A Comparison Estimate for Singular p-Laplace Equations and Its Consequences

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Abstract

Comparison estimates are an important technical device in the study of regularity problems for quasilinear possibly degenerate elliptic and parabolic equations. Such tools have been employed indispensably in the papers of Mingione, Duzaar–Mingione, and Kuusi–Mingione, etc. on certain measure datum problems to obtain pointwise bounds for solutions and their full or fractional derivatives in terms of appropriate linear or nonlinear potentials. However, a comparison estimate for p-Laplace type elliptic equations with measure data is still unavailable in the strongly singular case \(1 < p \leq \frac{3n-2}{2n-1}\), where \(n \geq 2\) is the dimension of the ambient space. This issue will be completely resolved in this work by proving a comparison estimate in a slightly larger range \(1 < p < \frac{3}{2}\). Applications include a ‘sublinear’ Poincaré type inequality, pointwise bounds for solutions and their derivatives by Wolff’s and Riesz’s potentials, respectively. Some global pointwise and weighted estimates are also obtained for bounded domains, which enable us to treat a quasilinear Riccati type equation with possibly sublinear growth in the gradient.

1. Introduction and Main Results

In this paper, we are concerned with the quasilinear elliptic equation with measure data

\[-\text{div}(A(x, \nabla u)) = \mu\]  \(1.1\)

in a bounded open subset \(\Omega\) of \(\mathbb{R}^n\), \(n \geq 2\). Here \(\mu\) is a finite signed measure in \(\Omega\) and the nonlinearity \(A = (A_1, \ldots, A_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is a vector valued function.

Our main assumptions on \(A\) are as follows. There exist \(\Lambda \geq 1\) and \(p \in (1, n)\) such that, for every \(x \in \mathbb{R}^n\) and every \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\},\)
\[ |A(x, \xi)| \leq \Lambda |\xi|^{p-1}, \quad |D_\xi A(x, \xi)| \leq \Lambda |\xi|^{p-2}, \quad (1.2) \]
\[ (D_\xi A(x, \xi) \eta, \eta) \geq \Lambda^{-1} |\xi|^{p-2} |\eta|^2. \quad (1.3) \]

A typical model for (1.1) is obviously given by the \( p \)-Laplace equation with measure data

\[-\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u) = \mu \quad \text{in} \quad \Omega.\]

For a reason that will be explained momentarily, unless otherwise stated, in this paper we only consider the ‘singular’ case

\[ 1 < p < 3/2. \]

An important technical tool in the study of regularity problems for equation (1.1) is a comparison estimate that connects the solution of measure datum problem to a solution of a homogeneous problem. To describe it, let \( Q_r(x_0) \) denote the open cube \( Q_r(x_0) := x_0 + (-(r, r)^n \text{ with center } x_0 \in \mathbb{R}^n \text{ and side-length } 2r \). Let \( u \in W^{1,p}_{\text{loc}}(\Omega) \) be a solution of (1.1). For a cube \( Q_r = Q_r(x_0) \subseteq \Omega \), we then consider the unique solution \( w \in W^{1,p}_{0}(Q_r(x_0)) + u \) to the local interior problem

\[ \begin{cases} 
-\text{div} (A(x, \nabla u)) = 0 & \text{in } Q_r(x_0), \\
 w = u & \text{on } \partial Q_r(x_0). 
\end{cases} \quad (1.4) \]

For \( p > \frac{3n-2}{2n-1} \), this comparison estimate reads as follows:

**Lemma 1.1.** [8,18,20] Assume that \( p > \frac{3n-2}{2n-1} \). Then we have

\[
\left( \int_{Q_r} |\nabla u - \nabla w|^{\gamma} \, dx \right)^{\frac{1}{\gamma}} \leq C \left[ \frac{|\mu|((Q_r))}{r^{n-1}} \right]^{\frac{1}{p-1}} + C \frac{|\mu|((Q_r))}{r^{n-1}} \left( \int_{Q_r} |\nabla w|^{\gamma} \, dx \right)^{\frac{2-p}{\gamma}}. \quad (1.5)
\]

Here \( \gamma = 1 \) in the case \( p > 2 - 1/n \) and when \( p \geq 2 \) the second term on the right can be dropped (see [8,18]). For \( \frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n} \), \( \gamma \in (\frac{n}{2n-1}, \frac{n(p-1)}{n-1}) \) (see [20]).

In fact, as demonstrated in our recent work [23, Lemma 2.1], the comparison bound (1.5) also holds for all \( 1 < p \leq \frac{3n-2}{2n-1} \), as long as we replace the quantity \( \left( \int_{Q_r} |\nabla u|^{\gamma} \, dx \right)^{(2-p)/\gamma} \) on the right-hand side with the quantity \( \int_{Q_r} |\nabla u|^{2-p} \, dx \). The power \( |\nabla u|^{2-p} \) arises naturally from the structure of \( p \)-Laplace type equations when \( p < 2 \), (see (2.6)). However, the quantity \( \int_{Q_r} |\nabla u|^{2-p} \, dx \) is useful only when \( 2 - p < \frac{n(p-1)}{n-1} \) as \( \frac{n(p-1)}{n-1} \) is the optimal (weak) integrability of the gradient of fundamental solution. This shows that the quantity \( \int_{Q_r} |\nabla u|^{2-p} \, dx \) is ‘super-critical’ in the singular case \( 1 < p \leq \frac{3n-2}{2n-1} \). As \( \frac{3n-2}{2n-1} < 3/2 \), it is enough to consider the
range $1 < p < 3/2$ mentioned above. In what follows, we shall fix a constant $\kappa$ defined by

$$\kappa = (p - 1)^2/2.$$  

**Theorem 1.2.** Suppose that $n \geq 2$ and $Q_{\Sigma r}(x_0) \subset \Omega$ for $r > 0$ and $\Sigma \in (1, 2]$. Let $u \in W^{1,p}_0(\Omega)$ be a solution of (1.1) and let $w$ be as (1.4). Then

$$
\left( \frac{\int_{Q_r(x_0)} |\nabla (u - w)|^\kappa}{|x'|^{n-1}} \right)^{\frac{1}{\kappa}} + \frac{1}{r} \left( \frac{\int_{Q_r(x_0)} |u - w|^\kappa}{|x'|^{n-1}} \right)^{\frac{1}{\kappa}} \\
\lesssim \left( \frac{|\mu|}{r^{n-p}} \right)^{\frac{1}{p-1}} + \frac{|\mu|}{r^{n-p}} \left( \frac{\int_{Q_{r\Sigma r}(x_0)} |\nabla u|^\kappa}{|x'|^{n-1}} \right)^{\frac{2-p}{\kappa}}, 
$$

and, for any $\lambda \in \mathbb{R}$,

$$
\left( \frac{\int_{Q_r(x_0)} |u - w|^\kappa}{|x'|^{n-1}} \right)^{\frac{1}{\kappa}} \lesssim \left( \frac{|\mu|}{r^{n-p}} \right)^{\frac{1}{p-1}} + \frac{|\mu|}{r^{n-p}} \left( \frac{\int_{Q_{r\Sigma r}(x_0)} |u - \lambda|^\kappa}{|x'|^{n-1}} \right)^{\frac{2-p}{\kappa}}.
$$

A boundary version of Theorem 1.2, where $x_0 \in \partial \Omega$ and $u \in W^{1,p}_0(\Omega)$, is also available; see Theorem 2.8 below. Theorem 1.2 follows from Lemmas 2.5, 2.6, and Corollary 2.4 below. Notice that we use the larger cube $Q_{\Sigma r}(x_0)$ on the right-hand side of the bounds in Theorem 1.2, which makes it slightly different from Lemma 1.1. But this is harmless in applications. As $\kappa < \frac{n(p-1)}{n-1}$ when $1 < p < 2$, we see that the quantity $(\int_{Q_{\Sigma r}(x_0)} |\nabla u|^\kappa)^{1/\kappa}$ is sub-critical in this case. In fact, as the proof goes, in the first step (see Lemma 2.5) we obtain an inequality similar to (1.6) but with a mixed norm quantity

$$
\left( \frac{\int_{Q_{\Sigma r}(x_0)} |\nabla u(x', x_n)|^{2-p} dx_n}{r^{n-1}} \right)^{\frac{p-1}{2-p}} \left( \frac{3-p}{p-1} \right)^{\frac{3-p}{2-p}}
$$

in place of $(\int_{Q_{\Sigma r}(x_0)} |\nabla u|^\kappa)^{1/\kappa}$, and no enlargement of the cube is needed in this step. Note that this is a substantial improvement of [23, Lemma 2.1] as the exponent $2 - p$ is used only in one direction and the exponent for the other directions goes to zero as $p$ approaches 1. Another way to look at this is to consider the fundamental solution $v(x) = c |x|^{(p-n)/(p-1)}$, $1 < p < n$, of the $p$-Laplace equation. Then $|\nabla v| \in L^{2-\frac{2}{n-1}}$ if and only if $p > \frac{3n-2}{2n-1}$, whereas the mixed norm above for $\nabla v$ is finite for all $p > 1$. We indeed show in Lemma 2.6 (see also Corollary 2.4) that the latter also holds for the gradient of any solution to (1.1) by using the first step and a suitable reverse Hölder type inequality for gradients of solutions of the homogeneous equation $\text{div}(A(x, \nabla v)) = 0$. Moreover, this enables us to control the quantity in (1.7) by $(\int_{Q_{\Sigma r}(x_0)} |\nabla u|^\kappa)^{1/\kappa}$ in some way to complete the proof.
For the rest of this section, we will present some selected consequences of Theorem 1.2. More consequences of Theorem 1.2 will be presented elsewhere in our future work.

First, we have a Poincaré type inequality with low integrability and with a ‘correction’ term. It is well-known that Poincaré inequality generally fails in the ‘sublinear’ range (see [4]).

**Corollary 1.3.** Suppose that $n \geq 2$ and $Q_{r}(x_0) \subset \Omega$ for $r > 0$ and $\Sigma \in (1, 2]$. Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a solution of (1.1). Then, for any $\varepsilon > 0$, we have

$$
\left( \inf_{q \in \mathbb{R}} \int_{Q_{r}(x_0)} |u - q|^\varepsilon \right)^{\frac{1}{\varepsilon}} \lesssim \left( \frac{|\mu|(Q_{r}(x_0))}{r^{n-p}} \right)^{\frac{1}{p-1}} + r \left( \int_{Q_{r}(x_0)} |\nabla u|^\varepsilon \right)^{\frac{1}{\varepsilon}}.
$$

Second, we have a result on potential estimates for solutions of (1.1). In particular, Theorem 1.4 below extends [14, Theorems 1.1 and 1.10] to all smaller values of $p$. For nonnegative measures $\mu$, (1.8) is a fundamental result due to Kilpeläinen and Malý [13] (see also [29]). Hereafter, the (truncated) Havin–Maz’ya–Wolff’s potential (often called Wolff’s potential) $W_{\gamma, p}^R(\nu)$, $R, \gamma > 0$, of a nonnegative measure $\nu$ is defined by

$$
W_{\gamma, p}^R(\nu)(x) := \int_0^R \left[ \frac{\nu(B_t(x))}{t^{n-\gamma p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.
$$

On the other hand, the truncated Riesz’s potential of order $\gamma > 0$ is defined as

$$
I_{\gamma}^R(v) := W_{\gamma/2, 2}^R(v)(x) = \int_0^R \frac{v(B_t(x))}{t^{n-\gamma}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.
$$

**Theorem 1.4.** Suppose that $u \in C^0(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega)$ solves (1.1) for a finite measure $\mu$ in $\Omega$. Then under (1.2)–(1.3), for any $B_R(x_0) \subset \Omega$ we have

$$
|u(x_0)| \lesssim W_{1, p}^R(|\mu|)(x_0) + \left( \int_{B_R(x_0)} |u(z)|^\kappa dz \right)^{\frac{1}{\kappa}}. \quad (1.8)
$$

Third, we discuss potential estimates for gradients of solutions to (1.1). Pointwise estimates for the gradient were obtained in [8,9,15,16] for the case $p > 2 - 1/n$. For the case $\frac{3n-2}{2n-1} < p \leq 2 - 1/n$, see the recent work [7] (see also [21]). The next theorem covers all of the remaining range of $p$.

**Theorem 1.5.** Assume that $A(x, \xi)$ satisfies the Dini condition

$$
\int_0^1 \omega(\rho) \frac{d\rho}{\rho} < +\infty \quad (1.9)
$$

for a non-decreasing function $\omega : [0, 1] \to [0, \infty)$ such that $\lim_{\rho \to 0^+} \omega(\rho) = \omega(0) = 0$ and

$$
|A(x, \xi) - A(y, \xi)| \leq \omega(|x - y|)|\xi|^{p-1}, \quad \forall x, y, \xi \in \mathbb{R}^n, |x - y| \leq 1.
$$
Suppose that \( u \in C^1(\Omega) \) solves (1.1) for a finite measure \( \mu \) in \( \Omega \). Then for any ball \( B_R(x_0) \subset \Omega \) we have

\[
|\nabla u(x_0)| \lesssim \left[ I_1^R(|\mu|)(x_0) \right]^{\frac{1}{p-1}} + \left( \int_{B_R(x_0)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}}.
\]

The proof of Theorem 1.5 is based mainly on Theorem 1.2 and the pioneering idea of [8] (see also [14]). However, some modifications are needed due to the fact that the gradient of solution may not belong to \( L^1_{\text{loc}}(\Omega) \). Such modifications have been carried out in [7] for \( \frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n} \) (see also [21]). The main point here is to replace the mean oscillations such as

\[
\int_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}| \, dy, \quad (\nabla u)_{B_\rho} := \int_{B_\rho} \nabla u \, dz,
\]

with the quantity

\[
\left( \inf_{q \in \mathbb{R}^n} \int_{B_\rho} |\nabla u - q|^p \, dy \right)^{\frac{1}{p}}.
\]

Thus we shall omit the details of the proof of Theorem 1.5.

Fourth, we discuss some global gradient estimates for renormalized solutions to the Dirichlet problem

\[
\begin{aligned}
-\text{div}(A(x, \nabla u)) &= \mu \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (1.10)

We refer to the paper [5] for several equivalent definitions of renormalized solutions along with their stability property. Here we mention that truncations of a renormalized solution \( u \) of (1.10) are stable even near the boundary in the sense that if \( u_k = T_k(u), \ k > 0 \), where

\[
T_k(s) = \max\{\min\{s, k\}, -k\}, \quad s \in \mathbb{R},
\] (1.11)

then \( u_k \in W^{1,p}_0(\Omega) \) is the unique solution of (1.10) with measure datum \( \mu_k \) in place of \( \mu \) such that \( \mu_k \) converges in the narrow topology of measures to \( \mu \), i.e.,

\[
\lim_{k \to \infty} \int_{\Omega} \varphi \, d\mu_k = \int_{\Omega} \varphi \, d\mu,
\]

for every bounded and continuous function \( \varphi \) on \( \Omega \).

Pointwise gradient estimates up to the boundary for renormalized solutions of (1.10) can be deduced from [7] in the case \( p \in \left( \frac{3n-2}{2n-1}, 2 \right) \). Here \( A(x, \xi) \) is assumed to have the special structure \( A(x, \xi) = a(x)|\xi|^{p-2}\xi \) for any \( x \) near \( \partial \Omega \), and the boundary \( \partial \Omega \) is assumed to be of class \( C^{1, \text{Dini}} \) (see [7]). With Theorem 1.2 and its boundary counterpart (Theorem 2.8) at hand, the method of [7] also applies to the case \( 1 < p < 3/2 \). For that reason, we shall not present the details of the proof.
Theorem 1.6. Assume that $A(x, \xi)$ satisfies the Dini condition \((1.9)\), and that $\partial \Omega$ is of class $C^{1,\alpha}$ or even $C^{1, \text{Dini}}$. Assume also that $A(x, \xi) = a(x) |\xi|^{p-2} \xi$ for any $x$ near $\partial \Omega$. Then for any renormalized solution $u$ of \((1.10)\) we have

$$|\nabla u(x)| \leq C_0 \left[ \int_{1}^{2 \text{diam}(\Omega)} (|\mu|)(x) \right]^{1 \over p-1} \quad \text{a.e. } x \in \Omega. \quad \text{(1.12)}$$

On the other hand, in many applications it is enough to use a weighted integral estimate of Muckenhoupt-Wheeden type for the gradient, which we shall describe next. For such weighted integral bounds, we can work with equation \((1.10)\) under a much weaker condition on the coefficients and the domain.

As for the coefficients, we recall the following definition from \cite{[25]}:

Definition 1.7. Given $\delta, R_0 > 0$, we say that $A(x, \xi)$ satisfies the $(\delta, R_0)$-BMO condition if

$$\sup_{y \in \mathbb{R}^n, 0 < r \leq R_0} \int_{B_r(y)} \Upsilon(A, B_r(y))(x) \, dx \leq \delta,$$

where

$$\Upsilon(A, B_r(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|A(x, \xi) - \int_{B_r(y)} A(z, \xi) \, dz|}{|\xi|^{p-1}}.$$

As for the domain, we shall use the notion of Reifenberg flat domain (see \cite{[28]}). This includes $C^1$ and Lipschitz domains (with sufficiently small Lipschitz constants), as well as certain fractal domains.

Definition 1.8. Given $\delta \in (0, 1)$ and $R_0 > 0$, we say that $\Omega$ is a $(\delta, R_0)$-Reifenberg flat domain if for every $x \in \partial \Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{z_1, z_2, \ldots, z_n\}$, which may depend on $r$ and $x$, so that in this coordinate system $x = 0$ and that

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}.$$

Our weighted integral estimate is obtained for the class of $A_\infty$ weights. This class consists of nonnegative functions $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that there are two positive constants $C$ and $c$ such that

$$w(E) \leq C \left( \frac{|E|}{|B|} \right)^c w(B),$$

for all balls $B$ and all measurable subsets $E$ of $B$. The pair $(C, c)$ is called the $A_\infty$ constants of $w$ and is denoted by $[w]_{A_\infty}$.

The following theorem extends the results of \cite{[20,25]} (see also \cite{[23]}) to the remaining case $1 < p \leq \frac{2n-2}{2n-1}$:
Theorem 1.9. For any $w \in A_{\infty}$ and $0 < q < \infty$, there exists $\delta = \delta(n, p, \Lambda, q, [w]_{A_{\infty}}) \in (0, 1)$ such that if $A(x, \xi)$ is $(\delta, R_0)$-BMO and $\Omega$ is $(\delta, R_0)$-Reifenberg flat for some $R_0 > 0$, then for any renormalized solution $u$ of (1.10), we have

$$
\|\nabla u\|_{L^q(\Omega)} \leq C\|([M_1(\mu)]^{p^{-1}})}_{L^q(\Omega)}.
$$

(1.13)

Here the constant $C = C(n, p, \Lambda, q, [w]_{A_{\infty}}, \text{diam}(\Omega)/R_0)$, and $M_1(\mu)$ is a fractional maximal function $\mu$ defined by

$$M_1(\mu)(x) := \sup_{r > 0} \frac{|\mu|(B_r(x))}{r^{n-1}}, \quad x \in \mathbb{R}^n. \tag{1.15}$$

We remark that the weighted bound (1.13) also holds if we replace the weighted space $L^q_0(\Omega)$ with the weighted Lorentz space $L^{q,s}_0(\Omega)$ for any $0 < s \leq \infty$.

A proof of Theorem 1.9 in the case $p > 2 - 1/n$ and the case $\frac{3n-2}{2n-1} < p \leq 2 - 1/n$ was presented in [20,25], respectively. Now with the availability of the interior and boundary comparison estimates (Theorems 1.2 and 2.8), the proof also works in the present case $1 < p < 3/2$. For that, we shall not repeat the proof here. We mention that, following the approach of [20], a ‘good-\lambda’ type inequality involving $M_1(\mu)^{1/(p-1)}$ and $M(|\nabla u|^\kappa)^{1/\kappa}$, where $M$ is the Hardy–Littlewood maximal function, can also be obtained as in [20, Theorem 1.5]. However, we take this opportunity to point out that the statement of [20, Theorem 1.5] contains an accuracy. The correct statement of [20, Theorem 1.5] requires the additional condition that $\lambda \geq c \text{diam}(\Omega)^{n/\gamma_0}\|\mu(\Omega)/\text{diam}(\Omega)\|^{1/(p-1)}$, where $c = c(n, p, \Lambda, \varepsilon, [w]_{A_{\infty}}, \text{diam}(\Omega)/R_0) > 0$. A similar adjustment should also be made to the statement of [23, Theorem 1.4].

The main results of [20,23] are unaffected.

Finally, as an application of Theorem 1.6 we obtain a sharp existence result for a quasilinear Riccati type equation with measure data:

$$
\begin{cases}
-\text{div}(A(x, \nabla u)) = |\nabla u|^q + \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.14)

To this end, we shall need the notion of capacity associated to the Sobolev space $W^{1,s}(\mathbb{R}^n)$, $1 < s < +\infty$, defined for each compact set $K \subset \mathbb{R}^n$ by

$$\text{Cap}_{1,s}(K) = \inf \int_{\mathbb{R}^n} (|\nabla \varphi|^s + \varphi^s)dx : 0 \leq \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 1 \text{ on } K.$$

Theorem 1.10. Let $q > p - 1$. Assume that (1.9) holds and $\partial \Omega$ is of class $C^{1, \text{Dini}}$. Also, assume that $A(x, \xi) = a(x)|\xi|^{p-2}\xi$ for any $x$ near the boundary $\partial \Omega$. Then there exists a constant $c_0 > 0$ such that if the measure $\mu$ satisfies

$$|\mu|(K) \leq c_0 \text{Cap}_{1, \frac{q}{q-p+1}}(K) \tag{1.15}$$

for all compact sets $K \subset \Omega$, then there exists a renormalized solution $u$ to the Riccati type equation (1.14) such that

$$|\nabla u(x)| \lesssim \left[1^{2\text{diam}(\Omega)}(|\mu|)(x)\right]^{\frac{1}{p-1}} \quad \text{a.e. } x \in \Omega.$$
We remark that condition (1.15) is sharp at least for nonnegative measures with compact support in $\Omega$ (see [12,24]). It is worth mentioning that Theorem 1.10 completely solve a problem raised in [3, pages 13–14]. On the other hand, it is new mainly in the case $n \geq 2$ and the case $q$ has been treated in [11] and [23], respectively. For other earlier work on equation (1.14), we refer to the papers [12,17,22,24–26] and the references therein.

2. Comparison Estimates

In what follows, for $x \in \mathbb{R}^n$ we write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. With this, we will use the notation $\|f\|_{L^q_{x'} L^p_{x_n}(Q, \rho(x_0))}$ for $s_1, s_2 > 0$ and $\rho > 0$, to indicate the mixed norm

$$
\|f\|_{L^q_{x'} L^p_{x_n}(Q, \rho(x_0))} = \left( \int_{x_0 + (-\rho, \rho)^{n-1}} \left( \int_{x_0 + (-\rho, \rho)} |f(x', x_n)|^{s_1} \, dx_n \right)^{s_2} \right)^{\frac{q}{s_2}}.
$$

On the other hand, we write $\|f\|_{L^q_{x'} L^p_{x_n}(Q, \rho(x_0))}$ for the normalized mixed norm

$$
\|f\|_{L^q_{x'} L^p_{x_n}(Q, \rho(x_0))} = \left( \int_{x_0 + (-\rho, \rho)^{n-1}} \left( \int_{x_0 + (-\rho, \rho)} |f(x', x_n)|^{s_1} \, dx_n \right)^{s_2} \right)^{\frac{q}{s_2}}.
$$

We begin with the following lemma.

**Lemma 2.1.** Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a solution of (1.1). Then for any function $\phi \in C^\infty_0(\Omega)$, $\phi \geq 0$, and $\varepsilon > 0$, we have

$$
\int_{\Omega} \left| \nabla \left[ (1 + |u|)^{\frac{p-1}{p} - \varepsilon} \phi \right] \right|^p \, dx \lesssim \int_{\Omega} \phi^p d|\mu| + \int_{\Omega} (1 + |u|)^{(p-1)(1+\varepsilon)} |\nabla \phi|^p \, dx.
$$

**Proof.** Note that as $u \in W^{1,p}_{\text{loc}}(\Omega)$, the function $g = \text{sign}(u)[1 - (1 + |u|)^{-\varepsilon}]$, $\varepsilon > 0$, also belongs to $W^{1,p}_{\text{loc}}(\Omega)$ with weak gradient $\nabla g = \varepsilon \nabla u (1 + |u|)^{-1-\varepsilon}$. This can be seen by approximating $g$ by the sequence $u(\sigma + u^2)^{-1/2}[1 - (1 + (\sigma + u^2)^{1/2})^{-\varepsilon}]$, $\sigma > 0$, and then sending $\sigma \to 0^+$ via Lebesgue’s Dominated Convergence Theorem.

Let $\phi$ be a function in $C^\infty_0(\Omega)$ such that $\phi \geq 0$. Then using $\text{sign}(u)[1 - (1 + |u|)^{-\varepsilon}] \phi^p$, $\varepsilon > 0$, as a test function in (1.1) we have

$$
\varepsilon \int_{\Omega} \frac{|\nabla u|^p}{(1 + |u|)^{1+\varepsilon}} \phi^p \, dx \leq \int_{\Omega} \phi^p d|\mu| + \Lambda p \int_{\Omega} |\nabla u|^{p-1} \phi^{p-1} |\nabla \phi| \, dx
$$

$$
= \int_{\Omega} \phi^p d|\mu| + \Lambda p \int_{\Omega} |\nabla u|^{p-1} \phi^{p-1} (1 + |u|)^{(1-\varepsilon)\frac{p-1}{p}} (1 + |u|)^{(\varepsilon+1)\frac{p-1}{p}} |\nabla \phi| \, dx.
$$
Thus by Young’s inequality we obtain
\[
\int_{\Omega} \frac{|\nabla u|^p}{(1 + |u|)^{1+\varepsilon}} \phi^p \, dx \lesssim \int_{\Omega} \phi^p \, d|\mu| + \int_{\Omega} (1 + |u|)^{(p-1)(1+\varepsilon)} |\nabla \phi|^p \, dx,
\]
where the constant in \( \lesssim \) depends only on \( \Lambda, p, \) and \( \varepsilon. \) Now in view of the identity
\[
\nabla \left[ (1 + |u|)^{p-1-\varepsilon} \phi \right] = (p-1-\varepsilon) (1 + |u|)^{-\frac{1-\varepsilon}{p}} \nabla u \, \text{sign}(\phi) + (1 + |u|)^{p-1-\varepsilon} \nabla \phi,
\]
we obtain the lemma. \( \square \)

Lemma 2.2 yields the following reverse Hölder type inequality for the solution:

**Lemma 2.2.** Let \( u \in W^{1,p}_{\text{loc}}(\Omega) \) be a solution of (1.1). Then for any cube \( Q_R(x_0) \subset \Omega \) and any \( q > q_1 > 0 \) we have
\[
\left( \int_{Q_{\sigma}(x_0)} |u|^q \, dx \right)^{\frac{1}{q}} \lesssim \left( \frac{\| \mu \|_{Q_R(x_0)}}{R^{n-p}} \right)^{\frac{1}{p-1}} + \left( \int_{Q_R(x_0)} |u|^{q_1} \, dx \right)^{\frac{1}{q_1}},
\]
provided that \( q < \frac{n(p-1)}{n-p} \) and \( \sigma \in (0, 1). \) The constant in \( \lesssim \) depends on \( \Lambda, p, q, q_1, n, \) and \( \sigma. \)

**Proof.** By translation and scale invariance, we may assume that \( x_0 = (0, \ldots, 0), \)
\( Q_R(x_0) = (-1, 1)^n, \) and
\[
|\mu|((-1, 1)^n)^{\frac{1}{p-1}} + \| u \|_{L^{q_1}((-1, 1)^n)} \leq 1. \tag{2.1}
\]
Moreover, we need to show that, for any \( \sigma < 1, \)
\[
\| u \|_{L^q((-\sigma, \sigma)^n)} \leq C(\Lambda, p, q, q_1, n, \sigma);
\]
see [8, Remark 4.1] for details.

Let \( \varepsilon > 0 \) and let \( \phi \) be a function in \( C^\infty_0(Q_r(0)) \) such that \( 0 \leq \phi \leq 1, \phi \equiv 1 \)
on \( Q_s(0), 1 \geq r > s > 0, \) and \( |\nabla \phi| \leq C/(r-s). \) By Lemma 2.1 and Sobolev embedding theorem we have
\[
\| (1 + |u|)^{p-1-\varepsilon} \phi \|_{L^{p^*}(Q_1(0))} \lesssim \int_{Q_{1}(0)} \phi^p \, d|\mu| + \int_{Q_{1}(0)} (1 + |u|)^{(p-1)(1+\varepsilon)} |\nabla \phi|^p \, dx,
\]
where \( p^* = \frac{np}{n-p}. \) This gives, in view of (2.1),
\[
\left( \int_{Q_{1}(0)} (1 + |u|)^{(p-1-\varepsilon)p^*} \, dx \right)^{\frac{1}{p^*}} \lesssim 1 + \frac{1}{(r-s)^p} \int_{Q_{r}(0)} (1 + |u|)^{(p-1)(1+\varepsilon)} \, dx.
\]
Let
\[
q_\varepsilon := \frac{(p-1-\varepsilon)p^*}{p}, \quad \tilde{q}_\varepsilon := (p-1)(1+\varepsilon),
\]
and choose $\varepsilon$ sufficiently small so that $q_{\varepsilon} \in \left[ q, \frac{n(p-1)}{n-p} \right)$, and $\bar{q}_{\varepsilon} < q_{\varepsilon}$.

Then we can rewrite the above inequality as
\[
\int_{Q_{r}(0)} (1 + |u|)^{q_{\varepsilon}} \, dx \lesssim 1 + \frac{1}{(r-s)^{p^*}} \left( \int_{Q_{r}(0)} (1 + |u|)^{\bar{q}_{\varepsilon}} \, dx \right)^{p^*/p}.
\]

Also, we may assume that $q_1 < p - 1 < \bar{q}_{\varepsilon}$ and then by Hölder’s inequality and (2.1) we have
\[
\left( \int_{Q_{r}(0)} (1 + |u|)^{\bar{q}_{\varepsilon}} \, dx \right)^{\frac{1}{\bar{q}_{\varepsilon}}} \leq \left( \int_{Q_{r}(0)} (1 + |u|)^{q_{1}} \, dx \right)^{\frac{\theta}{q_{\varepsilon}}} \left( \int_{Q_{r}(0)} (1 + |u|)^{\bar{q}_{\varepsilon}} \, dx \right)^{1 - \frac{\theta}{q_{1}}},
\]

where $\theta = \frac{(\bar{q}_{\varepsilon} - q_{1})q_{\varepsilon}}{(q_{\varepsilon} - q_{1})q_{\varepsilon}}$.

We next further restrict $\varepsilon < \frac{q_1}{n}$ so that
\[
\frac{\theta \bar{q}_{\varepsilon}}{q_{\varepsilon}} p^* = \frac{\bar{q}_{\varepsilon} - q_{1}}{q_{\varepsilon} - q_{1}} n - p < 1.
\]

Then by Young’s inequality with exponents $\frac{pq_{\varepsilon}}{\bar{q}_{\varepsilon} p^*}$, and $\frac{pq_{\varepsilon}}{pq_{\varepsilon} - \theta \bar{q}_{\varepsilon} p^*}$, we get
\[
\int_{Q_{r}(0)} (1 + |u|)^{q_{\varepsilon}} \, dx \leq C + \frac{1}{2} \int_{Q_{r}(0)} (1 + |u|)^{q_{\varepsilon}} \, dx + \frac{C}{(r-s)^{\mathcal{H}_0}},
\]

where
\[
\mathcal{H}_0 = \frac{p^*pq_{\varepsilon}}{pq_{\varepsilon} - \theta \bar{q}_{\varepsilon} p^*} = \frac{np(q_{\varepsilon} - q_{1})}{n(q_{\varepsilon} - \bar{q}_{\varepsilon}) - p(q_{\varepsilon} - q_{1})} > 0.
\]

Now we may apply Lemma 6.1 of [10] to obtain
\[
\int_{Q_{r}(0)} (1 + |u|)^{q_{\varepsilon}} \, dx \leq C,
\]

which of course yields the desired result. \hfill $\Box$

We next give a mixed norm estimate for the gradient.

**Lemma 2.3.** Let $Q_{r}(x_0) \subset \Omega$ and $1 - \frac{n-1}{p} < \alpha < 1/p$, $\alpha > 0$, $n \geq 2$. Then for any $\sigma \in (0, 1)$ and $q_1 > 0$ we have
\[
\|\nabla u\|_{L_{L^r_{\sigma}}^{P_1}(r) \cap L_{\mathcal{S}_{\sigma}}^{P_2}(Q_{\sigma r}(x_0))} \lesssim \left( \frac{1}{\mu(Q_r(x_0))} \right)^{\frac{1}{p-1}} + \frac{1}{r} \left( \int_{Q_{r}(0)} |u|^{q_1} \, dx \right)^{\frac{1}{q_1}},
\]

where $\gamma \in (1, \min\{p, n/(n-p)\})$, and
\[
p_1 = \frac{p - \gamma}{1 - \alpha \gamma}, \quad p_2 = \frac{(n-1)(p-\gamma)}{n - 1 - (1-\alpha)\gamma}.
\]
Proof. By translating and scaling considerations, we may assume that \( x_0 = (0, \ldots, 0) \), \( Q_r(x_0) = Q_1(0) = (-1, 1)^n \), and (2.1) holds. Moreover, we need to show that for any \( \sigma < 1 \)

\[
\|\nabla u\|_{L_{x_1}^p L_{x_n}^1(Q_\sigma(0))} \lesssim 1.
\]

By Lemma 2.1, Lemma 2.2, and (2.1), we find

\[
\int_{Q_\sigma(0)} |\nabla u|^p \frac{dx}{(1 + |u|)^\gamma} \leq C|\mu|(Q_1(0)) + C \int_{Q_1(0)} (1 + |u|)^{(p-1)\gamma} dx \leq C,
\]

(2.2)

provided \( \gamma \in (1, \min\{p, n/(n - p)\}) \). Here \( \sigma \in (0, 1) \) and the constant \( C \) may depend on \( \sigma \).

Also, by Lemma 2.1 and Sobolev embedding theorem for Lebesgue spaces of mixed norm (see [1, 2]), for any \( \alpha \) satisfying \( 1 - \frac{n-1}{p} < \alpha < 1/p, \alpha > 0 \), we have

\[
\| (1 + |u|)^{p-\gamma} \phi \|_{L_{x_1}^{p(1-\alpha)p} L_{x_n}^{1-\alpha p}} \lesssim \int \phi \, d|\mu| + \int (1 + |u|)^{(p-1)\gamma} |\nabla \phi|^p dx,
\]

where \( \phi \in C_0^\infty(Q_1(0)) \), \( \phi \geq 0 \), and \( 1 < \gamma < \min\{p, n/(n - p)\} \). By Lemma 2.2 and (2.1), this implies in particular that

\[
\| 1 + |u| \|_{L_{x_1}^{p(1-\alpha)p} L_{x_n}^{1-\alpha p}(Q_\sigma(0))} \lesssim 1.
\]

(2.3)

On the other hand, by Hölder’s inequality, we have

\[
\|\nabla u\|_{L_{x_1}^p L_{x_n}^1(Q_\sigma(0))} \leq \left\| \frac{\nabla u}{(1 + |u|)^{\frac{p}{\gamma}}} \right\|_{L^p(Q_\sigma(0))} \| 1 + |u| \|_{L_{x_1}^{\frac{pq_2}{p+q_2}} L_{x_n}^{\frac{q_2}{p+q_2}}(Q_\sigma(0))}^{\frac{q_2}{q_1}},
\]

(2.4)

where

\[
\frac{1}{p_1} = \frac{1}{p} + \frac{1}{q_1}, \quad \frac{1}{p_2} = \frac{1}{p} + \frac{1}{q_2}, \quad p_1, p_2, q_1, q_2 > 0.
\]

We now choose

\[
q_1 = \frac{p(p - \gamma)}{\gamma(1 - \alpha p)}, \quad q_2 = \frac{p(n - 1)(p - \gamma)}{\gamma[n - 1 - (1 - \alpha) p]},
\]

which gives that

\[
p_1 = \frac{p - \gamma}{1 - \alpha \gamma}, \quad p_2 = \frac{(n - 1)(p - \gamma)}{n - 1 - (1 - \alpha) \gamma}.
\]

At this point, we combine (2.2), (2.3), and (2.4) to deduce that

\[
\|\nabla u\|_{L_{x_1}^p L_{x_n}^1(Q_\sigma(0))} \lesssim 1.
\]

This completes the proof of the lemma.

\(\square\)
Again by translating and scaling, we may assume that

\[
\text{Corollary 2.4. Let } Q_r(x_0) \subset \Omega \text{ and } 1 < p < n, n \geq 2. \text{ Then for any } \sigma \in (0, 1) \text{ and } q_1 > 0, \text{ we have }
\]

\[
\| \nabla u \|_{L_{q_1}^{r_0} L_{q_1}^{r_1}(Q_{\sigma r}(x_0))} \leq \left( \frac{|\mu|(Q_r(x_0))}{r^{n-1}} \right)^{\frac{1}{p-1}} + \frac{1}{r} \left( \int_{Q_r(x_0)} |u|^{q_1} \, dx \right)^{\frac{1}{q_1}},
\]

provided

\[
p - 1 < s_1 < p, \quad 0 < s_2 < \frac{s_1(n-1)(p-1)}{s_1(n-1) - p + 1}.
\]

We now obtain a preliminary version of the comparison estimate.

\[
\text{Lemma 2.5. Suppose that } n \geq 2 \text{ and } Q_r(x_0) \subset \Omega. \text{ Let } u \in W^{1,p}_{\text{loc}}(\Omega) \text{ be a solution of (1.1) and let } w \text{ be as (1.4). Then for } \kappa = (p-1)^2/2 \text{ and } 1 < p < 3/2 \text{ we have }
\]

\[
\left( \int_{Q_r(x_0)} |\nabla (u - w)|^k \right)^{\frac{1}{k}} + \frac{1}{r} \left( \int_{Q_r(x_0)} |u - w|^k \right)^{\frac{1}{k}} \leq \left( \frac{|\mu|(Q_r(x_0))}{r^{n-1}} \right)^{\frac{1}{p-1}} \| \nabla u \|_{L_x^r, L_{s_1}^{s_2}(Q_{\sigma r}(x_0))}^{2-p} \| u \|_{L_x^{s_1} L_{s_2}^{s_3}(Q_{\sigma r}(x_0))}^{2-p}.
\]

\[
\text{Proof.} \text{ Again by translating and scaling, we may assume that } x_0 = (0, \ldots, 0), \quad Q_r(x_0) = (-1, 1)^n, \text{ and }
\]

\[
A := \| \mu \|((-1, 1)^n)^{\frac{1}{2-p}} + \| \mu \|((-1, 1)^n) \| \nabla u \|_{L_x^r, L_{s_1}^{s_2}((-1, 1)^n)}^{2-p} \| u \|_{L_x^{s_1} L_{s_2}^{s_3}((-1, 1)^n)}^{2-p} \lesssim 1.
\]

Moreover, we just need to show that

\[
\| \nabla u - \nabla w \|_{L^r((-1, 1)^n)} + \| u - w \|_{L^r((-1, 1)^n)} \lesssim 1.
\]

For any given } k > 0 \text{ we now set }
\[
E_k = (-1, 1)^n \cap \{ k < |u - w| < 2k \}, \quad F_k = (-1, 1)^n \cap \{|u - w| > k \}.
\]

We first recall that for } 1 < p < 2,
\[
|\nabla u(x) - \nabla w(x)| \leq C g(x)^{1/p} + C g(x)^{1/2} |\nabla u(x)|^{\frac{2-p}{2}},
\]

where
\[
g(x) := \frac{|\nabla u(x) - \nabla w(x)|^2}{(|\nabla u(x)| + |\nabla w(x)|)^{2-p}}.
\]
Let $T_k, k > 0$, be the truncation operator defined in (1.11). Then it follows from (2.6) that
\[
|\partial_{x_n}(T_{2k} - T_k)(u - w)(x', x_n)| \lesssim \left( 1_{E_k} g(x', x_n)^{\frac{1}{p}} + 1_{E_k} g(x', x_n)^{\frac{1}{2}} |\nabla u(x', x_n)|^{\frac{2-\nu}{p}} \right),
\]
where $1_{E_k}$ is the characteristic function of the set $E_k$.

Thus, for any $0 < \gamma \leq 1$, by Hölder’s inequality, we find that
\[
\int_{(-1,1)^n} |(T_{2k} - T_k)(u - w)(x')|^\gamma \, dx' \\
\leq 2 \int_{(-1,1)^n-1} \left( \int_{-1}^1 |\partial_{x_n}(T_{2k} - T_k)(u - w)(x', x_n)| \, dx_n \right)^\gamma \, dx'
\lesssim \left( \int_{(-1,1)^n} 1_{E_k} g(x)^{\frac{1}{p}} \, dx \right)^\gamma
\lesssim \left( \int_{E_k} g(x) \, dx \right)^\gamma |E_k|^{\frac{\gamma(p-1)}{p}}
\lesssim \left( \int_{E_k} g(x) \, dx \right)^\gamma \left( \int_{(-1,1)^n-1} \left( \int_{-1}^1 |\nabla u(x', x_n)|^{2-\nu} \, dx_n \right)^\frac{\gamma}{2-\nu} \, dx' \right)^{\frac{2-\nu}{2}}.
\]

On the other hand, using $T_{2k}(u - w)$ as a test function for (1.1) we have
\[
\int_{E_k} g(x) \, dx \leq \int_{(-1,1)^n \cap |u-w| < 2k} g(x) \, dx \lesssim k |\mu|((-1,1)^n) \lesssim k A^{p-1}. \quad (2.7)
\]

Thus we get
\[
\int_{(-1,1)^n} |(T_{2k} - T_k)(u - w)(x')|^\gamma \, dx' \lesssim k^\gamma A^\frac{\gamma(p-1)}{p} |E_k|^{\frac{\gamma(p-1)}{p}}
\lesssim k^\gamma |\mu|((-1,1)^n)^\gamma \left( \int_{(-1,1)^n-1} \left( \int_{-1}^1 |\nabla u(x', x_n)|^{2-\nu} \, dx_n \right)^\frac{\gamma}{2-\nu} \, dx' \right)^{\frac{2-\nu}{2}}.
\]

We now set
\[
\gamma = p - 1.
\]

Observe that $E_k \subset F_k$ and on $F_{2k}$ we have $|(T_{2k} - T_k)(u - w)| = k$. Then in view of (2.5), this gives
\[
k^\gamma |F_{2k}| \lesssim k^\gamma A^\frac{\gamma(p-1)}{p} |F_k|^{\frac{\gamma(p-1)}{p}} + k^{-\gamma} A^{\frac{p-1}{2}}.
\]

As $|F_{2k}| \lesssim |F_k|$ and $|F_{2k}| \lesssim 2^n$, we can write for any $\nu > 0$,
\[
k^{-\gamma} |F_{2k}|^{1+\nu} \lesssim k^{-\gamma} A^{-\frac{\gamma(p-1)}{p}} |F_k|^{\frac{\gamma(p-1)}{p} + \nu} + A^{\frac{p-1}{2}}.
\]
Then taking the supremum over $k > 0$ we obtain

$$\|w - u\|_{\frac{\gamma}{p}, \frac{\gamma}{p-1}, \infty} \lesssim \|w - u\|_{\frac{\gamma}{p}, \frac{\gamma}{p-1}, \infty} A^{\frac{\gamma(p-1)}{p}} + A^{\frac{p-1}{2}}.$$

We now choose $\nu \geq 0$ such that

$$\frac{\gamma}{2(1+\nu)} = \frac{\frac{\gamma}{p} - \gamma}{\frac{\gamma(p-1)}{p} + \nu} \iff \nu = \frac{2 - p - \gamma(p-1)}{2(p-1)} = \frac{2 - p - (p-1)^2}{2(p-1)} > 0.$$

Then using Young’s inequality, we obtain

$$\|w - u\|_{\frac{\gamma}{p}, \frac{\gamma}{p-1}, \infty} \lesssim A \lesssim 1, \quad \gamma = p - 1.$$

Note that by (2.7) and Chebyshev’s inequality,

$$\{g > \lambda\} = \{g > \lambda\} \cap \{|u - w| \geq k\} + \{g > \lambda\} \cap \{|u - w| < k\}$$

$$\lesssim k - \frac{\gamma(p-1)}{p} \|u - w\|_{\frac{\gamma(p-1)}{p}, \frac{\gamma(p-1)}{p-1}, \infty} + \frac{1}{\lambda} \int_{\{u - w < k\}} g dx$$

$$\lesssim k - \frac{\gamma(p-1)}{p} + \frac{k}{\lambda} |\mu|((-1, 1)^n).$$

Thus choosing $k = \lambda^{1-\frac{\gamma(p-1)}{p}} |\mu|((-1, 1)^n) - 1 + \frac{\gamma(p-1)}{p}$, one has

$$\{g > \lambda\} \lesssim \lambda^{1-\frac{\gamma(p-1)}{p}} |\mu|((-1, 1)^n)^{\frac{\gamma(p-1)}{p}} \lesssim \lambda^{-(p-1)^2} |\mu|((-1, 1)^n)^{\frac{(p-1)^2}{p}}.$$ 

It follows that, for any $\beta < \frac{(p-1)^2}{2p}$,

$$\|g\|_{p} \lesssim |\mu|((-1, 1)^n),$$

which, in view of (2.6) and (2.5), yields

$$\|\nabla (u - w)\|_{L^{p}} \lesssim \|g\|_{p}^{\frac{1}{p}} + \|g\|_{p}^{\frac{1}{p}} \|\nabla u\|_{L^{\frac{2p}{p}}} \lesssim A^{\frac{p-1}{p}} + |\mu|((-1, 1)^n)^{\frac{1}{p}} \|\nabla u\|_{L^{\frac{2p}{p}}}.$$

Next, we choose $\beta = \frac{(p-1)^2}{2p}$, and employ Holder’s inequality to deduce

$$\|\nabla u\|_{L^{\frac{(p-1)^2}{2}}} \lesssim \|\nabla u\|_{L^{\frac{(p-1)^2}{2}}} \lesssim \|\nabla u\|_{L^{\frac{(p-1)(2-p)}{2}}} L^{\frac{2p}{(2-p)}}(Q_1(0))$$

Here we used the fact that $\frac{(p-1)^2}{2} < \frac{(p-1)(2-p)}{3-p}$ provided $1 < p < 3/2$.

These inequalities now give

$$\|\nabla (u - w)\|_{L^{\frac{(p-1)^2}{2}}} \lesssim A^{\frac{p-1}{p}} + |\mu|((-1, 1)^n)^{\frac{1}{p}} \|\nabla u\|_{L^{\frac{(p-1)(2-p)}{2}}} L^{\frac{2p}{(2-p)}}(Q_1(0)) \lesssim 1,$$

which completes the proof of the lemma. □
Using Lemma 2.5, we next prove a reverse Hölder type inequality for the gradient.

**Lemma 2.6.** Suppose that $u$ is a $W^{1,p}_{\text{loc}}(\Omega)$ solution of (1.1). Let $Q_r(x_0) \subset \Omega$, $n \geq 2$, and $\kappa = (p-1)^2/2$, where $1 < p < 3/2$. Let

$$\theta \in \left(0, \frac{2\kappa (p-1)}{(2-p)(p-\kappa)}\right)$$

and define $s_1, s_2$ by the equations

$$\frac{1}{2-p} = \frac{\theta}{\kappa} + \frac{1-\theta}{s_1}, \quad \frac{3-p}{(p-1)(2-p)} = \frac{\theta}{\kappa} + \frac{1-\theta}{s_2}. \quad (2.8)$$

Then,

$$2-p < s_1 < p, \quad s_1 > s_2 > \frac{(p-1)(2-p)}{3-p}, \quad (2.9)$$

and moreover,

$$\|\nabla u\|_{L^{s_2}_{x'}L^{s_1}_{x}(Q_{\sigma r}(x_0))} \lesssim \left(\frac{|\mu|(Q_r(x_0))}{p^{n-1}}\right)^{\frac{1}{p-1}} + \left(\int_{Q_r(x_0)} |\nabla u|^\kappa \, dx\right)^{\frac{1}{\kappa}}, \quad (2.11)$$

provided $\sigma \in (0, 1)$.

**Remark 2.7.** By Hölder’s inequality, (2.11) also holds with

$$s_1 = s_2 = \frac{(p-1)(2-p)}{3-p} > \kappa,$$

and then a covering/iteration argument as in [10, Remark 6.12] implies that we can replace $\kappa$ with any $\varepsilon > 0$ in (2.11).

**Proof.** The proof of (2.9) and (2.10) is obvious. In order to show (2.11), we first show that

$$\|\nabla u\|_{L^{s_2}_{x'}L^{s_1}_{x}(Q_{\sigma r/2}(x_0))} \lesssim \left(\frac{|\mu|(Q_r(x_0))}{p^{n-1}}\right)^{\frac{1}{p-1}} + \|\nabla u\|_{L^{(p-1)(2-p)}_{x'}L^{2-p}_{x}(Q_r(x_0))} \quad (2.12)$$

for any $Q_r(x_0) \Subset \Omega$. Moreover, by scaling, to prove (2.12) we may assume that $Q_r(x_0) = Q_1(0) = (-1, 1)^n$,

$$|\mu|((-1, 1)^n)^{\frac{1}{p-1}} + \|\nabla u\|_{L^{(p-1)(2-p)}_{x'}L^{2-p}_{x}((-1, 1)^n)} \lesssim 1, \quad (2.13)$$

and prove that

$$\|\nabla u\|_{L^{s_2}_{x'}L^{s_1}_{x}((-1/2, 1/2)^n)} \lesssim 1. \quad (2.14)$$
Let $w$ be as in (1.4) with $Q_r(x_0) = (-1, 1)^n$. Then, by Lemma 2.5 and (2.13),

$$\left( \int_{Q_1(0)} |u - w|^k \right)^{\frac{1}{k}} \lesssim 1, \quad \kappa = \frac{(p - 1)^2}{2}.$$ 

Thus for $\lambda = \int_{Q_7/8(0)} w \, dx$, by Poincaré inequality we have

$$\left( \int_{Q_7/8(0)} |u - \lambda|^k \right)^{\frac{1}{k}} \lesssim \left( \int_{Q_7/8(0)} |w - \lambda|^k \right)^{\frac{1}{k}} + 1 \lesssim \int_{Q_7/8(0)} |\nabla w| \, dx + 1.$$

Then by the reverse Hölder property of $\nabla w$ and Lemma 2.5 we can now deduce

$$\left( \int_{Q_7/8(0)} |u - \lambda|^k \right)^{\frac{1}{k}} \lesssim \left( \int_{Q_1(0)} |\nabla w|^k \, dx \right)^{\frac{1}{k}} + 1 \lesssim \left( \int_{Q_1(0)} |\nabla u|^k \, dx \right)^{\frac{1}{k}} + 1 \lesssim \|\nabla u\|_{L_{x}^{\frac{n+1}{2-n}} L_{\mathbb{R}^n}^{2-p}(Q_1(0))} + 1 \lesssim 1.$$

On the other hand, it follows from Corollary 2.4 that

$$\|\nabla u\|_{L_{x}^{\frac{n+1}{2-n}} L_{\mathbb{R}^n}^{2-p}(Q_1/2(0))} \lesssim 1 + \left( \int_{Q_7/8(0)} |u - \lambda|^k \, dx \right)^{\frac{1}{k}}$$

for any $\lambda \in \mathbb{R}$. Thus we obtain (2.14) as desired.

Next, for any cube $Q_r(x_0) \subset \Omega$ we consider the cubes $Q_t(x_0) \subset Q_s(x_0)$ where $0 < t < s < r$. Recall that $Q_t(x_0) = \{x_0^' + (-t, t)^{n-1}\} \times \{x_0 + (-t, t)\}$. We can cover the interval $I_t(x_0) = x_0 + (-t, t)$ by a sequence of intervals $I_i = y_{in} + (-s - t)/2, (s - t)/2, y_{in} \in I_t(x_0)$, in such a way that any point $y \in \mathbb{R}$ belongs to almost $3$ intervals of the collection $\{I_i\} = \{y_{in} + (-s - t), (s - t)\}$. Note that we have $1 \leq i \leq M$, where $M \lesssim t/(s - t)$.

Likewise, we can cover the $(n - 1)$-dimensional cube $J_t(x_0^') = x_0^' + (-t, t)^{n-1}$ by a sequence of $(n - 1)$-dimensional cubes $J_j = y_j^' + (-s - t)/2, (s - t)/2, y_j^' \in J_t(x_0^')$, in such a way that any point $y \in \mathbb{R}^{n-1}$ belongs to almost $N(n)$ cubes of the collection $\{J_j\} = \{y_j^' + (-s - t), (s - t)\}$. Also, $1 \leq j \leq M'$, where $M' \lesssim t/(s - t)^{n-1}$.

Note that we have $2J_j \times 2I_i \subset Q_s(x_0)$ for any $i, j$.

Then applying (2.12), we get

$$\|\nabla u\|_{L_{x}^{\frac{n+1}{2-n}} L_{\mathbb{R}^n}^{2-p}(J_j \times I_i)} \lesssim \left( \frac{|\mu(Q_r(x_0))|}{(s - t)^{n-1}} \right)^{\frac{1}{p-1}} + \|\nabla u\|_{L_{x}^{\frac{n+1}{2-n}} L_{\mathbb{R}^n}^{2-p}(2J_j \times 2I_i)} \lesssim 1.$$ (2.15)

We now observe that, as $s_2 < s_1$,

$$\|\nabla u\|_{L_{x}^{\frac{n+1}{2-n}} L_{\mathbb{R}^n}^{2-p}(Q_r(x_0))} \lesssim \sum_{i, j} \int_{I_i} \left( \int_{J_j} |\nabla u|^{s_i} \, dx_n \right)^{\frac{s_2}{s_i}} \, dx' = \sum_{i, j} \|\nabla u\|_{L_{x}^{s_2} L_{\mathbb{R}^n}^{s}(J_j \times I_i)}.$$
Thus in view of (2.15) we find

$$
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))} \lesssim \left( \frac{r}{s - t} \right)^{n} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} \left( \frac{|\mu|(Q_{r}(x_0))}{(s-t)^{n-1}} \right)^{\frac{s_2}{p-1}} + r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} (s - t)^{-s_2 \left[ \frac{1}{2-p} + \frac{(n-1)(3-p)}{2-p(p-p)} \right]}
$$

$$
\sum_{i,j} \| \nabla u \|_{L^2_{s_2} L^1_{s_1}(2J_i \times 2J_j)}^{(p-1)(2-p)} L^{-p}_{s_2}(2J_i \times 2J_j).
$$

Since $s_2 > (p-1)(2-p)/(3-p)$ and $(p-1)/(3-p) < 1$, we then have

$$
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))} \lesssim \left( \frac{r}{s - t} \right)^{n} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} \left( \frac{|\mu|(Q_{r}(x_0))}{(s-t)^{n-1}} \right)^{\frac{s_2}{p-1}} + \left( \frac{r}{s - t} \right)^{2s_2} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} (s - t)^{-s_2 \left[ \frac{1}{2-p} + \frac{(n-1)(3-p)}{2-p(p-p)} \right]}
$$

$$
\times \left( \sum_{j} \int_{2J_i} \left( \sum_{i} \int_{2J_i} |\nabla u|^{2-p} \, dx_n \right)^{\frac{p-1}{2-p}} \, dx' \right)^{\frac{2s_2}{2-p}}.
$$

This gives

$$
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))} \lesssim \left( \frac{r}{s - t} \right)^{n} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} \left( \frac{|\mu|(Q_{r}(x_0))}{(s-t)^{n-1}} \right)^{\frac{s_2}{p-1}} + \left( \frac{r}{s - t} \right)^{2s_2} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} (s - t)^{-s_2 \left[ \frac{1}{2-p} + \frac{(n-1)(3-p)}{2-p(p-p)} \right]}
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))},
$$

and thus, by Hölder’s inequality,

$$
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))} \lesssim \left( \frac{r}{s - t} \right)^{n} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} \left( \frac{|\mu|(Q_{r}(x_0))}{(s-t)^{n-1}} \right)^{\frac{s_2}{p-1}} + \left( \frac{r}{s - t} \right)^{2s_2} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} (s - t)^{-s_2 \left[ \frac{1}{2-p} + \frac{(n-1)(3-p)}{2-p(p-p)} \right]}
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))} \| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))}.
$$

Then by Young’s inequality, we get

$$
\| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))} \leq C \left( \frac{r}{s - t} \right)^{n} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} \left( \frac{|\mu|(Q_{r}(x_0))}{(s-t)^{n-1}} \right)^{\frac{s_2}{p-1}} + \frac{1}{2} \| \nabla u \|_{L^2_{s_2} L^1_{s_1}(Q_{r}(x_0))}.
$$

$$
+ C \left( \frac{r}{s - t} \right)^{2s_2} r^{s_2 \left[ \frac{1}{s_1} + \frac{n-1}{s_2} \right]} (s - t)^{-s_2 \left[ \frac{1}{2-p} + \frac{(n-1)(3-p)}{2-p(p-p)} \right]}
\| \nabla u \|_{L^2_{s_2}(Q_{r}(x_0))}.
$$
This enables us to apply [10, Lemma 6.1] and use (2.8) to obtain
\[ \|\nabla u\|_{L^r_{\Sigma_1} L^q_{\Omega_1}(Q_{\sigma r}(x_0))}^{s_2} \lesssim r^{s_2\left[\frac{1}{r} + \frac{n-1}{2}\right]} \left( \frac{|\mu|(Q_{r}(x_0))}{r^{n-1}} \right)^{\frac{s_2}{p-\kappa}} + r^{s_2\left[\frac{1}{r} + \frac{n-1}{2}\right]} \frac{1}{r} \left( \frac{|\mu|(Q_{r}(x_0))}{r^{n-1}} \right)^{\frac{s_2}{p-\kappa}} \|\nabla u\|_{L^s_{\Sigma_1} L^t_{\Omega_1}(Q_{\sigma r}(x_0))}^{s_2}. \]

This completes the proof of the lemma. \(\square\)

We mention again that Theorem 1.2 follows from Lemmas 2.5, 2.6, and Corollary 2.4. We now describe the boundary version of Theorem 1.2. Let \( u \) be a \( W^{1,p}_0(\Omega) \) solution of (1.1), and let \( x_0 \in \partial \Omega, r < \text{diam}(\Omega)/10 \). We then extend both \( u \) and \( \mu \) by zero outside \( \Omega \) and consider the unique solution \( w \in W^{1,p}_0(Q_r(x_0) \cap \Omega) + u \) to
\[
\begin{align*}
&- \text{div} (A(x, \nabla w)) = 0 \quad \text{in} \quad Q_r(x_0) \cap \Omega, \\
&w = u \quad \text{on} \quad \partial (Q_r(x_0) \cap \Omega).
\end{align*}
\]

Then we have the following boundary counterpart of Theorem 1.2:

**Theorem 2.8.** Let \( x_0 \in \partial \Omega, u \in W^{1,p}_0(\Omega) \) and \( w \) be as in (2.16). Then with \( \kappa = (p - 1)/2 \) and \( 1 < p < 3/2 \), for any \( \Sigma \in (1, 2] \) we have
\[
\left( \int_{Q_r(x_0)} |\nabla (u - w)|^\kappa \right)^{\frac{1}{\kappa}} + \frac{1}{r} \left( \int_{Q_r(x_0)} |u - w|^\kappa \right)^{\frac{1}{\kappa}} \lesssim \left( \frac{|\mu|(Q_{\Sigma r}(x_0))}{r^{n-1}} \right)^{\frac{1}{p-\kappa}} + \frac{|\mu|(Q_{\Sigma r}(x_0))}{r^{n-1}} \left( \int_{Q_{\Sigma r}(x_0)} |\nabla u|^\kappa \right)^{\frac{2-p}{\kappa}},
\]
and
\[
\left( \int_{Q_r(x_0)} |u - w|^\kappa \right)^{\frac{1}{\kappa}} \lesssim \left( \frac{|\mu|(Q_{\Sigma r}(x_0))}{r^{n-p}} \right)^{\frac{1}{p-\kappa}} + \frac{|\mu|(Q_{\Sigma r}(x_0))}{r^{n-p}} \left( \int_{Q_{\Sigma r}(x_0)} |u|^\kappa \right)^{\frac{2-p}{\kappa}}.
\]

Note that if \( u \in W^{1,p}_0(\Omega) \) then in Lemma 2.1 we may take \( \phi \in C_0^\infty(\mathbb{R}^n) \) (i.e., \( \phi \) does not need to have compact support in \( \Omega \)). With this observation, we see that Theorem 2.8 can be proved in the same manner as Theorem 1.2.

We now devote the rest of this section to the proof of Corollary 1.3.

**Proof of Corollary 1.3.** By Remark 2.7, it is enough to consider the case \( \varepsilon = \kappa \).

Let \( w \) be as in (1.4) with \( Q_{\Sigma r}(x_0) \) in place of \( Q_r(x_0) \), where we choose \( \Sigma_0 \) so that \( 1 < \Sigma_0 < \Sigma \leq 2 \). By \( L^1 \) Poincaré inequality, Lemma 2.5, and Young’s inequality we find
\[
\left( \inf_{q \in \mathbb{R}} \int_{Q_r(x_0)} |u - q|^\kappa \right)^{\frac{1}{\kappa}} \lesssim \left( \inf_{q \in \mathbb{R}} \int_{Q_r(x_0)} |w - q|^\kappa \right)^{\frac{1}{\kappa}} + \left( \int_{Q_r(x_0)} |u - w|^\kappa \right)^{\frac{1}{\kappa}}.
\]
\[
\lesssim r \int_{Q_r(x_0)} |\nabla w| dx + \left( \frac{|\mu|(B_{\Sigma_0}(x_0))}{r^{n-p}} \right)^{\frac{1}{p-1}} + r \|\nabla u\|_{L^\infty} \frac{(p-1)(2-p)}{3-p} \|Q_{\Sigma_0}(x_0)\|^{\frac{n-1}{n}}.
\]

We now use the reverse Hölder property of $\nabla w$ to obtain from the above inequality that
\[
\left( \inf_{q \in \mathbb{R}} \int_{Q_r(x_0)} |u - q|^\kappa \right)^{\frac{1}{\kappa}} \lesssim r \left( \int_{Q_{\Sigma_0}(x_0)} |\nabla u - \nabla w|^\kappa dx \right)^{\frac{1}{\kappa}} + r \left( \int_{Q_{\Sigma_0}(x_0)} |\nabla u|^\kappa dx \right)^{\frac{1}{\kappa}} + \frac{(|\mu|(B_{\Sigma_0}(x_0)))^{\frac{1}{p-1}}}{r^{n-p}} + \frac{\|\nabla u\|_{L^\infty} (p-1)(2-p)}{3-p} \|Q_{\Sigma_0}(x_0)\|^{\frac{n-1}{n}}.
\]

At this point, we use Lemmas 2.5 and 2.6 to conclude the proof.

\[\square\]

3. Proof of Theorem 1.4

To prove Theorem 1.4, we shall need the following sharp quantitative regularity/decay estimates for homogeneous equations.

**Lemma 3.1.** Under (1.2)–(1.3), let $w \in W^{1,p}(\Omega_1)$ be a solution of $\text{div} (A(x, \nabla w)) = 0$ in $\Omega$. Then there exists $\alpha_0 \in (0, 1]$ such that for any $Q_\rho(x_0) \subset Q_R(x_0) \subset \Omega$, and $\epsilon \in (0, 1)$, we have
\[
\int_{Q_\rho(x_0)} |w - (w)|^{\rho} dx \lesssim \left( \frac{\rho}{R} \right)^{\alpha_0\rho} \int_{Q_R(x_0)} |w - (w)|^{\rho} dx, \tag{3.1}
\]
and
\[
\inf_{q \in \mathbb{R}} \int_{Q_\rho(x_0)} |w - q|^{\epsilon} dx \lesssim \left( \frac{\rho}{R} \right)^{\alpha_0\epsilon} \inf_{q \in \mathbb{R}} \int_{Q_R(x_0)} |w - q|^{\epsilon} dx. \tag{3.2}
\]

**Proof.** The proof of (3.1) follows from [10, Chapter 7], whereas the proof of (3.2) follows from (3.1) and the reverse Hölder property of $w$. \[\square\]

We are now ready for the proof of Theorem 1.4.

**Proof of (1.4).** We first observe that for each cube $Q_\rho(x_0) \subset \Omega$ and $f \in L^\kappa_{\text{loc}}(\Omega)$, there exists $q_{\rho, x_0} = q_{\rho, x_0}(f) \in \mathbb{R}$ such that
\[
\inf_{q \in \mathbb{R}} \left( \int_{Q_\rho(x_0)} |f - q|^{\kappa} dx \right)^{\frac{1}{\kappa}} = \left( \int_{Q_\rho(x_0)} |f - q_{\rho, x_0}|^{\kappa} dx \right)^{\frac{1}{\kappa}}.
\]
Then it is known that (see, e.g., [6, Lemma 4.1])
\[
\lim_{\rho \to 0} q_{\rho, x_0}(f) = f(x_0) \quad \text{a.e. with respect to } x_0 \in \mathbb{R}^n. \tag{3.3}
\]
For a cube $Q_\rho = Q_\rho(x_0) \subset \Omega$, we now define

$$J(\rho, x_0) = J(\rho, x_0, u) := \inf_{q \in \mathbb{R}} \left( \int_{Q_\rho(x_0)} |u - q|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}}.$$

This kind of quantity is a good substitution for the mean oscillation for functions that may not belong to $L^1_{\text{loc}}$ (see, e.g., [6, Section 4]).

Next, for any $\varepsilon \in (0, 1)$ and $Q_{\varepsilon r}(x_0) \subset \Omega$ by Lemma 3.1 and quasi-triangle inequality, we can find $\alpha_0 \in (0, 1]$ such that

$$J(\varepsilon r/2, x_0) \leq C \left( \int_{Q_{\varepsilon r/2}(x_0)} |u - q_{\varepsilon r/2, x_0}(w)|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}}$$

$$+ C \left( \int_{Q_{\varepsilon r/2}(x_0)} |u - w|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}} \leq C \varepsilon^{\alpha_0} \left( \int_{Q_{\varepsilon r/2}(x_0)} |w - q_{r, x_0}(w)|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}}$$

$$+ C \varepsilon^{-n/\kappa} \left( \int_{Q_{\varepsilon r/2}(x_0)} |u - w|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}}.$$

Here we choose $w$ as in (1.4) with $Q_{\varepsilon r/2}(x_0)$ in place of $Q_{\varepsilon r}(x_0)$. Thus after similar manipulations, we get that

$$J(\varepsilon r/2, x_0) \leq C \varepsilon^{\alpha_0} J(r/2, x_0) + C \varepsilon \left( \int_{Q_{r/2}(x_0)} |u - w|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}}.$$

We now apply Theorem 1.2 to bound the second term on the right-hand side of the above inequality. This yields that

$$J(\varepsilon r/2, x_0) \leq C \varepsilon^{\alpha_0} J(r/2, x_0) + C_\varepsilon \left( \int_{Q_{r/2}(x_0)} |u - w|^\kappa \mathrm{d}x \right)^{\frac{1}{\kappa}}.$$

Then by Young’s inequality we find

$$J(\varepsilon r/2, x_0) \leq C \varepsilon^{\alpha_0} J(r, x_0) + C_\varepsilon \left( \frac{\mu(Q_r(x_0))}{r^{n-p}} \right)^{\frac{1}{p+1}}.$$

We now choose $\varepsilon < \frac{1}{4\sqrt{n}}$ small enough so that $C(\varepsilon)^{\alpha_0} \leq \frac{1}{4}$, where $C$ is the constant in (3.4). Set $R_j = (\varepsilon/2)^j R$, $Q_j := Q_{R_j}(x_0)$. Applying (3.4) yields

$$J(R_{j+1}, x_0) \leq \frac{1}{4} J(R_j, x_0) + C \left( \frac{\mu(Q_j)}{R_j^{n-p}} \right)^{\frac{1}{p-1}}.$$
Summing this up over \( j \in \{2, 3, \ldots, m - 1\} \), we obtain

\[
\sum_{j=2}^{m} J(R_j, x_0) \leq C J(R_2, x_0) + \sum_{j=2}^{m-1} \left( \frac{\vert \mu \vert(Q_j)}{R_{j}^{n-p}} \right)^{\frac{1}{p-1}}. \tag{3.5}
\]

It is not hard to see that

\[
|q_{R_{j+1}, x_0}(u) - q_{R_j, x_0}(u)| \leq C \left( J(R_{j+1}, x_0) + J(R_j, x_0) \right)
\]

for all \( j \geq 1 \), and then, by iterating, we find that

\[
|q_{R_m, x_0}(u) - q_{R_2, x_0}(u)| \leq C \sum_{j=2}^{m} J(R_j, x_0).
\]

Note also that

\[
J(R_2, x_0) + |q_{R_2, x_0}(u)| \leq C J(R_2, x_0) + C \left( \int_{Q_2} \vert u \vert^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{B_{R}(x_0)} \vert u \vert^p \, dx \right)^{\frac{1}{p}}.
\]

Thus using these in (3.5) we get

\[
|q_{R_m, x_0}(u)| \leq C J(R_2, x_0) + C |q_{R_2, x_0}(u)| + C \sum_{j=2}^{m-1} \left( \frac{\vert \mu \vert(Q_j)}{R_{j}^{n-p}} \right)^{\frac{1}{p-1}}
\]

\[
\leq C \left( \int_{B_{R}(x_0)} \vert u \vert^p \, dx \right)^{\frac{1}{p}} + C \sum_{j=2}^{m-1} \left( \frac{\vert \mu \vert(Q_j)}{R_{j}^{n-p}} \right)^{\frac{1}{p-1}}.
\]

Note that

\[
\sum_{j=2}^{m-1} \left( \frac{\vert \mu \vert(Q_j)}{R_{j}^{n-p}} \right)^{\frac{1}{p-1}} \leq \sum_{j=2}^{m-1} \left( \frac{\vert \mu \vert(B_{\sqrt{n}R_j}(x_0))}{R_{j}^{n-p}} \right)^{\frac{1}{p-1}} \leq C W_{1,p}^R(\vert \mu \vert)(x_0).
\]

Thus in view of (3.3) we obtain (1.8) as desired. \( \square \)

4. Proof of Theorem 1.10

In this section, we provide a proof of Theorem 1.10.

Step 1: In this step we assume that \( \mu \in L^\infty(\Omega) \). Set

\[
E(\mu) := \left\{ u \in W_0^{1,p}(\Omega) : \vert \nabla u(x) \vert \leq 2C_0 \mathbf{P}[\mu](x) \text{ a.e.} \right\},
\]

where \( C_0 \) is the constant in (1.12), and

\[
\mathbf{P}[\mu](x) := \left[ \mathbf{I}_1^{2 \text{diam}(\Omega)}(\vert \mu \vert)(x) \right]^{\frac{1}{p-1}}, \quad x \in \mathbb{R}^n.
\]
Here we extend $u$ by zero outside $\Omega$. Note that by inequality (2.10) of [24] and (1.15), one has that

$$P[P[\mu]^q](x) \leq C_1 c_0^{\frac{q-p+1}{p-1}} P[\mu](x) \quad \forall x \in \mathbb{R}^n. \quad (4.1)$$

Now let $S : E(\mu) \rightarrow W_0^{1,p}(\Omega)$ be defined by $S(\nu) = u$ where $u \in W_0^{1,p}(\Omega)$ is the unique renormalized solution of

$$-\text{div}(A(x, \nabla u)) = |\nabla \nu|^q + \mu \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$

By (1.12) and (4.1), one obtains

$$|\nabla u(x)| \leq C_0 P[|\nabla u|^q + |\mu|](x) \leq C_0 \left( (2C_0)^q P[P[\mu]^q](x) + P[|\mu|](x) \right)$$

$$\leq C_0 \left( (2C_0)^q C_1 c_0^{\frac{q-p+1}{p-1}} + 1 \right) P[|\mu|](x) \leq 2C_0 P[|\mu|](x),$$

provided that $c_0 > 0$ is small enough. This means that $S(E(\mu)) \subset E(\mu)$. Then, similar to [20, Proof of Theorem 1.9], we obtain that $S$ has a fixed point in $E(\mu)$. Thus, there exists a solution $u \in E(\mu)$ to

$$-\text{div}(A(x, \nabla u)) = |\nabla u|^q + \mu \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega. \quad (4.2)$$

Step 2: We now extend $\mu$ be zero out side $\Omega$ and let $\mu_k = \rho_k * \mu$, where $\{\rho_k\}_{k>0}$ is a standard sequence of mollifiers: $\rho_k(\cdot) := k^n \rho(k \cdot)$ for a nonnegative and radial function $\rho \in C_0^\infty(B_1(0))$ such that $\int_{\mathbb{R}^n} \rho dx = 1$. It is not hard to see that there exists a constant $B > 0$ such that

$$|\mu_k|(K) \leq B c_0 \text{Cap}_1, \frac{q}{q-p+1}(K)$$

for any compact set $K \subset \Omega$; see [25, Lemma 5.7]. Thus, by Step 1, for each $k > 0$ there exists a renormalized solution $u_k \in E(\mu_k)$ to (4.2) with datum $\mu = \mu_k$. Also, by (5.1) in [20] we have

$$\int_1^{2\text{diam}(\Omega)}(|\mu_k|) \leq \rho_k * \int_1^{2\text{diam}(\Omega)}(|\mu|) \leq M \int_1^{2\text{diam}(\Omega)}(|\mu|).$$

Thus by the stability result for renormalized solutions of [5], we can find a subsequence of $\{u_k\}_k$ that converges to a renormalized solution $u$ to the equation (4.2) such that

$$|\nabla u(x)| \leq 2C_0 M^{\frac{1}{p-1}} P[\mu](x) \text{ a.e.}.$$  

This completes the proof of the theorem.

**Remark 4.1.** Finally, we remark that due to a comparison estimate obtained earlier in [20] for the case $\frac{2n-2}{2n-1} < p \leq 2 - \frac{1}{n}$, all results in this paper also hold in this case. Moreover, local and global estimates ‘below the duality exponent’ in the spirit of [19, 26, 27] for $1 < p \leq 2 - \frac{1}{n}$ can also be deduced from the comparison estimates of Theorems 1.2 and 2.8 and [20].
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Data availability Data will be made available on reasonable request.

Declarations

Conflict of interest We confirm that we do not have any conflict

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