GEODESIC FLOWS AND THEIR DEFORMATIONS IN BERTRAND SPACETIMES

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In this article we will discuss some features of a particular spacetime called Bertrand spacetime of Type II (BST-II). This spacetime is associated with multiple real parameters. The various energy conditions and geodesic equations of BST-II are used to find the limits of these parameters which can result in a meaningful and physical space-time. It will be shown that in certain circumstances where the weak and strong energy conditions hold BST-II can be thought of as a physically interesting spacetime. Further, the talk discusses about the ESR parameters in this spacetime. The properties of these parameters are numerically analyzed keeping an eye on the focusing property of radial timelike and radial null geodesics.

1. The energy conditions in BST-II spacetime

There has been an attempt to find out the properties of potentials which can produce closed, stable orbits using special relativistic techniques. Perlick introduced in Ref. [2] certain class of spherically symmetric and static spacetimes, in the general relativistic setting, called Bertrand spacetimes (BSTs) where one can have stable, bounded trajectories from each point. There are two kinds of spacetimes which satisfy the above properties of BSTs called BST of Type I which contains many parameters and is complicated to work with. The BST of Type II called BST-II in this article is more simple to work with. In this talk we will focus our attention to the various properties of BST-II.

The Bertrand spacetime of Type II (BST-II) is given as:

$$ds^2 = -\frac{dt^2}{G + \sqrt{r^{-2} + K}} + \frac{dr^2}{\beta^2(1 + Kr^2)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

The parameters $D$, $G$ and $K$ are real, and $\beta$ must be a real number. Out of these three parameters the parameter $\beta$ must be a positive rational number. The speciality of BST-II proposed by Perlick, in Ref. [2], is that this kind of spacetime admits closed, stable and periodic orbits from any point in the manifold. Unlike Schwarzschild solution BST-II is not a vacuum solution of Einstein’s equation. Assuming the validity of Einstein’s equation one can try to verify the various energy conditions in BST-II. The work presented in this talk rests heavily on Ref.

In BST-II the Ricci scalar diverges at $r \to 0$, and in the limit $r \to \infty$, $R_{\infty} = -6K\beta^2$. The energy density and the principal pressures can be calculated by using standard techniques. The energy density turns out to be

$$\rho = \frac{1 - \beta^2(3Kr^2 + 1)}{r^2} \quad (2)$$
For $r \gg 1$, this implies that the energy density is negative for positive values of $K$. The situation might be remedied by choosing a negative value of $K$. One can note that this necessitates, from Eq. (1) that for $K = -\kappa$ (where $\kappa$ is a positive real number), $r < 1/\sqrt{\kappa}$. We can thus choose $\kappa \ll 1$ so that the positivity of the energy density of space-time of Eq. (1) is guaranteed for a large range of $r$. The analysis of the weak energy condition (WEC) shows that a given choice of $\kappa$ (in accordance with the discussion above), the WEC is always satisfied for $r < 1/\sqrt{\kappa}$ when $G$ is a small positive real number.

Before we end this section, we will briefly comment on the strong energy condition (SEC): $\rho + \sum_i p_i \geq 0$, $\rho + p_i \geq 0$. We find that for the metric of eq.(1),

$$\rho + \sum_i p_i = \frac{3\beta^2}{2r^2 (Gr + \sqrt{Kr^2 + 1})^2}$$

so that the SEC is satisfied whenever $r < 1/\sqrt{\kappa}$, for positive values of $G$.

2. Geodesics flows in BST-II spacetime

Treating the geodesic congruence as a deformable fluid, one can write the evolution equation of the vector between two fluid points. This vector may get deformed as the geodesics flow and consequently the vector is called the deformation vector. Calling the deformation vector $\xi^\mu$ one can write $\dot{\xi}^\mu = B^\mu_{\nu} \xi^\nu$, where the affine parameter interval in which the rate is measured is supposed to be small. Here $B^\mu_{\nu}$ is a second rank tensor characterizing the time evolution of the deformation vector and its form is as, $B^\mu_{\nu} = \nabla_\nu u^\mu$. Here $u^\mu$ is a tangent vector on a geodesic. Specifically, $u^\nu \nabla_\nu u^\mu = 0$, and choosing a suitable affine parameter one can make $u^\mu u^\mu = -1$ for timelike geodesics while $u^\mu u^\mu = 0$ for a null geodesic. From the above definitions one gets:

$$\ddot{\xi}^\mu = (\dot{B}^\mu_{\nu} + B^\mu_{\tau} B^\tau_{\nu}) \xi^\nu.$$  \hspace{1cm} (4)

The Raychaudhuri equations are obtained by writing $\dot{\xi}^\mu = -R^\mu_{\kappa\tau\nu} u^\kappa u^\nu \xi^\tau$ and equating this with Eq. (4).

In $n$ space-time dimensions, the general form of the second rank tensor $B_{\mu\nu}$ can be decomposed into irreducible parts as $^{[5]}

$$B_{\mu\nu} = \frac{1}{n-1} \Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu},$$ \hspace{1cm} (5)

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ for $u_\mu$ time-like, and $\Theta$ is the expansion variable, $\sigma_{\mu\nu}$ is associated with shear and $\omega_{\mu\nu}$ signifies rotation. One can explicitly write

$$\Theta = B^\mu_{\mu},$$ \hspace{1cm} (6)

$$\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{n-1} \Theta h_{\mu\nu},$$ \hspace{1cm} (7)

$$\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu}).$$ \hspace{1cm} (8)
and the ESR variables are generally denoted by $\Theta$, $\sigma \equiv \sqrt{\sigma^2}$ and $\omega \equiv \sqrt{\omega^2}$. From the discussions in this section it is seen that if one knows the form of $u^\mu$ one can calculate $B_{\mu\nu}$, and hence the ESR parameters. These are expected to give us information about geodesic flows and their properties in BSTs. The general features of a spacetime as encoded in the geodesic structure it admits of is a richly studied subject, to get an idea of the various kinds of work done in this field one is referred to Refs. [6–8].

From now we will discuss the geodesics and their properties in the equatorial plane, i.e. for $\theta = \pi/2$ and $K$ will be assumed to be a negative real number unless it is set to zero. In general we will assume $G$ to be a positive real number. The choices are guided by the energy conditions discussed previously. Using the geodesic equations and the normalization condition of the 4-velocity $u^\mu \equiv dx^\mu/d\lambda$ (where $\lambda$ is the affine parameter) the expression for $u^r$ for the outgoing timelike radial geodesics comes out as:

$$u^r = \beta \sqrt{(1 + Kr^2) \left[ C^2 \left( \sqrt{K + r^{-2}} + G \right) - 1 \right]}. \quad (9)$$

In the above expression $C$ is an integration constant appearing in the geodesic equations. Assuming $Kr^2 + 1 > 0$ this implies that there is a turning point of the outgoing radial time-like geodesics, for $C^2 \left( \sqrt{K + r^{-2}} + G \right) = 1$. This implies that there is a maximum value of $r$ at which outgoing radial geodesics stop. One can also calculate the tangential velocity components for the radial null geodesics in BST-II. One finds that for the radial null geodesics there is no turning point. The $u^\mu$ components of timelike and null circular geodesics shows that there can be an

![Fig. 1. Numerical solutions for $r$ (solid blue), $\Theta$ (dotted red), and $\sigma$ (dashed magenta) for radial time-like geodesics in BST-II, as a function of $\lambda$ for $K = 0$.](image)
upper bound of the radial coordinate above which there cannot be stable circular geodesics. Generally this upper bound on $r$ depends upon the value of $K$ and $G$. For $K = 0$ the upper bound on $r$ tends to infinity.

2.1. The ESR variables for Type II BSTs

If $K = 0$ then the $\Theta$ parameter for the radial outgoing timelike geodesics looks like:

$$\Theta = \frac{(Gr + 1) \left(2C^2 - \frac{r(2Gr + 3)}{(Gr+1)^2}\right)}{2r^2 \sqrt{C^2(G + \frac{1}{r})} - 1}.$$  \hspace{1cm} (10)

One can solve Eq. (9) in terms of $\lambda$ and then plug in this solution in the above equation and get $\Theta$ as a function of the affine parameter. The functional dependence of $r$, $\Theta$ and $\sigma$ on $\lambda$ is shown in Fig. 1. To draw the above plots we have taken $G = 10^{-3}$, $C = 1$, and the lower and upper limits of the affine parameter have been set to $-0.2$ and $1.5$. The upper limit of $\lambda$ is chosen so that $r$ varies from zero to the turning point of $u^r$, which can be seen to be $r \sim 1$ in this case (these numbers are simply for illustration). In Fig. (1), where the solid, dotted and the dashed curves correspond to numerical solutions for $r$, $\Theta$ and $\sigma$ respectively, as a function of the affine parameter, with the chosen initial condition. From the figure we see that $d\theta/d\lambda$ is always negative. As in this case it turns out that the rotation parameter $\omega = 0$ this fact of confirms the focusing theorem. Also, $\Theta$ diverges at the upper and lower limits of $r$, signaling a true spacetime singularity at $r = 0$ and the turning point. The $K \neq 0$ case for radial time-like geodesics follow the same qualitative behavior. To illustrate the case of null geodesics, we have taken $K \neq 0$. We have followed a numerical procedure similar to that alluded to above, and chosen $K = -10^{-6}$, $G = 10^{-2}$, and $C$ and $\beta$ has been set to unity. Here, the lower and upper limits of the affine parameter has been set to $-0.2$ and $1.389$ respectively. Using the same numerical procedure as above, we solve for the expansion parameter for the null radial geodesics:

$$\Theta = \beta Cr^{-\frac{2}{3}} \left(Gr^2 + 1\right) \left(Gr + \sqrt{Kr^2 + 1}\right)$$  \hspace{1cm} (11)

and this is illustrated in Fig. 2, where we have multiplied $\Theta$ by a factor of $10^3$ to display the curves on the same graph. Note that in this case the expansion parameter diverges at $r = 0$ as expected from Eq. (11). The radial null geodesics converge with increasing radial distance but does not show any singular behaviour for finite $r$. This implies that there is no event horizon for BST-II like the blackhole solutions.

In our work we did not explicitely solve the Raychaudhuri equation in BST-II but have expressed the ESR parameters in terms of the affine parameter $\lambda$ by using the geodesic equations. As finally the ESR parameters are expressed in terms of $\lambda$ our treatment indirectly gives the solution of the Raychaudhuri equations in BST-II. Our analysis points to the fact that BST-II can be thought of as interesting realistic
examples of static, spherically symmetric space-times, which are asymptotically non-flat, and allow for stable, periodic orbits. This might be significant in astrophysical scenarios: for example, one might ask if a realistic space-time near a compact object can be modelled via BST-II. In this talk the main attention was given to the geodesics of BST-II and the ESR variables to understand the effect of spacetime curvature and probable singularities. To properly utilize BSTs, one must also have to think of the source of such kind of spacetime's and in future works one needs to look at this important aspect.

References

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