Local Statistics of Random Permutations from Free Products
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December 7, 2022

Abstract
Let \( \alpha \) and \( \beta \) be uniformly random permutations of orders 2 and 3, respectively, in \( S_N \), and consider, say, the permutation \( \alpha \beta \alpha^{-1} \). How many fixed points does this random permutation have on average? The current paper studies questions of this kind and relates them to surprising topological and algebraic invariants of elements in free products of groups.

Formally, let \( \Gamma = G_1 * \ldots * G_k \) be a free product of groups where each of \( G_1, \ldots, G_k \) is either finite, finitely generated free, or an orientable hyperbolic surface group. For a fixed element \( \gamma \in \Gamma \), a \( \gamma \)-random permutation in the symmetric group \( S_N \) is the image of \( \gamma \) through a uniformly random homomorphism \( \Gamma \to S_N \). In this paper we study local statistics of \( \gamma \)-random permutations and their asymptotics as \( N \) grows. We first consider \( \mathbb{E} [\text{fix}_\gamma (N)] \), the expected number of fixed points in a \( \gamma \)-random permutation in \( S_N \). We show that unless \( \gamma \) has finite order, the limit of \( \mathbb{E} [\text{fix}_\gamma (N)] \) as \( N \to \infty \) is an integer, and is equal to the number of subgroups \( H \leq \Gamma \) containing \( \gamma \) such that \( H \cong \mathbb{Z} \) or \( H \cong C_2 * C_2 \). Equivalently, this is the number of subgroups \( H \leq \Gamma \) containing \( \gamma \) and having (rational) Euler characteristic zero.

We also prove there is an asymptotic expansion for \( \mathbb{E} [\text{fix}_\gamma (N)] \) and determine the limit distribution of the number of fixed points as \( N \to \infty \). These results are then generalized to all statistics of cycles of fixed lengths.

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1 Introduction

Let us begin with a special case of the problem we study in this paper. Let $\alpha$ and $\beta$ be uniformly random permutations of orders 2 and 3, respectively, in $S_N$, or, almost equivalently, uniformly random permutations among all those satisfying $\alpha^2 = 1$ and $\beta^3 = 1$. Consider random permutations formed by some fixed word in $\alpha$ and $\beta$, e.g., the random permutation $\alpha\beta\alpha^{-1}$ $\beta$. This random permutation can also be described as the image of the element $xyxy^{-1}$ of $\Gamma = \langle x, y \mid x^2, y^3 \rangle \cong C_2 * C_3 \cong \text{PSL}(2, \mathbb{Z})$ through a uniformly random homomorphism to $S_N$. This paper studies the local statistics of such random permutations and shows that their limit distributions (as $N \to \infty$) can be completely extracted from certain algebraic and topological invariants of the corresponding element ($xyxy^{-1}$ in the above example) in the group $\Gamma$.

More generally, given a f.g. (finitely generated) group $\Gamma$, the set $\text{Hom}(\Gamma, S_N)$ of group homomorphisms from $\Gamma$ to the symmetric group $S_N$ is finite, and is a natural object of study, being the set of all permutation-representations (actions) of $\Gamma$ on a set of size $N$. This set also lies in one-to-one correspondence with all $N$-sheeted covering spaces of a “nice” topological space $\tilde{X}$ with fundamental group $\Gamma$. The set $\text{Hom}(\Gamma, S_N)$ also shows up in the study of residual properties of $\Gamma$, of its profinite topology, of its subgroup growth and so on.

In this paper we study $\text{Hom}(\Gamma, S_N)$ where $\Gamma$ is a free product of finite, free, and (orientable) hyperbolic surface groups. Namely,

**Assumption 1.1.** Throughout this paper, we let

$$\Gamma = G_1 * \ldots * G_k$$

for some $k \in \mathbb{Z}_{\geq 1}$, and for every $i = 1, \ldots, k$, the group $G_i$ is either a finite group, a f.g. free group, or the fundamental group $\Lambda_g \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ of a closed orientable surface of genus $g \geq 2$. Denote by $m(\Gamma) \in \mathbb{Z}_{\geq 1}$ the lcm of the orders of the finite factors in (1.1) (in particular, $m = 1$ if and only if $\Gamma$ is torsion free).

The case of $k = 1$, namely, when $\Gamma$ is simply a finite, free or surface group, was studied in previous works mentioned below and on which we build upon in the current paper. Indeed, the main innovation of the current paper is in treating non-trivial free products. Non-trivial free products in our setting include the modular group $\text{PSL}(2, \mathbb{Z}) \cong C_2 * C_3$ (we denote by $C_r$ the cyclic group of order $r$) and all its f.g. subgroups, as well as many other f.g. orientable Fuchsian groups: those with parabolic or hyperbolic boundary generators (see [LS04, pp. 553]).

The mere number of homomorphisms $\Gamma \to S_N$ is well understood – see Section 1.5. Another natural question is whether a uniformly random action of $\Gamma$ on $\{1, \ldots, N\}$ is transitive, or, equivalently, if a random $N$-cover of a corresponding space is connected. Here, known results are striking: in many of the cases covered by our setting, the image of a random homomorphism $\varphi: \Gamma \to S_N$ is not only a.a.s.\(^2\) a transitive subgroup of $S_N$, but actually a.a.s. contains the alternating group $A_N$. This is true for non-abelian free groups by the famous result of Dixon [Dix69] that two uniformly random permutations a.a.s. generate $A_N$ or $S_N$. It is true for hyperbolic surface groups and for free products of cyclic groups which are Fuchsian by [LS04, Thm. 1.12]. Of course, adding free factors to $\Gamma$ can only enlarge the image of a random homomorphism. If $\Gamma$ is a finite group, $\Gamma \cong \mathbb{Z}$ or $\Gamma \cong C_2 * C_2$, it is known (and easy) that the image of a random homomorphism to $S_N$ is not a.a.s. transitive. It is probable that in all remaining cases\(^3\), the image of a random homomorphism to $S_N$ should also contain $A_N$ a.a.s., but we do not know of a reference.

\(^1\)For this correspondence, $X$ needs to be connected, locally path-connected and semilocally simply-connected. Moreover, $X$ is equipped with a basepoint $x_0 \in X$, the group $\Gamma$ is identified with $\pi_1(X, x_0)$, and $X$’s $N$-sheeted covering spaces $\rho: \tilde{X} \to X$ are equipped with a bijection between $\{1, \ldots, N\}$ and the fiber $\rho^{-1}(x_0)$. See [Hat05, pp. 68-70].

\(^2\)We write a.a.s., or asymptotically almost surely, to describe an event which has probability tending to 1 as the implied parameter ($N$ in the current case) tends to infinity.

\(^3\)The remaining cases are non-trivial free products where all factors are finite groups or $\mathbb{Z}$, with at most one cyclic factor or precisely two cyclic factors both of which are $C_2$ (excluding, of course, the group $C_2 * C_2$ itself).
1.1 Fixed points in a $\gamma$-random permutation

In this paper, however, our focus is different. We fix an element $\gamma \in \Gamma$ and consider its image through a uniformly random homomorphism $\varphi : \Gamma \to S_N$. We call the resulting random permutation a $\gamma$-random permutation. In the topological setting, the image of $\gamma$ corresponds to the structure of the lifts of the corresponding closed curve in the space $X$ to a random $N$-sheeted cover. We concentrate on the local statistics of a $\gamma$-random permutation: the distribution of the number of cycles of given fixed lengths.

We begin by presenting our results for the distribution of the number of fixed points, and later generalize to cycles of arbitrary fixed lengths. Denote by $\text{fix}_\gamma(N)$ the random variable that counts the number of fixed points of a $\gamma$-random permutation in $S_N$.

There is a clear distinction between torsion elements and elements of infinite order. Any non-trivial torsion element $\gamma$ of $\Gamma$ is conjugate into one of the finite factors (the infinite factors are torsion-free). So the statistics of a $\gamma$-random permutation only depend on the particular factor it is conjugate into. In this case, the following proposition readily follows from results in [Mül97]:

**Proposition 1.2.** Let $\gamma \in \Gamma$ have finite order and let $|\gamma|$ denote its order. Then

$$
E[\text{fix}_\gamma(N)] = N^{1/|\gamma|} + O\left(N^{1/2|\gamma|}\right).
$$

(1.2)

For instance, Example 3.7 explains why $E[\text{fix}_\gamma(N)] = N^{1/2} + N^{1/4} + O(1)$ for $\gamma = x^2 \in \Gamma = C_4 = \langle x \rangle$.

(More general statistics of $\gamma$-random permutations when $\gamma$ has finite order can be derived from [MSP10].)

The picture is completely different for elements of infinite order. Consider first the case where $\Gamma \cong \mathbb{Z} = \langle x \rangle$ and $\gamma = x^q$. A $x^q$-random permutation is simply the $q$-power of a uniformly random permutation, and the local statistics here are well-understood: as $N \to \infty$ they converge in distribution to a sum of suitable independent Poisson variables – see [DS94]. In particular, $E[\text{fix}_\gamma(N)]$ converges to $d(q)$, the number of positive divisors of $q$. Nica showed in [Nic94] that the same is true for elements of a free group: if $\Gamma$ is a f.g. free group and $1 \neq \gamma \in \Gamma$, write $\gamma = \gamma_0^q$ with $q \in \mathbb{Z}_{\geq 1}$ and $\gamma_0 \in \Gamma$ a non-power. Then $\text{fix}_\gamma(N)$ converges in distribution, as $N \to \infty$, to the same sum of Poissons as $x^q \in \mathbb{Z}$ does. In particular, the limit distribution depends only on $q$ and not on $\gamma_0$.

The case of orientable surface groups was recently studied by Magee and the first author in [MP20]. While the presence of a relation makes the analysis in this case by far more complicated than in free groups, it is nevertheless shown in [MP20] that Nica’s results about free group elements hold in surface groups as well. In particular, for $g \geq 2$ and $1 \neq \gamma \in \Lambda_g$, if we write $\gamma = \gamma_0^q$ with $\gamma_0$ a non-power and $q \in \mathbb{Z}_{\geq 1}$, then $\text{fix}_\gamma(N)$ converges in distribution, as $N \to \infty$, to the same sum of Poissons as $x^q \in \mathbb{Z}$ does.

An interesting twist arises when one considers groups with torsion, and, in particular, free products of finite groups, as in the current paper. It turns out that the property of an element $\gamma \in \Gamma$ which determines the local statistics of a $\gamma$-random permutation in the limit is not only whether it is a power and the value of the exponent, but rather, the array of subgroups of Euler characteristic zero containing it and its powers. To explain this phenomenon, let us first recall what the Euler characteristic is for the groups in play in this paper.

**Definition 1.3** (Euler Characteristic of groups). The (rational) Euler characteristic of a group $\Gamma$, denoted $\chi(\Gamma)$, is a rational number defined for groups with a finite index subgroup of finite homological type – see [Bro82, Sec. IX.7]. For the sake of the current paper, it is enough to mention that

- For a finite group $G$, $\chi(G) = \frac{1}{|G|}$.

\[\text{To be precise, this result is not stated explicitly in [MP20]. The paper [MP20] is long as is and its main feature is the development of a new representation-theoretic method to compute integrals over $\text{Hom}(\Lambda_g, S_N)$. To keep that paper to a manageable size, it states explicitly only the result that } E[\text{fix}_\gamma(N)] \xrightarrow{N \to \infty} d(q). \text{ However, the stronger results about the limit distributions of } \text{fix}_\gamma(N) \text{ and other local statistics follow readily from [MP20]. At any rate, the proofs we give in the current paper heavily rely on [MP20] and encompass, as a special case, the case of } \gamma \in \Lambda_g.\]
Table 1: This table illustrates Theorem 1.4 and gives the limit value of $\mathbb{E} [\text{fix}_\gamma (N)]$ as $N \to \infty$ for various infinite-order elements $\gamma$ in various groups. The limit is the number of subgroups $H \leq \Gamma$ with $\chi (H) = 0$ containing $\gamma$, and their full list in each case is given in the rightmost column.

| $\Gamma$ | $\gamma$ | $\lim_{N \to \infty} \mathbb{E} [\text{fix}_\gamma (N)]$ | $\mathcal{H}_\gamma$ |
| --- | --- | --- | --- |
| $C_2 \ast C_2 = \langle x \rangle * \langle y \rangle$ | $[x, y]$ | 5 | $\langle \gamma \rangle, \langle xy \rangle, \langle x, yxy \rangle, \langle xyx, y \rangle, \Gamma$ |
| $C_2 \ast C_q = \langle x \rangle * \langle y \rangle$, $2 < q$ | | 2 | $\langle \gamma \rangle, \langle x, yxy^{-1} \rangle$ |
| $C_m \ast C_q = \langle x \rangle * \langle y \rangle$, $2 < m \leq q$ | | 1 | $\langle \gamma \rangle$ |
| $F_2 = \mathbb{Z} * \mathbb{Z} = \langle x \rangle * \langle y \rangle$ | | 1 | $\langle \gamma \rangle$ |
| $\Lambda_2 = \langle x, y, z, t \mid [x, y] [z, t] \rangle$ | | 1 | $\langle \gamma \rangle$ |
| $C_2 \ast C_2 = \langle x \rangle * \langle y \rangle$ | $(xy)^3$ | 6 | $\langle \gamma \rangle, \langle xy \rangle, \langle x, yxyy \rangle, \langle yxy, xy \rangle, \langle y, xyxyx \rangle, \Gamma$ |
| $C_3 \ast C_4 = \langle x \rangle * \langle y \rangle$ | $[x, y^2]$ | 2 | $\langle \gamma \rangle, \langle xy^2 x^{-1}, y^2 \rangle$ |

- For a rank-$r$ free group, $\chi (F_r) = 1 - r$.
- For a surface group $\Lambda_g \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$, $\chi (\Lambda_g) = 2 - 2g$.
- If $G_1$ and $G_2$ have a well-defined Euler characteristic, then so does $G_1 \ast G_2$, and

$$\chi (G_1 \ast G_2) = \chi (G_1) + \chi (G_2) - 1.$$ 

So, for example, $\chi (\text{PSL}_2 (\mathbb{Z})) = \chi (C_2 \ast C_3) = \frac{1}{2} + \frac{1}{3} - 1 = -\frac{1}{6}$. By Kurosh subgroup theorem, if $\Gamma$ is as in (1.1), then every subgroup of $\Gamma$ is a free product of (conjugates of) subgroups of the factors of $\Gamma$ together with, possibly, a free group factor. As every subgroup of a free group is free, and every subgroup of $\Lambda_g$ ($g \geq 2$) is either free or $\Lambda_h$ for some $h \geq g$ (e.g. [Sco78]), we get that every f.g. subgroup of $\Gamma$ is, too, of the form (1.1), and, in particular, has a well-defined EC (Euler characteristic) as in Definition 1.3. Note that when restricting to the groups considered in this paper, the only groups with positive EC are finite groups, and the only groups with EC zero are $\mathbb{Z}$ and $C_2 \ast C_2$.

For $\Gamma$ as in (1.1) and $\gamma \in \Gamma$, denote

$$\mathcal{H}_\gamma \overset{\text{def}}{=} \{ H \leq \Gamma \mid \gamma \in H \text{ and } \chi (H) = 0 \}. \quad (1.3)$$

Equivalently, this is the set of subgroups of $\Gamma$ containing $\gamma$ which are isomorphic to $\mathbb{Z}$ or to $C_2 \ast C_2$. It is not hard to show (and see Corollary 5.2) that this set is finite for every non-torsion $\gamma \in \Gamma$.

**Theorem 1.4.** Let $\Gamma$ be as in (1.1) and $\gamma \in \Gamma$ have infinite order. Then

$$\mathbb{E} [\text{fix}_\gamma (N)] \overset{N \to \infty}{\to} |\mathcal{H}_\gamma|.$$ \quad (1.4)

**More precisely,** writing $m = m (\Gamma)$ as in Assumption 1.1, we have

$$\mathbb{E} [\text{fix}_\gamma (N)] = |\mathcal{H}_\gamma| + O \left( N^{-1/m} \right).$$

In Table 1 we illustrate this result with some concrete examples. This generalizes the above-mentioned results in free groups and surface groups, as these groups are torsion free and have no embedded copies of $C_2 \ast C_2$. Thus, in this case $\mathcal{H}_\gamma$ contains only infinite cyclic groups: if $\gamma = \gamma_0^d$ with $\gamma_0$ a non-power, this set is $\mathcal{H}_\gamma = \{ \langle \gamma_0 \rangle^d \mid 1 \leq d \leq q \}$ (see Footnote 10 for some details).

One can give a unified statement encompassing both Proposition 1.2 and Theorem 1.4: for every $\gamma \in \Gamma$

$$\mathbb{E} [\text{fix}_\gamma (N)] = c_\gamma \cdot N^{1/|\gamma|} \left( 1 + O \left( N^{-1/m} \right) \right),$$ \quad (1.5)
where $|\gamma|$ is the order of $\gamma$, $\frac{1}{|\gamma|} \overset{\text{def}}{=} 0$, $c_\gamma$ is the number of subgroups $H \leq \Gamma$ containing $\gamma$ and of Euler characteristic $\frac{1}{|\gamma|}$, and $m = m(\Gamma)$. Indeed, (1.5) coincides with Theorem 1.4 when $|\gamma| = \infty$. If $\gamma$ is a torsion element, then $\langle \gamma \rangle$ is the sole subgroup of $\text{EC} \frac{1}{|\gamma|}$ containing $\gamma$, and so $c_\gamma = 1$. Both bounds on the error term $-O\left(N^{1/|\gamma|-1/m}\right)$ in (1.5) and $O\left(N^{1/(2|\gamma|)}\right)$ in (1.2) – hold in this case. See Section 3.2 for details.

In fact, the role of $\text{EC}$ of subgroups in local statistics of random homomorphisms $\Gamma \to S_N$ goes much further. Roughly, for a natural choice of a nice space $X_\Gamma$ with fundamental group $\Gamma$, let $p: X \to X_\Gamma$ be an arbitrary covering space, let $Y \leq X$ be a compact subspace, and for simplicity assume that $Y$ is connected. Let $\pi_1^{\text{lab}}(Y) \leq \Gamma$ be the (conjugacy class of the) subgroup corresponding to $Y$, namely, this is $\pi_1^{\text{lab}}(Y) \overset{\text{def}}{=} p_* (\pi_1(Y)) \leq \pi_1(X_\Gamma) = \Gamma$. Then the average number of embeddings of $Y$ in a random $N$-cover of $X_\Gamma$, or more precisely the average number of injective lifts of $p|_Y: Y \to X_\Gamma$ to a random $N$-cover of $X_\Gamma$, is of order $N^\chi(\pi_1^{\text{lab}}(Y))$. The precise statement is given in Theorem 2.6 below (and see Remark 2.7). This result is an important ingredient in the proof of Theorem 1.4 and the other main results.

The same method we use to prove Theorem 1.4 can be used to compute the limit of all moments of $\text{fix}_\gamma(N)$ and, by the method of moments, prove the following.

**Theorem 1.5.** Let $\Gamma$ be as in (1.1) and $\gamma \in \Gamma$ have infinite order. Let $H_1, \ldots, H_t$ be representatives of the conjugacy classes of subgroups represented in $\mathcal{H}_\gamma$. For $i = 1, \ldots, t$ let $\alpha_i = \left| \left\{ \mathcal{H}_\gamma \cap H_i^{-1} \right\} \right|$ be the number of conjugates of $H_i$ in $\mathcal{H}_\gamma$, and let $\beta_i \overset{\text{def}}{=} [N_\Gamma(H_i): H_i]$ be the index of $H_i$ in its normalizer. Then

$$\text{fix}_\gamma(N) \overset{\text{dis}}{\underset{N \to \infty}{\longrightarrow}} \sum_{i=1}^{t} \alpha_i \beta_i Z_{1/\beta_i}, \quad (1.6)$$

where $Z_\lambda \sim \text{Poi}(\lambda)$ (a random variable with Poisson distribution with parameter $\lambda$), the different $Z_\lambda$’s in the sum are independent, and “$\overset{\text{dis}}{\Rightarrow}$” denotes convergence in distribution.

**Example 1.6.** Consider the penultimate element from Table 1: $\gamma = (xy)^3 \in \Gamma = C_2 \ast C_2 = \langle x \rangle * \langle y \rangle$. In this case, the elements of $\mathcal{H}_\gamma$ belong to four different conjugacy classes: $\{\langle \gamma \rangle\}$, $\{\langle xy \rangle\}$, $\{\Gamma\}$ and $\{\langle x, yxyxy \rangle, \langle xy, xyx \rangle, \langle y, yxyxx \rangle\}$, so $t = 4$, $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 1$ and $\alpha_4 = 3$. In addition, $\langle \gamma \rangle \leq \Gamma$ and $\langle xy \rangle \leq \Gamma$ and so $\beta_1 = [\Gamma : \langle \gamma \rangle] = 6$, $\beta_2 = [\Gamma : \langle xy \rangle] = 2$ and $\beta_3 = [\Gamma : \Gamma] = 1$. Finally, $N_{\Gamma}(\langle x, yxyxy \rangle) = \langle x, yxyxy \rangle$ and so $\beta_4 = 1$. Hence in this case

$$\text{fix}_\gamma(N) \overset{\text{dis}}{\underset{N \to \infty}{\longrightarrow}} 6Z_{1/6} + 2Z_{1/2} + Z_1 + 3Z_1$$

(here the last two $Z_1$’s are two distinct, independent Poisson variables with parameter 1 each).

Given a non-torsion $\gamma \in \Gamma$, the set $\mathcal{H}_\gamma$ can be generated by following the procedure$^5$ in the proof of Theorem 1.4 in Section 5.

As a special case, we retrieve the known results when $\Gamma$ is free (originally due to Nica [Nic94]) or a hyperbolic orientable surface group (due to Magee-Puder [MP20] – and see Footnote 4). Recall that in these cases, if $\gamma = \gamma_0^q \in \Gamma$ with $\gamma_0$ a non-power, then $\mathcal{H}_\gamma = \{\langle \gamma_0^d \rangle : 1 \leq d | q\}$. Moreover, $N_{\Gamma}(\langle \gamma_0^d \rangle) = \langle \gamma_0^d \rangle$ and $\langle \gamma_0 \rangle$ is malnormal. Thus Theorem 1.5 translates to the following.

**Corollary 1.7.** [Nic94, MP20] Assume that $\Gamma$ is either free or a hyperbolic orientable surface group (so $\Gamma = \Lambda_\gamma = \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle$ with $g \geq 2$). Let $1 \neq \gamma = \gamma_0^q \in \Gamma$ with $\gamma_0$ a non-power and $q \in \mathbb{Z}_{\geq 1}$. Then

$$\text{fix}_\gamma(N) \overset{\text{dis}}{\underset{N \to \infty}{\longrightarrow}} \sum_{1 \leq d | q} dZ_{1/d}.$$ 

---

$^5$In short, this procedure involves constructing a 1-dimensional “sub-cover” corresponding to $\gamma$, producing all surjective morphisms from it (namely, construction all ’sub-covers’ which are its quotients, with the map between them), and recognizing the quotients with labeled fundamental group of Euler characteristic zero. See Sections 2 and 5 for details.
The following quantitative version of the residual finiteness of $\Gamma$, follows from Theorem 1.5 by a simple application of the Markov inequality (and see [MP20, Sec. 1.4] for some background).

**Corollary 1.8.** Given a non-torsion element $\gamma \in \Gamma$ and $r \in \mathbb{Z}_{\geq 1}$,

$$\frac{|\{\varphi \in \text{Hom} (\Gamma, S_N) \mid \varphi (\gamma) \neq \text{id}\}|}{|\text{Hom} (\Gamma, S_N)|} \geq 1 - \frac{c_r (\gamma)}{N^r} - O \left( \frac{1}{N^{r+1/m}} \right),$$

where $c_r (\gamma) = \mathbb{E} \left[ \left( \sum i \beta_i Z_1 / \beta_i \right)^r \right]$, and $\alpha_i$ and $\beta_i$ are the parameters from Theorem 1.5.

(The corollary follows from the fact that $\mathbb{E} [\text{fix}_\gamma (N)^r] = c_r (\gamma) + O \left( N^{-1/m} \right)$, which follows from Theorem 1.5, Equation (5.5) and Theorem 2.6.)

**Remark 1.9.** It is not clear to us to what extent the results in this paper can be extended to more general f.g. groups. There are certainly groups which behave very differently. As an example, consider the group $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$. The image of $x$ in a uniformly random homomorphism $\mathbb{Z}^2 \to G$ to some finite group $G$ is a uniformly random element in a uniformly random conjugacy class. So if $\varphi : \mathbb{Z}^2 \to S_N$ is uniformly random, $\varphi (x)$ has the cycle structure of a uniformly random conjugacy class. In particular, $\mathbb{E} \left[ \text{fix}_x (N) \right]$ is the average number of rows of length one in a uniformly random Young diagram with $N$ blocks. It it not hard to see that this number is

$$\mathbb{E} \left[ \text{fix}_x (N) \right] = \frac{p (0) + p (1) + \ldots + p (N - 1)}{p (N)},$$

where $p$ is the partition function. This number is of order $\sqrt{N}$. In fact, $\text{fix}_x (N) \cdot \frac{1}{\sqrt{2N}}$ converges in distribution to the exponential distribution with expectation $1 - \pi$ see [Fri93, Thm. 2.1]. Notice there are infinitely many EC-zero subgroups containing $x$: $\{ \langle x, y^j \rangle \mid j \in \mathbb{Z}_{\geq 0} \}$. See also Section 7.

### 1.2 Asymptotic expansion of $\mathbb{E} [\text{fix}_\gamma (N)]$

When $\Gamma$ is free and $\gamma \in \Gamma$, it is not hard to show that $\mathbb{E} [\text{fix}_\gamma (N)]$ is given by a rational function in $N$ for every large enough $N$ (see [Nic94, LP10]). For example, for $\gamma = [x, y] \in \mathbb{F}_2 = \mathbb{F} (x, y)$ we have $\mathbb{E} [\text{fix}_\gamma (N)] = \frac{N}{\mathbb{N} - 1}$ for every $N \geq 2$. Such a clean result does not hold for the other groups we consider here. Yet, asymptotic expansion, in the form of rational or “fractional rational” approximation, does exist.

**Definition 1.10** (Asymptotic expansion). Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$. Let $k_1 > k_2 > \ldots$ be a decreasing sequence of real numbers and $a_{k_1}, a_{k_2}, \ldots$ a sequence of real numbers. We say that $f$ has asymptotic expansion given by $a_{k_1}, a_{k_2}, \ldots$ and denote

$$f (N) \overset{\text{asymp. exp.}}{\sim} a_{k_1} N^{k_1} + a_{k_2} N^{k_2} + a_{k_3} N^{k_3} + \ldots,$$

or simply $f (N) \overset{\text{asymp. exp.}}{\sim} \sum_{j=0}^{\infty} a_{k_j} N^{k_j}$, if for every $\ell \in \mathbb{Z}_{\geq 1}$ we have

$$f (N) = a_{k_1} N^{k_1} + a_{k_2} N^{k_2} + \ldots + a_{k_\ell} N^{k_\ell} + O \left( N^{k_{\ell+1}} \right).$$

The most recent development here is the easiest to state:

**Theorem 1.11.** [MP20, Thm. 1.1] For any $\gamma \in \Lambda_\gamma$ there are rational numbers $a_i = a_i (\gamma)$ for $i = 1, 0, -1, -2, \ldots$ such that

$$\mathbb{E} [\text{fix}_\gamma (N)] \overset{\text{asymp. exp.}}{\sim} a_1 N + a_0 + a_{-1} N^{-1} + a_{-2} N^{-2} + \ldots \quad (1.8)$$
The case of finite groups has a long history. The expected number of fixed points is intimately related to the size of \( \text{Hom} (G, S_N) \): indeed, if \( \langle x \rangle = C_q \) is a cyclic group, then \( \mathbb{E} [\text{fix}_x (N)] = N \cdot \frac{|\text{Hom}(C_q, S_{N-1})|}{|\text{Hom}(C_q, S_N)|} \).

Already in 1951 it was conjectured by Chowla, Herstein and Moore [CHM51] that the asymptotic expansion of the form \( N^{1/2} + A + BN^{-1/2} + CN^{-1} + DN^{-3/2} + \ldots \), a conjecture proven slightly later by Moser and Wyman [MW55]. After many milestones, a complete solution for arbitrary finite groups was given by Müller in 1997.

**Theorem 1.12.** [Mül97, Thm. 6] Let \( G \) be a finite group of order \( m \geq 2 \). Then there are rational numbers\(^6\) \( Q_t = Q_t (G) \) for \( t = -1/m, -2/m, \ldots \) such that

\[
\frac{|\text{Hom} (G, S_N)|}{|\text{Hom} (G, S_{N-1})|} \xrightarrow{\text{asym. exp.}} N^{1-1/m} \cdot \left\{ 1 + Q_{-1/m} N^{-1/m} + Q_{-2/m} N^{-2/m} + \ldots \right\} \quad (1.10)
\]

Müller’s result can be translated into a similar asymptotic expansion for \( \mathbb{E} [\text{fix}_\gamma (N)] \) whenever \( \gamma \) is an element of a finite group (see Section 3.2 below). In the current paper we rely on Theorems 1.11 and 1.12 in order to generalize these results to arbitrary free products as in (1.1).

**Theorem 1.13.** Let \( \Gamma \) be a free product and let \( m = m (\Gamma) \) as in (1.1). Then for every \( \gamma \in \Gamma \) there are rational numbers \( a_t = a_t (\gamma) \) for \( t = 1, \frac{m-1}{m}, \frac{m-2}{m}, \ldots, \frac{1}{m}, 0, -\frac{1}{m}, \ldots \) so that

\[
\mathbb{E} [\text{fix}_\gamma (N)] \xrightarrow{\text{asym. exp.}} a_1 N + a_{1-1/m} N^{1-1/m} + a_{1-2/m} N^{1-2/m} + \ldots \quad (1.10)
\]

The leading non-vanishing term of (1.10) is given by Proposition 1.2 and Theorem 1.4. The value of the second non-zero term in (1.10), or, similarly, the order of \( \mathbb{E} [\text{fix}_\gamma (N)] - N^{1/|\Gamma|} \), may encode additional group-theoretic information about \( \gamma \): see Conjecture 7.1.

One may also consider joint local statistics of different elements in \( \Gamma \). We state our result for two elements, although it easily generalizes to any finite set of elements. Two variables with parameter \( N \) are asymptotically independent if they have a joint limit distribution as \( N \to \infty \) and the limit is that of two independent random variables.

**Theorem 1.14.** Let \( \Gamma \) be as in (1.1), let \( m = m (\Gamma) \) and let \( \gamma_1, \gamma_2 \in \Gamma \) have infinite order. Then the following three conditions are equivalent:

1. \( \text{fix}_{\gamma_1} (N) \) and \( \text{fix}_{\gamma_2} (N) \) are asymptotically independent as \( N \to \infty \).
2. \( \gamma_1 \) and \( \gamma_2 \) are not both conjugate into the same Euler-Characteristic-zero subgroup of \( \Gamma \).
3. \( \mathbb{E} [\text{fix}_{\gamma_1} (N) \cdot \text{fix}_{\gamma_2} (N)] = \mathbb{E} [\text{fix}_{\gamma_1} (N)] \cdot \mathbb{E} [\text{fix}_{\gamma_2} (N)] + O (N^{-1/m}) \).

In concrete terms, the condition from item (2) translates in our settings to that \((i)\) the non-power root of \( \gamma_1 \) is not conjugate to the non-power root of \( \gamma_2 \) nor of \( \gamma_2^{-1} \), and \((ii)\) \( \gamma_1 \) and \( \gamma_2 \) do not have conjugates belonging to the same subgroup isomorphic to \( C_2 \times C_2 \).

### 1.3 Statistical asymptotics of cycles of bounded lengths

The techniques used to study the asymptotic distribution of \( \text{fix}_\gamma (N) \), the number of fixed points in a \( \gamma \)-random permutation, can also be used to analyze the asymptotic distribution of \( \text{cyc}_{\gamma,L} (N) \): the number of \( L \)-cycles for any fixed \( L \) (in particular, \( \text{cyc}_{\gamma,1} = \text{fix}_\gamma \)). In addition, they lead to the asymptotic joint

\[^6\text{The statement of Theorem 6 in [Mül97] does not explicitly specify that the coefficients } Q_i \text{ are rational - the rationality is explicit only when } 1 \leq i \leq m + 3, \text{ in which case concrete formulas are given. However, the rationality of } Q_i \text{ for all } i \text{ does follow from the proof and was verified via personal communication with the author of [Mül97].}\]
distribution of \( \text{cyc}_{\gamma,1}, \text{cyc}_{\gamma,2}, \ldots, \text{cyc}_{\gamma,L} \). To state the results, define, for every \( \gamma \in \Gamma \) and \( L \in \mathbb{Z}_{\geq 1} \), a set analogous to \( \mathcal{H}_L \) from (1.3):

\[
\mathcal{H}_{\gamma,L} \overset{\text{def}}{=} \left\{ H \leq \Gamma \ \bigg| \ \gamma^L \in H, \ \chi(H) = 0 \forall 1 \leq L' < L \ \gamma^{L'} \notin H \right\}. \tag{1.11}
\]

So \( \mathcal{H}_{\gamma,L} \) is the set of EC-zero subgroups of \( \Gamma \) containing \( \gamma^L \) but not any smaller positive power of \( \gamma \). Note that \( \mathcal{H}_{\gamma,1} = \mathcal{H}_\gamma \) and \( \mathcal{H}_{\gamma,L} = \bigcup_{1 \leq d \leq L} \mathcal{H}_{\gamma,d} \). We summarize these results in the following theorem generalizing Theorems 1.4 and 1.5.

**Theorem 1.15.** Let \( \Gamma \) be as in (1.1), \( m = m(\Gamma) \) and \( \gamma \in \Gamma \) have infinite order, and fix \( L \in \mathbb{Z}_{\geq 1} \). Then

1. We have

\[
\mathbb{E} \left[ \text{cyc}_{\gamma,L}(N) \right] = \frac{1}{L} |\mathcal{H}_{\gamma,L}| + O \left( N^{-1/m} \right). \tag{1.12}
\]

2. Let \( H_1, \ldots, H_t \) be representatives of the conjugacy classes of subgroups represented in \( \mathcal{H}_{\gamma,L} \). For \( i = 1, \ldots, t \) let \( \alpha_i = \left| \{ \mathcal{H}_{\gamma,L} \cap H_i^\Gamma \} \right| \) be the number of conjugates of \( H_i \) in \( \mathcal{H}_{\gamma,L} \), and let \( \beta_i \overset{\text{def}}{=} [N_{\Gamma}(H_i) : H_i] \) be the index of \( H_i \) in its normalizer. Then,

\[
\text{cyc}_{\gamma,L}(N) \overset{\text{dis}}{\rightarrow} N \rightarrow \infty \frac{1}{L} \sum_{i=1}^{t} \alpha_i \beta_i Z_{1/\beta_i},
\]

(as in Theorem 1.5, \( Z_\lambda \sim \text{Poi}(\lambda) \), the different \( Z_\lambda \)'s in the sum are independent, and \( \overset{\text{dis}}{\rightarrow} \) denotes convergence in distribution).

3. The variables \( \text{cyc}_{\gamma,1}(N), \text{cyc}_{\gamma,2}(N), \ldots, \text{cyc}_{\gamma,L}(N) \) are asymptotically independent. In particular, for \( L_1 \neq L_2 \),

\[
\mathbb{E} \left[ \text{cyc}_{\gamma,L_1}(N) \cdot \text{cyc}_{\gamma,L_2}(N) \right] = \mathbb{E} \left[ \text{cyc}_{\gamma,L_1}(N) \right] \cdot \mathbb{E} \left[ \text{cyc}_{\gamma,L_2}(N) \right] + O \left( N^{-1/m} \right).
\]

In the case of free groups, parts 1 and 2 of Theorem 1.15 recover the full result of Nica [Nic94] which determines the limit distribution of \( \text{cyc}_{\gamma,L}(N) \) when \( \gamma \) is an element of a free group and shows the limit depends only on \( q \) where \( \gamma = \gamma_0^q \) with \( \gamma_0 \) a non-power as above (see also [LP10, Thm. 25] and [HP22, Thm. 1.3]).

### 1.4 Overview of the paper

**Outline of the proof of the main results**

Let us explain the ideas behind the proofs of the main results. First we construct a CW-complex, denoted \( X_\Gamma \), which is a graph of spaces (in the sense of Scott and Wall [SW79]) with fundamental group \( \Gamma \). The space \( X_\Gamma \) consists of a star with a central vertex \( o \) and, for every free factor \( G_i \) of \( \Gamma \) in (1.1), an edge \( e_i \) with one end at \( o \) and the other the basepoint of some pointed CW-complex \( X_{G_i} \) representing \( G_i \). For \( G \) a f.g. free group, \( X_G \) is a bouquet of circles; for \( G = \Lambda_g \) a surface group, \( X_G \) is a pointed, genus-\( g \) orientable surface with a given CW-structure specified below; and for \( G \) finite, \( X_G \) is some finite presentation 2-complex of \( G \). Clearly, \( \tau_1(X_G, o) \cong \Gamma \). See Figure 2.1.

Every covering space \( p: \hat{X} \rightarrow X_\Gamma \) inherits a CW-structure from \( X_\Gamma \). Let \( Y \subseteq \hat{X} \) be a sub-complex of \( \hat{X} \) with finitely many cells (so if some open cell belongs to \( Y \), then so do all the cells of smaller dimension it is attached to). We call such a sub-complex a compact sub-cover of \( X_\Gamma \). It is equipped with the restriction of the covering map \( p = p|_Y: Y \rightarrow X_\Gamma \). The main technical result of this paper is the following.
Let $p: Y \to X_\Gamma$ be a connected compact sub-cover of $X_\Gamma$. Let $\pi_1^{\text{lab}} (Y) \overset{\text{def}}{=} p_* (\pi_1 (Y)) \leq \pi_1 (X_\Gamma) = \Gamma$ be the corresponding conjugacy class of subgroups of $\Gamma$, and let $\chi^{\text{grp}} (Y) \overset{\text{def}}{=} \chi (\pi_1^{\text{lab}} (Y))$. Then the average number of injective lifts of $Y$ to a random $N$-cover of $X_\Gamma$, denoted $\mathbb{E}^{\text{emb}}_Y (N)$, satisfies

$$\mathbb{E}^{\text{emb}}_Y (N) = N \chi^{\text{grp}} (Y) \left( a_0 (Y) + O \left( N^{-1/m} \right) \right). \quad (1.13)$$

Here $m = m (\Gamma)$ and $a_0 (Y) \in \mathbb{Z}_{\geq 1}$ is a positive integer. Moreover, in many important cases $a_0 (Y) = 1$. A more precise statement is given in Theorem 2.6 below and applies to compact subcovers which are not necessarily connected. We first prove (1.13) for sub-covers of $X_G$ for each factor $G$ of $\Gamma$. This part is straightforward when $G$ is free, it relies on [Mül97] when $G$ is finite, and on [MP20] when $G = \Lambda_g$. We then integrate these results to obtain (1.13) for arbitrary sub-covers of $X_\Gamma$.

To analyze $\mathbb{E} [\text{fix}_\gamma (N)]$ for some $\gamma \in \Gamma$, recall that $\gamma$ corresponds to some loop $\overline{\gamma}: (S^1, 1) \to (X_\Gamma, o)$, and we may assume that the image of $\overline{\gamma}$ is a combinatorial closed path in the 1-skeleton of $X_\Gamma$. Given $\varphi: \Gamma \to S_N$, the fixed points of $\varphi (\gamma)$ are in bijection with the lifts of $\overline{\gamma}$ to the $N$-sheeted cover $\pi: X_{\varphi} \to X_\Gamma$ corresponding to $\varphi$.

\[
\begin{array}{ccc}
X_{\varphi} & \xrightarrow{\hat{\gamma}} & X_\Gamma \\
\downarrow \pi & & \downarrow \pi \\
S^1 & \xrightarrow{\overline{\gamma}} & X_\Gamma
\end{array}
\]

So we analyze the number of such lifts of $\overline{\gamma}$ into a random $N$-cover of $X_\Gamma$. In every such lift $\hat{\gamma}: (S^1, 1) \to (X_{\varphi}, y)$, the image $\hat{\gamma} (S^1)$ is a subcomplex of (the 1-skeleton of) $X_{\varphi}$ and in particular a sub-cover $Y$ of $X_\Gamma$. As $\overline{\gamma}$ is a finite path, there are finitely many such sub-covers.

\[
\begin{array}{ccc}
(Y, y) & \xrightarrow{\tilde{\gamma}} & (X_\Gamma, o) \\
\downarrow \pi & & \downarrow \pi \\
(S^1, 1) & \xrightarrow{\overline{\gamma}} & (X_\Gamma, o)
\end{array}
\]

Denote by $\mathcal{R}_\gamma$ the finite set of all such possible surjective lifts $\hat{\gamma}: S^1 \to Y$ to sub-covers. Such a set is called a resolution in the terminology of [MP20]. Figure 5.1 illustrates such a resolution for an element of $C_2 * C_4$. We obtain

$$\mathbb{E} [\text{fix}_\gamma (N)] = \sum_{Y \in \mathcal{R}_\gamma} \mathbb{E}^{\text{emb}}_Y (N), \quad (1.14)$$

and using (1.13) deduce that

$$\mathbb{E} [\text{fix}_\gamma (N)] = \sum_{Y \in \mathcal{R}_\gamma} N \chi^{\text{grp}} (Y) \left( a_0 (Y) + O \left( N^{-1/m} \right) \right).$$

It is clear that for every $Y \in \mathcal{R}_\gamma$, we have $\gamma \in \pi_1^{\text{lab}} (Y, y)$. It is not hard to show that if $|\gamma| = \infty$, then the subgroups in $\mathcal{H}_\gamma$ are precisely the subgroups $\pi_1^{\text{lab}} (Y, y)$ for $Y \in \mathcal{R}_\gamma$ with $\chi (\pi_1^{\text{lab}} (Y)) = 0$. Moreover, in all these elements of the resolution, $a_0 (Y) = 1$. This leads to Theorem 1.14.

Theorem 1.13 about the asymptotic expansion of $\mathbb{E} [\text{fix}_\gamma (N)]$ is proven along the same lines: we first prove asymptotic expansion of $\mathbb{E}^{\text{emb}}_Y (N)$ for sub-covers of $X_G$ separately for every factor $G$ of $\Gamma$ (again, heavily relying on [Mül97, MP20]), then establish this expansion for arbitrary sub-covers of $X_\Gamma$, and finally use (1.14) to establish the sought-after result of Theorem 1.13.

The proofs of the remaining results use similar techniques combined with the method of moments. In particular, to establish Theorem 1.5 about the limit distribution of $\text{fix}_\gamma (N)$, we study the moments $\mathbb{E} [\text{fix}_\gamma (N)^r]$ for every $r \in \mathbb{Z}_{\geq 1}$ by constructing a resolution for the union of $r$ disjoint copies of $\overline{\gamma}$. 

9
Every element in this resolution corresponds to a finite multiset of f.g. subgroups of $\Gamma$. The limit $\lim_{N \to \infty} E[\text{fix}_\gamma(N)]$ is given by the number of elements in this resolution corresponding to multisets of subgroup with total EC zero.

Paper organization

After mentioning some related works in Section 1.5, we formally construct the graph of spaces $X_\Gamma$ and introduce the notions of sub-covers and resolutions in Section 2. Section 3 studies sub-covers of a vertex-space $X_G$ of $X_\Gamma$, and analyzes them separately for every type of group $G$: free groups, finite groups, and surface groups. In particular, it proves our main technical result, Theorem 2.6, for all such sub-covers, and proves Proposition 1.2 concerning torsion elements of $\Gamma$. Then, Section 4 incorporates the results from Section 3 to prove Theorem 2.6 for arbitrary sub-covers. In Section 5 we complete the proof of Theorems 1.4, 1.5 and 1.13, and in Section 6 of Theorems 1.14 and 1.15. We end in Section 7 with two intriguing open questions which arise from our results.

Notation

We denote by $(N)_t$ the falling factorial, also known as Pochhammer symbol,
$$\sum_{k \in \mathbb{N}} \frac{1}{k!}$$
We write $[N]$ for the set $\{1, \ldots, N\}$. The notation $\text{asym, exp.}$ marks asymptotic expansion and is defined in Definition 1.10. Many repeating notions are formally defined in Section 2, such as sub-covers (Definition 2.1), resolutions (Definition 2.2), and $E_Y(N)$ and $E_{Y,\text{emb}}(N)$ (Definition 2.3).

1.5 Related works

The number of homomorphisms $\Gamma \to S_N$

As mentioned above, the size of $\text{Hom}(\Gamma, S_N)$ is well understood. Clearly, $|\text{Hom}(\Gamma, S_N)| = \prod_{i=1}^k |\text{Hom}(G_i, S_N)|$. If $G = F_r$ is a rank-$r$ free group, then $|\text{Hom}(F_r, S_N)| = |S_N|^{r} = (N!)^r$. If $G$ is finite, an asymptotic formula for $|\text{Hom}(G, S_N)|$ is given in [Mil97, Thm. 5]. Finally, if $G = \Lambda_g$ is a genus-$g$ surface group, then $|\text{Hom}(G, S_N)|$ is related to the “zeta function” of irreducible representations of $S_N$ and is equal to $|S_N|^{2g-1} (2 + O(N^{2-2g}))$ (see [Hur02, Lul96, LS04]).

Random Belyi surfaces

In [Gam06], Gamburd studies random Belyi surfaces glued from $N$ ideal triangles from the hyperbolic plane. Closed paths in these surfaces are related to closed paths in the dual graph, which is cubical. This cubical graph is completely determined by the cyclic order of half-edges around every vertex, so a permutation $\beta \in S_{3N}$ consisting of $N$ 3-cycles, and the perfect matching of the half edges which creates the edges, so a permutation $\alpha \in S_{3N}$ consisting of $1.5N$ transpositions. Gamburd analyzes the random permutation $\alpha\beta$ when $\alpha$ and $\beta$ are chosen uniformly at random, and proves it converges to the uniform distribution in total variation distance as $N \to \infty$. In fact, he proves the same when $3$ is replaced by an arbitrary $k \in \mathbb{N}$ [Gam06, Thm. 4.1]. Although similar, note that this model is different than ours, as $\alpha$ and $\beta$ are not allowed to have fixed points.

Measures induced by elements of finitely generated groups

There has been an extensive study of “word measures” – measures induced by elements of free groups – on various families of groups. As mentioned above, the asymptotics of word measures on $S_N$ were studied in [Nic94]. A precise result about the leading term of $E[\text{fix}_\gamma(N)] - 1$ (for $\gamma \in F$) was found in [Pud14, PP15].
– see Section 7, and more general results about all stable characters of $S_N$ were recently established in [HP22]. Additional works studied word measures on $U(N)$, $O(N)$, $Sp(N)$, $GL_N(\mathbb{F}_q)$ and generalized symmetric groups – see [HP22, Sec. 1.6] for a short survey. As for measures induced by elements of surface groups, aside for the above mentioned work [MP20], the recent works [Mag22, Mag21] study measures induced by elements of $\Lambda_g$ on $U(N)$ and establish results about the expected trace: its limit and its asymptotic expansion. Finally, Baker and Petri show in [BP20] how one can use results as in the current paper about measures induced on $S_N$ by elements of the free product $C_{p_1} \ast \ldots \ast C_{p_k}$ of cyclic group, in order to study such measures induced on $S_N$ by elements of the group

$$\Gamma_{p_1, \ldots, p_k} \overset{\text{def}}{=} \langle x_1, \ldots, x_m \mid x_1^{p_1} = x_2^{p_2} = \ldots = x_k^{p_k} \rangle.$$  

(Note that $C_{p_1} \ast \ldots \ast C_{p_k}$ is a quotient with kernel $\mathbb{Z}$ of $\Gamma_{p_1, \ldots, p_k}$ by the additional relation $x_1^{p_1} = 1$.) See also [HMP20] for a general discussion regarding “profinitely rigid” elements in finitely generated groups and the relation to measures induced by such elements on finite groups.

Spectral gap

Some of the works in this line of research are motivated, inter alia, by questions about expansion and spectral gap of random objects. For $r \in \mathbb{Z}_{\geq 2}$, a random $2r$-regular graph on $N$ vertices can be obtained as a random $N$-cover of the bouquet of one vertex and $r$ loops, thus corresponding to a random homomorphism $F_r \to S_N$. Relying on the trace method, word measures on $S_N$ can thus be used to show that random graphs are expanders. Indeed, this is the arranging idea in many works on the subject, starting from [BS87b] (and also in [Fri08, Pud15]). This also stands in the background for works about expansion of more general random Schreier graphs of $S_N$ [FJR+98, HP22]. Similarly, using Selberg’s trace formula, the results of [MP20] about measures induced on $S_N$ by elements of $\Lambda_g$, the fundamental group of a hyperbolic surface, were used in [MNP22] to yield results about spectral gap in random covers of closed hyperbolic surfaces. Results about $\Lambda_g$ stated in the current paper are used in the recent paper [Nau22], studying other statistics of the spectrum of the Laplacian on random covers of surfaces. Simpler techniques can also show that these random objects Benjamini-Schramm converge to the corresponding universal cover: see [MP20, Sec. 1.5] for random surfaces and [BP20] for random covers of torus-knot complements.

Core graphs and sub-covers

The notion of a sub-cover in this paper is very much related to Stallings core graphs. The original Stallings core graphs [Sta83] correspond to subgroups of a free group and they are only slightly generalized in sub-covers of $X_G$ where $G$ is a free group. Bass [Bas93] extends Stallings’ theory to a very general theory about geometric presentation of subgroups of the fundamental group of a graph of groups. Other authors developed a more specialized version for more specialized families of groups such as amalgams of finite groups [ME07] or the mere modular group $PSL_2(\mathbb{Z}) \cong C_2 \ast C_3$ [BNW21]. Some of the analysis in this paper is inspired by ideas in these works. We heavily rely here also on a theory of core surfaces developed for surface groups in [MP22, MP20].

Acknowledgements

We thank Nir Avni, Michael Magee, Chen Meiri, Thomas Müller and Ron Peled for beneficial discussions. Michal Buran carried out the computation leading to an estimate of $\mathbb{E} \left[ \text{fix}_{[a,b]}(N) \right]$ where $[a,b] \in \Lambda_2 = \langle a, b, c, d \mid [a, b] [c, d] \rangle$, an estimate used in Section 7. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 850956).
2 The space $X_\Gamma$, its covers and sub-covers

2.1 The graph of spaces representing $\Gamma$

Let us formally construct the space $X_\Gamma$ mentioned in Section 1.4. If $(X, x_0)$ and $(Y, y_0)$ are two pointed connected CW-complexes (or nice enough topological spaces), Seifert-Van-Kampen Theorem guarantees that their wedge sum at the points $x_0$ and $y_0$ has fundamental group isomorphic to the free product $\pi_1(X, x_0) \ast \pi_1(Y, y_0)$. Relying on this basic fact, we construct a CW-complex whose fundamental group is $\Gamma$ from smaller CW-complexes with fundamental groups $G_1, \ldots, G_k$ (we continue using here the notation from Assumption 1.1). To get a clearer picture which is somewhat easier to work with, we use a star-graph instead of a single wedge point. This leads to the following construction of $X_\Gamma$ as a graph of spaces.

For every $i = 1, \ldots, k$, let $X_{G_i}$ be a CW-complex with a marked vertex (in fact, a single vertex) $v_i$, so that $\pi_1(X_{G_i}, v_i) \cong G_i$. Moreover, the edges (1-cells) of $X_{G_i}$ are directed and labeled by a fixed set of generators of $G_i$. The construction of $X_{G_i}$ is as follows:

- If $G_i = F_r$ is a rank-$r$ free group, we fix a basis $B = \{b_1, \ldots, b_r\}$ and let $X_{G_i}$ be a bouquet made of one vertex, (named $v_i$) and $r$ directed loops labeled $b_1, \ldots, b_r$. This defines an isomorphism $\pi_1(X_{G_i}, v_i) \cong F_r$.

- If $G_i$ is a finite group, we let $X_{G_i}$ be some finite presentation complex of $G_i$: given some finite presentation $\langle S | R \rangle$ of $G_i$, the complex $X_{G_i}$ is made of the vertex $v_i$ together with a directed loop for every $s \in S$ and a 2-cell attached to the loops for every $r \in R$. We fix some isomorphism $\langle S | R \rangle \cong G_i$ and for every $s \in S$, label the $s$-loop by its image in $G_i$ through this isomorphism. For example, if $G_i = C_q = \langle a \rangle$ is a cyclic group, then $G_i \cong \langle s | s^q \rangle$ and so $X_{G_i}$ may consist of a single vertex, a single directed loop labeled "a", and a single 2-cell whose boundary wraps around the a-loop $q$ times.

- If $G_i = \Lambda_g$ is a surface group, we let $X_{G_i}$ be a genus-$g$ orientable surface with a CW-structure obtained from gluing the sides of a $4g$-gon according to the pattern $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$. So $X_{G_i}$ consists of a single vertex $v_i$, $2g$ directed loops labeled $a_1, b_1, \ldots, a_g, b_g$ and a single 2-cell. The elements $a_1, \ldots, b_g$ are elements of $\Lambda_g$ given by an isomorphism $\Lambda_g \cong \langle a_1, b_1, \ldots, a_g, b_g \rangle$.

Finally, $X_\Gamma$ consists of the complexes $X_{G_1}, \ldots, X_{G_k}$ together with a vertex $o$ and edges $e_1, \ldots, e_k$ so that $e_i$ connects $o$ and $v_i$. We have $\pi_1(X_\Gamma, o) \cong \Gamma$. Moreover, the labels of the 1-cells inside $X_{G_1}, \ldots, X_{G_k}$ form a generating set for $\Gamma$, and every word in this generating set corresponds to a closed loop in $X_\Gamma^{(1)}$, the 1-skeleton of $X_\Gamma$, based at $o$: simply follow the 1-cells according to the given word (transversing the corresponding edge backwards if the generator comes with a negative exponent), and going through $o$ and $e_1, \ldots, e_k$ to pass from one $X_{G_i}$ to another. This is illustrated in Figure 2.1.

2.2 Sub-covers and resolutions

Covering spaces of $X_\Gamma$ inherit the CW-structure from it (indeed, open cells are covered by disjoint homeomorphic sets), and we consider $N$-sheeted covering spaces $p: \tilde{X} \to X_\Gamma$ together with a bijection between $p^{-1}(o)$ and $\{1, \ldots, N\}$. This yields a bijection between these $N$-covers and $\text{Hom}(\Gamma, S_N)$ (see, for instance, [Hat05, pp. 68-70]). For $\varphi: \Gamma \to S_N$, we denote the corresponding $N$-cover of $X_\Gamma$ by $X_{\varphi}$.

**Definition 2.1 (Sub-covers).** A sub-cover $Y$ of $X_\Gamma$ is a (not necessarily connected) sub-complex of a (not necessarily finite degree) covering space of $X_\Gamma$. In particular, a sub-cover is endowed with the

---

7To make the notation complete, here and for the other types of factors, one needs the notation of the generators to formally reflect the factor they generate, for example $b_1, \ldots, b_r$, as there may be more than one factor which is a free group (or more than one factor which is a surface group and so on). However, we prefer to keep the notation a bit simpler and have the factor be understood from the context.
Figure 2.1: The graph of spaces $X_\Gamma$ when $\Gamma = \Lambda_2 \ast F_2 \ast C_2 \ast C_4$. The middle vertex $o$ is connected by four edges to the four basepoints of the spaces $X_{\Lambda_2}$, $X_{F_2}$, $X_{C_4}$ and $X_{C_2}$. The space $X_{\Lambda_2}$ is a genus-2 surface consisting of the vertex $v_1$, four edges labeled $a, b, c$ and $d$ and one disc. The space $X_{F_2}$ consists of one vertex $v_2$ and two loops labeled $z_1$ and $z_2$. The spaces $X_{C_2}$ and $X_{C_4}$ both consist of a single vertex ($v_3$ and $v_4$, respectively), a single edge (labeled $x$ and $y$, respectively), and a single 2-cell: the boundary of the 2-cell wraps around the edge twice (respectively, four times). Note that the latter part of the construction is not well reflected in the figure. Overall, $\{a, b, c, d, z_1, z_2, x, y\}$ is a generating set for $\Gamma$.

restricted covering map $p: Y \to X_\Gamma$, which is an immersion. Denote by $Y|_{G_i} \defeq p^{-1}(X_{G_i})$, $i = 1, \ldots, k$, the subcomplex of $Y$ lying above $X_{G_i}$, the subspace of $X_\Gamma$ corresponding to the factor $G_i$ of $\Gamma$. Let $p_i: Y_i \to X_\Gamma$ be sub-covers for $i = 1, 2$. A morphism of sub-covers $f: Y_1 \to Y_2$ is a combinatorial morphism of CW-complexes commuting with the restricted covering maps, namely, such that the following diagram commutes.

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{f} & Y_2 \\
p_1 \downarrow & & \downarrow p_2 \\
X_\Gamma & & \\
\end{array}
$$

This definition extends the definition of a tiled surface in the case where $\Gamma = \Lambda_g$ [MP22, Def. 3.1]. The covering space of which $Y$ is a sub-complex is not part of the data attached to the sub-cover: indeed, the same $Y$ can be a sub-cover of distinct covering spaces. In the following definition we adapt the terminology from [MP20], extend it to our more general setting and add a variant restricted to embeddings.

**Definition 2.2 (Resolutions).** Let $p: Y \to X_\Gamma$ be a sub-cover of $X_\Gamma$. A resolution $R$ of $Y$ is a collection of morphisms of sub-covers

$$
\{f: Y \to Z_f\}
$$

so that every morphism of sub-covers $h: Y \to \hat{X}$ to a full covering space $\hat{X}$ of $X_\Gamma$ decomposes uniquely as

$$
Y \xrightarrow{f} Z_f \xrightarrow{\bar{h}} \hat{X},
$$

with $f \in R$ and $\bar{h}$ an embedding.
Similarly, an embedding-resolution $\mathcal{R}^{\text{emb}}$ of $Y$ is a collection of injective morphisms of sub-covers $\{f: Y \hookrightarrow Z_f\}$ so that every injective morphism of sub-covers $h: Y \hookrightarrow \hat{X}$ to a full covering space $\pi: \hat{X} \rightarrow X_\Gamma$ of $X_\Gamma$ decomposes uniquely as $Y \overset{f}{\hookrightarrow} Z_f \overset{h}{\hookrightarrow} \hat{X}$ with $f \in \mathcal{R}^{\text{emb}}$ and $h$ an embedding.

Many of the resolutions we construct in this paper are the natural resolutions which consist of all possible surjective morphisms of sub-covers with domain $Y$. When $Y$ is compact, this resolution is finite. However, we sometimes need more involved resolutions. The identity map $\{\text{id}: Y \rightarrow Y\}$ constitutes a trivial embedding-resolution. But again, we will need below more involved embedding-resolutions.

**Definition 2.3** ($\mathbb{E}_Y$ and $\mathbb{E}_Y^{\text{emb}}$). Let $p: Y \rightarrow X_\Gamma$ be a sub-cover of $X_\Gamma$, and let $\pi: X_\varphi \rightarrow X_\Gamma$ be an $N$-cover of $X_\Gamma$ corresponding to a uniformly random $\varphi: \Gamma \rightarrow S_N$. Denote by $\mathbb{E}_Y(N)$ the expected number of lifts of $p$ to $X_\varphi$.

![Diagram](https://via.placeholder.com/150)

Namely

$$\mathbb{E}_Y(N) \overset{\text{def}}{=} \mathbb{E}_{\varphi \in \text{Hom}(\Gamma, S_N)} \left| \{ f: Y \rightarrow X_\varphi \mid \pi \circ f = p \} \right| . \quad (2.1)$$

Similarly, denote by $\mathbb{E}_Y^{\text{emb}}(N)$ the expected number of injective lifts of $p$ to the random $N$-cover $X_\varphi$.

![Diagram](https://via.placeholder.com/150)

Namely

$$\mathbb{E}_Y^{\text{emb}}(N) \overset{\text{def}}{=} \mathbb{E}_{\varphi \in \text{Hom}(\Gamma, S_N)} \left| \{ f: Y \hookrightarrow X_\varphi \mid \pi \circ f = p, \ f \text{ is injective} \} \right| . \quad (2.2)$$

As reflected in Theorem 2.6 below, the quantity $\mathbb{E}_Y^{\text{emb}}(N)$ has nice properties. In contrast, the quantity $\mathbb{E}_Y(N)$ does not share these properties, and we therefore study it via resolutions together with the following obvious lemma.

**Lemma 2.4.** Let $Y$ be a compact sub-cover of $X_\Gamma$. If $\mathcal{R}$ is a finite resolution of $Y$, then

$$\mathbb{E}_Y(N) = \sum_{f \in \mathcal{R}} \mathbb{E}_Y^{\text{emb}}(N). \quad (2.3)$$

If $\mathcal{R}^{\text{emb}}$ is a finite embedding-resolution of $Y$ then

$$\mathbb{E}_Y^{\text{emb}}(N) = \sum_{f \in \mathcal{R}^{\text{emb}}} \mathbb{E}_Y^{\text{emb}}(N). \quad (2.4)$$

**Definition 2.5** (Subgroups associated with sub-covers). Let $p: Y \rightarrow X_\Gamma$ be a compact\(^8\) sub-cover. If $Y$ is connected, then $p_*(\pi_1(Y)) \leq \pi_1(X_\Gamma) = \Gamma$ is a well-defined conjugacy class of f.g. subgroups of $\Gamma$ we denote by $\pi_1^{\text{lab}}(Y)$ if $y \in p^{-1}(o) \subseteq Y$ then $\pi_1^{\text{lab}}(Y, y)$ is the corresponding particular subgroup in the conjugacy class $\pi_1^{\text{lab}}(Y)$. If $Y$ is not necessarily connected, let $Y_1, \ldots, Y_\ell$ denote its connected components, and define

$$\pi_1^{\text{lab}}(Y) \overset{\text{def}}{=} \left\{ \pi_1^{\text{lab}}(Y_1), \ldots, \pi_1^{\text{lab}}(Y_\ell) \right\}$$

---

\(^8\)Some of the notions here can be defined for arbitrary, not-necessarily-compact sub-covers, but we only use them for compact ones.

\(^9\)The notation $\pi_1^{\text{lab}}(Y)$ is meant to hint that we consider closed cycles in the 1-skeleton of $Y$ (based at some vertex) as labeled cycles: every cycle represents the element of $\Gamma$ which is spelled by labels on the edges along the cycle.
to be the multiset of conjugacy classes of f.g. subgroups of $\Gamma$ corresponding to $Y_1, \ldots, Y_\ell$. Finally, denote by

$$\chi^{\text{grp}}(Y) \overset{\text{def}}{=} \chi \left( \pi_1^{\text{lab}}(Y) \right) \overset{\text{def}}{=} \sum_{i=1}^\ell \chi \left( \pi_1^{\text{lab}}(Y_i) \right)$$

the sum of Euler characteristics of the subgroups in the multiset.

The following theorem is at the heart of this paper.

**Theorem 2.6.** Let $p : Y \to X_\Gamma$ be a compact, not necessarily connected, sub-cover of $X_\Gamma$. Let $m = m(\Gamma)$. Then there are rational numbers $a_t = a_t(Y)$ for $t = 0, -\frac{1}{m}, -\frac{2}{m}, -\frac{3}{m}, \ldots$ so that

$$\mathbb{E}^{\text{emb}}_Y(N) \overset{\text{asym. exp.}}{\sim} N^{\chi^{\text{grp}}(Y)} \cdot \left\{ a_0 + a_{-1/m} N^{-1/m} + a_{-2/m} N^{-2/m} + \ldots \right\}. \tag{2.5}$$

Moreover, $a_0$ is a positive integer, so in particular,

$$\mathbb{E}^{\text{emb}}_Y(N) = N^{\chi^{\text{grp}}(Y)} \left( a_0 + O \left( N^{-1/m} \right) \right) = \Theta \left( N^{\chi^{\text{grp}}(Y)} \right).$$

Furthermore, whenever there are no surface groups involved, $a_0 = 1$, so

$$\mathbb{E}^{\text{emb}}_Y(N) = N^{\chi^{\text{grp}}(Y)} \left( 1 + O \left( N^{-1/m} \right) \right).$$

More details about the value of $a_0$ are given in Proposition 4.1. It is a product of positive integers determined by the sub-complexes $Y|_{G_i}$ of $Y$ lying above $X_{G_i}$ when $G_i$ is a surface group. In Section 3 we prove Theorem 2.6 for a compact sub-cover of $X_{G_i}$ for any factor $G$ of $\Gamma$. In Section 4 we complete the proof of Theorem 2.6 for arbitrary compact sub-covers of $X_\Gamma$.

**Remark 2.7.** There is a subtle issue in the central notion of $\mathbb{E}^{\text{emb}}_Y(N)$. In case $Y$ has non-trivial automorphisms, there may be different injective lifts with the same image in the $N$-cover. We count them separately. To illustrate, consider the somewhat degenerate case that $\Gamma = C_2 = \langle x \rangle$, $X_\Gamma$ is the presentation complex of $\langle x \ | \ x^2 \rangle$, and $Y$ is the graph \( \bullet \xymatrix{ \ar[r] & \bullet \ar@/^/[l] \ar@/_/[l] } \bullet \). Every $N$-cover of $X_\Gamma$ consists of connected components corresponding to the trivial subgroup (copies of $Y$ together with two discs attached, so 2-spheres), and connected components corresponding to $\Gamma$ (copies of $X_\Gamma$). By Theorem 1.12, a random $N$-cover has in expectation $\sqrt{N} + O(1)$ copies of $X_\Gamma$, and thus $\frac{N-\sqrt{N}}{2} + O(1)$ copies of the 2-sphere, and thus $\frac{N-\sqrt{N}}{2} + O(1)$ disjoint embeddings of $Y$. However, there are two different embeddings of $Y$ in every copy of the 2-sphere, and so $\mathbb{E}^{\text{emb}}_Y(N) = N - \sqrt{N} + O(1)$. This agrees with Theorem 2.6, as $\pi_1^{\text{lab}}(Y) = \{ 1 \}$ and so $\chi^{\text{grp}}(Y) = 1$. See also Remark 3.8.

Below we will repeatedly use the following standard fact from the theory of covering spaces (e.g., [Hat05, Prop. 1.33 and 1.34]).

**Proposition 2.8 (Lifting criterion).** Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map and $f : (Y, y_0) \to (X, x_0)$ a map with $Y$ path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of $f$ exists if and only if $f_*(\pi_1(Y, y_0)) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. In this case, the lift is unique.

Along the proof of Theorem 2.6, we need the following construction. Let $p : Y \to X_\Gamma$ be a connected compact sub-cover. Choose an arbitrary vertex $y \in Y$. The fundamental group $\pi_1(X_\Gamma, p(y))$ (isomorphic to $\Gamma$, of course) acts on the universal cover $(\tilde{X}_\Gamma, \tilde{x}_0) \to (X_\Gamma, p(y))$ by deck transformations. Let $(Y, u) \overset{\text{def}}{=} \pi_1^{\text{lab}}(Y, y) \sqcup (\tilde{X}_\Gamma, \tilde{x}_0)$ be the quotient of $\tilde{X}_\Gamma$ by the action of the subgroup $\pi_1^{\text{lab}}(Y, y)$. So $(Y, u)$ is the covering space of $(X_\Gamma, p(y))$ corresponding to the subgroup $\pi_1^{\text{lab}}(Y, y)$, and $\pi_1(Y, u) \cong \pi_1^{\text{lab}}(Y, y)$ (see [Hat05, Thm. 1.38]). By Proposition 2.8, there exists a unique lift $\tilde{p} : (Y, y) \to (Y, u)$ of $p$. 

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Definition 2.9 (Universal lift). Let \( p: Y \to X_\Gamma \) be a compact sub-cover. If \( Y \) is connected, the lift \( \hat{p}: Y \to \hat{Y} = \pi_1^{\text{lab}}(Y) \setminus \hat{X}_\Gamma \) of \( p \) is called the universal lift of \( Y \). If \( Y \) is not necessarily connected, let \( Y_1, \ldots, Y_\ell \) be its connected components. The universal lift of \( Y \) is the map

\[
\hat{p}: Y \to \pi_1^{\text{lab}}(Y_1) \setminus \hat{X}_\Gamma \sqcup \ldots \sqcup \pi_1^{\text{lab}}(Y_\ell) \setminus \hat{X}_\Gamma
\]

which maps every connected component of \( Y \) to its own connected cover of \( X_\Gamma \).

Lemma 2.10. The universal lift of any sub-cover is injective.

Proof. It is enough to show injectivity for every connected component of \( Y \) separately (as each is mapped to a different connected component of the codomain). So we may assume that \( Y \) is connected. Let \( \hat{p}: Y \to \hat{Y} \) be its universal lift. By the definition of a sub-cover, there is an embedding \( f: Y \to Z \) into a (full) covering map \( \pi: Z \to X_\Gamma \). Choose an arbitrary vertex \( y \in Y \). Then \( \pi_1^{\text{lab}}(\hat{Y}, \hat{p}(y)) = \pi_1^{\text{lab}}(Y, y) \leq \pi_1^{\text{lab}}(Z, f(y)) \), and so there is a map \( \phi: (\hat{Y}, \hat{p}(y)) \to (Z, f(y)) \). The uniqueness of lifts guarantees that \( f = \phi \cdot \hat{p} \), and we get the following commuting diagram.

\[
\begin{array}{ccc}
(Y, \hat{p}(y)) & \xrightarrow{\phi} & (Z, f(y)) \\
\downarrow{\hat{p}} & & \downarrow{\pi} \\
(Y, y) & \xrightarrow{p} & (X_\Gamma, p(y))
\end{array}
\]

The injectivity of \( f \) now implies the one of \( \hat{p} \). \( \square \)

3 Sub-covers of the factors of \( \Gamma \)

3.1 Sub-covers of \( X_G \) for \( G \) a free group

Consider a compact sub-cover \( Y \) of \( X_\Gamma \) projecting entirely into \( X_G = X_{G_i} \), where \( G = G_i \) is a rank-\( r \) free group with basis \( B = \{ b_1, \ldots, b_r \} \). So \( Y \) is a finite directed graph, not necessarily connected, equipped with a graph immersion to \( X_G \), the bouquet with \( r \) loops. Equivalently, \( Y \) is a directed finite graph with edges labeled by \( b_1, \ldots, b_r \), and at every vertex, at most one incoming \( b_j \)-edge and at most one outgoing \( b_j \)-edge for every \( j \). We call such a graph a \textbf{B-labeled graph}. Such graphs are closely related to Stallings core graphs [Sta83] and more generally to multi core graphs [HP22], but they may contain leaves and/or isolated vertices.

It is straight-forward to compute \( \mathbb{E}_Y^{\text{emb}}(N) \), the expected number of embeddings of \( Y \) into a random \( N \)-sheeted cover of \( X_\Gamma \), and there is no need here in any fancy resolution. Recall that \((N)_t \overset{\text{def}}{=} N(N-1)\cdots(N-t+1)\) denotes the falling factorial.

Lemma 3.1. Let \( Y \) be a finite B-labeled graph, \( v(Y) \) be the number of vertices in \( Y \) and \( e_j(Y) \) the number of \( b_j \)-edges. Then for every \( N \geq \max_j e_j(Y) \),

\[
\mathbb{E}_Y^{\text{emb}}(N) = \frac{(N)_{v(Y)}}{\prod_{j=1}^{r} (N)_{e_j(Y)}}.
\]

Proof. As \( Y \) is a sub-cover of \( X_\Gamma \) sitting exclusively above a particular component \( X_G = X_{G_i} \), where \( G = G_i \cong \mathbb{F}_r \), is a free group, it is enough to consider random \( N \)-covers \( \hat{X} \) of \( X_G \). Then \( \hat{X} \) is given by \( N \) vertices, labeled \( 1, \ldots, N \), above the unique vertex \( v = v_i \) of \( X_G \). Above the \( b_j \)-loop at \( v \), there are \( N \) \( b_j \)-edges in \( \hat{X} \), which are given by a uniformly random permutation \( \sigma_j \in S_N \). The random permutations \( \sigma_1, \ldots, \sigma_r \) are independent.

The number of possible embeddings of the vertices of \( Y \) into \( \hat{X} \) is precisely \( (N)_{v(Y)} \). Any given embedding of the vertices of \( Y \) extends to an embedding of the entire of \( Y \) if and only if for every
Proof. If $Y$ is a $B$-labeled graph, then the restricted covering map $p: Y \to X_G$ is an immersion of graphs, and therefore $p_*$ is injective. In particular, in every connected component $Y_j$ of $Y$, we have $\pi_1(Y_j) \cong p_*(\pi_1(Y_j)) = \pi_{1,\text{lab}}(Y_j)$, and so $\chi^{\exp}(Y_i) = \chi(Y_i)$. Thus $\chi^{\exp}(Y) = \chi(Y) = v(Y) - \sum_{j=1}^r e_j(Y)$, and (3.1) follows from Lemma 3.1.

3.2 Sub-covers of $X_G$ for $G$ a finite group

Here we prove Proposition 1.2 about $E[\text{fix}_\gamma(N)]$ for a torsion element $\gamma$, as well as the special case of Theorem 2.6 concerning a sub-cover $Y$ of $X_G$ projecting entirely into $X_G = X_{G_i}$ where $G = G_i$ is some finite group. Recall that $X_G$ is some finite presentation complex of $G$. For every sub-cover $p: Y \to X_G$, define the set

$$\mathcal{R}_Y \overset{\text{def}}{=} \left\{ f: Y \to Z_f \left| \begin{array}{c} Z_f \text{ is a (full) covering of } X_G, \text{ and} \\ f(Y) \text{ meets every connected component of } Z_f \end{array} \right. \right\},$$

(3.2)

where $f$ is a morphism of sub-covers, namely, it commutes with the immersions into $X_G$. We also denote

$$\mathcal{R}^{\text{emb}}_Y \overset{\text{def}}{=} \left\{ f: Y \leftrightarrow Z_f \left| \begin{array}{c} Z_f \text{ is a (full) covering of } X_G, f \text{ is injective, and} \\ f(Y) \text{ meets every connected component of } Z_f \end{array} \right. \right\} \subseteq \mathcal{R}_Y,$$

(3.3)

Note that there may be distinct elements of $\mathcal{R}_Y$ or of $\mathcal{R}^{\text{emb}}_Y$ with the same codomain $Z_f$.

Proposition 3.3. Let $G$ be a finite group and $Y$ a compact sub-cover of $X_G$. Then,

1. the set $\mathcal{R}_Y$ from (3.2) is a finite resolution of $Y$, and
2. the set $\mathcal{R}^{\text{emb}}_Y$ from (3.3) is a finite embedding-resolution of $Y$.

Proof. For every element $f: Y \to Z_f$ in $\mathcal{R}_Y$, the number of components in $Z_f$ is bounded by the number of components of $Y$, and because the number of connected covers of $X_G$ is finite (equal to the number of conjugacy classes of subgroups of $G$), we get that there are finitely many possibilities for $Z_f$. As there are finitely many morphisms of sub-covers between two given compact sub-covers, we conclude that $\mathcal{R}_Y$ is finite, and thus so is its subset $\mathcal{R}^{\text{emb}}_Y$.

The set $\mathcal{R}_Y$ is a resolution because every morphism $h: Y \to \hat{X}$ to a covering space $\pi: \hat{X} \to X_G$ decomposes uniquely to a map from $Y$ to the connected components of $\hat{X}$ that $h(Y)$ meets, followed by the embedding of these components in $\hat{X}$. The same argument shows that $\mathcal{R}^{\text{emb}}_Y$ is an embedding-resolution.
Proposition 3.4. Theorem 2.6 holds for (full) covers of $X_G$ when $G$ is a finite group. Namely, let $Z$ be a compact (full) covering space of $X_G$. Denote $\mu = |G|$. Then there are rational numbers $a_t = a_t(Z)$ for $t = -1/\mu, -2/\mu, \ldots$ so that

$$\mathbb{E}_Z^{\text{emb}}(N) \xrightarrow{\text{asym. exp.}} N^{\chi_{\operatorname{top}}(Z)} \cdot \left\{ 1 + a_{-1/\mu} N^{-1/\mu} + a_{-2/\mu} N^{-2/\mu} + \ldots \right\}. \quad (3.4)$$

Furthermore, $a_t = 0$ for $0 > t > -\frac{1}{2}$, so

$$\mathbb{E}_Z^{\text{emb}}(N) = N^{\chi_{\operatorname{top}}(Z)} \left( 1 + O \left( N^{-1/2} \right) \right).$$

Proof. Denote by $v = v(Z)$ the number of vertices in $Z$. Recall that $X_G$ contains a single vertex and so every $N$-cover of it contains exactly $N$ vertices. In every embedding $h: Z \hookrightarrow X$ of $Z$ into an $N$-cover $X$ of $X_G$, $h(Z)$ contains $v$ out of the $N$ vertices of $X$. Moreover, as $Z$ is a full cover of $X_G$, every embedding of $h: Z \hookrightarrow X$ into an $N$-cover of $X_G$ has the property that $h(Z)$ and its complement are disconnected. So the embeddings of $Z$ in all the $N$-covers of $X_G$ are in bijection with the embeddings of the vertices of $Z$ into $[N]$ along with an arbitrary $(N-v)$-cover which “uses” the remaining vertices in $[N]$. As the number of embeddings of the vertices of $Z$ in $[N]$ is $(N)_v$, we obtain

$$\mathbb{E}_Z^{\text{emb}}(N) = (N)_v \cdot \frac{\left| \operatorname{Hom} \left( G, S_{N-v} \right) \right|}{\left| \operatorname{Hom} \left( G, S_N \right) \right|} \quad (3.5)$$

By [Müll97, Thm. 6] (stated as Theorem 1.12 above), we have that $\frac{\left| \operatorname{Hom} \left( G, S_N \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-1} \right) \right|}$ has asymptotic expansion with rational coefficients

$$\frac{\left| \operatorname{Hom} \left( G, S_N \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-1} \right) \right|} \xrightarrow{\text{asym. exp.}} N^{1-1/\mu} \cdot \left\{ 1 + Q_{-1/\mu} N^{-1/\mu} + Q_{-2/\mu} N^{-2/\mu} + \ldots \right\}.$$

Moreover, [Müll97, pp. 552] specifies the precise values of the $\mu + 3$ first coefficients $Q_{-1/\mu}, \ldots, Q_{-(\mu+3)/\mu}$ in this asymptotic expansion. In particular, for $1 \leq \nu \leq \mu - 1$, if $\nu - 1/\mu$ then $Q_{-\nu/\mu} = 0$. Therefore, $Q_t = 0$ for $0 > t > -\frac{1}{2}$. We conclude that the inverse has asymptotic expansion with rational coefficients:

$$\frac{\left| \operatorname{Hom} \left( G, S_{N-1} \right) \right|}{\left| \operatorname{Hom} \left( G, S_N \right) \right|} \xrightarrow{\text{asym. exp.}} N^{-1+1/\mu} \cdot \left\{ 1 + \beta_{-1/\mu} N^{-1/\mu} + \beta_{-2/\mu} N^{-2/\mu} + \ldots \right\},$$

where, here too, $\beta_t = 0$ for $0 > t > -\frac{1}{2}$. For any $j \in \mathbb{Z}$, we have

$$\frac{\left| \operatorname{Hom} \left( G, S_{N-j-1} \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-j} \right) \right|} \xrightarrow{\text{asym. exp.}} (N-j)^{-1+1/\mu} \cdot \left\{ 1 + \beta_{-1/\mu} (N-j)^{-1/\mu} + \beta_{-2/\mu} (N-j)^{-2/\mu} + \ldots \right\}$$

$$\xrightarrow{\text{asym. exp.}} N^{-1+1/\mu} \cdot \left\{ 1 + \beta_{-1/\mu} N^{-1/\mu} + \beta_{-2/\mu} N^{-2/\mu} + \ldots \right\}$$

where the second equality follows from Taylor’s theorem, applied to the function $(N-j)^t$ at the point $N$, and the $\beta_t(j)$’s are rational numbers. Moreover, the Taylor expansion of $(N-j)^t$ at $N$ is of the form $N^t + c_1 N^{t-1} j + c_2 N^{t-2} j^2 + \ldots$, so for $0 > t > -1$, $\beta_t(j) = \beta_t$. In particular, $\beta_t(j) = 0$ for $0 > t > -\frac{1}{2}$. Therefore,

$$\frac{\left| \operatorname{Hom} \left( G, S_{N-v} \right) \right|}{\left| \operatorname{Hom} \left( G, S_N \right) \right|} \xrightarrow{\text{asym. exp.}} \frac{\left| \operatorname{Hom} \left( G, S_{N-1} \right) \right|}{\left| \operatorname{Hom} \left( G, S_N \right) \right|} \cdot \frac{\left| \operatorname{Hom} \left( G, S_{N-2} \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-1} \right) \right|} \cdot \frac{\left| \operatorname{Hom} \left( G, S_{N-v} \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-v+1} \right) \right|} \cdot \frac{\left| \operatorname{Hom} \left( G, S_{N-v+2} \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-v+1} \right) \right|} \cdot \frac{\left| \operatorname{Hom} \left( G, S_{N-v+3} \right) \right|}{\left| \operatorname{Hom} \left( G, S_{N-v+2} \right) \right|} \cdot \ldots$$

$$\xrightarrow{\text{asym. exp.}} N^{-1+1/\mu} \cdot \left\{ 1 + \beta_{-1/\mu} N^{-1/\mu} + \beta_{-2/\mu} N^{-2/\mu} + \ldots \right\} \cdot$$

$$\cdot N^{-1+1/\mu} \cdot \left\{ 1 + \beta_{-1/\mu} N^{-1/\mu} + \beta_{-2/\mu} N^{-2/\mu} + \ldots \right\} \cdot$$

$$\cdot \ldots$$

$$\xrightarrow{\text{asym. exp.}} N^{-v+v/\mu} \cdot \left\{ 1 + \delta_{-1/\mu} N^{-1/\mu} + \delta_{-2/\mu} N^{-2/\mu} + \ldots \right\}.$$
Lemma 2.10, the role of 3.3. By Lemma 2.4, Proof. 

Thus by duplicating the EC of finite groups is positive, we obtain thus there is a morphism of covering spaces and denote \( \chi \) of \( \text{lab} \) and that \( \phi \) is an isometry. Thus \( \mathcal{Y} \) is an isometry if and only if \( \tau \) is an isomorphism if and only if \( \hat{\mathcal{Y}} \) is the universal lift of \( \gamma \) and \( |\gamma| \) meets every component of \( \mathcal{V} \). Together with (3.5), this proves there is an asymptotic expansion for \( \chi_{\text{grp}} \). Consider the universal lift \( \hat{\mathcal{Y}} \) of \( \mathcal{Y} \) and \( \chi_{\text{lab}} \). As \( \pi_{1}^{\text{lab}}(\mathcal{Y}) \approx \pi_{1}^{\text{lab}}(\mathcal{Y}) \), we get \( \chi_{\text{grp}}(\mathcal{Y}) = \chi_{\text{grp}}(\mathcal{Y}) \). It remains to show that for any other element \( \hat{\mathcal{Y}} \neq f \in \mathcal{R}_{\mathcal{Y}}^{\text{emb}} \) we have \( \chi_{\text{grp}}(\mathcal{Y}) \leq \chi_{\text{grp}}(\mathcal{Y}) \).

First, we may reduce to the case where each connected component of \( \mathcal{Y} \) is mapped to its own connected component of \( \mathcal{Z}_{f} \). Indeed, if there are two distinct components \( \mathcal{Y}_{1} \) and \( \mathcal{Y}_{2} \) of \( \mathcal{Y} \) which are mapped to the same component \( \mathcal{Z}_{o} \) of \( \mathcal{Z}_{f} \), we may reduce to some \( f' \in \mathcal{R}_{\mathcal{Y}}^{\text{emb}} \) with \( \mathcal{Z}_{f} \) having more connected components by duplicating \( \mathcal{Z}_{o} \) to two copies and mapping \( \mathcal{Y}_{1} \) to one copy and \( \mathcal{Y}_{2} \) to another copy. Using the fact that the EC of finite groups is positive, we obtain \( \chi_{\text{grp}}(\mathcal{Z}_{f}) < \chi_{\text{grp}}(\mathcal{Z}_{f}) \).

So now it is enough to assume that \( \mathcal{Y} \) is connected and prove that for \( f \in \mathcal{R}_{\mathcal{Y}}^{\text{emb}} \), we have \( \chi_{\text{grp}}(\mathcal{Z}_{f}) \leq \chi_{\text{grp}}(\mathcal{Y}) \) with equality if and only if \( f \) is the universal lift of \( \hat{\mathcal{Y}} \) to \( X_{\mathcal{Y}} \) and denote \( J = \pi_{1}^{\text{lab}}(\mathcal{Y}, y) = \pi_{1}^{\text{lab}}(\mathcal{Y}, \hat{\mathcal{Y}}(y)) \leq \mathcal{G} \). The existence of \( f \) yields that \( J \leq \pi_{1}^{\text{lab}}(\mathcal{Z}_{f}, f(y)) \), and thus there is a morphism of covering spaces \( \tau: (\mathcal{Y}, \hat{\mathcal{Y}}(y)) \to (\mathcal{Z}_{f}, f(y)) \). By the classification of covering spaces (e.g. [Hat05, Thm. 3.18]), \( \tau \) is an isomorphism if and only if \( J = \pi_{1}^{\text{lab}}(\mathcal{Z}_{f}, f(y)) \). So if \( f \neq \hat{\mathcal{Y}} \) we obtain \( J \leq \pi_{1}^{\text{lab}}(\mathcal{Z}_{f}, f(y)) \), and

\[
\chi_{\text{grp}}(\mathcal{Y}) = \chi(J) = \frac{1}{|J|} > \frac{1}{|\pi_{1}^{\text{lab}}(\mathcal{Z}_{f})|} = \chi(\pi_{1}^{\text{lab}}(\mathcal{Z}_{f})) = \chi_{\text{grp}}(\mathcal{Z}_{f}).
\]

We can now also prove Proposition 1.2 stating that for any torsion element \( \gamma \in \Gamma \), we have \( E[\text{fix}_{\gamma}(\mathcal{N})] = N^{1/|\gamma|} + O(N^{1/(2|\gamma|)}) \). Along the way we also prove the existence of asymptotic expansion, as in Theorem 1.13, for torsion elements of \( \gamma \).

Proof of Theorem 1.13 for torsion elements and of Proposition 1.2. Assume that \( G \) is finite, that \( \gamma \in G \) and \( \varphi \in \text{Hom}(G, S_{\mathcal{X}}) \) is uniformly random. Let \( p: (X_{\gamma}, x) \to (X_{\mathcal{G}}, v) \) be the connected covering space with \( \pi_{1}^{\text{lab}}(X_{\gamma}, x) = \langle \gamma \rangle \leq \mathcal{G} \). Consider the \( N \)-cover \( X_{\mathcal{X}} \) of \( X_{\mathcal{G}} \) corresponding to \( \varphi \), with vertices...
labeled by \( [N] \). Then \( \varphi (\gamma) \) fixes \( i \in [N] \) if and only if there is a lift of \( p \) to \( (X, i) \). Thus \( \mathbb{E} [\text{fix}_\gamma (N)] = \mathbb{E} \hat{X}_{\langle \gamma \rangle} (N) \). By Proposition 3.3, the set \( \mathcal{R} = \mathcal{R}_{\hat{X}_{\langle \gamma \rangle}} \) from (3.2) is a finite resolution for \( \hat{X}_{\langle \gamma \rangle} \), and by Lemma 2.4,

\[
\mathbb{E} [\text{fix}_\gamma (N)] = \mathbb{E} \hat{X}_{\langle \gamma \rangle} (N) = \sum_{f \in \mathcal{R}} \mathbb{E}^{\text{emb}}_{Z_f} (N). \tag{3.7}
\]

Theorem 1.13 for torsion elements, namely, the fact that \( \mathbb{E} [\text{fix}_\gamma (N)] \) has asymptotic expansion as in (1.10), now follows from (3.7) together with Proposition 3.4.

Note that the identity map \( \text{id} : \hat{X}_{\langle \gamma \rangle} \to \hat{X}_{\langle \gamma \rangle} \) belongs to \( \mathcal{R} \). As \( \chi (\langle \gamma \rangle) = \frac{1}{|\gamma|} \), this element of \( \mathcal{R} \) satisfies

\[
\mathbb{E}^{\text{emb}}_{\hat{X}_{\langle \gamma \rangle}} (N) = N^{1/|\gamma|} \left( 1 + O \left( N^{-1/2} \right) \right) = N^{1/|\gamma|} + O \left( N^{1/|\gamma| - 1/2} \right)
\]

by Proposition 3.4. Note that \( \frac{1}{|\gamma|} - \frac{1}{2} \leq \frac{1}{2|\gamma|} \). It is left to show that for every other element \( \text{id} \neq f \in \mathcal{R} \), \( \chi^{\text{grp}} (Z_f) \leq \frac{1}{2|\gamma|} \). Indeed, as \( \hat{X}_{\langle \gamma \rangle} \) is connected, so is \( Z_f \) for every \( f \in \mathcal{R} \). As \( f \) is a lift of \( p \) but \( f \neq \text{id} \), we have \( \langle \gamma \rangle \leq \pi_1^{\text{lab}} (Z_f, f (x)) \). Thus

\[
\chi^{\text{grp}} (Z_f) = \frac{1}{|\pi_1^{\text{lab}} (Z_f, f (x))|} = \frac{1}{|\gamma| \cdot |\pi_1^{\text{lab}} (Z_f, f (x)) : \langle \gamma \rangle|} \leq \frac{1}{2 |\gamma|}.
\]

\[ \square \]

**Remark 3.6.** In Section 1 we claimed that torsion elements \( \gamma \in \Gamma \) satisfy also (1.5), namely, that

\[
\mathbb{E} [\text{fix}_\gamma (N)] = N^{1/|\gamma|} \left( 1 + O \left( N^{-1/m} \right) \right). \tag{3.8}
\]

Indeed, if \( \gamma \) is conjugated into the finite group \( G \) (one of the factors of \( \Gamma \)), then \( m (\Gamma) \geq m (G) = |G| \), and we may thus assume that \( m = |G| \). If \( |\gamma| = |G| \) then \( \langle \gamma \rangle = G \), namely, \( \gamma \) does not belong to any proper subgroup of \( G \). Thus, the only element of the resolution \( \mathcal{R} \) from the last proof is \( \hat{X}_{\langle \gamma \rangle} \), and \( \mathbb{E} [\text{fix}_\gamma (N)] = \mathbb{E}^{\text{emb}}_{\hat{X}_{\langle \gamma \rangle}} (N) = N^{1/|\gamma|} \left( 1 + O \left( N^{-1/2} \right) \right) \). This yields (3.8) as \( m \geq 2 \). Finally, if \( |\gamma| \leq \frac{m}{2} \), and (3.8) follows immediately from Proposition 1.2.

**Example 3.7.** Let \( G = C_4 = \langle x \rangle \) be the cyclic group of size 4 generated by \( x \), and consider the element \( x^2 \). There are two subgroups containing \( x^2 \): \( \langle x \rangle \) and \( \langle x^2 \rangle \), with corresponding coverings spaces \( \hat{X}_{\langle x \rangle} \) and \( \hat{X}_{\langle x^2 \rangle} \). The computations appearing above together with the some precise values of coefficients from [Mül97, p. 552], yield

\[
\mathbb{E}^{\text{emb}}_{\hat{X}_{\langle x \rangle}} \sim \text{asym. exp.} N^{1/4} \cdot \left\{ 1 - \frac{1}{4} N^{-1/2} - \frac{1}{4} N^{-3/4} + \ldots \right\}
\]

\[
\mathbb{E}^{\text{emb}}_{\hat{X}_{\langle x^2 \rangle}} \sim \text{asym. exp.} N^{1/2} \cdot \left\{ 1 - \frac{1}{2} N^{-1/2} - \frac{3}{4} N^{-1/4} + \ldots \right\},
\]

so

\[
\mathbb{E} [\text{fix}_{x^2} (N)] \sim \text{asym. exp.} N^{1/2} + N^{1/4} - \frac{1}{2} - \frac{3}{4} N^{-1/4} + \ldots.
\]

**Remark 3.8.** Some of the results of this subsection 3.2 also follow from [MSP10]. Let \( p : Z \to X_G \) be a connected (full) cover with \( m_1 \) vertices, and let \( m = |G| \), so \( \chi^{\text{grp}} (Z) = \frac{m_1}{m} \). Then [MSP10, Lem. 4] states that \( \mathbb{E}^{\text{emb}}_{\hat{Z}} (N) - N^{m_1/m} \) converges in distribution to a standard Gaussian \( \mathcal{N} (0, 1) \). Note that the statement of that lemma wrongly implies that what is being counted is the number of disjoint copies of \( Z \) in a random \( N \)-cover, whereas what is actually being counted there is \( \mathbb{E}^{\text{emb}}_{\hat{Z}} (N) \), namely the number of disjoint copies times \( |\text{Aut} (Z)| \), the number of automorphisms of \( Z \) as a covering map. See also Remark 2.7.
3.3 Sub-covers of $X_G$ for $G$ a surface group

We now assume that $G = \Lambda_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ with $g \geq 2$. Recall from Section 2.1 that $X_G$ is an orientable surface of genus $g$ endowed with a CW-structure of a single vertex $v$, $2g$ edges labeled $a_1, \ldots, b_g$, and a single 2-cell, which we think of as a $4g$-gon as its boundary is attached to a cycle of $4g$ edges. A sub-cover of $X_G$ is also called a tiled surface in [MP22, MP20]. See also [MP22, Prop. 3.3] for an intrinsic definition of a tiled surface.

We now introduce some further terminology from [MP22, MP20]. The definitions are laconic as they are only used in order to state some results from these two papers, and let $Y \subseteq Z$ be a sub-cover which is a subcomplex of the (full) covering space $p: Z \to X_G$. As $Y$ is embedded in a surface, we may take a small closed regular neighborhood of $Y$ in $Z$ and obtain the “thick version” of $Y$ which is a surface, possibly with boundary. The thick version of $Y$, which we sometimes denote by $\mathcal{Y}$, is a feature of $Y$ as a sub-cover, and does not depend on the particular $Z$ it is embedded in – see [MP22, Sec. 3.1]. We write $\partial Y$ for the boundary of the thick version of $Y$. This boundary is a finite collection of cycles. We pick an orientation on every boundary component (see below) to obtain a boundary cycle of $\mathcal{Y}$, and using the edge-labels along a boundary cycle, it corresponds to some cyclic word in the generators of $G$.

Every full cover $Z$ of $X_G$ consists of vertices, directed edges labeled by $a_1, \ldots, b_g$, and $4g$-gons. The cycle around every $4g$-gon reads the relation – the cyclic word $R = [a_1, b_1] \cdots [a_g, b_g]$. A boundary cycle of a sub-cover $Y$ is always oriented so that if $Y$ is embedded in the full cover $Z$, the cycle reads successive segments of the boundaries of the neighboring $4g$-gons (in $Z \setminus Y$) with the orientation of each $4g$-gon coming from $[a_1, b_1] \cdots [a_g, b_g]$ (and not from the inverse word).

If a boundary cycle of a sub-cover $Y$ contains a subword of $R$ of length $>\frac{1}{2} |R| = 2g$, then in every full cover $Z$ in which $Y$ is embedded, one may shorten the total boundary of $Y$ by annexing the $4g$-gon neighboring this subword. In this sense the boundary of $Y$ is not “reduced”. We call a subword of $R$ of length $\geq 2g + 1$ a long block.

There are further, more involved cases involving a sequence of a few consecutive $4g$-gons where $\partial Y$ is not reduced. For example, if $g = 2$ and $\partial Y$ contains the subword $a_1 b_1 a_1^{-1} b_1^{-1} a_1 b_1 a_1^{-1} b_1^{-1} a_2$ then there are three consecutive octagons neighboring this subword, and annexing them strictly reduces the total length of $\partial Y$. Such subwords are called long chains – see [MP22, Sec. 3.2] for the precise definition. This leads to the following definitions.

Definition 3.9 (Boundary reduced). [MP22, Def. 4.1] Let $G = \Lambda_g$ with $g \geq 2$. A sub-cover $Y$ of $X_G$ is called boundary reduced, or BR for short, if $\partial Y$ contains no long blocks nor long chains.

If $\partial Y$ contains a subword which constitutes half of the relation $R$, called a half-block, then in every full cover $Z$ in which $Y$ is embedded, the neighboring $4g$-gon can be annexed to $Y$ without increasing the total length of $\partial Y$. Likewise, there are cycles called half-chains so that annexing the sequence of consecutive $4g$-gons along them does not increase the length of the boundary. Again, see [MP22, Sec. 3.2] for the precise definitions.

Definition 3.10 (Strongly boundary reduced). [MP22, Def. 4.2] Let $G = \Lambda_g$ with $g \geq 2$. A sub-cover $Y$ of $X_G$ is called strongly boundary reduced, or SBR for short, if $\partial Y$ contains no half-blocks nor half-chains.

As explained in [MP22, Sec. 4], every SBR sub-cover is, in particular, BR. The case of Theorem 2.6 dealing with sub-covers of $X_G$ (where $G = \Lambda_g$ is a surface group), crucially relies on the following results from [MP20].

Theorem 3.11. [MP20] Let $Y$ be a compact sub-cover of $X_G$ where $G = \Lambda_g$ with $g \geq 2$.

1. If $Y$ is BR, there are rational number $a_t = a_t(Y)$ for $t = 0, -1, -2, \ldots$ with $a_0 > 0$, so that

$$\mathbb{P}^\text{emb}_{\mathcal{Y}}(N) \overset{\text{asym.}}{\sim} \exp. N^{a_t(Y)} \cdot \{a_0 + a_{-1}N^{-1} + a_{-2}N^{-2} + \ldots\}. \quad (3.9)$$
2. If $Y$ is moreover SBR, then $a_0 (Y) = 1$.

Although they probably should have been, these results are not written explicitly in [MP20]. However, they follow immediately from the results therein. In fact, as explained in [MP20, Sec. 1.6 and 5.1], the results of that paper immediately give (3.9) with $\chi^{\text{grp}} (Y)$ replaced with $\chi (Y)$. But then [MP22, Lem. 5.6] shows that if $Y$ is compact and BR, then $\chi^{\text{grp}} (Y) = \chi (Y).

**Theorem 3.12.** ([MP20, Thm. 2.14]) Let $Y$ be a compact sub-cover of $X_G$ where $G = \Lambda_g$ and let $\chi_0 \in \mathbb{Z}$. Then $Y$ admits a finite resolution $\mathcal{R} = \mathcal{R} (Y, \chi_0)$ such that for every $f : Y \to Z_f$ in $\mathcal{R}$, the following properties hold:

(i) the sub-cover $Z_f$ is compact and BR,
(ii) if $\chi^{\text{grp}} (Z_f) \geq \chi_0$, then $Z_f$ is SBR, and
(iii) the image of $f$ meets every connected component of $Z_f$.

The original statement of [MP20, Thm. 2.14] states the second condition as $\chi (Z_f) \geq \chi_0$, but as mentioned above, for compact BR sub-covers, $\chi (Z_f) = \chi^{\text{grp}} (Z_f)$. Part (iii) is not mentioned in ibid, but it follows from the specific construction of $\mathcal{R}$ in [MP20, Def. 2.13].

**Corollary 3.13.** Let $Y$ be compact sub-cover of $X_G$ where $G = \Lambda_g$ and let $\chi_0 \in \mathbb{Z}$. Then $Y$ admits a finite embedding-resolution $\mathcal{R}^{\text{emb}} = \mathcal{R}^{\text{emb}} (Y, \chi_0)$ for the injective lifts of $Y$ to (full) covers of $X_G$, and with the same three properties as in Theorem 3.12.

**Proof.** Take the subset of $\mathcal{R} (Y, \chi_0)$ from Theorem 3.12 consisting of all injective morphisms.

Let $\tilde{\Sigma}_g$ be the universal cover of the genus-$g$ orientable closed surface $\Sigma_g$, endowed with the CW-complex structure pulled-back from $X_G \cong \Sigma_g$. For every subgroup $J \leq \Lambda_g$, the corresponding covering space is $\bigcup \tilde{\Sigma}_g$ (see [MP22, Example 3.5]).

**Lemma 3.14.** If $f : Y \to Z$ is an embedding of compact sub-covers of $X_G$ with $G = \Lambda_g$ such that $f (Y)$ meets every component of $Z$, then $\chi^{\text{grp}} (Z) \leq \chi^{\text{grp}} (Y)$.

**Proof.** Let $Z_1, \ldots, Z_s$ be the connected components of $Z$ with $z_j \in Z_j$ some vertex. Denote $H_j = \pi_1^{\text{lab}} (Z_j, z_j)$, $Y_j = H_j \backslash \tilde{\Sigma}_g$ and $Y = Y_1 \cup \ldots \cup Y_s$. Then the universal lift of $Z$ has codomain $Y$ and $Z$ is embedded in $Y$ by Lemma 2.10. We may think of $f$ as an embedding of $Y$ inside $\tilde{\Sigma}_g$. Consider the thick part $\mathcal{Y}$ of $f (Y)$ in $\tilde{\Sigma}_g$, with $\mathcal{Y}_1, \ldots, \mathcal{Y}_t$ its connected components, and denote by $C_1, \ldots, C_q$ the connected components of the complement $\tilde{\Sigma}_g \setminus \mathcal{Y}$. We denote by $\overline{C}_i$ the closure of the component $C_i$, and the fact it is a component in the complement of $\mathcal{Y}$ and not of $f (Y)$ guarantees that $\overline{C}_i = C_i$. As $\pi_1^{\text{lab}} (Y_i) = \pi_1^{\text{lab}} (\mathcal{Y}_i)$, it is enough to prove that $\chi^{\text{grp}} (\mathcal{Y}) \leq \chi^{\text{grp}} (Y)$.

We may assume that none of the $\mathcal{Y}_i$'s and none of the $\overline{C}_j$'s are discs. Indeed, if some $\overline{C}_j$ is a disc, then we can replace $\mathcal{Y}$ with $\mathcal{Y} \cup C_i$: this does not change $\pi_1^{\text{lab}} (\mathcal{Y})$ nor $\chi^{\text{grp}} (\mathcal{Y})$. If any $\mathcal{Y}_i$ is a disc, then it is connected to a single $\overline{C}_j$. Assume that $\mathcal{Y}_j \subseteq \mathcal{Y}_t$. If $\mathcal{Y}_i \cup \overline{C}_j = \mathcal{Y}_t$, we may reduce to the case where this part is ignored completely, for $H_t \leq \Lambda_g$ and so $\chi^{\text{grp}} (Y_t) = 1 \geq \chi (H_t)$. If there are additional parts in $\mathcal{Y}_t$, we may reduce to the case where we remove $\mathcal{Y}_i$ from $\mathcal{Y}$ and replace $\overline{C}_j$ with $\mathcal{Y}_i \cup \overline{C}_j$, for then $\chi^{\text{grp}} (\mathcal{Y})$ is decreased by one and $\chi^{\text{grp}} (\mathcal{Y})$ does not change.

We obtained a decomposition of the space $\mathcal{Y}$ to a graph of spaces with vertex-spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_t, \overline{C}_1, \ldots, \overline{C}_q$ and all edge groups isomorphic to $Z$ (every edge connects some $\mathcal{Y}_i$ with some $\overline{C}_j$ and corresponds to some boundary component of $\mathcal{Y}$). As the vertex-spaces are not-a-disc surfaces and are embedded in hyperbolic surfaces, they have non-trivial fundamental groups. Furthermore, in every connected surface $S$ with boundary which is not a disc, the cyclic fundamental group of every boundary component is embedded in $\pi_1 (S)$. Thus, all edge-groups (which are infinite cyclic) are embedded in the corresponding vertex groups. By Bass-Serre theory of graph of groups, this means that $\pi_1 (\mathcal{Y}_i)$ is embedded in $H_t$ whenever $\mathcal{Y}_i \subseteq \mathcal{Y}_t$. If $\pi : \mathcal{Y}_t \to X_G$ is the covering map, then $\pi_* : \pi_1 (\mathcal{Y}_t) \to \pi_1 (X_G)$ is injective, which yields that so is $\pi_* \circ f_* : \pi_1 (\mathcal{Y}_i) \to \pi_1 (X_G)$. Thus $\pi_1^{\text{lab}} (\mathcal{Y}_i) \cong \pi_1 (\mathcal{Y}_i)$ and $\chi^{\text{grp}} (\mathcal{Y}_i) = \chi (\pi_1 (\mathcal{Y}_i))$. Finally, because all edge
groups are $\mathbb{Z}$ and all vertex groups of the graph of spaces are non-trivial groups with non-positive EC, we get

$$\chi^{\text{grp}}(Z) = \chi(H_1) + \ldots + \chi(H_s) = \chi(\pi_1(Y)) =$$

$$= \sum_{i=1}^{\ell} \chi(\pi_1(Y_i)) + \sum_{j=1}^{q} \chi(\pi_1(\overline{C}_i)) - \sum_{e \text{ edge of graph of spaces}} \chi(Z) \quad (3.10)$$

$$= \chi^{\text{grp}}(Y) + \sum_{i=1}^{q} \chi(\pi_1(\overline{C}_i)) \leq \chi^{\text{grp}}(Y). \quad (3.11)$$

We can now extend (3.9) to arbitrary sub-covers of $X_G$.

**Corollary 3.15.** Theorem 2.6 holds for sub-covers of $X_G$ when $G = \Lambda_g$ is a surface group. Namely, for every compact sub-cover $Y$ of $X_G$ there are rational numbers $a_t = a_t(Y)$ for $t = 0, -1, -2, \ldots$ so that

$$E_Y^{\text{emb}}(N) \overset{\text{asym., exp.}}{\sim} N^{\chi^{\text{grp}}(Y)} \cdot \{a_0 + a_{-1}N^{-1} + a_{-2}N^{-2} + \ldots\},$$

where $a_0 \in \mathbb{Z}_{\geq 1}$ is a positive integer.

**Proof.** Let $Y$ be an arbitrary compact sub-cover of $X_G$. Set $\chi_0 = \chi^{\text{grp}}(Y)$ and let $\mathcal{R}^{\text{emb}} = \mathcal{R}^{\text{emb}}(Y, \chi_0)$ be a finite embedding-resolution as in Corollary 3.13. By Lemma 3.14, $\chi^{\text{grp}}(Z_f) \leq \chi^{\text{grp}}(Y)$, and as $E_Y^{\text{emb}}(N) = \sum_{f \in \mathcal{R}^{\text{emb}}} E_{Z_f}^{\text{emb}}(N)$, it follows from Theorem 3.11 that $E_Y^{\text{emb}}(N)$ admits an asymptotic expansion as in (3.12), with some $a_0 \in \mathbb{Q}_{\geq 0}$. As every $f \in \mathcal{R}^{\text{emb}}$ with $\chi^{\text{grp}}(Z_f) \geq \chi_0 = \chi^{\text{grp}}(Y)$ is SBR, we get from Theorem 3.11(2) that each such $f$ contributes 1 to $a_0$ and so, in fact, $a_0 \in \mathbb{Z}_{\geq 0}$. It is thus left to show that there is an element of $\mathcal{R}^{\text{emb}}$ with $\chi^{\text{grp}}(Z_f) = \chi^{\text{grp}}(Y)$.

Let $\hat{p}: Y \to \Upsilon$ be the universal lift from Definition 2.9, which is injective by Lemma 2.10. By the definition of an embedding-resolution, this embedding $\hat{p}$ decomposes as

$$Y \xrightarrow{f} Z_f \xrightarrow{g} \Upsilon,$$

for some $f \in \mathcal{R}^{\text{emb}}$. Of course, $Z_f$ has the same number of connected components as $Y$ (and $\Upsilon$). For each connected component $Y_i$ of $Y$ with $y_i \in Y_i$ a vertex, we have

$$H_i \overset{\text{def}}{=} \pi_1^{\text{lab}}(Y_i, y_i) \leq \pi_1^{\text{lab}}(Z_f, f(y_i)) \leq \pi_1^{\text{lab}}(\Upsilon, \hat{p}(y_i)) = H_i$$

and so $\pi_1^{\text{lab}}(Z_f, f(y_i)) = H_j$. In particular, $\chi^{\text{grp}}(Z_f) = \chi(H_1) + \ldots + \chi(H_\ell) = \chi^{\text{grp}}(Y).$ \qed

**Example 3.16.** Figure 3.1 illustrates two different SBR sub-covers $Z_1$ and $Z_2$ in a possible resolution of a particular (BR) sub-cover $Y$. One of them is a torus with one boundary component, while the other is a pair of pants. In this example, $\pi_1^{\text{lab}}(Y) \cong \mathbb{F}_2$ and $\chi^{\text{grp}}(Y) = -1$. Both $Z_1$ and $Z_2$ have, too, $\chi(Z_1) = \chi(Z_2) = -1$. In fact, in an embedding-resolution $\mathcal{R}^{\text{emb}}$ of $Y$ which contains $Z_1$ and $Z_2$, they must be the only elements of $\text{EC} = -1$. This shows that $a_0(Y) = 2$, namely,

$$E_Y^{\text{emb}}(N) \overset{\text{asym., exp.}}{\sim} N^{-1} \cdot \{2 + a_{-1}N^{-1} + a_{-2}N^{-2} + \ldots\}.$$

To end this section, we characterize sub-covers $Y$ where $a_0(Y) = 1$. The characterization is stated in Proposition 3.19, and uses the following lemma (which could have fit in the paper [MP22] better than the current one).

**Lemma 3.17.** Let $Y \hookrightarrow Z$ be a SBR sub-cover $Y$ embedded in a full cover $Z$. Then there is no boundary component of $Y$ bounding a disc in $Z \setminus Y$, nor is there a pair of boundary components of $Y$ bounding an annulus in $Z \setminus Y$. 

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Figure 3.1: On the left hand side is a BR sub-cover $Y$ of $X_G$ where $G = \Lambda_2 = \langle a, b, c, d \mid [a, b] [c, d] \rangle$. This sub-cover consists of two hexagons connected by an additional edge, and it satisfies $\chi(Y) = \chi_{\text{grp}}(Y) = -1$. On the right there are two distinct SBR sub-covers in which $Y$ is embedded, and which can serve as part of an embedding-resolution of $Y$. Both of these have $\chi = \chi_{\text{grp}} = -1$ and $\pi_1^{\text{lab}} \cong F_2$, yet the upper one has $\pi_1^{\text{lab}} = \pi_1^{\text{lab}}(Y)$, while the bottom one has $\pi_1^{\text{lab}}$ which is an HNN-extension of $\pi_1^{\text{lab}}(Y)$.

Proof. If some boundary cycle $C$ of $Y$ bounds a disc, then $C$ spells a word which is equal to the trivial word in $\Lambda_g$. By the classical results of Dehn [Deh12], $C$ must contain a long block, contradicting the assumption that $Y$ is SBR.

Now assume that $C_1$ and $C_2$ are two boundary cycles of $Y$ bounding an annulus of $Z \setminus Y$. They both represent the same free-homotopy class in $Z$, and they are not null-homotopic (otherwise we get once again a contradiction to [Deh12]). One of the key features of SBR sub-covers such as $Y$ is that given a non-nullhomotopic loop $C$ in its 1-skeleton $Y^{(1)}$, one can greedily shorten $C$ by replacing a long block along some $4g$-gon with its complement on the other side of this $4g$-gon. Then, any two shortest representatives of the free homotopy class of $C$ can be obtained one from the other by “half-block switches” or a “half-chain switch” (see [MP22, Sec. 4]). All these switches take place inside $Y$. We conclude that $C_1$ and $C_2$ are freely-homotopic inside $Y$, which means that $Y$ is topologically an annulus, and $Z$ a genus-1 torus (or a Klein bottle). This contradicts the fact that $Z$ is a covering space of a genus-$g$ surface with $g \geq 2$.

Definition 3.18 (Matching boundary cycles). We say that two different boundary cycles $\gamma_1$ and $\gamma_2$ of a sub-cover $Y$ are matching if (i) there is an embedding $f: Y \hookrightarrow Z$ into a (full) cover $Z$ of $X_G$ such that one of the connected components of $Z \setminus f(Y)$ is an annulus bounded by $\gamma_1$ and $\gamma_2$, and (ii) $\gamma_1$ and $\gamma_2$ do not represent the trivial element of $\Lambda_g$.

Proposition 3.19. Let $G = \Lambda_g$ and let $p: Y \rightarrow X_G$ be a compact sub-cover. Then in the asymptotic expansion (3.12), $a_0(Y) = 1$ if and only if $Y$ does not admit matching boundary cycles.

Proof. Let $f: Y \hookrightarrow Z$ be an embedding of $Y$ into a full cover of $X_G$ with $f(Y)$ meeting every component of $Z$. As in the proof of Lemma 3.14, denote by $\overline{C_1}, \ldots, \overline{C_q}$ the connected components of $Z \setminus Y$. By that same proof, $\chi_{\text{grp}}(Z) = \chi_{\text{grp}}(Y)$ if and only if (i) every connected component $Y_i$ of $Y$ with trivial $\pi_1^{\text{lab}}(Y_i)$ is embedded in its own connected component $Z_j$ of $Z$ with $\pi_1^{\text{lab}}(Z_j) = 1$, and (ii) every $\overline{C_i}$ is either a disc or an annulus.
Now let $\mathcal{R}^{\mathrm{emb}} = \mathcal{R}^{\mathrm{emb}}(Y, \chi^{\mathrm{grp}}(Y))$ be the embedding-resolution from Corollary 3.13. Let $\hat{f} \in \mathcal{R}^{\mathrm{emb}}$ be the element taking part in the decomposition of the universal lift $\hat{p} : Y \hookrightarrow \Upsilon$ from Definition 2.9. As in the proof of Corollary 3.15, $\pi_1^{\mathrm{lab}}\left(Z_f\right) = \pi_1^{\mathrm{lab}}(Y)$ and $\chi^{\mathrm{grp}}\left(Z_f\right) = \chi^{\mathrm{grp}}(Y)$. So this $\hat{f}$ contributes 1 to the coefficient $a_0$ from (3.12). Note that if a pair of matching boundary cycles is realized in some embedding of $Y$ in a full cover, then $\pi_1^{\mathrm{lab}}$ of the codomain strictly contains that of $Y$ (it contains a non-trivial amalgamated product or HNN extension of $\pi_1^{\mathrm{lab}}(Y)$). In particular, $\hat{p} : Y \hookrightarrow \Upsilon$ does not realize any pair of matching boundary cycles.

Now assume that $a_0 \geq 2$, namely, that there exists another element $\hat{f} \neq g \in \mathcal{R}^{\mathrm{emb}}$ with $\chi^{\mathrm{grp}}(Z_g) = \chi^{\mathrm{grp}}(Y)$. Let $\overline{\varphi} : Y \to \Upsilon_g$ be the composition of $g$ with the universal lift of $Z_g$ (we let $\Upsilon_g$ denote the codomain of this lift). By the uniqueness in the definition of a resolution, $Z_g$ does not embed into $\Upsilon$ in a way compatible with the universal lift $\hat{p}$. So for some component $(Y_i, y_i)$ of $Y$, we must have $\pi_1^{\mathrm{lab}}(\Upsilon_g, f(y_i)) = \pi_1^{\mathrm{lab}}(Z_g, f(y_i)) \geq \pi_1^{\mathrm{lab}}(Y_i, y_i)$. But this can only happen, by the first paragraph of this proof applied to $\overline{\varphi}$, if some $C_i$ is an annulus which does not border any components of $Y$ with trivial $\pi_1^{\mathrm{lab}}$. This precisely means that $\overline{\varphi}$ realizes some pair of matching boundary cycles.

Conversely, if $Y$ admits a pair of matching boundary cycles, we may consider the embedding $f : Y \hookrightarrow Z$ from Definition 3.18 that realizes this pair. Let $C$ be the connected component of $Z - f(Y)$ which is an annulus bounded by the matching pair. Let $Y' = f(Y) \cup C \subseteq Z$. Then $\chi^{\mathrm{grp}}(Y') = \chi^{\mathrm{grp}}(Y)$. Let $h' : Y' \hookrightarrow \Upsilon'$ be the universal lift of $Y'$, and $j : Y \hookrightarrow \Upsilon'$ the resulting embedding of $Y$ in $\Upsilon'$. Then by the definition of $\mathcal{R}^{\mathrm{emb}}$, $j$ decomposes through some $g \in \mathcal{R}^{\mathrm{emb}}$,

$$Y \xrightarrow{\overline{\varphi}} Z_g \hookrightarrow \Upsilon'.$$

By Lemma 3.17, $Z_g$ must contain $Y'$. Thus $\pi_1^{\mathrm{lab}}(Y, y) \leq \pi_1^{\mathrm{lab}}(Z_g, j(y))$ for any vertex $y \in Y$. But $\pi_1^{\mathrm{lab}}(Y, y) = \pi_1^{\mathrm{lab}}\left(Z_f, \hat{f}(y)\right)$, and so $g \neq \hat{f}$. As $\chi^{\mathrm{grp}}(Z_g) = \chi^{\mathrm{grp}}(Y)$, we obtain that $a_0(Y) \geq 2$. \hfill \qed

Corollary 3.20. In the following cases, a compact sub-cover $Y$ of $X_G$ satisfies $a_0(Y) = 1$:

1. $\pi_1^{\mathrm{lab}}(Y_i)$ is trivial for every connected component $Y_i$ of $Y$.
2. $Y$ is a single topological annulus.
3. $Y$ is a disjoint union of several copies of the same topological annulus.
4. No two different boundary cycles $\gamma_1$ and $\gamma_2$ of $Y$ satisfy that $\gamma_1$ is conjugate to $\gamma_2$ or to $\gamma_2^{-1}$.
5. $Y$ is a disjoint union of topological annuli, where every two are either identical or have non-conjugate boundary cycles.
6. $Y$ is SBR.

Proof. Any matching pair of boundary cycles consists of boundary cycles corresponding to a non-trivial element of $\Lambda_g$, so part 1 follows immediately from Proposition 3.19. If an annulus has a matching pair of boundary cycles, then by definition, it can be embedded in a genus-one torus, which is impossible as a torus cannot cover $X_G$, and part 2 follows.

Now assume that $Y$ is a disjoint union of several copies of $A$, where the thick version $A$ of $A$ is an annulus with boundary cycles $\gamma_1$ and $\gamma_2$. If $\pi_1^{\mathrm{lab}}(A) = 1$ we reduce to part 1, so assume otherwise. Note that $A$ has a mirror symmetry swapping $\gamma_1$ and $\gamma_2$ if and only if it is a 1-dimensional simple cycle. Assume towards contradiction that $Y$ admits a pair of matching boundary cycles. If this pair involves one copy of $\gamma_1$ and one of $\gamma_2$, we can use the same annulus bounded between them to connect the two boundary components of the same copy of $A$ and thus obtain a torus which is a legitimate covering space of $X_G$ (see [MP22, Prop. 4.3]), which, as before, is impossible. If the matching involves two copies of $\gamma_1$ then, as no non-trivial element of $\Lambda_g$ is conjugate to its inverse, these two copies of $\gamma_1$ bound an annulus so that they
have matching orientations. If \( A \) has mirror symmetry, we return to the previous case where the matching involves \( \gamma_1 \) and \( \gamma_2 \). Otherwise, we get that on the same covering space of \( X_G \) we have two copies of \( A \) with opposite orientations, which is impossible (for example, the cyclic order of the half-edges at every vertex is determined by the edge-labels alone – see [MP22, Prop. 3.4]). This shows part 3.

Part 4 is immediate from Proposition 3.19 and the fact that the two boundary components of an annulus inside a cover of \( X_G \) must represent conjugates in \( \Lambda_g \). Part 5 follows from combining the arguments of parts 3 and 4. Finally, part 6 is Theorem 3.11(2).

Let us stress that part 6 also falls under the content of Proposition 3.19. Indeed, if \( Y \) is SBR and is embedded in a full cover \( Z \), then no connected component of \( Z \setminus Y \) is an annulus bounded by two non-nullhomotopic cycles of \( Z \), by Lemma 3.17.

### 4 Sub-covers of \( X_\Gamma \): expectations and asymptotic expansion

We can now prove Theorem 2.6 for an arbitrary compact sub-cover \( Y \) of \( X_\Gamma \). Namely, for \( m = m(\Gamma) \), we show that there are rational numbers \( a_t = a_t(Y) \) for \( t = 0, -\frac{1}{m}, -\frac{2}{m}, -\frac{3}{m}, \ldots \) so that

\[
E^{\text{emb}}_Y (N)^{\text{asym. exp.}} N^{\chi^{\text{grp}}(Y)} \cdot \left\{ a_0 + a_{-1/m}N^{-1/m} + a_{-2/m}N^{-2/m} + \ldots \right\},
\]

with \( a_0 \in \mathbb{Z}_{\geq 1} \).

Moreover, we need to show that whenever there are no surface groups involved, \( a_0 = 1 \). We show a bit more. Recall from Definition 2.1 that \( Y|_{G_i} \) denotes the subcomplex of \( Y \) sitting above \( X_{G_i} \) for \( i = 1, \ldots, k \). Also recall that Corollaries 3.2, 3.5 and 3.15 already established Theorem 2.6 for sub-covers of \( X_G \) where \( G \) is any single factor of \( \Gamma \).

**Proposition 4.1 (Addendum to Theorem 2.6).** For a compact sub-cover \( Y \) of \( X_\Gamma \), let \( a_0^{(i)} \in \mathbb{Z}_{\geq 1} \) denote the leading coefficient (the coefficient of \( N^{\chi^{\text{grp}}(Y|_{G_i})} \)) in the asymptotic expansion of \( E^{\text{emb}}_{Y|_{G_i}} (N) \). Then

\[
a_0(Y) = \prod_{i=1}^{k} a_0^{(i)}.
\]

In particular, \( a_0(Y) = 1 \) if and only if, whenever \( G_i \) is a surface group, the subcomplex \( Y|_{G_i} \) does not admit matching pairs of boundary cycles.

We will need the following lemma. Recall that \( o \) is the basepoint of \( X_\Gamma \) and \( e_i \) is the edge connecting \( o \) to \( X_{G_i} \).

**Lemma 4.2.** For any compact sub-cover \( p: Y \to X_\Gamma \) we have

\[
\chi^{\text{grp}}(Y) = |p^{-1}(o)| + \sum_{i=1}^{k} \left( \chi^{\text{grp}}(Y|_{G_i}) - |p^{-1}(e_i)| \right).
\]

**Proof.** We may assume that \( Y \) is connected: the general case follows immediately. We embed \( Y \) in a larger sub-cover \( Z \), where for every \( i = 1, \ldots, k \), \( Y|_{G_i} \) is embedded in a space \( Z|_{G_i} \) according to the following rules:

- if \( G_i \) is free, \( Z|_{G_i} = Y|_{G_i} \);
- if \( G_i \) is finite, \( f|_{G_i}: Y|_{G_i} \hookrightarrow Z|_{G_i} \) is the universal lift (Definition 2.9) of \( Y|_{G_i} \) as a sub-cover of \( X_{G_i} \) (in particular, \( Z|_{G_i} \) is a compact full cover of \( X_{G_i} \)).
• if $G_i$ is a surface group, $f|_{G_i} : Y|_{G_i} \hookrightarrow Z|_{G_i}$ is the element of $\mathcal{R}^{\text{emb}}_{Y|_{G_i}} = \mathcal{R}^{\text{emb}}_{Y|_{G_i}}(\chi^\text{grp}_{Y|_{G_i}})$ from Corollary 3.13 through which the universal lift of $Y|_{G_i}$ factors.

We let $Z$ be the union of the $Z|_{G_i}$’s together with $Y - \bigcup Y|_{G_i}$ (attached in the obvious manner), and $f : Y \hookrightarrow Z$ be the embedding obtained from the identity on $Y - \bigcup Y|_{G_i}$ and $f|_{G_i}$ on $Y|_{G_i}$.

For every vertex $y$ in $Y|_{G_i}$, we claim that

$$\pi_1^{\text{lab}}(Y|_{G_i}, y) = \pi_1^{\text{lab}}(Z|_{G_i}, f(y)) \cong \pi_1(Z|_{G_i}, f(y)).$$  \hfill (4.3)

Indeed, in the free case (4.3) is trivial. In the finite case the first equality in (4.3) follows from the definition of the universal lift and the second one from that $Z|_{G_i}$ is a full cover. Finally, in the surface case, the same argument gives (4.3) with $Z|_{G_i}$ replaced with the codomain of the universal lift $Y$ of the connected component of $y$ in $Y|_{G_i}$, but this implies (4.3) as $Z|_{G_i}$ is BR and so its embedding in $Y$ is $\pi_1$-injective [MP22, Cor. 4.11].

Now $Z$ has the structure of a graph of spaces with the edge-spaces being the ordinary edges in $\bigcup_{i=1}^k p^{-1}(e_i)$. Fix a vertex $y_0 \in p^{-1}(o)$. The sub-covering map $\phi : Z \to X_B$ induces an embedding on the fundamental group

$$\phi_* : \pi_1(Z, y_0) \hookrightarrow \pi_1(X_B, o) = \Gamma.$$

Indeed, every non-trivial element $g \in \pi_1(Z, y_0)$ can be described by an irreducible combinatorial path in the 1-skeleton of $Z$ based at $y_0$; this is a closed path where at each vertex-space it may “accumulate” an element of that vertex group, and if the path backtracks, the element of the vertex-group in the middle must be non-trivial. But then the $\phi$-image of this path is irreducible and thus non-trivial in $X_B$ by (4.3).

Finally, the embedding $f : Y \hookrightarrow Z$ induces a surjective homomorphism $f_* : \pi_1(Y, y_0) \to \pi_1(Z, y_0)$; this follows again from (4.3). As $p_* = \phi_* \circ f_*$, we conclude that $\pi_1^{\text{lab}}(Y, y_0) = p_*(\pi_1(Y, y_0)) \cong \pi_1(Z, y_0)$. Hence $\chi^\text{grp}(Y) = \chi(\pi_1(Z))$, and the latter is equal to the right hand side of (4.2).

\[\square\]

Proof of Theorem 2.6 and of Proposition 4.1. Let $p : Y \to X_B$ denote the sub-covering map. Denote by $\nu_o = |p^{-1}(o)|$ the number of vertices above the vertex $o \in X_B$. For $i = 1, \ldots, k$ denote by $\nu_i = v(Y|_{G_i})$ the number of vertices in $Y|_{G_i}$, and by $\varepsilon_i = |p^{-1}(e_i)|$ the number of edges projecting to the edge $e_i \in X_B$ which connects $o$ and $X_{G_i}$. In an $N$-cover $\hat{X}$ of $X_B$ in our model, the vertices above $o$ are labeled by $[N] = \{1, \ldots, N\}$. Every other vertex $u$ in $\hat{X}$ is a neighbor (in the 1-skeleton of $\hat{X}$) of exactly one vertex $u'$ in the fiber above $o$, and we label $u$ by the same label from $[N]$ as $u'$.

Let $q : Z \to X_{G_i}$ be a sub-cover of some $X_{G_i}$. Every embedding of the vertices of $Z$ to a cover of $X_{G_i}$ can be extended to an embedding of $Z$ in at most one way. Because we identified the vertices of an $N$-cover of $X_{G_i}$ with $[N]$, such an embedding of the vertices of $Z$ into an $N$-cover is an embedding into $[N]$. Let $p(Z)$ be the probability that a given embedding of the vertices of $Z$ to $[N]$ can be extended to an embedding of $Z$ to a random $N$-cover with vertices $[N]$. By symmetry, $p(Z)$ is independent of the embedding of vertices, so

$$\mathbb{E}^{\text{emb}}_{Z}(N) = (N)_{\nu(Z)} \cdot p(Z).$$  \hfill (4.4)

Consider an arbitrary embedding $f$ of the $\nu_o$ vertices $p^{-1}(o)$ of $Y$ into $[N]$, out of the $(N)_{\nu_o}$ possible ones. For every $i = 1, \ldots, k$, the embedding $f$ determines the embedding of the $\varepsilon_i$ vertices of $Y|_{G_i}$ incident to edges projecting to $e_i$, so there are

$$(N - \varepsilon_i)_{\nu_i - \varepsilon_i} = \frac{(N)_{\nu_i}}{(N)_{\varepsilon_i}}$$

possible extensions of the embedding $f$ to an embedding of the $\nu_i$ vertices of $Y|_{G_i}$. Because of the
independence of the random $N$-covers of every $X_{G_i}$, we obtain that

$$
\mathbb{E}_{Y_{\gamma}}^\text{emb} (N) = (N)_{y_0} \cdot \prod_{i=1}^{k} \left[ \frac{(N)_{\nu_i}}{(N)_{\nu_i}} p (Y | G_i) \right] \quad (4.4)
$$

We already know that each term $\mathbb{E}_{Y_{\gamma}}^\text{emb} (N)$ admits an asymptotic expansion with exponents in $\frac{1}{m(G_i)} \mathbb{Z}$ (where $m(G_i) = 1$ for torsion-free group and $m(G_i) = |G_i|$ if $G_i$ is finite). Of course, the product of these asymptotic expansions gives an asymptotic expansion with exponents in $\frac{1}{m(\Gamma)} \mathbb{Z}$. Together with the terms $(N)_{y_0} \cdot \prod_{i=1}^{k} \frac{1}{(N)_{\nu_i}}$, we get an asymptotic expansion as in (4.1), with leading coefficient $a_0 \defeq \prod_i a_0^{(i)}$, and with leading exponent $\nu_0 - \sum_i \varepsilon_i + \sum_i \chi^{\text{grp}}(Y^{(i)})$, which is equal to $\chi^{\text{grp}}(Y)$ by Lemma 4.2. The final statement of Proposition 4.1 now follows from Proposition 3.19.

5 The limit distribution and asymptotic expansion of $\text{fix}_\gamma (N)$

In this section we prove Theorems 1.4 and 1.5 about the limit distribution of $\text{fix}_\gamma (N)$ as $N \to \infty$ for non-torsion $\gamma \in \Gamma$, and Theorem 1.13 about the limit distribution of $\mathbb{E} [\text{fix}_\gamma (N)]$ for arbitrary $\gamma \in \Gamma$. For these results, we consider a natural sub-cover $p_\varphi : Y_\gamma \to X_\Gamma$ such that for every $\varphi \in \text{Hom} (\Gamma, S_N)$, the number of fixed points of $\varphi (\gamma)$ is equal to the number of lifts of $p_\gamma$ to $X_\varphi$, the corresponding $N$-cover of $X_\Gamma$. We then proceed by applying Theorem 2.6 to a natural resolution of $Y_\gamma$. All the results in this paper are immediate for the trivial element of $\Gamma$, so we may assume that $\gamma \neq 1$.

A canonical form of $\gamma$

Fix $1 \neq \gamma \in \Gamma$. We may write $\gamma$ in its canonical form as

$$
\gamma = h_1 h_2 \ldots h_{\ell (\gamma)},
$$

(5.1)

where $\ell = \ell (\gamma) \in \mathbb{Z}_{\geq 1}$, $h_j \in G_i \setminus \{1\}$ and $i_{j+1} \neq i_j$. We may further assume without loss of generality that $\gamma$ is cyclically reduced, namely, that if $\ell \geq 2$, then $i_2 \neq i_1$. Indeed, replacing $\gamma$ with a conjugate does not alter any of the local statistics of a $\gamma$-random permutation or the quantities appearing in our results (such as $|H_\gamma|$ or the integers $t$, $\alpha_1, \ldots, \alpha_t$ and $\beta_1, \ldots, \beta_t$ from Theorem 1.5).

Recall from Section 2.1 that each factor $G_i$ of $\Gamma$ is endowed with a fixed, finite set of generators -- those labeling the edges in $X_{G_i}$. For every $j = 1, \ldots, \ell$, let $w_j$ be a shortest word in these generators of $G_{i_j}$ representing $h_j$. Furthermore, we assume that whenever $i_j = i_s$ and $h_j = h_s$ or $h_j = h_s^{-1}$, then $w_j = w_s$ or $w_j = w_s^{-1}$, respectively.

Finally, if $\ell = 1$ and $G = G_{i_1}$ is free or a surface group, then there is a unique non-power $\gamma_0 \in G$ so that $\gamma = \gamma_0^q$ with $q \in \mathbb{Z}_{\geq 1}$, and the cyclic subgroups containing $\gamma$ are precisely $\langle \gamma_0^j \rangle$ for $1 \leq j < q$. As above, we may assume by conjugating $\gamma$ if needed, that $\gamma_0$ is represented by some word $w_0$ in the generators of $G$ which is a shortest representative of any element in the conjugacy class of $\gamma_0$. We then define $w = w_1$ to be the concatenation of $q$ copies of $w_0$. This $w$ represents $\gamma$, and is shortest among all words representing elements in the conjugacy class of $\gamma$: this is trivial if $G$ is free, and follows from [BS87a, Lem. 2.11] if $G$ is a surface group.

---

10 This fact is standard, but let us explain it for completeness: when $G$ is either free or a surface group with the generators from Section 2.1, and $w$ is any word in the generators that is a shortest representative of its conjugacy class, then the concatenation of $n$ copies of $w$ is also shortest in its conjugacy class (this is immediate for free groups and follows from [BS87a, Lem. 2.11] for surface groups). Therefore, there is a maximal $q \in \mathbb{Z}_{\geq 1}$ so that $\gamma$ has a $q$-th root in $G$. Finally, any two elements in a free or surface group generate a free subgroup, so every two roots of $\gamma$ belong to the same cyclic subgroup.
Consider the 'natural' resolution of the sub-cover \( p_\gamma : Y_\gamma \to X_\Gamma \):

\[
\mathcal{R}_\gamma \overset{\text{def}}{=} \{ f : Y_\gamma \to Z_f \mid f \text{ is a surjective morphism of sub-covers} \}.
\]

This is indeed a resolution as every morphism decomposes uniquely to a surjective one composed with an injective one. It is finite as \( Y_\gamma \) is compact. Figure 5.1 illustrates such a resolution.

**Proof of Theorem 1.13.** Let \( X_\varphi \) be the \( N \)-cover of \( X_\Gamma \) corresponding to the uniformly random \( \varphi : \Gamma \to S_N \). Recall that vertices in every fiber of \( X_\varphi \) are in a given bijection with \([N]\) (this is by definition for the vertices above \( o \), and we label every other vertex in the same label as its \( o \)-fiber neighbor). In the correspondence
between Hom(Γ, S_N) and N-covers of X_Γ, the fixed points of ϕ(γ) are precisely the elements i in [N] so that γ ∈ π^{lab}_1(X_ϕ, i). By Proposition 2.8, the number of fixed points of ϕ(γ) is thus precisely the number of lifts of p_γ: Y_γ → X_Γ to X_ϕ. Namely,

$$E[\text{fix}_γ(N)] = E_{Y_γ}(N) = \sum_{f ∈ R_γ} E^{emb}_Z(N),$$  \hspace{1cm} (5.2)

where the last equality is by Lemma 2.4. Theorem 1.13 now follows from Theorem 2.6. \hfill \square

The proof of Theorem 1.4

We now turn to proving Theorem 1.4. Now γ is a non-torsion element, so either ℓ(γ) = 1 and G = G_{i_1} is a free group or a surface group, or ℓ(γ) ≥ 2. We need to show that E[fix_γ(N)] →_{N→∞} |H_γ|

where H_γ = {H ≤ Γ | χ(H) = 0 and H ∋ γ}. Every morphism f: Y_γ → Z_f in the resolution R_γ satisfies π^{lab}_1(Z_f, f(y)) ≥ γ. (Here, if ℓ(γ) = 1, we consider π^{lab}_1(Z_f, f(y)) ≤ π_1(X_Γ, v_i_1), where v_i_1 is the vertex in X_{G_{i_1}}. We identify this group with Γ by conjugating with the edge e_{i_1}.) In particular, π^{lab}_1(Z_f, f(y)) is an infinite subgroup of Γ and therefore χ^{grp}(Z_f) ≤ 0 (see the discussion following Definition 1.3). Denote

$$R^0_γ \overset{\text{def}}{=} \{ f ∈ R_γ | χ^{grp}(Z_f) = 0 \}.$$

Theorem 2.6 and (5.2) now yield that

$$\lim_{N→∞} E[\text{fix}_γ(N)] = \sum_{f ∈ R^0_γ} a_0(Z_f),$$ \hspace{1cm} (5.3)

where a_0(Z_f) is the positive integer from Theorem 2.6. Consider the map

$$Ψ: R^0_γ \rightarrow H_γ,$$

$$f \mapsto H_f \overset{\text{def}}{=} π^{lab}_1(Z_f, f(y)).$$

Theorem 1.4 will be proved by showing that Ψ is a bijection and that a_0(Z_f) = 1 for all f ∈ R^0_γ.

First, we show that Ψ is injective. Let f_1, f_2 ∈ R^0_γ with π^{lab}_1(Z_{f_1}, f_1(y)) = π^{lab}_1(Z_{f_2}, f_2(y)). Then (Z_{f_1}, f_1(y)) and (Z_{f_2}, f_2(y)) have universal lifts to the same full covering p_γ: (Z_{f_1}, f_1(y)) → (Y, u) for i = 1, 2. Both are injective by Lemma 2.10, and so f_1 and f_2 coincide with (the surjective part in the decomposition of) the morphism f: (Y, y) → (Y, u), and are thus identical.

For the remainder of the proof, we need the following lemma. Recall that in any morphism f: (Y, y) → (Z, z) from a sub-cover to a full cover, π^{lab}_1(Y, y) ≤ π^{lab}_1(Z, z).

**Lemma 5.1.** If f: Y_Γ → Z is a morphism where Z is a connected full cover of X_Γ with χ^{grp}(Z) = 0, then it is π_1-surjective, namely, π^{lab}_1(f(Y_Γ), f(y)) = π^{lab}_1(Z, f(y)).

**Proof.** Denote by H = π^{lab}_1(Z, f(y)) ≤ Γ. By assumption, H ≅ Z or H ≅ C_2 * C_2. The existence of the morphism f guarantees that γ ∈ H.

Assume first that H ≅ Z. By the discussion above, there is a unique non-power γ_0 ∈ Γ and q ∈ Z_{≥1} so that γ = γ_0^q (if ℓ(γ) ≥ 2, then γ_0 = h_1 h_2 ··· h_{ℓ/q} and is the shortest period in γ), and H = ∪_{j=q}^{ℓ,q} γ_0^j for some j|q. By the way we defined the word w = w_1 ··· w_ℓ representing γ, the first \frac{ℓw}{q} letters of w represent γ_0^j, and thus the f-image in Z of these letters in Y_γ, is a loop at f(y) representing γ_0^j. We obtain π^{lab}_1(f(Y_γ), f(y)) ≥ γ_0^j = H.

Now assume that H ≅ C_2 * C_2. The cover Z, as all covers of X_Γ, is a graph of spaces itself (e.g., [SW79, Sec. 3]), with trivial edge-spaces which are the preimages of e_1, ..., e_k and vertex-spaces which
are the connected components of $Z|_{G_i}$ for $i = 1, \ldots, k$ and the vertices in the fiber above $o$. Furthermore, any decomposition of $H$ as a free product of indecomposable groups is of the form $C_2 \ast C_2$ (e.g., [SW79, Thm. 3.5]). Hence as a graph of spaces, $Z$ has no cycles (it is a tree), two of its vertex spaces have $\pi_1 \cong C_2$ and all remaining vertex spaces have trivial fundamental groups. Denote by $v_1$ and $v_2$ the two vertex spaces in $Z$ with $\pi_1 \cong C_2$. The cycle in the 1-skeleton of $Z$, based at $f(y)$, which spells out the word $w = w_1 \cdots w_k$, cannot enter a vertex-space and backtrack after reading the trivial element (this is by the assumption that $w$ is cyclically reduced). As $Z$ is a tree, this cycle must backtrack (cyclically) in at least two different vertex spaces. Thus, it must backtrack in $v_1$ and in $v_2$ (at least once in each of them), read the non-trivial element in the fundamental group in each of them, and traverse the entire path between them. Thus $\pi_1^{lab}(f(Y_\gamma), f(y)) \geq H$.

We can now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. It remains to show that $\Psi$ is surjective and that $a_0(Z_f) = 1$ for all $f \in \mathcal{R}_\gamma^0$. Let $H \in \mathcal{H}_\gamma$ and $(\Upsilon_H, u)$ the corresponding connected cover of $X_\Gamma$. By Lemma 5.1, the morphism $f : (Y_\gamma, y) \to (\Upsilon_H, u)$ satisfies $\pi_1^{lab}(f(Y_\gamma), f(y)) = H$. Hence its 'surjective part' $\overline{f} : Y_\gamma \to f(Y_\gamma)$ is an element of $\mathcal{R}_\gamma^0$ with $\Psi(\overline{f}) = H$. So $\Psi$ is surjective.

If $\ell(\gamma) \geq 2$, then all vertex spaces in $\Upsilon_H$ have fundamental groups trivial or $C_2$. So all vertex spaces of $\Upsilon_H$ projecting to $X_{G_i}$ where $G_i$ is a surface group must be trivial. By Corollary 3.20(1), $a_0(f(Y_\gamma)|_{G_i}) = 1$ in this case, by Theorem 2.6 $a_0(f(Y_\gamma)|_{G_i}) = 1$ whenever $G_i$ is not a surface group, and Proposition 4.1 now yields that $a_0(f(Y_\gamma)) = 1$.

Finally, if $\ell(\gamma) = 1$ and $G = G_{i_1}$ is a free or surface group\footnote{This case reduces to the results about free groups and surface groups due to [Nic94] and [MP20], respectively. We prove it here for completeness.} and $H = \langle \gamma_0^j \rangle$, then by our choice of the word $w = w_1$ above, $f(Y_\gamma)$ is a simple cycle: this is trivial for $G$ a free group, and for $G$ a surface group, $(\Upsilon_H, u)$ is the space $\langle \gamma_0^j \rangle \backslash \Sigma_g$, where $w_0$ is a simple cycle in the 1-skeleton (see the discussion in [MP22, Sec. 4 and 5]). So Proposition 4.1 and Corollary 3.20(2) yield that $a_0(f(Y_\gamma)) = a_0(f(Y_\gamma)|_{G_i}) = 1$.

Corollary 5.2. The set $\mathcal{H}_\gamma$ is finite for every non-torsion $\gamma \in \Gamma$.

Proof. This follows from the fact that $\Psi$ gives a bijection between the finite set $\mathcal{R}_\gamma^0$ and $\mathcal{H}_\gamma$.

We also record here the following lemma, which we need for the proof of Theorem 1.5.

Lemma 5.3. Let $H_1, H_2 \in \mathcal{H}_\gamma$ be conjugate and for $i = 1, 2$, $f_i : Y_\gamma \to Z_i$ the corresponding morphisms in $\mathcal{R}_\gamma^0$. Then $Z_1$ and $Z_2$ are identical.

(Of course, if $H_1 \neq H_2$, then $f_1$ and $f_2$ map the basepoint $y$ to two different vertices of $Z_1 = Z_2$.)

Proof. First assume that $\ell(\gamma) \geq 2$. Let $\Upsilon$ be the connected cover of $X_\Gamma$ corresponding to the conjugacy class of $H_1$ and $H_2$, with basepoints $u_1$ and $u_2$ corresponding to $H_1$ and $H_2$, respectively. Then $f_i$ is the surjective part of the morphism $\overline{f_i} : (Y_\gamma, y) \to (\Upsilon, u_i)$. Consider $\Upsilon$ as a graph of spaces. As explained in the proof of Lemma 5.1, the image of $\overline{f_i}$ goes precisely through the vertex-spaces and edge-spaces in the “core” of $\Upsilon$ (this is a cycle in the graph if $H_1 \cong Z$ or a path between the two non-$\pi_1$-trivial vertex spaces if $H_1 \cong C_2 \ast C_2$). Our choice of the words $w_1, \ldots, w_{\ell(\gamma)}$ – that $w_i$ and $w_j$ are identical or inverse of one another if so are the corresponding $h_i$ and $h_j$ – guarantees that the precise path traversed in every vertex space of the core is identical, and so, indeed, $\overline{f_1}(Y_\gamma) = \overline{f_2}(Y_\gamma)$.

If $\ell(\gamma) = 1$, then there are no (non-trivial) conjugates of $H_1 = \langle \gamma_0^j \rangle$ containing $\gamma$, so the Lemma is vacuous.
The method of moments

We now turn to prove Theorem 1.5, which describes the limit distribution of \( \text{fix}_\gamma (N) \) as \( N \to \infty \) for every fixed non-torsion element \( \gamma \in \Gamma \). Our proof is based on the method of moments. Some of the steps follow parallel steps in [LP10, Sec. 4].

A probability distribution \( \mu \) on \( \mathbb{R} \) is said to be determined by its moments if it has finite moments \( \alpha_r = \int_{-\infty}^{\infty} x^r \mu(dx) \) of all orders, and \( \mu \) is the only probability measure with these moments.

**Theorem 5.4** (Method of moments, e.g., [Bil95, Thm. 30.2]). Let \( X \) and \( X_N \ (N \in \mathbb{Z}_{\geq 1}) \) be random variables, and suppose that the distribution of \( X \) is determined by its moments, that the \( X_N \) have moments of all order, and that \( \lim_{N \to \infty} \mathbb{E}[X_N^r] = \mathbb{E}[X^r] \) for every \( r \in \mathbb{Z}_{\geq 1} \). Then \( X_N \xrightarrow{\text{dis}} X \).

where \( \xrightarrow{\text{dis}} \) denotes convergence in distribution.

**Theorem 5.5** (Sufficient condition for \( \mu \) to be determined by its moments, e.g., [Bil95, Thm. 30.1]). Let \( \mu \) be a probability measure on \( \mathbb{R} \) having finite moments \( \alpha_r = \int_{-\infty}^{\infty} x^r \mu(dx) \) of all orders. If the power series \( \sum_r \alpha_r \frac{t^r}{r!} \) has a positive radius of convergence, then \( \mu \) is determined by its moments.

Recall that Theorem 1.5 states that \( \text{fix}_\gamma (N) \) converges in distribution to \( \sum_i c_i Z_{1/\beta_i} \) — a finite linear combination of independent Poisson-distributed random variables with coefficients from \( \mathbb{Z}_{\geq 1} \). We first record the standard fact that such a sum is determined by its moments.

**Lemma 5.6.** Let \( Z_1, \ldots, Z_t \) be independent Poisson-distributed random variables with parameters \( \lambda_1, \ldots, \lambda_t > 0 \), respectively, and let \( c_1, \ldots, c_t \geq 1 \). Then the distribution of \( \sum_{i=1}^{t} c_i Z_i \) is determined by its moments.

**Proof.** For a single Poisson distribution with parameter \( \lambda \), the power series from Theorem 5.5 is \( \sum_r \alpha_r \frac{t^r}{r!} = e^{(e^t-1) \lambda} \) [Bil95, Eq. (21.22) and (21.27)] and, in particular, converges for all \( t \). The sum \( Z_1 + \ldots + Z_t \) is Poisson with parameter \( \lambda_1 + \ldots + \lambda_t \) and, in particular, the corresponding power series converges for all \( t \). Let \( c = \max\{c_1, \ldots, c_t\} \). Then \( \sum Z_i \leq \sum c_i Z_i \leq c \sum Z_i \). In particular, if the \( r \)-th moment of \( \sum Z_i \) is \( \alpha_r \) and of \( \sum c_i Z_i \) is \( \beta_r \), then \( \alpha_r \leq \beta_r \leq c^r \alpha_r \). Consequently, the series \( \sum \beta_r \frac{t^r}{r!} \) has radius of convergence that is \( \geq \frac{1}{c} \) that of the series \( \sum \alpha_r \frac{t^r}{r!} \). But the latter converges for all real \( t \), hence so does \( \sum \beta_r \frac{t^r}{r!} \). By Theorem 5.5 we conclude that \( \sum c_i Z_i \) is determined by its moments.

**The proof of Theorem 1.5**

Recall the statement of Theorem 1.5: \( H_1, \ldots, H_t \) are representatives of the conjugacy classes of subgroups represented in \( \mathcal{H}_\gamma = \{ H \leq \Gamma \mid \gamma \in H \text{ and } \chi (H) = 0 \} \), \( \alpha_i = |\{ \mathcal{H}_\gamma \cap H_i^\Gamma \}| \) and \( \beta_i = [N_{\Gamma} (H_i) : H_i] \). We need to show that as \( N \to \infty \)

\[
\text{fix}_\gamma (N) \xrightarrow{\text{dis}} \sum_{i=1}^{t} \alpha_i \beta_i Z_{1/\beta_i},
\]

where \( Z_{1/\beta_1}, \ldots, Z_{1/\beta_t} \) are independent Poisson random variables with parameters \( \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_t} \), respectively.

For every \( N \), the random variable \( \text{fix}_\gamma (N) \) is finitely supported and so has finite moments. By Theorem 5.4 and Lemma 5.6, it is enough to prove that for every \( r \in \mathbb{Z}_{\geq 1} \) we have

\[
\mathbb{E}[(\text{fix}_\gamma (N))^r] \to N \to \infty \mathbb{E}\left[ \left( \sum_{i=1}^{t} \alpha_i \beta_i Z_{1/\beta_i} \right)^r \right]. \tag{5.4}
\]

\footnote{The assumption that \( c_1, \ldots, c_t \geq 1 \) is not crucial; it only somewhat simplifies the notation in the proof and it holds anyway in the case we use.}
Recall that for every \( \varphi \in \text{Hom}(\Gamma, S_N) \), the number of fixed points of \( \varphi(\gamma) \) is equal to the number of lifts of \( p_\gamma: Y_\gamma \to X_\Gamma \) to \( X_\varphi \). Similarly, define

\[
Y_\gamma^{\text{Lifts}_r} = \bigcup_{r \text{ times}} Y_\gamma ,
\]

Then \( (\text{fix}(\varphi(\gamma)))^r \) is equal to \( (\# \text{ lifts of } p_\gamma)^r \), which is equal to the number of lifts to \( X_\varphi \) of

\[
p_\gamma^{\text{Lifts}_r}: Y_\gamma^{\text{Lifts}_r} \to X_\Gamma ,
\]

where \( p_\gamma^{\text{Lifts}_r} \) restricts to \( p_r \) on each of the \( r \) disjoint copies of \( Y_\gamma \). So

\[
\mathbb{E} [(\text{fix}_\gamma (N))^r] = \mathbb{E}_{Y_\gamma^{\text{Lifts}_r}} (N) = \sum_{f \in \mathcal{R}_{\gamma,r}} \mathbb{E}^{\text{emb}}_{Z_f} (N), \tag{5.5}
\]

where \( \mathcal{R}_{\gamma,r} \) is the standard resolution of \( p_\gamma^{\text{Lifts}_r} : \)

\[
\mathcal{R}_{\gamma,r} \overset{\text{def}}{=} \{ f: Y_\gamma^{\text{Lifts}_r} \to Z_f \mid f \text{ is a surjective morphism of sub-covers} \}. \nonumber
\]

As for the case \( r = 1 \), if we let

\[
\mathcal{R}_{\gamma,1}^0 \overset{\text{def}}{=} \{ f \in \mathcal{R}_{\gamma,r} \mid \chi^{\text{grp}} (Z_f) = 0 \} , \nonumber
\]

then

\[
\lim_{N \to \infty} \mathbb{E} [\text{fix}_\gamma (N)^r] = \sum_{f \in \mathcal{R}_{\gamma,1}^0} a_0 (Z_f). \tag{5.6}
\]

Let \( f_1, \ldots, f_t \in \mathcal{R}_{\gamma,1}^0 \) be the morphisms \( f_i: Y_\gamma \to Z_i = Z_{f_i} \) corresponding through the bijection \( \Psi \) to \( H_1, \ldots, H_t \) from the statement of Theorem 1.5, respectively. Namely, \( f_i \) is onto and \( \pi_1^{\text{lab}} (Z_i, f_i(y)) = H_i \).

**Lemma 5.7.** For every \( f \in \mathcal{R}_{\gamma,r}^0 \), the sub-cover \( Z_f \) is a disjoint union of copies of \( Z_1, \ldots, Z_t \).

**Proof.** Let \( f \in \mathcal{R}_{\gamma,r}^0 \). Every connected component of \( Z_f \) contains \( \gamma \) in its fundamental group, up to conjugation. Hence every connected component has non-positive \( \chi^{\text{grp}} \). We conclude that they all have \( \chi^{\text{grp}} = 0 \).

Let \( C \) be a connected component of \( Z_f \). Consider the universal lift \( \Upsilon \) of \( C \). Let \( g: Y_\gamma \to C \) be the restriction of \( f \) to one of the connected components of \( Y_\gamma^{\text{Lifts}_r} \) mapped to \( C \). By Lemma 5.1, the map \( g: Y_\gamma \to \Upsilon \) is \( \pi_1 \)-surjective. This shows that all components of \( Y_\gamma^{\text{Lifts}_r} \) mapped to \( C \) have \( \pi_1 \)-images which are identical, up to conjugation. By Lemma 5.3 \( C \) is identical to one of the \( Z_i \)'s (1 \( \leq i \leq t \)). \( \square \)

**Proof of Theorem 1.5.** Recall (5.4): it is enough to prove that \( \mathbb{E} [\text{fix}_\gamma (N)^r] \to_{N \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^t \alpha_i \beta_i Z_{1/\beta_i} \right)^r \right] \) for every \( r \in \mathbb{Z}_{\geq 1} \). By (5.6), \( \lim_{N \to \infty} \mathbb{E} [\text{fix}_\gamma (N)^r] = \sum_{f \in \mathcal{R}_{\gamma,1}^0} a_0 (Z_f) \). By Lemma 5.7, for every \( f \in \mathcal{R}_{\gamma,r}^0 \), the sub-cover \( Z_f \) is a disjoint union of copies of \( Z_1, \ldots, Z_t \). By Corollary 3.20(5) and Proposition 4.1, \( a_0 (Z_f) = 1 \) for all \( f \in \mathcal{R}_{\gamma,r}^0 \). With Lemma 5.7 we now obtain

\[
\lim_{N \to \infty} \mathbb{E} [\text{fix}_\gamma (N)^r] = \sum_{r_1 + \cdots + r_t = r} \left( \prod_{i=1}^t \# \{ \text{surjective maps } Y_\gamma^{\text{Lifts}_i} \to Z_i^{\text{Lifts}_i}, 0 \leq s_i \leq r_i \} \right) . \nonumber
\]

By assumption, the variables \( Z_{1/\beta_1}, \ldots, Z_{1/\beta_t} \) are independent, and so

\[
\mathbb{E} \left[ \left( \sum_{i=1}^t \alpha_i \beta_i Z_{1/\beta_i} \right)^r \right] = \sum_{r_1 + \cdots + r_t = r} \left( \prod_{i=1}^t \mathbb{E} \left[ (\alpha_i \beta_i Z_{1/\beta_i})^{r_i} \right] \right) , \nonumber
\]

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so it is enough to show that for all \( r \in \mathbb{Z}_{\geq 0} \), we have

\[
\# \left\{ \text{surjective maps } Y^{\text{li}}_{\gamma} \to Z^{\text{ls}}_{i}, \ 0 \leq s \leq r \right\} = \mathbb{E} \left[ (\alpha_{i} \beta_{i} Z_{1/\beta_{i}})^{r} \right]. \tag{5.7}
\]

Both sides equal 1 when \( r = 0 \), so assume that \( r \geq 1 \). Recall that \( \left\{ \begin{array}{c} r \\ j \end{array} \right\} \) denotes a Stirling number of the second kind, and is equal to the number of ways to partition a set of \( r \) objects into \( j \) non-empty subsets. The left hand side of (5.7) is equal to

\[
\sum_{j=1}^{r} \# \left\{ \text{surjective maps } Y^{\text{li}}_{\gamma} \to Z^{\text{ls}}_{i} \right\} = (\alpha_{i} \beta_{i})^{r} \sum_{j=1}^{r} \left\{ \begin{array}{c} r \\ j \end{array} \right\} \cdot \frac{1}{\beta_{i}^{j}}. \tag{5.8}
\]

Indeed, a surjective map \( Y^{\text{li}}_{\gamma} \to Z^{\text{ls}}_{i} \) is determined by a partition of the \( r \) copies of \( Y_{\gamma} \) into \( j \) non-empty subsets. For each subset, we map one element \((Y_{\gamma}, y)\) to one of \( \alpha_{i} \) non-isomorphic possible base points \( u \) in \( Z_{i} \) so that \( \pi^{\text{lab}}_{1} (Z_{i}, u) \equiv \gamma \). As \( \beta_{i} \) is the number of automorphisms of \( Z_{i} \), we get that each remaining element of the subset now has \( \alpha_{i} \beta_{i} \) possibilities for the image-vertex of \( y \). Together, these images of \( y \) completely determine the map \( Y^{\text{li}}_{\gamma} \to Z^{\text{ls}}_{i} \), and the total number of options is \( (\alpha_{i} \beta_{i})^{r-j} \cdot \alpha_{i}^{j} = (\alpha_{i} \beta_{i})^{r} \cdot \beta_{i}^{-j} \).

On the other hand, the right hand side of (5.7) is \( (\alpha_{i} \beta_{i})^{r} \mathbb{E} \left[ (Z_{1/\beta_{i}})^{r} \right] \), and it is a standard fact about the moments of Poisson variables that

\[
\mathbb{E} [(Z_{\lambda})^{r}] = \sum_{j=1}^{r} \left\{ \begin{array}{c} r \\ j \end{array} \right\} \cdot \lambda^{j}.
\]

\[ \Box \]

6 Asymptotic independence and statistics of small cycles

In this section we prove the remaining results: Theorem 1.14 giving a precise condition on when \( \text{fix}_{\gamma_{1}} (N) \) and \( \text{fix}_{\gamma_{2}} (N) \) are asymptotic independent for non-torsion \( \gamma_{1}, \gamma_{2} \in \Gamma \), and Theorem 1.15 about the statistics of cycles of bounded size. These two results are proven along similar lines to the results proven in Section 5, so we stress mostly some crucial points that did not appear in the previous results.

The multivariate method of moments

In both results we need the following classical extension of Theorem 5.4:

**Theorem 6.1** (Multivariate method of moments, e.g., [Bil95, Exer. 30.6]). Let \( X^{(1)}, \ldots, X^{(p)} \) and \( X^{(1)}_{N}, \ldots, X^{(p)}_{N} \) \((N \in \mathbb{Z}_{\geq 1})\) be random variables, and suppose that the distribution of \( X^{(1)}, \ldots, X^{(p)} \) on \( \mathbb{R}^{p} \) is determined by its moments (see [Bil95, Exer. 30.5] for the definition), that the \( X^{(i)}_{N} \) have moments of all order, and that

\[
\lim_{N \to \infty} \mathbb{E} \left[ (X^{(1)}_{N})^{r_{1}} \cdots (X^{(p)}_{N})^{r_{p}} \right] = \mathbb{E} \left[ (X^{(1)})^{r_{1}} \cdots (X^{(p)})^{r_{p}} \right]
\]

for every \( r_{1}, \ldots, r_{p} \in \mathbb{Z}_{\geq 0} \). Then

\[
\left( X^{(1)}_{N}, \ldots, X^{(p)}_{N} \right) \overset{\text{dip}}{\to} \left( X^{(1)}, \ldots, X^{(p)} \right).
\]

In particular, if \( X^{(1)}, \ldots, X^{(p)} \) are independent, then \( X^{(1)}_{N}, \ldots, X^{(p)}_{N} \) are asymptotically independent.
Proof of Theorem 1.14

Recall that we are given two non-torsion elements $\gamma_1, \gamma_2 \in \Gamma$, and we need to show that the following three conditions are equivalent: (i) $\text{fix}_{\gamma_1}(N)$ and $\text{fix}_{\gamma_2}(N)$ are asymptotically independent, (ii) $\gamma_1$ and $\gamma_2$ cannot be both conjugated into the same EC-zero subgroup of $\Gamma$, and (iii) $\mathbb{E} [\text{fix}_{\gamma_1}(N) \cdot \text{fix}_{\gamma_2}(N)] = \mathbb{E} [\text{fix}_{\gamma_1}(N)] \cdot \mathbb{E} [\text{fix}_{\gamma_2}(N)] + O\left(N^{-1/m}\right)$.

Proof of Theorem 1.14. We start by proving (ii) $\implies$ (i). So we assume that $\gamma_1$ and $\gamma_2$ cannot be both conjugated into the same EC-zero subgroup of $\Gamma$. For $j = 1, 2$, denote by $Y_j$ a random variable distributed as the linear combination of Poissons from Theorem 1.5 corresponding to $\text{fix}_{\gamma_j}(N)$. By Theorem 6.1, it is enough to show that

$$
\lim_{N \to \infty} \mathbb{E} \left[ (\text{fix}_{\gamma_1}(N))^r_1 (\text{fix}_{\gamma_2}(N))^r_2 \right] = \mathbb{E} \left[ (Y_1)^r_1 \right] \cdot \mathbb{E} \left[ (Y_2)^r_2 \right]
$$

for every $r_1, r_2 \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}$ be the natural resolution of $p: Y_{\gamma_1}^{\mathbb{L}r_1} \sqcup Y_{\gamma_2}^{\mathbb{L}r_2}$, and $\mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}^0$ the subset of morphisms $f \in \mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}$ with $\chi^\text{grp}(Z_f) = 0$. As in the proof of Theorem 1.5,

$$
\lim_{N \to \infty} \mathbb{E} \left[ (\text{fix}_{\gamma_1}(N))^r_1 (\text{fix}_{\gamma_2}(N))^r_2 \right] = \sum_{f \in \mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}^0} a_0(Z_f) = |\mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}^0|.
$$

The assumption on $\gamma_1$ and $\gamma_2$ guarantees that in every $f \in \mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}^0$, the copies of $Y_{\gamma_1}$ and those of $Y_{\gamma_2}$ are mapped to disjoint connected components of $Z_f$. Thus

$$
|\mathcal{R}_{\gamma_1,r_1,\gamma_2,r_2}^0| = |\mathcal{R}_{\gamma_1,r_1}^0| \cdot |\mathcal{R}_{\gamma_2,r_2}^0| = \mathbb{E} \left[ (Y_1)^r_1 \right] \cdot \mathbb{E} \left[ (Y_2)^r_2 \right].
$$

The implication (i) $\implies$ (iii): from the very definition of asymptotic independence it follows that $\mathbb{E} [\text{fix}_{\gamma_1}(N) \cdot \text{fix}_{\gamma_2}(N)] = \mathbb{E} [\text{fix}_{\gamma_1}(N)] \cdot \mathbb{E} [\text{fix}_{\gamma_2}(N)] + o_N(1)$. Applying Theorem 2.6 to $p_{\gamma_1}: Y_{\gamma_1} \to X_{\Gamma}$, to $p_{\gamma_2}: Y_{\gamma_2} \to X_{\Gamma}$ and to $p_{\gamma_1 \sqcup \gamma_2}: Y_{\gamma_1} \sqcup Y_{\gamma_2} \to X_{\Gamma}$, shows that the error term is $O\left(N^{-1/m}\right)$.

The implication (iii) $\implies$ (ii): Finally, assume that $\mathbb{E} [\text{fix}_{\gamma_1}(N) \cdot \text{fix}_{\gamma_2}(N)] = \mathbb{E} [\text{fix}_{\gamma_1}(N)] \cdot \mathbb{E} [\text{fix}_{\gamma_2}(N)] + O\left(N^{-1/m}\right)$. As in the proof of Theorem 1.4,

$$
\lim_{N \to \infty} \mathbb{E} \left[ \text{fix}_{\gamma_1}(N) \cdot \text{fix}_{\gamma_2}(N) \right] = |\mathcal{R}_{\gamma_1 \sqcup \gamma_2}^0|,
$$

the number of elements $f$ in the natural resolution of $p_{\gamma_1 \sqcup \gamma_2}$ with $\chi^\text{grp}(Z_f) = 0$. Inside $\mathcal{R}_{\gamma_1 \sqcup \gamma_2}^0$ there are all those morphisms in which $Y_{\gamma_1}$ and $Y_{\gamma_2}$ are mapped to two different connected components of $Z_f$. The number of such elements is $|\mathcal{R}_{\gamma_1}^0| \cdot |\mathcal{R}_{\gamma_2}^0|$. By the assumption in (iii), there are no further elements in $\mathcal{R}_{\gamma_1 \sqcup \gamma_2}^0$.

Assume towards contradiction that $\gamma_1$ and $\gamma_2$ are both conjugated into the same EC-zero subgroup $H \leq \Gamma$. In particular, $\ell(\gamma_1) \geq 2$ if and only if $\ell(\gamma_2) \geq 2$. We may assume that $Y_{\gamma_1}$ and $Y_{\gamma_2}$ were constructed in a coordinated manner: if $\ell(\gamma_1) \geq 2$, then in the word spelling $\gamma_1$ and the word spelling $\gamma_2$ we use the same subwords whenever the corresponding elements of the canonical forms are identical or inverse of one another, and if $\ell(\gamma_1) = 1$, then we take $\gamma_1$ and $\gamma_2$ to be powers of the same $\gamma_0$.

But then there is a map of $Y_{\gamma_1} \sqcup Y_{\gamma_2}$ into the connected cover $\mathcal{Y}_{\Gamma}$ corresponding to $H$. The images of both $Y_{\gamma_1}$ and $Y_{\gamma_2}$ in $\mathcal{Y}_{\Gamma}$ are identical, and the surjective part of this morphism constitutes another element of $\mathcal{R}_{\gamma_1 \sqcup \gamma_2}^0$, a contradiction. □

Proof of Theorem 1.15

We will need the following lemma. Recall that for non-torsion $\gamma \in \Gamma$ and any $L \in \mathbb{Z}_{\geq 1}$, we let $\mathcal{H}_{\gamma,L}$ mark the set of EC zero subgroups of $\Gamma$ containing $\gamma^L$ but not any smaller power of $\gamma$.

**Lemma 6.2.** Let $L_1 \neq L_2$ be positive integer. Then no subgroup in $\mathcal{H}_{\gamma,L_1}$ is conjugate to a subgroup in $\mathcal{H}_{\gamma,L_2}$.
Lemma 6.2 replaces the assumption in part 2 of Theorem 1.14. The subgroups isomorphic to \( \Z \) in \( \bigcup_L \mathcal{H}_{\gamma,L} \) are precisely \( \{(\gamma_0^j) \mid j \in \Z_{\geq 1}\} \), no distinct two of which are conjugate one to the other. Moreover, for every \( j \in \Z_{\geq 1} \), \( \gamma_0^j \) belongs to a single \( \mathcal{H}_{\gamma,L} \); exactly the \( L \) satisfying that it is the smallest positive integer with \( j \mid qL \).

For subgroups isomorphic to \( C_2 \ast C_2 \), the argument is similar, as we now explain. Let \( H \leq \Gamma \) with \( H \cong C_2 \ast C_2 \). If \( \gamma^L \in H \) for some \( L \in \Z_{\geq 1} \), then \( \ell(\gamma) \geq 2 \). Let \( f : Y_{\gamma,L} \to Z_f \) be the element of \( R^0_{\gamma,L} \) corresponding to \( H \). By the analysis in the proof of Lemma 5.1, as a graph of spaces, \( Z_f \) is a path of vertex-spaces with trivial groups, between two vertex-spaces representing order two subgroups. Assume that the path, excluding the vertex-spaces at the two ends, consists of \( s \) “vertex spaces”, namely, it spells out an element \( \delta \in \Gamma \) with \( \ell(\delta) = s \). Assume that \( H = \pi^\lab(Z_f,u) \) for some vertex \( u \). Because \( \gamma \) is cyclically reduced, the closed path at \( u \) corresponding to \( \gamma^L \) starts by leaving \( u \) to one direction (say, to the right), and ends by arriving to \( u \) from the other direction (say, from the left). Thus \( \ell(\gamma^L) = L \cdot \ell(\gamma) = Lq \cdot \ell(\gamma_0) \) is equal to some multiple of \( (2 + 2s) \). So knowing that \( H \in \mathcal{H}_{\gamma,L} \) for some \( L \), we may find \( L \) simply as the smallest positive integer \( L \) satisfying that \( 2 + 2s \mid Lq \cdot \ell(\gamma_0) \). Any conjugate of \( H \) has the same parameter \( s \) (it is the same graph-of-spaces, only, possibly, with a different basepoint), so if it belongs to any \( \mathcal{H}_{\gamma,L} \), it must belong to the same \( \mathcal{H}_{\gamma,L} \) as \( H \) does.

\[ \square \]

Proof of Theorem 1.15. We begin with the first part of the theorem: that \( \mathbb{E}[\cyc_{\gamma,L}(N)] = \frac{1}{L} |\mathcal{H}_{\gamma,L}| + O \left( N^{-1/m} \right) \), where \( \mathcal{H}_{\gamma,L} \) is the set of EC-zero subgroups of \( \Gamma \) containing \( \gamma^L \) but not any smaller power of \( \gamma \). Note that
\[
\fix_{\gamma,L}(N) = \sum_{1 \leq d \leq L} d \cdot \cyc_{\gamma,d}(N),
\]
so this part of the theorem follows from Theorem 1.4 by a simple induction on \( L \). Indeed, when \( L = 1 \) this is precisely Theorem 1.4. For general \( L \), we get by induction that
\[
L \cdot \mathbb{E}[\cyc_{\gamma,L}(N)] = \mathbb{E}[\fix_{\gamma,L}(N)] - \sum_{1 \leq d < L, d|L} d \cdot \mathbb{E}[\cyc_{\gamma,d}(N)].
\]
For the second part of Theorem 1.15, recall that \( H_1, \ldots, H_t \) are representatives of the conjugacy classes of subgroups represented in \( \mathcal{H}_{\gamma,L} \), and \( \alpha_1, \ldots, \alpha_t \) and \( \beta_1, \ldots, \beta_t \) are defined analogously to their definition in Theorem 1.5. By Lemma 6.2, \( H_1, \ldots, H_t \) can be taken to be a subset of the representatives of \( \mathcal{H}_{\gamma,L} \) from Theorem 1.5, which are not conjugate to any element of \( \mathcal{H}_{\gamma,L'} \) with \( L' < L \).

This part of Theorem 1.15 states that \( \cyc_{\gamma,L}(N) \xrightarrow{\text{dis}}_{N \to \infty} \frac{1}{L} \sum_{i=1}^t \alpha_i \beta_i Z_{1/\beta_i} \). As in the first part, this can be deduced from Theorem 1.5 applied to \( \fix_{\gamma,L}(N) \) by a simple induction on \( L \). Indeed,
\[
L \cdot \cyc_{\gamma,L}(N) = \fix_{\gamma,L}(N) - \sum_{1 \leq d < L, d|L} d \cdot \cyc_{\gamma,d}(N)
\]
and applying Theorem 1.5 to \( \fix_{\gamma,L}(N) \) and the induction hypothesis on \( d \cdot \cyc_{\gamma,d}(N) \) for \( 1 \leq d < L, d|L \), we obtain the result.

Finally, the third part of Theorem 1.15 states that \( \cyc_{\gamma,1}(N), \cyc_{\gamma,2}(N), \ldots, \cyc_{\gamma,L}(N) \) are asymptotically independent, and that for \( L_1 \neq L_2 \) we have \( \mathbb{E}[\cyc_{\gamma,L_1}(N) \cdot \cyc_{\gamma,L_2}(N)] = \mathbb{E}[\cyc_{\gamma,L_1}(N)] \cdot \mathbb{E}[\cyc_{\gamma,L_2}(N)] + O \left( N^{-1/m} \right) \). The argument here is the same as in the proof of Theorem 1.14, where Lemma 6.2 replaces the assumption in part 2 of Theorem 1.14.

\[ \square \]
7 Open questions

There are several questions the current paper raises. We discuss here two we find most appealing.

The leading term of \( E[\text{fix}_\gamma (N)] - 1 \)

Recall that when \( \Gamma \) is a free group and \( 1 \neq \gamma \in \Gamma \), Theorem 1.4, which is originally due to [Nic94] in this case, says that \( E[\text{fix}_\gamma (N)] = d(q) + O \left( N^{-1} \right) \), where \( q \in \mathbb{Z}_{\geq 1} \) is maximal so that \( \gamma \) is a \( q \)-th power, and \( d(q) \) the number of positive divisors of \( q \). In particular, when \( \gamma \) is a non-power, then \( E[\text{fix}_\gamma (N)] = 1 + O \left( N^{-1} \right) \).

But, in fact, much more is known. For \( \gamma \) in a free group \( \Gamma \), denote

\[
\chi^{\text{max}} (\gamma) \overset{\text{def}}{=} \max \{ \chi (H) \mid \gamma \in H \leq \Gamma, \text{ \( \gamma \) non-primitive in } H \}, \tag{7.1}
\]

and let \( \text{Crit} (\gamma) \) denote the number of subgroups attaining the maximum from (7.1). Then [PP15, Thm. 1.8] states that for every \( \gamma \in \Gamma \),

\[
E[\text{fix}_\gamma (N)] - 1 = |\text{Crit} (\gamma)| \cdot N^{\chi^{\text{max}}(\gamma)} \left( 1 + O \left( N^{-1} \right) \right).
\]

Notice that this estimate is true for proper powers as well and even for the identity element. We conjecture that the same phenomenon is true for the family of groups considered in this paper.

**Conjecture 7.1.** In the notation of Assumption 1.1, let \( \gamma \in \Gamma \) be a non-torsion element. Denote

\[
\chi^{\text{max}} (\gamma) \overset{\text{def}}{=} \max \left\{ \chi (H) \left| \begin{array}{c}
\gamma \in H \leq \Gamma, \\
\langle \gamma \rangle \text{ is not a free factor isomorphic to } \mathbb{Z} \text{ of } H
\end{array} \right. \right\},
\]

and let \( \text{Crit} (\gamma) \) denote the number of subgroups \( H \leq \Gamma \) satisfying the conditions in the definition of \( \chi^{\text{max}} (\gamma) \) with \( \chi(H) = \chi^{\text{max}}(\gamma) \). Then

\[
E[\text{fix}_\gamma (N)] - 1 = |\text{Crit} (\gamma)| \cdot N^{\chi^{\text{max}}(\gamma)} \left( 1 + O \left( N^{-1/m} \right) \right).
\]

The fact that \( \text{Crit} (\gamma) \) is finite can be shown using the techniques of the current paper. By Theorem 1.4, this conjecture is true for any element \( \gamma \) with \( |H_\gamma| \geq 1 \). Here are a few other examples illustrating the conjecture:

- Let \( \Gamma = \Lambda_2 = \langle a, b, c, d \mid [a, b] [c, d] \rangle \) be the genus-2 surface group. Consider \( \gamma = a \). It is possible to obtain the following estimate:

\[
E[\text{fix}_a (N)] = 1 + \frac{1}{N^2} + \frac{2}{N^3} + \frac{10}{N^4} + O \left( \frac{1}{N^5} \right).
\]

It seems that \( a \) is primitive in every free subgroup of \( \Gamma \) containing it, so the only subgroups containing it not inside a proper free factor are the finite-index subgroups, which are all surface groups. Among these, \( \Gamma \) itself has maximal Euler characteristic: \( \chi (\Gamma) = 2 - 2g = -2 \). So \( \chi^{\text{max}} (\gamma) = -2 \) and \( \text{Crit} (\gamma) = \{ \Gamma \} \). This agrees with the conjecture.

- Let \( \Gamma = \Lambda_2 = \langle a, b, c, d \mid [a, b] [c, d] \rangle \) again, and consider \( \gamma = [a, b] \). Using a computer, Michal Buran carried out a computation showing that most likely

\[
E[\text{fix}_{[a, b]} (N)] = 1 + \frac{2}{N} + O \left( \frac{1}{N^2} \right).
\]

This seems to agree with the conjecture as \( [a, b] \) is a non-primitive element in two free subgroups of Euler characteristic \(-1; \langle a, b \rangle \) and \( \langle c, d \rangle \). So \( \chi^{\text{max}} (\gamma) = -1 \), and most likely \( \text{Crit} (\gamma) = \{ \langle a, b \rangle, \langle c, d \rangle \} \).
Now consider \( \Gamma = C_3 \ast C_3 \ast C_3 = \langle x \rangle \ast \langle y \rangle \ast \langle z \rangle \), and let \( \gamma = xyz \). The resolution \( R_\gamma \) from Section 5 contains five elements, corresponding to the subgroups \( \langle \gamma \rangle, \langle x, yz \rangle, \langle xz, xyx^{-1} \rangle, \langle xy, z \rangle, \langle x, y, z \rangle \). It is possible to show that any critical subgroup of \( \gamma \) must be found inside this collection. But \( \langle \gamma \rangle \) is not critical by definition, \( \langle x, yz \rangle \cong C_3 \ast Z \) has also the decomposition \( \langle x \rangle \ast \langle xyz \rangle \) so \( \gamma \) belongs to a proper free factor. Similarly, \( \langle xz, xyx^{-1} \rangle = \langle xyz \rangle \ast \langle xyx^{-1} \rangle \) and \( \langle xy, z \rangle = \langle xyz \rangle \ast \langle z \rangle \). This leaves us with \( \langle x, y, z \rangle \) where \( \langle x, y, z \rangle \) does not seem to belong to a proper free factor. We conclude that most likely, \( \chi^{\max} (\gamma) = \chi ((x, y, z)) = \chi (\Gamma) = -1 \), and \( \mathrm{Crit} (\gamma) = \{ \Gamma \} \). We thus expect that \( \mathbb{E} [\mathrm{fix}_\gamma (N)] - 1 = N^{-1} + O \left( N^{-4/3} \right) \).

Using the results of [Mül97], we may compute the leading terms of \( \mathbb{E}^{\text{emb}}_Y (N) \) for these five sub-covers.

We get the following. For \( \langle \gamma \rangle \) we get \( 1 - 3N^{-2/3} + O \left( N^{-4/3} \right) \); For \( \langle x, yz \rangle \) we get \( N^{-2/3} + O \left( N^{-4/3} \right) \), and by symmetry, the same leading term apply to \( \langle xz, xyx^{-1} \rangle \) and to \( \langle xy, z \rangle \); For \( \langle x, y, z \rangle \) we get \( N^{-1} + O \left( N^{-5/3} \right) \). Overall the coefficients of \( N^{-2/3} \) cancel out and we get \( \mathbb{E} [\mathrm{fix}_\gamma (N)] = 1 + N^{-1} + O \left( N^{-4/3} \right) \), which agrees with the conjecture.

**Scope of the phenomena described in this paper**

We are curious as to what extent the results of this paper can be generalized to a larger family of groups. As noted in Remark 1.9, Theorem 1.4 does not hold when \( \Gamma = \mathbb{Z}^2 \). Other results in this paper, such as Theorem 2.6, require that \( \chi (H) \leq 1 \) for any subgroup of \( \Gamma \); indeed, the number of (injective) lifts of a connected sub-cover to an arbitrary \( N \)-cover is bounded from above by Assumption 1.1. For example, we suspect they are true for fundamental groups of non-orientable surfaces of negative Euler characteristic, and more generally to all Fuchsian groups. They may also hold for general amalgams of finite groups. What it the widest generality? Do some of the results, e.g., the asymptotic expansion of Theorem 1.13, apply nonetheless to groups such as \( \mathbb{Z}^2 \)?

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