Statistical Assessment of Shapes and Magnetic Field Orientations in Molecular Clouds through Polarization Observations

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ABSTRACT

We present a novel statistical analysis aimed at deriving the intrinsic shapes and magnetic field orientations of molecular clouds using dust emission and polarization observations by the Hertz polarimeter. Our observables are the aspect ratio of the projected plane-of-the-sky cloud image, and the angle between the mean direction of the plane-of-the-sky component of the magnetic field and the short axis of the cloud image. To overcome projection effects due to the unknown orientation of the line-of-sight, we combine observations from 24 clouds, assuming that line-of-sight orientations are random and all are equally probable. Through a weighted least-squares analysis, we find that the best-fit intrinsic cloud shape describing our sample is an oblate disk with only small degrees of triaxiality. The best-fit intrinsic magnetic field orientation is close to the direction of the shortest cloud axis, with small (\(\sim 24^\circ\)) deviations toward the long/middle cloud axes. However, due to the small number of observed clouds, the power of our analysis to reject alternative configurations is limited.

1 INTRODUCTION

Far-infrared and sub-millimeter emission from molecular clouds appears polarized, presumably as a result of the alignment of elongated dust grains with the cloud magnetic field (see, e.g., Curran & Chrysostomou 2007; Dotson et al. 2000, 2009; Hildebrand et al. 2000; Hoang & Lazarian 2008; Lazarian 2003, 2007). Measurable degrees of polarization, at the few percent level, are typical in many interstellar clouds (e.g., Curran & Chrysostomou 2007; Dotson et al. 2000, 2009). Additionally, theoretical advancements in understanding grain alignment (e.g., Bethell et al. 2007) indicate that even in relatively dense clouds, dust polarization traces the underlying magnetic field structure. The ubiquity of magnetic fields and dust in the interstellar medium makes polarization observations a powerful tool for distinguishing between theories of molecular cloud formation, support, and evolution.

From a theoretical point of view, the orientation of the mean magnetic field in molecular clouds is closely tied to the dynamical importance of magnetic forces compared to gravity, random motions (turbulence), and thermal pressure. If magnetic fields are dynamically important, and are responsible for a significant fraction of the support of molecular clouds against gravity, then the mean magnetic field is preferentially oriented parallel to the shortest axis of the molecular cloud (e.g., Mouschovias 1978). This is a result of an increased support against gravity in the direction perpendicular to the magnetic field compared to the direction parallel to it. The cloud contracts more in the direction parallel to the magnetic field than in the direction perpendicular to the field. An additional result of magnetic support is that the intrinsic shapes of molecular clouds in this case resemble mostly oblate (although not necessarily axisymmetric), flattened ellipsoids (one axis appreciably smaller than the other two).

If magnetic fields are dynamically unimportant compared to turbulence, then turbulent motions dominate the internal dynamics of clouds, and the magnetic fields are dragged around by turbulent eddies (e.g., Ballesteros-Paredes, Vázquez-Semadeni & Scalo 1999). In this case, the mean magnetic field has a random orientation with respect to the molecular cloud principal axes. The shape distribution for overdensities formed in a turbulent field is also random (Gammie et al. 2003). Molecular clouds forming out of magnetohydrodynamic turbulence in the weak-field regime therefore would be expected to have random shapes and random magnetic field orientations.

A third possibility is that magnetic fields have a helical configuration and thread prolate (filamentary) molecular clouds (Fiege & Pudritz 2000a). The most common outcome of such configurations is polarization patterns that may contain 90° flips of the polarization vector (Fiege & Pudritz 2000b). Additional ideas for molecular cloud shapes come from non-magnetic calculations; for example, finite, self-gravitating gaseous shells have been shown to collapse to filamentary structures with mass concentrations close to...
the edges or ends of the filaments (e.g., Burkert & Hartmann 2004; Hartmann & Burkert 2007).

If it were possible to determine the mean orientation of the ordered magnetic field in molecular clouds, then important constraints could be placed on theories of molecular cloud formation and support. However, such a task is not straightforward, due to various projection effects which prohibit us from knowing either the true orientation of the magnetic field or the principal axes of the cloud ellipsoid. Polarization measurements can only determine the orientation of the magnetic field on the plane-of-the-sky (POS), and, similarly, only the shape of the POS projection of the cloud can be measured. As a result, the magnetic field may be oriented, for example, along the shortest cloud axis, but its projection on the sky may appear closest to the major axis of the projected cloud ellipse. These difficulties were explicitly demonstrated by Basu (2000).

An effective way to overcome these difficulties and use polarization measurements to constrain theoretical models for the orientation of the mean magnetic field in molecular clouds is through a statistical treatment. Assuming that the orientation of the clouds themselves with respect to our LOS is random, we can assess the likelihood of different underlying distributions of molecular cloud shapes, and of orientations of the mean magnetic field in molecular clouds. In this work, we present a new method for such a treatment. We study a sample of 24 molecular clouds with measured apparent elongations and field orientations. By applying a weighted least-squares analysis, we derive the most probable intrinsic shapes and most probable magnetic field orientations for the parent population that the 24 clouds sample.

This paper is organized as follows. The observations and analysis used to derive elongations and apparent field orientations in the sample clouds are described in §2. The formalism used in our analysis, including a short discussion of projection effects and a description of our statistical analysis, are discussed in §3 (a detailed mathematical treatment of projection effects and the statistical analysis are given in Appendices A and B respectively). Our results are presented in §4 and discussed in §5.

2 OBSERVATIONS

The Hertz polarimeter (Schleuning et al. 1997; Dowell et al. 1998) deployed at the Caltech Submillimeter Observatory has been used to map the flux density and polarization of molecular clouds on scales of several arcminutes at a wavelength of 350 μm. In a complete sample of 56 Galactic clouds, 32 contain 2 or more polarization measurements satisfying the criterion $P \geq 3\sigma_p$, where $P$ is the measured polarization amplitude and $\sigma_p$ is its measurement uncertainty. The clouds are listed in Table I object positions and complete maps of flux density and polarization can be found in Dotson et al. (2009).

Figure 1. Mean polarization and cloud shape in DR21. Polarization measurements (small black line-segments) are plotted with their lengths proportional to $P$ (scale at lower-left); only $P \geq 3\sigma_p$ measurements are shown. The mean polarization direction ($E$-vector) and its dispersion are shown as the red line and the arcs at its end. The cloud principal axes are drawn as blue lines; their relative lengths are given by the aspect ratio $q = 0.26$. The lines corresponding to the mean polarization and cloud widths intersect at the cloud center ($x_0, y_0$) and are drawn with arbitrary absolute lengths. The gray arc labeled “$\lambda$” denotes the difference between the mean polarization angle and the cloud position angle. Coordinate offsets from $20^h39^m11^s, 42^d19^m31^s$ (J2000). Contours are 10, 20, ..., 90% of the peak flux density of 820 Jy/beam (at $(\Delta\alpha, \Delta\delta) = (0', 3.3')$). The shaded circle at lower-right is the $20''$ FWHM Hertz beam.

2.1 Polarization

Within each cloud, we limit the data points used to calculate mean polarizations to those satisfying the condition $P \geq 3\sigma_p$. The best estimate of the POS-projected mag-
nomic field direction is that of the measured polarization vector (rotated by 90°). The amplitude of the polarization $P$ yields some information about the inclination of the field to the line-of-sight (LOS). However, $P$ is also dependent on a number of other factors which are independent of the field, including the dust grain shape and alignment efficiency, and the magnetic field structure along the LOS and within the telescope beam (e.g., Draine & Lee 1985). Therefore, our analysis of the magnetic field will involve only the polarization position angle.

The mean polarization position angle $\langle \chi \rangle$ is defined by averaging Stokes parameters of unit magnitude such that

$$\langle \chi \rangle = \frac{\tan^{-1} \left( \frac{\sin 2 \chi}{\cos 2 \chi} \right)}{2},$$

(1)

where

$$\langle \sin 2 \chi \rangle = \frac{1}{N} \sum_{i=1}^{N} \sin 2 \chi_i,$$

(2)

$$\langle \cos 2 \chi \rangle = \frac{1}{N} \sum_{i=1}^{N} \cos 2 \chi_i,$$

(3)

and the $\chi_i$’s are the measured polarization angles at each position $i$ in the cloud. The sums are over all $N$ points in the map where the measured polarization $P \geq 3\sigma_p$.

We calculate the standard deviation of the mean $\langle \chi \rangle$ under the assumption of zero intrinsic dispersion (i.e., under the assumption that all of the dispersion in the observed $\chi_i$ is due to observational errors) as

$$\sigma_{\langle \chi \rangle} = \frac{1}{N} \left[ \sum_{i=1}^{N} \sigma_i^2 \right]^{1/2},$$

(4)

where $\sigma_i$ are the uncertainties on the measurements of $\chi_i$. The measurement uncertainty is limited to Hertz’s systematic uncertainties such that $\sigma_{\langle \chi \rangle} \geq 2^\circ$ (Dowell et al. 1998). In our case, the dispersion in the observed values of $\chi_i$ in individual clouds is substantial, so we also calculate the dispersion about the mean, $\sigma_{\chi}$, given by the standard deviation of the measurements

$$\sigma_{\chi} = \left[ \frac{1}{N-1} \sum_{i=1}^{N} (\chi_i - \langle \chi \rangle)^2 \right]^{1/2}.$$  

(5)

In using this equation we have accounted for the 180-degree periodicity in the polarization angles.

Each cloud’s mean polarization amplitude is given by

$$\langle P \rangle = \sqrt{\langle Q \rangle^2 + \langle U \rangle^2} / \langle I \rangle,$$

(6)

where $I$, $Q$, and $U$ are the measured Stokes parameters. The mean polarizations, angles, and uncertainties are all reported in Table I. The $\langle P \rangle$’s reported in Table I have been de-biased using the statistical measurement uncertainty on the mean (e.g., Vaillancourt 2006). However, the measurement uncertainties in the table are not those used to de-bias, but are limited to Hertz’s systematic uncertainties such that $\sigma_{\langle \chi \rangle} \geq 0.2^\circ$ (Dowell et al. 1998).

### 2.2 Cloud Shapes

In the case of a symmetric ellipsoidal cloud, the first- and second-moments of the flux density describe the cloud’s center and width. The location of the cloud’s center $(x_0, y_0)$ is given by the first-moments:

$$x_0 = \frac{\sum_i x_i F(x_i, y_i)}{\sum_i F(x_i, y_i)},$$

(7)

$$y_0 = \frac{\sum_i y_i F(x_i, y_i)}{\sum_i F(x_i, y_i)},$$

(8)

where the sum is over all points $i$ in the flux density map. No cuts are applied to the flux density data beyond those already made in the Dotson et al. (2009) archive. (The $P \geq 3\sigma_p$ criterion is not applied.)

The second moments compose the 3 elements of a symmetric $2 \times 2$ matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix},$$

(9)

with elements

$$I_{xx} = \sum_i (y_i - y_0)^2 F(x_i, y_i),$$

(10)

$$I_{xy} = -\sum_i (x_i - x_0)(y_i - y_0) F(x_i, y_i),$$

(11)

$$I_{yy} = \sum_i (x_i - x_0)^2 F(x_i, y_i).$$

(12)

The principal moments of the cloud are simply the eigenvalues, $I_1$ and $I_2$, of $\mathbf{I}$. Defining $I_1 \geq I_2$, the aspect ratio is given by $q = \sqrt{I_2/I_1}$ and the position angle of the long cloud axis is determined from the eigenvector corresponding to the eigenvalue $I_2$.

The first and second moments are well-defined and express the distribution of the flux density in the cloud even when the shape is irregular and cannot be well approximated by an ellipse. Therefore, we use these moments to define the aspect ratio and position angle of the principal axes for all clouds in our sample, with the caveat that we have ignored...
any higher-order moments which more precisely characterize the cloud shape. We note that some bias can be introduced through the finite size of each map.

Examples of the mean polarization and cloud shape calculations are shown in Figure 1 for the most elongated cloud, and Figure 2 for a nearly circular cloud. The measured cloud parameters are given in Table 1. The apparent cloud aspect ratio $q$ and the absolute value of the angle between mean field and short apparent axis, $\lambda$, are plotted in Fig. 3. Data shown as open circles in Fig. 3 and denoted by $h$ in Table 1 are not included in further analysis due to their large ($>30^\circ$) dispersion in $\lambda$.

Error bars on $\lambda$ correspond to the quadrature sum of $\sigma_{(\lambda)}$, the systematic angle uncertainty (2 degrees; Dowell et al. 1998), and the systematic polarization uncertainty ($\Delta \chi_{\text{sys}}$). We estimate the last quantity using the relation

$$\Delta \chi_{\text{sys}} = \frac{90^\circ}{\pi} \frac{0.2\%}{\langle P \rangle},$$

where 0.2% is Hertz’s systematic polarization uncertainty. We note that equation (13) assumes measurements of $P$ and $\chi$ follow normal distributions (equivalent to the assumption $P \gg \sigma_p$). While this is clearly not true for all measurements in Table 1, this fact has minimal effect on the apparent size of the error bars and has no effect on our subsequent analysis (as the uncertainties are not used).

Furthermore, these angle uncertainty estimates do not include any uncertainties in the estimate of the cloud shape, but include only the uncertainties on the measurement of the polarization angle. A visual comparison of Figures 1 and 2 should make it apparent that the uncertainty in the cloud orientation is smallest when $q = 0$ and must increase as the cloud becomes more circular ($q = 1$). An estimate of this uncertainty for all clouds in our sample is beyond the scope of this paper. Neglecting these uncertainties has no effect on our conclusions as this information is not used in our subsequent data analysis.

Data shown as open circles in Fig. 3 and denoted by $h$ in Table 1 are not included in further analysis due to their large ($>30^\circ$) dispersion in $\lambda$.

3 ANALYSIS

3.1 Projection Effects

Let us consider a triaxial ellipsoid model molecular cloud with semi-axes $a \geq b \geq c$, and axial ratios

$$\zeta = \frac{b}{a}$$

(14)

(middle-to-long axis ratio) and

$$\xi = \frac{c}{a}$$

(15)

(short-to-long axis ratio). When observed, the image of the molecular cloud appears on the POS as an ellipse. The observable quantity related to the cloud shape is the ellipse aspect ratio, $q$. Let us additionally assume that the molecular cloud is threaded by a magnetic field, the mean direction of which forms a polar angle $\theta_B$ with the short axis of the cloud ellipsoid, and an azimuthal angle $\phi_B$ with the long axis of the cloud ellipsoid (see Fig. 4). Through polarimetry observations, only the direction of the projection of the field on the POS can be measured, and the associated observable

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Figure 3. Elongations and angles between the magnetic field and the minor cloud axis for the 32 clouds in our sample. Note that the elongation $q$ is defined so that a very elongated cloud corresponds to $q \to 0$ and a circular cloud to $q = 1$. For angles, $\lambda = 0$ corresponds to a projected magnetic field aligned with the minor axis of the projected cloud. Each error bar represents the quadratic sum of the statistical uncertainty of the cloud’s mean angle uncertainty ($\sigma_{(\lambda)}$), the systematic angle uncertainty, and the systematic polarization uncertainty (see text for precise definitions of these terms). Data shown as open circles are not included in further analysis due to their large dispersion in $\lambda$ (all data are given in Table 1). Dotted horizontal lines are drawn at the boundaries of the allowed range $0^\circ < \lambda < 90^\circ$. The diagonal $\lambda/90^\circ = q$ is shown as a solid line.

Figure 4. Solid black lines: native cloud ellipsoid coordinate system ($x, y, z$). Dashed blue lines: observation coordinate system ($x', y', z'$). The LOS is along the $z'$ axis, and the POS is the $x'y'$ plane. The $y'$ axis represents the direction of the projection of the shortest ellipsoid axis onto the POS. The directions of the magnetic field and of its projection onto the $x$-$y$ plane are shown in red.

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2 Further study of polarization angle uncertainties at low $P/\sigma_p$ can be found in Naghizadeh-Khouei & Clarke (1993)
Table 1. Mean Polarization and Cloud Shape Parameters

| Object No. | Object Name | No. vectors | $\langle P \rangle$ (%) | $\sigma_{\langle P \rangle}$ (%) | $\langle \chi \rangle$ (degrees) | $\sigma_{\langle \chi \rangle}$ (degrees) | $\sigma_{\chi}$ (degrees) | Aspect Ratio | Angle $^d$ (degrees) | $|\lambda|^e$ (degrees) |
|------------|-------------|-------------|-------------------------|-------------------------------|-------------------------------|-------------------------------|----------------------------|---------------|---------------------|---------------------|
| 1 | W3$^h$ | 51 | 0.6 | 0.2 | 65 | 2 | 32 | 0.83 | −80 | 35 |
| 2 | NGC 1333 | 18 | 1.6 | 0.2 | 88 | 2 | 19 | 0.92 | −37 | 3 |
| 3 | IRAS 05327-0457 | 18 | 3.9 | 0.4 | −40 | 2 | 17 | 0.51 | 5 | 20 |
| 4 | OMC-1 | 437 | 2.0 | 0.2 | 25 | 2 | 26 | 0.44 | 14 | 53 |
| 5 | OMC-2$^h$ | 26 | 0.7 | 0.2 | −39 | 4 | 33 | 0.74 | −51 | 9 |
| 6 | OMC-3 MMS 6$^f$ | 43 | 2.1 | 0.2 | −42 | 2 | 12 | 0.95 | 11 | 73 |
| 7 | OMC-3 MMS 9$^f$ | 21 | 1.2 | 0.2 | −62 | 2 | 20 | 0.84 | 57 | 71 |
| 8 | OMC-4 | 16 | 1.7 | 0.2 | −14 | 2 | 13 | 0.65 | −15 | 8 |
| 9 | NGC 2024 | 54 | 0.7 | 0.2 | −23 | 2 | 24 | 0.98 | −50 | 49 |
| 10 | NCG 2068 LBS17 | 3 | 8.3 | 1.5 | 81 | 5 | 16 | 0.93 | 38 | 89 |
| 11 | NCG 2068 LBS10 | 43 | 3.8 | 0.2 | −51 | 2 | 20 | 0.82 | −18 | 48 |
| 12 | NGC 2071 | 6 | 0.6 | 0.2 | −27 | 4 | 25 | 0.97 | −13 | 14 |
| 13 | Mon R2$^h$ | 49 | 0.7 | 0.2 | 31 | 2 | 32 | 0.82 | 47 | 16 |
| 14 | GGD12 | 17 | 1.2 | 0.2 | 89 | 2 | 16 | 0.94 | −60 | 31 |
| 15 | S269 | 7 | 2.9 | 0.3 | 30 | 3 | 13 | 0.82 | −18 | 48 |
| 16 | Mon OB1 12$^g$ | 26 | 2.0 | 0.2 | −37 | 2 | 17 | 0.76 | 27 | 64 |
| 17 | NGC 2264 | 18 | 0.6 | 0.2 | 77 | 2 | 16 | 0.84 | −47 | 56 |
| 18 | $\rho$ Oph | 41 | 1.6 | 0.2 | −21 | 2 | 13 | 0.76 | 7 | 29 |
| 19 | IRAS 16293 | 7 | 0.6 | 0.2 | 90 | 3 | 25 | 0.67 | −65 | 26 |
| 20 | NGC 6334 V$^h$ | 8 | 0.3 | 0.2 | 90 | 3 | 34 | 0.93 | −53 | 37 |
| 21 | NGC 6334 A | 49 | 1.3 | 0.2 | 69 | 2 | 18 | 0.89 | −3 | 71 |
| 22 | NGC 6334 I | 54 | 0.9 | 0.2 | 41 | 2 | 19 | 0.51 | 11 | 30 |
| 23 | W33 C$^h$ | 29 | 0.4 | 0.2 | 48 | 2 | 37 | 0.88 | 82 | 35 |
| 24 | W33 A | 19 | 0.9 | 0.2 | −53 | 2 | 13 | 0.90 | −49 | 4 |
| 25 | M17 | 127 | 0.9 | 0.2 | −13 | 2 | 27 | 0.59 | −10 | 3 |
| 26 | W43-MM1 | 4 | 1.5 | 0.3 | 9 | 4 | 22 | 0.88 | 34 | 25 |
| 27 | G34$^h$ | 44 | 0.5 | 0.2 | −66 | 2 | 41 | 0.94 | 72 | 42 |
| 28 | G49 A | 32 | 0.6 | 0.2 | 59 | 2 | 29 | 0.87 | −80 | 41 |
| 29 | W51 A | 109 | 0.5 | 0.2 | 44 | 2 | 28 | 0.71 | −78 | 58 |
| 30 | W75 N$^b$ | 9 | 0.3 | 0.2 | 44 | 3 | 47 | 0.98 | 76 | 33 |
| 31 | DR21 | 142 | 1.2 | 0.2 | 5 | 2 | 20 | 0.26 | 0 | 5 |

$^a$ Mean polarization parameters defined in equations (1) – (6). Means are calculated using only the $P \geq 3\sigma_p$ vectors shown in “No. vectors” column. Angle $\chi$ is measured east of north.

$^b$ Object names follow the convention in Dotson et al. (2009); list is ordered with approximately increasing Right Ascension.

$^c$ Dispersion about mean polarization angle given by equation (5).

$^d$ Direction of long cloud axis, measured east of north.

$^e$ Angle between mean polarization angle and long cloud axis. This is equivalent to the angle between the inferred magnetic field direction and the short cloud axis.

$^f$ Part of OMC-3 in Dotson et al. (2009)

$^g$ a.k.a. IRAS 06382+0939

$^h$ Objects with angle dispersion larger than 30 degrees are not included in the angle analysis.

The observables $q$ and $\lambda$ can be calculated as a function of $\zeta$, $\xi$, $\theta_B$, $\phi_B$ and of the orientation angles of the observer’s LOS, $\theta$ and $\phi$. This calculation is described in detail in Appendix A.

However, the orientation of the LOS is unknown — in the absence of biases, the LOS orientation can be treated as uniformly distributed among different directions with respect to the observer. As a result, it is not generally possible to de-project the intrinsic cloud shape and magnetic field orientation for any single object. Instead, a statistical treatment must be used: the preferred intrinsic cloud shapes and magnetic field orientations in nature can only be obtained by observing a large number of clouds, and comparing the distribution of observables with expectations based on different intrinsic shapes and field orientations. In this work, we employ such a statistical analysis described below.

3.2 Statistical Analysis

As a first proof-of-concept, we test for the presence of possible degeneracies in the distribution of the observables ($q, \lambda$) which may limit the value of this analysis. The simplest
tests we can perform to verify that the data of Fig. 3 have nontrivial information content derive from the immediate observation that almost all datapoints lie below the diagonal extending from (0,0,0) to (1,0,90). We evaluate the probability to obtain this configuration by chance.

First, we consider the case in which both aspect ratios and angles are randomly drawn from uniform distributions. Under this assumption, the desired probabilities can be obtained analytically: the diagonal splits the parameter space in two parts of equal area. Therefore, the probability for any single point to lie above or below the diagonal is 0.5, and the outcome of multiple draws obeys the binomial distribution with probability for a positive outcome equal to 0.5. If we consider all 32 datapoints (including the ones that did not survive our quality cuts), then we have 2/32 points above the diagonal. The probability to obtain the observed outcome or one which is even more biased towards points below the diagonal is the sum of the probabilities to obtain 0/32, 1/32 or 2/32 points above the diagonal:

\[
P(\# \text{ above diagonal } \leq 1) = \sum_{i=0}^{2} \binom{32}{i} 0.5^{32-i} = 1.2 \times 10^{-7}
\]

If we consider only the 24 datapoints that survived our quality cuts, then we have 1/24 points above the diagonal, and the associated probability to have obtained such a result by chance is

\[
P(\# \text{ above diagonal } \leq 1) = \sum_{i=0}^{1} \binom{24}{i} 0.5^{24-i} = 1.2 \times 10^{-7}
\]

with the result being numerically approximately equal as in the previous case. This configuration is extremely unlikely to have been obtained by chance from a uniformly distributed parameter space.

Next, we consider the case in which the aspect ratio distribution in nature is identical to the observed aspect ratio distribution, but angles are randomly drawn from a uniform distribution. For each observed aspect ratio, we randomly select an angle, and we calculate the probability that the number of points below the diagonal is \( \leq 2 \) for the 32 datapoints and \( \leq 1 \) for the 24 datapoints surviving the cuts. We find that this probability is equal to 0.98% and 1.3% respectively. Although more likely than before, this scenario also has a very small probability of occurring.

A more sophisticated version of these qualitative arguments can be made starting from characteristic limiting cases for the distribution of intrinsic cloud features (shapes and magnetic field orientations). For each such case we compute, by performing “mock observations” along randomly selected and uniformly distributed LOS, the joint probability density function (PDF) of our two observables, the elongation \( q \) and the projected field - short ellipse axis angle \( \lambda \). The results are shown in Fig. 3 overplotted with the 24 observed points that survive the \( \lambda \) dispersion cut described in [2]. The parameters of the intrinsic shape and field orientation distributions that lead to each PDF are given in Table 2. These parameters are: in the case of the intrinsic shape distribution, the cloud ellipsoid axial ratios \( (\zeta_{\text{max}}, \zeta_{\text{max}}) \) of the most probable shape (the values that maximize the PDF of axial ratios); and in the case of the field orientation distribution, the most probable cosine of the field orientation polar angle \( \cos \theta_{B,\text{max}} \) and the standard deviation of the cosine of the field polar angle, \( \sigma(\cos \theta_B) \). The exact functional forms of the shape and field orientation distributions used are discussed in Appendices [11] and [12].

The qualitative distribution of the data points on the \( q-\lambda \) plane forms a triangular shape, with the lower-right part of the plot (almost circular cloud images, small angles of the magnetic field with the short ellipse axis) more populated than the upper left part of the plot (very elongated cloud images, large angles of the magnetic field with the short ellipse axis). In general, there are many fewer very elongated POS cloud ellipses than circular-looking ellipses, which qualitatively indicates oblate intrinsic shapes (since there are many more LOS that will yield a circularly-looking POS ellipse for a disk-like cloud than for a cigar-like cloud). Additionally, large angles with the short ellipse axis are only encountered for almost circular cloud images (giving rise to the triangular distribution of the data on the observables plane), which qualitatively hints to an orientation of the magnetic field at small angles with the short cloud axis (so that when the cloud is viewed edge-on, the observations yield an elongated POS ellipse and a small POS field angle with the short ellipse axis, cf. Fig. 1 while when viewed face-on the cloud looks circular and a larger variety of angles of the POS field with the short ellipse axis are possible, cf. Fig. 2).

The different PDFs plotted in Fig. 5 demonstrate how the behavior of the PDF of the observables responds to changes in the underlying distributions of shapes and field orientation. For oblate shapes, the contours of the joint PDF for \( \lambda \) and \( q \) for different field orientations are shown in panels a–d. As expected, when the field is oriented parallel to the short cloud axis, the \( \lambda-q \) PDF acquires a roughly triangular shape preferentially populating the high-\( q \) low-\( \lambda \) corner (panel a), while for a field oriented perpendicular to the short cloud axis the PDF is also roughly triangular, however now preferentially populating the high-\( q \) high-\( \lambda \) corner (panel b). For the intermediate situation of a field oriented at 45° from the short cloud axis, the PDF forms a “spike” at the mid-\( q \) mid-\( \lambda \) part of the plot (panel c). Finally, for a random distribution of field orientations (panel d), the contours are parallel to the \( \lambda \) axis, with low \( q \) values preferred. PDFs resulting from prolate intrinsic cloud shapes are shown in panels e and f, with panel e corresponding to a magnetic field oriented along the long axis (filamentary clouds formed along the magnetic field), and panel e corresponding to a magnetic field oriented along the short axis (filamentary clouds perpendicular to the magnetic field). In both cases, low values of \( q \) are preferred (a situation not seen in the data). The PDF of panel e preferentially populates the high-\( \lambda \) part of the plane, while the PDF of panel f preferentially populates the low-\( \lambda \) region.

We can therefore see that there is enough qualitative and quantitative variation between these PDFs so that, with a sufficiently large dataset, the effect of the random orientation angles can be overcome and we can draw conclusions regarding the underlying distributions of cloud shapes and magnetic field orientations in nature.

In order to determine which such intrinsic distributions of cloud parameters fit our data best, we bin the \( \lambda-q \) parameter space in \( 5 \times 5 \) bins. For each bin, we count the number of observed data points that fall within the bin limits. We parameterize the intrinsic shapes distribution as described
Figure 5. Contour plots of the joint probability density function of the observables \( q \) and \( \lambda \) obtained by convolving different intrinsic distributions of cloud shapes and magnetic field orientations with random LOS. The solid, dashed, and dotted lines correspond to the 1, 2, and 3\( \sigma \) contours, respectively. The blue data points correspond to our 24 data points surviving the quality cuts. Panel a: oblate clouds, magnetic field preferentially oriented parallel to short cloud axis; panel b: oblate clouds, magnetic field preferentially oriented parallel to the long cloud axis; panel c: oblate clouds, magnetic field preferentially oriented at a 45 degree angle with the short cloud axis; panel d: oblate clouds, uniformly distributed magnetic field orientations; panel e: prolate clouds, magnetic field preferentially oriented parallel to long cloud axis; panel f: prolate clouds, magnetic field preferentially oriented parallel to short cloud axis. The parameters of the intrinsic distributions resulting in each PDF are given in Table 2.

Table 2. Parameters of the intrinsic distributions giving rise to the PDF in each panel of Fig. 5.

| panel | \( \zeta_{\text{max}} \) | \( \xi_{\text{max}} \) | \( \cos \theta_{B_{\text{max}}} \) | \( \sigma(\cos \theta_B) \) | qualitative behavior |
|-------|----------------|----------------|----------------|----------------|-------------------|
| a     | 0.999          | 0.58           | 0.91           | 0.21           | oblate, \( B \parallel \) short axis |
| b     | 0.999          | 0.53           | 0.19           | 0.19           | oblate, \( B \perp \) short axis |
| c     | 0.999          | 0.62           | 0.70           | 0.04           | oblate, \( B \) at 45\(^\circ\) with short axis |
| d     | 1.00           | 0.62           | uniform in \( \cos \theta_B \) | oblate, \( B \) random |
| e     | 0.29           | 0.08           | 0.19           | 0.19           | prolate, \( B \perp \) short axis |
| f     | 0.29           | 0.08           | 0.91           | 0.16           | prolate, \( B \parallel \) short axis |

in Appendix B2 using a bi-parametric joint distribution in \( \zeta \) and \( \xi \) of non-zero spread, uniquely defined by the values of \( \zeta \) and \( \xi \) where the probability density becomes maximum, \( \zeta_{\text{max}} \) and \( \xi_{\text{max}} \). We parameterize the intrinsic distribution of magnetic fields orientations as described in Appendix B2 using a bi-parametric distribution in \( \cos \theta_B \), uniquely defined by the value of \( \cos \theta_B \) where the probability density becomes maximum.

in Appendix B3 indicates that the distribution of cloud shapes is strongly peaked on the \( \zeta = 1 \) axis, implying that the long and middle axes of the cloud ellipsoid are very close to being equal, the \( \phi_B \) angles are physically degenerate, and the only quantity of interest determining the orientation of the magnetic field is the angle with the shortest axis of the cloud ellipsoid, \( \theta_B \).
bility density becomes maximum, $\cos \theta_{B,\text{max}}$, and the standard deviation of the distribution, $\sigma(\cos \theta_B)$. Note that we wish to know the distribution of $\cos \theta_B$ rather than that of $\theta_B$, since the quantity of interest is the fraction of LOS per solid angle in any given direction $\theta_B$, which is expressed by $\cos \theta_B$ (see also discussion in Appendix B). Assuming uniformly distributed LOS, we determine the joint PDF for $q$, $\lambda$, and the expected counts for 24 observations in each bin, as well as the spread in the expected counts. We perform a weighted least-squares analysis to find the best-fit parameters $\xi_{\text{max}}, \xi_{\text{max}}$, $\cos \theta_{B,\text{max}}, \sigma(\cos \theta_B)$, using the inverse square of the spread of the expected counts as our weight. The details of this statistical analysis are discussed in Appendix B. Because of the nonlinearity of the problem, there are no applicable analytic solutions to the minimization of the weighted sum of squared residuals. For this reason, we have used a well-tested, Monte-Carlo-based approach, the “Simulated Annealing” algorithm (Corana et al. 1987) to sample the parameter space and identifying the location of the absolute minimum of the weighted sum of squared residuals.

4 RESULTS

The results of our weighted least-squares analysis are shown in Fig. 6. The upper panel shows contours of weighted squared residuals, $S_{\text{weighted}}$, as defined in Appendix B on the $\xi_{\text{max}}-\xi_{\text{max}}$ plane, with $\cos \theta_{B,\text{max}}$ and $\sigma(\cos \theta_B)$ fixed at their best-fit values. The lower panel shows contours of $S_{\text{weighted}}$ on the $\cos \theta_{B,\text{max}}-\sigma(\cos \theta_B)$ plane, with $\xi_{\text{max}}$ and $\xi_{\text{max}}$ fixed at their best-fit values. The allowable values of $\sigma(\cos \theta_B)$ vary between 0 (δ-function of B-field orientations) and 0.289 (uniform distribution of B-field orientations with all orientations equally probable, see Appendix B). The color scale corresponds to values of $S_{\text{weighted}}$ from 9.7 to 25, with the contours spaced by factors of 1.1 (so the suppressed z-axis in this plot which is visualized by the contours is in logarithmic scale). The location of the minimum is indicated in each case by the white point within the back region. The outermost contour corresponds to the $S_{\text{weighted}}$ value that is typically yielded by 24 observations drawn from the best-fit intrinsic distributions due to random fluctuations, as calibrated by Monte-Carlo simulations.

As expected from the qualitative arguments outlined above, oblate cloud shapes are preferred. In the case of the magnetic field orientation, there is a well-defined broad minimum in the weighted least-squares analysis, corresponding to a clear preference for a small angle from the short cloud ellipsoid axis and a moderate spread of angle about that. However, given the small number of data, the power of the test to reject models is limited, and a large fraction of the $\cos \theta_{B,\text{max}}-\sigma(\cos \theta_B)$ plane (excluding, however, distributions strongly peaked at large angles from the shortest cloud axis) could be consistent with the presently available data.

The best-fit values of the parameters are shown in the first line of Table 2 and the resulting PDF of our observables, $q$ and $\lambda$, is shown in panel a of Fig. 6. The best-fit distribution of intrinsic shapes (axial ratios of the model cloud ellipsoids) and magnetic field orientation angles are shown in the upper and lower panels of Fig. 6 respectively. The most probable shape is a moderately thick oblate disk. The best-fit shapes distribution is very strongly peaked close to $\xi \approx 1.0$ (so clouds are likely to have very small degrees of triaxiality). On the other hand, the distribution is very spread out in $\xi$, implying that many different disk thicknesses are possible. Although the most probable shape is a relatively thick disk, this does not mean that all clouds are thick disks. Much thinner disks [closer to $(\xi, \zeta) = (1.0)$], as well as very

![Figure 6. Contours of the weighted sum of squared residuals, $S_{\text{weighted}}$, for the $(\xi_{\text{max}}, \xi_{\text{max}})$ parameters (upper panel) and the $(\cos \theta_{B,\text{max}}, \sigma(\cos \theta_B))$ parameters (lower panel). The upper axis in the lower panel shows the most-probable orientation in degrees from the short cloud ellipsoid axis, corresponding to the value of $\cos \theta_{B,\text{max}}$ shown in the lower axis. The two parameters not shown in each plot are kept fixed at their least-squares values. The color scale corresponds to values of $S_{\text{weighted}}$ from 9.7 to 25, with the contours spaced by factors of 1.1 (so the suppressed z-axis in this plot which is visualized by the contours is in logarithmic scale). The minimum is indicated in each case by the white point within the back region. The final contour corresponds to the $S_{\text{weighted}}$ value that it typically yielded by 24 observations drawn from the best-fit intrinsic distributions due to random fluctuations, as calibrated by Monte-Carlo simulations. The solid line in the upper panel separates the domain of allowed values $(\phi \geq \zeta_d)$ from the rest of the plane, while the dotted line separates mostly oblate $[\xi_{\text{max}} \geq 0.5(1 + \xi_{\text{max}})]$ from mostly prolate $[\xi_{\text{max}} < 0.5(1 + \xi_{\text{max}})]$ most-probable ellipsoids.](image-url)
while the best-fit parameters of the $B$-field orientations distribution change from $[\cos \theta_{V_{\text{max}}, \sigma(\cos \theta_B)}] = (0.91, 0.21)$ to $[\cos \theta_{V_{\text{max}}, \sigma(\cos \theta_B)}] = (0.86, 0.19)$ - a change smaller than 10% in any one of the parameters, while the qualitative behavior of the underlying distributions remains unchanged. This result is not surprising: in the upper-left panel of Fig. 2 DR 21 is outside the 2$\sigma$ contour of the best-fit joint PDF for $q$ and $\lambda$, and barely within the 3$\sigma$ contour.

As indicated by the broadness of the minimum of $S_{\text{weighted}}$ as a function of our fitted parameters, the best-fit intrinsic distributions of shapes and field orientations are not unique in their ability to yield $q$-$\lambda$ PDFs that are acceptable representations of the data used in this analysis. However, both the qualitative arguments presented in [4.2] as well as the location of the minimum of $S_{\text{weighted}}$ indicate that oblate shapes and mean magnetic field orientations with small deviations from the shortest axis of the cloud are preferred.

5 DISCUSSION

In this paper, we have presented a new method for assessing quantitatively the intrinsic shapes and the orientations of the mean, ordered component of the magnetic field of molecular clouds, using observations of the apparent aspect ratios $q$ and angles $\lambda$ between the apparent mean magnetic field and the apparent short axis of the POS cloud images. Under the assumption that the LOS towards the observed clouds have random orientations with respect to the principal cloud axes, we have explored different statistical methods to evaluate the consistency of various models for the intrinsic shapes and intrinsic magnetic field orientations of clouds with our data. We have used data from 24 molecular clouds obtained through 350$\mu$m observations with the Hertz polarimeter (Dotson et al. 2009).

Based on our data sample we can exclude certain simple scenarios with high confidence. A scenario in which both cloud aspect ratios and magnetic field orientation angles are randomly drawn from uniform distributions can be excluded at the 10$^{-7}$ level. A less restrictive scenario, in which the distribution of cloud aspect ratios is identical to the one observed in the data but magnetic field orientations are drawn randomly from a uniform distribution, is also excluded at the 10$^{-2}$ level.

In order to examine more general cases, we have employed a weighted least-squares analysis to derive the best-fit intrinsic shapes and intrinsic magnetic field orientations of the clouds in our sample. We have found that the most probable intrinsic shape is a thick (short to long axis ratio $\sim 0.6$) oblate disk with a negligible degree of triaxiality (middle to long axis ratio $\sim 0.99$); the best-fit distribution of cloud thicknesses was found to be broad, so both thin and thick clouds are frequently encountered under this distribution. The most likely orientation of the magnetic field is close to the shortest axis of the cloud ($\sim 24^\circ$ offset toward the middle/long axis). The best-fit distribution of magnetic field orientations is clearly peaked around this value, but also features long tails. We have found that the best-fit distribution of shapes and orientations, when convolved with a random LOS distribution yields a distribution of observables which is in good agreement with the data. Our results are robust against the removal of any single point in our

**Figure 7.** Best-fit intrinsic distributions of cloud axial ratios (upper panel) and of magnetic field orientation angles (lower panel). In the upper panel, the red line extending from $\langle \zeta, \xi \rangle=(1,1)$ to $\langle \zeta, \xi \rangle=(0.5,0)$ separates the $p(\zeta, \xi) = 0$ plane in two sections: ellipsoids with axis ratios to the left of this line are mostly prolate, and ellipsoids to the right of the line are mostly oblate. Infinitesimally thin disks have $\langle \zeta, \xi \rangle=(1,0)$; infinitesimally thin cigars have $\langle \zeta, \xi \rangle=(0,0)$; and perfect spheres have $\langle \zeta, \xi \rangle=(1,1)$.

thick clouds [closer to $\langle \zeta, \xi \rangle=(1,1)$, labeled “sphere” on the $p(\zeta, \xi) = 0$ plane], are also frequently encountered although not equally common in this distribution.

The most probable magnetic field orientation has a small offset ($\theta_{\text{B0}} \sim 24^\circ$) from the short ellipsoid axis, and the standard deviation in $\cos \theta_B$ of $\sim 0.21$. These results are robust against the removal of any outliers from our dataset of 24 observations. For example, when repeating the analysis without our most elongated cloud (DR 21), the best-fit parameters of the shape distribution change from $\langle \zeta_{\text{max}}, \xi_{\text{max}} \rangle = (0.999, 0.58)$ to $\langle \zeta_{\text{max}}, \xi_{\text{max}} \rangle = (0.986, 0.64)$,
dataset (for example, if we repeat the analysis without our most elongated cloud, DR 21, the change in our best-fit parameters is smaller than 10%).

These results are in agreement with the qualitative trend seen in the data, where clouds with small apparent aspect ratio (apparently elongated clouds) have small angles between the mean projected magnetic field direction and the short axes of the cloud ellipses; clouds which are apparently circular on the other hand can have a large range of angles between mean magnetic field and short axis. Finally, apparently elongated clouds with large angles between mean field direction and short cloud axis are not seen in the data. As far as the distribution of apparent cloud shapes is concerned, more clouds are seen to have large aspect ratios (and are thus close to circular) while only a few clouds are apparently very elongated.

Indeed, clouds which are intrinsically disks with the magnetic field direction close to that of the shortest cloud axis would exhibit similar properties when viewed through random LOS. More LOS yield a disk seen almost face-on rather than edge-on, so most clouds would look almost circular and only a few would appear significantly elongated (in contrast to prolate clouds, most of which would appear elongated, and randomly shaped clouds, which would show no preference in apparent aspect ratio). In addition, when clouds are seen almost edge-on and appear elongated, the magnetic field will be aligned with the short axis of the projection of the cloud on the POS. In contrast, when clouds are seen face-on, the magnetic field projection could from any angle with respect to the short and long axes of the cloud ellipse, which in reality are, in this case, projections of the cloud ellipsoid middle and long axes.

Despite the preference of the data for such a scenario, and the clear potential of this test for discriminating between scenarios for intrinsic magnetic field orientations in molecular clouds and between theories for molecular cloud dynamics, we have shown quantitatively that a larger number of observations is needed for the test to conclusively reject alternative configurations. With the present, limited number of observed clouds, the distributions of LOS, magnetic field orientations, and shapes, are still sparsely sampled, and as a result random fluctuations in this sampling allows for a large number of parameters to yield acceptable representations of the observed $q$ and $\lambda$. However, our results explicitly demonstrate that there is no intrinsic degeneracy in the distributions of $q$ and $\lambda$ yielded by different classes of cloud shapes and magnetic field orientations in nature, so a large number of $q$, $\lambda$ observations should allow conclusive tests of different underlying distributions.

Our analysis has the advantage that it is not a priori tied to any theory or prediction regarding the dynamical processes in molecular clouds, but rather allows the data to pick freely the part of the shapes/orientations parameter space that best fits the observations. We have additionally tested that the location of the best-fit $q_{\text{max}}$, $\xi_{\text{max}}$, $\cos \theta_{\text{max}}$, and the spread of the orientations distribution are robust with respect to changes in the assumed functional form or the details of the analysis (for example, performing a joint analysis for the shapes and orientations, or performing an analysis for the shapes alone). Finally, observational uncertainties in the measured quantities are expected to have a limited effect in our analysis, as the binning of the $(q, \lambda)$ parameter space involves bins that are typically of the same order as or wider than such uncertainties.

The apparent orientation of the magnetic field in molecular clouds and cloud cores has been studied in the past by various authors. Kane et al. (1993), using 1.3 mm polarization measurements, claimed a strong preference for alignment of the polarized emission with with the structure in deconvolved IRAS maps. Glenn et al. (1999) used a sample of 7 elongated cloud cores selected to have a polarization detection greater than 3$\sigma$ at either 800 $\mu$m or 1.3 mm. They found that the orientation of the apparent magnetic field was random with respect to the apparent cloud axes. Vallée & Bastien (1999), using 760 $\mu$m observations, found the apparent magnetic field direction in molecular cloud cores (intensity peaks within clouds) to be parallel to the apparent minor axis in 3 out 10 cases. These studies all focused on polarization vectors measured at intensity peaks, contrary to our own survey, where polarization is extensively mapped throughout the clouds in our sample. The intrinsic shapes of molecular clouds were discussed by Kerton et al. 2003. Based on the Five College Radio Astronomy Observatory Outer Galaxy Survey in CO emission, they studied a sample of 15,000 clouds. They concluded that the intrinsic shapes of these objects are best described as intermediate between near-oblate and near-prolate ellipsoids.

Our results have important implications concerning the dynamical importance of magnetic fields in molecular clouds. If magnetic fields are dynamically unimportant compared to turbulent motions, the fields are expected to be carried around with turbulent eddies in the cloud, and show little correlation with the principal cloud axes. However, our data show clear indications that such a correlation exists. Our results are also unfavorable for the scenario of helical magnetic fields threading prolate clouds. Not only are the predicted 90$^\circ$ flips in the polarization vectors not observed (Dotson et al. 2009), but clouds in our sample are found to be oblate, rather than prolate. This result is also consistent with the argument against prolate cloud cores, which are also predicted in this scenario (Fiege & Pudritz 2000c), but are not observed in nature (Jones et al. 2001, Jones & Basu 2002, Tassis 2007).

If, on the other hand, magnetic fields are dynamically important, cloud shapes should be close to oblate disks, and the magnetic field should be closely aligned with the short axis of the cloud ellipsoid. This scenario seems to be the one which yields the best agreement with our data. Intrinsic cloud shapes are indeed very strongly peaked around oblate disks. The best-fit magnetic field orientation is close to the shortest cloud ellipsoid axis, with small offsets toward the middle/long axes. However, more data are required before a more robust quantitative statement can be made with regard to both the details of the best-fit underlying magnetic fields orientation distribution as well as the confidence with which alternative scenarios can be rejected.

In addition to support from our statistical treatment, this picture is also consistent with observations of individual well-studied systems where projection effects are minimal. In these cases, the hourglass morphology of the magnetic field, characteristic of dynamically important magnetic fields, is seen in very different scales, from clouds to dense cores (e.g., Schleuning 1998, in the case of the cloud OMC-1; Girart, Rao & Marrone 2006, in the case of protostellar...
Let us consider a triaxial ellipsoid model molecular cloud, the shape of which is described by Eq. (A1), will be coaxial ellipses of aspect ratio

\[ 1 \geq q(\theta, \phi, \zeta, \xi) = \frac{A + C - \sqrt{(A - C)^2 + B^2}}{A + C + \sqrt{(A - C)^2 + B^2}} \]  

where

\[ A \equiv \frac{\cos^2 \theta}{\xi^2} \left( \sin^2 \phi + \cos^2 \phi \right) + \frac{\sin^2 \theta}{\xi^2}, \]

\[ B \equiv \cos \theta \sin 2\phi \left( 1 - \frac{1}{\xi^2} \right) \frac{1}{\xi^2}, \]

\[ C \equiv \left( \frac{\sin^2 \phi}{\xi^2} + \cos^2 \phi \right) \frac{1}{\xi^2}. \]  

The orientation of the axes of the POS cloud ellipse with respect to the POS axes (x' and y') is given by the angle ψ, defined as

\[ \psi = \frac{1}{2} \arctan \left( \frac{B}{A - C} \right). \]  

In Eq. (A6), ψ can be the angle between either the minor or the major axis of the POS ellipse with the y' axis (the sum of the two angles is always equal to 90°, and ψ, which can take values between 0 and 45°, is always the smaller of the two). Which one of the two angles is represented by ψ in each case is determined by the sign of the quantity

\[ u = (A - C) \cos 2\psi + B \sin 2\psi. \]  

If u ≤ 0, ψ represents the angle between the minor ellipse axis and y'. If u > 0, ψ represents the angle between the major axis and y'. The sign of ψ indicates the quadrant in which the relevant axis (major or minor, depending on the sign of u) resides. If ψ ≤ 0 the relevant axis is located in the first/third quadrant, while if ψ > 0 the relevant axis is located in the second/fourth quadrant.

Additionally, let us consider that the cloud is threaded by a large-scale ordered magnetic field \( \vec{B} \), which has some orientation with respect to the cloud principal axes characterized by a set of angles \((\theta_B, \phi_B)\); here, \( \theta_B \) is the angle of the magnetic field with the shortest cloud axis and \( \phi_B \) is the angle of its projection onto the plane (x, y) defined by the directions of the middle and longest axes (see Fig. 3). We are interested in the orientation of the POS projection of \( \vec{B} \), since this is what can be measured through polarimetry observations. The components of \( \vec{B} \) in the (x, y, z) coordinate system are

\[ \vec{B} = B \left( \begin{array}{c} \sin \theta_B \cos \phi_B \\ \sin \theta_B \sin \phi_B \\ \cos \theta_B \end{array} \right). \]  

Performing the rotation to the (x', y', z') coordinate system we find that the x' and y' components of \( \vec{B} \) (the POS components) are

\[ B_{x'} = B \left[ -\sin \phi \sin \theta_B \cos \phi_B + \cos \phi \sin \theta_B \sin \phi_B \right] \]  

\[ B_{y'} = B \left[ \sin \theta_B \cos \phi_B + \cos \theta_B \sin \phi_B \right]. \]  

\[ ^4 \text{Note that the projection we are referring to here is not a "cut" of the cloud ellipsoid along some plane, but is rather the surface brightness along the observer’s LOS (and hence it represents an integral of the cloud density along the LOS).} \]
and

\[
B'_{\psi} = B \left[ -\cos \phi \cos \theta \sin \theta_B \cos \phi_B \\
- \sin \phi \cos \theta \sin \theta_B \sin \phi_B + \sin \theta \cos \theta_B \right] .
\]  

(A10)

The angle \( \omega \) between the POS projection of \( \vec{B} \) and the POS projection of the shortest cloud axis (the \( y' \) axis) is given by

\[
\tan \omega = \frac{\sin \phi \sin \theta_B \cos \phi_B - \cos \phi \sin \theta_B \sin \phi_B}{\cos \theta \sin \theta_B \cos \phi_B + \sin \theta \sin \phi_B - \sin \theta \cos \theta_B} .
\]  

(A11)

Note that, contrary to \( \psi \), the angle \( \omega \) is, by definition, positive in the first-third quadrant, and negative in the second-fourth quadrant, where \( x' \) and \( y' \) have opposite signs.

The angle that can be determined observationally, however, is not \( \omega \), but rather the magnitude of the angle \( \lambda \) between the POS projection of \( \vec{B} \) and one axis (for definiteness we will use the minor axis) of the POS cloud ellipse. The relation between \( \lambda, \omega, \) and \( \psi \) is different, depending on the value of \( u \) (see Fig. A1):

\[
\lambda = \begin{cases} 
90^\circ - |\psi + \omega| , & u \leq 0 \\
90^\circ - |\psi + \omega| , & u > 0
\end{cases} .
\]  

(A12)

We have therefore shown that the two observable quantities \( q \) and \( \lambda \) can be calculated using the formalism described above, provided that we know:

(a) the intrinsic shape of the cloud ellipsoid (i.e., its axial ratios \( \zeta \) and \( \xi \));
(b) the orientation of the ordered magnetic field with respect to the native coordinate system (i.e., the angles \( \theta_B \) and \( \phi_B \)); and
(c) the orientation of the LOS with respect to the native coordinate system (i.e., the angles \( \theta \) and \( \phi \)).

Note that the angles \( \theta_B \) and \( \phi_B \) only enter the calculation of \( \lambda \), while \( q \) is not dependent on them. Because the LOS is random, \( \lambda \) is not fixed, even if the true magnetic field direction is fixed with respect to the true cloud ellipsoid axes, and all molecular clouds have the exact same intrinsic shape (see e.g., Basu 2000). Rather, there is a distribution of such angles, which can be calculated analytically, given our hypothesis for the orientation of the true mean magnetic field in the clouds, as well as some knowledge of the distribution of intrinsic cloud shapes in nature.

**APPENDIX B: STATISTICAL ANALYSIS**

In this work, we use our set of observations of the aspect ratio \( q \) of the POS cloud ellipse and the magnitude of the angle \( \lambda \) between the POS projection of the magnetic field and the minor axis of the POS cloud ellipse to constrain the intrinsic shapes and magnetic field orientations in molecular clouds. Unfortunately, it is not possible to derive directly the intrinsic shape and the orientation of the magnetic field in each cloud, as this would require knowledge of the LOS orientation in each case. Instead, we employ a statistical analysis, assuming that the distribution of LOS orientations is uniform with respect to all directions (same fraction of LOS per unit solid angle in all directions). In practice, this means that the distribution of \( \cos \theta \) is uniform with values between \(-1\) and \(1\), and the distribution of \( \phi \) is also uniform with values between \(0\) and \(2\pi\).

**Figure A1.** POS angles between the POS projection of the magnetic field (red line), the POS projection of the shortest cloud ellipsoid axis (the \( y' \) axis), and the projected cloud ellipse axes (blue lines). Upper panel: \( u < 0 \) (\( \psi \) represents the angle between \( y' \) and the minor ellipse axis). Lower panel: \( u > 0 \) (\( \psi \) represents the angle between \( y' \) and the major ellipse axis). The magnitude of the angle \( \lambda \) is the observable angle between the POS cloud ellipse minor axis and the POS projection of the magnetic field. In this case, \( \psi \) and \( \omega \) are both positive, and \( |\psi + \omega| < 90^\circ \).

Under this assumption, we will perform a weighted least-squares analysis to deduce the best-fit probability density function of our observables, \( q \) and \( \lambda \), and the associated distributions of intrinsic shapes and orientations of the magnetic field with respect to the principal cloud ellipsoid axes that generate it.

Because the magnetic field orientation does not enter the calculation of \( q \), it is also possible to use only the observations of \( q \) to derive the best-fit distribution of the intrinsic molecular cloud shapes (an analysis similar to that of Tassis 2007 in the case of molecular cloud cores). The results obtained by this \( q \)-only analysis indicate that the distribution of intrinsic shapes is strongly peaked on the \( \xi = 1 \) axis (largest and middle axes equal) so that all azimuthal magnetic field angles \( \phi_B \) are physically equivalent. For this reason, we will treat the orientation angle \( \phi_B \) as random and uniformly distributed, and we will only attempt to determine the best-fit distribution of \( \cos \theta_B \), determining the most likely deviation of the magnetic field
orientation from the shortest cloud axis. Note that the relevant quantity is \( \cos \theta_B \), rather than \( \theta_B \), as we are not only interested in the most probable angle, but also on the spread of such angles: one of the theoretical cases we would like to test is that where all magnetic field orientations have an equal probability of occurring. This translates to a uniform distribution of magnetic field orientations per solid angle around the cloud center, which in turn is mathematically expressed as a uniform distribution in \( \cos \theta_B \).

Thus, the interpretation of a best-fit distribution in \( \cos \theta_B \) is intuitively straight-forward: the standard deviation corresponding to a uniform distribution of magnetic field orientations per unit solid angle is \( \sigma = \sqrt{3/4} = 0.625 \). For a distribution of finite width in each of the \( \zeta \) and \( \xi \) planes, if 24 observations are equally probable: the distribution of \( \cos \theta \) is uniform with values between \(-1\) and \(1\), and the distribution of \( \phi \) is uniform with values between \(0\) and \(360^\circ\).

Based on these assumptions, we calculate, for every set of distribution parameters \( \zeta_{\text{max}}, \xi_{\text{max}}, \cos \theta_{B,\text{max}}, \) and \( \sigma(\cos \theta_B) \), the expectation value of the number of observed points in each bin on the \( q \)-\( \lambda \) plane, if 24 observations are performed with random LOS: we also calculate the spread of the observed number of points. We do so using the following Monte-Carlo procedure.

(i) We randomly draw a pair of \( \zeta, \xi \) and a \( \cos \theta_B \) from their respective distributions, and a \( \phi_B \) from a uniform distribution between 0 and \(2\pi\).

(ii) We randomly draw a LOS orientation, by drawing a pair of \( \theta, \phi \), from a uniform probability distribution (equal probability for any value of \( \cos \theta \) or \( \phi \) in the intervals \((0,1)\) and \((0,2\pi)\), respectively).

(iii) We use the values of \( \zeta, \xi, \theta, \phi \) to calculate \( q, u \), and \( \psi \) through Equations (A2)-(A7).

(iv) We use the values of \( \theta_B, \phi_B, \theta, \phi \) and \( \psi \) to calculate \( \omega \) through \( \psi \) through Equations (A11).

(v) We use the values of \( \omega, \psi, u \), and \( \phi \) to calculate \( \lambda \) through Equations (A12).

(vi) We repeat steps (i)-(v) 24 times (as many as our observed clouds). For each bin in the \( q \)-\( \lambda \) plane, we calculate the number of observations that fall within its limits.

(vii) We repeat step (vi) 10,000 times. Using the 10,000 mock sets of 24 observations, we calculate for each bin \( i \) the mean number of observations, \( \langle N_i \rangle \) (which has to be \(0\leq \langle N_i \rangle \leq 24\)), and the sample variance of \( N_i \), \( \sigma_{N_i}^2 = \langle N_i^2 \rangle - \langle N_i \rangle^2 \). Because we only use 10,000 mock observations and we use the inverse of \( \sigma_{N_i}^2 \) as a weight in our least-squares analysis, for the cases wherein our 10,000 trials we obtain \( \sigma_{N_i}^2 = 0 \) we use instead a floor value of \( \sigma_{N_i}^2 = 10^{-4} \).

We finally obtain the best-fit set of parameters \( \zeta_{\text{max}}, \xi_{\text{max}}, \cos \theta_{B,\text{max}} \), and \( \sigma(\cos \theta_B) \) by minimizing the weighted squared residuals function,

\[
S_{\text{weighted}}(\zeta_{\text{max}}, \xi_{\text{max}}, \cos \theta_{B,\text{max}}, \sigma(\cos \theta_B)) = \sum_{i=1}^{5\times5} \frac{N_{\text{data},i} - \langle N_i \rangle}{\sigma_{N_i}^2}.
\]

### B1 Modified Lognormal Distribution of Intrinsic Cloud Shapes

We wish to use a distribution of finite width in each of the \( \zeta, \xi \) axes which is, however, smooth and has a relatively sharp maximum. Such a distribution is a modified lognormal, which we construct in the following way: we first seek an appropriate minimal transformation which will transform the domain of \( \zeta \) and \( \xi \) from \((0,1)\) to \((-\infty, \infty)\); we then take the transformed variable to follow a gaussian distribution.

An appropriate transformation is

\[
x = \ln \frac{\zeta}{1-\zeta}, \quad y = \ln \frac{\xi}{1-\xi}.
\]

It is easy to see that

\[
\lim_{\zeta \to 0} x = -\infty, \quad \lim_{\zeta \to 1} x = \infty, \quad \lim_{\xi \to 0} y = -\infty, \quad \lim_{\xi \to 1} y = \infty.
\]

If then the joint probability distribution function (PDF) of \( x \) and \( y \) is

\[
p(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} \exp \left[ -\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_0)^2}{2\sigma_y^2} \right],
\]

then the joint PDF of \( \zeta \) and \( \xi \) will be

\[
p(\zeta, \xi) = \frac{1}{2\pi \sigma_x \sigma_y} \exp \left[ -\frac{(\ln \frac{\zeta}{\xi} - x_0)^2}{2\sigma_x^2} \right] \times \exp \left[ -\frac{(\ln \frac{\xi}{1-\xi} - y_0)^2}{2\sigma_y^2} \right] \left(\frac{1}{\zeta(\zeta-\xi)(1-\xi)}\right).
\]

This is in principle a tetra-parametric distribution with parameters \( x_0, y_0, \sigma_x, \) and \( \sigma_y \). However, to reduce the number of parameters entering the problem and the associated computation time required for the analysis, while at the same time allowing the expected amount of variation in molecular cloud shapes, we fix the value of \( \sigma_x \) and \( \sigma_y \) to 0.2 and 0.7 respectively, determined from a likelihood analysis of the aspect ratios \( q \) alone.

\(^5\) For details on the methodology of this shapes-only likelihood analysis see Tassis 2007
The values $\xi = \zeta_{\text{max}}$ and $\xi = \xi_{\text{max}}$ that maximize $p(\zeta, \xi)$ are given by the system of equations

$$\frac{\partial p}{\partial \xi} = 0 \quad \text{(B6)}$$

$$\frac{\partial p}{\partial \zeta} = 0 \quad \text{(B7)}$$

or, equivalently,

$$0 = \sigma_y^2 \left( \ln \frac{\xi}{1 - \xi} - x_0 \right) (\xi - \zeta)$$

$$-\sigma_x^2 \left( \ln \frac{\zeta}{\xi - \zeta} - y_0 \right) (1 - \xi)$$

$$\xi + (1 - 2 \zeta + \xi) \zeta_\sigma \sigma_x^2 \sigma_y^2.$$  

$$0 = \left( \ln \frac{\xi}{\zeta - \zeta_\sigma} - y_0 \right) \frac{\xi}{\sigma_x^2} + \xi - 2 \xi.$$  

(B8)

With $\sigma_x$ and $\sigma_y$ fixed, $\zeta_{\text{max}}$ and $\xi_{\text{max}}$ can be used alternatively as the free parameters of $p(\zeta, \xi)$.

### B2 Modified Lognormal Distribution of Intrinsic Magnetic Field Orientations

The distribution of $\cos \theta_B$ follows a modified lognormal similar to the one discussed in Appendix B1: let the variable $w$ follow a Gaussian distribution with parameters $w_0, \sigma_w$, and let $\cos \theta_B$ be related to $w$ through the transformation

$$w = \ln \frac{\cos \theta_B}{1 - \cos \theta_B}. \quad \text{(B10)}$$

In this way, the PDF of $\cos \theta_B$ is

$$p(\cos \theta_B) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[ -\frac{\left( \ln \frac{\cos \theta_B}{1 - \cos \theta_B} - w_0 \right)^2}{2\sigma_w^2} \right] \frac{1}{\cos \theta_B(1 - \cos \theta_B)}. \quad \text{(B11)}$$

The value of $\cos \theta_B$ where $p(\cos \theta_B)$ is maximized, which we call $\cos \theta_{B_{\text{max}}}$, is obtained by requiring that $\partial p/\partial \cos \theta_B = 0$, or, equivalently, by solving the equation

$$-\ln \frac{\cos \theta_B}{1 - \cos \theta_B} w_0 - \sigma_w^2 + 2 \sigma_w^2 \cos \theta_B = 0. \quad \text{(B12)}$$

The standard deviation of this distribution is obtained through

$$\sigma(\cos \theta_B) = \sqrt{\int_{\cos \theta_B = 0}^{1} (\cos \theta_B - \langle \cos \theta_B \rangle)^2 p(\cos \theta_B) d \cos \theta_B}, \quad \text{(B13)}$$

with

$$\langle \cos \theta_B \rangle = \int_{\cos \theta_B = 0}^{1} \cos \theta_B p(\cos \theta_B) d \cos \theta_B. \quad \text{(B14)}$$

This is a bivariable distribution, with free parameters $w_0$ and $\sigma_w$, or, equivalently, $\cos \theta_{B_{\text{max}}}$ and $\sigma(\cos \theta_B)$.

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