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Numerical analysis of the neutron multigroup $SP_N$ equations

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Abstract

The multigroup neutron $SP_N$ equations, which are an approximation of the neutron transport equation, are used to model nuclear reactor cores. In their steady state, these equations can be written as a source problem or an eigenvalue problem. We study the resolution of those two problems with an $H^1$-conforming finite element method and a Discontinuous Galerkin method, namely the Symmetric Interior Penalty Galerkin method.

1 Introduction

The neutron transport equation describes the neutron flux density in a reactor core. It depends on 7 variables: 3 for the space, 2 for the motion direction, 1 for the energy (or the speed), and 1 for the time.

The energy variable is discretized using the multigroup theory [9]. In this method, the entire range of neutron energies is divided into $G$ intervals, called energy groups. In each energy group, the neutron flux density is lumped and all parameters are averaged. We denote by $I_G := \{1, \cdots, G\}$, the set of energy group indices.

Concerning the motion direction, the $P_N$ transport equations are obtained by developing the neutron flux on the spherical harmonics from order 0 to order

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Let us use these notations: for $E$, make the following abuse of notation:

For a set of functions $\psi$, the following space of piecewise regular functions:

We will denote by $\psi$ such that there are $\hat{N} := \frac{N+1}{2}$. We will denote by $\mathcal{I}_e$ (resp. $\mathcal{I}_o$) the subset of even (resp. odd) integers of the integer set $\{0, \cdots, N\}$.

Finally, the (motion direction and energy) discretization of the neutron flux is such that there are $\hat{N} \times G$ even and odd moments of the neutron flux.

We will denote by $\phi = ((\phi^g_m)_{m \in \mathcal{I}_e})_{g \in \mathcal{I}_G} \in \mathbb{R}^{\hat{N} \times G}$ the set of functions containing, for all energy group $g$, the even moments of the neutron flux.

Likewise, we will denote by $p = ((p^g_{x,m})_{m \in \mathcal{I}_e})_{g \in \mathcal{I}_G} \in \left(\mathbb{R}^{\hat{N} \times G}\right)^d$ the set of functions containing the odd moments of the neutron flux.

Note that while modelling the core of a pressurized water reactor, the number of groups if such that $2 \leq G \lesssim 30$, physicists usually choose $N = 1$ or $3$, more rarely $N = 5$.

## 2 Setting of the model

The reactor core is modelled by a bounded, connected and open subset $\mathcal{R}$ of $\mathbb{R}^d$, $d = 1, 2, 3$, having a Lipschitz boundary which is piecewise regular. The coefficients are piecewise regular, so that we split $\mathcal{R}$ into $\hat{N}$ open disjoint parts $(\mathcal{R}_i)_{i=1}^{\hat{N}}$ with Lipschitz, piecewise regular boundaries: $\mathcal{R} = \bigcup_{i=1}^{\hat{N}} \mathcal{R}_i$. For this reason, we will use the following space of piecewise regular functions:

$$\mathcal{PW}^{1,\infty}(\mathcal{R}) = \left\{ D \in L^\infty(\mathcal{R}) \mid \nabla D|_{\mathcal{R}_i} \in (L^\infty(\mathcal{R}_i))^d, i = 1, \cdots, \hat{N} \right\}.$$  

For a set of functions $\psi = (\phi^g_m)_{m \in G} \in \mathbb{R}^{\hat{N} \times G}$, we make the following abuse of notation: $\nabla \psi = ((\partial_x \psi^g_m)_{m \in G})_{x=1}^d \in \left(\mathbb{R}^{\hat{N} \times G}\right)^d$.

For a set of vector valued functions $q = ((g^g_{x,m})_{m \in G})_{x=1}^d \in \left(\mathbb{R}^{\hat{N} \times G}\right)^d$, we make the following abuse of notation:

$$\text{div } q = \left(\text{div } (g^g_{x,m})_{x=1}^d\right)_{m \in G}, \quad q \cdot p = \left(\sum_{x=1}^d q^g_{x,m} p^g_{x,m}\right)_{m \in G} \in \mathbb{R}^{\hat{N} \times G}.$$  

Let us use these notations: for $E \subset \mathbb{R}^d$, $L(E) = L^2(E)$; $L := L^2(\mathcal{R})$; $V := H^1_0(\mathcal{R})$; $V' := H^{-1}(\mathcal{R})$ its dual and $Q := H(\text{div }, \mathcal{R})$. For $W = L(E)$, $L$, $V$ or
The matrices \( H \) and \( k \) are defined such that the multiplication factor to the smallest eigenvalue, which in addition is simple \([7]\). In neutronics, the physical solution to Problem (3) corresponds to the eigenfunction associated with the smallest eigenvalue.

We also set \( V := (V')^{\tilde{N} \times G} \) and \( L(E) = (L(E))^{d} \) and \( L^{p}(\cdot) = (L^{p}(\cdot))^{\tilde{N} \times G} \).

Let \( q \in \left( (\tilde{R}^{\tilde{N} \times G})^{d} \right) \) and \( M \in \left( (\tilde{R}^{\tilde{N} \times \tilde{N}})^{G \times G} \right) \). We set \( q_{x} = (q_{g,m}^{e})_{m,g} \) and we use the notation \( M \cdot q = (M \cdot q_{x})^{d}_{x=1} \).

Given a source term \( S_{f} \in L \), the multigroup \( SP_{N} \) equations with zero-flux boundary conditions\(^{1}\) read as coupled diffusion-like equations set in a mixed formulation:

\[
\text{Solve in } (\phi, p) \in V \times Q \quad \left\{ \begin{array}{l}
T_{e} \phi + \vec{\nabla} (H \phi) = 0,
\text{div} p + T_{e} \phi = S_{f}.
\end{array} \right.
\]

When \( S_{f} \) depends on \( \phi \), the steady state multigroup \( SP_{N} \) equations read as the following generalized eigenproblem:

\[
\text{Solve in } (\lambda, \phi, p) \in \mathbb{R}^{*} \times V \times Q \quad \left\{ \begin{array}{l}
T_{e} \phi + \vec{\nabla} (H \phi) = 0,
\text{div} p + T_{e} \phi = \lambda^{-1} M_{f} \phi.
\end{array} \right.
\]

The physical solution to Problem (3) corresponds to the eigenfunction associated to the smallest eigenvalue, which in addition is simple \([7]\). In neutronics, the multiplication factor \( k_{eff} = \max \lambda \) characterizes the physical state of the core reactor: if \( k_{eff} = 1 \); the nuclear chain reaction is self-sustaining; if \( k_{eff} > 1 \); the chain reaction is diverging; if \( k_{eff} < 1 \); the chain reaction vanishes.

The matrices \( H, T_{e}, T_{o}, M_{f} \in \left( (\tilde{R}^{\tilde{N} \times \tilde{N}})^{G \times G} \right) \) are such that \( \forall (g, g') \in I_{G} \times I_{G}, \delta_{g,g'} \) is the Kronecker symbol:

- \( (H)_{g,g'} = \delta_{g,g'} H \in \tilde{R}^{\tilde{N} \times \tilde{N}} \), with \( \forall (i, j) \in \{1, \cdots, \tilde{N}\}^{2} \), \( \tilde{H}_{i,j} = \delta_{i,j} + \delta_{i,j-1} \).
- \( (T_{e})_{g,g'} := T_{e}^{g} \in \tilde{R}^{\tilde{N} \times \tilde{N}} \) denotes the even removal matrix, such that:
  \[ T_{e}^{g} = \text{diag} (t_{0} \sigma_{g,0}^{e}, t_{2} \sigma_{e,2}^{g}, ...) \, ; \]
- \( (T_{o})_{g,g'} := T_{o}^{g} \in \tilde{R}^{\tilde{N} \times \tilde{N}} \) denotes the odd removal matrix, such that:
  \[ T_{o}^{g} = \text{diag} (t_{1} \sigma_{r,1}^{g}, t_{3} \sigma_{r,3}^{g}, ...) \, ; \]

where \( \forall m \in I_{e,o}, \sigma_{r,m}^{g} := \sigma_{r}^{g} - \sigma_{s,m}^{g} \, , \forall m > 0, t_{m} > 0 \).

The coefficient \( \sigma_{r}^{g} \) is the macroscopic total cross section of energy group \( g \), and the coefficients \( \sigma_{s,m}^{g} \) denote the \( P_{N} \) moments of the macroscopic self scattering cross sections from energy group \( g \) to itself.

\(^{1}\)ie: for \( 1 \leq g \leq G, m \in I_{e}, (\phi_{m}^{g}) |_{\partial R} = 0 \).
The variational formulation of (5) writes:

\[
-\Delta \phi = \ell(\psi),
\]

where:

\[
\begin{align*}
&c : \mathbb{V} \times \mathbb{V} \to \mathbb{R}, \\
&c(\phi, \psi) = (\nabla \phi, \nabla \psi) - (\mathbb{T}_e \phi, \psi)_L - (\mathbb{T}_o \phi, \psi)_L,
\end{align*}
\]

and

\[
\ell : \mathbb{V} \to \mathbb{R}, \quad \ell(\psi) = (S_f, \psi)_L.
\]
Theorem 1 Suppose that $\mathbb{D}$ is positive definite. For a given source term $S_f \in L$, it exists a unique $\phi \in V$ that solves Problem 6. In addition, it holds: $\|\phi\|_V \lesssim \|S_f\|_L$.

Proof: The bilinear form $c$ and the linear form $\ell$ are continuous and under the hypothesis on $\mathbb{D}$, the bilinear form $c$ is coercive: we can apply Lax-Milgram theorem to conclude.

In the same way, Problem 3 can be written as:

$$\text{Solve in } (\lambda, \phi) \in \mathbb{R}^* \times V \setminus \{0\} | - \text{div} \left( \mathbb{D} \nabla \phi \right) + T_e \phi = \lambda^{-1} M_f \phi.$$ (7)

The variational formulation of (7) writes:

$$\text{Solve in } (\lambda, \phi) \in \mathbb{R}^* \times V \setminus \{0\} | \forall \psi \in V : c(\phi, \psi) = \lambda^{-1} \ell_f(\phi, \psi),$$ (8)

where:

$$\ell_f : L \times L \rightarrow \mathbb{R}, \ell_f(\phi, \psi) = (M_f \phi, \psi)_L.$$

Theorem 2 Suppose that $\mathbb{D}$ is positive definite. There exists a unique compact operator $T_f : L \rightarrow L$ such that $\forall (\phi, \psi) \in L \times V : c(T_f \phi, \psi) = \ell_f(\phi, \psi)$.

Proof: The bilinear form $c$ is a continuous and under the hypothesis on $\mathbb{D}$, it is coercive onto $V \times V$. The bilinear form $\ell_f$ is a continuous onto $L \times V$. Finally, $V$ is a subset of $L$ with a compact embedding. We can then apply the work of Babuška and Osborn in [2].

Thus, the couple $(\phi, \lambda^{-1})$ is a solution to Problem 8 iff the couple $(\phi, \lambda)$ is an eigenpair of operator $T_f$. Moreover, Problem 8 admits a countable number of eigenvalues.

We propose first to derive conditions on the macroscopic cross sections so that Problems 5 and 7 are well-posed. Then we obtain a priori error estimates for a discretization performed with some $H^1$-conforming FEM and a Discontinuous Galerkin method, namely the Symmetric Interior Penalty Galerkin method (SIPG) [8, Chapter 4]. The outline is as follows: in Section 3, we exhibit some conditions so that the matrix $T_e^{-1}$ and $T_e$ are positive definite. Then we study the discretization of the source problem (5) in Section 5, and the discretization of the eigenproblem in Section 6. Finally, we perform in Section 7 a numerical study of convergence on a benchmark representative of a nuclear core.

### 3 Properties of $T_e$ and $T_o^{-1}$

Consider the diagonal matrix containing the even (resp. odd) removal macroscopic cross sections: $T_{r,(e,o)} = \text{diag}(T_{r,e,o}^1, \ldots, T_{r,e,o}^G)$. We split $T_{e,o}$ so that:

$$T_{e,o} = T_{r,(e,o)}(I - \varepsilon U_{e,o}),$$

where $I \in \left(\mathbb{R}^{\hat{N} \times \hat{N}}\right)^{G \times G}$ is the identity matrix, and:

$$\forall g, g' \in \mathcal{I}_G, g' \neq g, (U_{e,o})_{g,g'} = \text{diag} \left( \frac{\sigma_{g'}^{g \rightarrow g}}{\varepsilon \sigma_{f,m}} \right)_{m \in \mathcal{I}_{e,o}} \in \mathbb{R}^{\hat{N} \times \hat{N}};$$

$$\forall g \in \mathcal{I}_G, (U_{e,o})_{g,g} = 0 \in \mathbb{R}^{\hat{N} \times \hat{N}}.$$
We have then: \( \|U_{r, o}\|_2 \lesssim \frac{\alpha_{s, (e,o)}}{\varepsilon} \) where: \( \alpha_{s, (e,o)} := (G - 1) \max_{m \in I_e, o} \max_{g \neq g' \in I_o} \sup_{\vec{x} \in \mathbb{R}} \frac{|\sigma_{r, e}^{g - g'}(\vec{x})|}{\sigma_{r, m}^2(\vec{x})} \).

Let us set \( \alpha_{r, (e,o)} = \frac{(\sigma_{r, (e,o)}^e)^*}{(\sigma_{r, (e,o)}^e)^*} > 1 \). We have the following properties.

**Property 3** Suppose that \( \alpha_{s, e} < \frac{1}{\alpha_{r, e}} \). The matrix \( T_e \) is such that:

\[
\forall X \in \mathbb{R}^{N \times G} \quad (T_e X X) \geq \tau_e \|X\|_2^2 \quad \text{where} \quad \tau_e = (\sigma_{r, e})^* (1 - \alpha_{r, e} \alpha_{s, e}) \tag{9}
\]

**Proof:** We have: \( \forall X \in \mathbb{R}^{N \times G}, (T_e X X) = (T_{r, e} X X) - \varepsilon(\mathbb{I} \times X T_{r, e} X), \) so that: \((T_{r, e} X X) \geq (\sigma_{r, e})^* - \varepsilon\|U_e\|_2 \|T_{r, e} \|_2 \|X\|_2, \) where \( \|T_{r, e}\|_2 \leq (\sigma_{r, e})^* \). \( \square \)

**Property 4** Suppose that \( \alpha_{s, o} < \frac{1}{\alpha_{r, o} + 1} \), the matrix \( T_o^{-1} \) is such that:

\[
\forall X \in \mathbb{R}^{N \times G} \quad (T_o^{-1} X X) \geq \tau_o \|X\|_2^2 \quad \text{where} \quad \tau_o = \frac{1}{(\sigma_{r, o})^*} \left( 1 - \frac{\alpha_{r, o} \alpha_{s, o}}{1 - \alpha_{s, o}} \right) \tag{10}
\]

**Proof:** The Taylor expansion of \( T_o^{-1} \) writes: \( T_o^{-1} = (\mathbb{I} + \sum_{l > 0} \varepsilon^l U^l_o) T_{r, o}^{-1} \).

We get that \( \forall X \in \mathbb{R}^{N \times G} \):

\[
(T_o^{-1} X X) = (T_{r, o}^{-1} X X) + \sum_{l > 0} \varepsilon^l (U^l_o T_{r, o}^{-1} X X)
\]

\[
\geq \frac{1}{(\sigma_{r, o})^*} \left( 1 - \alpha_{r, o} \sum_{l > 0} \varepsilon^l \|U^l_o\|_2 \right) \|X\|_2^2,
\]

\[
\geq \frac{1}{(\sigma_{r, o})^*} \left( 1 - \alpha_{r, o} \frac{\varepsilon \|U_o\|_2}{1 - \varepsilon \|U_o\|_2} \right) \|X\|_2^2,
\]

\[
\geq \frac{1}{(\sigma_{r, o})^*} \left( 1 - \alpha_{r, o} \alpha_{s, o} \frac{1}{1 - \alpha_{s, o}} \right) \|X\|_2^2.
\]

\( \square \)

Under assumptions of Properties 3 and 4 the matrices \( T_e \) and \( T_{o}^{-1} \) are positive definite. Moreover, one can show that \( \|H \vec{\nabla} \phi\|_L \gtrsim \|\vec{\nabla} \phi\|_L \) [12]. We infer that the matrix \( D \) is positive definite and that there exists a constant \( C_D > 0 \) such that for all \( \xi \in \mathbb{R}^{N \times G} \),

\[
(D \xi D \xi) \leq C_D \|\xi\|_2^2. \tag{11}
\]

From now on, we suppose that this property holds.
4 Discretizations

Let $\mathcal{T}_h$ be a shape-regular mesh of $\mathcal{R}$, with mesh size $h$. We denote by $K$ its elements and $F$ its facets. To simplify the presentation, we assume that the meshes are such that in every element, the cross-sections are regular. We define by $\mathcal{F}_i^h$ the set of interior faces of $\mathcal{T}_h$, $\mathcal{F}_b^h$ the set of boundary facets and $\mathcal{F}_h = \mathcal{F}_i^h \cup \mathcal{F}_b^h$. We denote by $N_\partial$ the maximum number of mesh faces composing the boundary of mesh elements

$$N_\partial := \max_{K \in \mathcal{T}_h} \text{Card}\{F \in \mathcal{F}_h, F \subset \partial K\}.$$  

We will first consider an $H^1$-conforming finite element method (FEM). For $k \in \mathbb{N}^*$, $V_h^k \subset V$ and $\mathcal{V}_h^k \subset \mathcal{V}$ are the finite dimension spaces defined by:

$$V_h^k = \{v_h \in V, \forall K \in \mathcal{T}_h, v_h|_K \in P_k\}, \quad \mathcal{V}_h^k := (V_h^k)^{N \times G}.$$  

The discrete variational formulation associated to Problem (6) writes:

$$\text{Solve in } \phi_h \in V_h^k \mid \forall \psi_h \in V_h^k : c(\phi_h, \psi_h) = \ell(\psi_h), \quad (12)$$

Similarly, the discrete variational formulation associated to Problem (7) writes:

$$\text{Solve in } (\lambda_h, \phi_h) \in \mathbb{R}^* \times \mathcal{V}_h^k \setminus \{0\} \mid \forall \psi \in \mathcal{V}_h^k : c(\phi_h, \psi_h) = \lambda_h^{-1} \ell_f(\phi_h, \psi_h), \quad (13)$$

Then, we will consider a non-conforming FEM. We define the broken spaces:

$$V_{NC} = \{v \in L^2(\mathcal{R}) \mid \forall K \in \mathcal{T}_h, v \in H^1(K)\}, \quad \mathcal{V}_{NC} = (V_{NC})^{N \times G}.$$  

For $(\phi, \psi) \in \mathcal{V}_{NC} \times \mathcal{V}_{NC}$, and $T \in \mathbb{R}^{N \times G}$, we set:

$$\left(\mathbb{D} \nabla_h \phi, \nabla_h \psi\right)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \left(\mathbb{D} \nabla \phi, \nabla \psi\right)_{L^2(K)}, \quad \left\|\nabla_h \psi\right\|_{\mathcal{T}_h} = \left(\nabla_h \psi, \nabla_h \psi\right)^{1/2}_{\mathcal{T}_h}.$$  

For $F \in \mathcal{F}_b^h$ such that $F = \partial K_1 \cap \partial K_2$, we define the average $\{\mathbb{D} \nabla_h \psi\}$ and the jump $[\psi]$ as:

$$[\mathbb{D} \nabla_h \psi]|_F = \frac{1}{2} \left( (\mathbb{D}_1 \nabla \psi_1)|_F + (\mathbb{D}_2 \nabla \psi_2)|_F \right) \in \left(\mathbb{R}^{N \times G}\right)^d, \quad [\psi]|_F = \psi_1|_F n_1 + \psi_2|_F n_2 \in \left(\mathbb{R}^{N \times G}\right)^d.$$  

where $n_i$ is is the unit outward normal to $K_i$ at face $F$ and $\mathbb{D}_i = \mathbb{D}|_{K_i}, \psi_i = \psi|_{K_i}$.

For $F \in \mathcal{F}_b^h$ such that $F \in K$, we set $\{\mathbb{D} \nabla_h \psi\}|_F = \mathbb{D}|_K \nabla \psi|_K$ and $[\psi]|_F = (\psi_K)|_F n$, where $\psi_K = \psi|_K$ and $n$ is the unit outward normal to $K$ at face $F$.

For $k \in \mathbb{N}^*$, $V_{NC}^k \subset H^1(\mathcal{T}_h)$ and $\mathcal{V}_{NC}^k$ are the finite dimension spaces defined by:

$$V_{NC}^k = \{v_h \in L^1(\mathcal{R}); \forall K \in \mathcal{T}_h, v_h|_K \in P_k\}, \quad \mathcal{V}_{NC}^k := (V_{NC}^k)^{N \times G}.$$  

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For $\phi_h, \psi_h \in V^k_{h,NC}$, we set: 
\[
\left\{ \{D \nabla_h \phi_h\}, [[\psi_h]] \right\}_{F_h} = \sum_{F \in F_h} \left( \{D \nabla_h \phi_h\}, [[\psi_h]] \right)_{L(F)}.
\]

Let us set
\[
c_h(\phi_h, \psi_h) = c_{T_h}(\phi_h, \psi_h) + c_{F_h}(\phi_h, \psi_h),
\]
with
\[
c_{T_h}(\phi_h, \psi_h) = \left( D \nabla_h \phi_h, D \nabla_h \psi_h \right)_{T_h} + (T_c \phi_h, \psi_h)_{L},
\]
\[
c_{F_h}(\phi_h, \psi_h) = \sum_{F \in F_h} \frac{\alpha}{h_F} (\|\phi_h\|, \|\psi_h\|)_{L(F)} - \left( \{D \nabla_h \phi_h\}, [[\psi_h]] \right)_{F_h} - \left( \{D \nabla_h \phi_h\}, [[\psi_h]] \right)_{F_h},
\]
where $\alpha$ is a stabilization parameter.

The Symmetric Interior Penalty Galerkin method (SIPG) associated to Problem (6) writes:
\[
\text{Solve in } \phi_h \in V^k_{h,NC} \text{ s.t. } \forall \psi_h \in V^k_{h,NC} : c_h(\phi_h, \psi_h) = \ell(\psi_h).
\]
Similarly, the SIPG method associated to Problem (8) writes:
\[
\text{Solve in } (\lambda_h, \phi_h) \in \mathbb{R}^* \times V^k_{h,NC} \setminus \{0\} \text{ s.t. } \forall \psi_h \in V^k_{h,NC} : c_h(\phi_h, \psi_h) = \lambda_h^{-1} \ell_f(\phi_h, \psi_h).
\]

5 The source problem

5.1 Conforming discretization

Theorem 5 Suppose that there exists $r_{\text{max}}$ in $[0,1]$ such that $\forall \mu \in [0, r_{\text{max}}]$, $\phi \in (H^{1+r}(\mathcal{R}))^N \times G$ ([6], Proposition 1). Let us set $\mu = \min(r_{\text{max}}, k)$. The solution of (12), $\phi_h$ is such that: $\|\phi - \phi_h\|_V \lesssim h^{p} \|S_f\|_L$ and $\|\phi - \phi_h\|_L \lesssim h^{2p} \|S_f\|_L$.

Proof: From Céa’s lemma and Aubin-Nitsche lemma as detailed in ([10], §2.3). \hfill $\square$

5.2 SIPG discretization

Assumption 5.1 (Regularity of exact solution and space $V^*$) Let us denote by $W^{2,p}(T_h)$ the broken Sobolev space spanned by those functions $v$ such that for all $K \in T_h$, $v|_K \in W^{2,p}(K)$. We set $W^{2,p}(T_h) = (W^{2,p}(T_h))^N \times G$.

We assume that $d \geq 2$ and that there is $2d/(d+2) < p \leq 2$ such that, for the exact solution $\phi \in V^*: = V \cap W^{2,p}(T_h)$. This holds for our assumptions on the coefficients, which are piecewise constant with respect to the triangulation [15].

This assumption requires $p > 1$ for $d = 2$ and $p > 6/5$ for $d = 3$. In particular, we observe that, in two space dimensions, $\phi \in W^{2,p}(T_h)$ in polygonal domains. Moreover, using Sobolev embeddings [4, Sect. IX.3], this implies
\[
\phi \in (H^{1+\alpha_p}(\mathcal{R}))^N \times G, \quad \alpha_p = \frac{d+2}{2} - \frac{d}{p} > 0.
\]

We state the following lemma [8, Lemma 1.46, p.27].

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Lemma 6 Suppose that \((T_h)_h\) is a shape- and contact-regular mesh sequence. Then, we have for all \(h > 0\):
\[
\forall \psi_h \in \mathcal{V}_{h,NC}^k, \forall K \in T_h, \forall F \in \partial K, \quad h^{-1/2}_K \|\psi_h\|_{L^2(F)} \leq C_{tr} \|\psi_h\|_{L^2(K)}, \tag{17}
\]
where \(h_K\) is the diameter of element \(K\).

We aim at asserting the discrete coercivity using the following norm:
\[
\forall \psi_h \in \mathcal{V}_{h,NC}^k, \quad \|\psi_h\|_{sp}^2 := c_{T_h}(\psi_h, \psi_h) + \|\psi_h\|_J^2
\]
with the jump semi-norm
\[
\|\psi_h\|_J^2 := \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|\mathcal{J} \psi_h\|_{L^2(F)}^2.
\]

Under assumption (4), there exists \(\beta > 0\) we have for all \(\psi_h \in \mathcal{V}_{h,NC}^k\)
\[
c_{T_h}(\psi_h, \psi_h) \geq \beta \left( \|\mathcal{V} \psi_h\|_{T_h}^2 + \|\psi_h\|_{L^2}^2 \right), \tag{18}
\]
so that
\[
\|\psi_h\|_{sp}^2 \geq \beta \left( \|\mathcal{V} \psi_h\|_{T_h}^2 + \|\psi_h\|_{L^2}^2 + \|\psi_h\|_J^2 \right).
\]

Lemma 7 (Discrete coercivity) Let \(\alpha := C_{tr}^2 N_\partial \frac{C_D}{\beta}\) where
- \(C_{tr}\) results from the discrete trace inequality (17),
- \(N_\partial\) is defined in Section 4,
- \(C_D\) is defined in (11).

For all \(\alpha \geq \alpha_c\), the SIP bilinear form defined by (14) is coercive on \(\mathcal{V}_{h,NC}^k\) with respect to the \(\|\cdot\|_{sp}\)-norm, i.e.,
\[
c_h(\psi_h, \psi_h) \geq C_\alpha \|\psi_h\|_{sp}^2,
\]
with \(C_\alpha := \left( \alpha - C_{tr}^2 N_\partial \frac{C_D}{\beta} \right) \min \left\{ \frac{1}{2}, \beta \left( \alpha + C_{tr}^2 N_\partial \frac{C_D}{\beta} \right)^{-1} \right\}.
\]

Proof: We follow the proof of [8, Lemma 4.12]. For all \(\psi_h \in \mathcal{V}_{h,NC}^k\),
\[
c_h(\psi_h, \psi_h) = c_{T_h}(\psi_h, \psi_h) + c_{\mathcal{F}_h}(\psi_h, \psi_h)
\]
\[
= c_{T_h}(\psi_h, \psi_h) + \sum_{F \in \mathcal{F}_h} \frac{\alpha}{h_F} \|\mathcal{J} \psi_h\|_{L^2(F)}^2 - 2 \left( \mathcal{D} \mathcal{V} \psi_h, [\psi_h] \right)_{\mathcal{F}_h}
\]
\[
\geq c_{T_h}(\psi_h, \psi_h) + \alpha \|\psi_h\|_J^2 - 2C_{tr}(N_\partial)^{1/2} \|\mathcal{D} \mathcal{V} \psi_h\|_{T_h} \|\psi_h\|_J
\]
\[
\geq C_{tr}(N_\partial)^{1/2} \|\mathcal{D} \mathcal{V} \psi_h\|_{T_h} \|\psi_h\|_J,
\]
\[
\geq C_{tr}(N_\partial)^{1/2} \|\mathcal{D} \mathcal{V} \psi_h\|_{T_h} \|\psi_h\|_J
\]
\[
\geq C_{tr}(N_\partial)^{1/2} \left( \|\mathcal{V} \psi_h\|_{T_h}^2 + \|\psi_h\|_{L^2}^2 \right),
\]
\[
\geq \frac{1}{2} \|\mathcal{V} \psi_h\|_{T_h}^2 + \frac{1}{2} \|\psi_h\|_{L^2}^2,
\]
\[
\geq \frac{1}{2} \|\mathcal{V} \psi_h\|_{T_h}^2 + \frac{1}{2} \|\psi_h\|_{L^2}^2 + \frac{1}{2} \|\psi_h\|_J^2.
\]
where we used Cauchy-Schwarz and Lemma 6 in the last line. Using the inequality $2ab \leq \varepsilon a + \varepsilon^{-1}b$ for any $\varepsilon > 0$, we obtain

$$2C_{tr}(N_\alpha)^{1/2} \left\| D\tilde{\nabla} h \psi_h \right\|_{T_h} \left\| \psi_h \right\|_J \leq \varepsilon C_{tr}^2 N_\alpha \left\| D\tilde{\nabla} h \psi_h \right\|_{T_h}^2 + \varepsilon^{-1} \left\| \psi_h \right\|_J^2$$

$$\leq \varepsilon C_{tr}^2 N_\alpha \left\| \tilde{\nabla} h \psi_h \right\|_{T_h}^2 + \varepsilon^{-1} \left\| \psi_h \right\|_J^2.$$  

Using (18), we obtain that there exists a constant $\beta > 0$ such that

$$c_h(\psi_h, \psi_h) \geq \beta (1 - \varepsilon \alpha) \left\| \tilde{\nabla} h \psi_h \right\|_{T_h}^2 + \beta \left\| \psi_h \right\|_L^2 + (\alpha - \varepsilon^{-1}) \left\| \psi_h \right\|_J^2.$$  

Choosing $\varepsilon = 2 (\alpha + \alpha)^{-1}$ yields the assertion. □

Thus, it only remains to prove boundedness. To this purpose, we need to define $V_{\ast,h} = V_{\ast} + V_{k,h,NC}$ and the following norm

$$\left\| \psi \right\|_{sip, \ast} := \left( \left\| \psi \right\|_{sip}^p + \sum_{K \in T_h} h_K^{1+\gamma_p} \left\| \tilde{\nabla} \psi \right\|_{K} \left\| n_K \right\|_{L^p(\partial K)} \right)^{1/p},$$  

where $\gamma_p = \frac{d(p - 2)}{2}$ and $n_K$ is the unit outward normal to $K$. Following [8, Section 4.2], we obtain the following results.

**Lemma 8 (Boundedness)** There is $C_{bnd}$, independent of $h$, such that for all $(\phi, \psi_h) \in V_{\ast,h} \times V_h$

$$c_h(\phi, \psi_h) \leq C_{bnd} \left\| \phi \right\|_{sip, \ast} \left\| \psi_h \right\|_{sip}$$

**Theorem 9 (Convergence)** Suppose that there exists $r_{\max}$ in $(0,1]$ such that $\forall r \in [0, r_{\max}], \phi \in (H^{1+r}(\mathcal{R}))^N \times G$ ([6], Proposition 1). Then the solution of (15), $\phi_h$ is such that:

$$\left\| \phi - \phi_h \right\|_{sip} \lesssim C \inf_{\psi_h \in V_{h,NC}} \left\| \phi - \psi_h \right\|_{sip, \ast},$$  

where $C$ is a constant independent of $h$. Moreover, under Assumption 5.1, there holds

$$\left\| \phi - \phi_h \right\|_{sip} \leq C |\phi|_{W^{2,\infty}(\partial T_h)} h^\mu,$$

where $\mu = r_{\max}$, $C$ is a constant independent of $h$ and $p$ is such that $\mu = \frac{d + 2}{d - \frac{d}{p}}$.

**Theorem 10 (L^2-norm estimate)** Suppose that there exists $r_{\max}$ in $(0,1]$ such that $\forall r \in [0, r_{\max}], \phi^0 \in H^{1+r}(\mathcal{R})$ ([6], Proposition 1). Under Assumption 5.1, the solution of (15), $\phi_h$ is such that: $\left\| \phi - \phi_h \right\|_L \lesssim h^{2\mu} |S_f|_L$, where $\mu = r_{\max}$.

**Proof:** We apply the Aubin-Nitsche similarly as in [8, Theorem 4.25]. □
6 The eigenproblem

6.1 Conforming discretization

Theorem 11 Let $\mu$ be the regularity of the eigenfunction $\varphi$ associated to $\lambda$, and $\omega = \min(\mu, k)$. Let $\lambda_h$ be the discrete eigenvalue associated to Problem (13). The following a priori error estimate holds: $|\lambda - \lambda_h| \lesssim h^{2\omega}$.

Proof: As in the continuous case (Theorem 2), since the discretization is conforming, there exists a unique compact operator $T_h : V^k_h \to V^k_h$ such that $\forall (\varphi_h, \psi_h) \in V^k_h \times V^k_h$, $c(T_h \varphi_h, \psi_h) = \ell_f(\varphi_h, \psi_h)$. According to Thm. 5, the sequence of the operators $(T_h)_h$ is pointwise converging towards $T$. As $T_h$ and $T$ are a compact operators, the sequence of the operators $(T_h)_h$ is then converging in $L(V)$ towards $T$: $\|T_h - T\|_{L(V)} \to 0$. The norm convergence guarantees that there is no spectral pollution (see [17]). Moreover, we can apply Theorem 8.3 in [2] to state the error estimate on the eigenvalue. We remark that $(M_f \varphi, \varphi)_L$ is a norm over $V_\lambda := \{\varphi \in V | \forall \psi \in V, c(\varphi, \psi) = \lambda \ell_f(\varphi, \psi)\}$ [12, Section 5.2.2 p. 78]. \hfill \Box

6.2 SIPG discretization

We recall that, in this section, we work under the assumption 5.1.

Theorem 12 Let $\mu$ be the regularity of the eigenfunction $\varphi$ associated to $\lambda$, and $\omega = \min(\mu, k)$. Let $\lambda_h$ be the discrete eigenvalue associated to Problem (16). The following a priori error estimate holds: $|\lambda - \lambda_h| \lesssim h^{2\omega}$.

Proof: We apply the theory developed in [1]. The proof is decomposed as follows. We first show that there is no spectral pollution. Then, we derive the error estimate.

Let $E : V + V^k_{h,NC} \to V + V^k_{h,NC}$ be the continuous spectral projector relative to $\lambda$ defined by

$$E = \frac{1}{2\pi i} \int_{\Gamma} (z - T|_{V + V^k_{h,NC}})^{-1} dz,$$

where $\Gamma$ is a circle in the complex plane centred at $\lambda$ which lies in $\rho(T|_{V + V^k_{h,NC}})$ and encloses no other points of $\sigma(T|_{V + V^k_{h,NC}})$. The absence of spectral pollution relies on two properties. First, using interpolation results [8, Assumption 4.31] we have for all $\phi \in E(V + V^k_{h,NC})$,

$$\inf_{\psi_h \in V^k_{h,NC}} \|\phi - \psi_h\|_{sip} \leq Ch^\mu,$$

where $\|\cdot\|_{sip}$ is the SIPG norm.

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where $C$ is a constant independent of $h$. Second, we have for all $\phi_h \in V^k_{h,NC}$,
\[
\|(T - T_h)\phi_h\|_{sip} \leq Ch^{\mu} \|T\phi_h\|_{W^{2,p}(\mathcal{T}_h)} \\
\leq Ch^{\mu} \|T\phi_h\|_{(H^{1+\alpha_p(\mathcal{R})})^{\mathcal{S} \times \mathcal{G}}} \\
\leq Ch^{\mu} \|\phi_h\|_{L} \\
\leq Ch^{\mu} \|\phi_h\|_{sip},
\]
where we used Theorem 9 in the second line and regularity results [15] in the third line. Applying [1, Theorem 3.7], we obtain that there is no spectral pollution. Moreover, we apply [1, Theorem 3.14] to state the error estimate on the eigenvalue,
\[
|\lambda - \lambda_h| \leq C\delta_h \delta_{*,h},
\]
where
\[
\delta_h = \gamma_h + \left\| (T - T_h)|_{E(V + V^k_{h,NC})} \right\|_{sip} \\
\delta_{*,h} = \gamma_{*,h} + \left\| (T_* - T_{*,h})|_{E(V + V^k_{h,NC})} \right\|_{sip},
\]
with
\[
\gamma_h = \delta(E(V + V^k_{h,NC}), V^k_{h,NC}), \\
\gamma_{*,h} = \delta(E_*(V + V^k_{h,NC}), V^k_{h,NC}),
\]
where
\[
\delta(Y, Z) = \sup_{y \in Y, \|y\|_{sip} = 1} \left( \inf_{z \in Z} \|y - z\|_{sip} \right),
\]
and $E_* : V + V^k_{h,NC} \to V + V^k_{h,NC}$ is the continuous spectral projector of the adjoint operator $T_*|_{V + V^k_{h,NC}}$ relative to $\bar{\lambda}$.

Using again elliptic regularity results [15] and Theorem 9, we obtain
\[
\left\| (T - T_h)|_{E(V + V^k_{h,NC})} \right\|_{sip} \leq Ch^{\mu} \\
\left\| (T_* - T_{*,h})|_{E(V + V^k_{h,NC})} \right\|_{sip} \leq Ch^{\mu}.
\]
Using elliptic regularity results, we get
\[
\|\varphi\|_{(H^{1+\alpha_p(\mathcal{R})})^{\mathcal{S} \times \mathcal{G}}} \leq C\|\varphi\|_{L} \leq C\|\varphi\|_{L}.
\]
Applying Theorem 9, we infer that
\[
\gamma_h \leq Ch^\mu
\]
\[
\gamma_{*,h} \leq Ch^\mu.
\]
This concludes the proof. \(\square\)

7 Numerical Results

We consider the test case Model 2, case 1 from the benchmark of Takeda and Ikeda [19]. The geometry of the core is three-dimensional and the domain is \(\{(x, y, z) \in \mathbb{R}^3; 0 \leq x \leq 140 \text{ cm}; 0 \leq y \leq 140 \text{ cm}; 0 \leq z \leq 150 \text{ cm}\}\). This test is defined with 4 energy groups, isotropic scattering and vacuum boundary conditions. Figure 1 represents the cross-sectional geometry on the plane \(z = 75 \text{ cm}\).

Since the scattering is isotropic, the \(SP_3\) formulation can easily be reformulated as a multigroup diffusion problem with 8 energy groups and an isotropic albedo boundary condition [3]. We then made the computations with the PRIAM solver from the code CRONOS2 [14] for the conforming case and with the MINARET solver [13] from the APOLLO3® code [18] for the SIPG discretization.

![Cross-sectional view of the core (z = 75 cm).](image)

In Figure 2, we consider the convergence of the fundamental mode where we used the \(SP_3\) formulation with \(Q^1\) finite elements and a regular cartesian mesh of size \(h\). The approximated order of convergence is 2.22.

In Figure 3, we consider the convergence of the fundamental mode for different the \(SP_N\) formulations with discontinuous \(P^1\) finite elements and a prismatic mesh of size \(h\). The approximated orders of convergence are given in Table 1.
8 Conclusion

We did the numerical analysis of the approximation with an $H^1$-conforming finite element method of the neutron multigroup $SP_N$ equations. We also studied the numerical analysis of the approximation with the Symmetric Interior Penalty Galerkin method of the neutron multigroup $SP_N$ equations. We then illustrated numerically the convergence results on a benchmark representative of a nuclear core. Those results can be extended to a mixed finite element method, see [5] for the diffusion case with an $H^1$-conforming finite element method.

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