Derivation of the effective action of a dilute Fermi gas in the unitary limit of the BCS-BEC crossover

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The effective action describing the gapless Nambu-Goldstone, or Anderson-Bogoliubov, mode of a zero-temperature dilute Fermi gas at unitarity is derived up to next-to-leading order in derivatives from the microscopic theory. Apart from a next-to-leading order term that is suppressed in the BCS limit, the effective action obtained in the strong-coupling unitary limit is proportional to that obtained in the weak-coupling BCS limit.

Keywords: Unitary Fermi gas; BCS-BEC crossover; Effective action; Anderson-Bogoliubov mode

I. INTRODUCTION

The ability to control the interaction between atoms is unique to experiments on optically trapped ultracold atomic gases, unmatched by other condensed matter experiments and unthought of even a few years ago. After successfully creating a degenerate Fermi sea in a trapped Fermi gas [1], Jin and collaborators were able to tune a dilute ultracold Fermi gas to be near a magnetic Feshbach resonance where small changes in magnetic field strength have pronounced effects on the two-particle scattering length characterizing the interatomic pair potential in vacuum [2]. Using this handle, the group produced in 2003 for the first time a condensate of pairs of fermionic atoms outside the weak-coupling Bardeen-Cooper-Schrieffer (BCS) limit of loosely bound Cooper pairs [3], and subsequently managed to realize the intriguing crossover from the BCS limit to the limit of tightly bound pairs that form a Bose-Einstein condensate [4]. This crossover, which was studied earlier by a number of theorists [5–11], is accompanied by a smooth change in chemical potential from positive in the BCS limit to negative in the Bose-Einstein condensate (BEC) limit.

The unitary, or strong-coupling, limit of the BCS-BEC crossover, which is in the region with positive chemical potential, is of particular interest [12]. This limit marks the threshold of a bound state in vacuum, where the scattering length diverges and changes from negative on the BCS side to positive on the BEC side. Because the scattering length diverges in this limit, the dilute system has no obvious scale parameter, other than the particle number density $n$. This implies, for example, the remarkable result that the ground-state energy of this strongly interacting system is proportional to that of a free Fermi gas, or equivalently, a neutral BCS superconductor at the same density. The absence of a coupling constant, on the one hand, precludes a standard perturbative approach, but on the other makes feasible an effective field theory approach. This is because, as for critical phenomena characterized by a diverging length scale, the absence of an intrinsic scale gives rise to universal behavior. The effective field theory program differs from perturbation theory in that it is carried out not by expanding in a small interaction-related parameter, but by expanding in powers of energy and momentum instead. Using general coordinate and conformal invariance as guiding principle, Son and Wingate [13], in an original paper, proposed the most general effective Lagrangian to leading and next-to-leading orders in a gradient, or momentum, expansion. The conformal transformations involve arbitrary reparametrizations of time $t \rightarrow t'(t)$, while the general coordinate transformations are restricted to curved space (as opposed to spacetime). By keeping the metric $g_{ij}$ ($i, j = 1, 2, 3$) arbitrary in intermediate steps, they were able to derive nontrivial results that survive the limit of flat space. At leading order, these new symmetry arguments are not more powerful than those based on just Galilei invariance, but they are claimed to be more powerful at the next-to-leading order [13].

The importance of Galilei invariance in describing (clean) BCS superconductors at the absolute zero of temperature was already stressed in the 1960ies by Kemoklidze and Pitaevskii [14], following a suggestion by Nozières. It has been used as a guiding principle for obtaining effective theories of zero-temperature neutral superconductors as well as superfluids by others since [15–18]. Such effective theories are expressed in terms of the Nambu-Goldstone mode emerging from the spontaneous breakdown of the global $U(1)$ phase symmetry in such systems. Since this mode is gapless, it constitutes the most important degree of freedom at low frequencies and long wave lengths. Its existence in neutral BCS superconductors was first pointed out by Anderson [19] and Bogoliubov [20]. Invariance under global

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phase transformations implies that a Nambu-Goldstone field is invariably accompanied by at least one derivative. Galilei invariance then restricts how gradient terms can appear in combination with terms involving time derivatives.

In this paper, the effective field theory (EFT) describing a dilute Fermi gas at unitarity and at the absolute zero of temperature up to next-to-leading order is derived from the microscopic theory using a derivative expansion method. This EFT was first carried out in Ref. [21] in the weak-coupling BCS limit and in Ref. [22] for a weakly interacting Bose gas, and is extended here to the strong-coupling unitary limit, for details see the textbook [23]. Due to a vanishing vertex and conspiring contributions, coefficients in the EFT at unitarity can, against common beliefs, be computed analytically beyond the Gaussian approximation—albeit approximately. By construction, the approach by Son and Wingate [13] is limited to an expansion in momentum (or inverse wave length). Since the derivative expansion method [24, 25] we adopt treats time and spatial derivatives on equal footing, it does not face this limitation and yields an expansion in both momentum and energy (or frequency). This leads to two additional terms in the EFT at next-to-leading order omitted in Ref. [13]. As far as static response functions are concerned, these additional terms are immaterial. But they are relevant for the spectrum of the Nambu-Goldstone mode, or phonons, at next-to-leading order and modify the result obtained in Ref. [13].

The paper is organized as follows. The next section sets the stage and briefly introduces the derivative expansion method we adopt. Sections III and V treat the effective theory at leading and next-to-leading order, respectively. Section IV discusses the one-dimensional case for which exact results are known. Finally, integrals and vertices needed in this study are collected in the Appendix.

For notational convenience, we adopt a relativistic notation. A spacetime point will be indicated by the four-vector $x = x^\mu = (t, \mathbf{x})$, $\mu = 0, 1, 2, 3$, while the frequency $\omega$ and momentum $k$ of a particle will be denoted by $k^\mu = (\omega, \mathbf{k})$. The time derivative $\partial_t = \partial/\partial t$ and the gradient $\nabla$ are combined in a single vector $\partial_\mu = (\partial_t, \nabla)$. Indices are raised and lowered with the help of the diagonal metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, so that, for example, $\partial^\mu = (\partial_t, -\nabla)$. We also write $k \cdot x = k_\mu x^\mu = k^0 x_0$ for $\omega t - \mathbf{k} \cdot \mathbf{x}$, and use Einstein’s summation convention. Natural units $\hbar = c = 1$ are adopted throughout.

II. DERIVATIVE EXPANSION

A dilute Fermi gas at the absolute zero of temperature can be modeled by the Lagrangian density

$$\mathcal{L} = \sum_{\sigma = \uparrow, \downarrow} \psi^*_\sigma \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \right) \psi_\sigma - \lambda \psi^*_\sigma \psi^*_\uparrow \psi_\downarrow \psi_\uparrow,$$  \hspace{1cm} (1)

where $\psi_{\uparrow \downarrow}$ is an anti-commuting field that describes the fermionic atoms of mass $m$ and spin up (down), and $\mu$ is the chemical potential. The true interatomic pair potential, which has typically a repulsive hard core of radius less than one nanometer and a weak long-range attractive tail, is approximated by a local, i.e., delta-function potential, characterized by a single interaction parameter $\lambda$. This parameter is related to the (s-wave) scattering length $a$ of the true potential, parameterizing two-particle scattering at low energy in vacuum, through

$$\frac{1}{\lambda} = \frac{\Gamma(D/2 - 1)}{4\pi D/2} \frac{m}{a^{D-2}}$$ \hspace{1cm} (2)

in dimensional regularization. For later convenience, we have recorded the result for arbitrary number of space dimensions $D$, and $\Gamma$ denotes the gamma function. For $D = 3$, Eq. (2) reduces to $1/\lambda = m/4\pi a$. The Lagrangian (1) is invariant under Galilei transformations, where it is recalled that under a Galilei boost with a constant velocity $\mathbf{u}$, the coordinates transform as

$$t \rightarrow t' = t, \quad \mathbf{x} \rightarrow \mathbf{x}' = \mathbf{x} - \mathbf{u} t$$ \hspace{1cm} (3)

so that

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'} \frac{\partial}{\partial t} + \frac{\partial \mathbf{x}}{\partial t'} \cdot \nabla = \partial_t + \mathbf{u} \cdot \nabla, \quad \nabla \rightarrow \nabla' = \nabla,$$ \hspace{1cm} (4)

and $\psi(x)$ picks up an extra phase factor

$$\psi(x) \rightarrow \psi'(x') = e^{im(\mathbf{x} + \frac{1}{2} \mathbf{u} t^2)} \psi(x).$$  \hspace{1cm} (5)

After a Hubbard-Stratonovich transformation, the zero-temperature partition function,

$$Z = \int D\Psi D\bar{\Psi} \exp \left( i \int d^4x \mathcal{L} \right),$$ \hspace{1cm} (6)
can be written in the standard form quadratic in the fermion fields at the expense of additional integrals over auxiliary fields $\Delta$ and $\Delta^*$:

$$Z = \int D\Psi^\dagger D\Psi \int D\Delta^* D\Delta \exp \left( \frac{i}{\lambda} \int d^4x |\Delta(x)|^2 \right) \times \exp \left[ i \int d^4x \Psi^\dagger \left( \begin{array}{cc} i\partial_t + \nabla^2/2m + \mu & -\Delta(x) \\ -\Delta^*(x) & i\partial_t - \nabla^2/2m - \mu \end{array} \right) \Psi \right],$$

(7)

where $\Psi$ stands for the two-component Nambu spinor

$$\Psi = \left( \begin{array}{c} \psi^\uparrow \\ \psi^\downarrow \end{array} \right), \quad \Psi^\dagger = \left( \begin{array}{c} \psi^\dagger_\uparrow \\ \psi^\dagger_\downarrow \end{array} \right).$$

(8)

The fermionic degrees of freedom can now be integrated out exactly with the result

$$Z = \int D\Delta^* D\Delta e^{iS[\Delta^*, \Delta]},$$

(9)

where $S[\Delta^*, \Delta]$ denotes the one-fermion-loop effective action

$$S[\Delta^*, \Delta] = \frac{1}{\lambda} \int d^4x |\Delta(x)|^2 - i \mathrm{Tr} \ln \left( \frac{p_0 - \xi(p)}{\Delta^*(x)} - \frac{\Delta(x)}{p_0 + \xi(p)} \right),$$

(10)

with $\xi(k) = k^2/2m - \mu$ the kinetic energy of noninteracting fermions measured relative to the chemical potential. The trace appearing here is evaluated by using plane waves as a basis

$$\mathrm{Tr} \ln \{K[p, \Delta(x)]\} = \mathrm{tr} \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \ln \{K[k, \Delta(x)]\} e^{-ik\cdot x},$$

(11)

with “$\mathrm{tr}$” denoting the trace over the discrete indices. We adopt the convention that the momentum operator $p^\mu = (i\partial_t, -i\nabla)$ acts on all the fields to its right, whereas the ordinary derivative $\partial^\mu = (\partial_t, -\nabla)$ acts only on the next field to its right. The integral $\int d^4k$ stands for the integral over loop momenta $k^\mu = (\omega, \mathbf{k})$.

For a static uniform system, where the order parameter $\Delta(x) = \Delta$ is independent of spacetime, the trace in Eq. (11) reduces to

$$\mathrm{Tr} \ln \{K[p, \Delta(x)]\} = \mathrm{tr} \int d^4x \int \frac{d^4k}{(2\pi)^4} \ln \{K[k, \Delta]\}$$

(12)

with the integral over spacetime giving just a volume factor. The frequency integral in Eq. (10) can be evaluated in closed form to give for the one-fermion-loop effective action $S[\Delta, \Delta^*] = -\int d^3x \mathcal{V}_{\text{eff}}$ with

$$\mathcal{V}_{\text{eff}} = -\frac{1}{\lambda} |\Delta|^2 - \int \frac{d^3k}{(2\pi)^3} \left[ E(k) - \xi(k) \right],$$

(13)

the effective energy density and

$$E(k) \equiv \sqrt{\xi^2(k) + |\Delta|^2}$$

(14)

the single-fermion excitation spectrum of the noninteracting system. The celebrated BCS gap equation,

$$-\frac{1}{\lambda} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{E(k)},$$

(15)

follows by minimizing the effective potential with respect to $\Delta$. The integral appearing here can be evaluated analytically to give

$$\frac{1}{k_F a} = \frac{4}{\pi} \frac{I_5(x_o) - x_o I_6(x_o)}{[x_o I_5(x_o) + I_6(x_o)]^{1/2}},$$

(16)
where $x_0 \equiv \mu/\Delta$, and $I_5$ and $I_6$ are two integrals which are related to the response functions $\partial^2 \chi_{\text{eff}} / \partial \mu^2$, or $\partial^2 \chi_{\text{eff}} / \partial \Delta^2$ and $\partial^2 \chi_{\text{eff}} / \partial \Delta \partial \mu$, respectively, and which can be expressed in terms of Legendre functions [26], see the Appendix. The momentum $k_F$, defined through

$$n \equiv \frac{1}{3\pi^2} k_F^3,$$

is used to remove the dimension of the scattering length introduced in Eq. (2). This parameter relates the particle number density $n$ of the interacting Fermi gas to the Fermi momentum of a free Fermi gas at the same density. The density $n$ of the interacting system is obtained from the effective potential [13] by differentiating it with (minus) the chemical potential. The result can be put in the form [11]

$$\Delta \equiv \frac{1}{\epsilon_F} \ln \left[ \frac{x_o I_5(x_o) + I_6(x_o)}{2} \right]^{2/3},$$

with $\epsilon_F \equiv k_F^2/2m = (3\pi^2 n)^{2/3}/2m$ the Fermi energy of a free gas. Equations (10) and (13) are valid in the entire BCS-BEC crossover, with the BCS limit ($\mu > 0$, $a < 0$) corresponding to $x_o > 1$, where

$$I_5(x_o) \approx x_o^{1/2}, \quad I_6(x_o) \approx \ln(8x_o)/2x_o^{1/2},$$

and the BEC limit ($\mu < 0$, $a > 0$) corresponding to $x_o << -1$, where

$$I_5(x_o) \approx \pi/16|x_o|^{3/2}, \quad I_6(x_o) \approx \pi/4|x_o|^{1/2}.$$

At unitarity, where a bound state appears and $1/k_F a$ tends to zero, $I_5 = x_o I_6$ by Eq. (10), so that $x_o = 0.8604 \ldots$, $I_5 = 0.8693 \ldots$, $I_6 = 1.010 \ldots$, and

$$\frac{\Delta}{\epsilon_F} = \frac{1}{[(1 + x_o^2 I_6)^{2/3}] = 0.6864 \ldots .}$$

For the ratio $\xi$ of the chemical potential and the Fermi energy $\epsilon_F = k_F^2/2m$ of the free gas, this gives the value

$$\xi \equiv \frac{\mu}{\epsilon_F} = x_o \frac{\Delta}{\epsilon_F} = \frac{x_o}{[(1 + x_o^2 I_6)]^{2/3}} = 0.5906 \ldots .$$

These mean-field values, which were first obtained numerically in Ref. [10], should be compared with, for example, the estimates $\Delta/\epsilon_F = 0.84(4)$ and $\xi = 0.42(1)$ obtained through quantum Monte Carlo simulations of systems of about 60 particles [27].

One of the observations of Ref. [13] is that all the leading order terms in the effective theory describing the Anderson-Bogoliubov mode at unitarity can be determined from the expression [13] for the static uniform system. This will be demonstrated in the following section by explicit calculation.

To determine the next-to-leading terms, the auxiliary fields $\Delta$ and $\Delta^\dagger$ can no longer be assumed to be constant in the formal expression [10]. It can then at best be evaluated in a derivative expansion. The method [24, 25] we adopt proceeds as follows. First, the logarithm is expanded in a Taylor series. Each term in the series contains powers of the derivative $\partial^\mu$ operating on every field appearing to the right. Second, all these operators are shifted to the left by repeated use of the identity

$$\phi_1(x)p^\mu \phi_2(x) = (p^\mu - i\partial^\mu)\phi_1(x)\phi_2(x),$$

where $\phi_1(x)$ and $\phi_2(x)$ are arbitrary fields, and the derivative $\partial^\mu = (\partial_t, -\nabla)$ acts by convention only on the next field to the right. Next, each term is integrated by parts so that all the $p^\mu$‘s act to the left where only a factor $\exp(ik \cdot x)$ is present and yield a factor of $k^\mu$. In this way, each occurrence of the operator $p^\mu$ is replaced with an integration variable $k^\mu$. Finally, the exponential integration function $\exp(-ik \cdot x)$ is moved to the left where it is multiplied with $\exp(ik \cdot x)$ to give unity. The momentum integration can now in principle be carried out to yield an effective action written as a spacetime integral over a Lagrangian density, $S = \int d^4 x \mathcal{L}$.

### III. LEADING ORDER

To derive the effective action, we write the complex field $\Delta(x)$ in terms of a spacetime-dependent amplitude and phase as

$$\Delta(x) = [\Delta + \sigma(x)] e^{2i\varphi(x)},$$

(24)
with $\Delta$ denoting a real solution of the gap equation \[15\]. The functional measure in Eq. \[9\] must be changed accordingly by expressing it in terms of the new fields. Physically, the phase $\varphi(x)$ of the order parameter describes the Nambu-Goldstone mode accompanying the spontaneous breakdown of global U(1) phase symmetry. The effective action governing these phonons obtains after integrating out the $\sigma$ field.

The phase can be removed from the order parameter by returning to the expression \[7\] for the partition function and decompose the fermion fields as

$$\psi_\sigma(x) = e^{i\varphi(x)}\chi_\sigma(x).$$ \[25\]

Instead of the one-fermion-loop effective action \[10\], one then obtains

$$S[\sigma, \varphi] = \frac{1}{\lambda} \int d^4 x (\Delta + \sigma)^2 - i \text{Tr} \ln \left( \begin{array}{cc} p_0 - V_0(x) - \xi [p + V(x)] & -[\Delta + \sigma(x)] \\ -[\Delta + \sigma(x)] & p_0 + V_0(x) + [p - V(x)] \end{array} \right),$$ \[26\]

where $V_\mu(x) \equiv \partial_\mu \varphi(x)$ formally plays the role of an Abelian gauge field. In this guise, the Nambu-Goldstone field is invariably accompanied by at least one derivative. The resulting effective theory is thus automatically invariant under global U(1) phase transformations, under which $\varphi(x)$ is shifted by a constant,

$$\varphi(x) \rightarrow \varphi(x) + \text{const.}$$ \[27\]

By construction, the $\sigma$ field appears only in the combination $\Delta + \sigma$.

The leading order (LO) terms in the effective theory can be obtained by ignoring derivatives on $\sigma$ and $V_\mu$ so that, after using Eq. \[12\], $\xi(k \pm V) = \xi(k) + V^2/2m \pm k \cdot V/m$ in Eq. \[28\]. We explicitly checked that the terms $\pm k \cdot V/m$, do not contribute in leading order. The constant field $V_\mu$ thus appears only in the combination $X \equiv \mu - V$ with

$$V \equiv V_0 + \frac{1}{2m}V^2 = \partial_i \varphi + \frac{1}{2m}(\nabla \varphi)^2.$$ \[28\]

By Eq. \[5\], the Nambu-Goldstone field transforms under a Galilei boost as

$$\varphi(x) \rightarrow \varphi'(x') = \varphi(x) - m \mathbf{u} \cdot \mathbf{x} + \frac{1}{2}m \mathbf{u}^2 t,$$ \[29\]

and the two terms at the right of Eq. \[28\] combine precisely so that $V$ is invariant.

Given these observations, the LO terms, i.e., terms without derivatives on $\sigma$ and $V$, in the one-fermion-loop effective action $S[\sigma, \varphi] = \int d^4 x L_{\text{LO}} + \ldots$ governing these fields, must be of the form

$$L_{\text{LO}}(\sigma, \varphi) = \sum_{i,j=0}^\infty \frac{1}{i! j!} \pi^{(i,j)} \sigma^i V^j$$ \[30\]

with $\pi^{(i,j)}$ the expansion coefficients, or vertices

$$\pi^{(i,j)} = (-1)^{i+1} \frac{\partial^{i+j}}{\partial \mu^i \partial \mu^j} V_{\text{eff}},$$ \[31\]

which depend on the parameters $m, \mu, \Delta$ of the theory. By dimensional analysis,

$$\pi^{(i,j)} = \frac{1}{2^{i+j/2} \pi^2} m^{3/2} \Delta^{5/2 - i - j} \pi^{(i,j)}(x_o)$$ \[32\]

with $x_o = \mu/\Delta$. The numerical prefactor is pulled out for later convenience. The expansion coefficients can be readily obtained from the expression \[13\] for the effective potential. The results up to order $i + j = 4$ are recorded in the Appendix. We iterate that the only terms generated at leading order are those dictated by symmetry and are precisely the one included in Eq. \[30\]. Returning to the original expression \[26\], we explicitly checked that terms spoiling any of the symmetries drop out.

To the order $i + j = 2$, i.e., in the Gaussian approximation, the $\sigma$ field can be integrated out by substituting the corresponding Euler-Lagrange equation for this field,

$$\sigma = -\frac{\pi^{(1,1)}}{\pi^{(0,2)}} V = -\frac{I_6}{I_5} V$$ \[33\]
back into the Lagrangian (30) with $i + j \leq 2$. This is tantamount to approximating the integral over $\sigma$ by the saddle point. It gives as effective theory governing solely the Anderson-Bogoliubov mode at LO

$$L_{LO}(\phi) = \pi^{(0,1)} V - \frac{1}{2} \left( \frac{\pi^{(1,1)}}{\pi^{(2,0)}} \right)^2 V^2. \tag{34}$$

From it, the speed of propagation $c$ of this gapless mode can be read off as

$$c^2 = \frac{1}{m} \frac{\pi^{(0,1)} \pi^{(2,0)}}{\left( \frac{\pi^{(1,1)}}{\pi^{(2,0)}} \right)^2 - \pi^{(0,2)} \pi^{(2,0)}}. \tag{35}$$

Substituting the explicit expressions (A7) for the coefficients $\pi^{(i,j)}$, we reproduce the result due to Marini, Pistolesi, and Strinati [11]

$$c^2 = 2 \cdot \frac{\mu}{3 m} \frac{I_5(x_o I_5 + I_6)}{x_o (I_5^2 + I_6^2)}. \tag{36}$$

In the weak-coupling BCS limit, obtained by letting $x_o \to \infty$, as well as in the strong-coupling unitary limit, where $I_5 = x_o I_6$ with $x_o = 0.8604\ldots$, it reduces to the same form

$$c^2 = \frac{2 \mu}{3 m}. \tag{37}$$

Both limits are in the regime where the chemical potential is positive. At unitarity,

$$\frac{c^2}{v_F^2} = \frac{1}{3} \xi = 0.1968\ldots, \tag{38}$$

with $v_F \equiv k_F/m$ the Fermi velocity of the free Fermi gas at the same density as the unitary gas, and $\xi$ the dimensionless parameter [22].

We next turn to LO terms of higher powers in the fields. In the BCS limit $x_o \to \infty$, where at leading order in energy and momentum, the $\sigma$ field decouples from $V$ and can be ignored, the effective theory (30) up to quartic order in $V$ reduces to

$$L_{LO}(\phi) = \frac{25/2}{15\pi^2} m^{3/2} \mu^{5/2} \left( 1 - \frac{5}{8} \hat{V} + \frac{15}{16} \hat{V}^2 - \frac{5}{16} \hat{V}^3 - \frac{5}{128} \hat{V}^4 + \ldots \right), \tag{39}$$

where we added the free Fermi gas contribution and introduced the abbreviation $\hat{V} \equiv V/\mu$. In this limit, $\mu \to \epsilon_F$. The terms in Eq. (39) form precisely the first in the Taylor series expansion of the full LO expression,

$$L_{LO}(\phi) = c_0 m^{3/2} X^{5/2}, \quad X \equiv \mu - V = \mu - \partial_t \phi - \frac{1}{2m} (\nabla \phi)^2, \tag{40}$$

proposed by Son and Wingate [13] with the coefficient

$$c_0 = \frac{25/2}{15\pi^2}. \tag{41}$$

The expression (40) sums up all terms where each Nambu-Goldstone field is accompanied by exactly one (space or time) derivative. For a static uniform system, where $V$ is zero, the right side of Eq. (40) reduces to the zero-temperature pressure expressed as a function of $\mu$, $P(\mu)$.

As an aside, a trapping or other external potential $U(x)$ can be readily included in the formalism by replacing $\mu$ with $\mu - U(x)$ in $X$.

To obtain the effective theory governing the field $V$ alone in the entire BCS-BEC region, the $\sigma$ field must be integrated out from the complete LO expression (30). Since it contains arbitrary powers of $\sigma$, this is in general impossible. At unitarity, however, the coefficients assume values that make this at least approximately possible, as we now demonstrate. The Lagrangian up to quartic order in $\sigma$ and $V$, with the vertices given in the Appendix, takes the
following form in this limit:

\[
L_{\text{LO}}(\sigma, \varphi) = \frac{2^{5/2}}{\pi^2} m^{3/2} \mu^{5/2} \frac{I_6}{x_o^{3/2}} \left[ \frac{2}{3} (1 + x_o^2) \dot{V} + \frac{1}{2} x_o^2 \ddot{V} - \frac{1}{12} x_o^2 \dot{V}^3 - \frac{1}{96} \frac{x_o^4}{1 + x_o^2} \dot{V}^4 
- x_o \sigma \dot{V} - \frac{1}{2} x_o^2 \dot{\sigma}^2 - \frac{1}{6} x_o^3 \dot{\sigma}^3 + \frac{1}{96} \frac{x_o^4 (3 + 4x_o^2)}{1 + x_o^2} \dot{\sigma}^4 
- \frac{1}{4} x_o^2 \ddot{\sigma}^2 \dot{V} + \frac{1}{24} \frac{x_o^3}{1 + x_o^2} \ddot{\sigma} \dot{V}^3 + \frac{1}{16} \frac{x_o^4}{1 + x_o^2} \ddot{\sigma}^2 \dot{V}^2 
+ \frac{1}{24} \frac{x_o^3 (1 + 2x_o^2)}{1 + x_o^2} \ddot{\sigma}^3 \dot{V} + \ldots \right].
\]

(42)

Note that the vertex \(\pi^{(1,2)}\) of the \(\sigma V^2\) term vanishes in this limit. Moreover, the coefficients are such that the Euler-Lagrange equation (33), which assumes the form \(\sigma = -V/x_o\) in the unitary limit, obtained in the Gaussian approximation remains unchanged after including the additional terms in Eq. (42). Put differently, the saddle point of the nonlinear theory (42) remains locked at \(\sigma = -V/x_o\) in the unitary limit. Approximating the integral over \(\sigma\) by this saddle point, we obtain an effective Lagrangian governing just \(\varphi\)

\[
L_{\text{LO}}(\varphi) = \frac{2^{5/2}}{15\pi^2} m^{3/2} \mu^{5/2} \frac{(1 + x_o^2)I_6}{x_o^{3/2}} \left( 1 - \frac{5}{2} \dot{V} + \frac{15}{8} \dot{V}^2 - \frac{5}{16} \dot{V}^3 - \frac{5}{128} \dot{V}^4 + \ldots \right),
\]

where we included the term for a static uniform unitary Fermi gas. The various contributions combine to exactly generate the first terms in the Taylor series expansion of the predicted form (40) with

\[
c_0 = \frac{2^{5/2}}{15\pi^2} \frac{1}{\xi^{3/2}}, \quad \frac{1}{\xi^{3/2}} \equiv \frac{(1 + x_o^2)I_6}{x_o^{3/2}} = 2.203 \ldots,
\]

(44)

so that \(c_0 = 0.0841 \ldots\). The dimensionless parameter \(\xi\), which was introduced in Eq. (22) as the ratio of the chemical potential and \(\epsilon_F\), gives here the ratio of the ground-state energy per particle \(\epsilon\) of the unitary gas and that of a free Fermi gas, or equivalently, a neutral BCS superconductor at the same density,

\[
\epsilon \equiv \xi \frac{3}{10} \frac{k_B^2}{m},
\]

(45)

as follows from taking the Legendre transform of \(P(\mu)\) and using that \(n = \partial P(\mu)/\partial \mu\) [13]. We emphasize that in determining the coefficients of the LO terms only the effective potential [13] describing a static uniform system is used. This validates the symmetry argument by Son and Wingate [13] that the LO effective theory (40) to all orders in the Nambu-Goldstone field can be obtained by simply replacing the chemical potential \(\mu\) with the Galilei-invariant combination \(X\) in the pressure \(P(\mu)\) of the static uniform system. It implies that the complete interaction between phonons at leading order in wave vector and frequency is determined by \(P(\mu)\).

The strong-coupling unitary limit is special as for no other point in the BCS-BEC crossover, the saddle point (33), obtained in the Gaussian approximation, constitutes a solution to the Euler-Lagrange equation when additional terms in the Lagrangian (40) are included.

IV. 1D

Although the coefficients of the LO effective Lagrangian in three space dimensions (3D) can only be determined approximately in the strong-coupling unitary limit, the form of the theory is precisely as predicted by Son and Wingate [13]. To provide further support for this prediction, we in this section consider the pairing theory in one space dimension (1D) for which exact results are available.

The 1D system of \(N\) spin-\(\frac{1}{2}\) fermions with attractive delta-function interactions is described exactly by the Gaudin integral equations [28]. Let, as in the three-dimensional case, \(\lambda(< 0)\) denote the coupling constant. It is related to the 1D scattering length through

\[
\frac{1}{\lambda} = -\frac{1}{2} m a,
\]

(46)

as follows from Eq. (2) with \(D = 1\). Whereas the 3D coupling constant is directly proportional to the scattering length, its 1D counterpart is inversely proportional to \(a\), and \(a \to +\infty\) in the limit \(\lambda \to 0^+\). This divergence of the scattering
length in the zero-coupling limit arises because an attractive delta-function potential possesses a two-particle bound state however small the attraction. In other words, this limit marks the threshold of a bound state in vacuum, where the scattering length diverges and changes from negative for repulsive interactions ($\lambda > 0$) to positive for attractive interactions ($\lambda < 0$).

In the weak-coupling BCS limit $\lambda \to 0^-$, the Gaudin integral equations yield for the ground-state energy per particle $\epsilon$, chemical potential $\mu$, and sound velocity $c$, the free Fermi gas expressions

$$\epsilon = \frac{1}{6} k_F^2, \quad \mu = \epsilon_F, \quad c^2 = \frac{2\mu}{m} = v_F^2,$$

as expected. In the strong-coupling limit $\lambda \to -\infty$, where $a \to 0^+$ in 1D, the fermions form tightly bound pairs with binding energy $\epsilon_a = 1/ma^2$, as in 3D. The Gaudin integral equations give in this limit

$$\epsilon_{\text{eff}} = \frac{1}{24} k_F^2, \quad \mu_{\text{eff}} = \frac{1}{8} k_F^2, \quad c^2 = \frac{2\mu_{\text{eff}}}{m} = \frac{1}{4} k_F^2,$$

where we removed the (diverging) binding energy from the ground-state energy and the chemical potential by introducing $\epsilon_{\text{eff}} \equiv \frac{1}{2} \epsilon_a + \epsilon$ and $\mu_{\text{eff}} \equiv \frac{1}{2} \epsilon_a + \mu$. The Fermi momentum $k_F$ is now defined through

$$n \equiv \frac{2}{\pi} k_F.$$

Note that as in 3D, the unitary limit is in the region with positive (effective) chemical potential. On comparison with the weak-coupling results (47), it follows that

$$\xi \equiv \frac{\mu_{\text{eff}}}{\epsilon_F} = \frac{1}{4}$$

exactly. As in 3D, $\xi$ also equals the ratio of the (effective) ground-state energy per particle $\epsilon$ of the unitary gas and that of a free Fermi gas at the same density. The exact 1D counterpart of Eq. (48) reads

$$\frac{c^2}{v_F^2} = \xi = \frac{1}{4},$$

whereas the full LO Lagrangian density is given by

$$\mathcal{L}_{\text{LO}}(\varphi) = c_0 m^{1/2} X^{3/2},$$

with

$$c_0 = \frac{2^{5/2}}{3\pi} \frac{1}{\xi^{1/2}} \approx \frac{2^{7/2}}{3\pi}$$

exactly in the unitary limit (and $c_0 = 2^{5/2}/3\pi$ in the weak-coupling limit).

The physical interpretation of the 1D unitary limit follows from comparison with the related problem of repulsively interacting bosons. The 1D bosonic system with a delta-function potential is described exactly by the Lieb-Liniger integral equations [20]. In the infinite-coupling limit, these equations coincide with the Gaudin integral equations at unitary, provided one identifies the boson mass $m_b$ with twice the fermion mass, $m_b = 2m$, and the boson number density $n_b$ with half the fermion number density, $n_b = n/2$ [29]. That is, the 1D unitary limit coincides with the BEC limit.

The two sets of integral equations can be mapped onto each other not only in the infinite-coupling limit, but also for finite coupling. With the above identifications, both sets become similar with one important distinction that the signs of the coupling constants differ, as already pointed out by Gaudin [28]. Specifically, the Gaudin integral equations can be obtained from the Lieb-Liniger integral equations by setting $m_b = 2m$ and $n_b = n/2$, and by replacing the bosonic coupling constant $\lambda_b(> 0)$ with $-2\lambda(< 0)$ [51], showing that the interaction between pairs is attractive. Whereas bosons must interact repulsively to guarantee stability, pairs of spin-$\frac{1}{2}$ particles can have attractive interactions, for the Pauli exclusion principle forbids two such pairs to form a four-fermion bound state. The fact that $c^2$ is positive in the unitary limit implies that the compressibility is also positive and that the system is mechanically stable even in this limit of infinite attraction between pairs.
V. NEXT-TO-LEADING ORDER

We next turn to the next-to-leading order (NLO) terms in the effective Lagrangian. These terms involve derivatives of the $\sigma$ field and $V$, or equivalently, $X = \mu - V$. To obtain the first NLO terms it suffice to consider quadratic terms in the fields $\sigma$ and $\varphi$ up to fourth order in derivatives. The possible independent terms of this form are given in 3D by

$$
L_{NLO}^{(2)}(\sigma, \varphi) = \frac{1}{2^{1/2} \pi} \left[ b_1 \left( \frac{m}{\Delta} \right)^{3/2} (\partial_\sigma)^2 + b_2 \left( \frac{m}{\Delta} \right)^{1/2} (\nabla \sigma)^2 
+ c_1 \left( \frac{m}{\Delta} \right)^{1/2} (\nabla \partial_\sigma)^2 + c_2 \left( \frac{m}{\Delta} \right)^{3/2} (\partial_\sigma^2)^2 + c_3 \left( \frac{\Delta}{m} \right)^{1/2} (\nabla^2 \varphi)^2 + c_4 \left( \frac{m}{\Delta} \right)^{1/2} \partial_\sigma^2 \varphi \nabla^2 \varphi 
+ d_1 \left( \frac{m}{\Delta} \right)^{1/2} \nabla \sigma \cdot \nabla \partial_\sigma + d_2 \left( \frac{m}{\Delta} \right)^{3/2} \partial_\sigma \partial_\sigma^2 \varphi + d_4 \left( \frac{m}{\Delta} \right)^{1/2} \partial_\sigma \nabla^2 \varphi \right],
$$

(54)

where the powers of $m/\Delta$ follow from dimensional analysis. In writing this general expression, with arbitrary coefficients $b_1, c_1$, and $d_1$, we also noted that the theory is invariant under time reversal, under which $t \rightarrow -t$ and $\varphi \rightarrow -\varphi$. Our choice of coefficients is such that at unitarity, the terms with coefficients $c_1, c_2$ and $c_4$, respectively after the $\sigma$ field has been integrated out. In addition to the $(\nabla^2 \varphi)^2$ term, there exists a second term quartic in derivatives, viz. $\partial_\sigma \partial_\sigma^2 \varphi \partial_\partial_\sigma \varphi$. In the quadratic approximation we are working, both differ by a total derivative and cannot be uniquely identified. However, the ratio of the two terms can be determined by the derivative expansion method we use and comes out to be two, leaving two possible combinations $(\nabla^2 \varphi)^2 + 2(\partial_\sigma \partial_\partial_\sigma \varphi)^2$ or $2(\nabla^2 \varphi)^2 + (\partial_\sigma \partial_\partial_\sigma \varphi)^2$. Through the study of two three-point correlation functions, it was shown in Ref. [32] that the former combination is in fact realized. This combination is also favored by symmetry, for

$$
(\nabla^2 \varphi)^2 + 2(\partial_\sigma \partial_\partial_\sigma \varphi)^2 = (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \partial_\sigma \partial_\partial_\sigma \varphi \partial_\partial_\sigma \varphi.
$$

(55)

We have computed the coefficients appearing in the Lagrangian (54) for the entire BCS-BEC crossover by applying the derivative expansion method to the formal expression (26). The results are collected in the Appendix. Although the terms $(\nabla \partial_\sigma)^2$ (with coefficient $c_1$) and $\partial_\sigma^2 \varphi \nabla^2 \varphi$ (with coefficient $c_4$) become identical after partial integration, they can be separately identified, for they originate from different parts in the effective action (26). Specifically, the term $(\nabla \partial_\sigma)^2$ arises as $(\nabla V_0)^2$ and can be calculated by setting $V$ to zero in Eq. (26) and ignoring further time derivatives, while the term $\partial_\sigma^2 \varphi \nabla^2 \varphi$ arises as $\partial_\partial \nabla \cdot \nabla$. We first consider the BCS limit, where the $\sigma$ field decouples in first approximation. The general expression (54) reduces in this limit, obtained by setting $x_0 \rightarrow \infty$, to the known result (in conventional notation) $\left[10, 21, 33\right]

$$
L_{NLO}^{(2)}(\varphi) = \frac{1}{6 \pi^2} \frac{\nu(0)}{\Delta^2} \left[ (\partial_\sigma^2)^2 + \frac{1}{5} \varphi^2 (\nabla^2 \varphi)^2 - \frac{2}{3} \partial_\sigma^2 \varphi \nabla^2 \varphi \right],
$$

(56)

with $\nu(0) \equiv m k_F / 2 \pi^2$ the density of states at the Fermi surface. Note that in this limit, the coefficient $c_1$ can be ignored in comparison to the coefficient $c_4$. These NLO terms together with the quadratic LO terms in Eq. (34) give rise to the phonon spectrum $\left[10, 32\right]

$$
\omega^2(k) = \frac{1}{3} \frac{v_F^2}{\Delta^2} k^2 \left( 1 - \frac{2 v_F^2}{45 \Delta^2} k^2 \right) = \frac{2 \mu}{3 m} k^2 \left( 1 - \frac{4 v_F^2}{45 m \mu} k^2 \right).
$$

(57)

The quadratic terms (54) can be put into Galilei-invariant form as

$$
L_{NLO}(\varphi) = \frac{5}{16} \kappa_0 \varphi_0 \left( \frac{m^{3/2}}{X^{3/2}} (D_\lambda X)^2 + \frac{4}{15} \frac{X^{1/2}}{m^{1/2}} ([\nabla^2 \varphi]^2 + 2(\partial_\sigma \partial_\partial_\sigma \varphi)^2] + \frac{4 m^{1/2}}{3 X^{1/2}} D_\lambda X \nabla^2 \varphi \right),
$$

(58)

with $\kappa_0$ given in Eq. (51) and $D_\lambda$ the material derivative,

$$
D_\lambda \equiv \partial_\lambda + \frac{1}{m} \nabla \varphi \cdot \nabla,
$$

(59)

which by the transformations (41) and (29) is invariant under Galilei boosts. We note that the coefficients of the NLO terms can be uniquely identified from the quadratic approximation (50) to this Lagrangian. Indeed, as prescribed by the derivative expansion method (24), each factor of $\mu$ is to be replaced with $X$ in Eq. (55). And each occurrence of $\partial_\sigma^2 \varphi$ is to be replaced with the Galilei-invariant form $-D_\lambda X$. 

For completeness, we mention that the Lagrangian governing the $\sigma$ field in the BCS limit reads to this order
\[ \mathcal{L}(\sigma) = \frac{2^{1/2}}{\pi^2} m^{3/2} \mu^{5/2} \frac{1}{x_o^6} \left\{ \frac{1}{24} \left[ (\partial_t \sigma)^2 - \frac{1}{3} \sigma^2 (\nabla \sigma)^2 \right] - \frac{1}{2} \sigma^2 - \frac{1}{27} \sigma^3 + \frac{1}{4!} \sigma^4 + \ldots \right\}, \]
where $\bar{\sigma} \equiv \sigma/\Delta$, and vanishes in the limit $x_o \to \infty$.

We continue by studying the strong-coupling unitary limit obtained by setting $I_5 = x_o I_6$ with $x_o = 0.8604 \ldots$. The general expression (54) assumes in this limit the explicit form:
\[ \mathcal{L}^{(2)}_{\text{NLO}}(\sigma, \varphi) = \frac{1}{2^{1/2} 3 \pi^2 x_0^2} I_6 \left[ \frac{1}{4} \left( \frac{m}{\mu} \right)^{3/2} x_0^2 (\partial_t \sigma)^2 - \frac{1}{24} \left( \frac{m}{\mu} \right)^{1/2} (7 + 4 x_0^2) (\nabla \sigma)^2 \right. \\
- \frac{1}{8} \left( \frac{m}{\mu} \right)^{1/2} (\nabla \partial_t \varphi)^2 + \frac{1}{2} x_0^2 \left( \frac{m}{\mu} \right)^{3/2} (\partial_t^2 \varphi)^2 + 2 \left( 1 + x_0^2 \right) \left( \frac{\mu}{m} \right)^{1/2} (\nabla^2 \varphi)^2 - \frac{1}{6} \left( 4 x_0^2 + 1 \right) \left( \frac{m}{\mu} \right)^{1/2} \partial_t^2 \varphi \nabla^2 \varphi \\
\left. - \frac{1}{4} \left( \frac{m}{\mu} \right)^{3/2} x_0^2 \sigma^2 \partial_t^2 \varphi + \frac{1}{2} \left( \frac{m}{\mu} \right)^{1/2} x_0 \partial_t \sigma \nabla^2 \varphi \right]. \]

Note that the vertex $d_1$ of the $\nabla \sigma \cdot \nabla \partial_t \varphi$ term vanishes in this limit. Approximating the integral over $\sigma$ by the LO saddle point (33), which is consistent to the order we are working, we obtain an effective theory of precisely the form (56) with an additional $(\nabla \partial_t \varphi)^2$ term included. As in the BCS limit, these quadratic terms fourth order in derivatives can be put into Galilei-invariant form as
\[ \mathcal{L}_{\text{NLO}}(\varphi) = \frac{5}{16} c_0 x_o^2 \left\{ -7 \frac{1}{12} \frac{m^{1/2}}{x_0^2} (\nabla X)^2 + \frac{m^{3/2}}{x_0^2} (D_t X)^2 + \frac{4}{15} X^{1/2} \left[ (\nabla^2 \varphi)^2 + 2 (\partial_t \varphi)^2 \right] + \frac{4 m^{1/2}}{3 x_0^2} D_t X \nabla^2 \varphi \right\}, \]
with $c_0$ now given by Eq. (44). Apart from overall normalization and the first term, which is suppressed in the BCS limit, these NLO terms are exactly as found in the BCS limit. As in that limit, the form and coefficients of the NLO terms uniquely follow from the quadratic approximation to this Lagrangian. In addition to the replacements already used in the BCS limit, $(\nabla \partial_t \varphi)^2$ is replaced with the Galilei-invariant form $(\nabla X)^2$ in Eq. (62). The second and last terms in that expression, both involving $D_t X$, were not considered by Son and Wingate (32) as their approach is limited to only gradients of $X$. Also the fourth term was omitted. With $r_i (i = 1, 2, 3, 4, 5)$ denoting the coefficients of the NLO terms, so that
\[ r_i = \frac{5}{16} c_0 x_o^2 \left( -7 \frac{1}{12} \frac{1}{x_0^2} \frac{1}{15} \frac{8}{15} \frac{4}{3} \right), \]
the spectrum of the gapless Anderson-Bogoliubov mode that follows when including the NLO terms (62) reads
\[ \omega^2(\mathbf{k}) = \frac{2}{3} \mu k^2 \left\{ 1 - \frac{4}{45 c_0} \left[ 6 r_1 + 4 r_2 + 9 (r_3 + r_4) - 6 r_5 \right] \frac{1}{m \mu} k^2 \right\}. \]
It reduces to the BCS expression (57) in the limit $x_o \to \infty$. In contrast to the BCS limit, the coefficient of the correction term is positive for the value $x_o = 0.8604 \ldots$ obtained in the saddle-point approximation. If it remains positive beyond this approximation, a low-energy phonon in a unitary Fermi gas can decay into two phonons.

The Son-Wingate result for the spectrum corresponds to setting $r_2 = r_5 = 0$ and also $r_4 = 0$ in Eq. (64a), which leads to an incorrect expression for the spectrum. The correct result (64b) was derived from a NLO Lagrangian of a form proposed by Son and Wingate, i.e., one without time derivatives in Ref. (32). This was achieved by eliminating the time derivatives in the NLO Lagrangian through the use of the leading-order field equations. This reduction leads to the following changes in the coefficients (63):
\[ r_3 = \frac{1}{12} c_0 x_o^2 \rightarrow r'_3 = -\frac{1}{18} c_0 x_o^2, \quad r_4 = 2 r_3 \rightarrow r'_4 = -3 r'_3 \]
which, when substituted in Eq. (64a) with $r_2 = r_5 = 0$, yields the correct spectrum (64b). The flip side of this reduction is that the static response functions come out incorrectly, for the coefficients of the static terms in the NLO Lagrangian now also include dynamic effects. The relation $r'_4 = -3 r'_3$ in the reduced NLO Lagrangian was argued in Ref. (32) to be a consequence of conformal invariance at unitarity.
VI. DISCUSSION

Its (relatively) simple form and the very fact that the effective action of a unitary Fermi gas up to next-to-leading order can be derived from the microscopic theory analytically underscores the special status of the unitary limit in the BCS-BEC crossover. Although the coefficients could only be computed approximately, using an (extended) saddle point, it is remarkable that the effective field program, which involves integrating out the fermionic degrees of freedom as well as the $\sigma$ field, can be carried out consistently up to the orders considered, featuring higher-order terms such as $$(\nabla \phi)^8$$ and $$(\partial^2 \phi)^2$$. Surprisingly, the effective actions obtained in the weak-coupling BCS and the strong-coupling unitary limits are proportional, save for a next-to-leading order term which is suppressed in the BCS limit.

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Appendix A: Integrals

All integrals encountered in this study can be expressed as linear combinations of two basic integrals introduced by Marini, Pistolesi, and Strinati \[11\]

$$I_5(x_o) \equiv \int_0^{\infty} dx \frac{x^2}{E_x}, \quad I_6(x_o) \equiv \int_0^{\infty} dx \frac{x^2 \xi_x}{E_x}, \quad (A1)$$

by integrating by parts and simple algebraic manipulations. Here, $x, \xi_x$, and $E_x$ denote the dimensionless variables

$$x^2 \equiv \frac{k^2/2m}{\Delta}, \quad \xi_x \equiv \frac{\xi}{\Delta} = x^2 - x_o, \quad x_o \equiv \frac{\mu}{\Delta}, \quad E_x \equiv \frac{E}{\Delta} = \sqrt{\xi_x^2 + 1}. \quad (A2)$$

These integrals can be expressed in terms of the complete elliptic integrals of the first and second kind, as was done in Ref. \[11\] or in terms of Legendre functions $P_\alpha$ \[20\] as ($\gamma_o \equiv x_o/\sqrt{1+x_o^2}$)

$$I_5(x_o) = \frac{\pi}{4} \frac{1}{(1+x_o^2)^{3/4}} \left[ (1-3x_o^2)P_{1/2}(-\gamma_o) - 3x_o(1+x_o^2)^{1/2}P_{3/2}(-\gamma_o) \right]$$

$$I_6(x_o) = -\frac{\pi}{4} \frac{1}{(1+x_o^2)^{3/4}} \left[ 4x_oP_{1/2}(-\gamma_o) + 3(1+x_o^2)^{1/2}P_{3/2}(-\gamma_o) \right], \quad (A3)$$

so that, for example,

$$I_5(x_o) - x_o I_6(x_o) = \frac{\pi}{4} (1+x_o^2)^{1/4} P_{1/2}(-\gamma_o). \quad (A4)$$

With the definition

$$I_{k,l,m} \equiv \int_0^{\infty} dx \frac{x^k \xi_x^l}{E_x^m}, \quad (A5)$$
so that $I_5 = I_{2,0,3}$ and $I_6 = I_{2,1,3}$, one readily verifies the relations

\[
I_{0,0,1} = 2I_6 \\
I_{0,0,3} = \frac{x_o I_5 + I_6}{1 + x_o^2} \\
I_{0,0,5} = \frac{I_{4,0,5} - 2I_{2,1,5} + I_{0,0,3}}{1 + x_o^2} \\
I_{0,1,3} = \frac{I_5 - x_o I_6}{1 + x_o^2} \\
I_{0,2,5} = I_{0,0,3} - I_{0,0,5} \\
I_{0,3,5} = I_{2,2,5} - x_o I_{0,2,5} \\
I_{2,0,5} = \frac{1}{6} x_o^2 \frac{x_o I_5 + I_6}{1 + x_o^2} + \frac{1}{2} I_5 \\
I_{2,0,7} = I_{2,4,7} + 2I_{2,0,5} - I_5 \\
I_{2,1,5} = \frac{1}{6} I_{0,0,3} \\
I_{2,1,7} = \frac{1}{10} I_{0,0,5} \\
I_{2,2,5} = \frac{1}{3} I_5 + \frac{1}{6} I_{0,1,3} \\
I_{2,4,7} = \frac{1}{10} (I_{0,3,5} + 6I_{2,2,5}) \\
I_{4,0,5} = (1 + x_o^2) I_{2,1,5} + \frac{1}{2} x_o I_5.
\]

These integrals are all a function of $x_o$ alone.

With the help of these integrals, the coefficients $\pi^{(i,j)}$ of the LO effective theory \[^{30}\] introduced in Eq. (32), can be readily evaluated in closed form, with the results up to $i + j = 4$

\[
\pi^{(0,1)} = -\frac{4}{3} (x_o I_5 + I_6) \\
\pi^{(0,2)} = 2I_5 \\
\pi^{(0,3)} = -\frac{x_o I_5 + I_6}{1 + x_o^2} \\
\pi^{(0,4)} = -\frac{1}{2} \left( -3 + x_o^2 \right) I_5 + 4x_o I_6 \\
\pi^{(2,0)} = -2I_5 \\
\pi^{(3,0)} = -\frac{(3 + 2x_o^2) I_5 - x_o I_6}{1 + x_o^2} \\
\pi^{(4,0)} = \frac{1}{2} \left( 3 + 3x_o^2 + 4x_o^4 \right) I_5 + 4x_o^3 I_6 \\
\pi^{(1,1)} = -2I_6 \\
\pi^{(1,2)} = \frac{I_5 - x_o I_6}{1 + x_o^2} \\
\pi^{(1,3)} = \frac{1}{2} \left( 4x_o I_5 + (1 - 3x_o^2) I_6 \right) \\
\pi^{(2,1)} = \frac{x_o I_5 - (1 + 2x_o^2) I_6}{1 + x_o^2} \\
\pi^{(2,2)} = -\frac{1}{2} \left( 1 - 3x_o^2 \right) I_5 + 2x_o (-1 + x_o^2) I_6 \\
\pi^{(3,1)} = \frac{1}{2} \left( 2x_o (-1 + x_o^2) I_5 + (1 + 5x_o^2) I_6 \right).
\]
Note that $\bar{\pi}^{(0,2)} = -\bar{\pi}^{(2,0)}$.

The coefficients of the NLO terms appearing in the quadratic Lagrangian [54], which are somewhat laborious to compute, can again be expressed as linear combinations of $I_5$ and $I_6$, with the results

\begin{align*}
b_1 &= \frac{1}{4} \frac{(3 + 2x_o^2)I_5 - x_oI_6}{1 + x_o^2} \quad \text{(A8a)} \\
b_2 &= -\frac{1}{12} \frac{x_o(1 + 4x_o^2)I_5 + (7 + 10x_o^2)I_6}{1 + x_o^2} \quad \text{(A8b)} \\
c_1 &= -\frac{1}{4} \frac{x_oI_5 + I_6}{1 + x_o^2} \quad \text{(A8c)} \\
c_2 &= \frac{1}{4} \frac{(3 + 4x_o^2)I_5 + x_oI_6}{1 + x_o^2} \quad \text{(A8d)} \\
c_3 &= \frac{1}{5} [(3 + 4x_o^2)I_5 + x_oI_6] \quad \text{(A8e)} \\
c_4 &= -\frac{1}{3} (4x_oI_5 + I_6) \quad \text{(A8f)} \\
d_1 &= \frac{1}{2} \frac{I_5 - x_oI_6}{1 + x_o^2} \quad \text{(A8g)} \\
d_2 &= -\frac{1}{2} \frac{x_oI_5 + I_6}{1 + x_o^2} \quad \text{(A8h)} \\
d_4 &= I_5 \quad \text{(A8i)}
\end{align*}

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