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Proceedings of the American Mathematical Society

DOI: 10.1090/proc/14825

Accepted Manuscript

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ON HYPERBOLIC SURFACE BUNDLES OVER THE CIRCLE AS BRANCHED DOUBLE COVERS OF THE 3-SPHERE

SUSUMU HIROSE AND EIKO KIN

Abstract. The branched virtual fibering theorem by Sakuma states that every closed orientable 3-manifold with a Heegaard surface of genus $g$ has a branched double cover which is a genus $g$ surface bundle over the circle. It is proved by Brooks that such a surface bundle can be chosen to be hyperbolic. We prove that the minimal entropy over all hyperbolic, genus $g$ surface bundles as branched double covers of the 3-sphere behaves like $1/g$. We also give an alternative construction of surface bundles over the circle in Sakuma’s theorem when closed 3-manifolds are branched double covers of the 3-sphere branched over links. A feature of surface bundles coming from our construction is that the monodromies can be read off the braids obtained from the links as the branched set.

1. Introduction

This paper concerns the branched virtual fibering theorem by Makoto Sakuma. To state his theorem let $\Sigma = \Sigma_{g,p}$ be an orientable, connected surface of genus $g$ with $p$ punctures possibly $p = 0$, and let us set $\Sigma_g = \Sigma_{g,0}$. The mapping class group $\text{Mod}(\Sigma)$ is the group of isotopy classes of orientation-preserving self-homeomorphisms on $\Sigma$ which preserve the punctures setwise. By Nielsen-Thurston classification, elements in $\text{Mod}(\Sigma)$ fall into three types: periodic, reducible, pseudo-Anosov [9]. To each pseudo-Anosov element $\phi$, there is an associated dilatation (stretch factor) $\lambda(\phi) > 1$ (see [4] for example). We call the logarithm of the dilatation $\log(\lambda(\phi))$ the entropy of $\phi$.

Choosing a representative $f : \Sigma \to \Sigma$ of $\phi$ we define the mapping torus $T_\phi$ by

$$T_\phi = \Sigma \times \mathbb{R} / \sim,$$

where $(x, t) \sim (f(x), t + 1)$ for $x \in \Sigma$, $t \in \mathbb{R}$. We call $\Sigma$ the fiber surface of $T_\phi$. The 3-manifold $T_\phi$ is a $\Sigma$-bundle over the circle with the monodromy $\phi$. By Thurston [10] $T_\phi$ admits a hyperbolic structure of finite volume if and only if $\phi$ is pseudo-Anosov.

The following theorem is due to Sakuma [8, Addendum 1]. See also [3, Section 3].

**Theorem 1** (Branched virtual fibering theorem). Let $M$ be a closed orientable 3-manifold. Suppose that $M$ admits a genus $g$ Heegaard splitting. Then there is a 2-fold branched cover $\widetilde{M}$ of $M$ which is a $\Sigma_g$-bundle over the circle.

It is proved by Brooks [3] that $\widetilde{M}$ in Theorem 1 can be chosen to be hyperbolic if $g \geq \max(2, g(M))$, where $g(M)$ is the Heegaard genus of $M$. See also [6] by Montesinos.
Let $D_g(M)$ be the subset of $\text{Mod}(\Sigma_g)$ consisting of elements $\phi$ such that $T_\phi$ is homeomorphic to a 2-fold branched cover of $M$ branched over some link. By Theorem 1 we have $D_g(M) \neq \emptyset$. By Brooks together with the stabilization of Heegaard splittings, there is a pseudo-Anosov element in $D_g(M)$ for each $g \geq \max(2, g(M))$.

The set of fibered 3-manifolds $T_\phi$ over all $\phi \in D_g(M)$ possesses various properties inherited under branched covers of $M$. It is natural to ask about the dynamics of pseudo-Anosov elements in $D_g(M)$. We are interested in the set of entropies of pseudo-Anosov mapping classes.

We fix a surface $\Sigma$ and consider the set of entropies

$$\{ \log \lambda(\phi) \mid \phi \in \text{Mod}(\Sigma) \text{ is pseudo-Anosov} \}$$

which is a closed, discrete subset of $\mathbb{R}$ ([1]). For any subset $G \subset \text{Mod}(\Sigma)$ let $\delta(G)$ denote the minimum of dilatations $\lambda(\phi)$ over all pseudo-Anosov elements $\phi \in G$. Then $\delta(G) \geq \delta(\text{Mod}(\Sigma))$. For real valued functions $f$ and $h$, we write $f \simeq h$ if there is a universal constant $c$ such that $h/c \leq f \leq ch$. It is proved by Penner [7] that

$$\log \delta(\text{Mod}(\Sigma_g)) \simeq \frac{1}{g}.$$ 

A question arises: what can we say about the asymptotic behavior of the minimal entropies $\log \delta(D_g(M))'$s for each closed 3-manifold $M$? In this paper we consider this question when $M$ is the 3-sphere $S^3$. Our main theorem is the following.

**Theorem A.** We have $\log \delta(D_g(S^3)) \simeq \frac{1}{g}$.

Let $q_L : M_L \to S^3$ denote the 2-fold branched covering map of $S^3$ branched over a link $L$ in $S^3$. Along the way in the proof of Theorem A we give an alternative proof of Theorem 1 when $M = M_L$ in Theorem B. A feature of surface bundles $M_L$ coming from our construction is that their monodromies can be read off the braids obtained from the links as the branched set. To state Theorem B, we need 3 ingredients.

1. **Involution** $\text{skew} : B_n \to B_n$. Let $B_n$ be the (planar) braid group with $n$ strands and let $\sigma_i$ denote the Artin generator of $B_n$ as in Figure 1(1). We define an
involution
\[
\text{skew} : B_n \rightarrow B_n
\]
\[
\sigma_{n_1}^{\epsilon_1} \sigma_{n_2}^{\epsilon_2} \cdots \sigma_{n_k}^{\epsilon_k} \rightarrow \sigma_{n-n_k}^{\epsilon_1} \cdots \sigma_{n-n_2}^{\epsilon_2} \sigma_{n-n_1}^{\epsilon_k}, \quad \epsilon_i = \pm 1.
\]
We say that \( b \in B_n \) is \textit{skew-palindromic} if \( \text{skew}(b) = b \). The braid \( \text{skew}(b) \cdot b \) is skew-palindromic for any \( b \in B_n \). (There is a skew-palindromic braid which cannot be written by \( \text{skew}(b) \cdot b \) for any \( b \), for example \( \sigma_1 \sigma_2 \sigma_3 \in B_4 \).) We write
\[
\tilde{b} = \text{skew}(b) \cdot b.
\]
Note that \( \text{skew} : B_n \rightarrow B_n \) is induced by the involution on the cylinder as shown in Figure 1(2) and skew-palindromic braids are invariant under such an involution.

In the latter section, the map \( \text{skew} \) is also regarded as a map from the braid group on the sphere or the annulus to itself. The above assertion for the braid \( \tilde{b} = \text{skew}(b) \cdot b \) also holds in this setting.

2. Homomorphism \( t : B_{2g+2} \rightarrow \text{Mod}(\Sigma_g) \). Let \( t_i \) denote the right-handed Dehn twist about the simple closed curve with the number \( i \) in Figure 2. Then there is a homomorphism
\[
t : B_{2g+2} \rightarrow \text{Mod}(\Sigma_g)
\]
which sends \( \sigma_i \) to \( t_i \) for \( i = 1, \ldots, 2g + 1 \), since \( \text{Mod}(\Sigma_g) \) has the braid relation. (We apply elements of mapping class groups from right to left.) The \textit{hyperelliptic mapping class group} \( \mathcal{H}(\Sigma_g) \) is the subgroup of \( \text{Mod}(\Sigma_g) \) consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution. By Birman-Hilden [2], \( \mathcal{H}(\Sigma_g) \) is generated by \( t_i \)'s. Thus
\[
\mathcal{H}(\Sigma_g) = t(B_{2g+2}).
\]

3. Circular plat closure \( C(b) \). We use two types of links in \( S^3 \) obtained from braids. One is the \textit{closure} \( \text{cl}(\beta) \) of \( \beta \in B_{g+1} \) as in Figure 3(1). The other is the \textit{circular plat closure} \( C(b) \) of \( b \in B_{2g+2} \) with even strands as in Figure 3(2)(3). We also use the link \( C(b) \cup W \), the union of \( C(b) \) and the trivial link \( W = O \cup O' \) with 2 components as shown in Figure 3(4).

Any link in \( S^3 \) can be represented by \( C(\beta') \) for some braid \( \beta' \). To see this, it is well-known that \( L \) is the closure \( \text{cl}(\beta) \) for some \( \beta \in B_{g+1} \) \((g \geq 1)\). The desired braid \( \beta' \in B_{2g+2} \) can be obtained from \( \beta \) by adding \( g + 1 \) straight strands as in Figure 3(1).

For a braid \( b \in B_{2g+2} \) let \( q = q_{C(b)} : M_{C(b)} \rightarrow S^3 \) be the 2-fold branched covering map of \( S^3 \) branched over \( C(b) \). There is a \((g + 1)\)-bridge sphere \( S \) for the link...
Figure 3. (1) $\text{cl}(\beta)$ for $\beta \in B_{g+1}$ and $\beta' \in B_{2g+2}$. (2) $C(b)$ for $b \in B_{2g+2}$. (3) $C(\sigma_3\sigma_4\sigma_5)$. (4) $C(b) \cup W$, where $W = O \cup O'$.

$C(b) \subset S^3$. Hence $M_{C(b)}$ admits a genus $g$ Heegaard splitting with the Heegaard surface $q^{-1}(S)$. Then we have the following result.

**Theorem B.** Let $\widetilde{M_{C(b)}}$ be the 2-fold branched cover of $M_{C(b)}$ branched over the link $q^{-1}(W)$. Then $M_{C(b)}$ is homeomorphic to the mapping torus $T_{(b)}$.

**Remark 2.** To be precise, $\widetilde{M_{C(b)}}$ is the 2-fold branched cover of $M_{C(b)}$ branched over $q^{-1}(W)$ obtained as the $\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$-cover of $S^3$ branched over the link $C(b) \cup W$ associated with the epimorphism $H_1(S^3 \setminus (C(b) \cup W)) \to \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$ which maps the meridians of $C(b)$ to the generator of the first factor and the meridians of $W$ to the generator of the second factor.

**Acknowledgments.** We thank Makoto Sakuma and Yuya Koda for helpful conversations and comments. We also thank the anonymous referee for valuable comments. The first author was supported by Grant-in-Aid for Scientific Research (C) (No. 16K05156), JSPS. The second author was supported by Grant-in-Aid for Scientific Research (C) (No. 18K03299), JSPS.

2. **Proof of Theorem B**

**Proof of Theorem B.** We construct the 3-sphere $S^3$ from two copies of the 3-ball $B^3$ by gluing their boundaries together. Consider the link $C(b) \cup W$ so that $C(b)$ is contained in one of the 3-balls, and $W$ is given by the union of the four thick segments in the two 3-balls, see Figure 4(2). Let $S$ be the sphere in $S^3$ which is the union of the two shaded disks in the same figure. The 2-sphere $S$ is a $(g+1)$-bridge sphere for $C(b)$, and the preimage $q^{-1}(S)$ is a genus $g$ Heegaard surface of $M_{C(b)}$.

Let $q_W : M_W \to S^3$ be the 2-fold branched covering map of $S^3$ branched over $W$ (Figure 4(1)). The preimage $q_W^{-1}(B^3)$ is homeomorphic to the solid torus $D^2 \times S^1$. Then $M_W$ is obtained from two copies of $D^2 \times S^1$ by gluing their boundaries together, and hence $M_W$ is homeomorphic to $S^2 \times S^1$. Observe that the link $q_W^{-1}(C(b))$ is the closure of the spherical braid $\widetilde{b} = \text{skew}(b) \cdot b$, i.e.,

$$q_W^{-1}(C(b)) = \text{cl}(\widetilde{b}) \subset S^2 \times S^1.$$
Let $p : N_{\text{cl}(\tilde{b})} \to S^2 \times S^1$ be the 2-fold branched covering map of $S^2 \times S^1$ branched over $\text{cl}(\tilde{b})$. The 2-fold branched cover of the level surface $S^2 \times \{u\}$ for $u \in S^1$ branched at the $2g + 2$ points in $\left((S^2 \times \{u\}) \cap \text{cl}(\tilde{b})\right)$ is a closed surface of genus $g$. Thus $N_{\text{cl}(\tilde{b})}$ is a $\Sigma g$-bundle over $S^1$ with the monodromy $t(\tilde{b})$, i.e., $N_{\text{cl}(\tilde{b})}$ is homeomorphic to $N_{\text{cl}(\tilde{b})}$.

We can observe that the composition

$$q_W \circ p : T_{\text{cl}(\tilde{b})} \cong N_{\text{cl}(\tilde{b})} \to S^3$$

is the $\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$-cover of $S^3$ branched over the link $C(b) \cup W$ described in Remark 2. Hence $T_{\text{cl}(\tilde{b})} \cong N_{\text{cl}(\tilde{b})}$ is identified with 2-fold branched cover $M_{C(b)}$. This completes the proof. \hfill \Box

3. Proof of Theorem A

Given a braid $b$ we first give a construction of a braid $b'$ (with more strands than $b$) such that $C(b)$ is ambient isotopic to $C(b')$.

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The bottom and top endpoints of a planar braid with \( n \) strands are denoted by \( l_1, \ldots, l_n \) and \( u_1, \ldots, u_n \) from left to right. For a braid \( \sigma \in B_{2g+2} \) with even strands, we choose a braid \( \sigma' \in B_{2g+3} \) obtained from \( \sigma \) by adding a strand, say \( \sigma'(g+2) \) connecting the middle of the two points \( l_{g+1} \) and \( l_{g+2} \) with the middle of the two points \( u_{g+1} \) and \( u_{g+2} \). Of course \( \sigma' \) is not unique. For example when \( b = b_1 = \sigma_3 \in B_4 \), one can choose \( \sigma'(=b') = \sigma_2^3 \sigma_4 \in B_5 \). See Figure 5.

We consider \( \sigma' = \text{skew}(\sigma') \cdot \sigma' \in B_{2g+3} \) with bottom endpoints \( l_1, \ldots, l_{2g+3} \) and top endpoints \( u_1, \ldots, u_{2g+3} \). The braid \( \sigma' \) has the strand \( \sigma'(g+2) \) with endpoints \( l_{g+2} \) and \( u_{g+2} \). If we remove this strand from \( \sigma' \), then we obtain \( \sigma = \text{skew}(\sigma') \cdot \sigma' \).

Now we construct \( S^2 \times S^1 \) from three pieces, two solid tori \( (D^2 \times S^1)_{\pm 1} \) and the product \( S^1 \times S^1 \times [-1, 1] \) of a torus \( S^1 \times S^1 \) and the interval \([ -1, 1] \) by gluing \( (\partial D^2 \times S^1)_i \) to \( S^1 \times S^1 \times \{i\} \) together for \( i = \pm 1 \). See Figure 6. We think of \( S^1 \) as
Clearly under the assumptions \((R \circ f)(\tilde{b}) = \text{cl}(\tilde{b}) \cup A\) in \(S^3\), we perform the 0-surgery along the braid axis \(A\). Then the image of \(\text{cl}(\tilde{b})\) forms a link in \(S^2 \times S^1\), which continues to denote by the same symbol \(\text{cl}(\tilde{b})\). We deform this link \(\text{cl}(\tilde{b})\) in \(S^2 \times S^1\) so that the knot \(\text{cl}(\tilde{b}(g + 2))\) becomes the core of \((D^2 \times S^1)_{-1}\) and \(\text{cl}(\tilde{b}) = \text{cl}(\tilde{b}) \setminus \text{cl}(\tilde{b}(g + 2))\) is contained in \(S^1 \times \tilde{S}^1 \times [-1, 1]\). One can regard \(\tilde{b}\) as a braid on the annulus \(\mathcal{A} := S^1 \times \{-1\} \times [-1, 1]\) which is embedded in \(S^1 \times [-1, 1] \times [-1, 1]\), and one can think of the link \(\text{cl}(\tilde{b})\) as the closure of the braid \(\tilde{b}\) on \(\mathcal{A}\). See Figure 5(4). Let

\[
R : S^2 \times S^1 \to S^2 \times S^1
\]

be the deck transformation of \(q_{W} : S^2 \times S^1 \to S^3\) in the proof of Theorem B. Then \(q_{W}\) sends the fixed point set of the involution \(R\) to the trivial link \(W\) (Figure 4(1)(2)). Let

\[
f : S^1 \times S^1 \to S^1 \times S^1
\]

be any orientation-preserving homeomorphism. We may assume that \(f\) commutes with the involution

\[
\iota : S^1 \times S^1 \to S^1 \times S^1
\]

\[(x, y) \mapsto (-x, -y).\]

We consider the homeomorphism

\[
\Phi_f = f \times \text{id}_{[-1, 1]} : S^1 \times S^1 \times [-1, 1] \to S^1 \times S^1 \times [-1, 1].
\]

The image of \(\text{cl}(\tilde{b})\) under \(\Phi_f\) may or may not be the closure of some braid on \(\mathcal{A}\). We assume the former case \((\ast)\):

\((\ast)\) \(\Phi_f(\text{cl}(\tilde{b})) = \text{cl}(\gamma)\) for some braid \(\gamma\) on \(\mathcal{A}\).

Then the involution \(R|_{S^1 \times S^1 \times [-1, 1]} = \iota \times \text{id}_{[-1, 1]}\) has the following property.

\(R(\text{cl}(\beta)) = \text{cl}(\text{skew}(\beta))\) for any braid \(\beta\) on \(\mathcal{A}\).

cf. Figure 1(2). Since \(f\) commutes with \(\iota\), it follows that \(\Phi_f\) commutes with \(R|_{S^1 \times S^1 \times [-1, 1]}\). Hence we have

\[
\text{cl}(\gamma) = \Phi_f(\text{cl}(\tilde{b})) = \Phi_f(\text{cl}(\text{skew}(\tilde{b}))) = \Phi_f R(\text{cl}(\tilde{b})) = R \Phi_f(\text{cl}(\tilde{b})) = R(\text{cl}(\gamma)).
\]

(The first and last equality come from the assumption \((\ast)\), and the second equality holds since \(\tilde{b}\) is skew-palindromic.) Thus \(\text{cl}(\gamma) = R(\text{cl}(\gamma))\). We further assume that

\((\ast\ast)\) \(\Phi_f(\text{cl}(\tilde{b})) = \text{cl}(\gamma) = \text{cl}(\tilde{b}_f)\) for some braid \(b_f\) on \(\mathcal{A}\).

**Remark 3.** Clearly \((\ast\ast)\) implies \(\text{cl}(\gamma) = R(\text{cl}(\gamma))\). It is likely the converse holds.

Now, we think of the braid \(b_f\) on \(\mathcal{A}\) as a planar braid as in Figure 5, and consider the link \(C(b_f)\) in \(S^3\). We have the following lemma.

**Lemma 4.** Under the assumptions \((\ast)\) and \((\ast\ast)\), \(C(b)\) and \(C(b_f)\) are ambient isotopic.
Proof. Note that the quotient \((S^1 \times S^1 \times [-1, 1])/R\) is homeomorphic to \(S^2 \times [-1, 1]\). Since \(\Phi_f\) commutes with \(R|_{S^1 \times S^1 \times [-1, 1]}\), \(\Phi_f\) induces a self-homeomorphism
\[
\Phi_f : S^2 \times [-1, 1] \rightarrow S^2 \times [-1, 1].
\]
Since \(\Phi_f(\text{cl}(\tilde{b})) = \text{cl}(\tilde{b}_f)\) we have \(\Phi_f(\text{cl}(b)) = \text{cl}(b_f)\). Any orientation-preserving self-homeomorphism on \(S^2\) is isotopic to the identity, and \(S^3\) is a union of \(S^2 \times [-1, 1]\) and two 3-balls by gluing the boundaries together. Thus \(\Phi_f\) extends to a self-homeomorphism on \(S^3\) which sends \(\text{cl}(b)\) to \(\text{cl}(b_f)\). This completes the proof. \(\square\)

Let us consider the mapping class group \(\text{Mod}(D_n)\) of the \(n\)-punctured disk \(D_n\) preserving the boundary \(\partial D\) of the disk setwise. We have a surjective homomorphism
\[
\Gamma : B_n \rightarrow \text{Mod}(D_n)
\]
which sends each generator \(\sigma_i\) to the right-handed half twist between the \(i\)th and \((i + 1)\)st punctures. We say that \(\beta \in B_n\) is pseudo-Anosov if \(\Gamma(\beta) \in \text{Mod}(D_n)\) is of the pseudo-Anosov type. When \(\beta\) is a pseudo-Anosov braid, the dilatation \(\lambda(\beta)\) is defined by the dilatation of \(\Lambda(\Gamma(\beta))\).

We consider the above homomorphism \(\Gamma : B_{2g+2} \rightarrow \text{Mod}(D_{2g+2})\) when \(n = 2g+2\). Recall the homomorphism \(t : B_{2g+2} \rightarrow \text{Mod}(\Sigma_g)\). The following lemma relates dilatations of \(\beta\) and \(t(\beta)\).

**Lemma 5.** Let \(\beta \in B_{2g+2}\) be pseudo-Anosov and let \(\Phi_\beta : D_{2g+2} \rightarrow D_{2g+2}\) be a pseudo-Anosov homeomorphism which represents \(\Gamma(\beta) \in \text{Mod}(D_{2g+2})\). Suppose that the stable foliation \(\mathcal{F}_\beta\) for \(\Phi_\beta\) defined on \(D_{2g+2}\) is not 1-pronged at the boundary \(\partial D\) of the disk. Then \(t(\beta) \in \text{Mod}(\Sigma_g)\) is pseudo-Anosov, and \(\lambda(t(\beta)) = \lambda(\beta)\) holds.

**Proof.** Since \(\mathcal{F}_\beta\) is not 1-pronged at \(\partial D\), \(\Phi_\beta : D_{2g+2} \rightarrow D_{2g+2}\) induces a pseudo-Anosov homeomorphism \(\Phi'_\beta : \Sigma_{0,2g+2} \rightarrow \Sigma_{0,2g+2}\) by filling \(\partial D\) with a disk. By Birman-Hilden [2], we have a surjective homomorphism
\[
q : \mathcal{H}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_{0,2g+2})
\]
sending the Dehn twist $t_i$ to the right-handed half twist $h_i$ between the $i$th and $(i + 1)$st punctures for $i = 1, \cdots, 2g + 1$. Consider the 2-fold branched cover $\Sigma_g \to \Sigma_{0,2g+2}$ branched at the $2g + 2$ marked points (corresponding to the punctures of $\Sigma_{0,2g+2}$). Then there is a lift $f_\beta : \Sigma_g \to \Sigma_g$ of $\Phi_\beta : \Sigma_{0,2g+2} \to \Sigma_{0,2g+2}$ which satisfies $t(\beta) = [f_\beta] \in \mathcal{H}(\Sigma_g)$. Note that the stable foliation $\mathcal{F}_\beta$ for $\Phi_\beta$ extends to the stable foliation $\mathcal{F}_\beta'$ for $\Phi_\beta'$ by the assumption that $\mathcal{F}_\beta$ is not 1-pronged at $\partial D$. The stable foliation $\mathcal{F}_\beta'$ defined on $\Sigma_{0,2g+2}$ is lifted to the stable foliation for $f_\beta$ defined on $\Sigma_g$. Thus $f_\beta$ is a pseudo-Anosov homeomorphism which represents $t(\beta) = [f_\beta]$, and we have
\[
\lambda([f_\beta]) = \lambda([\Phi_\beta]) = \lambda(\Phi_\beta) = \lambda(\beta).
\]
This completes the proof. \[ \square \]

**Proof of Theorem A.** For $g \geq 1$, we consider $b_g = \sigma_3 \sigma_4 \cdots \sigma_{2g+1} \in B_{2g+2}$ and
\[
b_g = \sigma_1 \sigma_2 \cdots \sigma_{2g-1} \cdot \sigma_3 \sigma_4 \cdots \sigma_{2g+1} \in B_{2g+2},
\]
see Figure 8. By Penner’s result it is enough to prove that $t(b_g)$ is a pseudo-Anosov element in $D_g(S^3)$ for large $g$ and $\log \lambda(t(b_g)) \asymp 1/g$ holds.

Applying Theorem B for the braid $b_g$ we have the 2-fold branched cover
\[
T_{t(b_g)} \to M_{C(b_g)}
\]
branched over $q^{-1}(W)$. We first prove that $M_{C(b_g)} \cong S^3$ for $g \geq 1$. Clearly $C(b_1) = C(\sigma_3)$ is a trivial knot, and hence $M_{C(b_1)} \cong S^3$. We add a strand to $b_1 = \sigma_3 \in B_3$ so that $b_2^g = \sigma_3^2 \sigma_4 \in B_5$, and think of $b_5^g$ as a braid on $A$, see Figure 5. Choose any $v \in S^1$ and consider the annulus $A$ in $S^1 \times S^1 \times [-1, 1]$ with boundary $\{v\} \times S^1 \times \{\pm 1\}$, see Figure 7(1). Let $f : S^1 \times S^1 \to S^1 \times S^1$ be the Dehn twist on $S^1 \times S^1$ about $\{v\} \times S^1$. Then the self-homeomorphism $\Phi_f$ on $S^1 \times S^1 \times [-1, 1]$ is an annulus twist about $A$, see Figure 7(2)(3). Observe that $\Phi_f(cl(b_j)) = cl(b_j)$ satisfies the assumptions (* and **). By repeating this process it is not hard to see that
\[
\Phi_f(cl(b_{j-1})) = cl(b_j) \text{ for each } j \geq 2.
\]
Thus $\Phi_f(\text{cl}(\overline{b_{j-1}}))$ satisfies $(\ast)$ and $(\ast\ast)$ for each $j$. Lemma 4 tells us that $C(b_g)$ is a trivial knot for $g \geq 1$ since so is $C(b_1)$. Thus $M_C(b_g) \cong S^3$, and $t(\tilde{b}_g) \in D_g(S^3)$ for $g \geq 1$ by Theorem B.

The proof of Theorem D in [5] says that $\Gamma(\tilde{b}_g) \in \text{Mod}(D_{2g+2})$ is pseudo-Anosov for $g \geq 2$ and $\log \lambda(\tilde{b}_g) \asymp \frac{1}{g}$ holds. Moreover the stable foliation of the pseudo-Anosov representive for $\Gamma(\tilde{b}_g)$ satisfies the assumption of Lemma 5, see the proof of Step 2 in [5, Theorem D]. Thus $t(\tilde{b}_g)$ is pseudo-Anosov with $\lambda(t(\tilde{b}_g)) = \lambda(\tilde{b}_g)$ by Lemma 5, and we obtain the desired claim $\log \lambda(t(\tilde{b}_g)) = \log \lambda(\tilde{b}_g) \asymp \frac{1}{g}$. This completes the proof.

We end this paper with a question.

**Question 6.** Let $M$ be a closed 3-manifold $M$ which is the 2-fold branched cover of $S^3$ branched over some link. Then does it hold $\log \delta(D_g(M)) \asymp \frac{1}{g}$?

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