Edge States in Canonical Gravity*

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Abstract

It is well-known that gauge fields defined on manifolds with spatial boundaries support states localized at the boundary. In this talk, we show how similar states arise in canonical gravity and discuss their physical relevance using their analogy to quantum Hall effect.

I. INTRODUCTION

Recently there has been a renewal of interest in the problems of black hole evaporation [1] and the information loss puzzle. Usually one studies quantum processes involving black holes in a semiclassical approximation and typically one notices that the situation calls for unknown physics involving the event horizon and its surroundings [2]. There have been proposals to circumvent the situation by hypothesizing a stretched membrane with certain classical properties, situated just outside the black hole horizon, which essentially captures most of the important physics of black holes [2–4]. In a sense, this is a phenomenological theory for black holes for an observer who is not falling into the black hole.

The presence of such a membrane leads to an inner boundary for spatial slices. We show that the presence of this boundary leads to an infinite set of observables which are completely localized at this boundary. These are obtained here in analogy to “edge” observables in gauge theories defined on manifolds with boundaries [6,7]. Such observables have important physical relevance in many examples of condensed matter physics, for instance, the quantum Hall effect (QHE) [8]. As described in a work under preparation [5], this analogy with the quantum Hall effect can be useful for understanding the origin of black hole entropy.

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II. EXISTENCE OF EDGE STATES

To show how edge states arise in gauge theories on manifolds, we discuss a very simple example which we will also use later for QHE, namely Chern-Simons theory for the $U(1)$ gauge group \[7\]. The action describing Abelian Chern-Simons theory is given by

$$S_{cs} = \frac{k}{4\pi} \int_\mathcal{M} A dA$$  \hspace{1cm} (1)

where $\mathcal{M}$ is the space-time manifold having the topology of $\mathbb{R} \times D$, with $D$ being the disk. Note also that we are using differential form notation for the sake of brevity. As $A_0$ is nothing but a Lagrange multiplier one immediately sees that the Gauss law constraint (up to a numerical factor) is

$$G(A) \equiv \epsilon^{ij} \partial_i A_j \approx 0. \hspace{1cm} (2)$$

Here it should be stressed that the existence of a Gauss law is the backbone of the subsequent analysis.

In evaluating the Poisson brackets of the constraints amongst themselves and for finding their action on the phase space it is necessary to smear them with test functions so that they become differentiable \[7\]. So, we smear the Gauss law with the test function $\Lambda$,

$$G_\Lambda = \int_D \Lambda G(A) d^2x = \int_D \Lambda dA \approx 0. \hspace{1cm} (3)$$

Now we require that this functional generates gauge transformations, which in turn requires that $(\frac{\delta G_\Lambda}{\delta A})$ exists. However, one notes from (3) that

$$G_\Lambda = \int_D Ad\Lambda + \int_{\partial D} \Lambda A, \hspace{1cm} (4)$$

and therefore differentiability requires the boundary condition

$$\Lambda|_{\partial D} = 0. \hspace{1cm} (5)$$

Hence the Gauss law really is

$$G_\Lambda = \int_D Ad\Lambda, \hspace{1cm} (6)$$

with the gauge parameter $\Lambda$ subjected to (4). Then, due to the boundary condition (3),

$$\{G_\Lambda, G_{\Lambda'}\} = \frac{2\pi}{k} \int_D \Lambda d\Lambda' = 0. \hspace{1cm} (7)$$

Recall that, any gauge invariant object is an observable and hence must have zero Poisson bracket with the Gauss law. Thus we can define the functional

$$Q(\xi) \equiv \int_D A d\xi \hspace{1cm} (8)$$
which is inspired by the form (6) with \( \xi \) however not subjected to the boundary condition (7), i.e.,

\[
\xi|_{\partial D} \text{ not necessarily equal to } 0.
\]

\( Q(\xi) \) is an observable because

\[
\{G_{\Lambda}, Q(\xi)\} = \frac{2\pi}{k} \int_{\partial D} \Lambda d\xi = 0.
\]

The fact that the observables \( Q(\xi) \) are really associated with the edge can be shown as follows. If \( \xi \) and \( \xi' \) are the test functions such that they coincide on the boundary so that

\[
\xi|_{\partial D} = \xi'|_{\partial D} \Rightarrow (\xi - \xi')|_{\partial D} = 0,
\]

then

\[
Q(\xi) - Q(\xi') = \int_D Ad (\xi - \xi') = G_{(\xi - \xi')}.
\]

So this weakly vanishes since \( (\xi - \xi') \) satisfies the condition (5) showing that \( Q(\xi) \) is localized at the edge.

Finally we see that these observables generate a \( U(1) \) affine Lie algebra at the edge,

\[
\{Q(\xi), Q(\xi')\} = \frac{2\pi}{k} \int_{\partial D} \xi d\xi'.
\]

Now we can perform a similar analysis for the edge variables in canonical gravity.

**III. EDGE STATES: A CASE FOR GRAVITY**

To demonstrate the existence of edge states for gravity we will follow the canonical treatment as before. The standard ADM phase space analysis for spacetimes foliated by Cauchy surfaces is discussed elsewhere in detail \[10\][11] and hence will not be repeated here. The constraints are

\[
D_a p^{ab} \approx 0,
\]

\[
-q^{1\over 2} R^{(3)} + q^{-{1\over 2}} (p^{ab} p_{ab} - {p^2 \over 2}) \approx 0.
\]

Here \( p^{ab} \) is the momentum density conjugate to the spatial metric \( q_{ab} \) and \( D_a \) is the (projected) covariant derivative compatible with \( q_{ab} \).

As before, the vector constraint is to be smeared with a form \( V_a \) that vanishes at the boundaries of the manifold while the scalar constraint is to be smeared with a test function \( S \) that vanishes (along with its derivatives) at the boundaries. The boundaries here are the boundary \( \partial B_3 \) of \( B_3 \) (the spatial 3-ball whose boundary is the stretched membrane enclosing the black hole) and the spatial infinity.

The smeared constraints are
\[ V_V(q, p) = -2 \int_\Sigma d^3 x V_a D_b p^{ab} \approx 0, \]  
(16)

\[ S_S(q, p) = \int_\Sigma d^3 x S[-q^{1/2} (3) R + q^{-1/2} (p^{ab} p_{ab} - \frac{p^2}{2})] \approx 0, \]  
(17)

where

\[ V_a|_{\partial \Sigma} = 0 \]  
(18)

\[ S|_{\partial \Sigma} = 0, \quad D_a S|_{\partial \Sigma} = 0 \]  
(19)

The above conditions on the form \( V_a \) and the function \( S \) follow purely from requiring differentiability in the phase space variables \( q_{ab} \) and \( p^{ab} \) of (16) and (17).

The PB’s among the constraints are

\[ \{V_{V_1}, V_{V_2}\} = V_{[V_1, V_2]}, \] 
\[ \{V_V, S_S\} = S_{[V, S]}, \] 
\[ \{S_{S_1}, S_{S_2}\} = V_{[S_1, S_2]}. \]  
(20)

The construction of edge observables uses the trick that we have already employed in the Chern-Simons theory. We can construct edge observables, analogous to \( S_S \) and \( V_V \), whose test functions/forms will not be subjected to the boundary conditions (17). These observables turn out to differentiable after adding suitable surface terms. The difference of two of these observables with different smearing forms/functions which coincide (along with derivatives in the case of the latter) only at the boundaries is a constraint and hence they are truly edge degrees of freedom.

To construct the edge observables arising from the vector (diffeomorphism) constraint we first rewrite the vector constraint in (14) after a partial integration as

\[ V_V = -\int_\Sigma d^3 x q_{ab} \mathcal{L}_V p^{ab}. \] 
(21)

In the above, let us replace \( V \) by \( W \) where \( W \) is any vector field. We require of \( W \) that, at the boundaries of the manifold, it is tangential to the boundary. Then it can be verified that the quantity so obtained, namely

\[ D_W = -\int_\Sigma d^3 x q_{ab} \mathcal{L}_W p^{ab}. \]  
(22)

continues to be differentiable in both \( q_{ab} \) and \( p^{ab} \). It furthermore has weakly zero PB’s with the constraints:

\[ \{D_W, V_V\} = V_{[W, V]}, \] 
\[ \{D_W, S_S\} = S_{[W, S]} \]  
(23)

The right hand sides in these equations are constraints and hence weakly zero because their respective test fields are easily verified to satisfy the conditions (18) and (19).

The algebra of observables generated by \( D_W \) is seen to be
\{D_{W_1}, D_{W_2}\} = D_{[W_1, W_2]}.

We are interested in observables which are supported at the edge corresponding to the membrane rather than those which are supported at spatial infinity. We will therefore hereafter assume that $W$ is non-zero only at the inner boundary and vanishes like $V$ at the boundary at infinity.

Next, let us look at the scalar constraint $S_S$:

$$S_S = \int_{\Sigma} d^3x \left[ \sum_{\{(3)R\}} - q^{\frac{1}{2}} R + q^{-\frac{1}{2}} \left( p^{ab} p_{ab} - \frac{p^2}{2} \right) \right].$$

The above is clearly differentiable in $p^{ab}$. As for differentiability in $q^{ab}$, it can be verified that a variation of $q_{ab}$ induces surface terms in its variation. They vanish only if the test functions $S$ satisfy (19). The condition on their derivatives emerges because variation of $(3)R$ contains second derivatives of the variation of the metric $q_{ab}$ [10]. The boundary condition in (19) on $S$ are in fact got from this requirement of differentiability of $S_S$.

Consider (25) with $S$ replaced by $T$, which however does not have to satisfy the boundary conditions satisfied by $S$. The only term in the expression that would require careful scrutiny for differentiability in $q_{ab}$ is

$$\int_{\Sigma} d^3x T \left[ - q^{\frac{1}{2}} (3)R \right].$$

The change in above term due to a variation $\delta q_{ab}$ is

$$\int_{\Sigma} d^3x T \left[ \frac{1}{2} (3)R q^{ab} - (3)R^{ab} \right] \delta q_{ab} - \int_{\Sigma} d^3x T q^{\frac{1}{2}} \left[ D^a D^b (\delta q_{ab}) - D^a (q^{cd} D_a \delta q_{cd}) \right].$$

Since the second term above contains derivatives of $\delta q_{ab}$, (26) is not differentiable with respect to $q_{ab}$.

Suppose now that

$$\delta q_{ab}|_{\partial \Sigma} = 0,$$

$$D_a T|_{\partial \Sigma} = 0.$$

[Note that (29) implies that $T$ at the boundary goes to a constant which can be non-zero.] The terms involving derivatives of $\delta q_{ab}$ in (27) give rise to surface terms in the variation. These surface terms are now exactly cancelled by the variation of

$$-2 \int_{\partial \Sigma} TK \sqrt{h}$$

where $K_{ab}$ and $h_{ab}$ are respectively the extrinsic curvature and the induced metric of the boundary $\partial \Sigma$ [14]. Thus so long as the conditions (28) and (29) above are met, we can define an edge observable of the form

$$H_T = \int_{\Sigma} d^3x T \left[ - q^{\frac{1}{2}} (3)R + q^{-\frac{1}{2}} \left( p^{ab} p_{ab} - \frac{p^2}{2} \right) \right] - 2 \int_{\partial \Sigma} d^2x h^{\frac{1}{2}} TK.$$

These edge observables, as presented, are independent of the observables defined in the bulk. It is then not clear how coarse-graining over the edge degrees of freedom can lead to an entanglement entropy for black holes [13]. We thus require a coupling between the edge and bulk degrees of freedom. Quantum Hall effect again provides us with the model where such a coupling occurs. This is what we discuss in the next section.
IV. THE QUANTUM HALL EFFECT: A MODEL FOR THE DYNAMICS OF EDGE DEGREES OF FREEDOM

A simple effective action that describes the physics of quantum Hall effect is the Chern-Simons action added on to the usual electromagnetic action:

\[ S_{\text{bulk}} = \int_M d^3x \left[ -\frac{t}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\sigma_H}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right], \quad (31) \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Here \( t \) is a constant related to the “effective thickness” of the Hall sample, while our metric is \((-1,+1,+1)_{\text{diagonal}}\). The \( \sigma_H \) that appears as the coefficient of the Chern-Simons term is the Hall conductivity.

The connection of the above system with edge observables is also well-known. The latter arise when we confine the above theory to a finite geometry (as is appropriate for any physical Hall sample). From very general arguments first articulated by Halperin [12], the existence of chiral edge currents at the boundary can then be established.

Naively, the theory in the bulk described by the action (31) does not communicate with the theory describing these chiral currents at the edge. It is then not clear how these edge currents can have any role in the description of bulk phenomena. However, gauge invariance [8,9] allows us to put them together. Thus the action (31) under the gauge transformation \( A \rightarrow A + d\alpha \) changes by the surface term

\[ -\frac{\sigma_H}{2} \int_{\partial M} d\alpha \wedge A. \quad (32) \]

But, physics is gauge invariant. Therefore it must be that there is a theory at the boundary describing the chiral edge currents which is also gauge non-invariant such that the total action \( (S_{\text{tot}} = S_{\text{bulk}} + S_{\text{edge}}) \) is itself gauge invariant. This line of argument [8] then leads us to the action

\[ S_{\text{tot}} = S_{\text{bulk}} + \frac{\sigma_H}{2} \int_{\partial M} d\phi \wedge A - \frac{\sigma_H}{4} \int_{\partial M} D_\mu \phi \, D^\mu \phi, \quad (33) \]

\[ D_\mu \phi = \partial_\mu \phi - A_\mu. \quad (34) \]

The field \( \phi \) under a gauge transformation transforms as

\[ \phi \rightarrow \phi + \alpha \quad \Rightarrow \quad D\phi \equiv d\phi - A \rightarrow D\phi \quad (35) \]

so that

\[ S_{\text{tot}} \rightarrow S_{\text{tot}}. \quad (36) \]

The second term in (33) is the term which restores gauge invariance. The last term is a kinetic energy term and is required if the theory at the edge is to give rise to a chiral theory.

The dynamics of the edge field on the boundary and its coupling to the gauge field allows one to calculate the entanglement entropy [13] arising due to a coarse-graining over the edge degrees of freedom [14]. One finds that the entropy scales as the perimeter of the disk. So it is natural to inquire whether the black hole entropy also arises due to a coarse-graining of the black hole edge states. However, to do this one needs to know the dynamics of the black hole edge states.
V. EDGE DYNAMICS FOR GRAVITY?

The ideas described in the previous sections give us hints about the dynamics of the edge degrees of freedom for black holes. In fact, for (2+1) dimensional gravity, which happens to be a Chern-Simons theory, one can find the edge action exactly \cite{15,5}. However, the situation for the (3+1) dimensional case is not so clear. One can write a kinetic energy term for the edge degrees of freedom though their coupling to the external fields would remain arbitrary. We hope to report on these matters in detail some time in the future.

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REFERENCES

[1] S. Hawking, Commun. Math. Phys. 43, 199 (1975).
[2] L. Susskind, Rutgers University preprint RU-93-44 (hep-th/9309143), L. Susskind, Phys. Rev. Lett. 71, 2367 (1993), L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D 48, 3743 (1993), L. Susskind and J. Uglum, Phys. Rev. D 50, 2700 (1994).
[3] K.S. Thorne, R.H. Price and D.A. Macdonald, Black Holes: The Membrane Paradigm [Yale University Press (1986)].
[4] M. Maggiore, Phys. Rev. D 49, 2918 (1993), Phys. Lett B 333,93(1994); Nucl. Phys. B 429, 205 (1994); University of Pisa preprint, IFUP-TH 43/94 (gr-qc/940716).
[5] For details see, A.P. Balachandran, L. Chandar and A. Momen, Syracuse University Preprint, SU-4240/590 (gr-qc/9412019).
[6] E. Witten, Comm. Math. Physics, 121, 351 (1989); S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B 326, 108 (1990).
[7] A.P. Balachandran, G. Bimonte, K.S. Gupta and A. Stern, Int. J. Mod. Phys. A 7, 4655 (1992); ibid. A 7, 5855 (1992); A.P. Balachandran, G. Bimonte and P. Teotonio-Sobrinho, Mod. Phys. Lett. A 8, 1305 (1993), A. P. Balachandran, L. Chandar and E. Ercolessi, Int. J. Mod. Phys. A 10, 1969 (1995); For a review see A. P. Balachandran, “Gauge Symmetries, Topology and Quantization” in AIP Conference Proceedings, No. 317, Vth Mexican School of Particles and Fields, Guanajuato, Mexico 1992, Editors J.L. Lucio and M. Vargas (1994), p.1.
[8] See for example, A.P. Balachandran, L. Chandar and B. Sathiapalan, preprint- SU-4240-578, PSU/TH/144. (hep-th/9405141) and Nucl. Phys. B (in press).
[9] The anomaly argument originates in the papers of C.G. Callan and J. Harvey, Nucl. Phys. B 250, 427 (1985); S. Naculich, Nucl. Phys. B 296, 837 (1988); F. Wilczek, Fractional Statistics and Anyon Superconductivity [World Scientific, Singapore(1990)]; M. Stone, Ann. Phys. 207, 38 (1991); J. Fröhlich and U.M. Studer, Rev. Mod. Phys., 65, 733 (1993) and references therein.
[10] R. Wald, General Relativity [University of Chicago Press (1984)].
[11] A. Ashtekar, New Perspectives in Canonical Gravity [Bibliopolis, Naples (1988)]; Lectures on Non-perturbative Canonical Gravity [World Scientific, Singapore (1991)]. See also Appendix E of [10].
[12] B. Halperin, Phys. Rev. B 25, 2165 (1984).
[13] L. Bombelli, R. Koul, J. Lee and R. Sorkin, Phys. Rev. D 34, 373 (1986); M. Srednicki, Phys. Rev. Lett. 71, 666 (1993); D. Kabat and M.J. Strassler, Phys. Lett. B 329, 46 (1994); C. G. Callan and F. Wilczek, Phys. Lett. B 333, 55 (1994); J.S. Dowker, Class. Quant. Grav. 11, L55 (1994); C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B 424, 443 (1994); F. Larsen and F. Wilczek, Princeton University preprint, PUPT-1480, IASSNS 94/51 (hep-th/9408089) and references therein.
[14] A.P.Balachandran, L. Chandar and A. Momen, (under preparation).
[15] S. Carlip, Phys. Rev. D 51, 632(1995).