Abstract

Let \( F \) be a number field, \( p \) a prime number. We construct the (multi-variable) \( p \)-adic L-function of an automorphic representation of \( GL_2(\mathbb{A}_F) \) (with certain conditions at places above \( p \) and \( \infty \)), which interpolates the complex (Jacquet-Langlands) L-function at the central critical point. We use this construction to prove that the \( p \)-adic L-function of a modular elliptic curve \( E \) over \( F \) has vanishing order greater or equal to the number of primes above \( p \) at which \( E \) has split multiplicative reduction, as predicted by the exceptional zero conjecture.

This is a generalization of analogous results by Spieß over totally real fields.
Introduction

Let $F$ be a number field (with adele ring $\mathbb{A}_F$), and $p$ a prime number. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of $\text{GL}_2(\mathbb{A}_F)$. Attached to $\pi$ is the automorphic $L$-function $L(s, \pi)$, for $s \in \mathbb{C}$, of Jacquet-Langlands [JL]. Under certain conditions on $\pi$, we can also define a $p$-adic $L$-function $L_p(s, \pi)$ of $\pi$, with $s \in \mathbb{Z}_p$. It is related to $L(s, \pi)$ by the interpolation property: For every character $\chi : G_p \to \mathbb{C}^*$ of finite order we have

$$L_p(0, \pi \otimes \chi) = \tau(\chi) \prod_{p \mid p} e(\pi_p, \chi_p) \cdot L(\frac{1}{2}, \pi \otimes \chi),$$

where $e(\pi_p, \chi_p)$ is a certain Euler factor (see theorem 4.12 for its definition) and $\tau(\chi)$ is the Gauss sum of $\chi$.

$L_p(s, \pi)$ was defined by Haran [Har] in the case where $\pi$ has trivial central character and $\pi_p$ is an ordinary spherical principal series representation for all $p \mid p$. For a totally real field $F$, Spieß [Sp] has given a new construction of $L_p(s, \pi)$ that also allows for $\pi_p$ to be a special (Steinberg) representation for some $p \mid p$.

Here, we generalize Spieß’ construction of $L_p(s, \pi)$ to automorphic representations $\pi$ of $\text{GL}_2$ over any number field, with arbitrary central character, and use it to prove a part of the exceptional zero conjecture on $p$-adic $L$-functions of elliptic curves (see below). For $F$ not totally real, $L_p$ can naturally be defined as a multi-variable function due to the existence of several $\mathbb{Z}_p$-extensions.

As in [Sp], we assume that $\pi$ is ordinary at all primes $p \mid p$ (cf. definition 2.3), that $\pi_v$ is discrete of weight 2 at all real infinite places $v$, and a similar condition at the complex places.

Throughout most of this paper, we follow [Sp]; for section 4.1, we follow Bygott [By], Ch. 4.2, who in turn follows Weil [We].

We define the $p$-adic $L$-function of $\pi$ as an integral, with respect to a certain measure $\mu_\pi$, on the Galois group $G_p$ of the maximal abelian extension that is unramified outside $p$ and $\infty$, specifically

$$L_p(s, \pi) := L_p(s_1, \ldots, s_t, \kappa_\pi) := \int_{G_p} \prod_{i=1}^t \exp_p(s_i \ell_i(\gamma)) \mu_\pi(d\gamma)$$

(for $s_1, \ldots, s_t \in \mathbb{Z}_p$), where $\kappa_\pi$ is a cohomology class attached to $\pi$ and the $\ell_i$ are $\mathbb{Z}_p$-valued homomorphisms corresponding to the $t$ independent $\mathbb{Z}_p$-extensions of $F$ (cf. section 4.7 for their definition).
Heuristically, $\mu_\pi$ is the image of $\mu_{\pi_p} \times W^p(\pi_p^\xi)$ under the reciprocity map $\mathbb{I}_F = F_p^* \times \mathbb{P} \to \mathcal{G}_p$ of global class field theory. Here $\mu_{\pi_p} = \prod_{p|p} \mu_{\pi_p}$ is the product of certain local distributions $\mu_{\pi_p}$ on $F_p^*$ attached to $\pi_p$, $d^x x^p$ is the Haar measure on the group $\mathbb{I}_p = \prod_{p|p} F_p^*$ of $p$-ideles, and $W^p = \prod_{p|p} W_p$ is a specific Whittaker function of $\pi^p := \otimes_{p|p} \pi_p$.

The structure of this work is the following: In chapter 2, we describe the local distributions $\mu_{\pi_p}$ on $F_p^*$; they are the image of a Whittaker functional under a map $\delta$ on the dual of $\pi_p$. For constructing $\delta$, we describe $\pi_p$ in terms of what we call the “Bruhat-Tits graph” of $F_p^2$, the directed graph whose vertices (resp. edges) are the lattices of $F_p^2$ (resp. inclusions between lattices). Roughly speaking, it is a covering of the (directed) Bruhat-Tits tree of $\text{GL}_2(F_p)$ with fibres $\cong \mathbb{Z}$. When $\pi_p$ is the Steinberg representation, $\mu_p$ can actually be extended to all of $F_p$.

In chapter 3, we attach a $p$-adic distribution $\mu_\phi$ to any map $\phi(U, x^p)$ of an open compact subset $U \subseteq F_p^*$ := $\prod_{p|p} F_p^*$ and an idele $x^p \in \mathbb{I}_p$ (satisfying certain conditions). Integrating $\phi$ over all the infinite places, we get a cohomology class $\kappa_\phi \in H^d(F_p^*, \mathbb{D}_f(\mathbb{C}))$ (where $d = r + s - 1$ is the rank of the group of units of $F$, $F^* \cong F^*/\mu_F$ is a maximal torsion-free subgroup of $F^*$, and $\mathbb{D}_f(\mathbb{C})$ is a space of distributions on the finite ideles of $F$). We show that $\mu_\phi$ can be described solely in terms of $\kappa_\phi$, and $\mu_\phi$ is a (vector-valued) $p$-adic measure if $\kappa_\phi$ is “integral”, i.e. if it lies in the image of $H^d(F^*, \mathbb{D}_f(R))$, for a Dedekind ring $R$ consisting of “$p$-adic integers”.

In chapter 4, we define a map $\phi_\pi$ by

$$\phi_\pi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}$$

($U \subseteq F_p^*$ compact open, $x^p \in \mathbb{I}_p$). $\phi_\pi$ satisfies the conditions of chapter 3 and we show that $\kappa_\pi := \kappa_{\phi_\pi}$ is integral by “lifting” the map $\phi_\pi \mapsto \kappa_\pi$ to a function mapping an automorphic form to a cohomology class in $H^d(\text{GL}_2(F)^+, \mathcal{A}_f)$, for a certain space of functions $\mathcal{A}_f$. (Here $\text{GL}_2(F)^+$ is the subgroup of $M \in \text{GL}_2(F)$ with totally positive determinant.) For this, we associate to each automorphic form $\varphi$ a harmonic form $\omega_\varphi$ on a generalized upper-half space $\mathcal{H}_\infty$, which we can integrate between any two cusps in $\mathbb{I}^1(F)$.

Then we can define the $p$-adic L-function $L_p(\underline{\zeta}, \pi) := L_p(\underline{\zeta}, \kappa_\pi)$ as above, with $\kappa_\pi := \kappa_{\phi_\pi}$. By a result of Harder [H], $H^d(\text{GL}_2(F)^+, \mathcal{A}_f)_\pi$ is one-dimensional, which implies that $L_p(\underline{\zeta}, \pi)$ has values in a one-dimensional $\mathbb{C}_p$-vector space.

We use our construction to prove the following result on the vanishing order of $p$-adic L-functions of elliptic curves:

If $E$ is a modular elliptic curve over $F$ corresponding to $\pi$ (i.e. the local L-factors of the Hasse-Weil L-function $L(E, s)$ and of the automorphic L-function $L(s - \frac{1}{2}, \pi)$ coincide at all places $v$ of $F$), we define the (multi-variable) $p$-adic L-function of $E$ as $L_p(E, \underline{s}) := L_p(\underline{s}, \pi)$. The condition that $\pi$ be ordinary at all $p|p$ means that $E$ must have good ordinary or multiplicative reduction at all places $p|p$ of $F$.

The exceptional zero conjecture (formulated by Mazur, Tate and Teitelbaum [MTT] for $F = \mathbb{Q}$, and by Hida [Hi] for totally real $F$) states that

$$\text{ord}_{s=0} L_p(E, s) \geq n,$$

(0.1)
where $n$ is the number of $p | p$ at which $E$ has split multiplicative reduction, and gives an explicit formula for the value of the $n$-th derivative $L_p^{(n)}(E, 0)$ as a multiple of certain L-invariants times $L(E, 1)$. The conjecture was proved in the case $F = \mathbb{Q}$ by Greenberg and Stevens [GS] and independently by Kato, Kurihara and Tsuji, and for totally real fields $F$ by Spieß [Sp].

In section 4.7 we formulate the exceptional zero conjecture and prove (0.1) for all number fields $F$.

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1 Preliminaries

Let $X$ be a totally disconnected locally compact topological space, $R$ a topological Hausdorff ring. We denote by $C(X, R)$ the ring of continuous maps $X \to R$, and let $C_c(X, R) \subseteq C(X, R)$ be the subring of compactly supported maps. When $R$ has the discrete topology, we also write $C^0(X, R) := C(X, R)$, $C^0_c(X, R) := C_c(X, R)$.

We denote by $\mathcal{C}o(X)$ the set of all compact open subsets of $X$, and for an $R$-module $M$ we denote by $\text{Dist}(X, M)$ the $R$-module of $M$-valued distributions on $X$, i.e. the set of maps $\mu : \mathcal{C}o(X) \to M$ such that $\mu(\bigcup_{i=1}^n U_i) = \sum_{i=1}^n \mu(U_i)$ for any pairwise disjoint sets $U_i \in \mathcal{C}o(X)$.

For an open set $H \subseteq X$, we denote by $1_H \in C(X, R)$ the $R$-valued indicator function of $H$ on $X$.

Throughout this paper, we fix a prime $p$ and embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\mathcal{O}$ denote the valuation ring of $\overline{\mathbb{Q}}$ with respect to the $p$-adic valuation induced by $\iota_p$.

We write $G := \text{GL}_2$ throughout the thesis, and let $B$ denote the Borel subgroup of upper triangular matrices, $T$ the maximal torus (consisting of all diagonal matrices), and $Z$ the center of $G$.

For a number field $F$, we let $G(F)^+ \subseteq G(F)$ and $B(F)^+ \subseteq B(F)$ denote the corresponding subgroups of matrices with totally positive determinant, i.e. $\sigma(\det(g))$ is positive for each real embedding $\sigma : F \hookrightarrow \mathbb{R}$. (If $F$ is totally complex, this is an empty condition, so we have $G(F)^+ = G(F)$, $B(F)^+ = B(F)$ in this case.) Similarly, we define $G(\mathbb{R})^+$ and $G(\mathbb{C})^+ = G(\mathbb{C})$.

1.1 $p$-adic measures

**Definition 1.1.** Let $X$ be a compact totally disconnected topological space. For a distribution $\mu : \mathcal{C}o(X) \to \mathbb{C}$, consider the extension of $\mu$ to the $\mathbb{C}_p$-linear map $C^0(X, \mathbb{C}_p) \to \mathbb{C}_p \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, $f \mapsto \int f d\mu$. If its image is a finitely-generated $\mathbb{C}_p$-vector space, $\mu$ is called a $p$-adic measure.

We denote the space of $p$-adic measures on $X$ by $\text{Dist}^b(X, \mathbb{C}) \subseteq \text{Dist}(X, \mathbb{C})$. It is easily seen that $\mu$ is a $p$-adic measure if and only if the image of $\mu$, considered as a map $C^0(X, \mathbb{Z}) \to \mathbb{C}$, is contained in a finitely generated $\mathcal{O}$-module. A $p$-adic measure can be integrated against any continuous function $f \in C(X, \mathbb{C}_p)$. 


2 Local results

For this chapter, let $F$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ its ring of integers, $\mathfrak{o}$ its uniformizer and $p = (\mathfrak{o})$ the maximal ideal. Let $q$ be the cardinality of $\mathcal{O}_F/p$, and set $U := U^{(0)} := \mathcal{O}_F^\times$, $U^{(n)} := 1 + p^n \subseteq U$ for $n \geq 1$.

We fix an additive character $\psi : F \to \mathbb{Q}_p^\times$ with $\ker \psi \supseteq \mathcal{O}_F$ and $p - 1 \not\subseteq \ker \psi$.

We let $| \cdot |$ be the absolute value on $F^\times$ (normalized by $|\mathfrak{o}| = q^{-1}$), $\text{ord} = \text{ord}_\mathfrak{o}$ the additive valuation, and $dx$ the Haar measure on $F$ normalized by $\int_{\mathcal{O}_F} dx = 1$. We define a (Haar) measure on $F^\times$ by $d x^\times := \frac{q}{q - 1} dx / |x|$ (so $\int_{\mathcal{O}_F^\times} d x^\times = 1$).

2.1 Gauss sums

Recall that the conductor of a character $\chi : F^\times \to \mathbb{C}^\times$ is by definition the largest ideal $p^{n}$, $n \geq 0$, such that $\ker \chi \supseteq U^{(n)}$, and that $\chi$ is unramified if its conductor is $p^0 = \mathcal{O}_F$.

Definition 2.1. Let $\chi : F^\times \to \mathbb{C}^\times$ be a quasi-character with conductor $p^f$. The Gauss sum of $\chi$ (with respect to $\psi$) is defined by

$$\tau(\chi) := [U : U^{(f)}] \int_{\mathfrak{o} - U} \psi(x) \chi(x) d^\times x.$$ 

For a locally constant function $g : F^\times \to \mathbb{C}$, we define

$$\int_{F^\times} g(x) dx := \lim_{n \to \infty} \int_{x \in F^\times, -n \leq \text{ord}(x) \leq n} g(x) dx,$$

whenever that limit exists. Then we have the following lemma of [Sp]:

Lemma 2.2. Let $\chi : F^\times \to \mathbb{C}^\times$ be a quasi-character with conductor $p^f$. For $f = 0$, assume $|\chi(\mathfrak{o})| < q$. Then we have

$$\int_{F^\times} \chi(x) \psi(x) dx = \begin{cases} 
1 - \chi(\mathfrak{o})^{-1} & \text{if } f = 0 \\
1 - \chi(\mathfrak{o})^{-1} q^{-1} & \text{if } f > 0.
\end{cases}$$

(Cf. [Sp], lemma 3.4.)

2.2 Tamely ramified representations of $\text{GL}_2(F)$

For an ideal $\mathfrak{a} \subset \mathcal{O}_F$, let $K_0(\mathfrak{a}) \subseteq G(\mathcal{O}_F)$ be the subgroup of matrices congruent to an upper triangular matrix modulo $\mathfrak{a}$.

Let $\pi : \text{GL}_2(F) \to \text{GL}(V)$ be an irreducible admissible infinite-dimensional representation on a $\mathbb{C}$-vector space $V$, with central quasicharacter $\chi$. It is well-known (e.g [Ge], Thm. 4.24) that there exists a maximal ideal $c(\pi) = \mathfrak{c} \subset \mathcal{O}_F$, the conductor

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So $\psi(p^{-1})$ is the set of all $p^e$-th roots of unity in $\overline{\mathbb{Q}}$, where $e$ is the ramification index of $F|\mathbb{Q}_p$. There is in general no $\psi$ such that $\ker(\psi) = \mathcal{O}_F$, since $p^{-1}/\mathcal{O}_F$ has more than $p$ points of order $p$ if $F|\mathbb{Q}_p$ has inertia index $> 1$. 

6
of \( \pi \), such that the space \( V^{K_0(\chi)} = \{ v \in V | \pi(g)v = \chi(a)v \ \forall g = (a \ b \ \ c \ d) \in K_0(\chi) \} \) is non-zero \( (\text{and in fact one-dimensional}). \) A representation \( \pi \) is called \textit{tamely ramified} if its conductor divides \( p \).

If \( \pi \) is tamely ramified, then \( \pi \) is the spherical resp. special representation \( \pi(\chi_1, \chi_2) \) (in the notation of \([Ge]\) or \([Sp]\)):

- If the conductor is \( \mathcal{O}_F, \pi \) is (by definition) spherical and thus a principal series representation \( \pi(\chi_1, \chi_2) \) for two unramified quasi-characters \( \chi_1 \) and \( \chi_2 \) with \( \chi_1 \chi_2^{-1} \neq 0 \).
- If the conductor is \( p \), then \( \pi = \pi(\chi_1, \chi_2) \) with \( \chi_1 \chi_2^{-1} = | \cdot |^{\pm 1} \).

For \( \alpha \in \mathbb{C}^* \), we define a character \( \chi_\alpha : F^* \to \mathbb{C}^* \) by \( \chi_\alpha(x) := \alpha^{\text{ord}(x)} \).

So let now \( \pi = \pi(\chi_1, \chi_2) \) be a tamely ramified irreducible admissible infinite-dimensional representation of \( \text{GL}_2(F) \); in the special case, we assume \( \chi_1 \) and \( \chi_2 \) to be ordered such that \( \chi_1 = | \cdot |^{\frac{1}{2}} \chi_2 \).

Set \( \alpha_i := \chi_i(\omega)\sqrt{q} \in \mathbb{C}^* \) \( (i = 1, 2) \). (We also write \( \pi = \pi_{\alpha_1, \alpha_2} \) sometimes.) Set \( \alpha := \alpha_1 + \alpha_2 \), \( \nu := \alpha_1 \alpha_2 / q \). Define a distribution \( \mu_{\alpha_1, \nu} := \mu_{\alpha_1, \nu} := \psi(x)\chi_{\alpha_1, \nu}(x)dx \) on \( F^* \).

For later use, we will need the following condition on the \( \alpha_i \):

**Definition 2.3.** Let \( \pi = \pi_{\alpha_1, \alpha_2} \) be tamely ramified. \( \pi \) is called \textit{ordinary} if \( \alpha \) and \( \nu \) both lie in \( \mathcal{O}^* \) \( (\text{i.e. they are } p\text{-adic units in } \mathcal{O}) \). Equivalently, this means that either \( \alpha_1 \in \mathcal{O}^* \) and \( \alpha_2 \in q\mathcal{O}^* \), or vice versa.

**Proposition 2.4.** Let \( \chi : F^* \to \mathbb{C}^* \) be a quasi-character with conductor \( p^f \); for \( f = 0 \), assume \( |\chi(\omega)| < |\alpha_2| \). Then the integral \( \int_{F^*} \chi(x)\mu_{\alpha_1, \nu}(dx) \) converges and we have

\[
\int_{F^*} \chi(x)\mu_{\alpha_1, \nu}(dx) = e(\alpha_1, \alpha_2, \chi)\tau(\chi)L(\frac{1}{2}, \pi \otimes \chi),
\]

where

\[
e(\alpha_1, \alpha_2, \chi) = \begin{cases} 
(1 - \alpha_1 \chi(\omega)q^{-1})(1 - \alpha_2 \chi(\omega)^{-1}q^{-1})(1 - \alpha_2 \chi(\omega)^{-1}q^{-1}), & f = 0 \text{ and } \pi \text{ spherical,} \\
(1 - \alpha_1 \chi(\omega)^{-1})(1 - \alpha_2 \chi(\omega)^{-1}q^{-1}), & f = 0 \text{ and } \pi \text{ special,} \\
(\alpha_1 / \nu)^{-f} = (\alpha_2 / q)^f, & f > 0,
\end{cases}
\]

and where we assume the right-hand side to be continuously extended to the potential removable singularities at \( \chi(\omega) = q / \alpha_1 \) or \( = q / \alpha_2 \).

**Proof.** Case 1: \( f = 0 \), \( \pi \) spherical

We have

\[
L(s, \pi \otimes \chi) = \frac{1}{(1 - \alpha_1 \chi(\omega)q^{-s+\frac{1}{2}})(1 - \alpha_2 \chi(\omega)^{-1}q^{-s+\frac{1}{2}})};
\]
so

\[
L(\frac{1}{2}, \pi \otimes \chi) \cdot \tau(\chi) \cdot e(\alpha_1, \alpha_2, \chi) = \frac{1 - \alpha_2 q^{-1} \chi(\omega)^{-1}}{1 - \chi(\omega) \alpha_2^{-1}}
\]

\[
= \frac{1 - \nu \alpha_1^{-1} \chi(\omega)^{-1}}{1 - \alpha_1 \chi(\omega) \nu^{-1} q^{-1}}
\]

\[
= \int_{F^*} \chi(x) \chi_{\alpha_1/\nu}(x) \psi(x) dx
\]

\[
= \int_{F^*} \chi(x) \mu_{\alpha_1/\nu}(dx)
\]

by lemma \[2.2\]

Case 2: \( f = 0, \pi \) special
Assuming \( \chi_1 = | \cdot | \chi_2 \), we have

\[
L(s, \pi \otimes \chi) = \frac{1}{1 - \alpha_1 \chi(\omega) q^{-s-\frac{1}{2}}}
\]

and thus

\[
L(\frac{1}{2}, \pi \otimes \chi) \cdot \tau(\chi) \cdot e(\alpha_1, \alpha_2, \chi) = \frac{1 - \nu \alpha_1^{-1} \chi(\omega)^{-1}}{1 - \alpha_1 \nu^{-1} \chi(\omega) q^{-1}}
\]

\[
= \int_{F^*} \chi(x) \chi_{\alpha_1/\nu}(x) \psi(x) dx
\]

\[
= \int_{F^*} \chi(x) \mu_{\alpha_1/\nu}(dx)
\]

by lemma \[2.2\]

Case 3: \( f > 0 \)
In this case, \( L(s, \pi \otimes \chi) = 1 \) for \( s > 0 \) and

\[
\int_{F^*} \chi(x) \mu_{\alpha_1/\nu}(dx) = \tau(\chi \cdot \chi_{\alpha_1/\nu})
\]

\[
= q^{f-1}(q-1) \int_{\omega^{-f}U} \psi(x) \chi(x) \chi_{\alpha_1/\nu}(x) d^\times x
\]

\[
= (\alpha_1/\nu)^{-f} q^{f-1}(q-1) \int_{\omega^{-f}U} \psi(x) \chi(x) d^\times x
\]

\[
= e(\alpha_1, \alpha_2, \chi) \cdot \tau(\chi) \cdot L(\frac{1}{2}, \pi \otimes \chi).
\]

\( \square \)

2.3 The Bruhat-Tits graph \( \tilde{T} \)

Let \( \tilde{V} \) denote the set of lattices (i.e. submodules isomorphic to \( \mathcal{O}_F^2 \)) in \( F^2 \), and let \( \tilde{E} \) be the set of all inclusion maps between two lattices; for such a map \( e : v_1 \hookrightarrow v_2 \) in \( \tilde{E} \), we define \( o(e) := v_1, t(e) := v_2 \). Then the pair \( \tilde{T} := (\tilde{V}, \tilde{E}) \) is naturally a directed
graph, connected, with no directed cycles (specifically, $\mathcal{E}$ induces a partial ordering on $\mathcal{V}$). For each $v \in \mathcal{V}$, there are exactly $q + 1$ edges beginning (resp. ending) in $v$, each.

Recall that the Bruhat-Tits tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ of $G(F)$ is the directed graph whose vertices are homothety classes of lattices of $F^2$ (i.e. $\mathcal{V} = \tilde{\mathcal{V}}/\sim$, where $v \sim w^i v$ for all $i \in \mathbb{Z}$), and the directed edges $e \in \mathcal{E}$ are homothety classes of inclusions of lattices. We can define maps $o, t : \mathcal{E} \to \mathcal{V}$ analogously. For each edge $e \in \mathcal{E}$, there is an opposite edge $\overline{e} \in \mathcal{E}$ with $o(\overline{e}) = t(e)$, $t(\overline{e}) = o(e)$; and the undirected graph underlying $\mathcal{T}$ is simply connected. We have a natural “projection map” $\pi : \mathcal{T} \to \mathcal{E}$, mapping each lattice and each homomorphism to its homothety class. Choosing a (set-theoretic) section $s : \mathcal{V} \to \tilde{\mathcal{V}}$, we get a bijection $\mathcal{V} \times \mathbb{Z} \to \tilde{\mathcal{V}}$ via $(v, i) \mapsto w^i s(v)$.

The group $G(F)$ operates on $\tilde{\mathcal{V}}$ via its standard action on $F^2$, i.e. $g v = \{gx \mid x \in v\}$ for $g \in G(F)$, and on $\mathcal{E}$ by mapping $e : v_1 \to v_2$ to the inclusion map $ge : gv_1 \to gv_2$. The stabilizer of the standard vertex $v_0 := \mathcal{O}_F^2$ is $G(\mathcal{O}_F)$.

For a directed edge $e \in \mathcal{E}$ of the Bruhat-Tits tree $\mathcal{T}$, we define $U(e)$ to be the set of ends of $e$ (cf. [Sel] / [Sp]); it is a compact open subset of $\mathbb{P}^1(F)$, and we have $gU(e) = U(g e)$ for all $g \in G(F)$.

For $n \in \mathbb{Z}$, we set $v_n := \mathcal{O}_F \oplus p^n \tilde{\mathcal{V}}$, and denote by $e_n$ the edge from $v_{n+1}$ to $v_n$; the “decreasing” sequence $(\pi(e_n))_{n \in \mathbb{Z}}$ is the geodesic from $\infty$ to $0$. (The geodesic from 0 to $\infty$ traverses the $\pi(v_n)$ in the natural order of $n \in \mathbb{Z}$.) We have $U(\pi(e_n)) = p^{-n}$ for each $n$.

Now (following [BL] and [Sp]), we can define a "height" function $h : \mathcal{V} \to \mathbb{Z}$ as follows: The geodesic ray from $v \in \mathcal{V}$ to $\infty$ must contain some $\pi(v_n)$ ($n \in \mathbb{Z}$), since it has non-empty intersection with $A := \{\pi(v_n) \mid n \in \mathbb{Z}\}$; we define $h(v) := n - d(v, \pi(v_n))$ for any such $v_n$; this is easily seen to be well-defined, and we have $h(\pi(v_n)) = n$ for all $n \in \mathbb{Z}$. We have the following lemma of [Sp]:

**Lemma 2.5.** (a) For all $e \in \mathcal{E}$, we have

\[
h(t(e)) = \begin{cases} h(o(e)) + 1 & \text{if } \infty \in U(e), \\ h(o(e)) - 1 & \text{otherwise.} \end{cases}
\]

(b) For $a \in F^*$, $b \in F$, $e \in \mathcal{E}$ we have

\[
h \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} e \right) = h(e) - \text{ord}_e(a).
\]

(Cf. [Sp], Lemma 3.6.)

### 2.4 Hecke structure of $\tilde{\mathcal{T}}$

Let $R$ be a ring, $M$ an $R$-module. We let $C(\tilde{\mathcal{V}}, M)$ be the $R$-module of maps $\phi : \tilde{\mathcal{V}} \to M$, and $C(\mathcal{E}, M)$ the $R$-module of maps $\mathcal{E} \to M$. Both are $G(F)$-modules via $(g \phi)(v) := \phi(g^{-1}v)$, $(ge)(e) := c(g^{-1}e)$.
We let \( C_c(\tilde{\mathcal{V}}, M) \subseteq C(\tilde{\mathcal{V}}, M) \) and \( C_c(\tilde{\mathcal{E}}, M) \subseteq C(\tilde{\mathcal{E}}, M) \) be the \((G(F))\)-stable sub-modules of maps with compact support, i.e. maps that are zero outside a finite set. We get pairings

\[
\langle \cdot, \cdot \rangle : C_c(\tilde{\mathcal{V}}, R) \times C(\tilde{\mathcal{V}}, M) \to M, \quad \langle \phi_1, \phi_2 \rangle := \sum_{v \in \tilde{\mathcal{V}}} \phi_1(v)\phi_2(v) \tag{2.1}
\]

and

\[
\langle \cdot, \cdot \rangle : C_c(\tilde{\mathcal{E}}, R) \times C(\tilde{\mathcal{E}}, M) \to M, \quad \langle c_1, c_2 \rangle := \sum_{e \in \tilde{\mathcal{E}}} c_1(v)c_2(v). \tag{2.2}
\]

We define Hecke operators \( T, \mathcal{R} : \mathcal{C}(\tilde{\mathcal{V}}, M) \to \mathcal{C}(\tilde{\mathcal{V}}, M) \) by

\[
T\phi(v) = \sum_{t(e)=v} \phi(o(e)) \quad \text{and} \quad \mathcal{R}\phi := \varpi\phi \quad \text{(i.e.} \ \mathcal{R}\phi(v) = \phi(\varpi^{-1}v))
\]

for all \( v \in \tilde{\mathcal{V}} \). These restrict to operators on \( C_c(\tilde{\mathcal{V}}, R) \), which we sometimes denote by \( T_c \) and \( \mathcal{R}_c \) for emphasis. With respect to (2.1), \( T_c \) is adjoint to \( TR \), and \( \mathcal{R}_c \) is adjoint to its inverse operator \( \mathcal{R}^{-1} : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{V}}, R) \).

\( T \) and \( \mathcal{R} \) obviously commute, and we have the following Hecke structure theorem on compactly supported functions on \( \tilde{\mathcal{V}} \) (an analogue of [BL], Thm. 10):

**Theorem 2.6.** \( C_c(\tilde{\mathcal{V}}, R) \) is a free \( R[T, \mathcal{R}^{\pm 1}] \)-module (where \( R[T, \mathcal{R}^{\pm 1}] \) is the ring of Laurent series in \( \mathcal{R} \) over the polynomial ring \( R[T] \), with \( \mathcal{R} \) and \( T \) commuting).

**Proof.** Fix a vertex \( v_0 \in \tilde{\mathcal{V}} \). For each \( n \geq 0 \), let \( C_n \) be the set of vertices \( v \in \tilde{\mathcal{V}} \) such that there is a directed path of length \( n \) from \( v_0 \) to \( v \) in \( \tilde{\mathcal{V}} \), and such that \( d(\pi(v_0), \pi(v)) = n \) in the Bruhat-Tits tree \( \mathcal{T} \). So \( C_0 = \{ v_0 \} \), and \( C_n \) is a lift of the ”circle of radius \( n \) around \( v_0 \)” in \( \mathcal{T} \), in the parlance of [BL].

One easily sees that \( \bigcup_{n \geq 0} C_n \) is a complete set of representatives for the projection map \( \pi : \tilde{\mathcal{V}} \to \mathcal{V} \); specifically, for \( n > 1 \) and a given \( v \in C_{n-1} \), \( C_n \) contains exactly \( q \) elements adjacent to \( v \) in \( \tilde{\mathcal{V}} \); and we can write \( \tilde{\mathcal{V}} \) as a disjoint union \( \bigcup_{j \in \mathbb{Z}} \bigcup_{n=0}^{\infty} \mathcal{R}^j(C_n) \).

We further define \( V_0 := \{ v_0 \} \) and choose subsets \( V_n \subseteq C_n \) as follows: We let \( V_1 \) be any subset of cardinality \( q \). For \( n > 1 \), we choose \( q-1 \) out of the \( q \) elements of \( C_n \) adjacent to \( v' \), for every \( v' \in C_{n-1} \), and let \( V_n \) be the union of these elements for all \( v' \in C_{n-1} \). Finally, we set

\[
H_{n,j} := \{ \phi \in C_c(\tilde{\mathcal{V}}, R) | \text{Supp}(\phi) \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i(C_i) \} \quad \text{for each} \ n \geq 0, j \in \mathbb{Z},
\]

\[
H_n := \bigcup_{j \in \mathbb{Z}} H_{n,j}, \quad \text{and} \quad H_{-1} := H_{-1,j} := \{ 0 \}. \quad (\text{For ease of notation, we identify} \ v \in \tilde{\mathcal{V}} \ \text{with its indicator function} \ 1_v \in C_c(\tilde{\mathcal{V}}, R) \ \text{in this proof}.)
\]

Define \( T' : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{V}}, R) \) by

\[
T'(\phi)(v) := \sum_{t(e)=v, \ o(e)\in \mathcal{R}^j(C_n)} \phi(o(e)) \quad \text{for each} \ v \in \mathcal{R}^j(C_{n-1}), j \in \mathbb{Z};
\]
$T'$ can be seen as the "restriction to one layer" $\bigcup_{n=0}^{\infty} R^j(C_n)$ of $T$. We have $T'(v) \equiv T(v) \mod H_{n-1}$ for each $v \in H_n$, since the "missing summand" of $T'$ lies in $\tilde{H}_{n-1}$.

We claim that for each $n \geq 0$, the set $X_{n,j} := \bigcup_{i=0}^{n} R^j T^{n-i}(V_i)$ is an $R$-basis for $H_{n,j}/H_{n-1,j}$. By the above congruence, we can replace $T$ by $T'$ in the definition of $X_{n,j}$.

The claim is clear for $n = 0$. So let $n \geq 1$, and assume the claim to be true for all $n' \leq n$. For each $v \in C_{n-1}$, the $q$ points in $C_n$ adjacent to $v$ are generated by the $q - 1$ of these points lying in $V_n$, plus $T'v$ (which just sums up these $q$ points). By induction hypothesis, $v$ is generated by $X_{n-1,0}$, and thus (taking the union over all $v$), $C_n$ is generated by $T'(X_{n-1,0}) \cup V_n = X_{n,0}$. Since the cardinality of $X_{n,0}$ equals the $R$-rank of $H_{n,0}/H_{n-1,0}$ (both are equal to $(q + 1)q^{n-1}$), $X_{n,0}$ is in fact an $R$-basis.

Analoguously, we see that $H_{n,j}/H_{n-1,j}$ has $R^j(X_{n,0}) = X_{n,j}$ as a basis, for each $j \in \mathbb{Z}$.

From the claim, it follows that $\bigcup_{j \in \mathbb{Z}} X_{n,j}$ is an $R$-basis of $H_n/H_{n-1}$ for each $n$, and that $V := \bigcup_{n=0}^{\infty} V_n$ is an $R[T, R^{+1}]$-basis of $C_c(\hat{V}, R)$.

For $a \in R$ and $\nu \in R^\ast$, we let $\tilde{B}_{a,\nu}(F, R)$ be the "common cokernel" of $T - a$ and $R - \nu$ in $C_c(\hat{V}, R)$, namely $\tilde{B}_{a,\nu}(F, R) := C_c(\hat{V}, R)/(\text{Im}(T - a) + \text{Im}(R - \nu))$; dually, we define $\tilde{B}^{a,\nu}(F, M) := \ker(T - a) \cap \ker(R - \nu) \subseteq C(\hat{V}, M)$.

For a lattice $v \in \hat{V}$, we define a valuation $\text{ord}_v$ on $F$ as follows: For $w \in F^2$, the set $\{x \in F|xw \in v\}$ is some fractional ideal $\varpi^m\mathcal{O}_F \subseteq F$ ($m \in \mathbb{Z}$); we set $\text{ord}_v(w) := m$. This map can also be given explicitly as follows: Let $\lambda_1, \lambda_2$ be a basis of $v$. We can write any $w \in F^2$ as $w = x_1\lambda_1 + x_2\lambda_2$; then we have $\text{ord}_v(w) = \min\{\text{ord}_w(x_1), \text{ord}_w(x_2)\}$. This gives a "valuation" map on $F^2$, as one easily checks. We restrict it to $F \cong F \times \{0\} \hookrightarrow F^2$ to get a valuation $\text{ord}_v$ on $F$, and consider especially the value at $e_1 := (1, 0)$.

**Lemma 2.7.** Let $\alpha, \nu \in R^\ast$, and put $a := \alpha + q\nu/\alpha$. Define a map $\varrho := \varrho_{a,\nu} : \hat{V} \to R$ by $\varrho(v) := \alpha h(\pi(v))\nu^{-\text{ord}_v(e_1)}$. Then $\varrho \in \tilde{B}^{a,\nu}(F, R)$.

**Proof.** One easily sees that $(v \mapsto \nu^{-\text{ord}_v(e_1)}) \in \ker(R - \nu)$. It remains to show that $\varrho \in \ker(T - a)$.

We have the Iwasawa decomposition $G(F) = B(F)G(\mathcal{O}_F) = \{(0, 1)\} \mathbb{Z}(F)G(\mathcal{O}_F)$; thus every vertex in $\hat{V}$ can be written as $\varpi^iv$ with $v = (\nu_0 1) v_0$, with $i \in \mathbb{Z}$, $a \in F^\ast$, $b \in F$.

Now the lattice $v = (\frac{a}{b} 1)v_0$ is generated by the vectors $\lambda_1 = (\frac{a}{b} 1)$ and $\lambda_2 = (\frac{b}{a} 1) \in \mathcal{O}_F^2$, so $e_1 = a^{-1}\lambda_1$ and thus $\text{ord}_v(e_1) = \text{ord}_w(a^{-1}) = -\text{ord}_w(a)$. The $q + 1$ neighbouring vertices $v'$ for which there exists an $e \in \hat{E}$ with $o(e) = v'$, $t(e) = v$ are given by $N_i v$, $i \in \{\infty\} \cup \mathcal{O}_F/\mathfrak{p}$, with $N_{\infty} := (1 \ 0)$, and $N_i := (\frac{a}{b} \ 1)$ where $i \in \mathcal{O}_F$ runs through a complete set of representatives mod $\varpi$. By lemma 2.5, $h(\pi(N_{\infty}v)) = h(\pi(v)) + 1$ and $h(\pi(N_i v)) = h(\pi(v)) - 1$ for $i \neq \infty$. By considering the basis $\{N_i\lambda_1, N_i\lambda_2\}$ of $N_i v$ for each $N_i$, we see that $\text{ord}_{N_{\infty}v}(e_1) = \text{ord}_v(e_1)$ and $\text{ord}_{N_i v}(e_1) = \text{ord}_v(e_1) - 1$ for $i \neq \infty$. Thus we have
(Tg)(v) = \sum_{t(e)=v} \alpha^{h(\pi(\alpha(e)))} v^{-\text{ord}_e(e_1)} = \alpha^{h(\pi(v))+1} v^{-\text{ord}_v e_1} + q \cdot \alpha^{h(\pi(v))-1} v^{1-\text{ord}_v(e_1)}
= (\alpha + q\alpha^{-1}v)\alpha^{h(\pi(v))} v^{-\text{ord}_v e_1} = ag(v),

and also (Tg)(\omega^iv) = (T\mathcal{R}^{-i}g)(v) = \mathcal{R}^{-i}(ag)(v) = ag(\omega^iv) for a general \omega^iv \in \hat{\mathcal{V}}, which shows that \varrho \in \ker(T-a).

If \alpha^2 \neq \nu(q+1)^2 (we will call this the “spherical case”), we put \mathcal{B}_{a,\nu}(F, R) := \hat{\mathcal{B}}_{a,\nu}(F, R) and \mathcal{B}^{a,\nu}(F, M) := \hat{\mathcal{B}}^{a,\nu}(F, M).

In the “special case” \alpha^2 = \nu(q+1)^2, we need to assume that the polynomial \begin{align*}
X^2 - \nu qX + q \nu^{-1} & \in \mathbb{R}[X]
\end{align*}
has a zero \alpha' \in \mathbb{R}. Then the map \varrho := g_{\alpha',\nu} \in C(\hat{\mathcal{V}}, R) defined as above lies in \mathcal{B}^{a,\nu^{-1}}(F, R) \cap \ker(\mathcal{R}^{-1} - \nu) by Lemma 2.7, since \nu q = \alpha' + q \nu^{-1}/\alpha'. In other words, the kernel of the map \begin{align*}
\langle \cdot, \varrho \rangle : C_{\text{c}}(\hat{\mathcal{V}}, R) & \to R
\end{align*}
contains \begin{align*}
(\text{Im}(T-a) + \text{Im}(\mathcal{R} - \nu))
\end{align*}
to be the quotient; evidently, it is an \begin{align*}
R\text{-submodule of codimension 1 of } \hat{\mathcal{B}}_{a,\nu}(F, R).
\end{align*}
Dually, \begin{align*}
T-a \text{ and } \mathcal{R} - \nu \text{ both map the submodule } \varrho M = \{ \varrho \cdot M, m \in M \} \text{ of } C(\hat{\mathcal{V}}, M) \text{ to zero and thus induce endomorphisms on } C(\hat{\mathcal{V}}, M)/\varrho M; \text{ we define } \begin{align*}
\mathcal{B}^{a,\nu}(F, M)
\end{align*}
to be the intersection of their kernels.

In the special case, since \nu = \alpha^2, Lemma 2.7 states that \begin{align*}
\varrho(gv_0) = \chi_{\alpha}(ad) \varrho(v_0) = \chi_{\alpha}(\text{det} g) \varrho(v_0) \text{ for all } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{B}(\mathcal{F}), \text{ and thus for all } g \in G(\mathcal{F}) \text{ by the Iwasawa decomposition, since } G(O_F) \text{ fixes } v_0 \text{ and lies in the kernel of } \chi_{\alpha} \circ \text{det}. \text{ By the multiplicity of det, we have } (g^{-1}g)(v) = \varrho(gv) = \chi_{\alpha}(\text{det} g) \varrho(v) \text{ for all } g \in G(\mathcal{F}), v \in \hat{\mathcal{V}}. \text{ So } \varrho \in \ker(\langle \cdot, \varrho \rangle) \implies \langle \varrho, g^{-1}g \rangle = \langle \varrho, g^{-1} \rangle = \chi_{\alpha}(\text{det} g) \varrho(\varrho) = 0, \text{ i.e. } \ker(\langle \cdot, \varrho \rangle) \text{ and thus } \mathcal{B}_{a,\nu}(F, R) \text{ are } G(\mathcal{F})\text{-modules.}

By the adjointness properties of the Hecke operators \begin{align*}
T \text{ and } \mathcal{R}, \text{ we have pairings } \begin{pmatrix} T_{\text{c}} - a \end{pmatrix} \times \ker(\mathcal{R} - a) & \to M \text{ and } \ker(\mathcal{R} - \nu) \times \ker(\mathcal{R}^{-1} - \nu) \to M, \text{ which }
\end{align*}
"combine" to give a pairing

\begin{align*}
\langle \cdot, \cdot \rangle : \mathcal{B}_{a,\nu}(F, R) \times \mathcal{B}^{a,\nu^{-1}}(F, M) & \to M
\end{align*}

(since \begin{align*}
\ker(\mathcal{R} - a) \cap \ker(\mathcal{R}^{-1} - \nu) = \ker(T - \nu q) \cap \ker(\mathcal{R}^{-1} - \nu)), \text{ and a corresponding isomorphism } \mathcal{B}^{a,\nu^{-1}}(F, M) \xrightarrow{\cong} \text{Hom}(\mathcal{B}_{a,\nu}(F, R), M).
\end{align*}

\textbf{Definition 2.8.} Let \begin{align*}
G \text{ be a totally disconnected locally compact group, } H \subseteq G \text{ an open subgroup. For a smooth } R[H]\text{-module } M, \text{ we define the (compactly) induced}
\end{align*}
\textsuperscript{a}We use this term since these pairs of } a, \nu \text{ will later be seen to correspond to a spherical representation of GL}_2(\mathcal{F}). \text{ The case } \alpha^2 = \nu(q+1)^2 \text{ means that there exists an } \alpha \in \mathbb{R}^* \text{ with } a = \alpha(q+1), \nu = \alpha^2, \text{ which will correspond to a special representation.}
By Theorem 2.6, $T_e - a$ (as well as $R_c - \nu$) is injective, and the induced map

$$
R_c - \nu: \ker(T_e - a) = C_e(\hat{V}, R)/ \text{Im}(T_e - a) \to \ker(T_e - a)
$$

(of $R[T, R^\pm 1]/(T - a) = R[R^\pm 1]$-modules) is also injective. Now since $G(F)$ acts transitively on $\hat{V}$, with the stabilizer of $v_0 := O_F^2$ being $K := G(O_F)$, we have an isomorphism $C_e(\hat{V}, R) \cong \text{Ind}^{G(F)}_K R$. Thus we have exact sequences

$$
0 \to \text{Ind}^{G(F)}_K R \xrightarrow{T-a} \text{Ind}^{G(F)}_K R \to \ker(T_e - a) \to 0 \quad (2.3)
$$

and (for $a, \nu$ in the spherical case)

$$
0 \to \ker(T_e - a) \xrightarrow{R-\nu} \ker(T_e - a) \to \mathcal{B}_{a,\nu}(F, R) \to 0, \quad (2.4)
$$

with all entries being free $R$-modules. Applying $\text{Hom}_R(\cdot, M)$ to them, we get:

**Lemma 2.9.** We have exact sequences of $R$-modules

$$
0 \to \ker(TR - a) \to \text{Coind}^{G(F)}_K M \xrightarrow{T-a} \text{Coind}^{G(F)}_K M \to 0
$$

and, if $\mathcal{B}_{a,\nu}(F, M)$ is spherical (i.e. $a^2 \neq \nu(q + 1)^2$),

$$
0 \to \mathcal{B}_{a,\nu-1}(F, M) \to \ker(TR - a) \xrightarrow{R-\nu} \ker(TR - a) \to 0.
$$

For the special case, we have to work a bit more to get similar exact sequences:

By [Sp], eq. (22), for the representation $St^{-}(F, R) := \mathcal{B}_{-(q+1),1}(F, R)$ (i.e. $\nu = 1$, $\alpha = -1$) with trivial central character, we have an exact sequence of $G$-modules

$$
0 \to \text{Ind}^{G}_K Z R \to \text{Ind}^{G}_{K^{'}} Z R \to St^{-}(F, R) \to 0, \quad (2.5)
$$

where $K' = \langle W \rangle K_0(p)$ is the subgroup of $KZ$ generated by $W := \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and the subgroup $K_0(p)Z$ is a subgroup of $K'$ of order 2.) Now $(\pi, V)$ can be written as $\pi = \chi \otimes St^{-}$ for some character $\chi = \chi_Z$ (cf. the proof of lemma 2.12 below), and we have an obvious $G$-isomorphism

$$(\pi, V) \cong (\pi \otimes (\chi \circ \text{det}), V \otimes_R R(\chi \circ \text{det})),
$$

where $R(\chi \circ \text{det})$ is the ring $R$ with $G$-module structure given via $gr = \chi(\text{det}(g))r$ for $g \in G, r \in R$. Tensoring (2.5) with $R(\chi \circ \text{det})$ over $R$ gives an exact sequence of $G$-modules

$$
0 \to \text{Ind}^{G}_{K^{'}} Z \chi \to \text{Ind}^{G}_{K^{'}} Z \chi \to V \to 0. \quad (2.6)
$$
It is easily seen that $R(\chi \circ \text{det})$ fits into another exact sequence of $G$-modules

$$0 \to \text{Ind}^G_H R \xrightarrow{(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}) - \chi(\psi) \text{id}} \text{Ind}^G_H R \xrightarrow{\psi} R(\chi \circ \text{det}) \to 0,$$

where $H := \{ g \in G \mid \det g \in \mathcal{O}_F^* \}$ is a normal subgroup containing $K$, $(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}) (f)(g) := f((\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix})^{-1} g)$ for $f \in \text{Ind}^G_H R = \{ f : G \to R | f(Hg) = f(g) \text{ for all } g \in G, \ g \in G \}$, is the natural operation of $G$, and where $\psi$ is the $G$-equivariant map defined by $1_u \mapsto 1$.

Now since $H \subseteq G$ is a normal subgroup, we have $\text{Ind}^G_H R \cong R[G/H]$ as $G$-modules (in fact $G/H \cong \mathbb{Z}$ as an abstract group). Let $X \subseteq G$ be a subgroup such that the natural inclusion $X/(X \cap H) \hookrightarrow G/H$ has finite cokernel; let $g_i H, i = 1, \ldots, n$ be a set of representatives of that cokernel. Then we have a (non-canonical) $X$-isomorphism $\bigoplus_{i=0}^n \text{Ind}^X_{X \cap H} \to \text{Ind}^G_H R$ defined via $(1_{(X \cap H)x_i}) \mapsto 1_{Hxg_i}$, for each $i = 1, \ldots, n$ (cf. [Br], III (5.4)).

Using this isomorphism and the “tensor identity” $\text{Ind}^G_H M \otimes N \cong \text{Ind}^G_H (M \otimes \text{res}^G_H N)$ for any groups $H \subseteq G$, $H$-module $M$ and $G$-module $N$ ([Br] III.5, Ex. 2), we have

$$\text{Ind}^G_K R \otimes_R \text{Ind}^H_K R \cong \text{Ind}^G_K (\text{res}^G_K (\text{Ind}^G_H R)) \cong \text{Ind}^G_K ((\text{Ind}^G_K \cap H) R)^2 \cong (\text{Ind}^G_K (\text{Ind}^G_K R))^2$$

(since $KZ/KZ \cap H \hookrightarrow G/H$ has index 2), and similarly

$$\text{Ind}^G_{K'} R \otimes_R \text{Ind}^H_{K'} R \cong \text{Ind}^G_{K'} (\text{res}^G_{K'} (\text{Ind}^G_H R)) \cong \text{Ind}^G_{K'} ((\text{Ind}^G_{K'} \cap H) R)^2 \cong (\text{Ind}^G_{K'} R)^2$$

and thus, we can resolve the first and second term of (2.6) into exact sequences

$$0 \to (\text{Ind}^G_K R)^2 \to (\text{Ind}^G_K R)^2 \to \text{Ind}^G_{KZ} \chi \to 0,$$

$$0 \to (\text{Ind}^G_K R)^2 \to (\text{Ind}^G_K R)^2 \to \text{Ind}^G_{(w)K_0(p)Z} \chi \to 0.$$

Dualizing (2.3) and these by taking $\text{Hom}(\cdot, M)$ for an $R$-module $M$, we get a “resolution” of $\mathcal{B}^{a \nu, \nu - 1}(F, M)$ in terms of coinduced modules:

**Lemma 2.10.** We have exact sequences

$$0 \to \mathcal{B}^{a \nu, \nu - 1}(F, M) \to \text{Coind}^{G}_{K'Z} M(\chi) \to \text{Coind}^{G}_{KZ} M(\chi) \to 0,$$

$$0 \to \text{Coind}^{G}_{KZ} M(\chi) \to (\text{Coind}^{G}_{K} R)^2 \to (\text{Coind}^{G}_{K} R)^2 \to 0,$$

$$0 \to (\text{Coind}^{G}_{K} R)^2 \to (\text{Coind}^{G}_{K} R)^2 \to (\text{Coind}^{G}_{K} R)^2 \to 0$$

for all special $\mathcal{B}_{a, \nu}(F, R)$ (i.e. $a^2 = \nu(g+1)^2$), where $\chi = \chi_Z$ is the central character.

It is easily seen that the above arguments could be modified to get a similar set of exact sequences in the spherical case as well (replacing $K'$ by $K$ everywhere), in addition to that given in lemma 2.9 but we will not need this.

14
2.5 Distributions on $\tilde{T}$

For $\varrho \in C(\tilde{\mathcal{V}}, R)$ we define $R$-linear maps

$$\tilde{\delta}_\varrho : C(\tilde{\mathcal{E}}, M) \to C(\tilde{\mathcal{V}}, M), \quad \tilde{\delta}_\varrho(c)(v) := \sum_{v = t(e)} \varrho(o(e))c(e) - \sum_{v = o(e)} \varrho(t(e))c(e),$$

$$\tilde{\delta}^\varrho : C(\tilde{\mathcal{V}}, M) \to C(\tilde{\mathcal{E}}, M), \quad \tilde{\delta}^\varrho(\phi)(e) := \varrho(o(e))\phi(t(e)) - \varrho(t(e))\phi(o(e)).$$

One easily checks that these are adjoint with respect to the pairings (2.1) and (2.2), i.e. we have $\langle \tilde{\delta}_\varrho(c), \phi \rangle = \langle c, \tilde{\delta}^\varrho(\phi) \rangle$ for all $c \in C_c(\tilde{\mathcal{E}}, R)$, $\phi \in C(\tilde{\mathcal{V}}, M)$. We denote the maps corresponding to $\varrho \equiv 1$ by $\delta := \delta_1$, $\delta^* := \delta^1$.

For each $\varrho$, the map $\tilde{\delta}_\varrho$ fits into an exact sequence

$$C_c(\tilde{\mathcal{E}}, R) \xrightarrow{\delta} C_c(\tilde{\mathcal{V}}, R) \xrightarrow{\langle \cdot, \phi \rangle} R \to 0$$

but it is not injective in general: e.g. for $\varrho \equiv 1$, the map $\tilde{\mathcal{E}} \to R$ symbolized by

$$
\begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix}
$$

(and zero outside the square) lies in ker $\delta$.

The restriction $\delta^*|_{C_c(\tilde{\mathcal{V}}, R)}$ to compactly supported maps is injective since $\tilde{T}$ has no directed circles, and we have a surjective map

$$\text{coker} \left( \delta^* : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{E}}, R) \right) \to C^0(\mathbb{P}^1(F), R)/R, \quad c \mapsto \sum_{e \in \tilde{\mathcal{E}}} c(e)1_{U(\pi(e))}$$

(which corresponds to an isomorphism of the similar map on the Bruhat-Tits tree $\mathcal{T}$). Its kernel is generated by the functions $1_{\{e\}} - 1_{\{e'\}}$ for $e, e' \in \tilde{\mathcal{E}}$ with $\pi(e) = \pi(e')$.

For $\varrho_1, \varrho_2 \in C(\tilde{\mathcal{V}}, R)$ and $\phi \in C(\tilde{\mathcal{V}}, M)$ it is easily checked that

$$\left( \delta^1_\varrho \circ \delta^2_\varphi \right)(\phi) = (T + TR)(\varrho_1 \cdot \varrho_2) \cdot \phi - \varrho_2 \cdot (T + TR)(\varrho_1 \cdot \phi).$$

For $a' \in R$ and $\varrho \in \ker(T + TR - a')$, applying this equality for $\varrho_1 = \varrho$ and $\varrho_2 = 1$ shows that $\tilde{\delta}_\varrho$ maps $\text{Im} \delta^*$ into $\text{Im}(T + TR - a')$, so we get an $R$-linear map

$$\tilde{\delta}_\varrho : \text{coker} \left( \delta^* : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{E}}, R) \right) \to \text{coker}(T_e + T_eR_e - a').$$

Let now again $\alpha, \nu \in R^*$, and $a := \alpha + q\nu/\alpha$. We let $\varrho := \varrho_{\alpha, \nu} \in \tilde{B}^{a, \nu}(F, R)$ as defined in lemma 2.7 and write $\delta_{\alpha, \nu} := \tilde{\delta}_{\varrho}$. Since $\delta_{\alpha, \nu}$ maps $1_{\{e\}} - 1_{\{e\nu\varepsilon\}}$ into $\text{Im}(R - \nu)$, it induces a map

$$\delta_{\alpha, \nu} : C^0(\mathbb{P}^1(F), R)/R \to B_{a, \nu}(F, R)$$
(same name by abuse of notation) via the commutative diagram

\[
\begin{array}{ccc}
\text{coker } \delta^* & \xrightarrow{\tilde{\delta}_{\alpha,\nu}} & \text{coker}(T_c + T_cR_c - a') \\
\downarrow & & \downarrow \text{mod } (R - \nu) \\
C^0(\mathbb{P}^1(F), R)/R & \xrightarrow{\tilde{\delta}_{\alpha,\nu}} & B_{a,\nu}(F, R)
\end{array}
\]

with \(a' := a(1 + \nu)\), since \(g \in B_{a,\nu}(F, R) \subset \ker(T + T'R - a')\).

**Lemma 2.11.** We have \(g(\nu) = \chi_{\alpha}(d/a')\chi_{\nu}(a')g(\nu)\), and thus

\[
\tilde{\delta}_{\alpha,\nu}(gf) = \chi_{\alpha}(d/a')\chi_{\nu}(a')g\tilde{\delta}_{\alpha,\nu}(f),
\]

for all \(v \in \tilde{\mathcal{V}}\), \(f \in C^0(\mathbb{P}^1(F), R)/R\) and \(g = \begin{pmatrix} a' & b \\ 0 & d \end{pmatrix} \in B(F)\).

**Proof.** (a) Using lemma 2.5(b) and the fact that \(\text{ord}_{\nu}(e_1) = -\text{ord}_{\nu}(a') + \text{ord}_{\nu}(e_1)\), we have

\[
g\begin{pmatrix} a' & b \\ 0 & d \end{pmatrix}v = \alpha^{h(\nu) - \text{ord}_{\nu}(a'/d)}f^{\text{ord}_{\nu}(a') - \text{ord}_{\nu}(e_1)} = \chi_{\alpha}(d/a')\chi_{\nu}(a')g(\nu)
\]

for all \(v \in \tilde{\mathcal{V}}\). For \(f\) and \(g\) as in the assertion, we thus have

\[
\tilde{\delta}_{\alpha,\nu}(gf)(v) = \sum_{v=t(e)} g(o(e))f(g^{-1}e) - \sum_{v=o(e)} g(t(e))f(g^{-1}e)
\]

\[
= \sum_{g^{-1}v=t(e)} g(o(ge))f(e) - \sum_{g^{-1}v=o(e)} g(t(ge))f(e)
\]

\[
= \chi_{\alpha}(d/a')\chi_{\nu}(a')g(\nu)\left(\sum_{g^{-1}v=t(e)} g(o(e))f(e) - \sum_{g^{-1}v=o(e)} g(t(e))f(e)\right)
\]

\[
= \chi_{\alpha}(d/a')\chi_{\nu}(a')\tilde{\delta}_{\alpha,\nu}(f)(v).
\]

\[
\square
\]

We define a function \(\delta_{\alpha,\nu} : C_c(F^*, R) \to B_{a,\nu}(F, R)\) as follows: For \(f \in C_c(F^*, R)\), we let \(\psi_0(f) \in C_c(\mathbb{P}^1(F), R)\) be the extension of \(x \mapsto \chi_{\alpha}(x)\chi_{\nu}(x)^{-1}f(x)\) by zero to \(\mathbb{P}^1(F)\). We set \(\delta_{\alpha,\nu} := \tilde{\delta}_{\alpha,\nu} \circ \psi_0\). If \(\alpha = \nu\), we can define \(\delta_{\alpha,\nu}\) on all functions in \(C_c(F, R)\).

We let \(F^*\) operate on \(C_c(F, R)\) by \((tf)(x) := f(t^{-1}x)\); this induces an action of the group \(T^1(F) := \{(0 1) | t \in F^*\}\), which we identify with \(F^*\) in the obvious way. With respect to it, we have

\[
\psi_0(tf)(x) = \chi_{\alpha}(t)\chi_{\nu}(t)^{-1}t\psi_0(f)(x)
\]

and

\[
\tilde{\delta}_{\alpha,\nu}(tf) = \chi_{\alpha}^{-1}(t)\chi_{\nu}(t)\tilde{\delta}_{\alpha,\nu}(f),
\]

16
so \( \delta_{\alpha,\nu} \) is \( T^1(F) \)-equivariant.

For an \( R \)-module \( M \), we define an \( F^* \)-action on \( \text{Dist}(F^*, M) \) by \( \int f d(t \mu) := t \int (t^{-1} f) d\mu \). Let \( H \subseteq G(F) \) be a subgroup, and \( M \) an \( R[H] \)-module. We define an \( H \)-action on \( \mathcal{B}^{\alpha,\nu^{-1}}(F, M) \) by requiring \( \langle \phi, h \lambda \rangle = h \cdot \langle h_i^{-1} \phi, \lambda \rangle \) for all \( \phi \in \mathcal{B}_{a,\nu}(F, M), \lambda \in \mathcal{B}^{\alpha,\nu^{-1}}(F, M), h \in H \). With respect to these two actions, we get a \( T^1(F) \cap H \)-equivariant mapping

\[
\delta^{\alpha,\nu} : \mathcal{B}^{\alpha,\nu^{-1}}(F, M) \to \text{Dist}(F^*, M), \quad \delta^{\alpha,\nu}(\lambda) := \langle \delta_{\alpha,\nu}(\cdot), \lambda \rangle
\]
dual to \( \delta_{\alpha,\nu} \).

**2.6 Local distributions**

Now consider the case \( R = \mathbb{C} \). Let \( \chi_1, \chi_2 : F^* \to \mathbb{C}^* \) be two unramified characters. We consider \( (\chi_1, \chi_2) \) as a character on the torus \( T(F) \) of \( \text{GL}_2(F) \), which induces a character \( \chi \) on \( B(F) \) by

\[
\chi \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} := \chi_1(t_1) \chi_2(t_2).
\]

Put \( \alpha_i := \chi_i(\varpi) \sqrt{q} \in \mathbb{C}^* \) for \( i = 1, 2 \). Set \( \nu := \chi_1(\varpi) \chi_2(\varpi) = \alpha_1 \alpha_2 q^{-1} \in \mathbb{C}^* \), and \( a := \alpha_1 + \alpha_2 = \alpha_1 + q \nu / \alpha_1 \) for each \( i \). When \( a \) and \( \nu \) are given by the \( \alpha_i \) like this, we will often write \( \mathcal{B}_{a,\nu}(F, R) := \mathcal{B}_{a,\nu}(F, R) \) and \( \mathcal{B}^{\alpha_1,\alpha_2}(F, M) := \mathcal{B}^{\alpha_1,\alpha_2}(F, M) \) (!) for its dual.

In the special case \( a^2 = \nu(q+1)^2 \), we assume the \( \chi_i \) to be sorted such that \( \chi_1 = | \cdot | \chi_2 \) (not vice versa).

Let \( \mathcal{B}(\chi_1, \chi_2) \) denote the space of continuous maps \( \phi : G(F) \to \mathbb{C} \) such that

\[
\phi \left( \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} g \right) = \chi_1(t_1) \chi_2(t_2) |t_1| \phi(g) \tag{2.7}
\]

for all \( t_1, t_2 \in F^*, u \in F, g \in G(F) \). \( G(F) \) operates canonically on \( \mathcal{B}(\chi_1, \chi_2) \) by right translation (cf. [Bu], Ch. 4.5). If \( \chi_1 \chi_2^{-1} \neq | \cdot |^{\pm 1} \), \( \mathcal{B}(\chi_1, \chi_2) \) is a model of the spherical representation \( \pi(\chi_1, \chi_2) \); if \( \chi_1 \chi_2^{-1} = | \cdot |^{\pm 1} \), the special representation \( \pi(\chi_1, \chi_2) \) can be given as an irreducible subquotient of codimension 1 of \( \mathcal{B}(\chi_1, \chi_2) \).\[iii\]

**Lemma 2.12.** We have a \( G \)-equivariant isomorphism \( \mathcal{B}_{a,\nu}(F, \mathbb{C}) \cong \mathcal{B}(\chi_1, \chi_2) \). It induces an isomorphism \( \mathcal{B}_{a,\nu}(F, \mathbb{C}) \cong \pi(\chi_1, \chi_2) \) both for spherical and special representations.

**Proof.** We choose a “central” unramified character \( \chi_Z : F^* \to \mathbb{C} \) satisfying \( \chi_Z^2(\varpi) = \nu \); then we have \( \chi_1 = \chi_Z \chi_0^{-1}, \chi_2 = \chi_Z \chi_0 \) for some unramified character \( \chi_0 \). We set \( a' := \sqrt{q} (\chi_0(\varpi)^{-1} + \chi_0(\varpi)) \), which satisfies \( a = \sqrt{a'} \).

For a representation \( (\pi, V) \) of \( G(F) \), by [Bu], Ex. 4.5.9, we can define another representation \( \chi_Z \otimes \pi \) on \( V \) via

\[
(g, v) \mapsto \chi_Z(\det(g)) \pi(g) v \quad \text{for all } g \in G(F), v \in V,
\]

\[iii\]Note that [Bu] denotes this special representation by \( \sigma(\chi_1, \chi_2) \), not by \( \pi(\chi_1, \chi_2) \).
and with this definition we have $\mathcal{B}(\chi_1, \chi_2) \cong \chi_Z \otimes \mathcal{B}(\chi_0^{-1}, \chi_0)$. Since $\mathcal{B}(\chi_0^{-1}, \chi_0)$ has trivial central character, [BL], Thm. 20 (as quoted in [Sp]) states that $\mathcal{B}(\chi_0^{-1}, \chi_0) \cong \mathcal{B}_a(a, F, C) \cong \text{Ind}_{KZ}^{G(F)} R/\text{Im}(T - a')$.

Define a $G$-linear map $\phi : \text{Ind}_K^G R \to \chi_Z \otimes \text{Ind}_{KZ}^G R$ by $1_K \mapsto (\chi_Z \circ \det) \cdot 1_{KZ}$. Since $1_K$ (resp. $(\chi_Z \circ \det) \cdot 1_{KZ}$) generates $\text{Ind}_K^G R$ (resp. $\chi_Z \otimes \text{Ind}_{KZ}^G R$) as a $\mathbb{C}[G]$-module, $\phi$ is well-defined and surjective.

$\phi$ maps $R1_K = (\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}) 1_K$ to

$$(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}) ((\chi_Z \circ \det) \cdot 1_{KZ}) = \chi_Z(\varpi)^2 \cdot ((\chi_Z \circ \det) \cdot 1_{KZ}) = \nu \cdot \phi(1_K).$$

Thus $\text{Im}(R - \nu) \subseteq \ker \phi$, and in fact the two are equal, since the preimage of the space of functions of support in a coset $KZg$ ($g \in G(F)$) under $\phi$ is exactly the space generated by the $1_{KZ} z \in Z(F) = Z(\mathfrak{o}_F) \{(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) \}$.

Furthermore, $\phi$ maps $T1_K = \sum_{i \in \mathcal{O}_{\mathfrak{r}}/(\varpi) \cup \{\infty\}} N_i 1_K$ (with the $N_i$ as in Lemma 2.7) to

$$\sum_i \chi_Z(\det(N_i)) \cdot ((\chi_Z \circ \det) \cdot N_i 1_{KZ}) = \chi_Z(\varpi) \cdot (\chi_Z \circ \det)T1_{KZ}$$

(since $\det(N_i) = \bar{\varpi}$ for all $i$), and thus $\text{Im}(T - a)$ is mapped to $\text{Im}\left((\chi_Z(\varpi)T - a) = \text{Im}\left((\chi_Z(\varpi)(T - a')) = \text{Im}(T - a').$$

Putting everything together, we thus have $G$-isomorphisms

$$C_{\chi}(V, C)/\left(\text{Im}(T - a) + \text{Im}(R - \nu)\right) \cong \text{Ind}_K^G R/\left(\text{Im}(T - a) + \text{Im}(R - \nu)\right) \cong \chi_Z \otimes \left(\text{Ind}_{KZ}^G R/\text{Im}(T - a')\right) \quad \text{(via } \phi)$$

$$\cong \chi_Z \otimes \mathcal{B}(\chi_0^{-1}, \chi_0) \cong \mathcal{B}(\chi_1, \chi_2).$$

Thus, $\mathcal{B}_{a, \nu}(F, C)$ is isomorphic to the spherical principal series representation $\pi(\chi_1, \chi_2)$ for $a^2 \neq \nu(q + 1)^2$.

In the special case, $\mathcal{B}_{a, \nu}(F, C)$ is a $G$-invariant subspace of $\tilde{\mathcal{B}}_{a, \nu}(F, C)$ of codimension 1, so it must be mapped under the isomorphism to the unique $G$-invariant subspace of $\mathcal{B}(\chi_1, \chi_2)$ of codimension 1 (in fact, the unique infinite-dimensional irreducible $G$-invariant subspace, by [Hu], Thm. 4.5.1), which is the special representation $\pi(\chi_1, \chi_2)$.

By [Hu], section 4.4, there exists thus for all pairs $a, \nu$ a Whittaker functional $\lambda$ on $\mathcal{B}_{a, \nu}(F, C)$, i.e. a nontrivial linear map $\lambda : \mathcal{B}_{a, \nu}(F, C) \to \mathbb{C}$ such that $\lambda \left((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \right) = \psi(x)\lambda(\phi)$. It is unique up to scalar multiples.

From it, we furthermore get a Whittaker model $\mathcal{W}_{a, \nu}$ of $\mathcal{B}_{a, \nu}(F, C)$:

$$\mathcal{W}_{a, \nu} := \{W_\xi : GL_2(F) \to \mathbb{C} | \xi \in \mathcal{B}_{a, \nu}(F, C)\},$$

where $W_\xi(g) := \lambda(g \cdot \xi)$ for all $g \in GL_2(F)$. (see e.g. [Hu], Ch. 3, eq. (5.6).)

Now write $\alpha := \alpha_1$ for short. Recall the distribution $\mu_{a, \nu} = \psi(x)\chi_{a, \nu}(x)dx \in \text{Dist}(F^*, \mathbb{C})$. For $\alpha = \nu$, it extends to a distribution on $F$.

**Proposition 2.13.** (a) There exists a unique Whittaker functional $\lambda = \lambda_{a, \nu}$ on $\mathcal{B}_{a, \nu}(F, C)$ such that $\delta^{a, \nu}(\lambda) = \mu_{a, \nu}$.  

18
(b) For every \( f \in C_c(F^*, \mathbb{C}) \), there exists \( W = W_f \in W_{a,\nu} \) such that

\[
\int_{F^*} (af)(x) \mu_{a,\nu}(dx) = W_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.
\]

If \( \alpha = \nu \), then for every \( f \in C_c(F, \mathbb{C}) \), there exists \( W_f \in W_{a,\nu} \) such that

\[
\int_F (af)(x) \mu_{a,\nu}(dx) = W_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.
\]

(c) Let \( H \subseteq U = O_F^\circ \) be an open subgroup, and write \( W_H := W_{1^H} \). For every \( f \in C_0^0(F^*, \mathbb{C})_H \) we have

\[
\int_{F^*} f(x) \mu_{a,\nu}(dx) = [U : H] \int_F f(x) W_H \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} dx.
\]

Proof. (a) By [Sp], proof of prop. 3.10, we have a Whittaker functional of the Steinberg representation given by the composite

\[
St(F, \mathbb{C}) := C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C} \xrightarrow{\delta} C_c(F, \mathbb{C}) \xrightarrow{\Lambda} \mathbb{C},
\]

where the first map is the \( F \)-equivariant isomorphism

\[
C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C} \to C_c(F, \mathbb{C}), \quad \phi \mapsto f(x) := \phi(x) - \phi(\infty),
\]

(with \( F \) acting on \( C_c(F, \mathbb{C}) \) by \( (x \cdot f)(y) := f(y - x) \), and on \( C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C} \) by \( x\phi := (\frac{\delta}{1}) \phi \), and the second is

\[
\Lambda : C_c(F, \mathbb{C}) \to \mathbb{C}, \quad f \mapsto \int_F f(x) \psi(x) dx.
\]

Let now \( \lambda : B_{a,\nu}(F, \mathbb{C}) \to \mathbb{C} \) be a Whittaker functional of \( B_{a,\nu}(F, \mathbb{C}) \). By lemma 2.11 for \( u = (\frac{\delta}{1}) \in B(F) \),

\[
(\lambda \circ \tilde{\delta}_{a,\nu})(u\phi) = \lambda(u\tilde{\delta}_{a,\nu}(\phi)) = \psi(x)\lambda(\tilde{\delta}_{a,\nu}(\phi)),
\]

so \( \lambda \circ \tilde{\delta}_{a,\nu} \) is a Whittaker functional if it is not zero.

To describe the image of \( \tilde{\delta}_{a,\nu} \), consider the commutative diagram

\[
\begin{array}{ccc}
C_c(\mathbb{E}, R) & \xrightarrow{\delta_{a,\nu}} & C_c(\hat{V}, R) \\
\downarrow^{\tilde{\delta}_{a,\nu}} & & \downarrow^{\phi \mapsto \phi \circ \rho} \\
C_c(\mathbb{E}, R) & \xrightarrow{\phi} & C_c(\hat{V}, R) \xrightarrow{\cdot 1} R \to 0
\end{array}
\]

where the vertical maps are defined by

\[
C_c(\mathbb{E}, R) \to C_c(\hat{E}, R), \quad c \mapsto (e \mapsto c(e) \rho(o(e)) \rho(t(e)))
\]

resp. by mapping \( \phi \) to \( v \mapsto \phi(v) \rho(v) \); both are obviously isomorphisms.
Since the lower row is exact, we have \( \text{Im} \delta = \ker (\cdot, 1) =: C^0_c(\mathcal{V}, R) \) and thus \( \text{Im} \tilde{\delta}_{\alpha, \nu} = \varrho^{-1} \cdot C^0_c(\mathcal{V}, R) \).

Since \( \lambda \neq 0 \) and \( \mathcal{B}_{a, \nu}(F, \mathbb{C}) \) is generated by (the equivalence classes of) the \( 1_{\{v\}} \), \( v \in \mathcal{V} \), there exists a \( v \in \mathcal{V} \) such that \( \lambda(1_{\{v\}}) \neq 0 \). Let \( \phi \) be this \( 1_{\{v\}} \), and let \( u = \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{T} & 1 \end{array} \right) \in B(F) \) such that \( x \notin \ker \psi \). Then

\[
\varrho \cdot (\varrho \phi - \phi) = \varrho \cdot (1_{\{u^{-1}v\}} - 1_{\{v\}}) = \varrho(v)(1_{\{u^{-1}v\}} - 1_{\{v\}}) \in C^0_c(\mathcal{V}, R)
\]

by lemma 2.11 so \( 0 \neq u\phi - \phi \in \text{Im} \tilde{\delta}_{\alpha, \nu} \), but \( \lambda(u\phi - \phi) = \psi(x)\lambda(\phi) - \lambda(\phi) \neq 0 \).

So \( \lambda \circ \tilde{\delta}_{\alpha, \nu} \neq 0 \) is indeed a Whittaker functional. By replacing \( \lambda \) by a scalar multiple, we can assume \( \lambda \circ \tilde{\delta}_{\alpha, \nu} = (2.8) \).

Considering \( \lambda \) as an element of \( \mathcal{B}^{\alpha, \nu}_{a, \nu} (F, \mathbb{C}) \cong \text{Hom}(\mathcal{B}_{a, \nu}(F, \mathbb{C}), \mathbb{C}) \), we have

\[
\delta^{\alpha, \nu}(\lambda)(f) = \langle \delta_{\alpha, \nu}(f), \lambda \rangle = \Lambda(\chi_{\alpha}\chi_{\nu}^{-1}f) = \int_{F^*} \chi_{\alpha}(x)\chi_{\nu}^{-1}(x)f(x)\psi(x)dx = \mu_{\alpha, \nu}(f).
\]

(b) For given \( f \), set \( W_f(g) := \lambda(g \cdot \delta_{\alpha, \nu}(f)) \). Then \( W_f \in \mathcal{W}_{a, \nu} \), and for all \( a \in F^* \) we have:

\[
W_f \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} \delta_{\alpha, \nu}(f) = \lambda(\delta_{\alpha, \nu}(af)) \quad \text{(by the \( T^1(F) \)-invariance of \( \delta_{\alpha, \nu} \))}
\]

\[
= \int_{F^*} (af)(x)\mu_{\alpha, \nu}(dx).
\]

(c) Without loss of generality we can assume \( f = 1_{aH} \) for some \( a \in F^* \). We have

\[
\int_{F^*} 1_{aH}(x)\mu_{\alpha, \nu}(dx) = \int_{F^*} 1_{H}(a^{-1}x)\mu_{\alpha, \nu}(dx) = \int_{F^*} (a \cdot 1_H)(x)\mu_{\alpha, \nu}(dx) = W_H \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} \quad \text{by (b)},
\]

and since the left-hand side is invariant under replacing \( a \) by \( ah \) (for \( h \in H \)), the
right-hand side also is, so we can integrate this constant function over $H$:

$$
\begin{align*}
\ &= [U : H] \int_H W_H \begin{pmatrix} ax & 0 \\ 0 & 1 \end{pmatrix} d^\times x \\
\ &= [U : H] \int_{F^*} 1_H(x) W_H \begin{pmatrix} ax & 0 \\ 0 & 1 \end{pmatrix} d^\times x \\
\ &= [U : H] \int_{F^*} 1_H(a^{-1} x) W_H \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x \\
\ &= [U : H] \int_{F^*} 1_{aH}(x) W_H \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x.
\end{align*}
$$

\[\square\]

### 2.7 Semi-local theory

We can generalize many of the previous constructions to the semi-local case, considering all primes $p | p$ at once.

So let $F_1, \ldots, F_m$ be finite extensions of $\mathbb{Q}_p$, and for each $i$, let $q_i$ be the number of elements of the residue field of $F_i$. We put $F := F_1 \times \cdots \times F_m$.

Let $R$ again be a ring, and $a_i \in R$, $\nu_i \in R^*$ for each $i \in \{1, \ldots, m\}$. Put $a := (a_1, \ldots, a_m)$, $\nu := (\nu_1, \ldots, \nu_m)$. We define $B_{\mathbb{A}, \nu}(F, R)$ as the tensor product

$$
B_{\mathbb{A}, \nu}(F, R) := \bigotimes_{i=1}^m B_{a_i, \nu_i}(F_i, R).
$$

For an $R$-module $M$, we define $\mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, M) := \text{Hom}_R(B_{\mathbb{A}, \nu}(F, R), M)$; let

$$
\langle \cdot, \cdot \rangle : B_{\mathbb{A}, \nu}(F, R) \times \mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, M) \to M \quad (2.10)
$$
denote the evaluation pairing.

We have an obvious isomorphism

$$
\bigotimes_{i=1}^m C^0_{\nu_i}(F_i^*, R) \to C^0(F^*, R), \quad \bigotimes_{i=1}^m f_i \mapsto \left( (x_i)_{i=1, \ldots, m} \mapsto \prod_{i=1}^m f_i(x_i) \right). \quad (2.11)
$$

Now when we have $\alpha_{i,1}, \alpha_{i,2} \in R^*$ such that $a_i = \alpha_{i,1} + \alpha_{i,2}$ and $\nu_i = \alpha_{i,1} \alpha_{i,2} q_i^{-1}$, we can define the $T^1(F)$-equivariant map

$$
\delta_{\alpha_{i,1}, \alpha_{i,2}} := \delta_{\alpha_{i,1}, \alpha_{i,2}} : C^0_{\nu_i}(F, R) \to B_{\mathbb{A}, \nu}(F, R)
$$
as the inverse of (2.11) composed with $\bigotimes_{i=1}^m \delta_{\alpha_{i,1}, \nu_i}$.

Again, we will often write $B_{\alpha_{i,1}, \alpha_{i,2}}(F, R) := \mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, R)$ and $B_{\alpha_{i,1}, \alpha_{i,2}}(F, M) := \mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, M)$.

If $H \subseteq G(F)$ is a subgroup, and $M$ an $R[H]$-module, we define an $H$-action on $\mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, M)$ by requiring $\langle \phi, h \lambda \rangle = h \cdot \langle h^{-1} \phi, \lambda \rangle$ for all $\phi \in B_{\mathbb{A}, \nu}(F, M)$, $\lambda \in \mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, M)$, $h \in H$, and get a $T^1(F) \cap H$-equivariant mapping

$$
\delta_{\alpha_{i,1}, \alpha_{i,2}} : \mathcal{B}_{\mathbb{A}, \nu}^{-1}(F, M) \to \text{Dist}(F^*, M), \quad \delta_{\alpha_{i,1}, \alpha_{i,2}}(\lambda) := \langle \delta_{\alpha_{i,1}, \alpha_{i,2}}(\cdot), \lambda \rangle.
$$

21
Finally, we have a homomorphism

\[
\bigotimes_{i=1}^{m} B^{\alpha_{i},\nu_{i},\nu_{i}^{-1}}_{i}(F_{i}, R) \xrightarrow{\cong} \bigotimes_{i=1}^{m} \text{Hom}_{R}(B^{\alpha_{i},\nu_{i},\nu_{i}^{-1}}_{i}(F_{i}, R), R)
\]

\[
\xrightarrow{\cong} \text{Hom}(B^{\alpha_{1},\nu_{1}}_{1}(F_{1}, R), \text{Hom}(B^{\alpha_{2},\nu_{2}}_{2}(F_{2}, R), \text{Hom}(\ldots, R))\ldots)
\]

\[
\xrightarrow{\cong} B^{\alpha_{\nu},\nu^{-1}}_{\nu}(F_{\nu}, R).
\]

(2.12)

where the second map is given by \( \otimes_{i} f_{i} \mapsto (x_{1} \mapsto (x_{2} \mapsto (\ldots \mapsto \prod_{i} f_{i}(x_{i}))\ldots) \), and the last map by iterating the adjunction formula of the tensor product.
3 Cohomology classes and global measures

3.1 Definitions

From now on, let $F$ denote a number field, with ring of integers $\mathcal{O}_F$. For each finite prime $v$, let $U_v := \mathcal{O}_v^*$. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adeles of $F$, and $\mathbb{I} = \mathbb{I}_F$ the group of ideles of $F$. For a finite subset $S$ of the set of places of $F$, we denote by $\mathbb{A}^S := \{x \in \mathbb{A}_F \mid x_v = 0 \text{ for } v \notin S\}$ the $S$-adeles and by $\mathbb{I}^S$ the $S$-ideles, and put $F_S := \prod_{v \in S} F_v$, $U_S := \prod_{v \in S} U_v$, $U^S := \prod_{v \in \mathbb{I}_F} U_v$ (if $S$ contains all infinite places of $F$), and similarly for other global groups.

For $\ell$ a prime number or $\infty$, we write $S_\ell$ for the set of places of $F$ above $\ell$, and abbreviate the above notations to $\mathbb{A}^\ell := \mathbb{A}^{S_\ell}$, $\mathbb{A}^{p,\infty} := \mathbb{A}^{S_{\infty}}$, and similarly write $\mathbb{I}^\ell$, $F_p$, $F_{\infty}$, $U_{\infty}$, $U_p$, $U^{p,\infty}$, $\mathbb{I}_{\infty}$ etc.

Let $F$ have $r$ real embeddings and $s$ pairs of complex embeddings. Set $d := r + s - 1$. Let $\{\sigma_0, \ldots, \sigma_{r-1}, \sigma_r, \ldots, \sigma_d\}$ be a set of representatives of these embeddings (i.e. for $i \geq r$, choose one from each pair of complex embeddings), and denote by $\infty_0, \ldots, \infty_d$ the corresponding archimedian primes of $F$. We let $S^0_\infty := \{\infty_1, \ldots, \infty_d\} \subseteq S_\infty$.

For each place $v$, let $dx_v$ denote the associated self-dual Haar measure on $F_v$, and $dx := \prod_v dx_v$ the associated Haar measure on $\mathbb{A}_F$. We define Haar measures $d^x x_v$ on $F_v$ by $d^x x_v := c_v \frac{dx_v}{|x_v|}$, where $c_v = (1 - \frac{1}{q_v})^{-1}$ for $v$ finite, $c_v = 1$ for $v|\infty$.

For $v|\infty$ complex, we use the decomposition $\mathbb{C}^* = \mathbb{R}_+^* \times S^1$ (with $S^1 = \{x \in \mathbb{C}^* \mid |x| = 1\}$) to write $d^x x_v = d^x r_v d\theta_v$ for variables $r_v, \theta_v$ with $r_v \in \mathbb{R}_+$, $\theta_v \in S^1$.

Let $S_1 \subseteq S_p$ be a set of primes of $F$ lying above $p$, $S_2 := S_p - S_1$. Let $R$ be a topological Hausdorff ring.

**Definition 3.1.** We define the module of continuous functions

$$\mathcal{C}(S_1, R) := C(F_{S_1} \times F_{S_2}^*, \mathbb{I}^p/\mathbb{I}^{p,\infty}, R);$$

and let $\mathcal{C}_c(S_1, R)$ be the submodule of all compactly supported $f \in \mathcal{C}(S_1, R)$. We write $\mathcal{C}^0(S_1, R)$, $\mathcal{C}^0_c(S_1, R)$ for the submodules of locally constant maps (or of continuous maps where $R$ is assumed to have the discrete topology).

We further define

$$\mathcal{C}^0_c(S_1, R) := \mathcal{C}_c(\varnothing, R) + \mathcal{C}^0_c(S_1, R) \subseteq \mathcal{C}_c(S_1, R)$$

to be the module of continuous compactly supported maps that are “constant near $(0_p, x^p)$” for each $p \in S_1$.

**Definition 3.2.** For an $R$-module $M$, let $\mathcal{D}_f(S_1, M)$ denote the $R$-module of maps

$$\phi : \mathcal{C}_c(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}_F^{p,\infty} \to M$$

that are $\mathbb{I}^{p,\infty}$-invariant and such that $\phi(\cdot, x^{p,\infty})$ is a distribution for each $x^{p,\infty} \in \mathbb{I}_F^{p,\infty}$.
Since $\mathbb{F}_F^{p,\infty}/U^{p,\infty}$ is a discrete topological group, $\mathcal{D}_f(S_1, M)$ naturally identifies with the space of $M$-valued distributions on $F_{S_1} \times F_{S_2}^* \times \mathbb{F}_F^{p,\infty}/U^{p,\infty}$. So there exists a canonical $R$-bilinear map

$$
\mathcal{D}_f(S_1, M) \times \mathcal{C}_c^0(S_1, R) \rightarrow M, \quad (\phi, f) \mapsto \int f \, d\phi,
$$

(3.1)

which is easily seen to induce an isomorphism $\mathcal{D}_f(S_1, M) \cong \text{Hom}_R(\mathcal{C}_c^0(S_1, R), M)$.

For a subgroup $E \subseteq F^*$ and an $R[E]$-module $M$, we let $E$ operate on $\mathcal{D}_f(S_1, M)$ and $\mathcal{C}_c^0(S_1, R)$ by $(a\phi)(U, x^{p,\infty}) := a\phi(a^{-1}U, a^{-1}x^{p,\infty})$ and $(a f)(x^{\infty}) := f(a^{-1}x^{\infty})$ for $a \in E, U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$, $x^{\infty} \in \mathbb{F}_F$; thus we have $\int (a f) \, d(\phi) = a \int f \, d\phi$ for all $a, f, \phi$.

When $M = V$ is a finite-dimensional vector space over a $p$-adic field, we write $\mathcal{D}_f^p(S_1, V)$ for the subset of $\phi \in \mathcal{D}_f(S_1, V)$ such that $\phi$ is even a measure on $F_{S_1} \times F_{S_2} \times \mathbb{F}_F^{p,\infty}/U^{p,\infty}$.

**Definition 3.3.** For a $\mathbb{C}$-vector space $V$, define $\mathcal{D}(S_1, V)$ to be the set of all maps $\phi : \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{P} \rightarrow V$ such that:

(i) $\phi$ is invariant under $F^*$ and $U^{p,\infty}$.

(ii) For $x^p \in \mathbb{P}$, $\phi(\cdot, x^p)$ is a distribution of $F_{S_1} \times F_{S_2}$.

(iii) For all $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$, the map $\phi_U : \mathbb{P} = F_{S_1}^* \times \mathbb{P} \rightarrow V, (x_p, x^p) \mapsto \phi(x_p U, x^p)$ is smooth, and rapidly decreasing as $|x| \rightarrow \infty$ and $|x| \rightarrow 0$.

We will need a variant of this last set: Let $\mathcal{D}'(S_1, V)$ be the set of all maps $\phi \in \mathcal{D}(S_1, V)$ that are "$(S^1)^s$-invariant", i.e. such that for all complex primes $\infty_j$ of $F$ and all $\zeta \in S^1 = \{x \in \mathbb{C}^* ; |x| = 1\}$, we have

$$
\phi(U, x^{p,\infty_j}, \zeta x_{\infty_j}) = \phi(U, x^{p,\infty_j}, x_{\infty_j}) \quad \text{for all } x^p = (x^{p,\infty_j}, x_{\infty_j}) \in \mathbb{P}.
$$

There is an obvious surjective map

$$
\mathcal{D}(S_1, V) \rightarrow \mathcal{D}'(S_1, V), \quad \phi \mapsto \left( (U, x) \mapsto \int_{(S_1)^s} \phi(U, x) \, d\theta_r \cdots d\theta_{r+s-1} \right)
$$

given by integrating over $(S^1)^s \subseteq (\mathbb{C}^*)^s \hookrightarrow \mathbb{P}_\infty$.

Let $F^{*'} \subseteq F^*$ be a maximal torsion-free subgroup (so that $F/F^{*'} \cong \mu_F$, the roots of unity of $F$). If $F$ has at least one real embedding, we specifically choose $F^{*'}$ to be the set $F^*_+ \subseteq F^*$ of all totally positive elements of $F$ (i.e. positive with respect to every real embedding of $F$). For totally complex $F$, there is no such natural subgroup available, so we just choose $F^{*'}$ freely. We set

$$
E' := F^{*'} \cap O_F^\times \subseteq O_F^\times,
$$

so $E'$ is a torsion-free $\mathbb{Z}$-module of rank $d$. $E'$ operates freely and discretely on the space

$$
\mathbb{R}_0^{d+1} := \left\{ (x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} x_i = 0 \right\}
$$

24
via the embedding
\[ E' \hookrightarrow \mathbb{R}^{d+1}_0 \]
\[ a \mapsto (\log |\sigma_i(a)|)_{i \in S_\infty} \]

(cf. proof of Dirichlet’s unit theorem, e.g. in [Neu, Ch. 1], and the quotient \( \mathbb{R}^{d+1}_0 / E' \) is compact. We choose the orientation on \( \mathbb{R}^{d+1}_0 \) induced by the natural orientation on \( \mathbb{R}^d \) via the isomorphism \( \mathbb{R}^d \cong \mathbb{R}^{d+1}_0, (x_1, \ldots, x_d) \mapsto (-\sum_{i=1}^d x_i, x_1, \ldots, x_d) \). So \( \mathbb{R}^{d+1}_0 / E' \) becomes an oriented compact \( d \)-dimensional manifold.

Let \( G_p \) be the Galois group of the maximal abelian extension of \( F \) which is unramified outside \( p \) and \( \infty \); for a \( \mathbb{C} \)-vector space \( V \), let \( \text{Dist}(G_p, V) \) be the set of \( V \)-valued distributions of \( G_p \). Denote by \( \varrho : \mathbb{I}_F / F^* \to G_p \) the projection given by global reciprocity.

### 3.2 Global measures

Now let \( V = \mathbb{C} \), equipped with the trivial \( F^* \)-action. We want to construct a commutative diagram

\[
\begin{align*}
\mathcal{D}(S_1, \mathbb{C}) & \xrightarrow{\phi \mapsto \mu_\phi} H^d(F^* E', \mathcal{D}_f(S_1, \mathbb{C})) \\
\mathcal{D}(G_p, \mathbb{C}) & \xrightarrow{\kappa \mapsto \mu_\kappa} H^0(F^* E', \mathcal{D}_f(S_1, \mathbb{C}))
\end{align*}
\]

First, let \( R \) be any topological Hausdorff ring. Let \( \overline{E}' \) denote the closure of \( E' \) in \( U_p \). The projection map \( \text{pr} : \mathbb{I}_\infty / U_{p, \infty} \to \mathbb{I}_\infty / (\overline{E}' \times U_{p, \infty}) \) induces an isomorphism

\[ \text{pr}^* : C_c(\mathbb{I}_\infty / (\overline{E}' \times U_{p, \infty}), R) \to H^0(E', C_c(\mathbb{I}_\infty / U_{p, \infty}, R)), \]

and the reciprocity map induces a surjective map \( \overline{\varrho} : \mathbb{I}_\infty / (\overline{E}' \times U_{p, \infty}) \to G_p \).

Now we can define a map

\[ \varrho^\#: H_0(F^* E', C_c(\mathbb{I}_\infty / (\overline{E}' \times U_{p, \infty}), R)) \to C(G_p, R) \]

\[ [f] \mapsto \left( \overline{\varrho}(x) \mapsto \sum_{\zeta \in F^* E'} f(\zeta x) \text{ for } x \in \mathbb{I}_\infty / (\overline{E}' \times U_{p, \infty}) \right). \]

This is an isomorphism, with inverse map \( f \mapsto [([f \circ \overline{\varrho}] \cdot 1_F], \) where \( 1_F \) is the characteristic function of a fundamental domain \( F \) of the action of \( F^* E' \) on \( \mathbb{I}_\infty / U_{p, \infty} \).

We get a composite map

\[
\begin{align*}
C(G_p, R) & \xrightarrow{(\varrho^\#)^{-1}} H_0(F^* E', C_c(\mathbb{I}_\infty / (\overline{E}' \times U_{p, \infty}), R)) \\
& \xrightarrow{\text{pr}^*} H_0(F^* E', H^0(E', C_c(\mathbb{I}_\infty / U_{p, \infty}, R))) \\
& \to H_0(F^* E', H^0(E', C_c(S_1, R))), \quad (3.3)
\end{align*}
\]

25
where the last arrow is induced by the “extension by zero” from \( C_c(\mathbb{P}^\infty/U_{p}^\infty, R) \) to \( C_c(S_1, R) \).

Now let \( \eta \in H_d(E', \mathbb{Z}) \cong \mathbb{Z} \) be the generator that corresponds to the given orientation of \( \mathbb{R}^{d+1}_0 \). This gives us, for every \( R \)-module \( A \), a homomorphism

\[
H_0\left(F^*/E', H^0(E', A)\right) \overset{\eta}{\longrightarrow} H_0\left(F^*/E', H_d(E', A)\right)
\]

Composing this with the edge morphism

\[
H_0\left(F^*/E', H^0(E', A)\right) \to H_d(F^*/E', A)
\] (3.4)

(and setting \( A = C_c(S_1, R) \)) gives a map

\[
H_0\left(F^*/E', H^0(E', C_c(S_1, R))\right) \to H_d\left(F^*/E', C_c(S_1, R)\right)
\] (3.5)

We define

\[
\partial : C(\mathcal{G}_p, R) \to H_d(F^*/E', C_c(S_1, R))
\]

as the composition of (3.3) with this map.

Now, letting \( M \) be an \( R \)-module equipped with the trivial \( F^* \)-action, the bilinear form (3.1)

\[
D_f(S_1, M) \times C_c(S_1, R) \to M
\]

\[
(\phi, f) \mapsto \int f \, d\phi
\]

induces a cap product

\[
\cap : H^d(F^*/E', D_f(S_1, M)) \times H_d(F^*/E', C_c(S_1, R)) \to H_0(F^*/E', M) = M.
\] (3.6)

Thus for each \( \kappa \in H^d(F^*/E', D_f(S_1, M)) \), we get a distribution \( \mu_\kappa \) on \( \mathcal{G}_p \) by defining

\[
\int_{\mathcal{G}_p} f(\gamma) \, \mu_\kappa(d\gamma) := \kappa \cap \partial(f)
\] (3.7)

for all continuous maps \( f : \mathcal{G}_p \to R \).

Now let \( M = V \) be a finite-dimensional vector space over a \( p \)-adic field \( K \), and let \( \kappa \in H^d(F^*/E', D^b_f(S_1, V)) \). We identify \( \kappa \) with its image in \( H^d(F^*/E', D_f(S_1, V)) \); then it is easily seen that \( \mu_\kappa \) is also a measure, i.e. we have a map

\[
H^d(F^*/E', D^b_f(S_1, V)) \to \text{Dist}^b(\mathcal{G}_p, V), \quad \kappa \mapsto \mu_\kappa.
\] (3.8)

Let \( L|F \) be a \( \mathbb{Z}_p \)-extension of \( F \). Since it is unramified outside \( p \), it gives rise to a continuous homomorphism \( \mathcal{G}_p \to \text{Gal}(L|F) \) via \( \sigma \mapsto \sigma|_L \). Fixing an isomorphism \( \text{Gal}(L|F) \cong p^{\ast} \mathbb{Z}_p \) (where \( \varepsilon_p = 2 \) for \( p = 2 \), \( \varepsilon_p = 1 \) for \( p \) odd), we obtain a surjective homomorphism \( \ell : \mathcal{G}_p \to p^{\ast} \mathbb{Z}_p \). Here we have chosen the target space such that the \( p \)-adic exponential function \( \exp_p(s\ell(\gamma)) \) is defined for all \( s \in \mathbb{Z}_p, \gamma \in \mathcal{G}_p \).
Example 3.4. Let \( L \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \). Then we can take \( \ell = \log_p \circ \mathcal{N} \), where \( \mathcal{N} : \mathcal{G}_p \to \mathbb{Z}_p^* \) is the \( p \)-adic cyclotomic character, defined by requiring \( \gamma \zeta = \zeta^{\mathcal{N}(\gamma)} \) for all \( \gamma \in \mathcal{G}_p \) and all \( p \)-power roots of unity \( \zeta \). It is well-known (cf. [Wa], par. 5) that \( \log_p(\mathbb{Z}_p^*) = p^{\mathbb{Z}} \mathbb{Z}_p \).

It is well-known that \( F \) has \( t \) independent \( \mathbb{Z}_p \)-extensions, where \( s + 1 \leq t \leq [F : \mathbb{Q}] \); the Leopoldt conjecture implies \( t = s + 1 \). We get a \( t \)-variable \( p \)-adic \( L \)-functions as follows:

Definition 3.5. Let \( K \) be a \( p \)-adic field, \( V \) a finite-dimensional \( K \)-vector space, \( \kappa \in H^d(F^{*\prime}, D^f(S_1, V)) \). Let \( \ell_1, \ldots, \ell_t : \mathcal{G}_p \to p^r \mathbb{Z}_p \) be continuous homomorphisms. The \( p \)-adic \( L \)-function of \( \kappa \) is given by

\[
L_p(\Sigma, \kappa) := L_p(s_1, \ldots, s_t, \kappa) := \prod_{i=1}^t \exp_p(s_i \ell_i(\gamma)) \mu_\kappa(d\gamma)
\]

for all \( s_1, \ldots, s_t \in \mathbb{Z}_p \).

Remark 3.6. Let \( \Sigma := \{ \pm 1 \}^r \), where \( r \) is the number of real embeddings of \( F \). The group isomorphism \( \mathbb{Z}/2\mathbb{Z} \cong \{ \pm 1 \}, \varepsilon \mapsto (-1)^\varepsilon \), induces a pairing

\[
\langle \cdot, \cdot \rangle : \Sigma \to \{ \pm 1 \}, \quad \langle ((-1)^{\varepsilon_i}), ((-1)^{\varepsilon'_i}) \rangle := (-1)^{\sum \varepsilon_i \varepsilon'_i}.
\]

For a field \( k \) of characteristic zero, a \( k[\Sigma] \)-module \( V \) and \( \mu = (\mu_0, \ldots, \mu_{r-1}) \in \Sigma \), we put \( V_\Sigma := \{ v \in \mathbb{V} | \langle \mu, \nu \rangle v = \nu v \ \forall \nu \in \Sigma \} \), so that we have \( \mathbb{V} = \bigoplus_{\mu \in \Sigma} V_\mu \). We write \( v_\Sigma \) for the projection of \( v \in \mathbb{V} \) to \( V_\Sigma \), and \( v_0 := v_{(1, \ldots, 1)} \).

We identify \( \Sigma \) with \( F^*/F^{*\prime} \) via the isomorphism \( \Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}_+^* \cong F^*/F^{*\prime} \). Then for each \( F^{*\prime} \)-module \( M \), \( \Sigma \) acts on \( H^d(F^{*\prime}, D^f(S_1, M)) \) and on \( H^d(F^{*\prime}, D^f(S_1, M)) \).

The exact sequence \( \Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}_+^* = \mathbb{I}_\infty/\mathbb{I}_0 \to \mathcal{G}_p \to \mathcal{G}_p^+ \to 0 \) of class field theory (where \( \mathbb{I}_0 \) is the maximal connected subgroup of \( \mathbb{I}_\infty \)) yields an action of \( \Sigma \) on \( \mathcal{G}_p \). We easily check that \( G_{\mathbb{I}_0} \) is \( \Sigma \)-equivariant, and that the maps \( \gamma \mapsto \exp_p(s\ell_i(\gamma)) \) factor over \( \mathcal{G}_p \to \mathcal{G}_p^+ \) (since \( \mathbb{Z}_p \)-extensions are unramified at \( \infty \)). Therefore we have \( L_p(\Sigma, \kappa) = L_p(\Sigma, \kappa^+ \kappa) \).

For \( \phi \in \mathcal{D}(S_1, V) \) and \( f \in C^0(\mathbb{I}/F^{*\prime}, \mathbb{C}) \), let

\[
\int_{\mathbb{I}/F^{*\prime}} f(x) \phi(d^x x_p, x^p) \ d^x x := [U_p : U] \int_{\mathbb{I}/F^{*\prime}} f(x) \phi_U(x) \ d^x x,
\]

where we choose an open set \( U \subseteq U_p \) such that \( f(x_p u, x^p) = f(x_p, x^p) \) for all \( (x_p, x^p) \in U \) and \( u \in U \); such a \( U \) exists by lemma 3.7 below.

Since this integral is additive in \( f \), there exists a unique \( V \)-valued distribution \( \mu_\phi \) on \( \mathcal{G}_p \) such that

\[
\int_{\mathcal{G}_p} f \ d\mu_\phi = \int_{\mathbb{I}/F^{*\prime}} f(q(x)) \phi(d^x x_p, x^p) \ d^x x^p
\]

for all functions \( f \in C^0(\mathcal{G}_p, V) \).
Lemma 3.7. Let $F : \mathbb{I}/F^* \to X$ be a locally constant map to a set $X$. Then there exists an open subgroup $U \subseteq \mathbb{I}$ such that $f$ factors over $\mathbb{I}/F^*U$.

Proof. (cf. [Sp], lemma 4.20)

Let $U_\infty^0 := \prod_{v \in S^0} \mathbb{R}^*_+$; the isomorphisms $U^0_\infty \cong \mathbb{R}^d$, $(r_v)_v \mapsto (\log r_v)_v$, and $\mathbb{R}^d \cong \mathbb{R}^{d+1}_+$ give it the structure of a $d$-dimensional oriented manifold (with the natural orientation). It has the $d$-form $d^x r_1 \ldots d^x r_d$, where (by slight abuse of notation) we choose $d^x r_i$ on $F_{\infty i}$ corresponding to the Haar measure $d^x x_i$ resp. $d^x r_i$ on $\mathbb{R}^*_+ \subseteq F^*_\infty$.

$E'$ operates on $U^0_\infty$ via $a \mapsto (|\sigma_i(a)|)_i \in S^0$, making the isomorphism $U^0_\infty \cong \mathbb{R}^{d+1}_+$ $E'$-equivariant.

For $\phi \in \mathcal{D}'(S_1, V)$, set

$$
\int_0^\infty \phi \ d^x r_0 : \mathfrak{a}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^{p,\infty_0} \to \mathbb{C} \\
(U, x^{p,\infty_0}) \mapsto \int_0^\infty \phi(U, r_0, x^{p,\infty_0}) \ d^x r_0,
$$

where we let $r_0 \in F_{\infty 0}$ run through the positive real line $\mathbb{R}^*_+$ in $F_{\infty 0}$. Composing this with the projection $\mathcal{D}(S_1, V) \to \mathcal{D}'(S_1, V)$ gives us a map

$$
\mathcal{D}(S_1, V) \to H^0(F^{*t}, \mathcal{D}_f(S_1, C^\infty(U^0_\infty, V))), \\
\phi \mapsto \int_{(S_1)^s} \left( \int_0^\infty \phi \ d^x r_0 \right) \ d\theta_1 \ d\theta_2 \ldots d\theta_{r+s+1} \tag{3.10}
$$

(3.12)

(where $C^\infty(U^0_\infty, V)$ denotes the space of smooth $V$-valued functions on $U^0_\infty$), since one easily checks that $\int_0^\infty \phi \ d^x r_0$ is $F^{*t}$-invariant.

Define the complex $C^* := \mathcal{D}_f(S_1, \Omega^*(U^0_\infty, V))$. By the Poincare lemma, this is a resolution of $\mathcal{D}_f(S_1, V)$. We now define the map $\phi \mapsto \kappa_\phi$ as the composition of $\kappa_{3.11}$ with the composition

$$
H^0(F^{*t}, \mathcal{D}_f(S_1, C^\infty(U^0_\infty, V))) \to H^0(F^{*t}, C^d) \to H^d(F^{*t}, \mathcal{D}_f(S_1, V)), \tag{3.11}
$$

where the first map is induced by

$$
C^\infty(U^0_\infty, V) \to \Omega^d(U^0_\infty, V), \quad f \mapsto f(r_1, \ldots, r_d) d^x r_1 \ldots d^x r_d, \tag{3.12}
$$

and the second is an edge morphism in the spectral sequence

$$
H^q(F^{*t}, C^p) \Rightarrow H^{p+q}(F^{*t}, \mathcal{D}_f(S_1, V)). \tag{3.13}
$$

Specializing to $V = \mathbb{C}$, we now have:
Proposition 3.8. The diagram (3.2) commutes, i.e., for each φ ∈ D(S₁, ℂ), we have

\[ \mu_\phi = \mu_\kappa \phi. \]

Proof. As in [Sp], we define a pairing

\[ \langle , \rangle : D(S_1, ℂ) \times C^0(G_p, ℂ) \to ℂ \]

as the composite of (3.10) × (3.3) with

\[ H^0(F^\ast', D_f(S_1, C^\infty(U_1^\infty, ℂ))) \times H_0(F^\ast/E', H^0(E, C^0_c(S_1, ℂ))) \]

\[ \xrightarrow{\cap} H_0(F^\ast/E', H^0(E', C^\infty(U_1^\infty, ℂ))) \to H_0(F^\ast/E', ℂ) \cong ℂ, \]

(3.14)

where \cap is the cap product induced by (3.1), and the second map is induced by

\[ H^0(E', C^\infty(U_1^\infty, ℂ)) \to ℂ, f \mapsto \int_{U_1^\infty/E'} f(r_1, \ldots, r_d) \, d^x r_1 \ldots d^x r_d. \]

(3.15)

Then we have

\[ \langle \phi, f \rangle = \int_{G_p} f(\gamma) \mu_\phi(d\gamma) \quad \text{for all } f \in C^0(G_p, ℂ), \]

and we can show that κ_φ ∩ ∂(f) = \langle φ, f \rangle by copyng the proof for the totally real case ([Sp], prop. 4.21, replacing \( F^\ast \) by \( F'^\ast \), \( E^\ast \) by \( E' \)), using the fact that for a \( d \)-form on the \( d \)-dimensional oriented manifold \( M := \mathbb{R}^{d+1}_0/\mathbb{Q} \), integration over \( M \) corresponds to taking the cap product with the fundamental class \( \eta \) of \( M \) under the canonical isomorphism \( H^d(M) \cong H^d(\mathbb{Q}) \).

3.3 Exceptional zeros

Now let \( \ell_1, \ldots, \ell_t : G_p \to ℤ_p \) be homomorphisms. Let \( S_1 \subseteq S_p \) be a set of primes, \( n := \#S_1 \).

Proposition 3.9. For each \( x = (x_1, \ldots, x_t) \in ℤ_p^t \) set \( |x| := \sum_{i=1}^t x_i \). Then

\[ \partial(\prod_{i=1}^t (\ell_i^x)) = 0 \text{ for all } x \text{ with } |x| \leq n - 1. \]

Proof. (cf. [Sp], Prop. 4.6)

For each \( i \in \{1, \ldots, t\} \) let \( \tilde{\ell}_i : \mathbb{P}^\infty \to ℚ_p \) be the composition

\[ \tilde{\ell}_i : \mathbb{P}^\infty \xrightarrow{\ell_i} G_p \xrightarrow{\ell_i} ℤ_p \hookrightarrow ℚ_p. \]

Let \( p_1, \ldots, p_m \) be the primes of \( F \) above \( p \). Since \( F^\ast/E' = F^\ast/O_F^\times \) is a free \( ℤ \)-module (it embeds into the group of fractional ideals of \( O_F \)), we can choose a subgroup \( T \subseteq F^\ast' \) such that \( F^\ast' = E' \times T \). By the finiteness of the class number, we can also find a subgroup \( T' \subseteq T \) of finite index such that

\[ T' = T_p \times T'^p = \langle t_1, \ldots, t_m \rangle \times T'^p, \]
where \( t_i \) generates some power \( p_i^{n_i} \) of \( p_i \) for all \( i \), and \( \text{ord}_{p_i}(t) = 0 \) for all \( t \in T_p \), \( i = 1, \ldots, m \).

Let \( F \subseteq \mathbb{I}^{\infty}/U^{p,\infty} \) be a fundamental domain for the action of \( \mathcal{T} \) such that \( \mathcal{T}/F = \mathcal{F} \). \((3.3)\) maps \( \tilde{\ell}_x := \prod_{i=1}^n \tilde{\ell}_x^i \) to the class \([\tilde{\ell}_x]_F \in H_0(\mathcal{T}, H^0(E', C_c^0(S_1, \mathbb{C}_p))) \subseteq H_0(\mathcal{T}, H^0(E', C_c^0(S_1, \mathbb{C}_p))). \) Thus, by the definition of \( \partial \), we have to show that \([\tilde{\ell}_x]_F \) is mapped to zero under the map \((3.5)\). Now we have a commutative diagram

\[
\begin{array}{ccc}
H_0(\mathcal{T}, H^0(E', C_c^0(S_1, \mathbb{C}_p))) & \xrightarrow{\iota} & H_0(\mathcal{T}, C_c^0(S_1, \mathbb{C}_p)) \\
\downarrow \gamma \eta & & \downarrow \gamma \eta \\
H_d(F^*, C_c^0(S_1, \mathbb{C}_p)) & \xrightarrow{\text{coinf}} & H_d(E', H_0(\mathcal{T}, C_c^0(S_1, \mathbb{C}_p)))
\end{array}
\]

where the upper horizontal map is induced by the inclusion \( H^0(\mathcal{T}, X) \to X \) (or equivalently, the projection \( X \to H_0(X) \)) and the lower horizontal map is the coinflation.

By prop. 3.1 of \([Sp]\), \( C_c^0(S_1, \mathbb{C}_p) \) is a free \( \mathbb{C}_p[\mathcal{T}] \)-module (the proof given in \([Sp]\) works verbatim for the case of an arbitrary number field \( F \)). So it is an induced \( \mathcal{T} \)-module and therefore homologically trivial. Thus the short exact sequence for group homology (or Shapiro’s lemma) shows that \( \text{coinf} : H_d(F^*, C_c^0(S_1, \mathbb{C}_p)) \to H_d(E', H_0(\mathcal{T}, C_c^0(S_1, \mathbb{C}_p))) \) is an isomorphism. So it suffices to prove that \( \iota * \) maps \([\tilde{\ell}_x]_F \) to zero, i.e. that

\[
\tilde{\ell}_x \in I(\mathcal{T}) C_c^0(S_1, \mathbb{C}_p), \tag{3.16}
\]

where \( I(\mathcal{T}) = (1 - t)_{t \in \mathcal{T}} \) is the augmentation ideal in the group ring \( \mathbb{C}_p[\mathcal{T}] \).

Again by prop. 3.1 of \([Sp]\), the restriction map

\[
\text{res} : H_0(\mathcal{T}, C_c^0(S_1, \mathbb{C}_p)) \to H_0(\mathcal{T}', C_c^0(S_1, \mathbb{C}_p)), \quad [f] \mapsto \sum_{[t] \in \mathcal{T}/T'} tf,
\]

is injective, and maps \([\tilde{\ell}_x]_F \) to \([\tilde{\ell}_x]_{F'} \), where \( F' = \bigcup_{[t] \in \mathcal{T}/T'} tF \subseteq \mathbb{I}^{\infty}/U^{p,\infty} \) is a fundamental domain for the action of \( \mathcal{T}' \). Thus we may replace \( \mathcal{T}, F \) by \( \mathcal{T}', F' \) in \((3.16)\).

We can specifically choose the fundamental domain \( F := \prod_{i=1}^n F_i \times F^p, \) where \( F^p \subseteq \mathbb{I}^{p,\infty}/U^{p,\infty} \) is a fundamental domain for \( \mathcal{T}^p \) and \( F_i := \mathcal{O}_{p_i} \setminus \mathcal{O}_{p_i} \).

Since the pro-\( q \)-part of \( \mathcal{G}_p \) is finite for every prime \( q \neq p \) and \( \mathcal{O}_p \) is torsion-free, \( \tilde{\ell}_j \) equals \( \tilde{\ell}_{ij} : \mathbb{I}^{p,\infty} \mathcal{O}_p^* \to \mathbb{I}^{\infty} \mathcal{O}_p^* \).

Similarly, we let \( \tilde{\ell}_{ij} \) be the restriction of \( \tilde{\ell}_j \) to \( F_{p_i}^* \) (considered as a function on \( \mathbb{I}^{\infty} \) or \( F_{p_i}^* \) as needed) for all \( i, j \).

For each subset \( \Xi \subseteq \{1, \ldots, r\} \) let

\[
\mathcal{F}_\Xi := \prod_{i \in \Xi} \mathcal{O}_{p_i} \times \prod_{i \in S_p \setminus \Xi} F_i \times F^p.
\]

For \( \Xi := (n_{i,j})_{i=1, \ldots, m; j=1, \ldots, t} \in \mathbb{N}_0^m \) with \( n_{i,j} = 0 \) for all \( i \in \Xi \) and all \( j \), we define \( \lambda(\Xi, \Xi) := \prod_{i,j} \tilde{\ell}_{ij} \cdot 1_{\mathcal{F}_\Xi} \in C_c^0(S_1, \mathbb{C}_p) \). Then by the multinomial formula,

\[
\tilde{\ell}_x = \sum_{|\Xi| = |\Xi|} N_{i,j} \lambda(\Xi, \Xi)
\]
for some \( N_{i,j} \in \mathbb{Z} \), and it suffices to show that \( \lambda(\emptyset, \underline{n}) \in I(\mathcal{P}_p)C_c^\ell(S_1, C_p) \subseteq I(\mathcal{T}_p)C_c^\ell(S_1, C_p) \) for all \( \underline{n} \) with \( |\underline{n}| = |\underline{x}| \). This follows from:

**Lemma 3.10.** If \( \#(\Xi) + |\underline{n}| < r \), then \( \lambda(\Xi, \underline{n}) \in I(\mathcal{P}_p)C_c^\ell(S_1, C_p) \).

**Proof.** (cf. [Sp], lemma 4.7)

For \( t \in F^* \) and \( f, g \in C_c^\ell(S_1, C_p) \), we have

\[
(1 - t)(f \cdot g) = ((1 - t)f) \cdot g + f \cdot ((1 - t)g) - ((1 - t)f) \cdot ((1 - t)g),
\]

where \( 1 - t \in C_p[\mathcal{T}'] \). Since \( (1 - t)\tilde{t}_i(x) = \tilde{t}_i(t) \), using this equation recursively shows that

\[
(1 - t) \prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} = \sum_{\underline{n}' < \underline{n}} a_{\underline{n}'} \prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}}
\]

for some \( a_{\underline{n}'} \in C_p \).

We prove the lemma by induction on \( |\underline{n}| \). Let \( \Xi^c := \{1, \ldots, r\} \setminus \Xi \). For \( \underline{n} = \underline{0} = (0, \ldots, 0) \) choose any \( i \in \Xi^c \) (which is nonempty since \( \#(\Xi) < r \)). Then we have

\[
\lambda(\Xi, \underline{0}) = 1_{F_p} = (1 - t_i)1_{F_{\Xi^c \cup \{i\}}} = (1 - t_i)\lambda(\Xi \cup \{i\}, \underline{0}) \in I(\mathcal{T}_p)C_c^\ell(S_1, C_p).
\]

For \( |\underline{n}| > 0 \), choose \( i' \in \Xi^c \) such that \( n_{i,j} = 0 \) for all \( j \) (such an \( i' \) exists because \( \# \Xi + |\underline{n}| < n \)). Put \( \Xi' := \Xi \cup \{i'\} \). Then we have

\[
\lambda(\Xi, \underline{n}) = \prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot (1 - t_{i'})1_{F_{\Xi'}}
\]

\[
= (1 - t_{i'})\lambda(\Xi', \underline{n}) - (1 - t_{i'})\prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot 1_{F_{\Xi'}} + (1 - t_{i'})\prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot 1_{F_{\Xi'}}
\]

\[
\equiv -((1 - t_{i'})\prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot 1_{F_{\Xi'}}) + ((1 - t_{i'})\prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot 1_{F_{\Xi}}) \mod I(\mathcal{T}_p)C_c^\ell(S_1, C_p).
\]

But by (3.17) and the induction hypothesis, we have

\[
(1 - t_{i'})\prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot 1_{F_{\Xi'}} \in \sum_{\underline{n}' < \underline{n}} C_p\lambda(\Xi', \underline{n}') \subseteq I(\mathcal{T}_p)C_c^\ell(S_1, C_p)
\]

and

\[
((1 - t_{i'})\prod_{i,j} \tilde{\ell}_{i,j}^{\underline{n}_{i,j}} \cdot 1_{F_{\Xi}}) \in \sum_{\underline{n} < \underline{n}} C_p\lambda(\Xi, \underline{n}) \subseteq I(\mathcal{T}_p)C_c^\ell(S_1, C_p),
\]

and thus the assertion for \( \lambda(\Xi, \underline{n}) \).

\[]

**Remark 3.11.** It would have been enough to show the proposition only for \( x \)-th powers of a single homomorphism \( \ell : G_p \to \mathbb{Z}_p \) (i.e. \( \partial(\ell^x) = 0 \) for all homomorphisms \( \ell \)) for all \( x \leq r - 1 \), since each product \( \prod_{i=1}^t \ell_i^{x_i} \) of degree \( x = |\underline{x}| \) can be written as a linear combination of \( x \)-th powers of some other homomorphisms \( \ell : G_p \to \mathbb{Z}_p \) by a simple algebraic argument (for a ring \( R \supseteq \mathbb{Q} \), each monomial \( \prod_{i=1}^t X_i^{n_i} \in R[X_1, \ldots, X_t] \) of degree \( n = \sum_i n_i \) can be written as a linear combination of \( n \)-th powers \( (X_i + r_{i,j}X_j)^n \); let \( \ell \) run through the \( \ell_i + r_{i,j}\ell_j \)).
Definition 3.12. A $t$-variable $p$-adic analytic function $f(s) = f(s_1, \ldots, s_t)$ ($s_i \in \mathbb{Z}_p$) has vanishing order $\geq n$ at the point $0 = (0, \ldots, 0)$ if all its partial derivatives of total order $\leq n - 1$ vanish, i.e. if

$$\frac{\partial^k}{\partial s_1^{k_1} \cdots \partial s_t^{k_t}} f(0) = 0$$

for all $k = (k_1, \ldots, k_t) \in \mathbb{N}_0^t$ with $k := |k| \leq n - 1$. We write $\text{ord}_{s=0} f(s) \geq n$ in this case.

The proposition implies the following result for the $p$-adic $L$-function:

Theorem 3.13. Let $n := \#(S_1)$, $\kappa \in H^d(F^s(S_1), \mathcal{D}_f(S_1, V))$, $V$ a finite-dimensional vector space over a $p$-adic field. Then $L_p(s, \kappa)$ is a locally analytic function, and we have

$$\text{ord}_{s=0} L_p(s, \kappa) \geq n.$$ 

Proof. We have

$$\frac{\partial^k}{\partial s_1^{k_1} \cdots \partial s_t^{k_t}} L_p(0, \kappa) = \int_{Sp} \left( \prod_{i=1}^t \ell_i(\gamma)^{k_i} \right) \mu_\kappa(d\gamma) = \kappa \cap \partial \left( \prod_{i=1}^t \ell_i(\gamma)^{k_i} \right)$$

for all $k = (k_1, \ldots, k_t) \in \mathbb{N}_0^t$. Thus the theorem follows from proposition 3.9. 

3.4 Integral cohomology classes

Definition 3.14. For $\kappa \in H^d(F^s(S_1), \mathcal{D}_f(S_1, \mathbb{C}))$ and a subring $R$ of $\mathbb{C}$, we define $L_{\kappa, R}$ as the image of

$$H^d(F^s(S_1), \mathcal{D}_f(S_1, \mathcal{C})) \to H^d(F^s(S_1), \mathcal{C}) = \mathbb{C}, \quad x \mapsto \kappa \cap x.$$ 

Lemma 3.15. Let $R \subseteq \mathbb{Q}$ be a Dedekind ring.

(a) For a subring $R' \supseteq R$ of $\mathbb{C}$, we have $L_{\kappa, R'} = R'L_{\kappa, R}$.

(b) If $\kappa \neq 0$, then $L_{\kappa, R} \neq 0$.

Proof. [Sp], lemma 4.15. 

Definition 3.16. A nonzero cohomology class $\kappa \in H^d(F^s(S_1), \mathcal{D}_f(S_1, \mathbb{C}))$ is called integral if $\kappa$ lies in the image of $H^d(F^s(S_1), \mathcal{D}_f(S_1, R)) \otimes_R \mathbb{C} \to H^d(F^s(S_1), \mathcal{C})$ for some Dedekind ring $R \subseteq \mathbb{Q}$. If, in addition, there exists a torsion-free $R$-submodule $M \subseteq H^d(F^s(S_1), \mathcal{D}_f(S_1, R))$ of rank $\leq 1$ (i.e. $M$ can be embedded into $R$, by the classification of finitely generated $R$-modules) such that $\kappa$ lies in the image of $M \otimes_R \mathbb{C} \to H^d(F^s(S_1), \mathcal{D}_f(S_1, \mathbb{C}))$, then $\kappa$ is integral of rank $\leq 1$.

The following results are straightforward generalizations of the corresponding results of Spieß for totally real $F$:

Proposition 3.17. Let $\kappa \in H^d(F^s(S_1), \mathcal{D}_f(S_1, \mathbb{C}))$. The following conditions are equivalent:

32
(i) $\kappa$ is integral (resp. integral of rank $\leq 1$).

(ii) There exists a Dedekind ring $R \subseteq \mathcal{O}$ such that $L_{\kappa, R}$ is a finitely generated $R$-module (resp. a torsion-free $R$-module of rank $\leq 1$).

(iii) There exists a Dedekind ring $R \subseteq \mathcal{O}$, a finitely generated $R$-module $M$ (resp. a torsion-free $R$-module of rank $\leq 1$) and an $R$-linear map $f : M \to \mathbb{C}$ such that $\kappa$ lies in the image of the induced map $f_* : H^d(F^*, D_f(S_1, M)) \to H^d(F^*, D_f(S_1, \mathbb{C}))$.

Proof. As in [Sp], prop. 4.17.

Corollary 3.18. Let $\kappa \in H^d(F^*, D_f(S_1, \mathbb{C}))$ be integral and $R \subseteq \mathcal{O}$ be as in proposition 3.14. Then
(a) $\mu_\kappa$ is a $p$-adic measure, and
(b) the map $H^d(F^*, D_f(S_1, L_{\kappa, R})) \otimes \mathbb{Q} \to H^d(F^*, D_f(S_1, \mathbb{C}))$ is injective and $\kappa$ lies in its image.

Proof. As in [Sp], cor. 4.18.

Remark 3.19. Let $\kappa$ be integral with Dedekind ring $R$ as above. By (b) of the corollary, we can view $\kappa$ as an element of $H^d(F^*, D_f(S_1, L_{\kappa, R})) \otimes \mathbb{Q}$. Put $V_\kappa := L_{\kappa, R} \otimes \mathbb{C}_p$; let $\overline{\kappa}$ be the image of $\kappa$ under the composition

$$H^d(F^*, D_f(S_1, L_{\kappa, R})) \otimes_R \mathbb{Q} \to H^d(F^*, D_f(S_1, L_{\kappa, R})) \otimes_R \mathbb{C}_p \to H^d(F^*, D^h_f(S_1, V_\kappa)),$$

where the second map is induced by $D_f(S_1, L_{\kappa, R}) \otimes_R \mathbb{C}_p \to D^h_f(S_1, V_\kappa)$. By lemma 3.15 (a), $\overline{\kappa}$ does not depend on the choice of $R$.

Since $\mu_\kappa$ is a $p$-adic measure, $\mu_{\overline{\kappa}}$ allows integration of all continuous functions $f \in C(G_p, \mathbb{C}_p)$, and by abuse of notation, we write $L_p(s, \kappa) := \int_{G_p} N(\gamma)^s \mu_\kappa(d\gamma) := L_p(s, \overline{\kappa})$ (cf. remark 3.6). So $L_p(s, \kappa)$ has values in the finite-dimensional $\mathbb{C}_p$-vector space $V_\kappa$.  

33
4 \textit{p}-adic L-functions of automorphic forms

We keep the notations from chapter 3 so \( F \) is again a number field with \( r \) real embeddings and \( s \) pairs of complex embeddings.

For an ideal \( 0 \neq m \subseteq \mathcal{O}_F \), we let \( K_0(m)_v \subseteq G(\mathcal{O}_F) \) be the subgroup of matrices congruent to an upper triangular matrix modulo \( m \), and we set \( K_0(m) := \prod_{v \mid \infty} K_0(m)_v \), \( K_0(m)^S := \prod_{v \notin S} K_0(m)_v \) for a finite set of primes \( S \). For each \( p \mid p \), let \( q_p = N(p) \) denote the number of elements of the residue class field of \( F_p \).

We denote by \( | \cdot |_v \) the square of the usual absolute value on \( \mathbb{C} \), i.e. \( |z|_C = z \bar{z} \) for all \( z \in \mathbb{C} \), and write \( | \cdot |_R \) for the usual absolute value on \( \mathbb{R} \) in context.

**Definition 4.1.** Let \( \mathcal{A}_0(G, 2, \chi_Z) \) denote the set of all cuspidal automorphic representations \( \pi = \otimes_v \pi_v \) of \( G(\mathbb{A}_F) \) with central character \( \chi_Z \) such that \( \pi_v \cong \sigma(\cdot, F_v, \cdot, \cdot^{-1/2}) \) at all archimedian primes \( v \). Here we follow the notation of \( \mathbb{L} \); so \( \sigma(\cdot, F_v, \cdot, \cdot^{-1/2}) \) is the discrete series of weight 2, \( D(2) \), if \( v \) is real, and is isomorphic to the principal series representation \( \pi(\mu_1, \mu_2) \) with \( \mu_1(z) = z^{1/2} \bar{z}^{1/2}, \mu_2(z) = z^{-1/2} \bar{z}^{1/2} \) if \( v \) is complex (cf. section 4.5 below).

We will only consider automorphic representations that are \( p \)-ordinary, i.e. \( \pi_p \) is ordinary (in the sense of chapter 2) for every \( p \mid p \).

Therefore, for each \( p \mid p \) we fix two non-zero elements \( \alpha_{p,1}, \alpha_{p,2} \in \mathcal{O} \subseteq \mathbb{C} \) such that \( \pi_{\alpha_{p,1}, \alpha_{p,2}} \) is an ordinary, unitary representation. By the classification of unitary representations (see e.g. [Ge], Thm. 4.27), a spherical representation \( \pi_{\alpha_{p,1}, \alpha_{p,2}} = \pi(\chi_1, \chi_2) \) is unitary if and only if \( \chi_1, \chi_2 \) are both unitary characters (i.e. \( |\alpha_{p,1}| = |\alpha_{p,2}| = \sqrt{q_p} \chi_1 \) or \( \chi_{1,2} = \chi_0 | \cdot |^{\pm s} \) with \( \chi_0 \) unitary and \( -\frac{1}{2} < s < \frac{1}{2} \). A special representation \( \pi_{\alpha_{p,1}, \alpha_{p,2}} = \pi(\chi_1, \chi_2) \) is unitary if and only if the central character \( \chi_1 \chi_2 \) is unitary. In all three cases, we have thus \( \max\{|\alpha_{p,1}|, |\alpha_{p,2}|\} \geq \sqrt{q_p} \). Without loss of generality, we will assume the \( \alpha_{p,i} \) to be ordered such that \( |\alpha_{p,1}| \leq |\alpha_{p,2}| \) for all \( p \mid p \).

As in chapter 2, we define \( a_p := \alpha_{p,1} + \alpha_{p,2}, \nu_p := \alpha_{p,1} \alpha_{p,2}/q_p \).

Let \( \alpha_i := (\alpha_{p,i}, p) \), for \( i = 1, 2 \). We denote by \( \mathcal{A}_0(G, 2, \chi_Z, \alpha_1, \alpha_2) \) the subset of all \( \pi \in \mathcal{A}_0(G, 2, \chi_Z) \) such that \( \pi_p = \pi_{\alpha_{p,1}, \alpha_{p,2}} \) for all \( p \mid p \).

Let \( S_1 \subseteq S_p \) be the set of places such that \( \pi_p \) is the Steinberg representation (i.e. \( \alpha_{p,1} = \nu_p = 1, \alpha_{p,2} = q \))\(^\text{IV}\).

For later use we note that \( \pi^\infty = \otimes_{v \mid \infty} \pi_v \) is known to be defined over a finite extension of \( \mathbb{Q} \), the smallest such field being the field of definition of \( \pi \) (cf. [Sp]).

\(^\text{IV}\)To avoid confusion: By \( |\alpha_{p,i}| \) we always mean the archimedian absolute value of \( \alpha_{p,1} \in \mathbb{C} \); whereas in the context of the \( p \)-adic characters \( \chi_1, | \cdot | \) always means the \( p \)-adic absolute value, unless otherwise noted.

\(^\text{V}\)Note that all \( p \mid p \) with \( \alpha_{p,2} = \nu_p \in \mathcal{O} \), i.e. \( \alpha_{p,2} = q \), already lie in \( S_1 \), since \( |\alpha_{p,2}| < q \) in the spherical case. \( L_p(s, \pi) \) should have an exceptional zero for each \( p \in S_1 \), according to the exceptional zero conjecture.
4.1 Upper half-space

Let \( \mathcal{H}_2 := \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \cong \mathbb{R} \times \mathbb{R}_+^* \) be the complex upper half-plane, and let \( \mathcal{H}_3 := \mathbb{C} \times \mathbb{R}_+^* \) be the 3-dimensional upper half-space. Each \( \mathcal{H}_m \) is a differentiable manifold of dimension \( i \). If we write \( x = (u, t) \in \mathcal{H}_m \) with \( t \in \mathbb{R}_+^* \), \( u \) in \( \mathbb{R} \) or \( \mathbb{C} \), respectively, it has a Riemannian metric 
\[
    ds^2 = dt^2 + du^2 \tag{4.11}
\]
which induces a hyperbolic geometry on \( \mathcal{H}_m \), i.e. the geodesic lines on \( \mathcal{H}_m \) are given by “vertical” lines \( \{u\} \times \mathbb{R}_+^* \) and half-circles with center in the line or plane \( t = 0 \).

We have the decomposition \( \text{GL}_2(\mathbb{C}) = B'_C \cdot Z(\mathbb{C}) \cdot K_C \), where \( B'_C \) is the subgroup of matrices \( \left( \begin{array}{cc} x & y \\ 0 & 1 \end{array} \right) \), \( Z \) is the center, and \( K_C = \text{SU}(2) \) (cf. [By], Cor. 43); and analogously \( \text{GL}_2(\mathbb{R})^* = B'_R \cdot Z(\mathbb{R}) \cdot K_R \) with \( B'_R = \{ \left( \begin{array}{cc} y & x \\ 0 & 1 \end{array} \right) | x \in \mathbb{R}, y \in \mathbb{R}_+^* \} \) and \( K_R = \text{SO}(2) \).

We can identify \( B'_C \) with \( \mathcal{H}_3 \) via \( \left( \begin{array}{cc} z \\ 0 \\ 1 \end{array} \right) \mapsto (z,t) \), and \( B'_R \) with \( \mathcal{H}_2 \) via \( \left( \begin{array}{cc} y \\ x \\ 0 \\ 1 \end{array} \right) \mapsto x + iy \).

This gives us natural projections
\[
    \pi_R : \text{GL}_2(\mathbb{R})^* \twoheadrightarrow \text{GL}_2(\mathbb{R})^*/\mathbb{R}_+^* \cong \mathcal{H}_2
\]
and
\[
    \pi_C : \text{GL}_2(\mathbb{C}) \twoheadrightarrow \text{GL}_2(\mathbb{C})/\mathbb{C}_+^* \cong \mathcal{H}_3.
\]

The corresponding left actions on cosets are invariant under the Riemannian metrics on \( \mathcal{H}_m \), and can be given explicitly as follows:

\( \text{GL}_2(\mathbb{R})^* \) operates on \( \mathcal{H}_2 \subseteq \mathbb{C} \) via Möbius transformations,
\[
    \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (z) := \frac{az + b}{cz + d},
\]
and \( \text{GL}_2(\mathbb{C}) \) operates on \( \mathcal{H}_3 \) by
\[
    \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (z, t) := \left( \frac{(az + b)(cz + d) + a\overline{ct}^2}{|cz + d|^2 + |ct|^2}, \frac{|ad - bc|t}{|cz + d|^2 + |ct|^2} \right),
\]
([By], (3.12)); specifically, we have
\[
    \left( \begin{array}{cc} t & z \\ 0 & 1 \end{array} \right) (0, 1) = (z, t) \quad \text{for} \quad (z, t) \in \mathcal{H}_3.
\]

A differential form \( \omega \) on \( \mathcal{H}_m \) is called left-invariant if it is invariant under the pullback \( L_g^* \) of left multiplication \( L_g : x \mapsto gx \) on \( \mathcal{H}_m \), for all \( g \in G \). Following [By], eqs. (4.20), (4.24), we choose the following basis of left invariant differential 1-forms on \( \mathcal{H}_3 \):
\[
    \beta_0 := -\frac{dz}{t}, \quad \beta_1 := \frac{dt}{t}, \quad \beta_2 := \frac{d\overline{z}}{t},
\]
and on \( \mathcal{H}_2 \) (writing \( z = x + iy \in \mathcal{H}_2 \)):
\[
    \beta_1 := \frac{dz}{y}, \quad \beta_2 := -\frac{d\overline{z}}{y}.
\]
We note that a form $f_1\beta_1 + f_2\beta_2$ is harmonic on $H_2$ if and only if $f_1/y$ and $f_2/y$ are holomorphic functions in $z$ ([By], lemma 60).

Let $k \in \{\mathbb{R}, \mathbb{C}\}$. The Jacobian $J(g, (0, 1))$ of left multiplication by $g$ in $(0, 1) \in H_m$ with respect to the basis $(\beta_i)$ gives rise to a representation

$$\varrho = \varrho_k : Z(k) \cdot K_k \to \text{SL}_m(\mathbb{C})$$

with $\varrho|_{Z(k)}$ trivial, which on $K_k$ is explicitly given by

$$\varrho_k \left( \begin{pmatrix} u \\ -\overline{v} \end{pmatrix} \right) = \begin{pmatrix} u^2 & 2uv & v^2 \\ -u\overline{v} & u\overline{v} - v\overline{u} & v\overline{u} \\ 2uv & -u\overline{v} & u\overline{v} \end{pmatrix},$$

resp.

$$\varrho_k \left( \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \right) = \begin{pmatrix} e^{2i\vartheta} & 0 \\ 0 & e^{-2i\vartheta} \end{pmatrix}$$

([By], (4.27), (4.21)). In the real case, we will only consider harmonic forms on $H_2$ that are multiples of $\beta_1$, thus we sometimes identify $\varrho_k$ with its restriction $\varrho_k^{(1)}$ to the first basis vector $\beta_1$.

$$\varrho_k^{(1)} : \text{SO}(2) \to S^1 \subseteq \mathbb{C}^*,$$  

$$\kappa_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \mapsto e^{2i\vartheta}.$$

For each $i$, let $\omega_i$ be the left-invariant differential 1-form on $\text{GL}_2(k)$ which coincides with the pullback $(\pi_C)^*\beta_i$ at the identity. Write $\omega$ (resp. $\beta$) for the column vector of the $\omega_i$ (resp. $\beta_i$). Then we have the following lemma from [By]:

**Lemma 4.2.** For each $i$, the differential $\omega_i$ on $G$ induces $\beta_i$ on $H_m$, by restriction to the subgroup $B'_k \cong H_m$. For a function $\phi : G \to \mathbb{C}^m$, the form $\phi \cdot \omega$ (with $\mathbb{C}^m$ considered as a row vector, so $\cdot$ is the scalar product of vectors) induces $f \cdot \beta$, where $f : H_m \to \mathbb{C}^m$ is given by

$$f(z, t) := \phi \left( \begin{pmatrix} t \\ z \\ 1 \end{pmatrix} \right).$$

(See [By], Lemma 57.)

To consider the infinite primes of $F$ all at once, we define

$$H_\infty := \prod_{i=0}^d H_{m_i} = \prod_{i=0}^{r-1} H_2 \times \prod_{i=r}^d H_3$$

(where $m_i = 2$ if $\sigma_i$ is a real embedding, and $= 3$ if $\sigma_i$ is complex), and let $H_0^0 := \prod_{i=1}^d H_{m_i}$ be the product with the zeroth factor removed.

For each embedding $\sigma_i$, the elements of $\mathbb{P}^1(F)$ are cusps of $H_{m_i}$: for a given complex embedding $F \hookrightarrow \mathbb{C}$, we can identify $F$ with $F \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}_{\geq 0}$ and define the "extended upper half-space" as $\overline{H}_3 := H_3 \cup F \cup \{\infty\} \subseteq \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$; \footnote{The choice of the 0-th factor is for convenience; we could also choose any other infinite place, whether real or complex.}
similarly for a given real embedding \( F \hookrightarrow \mathbb{R} \), we get the extended upper half-plane \( \overline{\mathcal{H}}_2 := \mathcal{H}_2 \cup F \cup \{ \infty \} \). A basis of neighbourhoods of the cusp \( \infty \) is given by the sets \( \{(u, t) \in \mathcal{H}_m | t > N \}, \ N \gg 0 \), and of \( x \in F \) by the open half-balls in \( \mathcal{H}_m \) with center \((x, 0)\).

Let \( G(F)^+ \subseteq G(F) \) denote the subgroup of matrices with totally positive determinant. It acts on \( \mathcal{H}_\infty^0 \) by composing the embedding

\[
G(F)^+ \hookrightarrow \prod_{v \mid \infty, v \neq 0} G(F_v)^+ \quad g \mapsto (\sigma_1(g), \ldots, \sigma_d(g)),
\]

with the actions of \( G(\mathbb{C})^+ = G(\mathbb{C}) \) on \( \mathcal{H}_3 \) and \( G(\mathbb{R})^+ \) on \( \mathcal{H}_2 \) as defined above, and on \( \Omega^{2\text{harm}}(\mathcal{H}_\infty^0) \) by the inverse of the corresponding pullback, \( \gamma \cdot \omega := (\gamma^{-1})^*\omega \). Both are left actions.

Denote by \( S_\mathbb{C} \) (resp. \( S_\mathbb{R} \)) the set of complex (resp. real) archimedean primes of \( F \). For each complex \( v \), we write the codomain of \( \varrho_{F_v} \) as

\[
\varrho_{F_v} : [Z(F_v) \cdot K_{F_v}] \to SL_3(\mathbb{C}) =: SL(V_v),
\]

for a three-dimensional \( \mathbb{C} \)-vector space \( V_v \). We denote the harmonic forms on \( GL_2(F_v) \), \( \mathcal{H}_{F_v} \) defined above by \( \omega_v, \beta_v \) etc.

Let \( V = \bigotimes_{v \in S_\mathbb{C}} V_v \cong (\mathbb{C}^3)^\otimes \), \( Z_\infty = \prod_{v \mid \infty} Z(F_v) \), \( K_\infty = \prod_{v \mid \infty} K_{F_v} \). We can merge the representations \( \varrho_{F_v} \) for each \( v \mid \infty \) into a representation

\[
\varrho = \varrho_\infty := \bigotimes_{v \in S_\mathbb{C}} \varrho_v \otimes \bigotimes_{v \in S_\mathbb{R}} \varrho_v^{(1)} : Z_\infty \cdot K_\infty \to SL(V),
\]

and define \( V \)-valued vectors of differential forms \( \omega := \bigotimes_{v \in S_\mathbb{C}} \omega_v \otimes \bigotimes_{v \in S_\mathbb{R}} \omega_v \), \( \beta := \bigotimes_{v \in S_\mathbb{C}} \beta_v \otimes \bigotimes_{v \in S_\mathbb{R}} \beta_v \) on \( GL_2(F_\infty) \) and \( \mathcal{H}_\infty \), respectively.

### 4.2 Automorphic forms

Let \( \chi_Z : \mathbb{A}_F^* / F^* \to \mathbb{C}^* \) be a Hecke character that is trivial at the archimedean places. We also denote by \( \chi_Z \) the corresponding character on \( Z(\mathbb{A}_F) \) under the isomorphism \( \mathbb{A}_F^* \to Z(\mathbb{A}_F), a \mapsto (\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) \).

**Definition 4.3.** An automorphic cusp form of parallel weight 2 with central character \( \chi_Z \) is a map \( \phi : G(\mathbb{A}_F) \to V \) such that

(i) \( \phi(z\gamma g) = \chi_Z(z)\phi(g) \) for all \( g \in G(\mathbb{A}), \ z \in Z(\mathbb{A}), \ \gamma \in G(F) \).

(ii) \( \phi(gk_\infty) = \phi(g)\phi(k_\infty) \) for all \( k_\infty \in K_\infty \), \( g \in G(\mathbb{A}) \) (considering \( V \) as a row vector).

(iii) \( \phi \) has “moderate growth” on \( B'_{\mathbb{A}} := \{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in G(\mathbb{A}) \} \), i.e. \( \exists C, \lambda \forall A \in B'_{\mathbb{A}} : \|\phi(A)\| \leq C \cdot \sup(|y|^\lambda, |y|^{-\lambda}) \) (for any fixed norm \( \|\cdot\| \) on \( V \));

and \( \phi|_{G(\mathbb{A}_\infty)} \cdot \omega \) is the pullback of a harmonic form \( \omega_\phi = f_\phi \cdot \beta \) on \( \mathcal{H}_\infty \).
(iv) There exists a compact open subgroup $K' \subseteq G(A^\infty)$ such that $\phi(gk) = \phi(g)$ for all $g \in G(A)$ and $k \in K'$.

(v) For all $g \in G(A_F)$,
\[
\int_{K_F/F} \phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \, dx = 0. \quad \text{("Cuspidality")}
\]

We denote by $\mathcal{A}_0(G, \text{harm}, 2, \chi_Z)$ the space of all such maps $\phi$.

For each $g^\infty \in A_F^\infty$, let $\omega_\phi(g^\infty)$ be the restriction of $\phi(g^\infty, \cdot) \cdot \omega$ from $G(A_F^\infty)$ to $\mathcal{H}_\infty$; it is a $(d + 1)$-form on $\mathcal{H}_\infty$.

We want to integrate $\omega_\phi(g^\infty)$ between two cusps of the space $\mathcal{H}_{m_0}$. (We will identify each $x \in \mathbb{P}^1(F)$ with its corresponding cusp in $\overline{\mathcal{H}}_{m_0}$ in the following.) The geodesic between the cusps $x \in F$ and $\infty$ in $\overline{\mathcal{H}}_{m_0}$ is the line $\{x\} \times \mathbb{R}^*_+ \subseteq \mathcal{H}_{m_0}$ and the integral of $\omega_\phi$ along it is finite since $\phi$ is uniformly rapidly decreasing:

**Theorem 4.4.** (Gelfand, Piatetski-Shapiro) An automorphic cusp form $\phi$ is rapidly decreasing modulo the center on a fundamental domain $\mathcal{F}$ of $GL_2(F) \backslash GL_2(A_F)$; i.e. there exists an integer $r$ such that for all $N \in \mathbb{N}$ there exists a $C > 0$ such that
\[
\phi(zg) \leq C|z|^r\|g\|^{-N}
\]
for all $z \in Z(A_F)$, $g \in \mathcal{F} \cap SL_2(A_F)$. Here $\|g\| := \max\{|g_{i,j}|, |(g^{-1})_{i,j}|\}_{i,j \in \{1,2\}}$.

(See [CKM], Thm. 2.2; or [Kur78], (6) for quadratic imaginary $F$.)

In fact, the integral of $\omega_\phi(g^\infty)$ along $\{x\} \times \mathbb{R}^*_+ \subseteq \mathcal{H}_{m_0}$ equals the integral of $\phi(g^\infty, \cdot) \cdot \omega$ along a path $g_t \in GL_2(F_{m_0})$, $t \in \mathbb{R}^*_+$, where we can choose
\[
g_t = \frac{1}{\sqrt{t}} \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix} = \left( \begin{pmatrix} \frac{1}{\sqrt{t}} & \frac{1}{\sqrt{t}} \\ 0 & \sqrt{t} \end{pmatrix} \right),
\]
and thus have $\|g_t\| = \sqrt{t}$ for all $t \gg 0$, $\|g_t\| = C\frac{1}{\sqrt{t}}$ for $t \ll 1$, so the integral $\int_x^\infty \omega_\phi(g^\infty) \in \Omega^d_{\text{harm}}(\mathcal{H}_{m_0}^0)$ is well-defined by the theorem.

For any two cusps $a, b \in \mathbb{P}^1(F)$, we now define
\[
\int_a^b \omega_\phi(g^\infty) := \int_a^\infty \omega_\phi(g^\infty) - \int_b^\infty \omega_\phi(g^\infty) \in \Omega^d_{\text{harm}}(\mathcal{H}_{m_0}^0).
\]

Since $\phi$ is uniformly rapidly decreasing ($\|g_t\|$ does not depend on $x$, for $t \gg 0$), this integral along the path $(a, 0) \to (a, \infty) = (b, \infty) \to (b, 0)$ in $\overline{\mathcal{H}}_{m_0}$ is the same as the limit (for $t \to \infty$) of the integral along $(a, 0) \to (a, t) \to (b, t) \to (b, 0)$; and since $\omega_\phi$ is harmonic (and thus integration is path-independent within $\mathcal{H}_{m_0}$) the latter is in fact independent of $t$, so equality holds for each $t > 0$, or along any path from $(a, 0)$ to $(b, 0)$ in $\mathcal{H}_{m_0}$. Thus $\int_a^b \omega_\phi(g^\infty)$ equals the integral of $\omega_\phi(g^\infty)$ along the geodesic from $a$ to $b$, and we have
\[
\int_a^b \omega_\phi(g^\infty) + \int_b^c \omega_\phi(g^\infty) = \int_a^c \omega_\phi(g^\infty)
\]
for any three cusps \(a, b, c \in \mathbb{P}^1(F)\). Let \(\text{Div}(\mathbb{P}^1(F))\) denote the free abelian group of divisors of \(\mathbb{P}^1(F)\), and let \(\mathcal{M} := \text{Div}_0(\mathbb{P}^1(F))\) be the subgroup of divisors of degree 0.

We can extend the definition of the integral linearly to get a homomorphism

\[
\mathcal{M} \to \Omega^d_{\text{harm}}(H_0^\infty), \quad m \mapsto \int_m \omega_\phi(g^\infty).
\]

For any three cusps \(a, b, c \in \mathbb{P}^1(F)\), let \(\mathbf{m} \in \mathcal{M}\), and \(g \in G(A_\infty)\), we have

\[
\gamma^* \left( \int_{\gamma m} \omega_\phi(\gamma g) \right)(x_\infty^0) = \int_{\gamma m} \omega_\phi(\gamma g)(\gamma x_\infty^0)
\]

\[
= \int_{\gamma m} \phi(\gamma g, \gamma x_\infty^0) \cdot \omega
\]

\[
= \int_{\gamma m} \phi(g, x_\infty^0) \cdot \omega \quad \text{ (by (i) of definition 4.3)}
\]

\[
= \int_m \omega_\phi(g)(x_\infty^0),
\]

i.e.

\[
\gamma^* \left( \int_{\gamma m} \omega_\phi(\gamma g) \right) = \int_m \omega_\phi(g).
\]

Now let \(m\) be an ideal of \(F\) prime to \(p\), let \(\chi_Z\) be a Hecke character of conductor dividing \(m\), and \(\alpha_1, \alpha_2\) as above.

**Definition 4.5.** We define \(S_2(G, m, \alpha_1, \alpha_2)\) to be the \(\mathbb{C}\)-vector space of all maps

\[
\Phi : G(A_p) \to \mathcal{B}_{\alpha_1, \alpha_2}(F_p, V) = \text{Hom}(\mathcal{B}_{\alpha_1, \alpha_2}(F_p, \mathbb{C}), V)
\]

such that:

(a) \(\phi\) is “almost” \(K_0(m)\)-invariant (in the notation of [Ge]), i.e. \(\phi(gk) = \phi(g)\) for all \(g \in G(A_p)\) and \(k \in \prod_{v|m} G(O_v)\), and \(\phi(gk) = \chi_Z(a)\phi(g)\) for all \(v|m\),

\[
k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(m) \quad \text{and} \quad g \in G(A_p).
\]

(b) For each \(\psi \in \mathcal{B}_{\alpha_1, \alpha_2}(F_p, \mathbb{C})\), the map

\[
\langle \Phi, \psi \rangle : G(A) = G(F_p) \times G(A_p) \to V, \quad (g_p, g^p) \mapsto \Phi(g^p)(g_p^p)\psi
\]

lies in \(\mathcal{A}_0(G, \text{harm} \circ \chi_Z)\).

Note that (a) implies that \(\phi\) is \(K'\)-invariant for some open subgroup \(K' \subseteq K_0(m)^p\) of finite index ([By] / [We]).
4.3 Cohomology of \( \text{GL}_2(F) \)

Let \( M \) be a left \( G(F) \)-module and \( N \) an \( R[H] \)-module, for a ring \( R \) and a subgroup \( H \subseteq G(F) \). Let \( S \subseteq S_p \) be a set of primes of \( F \) dividing \( p \); as above, let \( \chi = \chi_S \) be a Hecke character of conductor \( \mathfrak{m} \) prime to \( p \).

**Definition 4.6.** For a compact open subgroup \( K \subseteq K_0(\mathfrak{m})^S \subseteq \mathbb{A}^{S,\infty} \), we denote by \( A_f(K, S, M; N) \) the \( \mathcal{R} \)-module of all maps \( \Phi : \mathbb{A}^{S,\infty} \times M \to N \) such that

1. \( \Phi(gk, m) = \Phi(g, m) \) for all \( g \in \mathbb{A}^{S,\infty} \), \( m \in M \), \( k \in \prod_{v \mid \mathfrak{m}} G(O_v) \);
2. \( \Phi(gk) = \chi_Z(a)\Phi(g) \) for all \( v \mid \mathfrak{m} \), \( k = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_0(\mathfrak{m})_v \) and \( g \in \mathbb{A}^{S,\infty} \), \( m \in M \).

We denote by \( A_f(S, M; N) \) the union of the \( A_f(K, S, M; N) \) over all compact open subgroups \( K \).

\( A_f(S, M; N) \) is a left \( \mathbb{A}^{S,\infty} \)-module via \( (\gamma \cdot \Phi)(g, m) := \Phi(\gamma^{-1}g, m) \) and has a left \( H \)-operation given by \( (\gamma \cdot \Phi)(g, m) := \gamma\Phi(\gamma^{-1}g, \gamma^{-1}m) \), commuting with the \( G(\mathbb{A}^{S,\infty}) \)-operation.

In contrast to our previous notation, we consider two subsets \( S_1 \subseteq S_2 \subseteq S_p \) in this section. We put \( (\alpha_1, \alpha_2)_{S_1} := \{ (\alpha_{p,1}, \alpha_{p,2}) \mid \mathfrak{p} \in S_1 \} \), we set
\[
A_f((\alpha_1, \alpha_2)_{S_1}, S_2, M; N) = A_f(S_2, M; B^{(\alpha_1, \alpha_2)}_{S_1}(F_{S_1}, N));
\]
we write \( A_f(\mathfrak{m}, (\alpha_1, \alpha_2)_{S_1}, S_2, M; N) := A_f(K_0(\mathfrak{m}), (\alpha_1, \alpha_2)_{S_1}, S_2, M; N) \). If \( S_1 = S_2 \), we will usually drop \( S_2 \) from all these notations.

We have a natural identification of \( A_f(\mathfrak{m}, (\alpha_1, \alpha_2)_{S_1}, M; N) \) with the space of maps \( G(\mathbb{A}^{S,\infty}) \times M \times B^{(\alpha_1, \alpha_2)}_{S_1}(F_{S_1}, R) \to N \) that are “almost” \( K \)-invariant.

Let \( S_0 \subseteq S_1 \subseteq S_2 \subseteq S_p \) be subsets. The pairing \((2.10)\) induces a pairing
\[
\langle \cdot, \cdot \rangle : A_f((\alpha_1, \alpha_2)_{S_1}, S_2, M; N) \times B^{(\alpha_1, \alpha_2)}_{S_0}(F_{S_0}, R) \to A_f((\alpha_1, \alpha_2)_{S_0}, S_2, M; N),
\] (4.2)
which, when restricting to \( K \)-invariant elements, induces an isomorphism
\[
A_f(K, (\alpha_1, \alpha_2)_{S_1}, S_2, M; N) \cong B^{(\alpha_1, \alpha_2)}_{S_1-S_0}(F_{S_1-S_0}, A_f((\alpha_1, \alpha_2)_{S_0}, S_2, M; N)).
\] (4.3)
Putting \( S_0 := S_1 - \{ \mathfrak{p} \} \) for a prime \( \mathfrak{p} \in S_1 \), we specifically get an isomorphism
\[
A_f(K, (\alpha_1, \alpha_2)_{S_1}, S_2, M; N) \cong B^{q_{p,1},q_{p,2}}_{\mathfrak{p},\mathfrak{p}}(F_{\mathfrak{p}}, A_f((\alpha_1, \alpha_2)_{S_0}, S_2, M; N)).
\]

Lemmas \((2.9)\) and \((2.10)\) now immediately imply the following:

**Lemma 4.7.** Let \( S \subseteq S_p \), \( \mathfrak{p} \in S \), \( S_0 := S - \{ \mathfrak{p} \} \). Let \( K \subseteq \mathbb{A}^{S,\infty} \) be a compact open subgroup.
(a) If \( \pi_{\alpha_{p,1}, \alpha_{p,2}} \) is spherical, we have exact sequences
\[
0 \to A_f(K, (\alpha_1, \alpha_2)_S, M; N) \to \mathcal{Z} \overset{R-\mathfrak{p}}{\to} Z \to 0
\]
Using the five-lemma on the associated diagram of long exact cohomology sequences (4.4) holds for \( A \) and \( q \).

**Proof.** (cf. [Sp], Prop. 5.6)

Let \( \pi_{a,1,a_p,2} \) is special (with central character \( \chi_p \)), we have exact sequences

\[
0 \to A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; N) \to Z' \to Z \to 0
\]

and

\[
0 \to Z \to A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; N)^2 \to A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; N)^2 \to 0,
\]

\[
0 \to Z' \to A_f(K', (\underline{\alpha_1}, \underline{\alpha_2}), M; N)^2 \to A_f(K', (\underline{\alpha_1}, \underline{\alpha_2}), M; N)^2 \to 0,
\]

with \( Z := A_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2}), S, M; N(\chi_p)) \) and \( Z' := A_f(K'_0, (\underline{\alpha_1}, \underline{\alpha_2}), S, M; N(\chi_p)) \), where \( K_0 = K \times K_p \) and \( K'_0 = K \times K'_p \) are compact open subgroups of \( G(\mathbb{A}_S) \).

**Proposition 4.8.** Let \( S \subseteq S_p \) and let \( K \) be a compact open subgroup of \( G(\mathbb{A}_S) \).

**(a)** For each flat \( R \)-module \( N \) (with trivial \( G(F) \)-action), the canonical map

\[
H^q(G(F)^+, A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; R)) \otimes_R N \to H^q(G(F)^+, A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; N))
\]

is an isomorphism for each \( q \geq 0 \).

**(b)** If \( R \) is finitely generated as a \( \mathbb{Z} \)-module, then \( H^q(G(F)^+, A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; R) \) is finitely generated over \( R \).

**Proof.** (cf. [Sp], Prop. 5.6)

**(a)** The exact sequence of abelian groups \( 0 \to M \to \text{Div}(\mathbb{P}^1(F)) \cong \text{Ind}_{B(F)}^{G(F)} \mathbb{Z} \to Z \to 0 \) induces a short exact sequence of \( G(\mathbb{A}_S) \)-modules

\[
0 \to A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), Z; N) \to \text{Coind}_{B(F)}^{G(F)} A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), Z; N) \to A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; N) \to 0. \tag{4.4}
\]

Using the five-lemma on the associated diagram of long exact cohomology sequences \( H^q(\cdot, R) \otimes_R N \) (which is exact due to flatness) and \( H^q(\cdot, N) \), it is enough to show that (4.4) holds for \( A_f(K, (\alpha_1, \alpha_2), Z; \cdot) \) and \( \text{Coind}_{B(F)}^{G(F)} A_f(K, (\alpha_1, \alpha_2), Z; \cdot) \) instead of \( A_f(K, (\underline{\alpha_1}, \underline{\alpha_2}), M; \cdot) \). By lemma 4.7 it is furthermore enough to consider the case \( S = \emptyset \). Since \( A_f(K, Z; N) \cong \text{Coind}^{G(\mathbb{A}_S)}_K N \), we thus have to show that

\[
H^q(G(F)^+, \text{Coind}^{G(\mathbb{A}_S)}_K R) \otimes_R N \to H^q(G(F)^+, \text{Coind}^{G(\mathbb{A}_S)}_K N),
\]

\[
H^q(B(F)^+, \text{Coind}^{G(\mathbb{A}_S)}_K R) \otimes_R N \to H^q(B(F)^+, \text{Coind}^{G(\mathbb{A}_S)}_K N)
\]

are isomorphisms for all \( q \geq 0 \) and all flat \( R \)-modules \( N \).
Since every flat module is the direct limit of free modules of finite rank, it suffices to show that \( N \mapsto H^q(G(F)^+, \operatorname{Coind}_K^{G(k^{\infty})} N) \) and \( N \mapsto H^q(B(F)^+, \operatorname{Coind}_K^{G(k^{\infty})} N) \) commute with direct limits.

For \( g \in G(\mathbb{A}^{\infty}) \), put \( \Gamma_g := G(F)^+ \cap gKg^{-1} \). By the strong approximation theorem, \( G(F)^+ \backslash G(\mathbb{A}^{\infty}) / K \) is finite. Choosing a system of representatives \( g_1, \ldots, g_n \), we have

\[
H^q(G(F)^+, \operatorname{Coind}_K^{G(k^{\infty})} N) = \bigoplus_{i=1}^n H^q(\Gamma_{g_i}, N).
\]

Since the groups \( \Gamma_g \) are arithmetic, they are of type (VFL), and thus the functors \( N \mapsto H^q(\Gamma_g, N) \) commute with direct limits by [Se2], remarque on p. 101.

Similarly, the Iwasawa decomposition \( G(\mathbb{A}^{\infty}) = B(\mathbb{A}^{\infty}) \prod_{v \mid \infty} G(\mathcal{O}_v) \) implies that \( B(F)^+ \backslash G(\mathbb{A}^{\infty}) / K \) is finite. Therefore, the same arguments show that \( N \mapsto H^q(B(F)^+, \operatorname{Coind}_K^{G(k^{\infty})} N) \) commutes with direct limits.

(b) This follows along the same line of reasoning as (a), since \( H^q(\Gamma_g, R) \) is finitely generated over \( \mathbb{Z} \) by [Se2], remarque on p. 101.

With the notation as above, we define

\[
H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; R)) := \lim_{\to} H^q_i(G(F)^+, \mathcal{A}_f(K, (\alpha_1, \alpha_2)_S, M; R))
\]

where the limit runs over all compact open subgroups \( K \subseteq G(\mathbb{A}^{S, \infty}) \); and similarly define \( H^q_i(B(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; R) \). The proposition immediately implies

**Corollary 4.9.** Let \( R \to R' \) be a flat ring homomorphism. Then the canonical map

\[
H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; R)) \otimes_R R' \to H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; R') \]

is an isomorphism, for all \( q \geq 0 \).

If \( R = k \) is a field of characteristic zero, \( H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; R) \) is a smooth \( G(\mathbb{A}^{S, \infty}) \)-module, and we have

\[
H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; k)^K = H^q_i(G(F)^+, \mathcal{A}_f(K, (\alpha_1, \alpha_2)_S, M; k).
\]

We identify \( G(F)/G(F)^+ \) with the group \( \Sigma = \{\pm 1\}^r \) via the isomorphism

\[
G(F)/G(F)^+ \xrightarrow{\det} F^*/F^*_+ \approx \Sigma
\]

(with all groups being trivial for \( r = 0 \)). Then \( \Sigma \) acts on \( H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; k) \) and \( H^q(G(F)^+, \mathcal{A}_f(K, (\alpha_1, \alpha_2)_S, M; k) \) by conjugation.

For \( \pi \in \mathcal{A}_0(G, 2) \) and \( \underline{\mu} \in \Sigma \), we write \( H^q_i(G(F)^+, \cdot)_{\pi, \underline{\mu}} := \operatorname{Hom}_{G(\mathbb{A}^{S, \infty})}(\pi^S, H^q_i(G(F)^+, \cdot))_{\underline{\mu}} \).

Now we can show that \( \pi \) occurs with multiplicity \( 2^r \) in \( H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; k) \):

**Proposition 4.10.** Let \( \pi \in \mathcal{A}_0(G, 2, \chi_Z, \alpha_1, \alpha_2) \), \( S \subseteq S_p \). Let \( k \) be a field which contains the field of definition of \( \pi \). Then for every \( \underline{\mu} \in \Sigma \), we have

\[
H^q_i(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, M; k)_{\pi, \underline{\mu}} = \begin{cases} k, & \text{if } q = d; \\ 0, & \text{if } q \in \{0, \ldots, d - 1\} \end{cases}
\]
Proof. (cf. [Sp], prop. 5.8) First, assume \( S = \emptyset \). The sequence (4.3) induces a cohomology sequence

\[ \ldots \rightarrow H^2_f(G(F)^+, \mathcal{A}_f(Z, k)) \rightarrow H^2_f(B(F)^+, \mathcal{A}_f(Z, k)) \rightarrow H^2_f(G(F)^+, \mathcal{A}_f(M, k)) \rightarrow H^{2+1}_f(G(F)^+, \mathcal{A}_f(Z, k)) \rightarrow \ldots \]

Harder ([Ha], 3.6.2.2) has determined the action of \( G(A^\infty) \) on \( H^2_f(G(F)^+, \mathcal{A}_f(Z, k)) \) and \( H^2_f(G(F)^+, \mathcal{A}_f(Z, k)) \): For \( q < d \), \( H^2_f(G(F)^+, \mathcal{A}_f(Z, k)) \) is a direct sum of one-dimensional representations; for \( q = d \) there is a \( G(A^\infty) \)-stable decomposition

\[ H^{d+1}_f(G(F)^+, \mathcal{A}_f(Z, k)) = H^{d+1}_{\text{cusp}} \oplus H^{d+1}_{\text{res}} \oplus H^{d+1}_{\text{Eis}}, \]

with the last two summands again being direct sums of one-dimensional representations, and

\[ H^{d+1}_{\text{cusp}}(G(F)^+, \mathcal{A}_f(Z, k)) \mid_{\pi, \mu} \cong k \]

([Ha], 3.6.2.2); \( H^2_f(B(F)^+, \mathcal{A}_f(Z, k)) \) always decomposes into one-dimensional \( G(A^\infty) \)-representations. Since \( \pi^S \) does not map to one-dimensional representations, this proves the claim for \( S = \emptyset \).

Now for \( S = S_0 \cup \{ p \} \) and \( \pi_p \) spherical, lemma (4.7(a)) and the statement for \( S_0 \) give an isomorphism

\[ H^2_f(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, \mathcal{M}; k)) \mid_{\pi, \mu} \cong H^2_f(G(F)^+, \mathcal{A}_f((\alpha_1, \alpha_2)_S, \mathcal{M}; k)) \mid_{\pi, \mu} \]

since the Hecke operators \( T_p, R_p \) act on the left-hand side by multiplication with \( a_p \) or \( \nu_p \), respectively. If \( \pi_p \) is special, we can similarly deduce the statement for \( S \) from that for \( S_0 \), using the first exact sequence of lemma (4.7(b)) (cf. [Sp]), since the results of [Ha] also hold when twisting \( k \) by a (central) character. \( \square \)

### 4.4 Eichler-Shimura map

Given a subgroup \( K_0(m)^p \subseteq G(A^{p, \infty}) \) as above, there is a map

\[ I_0 : S_2(G, m, \alpha_1, \alpha_2) \rightarrow H^0(G(F)^+, \mathcal{A}_f(m, \alpha_1, \alpha_2, \mathcal{M}; \Omega^d_{\text{harm}}(H^0_\infty))) \]

given by

\[ I_0(\Phi) : (\psi, (g, m)) \mapsto \int_m \omega_{(\phi, \psi)}(1_p, g), \]

for \( \psi \in B_{\alpha_1, \alpha_2}(F_p, \mathbb{C}), g \in G(A^{p, \infty}), m \in \mathcal{M} \), where \( 1_p \) denotes the unity element in \( G(F_p) \).

This is well-defined since both sides are “almost” \( K_0(m) \)-invariant, and the \( G(F)^+ \)-invariance of \( I_0(\Phi) \) follows from the similar invariance for differential forms, and the definition of the \( G(F)^+ \)-operations on \( \mathcal{A}_f(M, N) \), \( B_{\alpha_1, \alpha_2}(F_p, N) \) and \( \Omega^d_{\text{harm}}(H^0_\infty) \): For each \( \psi \in B_{\alpha_1, \alpha_2}(F_p, \mathbb{C}), g \in G(A^{p, \infty}), m \in \mathcal{M} \), we have
\[(\gamma I_0(\Phi))(\psi, (g, m)) = \gamma I_0(\Phi)(\gamma^{-1}\psi, (\gamma^{-1}g, \gamma^{-1}m))\]

\[= \gamma \cdot \int_{\gamma^{-1}m} \omega_{\Phi, \gamma^{-1}\psi}(1_p, \gamma^{-1}g)\]

\[= (\gamma^{-1})^* \int_{\gamma^{-1}m} \omega_{\Phi, \gamma^{-1}\psi}(1_p, \gamma^{-1}g)\]

\[= \int_m \omega_{\Phi, \gamma^{-1}\psi}(1_p, g) \quad \text{ (by (1.1))}\]

\[= I_0(\Phi)(\psi, (g, m)).\]

We have a complex \(A_f(m, \alpha_1, \alpha_2, M; \mathbb{C}) \rightarrow C^* := A_f(m, \alpha_1, \alpha_2, M; \Omega^*_{\harm}(\mathcal{H}_\infty^0)).\)

Therefore we get a map

\[S_2(G, m, \alpha_1, \alpha_2) \rightarrow H^d(G(F)^+, A_f(m, \alpha_1, \alpha_2, M; \mathbb{C})) \quad \text{(4.6)}\]

by composing \(I_0\) with the edge morphism \(H^0(G(F)^+, C^d) \rightarrow H^d(G(F)^+, A_f(m, \alpha_1, \alpha_2, M; \mathbb{C}))\) of the spectral sequence

\[H^q(G(F)^+, C^p) \Rightarrow H^{p+q}(G(F)^+, C^*).\]

Using the map \(\delta^{\alpha_1, \alpha_2} : \mathcal{B}^{\alpha_1, \alpha_2}(F, V) \rightarrow \text{Dist}(F_p^*, V)\) from section 2.7, we next define a map

\[\Delta^{\alpha_1, \alpha_2}_{F^p} : S_2(G, m, \alpha_1, \alpha_2) \rightarrow \mathcal{D}(S_1, V) \quad \text{(4.7)}\]

by

\[\Delta^{\alpha_1, \alpha_2}_{F^p}(\Phi)(U, x^p) = \delta^{\alpha_1, \alpha_2} \left( \Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right) (U)\]

for \(U \in \mathcal{C}(F_{S_1} \times F_{S_2}), x^p \in \mathbb{P}^p\), and we denote by \(\Delta^{\alpha_1, \alpha_2} : S_2(G, m, \alpha_1, \alpha_2) \rightarrow \mathcal{D}(S_1, \mathbb{C})\) its \((1,\ldots,1)\)th coordinate function (i.e. corresponding to the harmonic forms \(\otimes_{v|\infty}^1(\omega_v)_1, \otimes_{v|\infty}^1(\beta_v)_1\) in section 1.1):

\[\Delta^{\alpha_1, \alpha_2}(\Phi)(U, x^p) = \delta^{\alpha_1, \alpha_2} \left( \Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right)_{(1,\ldots,1)} (U).\]

Since for each complex prime \(v\), \(S^1 \cong \text{SU}(2) \cap T(\mathbb{C})\) operates via \(g_v\) on \(\Phi\), \(\Delta^{\alpha_1, \alpha_2}\) is easily seen to be \(S^1\)-invariant, i.e. it lies in \(\mathcal{D}'(S_1, \mathbb{C})\).

We also have a natural (i.e. commuting with the complex maps of each \(C^*\)) family of maps

\[A_f(m, \alpha_1, \alpha_2, M, \Omega^i_{\harm}(\mathcal{H}_\infty^0)) \rightarrow \mathcal{D}_f(S_1, \Omega^i(U^0_\infty, \mathbb{C})) \quad \text{(4.8)}\]

for all \(i \geq 0\), and

\[A_f(m, \alpha_1, \alpha_2, M, \mathbb{C}) \rightarrow \mathcal{D}_f(S_1, \mathbb{C}) \quad \text{(4.9)}\]

(the \(i = -1\)-th term in the complexes), by mapping \(\Phi \in A_f(m, \alpha_1, \alpha_2, M, \cdot)\) first to

\[(U, x^{p, \infty}) \mapsto \Phi \begin{pmatrix} x^{p, \infty} & 0 \\ 0 & 1 \end{pmatrix}, \infty - 0 \quad (\delta^{\alpha_1, \alpha_2}(1_U)) \in \Omega^i_{\harm}(\mathcal{H}_\infty^0)\] resp. \(\in \mathbb{C},\)
and then for $i \geq 0$ restricting the differential forms to $\Omega^i(U_\infty^0)$ via

$$U_\infty^0 = \prod_{v \in S_\infty^0} \mathbb{R}^* \hookrightarrow \prod_{v \in S_\infty^0} \mathcal{H}_v = \mathcal{H}_\infty^0.$$ 

One easily checks that (4.8) and (4.9) are compatible with the homomorphism of "acting groups" $F'^{\prime} \hookrightarrow G(F)^{\prime}$ for $x = \left( \begin{smallmatrix} 1 \\ 0 \\ 0 \\ 1 \end{smallmatrix} \right)$, so we get induced maps in cohomology

$$H^0(G(F)^{\prime}, A_f(m, \alpha_1, \alpha_2, \mathcal{M}, \Omega^d_{\text{harm}}(\mathcal{H}_\infty^0))) \rightarrow H^0(D_f(S_1, \Omega^d(U_\infty^0, \mathbb{C}))) \quad (4.10)$$

and

$$H^d(G(F)^{\prime}, A_f(m, \alpha_1, \alpha_2, \mathcal{M}, \mathbb{C})) \rightarrow H^d(F'^{\prime}, D_f(S_1, \mathbb{C}))) \quad (4.11)$$

which are linked by edge morphisms of the respective spectral sequences to give a commutative diagram (given in the proof below).

**Proposition 4.11.** We have a commutative diagram:

$$\begin{array}{ccc}
S_2(G, m, \alpha_1, \alpha_2) & \xrightarrow{\Delta^G_{\alpha_1, \alpha_2}} & H^d(G(F)^{\prime}, A_f(m, \alpha_1, \alpha_2, \mathcal{M}, \mathbb{C})) \\
\downarrow \quad \Delta^G_{\alpha_1, \alpha_2} & & \downarrow \quad \Delta^G_{\alpha_1, \alpha_2} \\
\mathcal{D}'(S_1, \mathbb{C}) & \xrightarrow{\phi \mapsto \kappa_0} & H^d(F'^{\prime}, D_f(S_1, \mathbb{C})))
\end{array}$$

**Proof.** The given diagram factorizes as

$$\begin{array}{ccc}
S_2(G, m, \alpha_1, \alpha_2) & \xrightarrow{I_0} & H^0(G(F)^{\prime}, A_f(m, \alpha_1, \alpha_2, \mathcal{M}, \Omega^d_{\text{harm}}(\mathcal{H}_\infty^0))) \\
\downarrow \Delta^G_{\alpha_1, \alpha_2} & & \downarrow \Delta^G_{\alpha_1, \alpha_2} \\
\mathcal{D}'(S_1, \mathbb{C}) & \xrightarrow{\phi \mapsto \kappa_0} & H^d(F'^{\prime}, D_f(S_1, \mathbb{C})))
\end{array}$$

The right-hand square is the naturally commutative square mentioned above; the commutativity of the left-hand square can be checked by hand.

Let $\Phi \in S_2(G, m, \alpha_1, \alpha_2)$. Then $I_0(\Phi)$ is the map $(\psi, (g, m)) \mapsto \int_m \omega_{(\Phi, \psi)}(1_p, g)$, which is mapped under (4.10) to

$$(U, x^{p, \infty}) \mapsto \int_0^\infty \omega_{\Phi, \delta_{\alpha_1, \alpha_2}(1_U)} \left( 1_p, \begin{pmatrix} x_p^{U, \infty} & 0 \\ 0 & 1 \end{pmatrix} \right) |_{U_\infty^0}$$

along the other path, $\Phi$ is mapped under $\Delta^G_{\alpha_1, \alpha_2}$ to the map

$$(U, x^p) \mapsto \delta^{\alpha_1, \alpha_2}_{\alpha_1, \alpha_2}(\Phi x^p \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\Phi x^p \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

and then also to

$$(U, x^{p, \infty}) \mapsto \int_0^\infty \Phi_{(1, \ldots, 1)} \left( x^p \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \left( \delta_{\alpha_1, \alpha_2}(1_U) \right) d^x r_0 d^x r_1 \ldots d^x r_d$$

(with $x^p = (x^{p, \infty}, r_0, r_1, \ldots, r_d))$. □
4.5 Whittaker model

We now consider an automorphic representation $\pi = \otimes_v \pi_v \in \mathcal{A}_0(G, \mathbb{Z}, \chi, \alpha_1, \alpha_2)$. Denote by $c(\pi) := \prod_{v \text{ finite}} c(\pi_v)$ the conductor of $\pi$.

Let $\chi : \mathbb{I}^\infty \to \mathbb{C}^*$ be a unitary character of the finite ideles; for each finite place $v$, set $\chi_v = \chi|_{\mathbb{F}_v^\times}$. For each finite and each real prime, we choose $W_v \in \mathcal{W}_v$ such that the local $L$-factor equals the local zeta function at $g = 1$, i.e. such that

$$L(s, \pi_v \otimes \chi_v) = \int_{F_v^\times} \chi_v(x)|x|^{s-\frac{1}{2}} \, dx$$

for any unramified quasi-character $\chi_v : F_v^\times \to \mathbb{C}^*$ and $\Re(s) \gg 0$.

This is possible by [Ge], Thm. 6.12 (ii); and by loc.cit., Prop. 6.17, $W_v$ can be chosen such that $SO(2)$ operates on $W_v$ via $\varpi_v$ for real archimedian $v$, and is “almost” $K_0(c(\pi_v))$-invariant for finite $v$.

For complex primes $v$ of $F$, we can also choose a $W_v$ satisfying (4.12) and which behaves well with respect to the SU(2)-action $\varpi_v$, as follows:

By [Kur77], there exists a three-dimensional function

$$W_v = (W_v^0, W_v^1, W_v^2) : G(F_v) \to \mathbb{C}^3$$

such that $W_v^i \in \mathcal{W}_v$ for all $i$, and such that SU(2) operates by the right via $\varpi_v$ on $\mathcal{W}_v$; i.e. for all $g \in G(F_v)$ and $h = \begin{pmatrix} u & v \\ -\pi & \pi \end{pmatrix} \in SU(2)$, we have

$$W_v(gh) = W_v(g)M_3(h),$$

where

$$M_3(h) = \begin{pmatrix} u^2 & 2uv & v^2 \\ -u\overline{v} & u\overline{v} - v\overline{u} & v\overline{u} \\ \pi^2 & -2u\overline{v} & \pi^2 \end{pmatrix}.$$

Note that $W_v^1$ is thus invariant under right multiplication by a diagonal matrix $\begin{pmatrix} u & 0 \\ 0 & \pi \end{pmatrix}$ with $u \in S^1 \subseteq \mathbb{C}$. Since $\pi_v$ has trivial central character for archimedian $v$ by our assumption, a function in $\mathcal{W}_v$ is also invariant under $Z(F_v)$. Thus we have

$$W_v^1(g \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}) = W_v^1(g) \quad \text{for all } g \in G(F_v), \ u \in S^1.$$

$W_v^1$ can be described explicitly in terms of a certain Bessel function, as follows. The modified Bessel differential equation of order $\alpha \in \mathbb{C}$ is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2) y = 0.$$

Its solution space (on $\{\Re z > 0\}$) is two-dimensional; we are only interested in the second standard solution $K_v$, which is characterised by the asymptotics

$$K_v(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$$

(as defined in [We]; see also [DLMF], 10.25)\[vi\]

\[vi\]Note that [Kur77] uses a slightly different definition of the $K_v$, which is $\frac{2}{\pi}$ times our $K_v$.\[vii\]
By [Kur77], we have 
\[ W_1^v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{\pi} x^2 K_0(4\pi x). \]

\( W_0^v \) and \( W_2^v \) can also be described in term of Bessel functions; they are linearly dependent and scalar multiples of \( x^2 K_1(4\pi x). \)

By [JL], Ch. 1, Thm. 6.2(vi), \( \sigma(| \cdot |^{-1/2}_C, | \cdot |^{-1/2}_C) \cong \pi(\mu_1, \mu_2) \) with
\[ \mu_1(z) = z^{1/2} - \frac{1}{2} = |z|^{-1/2}_C, \quad \mu_2(z) = z^{-1/2} \cdot |z|^{-1/2}_C, \]
and the L-series of the representation is the product of the L-factors of these two characters:
\[ L_v(s, \pi_v) = L(s, \mu_1) L(s, \mu_2) = 2 (2\pi)^{-(s+\frac{1}{2})} \Gamma(s + \frac{1}{2}) \cdot 2 (2\pi)^{-(s+\frac{1}{2})} \Gamma(s + \frac{1}{2}) \]
\[ = 4 (2\pi)^{-2s+1} \Gamma(s + \frac{1}{2})^2. \]

On the other hand, letting \( d^\times x = \frac{dx}{|x|^2} = \frac{dx}{r} \) (for \( x = re^{i\theta} \)), we have for \( \text{Re}(s) > -\frac{1}{2} \):
\[ \int_{C^*} W_1^v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|^{s-\frac{1}{2}} d^\times x = \int_{S^1} \int_{\mathbb{R}^+} W_1^v \begin{pmatrix} re^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} |x|^{|s-\frac{1}{2}} d\frac{r}{r} d\theta \]
\[ = 4 \int_0^{\infty} x^2 K_0(4\pi x) x^{2s-1} \frac{dx}{x} \]
(invariance under \( \text{SU}(2) \cdot \mathbb{Z}(F_v) \) gives a constant integral w.r.t. \( \theta \))
\[ = 4 (4\pi)^{-2s+1} \int_0^{\infty} K_0(x) x^{2s} dx 
= 4 (4\pi)^{-2s+1} 2^{2s-1} \Gamma(s + \frac{1}{2})^2 \]  
(by [DLMF] 10.43.19)
\[ = 4 (2\pi)^{-2s+1} \Gamma(s + \frac{1}{2})^2 \]

Thus we have
\[ \int_{C^*} W_1^v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|^{s-\frac{1}{2}} d^\times x = (2\pi)^2 L_v(s, \pi_v) \]
for all \( \text{Re}(s) > -\frac{1}{2} \).

We set \( W_v := (2\pi)^{-2} W_1^v \); thus \((4.12)\) holds also for complex primes.

Now that we have defined \( W_v \) for all primes \( v \), put \( W^p(g) := \prod_{v \mid p} W_v(g_v) \) for all \( g = (g_v)_v \in G(A_F^p) \).

We will also need the vector-valued function \( \overline{W}^p : G(A_F) \to V \) given by
\[ \overline{W}^p(g) := \prod_{v \mid p \text{ finite or } v \text{ real}} W_v(g_v) \cdot \bigotimes_{v \text{ complex}} (2\pi)^{-2} W_v(g_v). \]
\section{\textit{p}-adic measures of automorphic forms}

Now return to our $\pi \in \mathfrak{A}_0(G, \xi, \alpha_1, \alpha_2)$. We fix an additive character $\psi : \mathbb{A} \rightarrow \mathbb{C}^*$ which is trivial on $F$, and let $\psi_v$ denote the restriction of $\psi$ to $F_v \rightarrow \mathbb{A}$, for all primes $v$. We further require that $\ker(\psi_v) \supseteq \mathcal{O}_p$ and $p^{-1} \not\subseteq \ker(\psi_p)$ for all $p | p$, so that we can apply the results of chapter 2.

As in chapter 2 let $\mu_\pi := \mu_{\alpha_p, \beta_p}$ denote the distribution $\chi_{D_p}(x)\psi(x)dx$ on $F_p$, and let $\mu_\pi := \prod_{p \mid p} \mu_\pi$ be the product distribution on $F_p := \prod_{p \mid p} F_p$.

Define $\phi = \phi : \mathfrak{C}(F_{S_1} \times F_{S_2}^*) \times \mathbb{P} \rightarrow \mathbb{C}$ by

$$\phi(U, x^p) := \sum_{\zeta \in F^*} \mu_\pi(\zeta U) W_p \left( \begin{array}{c} \zeta x^p \\ 0 \\ 1 \end{array} \right).$$

By proposition 2.13(a), we have for each $U \in \mathfrak{C}(F_{S_1} \times F_{S_2}^*)$:

$$\phi(x_p U, x^p) = \sum_{\zeta \in F^*} \mu_\pi(\zeta x_p U) W_p \left( \begin{array}{c} \zeta x^p \\ 0 \\ 1 \end{array} \right)$$

$$= \sum_{\zeta \in F^*} W_U \left( \begin{array}{c} \zeta x_p \\ 0 \\ 1 \end{array} \right) W_p \left( \begin{array}{c} \zeta x^p \\ 0 \\ 1 \end{array} \right)$$

$$= \sum_{\zeta \in F^*} W \left( \begin{array}{c} \zeta x \\ 0 \\ 1 \end{array} \right),$$

where $W(g) := W_U(g_p) W_p(g^p)$ lies in the global Whittaker model $\mathcal{W} = \mathcal{W}(\pi)$ for all $g = (g_p, g^p) \in G(\mathbb{A})$, putting $W_U := W_{A_U}$; so $\phi$ is well-defined and lies in $\mathcal{D}(S_1, \mathbb{C})$ (since $W$ is smooth and rapidly decreasing; distribution property, $F^*$- and $U^{\infty}$- invariance being clear by the definitions of $\phi$ and $W_p$).

Let $\mu := \mu_\phi$, be the distribution on $\mathcal{G}_p$ corresponding to $\phi$, as defined in \ref{3.5}, and let $\kappa := \kappa_\phi \in H^d(F^*, \mathcal{D}_f(S_1, \mathbb{C}))$ be the cohomology class defined by \ref{3.10} and \ref{3.11}.

\textbf{Theorem 4.12.} Let $\pi \in \mathfrak{A}_0(G, \xi, \alpha_1, \alpha_2)$; we assume the $\alpha_p$ to be ordered such that $|\alpha_{p,1}| \leq |\alpha_{p,2}|$ for all $p$.

(a) Let $\chi : \mathcal{G}_p \rightarrow \mathbb{C}^*$ be a character of finite order with conductor $\mathfrak{f}(\chi)$. Then we have the interpolation property

$$\int_{\mathcal{G}_p} \chi(\gamma) \mu_\pi(d\gamma) = \tau(\chi) \prod_{p \in \mathcal{P}} e(\pi_p, \chi_p) \cdot L \left( \frac{1}{2}, \pi \otimes \chi \right),$$

where

$$e(\pi_p, \chi_p) = \begin{cases} (1 - \alpha_{p,1} q_p^{-1})(1 - \alpha_{p,2} q_p^{-1})(1 - \alpha_{p,2} q_p^{-1}) \cdot \text{ord}_p(f(\chi)) = 0 \text{ and } \pi \text{ spherical}, \\ \text{ord}_p(f(\chi)) = 0 \text{ and } \pi \text{ special}, \\ \text{ord}_p(f(\chi)) > 0 \end{cases}$$

\footnote{So we have $\chi_p = | \cdot | \chi_{p,2}$ for all special $\pi_p$.}
and $x_p := \chi_p(\varpi_p)$.

(b) Let $U_p := \prod_{p \mid \mathfrak{p}} U_p$, put $\phi_0 := (\phi_\pi)_{U_p}$. Then

$$
\int_{1/F^*} \phi_0(x) d^\times x = \prod_{p \mid \mathfrak{p}} e(\pi_p, 1) \cdot L(\frac{1}{2}, \pi).
$$

(c) $\kappa_\pi$ is integral (cf. definition [J-I0]). For $\mu \in \Sigma$, let $\kappa_{\pi, \mu}$ be the projection of $\kappa_\pi$ to $H^d(F^{s'}, D_{f(S_1, \mathbb{C})})_{\pi, \mu}$. Then $\kappa_{\pi, \mu}$ is integral of rank $\leq 1$.

Proof. (a) We consider $\chi$ as a character on $\mathbb{I}_F/F^*$ (which is unitary and trivial on $\mathbb{I}_\infty$), and choose a subgroup $V \subseteq U_p$ such that $\chi_p|_V = 1$ (where $\chi_p := \chi|_{F_p}$) and $V$ is a product of subgroups $V_p \subseteq U_p$.

Let $W_V \in W_p$ be the product of the $W_{V_p}$, as defined in prop. [2.13] set $W(g) := W^p(g^p)W_V(y_p) \in W$, and let

$$
\phi_V(x) := \phi(x_p V, x^p) = \sum_{\xi \in F^*} W \begin{pmatrix} \xi x^p & 0 \\ 0 & 1 \end{pmatrix}.
$$

Since $\pi$ is unitary, we have $|\alpha_{p,2}| \geq \sqrt{q_p} > 1 = |\chi_p(\varpi_p)|$ for all $p$, thus $e(\pi_p, \chi_p : [\mathfrak{p}]^s)$ is always non-singular, and we will be able to apply proposition [2.4] locally below.

We want to show that the equality

$$
[U_p : V] \int_{1/F^*} \chi(x)|x|^s \phi_V(x) d^\times x = N(f(\chi))^s \tau(\chi) \prod_{p \mid \mathfrak{p}} e(\pi_p, \chi_p : [\mathfrak{p}]^s) \cdot L(s + \frac{1}{2}, \pi \otimes \chi)
$$

holds for $s = 0$. Since both the left-hand side and $L(s + \frac{1}{2}, \pi \otimes \chi)$ are holomorphic in $s$ (see [I-I], Thm. 6.18 and its proof), it suffices to show this equality for $\text{Re}(s) \gg 0$.

For such $s$, we have

$$
[U_p : V] \int_{1/F^*} \chi(x)|x|^s \phi_V(x) d^\times x = \int_{1/F_p} \chi(x)|x|^s W_U(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}) d^\times x
$$

$$
= \prod_{p \mid \mathfrak{p}} \int_{F_p^s} \chi_p(x)|x|^s \mu_\pi(dx) \cdot L_{\mathfrak{p}}(s + \frac{1}{2}, \pi \otimes \chi) \quad \text{(by prop. [2.13] and [4.12])}
$$

$$
= \prod_{p \mid \mathfrak{p}} (e(\pi_p, \chi_p : [\mathfrak{p}]^s) \tau(\chi_p : [\mathfrak{p}]^s)) \cdot L(s + \frac{1}{2}, \pi \otimes \chi) \quad \text{(by prop. [2.4])}
$$

$$
= N(f(\chi))^s \tau(\chi) \prod_{p \mid \mathfrak{p}} e(\pi_p, \chi_p : [\mathfrak{p}]^s) \cdot L(s + \frac{1}{2}, \pi \otimes \chi).
$$

For $s = 0$, we get the claimed statement, since by (3.5) we have

$$
\int_{\mathcal{G}_p} \chi(\gamma) \mu_\pi(d\gamma) = \int_{1/F^*} \chi(x) \phi(dx_p, x^p) d^\times x^p = [U_p : V] \int_{1/F^*} \chi(x) \phi_V(x) d^\times x.
$$
(b) This follows immediately from (a), setting $\chi = 1$, since $\tau(1) = 1$.

(c) Let $\lambda_{1,2} \in B_{\alpha_1,\alpha_2}(F_p, \mathbb{C})$ be the image of $\otimes_{v \mid p} \lambda_{\alpha_v \nu_v}$ under the map (2.12). For each $\psi \in B_{\alpha_1,\alpha_2}(F_p, \mathbb{C})$, define

\[
\langle \Phi, \psi \rangle \left( g^p, g_p \right) := \sum_{\zeta \in F^*} \lambda_{\alpha_1,\alpha_2} \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g_p \cdot \psi \right) W^p \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g^p
\]

for a $V$-valued function $W_\psi$ whose every coordinate function is in $W(\pi)$.

This defines a map $\Phi_v : G(H_p) \rightarrow B_{\alpha_1,\alpha_2}(F_v, V)$. In fact, $\Phi_v$ lies in $S_2(G, m, \alpha_1, \alpha_2)$, where $m$ is the prime-to-$p$ part of $f(\pi)$.

Condition (a) of definition 4.3 follows from the fact that the $W_v$ are almost $K_v \cdot c(\pi_v))$-invariant, for $v \nmid p$, $\infty$.

For condition (b), we check that $\langle \Phi_v, \psi \rangle$ satisfies the conditions (i)-(v) in the definition of $A_0(G, \text{harm}. \chi)$:

Each coordinate function of $\langle \Phi_v, \psi \rangle$ lies in (the underlying space of) $\pi$ by [Bu], Thm. 3.5.5, thus $\langle \Phi, \psi \rangle$ fulfills (i) and (v), and has moderate growth. (ii) and (iv) follow from the choice of the $W_v$ and $W_v$.

Now since $\pi_v \cong \sigma(\cdot \cdot | v^{1/2} | \cdot | v^{-1/2} )$ for $v \mid \infty$, it follows from those conditions that $\langle \Phi, \psi \rangle |_{B_{\pi_v} \cdot \beta_v} = C \sum_{\zeta \in F^*} W_\zeta \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot \beta_v$ is harmonic for each archimedian place $v$ of $F$: for real $v$, it is well-known that $f(z)/y$ is holomorphic for $f \in D(2)$, and thus $f \cdot (\beta_v)_1$ is harmonic; for complex $v$, this is also true, see e.g. [Kur78], p. 546 or [We].

Now we have

\[
\Delta^{\alpha_1,\alpha_2}(\Phi_{1,x}) = \delta^{\alpha_1,\alpha_2} \begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix}(U)
\]

\[
= \sum_{\zeta \in F^*} \lambda_{\alpha_1,\alpha_2} \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta^{\alpha_1,\alpha_2}(U) W^p \begin{pmatrix} \zeta \cdot x^p & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
(*) \sum_{\zeta \in F^*} \mu_{\pi_v}(\zeta U) W^p \begin{pmatrix} \zeta \cdot x^p & 0 \\ 0 & 1 \end{pmatrix} = \phi(U, x^p),
\]

where $(*)$ follows from the calculation (with $w_0$ as defined in Ch. 2)

\[
\lambda_{\alpha_1,\alpha_2} \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta^{\alpha_1,\alpha_2}(1_U) = \prod_{\nu \mid p} F_p \int \delta_{\alpha_1,\alpha_2}(1_U) \left( w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \psi_p(-x) dx
\]

\[
= \prod_{\nu \mid p} F_p \int \delta_{\alpha_1,\alpha_2}(1_U) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \psi_p(-x) dx
\]

\[
= \prod_{\nu \mid p} F_p \int \chi_{\alpha_2}(-x) \chi_{\alpha_1}(-1) \psi_p(-x) dx
\]

\[
= \prod_{\nu \mid p} \int \chi_{\alpha_2}(-x) \psi_p(-x) dx = \mu_{\pi_v}(\zeta U)
\]

50
for all $\zeta \in F^*$.

Let $R$ be the integral closure of $\mathbb{Z}[a_p, \nu_p; p|p]$ in its field of fractions; thus $R$ is a Dedekind ring $\subseteq \mathcal{O}$ for which $\mathcal{B}_{\mathcal{O}_1, \mathcal{O}_2}(F, R)$ is defined. $\mathbb{C}$ is flat as an $R$-module (since torsion-free modules over a Dedekind ring are flat); thus by proposition 4.8, the natural map

$$H^d(G(F)^+, \mathcal{A}_f(m, \alpha_1, \alpha_2, \mathcal{M}, R)) \otimes \mathbb{C} \to H^d(G(F)^+, \mathcal{A}_f(m, \alpha_1, \alpha_2, \mathcal{M}, \mathbb{C}))$$

is an isomorphism. The map (4.11) can be described as the "$R$-valued" map

$$H^d(G(F)^+, \mathcal{A}_f(m, \alpha_1, \alpha_2, \mathcal{M}, R)) \otimes \mathbb{C} \to H^d(F^*, \mathcal{D}_f(R))$$

tensored with $\mathbb{C}$. By proposition 4.11, $\kappa_\pi$ lies in the image of (4.11), and thus in $H^d(F^*, \mathcal{D}_f(R)) \otimes \mathbb{C}$; i.e. it is integral.

Similarly, it follows from propositions 4.8 and 4.10 that $\kappa_\pi, \mu$ is integral of rank $\leq 1$.

**Corollary 4.13.** $\mu_\pi$ is a $p$-adic measure.

**Proof.** By proposition 3.8, $\mu_\pi = \mu_\phi = \mu_\kappa$. Since $\kappa_\pi$ is integral, $\mu_\kappa_\pi$ is a $p$-adic measure by corollary 3.18. $\square$

### 4.7 Vanishing order of the $p$-adic L-function

Let $L_1, \ldots, L_t$ be independent $\mathbb{Z}_p$-extensions of $F$, and let $\ell_1, \ldots, \ell_t : \mathcal{G}_p \to \mathbb{F}\mathbb{Z}_p$ be the homomorphisms corresponding to them (as in section 3.2). Then we have the $p$-adic $L$-function

$$L_p(S, \pi) := L_p(s_1, \ldots, s_t, \kappa_\pi, +) := \int_{\mathbb{Q}_p} \prod_{i=1}^t \exp_p(s_i \ell_i(\gamma)) \mu_\pi(d\gamma)$$

of definition 3.5 with $s_1, \ldots, s_t \in \mathbb{Z}_p$. $L_p(S, \pi)$ is a locally analytic function with values in the one-dimensional $\mathbb{C}_p$-vector space $V_{\kappa_\pi, +} = L_{\kappa, \mathcal{O}, +} \otimes \mathbb{C}_p$.

By theorem 3.13 we have

**Theorem 4.14.** $L_p(S, \pi)$ is a locally analytic ($t$-varibled) function, and all partial derivatives of order $\leq n := \#(S_1)$ vanish; i.e. we have

$$\operatorname{ord}_{S_1} L_p(S, \pi) \geq n.$$
(resp. split multiplicative) reduction at $p$. For $v|\infty$, $\pi_v$ is “of weight 2” as assumed before.

We say that $E$ is $p$-ordinary if it has good ordinary or multiplicative reduction at all places $p|p$ of $F$. So $E$ is $p$-ordinary iff $\pi$ is ordinary at all $p|p$. In this case, we define the $p$-adic L-function of $E$ by $L_p(E, s) := L_p(\mathcal{L}_p, \pi)$.

For each $i \in \{1, \ldots, t\}$ and each prime $p|p$ of $F$, we write $\ell_{p,i}$ for the restriction of $\ell_i$ to $F_p \hookrightarrow \mathbb{I} \rightarrow G_p$. Let $q_p$ be the Tate period of $E|F_p$ and $\ord_p$ the normalized valuation on $F_p^*$. We define the $L$-invariants of $E|F_p$ with respect to $L_i$ by

$$L_{p,i}(E) := \frac{\ell_{p,i}(q_p)}{\ord_p(q_p)}$$

Then we can generalize Hida’s exceptional zero conjecture to general number fields:

**Conjecture 4.15.** Let $S_1$ be the set of $p|p$ at which $E$ has split multiplicative reduction, $n := \#S_1$, $S_2 := S_p \setminus S_1$. Then

$$\ord_{s=0} L_p(E, s) \geq n,$$

and we have

$$\frac{\partial^n}{\partial s^n} L_p(E, s)|_{s=0} = n! \prod_{p \in S_1} L_{p,i}(E) \prod_{p \in S_2} e(\pi_p, 1) \cdot L(E, 1),$$

for all $i = 1, \ldots, t$, where $e(\pi_p, 1) = (1 - \alpha_{p,1}^{-1})^2$ if $E$ has good ordinary reduction at $p$, and $e(\pi_p, 1) = 2$ if $E$ has (non-split) multiplicative reduction at $p$.

Note that the conjecture (considered for all sets of independent $\mathbb{Z}_p$-extensions of $F$) also determines the “mixed” partial derivatives $\frac{\partial^n}{\partial s^n} L_p(E, 0)$ of order $n$, since they can be written as $\mathbb{Q}$-linear combinations of $n$-th “pure” partial derivatives $\frac{\partial^n}{\partial s^n} L_p(E, 0)$ with respect to other choices of independent $\mathbb{Z}_p$-extensions of $F$ by remark 3.11.

Theorem 4.14 immediately implies the first part (4.13) of the conjecture:

**Corollary 4.16.** Let $E$ be a $p$-ordinary modular elliptic curve over $F$. Let $n$ be the number of places $p|p$ at which $E$ has split multiplicative reduction. Then we have

$$\ord_{s=0} L_p(E, s) \geq n.$$
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