Diffusion in a Half-Space: From Lord Kelvin to Path Integrals

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Many important transport phenomena are described by simple mathematical models rooted in the diffusion equation. Geometrical constraints present in such phenomena often have influence of a universal sort and manifest themselves in scaling relations and stable distribution functions. In this paper, I present a treatment of a random walk confined to a half-space using a number of different approaches: diffusion equations, lattice walks and path integrals. Potential generalizations and universal distribution functions can be obtained in each case through appropriate limiting procedures. We also discuss possible ways to extend and generalize the problem for more realistic systems.

I. INTRODUCTION

Diffusion is perhaps one of the most well-studied fields of modern physics. Diffusion processes are ubiquitous and extremely important in physics, biology, chemistry, material science and other disciplines of basic and applied research. Many famous names, such as Fourier, Kelvin and Einstein, are associated with the theory of diffusion. About two hundred years after the diffusion equation was first written down, the motivation for studying diffusion by both theorists and experimentalists has not faded. Modern research deals with questions of much greater complexity than the originally formulated diffusion equation. Fields such as pattern formation, non-equilibrium growth, reaction-diffusion processes and granular media keep challenging physicists and applied mathematicians and yield many interesting and often unexpected results.

Compared to these actively developing fields, "old-fashioned" linear diffusion may seem boring and trivial. Often it appears though, that newly discovered phenomena and incessantly developing techniques provide formal and conceptual contexts in which old and well-known questions appear in a new light and acquire additional scientific and pedagogical value. For instance, rapid progress in the physics of polymers and critical phenomena in the late 1960’s and 1970’s, stimulated by introduction of field-theoretic ideas and methods, has shifted the accents in diffusion studies towards scaling and universality. Many systems that appeared completely unrelated at first glance turned out to exhibit identical behavior in certain asymptotic ("critical") regions of the parameter space.

In this paper, we study the rather old problem of a random walk confined to a half-space, using a number of different approaches. First, we review the traditional methods – the diffusion equation and lattice walks. Next, we discuss the applications of these methods to Gaussian polymer chains, with emphasis on scaling. Finally, we show how the same problem can be solved using basic field-theoretic methods, i.e. functional integration in Fourier space. We shall see how important scaling relations and universal distribution functions can be obtained in each case through appropriate limiting procedures. We also discuss possible ways to extend and generalize the problem for more realistic systems.

II. DIFFUSION AND RANDOM WALKS

A. Boundary conditions

Consider a $N$-step random walk starting at $r_0 = 0$ in the three-dimensional (3D) space. Let $G(r,N)$ be the probability density for the walk to end at $r$. It is well known that for large $N$ and in the absence of obstacles and boundaries, $G(r,N)$ is a solution of the diffusion equation

$$ (\partial_N - \nabla^2)G(r,N) = \delta(r)\delta(N), $$

namely,

$$ G(r,N) = \frac{e^{-r^2/(2N)}}{(2\pi N)^{3/2}}. $$

It is reasonable therefore to assume that solutions of the same kind can as well be found for any bounded region $\mathbb{R}$.

Naturally, one has to specify the boundary conditions. This is not as trivial as it appears and, in fact, depends on the physical context of the problem. If, for example, the random walker is allowed to touch the boundary and then step back with probability one, the reflecting boundary conditions are appropriate. Formally, it means that the flux across the boundary vanishes

$$ \nabla G(r,N) \cdot n_{|r\in\partial\mathbb{R}} = 0. $$

Here $\partial\mathbb{R}$ denotes the boundary of the region $\mathbb{R}$ and $n$ is a unit vector locally normal to $\partial\mathbb{R}$. Another possible choice of boundary conditions corresponds to the case when the walker sticks to the boundary upon reaching it – the absorbing boundary conditions

$$ G(r,N)|_{r\in\partial\mathbb{R}} = 0. $$
For reasons that become clear below, we shall mainly be interested with the latter case.

**B. Method of images**

Consider a random walk starting at \( r_0 = za \) away from the plane \( z = 0 \) and confined to the \( z > 0 \) half-space. For the absorbing boundary, we expect the probability distribution for the end point to satisfy the following boundary value problem

\[
\begin{aligned}
(\partial_N - \nabla^2)G(r, N; a) &= \delta(r - za)\delta(N) \\
G(z = 0) &= 0, \\
G(|r| \to \infty) &= 0
\end{aligned}
\]  

Any introductory textbook on PDEs contains a straightforward solution of this problem, which consists of introducing a sink, or *negative image*, at \((0, 0, -a)\) and extending the problem to the entire space. The boundary condition at \( z = 0 \) is then automatically satisfied and the solution is

\[
G(r, N; a) = \frac{1}{(2\pi N)^{3/2}} \left[ e^{-|r-za|^2/(2N)} - e^{-|r+za|^2/(2N)} \right].
\]

For \( a \ll \sqrt{N} \), we can expand the expression in parentheses to obtain

\[
G(r, N; a) \simeq \left( \frac{2a^2}{N} \right) \left( \frac{e^{-r^2/(2N)}}{(2\pi N)^{3/2}} \right).
\]

We see that the probability distribution can be factorized into \( z \)-dependent and \( z \)-independent parts. The latter, which includes degrees of freedom parallel to the boundary, is not affected by the presence of the boundary. Thus, in what follows we will be predominantly occupied with the \( z \)-dependent part of \( G(r, N; a) \).

**C. A short historical digression**

The theory of diffusion was first developed in the beginning of the 19th century by Joseph Fourier; his work was summarized in the famous *Théorie analytique de la chaleur*, first published in 1822. It contains an extensive treatment of homogenous heat diffusion problems for a variety of geometries, mostly by the variable separation method.

The first generalized approach to solving non-homogenous diffusion problems was formulated by Sir William Thomson, more widely known as Lord Kelvin, in 1850. He realized that particular solutions can be obtained by superposition of solutions for “instantaneous simple point sources” (which are now called by physicists “Dirac’s delta–functions”). In short, what he did was to invent the Green’s function method for the diffusion equation; it was later used by E. W. Hobson to treat heat-conduction problems with a variety of sources and boundary conditions.

Kelvin was also the first to apply the method of images to account for boundary conditions for electricity conduction in a semi-infinite telegraph line.

**D. “Phantom” polymers and scaling**

As mentioned in the Introduction, random walks are often used to model long polymer molecules. Proper description includes a nontrivial requirement: the random walk must be self-avoiding, i.e. no point in space should be visited more than once. This requirement introduces long–range correlations and makes the problem tractable only approximately. However, if the self-avoidance is removed, the problem becomes much simpler. Such a polymer is called *phantom*; alternative names are *ideal* or *Gaussian* chain. The distribution function for the end–to-end radius vector \( r \) of a phantom polymer of length \( N \) obeys the diffusion equation.

Rapid developments in polymer physics and the theory of critical phenomena have revealed a number of universal properties that arise in all polymer chains beyond a certain level of coarse-graining. These properties are characterized by *scaling relations*. Perhaps the most widely known scaling law relates the mean end–to-end distance \( R \) of a polymer chain to its length: \( R \propto N^{\nu} \). The number \( 1/\nu \) thus plays the role of the polymer’s fractal dimension. It is universal in that it depends only on the dimensionality of the embedding space, e.g. in 3D, \( \nu \approx 0.59 \) for a self–avoiding polymer and \( \nu = 1/2 \) for a phantom one. Another important scaling relation describes the number of different configurations \( \mathcal{N} \) of a polymer

\[
\mathcal{N} = \text{const} \times \zeta^N N^{\gamma - 1},
\]

where \( \zeta \) is the “effective coordination number” that depends on the microscopic details (cf. Eq. 14), and \( \gamma \) is a universal exponent. The factor \( \zeta^N \) can be thought of as counting the configurations of an unconstrained \( N \)-step random walk with \( \zeta \) options available at each step, whereas \( N^{\gamma - 1} \) accounts for constraints such as self–avoidance, obstacles present etc.

If the distribution function \( G(r, N) \) is known, the exponent \( \gamma \) can be obtained in a very straightforward way, by simply integrating \( G(r, N) \) over the whole space. Thus, a phantom polymer has \( \gamma = 1 \). It turns out that incorporating self-avoidance constraint leads to \( \gamma = 1.1638 \). This can be interpreted as the enhancement of available space for a self–avoiding polymer which appears “swollen” compared to a phantom one.

Consider a phantom polymer anchored to the \( xy \) plane and confined to the \( z > 0 \) half-space. Assuming that the polymer is strongly repelled from the \( z = 0 \) plane, what is the corresponding boundary value problem? Intuitively,
it would seem that we must adopt reflecting (zero flux) boundary conditions at \( z = 0 \). However, a more careful analysis shows that this is incorrect. Note that \( G(r, N) \) is proportional to the number of \( N \)-step paths of length \( N \) ending at \( r \). Since the plane is repelling, while counting the paths contributing to \( G(r, N) \), we should discard those touching the \( z = 0 \) plane at least once. Thus, the probability and not the probability flux should vanish at the boundary, which corresponds to the absorbing boundary conditions. This is quite counter–intuitive: while a polymer, being an entire path, is repelled from the boundary, a fictitious random walker that we employ to model this path is absorbed there!

Chandrasekhar in his classic paper of 1943, considered both kinds of boundary conditions (we will touch on his derivation below). However, the abovementioned subtlety was overlooked by a number of subsequent authors which lead to incorrect calculation of average quantities in polymer adsorption problems (for a very lucid discussion of this topic see the paper by DiMarzio).

Thus, the anchored Gaussian chain should be described by Eq. (5). Apparently, we should take \( a = 0^+ \), which leads to

\[
G(r, N; 0^+) = 0. \tag{9}
\]

Of course, this result does not make sense. To obtain the correct result, we should have normalized the distribution function for a finite value of \( a \) and only then take the limit \( a \to 0^+ \), which would yield

\[
G(r, N; 0^+) = \left( \frac{z}{N} \right) e^{-r^2/(2N)} \left( \frac{2}{\pi N} \right). \tag{10}
\]

However, normalization eliminated any information connected with \( \gamma \); indeed, any normalized distribution function by definition has \( \int d^3r \, G(r, N) = 1 \). Again, one can see that the problem here is that of the order of taking limits. For any physical polymer \( a \) is finite, at least of the order of the smallest coarse-graining scale – the persistence length. For small but finite \( a \),

\[
\mathfrak{N} \propto \int_{-\infty}^{\infty} dx \, dy \int_{0}^{\infty} dz \, \frac{a^2 z}{N^{5/2}} e^{-r^2/2N} \propto aN^{-1/2}. \tag{11}
\]

From here, we can read off the value of \( \gamma \) which for the anchored random walk is denoted \( \gamma_a \):

\[
\gamma_a = \frac{1}{2}. \tag{12}
\]

Thus, we see that the presence of the plane reduces the number of accessible configurations compared to the unconstrained case which manifests itself in the scaling exponent \( \gamma_a \). Note that reflecting boundary conditions produce \( \gamma = 1 \) so that the number \( \mathfrak{N} \) is unchanged relative to the unconstrained case. This is yet another argument for incorrectness of reflecting boundary conditions in this problem.

### E. Counting walks on a lattice

Chandrasekhar suggested a direct way of counting the paths on a lattice when a reflecting or an absorbing boundary is present. We will briefly describe the derivation for an absorbing boundary. The reader is encouraged to read the original paper which despite being written more than half a century ago, remains one of the best introductions into random walks and stochastic processes in general.

Consider a one–dimensional random walker on a lattice (discrete \( z \)-axis) with absorbing boundary at \( z = 0 \). Suppose, the walk starts some distance \( n \) from the origin; our task is to calculate the number of paths leading from \( n \) to some other point \( m \), without touching the boundary. It turns out that it is easier to calculate the number of paths that do touch the boundary and then to subtract it from the total number of paths leading from \( n \) to \( m \). To do so, we make use of a very elegant theorem – the reflection principle.

![FIG. 1: The reflection principle.](image)

Let us extend our lattice to include the negative part of the \( z \)-axis as well. Then, the reflection principle states that the number of \( N \)-step paths originating at \( n \), ending at \( m \) and touching or crossing the boundary \( z = 0 \) is equal to the number of \( N \)-step paths that originate at \( -n \) and end at \( m \). Figure illustrates the reflection principle by presenting a way to build a one–to–one mapping between the two sets of paths. Thus, the number of paths not touching the boundary is

\[
\mathfrak{N} = \left( \frac{N}{2} [N + m - n] \right) - \left( \frac{N}{2} [N + m + n] \right). \tag{13}
\]

For the starting point near the boundary, and \( m \ll N \), we can expand the binomial coefficients using Stirling’s
formula to obtain
\[ \mathcal{Z} \approx 2^N \left( \frac{2}{\pi N} \right)^{1/2} m \frac{e^{-m^2/(2N)}}{N}. \]  

(14)

Dividing by the total number of paths of length \( N \) (which is \( 2^N \)), we obtain the probability density for a path to start near the boundary and to end at some point \( m \) without returning to the boundary
\[ G(m, N) \approx \frac{2m e^{-m^2/(2N)}}{(2\pi N)^{1/2}}. \]  

(15)

### III. RESULTS FROM PATH INTEGRALS

In this section, we employ the methods of functional (path) integration to build a field–theoretical model that describes a Gaussian chain anchored to an impenetrable plane.

#### A. Example: unconstrained Gaussian chain

In the field–theoretic approach, a flexible chain is described by a function (“path”) \( c(\tau) \), where \( \tau \) measures the position along the chain. The energy of a self–avoiding chain in an external potential is given by:
\[ H[c] = \frac{1}{2} \int_0^N \dot{c}^2(\tau) d\tau + \int_0^N U[c(\tau)]d\tau + \frac{v}{2} \int_0^N \int_0^N \delta[c(\tau) - c(\tau')]d\tau d\tau'. \]  

(16)

The first two terms can be viewed as a harmonic potential between neighboring segments of the chain and the external potential, respectively, whereas the last one accounts for excluded volume effects: each time the chain self–intersects, a penalty of \( v \) is paid. In what follows, we omit the self–avoidance constraint.

The partition function for such a chain is a sum over all possible paths \( c(\tau) \), given by a path integral
\[ \mathcal{Z}(N) = \int D[c(\tau)] e^{-H[c]}. \]  

(17)

If the external potential is set to zero, this expression simply counts the number of configurations of a Gaussian chain of length \( N \). The configurations are weighted with a weight \( e^{-H[c]} \). If we want to count only paths starting at the origin and leading to some point \( r \) then, after normalization by \( \mathcal{Z}(N) \), we obtain the probability density
\[ G(r, N) = \frac{1}{\mathcal{Z}(N)} \int D[c(\tau)] \delta[c(N) - r] e^{-H[c]}. \]  

(18)

To calculate this sum, one has to define a measure of integration. One way would be to discretize the chain and view \( \mathcal{Z}(N) \) as a limit of a multidimensional integral. In this case, the problem is almost identical to calculating a free particle propagator \( a \) à Feynman & Hibbs\textsuperscript{18}. Another way around is to count the Fourier–components of \( c(\tau) \): this way is somewhat easier, since for harmonic Hamiltonians, degrees of freedom decouple in Fourier space. Thus, setting
\[ \int D[c(\tau)] \rightarrow \int \prod_q \frac{d^3\tilde{c}(q)}{(2\pi)^3} \]  

(19)

and Fourier–transforming Equation (18), we obtain
\[ G(k, N) = \prod_q \int \frac{d^3\tilde{c}(q)}{(2\pi)^3} e^{ik \cdot \tilde{c}(q)} |e^{i(N-1)q} - \frac{1}{2}q^2| \tilde{c}(q)|^2 \]  

(20)

Both the numerator and the denominator contain products of Gaussian integrals that can be calculated by “completing the square.” The result is
\[ G(k, N) = \exp \left( -k^2 \int_{-\infty}^{+\infty} dq \frac{1 - \cos qN}{2\pi q^2} \right) = e^{-k^2N^2/2}, \]  

(21)

which is the Fourier–transform of
\[ G(r, N) = \frac{e^{-r^2/(2N)}}{(2\pi N)^{3/2}}. \]  

(22)

#### B. Anchored polymer - the partition function

Now that we are somewhat familiar with the methodology of path integrals, we return to the original problem – the anchored Gaussian chain in a half–space \( z > 0 \). Writing it in terms of path integrals, we immediately see that our task will not be as simple as before. The reason for this is that now possible values of \( c_z(\tau) \) should be positive. If we stay in real space and calculate the limit of a multidimensional integral, we see that “completing the square” does not work because of this constraint, since the resulting integrals cannot be calculated analytically. If we decide to move to Fourier space, it is not even clear how to define the measure of integration.

The suggested way out of this complication is as follows. We allow the polymer to cross the boundary and introduce a strong repulsive interaction between the plane at \( z = 0 \) and the chain. Each time the polymer crosses or touches the plane, it is “penalized” by a large amount of energy. The modified Hamiltonian is then
\[ H = H_0 + H_1, \]  

(23)
where
\[ H_0 = \frac{1}{2} \int_0^N \hat{c}^2 \, d\tau, \quad \text{(24)} \]
and
\[ H_1 = g \int_0^N \delta[c_z(\tau) - c_z(0)] \, d\tau. \quad \text{(25)} \]

Thus, we expect that when the coupling constant \( g > 0 \) becomes infinitely large, the polymer will be entirely on one (either positive or negative) side of the plane. The partition function is then
\[ Z(g, N) = \int D[c(\tau)] \, e^{-H_0[c]-H_1[c]}. \quad \text{(26)} \]

To evaluate \( Z(g, N) \), we expand the integrand in Eq. 26 in powers of \( g \). Such an expansion could be problematic when \( g \) is large, and this is the limit we are primarily interested in. However, if we are able to calculate the general term of the expansion and to perform the summation to infinity, this approach is valid.

The \( n \)-th \((n = 1, 2, \ldots)\) term of the expansion reads
\[ \left( \frac{-g}{2\pi} \right)^n \int D[c(\tau)] \, e^{-H_0[c]} \prod_{l=1}^n \int \delta[c_z(\tau_l) - c_z(0)] \, d\tau_l. \quad \text{(27)} \]

Ordering the set \( \{\tau_l\} \) and Fourier-transforming the \( \delta \)-functions, this can be rewritten as
\[ \left( \frac{-g}{2\pi} \right)^n \int D[c(\tau)] \, e^{-H_0[c]} \prod_{l=1}^n \int dk_l \, e^{-ik_l[c_z(\tau_l) - c_z(0)]}. \quad \text{(28)} \]
Henceforth, we focus on the \( c_z(\tau) \) and denote it \( c(\tau) \) for the sake of simplicity. Integrating it out by the method discussed above (i.e. “completing the square”), we are left with
\[ \left( \frac{-g}{2\pi} \right)^n \int_0^N d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1 \int_{-\infty}^{+\infty} dk_1 \cdots dk_l \exp \left[ -\frac{1}{2} \sum_{l,m} k_l^2 \tau_{lm}(n) k_m \right]. \quad \text{(29)} \]

Here,
\[ \tau_{lm}^{(n)} = \tau_{\text{min}[l,m]} = \begin{pmatrix} \tau_1 & \tau_1 & \cdots & \tau_1 \\ \tau_1 & \tau_2 & \cdots & \tau_2 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1 & \tau_2 & \cdots & \tau_n \end{pmatrix}. \quad \text{(30)} \]

To perform multiple integration over \( k_l \) we use the well-known formula
\[ \int_{-\infty}^{+\infty} \prod_j dk_j \exp \left( -\frac{1}{2} \sum_{l,m} k_l \tau_{lm}^{(n)} k_m \right) = \sqrt{\frac{(2\pi)^n}{\det\tau^{(n)}}}. \quad \text{(31)} \]
It is straightforward to show that
\[ \det\tau^{(n)} = (\tau_2 - \tau_1)(\tau_3 - \tau_2)\cdots(\tau_n - \tau_{n-1}), \quad \text{(32)} \]
so that after integration over \( k_1 \), Eq. 27 reduces to
\[ \left( \frac{-g}{\sqrt{2\pi}} \right)^n \int_0^N d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \frac{d\tau_1}{\tau_2} \cdots \frac{d\tau_2}{\tau_n} = \left( \frac{-g}{\sqrt{2\pi}} \right)^n \int_0^N \tau_n^{n/2-1} \prod_{m=1}^{n-1} \int_0^1 x^{m/2-1} (1-x)^{-1/2} \, dx = \frac{(-g/\sqrt{N/2})^n}{\Gamma \left( \frac{n}{2} + 1 \right)} = \frac{(-\hat{g})^n}{\Gamma \left( \frac{n}{2} + 1 \right)}. \quad \text{(33)} \]
where \( \hat{g} \equiv g\sqrt{N/2} \). Thus,
\[
\mathcal{Z}(g, N) = \mathcal{Z}(\hat{g}) = \sum_{n=0}^{\infty} \frac{(-\hat{g})^n}{n! (n/2 + 1)} = e^{\hat{g}^2}[1 - \Phi(\hat{g})],
\]
(34)
where
\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]
(35)
is the error function.

When \( N \to \infty \), so does \( \hat{g} \); expanding \( \mathcal{Z}(\hat{g}) \) for large \( \hat{g} \), we obtain
\[
\mathcal{Z}(\hat{g}) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{\hat{g}} - \frac{1}{2\hat{g}^3} + O(\hat{g}^{-5}) \right).
\]
(36)
Hence,
\[
\gamma_1 = 1 + \lim_{N \to \infty} \frac{\partial \ln \mathcal{Z}}{\partial \ln N} = \frac{1}{2},
\]
(37)
which is identical to the value we obtained in the previous section.

Several remarks should be made at this point. First, note that while calculating the general term of the expansion, we omitted the common factor
\[
\mathcal{Z}_0(N) = \int D[\mathbf{c}(\tau)] e^{-H_0[\mathbf{c}]},
\]
(38)
which actually has the form \( \zeta^N \) (c.f. Eq. [3]). Secondly, the coupling constant \( g \) plays here the role of the inverse cutoff length, which often occurs in field theory. Finally, we note that for any value of \( g > 0 \), we can find \( N \) large enough to make the non-dimensional coupling \( \hat{g} = g\sqrt{N/2} \gg 1 \), so that the number of accessible configurations scales as \( N^{-1/2} \) relative to the unconstrained case. This important observation is a signature of universality: no matter how small \( g \) is, for long enough polymer the overall repulsion is infinitely strong!

### C. Probability distribution

The Fourier–transform of the (unnormalized) probability distribution function (PDF) for the end-to-end distance of the random walk is given by
\[
G(\mathbf{q}, N) = \int D[\mathbf{c}(\tau)] e^{-i\mathbf{q}[\mathbf{c}(N) - \mathbf{c}(0)]} e^{-H_0[\mathbf{c}] - H_1[\mathbf{c}, \mathbf{e}]}.
\]
(39)
As above, we focus only on the \( z \)-dependent part of the PDF \( G(z, N) \). Expanding the path integral in powers of \( g \), and integrating out \( \mathbf{c}(\tau) \) we observe that the \( n \)-th term of the expansion reads
\[
\left( -\frac{g}{2\pi} \right)^N \int_0^N d\tau_n \int_0^{\tau_n} d\tau_{n-1} \ldots \int_0^{\tau_2} d\tau_1 \prod_i dk_i \exp \left[ -\frac{1}{2} (q^2N + \sum_l k_l\tau_l + \sum_{l,m} k_l T_{lm}^{(n)} k_m) \right]
\]
\[
= \left( -\frac{g}{2\sqrt{2}\pi} \right)^N \int_0^N d\tau_n \int_0^{\tau_n} d\tau_{n-1} \ldots \int_0^{\tau_2} d\tau_1 \frac{1}{\sqrt{\det T^{(n)}}} \exp \left[ -\frac{g^2}{2}(N - \sum_{l,m} \tau_l T_{lm}^{(n)} - 1 \tau_m) \right],
\]
(40)
where \( T^{(n)} \) is given by Eq. [32]. Now,
\[
[T^{(n)}]^{-1}_{lm} = \begin{cases} 
\frac{-\delta_{l+1,m}}{\tau_{l+1} - \tau_l} - \frac{\delta_{l-1,m}}{\tau_l - \tau_{l-1}} + \delta_{m,l} \left( \frac{1}{\tau_1 - \tau_1} + \frac{1}{\tau_{l+1} - \tau_{l+1}} \right) & l \neq n, 0 \\
\frac{-\delta_{l+1,m}}{\tau_{l+1} - \tau_l} - \frac{\delta_{l-1,m}}{\tau_l - \tau_{l-1}} + \delta_{m,l} \left( \frac{1}{\tau_1 - \tau_1} + \frac{1}{\tau_{l+1} - \tau_{l+1}} \right) & l = 1 \\
\frac{-\delta_{l-1,m}}{\tau_{l+1} - \tau_l} - \frac{\delta_{l-1,m}}{\tau_l - \tau_{l-1}} + \delta_{m,l} \left( \frac{1}{\tau_1 - \tau_1} + \frac{1}{\tau_{l+1} - \tau_{l+1}} \right) & l = n
\end{cases}
\]
(41)

Straightforward calculation yields
\[
\sum_{l,m} \tau_l [T^{(n)}]^{-1}_{lm} \tau_m = \tau_n.
\]
(42)
Equation [40] therefore reduces to
where we can take is the scaling function. Using a substitution 
\[ \tau = \frac{\tilde{z}}{\sqrt{N}} \]
so that
\[ \Phi(\tau) = \frac{e^{-\tau^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} \, dx \]
we finally obtain
\[ F(\hat{g}, \tilde{z}) \simeq A(\hat{g}) + \frac{|\tilde{z}|}{\sqrt{2\hat{g}}} \int_0^\infty \frac{du}{u^2} (1 - e^{-u^2}) \]
\[ = A(\hat{g}) + \frac{|\tilde{z}|}{\sqrt{2\hat{g}}}. \]

For large values of \( \hat{g} \), we have
\[ A(\hat{g}) = \frac{1}{2\sqrt{\pi \hat{g}^2}} + O(\hat{g}^{-4}). \]

Thus, when \( \hat{g} \to \infty \), the scaling function is linear in \( \tilde{z} \) and the normalized PDF has the form
\[ G(z, N) = \frac{|z|}{2N} e^{-z^2/2N}. \]

Apart from a factor of 2, this function is identical to the one obtained in the previous sections. This factor appears because now the chain can be either in the \( z < 0 \) or in the \( z > 0 \) half-space.

Figure 2 shows the numerically computed scaling function \( F(\hat{g}, \tilde{z}) \). Different curves are labeled by corresponding values of \( \hat{g} \). As expected, the larger \( \hat{g} \) is, the closer is \( F(\hat{g}, \tilde{z}) \) to the linear dependence.
IV. CONCLUSION

To say the least, functional integration is not the most effective way to obtain the probability distribution \( G(z, N) \). Why then has one to work so hard if it is possible to obtain the answer in just a few lines?

Beside its clear educational value, path integral analysis of random walks is a much more flexible and powerful tool when it comes to real systems such as polymers in solvents. The self-avoidance constraint that we omitted so readily introduces long-range correlations that make the traditional approaches loose their power and elegance. For example, factorization of \( G(r, N) \) into transverse and longitudinal parts is not valid anymore and integrating out one set of degrees of freedom introduces non-local interactions into the other. Whereas lattice walks are useful for Monte-Carlo simulations and diffusion equations could perhaps be modified to include mean-field corrections, field theory and primarily the renormalization group (RG) is today the only analytical tool to obtain universal scaling relations and phase diagrams using controlled approximations. For instance, generalizing our Hamiltonian to include self-interactions, interactions between a number of polymers, external potentials etc., it is possible to quantify the influence of space dimensionality and learn about the relevance of various interactions.

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13. A more concise term is random flight; we will use the first one as more widely known, since the difference between the two is inessential throughout this paper.
14. It is straightforward to verify that reflecting boundary conditions correspond to a positive image.
15. The exact form of \( G(r, N) \) for a self-avoiding polymer is not known, so \( \gamma \) is calculated either numerically, or by renormalization group methods.