Truncated matricial moment problems on a finite interval: the operator approach.

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1 Introduction.

In this paper we study the following problem: to find a non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $[a,b]$, which is left-continuous in $(a,b)$, $M(a) = 0$, such that

$$\int_a^b x^n dM(x) = S_n, \quad n = 0, 1, \ldots, l, \quad (1)$$

where $\{S_n\}_{n=0}^l$ is a given sequence of Hermitian ($N \times N$) complex matrices, $N \in \mathbb{N}$, $l \in \mathbb{Z}_+$. Here $a, b \in \mathbb{R}$: $a < b$.

In the scalar case this problem was solved by M.G. Krein, see [1]. Recently, a deep investigation of the matrix moment problem (1) was completed by A.E. Choque Rivero, Yu.M. Dyukarev, B. Frizsche and B. Kirstein, see [2],[3]. These authors used the Potapov method for interpolating problems which was enriched by the Sachnovich method of operator identities.

Set

$$\Gamma_k = (S_{i+j})_{i,j=0}^k = \begin{pmatrix} S_0 & S_1 & \cdots & S_k \\ S_1 & S_2 & \cdots & S_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_k & S_{k+1} & \cdots & S_{2k} \end{pmatrix}, \quad k \in \mathbb{Z}_+ : 2k \leq l; \quad (2)$$

$$\tilde{\Gamma}_k = (-abS_{i+j} + (a + b)S_{i+j+1} - S_{i+j+2})_{i,j=0}^{k-1}, \quad k \in \mathbb{Z}_+ : 2k \leq l. \quad (3)$$

If we choose an arbitrary element $f = (f_0, f_1, \ldots, f_{N-1})$, where all $f_k$ are some polynomials and calculate $\int_a^b f dM f^*$, one can easily deduce that

$$\Gamma_k \geq 0, \quad k \in \mathbb{Z}_+ : 2k \leq l. \quad (4)$$

In the case of an odd number of prescribed moments $l = 2d$, the strong result of A.E. Choque Rivero, Yu.M. Dyukarev, B. Frizsche and B. Kirstein is that conditions

$$\Gamma_d \geq 0, \quad \tilde{\Gamma}_d \geq 0, \quad (5)$$

are necessary and sufficient for the solvability of the matrix moment problem [1], see [3 Theorem 1.3, p. 106]. For the case $\Gamma_d > 0, \tilde{\Gamma}_d > 0$, they
parameterized all solutions of the moment problem via a linear fractional transformation where the set of parameters consisted of some distinguished pairs of meromorphic matrix-valued functions.

Set

\[ H_k = (-aS_{i+j} + S_{i+j+1})^k_{i,j=0}, \quad \tilde{H}_k = (bS_{i+j} - S_{i+j+1})^k_{i,j=0}, \quad k \in \mathbb{Z}_+ : 2k+1 \leq l. \]

(6)

In the case \( l = 2d+1 \), the analogous to the result for \( l = 2d \), the result of A.E. Choque Rivero, Yu.M. Dyukarev, B. Fritzsche and B. Kirstein states that conditions

\[ H_d \geq 0, \quad \tilde{H}_d \geq 0, \]

(7)

are necessary and sufficient for the solvability of the matrix moment problem \( (1) \), see [2, Theorem 1.3, p. 127]. For the case \( H_d > 0, \tilde{H}_d > 0 \), they parameterized all solutions of the moment problem via a linear fractional transformation. The set of parameters consisted of some distinguished pairs of meromorphic matrix-valued functions.

In this work we will study the matrix moment problem \( (1) \) by virtue of the operator approach based on the use of the generalized resolvents of some symmetric operators. In the study of the classical Hamburger moment problem this approach finds its origin in the papers of M.A. Neumark [4],[5] and M.G. Krein, M.A. Krasnoselskiy [6], see also [7]. All these authors used orthogonal polynomials connected with a Jacobi matrix related to the moment problem. Lately, we showed that this method in a general setting can be applied to the Hamburger moment problem both in the non-degenerate and degenerate cases, see [8]. Our goal here is to describe all solutions of the matrix moment problem \( (1) \) in a general case. This means that no conditions besides solvability of the moment problem will be assumed. At first, we study the case of an odd number of prescribed moments \( l = 2d \), and then we shall reduce the case of an even number of moments \( l = 2d+1 \) to the previous case \( (d \in \mathbb{Z}_+) \). In our study we shall use the basic results of M.G. Krein and I.E. Ovcharenko on generalized sc-resolvents of symmetric contractions, as well as M.G. Krein’s theory of self-adjoint extensions of semi-bounded symmetric operators, see [9],[10],[11].

Notations. As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \) the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively. The space of \( n \)-dimensional complex vectors \( a = (a_0, a_1, \ldots, a_{n-1}) \), will be denoted by \( \mathbb{C}^n, n \in \mathbb{N} \). If \( a \in \mathbb{C}^n \) then \( a^* \) means the complex conjugate vector. For a complex \( (n \times n) \) matrix \( A \), we denote by \( \ker A \) a set \( \{ x \in \mathbb{C}^n : Ax = 0 \} \). By \( \mathbb{P} \) we denote a set of all complex polynomials and by \( \mathbb{P}_d \) we mean all complex polynomials with degrees less or equal to \( d, d \in \mathbb{Z}_+ \), (including the zero
polynomial). Let $M(x)$ be a left-continuous non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $\mathbb{R}$, $M(-\infty) = 0$, and $\tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x)$; $\Psi(x) = (d m_{k,l}/d \tau_M)_{k,l=0}^{N-1}$. We denote by $L^2(M)$ a set (of classes of equivalence) of vector functions $f : \mathbb{R} \to \mathbb{C}^N$, $f = (f_0, f_1, \ldots, f_{N-1})$, such that (see, e.g., [12])

$$\|f\|_{L^2(M)}^2 := \int_{\mathbb{R}} f(x) \Psi(x) f^*(x) d\tau_M(x) < \infty.$$ 

The space $L^2(M)$ is a Hilbert space with the scalar product 

$$(f, g)_{L^2(M)} := \int_{\mathbb{R}} f(x) \Psi(x) g^*(x) d\tau_M(x), \quad f, g \in L^2(M).$$

For a separable Hilbert space $H$ we denote by $(\cdot, \cdot)_H$ and $\| \cdot \|_H$ the scalar product and the norm in $H$, respectively. The indices may be omitted in obvious cases.

For a linear operator $A$ in $H$ we denote by $D(A)$ its domain, by $R(A)$ its range, and by $A^*$ we denote its adjoint if it exists. If $A$ is bounded, then $\|A\|$ stands for its operator norm. For a set of elements $\{x_n\}_{n \in B}$ in $H$, we denote by Lin$\{x_n\}_{n \in B}$ and span$\{x_n\}_{n \in B}$ the linear span and the closed linear span (in the norm of $H$), respectively, where $B$ is an arbitrary set of indices. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of $M$ with respect to the norm of $H$. By $E_H$ we denote the identity operator in $H$, i.e. $E_H x = x$, $x \in H$. If $H_1$ is a subspace of $H$, by $P_{H_1} = P_{H_1}^H$ we denote the operator of the orthogonal projection on $H_1$ in $H$. A set of linear bounded operators which map $H$ into $H$ we denote by $[H]$.

2 The case of an odd number of given moments: solvability and a description of solutions.

We shall use the following important fact (see, e.g., [13], p.215):

Theorem 2.1 Let $K = (K_{n,m})_{n,m=0}^{r} \geq 0$ be a positive semi-definite complex $(r+1) \times (r+1)$ matrix, $r \in \mathbb{Z}_+$. Then there exist a finite-dimensional Hilbert space $H$ with a scalar product $(\cdot, \cdot)$ and a sequence $\{x_n\}_{n=0}^{r}$ in $H$, such that

$$K_{n,m} = (x_n, x_m), \quad n, m = 0, 1, \ldots, r,$$

and span$\{x_n\}_{n=0}^{r} = H$. 

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Proof. Let \( \{x_n\}_{n=0}^r \) be an arbitrary orthonormal basis in \( \mathbb{C}^n \). Introduce the following functional:

\[
[x, y] = \sum_{n,m=0}^r K_{n,m} a_n \overline{b_m},
\]

for \( x, y \in \mathbb{C}^n \),

\[
x = \sum_{n=0}^r a_n x_n, \quad y = \sum_{m=0}^r b_m x_m, \quad a_n, b_m \in \mathbb{C}.
\]

The space \( \mathbb{C}^n \) equipped with \( [\cdot, \cdot] \) will be a quasi-Hilbert space. Factorizing and making the completion we obtain the required space \( H \) (see, e.g., [14, p. 10-11]).

Consider the matrix moment problem (11) with \( l = 2d \), \( d \in \mathbb{N} \). Suppose that \( \Gamma_d \geq 0 \) (as we noticed in the Introduction, condition (4) is necessary for the solvability of the moment problem). Let \( \Gamma_d = (\gamma_{d,n,m})_{n,m=0}^{(d+1)N-1}, \gamma_{d,n,m} \in \mathbb{C} \). By Theorem 2.1 there exist a finite-dimensional Hilbert space \( H \) and a sequence \( \{x_n\}_{n=0}^{(d+1)N-1} \) in \( H \), such that

\[
(x_n, x_m) = \gamma_{d,n,m}, \quad n, m = 0, 1, \ldots, (d+1)N - 1,
\]

and \( \text{span}\{x_n\}_{n=0}^{(d+1)N-1} = \text{Lin}\{x_n\}_{n=0}^{(d+1)N-1} = H \). Notice that

\[
\gamma_{d,rN+j,tN+n} = s_{r+t}^{j,n}, \quad 0 \leq j, n \leq N - 1; \quad 0 \leq r, t \leq d,
\]

where

\[
S_n = (s_n^{k,l})_{k,l=0}^{N-1}, \quad n \in \mathbb{Z}_+,
\]

are the given moments. From (11) it follows that

\[
\gamma_{d:a+N,b} = \gamma_{d:a,b+N}, \quad a = rN + j, b = tN + n, 0 \leq j, n \leq N - 1; \quad 0 \leq r, t \leq d - 1.
\]

In fact, we can write

\[
\gamma_{d:a+N,b} = \gamma_{d:(r+1)N+j,tN+n} = s_{r+t+1}^{j,n} = \gamma_{d:rN+j,(t+1)N+n} = \gamma_{d:a,b+N}.
\]

Set \( H_a = \{x_n\}_{n=0}^{dN-1} \). We introduce the following operator:

\[
Ax = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \quad x \in H_a, \quad x = \sum_{k=0}^{dN-1} \alpha_k x_k.
\]

The following proposition shows when the operator \( A \) is correctly defined.

\[
\text{(12)}
\]
Theorem 2.2 Let a matrix moment problem $[1]$ with $l = 2d$, $d \in \mathbb{N}$, be given. The moment problem has a solution if and only if conditions $[2]$ are true and

$$\text{Ker } \Gamma_{d-1} \subseteq \text{Ker } \hat{\Gamma}_{d-1}, \quad (14)$$

where $\hat{\Gamma}_{d-1} = (S_{i+j+2})_{i,j=0}^{d-1}$.

If conditions $[3], [14]$ are satisfied then the operator $A$ in $(13)$ is correctly defined and the following operator:

$$Bx = \frac{2}{b-a} A - \frac{a+b}{b-a} E_H, \quad x \in H_a, \quad (15)$$

is a contraction in $H$ (i.e. $\|B\| \leq 1$). Moreover, operators $A$ and $B$ are Hermitian.

Proof. Let the matrix moment problem $(1)$ has a solution $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$. Consider the space $L^2(M)$ and let $Q$ be the operator of multiplication by an independent variable in $L^2(M)$. The operator $Q$ is self-adjoint and its resolution of unity is (see $[12]$)

$$E_b - E_a = E([a,b)) : h(x) \to \chi_{[a,b)}(x)h(x), \quad (16)$$

where $\chi_{[a,b)}(x)$ is the characteristic function of an interval $[a, b)$, $-\infty \leq a < b \leq +\infty$.

Set $e_k = (e_{k,0}, e_{k,1}, \ldots, e_{k,N-1})$, $e_{k,j} = \delta_{k,j}$, $0 \leq j \leq N - 1$, for $k = 0, 1, \ldots N - 1$. A set of (classes of equivalence of) functions $f \in L^2(M)$ such that (the corresponding class includes) $f = (f_0, f_1, \ldots, f_{N-1})$, $f_j \in \mathbb{P}_d$, we denote by $P^2_d(M)$ and call a set of vector polynomials of order $d$ in $L^2(M)$. Set $L^2_{d,0}(M) = P^2_d(M)$. Since $P^2_d(M)$ is finite-dimensional, we have $L^2_{d,0}(M) = P^2_d(M)$.

For an arbitrary polynomial (in a class) from $P^2_d(M)$ there exists a unique representation of the following form:

$$f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{d} \alpha_{k,j} x^j \bar{e}_k, \quad \alpha_{k,j} \in \mathbb{C}. \quad (17)$$

Let a polynomial $g \in P^2_d(M)$ have a representation

$$g(x) = \sum_{l=0}^{N-1} \sum_{r=0}^{d} \beta_{l,r} x^r \bar{e}_l, \quad \beta_{l,r} \in \mathbb{C}. \quad (18)$$

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We can write

\[
(f, g)_{L^2(M)} = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{d-1} \alpha_{k,j} \beta_{l,r} \int_{\mathbb{R}} x^{j+r} \tilde{e}_k dM(x) \tilde{e}_l^* = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{d-1} \alpha_{k,j} \beta_{l,r} \star \int_{\mathbb{R}} x^{j+r} dm_{k,l}(x) = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{d-1} \alpha_{k,j} \beta_{l,r} s_{j,r},
\]

where

\[
S_n = (s_n^{k,l})_{k,l=0}^{N-1}, \quad n \in \mathbb{Z}^+,
\]

are the given moments. On the other hand, we can write

\[
\begin{align*}
&\left( \sum_{j=0}^{d-1} \sum_{k=0}^{N-1} \alpha_{k,j} x^{j+N+k}, \sum_{r=0}^{d-1} \sum_{l=0}^{N-1} \beta_{l,r} x^{r+N+l} \right) \left( \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{d-1} \alpha_{k,j} \beta_{l,r} s_{j,r} \right) H = \\
&= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{d-1} \alpha_{k,j} \beta_{l,r} d_{j,N+k,r,N+l} = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{d-1} \alpha_{k,j} \beta_{l,r} s_{j,r}.
\end{align*}
\]

From relations (19), (21) it follows that

\[
(f, g)_{L^2(M)} = \left( \sum_{j=0}^{d-1} \sum_{k=0}^{N-1} \alpha_{k,j} x^{j+N+k}, \sum_{r=0}^{d-1} \sum_{l=0}^{N-1} \beta_{l,r} x^{r+N+l} \right)_H.
\]

Set

\[
V f = \sum_{j=0}^{d-1} \sum_{k=0}^{N-1} \alpha_{k,j} x^{j+N+k},
\]

for \( f(x) \in \mathbb{P}_d^2(M), f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{d-1} \alpha_{k,j} x^j \tilde{e}_k \), \( \alpha_{k,j} \in \mathbb{C} \).

If \( f, g \) have representations (17), (18), and \( \| f - g \|_{L^2(M)} = 0 \), then from (22) it follows that

\[
\| V f - V g \|_H^2 = (V(f-g), V(f-g))_H = (f-g, f-g)_{L^2(M)} = \| f - g \|_{L^2(M)}^2 = 0.
\]

Thus, \( V \) is a correctly defined operator from \( \mathbb{P}_d^2(M) \) to \( H \).

Relation (22) shows that \( V \) is an isometric transformation from \( \mathbb{P}_d^2(M) \) onto \( \text{Lin}\{x_n\}_{n=0}^{d(N+1) - 1} \). Thus, \( V \) is an isometric transformation from \( L^2_{d,0}(M) \) onto \( H \). In particular, we note that

\[
V x^j \tilde{e}_k = x^{j+N+k}, \quad 0 \leq j \leq d; \quad 0 \leq k \leq N - 1.
\]

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Set \( L^2_{d,1}(M) := L^2(M) \oplus L^2_{d,0}(M) \), and \( U := V \oplus E_{L^2_{d,1}(M)} \). The operator \( U \) is an isometric transformation from \( L^2(M) \) onto \( H \oplus L^2_{d,1}(M) =: \hat{H} \). Set

\[
\hat{A} := UQU^{-1}. \tag{25}
\]

The operator \( \hat{A} \) is a self-adjoint operator in \( \hat{H} \). Notice that

\[
UQU^{-1}x_{jN+k} = VQx_{jN+k} = VX^{j+1}e_k = x_{(j+1)N+k} = x_{jN+k+N},
\]

\[0 \leq j \leq d - 1; \quad 0 \leq k \leq N - 1.\]

By linearity we get

\[
UQU^{-1}x = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \quad x \in H_a, \quad x = \sum_{k=0}^{dN-1} \alpha_k x_k.
\]

Consequently, the operator \( A \) in (13) is correctly defined and

\[
A = \hat{A}|_{H_a}. \tag{26}
\]

Since \( A \) is correctly defined, from the equality

\[
\sum_{k=0}^{dN-1} \xi_k x_k = 0, \tag{27}
\]

with some complex numbers \( \xi_k \), it should follow the equality

\[
\sum_{k=0}^{dN-1} \xi_k x_{k+N} = 0. \tag{28}
\]

On the other hand, the equality (27) is equivalent to the equalities

\[
\sum_{k=0}^{dN-1} \xi_k(x_k, x_l) = \sum_{k=0}^{dN-1} \xi_k \gamma_{d,k,l} = 0, \quad l = 0, 1, \ldots, dN - 1. \tag{29}
\]

Analogously, the equality (28) is equivalent to the equalities

\[
\sum_{k=0}^{dN-1} \xi_k(x_{k+N}, x_{l+N}) = \sum_{k=0}^{dN-1} \xi_k \gamma_{d,k+N,l+N} = 0, \quad l = 0, 1, \ldots, dN - 1. \tag{30}
\]

If we shall use the matrix notations, the equality

\[
(\xi_0, \xi_1, \ldots, \xi_{dN-1}) (\gamma_{d,k,l})_{k,l=0}^{dN-1} = 0, \tag{31}
\]
implies the equality
\[(\xi_0, \xi_1, \ldots, \xi_{dN-1})(\gamma_{d,k+N,l+N})_{k,l=0}^{dN-1} = 0.\] (32)

Thus, relation (14) is true.
Consider the following operators:
\[R := \frac{2}{b-a} Q - \frac{a + b}{b-a} E_{L^2(M)},\] (33)
\[\hat{B} := U R U^{-1} = \frac{2}{b-a} \hat{A} - \frac{a + b}{b-a} E_{\hat{H}}.\] (34)

Define an operator \(B\) by the equality (15). From (26), (34) we get
\[B = \hat{B}|_{H_a}.\] (35)

For an arbitrary \(f \in D(R) = D(Q)\) we can write
\[\|Rf\|_{L^2(M)}^2 = \int_a^b \frac{2}{b-a} x - \frac{a + b}{b-a} f(x) dM(x) f^*(x) \leq \int_a^b f(x) dM(x) f^*(x) = \|f\|_{L^2(M)}^2,\]
and therefore the operators \(R, \hat{B}\) and \(B\) are contractions. Since \(R\) is Hermitian, the operators \(\hat{B}, B\) are Hermitian, as well. Choose an arbitrary \(x \in H_a, x = \sum_{k=0}^{dN-1} \alpha_k x_k,\) and write
\[0 \leq \|x\|^2 - \|Bx\|^2 = \sum_{k,j=0}^{dN-1} \alpha_k \overline{\alpha_j}(x_k, x_j) - \sum_{k,j=0}^{dN-1} \alpha_k \overline{\alpha_j}(Bx_k, Bx_j) =\]
\[= \sum_{k,j=0}^{dN-1} \alpha_k \overline{\alpha_j} \gamma_{d,k,j} - \sum_{k,j=0}^{dN-1} \alpha_k \overline{\alpha_j} \left( \frac{2}{b-a} x_k + N - \frac{a + b}{b-a} x_k, \frac{2}{b-a} x_j + N - \frac{a + b}{b-a} x_j \right) =\]
\[= \sum_{k,j=0}^{dN-1} \alpha_k \overline{\alpha_j} \left( \gamma_{d,k,j} - \frac{4}{(b-a)^2} \gamma_{d,k+N,j+N} + \frac{2(a + b)}{(b-a)^2} \gamma_{d,k+N,j} + \frac{2(a + b)}{(b-a)^2} \gamma_{d,k,j+N} - \frac{(a + b)^2}{(b-a)^2} \gamma_{d,k,j} \right).\] (36)

If we multiply the both sides of the latter inequality by \((b-a)^2\) and use (12) we get
\[0 \leq \sum_{k,j=0}^{dN-1} \alpha_k \overline{\alpha_j} \left( (b-a)^2 \gamma_{d,k,j} - 4 \gamma_{d,k+N,j+N} + 4(a + b) \gamma_{d,k+N,j} - (a + b)^2 \gamma_{d,k,j} \right) =\]
\[
\sum_{k,j=0}^{dN-1} \alpha_k \beta_j (x_{k+N}, x_j) = \sum_{k,j=0}^{dN-1} \alpha_k \beta_j (x_k, x_{j+N}) = (x, Ay) \]

Thus, operators \( A \) and \( B \) are Hermitian. If we use relation (36) (except the first inequality in it) and the second condition in (5), we obtain that \( B \) is a contraction. By Krein’s theorem [11, Theorem 2, p. 440], there exists a self-adjoint extension \( \tilde{B} \) of the operator \( B \) in \( H \) with the same norm as \( B \) (and therefore it is a contraction). Let

\[
\tilde{B} = \int_{-1}^{1} \lambda d\tilde{E}_\lambda \tag{37}
\]

where \( \{ \tilde{E}_\lambda \} \) be the left-continuous in \([-1, 1)\), right-continuous at the point 1, constant outside \([-1, 1)\), orthogonal resolution of unity of \( \tilde{B} \). Choose an arbitrary \( \alpha \), \( 0 \leq \alpha \leq d(N + 1) - 1 \), \( \alpha = rN + j \), \( 0 \leq r \leq d, 0 \leq j \leq N - 1 \). Notice that

\[
x_\alpha = x_{rN+j} = Ax_{(r-1)N+j} = \ldots = A^r x_j.
\]

Then choose an arbitrary \( \beta \), \( 0 \leq \beta \leq d(N + 1) - 1 \), \( \beta = tN + n \), \( 0 \leq t \leq d, 0 \leq n \leq N - 1 \). Using (11) we can write

\[
s_{r+t}^{j,n} = \gamma_{d,rN+j,tN+n} = (x_{rN+j}, x_{tN+n})_H = (A^r x_j, A^t x_n)_H = \left( \frac{b-a}{2} B + \frac{a+b}{2} E_H \right)^r x_j, \left( \frac{b-a}{2} B + \frac{a+b}{2} E_H \right)^t x_n \right)_H
\]
\[
\left( \frac{b-a}{2} \tilde{B} + \frac{a+b}{2} E_H \right)^{r+t} x_j, x_n \right)_H = \int_{-1}^{1} \left( \frac{b-a}{2} \lambda + \frac{a+b}{2} \right)^{r+t} d(\tilde{E}_{\lambda} x_j, x_n)_H.
\]

Set
\[
\tilde{m}_{j,n}(x) = (\tilde{E}_{\frac{2}{b-a}x - \frac{a+b}{b-a}} x_j, x_n)_H, \quad 0 \leq j, n \leq N - 1.
\]

Then
\[
\begin{align*}
\tilde{s}_{r+t}^{j,n} &= \int_{a}^{b} x^{r+t} d\tilde{m}_{j,n}(x), \quad 0 \leq j, n \leq N - 1, \quad 0 \leq r, t \leq d. \quad (38)
\end{align*}
\]

From relation (38) we derive that the matrix function \( \tilde{M}(\lambda) = (\tilde{m}_{j,n}(x))_{0 \leq j, n \leq N - 1} \) is a solution of the matrix Hamburger moment problem (11). (Properties of the orthogonal resolution of unity provide that \( \tilde{M}(\lambda) \) is left-continuous in \((a, b)\), non-decreasing and \( \tilde{M}(a) = 0 \)).

It remains to prove the last statement of the Theorem. If conditions (5),(14) are satisfied then we proved that the moment problem (11) has a solution. In this case we showed that the operator \( A \) in (13) is correctly defined and the operator \( B \) in (15) is a Hermitian contraction in \( H \). The fact that operators \( A \) and \( B \) are Hermitian was established, as well. \( \square \)

We shall continue our considerations before the statement of Theorem 2.2. We assume that conditions (5), (14) are true. Therefore the operators \( A \) in (13) and \( B \) in (15) are correctly defined Hermitian operators and \(|B| \leq 1 \).

Let \( \hat{B} \) be an arbitrary self-adjoint extension of \( B \) in a Hilbert space \( \hat{H} \supseteq H \). Let \( R_z(\hat{B}) \) be the resolvent of \( \hat{B} \) and \( \{\hat{E}_{\lambda}\}_{\lambda \in \mathbb{R}} \) be an orthogonal resolution of unity of \( \hat{B} \). Recall that the operator-valued function \( R_z = P_H \hat{R}_z(\hat{B}) \) is called a generalized resolvent of \( B \), \( z \in \mathbb{C} \setminus \mathbb{R} \). The function \( E_{\lambda} = P_H \hat{E}_{\lambda} \), \( \lambda \in \mathbb{R} \), is a spectral function of a symmetric operator \( B \) (e.g. [15]). There exists a one-to-one correspondence between generalized resolvents and (left-continuous or normalized in another way) spectral functions established by the following relation (16):

\[
(R_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(E_{\lambda} f, g)_H, \quad f, g \in H, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (39)
\]

To obtain the spectral function from relation (39), one should use the Stieltjes-Perron inversion formula (e.g. [7]).

In the case when \( \hat{B} \) is a self-adjoint contraction, the generalized resolvent \( R_z = P_H \hat{R}_z(\hat{B}) \) is called a generalized sc-resolvent of \( B \), see [9],[10]. The corresponding spectral function of \( B \) we shall call a sc-spectral function of \( B \).
Let
\[ \hat{B} = \int_{-1}^{1} \lambda d\hat{E}_\lambda, \] (40)
where \( \{\hat{E}_\lambda\} \) be the left-continuous in \([-1,1]\), right-continuous at the point 1, constant outside \([-1,1]\), orthogonal resolution of unity of \( \hat{B} \). In a similar manner as after (37) we obtain that
\[ s_{r+t}^{j,n} = \int_a^b x^{r+t} d\hat{m}_{j,n}(x), \quad 0 \leq j, n \leq N-1, \quad 0 \leq r, t \leq d, \] (41)
where
\[ \hat{m}_{j,n}(x) = (P_H \hat{E} \left[ \frac{2}{b-a} x - \frac{a+b}{b-a} \right] \hat{E}_\lambda, x_n)_H, \quad 0 \leq j, n \leq N-1. \] (42)

Thus, the function \( \hat{M}(x) = (\hat{m}_{j,n}(x))^{N-1}_{j,n=0} \) is a solution of the matrix moment problem.

**Theorem 2.3** Let a matrix moment problem \((1)\) with \( l = 2d, \quad d \in \mathbb{N} \), be given. Suppose that conditions \((5),(14)\) are true. All solutions of the moment problem have the following form
\[ M(x) = (m_{j,n}(x))^{N-1}_{j,n=0}, \quad m_{j,n}(x) = (E \left[ \frac{2}{b-a} x - \frac{a+b}{b-a} \right] \hat{E}_\lambda, x_n)_H, \quad 0 \leq j, n \leq N-1, \] (43)
where \( E_z \) is a left-continuous in \([-1,1]\), right-continuous at the point 1, constant outside \([-1,1]\) sc-spectral function of the operator \( B \) defined in (15). Moreover, the correspondence between all solutions of the moment problem and left-continuous in \([-1,1]\), right-continuous at the point 1, constant outside \([-1,1]\) sc-spectral functions of \( B \) in (43) is one-to-one.

**Proof.** Choose an arbitrary left-continuous in \([-1,1]\), right-continuous at the point 1, constant outside \([-1,1]\) sc-spectral function \( E_z \) of the operator \( B \) from (15). This function corresponds to a left-continuous in \([-1,1]\), right-continuous at the point 1, constant outside \([-1,1]\) resolution of unity \( \{\hat{E}_\lambda\} \) of a self-adjoint contraction \( \hat{B} \supseteq B \) in a Hilbert space \( \hat{H} \supseteq H \). Considerations before the statement of the Theorem show that formula (43) defines a solution of the moment problem.

On the other hand, let \( M(x) = (m_{k,l}(x))^{N-1}_{k,l=0} \) be an arbitrary solution of the matrix moment problem \((1)\). Proceeding like at the beginning of the Proof of Theorem 2.2 we shall construct a self-adjoint contraction \( \hat{B} \supseteq B \) in a space \( \hat{H} \supseteq H \). Repeating arguments before the statement of the last
theorem, we obtain that the function \( \hat{M}(x) = (\hat{m}_{j,n}(x))_{j,n=0}^{N-1} \), where \( \hat{m}_{j,n}(x) \) are given by (42), is a solution of the moment problem.

Choose an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \) and write

\[
\int_{-1}^{1} \frac{1}{\lambda - z} d(\hat{E}_\lambda x_k, x_j)_{\hat{H}} = \left( \int_{-1}^{1} \frac{1}{\lambda - z} d\hat{E}_\lambda x_k, x_j \right)_{\hat{H}} =
\]

\[
= \left( U^{-1} \int_{-1}^{1} \frac{1}{\lambda - z} d\hat{E}_\lambda x_k, U^{-1} x_j \right)_{L^2(M)} = \left( \int_{-1}^{1} \frac{1}{\lambda - z} dU^{-1} \hat{E}_\lambda U \hat{e}_k, \hat{e}_j \right)_{L^2(M)} =
\]

\[
= \left( \int_{-1}^{1} \frac{1}{\lambda - z} dE_{R;\lambda} \hat{e}_k, \hat{e}_j \right)_{L^2(M)} = \left( (R - zE_{L^2(M)})^{-1} \hat{e}_k, \hat{e}_j \right)_{L^2(M)} =
\]

\[
= \int_a^b \left( \frac{2}{b-a} - \frac{a+b}{b-a} \right)^{-1} d(E_u \hat{e}_k, \hat{e}_j)_{L^2(M)} =
\]

\[
= \int_{-1}^{1} \frac{1}{\lambda - z} d(E_{\frac{b-a}{2}\lambda + \frac{a+b}{2}} \hat{e}_k, \hat{e}_j)_{L^2(M)}, \quad 0 \leq k, j \leq N - 1,
\]

(44)

where \( \{\hat{E}_\lambda\} \) and \( \{E_{R;\lambda}\} \) are left-continuous in \([-1, 1]\), right-continuous at the point \( 1 \), constant outside \([-1, 1]\), orthogonal resolutions of unity of operators \( \hat{B} = U RU^{-1} \) and \( R \), respectively. Here \( \{E_\lambda\} \) is the orthogonal resolution of unity of \( Q \), given by (16). By the Stieltjes-Perron inversion formula we get

\[
(\hat{E}_\lambda x_k, x_j)_{\hat{H}} = (E_{\frac{b-a}{2}\lambda + \frac{a+b}{2}} \hat{e}_k, \hat{e}_j)_{L^2(M)}, \quad 0 \leq k, j \leq N - 1,
\]

(45)

for each \( \lambda \in [-1, 1] \), such that \( \lambda \) is a point of continuity for \( \hat{E}_\lambda \) and \( E_{\frac{b-a}{2}\lambda + \frac{a+b}{2}} \).

Using the change of variable we obtain that

\[
\hat{m}_{k,j}(x) = (P_H \hat{E}_{\frac{a+b}{b-a}x - \frac{a+b}{2}} x_k, x_j)_{\hat{H}} = (E_x \hat{e}_k, \hat{e}_j)_{L^2(M)},
\]

(46)

where \( 0 \leq k, j \leq N - 1 \), for \( x \in [a, b] \): \( x \) is a point of continuity of \( \hat{E}_{\frac{a+b}{b-a}x - \frac{a+b}{2}} \) and \( E_x \). Using (16) we can write

\[
(E_x \hat{e}_k, \hat{e}_j)_{L^2(M)} = m_{k,j}(x), \quad x \in (a, b),
\]

and therefore

\[
\hat{m}_{k,j}(x) = m_{k,j}(x),
\]

(47)
where \( x \in [a, b] \): \( x \) is a point of continuity of \( \hat{E}_{\frac{a+b}{2}} \) and \( E_x \). Since matrix functions \( \hat{M}(x) \) and \( M(x) \) are left-continuous in \( (a, b) \), they coincide in \( (a, b) \). It remains to note that \( \hat{M}(a) = M(a) = 0 \), and \( \hat{M}(b) = M(b) = S_0 \), to obtain
\[
\hat{M}(x) = M(x), \quad x \in [a, b].
\]
Consequently, all solutions of the truncated moment problem are generated by left-continuous in \([-1, 1)\), right-continuous at the point 1, constant outside \([-1, 1]\) sc-spectral functions of \( B \).

It remains to prove that different sc-spectral functions of the operator \( B \) produce different solutions of the moment problem \( (1) \). Suppose to the contrary that two different left-continuous in \([-1, 1)\), right-continuous at the point 1, constant outside \([-1, 1]\) sc-spectral functions produce the same solution of the moment problem. That means that there exist two self-adjoint contractions \( B_j \supseteq B \), in Hilbert spaces \( H_j \supseteq H \), such that
\[
P_{H_1} E_{1, \lambda} \neq P_{H_2} E_{2, \lambda}, \quad (48)
\]
where \( \{E_{n, \lambda}\}_{\lambda \in \mathbb{R}} \) are left-continuous in \([-1, 1)\), right-continuous at the point 1, orthogonal resolutions of unity of operators \( B_n \), \( n = 1, 2 \). Set \( L_N := \text{Lin}\{x_k\}_{k=0}^{N-1} \). By linearity we get
\[
(P_{H_1} E_{1, \lambda} x, y)_H = (P_{H_2} E_{2, \lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in [-1, 1]. \quad (49)
\]
From \((49)\) it follows that
\[
(R_{1, \lambda} x, y)_H = (R_{2, \lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (51)
\]
Choose an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \) and consider the space \( H_z := (B - zE_H)H_a \).

Since
\[
R_{j, z}(B - zE_H) x = (B_j - zE_{H_j})^{-1}(B_j - zE_{H_j}) x = x, \quad x \in H_a = D(B),
\]
we get
\[
R_{1, z} u = R_{2, z} u, \quad u \in H_z; \quad (52)
\]
\[
R_{1, z} u = R_{2, z} u, \quad u \in H_z, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (53)
\]
We can write
\[
(R_{n, z} x, u)_H = (R_{n, z} x, u)_{H_n} = (x, R_{n, z} u)_{H_n} = (x, R_{n, z} u)_H, \quad x \in H, \quad u \in H_z.
\]
\[ n = 1, 2, \quad (54) \]

and therefore we get
\[
(R_{1,z}x, u)_H = (R_{2,z}x, u)_H, \quad x \in H, \ u \in H_z. \quad (55)
\]

Choose an arbitrary \( u \in H, u = \sum_{k=0}^{dN+N-1} c_k x_k, c_k \in \mathbb{C} \). Consider the following system of linear equations:
\[
-a + \frac{b + z}{b - a} d_k = c_k, \quad k = 0, 1, \ldots, N - 1; \quad (56)
\]
\[
\frac{2}{b - a} d_{k-N} - \left( \frac{a + b}{b - a} + z \right) d_k = c_k, \quad k = N, N + 1, \ldots, dN + N - 1; \quad (57)
\]

where \( \{d_k\}_{k=0}^{dN+N-1} \) are unknown complex numbers, \( z \in \mathbb{C} \setminus \mathbb{R} \) is a fixed parameter, \( a, b \) are from (1). Set
\[
d_k = 0, \quad k = dN, dN + 1, \ldots, dN + N - 1;
\]
\[
d_{k-N} = \frac{b - a}{2} \left( \left( \frac{a + b}{b - a} + z \right) d_k + c_k \right), \quad k = dN+N-1, dN+N-2, \ldots, N; \quad (58)
\]

For such numbers \( \{d_k\}_{k=0}^{dN+N-1} \), all equations in (57) are satisfied. Equations (56) are not necessarily satisfied. Set \( v = \sum_{k=0}^{dN+N-1} d_k x_k = \sum_{k=0}^{dN-1} d_k x_k \). Notice that \( v \in H_a = D(B) \). We can write
\[
(B - zE_H)v = \left( \frac{2}{b - a} A - \frac{a + b}{b - a} E_H - zE_H \right) v =
\]
\[
= \sum_{k=0}^{dN-1} d_k \left( \frac{2}{b - a} x_{k+N} - \left( \frac{a + b}{b - a} + z \right) x_k \right) =
\]
\[
= \sum_{k=0}^{dN+N-1} \left( \frac{2}{b - a} d_{k-N} - \left( \frac{a + b}{b - a} + z \right) d_k \right) x_k,
\]

where \( d_{-1} = d_{-2} = \ldots = d_{-N} = 0 \). By the construction of \( d_k \) we have
\[
(B - zE_H)v - u = \sum_{k=0}^{N-1} \left( - \left( \frac{a + b}{b - a} + z \right) d_k - c_k \right) x_k;
\]
\[
u = (B - zE_H)v + \sum_{k=0}^{N-1} \left( \frac{a + b}{b - a} + z \right) d_k + c_k \right) x_k, \quad u \in H, \ z \in \mathbb{C} \setminus \mathbb{R}. \quad (59)\]
By (59) an arbitrary element $y \in H$ can be represented as $y = y_\pi + y'$, $y_\pi \in H_\pi$, $y' \in L_N$. Using (51) and (55) we get

$$(R_{1,z}x, y)_H = (R_{1,z}x, y_\pi + y')_H = (R_{2,z}x, y_\pi + y')_H = (R_{2,z}x, y)_H, \ x \in L_N, \ y \in H.$$ 

Thus, we obtain

$$R_{1,z}x = R_{2,z}x, \quad x \in L_N, \ z \in \mathbb{C}\setminus\mathbb{R}. \quad (60)$$

For an arbitrary $x \in L$, $x = x_z + x'$, $x_z \in H_z$, $x' \in L_N$, using relations (53), (60) we obtain

$$R_{1,z}x = R_{1,z}(x_z + x') = R_{2,z}(x_z + x') = R_{2,z}x, \quad x \in L, \ z \in \mathbb{C}\setminus\mathbb{R}, \quad (61)$$

and

$$R_{1,z}x = R_{2,z}x, \quad x \in H, \ z \in \mathbb{C}\setminus\mathbb{R}. \quad (62)$$

By (39) that means that the corresponding sc-spectral functions coincide and we obtain a contradiction. □

We shall recall some known facts about sc-resolvents (see [9, 10]). Let $A$ be a Hermitian contraction in a Hilbert space $H$ with a non-dense closed domain $D = D(A)$, and $\mathcal{R} = H \ominus D$. A set of all self-adjoint extensions of $A$ in $H$, which are contractions, we denote by $\mathcal{B}_H(A)$. A set of all self-adjoint extensions of $A$ in a Hilbert space $\tilde{H} \supseteq H$, which are contractions, we denote by $\mathcal{B}_{\tilde{H}}(A)$. The set $\mathcal{B}_H(A)$ is non-empty. Moreover, there are a "minimal" element $A^\mu$ and a "maximal" element $A^M$ in this set, such that $\mathcal{B}_H(A)$ coincides with the operator segment

$$A^\mu \leq \tilde{A} \leq A^M. \quad (63)$$

In the case $A^\mu = A^M$ the set $\mathcal{B}_H(A)$ consists of a unique element. This case is called determinate.

In the case $A^\mu \neq A^M$ the set $\mathcal{B}_H(A)$ consists of an infinite number of elements. This case is called indeterminate.

The case $A^\mu x \neq A^M x, x \in \mathcal{R}\setminus\{0\}$, is called completely indeterminate. The indeterminate case can be always reduced to the completely indeterminate. If $\mathcal{R}_0 = \{x \in \mathcal{R} : A^\mu x = A^M x\}$, we can set

$$A_e x = A x, \ x \in D; \quad A_e x = A^\mu x, \ x \in \mathcal{R}_0. \quad (64)$$

The sets of generalized sc-resolvents for $A$ and for $A_e$ coincide ([10] p. 1039).
Elements of $\mathcal{B}_H(A)$ are canonical (i.e. inside $H$) extensions of $A$ and their resolvents are called canonical sc-resolvents of $A$. On the other hand, elements of $\mathcal{B}_{\widetilde{H}}(A)$ for $\widetilde{H} \supseteq H$ generate generalized sc-resolvents of $A$. The set of all generalized sc-resolvents we denote by $\mathcal{R}^c(A)$.

Set
\[ C = A^M - A^\mu, \]
\[ Q_\mu(z) = \left( C \frac{1}{2} R^\mu \frac{1}{z} C + E_H \right) |_R, \]  
$z \in \mathbb{C}\setminus[-1,1], \tag{66} $  

where $R^\mu_z = (A^\mu - zE_H)^{-1}$.

An operator-valued function $k(z)$ with values in $[H]$ belongs to the class $\mathcal{R}^c([-1,1])$ if

1) $k(z)$ is analytic in $z \in \mathbb{C}\setminus[-1,1]$ and
\[ \frac{\text{Im} k(z)}{\text{Im} z} \leq 0, \quad z \in \mathbb{C} : \text{Im} z \neq 0; \]

2) For $z \in \mathbb{R}\setminus[-1,1]$, $k(z)$ is a self-adjoint positive contraction.

**Theorem 2.4 ([10, p. 1053]).** The following equality:
\[ \widetilde{R}_c^c z = R^\mu_z - R^\mu_z C \frac{1}{2} (E_H + (Q_\mu(z) - E)k(z))^{-1} C \frac{1}{2} R^\mu_z, \tag{67} \]

where $k(z) \in \mathcal{R}^c([-1,1])$, $\widetilde{R}_c^c \in \mathcal{R}^c(A)$, establishes a one-to-one correspondence between the set $\mathcal{R}^c([-1,1])$ and the set $\mathcal{R}^c(A)$.

Moreover, the canonical resolvents correspond in (67) to the constant functions $k(z) \equiv K, K \in [0, E_R]$.

Comparing Theorem 2.3 and Theorem 2.4 we obtain the following result.

**Theorem 2.5** Let a matrix moment problem (1) with $l = 2d$, $d \in \mathbb{N}$, be given. Suppose that conditions (5), (14) are true. Let the operator $B$ be defined by (15). The following statements are true:

1) If $B^\mu = B^M$, then the moment problem (1) has a unique solution. This solution is given by
\[ M(x) = (m_{j,n}(x))_{j,n=0}^{N-1}, \quad m_{j,n}(x) = (E^\mu_{\frac{x_j - x_n}{x_j - \frac{x_j + x_n}{2}}} x_j, x_n)_H, \quad 0 \leq j, n \leq N-1, \tag{68} \]

where $\{E^\mu_x\}$ is the left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$, orthogonal resolution of unity of the operator $A^\mu$. 

\[ \]
2) If $B^\mu \neq B^M$, we define the extended operator $B_e$ (see the construction in (64)). All solutions of the moment problem have the following form

$$M(x) = (m_{j,n}(x))_{j,n=0}^{N-1}, \quad m_{j,n}(x) = (E_{\frac{2}{b-a}} x - \frac{a+b}{b-a} x_j, x_n)_H, \quad 0 \leq j, n \leq N-1,$$

where $E_z$ is a left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$ functions given by

$$(R_z^H - R_z^\mu C_1^H (E_H + (Q_\mu(z) - E)k(z))^{-1} C_1^H R_z^\mu f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(E_\lambda f, g)_H,$$

where $k(z) \in R_{\mathcal{R}}[-1,1]$. Here $\mathcal{R} = H \ominus D(B_e)$, and $C, Q_\mu(z), R_z^\mu$ are defined by (62), (65) with $A = B_e$. Moreover, the correspondence between all solutions of the moment problem and $k(z) \in R_{\mathcal{R}}[-1,1]$ is one-to-one.

Proof. It remains to consider the case 1). In this case all self-adjoint contractions $\tilde{B} \supseteq B$ in a Hilbert space $\tilde{H} \supseteq H$ coincide on $H$ with $B^\mu$, see [10, p. 1039]. Thus, the corresponding sc-spectral functions are spectral functions of the self-adjoint operator $B^\mu$. However, a self-adjoint operator has a unique (normalized) spectral function. Thus, a set of sc-spectral functions of $B$ consists of a unique element. This element is a spectral function of $B^\mu$. □

Consider a matrix moment problem (1) with $l = 0$. In this case the necessary and sufficient condition of solvability is

$$S_0 \geq 0.$$  \hspace{1cm} (71)

The necessity is obvious. On the other hand, if relation (71) is true, we can choose

$$M(x) = \frac{x - a}{b - a} S_0, \quad x \in [a, b].$$  \hspace{1cm} (72)

This function is a solution of the moment problem. A set of all solutions consists of non-decreasing matrix functions $M(x)$ on $[a, b]$, left-continuous in $(a, b)$, with the boundary conditions $M(a) = 0, M(b) = S_0$.

3 The case of an even number of given moments: solvability and a description of solutions.

Consider the matrix moment problem (1) with $l = 2d + 1, d \in \mathbb{Z}_+$. 

\hspace{1cm} 17
Theorem 3.1 Let a matrix moment problem \(1\) with \(l = 2d + 1, d \in \mathbb{Z}_+\), be given. The moment problem has a solution if and only if
\[
\Gamma_d \geq 0; \quad \bar{\Gamma}_d \geq 0,
\]
and there exist matrix solutions \(X, Y\) of matrix equations
\[
\Gamma_d X = \begin{pmatrix} S_{d+1} \\ S_{d+2} \\ \vdots \\ S_{2d+1} \end{pmatrix}, \quad \bar{\Gamma}_d Y = \begin{pmatrix} -abS_d + (a + b)S_{d+1} - S_{d+2} \\ -abS_{d+1} + (a + b)S_{d+2} - S_{d+3} \\ \vdots \\ -abS_{2d-1} + (a + b)S_{2d} - S_{2d+1} \end{pmatrix},
\]
and for these solutions \(X, Y\) the following relation is true:
\[
X^* \Gamma_d X \leq -abS_{2d} + (a + b)S_{2d+1} - Y^* \bar{\Gamma}_d Y.
\] (75)

Proof. Consider a matrix moment problem \(1\) with \(l = 2d + 1, d \in \mathbb{Z}_+\).
It has a solution if and only if the moment problem with an odd number of moments
\[
\int_a^b x^n dM(x) = S_n, \quad n = 0, 1, \ldots, 2d + 1; \quad \int_a^b x^{2d+2} dM(x) = S_{2d+2},
\] (76)
with some complex \((N \times N)\) matrix \(S_{2d+2}\) has a solution. By \([5]\) the solvability of the moment problem \(76\) is equivalent to the matrix inequalities
\[
\Gamma_{d+1} \geq 0, \quad \bar{\Gamma}_{d+1} \geq 0.
\] (77)

If we apply to the latter inequalities the well-known block-matrix lemma (e.g. \([17, p.~223]\)), we obtain that solvability of \(76\) is equivalent to the condition \(73\), existence of solutions \(X, Y\) of \(74\) and inequalities
\[
S_{2d+2} \geq X^* \Gamma_d X, \quad S_{2d+2} \leq -abS_{2d} + (a + b)S_{2d+1} - Y^* \bar{\Gamma}_d Y.
\] (78)
Consequently, we get that the statement of the theorem is true. \(\Box\)

Theorem 3.2 Let a matrix moment problem \(1\) with \(l = 2d + 1, d \in \mathbb{Z}_+\), be given. Suppose that conditions \(73, 77\) and \(75\) are true. For an arbitrary \(S_{2d+2} \in [X^* \Gamma_d X \leq -abS_{2d} + (a + b)S_{2d+1} - Y^* \bar{\Gamma}_d Y]\), the set of all solutions of the matrix moment problem with an odd number of moments \(76\) we denote by \(\mathcal{V}(S_{2d+2})\). The set \(\mathcal{V}\) of all solutions of the moment problem \(1\) is given by the formula
\[
\mathcal{V} = \bigcup_{S \in [X^* \Gamma_d X \leq -abS_{2d} + (a + b)S_{2d+1} - Y^* \bar{\Gamma}_d Y]} \mathcal{V}(S).
\] (79)
The sets \(\mathcal{V}(S)\) in \(79\) for different \(S\) do not intersect. Each set \(\mathcal{V}(S)\) in \(79\) is parameterized by virtue of Theorem 2.5.
Proof. The proof of (79) follows obviously from the above considerations in the proof of Theorem 3.1. The sets $\mathcal{V}(S)$ in (79) for different $S$ do not intersect since the solutions in different sets have different $(2d + 2)$-th moments. □

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Truncated matricial moment problems on a finite interval: the operator approach.

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In this paper we obtain a description of all solutions of truncated matricial moment problems on a finite interval in a general case (no conditions besides solvability are assumed). We use the basic results of M.G. Krein and I.E. Ovcharenko about generalized sc-resolvents of Hermitian contractions.

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