Optimal Hierarchical Layouts for Cache-Oblivious Search Trees

Peter Lindstrom, Deepak Rajan

Center for Applied and Scientific Computing, Lawrence Livermore National Laboratory
7000 East Avenue, Livermore, California 94550, USA
pl@llnl.gov
rajan3@llnl.gov

Abstract—This paper proposes a general framework for generating cache-oblivious layouts for binary search trees. A cache-oblivious layout attempts to minimize cache misses on any hierarchical memory, independent of the number of memory levels and attributes at each level such as cache size, line size, and replacement policy. Recursively partitioning a tree into contiguous subtrees and prescribing an ordering amongst the subtrees, Hierarchical Layouts generalize many commonly used layouts for trees such as in-order, pre-order and breadth-first. They also generalize the various flavors of the van Emde Boas layout, which have previously been used as cache-oblivious layouts. Hierarchical Layouts thus unify all previous attempts at deriving layouts for search trees.

The paper then derives a new locality measure (the Weighted Edge Product) that mimics the probability of cache misses at each level, and shows that layouts that reduce this measure perform better. We analyze the various degrees of freedom in the construction of Hierarchical Layouts, and investigate the relative effect of each of these decisions in the construction of cache-oblivious layouts. Optimizing the Weighted Edge Product for complete binary search trees, we introduce the MINWEP layout, and show that it outperforms previously used cache-oblivious layouts by almost 20%.

I. INTRODUCTION

In today’s computer architectures, the memory hierarchy is becoming increasingly complex, both in terms of number of levels and in terms of the difference in performance from one level to the next. As a result, algorithms and data structures that are designed for flat (or even two-level) memory with uniform access times can result in significantly suboptimal performance. In this paper, we are interested in improving memory access locality for search trees via data reordering.

The classic search tree is the B-tree [1], which has been designed for a two-level cache hierarchy, and is usually optimized for a particular block transfer size (e.g., a cache line or disk block). It is not clear if B-trees can be successfully optimized for a multi-level cache hierarchy, with one level per transfer block size. Furthermore, B-trees are known to perform poorly when the nodes of the search trees are of different sizes (e.g., when the search keys are variable-length) [2]. As a result, cache-oblivious search trees have been suggested in the literature. In this paper, we present a new locality measure that can be used to derive cache-oblivious data structures. Focusing our attention on search trees, we show how optimizing our locality measure results in better cache-oblivious search tree layouts than prior layouts.

The fundamental structure commonly employed for cache-oblivious search trees is the van Emde Boas layout. First introduced by Prokop [3], these recursively defined layouts are similar to van Emde Boas trees, hence the name. In [4], this layout was shown to result in much better binary search times than simpler orderings such as breadth-first and depth-first pre- and in-order. Minor variants of these van Emde Boas layouts have since been used in a variety of other settings. In [5], the authors introduce a very similar layout that differs only in how the tree is partitioned at each branch of the recursion. Using this layout as the basic building block, they present dynamic search trees, and refer to these as cache-oblivious B-trees. In [6], the authors provide bounds on the asymptotic cost of cache-oblivious searching. By analyzing a generalized version of the van Emde Boas layout, they provide a modified version that is arbitrarily close to the asymptotic bound. In [2], [7], the authors address the problem of building cache-oblivious layouts of search trees with variable-sized search keys. They use a modified version of the van Emde Boas layout in which the tree is partitioned differently. In [8], the authors present two cache-oblivious streaming B-trees – data structures that implement cache-oblivious search trees optimized for dynamic insertions and deletions. Again, these rely on a version of the van Emde Boas layout with a slightly different partitioning scheme. We note that cache-oblivious data structures are not limited to search trees. They have been proposed in a variety of settings, some of which include hash tables [9], meshes [10], [11], and Bloom filters [12].

A. Contribution: Cache-oblivious Hierarchical Layouts

We describe a general framework for generating search tree layouts, and present new orderings from this framework that result in better cache-oblivious search tree layouts than those suggested in the literature. We refer to all layouts that fit the new framework as Hierarchical Layouts.

Consider a tree $T$ of height $h$, i.e., with $h$ levels of nodes. For ease of exposition, we restrict our discussion to complete binary trees; therefore the number of nodes is $2^h - 1$. Counting the levels from top to bottom, the root is on level 0 and the leaves are on level $h - 1$. Observe that level $i$ has $2^i$ nodes.

Any Hierarchical Layout can be described recursively as follows: Partition $T$ by cutting it horizontally between level $g - 1$ and $g$, which results in a top subtree $A$ of height $g$ with...
2^{h-1} leaves. Let $L_A$ be the set of leaves in the top subtree $A$. Given a leaf node $x$ in $L_A$, we say that a bottom subtree formed by a child $c$ of $x$ and the descendants of $c$ is a child subtree of $x$. With 2 child subtrees for each node in $L_A$, we have $2^h$ bottom subtrees of height $h-g$. Any relative ordering of the recursive subtrees that arranges them consecutively in memory constitutes a Hierarchical Layout. The node ordering within each subtree is given by recursive application of this decomposition, until each subtree consists of a single node.

The effectiveness of any particular Hierarchical Layout as a cache-oblivious search tree depends on the relative ordering of subtrees and on the height of the top subtree $g$. In this paper, we focus on finding optimal cuts and orderings of the subtrees to maximize locality and minimize cache misses, and propose a new cache-oblivious Hierarchical Layout.

We also show that the widely used van Emde Boas layouts are a special case of Hierarchical Layouts; therefore any cache-oblivious search tree data structure that utilizes a van Emde Boas layout can be improved by switching to our proposed layout. The main take-home message of this paper is that the widely used version of the van Emde Boas layout is not the best Hierarchical Layout. Significantly better cache-oblivious layouts can be obtained by considering Hierarchical Layouts that minimize our measure.

In Section II we motivate some simple improvements to the van Emde Boas layout. Section III describes a mathematical measure of locality for tree orderings that correlates well with cache-miss ratios, thus resulting in cache-oblivious layouts. We analyze which Hierarchical Layouts perform better with respect to this new measure, the Weighted Edge Product. We improve the layout further in Section IV by deriving MINWEP, the Hierarchical Layout that minimizes the Weighted Edge Product. Our experiments indicate that MINWEP on average improves performance by almost 20% compared to the layouts described in the literature.

B. Hierarchical Layouts: A Nomenclature

A Hierarchical Layout is given entirely by (1) the height at which the tree is partitioned, (2) the position of the top subtree relative to the bottom subtrees, and (3) the relative ordering of the bottom subtrees. This definition allows for a very large combination of cut heights and orderings. For this reason, we impose additional restrictions; the motivation behind some of them will become clear later in the paper. We refer to layouts belonging to this restricted set as Recursive Layouts because they can be categorized entirely using a small set of recursive rules and parameters, allowing for a more compact nomenclature than the more general Hierarchical Layout.

In a Recursive Layout, at any branch of the recursion that cuts a subtree into its top subtree $A$ and the corresponding bottom subtrees, we enforce the following restrictions. (a) $A$ is arranged either in the middle of all the bottom subtrees (in-order), or at one end (pre-order). (b) The top subtree obtained in the partitioning of $A$ must be arranged relative to the bottom subtrees in the same fashion as $A$. (c) If $A$ is arranged in-order, we choose the children of the leftmost $2^{h-2}$ leaves in $L_A$ to be the bottom subtrees on the left of the top subtree. (d) If any bottom subtree is arranged in-order, all bottom subtrees that are arranged further away from $A$ are also arranged in-order. (e) Looking outwards from $A$, the bottom subtrees are ordered in the same order as that of the parent leaves $L_A$, or in the reverse order. (f) If $A$ is arranged pre-order, then it is placed on the side of the bottom subtrees that is closer to its parent leaf. Thus, we use pre-order layouts to refer to both pre-order and post-order arrangements of the top subtree, depending upon the context. (g) The cut height $g$ is a function only of the height of the subtree.

Based on the preceding discussion, we present a new nomenclature for categorizing Recursive Layouts, an important subset of Hierarchical Layouts. A Recursive Layout is categorized as $P$ for pre-order and $I$ for in-order to indicate the arrangement for the outermost branch of the recursion (when we cut the tree $T$ itself). At each branch of the recursion, the position of the first in-order bottom subtree, counting outwards from the top subtree, is indicated as a subscript. If all the bottom subtrees are arranged pre-order, then we denote this by $\infty$. The cut height $g$ (as a function of the height of the subtree $h$) is indicated as a superscript. We indicate an layout where the bottom subtrees are arranged in reverse order of the leaves $L_A$ using the $\sim$ symbol on top.

Bringing this all together, we see that $\tilde{T}_L$, is the Recursive Layout that always cuts at height 1, arranges the top subtree in the outermost branch of the recursion in-order, and arranges all other subtrees pre-order in the reverse order of the top subtree leaves. In Table I we categorize all the layouts we consider in this paper using this nomenclature. All the layouts described in this paper belong to the restricted set of Recursive Layouts.

II. Cache-oblivious Hierarchical Layouts

In this Section, we motivate better cache-oblivious orderings within the framework of Hierarchical Layouts. First, we review the van Emde Boas layouts used in the literature, which are a special case of Hierarchical Layouts. In Prokop’s ordering [3], the subtrees are cut at height $g = \lfloor h/2 \rfloor$, the top subtree is placed before the bottom subtrees, and then this ordering strategy is applied recursively to each subtree. The bottom subtrees are arranged in the same order as the order of their parent leaves $L_A$. For the rest of this paper, we refer to this version of the van Emde Boas layout as the pre-order van Emde Boas layout, and denote it as PRE-VEB. In our nomenclature, PRE-VEB is $P^{\lfloor h/2 \rfloor}$ (see Table I).

Figure 5f illustrates PRE-VEB for a tree of height 6. The number inside each node is its position in the final layout, ranging from 1 to 63. Observe at every branch of the recursion, the top subtree is arranged pre-order. In the outermost branch of the recursion, the nodes in the top three levels are arranged first (positions 1 to 7). Figure 5f also indicates the length of each edge, i.e., the difference in position of its nodes, using lines whose thickness is inversely proportional to the length.

In Bender’s layout [5], the authors set $g = h - 2^{\lceil \log_2(h/2) \rceil}$. In other words, the height of the bottom subtrees is set to
the largest power of two smaller than \( h \). The authors refer to their layout as a van Emde Boas layout since it is similar to the layout introduced in [3]. Nevertheless, we make the distinction that only Hierarchical Layouts that set the cut height \( g = \lfloor h/2 \rfloor \) are van Emde Boas layouts. Bender is identical to \( \text{PRE-VEB} \) for trees whose height is a power of two. For all other tree heights, Bender layouts have smaller top subtrees, compared to \( \text{PRE-VEB} \). Figure 5 illustrates Bender’s layout. Observe that the nodes in the top two levels are arranged next to each other, indicating a cut height of \( 2 \) at the outermost branch of the recursion. In our nomenclature, Bender is \( p_{h-2^\lfloor \log_2(h/2) \rfloor} \) (see Table I).

We will see later that Hierarchical Layouts also include all the simple and commonly used layouts such as in-order, pre-order, and breadth-first. One can think of cut heights \( g = 1 \) and \( g = h - 1 \) as the extreme cases, corresponding to these simple layouts. We will also show that cache-oblivious layouts are obtained by cutting the tree near the center, with \( g \) approximately equal to \( h/2 \).

A. Evaluating layouts using block transitions

To compare Hierarchical Layouts, we will estimate the number of cache misses for a particular cache block size and layout as follows. Consider a cache consisting of a single block that can hold \( N \) data elements, and which is backed by a larger memory consisting of several such blocks. (In practice caches tend to hold more than one block, but that would unnecessarily complicate our derivation.) Let \( i \) and \( j \) be data elements stored in blocks \( B(i) \) and \( B(j) \), respectively, and let \( \ell_{ij} \) denote the difference in position of \( i \) and \( j \) on linear storage. For ease of exposition, we set \( \ell_{ij} = \ell \). Suppose \( i \) is accessed first, bringing \( B(i) \) into the cache. We wish to estimate the probability of a cache miss when \( j \) is accessed next. Clearly, if \( \ell \geq N \), then a cache miss is inevitable, since then \( i \) and \( j \) are stored in different blocks. When \( \ell < N \), the likelihood of a cache miss depends on the positions of \( i \) and \( j \) within their blocks. In absence of further information, we will assume that the position of \( i \) within \( B(i) \) is distributed uniformly, and similarly for \( j \). (Even in practice, modern operating systems allocate memory blocks with nearly arbitrary alignment.) Hence, there are \( \ell \) out of \( N \) possible alignments that separate \( i \) and \( j \) into different blocks, and the probability of a cache miss occurring when \( j \) is accessed is therefore

\[
M_N(\ell) = \begin{cases} 
\frac{\ell}{N} & \text{if } \ell \leq N \\
1 & \text{otherwise}
\end{cases}
\]

and transitioning to state \( j \) is \( a_{ij} \). If \( G \) is strongly connected, as is the case for binary trees, then it is well-known that the probability \( \Pr(X_t = i, X_{t+1} = j) \) of being in state \( i \) and transitioning to state \( j \) equals \( \frac{w_{ij}}{W} \), where \( W = \sum_{ij \in E} w_{ij} \). In other words, the probability of accessing two data elements in succession is proportional to the weight of the edge connecting them.

In a binary search tree \( T \), the affinity graph is the search tree itself, and the search for a particular element results in a walk from the root on level \( d = 0 \) of the tree to the node representing the element. Therefore, only the node searched for and its ancestors are visited, beginning with the root, and thus nodes near the top of the tree are more likely to be visited than nodes near the bottom. Assuming each node is equally likely to be searched for, the likelihood of traversing a given edge between levels \( d - 1 \) and \( d \) in a tree of height \( h \) is

\[
p_{d,h} = \frac{V_{h-d}}{V_h} = \frac{2^{h-d} - 1}{2^{h} - 1},
\]

where \( V_h \) is the number of nodes in a complete binary tree of height \( h \). For an edge \( ij \) between levels \( d - 1 \) and \( d \), we set \( w_{ij} = p_{d,h} \), which ensures that the probability of accessing two data elements in succession is proportional to the weight of the edge connecting them. Near the top of the tree, where \( 2^h > 2^{h-d} \gg 1 \), this likelihood decreases approximately geometrically by level, i.e. \( p_{d.h} \approx 2^{-d} \). We will use these approximate probabilities and corresponding edge weights for the rest of this paper, primarily for ease of analysis.

Given this probability of accessing any two nodes in succession, the expected percentage of consecutive accesses that will result in a cache miss for a particular block size \( N \) is

\[
\beta(N) = \frac{1}{W} \sum_{ij \in E} w_{ij} M_N(\ell_{ij})
\]

For any given layout and block size, we refer to \( \beta \) as the Percentage of Block Transitions. If one layout dominates another for all block sizes under this metric, then clearly it will result in a better cache-oblivious layout. For a particular block size, we can also calculate \( \beta \) for all tree heights \( h \).

Observe that for block sizes larger than the number of elements in the binary tree, \( M_N(\ell) \) reduces to \( \frac{\ell}{N} \), a linear function of the edge length \( \ell \). This implies that the probability of a cache miss \( \beta(N) \) reduces to \( \frac{1}{W} \sum_{ij \in E} w_{ij} \ell_{ij} \), a weighted sum of the edge lengths. Thus, for very large block sizes, the optimal ordering is one that minimizes a weighted sum of edge lengths, where the weights are approximately geometrically decreasing as a function of the level of the edge.

B. In-order Hierarchical Layouts

Consider the in-order van Emde Boas layout, denoted as \( \text{IN-VEB} \), and obtained by arranging all bottom subtrees in-order, and in the same relative order as that of their parent leaves \( L_A \). In our nomenclature, \( \text{IN-VEB} \) is \( \mathcal{F}_1^{[h/2]} \) (see Table I). Figure 5e illustrates \( \text{IN-VEB} \) for a tree of height 6. Observe that at each branch of the recursion, the top subtree
is arranged in-order. For instance, the nodes on the top three levels are ordered in the middle of the layout, from positions 29 to 35. To compare IN-VEB with the pre-order van Emde Boas layout that arranges all subtrees pre-order (PRE-VEB), we consider the percentage of block transitions $\beta$.

Illustrated in Figure 2, we see the same behavior as before. IN-VEB is significantly better than PRE-VEB, but is marginally worse than MINWEP. On average, MINWEP is about 5% better than IN-VEB and almost 20% better than PRE-VEB. The sudden uptick at $h = 32$ is due to NUMA misses. Our experiments were run on a machine with two memory banks of 48 GB each, and we need 64 GB of RAM to store a tree of height $h = 32$, generating a lot of traffic across the NUMA memory banks. The plots in Figure 2 therefore indicate that the percentage of block transitions ($\beta$) correlates very well with cache-miss ratios, and is therefore a good indicator of the quality of a layout. In Section III we mathematically derive a new locality measure, the Weighted Edge Product $\nu_0$, which is independent of the block size $N$ and correlates even better with these measures and performance metrics. In Figure 2 we see that IN-VEB has much lower $\nu_0$ values than PRE-VEB, but not as low as MINWEP.

III. A CACHE-OBLIVIOUS LOCALITY MEASURE

We have seen how the percentage of block transitions provides a quality measure for a layout given a particular cache block size $N$. We now remove this dependence on block size and derive a simple measure of locality for graph orderings in a cache-oblivious sense, i.e. with no knowledge of cache and line size. Continuing the discussion in Section II-A we here generalize the measure presented in [13] to weighted graphs.

The observation underlying our cache-oblivious measure is that most block-based caches employed in current computer architectures are hierarchical and nested, with a roughly geometric progression in size. That is, we may write $N = b^k$ for some base $b$ (usually $b = 2$) and positive integer $k$. We then estimate the total number of cache misses for all $k$ for a

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**Figure 1**: Two locality measures for several layouts of a tree of height $h = 20$. Left: Block transitions $\beta$ as a function block size (lower is better). Right: Cumulative distribution of edge weights as a function of edge length (higher is better).
Fig. 2: Clockwise from top left: weighted edge length product $w_0$; wall clock search time; L1 and L2 cache miss rate; and block transitions for blocks of $N \in \{2, 5, 16\}$ nodes as a function of tree height for several hierarchical layouts.

For a weighted graph, in particular edge length $\ell$ as

$$M(\ell) = \sum_{k=1}^{\infty} M_{bk}(\ell) = \sum_{k=1}^{[\log_b \ell]} 1 + \sum_{k=[\log_b \ell]+1}^{\infty} \frac{\ell}{b^k}$$

(4)

We note that when $\ell$ is an exact power of $b$, $M(\ell)$ simplifies to $\log_b \ell + \frac{1}{b^k-1}$; otherwise $M(\ell)$ increases monotonically with $\ell$. Our primary goal is not to estimate the exact number of cache misses incurred, but rather to assign a relative “cost” as a function of edge length $\ell$. We may thus ignore the value of $b$ (since it affects only the slope of $M$) and the constant term independent of $\ell$, and arrive at the approximation

$$M(\ell) \approx \log \ell$$

(5)

Intuitively, $\log \ell_{ij}$ measures the number of blocks smaller than $\ell_{ij}$ that cannot hold both $i$ and $j$, and thus captures the expected number of block transitions and cache misses associated with $\ell_{ij}$ in a memory hierarchy.

Finally, if we consider all edges $E$ of an affinity graph, then

$$M = \sum_{k=1}^{\infty} \beta(k) = \sum_{k=1}^{\infty} \frac{1}{W} \sum_{ij \in E} w_{ij} M_{bk}(\ell_{ij})$$

(6)

$$= \frac{1}{W} \sum_{ij \in E} w_{ij} \sum_{k=1}^{\infty} M_{bk}(\ell_{ij}) = \frac{1}{W} \sum_{ij \in E} w_{ij} M(\ell_{ij})$$

gives the average cache miss ratio, where $\nu_0$ denotes the weighted edge product functional

$$\nu_0 = \exp \left( \frac{1}{W} \sum_{ij \in E} w_{ij} \log \ell_{ij} \right) = \left( \prod_{ij \in E} \frac{w_{ij}}{\ell_{ij}} \right)^{1/W}$$

(7)

for a weighted graph. In other words, $\nu_0 \approx \exp(M)$. As a result, low values of $\nu_0$ imply good cache utilization across the whole memory hierarchy. As we shall see, this expected behavior is observed also in practice, with layouts optimized
for \( \nu_0 \) having excellent locality properties. (In the unweighted case, \( \nu_0 = 1 \) and \( W = |E| \). We denote the unweighted version of \( \nu_0 \) by \( \mu_0 \).) We call the Recursive Layout that minimizes \( \nu_0 \) for geometrically decreasing weights (as described in Section II-A) the MINWEP (short for minimum weighted edge product) layout of the tree.

A. Other edge-based locality measures

It is important to mention two other locality measures that have been considered in the literature: the average edge length, \( \mu_1 \), and the maximum edge length, \( \mu_\infty \). The small example in Figure 5 includes the layouts MINLA \( \nu_0 \) in Figure 5m which minimizes \( \mu_1 \), and MINBW \( \nu_0 \) in Figure 5n which minimizes \( \mu_\infty \). Similar to \( \nu_0 \), which measures the weighted edge length product, we may define the average weighted edge length \( \nu_1 \). This figure also presents these four statistics (\( \nu_0, \nu_1, \mu_1, \mu_\infty \)) for all layouts discussed in this paper. From the discussion in Section II-A we observe that the probability of a block transition for very large block sizes is given by \( \nu_1 \), i.e., a weighted version of \( \mu_1 \).

Based on an empirical study, we conjecture that among all Recursive Layouts \( \nu_1 \) is minimized by MINWLA, the layout that cuts at height \( g = 1 \), arranges the outermost top subtree in-order, and arranges every subsequent subtree preorder. In our nomenclature, MINWLA is \( \tilde{T}^1 \) (see Table 1). Restricting ourselves to Recursive Layouts with cut height \( g = 1 \), MINWLA provably minimizes \( \nu_1 \). We delegate all proofs to the Appendix.

Theorem 1. The MINWLA layout minimizes \( \nu_1 \) among all Recursive Layouts with cut height \( g = 1 \).

Our experiments on block transitions (Figure 3), observed cache misses, and timings indicate that these other layouts have significantly worse locality than MINWEP, lending support to our claim that the weighted edge product (represented by \( \nu_0 \)) is the correct measure to consider.

In this paper, we present a succession of Hierarchical Layouts that reduce \( \nu_0 \), and we see that these also tend to reduce \( \nu_1, \mu_1, \) and \( \mu_\infty \), suggesting that these might be good layouts in other settings that benefit from better locality. In [16], the authors show that minimizing \( \nu_0 \) results in compression-friendly layouts. We note that minimizing \( \nu_0 \) is likely to result in high locality layouts for all graphs, and not just trees. For algorithms designed to minimize \( \mu_0, \nu_0, \mu_1, \) and \( \mu_\infty \), respectively, on general graphs, see [13], [16], [17], and [18].

IV. MINIMIZING THE WEIGHTED EDGE PRODUCT

We have shown that layouts with lower Weighted Edge Product \( \nu_0 \) result in fewer block transitions (measured by \( \nu_0 \)). So far, we have presented IN-VEB, with lower \( \nu_0 \) values than PRE-VEB. Section IV-A shows that \( \nu_0 \) can be further reduced by alternating layouts. Ultimately, the goal is to find the Hierarchical Layout that minimizes \( \nu_0 - \) the MINWEP layout.

A. Ordering the subtrees: Alternating Hierarchical Layouts

In the discussion so far, we have not yet determined the optimal relative ordering of the bottom subtrees – we have prescribed it to be in the order of the top subtree leaves. A simple way to reduce \( \nu_0 \) is to reduce the product of edge lengths among all edges that have the same weight, without modifying the lengths of all other edges. If we consider the Hierarchical Layout at a particular branch of the recursion, all the edges between the top subtree and the bottom subtrees have the same weight. By considering such equal-weight edges, the next result proves that a layout that orders the bottom subtrees in the reverse order of the parent leaves reduces \( \nu_0 \). In such a layout, the order of the nodes appears to alternate between left-to-right and right-to-left. As a result, we refer to Hierarchical Layouts that arrange the bottom subtrees in the reverse order of the parent leaves as alternating Hierarchical Layouts.

Theorem 2. For any subtree in a particular branch of the recursion, suppose we fix the internal ordering of the leaves of the top subtree \( \mathcal{A} \) and the arrangement of all the bottom subtrees in subsequent branches of the recursion. Then, the product of all the edge lengths between the top subtree and the bottom subtrees is minimized by ordering the bottom subtrees in reverse order of that of the parent leaves \( \mathcal{A} \).

As a corollary of Theorem 2, we see that when the cut height \( g > 1 \), the optimal relative ordering of bottom subtrees is one that positions both the bottom subtrees of a particular parent leaf in \( \mathcal{A} \) adjacent to each other. This suggests that the initial orderings (PRE-VEB and IN-VEB) got the adjacency of the bottom subtrees right – they only had the order wrong.

By recursive application of Theorem 2 we order the bottom subtrees in the reverse ordering of the parent leaves \( \mathcal{A} \) at each level of recursion, converting any Hierarchical Layout to its alternating version, thus reducing \( \nu_0 \). We denote the alternating version of IN-VEB by IN-VEBA, and define PRE-VBA similarly. In our nomenclature, these two layouts are \( \tilde{T}^{[b/2]}_1 \) and \( \tilde{T}^{[\nu_0]}_\infty \), respectively (see Table 1).
Figure 5c illustrates IN-VEBA for a tree of height 6. Observe that the bottom subtrees are arranged in reverse order of their parent leaves. In the outermost branch of the recursion, the rightmost two leaves in the top subtree are arranged at positions 35 and 33, and the corresponding child subtrees are rooted at positions 39, 46, 53, and 60. That is, the child subtrees are arranged in reverse order of the parent leaves (39 and 46 connected to 35), compared to IN-VEB (see Figure 5c, where 39 and 46 are connected to 33). We see that by alternating, the sum of edge lengths between top and bottom subtrees remains the same, but we have increased their variance, thus reducing their product and consequently $\nu_0$. A similar argument holds for alternating pre-order trees (see Figure 5d and Figure 5f).

It is important to mention that alternating a particular layout has no effect on $\nu_1$. However, since the variance of the edge lengths is increased, alternating a layout will increase $\mu_\infty$, and may increase the number of unit-length edges. Since $\nu_0$ is a function of the product of the edge lengths, the Weighted Edge Product is reduced. As an example, we can see the effect of alternating a layout on the child subtrees in Figure 2 also plots $\nu$ as a function of block size for a tree of height 20. We see that IN-VEB is virtually indistinguishable from IN-VEB, whereas PRE-VEB dominates PRE-VEB for small block sizes. Figure 2 also plots $\beta$ for PRE-VEB and IN-VEB as a function of tree height for a variety of block sizes. Again, we see that IN-VEB is virtually indistinguishable from IN-VEB, but PRE-VEB dominates PRE-VEB for all tree heights. And we see the same pattern with cache miss rate. Figure 2 also plots the explicit search time for IN-VEB and PRE-VEB. Comparing with IN-VEB and PRE-VEB, we see the exact same pattern. IN-VEB and PRE-VEB are indistinguishable from each other, with approximately 5% worse explicit search times than MINWEP. On the other hand, PRE-VEB is about 5% better than PRE-VEB.

From these experiments, we see that the improvement from PRE-VEB to PRE-VEB is far less than the improvement from PRE-VEB to IN-VEB. This suggests that while an alternating version always improves the layout (we restrict our attention to alternating layouts for the rest of this paper), it is far more important to switch from pre-order to in-order. One should consider this the main take-home message of this paper: All data structures that use a pre-order Hierarchical Layout should, at the very least, switch to an in-order version of the same Hierarchical Layout. Later, in Section IV-C, we will see that this result may depend on the cut height, but not for the cut heights $\lfloor h/2 \rfloor$ that have been used in practice.

### B. Constructing hybrid layouts: The HALFWEP layout

We now analyze the impact of varying the position of the top subtree $A$ relative to all the bottom subtrees. Recursive Layouts restrict us to the two extremes represented by PRE-VEB and IN-VEB, wherein $A$ is positioned either at one end or in the middle of all the bottom subtrees. However, IN-VEB and PRE-VEB arrange all bottom subtrees identically, either in-order or pre-order, respectively. We can consider many more permutations by orderings some of the bottom subtrees in-order and others pre-order. As before, the locality measure $\nu_0$ guides us in these decisions. Clearly, IN-VEB results in smaller $\nu_0$ than PRE-VEB. Can a hybrid layout (by modifying IN-VEB, possibly) reduce $\nu_0$ even further?

To construct a hybrid layout, we must take into account the trade-offs involved. First, observe that any bottom subtree is arranged in a contiguous block in memory, and has only one edge connecting it to the rest of the tree – the edge from its root to a leaf in the top subtree. Therefore, rearranging any bottom subtree potentially results in two changes to its contribution to $\nu_0$: the length of the edge connecting its root to its parent, and the lengths of the edges in the subtree itself. Discounting the connection to the top subtree, a bottom subtree ordered as in PRE-VEB has a larger Weighted Edge Product than when it is ordered as in IN-VEB. However, the root of a pre-order bottom subtree is closer to its parent than the root of an in-order bottom subtree, and the weight of this edge is larger than the weight of any edge within the bottom subtree. So there are potential benefits to modifying IN-VEB by arranging some of the bottom subtrees pre-order. This nevertheless raises the question: Which bottom subtrees should we modify, if any? Also, observe that the in-order bottom subtrees are identical, and differ only in their distance to their parent leaf in the top subtree, and similarly for the pre-order trees. As we move further away from the top subtree, the proportional reduction in the length $\ell$ of the edge connecting the subtrees decreases (i.e. the slope of $\log \ell$ approaches zero), whereas the degradation in the $\nu_0$ value of the bottom subtree remains the same. As a result, the marginal benefit of converting an in-order bottom subtree into a pre-order bottom subtree decreases. Therefore, if arranging any bottom subtree in-order results in lower $\nu_0$ than arranging it pre-order, then this must also be true for all bottom subtrees further away from the top subtree.

To find the best layout, we undertook a detailed empirical study that evaluated all Recursive Layouts for trees up to height 20. We considered all possible cut heights $g \leq \lfloor h/2 \rfloor$ (we quickly determined that larger $g$ were not beneficial). We calculated $\nu_0$ for every layout for each tree height. We noticed that the optimal ordering always rearranged the bottom subtrees closest to the top subtree pre-order, arranged all other bottom subtrees in-order, and used an in-order arrangement for the outermost branch of the recursion. In comparison, IN-VEB arranges all bottom subtrees in-order, and PRE-VEB arranges all of them pre-order. We give the version of this layout with cut height $g = \lfloor h/2 \rfloor$ the special name HALFWEP. In our nomenclature, HALFWEP is $\tilde{T}_g^{[h/2]}$ (see Table 1).
Figure 4 shows that HALFWEP and MINWEP have almost indistinguishable values of \( \nu_0 \) and performance in explicit search times, further validating \( \nu_0 \) as the appropriate locality measure for deriving cache-oblivious layouts.

Figure 5b illustrates HALFWEP for a tree of height 6. Observe that the bottom subtrees closest to the top subtree are arranged pre-order. At the outermost branch of the recursion, the subtrees rooted at positions 28 and 36 are arranged pre-order in HALFWEP. These are arranged in-order in IN-VEBA (see Figure 5c). From the thickness of the edges, one can see that HALFWEP reduces some edge lengths for every branch of the recursion by replacing some in-order bottom subtrees by pre-order bottom subtrees. This does increase some distances within the bottom subtree (the next recursive branch), but deeper down the tree, where they contribute less to the Weighted Edge Product. This is confirmed by the \( \nu_0 \) values for HALFWEP (1.823) and IN-VEBA (2.184).

Our empirical analysis is also backed by theory, when restricted to certain cut heights. In Theorem 3, we show that when the cuts are made at the top of the tree (\( g = 1 \)) at all branches of the recursion, this HALFWEP-like layout provably minimizes \( \nu_0 \). We refer to this layout as the MINEP layout, because it also minimizes the edge product \( \mu_0 \) for unweighted trees. In our nomenclature, MINEP is \( I_2^R \) (see Table 1).

**Theorem 3.** The MINEP layout minimizes \( \nu_0 \) among all Recursive Layouts with cut height \( g = 1 \).

It can also be shown that for all Hierarchical Layouts where the closest bottom subtree is arranged pre-order, cutting a subtree that is arranged in-order at \( g = 1 \) results in the same layout as cutting it at \( g = 2 \). This is because cutting an in-order subtree of height \( h \) at height \( g = 1 \) results in two pre-order subtrees of height \( h - 1 \), the roots of which are adjacent to the top subtree, so long as the closest bottom subtree in either case is arranged pre-order. This explains why, for trees of small height, such as the example with \( h = 6 \) considered in Figure 5 MINEP is identical to MINWEP (see Figure 5a).

**C. Optimizing the cut height: The MINWEP layout**

In the discussion so far, we have ignored the effect of the cut height by restricting ourselves to the case where \( g = \lfloor h/2 \rfloor \). Before we find the optimal cut height, we describe other layouts that turn out to be part of the Hierarchical Layout framework, albeit with extreme cut height values.

Consider cut height \( g = 1 \). Analogous to how HALFWEP is a hybrid of IN-VEBA and PRE-VEBA, one can think of MINWEP as a hybrid of two other simple layouts: the common IN-ORDER and PRE-ORDER depth-first layouts. All three are Hierarchical Layouts that cut every subtree at height \( g = 1 \), but differ in how the bottom subtrees are arranged. IN-ORDER arranges all subtrees in-order, and PRE-ORDER arranges all subtrees pre-order. In our nomenclature, IN-ORDER is \( I_2^R \) and PRE-ORDER is \( P_0 \) (see Table 1). Observe that when the cut height \( g = 1 \), there is only one leaf node in \( L_A \) at every branch of the recursion, and therefore the notion of alternating layouts is not relevant. Furthermore, there are only 2 bottom subtrees at each branch of the recursion, and therefore in-order and pre-order are the only two options for positioning the top subtree. As a result, all Hierarchical Layouts that are described using cut height \( g = 1 \) at all branches of the recursion are in fact Recursive Layouts. This is not true for other cut heights.

Figure 5g illustrates IN-ORDER for a tree of height 6. Observe that IN-ORDER arranges the two bottom subtrees at the outermost recursion in-order, resulting in their roots being placed at positions 16 and 48. These roots are arranged pre-order at positions 31 and 33 in MINEP (see Figure 5a). On the other hand, PRE-ORDER (see Figure 5h) arranges all subtrees pre-order. The roots of the same bottom subtrees are arranged pre-order at positions 2 and 33. Observe that IN-ORDER and PRE-ORDER have (nearly exactly) the same number of short edge lengths (counting the number of thick lines), but these are at the bottom of the tree for in-order, where the weights are much smaller. This results in much larger \( \nu_0 \) values for IN-ORDER (4.000), when compared to PRE-ORDER (2.828).

At the other end of the spectrum in terms of cut height is \( g = h - 1 \), where each subtree is cut one level above the bottom. It turns out that the Hierarchical Layout with \( g = h - 1 \) that arranges all subtrees pre-order (similar to PRE-VEB with cut height \( g = \lfloor h/2 \rfloor \)) is the simple and commonly used breadth-first order. For this reason, we denote the breadth-first layout as PRE-BREADTH. Figure 5i illustrates the PRE-BREADTH layout for a tree of height 6. Observe that the nodes are arranged by level. Furthermore, one can now also consider in-order and/or alternating variants on the breadth-first ordering. We denote the in-order variant by IN-BREADTH. Observe that when \( g = h - 1 \), the bottom subtrees are single nodes, and therefore the notion of their arrangement into pre- or in-order is not relevant. In our nomenclature, IN-BREADTH is \( T_h^{IN-1} \) and PRE-BREADTH is \( P_h^{IN-1} \) (see Table 1).

In our detailed empirical analysis, which suggested that \( \nu_0 \) is minimized by layouts that fit the characterization \( I_2^R \) in our nomenclature, we noticed that the optimal cut height (denoted by \( \nu_0^{opt} \)) was different from HALFWEP for pre-order subtrees: \( g_1^{opt}(h) = \max\{1, \lfloor(h - 1)/2 \rfloor \} \). For in-order subtrees, it is the same as before, i.e., \( g_1^{opt}(h) = \lfloor h/2 \rfloor \). Furthermore, there is one exception to the above rule, with \( g_1^{opt}(6) = 2 \), and correspondingly \( g_1^{opt}(5) = 1 \).

Based on these experiments, we define MINWEP as the HALFWEP-like layout with the cut heights presented above, including the exception. In our nomenclature, MINWEP is \( I_2^{opt} \) (see Table 1). Figure 5a illustrates MINWEP for a tree of height 6. In the outermost branch of the recursion, the top two levels of the tree are arranged together in positions 31 to 33, indicating a cut of \( g = 2 \). This compares with a cut of \( g = 3 \) for HALFWEP (see Figure 5b). Also, we see that the pre-order subtree of height \( h = 4 \) rooted at position 34 in MINWEP is cut at a height \( g = \lfloor(h - 1)/2 \rfloor = 1 \).

The pre-order cut height exception at \( g_1^{opt}(5) \) can also be interpreted as part of the piece-wise function \( g_1^{opt}(h) = 1 \) if \( h \leq 5 \), and \( \lfloor(h - 1)/2 \rfloor \) otherwise. Furthermore, interpreting the in-order cut height for subtrees of height 3 as \( g = 2 \) instead of \( g = 1 \), which is an equally valid interpretation, the cut
height for an in-order subtree can be calculated directly from the cut height for an pre-order subtree as follows: \( g_O^\text{opt} (h) = 1 \) if \( h = 2 \), and \( g_O^\text{opt} (h - 1) + 1 \) otherwise.

Analyzing Hierarchical Layouts where the closest bottom subtree is arranged pre-order and the cut heights are chosen such that \( g_I (h) = g_P (h - 1) + 1 \), we see that cutting all in-order subtrees at height \( g_I (h) = 1 \) instead results in the same layout. This is because this in-order cut results in two pre-order bottom subtrees of height \( h - 1 \), each of which will subsequently be cut at the same height as they would have been if they had been part of an in-order subtree of height \( h \). Since the closest bottom subtree is pre-order in either case, the layouts are identical. As a result, we can set \( g_I^\text{opt} (h) = 1 \). As we shall see later, this is important since it simplifies the index computation for nodes in pointer-less trees. Note that this optimization cannot be applied to HALF/WEP.

D. The cost of cache-misses: Explicit pointer-based searches

We observed earlier that HALF/WEP and MIN/WEP are virtually indistinguishable in terms of explicit search time. This is because they are exactly the same ordering schemes, but with very slightly different cut heights. Larger differences in cut heights can make a significant difference. Consider the values of \( v_0 \) in Figure 4 for many of the layouts presented so far. Recall that BENDER and PRE-VEB differ from each other only in the choice of the cut height \( g \). For BENDER, the cut height \( g = h - 2 \left\lceil \log_2 (h/2) \right\rceil \), which is identical to PRE-VEB \( (g = [h/2]) \) only for subtree heights that are a power of two. As expected, we see identical values of \( v_0 \) for BENDER and PRE-VEB for trees of height 4, 8, 16, and 32. However, for all other tree heights, BENDER gives higher values for \( v_0 \); sometimes 20% worse. This manifests itself in similarly worse pointer-based search times compared to PRE-VEB. This suggests that for a particular ordering scheme, the optimal cut height is closer to halfway down the tree.

Cut heights \( g = 1 \) and \( g = h - 1 \) illustrate this further. MINEP, which is identical to MIN/WEP except in its choice of the cut height \( (g = 1) \), results in significantly different trees (especially for larger tree heights), and we observe a steep divergence in \( v_0 \) as the tree height increases. As expected,
MINEP’s performance (measured using pointer-based search times) also degrades significantly for large tree heights. At the other end of the spectrum, consider PRE-BREADTH and IN-BREADTH, which are identical to PRE-VEB and IN-VEB respectively, except in the choice of cut height. A cut height of \( g = h - 1 \) results in significantly different layouts, especially for large tree heights. Even for the small example in Figure 5, we see that PRE-BREADTH is quite different from PRE-VEB. In Figure 4, we see that the pointer-based search time is significantly worse for breadth-first layouts, when compared to PRE-VEB and IN-VEB.

From the \( \nu_0 \) values in Figure 4, we also observe that inorder is not always better than pre-order. For a cut height of \( g = 1 \), PRE-ORDER results in much smaller \( \nu_0 \) values than INORDER. The example in Figure 5 suggests why: All the short edges in IN-ORDER are near the bottom of the tree, where the contribution to \( \nu_0 \) is minimal. However, this behavior changes as we increase the cut height, and at some point, inorder layouts are better than pre-order layouts. When the cut is approximately near halfway down the tree, inorder layouts such as IN-VEB result in much smaller \( \nu_0 \) values than pre-order layouts such as PRE-VEB. As we increase the cut height all the way to \( g = h - 1 \), we observe that the inorder version of the breadth-first layout IN-BREADTH continues to be better than the pre-order version PRE-BREADTH.

E. The computational cost of layouts: Pointer-less searches

Based on explicit pointer-based search times, we have shown that MINEP is a cache-oblivious layout with almost 20% improvement in performance when compared to the best in the literature, represented by PRE-VEB. However, MINEP is a more complex layout than PRE-VEB. The natural question therefore is: If we considered implicit, pointer-less search times, would MINEP still compare favorably with PRE-VEB? In [4], the authors showed that for small tree heights, even layouts that have poor cache-performance such as IN-ORDER and PRE-BREADTH perform better than PRE-VEB in implicit search, simply because it is trivial to compute the position of a node in such layouts.

To understand the trade-offs involved, we first measured the time taken to compute the index of child nodes in a pointer-less search by excluding all memory accesses. Listing 1 lists the code that takes the PRE-BREADTH index for a node and computes its corresponding MINEP index. This code needs to be executed for every transition in the search tree. Here the depth (level) \( d = \lfloor \log_2 i \rfloor \) of the node is maintained together with \( i \) along the search path from the root. Observe that one of the two functions in this code segment (partition) calculates the cut height for any pre-order subtree.

In Figure 4, we see that the index computation time is almost constant for simple layouts (IN-ORDER, PRE-ORDER, IN-BREADTH, and PRE-BREADTH). The slow increase merely stems from the longer search paths as the height of the tree is increased. Furthermore, MINEP’s index computation time is usually 4 times that of the simple layouts. Comparing MINEP with the van Emde Boas layouts (IN-VEB, PREVEB, BENDER, HALFVEP) is more interesting. Not surprisingly, HALFVEP performs worse than IN-VEB on this metric (around 20% worse), since it is a more complex layout. Observe that PRE-VEB performs better than IN-VEB, and by almost 50%. It turns out that the index can be computed more quickly within a pre-order subtree, since one does not need to keep track of left and right, and also because some other optimizations unique to pre-order layouts can be performed. This observation is key, since it allows us to compute the index for MINEP in 30% less time than HALFVEP, which is very similar at first glance. This is because we can set \( g^{opt}(h) = 1 \), as shown in Section IV-C, reducing the computational burden significantly by converting any in-order computation to a pre-order computation. As a result of this optimization, mean index computation times for MINEP are also about 20% less than those of IN-VEB. Finally, observe that index computations take almost 60% more time in BENDER compared to PREVEB, because of the additional time spent computing BENDER’s complex cut heights.

Figure 4 also presents our results on implicit, pointer-less search times. One can think of these as a combination of the index computation times (which do not include memory accesses) and explicit search times (which include memory accesses, but avoid index computations using pointers). We see that for the more complex layouts, the implicit search times correlate very well with the index computation times. This is because of the relatively fast memory access times; if we added disk or even flash to the memory hierarchy, we would expect the relative order among the implicit times to be similar to the explicit times. The only perceptible difference in our experiments is that the pre-order layouts (PRE-VEB and BENDER) perform slightly worse, since they perform almost 20% worse on the explicit search times. Among the simpler layouts, the implicit search times diverge significantly from the index computation times due to their poor memory access times. For tress of height 28, IN-ORDER already performs worse than PRE-VEB, and we expect all of the simpler layouts to perform worse than MINEP as the height of the tree increases beyond 32.

F. Experimental Setup

Our experiments were run on a single core of a dual-socket 6-core 2.80 GHz Intel Xeon X5660 (Westmere-EP) processor with 96 GB of 3x DDR3-1333 RAM split over two 48 GB NUMA memory banks, 12 MB 16-way per-socket shared L3 cache, 256 KB 8-way L2 cache, and 32 KB 8-way L1 data cache. All three caches use 64-byte cache lines. To reduce noise in the timing measurements, we computed the median time of 15 runs. Each run searches for (up to) 10 million randomly selected nodes. We counted the number of L1 and L2 cache misses incurred in memory accesses to the binary tree (stored as a linear array) using valgrind-3.5.0. We also repeated our experiments on different architectures, from

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1 We achieved this by storing the keys \( \{1, \ldots, |V|\} \) in the tree, allowing them to be easily inferred without lookup via their in-order index.
powerful workstations to laptops, and observed similar results.

V. CONCLUSIONS

In this paper, we present MINWEP, a new layout for cache-oblivious search trees that outperforms layouts used in the literature by almost 20%. Using a general framework of Hierarchical Layouts, we showed that MINWEP minimizes a new locality measure (ν₀), which represents the Weighted Edge Product) that correlates very well with cache-misses in a multi-level cache hierarchy. All widely used cache-oblivious versions of search trees rely on van Emde Boas layouts, which are shown to be a special case of Hierarchical Layouts. Therefore, we suggest that the performance of all these data structures can be easily improved by switching to a layout that is derived from MINWEP.

While enumerating all possible orderings for small trees, we noticed that the optimal ν₀ value is sometimes obtained by layouts that do not place the top subtree at one end or in the middle of the bottom subtrees. This implies that Recursive Layouts do not necessarily optimize ν₀. One direction of future study is to generalize the notion of Recursive Layouts to include such Hierarchical Layouts, and to construct unrestricted layouts that optimize ν₀. We would also like to prove that, at least among all Recursive Layouts, MINEP and MINWEP minimize μ₀ and ν₀, respectively, since we believe this is true based on our extensive empirical study.

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Fig. 5: Layouts and locality functionals \(v_0\) (weighted edge product), \(v_1\) (weighted edge sum), \(\mu_1\) (mean edge length), and \(\mu_{\infty}\) (maximum edge length) of a tree with \(h = 6\) levels. For \(h \leq 6\), MinEP and MinWEP coincide. Edges \(ij\) are drawn with thickness inversely proportional to length \(\ell_{ij}\). Cuts are shown as dashed lines that span the width of subtrees with 3 or more levels. Colored vertices are roots of in- (blue) or pre-order (red) subtrees with 2 or more levels.
APPENDIX

Theorem 1. The \textit{MINWL}A layout minimizes $\nu_1$ among all Recursive Layouts with cut height $g = 1$.

\textbf{Proof:} Recall that the weight of an edge between level $d$ and level $d + 1$ is $2^{-d}$. Observe that when $g = 1$, the top subtree $A$ is a single node $x$, and there are only two bottom subtrees, which we denote as $B_1$ and $B_2$. Essentially, we have to prove the following for all Recursive Layouts with cut height $g = 1$.

1) $\nu_1$ is always minimized by arranging both bottom subtrees $B_1$ and $B_2$ pre-order.
2) For any subtree, the optimal in-order arrangement has lower $\nu_1$ than the optimal pre-order arrangement.

Consider the optimal pre-order arrangement for a subtree of height $h$. Without loss of generality, let $B_1$ be the bottom subtree closest to $A$ in a pre-order arrangement. We can obtain an in-order arrangement of lower cost by moving the second bottom subtree $B_2$ to the other side of the top subtree $A$, since this moves the root of $B_2$ closer to $x$ without changing the costs of either bottom subtree. Clearly, the optimal in-order arrangement must have lower cost than this in-order arrangement, which proves \textbf{item 2}.

Now, consider the optimal in-order arrangement of a bottom subtree, which has height $h - 1$. Let this be subtree $B_x$, since the analysis is valid for both bottom subtrees. By moving one of the bottom subtrees of height $h - 2$ to the other side of $B_x$'s root, we convert it to a pre-order arrangement, bringing $B_x$'s root closer to $x$ by $2^{h-2} - 1$. At the same time, if we flip the order of the nodes of the bottom subtree we just moved, we have increased the length of the edge connecting its root to $B_x$'s root by the same $2^{h-2} - 1$. However, since this edge is one level further down the tree, it has a lower weight, and therefore the overall cost has decreased. (Observe that no other edge lengths have changed.) As a result, we can obtain a pre-order arrangement for $B_x$ that is of lower cost than the optimal in-order arrangement for $B_x$. This implies that the optimal pre-order arrangement for $B_x$ has a lower $\nu_1$ value than the optimal in-order arrangement for $B_x$, proving \textbf{item 1}.

It is not too difficult to extend this proof for all weight distributions where the weights do not increase from one level to the next. As a result, among all Recursive Layouts with cut height $g = 1$, \textit{MINWL}A optimizes the unweighted measure $\mu_1$, and also optimizes $\nu_1$ for the exact weight distribution described in \textbf{Equation 2}.

Theorem 2. For any subtree in a particular branch of the recursion, suppose we fix the internal ordering of the leaves of the top subtree $A$ and the arrangement of all the bottom subtrees in subsequent branches of the recursion. Then, the product of all the edge lengths between the top subtree and the bottom subtrees is minimized by ordering the bottom subtrees in reverse order of that of the parent leaves $L_A$.

\textbf{Proof:} Let $\varphi$ represent the layout of the top subtree $A$. Consider two leaves $x$ and $y$ of the top subtree $A$. Let $X$ and $Y$ be bottom subtrees whose parents are $x$ and $y$, respectively.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig6}
\caption{The cost is reduced by swapping the positions of $X$ and $Y$. The bottom ordering is guaranteed to have a lower cost than the top ordering.}
\end{figure}

We use $x < y$ to mean $\varphi(x) < \varphi(y)$; $x < Y$ implies $x < y \ \forall y \in Y$; and $X < Y$ implies $x < y \ \forall x \in X, y \in Y$.

Without loss of generality, assume $x < y$. Consider the case in which the two bottom subtrees $X$ and $Y$ appear to the same side of the top subtree $A$ containing $x$ and $y$. Without loss of generality, assume that $X$ and $Y$ are to the right of $A$, i.e. $y < X$ and $y < Y$. We show that if $X < Y$, then there exist operations that will place $Y$ before $X$ such that the product of the edge lengths is lowered. Because the edges $\{x, X\}$ and $\{y, Y\}$ have the same weight, it is easy to see that the value of this weight does not affect the proof, and hence we assume that this weight is one.

Suppose $X < Y$. In $\varphi$, let the distance between $x$ and $y$ be $d_1$, between $y$ and the first node of $X$ be $d_2$, and between the first nodes of $X$ and $Y$ be $d_3$. Furthermore, let $r_x$ and $r_y$ denote the distances from the first node of $X$ and $Y$ to their roots, such that the lengths of the edges between $x$ and $y$ and their subtrees are $\ell_x = d_1 + d_2 + r_x$ and $\ell_y = d_2 + d_3 + r_y$, respectively. (See the top example in \textbf{Figure 6}.)

The edge product contribution of the two joining edges is $\ell_x \ell_y$. Now consider swapping the positions of the subtrees $X$ and $Y$ and permuting the nodes within $X$ and $Y$, keeping all other nodes fixed, so that $X$ takes on the relative ordering of $Y$ and vice versa. (See the bottom example in \textbf{Figure 6}.)

This permutation does not affect the internal contributions of $X$ and $Y$ to the cost since the two subtrees have the same structure (i.e. the internal costs of $X$ and $Y$ are interchanged, but their product remains the same). However, this operation results in new edge lengths $\ell'_x = \ell_y + d_1$ and $\ell'_y = \ell_x + d_3$ with respect to $x$ and $y$. The resulting cost difference is now $\ell_x \ell_y - \ell'_x \ell'_y = d_1 (d_1 + \ell_y - \ell_x)$. And since $d_1 \geq 1$ and by assumption $\ell_y > \ell_x$, the cost has been reduced. Consequently, we can always reduce the cost by enforcing $Y < X$ whenever $x < y$, which completes the proof.

Theorem 3. The \textit{MINEP} layout minimizes $\nu_0$ among all Recursive Layouts with cut height $g = 1$.

\textbf{Proof:} We introduce the following notation to prove that, for binary tree $T$, the \textit{MINEP} layout minimizes Weighted Edge Product among all Recursive Layouts with cut height $g = 1$. For trees of height $h$ cut at height $g = 1$, let $C^h_p$, $C^h_p$ be $W$
times the logarithm of the weighted edge product functional \( \nu_0 \) when the top subtree is arranged in-order and pre-order, respectively. Recall that the weight of an edge between level \( d \) and level \( d+1 \) is \( 2^{-d} \). Observe that when \( g = 1 \), the top subtree \( A \) is a single node \( x \), and there are only two bottom subtrees, which we denote as \( B_1 \) and \( B_2 \). Essentially, we have to prove the following for all Recursive Layouts with cut height \( g = 1 \).

1. If the top subtree \( A \) is in-order, then \( \nu_0 \) is minimized by arranging both bottom subtrees \( B_1 \) and \( B_2 \) pre-order.
2. If the top subtree \( A \) is pre-order, then \( \nu_0 \) is minimized by arranging the subtree closest to it \( (B_1) \) pre-order, and the other \( (B_2) \) in-order.
3. The optimal in-order arrangement has lower \( \nu_0 \) than the optimal pre-order arrangement.

Consider the optimal pre-order arrangement with cost \( C_P^h \), as prescribed by item 2. We can obtain an in-order arrangement of lower cost by moving the second bottom subtree \( B_2 \) to the other side of the top subtree \( A \), since this moves the root of \( B_2 \) closer to \( x \) without changing the costs of either bottom subtree.

Clearly, the optimal in-order arrangement, as prescribed by item 1, must have lower cost than this in-order arrangement, and therefore \( C_I^h \leq C_P^h \), which proves item 3. Therefore, we only need to prove item 1 and item 2.

First, by inspection, we prove that this is true for \( h = \{1, 2\} \). When \( h = 1 \), we trivially have the result, since there are no subtrees. When \( h = 2 \), item 1 and item 2 are trivial since the subtrees have only one node (and therefore in-order and pre-order is the same for them).

Observe that in the in-order case item 1, arranging one bottom subtree in-order and the other pre-order will be dominated either by ordering both the bottom subtrees in-order, or both pre-order. Therefore, we only need to compare arranging either bottom subtree pre-order with arranging either bottom subtrees in-order. The optimal choice for the in-order case applies for the bottom subtree closest to the top subtree in the pre-order case item 2. Therefore, we only need to need to compare the two choices for arranging the second bottom subtree \( (B_2) \) in this case.

We prove the general case \( (h \geq 3) \) by induction. Observe that the bottom subtrees \( B_1 \) and \( B_2 \) (of height \( h - 1 \)) have edges of weight \( 1/4 \) connecting its root to its children; this offset needs to be carefully accounted for. We can calculate the length of the edge between \( x \) and the root of the bottom subtrees \( B_1 \) and \( B_2 \) based on whether the subtrees are arranged in-order or pre-order. Comparing this with the optimal cost of the bottom subtrees, we need to prove that

\[
C_I^{h-1} + \log(2^{h-1} + 2^{h-2} - 1) \leq C_P^{h-1} + (h - 1) \quad (9)
\]

Inequality (5) proves item 1, and as a corollary, proves that \( C_I^h = C_P^h \). Used together, inequalities (8) and (9) prove item 2, and as a corollary, proves that \( C_P^h = \frac{1}{2}(C_I^{h-1} + C_P^{h-1} + \log(2^{h-1} + 2^{h-2} - 1)) \).

Let us assume the induction hypothesis. In other words, for \( h - 1 \), we have

\[
C_P^{h-2} \leq C_I^{h-2} + (h - 3) \quad (10)
\]

\[
C_I^{h-2} + \log(2^{h-2} + 2^{h-3} - 1) \leq C_P^{h-2} + (h - 2) \quad (11)
\]

which, in turn, imply

\[
C_I^{h-1} = C_P^{h-2} \quad (12)
\]

\[
C_P^{h-1} = \frac{1}{2}(C_I^{h-2} + C_P^{h-2} + \log(2^{h-2} + 2^{h-3} - 1)) \quad (13)
\]

Adding \( C_P^{h-2} \) to both sides of (11), and multiplying by \( \frac{1}{2} \), we have

\[
\frac{1}{2}(C_I^{h-2} + C_P^{h-2} + \log(2^{h-2} + 2^{h-3} - 1)) \leq C_P^{h-2} + \frac{1}{2}(h - 2). \quad (14)
\]

Substituting in \( C_I^{h-1} \) from (12) and (13), we get

\[
C_P^{h-1} \leq C_I^{h-1} + \frac{1}{2}(h - 2), \quad (15)
\]

which implies (8).

We define \( f(x) = \log(2^x + 2^{x-1} - 1) \). It is easy to see that for all \( x > 1 \), we have \( x < f(x) < x + 1 \). Therefore, \( -2 < f(h - 1) - f(h) \), and from (10), we have \( C_P^{h-2} < C_I^{h-2} + f(h - 1) + f(h) \). Adding \( C_P^{h-2} \) to both sides, and multiplying by \( \frac{1}{2} \), we have \( C_P^{h-2} < \frac{1}{2}(C_I^{h-2} + C_P^{h-2} + f(h - 1) + f(h - 1) - f(h)) \). Substituting in \( C_I^{h-1} \) from (12) and (13), we get

\[
C_P^{h-1} < C_I^{h-1} + \frac{1}{2}(f(h - 1) - f(h)) \quad (16)
\]

which implies \( C_P^{h-1} < C_I^{h-1} + \frac{1}{2}(f(h - 1) - f(h)) \), which in turn implies (9).

It is not too difficult to extend this proof for all weight distributions where the weights do not increase from one level to the next. As a result, among all Recursive Layouts with cut height \( g = 1 \), MINEP optimizes the unweighted measure \( \mu_0 \), and also optimizes \( \nu_0 \) for the exact weight distribution described in Equation 2.

\[ \blacksquare \]