Dynamos and anti-dynamos as thin magnetic flux ropes in Riemannian spaces

by

L.C. Garcia de Andrade

Departamento de Física Teórica – IF – Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

Two examples of magnetic anti-dynamos in magnetohydrodynamics (MHD) are given. The first is a 3D metric conformally related to Arnold cat fast dynamo metric: \( ds_A^2 = e^{-\lambda z} dp^2 + e^{\lambda z} dq^2 + dz^2 \) is shown to present a behaviour of non-dynamos where the magnetic field exponentially decay in time. The curvature decay as z-coordinates increases without bounds. Some of the Riemann curvature components such as \( R_{pzpz} \) also undergoes dissipation while component \( R_{qzqz} \) increases without bounds. The remaining curvature component \( R_{ppqq} \) is constant on the torus surface. The other anti-dynamo which may be useful in plasma astrophysics is the thin magnetic flux rope or twisted magnetic thin flux tube which also behaves as anti-dynamo since it also decays with time. This model is based on the Riemannian metric of the magnetic twisted flux tube where the axis possesses Frenet curvature and torsion. Since in this last example the Frenet torsion of the axis of the rope is almost zero, or the possible dynamo is almost planar it satisfies Zeldovich theorem which states that planar dynamos do not exist. Changing in topology of this result may result on a real dynamo as discussed. PACS numbers: 02.40.Hw-Riemannian geometries


I Introduction

Geometrical tools have been used with success [1] in Einstein general relativity (GR) have been also used in other important areas of physics, such as plasma structures in tokamaks as been clear in the Mikhailovskii [2] book to investigate the tearing and other sort of instabilities in confined plasmas [2], where the Riemann metric tensor plays a dynamical role interacting with the magnetic field through the magnetohydrodynamical equations (MHD). Recently Garcia de Andrade [3, 4] has also made use of Riemann metric to investigate magnetic flux tubes in superconducting plasmas. Thiffeault and Boozer [5] following the same reasoning applied the methods of Riemann geometry in the context of chaotic flows and fast dynamos. Yet more recently Thiffeault [6] investigated the stretching and Riemannian curvature of material lines in chaotic flows as possible dynamos models. An interesting tutorial review of chaotic flows and kinematical dynamos has been presented earlier by Ott [7]. Also Boozer [8] has obtained a geomagnetic dynamo from conservation of magnetic helicity. This can also be shown here in the generalization to non-holonomic Frenet frame [9]. In this paper we use the tools of Riemannian geometry, also user in GR, to obtain anti-dynamos in the conformal cat dynamo metric [10]. We also use the Euler equations for incompressible flows in Arnold metric [11]. Antidynamos or non-dynamos are also important in the respect that it is important to recognize when a topology or geometry of a magnetic field does force the field to decay exponentially for example. As we know planar dynamos does not exist and Anti-dynamos theorems are important in this respect. Thus in the present paper we also obtain antidynamos metrics which are conformally related to the fast dynamo metric discovered by Arnold. Levi-Civita connections [12] are found together Riemann curvature from the MAPLE X GR tensor package. The paper is organized as follows: In section II the the non-holonomic Frenet frame in MHD is briefly reviewed. In section III the flux rope thin antidynamo solution is presented. Curvature and connection are found for the Arnold metric in section IV. In section V the conformal antidynamo metric is considered. In section VI the dynamo solution is found from topological considerations. Conclusions are presented in section VII.
II MHD scalar equations for kinematical dynamos in nonholonomic Frenet frame

Let us now start by considering the MHD field equations

\[ \nabla \cdot \vec{B} = 0 \quad (\text{II.1}) \]

\[ \frac{\partial}{\partial t} \vec{B} - \nabla \times [\vec{u} \times \vec{B}] - \epsilon \nabla^2 \vec{B} = 0 \quad (\text{II.2}) \]

where \( \vec{u} \) is a solenoidal field while \( \epsilon \) is the diffusion coefficient. Equation (II.2) represents the induction equation. The magnetic field \( \vec{B} \) is chosen to lie along the filament and is defined by the expression \( \vec{B} = B(s, n)\vec{t} \) and \( \vec{u} = u\vec{b} \) is the speed of the flow. The remaining coordinate \( n \) is orthogonal to the filament all along its extension, and the arc length \( s \) measures distances along the filament itself. The vectors \( \vec{t} \) and \( \vec{n} \) along with binormal vector \( \vec{b} \) together form the Frenet frame which obeys the Frenet-Serret equations

\[ \vec{t}' = \kappa \vec{n} \quad (\text{II.3}) \]

\[ \vec{n}' = -\kappa \vec{t} + \tau \vec{b} \quad (\text{II.4}) \]

\[ \vec{b}' = -\tau \vec{n} \quad (\text{II.5}) \]

the dash represents the ordinary derivation with respect to coordinate \( s \), and \( \kappa(s, t) \) is the curvature of the curve where \( \kappa = R^{-1} \). Here \( \tau \) represents the Frenet torsion. We follow the assumption that the Frenet frame [7] may depend on other degrees of freedom such as that the gradient operator becomes

\[ \nabla = \vec{t} \frac{\partial}{\partial s} + \vec{n} \frac{\partial}{\partial n} + \vec{b} \frac{\partial}{\partial b} \quad (\text{II.6}) \]

The other equations for the other legs of the Frenet frame are

\[ \frac{\partial}{\partial n} \vec{t} = \theta_{ns} \vec{n} + [\Omega_b + \tau] \vec{b} \quad (\text{II.7}) \]

\[ \frac{\partial}{\partial n} \vec{n} = -\theta_{ns} \vec{t} - (\text{div} \vec{b}) \vec{b} \quad (\text{II.8}) \]

\[ \frac{\partial}{\partial n} \vec{b} = -[\Omega_b + \tau] \vec{t} - (\text{div} \vec{b}) \vec{n} \quad (\text{II.9}) \]
\[
\frac{\partial}{\partial b} \vec{t} = \theta_b \vec{b} - (\Omega_n + \tau) \vec{n} \quad \text{(II.10)}
\]
\[
\frac{\partial}{\partial b} \vec{n} = (\Omega_n + \tau) \vec{t} - \kappa + (\text{div} \vec{n}) \vec{b} \quad \text{(II.11)}
\]
\[
\frac{\partial}{\partial b} \vec{b} = -\theta_b \vec{t} - (\kappa + (\text{div} \vec{n})) \vec{n} \quad \text{(II.12)}
\]

Another set of equations which we shall need here is the time derivative of the Frenet frame given by
\[
\dot{\vec{t}} = [\kappa' \vec{b} - \kappa \tau \vec{n}] \quad \text{(II.13)}
\]
\[
\dot{\vec{n}} = \kappa \tau \vec{t} \quad \text{(II.14)}
\]
\[
\dot{\vec{b}} = -\kappa' \vec{t} \quad \text{(II.15)}
\]

III Thin magnetic flux ropes as antidynamos in Riemannian spaces

In this section we shall concerned with the presentation of a solution of the dynamo equation investigated in previous section which represents a antidynamo thin magnetic flux rope. Earlier Yoshimura [14] has investigated solar dynamos represented by magnetic flux ropes which are actually another name for twisted magnetic flux tubes. Let us now consider here the metric of magnetic flux
\[
ds^2 = dr^2 + r^2 d\theta_R^2 + K^2(s) ds^2 \quad \text{(III.16)}
\]
where the tube coordinates are \((r, \theta_R, s)\) [15] where \(\theta(s) = \theta_R - \int \tau ds\) where \(\tau\) is the Frenet torsion of the tube axis and \(K(s)\) is given by
\[
K^2(s) = [1 - r\kappa(s) \cos \theta(s)]^2 \quad \text{(III.17)}
\]

Computing the Riemannian Laplacian operator \(\nabla^2\) in curvilinear coordinates [16] one obtains
\[
\nabla^2 = \frac{1}{\sqrt{g}} \partial_i [\sqrt{g} g^{ij} \partial_j] \quad \text{(III.18)}
\]
\[
\nabla^2 = \partial_s^2 + \frac{1}{r^2} \partial_{\theta_R}^2 - \left[\frac{(\theta_R + \tau_0)}{r}\right] \partial_{\theta_R} \quad \text{(III.19)}
\]
where \( \partial_j := \frac{\partial}{\partial x^j} \) and \( g := det g_{ij} \) where \( g_{ij} \) is the covariant component of the Riemann metric of the flux rope. Applied to the above flux rope coordinates this yields Note also that we have considered that the flux rope magnetic field does not depend on the \( r \) and \( \theta_R \) coordinates. This avoids the problem of changing poloidal components into toroidal components of the field as in the dynamo solution of Chatterjee, Choudhuri and Petroyav [17] where his dynamo flux tube considers radial components of the magnetic fields involved. Thus the magnetic field here can be expressed as

\[
\vec{B}_s = e^{pt} \vec{B}_0 t
\]  

(III.20)

where from expression \( \nabla \cdot \vec{B} = 0 \) we note that \( B_0 \) is constant, where in general \( p = p(\epsilon) \). Thus substitution of this expression in the equation for the dynamo we obtain

\[
[k_0 \tau_0 \vec{b} - k_0^2 \epsilon \vec{t}] = pt \vec{t} - k_0 [\tau_0 + u_0] \vec{n}
\]  

(III.21)

where the subscript zero indicates constant physical quantities, and we consider the flow velocity \( \vec{u} = u_0 \vec{t} \) as constant in modulus. Splitting this last expression component by component we obtain the following scalar dynamo equations

\[
\tau_0 = -u_0
\]  

(III.22)

\[
p = -k_0^2 \epsilon
\]  

(III.23)

\[
k_0 \tau_0 \epsilon = 0
\]  

(III.24)

These three equations altogether yield that very slow dynamos imply that flux rope is almost planar and equation (III.23) already yields the following solution

\[
\vec{B}_s = e^{-k_0^2 t} \vec{B}_0 t
\]  

(III.25)

Since the resistivity \( \epsilon \geq 0 \), this expression tells us that the magnetic field cannot be sustained and decays in time which shows clearly that this solution does not represent a dynamo. This actually is in agreement with Zeldovich [18] theorem since the flux rope is almost planar.
IV Riemann dynamos and dissipative manifolds and Euler flows

Arnold metric can be used to compute the Levi-Civita-Christoffel connection

\[ \Gamma^p_{pz} = -\frac{\lambda}{2} \]  \hspace{1cm} (IV.26)

\[ \Gamma^q_{qz} = \frac{\lambda}{2} \]  \hspace{1cm} (IV.27)

\[ \Gamma^z_{pp} = \frac{\lambda}{2} e^{-\lambda z} \]  \hspace{1cm} (IV.28)

\[ \Gamma^z_{qq} = -\frac{\lambda}{2} e^{-\lambda z} \]  \hspace{1cm} (IV.29)

from these connection components one obtains the Riemann tensor components

\[ R_{pqpq} = -\frac{\lambda^2}{4} \]  \hspace{1cm} (IV.30)

Note that since this component is negative from the Jacobi equation [7] that the flow is unstable. The other components are

\[ R_{pzpz} = -\frac{\lambda^2}{2} e^{-\lambda z} \]  \hspace{1cm} (IV.31)

\[ R_{qzqz} = -\frac{\lambda^2}{2} e^{\lambda z} \]  \hspace{1cm} (IV.32)

one may immediatly notice that at large values of \( z \) the curvature component \( (zpzp) \) is bounded and vanishes, or undergoes a dissipative effect, while component \( (qzqz) \) of the curvature increases without bounds, component \( (pqpq) \) remains constant.

V Conformal anti-dynamo metric

Conformal metric techniques have been widely used as a powerful tool obtain new solutions of the Einstein’s field equations of GR from known solutions. By analogy, here we are using this method to yield new solutions of MHD anti-dynamo solutions from the well-known fast dynamo Arnold solution. We shall demonstrate that distinct physical features from the Arnold solution maybe obtained. The conformal metric line element can be defined as

\[ ds^2 = \lambda^{-2z} ds_A^2 = dx_+^2 + \lambda^{-4z} dx_-^2 + \lambda^{-2z} dz^2 \]  \hspace{1cm} (V.33)
where we have used here the Childress and Gilbert [5] notation for the Arnold metric in $\mathcal{R}^3$ which reads now

$$ds_A^2 = \lambda^{2z}dx_+^2 + \lambda^{-2z}dx_-^2 + dz^2$$  \hspace{1cm} (V.34)

where the coordinates are defined by

$$\vec{x} = x_+\vec{e}_+ + x_-\vec{e}_- + z\vec{e}_z$$  \hspace{1cm} (V.35)

where a right handed orthogonal set of vectors in the metric is given by

$$\vec{f}_+ = \vec{e}_+$$  \hspace{1cm} (V.36)

$$\vec{f}_- = \lambda^{2z}\vec{e}_-$$  \hspace{1cm} (V.37)

$$\vec{f}_z = \lambda^z\vec{e}_z$$  \hspace{1cm} (V.38)

A component of a vector in this basis, such as the magnetic vector $\vec{B}$ is

$$\vec{B} = B_+\vec{f}_+ + B_-\vec{f}_- + B_z\vec{f}_z$$  \hspace{1cm} (V.39)

The vector analysis formulas in this frame are

$$\nabla = [\partial_+, \lambda^{2z}\partial_-, \lambda^z\partial_z]$$  \hspace{1cm} (V.40)

$$\nabla^2 \phi = [\partial_+^2\phi, \lambda^{4z}\partial_-^2\phi, \lambda^{2z}\partial_z^2\phi]$$  \hspace{1cm} (V.41)

The MHD dynamo equations are

$$\nabla.\vec{B} = \partial_+B_+ + \lambda^{2z}\partial_-B_- + \lambda^z\partial_zB_z = 0$$  \hspace{1cm} (V.42)

$$\partial_t\vec{B} + (\vec{u}.\nabla)\vec{B} - (\vec{B}.\nabla)\vec{u} = \epsilon \nabla^2\vec{B}$$  \hspace{1cm} (V.43)

where $\epsilon$ is the conductivity coefficient. Since here we are working on the limit $\epsilon = 0$, which is enough to understand the physical behavior of the fast dynamo, we do not need to worry to expand the RHS of equation (V.43), and it reduces to

$$(\vec{u}.\nabla)\vec{B} = \partial_z[B_+\vec{e}_+ + B_-\epsilon^{2\mu z}\vec{e}_- + B_z\epsilon^{\mu z}\vec{e}_z]$$  \hspace{1cm} (V.44)

where we have used that $(\vec{B}.\nabla)\vec{u} = B_z\mu\epsilon^{\mu z}\vec{e}_z$ and that $\mu = \log\lambda$. This is one of the main differences between Arnold metric and ours since in his fast dynamo, this relation vanishes.
since in Arnold metric $\vec{u} = \vec{e}_z$ where $\vec{e}_z$ is part of a constant basis. Separating the equation in terms of the coefficients of $\vec{e}_+$, $\vec{e}_-$ and $\vec{e}_z$ respectively one obtains the following scalar equations

\[
\begin{align*}
\partial_z B_+ + \partial_t B_+ &= 0 \quad (V.45) \\
\partial_t B_- + \partial_t B_+ 2\mu B_- &= 0 \quad (V.46) \\
\partial_t B_z + \partial_z B_- &= 0 \quad (V.47)
\end{align*}
\]

Solutions of these equations allows us to write down an expression for the magnetic vector field $\vec{B}$ as

\[
\vec{B} = [B^0_z, \lambda^{-(t+z)}B^0_-, B^0_z](t - z, y, x + y) \quad (V.48)
\]

From this expression we can infer that the field is carried in the flow, stretched in the $\vec{f}_z$ direction and compressed in the $\vec{f}_-$ direction, while in Arnold’s cat fast dynamo is also compressed along the $\vec{f}_-$ direction but is stretched along $\vec{f}_+$ direction while here this direction is not affected. But the main point of this solution is the fact that the solution represents an anti-dynamo since as one can see from expression (V.48) the magnetic field fastly decays exponentially in time as $e^{\mu(t+z)}$. Let us now compute the Riemann tensor components of the new conformal metric to check for the stability of the non-dynamo flow. To easily compute this curvature components we shall make use of Elie Cartan [13] calculus of differential forms, which allows us to express the conformal metric as

\[
ds^2 = dp^2 + e^{4\lambda z} dq^2 + e^{\lambda z} dz^2 \quad (V.49)
\]

or in terms of the frame basis form $\omega^i$ is

\[
ds^2 = (\omega^p)^2 + (\omega^q)^2 + (\omega^z)^2 \quad (V.50)
\]

where we are back to Arnold’s notation for convenience. The basis form are write as

\[
\begin{align*}
\omega^p &= dp \quad (V.51) \\
\omega^q &= e^{\lambda z} dq \quad (V.52) \\
\omega^z &= e^{2\lambda z} dq \quad (V.53)
\end{align*}
\]
By applying the exterior differentiation in this basis form one obtains

\[ d\omega^p = 0 \]  \hspace{1cm} (V.54)

\[ d\omega^z = 0 \]  \hspace{1cm} (V.55)

and

\[ d\omega^q = \lambda e^{-\frac{1}{2}z}\omega^z \wedge \omega^q \]  \hspace{1cm} (V.56)

Substitution of these expressions into the first Cartan structure equations one obtains

\[ T^p = 0 = \omega^p \wedge \omega^q + \omega^p \wedge \omega^z \]  \hspace{1cm} (V.57)

\[ T^q = 0 = \lambda e^{-\frac{1}{2}z}\omega^z \wedge \omega^q + \omega^q \wedge \omega^p + \omega^q \wedge \omega^z \]  \hspace{1cm} (V.58)

and

\[ T^z = 0 = \omega^z \wedge \omega^p + \omega^z \wedge \omega^q \]  \hspace{1cm} (V.59)

where \( T^i \) are the Cartan torsion 2-form which vanishes identically on a Riemannian manifold. From these expressions one is able to compute the connection forms which yields

\[ \omega^p_q = -\alpha \omega^p \]  \hspace{1cm} (V.60)

\[ \omega^q_z = \lambda e^{-\frac{1}{2}z}\omega^q \]  \hspace{1cm} (V.61)

and

\[ \omega^z_p = \beta \omega^p \]  \hspace{1cm} (V.62)

where \( \alpha \) and \( \beta \) are constants. Substitution of these connection form into the second Cartan equation

\[ R^i_{\ j} = R^i_{\ jkl} \omega^k \wedge \omega^l = d\omega^i_{\ j} + \omega^i_{\ l} \wedge \omega^l_{\ j} \]  \hspace{1cm} (V.63)

where \( R^i_{\ j} \) is the Riemann curvature 2-form. After some algebra we obtain the following components of Riemann curvature for the conformal antidynamo

\[ R^{pq}_{\ pq} = \lambda e^{-\frac{1}{2}z} \]  \hspace{1cm} (V.64)

\[ R^{qz}_{\ qz} = \frac{1}{2} \lambda^2 e^{-\lambda z} \]  \hspace{1cm} (V.65)
and finally

\[ R^p_{zpq} = -\alpha \lambda e^{-\frac{\lambda}{2}z} \]  

(V.66)

We note that only component to which we can say is positive is \( R^p_{zpq} \) which turns the flow stable in this q-z surface. This component also dissipates away when \( z \) increases without bounds, the same happens with the other curvature components [13].

### VI Dynamos by topology change

A long and straightforward computation, specially due to the computation of \( \nabla^2 A \). and substituting these equations for the dynamics of the Frenet frame leads to the scalar MHD expressions

\[
\begin{align*}
\partial_t A &= -\partial_s \phi + [\partial^2_n A - A(\theta_{ns}^2 - \kappa_0^2)] \\
-\kappa \tau A &= -uB + \epsilon [2\partial_n A + (\Omega_s + \tau)\theta_{ns}A] \\
-\theta_{bs} A &= \epsilon [2\partial_n A\Omega_s + \Omega A] \tag{VI.69}
\end{align*}
\]

where \( \kappa_0 \) is the Frenet curvature of the streamlines. These equations have already been simplified by using the relations

\[
\nabla \times \vec{A} = \vec{B} \tag{VI.70}
\]

which yields the following differential scalar equations

\[
\begin{align*}
B &= -A[\Omega_s + \tau] \tag{VI.71} \\
\partial_n A + \kappa A &= 0 \tag{VI.72} \\
A(\Omega_n + \tau) &= 0 \tag{VI.73}
\end{align*}
\]

Where the \( \Omega_s \)’s represent the abnormalities of the streamlines of the flow. When the \( \Omega_s \) vanishes we note the geodesic streamlines are obtained. As we shall see below here we are not consider geodesic flows dynamos. By considering planar flows where torsion vanishes and the gauge condition

\[
\nabla.A + \frac{\partial}{\partial t} \phi = 0 \tag{VI.74}
\]
This equation can be expressed as

\[ \partial_s A + [\theta_{ns} + \theta_{bs}] A = 0 \]  

(VI.75)

Now by considering that \( A \) does not depend on the coordinate \( s \) this expression reduces to

\[ [\theta_{ns} + \theta_{bs}] A = 0 \]  

(VI.76)

which reduces to \( \theta_{ns} = -\theta_{bs} \). By making use of this expression and the assumption that \( \phi = 0 \) one simplifies the MHD scalar equations to

\[ \partial_t A = [\partial^2_n A - A(\theta_{ns}^2 - \kappa_0^2)] \]  

(VI.77)

\[ u A \Omega_b + \epsilon [2 \kappa_0 + \Omega_s] A = 0 \]  

(VI.78)

\[ \theta_{ns} A = \epsilon [-2 \kappa_0 \Omega_s + \Omega_s^2] \theta_{ns} A \]  

(VI.79)

Simple solution of these two last equations reads

\[ u = -\frac{\Omega_s e^{\kappa_0^2}}{\Omega_b} \]  

(VI.80)

and

\[ \theta_{ns} = [b^2 - 2 \kappa_0 b] \]  

(VI.81)

where \( b := \Omega_s \). From expression

\[ \partial_t A - \theta_{ns}^2 A = 0 \]  

(VI.82)

which yields the solution

\[ A = A_0(n) e^{[\theta_{ns}^2] t} \]  

(VI.83)

To simplify the analysis of Arnold’s theorem [8] in the next section we consider the geodesic flow assumption which simplifies this solution to \( A = A_0(n) \) which by solving the equation (VI.82) yields

\[ A = A_0^* e^{-[\kappa_0] n} \]  

(VI.84)

and finally we note that the magnetic field of streamlines becomes

\[ B = -\Omega_b A_0^* e^{-[\kappa_0] n} \]  

(VI.85)
We note from this expression that if the signs of Frenet curvature and coordinate $n$ coincides the magnetic field decays in space and a kinematical dynamo is not obtained. However if the signs do not agree such as $\kappa_0 > 0$ (positive curvature of the streamlines) and $n < 0$ the magnetic field increases with the distance from the streamlines and a kinematical dynamo is obtained.

VII Conclusions

In conclusion, we have used a well-known technique to find solutions of Einstein’s field equations of gravity namely the conformal related spacetime metrics to find a new anti-dynamo solution in MHD nonplanar flows. The stability of the flow is also analysed by using other tools from GR, namely that of Killing symmetries. Examination of the Riemann curvature components enable one to analyse the stretch and compression of the dynamo flow. Other interesting antidynamo metric equations in four-dimensional spacetime [1].

Acknowledgements

Thanks are due to CNPq and UERJ for financial supports. of a thin almost planar flux rope is also shown. Possibility of obtaining dynamos by changing topology in the nonholonomy Frenet frame is also discussed. It is shown that in this case dynamo solution is a realistic possibility.
References

[1] H. Stephani et al, Exact solutions of Einstein field equations (2003) Cambridge university press. G. Ricci, Tensor Analysis, Boston.

[2] A. Mikhailovskii, Instabilities in a Confined Plasma, (1998) IOP.

[3] L. C. Garcia de Andrade, Physics of Plasmas 13, 022309 (2006).

[4] L.C. Garcia de Andrade, Twist transport in strongly torsioned astrophysical flux tubes, Astrophysics and Space Science (2007) in press.

[5] J. Thiffeault and A.H. Boozer, Chaos 11, (2001) 16.

[6] J. Thiffeault, Stretching and Curvature of Material Lines in Chaotic Flows, (2004) Los Alamos arXiv:nlin. CD/0204069.

[7] E. Ott, Phys. Plasmas 5, (1998) 1636.

[8] A.H. Boozer, Phys. of Fluids B 5 (7), (1993) 2271.

[9] C. Thakur, Astrophysics and space science 149 (1988) 83.

[10] S. Childress and A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996) (Springer).

[11] V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematics Sciences 125 (1991) Springer.

[12] T. Kambe, The geometry of fluid flows, (2000) world scientific.

[13] E. Cartan, Riemannian geometry in an orthonormal Frame, (2001) Princeton University Press.

[14] Y. Yoshimura, ApJ. suppl. series 52:363.

[15] R. Ricca, Solar Physics 172, 241 (1997).

[16] W.D. D’haesseleer, W. Hitchon, J. Callen and J.L. Shohet, Flux Coordinates and Magnetic field Structure (1991) Springer.
[17] P. Chatterjee, A.R. Choudhuri and K. Petroyav, Astron and Astrophysics 449 (2006) 781.

[18] Ya B. Zeldovich, A.A. Ruzmaikin and D.D. Sokoloff, The Almighty Chance (1990) World sci. Press.