Cosmological consequences of a built-in cosmological constant model

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Abstract. We have analyzed the implications of a phenomenological model of a built-in cosmological constant of Rastall and Al-Rawaf-Taha type. We have produced the presently observed cosmic acceleration with the aid of introducing a positive cosmological constant. We have shown that the cosmological models of the type $\Lambda \propto \ddot{R}$, $\Lambda \propto H^2$ or $\Lambda \propto 8\pi G\rho$ are equivalent to a built-in cosmological constant model of Al-Rawaf and Taha type. Such models are compatible with the recent observational data and do not suffer from fine tuning problems. We predict a higher lensing probability than in Einstein de Sitter and Abdel Rahman models. We also predict a higher value for the luminosity and angular diameter distances than that of Abdel Rahman model. The lens distribution peaks at relatively smaller values of the lens red-shift and occurs at higher red-shift.

Key words: Cosmological constant, gravitational lensing, dark energy, cosmological tests.
1. Introduction

Gravitational lensing is currently known as one of the robust tests of cosmological models with variable cosmological constant ($\Lambda$) (Fukugita et al. 1992, Caroll et al. 1992). It is found by Ratra and Quillen (Ratra and Quillen 1998) that the optical depth is significantly smaller than that in a constant $\Lambda$ model with the same density parameter ($\Omega$). They also found that the red-shift of the maximum of the lens distribution falls between that in the constant-$\Lambda$ model and the Einstein-de Sitter model (ES).

It is recently shown that our present universe is accelerating. This apparent acceleration is attributed to a dark energy residing in space itself, which also balances the kinetic energy of expansion so as to give the universe zero spatial curvature, as deduced from the cosmic microwave background radiation (CMBR). This dark energy was important in the past as it is now. It might have played a part in limiting the formation of largest gravitationally bound structures. One can model the dark energy by a cosmological constant (or a vacuum decaying energy). However, not all vacuum decaying cosmological models predict this acceleration. We have recently proposed a cosmological model with a cosmological constant of the form $\Lambda = \beta \left( \frac{\ddot{R}}{R} \right)$, where $R$ is the scale factor of the universe and $\beta$ some constant. We have found for $\Lambda > 0$ cosmic acceleration is inevitable. This cosmological constant mimics a cosmic fluid (scalar field) with an equation of state of the form $p_{\phi} = \omega_{\phi} \rho_{\phi} \left( -1 < \omega_{\phi} < -1/3 \right)$. Such a field is sometimes known as *quintessence*. Quintessence models exhibit an event horizon which poses a serious problem for string theory (Fichler 2001). Quintessence models describe the present expansion without resort to the cosmological constant. The question whether this acceleration will last forever or changes is not yet known! If the present vacuum contribution ($\Omega_{\Lambda}$) is such that $\Omega_{\Lambda} = \frac{2}{3}$ then this cosmological constant mimics a string-like matter. Present observations indicate that the matter contribution ($\Omega_m$) to the present universe is such that $0.1 < \Omega_m < 0.4$. One possibility is that this $\Lambda$ may be an effective constant, and indeed could be varying slowly with time. Such a behavior can be modelled by the potential energy of a scalar field, in exactly the same way as inflation is.

In this work, we show the similarity between a phenomenological models of a type of $\Lambda \propto H^2$ and $\Lambda \propto \frac{\dot{R}}{R}$ and the one with $\Lambda \propto 8\pi G \rho$. These forms of $\Lambda$ are also equivalent to the Rastall (Rastall 1972) and Al-Rawaf and Taha (Al-Rawaf and Taha 1996) models of modified general relativity (MGR). Al Rawaf and Taha obtained a built-in cosmological constant that is related to the curvature scalar ($\mathcal{R}$). In MGR a flat universe has an age ($t_0$) of $t_0 < H_0^{-1}$, where $H_0$ is the present value of the Hubble constant. However, our model gives a resolution to both the age problem and the low energy density problem of the universe with present observational data. We also discuss the cosmological tests pertaining to diameter-distance, luminosity-distance and gravitational lensing effect. It is now known that the gravitational lensing probability is quite sensitive to the value of $\Lambda$ in low-$\Omega$, flat models. It has also been noted that for a source at a given red-shift, the red-shift of the peak of the expected normalized probability distribution for the lens red-shift depends on the value of $\Lambda$. We have found that our present model
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gives rather higher values for the luminosity and angular diameter distances with the presently preferred value for \( \Omega_m(=0.3) \). The constraints implied by these tests would place a limit on the value of \( \beta \).

2. The Field Equations

The Einstein field equations with a cosmological constant, and energy conservation law yield

\[
\left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} = \frac{8\pi}{3} G \rho + \frac{\Lambda}{3} \quad (1)
\]

\[
\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} \quad (2)
\]

and

\[
\dot{\rho} + 3 \frac{\dot{R}}{R} (p + \rho) = -\frac{\dot{\Lambda}}{8\pi G} \quad (3)
\]

Using the equation of the state

\[ p = (\gamma - 1)\rho, \quad 1 \leq \gamma \leq 2 \quad (4) \]

eq(2) can be written as

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \left( 1 - \frac{3}{2\gamma} \right) \rho + \frac{\Lambda}{3} \quad (5)
\]

Following (Arbab 2002), we consider

\[ \Lambda = \beta \left( \frac{\dot{R}}{R} \right) \quad (6) \]

where \( \beta \) is constant. From eqs.(1)-(5), one finds (Arbab 2002) for a flat universe \((k = 0)\)

\[
R(t) = Ct^{\frac{\beta - 2}{\beta - 3}}, \quad C = \text{const.} \, , \beta \neq 3 \quad (7)
\]

\[
\Lambda(t) = \frac{\beta(\beta - 2) \, 1}{(\beta - 3)^2 \, t^2}, \quad \beta \neq 3, \quad (8)
\]

\[
\rho(t) = \frac{(\beta - 2) \, 1}{(\beta - 3) \, 4\pi G t^2}, \quad \beta \neq 3 \quad (9)
\]

the deceleration parameter is given by

\[
q = -\frac{\ddot{R}R}{\dot{R}^2} = \left( \frac{1}{2 - \beta} \right), \quad \beta \neq 2 \quad (10)
\]

The matter contribution to density parameter of the universe is

\[
\Omega_m = \frac{2(\beta - 3)}{3(\beta - 2)}, \quad \beta \neq 2 \quad (11)
\]

while the vacuum contribution is

\[
\Omega_\Lambda = \frac{\beta}{3(\beta - 2)}, \quad \beta \neq 2, \quad (12)
\]
so that $\Omega_m + \Omega_\Lambda = 1$, as preferred by inflation. From eqs.(11) and (12) one finds

$$\beta = \frac{2\Omega_\Lambda}{\Omega_\Lambda - \frac{1}{3}} = \frac{1 - \Omega_m}{\frac{1}{3} - \frac{1}{2}\Omega_m}. \quad (13)$$

It is thus clear that the free parameter ($\beta$) can be determined from the present observational data from the knowledge of the vacuum or baryonic mass contribution to the total energy density of the universe. For $\beta > 0$ eq.(13) gives $\Omega_\Lambda > \frac{1}{3}$ (or $\Omega_m < \frac{2}{3}$). We see that $\beta = 6$, $\Omega_m = \Omega_\Lambda = \frac{1}{2}$. For $\beta > 6$, $\Omega_m > \Omega_\Lambda$ and for $\beta < 6$, $\Omega_m < \Omega_\Lambda$. Note that all other cosmological parameters depend on this constant. The case $\beta = 2$ defines a static universe, which is physically unacceptable for describing the present universe.

We see that the universe will be ultimately driven into a de-Sitter phase of exponential expansion as $\Omega_\Lambda \rightarrow 1$ (or $\beta \rightarrow 3$) (Johri 2000, Arbab 2002).

As suggested by many people, that $\Omega_\Lambda = \frac{2}{3}$ (or $\Omega_m = \frac{1}{3}$), one would obtain the following constraints:

$$\beta = 4, \quad H_0 t_0 = 2, \quad q_0 = -0.5. \quad (14)$$

These findings have to be confronted with current observation. Thus, if the universe is dominated by vacuum, today, then it should accelerate. Note that RT model does not exhibit an acceleration. This because a cosmic acceleration would mean a negative matter density parameter. This why RT predicts a shorter age ($H_0 t_0 = 0.85$) of the universe, unless Hubble constant assumes a smaller value.

We would like to remark that our cosmological constant is always positive ($\Lambda > 0$) irrespective of the sign of $\beta$. One only has a decelerating universe ($q > 0$) when $\beta < 2$.

### 3. A cosmological constant and matter content

In general, the most important difference of a dark energy component to a cosmological constant is that its equation of state can be different form $p = -\rho$, generally implying a time-variation. Using eqs.(4) and (5) one can write eq.(6) as

$$\Lambda = \left(\frac{\beta}{3 - \beta}\right) \left(1 - \frac{3}{2}\gamma\right) 8\pi G \rho, \quad \beta \neq 3. \quad (15)$$

It is evident that when $\gamma = \frac{2}{3}$ the cosmological constant vanishes ($\Lambda = 0$). Thus if the universe is dominated by strings today, the cosmological constant must vanish! Such a model is obtained by Sima and Sukenik (Sima and Sukenik 2001) for a non-decelerative universe.

For the matter-dominated universe ($\gamma = 1$) eq.(15) gives

$$\Lambda^{AR} = \left(\frac{\beta}{\beta - 3}\right) 4\pi G \rho^m. \quad (16)$$

It is evident from the above equation that an empty universe ($\rho = 0$) would imply a vanishing cosmological constant ($\Lambda = 0$).

For the radiation-dominated epoch ($\gamma = \frac{4}{3}$) eq.(15) gives

$$\Lambda = \left(\frac{\beta}{\beta - 3}\right) 8\pi G \rho^r. \quad (17)$$
If $\beta$ changes then eqs. (16) and (17) can be written as defining $\beta$ (with $\rho_v = \frac{\Lambda}{8\pi G}$) as

$$\beta^m = \frac{6\rho_v^m}{2\rho_v^m - \rho^m}, \quad \beta^r = \frac{3\rho_v^r}{\rho_v^r - \rho^r},$$

where the subscripts “r” and “m” denote the value of the quantity during radiation and matter epochs, respectively. We see that for $\beta > 0$, the vacuum contribution always surpasses the ordinary matter/radiation contribution to the total energy density of the universe in both radiation era ($\rho^r_v > \rho^r$) and matter era ($\rho^m_v > \frac{1}{2}\rho^m$). Therefore, the vacuum energy (cosmological constant) has played a very essential role in inflating the universe in the past, and accelerating it in the present epoch. Thus without the cosmological constant the observable universe would not have been produced. One then should not worry about the fine tuning problem infected other cosmologies inasmuch as the ratio between the vacuum energy to ordinary matter/radiation energy does not change with time.

4. A built-in cosmological constant

The curvature scalar $R$ is given by

$$R = 6 \left[ \left( \frac{\dot{R}}{R} \right)^2 + \frac{\ddot{R}}{R} \right].$$

Using eqs.(1), (5) and (15) the above equation reads

$$\Lambda^{AR} = \frac{\beta}{6(\beta - 1)} R.$$  \hspace{1cm} (20)

Recently, Al-Rawaf and Taha (RT), in an attempt to solve the entropy problem, proposed a theory in which the energy conservation law is relaxed. They have shown that

$$\Lambda^{RT} = \frac{(1 - \eta)}{3(2 - \eta)} R,$$

where $0 \leq \eta \leq 1$ and

$$\Omega_{m}^{RT} = \eta, \quad \eta = 2q.$$  \hspace{1cm} (22)

Similarly Majernik (Majernik 2001, 2002) proposed a cosmological model with the ansatz

$$\Lambda^{MJ} = 8\pi\kappa GT$$

where $T$ is the stress-energy scalar ($T_{\mu}^{\nu}$) and $\kappa$ is some constant. And since we know that $T$ is related to $R$, one can write for Majernik (MJ)

$$\Lambda^{MJ} = \frac{\kappa}{1 + 4\kappa} R.$$  \hspace{1cm} (24)

Thus eqs.(20), (21) and (24) are different types of a built-in cosmological constant. Though the three models are similar in the matter-dominated era (MD) they differ in both eras.
(i) In the radiation-dominated era (RD) eqs.(21) and (24) yield $\Lambda = 0$ whereas eq.(20) does not.

(ii) In the matter-dominated era eqs.(20) and (24) lead to cosmic acceleration of the universe whereas eq.(21) does not.

We see that our model interplays between the two. Comparison of eq.(20) with eq.(21) using eqs.(10) and (11) yields

$$\Omega_m = \frac{2}{3} + \frac{\eta}{3}, \quad \eta = 2q = \frac{2}{2 - \beta}. \quad (25)$$

We note that, unlike RT, our model can solve both the age and the low mass problems of the universe simultaneously. Our model gives

$$t_0 = \frac{2 H_0^{-1}}{3 \Omega_m} \quad (26)$$

while RT yields

$$t_0 = \frac{2 H_0^{-1}}{2 + \Omega_m}. \quad (27)$$

In RT the age of the universe is limited by the condition $H_0 t_0 < 1$, while in our model $H_0 t_0 \geq 1$, in a good agreement with the present observational data. This is an advantage over RT model, where the age problem and the low-mass problem can not be solved simultaneously. We remark that while our model can account for the presently accelerating universe, RT can not accommodate this possibility, as this is evident from eq.(25).

Substituting eq.(14) in eq.(2) one yields

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3} \frac{3}{(\beta - 3)} \rho. \quad (28)$$

For a positive energy density ($\rho > 0$), as seen from eq.(9), $\beta > 3$. This implies that $\dot{R} > 0$ (not with $p < -\frac{\rho}{\beta}$, as usual). Consequently, one obtains a new cosmic acceleration that has not been explored before. Hence, the observed acceleration, as indicated by supernovae of type Ia, may not require an exotic equation of state, as suggested by many scientists. A physically acceptable solution requires $\beta > 3$ and this represents the robust constraint for our model. However, this automatically give rise to cosmic acceleration. Hence cosmic acceleration could have started at any time whenever $\frac{1}{3} < \Omega_\Lambda < 1$.

A similar setting, as in eq.(28), is recently suggested by Gu and Hang (Gu and Hang 2001). However, we have found that this would imply $p = -\rho$. Therefore, ordinary matter in the presence of a positive cosmological constant can render cosmic acceleration. However, in Al-Rawaf and Taha the factor $\frac{3}{(3 - \beta)}$ in eq.(16) is absorbed in the definition of the gravitational constant $G$ making their field equations different from that of Einstein. This is done by replacing $G$ by $[G/\alpha]$ (or equivalently $G \rightarrow G/\alpha^2$). However, Al-Rawaf and Taha can describe an accelerating universe (with the normal equation of state) if the gravitational constant is negative!

In what follows we discuss the following cosmological tests:
5. Neoclassical tests

A photon emitted by a source with co-ordinate \( r = r_1 \) at time \( t = t_1 \) and received at time \( t_0 \) by an observer located at \( r = 0 \) will follow a null geodesic. The proper distance between the source and the observer is given by

\[
d(z) = R_0 \int_{R}^{R_0} \frac{dR}{RR'}
\]

From eq.(7) one obtains [Arbab 1998]

\[
H_0 d = \frac{1}{(\frac{3}{2}\Omega_m - 1)} \left[ 1 - (1 + z)^{(1 - \frac{3}{2}\Omega_m)} \right],
\]

where \( 1 + z = \frac{R_0}{R} \) defines the red-shift \( z \).

### 5.1. Luminosity distance

This is given by

\[
d_L = \left( \frac{L}{4\pi\ell} \right)^{1/2} = r_1 R_0 (1 + z) = d(z) (1 + z),
\]

where \( L \) is the total power emitted by the source and \( \ell \) is the apparent luminosity of the object at a distance \( r_1 \). Using eq.(30) one gets

\[
H_0 d_L = \frac{1 + z}{(\frac{3}{2}\Omega_m - 1)} \left[ 1 - (1 + z)^{(1 - \frac{3}{2}\Omega_m)} \right].
\]

We see that \( H_0 d_L \) is a decreasing function of \( \Omega_m \). \( H_0 d_L \) versus \( z \) is plotted in fig.(1) for \( \Omega_m = 0.3 \) for AM, ES and our model (AR). We observe that our model gives a higher value for \( H_0 d_L \) than AM and ES.

### 5.2. Angular diameter distance

The angular diameter \( (d_A) \) of a light source of proper distance \( d \) is given by

\[
d_A = \frac{d(z)}{(1 + z)},
\]

or

\[
d_A H_0 = \frac{1}{(\frac{3}{2}\Omega_m - 1)} \left[ 1 - (1 + z)^{(1 - \frac{3}{2}\Omega_m)} \right].
\]

This has a maximum at a red-shift \( (z_m) \) given by

\[
1 + z_m = \left( \frac{3}{2}\Omega_m \right)^{1/(\frac{3}{2}\Omega_m-1)}
\]

Again \( H_0 d_A \) is a decreasing function of \( \Omega_m \). \( H_0 d_A \) versus \( z \) is plotted in fig.(2) for \( \Omega_m = 0.3 \) for AM, ES and our model (AR). The maximum red-shift in our model (AR) occurs at \( z_m = 0.76 \).
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5.3. Gravitational Lensing

Following Abdel Rahman solution, one finds that the lensing probability equation for the present model is given by (Fukugita, et al. 1992; Carroll et al. 1992; Cohn 1992)

\[ P_{lens} = \frac{1}{8} \left( \frac{(1 - x_s^{(\frac{\Omega m}{2})-1})}{\left(\frac{\Omega m}{2} - 1\right)(1 - x_s^{1/2})} \right)^3, \]  

the normalized optical depth for lensing is given by (Ratra and Quillen 1992)

\[ \tau(z_s) = \frac{1}{30} \left( \frac{(1 - x_s^{(\frac{\Omega m}{2})-1})}{\left(\frac{\Omega m}{2} - 1\right)^3} \right)^3, \]  

and the normalized lens red-shift distribution is given by

\[ \tau^{-1} \frac{d\tau}{dz} = F \left( 30 \frac{x^{\Omega m}/(\frac{\Omega m}{2})-1}{}, \right), \]  

\[ F = \left( \frac{(1 - x_s^{(\frac{\Omega m}{2})-1})^2 - x_s^{(\frac{\Omega m}{2})-1} - x_s^{(\frac{\Omega m}{2})-1})^2}{\left(\frac{3}{2} \Omega m - 1\right)(1 - x_s^{(\frac{\Omega m}{2})-1})^5} \right), \]

where \( x = (1 + z)^{-1} \) and \( z_s \) is the source red-shift. In Abdel Rahman model \( \eta \) replaces \( \frac{2}{2-\beta} \) and both models coincide with the Einstein-de Sitter model (ES) when \( \eta = 1 \) (or \( \beta = 0 \), but differ otherwise. We plot the lensing probability (fig.(3)), the normalized optical depth (fig.(4)) and the normalized lens red-shift distribution (fig.(5)) against the source red-shift (\( z_s \)) and matter density (\( \Omega_m \)) for Abdel Rahman (AM) model and our present model (AR). We have found that in our model the normalized lens red-shift distribution peaks at relatively smaller values of the lens red-shift than that of AM and ES, and occurs at relatively high values of the lens red-shift. The lensing probability in our model is very high for a very low matter universe.

6. Discussion

All in all, since higher values for \( \Omega \) predict higher numbers of gravitational lenses, this technique offers a viable way of putting upper bounds on the value of the cosmological constant. The normalized lens red-shift distribution for \( \Omega_m = 0.3 \) peaks at smaller values in our model in comparison with AM, and occurs at a higher lens red-shift. For a low density universe the optical depth rises very quickly with the source red-shift than in AM model. The lens distribution peaks at considerably lower values of the lens red-shift than in AM. The diameter and luminosity distances are decreasing functions of \( \Omega_m \). Our model predicts rather higher values for these cosmological distances. We predict far more lens system for a low-density universe than might have been observed. We have shown that a cosmological model of the type \( \Lambda \propto \frac{\dot{R}}{R}, \Lambda \propto H^2 \) or \( \Lambda \propto 8\pi G \rho \) are equivalent to a built-in cosmological constant model of Al-Rawaf and Taha type. Different cosmological tests are worked out and some results are shown in fig.(1) to fig(5). The ensuing future results can limit the presently different cosmological scenarios to a few ones.
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