Properties of the spin liquid phase in the vicinity of the Néel - Spin-Spiral Lifshitz transition in frustrated magnets

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Three decades ago Ioffe and Larkin pointed out a generic mechanism for the formation of a gapped spin liquid. In the case when a classical two-dimensional (2D) frustrated Heisenberg magnet undergoes a Lifshitz transition between a collinear Néel phase and a spin spiral phase, quantum effects usually lead to the development of a spin-liquid phase sandwiched between the Néel and spin spiral phases. In the present work, using field theory techniques, we study properties of a universal spin liquid phase. We examine the phase diagram near the Lifshitz point and calculate the positions of critical points, excitation spectra, and spin-spin correlations functions. We argue that the spin liquid in the vicinity of 2D Lifshitz point (LP) is similar to the gapped Haldane phase in integer-spin 1D chains. We also consider a specific example of a frustrated system with the spiral-Néel LP, the \( J_1 - J_3 \) antiferromagnet on the square lattice that manifests the spin liquid behavior. We present numerical series expansion calculations for this model and compare results of the calculations with predictions of the developed field theory.

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I. INTRODUCTION

Quantum spin liquids (SL) are “quantum disordered” ground states of spin systems, in which zero-point fluctuations are so strong that they prevent conventional magnetic long-range order. The main avenues towards realizing SL phases in magnetic systems are frustration and quantum phase transitions. A particularly interesting example of SL is realized by tuning a frustrated magnetic system close to a Lifshitz point (LP) that separates collinear and spiral states. In the vicinity of the Lifshitz transition the quantum fluctuations are strongly enhanced, resulting in a plethora of novel intermediate quantum phases.

A general argument in favour of a universal gapped SL phase near LP in two-dimensional frustrated Heisenberg antiferromagnets (AF) was first proposed by Ioffe and Larkin. They showed that in the proximity of the LP quantum fluctuations destroy long-range spin correlations and create a region in the phase diagram with a finite magnetic correlation length. Subsequent studies found evidence for SL phases in various two-dimensional systems near the LP, including Heisenberg models on square and honeycomb lattices with second and third nearest neighbour antiferromagnetic couplings. However, the universality of the SL phase near LP, its ubiquitous properties, and the relation of the general argument to specific Heisenberg models has not previously been addressed.

In the present paper we revisit the Ioffe-Larkin scenario and consider a field theory for a quantum Lifshitz transition between collinear and spiral phases in \( D = 2+1 \). Disregarding microscopic properties of specific lattice models we focus on the generic infrared physics at the LP. We develop a field-theoretic description of the \( O(3) \) Lifshitz point based on the extended nonlinear sigma model. The nonlinear sigma model provides a unifying theoretical framework that allows us to analyze the phase diagram, calculate positions of critical points, excitation spectra, and static spin-spin correlations functions. We demonstrate universal scalings of observables (gaps, position of critical points, etc) in terms of the dimensionless SL gap at the LP, \( \delta_0 \), and show that the correlation length in the SL phase scales as \( \xi \sim 1/\sqrt{\delta_0} \). We also argue that the LP spin liquid has a similarity to the gapped Haldane phase in integer-spin 1D chains. However, for the 2D SL there is no significant difference between the integer and half-integer spin cases.

A particular example of a system that has a Néel-spiral LP and hence manifests the spin liquid behavior is the frustrated antiferromagnetic \( J_1 - J_2 - J_3 \) Heisenberg model on the square lattice with the second and third nearest neighbour couplings as well as it’s simplified version, the \( J_1 - J_3 \) model. We perform numerical series expansion calculations for the \( J_1 - J_3 \) model and compare results of the calculations with predictions of the developed field theory.

The structure of the paper is as follows. In Sec. III we introduce the effective field theory describing the Néel to Spin Spiral Lifshitz point. Section IV addresses the quantum LP, quantum fluctuations, and the criterion for quantum ‘melting’. Next, in Sec. V we calculate the spin-wave gap and positions of critical points. Section VI addresses the static spin-spin correlator in the spin liquid phase. In Sec. VII we describe our numerical series calculations for the \( J_1 - J_3 \) model with spin \( S = 1/2 \) and \( S = 1 \) and compare results of these calculations with predictions of the field theory. Finally our conclusions are presented in Sec. VII.

II. EFFECTIVE FIELD THEORY

We start with the following \( O(3) \) symmetric Lagrangian describing a transition from the Néel to a spiral
phase in two dimensional antiferromagnets:
\[ \mathcal{L} = \frac{X}{2} (\partial_t n_\mu)^2 - \frac{1}{2} n_\mu K(\partial_t) n_\mu, \quad (n_\mu)^2 = 1. \] (1)

Here \( \chi_c \) is the transverse magnetic susceptibility, \( n_\mu \) is a unit length vector with \( N = 3 \) components corresponding to the staggered magnetization, \( \partial_t \) are the spatial gradients. The general form of the “elastic energy" operator \( K(\partial_t) \) in inversion symmetric systems reads
\[ K(\partial_t) = -\rho(\partial_t)^2 + \frac{b_1}{2}(\partial_x^4 + \partial_y^4) + b_2 \partial_x^2 \partial_y^2 + \mathcal{O}(\partial_t^4), \] (2)

where we assume that the \( n \)-field is sufficiently smooth. The spin stiffness \( \rho \) is the tuning parameter that drives the system across the Lifshitz transition. The spin stiffness is positive in the Néel phase, negative in the spiral phase and vanishes at the Lifshitz point. The \( b \)-terms containing higher order spatial derivatives are necessary for stabilization of spiral order at negative \( \rho \), and we will assume that \( b_1, b_2 > 0 \). While the kinematic form of the Lagrangian (1) is dictated by global symmetries of the system, a formal derivation starting from a frustrated Heisenberg model can be found e.g. in Ref. Note that in Lagrangian (1) we do not take into account topological terms. We will discuss their possible role later in the text.

The Lagrangian (1) is relevant to a number of models and systems mentioned in the Introduction. Here we would like to mention another example motivated by rare-earth manganite materials (Tb, La, Dy)MnO\(_3\) (see Ref.(). These materials have a layered structure with the individual ferromagnetic layers coupled antiferromagnetically. Due to the antiferromagnetic interlayer coupling the dynamics of the system is described by the second-order time derivative as in usual antiferromagnets in agreement with Eq. (1). Within each plane there are ferromagnetic nearest neighbour and antiferromagnetic second nearest neighbour Heisenberg interactions leading to an in-plane frustration. These compounds could be tuned to the Néel-Spin-Spiral LP by performing chemical substitution. Of course real materials are three-dimensional and contain many planes, however thin films can manifest some physics considered here.

In the AF phase of (1), \( \rho > 0 \), the rotational symmetry is spontaneously broken and the Néel vector has a nonzero expectation value, e.g. is directed along the \( z \) axis \( (n) = e_z \). In the spin spiral phase, with \( \rho < 0 \), there is an incommensurate ordering
\[ n(r) = e_1 \cos(Qr) + e_2 \sin(Qr), \] (3)

where \( e_{1,2} \) are orthogonal unit vectors and \( Q \) is the pitch of the spiral. For \( b_1 \leq b_2 \) the spiral wave vector is directed along \( x \) or \( y \): \( Q = (\pm Q, 0), (0, \pm Q) \), where \( Q^2 = |\rho|/b_2 \). In the opposite case \( b_1 > b_2 \) the wave vector is directed along the main diagonals: \( Q = \sqrt[4]{2}(\pm Q, \pm Q) \), \( Q^2 = 2|\rho|/(b_1 + b_2) \). The relation between the coefficients \( b_1 \) and \( b_2 \) depends on the specific choice of the lattice model. In the “isotropic" case, \( b_1 = b_2 \), the system has additional rotational degeneracy in the momentum space due to the arbitrary orientation of wave vector \( Q \). The additional degeneracy can destabilize spiral states and result in quantum spin liquid states that have been predicted for 3D antiferromagnets. In the present paper we will stay away from this special critical point. The classical phase diagram is shown schematically in Fig.1.

We would like to make a comment regarding Lagrangian (1). Parameters of any field theory depend on the momentum and energy scales that is described by renormalization group procedure. We assume that parameters in (1),(2) are fixed at the ultraviolet cutoff \( \Lambda \approx 1 \), where unity corresponds to the inverse lattice spacing. Quantum fluctuations at scales larger than \( \Lambda \) but smaller than the boundary of magnetic Brillouin zone lead to a renormalization of the parameters \( \rho \to \rho^{\text{ren}}, b_1, b_2 \to b_1^{\text{ren}}, \ldots \). Therefore, the values of the parameters in (1),(2) can be different from those naively derived using spin wave theory. As was pointed out by Ioffe and Larkin this renormalization is especially relevant for the spin stiffness. The correction to the spin stiffness arises due to the the \( b \)-terms in (2). The easiest way to understand the correction\( b \) is to consider the Néel phase and decompose the order parameter into two transverse components and a longitudinal component
\[ n = (\pi, n_z), \quad n_z = \sqrt{1 - \pi^2} \approx 1 - \pi^2/2. \] (4)

Hence the following contribution from the \( b \)-term arises
\[ \partial^2 n_z \partial^2 n_z \sim b (\partial^2 \pi^2)(\partial^2 \pi^2). \] (5)
The field $\pi$ has fluctuations with momenta smaller than $\Lambda$, $\pi_\perp$, and fluctuations with momenta larger than $\Lambda$, $\pi_\parallel$, $\pi = \pi_\perp + \pi_\parallel$. Substitution in (10) and averaging over energy gives

$$b \left( \partial^2 \pi^2 \right) / \left( \partial^2 \pi \right) \rightarrow b \left( \partial \pi \right)^2 \left( \partial \pi \right)^2 = \delta \rho_\Lambda \left( \partial \pi \right)^2 \, .$$

Note, when averaging $\left( \partial^2 \pi \right) / \left( \partial \pi \right)$ each multiplier must contain the high ($\pi_\perp$) and the low ($\pi_\parallel$) energy components. The terms with one multiplier containing only the high energy and another only the low energy components give rise to a total derivative contributions to the Lagrangian and can be neglected. Equation (6) demonstrates a positive correction to the spin stiffness.

The elementary spin excitations in the AF phase are two gapless Goldstone modes - transverse spin-waves and a massive longitudinal ('Higgs') mode. Due to the unit length constraint $\left( \pi^2 = 1 \right)$ the Higgs mode has a very large energy and can be disregarded. In the spiral phase there are three Goldstone modes: a sliding mode and two out of plane excitations. These three modes correspond to the three Euler angles defining the orientation of the $\left( e_1, e_2, e_3 \right)$ triad, where $e_3 = \left[ e_1 \times e_2 \right]$. The excitation modes in the SL phase are three-fold degenerate due to $O(3)$ rotational invariance of the model. Above the LP $\left( \rho > 0 \right)$ the minimum of dispersion is located at $q = 0$, whereas below the LP $\left( \rho < 0 \right)$ the dispersion has four degenerate minima at the 'spiral' wave vectors $q = Q$. The evolution of the dispersion across the LP is schematically shown in Fig. 1i. The change of the shape of the dispersion indicates the Lifshitz point.

The location of this critical point $\rho_{cN}$ can be found by imposing the condition $\left( \langle n_z \rangle \rightarrow 0 \right)$, which naively provides the following criterion for the transverse spin fluctuations $\left( \pi^2 \right) \approx 2$. This critical value for $\left( \pi^2 \right)$ is largely overestimated and it is not consistent with the unit length constraint. One can find a more accurate value of $\left( \pi^2 \right)$ by accounting for the next order terms in the Taylor series expansion of $n_z = \sqrt{1 - \pi^2} \left( \text{see Appendix A} \right)$, or alternatively by using the $1/N$ expansion for $O(3)$ theory. The $1/N$ expansion has been extensively applied to describe quantum antiferromagnets. For the most relevant examples see Refs. 19, 21. In the $1/N$ expansion approach we lift the hard constraint $n^2 = 1$ by introducing a Lagrange multiplier

$$\mathcal{L} \rightarrow \mathcal{L} - \lambda (n^2 - 1). \quad (9)$$

After integrating out the $n$ field in the new Lagrangian $\mathcal{L}_N$, we obtain an effective Lagrangian depending only on the auxiliary field $\lambda$:

$$\mathcal{L}_N = N \text{tr} \ln \left( -\chi \partial_t - K(q) \right) - \lambda_1 \lambda + \lambda. \quad (10)$$

We can find the saddle point in the Lagrangian $\mathcal{L}_N$ by calculating the variational derivative in (10) with respect to $\lambda_1$ and regarding $\lambda$ as a constant, $\lambda = \chi_1 \Delta^2$:

$$\mathcal{L}_\lambda = \frac{1}{2} \chi_1 \left( \omega^2 - \Delta^2 \right) - K(q) = 1. \quad (11)$$

The Lagrange multiplier in Eq. (11) has the meaning of the spin gap. Equation (11) determines the evolution of the gap $\Delta \left( \rho \right)$ with the spin stiffness in the SL phase. Comparing Eq. (11) with Eq. (7) we conclude that at the boundary between SL and AF phases $\left( \pi^2 \right) \approx \left( N - 1 \right) / N = 2/3$. This criterion is quite natural for the $O(3)$ symmetric quantum critical point separating the spin liquid phase at which the long range AF order is lost and the order parameter correlations are exponentially decaying. Importantly, this is a generic gapped spin liquid originating from long range fluctuations and is unrelated to a spin-dimer ordering. The SL gap is zero, $\Delta = 0$, at the critical point $\rho_{cN}$ and the gap increases when we proceed deeper into the spin liquid phase. The SL phase stretches across a finite window $\left[ \rho_{cS}, \rho_{cN} \right]$ in the vicinity of the LP, as depicted in Fig. 1f.
Néel and SL states. Nevertheless, this criterion underestimates $(\pi^2)c$. One can see this from the example of the $S = 1/2$ 2D Heisenberg model on the square lattice. A textbook expression for the staggered magnetization is well known

$$\langle n_z \rangle = 2\langle S_z \rangle = 1 - 2 \int_{MBZ} \frac{d^2q}{(2\pi)^2} \left( \frac{1}{1 - \gamma_q^2} - 1 \right), \quad (12)$$

where $\gamma_q = \frac{1}{2}(\cos qx + \cos qy)$, and integration is performed over the magnetic Brillouin zone. In the limit $q < 1$ Eq. (12) is consistent with 7 since in this case $\gamma_q = 1/8J$ and $\omega_q/J \approx \sqrt{2q}$, where $J$ is the Heisenberg AF coupling. Integration over $q$ in (12) gives a well known result $\langle n_z \rangle \approx 2 \times 0.305$ which corresponds to $\langle \pi^2 \rangle \approx 0.78$ in the equation $\langle n_z \rangle \approx 1 - \frac{1}{2}(\pi^2)$. The integration in the corresponding long-wavelength approximation 7 with $N = 3, \chi_{\perp} = 1/8J$, $\omega_q \approx \sqrt{2q}J$ and the ultraviolet cutoff $\Lambda = 1$ gives a close value $\langle \pi^2 \rangle \approx 0.89$. Both values are above 2/3 and we know that the long range AF order in the unfrustrated Heisenberg model still persists. Based on this analysis we estimate the critical value of fluctuation as

$$\langle \pi^2 \rangle_c \approx 1. \quad (13)$$

Equation (13) is an analogue of the Lindemann criterion for quantum melting of long range magnetic order in 2D quantum magnets. Our approach implicitly violates rotational invariance, but it allows us to calculate approximately the positions of critical points and the value of the spin liquid gap.

The spin liquid gap $\Delta$ is determined by Eqs. (7) and 8 from the condition $\langle \pi^2 \rangle = \langle \pi^2 \rangle_c \approx 1$. At $\rho > 0$ (the Néel side of LP) $\Delta$ coincides with the physical gap. On the spiral side of LP, $\rho < 0$, the physical gap corresponds to the excitation energy at the “spiral” wave vector $Q$:

$$\Delta_{\rho < 0} = \min \omega_q = \sqrt{\Delta^2 + \frac{1}{\chi_{\perp}}K(Q)} \approx 0.$$  

This gap is closed at the spin-liquid-SL critical point. Therefore, the position of this critical point $\rho_{cS}$ is determined from the following two equations

$$\frac{1}{2} \sum_{q < \Delta} \int \frac{d\omega}{(2\pi)^2} \frac{1}{\chi_{\perp}(\omega^2 - \Delta^2) - K(Q) + \omega} = 1, \quad \Delta_{\rho = 0}^2 = \Delta^2 + \frac{1}{\chi_{\perp}}K(Q) = 0. \quad (14)$$

At $\rho < \rho_{cS}$, the magnon Green’s function acquires a pole at imaginary frequency $\omega = \pm \sqrt{\Delta^2 + K(Q)/\chi_{\perp}}$. This is the indication of an instability of the SL phase towards condensation of a static spiral with the wave vector $Q$.

It is instructive to draw an analogy between the SL physics at 2D Lifshitz point and the one-dimensional Haldane spin chain. A condition similar to (11) determines the value of the Haldane gap. Indeed, the integer spin $S$ Heisenberg model in the continuous limit can be mapped to the $O(3)$ relativistic nonlinear sigma model in $D = 1 + 1$. The model parameters are the speed of the magnon, $c = \sqrt{\rho/\chi_{\perp}} = 2JS$, and the transverse magnetic susceptibility, $\chi_{\perp} = 1/4J$ ($J$ is the Heisenberg coupling constant). Proceeding by analogy with (7) we find the fluctuations of the spin in the Haldane model

$$\langle \pi^2 \rangle_c = 2 \int_{0}^{\Lambda} \frac{dq}{2\pi} \frac{1}{\sqrt{c^2q^2 + \Delta^2}} \approx \frac{1}{2\pi c_{\perp}} \ln \frac{c_{\perp}}{\Delta}, \quad (15)$$

As we already discussed, the ultraviolet cutoff is $\Lambda \approx 1$. The logarithmically divergent $\langle \pi^2 \rangle$ in the Haldane model is analogous to the log-divergence in (7) at the LP. Numerical values of the Haldane gaps for $S = 1$ and $S = 2$ are known from DMRG calculations: see e.g. Ref. 22, $\Delta_{S = 1}/J \approx 0.41, \Delta_{S = 2}/J \approx 0.08$. Taking these values of the gap Eq. (15) we obtain the following critical values of fluctuations, $\langle \pi^2 \rangle_c \approx 0.5$ (for $S = 1$) and $\langle \pi^2 \rangle_c \approx 0.6$ (for $S = 2$), which are smaller than 13. We believe that the difference is due to different dimensionality. While DMRG is more reliable it is interesting to note that the renormalization group analysis 23 for the Haldane chain gives $\langle \pi^2 \rangle_c = 1$.

The differences in the values of $\langle \pi^2 \rangle_c$ is not crucial when making comparisons between 1D and 2D systems. However, it is well known that properties of the spin chains with half-integer and integer spins are very different. The gapped SL phase in 1D appears only in the integer spin chains, while in contrast the excitations of half-integer spin chains are gapless spinons in agreement with the Lieb-Shultz-Mattis theorem. We believe that the 2D spin liquid in the vicinity of LP point is generic and independent of the spin value. The Lieb-Shultz-Mattis theorem states that in systems with half-integer spin per unit lattice cell and full rotational $SU(2)$ symmetry the excitations are gapless or otherwise the ground state of the system is degenerate. The theorem was initially formulated for $D = 1 + 1$ systems and later generalized for higher spatial dimensions. Technically in $D = 1 + 1$ the dramatic difference between integer and half integer spin is due to the topological Berry phase term which is not included in the Lagrangian. Topological effects in $D = 2 + 1$ correspond to skyrmions or merons.

In principle topological configurations become more important when approaching the Lifshitz point. However such topological solutions are unstable within the model. Using scaling arguments one can see that due to the fourth spatial derivative term in the Lagrangian the energy of localized skyrmions at LP behaves as $\sim b_1R^2$, where $R$ is the skyrmion radius. Therefore any localized skyrmions energetically prefer to have large size $R \rightarrow \infty$ and only contribute to the boundary terms. Although the topological solutions might play a role to reconcile with the Lieb-Shultz-Mattis theorem, these configurations are statistically irrelevant in the bulk.
IV. POSITIONS OF NÉEL-SPIN LIQUID AND SPIN-SPIRAL-SPIN LIQUID CRITICAL POINTS

In order to make our calculations more specific and having in mind comparison with the $J_1 - J_3$ model, in this Section we set $b_2 = 0$. It is convenient to introduce dimensionless spin stiffness and dimensionless gap parameters

$$\bar{\rho} = \frac{2\rho}{b_1}, \quad \bar{\delta} = \sqrt{\frac{2\chi}{b_1}} \Delta.$$  \hspace{1cm} (16)

At negative $\rho$ the spiral wave vector is directed along the main diagonals $Q = \frac{1}{\sqrt{2}} (Q, \pm Q)$,

$$Q^2 = |\bar{\rho}|.$$  \hspace{1cm} (17)

As we already discussed in Section III the condition of criticality reads

$$\langle \pi^2 \rangle_c \approx 1 \approx \sqrt{\frac{2}{(4\pi^2)\sqrt{\chi b_1}}} \int \frac{d^2q}{\sqrt{\rho q^2 + q_x^2 + q_y^2 + \delta^2}}.$$  \hspace{1cm} (18)

First, we determine the gap exactly at the LP, $\delta_0 = \delta(\rho = 0)$. For $\delta_0 \ll 1$ the solution of (18) is

$$\delta_0 = 1.7 \Lambda^2 e^{-2\frac{\pi^2}{\chi b_1}}.$$  \hspace{1cm} (19)

The constant $\zeta$ in the exponent is given by the angular part of the $q$-integral $\zeta = \frac{2}{\pi} K \left( \frac{b_1}{b_2} \right)$, where $K(m) = \int_{0}^{\pi/2} d\phi \sqrt{1 - m \sin^2 \phi}$ is the complete elliptic integral. In the specific case under consideration, $b_2 = 0$, $\zeta = \frac{2}{\pi} K(1/2) \approx 1.18$. The numerical prefactor $A = 1.7$ in (19) is found by performing a least-squares fitting of the integral in Eq. (18). While Eq. (19) is derived for $\delta_0 \ll 1$, however direct numerical integration in (18) shows that (19) practically works up to $\delta_0 \leq 0.6 - 0.7$.

In order to determine the position of the Néel critical point $\rho_{cN}$ we evaluate the integral in (18) at $\delta \ll \bar{\rho} \ll 1$,

$$\frac{1}{2\pi} \int \frac{d^2q}{\sqrt{\rho q^2 + q_x^4 + q_y^4 + \delta^2}} \approx \zeta \ln \left( \frac{2.9 A^2 \rho}{\bar{\delta}} \right) - \frac{\delta}{\bar{\rho}}.$$  \hspace{1cm} (20)

The condition $\delta = 0$ gives the position of the Néel-SL critical point $\rho_{cS}$:

$$\rho_{cS} \approx 2.9 \Lambda^2 e^{-2\frac{\pi^2}{\chi b_1}} \approx 1.65 \delta_0.$$  \hspace{1cm} (21)

According to (20) in the vicinity of the Néel-SL critical point, $\rho < \rho_{cS}$, the gap grows linearly as $\delta \approx 0.64(\rho_{cS} - \rho)$, that corresponds to a mean-field prediction.

The spin stiffness $\rho_{cS}$ at the transition point from the Néel phase to the spin liquid phase is small but still finite. Therefore, we believe that the transition belongs to the standard O(3) universality class, the same as that in the bilayer quantum antiferromagnet, see e.g. Ref.25. The correct critical index for $O(3)$ transition is $\nu \approx 0.7$, which implies $\delta \propto (\rho_{cS} - \rho)^\nu$.

On the side of negative spin stiffness, $\rho_{cS} < \rho < 0$, the dimensionless physical gap reads

$$\alpha = \sqrt{2 \frac{\chi}{b_1}} \Delta_{ph} = \sqrt{\delta^2 - \bar{\rho}^2/2}.$$  \hspace{1cm} (22)

The condition $\alpha = 0$ determines the position of the spin-spiral to SL critical point $\rho_{cS}$. Calculating the integral in (18) at $\alpha \ll Q^2 \ll 1$ we find

$$\frac{1}{2\pi} \int \frac{d^2q}{\sqrt{Q^2/2 - Q^2 q^2 + q_x^4 + q_y^4 + \alpha^2}} \approx \zeta \ln \left( \frac{5.4 \Lambda^2}{Q} \right) - 2 \frac{\alpha}{Q^2}.$$  \hspace{1cm} (23)

The condition $\alpha = 0$ gives the position of the critical point $\rho_{cS}$:

$$\rho_{cS} = -Q^2 \approx -15\delta_0.$$  \hspace{1cm} (24)

The gap in the vicinity of this critical point is $\alpha = 0.27(\rho - \rho_{cS})$. This is a mean-field result and we believe that the transition at $\rho_{cS}$ does not belong to a standard universality class.

The dimensionless gap found by numerical solution of Eq. (18) for different values of $\delta_0$ in the entire SL region $\rho_{cS} < \rho < \rho_{cN}$ is presented in Fig. 2. From this figure we conclude that asymptotic solutions given by Eqs. (21) and (24) become valid only at sufficiently small values of $\delta_0$ (i.e large values of S): Eq. (24) is valid at $\delta_0 \lesssim 0.2$ and Eq. (21) is valid only for very small gaps, $\delta_0 \lesssim 0.02$. The asymmetry between $\rho_{cS}$ and $\rho_{cN}$ evident from Fig. 2 is due to stronger quantum fluctuations in the spiral ($\rho < 0$) region compared to the $\rho > 0$ domain.

An alternative method to determine $\rho_{cS}$ is to approach the spiral-SL critical point from the spiral phase and find the condition when quantum fluctuations melt the spiral. The fluctuations of spiral consist of the out-of-plane
\( h(r, t) \) and in-plane modes \( \phi(r, t) \), can be parametrized in the form
\[
\tilde{a} = (\sqrt{1 - h^2 \cos(Q \cdot r + \phi)}, \sqrt{1 - h^2 \sin(Q \cdot r + \phi)}, h) \tag{25}
\]
The total quantum fluctuation orthogonal to the spin alignment in the spiral state reads
\[
\langle \pi^2 \rangle = \langle \phi^2 \rangle + \langle h^2 \rangle, \tag{26}
\]
\[
\langle \phi^2 \rangle = \frac{1}{(4\pi^2)^2 \chi_{\perp} b_1} \int \frac{d^2q}{\sqrt{2Q^2q^2 + q_x^2 + q_y^2}},
\]
\[
\langle h^2 \rangle = \frac{1}{(4\pi^2)^2 \chi_{\perp} b_1} \int \frac{d^2q}{\sqrt{Q^2/2 - Q^2q^2 + q_x^4 + q_y^4}}.
\]

The denominators in the integrals for \( \langle \phi^2 \rangle \) and \( \langle h^2 \rangle \) in (26) represent the dispersions for the Nambu-Goldstone excitations: the sliding mode and the out of plane mode, see details in Appendix [4]. Evaluating the integrals with logarithmic accuracy, we obtain
\[
\langle \pi^2 \rangle \approx \frac{1}{(2\pi)^2 \chi_{\perp} b_1} \zeta \ln \left( \frac{6.5\Lambda^2}{Q^2} \right). \tag{27}
\]
Now, applying the same criterion for the critical point, \( \langle \pi^2 \rangle_c \approx 1 \), we find the critical \( \rho_{cS} \)
\[
\rho_{cS} \approx -6.5\Lambda^2 e^{-2\Delta/\chi_{\perp} b_1} \approx -4\delta_0. \tag{28}
\]

The prefactor in (28) is significantly smaller than the prefactor in Eq. (24). This emphasizes the fact that our calculation is only approximate. Pragmatically this uncertainty is not very significant. We already pointed out that Eq. (24) is valid only for extremely small gaps, \( \delta_0 \ll 0.02 \). At larger values of \( \delta_0 \) the position of the critical point \( \rho_{cS} \) is different from (24), see Fig. 4. Numerical evaluation of (26) combined with the criticality condition (13) gives the following locations of the critical points \( \rho_{cS} \) : \( \rho_{cS}/\delta_0 = -3.7 \) at \( \delta_0 = 0.06 \); \( \rho_{cS}/\delta_0 = -3.6 \) at \( \delta_0 = 0.2 \); \( \rho_{cS}/\delta_0 = -2.2 \) at \( \delta_0 = 0.7 \). Comparing these values with positions of the critical point that follow from Fig. 2 we conclude that, for the practically interesting case \( \delta_0 \gg 0.15 \), both methods give close positions of the critical point.

As was mentioned in Sec. 11 in the presence of inplane rotational symmetry \( b_1 = b_2 \) (e.g. frustrated Heisenberg model on the hexagonal lattice), quantum fluctuations become especially strong. In fact, when approaching the critical point \( \rho_{cS} \) the integral \( \int_q \frac{1}{\sqrt{\alpha^2 + (Q^2 - q^2)^2}} \) is logarithmically divergent for \( \alpha \to 0 \) at \( q = Q \). It implies that one has to keep higher order terms \( \mathcal{O}(q_6^2) \) in the expansion (2)
\[
K(q) = \rho q^2 + \frac{b}{2} q^4 + c(d_x^6 + q_y^6) + d(q_x^4 q_y^2 + q_x^2 q_y^4) \tag{29}
\]
which break the symmetry with respect to spatial rotations in the \( \{xy\} \) plane and remove the degeneracy with respect to the choice of the direction of \( Q \). After accounting for the higher order anisotropic terms \( \propto \mathcal{O}(q_6^2) \) the integral for \( \langle \pi^2 \rangle \) becomes convergent at \( |q| = Q \) and the value \( \rho_{cS} \) is well defined.

V. SPIN-SPIN CORRELATION FUNCTION

Spin-spin correlations of a standard tool to analyze quantum critical properties of a magnetic system. In the SL phase the correlator provides an essential information about the properties of the ground state. The equal time two-point spin-spin correlation function reads
\[
C(r) = \langle n^\alpha(r) n^\beta(0) \rangle = 1 + 2[L(r) - R(0)] + \ldots, \tag{30}
\]
where \( \langle n^\alpha(r) n^\beta(0) \rangle = \delta^{\alpha\beta} R(r) \) and indices \( \alpha, \beta \) refer only to the \( x \) and \( y \) spin components. The two-point correlator is normalized such that \( C(0) = \langle n_0^2 \rangle = 1 \). In the SL phase the correlation function should vanish at large distances \( C(r \to \infty) \to 0 \) and \( R(r \to \infty) \to 0 \). These conditions are consistent with the “melting criterion” in Eq. (13) if we truncate the asymptotic expansion in Eq. (30) keeping only the terms explicitly presented there.

The \( \langle \pi(r) \pi(0) \rangle \) correltion function in the SL phase reads
\[
R(r) = \int \frac{d\omega d^2q}{(2\pi)^3} \frac{e^{i\omega r}}{\chi_{\perp}(\omega^2 - \Delta^2) - K(q) + i\omega}. \tag{31}
\]
Calculating (31) and substituting the result in Eq. (30), we obtain the two-point spin-spin correlation function \( C(r) \); the numerical results are plotted in Fig. 3. Similar to the previous Section these plots correspond to the case \( b_2 = 0 \). Therefore, the correlator is somewhat anisotropic. There are two points to note, one is physical and another is technical. (i) The correlation length scales as one over the square root of the gap, \( \xi \propto 1/\sqrt{\delta_0} \), instead of the standard relation, \( \xi \propto 1/\delta_0 \). (ii) When integrating in Eq. (31) we use the soft ultraviolet cutoff by multiplying the integrand by \( e^{-q^2/(2\Lambda^2)} \). The soft cutoff allows us to avoid nonphysical oscillations in \( R(r) \) due to the Gibbs phenomenon. The Gibbs phenomenon results in spurious oscillations, which always exist for a sharp cutoff and are well known in Fourier analysis.

The asymptotic behaviour of the correlation function \( R(r \to \infty) \) in the spin liquid phase at \( \rho = 0 \) can be analytically obtained in the simplified isotropic approximation \( (b_1 = b_2) \):
\[
R(r) \sim e^{-r\sqrt{\delta_0}/\chi_{\perp}} \cos \left( r \sqrt{\sqrt{\delta_0} - \pi/4} \right), \tag{32}
\]
Using Eq. (32) we deduce the spin-spin correlation length \( \xi = \sqrt{\delta_0/\delta_0} \). In the case of negative spin-stiffness \( \rho_{cS} < \rho < 0 \) the correlation function \( R(r) \) becomes oscillating, see Fig. 3. In the vicinity of the critical
Formula (33) is consistent with the well known $\propto 1/r$ decay of correlations of transverse spin components in the Néel phase (see e.g. Ref. 4). We stress that the "isotropic approximation", $b_1 = b_2$, provides a qualitative and quantitative description of the correlation function $C(r)$ only away from the critical point $\rho_{cS}$. In the vicinity of the point $\rho_{cS}$ the isotropic model \( \| \) becomes unstable, see comments to Eq. (29).

Now we would like to make a comparison between $O(3)$ and $O(2)$ quantum Lifshitz transitions. The $O(2)$ version of Lagrangian \( \| \) describes the XY frustrated Heisenberg antiferromagnet in the continuous limit. The physics in the $O(2)$ model is quite different from the $O(3)$ model and the Ioffe-Larkin argument is inapplicable in this case. The $O(2)$ Lagrangian can be mapped to the scalar Lifshitz model described by a polar angle $\theta$: $n_x + in_y = e^{i\theta}$. This model has an exact solution for the correlation function $C(r)$ at the LP: $C(r)$ decays algebraically at the LP in contrast to the non-vanishing correlations at $r \rightarrow \infty$ in long-range ordered Néel or spin-spiral phase. Therefore we conclude that there exist a finite region in the vicinity of the LP with algebraically decaying correlations. The region with algebraic spin correlations in some extent is analogous to the SL phase in the $O(3)$ model addressed in the present paper.

VI. $J_1 - J_3$ MODEL ON THE SQUARE LATTICE

In the present Section we compare the field theory predictions with results of numerical calculations for the antiferromagnetic $J_1 - J_3$ Heisenberg model on the square lattice. Frustrated $J_1 - J_2$ and $J_1 - J_3 - J_2$ models have been discussed in numerous studies (see e.g. Refs. 4,6,29): some references are also presented in the Introduction. In the classical limit both models exhibit the spin spiral state at a sufficiently large frustration. Quantum versions of the models show a magnetically disordered state at a sufficiently large frustration. Classically the $J_1 - J_2$ model at $J_2/J_1 = 1/2$ has three degenerate ground states, the Néel, the spin-spiral, the spin-stripe. The tricritical point is somewhat special; the proximity of the columnar spin stripe phase enhances spin-dimer correlations and makes the physics of the $J_1 - J_2$ model different from that considered in the present work. On the other hand if we set $J_2 = 0$ and consider only the $J_3$ frustration then classically there is a Lifshitz point with a transition to the spin-spiral at $J_3 = J_1/4$, and the spin-stripe state has much higher energy than the spin-spiral and the Néel states. Therefore the $J_1 - J_3$ model is a good testing ground for the generic theory of a "soft" Lifshitz transition developed in the present work. The Hamiltonian of the $J_1 - J_3$ model reads

$$H = J_1 \sum_{i<j} S_i S_j + J_3 \sum_{\langle\langle ij\rangle\rangle} S_i S_j,$$

where $<ij>$ and $\langle\langle ij\rangle\rangle$ denotes first and third nearest neighbour interaction. The classical spin-spiral to Néel LP is located at $J_3/J_1 = 1/4$. As we already pointed out in Section III quantum fluctuations must shift the LP towards larger values $J_3/J_1 > 1/4$.

In the long-wavelength approximation we can map the Heisenberg model to the Lagrangian \( \| \). The magnetic susceptibility is well known,

$$\chi_\perp = \frac{1}{8J_1}. \quad (35)$$

The elasticity parameters of the Lagrangian can be found in two ways. (i) The first way is a straightforward expansion of the classical elastic energy at small wave number $q$, that gives

$$\rho = S^2(J_1 - 4J_3),$$

$$b_1 = S^2(16J_3 - J_1),$$

$$b_2 = 0. \quad (36)$$

(ii) An alternative way is to calculate the magnon dispersion in the Néel phase using the standard spin-wave
theory. The dispersion reads:

\[ \omega_q = 4S J_1 \sqrt{\left(1 - \frac{J_3}{J_1}(1 - \gamma_{2q}) \right)^2 - \gamma_q^2}, \]

\[ \gamma_q = \frac{1}{2}(\cos q_x + \cos q_y), \]

\[ \gamma_{2q} = \frac{1}{2}(\cos 2q_x + \cos 2q_y). \]

Expanding \( \omega_q \) at small \( q \) and comparing the results with Eq. (39) (at \( \Delta = 0 \)) we find

\[ \rho = S^2(J_1 - 4J_3), \]

\[ b_1 = 4J_1S^2 \left[ -\frac{5}{48} + \frac{2}{3} \left( \frac{J_3}{J_1} \right) + \left( \frac{J_3}{J_1} \right)^2 \right], \]

\[ b_2 = 4J_1S^2 \left[ -\frac{1}{8} + \frac{2}{3} \left( \frac{J_3}{J_1} \right)^2 \right]. \]

Expressions for \( b_1 \) and \( b_2 \) in Eqs. (36) and (39) do not coincide. At the LP, \( J_3 = J_1/4 \), both Eqs. give \( b_2 = 0 \), however, values of \( b_1 \) are different. Eq. (36) gives \( b_1 = 0.25S^2 J_1 \) while Eq. (39) gives \( b_1 = 0.5S^2 J_1 \). Of course the spin-wave theory value is more reliable.

We have performed extensive series calculations both in the Néel phase and the spin-spiral phase. Unfortunately the series expansion method does not allow to assess properties of the spin liquid phase directly. However, it allows to estimate the range of parameters where the spin liquid exists which can be compared with predictions of the field theory. In the Néel phase the series starts from the simple Ising antiferromagnetic state. In the spiral phase the calculation is more tricky. We first impose a classical diagonal spiral with some wave vector \( Q \) and find the total energy of this state \( E(Q) \). This includes the classical energy and the quantum corrections calculated by means of series expansions. We perform these calculations for many values of \( Q \) and then find numerically the minimum of \( E(Q) \). Such procedure gives us the ground state energy \( E_{gs} \) and the physical wave vector \( Q \). The ground state energy \( E_{gs} \) is plotted in Fig. 4 versus \( J_3 \). The plot of the wave vector squared, \( Q^2 \), versus \( J_3 \) is presented in Fig. 5. From the field theory description we expect that near the LP the wave vector behaves as

\[ Q^2 = \frac{2|\rho|}{b_1} = \frac{8S^2}{b_1}(J_3 - J_3^{LP}). \]

Therefore, from Fig. 5 we determine positions of Lifshitz points and, using Eq. (40) we find the values of the elastic constant \( b_1 \) at the LP:

\[ S = 1/2 : J_3^{LP} \approx 0.45J_1, \quad b_1/S^2 \approx 0.60J_1, \]

\[ S = 1 : J_3^{LP} \approx 0.3J_1, \quad b_1/S^2 \approx 0.74J_1. \]

As expected, (see the very end of Section II), quantum fluctuations extend the Néel phase in relation to the classical LP \( J_3^{LP} = 0.25J_1 \). Values of the elastic constant \( b_1 \) are larger than that given by Eq. (36) and smaller than that given by Eq. (39).

We have also calculated the magnon dispersion in the Néel phase. The series expansion becomes erratic at \( J_3 > 0.2J_1 \) and the errorbars in the calculations of \( \omega_q \) grow very quickly. The dispersion at \( J_3 = 0.2J_1 \) is shown in Fig. 5. We see that the shape of the dispersion is somewhat different from the prediction of the spin-wave theory (37). On the other hand the total bandwidth is consistent with the spin-wave theory. The situation is different in the case of a simple Heisenberg model \( (J_3 = 0) \), when the shape of magnon dispersion is consistent with the spin-wave theory but the total bandwidth is about 20% larger compared to the spin-wave theory value.

We also compute the static on-site magnetization in the Néel and spiral phases. The magnetization vanishes at \( J_3^SN \) and \( J_3^S \) critical points. We already pointed out that the Néel-SL transition at \( J_3^N \) belongs to the \( O(3) \) universality class. Therefore, we expect scaling \( \langle S_z \rangle \propto |J_3 - J_3^N|^\beta \) when approaching the critical point from the Néel phase, here \( \beta = (D - 2 + \eta)\nu/2 \approx \nu/2 \approx 0.35 \). Due to this reason in Fig. 5B we show series expansion results for the static on-site magnetization cubed. From here we locate the critical points:

\[ S = 1/2 : J_3^N \approx 0.35J_1, \quad J_3^S \approx 0.55J_1, \]

\[ S = 1 : J_3^N \approx J_3^S \approx 0.35J_1. \]
Nevertheless, we believe that Eq.(43) gives a reasonable estimate of the gaps. Knowing the dimensionless gaps and using Fig. 2 we can deduce the window $\delta \bar{\rho}$ occupied by the spin liquid phase. Combining this with Eq.(44) we find the spin liquid window $\Delta J_3 = |J_3^S - J_3^N|$ that follows from the field theory,

\[
\begin{align*}
\Delta J_3/J_1 & \approx 0.3, \quad (S = 1/2), \\
\Delta J_3/J_1 & \approx 0.1, \quad (S = 1).
\end{align*}
\]

These values while being slightly larger are in a reasonable agreement with the SL phase windows following from series expansion data in Fig. 7.

In conclusion of this Section we would like to comment on the anisotropic $J_1 - J_3$ model on square lattice. In this model $J_3$ frustrates $J_1$ only in one direction, say $J_3$ connects only the third nearest neighbours in the $y$-direction. This results in an anisotropic LP: the spin stiffness $\rho_y$ vanishes at some value of $J_3$ while $\rho_x$ remains finite and positive. The wave vector of the spin spiral is always directed along the $y$-axis. In this case quantum fluctuations at the LP are described as $\langle \pi^2 \rangle \propto \int \frac{d^q}{\sqrt{q_x^2 + q_y^2 + q_z^2}}$. The integral is infrared convergent unlike that in the isotropic LP. Therefore generically one cannot expect a spin liquid in this case. The fluctuations are still enhanced and there must be a suppression of the on-site magnetization at the LP. This is exactly what series expansions for the anisotropic $J_1 - J_3$ model with $S=1/2$ indicate. It is likely that a similar scenario is valid for thin films of frustrated manganites (Tb,La,Dy)MnO$_3$ tuned close to LP.

VII. CONCLUSION

In this work, using field theory techniques, we have studied properties of the universal spin liquid phase in a vicinity of an isotropic Lifshitz point in a system of localized frustrated spins. Our general analysis includes the phase diagram, positions of critical points, excitation spectra, and spin-spin correlations functions. In the semiclassical regime of large spin $S$ the spin liquid phase forms an exponentially narrow region in the vicinity of the Lifshitz point. The derivation of these results is accompanied with a thorough discussion of the criterion for quantum melting of long range magnetic order in two dimensions, an analogue of Lindemann criterion. We argue the 2D Lifshitz point spin liquid is similar to the gapped Haldane phase in integer-spin 1D chains. In order to check our general field theory results, and in particular to check the quantum melting criterion, we have performed numerical series expansion calculations for the $J_1 - J_3$ model on square lattice. We demonstrate that results of these two different approaches are in a good agreement.
VIII. ACKNOWLEDGMENTS

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Appendix A: The value of \( \langle \pi^2 \rangle_c \) derived from asymptotic Taylor expansion.

After expanding \( n_z = \sqrt{1 - \pi^2} \) in a Taylor series and using Wick’s theorem:

\[
\langle n_z \rangle = 1 - \sum_{k=1}^{\infty} \langle \pi^2 \rangle^k \frac{(2k - 2)!}{2^{2k-1}(k - 1)!} \approx 1 - \frac{1}{2} \langle \pi^2 \rangle - \frac{1}{4} \langle \pi^2 \rangle^2 - \frac{3}{8} \langle \pi^2 \rangle^3 + \ldots \]  

\[(A1)\]

The series \((A1)\) is asymptotic and the coefficients at large \( k \) diverge. Since the series is asymptotic we truncate it when the coefficients in front of \( \langle \pi^2 \rangle^k \) terms become larger then unity. Accounting for the leading terms in the expansion up to \( \langle \pi^2 \rangle^3 \) inclusive gives the critical value \( \langle \pi^2 \rangle_c \approx 0.93 \) for \( \langle n_z \rangle = 0 \).

Appendix B: Excitations in static spin-spiral phase

By considering fluctuations in the spin spiral state we find the condition when quantum fluctuations melt the spiral. Here we derive the dispersions of in plane and out of plane fluctuations in the spin-spiral state. To be specific let us assume that the spiral lies in \( \{xy\} \) plane:

\[
n = (\cos Qr, \sin Qr, 0) . \]  

\[(B1)\]

There are two different spin waves, the in-plane \( \varphi(r,t) \),

\[
n = (\cos(Qr + \phi), \sin(Qr + \phi), 0) , \]  

\[(B2)\]

and the out-of-plane \( h(r,t) \),

\[
n = (\sqrt{1 - h^2} \cos Qr, \sqrt{1 - h^2} \sin Qr, h) . \]  

\[(B3)\]

Substituting parametrization \((B2)\) and \((B3)\) in the Euler-Lagrangian equations of motion corresponding to the Lagrangian \((1)\) and linearising the equations with respect to \( \phi \) and \( h \) we obtain the dispersion of the in-plane and out of plane modes. The derivation is straightforward, see e.g. Ref.\([14]\). The dispersion of the in-plane mode is

\[
\omega_q^2 = \frac{1}{\chi} \left[ K(Q) - \frac{1}{2} (K(Q + q) + K(Q - q)) \right] \]  

\[= \frac{b_1}{2\chi} \left[ 2Q^2q^2 + q_x^4 + q_y^4 \right] , \]  

\[(B4)\]

and the dispersion of the out-of-plane mode is

\[
\Omega_q^2 = \frac{1}{\chi} \left[ K(q) - K(Q) \right] \]  

\[= \frac{b_1}{2\chi} \left[ Q^4/2 - Q^2q^2 + q_x^4 + q_y^4 \right] . \]  

\[(B5)\]
The total quantum fluctuation orthogonal to the spin alignment in the spiral phase reads

\begin{align}
\langle \pi^2 \rangle &= \langle \phi^2 \rangle + \langle h^2 \rangle, \\
\langle \phi^2 \rangle &= \int \frac{d^2q}{(2\pi)^2} \frac{1}{2\omega_q}, \\
\langle h^2 \rangle &= \int \frac{d^2q}{(2\pi)^2} \frac{1}{2\Omega_q}.
\end{align}

From the condition \( \langle \pi^2 \rangle = \langle \pi^2 \rangle_c \approx 1 \) we find the position of the spiral-SL critical point \( \rho_{cS} \), see Sec. IV in the main text.