DYNAMICAL ANALYSIS OF ANISOTROPIC INFLATION

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The inflaton coupling to a vector field via the \( f(\varphi)^2 F_{\mu\nu} F^{\mu\nu} \) term is used in several contexts in the literature, such as to generate primordial magnetic fields, to produce statistically anisotropic curvature perturbation, to support anisotropic inflation, and to circumvent the \( \eta \)-problem. In this work I perform dynamical analysis of this system allowing for the most general Bianchi I initial conditions. I also confirm the stability of attractor fixed points along phase-space directions that had not been investigated before.

1. Introduction

A vector field with a varying kinetic function of the form \( f(\varphi)^2 F_{\mu\nu} F^{\mu\nu} \), where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \), is often being used in the literature on cosmological inflation for several different purposes. This type of kinetic function breaks the conformal invariance of the vector field without introducing instabilities.\(^1\)

By identifying \( A_\mu \) with electromagnetic potential such a term for the first time was studied as a possible source of primordial magnetic fields\(^2\) and is still being used for that purpose (see e.g. Ref. 3). More generally, a vector field can affect cosmological perturbations or large scale dynamics of the universe during inflation. In Refs. 4–6 a massive vector field with \( f^2 F^2 \) kinetic term plays a role of the curvaton field. In Refs. 7 and 8 it was demonstrated that the contribution of the vector field to the primordial curvature perturbation can generate a new observable signature: statistical anisotropy. Moreover, it can affect large scale dynamics by supporting anisotropic inflation.\(^9\) Finally, if \( \varphi \) is an inflaton, this type of kinetic term can help to overcome the \( \eta \)-problem of inflation.\(^10,11\)

Generally vector field contribution to the curvature perturbation introduces quadrupolar angular dependence of the power spectrum. In Ref. 12 (see also 13–15) it was shown that observational constraints on such modulation put very stringent bounds on vector field models. This bound, however, was applied for a particular
model of Ref. 16. As it is shown in later sections, a much larger parameter space might be available for model building.

In this work I perform a classical dynamical analysis of the system with a scalar and a vector field which interacts via a $f^2(\varphi) F_{\mu\nu} F^{\mu\nu}$ coupling. A similar analysis has been performed in Ref. 17 and much more generally in Ref. 10. It has been also applied in many other contexts, for example a system of non-Abelian vector fields, a vector field plus a two-form field or DBI type action. Building on the former two works, especially Ref. 10, I extend their treatment by including more general initial conditions and by addressing several issues which have been overlooked in previous works. I also provide exact analytic expressions to determine the asymptotic behaviour of this dynamical system.

A vector field introduces anisotropic stress. If it contributes to the expansion of the universe, the expansion will be anisotropic. As the (homogeneous, space-like) vector field picks out only one preferred direction in space, we can assume that the anisotropy of the expansion will be of the axisymmetric Bianchi type I. Moreover, the preferred axis of such anisotropy will be aligned with the direction of the vector field. These two constraints were imposed in previous works on a single vector field. Even if such a configuration is self consistent, initial conditions can be much more general. In this work I relax both the constraints and confirm that the axial symmetry as well as the alignment are achieved dynamically if the vector field does not decay.

In cases where the vector field is absent a number of no-hair theorems have been proven, which state that the future attractor of any spatially homogeneous but anisotropic FRW universe if the energy density is dominated by the potential energy of a scalar field., See Ref. 23 for a recent review and Ref. 24 for the most recent work on no-hair theorems. The presence of a vector field introduces anisotropic stress into the energy-momentum tensor. For some parameter values, as we will see later in this letter, the anisotropic stress does not redshift away with the expansion of the universe, therefore violating some of the assumptions underlying no-hair theorems. In particular, due to the non-vanishing homogeneous vector field, the isotropic FRW universe is no longer an attractor but rather an axisymmetric Bianchi I universe is the attractor. This is indeed demonstrated to be the case starting from Bianchi type II and III as well as Kantowsk-Sachs metrics in Ref. 25. However, these authors assumed from the onset an alignment of the vector field with one of the preferred directions of the metric. More generally, the authors of Ref. 26 derived an expression which relates the spatial anisotropy to a general form of anisotropic stress, assuming an inflating universe. In this letter, I show by a concrete example how a generic Bianchi I space-time dynamically aligns itself with a direction of the vector field.

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\textsuperscript{a} An analysis of Bianchi IX type model with possible inflationary solutions is discussed in Ref. 21.

\textsuperscript{b} More generally, this statement is valid for any perfect fluid with an equation of state $-1 \leq w < -1/3$, where $P = w\rho$. 

and two perpendicular axes of the Bianchi I metric isotropise. For constant model parameters this attractor is reached from the most general initial conditions.

Even if one does restrict initial conditions to the axial symmetry and the orientation of that axis along the homogeneous vector field, effectively excluding some dynamical degrees of freedom, it still needs to be recognised that to ascertain the stability of a particular configuration, the stability of those degrees of freedom must be demonstrated. This is particularly relevant when dealing with inflation models where any fixed point is constantly being perturbed by quantum stochastic noise terms. Moreover, those degrees of freedom are more important in the present setup because anisotropic expansion couples modes of different spin. This is in contrast to FRW metric, where scalar, vector and tensor modes evolve independently. To account for this enlarged dynamical phase space six equations are added to the ones considered in previous works on anisotropic inflation with $f(\phi)^2 F_{\mu \nu} F^{\mu \nu}$ term.

2. Dynamical System Analysis

2.1. The action and the background metric

Consider a classical dynamical system described by the following action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} m_{Pl}^2 R - \frac{1}{2} \partial{}^\mu \varphi \partial{}^\mu \varphi - V(\varphi) - \frac{1}{4} f^2(\varphi) F_{\mu \nu} F^{\mu \nu} \right\}, \tag{1}$$

where the first term is the Einstein-Hilbert term, $\varphi$ is the scalar field with a potential $V(\varphi)$, $F_{\mu \nu}$ is the vector field strength tensor, and the kinetic function $f(\varphi)$ is modulated by the scalar field. The presence of the vector field provides anisotropic stress, which likely makes the spacetime metric anisotropic. The simplest anisotropic metric consistent with the symmetries of the energy-momentum tensor in this setup is the Bianchi type I. In some of the computations below I will use part of the formalism of Ref. 27 and their notation. In that work the Bianchi I metric is written

$$ds^2 = -dt^2 + 3 \sum_{i=1}^3 S_i^2(t) (dx_i)^2, \tag{2}$$

where $S_i(t)$ are scale factors along three spatial directions. To write the above expression, the orientation of the coordinate system is fixed to make the metric diagonal.

We can factor out the average scale factor from the spatial part of the metric

$$a(t) = \left[ S_1(t) S_2(t) S_3(t) \right]^{1/3}. \tag{3}$$

In that case the anisotropic part of that metric can be written as

$$\gamma_{ij} (t) = e^{2\beta_i(t)} \delta_{ij} \tag{4}$$

where $\sum_i^3 \beta_i(t) = 0$ and no summation implied. By rescaling the time coordinate as $d\tau = dt/a$ where $\tau$ is the so called conformal time, the metric in Eq. (2) becomes

$$ds^2 = a^2(\tau) \left[ -d\tau^2 + \gamma_{ij}(\tau) dx^i dx^j \right]. \tag{5}$$
The volume expansion rate and shear tensor can then be given as

\[ \mathcal{H} \equiv a' \quad \text{and} \quad \sigma_{ij} \equiv \frac{1}{2} \gamma'_{ij}, \]

where primes denotes differentiation with respect to \( \tau \).

It is also convenient to define a volume expansion rate with respect to the cosmic time \( t \), which is given by \( H \equiv \dot{a}/a \), where \( a = a(t) \) is defined in Eq. (3) and the dot represents a time derivative with respect to \( t \). One can easily check that \( \mathcal{H} = aH \).

### 2.2. Einstein equations

To write the Einstein equations, it is customary to decompose the energy-momentum tensor with respect to the average velocity vector \( u^\mu \), which is a time-like four vector field normalised \( u_\mu u^\mu = -1 \). Since we are interested in background quantities of non-tilted fluids\(^c\), the energy flux relative to \( u^\mu \) can be neglected and the general energy-momentum tensor can be written as

\[ T_{\mu\nu} = \rho u_\mu u_\nu + P (g_{\mu\nu} + u_\mu u_\nu) + \pi_{\mu\nu}, \]

where \( \rho \) is the total energy density, \( P \) is the total pressure, and \( \pi_{\mu\nu} \) is anisotropic stress satisfying \( \pi_{\mu\nu} u^\mu = 0 \) and \( \pi^\mu_{\mu} = 0 \).

The scalar and vector field energy-momentum tensors can be obtained by varying the action in Eq. (1) with respect to the metric \( g_{\mu\nu} \)

\[ T^{(\varphi)}_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \delta_\mu^\nu \left( \frac{1}{2} \partial_\rho \varphi \partial^\rho \varphi + V \right), \]

\[ T^{(A)}_{\mu\nu} = -f^2 \left( \frac{1}{4} \delta_\mu^\nu F_{\rho\sigma} F^{\rho\sigma} - F_{\mu\rho} F^{\nu\rho} \right). \]

Comparing these two expressions with Eq. (7), we easily find the energy density, pressure, and anisotropic stress for each component

\[ \rho_\varphi = \frac{1}{2} a^{-2} (\varphi')^2 + V, \quad P_\varphi = \frac{1}{2} a^{-2} (\varphi')^2 - V, \]

\[ \rho_A = \frac{1}{2} f^2 \gamma_{ij} \frac{A_i^A A_j^A}{a^4}, \quad P_A = \frac{1}{3} \rho_A, \quad \pi^A_{ij} = -f^2 \left( \delta^A_{ij} \delta_{jk} - \frac{1}{3} \delta^A_{ij} \delta_{jk} \right) \frac{A_i^A A_k^A}{a^4}. \]

The general expressions of the Einstein’s field equations with anisotropic energy-momentum tensor can be found in the Appendix A of Ref. 27. Plugging Eqs. (10)\(^c\)The flow lines of tilted fluids are not orthogonal to the surfaces of homogeneity. See, e.g., Refs. 28 and 29.
and (11) into those expressions, we get

\[ H^2 = \frac{1}{3m^2_{\text{Pl}}} \left[ \frac{1}{2} \dot{\varphi}^2 + a^2 V + \frac{1}{2} f^2 \frac{\gamma^{ij} A'_i A'_j}{a^2} \right] + \frac{1}{6} \sigma^2, \]  

(12)

\[ H' = -\frac{1}{3m^2_{\text{Pl}}} \left[ \dot{\varphi}^2 - a^2 V + \frac{1}{2} f^2 \frac{\gamma^{ij} A'_i A'_j}{a^2} \right] - \frac{1}{3} \sigma^2, \]  

(13)

\[ (\sigma^a_j)' = -2H \sigma^a_j + m^2_{\text{Pl}} \left( \frac{1}{3} \delta^a_j \gamma^{kl} - \delta^a_l \gamma^{jk} \right) f^2 \frac{A'_k A'_l}{a^2}. \]  

(14)

2.3. Decomposition of the shear tensor

\( \sigma_{ij} \) is a symmetric, trace free tensor. In the coordinate system in Eq. (2) \( \sigma_{ij} \) has only two components: the off-diagonal components of \( \sigma_{ij} \) vanish and the remaining two are given by the time derivative of functions \( \beta_i(\tau) \) in Eq. (4). In an arbitrary coordinate system, however, three additional parameters are needed to specify three angles of \( \sigma_{ij} \) with respect to that coordinate system. Thus in total there are five independent components.

In our case, the only source of the shear tensor \( \sigma_{ij} \) is anisotropic stress generated by the presence of the vector field. Thus, to study the evolution of \( \sigma_{ij} \) it is convenient to choose a reference frame related to the vector field \( A_i \). For that purpose we introduce a unit vector \( \hat{z}_i \) such that

\[ \hat{z}_i = \frac{A'_i}{A'} \quad \text{and} \quad A' = \sqrt{\gamma^{ij} A'_i A'_j} \]  

(15)

where \( A' \) is the magnitude of the \( A'_i \) vector. The shear tensor \( \sigma_{ij} \) can be decomposed into components relative to the orthogonal basis \( (\hat{z}, e^1, e^2) \), where

\[ \gamma^{ij} \hat{z}_i e^a_j = 0 \quad \text{for} \quad a = 1, 2. \]  

(16)

Adapting the expression in Eq. (3.1) of Ref. 27, we can write

\[ \sigma_{ij} = \frac{3}{2} \left( \hat{z}_i \hat{z}_j - \frac{1}{3} \gamma_{ij} \right) \sigma_{||} + 2 \sigma_a \hat{z}_i e^a_j + \sigma_\lambda \varepsilon^\lambda_{ij}. \]  

(17)

In this expression summation over the repeated indices \( a = 1, 2 \) and \( \lambda = +, \times \) is implied and parentheses in the indices denote symmetrization \( a_i b_j \equiv (a_i b_j + a_j b_i) / 2 \). The polarization tensor \( \varepsilon^\lambda_{ij} \) is defined as

\[ \varepsilon^\lambda_{ij} = \frac{1}{\sqrt{2}} \left( e^1_i e^1_j - e^2_i e^2_j \right) \delta^\lambda_+ + \frac{1}{\sqrt{2}} \left( e^1_i e^2_j + e^2_i e^1_j \right) \delta^\lambda_\times. \]  

(18)

I will use the notation where \( \sigma_a \) with a Latin letter index from the beginning of the alphabet denotes the two vector components of \( \sigma_{ij} \), while Greek letter index, as in \( \sigma_\lambda \), denotes the two tensor components.

As noted in the Introduction, none of the preferred directions of \( \sigma_{ij} \) are constrained to be aligned with the direction of \( \hat{z}_i \) initially. Neither \( \sigma_{ij} \) is assumed to be axisymmetric. Such an alignment and symmetry correspond to \( \sigma_{||} \neq 0 \) and
\[ \sigma_a = \sigma_\lambda = 0 \text{ for all } a \text{ and } \lambda. \] Instead, we find this configurations to be achieved dynamically in cases where the vector field does not vanish.

To derive equations of motion for \( \sigma_\parallel, \sigma_a, \) and \( \sigma_\lambda, \) we need to compute time derivatives of basis vectors \( e^a \) first. This can be done after fixing the remaining freedom in the definition of \( e^a. \) So far these vectors were defined up to a rotation around \( \hat{z}. \) Their definition is completed by imposing \( \mathcal{R}_{[ab]} = 0, \) where \( \mathcal{R} \) is defined as \( (e^a_i)' = \sum_b \mathcal{R}_{ab} e^b_i \) and \( [a, b] \equiv (ab - ba) / 2. \) Since the \( e^a \) are orthonormal by definition, they satisfy \( (e^a_i e^b_i)' = 0, \) which allows us to write \( (e^a_i)' = (2 \sigma_{ij} - \sigma_{ij} e^b_i e^b_j) e^i_a. \) Using this equation and plugging Eq. (17) into (14), we obtain equations of motion for each component of \( \sigma_{ij} \)

\[ \begin{align*}
\sigma_\parallel' &+ 2 \mathcal{H} \sigma_\parallel = -2 \sigma_\parallel^2 - \frac{2}{3} m_{\text{Pl}}^{-1} f^2 A'^2/a^2, \\
\sigma_a' &+ 2 \mathcal{H} \sigma_a = \frac{1}{2} \sigma_a \sigma_\parallel - \sigma^b \mathcal{M}_{ab}^\lambda, \\
\sigma_\lambda' &+ 2 \mathcal{H} \sigma_\lambda = 2 \sigma^a \sigma^b \mathcal{M}_{ab}^\lambda,
\end{align*} \]

where \( \sigma_\parallel^2 \equiv \sigma^a \sigma_a \) and \( \mathcal{M}_{ab}^\lambda \equiv \mathcal{E}_{\lambda ij} e^a_i e^b_j. \) To derive these equations, \( \dot{z} A_i' = A_i' \) and \( e^i_a A_i' = 0 \) were also used.

### 2.4. Scalar and vector field equations of motion

To close the system of equations we must specify equations of motion for the fields \( \varphi \) and \( A_i. \) The former can be obtained by varying the action in Eq. (1) with respect to \( \varphi, \) which gives

\[ \varphi'' + 2 \mathcal{H} \varphi' + a^2 V. \varphi = \frac{1}{2} a^{-2} f^2 \varphi A'^2, \]

where \( V. \varphi \) denotes the derivative with respect to a scalar field \( V. \varphi \equiv dV/d\varphi \) and similarly for \( f^2 \varphi \equiv d (f^2) /d\varphi. \) The equation for the vector field can be obtained by varying the same action with respect to \( A_i'. \) In this case we find

\[ A_i'' + 2 \left( \frac{f'}{f} - \sigma_\parallel \right) A_i' = 2 \sigma_a e^a_i A_i'. \]

### 2.5. Expansion-normalised variables

Equations (19)-(23) together with the energy constraint in Eq. (12) form a closed system of coupled dynamical equations. Equation (13) will be useful to determine which solutions correspond to an accelerated expansion, i.e. inflationary solutions. However, instead of analysing these equations in the form given above, it is more convenient to rewrite them in terms of expansion-normalised variables.\(^{29} \) For this purpose we define

\[ \begin{align*}
\Sigma_\parallel &\equiv \frac{\sigma_\parallel}{\mathcal{H}}, & \Sigma_a &\equiv \frac{\sigma_a}{\mathcal{H}}, & \Sigma_\lambda &\equiv \frac{\sigma_\lambda}{\mathcal{H}}, \\
x &\equiv \frac{\varphi'}{\sqrt{6} m_{\text{Pl}} \mathcal{H}}, & y &\equiv \frac{a \sqrt{V}}{\sqrt{6} m_{\text{Pl}} \mathcal{H}}, & z &\equiv \frac{f A'/a}{\sqrt{6} m_{\text{Pl}} \mathcal{H}},
\end{align*} \]

where \( \Sigma_\parallel \) is the scale factor, \( \Sigma_a \) is the anisotropic factor, \( \Sigma_\lambda \) is the directional factor, \( x \) is the scalar field, \( y \) is the inflaton field, and \( z \) is the vector field.
Hence each of the solutions $X \equiv (\Sigma_\parallel, \Sigma_a, \Sigma_\lambda, x, y, z)$ corresponds to a family of solutions in physical space, given by $(H, X)$. In addition to making expressions more compact, this normalisation decouples the Raychaudhury Eq. (13) from the rest of the equations and leads to a bounded dynamical system. Consequently, the closed system of dynamical equations become

$$
\epsilon_H = 3 \left( x^2 + \Sigma^2 + 2z^2 \right) \\
1 = \Sigma^2 + x^2 + y^2 + z^2 \\
\frac{d\Sigma_\parallel}{dN} = \Sigma_\parallel \left( \epsilon_H - 3 \right) - 2\Sigma_\parallel^2 - 4z^2 \\
\frac{d\Sigma_b}{dN} = \Sigma_b \left( \epsilon_H - 3 \right) + \frac{1}{2} \Sigma_b \Sigma_\parallel - \Sigma^a \Sigma_\lambda \mathcal{M}^a_{ab} \\
\frac{d\Sigma_\lambda}{dN} = \Sigma_\lambda \left( \epsilon_H - 3 \right) + 2\Sigma^a \Sigma^b \mathcal{M}^a_{ab} \\
\frac{dx}{dN} = x \left( \epsilon_H - 3 \right) - \lambda y^2 + \Gamma z^2 \\
\frac{dz}{dN} = z \left( \epsilon_H - 3 \right) + \lambda \Sigma_\parallel + 1 - \Gamma x
$$

where the slow-roll parameter is $\epsilon_H \equiv -\dot{H}/H^2$. In these equations the time variable is replaced by the number of e-folds of expansion $N \equiv \ln \left( \frac{a}{a_*} \right)$, where $a_*$ denotes an initial scale factor. $\Sigma_\parallel^2$ and $\Sigma^2$ in the above equations are defined as

$$
\Sigma_\parallel^2 \equiv \Sigma_a \Sigma^a \quad \text{and} \quad \Sigma^2 \equiv \frac{1}{6} \left( \frac{3}{2} \Sigma_\parallel^2 + 2\Sigma_\lambda \Sigma_\parallel + \Sigma^2 \right).
$$

$\lambda$ and $\Gamma$ are the free parameters of the model. Their values determine the asymptotic behaviour of the system. These parameters are given by

$$
\lambda \equiv \sqrt{\frac{3}{2} m_{Pl} V_{,\varphi} \frac{V_{,\varphi}}{V}} \quad \text{and} \quad \Gamma \equiv \sqrt{6 m_{Pl} f_{,\varphi} \frac{f_{,\varphi}}{f}}.
$$

Note that $\lambda$ is related to the standard slow-roll parameter $\epsilon_V$ by

$$
\epsilon_V \equiv \frac{m_{Pl}^2}{2} \left( \frac{V_{,\varphi}}{V} \right)^2 = \frac{\lambda^2}{3}.
$$

2.6. Fixed points

It is impossible to solve non-linear coupled Eqs. (28)–(32) in full generality exactly. However, it is enough to determine only the asymptotic behaviour of the solutions. This can be done by performing the dynamical analysis, that is, by finding fixed points and checking their stability. If a given fixed point is found to be a unique attractor in the sense that no other attractor or periodic orbit exist simultaneously, the system will asymptotically evolve to that fixed point starting from any initial conditions.

Fixed points (or fixed points) correspond to the state of the system which does not evolve in time. Hence, we want to find those points in the compact phase space
Table 1. Five fixed points of the system of equations (28)–(32).

| Fixed Point | $\Sigma_{||}$ | $\Sigma_\alpha$ | $\Sigma_\lambda$ | $x$ | $z$ |
|-------------|--------------|-----------------|------------------|-----|-----|
| $K_\pm$     | $\Sigma_{||}$ | 0               | 0                | $\pm\sqrt{1 - \Sigma^2}$ | 0   |
| $S$         | 0            | 0               | 0                | $-\frac{\lambda}{3}$ | 0   |
| $V_\pm$     | $-\frac{4\Delta_{III}}{\Delta I}$ | 0               | 0                | $-\frac{6(\Gamma + \lambda)}{\Delta I}$ | $\pm\frac{\sqrt{3\Delta_{II}\Delta_{III}}}{\Delta I}$ |

spanned by variables $X$, where l.h.s. of Eqs. (28)–(32) vanish. Setting $dX/dN = 0$, those equations reduce to a system of algebraic equations. It is much easier to find their solutions than to solve differential equations.

For the solutions to be physical, they have to satisfy two conditions. First, they have to be real and they have to satisfy the Hamiltonian energy constraint in Eq. (27), that is $\Sigma^2 + x^2 + z^2 \leq 1$.

To find fixed points let us assume the $\lambda$ and $\Gamma$ parameters to be constant. In that case five distinct fixed points exist which satisfy the above two conditions. They are listed in Table 1. For brevity $\Delta$’s in this table are defined as

$$\Delta_1 \equiv 12 + (\Gamma + \lambda) (3\Gamma + \lambda),$$

$$\Delta_{II} \equiv 12 + (\Gamma + \lambda) (3\Gamma - \lambda),$$

$$\Delta_{III} \equiv -6 + \lambda (\Gamma + \lambda).$$

The first two fixed “points” $K_\pm$ are not isolated points but rather a set of points which form a hypersphere given by $\Sigma_{K}^2 + x_{K}^2 = 1$ and $z_{K} = 0$. Subscripts will be used to denote values of dynamical variables in a given fixed point. The potential energy of the scalar field vanishes on $K_\pm$, i.e., $y_K = 0$. In Ref. 10 this hypersphere of fixed points was called the anisotropic kination solution ($AKS$).

The third fixed point $S$ corresponds to the case in which both the anisotropy and the vector field energy density vanishes, so that $\Sigma_S = 0$ and $z_S = 0$, and the universe is dominated by the scalar field. Its (expansion rate normalised) kinetic energy is given by $x_S = -\lambda/3$ and potential energy by $y_S = \sqrt{1 - x_S^2}$.

The last two fixed points, denoted by $V_\pm$, are the most interesting ones where the vector field and spatial anisotropy do not vanish, i.e., $z_V = \pm\sqrt{3\Delta_{II}\Delta_{III}/\Delta I}$ and $\Sigma_V = -2\Delta_{III}/\Delta I$.

2.7. Stability analysis

The asymptotic behaviour of the system depends on the stability of fixed points in Table 1. At linear level the stability can be determined by perturbing all the dynamical degrees of freedom as $X \rightarrow X + \delta X$, where $X$ on the right-hand side corresponds to the unperturbed values and $\delta X$ are small perturbations. Plugging this expression into Eqs. (26)–(32) we keep only terms up to the first order in $\delta X$.

It is important to note that $\delta X$ also includes the vector perturbation in the directions perpendicular to the homogeneous value $\hat{z}_i$, i.e.,

$$\delta z = \delta z_{||} \hat{z}_i + \delta z_{\alpha} e_i^{\alpha}.$$  (39)
The stability of fixed points in the $e_i^a$ directions has not been checked. However, the complete analysis requires dynamical equations to be perturbed along these directions too, even if initial conditions are constrained to be axisymmetric Bianchi I along $\hat{z}_i$. The same comment applies to perturbations along $\Sigma_a$ and $\Sigma_{\lambda}$ directions. Most of the analyses examined only the behaviour of the $\delta \Sigma_\parallel$ perturbation.

As homogeneous equations for the vector field along $e_i^a$ do not exist, linearised equations of motion for $\delta z_\parallel$ and $\delta z_a$ must be derived from Eq. (23) directly. The full system of linearised dynamical equations are provided in the Appendix. They can be written in symbolic form as

$$\frac{d}{dN} \begin{pmatrix} \delta \Sigma_\parallel & \cdots & \delta z_2 \end{pmatrix}^T = \mathcal{M} \begin{pmatrix} \delta \Sigma_\parallel & \cdots & \delta z_2 \end{pmatrix}^T,$$

where $\mathcal{M}(\lambda, \Gamma)$ is a $9 \times 9$ matrix composed of fixed point values given in Table 1. As this is a system of linear first order differential equations, the formal solution can be written as $\begin{pmatrix} \delta \Sigma_\parallel & \cdots & \delta z_2 \end{pmatrix}^T = C \begin{pmatrix} e^{m_1 N} & \cdots & e^{m_9 N} \end{pmatrix}^T$, where $C$ is the $9 \times 9$ matrix of integration constants and $m_i$’s are the eigenvalues of the matrix $\mathcal{M}$. If the real part of $m_i$ is positive, a small perturbation around the fixed point in the $v_i$ direction grows with time, where $v_i$ is an eigenvector corresponding to the eigenvalue $m_i$. And vice versa, if the real part of $m_i$ is negative, the perturbation decays exponentially. An fixed point is said to be an attractor if perturbations along all the $v_i$ directions decay. If some of the real parts of the $m_i$’s are positive, such an fixed point is not stable, as any small deviation from that point along corresponding $v_i$ directions will grow. If real parts of some eigenvalues $m_i$ are zero, the stability of the fixed point cannot be determined by linear analysis.

We also encounter non-isolated fixed points, i.e., $\mathcal{K}_\pm$. Such points form an $r$-dimensional set, for example, a curve of fixed points in a 1-dimensional case. Each of the non-isolated fixed points necessarily has $r$ vanishing eigenvalues. If real parts of all other eigenvalues are negative, such a set is an attractor because all the orbits approach this $r$-dimensional set as time grows. A much broader and more rigorous discussion of dynamical systems and their applications in cosmology can be found in Ref. 29. The full analytic expressions of all eigenvalues for each fixed point are given in the Appendix. Their most important aspects are also summarised on the lefthand side of Figure 1. Different colour regions (purple, blue and green) of this plot represent the $\lambda$ and $\Gamma$ parameter values for which a corresponding fixed point (or a set of points in the case of $\mathcal{K}_\pm$) is an attractor. For example, if the $\lambda$ and $\Gamma$ values fall within the blue region, the $S$ fixed point is an attractor.

It is important to notice is that none of the regions overlap. This means that no two fixed points in Table 1 are attractors simultaneously. The letter in each coloured region tells which fixed point is an attractor. For the values of $\lambda$ and $\Gamma$ within that region, other fixed points are either unstable (some or all of the real parts of eigenvalues are positive) or they do not exist (the fixed point is either imaginary or does not satisfy the Hamiltonian energy constraint). Hence, for fixed values of
Figure 1. Stability regions for fixed points of Table 1. Each colour represents a $(\lambda, \Gamma)$ parameter region where an indicated fixed point is an attractor. Equations for bifurcation curves separating these regions are given in Eqs. (41), (42) and (43). In the l.h.s. plot darker blue and green regions represent parameter space where the universe is inflating, i.e. $\epsilon_H < 1$. The r.h.s plot is a rescaled version of the l.h.s. one (a symmetric plot could be drawn for $\lambda < 0$ and $\Gamma < 0$ values). In this plot darker region represents the observationally allowed $\epsilon_H < 10^{-2}$ range (the origin of this bound is clarified before Eq. (46)). Contours show constant $\epsilon_H$ values. The orange curve in the lower left corner of this plot shows the trajectory of the model introduced in Ref. 16.

$\lambda$ and $\Gamma$ the asymptotic fate of the solution is uniquely determined, irrespective of initial conditions: it relaxes towards one of fixed points, depending on which region $\lambda$ and $\Gamma$ fall into.

The only parameter space where the above discussion is not entirely accurate is represented by narrow green bands between $\mathcal{K}_\pm$ and $\mathcal{V}_+$ regions. If the $\lambda$ and $\Gamma$ values fall within those bands, then both $\mathcal{V}_+$ as well as $\mathcal{V}_-$ fixed points are attractors simultaneously. Otherwise, $\mathcal{V}_-$ is a saddle point (real parts of some of $m_i$’s are negative and others are positive ) in the whole $\mathcal{V}_+$ region.

Different colour regions are separated by bifurcation curves. They signify boundaries where the stability of attractor fixed points changes. These bifurcation curves can be parametrised as follows. Curves separating $\mathcal{K}_\pm$ and $\mathcal{V}_+$ regions (purple and green) are given by a solution of the $\Delta_{\text{II}} = 0$ equation

$$\Gamma = -\frac{\lambda}{3} + 2\text{sg} (\lambda) \sqrt{\left(\frac{\lambda}{3}\right)^2 - 1} \text{ for } |\lambda| \geq 3, \quad (41)$$

where $\text{sg} (\lambda)$ is the sign of the parameter $\lambda$. The bifurcation curves separating $\mathcal{S}$ and $\mathcal{V}_+$ regions (blue and green) can be parametrised by a solution of $\Delta_{\text{III}} = 0$ equation

$$\Gamma = \frac{6 - \lambda^2}{\lambda} \text{ for } |\lambda| \leq 3. \quad (42)$$

Finally, $\mathcal{S}$ and $\mathcal{K}$ regions (purple and blue) are separated by lines

$$|\lambda| = 3 \text{ for } |\Gamma| \geq 1. \quad (43)$$
For example, if $|\Gamma|$ is larger than the values in Eqs. (41) and (42), the only attractor fixed point is $\mathcal{V}_+$ (disregarding the $\mathcal{V}_-$ bands). Other fixed points are either unstable or do not exist for these parameter values.

2.8. Inflationary solutions

The main goal of this work is to identify the parameter region in the $(\lambda, \Gamma)$ plane that accepts inflationary solutions as their attractor fixed points, i.e. $\epsilon_{HX} < 1$, where $X = \mathcal{K} \pm \mathcal{S}, \mathcal{V}_+$ denotes the values of $\epsilon_H$ in a given attractor. The general $\epsilon_H$ can be written in terms of dimensionless variables $\Sigma, x, \text{etc.}$ as in Eq. (26). By plugging the attractor values from Table 1 into this equation, we can find the parameter space which satisfies the condition $\epsilon_{HX} < 1$. It is represented by a darker region on the left-hand side of Figure 1.

From this figure we clearly see that the anisotropy and kinetic term dominated regions (denoted by $\mathcal{K}_{\pm}$) are non-inflationary. Indeed, the slow-roll parameter $\epsilon_{HF} = 3$ in the whole parameter space where fixed points $\mathcal{K}_{\pm}$ are attractors. On the other hand, the parameter space with inflationary attractors overlaps $\mathcal{S}$ and $\mathcal{V}_+$ regions.

In the $\mathcal{S}$ part the slow-roll parameter is given by

$$\epsilon_{HS} = -\lambda x_S = \frac{\lambda^2}{3}. \quad (44)$$

This fixed point corresponds to the case where the vector field as well as spatial anisotropy vanish. Furthermore, from the expression of $x_S$ in Table 1 and the energy constraint in Eq. (27) we find that the bound $\epsilon_{HS} < 1$ implies $y_S > x_S$, i.e., the universe is dominated by the potential energy of the scalar field. In this case, the inflating solution is indistinguishable (at least at the unperturbed level) from a single scalar field inflation scenario. If $\epsilon_{HS}$ satisfies a stronger condition $\epsilon_{HS} \ll 1$, then $y_S \simeq 1$ and we recover the standard slow-roll solution (SSR). Indeed, using Eq. (35) we can easily find that the condition $\epsilon_{HS} \ll 1$ is equivalent to the standard condition for the single field slow-roll inflation $\epsilon_V \ll 1$, where $\epsilon_V$ is defined in Eq. (35).

The darker region also extends into the $\mathcal{V}_+$ parameter space where the vector field and spatial anisotropy do not vanish. Such fixed points correspond to an inflating universe which circumvents the assumptions of no-hair theorems. This was first noticed in Ref. 16. The exact expression for the slow-roll parameter in the $\mathcal{V}_+$ region can be written as

$$\epsilon_{HV} = -\lambda x_V = 3 \left( 1 - \frac{\Delta_{\Pi}}{\Delta_{\Pi}} \right). \quad (45)$$

Even if inflationary attractor solutions can be obtained in the whole parameter space emphasised by the darker regions on the left-hand side of Figure 1, the value of $\epsilon_H$ when cosmological scales exit the horizon is bounded to be much smaller. $\epsilon_H$ can affect the spectral tilt of the primordial curvature perturbation, which is tightly constrained by the Planck satellite. Restricting only to a first order in Hubble...
flow parameters this bound is $\epsilon_H < 0.0068$ at 95% CL. One cannot, however, apply this bound directly as it was derived for a single scalar field models of inflation. The presence of a vector field could potentially modify this bound. However, I assume the bound is not changed drastically and use

$$\epsilon_H < 10^{-2}. \quad (46)$$

The parameter space where this condition is satisfied is shown by the darker region on the right-hand side of Figure 1. This plot is a rescaled version of the left-hand side making the relevant parameter space more pronounced.

As one can read from the figure, observational constraints require

$$|\Gamma| \gg |\lambda| \quad \text{and} \quad \Gamma^2 \gg 1$$

(47)

if inflation is to be realised in the $V_+$ attractor. In this limit expressions of $\Sigma_{||V}$, $x_V$ and $z_V$ can be written in a simplified form as

$$\Sigma_{||V} \simeq -\frac{4}{3} \frac{\Gamma \lambda - 6}{\Gamma^2}, \quad x_V \simeq \frac{2}{\Gamma}, \quad z_V \simeq \sqrt{\frac{\Gamma \lambda - 6}{\Gamma^2}}, \quad (48)$$

where terms up to first order in $\lambda/\Gamma$ and $1/\Gamma^2$ are displayed. Note that for $\Gamma \lambda < 6$ $V_+$ is not an attractor. The expression of $\epsilon_{HV}$ is also simplified in this region

$$\epsilon_{HV} \simeq \frac{2}{\Gamma} \lambda. \quad (49)$$

Using the definition of the standard slow-roll parameter $\epsilon_V$ in Eq. (35) and the above expression we can write $\epsilon_{HV} \simeq 2\sqrt{3} \epsilon_V / \Gamma$. It becomes clear from this result that in the $V_+$ attractor inflationary solutions are possible even if the scalar field potential is steep, that is, there is a large parameter space where $\epsilon_H \ll 1$ for $\epsilon_V > 1$.

The possibility of having an inflationary solution even in the steep region of the scalar field potential was first pointed out in Ref. 10. Based on this result one may hope that the scalar-gauge field interaction of the form $f^2(\varphi) F^2$ could be a viable candidate to solve the infamous $\eta$-problem, which plagues inflation model building in the framework of supergravity theories of particle physics (see e.g. Ref. 31 and references therein). This result suggests that if one includes the gauge sector into inflation model building, it might be possible to escape the $\eta$-problem while keeping a generic form of the Kähler potential. The gauge kinetic function is an integral part of supergravity theories, where $f$ is a holomorphic function of scalar fields. By including this sector it might be possible to achieve dynamical cancellation of large mass terms. A model along these lines, for example, was constructed in Ref. 11.

Another important consequence of the inflationary $V_+$ attractor solution is the scaling of the gauge kinetic function. Using Eqs. (25) and (34), we can write

$$\frac{f}{Hf} = \frac{\Gamma}{x}. \quad (50)$$

Plugging in the value of $x_V$ from Table 1 and using $|\Gamma| \gg |\lambda|$ and $|\Gamma| \gg 1$, we find $\Gamma x_V \simeq -2$ and therefore

$$f \propto a^{-2}.$$

(51)
As shown in Refs. 4 and 5 such scaling produces a flat vector field perturbation spectrum.

$f$ scales as a negative power of $a$ in the whole parameter region where $V_+$ is an attractor (the only attractor with a non-vanishing vector field). That is, $f$ can only decrease in time. This result is important if the current setup is to be used in constructing models motivated by particle physics theories, where $A_\mu$ is a gauge field. In that case the inverse of the gauge kinetic function $f^{-1}$ can be interpreted as the time-dependent gauge coupling “constant” and time-dependent self-coupling “constant” in the case of non-Abelian $A_\mu$. Once the inflaton is stabilised, $f$ is constant, and thus can be absorbed into the definition of the gauge field $A_\mu$. Therefore, without the loss of generality one can normalise $f_{\text{end}} = 1$. If $n < 0$ where $f \propto a^n$, the gauge kinetic function $f \gg 1$ during inflation and all couplings of the (canonically normalised) gauge field are exponentially suppressed, making it virtually a free Abelian field. In the opposite regime $n > 0$ and $f \ll 1$, which makes all coupling constants exponentially large. In that case the perturbative quantum field theory breaks down and we can no longer trust our computations. But as shown above, only $n < 0$ is possible in the regime with non-vanishing vector field, which does not present the strong coupling problem.

The spatial anisotropy does not vanish in $V_+$ attractor too. We can write it as

$$\Sigma_V = \frac{6 - \lambda(\Gamma + \lambda)}{3\lambda(\Gamma + \lambda)} \epsilon_{HV}, \quad (52)$$

where $\epsilon_{HV}$ is given in Eq. (49). An analogous relation was first noticed in Ref. 16 in the context of their model and in the case of constant $\lambda$ and $\Gamma$ it was given in Refs. 17 and 10. Looking at Eq. (42), we see that close to the $S/V_+$ bifurcation curve, the factor in front of $\epsilon_{HV}$ in Eq. (52) vanishes. Further away from that curve the second term in the numerator dominates and the spatial anisotropy becomes

$$\Sigma_V \simeq -\frac{1}{3} \epsilon_{HV}. \quad (53)$$

Hence close to the $S$ region in the parameter space, the spatial anisotropy can be arbitrarily small; otherwise it is of the order of slow-roll parameter.

By definition, the square of expansion rate normalised variables in Eqs. (25) and (33) are equal to their fractional contribution to the total energy density of the universe (c.f. Eq. (27)). Hence from Eq. (48) we find

$$z_V^2 \simeq \frac{\Gamma \lambda - 6}{\Gamma^2} \ll 1. \quad (54)$$

That is, the vector field energy density gives a constant but subdominant contribution to the total energy budget in the inflationary part of the parameter space. On the other hand, the scalar field potential energy $y_V^2 \simeq 1$ dominates in the same parameter region. More generally, we can find

$$\left(\frac{z_V}{y_V}\right)^2 = \frac{\lambda(\Gamma + \lambda) - 6}{\Gamma(\Gamma + \lambda) + 6}, \quad (55)$$

which is valid in the whole $V_+$ region.
2.9. Models with varying $\lambda$ and $\Gamma$

In the analysis above $\lambda$ and $\Gamma$ were assumed constant. As the slow-roll parameter $\epsilon_H$ is a function of these parameters only, it also is constant. Thus the model does not provide a graceful exit that is, a way to end inflation. However, this is not a big problem as one expects that at some critical value $\varphi_c$ the potential of the scalar field can be modified by other degrees of freedom in the theory which end inflation. Such a scenario could be realised similarly to the hybrid inflation, for example. Another consequence of constant $\epsilon_H$ is that the scale factor grows as $a \propto t^{1/\epsilon_H}$. A power law inflation driven by a single scalar field is excluded by observations.\textsuperscript{30} This does not, however, guarantee that power law models discussed in this work are all excluded. Their predicted spectral index might be modified due to the presence of the vector field.\textsuperscript{11}

Constant $\lambda$ and $\Gamma$ parameters imply that the scalar field potential and the gauge kinetic function are both exponential functions of $\varphi$. Indeed, from the definitions in Eqs. (34) we find $V = V_0 \exp \left(\sqrt{2/3} \lambda \varphi/m_{\text{Pl}}\right)$ and $f = f_0 \exp \left(1/\sqrt{6} \Gamma \varphi/m_{\text{Pl}}\right)$. The time dependence of $\lambda$ or $\Gamma$ implies the departure from purely exponential forms of $V(\varphi)$ and $f(\varphi)$. But even in this case the dynamical analysis presented in this work can still be applicable, subject to additional constraints. If $\lambda$ or $\Gamma$ change with time, the future asymptotic solution of Eqs. (28)–(32) will no longer be determined by a point in the plots of Figure 1. Rather, it will be a trajectory on those plots. Starting from initial values $(\lambda_*, \Gamma_*)$ the position of the $(\lambda, \Gamma)$ point in the plane will change. If the change is not too fast, the solution of Eqs. (28)–(32) will follow the attractor corresponding to that point. The conditions which $\lambda$ and $\Gamma$ have to satisfy were discussed in Ref. 10.

As an example, consider a model presented in Ref. 16. The scalar field potential in that work is taken to be of chaotic type $V(\varphi) = m^2 \varphi^2 / 2$ and the gauge kinetic function given by $f(\varphi) = \exp(c \varphi^2 / 2m_{\text{Pl}}^2)$. In this case we find

$$\lambda = \sqrt{6m_{\text{Pl}}^2/\varphi} \quad \text{and} \quad \Gamma = c \sqrt{6 \varphi / m_{\text{Pl}}^2}. \quad (56)$$

Let us take $c = 1$, which is also used in Ref. 12. Assuming that inflation proceeds in the $\mathcal{V}_+$ attractor and choosing that cosmological scales exit the horizon $N_* = 50$ e-folds before the end of inflation we find $\varphi_*/m_{\text{Pl}} \simeq 14$. Plugging this result into Eqs. (56) we get $\lambda_* \simeq 0.175$ and $\Gamma_* \simeq 34.3$. The value of $\Gamma_*$ is barely larger than the bound in Eq. (42). It is thus consistent with the assumption that inflation proceeds in the $\mathcal{V}_+$ attractor but it is very close to the $S$ attractor. The inflationary trajectory in the $(\lambda, \Gamma)$ plane of this model is represented by the short orange curve on the right-hand side of Figure 1. The orange dot marks the $(\lambda_*, \Gamma_*)$ values.

More variants of $V(\varphi)$ and $f(\varphi)$, which give varying $\lambda$ and $\Gamma$ parameters, are presented in Ref. 10. When building these types of models one should also note another subtlety. By looking at Eq. (53), we see that if the graceful exit is realised while the dynamics of the system is determined by the $\mathcal{V}_+$ attractor, the anisotropy of the universe increases towards the end of inflation until it becomes highly anisotropic...
$|\Sigma_V| \simeq 1$. Hence one needs to study how such a large anisotropy affects perturbations and also to ensure isotropisation of the universe in the post-inflationary period. However, this is not a concern if one introduces additional degrees of freedom to provide the graceful exit. In that case inflation can be terminated before $\epsilon_{H\nu}$, and hence $\Sigma_V$, become large.

3. Conclusions

In this work I present the dynamical analysis of a scalar-vector system which interacts via an $f^2(\varphi) F_{\mu\nu} F^{\mu\nu}$ term. The analysis is performed in full phase space, without restricting initial conditions to special alignments or additional symmetries apart from the Bianchi I spatial metric. The parameter space for each attractor region is shown in Figure 1. It is demonstrated that a unique attractor exists for each value of $\lambda$ and $\Gamma$. Bifurcation curves, where attractor points change their stability, are clearly delineated and their full analytic expressions are given in Eqs. (41)–(43). The stability of each fixed point is determined by the eigenvalues of the matrix $M$. Analytic expressions of those eigenvalues are presented in Eqs. (A.1)–(A.5) in full generality, without any approximations.

The parameter space, where potentially successful models of inflation can be build, are shown by the darker region in the right-hand side plot of Fig. 1. The analysis is performed for constant $\lambda$ and $\Gamma$, that is, for the exponential scalar field potential $V(\varphi)$ and kinetic function $f(\varphi)$. However, stability regions represented in Fig. 1 also apply if $\lambda$ and $\Gamma$ vary slowly, as formulated in Ref. 10. In that case parameters draw a trajectory in the $(\lambda, \Gamma)$ plane as inflation proceeds. As an example, I show the trajectory of the model presented in Ref. 16. A more rigorous study of allowed trajectories is left for future publications.

In this letter I indicate only the parameter range where $\epsilon_H$ of a given inflationary attractor is within the observationally allowed range. However, to find a realistic range, one has to compute the values of other observables. The model of Ref. 16, for example, is tightly constrained by the bounds on statistical anisotropy.12,13 Such analysis is beyond the scope of this work.

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Appendix A. Linearised Dynamical Equations

Using the procedure described in the main text to linearise Eqs. (28)–(32), we obtain

\[
\begin{align*}
\frac{d\delta \Sigma}{dN} &= (\epsilon - 3) \delta \Sigma + 2 \delta \epsilon H - 8z \delta z
\\
\frac{d\delta \Sigma_1}{dN} &= (\epsilon - 3) \delta \Sigma_1 + 2 \delta \epsilon H + \frac{\Sigma_1 \delta \Sigma_1 + \Sigma_1 \delta \Sigma_2}{\sqrt{2}} - 6 \delta z_1
\\
\frac{d\delta \Sigma_2}{dN} &= (\epsilon - 3) \delta \Sigma_2 + 2 \delta \epsilon H + \frac{\Sigma_2 \delta \Sigma_1 + \Sigma_2 \delta \Sigma_2}{\sqrt{2}} - 6 \delta z_2
\\
\frac{d\delta \Sigma_+}{dN} &= (\epsilon - 3) \delta \Sigma_+ + 2 \delta \epsilon H + 2 \sqrt{2} (\Sigma_1 \delta \Sigma_1 - \Sigma_2 \delta \Sigma_2)
\\
\frac{d\delta \Sigma_\times}{dN} &= (\epsilon - 3) \delta \Sigma_\times + 2 \delta \epsilon H + 2 \sqrt{2} (\Sigma_2 \delta \Sigma_1 + \Sigma_1 \delta \Sigma_2)
\\
\frac{d\delta x}{dN} &= (\epsilon - 3) \delta x + 2 \delta \epsilon H + 2 \lambda (\delta \Sigma + \Sigma \delta \Sigma) + 2 (\Gamma + \lambda) z \delta z
\\
\frac{d\delta z_\parallel}{dN} &= (\epsilon - 3) \delta z_\parallel + 2 \delta \epsilon H - \left(\Gamma x - 1 - \Sigma \parallel\right) \delta z_\parallel - z \left(\Gamma \delta x - \delta \Sigma_\parallel\right)
\\
\frac{d\delta z_1}{dN} &= (\epsilon - 3) \delta z_1 - \left(\Gamma x - 1 - \frac{5}{2} \Sigma \parallel\right) \delta z_1 - \frac{\delta \Sigma_1 + \Sigma_\times \delta z_2}{\sqrt{2}} + 2 \left(\delta \Sigma_1 + \Sigma_1 \delta z_\parallel\right)
\\
\frac{d\delta z_2}{dN} &= (\epsilon - 3) \delta z_2 - \left(\Gamma x - 1 - \frac{5}{2} \Sigma \parallel\right) \delta z_2 - \frac{-\delta \Sigma_2 + \Sigma_\times \delta z_1}{\sqrt{2}} + 2 \left(\delta \Sigma_2 + \Sigma_2 \delta z_\parallel\right)
\end{align*}
\]

where I used the notation

\[
\begin{align*}
\delta \epsilon H &\equiv 2 \left[3 (\delta \Sigma + \Sigma \delta \Sigma) + z \delta z_\parallel\right], \\
\Sigma \delta \Sigma &\equiv \frac{1}{6} \left(\frac{3}{2} \Sigma \parallel \delta \Sigma_\parallel + 2 \Sigma_1 \delta \Sigma_1 + 2 \Sigma_2 \delta \Sigma_2 + \Sigma_+ \delta \Sigma_+ + \Sigma_\times \delta \Sigma_\times\right)
\end{align*}
\]

Formally these equations can be written as in Eq. (40). The eigenvalues of the matrix \( \mathcal{M} \) for each fixed point in Table 1 can be compactly expressed as
\[ m_{1,2,3} = \epsilon_H - 3 \]  
\[ m_4 = \epsilon_H - 3 - \frac{1}{2} \left[ \Delta_a - \sqrt{\Delta_a^2 - 8 (6 + \Gamma (\Gamma + \lambda)) z^2} \right] + 2 (\epsilon_H + \lambda x) \]  
\[ m_5 = \epsilon_H - 3 - \frac{1}{2} \left[ \Delta_a + \sqrt{\Delta_a^2 - 8 (6 + \Gamma (\Gamma + \lambda)) z^2} \right] \]  
\[ m_{6,7} = \epsilon_H - 3 - \frac{1}{2} \left[ \Delta_b - \Sigma_{\parallel} - \sqrt{\Delta_b^2 - 48 z^2} \right] \]  
\[ m_{8,9} = \epsilon_H - 3 - \frac{1}{2} \left[ \Delta_b - \Sigma_{\parallel} + \sqrt{\Delta_b^2 - 48 z^2} \right] \]  

where \( \Delta_a \equiv \Gamma x - 1 - \Sigma_{\parallel} \) and \( \Delta_b \equiv \Gamma x - 1 - 2\Sigma_{\parallel} \). To find explicit expressions for each fixed point, one has to plug in the values of \( \Sigma_{\parallel}, x, z \) and \( \epsilon_H \) corresponding to that point and take the positive branch \( \sqrt{\Delta^2} = +\Delta \) where appropriate.

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