ELLIPTIC SOLITONS AND HEUN’S EQUATION

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Abstract. We find a new class of algebraic geometric solutions of Heun’s equation with the accessory parameter belonging to a hyperelliptic curve. Dependence of these solutions from the accessory parameter as well as their relation to Heun’s polynomials is studied. Methods of calculating the algebraic genus of the curve, and its branching points, are suggested. Monodromy group is considered. Numerous examples are given.

INTRODUCTION

The Heun equation (see, for instance, [1, 2, 3, 4])

\[
\frac{d^2 y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-a)} y = 0,
\]

(0.1)

where

\[1 + \alpha + \beta - \gamma - \delta - \varepsilon = 0,
\]

is a Fuchsian equation with four regular singularities at 0, 1, a and \(\infty\). Recently, it started to attract much attention due to a large number of its applications in mathematics and physics [5, 6, 7, 8, 9, 10].

As a matter of fact, this equation has been known for a very long time. Already in 1882 Darboux mentioned that it is the next one beyond the hypergeometric equation and that “it has already been studied by many geometers” [11]. In the works of Darboux this equation appears when he investigated the minimal surfaces in [12] and when he studied harmonic equations in [13]. In the work [11] Darboux points out at a close connection between the equation (0.1) and the generalization of Lamé’s equation

\[
\psi_{xx} - u(x)\psi = E\psi,
\]

(0.2)

\[
u(x) = m_0(m_0+1)\varphi(x) + \sum_{i=1}^{3} m_i(m_i+1)\varphi(x - \omega_i),
\]

(0.3)

\[
\varphi(\omega_i) = e_i, \quad \varphi(x - 2\omega_i) \equiv \varphi(x).
\]

Here \(\varphi(x)\) is the Weierstrass function [14],

\[
[\varphi'(x)]^2 = 4\varphi^3(x) - g_2\varphi(x) - g_3 = 4 \prod_{j=1}^{3} (\varphi(x) - e_j).
\]

(0.4)

In the same work Darboux remarks that the equation (0.2), (0.3) appeared already in the works of Hermite on solution of a special case (integer \(N\)) of Lamé’s equation:

\[
\psi_{xx} - N(N+1)\varphi(x)\psi = E\psi.
\]

(0.5)

Using the methods “proposed by Hermite in 1872 in the course of lectures in École Polytechnique”, Darboux solves the equation (0.2), (0.3) with integer \(m_i\) and points out that one can find solutions of the equation (0.2), (0.3) for any value of \(E\) in other special cases.
In particular, “when all \( m_i \) are half-odd-integer and \( N \) is odd integer” \(^1\). Unfortunately, until recently the work \([11]\) was not well-known\(^2\), in connection with this the equation \((0.2), (0.3)\) has the name of Treibich and Verdier who proved the finite-gapness of the potential \((0.3)\) for any integer \( m_i \) \([15, 16, 17]\).

In the present paper we study exact solutions of a special case of Heun’s equation:

\[
\frac{d^2 y}{dz^2} + \frac{1}{2} \left( \frac{1 - 2m_1}{z} + \frac{1 - 2m_2}{z - 1} + \frac{1 - 2m_3}{z - a} \right) \frac{dy}{dz} + \frac{N(N - 2m_0 - 1)z + \lambda}{4z(z - 1)(z - a)} y = 0, \tag{0.6}
\]

where

\[
N = m_0 + m_1 + m_2 + m_3, \tag{0.7}
\]

\(m_0, m_1, m_2, m_3 \in \mathbb{Z}, \quad \lambda, z \in \mathbb{C}.
\]

It is well-known (see, e.g., \([1, 2, 3, 4]\)) that by putting \( m_1 = m_2 = m_3 = 0, \quad m_0 = N \)

and changing the independent variable

\[
z = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad a = \frac{e_3 - e_1}{e_2 - e_1}, \tag{0.8a}
\]

\[
y(z) = \psi(x), \quad \lambda = \frac{E - N(N + 1)e_1}{e_1 - e_2}, \tag{0.8b}
\]

this equation is reduced to the familiar Lamé’s equation \((0.5)\). When \( N \in \mathbb{N} \) the solutions of the equation \((0.5)\) are the eigenfunctions of the Schrödinger operator \((0.2)\) with the \(N\)-gap elliptic potential

\[
u(x) = N(N + 1)\varphi(x).
\]

They can be found either in terms of solutions of a differential equation satisfied by products of these eigenfunctions \([4, 6, 18, 14]\) or by using the Krichever-Hermite ansatz \([19, 20, 21, 22]\).

In general, i.e. when \( m_i \in \mathbb{C} \), the change of variables (see, for instance, \([11, 12, 13]\))

\[
z = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad a = \frac{e_3 - e_1}{e_2 - e_1}, \tag{0.9a}
\]

\[
y(z) = \psi(x) \prod_{i=1}^{3} (\varphi(x) - e_i)^{m_i/2}, \quad \lambda = \frac{E}{e_1 - e_2} + \text{const}, \tag{0.9b}
\]

transforms Heun’s equation into Schrödinger’s equation \((0.2)\) with the potential \((0.3)\), which for integer \( m_i \) will be finite-gap \(^3\).

Because the Treibich-Verdier equation \((1.2), (1.3)\) with integer \( m_i \) has been actively studied over the years by many researchers \([21, 24, 15, 16, 17]\), including the author \([23, 24, 25]\), it became possible to apply the obtained results to the theory of solutions of Heun’s equation.

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\(^1\) so far we do not know by which method one can find solutions of this special case of Heun’s equation

\(^2\) the author learned about the work \([11]\) in 1999 from prof. V.B.Matveev

\(^3\) changes of variables \((0.8), (0.9)\) are easily generalized to the case of Schrödinger’s equation with any even elliptic potential. Based on our results in \([23, 25]\) we plan a further study of a special case of the Fuchsian equation with five singularities
In particular, in this paper it is proved that for any nonnegative integer $m_i$ and for all $\lambda \neq \lambda_j$ ($j = 1, \ldots, 2g + 1$) the solutions of the Heun equation (1.6) are the functions of the following form:

$$Y_{1,2}(m_0, m_1, m_2, m_3; \lambda; z) = \sqrt{\Psi_{g,N}(\lambda, z)} \exp \left( \pm \frac{i\nu(\lambda)}{2} \int \frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3} \, dz}{\Psi_{g,N}(\lambda, z) \sqrt{z(z-1)(z-a)}} \right).$$

Here $i^2 = -1$,

$$\Gamma: \quad \nu^2 = \prod_{j=1}^{2g+1} (\lambda - \lambda_j), \quad \lambda_j = \lambda(E_j),$$

$E_j$ are the gap edges of the finite-gap elliptic potential $u(x)$ (0.3) and $\Psi_{g,N}(\lambda, z)$ is some polynomial of the degree $N$ in $z$ and of the degree $g$ in $\lambda$. We develop the methods of calculating the polynomial $\Psi_{g,N}(\lambda, z)$, the branching points $\lambda_j$ of the algebraic curve $\Gamma = \{ (\nu, \lambda) \}$ and, respectively, the gap edges $E_j$ of the spectrum of the Treibich-Verdier potential. We find the dependence of the genus $g$ of the curves $\Gamma$ and $\tilde{\Gamma} = \{ (w, E) \}$,

$$\tilde{\Gamma}: \quad w^2 = \prod_{j=1}^{2g+1} (E - E_j),$$

as a function of the characteristics (exponents) $m_i$. The monodromy group of these solutions is also considered.

We also prove that for any integer $m_i$'s and for $\lambda = \lambda_j$ the solutions of Heun’s equation are expressed in terms of Heun polynomials [2, 3, 4], i.e. Heun polynomials with half-odd-integer $\beta - \alpha, \gamma, \delta, \epsilon$ are special cases of the found solutions.

The needed facts from the theory of finite-gap elliptic potentials for the Schrödinger operator are collected without proof in the first paragraph.

Because the exact solutions of the Heun equation are of much interest, in the Appendix we give several simplest solutions of the Heun equation belonging to this class.

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1. **Schrödinger operator with finite-gap elliptic potential**

Proposition 1 ([23, 27, 28]). For any $m_i \in \mathbb{Z}$ the Treibich-Verdier potential (0.3) is a finite-gap elliptic potential for the Schrödinger operator (0.2). For any nonnegative integer $m_i$ the following formula for the genus $g$ of the spectral curve $\tilde{\Gamma}$ (0.10) for the potential (0.3) is true:

$$g = \frac{1}{2} \max \left\{ 2 \max_{0 \leq i \leq 3} m_i, 1 + N - (1 + (-1)^N) \left( \min_{0 \leq i \leq 3} m_i + \frac{1}{2} \right) \right\}. \quad (1.1)$$

Proposition 2 ([23, 30, 31, 32]). Any $g$-gap potential $u(x)$ of the Schrödinger operator (0.2) satisfies the Novikov equation:

$$J_g + \sum_{m=1}^{g} c_m J_{g-m} = d, \quad (1.2)$$
or, which is the same, the stationary ‘higher’ Korteweg-de Vries (KdV) equation:

\[ \partial_x \left( J_g + \sum_{m=1}^{g} c_m J_{g-m} \right) = 0. \]

Here \( c_m, d \) are constants and the functions \( J_m \) are the flows of the ‘higher’ KdV equations

\[ \partial_{t_m} u = \partial_x J_m. \]

The expressions for the flows \( J_m \) are found from the relations

\[ L \psi = \psi_{xx} - 4u \psi + 2u_x \int_{x}^{\infty} \psi(\tau) d\tau, \]  
\[ (J_n)_x = L^n(u_x). \]

where \( u(x) \) is a potential decreasing fast at \( \infty \). In particular,

\[ J_0 = u, \quad J_1 = u_{xx} - 3u^2, \quad J_2 = u_{xxxx} - 10u_{xx}u - 5u_x^2 + 10u^3, \]
\[ J_3 = u_{xxxxxx} - 14u u_{xxxx} - 28u_x u_{xxx} - 21u^2_{xx} + 70u^2u_{xx} + 70uu_x^2 - 35u^4. \]

In the case of decreasing fast at \( \infty \) potential the variables

\[ C_j = \int_{-\infty}^{\infty} J_j(x,t) dx \]

constitute an infinite set of integrals of motion for the KdV equation

\[ \partial_t u = \partial_x J_1. \]

**Proposition 3** ([33, 32, 30]). The function

\[ \hat{\Psi}(x, E) = E^g + \sum_{j=1}^{g} \gamma_j(x) E^{g-j}, \]  
\[ (1.4) \]

where

\[ \gamma_j(x) = -\frac{2}{4^j} \left( J_{j-1} + \sum_{m=1}^{j-1} c_m J_{j-m-1} - \frac{c_j}{2} \right), \]  
\[ (1.5) \]

obeys the equation

\[ \hat{\Psi}_{xxx} = 4(u + E) \hat{\Psi}_x + 2u_x \hat{\Psi}, \]

the solutions of which are the products of any two solutions of the equation (0.2). Here \( c_j, J_j \) are the same as in (1.2) and \( u(x) \) is a \( g \)-gap potential of the Schrödinger operator (0.2).

**Proposition 4** ([33, 20, 21, 22]). The equation (0.2) with the potential of Treibich-Verdier (0.3) has a solution of the form (Krichever-Hermite ansatz)

\[ \psi(x; k, \alpha) = \sum_{i=0}^{3} \sum_{j=0}^{m_i-1} a_{ij}(k, \alpha) e^{kx} \frac{d^j}{dx^j} \Phi(x - \omega_i, \alpha). \]  
\[ (1.6) \]

Here \( k, \alpha \) are auxiliary spectral parameters, \( \omega_0 \equiv 0, \)

\[ \Phi(x, \alpha) = \frac{\sigma(\alpha - x)}{\sigma(x) \sigma(\alpha)} \exp\{\zeta(\alpha)x\}, \]  
\[ (1.7) \]

and \( \sigma(\alpha), \zeta(\alpha) \) are Weierstrass’ elliptic functions [14].
Substituting the ansatz (1.6) for the $\psi$-function into the equation (0.2) one can find the equation of the spectral curve $\Gamma$ for the finite-gap potential $u(x)$

$$\Gamma = \{(k, \alpha)\} : \quad R(k, \alpha) = 0,$$

as well as the dependence of the coefficients $a_{ij}(k, \alpha)$ and the main spectral parameter $E$

$$E(-k, -\alpha) = E(k, \alpha)$$

as functions of the auxiliary spectral parameters $k$ and $\alpha$.

The gap edges of the spectrum of the finite-gap potential $u(x)$ are the values of the spectral parameter $E$ calculated at the points $\mathcal{P}$ of the curve $\Gamma$

$$E_j = E(\mathcal{P}_j), \quad \tau \mathcal{P}_j = \mathcal{P}_j, \quad j = 1, \ldots, 2g + 2, \quad E_{2g+2} = \infty.$$

which are invariant with respect to the involution $\tau$

$$\tau : (k, \alpha) \rightarrow (-k, -\alpha).$$

Linear independent solutions of the equation (0.2) for all values of the spectral parameter $E$, except for the gap edges $E_j$, are the functions

$$\psi_1(x, E) = \psi(x; k, \alpha) \quad \text{and} \quad \psi_2(x, E) = \psi(x; -k, -\alpha). \quad (1.8)$$

A product of these functions is a doubly periodic function and, modulo the normalizing factor, it is equal to $\tilde{\Psi}(x, E)$ (1.4), (1.5).

## 2. Finite-gap solutions of Heun’s equation

Since the potential (0.3) is finite-gap, one can introduce the notion of ‘finite-gap’ solutions to the Heun equation which are obtained from the finite-gap solutions of the Treibich-Verdier equation by the inverse change of variables (see (0.8))

$$\varphi(x) = e_1 + (e_2 - e_1)z, \quad e_2 = \frac{a - 2}{a + 1}e_1, \quad e_3 = \frac{1 - 2a}{a + 1}e_1, \quad (2.1a)$$

$$E = (e_1 - e_2)\lambda + \text{const}, \quad u(x) = 4(e_2 - e_1)U(z) + \text{const}. \quad (2.1b)$$

**Definition.** By Novikov’s equation for the Heun equation of order $g$ we will call the following equality:

$$I_g + \sum_{j=1}^{g} \tilde{c}_j I_{g-j} = \tilde{d}, \quad (2.2)$$

where $\tilde{c}_j$, $\tilde{d}$ are some constants,

$$I_0 = U(z), \quad I_{j+1} = \mathcal{L}(I_j), \quad (2.3a)$$

$$\mathcal{L}(f) = z(z-1)(z-a) \frac{d^2 f}{dz^2} + \frac{3z^2 - 2(a+1)z + a}{2} \cdot \frac{df}{dz} - \int \left(4I_0 \frac{df}{dz} + 2f \frac{dI_0}{dz}\right) dz, \quad (2.3b)$$

$U(z)$ is the ‘potential’ (2.1b).

$$U(z) = \frac{m_0(m_0 + 1)}{4} \cdot \frac{a}{z} + \frac{m_1(m_1 + 1)}{4} \cdot \frac{a}{z} + a$$

The term “potential” attached to the function $U(z)$ is used here only in analogy with the function $u(x)$, and it does not carry any additional meaning.
\[ \frac{m_2(m_2 + 1)}{4} \cdot \frac{z - a}{z - 1} + \frac{m_3(m_3 + 1)}{4} \cdot \frac{a(z - 1)}{z - a}. \] (2.4)

**Theorem 1.** The ‘potential’ \( U(z) \) (2.4) satisfies Novikov’s equation for the Heun equation of order \( g \) (2.2), (2.3) if and only if the potential \( u(x) \) (1.3) satisfies Novikov’s equation (1.2), (1.3) of the same order.

**Definition.** ‘Potential’ \( U(z) \) will be called \( g \)-gap potential if it satisfies Novikov’s equation for the Heun equation of order \( g \) (2.2), (2.3).

**Remark 1.** The flows \( J_m \) (1.3) in the Novikov equation (1.2) were originally defined for the potentials decreasing fast at \( \infty \). Although for finite-gap as well as for any periodic potentials these functions can not be obtained from the equations (1.3) because of the divergent integral with an infinite upper limit, the expressions for \( J_m \) are still valid in the case of a finite-gap potential. The constant of integration in the definition of the ‘flows’ \( I_n \) (2.3) can not be easily fixed too. Because of that, all the ‘flows’ \( I_n \) are defined modulo a linear combination of lower order ‘flows’. However, it is easily seen that the property of ‘finite-gapness’ of the ‘potential’ \( U(z) \) (2.4) does not depend from the concrete values of integration constants in the definition of \( I_n \). The latter affect only the values of the constants \( \tilde{c}_m \) and \( \tilde{d} \) in the equation (2.2).

**Theorem 2.** For any \( m_i \in \mathbb{Z} \) there exists a nonnegative integer \( g \) such that the ‘potential’ \( U(z) \) (2.4) satisfies Novikov’s equation for the Heun equation of order \( g \) (2.2), (2.3).

**Proof.** Assume that all characteristics \( m_i \) in the equation (2.4) are nonnegative. If anyone \( m_i < 0 \) then one can always make the transformation \( m_i \to -m_i - 1 \) which does not change the ‘flows’ \( I_n \) (2.3), (2.4).

From the properties of the flows \( J_n \) (1.3) (see, e.g., [31]), from the properties of elliptic functions [14] and from the equation of the change of variable (2.1) it follows that all ‘flows’ \( I_n \) (2.3) are rational functions of the variable \( z \) (i.e. these functions do not have logarithmic singularities).

It is also not very difficult to check that the order of poles of the functions \( I_n(z) \) in the singularities of Heun’s equation does not exceed the corresponding characteristic \( m_i \).

Indeed, it is easy to verify that for \( m_1 = 0 \) neither \( I_0(z) = U(z) \) (2.4) nor all other \( I_n(z) \) have a pole at \( z = 0 \), but for \( m_1 \neq 0 \) the function \( I_0(z) \) has at this point a pole of first order.

If we now assume that the function \( I_n(z) \) has at the point \( z = 0 \) a pole of order \( \alpha \leq m_1 \) (which is undoubtedly true for \( n = 0 \) and \( m_1 \neq 0 \)):

\[ I_n(z) = \frac{A}{z^\alpha} + O(z^{1-\alpha}), \quad z \to 0, \]

then we obtain that the function \( I_{n+1}(z) \),

\[ I_{n+1}(z) = \mathcal{L}(I_n(z)) = \frac{(2\alpha + 1)(\alpha - m_1)(\alpha + m_1 + 1)}{2(\alpha + 1)} \cdot \frac{aA}{z^{\alpha+1}} + O(z^{-\alpha}), \]

has at the same point a pole of order \( \alpha' = \min\{m_1, \alpha + 1\} \). By similar means one can check the other poles of the function \( I_n(z) \).

Hence, for any \( n \) the dimension of the linear span of rational functions \( 1, I_0, \ldots, I_n \) does not exceed \( N + 1 \) and therefore there exists a number \( g \),

\[ \max_{0 \leq i \leq 3} m_i \leq g \leq N, \quad (2.5) \]
Corollary 1. For any \( m_i \in \mathbb{Z} \) the potentials \( U(z) \) \((2.4)\) and \( u(x) \) \((1.3)\) are finite-gap. The genus \( g \) of the spectral curve \( \Gamma \), associated with the finite-gap potential \( u(x) \) \((0.3)\), satisfies the inequality \((2.5)\).

Definition. Solutions of the Heun equation \((0.9)\) with integer characteristics \( m_i \) will be called finite-gap solutions of Heun’s equation.

Remark 2. As it was said above (cf. Proposition \[4\]) the proof of finite-gapness of the potential \( u(x) \) \((0.3)\) was first given in the works by Treibich and Verdier \([16, 17]\). Here we gave another proof which allowed us to specify the explicit form of the finite-gap solutions of Heun’s equation.

In order to find solutions of Heun’s equation \((0.6)\) let us use the results of the theory of finite-gap elliptic potentials for the Schrödinger operator and consider the equation

\[
\frac{d^3 \Psi}{dz^3} + 3P(z) \frac{d^2 \Psi}{dz^2} + \left( P'(z) + 4Q(z) + 2P^2(z) \right) \frac{d \Psi}{dz} + (2Q'(z) + 4P(z)Q(z)) \Psi = 0,
\]

\[(2.6)\]

\[
P(z) = \frac{1}{2} \left( \frac{1 - 2m_1}{z} + \frac{1 - 2m_2}{z - 1} + \frac{1 - 2m_3}{z - a} \right),
\]

\[
Q(z) = \frac{N(N - 2m_0 - 1)z + \lambda}{4z(z - 1)(z - a)},
\]

solutions of which are the products of any two solutions of Heun’s equation \((0.6)\).

Theorem 3. Equation \((2.6)\) with nonnegative integer characteristics \( m_i \) has as its solution the function \( \Psi_{g,N}(\lambda, z) \), which is a polynomial in \( \lambda \) of the degree \( g \) and in \( z \) of the degree \( N \) \((0.7)\). The leading coefficient of this function considered as a polynomial in \( \lambda \) is equal

\[
\tilde{a}_0(z) = z^{m_1}(z - 1)^{m_2}(z - a)^{m_3}.
\]

\[(2.7)\]

In other words,

\[
\Psi_{g,N}(\lambda, z) = a_0(\lambda)z^N + a_1(\lambda)z^{N-1} + \ldots + a_N(\lambda) = \]

\[
z^{m_1}(z - 1)^{m_2}(z - a)^{m_3} \lambda^g + \tilde{a}_1(z)\lambda^{g-1} + \ldots + \tilde{a}_g(z).
\]

\[(2.8)\]

Proof. From \((1.4)\)–\((1.8)\) it follows that the product \( \hat{\Psi}(x, E) \) of eigenfunctions of the Schrödinger operator with the Treibich-Verdier potential is an elliptic meromorphic function in the variable \( x \), and it is a polynomial of degree \( g \) in the spectral parameter \( E \). As a function of the variable \( x \) the function \( \hat{\Psi}(x, E) \) has poles of multiplicity \( 2m_i \) at the points \( x = \omega_j \) \((\omega_0 \equiv 0)\).

From \((1.4)\), \((1.3)\) and from the properties of the Weierstrass \( \wp \)-function it follows that \( \hat{\Psi}(x, E) \) is a rational function in \( \wp(x) \). Hence, the function

\[
\Psi_{g,N}(\lambda, z) = \text{const} \cdot \hat{\Psi}(x, E) \prod_{j=1}^{3} (\wp(x) - e_j)^{m_i},
\]

\[(2.9)\]
where $\lambda$ and $z$ are related with $E$ and $x$ by equalities (2.1), is a polynomial in $\lambda$ of degree $g$ and in $z$ of degree $N$ (i.e. it is a rational function in the variable $z$ with the unique pole of order $N$ at the point $z = \infty$). The constant in the equality (2.9) is chosen such that the leading coefficient of $\Psi_{g,N}(\lambda, z)$, considered as a polynomial in $\lambda$, to be equal (2.7).

**Corollary 2.** Coefficients $\tilde{a}_j(z)$ of the polynomial $\Psi_{g,N}(\lambda, z)$ have the form:

$$\tilde{a}_j(z) = z^{m_1}(z - 1)^{m_2}(z - a)^{m_3} \tilde{T}_{j-1}, \quad j = 1, \ldots, g,$$

(2.10)

where $\tilde{T}_j$ is a linear combination of the rational functions $I_j, \ldots, I_0, 1$ having poles in the singularities of Heun’s equation.

**Proof** follows from the equalities (1.5), (2.9) and from the change of variable (2.1).

Knowing the product of the solutions of Heun’s equation it is not difficult to find the solutions themselves (see, for instance, [18, §19.53, §23.7, §23.71]).

**Theorem 4.** Finite-gap solutions of Heun’s equation with nonnegative characteristics $m_i$ have the form

$$Y_{1,2}(m_0, m_1, m_2, m_3; \lambda; z) = \sqrt{\Psi_{g,N}(\lambda, z)} \exp \left( \pm \frac{\nu(\lambda)}{2} \int_{z_0}^z \frac{z^{m_1}(z - 1)^{m_2}(z - a)^{m_3}}{\Psi_{g,N}(\lambda, z) \sqrt{(z - 1)(z - a)}} dz \right),$$

(2.11)

Here $i^2 = -1$,

$$\Gamma = \nu^2 = \prod_{j=1}^{2g+1} (\lambda - \lambda_j), \quad \lambda_j = \lambda(E_j),$$

(2.12)

$E_j$ are the gap edges of the finite-gap elliptic potential $u(x)$ (0.3).

**Proof.** From the Liouville formula it follows that the Wronskian of two linearly independent solutions of the linear homogeneous differential equation,

$$y'' + P(z)y' + Q(z)y = 0,$$

has the following dependence from $z$:

$$W[y_1(\lambda, z), y_2(\lambda, z)] = W_0(\lambda) \exp \left\{- \int_{z_0}^z P(t) \, dt \right\},$$

where $W_0(\lambda)$ is Wronskian’s value at $z_0$. Hence, the Wronskian of two linearly independent solutions of Heun’s equation (1.6) is equal to

$$W[y_1(\lambda, z), y_2(\lambda, z)] = -i \nu(\lambda) \cdot \frac{z^{m_1}(z - 1)^{m_2}(z - a)^{m_3}}{\sqrt{(z - 1)(z - a)}},$$

(2.13)

where

$$\nu(\lambda) = i W_0(\lambda) \cdot \frac{\sqrt{z_0(z_0 - 1)(z_0 - a)}}{z_0^{m_1}(z_0 - 1)^{m_2}(z_0 - a)^{m_3}}.$$
we obtain a simple differential equation of the first order:

\[ \frac{y_2}{y_1} - \frac{y_1'}{y_1} = -i\nu(\lambda) \cdot \frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3}}{\Psi_{g,N}(\lambda, z)\sqrt{z(z-1)(z-a)}}. \]  

The equation (2.13) can be easily integrated:

\[ \frac{y_2(\lambda, z)}{y_1(\lambda, z)} = C(\lambda) \cdot \exp \left[ -i\nu(\lambda) \int_{z_1}^{z} \frac{t^{m_1}(t-1)^{m_2}(t-a)^{m_3} dt}{\Psi_{g,N}(\lambda, t)\sqrt{t(t-1)(t-a)}} \right], \]  

where \( C(\lambda) = \frac{y_2(\lambda, z_1)}{y_1(\lambda, z_1)} \).

If we now consider solutions \( Y_{1,2}(\lambda, z) \) with the same product (2.14) and Wronskian (2.13)

\[ Y_1(\lambda, z) = \sqrt{C(\lambda)} \cdot y_1(\lambda, z), \quad Y_2(\lambda, z) = \frac{y_2(\lambda, z)}{\sqrt{C(\lambda)}}, \]  

then from (2.14) and (2.16) we get (2.11).

Substituting the ansatz (2.11) into the Heun equation (0.6) we get

\[ \nu^2(\lambda) = \frac{2\Psi\Psi''-(\Psi')^2+2P(z)\Psi\Psi'+4Q(z)\Psi^2}{z^{2m_1-1}(z-1)^{2m_2-1}(z-a)^{2m_3-1}}, \]  

where

\[ \Psi = \Psi_{g,N}(\lambda, z), \quad \Psi' = \frac{d\Psi}{dz}, \quad \Psi'' = \frac{d^2\Psi}{dz^2}. \]

From (2.8), (2.13) it follows that \( \nu^2(\lambda) \) is a polynomial in \( \lambda \) of the degree \( 2g+1 \) with the leading coefficient equal to 1.

It is not difficult to show that under the change (0.9) the solutions \( Y_{1,2}(\lambda, z) \) of Heun’s equation (1.6) turn into Floquet solutions of the Treibich-Verdier equation (0.2), (0.3). Therefore, zeros \( \lambda_j \) \( (j = 1, \ldots, 2g+1) \) of the polynomial \( \nu(\lambda) \) or, which is the same, zeros of the Wronskian of the solutions \( Y_{1,2}(\lambda, z) \) correspond to zeros of the Wronskian of the Floquet solutions of the Treibich-Verdier equation, i.e. they correspond to the gap edges \( E_j \) \( (j = 1, \ldots, 2g+1) \) of spectrum of the Treibich-Verdier potential. Hence, the hyperelliptic curve \( \Gamma \) (2.12) is isomorphic to the spectral curve \( \tilde{\Gamma} \) (0.10) of the finite-gap elliptic potential \( u(x) \) (0.3).

**Remark 3.** The fact that the equation (2.6) for any nonnegative integer \( m_i \) and for any \( \lambda \) has as its solution a polynomial in \( z \) of degree \( N \) was known to Darboux who in the work [14] pointed out the method of finding some exact solutions of the generalized Lamé equation, which is now known as the ‘Treibich-Verdier equation’. However, he did not study the dependence of this polynomial and, respectively, solutions of the Heun equation, from \( \lambda \).

**Remark 4.** In order to find the ‘finite-gap’ solutions of Heun’s equation with negative characteristics one must use the equalities (see [14] [15] [16] [17]).

1) Negative \( m_1 \):

\[ Y(m_0, -m_1 - 1, m_2, m_3; \mu; z) = z^{-m_1-1/2}Y(m_0, m_1, m_2, m_3; \mu; z), \]  

where

\[ \mu = \lambda - (2m_1 + 1)(2m_2 - 1)a - (2m_1 + 1)(2m_3 - 1). \]
2) Negative $m_2$:

$$Y(m_0, m_1, -m_2 - 1, m_3; \lambda; z) = (z - 1)^{-m_2 - 1/2}Y(m_0, m_1, m_2, m_3; \mu; z), \quad (2.20)$$

where

$$\mu = \lambda - (2m_1 - 1)(2m_2 + 1)\alpha.$$

3) Negative $m_3$:

$$Y(m_0, m_1, m_2, -m_3 - 1; \lambda; z) = (z - a)^{-m_3 - 1/2}Y(m_0, m_1, m_2, m_3; \mu; z), \quad (2.21)$$

where

$$\mu = \lambda - (2m_1 - 1)(2m_3 + 1).$$

4) Negative $m_0$:

$$Y(-m_0 - 1, m_1, m_2, m_3; \lambda; z) = Y(m_0, m_1, m_2, m_3; \lambda; z). \quad (2.22)$$

3. Finite-gap solutions and Heun polynomials

Looking at the analytical properties of the solutions (2.11) it is not difficult to prove the following statements.

**Corollary 3.** For $\lambda \neq \lambda_j$ ($j = 1, \ldots, 2g + 1$), where $\lambda_j$ are the branching points of the hyperelliptic curve $\Gamma$ (2.12):

1) the polynomial $\Psi_{g,N}(\lambda, z)$ does not have zeros at the singularities of Heun’s equation;
2) the leading coefficient $a_0(\lambda)$ of the polynomial $\Psi_{g,N}(\lambda, z)$ is not equal to zero;
3) all zeros $z_k$ ($k = 1, \ldots, N$) of the polynomial $\Psi_{g,N}(\lambda, z)$ are simple zeros satisfying the relations

$$\prod_{j=1}^{N}(z_k - z_j)^2 = -\frac{\nu^2(\lambda)}{a_0^2(\lambda)}z_k^{2m_1-1}(z_k - 1)^{2m_2-1}(z_k - a)^{2m_3-1}.$$

**Corollary 4.** Zeros of the equations

$$a_0(\lambda) = 0, \quad (3.1a)$$
$$\Psi_{g,N}(\lambda, 0) = 0, \quad (3.1b)$$
$$\Psi_{g,N}(\lambda, 1) = 0, \quad (3.1c)$$
$$\Psi_{g,N}(\lambda, a) = 0, \quad (3.1d)$$
$$\Delta(\lambda) = 0, \quad (3.1e)$$

where $\Delta(\lambda)$ is the discriminant of the algebraic equation of $N$th order in $z$:

$$\Psi_{g,N}(\lambda, z) = 0,$$

are among the branching points $\lambda_j$ of the curve $\Gamma$ (2.12).

**Lemma 1.** For $\lambda = \lambda_j$ the solutions (2.11) cease to be linearly independent. In this case the linear independent ‘finite-gap’ solutions of Heun’s equation (1.6) are the functions

$$Y_1(m_0, m_1, m_2, m_3; \lambda_j; z) = \sqrt[\Psi_{g,N}(\lambda_j, z)](3.2a)$$
and
\[ Y_2(m_0, m_1, m_2, m_3; \lambda_j; z) = \sqrt{\Psi_{g,N}(\lambda_j, z)} \int \frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3} dz}{\Psi_{g,N}(\lambda_j, z) \sqrt{z(z-1)(z-a)}}. \] (3.2b)

**Corollary 5.** For \( \lambda = \lambda_j \) (\( j = 1, \ldots, 2g+1 \)), where \( \lambda_j \) are the branching points of the hyperelliptic curve \( \Gamma (2.12) \):

1) the polynomial \( \Psi_{g,N}(\lambda_j, z) \) has the form
\[ \Psi_{g,N}(\lambda_j, z) = z^{M_{1j}}(z-1)^{M_{2j}}(z-a)^{M_{3j}} F_{n_j}^2(z), \] (3.3)
where \( M_{ij} \in \{0, 2m_i + 1\} \) and \( F_{n_j}(z) \) is a polynomial in \( z \) of order \( n \);

2) the polynomial \( F_{n_j}(z) \) is a solution of Heun’s equation with the \( P \)-symbol
\[ P \left\{ \begin{array}{ccc} 0 & 1 & a \\ 0 & 0 & -n \\ 1/2 + \tilde{m}_{1j} & 1/2 + \tilde{m}_{2j} & -n + 1/2 + \tilde{m}_{0j} \end{array} \right\}, \] (3.4)
where
\[ \tilde{m}_{ij} = m_i - M_{ij}, \quad \tilde{m}_{ij} \in \{m_i, -m_i - 1\}, \quad i = 0, 1, 2, 3, \]
\[ n = (\tilde{m}_{0j} + \tilde{m}_{1j} + \tilde{m}_{2j} + \tilde{m}_{3j})/2, \]
\[ \tilde{\lambda}_j = \lambda_j + 2m_1M_{3j} + 2m_3M_{1j} - 2M_{1j}M_{3j} - M_{1j} - M_{3j} + \]
\[ + (2m_1M_{2j} + 2m_2M_{1j} - 2M_{1j}M_{2j} - M_{1j} - M_{2j})a, \]
i.e. the polynomial \( F_{n_j}(z) \) modulo a constant factor is a Heun polynomial (see, e.g., [4, 4]);

3) the order of the polynomial \( \Psi_{g,N}(\lambda_j, z) \), as a polynomial in \( z \), is equal to \( (N - M_{0j}) \);

4) all zeros \( z_k \) (\( k = 1, \ldots, n \)) of the polynomial \( F_{n_j}(z) \) are simple and satisfy the relations
\[ \sum_{i=1}^{n} \frac{1}{z_k - z_i} = 1 \left( \frac{2\tilde{m}_{1j} - 1}{z_k} + \frac{2\tilde{m}_{2j} - 1}{z_k - 1} + \frac{2\tilde{m}_{3j} - 1}{z_k - a} \right). \]

As a consequence, to every branching point \( \lambda_j \) (\( j = 1, \ldots, 2g+1 \)) there corresponds a set of numbers \( \tilde{m}_{ij} \) (\( i = 0, 1, 2, 3 \)) such that \( n \) \((3.5)\) is a nonnegative integer. Solutions of Heun’s equation for \( \lambda = \lambda_j \) are expressed in terms of Heun’s polynomials of order \( n \).

**Corollary 6.** For \( N \neq 0 \) every branching point \( \lambda_j \) of the hyperelliptic curve \( \Gamma (2.12) \) is a zero of one of the equations \((3.1)\).

**Proof** is done by assuming the opposite. Let there exists a branching point \( \lambda_j \) which is not a zero of anyone of the equations \((3.1)\). From the condition it follows that in this case the singularities of Heun’s equation are not the zeros of the polynomial \( \Psi_{g,N}(\lambda_j, z) \) \((3.3)\). Therefore
\[ M_{1j} = M_{2j} = M_{3j} = 0. \] (3.6)
From the equalities \((3.3)\), \((3.6)\) and from the condition \( \Delta(\lambda_j) \neq 0 \) it follows that \( \Psi_{g,N}(\lambda_j, z) = \text{const} \). But this, for \( N \neq 0 \), is in contradiction with the condition \( a_0(\lambda_j) \neq 0 \). Hence, every branching point \( \lambda_j \) of the hyperelliptic curve \( \Gamma (2.12) \) for \( N \neq 0 \) is a zero of one of the equations \((3.1)\).

\( \square \)
Lemma 2. For any set of integers \( m_i \) \((i = 0, 1, 2, 3)\) there exists not less than one and not more than four numbers \( n_k \) such that

\[
n_k = 1 + \frac{1}{2} \sum_{i=0}^{3} \tilde{m}_{ik}, \quad n_k \in \mathbb{N},
\]

\[
\tilde{m}_{ik} \in \{m_i, -m_i - 1\}, \quad i = 0, 1, 2, 3.
\]

Proof. Rewrite the equality (3.7) in the form

\[
n_k = \frac{1}{2} \sum_{i=0}^{3} (\tilde{m}_{ik} + \frac{1}{2}) = \frac{1}{2} \sum_{i=0}^{3} (-1)^{\varepsilon_{ik}} (m_i + \frac{1}{2}), \quad \varepsilon_{ik} \in \{0, 1\}.
\]

It is easy to see that from sixteen values which the number \( n_k \) can take there are only eight which are integer, and that to every positive number \( n_k \) there corresponds a sign-opposite negative one. Therefore the number of positive integer \( n_k \)'s can be not more than four.

On the other hand, all integer \( n_k \) can not be equal to zero because, as it is not difficult to show, this was possible only if all \( m_i = -1/2, \) \((i = 0, 1, 2, 3)\) which is in contradiction with the condition \( m_i \in \mathbb{Z}. \) Hence, among integer \( n_k \) there is at least one positive and one negative number.

Theorem 5. Let \( m_i \) \((i = 0, 1, 2, 3)\) be an arbitrary set of integers satisfying the condition:

\[
n = 1 + \frac{1}{2} \sum_{i=0}^{3} m_i, \quad n \in \mathbb{N}.
\]

Then:

1) there are exactly \( n \) values \( \lambda_j \) such that for \( \lambda = \lambda_j \) the Heun equation (1.6) has as its solution the Heun polynomial of order \( (n - 1) \);
2) the points \( \lambda = \lambda_j \) \((j = 1, \ldots, n)\) are the branching points of the hyperelliptic curve \( \Gamma \) (2.12) for the finite-gap solution of Heun’s equation with characteristics \( m_i \) (3.9).

Proof. For

\[
m_0 = m_1 = m_2 = m_3 = 0
\]

the statement of the Theorem is easily checked by the straightforward substitution of the polynomial of zero order into the Heun equation.

Now, let at least one of the characteristics \( m_i \) be different from zero. The first part of the statement is then a consequence of the general theory of Heun’s polynomials (see, e.g., [3, 4]).

The second part will be proved assuming the opposite. Let for some value \( \lambda = \lambda_1 \) a solution of Heun’s equation with the characteristics \( m_i \) (3.9) be a polynomial \( F(z) \) of order \( (n - 1) \). Also, assume that the point \( \lambda = \lambda_1 \) is not a branching point of the hyperelliptic curve \( \Gamma \) (2.12) for the finite-gap solution of Heun’s equation with these characteristics. Then one obtains that the 3-rd order equation (2.6) with nonnegative integer characteristics \( \tilde{m}_i \geq 0 \)

\[
\tilde{m}_i \in \{m_i, -m_i - 1\}, \quad i = 0, 1, 2, 3,
\]
has as its solution two polynomials of zero order:
\[
\mathcal{P}_1(z) = \Psi(\tilde{\lambda}_1, z),
\]
\[
\mathcal{P}_2(z) = z^{M_i}(z - 1)^{M_2}(z - a)^{M_3}F^2(z),
\]
\[
M_i = \tilde{m}_i - m_i, \quad i = 1, 2, 3,
\]
\[
\tilde{\lambda}_1 = \lambda_1 - 2m_1M_3 - 2m_3M_1 + 2M_1M_3 + M_1 + M_3 -
- (2m_1M_2 + 2m_2M_1 - 2M_1M_2 - M_1 - M_2)a.
\]

Moreover, because the point \( \lambda = \lambda_1 \) of the curve \( \Gamma \), as was assumed, is not a branching point, the polynomial \( \mathcal{P}_1(z) \) does not have multiple zeros and it also does not have zeros at the singularities of Heun’s equation (cf. Corollary 3). Hence the polynomials \( \mathcal{P}_1(z) \) and \( \mathcal{P}_2(z) \) are linearly independent. It is not difficult to show that the equation (2.6) cannot simultaneously have as its solutions the polynomials \( \mathcal{P}_1(z) \) and \( \mathcal{P}_2(z) \). This means that the point \( \lambda = \lambda_1 \) is a branching point of the hyperelliptic curve \( \Gamma \) (2.12) with the characteristics \( m_i \) (3.9).

Based on the statements of the Lemma 2, the Theorem 5 and Corollary 5 one can suggest one more method of finding the branching points of the curve \( \Gamma \) (2.12).

**Corollary 7.** In order to find all branching points \( \lambda_j \) of the hyperelliptic curve \( \Gamma \) (2.12) associated with ‘finite-gap’ solution of Heun’s equation with the characteristics \( m_i \) it is necessary and sufficient to find all Heun’s polynomials being the solutions of Heun equations (from one to four) with the characteristics \( \tilde{m}_i \) (3.7).

The algebraic genus \( g \) of the curve \( \Gamma \) (2.12) can be calculated by finding a sum of all \( n_k \in \mathbb{N} \) (3.7), i.e. by counting all the branching points \( \lambda_j \) (\( j = 1, \ldots, 2g + 1 \)).

**Corollary 8.** Let \( g \) be an algebraic genus of the hyperelliptic curves: a) \( \Gamma \) (2.12), which is associated with the ‘finite-gap’ solution of Heun’s equation, and b) \( \tilde{\Gamma} \) (0.10), which is a spectral curve of the Treibich-Verdier potential. If all characteristics \( m_i \) (\( i = 0, 1, 2, 3 \)) are nonnegative integer then:

1) in the case of even \( N \) (0.7)
\[
g = \max \left\{ \max_{0 \leq i \leq 3} m_i, \frac{N}{2} - \min_{0 \leq i \leq 3} m_i \right\}; \quad (3.10)
\]

2) in the case of odd \( N \)
\[
g = \max \left\{ \max_{0 \leq i \leq 3} m_i, \frac{N + 1}{2} \right\}. \quad (3.11)
\]

4. **Monodromy group of the finite-gap solutions**

Let \( a > 1, \lambda \neq \lambda_j, \text{Im} \nu(\lambda) = 0, m_i \geq 0 \). In the vicinity of points \( z = 0 \) and \( z = 1 \) let us consider the following ‘finite-gap’ solutions of Heun’s equation:

\[
y_1(z) = \sqrt{\Psi_{g,N}(\lambda, z)} \cos \left( \frac{\nu(\lambda)}{2} \int_0^z \frac{t^{m_1}(t - 1)^{m_2}(t - a)^{m_3}}{\Psi_{g,N}(\lambda, t) \sqrt{t(t - 1)(t - a)}} dt \right), \quad (4.1a)
\]
\[
y_1(z) = y_{10} + O(z), \quad z \to 0;
\]
\[
y_2(z) = \sqrt{\Psi_{g,N}(\lambda, z)} \sin \left( \frac{\nu(\lambda)}{2} \int_0^z \frac{t^{m_1} (t-1)^{m_2} (t-a)^{m_3} dt}{\Psi_{g,N}(\lambda, t) \sqrt{t(t-1)(t-a)}} \right),
\]
\[
y_2(z) = z^{m_1+1/2} (y_{20} + O(z)), \quad z \to 0;
\]
\[
y_3(z) = \sqrt{\Psi_{g,N}(\lambda, z)} \cos \left( \frac{\nu(\lambda)}{2} \int_1^z \frac{t^{m_1} (t-1)^{m_2} (t-a)^{m_3} dt}{\Psi_{g,N}(\lambda, t) \sqrt{t(t-1)(t-a)}} \right),
\]
\[
y_3(z) = y_{30} + O(1-z), \quad z \to 1;
\]
\[
y_4(z) = \sqrt{\Psi_{g,N}(\lambda, z)} \sin \left( \frac{\nu(\lambda)}{2} \int_1^z \frac{t^{m_1} (t-1)^{m_2} (t-a)^{m_3} dt}{\Psi_{g,N}(\lambda, t) \sqrt{t(t-1)(t-a)}} \right),
\]
\[
y_4(z) = (1-z)^{m_2+1/2} (y_{40} + O(1-z)), \quad z \to 1;
\]

where \(y_{10}, y_{20}, y_{30}\) and \(y_{40}\) are some functions of \(\lambda\).

From the Corollary \ref{corollary} it follows that the function \(\Psi_{g,N}(\lambda, z)\) does not have zeros in the real interval \(z \in [0, 1]\). Therefore, the connection matrix of the solutions \(y_1(z), y_2(z)\) and \(y_3(z), y_4(z)\),

\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = T_{01} \begin{pmatrix}
y_3 \\
y_4
\end{pmatrix}
\]

can be found relatively simple

\[
T_{01} = \begin{pmatrix}
\cos \varphi(\lambda) & -\sin \varphi(\lambda) \\
\sin \varphi(\lambda) & \cos \varphi(\lambda)
\end{pmatrix},
\]

where

\[
\varphi(\lambda) = \frac{\nu(\lambda)}{2} \int_0^1 \frac{t^{m_1} (t-1)^{m_2} (t-a)^{m_3} dt}{\Psi_{g,N}(\lambda, t) \sqrt{t(t-1)(t-a)}}.
\]

From (4.1b)-(4.3) it is not difficult to obtain the expressions for two generators of the monodromy group for ‘finite-gap’ solutions of Heun’s equation (1.6) which correspond to simple loops starting at the point \(z = 0\) and positively encircling the points \(z = 0\) and \(z = 1\), respectively

\[
M_0 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

\[
M_1 = T_{01} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} T_{01}^{-1} = \begin{pmatrix}
\cos 2\varphi(\lambda) & \sin 2\varphi(\lambda) \\
\sin 2\varphi(\lambda) & -\cos 2\varphi(\lambda)
\end{pmatrix}.
\]

Analogously, one can find the third generator

\[
T_{02} = \begin{pmatrix}
\cos \psi(\lambda) & -\sin \psi(\lambda) \\
\sin \psi(\lambda) & \cos \psi(\lambda)
\end{pmatrix},
\]

\[
\psi(\lambda) = \frac{\nu(\lambda)}{2} \int_0^1 \frac{t^{m_1} (t-1)^{m_2} (t-a)^{m_3} dt}{\Psi_{g,N}(\lambda, t) \sqrt{t(t-1)(t-a)}}.
\]

Here \(C\) is a path in the complex plane connecting the points \(z = 0\) and \(z = a\) and avoiding zeros of the polynomial \(\Psi_{g,N}(\lambda, z)\).
By exactly the same way one can find the generators of the monodromy group for the ‘finite-gap’ solutions of Heun’s equation with negative values of the characteristics $m_i$.

**Concluding remarks**

Recently, there have appeared papers [34, 35] in which, following [1, 2, 18], the solutions of Lamé (0.5) and Treibich-Verdier (0.2), (0.3) equations are studied by making use of periodic solution of a differential equation for the product of eigenfunctions of Schrödinger’s operator with periodic potential [1, 2, 18]. As seen from these papers, the author is not familiar with the results of Darboux [12, 13, 11], Krichever [36], Treibich and Verdier [37, 38, 15, 16, 17], Belokolos and Enol’skii [39, 19, 20, 21, 40, 22, 41, 42, 43], Gesztesy [26, 44, 45, 46, 47] who investigated in detail the finite-gap solutions of these equations as well as their connection with the Calogero-Moser system with elliptic interaction.

Despite this, in the part of studying the properties of solutions of Lamé and Treibich-Verdier equations with integer characteristics $m_i$ the author of those papers not only rediscovered already known facts, such as the finite number of eigenvalues for the periodic and anti-periodic problems, but for two special cases he calculated their number. In this part, the results of the papers [34, 35] coincide with those of the present paper (cf. Corollary 9), taking into account that the eigenvalues for periodic and anti-periodic problems are the gap edges of the spectrum of finite-gap periodic potential (see, for instance, [48]) or, which is the same, are the branching points of the hyperelliptic spectral curve $\Gamma (2.12)$. Unfortunately, in [34, 35] in the general case the formula for the number of eigenvalues is an unproven hypothesis (although the right one as follows from the results of the present work), and apart from that there are no methods suggested for calculating those eigenvalues.

Moreover, from our point of view, the analysis of finite-gap solutions of Lamé and Treibich-Verdier equations, in contrast to the case of Heun’s equation, should be carried out not by making use of the product of eigenfunctions of Schrödinger’s operator but by using the method of Krichever-Hermite ansatz [19, 20, 21, 22]. This is because, in the first case a solution is expressed in terms of elliptic functions in $N$ unknown parameters solving $N$ transcendental equations. In the second case, one needs to solve only an algebraic equation of $N$th order.

Although, perhaps the formulas used in [34, 35] serve best the problem considered there, i.e. to apply the method of Bethe ansatz to the Treibich-Verdier equation aiming to study the one-particle Inozemtsev’s model.

**Appendix A. Simplest ‘finite-gap’ solutions**

At the end of this paper we would like to list polynomials $\Psi(\lambda, z)$ and canonical equations (2.12) of the hyperelliptic curves $\Gamma = \{(\nu, \lambda)\}$ for some simplest ‘finite-gap’ solutions (2.11) of Heun’s equation with nonnegative characteristics $m_i$ ($i = 0, 1, 2, 3$). Our examples are indexed by the characteristics $(m_0, m_1, m_2, m_2)$.

$(0,0,0,0)$:

$$\Psi(\lambda, z) = 1,$$
$$\nu^2 = \lambda.$$

\footnote{the author learned about the work [35] from V.B.Kuznetsov}

\footnote{notice also that the whole volume 36 of Acta Appl.Math. for 1994 is dedicated to the memory of Verdier and contains many works directly or indirectly related to the Treibich-Verdier equation}
\[(1,0,0,0):\]
\[
\Psi(\lambda, z) = \lambda + z - a - 1, \\
\nu^2 = (\lambda - 1)(\lambda - a)(\lambda - a - 1). 
\]

\[(0,1,0,0):\]
\[
\Psi(\lambda, z) = z\lambda + a, \\
\nu^2 = \lambda(\lambda + a)(\lambda + 1). 
\]

\[(1,1,0,0):\]
\[
\Psi(\lambda, z) = z\lambda + z^2 + a, \\
\nu^2 = (\lambda + a + 1)(\lambda^2 - 4a). 
\]

\[(0,0,1,1):\]
\[
\Psi(\lambda, z) = (z - 1)(z - a)\lambda + (a + 1)z^2 - 4az + a(a + 1), \\
\nu^2 = (\lambda + a + 1)(\lambda^2 - 4a). 
\]

\[(1,1,1,0):\]
\[
\Psi(\lambda, z) = z(z - 1)\lambda^2 + \left(z^3 + 3(a - 1)z^2 - 3(a - 1)z - a\right)\lambda - 3a(a - 1), \\
\nu^2 = \lambda(\lambda + 3a)(\lambda + 3(a - 1))(\lambda^2 + 2(2a - 1)\lambda - 3). 
\]

\[(1,1,1,1):\]
\[
\Psi(\lambda, z) = z(z - 1)(z - a)\lambda + (z^2 - a)^2, \\
\nu^2 = \lambda(\lambda + 4a)(\lambda + 4). 
\]

\[(2,0,0,0):\]
\[
\Psi(\lambda, z) = \lambda^2 + (3z - 5(a + 1))\lambda + 9z^2 - 12(a + 1)z + (a + 4)(4a + 1), \\
\nu^2 = (\lambda - a - 1)(\lambda - a - 4)(\lambda - 4a - 1)(\lambda^2 - 4(a + 1)\lambda + 12a). 
\]

\[(0,2,0,0):\]
\[
\Psi(\lambda, z) = z\lambda^2 + \left(3z^3 - 3(a + 1)z + a\right)\lambda + 9z^3 - 9(a + 1)z^2 - 3a(a + 1), \\
\nu^2 = (\lambda + 3)(\lambda + 3a)(\lambda + 3(a + 1))(\lambda^2 + 4(a + 1)\lambda + 12a). 
\]

\[(2,1,0,0):\]
\[
\Psi(\lambda, z) = z^2\lambda^2 + 3\left((a + 1)z + a\right)z\lambda + 9az^2 + 9a^2, \\
\nu^2 = (\lambda - 3(a + 1))(\lambda^2 - 2\lambda - 3(4a + 1))(\lambda^2 - 2a\lambda - 3a(a + 4)). 
\]

\[(0,0,2,1):\]
\[
\Psi(\lambda, z) = (z - 1)^2(z - a)\lambda^2 + \\
+ \left(4az^2 - (a + 3)(3a - 1)z + 2a(3a - 1)\right)(z - 1)\lambda - \\
- 12a^2 + 9(a + 1)^2z^2 - 36az - 3a(3a^2 - 6a - 1), \\
\nu^2 = (\lambda - 3)(\lambda^2 + 4a\lambda - 12a)(\lambda^2 + 2(3a - 1)\lambda + 3(3a^2 - 6a - 1)). 
\]
\begin{align*}
(2,1,1,0): \\
\Psi(\lambda, z) &= z(z-1)\lambda^2 + \left(3z^3 + (3a-10)z^2 - (3a-8)z - a\right)\lambda + \\
&+ 9z^4 - 24z^3 - 2(3a-8)z^2 - a(3a-8), \\
\nu^2 &= (\lambda + 3a-8)(\lambda + 3a+1)(\lambda^3 + 4(a-2)\lambda^2 - 16a\lambda + 64a).
\end{align*}

\begin{align*}
(0,1,1,2): \\
\Psi(\lambda, z) &= z(z-1)(z-a)^2\lambda^2 + \\
&+ \left(3(2a+3)z^3 - (3a^2 + 18a + 8)z^2 + a(3a+10)z + a^2\right)(z-a)\lambda + \\
&+ 9a(a+3)z^4 - 24a(3a+1)z^3 + 18a^2(a+3)z^2 - 3a^3(a+3), \\
\nu^2 &= (\lambda + 3a)(\lambda + 3(a+3))(\lambda^3 + 4(a+4)\lambda^2 + 16(3a+4)\lambda + 192a).
\end{align*}

\begin{align*}
(2,1,1,1): \\
\Psi(\lambda, z) &= z(z-1)(z-a)\lambda^3 + \\
&+ \left(3z^4 - 4(a+1)z^3 + 2(a+1)^2z^2 - 2a(a+1)z + a^2\right)\lambda^2 + \\
&+ \left(9z^5 - 15(a+1)z^4 - 5(a^2 - 7a + 1)z^3 + 15(a-1)^2(a+1)z^2 - \\
&- 15a(a-1)^2z - 2a^2(a+1)\right)\lambda - 15a^2(a-1)^2, \\
\nu^2 &= \lambda(\lambda^2 - 2(a+1)\lambda - 15(a-1)^2) \times \\
&\times \left(\lambda^2 - 2(a-2)\lambda - 15a^2\right)(\lambda^2 + 2(2a-1)\lambda - 15).
\end{align*}

\begin{align*}
(3,0,0,0): \\
\Psi(\lambda, z) &= \lambda^3 + 2\left(3z - 7(a+1)\right)\lambda^2 + \\
&+ \left(45z^2 - 78(a+1)z + 49a^2 + 158a + 49\right)\lambda + \\
&+ 225z^3 - 405(a+1)z^2 + 9(3a + 8)(8a + 3)z - 12(a+1)(3a^2 + 26a + 3), \\
\nu^2 &= (\lambda - 4(a+1))(\lambda^2 - 10(a+1)\lambda + 3(3a^2 + 26a + 3)) \times \\
&\times \left(\lambda^2 - 2(2a+5)\lambda + 3(8a+3)\right)(\lambda^2 - 2(5a+2)\lambda + 3a(3a+8)).
\end{align*}

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