On Diffusion Processes with Drift in $L_{d+1}$

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Abstract
This paper is a natural continuation of Krylov (2020 and 2021) where strong Markov processes are constructed in time inhomogeneous setting with Borel measurable uniformly bounded and uniformly nondegenerate diffusion and drift in $L_{d+1}(\mathbb{R}^{d+1})$ and some properties of their Green’s functions and probability of passing through narrow tubes are investigated. On the basis of this here we study some further properties of these processes such as Harnack inequality, Hölder continuity of potentials, Fanghua Lin estimates and so on.

Keywords  Itô’s equations with singular drift · Markov diffusion processes · Harnack inequality

Mathematics Subject Classification (2010)  60H10 · 60J60

1 Introduction
Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, ..., x^d)$, $d \geq 2$. Fix some $p_0, q_0 \in [1, \infty)$ such that

$$\frac{d}{p_0} + \frac{1}{q_0} = 1. \quad (1.1)$$

It is proved in [11] that Itô’s stochastic equations of the form

$$x_t = x + \int_0^t \sigma(t_0 + s, x_s) \, dw_s + \int_0^t b(t_0 + s, x_s) \, ds \quad (1.2)$$

admit weak solutions, where $w_s$ is a $d$-dimensional Wiener process, $\sigma$ is a uniformly non-degenerate, bounded, Borel function with values in the set of symmetric $d \times d$ matrices, $b$ is a Borel measurable $\mathbb{R}^d$-valued function given on $\mathbb{R}^{d+1} = (-\infty, \infty) \times \mathbb{R}^d$ such that

$$\int_\mathbb{R} \left( \int_{\mathbb{R}^d} |b(t, x)|^{p_0} \, dx \right)^{q_0/p_0} \, dt < \infty \quad (1.3)$$
if $p_0 \geq q_0$ or

$$\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} |b(t, x)|^{q_0} dt \right)^{p_0/q_0} dx < \infty$$  \hspace{1cm} (1.4)$$

if $p_0 \leq q_0$. Observe that the case $p_0 = q_0 = d + 1$ is not excluded and in this case the condition becomes $b \in L_{d+1}(\mathbb{R}^{d+1})$.

The goal of this article is to study the properties of such solutions or Markov processes whose trajectories are solutions of Eq. 1.2. In particular, in Section 2 for more or less general processes of diffusion type we derive the power lower estimate for the probability to reach at time $T$ a given ball of radius $\rho$ (in case $t_0 = 0, x = 0$ in Eq. 1.2). This estimate plays a crucial role in proving the Harnack inequality in Section 4. In Section 2 we also prove that the probability to reach sets of almost full measure are strictly bigger than zero. This seemingly weak statement is also crucial for proving the Harnack inequality.

Section 3 is devoted to proving estimates from below for the average time spent in space-time sets of small measure in terms of a power of their measure. We also extract some consequences of these estimates, which help proving the Hölder continuity of potentials and harmonic functions in Section 4 and also allow us to establish the Fanghua Lin estimates in Section 5.

Section 4 is devoted to the case when our process is, actually, not just of diffusion type, but a Markov (time-inhomogeneous) process, whose existence is shown in [11]. We prove that their resolvent operators are bounded in $L_{p,q}$. We prove Harnack’s inequality for the caloric functions associated with these processes, establish that their resolvent operators map $L_{p,q}$ to the set of Hölder continuous functions, and give some other estimates for the resolvent in the whole space and in domains. These results extend some of those in [18].

In Section 5 we give some applications of our results to the theory of linear parabolic equations. In particular, we prove the Harnack inequality and Hölder continuity of their solutions, which in case $p = q = d + 1$ are known as Krylov-Safonov estimates and played an enormous role in the theory of linear and fully nonlinear elliptic and parabolic equations with bounded coefficients. The solutions we are dealing with are of class $W^{1,2}_{p,q}$ with $d_0/p + 1/q \leq 1$ and $d_0 < d$. We now have the opportunity to consider lower order coefficients in $L_{p,q}$-spaces and develop $W^{1,2}_{p,q}$-solvability theory. As is mentioned already, in this section we also derive the Fanghua Lin estimate, which is one of the starting point of the regularity theory of fully nonlinear equations as presented in [8].

The final section is an Appendix in which we collected some results from [12] frequently used in the main text.

To the best of the author’s knowledge our results are new even if $p_0 = q_0 = d + 1$, with such low integrability of $b$ and general $\sigma$ Hölder and Harnack properties were unknown.

It is worth mentioning that there is a vast literature about stochastic equations when Eq. 1.1 is replaced with $d/p_0 + 2/q_0 \leq 1$. This condition is much stronger than ours. Still we refer the reader to the recent articles [1, 15, 17] and the references therein for the discussion of many powerful and exciting results obtained under this stronger condition. There are also many papers when this condition is considerably relaxed on the account of imposing various regularity conditions on $\sigma$ and $b$ and/or considering random initial conditions with bounded density, see, for instance, [19, 20] and the references therein. Restricting the situation to the one when $\sigma$ and $b$ are independent of time allows one to relax the above conditions significantly further, see, for instance, [7] and the references therein.

Introduce

$$B_R = \{ x \in \mathbb{R}^d : |x| < R \}, \quad B_R(x) = x + B_R, \quad C_{T,R} = [0, T) \times B_R,$$$$

$$C_{T,R}(t, x) = (t, x) + C_{T,R}, \quad C_R(t, x) = C_{R^2,R}(t, x), \quad C_R = C_R(0, 0),$$
\[ D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j \quad \partial_t = \frac{\partial}{\partial t}. \]

For \( p, q \in [1, \infty] \) and domains \( Q \subset \mathbb{R}^{d+1} \) we introduce the space \( L_{p,q}(Q) \) as the space of Borel functions on \( Q \) such that

\[
\| f \|_{L_{p,q}(Q)}^q := \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} I_Q(t, x) |f(t, x)|^p \, dx \right)^{q/p} \, dt < \infty
\]

if \( p \geq q \) or

\[
\| f \|_{L_{p,q}(Q)}^p := \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} I_Q(t, x) |f(t, x)|^q \, dt \right)^{p/q} \, dx < \infty
\]

if \( p \leq q \) with natural interpretation of these definitions if \( p = \infty \) or \( q = \infty \). If \( Q = \mathbb{R}^{d+1} \), we drop \( Q \) in the above notation. Observe that \( p \) is associated with \( x \) and \( q \) with \( t \) and the interior integral is always elevated to the power \( \leq 1 \). In case \( p = q = d + 1 \) we abbreviate \( L^+_{d+1}(\mathbb{R}^{d+1}) = L_{d+1}(\mathbb{R}^{d+1}) \). For the set of functions on \( \mathbb{R}^d \) summable to the \( p \)th power we use the notation \( L^p(\mathbb{R}^d) \).

If \( \Gamma \) is a measurable subset of \( \mathbb{R}^{d+1} \) we denote by \( |\Gamma| \) its Lebesgue measure. The same notation is used for measurable subsets of \( \mathbb{R}^d \) with \( d \)-dimensional Lebesgue measure. We hope that it will be clear from the context which Lebesgue measure is used. If \( \Gamma \) is a measurable subset of \( \mathbb{R}^{d+1} \) and \( f \) is a function on \( \Gamma \) we denote

\[
\int_{\Gamma} f \, dxdt = \frac{1}{|\Gamma|} \int_{\Gamma} f \, dxdt.
\]

In case \( f \) is a function on a measurable subset \( \Gamma \) of \( \mathbb{R}^d \) we set

\[
\int_{\Gamma} f \, dx = \frac{1}{|\Gamma|} \int_{\Gamma} f \, dx.
\]

## 2 The Case of General Diffusion Type Processes with Drift in \( L_{p_0,q_0} \)

Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space, let \( \mathcal{F}_t, t \geq 0 \), be an increasing family of complete \( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \), let \( w_t \) be an \( \mathbb{R}^d \)-valued Wiener process relative to \( \mathcal{F}_t \). Fix \( \delta \in (0, 1) \) and denote by \( \mathbb{S}_\delta \) the set of \( d \times d \) symmetric matrices whose eigenvalues are between \( \delta \) and \( \delta^{-1} \). Assume that we are given an \( \mathbb{S}_\delta \)-valued \( \mathcal{F}_t \)-adapted process \( \sigma_t = \sigma_t(\omega) \) and an \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \)-adapted process \( b_t \), such that

\[
\int_0^T |b_t| \, dt < \infty
\]

for any \( T \in (0, \infty) \) and \( \omega \). Define

\[
x_t = \int_0^t \sigma_s \, dw_s + \int_0^t b_s \, ds
\]

and for \( R \in (0, \infty) \) define \( \tau_R(x) \) as the first time \( (t, x + x_t) \) exits from \( C_R \), \( \tau'_R(x) \) as the first time \( x_t \) exits from \( B_R \), \( \tau_R = \tau_R(0) \), \( \tau'_R = \tau'_R(0) \).

**Assumption 2.1.** We are given a function \( h \in L_{p_0,q_0,\text{loc}} \) such that

\[
|b_t| \leq h(t, x_t).
\]
Furthermore, there exists a bounded nondecreasing function $\bar{b}_R$, $R \in (0, \infty)$, such that for any $(t, x) \in \mathbb{R}^{d+1}$ and $R \in (0, \infty)$ we have

$$\|h\|_{L_{p_0,q_0}(C_R(t,x))}^{q_0} \leq \bar{b}_RR. \quad (2.1)$$

**Assumption 2.2.** We take $\bar{N} = \bar{N}(d, p_0, \delta)$ introduced in Theorem 6.1 and suppose that there exists $\bar{R} \in (0, \infty)$ such that

$$\bar{N}\bar{b}_R < 1. \quad (2.2)$$

This assumption as well as Assumption 2.1 is supposed to hold throughout the article.

**Remark 2.1.** Throughout the article we fix a number $\bar{\bar{R}} \in [\mathbb{R}, \infty)$. In some places we write that certain constants depend “only on... $\bar{\bar{R}}$ and $\bar{\bar{b}}\bar{\bar{R}}$” and the reader might notice that, actually, the constants depend “only on... $\bar{\bar{R}}$ and $\bar{\bar{b}}\bar{\bar{R}}^2$”. In such situations it is useful to note that, as is easy to see, $\bar{\bar{b}}\bar{\bar{R}}^2$ can always be chosen less than or equal to $\bar{\bar{N}}(d)\bar{\bar{b}}\bar{\bar{R}}$. Also note that if we take $\bar{\bar{R}} = \bar{R}$, then mentioning $\bar{\bar{b}}\bar{\bar{R}}$ becomes unnecessary, because $\bar{\bar{b}}\bar{\bar{R}} \leq \bar{\bar{N}}^{-1}$.

Our first big project is to prove a version of Theorem 4.17 of [9], which provides an important step toward establishing Harnack’s inequality for caloric functions. It is worth saying that in the case of bounded $b$ Theorem 2.1 is proved by constructing a rather simple barrier, see the PDE argument in the proof of Lemma 9.2.1 (“lemma on an oblique cylinder”) of [8] or the probabilistic argument in the proof of Lemma 2.3 of [18]. In our case for the same purpose, we need a rather tedious argument like in Theorem 4.17 of [9] just to get a good control of the spatial process $x_t$.

Below $\bar{\xi} = \bar{\xi}(d, \delta) \in (0, 1)$ is taken from Theorem 6.1.

**Theorem 2.1** Let $R \in (0, \bar{R})$, $\kappa, \eta \in (0, 1)$, $x, y \in B_{\kappa R}$, and $\eta^{-1}R^2 \geq t \geq \eta R^2$. Then there exist $N, \nu > 0$, depending only on $\kappa, \eta, \bar{\xi}, \bar{R}$, and $R$, such that, for any $\rho \in (0, 1)$,

$$NP(x + x_tR \in B_{\rho R}(y), \tau_R(x) > t) \geq \rho^\nu. \quad (2.3)$$

The proof of this theorem, given below after appropriate preparations, follows that of Theorem 4.17 of [9] and, roughly speaking, consists of splitting the interval $[0, t]$ into several parts, estimating the probability that on the first part the process will reach a neighborhood of $y$ without exiting from $B_R$, and then on the consecutive time intervals shrink the neighborhood with constant coefficient in such a way as to arrive at time $t$ in $B_{\rho R}(y)$ without exiting from $B_R$.

**Lemma 2.2** Let $R \in (0, \bar{R})$ and let $\rho_0 \in (0, 1)$, $\theta \in (0, \infty)$, and $\kappa \in [1/2, 1)$. Then there exists $\mu = \mu(\bar{\xi}, \bar{R}, \bar{R}, \kappa, \rho_0, \theta) > 0$ such that

$$P(x + x_\theta R^2 \in B_{\rho_0 \kappa R}(y), \tau_R(x) > \theta R^2) \geq \mu, \quad (2.4)$$

whenever $x, y \in B_{\kappa R}$. Furthermore, one can take $\mu > 0$ the same if $\bar{\xi}, \bar{R}, \kappa$ are fixed and $\rho_0$ and $\theta$ vary in compact subsets of their ranges.

**Proof** Observe that Eq. 2.4 becomes stronger if $\rho_0$ becomes smaller. Therefore we may assume that

$$\rho_0 \leq \min \left(\bar{R}/(16\bar{R}), \sqrt[1444]{\bar{\theta}/(14471)}, \kappa^{-1} - 1 \right) \quad (\leq 1/16), \quad (2.5)$$
where $T_1 = T_1(\xi)$ is taken from Theorem 6.3. Then we split the proof into two cases.

**Case 1:** $|x - y| \leq 3\rho_0^2 R$. In that case, owing to $R \leq R$ and $\rho_0 \leq R/(6R)$, we have

\[|x - y| \leq (1/2)\rho_0\kappa(R \wedge R).\]

By Corollary 6.11, which is applicable since $\rho_0\kappa(R \wedge R) \leq R$,

\[NP\left(\sup_{t \leq \theta R^2} |x + x_t - y| < \rho_0\kappa(R \wedge R)\right) \geq \exp(-4\nu \theta R^2 / [\rho_0\kappa(R \wedge R)]^2).\]

The probability here is less than the probability in Eq. 2.4 since $R \wedge R \leq R$ and $\rho_0\kappa R \leq (1 - \kappa)R$. Furthermore,

\[R^2 / (\theta R^2)^2 \leq \bar{R}^2 / \bar{R}^2\]

and this proves Eq. 2.4 in the first case.

**Case 2:** $|x - y| \geq 3\rho_0^2 R$. Set $R_0 = 16\rho_0^2 R$ and note that $|x| + R_0 < \kappa R + (1 - \kappa)16\rho_0 R < R$. Similarly, $|y| + R_0 < R$. Therefore, the sausage $S_{R_0}(x, y)$, defined as the open convex hull of $B_{R_0}(x) \cup B_{R_0}(y)$, belongs to $\bar{B}_R$. By Theorem 6.3 with probability not less than $\pi_0^n$ before time $nT_1 R_0^2$ the process $x + x_t$ will hit $\bar{B}_{R_{0}/16}(y)$ without exiting from $S_{R_0}(x, y)$, where

\[n \leq \frac{4|x - y|}{R_0} + \frac{1}{4}.\]

Since $R_0 < R$, $|x - y| < 2R$, and also thanks to $144T_1 \rho_0^2 \leq \theta$, we have

\[nT_1 R_0^2 \leq T_1 R_0^2(4|x - y| + R_0/4) \leq T_1 R_0^2 R = 144T_1 \rho_0^2 \kappa R^2 \leq \theta R^2.\]

By introducing $\gamma$ as the first time $x + x_t$ hits $\bar{B}_{R_{0}/16}(y)$ we conclude that

\[P(\tau_R(x) > \gamma, \gamma \leq \theta R^2) \geq \pi_0^n.\]

Observe also that $R_0/16 = \rho_0 R_1$, where $R_1 = \rho_0\kappa R \leq R$ and at time $\gamma$ the point $x + x_\gamma$ is in $\bar{B}_{\rho_0 R_1}(y)$. It follows from Corollary 6.11 that, given that $\gamma < \infty$, with probability $\pi_1 > 0$ depending only on $\xi, \rho_0$, and $\theta_1 = \theta R_{\gamma}^{-2} R^2 (\leq \theta R_{\gamma}^{-2} \kappa^{-2})$ the process $x + x_t$ will stay in $B_{R_1}(y)$ on the time interval $[\gamma, \gamma + \theta_1 R^2_1]$. Notice that $\gamma + \theta_1 R^2_1 \geq \theta_1 R^2_1 = \theta R^2$. Along with Eq. 2.6 this implies that

\[P(x + x_\theta R^2 \in B_{R_1}(y), \tau_R(x) > \theta R^2) \geq \pi_0^n \pi_1 > 0.\]

To prove Eq. 2.4, it only remains to recall that $R_1 = \rho_0\kappa R$.

This proves the first assertion of the lemma. The second one is obtained by just inspecting the above proof. The lemma is proved.

The following is a particular case of Theorem 2.1 for $t = \eta R^2$.

**Lemma 2.3** Let $\kappa, \eta \in (0, 1)$. Then there are constants $N, \nu > 0$, depending only on $\kappa, \eta, \xi, \bar{R}$, and $\bar{R}$, such that, for any $R \in (0, \bar{R})$, $\rho \in (0, 1)$, and $x \in B_{\kappa R}$,

\[NP\left(\tau_R(x) > \eta R^2, x + x_\eta R^2 \in B_{\rho R}\right) \geq \rho^\nu.\]

**Proof** We may assume that $\kappa \in (1/2, 1)$. Let $\rho_0$ be a positive solution of

\[\rho_0 = \min\left(\frac{R}{16\bar{R}}, \sqrt{\frac{\eta(1 - \rho_0)}{12\sqrt{T_1} \kappa - 1}}\right).\]
observe that it suffices to prove Eq. 2.7 for $\rho \leq \kappa$, and set 
\[ n(\rho) = \left\lfloor \frac{\ln(\rho/\kappa)}{\ln \rho_0} \right\rfloor + 1 \quad (\geq 1), \quad \tilde{\eta} = \frac{1 - \rho_0}{1 - \rho_0^{2n(\rho)}}. \]

Note that $\tilde{\eta} \in [\eta(1 - \rho_0), \eta(1 + \rho_0)^{-1}]$ and 
\[ \rho_0 \leq \min \left( \frac{R}{16R}, \frac{\sqrt{\tilde{\eta}}}{12\sqrt{T_1}}, \frac{1}{\kappa} - 1 \right) \]
so that by Lemma 2.2 estimate Eq. 2.4, with $\theta = \tilde{\eta}$, $y = 0$, is valid and means that 
\[ P(x + x\tilde{\eta}R^2 \in B_{\rho 0^k R}, \sup_{s \leq \tilde{\eta}R^2} |x + x_s| < R) \geq \mu, \quad (2.8) \]
whenever $R \in (0, \tilde{R}]$ and $x \in B_\kappa R$. For $n = 1, 2, \ldots$ introduce 
\[ R_n = \rho_0^{n-1} R, \quad s_n = \tilde{\eta}R_n^2 = \tilde{\eta}R^2 \rho_0^{2(n-1)}. \]
Then for each $n$ and $y \in B_{\kappa R_n}$ we get from Eq. 2.8 that 
\[ P\left(y + x_{s_n} \in B_{\kappa R_n+1}, \sup_{s \leq s_n} |y + x_s| < R_n\right) \geq \mu, \]
which in the conditional form yields that for any $t \geq 0$ (a.s.) 
\[ P\left(y + x_{t+s_n} - x_t \in B_{\kappa R_n+1}, \sup_{s \leq s_n} |y + x_{t+s} - x_t| < R_n \mid \mathcal{F}_t\right) \geq \mu, \quad (2.9) \]
whenever $y$ is $\mathcal{F}_t$-measurable and $y \in B_{\kappa R_n}$. Now set $(t_0 := 0)$ 
\[ t_n = \sum_{k=1}^{n} s_k, \quad A_n = \{ \sup_{s \leq s_n} |x + x_{s+t_n-1}| < R_n \}, \quad \Pi_n = \bigcap_{k=1}^{n} A_k \]
and observe that, according to Eq. 2.9, on the set $\{ y := x + x_{t_n-1} \in B_{\kappa R_n} \}$ we have (a.s.) 
\[ P\left(y + (x_{t_n} - x_{t_n-1}) \in B_{\kappa R_n+1}, \sup_{s \leq s_n} |y + (x_{t_n-1+s} - x_{t_n-1})| < R_n \mid \mathcal{F}_{t_n-1}\right) \geq \mu. \quad (2.10) \]
Furthermore, obviously, for $n \geq 2$, 
\[ P^n := P(x + x_{t_n} \in B_{\kappa R_n+1}, \Pi_n) \]
\[ \geq P(x + x_{t_{n-1}} \in B_{\kappa R_{n-1}}, \Pi_{n-1}, x + x_{t_n-1} + (x_{t_n} - x_{t_n-1}) \in B_{\kappa R_n+1}, \sup_{s \leq s_n} |x + x_{t_n-1} + (x_{t_n-1+s} - x_{t_n-1})| < R_n), \]
which in light of Eq. 2.10 yields $P^n \geq \mu P^{n-1}$ and, since for $|x| < \kappa R$ we have $P^1 \geq \mu$ by Eq. 2.8, it holds that for $|x| < \kappa R$ and all $n \geq 1$ 
\[ P\left(x + x_{t_n} \in B_{\kappa R_n+1}, \sup_{s \leq t_n} |x + x_s| < R\right) \geq \mu^n. \quad (2.11) \]

Observe that 
\[ t_n(\rho) = \tilde{\eta}R^2 \frac{1 - \rho_0^{2n(\rho)}}{1 - \rho_0} = \eta R^2, \quad \kappa R_{n(\rho)+1} = \kappa \rho_0 R^{n(\rho)} \leq \rho R. \]
Therefore, Eq. 2.11 implies that 
\[ P(\tau_1(x) > \eta R^2, |x + x\eta R^2| \leq \rho R) \geq \mu^{n(\rho)}. \]
Now to finish the proof, it only remains to note that 
\[ \mu^{n(\rho)} \geq \mu \exp \left( \frac{\ln(\rho/\kappa)}{\ln \rho_0} \ln \mu \right) = N\rho^\nu. \]

\[ \text{ Springer} \]
The lemma is proved. \hfill \Box

**Proof of Theorem 2.1** Let $R_1 = (1 - \kappa)R$ and note that $\xi := t/R_1^2 - \eta$ satisfies

$$\eta^{-1}(1 - \kappa)^{-2} > \xi \geq \eta[(1 - \kappa)^{-2} - 1].$$

By the conditional version of Lemma 2.3 on the set $\{z := x + x_{\xi R_1^2} \in B_\kappa R_1(y)\}$ we have (a.s.)

$$NP \left( \sup_{s \in [\xi R_1^2, \xi R_1^2 + \eta R_1^2]} |z + x_s - x_{\xi R_1^2} - y| < R_1, x + x_{\xi R_1^2 + \eta R_1^2} \in B_{\rho R_1}(y) \mid \mathcal{F}_{\xi R_1^2} \right) \geq \rho^\gamma.$$

By Lemma 2.2, where we take $\rho_0 = R_1/R$ and replace $\theta$ there with $\xi(1 - \kappa)^2$,

$$P \left( \sup_{s \leq \xi R_1^2} |x + x_s| < R, x + x_{\xi R_1^2} \in B_\kappa R_1(y) \right) \geq \mu.$$

By combining these two facts and using that $\xi R_1^2 + \eta R_1^2 = t$, we obviously come to Eq. 2.3. The theorem is proved. \hfill \Box

Theorem 2.4 originated in [13] in case $b$ is bounded and is one of two most important ingredients in the proof of the Harnack inequality. Observe that in this theorem we do not claim that $q(\xi) \neq 0$ for $\xi$ not close to one. This fact will be proved later.

**Theorem 2.4** For any $\kappa \in (0, 1)$ there is a function $q(\xi)$, $\xi \in (0, 1)$, depending only on $\kappa, \delta, d, R, p_0, \tilde{R}$, and, naturally, on $\xi$, such that for any $R \leq \tilde{R}$, $x \in B_{\kappa R}$, and closed $\Gamma \subset \bar{C}_R$ satisfying $|\Gamma| \geq \xi |C_R|$ we have

$$P(\tau_\Gamma(x) < \tau_R(x)) \geq q(\xi), \quad (2.12)$$

where $\tau_\Gamma(x)$ is the first time the process $(t, x + x_t)$ hits $\Gamma$ (and $\tau_R(x)$ is its first exit time from $C_R$). Furthermore, $q(\xi) \to 1$ as $\xi \uparrow 1$. Finally, for any closed $\Gamma' \subset B_R$ satisfying $|\Gamma'| \geq \xi |B_R|$ we have

$$P(\tau_{\Gamma'}(x) < \tau_R(x)) \geq q(\xi), \quad (2.13)$$

where $\tau_{\Gamma'}(x)$ is the first time the process $x + x_t$ hits $\Gamma$ (and $\tau_R(x)$ is its first exit time from $B_R$).

**Proof** By considering what is going on in $B_{(1 - \kappa)R}(x)$ we convince ourselves that we may assume that $x = 0$. Also, obviously we may assume that $R \leq \tilde{R}$.

In that case for any $\varepsilon \in (0, \xi/4)$ we have ($\tau_\Gamma = \tau_{\Gamma}(0)$)

$$P(\tau_\Gamma > \tau_R) \leq P \left( \tau_R = \int_0^{\tau_R} I_{C_R \setminus \Gamma}(t, x_t) \, dt \right) \leq P(\tau_R \leq \varepsilon R^2) + \varepsilon^{-1}R^{-2}E \int_0^{\tau_R} I_{C_R \setminus \Gamma}(t, x_t) \, dt.$$

In light of Theorems 6.2 and Lemma 6.10 we can estimate the right-hand side and then obtain

$$P(\tau_\Gamma > \tau_R) \leq Ne^{-1/(N\varepsilon)} + Ne^{-1}R^{d/(d+1) - 2}|C_R \setminus \Gamma|^{1/(d+1)} \leq Ne^{-1/(N\varepsilon)} + Ne^{-1}(1 - \xi)^{1/(d+1)}$$

where the constants $N$ depend only on $d, \delta, p_0$. By denoting

$$q(\xi) = 1 - \inf_{\varepsilon \in (0, \xi/4)} \left( Ne^{-1/(N\varepsilon)} + Ne^{-1}(1 - \xi)^{1/(d+1)} \right),$$
we get what we claimed about Eq. 2.12.

Estimate Eq. 2.13 follows from Eq. 2.12 if one takes in the latter \( \Gamma = [0, R^2] \times \Gamma' \) and observes that

\[
\{ \tau_{t}(x) < \tau_{R}(x) \} \subset \{ \tau'_{t}(x) < \tau'_{R}(x) \}.
\]

The lemma is proved. \( \square \)

3 Estimating Time Spent in Space-Time Sets of Small Measure

The central result of this section is Theorem 3.5 which needs some auxiliary constructions and assertions.

We present extensions to the case that \( b \in L_{d+1} \) of probabilistic versions of some PDE results found in [8, 14, 16]. Recall the notation introduced in the Introduction and also introduce

\[
C_{T,R}^0 = (0, T) \times B_R, \quad C_{T,R}^0(t,x) = (t,x) + C_{T,R}^0, \quad C_R^0(t,x) = C_{R^2,R}^0(t,x),
\]

\[
C_R^0 = C_R^0(0,0). \quad \text{Fix} \quad q, \eta, \kappa \in (0,1).
\]

For cylinders \( Q = C_{R}^0(t,x) \) define

\[
Q' = (t,x) - C_{\eta^{-1}\rho^2,\rho}, \quad Q'' = (t-\eta^{-1}\rho^2,x) + C_{\eta^{-1}\rho^2,\kappa^2,\rho,\kappa},
\]

\[
Q'_+ = Q \cup Q' \cup (\{t\} \times B_{\rho}(x)).
\]

Imagine that the \( t \)-axis is pointed up vertically. Then \( Q' \) is adjacent to \( Q \) from below, the two cylinders have a common base, and along the \( t \)-axis \( Q' \) is \( \eta^{-1} \) times longer than \( Q \). The cylinder \( Q'' \) is obtained by contracting \( Q' \) to the center of its lower base with the contraction factor \( \kappa^{-2} \) for the \( t \)-axis and \( \kappa^{-1} \) for the spatial axes.

**Remark 3.1.** If \( Q = C_{R}^0(t,x) \), then the distance between \( Q \) and \( Q'' \) along the \( t \) axis is

\[
\eta^{-1}\rho^2 - \eta^{-1}\rho^2\kappa^2 = \eta^{-1}\rho^2(1 - \kappa^2), \tag{3.1}
\]

which is 1 if \( \eta = 1 - \kappa^2 \).

Let \( \Gamma \) be a measurable subset of \( C_1 \) and introduce \( B = B(\Gamma, q) \) as the family of open cylinders \( Q \) of type \( C_{R}^0(t_0,x_0) \) such that

\[
Q \subset C_1 \quad \text{and} \quad |Q \cap \Gamma| \geq q|Q|.
\]

Finally, define

\[
\Gamma'' = \bigcup_{Q \in B} Q'', \quad \Gamma'_{\epsilon} = \bigcup_{Q \in B: |Q| \geq \epsilon} Q''.
\]

Observe that for \( Q \in B \) the set \( Q'' \) is open. Hence, \( \Gamma'' \) is open and measurable.

**Lemma 3.1** If \( |\Gamma| \leq q|C_1| \), then

\[
|\Gamma''| \geq \left( 1 - \frac{1-q}{3d+1} \right)^{-1} (1 + \eta)^{-1}\kappa^{d+2}|\Gamma|
\]

and, if the factor of \( |\Gamma| \) above is strictly bigger than one, there exists \( \theta = \theta(d,q,\eta,\kappa) > 1 \) such that for any sufficiently small \( \epsilon > 0 \) there exists a closed \( \Gamma_{\epsilon} \subset \Gamma'_{\epsilon} \) such that

\[
|\Gamma_{\epsilon}| \geq \theta|\Gamma|. \tag{3.2}
\]
The first assertion of the lemma originated in [14, 16], is presented, for instance as Lemma 9.3.6 in [8]. The second one is proved in the same way as the second assertion of Lemma 4.8 of [9].

**Lemma 3.2** There is a constant \( q_0 = q_0(d, \delta, p_0, R, \tilde{R}) \in (0, 1) \) such that for any \( R \leq \tilde{R} \), Borel set \( \Gamma \subset C_R \) satisfying \(|\Gamma| \geq q_0|C_R|\), and \( x \in B_{\bar{R}/2} \) we have

\[
E \int_0^{\tau_R(x)} I_{\Gamma}(t, x + x_t) \, dt \geq \mu_0 R^2,
\]

where \( \mu_0 = \mu_0(d, \delta, p_0, \tilde{R}, \bar{R}, \tilde{b}_R) \in (0, 1) \).

**Proof** Note that in light of Corollary 6.12 we have \( E\tau_R(x) \geq v R^2 \), where \( v = v(d, \delta, \bar{R}, \tilde{R}) > 0 \). By using Theorem 6.7 we get that

\[
E \tau_R(x) - E \int_0^{\tau_R(x)} I_{\Gamma}(t, x + x_t) \, dt = E \int_0^{\tau_R(x)} I_{C_R \setminus \Gamma}(t, x + x_t) \, dt
\]

\[
\leq NR^{2(d_0-d)/(d_0+1)}(|C_R| - |\Gamma|)^{1/(d_0+1)}
\]

\[
\leq NR^2(1-q_0)^{1/(d_0+1)} \leq N(1-q_0)^{1/(d_0+1)} E\tau_R(x),
\]

where the constants \( N \) depend only on \( d, \delta, \bar{R}, \tilde{R}, p_0, \) and \( \tilde{b}_R \) and \( d_0 = d_0(d, \delta, \bar{R}, p_0) \in (d/2, d) \) is taken from [12]. We see how to choose \( q_0 \) to get the desired result. The lemma is proved. \( \square \)

In Lemma 3.3 by \( q_0 \) we mean the one from Lemma 3.2.

**Lemma 3.3** Take \( Q = C^0_\rho(s, y) \) with \( \rho \leq \bar{R} \), use the notation \( Q', Q'', Q'_+ \) introduced above, assume that \( \eta = 1 - \kappa^2 \), and suppose that Borel \( \Gamma \subset Q \) is such that \(|\Gamma| \geq q_0|Q|\). Then there is a constant \( v_0 > 0 \), depending only on \( \kappa, d, \delta, \bar{R}, \tilde{R}, p_0, \) and \( \tilde{b}_R \), such that for any \( (t_0, x_0) \in Q'' \)

\[
E \int_0^\tau I_{\Gamma}(t_0 + t, x_0 + x_t) \, dt \geq v_0 E\tau,
\]

where \( \tau \) is the first exit time of \((t_0 + t, x_0 + x_t)\) from \( Q'_+ \).

**Proof** Thanks to Remark 3.1 we have \( s - t_0 \in (\rho^2, \eta^{-1}\rho^2) \). Also \(|y - x_0| < \kappa \rho\). It follows by Theorem 2.1 that

\[
P\left(\sup_{r \in [0, s-t_0]} |x_0 + x_r - y| < \rho, |x_0 + x_{s-t_0} - y| < (1/2)\rho \right) \geq v,
\]

where \( v = v(\kappa, d, \delta, \bar{R}, \tilde{R}) > 0 \).

Next, for \( y \) defined as the first exit time of \((t_0 + t, x_0 + x_t)\) from \( Q' \) in light of Lemma 3.2 we have

\[
E \int_0^\tau I_{\Gamma}(t_0 + t, x_0 + x_t) \, dt = EI_{y > s-t_0} \int_y^\tau I_{\Gamma}(t_0 + t, x_0 + x_t) \, dt
\]

\[
\geq EI_{y > s-t_0, |x_0 + x_{s-t_0} - y| < \rho/2} E\left( \int_0^\tau I_{\Gamma}(t_0 + t, x_0 + x_t) \, dt \mid \mathcal{F}_{s-t_0} \right)
\]

\[
\geq \mu_0 \rho^2 P\left(\sup_{r \in [0, s-t_0]} |x_0 + x_r - y| < \rho, |x_0 + x_{s-t_0} - y| < \rho/2 \right) \geq \mu_0 v \rho^2.
\]
On the other hand, the height of $Q'_+ \subset (1 + \eta^{-1})\rho^2$, so that $(t_0 + t, x_0 + x_i)$ cannot spend in $Q'_+$ more time than $(1 + \eta^{-1})\rho^2$. This proves the lemma. \qed

**Lemma 3.4** Denote

\[ G_R(\Gamma, x) := E \int_0^{r'_{\kappa}(x) \wedge (2R^2)} I_\Gamma(t, x + x_i) \, dt, \]

fix $q, \kappa \in (0, 1)$, and introduce $\mu_R(q)$ as $R^{-2}$ times the infimum of $G_R(\Gamma, x)$ over all Borel $\Gamma \subset C_R(R^2, 0)$ satisfying $|\Gamma| \geq q|C_R(R^2, 0)|$ over all $x \in B_{\bar{b}_R}$, and over all processes $x_i$, satisfying our assumptions with the same $\delta$ and $\bar{b}_R$. Then $\mu_R(q)$ is a decreasing function of $R$.

The proof of this lemma, left to the reader, is easily achieved by using the self-similar dilations: $x_i \to cx_i/\rho^2$, which preserves (actually, makes smaller) $\bar{b}_R$ (see, for instance, the proof of our Theorem 6.4 in [12]).

**Theorem 3.5** For any $\kappa \in (0, 1)$ there exist $\gamma \in (0, 1)$ and $N$, depending only on $\kappa, d, \delta, p_0, \bar{R}$, with $N$ also depending on $\bar{R}$ and $\bar{b}_R$, such that for any $R \in (0, \bar{R})$, $q \in (0, 1)$, Borel $\Gamma \subset C_R(R^2, 0)$ satisfying $|\Gamma| \geq q|C_R(R^2, 0)|$, and $x \in B_{\bar{b}_R}$ we have

\[ E \int_0^{r'_{\kappa}(x) \wedge (2R^2)} I_\Gamma(t, x + x_i) \, dt \geq N^{-1}q^{1/\gamma}R^2. \quad (3.5) \]

**Proof** By using the notation from Lemma 3.4, our assertion is rewritten as

\[ \mu_R(q) \geq N^{-1}q^{1/\gamma}R^2. \quad (3.6) \]

Fix $\kappa \in (0, 1)$, perhaps larger than the one in the statement of the theorem, such that for $\eta = 1 - \kappa^2$ and $q = q_0$ the factor of $|\Gamma|$ in Lemma 3.1 is strictly bigger than one and take $\theta = \theta(d, q_0, 1 - \kappa^2, \kappa) > 1$ from that lemma.

Next, observe that a combination of Lemma 3.2 and Theorem 2.1, as in the proof of Lemma 3.3, leads to the conclusion that there exists $\mu_0 \in (0, 1)$, depending only on $\kappa, d, \delta, p_0, \bar{R}, \bar{R}$, and $\bar{b}_R$, such that

\[ \mu_R(q) \geq \mu_0 \]

for $q \in [q_0, 1]$ and $R \leq \bar{R}$.

We will be comparing $\mu_R(q')$ and $\mu_R(q'')$ for $0 < q' < q'' < 1$ such that

\[ (1 + \theta)q' \geq 2q''. \quad (3.7) \]

We take a Borel $\Gamma \subset C_R(R^2, 0)$ satisfying $|\Gamma| \geq q'|C_R(R^2, 0)|$ and in the construction before Lemma 3.1 we replace $C_1$ by $C_R(R^2, 0)$, keep all other notation, and from the chosen $\Gamma, \kappa, \eta, and q_0 (not q')$ we build up the set $\Gamma_{\varepsilon}$ and take $\varepsilon$ so small that Eq. 3.2 holds. There are two cases:

(i) \[ |\Gamma_{\varepsilon} \setminus C_R(R^2, 0)| \leq (q'' - q')|C_R(R^2, 0)|, \]

(ii) \[ |\Gamma_{\varepsilon} \setminus C_R(R^2, 0)| > (q'' - q')|C_R(R^2, 0)|. \]

**Case (i)**. Our goal is to show that

\[ G_R(\Gamma, x) \geq \min (\mu_0, v_0\mu_R(q''))R^2, \quad |x| \leq \kappa R, \quad (3.8) \]

where $v_0$ depends only on $\kappa, d, p_0, \delta, \bar{R}, \bar{b}_R$. \qed
Observe that, if $|\Gamma| \geq q_0 |C_R|$, by definition $G_R(\Gamma, x) \geq \mu(q_0)R^2 \geq \mu_0 R^2$ for $|x| \leq \kappa R$. Hence, we may assume that 

$$|\Gamma| < q_0 |C_1|.$$ 

In that case define 

$$\hat{\Gamma}_\varepsilon = \Gamma_\varepsilon \cap C_R(R^2, 0).$$ 

Notice that by definition and Lemma 3.1 

$$q' |C_R| \leq |\Gamma| \leq \theta^{-1}|\Gamma_\varepsilon|.$$ 

Moreover, by assumption 

$$|\Gamma_\varepsilon| = |\Gamma_\varepsilon \setminus C_R(R^2, 0)| + |\hat{\Gamma}_\varepsilon| \leq (q'' - q') |C_R| + |\hat{\Gamma}_\varepsilon|.$$ 

Due to Eq. 3.7, it follows that 

$$|\hat{\Gamma}_\varepsilon| \geq q'' |C_R|,$$ 

so that 

$$G_R(\hat{\Gamma}_\varepsilon, x) \geq \mu_R(q'')R^2, \quad |x| \leq \kappa R.$$ 

We now estimate $G_R(\Gamma, x)$ from below by means of $G_R(\hat{\Gamma}_\varepsilon, x)$ using Lemma 3.3. Since $\Gamma_\varepsilon \subset \Gamma''$, the closed set $\Gamma_\varepsilon$ is covered by the family $\{Q'' : Q \in B, |Q| \geq \varepsilon\}$. Then there is finitely many $Q(1), ..., Q(n) \in B$ such that $|Q(i)| \geq \varepsilon$, $i = 1, ..., n$, and 

$$\Gamma_\varepsilon \subset \bigcup_{i=1}^{n} Q''(i) =: \Pi_\varepsilon.$$ 

Then for $(t, x) \in \Pi_\varepsilon$ define $i(t, x)$ as the first $i \in \{1, ..., n\}$ for which $(t, x) \in Q''(i)$. Also set $Q'_+(0) = C_2 R^2, R$ and $i(t, x) = 0$ if $(t, x) \in \partial C_2 R^2, R$. Now define recursively 

$$\gamma^0 = 0, \quad \tau^1$$ 

as the first time after $\gamma^0$ when $(t, x + x_\tau)$ exits from $C_2 R^2, \Gamma_\varepsilon, \gamma^1$ as the first time after $\tau^1$ when $(t, x + x_\tau)$ exits from $Q'_+(i(\tau^1, x + x_\tau))$, and generally, for $k = 2, 3, ...$ define $\tau^k$ as the first time after $\gamma^{k-1}$ when $(t, x + x_\tau)$ exits from $C_2 R^2, \Gamma_\varepsilon, \gamma^k$ as the first time after $\tau^k$ when $(t, x_\tau)$ exits from $Q'_+(i(\tau^k, x + x_\tau))$. It is easy to check that so defined $\tau^k$ and $\gamma^k$ are stopping times and, since $|Q(i)| \geq \varepsilon$ and the trajectories of $(t, x + x_\tau)$ are continuous, $\tau^k \uparrow \tau''(x) \land 2 R^2$ as $k \to \infty$. Furthermore, (a.s.) all the $\tau^k$’s equal $\tau''(x) \land 2 R^2$ for all large $k$.

For a domain $Q \subset \mathbb{R}^{d+1}$ we denote by $\gamma(s, y, Q)$ the first exit time of $(s + t, y + x_{s+t} - x_s)$ from $Q$ and obtain 

$$G_R(\Gamma, x) \geq \sum_{k=1}^{\infty} E \int_{\tau^k}^{\gamma^k} I_\Gamma(t, x + x_\tau) \, dt$$ 

$$= \sum_{k=1}^{\infty} E E \left( \int_0^{\gamma^k} I_\Gamma(s, x, x_{s+t} - x_s + y) \, dt \mid F_s \right) \bigg|_{i = i(s, y), y = x + x_\tau, s = \tau^k}.$$ 

We estimate the interior expectation from below by Lemma 3.3 and get that $G_R(\Gamma, x)/v_0$ is greater than or equal to 

$$\sum_{k=1}^{\infty} E \left( \int_0^{\gamma^k} I_{\Pi_\varepsilon}(s, x, x_{s+t} - x_s + y) \, dt \mid F_s \right) \bigg|_{i = i(s, y), y = x + x_\tau, s = \tau^k}$$ 

$$\geq \sum_{k=1}^{\infty} E \left( \int_0^{\gamma^k} I_{\Gamma_\varepsilon}(s, x, x_{s+t} - x_s + y) \, dt \mid F_s \right) \bigg|_{i = i(s, y), y = x + x_\tau, s = \tau^k}.$$
This proves Eq. 3.8.

**Case (ii).** Here the goal is to prove that

\[
G_R(\Gamma, x) \geq \mu_0 v \eta^n(q'' - q')^2, \quad |x| \leq \kappa,
\]

where \( v > 0 \) and \( n \geq 1 \) depend only on \( d, \delta, \tilde{R}, R, \) and \( \kappa \).

First we claim that for some \((t, x) \in \Gamma_\varepsilon^\prime\) it holds that \( t < (q' - q'' + 1)R^2 \). Indeed, otherwise

\[
\Gamma^\prime \subset C_R(R^2, 0) \subset C_{(q'' - q')}R^2, R((q' - q'' + 1)R^2, 0)
\]

and \(|\Gamma^\prime \setminus C_R(R^2, 0)| \leq (q'' - q')|C_R|\). It follows that there is a cylinder

\[
Q = C^\prime_R(s, y) \in \mathcal{B}
\]

such that \( Q' \) contains points in the half-space \( t < (q' - q'' + 1)R^2 \). Since \( q' < q'' \), we have \( q' - q'' + 1 < 1 \), and since \( Q' \) is adjacent to \( Q \subset C_R(R^2, 0) \), this implies that the height of \( Q' \) is at least \( (q'' - q')R^2 \), that is,

\[
\rho^2 \eta^{-1} \geq (q'' - q')R^2, \quad \rho^2 \geq \eta(q'' - q')R^2.
\]

(3.10)

On the other hand, \( Q \subset C_R(R^2, 0), s \geq R^2, \) and \( \rho < R \).

Moreover, by construction, \(|\Gamma \cap Q| \geq q_0|Q|\) and by Lemma 3.2 on the set where \(|z - y| \leq \rho/2\)

\[
I(z) := E \left( \int_0^\tau I_\varepsilon^\prime(s + t, z + x_{s+t} - x_s) dt \mid \mathcal{F}_s \right) \geq \mu_0 \rho^2 \geq \mu_0 \eta(q'' - q')R^2,
\]

where \( \tau \) is the first exit time of \((s + t, z + x_{s+t} - x_s)\) from \( C_\rho(s, y) \). Now by Theorem 2.1 for \( x \in B_{\kappa R} \)

\[
E \int_0^{\tau_\varepsilon^\prime(x) \wedge (2R^2)} I_\varepsilon^\prime(t, x + x_t) dt \geq E I_{\tau_\varepsilon^\prime(x) > s, |x + x_{s+t} - x_s| \leq \rho/2} I(x + x_s)
\]

\[
\geq \mu_0 \eta(q'' - q')R^2 P_x(\tau_\varepsilon^\prime(x) > s, |x + x_s - y| \leq \kappa \rho) \geq N^{-1}(\rho/R)^n \mu_0 \eta(q'' - q')R^2.
\]

This proves Eq. 3.9.

By combining the two cases (i) and (ii) we conclude that

\[
G_R(\Gamma, x) \geq \min \left( \mu_0, v_0 \mu_R(q''), \mu_0 v \eta^n(q'' - q')^2 \right), \quad |x| \leq \kappa R,
\]

and the arbitrariness of \( \Gamma \) allows us to conclude that

\[
\mu_R(q') \geq \min \left( \mu_0, v_0 \mu_R(q''), \mu_0 v \eta^n(q'' - q')^2 \right),
\]

(3.11)

whenever Eq. 3.7 holds. Observe that Eq. 3.11 is identical to (9.3.10) of [8] and by literally repeating what is in [8], just replacing \( \xi \) there with our \( \theta \), we come to Eq. 3.6 for any \( R \). By Lemma 3.4 the right-hand side in Eq. 3.6 can be taken the same for \( R \leq \tilde{R} \). The theorem is proved.

The following four results are derived from Theorem 3.5 in the same way as similar results are derived from Theorem 4.1 of [9].

**Corollary 3.6** For any \( \kappa \in (0, 1) \) there exists \( N \), depending only on \( \kappa, d, \delta, p_0, R, \tilde{R}, \) and \( B_R \), such that, for any \( R \in (0, \tilde{R}], x \in B_{\kappa R}, \) and closed set \( \Gamma \subset C_R(R^2, 0), \) the probability
that the process \((t, x + x_t)\) reaches \(\Gamma\) before exiting from \(C_{2R_2, R}\) is greater than or equal to \(N^{-1}(|\Gamma|/|C_R|)^{\mu-1/(d_0+1)}\):

\[
P(\tau^\Gamma(x) < \tau_{2R_2, R}(x)) \geq N^{-1}(|\Gamma|/|C_R|)^{\mu-1/(d_0+1)},
\]

(3.12)

where \(\tau^\Gamma(x)\) is the first time \((t, x + x_t)\) hits \(\Gamma\), \(\tau_{2R_2, R}(x)\) is the first exit time of \((t, x + x_t)\) from \(C_{2R_2, R}\), \(\mu = 1/\gamma\), and \(\gamma\) is taken from Theorem 3.5.

**Corollary 3.7** For any \(R \in (0, \bar{R})\), Borel nonnegative \(f\) vanishing outside \(C_R(R^2, 0)\), and \(x \in B_e R\)

\[
\int_{C_R(R^2, 0)} f^{1/(2\mu)}(t, y) \, dy \, dt \leq NR^{d+2-1/\mu} \left( E \int_0^{\tau_{2R_2, R}(x)} f(t, x + x_t) \, dt \right)^{1/(2\mu)},
\]

where \(N\) depends only on \(\kappa, d, \delta, p_0, R, \bar{R}, \tilde{b}_R\).

**Corollary 3.8** Let \(R \in (0, \bar{R}), \gamma \in (0, 1)\), and assume that a closed set \(\Gamma \subset B_R\) is such that, for any \(r \in (0, R)\), \(|B_r \cap \Gamma| \geq \gamma |B_r|\). Then there exist constants \(\alpha \in (0, 1)\) and \(N\), depending only on \(\kappa, d, \delta, p_0, R, \bar{R}, \tilde{b}_R, \gamma\), such that, for any \(x \in B_{R/2}\)

\[
P(\tau_R(x) < \tau^\Gamma(x)) \leq N(|x|/R)^\alpha,
\]

(3.13)

where \(\tau^\Gamma(x)\) is the first time \(x + x_t\) hits \(\Gamma\).

The fourth result has the same spirit as Theorem 4.11 of [9] and can be used in investigating the boundary behavior of solutions of parabolic equations with drift in \(L_{p,q}\).

We are going to use the following condition

\[
p, q \in [1, \infty], \quad \nu := 1 - \frac{d_0}{p} - \frac{1}{q} \geq 0,
\]

(3.14)

where \(d_0 \in (d/2, d)\), depending only on \(\delta, d, R, p_0\), is taken from [12].

**Theorem 3.9** Let Eq. 3.14 be satisfied with \(v = 0\), \(T \in (0, \infty)\), and let \(D\) be a bounded domain in \(\mathbb{R}^d\) with \(0 \in \partial D\). Assume that for some constants \(\rho, \gamma > 0\) and any \(r \in (0, \rho)\) we have \(|B_r \cap D| \geq \gamma |B_r|\). Then there exist \(\beta > 0\) and \(N\), depending only on \(d, \delta, p_0, R, \tilde{b}_\infty, \gamma\) with \(N\) also depending on \(\rho\) and the diameter of \(Q := [0, T] \times D\), such that, for any nonnegative \(f \in L_{p,q}(Q)\),

\[
u(x) := E \int_0^{\tau(x)} f(t, x + x_t) \, dt \leq N|x|^\beta \|f\|_{L_{p,q}(Q)},
\]

(3.15)

where \(\tau(x)\) is the first exit time of \((t, x + x_t)\) from \(Q\).

In the next section we will need the following fact of crucial importance, the origin of which lies in [14] and [16]. A few other related results below also have their origin in [14] and [16] where the drift is bounded.

**Theorem 3.10** Let \(\kappa, \eta, \xi, q \in (0, 1)\), \(R \in (0, \bar{R})\), \(T \in [\eta R^2, \eta^{-1} R^2]\), and closed \(\Gamma \subset C_{T,R}\) be such that \(|\Gamma \cap C_{T,R}((1 - \xi)T, 0)| \geq q|C_{T,R}|\). Then there exists \(\pi_0 = \pi_0(\kappa, \eta, \xi, q, d, \delta, p_0, R, \tilde{b}_R) > 0\), such that, for \((t, x) \in C_{(1-\xi)T, R}\),

\[
P(\tau^\Gamma(t, x) < \tau_{T,R}(t, x)) \geq \pi_0,
\]

(3.16)

where \(\tau^\Gamma(t, x)\) is the first time \((t + s, x + x_s)\) hits \(\Gamma\) and \(\tau_{T,R}(t, x)\) is its first exit time from \(C_{T,R}\).
Proof Observe that one can choose \( \rho \in (0, 1] \), depending only on \( d, \eta, \zeta, \) and \( q \), and one can find \((t^0, x^0) \in C_{T,R} \) with \( t^0 \geq \rho^2 R^2 + (1 - \zeta) T \) such that \( C_{\rho R}(t^0 + \rho^2 R^2, x^0) \subset C_{T,R} \) and \( |\Gamma \cap C_{\rho R}(t^0 + \rho^2 R^2, x^0)| \geq \bar{q}|C_{\rho R}| \), where \( \bar{q} > 0 \) depends only on \( d, \eta, \zeta, \) and \( q \). Then by Corollary 3.6, for \( t \in [P_t,x, \bar{P}_t,x] \) not preserved under such transformations if \( p \) is Markov and for any \((t, x) \in C_{2p^2 R^2, \rho R}(t^0, x^0) \) is estimated from below by a strictly positive constant depending only on \( \kappa, \bar{q}, d, \delta, p_0, R, \bar{R}, \) and \( \bar{h}_{\bar{R}} \). After that it only remains to invoke Theorem 2.1 recalling that \( t^0 \geq \rho^2 R^2 + (1 - \zeta) T \) and \( t < (1 - \zeta) T \). The theorem is proved. \( \square \)

4 The Case of Diffusion Processes

In this section, among other things, we generalize some recent results in [18] and extend them to processes with singular drift.

Fix a constant \( \delta \in (0, 1) \) and recall that by \( S_\delta \) we denote the set of \( d \times d \)-symmetric matrices whose eigenvalues are between \( \delta \) and \( \delta^{-1} \). In this section we impose the following.

Assumption 4.1. (i) On \( \mathbb{R}^{d+1} \) we are given Borel measurable \( \sigma(t, x) \) and \( b(t, x) \) with values in \( S_\delta \) and in \( \mathbb{R}^d \) respectively.

(ii) We are given \( p_0, q_0 \in [1, \infty) \) satisfying Eq. 1.1 and a function \( h(t, x) \) satisfying Eq. 2.1 and such that \( |b| \leq h \).

(iii) Assumption 2.2 is satisfied.

Let \( \Omega \) be the set of \( \mathbb{R}^{d+1} \)-valued continuous function \((t_0 + t, x_t), t_0 \in \mathbb{R}, \) defined for \( t \in [0, \infty) \). For \( \omega = \{(t_0 + t, x_t), t \geq 0\} \), define \( t_r(\omega) = t_0 + t, x_r(\omega) = x_t, \) and set \( \mathcal{N}_t = \sigma((t_s, x_s), s \leq t), \mathcal{N} = \mathcal{N}_\infty. \) In the following theorem which is Theorem 6.1 of [11] we use the terminology from [4].

Theorem 4.1 On \( \mathbb{R}^{d+1} \) there exists a strong Markov process

\[
X = \{(t_r, x_r), \infty, \mathcal{N}_t, P_{t,x}\}
\]

such that the process

\[
X_1 = \{(t_r, x_r), \infty, \mathcal{N}_{t+}, P_{t,x}\}
\]

is Markov and for any \((t, x) \in \mathbb{R}^{d+1} \) there exists a \( d \)-dimensional Wiener process \( w_t, t \geq 0 \), which is a Wiener process relative to \( \mathcal{N}_t \), where \( \mathcal{N}_t \) is the completion of \( \mathcal{N}_t \) with respect to \( P_{t,x} \), and such that with \( P_{t,x} \)-probability one, for all \( s \geq 0, t_r = t + s \) and

\[
x_s = x + \int_0^s \sigma(t + r, x_r) \, dw_r + \int_0^s b(t + r, x_r) \, dr. \quad (4.1)
\]

Remark 4.1. To be completely rigorous, to refer to [11] we should have \( b \in L_{p_0, q_0} \) (globally), and Eq. 2.1 is not needed. But with our \( b \), owing to Corollary 2.8 and Theorems 6.6, the arguments in [11] only simplify and do not require \( b \in L_{p_0, q_0} \). Still it is worth saying that the author believes that under only conditions in [11] Harnack’s inequality is true. Regarding the Hölder continuity of caloric functions in the same setting we have no guesses. The Hölder continuity seems to require some sort of self-similarity and the \( L_{p_0, q_0} \)-norm is not preserved under such transformations if \( p_0, q_0 \) are subject to Eq. 1.1.
Theorem 4.2 For any $\lambda \geq 1$, $p, q$ satisfying Eq. 3.14, and Borel nonnegative $f(t, x)$ and for
$$R_\lambda f(t, x) := E_{t, x} \int_0^\infty e^{-\lambda s} f(t + s, x_s) \, ds,$$
we have
$$\|R_\lambda f\|_{L^{p, q}(\mathbb{R}^{d+1})} \leq N^{\lambda^{-1}} \|f\|_{L^{p, q}(\mathbb{R}^{d+1})},$$
where $N = N(\delta, d, R, p, q, p_0, \bar{b}_\infty)$ and $\mathbb{R}^{d+1} = \mathbb{R}^d \cap \{t \geq 0\}$.

Proof By Theorem 6.5 we have
$$R_\lambda f(t, x) \leq N^{\lambda^{-1}} \eta \|\Psi_1^{1-\nu} f(t + \cdot, x + \cdot)\|_{L^{p, q}(\mathbb{R}^{d+1})},$$
where $\eta = \nu + (2d_0 - d)/(2p)$ and $\Psi_1(t, x) = \exp(-\sqrt{\lambda}(|x| + \sqrt{t})\bar{\xi}/16)$. The right-hand side here coincides with the right-hand side in (5.5) of [12] and by borrowing the result of arguments from there we get Eq. 4.2. The theorem is proved.

Definition 4.1. If $Q$ is a set in $\mathbb{R}^{d+1}$ and $u$ is a bounded Borel function on $Q$, we call it caloric (relative to the process $X$) if for any $(s, y)$ and $T, R \in (0, \infty)$ such that $\bar{C}_{T, R}(s, y) \subset Q$ and any $(t, x) \in C := C_{T, R}(s, y)$ we have
$$u(t, x) = E_{t, x} u(t + \tau_C, x_{\tau_C}),$$
where $\tau_C$ is the first exit time of $(t + s, x_s)$ from $C$.

First, we deal with Hölder norm estimates for caloric functions and potentials following [14] or [8]. If $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$, we define
$$\rho(z_1, z_2) = |x_1 - x_2| + |t_1 - t_2|^{1/2}$$
and call $\rho(z_1, z_2)$ the parabolic distance between $z_1$ and $z_2$.

Lemma 4.3 Let $R \in (0, \bar{R})$ and let $u$ be a caloric function in $\bar{C}_{2R}$. Then there exist constants $N$ and $\alpha_0 \in (0, 1)$, depending only on $\delta, d, p_0, R, \bar{R}, \bar{b}_{\bar{R}}$, such that, for any $\alpha \in (0, \alpha_0)$ and $z_1, z_2 \in C_R$, we have
$$|u(z_1) - u(z_2)| \leq NR^{-\alpha} \rho^\alpha(z_1, z_2) \sup_{\bar{C}_{2R}} |u|.$$
Furthermore, $\sup(|u|, \bar{C}_{2R})$ in Eq. 4.5 can be replaced by $\text{osc}(u, \bar{C}_{2R})$, where we use the notation
$$\text{osc}(g, \Gamma) = \sup_{\Gamma} g - \inf_{\Gamma} g.$$

Proof For $r$ such that $C_r \subset C_{2R}$, set
$$w(r) = \text{osc}(u, \bar{C}_r), \quad m(r) = \inf_{\bar{C}_r} u, \quad M(r) = \sup_{\bar{C}_r} u,$$
and
$$\mu(r) = (1/2)(m(r) + M(r)).$$
Take $r \leq R/2$ and suppose that
$$|C_{2r} \cap \{u \leq \mu(r)\}| \geq (1/2)|C_{2r}|.$$
Then there is a closed $\Gamma \subset C_{2r} \cap \{ u \leq \mu(r) \}$ such that

$$|C_{3r^2,r}(r^2, 0) \cap \Gamma| \geq (1/4)|C_{3r^2,r}|$$  \hspace{1cm} (4.6)

By Theorem 3.10 for any $(t, x) \in \tilde{C}_r$ we have

$$P_{t, x}(\tau_\Gamma < \tau_{2r}) \geq \pi_0,$$

where $\pi_0 > 0$ depends only on $\delta, d, p_0, R, \tilde{R}, b_R, \tau_\Gamma$ is the first time $(t + s, x_s)$ hits $\Gamma$, $\tau_{2r}$ is its first exit time from $C_{2r}$. Then by definition and the strong Markov property for $\tau = \tau_\Gamma \land \tau_{2r}$ we have

$$u(t, x) = E_{t, x} u(t + \tau_{2r}, x_{\tau_{2r}})$$

$$= E_{t, x} u(t + \tau_{2r}, x_{\tau_{2r}}) I_{\tau_\Gamma < \tau_{2r}} + E_{t, x} u(t + \tau_{2r}, x_{\tau_{2r}}) I_{\tau_\Gamma \geq \tau_{2r}}$$

$$= E_{t, x} u(t + \tau_\Gamma, x_{\tau_\Gamma}) I_{\tau_\Gamma < \tau_{2r}} + E_{t, x} u(t + \tau_{2r}, x_{\tau_{2r}}) I_{\tau_\Gamma \geq \tau_{2r}}$$

(we used that $\mu(r) \leq M(2r)$). It follows that

$$M(r) \leq \pi_0 \frac{1}{2} (m(r) + M(r)) + (1 - \pi_0) M(2r),$$

$$\left(1 - \frac{\pi_0}{2}\right) M(r) \leq \frac{\pi_0}{2} m(r) + (1 - \pi_0) M(2r).$$

Adding to this the obvious inequality

$$\left(\frac{\pi_0}{2} - 1\right) m(r) \leq -\frac{\pi_0}{2} m(r) + (\pi_0 - 1) m(2r),$$

we get

$$\left(1 - \frac{\pi_0}{2}\right) w(r) \leq (1 - \pi_0) w(2r), \quad w(r) \leq \varepsilon w(2r),$$

(4.7)

where $\varepsilon < 1, \varepsilon = \varepsilon(\pi_0)$. We may, certainly, assume that $\varepsilon > 1/2$.

We have proved (4.7) assuming that (4.6) is true. However if (4.6) is false, then $-u$ satisfies an inequality similar to Eq. 4.6 and this leads to (4.7) again.

Therefore, $w(r) \leq \varepsilon w(2r)$ for all $r \leq R/2$. Iterations then yield

$$w(r) \leq \varepsilon^2 w(4r) \quad \text{for} \quad r \leq R/4, \ldots, w(r) \leq \varepsilon^n w(2^n r) \quad \text{for} \quad r \leq 2^{-n} R.$$

If $r \leq R/2$ and we take $n := \lfloor \log_2 (R/r) \rfloor$, then $r \leq 2^{-n} R$ and

$$w(r) \leq \varepsilon^n w(2^n r) \leq \varepsilon^{-1} (r/R)^\alpha w(R) \leq 2 \varepsilon^{-1} (r/R)^\alpha \sup |u|, \tilde{C}_R,$$

where $\alpha = -\log_2 \varepsilon \in (0, 1)$. This provides an estimate of the oscillation of $u$ in any $C_r$ with $r \leq R/2$. The same estimate obviously holds for the oscillation of $u$ in any $C_r(t, x) \subset C_{2r}$ as long as $r \leq R/2$ and $(t, x) \in C_R$.

Now take $z_1 = (t_1, x_1), z_2 = (t_2, x_2) \in C_R$ such that $r := \rho(z_1, z_2) \leq R/2$ and define

$$t = t_1 \land t_2, \quad x = (x_1 + x_2)/2.$$

Then we have $z_i \in \tilde{C}_R(t, x), i = 1, 2$, and

$$|u(z_1) - u(z_2)| \leq 2 \varepsilon^{-1} (r/R)^\alpha \sup |u|, \tilde{C}_R(t, x)$$

$$\leq 2 \varepsilon^{-1} \rho^\alpha(z_1, z_2) R^{-\alpha} \sup |u|, \tilde{C}_2 R.$$

In the case that $\rho(z_1, z_2) \geq R/2$ we have

$$|u(z_1) - u(z_2)| \leq 2 \sup |u|, \tilde{C}_2$$

$$\leq 2^{1+\alpha} \rho^\alpha(z_1, z_2) R^{-\alpha} \sup |u|, \tilde{C}_2.$$

Thus, $N = 2^{1+\alpha} + 2 \varepsilon^{-1}$ in (4.5) is always a good choice with $\alpha$ found above. One can take any smaller $\alpha$ as well since $\rho(z_1, z_2) \leq N(d) R$. The lemma is proved.

\[ \square \]
Remark 4.2. The constant $N$ in Eq. 4.5, generally, depends on $\bar{R}$. However, if $b_\infty \leq \varepsilon$, where $\varepsilon > 0$ depends only on $d$ and $\delta$, then this constant is independent of $\bar{R}$. This is proved by using self-similar transformations which change the process but allow us to take any $R$ we wish. In such a situation the Liouville theorem is valid: If $u$ is bounded and caloric in $\mathbb{R}^{d+1}_+$, then $u$ is constant (just send $R \to \infty$ in Eq. 4.5).

Here is the statement of the Harnack inequality.

**Theorem 4.4** Let $R \in (0, \bar{R}]$, and let $u$ be a nonnegative caloric function in $\tilde{C}_{2R^2, R}$. Then there exists a constant $N$, which depends only on $\delta, d, R, \bar{R}, \mu_0, \bar{b}_R$, such that

$$u(R^2, 0) \leq Nu(0, x)$$

whenever $|x| \leq R/2$.

**Proof** We basically repeat the proof of Theorem 6.1 in [10] and, to exclude a trivial situation, additionally assume that

$$u(R^2, 0) > 0.$$ 

For $\kappa = 1/2, \eta = 1/2$, we take $N$ and $\nu$ from Theorem 2.1, call this $N_1$, and, having in mind Theorem 2.4, find $\gamma \in (0, 1)$ close to $1$ and $\varepsilon > 0$ close to zero, for which

$$1 - \varepsilon \geq q(\gamma)2^{-1} + (1 - q(\gamma))2^\nu. \quad (4.8)$$

Next, for $r \in [0, R)$, introduce

$$\mu(r) = u(R^2, 0)(1 - r/R)^{-\nu}, \quad n(r) = \sup\{u, \tilde{C}_r(R^2, 0)\} (n(0) = u(R^2, 0),$$

and define $r_0$ as the greatest number in $r \in [0, R)$ satisfying

$$n(r) = \mu(r).$$

Such a number does exist because $n(0) = \mu(0), \mu(r) \to \infty$ as $r \uparrow R$, and $n(r)$ is bounded, increasing, and (Hölder) continuous. Choose $(t^0, x^0) \in \tilde{C}_{r_0}(R^2, 0)$ such that

$$n(r_0) = u(t^0, x^0)$$

and consider the cylinder

$$Q := \{(t, x) : 0 \leq t \leq t^0 < \frac{(R - r_0)^2}{4}, |x - x^0| < \frac{R - r_0}{2}\}.$$ 

As is easy to see $\tilde{Q} \subset \tilde{C}_{r_1}(R^2, 0)$, where $r_1 = (R + r_0)/2$. By the definition of $r_0$, this implies that

$$\sup_{\tilde{Q}} u < \mu(r_1) = u(R^2, 0)\left(\frac{R - r_0}{2}\right)^{-\nu} \leq 2^\nu n(r_0).$$

We claim that owing to this and (4.8),

$$|Q \cap \{u > n(r_0)/2\}| \geq (1 - \gamma)|Q|. \quad (4.9)$$

To argue by contradiction, assume Eq. 4.9 is false. Then

$$|Q \cap \{u \leq n(r_0)/2\}| \geq \gamma|Q|$$

and there is a closed set $\Gamma \subset Q \cap \{u \leq n(r_0)/2\}$ such that $|\Gamma| > \gamma|Q|$. Introduce $\tau_\Gamma$ as the first time the process $(t^0 + s, x_t)$ hits $\Gamma$ and $\tau_{\tilde{Q}}$ as the first time it exits from $\tilde{Q}$. It follows by definition, the strong Markov property as in the proof of Lemma 4.3, and from Theorem 2.4 that (note that $n(r_0)/2 \leq \sup_{\tilde{Q}} u$)

$$u(t^0, x^0) = E_{t^0, x^0} I_{\tau_\Gamma < \tau_{\tilde{Q}}} u(t^0 + \tau_\Gamma, x_{\tau_\Gamma}) + E_{t^0, x^0} I_{\tau_\Gamma \geq \tau_{\tilde{Q}}} u(t^0 + \tau_{\tilde{Q}}, x_{\tau_{\tilde{Q}}})$$

via [Springer]
\[ \leq P_{\rho_0,x_0}(\tau_\Gamma < \tau_Q)n(r_0)/2 + (1 - P_{\rho_0,x_0}(\tau_\Gamma < \tau_Q)) \sup_{\tilde{Q}} u \]
\[ \leq q(\gamma)n(r_0)/2 + (1 - q(\gamma)) \sup_{\tilde{Q}} u \]
\[ \leq q(\gamma)n(r_0)/2 + (1 - q(\gamma))2^n n(r_0). \]

Owing to Eq. 4.8 we now have
\[ n(r_0) \leq (1 + \varepsilon)n(r_0) \left[ q(\gamma)2^{-1} + (1 - q(\gamma))2^{n} \right] \leq (1 - \varepsilon^2)n(r_0), \]
which is impossible. This proves (4.9).

Next we apply Theorem 3.10 and get that
\[ u(0, x) \geq \frac{1}{2}\pi_0 n(r_0) N^{-1} \left( \frac{R - r_0}{4} \right)^{\alpha} = 2^{-2}\pi_0 n N^{-1} u(4, 0). \]

The theorem is proved. \( \square \)

By using Lemma 4.3 and Theorem 6.7 one derives in three lines the following analog of Theorem 6.5 of [10].

**Theorem 4.5** Assume that Eq. 3.14 holds with \( \nu = 0 \). Let \( R \in (0, \tilde{R}/2] \) and let \( g \) be a Borel bounded function on \( \tilde{C}_{2R} \) and \( f \in L_{p,q}(C_{2R}) \). For \( (t, x) \in C_{2R} \) define
\[ u(t, x) = E_{t,x} \int_{0}^{\tau_{2R}} f(t + s, x_s) \, ds + E_{t,x} g(t + \tau_{2R}, x_{\tau_{2R}}), \]
where \( \tau_{2R} \) is the first exit time of \((t + s, x_s)\) from \( C_{2R} \). Then there exists a constant \( N \), which depends only on \( \delta, d, R, \tilde{R}, p, p_0, \) and \( \bar{b}_\infty \), such that
\[ |u(z_1) - u(z_2)| \leq N \left( R^{-\alpha} \rho^\alpha (z_1, z_2) \sup_{\tilde{C}_{2R}} |g| + R^{(2d_0 - d)/p} \|f\|_{L_{p,q}(C_{2R})} \right) \]
for \( z_1, z_2 \in C_R, \alpha \in (0, \alpha_0], \) and \( \alpha_0 \) is taken from Lemma 4.3.

By playing with \( R \) for fixed \( z_1, z_2 \in C_R \) as in the proof of Theorem 6.5 of [10] we get the following.

**Theorem 4.6** Under the conditions and notation from Theorem 4.5 there exists a constant \( N \), which depends only on \( \delta, d, R, \tilde{R}, p, p_0, \) and \( \bar{b}_\tilde{R} \), such that
\[ |u(z_1) - u(z_2)| \leq N R^{-\beta} \rho^\beta (z_1, z_2) \left( \sup_{\tilde{C}_{2R}} |u| + R^{(2d_0 - d)/p} \|f\|_{L_{p,q}(C_{2R})} \right) \]
for \( z_1, z_2 \in C_R \), where
\[ \beta = \frac{\alpha_0 (2d_0 - d)}{\alpha_0 \rho + 2d_0 - d}. \]

As a standard consequence of just continuity of \( u \) we have the following.

**Theorem 4.7** The process
\[ X_1 = \{(t, x_t), \infty, N_{t+}, P_{t,x}\} \]
is strong Markov.
5 Applications

Here we suppose that Assumption 4.1 is satisfied and set \( a = \sigma^2 \),

\[ Lu(t, x) = (1/2)\partial_{ij}^2 (t, x) D_{ij} u(t, x) + b^j(t, x) D_j u(t, x). \]

**Theorem 5.1** Let \( R \in (0, \tilde{R}/2) \) and assume that Eq. 3.14 holds with \( v = 0, p < \infty, q < \infty \) and that we are given a function \( u \in W^{1,2}_{p,q,\text{loc}}(C_{2R}) \cap C(\overline{C_R}) \). Then for \( -f = \partial_t u + Lu \) we have

\[ |u(z_1) - u(z_2)| \leq N R^{-\beta} p^\beta(z_1, z_2) \left( \sup_{C_{2R}} |u| + R^{(2d_0-d)/p} \| f \|_{L_{p,q}(C_{2R})} \right) \]  

(5.1)

for \( z_1, z_2 \in C_R \), where \( N \) and \( \beta \) are taken from Theorem 4.6.

**Proof** Approximating \( C_{2R} \) by \( C_{2R-\varepsilon} \) we see that we may assume that \( u \in W^{1,2}_{p,q,\text{loc}}(C_R) \cap C(\overline{C_R}) \). This gives us the opportunity to replace \( L \) in the definition of \( f \) with \( L_n := I_{|\theta| \leq n} \Delta + I_{|\theta| < n} L \) and then pass to the limit by the dominated convergence and monotone convergence theorems. Hence, we may assume that \( b \) is bounded. After that it only remains to use Itô’s formula (Theorem 6.8) for the Markov process from Section 4 (cf. Eq. 4.1) to see that \( u \) has form Eq. 4.10 for which Eq. 4.12 is valid. The theorem is proved. \( \square \)

**Remark 5.1.** A consequence of Theorem 5.1 is a rather weak statement that any \( u \in W^{1,2}_{p,q,\text{loc}}(C_R) \) admits a modification which is in \( C^\beta_{\text{loc}}(C_R) \).

Indeed, if \( u \in W^{1,2}_{p,q,\text{loc}}(C_R) \) then its mollifiers will belong to \( W^{1,2}_{p,q}(C_{R-\varepsilon}) \) and by Eq. 4.12 with \( L = \Delta \) will be in \( C^\beta(C_{R-\varepsilon}/2) \). Passing from the mollifiers to the function itself we find the modification in question in \( C_{R/2} \). After that scaling and shifting the origin takes care of the rest of \( C_R \).

**Remark 5.2.** If \( u \) is bounded in \( C_{2R} \), belongs to \( W^{1,2}_{p,q,\text{loc}}(C_{2R}) \), and is caloric (\( \partial_t u + Lu = 0 \)) in \( C_{2R} \), then Theorem 5.1 implies that it is Hölder continuous in \( C_R \) with the exponent and constant independent of any regularity of \( a \) and \( b \). This fact along with Harnack’s inequality was first proved in [14] for bounded \( b \) and \( p = q = d + 1 \) in the parabolic case and \( p = d \) in the elliptic case. They were generalized by Cabré [2], Escauriaza [5], and Fok [6] in the elliptic case when \( p < d \) (close to \( d \)) again when \( b \) is bounded. In [3] parabolic version of these results, extending some earlier results by Wang, are given for \( L_p \)-viscosity solutions with \( p < d + 1 \) (close to \( d + 1 \)) when \( b \) is bounded. In our situation we have some freedom in choosing \( p, q \) and \( b \in L_{p_0,q_0} \), but we only treat true solutions.

**Theorem 5.2** Let \( R \in (0, \tilde{R}) \) and assume that Eq. 3.14 holds with \( v = 0, p < \infty, q < \infty \). Let \( u \in W^{1,2}_{p,q,\text{loc}}(C_{2R^2} \cap C(\overline{C_{2R^2}})) \) be such that \( u > 0 \) on \( \partial C_{2R^2} \). Then there exists a constant \( N \), which depends only on \( \delta, d, \tilde{R}, R, p, p_0, \) and \( \tilde{b}_{\tilde{R}} \), such that

\[ u(R^2, 0) \leq N u(0, x) + N R^{(2d_0-d)/p} \| f \|_{L_{p,q}(C_{2R^2})} \]

whenever \( |x| \leq R/2 \), where \( -f = \partial_t u + Lu \). In particular, if \( \partial_t u + Lu = 0 \) in \( C_{2R^2} \) (a.e.), then (Harnack’s inequality)

\[ u(R^2, 0) \leq N u(0, x). \]

**Proof** As in the proof of Theorem 5.1, the general case is reduced to the one in which \( b \) is bounded and \( u \in W^{1,2}_{p,q}(C_{2R^2} \cap C(\overline{C_{2R^2}})) \). In that case, as in the proof of Theorem 5.1,
by Itô’s formula for the Markov process from Section 4

\[ u(t, x) = h(t, x) + F(t, x), \tag{5.2} \]

where

\[ h(t, x) = E_{t,x} u(t + \tau, x_\tau) \geq 0, \quad F(t, x) = E_{t,x} \int_0^\tau f(t + s, x_s) \, ds, \]

and \( \tau \) is the first exit time of \((t + s, x_s)\) from \(C_{2R^2, R}\). By Theorem 6.4, \( h(R^2, 0) \leq Nh(0, x) \) and it only remains to use Theorem 6.7 to estimate \( F \). The theorem is proved.

Here is a generalization of the Fanghua Lin estimate for operators with summable drift which is one of the main tools in the Sobolev space theory of fully nonlinear parabolic equations (see, for instance, [8]).

**Theorem 5.3** Let \( R \in (0, \bar{R}] \), \( p, q \) satisfy Eq. 3.14 with \( v = 0 \), \( p < \infty, q < \infty \). Let \( u \in W^{1,2}_{p,q,\text{loc}}(C_R) \cap C(\bar{C}_R) \), and \( c \in L_{p,q}(C_R), c \geq 0 \). Then

\[
\left( \int_{C_R} \left( |D^2u| + (|b| + R^{-1})|Du| \right)^{1/(2\mu)} \, dx dt \right)^{2\mu} \leq NR^{-d/p-2/q} \| f \|_{L_{p,q}(CR)} + NR^{-2} \sup_{\partial{\bar{C}_R}} |u|, \tag{5.3} \]

where \( f = \partial_t u + Lu - cu \), \( \mu \) is taken from Corollary 3.7 with \( \kappa = 1/2 \) and \( N \) depends only on \( d, \delta, \bar{R}, p, p_0, R^{2-d/p-2/q} \| c \|_{L_{p,q}(C_R)}, \bar{b}_R, \bar{b}_{\bar{R}}, \) and the function \( \hat{N}(d, p_0, \cdot) \) (see Eq. A.2).

**Proof** On the account of moving \( R \), we may assume that \( u \in W^{1,2}_{p,q}(C_R) \). After that we observe that in light of Theorem 6.9

\[
\| \partial_t u + Lu \|_{L_{p,q}(C_R)} \leq \| \partial_t u + Lu - cu \|_{L_{p,q}(C_R)} + \| c \|_{L_{p,q}(C_R)} \sup_{\partial{\bar{C}_R}} |u| \]

\[
\leq \left( 1 + NR^{-d/p-2/q} \| c \|_{L_{p,q}(C_R)} \right) \| \partial_t u + Lu - cu \|_{L_{p,q}(C_R)} + \| c \|_{L_{p,q}(C_R)} \sup_{\partial{\bar{C}_R}} |u| \]

and reduce the case of general \( c \) to the one with \( c = 0 \). As a few times before we may assume that \( b \) is bounded and then using approximations we see that assuming that \( u \in C^{1,2}(\bar{C}_R) \) and that the coefficients \( a^{ij} \) are infinitely differentiable do not restrict generality. In that case introduce \( L'(t, x) = L(t, x) \) for \( t \geq 0 \) and \( L'(t, x) = L(-t, x) \) for \( t < 0 \) and introduce \( v \) as a unique \( W^{1,2}_{d+1}(C_{2R^2, R}(\bar{R}, 0)) \)-solution of the equation

\[
\partial_t v + L'v = -f I_{C_R} \]

with boundary condition \( v = u \) on \( \{ t \geq 0 \} \cap \partial{\bar{C}_2R^2, R}(\bar{R}, 0) \) and \( v(t, x) = u(-t, x) \) on \( \{ t \leq 0 \} \cap \partial{\bar{C}_2R^2, R}(\bar{R}, 0) \). Owing to uniqueness \( v = u \) in \( C_R \) and by Theorem 6.9 in \( C_{2R^2, R}(\bar{R}, 0) \) we have

\[
|v| \leq NR^{-d/p-2/q} \| f \|_{L_{p,q}(C_R)} + \sup_{\partial{\bar{C}_R}} |u|. \tag{5.4} \]

Next, it is easy to see that for sufficiently small \( \varepsilon > 0 \), depending only on \( \delta \) and the function \( \hat{N}(d, p_0, \cdot) \), we have that \( \hat{a} := a - \varepsilon I_{C_R}(D_{ij} u)/|D^2 u| \in \mathbb{S}((\delta^2 - \varepsilon)/2) \) and

\[
\hat{N}(d, p_0, (\delta^2 - \varepsilon)^{1/2}) \bar{b}_R < 1 \tag{5.5} \]
(see Eq. A.2 and Assumption 2.2). Furthermore, for \( \tilde{b} = b - \varepsilon I_{CR}(|b| + 1)Du/|Du| \) any \( \rho > 0 \) and \((t, x) \in \mathbb{R}^{d+1}\) we have
\[
\|\tilde{b}\|_{L^{p_0,q_0}(C_p(t,x))} \leq (1 + \varepsilon)\|b\|_{L^{p_0,q_0}(C_p(t,x))} + \varepsilon N(d)(\rho \wedge \bar{R}).
\]
It follows that
\[
\|\tilde{b}\|_{q_0}^{q_0} L^{p_0,q_0}(C_p(t,x)) \leq \left(1 + \frac{\varepsilon}{q_0}\right)\|b\|_{L^{p_0,q_0}(C_p(t,x))} + N\varepsilon \rho.
\]
where \( N \) depends only on \( d, p_0, \) and \( \bar{R} \). It is seen that for a \( \varepsilon > 0 \), depending only on \( d, \delta, p_0, \) and the function \( \bar{N}(d,p_0,\cdot) \), not only Eq. 5.5 is satisfied but also
\[
\bar{N}(d,p_0,(\delta^2 - \varepsilon)^{1/2})(1 + \varepsilon)\frac{\bar{b}_1}{q_0}R^{1/q_0} + N\varepsilon < 1.
\]
Therefore the above theory is applicable to the operator
\[
\hat{L} = (1/2)\hat{a}^{i j}D_{i j} + \hat{b}^{i}D_{i}.
\]
Then set \( \hat{\sigma} = \frac{1}{2}\hat{a}^{i j}D_{i j} \) and consider the diffusion process \((-R^2 + t, x_t), t \geq 0\), starting from \((-R^2, 0)\) with diffusion matrix \( \hat{\sigma} \) and drift \( \hat{b} \). By Itô’s formula
\[
v(-R^2, 0) = -E \int_{0}^{\tau} (\partial_{t}v + \hat{L}v)(-R^2 + t, x_t) dt + Ev(-R^2 + \tau, x_\tau),
\]
where \( \tau \) is the first exit time of \((-R^2 + t, x_t)\) from \( C_{R^2,R}(-R^2, 0) \). Obviously,
\[
|Ev(-R^2 + \tau, x_\tau)| \leq \sup_{\partial^c C_{R^2,R}} |u|.
\]
Furthermore, on \( C_{R^2,R} \) we have
\[
\partial_{t}v + \hat{L}v = \partial_{t}u + \hat{L}u = -f - \varepsilon|D^2u| - \varepsilon(|b| + 1)|Du|.
\]
In the remaining part of \( C_{R^2,R}(-R^2, 0) \) we have \( \partial_{t}v + \hat{L}v = 0 \). It follows that
\[
E \int_{0}^{\tau} I_{C_{R^2,R}}(|D^2u| + (|b| + 1)|Du|)(-R^2 + t, x_t) dt
\]
\[
\leq v(-R^2, 0) - E \int_{0}^{\tau} I_{C_{R^2,R}} f(-R^2 + t, x_t) dt + \sup_{\partial^c C_{R^2,R}} |u|.
\]
After that it only remains to recall Eq. 5.4 and apply Theorem 6.9 and Corollary 3.7. The theorem is proved.

\[\square\]

**Appendix**

Here we present without proofs some results from [12] frequently used in the main text.

Set
\[
\tau'_R(x) = \inf\{t \geq 0 : x + x_t \notin B_R\}, \quad \gamma_R(x) = \inf\{t \geq 0 : x + x_t \in \bar{B}_R\}.
\]  \hspace{1cm} (A.1)

**Theorem A.1** (Theorem 2.3) *There are constants \( \tilde{\xi} = \tilde{\xi}(d, \delta) \in (0, 1) \) and \( \tilde{N} = \tilde{N}(d, p_0, \delta) \) continuously depending on \( \delta \) such that if, for an \( R \in (0, \infty) \), we have
\[
\tilde{N}R^2 \leq 1,
\]  \hspace{1cm} (A.2)
then for \( |x| \leq R \)
\[
P(\tau_R(x) = R^2) \leq 1 - \tilde{\xi}, \quad P(\tau_R = R^2) \geq \tilde{\xi}.
\]  \hspace{1cm} (A.3)
Moreover for \( n = 1, 2, \ldots \) and \(|x| \leq R\)

\[
P(\tau^n_R(x) \geq nR^2) = P(\tau^n_{nR^2,R}(x) = nR^2) \leq (1 - \tilde{\xi})^n,
\]

so that \( E\tau^n_R(x) \leq N(d, \delta)R^2 \).

Furthermore, for any \( x \in \overline{B}_{R/16} \)

\[
P(\tau^n_R(x) > \frac{\gamma R}{16}(x)) \geq \tilde{\xi}. \tag{A.5}
\]

**Theorem A.2** (Theorem 2.6) For any \( \lambda, R > 0 \) we have

\[
E e^{-\lambda \tau_R} \leq e^{\frac{\tilde{\xi}}{2} - \sqrt{\lambda}R\tilde{\xi}/2} = \begin{cases} 
  e^{\frac{\tilde{\xi}}{2} - \sqrt{\lambda}R\tilde{\xi}/2} & \text{if } \lambda \geq \hat{\lambda} \\
  e^{\frac{\tilde{\xi}}{2} - \lambda R\tilde{\xi}/2} & \text{if } \lambda \leq \hat{\lambda},
\end{cases}
\]

where

\[
\hat{\lambda} = \lambda \min(1, \lambda/\lambda), \quad \tilde{\lambda} = R^{-2}.
\]

In particular, for any \( R > 0 \) and \( t \leq \frac{RR\tilde{\xi}}{4} \) we have

\[
P(\tau_R \leq t) \leq e^{\frac{\tilde{\xi}}{2}} \exp\left(-\frac{\tilde{\xi}^2 R^2}{16t}\right). \tag{A.7}
\]

**Theorem A.3** (Theorem 2.9) Let \( R \in (0, \overline{R}] \), \( x, y \in \mathbb{R}^d \) and \( |x - y| \geq 3R \). For \( r > 0 \) denote by \( S_r(x, y) \) the open convex hull of \( B_r(x) \cup B_r(y) \). Then there exist \( T_0, T_1 \), depending only on \( \tilde{\xi} \), such that \( 0 < T_0 < T_1 < \infty \) and the probability \( \pi \) that \( x + xt \) will reach \( \overline{B}_{R/16}(y) \) before exiting from \( S_R(x, y) \) and this will happen on the time interval \([nT_0R^2, nT_1R^2]\) is greater than \( \pi_0^n \), where

\[
n = \left\lfloor \frac{16|x - y| + R}{4R} \right\rfloor
\]

and \( \pi_0 = \tilde{\xi}/3 \).

**Theorem A.4** (Theorem 4.3) There exists \( d_0 \in (1, d) \), depending only on \( \delta, d, R, p_0 \), such that for any \( p \geq d_0 + 1 \) and \( \lambda > 0 \)

\[
\int_0^\infty \int_{\mathbb{R}^d} G_\lambda^{p/(p-1)}(t, x) \, dx \, dt \leq N(\delta, d, R, p_0, \lambda, p).
\]

Furthermore, the above constant \( N(\delta, d, R, p_0, \lambda, p) \) can be taken in the form

\[
N(\delta, d, R, p_0, \lambda, p) = \hat{\lambda}_p^{(d+2)/(2p-2)},
\]

where

\[
\hat{\lambda}_p = \lambda(1 + \lambda)^{d/(2p-d-2)}.
\]

**Theorem A.5** (Theorem 4.8) Suppose

\[
p, q \in [1, \infty], \quad \nu := 1 - \frac{d_0}{p} - \frac{1}{q} > 0. \tag{A.8}
\]

Then there is \( N = N(\delta, d, R, p, q, p_0, \overline{b}_\infty) \) such that for any \( \lambda > 0 \) and Borel nonnegative \( f \) we have

\[
E \int_0^\infty e^{-\lambda t} f(t, x_t) \, dt \leq N\hat{\lambda}_{d_0+1}^{\nu+(d-2d_0)/(2p)} \| \Psi_1^{-\nu} f \|_{L_{p,q}(\mathbb{R}^{d+1})}. \tag{A.9}
\]
where \( \Psi_\lambda(x, \xi) = \exp(-\sqrt{\lambda}(|x| + \sqrt{t})\xi/16) \). In particular, if \( f \) is independent of \( t, p \geq d_0, \) and \( q = \infty \)

\[
E \int_0^\infty e^{-\lambda t} f(x_t) \, dt \leq N^{\lambda_{d_0+1}^d/(2p)} \| \Psi_\lambda \|_{L^p(\mathbb{R}^d)},
\]

where \( \Psi_\lambda(x) = \exp(-\sqrt{\lambda}|x|\xi/16) \).

**Theorem A.6** (Theorem 4.9) Assume that Eq. A.8 holds. Then

(ii) for any \( n = 1, 2, \ldots, \) nonnegative Borel \( f \) on \( \mathbb{R}^d_+ \), and \( T \leq 1 \) we have

\[
E \left[ \int_0^T f(t, x_t) \, dt \right]^n \leq n! N^n T^n \lambda_{n/\lambda_{1/\lambda}} \| \Psi_{1/\lambda} \|_{L^p(\mathbb{R}^d)}^n,
\]

where \( N = N(\delta, d, R, p, q, p_0, \beta_\infty) \) and \( \lambda = \nu + (2d_0 - d)/(2p) \);

(ii) for any nonnegative Borel \( f \) on \( \mathbb{R}_+^d \), and \( T \geq 1 \) we have

\[
I := E \int_0^T f(t, x_t) \, dt \leq NT^{1-1/q} \| \Psi_{1/\lambda} \|_{L^p(\mathbb{R}^d)}^n,
\]

where \( N = N(\delta, d, R, p, q, p_0, \beta_\infty) \).

**Theorem A.7** (Theorem 4.10) Assume that Eq. A.8 holds with \( \nu = 0 \). Then for any \( R \in (0, \bar{R}) \), \( x \), and Borel nonnegative \( f \) given on \( C_R \), we have

\[
E \int_0^{\tau_R(x)} f(t, x + x_t) \, dt \leq N R^{(2d_0 - d)/p} \| f \|_{L^p(\mathbb{R})},
\]

where \( N = N(\delta, d, R, p, q, p_0, \beta_\bar{R}, \bar{R}) \).

**Theorem A.8** (Theorem 4.11) Assume that Eq. A.8 holds with \( \nu = 0 \) and \( p < \infty, q < \infty \). Let \( Q \) be a bounded domain in \( \mathbb{R}_+^d \), \( 0 \in Q \), \( b \) be bounded, and \( u \in W_{1,2}^{1,2}(Q) \cap C(\bar{Q}) \). Then, for \( \tau \) defined as the first exit time of \( (t, x_t) \) from \( Q \) and for all \( t \geq 0 \),

\[
u(t \wedge \tau, x_{t \wedge \tau}) = u(0, 0) + \int_0^{t \wedge \tau} D_i u(s, x_s) \, dm^i_s
\]

\[
+ \int_0^{t \wedge \tau} \left[ \partial_i u(s, x_s) + a_{ij}^s D_{ij} u(s, x_s) + b_j^s D_j u(s, x_s) \right] \, ds
\]

and the stochastic integral above is a square-integrable martingale.

**Theorem A.9** (Theorem 5.1) Let \( 0 < R \leq \bar{R} \), domain \( Q \subset C_R \), and assume that Eq. 3.14 holds with \( \nu = 0 \), \( p < \infty, q < \infty \), and that we are given a function \( u \in W_{1,2}^{1,2}(Q) \cap C(\bar{Q}) \). Take a function \( c \geq 0 \) on \( Q \). Then on \( Q \)

\[
u \leq N R^{(2d_0 - d)/p} \| I_{Q \leq 0} (\partial_i u + Lu - cu) \|_{L^p(\mathbb{Q})} + \sup_{\partial'Q} u,
\]

where \( N = N(\delta, d, R, \bar{R}, p, p_0, \beta_\infty) \) and \( \partial'Q \) is the parabolic boundary of \( Q \). In particular (the maximum principle), if \( \partial_i u + Lu - cu \geq 0 \) in \( Q \) and \( u \leq 0 \) on \( \partial'Q \), then \( u \leq 0 \) in \( Q \).

**Lemma A.10** (Lemma 2.2) We have

\[
A := E \tau_R(x) \leq R^2,
\]
and, assuming that Eq. A.8 holds with \( v = 0 \), for any Borel nonnegative \( f \) we have

\[
E \int_0^{\tau_R(x)} f(t, x_t) \, dt \leq N(d, p_0, \delta) (1 + \tilde{b}_R)^{d/p} R^{d/p} \| f \|_{L_{p,q}}, \tag{A.16}
\]

**Corollary A.11** (Corollary 2.10) Let \( R \leq \tilde{R}, \kappa \in (0, 1), \text{and } |x| \leq \kappa R. \) Then for any \( T > 0 \)

\[
NP(\tau'_R(x) > T) \geq e^{-vT/(1-\kappa)R^2}, \tag{A.17}
\]

where \( N \) and \( v > 0 \) depend only on \( \tilde{\xi} \).

**Corollary A.12** (Corollary 2.7) Let \( \Lambda_1 \in (0, \infty) \). Then there is a constant \( N = N(R, R, \Lambda, \tilde{\xi}) \) such that for any \( R \in (0, \tilde{R}), \lambda \in [0, \Lambda] \)

\[
NE \tau_R \geq R^2, \quad NE \int_0^{\tau_R} e^{-\lambda t} \, dt \geq R^2. \tag{A.18}
\]

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