Advances in Metric Ramsey Theory and its Applications

Yair Bartal†

Abstract

Metric Ramsey theory is concerned with finding large well-structured subsets of more complex metric spaces. For finite metric spaces this problem was first studies by Bourgain, Figiel and Milman [21], and studied further in depth by Bartal et. al [10]. In this paper we provide deterministic constructions for this problem via a novel notion of metric Ramsey decomposition. This method yields several more applications, reflecting on some basic results in metric embedding theory.

The applications include various results in metric Ramsey theory including the first deterministic construction yielding Ramsey theorems with tight bounds, as well as stronger theorems and properties, implying appropriate distance oracle applications.

In addition, this decomposition provides the first deterministic Bourgain-type embedding of finite metric spaces into Euclidean space, and an optimal multi-embedding into ultrametrics, thus improving its applications in approximation and online algorithms.

The decomposition presented here, the techniques and its consequences have already been used in recent research in the field of metric embedding for various applications.

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*This is paper is still in stages of preparation, this version is not intended for distribution. A preliminary version of this article was written by the author in 2006, and was presented in the 2007 ICMS Workshop on Geometry and Algorithms [14]. The basic result on constructive metric Ramsey decomposition and metric Ramsey theorem has also appeared in the author’s lectures notes, e.g. [15].

†School of Engineering and Computer Science, Hebrew University, Israel. Email: yair@cs.huji.ac.il. Supported in part by a grant from the Israeli Science Foundation (1817/17).
1 Introduction

Metric embedding has played an important role in recent decades in the development of algorithms and efficient data structures, within Computer Science, and has also contributed significantly to the development of fundamental mathematical tools (see \[26\],[31],[27]\) for surveys).

Metric Ramsey Theory is asking whether complex metric spaces contain large well-structured subspaces, which embed with low distortion into a class of special metric spaces, e.g. Euclidean space. A basic theorem in this field \[10\],[32]\) states that this is indeed true when the target class is that of ultrametric spaces (which are in particular Euclidean).

In this paper we provide the first deterministic tight version of this theorem. Our method is based on a novel construct called metric Ramsey decomposition which we apply to obtain novel versions of this theorem its applications, as well as reflecting on other basic results in metric embedding theory.

- **Metric Ramsey Decompositions:** Our main contribution is a novel tool which can viewed as a deterministic counterpart to padded probabilistic partitions, a fundamental tool in many metric embedding results (e.g., \[11\],[34],[24],[12],[29],[5]\). The method of our the construction is very simple and combines ideas from \[13],[9],[10],[32]\) yielding a construction that is elementary and yet very powerful. Indeed, we show that this basic notion has many applications related to difference aspects of metric embedding theory. Essentially, they can replace probabilistic partitions in several fundamental metric embedding applications yielding first explicit deterministic and efficient constructions of embeddings into \(L_p\) and ultrametrics. In particular, we can obtain explicit constructions of embeddings of \(32\)[6][4][3]. Moreover, we obtain several new and improved embedding results.

- **Metric Ramsey Theorems and Distance Oracles:** The decomposition naturally arises within the context of the metric Ramsey problem: Given an arbitrary metric space, the goal is to find a large subspace that is highly structured, that is a subspace which embeds with low distortion into some natural class of highly structured metric spaces. Of particular interest is the class of ultrametrics (in particular, they embed isometrically in Euclidean space). This problem has been first addressed by Bourgain, Figiel and Milman \[21]\, motivated by its relation to Dvoretzky’s theorem. The problem has been further studied in a sequence of papers partially motivated by applications in computer science \[28],[20],[9]\. Bartal, Linial, Mendel, and Naor \[10\] obtained nearly tight bounds and Mendel and Naor \[32\] gave a randomized construction which obtained the asymptotically tight bound for large distortions. The theorem states that every \(n\) point metric space contains a subspace of size \(n^{1-1/t}\) which embeds in an ultrametric with distortion \(O(t), t > 1\).

In this paper we provide the first deterministic construction with tight bounds improving on the previously best known bounds of \[10\]. Our construction provides distortion of \(8t\) (for integer values), which is nearly the best known bound via a deterministic construction. Subsequent to our work a similar construction was shown to obtain distortion \(8t - 2\) \[8\]. The best probabilistic construction (also achieved following this work) obtains distortion \(2et\ [33]\).

- **Strong metric Ramsey theorems:**

Our constructions possesses additional stronger properties not provided by past constructions.

- **Linear subspaces with constant \(\ell_q\)-distortion:** One of our main contributions are Ramsey theorems with small average distortion and \(\ell_q\)-distortion. In \[5\] it was proved that every finite metric space embeds in Euclidean space with constant \(\ell_q\)-distortion, for all \(q < \infty\) (and the worst case
is $O(\log n)$. In [2] is was shown that every finite metric space embeds into an ultrametric with constant average distortion and $O(\sqrt{\log n})$ $\ell_2$-distortion (and the worst case is $O(n)$).

Here we prove that every finite metric space contains a linear size subspace which embeds in an ultrametric with constant $\ell_q$-distortion, for all $q < \infty$ (and the worst case is $O(\log n)$).

- **Linear subspaces with local distortion:**
  In [7] local embeddings of metric spaces where introduced. It is shown there how to achieve local Ramsey theorems. We give here explicit constructions of these theorems and prove that they work for arbitrary metric spaces.

- **Doubling and planar metrics:** We give new Ramsey theorems for decomposable metric spaces into $L_p$. This is the first example of a metric Ramsey theorem for a non-trivial family of metric spaces which obtains embeddings into $L_p$ that beat the best possible bounds given by embedding into ultrametrics.

- **Proximity Data Structures:** The metric Ramsey problem is closely related to the construction of proximity data structures including approximate distance oracles [35]. These are space efficient data structures that enable satisfying fast approximate distance queries. Our results yield the first deterministic construction of approximate distance oracles with asymptotically optimal space-distortion tradeoff and constant query time.

Our stronger metric Ramsey theorems also translate to new approximate distance oracles. In particular we provide distance oracles of linear size with constant $\ell_q$-distortion, for all $q < \infty$ (and the worst case is $O(\log n)$). Similarly, we obtain distance oracles of linear size with local stretch properties. In particular, we obtain the first construction of such data structure with $O(n)$ storage and $O(\vartheta(\log k))$ $k$-local stretch (i.e. this bounds the stretch for the $k$th nearest neighbor of a point).

We note that while constructing distance oracles using the basic metric Ramsey theorem can be done using the approach of [32] their method does not apply to the case that the distortion varies for different pairs of points as is the case with our strong metric Ramsey theorems. We therefore show directly how to apply our Ramsey decomposition to obtain Ramsey embeddings. This is a strengthening of the Ramsey type results which we define and have immediate application to proximity data structures.

- **Deterministic Embedding:** We provide a unified framework for deterministic Bourgain-type embedding of arbitrary metrics into $L_p$. This is achieved by applying our deterministic decomposition in place of the standard probabilistic padded decompositions. Moreover, builing ont the embedding of [5] this method implies a deterministic embedding in $O(\log n)$ dimension and distortion and constant $q$ moments, for all fixed $q < \infty$. This further provides optimal prioritized embeddings [23, 17].

- **Multi-Embedding:** In a multi-embedding [19] a metric space is embedded into a larger metric space so that the distortion of paths is preserved. These type of embeddings have applications in the context of approximation and online problems (in particular, the extensively studied metrical task systems and the group Steiner tree problems). Using the Ramsey decomposition we obtain optimal multi-embedding into ultrametrics, thereby improving bounds in the applications.

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1The randomized constructions that were presented in [7] needed a growth rate assumption on the space.
1.1 Further Related Work and Applications

The decomposition lemma, Ramsey theorems and methods provided in this paper have been basis for some further results in several papers. Bartal, Fandina and Neiman [16] study tree covers, bounding the number of trees necessary so that every pairwise distance is maintain in one of the trees within a given distortion bound. Among other results they show that Theorem 1 implies near tight bounds on Ramsey tree covers. Recently, Filtser and Le [25] make use of our decomposition and techniques to establish a distributional version of the multi-embedding theorem and thereby obtain a bound on the expected duplicity of points, which they then use to obtain certain “clan embedding” properties, which they show useful for compact routing. In the work of Abraham et. al [8] similar though somewhat more involved techniques are applied to obtain Ramsey metric theorems for spanning trees in graphs.

1.2 Preliminaries

**Definition 1.** Let $X, Y$ be metric spaces. An embedding of $X$ into $Y$ is a function $f : X \rightarrow Y$. The distortion of $f$ for the pair $\{u, v\} \in \binom{X}{2}$ is $\text{dist}_f(u, v) = \frac{d_Y(f(u), f(v))}{d_X(u, v)}$. The distortion of $f$ is given by $\frac{\max_{\{u, v\} \in \binom{X}{2}} \text{dist}_f(u, v)}{\min_{\{u, v\} \in \binom{X}{2}} \text{dist}_f(u, v)}$.

An embedding $f$ is non-contractive if for any $u, v \in X$: $d_Y(f(u), f(v)) \geq d_X(u, v)$ and non-expansive if for any $u, v \in X$: $d_Y(f(u), f(v)) \leq d_X(u, v)$.

For a vertex $v$ and $r \geq 0$, the ball at radius $r$ around $v$ is defined as $B(v, r) = \{u \in V | d(u, v) \leq r\}$.

1.3 Metric Ramsey Theorems

1.3.1 Definitions

We recall some definitions and notation from [10] that is useful in the context of the metric Ramsey problem.

**Definition 2** (Metric Ramsey functions). Let $\mathcal{M}$ be some class of metric spaces. For a metric space $X$, and $\alpha \geq 1$, $R_{\mathcal{M}}(X; \alpha)$ denotes the largest size of a subspace $Y$ of $X$ such that $Y$ embeds in a metric space in $\mathcal{M}$ with distortion $\alpha$.

Denote by $R_{\mathcal{M}}(\alpha, n)$ the largest integer $m$ such that any $n$-point metric space has a subspace of size $m$ that $\alpha$-embeds into a member of $\mathcal{M}$. In other words, it is the infimum over $X$, $|X| = n$, of $R_{\mathcal{M}}(X; \alpha)$.

In the most general form, let $\mathcal{N}$ be a class of metric spaces and denote by $R_{\mathcal{M}}(\mathcal{N}; \alpha, n)$ the largest integer $m$ such that any $n$-point metric space in $\mathcal{N}$ has a subspace of size $m$ that $\alpha$-embeds into a member of $\mathcal{M}$. In other words, it is the infimum over $X \in \mathcal{N}$, $|X| = n$, of $R_{\mathcal{M}}(X; \alpha)$.

It is useful to generalize the metric Ramsey problem to weighted metric spaces. Although the weighted Ramsey problem defined above is not necessary to obtain our basic results they are useful for extending them, e.g. by using results from [10].

Let a weighted metric space be a pair $(X, w)$, where $X$ is a metric space and $w : X \rightarrow \mathbb{R}^+$ is a weight function. For a subspace $Z \subseteq X$ let $w(Z) = \sum_{x \in Z} w(x)$. For $0 \leq \psi$, let $w^\psi$ denote the weight function defined by $w^\psi(x) = w(x)^\psi$, for every $x \in X$.

Note that for weight function $w(x) \equiv 1$: $w(Z) = w^\psi(Z) = |Z|$. We note that although we will use the weighted metric notation throughout, in most cases the uniform weight can be assumed.
Definition 3 (Weighted Ramsey Function). Let $\mathcal{M}, \mathcal{N}$ be classes of metric spaces. Denote by $\psi_{\mathcal{M}}(\mathcal{N}, \alpha)$ the largest $0 \leq \psi \leq 1$ such that for every metric space $X \in \mathcal{N}$ and any weight function $w : X \to \mathbb{R}^+$, there is a subspace $Y$ of $X$ that $\alpha$-embeds in $\mathcal{M}$ and satisfies: $w^\psi(Y) \geq w(X)^\psi$. When $\mathcal{N}$ is the class of all metric spaces, it is omitted from the notation.

The following is an immediate consequence of Definition 3.

Proposition 1. \[ R_{\mathcal{M}}(\mathcal{N}; \alpha, n) \geq n^{\psi_{\mathcal{M}}(\mathcal{N}, \alpha)}. \]

In particular, \[ R_{\mathcal{M}}(\alpha, n) \geq n^{\psi_{\mathcal{M}}(\alpha)}. \]

Let $\mathcal{U}M$ denote the class of ultrametrics. It is well-known (c.f. [30]) that ultrametrics embed isometrically in $\ell_2$. Therefore $\psi_{L_2}(\alpha) \geq \psi_{\mathcal{U}M}(\alpha)$.

Theorem 1. For any integer $t \geq 2$, \[ \psi_{\mathcal{U}M}(8t) \geq 1 - 1/t. \]

In particular, any $n$-point metric space contains a subspace of size $n^{1-1/t}$ which embeds in an ultrametric with distortion $8/t$.

We note that the constant in the theorem above can in fact be further improved at the price of proof elegance.

1.4 Stronger Ramsey Theorems: Subspaces of Linear Size

Recall the following definitions from [5]:

Definition 4 (Partial/Scaling Embedding). For $\epsilon > 0$, a $(1 - \epsilon)$-partial embedding $f$ has distortion $\alpha(\epsilon)$ if it is non-contractive and there exists a set $G_\epsilon \subset \binom{\mathcal{X}}{2}$ of size at least $(1 - \epsilon)\binom{n}{2}$ such that for every $u, v \in G_\epsilon$, \[ \text{dist}_f(u, v) \leq \alpha(\epsilon). \]

An embedding has scaling distortion $\alpha(\epsilon)$ if it is $(1 - \epsilon)$-partial for every $\epsilon > 0$.

Definition 5 ($\ell_q$-distortion). For $1 \leq q \leq \infty$, define the $\ell_q$-distortion of an embedding $f$ as:

\[ \text{dist}_q(f) = \|\text{dist}_f(u, v)\|_{L^q} = \mathbb{E}[\text{dist}_f(u, v)^q]^{1/q}, \]

where the expectation is taken according to the uniform distribution $\mathcal{U}$ over $\binom{\mathcal{X}}{2}$. The classic notion of distortion is expressed by the $\ell_\infty$-distortion and the average distortion is expressed by the $\ell_1$-distortion.

In [5] the notion of scaling embedding is shown to be closely related to the $\ell_q$-distortion of the embedding. They give partial and scaling distortion results for embedding into $L_p$. In [2] it was shown that every finite metric space embeds into an ultrametric with $O(\sqrt{1/\epsilon})$ scaling distortion (and this bound is tight).

Here we prove the following strengthened Ramsey theorems:

Theorem 2 (Partial Metric Ramsey Theorem). For every $\delta > 0$ and $\epsilon > 0$, any $n$-point metric space contains a subspace $Y$ of size $\geq \delta n$ such that $Y$ has a $(1 - \epsilon)$-partial embedding into an ultrametric with distortion $O(\log_{1/\delta} 1/\epsilon)$.

\footnote{A special type of partial embedding is called coarse where $G_\epsilon$ is composed of pairs $u, v$ where $v$ is not in the ball containing $\epsilon n/2$ points around $u$.}
Let \( \vartheta : \mathbb{R}^+ \to \mathbb{R}^+ \) be a function such that \( \int_1^\infty \frac{dx}{\vartheta(x)} = 1 \). In particular for any \( \xi > 0 \), we can have \( \vartheta(x) \leq cx \log^{1+\xi}(x) \).

**Theorem 3** (Scaling Metric Ramsey Theorem). For every \( \delta > 0 \), any \( n \)-point metric space contains a subspace \( Y \) of size \( \geq \delta n \) such that \( Y \) has a embedding into an ultrametric with scaling distortion \( O( [\vartheta(\log_{1/\delta} 1/\epsilon)] ) \). As a consequence \( \ell_q \)-distortion is bounded by \( O([\vartheta(q/\log(1/\delta))] ) \).

We also provide additional Ramsey theorems that give local distortion bounds. However, in this context the Ramsey embedding version of these theorems are more natural and will be discussed in the subsequent subsection below.

In addition we present below an improved Ramsey theorem for doubling and excluded-minor metrics.

### 1.4.1 Ramsey Theorems for Decomposable Metric Spaces

Recall that metric spaces \((X,d)\) can be characterized by their decomposability parameter \( \tau_X \) where it is known that \( \tau_X = O(\log \lambda_X) \), where \( \lambda_X \) is the doubling constant of \( X \), and for metrics of \( K_{s,s} \)-excluded minor graphs. \( \tau_X = O(s^2) \).

**Theorem 4** (Ramsey-type Theorem for Decomposable Metrics). Let \( X \) be a metric space. There exists \( C > 0 \) such that for every \( 1 \leq p \leq \infty \), and any \( \alpha > 1 \):

\[
\psi_{L_p}(X,\alpha) \geq 1 - C \left( \frac{\tau_X}{\alpha} \right)^p \log \left( \frac{\tau_X}{\alpha} \right).
\]

In particular, for every \( \varepsilon > 0 \), \( X \) contains a subspace of size \( n^{1-\varepsilon} \) which embeds in \( L_p \) space with distortion \( O(\tau_X/\varepsilon^{1/p}) \).

### 1.5 Ramsey Embedding, Ramsey Covers and Proximity Data Structures

We show that our algorithms for computing metric Ramsey constructions can be applied to obtain new results for proximity data structures, including distance oracles and approximate ranking.

To this aim we define the notion of a Ramsey embedding.

**Definition 6.** Given metric spaces \( X,Y \). A Ramsey embedding of \( X \) into \( Y \) is a pair composed of an embedding \( f : X \to Y \) and a subspace \( X' \subseteq X \). A non-contractive Ramsey embedding has distortion \( \alpha \) if for every \( x \in X' \) and \( y \in X \), \( \text{dist}(f(x),y) \leq \alpha \). We call the subspace \( X' \) the core subspace of the embedding.

We show that our Ramsey theorems can be extended to provide Ramsey embeddings. These can be further extended to obtain Ramsey covers which we then use to obtain first deterministic constructions distance oracles and approximate ranking data structures with optimal query, stretch and space tradeoffs.

Moreover we give the first construction of such data structures with \( O(n) \) storage, \( O(1) \) query time, and \( O(1) \) average distortion and \( \ell_q \) distortion for every fixed \( q < \infty \) (and the worst case distortion is \( O(\log n) \)).

### 1.5.1 \( k \)-Local Embeddings and Data Structures

**Definition 7.** For \( x \in X \) let \( r_k(x) \) the minimum \( r \) such that \( |B(x,r)| \geq k \). A Ramsey embedding \( f : X \to Y \) with core subspace \( X' \) has \( k \)-local distortion \( \alpha \) if it is non-expansive and for every \( x,y \in X \), \( d_Y(f(x),f(y)) \geq \min \{ d_X(x,y),r_k(x) \} / \alpha \). We say that \( f \) has local scaling distortion \( \alpha(k) \) if it is \( k \)-local for every \( 1 \leq k \leq n \).
We prove the following strengthened Ramsey embedding theorems:

**Theorem 5** (Local Metric Ramsey-type Theorem). For every $\varepsilon > 0$ and $k \in \mathbb{N}$, any $n$-point metric space has a Ramsey embedding with $k$-local distortion $O(1/\varepsilon)$ and core subspace $Y$ of size $\geq n \cdot k^{-\varepsilon}$.

**Theorem 6** (Scaling Local Metric Ramsey-type Theorem). For every $\varepsilon > 0$, any $n$-point metric space has a Ramsey embedding with scaling local distortion $O(\min\{\vartheta(\log k), 1/\varepsilon\})$ and core subspace $Y$ of size $\geq n^{1-\varepsilon}$.

By following the same procedure discussed before we can use these theorems to obtain new deterministic constructions of distance oracles and approximate ranking data structures with $k$-local stretch and scaling local stretch respectively. In particular, we obtain the first construction of such data structure with $O(n)$ storage and $O(\vartheta(\log k))$ $k$-local stretch (i.e. this bounds the stretch for the $k$th nearest neighbor of a point).

### 1.6 Multi-Embedding

Another application of our Ramsey decomposition is to obtain optimal multi-embeddings of metric spaces into ultrametrics [18]. A multi-embedding of a space $X$ into $Y$ is a mapping of points in $X$ to sets of points in $Y$. It is desirable that the size of $Y$ would be small. The path distortion of a multi-embedding is $\alpha$ if for every path $p$ in $X$ there is a corresponding path in $Y$ over the images of points in $p$ whose length is at most $\alpha$ times the length of $p$.

The following theorem gives a tight bound on multi-embedding into ultrametrics, improving the previous result of [18]. This implied improvements to algorithms for the online metrical task systems problem and for the group Steiner tree problem for metric spaces with small aspect ratio, and provides simpler algorithms for these problems.

**Theorem 7.** For any metric space on $n$ points and aspect ratio $\Phi$, and any $\varepsilon > 0$, there exists a multi-embedding into an ultrametric of size $n^{1+\varepsilon}$, whose path distortion is at most $O(\min\{\log n, \log \Phi\}/\varepsilon)$.

### 2 Notation

For sets $U, V \subseteq X$ let $d(U, V) = \min\{d(u, v) | u \in U, v \in V\}$. Let $\Delta(X) = \text{diam}(X)$ denote the diameter of $X$.

#### 2.1 Ultrametrics and Hierarchically Well-Separated Trees

Recall that an *ultrametric* is a metric space $(X, d)$ such that for every $x, y, z \in X$,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

We recall the following definition from [11]:

**Definition 8.** For $k \geq 1$, a $k$-hierarchically well-separated tree ($k$-HST) is a metric space whose elements are the leaves of a rooted tree $T$. To each vertex $u \in T$ there is associated a label $\Lambda(u) \geq 0$ such that $\Lambda(u) = 0$ iff $u$ is a leaf of $T$. It is required that if a vertex $u$ is a child of a vertex $v$ then $\Lambda(u) \leq \Lambda(v)/k$. The distance between two leaves $x, y \in T$ is defined as $\Lambda(\text{lca}(x, y))$, where $\text{lca}(x, y)$ is the least common ancestor of $x$ and $y$ in $T$. 


First, note that an ultrametric and a 1-HST are identical concepts. Any $k$-HST is also a 1-HST, i.e., an ultrametric. Any ultrametric is $k$-equivalent to a 1-HST [11].

When we discuss $k$-HSTs, we freely use the tree $T$ as in Definition 3 the tree defining the HST. Let UM to denote the class of ultrametrics, and $k$-HST denotes the class of $k$-HSTs.

3 Ramsey Decomposition

Define the spherical-weight of $Z \subseteq X$, $w^*(Z) = \max_{z \in Z} w(B(z, \Delta(Z)/4))$.

**Lemma 2.** Given a metric space $X$, and $0 < \hat{\Delta} \leq \Delta(X)/2$, and integer $t \geq 2$, then there exists a partition $(Q, \bar{Q})$ of $X$, and $P \subseteq Q$, such that: $\Delta(Q) \leq \hat{\Delta}$, $d(P, Q) \geq \hat{\Delta}/(4t)$, and

$$w(P) \geq w(Q) \cdot \left( \frac{w^*(X)}{w^*(Q)} \right)^{-1/t}.$$  

**Proof.** Let $v$ be a node that minimizes the ratio $w(B(v, \hat{\Delta}/2))/w(B(v, \hat{\Delta}))/4)$. We will choose $Q = B(v, r)$ for some $r \in [\hat{\Delta}/4, \hat{\Delta}/2]$. For $0 \leq i \leq t$, define $Q_i = B((1 + \frac{i}{t})\hat{\Delta}/4)$. Clearly there exist some $i > 0$ such that $w(Q_i) \leq w(Q_{i-1}) \left( \frac{w(Q_i)}{w(Q_{i-1})} \right)^{\frac{1}{t}}$. Then we set $Q = Q_i$ and $P = Q_{i-1}$. Therefore we have that $d(P, Q) \geq \hat{\Delta}/(4t)$ and

$$w(P) \geq w(Q) \cdot \left( \frac{w(B(v, \hat{\Delta}/2))}{w(B(v, \hat{\Delta}/4))} \right)^{-1/t}.$$  

Now, let $u$ be the node that maximizes $w(B(u, \Delta(Q)/4))$. Since $\Delta(Q) \leq \hat{\Delta}$ we have that $w^*(Q) \leq w(B(u, \Delta(Q)/4))$. Recall that $w^*(X) \geq w(B(u, \Delta(Q)/2))$, as $\Delta \leq \Delta(X)/2$. By the choice of $v$ we conclude that

$$w(P) \geq w(Q) \cdot \left( \frac{w(B(u, \Delta(Q)/2))}{w(B(u, \Delta(Q)/4))} \right)^{-1/t} \geq w(Q) \cdot \left( \frac{w^*(X)}{w^*(Q)} \right)^{-1/t}. \tag{11}$$

\[ \square \]

4 Metric Ramsey Theorems

**Theorem 1.** For any integer $t \geq 2$,

$$\psi_{UM}(8t) \geq 1 - 1/t.$$  

**Proof.** Let $X$ be an arbitrary metric space. Let $Z \subseteq X$. We will construct a subspace $S(Z) \subseteq Z$ and an ultrametric $U(S(Z))$ recursively as follows: use the decomposition described in Section 3 with $t = \Delta(Z)/2$ to obtain a partition of the graph $(Q, \bar{Q})$ and $P \subseteq Q$ satisfying Lemma 2. Run the algorithm on $P$ and $Q$ recursively, obtaining subspaces $S(P)$ and $S(Q)$ and ultrametrics $U(S(P))$ and $U(S(Q))$ respectively. Let $S(Z) = S(P) \cup S(Q)$. The ultrametric $U(Z)$ is constructed by creating a root $r$ labeled with $\Lambda(r) = \Delta(S(Z))$ with two children at which we root the trees defining $U(S(P))$ and $U(S(Q))$.

We first prove by induction on the size of $Z \subseteq X$ that $S(Z)$ is $4t$ equivalent to $U(S(Z))$ via a non-contractive embedding and $\Delta(U(S(Z)) = \Delta(S(Z))$. 

\[ 7 \]
If \( Z \) includes a single point \( z \) then the claim trivially holds. Assume by induction that the claim holds for strict subsets of \( Z \). Consider \( x, y \in S(Z) \). If \( x, y \in S(P) \) their distance in \( U(Z) \) is the same as in \( U(P) \) and therefore the claim follows from the induction hypothesis. If \( x, y \in \bar{Q} \) then a similar argument holds.

Let \( \eta = 1/(4t) \). Let \( x \in P \) and \( y \in \bar{Q} \) then by Lemma \( \ref{lemma2} \) \( d(x, y) \geq \eta/2 \cdot \Delta(Z) \geq \eta/2 \cdot \Delta(S(Z)) \). As \( d(x, y) \leq \Delta(S(Z)) \) it follows that \( d(x, y) \leq d_{U(Z)}(x, y) \leq 2/\eta \cdot d(x, y) \), and we conclude that \( S(Z) \) is \((8t)\text{-equivalent to} U(Z)\).

Let \( \psi = 1 - 1/t \). Next, we prove by induction on the size of \( Z \subseteq X \) that

\[
w^{\psi}(S(Z)) \geq w(Z) \cdot w^{\ast}(Z)^{-1/t}.
\]

If \( Z = \{z\} \) includes a single point \( z \) then the claim trivially holds since \( w^{\psi}(S(Z)) = w(Z) \cdot w^{\ast}(Z)^{-1/t} = w(z)^{\psi} \). By applying the induction hypothesis and Lemma \( \ref{lemma2} \) we obtain

\[
w^{\psi}(S(Z)) = w^{\psi}(S(P)) + w^{\psi}(S(\bar{Q})) \geq w(P) \cdot w^{\ast}(P)^{-1/t} + w(\bar{Q}) \cdot w^{\ast}(\bar{Q})^{-1/t}
\geq (w(Q) + w(\bar{Q})) \cdot w^{\ast}(Z)^{-1/t} = w(Z) \cdot w^{\ast}(Z)^{-1/t}.
\]

Noting that \( w^{\ast}(X) \leq w(X) \) we conclude that \( w^{\psi}(S(X)) \geq w(X)^{\psi} \).

### 4.1 Stronger Ramsey Theorems: Subspaces of Linear Size

In this section we will fix \( w(Z) = w^{\psi}(Z) = |Z| \) (however, the claims and proofs can be appropriately generalized to more general weight functions).

**Theorem 2.** For every \( \delta > 0 \) and \( \epsilon > 0 \), any \( n \)-point metric space contains a subspace \( Y \) of size \( \geq \delta n \) such that \( Y \) has a \((1 - \epsilon)\text{-partial embedding into an ultrametric with distortion } O(\lceil \log_{1/\delta} 1/\epsilon \rceil) \).

**Proof.** Let \( X \) be an arbitrary metric space. Let \( Z \subseteq X \). We construct a subspace \( S(Z) \subseteq \bar{Z} \) and an ultrametric \( U(S(Z)) \) recursively as follows. If \( w(Z) \leq \epsilon \cdot w(X) \) we let \( S(Z) = Z \) and define \( U(S(Z)) \) to be a rooted star with \( |Z| \) leaves and label the root with \( \Lambda(r) = \Delta(S(Z)) \). Otherwise, if \( w(Z) > \epsilon \cdot w(X) \) use the decomposition described in Section \( \ref{section3} \) with \( \eta = 1/[6 \log_{1/\delta} 1/\epsilon] \) and \( \Delta = \Delta(S(Z))/2 \) to obtain a partition of the graph \( (Q, \bar{Q}) \) and \( P \subset Q \) satisfying Lemma \( \ref{lemma2} \). Run the algorithm on \( P \) and \( \bar{Q} \) recursively, obtaining subspaces \( S(P) \) and \( S(\bar{Q}) \) and ultrametrics \( U(S(P)) \) and \( U(S(\bar{Q})) \) respectively. Let \( S(Z) = S(P) \cup S(\bar{Q}) \). The ultrametric \( U(Z) \) is constructed by creating a root \( r \) labeled with \( \Lambda(r) = \Delta(S(Z)) \) with two children at which we root the trees defining \( U(S(P)) \) and \( U(S(\bar{Q})) \).

We first prove by induction on the size of \( Z \subseteq X \) that there is a \((1 - \epsilon)\text{-partial embedding of } S(Z) \text{ into } U(S(Z)) \text{ with distortion } 2/\eta = O(\lceil \log_{1/\delta} 1/\epsilon \rceil), \text{ and that } \Delta(U(S(Z))) = \Delta(S(Z)). \)

If \( w(Z) \leq \epsilon \cdot w(X) \) then the number of pairs in \( Z \) is at most \( \lceil |Z|/2 \rceil \leq |Z|\epsilon(n - 1)/2 \). All such pairs are excluded from \( G_{\epsilon} \). It follows that the number of excluded pairs sums up to at most \( \epsilon(n - 1)/2 \) in total. Otherwise if \( w(Z) > \epsilon \cdot w(X) \) then the same argument in the proof of Theorem \( \ref{thm} \) holds.

Let \( \psi = 1 - 6\eta \). Next, we prove by induction on the size of \( Z \subseteq X \) that

\[
w^{\psi}(S(Z)) \geq w(Z) \cdot \left[ \frac{w^{\ast}(Z)}{\epsilon \cdot w(X)} \right]^{-6\eta}.
\]

If \( w(Z) \leq \epsilon \cdot w(X) \) then \( S(Z) = Z \) and the claim trivially holds since \( w^{\ast}(Z) \leq w(Z) \leq \epsilon \cdot w(X) \). Otherwise if \( w(Z) > \epsilon \cdot w(X) \) then essentially the same argument in the proof of Theorem \( \ref{thm} \) holds.

Noting that \( w^{\ast}(X) \leq w(X) \) we conclude that \( w^{\psi}(S(X)) \geq w(X) \cdot \left( \frac{1}{\epsilon} \right)^{-6/[6 \log_{1/\delta} 1/\epsilon]} \geq \delta \cdot w(X) \).
Theorem 3. For every $\delta > 0$, any $n$-point metric space contains a subspace $Y$ of size $\geq \delta n$ such that $Y$ has an embedding into an ultrametric with scaling distortion $O(\log(1/\delta))$. As a consequence its $\ell_q$-distortion is bounded by $O(\log(1/\delta))$).

Proof. Let $X$ be an arbitrary metric space. Let $Z \subseteq X$. Define $\ell(Z) = \max\{\log_{1/\delta}(w(X)/w(Z)), 1\}$. We will construct a subspace $S(Z) \subseteq Z$ and an ultrametric $U(S(Z))$ recursively as follows: use the decomposition described in Section 3 with $\eta = 1/(6\ell(\ell(Z)))$ and $\Delta = \Delta(Z)/2$ to obtain a partition of the graph $(Q, Q)$ and $P \subseteq Q$ satisfying Lemma 2. Run the algorithm on $P$ and $Q$ recursively, obtaining subspaces $S(P)$ and $S(Q)$ and ultrametrics $U(S(P))$ and $U(S(Q))$ respectively. Let $S(Z) = S(P) \cup S(Q)$. The ultrametric $U(Z)$ is constructed by creating a root labeled with $\Lambda(r) = \Delta(S(Z))$ with two children at which we root the trees defining $U(S(P))$ and $U(S(Q))$.

We first prove by induction on the size of $Z \subseteq X$ that there exists an embedding of $S(Z)$ into $U(S(Z))$ with coarsely scaling distortion $12[\log_{1/\delta} \ell(\ell(Z))]/\epsilon$, and that $\Delta(U(S(Z))) = \Delta(S(Z))$. More specifically, let $r_x(u)$ be the minimum $r$ such that $w(B(x, u)) \geq \epsilon \cdot w(X)$. That is, we let $G_x = \{(u, v) | d(u, v) \geq \max\{r_x/2(u), r_x/2(v)\}\}$.

If $Z$ includes a single point $z$ then the claim trivially holds. Assume by induction that the claim holds for strict subsets of $Z$. Consider $x, y \in S(Z)$. If $x, y \in S(P)$ their distance in $U(S(Z))$ is the same as in $U(P)$ and therefore the claim follows from the induction hypothesis. If $x, y \in Q$ then a similar argument holds. Let $x \in P$ and $y \in Q$, and assume $x, y \in G_x$. We may assume that $d(x, y) \leq \Delta(S)/4$, otherwise we can bound the distortion by $4$. It follows that $w^*(Z) \geq w(B(x, d(x, y)) \geq \epsilon \cdot w(X)$. Then by Lemma 2 $d(x, y) \geq \eta \Delta \geq \eta/2 \cdot \Delta(S(Z))$. As $d(x, y) \leq \Delta(S(Z))$ it follows that $d(x, y) \leq d_U(z)/2, y \leq 2\eta \cdot d(x, y)$, so that the distortion of $x$ and $y$ is bounded by $12[\log_{1/\delta} \ell(\ell(Z))]/\epsilon$.

We prove by induction on the size of $Z \subseteq X$ that

$$w^\ell(S(Z)) \geq \ell(S(Z)) \cdot \delta^{-\ell(\ell(Z))}.$$

If $Z = \{z\}$ includes a single point $z$ then the claim trivially holds since $w^\ell(S(Z)) = w(Z) = 1$. By applying the induction hypothesis and Lemma 2 we obtain

$$w^\ell(S(Z)) = w^\ell(S(P)) + w^\ell(S(Q)) \geq w(P) \cdot \delta^{\int_{1/\delta} \log_{1/\delta}(w(Z)/w(Q)) \cdot \int_{1/\delta} \log_{1/\delta}(w(Q)/w(z))} + w(Q) \cdot \delta^{\int_{1/\delta} \log_{1/\delta}(w(Q)/w(z)) \cdot \int_{1/\delta} \log_{1/\delta}(w(z)/w(z))}.$$

Since

$$\left(\frac{w^*(Z)}{w^*(Q)}\right)^{-\ell(\ell(Z))} = \delta^{\int_{1/\delta} \log_{1/\delta}(w(Z)/w(Q)) \cdot \int_{1/\delta} \log_{1/\delta}(w(Q)/w(z))} \geq \delta^{\int_{1/\delta} \log_{1/\delta}(w(Q)/w(z)) \cdot \int_{1/\delta} \log_{1/\delta}(w(z)/w(z))},$$

we get that

$$w^\ell(S(Z)) \geq w(Q) \cdot \delta^{\int_{1/\delta} \log_{1/\delta}(w(Q)/w(z)) \cdot \int_{1/\delta} \log_{1/\delta}(w(z)/w(z))} + \delta^{\int_{1/\delta} \log_{1/\delta}(w(Q)/w(z)) \cdot \int_{1/\delta} \log_{1/\delta}(w(z)/w(z))} + w(Q) \cdot \delta^{\int_{1/\delta} \log_{1/\delta}(w(Q)/w(z)) \cdot \int_{1/\delta} \log_{1/\delta}(w(z)/w(z))}.$$

Noting that $w^*(X) \leq w(X)$ and $\int_{1/\delta} \log_{1/\delta}(w(Q)/w(z)) \cdot \int_{1/\delta} \log_{1/\delta}(w(z)/w(z)) = 1$ we conclude that $w^\ell(S(X)) \geq \delta \cdot w(X).$ $\square$

The Ramsey theorems which obtain linear size spaces with local distortions are discussed in the next section.
5 Ramsey Embedding, Ramsey Covers and Proximity Data Structures

In this section we obtain first deterministic constructions of distance oracles and approximate ranking data structures with constant query time and asymptotically optimal storage-stretch tradeoffs.

5.1 Ramsey Embedding

The first step is to extend our Ramsey theorems to obtain Ramsey embeddings (see definition 6). As our construction must support distortions which vary as function of the pairs of points this does not follow directly from the Ramsey theorems themselves but the Ramsey embeddings can be derived by an appropriate modification of the algorithms and proofs as described below. We obtain the following theorems:

**Theorem 1.** For every $0 < \varepsilon$, and any $n$-point metric space $X$, there exists a Ramsey embedding of $X$ into an ultrametric with a core subspace of size $n^{1-\varepsilon}$ and distortion $O(1/\varepsilon)$.

**Theorem 2.** For every $\delta > 0$ and $\epsilon > 0$, and any $n$-point metric space $X$, there exists a Ramsey $(1 - \epsilon)$-partial embedding of $X$ into an ultrametric with a core subspace $Y$ of size $\geq \delta n$ and distortion $O(\log_{1/\delta} 1/\epsilon)$.

**Theorem 3.** For every $\delta > 0$, any $n$-point metric space $X$, there exists a Ramsey embedding of $X$ into an ultrametric with a core subspace $Y$ of size $\geq \delta n$ and scaling distortion $O(\log_{1/\delta} 1/\epsilon)$. As a consequence its $\ell_q$-distortion is bounded by $O(\log_{1/\delta} q)$.

To modify the proofs in Section 4 we need to build an ultrametric over the entire space $X$ rather than just on the subspace $S(X)$. Let us describe how to modify the constructions.

It would be useful to use a variation of Lemma 2 when replacing the function $\psi$ with $w_C$ which restricted to a subset $C$. Similar variation can be applied to the theorems of Section 4 (similarly replacing $w^\phi$ with $w_C^\phi$).

Let $X$ be an arbitrary metric space. The algorithm builds an ultrametric recursively. For a $Z \subseteq X$ we also maintain a core $C(Z) \subseteq Z$. Initially $Z = C(Z) = X$. We will construct a subspace $S(Z) \subseteq C(Z)$ and an ultrametric $U(S(Z))$ recursively as follows: use the decomposition described in Section 3 on $X$ with the variation described above for $C = C(Z)$, and with $\eta_Z$ defined appropriately as in the proofs of the theorems in Section 4 and $\Delta = \Delta(Z)/2$ to obtain a partition $(Q, Z \setminus Q)$ of the metric space on $Z$, and a set $P \subseteq Q \cap C(Z)$ satisfying Lemma 2. Let $R = \{x \in Z | d(x, P) \leq \eta_Z \Delta/2\}$. Run the algorithm recursively on $R$ with core $C(R) = P$ and on $\bar{R}$ with core $C(\bar{R}) = \bar{Q}$, obtaining subspaces $S(R)$ and $S(\bar{R})$ and ultrametrics $U(R)$ and $U(\bar{R})$ respectively. It is easy to verify that construction above is indeed valid satisfying $P \subseteq R$ and $Q \subseteq \bar{R}$, where the second condition follows as $d(P, Q) > \eta_Z \Delta$. Let $S(Z) = S(R) \cup S(\bar{R})$. The ultrametric $U(Z)$ is constructed by creating a root $r$ labeled with $\Delta(r) = \Delta(Z)$ with two children at which we root the trees defining $U(R)$ and $U(\bar{R})$.

We prove by induction on the size of $Z \subseteq X$ that there is a Ramsey embedding of $Z$ into $U(Z)$ with distortion $4/\eta_Z$.

If $Z$ includes a single point $z$ then the claim trivially holds. Assume by induction that the claim holds for strict subsets of $Z$. Consider $x \in S(Z)$ and $y \in Z$. If $x \in S(R)$ and $y \in R$ then their distance in $U(Z)$ is the same as in $U(R)$ and therefore the claim follows from the induction hypothesis. If $x \in S(\bar{R})$ and $y \in \bar{R}$ then a similar argument holds. Let $x \in S(R) \subseteq P$ and $y \in \bar{R}$ then by definition of $R$, $d(x, y) \geq \eta_Z \Delta/2$. Similarly if $x \in S(\bar{R}) \subseteq Q$ and $y \in R$ then by Lemma 2 $d(x, y) \geq \eta_Z \Delta/2$. Hence in both cases $d(x, y) \geq \eta_Z \Delta/2 \geq \eta_Z \Delta(Z)/4$. As $d(x, y) \leq \Delta(Z)$ it follows that $d(x, y) \leq d_{U(Z)}(x, y) \leq 4/\eta_Z \cdot d(x, y)$, and we conclude that our Ramsey embedding has distortion $4/\eta$. The rest of the proofs is the same as in Section 4.
5.2 Ramsey Covers

Let $\mathcal{M}$ be a class of metric spaces. Assume that given a metric space $X$ we can construct a Ramsey embedding of $X$ into $Y \in \mathcal{M}$ of size at least $\beta|X|$ which embeds with distortion $\alpha$ (possibly a function of pairs in $X$).

We build a Ramsey cover as follows. We apply this construction iteratively a follows: Let $X_0 = X$ and let $Z_0$ be the core subspace of $X$, and $Y_0$ be image of $X_0$ in $\mathcal{M}$ under the Ramsey embedding. For $i > 0$ let $X_i = X_{i-1} \setminus Z_{i-1}$ and let $Z_i$ be the core subspace of $X$, and $Y_i$ be the image of $X_i$ in $\mathcal{M}$ under the Ramsey embedding.

This construction yields a collection of spaces $Y_0, \ldots, Y_t \in \mathcal{M}$, where $|Y_i| \leq (1 - \beta)^i|X|$, such that for every $x \in X$ there exists $i \in [t]$ such that for every $y \in X_i$ the distortion of $x$ and $y$ in $Y_i$ is at most $\alpha(x, y)$. In particular, for $s \geq 1$, $\sum_i |Y_i|^s \leq |X|^s/\beta$.

By using Theorems 1, 2, and 3 in the above construction of Ramsey cover we obtain a cover by ultrametrics.

5.3 Application to Proximity Data Structures

An approximate distance oracle is a data structure for a given metric space $X$ of size $S$ (the space) such that for every $x, y \in X$ an approximation of the distance between them can be computed in time $Q$ (the query time) and distortion $D$ (also called stretch).

Thorup and Zwick [35] gave a randomized construction of approximate distance oracles of size $O(t \cdot n^{1+1/t})$, distortion $2t - 1$ and query time $O(k)$, for any $k \in \mathbb{N}$. Mendel and Naor [32] gave a different randomized construction of approximate distance oracles of size $n^{1+1/t}$, distortion $O(t)$ and query time $O(1)$. The preprocessing expected time is $O(n^{2+1/t} \log n)$. In [35] it is shown that this space-distortion tradeoff is best possible up to the constants.

One can naturally define $(1 - \epsilon)$-partial and scaling distortion distance oracles. These notions have been previously studied in [5, 22] where the results of [35] have been adapted to accommodate these notions. In particular, the scaling distortion constructions imply constant average distortion in $O(n \log n)$ space.

Using the construction of the Ramsey cover by ultrametrics described above we can obtain several new deterministic constructions of approximate distance oracles. The application follows since computing the distance in an ultrametric can be done by computing the LCA of the two leaves in $O(1)$ time. We get the following results:

**Theorem 4.** There exist deterministic constructions with the following properties:

1. For every $t \geq 1$, there exists an approximate distance oracle with space: $n^{1+1/t}$, distortion: $O(t)$, and query time $O(1)$.

2. For every $t \geq 1$ and $\epsilon \in (0, 1)$, there exists a $(1 - \epsilon)$-partial approximate distance oracle with space: $n \cdot (\frac{1}{t})^{1/t}$, distortion: $O(t)$, and query time: $O(1)$.

3. For every $t \geq 1$ and $\epsilon \in (0, 1)$, there exists a approximate distance oracle with space: $n^{1+1/t}$, distortion: $O(1)$, and query time: $O(1)$.

The preprocessing time can be bounded\(^3\) by $O(n^2)$.

\(^3\)The bound on the preprocessing time for the algorithm described here is larger by a factor of $n$. This can be improved by a more involved implementation and the details are left for the full version.
In particular we get the following corollary:

**Corollary 3.** There exists a deterministic construction of an approximate distance oracle with space: $O(n)$, distortion: $O(\vartheta(\log 1/\epsilon))$ and query: $O(1)$. In particular it has average distortion: $O(1)$ and $\ell_q$-distortion $O(\min\{\vartheta(q), \log n\})$.

Similar results can be derived for the approximate ranking problem via an approach similar to [32].

### 5.4 $k$-Local Embeddings and Data Structures

**Theorem 5.** For every $\epsilon > 0$ and $k \in \mathbb{N}$, any $n$-point metric space has a Ramsey embedding with $k$-local distortion $O(1/\epsilon)$ and core subspace $Y$ of size $\geq n \cdot k^{-\epsilon}$.

**Proof.** The proof is similar to the one described in the head of this section. The main difference is that the core subgraph to be partitioned $C(Z)$ is chosen as the subspace of $Z$ of maximum diameter amongst all subspaces of size at most $k$. To take care that the embedding is non-expansive we set the label of the constructed ultrametric $U(Z)$ to be $\Delta(C(Z))/ (C \cdot t)$ for some appropriate constant $C$. We omit the details of the proof.

**Theorem 6.** For every $\epsilon > 0$, any $n$-point metric space has a Ramsey embedding with scaling local distortion $O(\min\{\vartheta(\log k), 1/\epsilon\})$ and core subspace $Y$ of size $\geq n^{1-\epsilon}$.

**Proof.** Again, we follow the proof described in the head of the section. To obtain scaling local distortion we modify the subspace to be partitioned $C(Z)$ to be a subspace $H$ that maximizes $\Delta(H')/|H'|$ over all subspaces $H'$ such that $H' \cap H \neq \emptyset$. We set the label of the constructed ultrametric $U(Z)$ to be $\Delta(C(Z))/ (C \cdot \min\{t, \vartheta(\log k)\})$ for some appropriate constant $C$. We omit the details of the proof.

From the theorems above we can deduce the following new proximity data structure results. We give the distance oracles version below:

**Theorem 1.** There exist deterministic constructions with the following properties:

1. For every $t \geq 1$, and $1 \leq k \leq n$ there exists an approximate distance oracle with space: $n \cdot k^{1/t}$, local distortion: $O(t)$, and query time $O(1)$.

2. For every $t \geq 1$, and $1 \leq k \leq n$ there exists an approximate distance oracle with space: $n^{1+1/t}$, scaling local distortion: $O(\min\{\vartheta(\log k), t\})$, and query time $O(1)$.

### 6 Ramsey Theorems for Decomposable Metric Spaces

The main idea for achieving better metric Ramsey theorems for decomposable metric spaces is to reduce the problem to embedding metric spaces of small aspect ratio. Such metric spaces can be embedded with low distortion using Rao’s method [34]. Such a reduction is not quite possible but it is possible to reduce to the case of hierarchical metric spaces where each level has small aspect ratio. We will show that such metric spaces have low distortion embeddings as well. We recall the following definitions from [10]:

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*the worst case distortion can bounded by $O(\log n)$ by modifying slightly the definition of $\vartheta$. 

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**Corollary 5.** Let \( d \) be a finite metric space. Suppose that there is a collection of disjoint finite metric spaces \( N \) associated with the elements \( x \) of \( M \). Let \( N = \{ N_x \}_{x \in M} \). For \( \beta \geq 1/2 \), the \( \beta \)-composition of \( M \) and \( N \), denoted by \( C = M_{\beta}[N] \), is a metric space on the disjoint union \( \bigcup_x N_x \). Distances in \( C \) are defined as follows. Let \( x, y \in M \) and \( u \in N_x, v \in N_y \), then:

\[
d_C(u, v) = \begin{cases} 
  d_{N_x}(u, v) & x = y \\
  \beta \gamma d_M(x, y) & x \neq y.
\end{cases}
\]

where \( \gamma = \frac{\max_{x \in M} \text{diam}(N_x)}{\min_{x \neq y \in M} \text{diam}(N_x)} \).

**Definition 10 (Composition Closure).** Given a class \( \mathcal{M} \) of finite metric spaces, we consider \( \text{comp}_\beta(\mathcal{M}) \), its closure under \( \geq \beta \)-compositions. Namely, this is the smallest class \( \mathcal{C} \) of metric spaces that contains all spaces in \( \mathcal{M} \), and satisfies the following condition: Let \( M \in \mathcal{M} \), and associate with every \( x \in M \) a metric space \( N_x \) that is isometric to a space in \( \mathcal{C} \). Also, let \( \beta' \geq \beta \). Then \( M_{\beta'}[N] \) is also in \( \mathcal{C} \).

We prove the following general lemma:

**Lemma 4.** Let \( \mathcal{M} \) be a class of finite metric spaces such that every \( M \in \mathcal{M} \) has an embedding into \( L_p \) with distortion \( \alpha \) then every \( X \in \text{comp}_\beta(\mathcal{M}) \) embeds into \( L_p \) with distortion \( 2\alpha \).

*Proof.* Assume every \( M \in \mathcal{M} \) has a non-expansive embedding \( f_M : M \to L_p \) with distortion \( \alpha \). Let \( X \in \text{comp}_\beta(\mathcal{M}) \). We define an embedding \( \hat{f}_X : X \to L_p \) recursively on the structure of the metric composition. If \( X \in \mathcal{M} \) then \( \hat{f}_X = f_X \). Otherwise let \( X = M_{\beta'}[\mathcal{N}] \), \( \beta' \geq 2 \). Let \( u \in X \) such that \( u \in N_x \), \( x \in M \). Let \( \hat{f}_X(u) = (\beta' \gamma \cdot f_M(x)) \oplus \hat{f}_{N_x}(u) \).

W.l.o.g we may assume that \( \|f_M(x)\|_p \leq \text{diam}(M) \). We first claim by induction that \( \|\hat{f}_X(u)\|_p \leq 2\text{diam}(X) \). This follows as \( \|\hat{f}_X(u)\|_p \leq \beta' \gamma \cdot \text{diam}(M) + 2\text{diam}(N_x) \leq \text{diam}(X) + 2\gamma \cdot \text{diam}(M) \leq 2\text{diam}(X) \).

Consider \( u, v \in X \). Let \( X' = M_{\beta''}[\mathcal{N}'] \), \( \beta'' \geq 2 \), be the first level in the composition structure such that \( u \in N'_x \) and \( v \in N'_y \) for \( x \neq y \). Then

\[
\|\hat{f}_X(u) - \hat{f}_X(v)\|_p^p = \|\hat{f}_X'(u) - \hat{f}_X'(v)\|_p^p = \|f_{M'}(x) - f_{M'}(y)\|_p^p + \|\hat{f}_{N_x}(u) - \hat{f}_{N_y}(v)\|_p^p \\
\leq (\beta'' \gamma \cdot d(x, y))^p + (2 \max \{\text{diam}(N_x), \text{diam}(N_y)\})^p \\
\leq d(u, v)^p + (2\gamma \cdot d(x, y))^p \leq 2d(u, v)^p.
\]

On the other hand

\[
\|\hat{f}_X(u) - \hat{f}_X(v)\|_p \geq \|f_{M'}(x) - f_{M'}(y)\|_p \geq \beta'' \gamma \cdot d(x, y)/\alpha = d(u, v)/\alpha.
\]

Let \( \Phi \) denote the class of metric spaces \( M \) with aspect ratio at most \( \Phi \). Then we have the following:

**Corollary 5.** Let \( X \in \text{comp}_\Phi(\Phi) \) then \( X \) embeds into \( L_p \) with distortion \( O(\tau_X (\log \Phi)^{1/p}) \).

*Proof.* Apply Lemma 4 on Rao’s embeddings for metric spaces of aspect ratio \( \Phi \) to obtain the claimed distortion bound.

We are now ready to prove the metric Ramsey theorem:
Theorem 4. Let $X$ be a metric space. There exists $C > 0$ such that for every $1 \leq p \leq \infty$, and any $\alpha > 1$:

$$\psi_{L_p}(X, \alpha) \geq 1 - C \left( \frac{\tau_X}{\alpha} \right)^p \rho \log \left( \frac{\tau_X}{\alpha} \right).$$

*Proof.* Let $\alpha' = (\frac{\tau_X}{\alpha})^p / c$ (we may assume $\alpha' \geq 1$), where $c$ is a constant to be set later. We first use Theorem 1 to obtain a subspace $X'$ of $X$ which is $\alpha'$ equivalent to an ultrametric $Y'$ and satisfies the weighted Ramsey condition with $\psi_{UM}(X, \alpha') \geq 1 - \frac{c}{\alpha'}$.

We apply Lemma 2. However for the purpose of achieving a partition with properties similar to those of the probabilistic partitions of $[5]$ we need to choose the “padding parameter” $\eta$ more carefully. Specifically, we will set $\eta = \log(1/\delta) / \min\{\log(w'(X)/w'(Q)), 2^6\}$, where $0 < \delta < 1$. It follows that the decomposition creates a partition $(Q, \hat{Q})$ and a set $P$ such that $w(P) \geq w(Q) \cdot \delta$ and $d(P, Q) \geq \eta \cdot \hat{\Delta}$. We define $C = Q$ to be a cluster in the partition and let $\eta(C) = \eta$. We keep applying the lemma on $Q$ until $\Delta(\hat{Q}) \leq \hat{\Delta}$. This defines the first partition in the bundle. We then set $X'$ to be $X$ after all the core sets $P$ have been removed and repeat the process for $X'$. This is repeated until $X'$ is empty. It follows that this happens after repeating $O(\log n/\delta)$ times. We obtain the following lemma:

Lemma 6 (Explicit Padded Partitions). For every $n$-point metric space $X$ it is possible to efficiently construct deterministically a bundle of $O(\log n/\delta)$ $\hat{\Delta}$-bounded partitions such that for every $x \in X$ there is a cluster $C$ in the bundle such that $x \in C$ and $d(x, X/C) \geq \eta(C) \hat{\Delta}$.

This replaces the use of the uniform probabilistic partitions in $[5]$. It can be shown that the properties of these partitions provide similar qualities necessary for their main theorem. The other randomness used in
their proof is for choosing $O(\log n)$ independent 0, 1 valued random variables for each cluster. These can be replaces with assigning binary code words of length $O(\log n)$. Now, using this and lemma6 we can use the framework of [5] to define the partition-based embeddings of [3][7] in order to obtain deterministic constructions of embeddings into $L_p$. In particular we get a deterministic embedding of general metric spaces into $L_p$ in $O(\log n)$ dimensions with $O(\log n)$ distortion, $O(1)$ average distortion, and $O(q)$ $\ell_q$-distortion.

8 Multi-Embedding

In this section we obtain multi-embeddings of metric spaces into ultrametrics with optimal path-distortion. The construction is analogous to that of Theorem[3][1] combined with the proof in [18]. We first give the following variant of Lemma2

Lemma 7. Given a metric space $X$, and $0 < \hat{\Delta} \leq \Delta(X)/4$, and integer $t \geq 2$, then there exists a partition $(Q, \bar{Q})$ of $X$, and $P \subseteq Q$, such that: $|Q| \leq |X|/2$, $\Delta(Q) \leq \hat{\Delta}$, $d(P, Q) \geq \hat{\Delta}/(8t)$, and

$$w(P) \geq w(Q) \cdot \left(\frac{w^*(X)}{w^*(Q)}\right)^{-1/t}.$$  

Proof. Let $u, v \in X$ be two points such that $d(u, v) = \Delta(X)$. Then one of the open balls of radius $\Delta(X)/2$ around either $u$ or $v$ contains at most $|X|/2$ points. Assume w.l.o.g this holds for $v$ and let $B$ be the associated ball, and let $X'$ be composed of $B$ and the nearest neighbor of $v$ in $X \setminus B$. Then $\Delta(X') \geq \Delta(X)/2$, so that $\hat{\Delta} \leq \Delta(X')/2$. Now, apply Lemma2 on $X'$. Note that $\Delta(Q) \leq \hat{\Delta} < \Delta(X')$, so that $|Q| \leq |X'| - 1 \leq |X|/2$. \hfill $\square$

Theorem 7. For any metric space on $n$ points and aspect ratio $\Phi$, and any $\epsilon > 0$, there exists a multi-embedding into an ultrametric of size $n^{1+\epsilon}$, whose path distortion is at most $O(\min\{\log n, \log \Phi\}/\epsilon)$.

Proof. Let $X$ be an arbitrary metric space. Let $Z \subseteq X$. We will construct a multi-embedding of $Z$ into an ultrametric $U(Z)$ recursively as follows: use the decomposition described in Lemma7 with $t = \lceil 1/\epsilon \rceil$ and $\hat{\Delta} = \Delta(Z)/4$ to obtain a partition of the graph $(Q, \bar{Q})$ and $P \subseteq Q$ satisfying Lemma2. Run the algorithm on $Q$ and $\bar{P} = Z \setminus P$ recursively, obtaining multi-embedding into ultrametrics $U(Q)$ and $U(\bar{P})$ respectively. The ultrametric $U(Z)$ is constructed by creating a root $r$ labeled with $\Lambda(r) = \Delta(Z)$ with two children at which we root the trees defining $U(Q)$ and $U(\bar{P})$.

Let $\psi = 1 + 1/t \leq 1 + \epsilon$. Next, we prove by induction on the size of $Z \subseteq X$ that

$$w^\psi(U(Z)) \leq w(Z) \cdot w^*(Z)^\epsilon.$$  

If $Z = \{z\}$ includes a single point $z$ then the claim trivially holds since $w^\psi(U(Z)) = w(Z)w^*(Z)^\epsilon = w(z)^\psi$. By applying the induction hypothesis and Lemma2 we obtain

$$w^\psi(U(Z)) = w^\psi(U(Q)) + w^\psi(U(\bar{P})) \leq w(Q) \cdot w^*(Q)^\epsilon + w(\bar{P}) \cdot w^*(\bar{P})^\epsilon \leq w(P) \cdot \left(\frac{w^*(Z)}{w^*(Q)} \cdot w^*(Q)^\epsilon\right) + w(\bar{P}) \cdot w^*(Z)^\epsilon \leq (w(P) + w(\bar{P})) \cdot w^*(Z)^\epsilon = w(Z) \cdot w^*(Z)^\epsilon.$$

Noting that $w^*(X) \leq w(X)$ we conclude that $w^\psi(U(X)) \leq w(X)^\psi$. 

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In particular for \( w(X) = w^\psi(X) = |X| \) we get that the size of the ultrametric \( U(X) \) is bounded by \( n^{1+\varepsilon} \).

In addition the multi-embedding we constructed of \( X \) into \( U(X) \) has the property that the subtrees of \( U(X) \) correspond to subspaces of \( X \). At every level of \( U(X) \) we have a tree \( T = U(Z) \) for some subspace \( Z \) and \( T \) is split into two subtrees \( T_1 = U(Q) \) and \( T_2 = U(P) \) defined by the decomposition of Lemma\[7\]. Hence \( d(Q, P) \geq \hat{\Delta}/(8t) = \Delta(Z)/(32t) \geq \varepsilon/64 \cdot \Delta(U(Z)) \). We also have that \( |P| \leq |Z|/2 \) and \( \Delta(U(P)) = \Delta(P) \leq \hat{\Delta} = \Delta(Z)/4 = \Delta(U(Z))/4 \). In \[18\] it is shown that these properties imply that the path distortion of our multi-embedding is \( O(\min\{\log n, \log \Phi\}/\varepsilon) \).

\[ \square \]

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