Einstein-aether-axion theory:
How does dynamic aether regulate the state of axionic dark matter?

Alexander B. Balakin\textsuperscript{1} and Amir F. Shakirzyanov\textsuperscript{1,}\textsuperscript{†}

\textsuperscript{1}Department of General Relativity and Gravitation, Institute of Physics, Kazan Federal University, Kremlevskaya street 18, Kazan, 420008, Russia

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In the framework of extended Einstein-aether-axion theory we establish the model, which describes a stiff regulation of the behavior of axionic dark matter by the dynamic aether. The aether realizes this procedure via the modified Higgs potential, designed for modeling of nonlinear self-interaction of pseudoscalar (axion) field; the modification of this potential is that its minima are not fixed, and their positions and depths depend now on the square of the covariant derivative of the aether velocity four-vector. Exact solutions to the master equations, modified correspondingly, are obtained in the framework of homogeneous isotropic cosmological model. The analysis of the acceleration parameter has shown that it possesses one zero, thus guaranteeing the existence of one transition point in the Universe history, which separates the epoch of decelerated expansion and the late-time epoch of accelerated expansion: the asymptotic regime of expansion is of the quasi-de Sitter type (Pseudo-Rip). During the epoch of accelerated expansion the aether pressure is shown to be negative, i.e., the dynamic aether plays the role of dark energy. The effective equation of state for axionic dark matter is of the stiff type. Homogeneous perturbations of the pseudoscalar (axion) field, of the Hubble function and of the scale factor are shown to fade out with cosmological time, there are no growing modes, the model of stiff regulation is stable.

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I. INTRODUCTION

The Einstein-aether-axion theory is an example of a pseudoscalar-vector-tensor model of gravity. The first (tensorial) ingredient of this model is the gravitational field described by the Einstein-Hilbert Lagrangian \( \frac{1}{2} R \), with (or without) cosmological constant \( \Lambda \). The second element of the model, the unit time-like vector field \( U^i \), is associated with the velocity of some cosmic substratum, indicated as the aether in \([1–3]\). This unit vector field realizes the idea of a preferred frame of reference (see, e.g., \([10–12]\)), and its presence violates the Lorentz invariance of the model \([13, 15]\). The third element of the model, the pseudoscalar (axion) field \( \phi \), is associated with the axionic dark matter (see, e.g., \([16, 28]\) for review, references, historical and mathematical details). In many respects the model with pseudoscalar field is similar to the model with scalar field as the third ingredient (see, e.g., \([29, 33]\)); however, there is a number of dissimilarities \([34]\). The main difference between the models with scalar and pseudoscalar (axion) fields is that in the second case, respectively, only even functions of \( \phi \) can be introduced into the Lagrangian of the model. In particular, this means that the potential of the axion field has to be written in the form \( V(\phi^2) \); as a consequence, one can not use the linear function \( \alpha \phi \), the exponential potentials \( e^{\lambda \phi} \) (but, of course, one can consider \( \cos \mu \phi, \phi^2 \), etc...). Another difference appears, when we add terms with Levi-Civita pseudo-tensor \( \epsilon^{ikmn} \) into the Lagrangian of the model \([34]\). In this case, for instance, the convolution of the gradient four-vector of the pseudoscalar field \( \nabla_i \phi \) with the angular velocity pseudo-four vector \( \omega^i \) gives the true scalar \( (\omega^i = \frac{1}{2} \epsilon^{ikmn} U_k \nabla_m U_n) \). When the electromagnetic field is incorporated into the models with pseudoscalar (axion) field, a new invariant appears in the total Lagrangian, \( \phi F^*_a F^a_{mn} \) with the dual Maxwell tensor \( F^*_a F^a_{mn} \) [19].

The non-uniformly moving aether and the axionic dark matter can interact using three channels of coupling. The first and second channels of coupling are indirect; the first one is standardly realized via the gravitational field; the second one is mediated by the electromagnetic field \([32, 33]\). The third channel of coupling is direct, it includes many sub-channels with specific coupling constants (see \([34]\) for the classification of the terms up to the second order with respect to the effective field theory \([40, 41]\)). Based on that models we have shown in \([42]\) that the axion field can control the behavior of the aether through the guiding functions incorporated into the extended Jacobson’s constitutive tensor \( K^{abmn}(g_{ik}, U^p) \rightarrow K^{abmn}(g_{ik}, U^p, \phi^2) \). For the case of scalar field such idea was formulated in \([20]\): instead of Jacobson’s constants \( C_1, C_2, C_3, C_4 \) the authors of \([29]\) have introduced the functions \( \beta_1(\phi), \beta_2(\phi), \beta_3(\phi), \beta_4(\phi) \). Also, in \([42]\) we assumed that the potential of the pseudoscalar field \( V(\phi^2) \) is of the Higgs type, \( V = \frac{1}{2} \gamma (\phi^2 - \Phi_+^2)^2 \), where the quantities \( \pm \Phi_+ \), which correspond to the pair of stable basic states of the axion field, are constant. This potential appears in the class of models, which can be indicated

\*Electronic address: Alexander.Balakin@kpfu.ru
†Electronic address: shamir@mail.ru
as models with $\phi^4$-type self-interaction; this potential is widely used in cosmology for description of the axionic dark matter (see, e.g., \[24, 43\]). We have shown in \[42\] that the influence of the axionic dark matter on the dynamic aether can switch on or switch off the pp-wave modes. Now we focus on the inverse effect, when the dynamic aether guides the behavior of the axionic dark matter via the dynamic scalar $\Omega^2$ incorporated into the potential of the pseudoscalar field, $V(\phi^2, \Omega^2)$. For the case of scalar field this idea was used, for instance, in \[8, 32, 33\]. As for the Jacobson’s constitutive tensor, it holds the classical structure $K^{abmn}(g^i_{\kappa}, U^\kappa)$ \[1\]. In the cosmological context, the new variable in the potential of the scalar field was presented either by the expansion scalar $\Theta = \nabla_k U^k$ (isotropic models of the Friedmann type), or by the pair $\Theta$ and $\sigma^2$ (the last scalar is the square of the traceless shear tensor $\sigma_{mn}$, which is non-vanishing, e.g., in the Bianchi-I model).

In this paper the dynamic scalar $\Omega^2$ is reconstructed using the square of the covariant derivative of the vector field, $\Omega^2 \equiv \nabla_m U_n \nabla^n U^m$. Of course, we could introduce four new arguments: the square of the acceleration four-vector of the aether $d^2 = U^m \nabla_m U^n \nabla^n U^k$; the square of the shear tensor $\sigma^2 = \sigma_{mn} \sigma^{mn}$, the square of the vorticity tensor $\omega^2 = \omega_{mn} \omega^{mn}$; and the expansion scalar $\Theta$. Nevertheless, we add the term $\Omega^2 \equiv \nabla_m U_n \nabla^n U^m$ using the analogy and motivation given in the work \[44\]. We consider the potential $V(\phi^2, \Omega^2)$ to be again of the Higgs type, but now the quantity $\Phi$, depends on the dynamic scalar $\Omega^2$. This means that, formally speaking, we go beyond the framework of the effective field theory of the second order, however, in this paper we do not consider the corresponding complete version, so that only the potential $V$ contains the terms with covariant derivatives.

The paper is organized as follows. In Section II, we recall the basic elements of formalism, and derive master equations for pseudoscalar, unit vector and gravitational fields. In Section III we consider cosmological applications of the Einstein-aether-axion model, and obtain new exact solutions for the spatially isotropic homogeneous Universe with and without cosmological constant. Section IV contains discussion and conclusions.

II. FORMALISM OF THE EINSTEIN-AETHER-AXION THEORY

A. The action functional

We consider the action functional of the Einstein-aether-axion theory in the following form:

$$S_{(EAA)} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2k} \left[ R + 2\Lambda + \lambda (g_{mn} U^m U^n - 1) \right] + K^{abmn} \nabla_a U_m \nabla_b U_n \right\} + \frac{1}{2} \Psi_0^2 \left[ V(\phi^2, \Omega^2) - g^{mn} \nabla_m \phi \nabla_n \phi \right].$$

Here $g$ is the determinant of the metric tensor; $R$ is the Ricci scalar, $\Lambda$ is the cosmological constant, and $\kappa = 8\pi G$ is the Einstein constant with $c = 1$. The term $\lambda (g_{mn} U^m U^n - 1)$ stands to ensure that the vector field $U^i$ is normalized to one everywhere; $\lambda$ is the Lagrange multiplier. The tensor $\nabla_a U_m$ is the covariant derivative, and the Jacobson’s constitutive tensor $K^{abmn}$ is constructed using the metric tensor $g^{ij}$ and the aether velocity four-vector $U^k$ only:

$$K^{abmn} = C_1 g^{ab} g^{mn} + C_2 g^{am} g^{bn} + C_3 g^{an} g^{bm} + C_4 U^a U^b g^{mn}.$$ (2)

The parameters $C_1, C_2, C_3$ and $C_4$ are the Jacobson’s coupling constants \[1\]. The dimensionless quantity $\phi$ denotes the pseudoscalar (axion) field. The parameter $\Psi_0$ is reciprocal to the constant of the axion-photon coupling $g(\Lambda \gamma) = \frac{1}{\Psi_0}$; the constraint for the constant $g(\Lambda \gamma)$ is $g(\Lambda \gamma) < 1.47 \times 10^{-10} \text{GeV}^{-1}$ (see \[45\]).

The scalar $V(\phi^2, \Omega^2)$ is the potential of the pseudoscalar field, which is influenced by the aether vector field. We assume that the potential $V$ is the function of two scalar arguments: the first argument is the square of the axion field, $\phi^2$; the second argument $\Omega^2$ is defined as follows:

$$\Omega^2 \equiv (\nabla_m U_n) (\nabla^m U^n).$$ (3)

Using the standard decomposition of the covariant derivative of the velocity field $U^i$

$$\nabla_i U_k = U_i DU_k + \sigma_{ik} + \omega_{ik} + \frac{1}{3} \Delta_{ik} \Theta,$$ (4)

one can represent the scalar $\Omega^2$ as follows:

$$\Omega^2 = DU_m DU^m + \sigma_{mn} \sigma^{mn} + \omega_{mn} \omega^{mn} + \frac{1}{3} \Theta^2.$$ (5)

The acceleration four-vector $DU^i$, the symmetric shear tensor $\sigma_{mn}$, the skew-symmetric vorticity tensor $\omega_{mn}$, and the expansion scalar $\Theta$ are the irreducible elements of the decomposition of the covariant derivative of the velocity field:

$$DU_k \equiv U^m \nabla_m U_k, \quad \sigma_{ik} \equiv \frac{1}{2} \left( \nabla_i U_k + \nabla_k U_i \right) - \frac{1}{3} \Delta_{ik} \Theta,$$

$$\omega_{ik} \equiv \frac{1}{2} \left( \nabla_i U_k - \nabla_k U_i \right), \quad \Theta \equiv \nabla_m U^m,$$

$$D \equiv U^i \nabla_i, \quad \Delta^i_k = \delta^i_k - U^i U_k, \quad \nabla_i \equiv \Delta^j_i \nabla_j.$$ (6)

In these terms the scalar $K^{abmn}(\nabla_a U_m)(\nabla_b U_n)$ has the form

$$K^{abmn}(\nabla_a U_m)(\nabla_b U_n) =$$
\[ (C_1 + C_4) DU_k DU^k + (C_1 + C_3) \sigma_{ik} \sigma^{ik} + (C_1 - C_3) \omega_{ik} \omega^{ik} + \frac{1}{3} (C_1 + 3C_2 + C_3) \Theta^2. \] (7)

Clearly, the scalar \( m \) coincides with \( \Omega^2 \), when \( C_1 = 1 \) and \( C_2 = C_3 = C_4 = 0 \).

**B. Master equations for the pseudoscalar, unit vector, and gravitational fields**

1. Master equation for the axion field

Variation of the action functional \( S_{(EAA)} \) with respect to pseudoscalar field \( \phi \) gives the master equation in the standard form

\[ g^{mn} \nabla_m \nabla_n \phi + \phi \frac{\partial V}{\partial \phi} = 0. \] (8)

We assume the potential of the axion field is of the Higgs type

\[ V(\phi^2, \Omega^2) = \frac{1}{2} \gamma [\phi^2 - \Phi_*^2(\Omega^2)]^2, \] (9)

for which there are one local maximum located at \( \phi = 0 \), and two symmetric minima located at \( \phi = \pm \Phi_* \). The positions of these minima, as well as, their depths \( V_0 = \frac{1}{2} \gamma \Phi_*^2(\Omega^2) \) depend on the value of the function \( \Phi_*(\Omega^2) \). When \( \phi = \Phi_* + \psi, |\psi| << 1 \), i.e., the deviation of axion field from the minimal value is small, the potential reduces to \( 2 \gamma \Phi_*^2 \phi^2 \), so that the quantity \( 2 \gamma \Phi_*^2 \) plays the role of square of an effective mass, \( m^2 \) is now the constant, it is a scalar function of the argument \( \Omega^2 \). Since the scalar \( \Omega^2 \) depends, generally speaking, on time and spatial coordinates through the covariant derivative of the aether vector field, we deal with the case, when the dynamic aether controls the behavior of the axionic dark matter. To be more precise, the aether prescribes what is the basic state of the axion field; alternatively, one can say that the aether regulates the effective mass of the axion field. Thus the equation

\[ \nabla_m \nabla^m \phi + \gamma \phi [\phi^2 - \Phi_*^2(\Omega^2)] = 0 \] (10)

is the master equation for the pseudoscalar field.

2. Equations for the unit dynamic vector field

The aether dynamic equations can be found by variation of the action \( S_{(EAA)} \) with respect to the Lagrange multiplier \( \lambda \) and to the unit vector field \( U^i \). The variation of the action \( S_{(EAA)} \) with respect to \( \lambda \) yields the equation

\[ g_{mn} U^n U^m = 1, \] (11)

which is known to be the normalization condition of the time-like vector field \( U^k \). Then, variation of the functional \( S_{(EAA)} \) with respect to \( U^i \) yields the master equation in the standard form:

\[ \nabla_a J^{aj} = \lambda U^j + I^j, \] (12)

where \( I^j \) is the form

\[ I^j = C_4 (DU_m)(\nabla^j U^m). \] (13)

The essential difference from \([1]\) appears in the term \( J^{aj} \), which is defined now as follows:

\[ J^{aj} = \tilde{K}^{abjn}(\nabla_b U_n), \]

\[ \tilde{K}^{abjn} = K^{abjn} + \kappa \Psi_0^2 g^{ab} g^{jn} \frac{\partial V}{\partial \Omega^2}, \] (14)

where the tensor \( K^{abjn} \) is given by \([2]\). In fact, one can obtain the new constitutive tensor \( K^{abmn} \) from the Jacobson’s one \([2]\), if we replace the constant \( C_1 \) by the function \( h_1(\phi^2, \Omega^2) \) defined as

\[ h_1(\phi^2, \Omega^2) = C_1 - 2 \kappa \gamma \Psi_0^2 \left[ \phi^2 - \Phi_*^2(\Omega^2) \right] \Phi_* \frac{d \Phi_*}{d \Omega^2}. \] (15)

The Lagrange multiplier \( \lambda \) can be obtained standardly as

\[ \lambda = U^j \nabla_a J^{aj} - I^j. \] (16)

3. Equations for the gravitational field

The variation of the action \( S_{(EAA)} \) with respect to the metric \( g^{ik} \) yields the gravitational field equations, which can be presented in the following form

\[ R_{ik} - \frac{1}{2} R g_{ik} = \Lambda g_{ik} + T_{ik} + \kappa T^{(A)}_{ik}. \] (17)

\[ T_{ik} = \frac{1}{2} \tilde{g}_{ik} \tilde{K}^{abmn} \nabla_a U_m \nabla_b U_n + U_i U_k \nabla_a J^{aj} + \nabla^m \left[ U_i \nabla^j U_m - \nabla_m (U_i U_j) - J^{ij} U^m \right] + h_1 \left[ (\nabla_m U_i) (\nabla^m U_k) - (\nabla_i U_m) (\nabla_k U^m) \right] + C_4 \left[ DU_i DU_k - U_i U_k DU_m DU^m \right]. \] (18)

As usual, the symbol \( p_i(q_k) \) denotes symmetrization. The quantity \( T^{(A)}_{ik} \) written as

\[ T^{(A)}_{ik} = \Psi_0^2 \left[ \nabla_i \phi \nabla_k \phi + \frac{1}{2} \tilde{g}_{ik} (V - \nabla_n \phi \nabla^n \phi) \right] \] (19)

describes the extended stress-energy tensor of the pseudoscalar field. Compatibility conditions for the set of equations \([17]\)

\[ \nabla^k \left[ T_{ik} + \kappa T^{(A)}_{ik} \right] = 0, \] (20)

are satisfied automatically on the solutions to the master equations \([10], \[16]\).
III. APPLICATION: THE SPATIALLY ISOTROPIC AND HOMOGENEOUS COSMOLOGICAL MODEL WITH DYNAMIC AETHER AND AXIONIC DARK MATTER

A. Reduced master equations

In this Section we consider the master equations for the pseudoscalar, vector and gravitational field for the symmetry associated with spatially isotropic, homogeneous cosmological model of the Friedmann type. We assume the metric to be of the form

\[ ds^2 = dt^2 - a^2(t) [dx^2 + dy^2 + dz^2], \]

with the scale factor \( a(t) \) and the Hubble function \( H(t) = \frac{\dot{a}}{a}. \) The dot denotes the derivative with respect to cosmological time \( t; \) we use the units with \( c = 1. \) We use the ansatz that the pseudoscalar and unit dynamic vector fields, \( \phi \) and \( U^i, \) inherit the chosen symmetry. Mathematically, this requirement means that the pseudoscalar and vector fields are the functions of the cosmological time only, \( \phi(t) \) and \( U^i(t), \) and the velocity four-vector has to be of the form \( U^i = \delta^i_0, \) thus providing the absence of preferred spatial directions in the isotropic spacetime.

The covariant derivative \( \nabla_i U_k \) in this case is characterized by vanishing acceleration four-vector, shear and vorticity tensors:

\[ DU^i = 0, \quad \sigma_{mn} = 0, \quad \omega_{mn} = 0. \]

Only the expansion scalar is nonvanishing, and we can write

\[ \Theta = 3H(t), \quad \nabla_i U_k = \Delta_{ik} H(t), \quad \Omega^2 = 3H^2. \]

Our first task is to prove that for our ansatz the evolutionary equations for the unit vector field (12) are satisfied identically; then we will consider the reduced equations for pseudoscalar and gravitational fields, and obtain exact solutions to these equations.

1. Reduced equations for the unit vector field

Using the ansatz about the velocity four-vector, \( U^i = \delta^i_0, \) we can calculate explicitly all the necessary quantities. First, we see that the four-vector \( J^i \) vanishes. Second, we obtain that

\[ J^{ij} = H \left[ \Delta^{ij} (h_1 + 3C_2 + C_3) + 3C_2 U^a U^j \right], \]

and the divergence \( \nabla_a J^{aj} \) is parallel to the velocity four-vector

\[ \nabla_a J^{aj} = 3U_j \left[ C_2 \dot{H} - H^2 (h_1 + C_3) \right]. \]

Thus, three of four basic evolutionary equations (12) for the unit vector field are satisfied identically, and the last one defines the Lagrange multiplier (see (15)):

\[ \lambda(t) = -3H^2 (h_1 + C_3) + 3C_2 \dot{H}. \]

2. Reduced equation for the pseudoscalar (axion) field

The reduced evolutionary equation for the axion field

\[ \left[ \ddot{\phi} + 3H \dot{\phi} + \gamma \dot{\phi} (\dot{\phi}^2 - \dot{\Phi}^2) \right] = 0 \]

is, in general case, the non-linear one.

3. Reduced equations for the gravitational field

The stress-energy tensor describing the contribution of the unit dynamic vector field is

\[ T_{ik} = \Delta_{ik} \left\{ \frac{1}{3} \kappa \Psi_0^2 \frac{d}{dt} \left( \dot{\phi}^2 - \dot{\Phi}^2 \right) \frac{\Phi_* d\Phi_*}{H dH} \right\} + \frac{3}{2} H^2 \left( C_1 + 3C_2 + C_3 \right) - \left( \dot{H} + 3H^2 \right) (h_1 + 3C_2 + C_3) \]

\[ + \frac{3}{2} H^2 \left( C_1 + 3C_2 + C_3 \right) \].

The stress-energy tensor of the pseudoscalar field reads

\[ T_{ik}^{(A)} = \frac{1}{2} \Psi_0^2 \left( U_i U_k \left( V + \dot{\phi}^2 \right) + \Delta_{ik} \left( V - \dot{\phi}^2 \right) \right). \]

As usual, the scalars \( W_{(A)} \) and \( P_{(A)}, \) given by

\[ W_{(A)} = \frac{1}{2} \Psi_0^2 \left( V + \dot{\phi}^2 \right), \quad P_{(A)} = \frac{1}{2} \Psi_0^2 \left( \dot{\phi}^2 - V \right), \]

describe, respectively, the energy density and pressure of the axionic dark matter [40].

4. The key equation for the gravity field

As usual, only one of two non-trivial equations for the gravity field is independent for the symmetry associated with the Friedmann-type model; the second equation is the differential consequence, since the compatibility conditions are satisfied identically for the solutions to the axion field equation. This key equation can be written in the following form:

\[ \frac{1}{\kappa \Psi_0^2} \left\{ 3H^2 \left[ 1 + \frac{1}{2} (C_1 + 3C_2 + C_3) \right] - \Lambda \right\} = \]

\[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{4} (\dot{\phi}^2 - \dot{\Phi}^2)^2 + \gamma H \Phi_* (\dot{\phi}^2 - \dot{\Phi}^2) \frac{d\Phi_*}{dH}. \]
B. Exact solution, describing the basic state of axion field in the case \( \Lambda \neq 0 \)

We indicate the state of the pseudoscalar (axion) field as the basic one, when \( \phi = \pm \Phi_* \); in this sense the evolution of the function \( \Phi_* (t) \), guided by non-uniformly moving aether, describes the evolution of the axionic dark matter. For definiteness, below we assume that \( \phi = + \Phi_* \). Now the model is reduced to the following pair of evolutionary equations:

\[
\ddot{\Phi}_* + 3H \dot{\Phi}_* = 0 , \tag{32}
\]

\[
H^2 = H_\infty^2 + \frac{\kappa \Psi_0^2}{6\Gamma} \dot{\Phi}_*^2 . \tag{33}
\]

Here we introduced two convenient parameters: the first one is \( \Gamma \), the parameter containing the Jacobson’s constants only:

\[
\Gamma \equiv 1 + \frac{1}{2} (C_1 + 3C_2 + C_3) . \tag{34}
\]

The second parameter, \( H_\infty \), given by

\[
H_\infty \equiv \sqrt{\frac{\Lambda}{3\Gamma}} , \tag{35}
\]

plays the role of asymptotic value of the Hubble function, if the \( \Phi_* (t \to \infty) \to 0 \).

1. Searching for geometrical characteristics of the model

If we extract \( \ddot{\Phi}_* \) from (33) and put it into (32), we obtain the equation for the Hubble function

\[
\dot{H} + 3\left( H^2 - H_\infty^2 \right) = 0 , \tag{36}
\]

the solution of which is

\[
H(t) = H_\infty \cosh[3H_\infty(t - t_*)] , \tag{37}
\]

where \( t_* \) is an integration constant, which we will identify later. We assume, that the evolution of the model starts at the time moment \( t = t_0 \), and require that \( t_0 > t_* \); in other words for \( t \geq t_0 \) the Hubble function takes only finite values. Using (37) we obtain the scale factor to have the form

\[
a(t) = a(t_*) \left\{ \cosh[3H_\infty(t - t_*))] \right\}^{\frac{2}{3}} . \tag{38}
\]

The acceleration parameter

\[
- q(t) \equiv \frac{\ddot{a}}{aH^2} = 1 + \frac{\dot{H}}{H^2} = 1 - \frac{3}{c_\lambda^2[3H_\infty(t - t_*)]} \tag{39}
\]

takes zero value at \( t = t_* + \frac{1}{3H_\infty} \ln[\sqrt{3} \pm \sqrt{2}] \). Only one value of the cosmological time corresponds to inequality \( t > t_* \), and we indicate the cosmological event at the time moment \( t_T \):

\[
t_T \equiv t_* + \frac{1}{3H_\infty} \ln[\sqrt{3} + \sqrt{2}] \tag{40}
\]
as the transition point, which separates the epochs of decelerated expansion \( t_0 < t < t_T \) and of accelerated expansion \( t > t_T \).

2. Searching for the solution for the pseudoscalar field

The quantity \( \ddot{\Phi}_* (t) \) as the function of time at \( t > t_* \) can be extracted from (33) and (32):

\[
\ddot{\Phi}_* (t) = \pm \sqrt{\frac{6\Gamma}{\kappa \Psi_0^2}} \cosh[3H_\infty(t - t_*)]^{-1} \tag{41}
\]

clearly, it vanishes asymptotically, \( \dot{\Phi}_* (\infty) = 0 \). Since the function \( \dot{\Phi}_* (t) \) is positive, \( \Phi_* (t) \) is negative, and take the sign minus in (41). Thus the solution for \( \Phi_* (t) \) is

\[
\Phi_* (t) = \Phi_\infty + \frac{1}{3} \sqrt{\frac{6\Gamma}{\kappa \Psi_0^2}} \ln \left\{ \cosh \left[ \frac{3}{2} H_\infty (t - t_*) \right] \right\} , \tag{42}
\]

the constant of integration \( \Phi_\infty \) has to be chosen using some ansatz. We assume that \( \Phi_* (t \to \infty) = \Phi_\infty = 0 \), and now we can reconstruct the function \( \Phi_* (H) \) as follows:

\[
\Phi_* (H) = \frac{1}{3} \sqrt{\frac{6\Gamma}{\kappa \Psi_0^2}} \ln \left[ \frac{\sqrt{H^2 - H_\infty^2} + H}{H_\infty} \right] . \tag{43}
\]

In order to identify the parameter \( t_* \), we calculate \( \dot{\Phi}_* (t_0) \), and obtain finally that

\[
t_* = t_0 - \frac{1}{3H_\infty} \ln \left[ \frac{H_\infty}{\dot{\Phi}_* (t_0)} \right] \sqrt{\frac{6\Gamma}{\kappa \Psi_0^2}} \left( 1 + \sqrt{1 + \frac{\dot{\Phi}_*^2 (t_0) \kappa \Psi_0^2}{6\Gamma H_\infty^2}} \right) . \tag{44}
\]

It is clear from (41) with negative sign that

\[
\dot{\Phi}_* (t) = - \dot{\Phi}_* (\infty) \tag{45}
\]

3. Searching for the components of the stress-energy tensor

Since the axion potential vanishes on the solution \( \Phi = \Phi_* \), i.e., \( V(\phi^2, \Omega^2) = 0 \), the axion energy density and pressure (30) coincide

\[
W(A) = P(A) = \frac{1}{2} \Psi_0^2 \dot{\Phi}_*^2 = \lambda \kappa \cosh^2[3H_\infty(t - t_*)] . \tag{45}
\]
This means that the effective equation of state for the axions is of the stiff type. Similarly, one can obtain the aether energy density and pressure:

\[ W(U) = 3H^2(t)(1-\Gamma) , \]
\[ P(U) = 3(1-\Gamma)[H^2(t) - 2H^2_\infty] . \] (46)

The aether energy density is positive, when \( \Gamma < 1 \). The aether pressure changes the sign at the moment \( t_d \), when

\[ H(t_d) = \sqrt{2}H_\infty \rightarrow t_d = t_\ast + \frac{1}{3H_\infty} \ln \left( \sqrt{2}+1 \right) . \] (47)

At the same time moment \( t=t_d \) the effective parameter \( \zeta \), given by

\[ \zeta(t) = \frac{P(U)}{W(U)} = 1 - 2tq^2[3H_\infty(t-t_\ast)] , \] (48)

takes zero value and then changes the sign. The effective aether enthalpy

\[ P(U)+W(U) = 6(1-\Gamma)(H^2-H^2_\infty) = \frac{2\Lambda(1-\Gamma)}{\Gamma \sin^2[3H_\infty(t-t_\ast)]} \] (49)
is positive, when \( 0 < \Gamma < 1 \), and takes zero value asymptotically at \( t \rightarrow \infty \). This means that asymptotically the aether can play the role of the dark energy of the \( \Lambda \) type.

4. Asymptotic properties of the model functions

The obtained solution has a quasi-de Sitter asymptote

\[ H(t \rightarrow \infty) \rightarrow H_\infty , \quad -q(t \rightarrow \infty) \rightarrow 1 . \]
\[ a(t \rightarrow \infty) \propto e^{H_\infty t} . \] (50)
The value \(-q(t_0)\) is equal to

\[ -q(t_0) = \frac{3H^2_\infty}{H^2(t_0)} - 2 . \] (51)

Clearly, the transition point with \(-q(t_T) = 0\) exists, if \(-q(t_0)\) is negative, while \(-q(\infty) = 1\) is positive. This is possible, when \( H(t_0) > \sqrt{2}H_\infty \).

In the asymptotic limit \( t \rightarrow \infty \) the axion energy density and pressure, as well as, the aether energy density and pressure behave as follows:

\[ W(A) = P(A) \propto e^{-6H_\infty t} \rightarrow 0 , \]
\[ W(U) \rightarrow \Lambda \left( \frac{1}{\Gamma} - 1 \right) , \quad P(U) \rightarrow -\Lambda \left( \frac{1}{\Gamma} - 1 \right) . \] (52)

In order to complete the analysis let us consider the case with vanishing cosmological constant.

C. Exact solution, describing the basic state of axion field in the case \( \Lambda = 0 \)

When the cosmological constant vanishes, the solutions to the key equations for the Hubble function, scale factor and acceleration parameter take the form

\[ H(t) = \frac{H(t_0)}{1 + 3H(t_0)(t - t_0)} , \quad H(t_0) = \sqrt{\frac{\kappa \Psi_0^2}{6\Gamma}} \Phi_\ast(t_0) , \] (53)
\[ a(t) = a(t_0) \left[ 1 + 3H(t_0)(t - t_0) \right]^{\frac{1}{3}} , \quad -q(t) = -2 . \] (54)

The basic state for the axion field is described by the function

\[ \Phi_\ast(t) = \Phi_\ast(t_0) + \frac{\Phi_\ast(t_0)}{3K} \ln \left[ 1 + 3H(t_0)(t - t_0) \right] . \] (55)

Reconstruction of the function \( \Phi_\ast(H) \) yields

\[ \Phi_\ast(H) = \Phi_\ast(t_0) - \frac{\Phi_\ast(t_0)}{3H(t_0)} \ln \frac{H}{H(t_0)} . \] (56)

Now the axion energy density and pressure, and the corresponding aether quantities are of the form

\[ \kappa W(A) = \kappa P(A) = \Gamma F , \quad W(U) = P(U) = (1-\Gamma)F , \] (57)
where

\[ F = \frac{3H^2(t_0)}{[1 + 3H(t_0)(t - t_0)]^2} = \kappa W(A) + W(U) . \] (58)

Again, the aether energy density is positive, when \( \Gamma < 1 \).

D. Stability of the model with \( \Lambda \neq 0 \)

In order to solve the stability problem in a general form we have to go beyond the model in which all the quantities depend on time only, and thus we have to consider perturbations in an inhomogeneous universe. This problem is out of frame of this paper. However, one can answer the question: whether the homogeneous perturbations, depending on time only, can grow with cosmological time? For this purpose we assume that there are small variations of the pseudoscalar field and of the Hubble function:

\[ \phi(t) \rightarrow \Phi_\ast(t) + \psi(t) , \quad H(t) \rightarrow H(t) + h(t) . \] (59)

Then the equations (31) and (27) give, respectively:

\[ \frac{6\Gamma}{\kappa \Psi_0^2} Hh = \dot{\Phi}_\ast + 2\gamma \psi \Phi_\ast^2 H \frac{d\Phi_\ast}{dH} . \] (60)
\[ \ddot{\psi} + 3H \dot{\psi} + 3h \ddot{\Phi}_\ast + 2\gamma \psi \Phi_\ast^2 = 0 . \] (61)
When we extract \( h \) from (60), put it into (61) and use (37), (41) and (43) for \( H, \Phi_* \) and \( \Phi_* \), we obtain the key equation

\[
\dot{\psi} + 3H \psi \left[ \frac{2H}{H_{\infty}} - \frac{H_{\infty}}{H} \right] = 0. 
\]  

(62)

Surprisingly, this equation does not include \( \psi \); first integration of (62) yields

\[
\frac{\dot{\psi}(t)}{\dot{\psi}(t_0)} = \left[ \frac{sh[3H_{\infty}(t_0-t_*)]}{sh[3H_{\infty}(t-t_*)]} \right]^2 \frac{ch[3H_{\infty}(t-t_*)]}{ch[3H_{\infty}(t_0-t_*)]}. 
\]  

(63)

The ratio of the first derivatives of the perturbation and of the basic function

\[
\frac{\dot{\psi}(t)}{\Phi_* (t)} = \frac{\dot{\psi}(t)}{\Phi_* (t_0)} \sqrt{\frac{6t H_{\infty}^2 + \kappa \Psi_2^2 \Phi_*^2(t)}{6t H_{\infty}^2 + \kappa \Psi_2^2 \Phi_*^2(t)}} 
\]  

(64)

does not grow with time; similarly to the function \( \dot{\Phi}_*(t) \) (see (61)) the function \( \dot{\psi}(t) \) vanishes asymptotically. The second integration gives

\[
\psi(t) - \psi(t_0) = \frac{\dot{\psi}(t_0)}{3H(t_0)} \left[ 1 - \frac{\Phi_* (t)}{\Phi_* (t_0)} \right]. 
\]  

(65)

Clearly, the function \( \psi(t) \) tends monotonically to the constant at \( t \to \infty \):

\[
\psi(\infty) = \psi(t_0) + \frac{\dot{\psi}(t_0)}{3H(t_0)}, \quad |\psi(t) - \psi(\infty)| \sim e^{-3H_{\infty}t}. 
\]  

(66)

Linear approximation of the perturbation theory is valid, if not only \( |\psi(t_0)| << |\Phi_* (t_0)| \), but also if \( |\psi(\infty)| << |\Phi_* (\infty)| \); special case is when \( \Phi_* (\infty) = 0 \). Here we have to choose \( \psi(\infty) = 0 \) in this case. We see that there are no growing modes in the perturbations of pseudoscalar field. Moreover, keeping in mind that

\[
|\Phi_*(t \to \infty)| \sim e^{-3H_{\infty}t}, \quad \Phi_*^2 \sim e^{-6H_{\infty}t},
\]

we find that the asymptotic behavior of the function \( h(t) \) is \( |h(t \to \infty)| \sim e^{-3H_{\infty}t} \), when \( \psi(\infty) \neq 0 \), and \( |h(t \to \infty)| \sim e^{-6H_{\infty}t} \), when \( \psi(\infty) = 0 \). The corresponding estimations for the scale factor are, respectively

\[
|a(t \to \infty) - a_{\infty} e^{H_{\infty}t}| \sim e^{-2H_{\infty}t} \quad (\psi(\infty) \neq 0), 
\]

\[
|a(t \to \infty) - a_{\infty} e^{H_{\infty}t}| \sim e^{-5H_{\infty}t} \quad (\psi(\infty) = 0). 
\]  

(68)

We deal with asymptotically stable model.

IV. DISCUSSION

When we prepared the manuscript, we kept in mind two models of stiff control in moving physical systems. The first model deals with a moving dense plasma in external fields (gravitational and/or electromagnetic); in this model frequent particle collisions enforce the distribution function of plasma to follow the specific equilibrium function, which turns to zero the collision integral [37], and depends on parameters of external fields. The second model relates to the dynamics of large granules in the viscous fluid flow; the Stokes force in this model coerces the granules to have the velocity coinciding with the non-uniform macroscopic velocity of the fluid flow [18]. We tried to imagine, how the dynamic aether could realize the stiff regulation of the behavior of axionic dark matter. These two analogies hinted us, that the guidance of such kind is possible through the specific Higgs potential, \( V(\phi^2, \Omega^2) = \frac{1}{2} \gamma [\phi^2 - \Phi_*^2 (\Omega^2)]^2 \), describing non-linear self-interaction of pseudoscalar (axion) field. When the Higgs potential turns into zero, we obtain the analog of equilibrium for the axionic system in the aether flow. When the basic state \( \Phi_* (\Omega^2) \) is not constant and depends on the aether guiding function \( \Omega^2 \equiv g^{\mu\nu} g^{\alpha\beta} \nabla_\mu U_\mu \nabla_\nu U_\nu \), we face with the aether control over the state of the dark matter. The appropriate tool for this task is the extended Einstein-aether-axion model; the corresponding extension is connected with the fact that now the potential \( V(\phi^2, \Omega^2) \) includes not only the pseudoscalar field in square \( \phi^2 \), but also the vector field, the aether velocity four-vector, the metric and Christoffel symbols. The variation procedure gives the corresponding additional source terms into the master equations of the vector field [12], [14] and of the gravitational field (see, e.g., [18] with [15]).

Since the cosmology is the natural application of this model, we reduced the obtained master equations to the symmetry of homogeneous isotropic Friedmann - type model. In this model the guiding function \( \Omega^2 \) is proportional to the square of the Hubble function \( \Omega^2 = 3H^2 \), thus, just the rate of cosmological expansion predetermines the evolution of the basic state of the pseudoscalar field. We have found exact solutions of the reduced system of master equations in case of stiff regulation, i.e., when the value of the pseudoscalar field \( \phi(t) \) at any time \( t \) coincides with the basic state function \( \Phi_* (H(t)) \). When the cosmological constant is non-vanishing, \( \Lambda \neq 0 \), the Hubble function is presented by (37), and the scale factor is given by (48), when \( \Lambda = 0 \) we deal with the formulas (53) and (54), respectively. The quantity \( \Phi_* \) as a function of cosmological time is given, respectively, by (42) and (55); reconstruction of the function \( \Phi_* (H) \) gives, respectively, (13) and (50).

Finally, we have to formulate the following conclusions concerning the model in the framework of which the dynamic aether is shown to provide stiff regulation of the axionic dark matter behavior.
1. The Einstein-aether-axion model with cosmological constant $\Lambda \neq 0$ guarantees the existence of one transition point in the Universe history at $t = t_\nu$ [11], which separates the epochs of decelerated expansion and of the late-time accelerated expansion; as well as, it guarantees that the asymptotic regime of expansion is of the quasi-de Sitter type (Pseudo-Rip) with Hubble constant $H_\infty = \sqrt{\frac{\Lambda}{3}}$, where $\Gamma = 1 + \frac{1}{2}(C_1 + 3C_2 + C_3)$ is the parameter containing three Jacobson’s coupling constants $C_1, C_2, C_3$.

2. The scalars, which describe the effective energy density and pressure of the aether [16], demonstrate four interesting features: first, the aether energy density $W(U)$ is positive, when the effective Jacobson’s parameter $C_1 + 3C_2 + C_3$ is negative and satisfies the inequality $-2 < C_1 + 3C_2 + C_3 < 0$; second, the effective aether pressure $P(U)$ changes the sign and becomes negative at the time moment $t = t_\nu$ (see [17]); third, the function $W(U) + 3P(U)$ becomes negative, when $t > t_\nu > t_\nu$; fourth, the effective aether enthalpy $W(U) + P(U)$ tends to zero asymptotically at $t \to \infty$ (see [19]). In other words, the aether behaves as a dark energy, starting from the time moment $t_\nu$, and asymptotically becomes the dark energy of the $\Lambda$ type. When $\Lambda = 0$, the model can not explain the late-time accelerated expansion, since the acceleration parameter is negative for any time moment.

3. The scalars of energy density and pressure of the axionic dark matter [15] demonstrate that the corresponding effective equation of state, $W_A = P_A > 0$, can be indicated as the stiff one; this fact attracts the attention to the studies of the so-called stiff eras appeared in the framework of Modified Gravity (see, e.g., [49] and references therein).

4. When $\Lambda \neq 0$, homogeneous perturbations of the pseudoscalar (axion) field, of the Hubble function and of the scale factor fade out with cosmological time, i.e., we deal with the stable model of stiff regulation of the axionic dark matter behavior by the dynamic aether.

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