Casimir densities for two concentric spherical shells in the global monopole spacetime

A. A. Saharian 1* and M. R. Setare 2 †
1 Department of Physics, Yerevan State University, Yerevan, Armenia
and
2 Institute for Theoretical Physics and Mathematics, Tehran, Iran
Department of Science, Physics Group, Kurdistan University, Sanandeg, Iran
Department of Physics, Sharif University of Technology, Tehran, Iran

February 2, 2022

Abstract

The quantum vacuum effects are investigated for a massive scalar field with general curvature coupling and obeying the Robin boundary conditions given on two concentric spherical shells with radii \( a \) and \( b \) in the \( D + 1 \)-dimensional global monopole background. The expressions are derived for the Wightman function, the vacuum expectation values of the field square, the vacuum energy density, radial and azimuthal stress components in the region between the shells. A regularization procedure is carried out by making use of the generalized Abel-Plana formula for the series over zeros of combinations of the cylinder functions. This formula allows us to extract from the vacuum expectation values the parts due to a single sphere on background of the global monopole gravitational field, and to present the "interference" parts in terms of exponentially convergent integrals, useful, in particular, for numerical evaluations. The vacuum forces acting on the boundaries are presented as a sum of the self–action and interaction terms. The first one contains well known surface divergences and needs a further regularization. The interaction forces between the spheres are finite for all values \( a < b \) and are attractive for a Dirichlet scalar. The asymptotic behavior of the vacuum densities is investigated (i) in the limits \( a \to 0 \) and \( b \to \infty \), (ii) in the limit \( a, b \to \infty \) for fixed value \( b - a \), and (iii) for small values of the parameter associated with the solid angle deficit in global monopole geometry. We show that in the case (ii) the results for two parallel Robin plates on the Minkowski bulk are rederived to the leading order.

PACS number(s): 03.70.+k, 11.10.Kk

*E-mail: saharyan@www.physdep.r.am
†E-mail: rezakord@yahoo.com
1 Introduction

The Casimir effect is regarded as one of the most striking manifestations of vacuum fluctuations in quantum field theory. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. In particular, vacuum forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem (see, e.g., [2, 3, 4, 5, 6, 7, 8, 9] and references therein). Many different approaches have been used: mode summation method with combination of the zeta function regularization technique, Green function formalism, multiple scattering expansions, heat-kernel series, etc. The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. Another type of vacuum polarization arises in the case of external gravitational fields [10, 11]. In a previous paper [12] we have studied an example of a situation when both types of sources for the polarization are present. Namely, we have investigated the vacuum expectation values of the square of a scalar field and energy-momentum tensor induced by a spherical shell in the spacetime of a point-like global monopole.

Topological defects have attracted a great deal of attention because of their relevance to a number of different areas ranging from condensed matter to structure formation (for a review see [13]). In the context of hot big bang cosmology, the unified theories of the fundamental interactions predict that the universe passes through a sequence of phase transitions. These phase transitions might have given rise to several kinds of topological defects depending on the nature of the symmetry that is broken [14]. If a global SO(3) symmetry of a triplet scalar field is broken, the point like defects called global monopoles are believed to be formed. The simplified global monopole was introduced by Sokolov and Starobinsky [15]. The gravitational effects of the global monopole were studied in Ref. [16], where a solution is presented which describes a global monopole at large radial distances. The quantum vacuum effects of the matter fields on the global monopole background have been considered in [17, 18, 19, 20]. The effects produced by the non-zero temperature are investigated as well [21]. The zeta function and the Casimir energy for spherical boundaries in this background are considered in [22, 23].

In this paper, we continue the investigation started in [12] and consider the Casimir densities in the region between two concentric spherical shells on background of the $D + 1$-dimensional spacetime of a point-like global monopole. The positive frequency Wightman function, the vacuum expectation values of the field square and energy-momentum tensor are investigated for a massive scalar field with general curvature coupling parameter $\xi$. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modeling a self-consistent dynamics involving the gravitational field [10]. We study the general case of Robin boundary conditions with different coefficients for the inner and outer spheres. The Robin boundary conditions are an extension of the ones imposed on perfectly conducting boundaries and may, in some geometries, be useful for depicting the finite penetration of the field into the boundary with the ”skin-depth” parameter related to the Robin coefficient [24, 25]. It is interesting to note that the quantum scalar field satisfying Robin condition on the boundary of a cavity violates the Bekenstein's entropy-to-energy bound near certain points in the space of the parameter defining the boundary condition [26]. This type of conditions also appear in considerations of the vacuum effects for a confined charged scalar field in external fields [27] and in quantum gravity [28, 29, 30]. Mixed boundary conditions naturally arise for scalar and fermion bulk fields in the Randall-Sundrum model [31]. In this model the bulk geometry is a
slice of anti-de Sitter space and the corresponding Robin coefficient is related to the curvature scale of this space. For scalars with general curvature coupling the essential point is the relation between the mode sum energy, evaluated as a renormalized sum of the zero-point energies for each normal mode of frequency, and the volume integral of the renormalized energy density. In [32] it has been shown that in the discussion of this question for the Robin parallel plates it is necessary to include in the energy a surface term concentrated on the boundary (see [33, 34, 35] for similar issues in more complicated geometries and the discussion of the paper [32] in [36]).

We have organized the paper as follows. In the next section we derive a formula for the Wightman function in the region between two spheres. The reason for our choice of the Wightman function is that this function also determines the response of the particle detectors in a given state of motion. Following Refs. [33, 34, 35, 37], to evaluate the bilinear field products we use the mode sum method in combination with the summation formulae from [38] (see also [39]). These formulae allow (i) to extract from vacuum expectation values the parts due to a single sphere on the global monopole background, and (ii) to present the "interference" parts in terms of exponentially convergent integrals involving the modified Bessel functions. The vacuum expectation value of the field square is obtained computing the Wightman function in the coincidence limit, and is investigated in section 3. The various limiting cases are considered including the limit of the strong gravitational field corresponding to small values of the parameter associated with the solid angle deficit in the global monopole spacetime. Section 4 is devoted to the vacuum expectation values of the energy density and stresses. The formulae are derived for the interaction forces between the spheres. The behavior of these quantities is investigated in various limiting cases. Section 5 concludes the main results of the paper.

2 Wightman function

In this paper we consider a real scalar field $\varphi$ with curvature coupling parameter $\xi$ in the $D+1$-dimensional spacetime of a point-like global monopole. In the hyperspherical polar coordinates $(r, \theta, \phi) \equiv (r, \theta_1, \theta_2, \ldots, \theta_n, \phi)$, $n = D - 2$, the corresponding geometry is described by the line element

$$ds^2 = dt^2 - dr^2 - \sigma^2 r^2 d\Omega^2_D,$$

where $d\Omega^2_D$ is the line element on the surface of the unit sphere in $D$-dimensional Euclidean space, the parameter $\sigma$ is smaller than unity and is related to the symmetry breaking energy scale in the theory. In the spacetime given by line element (1) the solid angle deficit is $(1 - \sigma^2)S_D$, with $S_D = 2\pi^{D/2}/\Gamma(D/2)$ being the total area of the surface of the unit sphere in $D$-dimensional Euclidean space. The field equation has the form

$$\left(\nabla_i \nabla^i + m^2 + \xi R\right) \varphi = 0,$$

where $m$ is the mass for the field quanta, $\nabla_i$ is the covariant derivative operator associated with the metric given by line element (1), and

$$R = n(n+1)\frac{\sigma^2 - 1}{\sigma^2 r^2},$$

is the corresponding Ricci scalar (we adopt the convention of Birrell and Davies [10] for the curvature tensor). Note that for $\sigma \neq 1$ the geometry is singular at the origin (point-like monopole), $r = 0$. In (2) the values of the curvature coupling parameter $\xi = 0$, and $\xi = \xi_D$ with $\xi_D \equiv (D - 1)/4D$ correspond to the minimal and conformal couplings, respectively.

In this paper we are interested in the vacuum expectation values (VEVs) of the field bilinear products on background of the geometry described by (1), assuming that the field satisfies the
Robin boundary conditions
\[
\left( \hat{A}_r + \hat{B}_r \frac{\partial}{\partial r} \right) \varphi(x) = 0, \quad r = a, b, \tag{4}
\]
on two spheres with radii \(a\) and \(b\), \(a < b\), concentric with the monopole. Here the coefficients \(\hat{A}_r\) and \(\hat{B}_r\) are constants, in general, different for the inner and outer spheres. The imposition of this boundary condition on the quantum field \(\varphi(x)\) leads to the modification of the spectrum for the zero-point fluctuations and results in the shift in VEVs for physical quantities. In particular, vacuum forces arise acting on constraining boundary. This is the familiar Casimir effect. The VEVs for the physical quantities bilinear in the field can be evaluated if the corresponding Wightman function is known. For this reason we first concentrate on the positive frequency VEVs for the physical quantities. Bilinear in the field can be evaluated if the corresponding vacuum forces arise acting on the quantum field.

\[
\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \sum_{\alpha} \varphi_\alpha(x) \varphi^*_\alpha(x'), \tag{5}
\]
where \(\{\varphi_\alpha(x), \varphi^*_\alpha(x')\}\) is a complete orthonormal set of positive and negative frequency solutions to the field equation with quantum numbers \(\alpha\), satisfying boundary conditions (4). Note that for \(D = 1\) we have the standard Casimir-like geometry on Minkowski spacetime.

In the hyperspherical coordinates, for the region between two spheres the complete set of solutions to (2) with scalar curvature from (3), has the form
\[
\varphi_\alpha(x) = \beta_\alpha r^{-n/2} g_{\nu l}(\lambda \alpha, \lambda r) Y(l, \vartheta, \phi) e^{-i\omega t}, \quad l = 0, 1, 2, \ldots, \tag{6}
\]
where \(m_k = (m_0 \equiv l, m_1, \ldots, m_n)\), and \(m_1, m_2, \ldots, m_n\) are integers such that
\[
0 \leq m_{n-1} \leq m_{n-2} \leq \cdots \leq m_1 \leq l, \quad -m_{n-1} \leq m_n \leq m_{n-1}, \tag{7}
\]
\(Y(m_k; \vartheta, \phi)\) is the surface harmonic of degree \(l\) (see [40]). In (6)
\[
g_{\nu l}(\lambda \alpha, \lambda r) \equiv J_{\nu l}(\lambda \alpha) \bar{Y}_{\nu l}^{(a)}(\lambda a) - \bar{J}_{\nu l}^{(a)}(\lambda a) Y_{\nu l}(\lambda r), \quad \lambda = \sqrt{\omega^2 - m^2}, \tag{8}
\]
\(J_{\nu l}(z)\) and \(Y_{\nu l}(z)\) are the Bessel and Neumann functions, and the functions with overbars are defined in accordance with
\[
\bar{F}^{(a)}(z) \equiv A_\alpha F(z) + B_\alpha z^l F'(z), \quad A_\alpha = \hat{A}_\alpha - \hat{B}_\alpha n/2 \alpha, \quad B_\alpha = \hat{B}_\alpha / \alpha, \quad \alpha = a, b. \tag{9}
\]
Here and below we use the following notation
\[
\nu_l = \frac{1}{\sigma} \left[ (l + \frac{n}{2})^2 + (1 - \sigma^2) n(n + 1) (\xi - \xi_{D-1}) \right]^{1/2}, \tag{10}
\]
assuming that \(\nu_l^2\) is non-negative. This corresponds to the restriction on the values of the curvature coupling parameter for \(n > 0\), given by the condition
\[
\xi \geq -\frac{n \sigma^2}{4(n + 1)(1 - \sigma^2)}. \tag{11}
\]
This condition is satisfied by the most important special cases of the minimal and conformal couplings. The coefficients \(\beta_\alpha\) in (6) can be found from the normalization condition
\[
\int |\varphi_\alpha(x)|^2 \sqrt{-g} dV = \frac{1}{2\omega}, \tag{12}
\]
where the integration goes over the region between the spheres, \(a \leq r \leq b\). Substituting eigenfunctions (6), and using the relation

\[
\int |Y(m_k; \vartheta, \phi)|^2 d\Omega = N(m_k)
\]

(13)

(the explicit form for \(N(m_k)\) is given in [40] and will not be necessary for the following considerations in this paper) for the spherical harmonics and the value for the standard integral involving the square of a cylinder function [41], one finds

\[
\beta_a^2 = \frac{\pi^2 \lambda T_{\nu_1}(b/a, \lambda a)}{4N(m_k)\omega a \sigma^{D-1}},
\]

(14)

where we use the notation

\[
T_{\nu}^{ab}(\eta, z) = \left\{ \frac{\bar{J}_\nu^{(a)}(z)}{J_\nu^{(b)}(\eta z)} \right\} \left[ A_b^2 + B_b^2(\eta^2 z^2 - \nu^2) \right] - A_a^2 - B_a^2(z^2 - \nu^2) \right\}^{-1}, \quad \eta = \frac{b}{a}. \tag{15}
\]

The functions chosen in the form (8) satisfy the boundary condition on the sphere \(r = a\). From the boundary condition on \(r = b\) one obtains that the corresponding eigenmodes are solutions to the equation

\[
C_{\nu_i}^{ab}(b/a, \lambda a) \equiv \bar{J}_\nu^{(a)}(\lambda a)\bar{Y}_\nu^{(b)}(\lambda b) - \bar{J}_\nu^{(b)}(\lambda b)\bar{Y}_\nu^{(a)}(\lambda a) = 0. \tag{16}
\]

Below the roots to this equation will be denoted by \(\gamma_{\nu,k} = \lambda a\), \(k = 1, 2, \ldots\). The corresponding eigenfrequencies \(\omega = \omega_{\nu_i,k}\) are related to these zeros as \(\omega_{\nu_i,k} = \sqrt{\lambda_{\nu_i,k}^2/a^2 + m^2}\).

Substituting the eigenfunctions into the mode sum (5) and using the addition formula

\[
\sum_{m_k} \frac{1}{N(m_k)} Y(m_k; \vartheta, \phi)Y^*(m_k; \vartheta', \phi') = \frac{2l + n}{nS_D} C_l^{n/2}(\cos \theta), \tag{17}
\]

for the expectation value of the field product one finds

\[
\langle 0|\varphi(x)\varphi(x')|0 \rangle = \frac{\pi^2 (r^r')^{-n/2}}{4naS_D\sigma^{D-1}} \sum_{l=0}^{\infty} (2l + n)C_l^{n/2}(\cos \theta) \sum_{k=1}^{\infty} h(\gamma_{\nu,k})T_{\nu}^{ab}(b/a, \gamma_{\nu,k}), \tag{18}
\]

with the function

\[
h(z) = \frac{z}{\sqrt{z^2 + m^2}} g_\alpha(z, zr/a)g_\alpha(z, zr'/a)e^{i\sqrt{z^2 + m^2}(t'-t)}, \tag{19}
\]

and \(C_l^p(x)\) is the Gegenbauer or ultraspherical polynomial of degree \(l\) and order \(p\). To sum over \(k\) we will use the generalized Abel-Plana summation formula [38, 39, 33]

\[
\frac{\pi^2}{2} \sum_{k=1}^{\infty} h(\gamma_{\nu,k})T_{\nu}^{ab}(\eta, \gamma_{\nu,k}) = \int_0^\infty \frac{h(x)dx}{J_\nu^{(a)}(x)} - \frac{\pi}{4} \int_0^\infty \frac{K_\nu^{(a)}(x)K_\nu^{(b)}(x)\bar{I}_\nu^{(a)}(\eta x)\bar{I}_\nu^{(b)}(\eta x) - \bar{K}_\nu^{(b)}(\eta x)\bar{I}_\nu^{(a)}(x)}{K_\nu^{(a)}(x) - K_\nu^{(b)}(x)} dx, \tag{20}
\]

where \(I_\nu(x)\) and \(K_\nu(x)\) are the Bessel modified functions. The corresponding conditions for the function (19) are satisfied if \(r + r' + |t - t'| < 2b\). Note that this is the case in the coincidence limit for the region under consideration. Note that we have assumed values \(A_\alpha\) and \(B_\alpha\) for which all zeros for (16) are real. In case of existence of purely imaginary zeros we have to include
additional residue terms on the left of this formula (see [38, 39, 33]). Applying to the sum over \( k \) in (18) formula (20), for the corresponding Wightman function one obtains

\[
\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle - \frac{\sigma^{1-D}}{2\pi a S_D} \sum_{l=0}^{\infty} \frac{2l + n}{(r r') n/2} C_l^{n/2} (\cos \theta) \left\{ \int_0^{\infty} \frac{h(z) dz}{J_{l-1}^{(a)}(z) + J_{l+1}^{(a)}(z)} \right\},
\]

(21)

where we have introduced notations

\[
G_\nu^{(a)}(z, y) = I_\nu(y) K_\nu^{(a)}(z) - I_\nu^{(a)}(z) K_\nu(y), \quad \alpha = a, b.
\]

(22)

In the limit \( b \to \infty \) the second integral on the right of (21) tends to zero (for large \( b/a \) the subintegral is proportional to \( e^{-2b z/a} \)), whereas the first one does not depend on \( b \). It follows from here that the term with the first integral in the figure braces corresponds to the Wightman function for the region outside a single sphere with radius \( a \) on background of the global monopole geometry. As a result the Wightman function in the region between two spheres is presented in the form

\[
\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle - \frac{\sigma^{1-D}}{\pi a S_D} \sum_{l=0}^{\infty} \frac{2l + n}{(r r') n/2} C_l^{n/2} (\cos \theta) \times \int_0^{\infty} dz \frac{z \Omega_{\nu \nu}(az, bz)}{\sqrt{z^2 - m^2}} G_\nu^{(a)}(az, r z) G_\nu^{(a)}(az, r' z) \cosh \left( \sqrt{z^2 - m^2} (t' - t) \right),
\]

(23)

where

\[
\Omega_{\nu \nu}(az, bz) = \frac{\bar{\nu}^{(b)}(bz)/\nu^{(a)}(az)}{\bar{\nu}^{(b)}(bz)/\nu^{(a)}(az)},
\]

(24)

and \( \langle 0 | \varphi(x) \varphi(x') | 0 \rangle \) is the amplitude for the vacuum state in the case of a single sphere with radius \( a \). The Wightman function \( \langle 0 | \varphi(x) \varphi(x') | 0 \rangle \) is investigated in a previous paper [12], and below we will mainly concentrate on the terms induced by the presence of the second sphere. Note that in the coincidence limit, \( x' = x \), the second summand on the right hand side of (23) will give a finite result for \( a \leq r < b \), and is divergent on the boundary \( r = b \). In Ref. [12] the Wightman function for a single sphere in the global monopole spacetime is presented in the form

\[
\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \langle 0 | m | \varphi(x) \varphi(x') | 0 \rangle + \langle \varphi(x) \varphi(x') \rangle^a_b,
\]

(25)

where \( \langle 0 | m | \varphi(x) \varphi(x') | 0 \rangle \) is the amplitude for the vacuum state in the case of the boundary-free global monopole geometry. The expressions for the boundary induced part \( \langle \varphi(x) \varphi(x') \rangle^a_b \) for both regions inside and outside a single shell are given in [12].

Using the identity

\[
\Omega_{\nu \nu}(az, bz) G_\nu^{(a)}(az, rz) G_\nu^{(a)}(az, r' z) |_{\alpha = a}^{\alpha = b} = \frac{\bar{\nu}^{(a)}(az)}{\nu^{(a)}(az)} K_\nu(r z) K_\nu(r' z) - \frac{\bar{\nu}^{(b)}(bz)}{\nu^{(b)}(bz)} I_\nu(r z) I_\nu(r' z),
\]

(26)

it can be seen that for the case of two spheres the Wightman function in the intermediate region can also be presented in the form

\[
\langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle - \frac{\sigma^{1-D}}{\pi a S_D} \sum_{l=0}^{\infty} \frac{2l + n}{(r r') n/2} C_l^{n/2} (\cos \theta) \times \int_0^{\infty} dz \frac{z \Omega_{bz \nu}(az, bz)}{\sqrt{z^2 - m^2}} G_\nu^{(b)}(bz, rz) G_\nu^{(b)}(bz, r' z) \cosh \left( \sqrt{z^2 - m^2} (t' - t) \right),
\]

(27)
with \( \langle 0|\varphi(x)\varphi(x')|0 \rangle \) being the Wightman function for the vacuum inside a single sphere with radius \( b \), and

\[
\Omega_{b\nu}(az, bz) = \frac{\bar{I}^{(a)}_{\nu}(az)I^{(b)}_{\nu}(bz)}{K^{(a)}_{\nu}(az)\bar{K}^{(b)}_{\nu}(bz) - \bar{K}^{(a)}_{\nu}(bz)\bar{K}^{(b)}_{\nu}(az)}.
\]  

(28)

Note that formula (27) can be also derived by the procedure described above for (23), if in expression (6) for the eigenfunctions we replace the function \( g_{\nu}(\lambda a, \lambda r) \) (given by (8)) by the function \( J_{\nu}(\lambda r)Y^{(b)}_{\nu}(\lambda b) - J^{(b)}_{\nu}(\lambda b)Y_{\nu}(\lambda r) \). In the coincidence limit, the second summand on the right of formula (27) is finite for \( a < r < b \) and diverges on the boundary \( r = a \). It follows from here that if we write the regularized Wightman function in the form

\[
\langle 0|\varphi(x)\varphi(x')|0 \rangle = \langle 0_m|\varphi(x)\varphi(x')|0_m \rangle + \langle \varphi(x)\varphi(x') \rangle^{(a)}_b + \langle \varphi(x)\varphi(x') \rangle^{(b)}_b + \Delta W(x, x'),
\]  

(29)

then in the coincidence limit the "interference" term \( \Delta W(x, x') \) is finite for all values \( a < r < b \). Using the formula for the Wightman function for a single sphere (see [12]) and equation (23), it can be seen that this term may be presented as

\[
\Delta W(x, x') = -\frac{\sigma^{1-D}}{\pi n S_D} \sum_{l=0}^{\infty} \frac{2l + n}{(rr')^{n/2}} C_l^{n/2}(\cos \theta) \int_{m}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} W^{(ab)}(r, r') \cosh \left[ \sqrt{z^2 - m^2}(t' - t) \right],
\]  

(30)

where

\[
W^{(ab)}(r, r') = \frac{\bar{I}^{(a)}_{\nu}(az)\bar{K}^{(b)}_{\nu}(bz)}{I^{(b)}_{\nu}(bz)\bar{K}^{(a)}_{\nu}(az)} \left[ \frac{G^{(a)}_{\nu}(az, zr)G^{(b)}_{\nu}(bz, zr') - I^{(a)}_{\nu}(zr')K^{(b)}_{\nu}(zr)}{\bar{K}^{(a)}_{\nu}(az)\bar{I}^{(b)}_{\nu}(bz) - \bar{K}^{(b)}_{\nu}(bz)\bar{I}^{(a)}_{\nu}(az)} \right].
\]  

(31)

Formula (29) presents the Wightman function in the region between two spheres, \( a \leq r \leq b \). In the regions \( r \leq a \) and \( r \geq b \), the Wightman functions are given by the formulae corresponding to a single sphere with radius \( a \) and \( b \) respectively, and are investigated in [12].

3 Vacuum expectation value for the field square

The VEV of the field square is obtained computing the Wightman function in the coincidence limit \( x' \to x \). In the region between the spheres, from (23) and (27) we obtain two equivalent forms

\[
\langle 0|\varphi^2(x)|0 \rangle = \langle 0|\varphi^2(x)|0 \rangle^{(a)} - \frac{\sigma^{1-D}}{\pi n S_D r^n} \sum_{l=0}^{\infty} D_l \int_{m}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} G^{(a)}_{\nu}^2(az, rz),
\]  

(32)

with \( \alpha = a, b \), and

\[
D_l = (2l + D - 2) \frac{\Gamma(l + D - 2)}{\Gamma(D - 1) l!}
\]  

(33)

being the degeneracy of each angular mode with given \( l \), and \( \Gamma(x) \) is the gamma function. Using formula (29), the VEV for the field square can be also presented in the form

\[
\langle 0|\varphi^2(x)|0 \rangle = \langle 0_m|\varphi^2(x)|0_m \rangle + \langle \varphi^2(x) \rangle^{(a)}_b + \langle \varphi^2(x) \rangle^{(b)}_b + \langle \varphi^2(x) \rangle^{(ab)},
\]  

(34)

where \( \langle \varphi^2(x) \rangle^{(a)}_b \) is the VEV induced by a single sphere with radius \( a \), the "interference" term is given by the formula

\[
\langle \varphi^2(x) \rangle^{(ab)} = -\frac{\sigma^{1-D}}{\pi n S_D r^n} \sum_{l=0}^{\infty} D_l \int_{m}^{\infty} \frac{z dz}{\sqrt{z^2 - m^2}} \frac{\bar{I}^{(a)}_{\nu}(rz)\bar{K}^{(b)}_{\nu}(bz)}{\bar{I}^{(b)}_{\nu}(bz)\bar{K}^{(a)}_{\nu}(az)} \left[ \frac{I^{(a)}_{\nu}(az)K^{(b)}_{\nu}(bz) - K^{(a)}_{\nu}(az)I^{(b)}_{\nu}(bz)}{I^{(b)}_{\nu}(bz)\bar{K}^{(a)}_{\nu}(az)} \right],
\]  

(35)
and is finite for all values \( a \leq r \leq b \). In the case of the Dirichlet scalar, by using the relation \( I_\nu(x)K_\nu(y) > I_\nu(y)K_\nu(x) \) for \( y > x \), it can be seen that \( \langle \varphi^2(x) \rangle^{(ab)} > 0 \). Note that for this boundary condition the both boundary induced terms \( \langle \varphi^2(x) \rangle^{(a)}_a, \alpha = a, b, \) inside and outside of a single spherical shell are negative \([12]\).

Now let us consider the limiting cases of formula (35). In the limit \( a \to 0 \) the subintegrand behaves as \( a^{2\nu_1} \), and the dominant contribution to \( \langle \varphi^2(x) \rangle^{(ab)} \) comes from the summand with \( l = 0 \). Using the standard formulae for the Bessel modified functions \([42]\), noting that \( B_a = B_a/a \), and assuming \( \nu_0 > 0 \), to the leading order one has

\[
\langle \varphi^2(x) \rangle^{(ab)} \approx -\frac{\sigma^{1-D}}{\pi S_D r^n \nu_0 \Gamma^2(\nu_0)} \frac{n + 2\nu_0}{n + 2\nu_0} \left( \frac{n}{2} \right)^{2\nu_0} \int_0^\infty \frac{z^{2\nu_0+1} dz}{\sqrt{z^2 - m^2}} \frac{\tilde{K}_\nu^{(0)}(bz)}{I_\nu^{(0)}(bz)} \times I_\nu(rz) \left( \tilde{K}_\nu^{(b)}(b) I_\nu(rz) - 2K_\nu(rz) \right), \quad a \to 0, \tag{36}
\]

where we use the notation

\[
\eta_a = \begin{cases} 
1, & \text{for } \tilde{B}_a = 0 \\
-1, & \text{for } \tilde{B}_a \neq 0
\end{cases} \tag{37}
\]

For \( \nu_0 = 0 \) and \( a \to 0 \), the dominant contribution behaves as \( \langle \varphi^2(x) \rangle^{(ab)} \sim 1/\ln a \). Note that for a minimally coupled scalar (\( \xi = 0 \)) one has \( \nu_0 = D/2 - 1 \).

In the case of a massless scalar the asymptotic behavior of "interference" part (35) for large values \( b \) and fixed \( a \) and \( r \) can be obtained by changing the integration variable to \( y = zb \) and expanding the subintegrand in terms of \( a/b \) and \( r/b \). The dominant contribution for the summand with a given \( l \) has an order \( (a/b)^{2\nu_1} \) (assuming that \( A_a \neq B_a \nu_1 \)) and the main contribution comes from the \( l = 0 \) term. The leading term for the corresponding asymptotic expansion can be presented in the form

\[
\langle \varphi^2(x) \rangle^{(ab)} \approx -\frac{2\sigma^{1-D}}{\pi S_D a r^n \Gamma^2(\nu_0 + 1)} \left( \frac{a}{2b} \right)^{2\nu_0+1} \frac{A_a + B_a \nu_0}{A_a - B_a \nu_0} \left[ \frac{A_a + B_a \nu_0}{A_a - B_a \nu_0} \left( \frac{a}{r} \right)^{2\nu_0} - 2 \right] \times \int_0^\infty dz z^{2\nu_0} \frac{nK_\nu(rz) - 2\delta_0 \tilde{K}_\nu(z)}{nI_\nu(rz) - 2\delta_0 \tilde{I}_\nu(z)}, \tag{38}
\]

Hence, the "interference" part of the VEV of the square of the field operator vanishes in both limits \( a \to 0 \) and \( b \to \infty \).

Now we turn to the limit \( a, b \to \infty \) for fixed \( b-a \) and \( \sigma \). In this limit expression (35) diverges and the main contribution comes from large values of \( l \sim \sigma [2(1 - a/b)]^{-1} \). Introducing in (35) a new integration variable \( y = bz/\nu_1 \), we can replace the Bessel modified functions by their uniform asymptotic expansions for large values of the order \([42]\). To the leading order this gives

\[
\langle \varphi^2(x) \rangle^{(ab)} \approx -\frac{(b\sigma)^{1-D}}{\pi S_D \Gamma(D-1)} \sum_{l=0}^\infty l^{D-2} \int_{m_l}^\infty dy \frac{(y^2 - m^2)^{-1/2}}{c_b(y)c_a(y)} e^{2y/c_a(y) - 1} F(y, r), \tag{39}
\]

where we have introduced the notations

\[
m_l = (b - a) \sqrt{m^2 + (l/b\sigma)^2}, \quad c_a(y) = \frac{\tilde{A}_a + y\tilde{B}_a/(b-a)}{A_a - yB_a/(b-a)}, \quad \alpha = a, b, \tag{40}
\]

and

\[
F(y, r) = \frac{1}{c_b(y)} \exp \left( 2y \frac{r - b}{b - a} \right) + c_a(y) \exp \left( 2y \frac{a - r}{b - a} \right) - 2. \tag{41}
\]
In the limit under consideration in (39) we can make the replacement
\[ \sum_{l} l^{D-2} f \left( \frac{l(b - a)}{b \sigma} \right) \rightarrow \left( \frac{b \sigma}{b - a} \right)^{D-1} \int_{0}^{\infty} dt t^{D-2} f(t). \] (42)

Further, introducing instead of \( y \) a new integration variable \( u = \sqrt{y^2 - t^2 - \eta^2} \) and converting to polar coordinates on the plane \((u, t)\), the angular part of the resulting integral is easily evaluated. Using the standard relations for the gamma function, one receives
\[ \langle \varphi^2(x) \rangle^{(ab)} \approx -\frac{(b - a)^{1-D}}{(4\pi)^{D/2} \Gamma(D/2)} \int_{m_0}^{\infty} dy \frac{(y^2 - m_0^2)^{D/2-1}}{c_b(y) e^{-y^2/c_a(y)} - 1} F(y, r), \quad a, b \rightarrow \infty, b - a = \text{const}. \] (43)

This leading term of the corresponding asymptotic expansion does not depend on the parameter \( \sigma \) and coincides with the corresponding quantity for two parallel plates with the separation \( b - a \), on background of the Minkowski spacetime.

And finally, consider the “interference” term \( \langle \varphi^2(x) \rangle^{(ab)} \) in the limit of strong gravitational field, \( \sigma \ll 1 \), for fixed values \( a, b, r \). For \( \xi > 0 \), from (10) one has \( \nu_l \gg 1 \), and after introducing in (35) a new integration variable \( y = z/\nu_l \), we can replace the modified Bessel function by their uniform asymptotic expansions for large values of the order. The integral over \( y \) can be estimated by using the Laplace method. The main contribution to the sum over \( l \) comes from the summand with \( l = 0 \), and to the leading order we receive
\[ \langle \varphi^2(x) \rangle^{(ab)} \approx \frac{\sigma^{1-D} \eta_a \eta_b \exp[-2\tilde{\nu} \ln(b/a)]}{(2\pi \tilde{\nu})^{1/2} r^n S_D \sqrt{b^2 - a^2}}, \quad \tilde{\nu} = \frac{1}{\sigma \sqrt{n(n+1)\xi}}, \quad \sigma \ll 1. \] (44)

Note that, using the corresponding expressions for single sphere parts [12], we can see that in the limit under consideration
\[ \frac{\langle \varphi^2(x) \rangle^{(ab)}}{\langle \varphi^2(x) \rangle^{(ab)}_{(a)}} \approx -2\sqrt{\frac{\alpha^2 - r^2}{b^2 - a^2}} \frac{\eta_a \eta_b}{\eta_a} \exp[-2\tilde{\nu} \ln(\alpha/r)], \quad \alpha = a, b, \] (45)

and the “interference” term is suppressed compared to the single sphere contribution. For \( \xi = 0 \) and \( \sigma \ll 1 \) for the terms with \( l \neq 0 \) one has \( \nu_l \gg 1 \). The corresponding contribution can be estimated by the way similar to that in the previous case. This contribution is exponentially small. For the summand with \( l = 0 \) to the leading order over \( \sigma \) we have \( \nu_l = n/2 \) in (35), and \( \langle \varphi^2 \rangle \sim \sigma^{1-D} \). As we see, the behavior of the “interference” part in the VEV of the field square in the limit of the strong gravitational field is essentially different for minimally and non-minimally coupled scalars. This behavior is clearly seen from figure 1 where we have plotted the dependence of the “interference” term \( \langle \varphi^2(x) \rangle^{(ab)} \) on the ratio \( r/b \) in the cases of conformally \( (\xi = \xi_D, \text{left panel}) \) and minimally \( (\xi = 0, \text{right panel}) \) coupled massless Dirichlet scalars in \( D = 3 \) dimensions for \( a/b = 0.5 \). The separate curves correspond to the values \( \sigma = 1 \) (a), \( \sigma = 0.4 \) (b), \( \sigma = 0.2 \) (c).

4 Vacuum expectation values of the energy-momentum tensor and the interaction forces between the spheres

In this section we will consider the VEVs for the energy momentum tensor operator in the region between two spheres on background of the global monopole spacetime. Using the standard classical expression, we can write the vacuum energy-momentum tensor as the coincidence limit
\[ \langle 0 | T_{ik}(x) | 0 \rangle = \lim_{x' \rightarrow x} \partial_i \partial_{k} (0 | \varphi(x) \varphi(x') | 0) + \left[ (\xi - \frac{1}{4}) g_{ik} \nabla_i \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik} \right] \langle 0 | \varphi^2(x) | 0 \rangle, \] (46)
Figure 1: The "interference" part \( b^{D-1} \langle \varphi^2(x) \rangle^{(ab)} \) as a function on \( r/b \) in the cases of conformally (left panel) and minimally (right panel) coupled massless \( D = 3 \) Dirichlet scalars for \( a/b = 0.5 \). The curves are plotted for \( \sigma = 1 \) (a), \( \sigma = 0.4 \) (b), \( \sigma = 0.2 \) (c).

where for the point-like global monopole spacetime the nonzero components of the Ricci tensor are given by expressions

\[
R^2_2 = R^3_3 = \cdots = R^D_D = \frac{n \sigma^2 - 1}{\sigma^2 r^2},
\]

with the indices 2, 3, \ldots, \( D \) corresponding to the coordinates \( \theta_1, \theta_2, \ldots, \phi \) respectively. Substituting the Wightman function (23) into (46), we obtain that the vacuum energy-momentum tensor has the diagonal form (as expected by the symmetry of the model)

\[
\langle 0 | T^i_i | 0 \rangle = \text{diag} (\varepsilon, -p, -p_\perp, \ldots, -p_\perp),
\]

where the vacuum energy density \( \varepsilon \) and the effective pressures in radial, \( p \), and azimuthal, \( p_\perp \), directions are functions of the radial coordinate only. Using the Wightman function from (23) and the VEV for the field square from (32), the components of the vacuum energy-momentum tensor can be presented in the form

\[
q(a, b, r) = q(a, r) + q_a(a, b, r), \quad a < r < b, \quad q = \varepsilon, p, p_\perp,
\]

where \( q(a, r) \) are the corresponding functions for the vacuum outside a single sphere with radius \( a \). In (49) the additional components are in the form

\[
q_a(a, b, r) = -\frac{\sigma^{1-D}}{2\pi r^n S_D} \sum_{l=0}^\infty D_l \int_m^\infty dz \frac{z^3 \Omega_{anl}(az, bz)}{\sqrt{z^2 - m^2}} F^{(q)}_{v_l} \left[ G^{(a)}_{vl}(az, y), G^{(a)}_{vl}(az, y) \right]_{y=\sigma r},
\]

where for arbitrary functions \( f(y) \) and \( g(y) \) the functions \( F^{(q)}_{v_l} [f(y), g(y)] \) are defined by the relations

\[
F^{(\varepsilon)}_{v_l} [f(y), g(y)] = (1 - 4\xi) \left[ f'g' - \frac{n}{2y}(fg)' + \left( \frac{\nu^2}{y^2} - \frac{1 + 4\xi - 2(\nu r/y)^2}{1 - 4\xi} \right) fg \right]
\]

\[
F^{(p)}_{v_l} [f(y), g(y)] = f'g' + \frac{\xi}{2y} (fg)' - \left( 1 + \frac{\nu^2 + \xi n/2}{y^2} \right) fg, \quad \xi = 4(n+1)\xi - n
\]

\[
F^{(p_\perp)}_{v_l} [f(y), g(y)] = (4\xi - 1)f'g' - \frac{\xi}{2y} (fg)' + \left[ 4\xi - 1 + \frac{\nu^2 (1 + \xi) + \xi n/2}{(n+1)y^2} \right] fg,
\]
and the prime denotes the differentiation with respect to $y$. Quantities (50) are finite for $a \leq r < b$ and diverge at the surface $r = b$. Similar to the Wightman function, the components of the vacuum energy-momentum tensor for a single sphere case can be presented in the form

$$q(a, r) = q_m(r) + q_b(a, r),$$  \hspace{1cm} (54)

where $q_m(r)$ are the corresponding quantities for the boundary-free monopole geometry and the expressions for the sphere induced parts $q_b(a, r)$ are given in [12].

On the basis of formula (27), the vacuum energy-momentum tensor components may be written in another equivalent form:

$$q(a, b, r) = q(b, r) + q_b(a, b, r), \quad a < r < b, \quad q = \varepsilon, p, p_\perp,$$  \hspace{1cm} (55)

with $q(b, r)$ being the corresponding components for the vacuum inside a single sphere with radius $b$. Here the additional components are given by the formula

$$q_b(a, b, r) = -\frac{\sigma^{1-D}}{2\pi^D S_D} \sum_{l=0}^{\infty} D_l \int_{m}^{\infty} dz \frac{z^3 \Omega_{b\nu\lambda}(az, bz)}{\sqrt{z^2 - m^2}} F^{(q)}_{l\nu\lambda} \left[ G^{(b)}_{l\nu\lambda}(bz, y), G^{(b)}_{l\nu\lambda}(bz, y) \right]_{y=yr},$$  \hspace{1cm} (56)

This expressions are finite for all $a < r < b$ and diverge at the inner sphere surface $r = a$.

It follows from the above that if we present the vacuum energy-momentum tensor components in the form

$$q(a, b, r) = q_m(r) + q_b(a, r) + q_b(b, r) + \Delta q(a, b, r), \quad a < r < b,$$  \hspace{1cm} (58)

then the quantities

$$\Delta q(a, b, r) = q_a(a, b, r) - q(b, r) = q_b(a, b, r) - q(a, r)$$  \hspace{1cm} (59)

are finite for all $a \leq r \leq b$. Near the surface $r = a$ it is suitable to use the first equality in (59), as for $r \to a$ both summands are finite. For the same reason the second equality is suitable for calculations near the outer surface $r = b$. Using formula (30) for the corresponding part of the Wightman function, it can be seen that the following formula takes place for the "interference" parts

$$\Delta q(a, b, r) = \frac{\sigma^{1-D}}{2\pi^D S_D} \sum_{l=0}^{\infty} D_l \int_{m}^{\infty} dz \frac{z^3 \Omega_{b\nu\lambda}(az, bz)}{\sqrt{z^2 - m^2}} F^{(q)}_{l\nu\lambda} \left[ I^{(b)}_{l\nu\lambda}(bz, y), I^{(b)}_{l\nu\lambda}(bz, y) \right]_{y=yr}.$$  \hspace{1cm} (60)

It can be checked that this quantities satisfy the covariant continuity equation

$$r \frac{d\Delta p}{dr} + (D - 1)(\Delta p - \Delta p_\perp) = 0.$$  \hspace{1cm} (61)

Note that the ambiguities of the renormalization procedure for the VEV of the energy-momentum tensor in the form of an arbitrary mass scale (see [18]) are contained in the boundary-free parts $q_m(r)$ of the corresponding components. The boundary induced parts $q_b(\alpha, r)$, $\alpha = a, b$ and $\Delta q(a, b, r)$ are unambiguously defined for $a < r < b$. In particular, for the massless conformally coupled scalar they contain no conformal anomalies and are traceless.
Now we turn to the investigation of limiting cases of formula (60). In the limit $a \to 0$, the subintegrand behaves as $a^{2\nu_0}$ and the dominant contribution comes from the $l = 0$ term:

$$
\Delta q(a, b, r) \approx -\frac{\sigma^{-D}(a/2b)^{2\nu_0}}{2\pi S_D a r^{D/2} I_0^2(\nu_0)} \frac{n + 2\eta \nu_0}{n + 2\nu_0} \int_{mb}^\infty \frac{z^{2\nu_0+3} dz}{\sqrt{z^2 - m^2 b^2}} \tilde{K}_0^{(b)}(z)
\times \left\{ \frac{\tilde{K}_{(b)}(z)}{I_0^{(b)}(z)} F_q^{(b)}(I_0^{(b)}(y), I_0^{(b)}(y)) - 2F_q^{(b)}(I_0^{(b)}(y), K_0^{(b)}(y)) \right\}, \quad a \to 0, (62)
$$

assuming that $\nu_0 > 0$. For $\nu_0 = 0$ and $a \to 0$ the "interference" parts (60) behave as $1/\ln a$. Consider the limit $b \to \infty$ for fixed $a$ and $r$ in the case of a massless scalar field. By changing the integration variable to $y = bz$ and using the formula for the modified Bessel functions in the case of small values of the argument, we see that to the leading order the subintegrand in (60)

$$
\Delta q(a, b, r) \approx -\frac{\sigma^{-D}(a/2b)^{2\nu_0+1}}{\pi S_D a r^{D/2} I_0^2(\nu_0)} \frac{A_0 + B_0}{A_0 - B_0} \left[ (2\nu_0 + n) f_{1\nu_0}^{(q)} A_0 + B_0^{\nu_0} \left( \frac{a}{r} \right)^{2\nu_0} + 2f_{2\nu_0}^{(q)} \right]
\times \int_0^\infty dz z^{2\nu_0} \frac{nK_0(z)}{nI_0(z) - 2\delta_{0,0} z K_0(z)}, \quad b \to \infty, (63)
$$

with notations

$$
f_{1\nu}^{(e)} = \nu(1 - 4\xi), \quad f_{1\nu}^{(p)} = -\frac{\xi}{2}, \quad f_{1\nu}^{(p,\perp)} = \frac{\nu + 1/2}{n + 1} \xi, (64)
$$

$$
f_{2\nu}^{(e)} = 0, \quad f_{2\nu}^{(p)} = -n - 1, \quad f_{2\nu}^{(p,\perp)} = 4n(n + 1)\xi/\sigma^2. (65)
$$

As we see in both limits $a \to 0$ and $b \to \infty$ the "interference" parts for the energy-momentum tensor components vanish.

In the limit $a, b \to \infty$ for fixed values $b - a$ and $\sigma$, by the calculations similar to those for the "interference" part of the field square, one receives

$$
\Delta q(a, b, r) \approx -\frac{(b - a)^{-D-1}}{4\pi D/2 \Gamma(D/2 + 1)} \int_{m_0}^\infty dy \frac{(y^2 - m_0^2)^{D/2}}{c_b(y) c_a(y) y^{D/2}} \Delta F^q(y, r), (66)
$$

where

$$
\Delta F^e(y, r) = -\Delta F^{p,\perp}(y, r) = 1 - \frac{4D(\xi - \xi_D) y^2 - m_0^2}{2(y^2 - m_0^2)} [F(y, r) + 2], \quad \Delta F^p = 1, (67)
$$

with the function $F(y, r)$ defined by (41). These expressions are exactly the same as the corresponding expressions for the geometry of two parallel plates on the Minkowski background investigated in [32] (note that in [32], the notations $\tilde{B}_a/\tilde{A}_a = \beta_1$ and $\tilde{B}_b/\tilde{A}_b = -\beta_2$ are used) for a massless scalar and in Ref. [43] for the massive case.

And finally, let us consider the limit of the strong gravitational field, corresponding to $\sigma \ll 1$. In this limit for $\xi > 0$ one has $\nu_1 \gg 1$. After the change of variable to $y = bz/\nu_1$, we replace in (60) the Bessel modified functions by their uniform asymptotic expansions for large values of the order. The leading contribution comes from the $l = 0$ summand and we have the following limit for the "interference" parts of the vacuum energy-momentum components:

$$
\Delta p \approx -\frac{\Delta p_{1\perp}}{D - 1} \approx -\frac{\sigma^{-D} \eta_0 \eta_b \nu^{3/2}}{r^D S_D \sqrt{2\pi(b^2 - a^2)}} e^{-2\nu \ln(b/a)}, \quad \Delta \varepsilon/\Delta p \approx \sigma, (68)
$$

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where \( \tilde{\nu} \) is defined in (44). In this limit the "interference" parts are exponentially suppressed with respect to single spheres contributions.

Now we turn to the interaction forces between the spheres. The vacuum force acting per unit surface of the sphere \( r = \alpha, \alpha = a, b, \) is determined by the \( \frac{1}{r} \)–component of the vacuum energy-momentum tensor at this point. By virtue of relations (49) and (55), the corresponding effective pressures can be presented as a sum of the pressure for a single sphere with \( r = \alpha \) when the second sphere is absent and the pressure induced by the presence of the second sphere, \( p_\alpha(a, b, r = \alpha) \). The first term is divergent due to the well-known surface divergences and needs additional regularization. The second term is finite and can be termed as an interaction force between the spheres. This additional radial vacuum pressure on the sphere with \( r = \alpha, \alpha = a, b \) due to the existence of the second sphere can be found from (50) and (56), respectively. Using the relations

\[
G_\nu(r_z, r_z) = -B_r, \quad r_z \frac{\partial}{\partial y} G_\nu(r_z, y) \big|_{y=r_z} = A_r, \quad r = a, b,
\]

(69) they can be presented in the form

\[
p_\alpha(a, b, r = \alpha) = -\frac{\sigma^{1-D}}{2\pi \alpha^{D-1}} \sum_{l=0}^{\infty} D_l \int_0^\infty \frac{z dz}{\sqrt{z^2 - \alpha^2}} \Omega_{\alpha \nu_l}(az, bz) \times \left\{ \tilde{A}_\alpha^2 - 4(D-1)\xi \tilde{A}_\alpha B_\alpha - [z^2 \alpha^2 + \nu_l^2 - n^2/4] B_\alpha^2 \right\}, \quad \alpha = a, b.
\]

(70)

Unlike the self-action forces, these quantities are finite for \( a < b \) and need no further regularization. For the Dirichlet scalar one has \( \Omega_{\alpha \nu_l}(az, bz) > 0 \) and, hence, \( p_\alpha(a, b, r = \alpha) < 0 \). This means that in this case the vacuum interaction forces between the spheres are attractive. Using the Wronskian relation for the Bessel modified functions, it can be seen that

\[
[A_\alpha - B_\alpha(z^2 \alpha^2 + \nu^2)] \Omega_{\alpha \nu}(az, bz) = -n_\alpha \alpha \frac{\partial}{\partial \alpha} \ln \left[ 1 - \tilde{\nu}_\nu^{(a)}(az) \tilde{\nu}_\nu^{(b)}(bz) \right], \quad \alpha = a, b,
\]

(71)

where \( n_a = 1, n_b = -1 \). This allows us to write the expressions (70) for the interaction forces per unit surface in another equivalent form:

\[
p_\alpha(a, b, r = \alpha) = -\frac{n_\alpha \sigma^{1-D}}{2\pi \alpha^{D-1}} \sum_{l=0}^{\infty} D_l \int_0^\infty \frac{z dz}{\sqrt{z^2 - \alpha^2}} \left[ 1 - \tilde{\nu}_\nu^{(a)}(az) \tilde{\nu}_\nu^{(b)}(bz) \right] \times \frac{\partial}{\partial \alpha} \ln \left[ 1 - \tilde{\nu}_\nu^{(a)}(az) \tilde{\nu}_\nu^{(b)}(bz) \right], \quad \alpha = a, b,
\]

(72)

where \( \tilde{\xi} \) is defined in (52). For Dirichlet and Neumann scalars the second term in the square brackets is zero.

Let us consider the interaction forces between the spheres in limiting cases. For small values of the radius of the inner sphere, \( a \rightarrow 0 \), the leading contribution comes from the \( l = 0 \) summand. The corresponding terms are given by formulae

\[
p_a(a, b, r = a) \approx -\frac{\sigma^{1-D} a^{2\nu_0-D}}{2^{2\nu_0-1} \Gamma^2(\nu_0) \pi S_D} \frac{n + 2\eta_a \nu_0}{n + 2\nu_0} \int_0^\infty dz \frac{z^{2\nu_0+1}}{\sqrt{z^2 - \alpha^2}} \tilde{\nu}_\nu^{(b)}(bz),
\]

(73)

\[
p_b(a, b, r = b) \approx -\frac{\sigma^{1-D} a^{2\nu_0}}{2^{2\nu_0} \nu_0 \Gamma^2(\nu_0) \pi S_D b^{D-2}} \frac{n + 2\eta_b \nu_0}{n + 2\nu_0} \int_0^\infty dz \frac{z^{2\nu_0+1}}{\sqrt{z^2 - \alpha^2}} \tilde{\nu}_\nu^{(b)}(bz) \times \left\{ \tilde{A}_b^2 - 4(D-1)\xi \tilde{A}_b B_b - [z^2 b^2 + (1/\alpha^2 - 1)n(n+1)\xi] B_b^2 \right\},
\]

(74)
where $\eta_a$ is defined in (37).

For large values of the radius of the outer sphere, $b \to \infty$, in the case of a massless scalar to the leading order we find

$$p_a(a, b, r = a) \approx -\frac{2\sigma^{1-D}(A_a - B_a\nu_0)^2}{\pi a^D b S_D \Gamma^2(\nu_0)} \left( \frac{a}{2b} \right)^{2\nu_0} \left[ \tilde{A}_a^2 - 4(D - 1)\xi \tilde{A}_a B_a - (\nu_0^2 - n^2/4)B_a^2 \right]$$

$$\times \int_0^\infty dz z^{2\nu_0} nK_{\nu_0}(z) - 2\delta_{0,\lambda} K'_{\nu_0}(z)$$

$$p_b(a, b, r = b) \approx \frac{\sigma^{1-D}(a/2b)^{2\nu_0}}{\pi b^D a \Gamma^2(\nu_0) b^{D+1} S_D} \frac{A_a + B_a\nu_0}{A_a - B_a\nu_0} \int_0^\infty dz z^{2\nu_0} \frac{nI_{2\nu_0}(z)}{I_{2\nu_0}(z)}$$

(75)

In (76) we have assumed that $\tilde{A}_b \neq 0$. In the case $\tilde{A}_b = 0$ (Neumann boundary condition on the outer sphere) the subintegrand is equal to $-(z^2 + \nu_0^2 - n^2/4)z^{2\nu_0}[nI_{\nu_0}(z)/2 - zI'_{\nu_0}(z)]^{-2}$.

Now let us consider the limit $\sigma \ll 1$. For $\xi > 0$ one has $\nu_l \gg 1$ for all $l$, and to the leading order we have

$$p_{\alpha}(a, b, r = \alpha) \approx -\frac{\eta_a\eta_b\sigma^{1-D}\nu^{3/2}}{\sqrt{2\pi}a^D b S_D \sqrt{b^2 - a^2}} e^{-2\nu ln(b/a)}, \quad \alpha = a, b.$$  

(77)

In the case of a minimally coupled scalar, $\xi = 0$, the main contribution is due to the summand $l = 0$ with $\nu_l = n/2$ and is of an order $\sigma^{1-D}$. In this case the contributions coming from the $l > 0$ summands are exponentially suppressed.

As in the cases of the VEVs for the field square and energy-momentum tensor, the dependence of the interaction forces on the parameter $\sigma$ is essentially different for minimally and non-minimally coupled scalars. This feature is illustrated in figures 2 and 3 where we have plotted the dependence of the interaction forces $p_a(a, b, r = a)$ (left panel) and $p_b(a, b, r = b)$ (right panel) between the spheres on $a/b$ for conformally and minimally coupled $D = 3$ Dirichlet scalars on background of the global monopole background with $\sigma = 1$ (a), $\sigma = 0.4$ (b), $\sigma = 0.2$ (c).

![Figure 2](image_url)

Figure 2: The interaction forces between the spheres, $b^{D-1}p_a(a, b, r = a)$ (left panel) and $b^{D-1}p_b(a, b, r = b)$, (right panel) as functions on $a/b$ in the case of a conformally coupled massless $D = 3$ Dirichlet scalar on background of the global monopole geometry with $\sigma = 1$ (a), $\sigma = 0.4$ (b), $\sigma = 0.2$ (c).
Figure 3: The interaction forces between the spheres, $b^{D-1}p_a(a, b, r = a)$ (left panel) and $b^{D-1}p_b(a, b, r = b)$, (right panel) as functions on $a/b$ in the case of a minimally coupled massless $D = 3$ Dirichlet scalar on background of the global monopole geometry with $\sigma = 1$ (a), $\sigma = 0.4$ (b), $\sigma = 0.2$ (c).

5 Conclusion

In this paper, we present the quantum vacuum effects produced by two concentric spherical shells in the $D+1$-dimensional point-like global monopole spacetime defined by the line element (1). The case of a massive scalar field with general curvature coupling parameter and satisfying the Robin boundary conditions on the spheres is considered. To derive formulae for the vacuum expectation values of the square of the field operator and the energy-momentum tensor, we first construct the positive frequency Wightmann function. This function is also important in considerations of the response of a particle detector at a given state of motion through the vacuum under consideration [10]. The application of the generalized Abel-Plana formula to the mode sum over zeros of the combinations of the cylinder functions allows us to extract the part due to a single sphere on background of the global monopole geometry, formula (29). In this formula the "interference" part $\Delta W(x, x')$, given by formula (30), is finite in the coincidence limit for all values $a \leq r \leq b$. The VEV of the square of the scalar field, $\langle 0|\varphi^2(x)|0 \rangle$, is given by the evaluation of the Wightman function in the coincidence limit. The expectation values for the energy-momentum tensor are obtained by applying on the corresponding Wightman function a certain second-order differential operator and taking the coincidence limit. In both cases the expectation values can be presented as a sum of boundary-free global monopole, single sphere induced and "interference" terms. The VEVs related to the spacetime of a point-like global monopole without boundaries are considered in [17, 18, 19, 20], and the effects produced by a single sphere are investigated in a previous paper [12]. Note that for the points not lying on the spheres, the boundary induced terms are unambiguously defined and the ambiguities in the renormalization procedure in the form of an arbitrary mass scale are contained in the boundary-free parts only. In particular, for the massless conformally coupled scalar the boundary induced vacuum energy-momentum tensor contains no conformal anomalies and is traceless. In this paper we concentrate on the "interference" parts in the local Casimir densities and interaction forces between the spheres. The application of the generalized Abel-Plana formula allows us to derive closed expressions for these quantities, given by formula (35) in the case of the field square, and by formula (60) for the energy-momentum tensor. As bulk divergences are contained in the part corresponding to the global monopole geometry without boundaries, and the surface divergences are contained in the single sphere parts, the "interference" parts are finite for all
values $a \leq r \leq b$. In particular, the integrals in the corresponding formulae are exponentially convergent and they are useful for numerical evaluations. We have considered various limiting cases of the formulae for the "interference" parts. In the limits $a \to 0$ or $b \to \infty$ for a fixed value of $r$, these parts vanish as $a^{2\nu_0}$ and $b^{-(2\nu_0+1)}$ respectively, where $\nu_0$ is defined by relation (10) with $l = 0$. In the limit $a, b \to \infty$ for a fixed value $b - a$, the leading terms in the boundary produced parts do not depend on the parameter $\sigma$ associated with the solid angle deficit, and are exactly the same as the corresponding quantities for the geometry of two parallel Robin plates on the Minkowski background. We have also investigated the limit of strong gravitational fields corresponding to small values of the parameter $\sigma$, $\sigma \ll 1$. In this limit the behaviour of the Casimir densities is drastically different for minimally ($\xi = 0$) and non-minimally ($\xi \neq 0$) coupled scalars. In the minimal coupling case the leading terms of the corresponding asymptotic expansions for both field square and the energy-momentum tensor VEVs behave as $\sigma^{1-D}$. For a non-minimally coupled scalar, the "interference" parts behave as $\langle \varphi^2 \rangle^{(ab)} \sim \sigma^{3/2-D} \exp(-\gamma/\sigma)$ and $\Delta p \sim \Delta p_{\perp} \sim \Delta \varepsilon/\sigma \sim \sigma^{-D-1/2} \exp(-\gamma/\sigma)$, with $\gamma = 2\sqrt{n(n+1)}\sigma \ln(b/a)$. Note that in this case the "interference" parts are exponentially suppressed with respect to single sphere contributions. The vacuum forces acting on spheres contain two terms. The first ones are the forces acting on a single sphere then the second boundary is absent. Due to the well-known surface divergences in the VEV's of the energy-momentum tensor these forces are infinite and need an additional regularization. The another terms in the vacuum forces are finite and are induced by the presence of the second boundary. They correspond to the interaction forces per unit surface between the spheres and are determined by formula (70). For the Dirichlet scalar these forces are always attractive. In the limit of the strong gravitational field, $\sigma \ll 1$, for the minimally coupled scalar field the interactions forces behave as $\sigma^{1-D}$, whereas for a non-minimally coupled scalar they are exponentially small, $p_\alpha(a, b, r = a) \sim \sigma^{-D-1/2} \exp(-\gamma/\sigma)$.

**Acknowledgement**

We acknowledge support from the Research Project of the Kurdistan University. The work of AAS was supported in part by the Armenian Ministry of Education and Science (Grant No. 0887).

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