WEIGHTED DAVIS INEQUALITIES
FOR MARTINGALE SQUARE FUNCTIONS
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Abstract. For a Hilbert space valued martingale \((f_n)\) and an adapted sequence of positive random variables \((w_n)\), we show the weighted Davis type inequality

\[
E\left(|f_0|w_0 + \frac{1}{4} \sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} w_n\right) \leq E(f_N^* w_N^*).
\]

This inequality is sharp and implies several results about the martingale square function. We also obtain a variant of this inequality for martingales with values in uniformly convex Banach spaces.

1. Introduction
Throughout this article, \((f_n)\) denotes a martingale on a filtered probability space \((\Omega, (F_n)_{n \in \mathbb{N}})\) with values in a Banach space \((X, |\cdot|)\). A weight is a positive random variable on \(\Omega\). We denote martingale differences and running maxima by
\[
df_n = f_n - f_{n-1}, \quad f_n^* := \max_{n' \leq n} |f_{n'}|, \quad w_n^* := \max_{n' \leq n} w_{n'}.
\]

We begin with the Hilbert space valued case of our main result (Theorem 2.3).

Theorem 1.1. Let \((f_n)_{n \in \mathbb{N}}\) be a martingale with values in a Hilbert space \((X = H, |\cdot|)\). Let \((w_n)_{n \in \mathbb{N}}\) be an adapted sequence of weights (that need not be a martingale). Then, for every \(N \in \mathbb{N}\), we have

\[
(1.1) \quad E\left(|f_0| + \frac{1}{3} \sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*}\right) \leq E(f_N^*)
\]

and

\[
(1.2) \quad E\left(|f_0|w_0 + \frac{1}{4} \sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} w_n\right) \leq E(f_N^* w_N^*).
\]

A quantity similar to the left-hand side of (1.1), but with \(f_N^*\) in place of \(f_n^*\) and hence smaller, appeared in [Gar73b, §3]. In order to relate our result to the usual martingale square function

\[
Sf := \left(\sum_{n=1}^{N} |df_n|^2\right)^{1/2},
\]

we note that, by Hölder’s inequality,

\[
(1.3) \quad E Sf \leq E\left((f_N^*)^{1/2} \left(\sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*}\right)^{1/2}\right) \leq \left(E f_N^*\right)^{1/2} \left(E \sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*}\right)^{1/2}.
\]

By one of the Burkholder–Davis–Gundy inequalities [Dav70], we have \(E f_N^* \leq C E Sf\) for martingales with \(f_0 = 0\) (the optimal value of \(C\) does not seem to be known; the value \(C = \sqrt{10}\) was obtained in [Gar73a, II.2.8]). Assuming that both sides are finite, this implies

\[
E Sf \leq C E \sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*}
\]

with the same constant \(C\). Thus, we see that (1.1) adds a new equivalence to the \(L^1\) Burkholder–Davis–Gundy inequalities.

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The proof of Theorem 1.1 is based on Burkholder’s proof of the Davis inequality for the square function with the sharp constant [Bur02] and its weighted extension by Osękowski [Ose17b]. Note, however, that the weights in the latter article are assumed to be continuous in time, so that it does not yield weighted estimates in discrete time. The estimate (1.2) is instead motivated by [Ose17a], where the Davis inequality for the martingale maximal function was proved with a similar combination of weights \((w, w^\ast)\). Such weighted inequalities go back to [FS71], see also [HvNVW16, Theorem 3.2.3] for a martingale version.

1.1. Sharpness of the constants. Both estimates (1.1) and (1.2) are sharp, in the sense that the constants 1/3 and 1/4 cannot be replaced by any larger constants.

The sharpness of (1.1) is due to the fact that it implies the sharp version of the Davis inequality for the expectation of the martingale square function [Bur02]. To see this, for notational simplicity, suppose \(f_0 = 0\). By (1.3) and (1.1), we have

\[
\mathbb{E} Sf \leq \sqrt{3} \mathbb{E} f_N^*.
\]

Since the constant \(\sqrt{3}\) is the smallest possible in this inequality [Bur02, §5], also the constant in (1.1) is optimal.

The sharpness of (1.2) is proved in Section 4.

1.2. Consequences of the weighted estimate. Here, we show how Theorem 1.1 can be used to recover a number of known inequalities.

Let \(r \in [1, 2]\), \(w\) be an integrable weight, \(w_n = \mathbb{E}(w|\mathcal{F}_n), f^\ast = f^\ast_w\), and \(w^\ast = w^\ast_w\). For simplicity, we again assume \(f_0 = 0\). By Hölder’s inequality and (1.1), we obtain

\[
\mathbb{E} \left((Sf)^r \cdot w\right) \leq \mathbb{E} \left((f^\ast)^{(2-r)r/2}(\sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} (f_n^*)^{(r-1)/2} w)\right).
\]

\[
\leq \left(\mathbb{E} (f^\ast)^r w\right)^{1-r/2} \left(\mathbb{E} (\sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} (f_n^*)^{r-1} w)\right)^{r/2}
\]

\[
= \left(\mathbb{E} (f^\ast)^r w\right)^{1-r/2} \left(\mathbb{E} (\sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} (f_n^*)^{r-1} w)\right)^{r/2}
\]

\[
\leq 2^r \left(\mathbb{E} (f^\ast)^r w\right)^{1-r/2} \left(\mathbb{E} (f^\ast)^{r} w^\ast\right)^{r/2}.
\]

If we estimate \(w \leq w^\ast\) in the first term on the right-hand side, we recover a version of the main result in [Ose17b]. Our version has a worse constant, but does not require the weights to be continuous in time.

Recall that the \(A_1\) characteristic of a weight \(w\) is the smallest constant \([w]_{A_1}\) such that \(w^\ast \leq [w]_{A_1} w\). As a direct consequence of the estimate (1.4), we obtain the estimate

\[
\mathbb{E} \left(\sum_{n=1}^{N} |df_n|^2 \right)^{1/2} w \leq 2[w]_{A_1}^{1/2} \mathbb{E} (f^* w)
\]

for \(A_1\) weights \(w\). This improves the main result of [Ose18], where a similar estimate (with \(\sqrt{5}\) in place of 2) was proved for dyadic martingales. In view of [BO21, Theorem 1.3], it seems unlikely that the \(A_1\) characteristic in (1.5) can be replaced by a function of any \(A_p\) characteristic with \(p > 1\), although for dyadic martingales even the \(A_\infty\) characteristic suffices [GW74, Theorem 2].

By a version of the Rubio de Francia argument, one can deduce further \(L^p\) weighted estimates from (1.4) (with \(r = 1\)). Let \(p \in (1, \infty), w\) a weight, and \(\tilde{w} = w^{-p'/p}\) the dual weight, where \(p'\) denotes the Hölder conjugate that is determined by \(1/p+1/p' = 1\). Let also

\[
Mh := \sup_{n \in \mathbb{N}} |\mathbb{E}(h|\mathcal{F}_n)|
\]
denote the martingale maximal operator. Then, for any function $u$, by (1.4) and Hölder’s inequality, we obtain

\begin{equation}
\mathbb{E}(Sf \cdot u \cdot w) \leq 2\mathbb{E}(Mf \cdot M(uw))^{1/2} \cdot \mathbb{E}(Mf \cdot (uw))^{1/2}
\end{equation}

(1.6)

By duality, this implies

\begin{equation}
\|Sf\|_{L^p(w)} \leq 2\|M\|_{L^{p/2}(\tilde{w})}^{1/2} \|Mf\|_{L^{p/2}(\tilde{w})}^{1/2} \|f^*\|_{L^p(w)}
\end{equation}

(1.7)

Substituting this into (1.6), we obtain

\begin{equation}
\mathbb{E}(Sf \cdot u \cdot w) \leq 2\|M\|_{L^{p/2}(\tilde{w})}^{1/2} \|Mf\|_{L^{p/2}(\tilde{w})}^{1/2} \|Euf\|_{L^{p/2}(\tilde{w})}^{1/2}.
\end{equation}

By duality, this implies

In the case $w \equiv 1$, using Doob’s maximal inequality [HvNVW16, Theorem 3.2.2], this recovers the following version of the martingale square function inequality, which matches [Bur73, Theorem 3.2]:

\begin{equation}
\|Sf\|_{L^p} \leq 2\sqrt{p/p'} \|f\|_{L^{p'}}.
\end{equation}

More generally, an $A_p$ weighted BDG inequality can be obtained from (1.7) using the $A_{p'}$ weighted martingale maximal inequality proved in [DP16].

Another Rubio de Francia type extrapolation argument, see [Zor21, Appendix A], can be used to deduce UMD Banach space valued estimates from either (1.2) or (1.4). This recovers one of the estimates in [VY19, Theorem 1.1] (the other direction similarly follows from the weighted estimate in [Osę17a]).

2. Uniformly convex Banach spaces

In this section, we recall a few facts about uniformly convex Banach spaces that are relevant to the Banach space valued version of Theorem 1.1, Theorem 2.3.

**Definition 2.1.** Let $q \in [2, \infty)$. A Banach space $(X, \cdot \cdot)$ is called $q$-uniformly convex if there exists $\delta > 0$ such that, for every $x, y \in X$, we have

\begin{equation}
\frac{|x + y|}{2} + \delta \frac{|x - y|}{2} \leq \frac{|x|^q + |y|^q}{2}.
\end{equation}

We will use a different (but equivalent) characterization of uniform convexity, in terms of the convex function $\phi : X \to \mathbb{R}_{\geq 0}$, $\phi(x) = |x|^q$ and its directional derivative at point $x$ in direction $h$, given by

\begin{equation}
\phi'(x)_h := \lim_{t \to 0, t > 0} \frac{\phi(x + th) - \phi(x)}{t}.
\end{equation}

Convexity of $\phi$ is equivalent to the right-hand side of (2.2) being an increasing function of $t$ for fixed $x, h$. By the triangle inequality and Taylor’s formula, we have

\begin{equation}
|\phi'(x)_h| \leq |\phi'(x)|_h \leq q|x|^q - |x|^q \leq q|x|^{q-1}|h| + o_{|h| \to 0}(|h|).
\end{equation}

Therefore, the quotient on the right-hand side of (2.2) is bounded from below. Hence, the limit (2.2) exists, and we have

\begin{equation}
|\phi'(x)_h| \leq q|x|^{q-1}|h|.
\end{equation}

Moreover, for every $x \in X$, the function $h \mapsto \phi'(x)_h$ is convex, which follows directly from convexity of $\phi$.

**Lemma 2.2.** A Banach space $(X, \cdot \cdot)$ is uniformly convex if and only if, for every $x, h \in X$, we have

\begin{equation}
|x + h|^q \geq |x|^q + \phi'(x)_h + \tilde{\delta}|h|^q.
\end{equation}

(2.4)
Moreover, the largest $\delta, \tilde{\delta}$ for which (2.1) and (2.4) hold satisfy

\[
\frac{\delta}{2q - 1 - 1} \leq \tilde{\delta} \leq \delta.
\]

The estimate (2.4) can only hold with $\tilde{\delta} \leq 1$ (unless $X$ is 0-dimensional), as can be seen by taking $x = 0$. When $X$ is a Hilbert space, we can take $q = 2$ and $\delta = 1$ in (2.1) by the parallelogram identity and $\tilde{\delta} = 1$ in (2.4) by (2.5).

Proof. Clearly, the sets of $\delta$ and $\tilde{\delta}$ for which (2.1) and (2.4) hold are closed, so we may consider the largest such $\delta$ and $\tilde{\delta}$.

To see the first inequality in (2.5), let $C$ be the set of all constants $c \geq 0$ such that, for every $x, h \in X$, we have

\[
\phi(x + h) \geq \phi(x) + \phi'(x)h + c\phi(h).
\]

By convexity of $\phi$, we have $0 \in C$.

Let $c \in C$. For any $x, h \in X$, using the uniform convexity assumption (2.1) with $y = x + h$, we obtain

\[
|x + h/2|^q + \delta|h/2|^q \leq (|x|^q + |x + h|^q)/2.
\]

By the definition of $c \in C$, it follows that

\[
|x|^q + \phi'(x)h/2 + c|h/2|^q + \delta|h/2|^q \leq (|x|^q + |x + h|^q)/2.
\]

Rearranging this inequality, we obtain

\[
|x|^q + \phi'(x)h + 2c|h/2|^q + 2\delta|h/2|^q \leq |x + h|^q.
\]

Therefore, $2^{1-q}(c + \delta) \in C$. Since $c \in C$ was arbitrary, this implies

\[
\sup C \geq \delta/(2^{q-1} - 1).
\]

To see the second inequality in (2.5), note that convexity of $h \mapsto \phi'(x)h$ implies $\phi'(x)h + \phi'(x)(-h) \geq 0$. Applying (2.4) with $(z, h)$ and $(z, h)$, we obtain

\[
|z + h|^q + |z - h|^q \geq 2|z|^q + \phi'(z)h + \phi'(z)(-h) + \delta|h|^q + \delta|-h|^q
\]

\[
\geq 2|z|^q + 2\delta|h|^q.
\]

With the change of variables $x = z + h$, $y = z - h$, we obtain (2.1) with $\tilde{\delta}$ in place of $\delta$.

With the characterization of uniform convexity in (2.4) at hand, we can finally state our main result in full generality.

Theorem 2.3. For every $q \in [2, \infty)$, there exists $\gamma = \gamma(q) \in \mathbb{R}_{>0}$ such that the following holds.

Let $(X, |\cdot|)$ be a Banach space such that (2.4) holds. Let $(f_n)_{n \in \mathbb{N}}$ be a martingale with values in $X$, and $(w_n)_{n \in \mathbb{N}}$ an adapted sequence of weights. Then,

\[
\mathbb{E}\left(|f_0|w_0 + \delta \sum_{n=1}^{\infty} \frac{|df_n|^q}{(w^*)^q} w_n\right) \leq \gamma \mathbb{E}(f^*w^*)
\]

In the case $q = 2$, we can take $\gamma = 4$. In the case $q = 2$, $\tilde{\delta} = 1$, and $w_n = 1$ for all $n \in \mathbb{N}$, we can take $\gamma = 3$.

In order to see that the linear dependence on $\tilde{\delta}$ in (2.6) is optimal, we can apply this inequality with $w_n = (f_n^*)^{q-1}$, followed by Doob’s maximal inequality, which gives the estimate

\[
\mathbb{E}\left(|f_0|^q + \delta \sum_{n=1}^{\infty} |df_n|^q\right) \leq \gamma \mathbb{E}(f^*q) \leq (q')^q \sup_{n \in \mathbb{N}} \mathbb{E}|f_n|^q.
\]

By [Pis16, Theorem 10.6], linear dependence on $\tilde{\delta}$ is optimal in this inequality, and hence the same holds for (2.6).

Similarly as in Section 1.2, Theorem 2.3 implies several weighted extensions of the martingale cotype inequality [Pis16, Theorem 10.59]. We omit the details.
The proof of Theorem 2.3 is based on the Bellman function technique; we refer to the books [Ose12; VV20] for other instances of this technique. The particular Bellman function that we use here goes back to [Bur02]; the first weighted version of it was introduced in [Ose17b].

For $x \in X$ and $y, m, v \in \mathbb{R}_{\geq 0}$ with $|x| \leq m$, we define

$$U(x, y, m, v) := \delta y - \frac{|x|^q + (\gamma - 1)m^q}{m^{q-1}}v,$$

where $\gamma = \gamma(q)$ will be chosen later. The main feature of this function is the following concavity property.

**Proposition 3.1.** Suppose that $\gamma$ is sufficiently large depending on $q$ (see (3.8) for the precise condition). Let $(X, |\cdot|)$ be a Banach space such that (2.4) holds. Then, for any $x, h \in X$ and $y, m, w, v \in \mathbb{R}_{\geq 0}$ with $|x| \leq m$, we have

$$U(x + h, y + \frac{|w|h|^q}{(|x| + h|)q - 1}, |x + h| \vee m, v \vee w) \leq U(x, y, m, v) - \frac{v\phi'(x)h}{m^{q-1}}. \tag{3.1}$$

**Proof of Theorem 2.3 assuming Proposition 3.1.** Using (3.1) with

$$x = f_n, \quad y = \tilde{S}_n := \gamma|f_0|w_0 + \delta \sum_{j=1}^{n} \frac{|df_j|^q}{(f_j^*)^{q-1}}w_j, \quad m = f_n^*,$$

$$w = w_n, \quad v = w_n^*, \quad h = df_{n+1},$$

we obtain

$$U(f_{n+1}, \tilde{S}_{n+1}, f_{n+1}^*, w_{n+1}^*) \leq U(f_n, \tilde{S}_n, f_n^*, w_n^*) - \frac{\phi'(f_n)df_{n+1}}{(f_n^*)^{q-1}}w_n^*. \tag{3.2}$$

By convexity of $h \mapsto \phi'(x)h$, we have

$$\mathbb{E}(\frac{\phi'(f_n)df_{n+1}}{(f_n^*)^{q-1}}w_n^*|\mathcal{F}_n) = \frac{w_n^*}{(f_n^*)^{q-1}}\mathbb{E}(\phi'(f_n)df_{n+1}|\mathcal{F}_n) \geq 0.$$

Taking expectations, we obtain

$$\mathbb{E}U(f_{n+1}, \tilde{S}_{n+1}, f_{n+1}^*, w_{n+1}^*) \leq \mathbb{E}U(f_n, \tilde{S}_n, f_n^*, w_n^*).$$

Iterating this inequality, we obtain

$$\mathbb{E}\left(\gamma|f_0|w_0 + \delta \sum_{n=1}^{N} \frac{|df_n|^q}{(f_n^*)^{q-1}}w_n - \gamma f_N^*w_N^*\right) \leq \mathbb{E}U(f_N, \tilde{S}_N, f_N^*, w_N^*) \leq \mathbb{E}(f_0, \tilde{S}_0, f_0^*, w_0^*) = 0. \quad \square$$

**Remark 3.2.** The above proof in fact shows the pathwise inequality

$$\gamma|f_0|w_0 + \delta \sum_{n=1}^{N} \frac{|df_n|^q}{(f_n^*)^{q-1}}w_n \leq \gamma f_N^*w_N^* - \sum_{n=1}^{N} \frac{\phi'(f_n)df_{n+1}}{(f_n^*)^{q-1}}w_n^*.$$

This can be used to improve the first part of [BS15, Theorem 1.1]. For simplicity, consider the scalar case $X = \mathbb{C}$ (so that $q = 2$ and $\tilde{\delta} = 1$) with $f_0 = 0$ and $w_0 = 1$. The above inequality then simplifies to

$$\sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} \leq 3f_N^* - \sum_{n=1}^{N} \frac{2f_n|df_{n+1}|}{f_n^*}.$$
Using (1.3), the above inequality, and concavity of the function \( x \mapsto x^{1/2} \), we obtain

\[
S_N f \leq (f_N^*)^{1/2} \left( \sum_{n=1}^{N} \frac{|df_n|^2}{f_n^*} \right)^{1/2}.
\]

\[
\leq (f_N^*)^{1/2} \left( 3f_N^* - \sum_{n=1}^{N} \frac{2f_n df_{n+1}}{f_n^*} \right)^{1/2}.
\]

\[
\leq (f_N^*)^{1/2} \left( (3f_N^*)^{1/2} - \frac{1}{2} (3f_N^*)^{-1/2} \sum_{n=1}^{N} \frac{2f_n df_{n+1}}{f_n^*} \right)^{1/2}.
\]

\[
= \sqrt{3} f_N^* - \sum_{n=1}^{N} \frac{f_n df_{n+1}}{\sqrt{3} f_n^*}.
\]

**Proof of Proposition 3.1.** If \(|x + h| \leq m\), then

\[
U(x + h, y + \frac{w|h|^q}{(|x + h| \vee m)^q}, |x + h| \vee m, v \vee w) = \tilde{\delta}(y + \frac{w|h|^q}{m^{q-1}}) - |x + h|^q + (\gamma - 1)m^q (v \vee w)
\]

\[
\leq \tilde{\delta}(y + \frac{w|h|^q}{m^{q-1}}) - |x|^q + \phi'(x)h + \tilde{\delta}|h|^q + (\gamma - 1)m^q (v \vee w)
\]

\[
\leq \tilde{\delta} y - \frac{|x|^q + \phi'(x)h + (\gamma - 1)m^q}{m^{q-1}} (v \vee w)
\]

\[
= U(x, y, m, v, v) - \phi'(x)h \frac{m^{q-1}}{m^{q-1}} v.
\]

In the last inequality, we used

\[(3.3) \quad |\phi'(x)h| \leq q|x|^{q-1}|h| \leq q|x|^{q-1}(|x| + m) \leq |x|^q + (2q - 1)m^q \leq |x|^q + (\gamma - 1)m^q,
\]

which holds provided that \( \gamma \geq 2q \).

If \(|x + h| > m\), then we need to show

\[
\tilde{\delta}(y + \frac{w|h|^q}{m^{q-1}}) - \frac{|x + h|^q + (\gamma - 1)|x + h|^q}{|x + h|^{q-1}} (v \vee w)
\]

\[
\leq \tilde{\delta} y - \frac{|x|^q + (\gamma - 1)m^q}{m^{q-1}} v - \phi'(x)h \frac{m^{q-1}}{m^{q-1}} v.
\]

This is equivalent to

\[(3.4) \quad \frac{\delta |h|^q w - \gamma |x + h|^q (v \vee w)}{|x + h|^{q-1}} \leq -\frac{|x|^q v - (\gamma - 1)m^q v}{m^{q-1}} - \frac{\phi'(x)h}{m^{q-1}} v.
\]

Assuming that \( \gamma \geq 2\tilde{\delta} \), we have

\[(3.5) \quad \delta |h|^q \leq \tilde{\delta}(|x + h| + |x|)^q \leq \tilde{\delta}(2|x + h|)^q \leq \gamma |x + h|^q,
\]

and it follows that the left-hand side of (3.4) is

\[
\leq (v \vee w) \frac{\delta |h|^q - \gamma |x + h|^q}{|x + h|^{q-1}} \leq v \frac{\delta |h|^q - \gamma |x + h|^q}{|x + h|^{q-1}}.
\]

Hence, it suffices to show

\[(3.6) \quad \frac{\delta |h|^q - \gamma |x + h|^q}{|x + h|^{q-1}} \leq -\frac{|x|^q - (\gamma - 1)m^q}{m^{q-1}} - \frac{\phi'(x)h}{m^{q-1}} v.
\]

Let \( t := |x + h|/m > 1 \) and \( \tilde{t} := |h|/m \). Note that \( |t - \tilde{t}| = |(x + h)|/m \leq |h|/m \leq 1 \). We will show (3.6) in two different ways, depending on the values of \( t, \tilde{t} \).
Estimate 1. By (2.4), the inequality (3.6) will follow from
\[ \frac{\tilde{\delta} |h|^{q} - \gamma |x + h|^{q}}{|x + h|^{q-1}} \leq -|x + h|^{q} + \tilde{\delta} |h|^{q} - (\gamma - 1)m^{q}. \]
This is equivalent to
\[ \tilde{\delta} t^{q}/t^{q-1} - \gamma t \leq -t^{q} + \tilde{\delta} t^{q} - (\gamma - 1). \]
Equivalently,
\[ t^{q} - \gamma t + (\gamma - 1) \leq \tilde{\delta} t^{-q}(1 - 1/t^{q-1}). \]
so this would follow from
\[ \gamma \geq \frac{1}{t - 1}(t^{q} - 1 - \tilde{\delta} t^{q}(1 - 1/t^{q-1})). \]

Estimate 2. The inequality (3.6) is implied by
\[ \frac{\tilde{\delta} |h|^{q}}{|x + h|^{q-1}} + \frac{|x|^{q} + (\gamma - 1)m^{q}}{m^{q-1}} + \frac{\phi'(x)|h|}{m^{q-1}} \leq |x| + h. \]
This is equivalent to
\[ \frac{\tilde{\delta} t^{q}}{t^{q-1}} + \frac{|x|^{q}}{m^{q}} + (\gamma - 1) + \frac{\phi'(x)|h|}{m^{q}} \leq \gamma t. \]
Using (2.3), we see that the left-hand side is bounded by
\[ \frac{\tilde{\delta} t^{q}}{t^{q-1}} + 1 + (\gamma - 1) + \frac{q|x|^{q-1} |h|}{m^{q}} \leq \frac{\tilde{\delta} t^{q}}{t^{q-1}} + \gamma + qt. \]
Hence, it suffices to assume
\[ \frac{\tilde{\delta} t^{q}}{t^{q-1}} + \gamma + q(t + 1) \leq \gamma t, \]
or, in other words,
\[ \gamma \geq \frac{1}{t - 1} \left( \frac{\tilde{\delta} t^{q}}{t^{q-1}} + qt \right). \]
Combining the two estimates, we see that (3.6) holds provided that
\[ (3.7) \quad \gamma \geq \sup_{t > 1, |t - \tilde{t}| \leq 1} \frac{1}{t - 1} \min\left( t^{q} - 1 - \tilde{\delta} t^{q}(1 - 1/t^{q-1}), \frac{\tilde{\delta} t^{q}}{t^{q-1}} + qt \right). \]
In order to obtain a more easily computable bound, we estimate
\[ RHS(3.7) \leq \sup_{t \geq 1} \sup_{K \geq 0} \frac{1}{t - 1} \min\left( t^{q} - 1 - K(1 - 1/t^{q-1}), K/t^{q-1} + q(t + 1) \right). \]
Since we are taking the minimum of an increasing and a decreasing function in
\[ K, \]
the supremum over \( K \) is achieved for the value of \( K \) for which these functions
take equal values, or for \( K = 0 \) if the latter value in negative. Hence, substituting
\[ K = \max\{t^{q} - 1 - q(t + 1), 0\}, \]
we obtain
\[ RHS(3.7) \leq \sup_{t \geq 1} \frac{1}{t - 1} (t^{q} - 1 - \max(t^{q} - 1 - q(t + 1), 0)(1 - 1/t^{q-1})). \]
The function \( t \mapsto t^{q} - 1 - q(t + 1) \) is strictly monotonically increasing, so there is
a unique solution \( t_{0} \) to \( t_{0}^{q} - 1 - q(t_{0} + 1) = 0 \). The supremum is then assumed for\( t = t_{0} \), since
\[ \frac{d}{dt} \left( \frac{t^{q} - 1}{t - 1} \right) = \frac{(q - 1)t^{q} + 1 - qt^{q-1}}{(t - 1)^{2}} \geq 0 \]
by the AMGM inequality, and
\[ \frac{d}{dt} \left( \frac{1}{t - 1} \right) \left( q(t + 1) + \frac{t^{q} - 1}{t^{q-1}} \right) = -\frac{2q}{(t - 1)^{2}} - \frac{t^{q-2}}{(t^{q} - t^{q-1})^{2}} (t^{q} - qt + (q + 1)) \leq 0. \]
Hence,
\[ RHS(3.7) \leq \frac{q(t_{0} + 1)}{t_{0} - 1}. \]
Collecting the conditions on $\gamma$ in the proof, we see that it suffices to assume
\begin{equation}
\gamma \geq \max \left( \frac{q(t_0 + 1)}{t_0 - 1}, 2q, 2q' \right).
\end{equation}

For $q = 2$, we have $t_0 = 3$, so we can take $\gamma = 4$.

In the case $v = w = 1$, we do not use (3.3) and (3.5), so the only condition on $\gamma$ is given by (3.7). If we additionally assume $q = 2$ and $\hat{\delta} = 1$, that condition can be further simplified in the same way as in [Ose17b]. Namely, it suffices to ensure
\[ \gamma \geq \sup_{t > 1, |t - \tilde{t}| \leq 1} \frac{1}{t - 1} \left( t^2 - 1 - \tilde{t}(1 - 1/t) \right). \]

The supremum in $\tilde{t}$ is assumed for $\tilde{t} = (t - 1)$, so this condition becomes
\[ \gamma \geq \sup_{t > 1} \frac{1}{t - 1} \left( t^2 - 1 - (t - 1)^2(1 - 1/t) \right) = \sup_{t > 1} t + 1 - (t - 1)^2/t = \sup_{t > 1} 3 - 1/t = 3. \]

This is the bound used in (1.1).

\section{Optimality}

In this section, we show that the inequality (1.2) fails if $1/4$ is replaced by any larger number, already if the weights constitute a (positive) martingale.

Let $\Omega = \mathbb{N}_{\geq 1}$ with the filtration $\mathcal{F}_n$ such that the $n$-th $\sigma$-algebra $\mathcal{F}_n$ is generated by the atoms $\{1\}, \ldots, \{n\}$. Let $k \in \mathbb{R}_{\geq 0}$ be arbitrary. The measure on $(\Omega, \mathcal{F} = \bigvee_{n \in \mathbb{N}} \mathcal{F}_n)$ is given by $\mu(\{\omega\}) = (k + 1)^{-\omega}$. The martingale and the weights are given by
\[ f_n(\omega) = \begin{cases} (-1)^{n+1}k^{n+2}, & \omega \leq n, \\ (-1)^n, & \omega > n, \end{cases} \]
\[ w_n(\omega) = \begin{cases} 0, & \omega \leq n, \\ (k + 1)^n, & \omega > n. \end{cases} \]

Note that both these processes are indeed martingales. Their running maxima are given by
\[ f^*_n(\omega) = \begin{cases} k^{n+2}, & \omega \leq n, \\ 1, & \omega > n, \end{cases} \]
\[ w^*_n(\omega) = \begin{cases} (k + 1)^{n-1}, & \omega \leq n, \\ (k + 1)^n, & \omega > n. \end{cases} \]

Now, we compute both sides of (1.2):
\[ \mathbb{E} \sum_{n \leq N} \frac{|df_n|^2}{f_n^w} w_n = \sum_{n \leq N} \mu(\mathbb{N}_{>n}) \frac{2^2}{1} (k + 1)^n = \sum_{n \leq N} (k + 1)^{-n} \frac{2^2}{1} (k + 1)^n = 4N, \]
and
\[ \mathbb{E}(f^*_N w^*_N) = \sum_{\omega \leq N} \mu(\{\omega\}) \frac{k + 2}{k} (k + 1)^{-\omega - 1} + \mu(\mathbb{N}_{>N}) \cdot 1 \cdot (k + 1)^N \\
= \sum_{\omega \leq N} k(k + 1)^{-\omega} \frac{k + 2}{k} (k + 1)^{-\omega - 1} + (k + 1)^{-N} \cdot 1 \cdot (k + 1)^N \\
= N \frac{k + 2}{k + 1} + 1. \]

Since $k$ and $N$ can be arbitrarily large, we see that the constant in (1.2) is optimal.

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