Simple and Fast Algorithm for Binary Integer and Online Linear Programming

Xiaocheng Li† Chunlin Sun‡ Yinyu Ye†

†Department of Management Science and Engineering, Stanford University
‡ Institute for Computational and Mathematical Engineering, Stanford University
{chengli1, chunlin, yyye}@stanford.edu

Abstract

In this paper, we develop a simple and fast online algorithm for solving a general class of binary integer linear programs (LPs). The algorithm requires only one single pass through the input data, and is free of doing any matrix inversion. It can be viewed as both an approximate algorithm for solving binary integer LPs and a fast algorithm for solving online LP problems. The algorithm is inspired by an equivalent form of the dual problem of the relaxed LP and it essentially performs projected stochastic subgradient descent in the dual space. We analyze the algorithm in two different models, stochastic input model and random permutation model, with minimal assumptions on the input of the LP. The algorithm achieves $O(m^2\sqrt{n})$ expected regret under the stochastic input model and $O((m^2 + \log n)\sqrt{n})$ expected regret under the random permutation model, and it achieves $O(m\sqrt{n})$ expected constraint violation under both models, where $n$ is the number of decision variables and $m$ is the number of constraints. Furthermore, the algorithm is generalized to a multi-dimensional LP setting which covers a wider range of applications and features for the same performance guarantee. Numerical experiments illustrate the general applicability and the performance of the algorithms.

1 Introduction

In this paper, we present simple and fast online algorithms to approximately solve a general class of binary integer linear programs (LP). Specifically, we consider binary integer LPs that take the following form

$$\begin{align*}
\max \ & \ r^\top x \\
\text{s.t.} \ & \ Ax \leq b \\
& \ x_j \in \{0, 1\}, \ j = 1, \ldots, n,
\end{align*}$$

where $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$, $A = (a_1, \ldots, a_n) \in \mathbb{R}^{m \times n}$, and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$. The decision variables are $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$. Different specifications of the above formulation cover a wide range of
classic problems and modern applications: secretary problem (Ferguson et al., 1989), knapsack problem
(Kellerer et al., 2003), resource allocation problem (Vanderbei et al., 2015), generalized assignment
problem (Conforti et al., 2014), network routing problem (Buchbinder and Naor, 2009), matching problem
(Mehta et al., 2005), etc.

Our algorithm is a primal-dual algorithm inspired by an equivalent form of the dual problem of the
above integer LP. The key is to perform projected stochastic subgradient descent for the dual variable
and to decide the primal solution based on the dual variable in an online fashion. The algorithm requires
only one single pass through the input (r and A) of the problem, and is free of doing any matrix inversion.
When the right-hand-side b scales linearly with n, we show that the algorithm outputs a solution with
an expected optimality gap of $O(m^2 \sqrt{n})$ and constraint violation of $O(m \sqrt{n})$, under minimal statistical
assumptions (on r and A).

From the perspective of integer LP, our algorithm is an efficient approximate algorithm that features
for provable performance guarantee. In general, integer LP is NP-complete. The LP relaxation technique
has been widely used in designing integer LP algorithm. Our algorithm is also inspired by the relaxed
LP, and it directly outputs an integer solution to the relaxed LP. The solution can thus be viewed as an
approximate solution to both the integer LP and the relaxed LP.

From the perspective of online LP, to the best of our knowledge, our algorithm is the most simple
and efficient online LP algorithm so far. Furthermore, the algorithm analysis is conducted under the
two prevalent models: stochastic input model and random permutation model. The stochastic input
model assumes that the columns of the LP together with the corresponding coefficients in the objective
function are drawn i.i.d. from an unknown distribution. Our assumption is weaker than the previous
literature under this model in that the strong convexity is not assumed for the underlying stochastic
program. The random permutation model assumes that the columns together with the coefficients are
presented in a random permutation. It better captures the possible non-stationary and adversarial input
of the LP. Under this model, our assumption is weaker than all previous works in that we allow negative
data values for the input of the LP.

1.1 Related Literature

The algorithms developed in this paper can be viewed as a stochastic algorithm to solve large-scale
(integer) LPs. The literature on large-scale LP algorithms traced back to the early works on column
generation algorithm (Ford Jr and Fulkerson, 1958; Dantzig, 1963). In recent years, statistical struc-
tures underlying the input of LP have been taken into consideration. Sampling-based/randomized LP
algorithms are derived to handle large number of constraints in the LP of Markov Decision Processes
(De Farias and Van Roy, 2004; Lakshminarayanan et al., 2017), the standard form of LP (Vu et al.,
2018), robust convex optimization (Calafiore and Campi, 2005), etc. De Farias and Van Roy (2004)
studied an approximate LP problem arising from the approximate dynamic programming approach and
developed a sampling scheme to reduce the number of constraints under certain statistical assumptions.
A subsequent work (Lakshminarayanan et al., 2017) developed a soft approach by replacing the original
LP constraints by a smaller set of constraints that are constructed from positive linear combinations
of the original ones. Vu et al. (2018) discussed the standard LP formulation and introduced a random
projection method to approximately solve large-scale LP in the light of Johnson-Lindenstrauss Lemma.
Compared to this line of works, our algorithms utilize the dual LP and are free of solving any small-scale
or reduced-size LP. Recently, another stream of works studied first-order algorithms, mainly alternating
direction method of multipliers (ADMM) method, for solving large-scale LPs (Yen et al., 2015; Wang
and Shroff, 2017; Lin et al., 2018). Compared to the algorithms developed therein, our algorithms only
require one single pass through the inputs of the LP and do not involve any optimization sub-routine
nor matrix inversion. The design of ADMM algorithms is usually motivated from a careful study of the
optimization problem while our algorithms are built upon a statistical perspective.

Our algorithms and analyses also contribute to the literature of online linear programming. The
formulation studied in this paper is the same as the previous works (See (Molinaro and Ravi, 2013;
Agrawal et al., 2014; Kesselheim et al., 2014; Gupta and Molinaro, 2014; Li and Ye, 2019) among others).
Among all these algorithms, the algorithm proposed in this paper is the most simple and efficient
one. In terms of the assumptions, the online LP literature mainly consider two models – stochastic
input model and random permutation model. The key distinction between these two models lies in the
different assumptions put on the coefficients in the constraint matrix and in the objective function. In
Section 3 and Section 4, we analyze our algorithm under these two models respectively. Under stochastic
input model, our assumption on the distribution that generates the LP coefficients is minimal than
the previous works including (Li and Ye, 2019) and other work using LP resolving techniques for the
network revenue management problem (See (Jasin and Kumar, 2013; Jasin, 2015; Bumpensanti and
Wang, 2018) among others). Compared to the literature under random permutation (See (Molinaro and
Ravi, 2013; Agrawal et al., 2014; Kesselheim et al., 2014; Gupta and Molinaro, 2014) among others), we
allow the inputs of the LP to take negative data value and consider the regime for large right-hand-side.
Specifically, the previous works investigated the necessary and sufficient conditions on the right-hand-
side of the LP $b$ and the number of constraint $m$ for the existence of an $\epsilon$-competitive (near-optimal)
online LP algorithm. Alternatively, we research the question when the right-hand-side $b$ grows linearly
with the number of decision variables $n$, whether the algorithm could achieve a better performance than
$\epsilon$-competitiveness. More importantly, we are the first work on online LP under the random permutation
model that allows negative data values for the input of the LP which have wider applications such as
double-auction markets.

Similar or special forms of the online LP problem have been extensively studied in OM literature.
These problems include secretary problem (Kleinberg, 2005; Arlotto and Gurvich, 2019), auction problem (Zhou et al., 2008; Balseiro and Gur, 2019), network revenue management problem (Jasin and Kumar, 2013; Jasin, 2015; Bumpensanti and Wang, 2018), and resource allocation problem (Asadpour et al., 2019; Jiang and Zhang, 2019; Lu et al., 2020). The common point for all these problems is that there is an underlying LP and the coefficients of the LP are specified by the corresponding application context. Consequently, the algorithm design and analyses rely on the structure of the LP, such as all-one constraint matrix (in secretary problem), binary constraint matrix (in resource allocation problem), finite support of the random coefficients (in network revenue management problem), etc. Given that our work studies the general formulation, though it might not be able to degenerate and apply to every of the above problems, the design and analyses provide theoretical and algorithmic insights for these applications.

The problem of online LP seemingly can be viewed as a special form of online convex optimization with constraints (OCOwC). However, these two problems are studied separately in the literature. The paper establishes a connection between these two problems by identifying the dual form of online LP problem as a special form of the primal problem of OCOwC. The literature on OCOwC (Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018) that employs stochastic gradient descent methods thus can be applied to analyze the dual objective for the online LP problem under stochastic input model. Our contribution to this literature is two-fold: (i) We discuss the problem under random permutation model while the literature on OCOwC only studied the stochastic input model; (ii) Though its dual problem is related to the OCOwC problem, the online LP problem mainly concerns the primal objective. Our contribution here is to identify the connections between the primal and dual objectives, and the constraint violation of the online LP problem. Another elegant paper (Agrawal and Devanur, 2014) developed and analyzed fast algorithms for the problem of online convex programming. It differs from the online LP problem in that the formulation therein considered a simpler form of the constraint which requires the averaging of the decision variables chosen throughout the process belongs to a convex set. It thus corresponds to a setting in the online LP problem where the constraint matrix is an all-one matrix.

Another stream of literature studied the random reshuffling method for stochastic gradient descent (SGD) algorithm in minimizing a finite sum of convex component functions (Gürbüzbalaban et al., 2015; Ying et al., 2018; Safran and Shamir, 2019). The study of random reshuffling method is mainly focused on the question whether SGD will converge faster under sampling with or without replacement. The method shares a similar spirit with our algorithm under the random permutation model, but the speciality of our problem is the presence of the constraints. Also, the differentiation between the stochastic input model and the random permutation model in our paper emphasizes more on the generation mechanism for the inputs for the LP, whereas the study of random reshuffling method concerns more about the better way of sampling for SGD given the same data.
2 Integer Linear Program and Main Algorithm

2.1 Integer LP, Primal LP, and Dual LP

Consider the binary integer LP

\[
\begin{align*}
\text{max} & \quad r^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x_j \in \{0, 1\}, \ j = 1, ..., n
\end{align*}
\]

where \( r = (r_1, ..., r_n) \in \mathbb{R}^n, A = (a_{1j}, ..., a_{nj}) \in \mathbb{R}^{m \times n}, \) and \( b = (b_1, ..., b_m) \in \mathbb{R}^m. \) Here \( a_j = (a_{1j}, ..., a_{mj}) \) denotes the \( j \)-th column of the constraint matrix \( A. \) The decision variables \( x = (x_1, ..., x_n) \) are binary integers. An LP relaxation of the above problem is

\[
\begin{align*}
\text{max} & \quad r^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad 0 \leq x \leq 1.
\end{align*}
\]

The dual problem of (P-LP) is

\[
\begin{align*}
\text{min} & \quad b^T p + 1^T s \\
\text{s.t.} & \quad A^T p + s \geq r \\
& \quad p \geq 0, s \geq 0,
\end{align*}
\]

where the decision variables are \( p \in \mathbb{R}^m \) and \( s \in \mathbb{R}^n. \) Throughout this paper, \( 0 \) and \( 1 \) denote all-zero and all-one vector, respectively. We will use ILP, P-LP, and D-LP to refer to both the optimization problem and their optimal objective values. Evidently, we have the following relation between the optimal objective values,

\[
\text{ILP} \leq \text{P-LP} = \text{D-LP}.
\]

This natural relation provides the foundation for the wide usage of LP relaxation in solving integer linear programs (Conforti et al., 2014). Now, we start from the linear programs P-LP and D-LP to derive a simple algorithm for the ILP problem, by utilizing an underlying structure of the LPs.

We denote the optimal solutions to (P-LP) and (D-LP) with \( x^*, p_n^*, \) and \( s^*, \) and the optimal solutions
to (ILP) as $\mathbf{x}^*$. From the complementary condition, we know that

$$x_j^* = \begin{cases} 1, & r_i > a_j^T p_n^* \\ 0, & r_i < a_j^T p_n^* \end{cases}$$

for $j = 1, \ldots, n$. When $r_j = a_j^T p_n^*$, the optimal solution $x_j^*$ may be a non-integer value. The implication of this optimality condition is that the primal optimal solution $x^*$ can be largely determined by the dual optimal solution $p_n^*$. For the derivation of our algorithm, we first introduce an informal statistical assumption on the input of the LPs, and we will further detail the assumption in the later sections.

**Assumption 1.** (Informal). We assume the column-coefficient pair $(r_j, a_j)$’s are i.i.d. sampled from unknown distribution $\mathcal{P}$.

If we denote the right-hand-side $b = nd$, as noted by Li and Ye (2019), an equivalent form the dual problem that only involves decision variables $p$ can be obtained from (D-LP) by plugging the constraints into the objective and removing the decision variables $s$. Specifically, consider

$$\min_p f_n(p) = d^T p + \frac{1}{n} \sum_{j=1}^{n} (r_j - a_j^T p)^+$$

where $(\cdot)^+$ denotes the positive part function. Under Assumption 1, all the terms in the summation in (SAA) are independent with each other. Thus, the function $f_n(p)$ can be viewed as a sample average approximation of the stochastic program

$$\min_p f(p) = d^T p + E(r,a) \sim \mathcal{P} \left[ (r - a^T p)^+ \right]$$

Denote the optimal solution to (SP) as $p^*$. Then the optimal dual solution $p_n^*$ to $f_n(p)$ (equivalently, the original dual program D-LP) can be viewed as an approximate to $p^*$. We refer to the previous work (Li and Ye, 2019; Li, 2020) for an extensive discussion on the convergence analysis of $p_n^*$ to $p^*$.

### 2.2 Main Algorithm

Now, we present the main algorithm – Simple Online Algorithm. First, it is an online algorithm that observes the inputs of the LP sequentially and decides the value of decision variable $x_t$ immediately after each observation $(r_t, a_t)$. Second, the algorithm is a dual-based algorithm. It maintains a dual vector $p_t$ and determines $x_t$ in a similar way as the optimality condition (1). At each time $t$, it updates the vector...
with the new observation \((r_t, a_t)\) and projects to the non-negative cone to ensure the dual feasibility.

**Algorithm 1 Simple Online Algorithm**

1: Input: \(d\)
2: Initialize \(p_1 = 0\)
3: for \(t = 1, \ldots, n\) do
4:     Set 
5:     Compute 
6:     end for 
7: Output: \(x = (x_1, \ldots, x_n)\)

The key of the algorithm is the updating formula for \(p_t\), namely Step 5 in Algorithm 1. For two vectors \(u, v \in \mathbb{R}^m\), \(u \lor v = (\max\{u_1, v_1\}, \ldots, \max\{u_m, v_m\})^\top\) denotes the elementwise maximum operator. Specifically, the update from \(p_t\) to \(p_{t+1}\) can be interpreted as a projected stochastic subgradient descent method for optimizing the problem (SAA). Concretely, the subgradient of the \(t\)-th term in (SAA) evaluated at \(p_t\),

\[
\partial_p \left( d^\top p + (r_t - a_t^\top p)^+ \right) \bigg|_{p=p_t} = d - a_t I(r_t > a_t^\top p) \bigg|_{p=p_t}
\]

where the second line is due to the specification of \(x_t\) as the step 4 in the Algorithm 1. Throughout this paper, \(I(\cdot)\) denotes the indicator function. By setting the step size at time period \(t\) to be \(1/\sqrt{t}\), the dual updating rule precisely implements the stochastic subgradient descent in the dual space. We discuss some intuitions of the algorithm here and defer the rigorous analyses of the algorithm performance and the choice of the step size to later sections.

As for the computational aspect, Algorithm 1 requires only one pass through the data and is free of matrix multiplications. Generally, algorithms use LP relaxation to progressively solve integer LPs. In certain sense, the solution given by the optimal solution to the relaxed LP (P-LP) can be viewed as a non-integer approximation to the optimal solution of the according integer LP (ILP). In contrast, the integer solution output from Algorithm 1, though most likely not the optimal solution to the integer LP (ILP), can be viewed as an integer approximation to the (non-integer) optimal solution of the LP (P-LP). Consequently, Algorithm 1 can work as an approximate algorithm to solve the integer LP (ILP), and it is inspired by but not directly utilizing the corresponding LP (P-LP).
2.3 Performance Measures

We analyze the algorithm in two aspects – optimality gap (regret) and constraint violation. The optimality gap measures the difference in objective values for the algorithm output and the true optimal solution. Since Algorithm 1 does not ensure a feasible solution, we need to account the total amount of constraint violations for its output. In this paper, we focus on this bi-objective performance measure for two reasons. First, there may be ways to transform an infeasible solution to a feasible solution which absorbs the constraint violation into the regret (as Theorem 2 in Li and Ye (2019)), but it may require stronger assumptions on the inputs of the (integer) LP. In this paper, we aim to develop theoretical results under minimal assumptions on the input. In this light, it might be challenging to combine the two objectives into one. In Section 6.1, we elaborate more on this aspect and discuss a variant of Algorithm 1 that guarantees feasibility. Second, the bi-objective performance measure is aligned with the literature on the online convex optimization with constraints (OCOC); the same objective is considered in (Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018). Additionally, as we will see in the later sections, there is a natural connection between the primal optimality gap, dual optimality gap, and the constraint violation.

In the following two sections, we will formalize the assumptions and analyze the algorithm in two different settings.

3 Stochastic Input Model

In this section, we formalize and analyze the algorithm under the statistical assumption proposed in the last section. Concretely, we discuss the performance of Algorithm 1 when the inputs of an (integer) LP follows the stochastic input model which assumes the column-coefficient pair \((r_j, a_j)\)'s are i.i.d. generated. LPs and integer LPs that satisfy this model naturally arise from some application contexts where each pair represents a customer/order/request. In particular, at each time \(t\), \(a_t\) can be interpreted as a customer request for the resources while \(r_t\) represents the revenue that the decision maker receives from accepting this request. The binary decision variable \(x_t\) represents the decision of acceptance or rejection of the \(t\)-th request. In such context, the dual vector \(p_t\)'s convey a meaning of dual price and it assigns a value \(a_t^\top p_t\) to the \(t\)-th request. In Algorithm 1, the dual-based decision rule will accept this request if the revenue received \(r_t\) exceeds its assigned value. A more thorough discussion of online LP problem under the stochastic input model can be referred to (Li, 2020). For completeness, we include a self-contained discussion of the results related to Algorithm 1. We recently learned that Lu et al. (2020) also produced results for online optimization problem with a similar algorithm under the i.i.d. model where the random variable \((r_j, a_j)\) has a finite support.
### 3.1 Assumptions and Performance Measures

The following assumption formalizes the statistical assumption on \((r_j, a_j)\) in an i.i.d. setting.

**Assumption 2** (Stochastic Input). *We assume*

(a) The column-coefficient pair \((r_j, a_j)\)'s are i.i.d. sampled from an unknown distribution \(\mathcal{P}\).

(b) There exist constants \(\bar{r}\) and \(\bar{a}\) such that \(|r_j| \leq \bar{r} \) and \(\|a_j\|_\infty \leq \bar{a}\) for \(j = 1, \ldots, n\).

(c) The right-hand-side \(b = nd\) and there exist positive constants \(d\) and \(\bar{d}\) such that \(d \leq d_i \leq \bar{d}\) for \(i = 1, \ldots, m\).

We emphasize that the constants \(\bar{r}, \bar{a}, d\) and \(\bar{d}\) only serve for analysis purpose and are assumed unknown a priori. Also, we allow dependence between components in \((r_j, a_j)\)'s. Besides the boundedness, we have put minimal assumption on \(r_j\) and \(a_j\). This is different from the previous work (Li and Ye, 2019) where stronger assumptions are introduced to ensure a strong convexity for the stochastic program \(f(p)\) (SP). As a result, the convergence of \(p_t\) can be established under the assumptions here, as least not with the same convergence rate as (Li and Ye, 2019). For part (c), the assumption on right-hand-side side provides a service level guarantee, i.e., it ensures a fixed proportional of customers/orders can be fulfilled as the total number of customers (market size) \(n\) increases. We use \(\Xi\) to denote the family of distributions that satisfies Assumption 2 (b).

Next, we formally define the regret and the constraint violation. Denote the optimal objective values of the ILP and P-LP as \(Q_n^*\) and \(R_n^*\), respectively. The objective value obtained by the algorithm output is

\[
R_n = \sum_{j=1}^n r_j x_j.
\]

The quantity of interest is the optimality gap \(Q_n^* - R_n\), which has an upper bound

\[
Q_n^* - R_n \leq R_n^* - R_n.
\]

The expected optimality gap is

\[
\Delta_n^P = E[R_n^* - R_n]
\]

where the expectation is taken with respect to \((r_j, a_j)\)'s. Define regret as the worst-case optimality gap

\[
\Delta_n = \sup_{P \in \Xi} \Delta_n^P.
\]

Thus the regret bound derived in this paper has a two-fold meaning: (i) an upper bound for the optimality gap of solving the integer LP; (ii) a regret bound for the regret of solving online LP problem. Provided
that we do not assume any knowledge of the distribution $P$, this type of distribution-free bound is legitimate. Another performance measure for Algorithm 1 is the expected constraint violations,

$$
\mathbb{E} \left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{n} a_{it}x_{it} - b_i \right) ^+ \right]
$$

It quantifies the total amount of violations for all the constraints. Similar to the regret, we seek for an upper bound for the constraint violation that is not dependent on the distribution $P$.

3.2 Algorithm Analyses

First, we analyze the dual price sequence $p_t$’s. As stated in the following lemma, the dual price $p_t$’s under Algorithm 1 will remain bounded throughout the process, and this is true with probability $1$.

**Lemma 1.** Under Assumption 2, we have

$$
\|p^\ast\|_2 \leq \frac{\bar{r}}{d},
$$

$$
\|p_t\|_2 \leq \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}).
$$

with probability $1$ for $t = 1, \ldots, n$, where $p_t$’s are specified by Algorithm 1.

**Proof.** See Section A2.

Essentially, this boundedness property arises from the updating formula. The intuition is that if the dual price $p_t$ becomes very large, then most of the “buying” orders (with $a_i$ being positive) will not be rejected, and this will lead to a decrease of the dual price when computing $p_{t+1}$. As we will see later, the norm of $p_t$ will appear frequently in the algorithm performance analyses, in term of both the regret and the constraint violation. Therefore the implicit boundedness of $p_t$ becomes important in that it saves us from having to do explicit projection – by projecting $p_t$ into a compact set at every step – which requires the prior knowledge of the constants in Assumption 2.

**Theorem 1.** Under Assumption 2, the regret and expected constraint violation of Algorithm 1 satisfy

$$
\mathbb{E}[R_n^* - R_n] \leq 2 \left( \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right) ^2 \sqrt{n}
$$

$$
\mathbb{E} \left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{n} a_{it}x_{it} - b_i \right) ^+ \right] \leq \left( \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right) \sqrt{n}.
$$

hold for all $n \in \mathbb{N}^+$ and distribution $P \in \Xi$.

**Proof.** See Section A3.
The number of constraints \( m \) decides the dimension of the dual price vectors \( \mathbf{p}_t \)'s. The regret is \( O(m^2 \sqrt{n}) \) and this order matches the upper and lower bounds for online convex optimization (See Theorem 3.1 and 3.2 ([Hazan et al., 2016])). The expected constraint violation is \( O(m \sqrt{n}) \), but as discussed in the proof, this might not be a tight bound.

Algorithm 1 conducts subgradient descent updates in the dual space but the performance is measured by the primal objective. The key idea for the proof of Theorem 1 is to establish the connections between primal objective, dual objective, and constraints violation through the lens of the updating formula for \( \mathbf{p}_t \). This provides an explanation for why the seemingly related problems of online LP and online convex optimization with constraints (OCOwC) are studied separately in the literature. On one hand, the online LP literature has been focused on studying the primal objective value as the performance measure. On the other hand, the OCOwC problem ([Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018]) also studied mainly the primal objective under online stochastic subgradient descent algorithms. However, it is the dual problem of online LP that corresponds to a degenerated form of the primal problem in the OCOwC literature. Our contribution is to identify this correspondence and to establish the primal-dual connection for online LP problem when applying stochastic subgradient descent.

4 Random Permutation Model

In this section, we consider a random permutation model where the column-coefficient pair \( (r_j, a_j) \) arrives in a random order. The values of \( (r_j, a_j) \)'s can be chosen adversarially at the start. However, the arrival order of \( (r_j, a_j) \)'s is uniformly distributed over all the permutations. This characterizes a weaker condition than the previous stochastic input model and the analysis under this model allows more general application of the algorithm. There are two ways to interpret Algorithm 1 under this random permutation model. First, it can be interpreted as an online algorithm that solves an online LP problem under data generation assumptions that are weaker than the i.i.d. assumptions discussed in the last section. Hence, the stochastic input model can be viewed as a special case of the random permutation model. In particular, the latter captures the case when there exists possibly non-stationarity or adversary for the inputs of the LPs. Second, from the perspective of solving integer LPs, the permutation creates the randomness for integer LPs when there is no inherent randomness with the coefficients. As we will see, this artificially created randomness is sufficient for Algorithm 1 to provide provable performance guarantee that are comparable to the case of the stochastic input model.
Example 1. Consider a multi-secretary problem

\[
\begin{align*}
\max & \sum_{j=1}^{n} r_j x_j \\
\text{s.t.} & \sum_{j=1}^{n} x_j \leq b 
\end{align*}
\]

with \( b \in \mathbb{N}^+ \) and \( n = 2b \). Moreover, \( r_1 = ... = r_b = 1 \) and \( r_{b+1} = ... = r_n = 2 \).

This example of multi-secretary problem illustrates the idea and necessity of doing random permutation. This problem in its original form does not satisfy the i.i.d. assumption, and if one solves the problem in its original order, there is no way we can infer about the “good” candidates \( \{r_j\}_{j=b+1}^{n} \) in the later half by just observing the first half of the data \( \{r_j\}_{j=1}^{b} \). However, if we randomly permute the \( r_j \)'s, then the problem becomes

\[
\begin{align*}
\max & \sum_{j=1}^{n} r_{\sigma(j)} x_{\sigma(j)} \\
\text{s.t.} & \sum_{j=1}^{n} x_{\sigma(j)} \leq b 
\end{align*}
\]

where \( (\sigma(1), ..., \sigma(n)) \) is a random permutation of \( (1, ..., n) \). Intuitively, for this new problem, it is very likely that we obtain a good knowledge of the whole data \( \{r_j\}_{j=1}^{n} \) by simply observing the first few samples. More generally, this random permutation technique handles this type of problem where there is no inherent randomness and the theory developed in this section tells that we can achieve a comparable performance guarantee as the previous stochastic input model.

4.1 Assumption and Performance Measures

In parallel to the stochastic input model, we formalize the random permutation model as follows.

Assumption 3 (Random Permutation). We assume

(a) The column-coefficient pair \((r_j, a_j)\) arrives in a random order.

(b) There exist constants \( \bar{r} \) and \( \bar{a} \) such that \( |r_j| \leq \bar{r} \) and \( \|a_j\|_\infty \leq \bar{a} \) for \( j = 1, ..., n \).

(c) The right-hand-side \( b = nd \) and there exists positive constant \( d \) and \( \bar{d} \) such that \( d_i \leq d \leq \bar{d} \) for \( i = 1, ..., m \).

Assumption 3 part (b) and (c) are identical to stochastic input model in the previous section. Denote the input data set \( \mathcal{D} = \{(r_j, a_j) : 1 \leq j \leq n\} \). Part (a) in Assumption 3 states that we observe a permuted realization of the data set. Additionally, we make the following assumption on the data set \( \mathcal{D} \).
Assumption 4. The problem inputs are in a general position, namely for any price vector $p$, there can be at most $m$ columns such that $a_j^T p = r_j$.

This assumption is not necessarily true for all the data set $D$. However, as pointed out by (Devanur and Hayes, 2009), one can always randomly perturb $r_j$’s by arbitrarily small amount. In this way, the assumption will be satisfied, and the effect of this perturbation on the objective can be made arbitrarily small. Define

$$x_j(p) = \begin{cases} 
1, & r_j > a_j^T p, \\
0, & r_j \leq a_j^T p
\end{cases} \quad (2)$$

and $x(p) = (x_1(p),...,x_n(p))$. Lemma 2 tells that if $p_*^n$ is used in (2), the corresponding primal solution should be feasible and close to the primal optimal solution. The complementarity condition (1) does not imply anything about the primal optimal solution when $r_j = a_j^T p_*$. The thresholding rule (2), as it appears in Algorithm 1, takes a conservative standpoint by setting $x_t = 0$ if $r_j = a_j^T p$ when we use the dual price $p$. Essentially, the general position in Assumption 4 ensures that $r_j = a_j^T p$ will happen at most $m$ times for any $p$ and Lemma 2 justifies that the effect of being conservative on these points with the optimal dual price $p_*^n$ is marginal.

Lemma 2. $x_j(p_*^n) \leq x_j^* \quad \text{for all } j = 1,...,n \quad \text{and under Assumption 4, } x_j(p_*^n) \quad \text{and } x_j^* \quad \text{differs for no more than } m \quad \text{values of } j$. It states that, under Assumption 4, if one uses the optimal dual solution $p_*^n$ in the thresholding rule, the obtained solution will no greater than the primal optimal solution and they will be different for at most $m$ entries.

Proof. See Lemma 1 in (Agrawal et al., 2014).

As for the performance measure, we use the same notations as in Section 3.1. The expected optimality gap

$$\delta_n^D = R_*^n - \mathbb{E}[R_n]$$

Throughout this section, the expectation is always taken with respect to a random permutation on the data set $D$, unless otherwise stated. Given the data set $D$, $R_*^n$ is a deterministic quantity, so it is unnecessary to take an expectation for it. This also underlines the key difference between the stochastic input model and the random permutation model. That is, the randomness arises from the data (the LP input) in the stochastic input model, whereas it arises from the ordering of the data in the random permutation model. Define regret as the worst-case optimality gap

$$\delta_n = \sup_{D \in \Xi_D} \delta_n^D$$

where $\Xi_D$ denotes all the data sets that satisfy Assumption 3 (b) and Assumption 4. In this way, the regret quantifies the worst-case performance of the algorithm for all possible inputs data $D$. 

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4.2 Algorithm Analyses

First, the following lemma states that the boundedness property of the dual price remains the same as in the stochastic input model. Its proof is identical to the stochastic input model, since the proof of Lemma 1 only relies on the boundedness assumption on \((r_j, a_j)\)'s but not the statistical assumption about the data generation.

Lemma 3. Under Assumption 3 and Assumption 4, we have

\[
\|p^*_n\|_2 \leq \frac{\hat{r}}{d},
\]
\[
\|p_t\|_2 \leq \frac{2\hat{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}).
\]

with probability 1 for all \(t\), where \(p_t\)'s are specified by Algorithm 1.

To facilitate our derivation, we define a scaled version of the primal LP (P-LP),

\[
\begin{align*}
\max & \sum_{j=1}^{s} r_j x_j \\
\text{s.t.} & \sum_{j=1}^{s} a_{ij} x_j \leq \frac{sb_i}{n} \\
& 0 \leq x_j \leq 1 \text{ for } j = 1, \ldots, s.
\end{align*}
\]

for \(s = 1, \ldots, n\). Denote its optimal objective value as \(R^*_s\). The following lemma relates \(R^*_s\) with \(R^*_n\).

Lemma 4. For \(s > \max\{16\bar{a}^2, e^{16\bar{a}^2}, e\}\), the following inequality holds

\[
\frac{1}{s} \mathbb{E}[R^*_s] \geq \frac{1}{n} R^*_n - \frac{m\hat{r}}{n} - \frac{\hat{r}\log s}{d\sqrt{s}} - \frac{m\hat{r}}{s}. \tag{3}
\]

for all \(s \leq n \in \mathbb{N}^+\) and \(\mathcal{D} \in \Xi_D\).

Proof. See Section A4. □

Intuitively, in the random permutation model, the observations \(\{(r_j, a_j)\}_{j=1}^{s}\) collected until time \(s\) are less informative to infer the future observations than the case of the stochastic input model. However, Lemma 4 tells that the scaled LP \((s-S-LP)\) constructed based on the first \(s\) observations will achieve a similar expected optimal objective value (after scaling) compared with the original problem with all \(n\) observations. Note that \(\mathbb{E}[R^*_s]/s = \mathbb{E}[R^*_n]/n\) is evidently true in the stochastic input model, where the expectation is taken with respect to the distribution \(\mathcal{P}\). The additional terms on the right-hand-side of (3) captures the information toll for the assumption relaxation from the stochastic input model to the random permutation model. This lemma bridges the gap between past and future observations
in the random permutation model, i.e., what one can tell about the future samples based on the past observations. Comparatively, this gap between past and future observations is taken care by the sampling from same distribution $P$ in the stochastic input model.

The regret analyses in Theorem 2 builds on Lemma 4. The idea is that if $p_{s+1}$ from Algorithm 1 is a reasonably good dual solution to the scaled LP ($s$-S-LP), and plus that $E[R^*_n]/s \approx R^*_n/n$, $p_{s+1}$ should also be a good dual solution for the rest of inputs, and specifically for the upcoming sample $(r_{s+1}, a_{s+1})$.

Theorem 2. Under Assumption 3 and 4, the regret and expected constraint violation of Algorithm 1 satisfy

$$R^*_n - E[R_n] \leq O \left( \sqrt{n}(m^2 + \log n) \right)$$

$$E \left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{n} a_{it}x_{it} - b_i \right) \right] \leq \left( \frac{2r + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right) \sqrt{n}.$$

for all $n \in \mathbb{N}^+$ and $D \in \Xi_D$.

Proof. See Section A5.

Compared to the stochastic input model, the regret upper bound under random permutation model contains an extra term of $O(n \log n)$, while the constraint violation in two models enjoys the same upper bound. Moreover, Lemma 4 and Theorem 2 do not require the non-negativeness assumption of the LP input. As far as we know, this is the first online LP analysis under random permutation model without the non-negativeness assumption. In this light, the proof methodology of these two theorems contributes to the literature on the online LP problem under random permutation model, and similar theoretical results free of the non-negativeness assumption can be derived for algorithms in (Agrawal et al., 2014; Kesselheim et al., 2014).

5 Multi-dimensional Integer Linear Program

In this section, we discuss a multi-dimensional extension of (ILP)

$$\text{max } \sum_{j=1}^{n} r_j^\top x_j \quad \text{(Multi-ILP)}$$

$$\text{s.t. } \sum_{j=1}^{n} A_j x_j \leq b$$

$$1^\top x_j \leq 1, \quad x_j \in \{0,1\}^k, \quad j = 1, \ldots, n$$

where $r_j = (r_{j1}, \ldots, r_{jk}) \in \mathbb{R}^k$, $A_j = (a_{j1}, \ldots, a_{jk}) \in \mathbb{R}^{m \times k}$, and $a_{jl} = (a_{l1j}, \ldots, a_{lj})^\top$, for $j = 1, \ldots, n$ and $l = 1, \ldots, k$. The decision variables are $x = (x_1, \ldots, x_n)$ where $x_j = (x_{j1}, \ldots, x_{jk})$ for $j = 1, \ldots, n$. The right-hand-side capacity $b = (b_1, \ldots, b_m)$ is the same as the one-dimensional setting (ILP). The
formulation is called as multi-dimensional because the binary decision variable $x_j$ in (ILP) is replaced with a vector $x_j \in \{0,1\}^k$. It covers a wider range of applications than the previous setting, including adwords problem (Mehta et al., 2005), generalized assignment problem (Conforti et al., 2014), resource allocation problem (Asadpour et al., 2019), etc.

Algorithm 2 is a natural generalization of Algorithm 1 in the multi-dimensional setting. The idea is to maintain a dual price as Algorithm 1, and then to use the dual price to identify the most profitable dimension for each order. The decision of $x_t$ (Step 7 in Algorithm 2) arises from the complementarity condition of (Multi-ILP). Accordingly, Assumption 5 generalizes the stochastic input and random permutation assumptions in the previous sections.

**Algorithm 2** Simple Online Algorithm for Multi-dimensional ILP

1: Input: $d$
2: Initialize $p_1 = 0$
3: for $t = 1, \ldots, n$ do
4:   Set $v_t = \max_{l=1,\ldots,k} r_{tl} - a_{tl}^T p_t$
5:   if $v_t > 0$ then
6:      Pick an index $l_t$ randomly from the non-empty set
7:            \{$l : r_{tl} - a_{tl}^T p_t = v_t \}$
8:      Set $x_{tl} = \begin{cases} 1, & l = l_t \\ 0, & \text{otherwise} \end{cases}$
9:   else
10:      Set $x_t = 0$
11: end if
12: Compute
13: \[ p_{t+1} = p_t + \frac{1}{\sqrt{t}} (A_t x_t - d) \]
14: \[ p_{t+1} = p_{t+1} \lor 0 \]
15: end for
16: Output: $x = (x_1, \ldots, x_n)$

**Assumption 5.** We assume

(a) (Stochastic Input). The column-coefficient pair $(r_j, A_j)$’s are i.i.d. sampled from an unknown distribution $\mathcal{P}$.

(a') (Random Permutation). The column-coefficient pair $(r_j, A_j)$ arrives in a random order. The problem is in a general position; $x(p^*_n)$ and $x^*$ differs for no more than $m$ values of $t$.

(b) There exist constants $\bar{r}$ and $\bar{a}$ such that $|r_j| \leq \bar{r}$ and $\|A_j\|_\infty \leq \bar{a}$ for $j = 1, \ldots, n$.

(c) The right-hand-side $b = nd$ and there exist positive constants $\underline{d}$ and $\bar{d}$ such that $\underline{d} \leq d_i \leq \bar{d}$ for $i = 1, \ldots, m$. 

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Theorem 3. Under the stochastic input and random permutation model in Assumption 5, the regret and constraint violation of Algorithm 2 are the same as Theorem 1 and Theorem 2, respectively.

Theorem 3 states the regret and constraint violation of Algorithm 2 are the same as the previous one-dimensional setting and in particular, not dependent on the dimension \( k \) of \( x_t \)’s.

6 Algorithm Discussion

6.1 Feasible Algorithm

As discussed in the previous sections, Algorithm 1 and Algorithm 2 may output solutions that are not feasible. Here we present algorithm 3, a natural variant of the algorithm that outputs feasible solutions. Compared with Algorithm 1, Algorithm 3 sets \( x_t = 1 \) only when the constraints permit. This design is more aligned with the online LP algorithms that guarantees feasibility. Li and Ye (2019) provided a regret analysis framework for this type of feasible algorithms, and the key is to analyze the stopping time of constraint violation and the remaining resources for binding constraints. In this paper, the assumptions on \((r_j, a_j)\) are parsimonious and they seem not sufficient enough to derive an upper bound on these two key quantities. Numerically, we observe that this feasible algorithm, in comparison with Algorithm 1, will not compromise the performance in terms of the regret. We will elaborate more on its numerical performance in the next section and leave the regret analysis of this algorithm as an open question.

Algorithm 3 Simple Feasible Algorithm

1: Input: \( d \)
2: Initialize \( p_1 = 0 \)
3: for \( t = 1, \ldots, n \) do
4: \( \hat{x}_t = \begin{cases} 1, & r_t > a_t^\top p_t \\ 0, & r_t \leq a_t^\top p_t \end{cases} \)
5: Compute
6: \( p_{t+1} = p_t + \frac{1}{\sqrt{t}} (a_t x_t - d) \)
7: \( p_{t+1} = p_{t+1} \vee 0 \)
8: end for
9: Output: \( x = (x_1, \ldots, x_n) \)

6.2 Nonstationary Algorithm

We consider another variant of the algorithm that take into account the resource consumption while doing the subgradient descent. The intuition is similar to the action-history-dependent algorithm in (Li...
and Ye, 2019). If too much resources are consumed in the early periods, the remaining resource \( b_t \) will shrink, and this nonstationary algorithm will accordingly push up the dual price and be more inclined to reject an order. On the contrary, if we happen to reject a lot orders at the beginning and it results in too much remaining resources, the algorithm will lower down the dual price so as to accept more orders in the future. In numerical experiments (next section), this nonstationary algorithm performs better, but it is still on the same order of regret and constraint violation as Algorithm 1. The open question then is if there exists a first-order algorithm that is free of re-solving any linear programs and could achieve \( O(\log n) \) regret, possibly under stronger statistical assumptions.

**Algorithm 4** Simple Nonstationary Algorithm

1. Input: \( d \)
2. Initialize \( p_1 = 0, b_0 = b \)
3. for \( t = 1, \ldots, n \) do
4. Set
   \[
   x_t = \begin{cases} 
   1, & r_t > a_t^\top p_t \\
   0, & r_t \leq a_t^\top p_t 
   \end{cases}
   \]
5. Update
   \[
   b_t = b_{t-1} - a_t x_t
   \]
6. Compute
   \[
   p_{t+1} = p_t + \frac{1}{\sqrt{t}} \left( a_t x_t - \frac{b_t}{n-t} \right)
   \]
   \[
   p_{t+1} = p_{t+1} \vee 0
   \]
7. end for
8. Output: \( x = (x_1, \ldots, x_n) \)

### 6.3 Step Size

From the perspective of stochastic gradient descent, Algorithm 1 adopts a step size of \( 1/\sqrt{t} \). Some algorithms in the literature (Moulines and Bach, 2011; Lacoste-Julien et al., 2012) employed a step size of \( 1/t^\alpha \) for \( \alpha \in [1/2, 1] \) which commonly requires a strong convexity assumption. For the problem of online LP or integer LP, even if we can enforce a strong convexity assumption, it might be unrealistic to assume the knowledge of the strong convexity constant (necessary for a smaller step size) as a priori. There are also discussions on the usage of averaging method (Xiao, 2010; Juditsky and Nesterov, 2014) or an adaptive approach for choosing the step size (Flammarion and Bach, 2015; Lei and Jordan, 2019) for stochastic gradient descent. We leave it as an open question whether these designs will result in better online LP algorithms.
7 Numerical Experiments

The first experiment compares the performance of Algorithm 1, Algorithm 3, and Algorithm 4 in terms of regret and constraint violation. In this experiment, $m = 10$, $a_{ij}$’s and $r_j$’s are sampled i.i.d. from Unif[0, 2]. For each value of $n$, we run 100 simulation trials and in each trial, $d_i$’s are sampled i.i.d. from Unif[1/3, 2/3]. The average regret and constraint violation over all the simulation trials are shown in Figure 1. In the left panel, we plot the values of the average regret divided by $\sqrt{n}$. We observe that the non-stationary algorithm (Algorithm 4) performs better than the simple algorithm (Algorithm 1). Also, the feasible algorithm (Algorithm 3) guarantees feasibility, i.e. zero constraint violation; it produces slightly larger regret, but the regret is still on the order of $\sqrt{n}$. For Algorithm 1 and Algorithm 4, the regret might be negative because the constraint violation potential brings an objective value larger than the true optimal.

In the second experiment (Figure 2), we compare the three algorithms in a setting where the boundedness of the support of distribution $\mathcal{P}$ is violated. Specifically, $m = 10$, $a_{ij}$’s are generated i.i.d. from $\mathcal{N}(1, 1)$ and $r_j = \sum_{i=1}^{m} a_{ij} - \epsilon_j$ where $\epsilon_j \sim \text{Unif}(0, m)$. For each value of $n$, we run 100 simulation trials, and in each trial, $d_i$’s are sampled i.i.d. from Unif[1/3, 2/3]. In this experiment, the regret performances of Algorithm 1 and 4 are quite close to each other, while Algorithm 4 still performs better in respect with constraint violation. The feasible algorithm (Algorithm 3) still achieves regret on the order of $\sqrt{n}$. Note that our theoretical results, also all the previous literature on online LP problem, rely on the boundedness assumption for the LP input. An open question is how to generalize these results to the case when the distribution $\mathcal{P}$ has an unbounded support.

The third experiment (Figure 3) presents a negative result on all three algorithms. Specifically, $a_{ij}$’s are generated i.i.d. from Cauchy(1, 1) and $r_j = \sum_{i=1}^{m} a_{ij} - \epsilon_j$ where $\epsilon_j \sim \text{Unif}(0, m)$. As before, for each value of $n$, we run 100 simulation trials, and in each trial, $d_i$’s are sampled i.i.d. from Unif[1/3, 2/3]. We
observe that the performance is quite unstable for all three algorithms. This empirical finding suggests that a light-tail distribution may be necessary for an online LP algorithm to work.

Next, we illustrate the dependence of regret on $m$ with the same setup as the first experiment. We fix $n = 10000$ and change the value of $m$. From the results presented in Figure 4, we have two observations. First, the regret grows sub-linearly with $m$. This growth rate might be specific to this problem but it is an interesting question that how the structure of the LP input affect the dependence of regret on $m$. Also, we notice that under this experiment setup, the constraint violation is not dependent on $m$.

Figure 5 presents the algorithm performance under random permutation model. We first generate four groups of data with equal size from four different distributions and then combine these groups as the LP input: (i) $a_{ij}$’s are generated from $\text{Unif}[0, 2]$; (ii) $a_{ij}$ are generated from $\mathcal{N}(1, 1)$; (iii) $a_{ij}$ are generated from $\mathcal{N}(0, 1)$; (iv) $a_{i,j}$ are generated from uniform distribution on $\{-1, 1, 3\}$. $r_j$’s for all four groups are generated from $\text{Unif}[0, 1]$. Note this data set can not be generated from any distribution in the stochastic
input model. For each value of \( n \), we run 100 simulation trials, and in each trial, \( d_i \)'s are sampled i.i.d. from \text{Unif}[1/3, 2/3]. In this experiment, we observe that the three algorithms all achieve \( O(\sqrt{n}) \) regret. Algorithm 1 and 4 achieve \( O(\sqrt{n}) \) constraint violation but Algorithm 4 features for a smaller constant. Also, we emphasize that for all the experiments except for the Cauchy one, the constants before \( \sqrt{n} \) are usually small.

![Figure 4: Experiment with Uniform i.i.d. input](image1)

![Figure 5: Experiment with random permuted input](image2)

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Appendix

A1 Concentration Inequalities under Random Permutation

**Lemma 5.** Let $U_1, ..., U_n$ be a random sample without replacement from the real numbers $\{c_1, ..., c_N\}$. Then for every $s > 0$,

$$
\mathbb{P}(|U_n - \bar{c}_N| \geq s) \leq \begin{cases} 
2 \exp \left( -\frac{2ns^2}{\Delta_N^2} \right) & \text{(Hoeffding)}, \\
2 \exp \left( -\frac{2ns^2}{(1-(n-1)/N)\Delta_N^2} \right) & \text{(Serfling)}, \\
2 \exp \left( -\frac{ns^2}{2\sigma_N^2} \right) & \text{(Hoeffding-Bernstein)}, \\
2 \exp \left( -\frac{ns^2}{m\sigma_N^2} \right) & \text{if $N = mn$ (Massart)},
\end{cases}
$$

where $\bar{c}_N = \frac{1}{N} \sum_{i=1}^{N} c_i$, $\sigma_N^2 = \frac{1}{N} \sum_{i=1}^{N} (c_i - \bar{c}_N)^2$ and $\Delta_N = \max_{1 \leq i \leq N} c_i - \min_{1 \leq i \leq N} c_i$.

**Proof.** See Theorem 2.14.19 in van der Vaart (1996).

A2 Proof of Lemma 1

**Proof.** For $p^*$, the optimal solution of (SP), we have

$$
d\|p^*\|_1 \leq d^T p^* \overset{(a)}{\leq} Er \leq \bar{r},
$$

where inequality (a) is due to that if otherwise, $p^*$ cannot be the optimal solution because it will give a larger objective value of $f(p)$ than setting $p = 0$. Given the non-negativeness of $p^*$, we have $\|p^*\|_2 \leq \|p^*\|_1$. So we obtain the first inequality in the lemma.
For $p_t$ specified by Algorithm 1, we have,

$$
\|p_{t+1}\|_2^2 \leq \left\| p_t + \frac{1}{\sqrt{t}} (a_t x_t - d) \right\|_2^2
$$

$$
= \|p_t\|_2^2 + \frac{1}{t} \|a_t x_t - d\|_2^2 + \frac{2}{\sqrt{t}} (a_t x_t - d)^\top p_t
$$

$$
\leq \|p_t\|_2^2 + \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2}{\sqrt{t}} a_t^\top p_t x_t - \frac{2}{\sqrt{t}} d^\top p_t
$$

where the first inequality comes from the projection (into the non-negative cone) step in the algorithm.

Note that

$$
a_t^\top p_t x_t = a_t^\top p_t I (r_t > a_t^\top p_t) \leq r_t \leq \bar{r}.
$$

Therefore,

$$
\|p_{t+1}\|_2^2 \leq \|p_t\|_2^2 + \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2\bar{r}}{\sqrt{t}} - \frac{2}{\sqrt{t}} d^\top p_t,
$$

and it holds with probability 1.

Next, we establish that when $\|p_t\|_2$ is large enough, then it must hold that $\|p_{t+1}\|_2 \leq \|p_t\|_2$. Specifically, when $\|p_t\|_2 \geq \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d}$, we have

$$
\|p_{t+1}\|_2 - \|p_t\|_2 \leq \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2\bar{r}}{\sqrt{t}} - \frac{2}{\sqrt{t}} d^\top p_t
$$

$$
\leq \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2\bar{r}}{\sqrt{t}} - \frac{d}{\sqrt{t}} \|p_t\|_1
$$

$$
\leq \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2\bar{r}}{\sqrt{t}} - \frac{d}{\sqrt{t}} \|p_t\|_2
$$

$$
\leq 0.
$$

On the other hand, when $\|p_t\|_2 \leq \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d}$,

$$
\|p_{t+1}\|_2 \leq \left\| p_t + \frac{1}{\sqrt{t}} (a_t x_t - d) \right\|_2
$$

$$
\leq \|p_t\|_2 + \frac{1}{\sqrt{t}} \|a_t x_t - d\|_2
$$

$$
\leq \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d})
$$

where (b) comes from the triangle inequality of the norm.

Combining these two scenarios with the fact that $p_1 = 0$, we have

$$
\|p_t\|_2 \leq \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d})
$$

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for \( t = 1, \ldots, n \) with probability 1.

\[ \square \]

A3 Proof of Theorem 1

Proof. First, the primal optimal objective value is no greater than the dual objective with \( p = p^* \). Specifically,

\[ R^*_n = P-LP = D-LP \leq nd^T p^* + \sum_{j=1}^{n} (r_j - a_j^T p^*)^+ . \]

Taking expectation on both sides,

\[ \mathbb{E}[R^*_n] \leq \mathbb{E}[nd^T p^* + \sum_{t=1}^{n} (r_t - a_t^T p^*)^+] \leq nf(p^*). \]

Thus, the optimal objective value of the stochastic program (by a factor of \( n \)) is an upper bound for the expected value of the primal optimal objective. Hence

\[ \mathbb{E}[R^*_n - R_n] \leq nf(p^*) - \sum_{j=1}^{n} \mathbb{E}[r_t I(r_t > a_t^T p_t)] \]

\[ \leq \sum_{t=1}^{n} \mathbb{E}[f(p_t)] - \sum_{t=1}^{n} \mathbb{E}[r_t I(r_t > a_t^T p_t)] \]

\[ \leq \sum_{t=1}^{n} \mathbb{E}[d_t^T p_t + (r_t - a_t^T p_t)^+ - r_t I(r_t > a_t^T p_t)] \]

\[ = \sum_{t=1}^{n} \mathbb{E}[(d_t - a_t I(r_t > a_t^T p_t))^T p_t]. \]

where the expectation is taken with respect to \((r_t, a_t)\)'s. In above, the second line comes from the optimality of \( p^* \), while the third line is valid because of the independence between \( p_t \) and \((r_t, a_t)\).

The importance of the above inequality lies in that it relates and represents the primal optimality gap in the dual prices \( p_t \) – which is the core of Algorithm 1. From the updating formula in Algorithm 1, we know

\[ \|p_{t+1}\|_2^2 \leq \|p_t\|_2^2 - \frac{2}{\sqrt{t}} \left( d - a_t I(r_t > a_t^T p_t) \right)^T p_t + \frac{1}{t} \|d - a_t I(r_t > a_t^T p_t)\|_2^2 \]

\[ \leq \|p_t\|_2^2 - \frac{2}{\sqrt{t}} \left( d - a_t I(r_t > a_t^T p_t) \right)^T p_t + \frac{m(\bar{a} + \bar{d})^2}{t}. \]
Moving the cross-term to the right-hand-side, we have

\[
\sum_{t=1}^{n} (d - a_t I(r_t > a_t^T p_t))^\top p_t \leq \sum_{t=1}^{n} \left( \sqrt{t}\|p_t\|_2^2 - \sqrt{t}\|p_{t+1}\|_2^2 + \frac{m(\bar{a} + \bar{d})^2}{\sqrt{t}} \right)
\leq m(\bar{a} + \bar{d})^2\sqrt{n} + \sum_{t=1}^{n} (\sqrt{t + 1} - \sqrt{t})\|p_t\|_2^2
\leq m(\bar{a} + \bar{d})^2\sqrt{n} + \sum_{t=1}^{n} (\sqrt{t + 1} - \sqrt{t}) \left( \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right)^2
= 2 \left( \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right)^2 \sqrt{n}.
\]

Consequently,

\[
\mathbb{E}[R_n^* - R_n] \leq 2 \left( \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right)^2 \sqrt{n}
\]

hold for all \( n \) and distribution \( P \in \Xi \).

For the constraint violation, if we revisit the updating formula, we have

\[
p_{t+1} \overset{(c_1)}{=} p_t + \frac{1}{\sqrt{t}} (a_t x_t - d)
\]

where the inequality is elementwise. Therefore,

\[
\sum_{t=1}^{n} a_t x_t \overset{(c_2)}{\leq} nd + \sum_{t=1}^{n} \sqrt{t}(p_t - p_{t+1})
\leq b - \sqrt{n}p_{n+1} + \sum_{t=2}^{n} \frac{p_t}{\sqrt{t} + \sqrt{t - 1}}
\sum_{i=1}^{m} \left( \sum_{t=1}^{n} a_{it} x_{it} - b_i \right) \overset{(d)}{\leq} \left( \frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d}) \right) \sqrt{n}.
\]

In the second line, we remove the term involve \( p_1 \) with the algorithm specifying \( p_1 = 0 \). We point out the intuition that this bound on the constraint violation is far from tight. First, for binding constraints, intuitively, \( p_{t+1} \) would be positive with high probability; this makes the inequality (d) not tight and thus curbs the constraint violation for binding. Second, for non-binding constraints, the inequalities (c1) and (c2) are most likely not tight due to the projection of \( p_t \)'s (to non-negative cone). Consequently, the constraints violation for non-binding constraints might not be large as well. \( \square \)
Proof. Define $SLP(s, b_0)$ as the following LP

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{s} r_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{s} a_{ij} x_j \leq \frac{sb_i}{n} + b_{0i} \\
& 0 \leq x_j \leq 1 \quad \text{for } j = 1, ..., s.
\end{align*}
\]

where $b_0 = (b_{01}, ..., b_{0m})$ denotes the constraint relaxation quantity for the scaled LP. Denote the optimal objective value of $SLP(s, b_0)$ as $R^*(s, b_0)$. Also, denote $x(p) = (x_1(p), ..., x_n(p))$ and $x_j(p) = I(r_j > a_j^T p)$. It denotes the decision variables we obtain with a dual price $p$.

We prove the following three results:

(i) The following bounds hold for $R^*_n$,

\[
\sum_{j=1}^{n} r_j x_j(p_n^*) \leq R^*_n \leq \sum_{j=1}^{n} r_j x_j(p_n^*) + m\bar{r}.
\]

(ii) When $s \geq \max\{16\bar{a}^2, \exp\{16\bar{a}^2\}, e\}$, then the optimal dual solution $p_n^*$ is a feasible solution to $SLP\left(s, \frac{\log s}{\sqrt{s}} 1\right)$ with probability no less than $1 - \frac{m}{s}$.

(iii) The following inequality holds for the optimal objective values to the scaled LP and its relaxation

\[
R^*_s \geq R^*\left(s, \frac{\log s}{\sqrt{s}} 1\right) - \frac{\bar{r}\sqrt{s}\log s}{d}.
\]

For part (i), this inequality replace the optimal value with bounds containing the objective values obtained by adopting optimal dual solution. The left hand side of the inequality comes from the complementarity condition while the right hand side can be shown from Lemma 2.

For part (ii), the motivation to introduce a relaxed form of the scaled LP is to ensure the feasibility of $p_n^*$. The key idea for the proof is to utilize the feasibility of $p_n^*$ for (P-LP). To see that, let $\alpha_{ij} = a_{ij} I(r_j > a_j^T p^*)$ and

\[
\begin{align*}
\epsilon_\alpha &= \max_{i,j} \alpha_{ij} - \min_{i,j} \alpha_{ij} \leq 2\bar{a}, \\
\bar{\alpha}_i &= \frac{1}{n} \sum_{j=1}^{n} \alpha_{ij} = \frac{1}{n} \sum_{j=1}^{n} a_{ij} x_j(p_n^*) \leq d_i, \\
\sigma_i^2 &= \frac{1}{n} \sum_{j=1}^{n} (\alpha_{ij} - \bar{\alpha}_i)^2 \leq 4\bar{a}^2.
\end{align*}
\]
Here the first and third inequality comes from the bounds on $a_{ij}$'s while the second one comes from the feasibility of the optimal solution for (P-LP).

Then, when $k > \max\{16\bar{a}_2^2, \exp\{16\bar{a}_2^2\}, e\}$, by applying Hoeffding-Bernstein's Inequality

$$
P\left(\sum_{j=1}^{k} \alpha_{ij} - kd_i \geq \sqrt{k \log k}\right) \leq P\left(\sum_{j=1}^{k} \alpha_{ij} - k\bar{\alpha}_i \geq \sqrt{k \log k}\right)
$$

$$(f) \leq \exp\left(-\frac{k \log^2 k}{8k\bar{a}_2^2 + 2\bar{a}\sqrt{k \log k}}\right)
$$

$$(g) \leq \frac{1}{k}
$$

for $i = 1, \ldots, m$. Here inequality (e) comes from (4), (f) comes from applying Lemma 5, and (g) holds when $s > \max\{16\bar{a}_2^2, \exp\{16\bar{a}_2^2\}, e\}$.

Let event

$$E_i = \left\{\sum_{j=1}^{s} \alpha_{ij} - sd_i < \sqrt{s \log s}\right\}
$$

and $E = \bigcap_{i=1}^{m} E_i$. The above derivation tells $P(E_i) \geq 1 - \frac{1}{s}$ By applying union bound, we obtain $P(E) \geq 1 - \frac{m}{s}$ and it completes the proof of part (ii).

For part (iii), denote the optimal solution to SLP $\left(s, \frac{\log s}{\sqrt{s}} \mathbf{1}\right)$ as $\tilde{p}_s$.

$$R^* \left(s, \frac{\log s}{\sqrt{s}} \mathbf{1}\right) = s \left(d + \frac{\log s}{\sqrt{s}} \mathbf{1}\right)^\top \tilde{p}_s^* + \sum_{j=1}^{s} (r_j - a_\top \tilde{p}_s^*)^+
$$

$$\leq s \left(d + \frac{\log s}{\sqrt{s}} \mathbf{1}\right)^\top \tilde{p}_s^* + \sum_{j=1}^{s} (r_j - a_\top p_s^*)^+
$$

$$\leq \frac{\tilde{r} \sqrt{s \log s}}{d} + R_s^*/$

where the first inequality comes from dual optimality of $\tilde{p}_s^*$ and the second inequality comes from the upper bound of $\|p_s^*\|$ and the duality of the scaled LP $R_s^*$. Therefore,

$$R_s^* \geq R^* \left(s, \frac{\log s}{\sqrt{s}} \mathbf{1}\right) - \frac{\tilde{r} \sqrt{s \log s}}{d}.
$$

Finally, we complete the proof with the help of the above three results.

$$\frac{1}{s} \mathbb{E} \left[\mathbb{I}_E R_s^* \right] \geq \frac{1}{s} \mathbb{E} \left[\mathbb{I}_E R^* \left(s, \frac{\log s}{\sqrt{s}} \mathbf{1}\right) \right] - \frac{\tilde{r} \sqrt{s \log s}}{d}
$$

$$\geq \frac{1}{s} \mathbb{E} \left[\mathbb{I}_E \sum_{j=1}^{s} r_j x_j(p^*) \right] - \frac{\tilde{r} \sqrt{s \log s}}{d}
$$

where $\mathbb{I}_E$ denotes an indicator function for event $E$. The first line comes from applying part (iii) while
the second line comes from the feasibility of \( p^* \) on event \( E \). Then,

\[
\frac{1}{n} \mathbb{E} [R^*_n] \geq \frac{1}{s} \mathbb{E} \left[ \sum_{j=1}^{s} r_j x_j (p^*) \right] - \frac{\bar{r} \sqrt{s} \log s}{d} - \frac{m \bar{r}}{s}
\]

\[
= \frac{1}{n} \mathbb{E} \left[ \sum_{j=1}^{n} r_j x_j (p^*) \right] - \frac{\bar{r} \sqrt{s} \log s}{d} - \frac{m \bar{r}}{s}
\]

\[
\geq \frac{1}{n} R^*_n - \frac{\bar{r} \sqrt{s} \log s}{d} - \frac{m \bar{r}}{s} - \frac{m \bar{r}}{n}
\]

where the first line comes from part (ii) – the probability bound on event \( E \), the second line comes from the symmetry of the random permutation probability space, and the third line comes from part (i). We complete the proof.

A5 Proof of Theorem 2

Proof. For the regret bound,

\[
R^*_n - \mathbb{E} [R_n] = R^*_n - \sum_{t=1}^{n} \mathbb{E} [r_t x_t]
\]

where \( x_t \)'s are specified according to Algorithm 1. Then

\[
R^*_n - \mathbb{E} [R_n] = R^*_n - \sum_{t=1}^{n} \frac{1}{t} \mathbb{E} [R^*_t] + \sum_{t=1}^{n} \frac{1}{t} \mathbb{E} [R^*_t] - \sum_{t=1}^{n} \mathbb{E} [r_t x_t]
\]

\[
= \sum_{t=1}^{n} \left( \frac{1}{n} R^*_n - \frac{1}{t} \mathbb{E} [R^*_t] \right) + \sum_{t=1}^{n} \mathbb{E} \left[ \frac{1}{n+1-t} \hat{R}^*_n - r_t x_t \right]
\]

(5)

where \( \hat{R}^*_n - t+1 \) is defined as the optimal value of the following LP

\[
\begin{align*}
\text{max} & \quad \sum_{j=t}^{n} r_j x_j \\
\text{s.t.} & \quad \sum_{j=t}^{n} a_{ij} x_j \leq \frac{(n-t+1) b_i}{n} \\
& \quad 0 \leq x_j \leq 1 \quad \text{for } j = 1, ..., m.
\end{align*}
\]

For the first part of (5), we can apply Lemma 4. Meanwhile, the analyses of the second part takes a similar form as the previous stochastic input model. Specifically,

\[
\mathbb{E} \left[ \frac{1}{n+1-t} \hat{R}^*_n - t+1 - r_t x_t \right] \leq (d - a_i I(r_t > a_i^\top p_t))^\top p_t.
\]
Similar to the stochastic input model,
\[
\|p_{t+1}\|_2^2 \leq \|p_t\|_2^2 - \frac{2}{\sqrt{t}} (d - a_t I(r_t > a_t^\top p_t))^\top p_t + \frac{1}{t} \|d - a_t I(r_t > a_t^\top p_t)\|_2^2
\]
\[
\leq \|p_t\|_2^2 - \frac{2}{\sqrt{t}} (d - a_t I(r_t > a_t^\top p_t))^\top p_t + \frac{m(\bar{a} + \bar{d})^2}{t}.
\]

Thus, we have
\[
\sum_{t=1}^n \mathbb{E}\left[ (d - a_t I(r_t > a_t^\top p_t))^\top p_t \right] \leq \sum_{t=1}^n \mathbb{E}\left[ \sqrt{t}(\|p_t\|_2^2 - \|p_{t+1}\|_2^2) \right] + \sum_{t=1}^n \frac{m(\bar{a} + \bar{d})^2}{\sqrt{t}}
\]
\[
\leq m(\bar{a} + \bar{d})^2 \sqrt{n} + \sum_{t=1}^n (\sqrt{t+1} - \sqrt{t})\|p_t\|_2^2
\]
\[
\leq m(\bar{a} + \bar{d})^2 \sqrt{n} + \left(\frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d})\right)^2 \sqrt{n}.
\]

Combine two parts above, finally we have
\[
R_n^* - \mathbb{E}[R_n(\pi)] \leq m\bar{r} + \bar{r} \log n \sqrt{n} + m\bar{r} \log n + \frac{\max\{16\bar{a}^2, \exp\{16\bar{a}^2\}, e\}\bar{r}}{n}
\]
\[
+ m(\bar{a} + \bar{d})^2 \sqrt{n} + \left(\frac{2\bar{r} + m(\bar{a} + \bar{d})^2}{d} + m(\bar{a} + \bar{d})\right)^2 \sqrt{n}.
\]

Thus, we complete the proof for the regret. The proof for the constraint violation part follows exactly the same way as the stochastic input model. \(\square\)

A6 Proof for Theorem 3

Proof. The proof follows mostly the proof of Theorem 1 and Theorem 2. We only highlight the difference here. First, the sample average approximation form of the dual problem takes a slightly different form but it is still convex in \(p\).

\[
\min_p f_n(p) = d^\top p + \frac{1}{n} \sum_{j=1}^{n} \max_{s=1,\ldots,k} \left\{ r_{js} - a_j^\top p \right\}^+ \quad \text{(multi-D-SAA)}
\]
\[
\text{s.t. } p \geq 0.
\]

The updating formula for \(p_t\) is different but we can achieve the same relation between \(p_t\) and \(p_{t+1}\).
At time $t$, if $\max_{i=1,\ldots,k} \{ r_{ijl} - \bar{a}_{jl}^\top \bar{p} \} > 0$, we have

$$
\| p_{t+1} \|_2^2 \leq \| p_t + \frac{1}{\sqrt{t}} (A_t x_t - d) \|_2^2 \\
= \| p_t + \frac{1}{\sqrt{t}} (a_{t\ell_t} - d) \|_2^2 \\
= \| p_t \|_2^2 + \frac{1}{t} \| a_{t\ell_t} x_t - d \|_2^2 + \frac{2}{\sqrt{t}} (a_{t\ell_t} x_t - d)^\top p_t \\
\leq \| p_t \|_2^2 + \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2}{\sqrt{t}} a_{t\ell_t} p_{t\ell_t} - \frac{2}{\sqrt{t}} d^\top p_t \\
\leq \| p_t \|_2^2 + \frac{2r}{\sqrt{t}} - \frac{2}{\sqrt{t}} d^\top p_t,
$$

while if $\max_{i=1,\ldots,k} \{ r_{ijl} - \bar{a}_{jl}^\top \bar{p} \} \leq 0$, we have

$$
\| p_{t+1} \|_2^2 \leq \| p_t + \frac{1}{\sqrt{t}} (A_t x_t - d) \|_2^2 \\
= \| p_t - \frac{1}{\sqrt{t}} d \|_2^2 \\
\leq \| p_t \|_2^2 + \frac{m(\bar{a} + \bar{d})^2}{t} - \frac{2}{\sqrt{t}} d^\top p_t.
$$

Combining those two parts, we obtain

$$
\| p_{t+1} \|_2^2 \leq \| p_t \|_2^2 + \frac{m(\bar{a} + \bar{d})^2}{t} + \frac{2r}{\sqrt{t}} - \frac{2}{\sqrt{t}} d^\top p_t,
$$

which is the same formula as the one-dimensional setting. With the above results, the rest of the proof simply follows the same approach as the one-dimensional case.