STABLE EQUIMATCHABLE GRAPHS

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Abstract. A graph $G$ is equimatchable if every maximal matching of $G$ has the same cardinality. We are interested in equimatchable graphs such that the removal of any edge from the graph preserves the equimatchability. We call an equimatchable graph $G$ edge-stable if $G \setminus e$ is equimatchable for any $e \in E(G)$. After noticing that edge-stable equimatchable graphs are either 2-connected factor-critical or bipartite, we characterize edge-stable equimatchable graphs. This characterization yields an $O(\min(n^{3.376}, n^{1.5}m))$ time recognition algorithm. We also define vertex-stable equimatchable graphs and show that they admit a simpler characterization. Lastly, we introduce and shortly discuss the related notions of edge-critical and vertex-critical equimatchable graphs, pointing out the most interesting case in their characterization as an open question.

Introduction

The problem of finding an inclusion-wise maximal matching of minimum size in a graph has been a central problem for many researchers both for its practical and theoretical point of views. This minimum size is equal to the size of a minimum edge dominating set which is also widely studied [23]. One application is the following: let $A$ be a $0-1$ matrix, and consider the problem of finding a minimum set $C$ of 1’s in $A$ such that any other 1 of $A$ is in the same row or column with an element of $C$. Another application is about a telephone switching network built to route phone calls from incoming lines to outgoing trunks (assuming that a trunk can pass only one phone call at a time). The problem is to find the worst-case behavior of the network, i.e., the minimum number of routed calls when the network is saturated, thus no new call can be routed. Both applications can be modeled as the problem of finding a minimum maximal matching in a bipartite graph. However, as shown in [23], solving this problem is NP-hard even in restricted bipartite graphs. We note that finding a minimum maximal matching becomes a trivial task if all the maximal matchings of the graph under consideration have the same size. In this case, any maximal matching constructed greedily is a minimum (and maximum) one.

A graph $G$ is called equimatchable if every maximal matching of $G$ has the same cardinality. Equimatchable graphs has attracted a lot of attention in literature; they are mainly considered from structural point of view (see for instance [11, 5, 6, 11, 10, 4]). In this paper, we study equimatchable graphs from another structural perspective; we deal with the stability of this desired property of being equimatchable with respect to edge or vertex removals. Formally, we call an equimatchable graph $G$ edge-stable if $G \setminus e$ is equimatchable for each $e \in E(G)$. Edge-stable equimatchable graphs are denoted ESE-graphs as a shorthand. Similarly, an
equimatchable graph $G$ is called vertex-stable (VSE for short) if $G - v$ is equimatchable for each $v \in V(G)$.

We note that equimatchable graphs are closely related to a well-studied graph class; a graph is called well-covered if all its maximal independent sets have the same size. It is an easy observation to see that a graph $G$ is equimatchable if and only if its line graph $L(G)$ is well-covered where $L(G)$ is obtained by replacing every edge of $G$ with a vertex in $L(G)$ and where two vertices of $L(G)$ are adjacent if and only if their corresponding edges in $G$ share a common end-vertex. In other words, equimatchable graphs are well-covered line graphs. It is worth mentioning that although the recognition of well-covered graphs is co-NP-complete [2, 19], their line graphs can be recognized in polynomial time [3]. In 1979, Staples introduced the class of $W_2$ graphs, better known as 1-well-covered graphs; this class coincides with well-covered graphs that remain well-covered upon removal of any vertex [21, 20]. We note that, a graph is ESE if and only if its line graph is 1-well-covered graph. The excellent survey of Plummer on well-covered graphs dated 1992 already contains several results on 1-well-covered graphs [18]. After the study of some basic properties of 1-well-covered graphs in [21, 20], several papers, including some recent ones, studied this class. We note that like well-covered graphs, 1-well-covered graphs are also in co-NP [18], however, the complexity of their recognition is unknown to the best of our knowledge. Later on, several papers focused on subclasses of 1-well-covered graphs. In [15] and [16], Pinter gives the characterization of respectively 4-regular planar 3-connected 1-well-covered graphs, and 1-well-covered graphs which are planar and of girth 4. In [17], he also provides constructions of infinite families of 1-well-covered graphs of girth 4. Later in [7], Hartnell gives the characterization of 1-well-covered graphs with no 4-cycles. More recently, a characterization of 1-well-covered graphs where every triangle in $G$ is also a dominating set for $G$ is given in [8]. Lastly, Levit and Mandrescu give characterizations of all 1-well-covered graphs in terms of the existence of special independent sets [12]. However, these characterizations rely on the existence of independent sets with some properties that can not be checked in polynomial time. In addition, to the best of our knowledge, none of the above mentioned papers discuss the possibility of using the obtained characterizations for the development of a recognition algorithm for the related subclass of 1-well-covered graphs. In this paper, we characterize another subclass of 1-well-covered graphs which is 1-well-covered line graphs or equivalently, ESE-graphs. Our results are formulated and proved using the matching terminology and thus ESE-graphs. In addition, we show that our characterization yields an efficient recognition algorithm. We also consider some related notions such as vertex-stability, edge-criticality and vertex-criticality of equimatchable graphs.

We start in Section 1 with some definitions and preliminary results on ESE-graphs. As justified by Theorem 4 we describe ESE-graphs under three categories: 2-connected factor-critical ESE-graphs in Section 2 ESE-graphs with a cut-vertex in Section 3 (it turns out that these are all bipartite ESE-graphs), and bipartite ESE-graphs in Section 4. These results provide a full characterization of all ESE-graphs yielding an $O(\min(n^{3.376}, n^{1.5}m))$ time recognition algorithm (in Section 5) which is better than the most natural way of recognizing ESE-graphs by checking the equimatchability of $G \setminus e$ for every $e \in E$. In the same line as edge-stability, in Section 6 we study vertex-stable equimatchable graphs; it turns out that their characterization is much simpler than the characterization of ESE-graph. Lastly, in Section 7 after summarizing our findings, we introduce the opposite notions of edge-critical and vertex-critical equimatchable graphs (called respectively ECE-graphs and VCE-graphs) which are minimally equimatchable graphs with respect to edges and vertices, respectively. We conclude by giving some insight on our ongoing work about ECE-graphs and VCE-graphs.
1. Definitions and Preliminaries

Given a graph $G = (V, E)$ and a subset of vertices $I$, $G[I]$ denotes the subgraph of $G$ induced by $I$, and $G \setminus I = G[V \setminus I]$. When $I$ is a singleton $\{v\}$, we denote $G \setminus I$ by $G - v$. We also denote by $G \setminus e$ the graph $G(V, E \setminus \{e\})$. For a subset $I$ of vertices, we say that $I$ is complete to another subset $I'$ of vertices (or by abuse of notation, to a subgraph $H$) if all vertices of $I$ are adjacent to all vertices of $I'$ (respectively $H$). $K_r$ is a clique on $r$ vertices. For a vertex $v$, the neighborhood of $v$ in a subgraph $H$ is denoted by $N_H(v)$. We omit the subscript $H$ whenever it is clear from the context. For a subset $V' \subseteq V$, $N(V')$ is the union of the neighborhoods of the vertices in $V'$.

Given a graph $G$, the size of a maximum matching of $G$ is denoted by $\nu(G)$. A matching is maximal if no other matching properly contains it. A matching $M$ is said to saturate a vertex $v$ if $v$ is the endvertex of some edge in $M$, otherwise it leaves a vertex exposed. If every matching of $G$ extends to a perfect matching, then $G$ is called randomly matchable. Clearly, equimatchable graphs having a perfect matching are exactly randomly matchable graphs. These graphs have been characterized by Sumner [22].

**Lemma 1.** [22] A connected graph is randomly matchable if and only if it is isomorphic to either $K_{2r}$ or $K_{r,r}$ for some $r \geq 1$.

If $G - v$ has a perfect matching for each $v \in V(G)$, then $G$ is called factor-critical. For short, an equimatchable factor-critical graph is called an EFC-graph.

The analogous result for 1-well-covered graphs in [21] implies that $G$ is ESE with no connected components isomorphic to $K_2$ if and only if $\nu(G \setminus e) = \nu(G)$ and $G \setminus e$ is equimatchable for every $e \in E(G)$. Now, for the sake of completeness and since we heavily use this argument in our proofs, we give the proof of this result only in one direction.

**Proposition 2.** Let $G$ be an ESE-graph with no connected component isomorphic to $K_2$, then $\nu(G) = \nu(G \setminus e)$ for each $e \in E(G)$.

*Proof.* Let $G$ be an ESE-graph and assume for a contradiction that $\nu(G) = \nu(G \setminus e) + 1$ for some $e \in E(G)$. It follows that there is a maximal matching $M$ (of $G$) containing $e$ for $e = uv \in E(G)$ such that $M \setminus e$ is also a maximal matching in $G \setminus e$. Since no component of $G$ is a $K_2$, there exists $w \in N(u) \cup N(v)$, without loss of generality say $wu \in E(G)$. Moreover $wu$ can be extended to a maximal matching $M'$ of $G$, thus by equimatchability of $G$, $|M| = |M'|$. Besides, $M'$ is also a maximal matching in $G \setminus e$ of size $\nu(G)$, contradiction. □

The proof of Proposition 2 implies the following:

**Corollary 3.** Let $G$ be an ESE-graph with no connected component isomorphic to $K_2$ and $M$ be a maximal matching of $G$. Then the followings hold:

(i) $G$ has no perfect matching.

(ii) For every $e = uv \in M$, there exists a vertex $w$ exposed by $M$ which is adjacent to $u$ or $v$.

We notice that $G$ is ESE if and only if every connected component of $G$ is ESE. Therefore, in the remainder of this paper, we only consider connected ESE-graphs.

The following is a consequence of the results in [11] (although it is not explicitly mentioned in this paper) and will guide us through our characterization.

**Theorem 4.** [11] A 2-connected equimatchable graph is either factor-critical or bipartite or $K_{2t}$ for some $t \geq 1$. 

One can easily observe that the equimatchability of \( K_{2t} \) is lost when an edge is removed, hence \( K_{2t} \) is not ESE. It follows that ESE-graphs can be studied under three categories: 2-connected factor-critical, 2-connected bipartite and those having a cut vertex. Note that a bipartite graph can not be factor-critical, hence 2-connected factor-critical ESE-graphs and 2-connected bipartite ESE-graphs form a partition of 2-connected ESE-graphs into two disjoint subclasses. However, an ESE-graph with a cut-vertex can be either factor-critical, or bipartite, or none of them. Therefore a separate characterization of each one of these three categories would lead to a full characterization of all ESE-graphs with some overlaps (some graphs belonging to two categories). However, by showing in Section 2 that ESE-graphs with a cut vertex are bipartite, we provide a characterization containing only two exclusive cases: (2-connected) factor-critical ESE-graphs (Section 2) and bipartite ESE-graphs (Section 3).

2. Factor-critical ESE-graphs

Let us first underline that although we seek to characterize factor-critical ESE-graphs which are 2-connected, all the results in this section are valid for any factor-critical ESE-graph. Note that factor-critical graphs are connected but not necessarily 2-connected. However, it turns out that factor-critical ESE-graphs are also 2-connected (See Corollary 14).

Remind that a factor-critical graph \( G \) has \( \nu(G) = (|V(G)| - 1)/2 \) and if it is equimatchable then all maximal matchings have size \( (|V(G)| - 1)/2 \).

Lemma 5. Let \( G \) be a factor-critical graph. \( G \) is equimatchable if and only if there is no independent set \( S \) with 3 vertices such that \( G \setminus S \) has a perfect matching.

Proof. Let \( G \) be an EFC-graph. Then each maximal matching is of size \( (|V(G)| - 1)/2 \). If there is an independent set \( S \) with \( |S| \geq 3 \) such that \( G \setminus S \) has a perfect matching \( M \), then \( M \) is also a maximal matching in \( G \) and has size strictly less than \( (|V(G)| - 1)/2 \), a contradiction with being equimatchable.

Now, we suppose the converse. That is, for all independent set \( S \) with \( |S| \geq 3 \), \( G \setminus S \) has no perfect matching. Assume \( G \) is not equimatchable and admits therefore a maximal matching of size strictly less than \( \nu(G) = (|V(G)| - 1)/2 \). Remark that, if there is a maximal matching \( M \) such that \( |M| \leq \nu(G) - 2 \), then there is also a maximal matching \( M' \) of size \( \nu(G) - 1 \) which can be obtained from \( M \) by repetitively using augmenting chains (whose existence are guaranteed by the fact that the matching under consideration is not maximum). Hence, there are exactly 3 vertices exposed by the matching \( M' \). They form an independent set \( S \) in \( G \) and \( M' \) is a perfect matching in \( G \setminus S \). This is a contradiction. So, \( G \) is equimatchable. \( \square \)

Similarly, the following equivalence for an EFC-graph to be edge-stable will be very useful. Note that factor-critical graphs have no component isomorphic to \( K_2 \), thus Proposition 2 applies to edge-stable EFC-graphs.

Lemma 6. Let \( G \) be an EFC-graph. Then \( G \) is edge-stable if and only if there is no induced \( \overline{F_3} \) in \( G \) such that \( G \setminus \overline{F_3} \) has a perfect matching.

Proof. Let \( G \) be an EFC-graph which is also edge-stable. Assume \( G \) has an induced \( \overline{F_3} \) on vertices \( \{v, u_1, u_2\} \) with an edge between \( u_1 \) and \( u_2 \) such that \( G \setminus \{v, u_1, u_2\} \) has a perfect matching \( M \). Then, \( M \) is a maximal matching of \( G \setminus u_1u_2 \) of size one less than the matching \( M \cup \{uv\} \) of \( G \), contradicting the edge-stability of \( G \) by Proposition 2.

Now, let us consider the converse. Assume for a contradiction that \( G \) is not edge-stable. Then there is at least one edge \( u_1u_2 \) such that \( G \setminus u_1u_2 \) is not equimatchable. This means in particular that there is a maximal matching of \( G \setminus u_1u_2 \) leaving \( u_1, u_2 \) and one more vertex,
say $v$, exposed (remind that all maximal matchings of $G$ leave exactly one vertex exposed). Therefore $v$ is not adjacent to $u_1$ and $u_2$, thus $G[\{v, u_1, u_2\}] \cong \overline{P_3}$ and $M \setminus \{u_1u_2\}$ is a perfect matching of $G \setminus \{v, u_1, u_2\}$, a contradiction. \hfill $\square$

Although it is not directly related to further results, the following gives some insight about the structure of ESE-graphs which are factor-critical. Remind that $\text{diam}(G) = \max\{d(u, v) | u, v \in V(G)\}$ where $d(u, v)$ is the distance (i.e., the length of the shortest path) between vertices $u$ and $v$.

**Corollary 7.** Let $G$ be an EFC-graph. If $G$ is edge-stable, then $\text{diam}(G) \leq 2$.

**Proof.** Consider an EFC-graph $G$ which is also edge-stable. Then, for each vertex $v \in V(G)$, there is a matching $M_v$ leaving $v$ as the only exposed vertex. By Lemma 6, $v$ is adjacent to at least one of the endpoints of each edge in $M_v$, since otherwise $G$ contains an induced subgraph $\overline{P_3}$ such that $G \setminus \overline{P_3}$ has a perfect matching. Since $M_v$ saturates all vertices except $v$, we have $d(v, u) \leq 2$ for each $u \in V(G)$. By selecting $v$ arbitrarily, we have $d(v, u) \leq 2$ for every pair of vertices $u, v \in V(G)$ and therefore $\text{diam}(G) \leq 2$. \hfill $\square$

In what follows, we will be using a special decomposition of a factor-critical ESE-graph.

**Definition 8.** Let $G$ be a factor-critical ESE-graph. Then, a canonical decomposition $(N_1, N'_1, N_2)$ of $G$ with respect to a vertex $v$ and a perfect matching $M_v$ of $G - v$ is defined as follows (see Figure [1]):

- $N_1 = \{u_1, \ldots, u_t\}$ is the set of neighbors of $v$ which are matched with non-neighbors of $v$,
- $N'_1 = \{u'_1, \ldots, u'_t\}$ is the set of vertices matched to $N_1$, and
- $N_2 = \{x_1, x'_1, \ldots, x_p, x'_p\}$ is the set of neighbors of $v$ matched to other neighbors of $v$ with matching edges $\{x_1x'_1, \ldots, x_px'_p\}$,

where $V(M_v) = N_1 \cup N'_1 \cup N_2$ holds.

Indeed in a canonical decomposition of a factor-critical ESE-graph $G$ with respect to $v$ and any perfect matching $M_v$ of $G - v$, we necessarily have $V(M_v) = N_1 \cup N'_1 \cup N_2$. Note that otherwise there is an edge $yz$ such that $\{y, z\} \cap (N_1 \cup N'_1 \cup N_2) = \emptyset$ and $\{y, z\}$ induces a $\overline{P_3}$ whose removal leaves the perfect matching $M_v \setminus \{yz\}$ of $G \setminus \{v, y, z\}$, contradicting the edge-stability of $G$ by Lemma 8. Let us also emphasize that although the sets $N_1, N'_1$ and $N_2$ are defined with respect to $v$ and $M_v$, we do not adopt an additional index to denote this for the sake of simplicity.

The following lemma is an important intermediary result towards the achievement of our goal.

**Lemma 9.** Let $G$ be a factor-critical graph with at least 7 vertices. If $G$ is an ESE-graph which is not an odd clique, then there is a nontrivial independent set $S$ which is complete to $G \setminus S$.

**Proof.** Assume $G$ is a factor-critical ESE-graph with at least 7 vertices and which is not an odd clique, then let us show that $G$ contains an independent set $S$ with $|S| \geq 2$ which is complete to $G \setminus S$. Consider a canonical decomposition $(N_1, N'_1, N_2)$ of $G$ with respect to some vertex $v$ such that $d(v) < |V(G)| - 1$ and a perfect matching $M_v$ of $G - v$. As $d(v) < |V(G)| - 1$, clearly $N_1 \neq \emptyset$ and $N'_1 \neq \emptyset$.

We first note that $N_1$ is an independent set. Otherwise, there are $u_i$ and $u_j$ in $N_1$ such that $u_iu_j \in E(G)$. If $u'_iu'_j \notin E(G)$, then $\{u'_i, u'_j, v\}$ is an independent set $I$ of size 3 and $(M_v \setminus \{u_iu'_i, u_ju'_j\}) \cup \{u_iu_j\}$ is a perfect matching of $G \setminus I$, implying that $G$ is not equimatchable.
We first observe that for an edge $I$ is an independent set and $G$ has a contradiction, it follows that $N$ is false, that is, there exist $u \in N(G)$ and a perfect matching $M_u$ of $G - v$ which is shown with bold (red) edges.

by Lemma[5] If $u_i u_j' \in E(G)$, then $\{u_i', u_j', v\}$ induce a $P_3$ in $G$ such that $G \setminus P_3$ has the same perfect matching as previously, contradicting that $G$ is edge-stable by Lemma[6]. It follows that $N_1$ is an independent set.

**Claim 1.** If $N_1'$ is not an independent set then $N_2 = \emptyset$.

**Proof of the Claim.** Assume for a contradiction that $N_1'$ is not independent but $N_2 \neq \emptyset$. We first observe that for an edge $u_i u_j'$ in $N_1'$, both $u_i$ and $u_j$ is adjacent at least one of $x_1$ and $x_1'$ for an edge $x_1 x_1' \in M_v$. Clearly, $v_i$ is adjacent to $x_1$ or $x_1'$, since otherwise $G[\{v_i, x_1, x_1'\}] = P_3$ and $M_v \setminus \{x_1 x_1', u_i u_j', u_j u_j'\}$ gives a perfect matching in $G \setminus \{v_i, x_1, x_1'\}$, a contradiction by Lemma[6]. Similarly, $u_j$ is adjacent to $x_1$ or $x_1'$.

Now, there are two possible cases. Either $\{x_1, x_1'\} \subseteq N(u_i) \cup N(u_j)$, say without loss of generality $x_1 u_i, x_1' u_j \in E(G)$, then $G[\{v_i, u_i', u_j\}] = P_3$ and $(M_v \setminus \{x_1 x_1', u_i u_j', u_j u_j'\}) \cup \{x_1 u_i, x_1' u_j\}$ gives a perfect matching of $G \setminus \{v_i, u_i', u_j\}$, a contradiction by Lemma[6]. Or $\{x_1, x_1'\} \not\subseteq N(u_i) \cup N(u_j)$, and then without loss of generality $\{u_i, u_j\} \in N(x_1)$. In this case, $\{u_i, u_j, x_1'\}$ is an independent set $I$ and $(M_v \setminus \{x_1 x_1', u_i u_j', u_j u_j'\}) \cup \{x x_1, u_i u_j'\}$ gives a perfect matching in $G \setminus I$, a contradiction with $G$ being equimatchable by Lemma[5]. As both cases are concluded with a contradiction, it follows that $N_2 = \emptyset$.

**Claim 2.** $N_1'$ is complete to $N_1$.

**Proof of the Claim.** The case of $t = 1$ holds trivially. So assume that $t \geq 2$ and the claim is false, that is, there exist $u_k \in N_1$, $u'_k \in N_1'$ such that $u_k u'_k \notin E(G)$. If $u_k u'_k \notin E(G)$, then $G[\{u_k, u'_k, u'_k\}] = P_3$ and $G \setminus \{u_k, u'_k, u'_k\}$ has a perfect matching $(M_v \setminus \{u_k u'_k, u'_k u'_k\}) \cup \{x x_1, u'_k u'_k\}$, a contradiction by Lemma[3]. So $u_k u'_k \in E(G)$ and therefore $N'_1$ is not an independent set. We remark that in this case $t \geq 3$ since $|V(G)| = 2r + 1$ for $r \geq 3$ and $N_2 = \emptyset$ by Claim[1]. The followings hold by Lemma[6].
Claim 3. If $N'_1$ is not an independent set, then $N_1 = \{u_1, u_2, \ldots, u_t\}$ is a nontrivial independent set which is complete to $G \setminus N_1$.

Proof of the Claim. If $N'_1$ is not independent (thus $t \geq 2$), then $N_2 = \emptyset$ by Claim 1 and $t \geq 3$ since $G$ has at least 7 vertices. It follows from Claim 2 that $u_ku'_l \in E(G)$ for all $k, l \in [t]$. It suffices to notice that $S = N_1$ is an independent set which is complete $G \setminus S$. \\

From now on, we assume that $N'_1$ is an independent set and we have $u_ku'_l \in E(G)$ for all $k, l \in [t]$ by Claim 2. It can be observed that $N_2 \neq \emptyset$ since otherwise $G = K_{t-1,t}$ which is not factor-critical. If $ww_k' \in E(G)$ for all $w \in N_2$ and for all $k \in [t]$, then $S = N'_1 \cup \{v\}$ is an independent set complete to $G \setminus S$. So assume that there is $w \in N_2$ and $u_i' \in N'_1$ for $k \in [t]$ such that $ww_k' \notin E(G)$, let $w = x_1$ without loss of generality, that is $x_1u_k' \notin E(G)$. Note that, for every $k \in [t]$, $u'_k$ is adjacent to one of the endpoints of each edge in $\{x_1x'_1, x_2x'_2, \ldots , x_px'_p\}$; indeed if $\{x_i, x_i'\} \cap N(u'_k) = \emptyset$ for some $i \in [p]$, then $G[\{u'_k, x_i, x_i'\}] = \overline{P_3}$ and $G \setminus \{u'_k, x_i, x_i'\}$ has a perfect matching $(M_v \setminus \{x_i, x_i', u_ku'_k\}) \cup \{vv_i\}$, a contradiction. Let without loss of generality $\{x_1', x_2', \ldots , x_p'\}$ be the neighbours of $u'_k$. We remark that $x_1$ must be adjacent to each vertex of $N_1$, otherwise, let $x_1u_l \notin E(G)$ for $l \in [t]$, then $G[\{x_1, u_l, u'_k\}] = \overline{P_3}$ and $G \setminus \{x_1, u_l, u'_k\}$ has a perfect matching as $(M_v \setminus \{x_1', u'_k, u_ku'_k\}) \cup \{vx_k', u_ku'_k\}$, a contradiction. Similarly, if there is a vertex $x_i$ for $i \in [p]$, such that $x_iu'_k \notin E(G)$, then $x_i$ is adjacent to each vertex of $N_1$.

Now, we claim that $x_1, x_2, \ldots , x_p \notin N(u'_k)$. Otherwise, let $x_iu'_k \in E(G)$ for $i \neq 1$. If there is a perfect matching $P$ in $G[\{u_k, x'_1, x_i, x'_i\}]$, then consider the $\overline{P_3}$ induced by $\{v, x_i, u'_k\}$ and note that $G[\{v, x_1, u'_k\}]$ contains the perfect matching $(M_v \setminus \{x_1, x'_1, x_i, x_i', u_ku'_k\}) \cup P$, a contradiction. Assume $G[\{u_k, x_1, x_i, x_i'\}]$ has no perfect matching. Noting that $\nu(G[\{u_k, x_1, x_i, x_i'\}]) = 1$, this can only be the disjoint union of a star and (at most 2) isolated vertices, or a triangle and an isolated vertex, namely $K_{1,3}, K_3 \cup K_1, P_3 \cup K_1, P_2 \cup K_1 \cup K_1$ and each of these graphs contains either $\overline{P_3}$ or an independent set $I$ of size 3. For such $\overline{P_3}$’s and $I$’s, $G[\overline{P_3}$ or $G \setminus I$ has a perfect matching $(M_v \setminus \{x_1, x'_1, x_i, x_i', u_ku'_k\}) \cup \{vx_k, u_ku'_k\}$ where $w$ is the vertex remaining from $\{x_k, x'_1, x_i, x_i'\}$ after removing $\overline{P_3}$ or $I$ (note that $u_k$ is adjacent to each of $\{u_k, x'_1, x_i, x_i'\}$). This contradicts being ESE or equimatchable.

Remind that for every $j \in [p]$, $x_j$ is complete to $N_1$. Moreover, we now claim that for every $j \in [p]$, $x_j$ must be complete to $N(u'_k)$ (recall that $x_ju'_k \notin E(G)$, otherwise let $x_jx'_j \notin E(G)$, then $G[\{x_j, x'_j, u_ku'_k\}] = \overline{P_3}$ and $G \setminus \{x_j, x'_j, u_ku'_k\}$ has a perfect matching $(M_v \setminus \{x_jx'_j, x_i, x_i', u_ku'_k\}) \cup \{vx_j, x_iu_k\}$, a contradiction with $G$ being ESE. So $x_j$ must be complete to $N(u'_k) \setminus N_1$. As a result, for every $j \in [p]$, $x_j$ is adjacent to each vertex of $N(u'_k)$.

Besides, for every $j \in [p]$, $x_j$ has no neighbour in $N_1$ since otherwise let $x_j' \in E(G)$ then for some $k \in [p]$, $G[\{v, x_j, u_ku'_k\}]$ induces a $\overline{P_3}$ such that $G \setminus \{v, x_j, u_ku'_k\}$ has a perfect matching $(M_v \setminus \{x_jx'_j, u_ku'_k, u_ku'_k\}) \cup \{ux_j', u_ku'_k\}$, a contradiction.
Furthermore, for every \( j \in [p] \), \( x_j' \) is adjacent to each vertex of \( N'_1 \) since otherwise let \( x_j'u'_j \notin E(G) \) then \( G[x'_j, u'_j, u'_i] \) is a \( T_3 \) such that \( G \setminus \{x'_j, u'_j, u'_i\} \) has a perfect matching \((M_v \setminus \{x_jx'_j, u_ju'_j\}) \cup \{vx_j\})\). Now, we will show that any two \( x_i', x_j' \) can not be adjacent for \( i, j \in [p] \). Assume the contrary, let \( x_i'x_j' \in E(G) \) then \( G\{v, x_i, u'_i\} \cong T_3 \) and \( G \setminus \{v, x_i, u'_i\} \) has a perfect matching \((M_v \setminus \{x_ix'_i, x_jx'_j, u_ku'_k\}) \cup \{ux_jx', x_jx'_j\}\), it gives a contradiction with being ESE-graph. Hence, \( \{u_1, u_2, \ldots, u_t\} \cup \{x_1', x_2', \ldots, x_p\} \) is an independent set. On the other hand, \( x_i \) is complete to \( G \setminus (N_1 \cup N_2) \) for all \( i \in [p] \). Hence \( S = \{u_1, u_2, \ldots, u_t\} \cup \{x_1', x_2', \ldots, x_p\} \) is an independent set which is complete to \( G \setminus S \) as desired. \( \square \)

For later purpose, we need to show that there is a nontrivial independent set \( S \) complete to \( G \setminus S \) and having a special form with respect to a canonical decomposition of \( G \).

**Corollary 10.** Let \( G \) be a factor-critical graph with at least 7 vertices. If \( G \) is an ESE-graph which is not an odd clique, then for some \( v \in V(G) \) and some perfect matching \( M_v \) of \( G - v \), \( G \) has a canonical decomposition \((N_1, N'_1, N_2)\) where \( S = N'_1 \cup \{v\} \) is a nontrivial independent set which is complete to \( G \setminus S = N_1 \cup N_2 \), and \( N_1 \) is an independent set.

**Proof.** By Lemma 9 there is a nontrivial independent set \( S \) complete to \( G \setminus S \). Taking any vertex \( v \in S \), since \( G \) is factor-critical, \( G - v \) has a perfect matching \( M_v \). It is easy to see that \( M_v \) matches the vertices of \( S \setminus v \) to a subset \( S' \subset V(G \setminus S) \), since \( S \setminus v \) is independent. Moreover, both \( S \setminus v \) and \( S' \) are nonempty. We also remark that \( S' \) is an independent set, since otherwise let \( y'y' \in E(S') \) then \( y'y', z'z' \in M_v \) for \( y, z \in S \setminus v \) and \( y', z' \in S' \), then \( \{v, y, z\} \) is an independent set and \((M_v \setminus \{yy', zz'\}) \cup \{y'z'\}\) is a perfect matching on \( G \setminus \{v, y, z\}\), contradiction with equimatchability of \( G \).

Now, one can observe that \( G \) has a canonical decomposition where \( N'_1 = S \setminus v \), \( N_1 = S' \) and \( N_2 = G \setminus (S \cup S') \). Note that, \( N'_1 \) is complete to \( N_1 \cup N_2 \), and \( N_1 \) is an independent set. \( \square \)

We define two graph families \( G_1 \) and \( G_2 \) corresponding to the cases where the nontrivial independent set \( S \) described in Corollary 10 has respectively two or more vertices. A graph \( G \) belongs to \( G_1 \) if \( G \cong K_{2r+1} \setminus M \) for some nonempty matching \( M \) and \( r \geq 3 \). A graph \( G \) of \( G_1 \) is illustrated in Figure 2(a) where the edges in \( G[N_1 \cup N_2] \cong K_{2r-1} \setminus (M \setminus vu'_1) \) are not drawn, and \( S = \{v, u'_1\} \) is complete to \( G \setminus S \). Besides, \( G_2 \) is defined as the family of graphs \( G \) admitting an independent set \( S \) of size at least 3 which is complete to \( G \setminus S \) and such that \( \nu(G \setminus S) = 1 \). In Figure 2(b) we show an illustration of a graph \( G \) in \( G_2 \) where \( S = N'_1 \cup \{v\} \) with \( |S| \geq 3 \) and \( \nu(G \setminus S) = 1 \). Again, the edges in \( G[N_1 \cup N_2] \) are not drawn but just described by the property \( \nu(G \setminus S) = 1 \).

**Theorem 11.** Let \( G \) be a factor-critical graph with at least 7 vertices. Then, \( G \) is ESE if and only if either \( G \) is an odd clique or it belongs to \( G_1 \) or \( G_2 \).

**Proof.** Let \( G \) be a factor-critical graph with at least 7 vertices. It is clear that \( G \cong K_{2r+1} \) is an ESE-graph. Assume that \( G \) belongs to \( G_1 \) or \( G_2 \) and we will show that \( G \) is again ESE. First, let \( G \) be in \( G_1 \), that is \( G \cong K_{2r+1} \setminus M \) where \( M \) is a nonempty matching, then every maximal matching of \( G \) has \( r \) edges (indeed \( G \) has no independent set of size 3, hence no maximal matching of size \( r - 1 \)), thus \( G \) is equimatchable. In addition, \( G \) is \( P_3 \)-free, hence ESE by Lemma 3.

Now, let \( G \) be in \( G_2 \), then by definition of \( G_2 \), we have \( |S| \geq 3 \) and \( G \setminus S \) induces a graph whose matching number is equal to 1. Consider a vertex \( v \in S \). Since \( G \) is factor-critical,
show that $G$ has a canonical decomposition ($G$ two non-neighbors that it is not true, then, since $G$ First, let $v_1$ is in $G$ a nontrivial independent set which is complete to $G$ which do not form a matching. This means in particular that there is a vertex $v_1$ Now, let us show the converse. First assume that if $G$ is in $G$ $N_2$ $\subseteq$ $R$ $\subseteq$ $G$ $|S|$ $\subseteq$ $G$ $G$ $|$ $S$ $\subseteq$ $G$ $|$ $S$ $\neq$ $K_{2r-1}$ \ ($M \setminus vu_1$).

Figure 2. Factor-critical ESE-graph families $G_1$ and $G_2$ where the bold (red) edges illustrate a perfect matching $M_v$ of $G \setminus v$ which defines the canonical decomposition $(N_1, N_1', N_2)$.

$G \setminus v$ has a perfect matching where all vertices in $S$ are necessarily matched to some vertex in $G \setminus S$. If this perfect matching contains an edge in $G \setminus S$ then $|S| = |V(G \setminus S)| - 1$, otherwise $|S| = |V(G \setminus S)| + 1$. Applying the same argument to a vertex $v' \in G \setminus S$, we have either $|S| = |V(G \setminus S)| - 3$ (in case a perfect matching of $G - v'$ contains an edge in $G \setminus S$), or $|S| = |V(G \setminus S)| - 1$ (otherwise). It follows that the only possible case is $|V(G \setminus S)| = |S| + 1$.

We claim that $G$ is equimatchable. Since $S$ is complete to $G \setminus S$, we have either $I \subseteq S$ or $I \subseteq G \setminus S$ for independent set $I$ with 3 vertices. If $I \subseteq S$, then $G \setminus I$ has no perfect matching; indeed all vertices of $S' \setminus I$ has to be matched to vertices in $G \setminus S$, leaving 4 vertices in $G \setminus S$ exposed (because $|I| = 3$) that can not be all saturated since $\nu(G \setminus S) = 1$. Otherwise, if $I \subseteq G \setminus S$, then $G \setminus S$ has no perfect matching because there are $|S|$ vertices that can not be matched to remaining $|S| - 2$ vertices of $(G \setminus S) \setminus I$. Hence $G$ is equimatchable. Now, we claim that $G$ is edge-stable. Let $R \subset V(G)$ such that $G[R] \cong F_3$. It then follows that $R \subseteq V(G \setminus S)$ since every vertex of $S$ is adjacent to all vertices in $G \setminus S$. It can be observed that there is no perfect matching in $G \setminus R$ because there are $|S|$ vertices that can not be matched to remaining $|S| - 2$ vertices of $(G \setminus S) \setminus R$. Hence $G$ is edge-stable.

Now, let us show the converse. First assume that if $G$ is $K_{2r+1}$, it is clearly ESE. So, assume that $G$ is a factor-critical ESE-graph with $|V(G)| = 2r + 1$ and $G \not\cong K_{2r+1}$. We will show that $G$ is in $G_1$ or $G_2$, in other words, there is an independent set $S$ complete to $G \setminus S$ which satisfies the conditions. By Corollary \ref{corollary1}, for some $v \in V(G)$ and some perfect matching $M_v$ of $G - v$, $G$ has a canonical decomposition ($N_1, N_1', N_2$) where $S = N_1' \cup \{v\} = \{v, u'_1, u'_2, \ldots, u'_r\}$ is a nontrivial independent set which is complete to $G \setminus S$, and $N_1$ is an independent set (see Figure \ref{figure1}).

First, let $|S| = 2$, then $|N_1| = 1$. We note that in this case $p \geq 2$ since $2r + 1 \geq 7$. We will show that $G \cong K_{2r+1} \setminus M$, where $M$ is a matching of $K_{2r+1}$ containing $vu_1$. We suppose that it is not true, then, since $G \not\cong K_{2r+1}$, $G$ is obtained from $K_{2r+1}$ by removing an edge set which do not form a matching. This means in particular that there is a vertex $w_1$ with at least two non-neighbors $w_2$ and $w_3$. In addition, we know by Corollary \ref{corollary1} that $\{v, u'_1\}$ is complete.
to \( N_1 \cup N_2 \), implying that the missing edges are between vertices in \( N_1 \cup N_2 \). It follows that there exists \( w_1, w_2, w_3 \in N_1 \cup N_2 \) such that \( G\{w_1, w_2, w_3\} \cong P_3 \) or an independent set of size 3. Noting that \( S \) is complete to \( N_1 \cup N_2 \) and that without loss of generality, \( x_1, x'_1 \) can be considered for some \( x_1, x'_1 \) pair, the following cases cover all possibilities for \( w_1, w_2, w_3 \).

(a) If \( w_1 = u_1 \in N_1, w_2 = x_1 \in N_2, w_3 = x'_1 \in N_2 \), then \( P = \{vx_2, x'_2u'_1, x_3x'_3, \ldots, x_px'_p\} \) is a perfect matching in \( G \setminus \{w_1, w_2, w_3\} \).

(b) If \( w_1 = u_1 \in N_1, w_2 = x_1 \in N_2, w_3 = x_2 \in N_2 \), then \( P = \{vx'_1, x'_2u'_1, x_3x'_3, \ldots, x_px'_p\} \) is a perfect matching in \( G \setminus \{w_1, w_2, w_3\} \).

(c) If \( w_1 = x_1, w_2 = x_2, w_3 = x'_2 \), then \( P = \{u_1u'_1, vx'_1, x_3x'_3, \ldots, x_px'_p\} \) is a perfect matching in \( G \setminus \{w_1, w_2, w_3\} \).

(d) If \( w_1 = x_1, w_2 = x_2, w_3 = x_3 \), then there are two cases. If there is an edge \( e \in E(G\{x'_1, x'_2, x'_3\}) \), say without loss of generality \( e = x'_1x'_2 \), then \( P = \{u_1u'_1, e = x'_1x'_2, x'_3x'_4, \ldots, x_px'_p\} \) is a perfect matching in \( G \setminus \{w_1, w_2, w_3\} \). Assume now that \( G\{x'_1, x'_2, x'_3\} \) is a null graph. If one of the edges \( u_1x'_1, u_2x'_2 \) or \( u_1x'_3 \) exists, say without loss of generality \( u_1x'_1 \), then \( G \setminus \{w_1, w_2, w_3\} \) has a perfect matching \( P = \{vx'_2, u_1x'_1, u'_2x'_3, x'_4x'_4, \ldots, x_px'_p\} \). Otherwise, we can conclude in exactly the same manner as in Case (b) by considering \( \{u_1, x_1, x_2\} \) as \( \{w_1, w_2, w_3\} \).

In all these cases, we conclude by Lemma 5 and 6 that there is a contradiction, hence \( G \cong K_{2r+1} \setminus M \) where \( M \) is a matching of \( K_{2r+1} \) containing \( uu'_1 \) and consequently \( G \) belongs to \( G_1 \) (see Figure 2(a)).

Assume now \( |S| \geq 3 \). We need the following claim to show that \( G[N_1 \cup N_2] \) induces a graph whose matching number is equal to 1.

Claim 1. \(|N_2| = 2\)

Proof of the Claim. First, \(|N_2| \neq 0\), since otherwise for \( w \in N_1, G - w \) has no perfect matching (note that \( S = \{v\} \cup N_1 \) is an independent set of \( G - v \) of cardinality two more than \( (G - v) \setminus S \), hence \( G \) is not factor-critical. Now assume for a contradiction that \( p > 1 \). Note that \(|N_1| = |N'_1| = t \geq 2\) due to \(|S| \geq 3\), let \( u'_i, u'_j \in N'_1 \). First we observe that no vertex in \( N_2 \) forms an independent set with \( \{u_i, u_j\} \). Assume for a contradiction that there exists a vertex \( w \in N_2 \), say without loss of generality \( w = x_1 \) such that \( \{u_i, u_j\} \cap N(x_1) = \emptyset \), that is \( \{u_i, u_j, x_1\} \) is an independent set in \( G \), then we can obtain a perfect matching \( (M_0 \setminus \{x_1x'_1, x_2x'_2, u_iu'_1, u_ju'_j\}) \cup \{vx'_1, u'_2x'_2, u'_3x'_3, \ldots, x_px'_p\} \), a contradiction with being equimatchable. It follows that for all \( w \in N_2, N(w) \cap \{u_i, u_j\} \neq \emptyset \). If there exists \( u_i \in N_1 \) such that \( u_i \) is not adjacent to vertices \( \{x_k, x'_k\} \), say without loss of generality \( k = 1 \), then \( \{u_i, x_1, x'_1\} \) induces a \( P_3 \) and \( G \setminus \{u_i, x_1, x'_1\} \) has a perfect matching \( (M_0 \setminus \{x_1x'_1, x_2x'_2, u_iu'_1, u_ju'_j\}) \cup \{vu'_i, u'_jx'_j, u'_ix'_ix'_ix'_i\} \), contradiction with being edge-stable. Consequently, for any pair \( u_i, u_j \in N_1 \) and \( x_k, x'_k \in N_2 \), the graph induced by \( \{u_i, u_j, x_k, x'_k\} \) contains a perfect matching \( P \) since the only graphs on 4 vertices with matching number 1 are \( K_{1,3}, K_1 \cup K_1 \) and \( K_3 \cup K_1 \cup K_1 \), and \( G\{u_i, u_j, x_k, x'_k\} \) induces none of them by the above properties. Now, we notice that \( \{v, u'_i, u'_j\} \) is an independent set and \( G \setminus \{v, u'_i, u'_j\} \) has a perfect matching \( (M_0 \setminus \{x_kx'_k, u_iu'_i, u_ju'_j\}) \) \( P \). This is a contradiction with being equimatchable. So, \(|N_2| = 2\).

Let \( N_2 = \{x_1, x'_1\} \). Now, if \( \nu(G \setminus S) \geq 2 \), then there are 4 vertices \( u_i, u_j, x_1, x'_1 \) such that \( G\{u_i, u_j, x_1, x'_1\} \) has a perfect matching \( P \) (remind that \( u_iu_j \notin E(G) \) since \( N_1 \) is an independent set); now \( (M_0 \setminus \{x_1x'_1, u_iu'_i, u_ju'_j\}) \cup P \) is a maximal matching in \( G \) of size \((|V(G)| - 3)/2,\)
contradiction with equimatchability (see Figure 2(b)). Hence \( G \) belongs to \( \mathcal{G}_2 \). This completes the proof. \( \square \)

**Corollary 12.** For every \( r \geq 3 \), there are exactly \( 2r + 2 \) factor-critical ESE-graphs on \( 2r + 1 \) vertices.

**Proof.** A nonempty matching of \( K_{2r+1} \) has size between 1 and \( r \), implying that there are \( r \) non-isomorphic graphs of family \( \mathcal{G}_1 \) in Theorem 11 that is isomorphic to \( K_{2r+1} \setminus M \) for some nonempty matching \( M \). If \( G \) is of family \( \mathcal{G}_2 \) (see Figure 2(b)) then, \( \nu(G \setminus S) = 1 \) implies that \( G[N_1 \cup N_2] \) can only be a disjoint union of isolated vertices and one triangle or one star where the edge \( x_1x'_1 \) belongs to this unique triangle or star. Since all vertices of \( N_1 \cup N_2 \) are symmetric with respect to their neighborhoods outside of \( N_1 \cup N_2 \), there are exactly \( r \) ways of forming a star (note that \( N_1 \) has \( (2r + 1 - 3)/2 = r - 1 \) vertices) and just one way to form a triangle. Summing up all possibilities together being \( G \cong K_{2r+1} \), there are in total \( 2r + 2 \) factor-critical ESE-graphs. \( \square \)

**Remark 13.** We determined all factor-critical ESE-graphs whose orders are at most 5 vertices by using computer programming (Python-Sage). There are just 7 such graphs: \( K_1, K_3, K_5, C_5 \) and the following three graphs in Figure 3.

![Figure 3. Some factor-critical ESE-graphs with 5 vertices.](image)

By observing that all the graphs described in Theorem 11 and Remark 13 are 2-connected, we obtain as a byproduct the following:

**Corollary 14.** Factor-critical ESE-graphs are 2-connected.

### 3. ESE-graphs with a cut vertex

The main objective of this section is to show that ESE-graphs with a cut vertex are bipartite. Then, we will complete our characterization in the next section with bipartite ESE-graphs.

The following main result follows from Lemma 20 and Lemma 22.

**Theorem 15.** ESE-graphs with a cut vertex are bipartite.

In the sequel, we first show some results related to cut vertices. We will first extend the following known result to ESE-graphs.

**Lemma 16.** [1] Let \( G \) be a connected equimatchable graph with a cut vertex \( v \), then each connected component of \( G - v \) is also equimatchable.

**Lemma 17.** Let \( G \) be a connected ESE-graph with a cut vertex \( v \), then each connected component of \( G - v \) is also ESE.

**Proof.** Let \( v \) be a cut vertex and \( H_1, H_2, \ldots, H_k \) (\( k \geq 2 \)) be the connected components of \( G - v \). Assume \( H_i \) is not ESE for some \( i \in [k] \). However, we know from Lemma 16 that it is equimatchable. Then, there are two maximal matchings \( M_1 \) and \( M_2 \) in \( H_i \setminus w_1w_2 \) for some
$w_1w_2 \in E(H_i)$ of different sizes. Let $M$ be a maximal matching of $G \setminus H_i$ containing $uv$ where $u \in H_j$ for some $j \neq i$. Then $M_1' = M_1 \cup M$ and $M_2' = M_2 \cup M$ are maximal matchings of $G \setminus \{w_1w_2\}$ with different sizes, contradiction with equimatchability of $G \setminus \{w_1w_2\}$. \hfill $\Box$

Let us distinguish two types of vertices: a vertex $v \in V(G)$ is called strong (in $G$) if every maximal matching of $G$ saturates $v$ (or equivalently there is no maximal matching saturating all neighbors of $v$), otherwise it is called weak (in $G$).

**Lemma 18.** Let $G$ be an ESE-graph with a cut vertex $v$. Then the followings hold;

(i) if $v$ is a weak vertex in $G$, then every vertex in $N(v)$ is strong in $G - v$,

(ii) if $v$ is a strong vertex in $G$, then every vertex in $N(v)$ is weak in $G - v$ and also $G \setminus \{v, w\}$ is an ESE-graph for each $w \in N(v)$

**Proof.** Suppose that $G$ is an ESE-graph and $v$ is a cut vertex in $G$ with components $H_1, H_2, \ldots, H_k$ and $\nu(H_i) = m_i$. By Lemma 17, each component of $G - v$ is ESE.

(i) Let $v$ be a weak vertex in $G$. Then, there exists a maximal matching $M$ leaving $v$ exposed, so $M$ is the union of maximal matchings in each $H_i$, thus $|M| = \nu(G) = m_1 + m_2 + \ldots + m_k$. We claim that $N(v)$ contains only vertices which are strong in $G - v$. Otherwise, let $u \in N(v)$ be a weak vertex in $G - v$, then $uv \in E(G)$ can be extended to a maximal matching $M'$ where $|M'| = 1 + m_1 + m_2 + \ldots + m_k$ since each $H_i$ is equimatchable graph. This gives a contradiction with $G$ being equimatchable.

(ii) Assume that $v$ is a strong vertex, then $\nu(G) = \nu(G - v) + 1 = m_1 + m_2 + \ldots + m_k + 1$. Let $u \in N(v) \cap V(H_i)$ for $i \in [k]$ be a strong vertex in $G - v$, then $\nu(H_i) = \nu(H_i - u) + 1$ and consequently $\nu(G \setminus \{v, u\}) = \nu(G - v) - 1 = m_1 + m_2 + \ldots + m_k - 1$. Extending such a matching with $uv$ yields a maximal matching of $G$ of size $m_1 + m_2 + \ldots + m_k$, a contradiction. Hence $N(v)$ is a set of weak vertices. On the other hand, for every vertex $w \in N(v) \cap V(H_i)$ for $i \in [k]$, we have that $H_i - w$ is equimatchable, since otherwise $vw$ can be extended to two maximal matchings of $G$ with different sizes. In addition, $H_i - w$ is also edge-stable. Assume for a contradiction that there is an edge $e \in E(H_i - w)$ such that $(H_i - w) \setminus e$ is not equimatchable. Then, in a similar way, $vw$ can be extended to two maximal matchings of $G \setminus e$ of different sizes, contradiction with the edge stability of $G$. \hfill $\Box$

Lemma 17 and 18 together with the following well-known structural result on maximum matchings will guide us towards our objective.

**Theorem 19** (Gallai-Edmonds decomposition). \[\Box\] Let $G$ be a graph, $D(G)$ the set of vertices of $G$ that are not saturated by at least one maximum matching, $A(G)$ the set of vertices of $V(G) \setminus D(G)$ with at least one neighbor in $D(G)$, and $C(G) \overset{\text{def}}{=} V(G) \setminus (D(G) \cup A(G))$. Then:

(i) the connected components of $G[D(G)]$ are factor-critical,

(ii) $G[C(G)]$ has a perfect matching,

(iii) every maximum matching of $G$ matches every vertex of $A(G)$ to a vertex of a distinct component of $G[D(G)]$.

The following states that the Gallai-Edmonds decomposition of an equimatchable graph (which is not randomly matchable) is much more restricted. Indeed, it applies to ESE-graphs since they are equimatchable and they do not admit a perfect matching by Corollary 3(i).
Lemma 20. \[11\] Let \( G \) be a connected equimatchable graph with no perfect matching. Then \( C(G) = \emptyset \) and \( A(G) \) is an independent set of \( G \).

Now, in order to prove that ESE-graphs with a cut vertex are bipartite, it remains to show that if we use the property of being edge-stable in addition to equimatchability, each component of \( D(G) \) in the Gallai-Edmonds decomposition of an ESE-graph with a cut vertex restricts to a single vertex. We will show this in two steps.

Lemma 21. Let \( G \) be an ESE-graph and \( v \in V(G) \) be a cut vertex. Then every factor-critical component of \( G - v \) is trivial.

Proof. Let \( D \) be a factor-critical component of \( G - v \) with at least 3 vertices. We first note that since \( D \) is factor-critical, all vertices \( w \in V(D) \) are weak in \( D \). It follows from Lemma \[18\] that \( v \) is strong in \( G \) and for \( w \in N_D(v) \), \( D - w \) is ESE. However, \( D - w \) has a perfect matching since \( D \) is factor-critical, contradiction to Corollary \[3\] (i). \( \square \)

Lemma 22. Let \( G \) be an ESE-graph with a cut vertex. Then every component of \( D(G) \) in the Gallai-Edmonds decomposition is trivial.

Proof. Consider an ESE-graph \( G \) with a cut vertex. By Corollary \[14\] \( G \) is not factor-critical, moreover, it does not have a perfect matching by Corollary \[3\] (i). Consequently, \( G \) has a Gallai-Edmonds decomposition as described in Lemma \[20\] where \( A(G) \neq \emptyset \). Let \( D \) be a nontrivial component of \( D(G) \) (which is factor-critical by Theorem \[19\] (i)). By Theorem \[19\] (iii) and the equimatchability of \( G \), every maximal matching of \( G \) matches every vertex of \( A(G) \) to a vertex in a distinct component of \( G[D(G)] \). This implies that for \( w_1, w_2 \in V(D) \) and \( a_1, a_2 \in A(G) \), if \( w_1a_1 \in E(G) \) and \( w_2a_2 \in E(G) \), then we have \( w_1 = w_2 \) or \( a_1 = a_2 \). In other words, for some edge \( wa \) where \( w \in V(D) \) and \( a \in A(G) \), at least one of \( w \) and \( a \) is a cut vertex. If \( w \) is a cut vertex, each connected component of \( G - w \) must be also ESE by Lemma \[17\]. However, \( D - w \) is a connected component of \( G - w \) which has a perfect matching (remind that \( D \) is factor-critical by Theorem \[19\] (i)), thus \( D - w \) is not ESE by Corollary \[3\] (i), a contradiction. Otherwise \( a \) is a cut vertex and \( D \) is a connected component of \( G - a \) which is factor-critical by Theorem \[19\] (i); however \( D \) is not trivial, contradiction by Lemma \[21\]. \( \square \)

We obtain the main result of this section, namely Theorem \[15\] by combining Lemma \[20\] and Lemma \[22\]. Lemma \[20\] implies that an ESE-graph \( G \) has a Gallai-Edmonds decomposition where \( C(G) = \emptyset \) and \( A(G) \) is an independent set. Lemma \[22\] shows that if in addition \( G \) has a cut vertex, then every component in \( G[D(G)] \) is trivial. This structure clearly implies a bipartite graph.

4. Bipartite ESE-graphs

Having characterized all 2-connected factor-critical ESE-graphs in Section 2 and having shown that ESE-graphs with a cut vertex are bipartite (Theorem \[15\]), we now consider bipartite ESE-graphs to complete our characterization. We will see that bipartite ESE-graphs can be characterized in a way very similar to bipartite equimatchable graphs.

Lemma 23. \[11\] A connected bipartite graph \( G = (U \cup W, E) \), \( |U| \leq |W| \) is equimatchable if and only if for every \( u \in U \), there exists \( S \subseteq N(u) \) such that \( S \neq \emptyset \) and \( |N(S)| \leq |S| \).

Let us remind the well-known Hall’s Theorem in order to obtain a more intuitive reformulation of Lemma \[23\] in a bipartite graph \( G = (X \cup Y, E) \) with \( |X| \leq |Y| \), there exists a matching saturating all vertices in \( X \) if and only if for all subset \( A \subseteq X \), we have \( |N(A)| \geq |A| \). The
contrapositive of Lemma 23 states that \( G = (U \cup W, E) \), \(|U| \leq |W|\) is not equimatchable if and only if there is a vertex \( u \in U \) such that for all nonempty \( S \subseteq N(u) \), we have \( |N(S)| > |S| \). Hall’s condition applied to the bipartite subgraph induced by \( N(u) \) and \( N(N(u)) \) implies that the later condition is equivalent to the fact that there is a vertex \( u \in U \) such that there is a matching saturating all vertices of \( N(u) \), or alternatively leaving \( u \) exposed. The contrapositive of this equivalence suggests the following more intuitive statement:

**Corollary 24.** Let \( G = (U \cup W, E) \) be a connected bipartite graph with \(|U| \leq |W|\). Then \( G \) is equimatchable if and only if every vertex in \( U \) is strong (or equivalently every maximal matching of \( G \) saturates \( U \)).

Next we point out a remark which will allow us to restrict our attention to bipartite graphs with one part of the bipartition strictly less vertices than the other part.

**Remark 25.** Let \( G = (U \cup W, E) \), \(|U| \leq |W|\) be a connected bipartite ESE-graph with at least three vertices, then \(|U| < |W|\).

**Proof.** By Lemma 23 every maximal matching of \( G \) saturates \( U \). It follows that if \(|U| = |V|\) then \( G \) has a perfect matching, contradiction with Corollary 3 (i). \( \square \)

It should be noted that in addition to connected bipartite ESE-graphs with at least three vertices and \(|U| < |W|\), we also have \( K_1 \) and \( K_2 \) which are obviously bipartite ESE.

Now, let us introduce the following notion which we need for our characterization. A strong vertex \( u \) is called *square-strong* if for every \( v \in N(u) \), \( u \) is strong in \( G - v \). It follows from this definition that if \( u \) is a square-strong vertex, then for all \( v \in N(u) \), every maximal matching of \( G - v \) saturates \( u \), or equivalently, every maximal matching of \( G \) leaves at least one vertex of \( N(u) \) exposed.

The following characterizes bipartite ESE-graphs.

**Proposition 26.** Let \( G = (U \cup W, E) \) be a connected bipartite graph with \(|U| < |W|\). Then the followings are equivalent.

(i) \( G \) is ESE.

(ii) Every vertex of \( U \) is square-strong.

(iii) For every \( u \in U \), there exists nonempty \( S \subseteq N(u) \) such that \( |N(S)| \leq |S| - 1 \).

**Proof.** (i) \( \Rightarrow \) (ii) : Given a connected ESE graph \( G = (U \cup W, E) \) with \(|U| < |W|\), we suppose the converse for a contradiction with the claim. Then, there exists \( u \in U \) such that \( u \) is not square-strong. This means that there is a vertex \( v \in N(u) \) such that \( u \) is not strong in \( G - v \), that is, there is a maximal matching of \( G - v \) which leaves \( u \) exposed; let \( M' \) be such a maximal matching of \( G - v \). Then \( M' \cup \{uv\} \) is a maximal matching of \( G \) of size one more than \( M' \), contradiction with \( G \) being ESE by Proposition 2.

(ii) \( \Rightarrow \) (iii) : Assume every vertex of \( U \) is square-strong. It means that for each \( u \in U \), no matching in \( G \) saturates \( N(u) \). Therefore, Hall’s condition does not hold for \( N(u) \), and consequently there exists nonempty \( S \subseteq N(u) \) such that \( |N(S)| < |S| \).

(iii) \( \Rightarrow \) (i) : Suppose that for each \( u \in U \), there exists a nonempty set \( S \subseteq N(u) \) such that \( |N(S)| \leq |S| - 1 \). Remark that \( G \) is equimatchable by Lemma 23. It remains to show that \( G \setminus e \) is equimatchable for every \( e = w_1w_2 \in E(G) \). If the endpoints of \( e \) belong to \( S \cup \{u\} \), say \( w_1 \in S \) and \( w_2 = u \), let \( S' = S \setminus w_1 \), then we have \( |N_{G\setminus e}(S')| \leq |N_G(S')| \leq |N_G(S)| \leq |S| - 1 = |S'| \) and \( S' \subseteq N_{G\setminus e}(u) \), which implies by Lemma 23 that \( G \setminus e \) is equimatchable. For the other cases, we conclude similarly since \( |N_{G\setminus e}(S)| \leq |N_G(S)| \leq |S| \). As a result, \( G \setminus e \) is equimatchable. \( \square \)
One can reformulate Proposition 26 as follows: A connected bipartite graph $G = (U \cup W, E)$ with $|U| < |W|$ is ESE if and only if for every $u \in U$, every maximal matching of $G$ leaves at least one vertex of $N(u)$ exposed. Now, let us consider the contrapositive of the equivalence between Proposition 26 (i) and (iii). This suggests that $G$ is not ESE if and only if there exists a vertex $u \in U$ such that for every non-empty subset $S \subseteq N(u)$ we have $|N(S)| \geq |S|$. The later condition together with Hall’s Theorem implies the following characterization of graphs which are not ESE. This formulation is indeed the most convenient one for our recognition algorithm, and thus worth mentioning separately.

**Corollary 27.** A connected bipartite graph $G = (U \cup W, E)$ with $|U| < |W|$ is not ESE if and only if there exists $u \in U$ such that $N(u)$ is saturated by some (maximal) matching of $G$.

5. Recognition of ESE-graphs

The recognition of ESE-graphs is trivially polynomial since checking equimatchability can be done in time $O(n^2m)$ for a graph with $n$ vertices and $m$ edges (see [3]) and it is enough to repeat this check for every edge removal. This trivial procedure gives a recognition algorithm for ESE-graphs in time $O(n^2m^2)$. However, using the characterization of ESE-graphs, we can improve this time complexity in a significant way.

**Theorem 28.** ESE-graphs can be recognized in time $O(\min(n^{3.376}, n^{1.5}m))$.

*Proof.* Let us first note that Theorems 4 and 15 imply that an ESE-graph is either (2-connected) factor-critical or bipartite. Remark 13 exhibits all factor-critical ESE-graphs with at most 5 vertices; it is clear that one can check in linear time if the given graph is isomorphic to one of them simply by checking their degree sequences. Besides, one can also check whether a given graph with at least 7 vertices is factor-critical ESE in linear time ($O(n+m)$) using the characterization given in Theorem 11. Indeed, to decide whether $G$ is isomorphic to $K_{2r+1} \setminus M$ for some matching $M$, it is enough to check if the minimum degree is at least $2r - 1$. To decide whether $G$ admits an independent set $S$ of size at least 3 which is complete to $G \setminus S$ and $\nu(G \setminus S) = 1$, one can simply search for a connected component of the complement of $G$ which is a clique (in linear time); if yes it is the unique candidate for the set $S$. Then to check whether $\nu(G \setminus S) = 1$, it is enough to notice that a graph has matching number 1 if and only if it is the disjoint union of a triangle and isolated vertices, or the disjoint union of a star and isolated vertices. To recognize these graphs, one can check whether the degree sequence of $G \setminus S$ is one of $k, 1, \ldots, 1, 0, \ldots, 0$ (where $k \geq 1$ and there are $k$ times 1) or $2, 2, 2, 0, \ldots, 0$ where there is possibly no vertex of degree 0 at all. Clearly, these can be done in linear time.

Now, in order to decide whether a bipartite graph $G$ is ESE, we use Corollary 27. For every vertex $u \in U$ where $U$ is the small part of the bipartition, compute a maximum matching of the bipartite graph $G[N(u) \cup N(N(u))]$; if $\nu(G[N(u) \cup N(N(u))]) = |N(u)|$ for some $u \in U$, then it means that $G$ is not ESE; otherwise it is ESE. This check requires at most $n$ computation of a maximum matching in a bipartite graph, which can be done in time $O(n^{2.376})$ [14] or in time $O(\sqrt{nm})$ which runs in time $O(n^{2.5})$ in case of dense graphs but becomes near-linear for random graphs [9]. As this term dominates, it follows that the overall complexity of this recognition algorithm is $O(\min(n^{3.376}, n^{1.5}m))$. \(\square\)

6. Vertex-Stable Equimatchable Graphs

Similar to edge-stability, one can consider vertex-stability of equimatchable graphs and ask when an equimatchable graph remains equimatchable upon removal of any vertex. We
define a graph $G$ as vertex-stable equimatchable (VSE) if $G$ is equimatchable and $G - v$ is equimatchable for every $v \in V(G)$. In this section, we give a full description of VSE-graphs which turns out to be much simpler than the characterization of ESE-graphs. Like ESE-graphs, we are only interested in connected VSE-graphs since a graph is VSE if and only if every connected component of it is VSE.

If we require not only $G - v$ for some $v \in V(G)$ to remain equimatchable, but also all induced subgraphs of $G$, then this coincides with the notion of hereditary equimatchable graphs which is already studied in [4] where the following characterization is obtained:

**Theorem 29.** [4] Every induced subgraph of a connected equimatchable graph $G$ is equimatchable if and only if $G$ is isomorphic to a complete graph or a complete bipartite graph.

Since being hereditary equimatchable implies being VSE, we have the following.

**Corollary 30.** Complete graphs and complete bipartite graphs are VSE-graphs.

**Lemma 31.** Factor-critical VSE-graphs are odd cliques.

**Proof.** Assume there is a factor-critical VSE-graph $G$ which is not an odd clique. Since $G$ is factor-critical VSE, for every $u \in V(G)$, $G - u$ is equimatchable and has a perfect matching. It follows from Lemma [1] that for every $u \in V(G)$, $G - u$ is randomly matchable, which is either $K_{2r}$ or $K_{r,r}$ for some $r \geq 1$. Since $G \not\cong K_{2r+1}$ and $G$ has at least 3 vertices, there is a vertex $v \in V(G)$ such that $G - v$ is not a complete graph, therefore $G - v \cong K_{r,r}$. Note that factor-critical graphs can not be bipartite, implying that $v$ has at least one neighbour in each part of the complete bipartite graph $G - v$, say respectively $u_1$ and $w_1$. However, in this case, taking into account that $G$ is not a $K_3$, for a vertex $w \in V(G - \{v, u_1, w_1\})$ we note that $G - w$ is neither complete nor complete bipartite, contradiction. It follows that $G \cong K_{2r+1}$.

**Lemma 32.** Let $G$ be a VSE-graph which is not a complete graph. Then, $G$ is bipartite.

**Proof.** Let $G$ be a VSE-graph $G$ which is not complete. If $G$ admits a perfect matching, since $K_{2r}$ and $K_{r,r}$ are the only equimatchable graphs admitting a perfect matching (equivalently being randomly matchable, see Lemma [1]), $G$ is a $K_{r,r}$ and thus bipartite and we are done. Otherwise, $G$ does not admit a perfect matching and consequently, $G$ has a Gallai-Edmonds decomposition as described in Lemma [20] that is where $C(G) = \emptyset$ and $A(G)$ is independent. Since $G$ is different from a complete graph, by Lemma [31] it is not factor-critical and consequently we have $|A(G)| \geq 1$. We know by Theorem [19] $iii$ and the equimatchability of $G$ that every maximal matching of $G$ matches every vertex of $A(G)$ with a vertex of a distinct component of $G[D(G)]$. This implies that the number of components of $D(G)$ is greater than or equal to the number of vertices of $A(G)$. Moreover, we can show that $D(G)$ does not consist of only one component. Indeed if it was the case, then necessarily $|A(G)| = 1$ and $G$ has a perfect matching; let $a$ be the only vertex of $A(G)$ and $d \in V(D(G))$ such that $ad \in E(G)$, then $G[D(G) - d]$ has a perfect matching (because it is factor critical) which together with the edge $ad$ forms a perfect matching of $G$. Thus, $G$ is necessarily $K_{2r}$, because it can not be $K_{r,r}$ (remind that bipartite graphs are not factor-critical, thus $D(G)$ is not bipartite). This contradicts with our assumption that $G$ is not complete.

So there are at least two factor-critical components of $G[D(G)]$. Moreover, we can assume that at least one factor-critical component of $G[D(G)]$, say $D_1(G)$, has at least 3 vertices since otherwise $G$ is bipartite and we are done. Select another component of $G[D(G)]$, say $D_2(G)$, and let $a \in A(G)$ and $d_1, d_2 \in N(a)$ where $d_1 \in V(D_1(G))$ and $d_2 \in V(D_2(G))$. Let $v$ be a vertex in $D_1(G) - d_1$. Then $G - v$ has two matchings $M_1, M_2$ with different sizes obtained
respective by extending $ad_i$ to a maximal matching for $i = 1, 2$. It is a contradiction with the equimatchability of $G - v$, hence $G$ is bipartite.

Lemma 33. Let $G$ be a bipartite graph apart from $K_{t,t}$ for some $t \geq 2$. Then $G$ is VSE if and only if $G$ is ESE.

Proof. Assume that $G = (U \cup W, E)$ with $|U| \leq |W|$ is a VSE-graph. Then $G - w$ is equimatchable for every $w \in W$. It implies that every vertex of $U$ is strong in $G - w$ for each vertex $w \in W$ by Corollary 24. Therefore every vertex of $U$ is also square-strong in $G$. On the other hand $|U| \neq |W|$ since otherwise the graph $G$ has a perfect matching because every vertex of $U$ is strong, but in that case $G$ has to be $K_{t,t}$, a contradiction. Then $|U| < |W|$ and $G$ is ESE by Proposition 26.

We now assume that $G = (U \cup W, E)$ is a bipartite ESE-graph. Then by Remark 25 we have $|U| < |W|$ and by Proposition 26 every vertex of $U$ is square-strong. It means that every vertex $u \in U$ is strong in $G - w$ for each vertex $w \in N(u)$. Moreover, for every $w \notin N(u)$, every maximal matching of $G - w$ still saturate $u$ since $u$ is strong in $G$. Therefore, for every $w \in W$, $G - w$ is equimatchable by Corollary 24. Also, it is clear that $G - u$ is equimatchable for every $u \in U$, since every vertex of $U - u$ is still strong. Therefore, $G$ is also VSE.

As a consequence of Theorem 4 and Corollary 30 and Lemmas 31, 32 and 33, we have the following characterization of VSE-graphs.

Theorem 34. A graph $G$ is VSE if and only if $G$ is either a complete graph or a complete bipartite graph or a bipartite ESE-graph.

Since complete graphs can be recognized in linear time, the time complexity of recognizing VSE-graphs is equivalent to the time complexity of recognizing bipartite ESE-graphs, thus we have the following by the proof of Theorem 28.

Corollary 35. VSE-graphs can be recognized in time $O(\min(n^{3.376}, n^{1.5}m))$.

7. Concluding Remarks

Let us first summarize our findings in Figure 4. Note that complete bipartite graphs $K_{r,s}$ with $r \neq s$ are bipartite ESE-graphs; thus only bipartite VSE-graphs which are not ESE are $K_{t,t}$ for $t \geq 2$ is. As depicted in Figure 4, a natural consequence of the characterizations of ESE-graphs and VSE-graphs is the following:

Corollary 36. Let $G$ be an equimatchable graph. $G - v$ and $G \setminus e$ are equimatchable for all $v \in V(G)$ and all $e \in E(G)$ if and only if $G$ is either an odd clique or a bipartite ESE-graph.

![Figure 4. Illustration of the classes of connected ESE-graphs and VSE-graphs.](image)
graph, we say that \( e \in E(G) \) is a critical-edge if \( G \setminus e \) is not equimatchable. Note that if an equimatchable graph \( G \) is not edge-stable, then it has a critical-edge. A graph \( G \) is called edge-critical equimatchable, denoted ECE for short, if \( G \) is equimatchable and every \( e \in E(G) \) is critical. Similarly, one can introduce vertex-critical equimatchable graphs, denoted VCE-graphs for short, as equimatchable graphs which lose their equimatchability by the removal of any vertex.

We note that ECE-graphs can be obtained from any equimatchable graph by recursively removing non-critical edges. We also remark that if \( e \) is non-critical then \( \nu(G) = \nu(G - e) \) as already shown in Proposition 2. Some preliminary results show that ECE-graphs are either 2-connected factor-critical or 2-connected bipartite or \( K_{2t} \) for some \( t \geq 1 \). Moreover, connected bipartite ECE-graphs can be characterized using similar arguments as in the characterization of bipartite ESE-graphs. In order to complete the characterization of ECE-graphs, it remains to complete the case of factor-critical ECE-graphs. On the other hand, as for the VSE-graphs versus ESE-graphs, the description of VCE-graphs seems to be much simpler than the description of ECE-graphs.

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