On the charged boson gas model as a theory of high Tc superconductivity

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Abstract

An ideal charged boson gas is known to be a type 2 superconductor and the Bose-Einstein condensation is found to be closely related to the critical temperatures of high Tc superconductors. Motivated by these preceding partial successes we consider the possibility that the charged boson gas model might be not a mere approximation but the right framework for high Tc superconductivity. However this requires that the paired electrons are truly bound. We investigate the consequences of hypothesizing a bound state of two electrons and find that the bound electrons are capable of forming an ‘apparent’ Fermi surface in a solid. Then the solid can be a type 2 superconductor and therefore may be identified with a high Tc superconductor. The model appears compatible with the known Fermi surface phenomena of cuprates throughout the doping level from extremely underdoped to extremely overdoped. In particular the model predicts that superconductivity without a ‘real’ Fermi surface accompanied is possible, which seems supported by the fact that there is no evidence of Fermi surface below the critical temperature and at zero magnetic field in the underdoped regime. Nevertheless a validation of the model requires a more direct evidence not least because it is based on a very unlikely hypothesis. A low energy electron-electron scattering experiment appears to have the potential to settle the matter decisively.

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1. Introduction

An ideal charged boson gas, to be precise, one in a homogenous background of opposite charge which neutralize the system as a whole was considered as a natural model for superconductivity by Schafroth [1] for the first time in 1955. However the model was quickly replaced by the overwhelmingly successful BCS theory [2] published in 1957. Even if the original conclusion by Schafroth was
that the ideal charged boson gas should be a superconductor of type 1, a correction was made by Friedberg, Lee and Ren [3] in 1991 to conclude that the model should exhibit superconductivity of type 2. Since high Tc superconductors are of type 2, the work [3] makes the ideal charged boson gas model appear more relevant to high Tc superconductivity than to the conventional one. We will review their results more closely in Section 2 below.

On the other hand an important shortcoming of the ideal charged boson gas model was that, while the Bose-Einstein condensation (BEC) temperature was the obvious candidate for the superconducting critical temperature, it was many orders of magnitude higher than the critical temperatures of the real conventional superconductors when calculated assuming a sizable fraction of the carrier electrons are paired (cf. p. xii, [4] and [5]). However this failure is considerably mitigated in the case of high Tc superconductors which have much smaller carrier densities. In fact Uemura observes in [6] that the 3-dimensional BEC temperatures are only 4-5 times greater than the critical temperatures in the case of underdoped cuprates and predicts that the critical temperature of cuprates can be properly understood in terms of BEC when the two dimensional aspect is taken into account together with some other effects. This work also will be reviewed in more details in the next section.

The partial successes of the charged boson gas model, represented by respectively [3] and [6] as in the above, even if impressive, might not be a surprise if one expects that the electron pair should behave as if it is a boson. Indeed it is possible that the charged boson gas model turns out to be a good approximation, in each of the limited aspects, for a future successful theory of high Tc superconductivity in which the paired electrons are correlated but not bound. However it is not clear to what extent the expectation can be justified that the electron pair should behave as a boson. The BCS-BEC crossover theory (cf. [7]) provides a precise meaning to the idea that the pair can be viewed as a boson. It is a unified theory of fermion superfluids in which the attractive interaction between the fermions is of varying strength from extremely weak to extremely strong. However one should note that the arguments of Friedberg et al. proceed without any reference to BEC. In fact there is no reason to believe that BEC is directly related to type 2 superconductivity. Thus it appears that a separate justification is needed for one to treat the electron pairs as bosons in this case. On the other hand the relation between the critical temperature and BEC may be more readily justified by the BCS-BES crossover as Uemura himself emphasized in [6]. This nevertheless requires a sizable fraction of carrier electrons are paired at the critical temperature. Therefore the belief that the close relation between $T_c$’s and BEC can be understood in terms of BCS-BES crossover is equivalent to the prediction that the future theory should be a BCS type theory in which a sizable fraction of carrier electrons are paired at the critical temperature. It seems worth a mention that the original BCS theory, which represents the case when the attractive interaction between the fermions is extremely weak, is such that the critical temperature is the point where the pairs begin to be formed.
Instead, one may regard the partial successes of the charged boson gas model represented by \[3\] and \[6\] as indications that the charged boson gas model itself might be the right framework for high Tc superconductivity rather than a mere approximation. Indeed the main theme of this paper is to take this possibility seriously. Then we should consider the possibility that the pair might be in fact bound. However it is obvious that the electrons cannot be bound as far as we know of the electrons and their interaction in the lattice. In Section 3 we momentarily disregard the common sense to state a hypothesis that there might be a bound system of two electrons and then put the hypothesis into an investigation. An immediate consequence is that the bound electrons are able to form an ‘apparent’ Fermi surface in a solid which then can be a type 2 superconductor and therefore can be identified with a high Tc superconductor. This provides a reason for us to go on further with the hypothesis.

The model appears compatible with the known Fermi surface phenomena of cuprates throughout the doping level from extremely underdoped to extremely overdoped. In particular the model predicts that superconductivity without a ‘real’ Fermi surface accompanied is possible. Indeed there is no experimental evidence for existence of Fermi surface below \(T_c\) and under zero magnetic field in the underdoped regime while it is a common practice that one observes the Fermi surface in the overdoped regime under the same conditions. Therefore it appears as if the prediction actually has been realized, which is discussed in more details in Section 4. Nevertheless any existing experimental evidence will not suffice to validate the model decisively, considering in particular the fact that it is based on a very unlikely hypothesis. A low energy electron-electron scattering experiment described in Section 4 might have the potential to settle the matter decisively.

2. The charged boson gas model

Even if remarkable results, the work of Friedberg et al. in \[3\] and its subsequent work \[8\] consider only the case \(T = 0\) and lack an analysis at finite temperatures. The work begins with pointing at a flaw of Schafroth’s arguments in \[1\]. Then they show that there are two critical magnetic field \(H_{c_1}, H_{c_2}(H_{c_1} < H_{c_2})\) such that the boson gas shows a vortex behavior if \(H_{c_1} < H < H_{c_2}\) and is a perfect superconductor if \(H < H_{c_1}\) and normal if \(H > H_{c_2}\). Furthermore their work implies that \(H_{c_2}\) might be infinite. We should also mention that the arguments are not completely decisive(see Comment, \[8\]).

On the other hand we note that one of the conclusions of \[3\], \[8\] is that at a low density the ideal gas of charged boson is not a superconductor. The critical density \(\rho_c\) is given by Eqn. (51), \[8\]:

\[
r_b^3 \rho_c = \frac{6}{\pi^4} (K_n - \alpha_s)^3
\]

(1)

where \(r_b = 4\pi/me^2(= 4\pi\epsilon_0\hbar^2/me^2)\) is the Bohr radius and \(K_n\) and \(\alpha_s\) are
dimensionless constants such that $K_n \approx 0.759$ and $0.316 < \alpha_s < 0.558$. Here $m$ and $e$ denote respectively the mass and the charge of the boson. Unfortunately Eqn. (1) above leads to the conclusion that the pair density of a real high Tc superconductor is much smaller than $\rho_c$ when $m$ and $e$ are replaced with respectively twice the mass and twice the charge of an electron. In fact from Eqn. (1) we have:

$$5.58 \times 10^{22} \text{ cm}^{-3} < \rho_c < 5.99 \times 10^{23} \text{ cm}^{-3}.$$  

On the other hand the carrier density of a high Tc superconductor is known to be of the order $\sim 10^{21} \text{ cm}^{-3}$ (cf. [5]). Here we note that there is an estimation([9]) that about $20 \sim 25\%$ of the carriers are participating in the superfluid at the optimal doping at $T = 0$. If such a fraction of the carriers are paired the density of electron pairs in a high Tc superconductor is smaller than the critical density by more than 2 orders of magnitude.

However we would like to note that in the ideal charged boson gas model the boson has been treated as a point particle while the electron pair in a real high Tc superconductor is known to have the size $10 \sim 30 \text{ Å}$. The size and the density of the pair together imply that there are overlaps among the pairs that cannot be ignored(cf. §2, [6]). Therefore the Hamiltonian given by Eqn. (4) in [8] needs to be modified and accordingly Eqn. (1) above also can be corrected. Thus one may still hope that the works [3] and [8] not only successfully explain the type 2 superconductivity of high Tc superconductors but also may be compatible with their low pair densities.

On the other hand the 3-dimensional BEC temperature $T_c^{BE}$ is given by the equation(cf. [5]),

$$T_c^{BE} = 3.31 \left( \frac{\hbar^2 n_{B}^{2/3}}{m_B k_B} \right)$$  

where $\hbar$ is the reduced Plank constant, $n_B$, the density of the boson, $m_B$, the mass of the boson and $k_B$ is the Boltzmann constant. By assuming a sizable fraction of the carrier electrons are paired in a conventional superconductor, $T_c^{BE}$ was many orders of magnitude higher than the observed critical temperature as mentioned in the introduction above.

For a type 2 superconductor, the value $n_s/m^*(T)$ which is the superconducting carrier density divided by the effective mass of the electron at temperature $T$ can be measured by muon spin relaxation rate while it is in the vortex state. We denote the $T \rightarrow 0$ limit simply by $n_s/m^*$. Then Uemura observes in [6] that $T_c$ depends approximately linearly on $n_s/m^*$ for the underdoped cuprates, remains nearly constant of $n_s/m^*$ in the optimally doped region and a trends of linear dependence re-emerges in the overdoped region(cf. Fig. 1, [6]). Furthermore he notes that the BEC temperature $T_B$, when calculated by Eqn. (2) above assuming the boson mass and density are respectively $2m^*$ and $n_s/2$, is only 4-5 times greater than the $T_c$’s of underdoped cuprates and predicts that the Tc can be understood properly when the two dimensional aspect of cuprates is taken
into account together with some other effects including the overlap among the paired electrons. In any case the fact has been firmly established by [6] that the $T_c$ is closely related to BEC in high Tc superconductors.

3. Superconductivity by bound electrons

3.1. The hypothesis. If we are to examine the charged boson gas model as a true candidate for a theory of high Tc superconductivity rather than an approximation, we need to consider the possibility that the paired electrons might be truly bound. As the first step we articulate what we mean by a bound state of two electrons by stating the following hypothesis:

**Hypothesis B:** There is a bound state of two electrons which is short-lived in free space and has a size comparable to that of the electron pairs in a high Tc superconductor.

To be short-lived in free space, the bound system should have negative binding energy which we have omitted to avoid redundancy. The binding energy will of course have an intrinsic ambiguity because of the short lifetime. That the bound system has a finite size implies that it has an intrinsic structure which can be taken into account when one considers its interaction with the lattice or with any other system at short distance. We may not specify neither the lifetime nor the intrinsic structure not knowing the interaction that causes the binding.

The hypothesis looks far-fetched indeed. However we begin our deliberation by noting that the hypothesis provides the bound state with a property, which is a necessary, even if not a sufficient, condition for it not to have been easily noticed. That is, if the lifetime is short enough the process of its formation and decay cannot be easily distinguished from the usual scattering of two electrons.

Also we will estimate in §4.2 below the excess energy of the bound state to be less than 100 eV. This may partially explain how the bound state could have been unnoticed in high energy electron-electron scattering experiments by means of a resonance.

3.2. The superconductor. To claim any relevance of Hypothesis B to realistic superconductivity, we need to see first of all how the bound state, being unstable in free space, can be stable in a solid.

Let $E_0 < 0$ denote 2 times the energy of the lowest unoccupied electron state in the solid. On the other hand we denote by $E_l$ the sum $E_l + E_e$ where $E_l$ is the energy of the lowest state of the bound electrons in the solid and $E_e > 0$ is the excess energy of the bound system relative to two free electrons. Furthermore we may write $E_l = E_s + E_d$ where $E_s < 0$ is the electric potential energy of the two electron system in the lattice plus its center-of-mass kinetic energy and $E_d$ is the energy gain resulting from disturbing the intrinsic structure of the bound system by putting it in the lattice. We expect $E_d > 0$. Note that there is no such a thing as a filled state as far as the bound electrons are concerned. Therefore it
is possible that $E_s$ is significantly lower than $E_0$. Thus it may happen in some solid that $E_t < E_0$. Then the bound two electron system should be stable in the solid. This mechanism is the same as the one by which a neutron is stable in a nucleus. Thus we conclude:

If the inequality $E_t < E_0$ holds then the bound two electron system is stable in the solid.

Assume the inequality $E_t < E_0$ holds in a solid at absolute zero temperature. If there are any two electrons in states with energies above $\frac{1}{2}E_t$, they should be bound and fall into a state with apparent energy $\frac{1}{2}E_t$. Thus there can be no electron in a state with energy greater than $\frac{1}{2}E_t$ and the bound electrons ‘appear’ to be concentrated in one of the highest energy states of the solid.

Now assume further that $\frac{1}{2}E_t$ is the same as the energy of an electron state in a partially filled band. Since the band is partially filled there are filled states with energies infinitesimally close to $\frac{1}{2}E_0$. If the inequality $E_t < E_0$ holds those electrons should form a bound system so that they may apparently be in a state with energy $\frac{1}{2}E_t$. Thus we must have $E_t \geq E_0$. Since the inequality $E_t > E_0$ implies the bound two electrons cannot exist in the solid, we conclude that the bound electron exists in the solid if and only if $E_t = E_0$. In this case the bound electrons are not stable but in an equilibrium with the unbound electrons in states with energy near $\frac{1}{2}E_0$.

Only when all the bands which contain states with energy less than $\frac{1}{2}E_t$ are filled, the inequality $E_t < E_0$ may hold. In particular only in this case a bound two electron system is stable in the solid. This also means that $\frac{1}{2}E_t$ lies in the energy gap below which all bands are filled and above which no band is occupied.

We summarize the conclusions regarding the electronic structure of a solid in the above as follows:

**Conclusion E:** Consider a solid at absolute zero temperature. Then we have:

1. If $E_t \leq E_0$, the bound two electron system may exist in the solid and the bound electrons appear to be concentrated in one of the highest energy electron states in the solid.
2. $E_t = E_0$ if and only if the bound electrons are in an equilibrium with the unbound electrons. These two conditions are equivalent to the one that $\frac{1}{2}E_t$ lies in a partially filled band and the bound electrons exist in the solid.
3. $E_t < E_0$ if and only if the bound two electron system is stable in the solid. These two conditions are equivalent to the one that $\frac{1}{2}E_t$ lies in the energy gap of the band structure below which all electron states are occupied and above which no state is occupied.

One might have noticed that the assumption of a finite size for the bound two electron system has not played any role in reaching Conclusion E.
size is zero or can be regarded as zero compared to the atomic scale, then the lowest state for the bound two electron system in the solid is the lowest state in the most massive atom in the solid, being a bosonic point particle with twice the charge of an electron and with more than twice the mass of an electron. Therefore there is no chance that the bound electrons can be mobile and responsible for the superconductivity. Thus the assumption of finite size allows the bound electrons a chance to be the supercarrier.

The solids must be rare in which the inequality \( E_t \leq E_0 \) holds and at the same time the bound electrons are mobile. If these conditions are satisfied, it seems appropriate to say that the bound electrons have formed an ‘apparent’ Fermi surface. It follows that the solid will be a type 2 superconductor, assuming the results of [3] and [8] can be extended properly. Then one may identify the solid with a high Tc superconductor.

There are two immediate concerns that arise from the above description of high Tc superconductors:

Firstly, any two electrons whose energies are above \( \frac{1}{2} E_t \) should pairwise bind and fall into a state with apparent energy \( \frac{1}{2} E_t \). Therefore the density of bound electrons can be unrealistically high if \( \frac{1}{2} E_t \) happens to be a low value. However it is reasonable to assume that \( E_t \) depends not only on the lattice structure but also on the density of the bound electrons themselves. That is, as the density rises, \( E_t \) also rises. Then an unrealistically high density can be contained and we may come to terms with low density of pairs in real high Tc superconductors.

Secondly, the formation of the bound electrons does not appear to be directly related with lowering the temperature. It may be the case that once the condition \( E_t \leq E_0 \) is met, say, at the zero temperature, then the condition may persist in the solid at any temperature as long as the lattice structure is intact. However we do not know whether or not \( E_t \) depends on the temperature as well. Thus we may not be decisive on the temperature range in which the bound electrons may exist. Nevertheless it is clear that the density of the bound electrons should be large enough at the critical temperature for the superconductivity to be possible. Thus in our picture the bound electrons must exist above the critical temperature. Here we recall that many physicists suspect the electron pairs are preformed far above the transition temperature. Some of them think that the electron pair may exist up to \( T^* \), the temperature at which the pseudogap begins to appear(cf. [10]) or up to some other temperature \( T_{pair} \) such that \( T_c < T_{pair} < T^* \)(cf. [11]). Since the measured \( T^* \)'s are below 300 K, this may set the upper limit. However the origin of the pseudogap is not a settled issue(cf. [12]). Thus one may not be decisive regarding the matter even when the experimental results are taken into account.

3.3. Implications of the model. It is not easy to tell whether superconductivity is possible when the equality \( E_t = E_0 \) holds as in Conclusion E(2) since the bound electrons will be unstable under the condition. Clearly Schafroth and Friedberg et al. are treating the charged bosons as stable ones in their respec-
tive works, in [1] and in [3], [8]. However there is a reason for us to believe that superconductivity should be possible even when \( E_t = E_0 \) holds.

The inequality \( E_t < E_0 \) means that the only carriers are the bound electrons, as clearly implied by Conclusion E(3). Therefore if the inequality \( E_t < E_0 \) is a necessary condition for superconductivity then the density of bound electrons should increase as the carrier density of a superconductor are raised. However we know that the density \( n_s \) of superfluid should decrease as the doping level is raised beyond the optimal level(cf. [6]). Note that by definition \( n_s \) is the limit value as \( T \to 0 \) and also that the fraction of superfluid in a boson liquid approaches 1 as the liquid is cooled to zero temperature(cf. [13]). Thus in our model \( n_s \) can be regarded as the density of bound electrons. This means that the density of bound electrons should decrease even if the carrier density is increased by raising the doping level beyond optimal doping. We conclude that the inequality \( E_t < E_0 \) cannot be a necessary condition for superconductivity. Thus the equality \( E_t = E_0 \) must hold in the overdoped regime and therefore the inequality \( E_t < E_0 \) may hold only in the underdoped regime.

In particular we note that existence of a real Fermi surface is not a necessary condition for superconductivity in the model, which will be discussed in more details in §4.1 below.

There is also another unique feature of the electronic structure of a high Tc superconductor implied by the model which might be worth a mention. The energy \( E_s \), being the sum of the electric potential energy and the center-of-mass kinetic energy of the bound electrons, is equivalent to the usual energy of an unbound electron. However \( E_s \) should be lower than \( E_0 \) at least by \( E_e + E_d \) for the inequality \( E_t \leq E_0 \) to hold. This seems to imply that the bound electrons must exploit the electric potential of inner space of atoms much more than the electrons of the real Fermi surface. Therefore we may expect that the behavior of bound electrons reflects the lattice structure more closely than the electrons of Fermi surface.

Generally speaking the model presents the high Tc superconductor as a quite complex system in which the bound two electron systems, each with its own intrinsic structure, interact with the lattice and among themselves. In addition we do not know the interaction that binds the electrons pairwise.

4. Experiments

The description of a high Tc superconductor in the previous section is quite different from the more well-known views. First of all the electron pair is actually bound rather than merely correlated. Furthermore the apparent Fermi surface, which is the one responsible for the superconductivity, is not related to a real Fermi surface except that it appears to consist of the highest occupied electron states which are mobile. The picture is so unique that the resulting phenomena might have to be quite conspicuous even if we take the fact into account that the current form of the model is far from being ready to provide any quantitative
predictions. Indeed it appears as if the observed Fermi surface phenomena of cuprate superconductors has revealed the unique electronic structure predicted by the model.

The author has been unable to pinpoint any other experimental results which support or contradict the model in a significant way, which however can be attributed to the absence of any quantitative prediction in the model and also to the presence of other plausible explanations. For instance an anomaly of electronic specific heat can be expected because of the apparent Fermi surface even in the normal state above $T_c$ within the model. In fact existence of such an anomaly is well-known(cf. [14]). However there is already more than one explanation for this anomaly including those which promote paired electrons above $T_c$.

Even if the consistency with the known Fermi surface phenomena is suggestive, it is far from being a decisive evidence for the model. Since the model is based on the very unlikely hypothesis of a bound state of electrons its validity can be proved only by a more direct evidence. An electron-electron scattering experiment seems to have the potential to settle the matter decisively.

4.1. Supports from known Fermi surface phenomena. If the strict inequality $E_t < E_0$ is realized as in Conclusion E(3), the electronic structure of the solid will be such that there is an apparent Fermi surface consisting only of bound electrons without any real Fermi surface accompanied. We have observed in 3.3 above that the strict inequality may be realized only at the underdoped regime. If there is an experimental setup by which one may tell whether the true Fermi surface exists being unaffected by the apparent one, then it may help in determining whether $E_t < E_0$ is realized in the underdoped region in the following sense: If there is a Fermi surface then the strict inequality cannot hold and, otherwise, there is a chance for the inequality.

In fact existence of Fermi surface in the underdoped regime has been established by the experimental studies [15-7] based on quantum oscillation. However one should note that it is only under the magnetic field $H > H_{irr}$ where $H_{irr}$ denotes the irreversibility field at which resistive vortex phase begins. At zero magnetic field Fermi arcs are known to exist in the underdoped regime which are thought to be fragmented fermi surfaces at the temperature range $T_c < T < T^*$ by ARPES(cf. [18]). Since the Fermi surface of a two dimensional Fermi liquid should form a closed loop in the momentum space there has been a debate regarding the origin of the arcs. We note that there is an experimental study [18] which concludes that the Fermi arcs are in fact not related to true Fermi liquids. On the other hand what one can notice from these studies is the fact that there is no evidence what so ever for the existence of Fermi surface below $T_c$ and at zero magnetic field in the underdoped regime, which is well-known(cf. [19]). In contrast it seems a common practice in the overdoped regime for one to observe Fermi surface below $T_c$ and at zero magnetic field(cf. [20, 21]). Thus it appears as if superconductivity without a Fermi surface has been realized in
underdoped cuprate superconductors. In particular we conclude that there is a chance that the inequality \( E_t < E_0 \) could have been realized in the system.

The emergence of Fermi surface under a magnetic field \( H > H_{irr} \) can be understood under the two assumptions that the strict inequality \( E_t < E_0 \) has been realized in the underdoped superconductors and that a portion of the bound electrons has been dissolved by the strong magnetic field. Here we note that there is a report [19] that about 93% of superfluid density in an underdoped LSCO sample has been suppressed under the magnetic field \( H = 8 \) T and in the temperature \( T = 8 \) K, even if the suppression may not be directly interpreted as the result of dissolution of the bound states of electrons. If the magnetic field can dissolve the bound states then the freed electrons should occupy the band whose energy is just above \( \frac{1}{2}E_t \), that is, whose energy at the bottom is \( \frac{1}{2}E_0 \). Assuming this newly occupied band forms a Fermi liquid we have a Fermi surface. Thus the model may provide a simple explanation for the emergence of a Fermi surface at a magnetic field \( H > H_{irr} \).

In an overdoped superconductor, a Fermi surface can be expected since we have observed in 3.3 that the equality \( E_t = E_0 \) should hold as in Conclusion E(2) in the overdoped regime. Then there should be a partially filled band whose energy at the top is \( \frac{1}{2}E_t \). Assuming this partially filled band has formed a Fermi liquid, we have a Fermi surface whose energy level is \( \frac{1}{2}E_t \) in addition to the apparent Fermi surface whose 'apparent' energy level is also \( \frac{1}{2}E_t \). Note that the latter is the one which is responsible for the superconductivity. In particular our model will be invalidated if the superconductivity in an overdoped superconductor is proved to be a phenomena strictly on the Fermi surface.

We would like to mention also that the ARPES might be seeing the apparent Fermi surface, which must be invisible to the quantum oscillation. In particular we note that the bound electrons may constitute a source of the most energetic electrons in photoemission. If it is indeed the case then some aspects of ARPES data cannot be properly understood within the framework of modern condensed matter physics. The visibility apparently disappears below \( T_c \) once a coherence sets in the apparent Fermi surface. This recovery of invisibility poses by itself a puzzle which however is well-defined only in a world where our model is valid.

### 4.2. A rough estimation of \( E_c \)

It seems desirable for us to have an idea on the scale of \( E_c \) as a preparation for the experiment proposed in the next section.

We assume the interaction of the bound electrons with the lattice is mainly electric. Note that we have \( E_c \leq E_0 - E_l = E_0 - E_s - E_d \) and we are assuming \( E_d > 0 \). Thus \( E_0 - E_s \) is an upper bound for \( E_c \). In particular one may conjecture that the solids in which \( E_t \leq E_0 \) holds are such that \( E_d \) are relatively small and expect the upper bound \( E_0 - E_s \) for \( E_c \) may not be overly generous.

We interpret the mobility condition as implying that the wave function of the bound two electron system is such that the electrons are more or less evenly distributed throughout the solid regardless whether it is the inner space of the atoms or the outer space. It also implies the center-of-mass kinetic energy is
zero. Thus it suffices to calculate the potential energy. To make the calculation simple, we assume the electrons are evenly distributed. Furthermore we have chosen a simple model for the lattice, which reflects that of a simplest metal. The actual calculation implies that what is important for our purpose is not the precise lattice structure but the ratio of the volume of the inner space of ions to the total volume. Therefore we omit the details and simply report that the calculation has given $E_s = -40 \text{ eV}$. We assume $E_0 = 2 \times (-4 \text{ eV})$, where we take the value 4 eV as the typical work function of a metal. Then from $E_e < E_0 - E_s$ we have 32 eV as an upper bound of $E_e$.

Note that the extremely deep inner space of the atoms must be forbidden for the bound electrons since the screening of the positive charge of nucleus will raise the energy levels of the inner shells and any energy gain of the electron by nearing the nucleus will be overwhelmed by the screening effect. However the forbidden region will constitute only a small fraction of the solid and the required correction will not be significant. Even if our calculation is quite rough, we take the result as meaning that our assumptions lead to the energy scale $10 \sim 100 \text{ eV}$ for $E_e$.

4.3. Low energy electron-electron scattering. Let us consider an electron-electron beam scattering arrangement. If bound electrons as in Hypothesis B are real, one may expect that there will be a resonance for the formation of the bound two electron system when the kinetic energy of each beam is $\frac{1}{2}E_e$. Since we have estimated the scale of $E_e$ as $10 \sim 100 \text{ eV}$, one may pay a special attention to this scale of energy in the experiment. A bound two electron system will shortly decay into two free electrons. We do not know whether or not the decay may accompany emission of photons. In any case the event cannot be easily distinguished from the usual electron-electron scattering, which means that there is a strong background noise. It is not clear whether the resonance can be detected in spite of this noise.

We begin by noting that one may reduce the noise of usual scattering to some extent by concentrating on the events such that two electrons are scattered off from each other in directions perpendicular to the beam. This is because electrons are fermions as explained in most graduate texts of quantum mechanics. If we concentrate on the perpendicular scattering, then the noise can be estimated to be a quarter of the value when electrons were bosons instead of fermions.

However there is an arrangement by which one may reduce the noise much more significantly still concentrating on the perpendicular scattering. Unfortunately this method works only under the following assumptions: (1) a large fraction of bound electrons decays without any photon emission accompanied and (2) the possibility is not significantly suppressed that the two electrons from a decay might be in the same spin state. In fact the assumption (2) might appear suspicious. However we do not know the interaction which binds the two electrons and accordingly are completely uninformed regarding the process
which leads to the decay. Therefore one may say rather that if the arrangement works it will also tell something about the nature of the interaction that binds the electrons.

Assume we have arranged the two beams so that they are polarized respectively upward and downward when the z-axis is chosen perpendicular to the beams. Then we concentrate on the scattering events in which the electrons are scattered off elastically perpendicular to the beams. In addition we consider only the case when both of the scattered electrons are measured in spin up state with respect to the x-axis which is chosen as the beam line. Then the contribution of usual scattering to this event should vanish as explained in the below.

The basic fact which we exploit for the explanation is that parity is conserved in electromagnetic interaction. This implies that each relevant Feynman diagram FD represents an operator $S_{FD}$ invariant under the space inversion $P$, which is defined by $P(t, x, y, z) = (t, -x, -y, -z)$. That is, we have

$$\langle \alpha | P^* S_{FD} P | \beta \rangle = \langle \alpha | S_{FD} | \beta \rangle$$

where $P$ is the quantum transformation corresponding to $P$ and $|\alpha\rangle, |\beta\rangle$ are some relevant multi-particle states. Let $R$ denote the reflection of 3-space with respect to the $yz$-plane. Then we have $R = R_{x, \pi} P$ where $R_{x, \pi}$ is the rotation of 3-space around $x$-axis by angle $\pi$. Let $R, R_{x, \pi}$ be the corresponding quantum transformations. If $|\pm_z\rangle$ denotes the eigenstates for the spin operator $S_z$ and $|p, \pm_z\rangle$ is a free electron state, we have $R|p, \pm_z\rangle = \eta |Rp, \mp_z\rangle$ where $\eta$ is a complex constant such that $|\eta| = 1$. This follows from $R = R_{x, \pi} P$ (cf. p. 78, [22]). The choice of $\eta$ does not affect our argument and we choose for simplicity $\eta = 1$. Also we have that $R|p, +_z\rangle = |Rp, +_z\rangle$ and $R|p, -_z\rangle = -|Rp, -_z\rangle$. Let $p_1, p_2$ represent the initial electron 4-momentums which are related by $p_2 = Rp_1$ and $p'_1, p'_2$ be the final electron 4-momentums which satisfies $Rp'_i = p'_i, i = 1, 2$. Note that $S_{FD}$ is invariant under $R = R_{x, \pi} P$ since it is also invariant under the Lorentz transformation $R_{x, \pi}$. Now we have:

$$\langle p'_1, +_z; p'_2, +_z | S_{FD} | p_1, +_z; p_2, -_z \rangle = \langle p'_1, +_z; p'_2, +_z | R^* S_{FD} R | p_1, +_z; p_2, -_z \rangle = \langle p'_1, +_z; p'_2, +_z | S_{FD} | p_2, -_z; p_1, +_z \rangle.$$  

The last expression is the contribution of the Feynman diagram obtained by exchanging the initial electrons. Since electrons are fermions the contributions of the two Feynman diagrams cancel each other completely. Note that this cancellation should work also when both of the electrons are initially in $|+_z\rangle$ spin state, which illustrates the fact that spin is not a conserved quantity.

In reality the detector will be arranged so that it register two electrons scattered off from a small region within a short time interval in directions which make less than a small finite angle $\theta$ with the plane perpendicular to the beam. We may not improve the experiment for instance by simply making $\theta$ smaller.
since the electron kinetic energy cannot be perfectly homogeneous. Also an
electron may be scattered off in a perpendicular direction by a usual inelastic
scattering process, which means the other electron should come out in a non-
perpendicular direction. And it is possible that two independent such events
may occur at a small region within a short time interval. Therefore there should
still remain a small noise.

Even if the bound system happens to be a reality, it has not been guaranteed
that it should be detected by the experiment above. However if a resonance
should ever be found, it will surely imply a bound state of two electrons with
the negative binding energy corresponding to the resonance energy. Furthermore
even if the experiment cannot determine whether or not the bound system has a
finite size, a resonance at low energy will most likely be the signal of the bound
electrons described by Hypothesis B since existence of a bound two electron
system with zero size and with the excess energy, say, less than 1 keV will make
the known atomic phenomena impossible.

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