Abstract. In this paper, we propose a variational approach based on optimal transportation to study the existence and uniqueness of solution for a class of parabolic equations involving $q(x)$-Laplacian operator
\[
\frac{\partial \rho(t, x)}{\partial t} = \operatorname{div}v_x \left( \rho(t, x) \left| \nabla_x G'(\rho(t, x)) \right|^{q(x)-2} \nabla_x G'(\rho(t, x)) \right).
\]
The variational approach requires the setting of new tools such as appropriate distance on the probability space and an introduction of a Finsler metric in this space. The class of parabolic equations is derived as the flow of a gradient with respect the Finsler structure. For $q(x) \equiv q$ constant, we recover some known results existing in the literature for the $q$-Laplacian operator.

Key words: Finsler metric, variable exponent Lebesgue and Sobolev spaces, $q(x)$-Laplacian operator, $p(.)$-Wasserstein metric, gradient flow.

Mathematics Subject Classification: 35K57, 57K61, 58J35

1. Introduction

The purpose of this paper is twofold: Show that the Monge-Kantorovich problem associated with the Lagrangian $L(x, v) = |v|^{p(x)}$ induces a distance $W_{p(.)}$ and a Finsler metric $F_{p(.)}$ on the space of probability measures $P(\Omega)$ such that the induced distance function of $F_{p(.)}$ is equivalent to $W_{p(.)}$ where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is convex, and $p(.) : \Omega \rightarrow [1, +\infty[$, a variable exponent function.

Next, establish the existence of solutions for the following class of parabolic evolution equations involving the variable exponent $q(x)$ - operator.

\[
(1.1) \quad \frac{\partial \rho(t, x)}{\partial t} = \operatorname{div}v_x \left( a(t, x) \left| \nabla_x G'(\rho(t, x)) \right|^{q(x)-2} \nabla_x G'(\rho(t, x)) \right), \quad (t, x) \in [0, +\infty[ \times \Omega
\]
\[
\rho(0, x) = \rho_0(x) \quad \text{in} \quad \Omega,
\]

where $\Omega \subset \mathbb{R}^N$, is a bounded domain with smooth boundary $\partial \Omega$: $G : [0, +\infty[ \rightarrow \mathbb{R}$ is a convex function of class $C^2$; $a : [0, +\infty[ \times \Omega \rightarrow [0, +\infty[$ is a weight function and $q : \Omega \rightarrow [1, +\infty[$ is a bounded measurable function in $\Omega$, satisfying $\frac{d}{d\rho} a + \frac{\alpha}{\rho} = 1$. Here, the initial datum $\rho_0$ is a positive measurable function.

Finsler structure on the space of probability measures has been considered recently in [1] in the case of the constant exponent ($p(x) = p$). Our work generalizes the work of [1] and moreover we derive from the Finsler metric $F_{p(.)}$, the differential and gradient for functionals defined on the space of probability measures and then the gradient flows. Particularly, we show that the parabolic $q(x)$-Laplacian equation (1.1) is a gradient flows of the functional $E(\rho) = \int_{\Omega} G(\rho) dx$ in Finslerian manifold $(P(\Omega), F_{p(.)})$ and in the Wasserstein space $(P(\Omega), W_{p(.)})$.

Equation (1.1) presents some great interests since it is involved in the modeling of the evolution of many nonhomogeneous materials such as electrorational fluids, elastic mechanics, flow in porous media and image processing [5, 17, 19, 23].

Not too many works have been devoted to the study of parabolic equations involving variable exponent operator. In relation with our work, we can recall the works in [22] and in [24]. In [24], the authors established the existence and the uniqueness of weak solution of (1.1) in the case $a(t, x) = d^\alpha(x)$ and $G(t) = \frac{t^2}{2}$, where $\alpha > 0$ and $d(x) = \operatorname{dist}(x, \partial \Omega)$ is the distance function from the boundary. The technique used in their work, consist in approximating problem (1.1) by some regularized problems under the following assumptions: $\rho_0 \in L^\infty(\Omega)$, $d(x)^\alpha \left| \nabla \rho_0 \right|^{q^*} \in L^1(\Omega)$ and $q \equiv q(x)$ is continuous with $q^* := \max_{\Omega} q(x)$. Using similar variational approach as in [24], Huashui Zhan [22] studied existence and uniqueness of solutions of (1.1), when $G(t) = \frac{t^2}{2}$; $a(t, x) \equiv a(x)$ and the initial datum $\rho_0$ satisfies the following assumptions

\[
\alpha > 0 \quad \text{and} \quad \rho_0 \in L^\infty(\Omega).
\]
(H1) : \( a \in L^1(\Omega); a^{-\frac{1}{\theta(x)-1}} \in L^1(\Omega); a^{-s(x)} \in L^1(\Omega) \), with \( s(x) \in \left[ \frac{N}{\theta(x)}, +\infty \right] \) if \( \theta(x)-1 > 0 \), and \( +\infty \) otherwise.

(H2) : \( \rho_0 \in L^\infty(\Omega) \) and \( \rho_0 \in W^{1,q(x)}(\Omega) \).

Here, \( W^{1,q(x)}(\Omega) \) denote the variable exponent weight Sobolev space.

In this paper, we propose an approach based on optimal transportation, to study existence and uniqueness of solutions of \((1.1)\). From the best of our knowledge, this method is new and requires less regularity on the initial datum \( \rho_0 \) and on the variable exponent \( q \equiv q(x) \) than those imposed by the authors in \([22]\) and in \([24]\).

Optimal transportation method on the space of probability measures have been extensively used last two decades to investigate solution for parabolic partial differential equations of the form \((1.1)\), when \( q = q(x) \) being constant; see for instance the works in \([2, 13]\). In \([13]\), Jordan, Kinderlehrer and Otto have studied existence of solutions of the equation \((1.1)\) in the particular case where: \( G(t) = \frac{t^q}{q} \), \( a(t, x) \equiv 1 \) and \( q(x) \equiv 2 \), that is the heat equation:

\[
\frac{\partial \rho(t,x)}{\partial t} = \Delta_x \rho(t,x), \quad \text{in} \quad [0, +\infty[ \times \mathbb{R}^N.
\]

To do so, they use a descent algorithm in the probability space to construct the approximate solutions of \((1.2)\).

The descent of this algorithm is governed by the \( 2 \)-Wasserstein distance \((1.4) \), \( \partial \rho \) and \( \rho \) are continuous on \( \Omega \), then the dual problem of \((1.5) \):

\[
\inf_{\rho_0 \in L^\infty(\Omega)} \int_\Omega |x - T(x)| p(x) \rho_1 dx
\]

is still unknown. However, if the variable exponent \( p(x) \) is measurable, then the Kantorovitch problem associated to \((1.4) \):

\[
\inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y| p(x) d\gamma(x,y), \quad \gamma \in \Pi(\rho_1, \rho_2)
\]

admits a solution \( \gamma_0 \). Furthermore, if \( p(x) \) is continuous on \( \Omega \), then the dual problem of \((1.5) \):

\[
\sup_{u \oplus v \leq |x - y| p(x)} \left[ \int_\Omega u \rho_1 dx + \int_\Omega v \rho_2 dy \right]
\]

admits a solution \( (u, v) \) which satisfies the equality: \( u \oplus v = |x - y| p(x) \) on the support of \( \gamma_0 \).

We then use Kantorovich’s formulation to define the approximate solutions of the problem \((1.1)\).

This paper is organised as follows. Section 2 is devoted to the preliminary tools useful throughout the paper and section 3 deals the statement of the Wasserstein distance with variable exponent. In section 4, we define the Finsler structure, in section 5 we give the definition of the gradient flow in Finslerian space, and in section 6 our existence and unicity results are stated.

2. Preliminaries

2.1. Main assumptions. Throughout this work, we will assume the following:

- \((A_1)\) \( p(.) : \Omega \rightarrow [1, +\infty[ \) is measurable function such that:
  
  \[
  1 < p^* := \text{ess inf } p(x) \leq \text{ess sup } p(x) := p^+ < +\infty.
  \]

- \((A_2)\) \( G : [0, +\infty[ \) is convex function such that \( G(0) = 0 \) and \( G \in C^2([0, +\infty[) \).

To prove a maximum principle for the solutions of \((1.1)\), we assume that the initial datum \( \rho_0 \) is a probability density on \( \Omega \) and satisfies:

- \((A_3)\) \( \rho_0 \leq M_2 \), for some \( 0 < M_2 \).
(A₄) To prove the uniqueness of solutions of (1.1), we assume that
\[\inf_{t \in [0, M_2]} G'(t) > 0.\]

2.2. **Lebesgue-Sobolev spaces with variable exponent.** We recall in this section some definitions and fundamental properties of the Lebesgue and Sobolev space with variable exponents.

**Definition 2.1.** Let \( \rho \) be a probability measure on \( \Omega \), and \( p(\cdot) : \Omega \to ]1, +\infty[ \) a measurable function. We denote by \( L^{p(\cdot)}_\rho(\Omega) \) the Lebesgue space with variable exponent defined by:

\[
L^{p(\cdot)}_\rho(\Omega) := \left\{ u : \Omega \to \mathbb{R}; \int_\Omega \left| \frac{u(x)}{\rho(x)} \right|^{p(x)} \rho(x) \, dx \leq 1, \text{ for some } \lambda > 0 \right\},
\]

with the norm

\[
\|u\|_{L^{p(\cdot)}_\rho(\Omega)} = \inf \left\{ \lambda > 0, \int_\Omega \left| \frac{u(x)}{\rho(x)} \right|^{p(x)} \rho(x) \, dx \leq 1 \right\},
\]

for all \( u \in L^{p(\cdot)}_\rho(\Omega) \).

We denote by \( W^{1,p(\cdot)}_\rho(\Omega) \) the Sobolev space with variable exponent defined by

\[
W^{1,p(\cdot)}_\rho(\Omega) := \left\{ u \in L^{p(\cdot)}_\rho(\Omega), \ \nabla u \in L^{p(\cdot)}_\rho(\Omega) \right\}
\]

equipped with the norm

\[
\|u\|_{W^{1,p(\cdot)}_\rho(\Omega)} := \|u\|_{L^{p(\cdot)}_\rho(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}_\rho(\Omega)}
\]

It is well known from the work in [8] that \( L^{p(\cdot)}_\rho(\Omega) \) and \( W^{1,p(\cdot)}_\rho(\Omega) \) are Banach spaces respectively with the norms (2.1) and (2.3).

We denote by \( q(\cdot) : \Omega \to ]1, +\infty[ \) the conjugate function of \( p(\cdot) \) which is defined by

\[q(x) = \frac{p(x)}{p(x) - 1}, \ \text{for almost all } x \in \Omega.\]

**Proposition 2.1.** (Hölder inequality,[15]). Let \( \rho \in P(\Omega) \) and \( p(\cdot), q(\cdot) : \Omega \to ]1, +\infty[ \) two measurable functions such that \( \frac{1}{p(x)} + \frac{1}{q(x)} = 1 \), for all \( x \in \Omega \).

If \( u \in L^{p(\cdot)}_\rho(\Omega) \) and \( v \in L^{q(\cdot)}_\rho(\Omega) \), then:

\[
\int_\Omega |u(x)v(x)|\rho(x) \, dx \leq \left( \frac{1}{p} + \frac{1}{q} \right)\|u\|_{L^{p(\cdot)}_\rho(\Omega)}\|v\|_{L^{q(\cdot)}_\rho(\Omega)}.
\]

Furthermore, if \( p_1(\cdot), p_2(\cdot), p_3(\cdot) : \Omega \to ]1, +\infty[ \) are measurable functions such that \( \frac{1}{p_1(x)} = \frac{1}{p_2(x)} + \frac{1}{p_3(x)} \), for almost all \( x \in \Omega \), we have

\[
\|uv\|_{L^{p_1(\cdot)}_\rho(\Omega)} \leq 2\|u\|_{L^{p_2(\cdot)}_\rho(\Omega)}\|v\|_{L^{p_3(\cdot)}_\rho(\Omega)},
\]

for \( u \in L^{p_1(\cdot)}_\rho(\Omega) \) and \( v \in L^{p_3(\cdot)}_\rho(\Omega) \).

**Proposition 2.2.** (see [8]). Let \( \rho \in P(\Omega) \) and \( p_1(\cdot), p_2(\cdot) : \Omega \to ]1, +\infty[ \) two measurable functions such that \( p_1(x) \leq p_2(x) \) on \( \Omega \). Then, we have the following continuous injection:

\[
L^{p_2(\cdot)}_\rho(\Omega) \hookrightarrow L^{p_1(\cdot)}_\rho(\Omega).
\]

Furthermore,

\[
\|u\|_{L^{p_1(\cdot)}_\rho(\Omega)} \leq 2\|u\|_{L^{p_2(\cdot)}_\rho(\Omega)}
\]

**Theorem 2.1** ([8]). Assume that the variable exponent \( p(\cdot) : \Omega \to ]1, +\infty[ \) satisfies \( 1 < p^- \leq p(x) \leq p^+ < +\infty \) for almost all \( x \in \Omega \); with \( p^- = \text{ess inf} \ p(x) \), \( p^+ = \text{ess sup} \ p(x) \). Then the Banach spaces \( L^{p(\cdot)}_\rho(\Omega) \) and \( W^{1,p(\cdot)}_\rho(\Omega) \) are separable, reflexive and uniformly convex.
2.3. BV functions. In this section, we present some essential properties of bounded total variation functions.

**Definition 2.2.** Let \( \Omega \subset \mathbb{R}^N \) be an open subset of \( \mathbb{R}^N \) and \( u \in L^1(\Omega) \). The total variation of \( u \) in \( \Omega \) is defined by

\[
|D(u)| := \sup \left\{ \int_{\Omega} u \text{div}(\phi) \, dx, \quad \phi \in C^1_\text{c}(\Omega, \mathbb{R}^N), \ |\phi|_{\infty} \leq 1 \right\}
\]

If \( |D(u)| < \infty \), then we say that \( u \) has a bounded total variation. Let denote by \( BV(\Omega) \) the set of all functions having bounded total variation.

**Theorem 2.2.** The BV space equipped with the standard norm \( \|u\|_{BV} = |Du| + \|u\|_{L^1(\Omega)} \) is a Banach space. Moreover, the following injection

\[
BV(\Omega) \hookrightarrow L^r(\Omega), \quad \text{is continuous if } 1 \leq r \leq \frac{N}{N-1}
\]

and compact if \( 1 \leq r < \frac{N}{N-1}, \ N > 1 \)

2.4. Generalities on Finsler manifolds. Let \( M \) be a manifold, and denote by \( T_xM \) the tangent space at \( x \in M \) and by \( TM := \bigcup_{x \in M} \{ x \} \times T_xM \) the tangent bundle of \( M \), that is the set of all pairs \( (x, v) \in M \times T_xM \). A Finsler metric on \( M \) is a function \( F : TM \to [0, \infty) \) such that

(i) Positivity: \( F(x, v) > 0 \) for all \( x \in M \) and \( v \neq 0 \in T_xM \);

(ii) Positive homogeneity: \( F(x, \lambda v) = \lambda F(x, v) \) for all \( \lambda > 0 \), \( x \in M \) and \( v \in T_xM \);

(iii) Strong convexity: \( F(x, v + w) \leq F(x, v) + F(x, w) \) for all \( x \in M \) and \( v, w \in T_xM \), with equality (when \( v, w \neq 0 \)) if and only if \( v = \lambda w \) for some \( \lambda > 0 \).

In the framework of a general setting condition (iii) in the definition of a Finsler metric on a differentiable manifold, is replaced by the following more general condition:

(iii)' : For all \( x \in M \), and for all \( w \in T_xM \setminus \{0\} \), the symmetric bilinear form

\[
g_{x,w} : T_xM \times T_xM \to \mathbb{R}
\]

\[
(u, v) \mapsto \frac{1}{2} \left. \frac{\partial^2 F^2(x, w + tu + sv)}{\partial t \partial s} \right|_{t=s=0}
\]

is positive definite. For the convenience of our work, we use (i), (ii) and (iii) in our definition.

2.5. Duality mapping. Let \( X \) be a Banach space, \( X^* \) its dual and \( <, > \) is the duality between \( X^* \) and \( X \). As in [6], we defined the duality mapping between \( X^* \) and \( X \) corresponding to the normalization function \( \theta \) and the properties related.

**Definition 2.3.** A continuous function \( \theta : [0, +\infty] \to [0, +\infty] \) is called a normalization function if it is strictly increasing, \( \theta(0) = 0 \) and \( \theta(t) \to \infty \) if \( t \to \infty \).

**Definition 2.4.** Let \( \theta : [0, +\infty] \to [0, +\infty] \) be a normalization function. Let \( X \) be a Banach space and \( X^* \) its dual. The duality mapping corresponding to the normalization function \( \theta \) is the set values operator

\[
J_\theta : X \to P_a(X^*)
\]

defined by \( J_\theta(0) = 0 \) and

\[
J_\theta(x) := \{ x^* \in X^*, \quad <x^*, x> = \theta(\|x\|)\|x\|, \quad \|x^*\| = \theta(\|x\|) \} , \quad \text{if } x \neq 0.
\]

Here \( P_a(X^*) \) is the set of the subsets of \( X^* \). A such duality mapping exists because of the Hahn Banach homomorphism theorem.

**Theorem 2.3** ( Cf [6], [10]). Consider the normalization function \( \theta : [0, +\infty] \to [0, +\infty] \). Assume that \( X \) is reflexive, locally uniformly convex. Then the following assertions hold:

(i) \( \text{Dom}(J_\theta) = X \)

(ii) \( J_\theta \) is bijective.

(iii) Moreover if \( X^* \) is reflexive, \( \text{Card}(J_\theta(x)) = 1 \), for all \( x \in X \).

The proof of the theorem above can be founded in [6].
3. Wasserstein metric $W_{p(\cdot)}$

In this section, we define a distance on the space of the probability measures and then study its topology. This distance induces in the probability space, a metric space structure in which we obtain the solutions of the equation (1.1) as a gradient flow.

**Definition 3.1.** (Wasserstein metric $W_{p(\cdot)}$)

Let $p(\cdot) : \Omega \rightarrow [1, +\infty[$ a measurable function and $\rho_0, \rho_1$ two probability measures on $\Omega$. We define the Wasserstein distance $W_{p(\cdot)}$ between $\rho_0$ and $\rho_1$ by:

\[
W_{p(\cdot)}(\rho_0, \rho_1) := \inf_{\rho(t, \cdot)} \left\{ \|v(\cdot, \cdot)\|_{L_p^p([0,1] \times \Omega)}^p, \quad \frac{\partial \rho(t, x)}{\partial t} + \text{div}_x(\rho(t, x)v(t, x)) = 0, \quad \rho(0) = \rho_0, \quad \rho(1) = \rho_1 \right\},
\]

with

\[
\|v(\cdot, \cdot)\|_{L_p^p([0,1] \times \Omega)}^p := \inf \left\{ \Lambda > 0, \quad \int_{[0,1] \times \Omega} \frac{\|v(t, x)\|^p}{\Lambda} \rho(t, x)dxdt \leq 1 \right\}.
\]

In the following, we show that $W_{p(\cdot)}$ is a distance on $P(\Omega)$ and we study its topology.

**Lemma 3.1.** Let $p_1(\cdot), p_2(\cdot) : \Omega \rightarrow [1, +\infty[$ two mesurables functions such that $p_1(x) \leq p_2(x)$, for almost all $x \in \Omega$. Then:

\[
W_{p_1(\cdot)}(\rho_1, \rho_2) \leq 2W_{p_2(\cdot)}(\rho_1, \rho_2),
\]

for all $\rho_1, \rho_2 \in P(\Omega)$.

**Proof 3.1.** Let $\rho(t, \cdot) : [0,1] \rightarrow P(\Omega)$ be an arbitrary curve which is admissible for $W_{p_2(\cdot)}(\rho_1, \rho_2)$ and $v_t : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a velocity field in $\mathbb{R}^N$ satisfying $v_t \in L_p^{p_2(\cdot)}(\Omega)$, for all $t \geq 0$ and $\frac{\partial \rho(t, \cdot)}{\partial t} + \text{div}(\rho v_t) = 0$.

Then, $t \mapsto \rho(t, \cdot)$ is admissible for $W_{p_1(\cdot)}(\rho_1, \rho_2)$. Moreover,

\[
W_{p_1(\cdot)}(\rho_1, \rho_2) \leq \|v_t\|_{L_p^{p_1(\cdot)}([0,1] \times \Omega)} \leq 2\|v_t\|_{L_p^{p_2(\cdot)}([0,1] \times \Omega)}
\]

Then, we get

\[
W_{p_1(\cdot)}(\rho_1, \rho_2) \leq 2W_{p_2(\cdot)}(\rho_1, \rho_2)
\]

by taking the infimum. 

By using (3.1), we establish the following

**Theorem 3.1.** Let $p(\cdot) : \Omega \rightarrow [1, +\infty[$ be a mesurable function. Then $W_{p(\cdot)}$ is a distance on $P(\Omega)$. Furthermore, If $\{\rho_n\} \subset P(\Omega)$ is a sequence in $P(\Omega)$ and $\rho \in P(\Omega)$:

\[
W_{p(\cdot)}(\rho_n, \rho) \xrightarrow{n \to \infty} 0 \iff \rho_n \xrightarrow{n \to \infty} \rho \text{ narrowly},
\]

that is, $(\rho_n)_n$ converges to $\rho$ in the metric space $(P(\Omega), W_{p(\cdot)})$, if and only if $(\rho_n)_n$ converges narrowly to $\rho$ in $P(\Omega)$.

**Proof 3.2.** Let $\rho_0, \rho_1 \in P(\Omega)$ and $r > 1$, the Monge problem

\[
(M) : \inf_{T_{\rho_0 \rho_1}} \left\{ \int_{\Omega} |x - T(x)|^r \rho_0 dx \right\}
\]

admits a unique solution $T(x) = x - |\nabla \phi|^r \nabla \phi$, where $\phi$ is $C^1$ and $\frac{|x|^r}{r}$- convex, ie

\[
\phi(x) = \inf_{y \in \mathbb{R}^N} \left\{ \frac{|x - y|^r}{r} - \psi(y) \right\}
\]

with $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$, (see [1]).

Furthermore, if $t \in [0,1]$ and $T_t = (1-t)id + tT$ is Mc Cann’s interpolation, then the curve $\rho(t, \cdot) : t \in [0,1] \rightarrow T_t \rho_0$ is the unique (constant-speed) geodesic joining $\rho_0$ and $\rho_1$ in the $r$-Wasserstein space $(P(\Omega), W_r)$, (see [1]), and satisfies $\frac{\partial \rho}{\partial t} + \text{div}_x(\rho(T - id)) = 0$; $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$. 

5
Also, we have \( \int_{\Omega} \left| \frac{x - T(x)}{\lambda} \right|^{p(x)} \rho dx < +\infty \) for \( \lambda = \text{diam} (\Omega) \).

Assume that \( W_{p(\cdot)}(\rho_0, \rho_1) = 0 \). We use lemma (3.1) and we have
\[
W_1(\rho_0, \rho_1) \leq 2W_{p(\cdot)}(\rho_0, \rho_1).
\]
Then
\[
W_{p(\cdot)}(\rho_0, \rho_1) = 0 \implies W_1(\rho_0, \rho_1) = 0 \implies \rho_0 = \rho_1,
\]
so, we have
\[
W_1(\rho_0, \rho_1) = \inf_{T, \rho_0 = \rho_1} \left\{ \int_{\Omega} |x - T(x)|\rho_0 dx \right\}
\]
see [2].

To prove the axiom of symmetry, let \( \rho_0, \rho_1 \in P(\Omega) \) and \( \rho(t, \cdot) \), an arbitrary curve in \( P(\Omega) \) such that \( \rho(0) = \rho_0 \) and \( \rho(1) = \rho_1 \) and \( v_t : [0, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) a vector field in \( \mathbb{R}^N \) satisfying:
\[
v_t \in L^{p(t)}([0, 1] \times \Omega) \quad \text{and} \quad \frac{\partial \rho(t)}{\partial t} + \text{div}_x (\rho(t)v(t)) = 0.
\]

The reverse curve \( t \mapsto \bar{\rho}(t) := \rho(1 - t) \) of \( t \mapsto \rho(t, \cdot) \) satisfies: \( \bar{\rho}(0) = \rho_1 \) and \( \bar{\rho}(1) = \rho_0 \) and the velocity fields \( \bar{v}(t) : [0, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) defined by \( \bar{v}(t) = v(1 - t) \) satisfies
\[
\frac{\partial \bar{\rho}(t)}{\partial t} + \text{div}_x (\bar{\rho}(t)\bar{v}(t)) = 0.
\]

So, we have
\[
\|\bar{v}(t)\|_{L^{p(t)}([0, 1] \times \Omega)} = \|v(t)\|_{L^{p(t)}([0, 1] \times \Omega)}
\]
and next
\[
W_{p(t)}(\rho_0, \rho_1) \leq \|\bar{v}(t)\|_{L^{p(t)}([0, 1] \times \Omega)} = \|v(t)\|_{L^{p(t)}([0, 1] \times \Omega)}.
\]

Since \( \rho(t, \cdot) \) is arbitrary, we get
\[
W_{p(t)}(\rho_0, \rho_1) \leq W_{p(t)}(\rho_0, \rho_1)
\]
and similarly,
\[
W_{p(t)}(\rho_0, \rho_1) \leq W_{p(t)}(\rho_1, \rho_0),
\]
This concludes the symmetry axiom.

Let's now show the axiom of triangular inequality.

Let \( \rho_0, \rho_1 \) and \( \rho_2 \) three elements of \( P(\Omega) \) and \( \rho_1(t), \rho_2(t) \) two arbitrary curves of \( P(\Omega) \) such that: \( \rho_1(0) = \rho_0 \), \( \rho_1(1) = \rho_1 \), \( \rho_2(0) = \rho_1 \) and \( \rho_2(1) = \rho_2 \). Let \( v_1, v_2 : [0, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) two velocity fields in \( \mathbb{R}^N \) satisfying respectively
\[
\int_{[0, 1] \times \Omega} [\frac{\alpha}{p(x)}]^{p(x)} \rho_1^\alpha dt dx < +\infty, \quad \text{for some } \alpha > 0,
\]
and
\[
\int_{[0, 1] \times \Omega} \frac{v_1^2}{\beta} [p(x) \rho_2(t)] dt dx < +\infty, \quad \text{for some } \beta > 0,
\]
Consider the curve \( t \mapsto \bar{\rho}(t) \) defined by
\[
\begin{cases}
\bar{\rho}(t) = \rho_1(2t), & \text{if } t \in [0, \frac{1}{2}] \\
\bar{\rho}(t) = \rho_2(2t - 1), & \text{if } t \in [\frac{1}{2}, 1]
\end{cases}
\]
We have \( \bar{\rho}(0) = \rho_1(0) = \rho_0 \) and \( \bar{\rho}(1) = \rho_2(1) = \rho_2 \) and the velocity fields \( t \mapsto \bar{v}(t) \) defined by
\[
\begin{cases}
\bar{v}(t) = v_1(2t), & \text{if } t \in [0, \frac{1}{2}] \\
\bar{v}(t) = v_2(2t - 1), & \text{if } t \in [\frac{1}{2}, 1]
\end{cases}
\]
satisfies
\[ \frac{\partial \rho(t)}{\partial t} + \text{div}_x(\rho(\tilde{t},.)\tilde{v}(t)) = 0. \]

Furthermore
\[ \|\tilde{v}(t)\|_{L^p(\cdot)}([0, 1] \times \Omega) \leq \|v^1(t)\|_{L^p(\cdot)}([0, 1] \times \Omega) + \|v^2(t)\|_{L^p(\cdot)}([0, 1] \times \Omega). \]

We have that
\[ W_{p,\cdot}(\rho_0, \rho_2) \leq \|\tilde{v}(t)\|_{L^p(\cdot)}([0, 1] \times \Omega) \leq \|v^1(t)\|_{L^p(\cdot)}([0, 1] \times \Omega) + \|v^2(t)\|_{L^p(\cdot)}([0, 1] \times \Omega); \]

since the curves \( t \mapsto \rho_1(t) \) and \( t \mapsto \rho_2(t) \) are arbitrary, we obtain by taking the infimum that
\[ W_{p,\cdot}(\rho_0, \rho_2) \leq W_{p,\cdot}(\rho_0, \rho_1) + W_{p,\cdot}(\rho_1, \rho_2). \]

Hence, \( P_{p,\cdot}(\Omega) = (P(\Omega), W_{p,\cdot}) \) is a metric space with the Wasserstein distance \( W_{p,\cdot} \).

Let’s now study the topology of \( P_{p,\cdot}(\Omega) \).

Let \( \rho \in P(\Omega) \) and \( (\rho_n)_n \) be a sequence in \( P(\Omega) \) such that \( W_{p,\cdot}(\rho_n, \rho) \xrightarrow{n \to \infty} 0 \).

By using (3.11), we have
\[ W_1(\rho_n, \rho) \leq 2W_{p,\cdot}(\rho_n, \rho), \quad \text{and} \quad W_{p,\cdot}(\rho_n, \rho) \leq W_\infty(\rho_n, \rho), \]
for all \( n \in \mathbb{N} \), with
\[ W_\infty(\rho_1, \rho_2) = \inf \left\{ \|x - y\|_{L^\infty(\Omega \times \Omega)} : \gamma \in \Pi(\rho_1, \rho_2) \right\}. \]

Note that, \( W_\infty \) is Wasserstein metric defined in [3].

Using (3.12), we have
\[ W_{p,\cdot}(\rho_n, \rho) \xrightarrow{n \to \infty} 0 \quad \Rightarrow \quad W_1(\rho_n, \rho) \xrightarrow{n \to \infty} 0 \quad \Rightarrow \quad \rho_n \xrightarrow{n \to \infty} \rho \quad \text{narrowly}, \]

because \( W_1 \) induced the narrow topology on \( P(\Omega) \).

Inversely, if \( \rho_n \xrightarrow{n \to \infty} \rho \quad \text{narrowly} \), we have, by using (3.12)
\[ W_\infty(\rho_n, \rho) \xrightarrow{n \to \infty} 0 \quad \Rightarrow \quad W_{p,\cdot}(\rho_n, \rho) \xrightarrow{n \to \infty} 0. \]

\[ \square \]

4. Finsler metric on Wasserstein space \((P(\Omega), W_{p,\cdot})\)

In this section, we show that the space of probabilities measures can be endowed with a Finsler metric \( F_{p,\cdot} \) whose associated Minkowski norm is that of the Banach quotient space \( L^p(\Omega)/N_p \), with \( N_p \) the kernel of a linear and continuous application \( \Phi \) defined on \( L^p(\Omega) \) with values in the space of distributions. First of all, we specify the tangent space of \( P(\Omega) \) at the point \( \rho \) as well as its topological properties.

4.1. Tangent space \( T_pP_{p,\cdot}(\Omega) \) and its topology. In this part, we give a definition of the tangent space of the Wasserstein space \( P_{p,\cdot}(\Omega) = (P(\Omega), W_{p,\cdot}) \) at the point \( \rho \) and we study its topological properties later. It is known from the work of Ambrozie [3] that if \( r > 1 \) is a real number and if \( \rho(t,.) : [0, 1] \to P(\Omega) \) is a lipschitzian curve in the \( r \)-Wasserstein space \((P(\Omega), W_r)\), then there is a velocity field \( v(t) : \mathbb{R}^N \to \mathbb{R}^N \) in \( L^p_{p(t,.)}(\Omega) \) such that:
\[ \frac{\partial \rho}{\partial t} + \text{div}_x(\rho(t,.)v_1) = 0. \]

Using the continuous injection \( L^p_{p+}(\Omega) \hookrightarrow L^p_{p,\cdot}(\Omega) \), we obtain a similar result in the Wasserstein space \((P(\Omega), W_{p,\cdot})\)

**Lemma 4.1.** Assume that \( p(.) : \Omega \to [1, \infty[ \) satisfy \((A_1)\). Let \( \rho(t,.) : [0, 1] \to P_{p,\cdot}(\Omega) \) be a lipschitzian curve in \((P(\Omega), W_{p,\cdot})\). Then there is a velocity field \( v(t) : \mathbb{R}^N \to \mathbb{R}^N \) in \( L^p_{p(t,.)}(\Omega) \) such that
\[ \frac{\partial \rho}{\partial t} + \text{div}_x(\rho(t,.)v_1) = 0. \]

Using (4.11), we define the tangent space as follows
Lemma 4.2. Assume that \( p(\cdot): \Omega \rightarrow]1, +\infty[ \) be a variable exponent and \( \rho \in P_{p(\cdot)}(\Omega) \). We define the tangent space \( T_{\rho}P_{p(\cdot)}(\Omega) \) at \( \rho \), of the Wasserstein space \( P_{p(\cdot)}(\Omega) = (P(\Omega), W_{p(\cdot)}) \) as:

\[
T_{\rho}P_{p(\cdot)}(\Omega) := \{ \nu := -\text{div}(\rho v), \quad v \in L_{p(\cdot)}^1(\Omega) \}
\]

Proof 4.1. Define

\[
\Phi: L_{p(\cdot)}^1(\Omega) \rightarrow D'(\Omega)
\]

\[
v \mapsto -\text{div}(\rho v)
\]

The map \( \Phi \) is linear and continuous on \( L_{p(\cdot)}^1(\Omega) \). Then its kernel

\[
N_{\Phi} := \{ v \in L_{p(\cdot)}^1(\Omega) : -\text{div}(\rho v) = 0 \}
\]

is a closed subset of \( L_{p(\cdot)}^1(\Omega) \). Since \( L_{p(\cdot)}^1(\Omega) \) is reflexive and uniformly convex Banach space (see [16]), then the quotient space \( L_{p(\cdot)}^1(\Omega)/N_{\Phi} \) is reflexive and uniformly convex Banach space, with the norm

\[
\| \bar{v} \|_{L_{p(\cdot)}^1(\Omega)/N_{\Phi}} := \inf_{w \in N_{\Phi}} \{ \| v + w \|_{L_{p(\cdot)}^1(\Omega)} \},
\]

for all \( \bar{v} \in L_{p(\cdot)}^1(\Omega)/N_{\Phi} \).

The Tangent space \( X = T_{\rho}P_{p(\cdot)}(\Omega) \) can be identified to the image of \( \Phi \) via the isomorphism:

\[
L_{p(\cdot)}^1(\Omega)/N_{\Phi} \ni \bar{v} \mapsto -\text{div}(\rho v) \in T_{\rho}P_{p(\cdot)}(\Omega).
\]

Then \( T_{\rho}P_{p(\cdot)}(\Omega) \) endowed with the norm

\[
\| \nu \|_X := \inf \{ \| v + w \|_{L_{p(\cdot)}^1(\Omega)} : \quad w \in L_{p(\cdot)}^1(\Omega), \quad \text{div}(\rho w) = 0 \},
\]

for \( \nu := -\text{div}(\rho v) \in T_{\rho}P_{p(\cdot)}(\Omega) \) is reflexive and uniformly convex Banach space. \( \blacksquare \)

As in [7], we prove that the tangent space \( (X = T_{\rho}P_{p(\cdot)}(\Omega), \| \cdot \|_X) \) is a smooth Banach space, when \( \rho \) belong to \( L^\infty(\Omega) \).

Lemma 4.3. Assume that \( p(\cdot): \Omega \rightarrow]1, +\infty[ \) satisfy \( (A_1) \) and \( \rho \in L^\infty(\Omega) \). Then the tangent space \( X = T_{\rho}P_{p(\cdot)}(\Omega), \| \cdot \|_X \) is smooth Banach space. And for all \( \nu_1 \in X \setminus \{0\} \), the Gâteaux derivative of \( \| \cdot \|_X \) at \( \nu_1 \) is given by

\[
(4.2) \quad \| \cdot \|_X'(\nu_1)(\nu_2) := \| \cdot \|_{L_{p(\cdot)}^1(\Omega)}'(\nu_1)(\nu_2)
\]

where \( \nu_i \in L_{p(\cdot)}^1(\Omega), \quad i = 1, 2 \) satisfying \( \nu_i = -\text{div}(\rho \nu_i) \), and \( \nu_i \) is the unique solution of the variational problem

\[
(P_i) \quad \inf_{\nu_i \in L_{p(\cdot)}^1(\Omega)} \left\{ \| \nu_i \|_{L_{p(\cdot)}^1(\Omega)} : \quad \frac{\partial \rho_i(t)}{\partial t} \bigg|_{t=0} + \text{div}(\rho \nu_i) = 0 \quad \text{in} \quad \Omega, \quad \rho_i(0) = \rho, \quad \rho_i(\eta) = 0 \quad \text{on} \quad \partial \Omega \right\}
\]

for some curve \( t \mapsto \rho_i(t) \) in \( P(\Omega) \) such that \( \rho_i(0) = \rho, \) and \( \| \cdot \|_{L_{p(\cdot)}^1(\Omega)}'(\nu_1) \) is the Gâteaux derivative of the norm \( \| \cdot \|_{L_{p(\cdot)}^1(\Omega)} \) at \( \nu_1 \).

Proof 4.2. As in [7], the Banach space \( (L_{p(\cdot)}^1(\Omega), \| \cdot \|_{L_{p(\cdot)}^1(\Omega)}) \) is smooth if \( \rho \in L^\infty(\Omega) \), and at any \( u \in L_{p(\cdot)}^1(\Omega) \setminus \{0\} \), the Gâteaux derivative of the norm \( \| \cdot \|_{L_{p(\cdot)}^1(\Omega)} \) is given by

\[
(4.4) \quad \| \cdot \|_{L_{p(\cdot)}^1(\Omega)}'(u)(h) := \frac{\int_{\Omega} p(x) |u(x)| p(x) - 1 \text{sign}(u(x)) h(x) \rho dx}{\int_{\Omega} p(x) |u(x)| p(x) + 1 \rho dx}
\]

\[
\int_{\Omega} p(x) \| u(x) \|_{L_{p(\cdot)}^1(\Omega)} p(x) + 1 \rho dx.
\]
for all $h \in L^{p(\cdot)}_p(\Omega)$. Then, using the fact that the norm $\|\cdot\|_{L^{p(\cdot)}_p(\Omega)}$ is Gâteaux differentiable, we have

\begin{equation}
\|tv_1 + tw_2\| = \inf_{w \in N_{\Phi}} \{\|v_1 + tv_2 + tw\|_{L^{p(\cdot)}_p(\Omega)}\}
= \|v_1\| + t\|\|v_1\|\|'(v_1)(w)\| + 0(t)
\end{equation}

for $t \in \mathbb{R}$, and $0(t)$ tends to 0 when $t$ tends to 0.

Note that for $v_1 \in X$, there exists a unique element $v_1 \in L^{p(\cdot)}_p(\Omega)$ such that $v_1 = -\text{div}(\rho v_1)$ and $\|\|v_1\|\|'(v_1)(w)\| = 0$ for all $w \in \Phi$. Therefore $v_1 \in L^{p(\cdot)}_p(\Omega)$ is the unique solution of the variational problem:

\begin{equation}
(P) \quad \inf_{v_1 \in L^{p(\cdot)}_p(\Omega)} \left\{\|v_1\|_{L^{p(\cdot)}_p(\Omega)} : \frac{\partial \rho(t)}{\partial t}|_{t=0} + \text{div}(\rho v_1) = 0 \quad \text{in} \ \Omega \quad \text{and} \ \rho v_1 \eta = 0 \quad \text{on} \ \partial \Omega \right\}
\end{equation}

for some curve $t \mapsto \rho(t)$ in $P(\Omega)$ such that $\rho_0(0) = \rho$.

Then, we have:

\begin{equation}
\|v_1\|_X = \|v_1\|_{L^{p(\cdot)}_p(\Omega)} \quad \text{and} \quad \inf_{w \in \Phi} \{t\|\|v_1\|\|'(v_1)(w)\|\} = 0.
\end{equation}

We conclude that:

\begin{equation}
\|v_1 + tw_2\|_X = \|v_1\|_X + t\|\|v_1\|\|'(v_1)(w)\| + 0(t)
\end{equation}

this completes our proof. ■

We denote by $X^* = T^*_pP_{p(\cdot)}(\Omega)$ the dual of $X = T_pP_{p(\cdot)}(\Omega)$, and $X^*$ is identified with $(N_{\Phi})^\perp$ via the isomorphism

\begin{equation}
(N_{\Phi})^\perp \ni W \mapsto L_W \in X^*
\end{equation}

with

\begin{equation}
L_W(-\text{div}(\rho V)) := \int_\Omega WV \rho dx.
\end{equation}

Here $(N_{\Phi})^\perp$ is the orthogonal of $N_{\Phi}$ defined by:

\begin{equation}
(N_{\Phi})^\perp := \{W \in L^{p(\cdot)}_p(\Omega), \quad \int_\Omega WV \rho dx = 0, \quad \text{for all} \ V \in N_{\Phi}\}.
\end{equation}

$(N_{\Phi})^\perp$ is a closed subset of $L^{p(\cdot)}_p(\Omega)$.

We define a Finsler metric $F_{p(\cdot)}$ on Wasserstein space $(P(\Omega), W_{p(\cdot)})$ such that, for all $\rho \in P(\Omega)$, $F_{p(\cdot)}(\rho, .)$ coincides with $X$-norm. Thus, if $\rho(t, \cdot) : [0, 1] \longrightarrow P(\Omega)$ is an arbitrary curve of $P_{p(\cdot)}(\Omega)$ and if $v_t : [0, 1] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the velocity fields along of the curve $t \mapsto \rho(t, .)$, we have

\begin{equation}
F_{p(\cdot)}(\rho(t, .), -\text{div}(\rho v_t)) := \| - \text{div}(\rho v_t)\|_X.
\end{equation}

Using (4.3), we give the definition of the Finsler metric.

**Definition 4.2.** [Finsler metric] Let $p(\cdot) : \Omega \longrightarrow ]1, +\infty[ \quad \text{be a measurable function.}

The map

\begin{equation}
F_{p(\cdot)} : TP_{p(\cdot)}(\Omega) \longrightarrow [0, +\infty[\n\end{equation}

defined on the tangent bundle $TP_{p(\cdot)}(\Omega) := \bigcup_{\rho \in P_{p(\cdot)}(\Omega)} \{\rho\} \times T_pP_{p(\cdot)}(\Omega)$ of $P_{p(\cdot)}(\Omega)$, by

\begin{equation}
F_{p(\cdot)}(\rho, \nu) := \|\nu\|_X,
\end{equation}

with $\rho \in P_{p(\cdot)}(\Omega)$ and $\nu \in X := T_pP_{p(\cdot)}(\Omega)$ is a Finsler metric on $P_{p(\cdot)}(\Omega)$.

**Remark 4.1.** If $\rho \in P(\Omega)$, the Minkowski norm $F_{p(\cdot)}(\rho, .)$ can be identified with the $T_pP_{p(\cdot)}(\Omega)$-norm.

The space $P_{p(\cdot)}(\Omega)$ endowed with the Finsler metric $F_{p(\cdot)}$ is a Finsler manifold.
4.2. Relation between Finsler metric and Wasserstein distance. Let use the Finsler metric $F_{\rho(t)}$ to calculate the length of a curve of $P(\Omega)$. Indeed, if $\rho = \rho(t, \cdot) : [0, 1] \to P(\Omega)$ is a curve of $P(\Omega)$; the length of $\rho(t, \cdot)$ in Finslerian space $(P(\Omega), F_{\rho(t)})$ is defined by

\begin{equation}
L_{F_{\rho(t)}}(\rho(t, \cdot)) := \int_0^1 F_{\rho(t)}(\rho(t, \cdot), \dot{\rho}(t)) \, dt \quad \text{with } \dot{\rho}(t) = \frac{\partial \rho(t, \cdot)}{\partial t}.
\end{equation}

Then, the induced distance function $d_{F_{\rho(t)}}$ of $F_{\rho(t)}$ is a distance on $P(\Omega)$ defined by

\begin{equation}
b_{F_{\rho(t)}}(\rho_0, \rho_1) := \inf_{\rho(t, \cdot)} \{ L_{F_{\rho(t)}}(\rho(t, \cdot)), \; \rho(t, \cdot) : [0, 1] \to P(\Omega), \; \rho(0) = \rho_0; \; \rho(1) = \rho_1 \}.
\end{equation}

for $\rho_0, \rho_1 \in P(\Omega)$.

**Theorem 4.1.** Assume that $1 < p^- \leq p(x) \leq p^+ < +\infty$. For all $\rho_0, \rho_1 \in P(\Omega)$, we have

\begin{equation}
W_{p(t)}(\rho_0, \rho_1) \leq d_{F_{\rho(t)}}(\rho_0, \rho_1) \lesssim W_{p(t)}^\alpha(\rho_0, \rho_1) \quad \text{with } \alpha = \frac{p^-}{p^+}.
\end{equation}

**Proof 4.3.** Let $\rho_0, \rho_1 \in P_{p(t)}(\Omega)$ and $\rho(t, \cdot) : [0, 1] \to P_{p(t)}(\Omega)$ is an arbitrary curve of $P_{p(t)}(\Omega)$ such that $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$ and $\nu(t) : [0, 1] \times \Omega \to \mathbb{R}^N$ a velocity field of the curve $t \mapsto \rho(t, \cdot)$ which satisfy $\frac{\partial \rho}{\partial t} + \text{div}_x(\rho(t, \cdot) \nu(t)) = 0$ and $\nu(t) \in L^p_{p(t)}(\Omega)$, $t \geq 0$. We have

\begin{equation}
\int_{[0, 1] \times \Omega} \left| \frac{v(t, x) + w}{\lambda} \right|^{p(x)} \, dt \, dx = \leq 1 \quad \text{if } \lambda = \| \nu(t) \|_{L_{p(t)}^{p(t)}(\Omega)}, \text{ and } -\text{div}(\rho(t, \cdot) \nu) = 0.
\end{equation}

Then

\begin{equation}
W_{p(t)}(\rho_0, \rho_1) \quad \| \nu(t) \|_{L_{p(t)}^{p(t)}([0, 1] \times \Omega)} \quad \| \nu(t) \|_{L_{p(t)}^{p(t)}(\Omega)} \quad \text{for all } t \geq 0.
\end{equation}

\begin{equation}
W_{p(t)}(\rho_0, \rho_1) \quad \leq \int_0^1 \| \nu(t) \|_{L_{p(t)}^{p(t)}(\Omega)} \, dt \leq L_{F_{\rho(t)}}(\rho(t, \cdot))
\end{equation}

Then, if $\rho(t, \cdot) : [0, 1] \to P_{p(t)}(\Omega)$ is an curve of $P_{p(t)}(\Omega)$ such that $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$ and if $\nu_t = \Omega \to \mathbb{R}^N$ is a velocity field of the curve $t \mapsto \rho(t, \cdot)$ satisfying $\nu_t \in L^p_{p(t)}(\Omega)$ and $\frac{\partial \rho(t, \cdot)}{\partial t} + \text{div}_x(\nu(t, \cdot) \nu_t) = 0$, we have by using the continuous injection $L^p_{p(t)}(\Omega) \hookrightarrow L^p_{p(t)}(\Omega)$

\begin{equation}
d_{F_{\rho(t)}}(\rho_0, \rho_1) \quad \leq \int_0^1 F_{\rho(t)}(\rho(t, \cdot), \dot{\rho}(t)) \, dt \leq \int_0^1 \| \nu(t) \|_{L_{p(t)}^{p(t)}(\Omega)} \, dt \leq \int_0^1 \| \nu(t) \|_{L_{p(t)}^{p(t)}(\Omega)} \, dt \leq \left[ \int_0^1 \| \nu(t) \|_{L_{p(t)}^{p(t)}(\Omega)} ^{p(t)} \, dt \right]^{1/p(t)} \quad \text{for all } t \geq 0.
\end{equation}

The last relationship is a consequence of the Jensen’s inequality.

By taking the infimum, we obtain

\begin{equation}
d_{F_{\rho(t)}}(\rho_0, \rho_1) \lesssim W_{p(t)}(\rho_0, \rho_1).
\end{equation}

Notice that

\begin{equation}
W_{p(t)}(\rho_0, \rho_1) = \inf_{T \in \mathcal{T}_{p(t)}} \left[ \int_{\Omega} |x - T(x)|^{p(t)} \, d\rho_0 \right]^{1/p(t)} .
\end{equation}

Since $T(\Omega) \subseteq \Omega$, we have

\begin{equation}
|x - T(x)|^{p(t)} \leq (\text{diam}(\Omega))^{p(t) - p^-} |x - T(x)|^{p^-}, \text{ for all } x \in \Omega.
\end{equation}
where \( \text{diam}(\Omega) \) is the diameter of \( \Omega \).

Then, using (4.18), we get

\[
W_{p^+}(\rho_0, \rho_1) \leq \left[ \int_{\Omega} |x - T(x)|^{p^+} \rho_0 dx \right]^{\frac{1}{p^+}}
\]

\[
\leq (\text{diam}(\Omega))^{\frac{p^+ - p^-}{p^-}} \left[ \int_{\Omega} |x - T(x)|^{p^-} \rho_0 dx \right]^{\frac{1}{p^-}}
\]

for all \( T: \Omega \to \Omega \), \( T \neq \rho_0 = \rho_1 \) and \( \int_{\Omega} |x - T(x)|^{p^-} \rho_0 dx < \infty \).

Next, taking the infimum, we obtain

\[
W_{p^+}(\rho_0, \rho_1) \leq (\text{diam}(\Omega))^{\frac{p^+ - p^-}{p^-}} W_{p^+}^\alpha(\rho_0, \rho_1), \quad \text{with } \alpha = \frac{p^-}{p^+}.
\]

We use (3.12), (1.10) and (4.10) to derive that

\[
d_{F_{p^+}}(\rho_0, \rho_1) \leq W_{p^+}(\rho_0, \rho_1) \leq W_{p^+}^\alpha(\rho_0, \rho_1) \leq W_{p^+}^\alpha(\rho_0, \rho_1).
\]

Finally, from (4.14) and (4.21), we obtain

\[
W_{p^+}(\rho_0, \rho_1) \leq d_{F_{p^+}}(\rho_0, \rho_1) \leq W_{p^+}^\alpha(\rho_0, \rho_1), \quad \text{with } \alpha = \frac{p^-}{p^+} < 1.
\]

\[
\]

5. Gradient flows in Finslerian space \((P_{p^+}(\Omega), F_{p^+})\)

Now, we use the Finsler metric \( F_{p^+} \) to define the gradient flows of the functional defined on the space of probability measures.

Let \( \varrho \in P(\Omega) \) be a probability measure and let \( \varrho^+ : \Omega \to [1, +\infty[ \) be a measurable function as in \((A_1)\). For the tangent space \( X = T_\varrho P_{p^+}(\Omega) \) we consider the normalization function and the related duality mapping between it and its dual \( X^* = T_\varrho^* P(\Omega) \) as follows

Definition 5.1.

\[
J_\varrho : T_\varrho P_{p^+}(\Omega) \to P_\varrho(X^*)
\]

defined by \( J_\varrho(0) = 0 \) and

\[
J_\varrho(\nu^+) := \left\{ \nu^+ \in X^*, \nu^+, \nu^+ = \theta(\|V\|_{L^{p^+}_p(\Omega)})\|V\|_{L^{p^+}_p(\Omega)} : \|\nu^+\|^2 = \theta(\|V\|_{L^{p^+}_p(\Omega)}) = \|V\|_{L^{p^+}_p(\Omega)} \right\}
\]

if \( \nu^+ \neq 0 \) and with \( \nu = -\text{div}(\rho V) \), where \( V \in L^{p^+}(\Omega) \) is the velocity of a curve \( t \mapsto \rho(t, \cdot) \) of \( P(\Omega) \) at \( t = 0 \) and the unique solution of the variational problem:

\[
(P) : \inf_{V \in L^{p^+}_p(\Omega)} \left\{ \int_{\Omega} \frac{|V(x)|^{p^+} p(x)}{p(x) - 1} \rho(x) dx : \frac{\partial \rho(t, x)}{\partial t} = V(x) \frac{\partial \rho}{\partial t} + \varrho V, \varrho = 0 \text{ on } \partial \Omega \right\}.
\]

Here \( P_\varrho(X^*) \) is the set of the subsets of \( X^* \) and \( q(x) = \frac{p(x)}{p(x) - 1} \).

Clearly \( J_\varrho(\nu^+) = \left\{ \nu^+ \in X^*, \nu^+, \nu^+ = \theta(\|V\|_{L^{p^+}_p(\Omega)}) \right\} \) when \( \|V\|_{L^{p^+}_p(\Omega)} = 1 \)

Remark 5.1. It is worth noticing that \( J_\varrho(\nu) \) is well defined since such a \( V \) is unique. Besides, the set of strictly increasing functions \( \theta \) satisfying \( \theta(\|V\|_{L^{p^+}_p(\Omega)}) = \|V\|_{L^{p^+}_p(\Omega)} \) is not empty. Indeed \( \theta(\|V\|_{L^{p^+}_p(\Omega)}) > \theta(\|V\|_{L^{p^+}_p(\Omega)}) \) when \( \|V_1\|_{L^{p^+}_p(\Omega)} > \|V_2\|_{L^{p^+}_p(\Omega)} \) and for any \( t \in [0, +\infty[ , \) one can find \( V' \in L^{p^+}_p(\Omega) \) such that \( t = \|V'\|_{L^{p^+}_p(\Omega)} \) and \( \theta(t) = \|V'\|_{L^{p^+}_p(\Omega)} \).

Since \( X = T_\varrho P_{p^+}(\Omega) \) is reflexive, uniformly convex and smooth Banach space, the duality map \( J_\varrho : X \to X^* \) is bijective. Then, we obtain the gradient of all functional defined on \( P(\Omega) \) with respect to the Finsler metric \( F_{p^+} \) corresponding to the normalization function \( \theta : [0, +\infty[ \to [0, +\infty[ \) as the image by \( J_\varrho^{-1} \) of its differential at the point \( \rho \).
Definition 5.2. Let $E : P_{p(.)}(\Omega) \to \mathbb{R}$ be a functional and $\rho \in P_{p(.)}(\Omega)$.

The differential of $E$ at $\rho$ in the Finslerian space $(P_{p(.)}(\Omega), F_{p(.)})$ (if it exists) is the linear and bounded form $D_{F_{p(.)}}E(\rho)$ defined on $T_{\rho}P_{p(.)}(\Omega)$ by

\[
D_{F_{p(.)}}E(\rho)(\nu) := \frac{dE(\rho(t,.))}{dt}|_{t=0} \quad \text{where } \nu \in T_{\rho}P_{p(.)}(\Omega) \text{ and } p(t,.): [0, 1] \to P_{p(.)}(\Omega) \text{ is arbitrary curve in } P_{p(.)}(\Omega) \text{ satisfying:}
\]
\[
\rho(0) = \rho \quad \text{and} \quad \frac{\partial p(t,.)}{\partial t}|_{t=0} = \nu
\]

Definition 5.3. Let $E : P_{p(.)}(\Omega) \to \mathbb{R}$ be a functional and $\rho \in P_{p(.)}(\Omega)$.

The gradient of $E$ at $\rho$ in the Finslerian space $(P_{p(.)}(\Omega), F_{p(.)})$ corresponding to the normalization function $\theta : [0, +\infty] \to [0, +\infty]$ is a unique element $\nabla_{F_{p(.)}}^\theta E(\rho)$ (if it exists) of $T_{\rho}P_{p(.)}(\Omega)$ such that

\[
D_{F_{p(.)}}E(\rho) = J_\theta(\nabla_{F_{p(.)}}^\theta E(\rho)),
\]

where $J_\theta : T_{\rho}P_{p(.)}(\Omega) \to T^*_\rho P(\Omega)$ is the duality map corresponding to the normalization function $\theta : [0, +\infty] \to [0, +\infty]$.

Definition 5.4. Let $E : P_{p(.)}(\Omega) \to \mathbb{R}$ be a functional. The curve $t \mapsto \rho(t,. \in P(\Omega)$ is to say the gradient flows of $E$ in the Finslerian space $(P_{p(.)}(\Omega), F_{p(.)})$ corresponding to the normalization function $\theta : [0, +\infty] \to [0, +\infty]$ if it is solution of the Cauchy problem

\[
\begin{align*}
\frac{\partial \rho(t,x)}{\partial t} &= - \nabla_{F_{p(.)}}^\theta E(\rho(t,x)) \quad \text{in } [0, +\infty[ \times \Omega \\
\rho(t=0,x) &= \rho_0(x) \quad \text{in } \Omega
\end{align*}
\]

Theorem 5.1. Let $G : [0, +\infty] \to \mathbb{R}$ a convex and $C^2$ function. Assume that the variable exponent $p(.) : \Omega \to [1, +\infty]$ is measurable and satisfy $(A_1)$ and $\frac{1}{p} + \frac{1}{q} \leq 1$. The parabolic $q(x)$-Laplaceian equation

\[
\frac{\partial \rho(t,x)}{\partial t} = div_x \{ \rho(t,x) | \nabla_x (G'(\rho(t,x)))| q(x) - 2 \nabla_x G'(\rho(t,x)) \} \quad \text{in } [0, +\infty[ \times \Omega \\
\rho(t=0,x) &= \rho_0(x) \quad \text{in } \Omega \\
\rho(t,x)| \nabla_x (G'(\rho(t,x)))| p(x) - 2 \nabla_x G'(\rho(t,x)).x = 0 \quad \text{on } \partial \Omega,
\]

$q(x) = \frac{p(x)}{p(x)}$: is a gradient flows of the functional $E(\rho) = \int_\Omega G(\rho) dx$ in the Finslerian space $(P_{p(.)}(\Omega), F_{p(.)})$ corresponding to the normalization function $\theta$, with $\theta(|V| L_{p(.)}^{p(.)}(\Omega)) = ||V|_{p(.)}^{p(.)}(\Omega)$ for all $V \in L_{p(.)}^{p(.)}(\Omega)$.

Proof 5.1. Let $\rho \in P_{p(.)}(\Omega)$, determine $\nabla_{F_{p(.)}}^\theta E(\rho) := -div_x(\rho V)$, where $V \in L_{p(.)}^{p(.)}(\Omega)$ is the unique solution of the variational problem

\[
(P) : \inf_{V \in L_{p(.)}^{p(.)}(\Omega)} \left\{ \int_\Omega \frac{|V(x)| p(x)}{p(x)} \rho(x) dx : \frac{\partial \rho(t,x)}{\partial t}|_{t=0} = -div_x(\rho(t,x)V(x)), \rho V \eta = 0 \quad \text{on } \partial \Omega \right\}
\]

with $\rho(t,. : [0, 1] \to P_{p(.)}(\Omega)$ a curve in $P_{p(.)}(\Omega)$ such that $\rho(0) = \rho$ and $\frac{\partial \rho(t,.)}{\partial t}|_{t=0} = -div_x(\rho V)$.

We have

\[
\frac{dE(\rho(t,.))}{dt}|_{t=0} = \int_\Omega G'(\rho) \frac{\partial \rho(t,.)}{\partial t}|_{t=0} dx = \int_\Omega \rho(x) \nabla G'(\rho(x)).V dx
\]

Using definition of $\nabla_{F_{p(.)}}^\theta E(\rho)$, we have

\[
J_\theta(\nabla_{F_{p(.)}}^\theta E(\rho)) = D_{F_{p(.)}}E(\rho)
\]

Replace $V$ by $v = \frac{V}{\theta(\|V\|_{L_{p(.)}^{p(.)}(\Omega)}^{q(x)-1})}$, then $\|v\|_{L_{p(.)}^{p(.)}(\Omega)} = 1$ and consequently

\[
<D_{F_{p(.)}}E(\rho), -div_x(\frac{\rho V}{\theta(\|V\|_{L_{p(.)}^{p(.)}(\Omega)}^{q(x)-1})})>_x = \|D_{F_{p(.)}}E(\rho)\|_x.
\]
with $\theta(\|V\|_{L^p(\Omega)}^{\rho}) = \|\|V\|_{L^p(\Omega)}^{\rho-1}\|_{L^q(\Omega)}$. 

On the other hand, note that, if $\frac{1}{p} + \frac{1}{q} \leq 1$, 

$$
<DF_{\rho}(E(p), -\text{div}_x(\rho V \theta(\|V\|_{L^p(\Omega)}^{\rho-1})))> = \int_{\Omega} \frac{\nabla G'(\rho)V}{\theta(\|V\|_{L^p(\Omega)}^{\rho})} dx \\
\leq \|\nabla G'(\rho)\|_{L^q(\Omega)}
$$

In particular, if $V = |\nabla G'(\rho)|^{q(x)-2}\nabla G'(\rho)$, then $\theta(\|V\|_{L^p(\Omega)}^{\rho}) = \|\nabla G'(\rho)\|_{L^q(\Omega)}$, with $q(x) = \frac{p(x)}{p(x)-1}$, and we get 

$$
<DF_{\rho}(E(p), -\text{div}_x(\rho V \theta(\|V\|_{L^p(\Omega)}^{\rho-1})))> = \int_{\Omega} \frac{|\nabla G'(\rho)|^{q(x)}}{\theta(\|V\|_{L^p(\Omega)}^{\rho})} dx \\
= \theta(\|V\|_{L^p(\Omega)}^{\rho}) \int_{\Omega} \frac{|\nabla G'(\rho)|^{q(x)}}{\theta(\|V\|_{L^p(\Omega)}^{\rho})} dx = 1
$$

Since 

$$
\theta(\|V\|_{L^p(\Omega)}^{\rho}) = \|\nabla G'(\rho)\|_{L^q(\Omega)}, \\
\int_{\Omega} \frac{|\nabla G'(\rho)|^{q(x)}}{\theta(\|V\|_{L^p(\Omega)}^{\rho})} dx = 1
$$

and then 

$$
<DF_{\rho}(E(p), -\text{div}_x(\rho V \theta(\|V\|_{L^p(\Omega)}^{\rho-1})))> = \theta(\|V\|_{L^p(\Omega)}^{\rho}) = \|\nabla G'(\rho)\|_{L^q(\Omega)}.
$$

By using (5.13) and (5.10), we conclude that 

$$
<DF_{\rho}(E(p), -\text{div}_x(\rho V \theta(\|V\|_{L^p(\Omega)}^{\rho-1})))> = \|DF_{\rho}(E(p))X\| = \|\nabla G'(\rho)\|_{L^q(\Omega)}
$$

with $\theta(\|V\|_{L^p(\Omega)}^{\rho}) = \|\nabla G'(\rho)\|_{L^q(\Omega)}$ and $V = |\nabla G'(\rho)|^{q(x)-2}\nabla G'(\rho)$ 

Since the map $J_{\rho(x)} : X \rightarrow X^*$ is bijective, then $\nabla F_{\rho}(E(p)) = -\text{div}_x(\rho \nabla G'(\rho)|^{q(x)-2}\nabla G'(\rho))$ is the unique element of $X$ which satisfy (5.14). Then the gradient flow of $E$ is the parabolic $q(x)$-Laplacian equation (5.1). 

**Theorem 5.2.** The parabolic $q(x)$-Laplacian equation 

$$
\frac{\partial \rho(t, x)}{\partial t} = \text{div}_x\{\rho(t, x)|\nabla_x(G'(\rho(t, x)))|^{q(x)-2}\nabla_x G'(\rho(t, x))\} \quad \text{in} \quad [0, \infty) \times \Omega
$$

$$
\rho(t = 0, x) = \rho_0(x) \quad \text{in} \quad \Omega
$$

is the gradient flow of the functional $E(\rho) = \int_{\Omega} G(\rho(x)) dx$ with respect to the Wassertein distance $W_{P(\cdot)}$. 

**Proof 5.2.** Let $t \mapsto \rho(t, \cdot)$ be a solution of the parabolic equation involving $q(x)$-Laplacian operator (5.1). We have 

$$
-\frac{dE(\rho(t, \cdot))}{dt} = \int_{\Omega} |\nabla_x G'(\rho(t, x))|^{q(x)} \rho(t, x) dx.
$$

For $0 < h < 1$ let define 

$\bar{\rho} : [0, 1] \rightarrow P_{\rho(\cdot)}(\Omega)$, with $\bar{\rho}(0, \cdot) = \rho_0, \rho(h + t, \cdot)$, the curve joining $\rho(t, \cdot)$ and $\rho(h + t, \cdot)$, for such that $\bar{\rho}(0, \cdot) = \rho(t, \cdot)$ and $\bar{\rho}(1, \cdot) = \rho(h + t, \cdot)$. 

A velocity field along the curve $s \mapsto \bar{\rho}(s, \cdot)$ is $\bar{v}(s, t) = h\nu_0(hs + t, \cdot)$, where $v(t, x) = -|\nabla_x G'(\rho(t, x))|^{q(x)-2}\nabla_x G'(\rho(t, x))$, the velocity fields of the curve $t \mapsto \rho(t, \cdot)$.

So, we have 

$$
W_{P(\cdot)}(\rho(t, \cdot), \rho(h + t, \cdot)) \leq \|hv(sh + t, \cdot)\|_{L^p(\rho(h + t, \cdot))([0, 1] \times \Omega)} \\
= h\|v(sh + t, \cdot)\|_{L^p(\rho(h + t, \cdot))([0, 1] \times \Omega)}
$$

(5.17)
And then, we obtain

\[ |\rho^\prime(t)| := \lim_{{h \to 0}} \frac{W_{p(t)}(\rho(t+h),\rho(t))}{h} \leq \|v(t)\|_{{L^p_t(\Omega)}}. \]

Here, \( |\rho^\prime(t)| \) is the metric derivative of the curve \( t \mapsto \rho(t,.) \) at \( t \) and \( v(t,x) = -|\nabla_x G' (\rho(t,x))|^q(x) - \nabla_x G' (\rho(t,x)) \) is a velocity fields along the curve \( t \mapsto \rho(t,.) \). 

Note that if \( \int_\Omega |v(t,x)|^p(x)\rho(t,x)dx = \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx \leq 1 \) then 

\[ \|v(t)\|_{{L^p_t(\Omega)}} \leq \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx, \]

and if \( \int_\Omega |v(t,x)|^p(x)\rho(t,x)dx = \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx > 1 \), then 

\[ \|v(t,.)\|^q(t)|_{L^p_t(\Omega)} \leq \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx. \]

Write 

\[ \{ \begin{aligned} \alpha(t) &= p^+, & & \text{if} & & \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx \leq 1 \\ \alpha(t) &= p^- & & \text{if} & & \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx > 1 \end{aligned} \]

then we have 

\[ \|v(t,.)\|^\alpha(t)|_{L^p_t(\Omega)} \leq \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx = -\frac{dE(\rho(t,))}{dt}, \]

and recalling (5.18), we get 

\[ |\rho^\prime|^{\alpha(t)}(t) \leq -\frac{dE(\rho(t,))}{dt}. \]

Let \( \rho_0, \rho_1 \in P_1(\Omega), \gamma(t,.) : [0,1] \to P_1(\Omega) \) be an arbitrary curve such that \( \gamma(0,.) = \rho_0 \), and \( \gamma(1,.) = \rho_1 \) and \( V_1(.) : [0,1] \times \Omega \to \Omega \) the velocity fields along the curve \( t \mapsto \gamma(t) \) We have:

\[ E(\rho_1) - E(\rho_0) = \int_{[0,1]} \frac{dE(\gamma(t))}{dt}dt 
= \int_{[0,1] \times \Omega} \nabla_x G' (\gamma(t),v(t,x)\gamma(t,x)dt \dx 
\leq 2\|\nabla G' (\gamma(t,x))\|_{L^p_t(\gamma(t,.))} \|v(t,.)\|_{L^p_t(\gamma(t,.))}. \]

Since \( t \mapsto \gamma(t,.记载) \) is an arbitrary curve joining \( \rho_0 \) and \( \rho_1 \) in \( (P(\Omega),W_{p(.)}) \), we obtain:

\[ |E(\rho_1) - E(\rho_0)| \leq 2\|\nabla G' (\gamma(t,x))\|_{L^p_t(\gamma(t,.))} \|v(t,.)\|_{L^p_t(\gamma(t,.))}. \]

In the particular case we choose \( \gamma \) such that \( \gamma(t,.记载) = \rho_0 \) for \( t \in [0,1] \) and \( \gamma(1,.记载) = \rho_1 \), we obtain 

\[ \frac{|E(\rho_1) - E(\rho_0)|}{W_{p(.)}(\rho_0,\rho_1)} \leq 2\|\nabla G' (\rho_0)\|_{L^p(\rho_0)}. \]

And hence 

\[ \|\nabla W_{p(.)}(\rho_0)\| := \lim_{\rho_1 \to \rho_0} \frac{|E(\rho_1) - E(\rho_0)|}{W_{p(.)}(\rho_0,\rho_1)} \leq 2\|\nabla G' (\rho_0)\|_{L^p(\rho_0)}. \]

Where, \( |\nabla W_{p(.)}(\rho_0)| \) is the upper gradient of the functional \( E(\rho) = \int_\Omega G(\rho)dx \) at \( \rho_0 \) with respect to the Wasserstein distance \( W_{p(.)}. \)

When we suppose 

\[ \{ \begin{aligned} \beta(t) &= q^+, & & \text{if} & & \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx \leq 1 \\ \beta(t) &= q^- & & \text{if} & & \int_\Omega |\nabla_x G' (\rho(t,x))|^q(x)\rho(t,x)dx > 1 \end{aligned} \]

We have 

\[ \frac{1}{2\beta(t)} \|\nabla W_{p(.)}(\rho(t,.)\|^\beta(t) \leq -\frac{dE(\rho(t,))}{dt} = \int_\Omega |\nabla G' (\rho(t,x))|^q(x)\rho(t,x)dx. \]

Recalling inequalities (5.23) and (5.29), we conclude that the parabolic \( q(x) \)-Laplace equation (6.1) is a gradient flows of \( E(\rho) = \int_\Omega G(\rho)dx \) in Wasserstein space \( (P_{p(.)}(\Omega),W_{p(.)}) \).
6. Existence and uniqueness of solution for the parabolic \( q(x) \)-Laplacian equation

In this section, we prove the existence and uniqueness of solution for the class of parabolic \( q(x) \)-Laplacian equations

\[
\frac{\partial \rho(t, x)}{\partial t} = div_x \{ \rho(t, x) \nabla_x (G' (\rho(t, x))) | \nabla_x (G' (\rho(t, x))) |^{q(x)-2} \nabla_x (G' (\rho(t, x))) \} \quad \text{in } \Omega, \quad t \geq 0
\]

We prove in section 6.1 that the sequence \((\rho_k)\) admits a unique solution \(\rho(t, x)\) in \(\Omega\). Let \(\rho_0(x)\) be a probability density on \(\Omega\) and show that the sequence \((\rho_k)\) is compact in \(\Omega\). We use the steepest descent method in the Wasserstein space \((\mathcal{P}(\Omega), \mathcal{W}(\rho_0, \cdot))\). We assume that \(\mathcal{G} : [0, +\infty) \rightarrow \mathbb{R}\) is a convex function, which satisfies \((A_1)\), \(q(.) := \frac{\mathcal{G}(\cdot)}{\mathcal{F}(\cdot)} : \Omega \rightarrow [1, +\infty[\) is a measurable function and \(\Omega\) being an open, convex and smooth domain of \(\mathbb{R}^N\), \((N \geq 1)\).

We use analogue discrete scheme as in [2] to define a time discretization of the problem \((6.1)\). Indeed, we fix \(h > 0\) to be a time step and assume that \(\rho_0\) is a probability density on \(\Omega\). Define \(\rho^k, k \in \mathbb{N}^*\) as a solution of the variational problem

\[
(P_k) : \inf_{\rho \in \mathcal{P}(\Omega)} \left\{ I(\rho) := \int_{\Omega} G(\rho) \, dx + W^h_{\mathcal{P}(\cdot)}(\rho^{k-1}, \rho) \right\},
\]

where

\[
W^h_{\mathcal{P}(\cdot)}(\rho^{k-1}, \rho) := \inf_{\gamma \in \Pi(\rho^{k-1}, \rho)} \left\{ \int_{\Omega \times \Omega} \frac{|x - y|^q}{h^q(x) - 1} \rho(x) \, d\gamma(x, y) \right\}.
\]

Here, \(\Pi(\rho^{k-1}, \rho)\) is the set of all probability measures on \(\Omega \times \Omega\) whose marginals are \(\rho^{k-1}\) and \(\rho\).

We prove in section 6.1 that the sequence \((\rho^k)_k\), satisfies the equation

\[
\frac{\rho^k - \rho^{k-1}}{h} = div_x \left\{ \rho^k | \nabla_x (G' (\rho^k)) |^{q(x)-2} \nabla_x (G' (\rho^k)) \right\} + o(h),
\]

in a weak sense, where \(o(h)\) tends to 0 when \(h\) tends to 0 and accordingly equation \((6.3)\) shows that the sequence \((\rho^k)_k\) is a time discretization of \((6.1)\).

We define \(\rho^h\) as follow

\[
\begin{align*}
\rho^h(t, x) &= \rho(x) \quad \text{if } t \in [hk, h(k + 1)] \\
\rho^h(t = 0, x) &= \rho_0(x) \quad \text{if } t = 0
\end{align*}
\]

and we show that the sequence \((\rho^h)_h\) converges weakly to \(\rho(t, x)\) which solves the parabolic \(q(x)\)-Laplacian equation \((6.1)\) in a weak sense.

6.1. Euler Lagrange equation of the problem \(P^k\). Here, we establish the existence and uniqueness of the solution of problem \(P^k\) and show that the sequence \((\rho^k)_k\) is a time discretization of \((6.1)\).

**Proposition 6.1.** Let \(\rho_0\) be a probability density on \(\Omega\) such that \(\int_{\Omega} G(\rho_0) \, dx < +\infty\). The problem

\[
(P^1) : \inf_{\rho \in \mathcal{P}(\Omega)} \left\{ I(\rho) := \int_{\Omega} G(\rho) \, dx + W^h_{\mathcal{P}(\cdot)}(\rho_0, \rho) \right\}
\]

admits a unique solution \(\rho^1\) and \(\int_{\Omega} G(\rho^1) \, dx < +\infty\).

**Proof 6.1.** Let denote \(l\) the infimum of \(I\) over \(P(\Omega)\). Show that \(l\) is finite.

If \(\rho = \rho_0\), then \(\int_{\Omega} G(\rho_0) \, dx < +\infty\) and \(W^h_{\mathcal{P}(\cdot)}(\rho_0, \rho_0) = 0\).

Let \(\rho\) be a probability density on \(\Omega\). Since \(G\) is convex, we use Jessen’s inequality and obtain:

\[
\int_{\Omega} G(\rho) \, dx + W^h_{\mathcal{P}(\cdot)}(\rho_0, \rho) \geq |\Omega|G\left(\frac{1}{|\Omega|}\right).
\]

We deduce that \(l\) is finite.

Let \((\rho^* dx)_n\) be a minimizing sequence of \((P^1)\) in \(P(\Omega)\).
Since $P(\Omega)$ is tight, then $(\rho^n dx)_n$ converges narrowly to $\rho^1 dx$ in $P(\Omega)$, (up to a subsequence). Since $G$ is convex and $C^1$, we have

\begin{equation}
(6.8) \quad \int_{\Omega} G(\rho^n) dx \geq \int_{\Omega} G(\rho^1) dx + \int_{\Omega} (\rho^n - \rho^1) G'(\rho^1) dx
\end{equation}

By taking the limit in (6.8), we have

\begin{equation}
(6.9) \quad \liminf \int_{\Omega} G(\rho^n) dx \geq \int_{\Omega} G(\rho^1) dx
\end{equation}

Let $\gamma_n$ be a solution of Kantorovich problem

\begin{equation}
(6.10) \quad (K) : \inf_{\gamma \in \Pi(\rho_0, \rho^n)} \left\{ \int_{\Omega \times \Omega} \frac{|x-y|^{p(x)}}{h^{p(x)-1} p(x)} d\gamma(x,y) \right\}
\end{equation}

Since $P(\Omega \times \Omega)$ is tight, $(\gamma^n)_n$ converges narrowly to a probability measure $\gamma_1$ in $P(\Omega \times \Omega)$, (up to a subsequence), and $\gamma_1 \in \Pi(\rho_0, \rho^1)$. We also have

\begin{equation}
(6.11) \quad \liminf \int_{\Omega \times \Omega} \frac{|x-y|^{p(x)}}{h^{p(x)-1} p(x)} d\gamma^n(x,y) \geq \int_{\Omega \times \Omega} \frac{|x-y|^{p(x)}}{h^{p(x)-1} p(x)} d\gamma_1(x,y).
\end{equation}

By using (6.9) and (6.11), we obtain

\begin{equation}
(6.12) \quad l = \liminf \left\{ \int_{\Omega} G(\rho^n) dx + W^h_{p(1)}(\rho_0, \rho^n) \right\} \geq \int_{\Omega} G(\rho^1) dx + W^h_{p(1)}(\rho_0, \rho^1) \geq l.
\end{equation}

Then, $\rho^1$ is a solution of $(P^1)$ and $\int_{\Omega} G(\rho^1) dx \leq \int_{\Omega} G(\rho_0) dx < \infty$.

We obtain uniqueness of $\rho^1$ by using the convexity of $\rho \mapsto \int_{\Omega} G(\rho) dx$ and the strict convexity of the map $\rho \mapsto W^h_{p(1)}(\rho_0, \rho)$. □

By induction, we obtain existence and uniqueness of the sequence $(\rho^k)_k$ such that $\rho^k$ is a unique solution of the problem $(P^k)$.

**Lemma 6.1.** Let $\rho_0$ be a probability density on $\Omega$ such that $\int_{\Omega} G(\rho_0) dx < \infty$.

The Kantorovich problem

\begin{equation}
(6.13) \quad (K) : \inf_{\gamma \in \Pi(\rho_0, \rho^k)} \left\{ \int_{\Omega \times \Omega} \frac{|x-y|^{p(x)}}{h^{p(x)-1} p(x)} d\gamma(x,y) \right\}
\end{equation}

admits a unique solution $\gamma_k$, and

\[ \text{supp} \gamma_k \subset \{ (x, y) : y = x + h|\nabla_x G'(\rho^k)|q(x)^{-2} \nabla_x G'(\rho^k) \} . \]

**Proof 6.2.** We obtain existence and uniqueness of solutions of (6.13) because $(x, y) \mapsto |x - y|^{p(x)}$ is a Carathéodory function.

Let $\psi \in C_c^{\infty}(\Omega, \Omega)$ be a test function, and consider the flow map $(T_\varepsilon)_{\varepsilon \in \mathbb{R}}$ in $C_c^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$, such that

\begin{equation}
(6.14) \quad \left\{ \begin{array}{l}
\frac{\partial T_\varepsilon}{\partial \varepsilon} = T_\varepsilon \circ \psi \\
T_0 = \text{id}
\end{array} \right.
\end{equation}

Define: $\rho_\varepsilon = T_\varepsilon \# \rho^k$.

We have

\begin{equation}
(6.15) \quad \frac{d}{d\varepsilon} \left[ \int_{\Omega} G(\rho_\varepsilon) dx \right]_{\varepsilon=0} = \int_{\Omega} <\nabla_x G'(\rho^k), \psi> \rho^k dx, \quad \text{see} \ [2].
\end{equation}

Let $\gamma^\varepsilon$ be a probability measure on $\Omega \times \Omega$ defined by

\begin{equation}
(6.16) \quad \int_{\Omega \times \Omega} \phi(x, y) d\gamma^\varepsilon(x, y) = \int_{\Omega \times \Omega} \phi(x, T_\varepsilon(y)) d\gamma_k(x, y),
\end{equation}

for all $\phi \in C_b^0(\Omega \times \Omega)$. $\gamma^\varepsilon$ belongs to $\Pi(\rho^{k-1}, \rho_\varepsilon)$.

Let’s show that

\begin{equation}
(6.17) \quad \frac{d}{d\varepsilon} \left[ \int_{\Omega \times \Omega} \frac{|x-y|^{p(x)}}{h^{p(x)-1} p(x)} d\gamma^\varepsilon(x, y) \right]_{\varepsilon=0} = \int_{\Omega \times \Omega} \frac{1}{16} <\frac{x-y}{h} p(x)^{-2} \frac{x-y}{h}, \psi> d\gamma_k(x, y).
\end{equation}
By using the definition of $\gamma^\epsilon$, we have

\begin{equation}
\int_{\Omega \times \Omega} \frac{|x - y|^p(x)}{h^{p(x)}-1p(x)} d\gamma^\epsilon(x, y) = \int_{\Omega \times \Omega} \frac{|x - T_\epsilon(y)|^{p(x)}}{h^{p(x)}-1p(x)} d\gamma_k(x, y).
\end{equation}

Notice that

(i) For all $\epsilon \in \mathbb{R}$,

\begin{equation}
\int_{\Omega \times \Omega} \frac{|x - T_\epsilon(y)|^{p(x)}}{h^{p(x)}-1p(x)} d\gamma_k(x, y) < +\infty.
\end{equation}

(ii) For $(x, y) \in \Omega \times \Omega$ almost everywhere

\begin{equation}
\epsilon \to \frac{|x - T_\epsilon(y)|^{p(x)}}{h^{p(x)}-1p(x)}
\end{equation}

is differentiable

\begin{equation}
\frac{d}{d\epsilon} \left[ \frac{|x - T_\epsilon(y)|^{p(x)}}{h^{p(x)}-1p(x)} \right] = \frac{x - T_\epsilon(y)}{h^{p(x)}-1p(x)} | x - T_\epsilon(y) |^{\epsilon p(x) - 2} \left( \frac{x - T_\epsilon(y)}{h^{p(x)}-1p(x)} \right) \dot{\psi}(T_\epsilon(y)).
\end{equation}

Furthermore, for all $\epsilon \neq 0$

\begin{equation}
\left| \frac{d}{d\epsilon} \left[ \frac{|x - T_\epsilon(y)|^{p(x)}}{h^{p(x)}-1p(x)} \right] \right| \leq 2^{p(x)-2} \frac{|x - y|^{p(x) - 2}}{h^{p(x)}-1p(x)} + \frac{2^{p(x)-2}}{h^{p(x)}-1p(x)} \| \dot{\psi} \|_{\infty}
\end{equation}

Recalling (i), (ii), and the dominated convergence theorem yield

\begin{equation}
\int_{\Omega \times \Omega} \frac{|x - y|^p(x)}{h^{p(x)}-1p(x)} d\gamma(x, y) \big|_{\epsilon=0} = \int_{\Omega \times \Omega} \left\langle \frac{x - y}{h}, \psi \right\rangle \big|_{\epsilon=0} = 0.
\end{equation}

The solution $\rho^k$ of the problem $(P^k)$ satisfies

\begin{equation}
\frac{d}{d\epsilon} \int_{\Omega} G(\rho_\epsilon) dx + W^h_{\rho^k}(\rho^{k-1}, \rho_\epsilon) \bigg|_{\epsilon=0} = 0.
\end{equation}

Note that $\gamma^\epsilon$ is admissible for $(P^k)$, then

\begin{equation}
W^h_{\gamma(x)}(\rho^{k-1}, \rho_\epsilon) \leq \int_{\Omega \times \Omega} \frac{|x - y|^p(x)}{h^{p(x)}-1p(x)} d\gamma^\epsilon(x, y).
\end{equation}

By using the previous inequality, we obtain

\begin{equation}
I(\rho_\epsilon) = \int_{\Omega} G(\rho_\epsilon) dx + W^h_{\rho^k}(\rho^{k-1}, \rho_\epsilon) \leq \int_{\Omega} G(\rho_\epsilon) dx + \int_{\Omega \times \Omega} \frac{|x - y|^p(x)}{h^{p(x)}-1p(x)} d\gamma^\epsilon(x, y).
\end{equation}

So, for $\epsilon > 0$, we have

\begin{equation}
\frac{I(\rho_\epsilon) - I(\rho^k)}{\epsilon} = \int_{\Omega} G(\rho_\epsilon) dx - \int_{\Omega} G(\rho^k) dx + \frac{W^h_{\rho^k}(\rho^{k-1}, \rho_\epsilon) - W^h_{\rho^k}(\rho^{k-1}, \rho^k)}{\epsilon} \leq \int_{\Omega} G(\rho_\epsilon) dx - \int_{\Omega} G(\rho^k) dx + \int_{\Omega \times \Omega} \frac{|x - T_\epsilon(y)|^{p(x)}}{h^{p(x)}-1p(x)} d\gamma_k(x, y) - \int_{\Omega \times \Omega} \frac{|x - y|^p(x)}{h^{p(x)}-1p(x)} d\gamma_k(x, y).
\end{equation}

We use (6.15), (6.20), (6.21) and (6.23) and we tend $\epsilon$ to $0$

\begin{equation}
0 \leq \int_{\Omega \times \Omega} \left\langle \nabla_x G'(\rho^k), \psi \right\rangle \rho^k dx + \int_{\Omega \times \Omega} \left\langle \frac{x - y}{h}, \psi \right\rangle d\gamma_k(x, y).
\end{equation}

Changing $\psi$ by $-\psi$ in (6.23), we obtain the desired equality

\begin{equation}
\int_{\Omega} \left\langle \nabla_x G'(\rho^k), \psi \right\rangle \rho^k dx + \int_{\Omega \times \Omega} \left\langle \frac{x - y}{h}, \psi \right\rangle d\gamma_k(x, y) = 0
\end{equation}

Then

\begin{equation}
y = x + h |\nabla_x G'(\rho^k(x))|^{q(x)-2} \nabla_x G'(\rho^k(x)) \quad \gamma_k \quad a.e,
\end{equation}

with $p(x) = \frac{q(x)}{q(x)-1}$.
Now, let show that $(\rho^k)_k$ is a time discretization of \eqref{6.1}.
Let $\psi \in C_0^\infty(\Omega, \mathbb{R})$ be a test function, we have
\begin{equation}
\int_\Omega (\rho^k - \rho^{k-1})\psi(x)dx = \int_{\Omega \times \Omega} (\psi(y) - \psi(x))d\gamma_k(x, y)
\end{equation}

Using Taylor’s formula
\begin{equation}
\psi(y) = \psi(x) + (y - x) \cdot \nabla_x \psi(x) + (y - x)^T \nabla^2_x \psi(x + \theta(y - x)).(y - x),
\end{equation}
with $\theta \in [0, 1]$ and $(y - x)^T$ is the transpose of $y - x$

We use Jessen’s inequality in \eqref{6.29} and obtain:
\begin{equation}
\int_\Omega (\rho^k - \rho^{k-1})\psi(x)dx = -h \int_\Omega <|\nabla_x G'(\rho^k(x))|^{q(x)-2} \nabla_x G'(\rho^k(x)), \nabla_x \psi(x)\rangle \rho^k dx - \frac{h^2}{2} \int_\Omega \langle \nabla^2_x \psi(x + \theta \nabla_k \rho_k \rangle \nabla_x \psi(x + \theta \nabla_k \rho_k) \rangle \rho^k dx.
\end{equation}

In \eqref{6.30}, $V_k := |\nabla_x G'(\rho^k(x))|^{q(x)-2} \nabla_x G'(\rho^k(x))$.

Define $A_k(\psi) = h \int_\Omega \langle V_k, \nabla^2_x \psi(x + \theta \nabla_k \rho_k) \rangle \rho^k dx$ and show that $A_k(\psi)$ tends to 0 when $h$ tends to 0.

We have
\begin{equation}
|A_k(\psi)| \leq h \sup_{x \in \Omega} |\nabla^2_x \psi(x)| \int_\Omega |\nabla_x G'(\rho^k)| \frac{2q(x)}{p(x)} \rho^k(x)dx
\end{equation}

Since $\rho^k$ is the unique solution of $P^k$, we have
\begin{equation}
\int_\Omega G(\rho^k)dx + \int_{\Omega \times \Omega} \frac{|x - y|p(x)}{h^{p(x)-1}p(x)}d\gamma_k(x, y) \leq \int_\Omega G(\rho^{k-1})dx
\end{equation}

Using \eqref{6.27}, we obtain
\begin{equation}
\int_\Omega G(\rho^k)dx - \int_\Omega G(\rho^{k-1})dx \geq h \int_\Omega \frac{|\nabla_x G'(\rho^k)|^{q(x)}}{p(x)} \rho^k dx
\end{equation}

Using definition of $\rho^k$ and taking the sum over $k = 1, \ldots, \frac{T}{h}$ in \eqref{6.33}, we get
\begin{equation}
\int_\Omega G(\rho_0)dx - \int_\Omega G(\rho_T)dx \geq \int_{[0, T] \times \Omega} \frac{|\nabla_x G'(\rho^k)|^{q(x)}}{p(x)} \rho^k dtdx
\end{equation}

We use Jessen’s inequality in \eqref{6.34} and obtain:
\begin{equation}
\int_\Omega G(\rho_0)dx - \|\Omega\|G\left(\frac{1}{|\Omega|}\right) \geq \int_{[0, T] \times \Omega} \frac{|\nabla_x G'(\rho^k)|^{q(x)}}{p(x)} \rho^k dtdx
\end{equation}

Write
\begin{equation}
\Omega_1 = \{x \in \Omega, \quad p(x) \geq 2\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega, \quad 1 < p(x) < 2\}
\end{equation}

We have
\begin{equation}
\int_\Omega |\nabla_x G'(\rho^k)| \frac{2q(x)}{p(x)} \rho^k(x)dx = \int_{\Omega_1} |\nabla_x G'(\rho^k)| \frac{2q(x)}{p(x)} \rho^k(x)dx + \int_{\Omega_2} |\nabla_x G'(\rho^k)| \frac{2q(x)}{p(x)} \rho^k(x)dx
\end{equation}

Moreover,
\begin{equation}
\int_{\Omega_1} |\nabla_x G'(\rho^k)| \frac{2q(x)}{p(x)} \rho^k(x)dx \leq 2^{q+1} h^{\frac{1}{2}} \int_{\Omega_1} |\nabla_x G'(\rho^k)|^{q(x)} \rho^k(x)dx + 2^{q+1} h
\end{equation}

and
\begin{equation}
\int_{\Omega_2} |\nabla_x G'(\rho^k)| \frac{2q(x)}{p(x)} \rho^k(x)dx \leq \int_{\Omega_2} h^{1 - \frac{1}{p(x)}} |\nabla_x G'(\rho^k)|^{q(x)} \rho^k(x)dx
\end{equation}
So, by using the previous both inequalities and (6.34), we derive that

\[
|A_k(\psi)| \leq h \sup_{x \in \Omega} |\nabla_2^2 \psi(x)| p^+ \left( 2^{q-1} + (2^{q-1} + h^{1+\frac{q}{2}}) \left( \int_\Omega G(\rho_0) dx - |\Omega| G\left( \frac{1}{|\Omega|} \right) \right) \right),
\]

The inequality above proves that \( A_k(\psi) \) tends to 0 when \( h \) tends to 0. Hence, the sequence \( (\rho^k)_k \) is a time discretization of (6.1).■

Next, let’s show that the sequence \( (\rho^k)_k \) converges weakly (up to a subsequence) to a function \( \rho = \rho(t, x) \) which solves the parabolic \( q(x) \)-Laplacian equation (6.1).

### 6.2. Convergence of the sequence \((\rho^k)_k\), weak convergence of nonlinear term

\[
div_x \{ \rho^k \nabla_x G(\rho^k)|p(x)|^{-2} \nabla_x G(\rho^k) \}.
\]

In this section, we assume that the initial datum \( \rho_0 \) is a probability density which satisfies \( \rho_0 \leq M_2 \) with \( 0 < M_2 \) and we show that the function \( \rho^k \) satisfies also \( \rho^k \leq M_2 \), for all \( h > 0 \), (see (6.2)). Using (6.2) and the previous results, we prove that the sequence \( (H'(\rho^k(t, .)))_h \) is bounded in \( W^{1,q}(\Omega) \), for all \( r \geq 0 \), where \( H \) is a convex function such that \( H''(\rho) = tG''(\rho) \). Then, we use the fact that \( v \mapsto |v|^q(x) \) is coercive to deduce that \( (H'(\rho(t, .)))_h \) is bounded in \( W^{1,1}(\Omega) \), for all \( t \geq 0 \).

Using compactness argument on the BV spaces, we deduce that \( \rho^k(t, .) \) converges strongly to \( \rho(t, .) \) in \( L^1(\Omega) \).

Finally, we use the strong convergence of \( (\rho^k(t, .))_h \) to \( \rho(t, .) \), to prove the weak convergence of the nonlinear term \( \{ \text{div}_x \{ \rho^k \nabla_x G(\rho^k)|q(x)-2| \nabla_x G(\rho^k) \} \}_h \) to \( \text{div}_x \{ \rho \nabla_x G(\rho)|q(x)-2| \nabla_x G(\rho) \} \).

### Proposition 6.2. (maximum principle)

Assume that the initial datum \( \rho_0 \) satisfies \( 0 < \rho_0 \leq M_2 \).

Then the solution \( \rho^1 \) of the problem \( (P^1) \) satisfies: \( \rho^1 \leq M_2 \).

**Proof 6.3.** Assume by contradiction that the set \( E = \{ y \in \Omega, \ \rho^1(y) > M_2 \} \) has a positive Lebesgue measure.

Let \( \gamma_1 \) be a solution of Kantorovich problem

\[
(K) : \inf_{\gamma \in \Pi(\rho_0, \rho^1)} \int_{\Omega \times \Omega} \frac{|x - y|^{p(x)}}{h^{p(x)} - 1} p(x) d\gamma(x, y)
\]

where \( \Pi(\rho_0, \rho^1) \) is the set of all probability measures on \( \Omega \times \Omega \) whose marginals are \( \rho_0 dx \) and \( \rho^1 dy \). Denote \( E^c \) the complement of \( E \). If \( \gamma_1(E^c \times E) = 0 \), then we have

\[
|E|M < \int_E \rho^1(y) dy = \gamma_1(\Omega \times E) = \gamma_1(E \times E) \leq \gamma_1(E \times \Omega) = \int_E \rho_0(x) dx \leq |E|M_2
\]

which yields a contradiction. Then \( \gamma_1(E^c \times E) > 0 \).

Let \( \mu \) be a probability measure on \( \Omega \times \Omega \) defined by

\[
\int_{\Omega \times \Omega} \psi(x, y) d\mu(x, y) = \int_{E^c \times E} \psi(x, y) d\gamma_1(x, y),
\]

for all \( \psi \in C_0(\Omega \times \Omega) \).

Denote by \( \mu_0 \) and \( \mu_1 \) the marginals of \( \mu \). \( \mu_0 \) and \( \mu_1 \) are absolutely continuous with respect to the Lebesgue measure.

Denote by \( v_0 \) and \( v_1 \) the respective density functions of \( \mu_0 \) and \( \mu_1 \). We have \( v_0 = 0 \) on \( E \) and \( v_1 = 0 \) on \( E^c \).

Let \( \epsilon > 0 \) small, such that \( M_2 - \epsilon v_1 > 0 \) and define: \( \rho^* = \rho^1 + \epsilon (v_0 - v_1) \).

\( \rho^* \) is a probability density on \( \Omega \) and the measure \( \gamma^* \) defined by

\[
\int_{\Omega \times \Omega} \psi(x, y) d\gamma^*(x, y) = \int_{\Omega \times \Omega} \psi(x, y) d\gamma_1(x, y) + \epsilon \int_{E^c \times E} [\psi(x, x) - \psi(x, y)] d\gamma_1(x, y)
\]

belongs to \( \Pi(\rho_0, \rho^*) \).

Show that \( I(\rho^*) < I(\rho^1) \).

The measure \( \gamma^* \) is not necessarily a solution of Kantorovich problem

\[
(K) : \inf_{\gamma \in \Pi(\rho_0, \rho^*)} \int_{\Omega \times \Omega} \frac{|x - y|^{p(x)}}{h^{p(x)} - 1} p(x) d\gamma(x, y).
\]
However, we have

\begin{equation}
I(\rho') - I(\rho) \leq \int_{\Omega} G(\rho') dx - \int_{\Omega} G(\rho_1) dx + \left[ \int_{\Omega \times \Omega} \frac{|x-y|^{p(x)}}{h(x,y)^{-p(x)}} d\gamma(x,y) - \int_{\Omega \times \Omega} \frac{|x-y|^{q(x)}}{h(x,y)^{-q(x)}} d\gamma(x,y) \right].
\end{equation}

Using definition of $\gamma^*$, (6.46) becomes

\begin{equation}
I(\rho') - I(\rho) \leq \int_{\Omega} G(\rho') dx - \int_{\Omega} G(\rho_1) dx - \varepsilon \int_{E \times E} \frac{|x-y|^{q(x)}}{h(x,y)^{-q(x)}} d\gamma(x,y)
\end{equation}

Since $G$ is $C^2$ and convex, one obtains

\begin{equation}
\int_{\Omega} G(\rho') dx - \int_{\Omega} G(\rho_1) dx \leq -2\varepsilon^2 \int_{E \times E} G''(\rho_1 - \theta M_2 \varepsilon) d\gamma_1(x,y) < 0
\end{equation}

for $\theta \in ]0,1[$.

Using (6.40) and (6.43), we get $I(\rho') < I(\rho_1)$, this is a contradiction because $\rho_1$ is solution of problem $(P^1)$.

We deduce that $\rho^1 \leq M_2$.

We use now (6.35) and then we obtain $\rho^h \leq M_2$ for all $h > 0$.

We use now (6.35) and then

\begin{equation}
\int_{[0,T] \times \Omega} |f(x)|^2 \rho^h dtdx \leq p^+ \left[ \int_{\Omega} G(\rho_0) dx - \Omega |G(1/|\Omega|) | \right].
\end{equation}

Taking into account the coercivity of the map: $v \mapsto |v|^q(x)$ for any fixed $x$ when $\varepsilon$ is inf $q(x) > 1$, one can derive the following

\begin{equation}
\int_{[0,T] \times \Omega} |\nabla \xi H'(\rho^h)| dtdx \leq K + \int_{[0,T] \times \Omega} |\nabla \xi G'(\rho^h)|^q(x) \rho^h dtdx \leq p^+ \left[ \int_{\Omega} G(\rho_0) dx - \Omega |G(1/|\Omega|) | \right] + K.
\end{equation}

Where $K$ is a constant and $H$ is a convex function such that $H''(t) = t G''(t)$, $t > 0$.

We deduce that $(\nabla \xi H'(\rho^h))_h$ is bounded on $L^1([0,T] \times \Omega)$.

Note that $(\rho^h)_h$ is bounded in $L^\infty([0,T] \times \Omega)$, for $0 < T < \infty$ because $\rho^h \leq M_2$ and $[0,T] \times \Omega$ is bounded when $0 < T < \infty$.

Then $(H'(\rho^h(t,.)))_h$ is bounded in $W^{1,1}(\Omega)$ for all $t > 0$. This implies that, up to a subsequence, the vector valued measures $\nabla (H'(\rho^h(x,.))) dtdx$ converges weakly to a measure $\mu$ of finite mass.

Hence, we have

\begin{equation}
\infty > \lim_{h \to 0} \int_{[0,T] \times \Omega} |\nabla \xi G'(\rho^h)|^q(x) \rho^h dtdx
\end{equation}

\begin{equation}
\geq \lim_{h \to 0} \int_{[0,T] \times \Omega} \langle \nabla \xi G'(\rho^h), \omega(t,x) \rangle \rho^h dtdx - \lim_{h \to 0} \int_{[0,T] \times \Omega} |\omega(t,x)|^p(x) \rho^h dtdx
\end{equation}

\begin{equation}
= \int_{[0,T] \times \Omega} \langle w(t,x), \mu \rangle dtdx - \int_{[0,T] \times \Omega} |w(t,x)|^p(x) \rho(t,x) dtdx
\end{equation}

for every continuous function $w$. Consequently, $\mu$ is absolutely continuous with respect to $\rho(t,x) dtdx$ and then, there is a borel function $K_1 : [0,T] \times \Omega \to \mathbb{R}^N$ such that $\mu dtdx = K(t,x) \rho(t,x) dtdx$.

We conclude that $(H'(\rho^h))$ is bounded in $BV(\Omega)$.

So, up to a subsequence, there exists $\beta_1 \in L^1(\Omega)$ such that $H'(\rho^h(t,.))$ converge strongly to $\beta_1$ in $L^1(\Omega)$.

Since the Legendre transform $H^*$ of $H$ is convex, then we conclude that $\rho^h(t,.):= H^*(H'(\rho^h))$ converge strongly to $\rho(t,.)$ in $L^1(\Omega)$.

Since $G'$ is continuous $G'(\rho^h)$ converge strongly to $G'(\rho)$ in $L^1(\Omega)$.

Note that $|\nabla \xi G'(\rho^h)|^{q(x)-2} \nabla \xi G'(\rho^h)|^{p(x)} = |\nabla \xi G'(\rho^h)|^{q(x)}$, then $(|\nabla \xi G'(\rho^h)|^{q(x)-2} \nabla \xi G'(\rho^h))_h$ is bounded in $L^p(x)([0,\infty[\times \Omega)$.

$L^p(x)([0,T] \times \Omega)$ being reflexive, for $0 < T < \infty$, (see [8]), the sequence $(|\nabla \xi G'(\rho^h)|^{q(x)-2} \nabla \xi G'(\rho^h))_h$ converges weakly to some $\sigma$ in $L^p(x)([0,T] \times \Omega)$ up to a subsequence.

Arguing as in [2] we show, that $\sigma = |\nabla \xi G'(\rho)|^{q(x)-2} \nabla \xi G'(\rho)$ in the weak sense.
Theorem 6.1. Assume that the initial datum $\rho_0$ satisfy $\rho_0 \in L^\infty(\Omega)$ and that $G$ satisfy the above assumptions.
If $t \rightarrow u(t)$ is a positive test function whose support is in $[-T,T]$ for $0 < T < \infty$, then

$$
\lim_{h \to 0} \int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx = \int_{\Omega_T} \rho_0, \nabla_x G' (\rho^h) > u(t) dtdx,
$$

where $\Omega_T := [0, T] \times \Omega$, and $\rho$ and $\sigma$ are defined above. Furthermore, $\text{div}_x \{ \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h) \}$ converges weakly $\text{div}_x (\rho^s)$ in $[C^\infty(\mathbb{R} \times \Omega)]'$, and $\text{div}_x (\rho^s) = \text{div}_x (\rho |\nabla_x G' (\rho)|^{q(x)-2} \nabla_x G (\rho))$ in a weak sense.

Proof 6.4. The proof of (6.51) will be derived from the following three lemmas

Lemma 6.2. For $0 < T < +\infty$, we have

$$
\int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx \leq \liminf_{h \to 0} \int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx
$$

with $\Omega_T := [0, T] \times \Omega$.

Proof 6.5. Since $\rho^h$ and $u$ is positive and $y \rightarrow |y|^{q(x)-2} y$ is monotone, we have

$$
\int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h) - |\nabla_x G' (\rho)|^{q(x)-2} \nabla_x G' (\rho) = \rho^h |\nabla_x G' (\rho)|^{q(x)-2} \nabla_x G' (\rho) - \nabla_x G' (\rho) > u(t) dtdx \geq 0.
$$

By the previous inequality, we obtain

$$
\int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx \geq \int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx + \int_{\Omega_T} \rho^h |\nabla_x G' (\rho)|^{q(x)-2} \nabla_x G' (\rho), \nabla_x G' (\rho) - \nabla_x G' (\rho) > u(t) dtdx.
$$

Then, using the strong convergence of $\rho^h(t, \cdot)$ to $\rho(t, \cdot)$, the weak convergence of $(|\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h))_h$ to $\sigma$ and the weak convergence of $(\nabla_x G' (\rho^h))_h$ to $\nabla_x G' (\rho)$, we have

$$
\lim_{h \to 0} \int_{\Omega_T} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx = \int_{\Omega_T} \rho_0, \nabla_x G' (\rho^h) > u(t) dtdx.
$$

Also

$$
\lim_{h \to 0} \int_{\Omega_T} \rho^h |\nabla_x G' (\rho)|^{q(x)-2} \nabla_x G' (\rho), \nabla_x G' (\rho) - \nabla_x G' (\rho) > u(t) dtdx = 0.
$$

By tending $h$ to $0$ in (6.54) and using (6.53) and (6.51), we obtain the proof of (6.2).

Lemma 6.3. For $0 < T < \infty$, we have

$$
\lim_{h \to 0} \sup_{t \in [0, T]} \int_{\Omega} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) > u(t) dtdx \leq \int_{\Omega} \left[ \rho_0 G' (\rho_0) - G^*(G' (\rho_0)) \right] u(0) dx + \int_{\Omega_T} \left[ \rho G' (\rho) - G^*(G' (\rho)) \right] u'(t) dtdx,
$$

where $G^*$ is Legendre transform of $G$.

Proof 6.6. Since $\rho^k$ is solution of $(P^k)$, we use (6.2) and obtain

$$
\int_{\Omega} G(\rho^{k-1}) dx - \int_{\Omega} G(\rho^k) dx = \frac{h}{\rho^k} \int_{\Omega} \rho^h |\nabla_x G' (\rho^h)|^{q(x)-2} \nabla_x G' (\rho^k), \nabla_x G' (\rho^k) > dx.
$$
Multiplying the previous inequality by \( u \geq 0 \), we obtain after integration
\[
(6.59) \quad \sum_{k = 1}^{t_k} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left[ \frac{G'(\rho^{k-1}) - G'(\rho^k)}{h^2} \right] u(t) \, dt \, dx \geq \int_{[0,T] \times \Omega} \langle \rho^h | \nabla_x G' (\rho^h) \rangle_{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) \rangle \, u(t) \, dt \, dx.
\]

Notice that
\[
(6.60) \quad \sum_{k = 1}^{t_k} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left[ \frac{G'(\rho^{k-1}) - G'(\rho^k)}{h^2} \right] u(t) \, dt \, dx = \int_{[0,T] \times \Omega} G(\rho^h) \left[ \frac{u(t) - u(t - h^2)}{h^2} \right] \, dt \, dx + \frac{1}{h^2} \int_0^{t_k} \int_{\Omega} G(\rho^h) u(t - h^2) \, dt \, dx.
\]

We tend h to 0 in (6.60), and obtain
\[
(6.61) \quad \lim_{h \to 0} \sum_{k = 1}^{t_k} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left[ \frac{G'(\rho^{k-1}) - G'(\rho^k)}{h^2} \right] u(t) \, dt \, dx = \int_{[0,T] \times \Omega} G(\rho) u'(t) \, dt \, dx + \int_{\Omega} G(\rho_0) u(0) \, dx.
\]

We use (6.59) and (6.61), and obtain
\[
(6.62) \quad \limsup_{h \to 0} \int_{[0,T] \times \Omega} \langle \rho^h | \nabla_x G' (\rho^h) \rangle_{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) \rangle \, u(t) \, dt \, dx \leq \left[ \int_{[0,T] \times \Omega} G(\rho) u'(t) \, dt \, dx + \int_{\Omega} G(\rho_0) u(0) \, dx \right]
\]

From the definition of \( G^* \), we have \( G^*(a) \geq ab - G(b) \) for all \( a, b > 0 \) and we obtain the equality if \( a = G'(b) \). Then, using \( G(\rho_0) = \rho_0 G'(\rho_0) - G^*(\rho_0) \) and \( G(\rho) = \rho G'(\rho) - G^*(\rho) \) in (6.62), we obtain (6.57). \( \blacksquare \)

**Lemma 6.4.** For \( 0 < T < \infty \), we have
\[
(6.63) \quad \int_{[0,T] \times \Omega} \langle \rho \sigma, \nabla_x G' (\rho) \rangle \, u(t) \, dt \, dx \geq \int_{\Omega} \left[ \rho_0 G'(\rho_0) - G^*(G'(\rho_0)) \right] u(0) \, dx + \int_{[0,T] \times \Omega} \left[ \rho G'(\rho) - G^*(G'(\rho)) \right] u'(t) \, dt \, dx
\]

**Proof 6.7.** Define \( \psi(t, x) = G'(\rho(t, x)) u(t), \psi \in W^{1,q(x)}([0, + \infty[ \times \Omega) \). Approximating \( \psi \) by \( C^\infty_c(\Omega) \) functions and using (6.30), we have
\[
(6.64) \quad \int_{\Omega} \frac{\rho^k - \rho^k}{h} \psi(t, x) \, dx = - \int_{\Omega} \langle \rho^k | \nabla_x G' (\rho^k) \rangle_{q(x)-2} \nabla_x G' (\rho^k), \nabla_x G' (\rho^k) \rangle \, u(t) \, dx + o(h),
\]
where \( o(h) \) tends to 0 when \( h \) tends to 0.

By using the definition of \( \rho^h \), we obtain after integration
\[
(6.65) \quad \sum_{k = 1}^{t_k} \int_{t_{k-1}}^{t_k} \int_{\Omega} \frac{\rho^k - \rho^k}{h} \psi(t, x) \, dt \, dx = - \int_{[0,T] \times \Omega} \langle \rho^h | \nabla_x G' (\rho^h) \rangle_{q(x)-2} \nabla_x G' (\rho^h), \nabla_x G' (\rho^h) \rangle \, u(t) \, dt \, dx + o(h).
\]
Also

\begin{equation}
\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{\Omega} \rho(t-h) \frac{k}{h} \psi(t,x) dt dx = \int_{[0,T] \times \Omega} (\rho^{h} - \rho) \left[ \frac{\psi(t-h,x) - \psi(t,x)}{h} \right] dt dx + \int_{[0,T] \times \Omega} \rho u(t-h) \left[ \frac{G^{h}(\rho(t-h)) - G^{h}(\rho(t))}{h} \right] dt dx + \int_{[0,T] \times \Omega} \rho G^{h}(\rho) \left[ \frac{u(t-h) - u(t)}{h} \right] dt dx.
\end{equation}

Since \((\rho^{h}(t,))_{h}\) converges strongly to \(\rho(t,)\)

\begin{equation}
\lim_{h \to 0} \int_{[0,T] \times \Omega} (\rho^{h} - \rho) \left[ \frac{\psi(t-h,x) - \psi(t,x)}{h} \right] dt dx = 0.
\end{equation}

We tend \(h\) to 0 in (6.66), and using (6.67), we have

\begin{equation}
\lim_{h \to 0} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{\Omega} \rho \psi(t,x) dt dx = \lim_{h \to 0} \int_{[0,T] \times \Omega} \rho u(t-h) \left[ \frac{G^{h}(\rho(t-h)) - G^{h}(\rho(t))}{h} \right] dt dx + \int_{[0,T] \times \Omega} \rho G^{h}(\rho) \left[ u(t-h) - u(t) \right] dt dx.
\end{equation}

Since \(G^{*}\) is convex, then

\begin{equation}
\rho \left[ G^{*}(\rho(t-h)) - G^{*}(\rho(t)) \right] \leq G^{*}(G^{h}(\rho(t-h))) - G^{*}(G^{h}(\rho)).
\end{equation}

Using the previous inequality, we obtain after integration

\begin{equation}
\int_{[0,T] \times \Omega} \rho u(t-h) \left[ \frac{G^{h}(\rho(t-h)) - G^{h}(\rho)}{h} \right] dt dx \leq \int_{[0,T] \times \Omega} u(t-h) \left[ \frac{G^{*}(G^{h}(\rho(t-h))) - G^{*}(G^{h}(\rho))}{h} \right] dt dx.
\end{equation}

From (6.70) in (6.68) we obtain

\begin{equation}
\lim_{h \to 0} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{\Omega} \rho \psi(t,x) dt dx \leq - \int_{\Omega} \left[ \rho G^{h}(\rho_{0}) - G^{*}(G^{h}(\rho_{0})) \right] u(0) dx + \int_{[0,T] \times \Omega} \rho G^{h}(\rho) \left[ G^{h}(\rho) - G^{*}(G^{h}(\rho)) \right] u'(t) dt dx.
\end{equation}

Combining (6.71) and (6.68) and passing to the limit, we reach (5.63).

To get the proof of (5.51), we use the results in the three previous lemmas.

Now, let show that

\begin{equation}
(div_{x}(\rho^{h} | \nabla_{x} G^{h}(\rho^{h})|^{q(x)-2} \nabla_{x} G^{h}(\rho^{h}))_{h}
\end{equation}

converges to

\begin{equation}
div_{x}(\rho \sigma) = div_{x}(\rho \nabla_{x} G^{h}(\rho)|^{q(x)-2} \nabla_{x} G^{h}(\rho)) \quad \text{in} \quad [C^{n}_{c}([0,T] \times \Omega)]'.
\end{equation}
Let $\epsilon > 0$ be small and $\phi \in C_0^\infty(\Omega)$ be a test function. Define $\psi_\epsilon(t,x) = G'(\rho) - \epsilon \phi(x)$.

We use the fact that $y \mapsto |y|^{q(x)-2}y$ is monotone to derive

\begin{equation}
(6.72) \quad \int_{[0,T] \times \Omega} \rho^b < |\nabla_x G'(\rho^h)|^{q(x)-2} \nabla_x G'(\rho^h), \nabla_x \psi_\epsilon > u(t) dtdx \geq 0.
\end{equation}

Thus

\begin{equation}
(6.73) \quad \int_{[0,T] \times \Omega} \rho^b < |\nabla_x G'(\rho^h)|^{q(x)-2} \nabla_x G'(\rho^h), \nabla_x \psi_\epsilon > u(t) dtdx
- \int_{[0,T] \times \Omega} \rho^b < |\nabla_x G'(\rho^h)|^{q(x)-2} \nabla_x G'(\rho^h), \nabla_x \psi_\epsilon > u(t) dtdx
- \int_{[0,T] \times \Omega} \rho^b < |\nabla_x \psi_\epsilon|^{q(x)-2} \nabla_x \psi_\epsilon, \nabla_x G'(\rho^h) - \nabla_x \psi_\epsilon > u(t) dtdx \geq 0.
\end{equation}

We tend $h$ to 0 in the previous inequality, and we use (6.51), to get

\begin{equation}
(6.74) \quad \int_{[0,T] \times \Omega} \rho < \sigma, \nabla_x G'(\rho) > u(t) dtdx
- \int_{[0,T] \times \Omega} \rho < \sigma, \nabla_x \psi_\epsilon > u(t) dtdx
- \int_{[0,T] \times \Omega} \rho < |\nabla_x \psi_\epsilon|^{q(x)-2} \nabla_x \psi_\epsilon, \nabla_x G'(\rho) - \nabla_x \psi_\epsilon > u(t) dtdx \geq 0.
\end{equation}

By using definition of $\psi_\epsilon$, the previous inequality becomes

\begin{equation}
(6.75) \quad \int_{[0,T] \times \Omega} \rho < \sigma, \nabla_x \phi(x) > u(t) dtdx \geq \int_{[0,T] \times \Omega} \rho < |\nabla_x \psi_\epsilon|^{q(x)-2} \nabla_x \psi_\epsilon, \nabla_x \phi(x) > u(t) dtdx.
\end{equation}

We tend $\epsilon$ to 0, and we have

\begin{equation}
(6.76) \quad \int_{[0,T] \times \Omega} \rho < \sigma, \nabla_x \phi(x) > u(t) dtdx \geq \int_{[0,T] \times \Omega} \rho < |\nabla_x G'(\rho)|^{q(x)-2} \nabla_x G'(\rho), \nabla_x \phi(x) > u(t) dtdx.
\end{equation}

Replacing $\phi$ by $-\phi$ in the previous inequality, we obtain the equality:

\begin{equation}
(6.77) \quad \int_{[0,T] \times \Omega} \rho < \sigma, \nabla_x \phi(x) > u(t) dtdx = \int_{[0,T] \times \Omega} \rho < |\nabla_x G'(\rho)|^{q(x)-2} \nabla_x G'(\rho), \nabla_x \phi(x) > u(t) dtdx.
\end{equation}

Finally, we deduce that the sequence $\text{div}_x(\rho^h|\nabla_x G'(\rho^h)|^{q(x)-2} \nabla_x G'(\rho^h))$ converges to $\text{div}_x(\rho|\nabla_x G'(\rho)|^{q(x)-2} \nabla_x G'(\rho))$ in $[C_0^\infty([0,T] \times \Omega)]'$.

6.3. Existence and uniqueness of solution. In this section, we show existence and uniqueness of weak solutions of the parabolic $q(x)$-Laplacian equation (6.1).

**Theorem 6.2.** Assume that $G$ satisfies (A1) and the initial datum $\rho_0$ satisfy (A3). Then, the sequence $(\rho^h)_h$ converges strongly to a positive function $\rho(t,x)$ and $\rho \in L^\infty([0,\infty[ \times \Omega)$. Also $\rho$ is a weak solution of the equation (6.1), i.e., for all $\phi(t,x) \in C_0^\infty([0,\infty[ \times \Omega)$, $\text{supp}(\phi,\lambda) \subset [-T,T]$, for $0 < T < \infty$, we have:

\begin{equation}
(6.78) \quad \int_{[0,\infty[ \times \Omega} \rho \left[ \frac{\partial \phi(t,x)}{\partial t} - < |\nabla_x G'(\rho)|^{q(x)-2} \nabla_x G'(\rho), \nabla_x \phi(t,x) > \right] dtdx = - \int_0^T \rho(t_0) \phi(0,x) dx.
\end{equation}

**Proof 6.8.** Using (6.28):

\begin{equation}
(6.79) \quad \int_{[0,\infty[ \times \Omega} \rho^k - \rho^{k-1}_h \phi(t,x) dtdx + \int_{[0,\infty[ \times \Omega} \rho^b < |\nabla_x G'(\rho^h)|^{q(x)-2} \nabla_x G'(\rho^h), \nabla_x \phi(t,x) > dtdx = 0(h),
\end{equation}

where $\phi(h)$ tends to 0 when $h$ tends to 0.

**Note:**

\begin{equation}
(6.80) \quad \int_{[0,\infty[ \times \Omega} \rho^k - \rho^{k-1}_h \phi(t,x) dtdx = \int_{[0,\infty[ \times \Omega} \rho^b \left[ \frac{\phi(t-h,x) - \phi(t,x)}{h} \right] dtdx - \frac{1}{h} \int_0^h \int_{[0,\infty[ \times \Omega} \rho^b \phi(t-h,x) dtdx.
\end{equation}
Replacing the previous relation in (6.79), we have:

\begin{align}
\int_{[0,\infty]\times\Omega} \rho^h \left[ \frac{\phi(t-h,x) - \phi(t,x)}{h} \right] dt dx - \frac{1}{h} \int_0^h \int_\Omega \rho^h \phi(t-h) dt dx + \\
\int_{[0,\infty]\times\Omega} \rho^h \langle \nabla_x G'(\rho^h) \rangle (\rho^h, \nabla_x \phi(t,x)) dt dx = 0(h).
\end{align}

We tend \( h \) to 0 in (6.81) and use theorem (6.2) to obtain:

\begin{align}
\int_{[0,\infty]\times\Omega} \rho \left[ \frac{\partial \phi(t,x)}{\partial t} - \langle \nabla_x G'(\rho) \rangle (\rho^h, \nabla_x \phi(t,x)) \right] dt dx = - \int_\Omega \rho_0 \phi(0,x) dx.
\end{align}

We conclude that \( \rho \) is a weak solution of the parabolic \( q(x) \)-Laplacian equation (6.1). \( \square \)

**Theorem 6.3.** Assume that \( p(.) : \Omega \rightarrow [1, +\infty[ \) satisfy (A1), \( G \) satisfy (A2) and (A4); and \( \rho_0 \) satisfy (A3). Let \( \rho^1 \) and \( \rho^2 \) be two weak solutions of (6.1) satisfying \( \partial \theta_i / \partial t \in L^1(\Omega) \), for \( i = 1, 2 \), with initial datum \( \rho^1(0,.) \) and \( \rho^2(0,.) \) respectively satisfying (A3).

If \( \rho^1(0,.) = \rho^2(0,.) \), then:

\begin{align}
\int_\Omega [\rho^1(T,x) - \rho^2(T,x)]^+ dx \leq 0,
\end{align}

for all \( T \geq 0 \).

**Proof 6.9.** Define \( \theta_\delta : \mathbb{R} \rightarrow [0, 1] \), by:

\begin{align}
\theta_\delta(s) = \begin{cases}
0 & \text{if } s \leq 0 \\
\frac{s}{\delta} & \text{if } 0 \leq s \leq \delta \\
1 & \text{if } s \geq \delta.
\end{cases}
\end{align}

By using definition of the weak solution, we have:

\begin{align}
\int_{[0,T]\times\Omega} \rho \left[ \frac{\partial \phi(t,x)}{\partial t} - \langle \nabla_x G'(\rho^1) \rangle (\rho^1, \nabla_x \phi(t,x)) \right] dt dx = - \int_{[0,T]\times\Omega} \rho^1 |\nabla G'(\rho^1)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^1) - \rho^2 |\nabla G'(\rho^2)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^2), \nabla \phi dt dx.
\end{align}

We use \( \theta_\delta(G'(\rho^1) - G'(\rho^2)) \) in (6.85); we have:

\begin{align}
\int_{[0,T]\times\Omega} \theta_\delta(G'(\rho^1) - G'(\rho^2)) \frac{\partial}{\partial t} (\rho^1(t,x) - \rho^2(t,x)) dt dx = \\
- \int_{[0,T]\times\Omega} \rho^1 |\nabla G'(\rho^1)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^1) - \rho^2 |\nabla G'(\rho^2)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^2), \nabla (\theta_\delta(G'(\rho^1) - G'(\rho^2))) dt dx = \\
- \frac{1}{\delta} \int_{\Omega_T, \delta} \rho^1 |\nabla G'(\rho^1)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^1) - |\nabla G'(\rho^2)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^2), \nabla G'(\rho^1) - \nabla G'(\rho^2) > + \\
- \frac{1}{\delta} \int_{\Omega_T, \delta} \rho^2 |\nabla G'(\rho^2)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^2) >
\end{align}

Where

\( \Omega_T, \delta := \Omega_T \cap \{0 < G'(\rho^1) - G'(\rho^2) \leq \delta\} \)

and \( \Omega_T := [0, T] \times \Omega. \)

Since \( y \mapsto |y|^{\langle q(x) - 2 \rangle} y \) is monotone, we have

\begin{align}
- \frac{1}{\delta} \int_{\Omega_T, \delta} \rho^1 |\nabla G'(\rho^1)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^1) - |\nabla G'(\rho^2)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^2), \nabla G'(\rho^1) - \nabla G'(\rho^2) > \leq 0.
\end{align}

Furthermore, on \( \Omega_T, \delta \):

\begin{align}
|\rho^1 - \rho^2| = |(G'(\rho^1))' - (G'(\rho^2))'| \leq \delta \sup_{s \in [0, G'(M_2)]} (G')''(s)
\end{align}

Then,

\begin{align}
- \frac{1}{\delta} \int_{\Omega_T, \delta} <(\rho^1 - \rho^2)|\nabla G'(\rho^2)|^{\langle q(x) - 2 \rangle} \nabla G'(\rho^2) > \leq \Omega_T, \delta.
\end{align}
If $\delta \rightarrow 0^+$, then $|\Omega_{T_\delta}| \rightarrow 0$ and $\theta_1(G'(\rho^1) - G'(\rho^2)) \rightarrow \text{sign}^+(G'(\rho^1) - G'(\rho^2)) = \text{sign}^+(\rho^1 - \rho^2)$; with $\text{sign}(s) = \frac{s}{|s|}$ for all $s \in \mathbb{R}^+$. Then,

$$\int_{[0,T] \times \Omega} \frac{\partial (\rho^1 - \rho^2)}{\partial t}^{+} = \int_{[0,T] \times \Omega} \text{sign}^+(\rho^1 - \rho^2) \frac{\partial}{\partial t}(\rho^1 - \rho^2) \leq 0. \quad (6.88)$$

This implies

$$\int_{\Omega} (\rho^1(T,x) - \rho^2(T,x))^+ dx \leq 0 \quad (6.89)$$

for all $T \geq 0$. Then the solution of the $q(x)$-Laplacian equation (6.1) is unique. ■

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