On a \textit{KK}-Theoretic Counterpart of Relative Index Theorems

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Abstract. Relative index theorems, which deal with what happens with the index of elliptic operators when cutting and pasting, are abundant in the literature. It is desirable to obtain similar theorems for other stable homotopy invariants, not the index alone. In the spirit of noncommutative geometry, we prove a full-fledged “relative index” type theorem that compares certain elements of the Kasparov \textit{KK}-group $KK(A,B)$.

Introduction. Relative index theorems, which deal with what happens with the index of elliptic operators when cutting and pasting, are abundant in the literature. Here we only mention the famous Gromov–Lawson relative index theorem \cite{1} for Dirac operators on complete noncompact Riemannian manifolds and refer the reader for further examples and bibliography to the papers \cite{2,3}, where a locality theorem for the relative index was proved in a rather general setting, which not only covered many earlier-known special cases but also permitted one to obtain a number of index formulas for elliptic differential operators and even for Fourier integral operators on manifolds with singularities \cite{4}. It is however desirable to obtain similar theorems for other stable homotopy invariants, not the index alone. One of the first steps in this direction was made much earlier by Bunke \cite{5}, who considered Dirac operators on a complete noncompact Riemannian manifold in section spaces of bundles of projective Hilbert $B$-modules (e.g., see \cite{6}), where $B$ is a $C^*$-algebra, and obtained a relative index theorem for such operators, the index being an element of the $K$-group of $B$. In the present paper, in the spirit of noncommutative geometry, we prove a full-fledged “relative index” type theorem that compares certain elements of the Kasparov \textit{KK}-group $KK(A,B)$. In contrast to \cite{5}, where \textit{KK}-groups are used with $A$ being an algebra of functions on the manifold and the answers are only stated in terms of elements of $K_*(B) = KK(\mathbb{C},B)$, we admit an arbitrary noncommutative unital $C^*$-algebra $A$ and do not restrict ourselves to the index, even to the $K_*(B)$-valued one.

We freely use notions and notation related to $C^*$-algebras and \textit{KK}-theory (e.g., see \cite{7,8,9} and the literature cited therein). Full proofs will be given elsewhere.

1. Algebra $A$ and a partition of unity. Let $A$ be a unital $C^*$-algebra, and let $J_1, J_2 \subseteq A$ be two (closed, two-sided, +-)ideals such that $J_1 + J_2 = A$. By $J$ we denote the intersection of these ideals, $J = J_1 \cap J_2$.

**Lemma 1.** There exist self-adjoint positive elements $\psi_1 \in J_1$ and $\psi_2 \in J_2$ with

\begin{equation}
\psi_1^2 + \psi_2^2 = 1, \quad [\psi_1, \psi_2] = 0.
\end{equation}

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Proof. Since \(J_1 + J_2 = A\), it follows that there exists an element \(\chi \in J_1\) such that \(1 - \chi \in J_2\); taking the real part, we can assume that \(\chi = \chi^*\). By functional calculus, the element \(\chi^2 + (1 - \chi)^2\) is positive and invertible, and we set \(\psi_1 = |\chi^2 + (1 - \chi)^2|^{-1/2}\) and \(\psi_2 = |1 - \chi^2 + (1 - \chi)^2|^{-1/2}\). \(\square\)

2. Kasparov modules. Let \(B\) be another \(C^*\)-algebra, and let \(x = (H, \rho, F)\) be a Kasparov module for \((A, B)\), i.e., a triple consisting of a Hilbert module \(H\) over \(B\), a homomorphism \(\rho: A \to \mathbb{B}(H)\) of \(A\) into the \(C^*\)-algebra \(\mathbb{B}(H)\) of adjointable operators on \(H\), and an operator \(F \in \mathbb{B}(H)\) such that

\[
[F, \rho(a)] \sim 0, \quad \rho(a)(F - F^*) \sim 0
\]

for every \(a \in A\), where we write \(C \sim D\) if \(C - D\) lies in the ideal \(\mathbb{K}(H) \subset \mathbb{B}(H)\) of “compact” operators. We only consider Kasparov modules in which the homomorphism \(\rho\) is unital (and refer to these as unital Kasparov modules); in this case, the factor \(\rho(a)\) can be dropped in the second and third conditions in (2). The \((A, B)\)-sub-bimodules

\[
H_1 = \rho(J_1)H, \quad H_2 = \rho(J_2)H, \quad H_0 = \rho(J)H
\]

of \(H\) are automatically closed (the proof is similar to that in [8 pp. 25–26]) and hence are simultaneously Hilbert \(B\)-modules. It is easily seen that \(H_0 = H_1 \cap H_2\). Indeed, the inclusion \(H_0 \subset H_1 \cap H_2\) is obvious. Next, let \(\xi \in H_1 \cap H_2\). Then \(\rho(u_\alpha)\xi \to \xi\) and \(\rho(v_\mu)\xi \to \xi\), where \(\{u_\alpha\}\) and \(\{v_\mu\}\) are approximate units for \(J_1\) and \(J_2\), respectively. Now it follows from the inequality \(\|\rho(u_\alpha)\xi - \xi\| \leq \|\rho(v_\mu)\xi - \xi\| + \|\rho(u_\alpha)\xi - \rho(v_\mu)\xi\|\) that there exists a subsequence of \(\{\rho(u_\alpha)\xi\}\) that converges to \(\xi\), and hence \(\xi \in H_0\), because \(u_\alpha v_\mu \in J\) and \(\rho(u_\alpha v_\mu)\xi \in H_0\).

The \((A, B)\)-bimodule \(H\) is naturally isomorphic to the quotient \((H_1 \oplus H_2)/\Delta\), where \(\Delta = \{[\xi_1, \xi_2] \in H_1 \oplus H_2: \xi_1 = -\xi_2 \in H_0\}\). The isomorphism is induced by the mapping \(\alpha: H_1 \oplus H_2 \to H_1, [\xi_1, \xi_2] \mapsto \xi_1 + \xi_2\), with the inverse being induced by \(\beta: H \to H_1 \oplus H_2, \xi \mapsto (\rho(\psi_1^2)\xi, \rho(\psi_2^2)\xi)\).

3. Cutting and pasting. Let \(\tilde{x} = (\tilde{H}, \tilde{\rho}, \tilde{F})\) be another unital Kasparov \((A, B)\)-module, and let \(\tilde{H}_j, j = 0, 1, 2\), be the Hilbert submodules of \(\tilde{H}\) defined as in [3].

Definition 2. We say that \(x\) and \(\tilde{x}\) agree on \(J\) if there is a unitary (in the sense of Hilbert modules over \(B\)) isomorphism \(T: H_0 \to \tilde{H}_0\) of \((A, B)\)-bimodules such that, for arbitrary \(c, d \in J\), one has

\[
T \rho(c) F \rho(d) \sim \tilde{\rho}(c) \tilde{F} \tilde{\rho}(d) T.
\]

(Note that \(\rho(c) F \rho(d) \in \mathbb{B}(H_0)\) and \(\tilde{\rho}(c) \tilde{F} \tilde{\rho}(d) \in \mathbb{B}(\tilde{H}_0)\) are well defined.)

Assume that \(x\) and \(\tilde{x}\) agree on \(J\). Our aim is to use some sort of cutting-and-pasting procedure to define a unital Kasparov \((A, B)\)-module \(x \circ \tilde{x}\) that agrees with \(x\) on \(J_1\) and with \(\tilde{x}\) on \(J_2\). To this end, consider the \((A, B)\)-bimodule

\[
H \circ \tilde{H} = (H_1 \oplus \tilde{H}_2)/\{(\xi_1, \xi_2): \xi_1 \in H_0, \xi_2 \in \tilde{H}_0, T\xi_1 + \xi_2 = 0\}.
\]

The elements of \(H \circ \tilde{H}\) will be denoted by \(\xi = [(\xi_1, \xi_2)]\), and the action of \(A\) on \(H \circ \tilde{H}\) will be denoted by \(\rho \circ \tilde{\rho}\), \((\rho \circ \tilde{\rho})(\varphi)\xi = [\rho(\varphi)\xi_1, \rho(\varphi)\xi_2]\). Note that \(H_1\) and \(\tilde{H}_2\) are naturally embedded in \(H \circ \tilde{H}\) (the embeddings are induced by those of \(H_1\) and \(\tilde{H}_2\) in the direct sum \(H_1 \oplus \tilde{H}_2\)), and if we identify \(H_1\) and \(\tilde{H}_2\) with their images under these embeddings,

\[
H_1 \simeq (H \circ \tilde{H})_1 \equiv (\rho \circ \tilde{\rho})[J_1](H \circ \tilde{H}), \quad \tilde{H}_2 \simeq (H \circ \tilde{H})_2 \equiv (\rho \circ \tilde{\rho})[J_2](H \circ \tilde{H}),
\]
then, for arbitrary $\xi \in H \circ \tilde{H}$ and $\varphi_j \in J_j$, $j = 1, 2$, we have $(\rho \circ \tilde{\rho})(\varphi_1)\xi \simeq \rho(\varphi_1)\xi_1 + T^*\tilde{\rho}(\varphi_1)\xi_2 \in H_1$ and $(\rho \circ \tilde{\rho})(\varphi_2)\xi \simeq T\rho(\varphi_2)\xi_1 + \tilde{\rho}(\varphi_2)\xi_2 \in \tilde{H}_2$.

From now on, to simplify the notation, we identify $H_0$ and $\tilde{H}_0$ via $T$, accordingly suppress $T$ in all the formulas, and also write simply $\varphi$ instead of $\rho(\varphi)$, $\tilde{\rho}(\varphi)$, or $(\rho \circ \tilde{\rho})(\varphi)$. This will not lead to a misunderstanding even if several representations are involved, because which is meant is always clear from the context.

**Lemma 3.** The formula
\[
\langle \xi, \eta \rangle_{H \circ \tilde{H}} = \langle \psi_1^2 \xi, \psi_1^2 \eta \rangle_H + \langle \psi_2^2 \xi, \psi_2^2 \eta \rangle_{\tilde{H}} + 2\langle \psi_1 \psi_2 \xi, \psi_1 \psi_2 \eta \rangle_H,
\]
where $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{\tilde{H}}$ are the $B$-valued inner products on $H$ and $\tilde{H}$, respectively, specifies a well-defined $B$-valued inner product on $H \circ \tilde{H}$, which makes $H \circ \tilde{H}$ a Hilbert $B$-module and the action of $A$ on $H \circ \tilde{H}$ a unital $*$-homomorphism $\rho \circ \tilde{\rho}: A \to \mathbb{B}(H \circ \tilde{H})$.

Now set
\[
F \circ \tilde{F} = \psi_1 F \psi_1 + \psi_2 \tilde{F} \psi_2: H \circ \tilde{H} \to H \circ \tilde{H}.
\]
This is well defined. Indeed, for example, if $\xi \in H \circ \tilde{H}$, then, in view of our identifications, $\psi_1 \xi \in H \circ \tilde{H}_1 = H_1 \subset H$, hence $F \psi_1 \xi$ is a well-defined element of $H$, and hence $\psi_1 F \psi_1 \xi \in H_1 \subset H \circ \tilde{H}$ is a well-defined element of $H \circ \tilde{H}$.

**Theorem 4.** The triple $x \circ \tilde{x} = (H \circ \tilde{H}, \rho \circ \tilde{\rho}, F \circ \tilde{F})$ is a unital Kasparov $(A, B)$-module, which is independent modulo “compact” perturbations of the choice of the partition of unity \[^{[1]}\] and agrees with $x$ on $J_1$ and with $\tilde{x}$ on $J_2$.

In a similar way, one defines the Kasparov module $\tilde{x} \circ x$, which agrees with $\tilde{x}$ on $J_1$ and with $x$ on $J_2$.

**4. Main result.** Now we are in a position to state the main result of the paper. For a Kasparov $(A, B)$-module $y$, let $[y] \in KK(A, B)$ be the corresponding class in Kasparov’s $KK$-theory.

**Theorem 5.** Under the above assumptions, one has
\[
[x] + [\tilde{x}] = [x \circ \tilde{x}] + [\tilde{x} \circ x].
\]

Let us give a sketch of the proof. First, we note that, under our identifications, the formula
\[
\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \psi_1 \xi + \psi_2 \eta \\ -\psi_2 \xi + \psi_1 \eta \end{pmatrix}
\]
gives a well-defined unitary isomorphism $U: H \oplus \tilde{H} \to (H \circ \tilde{H}) \oplus (\tilde{H} \circ H)$ of Hilbert $B$-modules such that
\[
U(F \oplus \tilde{F})U^* \sim (F \circ \tilde{F}) \oplus (\tilde{F} \circ F).
\]
Moreover, this mapping is an isomorphism of $(A, B)$-bimodules, where the action of $A$ on $(H \circ \tilde{H}) \oplus (\tilde{H} \circ H)$ is the direct sum of the actions $\rho \circ \tilde{\rho}$ and $\tilde{\rho} \circ \rho$, while the action of $A$ on $H \oplus \tilde{H}$ is given by the matrix
\[
\tilde{\rho}(a) = \begin{pmatrix} \psi_1 a \psi_1 + \psi_2 a \psi_2 \\ \psi_1 a \psi_2 - \psi_2 a \psi_1 \end{pmatrix}, \quad a \in A.
\]

Note that the right-hand side is well defined because the off-diagonal entries lie in $J$ and hence take $H$ as well as $\tilde{H}$ to $H_0$, which lies in both. Moreover, $[\tilde{\rho}(a), F \oplus \tilde{F}] \sim 0$, because $F$ and $\tilde{F}$ agree on $J$. Thus, it remains to prove that the Kasparov modules $(H \oplus \tilde{H}, \rho \oplus \tilde{\rho}, F \oplus \tilde{F})$ and $(H \oplus \tilde{H}, \tilde{\rho}, F \oplus \tilde{F})$ define the same element in $KK(A, B)$. 
To this end, we construct a homotopy \((H \oplus \tilde{H}, \tilde{\rho}_t, F \oplus \tilde{F})\) of Kasparov modules such that \(\tilde{\rho}_0 = \rho \oplus \tilde{\rho}\) and \(\tilde{\rho}_1 = \tilde{\rho}\). Namely, for \(a \in A\) we set

\[
(11) \quad \tilde{\rho}_t(a) = \begin{pmatrix} \psi_1 t a \psi_1 t + \psi_2 t a \psi_2 t \\ \psi_1 t a \psi_2 t - \psi_2 t a \psi_1 t \\ \psi_1 t a \psi_1 t + \psi_2 t a \psi_2 t \end{pmatrix},
\]

where \(\psi_1 t = t \psi_1\) and \(\psi_2 t = \sqrt{1 - t^2} \psi_2\).

First, we should prove that the operator (11) is well defined on \(H \oplus \tilde{H}\). To this end, it suffices to show that the off-diagonal entries lie in \(J\). We have \(\psi_2 t = \sqrt{1 - t^2} + (\sqrt{1 - t^2} + t^2 \psi_2^2 - \sqrt{1 - t^2})\); the term in parentheses lies in \(J_2\) by functional calculus, and we obtain \(\psi_2 t a \psi_1 t - \psi_1 t a \psi_2 t \equiv t \sqrt{1 - F[a, \psi_1]} \mod J\). It remains to note that \([a, \psi_1] \in J_1\) and \([a, \psi_1] = [a, \sqrt{1 - \psi_2^2}] = [a, \sqrt{1 - \psi_2^2} - 1] \in J_2\), again by functional calculus, so that \([a, \psi_1] \in J\) and we are done.

Now the properties \(\psi_1^2 t + \psi_2^2 t = 1\) and \([\psi_1 t, \psi_2 t] = 0\) imply that \(\tilde{\rho}_t\) is indeed a homomorphism and that \([\tilde{\rho}_t(a), F \oplus \tilde{F}] \sim 0\). (Here we again use the fact that \(F\) and \(\tilde{F}\) agree on \(J\).) This completes the proof.

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