Logarithmic vector fields along smooth plane cubic curves

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Abstract

We study the sheaves of logarithmic vector fields along smooth cubic curves in the projective plane, and prove a Torelli-type theorem in the sense of Dolgachev–Kapranov [4] for those with non-vanishing $j$-invariants.

1 Introduction

K. Saito [6] introduced the notion of the sheaf of logarithmic vector fields along a divisor and proved that it is always reflexive. A divisor $D$ in a variety $S$ is said to be free if the sheaf of logarithmic vector field along $D$ is a free $\mathcal{O}_S$-module. He proved that the discriminant in the parameter space of the semi-universal deformation of an isolated hypersurface singularity is always free.

When the ambient space is the projective space $\mathbb{P}^\ell$, an $\mathcal{O}_{\mathbb{P}^\ell}$-module is said to be free if it is the direct sum $\bigoplus_i \mathcal{O}_{\mathbb{P}^\ell}(a_i)$ of invertible sheaves. The problem of characterizing free divisors in projective spaces has attracted much attention, especially when the divisor is given as an arrangement of hyperplanes. See e.g. [7]. If a divisor in $\mathbb{P}^\ell$ is free, then the passage from the divisor to the sheaf of logarithmic vector fields causes loss of information; only the sequence $\{a_i\}_{i=1}^\ell$ of integers is left, and it is impossible to reconstruct the divisor from this finite amount of information.

In the opposite extreme, Dolgachev and Kapranov [4] asked when the the sheaf $\mathcal{T}(-\log D)$ contains enough information to reconstruct $D$. A divisor $D$ in $\mathbb{P}^\ell$ is said to be Torelli if the isomorphism class of $\mathcal{T}(-\log D)$ as an $\mathcal{O}_{\mathbb{P}^\ell}$-module determines the divisor $D$. Their main result is the condition for an arrangement of sufficiently many hyperplanes in $\mathbb{P}^\ell$ to be Torelli.

In this paper, we discuss the case when $\ell = 2$ and $D$ is a smooth cubic curve. Our main result asserts that $D$ is Torelli precisely when the $j$-invariant of $D$ is not zero. The strategy of our proof is the following:
1. The set of jumping lines of the sheaf of logarithmic vector fields along a smooth cubic curve coincides with its Cayleyan curve.

2. For a smooth cubic curve with a non-vanishing \( j \)-invariant, the Cayleyan curve determines the original curve up to three possibilities.

3. The set of “jumping cubic curves” fixes this left-over ambiguity and the Torelli property holds.

4. When the \( j \)-invariant of \( D \) is zero, we can construct a family of divisors with isomorphic sheaves of logarithmic vector fields along them.

Smooth cubic curves with vanishing \( j \)-invariants provide examples of divisors which are neither free nor Torelli.

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2 Preliminaries

2.1 de Rham–Saito’s lemma

Let \( A \) be a Noetherian ring and \( M = \bigoplus_{i=1}^{n} A e_{i} \) be a free module over \( A \) generated by \( e_{1}, \ldots, e_{n} \). For \( \omega_{1}, \ldots, \omega_{r} \in M \), put

\[
\omega_{1} \wedge \cdots \wedge \omega_{r} = \sum_{1 \leq i_{1} < \cdots < i_{r} \leq n} a_{i_{1}, \ldots, i_{r}} e_{i_{1}} \wedge \cdots \wedge e_{i_{r}}.
\]

and define \( a \) to be the ideal generated by \( a_{i_{1}, \ldots, i_{r}} \) for \( 1 \leq r \leq n \) and \( 1 \leq i_{1} < \cdots < i_{r} \leq n \). We also define as follows:

\[
Z^{p} = \{ \varphi \in \wedge^{p} M \mid \omega_{1} \wedge \cdots \wedge \omega_{r} \wedge \varphi = 0 \},
\]

\[
B^{p} = \sum_{k=1}^{r} \omega_{k} \wedge (\wedge^{p-1} M),
\]

\[
H^{p} = Z^{p}/B^{p}.
\]

Theorem 1 (de Rham–Saito’s lemma [3, 5]). (1) There exists an integer \( \nu \in \mathbb{Z}_{\geq 0} \) such that \( a^{\nu} H^{p} = 0 \) for \( 0 \leq p \leq n \).

(2) For \( 0 \leq p < \text{depth}_{a} A \), we have \( H^{p} = 0 \).
2.2 Sheaf of logarithmic vector fields

Let \( A = \mathbb{C}[z_0, \ldots, z_\ell] \) be a polynomial ring and \( \text{Der}_A \) be the module of \( \mathbb{C} \)-derivations of \( A \), which is a free module of rank \( \ell + 1 \):

\[
\text{Der}_A = \sum_{i=0}^{\ell} A \frac{\partial}{\partial z_i}.
\]

**Definition 2.** For a homogeneous polynomial \( f \in A \), we define

\[
D(- \log f) = \{ \delta \in \text{Der}_A \mid \delta f \in (f) \},
\]

\[
D_0(- \log f) = \{ \delta \in \text{Der}_A \mid \delta f = 0 \}.
\]

We put \( \deg z_i = 1 \) and \( \deg (\partial/\partial z_i) = -1 \) for \( i = 0, \ldots, \ell \). The degree \( k \) part of \( D_0(- \log f) \) will be denoted by \( D_0(- \log f)_k \).

We have the direct sum decomposition

\[
D(- \log f) = D_0(- \log f) \oplus A \cdot E,
\]

where

\[
E = \sum_{i=0}^{\ell} z_i \partial/\partial z_i
\]

is the Euler vector field. Let \( \Omega_A \) be the module of differentials

\[
\Omega_A^1 = \bigoplus_{i=0}^{\ell} Adz_i,
\]

and \( \Omega_A^k \) be its \( k \)-th exterior power for \( k = 0, \ldots, \ell + 1 \). We have an isomorphism of \( A \)-modules

\[
D_0(- \log f) \cong \{ \omega \in \Omega^\ell \mid df \wedge \omega = 0 \}
\]

under the identification

\[
\begin{align*}
\text{Der}_A \ &\xrightarrow{\psi} \ \Omega^\ell \\
\sum_{i=0}^{\ell} f_i \frac{\partial}{\partial z_i} \ &\mapsto \ \sum_{i=0}^{\ell} (-1)^i f_i dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_\ell.
\end{align*}
\]

Let \( D \subset \mathbb{P}^\ell \) be the hypersurface defined by \( f \). If \( D \) is smooth, then the origin \( 0 \in \mathbb{C}^{\ell+1} \) is the only zero locus of the Jacobi ideal

\[
J(f) = \left( \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_\ell} \right),
\]
and hence we have

\[ \text{depth}_{J(f)} A = \ell + 1. \]

Let \( H^p \) be the \( p \)-th cohomology of the complex

\[ 0 \to \Omega^0_A \to \Omega^1_A \to \cdots \to \Omega^\ell_A \to \Omega^{\ell+1}_A \to 0. \]

If \( D \) is smooth, then we have \( H^p = 0 \) for \( p = 0, \ldots, \ell \) by de Rham–Saito’s lemma. Since

\[ D_0(-\log f) \cong \text{Ker} \left( df \wedge : \Omega^\ell \to \Omega^{\ell+1} \right), \]

the sequence

\[ 0 \to \Omega^0_A \to \Omega^1_A \to \cdots \to \Omega^{\ell-1}_A \to D_0(-\log f) \to 0 \quad (1) \]

gives a free resolution of \( D_0(-\log f) \).

The Euler sequence

\[ 0 \to \mathcal{O} \to \mathcal{O}(1) \to T_{P^\ell} \to 0, \]

shows that the sheafification \( T_{P^\ell}(-\log f) \) of \( D_0(-\log f) \) can be considered as a subsheaf of the tangent sheaf \( T_{P^\ell} \);

\[ T_{P^\ell}(-\log f) \subset T_{P^\ell}. \]

It is the sheaf of holomorphic vector fields tangent to the hypersurface \( D \) at smooth points of \( D \). If \( D \) is smooth, we have the short exact sequence

\[ 0 \to T_{P^\ell}(-\log f) \to T_{P^\ell} \to N_{D/P^\ell} \to 0, \]

where \( N_{D/P^\ell} \) is the normal bundle. We have an isomorphism

\[ df|_D : N_{D/P^\ell} \cong \mathcal{O}_D(d), \]

where

\[ d = \text{deg } f. \]

If \( D \) is smooth, then the sheaf \( T_{P^\ell}(-\log f) \) has the resolution

\[ 0 \to \mathcal{O}(1-(d-1)\ell) \to \cdots \to \mathcal{O}(3-2d)^{\oplus(\ell+1)}_{d-2} \to \mathcal{O}(2-d)^{\oplus(\ell+1)}_d \to T_{P^\ell}(-\log f) \to 0 \quad (2) \]

obtained by sheafifying the exact sequence (1). We also have

\[ \Gamma \left( \mathbb{P}^\ell, T_{P^\ell}(-\log f)(k) \right) = D_0(-\log f)_k \]

for \( k \in \mathbb{Z} \).
3 Plane curves

Now we set $\ell = 2$ to focus our attention on plane curves. Let $f \in \mathbb{C}[z_0, z_1, z_2]$ be a homogeneous polynomial of degree $d$ and $D \subset \mathbb{P}^2$ be the curve defined by $f$. Define $\mathcal{F}$ as the cokernel of $df \wedge : \mathcal{O}(3 - 2d) \to \mathcal{O}(2 - d)^{\oplus 3}$ so that we have the exact sequence

$$0 \to \mathcal{O}(3 - 2d) \xrightarrow{df \wedge} \mathcal{O}(2 - d)^{\oplus 3} \to \mathcal{F} \to 0. \quad (3)$$

The Chern polynomial of $\mathcal{F}(k)$ is given by

$$c_t(\mathcal{F}(k)) := 1 + c_1(\mathcal{F}(k))t + c_2(\mathcal{F}(k))t^2$$

$$= c_t(\mathcal{O}(2 - d + k))^3 c_t(\mathcal{O}(3 - 2d + k))^{-1}$$

$$= 1 + (3 - d + 2k)t + (d^2 - 3d + 3 + k^2 + (3 - d)k)t^2$$

for $k \in \mathbb{Z}$. If $D$ is smooth, then we have

$$\mathcal{F} := \text{Coker}(df \wedge : \mathcal{O}(3 - 2d) \to \mathcal{O}(2 - d)^{\oplus 3})$$

$$\cong \text{Coim}(df \wedge : \mathcal{O}(2 - d)^{\oplus 3} \to \mathcal{O}(1)^{\oplus 3})$$

$$\cong \text{Im}(df \wedge : \mathcal{O}(2 - d)^{\oplus 3} \to \mathcal{O}(1)^{\oplus 3})$$

$$\cong \text{Ker}(df \wedge : \mathcal{O}(1)^{\oplus 3} \to \mathcal{O}(d))$$

$$\cong \mathcal{T}_{\mathbb{P}^2}(- \log f).$$

**Lemma 3.** If $D$ is smooth, then $\mathcal{T}_{\mathbb{P}^2}(- \log f)$ is stable.

**Proof.** We consider $\mathcal{F}([(d - 3)/2])$ instead of $\mathcal{T}_{\mathbb{P}^2}(- \log f)$ whose first Chern number is normalized to either 0 (when $d$ is odd) or $-1$ (when $d$ is even). Then $\mathcal{F}([(d - 3)/2])$ is stable if and only if it has no global section. This follows from the cohomology long exact sequence associated with the short exact sequence (3) tensored with $\mathcal{O}_{\mathbb{P}^2}([(d - 3)/2])$. \qed

4 Smooth cubic curves

Let $f \in \mathbb{C}[z_0, z_1, z_2]$ be a homogeneous polynomial of degree three and $D \subset \mathbb{P}(V)$ be a cubic curve defined by $f$, where $V = \text{Spec} \mathbb{C}[z_0, z_1, z_2]$. We assume that $D$ is smooth.

4.1 Jumping lines

Let $L$ be a point in the dual projective plane $\mathbb{P}(V^*)$ defined by a linear form $\alpha = \alpha_0z_0 + \alpha_1z_1 + \alpha_2z_2 \in V^*$. We can think of $L$ as a line in $\mathbb{P}(V)$.
Restricting the short exact sequence (3) to \( L \) and taking the cohomology long exact sequence, we have
\[
0 \rightarrow H^0(\mathcal{F}|_L) \rightarrow H^1(\mathcal{O}_L(-3)) \rightarrow H^1(\mathcal{O}_L(-1))^3 \rightarrow H^1(\mathcal{F}|_L) \rightarrow 0.
\]
Since
\[
H^1(\mathcal{O}_L(-3)) \cong H^0(\mathcal{O}_L(1))^* \cong \mathbb{C}^2
\]
and
\[
H^1(\mathcal{O}_L(-1)) \cong H^0(\mathcal{O}_L(-1))^* = 0,
\]
we have
\[
\dim H^0(\mathcal{F}|_L) = 2.
\]
Hence \( \mathcal{F}|_L \) is either
\[
\mathcal{F}|_L = \begin{cases} 
\mathcal{O}_L \oplus \mathcal{O}_L & \text{L is generic,} \\
\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) & \text{L is a jumping line.}
\end{cases}
\]
In particular,
\[
L \text{ is a jumping line } \iff H^0(\mathcal{F}(-1)|_L) \neq 0.
\]
By tensoring \( \mathcal{O}_L(-1) \) with the short exact sequence (3) and taking the cohomology long exact sequence, we have
\[
0 \rightarrow H^0(\mathcal{F}(-1)|_L) \rightarrow H^1(\mathcal{O}_L(-4)) \xrightarrow{\partial f} H^1(\mathcal{O}_L(-2)^{\oplus 3}) \rightarrow H^0(\mathcal{F}(-1)|_L) \rightarrow 0
\]
\[
H^0(\mathcal{O}_L(2))^* \quad H^0(\mathcal{O}_L^{\oplus 3})^*.
\]
Since \( H^0(\mathcal{O}(2)|_L) \cong \text{Sym}^2 V^*/(z_0\alpha, z_1\alpha, z_2\alpha) \), the set \( S = S(\mathcal{T}_P(-\log f)) \subset \mathbb{P}(V^*) \) of jumping lines is characterized as follows;
\[
L \in S \iff (df \wedge)^* : H^0(\mathcal{O}_L^{\oplus 3} \rightarrow H^0(\mathcal{O}_L(2))) \text{ is not an isomorphism}
\]
\[
\iff z_0\alpha, z_1\alpha, z_2\alpha, \partial_0 f, \partial_1 f, \partial_2 f \text{ are linearly dependent in } \text{Sym}^2 V^*. \quad (4)
\]

### 4.2 Cayleyan curves

Here we prove the following:

**Proposition 4.** Let \( D \subset \mathbb{P}(V) \) be a smooth cubic curve defined by a polynomial \( f \). Then the set \( S = S(\mathcal{T}_P(-\log f)) \subset \mathbb{P}(V^*) \) of jumping lines of \( \mathcal{T}_P(-\log f) \) in the dual projective plane \( \mathbb{P}(V^*) \) is the Cayleyan curve of \( D \).
First we recall the definition of the Cayleyan curve of a plane cubic curve following Artebani and Dolgachev [1]. The first polar of a plane curve \( D = \{ f = 0 \} \) with respect to a point \( q = [a_0 : a_1 : a_2] \in \mathbb{P}(V) \) is the curve \( P_q(D) = \{ a_0 \partial_0 f + a_1 \partial_1 f + a_2 \partial_2 f = 0 \} \) whose degree is one less than that of \( D \). One can show that when \( D \) is a cubic curve, the Hessian curve \( \text{He}(D) = \{ \det ((\partial_i \partial_j f)_{i,j=1}^3 = 0) \subset \mathbb{P}(V) \} \) consists of points \( q \in \mathbb{P}(V) \) such that the polar curve \( P_q(D) \) decomposes into the union of two lines. For \( q \in \text{He}(D) \), let \( s_q \in \mathbb{P}(V^\ast) \) be the singular point of \( P_q(D) \) and \( L_q \in \mathbb{P}(V^\ast) \) be the line connecting \( q \) and \( s_q \). It is known that \( s_q \) lies on \( \text{He}(D) \) and the map \( s : \text{He}(D) \to \text{He}(D) \) is a fixed-point-free involution on \( \text{He}(D) \). The image of the map

\[
\begin{array}{ccc}
\text{He}(D) & \to & \mathbb{P}(V^\ast) \\
\psi & & \psi \\
q & \mapsto & L_q
\end{array}
\]

is called the Cayleyan curve of \( D \), which is known to be the quotient of \( \text{He}(D) \) by the involution \( s \). A linear form \( \alpha = \alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2 \in V^\ast \) represents a point in the Cayleyan curve of \( D \) if and only if there is a point \( [a_0 : a_1 : a_2] \in \mathbb{P}^2 \) such that

\[
a_0 \partial_0 f + a_1 \partial_1 f + a_2 \partial_2 f \in \alpha \cdot V^\ast.
\]

This is precisely the condition (4) for the line \([\alpha] \in \mathbb{P}(V^\ast)\) to be a jumping line of \( T_{\mathbb{P}^e}(-\log f) \).

### 4.3 The set of jumping lines and \( j \)-invariant

Here we prove the following:

**Proposition 5.** Let \( D \) be the smooth cubic curve defined by a polynomial \( f \). Then the set \( S(T_{\mathbb{P}^e}(-\log f)) \) of jumping lines is singular if and only if the \( j \)-invariant of \( D \) is zero.

**Proof.** Choose a coordinate of \( V \) so that \( f \) is a Hesse cubic

\[
f_t(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3 - 3t z_0 z_1 z_2,
\]

where \( t \in \mathbb{C} \setminus \{1, \zeta, \zeta^2\} \) and \( \zeta = \exp[2\pi \sqrt{-1}/3] \). Recall that \( D = \{ f_t = 0 \} \subset \mathbb{P}^2 \) is smooth if and only if \( t^3 \neq 1 \). The set \( S = S(T_{\mathbb{P}^e}(-\log f)) \) of jumping lines, which coincides with the Cayleyan curve of \( D \), is a Hesse cubic

\[
t(a_0^3 + a_1^3 + a_2^3) - (t^3 + 2) a_0 a_1 a_2 = 0
\]

This is precisely the condition (4) for the line \([\alpha] \in \mathbb{P}(V^\ast)\) to be a jumping line of \( T_{\mathbb{P}^e}(-\log f) \).
in the dual projective plane. It is the union of three lines in general position if \( t = 0 \) or \( (3t)^3 = (t^3 + 2)^3 \). Since

\[
(t^3 + 2)^3 - (3t)^3 = (t^3 - 1)^2(t^3 + 8)
\]

and the \( j \)-invariant \( j(D) \) of \( D \) is given by

\[
j(D) = \frac{1}{64}t^3(t^3 + 8)^3 \frac{1}{(t^3 - 1)^3},
\]

the Cayleyan curve of \( D \) is smooth if and only if \( j(D) \neq 0 \), and decomposes into the union of three lines in general position if \( j(D) = 0 \). \( \square \)

### 4.4 Restricting \( T_{\mathbb{P}^3}(-\log f) \) to other cubic curves

Here we consider the restriction of the sheaf \( T_{\mathbb{P}^3}(-\log f) \) to another cubic curve \( E \) defined by a polynomial \( g \). From the exact sequence (3), we have

\[
0 \rightarrow \mathcal{O}(-3)|_E \rightarrow \mathcal{O}(-1)^{\oplus 3}|_E \rightarrow \mathcal{F}|_E \rightarrow 0.
\]

Hence we have

\[
0 \rightarrow H^0(\mathcal{F}|_E) \rightarrow H^1(\mathcal{O}(-3)|_E \bigoplus H^1(\mathcal{O}(-1)^{\oplus 3}|_E) \rightarrow H^0(\mathcal{F}|_E) \rightarrow 0
\]

Since \( H^0(\mathcal{O}(3)|_E) = \text{Sym}^3 V^*/(g) \) and \( H^0(\mathcal{O}(1)|_E) = (V^*)^3 \), the map \( df \wedge \) is dual to the map induced by

\[
(V^*)^3 \bigoplus (F_0, F_1, F_2) \rightarrow \text{Sym}^3 V^*
\]

\[
\partial_0 f + \partial_1 f f + \partial_2 f f.
\]

This map is injective due to de Rham–Saito’s lemma, and the image can be identified with the degree 3 part \( J(f)_3 \) of the Jacobi ideal. Hence we have

\[
H^0(\mathcal{F}|_E) = \begin{cases} 
\mathbb{C} & g \in J(f)_3, \\
0 & g \notin J(f)_3.
\end{cases}
\]

By an explicit calculation, we obtain the following:

**Proposition 6.** Let \( f \) be the Hesse cubic in (2) and put

\[
g = \sum_{0 \leq i \leq j \leq k \leq 2} a_{ijk}z_iz_jz_k.
\]

Then the hyperplane \( J(f)_3 \subset \text{Sym}^3 V^* \) is given by

\[
J(f)_3 = \{ g \mid a_{012} + t(a_{000} + a_{111} + a_{222}) = 0 \}.
\]
5  Torelli theorem

Here we prove our main result:

**Theorem 7.** Let $C$ and $C'$ be smooth cubic curves with non-vanishing $j$-invariants. If $T(-\log C)$ is isomorphic to $T(-\log C')$ as an $\mathcal{O}_{\mathbb{P}^2}$-module, then $C = C'$.

**Proof.** Take a homogeneous coordinate of the dual projective plane so that the set of jumping lines of $T(-\log C)$ is a Hesse cubic. Since a smooth cubic whose Cayleyan curve is a smooth Hesse cubic must be a Hesse cubic, $C$ and $C'$ are Hesse cubics. Then Proposition 6 shows that $C$ must coincide with $C'$. □

**Remark 8.** The Torelli theorem fails for cubic curves with vanishing $j$-invariants. Indeed, the family

\[ az_0^3 + bz_1^3 + cz_0^3 = 0, \quad a, b, c \in \mathbb{C}^\times \]

consists of cubic curves with identical Cayleyan curves given by

\[ \alpha_0 \alpha_1 \alpha_2 = 0. \]

Since the set of jumping lines determines a unique stable bundle if it consists of three lines in general position by Barth [2], the sheaf of logarithmic vector fields does not depend on $a$, $b$, and $c$.

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