Horizontal symmetries of leptons with a massless neutrino

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Abstract

Residual symmetry $G_\nu$ of neutrino mass matrix with a massless neutrino and embedding of $G_\nu$ and the residual symmetry $G_l$ of the charged lepton mass matrix into finite discrete groups $G$ is discussed. Massless neutrino results if $G_\nu$ and hence $G$ are subgroups of $U(3)$ rather than of $SU(3)$. Structure of the resulting leptonic mixing matrix $U_{PMNS}$ is discussed in three specific examples based on groups (a) $\Sigma(3N^3)$, (b) $\Sigma(2N^2)$ and (c) $S_4(2) \equiv A_4 \rtimes Z_4$. $\Sigma(3N^3)$ groups are able to reproduce either the second or the third column of $U_{PMNS}$ correctly. $\Sigma(2N^2)$ groups lead to prediction $\theta_{13} = 0, \theta_{23} = \frac{\pi}{4}$ for the reactor and atmospheric mixing angles respectively if neutrino mass hierarchy is inverted. Solar angle remains undetermined in this case. This also gets determined when $G = S_4(2)$ which can give bi-maximal mixing for inverted hierarchy. Examples (b) and (c) provide a good zeroth order approximation to realistic leptonic mixing with a massless neutrino. We also present an example of the specific model based on $S_4(2)$ symmetry in which a massless neutrino and viable leptonic mixing angles are obtained.

PACS numbers: 11.30.Hv, 14.60.Pq, 11.30.Er

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I. INTRODUCTION

The observed leptonic mixing angles are known to be close to special values. The atmospheric mixing angle is close to maximal with $\sin^2 \theta_{23} \approx 0.44 \sim \frac{1}{2}$, the solar angle $\theta_{12}$ and the reactor angle $\theta_{13}$ satisfy $\sin^2 \theta_{12} \approx 0.31 \sim \frac{1}{3}$ and $\sin^2 \theta_{13} \approx 0.023 \sim 0$. It is natural to look for group theoretical explanations for such special values as has been extensively done, see [2, 3] for reviews. In this approach, it is assumed that underlying theory of leptonic flavor possesses some discrete symmetry $G$. The group $G$ breaks to smaller non-commuting subgroups $G_\nu$ and $G_l$ which correspond to unbroken symmetries respectively of the neutrino and the charged lepton mass matrices $M_\nu$ and $M_l$, more precisely of $M_l M_l^\dagger$.

While possible choices of $G$ are a priori unknown and numerous, one can relate $G_\nu$ and $G_l$ to the known structure of the mixing matrix. Thus it becomes more profitable to start with possible choices of $G_\nu$ and $G_l$ dictated from physical considerations and search for groups which contain them as subgroups. In this way, Lam [5] argued that minimal group which combines symmetries of $M_\nu$ with tri-bimaximal (TBM) mixing [7] structure and a diagonal $M_l M_l^\dagger$ is $S_4$.

In all these analysis, basic but implicit assumption is that neutrinos are Majorana particles and all three of them are massive. The present neutrino data are however quite consistent with one of the neutrinos being exactly massless both in case of the normal and inverted hierarchy for neutrino masses. The underlying symmetry $G_\nu$ and hence possible choice of $G$ become quite different in this case. In this note, we discuss possible symmetry groups $G_\nu$ and embedding of $G_\nu$ and $G_l$ into some bigger group $G$ assuming that one of the three neutrinos is massless.

Let us quickly summarize the steps [4, 5, 9–11] used in relating mixing angles to symmetry groups $G$. Let $U_\nu$ and $U_l$ respectively diagonalize $M_\nu$ and $M_l M_l^\dagger$:

\begin{align}
U_\nu^T M_\nu U_\nu &= \text{Diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) , \\
U_l^T M_l M_l^\dagger U_l &= \text{Diag}(m_2^e, m_2^\mu, m_2^\tau) .
\end{align}

Assume that $M_\nu$ ($M_l M_l^\dagger$) is invariant under some set of discrete symmetries $S_i$ ($T_i$):

\begin{align}
S_i^T M_\nu S_i &= M_\nu , & T_i^T M_l M_l^\dagger T_i &= M_l M_l^\dagger .
\end{align}

It is assumed that elements within $S_i$ and $T_i$ commute among themselves and hence can be simultaneously diagonalized by unitary matrices $V_\nu$ and $V_l$ respectively:

\begin{align}
V_\nu^T S_i V_\nu &= s_i , & V_l^T T_i V_l &= t_l ,
\end{align}

where $s_i$ and $t_l$ correspond to diagonal matrices. Eqs. [4, 5, 9–11] can be used to show that

\begin{align}
U_\nu &= V_\nu P_\nu , & U_l &= V_l P_l ,
\end{align}

\footnote{A recent analysis in [8] addresses the problem of relating mixing to symmetries in case of completely or partially degenerate neutrino masses.}
where \( P_{l,\nu} \) are diagonal phase matrices. Therefore, one can write:

\[
U \equiv U_{PMNS} = U_{l}^\dagger U_{\nu} = P_{l}^\dagger V_{l}^\dagger V_{\nu} P_{\nu} .
\]

(5)

Note that \( S_i \) and \( T_l \) denote the 3-dimensional representations of elements of some symmetry group \( G \) in this approach. The structure of these symmetries and the matrices \( V_{l,\nu} \) diagonalizing them is thus determined by group theory and Eq. (5) provides a direct link between leptonic mixing and group theory.

In a bottom up approach, one first determines groups of \( S_i \) and \( T_l \) and then uses them to find suitable group \( G \). A complete set of \( S_i \) and \( T_l \) may depend on underlying dynamics. However one can define a minimal set which can always be taken as symmetries of mass matrices. A field corresponding to a massive Majorana neutrino is arbitrary up to a change of sign in the mass basis. If all three neutrinos are massive then the corresponding diagonal mass matrix is trivially invariant under

\[
s_1 = \text{Diag.}(1, -1, -1) , \quad s_2 = \text{Diag.}(-1, 1, -1) \quad \text{and} \quad s_3 = s_1 s_2 ,
\]

(6)

where \( \text{Det}(s_i) \) is chosen +1. Any two of these define a \( Z_2 \times Z_2 \) symmetry. One can go to arbitrary basis and define corresponding \( G_{\nu} = Z_2 \times Z_2 \) symmetry transformation \( S_i = V_{\nu} s_i V_{\nu}^\dagger \) as a symmetry of the general neutrino mass matrix. The fields corresponding to the charged lepton mass eigenstates are invariant under three independent \( U(1) \) symmetries and Eq. (6) gets replaced by

\[
t_l = \text{Diag.}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}) .
\]

(7)

Assuming that \( t_l \) is an element of some discrete group, the phases \( \phi_{e,\mu,\tau} \) would be restricted to some discrete values and the most general \( G_l \) would be a discrete sub-group of \( U(1)^3 \). Conversely, one can start with a group \( G \), identify its \( Z_2 \times Z_2 \) subgroup corresponding to \( G_{\nu} \) and appropriate \( G_l \) and use them to predict the observed mixing. In this way, \( G = A_4, S_4, A_5, PSL(2, Z_7), \Delta(96), \Delta(384), \Delta(150), \Delta(600) \) and general \( \Delta(6n^2) \) are studied for their predictions of the mixing angles. A complete scan over large number of groups is performed \[11] \[15] and it is found that only three of about a million groups analyzed in \[11] can predict all the mixing angles within 3\( \sigma \). However there exist many choices which lead to very good zeroth order approximation. In particular, groups leading to democratic, bi-maximal (BM) or a TBM mixing matrix are identified. A summary of various cases is given in \[12] . Alternative approach is also proposed in which one relates mixing matrix elements directly to group theoretical parameters using various von-Dyck groups \[10, 16]\.

II. RESIDUAL SYMMETRY WITH A MASSLESS NEUTRINO

Let us now discuss the situation when one of the neutrinos is massless. It follows from Eq. (2) that \( \text{Det}(M_{\nu}) = 0 \) if \( \text{Det}(S_i) \neq \pm 1 \). Thus if neutrino mass matrix is invariant under an element of a subgroup of \( U(3) \) which is not in \( SU(3) \) then such invariance automatically
implies the presence of at least one massless state. The underlying group $G$ necessarily belongs to $U(3)$ and one must look for groups different from the ones used in the existing studies \([2]\).

Residual symmetry of neutrino has to be very specific if one further requires that only one of the neutrinos is massless. While corresponding mass eigenstate appears in the definition of flavor eigenstates and hence in Lagrangian, the neutrino mass term is trivially invariant under a $U(1)$ symmetry which corresponds to an arbitrary change of phase of the massless field. In addition, one can independently change the signs of other two massive states which correspond to a $Z_2 \times Z_2$ symmetry. Thus the full residual symmetry of the neutrino mass matrix is now $Z_2 \times Z_2 \times U(1)$ instead of $Z_2 \times Z_2$. We shall restrict ourselves to a discrete subgroup $Z_N$ (with $N \geq 3$) of $U(1)$ and take the residual symmetry as $Z_2 \times Z_2 \times Z_N$ with the following definition for $Z_N$ in an arbitrary basis

$$
S = V_\nu \text{Diag} \left( e^{\frac{2\pi i k}{N}}, 1, 1 \right) V_\nu^\dagger .
$$

with $k = 1, 2, \ldots, N - 1$ and $e^{\frac{2\pi i k}{N}} \neq -1$. This describes normal hierarchy while the inverted hierarchy can be obtained by replacement $V_\nu \rightarrow V_\nu P_{13}$ in the above $S$, where $P_{13}$ is a permutation matrix in 1-3 plane.

We start with an example which brings out clear differences between situation with a massless state compared to all three neutrinos being massive. Consider the following $Z_2$ and $Z_3$ as residual symmetries respectively for the neutrino and the charged lepton mass matrices:

$$
s_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} .
$$

It is known \([4, 6]\) that $s_1$ and $T$ together generate the $A_4$ group with presentations $S = s_1$ and $T$ satisfying $S^2 = T^3 = (ST)^3 = 1$. The leptonic mixing matrix $U$ in this case can be worked out using Eq. (5) and is given by the following apart from phase matrices.

$$
U \equiv U_\omega^T U_{23}(\theta) ,
$$

where

$$
U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega^2 & \omega \\
1 & \omega & \omega^2
\end{pmatrix} \quad \text{(11)}
$$

with $\omega^3 = 1$ diagonalizes $T$ and $U_{23}(\theta)$ is a unitary rotation in 2-3 plane arising due to degeneracy of two eigenvalues in $s_1$. The mixing matrix is democratic if $\theta = 0$. For $\theta = \pi/4$, absolute values of columns of $U$ coincide with columns of $|U_{TBM}|$ describing the TBM mixing. Whether such $U$ can describe a good zeroth order approximation to the observed mixing further depends on the neutrino mass hierarchy which is not predicted by group theory and depends on the model parameters. There exist variety of models based on $A_4$ giving correct hierarchy and TBM \([2]\). Note that $\theta$ is a free parameter if only $A_4$ is used. It can be fixed
at a value $\pi/4$ by adding another $Z_2$ symmetry corresponding to $\mu$-$\tau$ interchange on the neutrino mass matrix. The group containing $s_1$, $T$ and $\mu$-$\tau$ interchange symmetry is $S_4$ [5].

Let us now demand one of the states to be massless and replace $s_1$ of Eq. (9) with an analogous symmetry

$$s_1(\omega) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(12)

The multiple products of $s_1(\omega)$ as given in Eq. (12) and $T$ as given in Eq. (9) together form a finite group $\Sigma(81)$. This is seen as follows. Define, $a''' = s_1(\omega)$, $b = T$, $a' = T^{-1}a''T$ and $a = T^{-1}a'T$. The matrices $a$, $a'$, $a''$, $b$ define [3] generators of the group $\Sigma(81)$ whose elements are labeled as $g = b^k a^l a''^m a'''^n$ with $k, l, m, n = 0, 1, 2$. $\Sigma(81)$ is known to be a subgroup of $U(3)$ and not of $SU(3)$ [3, 17, 18]. It has been used as a flavor symmetry in [19] entirely for different reasons. While the group $A_4$ obtained in case with all massive neutrinos gets replaced by $\Sigma(81)$, the leptonic mixing matrix is still formally given by Eq. (10) obtained in case of $A_4$. But unlike in case of $A_4$, now the neutrino mass hierarchy is also partly determined by the symmetry due to one vanishing mass. By construction, the massless state in the mass basis corresponds to an eigenvector $|\psi_0\rangle = (1, 0, 0)^T$ which becomes in the flavor basis $U|\psi_0\rangle = \frac{1}{\sqrt{3}}(1, 1, 1)^T$ independent of $\theta$ appearing in Eq. (10). This state being massless cannot be associated with heavier of the solar pair and above state must correspond to the first column of $U$ and not the second column. Thus the TBM pattern cannot be realized in this simple example.

III. EXAMPLES OF MIXING PATTERNS WITH A MASSLESS NEUTRINO

We now follow the strategy as given in the previous section and study several examples of groups accommodating a massless neutrino and derive various mixing patterns implied by them. While all finite subgroups of $SU(3)$ are systematically classified, see [20] for a review, not all subgroups of $U(3)$ are known. Ref. [15] has listed all such subgroups of order less than 100 and this analysis is extended in [18] where all finite subgroups of $U(3)$ of order 512 or less possessing a faithful three dimensional irreducible representation are listed. In the following, we consider three different class of examples based on groups $\Sigma(3N^3)$, $\Sigma(2N^2)$ and $S_4(2) \equiv A_4 \rtimes Z_4$ and study possible mixing patterns implied by them.

A. Mixing pattern with $\Sigma(3N^3)$

The example of $\Sigma(81)$ studied above admits a straightforward generalization to a group series $\Sigma(3N^3)$. Properties of these groups are studied in [3, 17]. These groups are semi-direct product of product $Z_N \times Z_N' \times Z_N''$ of three cyclic groups with a $Z_3$ group. Adopting notation of [3], the $Z_N$ groups are generated by $a$, $a'$, $a''$ satisfying $a^N = a'^N = a''^N = 1$ and commuting with each other. The $Z_3$ generator corresponding to cyclic permutation
transforms $a$, $a'$ and $a''$ among each other as follows:

$$b^{-1}ab = a'', \quad b^{-1}a''b = a' \quad \text{and} \quad b^{-1}a'b = a.$$  \hspace{1cm} (13)

As a result of the above equation, all $3N^3$ elements of the group can be written as $g(k, l, m, n) = b^k a^m a^n$ with $k = 0, 1, 2$ and $l, m, n = 0, 1, ..., N - 1$.

A specific 3-dimensional representation for the generators is given by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}, \quad a' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a'' = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (14)

with $\rho = e^{2\pi i/N}$. Group elements $g(0, l, m, n) = a^l a^m a^n$ are diagonal in this specific representation. From these one can choose $g(0, 0, 0, p)$ as the residual symmetry of the neutrino mass matrix with $p = 1, 2, ..., N - 1$. The residual symmetry of the charged lepton has to be non-diagonal and only allowed choices are $g(1, l, m, n)$ and $g(2, l, m, n)$. We choose $g(1, l, m, n)$ but other choice also gives identical mixing pattern. Let’s choose

$$S = \begin{pmatrix} \rho^p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & \rho^m & 0 \\ 0 & 0 & \rho^l \\ \rho^p & 0 & 0 \end{pmatrix}. \hspace{1cm} (15)$$

Now $T$ is diagonalized by a unitary matrix $U_\rho$ such that $U_\rho^\dagger TU_\rho = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$, where

$$U_\rho = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 \rho^{-m} & \lambda_2 \rho^{-m} & \lambda_3 \rho^{-m} \\ \lambda_1^2 \rho^{-l-m} & \lambda_2^2 \rho^{-l-m} & \lambda_3^2 \rho^{-l-m} \end{pmatrix}. \hspace{1cm} (16)$$

Here, the eigenvalues $\lambda_i \ (i = 1, 2, 3)$ satisfy

$$\lambda_i^3 = \rho^{m+n+l},$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 0,$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0. \hspace{1cm} (17)$$

The solutions to these equations are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \delta\{1, \omega^2, \omega\} \quad \text{or} \quad \delta\{1, \omega, \omega^2\}. \hspace{1cm} (18)$$

where $\delta^3 = \rho^{n+m+l}$ and $\omega^3 = 1$. Thus one can rewrite all possible $U_\rho$ as

$$U_\rho = P_3 U_\omega \quad \text{or} \quad U_\rho = P_8 U^*_\omega, \hspace{1cm} (19)$$

where $P_3 = \text{Diag}(1, \delta \rho^{-m}, \delta^2 \rho^{-l-m})$ and $U_\omega$ is given in Eq. (11). Given the above, one can now work out the predicted mixing pattern using Eq. (5):

$$U = U_\rho^\dagger U_{23}(\theta). \hspace{1cm} (20)$$
This is very similar to the structure obtained in case of $\Sigma(81)$ which is a special case of these groups with $N = 3$. Once again, the state which remains massless is given by $|\psi_0\rangle = (1, 0, 0)^T$ in the mass basis. The composition of the corresponding state in the flavor basis $|\psi_f\rangle \equiv U|\psi_0\rangle = U^\dagger|\psi_0\rangle$ is independent of the angle and phase in $U_{23}$. Also, it does not depend on the charges $l, m, n$ of the residual symmetry of the charged leptons since $P_3|\psi_0\rangle = |\psi_0\rangle$. As a result, one finds $|\psi_f\rangle = \frac{1}{\sqrt{3}} (1, 1, 1)^T$ from Eqs. (16, 19).

The choice of $S$ in Eq. (15) leads to normal hierarchy for which the state $|\psi_f\rangle$ has to be identified with the first column of $U$. The second and the third columns depend on the choice of angle $\theta$ and one has two possibilities. (1) $\theta$ is such that $|U e_2| = \frac{1}{\sqrt{3}}$ so that one gets the TBM value. Then by orthogonality one also predicts $|U e_3| = \frac{1}{\sqrt{3}}$ which is far from the observed value. (2) $\theta$ may be chosen to get $|U e_3| = 0$. Then one automatically gets $\sin^2 \theta_{12} = \frac{2}{3}$. Both these choices are possible. In general, Eq. (20) leads to

$$
\sin \theta_{13} = \frac{1}{\sqrt{3}} |s - c \delta^* \rho|,
$$

$$
\cos \theta_{13} \sin \theta_{12} = \frac{1}{\sqrt{3}} |c + s \delta^* \rho|,
$$

$$
\cos \theta_{13} \sin \theta_{23} = \frac{1}{\sqrt{3}} |s - c \omega \delta^* \rho|, \quad (21)
$$

where $s = \sin \theta$, $c = \cos \theta$ and we have neglected phase in $U_{23}$ for simplicity. It does not alter the result. For the second solution in Eq. (19), $\omega^2$ replaces $\omega$ in the last expression in Eq. (21). Now if we require $\sin \theta_{13} = 0$ in above equation then $s = c$ and $\delta = \rho^*$. This then implies

$$
\sin^2 \theta_{12} = 2/3 \quad \text{and} \quad \sin^2 \theta_{23} = 1/2.
$$

By deviating away from $s = c$ or $\delta = \rho^*$ one either generates large $\theta_{13}$ or small deviation in prediction for the solar mixing angle. Above equations (or equivalently composition of the massless state) can be used to show that irrespective of the choice of $\theta, \delta$ and $\rho$ one has $\sin^2 \theta_{12} \cos^2 \theta_{13} + \sin^2 \theta_{13} = \frac{2}{3}$. This substantially differs from the experimentally required value $\approx \frac{1}{3}$. The choice $s = 0$ gives democratic mixing which is also found to emerge from other groups [12] like $A_4$ in the presence of all three massive neutrinos.

The same argument holds even if one chooses $g(2, l, m, n)$ as $T$ or $S$ is chosen with a phase in either (2,2) or (3,3) entry instead of Eq. (15). In all these cases, structure of the massless state is independent of the unknown angle $\theta$ of the mixing matrix and has a trimaximal form with equal mixture of all three flavours in it. Thus this class of groups can predict either the third or the second column of the PMNS matrix correctly but not both of them simultaneously. We now turn to another choices which can lead to mixing patterns close to reality.

**B. Mixing pattern with $\Sigma(2N^2)$**

The example in this section is based on the $\Sigma(2N^2)$ groups whose properties are listed in [3]. Unlike many examples considered in literature [2] these groups do not admit a
faithful 3-dimensional irreducible representation and we will use reducible 2 + 1 dimensional representation to describe leptonic doublets. As we will see, this too leads to a realistic mixing pattern modulo one free parameter.

Σ(2N^2) is constructed from two Z_N and a Z_2 group and it is isomorphic to Z_N × Z'_N × Z_2. Two commuting Z_N generators a and a' satisfy

\[ a^N = a'^N = 1. \]

Z_2 generator b transforms them into each other:

\[ b^2 = 1 \quad \text{and} \quad bab = a'. \]

Because of the above defined semi-direct product structure all the groups elements can be written as \( b^k a^p a'^q \) with \( k = 0, 1 \) and \( p, q = 0, 1, \ldots, N - 1 \). All the irreducible representations of the group are either 1 or 2 dimensional. The two dimensional representations are labeled by two \( Z_N \) charges \( (p, q) \). One can represent the generators in a 2-dimensional representation \((1, 0)\) by

\[
\begin{align*}
a &= \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, & a' &= \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}, & b &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

where \( \rho = e^{2\pi i/N} \). Using the above representation all the elements of \( \Sigma(2N^2) \) can be written as \(3\)

\[ s_{mn} = \begin{pmatrix} \rho^m & 0 \\ 0 & \rho^n \end{pmatrix} \quad \text{and} \quad t_{mn} = \begin{pmatrix} 0 & \rho^m \\ \rho^n & 0 \end{pmatrix}, \]

where \( m, n = 0, 1, \ldots, N - 1 \). The elements \( t_{mn} \) in the above equation can be diagonalized as \( v_\rho^T t_{mn} v_\rho = \text{Diag}(\lambda_1, \lambda_2) \). Such \( v_\rho \) can be written as

\[
v_\rho = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \lambda_1 \rho^{-m} & \lambda_2 \rho^{-m} \end{pmatrix},
\]

where \( \lambda_1 = -\lambda_2 = \sqrt{\rho^{m+n}} \).

Let’s now apply this symmetry to the lepton sector by constructing its 3-dimensional reducible representation from the above. Let’s define

\[
S_{qp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^p & 0 \\ 0 & 0 & \rho^p \end{pmatrix} \quad \text{and} \quad T_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho^m \\ 0 & \rho^n & 0 \end{pmatrix}
\]

\( S_{0p} \) would be the appropriate symmetry of the neutrino mass matrix with inverted hierarchy. In order to get non-trivial mixing pattern, the charged lepton mass matrix should posses a symmetry which is generated by any of \( T_{mn} \) such that

\[
S_{0p}^T M_\nu S_{0p} = M_\nu \quad \text{and} \quad T_{mn}^T M_1 T_{mn} = M_1 M_1^T.
\]
The above symmetry ensures $m_{\nu_3} = 0$ for $N \geq 3$. The $3 \times 3$ matrix $V_\rho$ diagonalizing $T_{mn}$ has a block diagonal form with the lower $2 \times 2$ block given by the matrix $v_\rho$ in Eq. (25) and $(V_\rho)_{11} = 1$. Once again using Eq. (5), the PMNS matrix can be written as

$$U = V_\rho^t U_{12}(\theta_\nu) = R_{23}(\pi/4)U_{12}(\theta_\nu)P_\nu,$$  \hspace{1cm} (28)

where $P_\nu = \text{Diag} (1, 1, \lambda^*_2 \rho^m)$. The mass eigenstate $|\psi_0\rangle = (0, 0, 1)^T$ of $S_{0\rho}$ with zero eigenvalue goes to $|\psi_f\rangle \equiv U|\psi_0\rangle = V_\rho^t|\psi_0\rangle = \lambda_2^* \rho^m \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T$ and is independent of the unknown angle $\theta_\nu$. Since we are considering inverted hierarchy, this state has to be associated with the third column of $U$. This leads to two predictions namely, $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. The same eigenvector would describe the first column of $U$ in case of the normal hierarchy and would lead to prediction $\theta_{12}$ or $\theta_{13} = \pi/2$ not realized by data. Thus the above example cannot accommodate normal hierarchy.

The above predictions are known to follow if neutrino mass matrix $M_{\nu f}$ in the flavor basis possesses a $\mu$-$\tau$ symmetry. In the present case, $M_{\nu f}$ is invariant under the symmetry $S_{\nu f} = U_{l}^\dagger S_{0\rho} U_{l}$ with $U_l = V_\rho$ and $S_{0\rho}$ given by Eq. (26). Requiring this invariance, one arrives at

$$M_{\nu f} = \begin{pmatrix}
X & A & A \\
A & B & B \\
A & B & B 
\end{pmatrix}.$$  \hspace{1cm} (29)

This form is independent of the integer $m, n, p$ used in defining $S_{0\rho}$ and $T_{mn}$. It displays the scaling form with a massless neutrino which is studied in a number of papers [21].

The above form for neutrino mass matrix provides a good zeroth order approximation to realistic pattern with non-zero $\theta_{13}$ for the following reason. In order to be able to do so, $M_{\nu f}$ must be such that a small perturbation to it can generate the correct mixing angles. It is found [22] that this does not happen in case of an arbitrary $\mu$-$\tau$ symmetric neutrino mass matrices. Only those possessing inverted or quasi degenerate spectrum can lead to correct mixing pattern with small perturbation. Since the above mass matrix implies inverted hierarchy, small perturbations in $A$ and $B$ which can arise from the residual symmetry breaking are expected to generate the correct mixing pattern.

C. Mixing pattern with $S_4(2)$

$S_4(2)$ is a member of the group series $S_4(m) \equiv A_4 \rtimes Z_{2m}$ which are subgroups of $U(3)$ admitting a three dimensional faithful irreducible representation [18]. Properties of $S_4(2)$ are studied by Ludl [18] and we use this group here to show that it can be used to predict BM mixing in case of the inverted hierarchy.

The first member of the series $S_4(1)$ is isomorphic to $S_4$. This group has been used to predict both TBM [2] and BM [12, 23] mixing patterns. It is useful to briefly recapitulate how this is achieved. The presentations of $S_4$ are given in a three dimensional representation
by the matrices $A \equiv s_1$ and $B \equiv T$ defined in Eq. (39) and the $\mu$-$\tau$ interchange symmetry $C$:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

The TBM mixing arises when $S_4$ gets broken to $G_\nu = Z_2 \times Z_2$ and $G_l = Z_3$ with $S_1 = A$ and $S_2 = C$ generating $Z_2 \times Z_2$ and $T = B$ generating $Z_3$. The BM mixing is obtained with $G_\nu = Z_2 \times Z_2$ and $G_l = Z_4$. The corresponding generators are give by:

$$S_1 \equiv BAB^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S_2 \equiv BCB^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T \equiv BAB^{-1}C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (30)$$

Let us now discuss $S_4(2)$. It contains $A_4$ subgroup which is given by $A$ and $B$ defined above in case of $S_4$. The $\mu$-$\tau$ symmetry operator $C$ is replaced by $C' \equiv iC$ generating a $Z_4$ group used in defining the semi-direct product $A_4 \times Z_4$. $A$, $B$ and $C'$ provide a presentation of $S_4(2)$ \[18\]. In order to obtain a massless state at least one of the generator in $G_\nu$ has to be $Z_N$ with $N > 2$. After a systematic search over various $Z_N \in S_4(2)$ we find that getting TBM mixing along with the right mass hierarchy is not possible but one can obtain the BM mixing with inverted hierarchy. This is obtained when both $G_\nu$ and $G_l$ correspond to $Z_4$ symmetry with the generators $S_\nu$, $T_l \in S_4(2)$:

$$S_\nu \equiv A B^{-1} C' = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \quad \text{and} \quad T_l \equiv B A B^{-1} C' = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \quad (31)$$

Not that $S_\nu$ and $T_l$ are permutations of each other. $T_l$ is diagonalized by a rotation $U_l = P_2 R_{23}(\pi/4)$ in the 2-3 plane and $S_\nu$ by $U_\nu = P_1 R_{12}(\pi/4)$ where $P_2 = \text{Diag.}(1, i, 1)$ and $P_1 = \text{Diag.}(i, 1, 1)$. As a result, one obtains the PMNS mixing matrix

$$U = R_{23}^T(\pi/4) P_2^T P_1 R_{12}(\pi/4)$$

with predictions $\theta_{12} = \theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$. The $S_\nu$ has the eigenvalues $(1, -1, -i)$ and this choice describes inverted hierarchy with the eigenvector of a massless state in flavor space $|\psi_f\rangle = \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right)^T$. The neutrino mass matrix in the flavor basis invariant under
the symmetry $S_{\nu f} = U^\dagger_{l} S_{\nu} U_{l}$ in this case is given by

$$\mathcal{M}_{\nu f} = \begin{pmatrix} 2B & A & A \\ A & B & B \\ A & B & B \end{pmatrix}. \quad (33)$$

This leads to a massless neutrino as expected. It also gives the BM mixing except in a special case with $\text{Re}(AB^*) = 0$. One gets degenerate neutrino pair in this case and the solar angle is undefined in this limit.

One can obtain normal hierarchy by interchanging the role of $S_{\nu}$ and $T_{l}$ in Eq. (31). However in this case, the first and the third columns of $U$ in Eq. (32) get interchanged and it predicts $\theta_{13} = \pi/4$ and $\theta_{12} = \pi/2$ which are far away from their experimental values. We also searched for alternative solutions with normal hierarchy within $S_{4}(2)$. A solution which comes closest to the observed mixing pattern is given by the choice $S_{\nu} = BAB^{-1}C' \in Z_{4}$ and $T_{l} = B \in Z_{3}$. This predicts $\theta_{23} \approx 36.2^\circ$, $\theta_{12} \approx 53.8^\circ$ and $\theta_{13} \approx 12.2^\circ$. This solution accommodates non-zero $\theta_{13}$ and non-maximal $\theta_{23}$ at the zeroth order approximation itself but it requires rather large corrections in order to get viable solar mixing angle. In the next section, we provide a specific model realization of the above solution and discuss the suitable corrections which can lead to viable mixing angles.

IV. AN $S_{4}(2)$ MODEL OF A MASSLESS NEUTRINO

The $S_{4}(2)$ group contains four 1-dimensional $(1_1, 1_2, 1_3, 1_4)$, two 2-dimensional $(2_1, 2_2)$ and four 3-dimensional $(3_1, 3_2, 3_3, 3_4)$ irreducible representations (irreps) [18]. The above set includes the irreps of $S_{4}$ which are $1_1, 1_2, 2_1, 3_1$ and $3_2$. The remaining irreps of $S_{4}(2)$ can be obtained by multiplying each irrep of $S_{4}$ with 1-dimensional irreps of $Z_{4}$ (see the Appendix for more details). The structure of tensor product decomposition can easily be obtained using those of $S_{4}$ as discussed in [18] and in the Appendix here. We also give in the Appendix representation matrices in a chosen basis and multiplication rules relevant for the model discussed here. Note that in our basis all the irreps of $S_{4}$ are real. Further, one can see that $1_{3}^* = 1_4$ and $3_{3}^* = 3_{4}$.

We now present an extension of the minimal supersymmetric standard model (MSSM) based on $S_{4}(2)$ symmetry which can lead to a massless neutrino and bi-maximal mixing. Let’s consider the left-handed lepton doublet $L$ transforming as a triplet $3_{3}$ which does not belong to $S_{4}$. The right-handed charged leptons are assigned to $e^c \sim 1_{3}$, $\mu^c \sim 1_{2}$ and $\tau^c \sim 1_{1}$ representations of $S_{4}(2)$. We require five flavon fields $\phi_{1} (\sim 3_{2}), \phi_{2} (\sim 3_{3}), \phi_{3} (\sim 3_{4}), \phi^c (\sim 3_{2})$ and $\chi^c (\sim 2_{1})$ in order to break $S_{4}(2)$ into $G_{l} = Z_{4}$ in the charged lepton sector and $G_{\nu} = Z_{4}$ in the neutrino sector as discussed in the previous section. The light neutrino masses arise through a dim-5 operator $LLH_u H_u$ where $H_u$ and $H_d$ are the MSSM Higgs doublets which are singlets under $S_{4}(2)$. In order to distinguish between $\phi_{1}$ and $\phi^c$, we impose an additional $Z_{2}$ symmetry under which $\phi_{1,2,3}$ and right-handed charged leptons are
odd and the remaining fields are even. The $S_4(2) \times Z_2$ invariant Yukawa superpotential at the leading order can be written as

$$\mathcal{W}_Y = (y_e L \phi_1^c e^c + y_\mu L \phi_2^c \mu^c + y_\tau L \phi_3^c \tau^c) \frac{H_d}{\Lambda} + (y_{1L} L \phi^\nu + y_{2L} L \chi^\nu) \frac{H_u H_u}{\Lambda^2}. \quad (34)$$

Let’s now discuss the breaking of $S_4(2)$ symmetry. The bi-maximal mixing and inverted hierarchy can be achieved if $S_4(2)$ is broken to $G_i = Z_4$ in the charged lepton sector and $G_\nu = Z_4$ in the neutrino sector as already mentioned in Eq. \[31\]. In order to ensure that $S_4(2)$ breaks into $G_i$ in the charged lepton sector, the vevs of $\phi^i_{1,2,3}$ have to be invariant under $G_i$. This can be achieved by taking $T_i$ in the representation corresponding to the flavon field $\phi^i$ and demanding $T_i \langle \phi^i \rangle = \langle \phi^i \rangle$. For example, in case of $\phi^1_1$ ($\sim 3_2$) one gets $T_i(3_2) \equiv T_i(A \rightarrow A, B \rightarrow B, C' \rightarrow iC') = BAB^{-1}(iC')$ and an invariance under $T_i(3_2)$ requires $\langle \phi^1_1 \rangle = (v_{\phi_1}, 0, 0)^T$. Following the same strategy for $\phi^2_1$ and $\phi^3_1$, one obtains the vacuum structures

$$\langle \phi^1_1 \rangle = v_{\phi_1}(1, 0, 0)^T, \quad \langle \phi^2_1 \rangle = v_{\phi_2}(0, 1, i)^T, \quad \text{and} \quad \langle \phi^3_1 \rangle = v_{\phi_3}(0, 1, -i)^T. \quad (35)$$

We similarly find the vacuum structures of $\phi^\nu$ and $\chi^\nu$ invariant under $G_\nu$:

$$\langle \phi^\nu \rangle = v_\phi(0, 0, 1)^T, \quad \text{and} \quad \langle \chi^\nu \rangle = v_\chi(1, -\sqrt{3})^T. \quad (36)$$

After the flavor symmetry is broken by the vevs of flavons as given in Eqs. \[35, 36\] and after the breaking of electroweak symmetry, we obtain the following mass matrices for charged leptons and neutrinos:

$$M_l = v_d \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & y_\tau \\ iy_\mu & -iy_\tau & 0 \end{pmatrix} \quad \text{and} \quad M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} y_2 & y_1 & 0 \\ y_1 & -y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (37)$$

where $v_{u,d} = \langle H_{u,d} \rangle$ and the Yukawa couplings $y_i$ are suitably redefined by absorbing the flavon vevs. In the derivation of the above mass matrices from the Lagrangian \[34\], we have used the multiplication rules listed in Eqs. \[A6, A9\] in the Appendix. As it is set by the vacuum structures of the flavon fields, the above mass matrices satisfy

$$S_\nu^T M_\nu S_\nu = M_\nu \quad \text{and} \quad T_i^\dagger M_l M_l^\dagger T_i = M_\nu M_\nu^\dagger \quad (38)$$

and, at the leading order, lead to a massless neutrino with bi-maximal mixing in the lepton sector.

The above predictions can be corrected and made viable in the following two possible ways. First, the next to leading order effects in general break both $G_i$ and $G_\nu$ generating corrections to the bi-maximal mixing pattern as well as generating a mass for the neutrino. However such corrections are generically assumed to be very small and they may not induce
relatively large corrections required in the solar and reactor angles. Second, a breaking in
the residual symmetry of the charged leptons only, namely in $G_l$, may arise at the leading
order itself due to the presence of additional fields in the spectrum. In this scenario, the
leading order prediction of a massless neutrino is not perturbed by such corrections. Here,
we provide an example of the second type. Consider the presence of an additional
Z\textsubscript{2} odd flavon field $\phi'^l$ which transforms as 3\textsubscript{3} of $S_4(2)$. This adds a piece of interaction in Eq. (34)
proportional to $\epsilon L \phi'^l u^c$. If $\phi'^l$ takes vev in the direction $(1,0,0)^T$ which does not respect
the $G_l$ symmetry characterized by $T_l$ in Eq. (31). This leads to the following correction in
$M_l$:

$$M_l = v_d \begin{pmatrix} y_e & \epsilon & 0 \\ 0 & y_\mu & y_\tau \\ 0 & iy_\mu & -iy_\tau \end{pmatrix}.$$  

The resulting $M_l M_l^\dagger$ takes the following form

$$M_l M_l^\dagger = v_d^2 \begin{pmatrix} x & a & -ia \\ a^* & y & -iz \\ ia^* & iz & y \end{pmatrix},$$  

where $x$, $y$ and $z$ are real parameters. The group $Z_4$ breaks completely once the $\phi'^l$ acquires
vev. However, the matrix $M_l M_l^\dagger$ now possess an accidental $Z_2$ symmetry generated by

$$T'_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$  

Note that $T'_l$ is not even a subgroup of $S_4(2)$ since all generators of $S_4(2)$ in our basis are
either purely real or purely imaginary. The $M_l M_l^\dagger$ in Eq. (40) can be diagonalized by

$U_l = P_3 R_{23}(\pi/4) U_{12}(\theta, \alpha)$ where $P_3 = \text{Diag.}(1,1,i)$,

$$U_{12}(\theta, \alpha) = \begin{pmatrix} \cos \theta & -e^{i\alpha} \sin \theta & 0 \\ e^{-i\alpha} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  

where $\tan 2\theta = 2\sqrt{2}|a|/(x - y - z)$ and $\alpha = \text{arg}(a)$. The neutrino mass matrix in Eq. (37)
is diagonalized by $U_\nu = P_2 R_{12}(\pi/4)$ where $P_2 = \text{Diag.}(1,i,1)$. The resulting $U_{PMNS} \equiv U = U_l^\dagger U_\nu$ predicts the following correlations among the leptonic mixing angles.

$$U_{e3} = -i e^{i\alpha} \sin \theta \sqrt{2},$$

$$U_{e2} = -\frac{1}{\sqrt{2}} \left( \cos \theta - i e^{i\alpha} \sin \theta \sqrt{2} \right),$$

$$U_{\mu3} = -\frac{i}{\sqrt{2}} \cos \theta.$$  

(43)
For $\alpha \approx -\pi/2$ and $\theta \approx 0.23$, it predicts $\sin^2 \theta_{13} \approx 0.026$, $\sin^2 \theta_{12} \approx 0.339$ and $\sin^2 \theta_{23} \approx 0.487$ which are in agreement within the $3\sigma$ ranges of their global fit values. Interestingly, the angle $\theta$ is close to the Cabibbo angle. Also, the amount of perturbation $\epsilon$ required to correct the mixing angles is quite small

$$\frac{|\epsilon| \nu_d}{m_\tau} \approx \frac{m_\mu}{m_\tau} \frac{\theta}{\sqrt{2}} \approx 10^{-2}. \quad (44)$$

Having fixed $\alpha$ and $\theta$ one now predicts $\delta \sim \pi$ for the Dirac CP phase implying the near absence of CP violation in neutrino oscillations. The above values of $\theta$ and $\alpha$ fix the parameter $a$ while the three charged lepton masses are determined using the remaining three free parameters in Eq. (40). Similarly, the solar and atmospheric mass squared differences determine the complex parameters $y_1$ and $y_2$ in $M_\nu$ in Eq. (37). The above corrections generate viable mixing pattern at the leading order maintaining the prediction of a massless neutrino.

V. DISCUSSIONS

We have addressed here the problem of finding appropriate groups $G$ which can lead to a massless neutrino by identifying residual symmetry $G_\nu$ of the neutrino mass matrix in this case. Resulting $G_\nu$ is larger than the conventional $Z_2 \times Z_2$ groups used extensively in case of massive Majorana neutrinos. As argued here, the groups $G$ which contain $G_\nu$ are subgroups of $U(3)$ rather than of $SU(3)$ and hence are quite different from the ones used so far in literature [2, 3]. We have considered group series $\Sigma(3N^3)$, $\Sigma(2N^2)$ and the group $S_4(2) \equiv A_4 \times Z_4$ as possible examples which contain $G_\nu$ implying a massless neutrino. It is shown that last two of these lead to good zeroth order approximation to realistic mixing.

Suitable perturbations are required in the discussed examples to generate the viable leptonic mixing angles. Depending on the nature of corrections, the following possibilities may arise: (1) If perturbation only breaks the symmetry group $G_I$ and keeps $G_\nu$ intact then we can generate correction to mixing without generating a mass for a neutrino. An explicit example of this case is presented in the section IV where we discuss a breaking of $G_I$ which can lead to a realistic mixing pattern maintaining a massless neutrino. Similar argument works in the case of $\Sigma(2N^2)$ where the solar angle is not fixed by the symmetry. (2) Even if perturbation is such that $G_\nu$ is affected and massless neutrino picks a mass, we would have found a new flavor symmetry to describe realistic masses and mixing which cannot be arrived at by insisting that $G_\nu$ contains a $Z_2 \times Z_2$ symmetry. Clearly, these possibilities require separate and detailed investigations of the specific models based on the symmetry groups proposed here.

The approach pursued here shares advantages and disadvantages of similar approach used extensively [2, 3] for relating leptonic mixing to symmetry in case of massive neutrinos. Advantage is that residual symmetries $G_\nu$ and $G_I$ inferred from experimental knowledge tell us what could be the underlying flavor symmetry groups which when broken to $G_\nu$ and $G_I$ lead to definite mixing pattern and in our case also a massless neutrino. Disadvantage
is that other groups $G'$ which may not contain $G_\nu$ fully may also lead to the same mixing pattern as $G$ and the remaining symmetry in $G_\nu$ may arise accidentally. Well-known examples are models based on $A_4$ \cite{2,3} leading to TBM mixing in spite of the fact that $A_4$ contains only one of the two $Z_2$ symmetries required to obtain TBM. The other one arises as accidental symmetry in these models. Similarly, in our case also, a massless neutrino may result from an accidental symmetry of the mass matrix. A simple example would be the type-I seesaw model with three active and two massive right handed neutrinos. In this model, a massless neutrino and hence the corresponding residual symmetry would arise purely for ‘kinematical’ reasons without imposing any flavor symmetry. But barring such cases, the approach studied here allows a systematic way of identifying flavor groups leading to a massless neutrino and some definite mixing patterns.

Acknowledgments: A.S.J. thanks the Department of Science and Technology, Government of India for support under the J. C. Bose National Fellowship programme, grant no. SR/S2/JCB-31/2010. K.M.P. thanks the Theory Division of Physical Research Laboratory for hospitality and support.

Appendix A: The group $S_4(2)$

The group $S_4(2)$ is a subgroup of $U(3)$ and it has a structure equivalent to $A_4 \rtimes Z_4$. The properties of $S_4(2)$ are studied in \cite{18} and we use many of the results obtained there. In the basis considered in the text, the generators of a faithful 3-dimensional irrep of $S_4(2)$ are given by

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C' = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
$$

(A1)
The irreps of $S_4(2)$ in the above basis are given by [18]

$$
1_1 : A = 1, \ B = 1, \ C' = 1; \\
1_2 : A = 1, \ B = 1, \ C' = -1; \\
1_3 : A = 1, \ B = 1, \ C' = i; \\
1_4 : A = 1, \ B = 1, \ C' = -i; \\
2_1 : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \ C' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\
2_2 : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \ C' = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\
3_1 : A = A, \ B = B, \ C' = -iC'; \\
3_2 : A = A, \ B = B, \ C' = iC'; \\
3_3 : A = A, \ B = B, \ C' = C'; \\
3_4 : A = A, \ B = B, \ C' = -C'.
$$

(A2)

All the irreps of $S_4$ ($1_1, 1_2, 2_1, 3_1, 3_2$) are irreps of $S_4(2)$ too as well as all the irreps of $Z_4$ ($1_1, 1_2, 1_3, 1_4$) are irreps of $S_4(2)$. Further, one can obtain all the irreps of $S_4(2)$ from the irreps of $S_4$ by multiplying them with the irreps of $Z_4$. For example, one gets

$$
2_1 \equiv 2_1 \otimes 1_3, \ 3_3 \equiv 3_1 \otimes 1_3, \ 3_4 \equiv 3_1 \otimes 1_4
$$

(A3)

1. Tensor products

The tensor products for irreps of $S_4$ are given as:

$$
1_1 \otimes \chi_i = \chi_i; \\
1_2 \otimes 1_2 = 1_1; \ 1_2 \otimes 2_1 = 2_1; \ 1_2 \otimes 3_1 = 3_2; \ 1_2 \otimes 3_2 = 3_1; \\
2_1 \otimes 2_1 = 1_1 \oplus 1_2 \oplus 2_1; \ 2_1 \otimes 3_1 = 2_1 \oplus 3_2 = 3_1 \oplus 3_2; \\
3_1 \otimes 3_1 = 3_2 \oplus 3_2 = 1_1 \oplus 2_1 \oplus 3_1 \oplus 3_2; \\
3_1 \otimes 3_2 = 1_2 \oplus 2_1 \oplus 3_1 \oplus 3_2
$$

(A4)

The tensor products of remaining irreps of $S_4(2)$ can be obtained in a straightforward way by multiplying the irreps of $S_4$ with suitable 1-dimensional irrep of $Z_4$. For example, one obtains using Eqs. (A3)

$$
1_3 \otimes 1_3 = 1_4 \otimes 1_4 = 1_2; \ 1_3 \otimes 1_4 = 1_1; \\
3_1 \otimes 3_3 = 3_1 \otimes (3_1 \otimes 1_3) = 1_3 \oplus 2_2 \oplus 3_3 \oplus 3_4; \\
3_2 \otimes 3_3 = 3_2 \otimes (3_1 \otimes 1_3) = 1_4 \oplus 2_2 \oplus 3_3 \oplus 3_4; \\
3_3 \otimes 3_3 = (3_1 \otimes 1_3) \otimes (3_1 \otimes 1_3) = 1_2 \oplus 2_1 \oplus 3_1 \oplus 3_2; \\
3_3 \otimes 3_4 = (3_1 \otimes 1_3) \otimes (3_1 \otimes 1_4) = 1_1 \oplus 2_1 \oplus 3_1 \oplus 3_2
$$

(A5)
2. Multiplication rules

The multiplication rules for tensor products in the basis we have chosen are as the below. Here, we give only those multiplication rules relevant for the model presented in the text.

\[(x_1)_i \otimes (y_1)_j = (xy)_k \quad (A6)\]

\[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{i} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{j} = \frac{1}{\sqrt{2}}(x_1y_1 + x_2y_2)_{k} + \ldots \quad (A7)\]

\[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{i} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{j} = \frac{1}{\sqrt{3}}(x_1y_1 + x_2y_2 + x_3y_3)_{k} + \ldots \quad (A8)\]

\[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{i} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{j} = \frac{1}{\sqrt{3}}(x_1y_1 + x_2y_2 + x_3y_3)_{1} + \frac{1}{\sqrt{2}}(x_2y_2 - x_3y_3)_{2} + \frac{1}{\sqrt{2}}(2x_1y_1 - x_2y_2 - x_3y_3)_{2i} + \frac{1}{\sqrt{2}}(x_2y_3 - x_3y_1)_{31} + \frac{1}{\sqrt{2}}(x_3y_1 - x_1y_3)_{31} + \frac{1}{\sqrt{2}}(x_1y_2 - x_2y_1)_{31} + \frac{1}{\sqrt{2}}(x_2y_3 + x_3y_1)_{32} + \frac{1}{\sqrt{2}}(x_3y_1 + x_1y_3)_{32} + \frac{1}{\sqrt{2}}(x_1y_2 + x_2y_1)_{32} \quad (A9)\]

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