Many-body dynamics in long-range hopping model in the presence of correlated and uncorrelated disorder

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Much have been learned about universal properties of entanglement entropy (EE) and participation ration (PR) for Anderson localization. We find a new sub-extensive scaling with system size of the above measures for algebraic localization as noticed in one-dimensional long-range hopping models in the presence of uncorrelated disorder. While the scaling exponent of EE seems to vary universally with the long distance localization exponent of single particle states (SPSs), PR does not show such universality as it also depends on the short range correlations of SPSs. On the other hand, in presence of correlated disorder, an admixture of two species of SPSs (ergodic delocalized and non-ergodic multifractal or localized) is observed, which leads to extensive (sub-extensive) scaling of EE (PR). Considering typical many-body eigenstates, we obtain above results that are further corroborated with the asymptotic dynamics. Additionally, a finite time secondary slow growth in EE is witnessed only for correlated case while for uncorrelated case there exists only primary growth followed by the saturation. We believe that our findings from typical many-body eigenstate would remain unaltered even in the weakly interacting limit.

Introduction: In one and two dimensions, an arbitrarily weak amount of disorder is sufficient to exponentially localize all eigenstates of a system of non-interacting particles, known as Anderson localization [1–3]. However, correlated disorder in one dimensional system can lead to a coexistence of exponentially localized and delocalized states, separated by mobility edge (ME) [4–6]. Interestingly, in the presence of interactions a transition between delocalized (ergodic) to many-body localized (MBL) phase can be observed [7, 8]. Algebraic localization is another variety of localization that draws significant attention in recent times [9–24]. Entanglement entropy (EE), estimating the bipartite quantum correlations, and participation ratio (PR), quantifying the information about the localization properties of wavefunction, happen to be the primary measures of localized and delocalized phases [25–28].

There is an upsurge of studies with disordered models in presence of long range hopping that decays with distance $l$ as a power-law $1/l^a$ can show algebraic localization [29–32]. Interestingly, Levitov’s conjecture [9–11, 14] about the absence of localization in $d$ dimensional model with $a < d$, is violated in one of such non-interacting long range model where also single particle states (SPSs) are found to be algebraically localized [9, 32, 33]. Recent advancement in experiments with atomic, molecular and optical systems [34, 35], power-law spin interactions with tunable exponent $0 < a < 3$ can be realized in laser-driven cold atom setup [36, 37]. The dipolar ($a = 3$) and van-der-Waals ($a = 6$) couplings have also been experimentally observed for the ground-state of neutral atoms and Rydberg atoms [38–41].

Thanks to the availability of analytical and computational methods, many compelling results have been obtained for EE e.g., it satisfies area (volume) law for exponentially localized (delocalized) phase [25, 26, 42–47]. Another important diagnostic PR is expected to follow the similar behavior [27, 48]. While turning into dynamics, EE for clean (disordered) systems show a faster power law (a slower double log-type) growth with time [49–51]. Having known all of these, we here pose the question that what would be the nature of EE and PR in an algebraically localized phase and do they scale identically? Additionally, the big underlying quest is to predict the Fock space picture for weakly interacting model by performing a many-body analysis (statics and dynamics) on the non-interacting system.

In particular, we study EE and PR for non-interacting power-law hopping model in the presence of uncorrelated disorder (referred as model I), which supports algebraically localized phase, and correlated disorder (referred as model II), that contains ME and multifractal phases. We do find that algebraic localization leads to new sub-extensive scaling of EE and PR with system size for model I, while EE (PR) satisfies extensive (sub-extensive) scaling for model II. Sub-extensive nature of PR is possibly a manifestation of multifractality of many-body wave-functions in Fock space [52, 53]. Additionally, probing the associated exponents, we can convey that EE (PR) can capture the long-range (short-range) correlation of SPSs.

Model: We study noninteracting fermions in 1D lattice in the presence of disordered potential. The system is described by the following long-range power-law hopping Hamiltonian,

$$H = - \sum_{i,j \neq i} \frac{1}{|i-j|^a} (\hat{c}_i \hat{c}_j + H.c.) + \sum_i \epsilon_i \hat{n}_i$$

(1)

where $\hat{c}_i^\dagger$ ($\hat{c}_i$) is the fermionic creation (annihilation) operator at site $i$, $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$ is the number operator, and $L$ is the size of the system.
We consider two cases here. 1) Model I (with uncorrelated disorder): $\epsilon_i$ are chosen randomly from a uniform distribution between $[-W, W]$. In this paper, we choose $W = 20$, for which a very tiny fraction of states are delocalized for $a < 3/2$ and all states are localized for $a > 3/2$ (see Ref. [54] for details). It has been shown that the single-particle wave function $\psi(x)$ of this model displays power-law localization [9] (not exponential) $|\psi(x)|^2 \sim 1/|x-x_0|^{\nu}$ in the limit $x >> x_0$, where, $x_0$ is the localization center and $\nu$ is the localization exponent. $\nu$ shows duality $\nu(a) = \nu(2-a)$ around $a = 1$ for $0 < a < 2$ as investigated numerically [33] and analytically [32]. However, we like to point out that near $x_0$, in the limit $|x-x_0| << L$, the SPs are completely different in both sides of $a$, while one finds an exponential decay of wave-function for $a > 1$, the decay is still algebraic for $a < 1$. $a = 1$ point has been shown to be critically localized [9]. We note that as $a \rightarrow \infty$, SPS becomes completely exponentially localized with $L \rightarrow \infty$. On the other hand, at $a = 0$ the model is exactly solvable and wave-functions in the bulk of the spectrum are critically multifractal [55–58]. We note that the algebraic localization is also observed for Hamiltonian residing in the family of power-law random banded matrix model e.g., model III [10, 11] (see Ref. [54] for details).

2) Model II (with correlated disorder): $\epsilon_i = h \cos(2\pi \sigma_i + \phi)$, where $\sigma = (\sqrt{5} - 1)/2$ and $\phi$ is an offset chosen from a uniform random distribution [0,1] and closely related to the self-dual quasiperiodic model [19, 59]. This seemingly innocent difference has drastic consequences on the physics of this model compared to the previous one. Interestingly, for $a < 1$, all SPs are extended. However, depending on the choice of parameters ($h$ and $a$), there are different phases where ergodic and multifractal (MF) states coexist. We will refer this phase as MF phase. On the other hand, for $a > 1$ there is a coexistence of delocalized and localized SPs, hence, mobility edge (ME) exists (we will refer this phase as ME phase). In either side of $a$, different regimes, denoted by $P_a$, are characterized by a fraction $\sigma^a < 1$ of ergodic SPs at the bottom of the band and the rest are either localized (for $a > 1$) or multifractal (for $a \leq 1$) (see [54] for details) [60].

For all calculations in this paper, we restrict ourselves to half-filling. All quantities are obtained after algebraically averaging over $10^3$ disorder realizations (see Ref. [54] for details). All time evolution calculations are done starting from an initial product state $|\Psi_0\rangle = \prod_{i=1}^{L/2} c_{2i}^\dagger |0\rangle$.

**ENTANGLEMENT ENTROPY (EE)**

In this section we will discuss the eigenstate EE and also the non-equilibrium dynamics of EE after a global quench starting from above product state. We note that a typical measure of the entanglement in a quantum system is bipartite von Neumann entanglement entropy $S$ defined as, $S = -\text{Tr}_A[\rho_A \ln \rho_A]$, where $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$ is the reduced density matrix of a sub-system $A$ after dividing the system into two equal adjacent parts $A$ and $B$, both comprised of $L/2$ sites. $|\Psi\rangle$ is many-body wave function of the composite system.

**Eigenstate EE:** For model I, we notice that the typical eigenstate EE [61–64] (for details, see Ref. [54]) marked by lines in Fig. 1 (a) shows the absence of data collapse in $S/L$. However, the data collapse appears when we replace $S/L$ by $S/L^\gamma(a)$ with $\gamma(a) < 1$ as shown in Fig. 1(c). Interestingly, $\gamma(a)$ exhibits the duality around $a = 1$: $\gamma(a) \simeq \gamma(2-a)$ as shown in Fig. 1(d) (see Ref. [54] for more details). This duality is the consequence of the duality present in the spatial exponent associated with the algebraically decaying long tail of SPs for this model in either side of $a < 1$ and $a > 1$ [65]. Furthermore, the inset of Fig. 1(d) suggests that the exponent $\gamma$ follows a universal behavior with spatial exponent $a$ as long as SPs are algebraically localized. The sub-extensive $L^\gamma$ law can be naively understood from the spatial algebraic structure of SPs. The total probability of finding a particle at any site $x \in B$ while its localization center is at site $x_0 \in A$ becomes $p \sim \sum_{x \in B} |x-x_0|^{-2a} \sim L^{-2a} f(x_0, a, L)$ where $f$ is a non-linear function of $x_0, a, L$. This type of fractional scaling with $L$ is absent for exponentially localized

![FIG. 1. (Color online) Entanglement entropy density $S/L$ vs $a$ for Model I in (a) and Model II with $h = 0.0$ in (b). Here, P3 and P4 represent different multifractal (mobility edge) phases for $a < 1$ ($a > 1$). A perfect data collapse is observed in (b) and (c) for different $L$ suggesting volume law of EE in model II and sub-extensive scaling of EE in model I respectively. The variation of sub-extensive exponent $\gamma(a)$ (with a showing duality around $a = 1$ is depicted in (d). Lines (symbols) correspond to EE obtained from eigenstates (dynamics). Inset in (d) shows that $\gamma$ for model I (depicted by red circle) and model in Ref. [54] (depicted by blue triangles) behave identically (referenced with stretched exponential, green dashed line) with $a$, which is related to localization exponent $\nu \simeq 2a$.](image-url)
SPSs where \( p \sim \xi f(x_0, \xi, L) \) and \( \xi \) being the localization length. Moreover, in the many-body case, EE becomes a complex function of \( x_0 \) as the different SPSs have different localization centres. The absence of the coherent length scale \( \xi \) in algebraic localization can lead to non-trivial sub-extensive behavior in many-body EE.

On the other hand, Fig. 1(b) shows the presence of data collapse in \( S/L \) for model II. There exists a fraction of delocalized ergodic phase yielding \( S \) extensive. Even though, in both MF \((a < 1)\) and ME phase \((a > 1)\), \( S \sim L \) [as shown in Fig. 1(b)], one can distinguish them by their corresponding numerical values of \( S/L \). This is higher in MF phase compared to ME phase because all SPSs are essentially delocalized in MF phase. Moreover, in both sides of \( a = 1 \), transition between different \( P_s \) phases are clearly visible. The volume law of eigenstate EE has also been recently observed for 1D short-range noninteracting model in the presence of correlated disorder, where there exists a mobility edge in single particle spectrum [65, 66].

**Asymptotic EE:** We shall now extensively investigate the scaling of asymptotic saturation value \( S(L, t \to \infty) \equiv S_\infty \) with \( a \) as marked by symbols, starting from an initial product state \( |\Psi_0\rangle \). Figure 1(b) and (a) show the presence of data collapse in \( S_\infty/L \) for model II and absence of it in model I, respectively, for different values of \( L \). Similar to the typical eigenstates, we here in dynamics find sub-extensive (extensive) nature of EE and duality in \( \gamma \) for model I (model II) as shown in Fig. 1(c) and (d) (Fig. 1(b)). However, the proportionality factor associated with \( S/L \) changes from its eigenstate value. Since, these are noninteracting systems, we expect that \( S_\infty \) obtained from dynamics should show similar behavior as typical eigenstates. Note that interaction leads to dephasing mechanism via scattering in the system and hence, for interacting system the above expectation may not hold true. MBL systems are one such examples, where we see that EE of many-body eigenstates obey area law, however, \( S_\infty \sim L^{[20, 44]} \). Since, for both models, we do not have a parameter regime where all states are ergodic (delocalized), hence, \( S_\infty/L \) and also the eigenstates EE density always become less than the Page value \([67]\).

The EE associated with the mid-spectrum eigenstates of a generic interacting non-integrable systems obeying ETH, \([46, 68, 69]\) satisfy this bound.

**Finite time rise:** Having studied finite size scaling of asymptotic EE \( S_\infty \), now we analyze the finite time growth of EE. The results are shown for model I in Fig. 2(a), and (b). We observe for \( a \leq 1 \), a power law rise occurs, \( S \sim t^\alpha \). This growth exponent \( \alpha \) becomes larger near the point \( a = 1 \) (see Ref. [54] for details). For \( a > 1 \), growth exponent \( \alpha \) decreases and for \( a \sim 2 \), EE shows a logarithmic rise. Note that in the case of Hamiltonian (1) even without disorder, initial growth of \( S(t) \) is sub-linear in \( t \) [54, 70]. Since, SPSs behave differently in either side of \( a \): the presence of rapid fall of single particle wave function near the localization center causes relatively slow rise of EE for \( 1 < a < 2 \) compared to \( a < 1 \) regime [29]. We note that our power law growth of EE resembles with the out of time ordered correlator showing a deviation from light-cone like behavior in the context of long range models [22].

Turning into model II as shown in Fig. 2(c), and (d), one can see a fast power law rise \( (S \sim t^{\alpha_1}) \) with \( \alpha_1 < 1 \) followed by a much slower rise in EE. We observe the value of \( \alpha_1 \) is larger for \( a > 1 \) in comparison to \( a < 1 \) case. We also note that in the MF phase, the growth exponent depends on \( P_s \) phases, where as, in the ME phase, \( \alpha \) remains almost same in different \( P_s \) phases (see Ref. [54] for details). This is presumably the consequence of the fact that the spatial structure of the multifractal SPS are different in different \( P_s \) phases. Contrastingly, for \( a > 1 \) even in different \( P_s \) ME phases, the spatial structure of localized states wave function are different from \( a < 1 \). For \( a > 1 \), the long time slow growth, visible in a reasonably large time window, is found to be logarithmic. On the other hand, for \( a < 1 \), we see a similar secondary slow rise of \( S \), since the time window is much smaller we can not comment on it whether this is a power-law with exponent \( \alpha_2 < \alpha_1 \) or logarithmic. One can naively connect our results with non-interacting central site model [71], where multifractality, appeared due to the coupling of a single central bound state with all Anderson localized states, can give rise to a slow logarithmic growth in entanglement dynamics. Previously, logarithmic growth of \( S \) was thought to be a unique feature of MBL systems[44, 72]. Interestingly, our results indicate that the presence of two different types of SPS, there exists a secondary slow rise in the finite time evolu-
tion of $S$ for model II; this growth is completely missing for the uncorrelated disordered model I. Based on our analysis of finite time EE in Model I and II, we can convey that the sub-linear temporal growth of $S$ is related to the detail (structure and fraction of delocalized states) of SPSs in these systems: 1) This is an apparent evident form the behavior of $S$ for $a < 3/2$ in Model I, and 2) for model II, $\alpha_1$ remains unaltered with $a$ as long as one stays inside a fixed $P_s$ phase. However, the value of $\alpha$ interestingly changes as one varies $a$; similarly, $\alpha_1$ changes as one goes from MF side to ME side even within the same $P_s$ phase. A finite fraction of delocalized SPSs can also cause the two-stage growth of EE in model II. On the other hand, this fraction becomes vanishingly small for model I originating the single stage growth (see Ref. [54] for detailed analysis).

**PARTICIPATION RATIO (PR)**

Having examined the scaling of eigenstate EE and $S_\infty$ with $L$, in the similar spirit, we now look for the scaling of eigenstate PR and the saturation values of PR (designated by $PR_\infty$). We shall use the definition of multi-body PR as introduced in Ref. [46]. It is defined for half-filling case as, $PR = L^{1/2} \sum_{\alpha=1}^L n_\alpha \sum_{j=1}^L \left| \phi_\alpha(j) \right|^2$, where, $\left| \phi_\alpha \right>$ are eigenstates and eigenvalues of on-body density matrix $\rho_{ij} = \langle \phi_i^\dagger \phi_j \rangle$ respectively. PR $\sim L$ for delocalized ergodic systems and $PR \sim \xi$ for exponentially localized many-body states [27, 48].

We study the characteristics of $PR_\infty$ and eigenstate PR in Fig. 3. From the data collapse of $PR/L^\beta$ as a function of $a$ with different system sizes (see Fig. 3 (a) and (b)), we show for both the models that PR exhibits a sub-extensive scaling with $L$ (see Ref. [54] for detail). In order to analyze the exponent $\beta$ more concretely, we show the variation of $\beta$ with $a$ for model I and II in Fig. 3 (c) and (d), respectively. $\beta$ remains fixed at a higher value for $a < 1$ while it decreases monotonically for $a > 1$ in model I. Very surprisingly, unlike the EE, PR does not exhibit any duality with $a$ around $a = 1$. The reason being PR is a local quantity, it is not able to capture the duality of SPS in the long-distance scale where power law tail is observed in either side of $a = 1$. Precisely, PR accounts for the short distance behavior of SPSs where exponential and algebraic decay are present for $a > 1$ and $a < 1$, respectively, hence, $\beta$ is completely asymmetric around $a = 1$.

On the other hand, for model II, $\beta$ shows a kind of symmetric behavior around $a = 1$ (see Fig. 3). This result may be counter intuitive in the sense that phases in both sides of $a = 1$ are completely different i.e., MF phase for $a < 1$ and ME phase for $a > 1$. Multifractal SPSs in this model have a form of multiple sharp peak on the top of almost flat background in contrast to the ergodic delocalized SPSs that are extended all over the lattice [54]. The structure of SPSs for MF phase is kind of similar to the exponentially localized SPSs having only one peak and the background is suppressed exponentially with distance from that peak. Since, PR is an inappropriate measure to identify the fine tuned long distance structure of SPSs for ME and MF phases, we find similar behavior of $\beta$ in either side of $a$. However, we note that $\beta$ is much closer to 1 in MF phase compared to ME phase. Moreover, from the variation of $\beta$ with $a$, we can roughly identify different $P_s$ phases in either side of $a = 1$ [see Fig. 3(d)]. Moreover, we note that in the calculation with typical eigenstates, we discard a few bottom spectrum delocalized states to minimize their effect (see Ref. [54] for details). On the other hand, all the energy states come automatically into the non-eignestate dynamics. We believe that this is the origin of the apparent dissimilarities between the predictions from typical eigenstate and long time dynamics as observed in Fig. 1 and Fig. 3.

**Conclusion:** We summarize our main results in Table 1. One of the most intriguing finding is to show the

**TABLE I. Summary of the main differences between different phases in the non-interacting long range systems.**

| SPS Type                  | $S_\infty$ | $PR_\infty$ |
|---------------------------|------------|-------------|
| Exponential localization  | $L^\theta$ | $L^\theta$  |
| Algebraic localization    | $L^\gamma$, $\gamma < 1$ | $L^\beta$, $\beta < 1$ |
| Ergodic (delocalized)     | $L$        | $L$         |
| Multi-fractal (non-ergodic)| $L$       | $L^\theta$, $0 < \beta < 1$ |
| Mobility-edge             | $L$        | $L^\theta$, $0 < \beta < 1$ |
sub-extensive scaling in EE and PR, when SPSs are algebraically localized as observed in model I. This is firmly evident from both the eigenstate and long time dynamics. The absence of length scale thus imprints its’ signature unlike the exponentially localized phase. Moreover, these behaviors are not the artifact of delocalized states present in Model I (at least for $a < 3/2$) as the number of such states has measure zero for $L \rightarrow \infty$ [33, 56, 73]. Turning to model II, asymptotic and eigenstate EE both obey volume law due to the presence of ergodic SPSs; however, interestingly, the proportionality factors change in different $P, \gamma$ phases. The adiabatic connectivity allows us to conjecture that algebraically localized quasi-local integrals of motion would survive even in the weakly interacting limit [23, 26, 57, 74] and hence, the eigenstate EE scaling should remain unaltered even in the above limit. One might not expect the similar scaling of asymptotic EE, obtained from the long time dynamics, due to the dephasing mechanism caused by the interaction.

Our study further reveals the connection between the exponents ($\gamma$ for EE and $\beta$ for PR), and the spatial structure of the SPSs. The EE is maximally governed by the long distance nature of the SPSs and thus the duality in $\gamma$ is closely connected to the duality of the localization exponent $\nu$ as noticed for model I [33]. Moreover, $\gamma$ follows a universal behavior as far as the algebraically localized SPSs are concerned. In contrary, PR captures the short distance nature of correlation leading to the fact that exponent $\beta$ does not show duality around $a = 1$. The short distance behavior of SPSs are very different for $a < 1$ and $a > 1$ for model I. Surprisingly, model II shows duality like behavior within a small window around $a = 1$. This can be related to the peculiar spatial distribution of multifractal SPSs at short distance (see Ref. [54] for detail).

Another important contribution of our work is to show how the structure of SPSs can influence the finite time rise of EE. An unprecedented two-stage growth of EE for model II is exclusively observed while model I exhibits single-stage growth. The secondary rise in EE for model II might be related to the fact that there exist finite fraction of two types of SPSs i.e., multifractal and delocalized or localized and delocalized. The initial algebraic temporal growth is common in both the models. Recent studies also find signatures of temporal power law growth of EE in long range interacting models [23, 75]. The connection between the temporal power-law growth of EE and algebraical SPSs (LIOM) for non-interacting (interacting) model is still an open field of research. Given the experimental realizability of spin models [38–41], we believe that our study would initiate a plethora of work in this direction.

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