THE TORSION FLOW ON A CLOSED PSEUDOHERMITIAN 3-MANIFOLD

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Abstract. In this paper we define the torsion flow, a CR analogue of the Ricci flow. We show that the torsion flow on a closed, strictly pseudoconvex pseudohermitian manifold has short time existence for some favorable choice of initial conditions. For homogeneous CR manifolds we give explicit solutions to the torsion flow illustrating various kinds of behavior. We also derive monotonicity formulas for CR entropy functionals. As an application, we classify torsion breathers.

1. Introduction

The Ricci flow, introduced by Hamilton, is a geometric flow for metrics on 3-manifolds, and has played a decisive role in the proof of the Poincaré conjecture and Thurston’s geometrization conjecture for 3-manifolds. It is natural to then investigate a corresponding problem for contact 3-manifolds. One of way of doing this is to find a CR analogue of the Ricci flow on a pseudohermitian 3-manifold (see Section 2 for definitions and basic notions in pseudohermitian geometry).

Recall that a strictly pseudoconvex CR structure on a 3-manifold $M$ is given by a cooriented plane field $\ker \theta$, where $\theta$ is a contact form, together with a compatible complex structure $J$. This gives rise to a natural metric $g = \theta \otimes \theta + d\theta(\cdot, J\cdot)$ for $M$. Given this data, there is a natural connection, the so-called Tanaka-Webster connection or pseudohermitian connection. We denote the torsion of this connection by $A_{J,\theta}$, and the Webster curvature, a kind of scalar curvature, by $W$. The torsion flow is then the following PDE,

\[
\begin{align*}
\partial_t J(t) &= 2A_{J(t),\theta(t)}, \\
\partial_t \theta(t) &= -2W_{\theta(t)}.
\end{align*}
\]

It seems to us that the torsion flow (1.1) is the right CR analogue of the Ricci flow.

The torsion flow greatly simplifies if the torsion vanishes. This only happens in very special setups. Indeed, CR 3-manifolds with vanishing torsion are $K$-contact, meaning that the Reeb vector field is a Killing vector field for the metric $g$. In general, one can still hope that the torsion flow improves properties of the contact manifold underlying the CR-manifold. Like the Ricci flow, the torsion flow is a weakly parabolic PDE, provided some suitable initial conditions hold, and is hard to study directly.

The mostly used tools in the study of Hamilton’s Ricci flow [H1] consist of maximum principles. Exceptions are formed by Hamilton’s entropy formula which holds for closed surfaces with positive Gaussian curvature [H2], and also by Perelman’s entropy formulas [Pe]. These formulas can be thought of as monotonicity formulas for integrals of local geometric quantities.

In this paper, we try to do the same for the torsion flow by setting up some monotonicity formulas for Perelman-type functionals. We also give a short time existence result which works in any dimension, but unfortunately it requires additional assumptions in dimension 3.

We conclude this introduction with a brief plan of the paper.

- In Section 1.1 and Section 1.2 we motivate the definition of the torsion flow and give more precise statements of our results.
- In Section 2 we survey basic notions in CR geometry.
- We prove a short time existence result for the torsion flow in Section 3.

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The CR Einstein-Hilbert functional is defined by a fixed contact structure, i.e., \( \xi \). Here \( \xi \) and notions involved, we refer the reader to Section 2. Consider a closed 2-manifold whose first Chern class as \( \theta \). Let \( \Xi \) be a nowhere vanishing closed section \( \zeta \) of the canonical bundle \( \Lambda^{2,0} M \) with the same conformal class as \( \theta \). This conjecture is known to hold in the several cases, and in the following proposition we have collected some of them.

\textbf{Proposition 1.2.} The following criteria for the existence of volume-normalized contact forms hold true.

\begin{itemize}
  \item In Sections 4 and 5 we describe CR manifolds with a global coframe and we also define homogeneous CR manifolds. On the latter class the torsion flow reduces to an ODE if we start with some appropriate initial conditions. These computations illustrate the behavior of the torsion flow in special cases, and in these cases the torsion flow behaves as can be expected from a Ricci-like flow.
  \item Finally, in Section 6 we discuss analogues of Perelman’s entropy formulas.

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\textbf{1.1. Motivation for the torsion flow and statement of results.} For the basic definitions and notions involved, we refer the reader to Section 2. Consider a closed 2n+1-manifold \( M \), with a smooth family of pseudohermitian structures \( (J(t), \theta(t)) \) for which \( J(t) \) is compatible with \( d\theta(t) \); this means that

\begin{equation}
H(t) := d\theta(t)(\cdot, J(t)\cdot) - i d\theta(\cdot, \cdot)
\end{equation}

forms a hermitian metric on the complex vector bundle \( (\xi(t) = \ker \theta(t), J(t)) \).

Furthermore, \( H(t) \) induces a metric on all tensor fields. We shall use these metrics and the induced norms without explicitly referring to \( H(t) \). Throughout the paper, we only consider a fixed contact structure, i.e. \( \xi(t) = \ker \theta(t) \) is independent of \( t \). Henceforth, we just write \( \xi \).

Take a local orthonormal frame \( \{T, Z_\alpha, Z_\beta\} \), where \( T \) is the Reeb field, \( \{Z_\alpha\} \) is a basis of \( (\xi \otimes \mathbb{C})^{1,0} \), and \( \{Z_\beta\} \) is a basis of \( (\xi \otimes \mathbb{C})^{0,1} \). Then we write \( J = i \theta^\alpha \otimes Z_\alpha - i \theta^\beta \otimes Z_\beta \). Define \( E = E_\alpha^\beta \theta^\alpha \otimes Z_\beta + E_\alpha^\beta \theta^\beta \otimes Z_\beta \), and consider the general flow on \( (M, J, \theta) \times [0, T) \) given by

\begin{equation}
\begin{cases}
\partial_t J(t) = 2E, \\
\partial_t \theta(t) = 2\eta(t) \theta(t).
\end{cases}
\end{equation}

The CR Einstein-Hilbert functional is defined by

\[ \mathcal{E}(J(t), \theta(t)) = \int_M W d\mu. \]

Here \( d\mu = \theta \wedge d\theta^n \) is the volume form and \( W \) denotes Tanaka-Webster curvature. From computations later on, namely (5.8) and (6.2), it follows that

\[ \frac{d}{dt} \mathcal{E}(J(t), \theta(t)) = - \int_M \left( A^\alpha_\beta E^\beta_\alpha + A^\alpha_\beta E^\beta_\alpha \right) - 2 \eta W d\mu \leq 0 \]

if we put \( E = A_{J,\theta} \) and \( \eta(t) = -W(t) \). Here \( A_{J,\theta} := A^\beta_{\alpha}Z_\beta \otimes \theta^\alpha + A^\beta_{\alpha}Z_\alpha \otimes \theta^\beta \) denotes the torsion tensor. It is therefore natural to consider the torsion flow on \( M \times [0, T) \) as defined in (1.1). Unfortunately, we do not know whether a short-time solution to the torsion flow (1.1) exists in general. We use the following conjecture to guarantee a suitable initial condition for the flow.

\textbf{Conjecture 1.1.} Let \( (M, J, \theta) \) be a closed, strictly pseudoconvex CR 3-manifold whose first Chern class vanishes, \( c_1(T^{1,0} M) = 0 \). Then there exists a volume-normalized contact form with respect to a nowhere vanishing closed section \( \zeta \) of the canonical bundle \( \Lambda^{2,0} M \) with the same conformal class as \( \theta \).
Proposition 1.3. Let \((S^3, J, \theta)\) be a closed, strictly pseudoconvex CR 3-sphere. Then \(\hat{\theta} = e^{2t} \theta\) is a contact form with vanishing CR Q-curvature if and only if \(\theta\) is a volume-normalized contact form with respect to a nowhere vanishing closed section \(\zeta\) of the bundle \(\Lambda^{2,0}(S^3)\).

(2) (HI) Suppose that \((M, J, \xi = \ker \theta_0)\) has a transverse symmetry and \(\theta_0\) is a volume-normalized contact form. Then a contact form \(\theta\) for \(\xi\) has vanishing CR Q-curvature if and only if \(\theta\) is a volume-normalized contact form. In particular, this holds if \((M, J, \theta_0)\) is a closed hypersurface in \(\mathbb{C}^2\) with transverse symmetry.

(3) (CC) Let \((M, J, \xi = \ker \theta_0)\) be a CR 3-manifold that is Stein fillable. Then there is contact form \(\theta\) for \(\xi\) with vanishing CR Q-curvature.

(4) For a homogeneous CR 3-manifold \((M, J, \theta)\) as defined in Section 5 the homogeneous contact form \(\theta\) is volume-normalized.

The notions in this proposition are defined and discussed in Section 2.9. Further evidence for this conjecture is the following.

Theorem 1.4. Let \((M, J_0, \theta_0)\) be a closed, strictly pseudoconvex CR-manifold of dimension \(2n+1\) satisfying \(c_1(T^{1,0}M) = 0\) and \(n \geq 2\). Then there is a smooth real-valued function \(u\) such that \(e^{2u} \theta\) is a contact form of pseudo-Einstein type. In other words, \(e^{2u} \theta\) is a volume-normalized contact form with respect to a nowhere vanishing closed section \(\zeta\) of the bundle \(\Lambda^{n+1,0}(M)\).

Other related statements are discussed in Section 2.2. Note that the contact structure is fixed under the torsion flow. Since the Chern class of a contact structure is independent of the choice of compatible complex structure, we conclude that the condition \(c_1(T^{1,0}M) = 0\) is preserved under the torsion flow (1.1). Furthermore, Proposition 1.3 tells us that the higher-dimensional analogue of Conjecture 1.1 holds for a strictly pseudoconvex CR-manifold of dimension \(2n+1\) with \(c_1(T^{1,0}M) = 0\) and \(n \geq 2\). This leads us to a suitable initial condition that guarantees the following short-time existence result for the torsion flow.

Theorem 1.5. Assume that \((M, J_0, \theta_0)\) is a closed hypersurface in \(\mathbb{C}^2\) (so \(c_1(T^{1,0}M) = 0\)) and has a transverse symmetry. Then there exists \(\delta > 0\) and a unique smooth solution \((J(t), \theta(t))\) to the torsion flow (1.1) on the interval \([0, \delta]\) such that \((J(0), \theta(0)) = (J_0, \theta_0)\).

From Lemma 3.1 and part (ii) of Proposition 2.9 we deduce the following corollary,

Corollary 1.6. Let \((M, J_0, \theta_0)\) be a closed homogeneous pseudohermitian 3-manifold, so \(c_1(T^{1,0}M) = 0\). Then there exists \(\delta > 0\) and a unique smooth solution \((J(t), \theta(t))\) to the torsion flow (1.1) on the interval \([0, \delta]\) such that \((J(0), \theta(0)) = (J_0, \theta_0)\).

Also, part (4) of Proposition 1.2 implies the corollary.

Corollary 1.7. Let \((M, J, \theta)\) be a closed homogeneous pseudohermitian 3-manifold, so \(c_1(T^{1,0}M) = 0\). Then there exists \(\delta > 0\) and a unique smooth solution \((J(t), \theta(t))\) to the torsion flow (1.1) on the interval \([0, \delta]\) such that \((J(0), \theta(0)) = (J_0, \theta_0)\).

For the homogeneous CR manifolds we define in Section 5 we also show the following convergence result.

Theorem 1.8 (Convergence to torsion free CR structure). Let \((M, \{\omega^i\}, \theta = \omega^1)\) be a homogeneous contact manifold whose Lie algebra is isomorphic to \(\mathfrak{su}(2)\). Then there is a unique homogeneous complex structure \(J_{n, b=1, c=\infty}\) that is torsion free. Moreover, for any choice of homogeneous complex structure \(J_{a,b=1,c}\), the normalized torsion flow converges to this unique CR-structure (\(\ker \theta, J_{a,b=1,c}\)).

In particular, for any choice of homogeneous complex structure on \(\mathfrak{su}(2)\), the normalized torsion flow converges to the standard CR-structure.

In the next section we shall discuss somewhat technical results concerning monotonicity properties of Perelman type functionals. As an application of these monotonicity results from Section 1.2 we classify the torsion breathers and solitons. The classification of torsion solitons is a necessary
step in understanding the singularity formation in the torsion flow. Indeed, one expects the torsion soliton solutions to model finite time singularities of the torsion flow. In view of the flow (1.9) and original definition in [Pe], it is natural to define the soliton solutions for torsion flow (1.1) as follows.

**Definition 1.8.** (i) A family \( J(t) \) of CR structures on \((M, \theta, J)\) evolving by the torsion flow (1.1) is called a breather if for some \( t_1 < t_2 \) and \( \delta > 0 \), the CR structures \( J(t_1) \) and \( J(t_2) \) differ only by a contact diffeomorphism \( \Phi \) with \( \theta(t_2) = \Lambda \Phi^* \theta(t_1) \); the cases \( \lambda = 1, \lambda < 1, \lambda > 1 \) correspond to steady, shrinking or expanding breathers, respectively.

(ii) A breather satisfying the above properties for all pairs of \( t_1 \) and \( t_2 \) of real numbers is called a torsion soliton.

Ideas of Perelman [Pe] (see also [Ca] and [Li]) can be combined with Theorem 1.10, Theorem 1.13 and Theorem 1.14 to show the following classification result.

**Corollary 1.9.** Let \((M, J, \theta)\) be a closed pseudohermitian 3-manifold. Then

(i) there is no closed steady torsion soliton other than the one which admits zero Tanaka-Webster curvature and vanishing pseudohermitian torsion up to a contact transformation.

(ii) there is no closed expanding torsion soliton other than the one which admits negative Tanaka-Webster curvature and vanishing pseudohermitian torsion.

(iii) there is no closed shrinking torsion soliton other than the one which admits positive Tanaka-Webster curvature and vanishing pseudohermitian torsion.

1.2. Some monotonicity results for Perelman-type functionals. The statements in this section are of a more technical nature: we will derive the CR analogue of Perelman’s monotonicity formulas for the so-called coupled torsion flows (1.4), (1.6) and (1.7) in dimension 3.

In Section 6.1 we define the CR analogue of Perelman’s \( F \)-functional by

\[
F(J(t), \theta(t), \varphi(t)) = \int_M (W + |\nabla_b \varphi|^2) e^{-\varphi} d\mu
\]

with the constraint

\[
\int_M e^{-\varphi} d\mu = 1.
\]

Under the flow (1.3), this is equivalent to

\[
\int_M (\varphi_t - 4\eta(t)) e^{-\varphi} d\mu = 0.
\]

Therefore, the following coupled torsion flow is natural,

\[
\begin{aligned}
\partial_t J(t) &= 2E, \\
\partial_t \theta(t) &= 2\eta(t) \theta(t), \\
\partial_t \varphi(t) &= 4\eta(t),
\end{aligned}
\]

with \( E_{11} = e^\varphi (A_{11} - i\varphi_{11} - i\varphi_{1\varphi_1}) \) and \( \eta(t) = e^\varphi (2\Delta_b \varphi - |\nabla_b \varphi|^2_{J,\theta} + W) \).

**Theorem 1.10.** Let \((M, J, \theta)\) be a closed 3-dimensional pseudohermitian manifold and \( J(t), \theta(t), \varphi(t) \) be a solution of the coupled torsion flow (1.4) on \( M \times [0, T) \). Then

\[
\frac{d}{dt} F(J(t), \theta(t), \varphi(t)) = -2\int_M (2\Delta_b \varphi - |\nabla_b \varphi|^2_{J,\theta} + W)^2 d\mu \\
-2\int_M |A_{11} - i\varphi_{11} - i\varphi_{1\varphi_1}|^2 d\mu \\
\leq 0.
\]

The monotonicity formula is strict unless

\[
A_{11} - i\varphi_{11} - i\varphi_{1\varphi_1} = 0 \quad \text{and} \quad 2\Delta_b \varphi - |\nabla_b \varphi|^2_{J,\theta} + W = 0.
\]

That is, up to a contact transformation \( \tilde{\theta} = e^{-\varphi} \theta \)

\[
A_{11} = 0 \quad \text{and} \quad \tilde{W} = 0.
\]

**Remark 1.11.** Observe that for \( \tilde{\theta} = e^{-\varphi} \theta \),

\[
\begin{aligned}
\tilde{A}_{11} &= e^\varphi (A_{11} - i\varphi_{11} - i\varphi_{1\varphi_1}), \\
\tilde{W} &= e^\varphi (2\Delta_b \varphi - |\nabla_b \varphi|^2_{J,\theta} + W).
\end{aligned}
\]
Then the coupled torsion flow \(1.4\) on \((M, J(t), \theta(t))\) is equivalent to the following system of coupled torsion flows on \((M, J(t), \tilde{\theta}(t))\)

\[
\mathcal{F}(\tilde{J}(t), \tilde{\theta}(t), \varphi(t)) = \mathcal{E}(\tilde{J}(t), \tilde{\theta}(t)) = \int_M \tilde{W} \, d\tilde{\mu}
\]

and

\[
\begin{cases}
\partial_t \tilde{J}(t) = 2 A_{\tilde{J}(t)} \tilde{\theta}(t), \\
\partial_t \tilde{\theta}(t) = -2 \tilde{W}(t) \tilde{\theta}(t), \\
\partial_t \varphi(t) = 4 \tilde{W}(t).
\end{cases}
\]

In Section 3.2, we define two functionals analogous to Perelman’s \(W\)-functional, namely the \(W^+\)-functional,

\[
W^+(J(t), \theta(t), \varphi(t), \tau(t)) = \int_M [\tau(W + |\nabla_b \varphi|_{J,\theta}^2 + \frac{1}{2} \varphi - 1)](4\pi \tau)^{-2} e^{-\varphi} d\mu
\]

and the \(W^-\)-functional

\[
W^-(J(t), \theta(t), \varphi(t), \tau(t)) = \int_M [\tau(W + |\nabla_b \varphi|_{J,\theta}^2 - \frac{1}{2} \varphi + 1)](4\pi \tau)^{-2} e^{-\varphi} d\mu.
\]

Remark 1.12. Note that \(W^+\) and \(W^-\) are invariant under the rescaling \(\tau \mapsto c \tau\) and \(\theta \mapsto c \theta\). Furthermore, we have \(W^\pm(J, \theta, \varphi, \tau) = W^\pm(\Phi^* J, \Phi^* \theta, \varphi \circ \Phi, \tau)\) for a contact diffeomorphism \(\Phi : M \to M\).

In view of Theorem 1.10, we first study the monotonicity property of \(W^+\)-functional. By the same discussion as before, the constraint

\[
\int_M (4\pi \tau)^{-2} e^{-\varphi} d\mu = 1
\]

is equivalent to another constraint, namely

\[
\int_M (\varphi_t + 2\tau^{-1} \frac{d\tau}{d\xi} - 4\eta(t))(4\pi \tau)^{-2} e^{-\varphi} d\mu = 0
\]

under the flow \(1.3\). Therefore we consider the following coupled torsion flow:

\[
\begin{cases}
\partial_t J(t) = 2 E, \\
\partial_t \theta(t) = 2 \eta(t) \theta(t), \\
\partial_t \varphi(t) = 4(\eta(t) - \tau^{-1}), \\
\partial_t \tau = 2,
\end{cases}
\]

with \(E_{11} = (A_{11} - i\varphi_{11} - i\varphi_{1} \varphi_1)\) and \(\eta(t) = (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W)\).

**Theorem 1.13.** Let \((M, J, \theta)\) be a closed 3-dimensional pseudohermitian manifold and \(J(t), \theta(t), \varphi(t)\) and \(\tau(t)\) be a solution of the coupled torsion flow \(1.7\). Then

\[
\frac{d}{d\tau} W^+(J(t), \theta(t), \varphi(t), \tau(t)) = -2\tau \int_M (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W - \tau^{-1})^2 (4\pi \tau)^{-2} e^{-\varphi} d\mu \\
-2\tau \int_M |A_{11} - i\varphi_{11} - i\varphi_1 \varphi_1|^2 (4\pi \tau)^{-2} e^{-\varphi} d\mu \\
\leq 0.
\]

The monotonicity formula is strict unless

\[
A_{11} - i\varphi_{11} - i\varphi_1 \varphi_1 = 0 \quad \text{and} \quad 2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W - \tau^{-1} = 0.
\]

That is, up to a contact transformation \(\tilde{\theta} = e^{-\varphi} \theta\)

\[
\tilde{A}_{11} = 0 \quad \text{and} \quad \tilde{W} - \tau^{-1} e^\varphi = 0.
\]

Next we study the monotonicity property of \(W^-\)-functional

\[
W^-(J(t), \theta(t), \varphi(t), \tau(t)) = \int_M [\tau(W + |\nabla_b \varphi|_{J,\theta}^2 - \frac{1}{2} \varphi + 1)](4\pi \tau)^{-2} e^{-\varphi} d\mu.
\]

By the same discussion as before, the constraint

\[
\int_M (4\pi \tau)^{-2} e^{-\varphi} d\mu = 1
\]

is equivalent to

\[
\int_M (\varphi_t + 2\tau^{-1} \frac{d\tau}{d\xi} - 4\eta(t))(4\pi \tau)^{-2} e^{-\varphi} d\mu = 0
\]
under the flow (1.3). Therefore we consider the following coupled torsion flow:

\begin{align*}
&\partial_t J(t) = 2E, \\
&\partial_t \theta(t) = 2\eta(t) \theta(t), \\
&\partial_t \varphi(t) = 4(\eta(t) + \tau^{-1}), \\
&\partial_t \tau = -2,
\end{align*}

with \( E_{11} = (A_{11} - i\varphi_{11} - i\varphi_1\varphi_1) \) and \( \eta(t) = (2\Delta_b \varphi - |\nabla_b \varphi|_{j,\theta}^2 + W) \).

**Theorem 1.14.** Let \((M, J, \theta)\) be a closed 3-dimensional pseudohermitian manifold and \(J(t), \theta(t), \varphi(t)\) and \(\tau(t)\) be a solution of the coupled torsion flow (1.7). Then

\[
\frac{d}{dt} W^-(J(t), \theta(t), \varphi(t), \tau(t)) = -2 \tau \int_M (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W + \tau^{-1})^2 (4\pi\tau)^{-2} e^{-\varphi} d\mu \\
\leq 0.
\]

The monotonicity formula is strict unless

\[
A_{11} - i\varphi_{11} - i\varphi_1\varphi_1 = 0 \quad \text{and} \quad 2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W + \tau^{-1} = 0.
\]

That is, up to a contact transformation \( \widetilde{\theta} = e^{-\varphi} \theta \)

\[
\widetilde{A}_{11} = 0 \quad \text{and} \quad \widetilde{W} + \tau^{-1} e^\varphi = 0.
\]

**Remark 1.15.** Note that for \( \tilde{\theta}(t) = e^{-\varphi(t)} \theta(t) \), we may reparametrize the time \( t \) by the formula \( \tilde{t} = \int_0^t e^{-\varphi(s)}(x(s)) ds \). We have \( \frac{d\tilde{t}}{dt} = e^{-\varphi(t)}(x(t)) \), so the coupled torsion flows (1.6) and (1.7) on \((M, J(t), \theta(t))\) are equivalent to the following coupled torsion flows on \((M, \tilde{J}(\tilde{t}), \tilde{\theta}(\tilde{t}))\), respectively:

\[
\begin{align*}
&\partial_{\tilde{t}} \tilde{J}(\tilde{t}) = 2A_{\tilde{J}(\tilde{t}), \tilde{\theta}(\tilde{t})}, \\
&\partial_{\tilde{t}} \tilde{\theta}(\tilde{t}) = -2W \tilde{\theta}(\tilde{t}), \\
&\partial_{\tilde{t}} \varphi(\tilde{t}) = 4(\tilde{W} - \tau^{-1} e^\varphi), \\
&\partial_{\tilde{t}} \tau = 2,
\end{align*}
\]

and

\[
\begin{align*}
&\partial_{\tilde{t}} \tilde{J}(\tilde{t}) = 2A_{\tilde{J}(\tilde{t}), \tilde{\theta}(\tilde{t})}, \\
&\partial_{\tilde{t}} \tilde{\theta}(\tilde{t}) = -2W \tilde{\theta}(\tilde{t}), \\
&\partial_{\tilde{t}} \varphi(\tilde{t}) = 4(\tilde{W} + \tau^{-1} e^\varphi), \\
&\partial_{\tilde{t}} \tau = -2.
\end{align*}
\]

Recall that \( X_f \) is called a contact vector field or an infinitesimal contact diffeomorphism if the Lie derivative \( \mathcal{L}_{X_f} \theta = \eta \theta \) for some function \( \eta \). Such a contact vector field has the form \( X_f = i f_1 Z_T - i f_2 Z_1 - f T \) for some smooth function \( f \) on \( M \). Furthermore

\[
\mathcal{L}_{X_f} J = 2B'_jf := (f_{11} + iA_{11}f)\theta_1 \otimes Z_T + (f_{11} - iA_{11}f)\theta_T \otimes Z_1
\]

so (1.4) is equivalent to

\[
\begin{align*}
&\partial_t J(t) = 2JB'_jf = J\mathcal{L}_{X_f} J, \\
&\partial_t \theta(t) = 2\eta(t) \theta(t), \\
&\partial_t \varphi(t) = 4\eta(t),
\end{align*}
\]

with \( f = e^\varphi \). Similar results hold for (1.6) and (1.7).

2. Preliminaries and definitions

In this section we introduce some basic notions from pseudohermitian geometry. We learned many of these notions from [L1, L2], and we refer to these papers for proofs and more references.

**Definition 2.1.** Let \( M \) be a smooth manifold and \( \xi \subset TM \) a subbundle. A **CR structure** on \( \xi \) consists of an endomorphism \( J : \xi \to \xi \) with \( J^2 = -\text{Id} \) such that the following integrability condition holds.
(1) if $X, Y \in \xi$, then so is $[JX, Y] + [X, JY]$.
(2) $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.

The CR structure $J$ can be extended to $\xi \otimes \mathbb{C}$, which we can then decompose into the direct sum of eigenspaces of $J$. The eigenvalues of $J$ are $i$ and $-i$, and the corresponding eigenspaces will be denoted by $T^{1,0}$ and $T^{0,1}$, respectively. The integrability condition can then be reformulated as

$$X, Y \in T^{1,0} \implies [X, Y] \in T^{1,0}.$$ 

Now consider a closed $2n + 1$-manifold $M$ with a cooriented contact structure $\xi = \ker \theta$. This means that $\theta \wedge d\theta^n \neq 0$. The **Reeb vector field** of $\theta$ is the vector field $T$ uniquely determined by the equations

$$(2.1) \quad \theta(T) = 1, \quad d\theta(T, \cdot) = 0.$$

**Definition 2.2.** A pseudohermitian manifold is a triple $(M^{2n+1}, \theta, J)$ where

- $\theta$ is a contact form on $M$.
- $J$ is a CR structure on $\ker \theta$.

**Definition 2.3.** The **Levi form** $\langle \cdot, \cdot \rangle$ is the Hermitian form on $T^{1,0}$ defined by

$$H(Z, W) = \langle Z, W \rangle = -i \langle d\theta, Z \wedge W \rangle.$$

We can extend this Hermitian form $\langle \cdot, \cdot \rangle$ to $T^{0,1}$ by defining $\langle Z, W \rangle = \langle Z, W \rangle$ for all $Z, W \in T^{1,0}$. Furthermore, the Levi form naturally induces a Hermitian form on the dual bundle of $T^{1,0}$, and hence on all induced tensor bundles.

We now restrict ourselves to strictly pseudoconvex manifolds, or in other words compatible complex structures $J$. This means that the Levi form induces a Hermitian metric $\langle \cdot, \cdot \rangle_{J,\theta}$ by

$$\langle V, U \rangle_{J,\theta} = d\theta(V, JU).$$

The associated norm is defined as usual: $|V|_{J,\theta}^2 = \langle V, V \rangle_{J,\theta}$. It follows that $H$ also gives rise to a Hermitian metric for $T^{1,0}$, and hence we obtain Hermitian metrics on all induced tensor bundles.

By integrating this Hermitian metric over $M$ with respect to the volume form $d\mu = \theta \wedge d\theta^n$, we get an $L^2$-inner product on the space of sections of each tensor bundle.

**Definition 2.4.** Let $(M, J, \xi = \ker \theta_0)$ be a CR 3-manifold that is the smooth boundary of a bounded, strictly pseudoconvex domain in a complete Stein manifold $V^4$. We shall call such a CR 3-manifold **Stein fillable**.

2.1. **Pseudohermitian connection.** Let $\{T, Z_\alpha, Z_\bar{\beta} \}$ be a frame of $TM \otimes \mathbb{C}$, where $\{Z_\alpha \}$ is any local frame of $T^{1,0}$, and $Z_\bar{\beta} = Z_{\bar{\beta}} \in T^{0,1}$. Then $\{\theta, \theta^\alpha, \theta^{\bar{\beta}}\}$, the coframe dual to $\{T, Z_\alpha, Z_\bar{\beta}\}$, satisfies

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}},$$

where $h_{\alpha\bar{\beta}}$ is a positive definite matrix. By the Gram-Schmidt process we can always choose $Z_\alpha$ such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$; throughout this paper, we shall take such a frame.

The **pseudohermitian connection** or **Tanaka-Webster connection** of $(J, \theta)$ is the connection $\nabla$ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $\{Z_\alpha\}$ for $T^{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where $\omega_\alpha^\beta$ is the 1-form uniquely determined by the following equations:

$$(2.2) \quad d\theta^{\beta} = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta$$

$$\tau_\alpha \wedge \theta^\alpha = 0$$

$$\omega_\alpha^\beta + \omega_{\bar{\beta}}^{\bar{\alpha}} = 0.$$
Here $\tau^\alpha$ is called the **pseudohermitian torsion**, which we can also write as

$$\tau_\alpha = A_{\beta\alpha} \theta^\beta.$$  

The components $A_{\alpha\beta}$ satisfy

$$A_{\alpha\beta} = A_{\beta\alpha}.$$  

We often consider the **torsion tensor** given by

$$A_{f,\theta} = A^\alpha_{\beta\alpha} Z_\alpha \otimes \theta^\beta + A^\alpha_{\beta\bar{\alpha}} \bar{Z}_\alpha \otimes \theta^\beta.$$  

The following remark gives some geometric meaning to the pseudohermitian torsion.

**Remark 2.5.** Let $X_f$ be the contact vector field for the real-valued function $f \in C^2(M)$. Then we have

$$\mathcal{L}_{X_f} \theta = -(Tf) \theta,$$  

and

$$\mathcal{L}_{X_f} J = 2B'_f f := 2(f^3 - iA_{\alpha} \bar{\beta}) \theta^\alpha \otimes Z_{\beta} + 2(f^\beta - iA_{\alpha} \bar{\beta}) \bar{\theta}^\alpha \otimes Z_{\beta}.$$  

See for instance [CL1, Lemma 3.4]. In particular, we have $X_f = T$ for $f = 1$, and the above equation reduces to

$$\mathcal{L}_T J = 2B_{J,\theta},$$  

so we see that the torsion tensor measures to what extent the complex structure $J$ is invariant under the Reeb flow.

We now consider the curvature of the Tanaka-Webster connection in terms of the coframe $\{\theta = \theta^\alpha, \theta^\beta, \theta^\gamma\}$. The second structure equation gives

$$\Omega^\alpha_\beta = \Omega^\beta_\alpha = d\omega^\alpha - \omega^\beta \wedge \omega^\gamma,$$

$$\Omega^0_\alpha = \Omega^0_\beta = \Omega^0_\gamma = \Omega^0_0 = 0.$$  

In [We, Formulas 1.33 and 1.35], Webster showed that the curvature $\Omega^\alpha_\beta$ can be written as

$$\Omega^\alpha_\beta = R^\alpha_\beta \rho \theta^\rho \wedge \theta + W^\alpha_\beta \rho \theta^\rho \wedge \theta - W^\alpha_\beta \bar{\theta} \wedge \theta + i\beta \wedge \tau^\alpha - i\tau^\beta \wedge \theta^\alpha,$$  

where the coefficients satisfy

$$R_{\bar{\beta}\alpha \beta \alpha} = R^\alpha_{\beta \alpha \beta} = R_{\alpha \beta \bar{\alpha} \beta} = R_{\bar{\alpha} \beta \bar{\alpha} \beta}, \quad W_{\bar{\beta} \alpha \gamma} = W_{\gamma \bar{\alpha} \beta}.$$  

In addition, by [L2] (2.4) the coefficients $W^\alpha_{\beta \rho}$ are determined by the torsion,

$$W^\alpha_{\beta \rho} = A^\alpha_{\alpha \rho \beta}.$$  

**Contraction of (2.3)** yields

$$\Omega^\alpha_\alpha = d\omega^\alpha = R^\alpha_{\rho \alpha} \theta^\rho \wedge \theta^2 + W^\alpha_{\alpha \rho} \theta^\rho \wedge \theta - W^\alpha_{\bar{\alpha} \rho} \bar{\theta} \wedge \theta$$

$$= R^\beta_{\beta \rho \sigma} \theta^\rho \wedge \theta^2 + A_{\alpha \rho \sigma} \theta^\rho \wedge \theta - A_{\bar{\alpha} \rho} \bar{\theta} \wedge \theta$$

**Definition 2.6.** The **pseudohermitian Ricci tensor** is the tensor with components $R_{\rho \beta}$. Its trace $W := R_{\rho \rho}$ is called the **Webster curvature**. If the pseudohermitian Ricci tensor is a scalar multiple of the Levi form, then we say that the pseudohermitian structure is **pseudo-Einstein**.

**Remark 2.7.** From the definition it is clear that the Webster curvature is the analogue of the scalar curvature in Riemannian geometry, and we also see that the pseudo-Einstein condition mimics the Einstein condition. Unlike the Riemannian case, pseudo-Einstein structures do not necessarily have constant Webster curvature, even in higher dimensions.

We will denote components of covariant derivatives by indices preceded by a comma. For instance, we write $A_{\alpha \beta \gamma}$. Here the indices $\{0, \alpha, \beta\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_\beta\}$. For derivatives of a scalar function, we will often omit the comma. For example, $\varphi_\alpha = Z_\alpha \varphi$, $\varphi_{\alpha \beta} = Z_\beta Z_\alpha \varphi - \omega_\gamma(Z_\beta) Z_\gamma \varphi$, $\varphi_0 = T \varphi$ for a (smooth) function $\varphi$.

For a real-valued function $\varphi$, the **subgradient** $\nabla_b \varphi$ is defined as the unique vector field $\nabla_b \varphi \in \xi$ such that $\langle Z, \nabla_b \varphi \rangle = d\varphi(Z)$ for all vector fields $Z$ tangent to the contact distribution $\xi$. Locally $\nabla_b \varphi = \varphi^\alpha Z_\alpha + \varphi^\alpha Z_\alpha$. Define the **sublaplacian** $\Delta_b$ by

$$\Delta_b \varphi = \varphi^\alpha + \varphi^\alpha.$$  


Finally, we also need commutation relations, sometimes called Ricci identities. We list a few of them. These cover 3-dimensional CR-manifolds, but we also need higher-dimensional versions which we do not list here. For the idea of the derivation, see [L2] Equation 2.15].

\[
\begin{align*}
C_{I,01} - C_{I,10} &= C_{I,\bar{T}}A_{11} - kC_{I,A_{11}}T, \\
C_{I,0T} - C_{I,10T} &= C_{I,\bar{T}}A_{11} + kC_{I,\bar{T}}A_{11}, \\
C_{I,1\bar{T}} - C_{I,11\bar{T}} &= iC_{I,0} + kWC_{I}.
\end{align*}
\]

Here \(C_I\) denotes a coefficient of a tensor with multi-index \(I\) consisting of only 1 and \(\bar{T}\), and \(k\) is the number of 1’s minus the number of \(\bar{T}\)’s in \(I\).

Next we mention some curvature identities. The ones that are relevant for us are the **contracted CR Bianchi identities** from [L2] Lemma 2.2.

\[
\begin{align*}
R_{\rho\sigma,\gamma} - R_{\gamma\sigma,\rho} &= iA_{\alpha\gamma,\rho} - iA_{\alpha\rho,\gamma}, \\
W_{\gamma} - R_{\gamma\bar{\sigma}} &= i(n - 1)A_{\alpha\gamma,\rho}, \\
R_{\rho\sigma,0} &= A_{\alpha\rho,\alpha} + A_{\beta\bar{\sigma},\beta}, \\
W_{0} &= A_{\alpha\rho,\alpha} + A_{\beta\bar{\sigma},\beta}.
\end{align*}
\]

2.2. **Volume-normalized contact forms and pseudo-Einstein CR-manifolds.** To give our results some geometric meaning, we recall some results concerning volume-normalized contact forms and related notions. First recall the following definitions, taken from [L1].

**Definition 2.8.** Let \((M, \theta, J)\) be a closed pseudohermitian 3-manifold. Using Chern-Weil theory, we define the **real first Chern class** \(c_1(T_{1,0}(M))\) as

\[
c_1(T_{1,0}(M)) = \frac{i}{2\pi} [d\omega]^{\alpha}
\]

where \(\omega^{\alpha\beta}\) is the connection form of the Tanaka-Webster connection in some local frame.

Let \((M^{2n+1}, \theta, J)\) be a (strictly pseudoconvex) CR-manifold. The **canonical bundle** of \((M^{2n+1}, \theta, J)\) is the complex line bundle

\[
\Lambda^{n+1,0}(M) = \{ \eta \in \Omega^{n+1} \otimes \mathbb{C} \mid i_V \eta = 0 \text{ for all } V \in T^{0,1}M \}.
\]

A contact form \(\theta\) in a pseudohermitian 3-manifold \(M\) is said to be **volume-normalized** with respect to a nowhere vanishing section \(\zeta\) of the canonical bundle \(\Lambda^{n+1,0}(M)\) if

\[
\theta \wedge d\theta^n = i^n n! \theta \wedge (T^1 \zeta) \wedge (T^1 \bar{\zeta})
\]

where \(T\) is the Reeb vector field with respect to \(\theta\).

We say that a CR-manifold \((M, J, \xi = \ker \theta_0)\) has a **transverse symmetry** if there exists a contact form \(\theta\) in the same conformal class as \(\theta_0\) such that \(J\) is invariant under the flow of the Reeb vector field \(T_\theta\) of \(\theta\),

\[
\mathcal{L}_{T_\theta}J = 0.
\]

Equivalently \(A_{J,\theta} = 0\).

### 2.2.1. Notions specific to dimension 3.

For the remainder of this section we specialize to 3-dimensional CR-manifolds.

**Proposition 2.9.** (i) ([L1]) Let \((M, \theta, J)\) be a closed pseudohermitian 3-manifold. A contact form \(\theta\) is volume-normalized with respect to a non-vanishing closed section \(\zeta\) in a closed pseudohermitian 3-manifold \((M, J, \theta)\) if and only if

\[
d\omega = -id(W\theta).
\]

That is,

\[
W_1 - iA_{1,\bar{T}} = 0.
\]

(ii) ([L1]) If \(M\) is a closed CR embedding of a hypersurface in \(\mathbb{C}^2\), then \(\theta = e^{2t} \theta_0\) is volume-normalized with respect to a closed section \(\zeta = j^*(dz^1 \wedge dz^2)\) for \(j : (M, \theta_0, J) \hookrightarrow \mathbb{C}^2\) and some smooth function \(f(x)\).
Remark 2.10. The condition
\[ W_1 - iA^{1,1}_1 = 0 \]
corresponds to the pseudo-Einstein condition for higher dimensional CR-manifolds. Indeed, suppose that \( n \geq 2 \). Then according to [2] Lemma 4.1, a contact form \( \theta \) on a pseudohermitian manifold \((M^{2n+1}, J, \ker \theta)\) is pseudo-Einstein if for every admissible coframe, we have
\[ d\omega^{\alpha} = -\frac{i}{n}d(W\theta). \]
By writing out this equation, we see that the following holds for a higher-dimensional pseudo-Einstein manifold,
\[ W_{,\alpha} - iA_{\beta\alpha} = 0. \]
However, in dimension 3 any CR-manifold is pseudo-Einstein, since the metric tensor has only one component. For us, a modification of this condition will serve as an integrability condition for the torsion flow.

Define the tensor
\[ R_1 \theta^1 := (W_1 - iA^{1,1}_1)\theta^1. \]
Use this to define the CR Q-curvature, see [11] CCC.
\[ Q := -R_1^{1} = -\frac{1}{2}(\Delta_\theta W - 2\text{Im} A^{1,1}_1). \]
The last equality follows from [11] Lemma 5.4. Clearly, vanishing of \( R_1 \) guarantees vanishing of the CR Q-curvature. For a converse, we already mentioned Proposition 1.2.

3. Short time Existence

In this section we give some results concerning the short-time existence of the torsion flow. As mentioned, the motivation for the integrability condition comes from higher-dimensional CR-manifolds, so we give proofs that also hold for these high-dimensional cases. We thank Jih-Hsin Cheng for valuable contributions in early computations in the 3-dimensional case, in particular involving the variation formulas and linearized operator. See also his paper for a related flow, [C]. Let \( \theta(t) \) be a family of smooth contact forms and \( J(t) \) be a family of CR structures on \((M, J, \theta)\). We consider the following flow on a closed pseudohermitian \((2n+1)\)-manifold \((M, J, \theta) \times [0, T)\):
\[ \begin{cases} 
\partial_t J(t) = 2E, \\
\partial_t \theta(t) = 2\eta(t)\theta(t).
\end{cases} \]
Here \( J = i\theta^a \otimes Z_a - i\theta^\tau \otimes Z_\tau \) and \( E = E^{\alpha}_a \overline{\theta}^a \otimes Z_\overline{\alpha} + E^{\beta}_\overline{\alpha} \theta^\beta \otimes Z_\alpha \).

We start by deriving some evolution equations under the general flow [2] before specifying to the torsion flow, for which \( E = A_J \) (the torsion tensor), and \( \eta = -W \) (the Webster curvature). All computations will be done in a local frame. Fix a unit-length local frame \( \{Z_a\} \) and let \( \{\theta^a\} \) be its dual admissible 1-form. Let \( Z_{\alpha(t)}, \theta^\alpha(t) \) denote a unit-length frame and dual admissible 1-form with respect to \((J(t), \theta(t))\). Since \( \theta^a(Z_{\beta(t)}) \) is a positive real function, we can write \( \hat{Z}_a = F^{\alpha}_a Z_\alpha + G^{\beta}_\alpha Z_\beta \) where \( F^{\alpha}_a \) are real and \( G^{\beta}_\alpha \) are complex. The fact that \( Z_{\alpha(t)} \) is an orthonormal frame means that
\[ \delta_{\alpha\beta} = -i\eta(t)(Z_{\alpha(t)} \wedge Z_{\beta(t)}). \]
By differentiating and substituting the above expression for \( \hat{Z}_a \), we obtain \( F^{\alpha}_a = -\overline{\eta}\delta^\alpha_a \). By differentiating \( J(t)Z_{\alpha(t)} = iZ_{\alpha(t)} \) we find
\[ 0 = \dot{J}Z_\alpha + J\hat{Z}_\alpha - i\hat{Z}_\alpha = 2E^{\alpha}_\alpha \overline{Z}_\alpha - 2iG^{\beta}_\alpha Z_\overline{\beta}, \]
so
\[ \hat{Z}_\alpha = -\eta Z_\alpha - iE^{\beta}_\alpha \overline{Z}_\beta. \]
Now differentiate the identities
\[ d\theta_{(t)} = i\alpha_{\beta} \theta_{(t)}^\alpha \wedge \theta_{(t)}^\beta, \quad \theta_{(t)}^\alpha(Z_{(t)}) = \delta_\beta^\alpha, \quad \text{and} \quad \theta_{(t)}^\alpha(Z_{\pi(t)}) = 0, \]
to deduce that
\begin{equation}
\dot{\theta}^\alpha = 2i\eta^\alpha \theta + \eta \theta^\alpha - iE^\alpha \pi \theta^\pi.
\end{equation}

Now we differentiate \((2.2)\) to obtain
\begin{equation}
d\dot{\theta}^\alpha = \dot{\theta}^\gamma \wedge \omega_{\gamma}^\alpha + \dot{\gamma}^\gamma \wedge \omega_{\gamma}^\alpha + \dot{A}_\alpha \theta \wedge \theta^\alpha + A^\alpha \theta \wedge \theta^\pi + A^\pi \theta \wedge \theta^\pi.
\end{equation}

Since we will derive an identity involving tensors, we will take an adapted frame satisfying \(\omega_{\gamma}^\alpha = 0\) at a point. Plug in \((3.3)\) and consider the \(\theta \wedge \theta^\pi\) terms to obtain
\begin{equation}
\dot{A}_\alpha = -2i(n\eta_\pi + nA_{\pi}) - iE_{\pi,0}.
\end{equation}

On the other hand, contracting \((3.4)\) with \(Z_\beta\) and then contracting with \(h^\pi\), computing modulo \(\theta^\pi\) yields
\(\dot{\omega}_{\alpha} = i(A_\alpha \gamma E_{\pi}^\gamma + A^\gamma \gamma E_{\alpha}^\pi + \eta_\pi \eta^\gamma)\theta - [(n + 2)\eta_\pi + iE_{\pi,0}]\theta^\pi \mod \theta^\pi.\)

Since \(\dot{\omega}_{\alpha}\) is pure imaginary, we have
\begin{equation}
\dot{\omega}_{\alpha} = i(A_\alpha \gamma E_{\pi}^\gamma + A^\gamma \gamma E_{\alpha}^\pi + \Delta_\alpha \eta)\theta + [(n + 2)\eta_\alpha - iE_{\pi \gamma}^\gamma] \theta^\alpha - [(n + 2)\eta_\pi + iE_{\pi,0}]\theta^\pi.
\end{equation}

Differentiate the structure equation \((2.4)\) with respect to \(t\) and consider only the \(\theta^\alpha \wedge \theta^\pi\) terms. This gives
\begin{equation}
\dot{R}_{\rho\sigma} = -(A_\alpha \gamma E_{\pi}^\gamma + A^\gamma \gamma E_{\alpha}^\pi + \Delta_\alpha \eta)h_{\rho\sigma} - 2\eta_{\rho\sigma} - [(n + 2)\eta_\rho - iE_{\pi,\sigma}]\pi - [(n + 2)\eta_\pi + iE_{\pi,0}]\pi.
\end{equation}

After contracting with \(h^\rho\pi\) we get
\begin{equation}
\dot{W} = i(E_{\pi \gamma} \gamma - E_{\pi \pi}^\pi) - n(A_\alpha \gamma E_{\pi}^\gamma + A^\gamma \gamma E_{\alpha}^\pi) - 2(n + 1)\Delta_\alpha \eta + 2W_\eta.
\end{equation}

Recall that the transformation law of the connection under a change of pseudohermitian structure was computed in [11, Sec. 5]. Let \(\tilde{\theta} = e^{i\theta} \theta\) be another pseudohermitian structure. Then we can define an admissible coframe by \(\tilde{\theta}^\alpha = e^{i}(\theta^\alpha + 2i\eta^\alpha \theta)\). With respect to this coframe, the connection 1-form and the pseudohermitian torsion are given by
\begin{equation}
\tilde{\omega}_{\beta} = \omega_\beta + 2(f_\beta \theta^\alpha - f^\alpha \theta_\beta) + \delta_\beta^\alpha(f_\gamma \theta^\gamma - f^\gamma \theta_\gamma) + i(f^\alpha \beta + f_\beta \alpha + 4\delta_\beta^\alpha f_\gamma f^\gamma)\theta,
\end{equation}
and
\begin{equation}
\tilde{A}_{\alpha \beta} = e^{-2f}(A_{\alpha \beta} + 2i\eta_{\alpha \beta} - 4if_{\alpha} f_{\beta}),
\end{equation}
respectively. Thus the Webster curvature transforms as
\begin{equation}
\dot{W} = e^{-2f}(W - 2(n + 1)\Delta f - 4n(n + 1)f_\gamma f^\gamma).
\end{equation}

Here covariant derivatives on the right side are taken with respect to the pseudohermitian structure \(\theta\) and an admissible coframe \(\theta^\alpha\). Note also that the dual frame of \(\{\tilde{\theta}, \tilde{\theta}^\alpha, \tilde{\theta}^\pi\}\) is given by \(\{\tilde{T}, \tilde{Z}_\alpha, \tilde{Z}_\pi\}\), where
\begin{equation}
\tilde{T} = e^{-2f}(T + 2if^\gamma \tilde{Z}_\gamma - 2if_\gamma \tilde{Z}_\gamma), \quad \tilde{Z}_\alpha = e^{-f} Z_\alpha.
\end{equation}

We need one more ingredient before we can prove Theorem 1.4. Let \(f\) be a smooth function, and define the CR pluriharmonic operator \(P_\alpha f\) by
\[ P_\alpha f = f^\alpha \theta^\alpha + inA_{\alpha \gamma} f^\gamma. \]
Define the CR Paneitz real operator by

\[ P_\theta f = (P_\alpha f)^{\alpha}. \]

We have the following lemma.

**Lemma 3.1.** Let \((M^{2n+1}, J, \xi = \ker \theta)\) be a pseudohermitian manifold, and suppose, if \(n = 1\), that Conjecture 1.1 holds for this manifold. Then there is a smooth function \(f\) such that

\[ W_\alpha - inA_{\alpha\beta}, \hat{\beta} - 2(n + 2)P_\alpha f = 0, \tag{3.12} \]

and thus

\[ W_\alpha ^{\alpha} - inA_{\alpha\beta}, \hat{\beta} - 2(n + 2)P_\beta f = 0. \]

**Proof.** By Conjecture 1.1 in case \(n = 1\) and Proposition 1.3 in case \(n \geq 2\) there is a smooth function \(f\) such the rescaled contact form \(\hat{\theta} = e^{2f}\theta\) is volume-normalized. By Proposition 2.9 and Remark 2.10, this volume-normalized contact form satisfies

\[ \hat{\bar{W}}_\alpha - in\hat{A}_{\alpha\beta}, \hat{\beta} = 0. \]

The second contracted Bianchi identity from (2.7) implies then that

\[ 0 = \frac{n+1}{n}(\hat{\bar{W}}_\alpha - in\hat{A}_{\alpha\beta}, \hat{\beta}) = (\bar{R}_{\alpha\beta} - \frac{\nabla h_{\alpha\beta}}{n})^{\beta}. \tag{3.13} \]

Also by [2, page 172], we have

\[ (R_{\alpha\beta} - \frac{\nabla h_{\alpha\beta}}{n}) - 2(n + 2)(f_{\alpha\beta} - \frac{1}{n}f_{\gamma\delta}h_{\alpha\beta}) = \bar{R}_{\alpha\beta} - \frac{\nabla h_{\alpha\beta}}{n} = 0. \tag{3.14} \]

Following the same computation as in the proof of Lemma 5.4 in [1], we find using (3.9), (3.10) and (3.11), that

\[ \hat{\bar{W}}_\alpha = \hat{Z}_\alpha \hat{W} = e^{-f}Z_\alpha e^{-2f}(W - 2(n+1)\Delta f - 2n(\Delta f + n\nabla f)^2). \]

\[ = e^{-f}[W_\alpha - 2Wf_\alpha + 4(n+1)(\Delta f + n\nabla f)^2]f_\alpha - 2(n+1)(f_{\gamma\alpha} + f_{\gamma\beta} - 4n(n+1)(f_{\gamma\alpha}f_\gamma + f_\gamma f_\alpha)], \]

\[ i\hat{A}_{\alpha\beta, \gamma} = i(\hat{Z}_\alpha \hat{A}_{\alpha\beta} - \hat{\omega}_\alpha^{\beta}(\hat{Z}_\gamma) \hat{A}_{\beta\gamma} - \hat{\omega}_\alpha^{\beta}(\hat{Z}_\gamma) \hat{A}_{\alpha\gamma}) \]

\[ = ie^{-f}[(Z_\gamma + 2f_{\gamma\beta})\hat{A}_{\alpha\beta} + 2(\delta_{\alpha\gamma} \hat{A}_{\beta\gamma} + \delta_{\beta\gamma} \hat{A}_{\alpha\gamma})f_{\gamma}] \]

\[ = ie^{-f}[(Z_\gamma + 2f_{\gamma\beta})e^{2f}(A_{\alpha\beta} + 2f_{\alpha\beta} + 2f_{\alpha\beta}f_\beta)] \]

\[ + 2e^{-f}[(\delta_{\beta\gamma} + \delta_{\gamma\beta})f_{\alpha\beta} + (\delta_{\beta\gamma} + \delta_{\gamma\beta})f_{\alpha\gamma}] \]

\[ = e^{-f}[iA_{\alpha\beta, \gamma} - 2f_{\alpha\beta}(\gamma) + 4f_{\alpha\gamma}f_{\beta} + 4f_{\alpha\gamma}f_{\beta}f_\gamma] \]

\[ + 2e^{-f}[(\delta_{\beta\gamma} + \delta_{\gamma\beta})f_{\alpha\beta} + (\delta_{\beta\gamma} + \delta_{\gamma\beta})f_{\alpha\gamma}] \]

Contract the second equation with respect to the Levi metric \(\hat{h}_{\alpha\beta} = h_{\alpha\beta}\). This yields

\[ i\hat{A}_{\alpha\beta, \gamma} = e^{-3f}[iA_{\alpha\beta, \gamma} - 2f_{\alpha\beta}(\gamma) + 4f_{\alpha\gamma}f_{\beta} + 4f_{\alpha\gamma}f_{\beta}f_\gamma) \]

\[ + 2(n+1)(iA_{\alpha\beta} - 2f_{\alpha\beta} + 4f_{\alpha\gamma}f_{\beta})f_{\gamma}. \]

Thus we get

\[ \hat{\bar{W}}_\alpha - in\hat{A}_{\alpha\beta, \gamma} = e^{-3f}[W_\alpha - inA_{\alpha\beta, \gamma} - 2(n+1)(f_{\beta\gamma} + f_{\gamma\beta}) + 2nf_{\alpha\beta} \]

\[ - 2Wf_\alpha - 2n(\Delta f + n\nabla f)^2 + 4n(\Delta f + n\nabla f)^2]f_\alpha - 4n(n+1)f_{\alpha\beta}f_\beta - 4n(n+1)f_{\alpha\gamma}f_{\beta}f_\gamma). \]

We will apply the following commutation relations, which are similar to those in (2.0)

\[ -2(n+1)f_{\beta\gamma} + 2nf_{\alpha\beta} = -2f_{\gamma\beta} + 2nR_{\alpha\beta}f_\alpha - 2niA_{\alpha\beta}f_\gamma, \]

\[ f_{\gamma\beta} - f_{\beta\gamma} = ih_{\alpha\beta}f_\alpha, \]

and

\[ 2n(R_{\alpha\beta} - \frac{W}{n}h_{\alpha\beta})f_\gamma = 4n(n+2)(f_{\alpha\beta}f_\gamma - \frac{1}{n}f_{\alpha\beta}f_\alpha). \]
The latter relation follows from (3.14). Together with the above we obtain the following transformation law

\[ \hat{W}_\alpha - in\hat{A}_{\alpha\beta} = e^{-3f}[W_\alpha - inA_{\alpha\beta} - 2(n + 2)(f_\alpha + nA_{\alpha\beta}f)] \]

Apply (3.13) to complete the proof of (3.12). □

The following lemma is needed to prove Corollary [1,5] We use the observation that CR 3-manifolds with transverse symmetry are Stein fillable.

**Lemma 3.2.** Suppose that \((M^3, J, \xi = \ker \theta_0)\) has a transverse symmetry and \(\theta_0\) is volume-normalized with respect to a non-vanishing closed section \(\zeta\). Assume furthermore that \((J(t), \theta(t))\) is a solution of the torsion flow (1.1) on the interval \([0, T]\). Then for each \(t \in [0, T]\), there exists a volume-normalized contact form \(\hat{\theta}(t)\) for \((\xi, J(t))\).

**Proof.** Since \((M, J(t), \xi = \ker \theta(t))\) is Stein fillable, there exists a contact form \(\hat{\theta}(t)\) with vanishing CR \(Q\)-curvature by (3) of Proposition [1,2] Write

\[ \hat{\theta}(t) = e^{2f}\theta_0 \]

where \(\theta_0\) is volume-normalized. By [H] Lemma 5.4, formula 5.7] the CR \(Q\)-curvature of \(\hat{\theta}(t)\) is given by

\[ \hat{Q} = e^{4f}(Q_0 - 6P_{\theta_0}f) \]

and, because \(Q_0 = 0\) by (2) of Proposition [1,2] (also a result of Hirachi), we find

\[ P_{\theta_0}f = 0. \]

Since \(M\) has a transverse symmetry, it follows from [H] Proposition 7.3 that \(f\) is a CR pluriharmonic function, and \(Pf = 0\). This implies \(\hat{\theta}(t)\) is volume-normalized. □

**Proof of Theorem [1,4]** We follow the arguments of Hamilton, [H] Section 4], and adapt them to our setup. First rewrite the torsion flow (3.1) as

(3.15)\[ \frac{\partial}{\partial t}(J \oplus \theta) = 2(A_{J,\theta} \oplus -W\theta) = 2Q(J, \theta), \]

with \(Q(J, \theta) = A_{J,\theta} \oplus -W\theta\).

**Step 1: determining the linearization of the torsion flow**

We use \(\delta J, \delta \theta\) to denote the variations of \(J\) and \(\theta\), respectively. Set

\[ \delta J = 2E \text{ and } \delta \theta = 2h\theta, \]

where \(E\) is an endomorphism: \(\xi \to \xi\) satisfying \(J \circ E + E \circ J = 0\) and \(h\) is a smooth function. Let \(\delta J\) and \(\delta \theta\) denote the linearization operators with respect to \(J\) and \(\theta\). From (3.5) and (3.8), we see that

(3.16)\[ \delta J A_{\alpha\beta} = iE_{\alpha\beta} \quad \delta \theta A_{\alpha\beta} = 2(ih_{\alpha\beta} - hA_{\alpha\beta}) \]

and

(3.17)\[ \delta J W = i(E_{\alpha\gamma} \gamma_{\alpha} - E_{\gamma\alpha} \gamma_{\alpha} ) - n(A_{\alpha\gamma} E_{\gamma} + A_{\gamma\alpha} E_{\gamma}) \quad \delta \theta W = -2(n + 1)\delta h - 2Wh. \]

We first compute the linearization of \(2A_{J,\theta}\). Let \(O_m\) denote an operator of weight \(\leq m\) (see [CT] page 234]), and define the total variation as \(\delta = \delta J + \delta \theta\). Put \(\bar{E} := E + h\theta\). With (3.16) we compute the variation of the torsion as

(3.18)\[ \delta(2A_{J,\theta}) = (2\delta A_{\alpha} \gamma^{\alpha} \otimes \gamma^{\alpha} + 2A_{\alpha} \gamma^{\alpha}(\delta \theta^{\alpha}) \otimes \gamma^{\alpha}) \]

\[ + 2Re([2iE_{\alpha} \gamma^{\alpha} + 4ih_{\alpha} \gamma^{\alpha}] \theta^{\alpha} \otimes \gamma^{\alpha}) + O_1(\bar{E}). \]
The linearization of \(-2W\theta\) can be computed with (3.17).
\[
\delta(-2W\theta) = -2(\delta W)\theta - 2W(\delta\theta)
\]
(3.19)
\[
= (2i(E_{\pi,\delta} - E_{\gamma},_{\alpha},\gamma,_{\alpha}) + 2n(A_{\alpha,\gamma}E_{\pi} + A_{\gamma,\alpha}E_{\gamma,\pi})
+ 4(n + 1)\Delta_b h)\theta.
\]
With these linearizations (3.18) and (3.19) we write the highest weight operator \(L_{J,\theta}(\tilde{E})\) of the linearization \(\delta(2Q(J,\theta))\) as
\[
L_{J,\theta}(\tilde{E}) = 2\text{Re} \left( 2iE_{\alpha,\gamma} + 4ih_{\alpha,\gamma} \right) \theta \otimes Z_{\pi}
+ [2i(E_{\pi,\delta} - E_{\beta,\alpha},_{\beta,\alpha}) + 4(n + 1)\Delta_b h]\theta.
\]

**Step 2: finding integrability conditions**

Following Hamilton’s ideas from ([11] section 5), we find an integrability condition of the form \(H(J,\theta)Q(J,\theta) = 0\) for the torsion flow (3.15). By Lemma 3.1 we find a function \(f\) satisfying equation (3.12). We then define \(O_\theta(A_{J,\theta} \oplus -W\theta) := 2(n + 2)P_{\alpha}f\). We use this function to get an integrability condition. Define the linear operator \(H(J,\theta)\) by
\[
H(J,\theta)(\tilde{E}) := h_\alpha + inE_{\alpha,\beta},_{\beta} + O_\theta(\tilde{E}).
\]
This operator has degree 1 in \(\tilde{E} := E \oplus h\theta\); the term \(O_\theta\) has lower order. By (3.12) we see that \(Q(J,\theta)\) satisfies the integrability condition
\[
H(J,\theta)Q(J,\theta) = -W_\alpha + inA_{\alpha,\beta},_{\beta} + O_\theta(A_{J,\theta} \oplus -W\theta) = 0.
\]
By taking first variations, we find
\[
H(J,\theta)\delta Q(J,\theta)(\tilde{E}) + (\delta H(J,\theta))Q(J,\theta)(\tilde{E}) = 0.
\]

We continue to argue as in ([11] section 4). The operator in \(\tilde{E}\) given by \(\delta H(J,\theta)\) has degree 1, so its symbol, which we denote by \(\sigma H\), of maximal degree vanishes. The other term, \(H(J,\theta)\delta Q(J,\theta)\), has maximal degree since \(H\) has degree 1 and \(\delta Q\) has positive degree. The symbols of linear differential operators \(H(J,\theta)\) and \(Q(J,\theta)\) therefore satisfy
\[
[\sigma H(J,\theta)](\cdot) \circ [\sigma [\delta Q(J,\theta)](\cdot)](\tilde{E}) = 0,
\]
so we conclude that the image of \(\sigma[\delta Q(J,\theta)]\) must lie in the kernel of \(\sigma H(J,\theta)\). We now want to show that the restriction of the symbol \(\sigma H(J,\theta)](\xi)\) to \(\ker \sigma H\) has only positive eigenvalues when \(\xi \neq 0\). To see this, we first need additional commutation relations, again similar to those in (2.6), namely
\[
E_{\alpha,\gamma},_{\beta} - E_{\alpha,\gamma},_{\beta} = i(n - 1)E_{\alpha,\gamma,\beta} + R_{\alpha,\gamma}E_{\pi,\gamma} - R_{\alpha,\gamma,\beta}E_{\pi,\gamma,\beta},
E_{\alpha,\gamma,\beta} - E_{\alpha,\gamma},_{\beta} = inE_{\alpha,\gamma,\beta} + R_{\gamma,\beta}E_{\alpha,\gamma} + R_{\alpha,\gamma}E_{\gamma,\beta}.
\]
From these relations we derive that on the subspace
\[
\ker H(J,\theta)(\tilde{E}) = \{ \tilde{E} = E \oplus h\theta || h_\alpha + inE_{\alpha,\beta},_{\beta} + O_\theta(\tilde{E}) = 0 \}
\]
the following identities hold true,
\[
\Delta_b h = in(E_{\alpha,\gamma,\beta} - E_{\alpha,\gamma},_{\beta,\alpha}) + O_1(\tilde{E}),
\Delta_b E_{\alpha,\gamma} = (2 - n)E_{\alpha,\gamma,\beta} + \frac{2}{n}ih_{\alpha,\gamma} + O_1(\tilde{E}).
\]
Therefore it follows from (3.20) that the highest weight operator \(L_{J,\theta}(\tilde{E})\) of the linearization \(\delta(2Q(J,\theta))\) is
\[
L_{J,\theta}(\tilde{E}) = 4n \text{Re} [\mathcal{L}_{\alpha} E_{\alpha,\gamma} \theta \otimes Z_{\pi}] \oplus 2(2n + 2 + \frac{1}{n})(\Delta_b h)\theta,
\]
on \(\ker H(J,\theta)(\tilde{E})\). Note here that both the Folland-Stein operator \(\mathcal{L}_{\alpha}\), given for \(n \geq 2\) by
\[
\mathcal{L}_{\alpha} = \Delta_b - i\alpha T, \quad \alpha = \frac{(n - 1)^2}{n} = -(n - 2 + \frac{1}{n}),
\]
and the sublaplacian \(\Delta_b\) are subelliptic, see [FS]. Therefore it follows from (3.22) and (3.25) that all eigenvalues of the eigenspaces of \(\sigma[\delta Q(J,\theta)]\) in \(\ker \sigma H(J,\theta)\) are strictly positive.
Alternatively, note that the proof of Lemma 3.1 can also be used to show that there are no degeneracies for the torsion flow (3.1) other than those implied by the second contracted Bianchi identity from (2.7).

**Step 3: solving the linearized equations and handling the non-linear problem**

To complete the argument, we follow [H1, Section 5 and Section 6]. The linearized equations can be solved using the regularization trick from [H1, Section 6]. Theorem 5.1 of [H1] can then be used to obtain short-time existence of the torsion flow (3.1). The main ingredient is the Nash-Moser inverse function theorem ([H3]).

□

4. Contact 3-manifolds with a global frame and pseudohermitian structures

We now specialize to 3-dimensional pseudohermitian manifolds. Let \((M, \xi = \ker \theta)\) be a cooriented contact 3-manifold. Denote the Reeb field by \(T\). Furthermore, in this section and in the next, Section 5, we shall assume that \(\xi\) admits a global symplectic trivialization, i.e. there are vector fields \(U, V\) such that \(\xi = \text{Span}(U, V)\) and \(d\theta(U, V) = 1\).

**Lemma 4.1.** Let \((M, \xi = \ker \theta)\) be a contact 3-manifold. Then there is a global trivialization \(U, V\) of its contact structure if and only if \(c_1(\xi) = 0\).

**Proof.** The contact structure \(\xi\) admits the structure of a symplectic vector bundle \((\xi, d\theta)\). By choosing a compatible complex structure \(J\), we obtain a complex line bundle \((\xi, J)\). It is well-known that smooth complex line bundles are trivial if and only if their first Chern class vanishes, see [W, Chapter III, Section 4]. □

The last step needs the first Chern class with integer coefficients. Chern-Weil theory will not suffice in general. Henceforth, we shall assume that the globally defined vector fields \(U, V\) form a symplectic basis of \((\xi, d\theta)\). Consider the coframe \(\theta, \alpha, \beta\) dual to \(T, U, V\). Then \(d\theta(U, V) = 1\), so

\[
\theta = \alpha \wedge \beta.
\]

**Lemma 4.2.** Let \(J\) be a compatible complex structure for the symplectic vector bundle \((\xi = \ker \theta, d\theta)\). Then there are smooth functions \(a : M \to \mathbb{R}\) and \(c : M \to \mathbb{R}_{>0}\) such that, with respect to the frame \(U, V\), the complex structure \(J\) is represented by the matrix

\[
J = \begin{pmatrix}
a & \frac{-1 + \alpha^2}{c} \\
c & -a
\end{pmatrix}.
\]

**Proof.** With respect to the global frame \(U, V\), the endomorphism \(J\) is represented by a \(2 \times 2\)-matrix. Writing out the condition \(J^2 = -\text{id}\) shows that the matrix representation for \(J\) has the above form. The compatibility condition means that \(d\theta(\cdot, J\cdot)\) is a metric, so it is represented by a positive definite matrix. Writing out this matrix shows that \(c\) is a positive function. □

The following is motivated by our goal to convert the torsion flow (a PDE for tensors) into a PDE for functions. Choose real-valued functions \(a, b\) and \(c\) where \(b\) and \(c\) are positive. We attach super- and subscripts to indicate the dependence on these functions. In order to keep track of deformations of the contact form, we express all data in the given frame \(T, U, V\). Define

\[
\begin{aligned}
\theta_b &= b^2 \theta, & \alpha_b &= b\alpha - 2V(b)\theta, & \beta_b &= b\beta + 2U(b)\theta, \\
T_b &= \frac{1}{b^2} T + \frac{2V(b)}{b^3} U - \frac{2U(b)}{b^3} V, & U_b &= \frac{1}{b} U, & V_b &= \frac{1}{b} V.
\end{aligned}
\]

**Lemma 4.3.** The vector field \(T_b\) is the Reeb vector field for \(\theta_b\). Furthermore, we have \(d\theta_b = \alpha_b \wedge \beta_b\).

This can be checked by plugging in the vector field into the defining equations (2.1). The second assertion is obtained by writing out the terms.

**Remark 4.4.** If \(b\) is a constant function, then the deformation from (4.2) corresponds to a \(\xi\)-homothetic deformation as defined in [H1, Section 10.4]. We take \(b^2\) in \(\theta_b\) to have fewer expressions with square roots.
Define a complex structure by
\[ J_{abc}(U_b) = a U_b + c V_b, \]
\[ J_{abc}(V_b) = \frac{-1 + a^2}{c} U_b - a V_b. \]

By Lemma 4.2 this is the most general choice.

**Remark 4.5.** In higher dimensions, strictly pseudoconvex CR-manifolds require an integrability condition, see Definition 2.1 which is trivially satisfied in dimension 3.

We now compute the Tanaka-Webster connection as in Section 2.1. We use the coframe \( \theta, \theta^1, \theta^\dagger \), where
\[
\theta_{abc}^1 = \sqrt{2c(a^2 + 1)} \left( \frac{-i}{2(a - i)} \alpha_b + \frac{i}{2c} \beta_b \right),
\]
\[
\theta_{abc}^\dagger = \sqrt{2c(a^2 + 1)} \left( \frac{i}{2(a + i)} \alpha_b - \frac{i}{2c} \beta_b \right).
\]

This satisfies our normalization condition \( d\theta = \alpha_b \wedge \beta_b = i \theta_{abc}^1 \wedge \theta_{abc}^\dagger \). The corresponding eigenvectors of \( J_{abc} \) are
\[
Z_{abc}^1 = \frac{1}{\sqrt{2c(a^2 + 1)}} \left( (a^2 + 1) U_b + c(a - i) V_b \right),
\]
\[
Z_{abc}^\dagger = \frac{1}{\sqrt{2c(a^2 + 1)}} \left( (a^2 + 1) U_b + c(a + i) V_b \right).
\]

### 4.1. Converting the torsion flow into a system of PDE’s for the functions \( a, b, c \). To write down the equations of the torsion flow, we need the work out the torsion tensor. We have
\[
Z_{abc}^1 \otimes \theta_{abc}^1 = \left( \frac{-i(a + i)}{2} \right) U_b \otimes \alpha_b + \left( \frac{i(a^2 + 1)}{2c} \right) U_b \otimes \beta_b + \left( \frac{-i(a + i)c}{2(a - i)} \right) V_b \otimes \alpha_b + \left( \frac{i(a + i)}{2} \right) V_b \otimes \beta_b,
\]
so we find
\[
A_{abc}^{abcd} = A_{abc}^1 Z_{abc}^1 \otimes \theta_{abc}^1 + A_{abc}^\dagger Z_{abc}^\dagger \otimes \theta_{abc}^\dagger = +2 \text{Re}(A_{abc}^1 Z_{abc}^1 \otimes \theta_{abc}^1) + \text{Im}(A_{abc}^1 Z_{abc}^1 \otimes \theta_{abc}^1)
\]
\[
= \left( \text{Re}(A_{abc}^1) + a \text{Im}(A_{abc}^1) \right) U_b \otimes \alpha_b - \frac{\text{Im}(A_{abc}^1)(a^2 + 1)}{c} U_b \otimes \beta_b
\]
\[
- \left( \text{Re}(A_{abc}^\dagger) \left( \frac{-2ac}{a^2 + 1} \right) + \text{Im}(A_{abc}^\dagger) \left( \frac{1 - a^2}{a^2 + 1} \right) \right) V_b \otimes \alpha_b - \left( \text{Re}(A_{abc}^\dagger) + a \text{Im}(A_{abc}^\dagger) \right) V_b \otimes \beta_b.
\]

Hence the first equation of the torsion flow (1.11), \( \dot{J}_{abc} = 2A_{abc}^{abcd} \) is equivalent to the system
\[
\dot{a} = 2 \left( \text{Re}(A_{abc}^1) + a \text{Im}(A_{abc}^1) \right)
\]
\[
\dot{c} = -2 \left( \text{Re}(A_{abc}^1) \left( \frac{-2ac}{a^2 + 1} \right) + \text{Im}(A_{abc}^1) \left( \frac{1 - a^2}{a^2 + 1} \right) \right)
\]
\[
\frac{d}{dt} \left( \frac{(1 + a^2)}{c} \right) = -2 \text{Im}(A_{abc}^1) \frac{a^2 + 1}{c}
\]
\[
\frac{d}{dt} \left( -a \right) = -2 \left( \text{Re}(A_{abc}^1) + a \text{Im}(A_{abc}^1) \right)
\]

Indeed, modulo \( \theta \) we have
\[
U_b \otimes \alpha_b \equiv U \otimes \alpha
\]
\[
U_b \otimes \beta_b \equiv U \otimes \beta
\]
\[
V_b \otimes \alpha_b \equiv V \otimes \alpha
\]
\[
V_b \otimes \beta_b \equiv V \otimes \beta,
\]
so we obtain the above system by looking at the coefficients of \( U \otimes \alpha, U \otimes \beta, V \otimes \alpha, \) and \( V \otimes \beta \). This works since these tensors are time-independent.
Lemma 4.6 (A smaller system for the J-part of the torsion flow). The system given by (4.5) is equivalent to system given by

\[
\dot{a} = 2 \left( \text{Re}(A_{11}^{abc}) + a \text{Im}(A_{11}^{abc}) \right) \\
\dot{c} = -2 \left( \text{Re}(A_{11}^{abc}) \left( \frac{-2ac}{a^2 + 1} \right) + \text{Im}(A_{11}^{abc}) \left( \frac{1 - a^2}{a^2 + 1} \right) \right).
\]

Proof. The first equation of (4.5) implies the fourth. We now verify that the first and second equation of (4.5) imply the third.

\[
\frac{d}{dt} \left( -\frac{1 + a^2}{c} \right) = -\frac{2a\dot{a}}{c} + \frac{1 + a^2}{c^2} \dot{c} \\
= -\frac{4a}{c} \text{Re}(A_{11}^{abc}) - \frac{4a^2}{c} \text{Im}(A_{11}^{abc}) + \frac{1 + a^2}{c^2} \text{Re}(A_{11}^{abc}) \frac{4ac}{a^2 + 1} - \frac{1 + a^2}{c^2} \text{Im}(A_{11}^{abc}) \frac{2c(1 - a^2)}{a^2 + 1} \\
= -2 \text{Im}(A_{11}^{abc}) \left( \frac{2a^2}{c} + \frac{1 - a^2}{c} \right) = -2 \text{Im}(A_{11}^{abc}) \frac{a^2 + 1}{c}.
\]

On the other hand, the second equation of the torsion flow (1.1) reduces to

\[
\frac{d}{dt}(\theta b) = \frac{d}{dt}(b^2)\theta = -2W^{abc}b^2\theta,
\]

so we can reduce the torsion flow to a system of PDE’s for the functions a, b, c, giving us the following proposition.

Proposition 4.7. Let \((M^3, \theta, J)\) be a CR-manifold with \(c_1(\xi, J) = 0\). Then there exists a basis of \(T^*M\), and functions \(a, b, c\) such that

\begin{itemize}
  \item the complex structure \(J\) can be written as \(J_{abc}\).
  \item the torsion flow (1.1) is equivalent to the system
  \[
  \dot{a} = 2 \left( \text{Re}(A_{11}^{abc}) + a \text{Im}(A_{11}^{abc}) \right) \\
  \dot{c} = -2 \left( \text{Re}(A_{11}^{abc}) \left( \frac{-2ac}{a^2 + 1} \right) + \text{Im}(A_{11}^{abc}) \left( \frac{1 - a^2}{a^2 + 1} \right) \right)
  \]
  \end{itemize}

\[
\frac{d}{dt}(b^2) = -2W^{abc}b^2.
\]

Remark 4.8. Spatial derivatives of \(a, b, c\) come in via the definition of torsion and Webster curvature.

5. 3-manifolds with constant structure constants and the Tanaka connection

In this section we consider manifolds \(M^3\) that admit global 1-forms \(\omega^1, \omega^2, \omega^3\) such that

\begin{itemize}
  \item \(\omega^1, \omega^2, \omega^3\) form a basis of \(T^*M\).
  \item The structure coefficients are constant, i.e.
  \[
d\omega^i = \sum_{j<k} c_{j}^{i} \omega^j \wedge \omega^k
  \]
  with \(c_{j}^{i}\) constant.
  \item There is a constant form \(\theta\) of the form
  \[
  \theta = \sum_i c_i \omega^i,
  \]
  where \(c_i\) are constant.
\end{itemize}

We shall call such a contact manifold a homogeneous contact manifold. This terminology is not standard, but it serves a useful purpose in this note. Let us point out that a related, but not equivalent notion, also referred to as homogeneous contact, was used by Perrone. [Pr].
Remark 5.1. The structure coefficients are the structure constants of some 3-dimensional Lie-algebra. Indeed, the dual frame \{X_1, X_2, X_3\} satisfies

\[ [X_j, X_k] = - \sum_i c^i_{jk} X_i, \]

the Lie bracket on vector fields satisfies the Jacobi identity. We shall call this the Lie algebra of a homogeneous contact manifold.

From Lemma 4.1 we get immediately.

Lemma 5.2. Homogeneous contact manifolds have trivial Chern class.

Before we define a CR structure on such manifolds, we use the following lemma to provide a better coframe. In many cases, this lemma can be improved upon, but this version is sufficiently convenient.

Lemma 5.3. Let \((M, \{\omega_i\}, \theta)\) be a homogeneous contact manifold. Then there is a basis \{\tilde{\omega}^i\}, such that

- \(\tilde{\omega}^1\) is contact, and \(c^1_{23} = 1\).
- The structure coefficients \(d\tilde{\omega} = \sum_{j,k} c^i_{jk} \tilde{\omega}^j \wedge \tilde{\omega}^k\) satisfy \(c^1_{12} = c^1_{13} = c^1_{23} = 0\).

Proof. Choose a compatible complex structure \(J\) for \((\xi, d\theta)\). Consider the operator \(h := \frac{1}{2} \mathcal{L}_T J\). From [B] Lemma 6.2 we see that \(h\) is self-adjoint with respect to the metric \(d\theta(\cdot, J\cdot)\), and we also get the identity

\[ 0 = \frac{1}{2} \mathcal{L}_T J^2 = Jh + hJ. \]

Since \(h\) is self-adjoint, we can find a basis of eigenvectors \(X, Y\) of \(h\) for \(\xi\). If the eigenvalue of \(X\) is \(\lambda\), then \(JX\) is an eigenvector with eigenvalue \(-\lambda\),

\[ hJX = -JhX = -\lambda JX, \]

so we can assume that \(Y = JX\). We consider the Levi-Civita connection \(\nabla\) for \(g = \theta \otimes \theta + d\theta(\cdot, J\cdot)\). Take the basis \(e_1 = T, e_2 = X, e_3 = JX\), where \(T\) is the Reeb field of \(\theta\). Then

\[ [e_1, e_2] = [T, X] = \nabla_T X - \nabla_X T = +JX + \lambda JX - \mu JX. \]

for some \(\mu \in \mathbb{R}\). In the last step we have used the identity (see [B] Lemma 6.2)

\[ \nabla_X T = -JX - JhX. \]

The same steps work for \([e_1, e_3]\), so we conclude that there are constants \(C_1, C_2\) such that

\[ [e_1, e_2] = C_1 e_3 \quad [e_1, e_3] = C_2 e_2. \]

Consider the dual basis \(\{\omega^i\}\). Then \(\omega^1\) is a contact form, and since \(T\) is the Reeb field, we have

\[ 0 = i_{e_i} d\omega^1 = c^1_{i2} \omega^2 + c^1_{i3} \omega^3. \]

Hence \(c^1_{12} = c^1_{13} = 0\), and from Lie bracket computations we see that \(c^1_{23} = c^1_{33} = 0\). By rescaling, we see that the claim holds. \(\square\)

We assume now that \(\omega^1, \omega^2, \omega^3\) is a basis that is provided by this lemma. Take the basis \(X_1, X_2, X_3\) that is dual to \(\omega^1, \omega^2, \omega^3\), i.e.

\[ \omega^i(X_j) = \delta^i_j. \]

We have \(i_{X_1} \omega^1 = 1\) and \(i_{X_1} d\omega^1 = \sum_{k} c^1_{k} \omega^k = 0\), so \(X_1\) is the Reeb vector field for \(\omega^1\). Note that \(\xi = \ker \omega^1 = \text{Span}(X_2, X_3)\). Also, if \(c^1_{23} = 1\), then \(d\omega^1 = \omega^2 \wedge \omega^3\), so we have the right normalization convention for the setup of the Tanaka connection described in Formula 4.1.

Choose constants \(a \in \mathbb{R}\) and \(b, c > 0\), and define a complex structure on \(\xi\) (or CR-structure on \(M\)) following Lemma 4.2 by

\[ J_{abc} = aX_2 \otimes \omega^2 + cX_3 \otimes \omega^2 - \frac{1}{c} X_2 \otimes \omega^3 - aX_3 \otimes \omega^3. \]
We call such an endomorphism a **homogeneous complex structure**, and we refer to a homogeneous contact manifold together with the above complex structure as a **homogeneous CR-manifold** or a **homogeneous CR structure**. By direct computation, we obtain the following result for the Tanaka-Webster connection, its torsion and the Webster curvature.

**Proposition 5.4.** Let \((M, \{\omega^i\}, \theta)\) be a homogeneous contact manifold with basis provided by Lemma \(5.3\). Fix \(a \in \mathbb{R}\) and \(b, c > 0\) and define \(J_{abc}\) as in formula \((5.1)\). Then the connection form for the Tanaka connection of the pseudohermitian manifold \((M, \theta_b = b^2 \theta, J_{abc})\) is given by

\[
\omega_1 = \frac{i}{b^2} \left( -\frac{a^2 + 1}{2c} c_{12}^3 + c_{23}^2 \right) \theta_b + \frac{i}{b} \left( c \cdot c_{23}^2 - a \cdot c_{23}^3 \right) \alpha_b + \frac{i}{b} \left( \frac{a^2 + 1}{c} c_{23}^3 - a \cdot c_{23}^2 \right) \beta_b
\]

Its torsion is given by

\[
A_1^{abc} = \frac{1}{b^2} \left( i \frac{a^2 + 1}{2c} c_{12}^3 - i \frac{c(a + i)}{2(a - i)} c_{13}^2 \right),
\]

and its Webster curvature by

\[
W^{abc} = \frac{1}{b^2} \left( \frac{a^2 + 1}{2c} c_{12}^3 - \frac{c}{2} c_{23}^2 - c \cdot (c_{23}^2)^2 + 2a \cdot c_{23}^2 c_{23}^3 - \frac{a^2 + 1}{c} (c_{23}^2)^2 \right).
\]

**Proof.** See the computations in the appendix. Alternatively, these computations are essentially also contained in [Pr]. Note that Perrone uses a \(J\)-basis, that is \(e_1 = T, e_2, e_3 = J e_2\). \(\square\)

We can now reduce the torsion flow for homogeneous CR-manifolds to an ODE by plugging in the results of Proposition 5.4 into Proposition 5.7. The general system is fairly complicated, so we will work out some interesting case in Section 5.2.

We shall also consider the **normalized torsion flow** which, in general, is given by the system

\[
\begin{align*}
\frac{\partial J}{\partial t} &= 2A_{J(t), \theta(t)}, \\
\frac{\partial \theta}{\partial t} &= -2(W - \bar{W}) \theta(t),
\end{align*}
\]

where \(\bar{W} = \int_M W \theta \wedge d\theta / \int_M \theta \wedge d\theta\). Since the Webster curvature is constant in space for a homogeneous CR-manifold, then second equation of the normalized is flow is trivial. Inserting the result of Proposition 5.4 into the explicit system provided by Proposition 4.7 gives the following.

**Proposition 5.5** (Normalized torsion flow for homogeneous CR-manifolds). Let \((M, \{\omega^i\}, \theta)\) be a homogeneous contact manifold with \(\theta = \omega^1\). Set \(b = 1\), and let \(a, c, t\) be real valued functions. Then for a complex structure \(J_{abc}\) as defined in \((5.1)\), the normalized torsion flow satisfies the ODE

\[
\begin{align*}
\dot{a}_t &= c_{12}^3 c_{13}^2 - c_{13}^2 a_t^3 + \frac{a_t}{c_t} a_t \\
\dot{c}_t &= c_{12}^3 c_{23}^2 + c_{23}^2 (1 - a_t^2).
\end{align*}
\]

**5.1. Homogeneous CR-manifolds and vanishing torsion.** Observe that homogeneous CR-manifolds need not be compact. In Section 5.2 we give some compact examples, but we start out by characterizing homogeneous CR-manifolds with vanishing torsion directly in terms of the structure coefficients and coefficients for the CR-structure \(a, b, c\).

**Proposition 5.6.** Let \((M, \{\omega^i\}, \theta)\) be a homogeneous contact manifold with \(\theta = \omega^1\) and basis provided by Lemma \(5.3\). For \(a \in \mathbb{R}, b = 1\) and \(c > 0\), a complex structure \(J_{abc}\) as defined in \((5.1)\) has vanishing torsion precisely when one of the following conditions is satisfied.

1. if \(a = 0\), \(c_{12}^2 = 0\), and \(c = \sqrt{-\frac{c_{13}^2}{c_{12}^3}} > 0\) (not imaginary).
2. if \(a \neq 0\), \(c_{13}^2 = c_{12}^3 = 0\).

**Proposition 5.7** (Convergence of the torsion flow). Let \((M, \{\omega^i\}, \theta)\) be a homogeneous contact manifold with \(\theta = \omega^1\) and a basis provided by Lemma \(5.3\). Take \(a \in \mathbb{R}, b = 1\) and \(c > 0\), and let \(J_{abc}\) be a complex structure as defined in \((5.1)\).

1. If \(c_{12}^2 > 0\) and \(c_{13}^2 < 0\), then the normalized torsion flow converges to the unique torsion free complex structure of Proposition 5.6.
If \( c_{12}^3 < 0 \) and \( c_{13}^2 > 0 \), then the normalized torsion flow blows up in finite time and the complex structure does not converge unless \( a_0 = 0 \) and \( c_0 = \sqrt{-c_{12}^3 / c_{13}} \). In the latter case, the torsion vanishes and the torsion flow is constant.

Proof. In both cases, the ODE describing the normalized torsion flow has a unique fixed point \((0, \sqrt{-c_{12}^3 / c_{13}})\) in the upper half-plane with coordinates \((a, c)\).

The phase diagram for the repelling (R) case is given in Figure 1. The phase diagram for the attracting (A) case is similar, but the arrows are reversed. It is clear that the attracting case converges to \((0, \sqrt{-c_{12}^3 / c_{13}})\).

From the second equation we then deduce that the function \( c \) converges to \( c_\infty = \sqrt{-c_{12}^3 / c_{13}} \).

To see that solutions in the repelling case blow up unless \((a_0, c_0) = (0, \sqrt{-c_{12}^3 / c_{13}})\), we make two observations.

- any initial condition \((a_0, c_0)\) that starts in the set given by \( |c_{13}^2|a_0^2 + |c_{12}^3|a_0^2 - |c_{12}^3| > 0 \) blows up in finite time. Indeed, the \( c \)-coordinate is strictly increasing in that case, so there is \( t_0 \) such that \( c_{t_0} > 1 \) or the solution blows up before \( t_0 \). If \( c_{t_0} > 1 \) then the solution blows up in finite by the following argument. We have

  \[
  \dot{c} = |c_{12}^3|c^2 + |c_{13}^2|(a^2 - 1) \geq |c_{12}^3|c^2 - |c_{13}^2|,
  \]

  which must blow up in finite time.

- an initial condition \((a_0, c_0)\) with

  \[
  |c_{13}^2|a_0^2 + |c_{12}^3|a_0^2 - |c_{12}^3| \leq 0 \text{ and } (a_0, c_0) \neq (0, \sqrt{-c_{12}^3 / c_{13}})
  \]

  with \( c_0 > 0 \) exits the half-disk \( \{|c_{13}^2|a_0^2 + |c_{12}^3|a_0^2 - |c_{12}^3| \leq 0\} \) in some finite time \( t_1 \). Then either \( c_{t_1} = 0 \), meaning that \( a \) blows up in finite time, or we can reduce to the first case.

An alternative proof can be given by starting with a \( J \)-basis. Then \( a_0 = 0 \) and \( c_0 = 1 \). Such a basis is not preserved under the torsion flow, but the condition \( a_t = 0 \) holds. The resulting ODE is simpler to analyze. This method is applied in Section 5.2.

Remark 5.8. By Proposition 5.4 there is a relation between the Webster curvature and the sign of \( c_{12}^3 \). However, it is not true that repelling is equivalent to negative curvature.
5.1.1. Homogeneous CR structures on unimodular Lie groups. We recall the definitions involved.

**Definition 5.9.** A Lie group $G$ is called **unimodular** if the left-invariant Haar measure is also right invariant. A Lie algebra $(\mathfrak{g}, [, .])$ is called **unimodular** if $Tr \text{ad}_X = 0$ for all $X \in \mathfrak{g}$.

Note that the Lie algebra of a unimodular Lie group is unimodular. The unimodularity of a Lie group can be used to simplify the structure coefficients of the Lie algebra.

**Lemma 5.10.** Let $G$ be a unimodular, 3-dimensional Lie group admitting a homogeneous contact structure $\xi = \ker \theta$. Then there is a basis $\{e_i\}_i$ of the Lie algebra $\mathfrak{g}$ such that

- $e_1$ is the Reeb vector field, and $e_2, e_3$ lie in the contact structure $\xi$.
- There are $\lambda, \mu \in \mathbb{R}$ such that $[e_1, e_2] = \lambda e_3$, $[e_1, e_3] = \mu e_2$. Furthermore, $[e_2, e_3] = e_1$.

**Proof.** First apply Lemma 5.3. Note that unimodularity implies that $\sum_j c_{ij}^k = 0$ for all $i$. We conclude that $c_{11}^1 + c_{12}^2 + c_{13}^3 = 0$. By putting $i = 2, 3$ we deduce that $c_{22}^3 = c_{33}^2 = 0$, so $[e_2, e_3] = -c_{23}^1 e_1 = -e_1$.

Here is a table with all possible unimodular Lie groups admitting a homogeneous contact structure,

| $c_{11}^2$ | $c_{12}^3$ | Geometry | $\exists J$ with $A_{J, \theta} = 0$ |
|------------|------------|----------|---------------------------------|
| +          | +          | $SU(2) = S^3$ | yes |
| -          | -          | $SL(2, \mathbb{R})$ | yes |
| -          | +          | $SL(2, \mathbb{R})$ | no |
| 0          | +          | $E(2)$ | no |
| 0          | 0          | Heisenberg | yes |

**Remark 5.11.** We point out that the topology or geometry of the underlying CR-manifold does not uniquely determine the underlying contact structure. In particular, for some compact quotients of $SL(2, \mathbb{R})$ the above contact structures are not isomorphic. Explicit examples are given in Sections 5.2.1 and 5.2.2.

**Theorem 5.12** (Convergence to torsion free CR structure). Let $(M, \{\omega^i\}_i, \theta = \omega^1)$ be a homogeneous contact manifold whose Lie algebra is isomorphic to $su(2)$. Then there is a unique homogeneous complex structure $J_{a, b = 1, c = \infty}$ that is torsion free. Moreover, for any choice of homogeneous complex structure $J_{a, b = 1, c = 1}$, the normalized torsion flow converges to this unique CR-structure $(\ker \theta, J_{a, b = 1, c = \infty})$.

In particular, for any choice of homogeneous complex structure on $SU(2)$, the normalized torsion flow converges to the standard CR-structure.

**Example 5.13** (Rossi’s examples). We recall Rossi’s examples of non-embeddable CR-manifolds. Define the strictly plurisubharmonic function

$$f : \mathbb{C}^2 \rightarrow \mathbb{R}$$

$$z \mapsto \frac{1}{2} \|z\|^2. $$

Let $S^3 := f^{-1}(\frac{1}{2})$, and put $\omega^1 = -df \circ i$, $\omega^2 = -df \circ j$, $\omega^3 = -df \circ k$, where $i, j, k$ are the standard quaternions. With $\theta = \omega^1$, this gives $S^3$ the structure of a homogeneous contact manifold. Its structure constants are $c_{13}^2 = 1, c_{23}^1 = -1$ and $c_{12}^3 = 1$. Recall that the standard CR structure on $S^3$ is then given by $(\theta, J = i)$. Put $\theta^1 := \frac{1}{\sqrt{2}} (\omega^2 + i \omega^3)$. Then $\{\theta, \theta^1, \theta^1\}$ is an admissible frame. Following [CCY] we define the CR structure via the deformed coframe

$$\theta_1^1 = \frac{1}{\sqrt{1 - t^2}} \left( \theta^1 - t \theta^3 \right).$$
We specialize to the case that a curvature with the formulas from Proposition 5.4:

\[ \theta^i_t = \frac{1}{2} \left( \sqrt{1-t} \alpha^i + i \sqrt{1+t} \beta^i \right). \]

Comparing this with Equation (4.3) shows that the examples of Rossi are homogeneous CR structures with

\[ J_{abc} = J_{a=0,b=1/c} = \left( \begin{array}{ccc} 0 & -\frac{1+t}{1-t} & 0 \\ \frac{1+t}{1-t} & 0 & 0 \end{array} \right). \]

For \( t > 0 \), these CR-manifolds are not embeddable.

On the other hand, Theorem 5.12 applies to Rossi’s examples, so we have.

**Corollary 5.14.** Under the normalized torsion flow, Rossi’s examples flow to the standard CR structure on \( S^3 \), which is embeddable.

### 5.2. Examples of compact homogeneous CR-manifolds: different CR structures on \( ST^\ast \Sigma \).

We describe the torsion flow on several geometries, namely \( SU(2) \), \( E(2) \), \( SL(2,\mathbb{R}) \) and Heisenberg geometry.

As an explicit, compact model covering the first three cases we consider a compact orientable surface Riemann surface \( (\Sigma, g) \). According to a standard theorem in Riemannian geometry, the unit cotangent bundle \( ST^\ast \Sigma \) admits a canonical coframe \( \omega^1, \omega^2, \omega^3 \) (see for instance [BCS], Chapter 4.4 for the more general Finsler case with a different ordering of the coframe) satisfying

\[
\begin{align*}
\omega^1 &= -\omega^2 \wedge \omega^3 \\
\omega^2 &= -\omega^3 \wedge \omega^1 \\
\omega^3 &= -K \omega^1 \wedge \omega^2,
\end{align*}
\]

where \( K \) is the Gauss curvature of \( (\Sigma, g) \). Assume that \( g \) is a metric of constant Gauss curvature. Then these manifolds provide models of homogeneous contact manifolds.

#### 5.2.1. Homogeneous contact structure associated with the canonical contact structure “pdq”.

We consider the standard contact structure (“pdq”) on the unit cotangent bundle of \( (\Sigma, g) \). With respect to the canonical coframe (4.4), the defining form for this contact structure is \( \omega^3 \).

Consider time-dependent functions \( a_t, b_t, c_t \) that are constant in space, and define the coframe

\[
\theta^i = b_t \omega^i, \quad \alpha^i = b_t \omega^i, \quad \beta^i = b_t \omega^i.
\]

With this ordering, we obtain the structure coefficients \( c_{13}^1 = 1 \), \( c_{32}^1 = -K \), and \( c_{13}^2 = 1 \) (take \( b_0 = 1 \)), and all other coefficients vanish. With the standard choice of complex structure \( J_{abc} \), we obtain a pseudohermitian manifold \( (ST^\ast \Sigma, \theta_t, J_{a_t,b_t,c_t}) \). We compute the torsion and Webster curvature with the formulas from Proposition 5.4

\[
A^{11}_{a_t b_t c_t} = \frac{i}{b_t} \left( \frac{a_t^2 + 1}{2 c_t} + \frac{c_t a_t}{2 a_t^2} - i K \right) = -\frac{a_t c_t}{2} - K + \left( \frac{a_t^2 + 1}{2 c_t} + \frac{c_t K}{2} \right) \frac{a_t^2 - 1}{2 a_t^2 + 1},
\]

\[
W_{a_t b_t c_t} = \frac{1}{b_t} \left( \frac{a_t^2 + 1}{2 c_t} + \frac{c_t K}{2} \right).
\]

We specialize to the case that \( a = 0 \) and substitute \( B(t) = b(t)^2 \). By Proposition 4.7 the (unnormalized) torsion flow reduces to ODE

\[
\dot{c} = -\left( \frac{2ac}{a^2 + 1} \right) K b_t^2 + \frac{1}{b_t^2} \left( \frac{a^2 + 1}{c} + c \frac{a^2 - 1}{a^2 + 1} K \right) (1 - a^2)c = \frac{1 - c^2 K}{B}
\]

\[
\dot{B} = -c K - \frac{1}{c}
\]

\[
c(0) = c_0
\]

\[
B(0) = (b_0)^2.
\]
If $Kc_0^2 \neq 1$, then the solution to this system is given by

$$c(t) = c_0 e^{\frac{(1-Kc_0^2)t}{c_0}}$$

$$B(t) = \frac{Kc_0^2 \left( e^{\frac{(1-Kc_0^2)t}{c_0}} \right)^2 - 1}{(Kc_0^2 - 1)e^{\frac{(1-Kc_0^2)t}{c_0}} - (b_0)^2}.$$ 

If $Kc_0^2 = 1$, which can only happen if $K > 0$, then

$$c(t) = c_0$$

$$B(t) = b_0^2 - \frac{t}{c_0} \left( c_0^2 \cdot K + 1 \right).$$ 

We draw some conclusions:

- For $K \leq 0$, the solution exists for all time. For $K = 0$ (torus case), one has the curious property that the Webster curvature is constant. The torsion is also constant in that case, when measured in our coframe $\theta_b, \alpha_b, \beta_b$. For all $K \leq 0$, the torsion flow skews the complex structure more and more. The limit

$$\lim_{t \to \infty} c(t) = \infty,$$

so in the limit, the complex structure blows up.

- For $K > 0$, the solution blows up in finite time because of shrinking: $b(t) = \sqrt{B(t)}$ reaches 0 in finite time. The special case $Kc_0^2 = 1$ corresponds to vanishing torsion.

**Remark 5.15.** We point out that, with its canonical contact structure “$pdq$”, only the unit cotangent bundle of $S^2$ admits a complex structure for which the torsion vanishes. Indeed, all other unit cotangent bundles of surfaces with constant Gauss curvature are not K-contact, which is a necessary requirement by the appendix of Weinstein in [CH].

In this specialized case $a = 0$, the volume-normalized flow is particularly simple. We have

$$\dot{c}(t) = 1 - Kc_0^2.$$

We see the following

1. if $K > 0$, then there exists a torsion free complex structure, namely for $c_\infty = 1/\sqrt{K}$.

2. if $K = 0$ (the torus case), then $c$ increases linearly. The flow exists for all time, and that it converges to this torsion free complex structure.

3. if $K < 0$, then $c$ blows up in finite time. Geometrically, we see by (5.5) that torsion grows in norm, and the Webster curvature becomes more and more negative. Accordingly, the complex structure blows up.

Note that $ST^*S^2 \cong SO(3)$, so alternatively we can apply Proposition 5.7 to the case $K > 0$.

5.2.2. Prequantization structures on $ST^*\Sigma$. We consider again the canonical coframe on the unit cotangent bundle with structure coefficients as in (5.4) for a surface with constant Gauss curvature. If $\Sigma$ is not a torus, then we define the following coframe

$$\theta_b = b^2 \omega^2, \quad \alpha_b = -bK \omega^3, \quad \beta_b = b \omega^3.$$ 

The resulting contact manifold is known as a prequantization bundle, a circle bundle over a symplectic manifold (here $\Sigma$) whose fibers are periodic Reeb orbits. The corresponding structure coefficients are now $c_{23}^1 = 1, c_{31}^2 = -K, c_{12}^3 = \frac{1}{K}$, and all other coefficients vanish. By defining
In this section, we consider a pseudohermitian manifold \((ST^*\Sigma, \theta_b, J_{abc})\). Its torsion and Webster curvature are given by

\[
A_{11}^{abc} = \frac{i}{b^2} \left( \frac{a^2 + 1}{2c} \frac{1}{K} + \frac{c(a+i)}{2(a-i)}K \right),
\]

\[
W^{abc} = \frac{1}{b^2} \left( \frac{a^2 + 1}{2c} \frac{1}{K} + \frac{c}{2}K \right).
\]

**Remark 5.16.** If \(\Sigma \neq S^2\), then the resulting contact structure is not contactomorphic to the “pqd”-structure from the previous section. Also, the contact structure is now K-contact, and we can choose a complex structure with vanishing torsion. Indeed, choose \(a = 0\), and \(c = \frac{1}{|K|}\), and the torsion tensor will vanish.

As in the previous section we specialize to the case that \(a = 0\) and substitute \(B(t) = b(t)^2\). By Proposition 4.7 the (unnormalized) torsion flow reduces to ODE

\[
\dot{c} = -Kc^2 + \frac{1}{K}
\]

\[
\dot{B} = -\left( \frac{1}{Kc} + cK \right)
\]

\(c(0) = c_0\)

\(B(0) = (b_0)^2\)

The solution is given by

\[
c(t) = c_0 e^{1-Kc_0^2 t}\]

\[
B(t) = B_0 \frac{1 - Kc_0^2 e^{1-Kc_0^2 t}}{(1 - Kc_0^2 e^{1-Kc_0^2 t}) e^{Kc_0^2 t}}
\]

If we start the flow at \(a_0 = 0\) and \(c_0 = \frac{1}{|K|}\), then we have vanishing torsion, and the torsion flow just contracts or expands depending on the sign of the Webster curvature. We have

\[
B(t) = B_0 - \frac{2|K|}{K} t.
\]

The normalized torsion for these homogeneous contact manifolds are covered by Proposition 5.7.

**5.2.3. Heisenberg geometry.** As an explicit, compact example with Heisenberg geometry, consider the 2-torus with standard symplectic form \((T^2, \Omega = d\phi_1 \wedge d\phi_2)\). There is a principal circle bundle \(p : P \to T^2\) whose connection form \(\theta\) satisfies \(d\theta = p^*\Omega\). We see that \((P, \theta, \alpha = d\phi_1, \beta = d\phi_2)\) is a homogeneous contact manifold of Heisenberg type. Indeed, all structure coefficients except \(c_{123}\) vanish.

Hence any homogeneous CR-structure has vanishing torsion and Webster curvature. It follows that the torsion flow is constant, so this is an explicit example of a torsion soliton, namely a steady breather, see Corollary 1.9 case (i).

**6. Entropy functionals**

The following section discusses entropy functionals on a closed 3-dimensional pseudohermitian manifold \((M, J, \theta)\).

**6.1. The Entropy \(F\)-Functional.** Let \((M, J, \theta)\) be a closed pseudohermitian 3-manifold. In this section, we study the monotonicity property of the \(F\)-functional

\[
F(J(t), \theta(t), \varphi(t)) = \int_M (W + |\nabla_b \varphi|^2_{J(t)}) e^{-\varphi} d\mu
\]

with the constraint

\[
\int_M e^{-\varphi} d\mu = 1
\]

under the coupled torsion flow 1.4.
Proof. We compute $\frac{\partial}{\partial t} |\nabla_b \varphi|_{J,\theta}^2$ with Equation (3.2) and use the result to obtain

\begin{equation}
\frac{\partial}{\partial t} |\nabla_b \varphi|_{J,\theta}^2 = 4 \Re (iE_{TT} \varphi_1) + 2(\nabla_b \varphi_1, \nabla_b \varphi_1)_{J,\theta} - 2\eta(t) |\nabla_b \varphi|_{J,\theta}^2.
\end{equation}

By (1.4) we find

\begin{equation}
\frac{\partial}{\partial t} d\mu = 4 \eta(t) d\mu.
\end{equation}

Use these formulas together with Equation (3.8) to compute the variation of the $F$-functional,

\begin{equation}
-\frac{1}{2} \frac{d}{dt} F(J(t), \theta(t), \varphi(t)) = -\int_M \eta(t) [W + |\nabla_b \varphi|_{J,\theta}^2] e^{-\varphi} d\mu + 2 \int_M (\Delta_b \eta(t)) e^{-\varphi} d\mu
+ \frac{1}{2} \int_M (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W) \varphi e^{-\varphi} d\mu
- 2 \int_M \Re (iE_{TT} \varphi_1) e^{-\varphi} d\mu
= \int_M \left( \frac{7}{2} \varphi_t - \eta(t) \right) (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W) e^{-\varphi} d\mu
+ \int_M \Re [(A_{11} - i\varphi_1) E_{TT}] e^{-\varphi} d\mu
= \int_M \eta(t) (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W) e^{-\varphi} d\mu
+ \int_M \Re [(A_{11} - i\varphi_1) E_{TT}] e^{-\varphi} d\mu.
\end{equation}

We first set $E_{11} = e^{\varphi} (A_{11} - i\varphi_1 - i\varphi_1 \varphi_1)$ and $\eta(t) = e^{\varphi} (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W)$, then

\begin{equation}
-\frac{1}{2} \frac{d}{dt} F(J(t), \theta(t), \varphi(t)) = \int_M (2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W)^2 d\mu
+ \int_M |A_{11} - i\varphi_1 - i\varphi_1 \varphi_1|^2 d\mu
\geq 0.
\end{equation}

The monotonicity formula is strict unless

\begin{equation}
A_{11} - i\varphi_1 - i\varphi_1 \varphi_1 = 0 \quad \text{and} \quad 2\Delta_b \varphi - |\nabla_b \varphi|_{J,\theta}^2 + W = 0.
\end{equation}

Moreover, up to a contact transformation $\tilde{\theta} = e^{-\varphi} \theta$

\begin{equation}
\tilde{A}_{11} = 0, \quad \tilde{W} = 0.
\end{equation}

This completes the proof of Theorem 1.10.


6.2. The Entropy $W^\pm$-Functionals. We study the monotonicity property of the $W^+$-functional

\begin{equation}
W^+(J(t), \theta(t), \varphi(t), \tau(t)) = \int_M [\tau(W + |\nabla_b \varphi|_{J,\theta}^2 + \frac{1}{2} \varphi - 1) (4\pi \tau)^{-2} e^{-\varphi} d\mu
\end{equation}

with the constraint

\begin{equation}
\int_M (4\pi \tau)^{-2} e^{-\varphi} d\mu = 1
\end{equation}

under the coupled torsion flow (1.6).
Proof. Following the same computations as in the proof of Theorem 1.10, we can derive that

\[
\frac{d}{d\tau} \int_M [\tau(W + |\nabla_b\varphi|^2_{J,\theta}) - 1](4\pi\tau)^{-2}e^{-\varphi}d\mu = \frac{d}{d\tau} \int_M \frac{d}{d\tau}(W + |\nabla_b\varphi|^2_{J,\theta})(4\pi\tau)^{-2}e^{-\varphi}d\mu
\]

\[
- 2\tau \int_M \eta(t)(W + |\nabla_b\varphi|^2_{J,\theta}) + 2\Delta_b\eta(t) - 4(|\nabla_b\varphi, \nabla_b\eta(t)|)(4\pi\tau)^{-2}e^{-\varphi}d\mu + 2\tau \int_M \text{Re}(E_{11,\Pi} - A_{11}E_{\Pi}) + 2iE_{\Pi}\varphi_1(\varphi_1)(4\pi\tau)^{-2}e^{-\varphi}d\mu
\]

\[
+ \int_M [\tau(W + |\nabla_b\varphi|^2_{J,\theta}) - 1](4\eta(t) - 2\tau - 1\frac{d}{d\tau} - \varphi_1)(4\pi\tau)^{-2}e^{-\varphi}d\mu
\]

\[
= 2\int_M (W + |\nabla_b\varphi|^2_{J,\theta})(4\pi\tau)^{-2}e^{-\varphi}d\mu - 2\int_M (\eta(t) - \tau^{-1})(4\pi\tau)^{-2}e^{-\varphi}d\mu
\]

Here we have used the following identities:

\[
\int_M (\Delta_b\eta(t))(4\pi\tau)^{-2}e^{-\varphi}d\mu = \int_M \eta(t)(|\nabla_b\varphi|^2_{J,\theta} - \Delta_b\varphi)(4\pi\tau)^{-2}e^{-\varphi}d\mu
\]

and

\[
\int_M (W + |\nabla_b\varphi|^2_{J,\theta})(4\pi\tau)^{-2}e^{-\varphi}d\mu = \int_M (2\Delta_b\varphi - |\nabla_b\varphi|^2_{J,\theta} + W)(4\pi\tau)^{-2}e^{-\varphi}d\mu.
\]

On the other hand,

\[
\frac{1}{2} \frac{d}{d\tau} \int_M \varphi(4\pi\tau)^{-2}e^{-\varphi}d\mu = 2\int_M (2\Delta_b\varphi - |\nabla_b\varphi|^2_{J,\theta} + W - \tau^{-1})(4\pi\tau)^{-2}e^{-\varphi}d\mu.
\]

It follows that

\[
-\frac{1}{2} \frac{d}{d\tau} W^+(J(t), \theta(t), \varphi(t), \tau(t)) = \tau \int_M (2\Delta_b\varphi - |\nabla_b\varphi|^2_{J,\theta} + W - \tau^{-1})(4\pi\tau)^{-2}e^{-\varphi}d\mu + \tau \int_M |A_{11} - i\varphi_{11} - i\varphi_1(\varphi_1)|^2(4\pi\tau)^{-2}e^{-\varphi}d\mu.
\]

Moreover, the monotonicity formula is strict unless

\[
A_{11} - i\varphi_{11} - i\varphi_1(\varphi_1) = 0 \quad \text{and} \quad 2\Delta_b\varphi - |\nabla_b\varphi|^2_{J,\theta} + W - \tau^{-1} = 0.
\]

This completes the proof of Theorem 1.13 \(\square\)

Next we study the monotonicity property of \(W^\text{-}\)-functional

\[
W^-(J(t), \theta(t), \varphi(t), \tau(t)) = \int_M [\tau(W + |\nabla_b\varphi|^2_{J,\theta}) - \frac{1}{2}\varphi + 1](4\pi\tau)^{-2}e^{-\varphi}d\mu
\]

with the constraint

\[
\int_M (4\pi\tau)^{-2}e^{-\varphi}d\mu = 1
\]

under the coupled torsion flow \(1.7\).
Proof. Following the same computations as in the proof of Theorem 1.13, we can derive that
\[
\frac{d}{dt} \int_M \left[ \tau(W + |\nabla_b \phi|^2) + 1 \right] (4\pi \tau)^{-2} e^{-\phi} d\mu = \int_M \frac{d}{dt} \left[ W + |\nabla_b \phi|^2 \right] (4\pi \tau)^{-2} e^{-\phi} d\mu - 2 \tau \int_M \eta(\tau(W + |\nabla_b \phi|^2) + 2\Delta_b \eta - 4(\nabla_b \phi, \nabla_b \eta)) (4\pi \tau)^{-2} e^{-\phi} d\mu + 2 \tau \int_M \text{Re}(iE_{11,TT} - A_{11} E_{TT} + 2iE_{TT}\varphi_1 \varphi_1) (4\pi \tau)^{-2} e^{-\phi} d\mu + \int_M \frac{d}{dt} \left[ \tau(W + |\nabla_b \phi|^2) - 1 \right] (4\pi \tau)^{-2} e^{-\phi} d\mu - \tau \int_M \theta(\tau(W + |\nabla_b \phi|^2) + 13) (4\pi \tau)^{-2} e^{-\phi} d\mu + \int_M (2\Delta_b \varphi - |\nabla_b \phi|^2 + W + \tau^{-1}) (4\pi \tau)^{-2} d\mu.
\]

On the other hand,
\[
- \frac{1}{2\pi} \frac{d}{dt} \int_M \varphi (4\pi \tau)^{-2} e^{-\phi} d\mu = -2 \int_M (2\Delta_b \varphi - |\nabla_b \phi|^2 + W + \tau^{-1}) (4\pi \tau)^{-2} d\mu.
\]
It follows that
\[
- \frac{1}{2\pi} \frac{d}{dt} W - (J(\tau), \varphi(\tau), \tau(\tau)) = \tau \int_M (2\Delta_b \varphi - |\nabla_b \phi|^2 + W + \tau^{-1}) (4\pi \tau)^{-2} d\mu + \int_M |A_{11} - i\varphi_{11} - i\varphi_1 \varphi_1|^2 (4\pi \tau)^{-2} d\mu.
\]
Moreover, the monotonicity formula is strict unless
\[
A_{11} - i\varphi_{11} - i\varphi_1 \varphi_1 = 0 \quad \text{and} \quad 2\Delta_b \varphi - |\nabla_b \phi|^2 + W + \tau^{-1} = 0.
\]
This completes the proof of Theorem 1.14. \qed

7. APPENDIX

7.1. Computations.

Proof of Proposition 5.4: As in Formula (4.2), put
\[
\theta_b = b^2 \omega_1, \quad \alpha_b = b \omega_2, \quad \beta_b = b \omega_3, \quad T_b = \frac{1}{b^2} X_1, \quad U_b = \frac{1}{b} X_2, \quad V_b = \frac{1}{b} X_3.
\]
We first work out some general formulas. Given the second equation of (2.2), we can assume without loss of generality that
\[
\omega_1 = ic_\theta \omega_1 + ic_2 \theta^1 + i\bar{c}_2 \bar{\theta}^1,
\]
with \(c_\theta\) a real function. Now we insert our frame to get explicit equations
\[
d\theta^1(Z_1, Z_1) = \omega_1^1(Z_1) = i\bar{c}_2
\]
(7.2)
\[
d\theta^1(T, Z_1) = -i c_\theta
\]
\[
d\theta^1(T, Z_1) = A_1.
\]
The Webster curvature is determined by the second structure equation,
\[
d\omega_1 = W \theta^1 \wedge \theta^1 + 2i \text{Im}(A^1_{1,1} \theta^1 \wedge \theta).
\]
Wedges the equation with \(\theta\) gives \(\theta \wedge d\omega_1 = W \theta^1 \wedge \theta^1 \wedge \theta^3\), which can be rewritten by using (7.1) to write out \(d\omega_1\). We find \(\theta \wedge d\omega_1 = (-c_\theta - 2|c_2|^2 - 2 \text{Im}(Z_1(\bar{c}_2))) \theta \wedge \theta^1 \wedge \theta^3\), and conclude
\[
W = -c_\theta - 2|c_2|^2 - 2 \text{Im}(Z_1(\bar{c}_2)).
\]
We now determine these coefficients. For \(c_{\theta^a}^{bc}\), we use the second equation from (7.2) to get
\[
-ie_{\theta^a}^{bc} = -\frac{i(a + i)}{2} d\alpha_b(T_b, U_b) + \frac{a^2 + 1}{2c} \text{Id} \beta_b(T_b, U_b) - \frac{ci}{2} d\alpha_b(T_b, V_b) + \frac{i}{2}(a - i) d\beta_b(T_b, V_b)
\]
\[
= -\frac{i(a + i)}{2b^2} c_1^2 + i \frac{a^2 + 1}{2c b^2} c_2 - \frac{ci}{2b^2} c_3^1 + \frac{i}{2b^2} (a - i) c_3^3.
\]
Use Lemma 5.3 to see that $c_{12}^2 = c_{13}^3 = 0$. For the $c_{2}$-component, we use the first equation from (7.2) we compute

\[
d i c_2 = d \theta^1 (Z_1, Z_1) = \frac{1}{\sqrt{2c(a^2 + 1)}} \left( -\frac{i}{2(a-i)} d \alpha_b(U_b, V_b) \left( (a^2 + 1)c(a+i) - c(a-i)(a^2 + 1) \right) \\
+ \frac{i}{2c} d \beta_b(U_b, V_b) \left( (a^2 + 1)c(a+i) - c(a-i)(a^2 + 1) \right) \right) \\
= \frac{1}{\sqrt{2c(a^2 + 1)}} \left( c(a+i)c_{23}^3 - (a^2 + 1)c_{33}^3 \right).
\]

By the above formulas, these coefficients determine the connection form $\omega_1^1$ and the curvature $W$. For the torsion we use formulas (4.4), (4.3) and (7.2), and find

\[
A_1 = d \theta^1 (T_b, Z_b, i) = \frac{i}{2(a-i)} d \alpha_b(T_b, (a^2 + 1)U_b + c(a+i)V_b) + \frac{i}{2c} d \beta_b(T_b, (a^2 + 1)U_b + c(a+i)V_b) \\
= \frac{i}{2(a-i)b^2} \left( (a^2 + 1)c_{12}^2 + c(a+i)c_{13}^2 \right) + \frac{i}{2cb^2} \left( (a^2 + 1)c_{12}^2 + c(a+i)c_{13}^2 \right).
\]

Combining this with $c_{12}^2 = c_{13}^3 = 0$ gives the desired expression for the torsion. \qed

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