Higher-order Convergence Statistics for Three-dimensional Weak Gravitational Lensing

Dipak Munshi\textsuperscript{1,2}, Alan Heavens\textsuperscript{1} and Peter Coles\textsuperscript{2}

\textsuperscript{1}Scottish Universities Physics Alliance (SUPA), Institute for Astronomy, University of Edinburgh, Blackford Hill, Edinburgh EH9 3HJ, UK
\textsuperscript{2}School of Physics and Astronomy, Cardiff University, Queen's Buildings, 5 The Parade, Cardiff, CF24 3AA, UK

Mon. Not. R. Astron. Soc. 000, 000–000 (0000) Printed 10 February 2010 (MN LaTeX style file v2.2)

\section*{ABSTRACT}

Weak gravitational lensing on a cosmological scales can provide strong constraints both on the nature of dark matter and the dark energy equation of state. Most current weak lensing studies are restricted to (two-dimensional) projections, but tomographic studies with photometric redshifts have started, and future surveys offer the possibility of probing the evolution of structure with redshift. In future we will be able to probe the growth of structure in 3D and put tighter constraints on cosmological models than can be achieved by the use of galaxy redshift surveys alone. Earlier studies in this direction focused mainly on evolution of the 3D power spectrum, but extension to higher-order statistics can lift degeneracies as well as providing information on primordial non-gaussianity. We present analytical results for specific higher-order descriptors, the bispectrum and trispectrum, as well as collapsed multi-point statistics derived from them, i.e. cumulant correlators. We also compute quantities we call the power spectra associated with the bispectrum and trispectrum, the Fourier transforms of the well-known cumulant correlators. We compute the redshift dependence of these objects and study their performance in the presence of realistic noise and photometric redshift errors.

\section*{Key words:} Cosmology– Weak Lensing Surveys- Large-Scale Structure of Universe – Methods: analytical, statistical, numerical

\section*{INTRODUCTION}

Until very recently, the best information about the power spectrum of cosmological density perturbations has been obtained from large-scale galaxy surveys and Cosmic Microwave Background (CMB) observations. However, galaxy surveys only probe directly the clustering of luminous matter, while CMB observations mainly explore the power spectrum at a very early linear stage of their evolution. Weak gravitational lensing studies provide a complementary approach for probing the cosmological power spectrum at modest redshift in an unbiased way; for a recent review, see Munshi et al. (2008). Weak lensing is a relatively young subject; the first measurements were published within the last decade (Beacon, Refregier & Ellis 2000; Wittman et al 2000; Kaiser, Wilson & Luppino 2000; Waerbeke et al 2000). Since then rapid progress has been made on analytical modelling, technical specification and the control of systematics. Over the course of the next few years, photometric redshift surveys are going to be increasingly prevalent. Such deep imaging surveys combined with resulting photometric redshift information will mean that there will be a considerable scope for using weak lensing studies to map the dark matter in the universe in three dimensions.

Ongoing and future weak lensing surveys such as the CFHT legacy survey\textsuperscript{1}, Pan-STARRS\textsuperscript{2}, the Dark Energy Survey, and further in the future, the Large Synoptic Survey Telescope\textsuperscript{3}, JDEM and Euclid will provide a wealth of information in terms of mapping the distribution of mass and energy in the universe. With these surveys, like the CMB, the study of weak lensing is entering a golden age. However to achieve its full scientific potential, control is needed of systematics such as arise from shape measurement errors, photometric redshift errors, and intrinsic alignments.

The use of photometric redshifts to study weak lensing in three dimensions was introduced by Heavens (2003). It was later developed by many authors (Heavens et al 2004; Heavens et al 2006; Heavens, Kitching & Verde 2007; Castro et al 2005), and was shown to be a vital tool in constraining dark energy equation of state (Heavens et al 2006), neutrino mass (Kitching et al 2008) and many other possibilities. While the traditional approach deals with projected surveys or with tomographic information, there has been substantial recent progress in the development of studying weak lensing in 3D.

Early analytical work in weak lensing mainly adopted a 2D approach (Jain, Seliak & White 2000), due to the lack of any photometric redshift information about the source galaxies. It is also interesting to note that most of these studies also employed a flat-sky approach (Munshi & Jain 2001; Munshi 2000; Munshi & Jain 2000), as the first generations of weak lensing surveys mainly focused on small patches of the sky (Munshi et al. 2008).

\begin{footnotesize}
\begin{itemize}
\item[1] \url{http://www.cfht.hawaii.edu/Sciences/CFHTLS/}
\item[2] \url{http://pan-starrs.ifa.hawaii.edu/}
\item[3] \url{http://www.lsst.org/lsst_home.shtml}
\end{itemize}
\end{footnotesize}
These works made analytical predictions for lower-order moments as well as the entire probability distribution function of convergence field $\kappa$ or shear $\gamma$ (Munshi & Jain 2001; Valageas 2000, Munshi & Valageas 2003; Valageas, Barber, & Munshi 2004; Valageas, Munshi & Barber 2005). A tomographic ("2.5D") approach has also been developed, wherein the sources are divided into a few redshift slices (Takada & White 2003; Takada & Jain 2004; Massey et al. 2005; Schrabback et al. 2009) and these slices are then analyzed jointly, essentially using the 2D approach but keeping the information regarding the correlation among these redshift slices. A notable exception however in this trend was Siebenmorgen (1996) who developed the analysis techniques for weak lensing surveys covering the entire sky. In recent years there has been a lot of interest in developing analysis techniques and predicting the cosmological impact of future generations of weak lensing surveys with large sky coverage which are naturally analyzed in the spherical harmonic domain. In this paper, we present a very general study of 3D Weak lensing beyond the power spectrum. Extending the formalism developed in Heavens (2003) and Castro et al. (2005) we use higher-order statistics to probe non-Gaussianity present in primordial anisotropy as well as that induced by gravity.

While power spectrum analysis does provide the bulk of the information regarding background cosmology, higher-order statistics are useful to lift degeneracies, allowing determination of $\Omega_m$ and $\sigma_8$ independently - see, e.g., Bernardeau, Van Waerbeke & Mellier (1997); Jain & Seljak (1997); Hui (1999); Schneider et al. (2002); Takada & Jain (2003). Some of these studies also carried out tomographic analysis using the bispectrum. Detailed Fisher matrix analysis found that the level of accuracy with which various cosmological parameters including the dark energy equation of state parameters can be enhanced considerably by using bispectrum data in combination with power spectrum measurements. Higher-order studies also are important for evaluating scatter in lower-order estimates; e.g. the trispectrum is important for computing error bars in power spectrum estimators (Takada & Jain 2004). Most studies involving higher-order correlations however mainly concentrate on projected convergence. The aim of this paper is to extend higher-order analysis to 3D by taking into account the radial distance as inferred through photometric redshift information.

Modelling of the underlying mass distribution is necessary for predictions of weak lensing multipoles. In earlier studies, the hierarchical ansatz was found to be very useful in modelling higher-order statistics of weak lensing observables (Fry 1984; Scannapieco & Szalay 1993, 1997; Munshi et al. 1999; Munshi, Coles & Melott 1999a,b, 1999; Munshi, Melott & Munshi 1999; Munshi & Coles 2000, 2002, 2003). The hierarchical ansatz models the higher-order correlations as a hierarchy, with higher-order correlation functions being expressed as products of correlation functions. The diagrammatic representation of these expressions resembles perturbative models of the correlation hierarchy which typically develops with the onset of gravitational clustering in collisionless media. The amplitudes of these “Feynman diagrams” are of course different in the perturbative regime and in the quasi-linear regime. Various hierarchical ansätze differ in the way they assign amplitudes to diagrams with different topologies. We will employ the most generic hierarchical ansatz in modelling the underlying mass distribution, and the method can be modified to take into account any other specific forms of correlation hierarchy in a relatively simple way.

Higher-order correlation functions have also been detected observationally (Hui 1999; Bernardeau, Van Waerbeke & Mellier 1997; Bernardeau, Mellier & Van Waerbeke 2002). As expected though these studies are more difficult than two-point as they can be dominated by noise. There have been several studies in this direction which focuses mainly on projected surveys as well as using tomographic information (Hu 1999; Takada & Jain 2004, 2005; Sembolini et al. 2008). Typically one-point cumulants or lower-order moments are employed to compress information in higher-order correlation functions into a single number. This is indeed due to the related gain in signal-to-noise. A full analysis of multi-point correlation function (or their respective Fourier transforms, the multipoles) is relatively difficult because of the low signal-to-noise ratio of individual modes. In their recent study Munshi & Heavens (2009) we used an intermediate option: they found that a better approach, one that is even optimal in certain cases, is to use the power spectra associated with the various multi-spectra. These objects combine various individual modes of the multi-spectra in a specific way and can be computed from numerical simulations or observed data relatively easily. While their approach has been motivated from a semi-analytical analysis of the CMB, here we generalize it to 3D weak lensing studies. We primarily focus here on convergence studies but the results can be generalized to shear statistics. We focus on three- and four-point statistics, but the formalism is general. We develop the analysis tools and provide results both in all-sky limit using a harmonics treatment (useful for future surveys with large sky coverage) as well as using a patch-sky approach using flat-sky Fourier transforms (for surveys with relatively small sky coverage). In addition to full analytical results we also provide results using the extended Limber approximation which can drastically reduce the computational cost with very little loss of accuracy at high wavenumbers.

The paper is arranged as follows: In §2 we discuss the basic formalism of 3D weak lensing and how it can be used to estimate the power spectrum in 3D. It introduces the notations which will be used in the following sections. In §3 we introduce the models describing higher-order clustering and their associated hierarchical amplitudes which are then used to construct a model for the bispectrum and higher-order multipoles in the nonlinear regime. In §4 we focus on representation of bispectrum and trispectrum respectively in various coordinate systems. In §5 we relate observed convergence statistics with the underlying statistics of mass distribution. Sections §6 and §7 are devoted to development of the skew spectrum (3-point) and the kurt-spectrum (4-point). In §7 we develop the formalism for surveys with small sky coverage and §8 is reserved for discussion of results and future prospects. Throughout we will borrow the notations from Castro et al. (2005) wherever possible. We have analysed the effect of photometric redshift errors as an appendix.

The general formalism developed in this paper will have applicability in other areas of cosmology, where 3D information can be used effectively, including future generations of 21cm surveys as well as near all-sky redshift surveys.
Derivation of the above expression assumes the Born approximation (Bernardeau, Van Waerbeke & Mellier 1997; Schneider et al 2002; Waerbeke et al 2002), which evaluates the line of sight integral along the unperturbed photon trajectory. Note that the lensing potential $\phi(r, \Omega)$ is radially dependent. Throughout these papers we denote vectors in bold letters, c denotes the speed of light. $r = r(t)$ is the comoving distance to the source at a given instant of time $t$ from the observer who is situated at the origin ($r = 0$). Depending on the background cosmology $f_k(r)$ can be $\sin r$, $r$, or $\sinh r$ for a closed ($K = 1$), flat ($K = 0$) or open ($K = -1$) universes respectively. Our convention for the Fourier transform for the 3D fields closely resemble that of Castro et al (2005). The eigenfunctions of the Laplacian operator in flat space when expressed in spherical coordinates turn out to be a product of spherical Bessel functions $j_l(kr)$ in the radial direction and the spherical harmonics on the surface of a unit sphere i.e. $Y_{lm}(\Omega) = Y_{lm}(\theta, \phi)$. The eigenfunctions $j_l(kr)Y_{lm}(\theta, \phi)$ are associated with eigenvalues $-k^2$. The eigendecomposition and its inverse transformation can be expressed as:

$$
\Phi_{lm}(k) = \sqrt{\frac{2}{\pi}} \int d^3r \Phi(r) k j_l(kr) Y_{lm}^*(\Omega); \quad \Phi(r) = \sqrt{\frac{2}{\pi}} \int dk k \sum_{l=0}^{\infty} \sum_{m=0}^{l} \Phi_{lm}(k) j_l(kr) Y_{lm}(\Omega).
$$

(2)

The choice for this eigendecomposition is determined by various factors. First it can deal with large areal sky coverage, and secondly as the lensing is related to gravitational potentials, the expansion allows us easily to express the coefficients of expansion of the convergence (or shear) in terms of the expansion of the density field through the Poisson equation (Heavens 2003). $\Phi_{lm}(k)$ here is the spherical harmonic decomposition of $\Phi(r)$, and similarly for $\phi(r)$. The orthogonality properties for the harmonic modes for the 3D potential can be used to define the 3D all sky power spectra for the following expressions.

$$
\left\langle \Phi_{lm}(k)\Phi_{l'm'}(k') \right\rangle = C_{l}^{\phi\phi}(k) \delta_{lD}(k + k') \delta_{m'm'}^{K}; \quad \left\langle \phi_{lm}(k)\phi_{l'm'}(k') \right\rangle = C_{l}^{\delta\delta}(k) \delta_{lD}(k + k') \delta_{m'm'}^{K}.
$$

Here the power spectrum $C_{l}^{\phi\phi}$ represents the 3D power spectrum associated with the 3D potential field $\Phi_{lm}(k)$ and $\delta_{lD}$ and $\delta^{K}$ represent the n-dimensional Dirac and Kronecker delta functions respectively. The corresponding all-sky power spectra for the lensing potential $\phi$ is denoted by $C_{l}^{\phi\phi}$. In comoving coordinates we can write:

$$
\Delta \Phi(r) = \frac{3}{2a} \Omega_{m0} H_0^2 \delta(r); \quad \Phi_{lm}(k) = -\frac{3}{8a(k)^2} \Omega_{m0} H_0^2 \delta_{lm}(k; r) + \frac{C}{a(k)^2} \delta_{lm}(k; r).
$$

(4)

Here $a(z) = 1/(1 + z)$ is the scale factor at redshift $z$, $\Omega_{m0}$ is the total matter density at $z = 0$, and $H_0$ is the Hubble constant today. $\delta_{lm}(k; r)$ is the eigendecomposition of $\delta(r)$. When appearing after the semi-colon, the $\delta_{lm}(k; r)$ really expresses the time-dependence of the potentials; see Castro et al (2005) for a discussion of the subtleties of this. Any model we assume for describing non-linear growth of perturbations for $\delta(r)$ will thus have direct impact on statistics of observed weak lensing convergence $\kappa$ as it depends on $\delta(r)$ through its dependence on $\Phi(r)$. The harmonic decomposition of the lensing potential and the 3D gravitational potential are related by the following expression.

$$
\phi_{lm}(k) = \frac{4k^4}{\pi c^2} \int_0^\infty dk' k'^2 \int_0^\infty rdr j_l(kr) \int_0^r dr' \left( \frac{r - r'}{r} \right) j_l(k' r') \Phi_{lm}(k'; r').
$$

(5)

We ignore complications of varying radial selection function for the sources, and distance errors in the main text for clarity. These are considered in Appendix A.

Using the relation which relates the density and the gravitational potential as well as expressing the convergence field in terms of the lensing potential $\kappa_{lm}(k) = -\frac{1}{2}(l + 1) \phi_{lm}(k)$; we can express the convergence coefficients in terms of those describing the density field. This is important because we can then relate the statistics of the density field with that of the convergence field directly which is potentially observable:

$$
\kappa_{lm}(k) = \frac{16kA}{\pi c^2} \int_{l (l+1)} \frac{1}{l (l+1)} \int_0^{\infty} dk' k' \int_0^\infty rdr j_l(kr) \int_0^r dr' \left( \frac{r - r'}{r} \right) j_l(k' r') \delta_{lm}(k'; r').
$$

(6)

We have absorbed the cosmological constants in the constant $A = -3\Omega_{m0} H_0^2/2$. We will also introduce the quantity $I_l(k_i, k)$, which will be useful in displaying the future results:

$$
I_l(k_i, k) \equiv k_i \int_0^\infty dr j_l(k_i, r) \int_0^r dr' \left( \frac{r - r'}{r} \right) j_l(k' r') \sqrt{P^{\phi\phi}(k; r')}
$$

(7)

We can now write down the power spectrum associated with the lensing potential $\phi$ in a more compact form. Next we relate the convergence power spectra in terms of the $C_{l}^{\phi\phi}$ using their relationship in the harmonic domain.

$$
C_{l}^{\phi\phi}(k_1, k_2) = \frac{16 A^2}{\pi c^2} \int_0^{\infty} k^2 I_l(k_1, k) I_l(k_2, k) dk; \quad C_{l}^{\delta\delta}(k_1, k_2) = \frac{1}{4} k^2 (l + 1)^2 C_{l}^{\phi\phi}(k_1, k_2).
$$

(8)

In deriving the above expression it was assumed that gravitational potential power spectrum can be accurately approximated by $P^{\phi\phi} \sim \sqrt{P^{\phi\phi}(k; r)P^{\phi\phi}(k; r')}$ for a detailed description and range of validity see Castro et al (2005). Clearly the analysis outlined above follows three different steps. First we relate the lensing potential $\phi_{lm}(k; r)$, or equivalently $\kappa_{lm}(k; r)$, to the 3D potential $\Phi_{lm}(k; r)$. Next the statistics of $\Phi_{lm}(k; r)$ are used to make concrete predictions about the statistics of $\kappa_{lm}(k; r)$. However an intermediate step is required to connect the 3D Fourier decomposition $\delta(k)$ and its harmonic counterpart $\delta_{lm}(k; r)$. See Castro et al (2005) for details.

Throughout this paper, the analysis will rely on various assumptions. For an arbitrary field $\Psi(r; r')$ which is assumed isotropic and homogeneous with spherical harmonics decomposition $\Psi_{lm}(k)$ can be characterized by a power spectrum $C_{l}(k; r)$. It is important to realise that $r$, the comoving distance, also plays the dual role of cosmic epoch, and we will consider $r$ in representations to label the cosmic epoch. The cross-power spectrum related
to an arbitrary 3D field at two different radial distances (redshifts) $C_l(k; r, r')$ will be expressed as $\langle \Psi_{lm}(k; r)\Psi_{km}(k'; r') \rangle = C_l(k; r, r')\delta_{1D}(k - k') \delta^K_{lm}$. It was shown by Castro et al. (2005) that $C_l(k; r)$ is simply the 3D power spectrum $P(k; r)$, $C_l(k; r, r') = P(k; r, r')$. For the derivation we need to expand the Fourier decomposition of $\Psi(r; r)$ and exploit the fact that $\langle \Psi_k(k; r)\Psi_k(k'; r') \rangle = (2\pi)^3 P(k; r)\delta_{1D}(k - k')$. In our present analysis we will focus on extending these results to higher-order statistics. We model the underlying statistics of the density field by using non-perturbative results and relate these to the statistics of projected field such as convergence. We also introduce power spectra related to multispectra to effectively compress the information content. The results are presented both in harmonic space as well as in Fourier domain using the Fourier approximation.

2.1 3D Convergence Power Spectrum Using the Limber Approximation

Computations of higher-order multi-spectra are often difficult, given the multi-dimensional integrals involved, which often make numerical computations time consuming if not prohibitive. The Limber approximation (Limber 1954) or its generalization to Fourier space is often used to simplify the evaluation numerically by reducing the dimensionality of the integrals. Typically implementation of the Limber approximation is valid at small angular separations which correspond to large multipole moments $l$ in harmonic domain. It also requires smooth variations of the integrand compared to the Bessel functions of relevant $l$. For a detailed description of various issues and calculations of next order correction terms see a recent discussion by LoVerde & Afshordi (2008).

We start by the following expression Eq. (5) and Eq. (8) from the previous section:

\[
C_{l}^{\phi\phi}(k_1, k_2) = \frac{16A^2}{\pi^22^5} \int_{0}^{\infty} k^2 dk I_1(k_1, k) I_1(k_2, k) \times \int_{0}^{\infty} k_1^2 dk \frac{P_{\phi\phi}(k; r_1^a) P_{\phi\phi}(k; r_1^b) j_i(k r_1) j_i(k r_2)}{r_1^a r_1^b}
\]

(9)

We now use the Limber approximation to simplify the $k$ integral which produces a $\delta_{1D}(r_1^a - r_1^b)$ function. Integrating out $r_1^a$ with the help of the delta function and renaming the dummy variable $r_1^b$ to $r'$ we can finally write:

\[
C_{l}^{\phi\phi}(k_1, k_2) = \frac{\pi^2}{2} \frac{(l + 1)^2 A^2}{4} k_1 k_2 \int_{0}^{\infty} dr' j_l(k_1 r_1) j_l(k_2 r_2) I_1(r_1, r_2); I_1(r_1, r_2) = \frac{16}{\pi^22^5} \int_{0}^{r_{min}} \frac{dr'}{r'} \left( \frac{r_1 - r'}{r} \right) \left( \frac{r_2 - r'}{r} \right) \left( \frac{l}{r'} \right)^2 P_{\phi\phi} \left( \frac{l}{r'} r' ; r \right); \quad r_{min} = min(r_1, r_2).
\]

(10)

The limits of the integral only cover the overlapping region. We notice here that if we use the Limber approximation Eq. (10), then this equation reduces to simpler form as higher harmonics at different radial distances $r_i$ becomes uncorrelated.

\[
C_{l}^{\phi\phi}(k_1, k_2; r_1, r_2) = \delta_{1D}(r_1 - r_2) \frac{\pi^2}{2r_i^2} \frac{(l + 1)^2 A^2}{4} k_1 k_2 I_1(r_1, r_2).
\]

(11)

The convergence power spectrum now can be computed using Eq. (10). It is worth mentioning that this equation establishes a direct link of convergence power spectra and the underlying mass distributions. In later sections we will extend this result to higher-order multispectra.

The signal-to-noise associated with various estimators from all-sky weak lensing surveys and the issues related to optimization will be dealt with in a separate work. In this paper we will focus mainly on development of statistics which can be employed to study gravity-induced non-Gaussianity using higher-order statistics. We will use the expressions for the $C_{l,8}$ derived here for the construction of the optimal estimators for the bispectrum and trispectrum in the following sections.

3 MODELLING GRAVITY-INDUCED NON-GAUSSIANITY

Two point statistics are useful to constrain cosmological parameters. However the convergence power spectrum depends principally on a specific combination of cosmological parameters. Typical additional inputs in the form of external data sets such as the CMB, or higher-order statistics can lift the degeneracy. While the use of the 3D power spectrum already tightens the constrains further gain is anticipated with the help of non-Gaussianity studies in 3D.

It is already known from earlier studies (Takada & Jain 2004) that lensing tomography with the power spectrum and bispectrum can act as a probe of dark energy and mass power spectrum. The lensing bispectrum has a different dependence on the lensing weight function and the growth rate of perturbations. This is the main reason why bispectrum tomography can provide complementary constraints to the power spectrum. In fact it was found in previous studies that constraints from the bispectrum can be as tight as that from the power spectrum.

We will model the non-Gaussianity using the hierarchical ansatz which is known to be a reasonable approximation at small scales in the highly nonlinear regime. This approach has been used previously to model the statistics of the convergence and shear fields (Munshi & Jain 2001; Valageas 2000; Munshi & Valageas 2005; Valageas, Barber, & Munshi 2004; Valageas, Munshi & Barber 2005).

The convergence power spectrum now can be computed using Eq. (6). It is worth mentioning that this equation establishes a direct link of convergence power spectra and the underlying mass distributions. In later sections we will extend this result to higher-order multispectra.

The signal-to-noise associated with various estimators from all-sky weak lensing surveys and the issues related to optimization will be dealt with in a separate work. In this paper we will focus mainly on development of statistics which can be employed to study gravity-induced non-Gaussianity using higher-order statistics. We will use the expressions for the $C_{l,8}$ derived here for the construction of the optimal estimators for the bispectrum and trispectrum in the following sections.
Higher-order Statistics for 3D Weak Lensing

3.1 The Hierarchical Ansatz in the Highly Non-linear Regime

We need a reliable technique for modelling weak lensing statistics beyond the power spectrum. On larger scales, where the density field is only weakly non-linear, perturbative treatments are known to be valid. For a statistical description of dark matter clustering in collapsed objects on small scales, the standard approach is to use the halo model [Cooray & Sheth 2002]. However, an alternative approach on small scales is to employ various ansatze which trace their origin to field theoretic techniques used to probe gravitational clustering. For our work, we will use a hierarchical ansatz where the higher-order correlation functions are constructed from the two-point correlation functions. Assuming a tree model for the matter correlation hierarchy (typically used in the highly non-linear regime) one can write the most general case, the N point correlation function, $\xi_N(r_1, \ldots, r_N)$ as a product of two-point correlation functions $\xi(|r_i - r_j|)$ [Bernardeau et al 2002]. Equivalently in the Fourier domain the multispectra can be written as products of the matter power spectrum $P_\delta(k_i)$. The temporal dependence is implicit here.

$$\xi_N(r_1, \ldots, r_N) = \sum_{\alpha, N=\text{trees}} Q_{N,\alpha} \sum_{\text{labellings/edges}(i,j)} (N-1)! \xi(|r_i - r_j|).$$

(12)

It is however very interesting to note that a similar hierarchy develops in the quasi-linear regime at tree-level in the limiting case of vanishing variance. The hierarchical amplitudes become configuration-independent again as has been shown by high resolution studies for the lowest order case $Q_3 = Q$ [Scoccimarro et al 1998, Bernardeau et al 2002]. In the Fourier space however such an ansatz means that the entire hierarchy of the multi-spectra can be written in terms of sums of products of power spectra with different amplitudes $Q_{N,\alpha}$ etc., e.g. in the lowest order we can write:

$$\langle \delta(k_1)\delta(k_2) \rangle_c = (2\pi)^3 \delta_{3D}(k_1 + k_2) P(k_1)$$

(13)

$$\langle \delta(k_1)\delta(k_2)\delta(k_3) \rangle_c = (2\pi)^3 \delta_{3D}(k_1 + k_2 + k_3) B(k_1, k_2, k_3)$$

(14)

$$\langle \delta(k_1) \cdots \delta(k_4) \rangle_c = (2\pi)^3 \delta_{3D}(k_1 + k_2 + k_3 + k_4) T(k_1, k_2, k_3, k_4).$$

(15)

The subscript $c$ here represents the connected part of the spectra. The Dirac delta functions $\delta_{3D}$ ensure the conservation of momentum at each vertex representing the multispectrum.

$$B_3(k_1, k_2, k_3) \sum_{k_i = 0} = Q_3 [P(k_1)P(k_2) + P(k_1)P(k_3) + P(k_2)P(k_3)]$$

(16)

$$T_4(k_1, k_2, k_3, k_4) \sum_{k_i = 0} = R_\text{trees} [P(k_1)P(k_2)P(k_3) + \text{cyc.perm.}] + R_\text{lab} [P(k_1)P(k_2) + \text{cyc.perm.}] + \text{cyc.perm.}.$$ 

(17)

Different hierarchical models differ in the way numerical values are allotted to various amplitudes. Bernardeau & Schaeffer [1992] considered “snake”, “hybrid” and “star” diagrams with differing amplitudes at various order. A new “star” appears at each order, higher-order “snakes” or “hybrid” diagrams are built from lower-order “star” diagrams. In models where we only have only star diagrams [Valageas, Barber, & Munshi 2004] the expressions for the trispectrum takes the following form:

$$T(k_1, k_2, k_3, k_4) \sum_{k_i = 0} = Q_4 [P(k_1)P(k_2)P(k_3) + \text{cyc.perm.}] + \text{cyc.perm.}.$$ 

(18)

Following Valageas, Barber, & Munshi [2004] we will call these models “stellar models”. Indeed it is also possible to use perturbative calculations which are however valid only at large scales. While we still do not have an exact description of the non-linear clustering of a self-gravitating medium in a cosmological scenario, these approaches do capture some of the salient features of gravitational clustering in the highly non-linear regime and have been tested extensively against numerical simulation in 2D statistics of convergence of shear [Valageas, Barber, & Munshi 2004]. These models were also used in modelling of the covariance of lower-order cumulants [Munshi & Valageas 2005].

4 THEORETICAL MODEL LING OF 3D CONVERGENCE BISPECTRUM FOR 3D WEAK LENSING SURVEYS

Previous studies of the bispectrum involving weak lensing include work on projection (2D) as well as tomography. In most of these studies the prediction of the bispectrum is tied to a specific assumption regarding the growth of instability in the underlying density distribution. The main motivation for most of these studies was to put tighter constraints on the dark energy equation of state using weak lensing surveys by lifting the degeneracy involved in power spectrum analysis alone. The three-point correlation function in real space (or equivalently its harmonic transform the bispectrum) encodes information about the departure from Gaussianity, and this departure can be induced by non-linear gravity or by non-Gaussian initial conditions. We will focus on the gravity-induced bispectrum here and plan to present a complete treatment of the effect of initial non-Gaussianity on 3D weak lensing statistics elsewhere.

4.1 Linking the Density Bispectrum in Various Representations: $\delta(k)$, $\delta_1(r)$ and $\delta_1(k)$

The convergence bispectrum in 3D will depend on modelling of underlying density bispectrum. We start by quoting the relation of the 3D density bispectrum expressed in Cartesian coordinate and in the harmonic space, we refer the reader to Castro et al [2006] for detailed derivation of the following equation:

$$\delta_{lm}(k; r) = \frac{1}{\sqrt{2\pi}} k \int d\Omega_k \delta(k; r) Y_{lm}(\Omega_k).$$

(19)
Using this definition, we can link the bispectrum defined in Fourier space with the one in the harmonic domain:

\[
\langle \delta_{l_1m_1}(k_1;r_1)\delta_{l_2m_2}(k_2;r_2)\delta_{l_3m_3}(k_3;r_3) \rangle_c = \left( \frac{1}{2\pi} \right)^3 \int d\Omega_{k_1} Y_{l_1m_1}(\Omega_{k_1}) \int d\Omega_{k_2} Y_{l_2m_2}(\Omega_{k_2}) \int d\Omega_{k_3} Y_{l_3m_3}(\Omega_{k_3}) \langle \delta(k_1;r_1)\delta(k_2;r_2)\delta(k_3;r_3) \rangle.
\]  

Using this definition, we can link the bispectrum defined in Fourier space with the one in the harmonic domain:

\[
\langle \delta_{l_1m_1}(k_1;r_1)\delta_{l_2m_2}(k_2;r_2)\delta_{l_3m_3}(k_3;r_3) \rangle_c = \left( \frac{1}{2\pi} \right)^3 \int d\Omega_{k_1} Y_{l_1m_1}(\Omega_{k_1}) \int d\Omega_{k_2} Y_{l_2m_2}(\Omega_{k_2}) \int d\Omega_{k_3} Y_{l_3m_3}(\Omega_{k_3}) \langle \delta(k_1;r_1)\delta(k_2;r_2)\delta(k_3;r_3) \rangle.
\]  

Let us introduce the following notation for the bispectrum associated with the density field:

\[
\langle \delta(k_1;r_1)\delta(k_2;r_2)\delta(k_3;r_3) \rangle = B(k_1, k_2, k_3; r_1, r_2, r_3) \delta_{3D}(k_1 + k_2 + k_3).
\]  

Expanding the Dirac delta function \(\delta_{3D}(k_1 + k_2 + k_3)\) and using Rayleigh’s expansion of the exponentials:

\[
\delta_{3D}(k_1 + k_2 + k_3) = \frac{1}{(2\pi)^3} \int e^{i(k_1+k_2+k_3)\cdot r} d^3r = \sum_{l_1, l_2, l_3} \int d^3r \, i^{l_1+l_2+l_3} j_{l_1}(k_1r)j_{l_2}(k_2r)j_{l_3}(k_3r) Y_{l_1m_1}(\Omega_{k_1}) Y_{l_2m_2}(\Omega_{k_2}) Y_{l_3m_3}(\Omega_{k_3}) Y_{l_4m_4}(\bar{\Omega}) Y_{l_5m_5}(\bar{\Omega}) Y_{l_6m_6}(\bar{\Omega}).
\]  

Next, we use the orthogonality property of the spherical harmonics, Eq. (15) to carry out the integrals to simplify the expression. We have introduced the notation \(d^3k \equiv k^2 dk d\Omega_k \equiv k^2 \sin \theta_k dk d\theta_k d\phi_k\). This allows us to write the bispectrum in a spherical harmonics representation to its Fourier counterpart. Spherical coordinates are natural choice for various reasons. The line of sight integration can be treated quite separately with sky coverage issues. As we will see the partial sky coverage issues can also be dealt with a natural way in harmonic expansions. It is also important to notice that (radial) errors due to photometric redshift can also be incorporated naturally (see Appendix A for more details).

\[
\langle \delta_{l_1m_1}(k_1;r_1)\ldots\delta_{l_3m_3}(k_3;r_3) \rangle = \left( \frac{1}{2\pi} \right)^3 G_{l_1l_2l_3}^{m_1m_2m_3} \int r^2 dr j_{l_1}(k_1r)j_{l_2}(k_2r)j_{l_3}(k_3r)B(k_1; r_1).
\]  

The directional dependence through the azimuthal quantum number \(m\) is encapsulated through the Gaunt integral \(G\) and is defined by the following expressions (we also introduce the quantity \(I_{l_1l_2l_3}\) which we will find useful later):

\[
G_{l_1l_2l_3}^{m_1m_2m_3} = \int d\Omega Y_{l_1m_1}(\Omega) Y_{l_2m_2}(\Omega) Y_{l_3m_3}(\bar{\Omega}) = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} I_{l_1l_2l_3};
\]

\[
I_{l_1l_2l_3} = \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]  

Here the matrices are the \(3J\) symbols, which are nonzero only if the triplets of harmonics \((l_1, l_2, l_3)\) satisfy the triangle equality, including the condition that the sum \(l_1 + l_2 + l_3\) is even which ensures the parity invariance of the bispectrum. We will also need the shorthand notation \(I_{l_1l_2l_3}\) in our following derivations. The rotationally invariant bispectrum \(B_{l_1l_2l_3}\) can now be written in terms of \(B_{l_1l_2l_3}^{m_1m_2m_3}\) as:

\[
B_{l_1l_2l_3}^{m_1m_2m_3}(k_1; r_1) = \sum_{m_1, m_2, m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{l_1l_2l_3}^{m_1m_2m_3}(k_1; r_1).
\]  

We will also need the reduced bispectrum commonly used in the literature which has direct correspondence to the flat-sky bispectrum. In terms of the bispectrum \(B_{l_1l_2l_3}\) we can define \(b_{l_1l_2l_3} = I_{l_1l_2l_3} B_{l_1l_2l_3}\). Finally we can write the general correspondence between the spherical harmonics representation of the angular bispectrum \(B_{l_1l_2l_3}(k_1; r_1) = B_{l_1l_2l_3}(k_1, k_2, k_3; r_1, r_2, r_3)\), and its Fourier representation \(B_{l_1l_2l_3}(k_1; r_1) = B(k_1, k_2, k_3; r_1, r_2, r_3)\). We will suppress the explicit display of the radial coordinates to simplify notations. We need to keep in mind in the Fourier representations the radial coordinates simply denote the cosmic epoch.
In certain applications it is also interesting to work in a basis where the harmonic decomposition is carried out only on the surface of the sky, retaining the radial dependence in configuration space. In such circumstances, the following transformations are useful in relating the bispectrum expressed in this mixed coordinate with bispectrum in full spherical coordinate.

\[ B^{\text{mixed}}_{l_i l_2 l_3}(r_1, r_2, r_3) = \int r^2 dr B^{\text{rect}}(k_1, k_2, k_3; r_1, r_2, r_3) j_i(k_1 r_1) j_i(k_2 r_2) j_i(k_3 r_3). \]  

In certain applications it is also interesting to work in a basis where the harmonic decomposition is carried out only on the surface of the sky, retaining the radial dependence in configuration space. In such circumstances, the following transformations are useful in relating the bispectrum expressed in this mixed coordinate with bispectrum in full spherical coordinate.

\[
J_{l_1 l_2 l_3}(k_i; r_i) = \left( \frac{1}{\sqrt{2\pi}} \right)^3 \{k_1 k_2 k_3\} J_{l_1 l_2 l_3}(k_i; r_i) I_{l_1 l_2 l_3}; \]

\[
b^{\text{mixed}}_{l_i l_2 l_3}(k_i; r_i) = \left( \frac{1}{\sqrt{2\pi}} \right)^3 \{k_1 k_2 k_3\} J_{l_1 l_2 l_3}(k_i; r_i). \]

The expression \( J_{l_1 l_2 l_3}(k_i; r_i) \) encapsulates the dependence on \( k_i \) and \( r_i \) with \( i = 1, 2, 3 \).

\[
J_{l_1 l_2 l_3}(k_i; r_i) = \int r^2 dr B^{\text{rect}}(k_1, k_2, k_3; r_1, r_2, r_3) j_i(k_1 r_1) j_i(k_2 r_2) j_i(k_3 r_3). \]  

The above analysis relates the underlying 3D bispectrum to the 3D convergence bispectrum for the most general case. However numerical computation involving such multiple integrals can be prohibitive. To make further progress we will use specific models of gravity-induced bispectrum to simplify the calculations. We will also use the Limber approximation to simplify our results. The Limber approximation is known to be a very good approximation for smaller angular scales or high \( l \). We would like to stress however that although the results presented here are for a specific models for hierarchical clustering it is nevertheless possible to extend the results of our analysis to other models. Assuming the specific form of hierarchical ansatz, introduced before, we can have:

\[
B^{\text{rect}}(k_1, k_2, k_3; r_1, r_2, r_3) = Q_3 [P(k_1; r_1) P(k_2; r_2) + \text{cyc.perms}]. \]  

4.2 Linking the Convergence Bispectrum to the Underlying Matter Bispectrum

To make contact with the observables we use the fact that the projected convergence (which is related to the lensing potential) can be related directly to the 3D density field. We will start by linking the 3D convergence bispectrum \( B \) and the 3D density bispectrum expressed in harmonic coordinates. In the next section we will express the bispectrum in spherical coordinate in terms of the bispectrum in rectangular coordinates and use some well-motivated approximations to simplify the results. Using Eq. (32) we can write the following expression:

\[
B^{\text{ph}}_{l_1 l_2 l_3}(k_i; r_i) = A^3 \left[ \frac{4 k_i^3}{\pi c^2} \right] \left[ \frac{4 k_j^3}{\pi c^2} \right] \left[ \frac{4 k_l^3}{\pi c^2} \right] \int_0^{\infty} \frac{dk_1^\prime}{k_1^\prime} \int_0^{\infty} \frac{dk_2^\prime}{k_2^\prime} \int_0^{\infty} \frac{dk_3^\prime}{k_3^\prime} \int_0^{r_1} \frac{dr_1}{a(r_1)} \left[ \frac{r_1 - r_1^\prime}{r_1^\prime} \right] \int_0^{r_2} \frac{dr_2}{a(r_2)} [r_2 - r_2^\prime]^2 \int_0^{r_3} \frac{dr_3}{a(r_3)} [r_3 - r_3^\prime]^2 B^{\text{rect}}_{l_1 l_2 l_3}(k_i^\prime; r_i^\prime). \]

The bispectrum \( B_{l_1 l_2 l_3}(k_i; r_i) \) is now expressed in terms of the bispectrum \( B^{\text{rect}}_{l_1 l_2 l_3}(k_i; r_i) \). This relation mixes modes only in radial directions. On the surface of the sky there is no mixing of angular harmonics corresponding to various \( l \) values. While expressing the density harmonics in terms of the 3D potential harmonics, we pick up additional scale factor \( a(r_i) \) and \( k_i \) dependence in the denominator (see Eq. (9) for more on notational details).

We have so far ignored the presence of noise. Indeed because of the limited number of galaxies available it may not be possible to probe individual modes of the bispectrum at high signal-to-noise ratio. In later sections we will be able to address issues related to optimum combinations of individual modes which may be better suited for observational studies.

4.3 Specific Models for underlying bispectrum and Limber’s Approximation to the Exact Analysis

The above analysis relates the underlying 3D bispectrum to the 3D convergence bispectrum for the most general case. However numerical computation involving such multiple integrals can be prohibitive. To make further progress we will use specific models of gravity-induced bispectrum to simplify the calculations. We will also use the Limber approximation to simplify our results. The Limber approximation is known to be a very good approximation for smaller angular scales or high \( l \). We would like to stress however that although the results presented here are for a specific models for hierarchical clustering it is nevertheless possible to extend the results of our analysis to other models. Assuming the specific form of hierarchical ansatz, introduced before, we can have:

\[
B^{\text{rect}}(k_1, k_2, k_3; r_1, r_2, r_3) = Q_3 [P(k_1; r_1) P(k_2; r_2) + \text{cyc.perms}]. \]  

Using this notation for the function \( J_{l_1 l_2 l_3}(r_1, r_2, r_3) \) we introduced in Eq. (28) takes the following form:

\[
J_{l_1 l_2 l_3}(r_1, r_2, r_3) = Q_3 I_{l_1 l_2 l_3} \int r^2 dr j_i(k_1 r_1) j_i(k_2 r_2) j_i(k_3 r_3) [P(k_1; r_1) P(k_2; r_2) + \text{cyc.perms}]. \]
We use the extended Limber approximation (see Eq. (B2)) to simplify the integrals involving \( k' \). The delta functions simplify the resulting \( r' \) integrations, Finally the observable convergence bispectrum can be written in terms of directly the density bispectrum as follows:

\[
B_{l_1l_2l_3}(k_1; r_1) = \int_0^\infty dr_1 j_{l_1}(k_1 r_1) \int_0^\infty dr_2 j_{l_2}(k_2 r_2) \int_0^\infty dr_3 j_{l_3}(k_3 r_3) B_{l_1l_2l_3}(r_1, r_2, r_3),
\]

\[
I_{l_1l_2l_3}(r_1, r_2, r_3) = \left[ \frac{\pi}{2} \right]^3 3^l 4^{-l} \frac{\pi}{2} \int_0^{r_{\text{min}}} dr \left( \frac{r_1 - r}{a(r)r^3} \right) \left( \frac{r_2 - r}{a(r)r^3} \right) \left( \frac{r_3 - r}{a(r)r^3} \right) \left\{ P \left( \frac{r_1}{r} \right) P \left( \frac{r_2}{r} \right) + \text{cyc. perm.} \right\}. \tag{35}
\]

The integral here extends to the overlapping region i.e. \( r_{\text{min}} = \min(r_1, r_2, r_3) \), and the final result is not specific to the assumed non-Gaussianity, but assumes the hierarchical ansatz. For an arbitrary bispectrum the result can be expressed by a suitable change in \( I_{l_1l_2l_3}(r_1, r_2, r_3) \):

\[
I_{l_1l_2l_3}(r_1, r_2, r_3) = \left[ \frac{\pi}{2} \right]^3 3^l 4^{-l} \frac{\pi}{2} \int_0^{r_{\text{min}}} dr \left( \frac{r_1 - r}{a(r)r^3} \right) \left( \frac{r_2 - r}{a(r)r^3} \right) \left( \frac{r_3 - r}{a(r)r^3} \right) B \left( \frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r} \right). \tag{36}
\]

For computation of the bispectrum in scenarios with a specific model for the primordial non-Gaussianity we will have to replace the kernel that appears in the expression for \( I_{l_1l_2l_3}(r_1, r_2, r_3) \) and similar results will follow. In particular we can replace the gravity-induced bispectrum with models of the primordial bispectrum, e.g. \( B^{\text{loc}} \) or \( B^{\text{quad}} \), to compute the related bispectrum for convergence \( \kappa \).

## 5 THEORETICAL MODELLING OF THE CONVERGENCE TRISPECTRUM FOR 3D WEAK LENSING SURVEYS

As before we start by linking the trispectrum in spherical coordinates with the spatial trispectrum in rectangular coordinates. The procedure we will follow will be very similar to what we have done for the case of the bispectrum. We start by introducing the trispectrum in the Cartesian coordinate \( \delta(k_1; r_1) \ldots \delta(k_4; r_4) = T^{\text{rect}}(k_1; r_1) \delta^{3 \text{D}}(k_1 + k_2 + k_3 + k_4) \) and in radial and polar coordinates as:

\[
(\delta_{l_1m_1}(k_1; r_1) \ldots \delta_{l_4m_4}(k_4; r_4))_c = \sum_{L,M} (-1)^M \frac{L}{2^m} \left( \begin{array}{ccc} l_1 & m_1 & L \\ m_1 & m_2 & M \\ l_3 & m_3 & m_4 \end{array} \right) \frac{T^{\text{sph}}_{l_1l_2l_3}(L; k_1; r_1)}{r^{\text{rect}}(k_1; r_1)}. \tag{37}
\]

The vectors \( l_1, l_2, l_3, l_4 \) represents the sides of a quadrilateral and \( L \) is the length of the diagonal. The matrices as before are the Wigner 3J symbols. The symbols are only non-zero when they satisfy several conditions; which are \( |l_1 - l_2| \leq L \leq l_1 + l_2, |l_3 - l_4| \leq L \leq l_3 + l_4, l_1 + l_2 + L = \text{even}, l_3 + l_4 + L = \text{even} \) and \( m_1 + m_2 = M \) as well as \( m_3 + m_4 = -M \).

In our notation for the trispectrum, \( T^{\text{sph}}_{l_1l_2l_3}(k_1, r_1; L) \), the indices \( (k_1, r_1) \) encodes their dependence on various Fourier modes of the density harmonics in the radial direction, used in their construction. No summation will assumed over these variable unless explicitly specified. We need also to subtract the Gaussian or the disconnected part from the estimated trispectrum to compute the connected part of the trispectrum, denoted by the subscript \( \gamma \), in ensemble averaging. By expanding the Dirac delta function in spherical harmonics and going through the same algebra as above we can finally express \( T^{\text{sph}}_{l_1l_2l_3}(k_1; r_1) \) in terms of the following integrals:

\[
(\delta_{l_1m_1}(k_1; r_1) \ldots \delta_{l_4m_4}(k_4; r_4))_c \equiv \sum_{L,M} (-1)^M \frac{L}{2^m} \left( \begin{array}{ccc} l_1 & m_1 & L \\ m_1 & m_2 & M \\ l_3 & m_3 & m_4 \end{array} \right) \frac{T^{\text{sph}}_{l_1l_2l_3}(L; k_1; r_1)}{r^{\text{rect}}(k_1; r_1)}. \tag{38}
\]

Next we express the four-point correlation function in terms of \( T^{\text{sph}}_{l_1l_2l_3}(k_1; r_1) \). Finally using the orthogonality properties of the 3J functions, we can finally connect the two representations. It involves the functions \( I_{l_1l_2l_3} \) we have introduced before. The prefactor involving \( k_1 \) is an artifact of the normalization which we have adopted.

\[
T^{\text{sph}}_{l_1l_2l_3}(k_1; r_1) = \left( \frac{1}{\sqrt{2\pi}} \right)^4 \left\{ k_1 k_2 k_3 k_4 \right\} \sum_{L} I_{l_1l_2l_3}(I_{l_1l_2l_3}(k_1; r_1) \ldots j_4(k_4 r_4) T^{\text{sph}}_{l_1l_2l_3}(L; k_1; r_1)). \tag{39}
\]

In our derivation we have used the following identity to simplify the integration involving four spherical harmonics:

\[
\int d\Omega Y_{l_1m_1}(\hat{\Omega}) Y_{l_2m_2}(\hat{\Omega}) Y_{l_3m_3}(\hat{\Omega}) Y_{l_4m_4}(\hat{\Omega}) = \sum_{L,M} (-1)^M \left( \begin{array}{ccc} m_1 & m_2 & M \\ m_1 & m_2 & M \\ m_3 & m_4 & -M \end{array} \right) \frac{C^{l_1l_2l_3}_{l_1l_2l_3}}{C^{l_1l_2l_3}_{l_1l_2l_3}}.
\tag{40}
\]

We will also add the following expressions for the sake of completeness. As before we will relate the trispectrum defined from the harmonics \( \delta_{l_1m_1}(k_1; r_1) \) i.e. \( T^{\text{sph}}_{l_1l_2l_3}(L; k_1; r_1) \) with \( T^{\text{sph}}_{l_1l_2l_3}(L; r_1) \) which is defined from the harmonics \( \delta_{l_1m_1}(k_1) \).

\[
T^{\text{sph}}_{l_1l_2l_3}(L; k_1; r_1) = \left( \frac{2}{\pi} \right)^2 \int r_1^2 dr_1 j_{l_1}(k_1 r_1) \ldots j_{l_4}(k_4 r_4) \frac{T^{\text{sph}}_{l_1l_2l_3}(L; k_1; r_1)}{r^{\text{mixed}}(k_1; r_1)}. \tag{41}
\]

The inverse relation which relates \( T^{\text{sph}}_{l_1l_2l_3}(L; r_1) \) with \( T^{\text{sph}}_{l_1l_2l_3}(L; k_1, r_1) \) is given by following expression:

\[
T^{\text{sph}}_{l_1l_2l_3}(L; r_1) = \left( \frac{2}{\pi} \right)^2 \left\{ k_1 k_2 k_3 k_4 \right\} \sum_{L} k_1 d_{k_1j_1}(k_1 r_1) \ldots k_4 d_{k_4j_4}(k_4 r_4) T^{\text{sph}}_{l_1l_2l_3}(L; k_1, r_1). \tag{42}
\]

The relation of the full 3D trispectrum in spherical coordinates and its Fourier decomposition, which generalizes our previous results for the bispectrum, is written as follows:
Along with the bispectrum expression this generalizes the previously-obtained relationship at the level of the power spectrum in Castro et al. [2009]. Clearly for practical purposes we will need to devise an approximation to the exact result. We will use the Limber approximation to approximate the spherical Bessel functions. Note that numerical evaluation of trispectra is considerably more involved than the bispectrum. It is also important to note that as we climb upwards in the hierarchy realistically it gets more difficult to extract signals from observational data because of the presence of noise.

5.1 Linking the convergence trispectrum with the underlying matter trispectrum

Finally the observable trispectrum $\mathcal{T}$ for the convergence $\kappa$ (defined through an equivalent expression as in Eq. (37)) can be expressed in terms of the underlying trispectrum of the mass distribution:

$$T_{i j k}^{L, r} = \frac{A}{L^3} \int_0^\infty dk_1 P(k_1) P(k_2) P(k_3) \int_0^{\infty} dr_1 j_1(k_1 r_1) \cdots j_4(k_4 r_4) T_{i j k}^{L, r} \text{rect}. \quad (43)$$

The mode-mixing in spherical coordinates happens only in the radial direction. It is expected that the estimation of the trispectrum from a realistic sky will be noise-dominated in the near future. This means estimation will be difficult for individual modes. We will develop methods to compress the information content in individual modes in an optimal way elsewhere. The trispectrum is dominated by the noise from galaxy intrinsic ellipticity as well as shot noise from the Poissonian nature of the galaxy distribution. To determine this we need to take into fact that a contribution to the trispectrum not only comes from the non-Gaussian signal but also from disconnected Gaussian terms too.

5.2 Specific Forms for Underlying Matter Trispectrum and the Limber Approximation

We will derive the result quoted above for the case of the hierarchical ansatz with a “stellar” approximation we make further use of the extended Limber approximation to simplify. The hierarchical ansatz as well as the Limber approximation are both valid at the small angular scale, which justifies their joint use to simplify the results. The result takes the following form:

$$T_{i j k}^{L, r} = \int_0^\infty dr_1 j_1(k_1 r_1) \cdots j_4(k_4 r_4) T_{i j k}^{L, r} \text{rect}, \quad (44)$$

The mode-mixing in spherical coordinates happens only in the radial direction. It is expected that the estimation of the trispectrum from a realistic sky will be noise-dominated in the near future. This means estimation will be difficult for individual modes. We will develop methods to compress the information content in individual modes in an optimal way elsewhere. The trispectrum is dominated by the noise from galaxy intrinsic ellipticity as well as shot noise from the Poissonian nature of the galaxy distribution. To determine this we need to take into fact that a contribution to the trispectrum not only comes from the non-Gaussian signal but also from disconnected Gaussian terms too.

These results are extensions of analogous relations obtained for the bispectrum. We will introduce contributions from star topology under the stellar approximation (for other hierarchical ansätze see e.g. Szapudi & Szalay [1993, 1997] which assumes that the amplitudes associated with all topologies are the same).

We are only concerned with the connected part of the trispectrum here. Next we use the hierarchical ansatz to model the four-point correlation function. We will use the hierarchical ansatz to model the four-point correlation function. The trispectrum as outlined before in hierarchical approximation can be written as a product of three power spectra:

$$\langle \delta(k_1; r_1) \delta(k_2; r_2) \delta(k_3; r_3) \delta(k_4; r_4) \rangle \equiv R_0 \left[ \int d^3 k P(k_1) P(k_2) P(k_3) \delta_{3D}(k_1 + k_2 - k) \delta_{3D}(k_3 + k_4 + k) + \text{cyc.perm.} \right] \quad + R_0 \left[ \int d^3 k P(k_1) P(k_2) P(k_3) \delta_{3D}(k_1 + k_2 - k) \delta_{3D}(k_3 + k_4 + k) + \text{cyc.perm.} \right]. \quad (46)$$

In general the hierarchical amplitudes $R_0$ (associated with the snake topology) and $R_0$ (associated with star topology) will have different amplitudes. There are 12 terms with snake topology and 4 terms with star topology which are represented by the “cyc.perm.”. Various hierarchical models differ in the way they ascribe values to various amplitudes. It is possible also to employ Hyper Extended Perturbation Theory (Scoccimarro et al. 1998) to compute these amplitudes. For our purpose we will assume:

$$\langle \delta(k_1; r_1) \delta(k_2; r_2) \delta(k_3; r_3) \delta(k_4; r_4) \rangle \equiv Q_4 \left[ \int d^3 k P(k_1) P(k_2) P(k_3) \delta_{3D}(k_1 + k_2 - k) \delta_{3D}(k_3 + k_4 + k) + \text{cyc.perm.} \right]. \quad (47)$$

In this case, the analysis is essentially the same as that of the bispectrum and it simplifies the results considerably. The stellar approximation consists of approximating the four-point correlation only with stellar diagrams. This model has been checked in considerable detail in 2D in previous work (Barber, Munshi & Valageas 2004, Munshi, Valageas & Barber 2004, Valageas, Munshi & Barber 2005). We will assume a “stellar” model from this point onward. However the method outlined can also be generalised to take into account the “snake” diagrams. The cyclic permutations for the stellar model now represents all 16 diagrams.
We start by expanding the Dirac delta functions $\delta_{il}(k)$ using two dummy positional variables $x$ and $y$. Next following the same procedure as we have followed for the case of the bispectrum we can express the star part of the trispectrum in spherical coordinates. This will next be needed for the derivation of the convergence trispectrum.

\[
\langle \delta_{l_1 m_1}(k_1; r_1) \ldots \delta_{l_4 m_4}(k_4; r_4) \rangle_c \equiv \int_0^\infty dr_1 r_1 j_{l_1}(k_1 r_1) \ldots \int_0^\infty dr_4 r_4 j_{l_4}(k_4 r_4) J_{l_1 l_2 l_3 l_4}^{(4)}(r_1, r_2, r_3, r_4),
\]

\[
J_{l_1 l_2 l_3 l_4}^{(4)}(r_1, r_2, r_3, r_4) \equiv Q_4 \left( \frac{4 \pi}{c^2} \right)^4 k_1 k_2 k_3 k_4 \sum_{LM} G_{l_1 l_2 L}^{m_1 m_2 M} G_{l_3 l_4 L}^{m_3 m_4 M} \int r^2 dr j_{l_1}(k_1 r) \ldots j_{l_4}(k_4 r) \left\{ P(k_1) P(k_2) P(k_3) + \text{cyc. perm.} \right\},
\]

where the definition for the kernel $J_{l_1 l_2 l_3 l_4}^{(4)}(r_1, r_2, r_3, r_4)$ is similar to its counterpart we introduced for the bispectrum. In our notation, $J_{l_1 l_2}^{(4)}$ denotes the star contribution with the corresponding amplitude $R_0$. It is now possible to express the star contribution to the 3D convergence trispectrum using the following relation:

\[
\langle \kappa_{l_1 m_1}(k_1; r_1) \ldots \kappa_{l_4 m_4}(k_4; r_4) \rangle_c = \int_0^\infty dr_1 r_1 j_{l_1}(k_1 r_1) \ldots \int_0^\infty dr_4 r_4 j_{l_4}(k_4 r_4) \mathcal{I}_{l_1 l_2 l_3 l_4}(r_1, r_2, r_3, r_4)
\]

\[
\mathcal{I}_{l_1 l_2 l_3 l_4}(r_1, r_2, r_3, r_4) = C_{l_1}^4 \left( \frac{4 \pi k_1^3}{c^2} \right)^4 \left( \frac{4 \pi k_4^3}{c^2} \right)^4 \pi \sum_L I_{l_1 l_2 L} I_{l_3 l_4 L} \int r_{\text{min}}^\infty dr \left[ \frac{(r_1 - r)}{a(r) r^3} \right] \ldots \left[ \frac{(r_4 - r)}{a(r) r^3} \right] \left\{ P \left( \frac{l_1}{r}, r \right) P \left( \frac{l_2}{r}, r \right) P \left( \frac{l_3}{r}, r \right) + \text{cyc. perm.} \right\}.
\]

We simplify the expression using the Limber approximation as before. It effectively replaces the wavenumber $k_i$ with the corresponding $l_i/r$ while reducing the dimensionality of the integrals.

\[
\langle \kappa_{l_1 m_1}(k_1; r_1) \ldots \kappa_{l_4 m_4}(k_4; r_4) \rangle_c = \int_0^\infty dr_1 r_1 j_{l_1}(k_1 r_1) \ldots \int_0^\infty dr_2 r_2 j_{l_2}(k_2 r_2) \ldots \int_0^\infty dr_3 r_3 j_{l_3}(k_3 r_3) \mathcal{I}_{l_1 l_2 l_3 l_4}(r_1, r_2, r_3)
\]

\[
\mathcal{I}_{l_1 l_2 l_3 l_4}(r_1, r_2, r_3) = \left[ \frac{\pi}{2} \right]^3 Q_4 \sum_L I_{l_1 l_2 L} I_{l_3 l_4 L} \int r_{\text{min}}^\infty dr \left[ \frac{(r_1 - r)}{a(r) r^3} \right] \ldots \left[ \frac{(r_4 - r)}{a(r) r^3} \right] \left\{ P \left( \frac{l_1}{r}, r \right) P \left( \frac{l_2}{r}, r \right) P \left( \frac{l_3}{r}, r \right) + \text{cyc. perm.} \right\}.
\]

Here the upper limit of integration along the radial direction is $r_{\text{min}} = \min(r_1, r_2, r_3, r_4)$. In our analysis above we have taken the hierarchical ansatz as an example, but it is quite general and the expression for the density trispectrum only affects the expression for $J_{l_1 l_2 l_3 l_4}^{(4)}(r_1, r_2, r_3, r_4)$. Perturbative results are in general more complicated to deal with because of configuration angle-dependence, but will result in similar signal-to-noise ratio. In stellar models the star topologies at various orders carry all the weights, diagrams with snake topologies are ignored and arbitrary order in correlation functions are simply expressed in terms of the star contributions at that order. This approximation, as we will see, can simplify the calculations immensely.

We have concentrated here in modelling of the trispectrum and stressed its importance in confirmation of detection of non-Gaussianity determined using the bispectrum. The analytical modelling of the trispectrum is also important in itself for calculation of the error-covariance of the power spectrum.

## 6 CONVERGENCE SKEW-SPECTRUM

Because of signal-to-noise issues it is difficult to study the bispectrum for all possible configuration of triplets. The skewness compresses all the information contents of the bispectrum into a single number. Such aggressive data compression may be elegant but it fails to distinguish various contributions which might have different shape dependence. These issues have been discussed extensively in recent literature (see Munshi & Heavens [2009] and references therein for detailed discussion of related issues). The cumulant correlators which were introduced in the literature are the two-point objects and are well studied in the case of galaxy surveys. These are multi-point statistics collapsed to two-point objects. In harmonic space they correspond to power spectra associated with multispets of various orders. At the lowest order there is only one power spectrum (coined the skew-spectrum) related to the bispectrum. Cooray [2003] had earlier introduced the unoptimised versions of power spectra associated with bispectra and used them to study lensing-secondary cross correlation. Later studies by Cooray [2003], Cooray, Li & Melchiorri [2003] used power spectra associated with the bispectra and trispectra for redshifted 21cm studies. These power spectra retain some of the information of the multispets that they are associated with, and the numbers increase with the order. We will generalize and use the idea of cumulant correlators here to study the bispectrum and trispectrum associated with the 3D convergence field.

The squared convergence field $\kappa^2(r_1)$ is constructed at a radial distance $r_1$; its harmonic transform on the surface of the sky is denoted by $\kappa_{lm}^{(2)}(r_1)$ (we will be using the mixed representation throughout). Let us start by expressing the spherical harmonics transform $\kappa_{lm}^{(2)}(r_1)$ of the squared convergence field $\kappa^2(\hat{\Omega}, r_1)$ in terms of the spherical harmonics of the original convergence map $\kappa_{lm}(r_1)$.

\[
\kappa_{lm}^{(2)}(r_1) = \int Y_{lm}^*(\hat{\Omega}) \kappa^2(\hat{\Omega}, r_1) d\Omega = \sum_{l_1 m_1} \sum_{l_2 m_2} \kappa_{l_1 m_1}(r_1) \kappa_{l_2 m_2}(r_1) \int d\Omega Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) Y_{lm}^*(\hat{\Omega}).
\]

\[\text{4 Detailed modeling of a multispectra is not important for defining the associated power spectra}\]
The above expression assumes all-sky coverage. In practice the surveys will cover only a fraction of the sky. The mask used in the survey \( w(\hat{\Omega}) \) will affect the estimators that introduces a multiplicative bias which needs to be properly accounted for. If we denote the masked sky harmonics of the squared field by \( \tilde{\kappa}_{lm}^{(2)}(r_1) \) then we can express them in terms of the original convergence harmonics of the all-sky and the harmonics of the mask:

\[
\tilde{\kappa}_{lm}^{(2)}(r_1) = \int w(\hat{\Omega}) Y_{lm}(\hat{\Omega}) \kappa^2(\hat{\Omega}, r_1) d\Omega = \sum l_1 m_1 \sum l_2 m_2 \sum l_3 m_3 \kappa_{l_1 m_1}(r_1) \kappa_{l_2 m_2}(r_1) w_{l_3 m_3} \int d\Omega Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) Y_{l_3 m_3}(\hat{\Omega}) Y_{lm}^*(\Omega)
\]

(52)

\[
\equiv \sum_{(l'm')} K_{l m' m'}(w) \kappa_{l m}^{(2)}(r_1).
\]

(53)

Here the mixing matrix \( K_{l m' m'}(w) \) denotes the mixing of harmonics modes due to the presence of the sky mask whose harmonic transform is \( w_{l m} \). We will use the same mask at both radial distances, but the results can easily be generalized to two different masks. Using these harmonics we can now define \( C_l \) while the bispectrum is invariant under the permutations of the angular harmonics \( \kappa_l \), \( \tilde{\kappa}_l \) and \( \hat{\kappa}_l \) is the power spectrum of the mask, which is completely general. It is easy to see that in the absence of any correlation between signal and noise, the estimator \( \tilde{\kappa}_l \) is unbiased and no noise subtraction is needed as long as it is Gaussian. Using the definition of the coupling matrix \( M \), introduced above, we express the pseudo-CI (PCL) estimator as:

\[
\tilde{C}_l^{(2,1)}(r_1, r_2) = \sum (2l'+1) \sum (2l''+1) \frac{(2l''+1)}{4\pi} \left( \begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right)^2 |w_l|^2 \sum_{l_1 l_2} B_{l_1 l_2} (r_1, r_2, r_3) \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \left( \begin{array}{ccc} l_1 & l_2 & l'_2 \\ 0 & 0 & 0 \end{array} \right) \equiv M_{ll'} C_{l'}^{(2,1)}(r_1, r_2).
\]

(54)

\[
\tilde{C}_l^{(2,1)}(r_1, r_2) \equiv \frac{1}{2l+1} \sum_m \text{Real} \left\{ \kappa_{l m}^{(2)}(r_1) \kappa_{l m}^{(2)}(r_2) \right\} ; \quad \tilde{C}_l^{(2,1)}(r_1, r_2) \equiv \frac{1}{2l+1} \sum_m \text{Real} \left\{ \kappa_{l m}^{(2)*}(r_1) \kappa_{l m}^{(2)}(r_2) \right\}.
\]

(55)

The skew-spectrum \( C_l^{(2,1)}(r_1, r_2) \) probes directly the bispectrum \( B_{l_1 l_2 l_3} \). Though it does not encode the entire shape dependence for each triangular configuration it encodes more information compared to its one-point counterpart the skewness \( S_l(r_1, r_2) \), and has the ability to distinguish different contributions to non-Gaussianity, both primordial as well as gravity-induced. We will focus on gravity-induced non-Gaussianity here and issues related to primordial non-Gaussianity will be addressed elsewhere.

\[
C_l^{(2,1)}(r_1, r_2) = \sum_{l_1 l_2} B_{l_1 l_2} (r_1, r_2, r_3) \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) = \frac{1}{2l+1} \sum_{l_1 l_2} I_{l_1 l_2} B_{l_1 l_2} (r_1, r_2, r_3). \]

(56)

\[
\]
7 KURT-SPECTRUM, OR THE POWER SPECTRUM ASSOCIATED WITH THE TRISPECTRUM

The four-point correlation function, or its harmonic counterpart the trispectrum, has been studied in the literature extensively. This contains the information about the non-Gaussianity beyond the lowest level \cite{Hu1999, Okamoto&Hu2002}. For the case of weak lensing studies clearly the gravity-induced non-Gaussianity is the main motivation. Studies in trispectrum analysis have also been pursued using various other probes e.g. using 21cm surveys \cite{Cooray&Melchiorri2003} or more extensively in several CMB studies; see \cite{Bartoloetal2004} for a review. However these studies typically probe the trispectrum induced by primordial non-Gaussianity.

It is important to note however that at the level of four-point studies, the Gaussian fluctuations from the signal as well as the from the (Gaussian) noise too carry a non-zero (unconnected) trispectrum. This degrades the signal-to-noise for various estimators and clearly needs to be subtracted out before an unbiased comparison with the theoretical predictions can be made.

It is obvious that detection of the trispectrum from noisy data is far more nontrivial than the estimation of the bispectrum. Previous studies have mainly concentrated on one-point estimators which collapse the data to a single number - known as the kurtosis. We extend studies involving kurtosis \(<\kappa^4(\Omega))\) to its two-point counterparts: \(<\kappa^2(\Omega,r_1)\kappa^2(\Omega',r_2))\) and \(<\kappa^4(\Omega,r_1)\kappa(\Omega',r_2))\). In practice however we will consider the Fourier transforms of these objects which are the power spectra associated with the trispectra, \(C^{(3,1)}_l\) and \(C^{(2,2)}_l\). Indeed the radial coordinates associated with two different fields being cross-correlated can be different and will be denoted as \(C^{(3,1)}_{l1}(r_1,r_2)\) or \(C^{(2,2)}_{l1}(r_1,r_2)\).

We start by defining the all-sky harmonic transform \(\kappa^{(3)}_{lm}(r_1)\) for the convergence field \(\kappa^2(\Omega, r_1)\) and cross-correlate it against \(\kappa_{lm}(r_2)\). In the presence of a mask \(\omega(\Omega)\) which we assume to be the same at both radial distances, the harmonic transforms of the cubic field \(\kappa^{(3)}_{lm}(r_1)\) will depend also on the spherical transforms of the mask \(\omega_{lm}\) too:

\[
\kappa^{(3)}_{lm}(r_1) = \sum_{l_1m_1,l_2m_2,l_3m_3} \kappa_{l_1m_1}(r_1)\kappa_{l_2m_2}(r_1)\kappa_{l_3m_3}(r_1) \int d\Omega Y_{l_1m_1}(\Omega)Y_{l_2m_2}(\Omega)Y_{l_3m_3}(\Omega)Y^*_m(\Omega)
\]

\[
\kappa^{(3)}_{lm}(r_1) = \sum_{l_1m_1,l_2m_2,l_3m_3} \kappa_{l_1m_1}(r_1)\kappa_{l_2m_2}(r_1)\kappa_{l_3m_3}(r_1) \omega_{l_4m_4} \int d\Omega Y_{l_1m_1}(\Omega)Y_{l_2m_2}(\Omega)Y_{l_3m_3}(\Omega)Y_{l_4m_4}(\Omega)Y^*_m(\Omega)
\]

We will use these results to derive expressions for \(\tilde{C}^{(3,1)}_l(r_1, r_2)\). The other cut-sky power spectra \(\tilde{C}^{(2,2)}_l(r_1, r_2)\) and all-sky counterparts \(C^{(2,2)}_l(r_1, r_2)\) are given by the following expressions:

\[
\tilde{C}^{(2,2)}_l(r_1, r_2) = \frac{1}{2l+1} \sum_m \kappa^{(2)+}_{lm} \kappa^{(2)}_{lm} \quad C^{(2,2)}_l = \frac{1}{2l+1} \sum_m \kappa^{(2)+}_{lm} \kappa^{(2)}_{lm}.
\]

These power spectra directly probe \(T^{\text{mixed}}_{l1l2l3l4}(l_1, r_1, r_2)\). It compresses all the available information in quadruplet of modes specified by \((l_1, l_2, l_3, l_4)\) to a power spectrum. The power spectra \(C^{(2,2)}_l(r_1, r_2)\) and \(C^{(3,1)}_l(r_1, r_2)\) differ in the way they associate weights to various modes:

\[
C^{(2,2)}_l(r_1, r_2) = \sum_{l_1,l_2,l_3,l_4} T^{l1l2}_{l3l4}(l_1, r_1, r_2) \text{mixed}\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \sqrt{\frac{(2l_3+1)(2l_4+1)}{4\pi(2l+1)}} \left( \begin{array}{c} l_1 \ 0 \ 0 \ 0 \\ l_2 \ 0 \ 0 \ 0 \\ l_3 \ 0 \ 0 \ 0 \\ l_4 \ 0 \ 0 \ 0 \end{array} \right) \left( \begin{array}{c} 1 \ 0 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \\ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 1 \end{array} \right) 
\]

Here the reduced trispectrum \(T^{l1l2}_{l3l4}(r_1; L)\)\text{mixed} is defined in terms of \(\langle \kappa_{l_1m_1}(r_1)\kappa_{l_2m_2}(r_2)\kappa_{l_3m_3}(r_3)\kappa_{l_4m_4}(r_4)\rangle\), as follows. We have added the radial distances \(r_i\) associated with each spherical harmonic in the argument with \(L\), which specifies the diagonal formed by the quadruplet of four quantum numbers \(l_i\).

\[
\langle \kappa_{l_1m_1}(r_1)\kappa_{l_2m_2}(r_2)\kappa_{l_3m_3}(r_3)\kappa_{l_4m_4}(r_4)\rangle = \sum_{LM} T^{l1l2}_{l3l4}(r_1; L)\text{mixed}\left( \begin{array}{c} l_1 \ m_1 \ l_2 \ m_2 \\ L \ M \ l_3 \ m_3 \ l_4 \ m_4 \end{array} \right). 
\]

For partial sky coverage we can express the cut-sky version of the estimator \(\tilde{C}^{(2,2)}_l(r_1, r_2)\) in the following way. The resulting pseudo-\(C_l\)s can then be expressed in terms of \(C^{(2,2)}_{l'}(r_1, r_2)\) through the mixing matrix \(M_{ll'}\):

\[
\tilde{C}^{(2,2)}_l(r_1, r_2) = \sum_{l_1,l_2,l_3,l_4} S_{l_1l_2l_3l_4} S'_{l_1l_2l_3l_4} \left( \begin{array}{c} l \ 0 \ 0 \ 0 \\ l' \ 0 \ 0 \ 0 \end{array} \right) T^{l1l2}_{l3l4}(l_1, r_1, r_2) \text{mixed} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \sqrt{\frac{(2l_3+1)(2l_4+1)}{4\pi(2l+1)}} \left( \begin{array}{c} l_1 \ 0 \ 0 \ 0 \\ l_2 \ 0 \ 0 \ 0 \\ l_3 \ 0 \ 0 \ 0 \\ l_4 \ 0 \ 0 \ 0 \end{array} \right) \left( \begin{array}{c} 1 \ 0 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \\ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 1 \end{array} \right) 
\]

\[
= \frac{1}{2l+1} \sum_{l_1l_2l_3l_4} M_{l'l''} C^{(2,2)}_{l''}(r_1, r_2)\text{mixed}.
\]
Higher-order Statistics for 3D Weak Lensing

There are two cumulant correlators at four-point level as explained above. Following the discussion above we now focus on the other degenerate power spectra associated with the cumulant correlator $\kappa^3(\Omega)\kappa^3(\hat{\Omega})$. This is of the same order as $\kappa^2(\hat{\Omega})\kappa^2(\hat{\Omega})$ and contains information about trispectra as well.

The compression of the information is done with different weighting for different modes:

\[
C_i^{(3,1)}(r_1, r_2) = \frac{1}{2l+1} \sum_r \text{Real} \left\{ \kappa_{im}^*(r_1) \kappa_{im}^*(r_2) \right\},
\]

\[
C_i^{(3,1)}(r_1, r_2) = \frac{1}{2l+1} \sum_r \text{Real} \left\{ \kappa_{im}^*(r_1) \kappa_{im}^*(r_2) \right\}.
\]  

(67)

We can now use the definition of the trispectrum $T_{l_1l_2}(L; r_1, r_2)$ to express $C_i^{(3,1)}(r_1, r_2)$ in terms of the trispectra. The main difference with the previous spectrum $C_i^{(2,2)}(r_1, r_2)$ is that, it sums over all possible configuration of the quadrilateral keeping one of the sides fixed, whereas $C_i^{(2,2)}(r_1, r_2)$ keeps one of the diagonal fixed but sums over all possible configuration of the quadrilateral.

\[
C_i^{(3,1)}(r_1, r_2) = \sum_{l_1, l_2, L} T_{l_1l_2}^{(3,1)}(L, r_1, r_2) \text{mixed} \left( \frac{2l_1+1)(2l_2+1)}{4\pi (2L+1)} \right) \left( \begin{array}{ccc} l_1 & l_2 & L \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} L & l_3 & l \\ 0 & 0 & 0 \end{array} \right)
\]

(68)

\[
C_i^{(3,1)}(r_1, r_2) = \frac{1}{2l+1} \sum_{l_1, l_2, L} \frac{1}{2L+1} I_{l_1l_2 L} I_{Ll_1l_2} T_{l_1l_2}^{(3,1)}(L, r_1, r_2) \text{mixed}.
\]  

(69)

The partial sky coverage will mean that the measured power spectrum $C_i^{(3,1)}(r_1, r_2)$ is not the same as theoretical expectation, but is related as before by $C_i^{(3,1)}(r_1, r_2) = M_{P, P} C_i^{(3,1)}(r_1, r_2)$.

In fact it can show that for arbitrary sky coverage with arbitrary mask the above analysis can be generalized to arbitrary order of correlation hierarchy. If we consider a correlation function at $p + q$ order, for every possible combination of $(p, q)$ we will have an associated power spectrum. Using the same expression for the mode mixing matrix, we can invert the observed $C_i^{(p,q)}(r_1, r_2)$ to $C_i^{(p,q)}(r_1, r_2)$. Hence for arbitrary mask with arbitrary weighting functions the deconvolved set of estimators can be written as:

\[
C_i^{(p,q)}(r_1, r_2) = M^{-1}_{P, P} C_i^{(p,q)}(r_1, r_2).
\]  

(70)

The Gaussian components of the corresponding multiscoptra at that order need to be subtracted out, and can be written in terms of the $C_i$. The noise contribution too is assumed Gaussian and hence only contributes to the unconnected components. As before we can collapse the two-point objects and reduce them to a one-point number, the cross-kurtosis, which will be a function of both radial distances $r_1, r_2$.

\[
K_4(r_1, r_2) = \sum_{l} (2l+1)C_i^{(3,1)}(r_1, r_2) = \sum_{l} (2l+1)C_i^{(2,2)}(r_1, r_2).
\]  

(72)

As we demonstrated with the cross-skewness, $K_4(r_1, r_2)$ can be decomposed in Fourier modes in the radial direction and an associated power spectrum can be defined.

The Gaussian contribution to the trispectrum can be written as:

\[
G_{l_1l_2l_3}(r_1, r_2, r_3, r_4; L) = (-1)^{l_1+l_2} \sqrt{(2l_2+1)(2l_3+1)} C_i^{(3,1)}(r_1, r_2) C_i^{(3,1)}(r_3, r_4) \delta_{L,0} \delta_{l_1l_2} \delta_{l_3l_4} + (2L+1) (-1)^{l_1+l_2} \delta_{l_1l_2} \delta_{l_3l_4} C_i^{(3,1)}(r_1, r_2) C_i^{(3,1)}(l_2, r_4) + (2L+1) \delta_{l_1l_4} \delta_{l_2l_3} C_i^{(3,1)}(r_1, r_2) C_i^{(3,1)}(r_3, r_4) C_i^{(3,1)}(l_2, r_4) + (2L+1) \delta_{l_1l_4} \delta_{l_2l_3} C_i^{(3,1)}(r_1, r_2) C_i^{(3,1)}(r_3, r_4) C_i^{(3,1)}(l_2, r_4).
\]  

(73)

Next we can compute the Gaussian contributions to $C_i^{(3,1)}$ and $C_i^{(2,2)}$ following the same procedure as before just by replacing the trispectrum $T_{l_1l_2}$ with its Gaussian counterpart $G_{l_1l_2l_3}(r_1; L)$. Indeed we will have to keep in mind the ordering correct for various $l_i$ and their $r_i$ counterparts. It is also important to realize that in computing the Gaussian contribution we will have to take into account both the signal and the noise $C_i$s (assumed to be Gaussian), i.e., $C_i = C_i^S + C_i^N$.

\[
G_i^{(3,1)}(r_1, r_2) = \frac{1}{2l+1} \sum_{l_1l_2l_3L} \frac{1}{2L+1} I_{l_1l_2 L} I_{Ll_1l_2} G_i^{(3,1)}(L, r_1, r_2) \text{mixed},
\]

\[
G_i^{(2,2)}(r_1, r_2) = \frac{1}{(2l+1)^2} \sum_{l_1l_2l_3l_4} I_{l_1l_2 L} I_{Ll_1l_2} G_i^{(2,2)}(l_1, r_1, r_2) \text{mixed}.
\]  

(74)

For realistic surveys with a mask, results identical to Eq. (70) and Eq. (66) will hold true for the Gaussian contributions. From the estimated $C_i^{(3,1)}(r_1, r_2)$ and $C_i^{(2,2)}(r_1, r_2)$ these contributions need to be subtracted out before comparing them against the theoretical expectations.

8 OPTIMAL ESTIMATES

The estimators introduced above are not optimal as they are not inverse-variance weighted. In this section we discuss the signal-to-noise of the estimators introduced above, namely $C_i^{(2,1)}$ at the level of bispectrum and $C_i^{(3,1)}$ and $C_i^{(2,2)}$ at the level of trispectrum.
For construction of the optimum estimates the harmonics $a_{lm}$ needs to be inverse covariance weighted (Smith & Zaldarriaga 2006). We refer to Munshi & Heavens (2009), Munshi et al. (2009) for a complete discussion which also requires presence of a linear term (Creminelli et al. 2006) in the case of absence of spherical symmetry. The optimal estimates at the three-point level in the all-sky limit and in the presence of constant variance noise can be expressed as:

$$
\hat{S}^{(2,1)}_l(r_1, r_2) = \frac{1}{2l+1} \sum_{i_l i_2} \tilde{B}_{i_l i_2}(r_1, r_1, r_2) \tilde{B}_{i_l i_2}(r_1, r_1, r_2) \left\{ C_l(r_1, r_1)^{-1} C_{l_1}(r_2, r_2)^{-1} C_{l_2}(r_3, r_3)^{-1} + \text{cyc.perm.} \right\}.
$$

(75)

We have denoted the estimators by $\tilde{B}$. The other terms can be obtained by circular permutation. Different choices of $r_1, r_2, r_3$ can give us different skew spectra. In our discussion of the unoptimized version we have developed one specific example which corresponds to $S_l(r_1, r_2, r_2)$ and was denoted by $C_l^{(2,1)}(r_1, r_2)$. The other choices that we can construct are $S_l(r_1, r_2, r_1)$ and $S_l(r_1, r_2, r_2)$. It is important to note that $C_l^{(2,1)}(r_1, r_2)$ is not invariant under permutation of its indices.

Similar results hold for the case of trispectrum. The optimal estimates for the case of the trispectrum, we can write for the full sky estimates with uniform noise:

$$
\hat{K}^{(3,1)}_l(r_1, r_2) = \frac{1}{2l+1} \sum_{i_l i_2 i_3} \tilde{T}_{i_l i_2 i_3}(L; r_1) \tilde{T}_{i_l i_2 i_3}(L; r_2) \left\{ C_l(r_1, r_1)^{-1} C_{l_1}(r_2, r_1)^{-1} C_{l_2}(r_1, r_2)^{-1} C_{l_3}(r_2, r_2)^{-1} + \text{cyc.perm.} \right\}
$$

(76)

$$
\hat{K}^{(2,2)}_l(r_1, r_2) = \frac{1}{2l+1} \sum_{i_l i_2 i_3 i_4} \tilde{T}_{i_l i_2 i_3 i_4}(L; r_1) \tilde{T}_{i_l i_2 i_3 i_4}(L; r_2) \left\{ C_l(r_1, r_1)^{-1} C_{l_1}(r_2, r_1)^{-1} C_{l_2}(r_1, r_2)^{-1} C_{l_3}(r_2, r_2)^{-1} + \text{cyc.perm.} \right\}.
$$

(77)

The one-point estimator with inverse covariance weighting is simply the sum over the free index of the respective two-point estimators and is expressed as follows:

$$
\hat{S}_l(r_1, r_2) = \sum_i (2l+1) \hat{S}^{(2,1)}_l(r_i); \quad \hat{K}_l(r_1, r_2) = \sum_i (2l+1) \hat{K}^{(3,1)}_l(r_i) = \sum_i (2l+1) \hat{K}^{(2,2)}_l(r_i).
$$

(78)

Depending on various choices to identify the quadruplet of the radial distances $r_i$ that are associated with each harmonic $l$, we will have a different estimator which can provide complementary information. These estimators can help us to optimize survey depth and width for the study of non-Gaussianity at a given order.

## 9 FLAT SKY TREATMENT

As pointed out before for surveys with large opening angle, the all sky expressions developed so far involve the expansion in terms of the spherical harmonics and spherical Bessel functions. However for surveys which cover only a small patch of the sky most of the signal comes from higher harmonics. In such a situation, a natural choice would be to directly deal with a two-dimensional Fourier expansion suitable for flat space. This makes the analysis more straightforward. Our analysis here closely follows that of Hu (1999) and Okamoto & Hu (2002). We are however required to take into account the additional radial coordinate in our analysis. We expand the 3D convergence field as before both in the line-of-sight direction as well as on the surface of the sky:

$$
\kappa(1, k) = \sqrt{\frac{2}{\pi}} \int_0^\infty dr_\parallel \int \frac{d^2 r_\perp}{2\pi} \kappa(r_\parallel, r_\perp) k_{j_1}(kr_\parallel) \exp(-il \cdot r_\perp).
$$

(79)

In this expansion 1 depicts a 2D angular wavenumber and $k$ represents a wavenumber in the radial direction. It will also be advantageous in certain situations when a harmonic expansion is only performed on the surface of the sky, but the radial dependence is kept in configuration space. As before we have assumed in the above expansion that Universe is flat. Alternatively the eigenfunctions for the expansion needs to be suitably modified. The real space correlation functions and their Fourier counterparts are related by the following expression:

$$
\langle \kappa(r_{\parallel 1}, r_{\perp 1}) \cdots \kappa(r_{\parallel n}, r_{\perp n}) \rangle_c = \int \frac{d^2 l_1}{2\pi} \cdot \cdots \cdot \frac{d^2 l_n}{2\pi} \langle \kappa(L, r_{\parallel 1}) \cdots \kappa(0, r_{\parallel n}) \rangle_c \exp[i(l_1 \cdot r_{\perp 1} + \cdots + l_n \cdot r_{\perp n})].
$$

(80)

The flat sky correlation hierarchy, which ensures translational symmetry of the 2D patch sky is expressed by the following equations. The $r$ label is retained as we have not performed the Fourier expansion in the radial direction.

$$
\langle \kappa(l_1, r_{\parallel 1}) \kappa(l_2, r_{\parallel 2}) \rangle_c = (2\pi)^2 \delta_{2D}(l_1 + l_2) \mathcal{P}(l_1, r_{\parallel 1})
$$

(81)

$$
\langle \kappa(l_1, r_{\parallel 1}) \kappa(l_2, r_{\parallel 2}) \rangle_c = (2\pi)^2 \delta_{2D}(l_1 + l_2 + l_3) \mathcal{B}_3(l_1, l_2, l_3; r_{\parallel 1})
$$

(82)

$$
\langle \kappa(l_1, r_{\parallel 1}) \kappa(l_2, r_{\parallel 2}) \kappa(l_3, r_{\parallel 3}) \rangle_c \kappa(l_4, r_{\parallel 4}) \rangle_c = (2\pi)^2 \delta_{2D}(l_1 + l_2 + l_3 + l_4) \mathcal{T}_4(l_1, l_2, l_3, l_4; r_{\parallel 1}).
$$

(83)

The labels $r$ which appears as the arguments for multispectra denote all the radial coordinates which are involved in their definition, e.g. $r_i = r_1, \ldots, r_3$ for the bispectrum. The treatment for trispectra is more complicated as it also gets disconnected Gaussian contributions (which are non-zero even in the absence of any non-Gaussianity) and the contribution from the reduced segment, discussed above, which carries all the information about non-Gaussianity at the level of fourpoint.
\( \langle \kappa(1_1, r_{||}) \kappa(1_2, r_{||}) \kappa(1_3, r_{||}) \kappa(1_4, r_{||}) \rangle_G = (2\pi)^2 \delta_{2D}(1_1 + 1_2)(2\pi)^2 \delta_{2D}(1_3 + 1_4)P(l_1, r_{||})P(l_3, r_{||}) \\
+ (2\pi)^2 \delta_{2D}(1_1 + 1_3)(2\pi)^2 \delta_{2D}(1_2 + 1_4)P(l_1, r_{||})P(l_2, r_{||}) + (2\pi)^2 \delta_{2D}(1_1 + 1_4)(2\pi)^2 \delta_{2D}(1_2 + 1_3)P(l_1, r_{||})P(l_4, r_{||}) \\
\langle \kappa(1, r_{\perp}) \kappa(1_2, r_{\perp}) \kappa(1_3, r_{\perp}) \kappa(1_4, r_{\perp}) \rangle_c = (2\pi)^2 \delta_{2D}(1_1 + 1_2 + 1_3 + 1_4)T_4(l_1, l_2, l_3, l_4, L; r_{||}). \) \( (84) \)

Following the discussion in \cite{Hui1999} and \cite{OkamotoHui2002} we write the tri spectra \( T_{l_1 l_2 l_3} \) in terms of the reduced tri spectra as follows:

\[ T_{l_1 l_2 l_3} = R_{l_1 l_2 l_3}(l_{12}) + R_{l_1 l_2 l_3}(l_{13}) + R_{l_1 l_2 l_3}(l_{14}). \] \( (85) \)

We have used the notation \( l_{12} = l_1 + l_2 \). We will quote the results obtained in \cite{Hui1999} which relate the multi-spectra defined in the full-sky analysis to their flat-sky counterpart.

\[ C_3(k; r) = P(k; r); \quad B_{l_1 l_2 l_3}(r_{||}) = I_{l_1 l_2 l_3}B(l_1, l_2, l_3; r_{||}); \quad R_{l_1 l_2 l_3}^{(l_2 l_3)}(L, r_{||}) = I_{l_1 l_2 l_3}L_3(l_{12})R_{l_1 l_2 l_3}^{(l_1 l_2)}(L, r_{||}). \] \( (86) \)

While the results for the power spectrum and bispectrum are straightforward the reduced tri spectrum is more involved as it depends on the choice of the diagonal of the quadrilateral constructed out of the vectors \( l_i \). The implementation of the momentum conservation is imposed by the translational symmetry is built in the definition is

\[ \langle \kappa(1_1, r_{||}) \kappa(1_2, r_{||}) \kappa(1_3, r_{||}) \kappa(1_4, r_{||}) \rangle_c = (2\pi)^2 \int \delta_{2D}(1_1 + 1_2 + 1_l)\delta_{2D}(1_3 + 1_4 - 1_l)T_4(l_1, 1_2, 1_3, 1_4; r_{||}) \]

\[ = \int \left[ \delta_{2D}(1_1 + 1_2 + 1_3 + 1_4 - 1_l)R_{l_1 l_2 l_3}^{(l_1 l_2 l_3)}(l) + \delta_{2D}(1_1 + 1_3 + 1_4 - 1_l)R_{l_1 l_2 l_3}^{(l_1 l_3 l_4)}(l) + \delta_{2D}(1_1 + 1_4 + 1_3 - 1_l)R_{l_1 l_2 l_3}^{(l_1 l_4 l_3)}(l) \right] d^2l. \] \( (87) \)

The flat patch wave numbers \( (l_i) \) are used within the parentheses which appear on the r.h.s. of the equations, whereas their all-sky versions appear in the l.h.s. without the parentheses.

The radial independence in our calculation can also be displayed by doing a Fourier transform in the radial direction. The transformations from all-sky to the Fourier representation are given by the following expressions. We use the same notations \( B_3 \) or \( T_3 \) in both representations.

\[ B_3(l_1, l_2, l_3; r_{||}) = \left( \frac{2}{\pi} \right)^3 \int dr_{||} j_i(k_1 r_{||}) \ldots \int dr_{||} j_i(k_3 r_{||})B_3(l_1, l_2, l_3; k_i). \] \( (88) \)

A similar expression holds for the tri-spectrum:

\[ T_3(l_1, \ldots, l_4; r_{||}) = \left( \frac{2}{\pi} \right)^4 \int dr_{||} j_i(k_1 r_{||}) \ldots \int dr_{||} j_i(k_3 r_{||})T_3(l_1, \ldots, l_4; k_i). \] \( (89) \)

For a flat sky we can work with various representations for the multispectra as before. The real space variables \( r = (r_{||}, r_{\perp}) \) and their Fourier representations using variables \( k = (k, 1) \) are both useful.

\[ \langle \kappa(1, k_1) \kappa(2, k_2) \rangle = (2\pi)^3 \delta_{2D}(1_1 + 1_2)\delta_{1D}(k_1 + k_2)P(k_1, k_2) \]

\[ \langle \kappa(1, k_1) \kappa(2, k_2) \rangle = (2\pi)^3 \delta_{2D}(1_1 + 1_2)\delta_{1D}(k_1 + k_2)B_3(k_1, k_2) \]

\[ \langle \kappa(1, k_1) \kappa(2, k_2) \rangle = (2\pi)^3 \delta_{2D}(1_1 + 1_2)\delta_{1D}(k_1 + k_2)T_3(k_1, k_2). \]

The relations that we will be using most in our derivations are the orthogonality relationship of the Bessel functions Eq. \( (37) \) and the representation of the 2D Dirac delta function

\[ \int j_i(l_1 r_{||}) j_i(l_2 r_{||}) dr_{||} = \left( \frac{\pi}{2 r_{||}^2} \right) \delta_{2D}(l_1 - l_2); \quad \int \exp(i r_{\perp} \cdot (1_1 - 1_2)) d^2r_{\perp} = (2\pi)^2 \delta_{2D}(1_1 - 1_2). \] \( (93) \)

Our convention for the Fourier transform of an arbitrary real-space function \( \kappa(r_{||}, r_{\perp}) \) to Fourier space \( \kappa(k, 1) \) for small angular scale approximation is given by:

\[ \kappa(k, 1) = \sqrt{\frac{2}{\pi}} \int dr_{||} \int \frac{d^2r_{\perp}}{2\pi} k_ji(k r_{||}) \exp(-i \cdot r_{\perp}) \kappa(r_{||}, r_{\perp}); \quad \kappa(r_{||}, r_{\perp}) = \sqrt{\frac{2}{\pi}} \int dk \int \frac{d^4k}{(2\pi)^3} k_ji(k r_{||}) \exp(-i \cdot r_{\perp}) \kappa(k, 1). \] \( (94) \)

We will also be working with partial transforms such as \( \kappa(k_l, r_{\perp}) \) and \( \kappa(r_{||}, l) \) which are defined in an obvious way.

### 9.1 Flat Sky Without Mask

In this section we will consider the power spectra associated with multi-spectra in a flat patch of sky suitable for smaller surveys. These are cross power spectra \( P_{\alpha \beta}(l) \), that are the Fourier transforms of cross correlation function of fields constructed from the moments of the original field \( \kappa(r) \), e.g. \( \kappa^2(r) \) and \( \kappa^2(r) \). Being collapsed multipoint statistics, they carry information about the associated multispectra of order \( p + q \), although they themselves are just two-point objects in real space. We will be using the same convention for the Fourier transform as the previous section:
\[ \kappa(r_\perp) = \frac{1}{2\pi} \int \kappa(l) \exp(i l \cdot r_\perp) d^2l; \quad \kappa(l) = \frac{1}{2\pi} \int \kappa(l) \exp(-i l \cdot r_\perp) d^2r_\perp; \quad \langle \kappa(l_1) \kappa(l_2) \rangle_c = (2\pi)^2 P(l) \delta_{2D}(l_1 - l_2) \]  

In these expressions we have suppressed the explicit radial dependence - which will be introduced at the end of sections. We will first consider the skew spectrum. The cumulant correlator of interest in this case is \( \langle \kappa(r_1) \kappa(r_2) \rangle_c \). This is related to the underlying bispectrum \( B(l_1, l_2, l_3) \) and we denote the associated power spectrum by \( P_{21}(l) \).

\[ \kappa^{(2)}(l) = \frac{1}{(2\pi)^2} \int \kappa(l) \exp(-i l \cdot r_\perp) d^2r_\perp; \quad \kappa^2(l) = \frac{1}{(2\pi)^2} \int \kappa(l_1) \kappa(l_2) \exp((l_1 + l_2 - 1) \cdot r_\perp) d^2l_1 d^2l_2; \]  

\[ \kappa^{(2)}(l) = \frac{1}{(2\pi)^2} \int \kappa(l_1) \kappa(l_2) \delta_{2D}(l_1 + l_2 - 1)d^2l_1 d^2l_2. \]  

Using the above expressions we can write down the flat-sky version of the skew spectrum:

\[ P_{21}(l) \equiv \langle \kappa^{(2)}(l) \kappa(l) \rangle_c = \int B_3(l_1, l_2, l) S(l_1, l_2, l) l_1 l_2 dl_1 dl_2. \]  

In deriving this we have carried out the angular integrals \( \phi_1 \) and \( \phi_2 \) by using the following equation (Hivon et al. 2002):

\[ \int d\phi_1 \int d\phi_2 \delta_{2D}(l_1 + l_2 + 1) = 2\pi S(l_1, l_2, l), \]  

where

\[ S(l_1, l_2, l_3) \equiv \frac{2}{\pi} (l_1^2 + l_2^2 + l_3^2 - 2l_1 l_2 - 2l_1 l_3 - 2l_2 l_3)^{-1/2}. \]  

The derivation outlined above implicitly assumes that the bispectrum has no angular dependence in Fourier space. This is valid for the “stellar” model we will be considering. Carrying through the analysis in a very similar way we can write down the first of a set of two degenerate power spectra associated with the trispectrum \( P_{31}(l) \):

\[ P_{31}(l) = \langle \kappa^{(3)}(l) \kappa^2(l) \rangle_c = \int \langle \kappa(l_1) \kappa(l_2) \kappa(l_1) \kappa(l_4) \rangle_c \delta_{2D}(l_1 + l_2 - 1) \delta_{2D}(l_3 + l_4 + 1) d^2l_1 d^2l_2 d^2l_3 d^2l_4. \]  

Using Eq. (99) again to simplify the angular integrals we find

\[ P_{32}(l) = \int l_1 dl_1 \int l_2 dl_2 \int l_3 dl_3 \int l_4 dl_4 T_4(l_1, l_2, l_3) S(l_1, l_2, l) S(l_3, l_4, l). \]  

In an analogous way, going through the same algebra for the other degenerate power spectra associated with trispectrum we find

\[ \kappa^{(3)}(l) = \int \kappa^3(r_{pec}) \exp(i l \cdot r_{pec}) d^2r_{pec}; \quad \kappa^{(3)}(l) = \int \kappa(l_1) \kappa(l_2) \kappa(l_3) \delta_{2D}(l_1 + l_2 + l_3 - 1) d^2l_1 d^2l_2 d^2l_3 \]  

\[ P_{31}(l) = \langle \kappa^{(3)}(l) \rangle_c = \int T_4(l_1, l_2, l_3) S(l_1, l_2, l') S(l_3, l_4, l) l_1 l_2 dl_1 dl_2 dl_3 dl_4 \]  

In our derivation we have decomposed the \( \delta_{2D} \) function \( \delta_{2D}(l_1 + l_2 + l_3 - 1) \) in terms of two \( \delta_{2D} \) function \( \delta_{2D}(l_1 + l_2 + l_3 - 1) = \int \delta_{2D}(l_1 + l_2 + l_3 - 1) \) used Eq. (99) individually on each of them. We assume no angular dependence for the trispectrum, valid for the stellar model that we consider. For the radial dependence we need to replace the \( T_4(l_1, l_2, l_3) \) with \( T_4(l_1, l_2, l_3, l_4; r_i) \) which will not affect the rest of the analysis.

The power spectra \( P_{31}(l) \) and \( P_{32}(l) \) both probe the configuration dependence of the trispectrum \( T_4(l_1, l_2, l_3, l_4) \) in a restricted sense. While the power spectrum \( P_{32}(l) \) considers all possible configurations with the diagonal of the quadrangle (formed by momentum vectors \( l_1 \) constant, \( P_{31}(l) \) gives an estimate where one side of the quadrangle is kept fixed while all other sides as well as both diagonals vary.

Given a specific model for the multispectra now we can compute the associated power spectra and compare them with simulated and observed data. To consider radial dependence we need to start our analysis from Eq. (88) for the bispectrum and Eq. (89) for the trispectrum. Following exactly similar analysis we will recover the flat sky versions Eq. (63) and Eq. (69).

### 9.2 Flat Sky With a Mask

We will consider a very general mask \( w(r) \) without enforcing any symmetry. We will show that the estimated power spectra from a masked data is a convolved estimate of the underlying true estimates. We will derive the convolution function and how it is determined by the properties of the mask and develop procedures to deconvolve the effect of the mask to have an unbiased estimator. The analysis will parallel our discussion for the spherical sky. Let us start by defining a convolved scalar field \( \tilde{\kappa}(r) \) which depends both on the original field \( \kappa(r) \) as well as the mask \( w(r) \), \( \tilde{\kappa}(r) = \kappa(r) w(r) \) In the Fourier domain this takes the form of a convolution. If we represent the Fourier transform of \( \tilde{\kappa}(r) \) by \( \tilde{\kappa}(l) \) we can express it in terms of the following expression using the Fourier transform of the mask \( w(l) \) and the unsmoothed convergence field \( \kappa(l) \):

\[ \tilde{\kappa}(l) = \int \kappa(l_1) w(l_2) \delta_{2D}(l_1 + l_2 - 1) d^2l_1 d^2l_2. \]
which with the help of a kernel \( K_{\mu} \), which encodes the effect of the mask, takes a more compact form:

\[
\tilde{k}(l_1) = \int d^2l_2 K_{l_1 l_2}[w]; \quad \tilde{K}_{l_1 l_2}[w] = \int d^2l_3 w(l_3) \delta_{2D}(l_1 - l_2 + l_3).
\] (106)

The power spectrum associated with the masked fields is given in terms of the kernel \( S(l, l_1, l_2) \) and the power spectrum of the unmasked field \( P(l_1) \) and the power spectrum of the mask \( P_m(l) = \frac{1}{2\pi} \int d\phi w(l)w(l)^* \) \cite{Hivon2002}:

\[
\tilde{P}_{11}(l) = \frac{1}{2\pi} \int d\phi \langle \tilde{k}(l) \tilde{k}(l) \rangle = \int P(l_1)P_m(l_2)S(l_1, l_1, l_2)l_2dl_2dl_1.
\] (107)

We can rewrite the same equation in a compact form \cite{Hivon2002}:

\[
\tilde{P}_{11}(l_2) = \int P(l_1)M_{l_1 l_2}dl_1; \quad M_{l_1 l_2} = 2\pi \int P_m(l)S(l_1, l_1, l_2)dl_1.
\] (108)

Here we have introduced the kernel \( M_{l_1 l_2} \) and used the notation \( d^2l = dl_1dl_2 \) for the integration variables on the surface of the sky (2D flat patch). \( S(l, l_1, l_2) \) is defined in Eq. (109). This result has general applicability and does not depend on any specific mask properties. Hence \( \tilde{P}_{11}(l) \) can be used as an estimator for the deconvolved power spectrum \( P(l_1) \). Carrying out a exactly similar procedure we can obtain the results for the skew spectrum or the kurt-spectrum. If we denote the power spectrum associated with the correlation function \( \langle \kappa^2(r_1) \kappa^2(r_2) \rangle \), by \( P_{\kappa\kappa}(l) \) and the deconvolved power spectrum \( P_{\kappa\kappa}(l) \) which is the Fourier representation of the correlation function \( \langle \kappa^2(r_1) \kappa^2(r_2) \rangle \), then they are related by the same expression:

\[
\tilde{P}_{\kappa\kappa}(l_1) = \frac{1}{2\pi} \int d\phi \langle \tilde{k}(l_1)^p \tilde{k}(l_1)^q \rangle = \int M_{l_1 l_2} P_{\kappa\kappa}(l_2)l_2dl_2.
\] (109)

The Fourier transforms of the squared field \( \kappa^2 \) with a mask are related by the usual coupling matrix, \( \tilde{k}^{(2)}(l_2) = \int K_{l_1 l_2}[w]|k^{(2)}(l_1)|^2dl_1; \quad \tilde{k}^{(3)}(l_2) = \int K_{l_1 l_2}[w]|k^{(3)}(l_1)|^2dl_1. \) (110)

\[
\tilde{P}_{22}(l_1) = \int M_{l_1 l_2} P_{22}(l_2)l_2dl_2; \quad \tilde{P}_{22}(l_1) = \int M_{l_1 l_2} P_{22}(l_2)l_2dl_2; \quad \tilde{P}_{31}(l_1) = \int M_{l_1 l_2} P_{31}(l_2)l_2dl_2.
\] (111)

These equations represent the flat-sky version of the all-sky expressions, Eq. (65) and Eq. (70). They generalise the results obtained for the flat-sky power spectrum by \cite{Hivon2002}. The coupling matrix \( M_{l_1 l_2} \) introduced in this section is the flat-sky analogue of its all-sky counterpart. The power spectra described here, i.e. \( P_{11}(l), P_{22}(l), P_{23}(l), P_{31}(l) \) are associated with relevant multispectra of the same order. These are useful probes of associated multispectra as they do not compress the available information to a single number and retain some of the relevant shape dependence. The fact that unbiased estimators can be constructed by simple inversion means estimation of such multispectra from simulations and observational data may be realistically possible even in the presence of complicated masks with non-trivial topology. The issues of analysis of noise subtraction can be dealt with in a similar manner.

10 CONCLUSIONS

Future weak lensing surveys will play a big part in further reducing the uncertainty in fundamental parameters, including those that describe the evolution of equation of state of dark energy \cite{Refregier2010}. Weak lensing surveys can exploit both the angular diameter distance and the growth of structure to constrain cosmological parameters, and can test the gravity model \cite{Heavens2007, Amendola2008, Benoy2009}. For recent results, see \cite{Schrabback2009, Kilbinger2009}. Such constraints from weak lensing are complementary to those obtained from cosmic microwave background studies and from galaxy surveys as they probe structure formation in the dark sector at a relatively low redshift range. Initial studies in weak lensing were restricted to studying two-point functions in projection for the entire source distribution. It was, however, found that binning sources in a few photometric redshift bins can improve the constraints \cite{Hu1999}. More recently a full 3D formalism has been developed which uses photometric redshift of all sources without any binning \cite{Heavens2003, Castro2008, Heavens2006}. These studies have demonstrated that 3D lensing can provide more powerful and tighter constraints on the dark energy equation of state parameter, on neutrino masses \cite{deBernardis2009}, as well as testing braneworld and other alternative gravity models. Most of these 3D works have primarily focussed on power spectrum analysis, but in future accurate higher-order statistic measurement should be possible (e.g. \cite{Takada2004, Sembolini2009}).

In this paper we have generalized these studies analytically to multi-spectra which takes us beyond conventional power spectrum analysis. The previously obtained analytical results were developed for the statistical study of weak lensing observable using generic models for the multispectra of the underlying mass distribution. Later on we specialize the results for the case of specific examples using the hierarchical ansatz, where higher-order multispectra are constructed from various products of power spectra organized in all possible topological diagrams with different amplitudes. The analytical results are developed both for near all-sky surveys as well as for flat patches of the sky. The formalism developed does not depend on the background cosmology and can be used to predict level of non-Gaussianity for both primary as well as secondary non-Gaussianity.

The higher-order multispectra contain a wealth of information in through their shape dependence. Though partly degenerate, this information can be invaluable for constraining structure formation scenarios. However determination of the multispectra and their complete shape dependence is not an easy task from noisy data. In this paper we advocate a set of statistics called “cumulant correlators” which were first used in real space in the context of galaxy surveys and later extended to CMB studies. Here we have presented a general formalism for the study of the power spectra or the Fourier transforms of...
these correlators. We present a 3D analysis which takes into account the radial as well as on the surface of the sky decomposition. We start by relating various representation of multspectra in three dimensions. We relate the spherical representation and the Fourier representations with other possibilities: mixed modes of representations. These allow us to relate the harmonic decomposition of convergence directly with that of underlying mass distribution.

We have restricted this study to the third and fourth order, though it can be generalised to higher order and some of our results are valid at arbitrary order. At third order, we define a power spectrum which compresses information associated with a bispectrum to a power spectrum. This power spectrum $C^{(3)}_{l}^2(r_2, r_1)$ is the cross-power spectrum associated with squared convergence maps $\kappa^2(r_1, \Omega)$ constructed at a specific radial distance $r_1$ against $\kappa^2(r_2, \Omega)$ at $r_2$. In a similar manner we also associate power spectra $C^{(2)}_l$ and $C^{(3)}_l$ with associated trispectra $T_{l}^{(2)}(L; r_1)$. There are two different power spectra at the level of trispectra which are related to the respective real-space correlation functions $\langle \kappa^2(r_1, \Omega)\kappa^2(r_2, \Omega') \rangle$ and $\langle \kappa^3(r_1, \Omega)\kappa(r_2, \Omega') \rangle$. We expressed these real-space correlators in terms of their Fourier space analogue which take the form of $C^{(3)}_{l}^2(r_2, r_1)$ and $C^{(3)}_{l}^2(r_2, r_1)$. We develop analytical expressions to take into account the photometric redshift errors in these power spectra. While we present formalisms which are completely analytical, we also use the Limber approximation to reduce the dimensionality of the relevant integrations. These when combined with specific hierarchical models of gravitational clustering can make analytical results remarkably simpler.

The statistics presented here will be a useful tool in studying non-Gaussianity in alternative theories of gravity; which are one of the important science drivers for the future generations of weak lensing surveys. We plan to present detailed results elsewhere in future.

11 ACKNOWLEDGEMENTS

Initial phase of this work was completed when DM was supported by a STFC rolling grant at the Royal Observatory, Institute for Astronomy, Edinburgh. DM also acknowledges support from STFC standard grant ST/G002231/1 at the School of Physics and Astronomy at Cardiff University where this work was completed. It is a pleasure to thank Asantha Cooray and Patrick Valageas for many useful discussions.

REFERENCES

Amendola, L., Kunz M., Sapone D., 2008, JCAP, 04, 13
Barber A.J., Munshi D., Valageas P., 2004, MNRAS, 347, 667
Bartolo N., Komatsu E., Matarrese S., Riotto A., 2004, Phys.Rept., 402, 103
Beacon D.J., Refregier A., Ellis R.S., 2000, MNRAS, 318,625
Benyon E., Bacon D.J., Koyama K., 2009, astroph/0910.1480
Bernardeau F., Schaeffer R., 1992, A&A, 255, 1
Bernardeau F., Valageas P., 2000, A&A, 364, 1
Bernardeau F., Van Waerbeke L., Mellier Y., 1997, A&A, 322, 1
Bernardeau F., Mellier Y., Van Waerbeke L., 2002, A&A, 389, L28
Bernardeau F., Mellier Y. van Waerbeke L., 2003, A&A, 389, L28
Bernardeau F., van Waerbeke L., Mellier Y., 2003, A&A, 397, 405
Bernardeau F., Colombi S., Gaztanaga E., Scoccimarro R., 2002, Phys.Rept.,367, 1
Castro P.G., Heavens A.F., Kitching T.D., 2005, Phys.Rev. D72, 023516
Coles P., Melott A.L., Munshi D., 1999, ApJ, 521, L5
Cooray A., 2001, Phys.Rev. D, 64, 043516
Cooray A., Seth R., 2002, Phys. Rep. 372, 1
Cooray A., 2006, PRL, 97, 261301
Cooray A., C., Li C., Melchiorri A., 2008, Phys.Rev.D, 77,103506
Creminelli P., Nicolis A., Senatore L., Tegmark M., Zaldarriaga M., 2006, JCAP, 5, 4
de Bernardis F., Kitching T. D., Heavens, A., Melchiorri, A., 2009, Phys. Rev. D80, 123509
Fry J.N., 1984, ApJ, 279, 499
Heavens A.F., 2003, MNRAS, 343, 1327
Heavens A. F., Refregier A., Heymans C.E., 2000, MNRAS, 319, 649
Heavens A. F., Kitching T. D., Taylor A.N., 2006, MNRAS, 373, 105
Heavens A. F., Kitching T. D., Verde L., 2007, MNRAS, 380, 1029
Hivon E., Górski K. M., Netterfield C. B., Crill B. P., Prunet S., Hansen F., 2002, ApJ, 577, 2
Hoekstra H., Yee H. K. C., Gladders M. D., 2002, ApJ, 577, 595
Hu W., ApJ., 1999, 522, L21
Hui L., ApJ.,1999, 519, L9
Jain B., Seljak U., White S. Astrophys.J., 2000, 530, 547
Jain B., Seljak U., 1997, ApJ, 484, 560
Kaiser N. 1992, ApJ, 388, 272
Kaiser N., Wilson G., Luppino G.A., astro-ph/0003338
Kilbinger M., et al., 2009, A & A, 497, 677
Kitching T.D., Heavens A. F., Verde L., Serra P., Melchiorri A., Phys.Rev. 2008, D77, 103008
Limber D.N., 1954, ApJ, 119, 665
LoVerde M., Afshordi N. 2008, Phys.Rev.D78, 123506
APPENDIX A: REALISTIC SELECTION FUNCTION AND PHOTOMETRIC REDSHIFT ERRORS

The results obtained in the main text was simplified for clarity, ignoring the fact that in a realistic survey, the average number density of sources will decline with distance, and the distances estimated from photometry will include errors. We consider these here.

A1 All Sky results

The lensing potential can only be sampled at the position of galaxies. Hence it can be written as sum over galaxy positions. This discrete sum can be expressed as

\[ \kappa^O_{lm}(k; r) = \frac{\sqrt{2}}{\pi} \sum_g \kappa(r) k_{ji} (kr_g^0) Y_{lm}(\hat{\Omega}_g) W(r_g^0). \]  

(A1)

Here \( W \) is an arbitrary weight function, and \( r_g^0 \) is the distance to galaxy \( g \) assuming a fiducial cosmology. The convergence depends of course on the correct distance in the true cosmology, \( r \). Replacing the discrete sum over the galaxy positions with an integral we can write

\[ \kappa^O_{lm}(k; r) = \frac{\sqrt{2}}{\pi} \int d^3r \ k(r) k_{ji} (kr_g^0) Y_{lm}(\hat{\Omega}_g) W(r_g^0). \]  

(A2)
The quantity \( n(r) \) comprised of sum of delta functions which peaks at observed positions of the galaxies. Ensemble averaging of these quantities will reduce the equation to

\[
\kappa^O_{l,m}(k; r) = \frac{1}{2\pi} \int d^3r \tilde{n}(r) \kappa(r) k_j(kr^0) Y_{lm}(\hat{\Omega}) W(r^0)
\]

(A3)

Because of the discrete nature of source galaxies the estimator will have a scatter due to the shot noise. It will also have contribution from source clustering. While we will investigate the effects of photometric redshift errors, we will ignore the uncertainties in the photometric redshift distribution of sources which means we can write \( n(r^0) d^3r^0 = \tilde{n}_s(z_p) dz_p d\Omega/4\pi \). We will ignore the effect of source clustering which does not play a dominant role in the error budget for deep surveys.

\[
\kappa^O_{l,m}(k; r) = \sqrt{\frac{2}{\pi}} \int dz \bar{\Omega} \tilde{n}_s(z_p) \kappa(r) k_j(kr^0) Y_{lm}(\hat{\Omega}) W(z_p)
\]

(A4)

where \( r^0 \) is the fiducial distance at redshift \( z_p \).

The primary effect of photometric redshifts is to smooth the source distributions along the line of sight distribution. If \( p(z|z_p) \) denotes the probability of the true redshift being \( z \) given the photometric redshift \( z_p \), the above equation, when modified to take into account the effect of photometric redshift error can be written as:

\[
\kappa^O_{l,m}(k; r) = \sqrt{\frac{2}{\pi}} \int dz_p \int dz \bar{\Omega} \tilde{n}_s(z_p)p(z|z_p) \kappa(r) k_j(kr^0) Y_{lm}(\hat{\Omega}) W(z_p)
\]

(A5)

After expanding the \( \kappa(r) \) and carrying out the angular integrations we can eventually arrive at the following expression:

\[
\kappa^O_{l,m}(k; r) = \sqrt{\frac{2}{\pi}} \int dz_p \int dz \tilde{n}_s(z_p) p(z|z_p) k_j(kr^0) \int dk' k' j_i(k'r^0) W(z_p) \kappa_{l,m}(k'; r)
\]

(A6)

Typically \( p(z|z_p) \) is modelled as a Gaussian for simplicity, though it may have catastrophic failures. These can be included by modification of \( p(z|z_p) \).

\[
p(z|z_p) = \frac{1}{\sqrt{2\pi} \sigma_z(z)} \exp \left[ -\frac{(z_p - z + z_{bias})^2}{2\sigma_z^2(z)} \right]
\]

(A7)

In this expression \( z_{bias} \) is the possible bias in the photometric redshift calibration. The dispersion in error \( \sigma_z(z) \) depends on the redshift. The photometric redshift errors evidently introduces error in the radial direction. Having expressed the \( \kappa^O_{l,m}(k; r) \) by taking into account the photometric redshift errors in terms of \( \kappa_{l,m}(k, r) \), we can now construct the multiscopra for the observed harmonics by relating \( \kappa_{l,m}(k; r) \) to \( \delta_{l,m}(k; r) \) as outlined in the main text.

### A2 Flat Sky Expressions

Here we give flat-sky results. We start by decomposing the convergence field \( \kappa(k, 1) \) as before. Here \( \kappa^O(k, 1) \) is the observed convergence assuming a fiducial cosmology.

\[
\kappa^O(k, 1) = \frac{2}{\pi} \int d^2r_\perp \int dz_p \tilde{n}_s(z_p) \kappa(r) k_j(kr^0) \exp[-i \cdot \theta].
\]

(A8)

By expressing \( \kappa(r, \perp, r_\perp) \) in terms of \( \kappa(k, l) \) we find

\[
\kappa^O(k, 1) = \frac{2}{\pi} \int \frac{dr_\parallel}{\sqrt{2\pi}} \int dz_p \int \frac{dk'}{\sqrt{2\pi}} k_j(kr^0) k' j_i(k'r^0) \tilde{n}_s(z_p) \kappa(k', 1)
\]

(A9)

\[
\kappa^O(k, 1) = \frac{2}{\pi} \int dz \int dz_p p(z|z_p) \tilde{n}_s(z_p) k_j(kr^0) \int \frac{dr_\parallel}{\sqrt{2\pi}} \int \frac{dk'}{\sqrt{2\pi}} k' j_i(k'r^0) \kappa(k', 1).
\]

(A10)

This expression enables us to relate the theoretical predictions for a fiducial cosmology to the observed \( \kappa^O(k, 1) \). We have assumed \( n(r) dr_\perp d^2r_\parallel = \tilde{n}_s(z_p) dz_p d\Omega \), where \( A \) is the solid angle of the sky covered. The statistical properties such as the bispectrum and trispectrum of the field \( \kappa(k, l) \) derived in the main text can now be used to predict the observed statistics of \( \kappa^O(k, 1) \). Mixing of modes due to the photometric redshift error will couple the radial modes, whereas the partial sky coverage mixes angular modes.

### APPENDIX B: USEFUL MATHEMATICAL RELATIONS

#### B1 Spherical Bessel Functions

The orthogonality relationship for the spherical Bessel functions is given by the following expression:

\[
\int k^2 j_i(kr_1) j_i(kr_2) dk = \left[ \frac{\pi}{2(l + 1/2)^2} \right] \delta_{1D}(r_1 - r_2).
\]

(B1)

The extended Limber approximation is also implemented through the following approximate relation [LoVerde & Afshordi (2008)].
\[
\int F(k)ji(kr_1)ji(kr_2)dk \sim \left[ \frac{\pi}{2r_1^2} \right] F\left(\frac{l}{r_1}\right) \delta_{1D}(r_1 - r_2).
\]
\hspace{1cm} (B2)

Thus for high \( l \) the spherical Bessel functions can be replaced by a Dirac delta function \( \delta_{1D} \):

\[
\lim_{x \to \infty} ji(x) = \sqrt{\frac{\pi}{2l + 1}} \left( l + \frac{1}{2} - x \right).
\]
\hspace{1cm} (B3)

### B2 Spherical Harmonics

The completeness relationship for the spherical harmonics is given by:

\[
\sum_{lm} Y_{lm}(\hat{\Omega}) Y_{l'm'}(\hat{\Omega}') = \delta_{2D}(\hat{\Omega} - \hat{\Omega}').
\]
\hspace{1cm} (B4)

The orthogonality relationship is as follows:

\[
\int d\Omega Y_{lm}(\hat{\Omega}) Y_{l'm'}(\hat{\Omega}) = \delta^K_{ll'}\delta^K_{mm'}.
\]
\hspace{1cm} (B5)

### B3 3J Symbols

The following properties of 3J symbols were used to simplify various expressions.

\[
\sum_{l_3m_3} (2l_3 + 1) \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & l \\ m_1' & m_2' & m \end{array} \right) = \delta^K_{m_1m_1' \delta^K_{m_2m_2'}}
\]
\hspace{1cm} (B6)

\[
\sum_{m_1m_2} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & l_3' \\ m_1 & m_2 & m_3' \end{array} \right) = \delta^K_{ll'}\delta^K_{m_3m_3'}
\frac{2l_3 + 1}{2l_3 + 1}
\]
\hspace{1cm} (B7)

\[
(-1)^m \left( \begin{array}{ccc} l & l & l' \\ m & -m & 0 \end{array} \right) = \frac{(-1)^d}{\sqrt{2l + 1}}\delta^K_{ll'}
\]
\hspace{1cm} (B8)