STABILITY OF THE DENSITY PATCHES PROBLEM WITH VACUUM FOR
INCOMPRESSIBLE INHOMOGENEOUS VISCOUS FLOWS

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Abstract. We consider the inhomogeneous incompressible Navier-Stokes system in a smooth two or
three dimensional bounded domain, in the case where the initial density is only bounded. Existence
and uniqueness for such initial data was shown recently in [10], but the stability issue was left open.
After observing that the solutions constructed in [10] have exponential decay, a result of independent
interest, we prove the stability with respect to initial data, first in Lagrangian coordinates, and then
in the Eulerian frame. We actually obtain stability in $L_2(R_+; H^1(\Omega))$ for the velocity and in a
negative Sobolev space for the density.

Let us underline that, as opposed to prior works, in case of vacuum, our stability estimates are
not weighted by the initial densities. Hence, our result applies in particular to the classical density
patches problem, where the density is a characteristic function.

Keywords: Stability, inhomogeneous flows, rough density, vacuum.

AMS subject classification: 35Q30, 76N10

1. Introduction

We are interested in the following inhomogeneous incompressible Navier-Stokes system:

\[
\begin{aligned}
\rho_t + v \cdot \nabla \rho &= 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\
\rho v_t + \rho v \cdot \nabla v - \Delta v + \nabla P &= 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\
\text{div } v &= 0 \quad \text{in } \mathbb{R}_+ \times \Omega.
\end{aligned}
\]

This system describes the motion of incompressible fluids with variable density and originates from
simplified models in geophysics. The unknowns in the above system are the velocity $v$, the density
$\rho$ and the pressure $P$, depending on the time variable $t \geq 0$ and on the space variable $x \in \Omega$ where
the fluid domain $\Omega$ is a smooth bounded subset of $\mathbb{R}^d$ with the physical dimensions $d = 2, 3$.

The system is supplemented with the initial data

\[
\begin{aligned}
\rho|_{t=0} &= \rho_0 \quad \text{and} \quad v|_{t=0} = v_0.
\end{aligned}
\]

At the boundary, we prescribe the no-slip condition

\[
\begin{aligned}
v|_{\partial \Omega} &= 0.
\end{aligned}
\]

The existence of weak solutions to (1.1) is nowadays well understood and the state of the art
on this issue is rather similar to that of the classical incompressible Navier-Stokes system (i.e.
with constant density). The analysis goes back to the work of A. Kazhikhov [25] who showed global
existence of weak solutions for initial density bounded away from zero. This constraint was removed
by J. Simon in [39]. Later, P.-L. Lions [31] showed that the density is a renormalized solution to
the continuity equation, this allowed in particular to treat the case of density-dependent viscosity
in [18]. Still in the framework of weak solutions, F. Fanelli and I. Gallagher investigated recently
in [19] the fast rotation limit of (1.1) supplemented with a Coriolis force.

Producing 'strong solutions' (by strong, we mean solutions having the uniqueness property) re-
quires more constraints on the data: enough regularity and no vacuum, typically. Roughly speaking,
according to the classical literature, for smooth enough data and provided the density does not van-
ish, we have global existence of strong solutions even for large data in dimension two, and, like
for the constant density case, for small enough initial velocity in dimension three. For such results
in the bounded domain case, one can refer to the pioneering work by O.A. Ladyzhenskaya and
V. Solonnikov in [26] (further extended to less regular data by the first author in [7]).

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A number of works have been dedicated to solving (1.1) in $\Omega = \mathbb{R}^d$ in so-called ‘critical regularity frameworks’. The underlying idea (that originates from Fujita and Kato’s work [20] for the constant density case) is that ‘optimal’ functional spaces for well-posedness of (1.1) have to share its scaling invariance, namely, for all $\ell > 0$,

\begin{equation}
(\varrho, v, P)(t, x) \rightsquigarrow (\varrho, \ell v, \ell^2 P)(\ell^2 t, \ell x) \quad \text{and} \quad (\varrho_0, v_0)(x) \rightsquigarrow (\varrho_0, \ell v_0)(\ell x).
\end{equation}

Observing that the couple of homogeneous Besov space $\dot{B}_{2,1}^s(\mathbb{R}^d) \times (\dot{B}_{2,1}^{s-1}(\mathbb{R}^d))^d$ indeed possesses this invariance, the first author proved in [6] the well-posedness of (1.1) supplemented with initial velocity $v_0$ in $\dot{B}_{2,1}^{-1}(\mathbb{R}^d)$ and initial density $\varrho_0$ close to some positive constant in $\dot{B}_{2,1}^2(\mathbb{R}^d)$. Note that, owing to the embedding $\dot{B}_{2,1}^2(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$, this forces the density to be continuous. Subsequent improvements have been brought to this approach (see e.g. [1]) but, still, the density has to be ‘almost’ continuous. In particular, one cannot consider initial densities that have a jump across an interface, even a smooth one.

Toward considering less regular densities, a first breakthrough has been brought by the first two authors in [8, 9]: taking advantage of Lagrangian coordinates (that will be presented below), they established well-posedness results for densities that are possibly discontinuous along interfaces, provided the jump is small enough.

Then, in [35], by a totally different approach, M. Paicu, P. Zhang and Z. Zhang succeeded in proving the global existence in $\mathbb{R}^2$ for $v_0 \in H^s$, $s > 0$ and in $\mathbb{R}^3$ for $v_0 \in H^1$ with $\|v_0\|_2 \|\nabla v_0\|_2$ sufficiently small, provided the initial density satisfies

$$0 < c_0 \leq \varrho_0 \leq C_0 < \infty.$$  

In dimension 3, this work was extended in [3] to initial velocities that are only in $H^s$ for some $s > 1/2$. Still, the density has to be bounded away from zero, and the solution in not time continuous with values in $H^s$. Very recently, P. Zhang in [41] achieved the critical regularity $\dot{B}_{2,1}^{1/2}$ for the initial velocity, but did not address the uniqueness issue in this setting.

For more results where the initial density is allowed to be discontinuous but, still, strictly positive, the reader may refer among others to [21, 24, 18] and to a recent result [2], where an inflow boundary condition is considered. Let us also mention that global well-posedness in the half-space $\mathbb{R}^+_d$ with initial density only bounded but close to a positive constant was shown in [13].

All the above results require the strict positivity of the initial density. To our knowledge, the existence of unique solutions in presence of vacuum has been first proved in [4] for rather high regularity of the initial density and velocity, (namely $\varrho_0 \in L^{3/2} \cap H^2$ and $u_0 \in H^2$) and provided the following compatibility condition is satisfied:

\begin{equation}
- \Delta v_0 + \nabla P_0 = \sqrt{\varrho_0} g \quad \text{for some} \ g \in L_2 \text{ and } P_0 \in H^1.
\end{equation}

Global existence of unique solutions in a 3D bounded domain or in $\mathbb{R}^3$ under the same compatibility condition and smallness of $\|u_0\|_{H^{1/2}}$ was shown in [5].

Condition (1.5) was removed in [27], where local well-posedness in a bounded domain is shown, but still for sufficiently smooth initial density. Global existence in the whole space $\mathbb{R}^3$, again under sufficient regularity of initial density, was proved recently in [22].

An important place in the theory of (1.1) is taken by the so-called density patch problem: assuming that

\begin{equation}
\varrho_0 = \alpha_1 \chi_{A_0} + \alpha_2 \chi_{\Omega \setminus A_0}
\end{equation}

for some nonnegative constants $\alpha_1, \alpha_2$ and a measurable set $A_0$, can we say that $\varrho(t)$ has the same structure for all time, with persistence of the regularity of the interface? This problem seems to have been first raised by P.-L. Lions in [31] in the specific case where $\varrho_0 = \chi_{A_0}$ with $A_0 \in \mathbb{R}^2$, and $\sqrt{\varrho_0} u_0 \in L_2$. The original question was whether for all time $\varrho(t) = \chi_{A(t)}$ for some domain $A(t)$ with the same regularity as $A_0$.

It turns out that a positive answer is obtained for the $C^1$ regularity as a consequence of the works of the first two authors in [8, 9] if $\alpha_1, \alpha_2 > 0$ are close to each other. Much more complete results have been obtained in the two-dimensional case in [29] (case $\alpha_1 - \alpha_2$ small), and then in [30] for any positive constants. There, the authors actually establish the persistence of high ‘striated’ Sobolev regularity for the density. Similar results have been proved in the 3D case in [28]. The propagation
of striated regularity has been adapted to the case where the viscosity depends on the density in [34]. By a different approach, persistence of Hölder continuity of the interface if $\alpha_1, \alpha_2$ are close to each other was shown in [14].

Requiring that the initial density is away from zero precludes to consider the original Lions’ problem, namely the case when $\alpha_2 = 0$ in (1.6). Recently in [10], the first and second authors proved the well-posedness of (1.1) for only bounded initial density

$$0 \leq \rho_0 \leq \rho^*$$

and initial velocity satisfying

$$v_0 \in H^1_0(\Omega), \quad \text{div} v_0 = 0.$$  

In the two-dimensional case, the solutions are global without any additional condition while, in the 3D case, $v_0$ has to satisfy some smallness condition (as the results of [10] are of particular importance for our analysis, they will be recalled precisely below). As a by-product, the authors obtained a positive answer to the Lions question in the case $\rho_0 = \chi_{A_0}$: persistence of Hölder regularity $C^{1,\alpha}$ holds true for any $0 < \alpha < 1$ in 2D and $0 < \alpha < \frac{1}{2}$ in 3D.

However, the question concerning the stability of the solutions was left open in [10]. In fact, if the density is bounded away from zero then the stability can be proved in the same way as uniqueness, but this is no longer the case if the initial density is allowed to vanish (this has to do with the parabolic character of the momentum equation, which degenerates if the density vanishes). This typically happens if we consider the following model configuration: the original density is $\rho^{\text{org}}_0$ and the perturbation is $\rho^{\text{per}}_0$, that is

$$\rho^{\text{org}}_0 = \chi_{A(0)} \quad \text{and} \quad \rho^{\text{per}}_0 = \chi_{A(0)} + \phi.$$  

![Figure 1.1. Support of $\rho^{\text{org}}_0 = A$ and support of $\rho^{\text{per}}_0 = B$.](image)

The problem happens whenever

$$(\text{supp } \rho^{\text{org}}_0 \setminus \text{supp } \rho^{\text{per}}_0) \cup (\text{supp } \rho^{\text{per}}_0 \setminus \text{supp } \rho^{\text{org}}_0)$$  

is not empty.

Indeed, unless $\phi = 0$, the perturbation is large in $L_\infty(\Omega)$ but small in $L_p(\Omega)$ for $p < \infty$, and we need a functional framework for stability that captures this situation.

The goal of the present paper is to show the stability of solutions to the inhomogeneous Navier-Stokes (1.1) with respect to initial condition of type (1.2), in a regularity setting that includes the density patches problem (1.6) even if one of the parameters $\alpha_1, \alpha_2$ vanishes. In particular, we allow the density to be a characteristic function of some set. In the most pathological case the supports of $\rho^{\text{org}}_0$ and $\rho^{\text{per}}_0$ can be even disjoint.

Before stating our main stability result, we have to recall the state-of-the-art concerning the global well-posedness for (1.1) supplemented with general data satisfying (1.7) and (1.8).

In dimension $d = 2$, [10] Thm 2.1 states the following result:
Theorem 1. Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^2 \), or a two-dimensional torus. Let \( \varrho_0 \in L_\infty(\Omega) \) satisfy (1.7) and let \( v_0 \) satisfy (1.8). Then, System (1.1) admits a unique solution \( (\varrho, v, P) \) such that
\[
\begin{align*}
\varrho &\in L_\infty(\mathbb{R}^+; L_\infty), \quad v \in L_\infty(\mathbb{R}^+; H^1), \\
\sqrt{\varrho} v_t, \nabla^2 v, \nabla P &\in L_2(\mathbb{R}^+; L_2), \quad \nabla v \in L_{1,loc}(\mathbb{R}^+; L_\infty), \\
\sqrt{\varrho} v &\in C(\mathbb{R}^+; L^2) \quad \text{and} \quad \varrho \in C(\mathbb{R}^+; L^{p}) \quad \text{for all } 1 \leq p < \infty.
\end{align*}
\]
For arbitrarily large but finite time \( T > 0 \) these solutions satisfy in addition for all \( 1 \leq r < 2, 1 \leq m < \infty, s < 1/2 \) and \( 1 \leq p < \infty \),
\[
\begin{align*}
\nabla(\sqrt{\varrho} P), \nabla^2(\sqrt{\varrho} v) &\in L_\infty(0, T; L_r(\Omega)) \cap L_2(0, T; L_m(\Omega)), \quad v \in H^s(0, T; L_p(\Omega)), \\
\sqrt{\varrho} v_t &\in L_\infty(0, T; L_2) \quad \text{and} \quad \nabla v_t \in L_2(0, T; L_2).
\end{align*}
\]

In the three dimensional case, we know from [10] Thm 2.2 the following result:

Theorem 2. Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^3 \), or a three-dimensional torus. Let \( \varrho_0 \in L_\infty(\Omega) \) satisfy (1.7) and \( v_0 \) satisfy (1.8). There exists \( c > 0 \) such that if, in addition,
\[
(\varrho^*)^{3/2} \left\| \sqrt{\varrho} v_0 \right\|_2 \left\| \nabla v_0 \right\|_2 \leq c \mu^2,
\]
then System (1.1) admits a unique solution \( (\varrho, v, P) \) satisfying (1.10) and, for any finite \( T > 0 \), \( s < 1/2 \) and \( 1 \leq p < \infty \),
\[
\begin{align*}
\nabla(\sqrt{\varrho} P), \nabla^2(\sqrt{\varrho} v) &\in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; L_6(\Omega)), \quad v \in H^s(0, T; L_p(\Omega)), \\
\sqrt{\varrho} v_t &\in L_\infty(0, T; L_2) \quad \text{and} \quad \nabla v_t \in L_2(0, T; L_2).
\end{align*}
\]

Although the above solutions are unique, the question of their stability remains open so far. Here we aim at supplementing the above statements with a stability result. In order to obtain the most accurate information, it is natural to use Lagrangian coordinates since, in this setting, the density is time independent (it only depends on the position of the particles initially). Therefore, the problem is reduced to the control of the difference of the velocities which, somehow, satisfies a parabolic equation.

Let us shortly recall how to define Lagrangian coordinates in our setting. First, we introduce the flow \( X : (t, y) \mapsto X(t, y) \) of \( v \), that is the unique solution to the following ODE:
\[
\begin{align*}
\frac{dX}{dt} &= v(t, X(t, y)) \quad \text{in } \mathbb{R}^+ \times \Omega, \\
X(0, y) &= y \quad \text{in } \Omega.
\end{align*}
\]

Integrating (1.14) yields the following relation between the Eulerian “\( x \)” and Lagrangian “\( y \)” coordinates:
\[
x = X(t, y) = y + \int_0^t v(t', X(t', y)) \, dt'.
\]

By the standard theory of ODEs, the above change of coordinates is well defined whenever \( v \in L_{1,loc}(\mathbb{R}^+; C^{1,0}) \). In the coordinates system \( (t, y) \), the unknown functions are named as follows:
\[
u(t, y) = v(t, X(t, y)), \quad \eta(t, y) = \varrho(t, X(t, y)), \quad Q(t, y) = P(t, X(t, y)).
\]

Let us denote
\[
A_u(t) = \left( \frac{dX}{dy} \right)^{-1} = \left( \text{Id} + \int_0^t \nabla y u(t', y) \, dt' \right)^{-1} = \left[ \text{cof} \left( \text{Id} + \int_0^t \nabla y u(t', y) \, dt' \right) \right]^T,
\]
where \( \text{cof}(\cdot) \) denotes the cofactor matrix and, in the second equality, we used the fact that \( \det A_u = 1 \) (see e.g. [10]). For a function \( f(t, x) \), denote \( \tilde{f}(t, y) = f(t, X(t, y)) \). Then, owing to the chain rule,
\[
\nabla_x f(t, x) = A_u^T \nabla_y \tilde{f}(t, y) =: \nabla_u \tilde{f}(t, y), \quad \partial_t f(t, x) + v(t, x) \cdot \nabla_x f(t, x) = \partial_t \tilde{f}(t, y).
\]

In order to transform the divergence operator, observe that \( \text{div}_y A_u = 0 \) since \( \det A_u = 1 \). Therefore, for any vector field \( z(t, y) \), one may write
\[
\text{div}_y (A_u z) = A^T : \nabla_y z + z : \text{div}_y (A_u) = A^T : \nabla_y z.
\]
Hence, if we denote \( \tilde{w}(t,y) = w(t,X(t,y)) \) for any vector field \( w(t,x) \), we discover that
\[
(1.19) \quad \text{div}_x w(t,x) = A_u : \nabla_y \tilde{w}(t,y) = \text{div}_y (A_u \tilde{w}) =: \text{div}_u \tilde{w}.
\]
Taking all above into account we see that in coordinates \((t,y)\) system (1.1) reads
\[
(1.20) \quad \begin{cases}
\eta_t = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
\eta u_t - \text{div}_u \nabla u + \nabla u Q = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
\text{div}_u u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
u|_{t=0} = v_0, \quad \eta|_{t=0} = \varrho_0 & \text{in } \Omega.
\end{cases}
\]
The main achievement of this paper is the following stability result in the Lagrangian coordinates setting.

**Theorem 3.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^d \) with \( d = 2, 3 \). Let \((g^1, v^1)\) and \((g^2, v^2)\) be two solutions of (1.1) with initial data \((\varrho^1_0, v^1_0)\) and \((\varrho^2_0, v^2_0)\), respectively, with non identically zero bounded \( \varrho^1_0 \) and \( \varrho^2_0 \), and \( v^1_0, v^2_0 \) in \( H^1_0(\Omega) \), given either by Theorem 1 or by Theorem 2 (depending on the dimension). Denote by \( u^1 \) and \( u^2 \) the corresponding velocities in Lagrangian coordinates. Finally, set \( \delta u := u^2 - u^1, \quad \delta v := v^2 - v^1 \) and \( \delta \varrho := \varrho^2 - \varrho^1 \).

Then, there exists a positive constant \( \beta \) depending only on the shape of \( \Omega \), and such that
\[
(1.21) \quad \sup_{t \in \mathbb{R}_+} \left( e^{\beta \varrho^*} \left[ \min \left( \sqrt{\varrho^1_0}, \sqrt{\varrho^2_0} \right) \delta u(t) \right]_2 + \left\| e^{\beta \varrho^*} \nabla \delta u \right\|_{L_2(\mathbb{R}_+, L_2)} \right) \leq C \left( \left\| \sqrt{\varrho^1_0} \delta v_0 \right\|_2 + \left\| \delta \varrho_0 \right\|_2^{1/2} \right),
\]
where \( \delta \) stands for the diameter of \( \Omega \), and \( \varrho^* := \left\| \varrho_0 \right\|_\infty \).

Coming back from Lagrangian to Eulerian coordinates, we obtain

**Corollary 1.** Under the assumptions of Theorem 3 in Eulerian coordinates, we have
\[
\sup_{t \in \mathbb{R}_+} \left\| \delta \varrho(t) \right\|_{W^{-1}_p} + e^{\beta \varrho^*} \left\| \delta \varrho \right\|_{L_2(\mathbb{R}_+, L_2)} \leq C_0 \left( \left\| \sqrt{\varrho^1_0} \delta v_0 \right\|_2 + \left\| \delta \varrho_0 \right\|_2^{1/2} \right)
\]
for \( 1 < p < \infty \) if \( d = 2 \) and \( 1 < p \leq 6 \) if \( d = 3 \).

In our low regularity setting, the key difficulty for proving stability of solutions to (1.1) comes from the partially hyperbolic nature of the system. In fact, if writing the system satisfied by the difference \((\delta \varrho, \delta v, dP)\) of two solutions \((g^1, v^1, P^1)\) and \((g^2, v^2, P^2)\) of (1.1), then the mass equation gives:
\[
(\delta_t + v^2 \cdot \nabla) \delta v = \delta v \cdot \nabla v^2.
\]
In our framework where the density is only in \( L_\infty(\Omega) \), this forces us to perform estimates for \( \delta \varrho \) in a space with regularity index equal to \(-1\). Following the duality approach initiated by D. Hoff in [23] and recently renewed in [11] and [33] (for the related compressible Navier-Stokes system), we shall actually prove stability estimates for the density in \( W^{-1}_p \), and in \( L_2 \) for the velocity.

A key point in order to get rid of any smallness condition, is to first establish a sufficiently strong time-decay of the solutions that have been constructed in Theorems 1 and 2. This will be achieved in Section 3 where by means of rather classical energy arguments, we will even get exponential decay, owing to the boundedness of the fluid domain. Before that, in Section 2 we shall compare two global solutions satisfying a priori those decay properties, and estimate their difference in Lagrangian coordinates in terms of the difference of the data, getting the result of Theorem 3.

Finally, we rewrite those estimates in the Eulerian coordinates to obtain Corollary 1.

Proving the exponential decay estimates of Section 3 can be achieved by means of a remarkably simple energy method that is performed directly on the nonlinear problem. This is in sharp contrast with the proof of decay estimates for compressible Navier-Stokes and related models which requires a refined analysis of the linearized system combined with a perturbation argument (see among others [15, 38, 17, 36] and the references therein).

When comparing Theorem 3 with results of [10], a remark is in order concerning the domain. Although Theorems 1 and 2 hold both for a bounded domain with Dirichlet condition and a torus, we here restricted our analysis to the case of a bounded domain to avoid further technical complications. In fact, in the case of Dirichlet boundary conditions, we have the basic Poincaré inequality at our disposal, which is fundamental to close the estimates globally in time. In the torus case, the corresponding Poincaré inequality has an additional term (namely the total momentum of the
solution, see Lemma A.1 in [10] which, although probably harmless, entails serious complications in the proof of decay estimates. Therefore, we leave the torus case for future research.

Notation. We use standard notations $L_p$ and $W^k_p$ for Lebesgue and Sobolev spaces (the dependency with respect to the fluid domain $Ω$ is omitted). For the corresponding norms, we use the short notation:

$$\|\cdot\|_p := \|\cdot\|_{L_p}, \quad \|\cdot\|_{k,p} := \|\cdot\|_{W^k_p}.$$ 

In some computations, we will agree that $f_p(t)$ denotes a generic function of time which is in $L_p(\mathbb{R}_+)$, and that $f_{pq}(t)$ stands for a function which is in $L_p(\mathbb{R}_+) \cap L_q(\mathbb{R}_+)$. The precise form of these functions may vary from line to line, but the property of integrability is preserved.

2. Stability under given decay properties

Throughout this section, we are given two solutions $(\varrho^1, v^1, P^1)$ and $(\varrho^2, v^2, P^2)$ pertaining to data $(\varrho^1_0, v^1_0)$ and $(\varrho^2_0, v^2_0)$, and satisfying the following properties for some $\beta_0 > 0$:

\[(2.1a) \quad \sqrt{\varrho} e^{\beta_0 t} v^1_t \in L_2(\mathbb{R}_+, L_2),\]
\[(2.1b) \quad e^{\beta_0 t} \nabla v^1 \in L_1(\mathbb{R}_+, L_\infty) \cap L_4(\mathbb{R}_+, L_3) \cap L_2(\mathbb{R}_+, L_6),\]
\[(2.1c) \quad \sqrt{t} e^{\beta_0 t} v^1 \in L_1 \cap L_\infty(\mathbb{R}_+, L_\infty),\]
\[(2.1d) \quad \sqrt{t} e^{\beta_0 t} (\nabla^2 v^1, \nabla P^1) \in L_2(\mathbb{R}_+, L_6),\]
\[(2.1e) \quad \sqrt{t} e^{\beta_0 t} v^1_t \in L_{4/3}(\mathbb{R}_+, L_6),\]
\[(2.1f) \quad e^{\beta_0 t} v^1 \in L_1 \cap L_\infty(\mathbb{R}_+, L_6),\]
\[(2.1g) \quad \sqrt{t} e^{\beta_0 t} \nabla v^1 \in L_2(\mathbb{R}_+, L_\infty),\]
\[(2.1h) \quad e^{\beta_0 t} \nabla^2 v^1 \in L_1(\mathbb{R}_+, L_r) \text{ with } r > d.\]

We denote by $(\eta^t, u^t, Q^t)$ the corresponding solutions in Lagrangian coordinates (hence $\eta^t = \varrho^1_0$).

Lemma 1. Let $(\varrho, v, P)$ solve (1.1) and satisfy Conditions (2.1a) to (2.1h). Then, the corresponding Lagrangian solution $(\eta, u, Q)$ also satisfies (2.1a) to (2.1h).

Proof. All properties except for the ones involving a time derivative and the last one just follow from the corresponding ones for $v$, and from the fact that the matrix $A_u$ is bounded.

For proving (2.1a), we start from the identity

$$\sqrt{t} e^{\beta_0 t} u_t = \sqrt{\rho} \varrho^1_0 \nabla v^1 - \nabla v \circ X_t.$$ 

As $X_t$ is measure preserving, the term with $v_t$ may be bounded by means of (2.1a). For bounding the other term, it suffices to observe that

$$\|e^{\beta_0 t} \sqrt{\rho} \varrho^1_0 \nabla v \circ X_t\|_{L_2(\mathbb{R}_+ \times \Omega)} \leq \rho^1 \|e^{\beta_0 t/2} \nabla v\|_{L_\infty(\mathbb{R}_+, L_3)} \|e^{\beta_0 t/2} \nabla v\|_{L_2(\mathbb{R}_+, L_3)}.$$ 

The first term of the right-hand side may be bounded thanks to (2.1f) while the second one, according to (2.1h) and the boundedness of $\Omega$. As regards (2.1e), one can use again $u_t = (v_t + v \cdot \nabla) \circ X_t$, and properties (2.1f) to (2.1g) for $v$.

In order to prove (2.1h), we differentiate the identity $\nabla y u = \nabla y v (t, X(t, y))$ with respect to $y$. By (1.16) we obtain

$$\|\nabla^2 y u\|_{L_1(0, T; L_r)} \leq C\|\nabla^2 y v\|_{L_1(0, T; L_r)} + C\|\nabla y A\|_{L_\infty(0, T; L_r)} \|\nabla y u\|_{L_1(0, T; L_\infty)},$$ 

which implies (2.1h) for $u$ due to (2.1h) for $v$ and to (2.1b) for $u$. \qed

In the rest of this section, we aim at estimating

$$\delta u := u^2 - u^1 \text{ and } \delta Q := Q^2 - Q^1$$
in terms of the difference of the data. Obviously, denoting \( \Delta u := \text{div}_u \nabla u \) and \( \delta u_0 = v^2_0 - v^1_0 \), the couple \((\delta u, \delta Q)\) satisfies
\[
\begin{align*}
\varrho_0^1 \delta u_t - \Delta u_1 \delta u + \nabla u_1 \delta \hat{Q} &= (\Delta u^2 - \Delta u^1) u^2 - (\nabla u^2 - \nabla u^1) \hat{Q}^2 - \delta \varrho_0 u^1_t, \\
\text{div}_u \delta \hat{u} &= (\text{div}_u - \text{div}_u^2) u^2, \\
\delta u|_{t=0} &= \delta v_0.
\end{align*}
\]
(2.2)

Note that
\( \delta \varrho_0 := \varrho_0^2 - \varrho_0^1 = \eta^2 - \eta^1 \).

By (1.20)\(_1\), functions \( \eta^i \) are constant in time, so the perturbation of the density stays the same in time. It is one of the main reasons why we chose the Lagrange coordinates approach to deal with the stability issue of system (1.1).

2.1. The case of a nice control of vacuum. Compared to the proof of uniqueness that has been performed in [10], the troublemaker is the term \( \delta \varrho_0 u^2_t \) in the equation for \( \delta \hat{u} \) since Theorems [1] or [2] only provide us with an information on \( \sqrt{\varrho_0^2 u^2_t} \) (hence on \( \sqrt{\varrho_0^2 u^2_t} \)) while we do not necessarily have
\[
\supp \delta \varrho_0 \subset \text{supp } \varrho_0^2.
\]
(2.3)

In this part, we assume that the initial densities satisfy
\[
\| \delta \varrho_0 \|_X := \| \delta \varrho_0 \|_X / \| \varrho_0^2 \|_2 < \infty
\]
and derive a differential inequality which is crucial for proving Theorem [2.4]. The general case, when (2.3) is not valid, will be investigated in the next subsection.

The idea is to decompose \( \delta u \) into
\[
\delta u = w + z,
\]
(2.5)

where \( w \) stands for a suitable solution to the divergence equation
\[
\text{div}_u w = (\text{div}_u - \text{div}_u^2) u^2 = T \delta A : \nabla u^2 = \text{div} (\delta A u^2),
\]
with \( \delta A := A^1 - A^2 \) and \( A^i := A^i_\mu \).

Since at \( t = 0 \) by (1.16) we have \( \delta A = 0 \), we put \( w = 0 \) at \( t = 0 \). Note that \( A^1 \) and \( A^2 \) need not to be close to Id, but are invertible and uniformly bounded in time.

Lemma 2. There exists a solution to (2.6) such that
\[
\begin{align*}
\| e^{\beta_0 t} w(t) \|_2 &\leq f_{1,\infty}(t) \left( \int_0^t \| \nabla \delta u_0(\tau) \|_2^2 d\tau \right)^{1/2}, \\
\| e^{\beta_0 t} \nabla w(t) \|_2 &\leq f_{2}(t) \left( \int_0^t \| \nabla \delta u_0(\tau) \|_2^2 d\tau \right)^{1/2}, \\
\| e^{\beta_0 t} w_1(t) \|_{3/2} &\leq f_{4/3}(t) \left( \int_0^t \| \nabla \delta u_0(\tau) \|_2^3 d\tau \right)^{1/2} + f_{4}(t) \| \nabla \delta u_0(\tau) \|_2,
\end{align*}
\]
(2.7)

where the notation \( f_p(t) \) and \( f_{p,q}(t) \) is explained at the end of Section [1].

Proof. The vector \( w \) will be sought in the form \( w = (A^1)^{-1} A^1 w = (A^1)^{-1} \bar{w} \), where \( \bar{w} \) is given as a solution to
\[
\text{div} \bar{w} = (\text{div} A^1 w) = \text{div}_u w = (\text{div}_u - \text{div}_u^2) u^2 = T \delta A : \nabla u^2 = \text{div} (\delta A u^2).
\]
(2.8)

As a first step in the proof of our claim, let us establish the following bounds:
\[
\begin{align*}
\| e^{\beta_0 t} \bar{w} \|_{L_4(0,T;L_2)} &\leq C \| e^{\beta_0 t} \delta A u^2 \|_{L_4(0,T;L_2)}, \\
\| e^{\beta_0 t} \nabla \bar{w} \|_{L_2(0,T;L_2)} &\leq C \| e^{\beta_0 t} T \delta A : \nabla u^2 \|_{L_2(0,T;L_2)}
\end{align*}
\]
and
\[
\| e^{\beta_0 t} \bar{w} \|_{L_{4/3}(0,T;L_{3/2})} \leq C \| e^{\beta_0 t} (\delta A u^2)_1 \|_{L_{4/3}(0,T;L_{3/2})}.
\]
(2.9)

The existence of a vector field \( \bar{w} \) satisfying (2.8)–(2.9) is assured by the following Lemma:
Let $A$ be a matrix valued function with $\det A \equiv 1$. Consider the following divergence equation in a bounded domain with smooth boundary:
$$\text{div} b = f \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \quad b = 0 \quad \text{at} \quad \mathbb{R}_+ \times \partial \Omega,$$
where $f = \Pi^T : \nabla d = \text{div}(Ad)$ and the average of $f$ is equal zero.

Then, there exists a constant $C$ such that for any $\beta \geq 0$, there exists a solution $b$ to the above equation, such that for all $t \geq 0$, we have
$$\|e^{\beta t}b(t)\|_2 \leq C\|e^{\beta t}A(t)d(t)\|_2, \quad \|e^{\beta t}\nabla b(t)\|_2 \leq C\|e^{\beta t}A(t)^T \nabla d(t)\|_2,$$
$$\|e^{\beta t}b(t)\|_{3/2} \leq C\|e^{\beta t}(Ad_t(t))\|_{3/2}.$$

Lemma 3 has been proved without exponential weight by the first two authors in [12] (see also [10, Lemma A.3]). In their proof the function $b$ is given by an explicit formula, so a time weight can be treated as a multiplicative parameter (as the norms in Lemma 3 only involve the space variable).

Now, let us bound the right-hand sides of (2.9). In order to emphasize that we do not need any smallness of $\int_0^t \nabla_y u \, d\tau$, let us derive an explicit formula for $\delta A$.

In the two dimensional case, starting from (1.16), we immediately obtain
$$\delta A(t) = \left[ \int_0^t \delta u_{2,2} \, dt' - \int_0^t \delta u_{1,2} \, dt' \right].$$

In three dimensions we can also use (1.16), let us compute precisely one entry of $\delta A$. We have
$$(A_{i1}^t)_{11}(t) = \left( 1 + \int_0^t u_{1,2,2} \, dt' \right) \left( 1 + \int_0^t u_{3,3,2} \, dt' \right) - \int_0^t u_{2,2,2} \, dt' \int_0^t u_{2,3,2} \, dt'$$
for $i = 1, 2$, therefore
$$\delta A(t)_{11} = \int_0^t \delta u_{2,2,2} \, dt' + \int_0^t \delta u_{3,3,2} \, dt' + \int_0^t \delta u_{2,3,2} \, dt' \int_0^t u_{3,3,2} \, dt' + \int_0^t \delta u_{3,2,2} \, dt' \int_0^t u_{2,3,2} \, dt'$$
$$- \int_0^t \delta u_{3,3,2} \, dt' \int_0^t u_{2,2,2} \, dt' - \int_0^t \delta u_{2,3,2} \, dt' \int_0^t u_{2,3,2} \, dt'$$

Other entries will have a similar structure, we can write them as
$$\delta A(t)_{ij} = \sum_{1 \leq k,l \leq 3} a_{ij}^{kl} \int_0^t \delta u_{k,2} \, dt' + \sum_{1 \leq k,l,m,n \leq 3, s \in \{1,2\}} b_{ij}^{kl,m,n} \int_0^t \delta u_{k,2} \, dt' \int_0^t u_{m,2} \, dt',$$
where $a_{ij}^{kl}, b_{ij}^{kl,m,n,s} \in \{0,1\}$. Now if $u^1, u^2$ satisfy (2.1b), then, by H"older inequality, we obtain
$$\|t^{-1/2} \delta A(t)\|_2 \leq C \left( t^{-1/2} \int_0^t \nabla \delta u \, d\tau \right)^{1/2}.$$

Therefore, by (2.1g), we have
$$\|e^{\beta t} T \delta A : \nabla u^2(t)\|_2 \leq \|t^{-1/2} \delta A(t)\|_2 \|e^{\beta t} t^{1/2} \nabla u^2\|_\infty \leq C f_2(t) \left( \int_0^t \|\nabla \delta u\|_2^2 \, d\tau \right)^{1/2},$$
which implies (2.7) for $\|\nabla \bar{w}\|_{L_2}$. Similarly, by (2.1c) and (2.12)
$$\|e^{\beta t} \delta A u^2\|_2 \leq \|t^{-1/2} \delta A\|_2 \|e^{\beta t} t^{1/2} u^2\|_\infty \leq f_{1,\infty}(t) \left( \int_0^t \|\nabla \delta u(\tau)\|_2^2 \, d\tau \right)^{1/2},$$
whence, applying (2.9) gives
$$\|e^{\beta t} \bar{w}_t(t)\|_2 \leq f_1(t) \left( \int_0^t \|\nabla \delta u(\tau)\|_2^2 \, d\tau \right)^{1/2}.$$

In order to bound $\bar{w}_t$, it suffices to derive an appropriate estimate in $L_{4/3}(0, T; L_{3/2})$ for
$$(\delta A u^2)_t = \delta A u^2 + (\delta A)_t u^2.$$

For the first term, thanks to (2.1e) and (2.12) we have
$$\|e^{\beta t} \delta A u^2\|_{3/2} \leq \|t^{-1/2} \delta A\|_2 \|e^{\beta t} t^{1/2} u^2\|_6 \leq f_{4/3}(t) \left( \int_0^t \|\nabla \delta u(\tau)\|_2^2 \, d\tau \right)^{1/2}.$$
The other term can be bounded as follows:
\[ \|e^{\beta t}\tilde{A}t u^2\|_{L^2} \leq \|\tilde{A}\|_2 \|e^{\beta t} u^2\|_{L^2}. \]
Differentiating (2.11) with respect to \( t \) and using (2.1g) for \( u^1 \) and \( u^2 \), we see that
\[ \|\tilde{A}t(t)\|_2 \leq C \left( \|\nabla\tilde{u}(t)\|_2 + \|t^{-1/2} \int_0^t \nabla\tilde{u}(\tau) \, d\tau\|_2 \right) \left( \|t^{1/2}\nabla u^1(t)\|_{L^\infty} + \|t^{1/2}\nabla u^2(t)\|_{L^\infty} \right). \]
In the 2D case, owing to (2.10), one can skip the second term on the right-hand side of the above inequality. Thus, owing to (2.11), for both \( d = 2, 3 \) we obtain
\[ \|\tilde{A}t(u^2(t))\|_{L^2} \leq f_{1/3}(t) \left( \int_0^t \|\nabla\tilde{u}(\tau)\|_2^2 \, d\tau \right)^{1/2} + f_4(t)\|\nabla\tilde{u}(t)\|_2 \]
which, by (2.9) implies (2.7) for \( \|\tilde{w}_t\|_{L^3} \). Altogether, this gives the thesis of Lemma 2 for \( \tilde{w} \), but not yet for \( w \).

In order to get (2.7) for \( w \) from the estimates of \( \tilde{w} \), we first observe that
\[ \sup_{t \leq T} \| (A^1)^{-1}(t)\|_{L^\infty} \leq C \|\nabla u\|_{L^1(0,T;L^\infty)}. \]
Therefore
\[ \|e^{\beta t}w\|_{L^4(0,T;L^2)} \leq C\|A^1\|_{L^\infty} \|e^{\beta t}\tilde{w}\|_{L^4(0,T;L^2)} \leq C\|e^{\beta t}\tilde{w}\|_{L^4(0,T;L^2)}. \]
In order to estimate \( \|\nabla w\|_{L^2(0,T;L^2)} \) we proceed as follows:
\[ \|e^{\beta t}\tilde{w}\nabla((A^1)^{-1})\|_{L^2(0,T;L^2)} \leq \|\nabla((A^1)^{-1})\|_{L^\infty(0,T;L^r)} \|e^{\beta t}\tilde{w}\|_{L^2(0,T;L^r)} \]
\[ \leq C\|\nabla^2 u^1\|_{L^1(0,T;L^r)} \|e^{\beta t}\tilde{w}\|_{L^2(0,T;L^r)} \]
with \( \frac{1}{2} = \frac{1}{r} + \frac{1}{r} \), where in the last passage we used (1.16). Therefore by (2.1b) and (2.1h)
\[ \|e^{\beta t}\nabla w\|_{L^2(0,T;L^{2})} \leq \|e^{\beta t}(A^1)^{-1} \nabla\tilde{w}\|_{L^2(0,T;L^2)} + \|e^{\beta t}\nabla((A^1)^{-1})\|_{L^2(0,T;L^r)} \]
\[ \leq C\|e^{\beta t}\nabla\tilde{w}\|_{L^2(0,T;L^{2})} + \|e^{\beta t}\tilde{w}\|_{L^2(0,T;L^{r})} \]
Since, in (2.1h), one can take \( r > d \), one can always ensure that \( r^* < 6 \).

Hence, Finally we have
\[ \|((A^1)^{-1})_t\tilde{w}\|_{L^2} \leq \|((A^{-1})^t)_0\|_{L^6} \|\tilde{w}\|_{L^2} \leq \|\nabla u^1\|_{L^6} \|\tilde{w}\|_{L^2}, \]
which together with (2.15) implies
\[ \|e^{\beta t}w_t\|_{L^4(0,T;L^2)} \leq C\|A^1\|_{L^\infty} \|e^{\beta t}\tilde{w}_t\|_{L^4(0,T;L^2)} + C\|\nabla u^1\|_{L^2(0,T;L^6)} \|e^{\beta t}\tilde{w}\|_{L^4(0,T;L^2)} \]
\[ \leq C\|e^{\beta t}\tilde{w}_t\|_{L^4(0,T;L^2)} \|e^{\beta t}\tilde{w}\|_{L^4(0,T;L^2)} \]
which completes the proof of Lemma 2.

Next, let us restate the equations (2.2) as the following system for \( (z, \tilde{Q}) \):
\[ \begin{cases}
\rho_0 \delta t \Delta_{\delta t} z - \Delta_{u^1} z + \nabla u^1 \tilde{Q} = (\Delta_{u^2} - \Delta_{u^1}) u^2 + (\nabla u^1 - \nabla u^2) Q^2 - \rho_0^u w_t + \Delta_{u_1} w - \delta \tilde{Q}_0 u^2,

\text{div}_{u^1} z = 0, \quad z|_{t=0} = \delta \tilde{Q}_0.
\end{cases} \]
(2.16)
Observe that for a vector field \( z \) and functions \( f, g \) defined in Lagrangian coordinates we have, according to (1.17) and integration by parts,
\[ - \int_{\Omega} f \, \text{div}_{u^1} z \, dy = - \int_{\Omega} f \, \text{div} (A_{u^1} z) \, dy = \int_{\Omega} A_{u^1} z \cdot \nabla_y f \, dy \]
(2.17)
which implies
\[ - \int_{\Omega} f \, \Delta_{u^1} g \, dy = - \int_{\Omega} f \, \text{div}_{u^1} (\nabla_u g) \, dy = \int_{\Omega} \nabla u \cdot \nabla_y f \, dy. \]
(2.18)
These identities allow to test (2.16) by \( z \) while (2.17) implies the following crucial property thanks to which one does not have to care about the difference of the pressures:
\[ \int_{\Omega} (\nabla u \cdot \tilde{Q}) \cdot z \, dy = - \int_{\Omega} \text{div}_{u^1} z \, \text{div} \tilde{Q} \, dy = 0. \]
(2.19)
Therefore, using also (2.18), we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \phi_0^1 |z|^2 \, dx + \int_{\Omega} |\nabla u^1 z|^2 \, dx = \sum_{k=1}^{5} I_k,
$$

where

$$
I_1 := \int_{\Omega} (\Delta u^2 - \Delta u^1) u^2 \cdot z \, dy,
I_2 := \int_{\Omega} (\nabla u^1 - \nabla u^2) Q^2 \cdot z \, dy,
I_3 := -\int_{\Omega} \phi_0^1 w \cdot z \, dy,
I_4 := \int_{\Omega} (\Delta u^1 w) \cdot z \, dy,
I_5 := -\int_{\Omega} \delta \phi_0 u^2 \cdot z \, dy.
$$

In order to bound $I_1$, we combine (2.1g) and (2.12) to write for all $\varepsilon > 0$,

$$
|I_1| = \left| \int_{\Omega} \text{div} \left( (\delta A)^T A_2 + (A_1^T) \delta A \right) \nabla u^2 \cdot z \, dy \right|
\leq \int_{\Omega} \left| (\delta A)^T A_2 + (A_1^T) \delta A \right| |\nabla u^2| |\nabla z| \, dx
\leq C \|t^{-1/2} \delta A\|_2 \|t^{1/2} \nabla u^2\|_\infty \|\nabla z\|_2
\leq \varepsilon \|\nabla z\|_2^2 + C \varepsilon^{-1} \|t^{1/2} \nabla u^2\|_\infty^2 \left( \int_0^t \|\nabla \delta u\|_2^2 \, d\tau \right).
$$

Next, according to (2.14), (2.13) and to Sobolev embedding $H^1_0 \hookrightarrow L_4$,

$$
|I_2(t)| \leq \left| \int_{\Omega} \delta A \nabla Q^2 \cdot z \, dx \right| \leq C \|t^{-1/2} \delta A\|_2 \|t^{1/2} \nabla Q^2\|_4 \|z\|_4,
$$

Therefore, for all $\varepsilon > 0$,

$$
I_2(t) \leq \varepsilon \|\nabla z\|_2^2 + C \varepsilon^{-1} \|t^{1/2} \nabla Q^2\|_4^2 \left( \int_0^t \|\nabla \delta u\|_2^2 \, d\tau \right).
$$

Note that from Hölder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L_6(\Omega)$, we have

$$
\|\phi_0^{1/2} z\|_3 \leq \|\sqrt{\phi_0} z\|^{1/2}_6 \leq C \|\sqrt{\phi_0} z\|^{1/2}_2 \|z\|^{1/2}_{\mathcal{H}^1}.
$$

Therefore using Hölder inequality, one can write that

$$
I_3(t) \leq \|w t\|_{3/2} \|\phi_0^{1/2} z\|_3
\leq C \|w t\|_{3/2} \|\phi_0^{1/4} z\|_3
\leq \|w t\|_{3/2} \sqrt{\phi_0^{1/2}} \|z\|^{1/2}_{\mathcal{H}^1}.
$$

Next, integrating by parts, we get for all $\varepsilon > 0$,

$$
I_4 \leq \int \|
abla u^1 w\| \|
abla u^1 z\| \, dx \leq \varepsilon \|
abla u^1 z\|_2^2 + C \varepsilon^{-1} \|
abla u^1 w\|_2^2.
$$

Finally, in order to estimate $I_5$, we apply Hölder and Young inequality, as well as the embedding $H^1_0 \hookrightarrow L_4$ to get

$$
I_5 \leq \left| \int_{\Omega} \delta \phi_0 \sqrt{\phi_0} \phi_0^{1/2} u^2 z \, dy \right|
\leq \ell \left( \delta \phi_0 \sqrt{\phi_0} \phi_0^{1/2} \right) \|\phi_0\|_2 \|z\|_4
\leq \varepsilon \|
abla z\|_2^2 + C \varepsilon^{-1} \|
abla u^1 w\|_2^2,
$$

where $\|\delta \phi_0\|_2$ is defined in (2.4). Let us point out that our ‘nice control of vacuum’ hypothesis (2.4) comes into play only for handling $I_5$. 


In the end, plugging (2.21), (2.22), (2.23), (2.24) and (2.25) in (2.20), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \partial_0 z(t) \|^2 + \| \nabla u_1 z \|^2_2 \leq \varepsilon \| \nabla z \|^2_2 + \left( \| t^{1/2} \nabla u_2 \|_\infty^2 + \| t^{1/2} \nabla Q \|_2^2 \right) \int_0^t \| \nabla \delta u \|_2^2 \, dt \\
+ \| \partial_0^2 u_1^2 \|_2^2 \| \partial_0 \|_X^2 + C \| \nabla u_1 w \|_2^2 + \| w_t \|_{3/2} \| \partial_0 z \|_{1/2}^2 \| z \|_{H^1,1}^{1/2}.
\]
Since $\nabla u_1 = T_{u_1} \nabla z$, we have
\[
\| \nabla u_1 z \|_2 \geq \| (A_{u_1})^{-1} \|_\infty^{-1} \| \nabla z \|_2.
\]
Hence, taking $\varepsilon$ small enough, the above inequality implies for some $c_0 > 0$,
\[
(2.26) \quad \frac{d}{dt} \| \partial_0 z(t) \|^2_2 + c_0 \| \nabla z \|^2_2 \leq \left( \| t^{1/2} \nabla u_2 \|_\infty^2 + \| t^{1/2} \nabla Q \|_2^2 \right) \int_0^t \| \nabla \delta u \|_2^2 \, dt \\
+ \| \partial_0^2 u_1^2 \|_2^2 \| \partial_0 \|_X^2 + C \| \nabla u_1 w \|_2^2 + \| w_t \|_{3/2} \| \partial_0 z \|_{1/2}^2 \| z \|_{H^1,1}^{1/2}.
\]
Now, using the fact that $\| \nabla \delta u \|_2^2 \leq 2 (\| \nabla z \|^2_2 + \| \nabla w \|^2_2)$, multiplying (2.26) with $e^{2\beta t}$ (with $\beta \leq \beta_0$) and adding up the second inequality of (2.7), we arrive at
\[
(2.27) \quad \frac{d}{dt} \| e^{\beta t} \partial_0 z(t) \|^2_2 + (\| e^{\beta t} \nabla z \|^2_2 + \| e^{\beta t} \nabla w \|^2_2) \leq 2 \beta e^{2\beta t} \| \partial_0 z(t) \|^2_2 \\
\left( \| e^{\beta t} t^{1/2} \nabla u_2 \|_\infty^2 + \| e^{\beta t} t^{1/2} \nabla Q \|_2^2 + f_1(t) \right) \int_0^t (\| \nabla z \|^2_2 + \| \nabla w \|^2_2) \, dt \\
+ \| e^{\beta t} w_t \|_{3/2} \| e^{\beta t} \| \partial_0 z \|_{1/2}^2 \| e^{\beta t} z \|_{1/2}^{1/2}.
\]
For small $\beta$ one can absorb the first term on the right-hand side due to Poincaré inequality. By (2.1), provided $\beta \leq \beta_0$ we have
\[
(2.28) \quad \| e^{\beta t} t^{1/2} \nabla u_2 \|_\infty^2 + \| e^{\beta t} t^{1/2} \nabla Q \|_2^2 + \| e^{\beta t} \| \partial_0^2 u_1^2 \|_2^2 \| e^{\beta t} z \|_{1/2}^{1/2} \leq f_1(t),
\]
while, by Lemma 2
\[
\| e^{\beta t} w_t \|_{3/2} \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2} \| e^{\beta t} z \|_{1/2}^{1/2} \leq \left( f_{4/3}(t) \left( \int_0^t \| \nabla \delta u \|^2_2 \, d\tau \right)^{1/2} + f_1(t) \| \nabla \delta u \|_2 \right) \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2} \| e^{\beta t} z \|_{1/2}^{1/2}.
\]
Taking advantage of Poincaré inequality and Lemma 2, we bound the right-hand side of the above inequality as follows:
\[
f_{4/3}(t) \left( \int_0^t \| \nabla \delta u \|^2_2 \, d\tau \right)^{1/2} \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2} \| e^{\beta t} \| \nabla z \|_{1/2}^{1/2} \\
\leq f_{4/3}^{4/3}(t) \int_0^t \| \nabla \delta u \|^2_2 \, d\tau + f_{4/3}^{2/3} \| e^{\beta t} \| \partial_0 z \|_{1/2} \| e^{\beta t} \| \nabla z \|_{1/2}^{1/2},
\]
whence
\[
(2.29) \quad f_{4/3}(t) \left( \int_0^t \| \nabla \delta u \|^2_2 \, d\tau \right)^{1/2} \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2} \| e^{\beta t} \| \nabla z \|_{1/2}^{1/2} \\
\leq f_1(t) \int_0^t \| \nabla \delta u \|^2_2 \, d\tau + \varepsilon \| e^{\beta t} \nabla z \|^2_2 + f_1(t) \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2},
\]
and
\[
f_4(t) \| \nabla \delta u \|_{1/2} \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2} \| e^{\beta t} \| \nabla z \|_{1/2}^{1/2} \leq \varepsilon \| \nabla \delta u \|_{1/2}^{1/2} + f_4^2(t) \| e^{\beta t} \| \partial_0 z \|_{1/2} \| e^{\beta t} \| \nabla z \|_{1/2}^{1/2} \\
\leq f_4(t) \| e^{\beta t} \| \partial_0 z \|_{1/2}^{1/2} + \varepsilon (\| e^{\beta t} \| \nabla z \|_{1/2}^{1/2} + \| \nabla \delta u \|_{1/2}^{1/2}).
\]
The \( \varepsilon \) terms from (2.29) and (2.30) can again be absorbed by the left-hand side of (2.27). Then, plugging (2.28)-(2.30) in (2.27) we obtain

\[
\frac{d}{dt}(e^{2\beta t} \|\nabla \delta \rho(t)\|_2^2) + 2e^{2\beta t}(\|\nabla \z\|_2^2 + \|\nabla w\|_2^2) \\
\leq f_1(t) \left( e^{2\beta t} \|\nabla \delta \rho(t)\|_2^2 + \int_0^t e^{2\beta \tau}(\|\nabla \z\|_2^2 + \|\nabla w\|_2^2) \, d\tau \right) + f_1(t)\|\delta \rho_0\|_X^2.
\]

Denoting

\[
G(t) = e^{2\beta t} \|\nabla \delta \rho(t)\|_2^2 + \int_0^t e^{2\beta \tau}(\|\nabla \z\|_2^2 + \|\nabla w\|_2^2) \, d\tau,
\]

we rewrite (2.31) as

\[
\frac{d}{dt}G(t) \leq f_1(t)G(t) + f_1(t)\|\delta \rho_0\|_X.
\]

We have \( G(0) = \|\nabla \delta \rho_0\|_2^2 \), therefore (2.32) implies

\[
G(t) \leq \|\nabla \delta \rho_0\|_2^2 e^{f_1(t)\tau} + \|\delta \rho_0\|_X \int_0^t e^{f_1(t)\tau} f(s) \, ds.
\]

Notice that the first inequality of (2.7) implies that

\[
\sup_{t \in \mathbb{R}^+} e^{2\beta t}\|w(t)\|_2^2 \leq f_1(t) \int_0^t \|\nabla \delta u(\tau)\|_2^2 \, d\tau \leq f_1(t)G(t).
\]

Combining (2.33) and (2.34), we obtain the following result:

**Lemma 4.** Assume that (2.3) holds. Then, there exist a positive constant \( \beta < \beta_0 \) and \( C_0 \) depending only on the data such that

\[
\sup_{t \in [0, \infty)} e^{2\beta t}\|\nabla \delta u\|_2^2 + \int_0^\infty e^{2\beta t}(\|\nabla \delta u\|_2^2 \, dt \leq C_0(\|\nabla \delta \rho_0\|_2^2 + C_2\|\delta \rho_0\|_X^2).
\]

2.2. **The general case.** The estimate (2.35) has been obtained under assumption (2.4). It may happen however that the denominator of (2.4) is zero on a subset with positive measure, while the numerator is not. To overcome this obstacle, instead of comparing solutions emanating from \((\rho_0^1, v_0^1)\) and \((\rho_0^2, v_0^2)\) directly we compare each of them to an appropriate intermediate solution satisfying (2.4).

To proceed, let us denote by \((\rho^{3/2}, \varphi^{3/2}, P^{3/2})\) the velocity solution to (1.1) emanating from \((\frac{1}{2}(\rho_0^1 + \rho_0^2), v_0^1)\) and by \((\eta^{3/2}, \varphi^{3/2}, Q^{3/2})\) the corresponding solution in Lagrangian coordinates. Then, we look at the following differences between solutions (recall that the density component of the solution in Lagrangian coordinates is constant in time):

\[
(\delta \rho^I, \delta u^I) = \left( \frac{1}{2}(\rho_0^1 + \rho_0^2) - \rho_0^1, u^{3/2} - u^1 \right)
\]

and

\[
(\delta \rho^{II}, \delta u^{II}) = \left( \frac{1}{2}(\rho_0^1 + \rho_0^2) - \rho_0^2, u^{3/2} - u^2 \right).
\]

Clearly, we have

\[
\delta \rho^I = -\delta \rho^{II} = \frac{1}{2}(\rho_0^1 - \rho_0^2) \quad \text{and} \quad \delta u = \delta u^I - \delta u^{II},
\]

and the condition (2.3) is fulfilled in both cases, i.e.

\[
\text{supp} \delta \rho^{II} \subset \text{supp} \frac{1}{2}(\rho_0^1 + \rho_0^2).
\]

Let us define

\[
\|\delta \rho^I\|_X = \left\| \frac{\delta \rho^I}{\sqrt{\rho_0^1 + \rho_0^2}} \right\|_X = \left\| \frac{\delta \rho^{II}}{\sqrt{\rho_0^1 + \rho_0^2}} \right\|_X.
\]

Note that, obviously

\[
\left| \frac{\rho_0^1 - \rho_0^2}{\sqrt{\rho_0^1 + \rho_0^2}} \right|^2 = \left| \frac{1}{2} - 1 \right| \left( \frac{\rho_0^1 - \rho_0^2}{\rho_0^1 + \rho_0^2} \right) \leq \left| \rho_0^1 - \rho_0^2 \right|
\]

whence

\[
\|\delta \rho^I\|_X \leq \|\rho_0^1 - \rho_0^2\|_2^{1/2}.
\]
Denoting $\delta Q^I = Q^{3/2} - Q^1$, we obtain the following system for $(\delta u^I, \delta Q^I)$:
\begin{align}
\begin{cases}
\delta u^I + \Delta u^I \delta u^I + \nabla u^I \delta Q^I = (\Delta u^{3/2} - \Delta u^I)u^{3/2} - (\nabla u^{3/2} - \nabla u^I)Q^{3/2} - \delta Q^I u_{t}^{3/2}, \\
\text{div}_{u^I} \delta u^I = (\text{div}_{u^I} - \text{div}_{u^I})u^{3/2},
\end{cases}
\end{align}
(2.38)
\begin{align}
\delta u^I |_{t=0} = \delta v_0.
\end{align}

Setting $\delta Q^II = Q^{3/2} - Q^2$, we see that $(\delta u^II, \delta Q^II)$ satisfies an analogous system with $(\delta u^I_0, u^1, \delta Q^I)$ replaced by $(\delta u^I_0, u^2, \delta Q^I)$ and with the initial condition $\delta u^II |_{t=0} = 0$.

Let us consider the decompositions
\begin{align}
\delta u^I = z^I + w^I \quad \text{and} \quad \delta u^II = z^II + w^II,
\end{align}
where the components are defined by (2.6) and (2.16) with obvious replacements of $u^1, u^2$. Then, one can repeat the estimates from the previous subsection. In the end, defining
\begin{align}
G^I(t) &= e^{2\beta t} \| \sqrt{\delta u^I_0} z^I(t) \|_2^2 + \int_0^t e^{2\beta s} (\| \nabla z^I(s) \|_2^2 + \| \nabla w^I(s) \|_2^2) \, ds, \\
G^{II}(t) &= e^{2\beta t} \| \sqrt{\delta u^{II}_0} z^{II}(t) \|_2^2 + \int_0^t e^{2\beta s} (\| \nabla z^{II}(s) \|_2^2 + \| \nabla w^{II}(s) \|_2^2) \, ds,
\end{align}
(2.39)
one obtains the following analogs of (2.33) (recall that $\delta u^I_0 = -\delta u^{II}_0$ and that the velocity component of the initial data for $\delta u^{II}$ is zero) for some $f^I, f^{II} \in L^1(\mathbb{R}^+)$:
\begin{align}
G^I(t) &\leq \| \sqrt{\delta u^I_0} \|_2^2 \int_0^t \int e^{2\beta s} f^I(r) dr f^{I}(s) ds, \\
G^{II}(t) &\leq \| \delta u^{II}_0 \|_2^2 \int_0^t \int e^{2\beta s} f^{II}(r) dr f^{II}(s) ds.
\end{align}

Summing up the above inequalities and using (2.36) and (2.37) we obtain
\begin{align}
\sup_{t \in \mathbb{R}^+} \left\{ e^{2\beta t} \left[ \| \sqrt{\delta u^I_0} z^I(t) \|_2 + \| \sqrt{\delta u^{II}_0} z^{II}(t) \|_2 \right] \right\} + \| e^{\beta \Delta} \delta u^I \|_{L^2(\mathbb{R}^+, L^2)} &\leq C \left( \| \sqrt{\delta u^I_0} \|_2 + \| \delta u^{II}_0 \|_2^{1/2} \right).
\end{align}

Combining this estimate with analogs of (2.34), where $(w, \delta u, G)$ has been replaced by $(w^I, \delta u^I, G^I)$ and $(w^{II}, \delta u^{II}, G^{II})$, we obtain (1.21), which completes the proof of Theorem 3.

2.3. Back to the Euler perspective. In this part, we want to translate the stability result obtained in Theorem 3 in terms of Eulerian coordinates, proving Corollary 1. As already pointed out in the introduction, in the case of only bounded initial densities, getting a relevant information on $\varrho^I(t, \cdot) - \varrho^J(t, \cdot)$ in some $L^p$ space (even for $p = 1$) is hopeless; it is more reasonable to compare functions along different characteristics fields. Here we want to adopt the language of kinetic theory, considering quantities along characteristics/trajectories, defined in terms of Wasserstein metrics like in e.g. [37, 32].

More precisely, denote by $X^1$ and $X^2$ the flows defined by (1.14) for, respectively, $v^1$ and $v^2$, and take a smooth function $\phi : \Omega \to \mathbb{R}$. Then we consider, for each $t \geq 0$, the quantity
\begin{align}
I_\phi(t) := \int_\Omega \left( \varrho^1(t, x) - \varrho^2(t, x) \right) \phi(x) \, dx.
\end{align}

Then, using the change of variables (of Jacobian 1) $x = X^1(t, y)$ and $x = X^2(t, y)$ for $\varrho^1$ and $\varrho^2$, respectively, yields
\begin{align}
I_\phi(t) &= \int_\Omega \left[ \varrho^1_0(y) \phi(X^1(t, y)) - \varrho^2_0(y) \phi(X^2(t, y)) \right] \, dy \\
&= \int_\Omega \left( \varrho^1_0(y) - \varrho^2_0(y) \right) \phi(X^1(t, y)) \, dy + \int_\Omega \varrho^2_0(y) \phi(X^2(t, y)) \, dy.
\end{align}
(2.40)

In order to find out the right level of regularity for $\phi$, let us first examine the term $A_2(t)$. For suitable functions $\eta^i$, we set
\begin{align}
\eta^1(t, x) := \eta^1(t, Y^1(t, x)) \quad \text{and} \quad \eta^2(t, x) := \eta^2(t, Y^2(t, x)) \quad \text{with} \quad Y^i(t, \cdot) := (X^i(t, \cdot))^{-1}.
\end{align}
Then, using again the fact that $X^1$ and $X^2$ are measure preserving, we have

\[
\|\eta^1(t) - \eta^2(t)\|_{W_p^{-1}} = \sup \left\{ \int_{\Omega} (\eta^1(t, x) - \eta^2(t, x)) \phi(t, x) \, dx : \phi \in W_p^1(\Omega), \|\phi\|_{W_p^1} \leq 1 \right\}
\]

(2.41)

\[
= \sup \left\{ \int_{\Omega} \eta(y) [\phi(X^1(t, y)) - \phi(X^2(t, y))] \, dy : \phi \in W_p^1(\Omega), \|\phi\|_{W_p^1} \leq 1 \right\}.
\]

Following [40], in order to handle the term $\phi(X^1(t, y)) - \phi(X^2(t, y))$, we consider the family of intermediate measure preserving flows $(X^s)_{1 \leq s \leq 2}$ between $X^1$ and $X^2$ defined as follows:

\[
d\frac{X^s}{dt}(t, y) = (2 - s)v^1(t, X^s(t, y)) + (s - 1)v^2(t, X^s(t, y)), \quad X^s(0, y) = y.
\]

By the chain rule and the definition of $X^s$, we have

\[
\phi(X^2(t, y)) - \phi(X^1(t, y)) = \int_1^2 \frac{d}{ds} \phi(X^s(t, y)) \, ds
\]

\[
= \int_1^2 \left( \frac{d}{ds} X^s(t, y) \right) \cdot \nabla \phi(X^s(t, y)) \, ds.
\]

From the definition of $X^s$, we discover that

\[
\frac{d}{ds} X^s(t, y) = \int_0^t \left( v^2(t', X^s(t', y)) - v^1(t', X^s(t', y)) \right) \, dt',
\]

which implies that

\[
\frac{d}{ds} X^s(t, \cdot) \leq \int_0^t \|\delta v\|_p \, dt'.
\]

(2.42)

Furthermore, as said before, $X^s(t, \cdot)$ is measure preserving for all $t \geq 0$, and thus $\|\nabla \phi(X^s(t, \cdot))\|_p = \|\nabla \phi\|_p$. In the end, using the embedding $H_0^d \hookrightarrow L_p$ for all $1 \leq p < \infty$ if $d = 2$, and all $1 \leq p \leq 6$ if $d = 3$, we obtain for these values of $p$ and all $t \geq 0$,

\[
A_2(t) \leq C\|\phi_0^2\|_\infty \left( \int_0^\infty e^{2\beta t} \|\nabla \delta v\|_2^2 \, dt \right)^{1/2} \|\phi\|_{1,p'}.
\]

(2.43)

The above inequality reveals that one can take the functions $\phi$ in the space

$\phi \in W_{1+}^1(\Omega)$ if $d = 2$ and $\phi \in W_{5/3}^1(\Omega)$ if $d = 3$.

Bounding the term $A_1(t)$ is simpler. Under the above assumptions on $p$, we have $W_p^1 \hookrightarrow L_2$. Hence, using again that $X^1$ is measure preserving, we may write by Cauchy-Schwarz inequality,

\[
A_1(t) \leq \|\delta u\|_2 \|\phi\|_2 \leq C\|\phi_0\|_2 \|\phi\|_{1,p'}.
\]

(2.44)

Altogether, by (2.40), (2.41), (2.43), (2.44) and (1.21), we get for any finite $p > 1$ if $d = 2$ and for $p \leq 6$ if $d = 3$,

\[
\sup_{t \in \mathbb{R}^+} \|\phi^1(t, \cdot) - \phi^2(t, \cdot)\|_{W_p^{-1}} \leq C \left( \|\delta u\|_2^2 \left( \int_0^\infty e^{2\beta t} \|\nabla \delta v\|_2^2 \, dt \right)^{1/2} + \|\phi_0\|_2^2 \right).
\]

(2.45)

Next, let us turn to the velocity estimate. We start from the relation

\[
\nabla \delta u(t, y) = v^2(t, X^2(t, y)) - v^1(t, X^1(t, y)),
\]

so

\[
\nabla \delta u(t, y) = \nabla_y v^2(t, X^2(t, y)) \cdot \nabla_x v^2(t, X^2(t, y)) - \nabla_y X^1(t, y) \cdot \nabla_x v^1(t, X^1(t, y)).
\]

Hence, denoting $\delta X := X^2 - X^1$, we may write

\[
\nabla \delta u(t, y) = \nabla_y \delta X(t, y) \cdot \nabla v^2(t, X^2(t, y)) + \nabla_y X^1(t, y) \cdot \nabla \delta v(t, X^2(t, y))
\]

\[
+ \nabla_y X^1(t, y) \cdot [\nabla v^1(t, X^2(t, y)) - \nabla v^1(t, X^1(t, y))]
\]

\[
=: K_1 + K_2 + K_3.
\]

First, we observe that

\[
\nabla \delta v(t, X^2(t, y)) = T_A^1 K_2.
\]

Hence, taking the $L_2(\Omega)$ norm, and using that $X^2(t, \cdot)$ is measure preserving, we get

\[
\|\nabla \delta u\|_2 \leq \|A^1\|_\infty \|K_2\|_2 \leq \|A^1\|_\infty (\|\nabla \delta u\|_2 + \|K_1\|_2 + \|K_3\|_2).
\]

(2.46)
Next, from the definition of \(X^1\) and \(X^2\) in terms of the Lagrangian velocity, we have

\[
\|K_1\|_2 \leq C t^{1/2} \left( \int_0^t \|\nabla \delta u\|_2^2 \, dt \right)^{1/2} \|\nabla v\|_\infty.
\]

For bounding \(K_3\), we first notice that by the mean value theorem and the definition of the intermediate flow \(X^s\), we have

\[
K_3 = \nabla_y X^1(t,y) \cdot \left( \int_1^2 \nabla^2 v^1(t,X^s(t,y)) \, ds \right) \cdot \left( \frac{d}{ds} X^s(t,y) \right).
\]

Hence, since \(X^s\) is measure preserving, using Hölder inequality and (2.42) allows to get

\[
\|K_3\|_2 \leq \|\nabla_y X^1(t)\|_\infty \int_1^2 \|\nabla^2 v^1(t,X^s(t))\|_3 \left\| \frac{d}{ds} X^s(t) \right\|_6 \, ds
\]

\[
\leq C \|\nabla_y X^1(t)\|_\infty \|\nabla^2 v^1(t)\|_3 \int_0^t \|\delta v(t')\|_6 \, dt'
\]

\[
\leq C \|\nabla_y X^1(t)\|_\infty \|t^{1/2} \nabla^2 v^1(t)\|_3 \left( \int_0^t \|\nabla \delta v\|_2^2 \, dt' \right)^{1/2}.
\]

Altogether, remembering (2.40), we obtain for all \(t \geq 0\),

\[
\|\nabla \delta v(t)\|_2^2 \leq C \left[ \|\nabla \delta u(t)\|_2^2 + \|t^{1/2} \nabla v\|_\infty^2 \int_0^t \|\nabla \delta u\|_2^2 \, dt' + \|t^{1/2} \nabla^2 v\|_3^2 \int_0^t \|\nabla \delta v\|_2^2 \, dt' \right].
\]

First multiplying both sides by \(e^{2\beta t}\), next integrating in time and using Theorem 3 combined with the properties (2.1), and finally applying Gronwall lemma, we obtain for all \(t \geq 0\),

\[
\int_0^t e^{2\beta t'} \|\nabla \delta v\|_2^2 \, dt' \leq C_0 \left( \|\nabla \rho_0 \|_2^2 + \|\delta \varrho_0\|_2 \right) \exp \left( C \int_0^t e^{2\beta t'} \|t^{1/2} \nabla^2 v\|_3^2 \, dt' \right).
\]

The term in the exponential may be bounded thanks to (2.14). Hence, reverting to (2.45) completes the proof of Corollary 1.

3. Decay estimates

The goal of this section is to show that the solutions provided by Theorems 1 and 2 indeed satisfy Properties (2.1a) to (2.1h). This will be an immediate consequence of Lemmas 6, 7 and 8 below.

Before tackling the proof, we need to recall elementary properties of the solutions to (1.1). The first one is the conservation of any \(L_p\) norm of the density, a consequence of the divergence free property of the velocity field: we have \(\|\varrho(t)\|_p = \|\varrho_0\|_p\).

Of course, as \(v\in H^1_0(\Omega)\), we have the Poincaré inequality:

\[
\|v\|_2 \leq C_P \|\nabla v\|_2.
\]

3.1. Decay of space derivatives. The starting point is the following decay estimate which is a direct consequence of diffusion and Poincaré inequality:

**Lemma 5.** Let \((\varrho, v)\) be a solution to (1.1) given either by Theorem 2 or by Theorem 3. Then

\[
\forall t \geq 0, \int_\Omega \varrho(t)|v(t)|^2 \, dx \leq e^{-\beta t} \int_\Omega \varrho_0|v_0|^2 \, dx, \quad \text{where} \quad \beta = \frac{2\mu}{\varrho_0 C_P}.
\]

**Proof.** The proof is independent of the space dimension. We start with the classical energy estimate that is obtained testing the momentum equation by \(v\), namely

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \varrho|v|^2 \, dx + \mu \int_\Omega |\nabla v|^2 \, dx = 0.
\]

Hence, remembering (3.1), we get

\[
\frac{C_P^2}{2\mu} \frac{d}{dt} \int_\Omega \varrho|v|^2 \, dx + \int_\Omega |v|^2 \, dx \leq 0.
\]

Multiplying both sides by \(\varrho^*\) and using the obvious fact that \(\varrho^*|v|^2 \geq \varrho|v|^2\), we obtain

\[
\frac{\varrho^* C_P^2}{2\mu} \frac{d}{dt} \int_\Omega \varrho|v|^2 \, dx + \int_\Omega \varrho|v|^2 \, dx \leq 0,
\]

where \(\varrho^* = \max(\varrho, 0)\).
from which we conclude to \((3.2)\).

Our next aim is to establish a decay estimate for the gradient of the velocity.

**Lemma 6.** Let \((\varrho, v)\) be a solution to \((1.1)\) given either by Theorem \(2\) or by Theorem \(3\). Then, there exist positive constants \(C_0\) and \(C_{0,p}\) depending only on the data \((and on p for second one)\), and \(\beta_2 < \beta_1\) such that for all \(t \geq 0\), we have

\[
\|\nabla v(t)\|_2 \leq C_0 e^{-\beta_2 t},
\]

\[
\|v(t)\|_p \leq C_{0,p} e^{-\beta_2 t},
\]

\[
\int_0^\infty e^{2\beta_2 t} \left[ \|\sqrt{\varrho(t)} v_1(t)\|_2^2 + \|\nabla^2 v(t)\|_2^2 + \|\nabla P(t)\|_2^2 \right] \, dt \leq C_0,
\]

where, in \((3.5)\), one can take any \(p \in [1, \infty)\) if \(d = 2\), and any \(p \in [1, 6]\) if \(d = 3\).

**Proof.** Clearly, performing a suitable time, space, density and velocity rescaling reduces the proof to the case \(\varrho^* = \mu = 1\), in a domain of diameter \(1\). Hence, we shall only prove the lemma in this case for notational simplicity.

Now, testing the momentum equation by \(v_t\) yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\Omega} \varrho |v_t|^2 \, dx = - \int_{\Omega} \varrho v \cdot (v \cdot \nabla v) \, dx \leq \frac{1}{2} \int_{\Omega} \varrho |v_t|^2 \, dx + \frac{1}{2} \int_{\Omega} \varrho |v \cdot \nabla v|^2 \, dx.
\]

The classical theory for the Stokes system yields for some \(C_1\) depending only on the shape of \(\Omega\),

\[
\|\nabla^2 v\|_2^2 + \|\nabla P\|_2^2 \leq C_1 (\|\varrho v \cdot \nabla v\|_2^2 + \|v_t\|_2^2).
\]

Hence we have (remember that \(\varrho^* = 1\)):

\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\Omega} (\varrho |v_t|^2 + \frac{1}{C_1} (\|\nabla^2 v\|_2^2 + |\nabla P|_2^2)) \, dx \leq \frac{3}{2} \int_{\Omega} \varrho |v \cdot \nabla v|^2 \, dx.
\]

Adding this inequality to \((3.3)\), we get (up to a change of \(C_0\))

\[
\frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + \frac{1}{2} \varrho |v|^2) \, dx + \int_{\Omega} (|\nabla^2 v|^2 + \frac{1}{2} \varrho |v|^2) \, dx + \frac{1}{2} \int_{\Omega} (\varrho |v_t|^2 + \frac{1}{C_1} (\|\nabla^2 v\|_2^2 + |\nabla P|_2^2)) \, dx \leq \frac{3}{2} \int_{\Omega} \varrho |v \cdot \nabla v|^2 \, dx + \frac{1}{2} \int_{\Omega} \varrho |v|^2 \, dx.
\]

In order to estimate the first term on the right in the two dimensional case we first write

\[
\int_{\Omega} \varrho |v \cdot \nabla v|^2 \, dx \leq \|\sqrt{\varrho}\|_2 \|v\|_\infty \|\nabla v\|_4^2.
\]

Hence, using the following two interpolation inequalities:

\[
\|z\|_4 \leq C \|z\|_2^{1/2} \|\nabla z\|_2^{1/2} \quad \text{and} \quad \|z\|_\infty \leq C \|z\|_2^{1/2} \|\nabla^2 z\|_2^{1/2},
\]

and remembering that \(\varrho^* = 1\), we get

\[
\frac{3}{2} \int_{\Omega} \varrho |v \cdot \nabla v|^2 \, dx \leq C \|\sqrt{\varrho}\|_2 \|v\|_2^{1/2} \|\nabla v\|_2 \|\nabla^2 v\|_2^{3/2} \leq \frac{1}{4C_1} \|\nabla^2 v\|_2^2 + C \|\sqrt{\varrho}\|_2 \|v\|_2 \|\nabla v\|_4^2.
\]

The first term can be absorbed by the left-hand side of \((3.8)\) and one can bound \(\|v\|_2\) from \((3.1)\).

Hence, we get for some \(\gamma > 0\) depending only on the shape of the domain,

\[
\frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + \frac{1}{2} \varrho |v|^2) \, dx + \int_{\Omega} (|\nabla v|^2 + \frac{1}{2} \varrho |v|^2) \, dx + \frac{1}{2} \int_{\Omega} (\varrho |v_t|^2 + \frac{1}{2C_1} (|\nabla^2 v|^2 + |\nabla P|^2)) \, dx \leq C e^{-\gamma t}[1 + \|\nabla v\|_2^2].
\]

The crucial observation is that \(\|\nabla v(t)\|_2\) may bounded uniformly in time in terms of the data (see Propositions 3.1 and 3.3 in \((10)\)). Therefore, we obtain

\[
\frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + \frac{1}{2} \varrho |v|^2) \, dx + \frac{1}{2} \int_{\Omega} (|\nabla v|^2 + \frac{1}{2} \varrho |v|^2 + \varrho |v|_t^2 + \frac{1}{2C_1} (|\nabla^2 v|^2 + |\nabla P|^2)) \, dx \leq C_0 e^{-\gamma t}.
\]

\(^{1}\text{One can take } \beta_2 \text{ of the form } \beta_2 = c_0 \mu/(\varrho^* \delta) \text{ where } \delta \text{ stands for the diameter of } \Omega \text{ and } c \text{ depends only on the shape of } \Omega.\)
This leads for any $\gamma_1 \in \mathbb{R}$ to
\begin{equation}
\frac{d}{dt} \left[ e^{\gamma_1 t} \int_{\Omega} \left( |\nabla v|^2 + \frac{1}{2} \varrho |v|^2 \right) dx \right] + \left( \frac{1}{2} - \gamma_1 \right) \int_{\Omega} e^{\gamma_1 t} (|\nabla v|^2 + \frac{1}{2} \varrho |v|^2 + \varrho |v_t|^2 + \frac{1}{2C_{\Omega}} (|\nabla v|^2 + |\nabla P|^2)) dx \leq C_0 e^{-(\gamma-\gamma_1)t}.
\end{equation}

In the three dimensional case, the only difference is the slightly more complicated treatment of the right-hand side of (3.8). Nevertheless, by using Hölder inequality and Gagliardo-Nirenberg inequality, we arrive (still using that $\varrho^* = 1$) at:
\begin{align*}
\int_{\Omega} \varrho |v \cdot \nabla v|^2 &\leq \|\varrho^{1/4} v\|_4^2 \|\nabla v\|_2^2 \\
&\leq C \|\sqrt{\varrho} e\|_2^{1/2} \|v\|_6^{3/2} \|\nabla v\|_2 \|\nabla^2 v\|_2^{3/2} \\
&\leq C \|\sqrt{\varrho} v\|_2 \|\nabla v\|_2 \|\nabla^2 v\|_2^{3/2} \\
&\leq \frac{1}{2C_{\Omega}} \|\nabla v\|_2^2 + C \|\sqrt{\varrho} v\|_2^2 \|\nabla v\|_2^2 \leq \frac{1}{2C_{\Omega}} \|\nabla v\|_2^2 + C_0 e^{-\beta t},
\end{align*}
where in the last passage we have used (3.2) and uniform boundedness of $\|\nabla v(t)\|_2$. Again, the first term can be absorbed by the left hand side of (3.8) and we obtain (3.12).

Now, choosing e.g. $\gamma_1 = \min(\gamma, 1/4)$ and integrating (3.12) on $[0, t]$ yields
\begin{equation}
\frac{c_0}{\gamma_1} \left( |\nabla v(t)|_2^2 + \frac{1}{2} \|\sqrt{\varrho}(t) v(t)\|_2^2 \right) + \left( \frac{1}{2} - \gamma_1 \right) \int_0^t \frac{c_0}{\gamma_1} \left( |\nabla v|_2^2 + \frac{1}{2} \|\sqrt{\varrho} v\|_2^2 \right. \\
+ \left. \frac{1}{2} \|\sqrt{\varrho} v_t\|_2^2 + \frac{1}{2C_{\Omega}} (|\nabla v|^2 + |\nabla P|^2) \right) ds \leq \frac{C_0}{\gamma - \gamma_1} + \|\nabla v_0\|_2^2 + \frac{1}{2} \|\sqrt{\varrho} v_0\|_2^2,
\end{equation}
which readily gives (3.4) and (3.6). As for (3.5), it just results from Poincaré inequality and Sobolev embedding.

**3.2. Decay of time derivatives.** Here we estimate time and time-space derivatives.

**Lemma 7.** Let $(\varrho, v)$ be a solution to (1.1) given either by Theorem 1 or by Theorem 2. Then, there exists a positive constant $\beta_3 < \beta_2$ (still of the form $\beta_3 = c\mu/(\varrho^* \delta^2)$) such that
\begin{equation}
\sup_{t \in \mathbb{R}} \int_{\Omega} te^{2\beta_3 t} \varrho |v_t|^2 dx + \int_0^{\infty} e^{2\beta_3 t} \left[ \|\sqrt{\varrho} v_t\|_2^2 + \|\sqrt{v} v_t\|_2^2 \right] dt \leq C_0
\end{equation}
for some positive constant $C_0$ depending only on the data.

**Proof.** We keep the assumption $\delta = \mu = \varrho^* = 1$. Now, differentiating the momentum equation in time and multiplying by $\sqrt{\varrho} v_t$, we obtain
\begin{equation}
\varrho(\sqrt{\varrho} v_t)_t + \sqrt{\varrho} v_t = -\frac{1}{2} e^{\beta t} \varrho v_t - \beta \sqrt{\varrho} v_t + \sqrt{\varrho} \varrho_t v \cdot \nabla v \\
+ \sqrt{\varrho} \varrho_t v \cdot \nabla v + \sqrt{\varrho} \varrho v \cdot \nabla v_t - \Delta(\sqrt{\varrho} v_t) + \nabla((\sqrt{\varrho}) P_t) = 0.
\end{equation}
Testing (3.14) with $\sqrt{\varrho} v_t$ yields
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} te^{2\beta t} \varrho |v_t|^2 dx + \int_{\Omega} te^{2\beta t} |\nabla v_t|^2 = \sum_{i=1}^5 I_i,
\end{equation}
where, using $\varrho_t = -\text{div}(\varrho v)$ in the third term on the left-hand side of (3.14), we have
\begin{align*}
I_1 &= \frac{1}{2} \int_{\Omega} e^{2\beta t} \varrho |v_t|^2 dx, \\
I_2 &= -\int_{\Omega} (\sqrt{\varrho} \varrho_t v \cdot \nabla v) \cdot (\sqrt{\varrho} v_t) dx, \\
I_3 &= -\int_{\Omega} (\sqrt{\varrho} \varrho v_t \cdot \nabla v) \cdot (\sqrt{\varrho} \varrho v_t) dx, \\
I_4 &= -2 \int_{\Omega} (\sqrt{\varrho} \varrho v \cdot \nabla v) \cdot (\sqrt{\varrho} v_t) dx, \\
I_5 &= \beta \int_{\Omega} te^{2\beta t} \varrho |v_t|^2.
\end{align*}
Now, we estimate the right-hand side of (3.15). From the continuity equation, we have
\[ I_2 = \int_{\Omega} te^{2\beta t} \text{div} (g v)(v \cdot \nabla v) \cdot v_t \, dx = - \int_{\Omega} te^{2\beta t} g v \cdot \nabla [(v \cdot \nabla v) \cdot v_t] \, dx. \]
Therefore,
\[ |I_2| \leq \int_{\Omega} te^{2\beta t} \rho |v| \left( |\nabla v|^2 |v_t| + |v| |\nabla^2 v||v_t| + |v| |\nabla v| |\nabla v_t| \right) \, dx =: I_{21} + I_{22} + I_{23}. \]
For $I_{21}$ in the two-dimensional case, we have
\[ |I_{21}| \leq \int_{\Omega} \left( \sqrt{t \rho} |v| |\nabla v|^2 \sqrt{t \rho} |v_t| e^{2\beta t} \right) \, dx \leq \|v\|^2 \|\sqrt{t \rho} e^{\beta t} v_t\|^2_2 + te^{2\beta t} \|\nabla v\|^4_4, \]
so using (3.9) and (3.4) we get
\[
|I_{21}| \leq \|v\|^2_\infty \left( 2\|\sqrt{t \rho} e^{\beta t} v_t\|^2_2 + \sqrt{2} \sqrt{\rho \|v\| \|\nabla v\| \|\nabla v_t\|} \right),
\]
(3.16)
In the three dimensional case we do not have (3.9), but we can write using Hölder inequality and the embedding $H^1_0 \hookrightarrow L^6$,
\[
|I_{21}| \leq \sqrt{t e^{2\beta t}} \int_{\Omega} \sqrt{t \rho} |v_t| |v| |\nabla v|^2 \, dx \leq \sqrt{t e^{2\beta t}} \|\sqrt{t \rho} v_t\|_4 \|v\|_6 \|\nabla v\|^{2/7}_{24/7}
\leq \sqrt{t e^{\beta t}} \|\sqrt{t \rho} v_t\|_2^{1/4} \|\sqrt{t \rho} e^{\beta t} v_t\|_6^{3/4} \|v\|_6 \|\nabla v\|^{2/7}_{24/7}
\leq \frac{1}{10} \|\sqrt{t e^{\beta t}} |\nabla v_t|^2 + C t^{4/5} e^{3\beta t/5} \|\sqrt{t \rho} v_t\|_2^{2/5} \|\nabla v\|^{16/5}_{24/7} \|\nabla v\|^{8/5}_{2}.
\]
Then, using the Gagliardo-Nirenberg inequality $\|\nabla v\|^{16/5}_{24/7} \leq C \|\nabla v\|^{6/5}_2 \|\nabla^2 v\|^{2}_2$ and (3.4), we discover that
\[
|I_{21}| \leq \frac{1}{10} \|\sqrt{t e^{\beta t}} |\nabla v_t|^2 + C e^{-ct} \|e^{\beta t} \nabla^2 v\|^{2}_2 \left( 1 + \|\sqrt{t \rho} e^{\beta t} v_t\|^{2}_2 \right).
\]
(3.17)
Therefore, there exists $c > 0$ such that
\[
|I_{21}| \leq \frac{1}{10} \|\sqrt{t e^{\beta t}} |\nabla v_t|^2 + C e^{-ct} \|e^{\beta t} \nabla^2 v\|^{2}_2 \left( 1 + \|\sqrt{t \rho} e^{\beta t} v_t\|^{2}_2 \right).
\]
The remaining two parts of $I_2$ are simpler: we have
\[
I_{22} = \int_{\Omega} \sqrt{t \rho} e^{\beta t} |\nabla^2 v||v|^2 \sqrt{t \rho} e^{\beta t} |v_t| \leq \|\sqrt{t \rho} e^{\beta t} \nabla^2 v\|_2 \|v\|^3_6 \|\sqrt{t \rho} e^{\beta t} v_t\|_6
\leq C \|\sqrt{t e^{\beta t}} \nabla^2 v\|_2 \|v_t\|_2 \|\sqrt{t e^{\beta t}} \nabla v_t\|_2
\leq \frac{1}{15} \|\sqrt{t e^{\beta t}} |\nabla v_t|^2 + C e^{-4\beta t} \|\sqrt{t e^{\beta t}} \nabla^2 v\|^{2}_2,
\]
(3.18)
and
\[
I_{23} \leq \|\sqrt{t \rho} e^{\beta t} \nabla v_t\|_2 \|v\|^2_6 \|\sqrt{t \rho} e^{\beta t} \nabla v\|_6
\leq C \|\sqrt{t e^{\beta t}} \nabla v_t\|_2 \|v_t\|_2 \|\sqrt{t e^{\beta t}} \nabla^2 v\|_2
\leq \frac{1}{15} \|\sqrt{t e^{\beta t}} |\nabla v_t|^2 + C e^{-4\beta t} \|\sqrt{t e^{\beta t}} \nabla^2 v\|^{2}_2.
\]
(3.19)
Hence, putting (3.17), (3.18) and (3.19) together, we obtain for some $c > 0$,
\[
|I_2| \leq \frac{1}{5} \|\sqrt{t e^{\beta t}} |\nabla v_t|^2 + C e^{-ct} \|e^{\beta t} \nabla^2 v\|^{2}_2 \left( 1 + \|\sqrt{t \rho} e^{\beta t} v_t\|^{2}_2 \right).
\]
(3.20)
Next, we estimate $I_3$ as follows:

$$I_3 \leq \|\nabla v\|_2 \sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_4^2$$

$$\leq C \|\nabla v\|_2 \sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_2^{1/2} \sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_6^{3/2}$$

$$\leq C \|\nabla v\|_2 \sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_2^{1/2} \sqrt{\bar{\rho} e^{\beta t} \nabla v_t} \|_2^{3/2}$$

$$\leq \frac{1}{10} \|\sqrt{\bar{\rho} e^{\beta t} \nabla v_t} \|_2^2 + C \|\sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_2 \|\nabla v\|_2$$

(3.21)

$$\leq \frac{1}{10} \|\sqrt{\bar{\rho} e^{\beta t} \nabla v_t} \|_2^2 + C_0 e^{-\beta t} \|\sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_2^2$$

and

(3.22)  \[ I_4 \leq 2 \int_\Omega \rho |v|\|\sqrt{\bar{\rho} e^{\beta t} \nabla v_t} \|_2 \|\sqrt{\bar{\rho} e^{\beta t} v_t} \|_2 \leq \frac{1}{10} \|\sqrt{\bar{\rho} e^{\beta t} \nabla v_t} \|_2^2 + C \|v\|_\infty^2 \sqrt{\bar{\rho} e^{\beta t} v_t} \|_2^2. \]

Finally, as (3.1) also applies to $v_t$, one may write for sufficiently small $\beta$,

(3.23)  \[ I_5 \leq \frac{1}{10} \|\sqrt{\bar{\rho} e^{\beta t} \nabla v_t} \|_2^2. \]

Combining (3.15), (3.20), (3.21), (3.22) and (3.23) we arrive at

(3.24)  \[ \frac{1}{2} \frac{d}{dt} \int_\Omega \rho e^{2\beta t} |v|^2 \, dx + \frac{1}{2} \int_\Omega e^{2\beta t} \|\nabla v_t\|^2 \, dx \leq C_0 (e^{-ct} \|e^{\beta t} \nabla v_t\|_2^2 + \|v\|_\infty^2) \|\sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_2^2 + C_0 (\|\sqrt{\bar{\rho} e^{\beta t} v_t^2} \|_2^2 + e^{-ct} \|e^{\beta t} \nabla v_t\|_2^2). \]

By virtue of Lemma 6, the last line is integrable on $\mathbb{R}_+$ for any $\beta \leq \beta_2$, as well as the prefactor of the second line (observe that $H^2 \hookrightarrow L_\infty$). Hence, Gronwall inequality ensures that

(3.25)  \[ \sup_{t \in \mathbb{R}_+} \int_\Omega e^{2\beta t} |v|^2 \, dx + \int_\mathbb{R}_+ e^{2\beta t} \|\nabla v_t\|_2^2 \, dt < \infty. \]

Now, Poincaré inequality implies the bound for $\|e^{\beta t} \nabla v_t\|_{L_2(\mathbb{R}_+; L_2)}$, which completes the proof. □

3.3. Shift of integrability and control of $\|\nabla v\|_\infty$. Using the decay estimates we proved so far will finally enable us to establish similar properties for higher order norms:

**Lemma 8.** Let $(\rho, v)$ be a solution to (1.1) given either by Theorem 7 or by Theorem 2. Then, the following properties hold true:

(3.26)  \[ \sqrt{\bar{\rho} e^{\beta t}} (\nabla^2 v, \nabla P) \in L_p(\mathbb{R}_+; L_s(\Omega)) \quad 2 \leq s \leq 6 \quad \text{and} \quad p = \frac{4s}{3s - 6} \quad \text{if} \quad d = 3, \]

(3.27)  \[ \sqrt{\bar{\rho} e^{\beta t}} (\nabla^2 v, \nabla P) \in L_p(\mathbb{R}_+; L_s(\Omega)) \quad 2 \leq s < \infty \quad \text{and} \quad p = \frac{2s}{s - 2} \quad \text{if} \quad d = 2, \]

(3.28)  \[ e^{\beta t} \nabla^2 v \in L_1(\mathbb{R}_+; L_r(\Omega)) \quad \text{for some} \quad r > d, \]

(3.29)  \[ e^{\beta t} \nabla v \in L_1(\mathbb{R}_+; L_\infty(\Omega)), \]

for some $0 < \beta_4 = \frac{a_0}{e^{\beta_2}} < \beta_2$, where $\beta_2$ is from Lemma 6.

**Proof.** We multiply (1.1) by $\sqrt{\bar{\rho} e^{\beta t}}$, where $\beta = \beta_2$, and rewrite it as a Stokes system:

(3.30)  \[ -\Delta \sqrt{\bar{\rho} e^{\beta t}} v + \nabla \sqrt{\bar{\rho} e^{\beta t}} P = -\sqrt{\bar{\rho} e^{\beta t}} g v_t - \sqrt{\bar{\rho} e^{\beta t}} g v \cdot \nabla v, \quad \text{div} \sqrt{\bar{\rho} e^{\beta t}} v = 0. \]

We start with proving (3.26). By the interpolation inequality

$$\|f\|_q \leq \|f\|_{2}^{(6-q)/2q} \|f\|_{6}^{(3q-6)/2q}, \quad 2 \leq q \leq 6,$$

it is enough to prove

(3.31)  \[ \sqrt{\bar{\rho} e^{\beta t}} (\nabla^2 v, \nabla P) \in L_\infty(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; L_6(\Omega)). \]

By (3.13) and Sobolev embedding, we have

(3.32)  \[ \sqrt{\rho} e^{\beta t} v_t \in L_\infty(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; L_6(\Omega)). \]
Therefore, the elliptic regularity of (3.30) implies
\[
\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L^\infty(\mathbb{R}^+; L^2)} \leq C_0 + C\|\nabla \sqrt{t}e^{\beta t}v\|_{L^\infty(\mathbb{R}^+; L^2)}
\]
\[
\leq C_0 + C\|e^{\beta t}v\|_{L^\infty(\mathbb{R}^+; L^6)}\|\sqrt{t}v\|_{L^\infty(\mathbb{R}^+; L^2)}
\]
\[
\leq C_0 + C\sqrt{t}e^{-d\beta t/6}\|e^{\beta t}v\|_{L^\infty(\mathbb{R}^+; L^6)}\|\nabla v\|^{1-d/6}_{L^\infty(\mathbb{R}^+; L^2)}\|\sqrt{t}e^{\beta t}\nabla^2 v\|^{d/6}_{L^\infty(\mathbb{R}^+; L^2)}.
\]
So, by Young inequality and (3.5), we get
\[
(3.33)\quad \|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L^\infty(\mathbb{R}^+; L^2)} \leq C_0.
\]
Similarly, starting from (3.30) and thanks to Inequality (3.13) and embedding $H^1_\Omega \rightarrow L^6$, we have
\[
\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L^2(\mathbb{R}^+; L^6)} \leq C_0 + C\|\sqrt{t}e^{\beta t}v \cdot \nabla v\|_{L^2(\mathbb{R}^+; L^6)}
\]
\[
\leq C_0 + C\|e^{\beta t/2}\sqrt{t}v\|_{L^\infty(\mathbb{R}^+; L^2)}\|e^{\beta t/2}\nabla v\|_{L^2(\mathbb{R}^+; L^6)}.
\]
In three-dimensional case, we have
\[
\|\sqrt{t}e^{\beta t/2}v\|_{L^\infty(\mathbb{R}^+; L^6)} \leq C\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L^\infty(\mathbb{R}^+; L^2)}\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L^\infty(\mathbb{R}^+; L^6)},
\]
which implies
\[
\|\sqrt{t}e^{\beta t}\nabla^2 v\|_{L^2(\mathbb{R}^+; L^6)} \leq C_0 + C\|e^{\beta t/2}\sqrt{t}\nabla v\|^{1/2}_{L^\infty(\mathbb{R}^+; L^2)}\|e^{\beta t/2}\nabla v\|^{1/2}_{L^\infty(\mathbb{R}^+; L^6)}
\]
\[
\leq C_0 + C\|e^{\beta t}\nabla v\|^{1/2}_{L^\infty(\mathbb{R}^+; L^2)}\|\sqrt{t}e^{\beta t/2}\nabla v\|^{1/2}_{L^\infty(\mathbb{R}^+; L^6)}
\]
\[
\leq C_0 + C\|e^{\beta t}\nabla v\|_{L^\infty(\mathbb{R}^+; L^2)} + \|\sqrt{t}e^{\beta t}\nabla^2 v\|_{L^\infty(\mathbb{R}^+; L^2)},
\]
where in the first passage we used (3.6) and Sobolev embedding to estimate $\|e^{\beta t/2}\nabla v\|_{L^2(\mathbb{R}^+; L^6)}$.

Therefore, by (3.4) and (3.33), choosing $\beta$ small enough, we obtain
\[
\|\sqrt{t}e^{\beta t}(\nabla^2 v, \nabla P)\|_{L^2(\mathbb{R}^+; L^6)} \leq C_0.
\]
This completes the proof of (3.31), and thus also of (3.26). The easier two-dimensional case (that is (3.27)) is left to the reader.

In order to show the next part of the lemma, we note that for $\beta_4, \delta > 0$ such that $\beta_4 + \delta < \beta_2$ we have
\[
\int_{1}^{+\infty} e^{\beta_4 t}\|\nabla^2 v\|_{L^p} \leq \left(\int_{1}^{+\infty} e^{-\delta p t}dt\right)^{1/p'} \left(\int_{1}^{+\infty} e^{(\beta_4 + \delta) p t}\|\nabla^2 v(t)\|_{L^p}^p dt\right)^{1/p}
\]
\[
\leq C\|e^{\beta_4 t}\sqrt{t}\nabla^2 v\|_{L^p(\mathbb{R}^+; L^p)}.
\]
For small times we can write
\[
\int_{0}^{1} \|\nabla^2 v(t)\|_{L^p} dt \leq \left(\int_{0}^{1} t^{-\alpha p}dt\right)^{1/p'} \int_{0}^{1} t^{\alpha p} \|\nabla^2 v(t)\|_{L^p}^p dt.
\]
We can choose for instance $p = \frac{8}{5}$, which corresponds to $s = 4$ in (3.26), then $\alpha = \frac{5}{16}$ so that $\alpha p = \frac{5}{8}$ and $\alpha p' < 1$, so the first integral is again finite. This completes the proof of (3.28).

As for (3.29), it results directly from (3.4) and (3.28) owing to a suitable Gagliardo-Nirenberg inequality that yields:
\[
\|e^{\beta t}\nabla v\|_{L^1(\mathbb{R}^+; L^\infty)} \leq C(\|e^{\beta_4 t}\nabla v\|_{L^4(\mathbb{R}^+; L^2)} + \|e^{\beta_4 t}\nabla^2 v\|_{L^1(\mathbb{R}^+; L^p)}).
\]
The right-hand side is finite whenever $\beta_4 < \beta_2$.

**Acknowledgments:** The first two authors have been partially supported by the ANR project INFAMIE (ANR-15-CE40-0011). The second and third author were partially supported by National Science Centre grant No2018/29/B/ST1/00339 (Opus).
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