Skewon-Axion Medium and Soft-and-Hard/DB Boundary Conditions

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Abstract

The class of skewon-axion media can be defined in a simple and natural manner applying four-dimensional differential-form representation of electromagnetic fields and media. It has been recently shown that an interface of a uniaxial skewon-axion medium acts as a DB boundary requiring vanishing normal components of the D and B vectors. In the present paper a more general skewon-axion medium is considered. It is shown that a planar interface of such a medium acts as a boundary generalizing both soft-and-hard (SH) and DB boundary conditions to SHDB conditions. Reflection of a plane wave from a planar SHDB boundary is studied. It is shown that for the two eigenpolarizations the boundary can be replaced by equivalent PEC or PMC boundaries. The theory is tested with a numerical example.

1 Introduction

The most general electromagnetic medium can be defined in terms of 36 parameters, either in terms of four medium dyadics as

\[
\begin{pmatrix}
    D_g \\
    B_g
\end{pmatrix}
= \begin{pmatrix}
    \frac{\xi}{\eta} & \frac{\bar{\eta}}{\bar{\xi}} \\
    \frac{\bar{\xi}}{\bar{\eta}} & \frac{\eta}{\xi}
\end{pmatrix}
\begin{pmatrix}
    E_g \\
    H_g
\end{pmatrix},
\]

(1)
in three-dimensional representation for Gibbsian vector fields denoted by \( \mathbf{D}_g \), \( \mathbf{B}_g \), \( \mathbf{E}_g \), and \( \mathbf{H}_g \), or in terms of a single medium dyadic as
\[
\Psi = \overline{M} \Phi,
\]
(2)
applying four-dimensional differential-form representation for electromagnetic fields and media [1, 2]. In the latter case the field two-forms \( \Psi \) and \( \Phi \) can be expressed as
\[
\Psi = \mathbf{D} - \mathbf{H} \wedge d\tau, \quad \Phi = \mathbf{B} + \mathbf{E} \wedge d\tau,
\]
(3)
in terms of three-dimensional (spatial) two-forms \( \mathbf{D} \), \( \mathbf{B} \) and one-forms \( \mathbf{H} \), \( \mathbf{E} \) when \( d = \sum \varepsilon_i \partial x_i \), \( i = 1..4 \) is the differential operator and \( \tau = ct \) is the normalized time. The medium dyadic \( \overline{M} \) mapping two-forms to two-forms corresponds to a \( 6 \times 6 \) matrix in any basis expansion of two-forms. The notation in this paper follows that given in [2].

The medium dyadic \( \overline{M} \) has a natural (independent of any basis system) decomposition in three parts as
\[
\overline{M} = \overline{M}_1 + \overline{M}_2 + \overline{M}_3
\]
(4)
and the dyadics \( \overline{M}_i \) are respectively called principal, skewon and axion components of \( \overline{M} \) [3]. The axion part is a multiple of the unit dyadic which in the notation of [2] can be expressed as
\[
\overline{M}_3 = M_3 \mathbf{1}^{(2)T},
\]
(5)
while both \( \overline{M}_1 \) and \( \overline{M}_2 \) are trace-free dyadics. The components \( \overline{M}_1 \) and \( \overline{M}_2 \) can be defined so that the dyadics contracted by a quadrivector \( \mathbf{e}_N = e_{1234} \) as \( \mathbf{e}_N | \overline{M}_1 \) and \( \mathbf{e}_N | \overline{M}_2 \) are respectively symmetric and antisymmetric. Physical properties of these components are discussed in [3].

The total number of 36 parameters is distributed by the three components so that the principal part \( \overline{M}_1 \), corresponding to a trace-free symmetric \( 6 \times 6 \) matrix has 20 parameters, the skewon part \( \overline{M}_2 \), corresponding to an antisymmetric \( 6 \times 6 \) matrix, has 15 parameters, and the axion part \( \overline{M}_3 \), has 1 parameter. A medium consisting only of its axion parameter, \( \overline{M} = \overline{M}_3 \) in (5), has been called PEMC, perfect electromagnetic conductor, because it is a generalization of both PMC \( (M_3 = 0) \) and PEC \( (1/M_3 = 0) \), [4, 5, 6]. The medium equations for the PEMC can be expressed in the simple form
\[
\mathbf{D} = \overline{M} \mathbf{B}, \quad \mathbf{H} = -\overline{M} \mathbf{E}.
\]
(6)
2 Skewon-Axion Medium

One can show that the most general medium with no principal component, the skewon-axion medium [8, 9, 10] (also called as the IB-medium [7]), can be expressed in terms of a dyadic $\mathbf{B}$, corresponding to a $4 \times 4$ matrix involving only 16 parameters, as

$$
\mathbf{M} = (\mathbf{I} \wedge \mathbf{B})^T = (\mathbf{B} \wedge \mathbf{I})^T. 
$$

(7)

It will turn out that a plane wave in the most general skewon-axion medium is not governed by a dispersion equation, normally restricting the choice of the $\mathbf{k}$ vector in a given medium. Thus, basically, one can choose any $\mathbf{k}$ vector for a plane wave in such a medium.

In [11] the problem of plane wave reflection from the planar interface of a uniaxial skewon-axion medium, defined by six medium parameters, was analyzed. It was shown that, despite the absence of any dispersion equation, the $\mathbf{k}$ vector in the skewon-axion medium was uniquely determined by the interface conditions. Moreover, it was shown that the interface conditions serve as boundary conditions

$$
\mathbf{n} \cdot \mathbf{D}_g = 0, \quad \mathbf{n} \cdot \mathbf{B}_g = 0, 
$$

(8)

for Gibbsian fields at the interface of the uniaxial skewon-axion medium, dubbed as the DB conditions [12, 13]. It subsequently turned out that such conditions were defined already in 1959 [14] and that they are known to have unique mathematical properties [15, 16]. The DB conditions have been shown to play a central role in electromagnetic cloaking problems [17, 18, 19, 20]. More recently, the same DB conditions were shown to emerge at the interface of a simple skewon medium defined by just one parameter in the medium conditions [22]

$$
\mathbf{D} = \mathbf{N}\mathbf{B}, \quad \mathbf{H} = \mathbf{N}\mathbf{E}, 
$$

(9)

which differ slightly those of the PEMC (axion medium) in (6).

Fields in the skewon-axion medium can be compactly handled applying the four-dimensional formalism [2]. The Maxwell equations outside sources are

$$
\mathbf{d} \wedge \Phi = 0, \quad \mathbf{d} \wedge \Psi = 0, 
$$

(10)

whence the field two-form $\Phi$ can be (locally) represented in terms of a potential one-form $\phi$ as

$$
\Phi = \mathbf{d} \wedge \phi. 
$$

(11)
Substituting (2) and (7), the second Maxwell equation can now be expanded as

\[ d \wedge \Psi = d \wedge (B^T \wedge d^T)(d \wedge \phi) \]  \hspace{1cm} (12)

\[ = d \wedge (B^T \wedge d \wedge \phi) + d \wedge d \wedge (B^T \wedge \phi) \]  \hspace{1cm} (13)

\[ = d \wedge ((B^T d) \wedge \phi) \]  \hspace{1cm} (14)

\[ = 0. \]  \hspace{1cm} (15)

This shows us that the general solution can be represented as

\[ \phi = d \psi + B^T d \varphi, \]  \hspace{1cm} (16)

in terms of two scalar functions \( \psi \) and \( \varphi \), whence the field two-forms have the form

\[ \Phi = d \wedge \phi = d \wedge (B^T d) \varphi, \]  \hspace{1cm} (17)

\[ \Psi = (B^T d) \wedge \phi = d \wedge (B^{2T} d) \varphi. \]  \hspace{1cm} (18)

Instead of studying the general case, let us assume that the skewo-axion medium is defined by the dyadic

\[ \overline{B} = B \tilde{I} + a \varepsilon_3 + b(\varepsilon_1 + A \varepsilon_4), \]  \hspace{1cm} (19)

where \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are spatial one-forms and \( \varepsilon_4 = d \tau \) a temporal one-form, and together they form a basis. The corresponding reciprocal basis vectors are denoted by \( e_i \) with the property \( e_i | e_j = \delta_{ij} \) [2].

Substituting (19) in (17) and (18), the field two-forms can be shown to satisfy the conditions

\[ \varepsilon_3 \wedge (\varepsilon_1 + A \varepsilon_4) \wedge \Phi = 0, \]  \hspace{1cm} (20)

\[ \varepsilon_3 \wedge (\varepsilon_1 + A \varepsilon_4) \wedge \Psi = 0, \]  \hspace{1cm} (21)

for the special skewon-axion medium considered here. Inserting the expansions (3), the conditions become

\[ A \varepsilon_3 \wedge \varepsilon_4 \wedge B + \varepsilon_3 \wedge \varepsilon_1 \wedge E \wedge \varepsilon_4 = 0, \]  \hspace{1cm} (22)

\[ A \varepsilon_3 \wedge \varepsilon_4 \wedge D - \varepsilon_3 \wedge \varepsilon_1 \wedge H \wedge \varepsilon_4 = 0, \]  \hspace{1cm} (23)
and they are equivalent with the spatial conditions
\[ A\varepsilon_3 \land \mathbf{B} + \varepsilon_3 \land \varepsilon_1 \land \mathbf{E} = 0, \quad (24) \]
\[ A\varepsilon_3 \land \mathbf{D} - \varepsilon_3 \land \varepsilon_1 \land \mathbf{H} = 0. \quad (25) \]

To conclude, any fields in a skewon-axion medium defined by (19) and (7) must satisfy the conditions (24) and (25).

### 3 SHDB Boundary Conditions

Let us interprete the result (24) and (25) in terms of three-dimensional Gibb-
sian vectors applying the rules given in the Appendix. Denoting again t he
Gibbsian vectors by the subscript \(g\) we can write
\[ e_{123} |(\varepsilon_3 \land \mathbf{B}) = \varepsilon_3 |(e_{123} | \mathbf{B}) = e_3 \cdot \mathbf{B}_g, \quad (26) \]
\[ e_{123} |(\varepsilon_3 \land \mathbf{D}) = \varepsilon_3 |(e_{123} | \mathbf{D}) = e_3 \cdot \mathbf{D}_g, \quad (27) \]
\[ e_{123} |(\varepsilon_3 \land \varepsilon_1 \land \mathbf{E}) = e_3 \cdot (e_1 \times \mathbf{E}_g) = e_2 \cdot \mathbf{E}_g, \quad (28) \]
\[ e_{123} |(\varepsilon_3 \land \varepsilon_1 \land \mathbf{H}) = e_3 \cdot (e_1 \times \mathbf{H}_g) = e_2 \cdot \mathbf{H}_g. \quad (29) \]

Here we have applied the property \(e_i | \varepsilon_j = \delta_{ij}\) transformed to \(e_i \cdot e_j = \delta_{ij}\) for
the spatial basis vectors \(i,j = 1,2,3\).

Since the coefficient \(A\) in the conditions (24) and (25) has the dimension
of velocity (due to the normalized time parameter \(\tau = ct\)), let us further
express it as
\[ A = Tc = \frac{T}{\sqrt{\mu_o \varepsilon_o}} = \frac{\omega T}{k_o}, \quad (30) \]
in terms of a dimensionless scalar \(T\). Here we have tacitly assumed that the
fields are time harmonic as \(\exp(j\omega t)\) with \(k_o = \omega \sqrt{\mu_o \varepsilon_o}\). Thus, the conditions
(24) and (25) can be represented in the following form for the Gibbsian field
vectors,
\[ T\omega e_3 \cdot \mathbf{B}_g + k_o e_2 \cdot \mathbf{E}_g = 0, \quad (31) \]
\[ T\omega e_3 \cdot \mathbf{D}_g - k_o e_2 \cdot \mathbf{H}_g = 0, \quad (32) \]
and they are satisfied by any fields in the skewon-axion medium under consid-
eration. One may note that the pair of conditions is invariant in the duali-
ty transformation \(\mathbf{D}_g \leftrightarrow \mathbf{B}_g, \mathbf{E}_g \leftrightarrow -\mathbf{H}_g\) which leave the Maxwell equations
invariant.
Assuming now a planar interface at $e_3 \cdot r = 0$, with normal unit vector $e_3$, between the skewon-axion medium and an isotropic medium (parameters $\mu_0, \epsilon_0$), the conditions (24) and (25), and their Gibbsian counterparts (31) and (32), are actually continuous across the interface. Since these conditions at the interface are enough to determine the fields in the isotropic medium without having to solve for the fields behind the interface, they can be considered as boundary conditions. Although (31) and (32) were introduced through a consideration of a planar interface, their form is applicable to curved boundaries as well, with $e_3$ and $e_2$ denoting respective unit vectors normal and tangential to the boundary surface.

The boundary defined by (31), (32) appears as a generalization of both the DB boundary defined by the conditions (8) and the soft-and-hard (SH) boundary [23, 24, 20] defined by the conditions

$$e_2 \cdot E_\text{g} = 0, \quad e_2 \cdot H_\text{g} = 0.$$ \hspace{1cm} (33)

In fact, the SH conditions are obtained for the limiting parameter value $T \to 0$ while the DB conditions are obtained from (31), (32) for $T \to \infty$. For convenience, the more general boundary defined by (31) and (32) and combining both SH and DB conditions will be called the SHDB boundary.

One can immediately observe that, due to its self-dual property, an object with rotational symmetry and SHDB boundary should appear invisible to the radar because the back-scattering cross section of such an object is zero [21].

4 Gibbsian Medium Conditions

The skewon-axion medium was defined in terms of four-dimensional quantities through (7), (19). For the convenience of readers not familiar with the four-dimensional formalism and for double-checking the results, let us now find the relation between the skewon medium interface and the boundary conditions (31), (32) in terms of three-dimensional Gibbsian vectors and dyadics. For that purpose we first find the Gibbsian medium conditions by expanding

$$(\overline{B} \wedge \tilde{I})^T = 2B^{(2)T} - (\epsilon_3 \wedge \tilde{I} \wedge a + (\epsilon_1 + A\epsilon_4) \wedge \tilde{I} \wedge b),$$ \hspace{1cm} (34)

and expand

$$(\epsilon_3 \wedge \tilde{I} \wedge a) \Phi = (\epsilon_3 \wedge \tilde{I} \wedge a) (\tilde{B} + E \wedge \epsilon_4)$$
\[ \varepsilon_3 \wedge (a \| B) + (\varepsilon_3 \wedge E)(a \| \varepsilon_4) - (\varepsilon_3 \wedge \varepsilon_4)(a \| E). \]  

The expansion for the second term is obtained similarly, replacing \( \varepsilon_3 \) by \( \varepsilon_1 + A \varepsilon_4 \) and \( a \) by \( b \). Let us also expand the two vectors in their spatial and temporal components as

\[ a = a_s + a_4 e_4, \quad b = b_s + b_4 e_4, \]  

With these we can split the medium equation

\[ D - H \wedge \varepsilon_4 = (\overline{B} \wedge \varepsilon)^T (B + E \wedge \varepsilon_4) \]  

in its spatial and temporal parts and identify the fields as

\[ D = 2B \varepsilon_3 \wedge (a_s \| B) - \varepsilon_1 \wedge (b_s \| B) - (a_4 \varepsilon_3 + b_4 \varepsilon_1) \wedge E, \]  

\[ H = -A b_s \| B - (A b_4 + 2B) E - (\varepsilon_3 a_s + \varepsilon_1 b_s) \cdot E. \]

Applying the transformation rules given in the Appendix, the medium conditions can be written for the Gibbsian vectors as

\[ D_g = 2B B_g - \varepsilon_3 \wedge (a_s \times B_g) - \varepsilon_1 \wedge (b_s \times B_g) - (a_4 \varepsilon_3 + b_4 \varepsilon_1) \times E_g, \]  

\[ H_g = -A b_s \times B_g - (A b_4 + 2B) E_g - (\varepsilon_3 a_s + \varepsilon_1 b_s) \cdot E_g. \]

Expressing the Gibbsian medium conditions in the form

\[ \begin{pmatrix} D_g \\ H_g \end{pmatrix} = \begin{pmatrix} \overline{a} \\ \overline{\varepsilon} \end{pmatrix} \begin{pmatrix} \overline{\varepsilon}^{-1} \\ \overline{\beta} \end{pmatrix} \begin{pmatrix} B_g \\ E_g \end{pmatrix}, \]

where the medium dyadics are understood in the Gibbsian sense, their expressions become

\[ \overline{a} = 2B \overline{I}_g - \varepsilon_3 \times (a_s \times \overline{I}_g) - \varepsilon_1 \times (b_s \times \overline{I}_g), \]  

\[ \overline{\varepsilon} = -(a_4 \varepsilon_3 + b_4 \varepsilon_1) \times \overline{I}_g, \]  

\[ \overline{\varepsilon}^{-1} = -A b_s \times \overline{I}_g, \]  

\[ \overline{\beta} = -(A b_4 + 2B) \overline{I}_g - (\varepsilon_3 a_s + \varepsilon_1 b_s). \]
One must notice that the dyadics $\tilde{\epsilon}'$ and $\tilde{\mu}^{-1}$ are antisymmetric and, consequently, do not have inverses. Thus, it is not possible to express the skewon-axion medium equations in the form (1).

The most general skewon-axion medium can be defined in terms of three-dimensional Gibbsian vectors and dyadics in the form [7, 11]

$$
\mathbf{D}_g = \text{tr} \mathbf{A} \mathbf{B}_g - \mathbf{A} \cdot \mathbf{B}_g - \mathbf{c} \times \mathbf{E}_g, \quad (47)
$$

$$
\mathbf{H}_g = -\mathbf{g} \times \mathbf{B}_g - \mathbf{A}^T \cdot \mathbf{E}_g - a \mathbf{E}_g, \quad (48)
$$

where $\mathbf{A}$ is a dyadic, $\mathbf{c}, \mathbf{g}$ are two vectors and $a$ is a scalar. Together they make $9 + 3 + 3 + 1 = 16$ scalar parameters. Comparing with the representation (43) - (46), we can identify the quantities in (47) and (48) as

$$
\mathbf{A} = B \mathbf{e}_2 \mathbf{e}_2 + (B \mathbf{e}_3 + a) \mathbf{e}_3 + (B \mathbf{e}_1 + b) \mathbf{e}_1, \quad (49)
$$

$$
\mathbf{c} = b_4 \mathbf{e}_1 + a_4 \mathbf{e}_3, \quad (50)
$$

$$
\mathbf{g} = A \mathbf{b}_4, \quad (51)
$$

$$
A = B + A b_4, \quad (52)
$$

which shows us that the number of free parameters of the medium defined by (19) must be less than 16. Expressing the dyadic $\mathbf{A}$ as

$$
\mathbf{A} = B \mathbf{e}_2 \mathbf{e}_2 + (B \mathbf{e}_3 + a) \mathbf{e}_3 + (B \mathbf{e}_1 + b) \mathbf{e}_1, \quad (53)
$$

its definition is seen to require $1 + 3 + 3 = 7$ parameters in a given basis $\mathbf{e}_i$, while $\mathbf{c}$ and $\mathbf{g}$ require 2 and 1 parameters, respectively. Thus, the total number of free parameters in the medium under consideration appears to be 10. The pure axion (PEMC) medium (6) special case is obtained for

$$
\mathbf{A} = \frac{M}{2} \mathbf{I}_s, \quad \mathbf{c} = 0, \quad \mathbf{g} = 0, \quad a = M/2. \quad (54)
$$

### 5 Reflection from SHDB Boundary

Let us consider the basic problem, reflection of a plane wave from the planar interface of the skewon-axion half space $z < 0$, in terms of Gibbsian quantities corresponding to time-harmonic fields.
5.1 Dispersion Equation

Denoting for simplicity

\[ p = \frac{k}{\omega}, \]  

(55)

a plane wave with the dependence \( \exp(-j \omega p \cdot r) \) satisfies

\[ p \times E_g = B_g, \quad p \times H_g = -D_g. \]  

(56)

To obtain a condition for a plane wave in the skewon-axion medium, substituting (47) and (48) and eliminating \( B_g \) in (56) leaves us with an equation of the form [7]

\[ \overline{D}(p) \cdot E_g = 0, \]  

(57)

where we denote

\[ \overline{D}(p) = q(p) \times \tilde{t}_g, \]  

(58)

\[ q(p) = (g \cdot p - a)p - c + p \cdot \overline{A}. \]  

(59)

Because \( \det(\overline{D}(p)) = 0 \) for any \( p \), there is no dispersion equation to restrict the choice of \( p \). Thus, a plane wave with any vector \( k \) is possible in the skewon-axion medium. One can further show that \( q(p) = 0 \) for any \( p \) exactly when the medium has no skewon component, i.e., when it is a pure axion (PEMC) medium with \( c = g = 0 \) and \( \overline{A} = a \tilde{s} \). In the following we exclude the pure axion medium from the analysis.

5.2 Field Conditions

The field vectors \( E_g \) and \( B_g \) of the plane wave can be expressed as

\[ E_g = q(p) = (g \cdot p - a)p - c + p \cdot \overline{A}, \]  

(60)

\[ B_g = p \times E_g = -p \times c - p \cdot \overline{A} \times p. \]  

(61)

Inserting (49) - (52), the field expressions for the special skewon-axion medium become

\[ E_g = (b_s \cdot p - b_4)(Ap + e_1) + (a_s \cdot p - a_4)e_3, \]  

(62)

\[ B_g = (b_s \cdot p - b_4)p \times e_1 + (a_s \cdot p - a_4)p \times e_3. \]  

(63)
The form of the components
\[ E_g \cdot e_2 = A(b_s \cdot p - b_4)(p \cdot e_2), \] (64)
\[ B_g \cdot e_3 = -(b_s \cdot p - b_4)(p \cdot e_2), \] (65)
implies that, for the general vector \( p \), the fields satisfy the condition
\[ Ae_3 \cdot B_g + e_2 \cdot E_g = 0, \] (66)
which coincides with (31) when (30) is taken into account. If \( p \) satisfies \( p \cdot e_2 = 0 \), we obtain both \( B_g \cdot e_3 = 0 \) and \( E_g \cdot e_2 = 0 \), which are actually the same condition because they are connected by
\[ e_3 \cdot (p \times E_g - B_g) = (e_1 \cdot p)(e_2 \cdot E_g) - e_3 \cdot B_g = 0. \] (67)

Inserting (60) and (61) in (47) and (48), with (49) - (52) and after some algebraic steps, the following expansions for the other field components are obtained,
\[ e_3 \cdot D_g = -(p \cdot e_2)((a_s \cdot p - a_4)(b_s \cdot e_3) + (b_s \cdot p - b_4)(Ab_4 + 2B + b_s \cdot e_1)) \] (68)
\[ e_2 \cdot H_g = -A(p \cdot e_2)((a_s \cdot p - a_4)(b_s \cdot e_3) + (b_s \cdot p - b_4)(Ab_4 + 2B + b_s \cdot e_1)). \] (69)
Thus, in the general case, the fields satisfy
\[ Ae_3 \cdot D_g - e_2 \cdot H_g = 0, \] (70)
which coincides with (32) when (30) is taken into account. Again, for \( p \cdot e_2 = 0 \) we obtain \( e_3 \cdot D_g = 0 \) and \( e_2 \cdot H_g = 0 \), which are the same condition.

It now follows that, inserting (30), the sum of incident and reflected plane-wave fields at the boundary of the isotropic medium satisfy the conditions
\[ T\omega(B^i_g + B^r_g) \cdot e_3 = -k_o(E^i_g + E^r_g) \cdot e_2, \] (71)
\[ T\omega(D^i_g + D^r_g) \cdot e_3 = k_o(H^i_g + H^r_g) \cdot e_2, \] (72)
which can be rewritten as
\[ T(k^i \times E^i_g + k^r \times E^r_g) \cdot e_3 = -k_o(E^i_g + E^r_g) \cdot e_2, \] (73)
\[ T(k^i \times H^i_g + k^r \times H^r_g) \cdot e_3 = -k_o(H^i_g + H^r_g) \cdot e_2. \] (74)
Applying the relations

\[ H_g^i = \frac{1}{\omega \mu_o} k^i \times E_g^i, \quad H_g^r = \frac{1}{\omega \mu_o} k^r \times E_g^r, \quad (75) \]

\[ k^r = (e_1 e_1 + e_2 e_2 - e_3 e_3) \cdot k^i, \quad (76) \]

\[ k^i \cdot E_g^i = 0, \quad k^r \cdot E_g^r = 0, \quad (77) \]

\[ k^r \cdot k^r = k^i \cdot k^i = k_o^2, \quad (78) \]

one can find relations between the fields \( E_g^i \) and \( E_g^r \) which depend on the medium parameters and the vector \( k^i \) of the incident wave. Skipping the algebraic details, we obtain the following result for the field components parallel to the boundary:

\[ A \begin{pmatrix} E_r^1 \\ E_r^2 \end{pmatrix} = B \begin{pmatrix} E_i^1 \\ E_i^2 \end{pmatrix}, \quad (79) \]

with matrices defined by

\[ A = \begin{pmatrix} \alpha & -\beta \\ \gamma & \delta \end{pmatrix}, \quad B = \begin{pmatrix} -\alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (80) \]

\[ \alpha = T k^i_2, \quad (81) \]

\[ \beta = k_o + T k^i_1, \quad (82) \]

\[ \gamma = k_o^2 + T k_o k^i_1 - k^i_2, \quad (83) \]

\[ \delta = (k^i_1 + T k_o) k^i_2. \quad (84) \]

The determinants of the two matrices become

\[ \Delta = \det A = -\det B = k_o ((Tk_o + k^i_1)^2 - (T^2 - 1)k^i_3^2). \quad (85) \]

The reflected field components can be solved as

\[ E_r^1 = -E_i^1 + \frac{2\beta}{\Delta} (\gamma E_i^1 + \delta E_i^2), \quad (86) \]

\[ E_r^2 = -E_i^2 + \frac{2\alpha}{\Delta} (\gamma E_i^1 + \delta E_i^2), \quad (87) \]

\[ E_r^3 = E_i^3 + \frac{2}{k^i_3 \Delta} (\alpha k^i_2 + \beta k^i_1)(\gamma E_i^1 + \delta E_i^2). \quad (88) \]
As a check, for \( T = 0 \) we obtain
\[
E_r^2 = -E_i^2, \quad E_r^1 = E_i^1 + \frac{2k_i^1 k_i^2}{k_1^{i2} + k_3^2} E_i^2.
\] (89)

These coincide with the SH conditions because the latter relation equals \( \mathbf{e}_2 \cdot \mathbf{H}_g^i = -\mathbf{e}_2 \cdot \mathbf{H}_g^i \), as can be verified. Also, for \( T \to \infty \) we obtain \( E_r^3 \to -E_i^3 \) and \( H_r^3 \to -H_i^3 \) which correspond to the respective DB conditions \( \mathbf{e}_3 \cdot \mathbf{D}_g \to 0 \) and \( \mathbf{e}_3 \cdot \mathbf{B}_g \to 0 \) for the total fields.

As another check we may consider the normally incident plane wave with \( \mathbf{k}^i = -k_o \mathbf{e}_3 \). In this case we obtain \( E_r^1 = E_i^1 \) and \( E_r^2 = -E_i^2 \), which coincide with the SH conditions. This is independent of any finite value of the parameter \( T \). Thus, if \( 1/T \) is a small quantity, the SHDB boundary acts as a DB boundary, except for normal incidence where the SH condition suddenly takes over.

Finally, for a plane wave satisfying \( \mathbf{e}_2 \cdot \mathbf{k}^i = 0 \) both SH and DB conditions are simultaneously satisfied for any \( T \) and \( k_1^i \) values.

### 5.3 Eigenfields

To find the eigenpolarizations satisfying
\[
E_r^1 = \lambda E_i^1, \quad E_r^2 = \lambda E_i^2,
\] (90)
we must solve
\[
\det(\mathbf{A} - \lambda \mathbf{B}) = 0,
\] (91)
for \( \lambda \). The solutions are simply
\[
\lambda_{\pm} = \pm 1,
\] (92)
whence the eigenfields satisfy either PMC \( (\lambda_+) \) or PEC \( (\lambda_-) \) conditions at the SHDB boundary. The eigenfields satisfy
\[
\alpha_1 E_{1+}^i = \beta E_{2+}^i, \quad \gamma E_{1-}^i = -\delta E_{2-}^i.
\] (93)
(94)

One can show that these equal the conditions
\[
T \omega \mathbf{e}_3 \cdot \mathbf{B}_g^i + k_o \mathbf{e}_2 \cdot \mathbf{E}_g^i = 0
\] (95)
\[
T \omega \mathbf{e}_3 \cdot \mathbf{D}_g^i - k_o \mathbf{e}_2 \cdot \mathbf{H}_g^i = 0,
\] (96)
which restrict the polarization of the two eigenwaves. Considering the two special cases, for $T = 0$ (SH boundary) the eigenfields are either TE or TM with respect to $e_2$ while for $T \to \infty$ (DB boundary) they are TE or TM with respect to $e_3$. Actually, in an isotropic medium, the polarization conditions are of the form

\begin{align}
\mathbf{a}_+ \cdot \mathbf{E}_{g+} + \mathbf{b}_+ \cdot \mathbf{H}_{g+} &= 0 \\
\mathbf{a}_- \cdot \mathbf{E}_{g-} + \mathbf{b}_- \cdot \mathbf{H}_{g-} &= 0,
\end{align}

which can be understood as generalizations of TE/TM conditions. Such polarization conditions have been previously encountered in media known by the name decomposable media, in which any fields can be decomposed in two non-interacting components \[25\].

Since (95) and (96) are also valid for the respective reflected eigenwaves, they are valid for the total eigenfields. The conditions are linear in the field vectors and independent of the $\mathbf{k}$ vector of the plane wave. Thus, they are valid for linear combinations of plane waves and, actually, for any fields. Thus, any field can be decomposed in two eigenfields satisfying either (95) or (96) for which the SHDB boundary can be replaced by respective PMC and PEC boundaries.

To show that any given field can be expanded as a sum of eigenfields, let us consider the general plane wave and express the conditions (95) and (96) as

\begin{align}
T e_3 \cdot \mathbf{k} \times \mathbf{E}_{g+} + k_o e_2 \cdot \mathbf{E}_{g+} &= 0 \\
Tk_o e_3 \cdot \mathbf{E}_{g-} - e_2 \cdot (\mathbf{k} \times \mathbf{E}_{g-}) &= 0,
\end{align}

which are of the form

\begin{align}
\mathbf{c}_3 \cdot \mathbf{E}_{g+} = 0, & \quad \mathbf{c}_2 \cdot \mathbf{E}_{g-} = 0,
\end{align}

with

\begin{align}
\mathbf{c}_3 &= Te_3 \times \mathbf{k} + k_o e_2, \\
\mathbf{c}_2 &= Tk_o e_3 - e_2 \times \mathbf{k}.
\end{align}

These vectors satisfy

\begin{align}
e_3 \cdot \mathbf{c}_3 = 0, & \quad e_2 \cdot \mathbf{c}_2 = 0.
\end{align}
Expanding \( \mathbf{a} = \mathbf{k} \times ((\mathbf{c}_3 \times \mathbf{c}_2) \times \mathbf{E}_g) \) in two ways as

\[
\mathbf{a} = \mathbf{k} \times \mathbf{c}_2 (\mathbf{c}_3 \cdot \mathbf{E}_g) - \mathbf{k} \times \mathbf{c}_3 (\mathbf{c}_2 \cdot \mathbf{E}_g) \tag{105}
\]

and applying \( \mathbf{k} \cdot \mathbf{E}_g = 0 \), the eigenfield representation

\[
\mathbf{E}_g = \mathbf{E}_{g+} + \mathbf{E}_{g-}, \tag{107}
\]

is obtained by defining

\[
\mathbf{E}_{g+} = \frac{\mathbf{k} \times \mathbf{c}_3 \mathbf{c}_2}{\mathbf{k} \cdot (\mathbf{c}_2 \times \mathbf{c}_3)} \cdot \mathbf{E}_g, \tag{108}
\]

\[
\mathbf{E}_{g-} = \frac{\mathbf{k} \times \mathbf{c}_2 \mathbf{c}_3}{\mathbf{k} \cdot (\mathbf{c}_3 \times \mathbf{c}_2)} \cdot \mathbf{E}_g. \tag{109}
\]

The expansion requires \( \mathbf{k} \cdot (\mathbf{c}_3 \times \mathbf{c}_2) \neq 0 \) which corresponds to

\[
T^2(k_1^2 + k_2^2) + k_1^2 + k_3^2 \neq 0. \tag{110}
\]

For the magnetic field \( \mathbf{H}_g = \mathbf{H}_{g+} + \mathbf{H}_{g-} \), the polarization conditions can be shown to take the form

\[
\mathbf{c}_2 \cdot \mathbf{H}_{g+} = 0, \quad \mathbf{c}_3 \cdot \mathbf{H}_{g-} = 0, \tag{111}
\]

whence the magnetic eigenfields obey the expressions

\[
\mathbf{H}_{g+} = \frac{\mathbf{k} \times \mathbf{c}_3 \mathbf{c}_2}{\mathbf{k} \cdot (\mathbf{c}_2 \times \mathbf{c}_3)} \cdot \mathbf{H}_g, \tag{112}
\]

\[
\mathbf{H}_{g-} = \frac{\mathbf{k} \times \mathbf{c}_2 \mathbf{c}_3}{\mathbf{k} \cdot (\mathbf{c}_3 \times \mathbf{c}_2)} \cdot \mathbf{H}_g. \tag{113}
\]

Obviously, the two eigenfields are dual to each other because the transformation \( \mathbf{E}_g \rightarrow \mathbf{H}_g, \mathbf{H}_g \rightarrow \mathbf{E}_g \) corresponds to \( \mathbf{E}_{g+} \rightarrow \mathbf{E}_{g-}, \mathbf{E}_{g-} \rightarrow \mathbf{E}_{g+} \).

### 6 Numerical example

To have an idea of SHDB boundary in action, let us consider as an example a plane wave incident to the SHDB plane with \( \varphi = \pi/2 \) in Figure 1. Thus, the incident wave vector satisfies

\[
\mathbf{e}_1 \cdot \mathbf{k}^i = 0, \quad \mathbf{k}^i = k_o (\mathbf{e}_2 \sin \theta + \mathbf{e}_3 \cos \theta). \tag{114}
\]
Figure 1: Geometry of the reflection problem. A plane wave is incident on the SHDB boundary with directions defined by the angles $\theta, \varphi$ of the spherical coordinate system. The fields satisfy $k^i \cdot E^i_g = 0$ and $k^r \cdot E^r_g = 0$.

The relation between the incident and reflected field components transverse to $e_3$ can be represented by the planar reflection dyadic $\overline{R}$ as

$$e_1 E^r_1 + e_2 E^r_2 = \overline{R} \cdot (e_1 E^i_1 + e_2 E^i_2),$$
(115)

$$\overline{R} = e_1 e_1 R_{11} + e_1 e_2 R_{12} + e_2 e_1 R_{21} + e_2 e_2 R_{22},$$
(116)

with

$$R_{11} = \frac{\cos^2 \theta - T^2 \sin^2 \theta}{\cos^2 \theta + T^2 \sin^2 \theta},$$
(117)

$$R_{12} = \frac{2T \sin \theta}{\cos^2 \theta + T^2 \sin^2 \theta},$$
(118)

$$R_{21} = \frac{2T \sin \theta \cos^2 \theta}{\cos^2 \theta + T^2 \sin^2 \theta},$$
(119)

$$R_{22} = -R_{11}. $$
(120)

The reflection dyadic satisfies

$$\text{tr} \overline{R} = R_{11} + R_{22} = 0,$$
(121)

$$\det \overline{R} = -R_{11}^2 - R_{12}R_{21} = -1,$$
(122)
as is also obvious from the eigenvalues $\lambda_{\pm} = \pm 1$.

Dependence of each component $R_{ij}$ of the reflection dyadic on the angle of incidence $\theta$ is depicted in Figure 2 for three parameter values $T = (0.1, 1, 10)$. According to the previous theory, for $T = 0.1$ the SHDB boundary should act almost like the DB boundary while for $T = 10$ it should rather resemble the SH boundary. This is clearly seen from the graphs of $R_{11}$ and $R_{22}$ because for $T = 0.1$ (solid curves) the component $E_2$ reflects almost from PEC with $R_{22} \approx -1$ and the component $E_1$ almost from PMC with $R_{11} \approx +1$. There is a deviation for angles close to grazing. The opposite is the case for $T = 10$ with deviation close to normal incidence which is a known effect for the DB boundary. There are small cross-polarized reflections for small and large values of $T$.

For $T = 1$ the SHDB boundary exhibits properties not shared by SH or DB boundaries. In this case we have

$$R_{11} = -R_{22} = \cos 2\theta, \quad R_{12} = 2 \sin \theta, \quad R_{21} = \sin 2\theta \cos \theta. \quad (123)$$

For example, for the incidence angle $\theta = \pi/4$ the reflected field appears totally cross polarized, $R_{11} = R_{22} = 0$. This is due to the nonreciprocal character of the SHDB boundary, which stems from the gyrotropic medium dyadics of the skewon-axion medium, (43) – (46).

As an another example we could consider the case $e_2 \cdot k_i = 0$, or $\varphi = 0$ in Figure 1. Forming the reflection dyadic, we would obtain $R_{11} = -R_{22} = 1$ and $R_{12} = R_{21} = 0$ for all $\theta$ and $T$ values. This would make a dull diagram. As was already pointed out, the case $e_2 \cdot k_i = 0$ corresponds to reflection from either an SH boundary or a DB boundary, both of which yield the same result. Actually, the DB boundary can be characterized as an isotropic soft surface [11], while the SH boundary is a soft surface for waves arriving from a certain direction, which in the present case corresponds to $e_2 \cdot k_i = 0$.

7 Conclusion and Discussion

The class of skewon-axion media can be described mathematically in natural (coordinate-independent) manner applying four-dimensional formalism. Instead of the 36 medium parameters of the most general linear medium, represented by a $6 \times 6$ matrix, the skewon-axion medium only involves 16 parameters defined by its characteristic $4 \times 4$ matrix. General solutions to the Maxwell equations can be expressed in simple form in the four-dimensional
Figure 2: Components of the reflection dyadic $R_{ij}$ for a plane wave incident as $\varphi = \pi/2$. The angle of incidence $\theta$ is shown in degrees. Curves in each figure correspond to parameter values $T = 0.1$ (solid blue line); $T = 1$ (long-dashed red line); and $T = 10$ (short-dashed green line). Note that the co-polarized reflection coefficients satisfy $R_{11} = -R_{22}$.

form. Considering a plane wave in terms of three-dimensional Gibbsian vector fields, it is shown that there is no characteristic equation for the vector $\mathbf{k}$ in the medium which leaves a lot of freedom for the definition of the wave.

In this paper a certain class of skewon-axion media is studied whose number of parameters is limited to 10. Such a medium has three specified axes denoted by the constant orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$. It is shown that, in such a medium, two scalar conditions (31), (32) are satisfied by the four Gibbsian field vectors. The form of the conditions suggests that, if there is a planar interface orthogonal to the vector $\mathbf{e}_3$, (31) and (32) serve as boundary conditions for the fields. Since the conditions generalize those obtained for the soft-and-hard (SH) boundary and for the DB boundary for the limiting cases of the free parameter $T$, the novel boundary has been dubbed as the SHDB boundary.

Plane wave reflection from a SHDB boundary was considered and analytical expressions for the reflected field components were derived. It was shown that, for two eigenpolarizations, the SHDB boundary can be replaced by effective PEC and PMC boundaries just like for the SH and DB boundaries. The theory was tested by a numerical example demonstrating the nonreciprocal property of the SHDB boundary.
Although it appears quite easy to define the generalized boundary conditions (31) and (32), quite simple in form, one may note that their realization in terms of a medium interface is not so obvious. In fact, it appears that, without the four-dimensional formalism, the introduction of the skewon-axion medium is a difficult task. Also, it is not at all obvious that a medium defined by (47) and (48) with (49) – (52) will yield a set of boundary conditions since it does not pop out directly from the medium equations but requires consideration of fields in the medium.

Realization of the skewon-axion medium by some metamaterial is a challenge for which there may not exist an easy solution. However, since the boundary conditions (31) and (32) appear more important than the medium itself, it should be easier to realize the SHDB boundary, whenever a useful engineering application for the boundary has been discovered. A corresponding realization for the DB boundary was recently found in terms of a planar structure containing metamaterial inclusions [26].

Appendix: Gibbsian Representation of Forms

The connection between spatial multiforms and Gibbsian vectors can be expressed according to the following rules which are based on the spatial metric dyadic

$$\mathbf{I}_g = e_1e_1 + e_2e_2 + e_3e_3,$$  \hfill (124)

which serves as the Gibbsian unit dyadic. Thus, a spatial one-form $\alpha_s = \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3$ is transformed to a Gibbsian vector as

$$\alpha_s \rightarrow \alpha_g = \mathbf{I}_g|\alpha_s = \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3.$$  \hfill (125)

Similarly, a spatial two-form $B = B_{12}e_{12} + B_{23}e_{23} + B_{31}e_{31}$ is transformed as

$$B \rightarrow B_g = e_{123}|B = B_{12}e_3 + B_{23}e_1 + B_{31}e_2.$$  \hfill (126)

The bar product of a spatial vector $a_s$ and a spatial one-form $\alpha_s$ transforms to a dot-product of Gibbsian vectors,

$$a_s|\alpha_s \rightarrow a_s \cdot \alpha_g = a_s \cdot \mathbf{I}_g|\alpha_s = a_s|\alpha_s.$$  \hfill (127)

The wedge product of two spatial one-forms $\alpha_s, \beta_s$ becomes the cross product of two Gibbsian vectors,

$$\alpha_s \wedge \beta_s \rightarrow \alpha_g \times \beta_g = e_{123}|(\alpha_s \wedge \beta_g),$$  \hfill (128)
and the wedge product of three one-forms transforms to a scalar,
\[ \alpha_s \wedge \beta_s \wedge \gamma_s \rightarrow \alpha_g \cdot (\beta_g \times \gamma_g) = \alpha_s|((e_{123})(\beta_s \wedge \gamma_s)). \]  
(129)

Finally, the contraction product of a spatial vector \(a_s\) and a spatial two-form \(B\) is transformed to the cross product of the Gibbsian vectors \(a_s\) and \(B_g\) as
\[ a_s|B \rightarrow a_s \times B_g = \tilde{I}_g|(a_s|B). \]  
(130)

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