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Emergent Spacetime and Gravitational Nieh-Yan Anomaly in Chiral $p + ip$ Weyl Superfluids and Superconductors

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Momentum transport is anomalous in chiral $p + ip$ superfluids and superconductors in the presence of textures and superflow. Using the gradient expansion of the semiclassical approximation, we show how gauge and Galilean symmetries induce an emergent curved spacetime with torsion and curvature for the quasirelativistic low-energy Majorana-Weyl quasiparticles. We explicitly show the emergence of the spin connection and curvature, in addition to torsion, using the superfluid hydrodynamics. The background constitutes an emergent quasirelativistic Riemann-Cartan spacetime for the Weyl quasiparticles which satisfy the conservation laws associated with local Lorentz symmetry restricted to the plane of uniaxial anisotropy of the superfluid (or superconductor). Moreover, we show that the anomalous Galilean momentum conservation is a consequence of the gravitational Nieh-Yan (NY) chiral anomaly the Weyl fermions experience on the background geometry. Notably, the NY anomaly coefficient features a nonuniversal ultraviolet cutoff scale $\Lambda$, with canonical dimensions of momentum. Comparison of the anomaly equation and the hydrodynamic equations suggests that the value of the cutoff parameter $\Lambda$ is determined by the normal state Fermi liquid and nonrelativistic uniaxial symmetry of the $p$-wave superfluid or superconductor.

Introduction.—Topological phases can be classified in terms of quantum anomalies that are robust to interactions and other perturbations [1–3]. Protected emergent quasirelativistic Fermi excitations coupled to gauge fields and geometry arise as dictated by topology and anomaly inflow [4–6]. In particular, gapless fermions with Weyl spectrum and chiral anomalies are a recent prominent example [7–17]. On the other hand, topological phases and their coupling to geometry (and gravity) is currently a rapidly advancing subject. The well-established results concern the Hall viscosity [18] and chiral central charge [19–21] related to gravitational anomalies [22–24] and thermal transport in quantum Hall systems, topological superfluids (SFs) and superconductors (SCs), as well as semimetals [25–55], the case of SFs and SCs being especially important due to the lack of conserved charge. Any purported topological response of geometrical origin is necessarily more subtle than that based on gauge fields with conserved charges due to the inherent dichotomy between topology and geometry.

Topologically protected Weyl quasiparticles also arise in three-dimensional chiral $p$-wave SFs and SCs from the gap nodes [5,56] and lead to nonzero normal density close to the Fermi surface zeros, even at zero temperature [57,58]. As any chiral fermion in three dimensions, they suffer from the chiral anomaly in the presence of nontrivial background fields, now as an anomaly where momentum is transferred from the order parameter fluctuations to the quasiparticles [56–62].

\[ \partial_t \Pi_{\text{vac}} + \nabla \cdot \Pi_{\text{vac}} = -\partial_t P_{\text{qp}} - \nabla \cdot \Pi_{\text{qp}} \neq 0, \quad (1) \]

where $P_{\text{vac}},\Pi_{\text{vac}}$ and $P_{\text{qp}},\Pi_{\text{qp}}$ refer, respectively, to the momentum and stress-tensor of the order parameter vacuum and quasiparticles (qps). Because of textures and superflow, the quasiparticles and -holes flow through the gap nodes transferring net momentum. The total momentum is conserved. The chiral anomaly (1) on vortices [63] has been measured in the early landmark experiment in $^3$He-A [64]. See the Supplemental Material (SM) [65] for a detailed review of Eq. (1) in terms of SF hydrodynamics.

The anomaly (1) relates to the famous angular momentum “paradox” of the chiral superfluid or superconductor [57,59–61,66,67]; even though each Cooper pair carries net angular momentum $h$, the pairs overlap substantially. In a fixed volume, the overlap is of the order $\sim a^2/\xi^2$ with mean pair separation $a$ and coherence length $\xi$. This makes the local angular momentum contribution very small, of the order $\sim (\Delta_0/E_F)^2$, instead of (half) the total fermion density, where $\Delta_0$ is the gap and $E_F$ the Fermi energy. At equilibrium, the local variation of angular momentum is still well defined. Similarly, the total linear momentum density is well defined, but it is no longer conserved separately between the condensate and normal component, producing the anomaly [57,58]. This is in contrast to a system of nonoverlapping Bose Cooper pair “molecules” where the anomaly due to Weyl nodes also naturally vanishes [56,57].
Here, we show that the momentum anomaly (1) is a manifestation of the so-called chiral gravitational Nieh-Yan (NY) anomaly [68–74] on emergent spacetime with torsion (and curvature) coupling to the low-energy Weyl fermions. The subtle role of this gravitational (geometric) anomaly term due to torsion has been debated in the literature since its discovery [75]. We show how it arises through the Galilean symmetries and hydrodynamics of the nonrelativistic system with an explicit ultraviolet completion. Therefore, taken at face value, the NY anomaly has been experimentally verified in the late 90’s, albeit in the context of emergent condensed matter fermions and spacetime induced by the SF order parameter. Emergent spacetime in two-dimensional topological SCs was recently carefully discussed in Ref. [53]. Related recent work discusses emergent tetrads, gauge fields, and anomaly terms in Weyl SFs [67,76] and semimetals [54,77–81], without the framework of emergent conservation laws and geometry.

Model.—We consider an equal spin pairing p-wave SF at zero temperature on flat Euclidean space in the mean-field (MF) approximation [5,82] (see the SM [65] for more details). We comment below how to extend our results to SCs. The action for the spinless Grassmann fermion \( \{\Psi(x), \Psi^\dagger(x)\} = 0 \) is \((h = 1, \text{ summation over repeated indices})\)

\[
S[\Psi, \Psi^\dagger, \Delta, \Delta^\dagger] = \int d^4x \Psi^\dagger_i \partial^4 \Psi - \mathcal{H}_{MF}.
\]

\[
\mathcal{H}_{MF} = c(\Psi, \partial^4 \Psi) + \frac{1}{2l}(\Delta^\dagger \Psi \partial^4 \partial^4 \Psi + \Delta \partial^4 \Psi \partial^4 \Psi).
\] (2)

The normal state energy is \( c(\Psi, \partial^4 \Psi) = (\partial^4 \Psi \partial^4 \Psi/2m) - \mu^4 \Psi \Psi^\dagger \Psi \), where \( m \) is the constituent mass, \( \mu^4 \) is the normal state Fermi level. The spinless MF gap amplitude \( \Delta^\dagger(x) \propto (1/2l)\Psi^\dagger \Psi \) is \( \Delta^\dagger = (\partial \phi / m \hat{r}) \hat{m} \cdot \hat{n} = 1 \) the unit vector \( \hat{e}_3 = \hat{m} \times \hat{n} = \hat{l} \) being the axis of orbital angular momentum of the Cooper pairs. The dynamics of the SF free energy is ignored, and \( \Delta(x), \Delta^\dagger(x) \) represent given background fields. We ignore Fermi liquid corrections [82] for simplicity as we expect that these will not affect the arguments which are based on the symmetries, hydrodynamics, and anomalies of the system. For SCs, we perform minimal substitution in \( c(\Psi, \partial^4 \Psi) \).

Symmetries and Galilean transformations.—The unbroken continuous symmetries of the normal Fermi liquid state are \( U(1)_N \times \mathbb{R}^3 \times SO(3)_L \), where the translations and rotations \( \mathbb{R}^3 \times SO(3)_L \) form a subgroup of the Galilean group. The gauge and rotational symmetries are broken to the combined gauge symmetry \( U(1)_{N/2} \times \mathbb{R}^3 \). In addition, time-reversal symmetry is broken allowing for the emergence of Weyl quasiparticles.

In the SF, the global \( U(1)_N \) gauge symmetry leads to the conservation law \( \partial \mu \Phi^\dagger = -\Delta^\dagger \partial \Phi^\dagger + \Delta \partial \Phi \Phi^\dagger \), where \( \mu^\dagger = (\rho, J^i) \) is the normal state fermion current. A Galilean transformation from the SF comoving frame (cmf) is given as \( x' = x + v_s t \) and \( t' = t \). In terms of SF velocity \( v_s \) and chemical potential \( \mu_m \), we transform \[ \Psi(x) \to \Psi'(x') = e^{-i(mv_s \cdot x + \frac{1}{2}(mv_s^2 - 2\mu_m))}(x + v_s t, t), \]

\[ \Delta(x) \to e^{-i(2mv_s \cdot x + \frac{1}{2}(mv_s^2 - 2\mu_m))}(x + v_s t, t). \] (3)

The gauge and Galilean transformations are not independent for coordinate dependent transformation parameters. For infinitesimal constant velocity, the action changes in the cmf with \( \Psi'(x') \) as \( \delta S_{cmf} = \int d^4x \Psi \cdot J' - \mu m\Psi + O(v_s^2) \). Equivalently, \( \partial \mu \Phi^\dagger = (\partial \mu - imv_{p}) \Psi \), where \( v_{p} = (-\mu_m/m, v_s) \). Rotations along \( \hat{e}_3 \) act as \( \Delta^\dagger \to e^{i\varphi} \Delta^\dagger \) and lead to the combined gauge symmetry [83]. The SF velocity is determined as

\[
2mv_s = -\hat{e}_3 \cdot \partial \varphi \hat{e}_3 = -\partial \varphi. \] (4)

where \(-\varphi \) is the rotation angle. In addition, the SF velocity satisfies the Mermin-Ho relations [56,85]

\[
\nabla \times \mathbf{v}_s = -\frac{k}{4\pi} e^{ikj} \hat{r} \mathbf{v}_j \times \hat{r} \mathbf{v}_k,
\]

\[
\partial \mathbf{v}_s + \nabla \mu_m = -\frac{k}{2\pi} e^{ikj} \hat{r} \mathbf{v}_j \times \hat{r} \mathbf{v}_k. \] (5)

where \( k = h/2m \) is the circulation quantum. The translation symmetry corresponds to the energy-momentum tensor conservation law, \( S = \int d^4x \left[ \nabla \times \mathbf{v}_s + \frac{1}{2}\nabla \mu_m \right] \).

\[
\partial \mu \Pi^\mu = \partial \mu \mathcal{L}_{\text{explicit}}. \] (6)

where \( \Pi^\mu = (\partial \mathcal{L} / \partial (\partial_\nu \Psi^\dagger \Psi \)) \partial_\nu \Psi^\dagger \Psi + \text{H.c.} + \mathcal{L}^\Phi \). In particular, this conservation law is broken by the anomaly (1) for \( \nu = i \) along \( \hat{r} \). Moreover, \( \Pi^\mu \) is not symmetric in the plane determined by \( \Delta^\dagger \), as is well known in \( \text{He-A} [53,56,82] \). See the SM [65] for more details on the symmetries of the model.

Linearized comoving quasiparticle action.—We define the Bogoliubov transformation \( \Phi^\dagger(x) = \sum_s u_s(x) a_s + v_s^\dagger(x) a_s^\dagger \), where \( s \) is a generalized index in quasiparticle (particle-hole) space with \( \{a_s^\dagger, a_s\} = \delta_{st} \). The Bogoliubov-de Gennes (BDG) quasiparticles form a spinless two-component Grassman Nambu spinor \( \Phi^\dagger(x) \sim \sum_s (u_s(x) v_s(x))^T a_s \), with the Lagrangian

\[
\mathcal{L}_{BDG} = \Phi^\dagger(x) \left( \begin{array}{c} i \partial_\tau - e^{i(i\partial_\tau)} & \frac{1}{2} \{ \Delta^\dagger, i \partial_\tau \} \\ \frac{1}{2} \{ \Delta^\dagger, i \partial_\tau \} & i \partial_\tau + e^{i(i\partial_\tau)} \end{array} \right) \Phi(x). \] (7)

If \( (u_s v_s) \) is a solution to the equation of motion with energy \( \epsilon_s, (v_s^\dagger u_s^\dagger) \) satisfies the particle-hole conjugate solution \( \epsilon_s = -\epsilon_s \). This implies the Majorana relation \( e^{i\Phi^\dagger} \sim \Phi \). For a homogeneous state, the dispersion vanishes on the
Fermi surface at $k = \pm p_F \hat{1}$. The same BDG action applies for SCs with the replacements $\partial_i \to \partial_i - i A_\mu r^\mu$, $\tau_3 e^{\mp i \partial_i} \to \tau^2 e^{\mp (i \partial_i + A_\mu)}$, where the signs are according to eigenstates of $\tau^2$ and $v_\mu \to v_\mu - A_\mu$, where $A_\mu$ is the electromagnetic gauge potential.

Now, we consider the low-energy fermions in the presence of slowly varying order parameter texture and Galilean transformation parameters $v_\mu(x), \mu(x)$ in the semiclassical approximation, see, e.g., [86]. We assume that the BDG fermions with gap $\gtrsim mc_1^2$ have been integrated out and restrict ourselves to the linear expansion close to the Weyl nodes $\pm p_F \hat{1}$. The dispersion is

$$\Phi^\dagger(x)e^{(-i\partial)}\Phi(x) \approx \sum_{\pm} \chi^\dagger_{\pm}(x) \left\{ \frac{v_F}{2} \bar{\chi}^\dagger_{\pm}(x) - i \partial_\mu \right\} \chi_{\pm}(x), \quad (8)$$

and we neglect terms of order $O(\partial^2 \chi)$, where $\Phi(x) = \sum_{\pm} e^{\pm i p_F \int x^\nu \partial_\nu} \chi^\dagger_{\pm}(x)$ and $\chi_{\pm}(x)$ is slowly varying compared to $p_F$. The order parameter vectors form the spatial part of an inverse tetrad $e_d^i = \{(c_\perp/c_{\parallel})\hat{\mathbf{m}}, (c_\parallel/c_{\perp})\hat{\mathbf{m}}\}$ with uniaxial symmetry. Note that, after linearization of $\epsilon(-i\partial)$, we perform in (8) the transformation $\Phi \to \tilde{\chi}$, i.e., $\{e^\dagger_{\parallel}, (-i\partial_\mu - \epsilon^\dagger_{\parallel} p_F)\} \to \{e^\dagger_{\parallel}, \partial_\mu\}$ corresponding to a chiral rotation in momentum space due to the node. This is precisely the anomalous chiral symmetry in the system and produces the quasirelativistic anomaly (18) below, proportional to $\Lambda \propto p_F$ in the path-integral representation [72,73,87].

Further, in the presence of SF velocity $v_\mu = \partial_\mu \varphi = 2m(\mu_\mu/m, - v_\parallel)$, the fermions transform in the comoving frame as $\tilde{\chi} \to e^{-i q \partial^\mu/2 \chi} e^\dagger_{\parallel} - i e^{-i \varphi}(e^\dagger_{\parallel} - i e_\perp)$ where $\varphi(x)$ is slowly varying compared to $p_F$. The derivative operator transforms to $\tau^\nu \partial_\nu \tilde{\chi} \to \tau^\nu i e^\dagger_{\parallel} \partial_\nu \tilde{\chi} + \tau^\nu i e_\perp \partial_\nu \tilde{\chi}$. These coincide with local spin-1/2 Lorentz transformations in the 12 plane, and we can attempt to associate the linearized action to a nontrivial spacetime. Denoting the Pauli matrices $\tau^\mu, \tau^\nu = (1, \pm r^\mu)$ in Nambu space, velocities $c_{\parallel} = v_F = p_F/m$ and $c_{\perp} = (\Delta_0/p_F)$, and $\partial_\mu = (\partial_\mu, c_{\parallel} \partial_\nu)$, the explicitly Hermitian form, becomes

$$S[\chi, \tilde{\chi}, \Delta, \mu_\mu, v_\parallel] = \int d^4 x \sqrt{\gamma} \left[ \frac{1}{2} \left( \chi^\dagger \partial_\nu \chi - i \partial_\nu \tilde{\chi}^\dagger \tilde{\chi} \right) + (\mu_\mu - \mu_{\perp}) \tilde{\chi}^\dagger \tau^\mu \chi - \frac{p_F}{2} \tilde{\chi}^\dagger v_F \tau^\mu \chi \right]$$

Emergent Riemann-Cartan spacetime.—The action (9) is (after a rotation of $r^\mu$ in the 12 plane) equivalent to a relativistic chiral (right-handed) Majorana-Weyl fermion on Riemann-Cartan spacetime [69,88,89], see the SM [65] for a review of Riemann-Cartan spacetimes and our conventions on relativistic fermions.

$$S_{\text{weyl}}[\chi, \tilde{\chi}, e, \omega, a] = \frac{1}{2} \int d^4 x \sqrt{\gamma} \bar{\epsilon} e^\mu a\sigma_\mu D_\mu \chi + \text{H.c.}, \quad (10)$$

with the identifications $\chi = e^{-1/2} \bar{\chi}$, $e = \text{det} e^\mu_\nu = (c_{\perp}^2/c_{\parallel}^2$),

$$e^\mu_0 = (1, - v_\parallel), \quad e^\mu_1 = \frac{c_{\perp}}{c_{\parallel}} (0, \hat{\mathbf{m}}),$$

$$e^\mu_2 = \frac{c_{\perp}}{c_{\parallel}} (0, \hat{\mathbf{n}}), \quad e^\mu_3 = (0, \hat{\mathbf{l}}), \quad (11)$$

and $\omega^\mu_{12} = 2m v_\parallel = 2m(\mu_{\perp}/m, v_\parallel) = \partial_\mu \varphi$. The covariant derivate is $D_\mu = \partial_\mu - (i/4) a^\mu_{\alpha \beta} \sigma_{\alpha \beta}$. $\sigma_{\alpha \beta} = (i/2) [\tau^\alpha, \tau^\beta]$. Note that the Galilean invariance leads to the shift $e^\mu_0 = - v^\mu_1$, although it is $O(\partial^2)$ in the action, and the spin connection term $e^\mu D^\nu \tilde{\chi} \propto p_F \hat{1} \cdot \mathbf{v}$, in Eq. (9). We also emphasize that we explicitly retained the momentum density $p_F^2 \tilde{\chi}^\dagger/2m \tilde{\chi} \tau^\mu \chi$ in the action, which cancels with the Fermi level $\mu_\mu \tau^\mu \tilde{\chi}$, since this term represents nonzero physical Galilean momentum present at the Weyl node, see Eq. (16) below.

Equations (10), (11), and the spin-connection $\omega^\mu_{12}$ define a Weyl fermion on an emergent Riemann-Cartan spacetime [69,88], with the uniaxial metric $g^\mu\nu = e^\mu_\alpha e^\nu_\beta g^\alpha\beta$, $e^\mu_\alpha \partial_\mu = \delta^\mu_\alpha$, and $\nabla_\mu e^\nu_\beta = \partial_\mu e^\nu_\beta + \omega^\nu_{\alpha \beta} e^\mu_\alpha - \Gamma^\nu_{\alpha \beta} e^\mu_\alpha = 0$. The nonzero torsion and curvature tensors are given by $T^\mu_{\nu \rho} = \partial_\nu e^\mu_\rho - \partial_\rho e^\mu_\nu + \omega^\mu_{\nu \rho} e^\nu_\theta - \omega^\mu_{\rho \theta} e^\nu_\nu$ and $R^\nu_{\mu \rho} = \partial_\rho \omega^\nu_{\mu \lambda} - \partial_\nu \omega^\rho_{\mu \lambda} = 2m(\partial_\mu v_\nu - \partial_\nu v_\mu)$. Although seemingly pure gauge, the latter is, in general, nonzero by the Mermin-Ho relations Eq. (5). This correspondence is the first major result of the Letter. See the SM [65] for a glossary of Riemann-Cartan spacetimes [88,89].

Conservation laws on curved spacetime.—Now, we formulate the SF conservation laws in terms of the Weyl fermions coupled to emergent spacetime [38,53,69]. The current ($p, J^\mu$) in terms of the Weyl quasiparticles is $\langle \tilde{\chi} \tau^\mu \chi, p_F \tilde{\chi}^\dagger \chi - \frac{1}{2} [\tilde{\chi}^\dagger \partial_\mu \chi - i \tilde{\chi}^\dagger \partial_\mu \tilde{\chi}] \rangle$. To first order in gradients, $\partial_\mu J^\mu$ is equal to $\frac{1}{2} \langle \tilde{\chi}^\dagger \chi \rangle$, in Eq. (9), which becomes (assuming only $\omega^\mu_{12} \neq 0$ and $e = \text{const}$)
in terms of the Weyl fermions [53]. Here, the currents
\[ T^\mu_a = \frac{1}{2} \partial^\mu \left( i\gamma^\nu e_a^{\nu} \right), \]
are derived from the relativistic Weyl action Eq. (10). In particular \( S_{12}^w = \frac{1}{2} \chi^4 \gamma^\mu \{ e^\mu, \chi^4 \} \) and \( T^w_{ab} = e^\mu_a T^\mu_a \), the relativistic conservation law Eq. (12) follows from the local Lorentz symmetries
\[ \delta \chi = -\frac{i}{4} \Lambda_{ab} \gamma^a \gamma^b \chi, \quad \delta e^a = \Lambda^a_b e^b, \]
which, when restricted to the 12 plane, coincide with the Galilean transformations. Indeed, the energy momentum tensor \( \Pi^w_\mu \) was not symmetric either in this plane, the linearization of which is equal to, see the SM [65],
\[ \Pi^w_\mu = (\Pi^{(1)}_\mu + \Pi^{(2)}_\mu) = p_F \hat{\mathbf{v}} e^\mu_a \delta_{ab} - e^\mu_a T^\mu_a + e^a_{\mu} S^w_{ab}. \]
The Galilean term \( \Pi^{(1)}_\mu \) proportional to \( e^\mu_a \) transfers due to the finite momentum density \( +p_F \hat{\mathbf{v}} \) at the node and, therefore, contributes to energy momentum. The corresponding relativistic conservation law related to \( \Pi^w_\mu \) of the linearized Weyl action follows from spacetime diffeomorphisms and leads to
\[ \partial_\mu \left( -e^\mu_a T^w_a + \frac{1}{2} \partial_{\alpha b} S^w_{ab} \right) = \partial_\mu e^\mu_a T^w_a + \frac{1}{2} \partial_{\alpha b} e^\alpha_b S^w_{ab}, \]
\[ \partial_\mu \left( -e^\mu_a T^w_a + \frac{1}{2} \partial_{\alpha b} S^w_{ab} \right) \]
i.e., \( \partial_\mu \Pi^{(2)}_w = \partial_\mu \mathcal{L} \). The field theory conservation equation for the energy-momentum Eq. (6) is, then, equivalent to Eq. (17) and the conservation of the quasiparticle current density \( e^\mu_a \) at the node and leads to the observed momentum nonconservation Eq. (1) in the system. The gravitational NY anomaly is [34,68,70,72], for a chiral pair of Weyl fermions, with \( e^a = e^a_{\mu} dx^\mu \),
\[ \partial_\mu (e^\mu_a dx^\mu) = \frac{\Lambda^2}{4\pi^2} (T^w_a \wedge T^w_a - e^a \wedge e^b \wedge R_{ab}), \]
where the higher order term \( O(R^2) \) is neglected. For the \( p + i\rho \) SF, the anomalous chiral Weyl action is (10) with spacetime defined by Eq. (11). Note that, although the quasiparticles \( \tilde{\chi} \) are Majorana-Weyl contributing one-half of Eq. (18) per node, a factor of 2 comes from accounting for spin degeneracy. The temporal torsion \( T^0 = 0 \), and we compute the spatial contribution,
\[ T^1 \wedge T^1 + T^2 \wedge T^2 \]
\[ = 2 \left( \frac{c_1}{c_{\perp}} \right)^2 e^{\partial_{ijkl}} \left( \hat{\mathbf{I}} \cdot \mathbf{v}_j \right) \partial_{i} \hat{\mathbf{I}}_k - \partial_{j} \hat{\mathbf{I}}_i \partial_{k} \hat{\mathbf{I}}_l d^4x \]
\[ \approx 2 \left( \frac{c_1}{c_{\perp}} \right)^2 e^{\partial_{ijkl}} \left( \hat{\mathbf{I}} \cdot \mathbf{v}_j \right) \partial_{i} \hat{\mathbf{I}}_k - \partial_{j} \hat{\mathbf{I}}_i \partial_{k} \hat{\mathbf{I}}_l d^4x. \]

The curvature term \( -e^a \wedge e^b \wedge R_{ab} \) is
\[ -\frac{4\pi}{\kappa} \left( \frac{c_1}{c_{\perp}} \right)^2 e^{\partial_{ijkl}} \left( \hat{\mathbf{I}} \cdot \mathbf{v}_j \right) \partial_{i} \hat{\mathbf{I}}_k - \partial_{j} \hat{\mathbf{I}}_i \partial_{k} \hat{\mathbf{I}}_l - \left( \mathbf{v}_j \cdot \nabla \right) \left( \hat{\mathbf{I}} \cdot \mathbf{v}_k \right) d^4x \]
\[ = \left( \frac{c_1}{c_{\perp}} \right)^2 e^{\partial_{ijkl}} \left( \hat{\mathbf{I}} \times \mathbf{v}_j \right) \left( \hat{\mathbf{I}} \cdot \mathbf{v}_j \right) \partial_{i} \hat{\mathbf{I}}_k - \partial_{j} \hat{\mathbf{I}}_i \partial_{k} \hat{\mathbf{I}}_l d^4x. \]
To lowest order in gradients, we arrive to
\[ e \partial_\mu \partial^\mu_i \right. \]
\[ \left. \frac{\Lambda^2}{2\pi^2} e \left( 1 - \frac{c_1}{c_{\perp}} \right)^2 \right. \]
\[ \left. e^{\partial_{ijkl}} \left( \hat{\mathbf{I}} \cdot \mathbf{v}_j \right) \partial_{i} \hat{\mathbf{I}}_k, \right. \]
where \( e = \left( c_1 / c_{\perp} \right)^2 \). Matching the expression with the hydrodynamic anomaly [56,57,62] in the SM [65], the anisotropic cutoff is \( \Lambda = \left( c_{\perp} / c_1 \right) p_F \) and applies in a Weyl SF with the nodes at \( \frac{1}{2} \mathbf{p}_F \), such as He-A, or in a Weyl SC after minimal substitution [53,62]. The expression is a Galilean invariant and the coefficient is proportional to the weak-coupling normal state density [without the logarithm \( \ln(E_F / \Delta_0) \) due to the neglected gapped fermions [57]]. This the central result of the Letter. The NY anomaly equation can also be derived with simple arguments using
Landau levels and spectral flow in the case of a torsional magnetic field $T_{xz}$ [9,11,34,58]. In general, the dimensional coefficient $\Lambda$ is seen simply to follow from the fact that torsion couples to momentum and that the density of states of the anomalous chiral lowest Landau level branches is momentum dependent. Lorentz invariance would require that the Weyl nodes are symmetrically at $p^0 = 0$ which leads to $\Lambda = 0$ at the node. For chiral Weyl nodes with a nonzero separation $2p^0$ in momentum space, the coefficient of the torsion anomaly is $\Lambda \propto |p|$ according to the spectral flow calculation.

In condensed matter systems, however, the Weyl description of the quasiparticles and the chiral anomaly breaks down at some cutoff scale. This is in contrast to fundamental Weyl fermions, where the conventional chiral anomalies satisfy IR-UV independence: the anomaly is the same at each energy scale since it can be computed by comparing to a theory with no anomaly simply by adding a high-energy chiral fermion that cancels the anomaly of the original theory [23]. On the other hand, for $^3$He-A the UV completion is fully known in terms of the Fermi-liquid theory and the anomalous SF hydrodynamics of $^3$He-A [5]. In the idealized $p$-wave BCS pairing model (2), the cutoff energy scale $E_{\text{IR}} = \Delta$ for the SF is determined from the MF gap equation $c_{\perp} \sim (E_{\text{UV}}/p_F) e^{-m_n/g} \sim (\Delta/p_F)$, where $g$ is the $\delta$-function interaction coupling constant and $E_{\text{UV}} \sim \tau_F p_F = c_{\parallel} p_F$ the normal state Fermi energy. The linear quasirelativistic Weyl regime emerges when $E \ll E_W = m c_{\perp}^2 = (c_{\perp}/c_{\parallel}) \Delta$. Therefore, the uniaxial anisotropy is simply the relative scale $(c_{\perp}/c_{\parallel}) = (E_{\text{IR}}/E_{\text{UV}})$, while the linear Weyl regime is suppressed by an additional factor of $c_{\perp}/c_{\parallel}$ compared to $E_{\text{IR}}$ leading to the value of $\Lambda = (c_{\perp}/c_{\parallel}) p_F$. In $^3$He-A, $c_{\perp}/c_{\parallel}$ is of the order $10^{-5}$ [82]. Remarkably, the hydrodynamic anomaly (1) is the same as in Eq. (18) when all states beyond the linear quasirelativistic Weyl approximation are taken into account.

Outlook.—We have revisited the anomalous momentum transport in chiral $p$-wave SFs and SCs in terms of a consistent hydrodynamic and low-energy effective theory description. Using the gauge and Galilean symmetries of the system, we have shown how the quasirelativistic Weyl approximation, emergent spacetime, and symmetries appear in the semiclassical derivative expansion. The anomalous transport is a consequence of the axial gravitational NY anomaly due to the chiral Weyl fermions on an emergent Riemann-Cartan spacetime with torsion.

Here, we have shown that the emergent spacetime formulation satisfies all the symmetries and conservation laws of the effective field theory required for the gravitational NY anomaly and saturates the nonzero value from SF hydrodynamics [64]. The early papers [9–12] treat the anomaly in terms of a momentum space axial gauge field; this follows from our formalism via the substitution of the tetrad $e_i^J = \hat{z} + \delta e_i^J$, formally equivalent to gauge field $\sim p_F \delta^J_p$ in the Hamiltonian [34,54,90]. In contrast, Refs. [67,76] inconsistently consider both contributions independently. Moreover, the emergent gauge field does not correspond to physical symmetry in the system and $p_F$ is an explicit UV scale. This is distinct from the emergent spacetime (11), which, in addition, is valid for arbitrary (semiclassical) textures and superflow, or considerations of other Weyl systems, where the emergent tetrad or gauge fields (e.g., elastic deformations, Fermi velocities, node separation) are independent [54,77,78]. On the other hand, Ref. [54] sets the NY cutoff to the lattice scale and neglects the breakdown of the linear Weyl spectrum.

Interestingly, for the emergent spacetime in Eq. (11), the anomaly coefficient seems to vanish in the relativistic case $c_{\perp} = c_{\parallel}$, but this is probably an artefact of the breakdown of the (weak-coupling) BCS model. Our findings corroborate the subtle interplay of broken Lorentz invariance, anisotropic dispersion, renormalization, and the NY-anomaly coefficient $\Lambda$ and should be verified by detailed field theory computations [34,53,55,72,91,92]. Similarly, the relation of the emergent quasirelativistic (or uniaxial) spacetime to Newton-Cartan geometries should be clarified [84,93–96]. We did not consider the dynamics of the SF order parameter or Goldstone modes, orbital nonanalyticity [5,56,97], nor derive the Wess-Zumino consistency equation and action for the chiral NY anomaly. This will be a gravitational Chern-Simons term for the tetrad and spin connection [16,98–100]. Likewise, we did not consider singular vortices, which will lead to additional curvature and zero modes, as well as the Iordanskii force and gravitational Aharonov-Bohm phase [5] for the quasiparticles. The connection of emergent spacetime and thermal transport should be explored [40,76]. In particular, it is possible that the UV scale $\Lambda$ is supplemented by the IR temperature scale in the anomaly, which can be universal [101]. These, and other considerations extending previous results in the literature, will be left for the future.

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[1] S. Ryu, J. E. Moore, and A. W. W. Ludwig, Electromagnetic and gravitational responses and anomalies in topological insulators and superconductors, Phys. Rev. B 85, 045104 (2012).
[2] X.-G. Wen, Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders, Phys. Rev. D 88, 045013 (2013).
[3] A. Kapustin, Symmetry protected topological phases, anomalies, and cobordisms: Beyond group cohomology, arXiv:1403.1467.
[4] C. G. Callan and J. A. Harvey, Anomalies and fermion zero modes on strings and domain walls, Nucl. Phys. B250, 427 (1985).
[5] G. E. Volovik, The Universe in a Helium Droplet (Oxford University Press, Oxford, 2003).
[6] P. Horava, Stability of Fermi Surfaces and K Theory, Phys. Rev. Lett. 95, 016405 (2005).
[7] S. L. Adler, Axial-vector vertex in spinor electrodynamics, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, A PCAC puzzle; \( p^0 \to \gamma \gamma \) in the \( \sigma \)-model, Nuovo Cimento A 60, 47 (1969).
[8] H. B. Nielsen and M. Ninomiya, The Adler-Bell-Jackiw anomaly and Weyl fermions in a crystal, Phys. Lett. 130B, 389 (1983).
[9] G. E. Volovik, Normal Fermi liquid in a superfluid \(^3\)He-A at \( T = 0 \) and the anomalous current, Pis’ma Zh. Eksp. Teor. Fiz. 42, 294 (1985) [JETP Lett. 42, 363 (1985)], http://www.jetpletters.ac.ru/ps/1435/article_21833.shtml.
[10] G. E. Volovik, Chiral anomaly and the law of conservation of momentum in \(^3\)He-A, Pis’ma Zh. Eksp. Teor. Fiz. 43, 428 (1986) [JETP Lett. 43, 551 (1986)], http://195.178.214.34/ps/1407/article_21374.shtml.
[11] A. V. Balatskii, G. E. Volovik, and A. V. Konychev, On the chiral anomaly in superfluid \(^3\)He-A, Zh. Eksp. Teor. Fiz. 90, 2038 (1986) [Sov. Phys. JETP 63, 1194 (1986)], http://jpt.ac.ru/cgi-bin/e/index/e/63/6/p1194?l=a=list.
[12] G. E. Volovik, Peculiarities in the dynamics of superfluid \(^3\)He-A: Analog of chiral anomaly and of zero-charge, Zh. Eksp. Teor. Fiz. 92, 2116 (1987) [Sov. Phys. JETP 65, 1193 (1987)], http://jetp.ac.ru/cgi-bin/e/index/e/65/6/p1193?l=a-list.
[13] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Topological semimetal and Fermi-arc surface states in the electronic structure of pyrochlore iridates, Phys. Rev. B 83, 205101 (2011).
[14] A. A. Zyuzin and A. A. Burkov, Topological response in Weyl semimetals and the chiral anomaly, Phys. Rev. B 86, 115133 (2012).
[15] D. T. Son and N. Yamamoto, Berry Curvature, Triangle Anomalies, and the Chiral Magnetic Effect in Fermi Liquids, Phys. Rev. Lett. 109, 181602 (2012); Kinetic theory with Berry curvature from quantum field theories, Phys. Rev. D 87, 085016 (2013).
[16] I. Zahed, Anomalous Chiral Fermi Surfaces, Phys. Rev. Lett. 109, 091603 (2012).
[17] D. T. Son and B. Z. Spivak, Chiral anomaly and classical negative magnetoresistance of Weyl metals, Phys. Rev. B 88, 104412 (2013).
[18] J. E. Avron, R. Seiler, and P. G. Zograf, Viscosity of Quantum Hall Fluids, Phys. Rev. Lett. 75, 697 (1995); P. Lévy, Berry phases for Landau Hamiltonians on deformed tori, J. Math. Phys. (N.Y.) 36, 2792 (1995).
[19] G. E. Volovik, Analog of gravity in superfluid \(^3\)He-A, Pis’ma Zh. Eksp. Teor. Fiz. 44, 388 (1986) [JETP Lett. 44, 498 (1986)], http://jetpletters.ac.ru/ps/1393/article_21138.shtml.
[20] G. E. Volovik, The gravitational topological Chern-Simons term in a fluid of superfluid \(^3\)He-A, Pis’ma Zh. Eksp. Teor. Fiz. 51, 111 (1990) [JETP Lett. 51, 125 (1990)], http://www.jetpletters.ac.ru/ps/1136/article_17200.shtml.
[21] N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time reversal symmetries and the fractional quantum Hall effect, Phys. Rev. B 61, 10267 (2000).
[22] T. Eguchi and P. Freund, Quantum Gravity and World Topology, Phys. Rev. Lett. 37, 1251 (1976).
[23] L. Alvarez-Gaumé and E. Witten, Gravitational anomalies, Nucl. Phys. B234, 269 (1984).
[24] L. Alvarez-Gaumé and P. Ginsparg, The structure of gauge and gravitational anomalies, Ann. Phys. (N.Y.) 161, 423 (1985).
[25] N. Read, Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and \( p_x + ip_y \) paired superfluids, Phys. Rev. B 79, 045308 (2009).
[26] K. Landsteiner, E. Megías, and F. Pena-Benitez, Gravitational Anomaly and Transport Phenomena, Phys. Rev. Lett. 107, 021601 (2011).
[27] M. Stone, Gravitational anomalies and thermal Hall effect in topological insulators, Phys. Rev. B 85, 184503 (2012).
[28] B. Bradlyn, M. Goldstein, and N. Read, Kubo formulas for viscosity: Hall viscosity, Ward identities, and the relation with conductivity, Phys. Rev. B 86, 245309 (2012).
[29] C. Hoyos and D. T. Son, Hall Viscosity and Electromagnetic Response, Phys. Rev. Lett. 108, 066805 (2012).
[30] G. Basar, D. E. Kharchev, and I. Zahed, Chiral and Gravitational Anomalies on Fermi Surfaces, Phys. Rev. Lett. 111, 161601 (2013).
[31] T. L. Hughes, R. G. Leigh, and O. Parrikar, Torsional anomalies, Hall viscosity, and bulk-boundary correspondence in topological states, Phys. Rev. D 88, 025040 (2013).
[32] T. Can, M. Laskin, and P. Wiegmann, Fractional Quantum Hall Effect in a Curved Space: Gravitational Anomaly and Electromagnetic Response, Phys. Rev. Lett. 113, 046803 (2014).
[33] A. G. Abanov and A. Gromov, Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in a background magnetic field, Phys. Rev. B 90, 014435 (2014).
[34] O. Parrikar, T. L. Hughes, and R. G. Leigh, Torsion, parity-odd response and anomalies in topological states, Phys. Rev. D 90, 105004 (2014).
[35] L. Sun and S. Wan, Chiral viscoelastic response in Weyl semimetals, Europhys. Lett. 108, 37007 (2014).
[36] M. A. Zhubkov, Emergent gravity and chiral anomaly in Dirac semimetals in the presence of dislocations, Ann. Phys. (Amsterdam) 360, 655 (2015).
[37] M. Valle, Torsional response of relativistic fermions in 2 + 1 dimensions, J. High Energy Phys. 07 (2015) 006.
[38] B. Bradlyn and N. Read, Low-energy effective theory in the bulk for transport in a topological phase, Phys. Rev. B 91, 125303 (2015); Topological central charge from Berry
curvature: Gravitational anomalies in trial wave functions for topological phases, Phys. Rev. B 91, 165306 (2015).

[39] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, Framing Anomaly in the Effective Theory of the Fractional Quantum Hall Effect, Phys. Rev. Lett. 114, 016805 (2015).

[40] A. Gromov and A. G. Abanov, Thermal Hall Effect and Geometry with Torsion, Phys. Rev. Lett. 114, 016802 (2015).

[41] A. Lucas, R. A. Davison, and S. Sachdev, Probing topological superconductors, Ann. Phys. (Amsterdam) 372, 175 (2016).

[42] J. Gooth et al., Experimental signatures of the mixed axial-gravitational anomaly in the Weyl semimetal NbP, Nature (London) 547, 324 (2017).

[43] Z. V. Khaidukov and M. A. Zubkov, Chiral torsional effect, JETP Lett. 108, 670 (2018).

[44] P. Maraner, J. K. Pachos, and G. Palumbo, Specific heat of 2D interacting Majorana fermions from holography, Sci. Rep. 9, 17308 (2019).

[45] G. Palumbo and J. K. Pachos, Holographic correspondence in topological superconductors, Ann. Phys. (Amsterdam) 372, 175 (2016).

[46] A. Gottlieb et al., Experimental signatures of the mixed axial-gravitational anomaly in the Weyl semimetal NbP, Nature (London) 547, 324 (2017).

[47] Z. V. Khaidukov and M. A. Zubkov, Chiral torsional effect, JETP Lett. 108, 670 (2018).

[48] P. Maraner, J. K. Pachos, and G. Palumbo, Specific heat of 2D interacting Majorana fermions from holography, Sci. Rep. 9, 17308 (2019).

[49] A. Gromov and D. Thanh Son, Bimetric Theory of Fractional Hall States, Phys. Rev. X 7, 041032 (2017); A. Gromov, S. D. Geraedts, and B. Bradlyn, Investigating Anisotropic Quantum Hall States with Bimetric Geometry, Phys. Rev. Lett. 119, 146602 (2017).

[50] P. Wiegemann, Inner Nonlinear Waves and Inelastic Light Scattering of Fractional Quantum Hall States as Evidence of the Gravitational Anomaly, Phys. Rev. Lett. 120, 086601 (2018).

[51] G. Fujii and Y. Nishida, Low-energy effective field theory of superfluid $^3$He-B and its gyromagnetic and Hall responses, Ann. Phys. 395, 170 (2018).

[52] O. Golan, C. Hoyos, and S. Moroz, Boundary central charge from bulk odd viscosity—chiral superfluids, Phys. Rev. B 100, 104512 (2019).

[53] Z.-M. Huang, L. Li, J. Zhou, and H.-H. Zhang, Torsional response and Liouville anomaly in Weyl semimetals with dislocations, Phys. Rev. B 99, 155152 (2019).

[54] J. Nissinen and G. E. Volovik, Elasticity tetrads, mixed axial-gravitational anomalies, and (3 + 1)-d quantum Hall effect, Phys. Rev. Research 1, 023007 (2019).

[55] T. Kvorning, T. H. Hansson, A. Queelle, and C. M. Smith, Proposed Spontaneous Generation of Magnetic Fields by Curved Layers of a Chiral Superconductor, Phys. Rev. Lett. 120, 217002 (2018).

[56] G. E. Volovik, Exotic Properties of Superfluid $^3$He (World Scientific, Singapore, 1992).

[57] G. E. Volovik and V. P. Mineev, Orbital angular momentum and orbital dynamics: $^3$He-A and the Bose liquid, Zh. Eksp. Teor. Fiz. 81, 989 (1981) [Sov. Phys. JETP 54, 524 (1981)], http://jetp.ac.ru/cgi-bin/e/index/e/54/3/p524a.html.

[58] R. Comesbuct and T. Dombre, Twisting in superfluid $^3$He-A and consequences for hydrodynamics at $T = 0$, Phys. Rev. B 33, 79 (1986).

[59] N. D. Mermin and Paul Muzikar, Cooper pairs versus Bose condensed molecules: The ground-state current in superfluid $^3$He $\sim$ A, Phys. Rev. B 21, 980 (1980).

[60] R. Comesbuct and T. Dombre, Superfluid current in $^3$He-A at $T = 0$, Phys. Rev. B 28, 5140 (1983).

[61] R. Comesbuct and T. Dombre, Excitation spectrum and superfluid density of $^3$He-A at $T = 0$, Phys. Rev. B 30, 3765 (1984).

[62] G. E. Volovik, Superfluid properties of $^3$He-A, Sov. Phys. Usp. 27, 363 (1984).

[63] G. E. Volovik, Orbital angular momentum of vortices and textures due to spectral flow through the gap nodes: Example of the $^3$He-A continuous vortex, Pis’ma Zh. Eksp. Teor. Fiz. 61, 935 (1995) [JETP Lett. 61, 958 (1995)], http://www.jetpletters.ac.ru/ps/1210/article_18297.shtml.

[64] T. D. C. Bevan, A. J. Manninen, J. B. Cook, J. R. Hook, H. E. Hall, T. Vachaspati, and G. E. Volovik, Momentum creation by vortices in $^3$He as a model of primordial baryogenesis, Nature (London) 368, 689 (1997).

[65] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.117002 for further details on the model, symmetries and conservation laws, the momentum anomaly of Eq. (1), as well as for a glossary on Riemann-Cartan spacetimes and our conventions regarding relativistic fermions.

[66] M. G. McClure and S. Takagi, Angular Momentum of Anisotropic Superfluids, Phys. Rev. Lett. 43, 596 (1979).

[67] Y. Ishihara, T. Mizushima, A. Tsuruta, and S. Fujimoto, Torsional chiral magnetic effect due to skyrmion textures in a Weyl superfluid $^3$He-A, Phys. Rev. B 99, 024513 (2019).

[68] H. T. Nieh and M. L. Yan, An identity in Riemann-Cartan geometry, J. Math. Phys. (N.Y.) 23, 373 (1982); H. T. Nieh, A torsional topological invariant, Int. J. Mod. Phys. A 22, 5237 (2007).

[69] H. T. Nieh and M. L. Yan, Quantized Dirac field in curved Riemann-Cartan background. I. Symmetry properties, Ann. Phys. (Amsterdam) 138, 237 (1982).

[70] S. Yajima, Evaluation of the heat kernel in Riemann—Cartan space, Classical Quantum Gravity 13, 2423 (1996).

[71] Y. N. Obukhov, E. W. Mielke, J. Budczies, and F. W. Hehl, On the chiral anomaly in non-Riemannian spacetimes, Found. Phys. 27, 1221 (1997).

[72] O. Chandia and J. Zanelli, Topological invariants, instantons, and the chiral anomaly on spaces with torsion, Phys. Rev. D 55, 7580 (1997); Torsional topological invariants (and their relevance for real life), Phys. Lett. B 416, 85 (1998); Supersymmetric particle in a spacetime with
torsion and the index theorem, Phys. Rev. D 58, 045014 (1998).

[73] C. Soo, Adler-Bell-Jackiw anomaly, the Nieh-Yan form, and vacuum polarization, Phys. Rev. D 59, 045006 (1999).

[74] K. Peeters and A. Waldron, Spinors on manifolds with boundary: APS index theorems with torsion, J. High Energy Phys. 02 (1999) 024.

[75] D. Kreimer and E. W. Mielke, Comment on “Topological invariants, instantons, and the chiral anomaly on spaces with torsion”, Phys. Rev. D 63, 048501 (2001); O. Chandra and J. Zanelli, Reply to “Comment on “Topological invariants, instantons, and the chiral anomaly on spaces with torsion””, Phys. Rev. D 63, 048502 (2001).

[76] T. Kobayashi, T. Matsushita, T. Mizushima, A. Tsuruta, and S. Fujimoto, Negative Thermal Magnetoresistivity as a Signature of a Chiral Anomaly in Weyl Superconductors, Phys. Rev. Lett. 121, 207002 (2018).

[77] A. G. Grushin, J. W. F. Venderbos, A. Vishwanath, and R. Ilan, Inhomogeneous Weyl and Dirac Semimetals: Transport in Axial Magnetic Fields and Fermi Arc Surface States from Pseudo-Landau Levels, Phys. Rev. X 6, 041046 (2016).

[78] D. I. Pikulin, A. Chen, and M. Franz, Chiral Anomaly from Strain-Induced Gauge Fields in Dirac and Weyl Semimetals, Phys. Rev. X 7, 041021 (2016).

[79] A. Westström and T. Ojanen, Designer Curved-Space Geometry for Relativistic Fermions in Weyl Metamaterial, Phys. Rev. X 7, 041026 (2017).

[80] L. Liang and T. Ojanen, Curved spacetime theory of inhomogeneous Weyl materials, Phys. Rev. Research 1, 032006 (2019).

[81] R. Ilan, A. G. Grushin, and D. I. Pikulin, Pseudo-electromagnetic fields in topological semimetals, Nat. Rev. Phys. 2, 29 (2020).

[82] D. Vollhardt and P. Wölfle, The Superfluid Phases of Helium 3 (Taylor & Francis, London, 1990).

[83] M. Liu and M. C. Cross, Gauge Wheel of Superfluid $^3$He, Phys. Rev. Lett. 43, 296 (1979).

[84] D. Son and M. Wingate, General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas, Ann. Phys. (Amsterdam) 321, 197 (2006).

[85] N. Mermin and T.-L. Ho, Circulation and Angular Momentum in the A Phase of Superfluid Helium-3, Phys. Rev. Lett. 36, 594 (1976).

[86] N. Kopnin, Theory of Nonequilibrium Superconductivity (Oxford University Press, Oxford, 2001).

[87] K. Fujikawa, Path-Integral Measure for Gauge-Invariant Fermion Theories, Phys. Rev. Lett. 42, 1195 (1979); Chiral anomaly and the Wess-Zumino condition, Phys. Rev. D 31, 341 (1985).

[88] I. L. Shapiro, Physical aspects of the space-time torsion, Phys. Rep. 357, 113 (2002).

[89] R. T. Hammond, Torsion gravity, Rep. Prog. Phys. 65, 599 (2002).

[90] This is part of the ambiguity of assigning emergent fields to the Weyl quasiparticles without the UV completion. In particular, the UV scale $\Lambda$ is undetermined in the quasirelativistic low-energy theory. I thank G. E. Volovik for repeatedly emphasizing this point to me. The Landau level spectral flow calculations in [9,11] arise from the torsional magnetic field due to a nontrivial tetrad [34].

[91] E. V. Castro, A. Flachi, P. Ribeiro, and V. Vitagliano, Symmetry Breaking and Lattice Kirigami, Phys. Rev. Lett. 121, 221601 (2018).

[92] S. Imaki and A. Yamamoto, Lattice field theory with torsion, Phys. Rev. D 100, 054509 (2019).

[93] C. Duval, G. Burdet, H. P. Künzle, and M. Perrin, Bargmann structures and Newton-Cartan theory, Phys. Rev. D 31, 1841 (1985).

[94] D. T. Son, Newton-Cartan geometry and the Quantum Hall Effect, arXiv:1306.0638.

[95] M. H. Christensen, I. Hartong, N. A. Obers, and B. Rollier, Torsional Newton-Cartan geometry and Lifshitz holography, Phys. Rev. D 89, 061901(R) (2014).

[96] C. Copetti, Torsion and anomalies in the warped limit of Lifshitz theories, J. High Energy Phys. 01 (2020) 190 (2020).

[97] J. Nissinen and G. E. Volovik, Dimensional crossover of effective orbital dynamics in polar distorted $^3$He-A: Transitions to antispacetime, Phys. Rev. D 97, 025018 (2018).

[98] G. E. Volovik, Wess-Zumino action for the orbital dynamics of $^3$He-A, Pis'ma Zh. Eksp. Teor. Fiz. 44, 144 (1986); [JETP Lett. 44, 185 (1986)], http://www.jetpletters.ac.ru/ps/1376/article_20873.shtml.

[99] A. V. Balatsky, Microscopically derived Wess-Zumino action for $^3$He-A, Phys. Lett. A 123, 27 (1987).

[100] J. Dziarmaga, Low-temperature effective electromagnetism in superfluid $^3$He-A, Pis'ma Zh. Eksp. Teor. Fiz. 75, 323 (2002) [JETP Lett. 75, 273 (2002)].

[101] J. Nissinen and G. E. Volovik, On thermal Nieh-Yan anomaly in Weyl superfluids, arXiv:1909.08936.