Nature of excitations of the 5/2 fractional quantum Hall effect

Csaba Tőke*, Nicolas Regnault†, and Jainendra K. Jain*
*Department of Physics, 104 Davey Laboratory, The Pennsylvania State University, Pennsylvania, 16802 and
† Laboratoire Pierre Aigrain, Département de Physique, 24 rue Lhomond, 75005 Paris, France

It is shown, with the help of exact diagonalization studies on systems with up to sixteen electrons, in the presence of up to two delta function impurities, that the Pfaffian model is inadequate for the actual quasiholes and quasiparticles of the 5/2 fractional quantum Hall effect. Implications for non-Abelian statistics are discussed.

Comparisons with exact diagonalization studies on finite size systems have served as a litmus test for theoretical proposals on fractional quantum Hall effect. This paper reports on such tests for the Pfaffian model of the 5/2 FQHE, which describes a paired state of composite fermions $|\Psi\rangle_0$. The Pfaffian wave function for the incompressible ground state has been studied before; the focus in this paper is on quasiparticles and quasiholes of this state. A case has been made, both from analytical arguments and numerical calculations, that the Pfaffian quasiholes exhibit non-Abelian braiding statistics. However, the relevance of this result to the actual Coulomb quasiholes has not been established, and there has been a debate in the literature on whether the Coulomb state is adiabatically connected to the Pfaffian state $|\Psi\rangle_0$. This question is interesting in its own right, and also because of recent proposals of exploiting non-trivial braiding properties of certain FQHE quasiparticles for quantum computation. We investigate below how well the Pfaffian wave functions represent the solutions of the Coulomb interaction, with and without the presence of weak impurities.

Our calculations are performed in the standard spherical geometry $\mathbb{R}^2$, with $N$ electrons on the surface of a sphere subjected to a magnetic field that produces a flux $2Q\phi_0$ through the surface. Here $\phi_0 = \hbar c/e$ and $2Q$ is an integer. Only electrons in the second Landau level (LL) are considered, assumed to be fully spin polarized. The eigenstates are labeled by an orbital angular momentum.

The system of Coulomb electrons at $\nu = 1/2$ in the second LL is equivalent to the system of lowest-LL electrons at $\nu = 1/2$ interacting via an effective interaction. The Pfaffian wave function in the lowest LL is given by (Moore and Read $|\Psi\rangle_0$)

$$|\Psi\rangle_0 = \text{Pf} \left( \frac{1}{u_i v_j - v_i u_j} \right) \prod_{i<j} (u_i v_j - v_i u_j)^2, \quad (1)$$

where $u_i = \cos \frac{\theta_i}{2} e^{-i\phi_i/2}$ and $v_i = \sin \frac{\theta_i}{2} e^{i\phi_i/2}$. The wave function for two quasiholes at $(U_1, V_1)$ and $(U_2, V_2)$ is given by

$$|\Psi\rangle_{2qh} = \text{Pf} (M_{ij}) \prod_{i<j} (u_i v_j - v_i u_j)^2, \quad (2)$$

where $M_{ij} = (u_i V_1 - v_i U_1) (U_2 v_j - V_2 u_j) + (i \leftrightarrow j)$. (3)

For two coincident quasiholes, $(U_1, V_1) = (U_2, V_2) \equiv (U, V)$, it reduces to a charge 1/2 vortex,

$$|\Psi\rangle_V = \prod_i (u_i V - v_i U) |\Psi\rangle_0. \quad (4)$$

Separately, each quasihole has a charge deficiency of 1/4 associated with it. Unlike for the vortex, the density does not vanish at the position of a quasihole. Analogous wave function can be written for many quasiholes. No simple wave functions presently exist for quasiparticles.

The above wave functions are the exact, zero-energy ground state of an unphysical short-range three-body interaction in the lowest LL $|\Psi\rangle_0$. (5)

$$H^{(3)} = \left( \frac{e^2}{\epsilon l_B} \right) \sum_{i<j<k} P_{ijk}(L_{\max})$$

where $P_{ijk}(L_{\max})$ is the projection operator onto a triplet of orbital angular momentum $L_{\max} = 3Q - 3$, $l_B = \sqrt{\hbar c/\epsilon B}$ is the magnetic length and $\epsilon$ is the static dielectric constant of the background semiconductor. There is no interaction when two electrons approach one another, but an energy cost is associated with electron triplets in their closest configuration. When two quasiholes are present, a diagonalization of $H^{(3)}$ produces many zero energy states, which can be chosen to be eigenstates of $L_z$; we refer to the subspace of zero-energy wave functions as the “Pfaffian quasihole (PfQH) sector.” Spatial localization of quasiholes, as described by the wave function $|\Psi\rangle_{2qh}$, breaks rotational invariance, but the wave function still lives entirely in the PfQH sector. The origin of non-Abelian statistics lies in the degeneracy of states in the PfQH sector, which produces, in general, several degenerate wave functions for a given quasihole configuration, thereby allowing for the possibility that quasihole braidings can produce different linear combinations of PfQH states, hence non-Abelian statistics.

Tables I and III show the overlaps between the PfQH basis and the corresponding number of lowest energy states for the Coulomb interaction, for two as well as four quasiholes. The overlaps are low by the FQHE standards.
TABLE I: Overlaps between the PfQH basis for two quasiholes and the lowest energy states for Coulomb interaction in the second Landau level at $2Q = 2N - 2$. The overlaps are defined as $O = |\langle \Psi_{2-qh}^{(3)} | \Psi_{2-qh}^{(3)} \rangle |^2$. The wave function $|\Psi_{2-qh}^{(3)}\rangle$ at orbital angular momentum $L$ refers to the two quasihole eigenstate of $H^{(3)}$ with quantum number $L_z = L$, and $|\Psi_{2-qh}^{(3)}\rangle$ is the lowest energy state for the Coulomb interaction with the same quantum numbers. We note that for two quasiholes, there is a single zero energy multiplet at alternate values of $L$ for $H^{(3)}$ ($L = 0, 2, \cdots, N/2$ for even $N/2$, and $L = 1, 3, \cdots, N/2$ for odd $N/2$).

| $N$ | $L = 0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|--------|---|---|---|---|---|---|---|
| 8   | 0.64   | - | 0.48| - | 0.52| - | - | - |
| 10  | -      | 0.05| - | 0.56| - | 0.61| - | - |
| 12  | 0.59   | - | 0.30| - | 0.49| - | 0.39| - |
| 14  | -      | 0.39| - | 0.13| - | 0.39| - | 0.27|

TABLE II: Overlaps between the PfQH basis for four quasiholes and the lowest energy states for Coulomb interaction in the second Landau level at $2Q = 2N - 1$. The overlap at a given $L$ is defined as $O = \sum_{i,j} |\langle \Psi_{4-qh}^{(3)} | \Psi_{4-qh}^{(3)} \rangle |^2/N$, where $N$ is the number of degenerate multiplets of $H^{(3)}$ at $L \in \mathbb{Z}$, and $i,j = 1, \cdots, N$. The states $\Psi_{4-qh,i}^{(3)}$ represent the $N$ lowest energy eigenstates of the Coulomb interaction.

| $N$ | $L = 0$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|-----|--------|---|---|---|---|---|---|---|---|---|---|
| 8   | 0.78   | 0.54| 0.63| 0.47| 0.36| 0.45| - | 0.21| - | - | - |
| 10  | 0.67   | 0.48| 0.49| 0.47| 0.21| 0.34| 0.26| 0.32| - | 0.02| - |
| 12  | 0.42   | 0.32| 0.27| 0.32| 0.17| 0.28| 0.21| 0.23| 0.23| 0.24| 0.07|

Also, in general, no distinct quasihole band analogous to the PfQH band is identifiable in the exact Coulomb spectrum. The Coulomb interaction does not simply lift the degeneracy of the PfQH states but changes the structure of the low energy sector in a fundamental manner.

We next investigate spatially localized quasiholes and quasiparticles, as needed for an evaluation of their braiding phases. It is natural to attempt to localize them with the help of weak delta function potentials of appropriate sign, which can serve as a model of an STM tip for manipulating them. We have studied a range of strengths for the delta function potential, but we show results below for weak delta functions of strength $(0.005/\sqrt{Q})e^2/\ell_B$. We place a delta function at one or both poles of the sphere, so $L_z$ remains a good quantum number [10].

We first consider the $H^{(3)}$ model. The impurity, being weak, does not cause significant mixing with states outside the PfQH sector. A single delta function impurity in the lowest LL localizes a vortex rather than a Pfaffian quasihole for the following reason. The energy of a given wave function is equal to a properly weighted average of the densities at the positions of the delta functions (for weak impurity strengths). For a delta function at $(U, V)$, the lowest energy state (which has zero energy independent of the strength of the delta impurity) is the one in which both quasiholes localize at $(U, V)$, producing a vortex $\Psi_V$ with vanishing density at $(U, V)$. (The binding of quasiholes remains valid also for a delta function in the second LL, although the density has more complicated structure than a simple vortex, as seen in Fig. III.) We ask if it is possible to split the vortex into two quasiholes with the help of two delta function impurities, placed at the two poles of the sphere. The diagonalization is performed in the full $L_z$ subspace, including states within and outside the PfQH band; for sufficiently weak
We next turn to two Coulomb quasiholes or quasiparticles. Here, at first sight, one may expect that even a single delta function should produce well separated quasiholes or quasiparticles, because it can bind one of them, which then should repel the other. As seen in Figs. (1d) and (2c), neither one nor two delta functions produce separated quasiholes or quasiparticles. In fact, the charge profile is practically identical for the two cases. The situation is more restrictive for the Coulomb interaction because, instead of many degenerate states, we have a single ground state multiplet with a definite $L$. All that
weak disorder can do is cause a mixing between the different $L_z$ components of the ground state multiplet. For the case of two delta functions at the two poles, $L_z$ is a good quantum number, so the delta functions only lift the degeneracy of the $L_z$ states. The lack of quasiparticle or quasihole separation in space is attributable to the fact that the ground state now has a more or less definite $L_z$. The absence of exact degeneracy, as found for the $H^{(3)}_\nu$ model, thus inhibits quasihole localization. The overlaps of the Coulomb quasiholes with the Pfaffian quasiholes (Table II) are very low and rapidly decreasing with $N$
[11].

While our study does not rule out well separated charge-1/4 quasiholes for the Coulomb problem for larger systems, it is rather striking that no well defined quasiholes are seen even for systems with as many as 16 electrons, which is sufficiently large at least for the Pfaffian quasiholes to be well separated. The Pfaffian model thus fails to capture the long range correlations present in the true state. Eigenstates with separated quasiholes and quasiparticles do exist in some of our finite size systems, but are not the lowest energy states for our disorder potential. Separating them with more complicated impurity potentials is thus possible, but our calculations show that it does not happen generically.

The braiding properties of the Pfaffian quasiholes have been studied numerically by Tserkovnyak and Simon[8]. In view of the above results, it is crucial to carry out similar calculations directly for the braiding properties of the Coulomb quasiholes and quasiparticles of the 5/2 state. That, unfortunately, is beyond our present capabilities. The systems accessible in exact diagonalization study are too small for this purpose, as they do not even show charge-1/4 quasiholes. A computation for larger systems would require a knowledge of accurate trial wave functions for the Coulomb quasiparticles and quasiholes, not currently available. Nonetheless, to the extent that non-Abelian statistics is a consequence of the Pfaffian structure, our study questions its validity for the Coulomb problem. One may ask if the deviation between Pfaffian wave functions and the Coulomb solutions is a finite size effect and if the Pfaffian physics can be recovered in the thermodynamic limit. We see no reason for that to happen; as seen in Tables I-III the Pfaffian wave functions rapidly deteriorate with increasing $N$
[12]. Larger systems will surely produce many quasi degenerate states for quasiholes, but they are unlikely to bear any relation to the zero energy states of the PfQH sector. Also, a gap separating “quasihole states” from rest of the states is crucial for maintaining their “topological” integrity; such a gap is present for the $H^{(3)}_\nu$ model, but not for the Coulomb Hamiltonian. These considerations suggest that the braiding properties of the Pfaffian quasiholes are unlikely to carry over to the Coulomb quasiholes.

The existence of charge-1/4 quasiholes is necessary (charge-1/2 vortices have abelian braiding statistics) but not sufficient for non-Abelian braiding statistics. The braiding statistics is much more sensitive to subtle long range correlations than the fractional charge. As an example, Laughlin’s wave function for the quasiparticle at 1/3 has the correct charge but an incorrect abelian braiding statistics[13].

If the excitations of the 5/2 state are not well described by Pfaffian wave function, what describes their physics? Further work will be needed to answer this question. The CF theory may shed some light on that. It has been shown[14] that the residual interaction between composite fermions opens a gap at $\nu = 5/2$. In this picture, the quasiparticles are excited composite fermions, although heavily renormalized by interaction.

We thank IDRIS-CNRS for a computer time allocation, and the High Performance Computing (HPC) group at Penn State University ASET (Academic Services and Emerging Technologies) for assistance and computing time on the Lion-XI and Lion-XO clusters. JKJ thanks Hans Hansson for stimulating discussions. Partial support of this research by the National Science Foundation under grant No. DMR-0240458 is gratefully acknowledged.

[1] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[2] M. Greiter, X.G. Wen, and F. Wilczek, Phys. Rev. Lett. 66, 3205 (1991); Nucl. Phys. B 374, 567 (1992).
[3] M. Greiter, X.G. Wen, and F. Wilczek, Phys. Rev. B 46, 9586 (1992).
[4] C. Nayak and F. Wilczek, Nucl. Phys. B 479, 529 (1996).
[5] N. Read and E. Rezayi, Phys. Rev. B 54, 16864 (1996).
[6] Y. Tserkovnyak and S.H. Simon, Phys. Rev. Lett. 90, 016802 (2003).
[7] T.-L. Ho, Phys. Rev. Lett. 75, 1186 (1995); N. Read and E.H. Rezayi, Phys. Rev. B 54, 16864 (1996); C. Nayak and F. Wilczek, Nucl. Phys. B 479, 529 (1996).
[8] F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
[9] T.T. Wu and C.N. Yang, Nucl. Phys. B 107, 365 (1976).
[10] The results depend slightly (though not qualitatively) on whether the impurity is a delta function in the lowest or the second LL. With the exception of Fig. 1a, the results in Figs. 1 and 2 depict the electron density in the second LL for a delta function impurity in the second LL.
[11] Slight modification in the pseudopotentials does not affect the picture qualitatively. For two quasiholes at $N = 14$, changing $V_1$ pseudopotential by $\delta V_1 = 0.03$, which results in the maximum overlap with the PfQH sector, produces a ground state which has an overlap of 0.998 with the pure Coulomb ground state.
[12] For odd-denominator FQHE states at $\nu = \frac{n}{2pn+1}$, similar system sizes show a quasi-degenerate band of states that is consistent with the effective magnetic field, and the Coulomb eigenfunctions are accurately described by the CF theory. See, J.K. Jain, Phys. Rev. Lett. 63, 199 (1989); J.K. Jain and R.K. Kamilla, Int. J. Mod. Phys. B11, 2621 (1997).
[13] H. Kjønsberg and J. Myrheim, Int. J. Mod. Phys. A 14,
537 (1999).

[14] C. Töke and J.K. Jain, Phys. Rev. Lett. 96, 246805 (2006).