Domains of Holomorphy

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Dedicated to my friend Professor Paul M. Gauthier

Abstract

We give a simple proof that the notions of Domain of Holomorphy and Weak Domain of Holomorphy are equivalent. This proof is based on a combination of Baire’s Category Theorem and Montel’s Theorem. We also obtain generalizations by demanding that the non-extensible functions belong to a particular class of functions $X = X(\Omega) \subset H(\Omega)$. We show that the set of non-extensible functions not only contains a $G_\delta$-dense subset of $X(\Omega)$, but it is itself a $G_\delta$-dense set. We give an example of a domain in $\mathbb{C}$ which is a $H(\Omega)$-domain of holomorphy but not a $A(\Omega)$-domain of holomorphy.

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1 Introduction

It is well-known that the notions of domain of holomorphy and of weak-domain of holomorphy are equivalent. The original proof is constructive, technical and by no means elementary ([9]). A simpler proof was obtained in [7] by a combination of Baire’s theorem and a theorem of Banach. Furthermore in [7] it was proven that the set of non-extensible functions contains a $G_\delta$-dense subset of $H(\Omega)$ of holomorphic functions on a domain $\Omega$, or more generally in a space $X(\Omega) \subset H(\Omega)$ satisfying some assumptions. The use of Banach’s theorem does not allow to conclude that the set of non-extensible functions is itself a $G_\delta$-dense subset of $X(\Omega)$; in fact the sets appearing in Banach’s theorem can be very high in Borel’s hierarchy ([10], [3]).
In the present paper we replace Banach’s theorem by Montel’s theorem and combining it with Baire’s theorem we obtain a new, very simple proof of complex analytic nature of a slightly stronger result; that is, we prove that the set of non-extendable functions is itself $G_\delta$-dense and not only that it contains a $G_\delta$-dense set.

At present, $\Omega$ is a domain in the finite dimensional space $\mathbb{C}^d$. In future papers we will discuss the infinite dimensional care. We mention that in [8] Montel’s theorem was used to treat the case $X(\Omega) = H(\Omega)$, where $\Omega$ is a domain in a separable Banach space, even infinite dimensional. For finite dimensional holomorphy Montel’s theorem towards generic results has also been used in the works of Paul M. Gauthier; see for instance [5].

Some generic results for particular choices of $X(\Omega)$ have already been obtained in [1], [2], [6].

It is well-known that in $\mathbb{C}$ every domain is a $H(\Omega)$-domain of holomorphy. We give an example of a domain in $\mathbb{C}$ which is a $H(\Omega)$-domain of holomorphy but it is not a $A(\Omega)$-domain of holomorphy, where $A(\Omega) = \{f : \overline{\Omega} \rightarrow \mathbb{C}, \text{continuous on } \overline{\Omega} \text{ and holomorphic in } \Omega\}$. Such an example is the domain $\Omega = \{z \in \mathbb{C} : |z| < 1, z \notin [0,1/2]\}$. Our proof (in several variables) implies in particular that if $\Omega \subset \mathbb{C}^d$ is not a $X(\Omega)$-domain of holomorphy, then there exist two balls $B_1, B_2, B_2 \cap \Omega \neq \emptyset, B_2 \cap \Omega^c \neq \emptyset, B_1 \subset \overline{B}_1 \subset B_2 \cap \Omega$ such that every $f \in X(\Omega)$ restricted to $B_1$ admits a bounded holomorphic extension in $B_2$.

Possible particular choices of $X(\Omega)$ are the spaces $A^p(\Omega), H^\infty_p(\Omega), p \in \{\infty\} \cup \{0,1,2,\ldots\}$, the Bergman spaces $B_p(\Omega), 0 < p < +\infty$, the Hardy spaces and variations or combinations of them in one or several complex variables. Here $A^p(\Omega) = \{f : \Omega \rightarrow \mathbb{C}, \text{ holomorphic, such that the derivative } f^{(\ell)} \text{ extends continuously in } \overline{\Omega} \text{ for all } \ell \in \{0,1,2,\ldots\}, \ell \leq p \}$ and $H^\infty_p(\Omega) = \{f : \Omega \rightarrow \mathbb{C}, \text{ holomorphic such that each derivative } f^{(\ell)} \text{ is bounded on } \Omega \text{ for all } \ell \in \{0,1,2,\ldots\}, \ell \leq p \}$. In the case of $X = A^p(\Omega)$, under some assumptions, there is a relation with the $p$-continuous analytic capacity $\alpha_p$ introduced in [2]. Similarly for the case $X = H^\infty_p(\Omega)$, I believe that it is possible to define an analogous notion of $p$-analytic capacity $\gamma_p$ which relates to our situation. It suffices to replace the space $A^p(\Omega)$ by the space $H^\infty_p(\Omega)$ in the definition of $\alpha_p$. 

2
2 Preliminaries

In [9] we find the following definition.

**Definition 2.1.** Let $\Omega \subset \mathbb{C}^d$ be open and connected and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, $f$ is called extendable if there exist an open and connected set $U \subset \mathbb{C}^d$ with $U \cap \Omega \neq \emptyset$ and $U \cap \Omega^c \neq \emptyset$, a holomorphic function $F : U \to \mathbb{C}$ and a component $V$ of $U \cap \Omega$ such that $F|_V = f|_V$. Otherwise $f$ is called non-extendable.

The following definition comes from the theory of analytic continuation mainly in one complex variable.

**Definition 2.2.** Let $\Omega \subset \mathbb{C}^d$ be open and connected and $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, $f$ is called extendable in the sense of Riemann domains, if there exist two open Euclidean balls $B_1, B_2 \subset \mathbb{C}^d$, with $B_1 \subset B_2 \subset B_2 \cap \Omega$, $B_2 \cap \Omega \neq \emptyset$, $B_2 \cap \Omega^c \neq \emptyset$, and a bounded holomorphic function $F : B_2 \to \mathbb{C}$ such that $F|_{B_1} = f|_{B_1}$. Otherwise the function $f$ is called non-extendable in the sense of Riemann domains.

**Proposition 2.3.** Definitions 2.1 and 2.2 are equivalent.

**Proof.** Suppose that $f$ satisfies Definition 2.2. Set $U = B_2$ and let $V$ the component of $U \cap \Omega$ containing $B_1$. Since $f|_{B_1} = F|_{B_1}$, by analytic continuation it follows $f|_V = F|_V$. Thus, Definition 2.1 is also satisfied.

Conversely, suppose that $f$ satisfies Definition 2.1. We claim that $\overline{V} \cap \partial \Omega \neq \emptyset$. Let $z_1 \in V \cap \Omega \subset U \cap \Omega$ and $z_2 \in U \cap \Omega^c$. Since $U \subset \mathbb{C}^d$ is open and connected, there is a polygonal line $\Gamma$ in $U$ joining $z_1$ and $z_2$. Since $z_1 \in V$ and $z_2 \notin V$ it follows that $\Gamma$ meets $\partial V$. Let $w \in \Gamma \cap \partial V$. Since $\Gamma \subset U$ it follows easily that $w \in \partial \Omega$. Thus, $w \in \partial V \cap \partial \Omega \subset \overline{V} \cap \partial \Omega$ and the claim is proven.

Since $w \in U$ there is $r > 0$, so that $B(w, r) \subset U$, where $B(w, r)$ denotes the Euclidian ball centered at $w$ with radius $r$. We set $B_2 = B\left(w, \frac{r}{2}\right)$. Then $F$ is holomorphic and bounded on $B_2$. Since $w \in \overline{V}$ it follows that $B_2 \cap V \neq \emptyset$ and let $z \in B_2 \cap V$. Then there is $\delta > 0$ so that $B(z, \delta) \subset B_2 \cap V$. We set $B_1 = B\left(z, \frac{\delta}{2}\right)$. Thus, $B_1 \subset \overline{B_1} \subset B_2 \cap V \subset B_2 \cap \Omega$, $F|_{B_1} = f|_{B_1}$ and $F$ is holomorphic and bounded on $B_2$. Thus, $f$ satisfies Definition 2.2. ■
Remark. Often a function which is non extendable is called nowhere extendable, and a function which is extendable is called somewhere extendable.

3 The result

Let $\Omega \subset \mathbb{C}^d$ be open and connected and $X = X(\Omega)$ let be a set of holomorphic functions $f : \Omega \to \mathbb{C}$; that is, $X \subset H(\Omega)$.

Definition 3.1. The open connected set $\Omega \subset \mathbb{C}^d$ is called an $X$-domain of holomorphy if there exists $f \in X$ which is non-extendable.

Definition 3.2. The open connected set $\Omega \subset \mathbb{C}^d$ is called weak $X$-domain of holomorphy if for every pair of open Euclidean balls $B_1, B_2$ with $B_2 \cap \Omega \neq \emptyset$, $B_2 \cap \Omega^c \neq \emptyset$, $B_1 \subset \overline{B_1} \subset B_2 \cap \Omega$ there exists a function $f_{B_1,B_2} \in X$ such that the restriction of $f_{B_1,B_2}$ on $B_1$ does not have any bounded holomorphic extension on $B_2$.

Theorem 3.3. We suppose that $X = X(\Omega) \subset H(\Omega)$ is a topological vector space endowed with the usual operations $+, \cdot$ and that its topology is induced by a complete metric. We also suppose that the convergence $f_n \to f$ in $X$ implies the pointwise convergence $f_n(z) \to f(z)$ for all $z \in \Omega$. Then definitions 3.1 and 3.2 are equivalent. If the above assumptions hold and $\Omega$ satisfies definitions 3.1 and 3.2, then the set $\{f \in X : f \text{ is non-extendable}\}$ is a dense and $G_\delta$ subset of $X$.

Proof. It is obvious that Definition 3.1 implies Definition 3.2; it suffices to set $f_{B_1,B_2} = f$. In order to prove the rest it suffices to assume that $\Omega$ satisfies Definition 3.2 and prove that the set $A = \{f \in X : f \text{ is non-extendable}\}$ is dense and $G_\delta$ in $X$. Equivalently, it suffices to show that $A^c$ is a denumerable union of closed sets in $X$ with empty interiors.

Consider the set of couples $(B_1, B_2)$ of open Euclidean balls such that $B_2 \cap \Omega \neq \emptyset$, $B_2 \cap \Omega^c \neq \emptyset$, $B_1 \subset \overline{B_1} \subset B_2 \cap \Omega$ and the centers of $B_1, B_2$ belong to $Q^{2d}$ and the radii of $B_1, B_2$ belong to $(0, +\infty) \cap Q$, where the set $Q$ denotes the set of rational numbers. This set $Y$ is denumerable.

It is easy to see that

$$A^c = \bigcup_{(B_1,B_2) \in Y} \bigcup_{M \in \{1, 2, \ldots\}} \{f \in X : \exists F \text{ holomorphic on } B_2 \text{ and bounded by } M \text{ so that } F|_{B_1} = f|_{B_1}\}.$$
Thus, $A^c$ is a denumerable union of sets of the form $T(B_1, B_2, M) = \{ f \in X : \exists F$ holomorphic on $B_2$ bounded by $M$ such that $F|_{B_1} = f|_{B_1} \}$. By Baire’s Theorem it suffices to prove that for any fixed choice of $B_1, B_2$ and $M$ the set $T(B_1, B_2, M)$ is closed in $X$ and its interior is empty.

Suppose $f_n \in T(B_1, B_2, M)$ and $f_n \to f$ in the topology of $X$, where $f \in X$. For each $n \in \{1, 2, \ldots\}$ there exists a holomorphic function $F_n$ on $B_2$ bounded by $M$ such that $F_n|_{B_1} = f_n|_{B_1}$. By Montel’s theorem [4] a subsequence $F_{k_n}$ of $F_n$ converges uniformly on compact subsets of $B_2$ towards a function $F$ holomorphic on $B_2$ and bounded by $M$.

Since the convergence $f_n \to X$ in the topology of $X$ implies pointwise converge in $\Omega$ by assumption, it follows that $f|_{B_1} = \lim_n f_n|_{B_1} = \lim_n F_{k_n}|_{B_1} = \lim_n F_{k_n}|_{B_1} = F|_{B_1}$, where $f \in X$ and $F$ is holomorphic in $B_2$ and bounded by $M$. Thus, $f \in T(B_1, B_2, M)$. This proves that $T(B_1, B_2, M)$ is closed in $X$.

Finally, we shall show that the interior of $T(B_1, B_2, M)$ in $X$ is void. Assume that $T(B_1, B_2, M)^0 \ni f$ to arrive at a contradiction. By assumption there exists a function $f_{B_1, B_2} \in X$ such that its restriction on $B_1$ does not have any bounded holomorphic extension on $B_2$. Since $f + \frac{1}{n} f_{B_1, B_2} \to f$ in the topology of $X$ and $f$ is in the interior of $T(B_1, B_2, M)$ it follows that for some $n \in \{1, 2, \ldots\}$ the function $f + \frac{1}{n} f_{B_1, B_2}$ belongs to $T(B_1, B_2, M)$. The same holds for the function $f$. Thus, both functions restricted to $B_1$, admit holomorphic extensions on $B_2$ bounded by $M$. Thus, their difference $\frac{1}{n} f_{B_1, B_2}$ restricted to $B_1$ admits a holomorphic extension on $B_2$ bounded by $2M$. It follows that the function $f_{B_1, B_2}$ restricted to $B_1$ admits a holomorphic extension on $B_2$ bounded by $2nM$. This contradicts the fact that $f_{B_1, B_2}|_{B_1}$ does not admit any bounded holomorphic extension on $B_2$. Thus, $T(B_1, B_2, M)^0 = \emptyset$ and the proof is complete. \[\square\]

4 Remarks

If $X = H(\Omega)$ we have the notions of Domain of holomorphy and Weak Domain of holomorphy. Their equivalence in this case is well known [9], [7]. Our proof is very simple.

It is also known that for $d = 1$ every domain $\Omega \subset \mathbb{C}$ is a domain of holomorphy. It is very simple to see that it is a weak domain of holomorphy; it suffices to consider the function $\frac{1}{z - \zeta}$ for each $\zeta \in \partial \Omega$ [9], [7].
For $d = 1$ the domain $\Omega = \{ z \in \mathbb{C} : |z| < 1, z \notin [0, 1] \}$ is an $H(\Omega)$-domain of holomorphy but not a $A(\Omega)$-domain of holomorphy, where $A(\Omega) = \{ f : \overline{\Omega} \rightarrow \mathbb{C},$ continuous an $\overline{\Omega}$ and holomorphic in $\Omega \}$. This follows easily by Morera’s theorem, because if $E$ is a straight line and $\mathcal{U} \subset \mathbb{C}$ an open set, every function continuous on $\mathcal{U}$ and holomorphic in $\mathcal{U} - E$ is holomorphic in $\mathcal{U}$ [11].

Examples of sets $X = X(\Omega)$ satisfying the assumptions of Theorem 3.3 (and therefore, its conclusion, as well) are the following in one complex variable.

$A(\Omega), A^p(\Omega) = \{ f \in H(\Omega) : f^{(\ell)} \text{ extends continuously on } \overline{\Omega} \text{ for all } \ell \in \mathbb{N}, \ell \leq p \},$  $p \in \{ \infty \} \cup \{ 0, 1, 2, \ldots \}$ $H^\infty_p(\Omega) = \{ f \in H(\Omega) : f^{(\ell)} \text{ is bounded in } \Omega \text{ for all } \ell \in \mathbb{N}, \ell \in p \}$ and mixtures of them, where $N = \{ 0, 1, 2, \ldots \}$. Also the Bergman and Hardy spaces and variants of them.

Let $L \subset \mathbb{C}$ be a compact set satisfying some assumptions. We set $\Omega = \mathbb{C} - L$. Then the fact that $\Omega$ is an $A^p(\Omega)$-domain of holomorphy or not, relates to the $p$-continuous analytic capacity $\alpha_p(L)$ [2]. The analogous question for $H^\infty_p(\Omega)$ should relate to the $p$-analytic capacity $\gamma_p(L)$ which, I believe, can be defined in a similar way with the definition of $\alpha_p(L)$; it suffices to replace the space $A^p(\mathbb{C} \setminus L)$ by the space $H^\infty_p(\mathbb{C} \setminus L)$.

In several variables one can examine the situation for the analogous spaces $A^p(\Omega), H^\infty_p(\Omega)$, Bergman and Hardy spaces and variants of them.

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