Avoiding small subgraphs in Achlioptas processes

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Joint work with Michael Krivelevich and Benny Sudakov
Erdős-Rényi random graph $G(n, p)$: each of $\binom{n}{2}$ edges appears independently with probability $p$. 

**Definition**

Monotone property: closed under edge addition.
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**Natural question**

For what range of $p$ does $G(n, p)$ typically satisfy a certain property? e.g.:  
- containment of a triangle?  
- containment of a “giant component”?  
(say with 1% of the vertices)
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*Monotone property:* closed under edge addition.
Every monotone property $Q$ has a threshold $p^*(n)$, i.e. as $n \to \infty$:

$$\mathbb{P}[G(n, p(n)) \text{ satisfies } Q] \to \begin{cases} 
0 & \text{ when } p(n) \ll p^*(n) \\
1 & \text{ when } p(n) \gg p^*(n)
\end{cases}$$
Bollobás-Thomason, 1987:

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**Proof sketch:** The key observation is that $G(n, 30p)$ contains the union of 30 independent copies of $G(n, p)$.

So if $p(n)$ is such that $\mathbb{P}[G(n, p) \text{ satisfies } Q] = 0.1$, monotonicity gives us:

$$\mathbb{P}[G(n, 30p) \text{ satisfies } Q] \geq 1 - 0.9^{30} > 0.9.$$
Threshold phenomena

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Threshold for containment of triangle

Simple first-moment calculation:

$$\mathbb{E}[\#K_3 \text{ in } G(n, p)] = 3! \binom{n}{3} p^3 \sim n^3 p^3$$

Thus, when $p \ll n^{-1}$, we have $\mathbb{P}[G(n, p) \text{ contains triangle}] \to 0$.

When $p \gg n^{-1}$, we do have $\mathbb{E}[\#K_3] \to \infty$, but this is insufficient in general. However, it can be shown that $n^{-1}$ is the threshold.
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Many thresholds from $G(n, p)$ transfer over to $G(n, M)$ because

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**Equivalent definition of $G(n, M)$**

- Sample a random permutation of the $\binom{n}{2}$ potential edges.
- Then, keep the first $M$ edges of the permutation.
Classical result

- Each of $n$ balls is randomly distributed into one of $n$ bins.
- Then w.h.p., the highest occupancy is $\sim \frac{\log n}{\log \log n}$.
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Application to hashing character strings:
Consider the Perl hash function from \{character strings\} to, say, \{0, 1, \ldots, 999\}:

$$h(\text{adam}) := 1 + 4 \cdot 33 + 1 \cdot 33^2 + 13 \cdot 33^3 \mod 1000.$$  

This can be used to implement a fast lookup table with 1000 bins. Given a record associated with a person’s name $N$, we store it in the bin labeled $h(N)$.

Worst-case running time is proportional to the highest occupancy.
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Azar-Broder-Karlin-Upfal, 1994

- Suppose balls come sequentially, and each receives \textbf{two} independent random options for bins.
- For each ball, you choose one of the options in a deterministic, on-line fashion.
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- Then *whp*, the highest occupancy is $\sim \frac{\log n}{\log \log n}$.

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- For each ball, you choose one of the options in a deterministic, on-line fashion.
- Then, there is a strategy for which *whp*, the highest occupancy is only $\sim \log \log n$.
- In fact, strategy is simple: pick lesser-occupied bin.
- This is also the optimal strategy.
**Achlioptas process**

**Question (of Dimitris Achlioptas)**

Can the power of two random choices substantially delay the appearance of the giant component in the random graph?

Recall the equivalent definition of $G(n, M)$: keep the first $M$ edges in a random permutation of all edges.
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Can the power of two random choices substantially delay the appearance of the giant component in the random graph?

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**Definition**

- An *Achlioptas process* consists of sequential rounds.
- Each round, you receive **two** independent random choices of potential edges, and select one with in a deterministic, on-line fashion.
Can the power of two random choices substantially delay the appearance of the giant component in the random graph?

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**Definition**

- An Achlioptas process consists of sequential rounds.
- Each round, you receive **two** independent random choices of potential edges, and select one with in a deterministic, on-line fashion.

**Remark:** if the strategy is to pick the first edge in each pair, then this gives \( G(n, M) \) after \( M \) rounds.
Erdős-Rényi, 1960

- If $M(n) = cn$ for any $c < \frac{1}{2}$, then \textbf{whp} all connected components of $G(n, M)$ are of size $O(\log n)$.
- If $M(n) = cn$ for any $c > \frac{1}{2}$, then \textbf{whp} $G(n, M)$ has a “giant” component of size $\Omega(n)$. 
Delaying the giant

Erdős-Rényi, 1960

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Series of papers, by Bohman, Frieze, Spencer, and Wormald . . .

Current best results

- (Spencer-Wormald) There is a strategy that can last for \( 0.829n \) rounds while all components are \( O(\log n) \) \textit{whp}.
- (Bohman-Frieze-Wormald) Regardless of the choice of strategy, \textit{whp} the first \( 0.964n \) rounds will offer a sequence of choices that cause the strategy to create a component of size \( \Omega(n) \).
First problem in Erdős-Rényi paper . . .

**Small subgraph problem**

What is the threshold for $G(n, p)$ containing a given graph $H$?

Simple first-moment calculation:

$$E[#H in G(n, p)] = \frac{v(H)!}{n^{v(H)}} p^{e(H)} \sim n^{v(H)} p^{e(H)}$$

So if $p \ll \frac{n - v(H)}{e(H)}$, then $whp$ $G(n, p)$ does not contain $H$.

But:

if $H' \subseteq H$ and $-v(H) < -v(H')$, then $G(n, p)$ still has no $H$ for $n - v(H)/e(H) \ll p \ll n - v(H')/e(H')$.

**Definition**

A graph $H$ is balanced if the quantity $-v(H') e(H')$ is maximized by $H' = H$, for subgraphs $H' \subseteq H$. 
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\mathbb{E}[\#H \text{ in } G(n, p)] = \nu(H)! \binom{n}{\nu(H)} p^{e(H)} \sim n^{\nu(H)} p^{e(H)}
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So if \( p \ll n^{-\nu(H)/e(H)} \), then \textbf{whp} \( G(n, p) \) does not contain \( H \).
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\textbf{But}: if \( H' \subseteq H \) and \( -\frac{v(H)}{e(H)} < -\frac{v(H')}{e(H')} \), then \( G(n, p) \) still has no \( H \) for

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**But:** if $H' \subseteq H$ and $-\frac{v(H)}{e(H)} < -\frac{v(H')}{e(H')}$, then $G(n, p)$ still has no $H$ for

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**Definition**

A graph $H$ is *balanced* if the quantity $-\frac{v(H')}{e(H')}$ is maximized by $H' = H$, for subgraphs $H' \subseteq H$. 
For any given graph $H$, its threshold of appearance in $G(n, M)$ is $M \sim n^{2 + m(H)}$, where

$$m(H) = \max \left\{ -\frac{\nu(H')}{e(H')} : H' \subseteq H \right\}.$$
**Avoiding small subgraphs**

**Bollobás, 1981**

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**Krivelevich-L.-Sudakov, 2007**

Suppose that $H$ is a cycle, clique, or complete bipartite graph $K_{t,t}$. Then there is an explicit constant $\delta > 0$ such that:

- There is a strategy that can avoid creating a copy of $H$ for any $M \ll n^{2+m(H)+\delta}$ rounds \textit{whp}. 
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Remarks:

- There is no first-moment calculation to guess the bound. The first hurdle is to decide what to prove!
- Even the threshold existence does not follow from some general argument.
To avoid $K_3$:

- If there is a choice that does not create $K_3$, pick it.
- Otherwise, forced loss.

In general: for a given $H$ to avoid, there is a parameter $s$ for which the strategy considers precursors of $H$ up to $s$ steps back.

For both $K_t$ and $K_t'$, the dependence of $s$ on $t$ is magically explicit.

$s + 1 = \lfloor \log_2 (t + 1) \rfloor$. 
Avoidance strategy

**To avoid** $K_3$:
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**To avoid** $K_4$:
- If there is a choice that does not create $K_4 \setminus e$, pick it.
- Otherwise, if there is a choice that does not create $K_4$, pick it.
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- If there is a choice that does not create $K_4 \setminus e$, pick it.
- Otherwise, if there is a choice that does not create $K_4$, pick it.
- Otherwise, forced loss.

In general: for a given $H$ to avoid, there is a parameter $s$ for which the strategy considers precursors of $H$ up to $s$ steps back.

For both $K_t$ and $K_{t,t}$, the dependence of $s$ on $t$ is magically explicit:

$$s + 1 = \lfloor \log_2(t + 1) \rfloor.$$
Fix a strategy. Then the graph $G_m$ after $m$ rounds has some distribution. We sandwich $G_m$ between two known objects.
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Fix a strategy. Then the graph $G_m$ after $m$ rounds has some distribution. We sandwich $G_m$ between two known objects.

- $G_m \subset G(n, 2m)$, which is itself in $G(n, 4p)$ whp.
- We use extremal results (based on $e(G_m) = m$) to find “dangerous” structures in $G_m$.

**Classical theorems**

For $n$-vertex graphs with $\binom{n}{2} p$ edges:

- (Kővári-Sós-Turán) The number of copies of $K_{s,t}$ is $\gtrsim n^{s+t} p^{st}$.
- (Erdős-Simonovits) The number of copies of $P_t$ is $\gtrsim n^t p^{t-1}$.
Let $M \ll n^{6/5}$. We show that \textbf{whp}, our strategy will avoid $K_3$ for $M$ rounds. Let $p = M/\binom{n}{2} \ll n^{-4/5}$.
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- Even in $G_M$, the number of copies of $K_3 \setminus e$ is $\lesssim n^3 p^2$ whp.
Avoiding $K_3$: Lower bound

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- Then, the probability of creating a copy of $K_3$ in $M$ rounds is:

$$
\mathbb{P} \lesssim M \cdot \left( \frac{n^3 p^2}{n^2} \right)^2 \\
\sim \left( n^2 p \right) \left( \frac{n^3 p^2}{n^2} \right)^2 \\
= n^4 p^5 \\
= o(1).
$$
Let $M \gg n^{6/5}$. We show that \textbf{whp}, any strategy will make $K_3$ within $2M$ rounds. Let $p = M/(\binom{n}{2}) \gg n^{-4/5}$. 
Avoiding $K_3$: upper bound

Let $M \gg n^{6/5}$. We show that \textbf{whp}, any strategy will make $K_3$ within $2M$ rounds. Let $p = M/\binom{n}{2} \gg n^{-4/5}$.

- By extremal combinatorics, $G_M$ has $\gtrsim n^3 p^2$ copies of $K_3 \setminus e$; also, \textbf{whp} there are $\lesssim 1$ of these sitting on each vertex pair.
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- Then, probability that each of turns $M \ldots 2M$ do not have both choices closing $K_3$ is:

\[
P \leq \left(1 - \Omega \left(\frac{n^3 p^2}{n^2}\right)^2\right)^M
\]
\[
\leq \exp\left\{-\Omega \left(M \cdot \frac{n^6 p^4}{n^4}\right)\right\}
\]
\[
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- By extremal combinatorics, $G_M$ has $\gtrsim n^3 p^2$ copies of $K_3 \setminus e$; also, \textbf{whp} each pair extends $\lesssim 1$ of these into $K_3$.
- Then, probability that each of turns $M \ldots 2M$ do not have both choices closing $K_3$ is:

$$P \leq \left(1 - \Omega \left(\frac{n^3 p^2}{n^2}\right)^2\right)^M = o(1).$$
Avoiding $K_4$: upper bound

Let $M \gg n^{28/19}$. We show that \textbf{whp}, any strategy will make $K_4$ within $4M$ rounds. Let $p = M/(\binom{n}{2}) \gg n^{-10/19}$.

- By extremal combinatorics, $G_M$ has $\gtrsim n^4 p^3$ copies of $P_4$; also, \textbf{whp} each pair extends $\lesssim n^2 p^3$ of these into $C_4$.
- Then, probability that each of turns $3M \ldots 4M$ do not have both choices closing $K_4$ is:

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- whp, $G_{2M}$ has $\gtrsim n^4 p^4$ copies of $C_4$; also, whp each pair extends $\lesssim 1$ of these into $K_4 \setminus e$.

- whp, $G_{3M}$ has $\gtrsim n^6 p^9$ copies of $K_4 \setminus e$; also, whp each pair extends $\lesssim 1$ of these into $K_4$.

- Then, probability that each of turns $3M \ldots 4M$ do not have both choices closing $K_4$ is:

$$
\mathbb{P} \leq \left( 1 - \Omega \left( \frac{n^6 p^9}{n^2} \right)^2 \right)^M
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Concluding remarks

- We found precise thresholds for avoiding $C_t$, $K_t$, and $K_{t,t}$ in the Achlioptas process.
- Limitation of our method was reliance on balancedness.
- We conjecture that for general graphs, our simple strategies are still optimal.
Concluding remarks

- We found precise thresholds for avoiding $C_t$, $K_t$, and $K_{t,t}$ in the Achlioptas process.
- Limitation of our method was reliance on balancedness.
- We conjecture that for general graphs, our simple strategies are still optimal.
- Off-line case is still open. An off-line strategy can avoid $K_3$ for $n^{4/3} \gg n^{6/5}$ rounds: if one edge in a pair is not part of any triangles, pick it.

Only can fail if some triangle has all 3 partner edges also in some triangle.

$$\mathbb{P}[\text{failure}] \lesssim (n^3 p^3) \left( \frac{n^3 p^3}{n^2 p} \right)^3 = n^6 p^9$$