

**Lifelong Matrix Completion with Sparsity-Number**

Ilqar Ramazanli  
Carnegie Mellon University  
Pittsburgh, USA  
iramazan@alumni.cmu.edu

**ABSTRACT**

Matrix completion problem has been previously studied under various adaptive and passive settings. Previously, researchers have proposed passive, two-phase and single-phase algorithms using coherence parameter, and multi phase algorithm using sparsity-number. It has been shown that the method using sparsity-number reaching to theoretical lower bounds in many conditions. However, the aforementioned method is running in many phases through the matrix completion process, therefore it makes much more informative decision at each stage. Hence, it is natural that the method outperforms previous algorithms. In this paper, we are using the idea of sparsity-number and propose and single-phase column space recovery algorithm which can be extended to two-phase exact matrix completion algorithm. Moreover, we show that these methods are as efficient as multi-phase matrix recovery algorithm. We provide experimental evidence to illustrate the performance of our algorithm.

1 INTRODUCTION

Lifelong Adaptive Matrix Completion problem has been the center of the attention of research for many recent years. Previously matrix completion has been mainly studied in passive sampling setting by uniformly sampling and observing some subset of entries of the matrix in the beginning of the algorithm and applying nuclear norm minimization. In a series of work in Gross (2011), Candés & Recht (2009) and Recht (2011) it has been shown that uniformly sampling and observing \( \Omega((m+n)r \max(\mu_0^2, \mu_1^2) \log^2 n_2) \) many entries are enough to recover an \( m \times n \) matrix of rank \( r \) with \( \mu_0 \) and \( \mu_1 \) column and row space coherence parameters using this idea. Moreover, Candés & Tao (2010) showed this passive uniform sampling setting we need at least \( \Omega(mr\mu_0 \log n) \) observations to recover the matrix exactly. Therefore, this result was concluding almost optimality of the nuclear norm minimization where sampling method is passive.

However, in modern data analysis it has been show that adaptive sensing and sampling methods are outperforming passive sampling methods Ramazanli (2022); Haupt et al. (2011); Warmuth & Kuzmin (2008); Ramazanli et al. (2020). In a follow up work the paper Krishnamurthy & Singh (2013) has proposed an adaptive matrix completion algorithm that can actually recover underlying matrix using at most \( \mathcal{O}(\mu r^{1.5} \log \frac{n}{\epsilon}) \) many observation. Later authors further improved their existing results in Krishnamurthy & Singh (2014) to show that the adaptive matrix completion algorithm can actually successfully perform by just observing \( \mathcal{O}(\mu r \log^2 \frac{n}{\epsilon}) \) many observation.

The latest result has been even further optimized in the paper Balcan & Zhang (2016) to present an adaptive algorithm which performs only with observation complexity of just \( \mathcal{O}(\mu_0 r \log \frac{n}{\epsilon}) \). Recently Ramazanli & Poczos (2020) proposed a new parametrization method called sparsity-number and a multi-phase algorithm which shows that using this new method, the observation complexity can go as low as \( \mathcal{O}((m + n - r)r) \). The idea of this method to use the sparsest vector of column and row spaces instead of coherence in the recovery process. Studying the sparsest vector of the column and row space has been focus in many research problems (McCormick (1983), Qu et al. (2015), Coleman & Pothen (1986)), and Ramazanli & Poczos (2020) showed how to use this parameter efficiently for adaptive matrix completion. In this paper, we propose a matrix completion method which merges ideas of single phase algorithms such as Balcan & Zhang (2016) with the idea of sparsity number and multi-phase method Ramazanli & Poczos (2020) which is resulting in a single phased algorithm which is as efficient as multi-phased one.

Low-rank matrix completion plays a significant role in many real-world applications, including camera motion inferencing, multi-class learning, positioning of sensors, and gene expression analysis Balcan & Zhang (2016);
Krishnamurthy & Singh (2013). In gene expression analysis, the target matrix represents expression levels for various genes across several conditions. Measuring gene expression, however, is expensive, and we would like to estimate the target matrix with a few observations as possible. In this paper, we provide an algorithm that can be used for matrix completion from limited data. Roughly speaking, to find each unknown expression level, we are supposed to target matrix with a few observations as possible. In this paper, we provide an algorithm that can be used for matrix completion from limited data. Roughly speaking, to find each unknown expression level, we are supposed to do multiple measurements. Each of the additional measurements has its extra cost. Naturally, we aim to solve the entire problem using the least possible measurement cost.

2 Preliminaries

In this section we start by providing notations and definitions those are used throughout the paper. Then we will provide a single phase Balcan & Zhang (2016) and multi-phase Poczos et al. (2020) matrix recovery algorithms.

Throughout the paper, we denote by $M$ the target underlying $m \times n$ sized rank-$r$ matrix that we want to recover. $\|x\|_p$ denote the $L_p$ norm of a vector $x \in \mathbb{R}^n$. We call $x_i$ the $i$'th coordinate of $x$. For any, $\Omega \subset [n]$ let $x_\Omega$ denote the induced subvector of $x$ from coordinates $\Omega$. For any $R \subset [m]$, $M_{R \Omega}$ stands for an $|R| \times n$ sized submatrix of $M$ that rows are restricted by $R$. We define $M_{C R}$ in a similar way for restriction with respect to columns. Intuitively, $M_{R \subset C}$ defined for $|R| \times |C|$ sized submatrix of $M$ with rows restricted to $R$ and columns restricted to $C$. Moreover, for the special case $M_{iR}$ stands for $i$-th row and $M_{jC}$ stands for the $j$'th column. Similarly, $M_{i,C}$ will represent the restriction of the row $i$ by $C$ and $M_{R,j}$ represents restriction of the column $j$ by $R$.

The following definitions are key parameters in this paper, first we define coherence which is an important parameter in the matrix completion research domain. Then, we visit the definition of the sparsity-number.

**Definition 1.** Coherence parameter of a matrix $M$ with column space $U$ is defined as following where $P_U$ represents the orthogonal projection operator onto the subspace $U$.

$$\mu(U) = \frac{n}{r} \max_{1 \leq j \leq n} ||P_U e_j||^2,$$

One can observe that if $e_j \in U$ for some $j \in [n]$, then the coherence will attain its maximum value: $\mu(U) = \frac{n}{r}$.

**Definition 2.** Nonsparsity-number of a vector $x \in \mathbb{R}^m$ represented by $\psi(x)$ and defined as $\psi(x) = \|x\|_0$. Moreover, nonsparsity-number of an $m \times n$ sized matrix $M$ and a subspace $U \subseteq \mathbb{R}^m$ also represented by $\psi$ and defined as:

$$\psi(M) = \min\{\psi(x) | x = Mz \text{ and } z \notin \text{null}(M)\} \quad \psi(U) = \min\{\psi(x) | x \in U \text{ and } x \neq 0\}$$

Sparsity-number is just completion of nonsparsity-number and is donated by $\overline{\psi}$:

$$\overline{\psi}(x) = m - \psi(x) \quad \overline{\psi}(M) = m - \psi(M) \quad \overline{\psi}(U) = m - \psi(U)$$

2.1 Single Phase Matrix Completion

Here we provide the generic framework for single-phase matrix completion algorithm that has been used by Krishnamurthy & Singh (2013, 2014); Balcan & Zhang (2016). The algorithm uniformly sample $d$-many entry in each column and observe them. Based on the observation, if it has been decided the column is linearly independent with previous columns then algorithm decides to observe entire column. Initially, it has been shown that choosing $d = \mathcal{O}(n\mu_0 r^{1.5} \log r)$ would be enough this algorithm to perform successfully in Krishnamurthy & Singh (2013). Then, authors showed setting $d = \mathcal{O}(n\mu_0 r \log^2 r)$ would also be enough Krishnamurthy & Singh (2014). In a following work Balcan & Zhang (2016) optimized the complexity by reducing one of the $\log r$ factors from $d$ and proving $d = \mathcal{O}(n\mu_0 r \log r)$ still works successfully with probability $1 - \epsilon$. 

2
**HN2016: Exact recovery Balcan & Zhang (2016).**

**Input:** $d = O(\mu_0 r \log \frac{1}{\epsilon})$

**Initialize:** $k = 0$, $\hat{U}^0 = \emptyset$

1. Draw uniformly random entries $\Omega \subset [m]$ of size $d$
2. for $i$ from 1 to $n$ do
3. if $\|M_{ij} - \hat{P}_{\hat{U}^k} M_{\Omega : i}\| > 0$
4. Fully observe $M_{ij}$
5. $\hat{U}^{k+1} \leftarrow \hat{U}^k \cup M_{ij}$. Orthogonalize $\hat{U}^{k+1}$, $k = k + 1$
6. otherwise: $M_{ij} = \hat{U}^k \hat{U}^k^\perp M_{\Omega : i}$

**Output:** $\hat{M}$

### 2.2 Multi Phase Matrix Completion

Here we provide multi-phase matrix completion algorithm from Ramazanli & Poczos (2020). There are two main differences between this algorithm and previous ones. First, previous algorithms go through the columns of the matrix just once, however, here we go through each column many times. Second, there is a difference in terms of available pre-information. One can see that, in order to perform previous algorithms, we need to have at least estimated information of rank $r$ and column space coherence $\mu_0$. However, in order to perform ERRE all we need to know is either column or row space is not highly coherent.

**ERRE: Exact recovery while rank estimation. Ramazanli & Poczos (2020)**

**Input:** $T$-delay parameter at the end of algorithm

**Initialize:** $R = \emptyset$, $C = \emptyset$, $\hat{r} = 0$, $delay = 0$

1. while $delay < T$ do
2. $delay = delay + 1$
3. for $j$ from 1 to $n$ do
4. Uniformly pick an unobserved entry $i$ from $M_{j}$
5. $\hat{R} = R \cup \{i\}$, $\hat{C} = C \cup \{j\}$
6. If $M_{R : \hat{C}}$ is nonsingular:
7. Fully observe $M_{R : \hat{C}}$ and $M_{i : \hat{R}}$, and set $R = \hat{R}$, $C = \hat{C}$, $\hat{r} = \hat{r} + 1$, $delay = 0$
8. Orthogonalize column vectors in $C$ and assign to $U$
9. for each column $j \in [n] \setminus C$ do
10. $\hat{M}_{j : \hat{R}} = \hat{U} U^\perp R : \hat{M}_{R : j}$

**Output:** $\hat{M}$, $\hat{r}$

The authors proved that the observation complexity could be bounded by the following expression which is more optimal than previous methods and many times it matches $O((m + n - r)r)$ which is theoretical lower bound for any kind of rank-$r$ matrix recovery algorithm.

**Theorem 1.** Let $r$ be the rank of underlying $m \times n$ sized matrix $M$ with column space $U$ and row space $V$. Then, ERRE exactly recovers the underlying matrix $M$ while estimating rank with probability at least $1 - (\epsilon + e^{-T \frac{\psi(U) \psi(V)}{m}})$ using number of observations at most:

$$(m + n - r) r + T n + \min \left( 2 \frac{m n}{\psi(U)} \log \left( \frac{r}{\epsilon} \right), \frac{2m}{\psi(U)} \frac{(r + 2 + \log \frac{1}{\epsilon})}{\psi(V)} n \right)$$
3 MAIN RESULTS

In this section, we provide single-phase column space recovery algorithm using \textit{sparsity-number} which can be easily extended to two-phase matrix recovery algorithm. The algorithm works in a similar flavor to Balcan & Zhang (2016) that it requires a preinformation regarding properties of the matrix in order to execute.

However, there is one clear difference in terms of preinformation that, here we require estimated information for sparsity number rather than coherence, and also we use the information regarding row-space as well rather than simply columns space. The reason behind this is, in deep mathematical analysis of aforementioned methods shows us all these methods work efficiently under the condition row spaces are highly coherent and in-depth analysis of these methods tells us that these algorithms designed for the highest value of row space coherence, and they also work perfectly well for remaining cases.

In the following algorithm, we extend these algorithms to a method that can enjoy the properties of row space; meanwhile, performing similarly good for highly coherent row space matrices. Note that, the algorithm below designed and analyzed under the condition that estimated $r$, $\psi(U)$, and $\psi(V)$ provided. However, under the condition just estimated $r$ and $\mu_0$ provided, we can set $\psi(V)$ to be equal to 1 and use the inequality $\mu_0 r > \frac{m}{2}$ to transfer the information of $\mu_0$ to $\psi(U)$. Then the algorithm below will perform as good as Balcan & Zhang (2016) with observation complexity of $nr + \Theta(nr\mu_0 \log \frac{1}{\epsilon})$.

\textbf{EREI: Exact recovery with estimated information}

\begin{itemize}
\item \textbf{Input:} $r$, $\psi(U)$, $\psi(V)$
\item \textbf{Initialize:} $R = 0, k = 0, \hat{U}\hat{0} = 0, d = \min\left(2 \frac{m}{\psi(U)} \log \left(\frac{e}{\epsilon}\right), \frac{2m\epsilon (r + 2 + \log \frac{1}{\psi(V)})}{\psi(V)}\right)$
\item 1: Draw uniformly random entries $\Omega \subset [m]$ of size $d$
\item 2: \textbf{for} $i$ from 1 to $n$ \textbf{do}
\item 3: \textbf{if} $||M_{\Omega,i} - \hat{P}_{\Omega_i} M_{\Omega,i}|| > 0$
\item 4: Fully observe $M_{\hat{R}}$
\item 5: $\hat{U}^{k+1} \leftarrow \hat{U}^k \cup M_i$, Orthogonalize $\hat{U}^{k+1}$, $k = k + 1$
\item 6: Select a row $a \in \Omega \setminus R$ that, $\hat{U}^{k+1}_{\hat{R}(a)}$ is rank $k + 1$ then $R \leftarrow R \cup \{a\}$
\item 7: Draw uniformly random entries $\Delta \subset [m] \setminus R$ of size $d$ and $\Omega = \Delta \cup R$
\item 8: Observe entire $\hat{M}_{\hat{R}}$
\item 9: \textbf{for} $i$ from 1 to $n$ \textbf{do}
\item 10: \textbf{if} $M_{\hat{R}}$ not fully observed \textbf{then}: $\hat{M}_i = \hat{U}^k \hat{U}^{k+1}_{\hat{R}(a)} \hat{M}_{\hat{R},i}$
\end{itemize}

\textbf{Output:} $\hat{M}_i$

Below we show the execution of the algorithm for $d = 2$ and $r = 1$. \(a\) stands for entries that is observed randomly, \(d\) stands for entries that is observed deterministically and \(a\) stands for recovered entries without observing. Note that $\Omega_1 = \{1, 5\}$, $\Omega_2 = \{2, 5\}$, $\Omega_3 = \{1, 3\}$ and after 3-rd iteration $R$ becomes $\{3\}$ therefore, in fourth column $M_{3,4}$ observed deterministically, besides together random observations $\Omega_4 = \{5, 6\}$. After all of the iterations completed, we observe the entire $M_3$, and in the next step, we recover remaining entries.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 6 & 4 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 6 & 4 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 6 & 4 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 6 & 4 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 6 & 4 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 6 & 4 & 6 \\
\end{array}
\]
Theorem 2. Let $r$ be the rank of underlying $m \times n$ sized matrix $M$ with column space $U$ and row space $V$. Then, with probability $1 - \epsilon$ the algorithm EREI exactly recovers the underlying matrix $M$ using number of observations at most

$$(m + n - r)r + \min\left(2mn \psi(U) \log \left(\frac{r}{\epsilon}\right), \frac{2m}{\psi(U)}(r + 2 + \log \left(\frac{1}{\epsilon}\right))n\right)$$

Proof. We show that the observation complexity is upper bounded both by

$$(m + n - r)r + 2m \psi(U) \log \left(\frac{r}{\epsilon}\right)$$

and also

$$(m + n - r)r + \frac{2m}{\psi(V)}(r + 2 + \log \frac{1}{\epsilon})$$

Throught the proof we will refer to the proof Theorem 4 - complexity bound regarding ERRE of the paper Ramazanli & Poczos (2020) also we are denoting $\psi(U)$ by $k$ to have the consistency with that proof.

3.1 $$(m + n - r)r + 2m \psi(U) \log \left(\frac{1}{\epsilon}\right)$$

We first start with the case that if minimum of these two quantities is $2m \psi(U) \log \left(\frac{1}{\epsilon}\right)$. As the matrix has rank $r$, there exists at least one set of linearly independent columns with $r$ columns. We select the set of linear independent columns $-C$ that has lexicographically smallest indices. We show sampling $2m \psi(U) \log \left(\frac{1}{\epsilon}\right)$ entries from each column will give us the probability of at least $1 - r\epsilon$ correctly recovery.

From the step 5 of proof of the theorem ERRE we can see sampling $2m \psi(U) \log \left(\frac{1}{\epsilon}\right)$ entries from an active column, would give guarantee of probability of at least $1 - \epsilon$ detection of independence. As $C$ is lexicographically smallest, each column is active on the time entries sampled from it, and each of $r$ columns will succeed with probability at least $1 - \epsilon$. Therefore, using union bound, with probability $1 - r\epsilon$ all of the columns in $C$ will succeed, which guarantees the exact recovery.

Replacing $\epsilon$ by $\frac{1}{\epsilon}$ will conclude the result that sampling $2m \psi(U) \log \left(\frac{1}{\epsilon}\right)$ from each column will guarantees the correctness of the algorithm with probability at least $1 - \epsilon$.

3.2 $$(m + n - r)r + \frac{2m}{\psi(V)}(r + 2 + \log \frac{1}{\epsilon})$$

From the follow up of lemma 13 in Ramazanli & Poczos (2020), we conclude that in a process of $\frac{k}{m}$ probability success and $1 - \frac{k}{m}$ probability of failure, having $\frac{2m}{k}(r + 2 + \log \frac{1}{\epsilon})$ trial is enough to guarantee getting $r$ many success with at least probability $1 - \epsilon$.

Failure probability of the algorithm EREI is equal to failing finding $r$ linearly independent columns. Consider following equation:

$$P(\text{EREI fails}) = P(\text{EREI fails and TNAO} \geq \frac{2m}{k}(r + 2 + \log \frac{1}{\epsilon})) + P(\text{EREI fails and TNAO} < \frac{2m}{k}(r + 2 + \log \frac{1}{\epsilon}))$$

where we denote TNAO as total number of active observations. Recall that we call an observation active, if it is active in the execution time (the column is still not contained in the current column space). Intuitively we represent NAO by number of active observations executed by the algorithm EREI in the given specific time.
From lemma 18, if there exists an active column, then there exists at least \( t \) many active columns. Therefore, failure of the algorithm is equivalent to existence of an active column when algorithm terminates. Moreover each of our observations in those columns were active observations and considering the fact that we observed \( \frac{2m}{t} \left( r + 2 + \log \frac{1}{\epsilon} \right) \) many entries in each of them, total number of active observations is at least

\[
\frac{2m}{t} \left( r + 2 + \log \frac{1}{\epsilon} \right) = \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right)
\]

Therefore \( P \left( \text{NAO} < \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) | \text{EREI fails} \right) = 0 \) and using Bayesian rule we conclude

\[
P \left( \text{EREI fails and TNAO} \leq \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \right) = 0
\]

Then, following equation simply satisfied:

\[
P \left( \text{EREI fails} \right) = P \left( \text{EREI fails and TNAO} \geq \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \right)
\]

We can observe the following inequality as \( \text{EREI} \) may temporarily fail at the point that the number of active observations is \( \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \) but it can succeed finding remaining independent columns later during the execution:

\[
P \left( \text{EREI currently fail when NAO} = \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \right) \geq \]

\[
P \left( \text{EREI fails and TNAO} \geq \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \right)
\]

Therefore we conclude that:

\[
P \left( \text{EREI fails} \right) \leq P \left( \text{EREI currently fail when NAO} = \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \right)
\]

Remember the fact that at each active observation probability of \( \text{EREI} \) detecting linear independence of is larger or equal than \( \frac{k}{m} \). From the previous discussion if the probability is exactly equal to \( \frac{k}{m} \) then still not finding \( r \) linearly independent column at \( \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \) observations is less than \( \epsilon \). Therefore, \( \text{EREI} \) not detecting \( r \) linearly independent column after \( \frac{2m}{k} \left( r + 2 + \log \frac{1}{\epsilon} \right) \) observations is smaller than \( \epsilon \), which is equivalent to

\[
P \left( \text{EREI fails} \right) \leq \epsilon
\]

as desired. \( \Box \)

**Corollary 3.** Let's assume that we have estimated values of rank \( r \), column space coherence number \( \mu_0 \) and estimated row space sparsity-number is \( \psi(V) \). Then, if \( \psi(V) \) is \( \mathcal{O}(1) \) then observation complexity is bounded by \( (m + n - r) r + \mathcal{O}(n\mu_0 \log \frac{1}{\epsilon}) \) and if \( \psi(V) \) is \( \mathcal{O}(n) \) then with probability \( \frac{1}{2^{2n}} \), the bound is \( \mathcal{O}((m + n - r) r) \) which is theoretical lower bound for exact completion problem.
REFERENCES

M. F. Balcan and H. Zhang. Noise-tolerant life-long matrix completion via adaptive sampling. *Advances in Neural Information Processing Systems*, 2016.

E. J. Candès and B. Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 2009.

E. J. Candés and T. Tao. The power of convex relaxation: Near-optimal matrix completion. *IEEE Transactions on Information Theory*, 2010.

T. F. Coleman and A. Pothen. The null space problem i. complexity. *SIAM Journal on Algebraic Discrete Methods*, 1986.

D. Gross. Recovering low-rank matrices from few coefficients in any basis. *Information Theory, IEEE Transactions on*, 2011.

J. Haupt, R. Castro, and R. Nowak. Adaptive sampling for sparse detection and estimation. *IEEE Transactions on Information Theory*, 2011.

A. Krishnamurthy and A. Singh. Low-rank matrix and tensor completion via adaptive sampling. In *Advances in Neural Information Processing Systems*, 2013.

A. Krishnamurthy and A. Singh. On the power of adaptivity in matrix completion and approximation. *arXiv preprint*, 2014.

S. T. McCormick. A combinatorial approach to some sparse matrix problems. *DTIC Document*, 1983.

Barnabas Poczos et al. Optimal adaptive matrix completion. *arXiv preprint arXiv:2002.02431*, 2020.

Qing Qu, Ju Sun, and John Wright. Finding a sparse vector in a subspace: linear sparsity using alternating directions. *IEEE Transactions on Information Theory*, 2015.

Ilqar Ramazanli. Performance of distribution regression with doubling measure under the seek of closest point. *arXiv preprint arXiv:2203.00155*, 2022.

Ilqar Ramazanli and Barnabas Poczos. Optimal exact matrix completion under new parametrization. *arXiv preprint arXiv:2002.02431*, 2020.

Ilqar Ramazanli, Han Nguyen, Hai Pham, Sashank J Reddi, and Barnabas Poczos. Adaptive sampling distributed stochastic variance reduced gradient for heterogeneous distributed datasets. *arXiv preprint arXiv:2002.08528*, 2020.

B. Recht. A simpler approach to matrix completion. *The Journal of Machine Learning Research*, 2011.

M. K. Warmuth and D. Kuzmin. Randomized online pca algorithms with regret bounds that are logarithmic in the dimension. *Journal of Machine Learning Research*, 2008.