A FIRST-ORDER THEORY OF ULM TYPE

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Abstract. The class of abelian $p$-groups are an example of some very interesting phenomena in computable structure theory. We will give an elementary first-order theory $T_p$ whose models are each bi-interpretable with the disjoint union of an abelian $p$-group and a pure set (and so that every abelian $p$-group is bi-interpretable with a model of $T_p$) using computable infinitary formulas. This answers a question of Knight by giving an example of an elementary first-order theory of “Ulm type”: Any two models, low for $\omega_1^{CK}$, and with the same computable infinitary theory, are isomorphic. It also gives a new example of an elementary first-order theory whose isomorphism problem is $\Sigma_1^1$-complete but not Borel complete.

1. Introduction

The class of abelian $p$-groups is a well-studied example in computable structure theory. A simple compactness argument shows that abelian $p$-groups are not axiomatizable by an elementary first-order theory, but they are definable by the conjunction of the axioms for abelian $p$-groups (which are first-order $\forall \exists$ sentences) and the infinitary $\Pi^0_2$ sentence which says that every element is torsion of order some power of $p$.

Abelian $p$-groups are classifiable by their Ulm sequences [Ulm33]. Due to this classification, abelian $p$-groups are examples of some very interesting phenomena in computable structure theory and descriptive set theory. We will define a theory $T_p$ whose models behave like the class of abelian $p$-groups, giving a first-order example of these phenomena. In particular, Theorem 1.6 below answers a question of Knight.

1.1. Infinitary Formulas. The infinitary logic $L_{\omega_1 \omega}$ is the logic which allows countably infinite conjunctions and disjunctions but only finite quantification. If the conjunctions and disjunctions of a formula $\varphi$ are all over computable sets of indices for formulas, then we say that $\varphi$ is computable. We use $\Sigma^m_\alpha$ and $\Pi^m_\alpha$ to denote the classes of all infinitary $\Sigma_\alpha$ and $\Pi_\alpha$ formulas respectively. We will also use $\Sigma^c_\alpha$ and $\Pi^c_\alpha$ to denote the classes of computable $\Sigma_\alpha$ and $\Pi_\alpha$ formulas, where $\alpha < \omega_1^{CK}$ the least non-computable ordinal. See Chapter 6 of [AK00] for a more complete description of computable formulas.

1.2. Bi-Interpretability. One way in which we will see that the models of $T_p$ are essentially the same as abelian $p$-group is using bi-interpretations using infinitary formulas [Mon, HTMMM, HTMM]. A structure $A$ is infinitary interpretable in a structure $B$ if there is an interpretation of $A$ in $B$ where the domain of the interpretation is allowed to be a subset of $B^{<\omega}$ and where all of the sets in the interpretation are definable using infinitary formulas. This differs from the classical notion of interpretation, as in model theory [Mar02, Definition 1.3.9], where the
domain is required to be a subset of $B^n$ for some $n$, and the sets in the interpretation are first-order definable.

**Definition 1.1.** We say that a structure $A = (A; P^A_0, P^A_1, \ldots)$ (where $P^A_i \subseteq A^{n(i)}$) is infinitary interpretable in $B$ if there exists a sequence of relations $(\text{Dom}_A^B, \sim, R_0, R_1, \ldots)$, definable using infinitary formulas (in the language of $B$, without parameters), such that

1. $\text{Dom}_A^B \subseteq B^{\omega}$,
2. $\sim$ is an equivalence relation on $\text{Dom}_A^B$,
3. $R_i \subseteq (B^{\omega})^{a(i)}$ is closed under $\sim$ within $\text{Dom}_A^B$.

and there exists a function $f_A^B : \text{Dom}_A^B \to A$ which induces an isomorphism:

$$(\text{Dom}_A^B / \sim; R_0 / \sim, R_1 / \sim, \ldots) \cong (A; P^A_0, P^A_1, \ldots),$$

where $R_i / \sim$ stands for the $\sim$-collapse of $R_i$.

Two structures $A$ and $B$ are infinitary bi-interpretable if they are each effectively interpretable in the other, and moreover, the composition of the interpretations—i.e., the isomorphisms which map $A$ to the copy of $A$ inside the copy of $B$ inside $A$, and $B$ to the copy of $B$ inside the copy of $A$ inside $B$—are definable.

**Definition 1.2.** Two structures $A$ and $B$ are infinitary bi-interpretable if there are infinitary interpretations of each structure in the other as in Definition 1.1 such that the compositions

$$f_A^B \circ f_B^A : \text{Dom}_B^{\text{Dom}_A^B} \to A$$

and

$$f_A^B \circ f_B^A : \text{Dom}_A^{\text{Dom}_B^B} \to A$$

are definable in $B$ and $A$ respectively. (Here, we have $\text{Dom}_B^{\text{Dom}_A^B} \subseteq (\text{Dom}_A^B)^{\omega}$, and $f_A^B : (\text{Dom}_B^B)^{\omega} \to A^{\omega}$ is the obvious extension of $f_A^B : \text{Dom}_A^B \to A$ mapping $\text{Dom}_B^{\text{Dom}_A^B}$ to $\text{Dom}_B^B$.)

If we ask that the sets and relations in the interpretation (or bi-interpretation) be (uniformly) relatively intrinsically computable, i.e., definable by both a $\Sigma^0_1$ formula and a $\Pi^0_1$ formula, then we say that the interpretation (or bi-interpretation) is effective. Any two structures which are effectively bi-interpretable have all of the same computability-theoretic properties; for example, they have the same degree spectra and the same Scott rank. See [Mon, Lemma 5.3].

Here, we will use interpretations which use (lightface) $\Delta^0_2$ formulas. It is no longer true that any two structures which are $\Delta^0_2$-bi-interpretable have all of the same computability-theoretic properties, but it is true, for example, that any two such structures either both have computable, or both have non-computable, Scott rank.

**Theorem 1.3.** Each abelian $p$-group is effectively bi-interpretable with a model of $T_p$. Each model of $T_p$ is $\Delta^0_2$-bi-interpretable with the disjoint union of an abelian $p$-group and a pure set.

This theorem will follow from the constructions in Sections 3 and 4. Given a model $\mathcal{M}$ of $T_p$, $\mathcal{M}$ is bi-interpretable with an abelian $p$-group $G$ and a pure set. The domain of the copy of $G$ inside of $\mathcal{M}$ is definable by a $\Sigma^0_1$ formula but not by a $\Pi^0_1$ formula. This is the only part of the bi-interpretation which is not effective.
1.3. Classification via Ulm Sequences. Let $G$ be an abelian group. For any ordinal $\alpha$, we can define $p^\alpha G$ by transfinite induction:

- $p^0 G = G$;
- $p^{\alpha+1} G = p(p^\alpha G)$;
- $p^{\beta} G = \bigcap_{\alpha < \beta} p^\alpha G$ if $\beta$ is a limit ordinal.

These subgroups $p^\alpha G$ form a filtration of $G$. This filtration stabilizes, and we call the smallest ordinal $\alpha$ such that $p^\alpha G = p^{\alpha+1} G$ the length of $G$. We call the intersection $p^\infty G$ of these subgroups, which is a $p$-divisible group, the $p$-divisible part of $G$. Any countable $p$-divisible group is isomorphic to some direct product of the Pr"ufer group

$$\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p, 1/p^2, 1/p^3, \ldots]/\mathbb{Z}.$$  

Denote by $G[p]$ the subgroup of $G$ consisting of the $p$-torsion elements. The $\alpha$th Ulm invariant $u_\alpha(G)$ of $G$ is the dimension of the quotient

$$\left(p^\alpha G\right)[p] / \left(p^{\alpha+1} G\right)[p]$$

as a vector space over $\mathbb{Z}/p\mathbb{Z}$.

**Theorem 1.4** (Ulm’s Theorem, see [Fuc70]). Let $G$ and $H$ be countable abelian $p$-groups such that for every ordinal $\alpha$ their $\alpha$th Ulm invariants are equal, and the $p$-divisible parts of $G$ and $H$ are isomorphic. Then $G$ and $H$ are isomorphic.

1.4. Scott Rank and Computable Infinitary Theories. Scott [Sco65] showed that if $M$ is a countable structure, then there is a sentence $\varphi$ of $L_{\omega_1\omega}$ such that $M$ is, up to isomorphism, the only countable model of $\varphi$. We call such a sentence a Scott sentence for $M$. There are many different definitions [AK00, Sections 6.6 and 6.7] of the Scott rank of $M$, which differ only slightly in the ranks they assign. The one we will use, which comes from [Mon15], defines the Scott rank of $A$ to be the least ordinal $\alpha$ such that $A$ has a $\Pi^0_{\alpha+1}$ Scott sentence. We denote the Scott rank of a structure $A$ by $\text{SR}(A)$. It is always the case that $\text{SR}(A) \leq \omega^A_1 + 1$ [Nad74]. We could just as easily use any of the other definitions of Scott rank; for all of these definitions, given a computable structure $A$:

1. $A$ has computable Scott rank if and only if there is a computable ordinal $\alpha$ such that for all tuples $\bar{a}$ in $A$, the orbit of $\bar{a}$ is defined by a computable $\Sigma_\alpha$ formula.
2. $A$ has Scott rank $\omega^A_1 C^K$ if and only if for each tuple $\bar{a}$, the orbit is defined by a computable infinitary formula, but for each computable ordinal $\alpha$, there is a tuple $\bar{a}$ whose orbit is not defined by a computable $\Sigma_\alpha$ formula.
3. $A$ has Scott rank $\omega^A_1 C^K + 1$ if and only if there is a tuple $\bar{a}$ whose orbit is not defined by a computable infinitary formula.

Given a structure $M$, define the computable infinitary theory of $M$, $\text{Th}_\infty(M)$, to be collection of computable $L_{\omega_1\omega}$ sentences true of $M$. We can ask, for a given structure $M$, whether $\text{Th}_\infty(M)$ is $\aleph_0$-categorical, or whether there are other countable models of $\text{Th}_\infty(M)$. For $M$ a hyperarithmetic structure:

1. If $\text{SR}(M) < \omega^M_1 C^K$, then $\text{Th}_\infty(M)$ is $\aleph_0$-categorical. Indeed, $M$ has a computable Scott sentence [Nad74].
2. If $\text{SR}(M) = \omega^M_1 C^K$, then $\text{Th}_\infty(M)$ may or may not be $\aleph_0$-categorical [HTIK].
3. If $\text{SR}(M) = \omega^M_1 C^K + 1$, then $\text{Th}_\infty(M)$ is not $\aleph_0$-categorical as $M$ has a non-principal type which may be omitted.
In the case of abelian $p$-groups, we can say something even when we replace the assumption that $\mathcal{M}$ is hyperarithmetic with the assumption that $\omega_1^G = \omega_1^{CK}$.

**Definition 1.5** (Definition 6 of [FKM’11]). A class of countable structures has *Ulm type* if for any two structures $A$ and $B$ in the class, if $\omega_1^A = \omega_1^B = \omega_1^{CK}$ and $\text{Th}_\infty(A) = \text{Th}_\infty(B)$, then $A$ and $B$ are isomorphic.

It is well-known that abelian $p$-groups are of Ulm type; however, we do not know of a good reference with a complete proof, so we will give one in Section 2. We also note that there are indeed non-hyperarithmetic abelian $p$-groups $G$ with $\text{SR}(G) < \omega_1^{CK}$.

Knight asked whether there was a (non-trivial) first-order theory of Ulm type. By a non-trivial example, we mean that the elementary first-order theory should have non-hyperarithmetic models which are low for $\omega_1^{CK}$. Our theory $T_p$ is such an example.

**Theorem 1.6.** The models of $T_p$ are of Ulm type. Moreover, given $\mathcal{M} \models T_p$ with $\omega_1^{CK} = \omega_1^M$ and $\text{SR}(\mathcal{M}) < \omega_1^{CK} = \omega_1^M$, $\text{Th}_\infty(\mathcal{M})$ is $\aleph_0$-categorical.

**Proof.** Let $\mathcal{M}$ be a model of $T_p$. Now $\mathcal{M}$ is bi-interpretable, using computable infinitary formulas, with the disjoint union of an abelian $p$-group $G$ and a pure set. Thus $\mathcal{M}$ inherits these properties from $G$ (see Theorem 2.1). □

Of course, there will be non-hyperarithmetic models of $T_p$ with Scott rank below $\omega_1^{CK}$.

### Borel Incompleteness

In their influential paper [FS89], Friedman and Stanley introduced Borel reductions between invariant Borel classes of structures with universe $\omega$ in a countable language. Such classes are of the form $\text{Mod}(\varphi)$, the set of models of $\varphi$ with universe $\omega$, for some $\varphi \in L_{\omega_1\omega}$. A Borel reduction from $\text{Mod}(\varphi)$ to $\text{Mod}(\psi)$ is a Borel map $\Phi : \text{Mod}(\varphi) \to \text{Mod}(\psi)$ such that $\mathcal{M} \cong \mathcal{N} \iff \Phi(\mathcal{M}) \cong \Phi(\mathcal{N})$.

If such a Borel reduction exists, we say that $\text{Mod}(\varphi)$ is Borel reducible to $\text{Mod}(\psi)$ and write $\varphi \preceq_B \psi$. If $\varphi \preceq_B \psi$ and $\psi \preceq_B \varphi$, then we say that $\text{Mod}(\varphi)$ and $\text{Mod}(\psi)$ are Borel equivalent and write $\varphi \equiv_B \psi$. Friedman and Stanley showed that graphs, fields, linear orders, trees, and groups are all Borel equivalent, and form a maximal class under Borel reduction.

If $\text{Mod}(\varphi)$ is Borel complete, then the isomorphism relation on $\text{Mod}(\varphi) \times \text{Mod}(\varphi)$ is $\Sigma^1_1$-complete. The converse is not true, and the most well-known example is abelian $p$-groups, whose isomorphism relation is $\Sigma^1_1$-complete but not Borel complete. Until very recently, they were one of the few such examples, and there were no known examples of elementary first-order theories with similar properties. Recently, Laskowski, Rast, and Ulrich [URL] gave an example of a first-order theory which is not Borel complete, but whose isomorphism relation is not Borel. Our theory $T_p$ is another such example.

**Theorem 1.7.** The class of models of $T_p$ is Borel equivalent to abelian $p$-groups.

Because abelian $p$-groups are not Borel complete, but their isomorphism relation is $\Sigma^1_1$-complete, we get:

**Corollary 1.8.** The class of models of $T_p$ is not Borel complete but the isomorphism relation is $\Sigma^1_1$-complete.
Theorem 1.7 is a specific instance of the following general question asked by Friedman:

**Question 1.9.** Is it true that for every \( L_{\omega_1 \omega} \) sentence there is a Borel equivalent first-order theory?

## 2. Abelian \( p \)-groups are of Ulm type

In this section we will describe a proof of the following well-known theorem, which shows that abelian \( p \)-groups are of Ulm type.

**Theorem 2.1.** Let \( G \) be an abelian \( p \)-group with \( \omega_1^{CK} = \omega_1^G \). Then:

1. \( G \) is the only countable model of Th\( _\infty(G) \) with \( \omega_1^G = \omega_1^{CK} \), and
2. if SR\( _\infty(G) \) \( < \omega_1^{CK} = \omega_1^G \), then Th\( _\infty(G) \) is \( \aleph_0 \)-categorical.

The proof of Theorem 2.1 consists essentially of expressing the Ulm invariants via computable infinitary formulas.

**Definition 2.2.** Let \( G \) be an abelian \( p \)-group. For each ordinal \( \alpha < \omega_1^{CK} \), there is a computable infinitary sentence \( \psi_\alpha(x) \) which defines \( p^n G \) inside of \( G \):

- \( \psi_\alpha(x) \) is just \( x = x \);
- \( \psi_{\alpha+1}(x) \) is \((\exists y)[\psi_\alpha(y) \land py = x] \);
- \( \psi_\beta(x) \) is \( \bigwedge_{\alpha < \beta} \psi_\alpha(x) \) for limit ordinals \( \beta \).

**Definition 2.3.** For each ordinal \( \alpha < \omega_1^{CK} \) and \( n \in \omega \cup \{ \omega \} \), there is a computable infinitary sentence \( \varphi_{\alpha,n} \) such that, for \( G \) an abelian \( p \)-group,

\[
G \models \varphi_{\alpha,n} \leftrightarrow u_\alpha(G) = n.
\]

For \( n \in \omega \), define \( \varphi_{\alpha,\geq n} \) to say that there are \( x_1, \ldots, x_n \) such that:

- \( \psi_\alpha(x_1) \land \cdots \land \psi_\alpha(x_n) \),
- \( px_1 = \cdots = px_n = 0 \), and
- for all \( c_1, \ldots, c_n \in \mathbb{Z}/p\mathbb{Z} \) not all zero, \( \neg \psi_{\alpha+1}(c_1x_1 + \cdots + c_nx_n) \).

Then for \( n \in \omega \), \( \varphi_{\alpha,n} \) is \( \varphi_{\alpha,\geq n} \land \neg \varphi_{\alpha,\geq n+1} \), and \( \varphi_{\alpha,\omega} \) is \( \bigwedge_{n \in \omega} \varphi_{\alpha,\geq n} \).

**Lemma 2.4** (Theorem 8.17 of [AK00]). Let \( G \) be an abelian \( p \)-group. Then:

1. the length of \( G \) is at most \( \omega_1^G \), and
2. if \( G \) has length \( \omega_1^G \) then \( G \) is not reduced (in fact, its \( p \)-divisible part has infinite rank) and SR\( _\infty(G) = \omega_1^G + 1 \).

We are now ready to give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Since \( \omega_1^{CK} = \omega_1^G \), \( G \) has length at most \( \omega_1^{CK} \). Note that Th\( _\infty(G) \) contains the sentences \( \varphi_{\alpha,u_\alpha(G)} \) for \( \alpha < \omega_1^{CK} \). Thus any model of Th\( _\infty(G) \) has the same Ulm invariants as \( G \), for ordinals below \( \omega_1^{CK} \).

If \( \text{SR}(G) \) \( < \omega_1^{CK} \), let \( \lambda \) be the length of \( G \). Then Th\( _\infty(G) \) includes the computable formula \( (\forall x)[\psi_\lambda(x) \leftrightarrow \psi_{\lambda+1}(x)] \), so that any countable model of Th\( _\infty(G) \) has length at most \( \lambda \). Note that in such a model, \( \psi_\lambda \) defines the \( p \)-divisible part. Let \( n \in \omega \cup \{ \omega \} \) be such that \( p^n G \) is isomorphic to \( \mathbb{Z}(p^n)^n \). Then, if \( n \in \omega \), Th\( _\infty(G) \) contains the formula which says that there are \( x_1, \ldots, x_n \) such that:

- \( \psi_\lambda(x_1) \land \cdots \land \psi_\lambda(x_n) \),
- for all \( c_1, \ldots, c_n < p \) not all zero and \( k_1, \ldots, k_n \in \omega \),

\[
\frac{c_1}{p^{k_1}}x_1 + \cdots + \frac{c_n}{p^{k_n}}x_n \neq 0,
\]
• for all $y$ with $\psi(\lambda(y))$, there are $c_1, \ldots, c_n < p$ and $k_1, \ldots, k_n \in \omega$ such that
  \[ y = \frac{c_1}{p^{k_1}} x_1 + \cdots + \frac{c_n}{p^{k_n}} x_n. \]

If $n = \omega$, then $\text{Th}_\omega(G)$ contains the formula which says that for each $m \in \omega$, there are $x_1, \ldots, x_m$ such that
• $\psi(x_1) \land \cdots \land \psi(x_m)$, and
• for all $c_1, \ldots, c_n < p$ not all zero and $k_1, \ldots, k_m \in \omega$,
  \[ \frac{c_1}{p^{k_1}} x_1 + \cdots + \frac{c_m}{p^{k_m}} x_m \neq 0. \]

Any countable model of $\text{Th}_\omega(G)$ has $p$-divisible part isomorphic to $\mathbb{Z}(p^{\infty})^n$. So any countable model of $\text{Th}_\omega(G)$ has the same Ulm invariants and $p$-divisible part as $G$, and hence is isomorphic to $\text{Th}_\omega(G)$. Hence $\text{Th}_\omega(G)$ is $\aleph_0$-categorical. This gives (2), and (1) for the case where $\text{SR}(G) < \omega_1^{CK}$.

If $\text{SR}(G) = \omega_1^{CK} + 1$, let $H$ be any other countable model of $\text{Th}_\omega(G)$ with $\omega_1^H = \omega_1^G = \omega_1^{CK}$. Thus $G$ and $H$ both have length $\omega_1^{CK}$ and their $p$-divisible parts have infinite rank. As remarked before, they have the same Ulm invariants, and so they must be isomorphic. This completes the proof of (1).

### 3. The Theory $T_p$

Fix a prime $p$. The language $\mathcal{L}_p$ of $T_p$ will consist of a constant 0, unary relations $R_n$ for $n \in \omega$, and ternary relations $P_{\ell,m}^n$ for $\ell, m \in \omega$ and $n \leq \max(\ell, m)$. The following transformation of an abelian $p$-group into an $\mathcal{L}_p$-structure will illustrate the intended meaning of the symbols.

**Definition 3.1.** Let $G$ be an abelian $p$-group. Define $\mathfrak{M}(G)$ to be an $\mathcal{L}_p$-structure obtained as follows, with the same domain as $G$, and the symbols of $\mathcal{L}_p$ interpreted as follows:

- Set $0^{\mathfrak{M}(G)}$ to be the identity element of $G$.
- For each $n$, let $R_n^{\mathfrak{M}(G)}$ be the elements which are torsion of order $p^n$.
- For each $\ell, m \in \omega$ and $n \leq \max(\ell, m)$, and $x, y, z \in G$, set $P_{\ell,m}^{n,\mathfrak{M}(G)}(x, y, z)$ if and only if $x + y = z$, $x \in R_\ell^{\mathfrak{M}(G)}$, $y \in R_m^{\mathfrak{M}(G)}$, and $z \in R_n^{\mathfrak{M}(G)}$.

One should think of such $\mathcal{L}_p$-structures as the canonical models of $T_p$. The theory $T_p$ will consist of following axiom schemata:

(A1) For all $\ell, m, n \in \omega$:

\[
(\forall x)(\forall y, z) \left[ P_{\ell,m}^n(x, y, z) \rightarrow (R_\ell(x) \land R_m(x) \land R_n(z)) \right].
\]

(A2) $(R_n$ contains the elements which are torsion of order $p^n.)$

\[
(\forall x)[R_0(x) \leftrightarrow x = 0].
\]

and, for all $n \geq 1$:

\[
(\forall x) \left[ x \in R_n \leftrightarrow (\exists x_2 \cdot \exists x_{p-1}) \left[ P_{n,n}^n(x, x_2) \land P_{n,n}^n(x, x_3) \land \cdots \land P_{n,n}^n(x, x_{p-1}, x_p) \right] \right].
\]

(A3) $(P$ defines a partial function.) For all $\ell, m, n, n' \in \omega$:

\[
(\forall x)(\forall y)(\forall z)(\forall z') \left[ (P_{\ell,m}^n(x, y, z) \land P_{\ell,m}^{n'}(x, y, z')) \rightarrow z = z' \right].
\]
(A4) (P is total.) For all \( \ell, m \in \omega \):
\[
(\forall x, y) \left[ \left( R_\ell(x) \land R_m(y) \right) \rightarrow \bigvee_{n \leq \max(\ell, m)} (\exists z) P_{\ell, m}^n(x, y, z) \right].
\]

(A5) (Identity.) For all \( \ell \in \omega \):
\[
(\forall x) [R_\ell(x) \rightarrow [P_{\ell, \ell}^0(0, x, x) \land P_{\ell, 0}^0(x, 0, x)]].
\]

(A6) (Inverses.) For all \( \ell \in \omega \):
\[
(\forall x) (\exists y) [R_\ell(x) \rightarrow [P_{\ell, \ell}^0(x, y, 0) \land P_{\ell, 0}^0(y, x, 0)]].
\]

(A7) (Associativity.) For all \( \ell, m, n \in \omega \):
\[
(\forall x, y, z) [[R_{\ell, m}(x) \land R_m(y) \land R_n(z)] \rightarrow \\
\left( \bigvee_{r \leq \max(\ell, m), \ell \leq \max(m, n), s \leq \max(r, n), t \leq \max(\ell, s)} (3u3v3w) \right) \left[ P_{\ell, m, n}^r(x, y, u) \land P_{\ell, n}^s(u, z, v) \land P_{\ell, s}^t(v, w) \right].
\]

(A8) (Abelian.) For all \( \ell, m \in \omega \) and \( n \leq \max(\ell, m) \):
\[
(\forall x, y, z) \left[ R_{\ell, m}(x) \land R_m(y) \land R_n(z) \land P_{\ell, m, n}^x(x, y, z) \right] \rightarrow P_{\ell, m, n}^n(y, x, z).
\]

We must now check that the definition of \( T_p \) works as desired, that is, that if \( G \) is an abelian \( p \)-group, then \( \mathfrak{M}(G) \) is a model of \( T_p \).

Lemma 3.2. If \( G \) is an abelian \( p \)-group, then \( \mathfrak{M}(G) \models T_p \).

Proof. We must check that each instance of the axiom schemata of \( T_p \) holds in \( \mathfrak{M}(G) \).

(A1) Suppose that \( x, y, \) and \( z \) are elements of \( G \) with \( P_{m, \ell}^{n, \mathfrak{M}(G)}(x, y, z) \). Then, by definition, \( x + y = z, x \in R_{\ell}^{\mathfrak{M}(G)}, y \in R_{m}^{\mathfrak{M}(G)}, \) and \( z \in R_{n}^{\mathfrak{M}(G)} \).

(A2) \( R_{0}^{\mathfrak{M}(G)} \) contains the elements of \( G \) which are torsion of order \( p^0 = 1 \), so \( R_0 \) contains just the identity. For each \( n > 0 \), \( F_{n}^{\mathfrak{M}(G)} \) contains the elements of order \( p^n \). An element \( x \) has order \( p^n \) if and only if \( px \) has order \( p^{n-1} \). It remains only to note that if \( x \) has order \( p^n \), then \( x, 2x, 3x, \ldots, (p-1)x \) all have order \( p^n \) as well. The existential quantifier is witnessed by \( x_2 = 2x, x_3 = 3x, \) and so on.

(A3) If, for some \( x, y, z, \) and \( z' \), \( P_{\ell, m}^{n, \mathfrak{M}(G)}(x, y, z) \) and \( P_{\ell, m}^{n', \mathfrak{M}(G)}(x, y, z') \), then \( x + y = z \) and \( x + y = z' \), so that \( z = z' \).

(A4) Given \( x \) and \( y \) in \( G \) which are of order \( p^m \) and \( p^\ell \) respectively, \( x + y \) is of order \( p^n \) for some \( n \leq \max(m, \ell) \), and so we have \( P_{m, \ell}^{n, \mathfrak{M}(G)}(x, y, x + y) \).

(A5) If \( x \in G \) is of order \( p^\ell \), then \( x + 0 = 0 + x = x \) and so we have \( P_{\ell, 0}^{0, \mathfrak{M}(G)}(x, 0, x) \).

(A6) If \( x \in G \) is of order \( p^\ell \), then \( -x \) is also of order \( p^\ell \), and \( x + (-x) = 0 = (-x) + x \). So we have \( P_{\ell, \ell}^{0, \mathfrak{M}(G)}(x, -x, 0) \).

(A7) Given \( x, y, z \in G \) of order \( p^\ell, p^m \), and \( p^n \) respectively, there are \( r \leq \max(\ell, m) \) and \( s \leq \max(m, n) \) such that \( x + y \) and \( y + z \) are of order \( p^r \) and \( p^s \) respectively. Then there is \( t \) such that \( x + y + z \) is of order \( p^t \); \( t \leq \max(r, n) \) and \( t \leq \max(\ell, s) \).
(A8) Given \( x, y, z \in G \) of order \( p^\ell, p^m, \) and \( p^n \) respectively, \( n \leq \max(\ell, m), \) and with \( x + y = z, \) we have \( y + x = z \) as \( G \) is abelian.

Thus we have shown that \( \mathfrak{M}(G) \) is a model of \( T_p. \)

\[ \square \]

Note that \( G \) and \( \mathfrak{M}(G) \) are effectively bi-interpretable, proving one half of Theorem 4.3.

4. FROM A MODEL OF \( T_p \) TO AN ABELIAN \( p \)-GROUP

Given an abelian \( p \)-group \( G, \) we have already described how to turn \( G \) into a model of \( T_p. \) In this section we will do the reverse by turning a model of \( T_p \) into an abelian \( p \)-group.

**Definition 4.1.** Let \( \mathcal{M} \) be a model of \( T_p. \) Define \( \mathfrak{S}(\mathcal{M}) \) to be the group obtained as follows.

- The domain of \( \mathfrak{S}(\mathcal{M}) \) will be the subset of the domain of \( \mathcal{M} \) given by \( \bigcup_{n\in\omega} R^\mathcal{M}_n. \)
- The identity element of \( \mathfrak{S}(\mathcal{M}) \) will be \( 0^\mathcal{M}. \)
- We will have \( x + y = z \) in \( \mathfrak{S}(\mathcal{M}) \) if and only if, for some \( \ell, m, \) and \( n, P^\mathcal{M}_{\ell,m}(x,y,z). \)

We will now check that \( \mathfrak{S}(\mathcal{M}) \) is always an abelian \( p \)-group.

**Lemma 4.2.** If \( \mathcal{M} \) is a model of \( T_p, \) then \( \mathfrak{S}(\mathcal{M}) \) is an abelian \( p \)-group.

**Proof.** First we check that the operation \( + \) on \( \mathfrak{S}(\mathcal{M}) \) defines a total function. Given \( x, y \in \mathfrak{S}(\mathcal{M}), \) choose \( \ell \) and \( m \) such that \( x \in R^\mathcal{M}_\ell \) and \( y \in R^\mathcal{M}_m. \) Then by (A3) and (A4), there is a unique \( n \leq \max(\ell, m) \) and a unique \( z \) such that \( P^\mathcal{M}_{\ell,m}(x,y,z). \) Thus \( x + y = z, \) and \( z \) is unique.

Second, we check that \( \mathfrak{S}(\mathcal{M}) \) is in fact a group. To see that \( 0^\mathcal{M} \) is the identity, given \( x \in \mathfrak{S}(\mathcal{M}), \) there is \( \ell \) such that \( x \in R^\mathcal{M}_\ell. \) By (A5), \( P^\mathcal{M}_0(x,0^\mathcal{M},x) \) and \( P^\mathcal{M}_0(0^\mathcal{M},x,0^\mathcal{M}). \) Thus \( x + 0^\mathcal{M} = 0^\mathcal{M} + x = x, \) and \( 0^\mathcal{M} \) is the identity of \( \mathfrak{S}(\mathcal{M}). \)

To see that \( \mathfrak{S}(\mathcal{M}) \) has inverses, given \( x \in \mathfrak{S}(\mathcal{M}), \) there is \( \ell \) such that \( x \in R^\mathcal{M}_\ell, \) and by (A6) there is \( y \in R^\mathcal{M}_\ell \) such that \( P^\mathcal{M}_\ell(x,y,0^\mathcal{M}) \) and \( P^\mathcal{M}_\ell(y,x,0^\mathcal{M}). \) Thus \( x + y = y + x = 0^\mathcal{M}, \) and \( y \) is the inverse of \( x. \) Finally, to see that \( \mathfrak{S}(\mathcal{M}) \) is associative, given \( x, y, z \in \mathfrak{S}(\mathcal{M}), \) there are \( \ell, m, \) and \( n \) such that \( x \in R^\mathcal{M}_\ell, y \in R^\mathcal{M}_m, \) and \( z \in R^\mathcal{M}_n. \) Then by (A7) there are \( r, s, \) and \( t, u, v, \) and \( w, \) such that \( P^\mathcal{M}_{r,m}(x,y,u), P^\mathcal{M}_{r,n}(u,z,w), P^\mathcal{M}_{s,n}(y,z,v), \) and \( P^\mathcal{M}_{s,t}(x,v,w). \) Thus \( x + y = u, \)

\[ u + z = w, \]

\[ y + z = v, \]

\[ x + v = w. \]

So \( (x + y) + z = x + (y + z). \) Thus \( \mathfrak{S}(\mathcal{M}) \) is associative.

Third, to see that \( \mathfrak{S}(\mathcal{M}) \) is abelian, let \( x, y \in \mathfrak{S}(\mathcal{M}). \) There are \( \ell \) and \( m \) such that \( x \in R^\mathcal{M}_\ell \) and \( y \in R^\mathcal{M}_m. \) Let \( n \leq \max(\ell, m) \) be such that \( z = x + y \in R^\mathcal{M}_n. \) (Such an \( n \) and \( z \) exist by the arguments above that \( + \) is total, via (A3) and (A4).) Then \( P^\mathcal{M}_{n,m}(x,y,z), \) and so by (A8), \( P^\mathcal{M}_{n,\ell}(y,x,z). \) Thus \( y + x = z \) and \( \mathfrak{S}(\mathcal{M}) \) is abelian.

Finally, we need to see that \( \mathfrak{S}(\mathcal{M}) \) is a \( p \)-group. We claim, by induction on \( n \geq 0, \)

that \( R^\mathcal{M}_n \) consists of the elements of \( \mathfrak{S}(\mathcal{M}) \) which are of order \( p^n. \) From this claim, it follows that \( \mathfrak{S}(\mathcal{M}) \) is a \( p \)-group. For \( n = 0, \) the claim follows directly from (A2). Given \( n > 0, \) suppose that \( x \in R^\mathcal{M}_n. \) Then the witnesses \( x_2, x_3, \ldots, x_p \) to (A2) must be \( 2x, 3x, \ldots, px. \) Note that since \( P^{n-1,\mathcal{M}}(p,0) \) \( \in R^\mathcal{M}_{n-1}. \) Thus \( px \) is of
order $p^{n-1}$, and so $x$ is of order $p^n$. On the other hand, if $x$ is of order $p^n$, then $px$ is of order $p^{n-1}$ and so $px \in R_{n-1}^M$. Moreover, $x_2 = 2x, x_3 = 3x, \ldots, x_{p-1} = (p-1)x$ are all of order $p^n$. So we have $P_{n,n}^M(x, x, x_2), P_{n,n}^M(x, x_2, x_3), \ldots, P_{n,n}^M(x, x_{p-1}, x_p)$. By (A2), $x \in R_n^M$. This completes the inductive proof.

We now have two operations, one which turns an abelian $p$-group into a model of $T_p$, and another which turns a model of $T_p$ into an abelian $p$-group. These two operations are almost inverses to each other. If we begin with an abelian $p$-group, turn it into a model of $T_p$, and then that model into an abelian $p$-group, we will obtain the original group. However, if we start with a $M$ model of $T_p$, turn it into an abelian $p$-group, and then turn that abelian $p$-group into a model of $T_p$, we may obtain a different model of $T_p$. The problem is that the of elements of $M$ which are not in any of the sets $R_n^M$ are discarded when we transform $M$ into an abelian $p$-group. However, these elements form a pure set, and so the only pertinent information is their size.

**Definition 4.3.** Given a model $M$ of $T_p$, the size of $M$, $\#M \in \omega \cup \{\infty\}$, is the number of elements of $M$ not in any relation $R_n$.

**Lemma 4.4.** Given an abelian $p$-group $G$, $\mathfrak{S}(\mathfrak{M}(G)) = G$.

*Proof.* Since $\#\mathfrak{M}(G) = 0$, we see that $G$, $\mathfrak{M}(G)$, and $\mathfrak{S}(\mathfrak{M}(G))$ all have the same domain. The identity of $\mathfrak{S}(\mathfrak{M}(G))$ is $0^{\mathfrak{M}(G)}$ which is the identity of $G$. If $x + y = z$ in $G$, then, for some $\ell, m, n \in \omega$, we have $P_{\ell,m}^{\mathfrak{M}(G)}(x, y, z)$. Thus, in $\mathfrak{S}(\mathfrak{M}(G))$, we have $x + y = z$. So $\mathfrak{S}(\mathfrak{M}(G)) = G$. □

We make a simple extension to $\mathfrak{M}$ as follows.

**Definition 4.5.** Let $G$ be an abelian $p$-group and $m \in \omega \cup \{\infty\}$. Define $\mathfrak{M}(G, m)$ to be $\mathcal{L}_p$-structure with domain $G \cup \{a_1, \ldots, a_m\}$ with the relations interpreted as in $\mathfrak{M}(G)$. Thus, no relations hold of any of the elements $a_1, \ldots, a_m$.

**Lemma 4.6.** Given a model $M$ of $T_p$, $\mathfrak{M}(G(M), \#M) \cong M$.

*Proof.* We will show that if $\#M = 0$, then $\mathfrak{M}(\mathfrak{S}(M)) = M$. From this one can easily see that $\mathfrak{M}(G(M), \#M) \cong M$ in general.

If $\#M = 0$, then $M$, $\mathfrak{S}(M)$, and $\mathfrak{M}(\mathfrak{S}(M))$ all share the same domain. It is clear that $0^M = 0^{\mathfrak{S}(M)} = 0^{\mathfrak{M}(\mathfrak{S}(M))}$. From the proof of Lemma 4.2, we see that for each $n$, $R_n^M$ defines the set of elements of $\mathfrak{S}(M)$ which are torsion of order $p^n$, and so $R_n^M = R_n^{\mathfrak{M}(\mathfrak{S}(M))}$. Given $\ell, m \in \omega$ and $n \leq \max(\ell, m)$, and $x, y, z$ elements of the shared domain, we have $P_{\ell,m}^{\mathfrak{M}(M)}(x, y, z)$ if and only if

$$x + y = z \text{ in } \mathfrak{S}(M) \text{ and } x \in R_{\ell}^M, y \in R_{m}^M, \text{ and } z \in R_{n}^M.$$ 

Since $R_i^M = R_i^{\mathfrak{M}(\mathfrak{S}(M))}$ for each $i$, this is the case if and only if $P_{\ell,m}^{\mathfrak{M}(\mathfrak{S}(M))}(x, y, z)$. Thus we have shown that $\mathfrak{M}(\mathfrak{S}(M)) = M$. □

Note that $M$ and the disjoint union of $\mathfrak{S}(M)$ with a pure set of size $\#M$ are bi-interpretable, using computable infinitary formulas, completing the proof of Theorem 1.3.
5. Borel Equivalence

In this section we will prove Theorem 1.7 by showing that the class of models of $T_p$ and the class of abelian $p$-groups are Borel equivalent. $G \mapsto \mathfrak{G}(\mathfrak{M}(G)) = \mathfrak{G}(\mathfrak{M}(G,0))$ is a Borel reduction from isomorphism on abelian $p$-groups to isomorphism on models of $T_p$. However, $\mathcal{M} \mapsto \mathfrak{G}(\mathcal{M})$ is not a Borel reduction in the other direction, because two non-isomorphic models of $T_p$ might be mapped to isomorphic groups. We need to find a way to turn $\mathfrak{G}(\mathcal{M})$ and $\#\mathcal{M}$ into an abelian $p$-group $\mathfrak{H}(\mathfrak{G}(\mathcal{M}), \#\mathcal{M})$, so that $\mathcal{M}$ and $\#\mathcal{M}$ can be recovered from $\mathfrak{H}(\mathfrak{G}(\mathcal{M}), \#\mathcal{M})$.

We will define $\mathfrak{H}(G,m)$ for any abelian $p$-group $H$ and $m \in \Omega \cup \{\infty\}$. It is helpful to think about what this reduction will do to the Ulm invariants: The first Ulm invariant of $\mathfrak{H}(G,m)$ will be $m$, and for each $\alpha$, then $1 + \alpha$ Ulm invariant of $\mathfrak{H}(G,m)$ will be the same as the $\alpha$th Ulm invariant of $G$.

Definition 5.1. Given an abelian $p$-group $G$, and $m \in \Omega \cup \{\infty\}$, define an abelian $p$-group $\mathfrak{H}(G,m)$ as follows. Let $\mathcal{B}$ be a basis for the $\mathbb{Z}_p$-vector space $G/pG$. Let $\mathcal{B} \subseteq G$ be a set of representatives for $\mathcal{B}$. Let $G^*$ be the abelian group $\langle G, a_b : b \in B | p\alpha_b = b \rangle$. Then define $\mathfrak{H}(G,m) = G^* \oplus (\mathbb{Z}_p)^m$.

To make this Borel, we can take $\mathcal{B}$ to be the lexicographically first set of representatives for a basis. It will follow from Lemma 5.2 that the isomorphism type of $\mathfrak{H}(G,m)$ does not depend on these choices. First, we require a couple of lemmas.

Lemma 5.2. Each element of $G$ can be written uniquely as a (finite) linear combination $h + \sum_{b \in \mathcal{B}} x_b b$ where $h \in pG$ and each $x_b < p$.

Proof. Given $g \in G$, let $\hat{g}$ be the image of $g$ in $G/pG$. Then, since $\mathcal{B}$ is a basis for $G/pG$, we can write

$$\hat{g} = \sum_{b \in \mathcal{B}} x_b \hat{b}$$

with $x_b < p$, where $\hat{b}$ is the image of $b$ in $G/pG$. Thus setting

$$h = g - \sum_{b \in \mathcal{B}} x_b b \in pG$$

we get a representation of $g$ as in the statement of the theorem.

To see that this representation is unique, suppose that

$$h + \sum_{b \in \mathcal{B}} x_b b = h' + \sum_{b \in \mathcal{B}} y_b b.$$

Then, modulo $pG$,

$$\sum_{b \in \mathcal{B}} x_b \hat{b} = \sum_{b \in \mathcal{B}} y_b \hat{b}.$$

Since $\mathcal{B}$ is a basis, $x_b = y_b$ for each $b \in \mathcal{B}$. Then we get that $h = h'$ and the two representations are the same. \qed

Lemma 5.3. Each element of $G^*$ can be written uniquely in the form $h + \sum_{b \in \mathcal{B}} x_b a_b$ where $h \in G$ and each $x_b < p$.

Proof. It is clear that each element of $G^*$ can be written in such a way. If

$$h + \sum_{b \in \mathcal{B}} x_b a_b = h' + \sum_{b \in \mathcal{B}} y_b a_b$$

then, in $G$,

$$ph + \sum_{b \in \mathcal{B}} x_b b = ph' + \sum_{b \in \mathcal{B}} y_b b.$$

\qed
Lemma 5.6. H is of order pg such that \( \in \). Note that Proof. We will show that \((H)\subseteq(G)\subseteq(B)\). Hence \(pG\), \(G\). This representation is unique, so \(x_b = y_b\) for each \(b \in B\), and so \(h = h'\). □

Lemma 5.4. The isomorphism type of \(\hat{\mathcal{H}}(G,m)\) depends only on the isomorphism type of \(G\), and not on the choice of \(B\).

Proof. It suffices to show that if \(\mathcal{C}\) is another choice of representatives for a basis of \(G/pG\), then \(G'_B = G'_\mathcal{C}\), where the former is constructed using \(B\), and the later is constructed using \(\mathcal{C}\). Let \(f: B \to \mathcal{C}\) be an bijection.

Given \(g \in G'_B\), write \(g = g' + \sum_{b \in B} x_b a_b\) with \(g' \in G\) and \(0 \leq x_b < p\). This representation of \(g\) is unique by Lemma 5.3. Define \(\varphi(g) = g' + \sum_{b \in B} x_b a_{f(b)}\). It is not hard to check that \(\varphi\) is a homomorphism. The inverse of \(\varphi\) is the map \(\psi\) which is defined by \(\psi(h) = h' + \sum_{c \in C} y_c a_{f^{-1}(c)}\) where \(h = h' + \sum_{c \in C} y_c a_c\). □

The next two lemmas will be used to show that if \(G\) is not isomorphic to \(G'\), or if \(m\) is not equal to \(m'\), then \(\hat{\mathcal{H}}(G,m)\) will not be isomorphic to \(\hat{\mathcal{H}}(G',m')\).

Lemma 5.5. \(G = pG^*\).

Proof. Each element of \(G\) can be written as \(g + \sum_{b \in B} x_b b\) with \(g \in pG\). Let \(g' \in G\) be such that \(pg' = g\). Then
\[
p(g' + \sum_{b \in B} x_b a_b) = g + \sum_{b \in B} x_b b.
\]
Hence \(G \subseteq pG^*\). Given \(h \in G^*\), write \(h = g + \sum_{b \in B} x_b a_b\). Then \(ph = pg + \sum_{b \in B} x_b b \in G\). So \(pG^* \subseteq G\), and so \(G = pG^*\). □

If \(G\) is a group, recall that we denote by \(G[p]\) the elements of \(G\) which are torsion of order \(p\).

Lemma 5.6. \(\hat{\mathcal{H}}(G,m)[p] / (p\hat{\mathcal{H}}(G,m))[p] \cong (\mathbb{Z}_p)^m\).

Proof. Note that
\[
\hat{\mathcal{H}}(G,m)[p] / (p\hat{\mathcal{H}}(G,m))[p] \cong (G^*[p] / (pG^*)[p]) \oplus (\mathbb{Z}_p)^m[p] / (p(\mathbb{Z}_p)^m)[p]
\]
\[
\cong (G^*[p] / G[p]) \oplus (\mathbb{Z}_p)^m.
\]
We will show that \((G^*[p] / G[p])\) is the trivial group by showing that if \(g \in G^*\), \(pg = 0\), then \(g \in G\). Indeed, write \(g = g' + \sum_{b \in B} y_b a_b\) with \(g' \in G\). Then
\[
0 = pg = pg' + \sum_{b \in B} py_b a_b = pg' + \sum_{b \in B} y_b b.
\]
Since \(0 \in pG\) has a unique representation (by Lemma 5.2) \(0 = 0 + \sum_{b \in B} 0b\), we get that \(y_b = 0\) for each \(b \in B\), and so \(g = g' \in G\). □

By the previous lemma, we can recover \(m\) from \(\hat{\mathcal{H}}(G,m)\). We have
\[
p\hat{\mathcal{H}}(G,m) = pG^* \oplus p(\mathbb{Z}_p)^m \cong pG^* = G
\]
so that we can also recover \(G\).

Thus, using Lemma 4.6 \(\mathcal{M} \to \hat{\mathcal{H}}(\mathfrak{G}(\mathcal{M}), \#\mathcal{M})\) gives a Borel reduction from \(T_p\) to abelian \(p\)-groups. This completes the proof of Theorem 1.7.
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