Covariant Phase Space Formulations of Superparticles and Supersymmetric WZW Models

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Abstract

We present new covariant phase space formulations of superparticles moving on a group manifold, deriving the fundamental Poisson brackets and current algebras. We show how these formulations naturally generalise to the supersymmetric Wess-Zumino-Witten models.

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1 Introduction

The Wess-Zumino-Witten (WZW) models \([1]\) are fundamental rational conformal field theories, and have a rich structure which has occasioned much interest. As the WZW models are expected to possess most of the characteristic features of rational conformal field theories, they are of considerable interest in the study of conformal field theory and string theory in general. In particular, there has been some inquiry of late into the phenomenon of the emergence of quantum groups in WZW models \((2) - (11)\). WZW models are also of interest due to the fact that Hamiltonian reduction of these models gives another important class of integrable two-dimensional systems - the Toda theories (see \([12]\) for a review).

A new covariant phase space description of the non-supersymmetric WZW model has been found recently \([13]\). This description makes possible a particularly simple derivation of the fundamental Poisson brackets of the theory. It is also clear from this formulation that the differences between it and other approaches in this field arise due to the differing topologies of the phase spaces involved. An appealing feature of the formulation in reference \([13]\) is that the parameterisation of the solutions of the theory given there is such as to guarantee that the covariant phase space is diffeomorphic to the Hamiltonian phase space, and thus this covariant phase space formulation gives agreement with the usual Hamiltonian methods. With regard to the further development of the formulation of this approach, as well as to the various applications of supersymmetric WZW models in superstring theories, it is of interest to consider the question of whether one can generalise this covariant phase space formulation to the *supersymmetric* WZW models. In this paper, we will present such formulations.

In section 2 we will discuss superparticles moving upon group manifolds. These systems share many of the important features of the supersymmetric WZW models. The WZW models are then discussed in section 3, taking advantage of the results of the earlier particle discussion. We show how the full current algebras for all these models arise naturally in our approach. We then show that the topological issues which arose in the bosonic case also are found here, with the same resolutions. We finish with some concluding remarks in section 4.
2 Superparticles on Group Manifolds

An analysis of the one-dimensional supersymmetric non-linear sigma models has been given in ref. [14]. The class of such models defined on group manifolds is more general than those considered here - as we are interested in the two-dimensional supersymmetric WZW models, here we will restrict our attention only to those particle models which arise from reducing the two-dimensional WZW models to one dimension. For ease of comparison, the one-dimensional particle model which arises from dimensional reduction of the two-dimensional \((p,q)\) supersymmetric WZW model \((p,q = 0, 1, 2, \ldots)\) will be called the “\((p,q)\) superparticle”.

For simplicity, in this paper we will restrict ourselves to considering a simple, compact, simply-connected Lie group \(G\), with generators \(t^a, a = 1, 2, \ldots, \dim G\), satisfying \([t^a, t^b] = f^{ab}_{\phantom{ab}c} t^c\), with \(f^{ab}_{\phantom{ab}c}\) the structure constants of \(G\). We also have \(\text{Tr}[t_a t_b] = \delta_{ab}\). The group manifold has coordinates \(X^i, i = 1, 2, \ldots, \dim G\). We denote by \(g_{ij}\) the Cartan-Killing metric on \(G\). For group elements \(h\), we define the one-forms \(l = h^{-1} dh\), satisfying \(l^a_{[i,j]} = \frac{1}{2} f^{ab}_{\phantom{ab}c} t^c_{ij}\), \([l^a, l^b]^i = f^{ab}_{\phantom{ab}c} t^c_i\), and \(g_{ij} l^a_j b^b_i = \delta_{ab}\). The torsion on the group manifold is given by \(H_{ijk} = f_{abc} l^a_{ij} l^b_{jk}\), and is given locally in terms of an antisymmetric tensor potential \(b_{ij}\) by \(H_{ijk} = 3 \partial_i (b_{jk})\).

2.1 The \((1,0)\) Superparticle

We begin by considering the ‘\((1,0)\)’ superparticle moving upon a group manifold. The superspace for this particle has coordinates \((x, \theta)\), and we define the covariant derivative

\[
D \equiv \partial_\theta + \theta \partial_t = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t}.
\]

(1)

Our superfields \(\Phi^i(t, \theta)\) will be maps from the superspace into the group \(G\), with a component expansion

\[
\Phi^i(t, \theta) = X^i(t) + \theta \psi^i(t).
\]

(2)

The action for the \((1,0)\) superparticle on a group manifold is then

\[
S = \int dt \, d\theta \left( g_{ij}(\Phi) - b_{ij}(\Phi) \right) D\Phi^i \partial_t \Phi^j.
\]

(3)
The action \((3)\) can be conveniently written in terms of maps \(G = G(t, \theta)\) from the superspace into the group manifold. This form of the action is

\[
S = \text{Tr} \int dt \, d\theta \left( (DG^{-1})(\dot{G}G^{-1}) - \int_0^1 ds \, \bar{G}^{-1} \frac{\partial \bar{G}}{\partial s} \left[ \bar{G}^{-1} D\bar{G}, \bar{G}^{-1} \bar{G}' \right] \right),
\]

where \(\bar{G}\) is an extension of \(G\) to a map from the product of the interval \([0, 1]\) with the superspace, to the group \(G\). We impose the boundary conditions \(\bar{G}|_{s=1} = G, \bar{G}|_{s=0} = e\), where \(e\) is the identity element of the group \(G\). We introduce a matrix representation for \(G\), and all multiplications are taken to be matrix multiplications.

The equation of motion obtained from the action \((5)\) is

\[
D(G^{-1}\dot{G}) = 0.
\]

The solution to this equation is

\[
G = U(\theta)e^{i\hat{a}V},
\]

where \(U\) and \(\hat{a}\) are time independent, with \(U\) a superfield group element and \(\hat{a}\) an element of the Lie algebra \(\text{Lie}G\). The parameterisation \((7)\) of the solutions of the equation of motion is invariant under the replacements \(U \rightarrow Uh, \hat{a} \rightarrow h^{-1}\hat{a}h, V \rightarrow hV\), where \(h\) is any element of the group \(G\). This symmetry can be fixed by choosing (for example) \(V = e\).

We now turn to consider the phase space of this system. There are two ways in which to define the phase space of a classical system. The first is the Hamiltonian definition as the space of positions and their conjugate momenta. A second definition of the phase space, here denoted \(P_C\), is as the space of solutions of the Euler-Lagrange equations of motion. We will consider the space \(P_C\), and describe it as the ‘covariant phase space’. From the explicit solution \((7)\) above, taking into account the gauge-fixing, we see that this space is just \(G \times \text{Lie}G \times \Psi\), where \(\Psi\) is the space of anticommuting variables \(\psi\). In order to deduce the Poisson brackets of the theory, we seek on the phase space \(P_C\) a differential two-form which is closed, time-independent, and supersymmetric. The unique such form is

\[
\omega = \text{Tr} \int d\theta \left( \delta G \delta(G^{-1}DG^{-1}) \right) - \frac{1}{2} \text{Tr} \left( \delta(G^{-1}DG)\delta(G^{-1}DG) \right) |_{\theta=0}.
\]
It will simplify our discussion to use the variable \(a \equiv \hat{a} + \psi^2\) instead of \(\hat{a}\). Writing \(\omega\) in terms of the superfield component fields \(\{u, a, \psi\}\) then gives

\[
\omega = \omega(u, a) + \frac{1}{2} \text{Tr} (\delta \psi \delta \psi) \tag{8}
\]

\[
\omega(u, a) \equiv \text{Tr} \left( (u^{-1} \delta u) \delta a + (u^{-1} \delta u)^2 a \right). \tag{9}
\]

One can check that this form \(\omega\) is invariant under the supersymmetry transformations \(\Delta u = \epsilon u \psi, \Delta \psi = \epsilon (\hat{a} - \psi^2), \Delta \hat{a} = 0\), with \(\epsilon\) a constant anticommuting parameter. The symplectic form \(\chi\) factorises into fermionic and bosonic sectors, and we can invert each sector separately in order to find the Poisson brackets. (That this factorisation must occur follows from the fact that a redefinition of the component fields \(\hat{a}\) in the action (4) decouples the fermions. We will be presenting here new canonical formulations of the superspace WZW models, which provide a basis for the calculation of the Poisson brackets of arbitrary superfield group elements. Hence we will work with the superfield group elements \(G\), rather than the component field coordinates \(X\) and \(\psi\), and will express \(G\) in terms of our new parameterisations of the solutions of the supersymmetric WZW models.)

After converting to group manifold coordinates, inverting the bosonic form follows ref. [13], and using this we obtain the fundamental Poisson brackets for the \((1, 0)\) superparticle on a group manifold

\[
\{X^i, X^j\} = 0, \quad \{X^i, a_b\} = R^i_b, \quad \{a_a, a_b\} = f^{c}_{ab} a_c, \tag{10}
\]

and

\[
\{\psi^i, \psi^j\} = \delta^{ij}, \tag{11}
\]

If we use group elements \(u\) rather than coordinates \(X\), then the brackets (10) become

\[
\{u, u\} = 0, \quad \{u, a_b\} = -u t_b, \quad \{a_a, a_b\} = f^{c}_{ab} a_c. \tag{12}
\]

In order to simplify calculations as well as the appearance of equations, we will henceforth adopt a condensed notation. We define elements of the tensor product space \(G \otimes G\) as \(u_1 = u \otimes e\) and \(u_2 = e \otimes u\). We similarly define \(t^a_1 = t^a \otimes 1, t^a_2 = 1 \otimes t^a\) and define the Casimir tensor \(C_{12} = t^a_1 t_{2a} = t^a \otimes t_a\). In this notation, equation (12) may be written

\[
\{u_1, u_2\} = 0, \quad \{u_1, a_2\} = -u_1 C_{12}, \quad \{a_1, a_2\} = -[C_{12}, a_2]. \tag{13}
\]
(Note that $[C_{12}, a_1 + a_2] = 0$.)

Now we will use these fundamental Poisson brackets to show that the supercurrents of this model satisfy the particle version of the $(1, 0)$ super Kač-Moody algebra. The currents of the model are $J_L = G^{-1}\dot{G}$ and $J_R = DG G^{-1}$, satisfying the conservation laws $DJ_L = 0$ and $\partial_t J_R = 0$. In the gauge $V = e$ these currents take the form

$$
\begin{align*}
J_L &= \dot{a} = a - \psi^2, \\
J_R &= u(\psi + \theta a) u^{-1}.
\end{align*}
$$

A straightforward calculation, using the brackets (13), then yields the current superalgebra

$$
\begin{align*}
\{J_{L1}, J_{L2}\} &= -[C_{12}, J_{L2}], \\
\{J_{L1}, J_{R2}\} &= 0, \\
\{J_{R1}, J_{R2}\} &= C_{12} + [C_{12}, \theta_1 J_{R2} + \theta_2 J_{R1} + \theta_1 \theta_2 \partial_\theta J_{R2}],
\end{align*}
$$

which is the particle version of the classical $(1, 0)$ super Kač-Moody algebra (i.e., the zero-mode subalgebra).

### 2.2 The $(1, 1)$ Superparticle

The $(1, 1)$ particle has superspace coordinates $(x, \theta^+, \theta^-)$, and we define the covariant derivatives $D_{\pm} \equiv \partial_{\theta^\pm} \pm \theta^\pm \partial_t$. The superfields $\Phi^i(t, \theta^+, \theta^-)$ are maps from the superspace into the group $G$. The action, written in terms of the group elements $G(t, \theta^+, \theta^-)$ corresponding to the coordinates $\Phi^i$, is

$$
S = \text{Tr} \int dt \, d^2\theta \left( (D_+ GG^{-1})(D_- GG^{-1}) - \int_0^1 ds \left( \tilde{G}^{-1} \frac{\partial \tilde{G}}{\partial s} [\tilde{G}^{-1} D_+ \tilde{G}, \tilde{G}^{-1} D_- \tilde{G}] \right) \right).
$$

($\tilde{G}$ is defined in the same way as in the $(1, 0)$ case above.) The equation of motion is

$$
D_+ [G^{-1} D_- G] = 0,
$$

for which the general solution is

$$
G = U(\theta^+) e^{\hat{a}} V(\theta^-),
$$
with $U$ and $V$ group elements and $\hat{a}$ in the Lie algebra of $G$. We will expand $U(\theta^+) = u(1 + \theta^+ \psi_+)$ and $V = (1 + \theta^- \psi_-)v$, with $u, v \in G$ and $\psi_\pm \in \text{Lie}G$.

The symplectic form of this model is readily derived, and written in terms of the parameters $(u, v, \hat{a}, \psi_\pm)$ of the solutions, this form becomes

$$\omega = \omega(u, \hat{a} + \psi_+^2) - \omega(v^{-1}, \hat{a} - \psi_-^2) + \frac{1}{2} \text{Tr}(\delta \psi_+ \delta \psi_+ - \delta \psi_- \delta \psi_-),$$

(19)

where $\omega(u, a)$ is defined in equation (10). The form given in equation (19) is degenerate along the directions of the gauge symmetry $u \rightarrow uh, \hat{a} \rightarrow h^{-1} \hat{a}h, v \rightarrow hv, \psi_\pm \rightarrow h^{-1} \psi_+ h$, where $h$ is any element of the group $G$. We may fix this symmetry and eliminate the degeneracy by choosing $v = e$. (Note, however, that this gauge fixing breaks explicit supersymmetry.) In this gauge the symplectic form is

$$\omega = \text{Tr} \left( (u^{-1} \delta u) \delta a + (u^{-1} \delta u)^2 a + \frac{1}{2} \delta \psi_+ \delta \psi_+ - \delta \psi_- \delta \psi_- \right),$$

(20)

with $a \equiv \hat{a} + \psi_+^2$. The bosonic sector Poisson brackets are then the same as in the $(1, 0)$ case, and the fermionic sector brackets are

$$\{\psi_1^+, \psi_2^+\} = C_{12}, \quad \{\psi_1^-, \psi_2^-\} = -C_{12},$$

(21)

with other Poisson brackets vanishing. The covariant phase space $P_C$ has coordinates $(u, a, \psi_+, \psi_-)$, and equals $G \times \text{Lie}G \times \Psi \times \Psi$.

The supercurrents of this model are $J_L = G^{-1}D_- G$ and $J_R = D_+ GG^{-1}$, satisfying the conservation laws $D_+ J_L = 0$ and $D_- J_R = 0$. In terms of the covariant phase space coordinates, in the gauge $v = e$, these currents take the form

$$J_L = \psi_+ - \theta^- (a - \psi_+^2 + \psi_-^2),$$
$$J_R = u(\psi_+ + \theta^+ a)u^{-1}.$$  

(22)

A straightforward calculation, using the brackets (23) then yields the current algebra

$$\{J_{R1}, J_{R2}\} = C_{12} + \left[ C_{12}, \theta_1^+ J_{R2} + \theta_2^+ J_{R1} + \theta_1^+ \theta_2^+ \partial_{\theta_2^+} J_{R2} \right],$$
$$\{J_{L1}, J_{R2}\} = 0,$$
$$\{J_{L1}, J_{L2}\} = -C_{12} + \left[ C_{12}, \theta_1^- J_{L2} + \theta_2^- J_{L1} + \theta_1^- \theta_2^- \partial_{\theta_2^-} J_{L2} \right].$$

(23)

which is the particle version of the classical $(1, 1)$ super Kač-Moody algebra.
2.3 The (2, 0) Superparticle

This particle model is that obtained from a dimensional reduction of the (2, 0) WZW model. The latter is defined only on manifolds admitting a suitable complex structure (the classical formulation of the (2, 0) WZW model is given in reference [16], and the quantum theory is discussed in [17]. The reader is referred to these references for more discussion on the formulation of this model.) The particle superspace has coordinates \((t, \theta^+, \bar{\theta}^+), (\theta^-, \bar{\theta}^-)\), with supercovariant derivatives
\[
D_+ = \frac{\partial}{\partial \theta^+} + \theta^+ \frac{\partial}{\partial t}, \quad \bar{D}_+ = \frac{\partial}{\partial \bar{\theta}^+} + \bar{\theta}^+ \frac{\partial}{\partial t}.
\]
The complex structure on the group manifold may be used to separate the group manifold coordinate superfield components \(\phi^M, M = 1, \ldots, \dim G\), into two sets \(\phi^m, \phi^\bar{m}\), \((m, \bar{m} = 1, \ldots, \frac{1}{2}\dim G)\). These coordinate superfields are required to satisfy the chiral constraints
\[
\bar{D}_+ \phi^m = 0 = D_+ \phi^\bar{m}. \tag{24}
\]
The equations of motion of the (2, 0) superparticle are
\[
D_+(G^{-1}\dot{G}) = 0 = \bar{D}_+(G^{-1}\dot{G}), \tag{25}
\]
where \(G(t, \theta^+, \bar{\theta}^+)\) are maps from the superspace into the group. The equations (23) are solved by
\[
G = U(\theta^+, \bar{\theta}^+) e^{ta}, \tag{26}
\]
where \(U\) is a superfield group element, \(v\) a group element, and \(a \in \text{Lie}G\).
We will expand \(U\) in component fields as
\[
U = u(1 + \theta^+ \psi + \bar{\theta}^+ \bar{\psi} + \theta \bar{\theta} b).
\]
The conditions (24) are then solved if we impose the following consistency conditions upon these component fields
\[
\psi^c = 0 = \bar{\psi}^c, \quad u = e, \quad b^c = a^c + (\bar{\psi} \psi)^c, \quad b^\bar{c} = -a^\bar{c} - (\bar{\psi} \bar{\psi})^\bar{c}. \tag{27}
\]
(It will turn out that products of the Lie algebra-valued fields \(\psi, \bar{\psi}\) only appear in the form of commutators, when the above expressions for the \(b\) field components are substituted in expressions below.) Note that in this model there is no redundancy in the parameterisation of the solutions, due to the constraints (24) imposing the relations (27).
To find the Poisson brackets of this model, it is easiest to note that the
\((2, 0)\) model is just a \((1, 0)\) model defined upon a special manifold (i.e., one
admitting a suitable complex structure). Hence, on such a manifold the
symplectic form, and the consequent Poisson brackets, become those of the
\((1, 0)\) model. In this case this is just
\[ \omega = -\omega(v^{-1}, a) + \frac{1}{2} \text{Tr}(\delta \Psi \delta \Psi), \]
where \(\Psi = (\psi^a, \bar{\psi}^\bar{a})\). Using \(2 \omega\) rather than \(\omega\) for later convenience, we are thus led
to the Poisson brackets

\[ \{v, v\} = 0, \quad \{v, a^c\} = \frac{1}{2} f^{c} v, \quad \{v, \bar{a}^\bar{c}\} = \frac{1}{2} f^{c} \bar{v}, \]

\[ \{a^b, a^c\} = \frac{1}{2} f^{bc} a^{d\bar{d}} + \frac{1}{2} f^{bc} a^{d\bar{d}}, \]

\[ \{a^\bar{b}, a^\bar{c}\} = \frac{1}{2} f^{\bar{b}\bar{c}} a^{d\bar{d}}, \quad \{a^b, a^\bar{c}\} = \frac{1}{2} f^{bc} a^{d\bar{d}}, \]

\[ \{\psi^b, \bar{\psi}^{\bar{c}}\} = -\delta^{bc}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (28) \]

with the other Poisson brackets vanishing.

The currents of this model are \(J_L = G^{-1} \dot{G}\) and \(J_R = D_+G G^{-1}\), \(\bar{J}_R = \bar{D}_+ \bar{G} G^{-1}\). (The right currents automatically satisfy the constraints discussed
in refs. [16], [17], which are \(D_+ J_R = J_R^2\) and its conjugate.) The left current
is simply \(v^{-1} a v\) in the covariant phase space coordinates, and its Poisson
bracket with itself just gives a copy of the Lie algebra. The right currents
are given explicitly by

\[ J_R = \psi + \theta^+ \psi^2 + \bar{\theta}^+ (a + b + \psi \bar{\psi}) + \theta^+ \bar{\theta}^+ [a + b + \psi \bar{\psi}, \psi], \]

\[ \bar{J}_R = \bar{\psi} + \bar{\theta}^+ \bar{\psi}^2 + \theta^+ (a - b + \bar{\psi} \psi) + \theta^+ \bar{\theta}^+ [-a + b - \bar{\psi} \psi, \bar{\psi}], \quad (29) \]

where the expressions for the components of the field \(b\) given in equation (27)
are to be inserted in equation (29). Note that the components \(J_R^a\) and \(\bar{J}_R^a\)
vanish due to the constraints (24), as expressed by the relations (27).

Since the expression \(v^{-1} a v\) has vanishing Poisson brackets with the variables \(a, \psi, \text{and} \bar{\psi}\), we see immediately that the left current \(J_L\) Poisson
commutes with the right currents \(J_R, \bar{J}_R\). The calculation of the Poisson brackets of the right currents amongst themselves is rather tedious. However, if we
refer to ref. [17], we note that this calculation is precisely the classical, particle
analogue of the quantum operator product expansions calculated there.
(The classical calculation corresponds to taking only single operator products
of elementary fields in calculating the operator product of composite fields.)
The field redefinitions $\psi^c \rightarrow i\sqrt{2}\psi^c$, $\bar{\psi}^c \rightarrow -i\sqrt{2}\bar{\psi}^c$, $j \rightarrow 2a$ relate the fields of ref. [17] to those used here, and the constant $k$ of ref. [17] equals $4i$ here.) Whence we deduce that the Poisson brackets of the right currents given by equation (29) yields the classical, particle version of the non-linear $N = 2$ Kač-Moody algebra presented in ref. [17]. This is ($J$ is $J_R$ in these relations)

\begin{align}
\{J^a_1, J^b_2\} &= \bar{\theta}_{12} f^{ab}_{\ c} J^c_2 - \theta_{12} \bar{\theta}_{12} f^{a}_{\ ec} f^{be}_{\ d} J^c_2 J^d_2, \\
\{\bar{J}^\bar{a}_1, \bar{J}^\bar{b}_2\} &= \theta_{12} f^{\bar{a}b}_{\ \bar{c}} \bar{J}^\bar{c}_2 - \bar{\theta}_{12} \theta_{12} f^{\bar{a}}_{\ \bar{e}c} f^{\bar{b}e}_{\ \bar{d}} \bar{J}^\bar{c}_2 \bar{J}^\bar{d}_2, \\
\{J^a_1, \bar{J}^\bar{b}_2\} &= -\delta^{ab} + \theta_{12} f^{ab}_{\ \bar{c}} J^\bar{c}_2 + \bar{\theta}_{12} f^{ab}_{\ c} J^c_2 \\
&\quad + \theta_{12} \bar{\theta}_{12} \left( f^{ab}_{\ \bar{c}} D_+ J^\bar{c}_2 - f^{\bar{b}}_{\ \bar{e}c} f^{\bar{a}c}_{\ \bar{d}} \bar{J}^\bar{d}_2 J^\bar{c}_2 \right),
\end{align}

(30)

where $\theta_{12} = \theta_1 - \theta_2$, $\bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2$, $J_1 = J(\theta_1, \bar{\theta}_1)$ and $J_2 = J(\theta_2, \bar{\theta}_2)$. It is interesting that even this classical $(2,0)$ particle algebra is not a Lie algebra. That this must occur can be guessed immediately by realising that the algebra of the right currents must be consistent with the non-linear constraints, mentioned above, which they must satisfy.

3 The Supersymmetric WZW Models

We now turn our attention to the two-dimensional supersymmetric WZW models. We will work in Minkowski space, on a cylinder with coordinates $(x,t)$, with $0 \leq x \leq 1$, $x^\pm = \frac{1}{2}(x \pm t)$, and $\partial_\pm = \partial_x \pm \partial_t$.

3.1 The $(1,0)$ WZW Model

The $(1,0)$ superparticle discussed above is the one-dimensional reduction of the two-dimensional $(1,0)$ supersymmetric WZW model. The two-dimensional $(1,0)$ superspace has co-ordinates $z = (\sigma^+, \theta^+) = (t, x, \theta^+)$, with $\theta^+$ a real anti-commuting co-ordinate. The supercovariant derivative is $D_+ = \frac{\partial}{\partial \theta^+} + \theta^+ \partial_+$. With $\Phi^i(x,t,\theta^+)$ coordinates on the group manifold, the action for the $(1,0)$ supersymmetric WZW model is

$$S = \int d^2 \sigma \ d\theta^+ (g_{ij} - b_{ij}) D_+ \Phi^i \partial_- \Phi^j.$$  

(31)
Using a group manifold superfield $G = G(x, t, \theta^+)$ instead of the coordinates $\phi^i$, the action can be written in the form

$$S = \mathrm{Tr} \int d^2 \sigma d\theta^+ \left( (G^{-1} \partial_- G)(G^{-1} D_+ G) - \int_0^1 ds \left[ \bar{G}^{-1} \partial_- \bar{G}, \bar{G}^{-1} \partial_\theta \bar{G} \right] \bar{G}^{-1} D_+ \bar{G} \right),$$

(32)

where $\bar{G}(z, s)$ is an extension of the map $G$ to a map from the product of the superspace with the unit interval, into the group $G$, with boundary conditions $\bar{G}(z, 0) = e$ and $\bar{G}(z, 1) = G(z)$. The equation of motion following from the action (25) is

$$D_+ (G^{-1} \partial_- G) = 0.$$  

(33)

The left and right currents for this model are given by

$$J_L(x^-) = -G^{-1} \partial_- G,$$

$$J_R(x^+) = D_+ G G^{-1}.$$  

The closed, supersymmetric, time-independent symplectic form for the $(1, 0)$ model is found to be

$$\Omega = \mathrm{Tr} \int_0^1 dx \left( \int d\theta^+ \delta G \delta (G^{-1} D_+ G G^{-1}) - \frac{1}{2} \delta (G^{-1} D_+ G) \delta (G^{-1} D_+ G |_{\theta^+ = 0}) \right).$$

(34)

The solutions to the field equations (33) may be written

$$G(t, x; \theta^+) = U(x^+, \theta^+) W(\hat{A}; x^+, x^-) V(x^-),$$

$$W(\hat{A}; x^+, x^-) = P \exp \int_{x^-}^{x^+} \hat{A}(s) ds,$$

(35)

with $\hat{A}$ a $(\text{Lie} G)^*$-valued periodic one-form on the real line, ‘$P$’ denoting path ordering. The superfields $U$ and $V$ are periodic in $x$, and hence so is $G$. The parameterisation (35) of the solutions is invariant under the transformations

$$U(x, \theta^+) \longrightarrow U(x, \theta^+) h(x), \quad V(x, \theta^+) \longrightarrow h^{-1}(x) V(x, \theta^+),$$

$$\hat{A}(x) \longrightarrow -h^{-1}(x) \partial_\theta h(x) + h^{-1}(x) \hat{A}(x) h(x),$$

(36)

where $h$ is an element of the loop group of $G$, $LG$. The symplectic form (34) is degenerate along the directions of the action of the transformations (36). This symmetry may be fixed by setting $V = e$. We will expand $U = u(1 + \theta^+ \psi_+).$ It is also convenient to use the variable $A = \hat{A} - \psi^2$ instead of $\hat{A}$. Substituting the gauge-fixed solution into the symplectic form then gives

$$\Omega = \mathrm{Tr} \int_0^1 dx \left( \frac{1}{2} (u^{-1} \delta u) \partial_\theta (u^{-1} \delta u) + \frac{1}{2} \delta \psi_+ \delta \psi_+ + (u^{-1} \delta u)^2 A + (u^{-1} \delta u) \delta A \right).$$

(37)
Following the approach of [13], it is straightforward to invert this form to obtain the Poisson brackets

\[
\{u_1, u_2\} = 0, \quad \{u_1, A_2\} = -u_1 C_{12} \delta_{12}, \\
\{A_1, A_2\} = (C_{12} \partial_1 - [C_{12}, A_2]) \delta_{12}, \\
\{\psi_1, \psi_2\} = C_{12} \delta_{12}, \quad \{\psi_1, A_2\} = 0,
\]

(38)

where \(u_1 = u(x_1) \otimes e, A_2 = 1 \otimes A(x_2), \partial_1 = \frac{\partial}{\partial x_1}, \) etc., with \(C_{12} = t^a \otimes t_a\) again and \(\delta_{12} = \delta(x_1, x_2)\), the delta function on \(S^1\). In the gauge \(V = e\), the WZW currents become

\[
J_L = A - \psi^2, \\
J_R = u \left( \psi + \theta^+ (A + u^{-1} \partial_+ u) \right) u^{-1}.
\]

(39)

Using the fundamental brackets (38), we deduce the current algebra

\[
\{J_{L1}, J_{L2}\} = (C_{12} \partial_1 - [C_{12}, J_{L2}]) \delta_{12}, \\
\{J_{L1}, J_{R2}\} = 0, \\
\{J_{R1}, J_{R2}\} = (1 - \theta_1^+ \theta_2^+) C_{12} \partial_1 \delta_{12} + [C_{12}, \theta_1^+ J_{R2} + \theta_2^+ J_{R1} + \theta_1^+ \theta_2^+ \partial_+ J_{R1}] \delta_{12},
\]

(40)

which is the classical \((1, 0)\) super Kač-Moody algebra.

### 3.2 The \((1, 1)\) WZW Model

The analysis of this model follows along the lines of the models considered above, and the reader may derive it using the above methods. The equation of motion for the \((1, 1)\) model is

\[
D_+ (G^{-1} D_- G) = 0,
\]

(41)

where \(G(x, t, \theta^+, \theta^-)\) is a map from the \((1, 1)\) superspace into the group \(G\), and the supercovariant derivatives are given by \(D_\pm = \partial/\partial \theta^\pm + \theta^\pm \partial_\pm\). The solutions of the equation of motion (41) are

\[
G = u(1 + \theta^+ \psi_+) P \exp \left( \int_{x^-}^{x^+} \hat{A}(s) ds \right) (1 + \theta^- \psi_-) v,
\]

(42)
where \( u, v \in LG \), \( \psi_\pm \) are maps from the circle into \( \Psi \), and \( \hat{\mathcal{A}} \) is a Lie-algebra valued periodic one-form on the real line. We will define \( A = \hat{\mathcal{A}} + \psi_+^2 \). The gauge symmetry in the parameterisation (42) will be fixed by setting \( v = e \).

The covariant phase space \( P_C \) for the \((1, 1)\) supersymmetric WZW model has coordinates \((u, A, \theta^+, \theta^-)\), with \( u \in LG \), \( A \) a \((\text{Lie} G)^*\)-valued periodic one-form on the real line, and \( \psi_+, \psi_- \) anticommuting elements of \( L(\text{Lie} G) \).

The fundamental Poisson brackets on \( P_C \) are then the brackets for \( u \) and \( A \) given in equation (38), together with the further brackets

\[
\{ \psi_+, \psi_+ \} = C_{12} \delta_{12}, \quad \{ \psi_-, \psi_- \} = -C_{12} \delta_{12}, \quad \text{(43)}
\]

with all other brackets vanishing. The conserved currents are \( J_L = -G^{-1} D_G \), \( J_R = D_+ G G^{-1} \) and the algebra of these currents may then be shown to be the classical \((1, 1)\) super Kač-Moody algebra

\[
\begin{align*}
\{ J_{L1}, J_{L2} \} &= - \left( 1 - \theta_1^- \theta_2^- \right) C_{12} \partial_1 \delta_{12} - \left[ C_{12}, \theta_1^- J_{L2} + \theta_2^- J_{L1} + \theta^- \theta_2^- \partial \theta_2^- J_{L2} \right] \delta_{12}, \\
\{ J_{L1}, J_{R2} \} &= 0, \\
\{ J_{R1}, J_{R2} \} &= \left( 1 - \theta_1^+ \theta_2^+ \right) C_{12} \partial_1 \delta_{12} + \left[ C_{12}, \theta_1^+ J_{R2} + \theta_2^+ J_{R1} + \theta_1^+ \theta_2^- \partial \theta_2^+ J_{R2} \right] \delta_{12}.
\end{align*}
\]

As we have seen above, the covariant phase spaces of the WZW models considered all take the form \( T^*(LG) \times Y \), where \( Y \) is a space of anticommuting variables. The topology of these covariant phase spaces is concentrated in the subspace \( T^*(LG) \). Thus the same topological issues which arose for the bosonic WZW model will occur here. Alternative parameterisations of the solution spaces of the supersymmetric WZW models which could be considered in this context are those corresponding to taking the connection \( \hat{\mathcal{A}} \) to be a constant connection. These parameterisations can be reached from ours by a partial gauge-fixing, however this fixing encounters the Gribov problem. As a consequence, these alternative covariant phase spaces have differing topologies to the ones presented here (they will also have difficulties associated with choosing a proper gauge-fixing of the redundancy in the solution parameterisations). The covariant phase spaces presented here have the advantage that they are diffeomorphic to the spaces of initial data (this is straightforward to check). These issues are discussed fully in references [13], [17], and this discussion carries over entirely to the supersymmetric models considered here.
3.3 Other Supersymmetric WZW Models

We expect that there are covariant phase space formulations of the WZW models with more supersymmetries than those studied above. As we have discussed with regard to the $(2,0)$ superparticle, these models have superfield group elements which have to satisfy constraints, and because of this the analysis involving superfield group elements becomes more involved. A study of the $(2,0)$ WZW model is possible along these lines, following the analogous particle discussion given above. WZW models with more supersymmetries exist - for example, the $N = 4$ models exist on the group manifolds $SU(2n + 1), (n = 1, 2, \ldots)$ [18], and a similar covariant phase space analysis should apply in such cases. For these models, however, due to the constraints involved a component-field analysis is preferable to a superfield analysis, although some simplifications can be realised by working in $N = 1$ superspace.

4 Discussion

Our solutions of the supersymmetric WZW theories involve an integral $\int_{x^-}^{x^+} ds \hat{A}(s)$. By choosing some point $x_0 \in R$, one may split this integral into $\int_{x_0}^{x^-} ds \hat{A}(s) + \int_{x^-}^{x_0} ds \hat{A}(s)$. Using this, one may write our solutions as products of chiral fields depending upon only one of the variables $x^+, x^-$ (note, however, that these chiral fields also depend upon the arbitrary point $x_0$). One can use our fundamental Poisson brackets to study the Poisson brackets of these chiral fields. To do this, one needs to choose a regularisation carefully, as divergences arise from the delta functions in the fundamental Poisson brackets. This problem has been discussed in a general context recently in ref. [19]. It would be pleasing if it would be possible to define the Poisson brackets of chiral fields in such a way that the emergence of $r$-matrix relations appears naturally in this approach. One might expect that zero-mode subtleties would be taken care of by a proper treatment of the dependence upon the point $x_0$. (These comments also apply to the bosonic WZW model.)

The above analysis of the covariant phase spaces of supersymmetric WZW models makes possible a corresponding discussion of the supersymmetric Toda theories. This will follow by supersymmetric Hamiltonian reductions [20] of the formulations given here, along the lines of the bosonic results.
given in reference [21]. Furthermore, it has been found recently that these methods apply to the affine Toda and conformal affine Toda theories [22], so that the supersymmetric versions of these theories will also be amenable to this approach.

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