A2) for all \( \psi \) in a dense subset of \( \mathcal{A} \cdot \text{Dom} \mathcal{D}_1 \), we have
\[
\widetilde{T}_{\psi} := \begin{pmatrix} 0 & T_{\psi} \\ T_{\psi}^* & 0 \end{pmatrix} \in \text{Lip}(\mathcal{T} \oplus \mathcal{D}_2);
\]
(A3) the ∗-subalgebra $A \subset A$ contains an almost idempotent approximate unit $\{u_n\}_{n \in \mathbb{N}}$ for $A$, such that we have the inclusion

$$u_n \cdot \text{Dom}(D) \subset \text{Dom}(S) \cap \text{Dom}(T).$$

**Proposition 6.16.** Consider the setting of Assumption 6.15. We assume furthermore that for each $n \in \mathbb{N}$ there exists $c_n \in [0, \infty)$ such that for all $\psi \in \text{Dom}D$ we have the inequality

$$\pm \left( \langle Su_n \psi \mid Tu_n \psi \rangle \right) \leq c_n \langle u_n \psi \mid (1 + D^*D)\frac{1}{2}u_n \psi \rangle. \quad (6.4)$$

Then $(A, \pi E C, D)$ represents the Kasparov product of $(A, \pi_1 (E_1)_B, D_1)$ and $(B, \pi_2 (E_2)_C, D_2)$.

**Proof.** For any $\psi \in \text{Dom}D$, we have bounded operators $ST_{\psi} = T_{D_1} \psi$ and $T_{\psi}^* S = T_{D_1}^* \psi$, which means that $T_{\psi}$ preserves $(\text{Dom}(S) \cap \text{Dom}(T)) \oplus \text{Dom}(D_2)$, and that $[D \oplus D_2, T_{\psi}]$ is bounded on this domain. Since $\text{Dom}(S) \cap \text{Dom}(T)$ is a core for $D$, it follows that the connection condition (Definition 2.10) is satisfied.

Next, using Eq. (6.4) we obtain the inequality

$$\langle Su_n \psi \mid Du_n \psi \rangle + \langle Du_n \psi \mid Su_n \psi \rangle = 2\left( \langle Su_n \psi \rangle + \langle Tu_n \psi \rangle + \langle Tu_n \psi \mid Su_n \psi \rangle \right) \geq 2\langle Su_n \psi \rangle - c_n \langle u_n \psi \rangle \left( (1 + D^*D)\frac{1}{2}u_n \psi \right).$$

We conclude that the strong local positivity condition (Definition 6.9) is also satisfied (with $\nu_n = 2$ for all $n \in \mathbb{N}$). The statement then follows from Theorem 6.11. \hfill \Box

The following result provides a sufficient condition which ensures that the hypothesis of the above proposition is satisfied, by considering the anti-commutator $[S, T]$ on a suitable domain.

**Theorem 6.17.** Consider the setting of Assumption 6.15. We assume furthermore that there exists a core $F \subset \text{Dom}D$ such that for each $n \in \mathbb{N}$ we have $u_n \cdot F \subset \text{Dom}(ST) \cap \text{Dom}(TS)$, and there exists $C_n \in [0, \infty)$ such that for all $\eta \in F$ we have

$$\| [S, T] u_n \eta \| \leq C_n \| u_n \eta \|_D. \quad (6.5)$$

Then $(A, \pi E C, D)$ represents the Kasparov product of $(A, \pi_1 (E_1)_B, D_1)$ and $(B, \pi_2 (E_2)_C, D_2)$.

**Proof.** We consider the closed symmetric operator $[S, T]$. Fix $n \in \mathbb{N}$, and let $\psi \in \text{Dom}(D)$. Since $F$ is a core for $D$, we can choose a sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset F$ such that $\| \psi_k - \psi \|_D \to 0$ as $k \to \infty$. The inequality

$$\| u_n \psi_k - u_n \psi \|_D^2 \leq 2(\| u_n \psi \|^2 + \| [D, u_n] \|^2) \| \psi_k - \psi \|^2_D$$

ensures that we also have the convergence $\| u_n \psi_k - u_n \psi \|_D \to 0$ as $k \to \infty$. In particular, $u_n \psi_k$ is Cauchy with respect to $\| \cdot \|_D$, and from Eq. (6.5) we see that $u_n \psi_k$ is also Cauchy with respect to the graph norm of $[S, T]$. Hence $u_n \psi = \lim_{k \to \infty} u_n \psi_k$ lies in the domain of the closure $[S, T]$, and we have shown the inclusion $u_n \cdot \text{Dom}(D) \subset \text{Dom}([S, T])$. \hfill \Box
Furthermore, since \( u_n \psi \in \text{Dom}(\mathcal{D}) \), we can also choose a sequence \( \{ \eta_k \}_{k \in \mathbb{N}} \subset \mathcal{F} \) such that \( \| \eta_k - u_n \psi \|_\mathcal{D} \to 0 \) as \( k \to \infty \). The inequality
\[
\| u_{n+1} \eta_k - u_n \psi \|_\mathcal{D}^2 = \| u_{n+1}(\eta_k - u_n \psi) \|_\mathcal{D}^2 \leq 2 \left( \| u_{n+1} \|_\mathcal{D}^2 + \| [\mathcal{D}, u_{n+1}] \|_\mathcal{D}^2 \right) \| \eta_k - u_n \psi \|_\mathcal{D}^2
\]
ensures that we then have the convergence \( \| u_{n+1} \eta_k - u_n \psi \|_\mathcal{D} \to 0 \) as \( k \to \infty \). Since \( u_{n+1} \cdot \text{Dom}(\mathcal{D}) \subset \text{Dom}(\mathcal{S}) \) by condition (A3), we can estimate
\[
\| \mathcal{S} u_{n+1} \eta_k - \mathcal{S} u_n \psi \| \leq \| \mathcal{S} u_{n+1} (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} \| \| \eta_k - u_n \psi \|_\mathcal{D}.
\]
This ensures that we have norm-convergence \( \mathcal{S} u_{n+1} \eta_k \to \mathcal{S} u_n \psi \) as \( k \to \infty \). Similarly we also have norm-convergence \( \mathcal{T} u_{n+1} \eta_k \to \mathcal{T} u_n \psi \) and \( \| \mathcal{S}, \mathcal{T} \| u_{n+1} \eta_k \to \| \mathcal{S}, \mathcal{T} \| u_n \psi \). Hence we have
\[
\langle \mathcal{S} u_n \psi \mid \mathcal{T} u_n \psi \rangle + \langle \mathcal{T} u_n \psi \mid \mathcal{S} u_n \psi \rangle = \lim_{k \to \infty} \langle \mathcal{S} u_{n+1} \eta_k \mid \mathcal{T} u_{n+1} \eta_k \rangle + \langle \mathcal{T} u_{n+1} \eta_k \mid \mathcal{S} u_{n+1} \eta_k \rangle
= \lim_{k \to \infty} \langle u_{n+1} \eta_k \mid \mathcal{S}, \mathcal{T} \mid u_{n+1} \eta_k \rangle
= \langle u_n \psi \mid \mathcal{S}, \mathcal{T} \mid u_n \psi \rangle.
\]
Finally, we can estimate the right-hand-side as follows. Since \( \| \mathcal{S}, \mathcal{T} \| u_{n+1} (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} \| \) is bounded, and writing \( c_n := \| \mathcal{S}, \mathcal{T} \| u_{n+1} (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} \| \), we know from Lemma 2.7 that
\[
\| (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} \| \mathcal{S}, \mathcal{T} \| u_{n+1} (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} \| \leq c_n.
\]
For any \( \psi \in \text{Dom}(\mathcal{D}) \) we then have the inequality
\[
\pm \langle u_n \psi \mid \mathcal{S}, \mathcal{T} \mid u_n \psi \rangle
= \pm \langle (1 + \mathcal{D}^* \mathcal{D})^{\frac{1}{2}} u_n \psi \mid (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} \mathcal{S}, \mathcal{T} \mid u_{n+1} (1 + \mathcal{D}^* \mathcal{D})^{-\frac{1}{2}} (1 + \mathcal{D}^* \mathcal{D})^{\frac{1}{2}} u_n \psi \rangle
\leq c_n \langle u_n \psi \mid (1 + \mathcal{D}^* \mathcal{D})^{\frac{1}{2}} u_n \psi \rangle.
\]
The statement then follows from Proposition 6.16.

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