Quantum Measured Information

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A framework for a quantum information theory is introduced that is based on the measure of quantum information associated with probability distribution predicted by quantum measuring of state. The entanglement between states of measured system and "pointer" states of measuring apparatus, which is generated by dynamical process of quantum measurement, plays a dominant role in expressing quantum characteristics of information theory. The quantum mutual information of transmission and reception of quantum states along a noisy quantum channel is given by the change of quantum measured information. In our approach, it is not necessary to purify the transmitted state by means of the reference system. It is also clarified that there exist relations between the approach given in this letter and those given by other authors.

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Quantum information theory is a new field with potential implication for the conceptual foundations of quantum mechanics. It appears to be the base for a proper understanding of the emerging fields of quantum computation [1-4], quantum communication, and quantum cryptography [5,6]. Recently, a correlated state in quantum systems, so-called quantum entangled or quantum entanglement, is utilized to study quantum information [7,8], in particular, quantum teleportation. A theory of quantum information has emerged which shows striking parallels with, but also fascinating differences from, classical information theory entirely based on the von Neumann entropy of quantum state [9]. Although some useful fundamental results about quantum information theory, e.g., quantum noiseless coding theorem [10] and the capacity of quantum noisy channels [11], have been obtained recently, quantum information is still mystery in many respects. To our knowledge, there exist several approaches for quantum information theory [12-14]. However, the relation among these approaches is obscure. We argue in this letter that it is necessary to introduce quantum measured information which is regarded as the measure of information for quantum input and output. This leads us to propose a scheme for quantum information theory. Our approach can give rise to a unified description of classical correlation and quantum entanglement. Furthermore, we shall clarify the relations between our approach and those given by other authors for quantum information theory.

In classical information theory, there is a set of mutually exclusive classical states. In quantum mechanics, a quantum state is represented by a vector in a Hilbert space, or a density operator on that space. Classically, the input system may retain its original state, while the no-cloning theorem [12] implies that in the quantum case the input system cannot in general remain in its initial state. However, in many quantum applications, one is interested not only in transmitting a discrete set of states, but also in arbitrary superpositions of those states. That is, one wants to transmit entire subspace of states. It is well known that an arbitrary state can be represented as a mixture of pure states, i.e., by imposing classical randomness on pure states. In this sense pure states are "noiseless", i.e., they contain no classical source of randomness. In our point of view, this does not imply that the pure states contain no quantum source of randomness. Such randomness comes from probability distribution predicted by quantum measuring about quantum states.

We now consider the question of measurement in quantum mechanics. From our point of view there is no fundamental distinction between measuring apparatus and other physical systems. Therefore, a measurement is simply a special case of interaction between physical systems, which has the property of correlating a quantity in one system with a quantity in the other. Nearly every interaction between systems produces some correlation however. Suppose that at some instant a pair of systems are independent, so that the composite system state function is a product of subsystem states. Then this condition obviously holds only instantaneously, since the systems are interacting, the independence will be immediately destroyed and the systems will become correlated. There is still one more requirement that we must impose on an interaction before we shall call it a measurement. If the interaction is to produce a measurement of mechanical quantity A of subsystem S1 by the quantity P of another one S2, we require that such interaction shall never decrease the information in the reduced distribution about A. Furthermore, we also expect that a knowledge of P shall give us more information about A than we had before the measurement took place, since otherwise the measurement would be useless. The restriction that the interaction shall not decrease the information of the reduced system S1 has the interacting consequence that the eigenstates of A will not be disturbed, since otherwise the information of A would be decreased. The time evolution of the composed system made of the measured system and the apparatus system should exhibit the following properties. The initial states of systems S1 and S2
are the form of superposition \( \sum_i a_i |\phi_i > \) and \( |P(0) > \), respectively. Then, after a specified time of interaction the total state \( \sum_i a_i |\phi_i > |P(0) > \) will be transformed into a form of the superposition states \( \sum_i a_i |\phi_i > |P_i > \), i.e., the initial independent state is evolved into entanglement state.

Information concepts have been used in the context of quantum measurements long ago \cite{13}, and various quantities, all labeled entropies, have been introduced to characterize uncertainties about events or about states of a system \cite{14}. For a completed orthonormal set \( \{|\phi_i >\} \) and a pure state \( |\psi > := \sum a_i |\phi_i > \), we have a square-amplitude distribution \( |a_i|^2 \) called the distribution of \( |\psi > \) over \( \{|\phi_i >\} \). In the probabilistic interpretation this distribution represents the probability distribution over the results of a measurement with eigenstates \( \{|\phi_i >\} \) performed upon the measured system in the state \( |\psi > \).

An entropy of information depends not only on the a priori probabilities predicted. Such entropy of information is given by Shannon entropy \cite{15} \( S = -\sum |a_i|^2 \log |a_i|^2 \). Considering the dynamical process of quantum measurement as mentioned above, we can arrange the distribution corresponding to the reduced density matrix of measured system into the a priori probabilistic distribution predicted by quantum measurement. While for the composite system made of the measured system and the apparatus system, its state is evolved into such a state \( |\psi > := \sum a_i |\phi_i > |P_i > \). The reduced density matrix can be obtained by tracing the state out the degrees of freedom of the apparatus system, i.e., \( \rho_M = Tr_P (|\psi > < \psi|) = \sum_i |a_i|^2 |\phi_i > < \phi_i| \). The quantum measured information carried by the quantum state \( |\psi > \) can be read as

\[
S_M(\rho) = S(\rho_M) = -Tr \rho_M \log \rho_M.
\]

In order to investigate the quantum mutual information, we shall consider the mathematical description of model about a noisy quantum channel following Schumacher \cite{16}. Suppose a quantum system \( S \) is subjected to a dynamical evolution, which may represent the transmission of \( S \) along a noisy quantum channel. In general, the evolution of \( S \) will be represented by a superoperator \( \mathcal{E} \) which gives the mapping from the initial states of the system \( \rho \) to the final states after the evolution of the system \( \rho' \), i.e., \( \rho' = \mathcal{E}(\rho) \). The mapping represented by \( \mathcal{E} \) is a linear, trace-nonincreasing and completely positive map. The evolution of system will be unitary only if it is isolated from other systems. The input quantum state, after interaction with an environment, is lost, having become the output state. Any attempt at copying the quantum state before decoherence will result in a classical channel. Thus a joint probability for input and output symbols does not exist for quantum channels. However, this is not essential, as the quantity of interest is the quantum measured information associated with the probabilistic distribution predicted by quantum measurement, which is regarded as a measure of information carried by a quantum state. Let us recall the definition of the mutual information in classical information theory. The mutual information is a measure of an amount of information that one random variable contains about another random variable, and is the reduction in the uncertainty of one random variable due to the knowledge of the other. In the quantum case, the corresponding quantity is the entropy of information given by that the change of quantum measured distribution, which is resulted in by the evolution along a noisy quantum channel.

After the interaction with the environment, the quantum measured distribution \( \rho_M \) of quantum state becomes \( \rho^E_M = \mathcal{E}(\rho_M) = \sum |a_i|^2 \mathcal{E} (|\phi_i > < \phi_i|) \). In fact, The \( \rho^E_M \) can be equivalently expressed as \( \rho^E_M = Tr_P \rho^E \otimes \mathbb{1}_P (|\psi > < \psi|) \). It should be noticed that the tracing process implies the determination of probabilistic distribution predicted by quantum measurement. In general, the state \( \rho^E_M = \mathcal{E} \otimes \mathbb{1}_P (|\psi > < \psi|) \) is a quantum mixing state, which carries the quantum information given by the von Neumann entropy, i.e., \( S(\rho^E_M) = -Tr \rho^E_M \log \rho^E_M \). In physics, it is known that the entropy change in the quantum states represents the amount of the information obtained by the quantum measurement. The amount of information of gain is equal to subtracting the information \( S(\rho^E_M) \) before quantum measurement from the quantum measured information \( S(\rho^E_M) \). This leads to the quantum mutual information represented by

\[
I_E = S(\rho^E_M) - S(\rho^F_M),
\]

based on the quantum measured information.

Up to now, we have discussed how the quantum mutual information is measured when one transmits a pure quantum state along a noisy quantum channel by means of concept of quantum measured information. It is well known that in general case we should study the problem of transmission of some quantum mixed states because the states of a quantum system are fragile. Based on the following propositions \cite{14}, we can easily generalize the above idea to the case of quantum mixed states. The first proposition is that if \( \rho \) is a pure state, there exists a composite system made of two subsystems of which \( \rho \) is the state and the von Neumann entropies of the subsystems satisfy \( S(\rho_1) = S(\rho_2) \), where \( \rho_1 \) and \( \rho_2 \) are the density matrices of the subsystems. Moreover, the positive spectra of \( \rho_1 \) and \( \rho_2 \) coincide. Secondly, given \( \rho_1 \), one can always find a Hilbert space \( \mathcal{H}_2 \) and a pure density matrix \( \rho \) in the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) such that \( \rho = Tr_{\mathcal{H}_2} \rho \). Now, let us suppose that a transmitted state in the quantum mixed state \( \rho_1 = \sum_m p_m |\psi_m > < \psi_m| \). We can take an auxiliary Hilbert space \( \mathcal{H}_2 \) of which the dimensions are the same as those of \( \mathcal{H}_1 \). Thus, a pure state of the composite system can be constructed as \( \rho = |\chi > < \chi| \), here
about the entropy of information \[14\], we can lead to the

\[ \langle \phi_1^{i} \rangle \]

The dynamical evolution of quantum measurement is that of combining the measured system with the apparatus system of measuring, which leads to the state

\[ \chi > = \sum_{m,i} \sqrt{p_m c_m} |\phi_i^{m} > | \phi_i^{2} > \sum_i \tilde{c}_i |\phi_i^{1} > | \phi_i^{2} > \cdot (3) \]

The noisy quantum channel leads to the transmitted state going completely entangled with those of the auxiliary subsystem \( S_2 \). Hence, we can regard the states \( \tilde{P}_i \) as the "pointer" basis of quantum measurement. This implies that the subsystem \( S_2 \) is completely auxiliary, and can be absent in our formalism of quantum information theory. Consequently, it is not necessary in our approach of quantum measurement theory based on the quantum measured information to purify the initial transmitted state in the Schumacher’s approach.\[7\]. By means of the previous discussion about the quantum mutual information of transmission of quantum pure state and the corresponding reception, we can immediately write the expression of quantum mutual information of transmission of quantum mixed states in the noisy quantum channel mentioned above. The result is

\[ \hat{I}_E = S(\tilde{\rho}_E) - S(\tilde{\rho}_E). \] (5)

\( \tilde{\rho}_E \) denotes the dynamically evolved state of the composite system made of the measured system and the apparatus system, i.e., \( \tilde{\rho}_E = [\tilde{\psi} >> \psi] \). Through the noisy quantum channel, the states \( \tilde{\rho}_E \) and \( \tilde{\rho}_M \) become \( \tilde{\rho}_E = \mathcal{E} \otimes \rho_E \) and \( \tilde{\rho}_M = \mathcal{T}_M \tilde{\rho}_E \), respectively. It should be emphasized again that the \( \tilde{\rho}_M \) stands for the change of the probabilistic distribution predicted by quantum measurement after the transmitted state going through the noisy quantum channel.

If a quantum channel is trivial, i.e., \( \mathcal{E} = \text{identity map} \), then the quantum mutual information equals to the quantum measured information of inputs. This is easily seen from the relation that \( \hat{I}_E \mid \varepsilon = 1 = S(\tilde{\rho}_M) \mid \varepsilon = 1 = S(\tilde{\rho}_M). \) According to the Araki-Lieb triangle inequality about the entropy of information [3], we can lead to the quantum mutual information presented here satisfying \( \hat{I}_E \leq S(\tilde{\rho}_M) \) which is the first part of the data processing inequality. Now, we shall consider a more complicated quantum channel. Suppose the initial state of the measured system \( \rho \) and further suppose \( \rho \) undergoes two successive dynamical evolutions described by superoperators \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). This scheme of the noisy quantum channel can be represented by the evolutions of quantum state \( \rho \rightarrow \mathcal{E}_1(\rho) \rightarrow \mathcal{E}_2 \circ \mathcal{E}_1(\rho) \). Following Schumacher and Neilsen \[6\] and applying the strong subadditivity inequality, we can prove the second part of the data processing inequality \( \hat{I}_{E_2} \geq \hat{I}_{E_2 \circ E_1} \).

On the other hand, by writing the superoperator \( \mathcal{E} \) as a unitary evolution \( U_{SE} \) on an extended system \( SE \) followed by a partial trace over an environment system \( E \), we can investigate the reverse data processing inequality in the quantum information theory, which reflects the fact that any quantum channel used in a forward manner can be used in a backward manner. If we consider the case of transmission of quantum states in a backward manner and exchange the input state with the output state, the order of time in the unitary evolution \( U_{SE} \) is reverse, which leads to the evolution operator to be changed into \( U_{SE}^\dagger \). So, in the transmission of backward manner, we should substitute the evolution of state \( \mathcal{E}^\dagger(\rho_S) \) for the \( \mathcal{E}(\rho_S) \). Since the "pointer" state of apparatus system is completely entangled with the input states, the exchange between the input states and the output states is equivalent to the change of \( |\phi_i^{1} > | \tilde{P}_i > \rightarrow | \tilde{P}_i > | \phi_i^{2} > \) in the states \( \tilde{\rho}_E \) and \( \tilde{\rho}_E \). Noticing that the reverse of the noisy quantum channel leads \( \mathcal{E} \rightarrow \mathcal{E}^\dagger \), so we should use the superoperator \( \mathcal{E} \otimes \mathcal{E}^\dagger \) as a substitute for \( \mathcal{E} \) in the expression of the quantum mutual information in the backward manner. Based on the fact that the spectra of general quantum states \( \rho \) and \( \rho^\dagger \) coincide, we can obviously see that the quantum mutual information Eq.(5) is invariant under the transformations of \( |\phi_i^{1} > | \tilde{P}_i > \rightarrow | \tilde{P}_i > | \phi_i^{2} > \) and \( \mathcal{E} \otimes \mathcal{E}^\dagger \rightarrow \mathcal{E} \otimes \mathcal{E}^\dagger \). From these, we obtain the reverse data processing inequality about the quantum mutual information, \( \hat{I}_{E_2 \circ E_1} = \hat{I}_{E_1 \circ E_2} \leq \hat{I}_{E_2} \leq S(\rho_M) \).

The appearance of quantum characteristics in a quantum state is related to quantum non-separability. In fact, in general, the states of apparatus system and the states of measured system are not separable. If the general quantum state constructed in terms of these states is restricted within the separable case, i.e., \( \rho_e = \sum w_i \rho_i^1 \otimes \rho_i^2 \) where \( \rho_i^1 \) is from the Schatten decomposition of the state of the measured system \( \rho_1 = \sum w_i \rho_i^1 \), we find that the quantum mutual information Eq.(5) is reduced to

\[ \hat{I}_E = S(\mathcal{T}_M \rho \mathcal{E} \otimes 1(\rho_e)) - S(\mathcal{E} \otimes 1(\rho_e)) = S(\mathcal{T}_M \rho \mathcal{E} \otimes 1(\rho_e)) - \sum_i w_i S(\mathcal{E}(\rho_i^1)). \]

The change of the unmeasured information is present in the square bracket of the above expression because the separable state \( \rho_e \) is a quantum mixed state. The input state can be equivalently described by the "pointer" states of input exploring the property of quantum measurement. Then, for the case of separability, our definition of quantum nu-
tual information based on the quantum measured information deduces to that of Ohya’s approach [11] which is established by means of the compound state describing the correlation between an input state $\rho_1$ and the output state $\mathcal{E}(\rho_1)$. However, we should emphasize that Ohya’s approach is not complete because the quantum coherence of the entanglement states, which plays the important roles in quantum computation, quantum teleportation and quantum cryptography, is restricted out in his approach.

In our scheme, the quantum mutual information reduces to the classical one when the system is classical. When the input system is classical, an input state is given by a probability distribution or a probability measure. For the case of probability distribution, the input state is established by means of the compound state describing the correlation between an input state $\rho_1$ and the output state $\mathcal{E}(\rho_1)$. However, we should emphasize that Ohya’s approach is not complete because the quantum coherence of the entanglement states, which plays the important roles in quantum computation, quantum teleportation and quantum cryptography, is restricted out in his approach.

Since the measure of quantum information is the quantum measured information in our formalism, we can communicate the same amount of quantum informations by using the transmissions of the pure state $\rho = \sum_{i,j} a_i a_j^\dagger |\phi_i>|<\phi_j|$ or the mixed state $\tilde{\rho} = \sum_i |\phi_i>|<\phi_i|$ in a Hilbert space. However, in the process of transmissions, the fidelities for the two type of quantum states are different. The quantum states $\rho$ and $\tilde{\rho}$ of transmissions along the noisy quantum channel $\mathcal{E}$ are evolved into $\hat{\rho} = \mathcal{E}(\rho)$ and $\tilde{\rho} = \mathcal{E}(\tilde{\rho})$, respectively. For the case of transmission of the pure state, its fidelity is read as $F(\rho, \hat{\rho}) = Tr(\rho \hat{\rho})$. But, the fidelity for the mixed quantum states, which is defined in terms of Uhlmann’s formula of transition probability [17], is given by $F(\tilde{\rho}, \tilde{\rho}) = (Tr[(\sqrt{\tilde{\rho}} \hat{\rho} \sqrt{\tilde{\rho}})^{\dagger}])^2$. It is necessary to analyze the quantitative properties of these fidelities although we do not extensively discuss this topic here. It is hoped that such analyzing results may be applicable to developing a quantum analogue of Shannon’s channel capacity theorem.

Summarizing, we have introduced the quantum measured information to measure the quantum information of a quantum state. Using this point of view we have consistently decided the quantum mutual information, which measures the amount of quantum information conveyed in the noisy quantum channel. We has proven that such quantum mutual information obeys the data processing inequality in both forward manner and backward manner.

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