Global existence of strong solutions with large oscillations and vacuum to the compressible nematic liquid crystal flows in 3D bounded domains

Yang Liu¹, Xin Zhong²†

¹ College of Mathematics, Changchun Normal University, Changchun 130032, P. R. China
² School of Mathematics and Statistics, Southwest University, Chongqing 400715, P. R. China

Abstract

We investigate compressible nematic liquid crystal flows in three-dimensional (3D) bounded domains with slip boundary condition for velocity and Neumann boundary condition for orientation field. By applying piecewise-estimate method and delicate analysis based on the effective viscous flux and vorticity, we derive the global existence and uniqueness of strong solutions provided that the initial total energy is suitably small. Our result is an extension of the works of Huang–Wang–Wen (J. Differential Equations 252: 2222–2265, 2012) and Li–Xu–Zhang (J. Math. Fluid Mech. 20: 2105–2145, 2018), where the local strong solutions in three dimensions and the global strong solutions for 3D Cauchy problem were established, respectively. Moreover, it also shows that blow up mechanism for local strong solutions obtained by Huang–Wang–Wen (Arch. Ration. Mech. Anal. 204: 285–311, 2012) cannot occur if the initial total energy is sufficiently small.

Key words and phrases. Compressible nematic liquid crystal flows; global strong solutions; slip boundary condition; large oscillations.

2020 Mathematics Subject Classification. 76A15; 76N10; 35Q35.

1 Introduction

Liquid crystals can form and remain in an intermediate phase of matter between liquids and solids. When a solid melts, if the energy gain is enough to overcome the positional order but the shape of the molecules prevents the immediate collapse of orientational order, liquid crystals are formed. The nematic liquid crystals exhibit long-range ordering in the sense that their rigid rod-like molecules arrange themselves with their long axes parallel to each other. Their molecules float around as in a liquid, but have the tendency to align along a preferred direction due to their orientation. The continuum theory of the nematic liquid crystals was first developed by Ericksen [5] and Leslie [15] during the period of 1958 through 1968. The rigorous mathematical analysis of the Ericksen-Leslie model was first made by Lin [18] and Lin-Liu [19–21], in which they introduced a considerably simplified version and proved the existence of global weak solutions and their partial regularities. Regarding modeling and analysis of the Ericksen-Leslie equations describing nematic liquid crystals, please refer to the survey papers [9,22,36] as well as the references therein for more discussions on the physics and mathematical results.

In this paper, let Ω ⊂ R³ be a bounded domain, we consider a simplified hydrodynamic flow modeling compressible nematic liquid crystal materials in Ω × (0, T):

\[\rho_t + \text{div}(\rho u) = 0,\]  
\[\rho u_t + \rho u \cdot \nabla u + \nabla P = \mathcal{L}u - \Delta d \cdot \nabla d,\]  
\[d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d,\]

*This research was partially supported by National Natural Science Foundation of China (No. 12271227), Scientific Research Foundation of Jilin Provincial Education Department (No. JJKH20210873KJ), and Postdoctoral Science Foundation of China (No. 2021M691219).

†Corresponding author. E-mail addresses: liuyang0405@ccsu.edu.cn (Y. Liu), xzhong1014@amss.ac.cn (X. Zhong).
where $\rho : \Omega \times (0,T) \rightarrow \mathbb{R}^+$ is the density, $u : \Omega \times (0,T) \rightarrow \mathbb{R}^3$ is the velocity field, $d : \Omega \times (0,T) \rightarrow S^2$ represents the macroscopic average of the nematic liquid crystal orientation field, $P(\rho) = a\rho^\gamma$ ($a > 0, \gamma > 1$) is the pressure, while $\mathcal{L}$ denotes the Lamé operator defined by

$$
\mathcal{L}u = (\lambda + \mu) \nabla \div u + \mu \Delta u,
$$

where $\mu$ and $\lambda$ are the shear viscosity and the bulk viscosity coefficients of the fluid, respectively, which satisfy the physical restrictions

$$
\mu > 0, \quad 3\lambda + 2\mu \geq 0.
$$

The system (1.1)–(1.3) will be studied along with the initial condition

$$(\rho, \rho u, d)(x,0) = (\rho_0, \rho_0 u_0, d_0)(x), \quad x \in \Omega,$$

and boundary condition for $(u, d)$:

$$
\begin{align*}
 u \cdot n = 0, \quad \curl u \times n &= 0, \quad \frac{\partial d}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0,T),
\end{align*}
$$

where $n = (n^1, n^2, n^3)$ is the unit outward normal vector to $\partial \Omega$.

From the mathematical point of view, system (1.1)–(1.3) is a strongly coupled system between the compressible Navier-Stokes equations (see, e.g., [3, 26]) and the transported heat flows of harmonic map (see, e.g., [29]). In the last few years, there have been substantial developments on the mathematical study for such a model. Among them, Wang and Yu [30] obtained existence and large-time behavior of a global weak solution through a three-level approximation, energy estimates, and weak convergence for the adiabatic exponent $\gamma > \frac{2}{3}$. Lin–Lai–Wang [23] established the existence of finite energy weak solutions with the large initial data provided that the initial orientational director field $d_0$ lies in $S^2$. When the initial data are of small energy and initial density is positive and essentially bounded, Wu and Tan [34] proved the global existence of a weak solution in $\mathbb{R}^3$ and established the large-time behavior of such a weak solution. At the same time, for 2D case, Jiang–Jiang–Wang [13] investigated the existence of global weak solutions in a bounded domain under a restriction imposed on the initial energy. Moreover, they [14] also obtained the global existence of finite energy weak solutions to the 2D Cauchy problem under the assumption that the second component of initial data of direction field satisfied some geometric angle condition.

On the other hand, there are some results devoting to the existence of strong solutions. Applying local well-posedness and uniform estimates for solutions of proper linearized systems with convective terms, Hu and Wu [10] showed global existence and uniqueness of strong solutions in critical Besov spaces provided that the initial data are close to an equilibrium state $(1, 0, \vec{d})$ with a constant vector $\vec{d} \in S^2$. Gao–Tao–Yao [7] investigated the global existence of classical solution under the assumption that the initial data are close to the constant equilibrium state in $H^N(\mathbb{R}^3)$ ($N \geq 3$)-framework and proved algebraic time decay for the classical solution by weighted energy method. Schade and Shibata [27] established the unique existence of local-in-time strong solutions in a uniform $W_2^{3-\frac{q}{N}}$-domain $\Omega \subseteq \mathbb{R}^N$ for $N < q < \infty$. Furthermore, if $\Omega$ is bounded and initial data are chosen suitably small, they derived global-in-time strong solutions. When the initial density allows for vacuum (i.e., the initial density may vanish in some open sets), the issue of the existence of solutions becomes much more complicated due to the possible degeneracy near vacuum. Under the condition that the initial data satisfies a compatibility condition

$$
-\mu \Delta u_0 - (\lambda + \mu) \nabla \div u_0 + \nabla P_0 + \nabla d_0 \cdot \Delta d_0 = \sqrt{\rho_0} g
$$

for some $g \in L^2$, Huang-Wang-Wen [11] obtained the short existence and uniqueness of strong solutions to the 3D initial boundary value problem with vacuum. Later on, Li–Xu–Zhang [16] established the unique global classical solutions to the 3D Cauchy problem with smooth initial data which are of small energy but possibly large oscillations with the constant state as the far-field condition. By weighted energy method, the global well-posedness of strong solutions containing vacuum in $\mathbb{R}^2$ was studied by Wang [32]. There are also other interesting studies on the compressible nematic liquid crystal flows, such as singularity formation of strong solutions [12, 33], incompressible limit [4, 31], and the optimal time-decay rates [2] and so on.
However, all of the results on global strong solutions mentioned above only concern with the whole space or with non-vacuum. It remains completely open for the global existence of strong solutions of compressible nematic liquid crystal flows with density containing vacuum initially in general bounded domains. So, motivated by the recent work due to Cai and Li [3], where the authors proved global classical solutions with vacuum and small initial energy for the 3D initial-boundary-value problem of compressible isentropic Navier-Stokes equations with slip boundary condition, the main aim of this paper is to investigate the global well-posedness of strong solutions with large oscillations and vacuum to the initial-boundary-value problem (1.1)–(1.5) when the initial energy is suitably small.

Before stating our main result, we first explain the notation and conventions used throughout the paper. We denote the initial total energy of (1.1) by

\[ E_0 \triangleq \int \left( \frac{1}{2} \rho_0 u_0^2 + \frac{1}{2} | \nabla d_0 |^2 + G(\rho_0) \right) dx, \] (1.7)

where

\[ G(\rho) \triangleq \rho \int_\rho^{\tilde{\rho}} \frac{P(\xi) - \tilde{P}}{\xi^2} d\xi, \quad \tilde{\rho} \triangleq \frac{1}{|\Omega|} \int \rho_0 dx, \quad \tilde{P} \triangleq P(\tilde{\rho}). \] (1.8)

Moreover, we write

\[ H^1_\omega \triangleq \{ v \in H^1(\Omega) : v \cdot n = 0 \text{ and } \text{curl} v \times n = 0 \text{ on } \partial \Omega \}, \]

\[ H^2_\omega \triangleq \{ v \in H^2(\Omega) : \nabla v \cdot n = 0 \text{ on } \partial \Omega \}. \]

Now we state our main result concerning global strong solutions to the problem (1.1)–(1.5).

**Theorem 1.1.** Let \( \Omega \) be a bounded simply connected smooth domain in \( \mathbb{R}^3 \) and its boundary \( \partial \Omega \) has a finite number of two-dimensional connected components. For some given constants \( M_1, M_2 > 0 \) (not necessarily small) and \( \tilde{\rho} \geq \rho \geq 1 \), suppose that the initial data \((\rho_0, u_0, d_0)\) satisfying, for \( q \in (3, 6) \),

\[
\begin{cases}
0 \leq \rho_0 \leq \tilde{\rho}, \quad (\rho_0, P(\rho_0)) \in W^{1,q}, \quad u_0 \in H^1_\omega, \quad d_0 \in H^2_\omega, \\
\| \nabla u_0 \|_{L^2}^2 \leq M_1, \quad \| \Delta d_0 \|_{L^2}^2 \leq M_2, \quad |d_0| = 1.
\end{cases}
\] (1.9)

There exists a positive constant \( \varepsilon \) depending only on \( \mu, \lambda, \gamma, a, \tilde{\rho}, \Omega, M_1, \) and \( M_2 \) such that if

\[ E_0 \leq \varepsilon, \] (1.10)

the problem (1.1)–(1.5) has a unique global strong solution \((\rho, u, d)\) in \( \Omega \times (0, \infty) \) satisfying

\[ 0 \leq \rho(x,t) \leq 2\tilde{\rho}, \quad (x,t) \in \Omega \times (0, \infty), \] (1.11)

and for any \( 0 < \tau < T < \infty \),

\[
\begin{cases}
(\rho, P) \in C([0,T]; W^{1,q}), \\
\nabla u \in C([\tau,T]; H^1) \cap L^2(\tau,T; W^{2,q}), \\
u_t \in L^2(\tau,T; H^1), \quad \sqrt{\rho} u_t \in L^\infty(\tau,T; L^2), \\
\nabla d \in C([\tau,T]; H^2) \cap L^2(\tau,T; H^3), \quad |d| = 1, \\
\quad d_t \in C([0,T]; L^2) \cap L^2(\tau,T; H^2).
\end{cases}
\] (1.12)

**Remark 1.1.** Our Theorem 1.1 generalizes the Cauchy problem [16] to the case of bounded domains. However, this is a non-trivial generalization because we need to deal with many surface integrals caused by the boundary condition (1.5). Moreover, it should be noted that there is no need to require the compatibility condition (1.6) for the global existence of strong solutions via appropriate time-weighted techniques.

**Remark 1.2.** If the initial data \((\rho_0, u_0, d_0)\) satisfies some additional regularity and compatibility conditions, the global strong solutions obtained by Theorem 1.1 become classical ones.
Remark 1.3. Compared with [3], where Cai and Li established the unique global classical solution to the compressible Navier-Stokes equations with slip boundary condition in 3D bounded domains, we cannot obtain exponential decay rates of solutions. This means that the orientation field acts some significant roles on the large-time behavior of the solutions.

We now comment on the key analysis for the proof of Theorem 1.1. As emphasized in many related papers (see, e.g., [3,16]), the key issue is to derive the uniform-in-time lower-order estimates and uniform upper bound of the density. Compared with the Cauchy problem [16], the main difficulty lies in dealing with many surface integrals (see (3.21), (3.22), and (3.33) for example) caused by the boundary condition (1.5). To overcome this obstacle, motivated by [3], we see that \((v \cdot \nabla v \cdot n)|_{\partial \Omega} = -(v \cdot \nabla n \cdot v)|_{\partial \Omega}\) for any smooth vector field \(v\) satisfying \((v \cdot n)|_{\partial \Omega} = 0\). Moreover, the trace theorem (see Lemma 2.3) and Gagliardo-Nirenberg inequality (see Lemma 2.2) as well as \(L^p\)-estimates based on the effective viscous flux (see (2.6) for the definition) and the vorticity (see Lemma 2.7) play crucial roles. Nevertheless, compared with the compressible Navier-Stokes equations [3], some additional difficulties arise due to the presence of the liquid crystal director field \(d\). Especially, the supercritical nonlinearity \(|\nabla d|^{2}d\) in the transported heat flow of harmonic map equation (1.3) and the strong coupling nonlinear term \(\Delta d \cdot \nabla d\) in the momentum equations (1.2) will cause serious difficulties in the proofs of the time-independent estimates. To this end, we need to control the weighted \(L^2\)-estimate of the second-order spatial derivatives of \(d\) (see Lemmas 3.2 and 3.3). Then we adopt the piecewise-estimate method introduced by Yu and Zhao [35], which enables us to derive the weight estimates of \(L^\infty(0,T)\)-norm of \(|\nabla u|^2_2 + |\Delta d|^2_2\) and \(|\sqrt{p}\hat{u}|^2_2 + |\nabla d|^2_2\) step by step (see (3.48) and (3.52)). Another key observation lies in the fact that the estimate of \(|\nabla \hat{u}|_2(L^2(t_1,t_2;L^2))\) can be bounded by the initial energy and the factor \(t_2 - t_1\) for any \(0 \leq t_1 < t_2 \leq T\) (see (3.53)). Once we overcome these difficulties, by virtue of Zlotnik’s inequality (see Lemma 2.9), it allows us to obtain the uniform upper bound of the density at large time provided that the initial energy is properly small (see Lemma 3.8).

The rest of the paper is arranged as follows. In Section 2, we collect some known facts and give crucial \(L^p\)-estimates involving the effective viscous flux and the vorticity. In Subsection 3.1, we make some \(a priori\) assumptions and show the uniformly \(a priori\) estimates of local strong solutions independent of the time, while the energy estimates for the higher order derivatives are obtained in Subsection 3.2. Finally, we give the proof of Theorem 1.1 in Section 4.

2 Preliminaries

In this section, we recall some known facts and inequalities which will be used later.

First of all, by time-weighted techniques used in [8] and arguments as in [11], we can obtain the following local existence theorem of strong solutions of (1.1)–(1.5). Here we omit the details for simplicity.

Lemma 2.1. Assume that the initial data \((\rho_0, u_0, d_0)\) satisfies the condition (1.9). Then there exists a positive time \(T_0 > 0\) and a unique strong solution \((\rho, u, d)\) of the problem (1.1)–(1.5) in \(\Omega \times (0,T_0]\).

Next, the well-known Gagliardo-Nirenberg inequality (see [25]) will be used frequently.

Lemma 2.2. Assume that \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^3\). For \(p \in [2,6]\), \(q \in (1,\infty)\), and \(r \in (3,\infty)\), there exist two generic constants \(C_1, C_2 > 0\), which may depend on \(p, q, r, \) and \(\Omega\) such that, for any \(f \in H^1(\Omega)\) and \(g \in L^q(\Omega) \cap W^{1,r}(\Omega)\),

\[
\|f\|_{L^p} \leq C_1 \|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3q-6}{2q}} + C_2 \|f\|_{L^2}, \quad (2.1)
\]

\[
\|g\|_{L^\infty} \leq C_1 \|g\|_{L^q}^{\frac{6(p-r)}{3q+6(q-r-3)}} \|\nabla g\|_{L^r}^{\frac{3q+6(q-r-3)}{3q}} + C_2 \|g\|_{L^2}. \quad (2.2)
\]

Moreover, if \((f \cdot n)|_{\partial \Omega} = 0\) and \((g \cdot n)|_{\partial \Omega} = 0\), then \(C_2 = 0\).

Next, the following trace theorem (see [6, p. 272]) plays an important role in dealing with the boundary integral in the next section.
Lemma 2.3. Assume that $\Omega$ is a bounded domain and $\partial \Omega$ is $C^1$. Then there exists a bounded linear operator

$$T : W^{1,p}(\Omega) \to L^p(\partial \Omega), \quad 1 \leq p < \infty$$

such that

$$Tu = u|_{\partial \Omega} \quad \text{for} \quad u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}),$$

and

$$\|Tu\|_{L^p(\partial \Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \quad \text{for} \quad u \in W^{1,p}(\Omega),$$

with the constant $C$ depending only on $p$ and $\Omega$.

The following two lemmas are given in [1, 28].

Lemma 2.4. Let $k \geq 0$ be an integer and $1 < r < \infty$. Assume that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^3$ with $C^{k+1,1}$ boundary $\partial \Omega$. Then, for $v \in W^{k+1,r}(\Omega)$ with $(v \cdot n)|_{\partial \Omega} = 0$, it holds that

$$\|v\|_{W^{k+1,r}} \leq C(\|\text{div } v\|_{W^{k,r}} + \|\text{curl } v\|_{W^{k,r}}).$$

In particular, for $k = 0$, we have

$$\|\nabla v\|_{L^r} \leq C(\|\text{div } v\|_{L^r} + \|\text{curl } v\|_{L^r}).$$

Lemma 2.5. Let $k \geq 0$ be an integer and $1 < r < \infty$. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^3$ and its $C^{k+1,1}$ boundary $\Omega$ has a finite number of two-dimensional connected components. Then, for $v \in W^{k+1,r}(\Omega)$ with $(v \times n)|_{\partial \Omega} = 0$, we have

$$\|v\|_{W^{k+1,r}} \leq C(\|\text{div } v\|_{W^{1,r}} + \|\text{curl } v\|_{W^{k,r}} + \|v\|_{L^r}).$$

In particular, if $\Omega$ is a simply connected bounded domain, then it holds that

$$\|v\|_{W^{k+1,r}} \leq C(\|\text{div } v\|_{W^{k,r}} + \|\text{curl } v\|_{W^{k,r}}).$$

When $v$ satisfies $v \cdot n = 0$ on $\partial \Omega$, we will also use the identity

$$(v \cdot \nabla)v \cdot n = -(v \cdot \nabla)n \cdot v \quad \text{on} \quad \partial \Omega \quad (2.3)$$

for any smooth vector field $v$.

The following estimates (see [3, Lemma 2.10]) on the material derivative of $u$ will be useful.

Lemma 2.6. If $(\rho, u, d)$ is a smooth solution of (1.1)–(1.5). Then there exists a positive constant $C$ depending only on $\Omega$ such that

$$\|\dot{u}\|_{L^6} \leq C(\|\nabla u\|_{L^2} + \|u\|_{L^2}^2), \quad (2.4)$$

$$\|\nabla \dot{u}\|_{L^2} \leq C(\|\text{div } u\|_{L^2} + \|\text{curl } u\|_{L^2} + \|u\|_{L^4}^2), \quad (2.5)$$

where $\dot{f} \triangleq f_t + u \cdot \nabla f$.

Next, we introduce the effective viscous flux of the system (1.1)–(1.3) as the following

$$F \triangleq (2\mu + \lambda) \text{div } u - (P - \bar{P}). \quad (2.6)$$
Lemma 2.7. Let \((\rho, u, d)\) be a strong solution of (1.1)–(1.5) in \(\Omega \times (0, T)\). Then, for any \(p \in [2, 6]\) and \(1 < r < \infty\), there exists a positive constant \(C\) depending only on \(p, r, \mu, \lambda,\) and \(\Omega\) such that

\[
\|\nabla u\|_{L^r} \leq C \left( \|\text{div} u\|_{L^r} + \|\text{curl} u\|_{L^r} \right), \\
\|\nabla F\|_{L^r} \leq C \left( \|\rho \dot{u}\|_{L^r} + \|\nabla d\|_{L^r} \right), \\
\|F\|_{L^p} \leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} \right) \frac{3p-6}{2p} \left( \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} \right)^{\frac{6-p}{2p}} \\
\quad + C \left( \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} \right),
\]

\[
\|\text{curl} u\|_{L^p} \leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} \right) \frac{3p-6}{2p} \|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2},
\]

and

\[
\|\nabla \text{curl} u\|_{L^p} \leq C \left( \|\rho \dot{u}\|_{L^p} + \|\nabla d\|_{L^p} + \|\nabla u\|_{L^2} \right).
\]

Moreover, one also has

\[
\|F\|_{L^p} + \|\text{curl} u\|_{L^p} \leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} + \|\nabla u\|_{L^2} \right),
\]

and

\[
\|\nabla u\|_{L^p} \leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} \right) \frac{3p-6}{2p} \left( \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} \right)^{\frac{6-p}{2p}} \\
\quad + C \left( \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} \right).
\]

Proof. 1. Due to (1.5), we obtain (2.7) from Lemma 2.4. Moreover, by (1.2), (1.5), and (2.6), one finds that \(F\) satisfies

\[
\begin{aligned}
\Delta F &= \text{div}(\rho \dot{u}) + \text{div} \text{div}(M(d)) \quad \text{in} \ \Omega, \\
\partial F/\partial n &= (\rho \dot{u} + \text{div}(M(d))) \cdot n \quad \text{on} \ \partial \Omega,
\end{aligned}
\]

where

\[
M(d) \triangleq \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3,
\]

\(\nabla d \odot \nabla d\) denotes a matrix whose \(ij\)th entry \((1 \leq i, j \leq 3)\) is \(\partial_i d \cdot \partial_j d\) and \(\mathbb{I}_3\) is the \(3 \times 3\) identity matrix. Then we obtain (2.8) from (2.14) and [26, Lemma 4.27].

2. By (2.6), we rewrite (1.2) as

\[
\mu \text{curl} \text{curl} u = \nabla F - \rho \dot{u} + \text{div}(M(d)),
\]

Noticing that \(\text{div}(\nabla \times \nabla u) = 0\), we get from (1.5) and Lemma 2.5 that

\[
\|\nabla \text{curl} u\|_{L^r} \leq C \left( \|\text{curl} \text{curl} u\|_{L^r} + \|\text{curl} u\|_{L^r} \right) \\
\leq C \left( \|\rho \dot{u}\|_{L^r} + \|\nabla d\|_{L^r} \right). \\
\]

One deduces from (2.1), (2.8), and (2.6) that, for \(p \in [2, 6]\),

\[
\|F\|_{L^p} \leq C \|F\|_{L^2} \frac{6-p}{2p} \|\nabla F\|_{L^2} + C \|F\|_{L^2} \\
\leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} \right) \frac{3p-6}{2p} \left( \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} \right)^{\frac{6-p}{2p}} + C \left( \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} \right).
\]

By (2.1) and (2.16), we arrive at

\[
\|\text{curl} u\|_{L^p} \leq C \|\text{curl} u\|_{L^2} \frac{6-p}{2p} \|\nabla u\|_{L^2} + C \|\text{curl} u\|_{L^2} \\
\leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} \right) \frac{3p-6}{2p} \|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2}.
\]
In this section, we will establish some necessary \( A \) priori bounds for strong solutions to the problem (1.1)–(1.5) in order to extend the local strong solutions guaranteed by Lemma 2.1. Let \( T > 0 \) be a fixed time and \((\rho, u, d)\) be a strong solution to (1.1)–(1.5) in \( \Omega \times (0,T) \) with initial data \((\rho_0, u_0, d_0)\) satisfying (1.9).
3.1 Lower-order estimates

Throughout this subsection, we will use $C$ or $C_i$ ($i = 1, 2, \cdots$) to denote the generic positive constants, which may depend on $\mu$, $\lambda$, $\gamma$, $a$, $\rho$, $\Omega$, $M_1$, $M_2$, and $\tilde{\rho}$. In particular, they are independent of $T$. Sometimes we use $C(\alpha)$ to emphasize the dependence of $C$ on $\alpha$.

Set $\sigma = \sigma(t) \triangleq \min\{1, t\}$, we define

$$A_1(T) \triangleq \sup_{0 \leq t \leq T} \left[ \sigma(t)(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \right] \quad A_2(T) \triangleq \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \quad (3.1)$$

The main aim of this subsection is to obtain the following key \textit{a priori} estimates, which give the uniform upper bound of the density.

**Proposition 3.1.** Under the conditions of Theorem 1.1, there exist positive constants $\varepsilon$ and $K$ both depending on $\mu$, $\lambda$, $\gamma$, $a$, $\rho$, $\tilde{\rho}$, $\Omega$, $M_1$, and $M_2$ such that if $(\rho, u, d)$ is a strong solution of (1.1)–(1.5) in $\Omega \times (0, T]$ satisfying

$$\sup_{\Omega \times [0, T]} \rho \leq 2\tilde{\rho}, \quad A_1(T) \leq 2E_0^{\frac{1}{2}}, \quad A_2(\sigma(T)) \leq 4K, \quad (3.2)$$

then the following estimates hold

$$\sup_{\Omega \times [0, T]} \rho \leq \frac{7}{4}\tilde{\rho}, \quad A_1(T) \leq E_0^{\frac{1}{2}}, \quad A_2(\sigma(T)) \leq 3K, \quad (3.3)$$

provided that $E_0 \leq \varepsilon$.

**Remark 3.1.** Recalling the definition of $\sigma(t)$, we then obtain from (3.2) that

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \leq C. \quad (3.4)$$

Before proving Proposition 3.1, we show some necessary \textit{a priori} estimates, see Lemmas 3.1–3.8 below.

**Lemma 3.1.** Let $(\rho, u, d)$ be a strong solution of (1.1)–(1.5) in $\Omega \times [0, T]$, then it holds that

$$\sup_{0 \leq t \leq T} \left( \frac{1}{2}\|\sqrt{\rho}u\|_{L^2}^2 + \frac{1}{2}\|\nabla d\|_{L^2}^2 + \|G(\rho)\|_{L^1} \right) + \int_0^T [\mu||\nabla u||_{L^2}^2 + (\mu + \lambda)||\nabla d||_{L^2}^2 + \|\Delta d + ||\nabla d||_{L^2}^2\|]dt \leq E_0, \quad (3.5)$$

$$\sup_{0 \leq t \leq T} (||d_t||_{L^2}^2 + ||\nabla d||_{L^2}^2) \leq C. \quad (3.6)$$

Moreover, for any integer $1 \leq i \leq [T] - 1$, one has

$$\sup_{0 \leq t \leq T} ||\rho - \bar{\rho}||_{L^2}^2 + \int_{i-1}^{i+1} (||d_t||_{L^2}^2 + ||\nabla d||_{L^2}^2)dt \leq CE_0^{\frac{1}{2}}, \quad (3.7)$$

provided that $E_0 \leq 1$.

**Proof.** 1. Due to

$$-\Delta u = -\nabla \text{div} u + \text{curl} \text{curl} u, \quad (3.8)$$

we rewrite (1.2) as

$$\rho u_t + \rho u \cdot \nabla u - (2\mu + \lambda)\nabla \text{div} u + \mu \text{curl} \text{curl} u + \nabla (P - \tilde{P}) + \Delta d \cdot \nabla d = 0. \quad (3.9)$$

Multiplying (3.9) by $u$ and (1.1) by $G'(\rho)$, respectively, summing up, and integrating the resulting equality over $\Omega$, we get that

$$\frac{d}{dt} \int \left( \frac{1}{2} |u|^2 + G(\rho) \right) dx + (2\mu + \lambda) \int (\nabla u)^2 dx + \mu \int |\text{curl} u|^2 dx \quad (3.9)$$
\[
\begin{align*}
&= - \int \text{div}(\rho u)G'(\rho)dx - \int u \cdot \nabla (P - \bar{P})dx - \int u \cdot \nabla d \cdot \Delta ddx \\
&= \int \rho u \cdot \nabla Q(\rho)dx - \int u \cdot \nabla Pdx - \int u \cdot \nabla d \cdot \Delta ddx \\
&= - \int u \cdot \nabla d \cdot \Delta ddx,
\end{align*}
\]
where we have used (3.8) and (1.5) to obtain
\[
\int \mathcal{L}u \cdot u\,dx = \int [(2\mu + \lambda)\nabla \text{div} u - \mu \text{curl curl} u \cdot u]\,dx
\]
and
\[
G'(\rho) = Q(\rho) - Q(\bar{\rho}), \quad Q'(\rho) = P'(\rho)/\rho.
\]
Multiplying (1.3) by $\Delta d + |\nabla d|^2 d$ and integrating by parts, we derive after using $|d| = 1$ and $\frac{\partial d}{\partial n}|_{\partial \Omega} = 0$ that
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int |\Delta d + |\nabla d|^2 d|^2 dx
\]
\[
= \int u \cdot \nabla d \cdot \Delta ddx + \int (|\nabla d|^2 d \cdot d_t + |\nabla d|^2 (u \cdot \nabla) d \cdot d)\,dx
\]
\[
= \int u \cdot \nabla d \cdot \Delta ddx + \frac{1}{2} \int (|\nabla d|^2 \partial_t |d|^2 + |\nabla d|^2 u \cdot |\nabla d|^2)\,dx
\]
\[
= \int u \cdot \nabla d \cdot \Delta ddx,
\]
which together with (3.10) yields (3.5).

2. Integration by parts, we deduce from (1.5), (2.3), and Lemma 2.3 that
\[
\|\Delta d\|^2_{L^2} = \sum_{i,j=1}^3 \int \partial_i d \cdot \partial_i \partial_j ddx = - \sum_{i,j=1}^3 \int \partial_i d \cdot \partial_i \partial_j ddx
\]
\[
= \sum_{i,j=1}^3 \int |\partial_{ij} d|^2 dx - \sum_{i,j=1}^3 \int \partial_i d \cdot \partial_i \partial_j dn^i dS
\]
\[
= \sum_{i,j=1}^3 \int |\partial_{ij} d|^2 dx + \sum_{i,j=1}^3 \int \partial_i d \partial_i n^i \partial_j ddS
\]
\[
\geq \|\nabla^2 d\|^2_{L^2} - C\|\nabla d\|^2_{W^{1,1}}
\]
\[
\geq \|\nabla^2 d\|^2_{L^2} - C\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2} - C\|\nabla d\|^2_{L^2}
\]
\[
\geq \frac{1}{2}\|\nabla^2 d\|^2_{L^2} - C\|\nabla d\|^2_{L^2},
\]
which combined with (3.4) and (3.5) implies that
\[
\sup_{0 \leq t \leq T} \|\nabla^2 d\|^2_{L^2} \leq C.
\]

It follows from (1.3), (2.1), and (1.5) that
\[
\|d_t\|^2_{L^2} \leq C (\|u\|_2 \|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)
\]
\[
\leq C (\|u\|_2 \|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)
\]
\[
\leq C \|\nabla u\|_2^2 \|\nabla d\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 + C \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2}^3 + C \|\Delta d\|_{L^2}^2,
\]

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which together with (3.5) and (3.12) leads to (3.6).

3. In view of (1.8), we see that there exists a positive constant $C$ depending only on $a$, $\gamma$, and $\hat{\rho}$ such that
\[ |P - \hat{P}| \leq C|\rho - \hat{\rho}|, \quad C^{-1}(\rho - \hat{\rho})^2 \leq G(\rho) \leq C(\rho - \hat{\rho})^2, \]
which along with (3.5) gives that
\[ \sup_{0 \leq t \leq T} \|\rho - \hat{\rho}\|_{L^2}^2 \leq CE_0. \] (3.13)

We derive from (1.3), (1.5), (2.1), (3.4), and (3.12) that
\[ \frac{d}{dt}\|\nabla d\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 = \int |d_t - \Delta d|^2dx \]
\[ = \int |u \cdot \nabla d - |\nabla d||^2dx \]
\[ \leq C\|u\|_{L^6}^2\|\nabla d\|_{L^6}^2 + C\|\nabla d\|_{L^4}^4 \]
\[ \leq C\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^2}^2 + C\|\nabla d\|_{L^2}\||\nabla d||_{L^2}^3 \]
\[ \leq CE_0^\frac{5}{2}, \]
which together with (3.11) and (3.5) leads to
\[ \frac{d}{dt}\|\nabla d\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \leq CE_0^\frac{5}{2}, \] (3.14)
provided that $E_0 \leq 1$. Integrating (3.14) over $[0, \sigma(T)]$ and using (3.5), we have
\[ \int_0^{\sigma(T)} (\|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)dt \leq CE_0^\frac{5}{2}. \] (3.15)

Denote $\sigma_i \triangleq \sigma(t + 1 - i)$. For any integer $1 \leq i \leq \lfloor T \rceil - 1$, multiplying (3.14) by $\sigma_i$, we arrive at
\[ \frac{d}{dt}(\sigma_i\|\nabla d\|_{L^2}^2) + \sigma_i(\|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \leq \sigma'_i\|\nabla d\|_{L^2}^2 + CE_0^\frac{5}{2}\sigma_i \leq \|\nabla d\|_{L^2}^2 + CE_0^\frac{5}{2}. \] (3.16)

Integrating (3.16) over $(i - 1, i + 1]$, we obtain (3.7) from (3.5) and (3.13).

**Lemma 3.2.** Let $(\rho, u, d)$ be a strong solution of (1.1)–(1.5) satisfying (3.2). Assume that $\eta(t) \geq 0$ is a piecewise differentiable function, then it holds that
\[ \frac{d}{dt}\left(\frac{2\mu + \lambda}{2}\eta(t)\|\text{div } u\|_{L^2}^2 + \frac{\mu}{2}\eta(t)\|\text{curl } u\|_{L^2}^2 + \eta(t)\|\Delta d\|_{L^2}^2\right) + \frac{1}{2}\eta(t)\|\nabla u\|_{L^2}^2 + \frac{1}{2}\eta(t)\|\nabla d\|_{L^2}^2 \]
\[ \leq \frac{d}{dt}\int \eta(t)(P - \hat{P})\text{div } u dx + \frac{d}{dt}\int \eta(t)M(d) : \nabla u dx + C(\eta(t) + |\eta'(t)|)\|\nabla u\|_{L^2}^2 \]
\[ + C\eta(t)\|\nabla u\|_{L^2}^3 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla d\|_{H^1}^6 + \|\nabla u\|_{L^2}^2\|\nabla d\|_{H^1}^4 \]
\[ + C|\eta'(t)|\|\Delta d\|_{L^2}^2 + C|\eta'(t)|(|\|\nabla d\|_{L^2}\|\nabla d\|_{H^1}^3 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2\|\nabla d\|_{H^1}^4 + \|\nabla u\|_{L^2}^2\|\nabla d\|_{H^1}^4 + C|\eta'(t)|E_0, \] (3.17)
provided that $E_0 \leq \varepsilon_2$.

**Proof.** 1. Multiplying (3.9) by $\eta(t)\hat{u}$ and integrating the resulting equality over $\Omega$ lead to
\[ \int \eta(t)\rho|\hat{u}|^2dx = -\int \eta(t)\hat{u} \cdot \nabla(P - \hat{P})dx + (2\mu + \lambda)\int \eta(t)\nabla \text{div } u \cdot \hat{u} dx \]
\[ - \mu\int \eta(t) \text{curl } u \cdot \hat{u} dx - \int \eta(t)|\Delta d \cdot \nabla dx \triangleq \sum_{i=1}^{4} I_i. \] (3.18)
By (1.1) and $P = \rho \gamma$, we have

$$P_t + \text{div}(Pu) + (\gamma - 1)P \text{ div } u = 0,$$

which together with integration by parts and (3.2) shows that

$$I_1 = -\int \eta(t)u_t \cdot \nabla (P - \bar{P})dx - \int \eta(t)u \cdot \nabla u \cdot \nabla Pdx$$

$$= \frac{d}{dt} \int \eta(t)(P - \bar{P})\text{div} udx - \eta'(t) \int (P - \bar{P})\text{div} udx - \int \eta(t)u \text{div} P_t dx - \int \eta(t)u \cdot \nabla u \cdot \nabla Pdx$$

$$= \frac{d}{dt} \int \eta(t)(P - \bar{P})\text{div} udx - \eta'(t) \int (P - \bar{P})\text{div} udx + \int \eta(t)u \text{div} (Pu)dx$$

$$+ (\gamma - 1) \int \eta(t)P(\text{div} u)^2 dx - \int \eta(t)u \cdot \nabla u \cdot \nabla Pdx$$

$$= \frac{d}{dt} \int \eta(t)(P - \bar{P})\text{div} udx - \eta'(t) \int (P - \bar{P})\text{div} udx + \int \eta(t)P\nabla u : \nabla udx$$

$$+ (\gamma - 1) \int \eta(t)P(\text{div} u)^2 dx - \int \eta(t)P\text{ div} u \cdot \nabla u \cdot ndS$$

$$\leq \frac{d}{dt} \int \eta(t)(P - \bar{P})\text{div} udx + C(\eta(t))||\nabla u||^2_{L^2} + ||\eta'(t)|| ||P - \bar{P}||_{L^2}||\nabla u||_{L^2}$$

$$\leq \frac{d}{dt} \int \eta(t)(P - \bar{P})\text{div} udx + C(\eta(t) + |\eta'(t)|)||\nabla u||^2_{L^2} + C|\eta'(t)||P - \bar{P}||^2_{L^2}$$

$$\leq \frac{d}{dt} \int \eta(t)(P - \bar{P})\text{div} udx + C(\eta(t) + |\eta'(t)|)||\nabla u||^2_{L^2} + C|\eta'(t)|E_0,$$

(3.20)

where we have used

$$\int \eta(t)\text{div} (Pu)\text{div} udx = -\int \eta(t)Pu^2 \partial_{j_i}u^i dx = \int \eta(t)\partial_i(\partial_j(Pu^j))\partial_j u^i dx$$

$$= -\int \Omega \eta(t)Pu \cdot \nabla u \cdot ndS + \int \Omega \eta(t)\partial_i(Pu^j)\partial_j u^i dx + \int \Omega \eta(t)P\partial_i u^j \partial_j u^i dx$$

$$= -\int \Omega \eta(t)Pu \cdot \nabla u \cdot ndS + \int \Omega \eta(t)u \cdot \nabla u \cdot \nabla Pdx + \int \Omega \eta(t)P\nabla u : \nabla u dx,$$

and

$$-\int \Omega \eta(t)Pu \cdot \nabla u \cdot ndS = \int \Omega \eta(t)Pu \cdot \nabla u \cdot ndS \leq C(\eta(t) \int_{\Omega} |u|^2 dS \leq C(\eta(t)||\nabla u||^2_{L^2},$$

(3.21)

due to (2.3), (3.2), Lemma 2.3, and (2.1). Here and in what follows, we use the Einstein convention that the repeated indices denote the summation.

2. By (1.5) and (2.3), we derive from integration by parts that

$$I_2 = (2\mu + \lambda) \int_{\partial \Omega} \eta(t)\text{div} u(\dot{u} \cdot n) dS - (2\mu + \lambda) \int \eta(t)\text{div u} d\dot{u} dx$$

$$= (2\mu + \lambda) \int_{\partial \Omega} \eta(t)\text{div} u(\dot{u} \cdot \nabla u \cdot n) dS - \frac{2\mu + \lambda}{2} \int \eta(t)(\text{div} u)^2 dx$$

$$- (2\mu + \lambda) \int \eta(t)\text{div} u\text{div} (u \cdot \nabla u) dx + \frac{2\mu + \lambda}{2} \eta(t) \int (\text{div} u)^2 dx$$

$$= -\frac{2\mu + \lambda}{2} \int \eta(t)(\text{div} u)^2 dx - (2\mu + \lambda) \int \eta(t)\text{div} u(\dot{u} \cdot \nabla n \cdot u) dS$$

$$- (2\mu + \lambda) \int \eta(t)\text{div} u\partial_i(u^j \partial_j u^i) dx + \frac{2\mu + \lambda}{2} \eta(t) \int (\text{div} u)^2 dx$$

$$= -\frac{2\mu + \lambda}{2} \int \eta(t)(\text{div} u)^2 dx - (2\mu + \lambda) \int \eta(t)\text{div} u(\dot{u} \cdot \nabla n \cdot u) dS$$
\[-(2\mu + \lambda) \int \eta(t) \text{div} w \nabla u \cdot \nabla u \, dx - (2\mu + \lambda) \int \eta(t) \text{div} \, w u^i \partial_j u^j \, dx + \frac{2\mu + \lambda}{2} \eta'(t) \int (\text{div} \, u)^2 \, dx \]
\[= -\frac{2\mu + \lambda}{2} \frac{d}{dt} \int \eta(t) (\text{div} \, u)^2 \, dx - (2\mu + \lambda) \int_{\partial \Omega} \eta(t) \text{div} \, u (u \cdot \nabla n \cdot u) \, dS \]
\[-(2\mu + \lambda) \int \eta(t) \text{div} w \nabla u \cdot \nabla u \, dx + \frac{2\mu + \lambda}{2} \int \eta(t) \text{div} \, u^3 \, dx + \frac{2\mu + \lambda}{2} \eta'(t) \int (\text{div} \, u)^2 \, dx \leq -\frac{2\mu + \lambda}{2} \frac{d}{dt} \int \eta(t) (\text{div} \, u)^2 \, dx + \frac{1}{2} \eta(t) \| \sqrt{\rho} u \|_{L^2}^2 + \delta \eta(t) \| \nabla^3 d \|_{L^2}^2 + C|\eta'(t)| \| \nabla u \|_{L^2}^2 \]
\[+ C|\eta(t)| (\| \nabla u \|_{L^3}^3 + \| \nabla u \|_{L^2}^4 + \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2), \]
where we have used
\[\int \eta(t) \text{div} w^2 \partial_j u^j \, dx = -\int \eta(t) \partial_j (\partial_k w^k) \partial_i u^j \, dx \]
\[= -\int \eta(t) \delta_{jk} w^k \partial_j u^j \, dx - \int \eta(t) \text{div} \, w \partial_j u^j \, dx \]
\[= -\int \eta(t) \delta_{jk} w^k \text{div} \, u \, dx - \int \eta(t) (\text{div} \, u)^3 \, dx, \]
and
\[\left| -(2\mu + \lambda) \int_{\partial \Omega} \text{div} \, u (u \cdot \nabla n \cdot u) \, dS \right| \]
\[= \left| -\int_{\partial \Omega} (F + (P - \bar{P}))(u \cdot \nabla n \cdot u) \, dS \right| \]
\[\leq \left| \int_{\partial \Omega} F \cdot \nabla u \cdot u \, dS \right| + \left| \int_{\partial \Omega} (P - \bar{P})(u \cdot \nabla n \cdot u) \, dS \right| \]
\[\leq C \int_{\partial \Omega} |F| \|u\|^2 \, dS + C \int_{\partial \Omega} |u|^2 \, dS \]
\[\leq C (\| \nabla F \|_{L^2} \| u \|_{L^4}^2 + \| F \|_{L^6} \| u \|_{L^4} \| \nabla u \|_{L^2} + \| F \|_{L^2} \| u \|_{L^4}^2) + C \| \nabla u \|_{L^2}^2 \]
\[\leq C \| F \|_{H^1} \| u \|_{H^1}^2 + C \| \nabla u \|_{L^2}^2 \]
\[\leq \frac{1}{2} \| \sqrt{\rho} u \|_{L^2}^2 + \delta \| \nabla^3 d \|_{L^2}^2 + C(\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla d \|_{H^1}^2), \quad (3.22) \]
due to Lemma 2.3, (2.1), Lemma 2.7, and
\[\| \nabla d \| \| \nabla^2 d \|_{L^2} \leq C \| \nabla d \|_{L^6} \| \nabla^2 d \|_{L^3} \leq C \| \nabla d \|_{H^1} \| \nabla^2 d \|_{L^3} \]
\[\leq C \| \nabla d \|_{H^1}^2 \| \nabla^2 d \|_{L^2}^2 \leq C \| \nabla d \|_{H^1}^2 \| \nabla^2 d \|_{L^2}^2 \leq C \| \nabla d \|_{H^1}^2 \| \nabla^3 d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 \] (by Lemma 2.5)
\[\leq C \| \nabla d \|_{H^1}^2 \| \nabla^3 d \|_{L^2}^2 + \| \nabla d \|_{H^1}^2. \]

3. Noting that
\[\int \text{curl} \, u \cdot (u^i \partial_i \text{curl} \, u) \, dx = -\int \text{curl} \, u \cdot (u^i \partial_i \text{curl} \, u) \, dx - \int |\text{curl} \, u|^2 \text{div} \, u \, dx. \]
Thus, we have
\[\int \text{curl} \, u \cdot (u^i \partial_i \text{curl} \, u) \, dx = -\frac{1}{2} \int |\text{curl} \, u|^2 \text{div} \, u \, dx. \]
This implies that
\[\mu \int \text{curl} \, u \cdot \nabla u \, dx = \mu \int \text{curl} \, u \cdot \nabla u \, dx. \]
which combined with (1.5) and integration by parts leads to

\[ I_3 = -\mu \int \eta(t) \nabla u \cdot \nabla \dot{u} dx \]

\[ = -\frac{\mu}{2} \frac{d}{dt} \int \eta(t) \left( \nabla u \cdot \nabla \right)^2 dx + \frac{\mu}{2} \int \nabla u \cdot \nabla \eta \right) dx \]

\[ = -\frac{\mu}{2} \frac{d}{dt} \int \eta(t) \left( \nabla u \cdot \nabla \right)^2 dx + \frac{\mu}{2} \int \nabla u \cdot \nabla \eta \right) dx \]

\[ \leq -\frac{\mu}{2} \frac{d}{dt} \int \eta(t) \left( \nabla u \cdot \nabla \right)^2 dx + C|\eta'(t)| \left( \left\| \nabla u \right\|^2_{L^2} + C\eta(t)\left\| \nabla u \right\|^2_{L^3} \right). \tag{3.23} \]

4. Noticing that

\[ I_4 = -\int \eta(t) u_t \cdot \Delta d \cdot \nabla dx - \int \eta(t) u \cdot \nabla u \cdot \Delta d \cdot \nabla dx \triangleq I_{41} + I_{42}. \]

Using (1.5), Hölder’s inequality, Sobolev’s inequality, and (2.1), we have

\[ I_{41} = \int \eta(t) M(d) : \nabla u_t dx \]

\[ = \frac{d}{dt} \int \eta(t) M(d) : \nabla u dx - \int \eta(t) M(d)_t : \nabla u dx \]

\[ \leq \frac{d}{dt} \int \eta(t) M(d) : \nabla u dx + C\eta(t)\left( \left\| \nabla u \right\|^2_{L^2} + C\eta(t)\left\| \nabla u \right\|^2_{L^3} \right) \]

\[ + \delta\eta(t)\left( \left\| \nabla u \right\|^2_{L^2} + C\eta(t)\left( \left\| \nabla u \right\|^3_{H^1} + \left\| \nabla u \right\|^3_{H^1} \right), \tag{3.24} \]

and

\[ I_{42} \leq C\eta(t)\left( \left\| u \right\|^2_{L^6} + \left\| \nabla u \right\|^2_{L^2} + \left\| \Delta d \right\|^2_{L^6} + \left\| \nabla d \right\|^2_{L^6} \right) \]

\[ \leq \delta\eta(t)\left( \left\| \nabla u \right\|^2_{L^2} + C\eta(t)\left( \left\| \nabla u \right\|^3_{H^1} + \left\| \nabla u \right\|^3_{H^1} \right) \right). \tag{3.25} \]

Combining (3.24) and (3.25), we deduce that

\[ I_4 \leq \frac{d}{dt} \int \eta(t) M(d) : \nabla u dx + \delta\eta(t)\left( \left\| \nabla d \right\|^2_{L^2} + \left\| \nabla^3 d \right\|^2_{L^2} \right) + C\eta(t)\left( \left\| \nabla d \right\|^2_{L^2} + \left\| \nabla d \right\|^2_{H^1} + \left\| \nabla u \right\|^2_{L^2} \right) \]

\[ + C\eta(t)\left( \left\| \nabla u \right\|^3_{L^3} + \left\| \nabla u \right\|^3_{H^1} + \left\| \nabla u \right\|^3_{H^1} + \left\| \nabla u \right\|^3_{H^1} \right). \]

Putting the above estimates on \( I_i \) (\( i = 1, 2, 3, 4 \)) into (3.18), one obtains that

\[ \frac{d}{dt} (\frac{2\mu + \lambda}{2} \eta(t) \left\| \nabla u \right\|^2_{L^2} \frac{\eta(t)}{\left\| \nabla u \right\|^2_{L^2}} + \frac{\mu}{2} \eta(t) \left\| \nabla u \right\|^2_{L^2}) + \frac{1}{2} \eta(t) \left\| \nabla u \right\|^2_{L^2} \]

\[ \leq \frac{d}{dt} \int \eta(t) (P - P) \nabla u dx + \frac{d}{dt} \int \eta(t) M(d) : \nabla u dx + \delta\eta(t)\left( \left\| \nabla^3 d \right\|^2_{L^2} + \left\| \nabla d \right\|^2_{L^2} \right) \]

\[ + C\eta(t)\left( \left\| \nabla d \right\|^2_{L^2} + \left\| \nabla d \right\|^2_{H^1} + \left\| \nabla u \right\|^2_{L^2} \right) \]

\[ + C\eta(t)\left( \left\| \nabla u \right\|^3_{L^3} + \left\| \nabla u \right\|^3_{H^1} + \left\| \nabla u \right\|^3_{H^1} + \left\| \nabla u \right\|^3_{H^1} \right). \]
+ C\eta(t) (\|\nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla d\|^2_{H^1} + \|\nabla d\|^2_{H^1})
+ C\eta(t)\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{H^1} + C|\eta'(t)|E_0.
\tag{3.26}

5. It remains to estimate \|\nabla d_t\|^2_{L^2}. To this end, applying the operator \nabla to (1.3) gives that
\begin{equation}
\nabla d_t - \nabla \Delta d = -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d).
\end{equation}
Multiplying (3.27) by \nabla d_t and integrating by parts, we find that
\begin{equation}
\frac{d}{dt} \int |\Delta d|^2 dx + \int |\nabla d_t|^2 dx = \int (\nabla (|\nabla d|^2 d) - \nabla (u \cdot \nabla d)) \nabla d_t dx
\leq \frac{1}{\delta} \|\nabla d_t\|^2_{L^2} + C \int (|\nabla d|^2 |\nabla^2 d|^2 + |\nabla d|^6 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2) dx
\leq \delta \|\nabla d_t\|^2_{L^2} + C\|\nabla d\|^2_{L^2}\|\nabla d\|^2_{L^2} + C\|\nabla d\|^2_{H^1} + C\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{L^2} + C\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{L^2} + C\|\nabla u\|^4_{L^2}\|\nabla d\|^2_{L^2} + C\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{H^1} + C\|\nabla d\|^6_{H^1},
\end{equation}
which implies that
\begin{equation}
\frac{d}{dt} (\eta(t) \|\Delta d\|^2_{L^2}) + C\eta(t)\|\nabla d_t\|^2_{L^2}
\leq \eta'(t) \|\Delta d\|^2_{L^2} + \frac{1}{4}\eta(t)\|\nabla d_t\|^2_{L^2} + (CE_0^2 + \delta)\eta(t)\|\nabla^3 d\|^2_{L^2} + C\eta(t)\|\nabla d\|^2_{H^1}
+ C\eta(t)\|\nabla u\|^2_{L^2}\|\nabla^2 d\|^2_{L^2} + C\eta(t)\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{L^2}
+ C\eta(t)\|\nabla u\|^2_{L^2} + C\eta(t)\|\nabla d\|^2_{H^1} + C\eta(t)\|\nabla d\|^6_{H^1}.
\end{equation}
Applying the \(L^2\)-theory to the Neumann boundary value problem of elliptic equations (see [17]), we infer from (3.27), (3.5), and (3.6) that
\begin{equation}
\begin{aligned}
\|\nabla^3 d\|^2_{L^2} &\leq C\|\nabla \Delta d\|^2_{L^2} + \|\nabla d\|^2_{H^1},
\leq C\|\nabla d_t\|^2_{L^2} + C\|\nabla (u \cdot \nabla d\)\|^2_{L^2} + C\|\nabla (|\nabla d|^2 d)\|^2_{L^2} + C\|\nabla d\|^2_{H^1},
\leq C\|\nabla d_t\|^2_{L^2} + (CE_0^2 + \frac{1}{4})\|\nabla^3 d\|^2_{L^2} + C\|\nabla d\|^2_{H^1} + C\|\nabla u\|^2_{L^2}\|\nabla^2 d\|^2_{L^2} + C\|\nabla u\|^4_{L^2}\|\nabla d\|^2_{L^2},
\end{aligned}
\end{equation}
which leads to
\begin{equation}
\|\nabla^3 d\|^2_{L^2} \leq C\|\nabla d_t\|^2_{L^2} + C\|\nabla d\|^2_{H^1} + C\|\nabla u\|^2_{L^2}\|\nabla^2 d\|^2_{L^2} + C\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{L^2},
\end{equation}
provided that \(E_0 \leq \varepsilon_1\) is suitably small. Substituting (3.29) into (3.28), one has
\begin{equation}
\begin{aligned}
\frac{d}{dt} (\eta(t) \|\Delta d\|^2_{L^2}) + \frac{1}{2}\eta(t)\|\nabla d_t\|^2_{L^2}
\leq \eta'(t) \|\Delta d\|^2_{L^2} + C\eta(t)\|\nabla u\|^2_{L^2}\|\nabla^2 d\|^2_{L^2} + C\eta(t)\|\nabla u\|^2_{L^2}\|\nabla d\|^2_{L^2}
+ C\eta(t)\|\nabla u\|^2_{L^2} + C\eta(t)\|\nabla d\|^2_{H^1} + C\eta(t)\|\nabla d\|^6_{H^1}.
\end{aligned}
\end{equation}
This together with (3.26) and (3.29) gives (3.17) after choosing \(E_0 \leq \varepsilon_2 \leq \varepsilon_1\) and \(\delta\) sufficiently small. 
\end{proof}

\begin{lemma}
Let \((\rho, u, d)\) be a strong solution of (1.1)–(1.5) satisfying (3.2) and \(\eta(t)\) be as in Lemma 3.2, then it holds that
\begin{equation}
\frac{d}{dt} \left( \frac{\eta(t)}{2}\|\sqrt{\rho}u\|^2_{L^2} + \frac{\eta(t)}{2}\|\nabla d\|^2_{L^2} \right) + (2\mu + \lambda)\eta(t)\|\text{div }u\|^2_{L^2} + \mu\eta(t)\|\text{curl }u\|^2_{L^2} + \eta(t)\|d_t\|^2_{L^2}
\end{equation}
\end{lemma}
\[
\begin{align*}
\leq & -C \frac{d}{dt} \int_{\partial \Omega} \eta(t)(u \cdot \nabla n \cdot u) F dS + C|\eta'(t)| (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4 + E_0) \\
+ & C|\eta'(t)| (\|\sqrt{\mu} \hat{\rho} u\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 + \|\nabla^3 d\|_{L^2}^4 + C E_0^2 \|\nabla^2 d\|_{L^2}^4 + \|\nabla d_t\|_{L^2}^4) \\
+ & C \eta(t) (\|\sqrt{\rho} \hat{\rho} u\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 \|\nabla u\|_{L^2}^4) \\
+ & C \eta(t) (\|\nabla u\|_{L^2}^4 + C E_0^2 \|\nabla u\|_{L^2}^4 + \|\nabla d_t\|_{L^2}^4) \\
+ & C \eta(t) (\|\nabla d_t\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \\
+ & C \eta(t) (\|\nabla u\|_{L^2}^4 \|\nabla d_t\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 \|\nabla u\|_{L^2}^4) + C \eta(t) (\|\nabla d_t\|_{L^2}^4 + \|\nabla d_t\|_{L^2}^4),
\end{align*}
\] (3.30)

provided that \( E_0 \leq \varepsilon_3 \).

**Proof.** 1. By (2.6) and (3.8), we rewrite (1.2) as

\[
\rho \hat{u} = \nabla F - \mu \text{curl} \, \text{curl} \, u - \Delta d \cdot \nabla d.
\] (3.31)

Applying \( \eta(t) \hat{u}|^j \partial/\partial t + \text{div}(u \cdot \cdot) \) to the \( j \)-th component of (3.31), and then integrating the resulting equality over \( \Omega \), we get that

\[
\frac{1}{2} \left( \frac{d}{dt} \int \eta(t) |\hat{u}|^2 dx - \eta'(t) \int \rho |\hat{u}|^2 dx \right) = \int \eta(t) \left( \hat{u} \cdot \nabla F_1 + \hat{u}^j \text{div} (u \hat{\omega}_j F) \right) dx \\
- \mu \int \eta(t) \left( \hat{u} \cdot \text{curl} \, \text{curl} \, u_t + \hat{u}^j \text{div} \left( (\text{curl} \, \text{curl} \, u) \hat{u}^j \right) \right) dx \\
+ \int \eta(t) (M(d) : \nabla u_t + \text{div} \, M(d) \cdot (u \cdot \nabla \hat{u})) dx = \sum_{i=1}^3 J_i.
\] (3.32)

We denote by \( h \triangleq u \cdot (\nabla n + (\nabla n)^T) \) and \( u^+ \triangleq -u \times n \), then it deduces from Lemma 2.3 that

\[
- \int_{\partial \Omega} \eta(t) F_1 (u \cdot \nabla n \cdot u) dS \\
= -\frac{d}{dt} \int_{\partial \Omega} \eta(t) (u \cdot \nabla n \cdot u) F dS + \int_{\partial \Omega} \eta(t) F h \cdot u t dS - \int_{\partial \Omega} \eta(t) F h \cdot (u \cdot \nabla u) dS \\
+ \eta'(t) \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F dS \\
= -\frac{d}{dt} \int_{\partial \Omega} \eta(t) (u \cdot \nabla n \cdot u) F dS + \int_{\partial \Omega} \eta(t) F h \cdot u t dS + \eta'(t) \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F dS \\
- \int_{\partial \Omega} \eta(t) F h (\nabla u^+ \times u^+) \cdot ndS \\
= -\frac{d}{dt} \int_{\partial \Omega} \eta(t) (u \cdot \nabla n \cdot u) F dS + \int_{\partial \Omega} \eta(t) F h \cdot u t dS + \eta'(t) \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F dS \\
- \int_{\partial \Omega} \eta(t) \nabla u^+ \times u^+ \cdot \nabla (F h) \, dx + \int \eta(t) F h \nabla \times u^+ \cdot \nabla u \, dx \\
\leq -\frac{d}{dt} \int_{\partial \Omega} \eta(t) (u \cdot \nabla n \cdot u) F dS + C \eta(t) \|\nabla F\|_{L^2} \|u\|_{L^3} \|\hat{u}\|_{L^6} \\
+ C \eta(t) (\|F\|_{L^3} \|u\|_{L^6} \|\nabla \hat{u}\|_{L^2} + \|F\|_{L^3} \|u\|_{L^6} \|\hat{u}\|_{L^2} + \|F\|_{L^3} \|\nabla u\|_{L^2} \|\hat{u}\|_{L^6}) \\
+ C \eta(t) (\|\nabla u\|_{L^2} \|\hat{u}\|_{L^6} \|\nabla F\|_{L^6} + \|\nabla u\|_{L^2} \|u\|_{L^6} \|F\|_{L^3}) \\
+ \eta'(t) \|\nabla u\|_{L^2} \|u\|_{L^6} \|F\|_{L^3} \|u\|_{L^2} + \|\nabla F\|_{L^2} \|u\|_{L^6}^2) \\
\leq -\frac{d}{dt} \int_{\partial \Omega} \eta(t) (u \cdot \nabla n \cdot u) F dS + C |\eta'(t)| \|\nabla u\|_{L^2} (\|\rho \hat{u}\|_{L^2} + \|\nabla d\|_{L^6} \|\nabla^2 d\|_{L^3} + \|\nabla u\|_{L^2} + \|P - P\|_{L^2}) \\
+ C \eta(t) (\|\rho \hat{u}\|_{L^2} + \|\nabla d\|_{L^6} \|\nabla^2 d\|_{L^3} + \|\nabla u\|_{L^2} + \|P - P\|_{L^2}) \|\nabla u\|_{L^2} (\|\nabla \hat{u}\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)
\begin{align*}
&+ C\eta(t)\|\nabla u\|_{L^2}^2 \|\nabla F\|_{L^6} \\
\leq -\frac{d}{dt} \int_{\Omega} \eta(t)(u \cdot \nabla n \cdot u) FdS + C\eta'(t)(\|\sqrt{\mu} \nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + CE_0^2 \|\nabla^2 d\|_{L^2}^2) \\
&+ C\eta'(t)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + E_0) + \delta(t)(\|\nabla u\|_{L^2}^2 + \|\nabla F\|_{L^6}^2) \\
&+ C\eta(t)(\|\sqrt{\mu} \nabla u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) \\
&+ C\eta(t)(\|\nabla u\|_{L^2}^4 + E_0^2 \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^6 + \|\nabla^2 d\|_{L^2}^2 \|\nabla u\|_{L^2}^2),
\end{align*}
\tag{3.33}

due to

$$
\nabla \cdot \nabla u' = u' \cdot \nabla u' - \nabla u' \cdot \nabla u' = -\nabla u' \cdot \nabla u',
$$
\|\nabla u\|_{L^6} \leq C(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}), \text{ (see \eqref{2.4}),}

and

\begin{align*}
&\|\nabla d\|_{L^6} \|\nabla^2 d\|_{L^3} \leq C(\|\nabla d\|_{L^2} + \|\nabla^2 d\|_{L^2})(\|\nabla^2 d\|_{L^2}^\frac{3}{4} \|\nabla^3 d\|_{L^2}^\frac{1}{4} + \|\nabla^2 d\|_{L^2}) \\
&\leq CE_0^\frac{3}{4} \|\nabla^2 d\|_{L^2}^\frac{3}{4} \|\nabla^3 d\|_{L^2}^\frac{1}{4} + C\|\nabla^2 d\|_{L^2}^3 \|\nabla^3 d\|_{L^2}^\frac{1}{4} + CE_0.
\end{align*}
\tag{3.34}

Thus, it follows from integration by parts, \eqref{1.5}, \eqref{2.3}, \eqref{2.2}, Hölder’s inequality, \eqref{2.1}, Lemma 2.7, and \eqref{3.33} that

\begin{align*}
J_1 &= \int_{\Omega} \eta(t) F_i \hat{u} \cdot ndS - \int_{\Omega} \eta(t) F_i \nabla \hat{u} \cdot \nabla Fdx \\
&= -\int_{\Omega} \eta(t) F_i (u \cdot \nabla n \cdot u) ds - (2\mu + \lambda) \int_{\Omega} \eta(t) (\text{div } \hat{u}^2) dx + (2\mu + \lambda) \int_{\Omega} \eta(t) \text{div } \hat{u} \nabla u \cdot \nabla u dx \\
&- \gamma \int_{\Omega} \eta(t) P \text{div } \hat{u} \cdot u dx + \int_{\Omega} \eta(t) \text{div } \hat{u} \nabla F dx - \int_{\Omega} \eta(t) u \cdot \nabla \hat{u} \cdot \nabla F dx \\
&\leq -\int_{\Omega} \eta(t) F_i (u \cdot \nabla n \cdot u) ds - (2\mu + \lambda) \int_{\Omega} \eta(t) (\text{div } \hat{u}^2) dx + \delta(t) \|\nabla \hat{u}\|_{L^2}^2 \\
&+ C(\delta(t) \|\nabla u\|_{L^2}^2 \|\nabla F\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^2) \\
&\leq -\int_{\Omega} \eta(t) F_i (u \cdot \nabla n \cdot u) ds - (2\mu + \lambda) \int_{\Omega} \eta(t) (\text{div } \hat{u}^2) dx + \delta(t) \|\nabla \hat{u}\|_{L^2}^2 \\
&+ C\eta(t)(\|\nabla u\|_{L^2}^2 \|\nabla F\|_{L^6} + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^2) \\
&\leq -\int_{\Omega} \eta(t) F_i (u \cdot \nabla n \cdot u) ds - (2\mu + \lambda) \int_{\Omega} \eta(t) (\text{div } \hat{u}^2) dx + \delta(t)(\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla F\|_{L^6}^2) \\
&+ C\eta(t)(\|\nabla u\|_{L^2}^2 \|\nabla F\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^2) \\
&\leq -\frac{d}{dt} \int_{\Omega} (u \cdot \nabla n \cdot u) FdS - (2\mu + \lambda) \int_{\Omega} \eta(t) (\text{div } \hat{u}^2) dx + \delta(t)(\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla F\|_{L^6}^2) \\
&+ C\eta'(t)(\|\sqrt{\mu} \nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + CE_0^2 \|\nabla^2 d\|_{L^2}^2) \\
&+ C\eta'(t)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) \\
&+ C\eta(t)(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2) \\
&+ C\eta(t)(\|\nabla u\|_{L^2}^2 + E_0^2 \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^6 + \|\nabla^2 d\|_{L^2}^2 \|\nabla u\|_{L^2}^2) \\
&\leq -\frac{d}{dt} \int_{\Omega} (u \cdot \nabla n \cdot u) FdS - (2\mu + \lambda) \int_{\Omega} \eta(t) (\text{div } \hat{u}^2) dx + \delta(t)(\|\nabla \hat{u}\|_{L^2}^2) \\
&+ C\eta'(t)(\|\sqrt{\mu} \nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + CE_0^2 \|\nabla^2 d\|_{L^2}^2) \\
&+ C\eta'(t)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) \\
&+ C\eta(t)(\|\nabla u\|_{L^2}^2 + E_0^2 \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^6 + \|\nabla^2 d\|_{L^2}^2 \|\nabla u\|_{L^2}^2) \\
&+ C\eta(t)(\|\nabla^2 d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^3 \|\nabla^2 d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2),
\end{align*}
\tag{3.35}
where we have used

\[
F_t = (2\mu + \lambda)\text{div } u_t - (P - \bar{P})_t
= (2\mu + \lambda)\text{div } \dot{u} - (2\mu + \lambda)\text{div } (u \cdot \nabla u) + u \cdot \nabla P + \gamma P \text{div } u
= (2\mu + \lambda)\text{div } \dot{u} - (2\mu + \lambda)u \cdot \nabla \text{div } u - (2\mu + \lambda)\nabla u : \nabla u + u \cdot \nabla P + \gamma P \text{div } u
= (2\mu + \lambda)\text{div } \dot{u} - (2\mu + \lambda)\nabla u : \nabla u + \gamma P \text{div } u - u \cdot \nabla F,
\]

and

\[
\|\nabla F\|_{L^6} \leq C\left(\|\rho \dot{u} \|_{L^6} + \|\nabla d\|_{L^6} \|\nabla^2 d\|_{L^6}\right)
\leq C(\dot{\rho})\left(\|\dot{u} \|_{L^6} + \|\nabla d\|_{L^\infty} \|\nabla^2 d\|_{L^6}\right)
\leq C\left(\|\nabla^2 d\|_{L^2}^{\frac{3}{2}} \|\nabla^3 d\|_{L^2}^{\frac{1}{2}} + \|\nabla^2 d\|_{L^2} \right)\left(\|\nabla^3 d\|_{L^2} + \|\nabla^2 d\|_{L^2} + \|\nabla^2 d\|_{L^2}^{\frac{1}{2}}\right)
\leq C\left(\|\nabla^2 d\|_{L^2}^{\frac{3}{2}} \|\nabla^3 d\|_{L^2}^{\frac{1}{2}} + \|\nabla d\|_{L^2} + \|\nabla u\|_{L^2}^2\right).
\]

2. By a direct calculation, one obtains that

\[
J_2 = -\mu \int \eta(t)\dot{u} \cdot (\text{curl } \text{curl } u_t)dx - \mu \int \eta(t)\dot{u} \cdot (\text{curl } \text{curl } u) \text{div } u dx
- \mu \int \eta(t)u^i \dot{u} \cdot \text{curl}(\partial_i \text{curl } u) dx
= -\mu \int \eta(t)|\text{curl } \dot{u}|^2 dx + \mu \int \eta(t)\text{curl } \dot{u} \cdot (u \cdot \nabla u) dx
+ \mu \int \eta(t)(\text{curl } u \times \dot{u}) \cdot \nabla \text{div } u dx - \mu \int \eta(t)\text{div } (\text{curl } u \cdot \text{curl } u) dx
- \mu \int \eta(t)u^i \text{div } (\partial_i \text{curl } u \times \dot{u}) dx - \mu \int \eta(t)u^i \text{curl } u \cdot \text{curl } \dot{u} dx
= -\mu \int \eta(t)|\text{curl } \dot{u}|^2 dx + \mu \int \eta(t)\text{curl } \dot{u} \partial_i u \times \nabla u^i dx
+ \mu \int \eta(t)(\text{curl } u \times \dot{u}) \cdot \nabla \text{div } u dx - \mu \int \eta(t)\text{div } (\text{curl } u \cdot \text{curl } \dot{u}) dx
- \mu \int \eta(t)u \cdot \nabla \text{div } (\text{curl } u \times \dot{u}) dx + \mu \int \eta(t)u^i \text{div } (\text{curl } u \times \partial_i \dot{u}) dx
= -\mu \int \eta(t)|\text{curl } \dot{u}|^2 dx + \mu \int \eta(t)\text{curl } \dot{u} \partial_i u \times \nabla u^i dx
- \mu \int \eta(t)\text{div } (\text{curl } u \cdot \text{curl } \dot{u}) dx - \mu \int \eta(t)\text{div } (\text{curl } u \times \partial_i \dot{u}) dx
\leq \delta \eta(t)\|\nabla \dot{u}\|_{L^2}^2 + C\eta(t)\|\nabla u\|_{L^4}^4 - \mu \eta(t)\|\text{curl } \dot{u}\|_{L^2}^2,
\]

due to

\[
\text{curl}(u \text{div } u) = \text{div } u \text{curl } \dot{u} + \nabla \text{div } u \times \dot{u},
\]

\[
\text{div } (\partial_i \text{curl } u \times \dot{u}) = \dot{u} \cdot \text{curl}(\partial_i \text{curl } u) - \partial_i \text{curl } u \cdot \text{curl } \dot{u},
\]

\[
\int \text{curl } u \cdot (\nabla \text{div } u \times \dot{u}) dx = - \int (\text{curl } u \times \dot{u}) \cdot \nabla \text{div } u dx,
\]
\[ \int \text{curl} \, \hat{u} \cdot \text{curl} (u \cdot \nabla u) \, dx = \int \text{curl} \, \hat{u} \cdot \text{curl} (u^t \partial_t u) \, dx \]
\[ = \int \text{curl} \, \hat{u} \left( u^t \text{curl} \partial_t u + \partial_t u \cdot \nabla u^t \right) \, dx \]
\[ = \int u^t \partial_t \text{curl} \, u \cdot \text{curl} \, \hat{u} \, dx + \int \text{curl} \, \hat{u} \partial_t u \times \nabla u^t \, dx, \]
and
\[ \int u \cdot \nabla \text{div} \left( \text{curl} \, u \times \hat{u} \right) \, dx = \int u^t \partial_t \text{div} \left( \text{curl} \, u \times \hat{u} \right) \, dx \]
\[ = \int u^t \partial_t (u \cdot \text{curl} \, u - \text{curl} \, \hat{u} \cdot \text{curl} \, u) \, dx \]
\[ = \int u^t (\partial_t \hat{u} \cdot \text{curl} \, u \, dx - \partial_t \text{curl} \, \hat{u} \cdot \text{curl} \, u) \, dx \]
\[ + \int u^t (\partial_t \hat{u} \cdot \text{curl} \, u \, dx - \partial_t \text{curl} \, \hat{u} \cdot \text{curl} \, u) \, dx \]
\[ = \int u^t \text{div} (\partial_t \text{curl} \, u \times \hat{u}) \, dx + \int u^t \text{div} (\text{curl} \, u \times \partial_t \hat{u}) \, dx. \]

By Hölder’s inequality, (2.4), Sobolev’s inequality, (2.1), and (3.5), we derive from Lemmas 2.4 and 2.5 that
\[ J_3 \leq C\eta(t) \| \nabla \hat{u} \|_{L^2} \left( \| \nabla d \|_{L^6} \| \nabla d_t \|_{L^3} + \| \nabla d_t \|_{L^2} \| \nabla^2 d \|_{L^6} \| u \|_{L^6} \right) \]
\[ \leq C\eta(t) \| \nabla \hat{u} \|_{L^2} \left( \| \nabla d_t \|_{L^2} \| \nabla^2 d_t \|_{L^2} + \| \nabla d_t \|_{L^2} \right) \]
\[ + C\eta(t) \| \nabla \hat{u} \|_{L^2} \| \nabla u \|_{L^2} \left( \| \nabla d \|_{L^2} + \| \nabla^2 d \|_{L^2} \right) \]
\[ \leq \| \nabla \hat{u} \|_{L^2} \left( \| \nabla d_t \|_{L^2} + \| \nabla^2 d_t \|_{L^2} \right) \]
\[ + C\eta(t) \left( \| \nabla^3 d \|_{L^2} + \| \nabla \hat{u} \|_{L^2} \right). \]

Putting above estimates on \( J_1, J_2, \) and \( J_3 \) into (3.32), we obtain after choosing \( \delta \) suitably small that
\[ \frac{d}{dt} \left( \frac{\eta(t)}{2} \| \sqrt{\rho} \hat{u} \|_{L^2}^2 \right) + (2\mu + \lambda) \eta(t) \| \text{div} \, \hat{u} \|_{L^2}^2 + \mu \eta(t) \| \text{curl} \, \hat{u} \|_{L^2}^2 \]
\[ \leq -\frac{d}{dt} \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F \, dS + C \eta(t) \left( \| \nabla u \|_{L^4}^4 \| \nabla d \|_{L^4}^4 + \| \nabla^2 d \|_{L^4}^4 + \| \nabla^2 d \|_{L^6}^6 + E_0 \right) \]
\[ + C \eta(t) \left( \| \nabla^3 d \|_{L^2} + \| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + CE_0^2 \| \nabla^2 d \|_{L^2}^2 \right) \]
\[ + C \eta(t) \left( \| \nabla^2 d \|_{L^2}^2 \| \nabla^3 d \|_{L^2} + \| \nabla^2 d \|_{L^2} \| \nabla^3 d \|_{L^2} \right) \]
\[ + C \eta(t) \left( \| \nabla^2 d \|_{L^2}^2 \| \nabla^4 d \|_{L^2} + \| \nabla^2 d \|_{L^2} \| \nabla^3 d \|_{L^2} \right) \]

3. Differentiating (1.3) with respect to \( t \) and multiplying the resulting equations by \( d_t \), we obtain from integration by parts, \( \frac{\partial}{\partial t} |\partial_t| = 0 \), Sobolev’s inequality, and Hölder’s inequality that
\[ \frac{1}{2} \frac{d}{dt} \int |\nabla d_t|^2 \, dx + \int |d_t|^2 \, dx \]
\[ = \int \langle \partial_t (|\nabla d|^2 - u \cdot \nabla d), d_t \rangle \, dx \]
\[ \leq C \int |d_t| u_t |\nabla d| \, dx + C \int |d_t| |u_t| |\nabla d_t| \, dx + C \int |d_t| |d_t| |\nabla d| \, dx + C \int |d_t| |\nabla d| |\nabla d| \, dx \]
\[ \leq C \int |d_t| u_t |\nabla d| \, dx + C \int |d_t| |u_t| |\nabla d_t| \, dx + C \int |d_t| |d_t| |\nabla d|^2 \, dx + C \int |d_t| |\nabla d| |\nabla d| \, dx \]

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\[
+C \int |d_{tt}||u||\nabla u||\nabla d||dx \triangleq \sum_{i=1}^{5} U_i.
\]

By a direct computation, one has

\[
U_1 \leq \delta |d_{tt}|_{L^2}^2 + C|\dot{u}||\dot{u}|_{L^4_x}||\nabla d||_{L^2}^2,
\]

\[
\leq \delta |d_{tt}|_{L^2}^2 + C\left(||\nabla u||_{L^2}^4 + ||\nabla u||_{L^6}^6\right)\left(||\nabla d||_{L^2}^2||\nabla^2 d||_{L^2}^2 + ||\nabla d||_{L^2}^2\right),
\]

\[
\leq \delta |d_{tt}|_{L^2}^2 + CE_{0}^4 ||\nabla u||_{L^2}^2 + C||u||_{L^2}^4,
\]

\[
U_2 \leq \delta |d_{tt}|_{L^2}^2 + C|u|_{L^6}^4 ||\nabla d||_{L^2}^2,
\]

\[
\leq \delta |d_{tt}|_{L^2}^2 + C\left(||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right),
\]

\[
\leq \delta \left(||d_{tt}\right)_{L^2}^2 + ||\nabla d||_{L^2}^2\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)
\]

\[
U_3 \leq \delta |d_{tt}|_{L^2}^2 + C|u||\nabla d||_{H^1}^2\left(||u||_{L^2}^2 + ||\nabla d||_{L^2}^2\right),
\]

\[
U_4 \leq \delta |d_{tt}|_{L^2}^2 + C||\nabla d||_{L^2}^2||\nabla d||_{L^2}^2,
\]

\[
\leq \delta \left(||d_{tt}\right)_{L^2}^2 + \left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right),
\]

\[
U_5 \leq \delta |d_{tt}|_{L^2}^2 + C|u||\nabla u||_{L^4}^4 + ||\nabla d||_{L^2}^2||\nabla d||_{L^2}^2,
\]

where we have used (2.3), (3.5), (3.4), and

\[
||\nabla d||_{L^2}^2 \leq C\left(||\nabla d||_{L^2}^2||\nabla^2 d||_{L^2}^2 \leq C||\nabla d||_{L^6}^2 ||\nabla^2 d||_{L^2}^2\right)
\]

\[
\leq C\left(||\nabla d||_{H^1}^2||\nabla^3 d||_{L^2}^2 + ||\nabla d||_{L^2}^2\right),
\]

due to Lemma 2.1. Thus, substituting the above estimates on $U_i$ ($i = 1, 2, \ldots, 5$) into (3.39) shows that

\[
\frac{d}{dt} \left(\frac{\eta(t)}{2} ||\nabla d||_{L^2}^2\right) + \eta(t)||d_{tt}||_{L^2}^2 - \frac{1}{2} \eta'(t)||\nabla d||_{L^2}^2
\]

\[
\leq C\delta\eta(t)\left(||d_{tt}\right)_{L^2}^2 + ||\nabla^2 d||_{L^2}^2 + CE_{0}^{\frac{1}{2}} \eta(t)||\nabla u||_{L^4}^2 + C\eta(t)||\nabla u||_{L^2}^4 + C\eta(t)||\nabla d||_{L^2}^6 + C\eta(t)||\nabla u||_{L^2}^4 + C\eta(t)||\nabla d||_{L^2}^6
\]

\[
+ C\eta(t)\left(||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)
\]

\[
+ C\eta(t)\left(||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right)\left(||\nabla d||_{L^2}^2||\nabla u||_{L^2}^4 + ||\nabla d||_{L^2}^2\right).
\]

4. It remains to estimate $||\nabla^2 d||_{L^2}$. In fact, by applying the standard $L^2$-estimate of (1.3), we obtain from (2.1) and (3.5) that

\[
||\nabla^2 d||_{L^2}^2 \leq C\left(||\nabla d||_{L^2}^2 + ||d_{tt}||_{L^2}^2 + ||\partial_t(u \cdot \nabla d)||_{L^2}^2 + ||\partial_t(||\nabla^2 d||_{L^2}^2)\right)
\]

\[
\leq C||\nabla d||_{L^2}^2 + C||d_{tt}||_{L^2}^2 + C||\nabla u||_{L^2}^4 ||\nabla d||_{L^2}^2 + C||d_{tt}||_{L^2}^2 ||\nabla d||_{L^2}^2
\]

\[
+ C||\nabla d||_{L^2}^2 ||\nabla d||_{L^2}^2 + C||u||_{L^2}^4 ||\nabla u||_{L^2}^4 ||\nabla d||_{L^2}^2
\]

\[
\leq \frac{1}{2}||\nabla^2 d||_{L^2}^2 + C||d_{tt}||_{L^2}^2 + CE_{0}^{\frac{1}{2}} ||\nabla u||_{L^2}^2 + C||\nabla u||_{L^2}^2 + C||\nabla d||_{L^2}^2 + C||\nabla u||_{L^2}^2 + C||\nabla d||_{L^2}^2
\]

\[
+ C||\nabla d||_{H^1}^2 ||\nabla d||_{L^2}^2 + C||\nabla u||_{L^2}^4 ||\nabla^3 d||_{L^2}^2.
\]

This together with (3.40) gives that

\[
\frac{d}{dt} \left(\frac{\eta(t)}{2} ||\nabla d||_{L^2}^2\right) + \eta(t)||d_{tt}||_{L^2}^2 - \frac{1}{2} \eta'(t)||\nabla d||_{L^2}^2
\]
\begin{align}
\frac{1}{2} \| \phi \|_{L^2}^2 & \leq C \delta \eta(t) \| \partial_t \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 \\
& + C \eta(t) \| \nabla \phi \|_{L^2}^2 \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 \\
& + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 \\
& + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 \\
& + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2 + C \eta(t) \| \nabla \phi \|_{L^2}^2,
\end{align}

(3.42)

which combined with (3.38) leads to (3.30) after choosing $E_0 \leq \varepsilon_3$ and $\delta$ sufficiently small.

\section*{Lemma 3.4}

Let the assumptions of Proposition (3.1) hold. Then there exists a positive constant $\varepsilon_4$ such that

\begin{equation}
A_2(\sigma(T)) + \int_0^{\sigma(T)} \left( \| \sqrt{\rho} + \| \nabla \phi \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \right) dt \leq 3K,
\end{equation}

(3.43)

provided that $E_0 \leq \varepsilon_4$.

\textbf{Proof.} In view of Lemma 2.7 and (3.5), we have

\begin{align}
\| \nabla \phi \|_{L^2}^2 & \leq C \left( \| \rho \nabla \phi \|_{L^2} + \| \nabla \phi \|_{L^2}^2 + \| \phi \|_{L^2}^2 + \| \phi \|_{L^2}^2 + \| \phi \|_{L^2}^2 + \| \phi \|_{L^2}^2 \right) \\
& \leq \delta \| \rho \nabla \phi \|_{L^2}^2 + C \| \nabla \phi \|_{L^2}^2 + C \| \phi \|_{L^2}^2 + C \| \phi \|_{L^2}^2 + C \| \phi \|_{L^2}^2 + C \| \phi \|_{L^2}^2
\end{align}

(3.44)

due to

\begin{equation}
\| \phi \|_{L^2}^2 \leq C \| \phi \|_{L^2}^2 \left( \int \| \phi \|_{L^2}^2 \right)^{\frac{1}{2}} \leq C \| \phi \|_{L^2}^2.
\end{equation}

(3.45)

By (3.44), (3.5), (3.4), and Lemma 3.2, one gets that

\begin{align}
\int_0^{\sigma(T)} \| \nabla \phi \|_{L^2}^2 dt & \leq C(\delta) \int_0^{\sigma(T)} \| \sqrt{\rho} \|_{L^2}^2 dt + C(\delta) \int_0^{\sigma(T)} \| \nabla d \|_{L^2}^2 dt \\
& \quad + C \int_0^{\sigma(T)} \left( \| \nabla \phi \|_{L^2}^2 + \| \phi \|_{L^2}^2 \right) dt + C \int_0^{\sigma(T)} \| \nabla \phi \|_{L^2}^2 dt + C E_0.
\end{align}

(3.45)

Taking $\eta(t) = 1$ and integrating (3.17) over $[0, t]$ for $0 < t \leq \sigma(T)$, we deduce from (3.1), (3.2), and (3.45) that

\begin{align}
A_2(t) + \int_0^t \left( \| \sqrt{\rho} \|_{L^2}^2 + \| \nabla d \|_{L^2}^2 \right) d\tau \\
& \leq \int (P - \bar{P}) \text{div} \, u dx \bigg|_{r=0}^t - \int M(d) : \nabla u dx \bigg|_{r=0}^t + C \int_0^t \| \nabla \phi \|_{L^2}^2 dt \\
& \quad + C \int_0^t \left( \| \nabla d \|_{L^2}^2 + \| \nabla d \|_{H^1}^2 + \| \nabla \phi \|_{L^2}^2 + \| \nabla \phi \|_{H^1}^2 + \| \nabla \phi \|_{L^2}^2 \right) dt + C E_0 \\
& \leq K + \frac{1}{4} A_2(t) + C E_0 A_2(t) + C E_0 A_2(t) + C E_0,
\end{align}

(3.46)

due to

\begin{equation}
\| \nabla d \|_{L^2}^2 \leq C \| \nabla d \|_{L^2}^2 + C \| \nabla d \|_{L^2}^2.
\end{equation}
We infer from (3.46) that
\[
A_2(\sigma(T)) + \int_0^{\sigma(T)} \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \right) dt \
\leq 2K + CE_0 A_2^3(t) + CE_0 \leq \frac{5}{2} K + CE_0 K A_2(\sigma(T)),
\]
provided that \(E_0 \leq \varepsilon_4\) is suitably small. We immediately obtain (3.43) from (3.47).

**Lemma 3.5.** Let the assumptions of Proposition 3.1 be satisfied. For \(\sigma_i \triangleq \sigma(t + 1 - i)\) with \(i\) being an integer satisfying \(1 \leq i \leq [T] - 1\), then there exists a positive constant \(\varepsilon_5\) such that
\[
A_1(T) + \int_{i-1}^{i+1} \sigma_i \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \right) dt \leq E_0^{\frac{3}{4}},
\]
provided that \(E_0 \leq \varepsilon_5\).

**Remark 3.2.** For simplicity, we only prove the case \(T > 2\). Otherwise, the same thing can be done by choosing a suitably small step size.

**Proof.** For integer \(i\) \((1 \leq i \leq [T] - 1)\), taking \(\eta(t) = \sigma_i\) and integrating (3.17) over \((i-1, i+1]\), we derive
\[
\sup_{i-1 \leq t \leq i+1} \left[ \sigma_i \left( \| \nabla u \|^2_{L^2} + \| \Delta d \|^2_{L^2} \right) \right] + \int_{i-1}^{i+1} \sigma_i \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \right) dt \
\leq \sigma_i \int (P - \bar{P}) \text{div} \, u dx + \sigma_i \int \left[ \delta \nabla u \cdot \nabla x + C \int_{i-1}^{i+1} (\sigma_i + \sigma_i') \| \nabla u \|^2_{L^2} dt \
+ C \int_{i-1}^{i+1} \sigma_i' \left( \| \nabla^2 d \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right) + E_0^2 \right) dt 
+ C \int_{i-1}^{i+1} \sigma_i \| \nabla u \|^3_{L^2} dt 
+ C \int_{i-1}^{i+1} \sigma_i \left( \| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2} + \| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2} \right) dt + CE_0 \
\leq \frac{1}{2} \sup_{i-1 \leq t \leq i+1} \left( \sigma_i \| \nabla u \|^2_{L^2} \right) + C \sup_{i-1 \leq t \leq i+1} \left( \sigma_i \| \nabla d \|^4_{L^2} \right) + C \int_{i-1}^{i+1} \| \nabla u \|^2_{L^2} dt 
\]
\[
+ C \sup_{i-1 \leq t \leq i+1} \left( \| \nabla u \|^2_{L^2} + \| \Delta d \|^2_{L^2} + \| \nabla d \|^2_{L^2} \right) \int_{i-1}^{i+1} \left( \| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2} \right) dt 
\]
\[
+ C \sup_{i-1 \leq t \leq i+1} \left( \| \nabla u \|^2_{L^2} + \| \Delta d \|^2_{L^2} + \| \nabla d \|^2_{L^2} \right) \int_{i-1}^{i+1} \left( \| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2} \right) dt 
\]
\[
+ C \sup_{i-1 \leq t \leq i+1} \| \nabla u \|^2_{L^2} \int \| \nabla u \|^2_{L^2} dt + CE_0 + \delta \int_{i-1}^{i+1} \sigma_i \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \right) dt 
\]
\[
\leq \frac{1}{2} \sup_{i-1 \leq t \leq i+1} \left( \sigma_i \| \nabla u \|^2_{L^2} \right) + CE_0^{\frac{3}{4}} \sup_{i-1 \leq t \leq i+1} \left( \sigma_i \| \Delta d \|^2_{L^2} \right) 
\]
\[
+ CE_0 \left( A_1(T) + A_2(\sigma(T) + E_0) \right) + CE_0 \left( A_2^2(T) + A_2^2(\sigma(T)) + E_0^2 \right) 
\]
\[
+ \delta \int_{i-1}^{i+1} \sigma_i \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \right) dt + CE_0 
\]
\[
\leq \frac{1}{2} \sup_{i-1 \leq t \leq i+1} \left( \sigma_i \| \nabla u \|^2_{L^2} \right) + CE_0^{\frac{3}{4}} \sup_{i-1 \leq t \leq i+1} \left( \sigma_i \| \Delta d \|^2_{L^2} \right) 
\]
\[
+ \delta \int_{i-1}^{i+1} \sigma_i \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \right) dt + CE_0,
\]
where we have used (3.2), (3.5), and (3.12). Choosing \(\delta\) and \(E_0\) suitably small, we deduce from (3.49) that
\[
\sup_{0 \leq t \leq \sigma(T)} \left[ \sigma \left( \| \nabla u \|^2_{L^2} + \| \Delta d \|^2_{L^2} \right) \right] \leq CE_0 \leq E_0^{\frac{3}{4}}.
\]
due to $\sigma_1(t) = \sigma(t)$ and

$$\sup_{i \leq t \leq i+1} \left[ \sigma_i(\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}) \right] + \int_{i-1}^{i+1} \sigma_i(\|\sqrt{\rho} \dot{u}\|^2_{L^2} + \|\nabla d_t\|^2_{L^2}) dt \leq C_3 E_0 \leq E_0^{\frac{1}{2}},$$

(3.51)

provided that $E_0 \leq \varepsilon_5$ is properly small. Note that the constant $C_3$ is independent of $i$. Thus, the desired (3.48) follows from (3.50) and (3.51).

**Lemma 3.6.** Let the assumptions of Proposition 3.1 be satisfied. Then there exists a positive constant $\varepsilon_6$ such that, for $\sigma(T) \leq t_1 < t_2 \leq T$,

$$\sup_{0 \leq t \leq T} \left[ \sigma^2(\|\nabla u\|^2_{L^2} + \|\nabla d_t\|^2_{L^2}) \right] \leq CE_0^\frac{1}{2},$$

(3.52)

and

$$\int_{t_1}^{t_2} \sigma^2(\|\nabla u\|^2_{L^2} + \|d_t\|^2_{L^2}) dt \leq CE_0^\frac{1}{2} + CE_0(t_2 - t_1),$$

(3.53)

provided that $E_0 \leq \varepsilon_6$.

**Proof.** 1. We get from Lemma 2.7 that

$$\|\nabla u\|^4_{L^4} \leq C(||\rho \dot{u}\|_{L^2} + ||\nabla d||\nabla^2 d||_{L^2})^3(||\nabla u\|_{L^2} + ||P - \bar{P}||_{L^2}) + C(||\nabla u\|^4_{L^4} + ||P - \bar{P}||_{L^4}),$$

which implies that

$$\int_{i-1}^{i+1} \sigma_i^2(\|\nabla u\|^4_{L^4} dt \leq C \int_{i-1}^{i+1} \sigma_i^2(\|\nabla u\|_{L^2} + E_0^{\frac{1}{2}})^2(\|\sqrt{\rho} \dot{u}\|^3_{L^2} + \|\nabla d\|^3_{L^1}||\nabla^2 d||^3_{L^3}) dt$$

$$+ C \int_{i-1}^{i+1} \sigma_i^2(\|\nabla u\|_{L^2} + E_0^{\frac{1}{2}}) \int_{i-1}^{i+1} \sigma_i^2(\|\sqrt{\rho} \dot{u}\|^3_{L^2} + \|\nabla d\|^3_{L^1}) dt$$

$$+ C \int_{i-1}^{i+1} \sigma_i^2(\|\nabla u\|_{L^2} + \|\nabla d\|^4_{L^2}) dt + CE_0,$$

(3.54)

$$\int_{0}^{T} \|\nabla u\|^4_{L^2} dt = \int_{0}^{\sigma(T)} \|\nabla u\|^4_{L^2} dt + \int_{\sigma(T)}^{T} \|\nabla u\|^4_{L^2} dt$$

$$\leq \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|^4_{L^2} \int_{0}^{\sigma(T)} \|\nabla u\|^4_{L^2} dt + \sup_{\sigma(T) \leq t \leq T} (\sigma \|\nabla u\|^4_{L^2}) \int_{\sigma(T)}^{T} \|\nabla u\|^4_{L^2} dt$$

$$\leq C(K) E_0,$$

(3.55)

and

$$\|P - \bar{P}\|_{L^4} \leq C \|P - \bar{P}\|^\frac{1}{2}_{L^\infty} \left( \int \|P - \bar{P}\|^2 dx \right)^\frac{1}{2} \leq C(\rho) \|\rho - \bar{\rho}\|^\frac{1}{2}_{L^2}.$$ 

(3.56)

For any integer $1 \leq i \leq |T| - 1$, integrating (3.30) with $\eta(t) = \sigma_i^2$ over $(i - 1, i + 1]$, we deduce from (3.2), (3.7), (3.43), and Young’s inequality that

$$\sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2(\|\sqrt{\rho} \dot{u}\|^2_{L^2} + \|\nabla d_t\|^2_{L^2}) \right] + \int_{i-1}^{i+1} \sigma_i^2(\|\nabla u\|^2_{L^2} + \|d_t\|^2_{L^2}) dt$$

$$.$$
\[\begin{align*}
&\leq - \int_{\partial \Omega} \sigma_i^2(u \cdot \nabla n \cdot u) F dS \bigg|_{i-1}^{i+1} + C \int_{i-1}^{i+1} \sigma_i^2 \|
abla u\|_{L^2}^2 \, dt \\
&+ C \int_{i-1}^{i+1} \sigma_i \sigma_i' (\|\sqrt{\rho} u\|_{L^2}^2 + \|
abla d_t\|_{L^2}^2 + \|
abla u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|
abla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) \, dt \\
&+ C \int_{i-1}^{i+1} \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|
abla u\|_{L^2}^2 + \|
abla d_t\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) \, dt \\
&+ C \int_{i-1}^{i+1} \sigma_i^2 (\|\nabla d\|_{H^1}^2 + \|
abla u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \, dt \\
&+ C \int_{i-1}^{i+1} \sigma_i^2 (\|
abla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) \|
abla d_t\|_{L^2}^2 \, dt \\
&\leq C \sup_{i-1 \leq t \leq i+1} (\sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|
abla d_t\|_{L^2}^2)) \int_{i-1}^{i+1} (\|
abla u\|_{L^2}^2 + \|
abla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) \, dt \\
&+ C \sup_{i-1 \leq t \leq i+1} (\sigma_i (\|\sqrt{\rho} u\|_{L^2}^2 + \|
abla d\|_{L^2}^2 + E_0^2 + \delta) \int_{i-1}^{i+1} \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \, dt \\
&+ C \sup_{i-1 \leq t \leq i+1} (\sigma_i (\|
abla d\|_{H^1}^2) \int_{i-1}^{i+1} \sigma_i \|
abla d_t\|_{L^2}^2 \, dt + C \int_{i-1}^{i+1} \sigma_i^2 (\|
abla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) \, dt \\
&+ \frac{1}{4} \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + CE_0 \\
&\leq \frac{1}{4} \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] \\
&+ C \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) \right] \int_{i-1}^{i+1} \sigma_i (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \, dt \\
&+ CE_0 \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + CE_0^{\frac{1}{2}} \\
&\leq \frac{1}{4} \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + CE_0^{\frac{1}{2}} \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) \right] \\
&+ CE_0 \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + CE_0^{\frac{1}{2}} \\
&\leq \frac{1}{4} \sup_{i-1 \leq t \leq i+1} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + CE_0^{\frac{1}{2}}, \quad \text{(3.57)}
\end{align*}\]

where we have used

\[\int_{\partial \Omega} (u \cdot \nabla n \cdot u) F dS \leq C \|u\|_{W^{1,1}} \leq C \|
abla u\|_{L^2}^2 \|F\|_{H^1} \]

\[\leq \frac{1}{4} (\|\sqrt{\rho} u\|_{L^2}^2 + \|
abla d_t\|_{L^2}^2) + C (\|
abla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2 + \|\nabla d\|_{H^1}^2).\]

According to (3.57), we get that

\[\sup_{0 \leq t \leq \sigma(T)} \left[ \sigma_i^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] \leq CE_0^{\frac{1}{2}}, \quad \text{(3.58)}\]

and

\[\sup_{i-1 \leq t \leq i+1} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \leq CE_0^{\frac{1}{2}}. \quad \text{(3.59)}\]

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Hence, we deduce (3.52) from (3.2), (3.58), and (3.59).

2. We integrate (3.17) over \([t_1, t_2] \subseteq [\sigma(T), T]\) and take \(\eta(t) = \sigma\) to obtain, from (3.2) and (3.7), that

\[
\int_{t_1}^{t_2} \sigma(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \leq CE_0 + CE_0(t_2 - t_1) + C \int_{t_1}^{t_2} \sigma(\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) dt \\
\quad + C \int_{t_1}^{t_2} \sigma(\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) dt \\
\leq CE_0^\frac{1}{2} + CE_0(t_2 - t_1) + C \int_{t_1}^{t_2} \sigma(\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{H^1}^2) dt \\
\leq CE_0^\frac{1}{2} + CE_0(t_2 - t_1),
\]

(3.60)

Similarly to (3.57), integrating (3.30) over \([t_1, t_2]\) and taking \(\eta = \sigma^2\), we find that

\[
\int_{t_1}^{t_2} \sigma^2(\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2) dt \\
\leq C \sup_{0 \leq t \leq \sigma(T)} \left[ \sigma(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] \int_{t_1}^{t_2} \sigma(\|\nabla u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt + CE_0^\frac{1}{2} \\
\leq CE_0^\frac{1}{2} + CE_0 \int_{t_1}^{t_2} \sigma(\|\nabla u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \\
\leq CE_0^\frac{1}{2} + CE_0(t_2 - t_1),
\]

(3.61)

owing to (3.2), (3.7), (3.52), and (3.60). The conclusion follows.

We still need the following result before showing the upper bounds of the density.

**Lemma 3.7.** Let the assumption of Proposition 3.1 be satisfied. Then there exists a positive constant \(\varepsilon_7\) such that

\[
\sup_{0 \leq t \leq \sigma(T)} \left[ \sigma(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + \int_0^{\sigma(T)} \sigma(\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2) dt \leq C,
\]

(3.62)

provided that \(E_0 \leq \varepsilon_7\).

**Proof.** Taking \(\eta(t) = \sigma\) and integrating (3.30) over \([0, \sigma(T)]\), we get from (3.2), (3.7), (3.43), and Young’s inequality that

\[
\sup_{0 \leq t \leq \sigma(T)} \int_0^{\sigma(T)} \sigma(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \leq C \int_0^{\sigma(T)} \sigma' \left(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\right) dt + CE_0^\frac{1}{2} \\
\quad + C \int_0^{\sigma(T)} \sigma(\|\nabla u\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2 + E_0^\frac{1}{2} + \delta) \left(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\right) dt \\
\leq CE_0^\frac{1}{2} \sup_{0 \leq t \leq \sigma(T)} \left[ \sigma^\frac{1}{2}(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] \int_0^{\sigma(T)} \left(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\right) dt + C(K) \\
\leq C(K) + C(K) \sup_{0 \leq t \leq \sigma(T)} \left[ \sigma^\frac{1}{2}(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] \\
\leq \frac{1}{2} \sup_{0 \leq t \leq \sigma(T)} \left[ \sigma(\|\sqrt{\rho u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right] + C,
\]

from which, the conclusion follows.

With Lemmas 3.6 and 3.7 at hand, we derive the uniform upper bounds of the density, which is the key to obtain all the higher-order estimates and thus to extend the classical solution globally.
Lemma 3.8. There exists a positive constant $\varepsilon$ as in Theorem 1.1 such that if $(\rho, u, d)$ is a strong solution of (1.1)–(1.5) in $\Omega \times (0, T)$ satisfying (3.2), then

$$\sup_{0 \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \frac{7}{4} \rho$$

(3.63)

provided that $E_0 \leq \varepsilon \triangleq \min\{1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$.

Proof. 1. We rewrite (1.1) as

$$D_t \rho = g(\rho) + b'(t),$$

(3.64)

where

$$D_t \rho = \rho_t + u \cdot \nabla \rho, \quad g(\rho) = -\frac{\rho(P - \bar{P})}{2\mu + \lambda}, \quad b(t) = -\frac{1}{2\mu + \lambda} \int_0^t \rho F d\tau.$$

For $t \in [0, \sigma(T)]$, we deduce from Hölder’s inequality, Lemma 2.7, and (3.43) that, for $0 \leq t_1 < t_2 \leq \sigma(T)$,

$$|b(t_2) - b(t_1)| \leq C(\rho) \int_0^{\sigma(T)} \|F\|_{L^\infty} dt + C \int_0^{\sigma(T)} \|F\|_{L^2} dt$$

$$\leq C \int_0^{\sigma(T)} \left(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^\frac{3}{2} \|\nabla^3 d\|_{L^2}^\frac{3}{2} + \|\nabla^2 d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^\frac{1}{8} + \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2} \|\nabla^2 d\|_{H^1}^2\right) dt + C \int_0^{\sigma(T)} \left(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2}^\frac{1}{8}\right) dt$$

$$\leq C E_0^\frac{3}{4} + C \int_0^{\sigma(T)} \left(\|\nabla \dot{u}\|_{L^2}^\frac{3}{4} + E_0^\frac{3}{8} \|\nabla u\|_{L^2}^\frac{1}{8} + E_0^\frac{1}{4} \|\nabla u\|_{L^2}^\frac{3}{8}\right) dt$$

$$\leq C \left[ E_0^\frac{3}{4} \left(\int_0^{\sigma(T)} \sigma^{-\frac{3}{8}} dt \right)^{\frac{3}{8}} + E_0^\frac{1}{4} \left(\int_0^{\sigma(T)} \sigma^{-\frac{3}{8}} dt \right)^{\frac{3}{8}} \right] \left(\int_0^{\sigma(T)} \|\nabla \dot{u}\|_{L^2}^2 dt \right)^{\frac{3}{8}}$$

$$+ C \left[ E_0^\frac{3}{4} + E_0^\frac{1}{4} \left(\int_0^{\sigma(T)} \sigma^{-\frac{3}{8}} dt \right)^{\frac{3}{8}} \right] \left(\int_0^{\sigma(T)} \|\nabla d\|_{L^2}^2 dt \right)^{\frac{3}{8}}$$

$$+ C \left[ E_0^\frac{3}{4} + E_0^\frac{1}{4} \left(\int_0^{\sigma(T)} \sigma^{-\frac{3}{8}} dt \right)^{\frac{3}{8}} \right] \left(\int_0^{\sigma(T)} \|\nabla d\|_{L^2}^2 dt \right)^{\frac{3}{8}} + CE_0^\frac{3}{8},$$

(3.65)

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provided that $E_0 \leq \varepsilon$. Thus, for $t \in [0, \sigma(T)]$, we can choose $N_0$ and $N_1$ in Lemma 2.9 as
\[
N_1 = 0, \quad N_0 = CE_0^{\frac{1}{\mu}},
\] (3.66)
and $\xi_0 = \hat{\rho}$. Then, one has
\[
g(\xi) = -\frac{\xi}{2\mu + \lambda} (\xi^7 - \hat{\rho}^7) \leq -N_1 = 0 \quad \text{for all } \xi \geq \xi_0 = \hat{\rho},
\] (3.67)
which together with Lemma 2.9 implies that
\[
\sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^\infty} \leq \max\{\hat{\rho}, \xi_0\} + N_0 \leq \hat{\rho} + CE_0^{\frac{1}{\mu}} \leq \frac{3\hat{\rho}}{2},
\] (3.68)
provided that $E_0 \leq \varepsilon$.

2. For $t \in [\sigma(T), T]$, we derive from Lemma 2.7, (3.2), (3.5), (3.7), (3.29), (3.48), and (3.53) that, for $\sigma(T) \leq t_1 < t_2 \leq T$,
\[
|b(t_2) - b(t_1)| \leq C(\hat{\rho}) \int_{t_1}^{t_2} \|F\|_{L^\infty} dt
\]
\[
\leq C \int_{t_1}^{t_2} \|F\|_{L^\infty}^2 dt + \frac{a}{4\mu + 2\lambda} (t_2 - t_1)
\]
\[
\leq C \int_{t_1}^{t_2} \|F\|_{L^2}^2 \|\nabla F\|_{L^2}^2 dt + \int_{t_1}^{t_2} \|F\|_{L^2}^2 dt + \frac{a}{4\mu + 2\lambda} (t_2 - t_1)
\]
\[
\leq C \int_{t_1}^{t_2} \left( \|\nabla u\|_{L^2}^2 + \|
abla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2} \|\nabla^3 d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2 \right)
\]
\[
+ \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2} \|\nabla^3 d\|_{L^2} \left( \|\nabla u\|_{L^2}^2 + \|P - \bar{P}\|_{L^2}^2 \right) dt
\]
\[
+ C \int_{t_1}^{t_2} \left( \|\nabla u\|_{L^2}^2 + \|P - \bar{P}\|_{L^2}^2 \right) dt + \frac{a}{4\mu + 2\lambda} (t_2 - t_1)
\]
\[
\leq \left( \frac{a}{4\mu + 2\lambda} + CE_0^{\frac{2}{\mu}} \right) (t_2 - t_1) + CE_0^{\frac{1}{\mu}} \int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2 dt
\]
\[
+ CE_0^{\frac{1}{\mu}} \left( \int_{t_1}^{t_2} \|\nabla^2 d\|_{L^2}^2 \right) \left( \frac{1}{2} \int_{t_1}^{t_2} \|\nabla d\|_{L^2}^2 dt \right)^{\frac{1}{2}} + CE_0^{\frac{1}{\mu}} \int_{t_1}^{t_2} \|\nabla^2 d\|_{H^1}^2 + \|\nabla u\|_{L^2}^2 dt
\]
\[
+ CE_0^{\frac{1}{\mu}} \sup_{t_1 \leq t \leq t_2} \|\nabla d\|_{L^2} \int_{t_1}^{t_2} \|\nabla d\|_{L^2}^2 dt + CE_0^{\frac{1}{\mu}} \int_{t_1}^{t_2} \|\nabla d\|_{L^2}^2 dt
\]
\[
\leq \left( \frac{a}{4\mu + 2\lambda} + CE_0^{\frac{2}{\mu}} + CE_0^{\frac{17}{2}} + CE_0^{\frac{7}{2}} \right) (t_2 - t_1) + CE_0^{\frac{1}{\mu}}
\]
\[
\leq \frac{2a}{4\mu + 2\lambda} (t_2 - t_1) + CE_0^{\frac{1}{\mu}},
\] (3.69)
provided that $E_0 \leq \varepsilon$. Thus, for $t \in [\sigma(T), T]$, we can choose $N_0$, $N_1$, and $\xi_0$ in Lemma 2.9 as follows:
\[
N_0 = CE_0^{\frac{1}{\mu}}, \quad N_1 = \frac{a}{2\mu + \lambda}, \quad \xi_0 = \frac{3\hat{\rho}}{2}.
\]
Since for all $\xi \geq \xi_0 = \frac{3\hat{\rho}}{2} > \hat{\rho} + 1$,
\[
g(\xi) = -\frac{\xi}{2} (\xi^7 - \hat{\rho}^7) \leq -N_1 \quad \text{for all } \xi \geq \xi_0 = \frac{3\hat{\rho}}{2}.
\] (3.70)
Thus, due to Lemma 2.9, we arrive at
\[
\sup_{\sigma(T) \leq t \leq T} \|\rho\|_{L^\infty} \leq \frac{3\hat{\rho}}{2} + N_0 \leq \frac{3\hat{\rho}}{2} + CE_0^{\frac{1}{\mu}} \leq \frac{7\hat{\rho}}{4},
\] (3.71)
provided that $E_0 \leq \varepsilon$. The combination of (3.68) and (3.71), we obtain (3.63).

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** Proposition 3.1 follows from Lemma 3.4, Lemma 3.5, and Lemma 3.8. □
3.2 Higher-order estimates

In this subsection, we establish the time-dependent higher-order estimates of solutions, which are necessary for the global existence of strong solutions. In what follows, we denote by \( C \) or \( C_i \) \((i = 1, 2, \ldots)\) the various positive constants, which may depend on the initial data, \( \mu, \lambda, \gamma, a, \rho, \Omega, M_1, M_2, \rho, \) and \( T \) as well.

Lemma 3.9. Under the conditions of Theorem 1.1, it holds that

\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^r} + \int_0^T \| \nabla u \|_{L^\infty} \, dt \leq C, \tag{3.72}
\]

\[
\sup_{0 \leq t \leq T} \left[ t \left( \| \sqrt{\rho} \alpha \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2 \right) \right] + \int_0^T t \left( \| \nabla \dot{u} \|_{L^2}^2 + \| d_t \|_{L^2}^2 \right) dt \leq C, \tag{3.73}
\]

\[
\sup_{0 \leq t \leq T} \left[ t \left( \| u \|_{H^2}^2 + \| \nabla^3 d \|_{L^2}^2 \right) \right] + \int_0^T \left( \| \nabla^2 d_t \|_{L^2}^2 + \| \nabla^4 d \|_{L^2}^2 \right) dt \leq C. \tag{3.74}
\]

Proof. First, based on the Beale-Kato-Majda type inequality (see Lemma 2.8), we can obtain (3.72). Since the argument is similar to [3, Lemma 4.1], we omit the details for simplicity.

By virtue of (3.62), it is easy to check that

\[
\sup_{0 \leq t \leq \sigma(T)} \left[ t \left( \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2 \right) \right] + \int_0^{\sigma(T)} t \left( \| \nabla \dot{u} \|_{L^2}^2 + \| d_t \|_{L^2}^2 \right) dt \leq C,
\]

which together with (3.53) leads to (3.73).

Next, by Lemma 2.4, (2.8), (2.6), and (2.11), we obtain that

\[
\| \nabla^2 u \|_{L^2} \leq C \left( \| \text{div} \, u \|_{H^1} + \| \text{curl} \, u \|_{H^1} \right)
\leq C \left( \| F + P - \bar{P} \|_{H^1} + \| \text{curl} \, u \|_{H^1} \right)
\leq C \left( \| \rho \dot{u} \|_{L^2} + \| \nabla d \|_{L^2} \| \nabla^2 d \|_{L^2} + \| \nabla P \|_{L^2} + \| P - \bar{P} \|_{L^2} + \| \nabla u \|_{L^2} \right).
\]

This along with (3.73), (3.29), (3.5), (3.6), and (3.2) yields that

\[
\sup_{0 \leq t \leq T} \left[ t \left( \| u \|_{H^2}^2 + \| \nabla^3 d \|_{L^2}^2 \right) \right] \leq C. \tag{3.75}
\]

Furthermore, taking the operator \( \nabla \) to (3.27), we get that

\[
-\nabla^2 \Delta d = \nabla^2 (|\nabla d|^2 d - u \cdot \nabla d - d_t).
\tag{3.76}
\]

Applying the standard \( L^2 \)-estimate to (3.76), we derive that

\[
\| \nabla^4 d \|_{L^2}^2 \leq C \left( \| \nabla^2 d_t \|_{L^2}^2 + \| \nabla^2 (u \cdot \nabla d) \|_{L^2}^2 + \| \nabla^2 (|\nabla d| d)^2 \|_{L^2}^2 \right) + C \| \nabla d \|_{H^1}^2
\leq C \| \nabla^2 d_t \|_{L^2}^2 + C \| \nabla d \|_{L^2}^2 + \| \nabla u \|_{H^2}^2 + C \| \nabla d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + C \| \nabla^2 u \|_{L^2}^2 + C \| \nabla d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + C \| \nabla^3 d \|_{L^2}^2 + C \| \nabla^4 d \|_{L^2}^2 + C
\]

\[
\leq \frac{1}{2} \| \nabla^4 d \|_{L^2}^2 + C \| \nabla d \|_{H^2}^2 + C \| \nabla u \|_{H^2}^2 \| \nabla^2 d \|_{H^1}^2 + C \| \nabla^2 d \|_{L^2}^2 + \| \nabla^3 d \|_{L^2}^2 + C \| \nabla^4 d \|_{L^2}^2 + C
\]

which leads to

\[
\| \nabla^4 d \|_{L^2}^2 \leq C \| \nabla d \|_{H^2}^2 \| \nabla^3 d \|_{L^2}^2 + C \| \nabla u \|_{H^2}^2 \| \nabla^2 d \|_{H^1}^2 + C \| \nabla^3 d \|_{L^2}^2 + C \| \nabla^4 d \|_{L^2}^2 + C.
\]

This together with (3.75), (3.73), (3.29), (3.2), (3.5), (3.43), and (3.32) implies (3.74).
Lemma 3.10. Under the conditions of Theorem 1.1, it holds that
\[
\sup_{0 \leq t \leq T} (t\|\sqrt{\rho}u_t\|_{L^2}^2) + \int_0^T t\|\nabla u_t\|_{L^2}^2 dt \leq C.
\]

Proof. By Lemma 3.9, Proposition 3.1, (2.1), and Sobolev’s inequality, we deduce that
\[
t\|\sqrt{\rho}u_t\|_{L^2}^2 \leq t\|\sqrt{\rho}u\|_{L^2}^2 + t\|\sqrt{\rho} \cdot \nabla u\|_{L^2}^2 \\
\leq C(T) + Ct\|u\|_{L^2}^2\|\nabla u\|_{L^4}^2 \\
\leq C(T) + C\|\nabla u\|_{L^2}^2 (t\|u\|_{H^2}^2)
\]
and
\[
\int_0^T t\|\nabla u_t\|_{L^2}^2 dt \leq \int_0^T t\|\nabla u\|_{L^2}^2 dt + \int_0^T t\|\nabla (u \cdot \nabla u)\|_{L^2}^2 dt \\
\leq C(T) + \int_0^T t(\|u\|_{L^4}^4 + \|u\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2}^2) dt \\
\leq C(T) + C\int_0^T t\|u\|_{H^1}^2 \|\nabla u\|_{H^2}^2 dt \\
\leq C.
\]
This finishes the proof of Lemma 3.10.

4 Proof of Theorem 1.1

With all the a priori estimates in Section 3 at hand, we are going to prove the main result of the paper.

Proof of Theorem 1.1. By Lemma 2.1, there exists a \( T_* > 0 \) such that the system (1.1)–(1.5) has a unique classical solution \((\rho, u, d)\) in \( \Omega \times (0, T_*) \). By the definitions of (3.1) and (1.9), it is easy to check that
\[
0 \leq \rho_0 \leq \hat{\rho}, \quad A_1(0) = 0, \quad A_2(0) \leq K.
\]
Therefore, there exists a \( T_1 \in (0, T_*) \) such that
\[
0 \leq \rho_0 \leq 2\hat{\rho}, \quad A_1(T) \leq 2E_0^{\frac{1}{q}}, \quad A_2(\sigma(T)) \leq 4K \\
\text{ (4.1)}
\]
holds for \( T = T_1 \).

Next, we set
\[
T^* = \sup\{T | (4.1) \text{ holds}\}. \quad \text{(4.2)}
\]
Then \( T^* \geq T_1 > 0 \). Hence, for any \( 0 < \tau < T < T^* \) with \( T \) finite, it follows from Lemmas 3.9 and 3.10 that
\[
\rho \in C([0, T]; W^{1,q}), \quad (\nabla u, \nabla^2 d) \in C(\tau, T; L^q), \quad \text{ (4.3)}
\]
where one has taken advantage of the standard embedding
\[
L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q), \quad \text{ for any } q \in (3, 6).
\]
Finally, we claim that
\[
T^* = \infty. \quad \text{(4.4)}
\]
Otherwise, $T^* < \infty$. Then, by Proposition 3.1, (3.3) holds for $T = T^*$. It follows from (3.3), (3.5), and (4.3) that $(\rho(x, T^*), u(x, T^*), d(x, T^*))$ satisfies
\[
\begin{cases}
0 \leq \rho(x, T^*) \leq \hat{\rho}, & (\rho(x, T^*), P(\rho(x, T^*))) \in W^{1,q}, \ u(x, T^*) \in H^1, \ d(x, T^*) \in H^2, \\
\|\nabla u(x, T^*)\|^2_{L^2} \leq M_1, \ |\Delta d(x, T^*)|^2_{L^2} \leq M_2,
\end{cases}
\]
and $|d(x, T^*)| = 1$ due to $|d_0| = 1$ (see (3.7) in [24]). Thus, Lemma 2.1 implies that there exists some $T^{**} > T^*$ such that (4.1) holds for $T = T^{**}$, which contradicts the definition of $T^*$. As a result, (4.4) follows. By Lemma 2.1 and Lemmas 3.9 and 3.10, it indicates that $(\rho, u, d)$ is in fact the unique strong solution defined in $\Omega \times (0, T]$ for any $0 < T < T^* = \infty$. The proof of Theorem 1.1 is finished. \qed

Acknowledgments

The authors would like to express their gratitude to the reviewers for careful reading and helpful suggestions which led to an improvement of the original manuscript.

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