Abstract. — We start the study of the family of birational maps \((f_{\alpha, \beta})\) of \(\mathbb{P}^2_{\mathbb{C}}\) in [14]. For generic \(\alpha\) and \(\beta\) of modulus 1 the centraliser of \(f_{\alpha, \beta}\) is trivial, the topological entropy of \(f_{\alpha, \beta}\) is 0, there exist two areas of linearisation: in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On \(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}\) any \(f_{\alpha, \beta}\) can be viewed as a cocycle; using recent results about \(\text{SL}(2; \mathbb{C})\)-cocycles ([1, 2]) we determine the LYAPUNOV exponent of the cocycle associated to \(f_{\alpha, \beta}\).

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Introduction

In this article we deal with a family of birational maps \((f_{\alpha, \beta})\) given by
\[
f_{\alpha, \beta} : \mathbb{P}^2_{\mathbb{C}} \rightarrow \mathbb{P}^2_{\mathbb{C}} \quad (x : y : z) \rightarrow ((\alpha x + y)z : \beta y(x + z) : z(x + z))
\]
where \(\alpha, \beta\) denote two complex numbers with modulus 1, case where we know almost nothing about the dynamics. This family of maps satisfies the following properties ([14]):

- for \(\alpha\) and \(\beta\) generic the centraliser
  \[
  \text{Cent}(f_{\alpha, \beta}) = \{ g \in \text{Bir}(\mathbb{P}^2_{\mathbb{C}}) \mid g \circ f_{\alpha, \beta} = f_{\alpha, \beta} \circ g \}
  \]
of \(f_{\alpha, \beta}\) is isomorphic to \(\mathbb{Z}\);
- the topological entropy of \(f_{\alpha, \beta}\) is 0;
- rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under \(f_{\alpha, \beta}\) is a torus, and there is an other domain of linearisation where the orbit of a generic point under \(f_{\alpha, \beta}^2\) is a circle.

We can also see \(f_{\alpha, \beta}\) on \(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}\) (since all the computations of [14] have been done in an affine chart they may all be carried on \(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}\)); the sets \(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{S}^1_{\mathbb{C}}\), where \(\mathbb{S}^1_{\mathbb{C}} = \{ y \in \mathbb{C} \mid |y| = \rho \}\), are invariant.
Let us define $A_n^{\alpha,\rho}: \mathbb{S}^1 \to \text{M}(2;\mathbb{C})$ given in terms of $A^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y \\ 1 & 1 \end{bmatrix}$ by

$$A_n^{\alpha,\rho}(\cdot) = A^{\alpha,\rho}(\beta^n \cdot)A^{\alpha,\rho}(\beta^{n-1} \cdot) \ldots A^{\alpha,\rho}(\beta \cdot)A^{\alpha,\rho}(\cdot).$$

To compute $f_{\alpha,\beta}^n(x,y)$ is equivalent to compute $A_n^{\alpha,\rho}(y)$ as soon as $f_{\alpha,\beta}^k(x,y) \neq (-1,\alpha)$ for any $1 \leq k \leq n$.

Using [1] we are able to determine the Lyapunov exponent of the cocycle $(A^{\alpha,\rho},\beta)$:

**Theorem A.** — The Lyapunov exponent of $(A^{\alpha,\rho},\beta)$ is

- positive as soon as $\rho > 1$;
- zero as soon as $\rho \leq 1$.

More precisely $f_{\alpha,\beta}$ is semi conjugate to $(\frac{\alpha + y^2}{x+1},\frac{\beta^{1/2} y}{x+1})$ and the Lyapunov exponent of the cocycle $(B^{\alpha,\rho},\beta^{1/2})$, where

$$B^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix},$$

is equal to $\max(0,\ln\rho)$.

In the next section we introduce the family $(f_{\alpha,\beta})$ and its properties (§1). Then we deal with the recent works of Avila on SL(2;\mathbb{C})-cocycles. In the last section we give the proof of Theorem A (see §2). Let us explain the sketch of it. We associate to $(B^{\alpha,\rho},\beta^{1/2})$ a cocycle $(\tilde{B}^{\alpha,\rho},\beta^{1/2})$ that belongs to SL(2;\mathbb{C}). We first determine

$$\lim_{\rho \to 0} L(\tilde{B}^{\alpha,\rho},\beta^{1/2}),$$

and then

$$\lim_{\rho \to +\infty} L(\tilde{B}^{\alpha,\rho},\beta^{1/2})$$

where $L(C,\gamma)$ denotes the Lyapunov exponent of the SL(2;\mathbb{C})-cocycle $(C,\gamma)$. In both cases, we get 0. Using [1, Theorem 5] we obtain that $L(\tilde{B}^{\alpha,\rho},\beta^{1/2})$ vanishes everywhere; it allows us to determine $L(A^{\alpha,\rho},\beta)$ since

$$L(B^{\alpha,\rho}(y),\beta^{1/2}) = L(\tilde{B}^{\alpha,\rho}(y),\beta^{1/2}) + \max(0,\ln\rho),$$

and since $(A^{\alpha,\rho},\beta)$ and $(\beta^{1/2},B^{\alpha,\rho})$ are conjugate.

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1. Some properties of the family \((f_\alpha, \beta)\)

A rational map \(\phi\) from \(\mathbb{P}^2_C\) into itself is a map of the form
\[
(x : y : z) \mapsto (\phi_0(x,y,z) : \phi_1(x,y,z) : \phi_2(x,y,z)),
\]
where the \(\phi_i\)'s are some homogeneous polynomials of the same degree without common factor; \(\phi\) is birational if it admits an inverse of the same type. We will denote by Bir\((\mathbb{P}^2_C)\) the group of birational maps of \(\mathbb{P}^2_C\), also called the Cremona group. The degree of \(\phi\), denoted \(\deg \phi\), is the degree of the \(\phi_i\)'s. The degree is not a birational invariant: \(\deg \psi \phi \psi^{-1} \neq \deg \phi\) for generic birational maps \(\phi\) and \(\psi\). The first dynamical degree of \(\phi\) given by
\[
\lambda(\phi) = \lim_{n \to +\infty} \left(\deg \phi^n\right)^{1/n},
\]
is a birational invariant; it is strongly related to the topological entropy \(h_{\text{top}}(\phi)\) of \(\phi\) (see [19, 22])
\[
h_{\text{top}}(\phi) \leq \log \lambda(\phi) \tag{1.1}
\]

Any birational map \(\phi\) admits a resolution
\[
\begin{array}{c}
\pi_1 \\
\downarrow \quad \quad \quad \downarrow \pi_2 \\
\mathbb{P}^2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \mathbb{P}^2 \\
\phi \quad \\
\end{array}
\]
where \(\pi_1, \pi_2 : S \to \mathbb{P}^2_C\) are sequences of blow-ups (see [4] for example). The resolution is minimal if and only if no \((-1)\)-curve of \(S\) is contracted by both \(\pi_1\) and \(\pi_2\). The base-points of \(\phi\) are the points blown-up in \(\pi_1\), which can be points of \(\mathbb{P}^2_C\) or infinitely near points. We denote by \(b(\phi)\) the number of such points, which is also equal to the difference of the ranks of Pic\((S)\) and Pic\((\mathbb{P}^2_C)\), and thus equals to \(b(\phi^{-1})\). The dynamical number of base-points of \(\phi\) introduced in [10] is by definition
\[
\mu(\phi) = \lim_{n \to +\infty} \frac{b(\phi^n)}{n},
\]
it is a real positive number that satisfies \(\mu(\phi^n) = |n\mu(\phi)|\) for any \(n \in \mathbb{Z}\), \(\mu(\psi \phi \psi^{-1}) = \mu(\phi)\), and allows us to give a characterization of birational maps conjugate to automorphisms:

**Theorem 1.1 ([10]).** — Let \(S\) be a smooth projective surface; the birational map \(\phi \in \text{Bir}(S)\) is conjugate to an automorphism of a smooth projective surface if and only if \(\mu(\phi) = 0\).

The behavior of \(\phi \in \text{Bir}(\mathbb{P}^2_C)\) is strongly related to the behavior of \(\left(\deg \phi^n\right)_{n \in \mathbb{N}}\) (see [18, 17, 10]); up to birational conjugacy exactly one of the following holds:
1. the sequence \(\left(\deg \phi^n\right)_{n \in \mathbb{N}}\) is bounded and either \(\phi\) is of finite order, or \(\phi\) is an automorphism of \(\mathbb{P}^2_C\);
2. there exists an integer \(k\) such that
\[
\lim_{n \to +\infty} \frac{\deg \phi^n}{n} = k^2 \frac{\mu(\phi)}{2}
\]
and \(\phi\) is not an automorphism;
3. there exists an integer $k \geq 3$ such that
\[
\lim_{n \to +\infty} \frac{\deg \phi^n}{n^2} = k \cdot \frac{\kappa(\phi)}{9}
\]
where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant, and $\phi$ is an automorphism;
4. the sequence $\left(\deg \phi^n\right)_{n \in \mathbb{N}}$ grows exponentially (see [17] for more precise dynamical properties).

In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$. In case 2. (resp. 3.) the map $\phi$ preserves a unique fibration which is rational (resp. elliptic).

In case 1. (resp. 2., resp. 3, resp. 4) we say that $\phi$ is elliptic (resp. a JONQUIÈRES twist, resp. an HALPHEN twist, resp. hyperbolic).

Let us give some examples. Let
\[
\phi(x, y) = \left(\frac{a(y)x + b(y)}{c(y)x + d(y)}, \frac{\alpha y + \beta}{\gamma y + \delta}\right)
\]
be an element of the JONQUIÈRES group $\text{PGL}(2; \mathbb{C}(y)) \rtimes \text{PGL}(2; \mathbb{C})$; either $\phi$ is elliptic (for instance $\phi: (x : y : z) \mapsto (yz : xz : xy)$), or $\phi$ is a JONQUIÈRES twist (for example $\phi: (x : y : z) \mapsto (xz : xy : x^2)$ for which the unique invariant fibration is $y/z = \text{constant}$). The map
\[
\phi: \mathbb{P}^2 \to \mathbb{P}^2 \quad (x : y : z) \mapsto ((2y + z)(y + z) : x(2y - z) : 2z(y + z))
\]
is an HALPHEN twist ([17, Proposition 9.5]). HÉNON automorphisms give by homogeneization examples of hyperbolic maps.

Clearly elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of $\mathbb{P}^2$:

**Theorem 1.2 ([11]).** — Let $S$ be a compact complex surface that carries an automorphism $\phi$ of positive topological entropy.

- Either the KODAIRA dimension of $S$ is zero and $\phi$ is conjugate to an automorphism on the unique minimal model of $S$ that necessarily is a torus, or a K3 surface or an ENRIQUES surface;
- or the surface $S$ is a non-minimal rational one, isomorphic to $\mathbb{P}^2$ blown up at $n$ points, $n \geq 10$, and $\phi$ is conjugate to a birational map of $\mathbb{P}^2$.

This yields many examples of hyperbolic birational maps for which we can establish a lot of dynamical properties ([20, 5, 6, 7, 8, 16, 15]).

Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of $\mathbb{C}^2$ having rich dynamics are HÉNON maps; furthermore a polynomial automorphism of $\mathbb{C}^2$ is a HÉNON one if and only if its centraliser is countable. Let us now consider rational maps on $\mathbb{S}^1$; if the centraliser of such maps is not trivial ([1]), then the JULIA set is "special". The centraliser of an elliptic birational map of infinite order is uncountable ([10]). The centraliser of HALPHEN twists are described in [18]. The centraliser of an hyperbolic map is countable ([12]). In [13] we end the story by studying centralisers of JONQUIÈRES twists. If the centraliser of a map $\phi$ is trivial if it coincides with the iterates of $\phi$.

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1. The centraliser of a map $\phi$ is trivial if it coincides with the iterates of $\phi$. 
fibration is fiberwise invariant, then the centraliser is uncountable; but if it isn’t, then generically the centraliser is isomorphic to \( \mathbb{Z} \). We don’t know a lot about dynamics of these maps, in this article we will thus focus on a family of such maps. We consider the JONQUIÈRES maps

\[
f_{\alpha, \beta} : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}} \\
(\alpha x + y)z : \beta y(x + z) : z(x+z)
\]

where \( \alpha, \beta \) denote two complex numbers with modulus 1. The base-points of \( f_{\alpha, \beta} \) are

\[(1 : 0 : 0), \quad (0 : 1 : 0), \quad (-1 : \alpha : 1).
\]

Any \( f_{\alpha, \beta} \) preserves a rational fibration (the fibration \( y = \) constant in the affine chart \( z = 1 \)). Each element of the family \( (f_{\alpha, \beta}) \) has first dynamical degree 1 hence topological entropy zero (1.1); more precisely one has (10, Example 4.3)

\[
\mu(f_{\alpha, \beta}) = \frac{1}{2}
\]

so \( f_{\alpha, \beta} \) is not conjugate to an automorphism (Theorem 1.1). The centralizer of \( f_{\alpha, \beta} \) is isomorphic to \( \mathbb{Z} \) (see [14, Theorem 1.6]). The idea of the proof is the following: the point \( \psi(p) \) is blown-up onto a fiber of the fibration \( y = \) constant. Let \( \psi \) be an element of

\[
\text{Cent}(f_{\alpha, \beta}) = \{ g \in \text{Bir}(\mathbb{P}^2_{\mathbb{C}}) \mid g \circ f_{\alpha, \beta} = f_{\alpha, \beta} \circ g \};
\]

since \( \psi \) blows down a finite number of curves there exists a positive integer \( k \) (chosen minimal) such that \( f_{\alpha, \beta}^k(p) \) is not blown down by \( \psi \). Replacing \( \psi \) by \( \tilde{\psi} = \psi f_{\alpha, \beta}^k \) one gets that \( \tilde{\psi}(p) \) is an indeterminacy point of \( f_{\alpha, \beta} \). In other words \( \tilde{\psi} \) permutes the indeterminacy points of \( f_{\alpha, \beta} \). A more precise study allows us to establish that \( p \) is fixed by \( \tilde{\psi} \). The parameters \( \alpha, \beta \) being generic, the closure of the negative orbit of \( p \) under the action of \( f_{\alpha, \beta} \) is ZARISKI dense; since \( \tilde{\psi} \) fixes any element of the orbit of \( p \) one obtains \( \tilde{\psi} = \text{id} \).

Let us recall that if \( \psi \) is an automorphism on a compact complex manifold \( M \), the \textbf{Fatou set} \( \mathcal{F}(\psi) \) of \( \psi \) is the set of points that own a neighborhood \( \mathcal{V} \) such that \( \{ f_{\psi}^n \mid n \in \mathbb{N} \} \) is a normal family. Set

\[
G(\mathcal{U}) = \{ \phi : \mathcal{U} \to \overline{\mathcal{U}} \mid \phi = \lim_{n \to +\infty} \psi^n \};
\]

we say that \( \mathcal{U} \) is a \textbf{rotation domain} if \( G(\mathcal{U}) \) is a subgroup of \( \text{Aut}(\mathcal{U}) \). An equivalent definition is the following: a component \( \mathcal{U} \) of \( \mathcal{F}(\psi) \) which is invariant by \( \psi \) is a rotation domain if \( \psi|_{\mathcal{U}} \) is conjugate to a linear rotation. If \( \mathcal{U} \) is a rotation domain, \( G(\mathcal{U}) \) is a compact Lie group, and the action of \( G(\mathcal{U}) \) on \( \mathcal{U} \) is analytic real. Since \( G(\mathcal{U}) \) is a compact, infinite, abelian Lie group, the connected component of the identity of \( G(\mathcal{U}) \) is a torus of dimension \( 0 \leq d \leq \dim_{\mathbb{C}} M \). The integer \( d \) is the \textbf{rank of the rotation domain}. The rank coincides with the dimension of the closure of a generic orbit of a point in \( \mathcal{U} \).

We can also see \( f_{\alpha, \beta} \) on \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \) and that is what we will do in the sequel (since all the computations of [14] have been done in an affine chart they may all be carried on \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \)); the sets \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{S}^1_{\mathbb{R}} \) are invariant. In [14] we show that there are two rotation domains for \( f_{\alpha, \beta}^2 \), one of
Theorem 1.3. — Assume that α and β are generic.

There exists a strictly positive real number r such that \( f_{\alpha, \beta} \) is conjugate to \((\alpha x, \beta y)\) on \( \mathbb{P}^1 \times \mathbb{D}(0, r) \) where \( \mathbb{D}(0, r) \) denotes the disk centered at the origin with radius r.

There exists a strictly positive real number \( \tilde{r} \) such that \( \left( \frac{1}{x}, \frac{1}{y} \right) \) is conjugate to \( \left( \frac{x}{\beta}, \frac{y}{\beta^2} \right) \) on \( \mathbb{P}^1 \times \mathbb{D}(0, \tilde{r}) \).

Remark 1.4. — The point \((\alpha - 1, 0)\) is also a fixed point of \( f_{\alpha, \beta} \) where the behavior of \( f_{\alpha, \beta} \) is the same as near \((0, 0)\).

Proof. — The first assertion is proved in [14].

Let us consider the map \( \psi(x, y) = \left( \frac{a(x)y + b(y)}{c(y)x + 1}, y \right) \). The equation

\[
\psi^{-1} \left( \frac{1}{x}, \frac{1}{y} \right) f_{\alpha, \beta} \left( \frac{1}{x}, \frac{1}{y} \right) \psi = \left( \frac{x}{\beta}, \frac{y}{\beta^2} \right)
\]

yields to

\[
\beta a(\beta^{-2}y)c(y) + \beta a(\beta^{-2}y)a(y) - c(\beta^{-2}y)a(y) + \alpha a(\beta^{-2}y)c(y) + y(\alpha^2 a(\beta^{-2}y)c(y) - c(\beta^{-2}y)c(y) - c(\beta^{-2}y)a(y)) = 0, \tag{1.2}
\]

\[
\beta a(\beta^{-2}y) - \beta a(y) + y(\alpha^2 a(\beta^{-2}y) - \alpha \beta c(y) - \beta c(y) - \beta a(y) - \alpha c(\beta^{-2}y) - c(\beta^{-2}y)) + \beta(\alpha + \beta) b(\beta^{-2}y) + (\alpha + \beta) b(\beta^{-2}y) + \beta^2 b(\beta^{-2}y)c(y) - b(y)c(\beta^{-2}y)
\]

\[
+ y(\alpha^2 b(\beta^{-2}y)c(y) - b(y)c(\beta^{-2}y)) = 0 \tag{1.3}
\]

and

\[
(\alpha + 1) y + b(y) - \beta b(\beta^{-2}y) - \alpha^2 y b(\beta^{-2}y) + y b(y) - (\alpha + \beta) b(\beta^{-2}y) b(y) = 0 \tag{1.4}
\]

Let us set

\[
a(y) = \sum_{i \geq 0} a_i y^i, \quad b(y) = \sum_{i \geq 0} b_i y^i, \quad c(y) = \sum_{i \geq 0} c_i y^i.
\]

We easily get \( a_0 = 1 - \beta \), \( b_0 = 0 \) and \( c_0 = \alpha + \beta \).

Relation (1.4) implies that

\[
b_1 = \frac{\beta(1 + \alpha)}{1 - \beta} \quad & \quad \beta b_1 \left( 1 - \beta^{1-2v} \right) + F_i(b_i | i < v) = 0 \quad \forall v > 1.
\]

Equality (1.3) yields to

\[
a_v \left( \beta^{1-2v} - \beta \right) + b_v \left( (\alpha + \beta) a_0 \left( 1 + \beta^{1-2v} \right) + c_0 \left( \beta^{2-2v} - 1 \right) \right) + F_i(a_i, b_i, c_i | i < v) = 0
\]

and (1.2) to

\[
c_v a_0 \left( \beta - \beta^{-2v} \right) + a_v \left( (\alpha + \beta) a_0 \left( 1 + \beta^{-2v} \right) + c_0 \left( \beta^{1-2v} - 1 \right) \right) + F_i(a_i, b_i, c_i | i < v) = 0
\]

2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (see [6]), but \( f_{\alpha, \beta} \) is not conjugate to an automorphism on a rational surface.
where the $F_i$’s denote universal polynomials; this allows to compute $b_\nu$, $a_\nu$ and $c_\nu$. Thus we get a formal conjugacy. According to Siegel’s theorem ([21]) any linearizing map is convergent on a polydisc; so $a(y)$, $b(y)$ and $c(y)$ are convergent which gives the result. □

2. About $SL(2;\mathbb{C})$-cocycles

A one-dimensional Schrödinger operator with an analytic one-frequency potential is given by

\[ H = H_{\beta,v} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (Hu)_n = u_{n+1} + u_{n-1} + v(n\beta)u_n \]

where $v : S^1 \to \mathbb{R}$ is an analytic function, called the potential, and $\beta \in \mathbb{R} \setminus \mathbb{Q}$, called the frequency. Let $\Sigma = \Sigma_{\beta,v}$ be the spectrum of $H$. Fix an energy $E \in \mathbb{R}$; consider the analytic functions defined by

\[
A(y) = A(E - v(y)) = \begin{bmatrix} E - v(y) & -1 \\ 1 & 0 \end{bmatrix},
A_n(y) = A(y + (n-1)\beta) \ldots A(y) n \geq 1, \quad A_0(y) = id, \quad A_{-n}(y) = A_n(y - n\beta)^{-1} \tag{2.1}
\]

They are strongly related to $H$: a formal solution of $Hu = Eu$ satisfies

\[
A_n(k\beta) \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} u_{k+n} \\ u_{k+n-1} \end{bmatrix}.
\]

The Lyapunov exponent at energy $E$ is given by the following formula

\[
L(E) = \lim_{n \to +\infty} \frac{1}{n} \int_{S^1} \ln \|A_n(y)\| \, dy \geq 0.
\]

An energy $E \in \Sigma$ is

- **supercritical** if $L(E) > 0$;
- **subcritical** if there is a uniform subexponential bound $\ln \|A_n(z)\| = o(n)$ through some band $|\text{Im} z| < \varepsilon$;
- **critical** otherwise.

Progress have been made into the understanding of the behavior of supercritical and subcritical energies; but until [1, 2] there was no global theory of such operators, and the transition between supercritical and subcritical energies was not understood. Subcritical and supercritical regimes can coexist in the spectrum of the same operator ([9]) (more precisely in [2] the author exhibits an example with arbitrarily many alternances between subcritical and supercritical regimes). However it may not be necessary to pass through the critical regime to go from the subcritical and supercritical ones.

In the dynamical approach the understanding of the Schrödinger operator is obtained through the detailed description of the analytic functions defined by (2.1). A (one-frequency, analytic) quasiperiodic $SL(2;\mathbb{C})$-cocycle is a pair $(A, \beta)$, where $\beta \in \mathbb{R}$ and

\[ A : S^1 \to SL(2;\mathbb{C}) \]

is analytic, and defines a linear skew product acting on $\mathbb{C}^2 \times S^1$ by

\[ (x,y) \mapsto (A(y) \cdot x, \beta y). \]
The iterates of the cocycle are given by \( (A_n, n\beta) \) where \( A_n \) is given by
\[
A_n(y) = A(\beta^n y) \cdots A(y) \quad n \geq 1, \quad A_0(y) = \text{id}, \quad A_{-n}(y) = A_n(\beta^{-n} y)^{-1}.
\]
The **Lyapunov exponent** \( L(A, \beta) \) of a quasiperiodic \( \text{SL}(2; \mathbb{C}) \)-cocycle \( (A, \beta) \) is given by
\[
\lim_{n \to +\infty} \frac{1}{n} \int_{S^1} \ln |A_n(y)| \, dy.
\]
A quasiperiodic \( \text{SL}(2; \mathbb{C}) \)-cocycle \( (A, \beta) \) is **uniformly hyperbolic** if there exist analytic functions \( u, s : S^1 \to \mathbb{P}^2_{\mathbb{C}}, \) called the **unstable and stable directions**, and \( n \geq 1 \) such that for any \( y \in S^1, \)
\[
A(y) \cdot u(y) = u(By) \quad A(y) \cdot s(y) = s(By),
\]
and for any unit vector \( x \in s(y) \) (resp. \( x \in u(y) \)) we have \(|A_n(y) \cdot x| < 1\) (resp. \(|A_n(y) \cdot x| > 1\)).
The unstable and stable directions are uniquely characterized by those properties, and clearly \( u(y) \neq s(y) \) for any \( y \in S^1 \). If \( (A, \beta) \) is uniformly hyperbolic, then \( L(A, \beta) > 0 \). Let us denote by
\[
\mathcal{UH} \subset C^0(\text{SL}(2; \mathbb{C}), S^1)
\]
the set of \( A \) such that \( (A, \beta) \) is uniformly hyperbolic. Uniform hyperbolicity is a stable property: \( \mathcal{UH} \) is open, and \( A \to L(A, \beta) \) is analytic over \( \mathcal{UH} \) (regularity properties of the Lyapunov exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

**Definition.** — Let \( (A, \beta) \) be a quasiperiodic \( \text{SL}(2; \mathbb{C}) \)-cocycle. If \( L(A, \beta) > 0 \) but \( (A, \beta) \notin \mathcal{UH} \), then \( (A, \beta) \) is **nonuniformly hyperbolic**.

Supercritical means nonuniformly hyperbolic in dynamical systems terminology; critical energies are in the boundary of the supercritical regime \(^3\), while subcritical ones are far away.

Most important examples are **Schrödinger** cocycles and \( L(E) = L(A^{(E-v)}, \beta) \). One of the most basic aspects of the connection between spectral and dynamical properties is that \( E \notin \Sigma_{\beta,v} \) if and only if \( (A^{(E-v)}, \beta) \) is \( \mathcal{UH} \).

If \( A \in C^0(\text{SL}(2; \mathbb{C}), S^1) \) admits a holomorphic extension to \( |\text{Im} y| < \delta \) then for \( |\varepsilon| < \delta \) we can define \( A_{\varepsilon} \in C^0(\text{SL}(2; \mathbb{C}), S^1) \) by
\[
A_{\varepsilon}(y) = A(y + i\varepsilon).
\]
The Lyapunov exponent \( L(A_{\varepsilon}, \beta) \) is a convex function of \( \varepsilon \). We can thus introduce the following notion. The **acceleration** of a quasiperiodic \( \text{SL}(2; \mathbb{C}) \)-cocycle \( (A, \beta) \) is given by
\[
\omega(A, \beta) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi \varepsilon} \left( L(A_{\varepsilon}, \beta) - L(A, \beta) \right).
\]
Since the Lyapunov exponent is a convex and continuous function the acceleration is an upper semi-continuous function in \( \mathbb{R} \setminus \mathbb{Q} \times C^0(\text{SL}(2; \mathbb{C}), S^1) \). The acceleration is quantized:

\(^3\) The critical regime in fact equals the boundary of the nonuniformly hyperbolic regime ((2)): if \( E \) is critical, we can perturb \( v \) so that \( E \) still belongs to the perturbed spectrum (with the same \( \beta \)), and becomes nonuniformly hyperbolic.
**Theorem 2.1** ([1]). — If \((A, \beta)\) is a \(SL(2; \mathbb{C})\)-cocycle with \(\beta \in \mathbb{R} \setminus \mathbb{Q}\), then \(\omega(A, \beta)\) is always an integer.

A direct consequence is the following:

**Corollary 2.2.** — The function \(\varepsilon \mapsto L(A_\varepsilon, \beta)\) is a piecewise affine function of \(\varepsilon\).

It is thus natural to introduce the notion of regularity. A cocycle \((A, \beta)\) is **regular** if \(L(A_\varepsilon, \beta)\) is affine for \(\varepsilon\) in a neighborhood of 0. In other words \((A, \beta)\) is regular if the equality

\[ L(A_\varepsilon, \beta) - L(A, \beta) = 2\pi \varepsilon \omega(A, \beta) \]

holds for all \(\varepsilon\) small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near \((A, \beta)\). It is an open condition in \(C^{\infty}(SL(2; \mathbb{C}), S^1) \times \mathbb{R} \setminus \mathbb{Q}\). The following statement gives a characterization of the dynamics of regular cocycles with positive Lyapunov exponent:

**Theorem 2.3** ([1]). — Let \((A, \beta)\) be a \(SL(2; \mathbb{C})\)-cocycle with \(\beta \in \mathbb{R} \setminus \mathbb{Q}\). Assume that \(L(A, \beta) > 0\); then \((A, \beta)\) is regular if and only if \((A, \beta)\) is \(\mathcal{UH}\).

One striking consequence is the following:

**Corollary 2.4** ([1]). — For any \((A, \beta)\) in \(C^{\infty}(SL(2; \mathbb{C}), S^1) \times \mathbb{R} \setminus \mathbb{Q}\) there exists \(\varepsilon_0\) such that

- \(L(A_\varepsilon, \beta) = 0\) (and \(\omega(A, \beta) = 0\)) for every \(0 < \varepsilon < \varepsilon_0\),
- or \((A_\varepsilon, \beta) \in \mathcal{UH}\) for every \(0 < \varepsilon < \varepsilon_0\).

**Remark 2.5.** — Furthermore let us mention the fact that almost constant cocycles are regular with zero acceleration (see [1]).

### 3. Proof of Theorem A

Suppose that \(\rho \neq 1\), and let us consider the cocycle \((B^{\alpha, \rho}, \beta^{1/2})\) where

\[
B^{\alpha, \rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix}.
\]

Since

\[
\left( \frac{\alpha x + y}{x + 1}, \beta y \right)(x, y^2) = (x, y^2) \left( \frac{\alpha x + y}{x + 1}, \beta^{1/2} y \right)
\]

the cocycles \((A^{\alpha, \rho}, \beta)\) and \((B^{\alpha, \rho}, \beta^{1/2})\) have the same behavior. Using two different arguments of monodromy (one for \(\rho < 1\), and the other one for \(\rho > 1\)) we see that there is a continuous determination for the square root of \(\det B^{\alpha, \rho}(y) = \alpha - y^2\). Let us thus set

\[
\tilde{B}^{\alpha, \rho}(y) = \frac{1}{\sqrt{\alpha - y^2}} B^{\alpha, \rho}(y) \in SL(2; \mathbb{C}).
\]
We first study \(L(\tilde{B}^{\alpha, \rho}, \beta^{1/2})\) when \(\rho\) is near 0. Note that \(\tilde{B}^{\alpha, \rho}\) is almost constant when \(\rho\) is near 0. Hence \(\tilde{B}^{\alpha, \rho}\) is regular with \(\omega(\tilde{B}^{\alpha, \rho}, \beta^{1/2}) = 0\) (Remark 2.5), and

\[
\lim_{\rho \to 0} L(\tilde{B}^{\alpha, \rho}, \beta^{1/2}) = 0;
\]

so \(L(\tilde{B}^{\alpha, \rho}, \beta^{1/2})\) does not depend on \(\rho\), and \(L(\tilde{B}^{\alpha, \rho}, \beta^{1/2}) = 0\) when \(\rho\) is close to 0.

Let us now determine the behavior of \(L(\tilde{B}^{\alpha, \rho}, \beta^{1/2})\) when \(\rho\) is near \(+\infty\). Remark that

\[
\tilde{B}_2^{\alpha, \rho}(y) = \frac{1}{\sqrt{\alpha - y^2} \sqrt{\alpha - \beta^2 y^2}} \left[ \frac{\alpha^2 + \beta y^2}{\alpha + 1} \left( \frac{\alpha + \beta}{y^2 + 1} \right) \right]
\]

is almost constant when \(\rho \to +\infty\), its limit is

\[
-\frac{1}{\beta} \left[ \begin{array}{c} \beta \\ \alpha + \beta \\ 0 \\ 1 \end{array} \right],
\]

and its Lyapunov exponent is 0. So as previously \(L(\tilde{B}_2^{\alpha, \rho}, \beta^{1/2}) = 0\) when \(\rho\) is near \(+\infty\), and the same holds for \(L(\tilde{B}^{\alpha, \rho}, \beta^{1/2})\).

Assume that \(L(\tilde{B}^{\alpha, \rho}, \beta^{1/2})\) is non constant. As the acceleration is non decreasing everywhere except maybe at 1, and as \(L\) is continuous, \(\omega(\tilde{B}^{\alpha, \rho}, \beta^{1/2})\) is positive for \(\rho < 1\) and negative for \(\rho > 1\); Theorem 2.1 implies the following inequality

\[
\omega(\tilde{B}_1^{\alpha, 1}, \beta^{1/2}) - \omega(\tilde{B}_1^{\alpha, 1}, \beta^{1/2}) \leq -2.
\]

By definition of \(\tilde{B}^{\alpha, \rho}\) we have

\[
L(\tilde{B}^{\alpha, \rho}(y), \beta^{1/2}) = L(B^{\alpha, \rho}(y), \beta^{1/2}) - \int_{S^1} \ln \sqrt{\alpha - y^2} \ dy
\]

\[
= L(B^{\alpha, \rho}(y), \beta^{1/2}) - \max(0, \ln \rho).
\]

Even though \((B^{\alpha, \rho}(y), \beta^{1/2})\) is not a SL(2; \(\mathbb{C}\))-cocycle, the Lyapunov exponent is still a convex function of \(\log \rho\) (see for example [3]). The jump of \(\omega(B^{\alpha, \rho}(y), \beta^{1/2})\) is thus \(\geq 0\), and the jump for the second term of the right member is \(-1\). Therefore the jump of \(L(\tilde{B}^{\alpha, \rho}(y), \beta^{1/2})\) is \(\geq -1\): contradiction.

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