Restrictions of rainbow supercharacters

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Abstract

The maximal subgroup of unipotent upper-triangular matrices of the finite general linear groups are a fundamental family of $p$-groups. Their representation theory is well-known to be wild, but there is a standard supercharacter theory, replacing irreducible representations by super-representations, that gives us some control over its representation theory. While this theory has a beautiful underlying combinatorics built on set partitions, the structure constants of restricted super-representations remain mysterious. This paper proposes a new approach to solving the restriction problem by constructing natural intermediate modules that help “factor” the computation of the structure constants. We illustrate the technique by solving the problem completely in the case of rainbow supercharacters (and some generalizations). Along the way we introduce a new $q$-analogue of the binomial coefficients that depend on an underlying poset.

1 Introduction

Let $N$ be a set with a total order so that we can construct the group $\text{GL}_N(\mathbb{F}_q)$ of invertible matrices with rows and columns indexed by $N$ and entries in the finite field $\mathbb{F}_q$ with $q$ elements. The supercharacter theory of the finite uniprotial groups

$$\text{UT}_N = \{u \in \text{GL}_N(\mathbb{F}_q) \mid (u - \text{Id}_N)_{ij} \neq 0 \text{ implies } i < j\},$$

where $\text{Id}_N$ is the multiplicative identity of $\text{GL}_N(\mathbb{F}_q)$, has developed into a rich combinatorics based on set partitions. In particular, [1] showed that — taken as a family — they give a representation theoretic realization of the Hopf algebra of symmetric functions in noncommuting variables (also studied in [18], for example), where the product comes from inflation and the coproduct from restriction. Thus, the representation theory of unipotent $p$-groups gives a noncommuting analogue to the classical combinatorial representation theory of the symmetric groups. The supercharacters of these groups give a new basis for this Hopf algebra.

One obstruction to making use of this connection is that while the inflation functor is straightforward for supercharacters, the restriction functor is still somewhat mysterious. That is, given a subset $K \subseteq N$, we want to decompose a supercharacter of $\text{UT}_N$ as a linear combination of supercharacters of the subgroup

$$\text{UT}_K = \{u \in \text{UT}_N \mid (u - \text{Id}_N)_{ij} \neq 0 \text{ implies } i, j \in K\} \subseteq \text{UT}_N.$$

The paper [19] gives an iterative algorithm for computing restrictions of supercharacters, but this gives us little information about the coefficients that occur (though it does imply that they will
be polynomial in the size of the underlying field $q$). As a preliminary step, [16] uses matchings in bipartite graphs to give a combinatorial characterization of when such a coefficient is nonzero; however, only a small set of examples have a direct computation of the coefficients.

The supercharacters of $\text{UT}_N$ are indexed by set partitions of the set $N$. In this subject, it seems preferable to view set partitions as a set of pairs, as follows. Given a set partition $\bl(\lambda)$ of $N$, we can store the block information as a set of pairs $\lambda = \left\{ i \sim j \mid i < j \text{ with } i, j \text{ in the same block of } \bl(\lambda), \quad i < j' < j \text{ implies } j' \text{ is in a different block} \right\}$.

Let $V^\lambda$ denote the $\text{UT}_N$-module whose trace is the supercharacter $\chi^\lambda$ indexed by $\lambda$ (these modules are explicitly described in, for example, [12]). These modules have a convenient factorization

$$V^\lambda = \bigotimes_{i \sim j \in \lambda} V^{i \sim j},$$

When restricting, these pairs then fall into three cases:

**Case 1.** $i \sim j \in \lambda$ satisfies $|\{i, j\} \cap K| = 0$,

**Case 2.** $i \sim j \in \lambda$ satisfies $|\{i, j\} \cap K| = 1$,

**Case 3.** $i \sim j \in \lambda$ satisfies $|\{i, j\} \cap K| = 2$.

It follows from [19] that Case 3 is easy to deal with (the pair restricts to the same pair with a $q$-power coefficient). Case 2 seems reasonably manageable (the one endpoint in $K$ acts as an anchor). Case 1 is the most unpredictable, and it is the goal of this paper to begin developing a theory to tackle this case.

In particular, a natural first case to consider is the case where

$$N = \{1, 2, \ldots, 2m + k\} = \{1, 2, \ldots, m\} \sqcup K \sqcup \{m + k + 1, \ldots, 2m + k\},$$

and $\lambda = \{(1, 2m + k), (2, 2m + k - 1), \ldots, (m, m + k + 1)\}$, known as a rainbow set partition (see the picture (1.1) with $\ell = 1$ below). In this case, it follows easily from [16] that

$$\text{Res}_{\text{UT}_K}^{\text{UT}_N} (V^\lambda) = \sum_{\text{set partition } \nu \text{ of } K, |\nu| \leq m} c_\nu V^\nu,$$

where all the $c_\nu$ are nonzero. However, trying to compute these coefficients with the techniques of [19] is prohibitively complicated. It is also worth mentioning that if one extends the action from $\text{UT}_N$ to the Borel subgroup $B_N$ of upper-triangular matrices, then the rainbow modules are irreducible $B_N$-modules.

This paper proposes an alternative approach by observing that as a class function of $\text{UT}_K$, the rainbow supercharacter $\chi^\lambda$ does not distinguish well between the superclasses of $\text{UT}_K$. We can therefore find a set of characters $\{\psi^0_K, \psi^1_K, \ldots, \psi^{\ell K}_K\}$, with natural and explicit modules $V^\psi_K$, that span a subspace of superclass functions in which the restrictions land. Conveniently, the rainbow modules decompose nicely in this space; in particular, for some explicit polynomials $\varphi^m_K(q)$ in $q$, we prove the following theorem.

**Theorem 3.10.** Let $N' = N \cup \{n_-, n_+\}$ with $\{n_-\} < N < \{n_+\}$. Then

$$\text{Res}_{\text{UT}_N}^{\text{UT}_{N'}} (V^{n_-} \otimes \cdots \otimes V^{n_-} \otimes V^{n_+} \otimes \cdots \otimes V^{n_+} \otimes V^{n_+}) \cong (q - 1)^m \bigoplus_{k=0}^m \varphi^m_K(q) V_N^k.$$
We call these intermediate modules $V_{j,K}^j$ core modules and they also decompose nicely in terms of the supercharacter modules; Proposition 3.6 gives the multiplicities of the modules $V_{j,K}^j$ in the decomposition of $V_{j,K}^j$ explicitly as polynomials in $q$. Combining these two decompositions in Corollary 3.11 gives coefficients in the case of the rainbow supercharacters.

The decompositions of the core modules into supercharacter modules leads into an apparently new variation of $q$-binomial coefficients that depend on a poset (the total order on a set gives the usual $q$-binomial). We define these combinatorial gadgets, and explore some of their properties.

Another bi-product of our arguments is a pleasing decomposition of the tensor product of two of the core modules, as given proved in the following theorem.

**Theorem 3.8.** For $0 \leq j \leq k \leq |N|$,\n
$$V_{j,N}^j \otimes V_{k,N}^k \cong \bigoplus_{m=0}^{j} q^{\binom{j-m}{2}} \left[ \binom{k+m}{k+m-j,m,j-m} \right] q^{k+m} V_{N}^{k+m}.\n$$

We expand the rainbow program further and consider multi-rainbows or set partitions of the form

\[ 1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 8 \quad 9 \quad 10 \quad 12 \quad 13 \quad 14 \quad 16 \quad \ldots \quad K_1 \quad \ldots \quad K_2 \quad \ldots \quad K_{\ell-1} \quad K_\ell, \quad (1.1) \]

where $K = K_1 \cup K_2 \cup \cdots \cup K_{\ell} \subseteq N$. We study the case $\ell = 3$ in the most depth (as the most natural next generalization). In this case, we mix metaphors and develop a notion of “peel” module. In the general $\ell$ case, the decomposition will be in terms of products of peel modules (nested in a way reminiscent of an onion), as is worked out in Corollary 4.5. Our main result here is to decompose the double rainbow supercharacter modules in terms of tensor products of peel modules and core modules in Theorem 4.3. Again, in this case, the decomposition from peel modules to supercharacter modules is relatively straightforward in Proposition 4.2.

The paper is organized as follows. In Section 2, we introduce our notation for set partitions, the relevant supercharacter theory of $\text{UT}_N$, and their corresponding modules. In Section 3 we study the case of rainbow characters; we introduce a notion of poset $q$-binomial and define the modules corresponding to the characters $\{\psi^j_K\}$. We decompose the rainbow modules in terms of these new modules, and then decompose further into supercharacters. Section 4 then explores the case where $\ell = 3$ in (1.1), and indicates how one might generalize to arbitrary $\ell$.

**Some remarks on notation.** This paper has made several somewhat nonstandard notational choices. The first choice is to study $\text{UT}_N$ for a set $N$ rather than $\text{UT}_{|N|}$ for a number $|N|$. There are two reasons for this choice. First, the representation theory of $\text{UT}_N$ seems most natural in the context of Hopf monoids in species, which associate sets $N$ to groups [2]. Second, for restriction problems it is critical to see the exact subset $K \subseteq N$. That is, up to isomorphism the subgroups only depend on the cardinality of the subset, but the restriction problem depends on the actual embedding.
The second choice is to express the main results as isomorphisms of modules, rather than
corresponding formulas. We do this to stress that all our characters have explicit associated modules,
and these modules actually help in finding the right decompositions.

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2 Preliminaries

This section introduces the notation we will use for set partitions, the notion of a supercharacter
theory, and gives both the supercharacters and their modules for the supercharacter theory of UT
used by this paper.

Fix a finite set \( N \) and a total order \( \leq \) on \( N \) (for example, \( N = \{1, 2, \ldots, n\} \) with \( 1 < 2 < \cdots < n \)). We say that a subset \( K \subseteq N \) is an interval if for all \( k, l \in K \) with \( k < l \), we have \( \{n \in N \mid k < n < l\} \subseteq K \).

2.1 Set partitions

A set partition \( \lambda \) of \( N \) is a set of pairs \((i, j) \in N \times N\) with \( i < j \) such that if \((i, k), (j, l) \in \lambda\), then \( i = j \) if and only if \( k = l \).

Let

\[
\mathcal{S}_N = \{\text{set partitions of } N\}.
\]

We typically view these set partitions diagrammatically as a family of arcs on a row of \(|N|\) nodes
so that if \((i, j) \in \lambda\), then there is an arc connecting the \(i\)th node to the \(j\)th node. For example,

\[
\lambda = \{(1, 5), (2, 4), (4, 6)\} \quad \longleftrightarrow \quad \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \quad \text{or} \quad \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Remark. There are at least two natural choices for the arcs, either above or below the nodes. We
will use both, below, letting the two choices distinguish between indexing sets for the superclasses
and supercharacters.

We typically refer to \((i, j)\) as an arc of \( \lambda \) and write \((i, j) = i \sim j\) or \((i, j) = i \rightleftharpoons j\), depending
on our desired orientation. For each \( i \sim j \in \lambda \), we call \( i \) the left endpoint of \( i \sim j \) and \( j \) the right endpoint of \( i \sim j \). In general,

\[
\begin{align*}
\lambda_\sim & = \{i \in N \mid i \sim j \in \lambda, \text{ for some } j \in N\} \\
\lambda_\rightleftharpoons & = \{j \in N \mid i \sim j \in \lambda, \text{ for some } i \in N\}
\end{align*}
\]

give the full sets of left and right endpoints of \( \lambda \).

We obtain the more traditional version of set partitions by defining the blocks \( \text{bl}(\lambda) \) of \( \lambda \in \mathcal{S}_N \)
to be the set of equivalence classes on \( N \) given by the transitive closure of \( i \sim j \) if \( i \sim j \in \lambda \). For example,

\[
\text{bl}\left( \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \right) = \{\{1, 5\}, \{2, 4, 6\}, \{3\}\}.
\]
There is an involution $\dagger : S_N \to S_N$ given by flipping the diagram across the middle, or if $w_0 \in S_N$ is the permutation of $N$ that reverses the order of the elements, then
\[ \dagger (\lambda) = \{ w_0(j) \sim w_0(i) \mid i \sim j \in \lambda \}, \] (2.1)
so for example,
\[ \dagger \left( \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \right) = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}. \]

There are a number of natural statistics on set partitions, and they turn out to have a nice algebraic structure [11]; the most important ones for us come from nestings and crossings. A crossing in a set-partition $\lambda$ is a pair of arcs $i \sim k, j \sim l \in \lambda$ such that $i < j < k < l$. A nesting in a set partition $\lambda$ is a pair of arcs $i \sim l, j \sim k \in \lambda$ such that $i < j < k < l$. For $\lambda, \mu \in S_N$ and $A \subseteq N$, let
\[
\text{nst}_\mu^\lambda = \# \{(i \sim l, j \sim k) \in \lambda \times \mu \mid i < j < k < l\}
\]
\[
\text{nst}_A^\lambda = \# \{(i \sim l, j) \in \lambda \times A \mid i < j < l\}
\]
\[
\text{crs}(\lambda) = \# \{(i \sim k, j \sim l) \in \lambda \times \lambda \mid i < j < k < l\}.
\]

Define the set of noncrossing set-partitions to be
\[ S_N^\equiv = \{ \lambda \in S_N \mid \text{crs}(\lambda) = 0 \}. \]

We obtain a projection,
\[ \cong : S_N \longrightarrow S_N^\equiv \]
\[ \lambda \mapsto \lambda^\equiv, \]
where $\lambda$ is the unique set partition in $S_N^\equiv$ that has the same left endpoints and the same right endpoints as $\lambda$ (in $\lambda$, the first right end-point from the left must be connected to the closest left endpoint, etc.). For example,
\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}^\equiv = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}, \]
where we have marked the left endpoints with $\leftarrow$ and the right endpoints with $\rightarrow$.

**Remark.** One can also obtain $\lambda$ from $\lambda$ by iteratively uncrossing each crossing $\{ i \sim k, j \sim l \}$ into a nesting $\{ i \sim l, j \sim k \}$. Since this map changes neither the set of left endpoints nor the set of right endpoints, the order in which we “resolve” the crossings does not matter.

For $A, C \subseteq N$, let
\[ \text{wt}_C^A(\lambda) = \# \{(a, c) \in A \times C \mid a < c\} = \text{wt}_A^C(\lambda). \]

Note that for $A \subseteq N$ and $\lambda \in S_N$,
\[ \text{nst}_A^\lambda = \text{wt}_A^\lambda(\lambda) - \text{wt}_A^\lambda(A). \] (2.2)

It follows that $\text{nst}_A^\lambda$ only depends on the left and right endpoints of $\lambda$, so
\[ \text{nst}_A^\lambda = \text{nst}_A^A. \]
2.2 Supercharacters of $\text{UT}_N$

Supercharacter theories were originally developed to study the representation theory of $\text{UT}_N$. The first such theory was developed by [3], and it was generalized to algebra groups in [12]. The theory we use below is slightly coarser (and more combinatorial), and it was first used in [10]. The study of supercharacters has seen a fair amount of interest from a variety of points of view in recent years, including the Hopf structure [2, 9, 17], good supercharacter theories for unipotent groups [5, 7], and generalizations to structures other than finite groups [6, 14].

A supercharacter theory $\langle K, X \rangle$ of a finite group $G$ is a set partition $K$ of $G$ with a set $X$ of characters of $G$ such that

(SC1) The number of blocks of $K$ is the same as the number of characters in $X$,

(SC2) Each block $K$ is a union of conjugacy classes of $G$,

(SC3) Each irreducible character $\psi \in \text{Irr}(G)$ is a nonzero constituent of exactly one element of $X$,

(SC4) $X \subseteq \{ \varphi : G \to \mathbb{C} \mid \varphi(g) = \varphi(h), g, h \in K, K \in K \}$.

We refer to the blocks of $K$ as superclasses and the elements of $X$ as supercharacters.

Let $\text{UT}_N$ be the subgroup of unipotent upper-triangular matrices of the general linear group $\text{GL}_N(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$ with $q$ elements, and let $\text{UT}_N \subseteq B_N \subseteq \text{GL}_N(\mathbb{F}_q)$ be the Borel subgroup of uppertriangular matrices. While there are many supercharacter theories of $\text{UT}_N$, we will focus on the supercharacter theory that is a slight coarsening of the canonical supercharacter theory for algebra groups given by [12]. Let

$$\text{ut}_N = \text{UT}_N - \text{Id}_N$$

be the nilpotent $\mathbb{F}_q$-algebra of strictly uppertriangular matrices. Then the superclasses are given by the two-sided orbits

$$K_{B_N} = \{ \text{Id}_N + B_N \cdot x/B_N \mid x \in \text{ut}_N \}.$$

The number of superclasses $|K_{B_N}|$ is the Bell number $|\mathcal{S}_N|$. In fact, for every superclass of $\text{UT}_N$ there exists $\mu \in \mathcal{S}_N$ and a distinguished element $u_{\mu}$ in the superclass such that

$$(u_{\mu})_{ij} = \begin{cases} 1, & \text{if } i \sim j \in \mu \text{ or } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

We will construct the supercharacters explicitly in the next section. However, the following character formula gives them explicitly and will be useful, below. It was first proved for a slightly finer supercharacter theory (indexed by labeled set partitions) for char$(\mathbb{F}_q)$ sufficiently large in [4] and then for general $q$ in [8].

**Proposition 2.1.** For $\lambda, \mu \in \mathcal{S}_N$,

$$\chi^{\lambda}(u_{\mu}) = \begin{cases} (q - 1)^{||\lambda - \mu||}(-1)^{||\lambda \cap \mu||}q^{\text{nst}_N^\lambda - \text{nst}_N^\mu} & \text{if } i < j < k, i \sim k \in \lambda \\
0 & \text{implies } i \sim j, j \sim k \notin \mu, \text{ otherwise.} \end{cases}$$

In particular, note that the trivial character $1$ is the supercharacter $\chi^\emptyset$ associated with the empty set partition of $N$, and

$$\chi^{\lambda}(1) = (q - 1)^{||\lambda||}q^\text{nst}_N^\lambda.$$
The space $f(G) = \{ \varphi : G \to \mathbb{C} \}$ has a point-wise product
\[
\odot : f(G) \otimes f(G) \to f(G) \\
(\varphi, \theta) \mapsto \varphi \odot \theta : G \to \mathbb{C} \\
g \mapsto \varphi(g) \theta(g).
\]

It also follows from the character formula that
\[
\chi^\lambda = \bigcirc_{i \sim j \in \Lambda} \chi^{i \sim j}.
\]

Let
\[
\mathcal{M}_N = \{ \text{multi-sets of arcs with endpoints in } N \}.
\]

For each $\lambda \in \mathcal{M}_N$, define the character
\[
\chi^\lambda = \bigcirc_{i \sim j \in \Lambda} \chi^{i \sim j}.
\]

If $K \subseteq N$, then an explicit restriction formula for restricting supercharacters from $\text{UT}_N$ to $\text{UT}_K \cong \{ u \in \text{UT}_N \mid (u - \text{Id}_N)_{i,j} \neq 0 \text{ implies } i, j \in K \} \subseteq \text{UT}_N$
seems to be difficult. The paper [19] gives iterative algorithms for both computing restrictions and pointwise products of supercharacters; heuristically, arcs with illegal endpoints shrink in all possible ways. The following lemma gives the first steps of this algorithm.

**Lemma 2.2.** Let $N$ be a set

(a) If $N = K \cup \{ n \}$ with $K < \{ n \}$, then for $k \in K$,
\[
\text{Res}^{\text{UT}_N}_{\text{UT}_K}(\chi^{k \sim n}) = (q - 1) \left( \chi^{\emptyset} + \sum_{k \in K \atop k < l} \chi^{k \sim l} \right).
\]

(b) If $i < j < l$, then
\[
\chi^{i \sim l} \odot \chi^{j \sim l} = (q - 1) \left( \chi^{i \sim l} + \sum_{j < k < l} \chi^{i \sim l, j \sim k} \right).
\]

**Remark.** In Section 2.3 below we discuss a natural involution $\dagger$ on the supercharacters of $\text{UT}_N$, corresponding to the combinatorial $\dagger$ on set partitions. Using this involution one can also obtain a “flipped” version of the above lemma.

### 2.3 The supercharacter $\text{UT}_N$-modules

Fix a nontrivial homomorphism
\[
\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times.
\]

The $\mathbb{F}_q$-vector space of $|N| \times |N|$ matrices $\mathfrak{gl}_N$ with entries in $\mathbb{F}_q$ decomposes in terms of uppertriangular matrices $\mathfrak{b}_N$ and strictly lower triangular matrices $\mathfrak{lt}_N$, so
\[
\mathfrak{gl}_N = \mathfrak{lt}_N \oplus \mathfrak{b}_N.
\]

Define the $\mathbb{C}$-vector space
\[
V_N = \mathbb{C}\text{-span}\{\mathfrak{lt}_N\}.
\]
We can define several left $\mathfrak{U}T_N$-module structures on $V_N$ using the fact that $\mathfrak{lt}_N$ is a canonical set of coset representatives in $\mathfrak{gl}_N/\mathfrak{b}_N$. For $v \in \mathfrak{gl}_N$, define

$$\bar{v} \in (v + \mathfrak{b}_N) \cap \mathfrak{lt}_N.$$ 

We use the left multiplication action on $\mathfrak{gl}_N/\mathfrak{b}_N$ to confer a $\mathfrak{U}T_N$-module $\bar{V}_N$ structure on the space $V_N$ by defining

$$u \triangleright v = \vartheta\left(\text{tr}(u (v I - 1))\right)(\bar{v} v) \text{ for } u \in \mathfrak{U}T_N, \quad v \in \mathfrak{lt}_N. \quad (2.4)$$

Similarly, the right multiplication action on $\mathfrak{gl}_N/\mathfrak{b}_N$ gives a $\mathfrak{U}T_N$-module $\bar{V}_N$ structure on the space $V_N$ by

$$u \triangleright v = \vartheta\left(\text{tr}(v (u I - 1))\right)(v u) \text{ for } u \in \mathfrak{U}T_N, \quad v \in \mathfrak{lt}_N. \quad (2.5)$$

**Remark.** In constructing the supercharacters of $\mathfrak{U}T_N$ it is more common to construct a module structure on $\mathfrak{ut}_N$, where $\mathfrak{ut}_N \subseteq \mathfrak{gl}_N$ is as in (2.3) [12]. The two actions above correspond to the two canonical actions on $\mathfrak{ut}_N$. This paper translates this picture to matrices to make studying submodules more straight-forward.

Let $\dagger : \mathfrak{gl}_N \to \mathfrak{gl}_N$ be the anti-involution obtained by transposing across the anti-diagonal, or for $g \in \mathfrak{gl}_N$,

$$g^\dagger = w_0 \text{Transpose}(g) w_0, \quad \text{where} \quad w_0 = \begin{bmatrix} 0 & 1 & \cdots \ 1 & 0 \end{bmatrix}.$$ 

Note that $\dagger$ restricts to an anti-involution $\dagger : \mathfrak{lt}_N \to \mathfrak{lt}_N$ which we may extend linearly to $V_N$. While $\bar{V}_N \cong \overline{V}_N$ are always isomorphic to the regular module of $\mathfrak{U}T_N$, subspaces are not necessarily invariant under both actions. However, we get a bijection

$$\dagger : \{ \text{Submodules of } \bar{V}_N \} \mapsto \{ \text{Submodules of } \overline{V}_N \} \quad (2.6)$$

For $\lambda \in S_N$, we define a canonical basis element $v_\lambda \in \mathfrak{lt}_N$ given by

$$(v_\lambda)_{kj} = \begin{cases} 1 & \text{if } j \sim k \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathfrak{lt}_N = \bigcup_{\lambda \in S_N} \{ \bar{a} v_\lambda b \mid a, b \in B_N \}.$$ 

In this notation, for a fixed $b_0 \in B_N$, the character of the submodule

$$V^\lambda \cong \mathbb{C}\text{-span}\{ \bar{a} v_\lambda b_0 \mid a \in B_N \}$$

is the supercharacter

$$\chi^\lambda = \frac{|B_N v_\lambda|}{|B_N v_\lambda B_N|} \sum_{v \in B_N v_\lambda B_N} \vartheta(\text{tr}(\cdot, v))$$

of $\mathfrak{U}T_N$ (though the formula in Proposition 2.1 is more useful for our purposes).

The notation of $\dagger : S_N \to S_N$ in (2.1) now matches up with the notation in (2.6).
Proposition 2.3. We have
\[ M \cong \bigoplus_{\lambda \in \mathcal{S}_N} m_\lambda V^\lambda \subseteq \overline{V}_N \quad \text{if and only if} \quad \uparrow(M) \cong \bigoplus_{\lambda \in \mathcal{S}_N} m_\lambda V^{\uparrow(\lambda)} \subseteq \overline{V}_N. \]

Proof. Since \( \uparrow(v_\lambda) = v_{\uparrow(\lambda)} \), the definition of \( V^\lambda \) gives the result. \( \square \)

Remark. Since most of our work will be studying submodules of \( \overline{V}_N \), we will typically omit the arrow from the notation. However, in Section 4 we will require both kinds of submodules, so in that case we will use the arrows to help differentiate the actions.

3 Rainbow supercharacter restrictions

This section studies the special case where we have \( N \subseteq N' \), where \( N' = N_+ \sqcup N \sqcup N_+ \) with \( N_- < N < N_+ \) and \( |N_-| = |N_+| = \ell \). Let \( w : N_- \to N_+ \) be the unique bijection such that \( i < j \) if and only if \( w(j) < w(i) \) for all \( i, j \in N_- \). Then the set partition
\[
\{ i \sim w(i) \mid i \in N_- \} = \{ \cdot \ldots \cdot \} \quad \text{and} \quad \{ \cdot \ldots \cdot \} \quad \text{for all} \quad N_- \quad \text{and} \quad N_+ \in N_-.
\]

looks a little like an upside-down rainbow, so we’ve informally dubbed the corresponding supercharacter a rainbow supercharacter. For our purposes, however, it is essentially equivalent to contract \( N_- \) and \( N_+ \) into single points \( \{ n_- \} \) and \( \{ n_+ \} \) and have a multi-set of arcs passing between these two points. In fact, by comparing their values on \( UT_N \) using Proposition 2.1, we see that
\[
\text{Res}_{UT_N}^{UT_{N'}} (\chi_{\{ i \sim w(i) \mid i \in N_- \}}) = q^{2(k)} \text{Res}_{UT_{N \cup \{ n_-, n_+ \}}}^{UT_N} (\chi_{n_- \sim n_+} \cdots \chi_{n_- \sim n_+}).
\]

We will write
\[
n_- \sim \ell \sim n_+ = \{ n_- \sim n_+, \ldots, n_- \sim n_+ \} \in \mathcal{M}_{N \cup \{ n_-, n_+ \}} \quad \text{for all} \quad \ell \text{ terms}.
\]

The fundamental idea of this paper is to construct intermediate modules to make the restriction of supercharacters more manageable. This section is meant to serve as a model for this approach. We begin in Section 3.1 by developing a poset analogue of \( q \)-binomial coefficients that will be helpful in understanding the modules we construct in Sections 3.2 and 3.3. Section 3.4 then uses the intermediate modules to decompose the rainbow supercharacters.

3.1 Poset \( q \)-binomial coefficients

For \( n, k \in \mathbb{Z}_{\geq 0} \), let
\[
[n] = \frac{q^n - 1}{q - 1} \quad \text{and} \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]!}{[k]! [n-k]!} = \# \left\{ \text{\( k \)-dimensional subspaces of} \ \mathbb{F}_q^n \right\}
\]

be the usual \( q \)-integer and \( q \)-binomial.
Let \( P \) be a (finite) poset. Given a subset \( A \subseteq P \), let
\[
\mathrm{wt}^P(A) = \# \{(a, b) \in P \times P \mid a <_P b\}.
\]

For \( n, k \in \mathbb{Z}_{\geq 0} \), the \( P \)-binomial coefficient is
\[
\left[ \begin{array}{c} P \\ k \end{array} \right]_q = \sum_{A \subseteq P \atop |A| = k} q^{\mathrm{wt}^P(A)}.
\]

Note that for \( a \in A \), \( \mathrm{wt}_P^P(a) \) can also be expressed in terms of the size of the upper ideal containing \( a \), and if \( T \) is the usual order on \( N \), then \( \mathrm{wt}_T^T(a) = \mathrm{wt}_P^P(a) \).

Examples.

- If \( P \) is the poset with no relations on \( n \) elements, then
\[
\left[ \begin{array}{c} P \\ k \end{array} \right]_q = \sum_{A \subseteq P \atop |A| = k} q^0 = \binom{|P|}{k}.
\]

- If \( P \) is a total order (say \( 1 < 2 < \cdots < n \)), then
\[
\left[ \begin{array}{c} P \\ k \end{array} \right]_q = q^{1+2+\cdots+(k-1)} \sum_{1 \leq a_1 < a_2 < \cdots < a_k \leq n} q^{n-a_1-(k-1)+n-a_2-(k-2)+\cdots+n-a_k-0} = q(k) \left[ \begin{array}{c} |P| \\ k \end{array} \right]_q.
\]

Note that this is also \( e_k(1, q, \ldots, q^{n-1}) \), where \( e_k(X_1, \ldots, X_n) \) is the \( k \)th elementary symmetric polynomial [15, Exercise I.2.3].

Remark. In general \( \left[ \begin{array}{c} P \\ k \end{array} \right]_q \neq \left[ \begin{array}{c} n-k \\ k \end{array} \right]_q \), though the coefficient sequences of the polynomial of one is the reverse coefficient sequence of the other. That is, if
\[
b_P(n, k; r, s) = \sum_{A,B=\{1,2,\ldots,n\} \atop |A|=k, |B|=n-k} r^{\mathrm{wt}^P(A)} s^{\mathrm{wt}^P(B)},
\]
then
\[
\left[ \begin{array}{c} P \\ k \end{array} \right]_q = b_P(n, k, q, 1) \quad \text{and} \quad \left[ \begin{array}{c} P \\ n-k \end{array} \right]_q = b_P(n, k, 1, q).
\]

The usual method of defining a \( q \)-multinomial coefficient does not seem to have a useful analogue in our case. However, there is a different way to pick multiple disjoint subsets of a poset that is relevant. Fix a set partition \( P_1 \sqcup P_2 \sqcup \cdots \sqcup P_\ell = \mathcal{P} \), and define
\[
\left[ \begin{array}{c} \mathcal{P} \\ k \end{array} \right]_q = \prod_{j=1}^{\ell} \left( \sum_{A_j \subseteq P_j \atop |A_j|=k_j} q^{\mathrm{wt}^P(A_j)} \right).
\]

Now, if \( A \sqcup B = \mathcal{P} \), then we obtain a kind of symmetry
\[
\left[ \begin{array}{c} \mathcal{P} \\ k \end{array} \right]_q = \left[ \begin{array}{c} \mathcal{P} \\ |\mathcal{P}|-k \end{array} \right]_q = \left[ \begin{array}{c} \mathcal{P} \\ |\mathcal{P}| - k \end{array} \right]_q = \left[ \begin{array}{c} \mathcal{P} \\ B \end{array} \right]_q.
\]
Remark. The notation \( k \subseteq P \) is somewhat odd, since \( k \notin P \), nor is it in fact a subset. However, this notation is meant to indicate that we are picking a subset with \( k \) items from \( P \).

These coefficients satisfy a family of recursive relations.

**Proposition 3.1.** Let \( a \in P \), \( P' = P - \{a\} \) and let \( P_a \) be the restriction of \( P \) to the set \( \{a' \mid a' < a\} \). Then

\[
\begin{bmatrix} P \end{bmatrix}_q = \sum_{j=0}^{k} q^j \left( \begin{bmatrix} \mathcal{P'} \end{bmatrix}_{q, j} + \begin{bmatrix} P' \end{bmatrix}_{q, k-j} \right). 
\]

**Proof.** We sort subsets of \( P \) into \( k \)-subsets that contain \( a \) and \( k \)-subsets that do not contain \( a \). If a \( k \)-subset \( A \subseteq P \) then \( A - \{a\} \subseteq P' \) is a subset of size \( k - 1 \). In this case,

\[
wt^P(A) = |A \cap P_a| + wt^P(A \cap P_a) + wt^P(a) + wt^P(A - P_a).
\]

If a \( k \)-subset \( A \subseteq P \), then \( A \subseteq P' \), and

\[
wt^P(A) = |A \cap P_a| + wt^P(A \cap P_a) + wt^P(A - P_a).
\]

The result now follows from summing over possible cardinalities of \( A \cap P_a \).

Perhaps the most useful recursion corresponds to removing some minimal element in the poset.

**Corollary 3.2.** Let \( a \) be a minimal element of a poset \( P \). Let \( P' = P - \{a\} \). Then

\[
\begin{bmatrix} n \end{bmatrix}_P = q^{wt(a)} \begin{bmatrix} \mathcal{P'} \end{bmatrix}_{q, k-1} + \begin{bmatrix} P' \end{bmatrix}_q.
\]

**Example.** For the posets with nearly no relations we obtain nearly familiar recursions,

\[
\begin{bmatrix} \begin{array}{c} n \\ 1 \end{array} \end{bmatrix}_{q, k} = \sum_{|A|=k, n \in A} q^{k-1} + \sum_{|A|=k, n \notin A} q^k =  q^{k-1} \left( \binom{n-1}{k-1} + \binom{n-1}{k} \right),
\]

and

\[
\begin{bmatrix} \begin{array}{c} n \\ 1 \end{array} \end{bmatrix}_{q, k} = \sum_{|A|=k, 1 \in A} q^{n-1} + \sum_{|A|=k, 1 \notin A} q^0 = q^{n-1} \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

### 3.1.1 Main example

Let \( \lambda \in \mathcal{P}_N \). Since there are no crossings in \( \lambda \), we can define a poset depending on \( \lambda \) based on whether blocks are nested or not. Let \( P(\lambda) \) be the poset on \( \text{bl}(\lambda) \) given by \( a < b \) if either

- \( |a| > 1 \) and there exist \( j, k \in a \) and \( i, l \in b \) such that \( i < j < k < l \), or
- \( a = \{j\} \) and there exist \( i, k \in b \) such that \( i < j < k \).
For example,

\[ \lambda = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \]

and \( \mathcal{P}(\lambda) = \begin{array}{ccc} 2 & 4 & 5 \\ 3 \end{array} \).

Note that for each \( \lambda \in \mathcal{S}_N^\infty \) the poset \( \mathcal{P}(\lambda) \) is a forest where each connected component has a unique maximal element. In fact, all such forests arise in this way (as \( N \) and \( \lambda \) vary).

Thus, we obtain a function

\[ \mathcal{S}_N^\infty \xrightarrow{=} \mathcal{S}_N^\infty \xrightarrow{\mathcal{P}} \{ \text{rooted forests} \} . \]

### 3.2 Column set submodules

For \( K \subseteq N \), the submodule

\[ V^K_N = \mathbb{C}\text{-span}\{ t^K_N \} \subseteq V_N, \]

where \( t^K_N = \{ v \in t_N \mid v_{ji} \neq 0 \text{ implies } i \in K \} \),

decomposes completely into super-modules.

**Proposition 3.3.** For \( K \subseteq N \),

\[ V^K_N \cong \bigoplus_{\lambda \in \mathcal{S}_N^\infty \atop \lambda \subseteq K} q^{\text{nst}_K^\lambda + \text{nst}_K^- \lambda} V^\lambda. \]

**Proof.** The proof takes every basis element in \( V^K_N \) and finds it as a basis element for some copy of \( V^\lambda \). Note that

\[ V^K_N = \mathbb{C}\text{-span}\{ av_{\lambda} b \mid a \in B_N, b \in B_K, \lambda \subseteq K \} . \]

For a fixed \( b_0 \in B_K \) and \( \lambda \),

\[ \mathbb{C}\text{-span}\{ av_{\lambda} b_0 \mid a \in B_N \} \cong V^\lambda \]

so for each \( \lambda \) it suffices to determine the size of

\[ \frac{|B_N v_{\lambda} B_K|}{|B_N v_{\lambda}|} = \frac{|B_N v_{\lambda}||B_N v_{\lambda} \cap v_{\lambda} B_K|}{|B_N v_{\lambda} \cap v_{\lambda} B_K|} = \frac{|v_{\lambda} B_K|}{|B_N v_{\lambda} \cap v_{\lambda} B_K|}. \]

The elements of \( K \) fall into two categories:

**Case 1.** \( k \in K \setminus \lambda \).

**Case 2.** \( k \in K \cap \lambda \). If \( k \sim j \in \lambda \) and \( l \sim i \in \lambda \) with \( l > k > i \), then either \( k \sim j \) crosses or is nested in \( l \sim i \).

Therefore, we conclude that

\[ \frac{|v_{\lambda} B_K|}{|B_N v_{\lambda} \cap v_{\lambda} B_K|} = q^{\text{nst}_K^\lambda} q^{\#\{ i < j < k < l \mid i \sim k, j \sim l \in \lambda, j \in K \}} = q^{\text{nst}_K^\lambda + \text{nst}_K^- \lambda}, \]

as desired.
By the previous proposition the trace of $V_N^K$ will be a superclass function
\[ \psi_N^K = \text{tr}(\cdot, V_N^K). \]
of $\text{UT}_N$. We can easily compute its trace directly on superclass representatives.

**Proposition 3.4.** For $K \subseteq N$ and $\mu \in \mathcal{I}_N$,
\[
\psi_N^K(u_\mu) = \begin{cases} 
\text{wt}_{N-\mu}(K) & \text{if } |\mu \cap K| = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Let $v \in V_N^K$. Then
\[
u_\mu \odot v = \partial(\text{tr}((u_\mu - 1)v)) \left[ \sum_{l=1}^{|N|} (u_\mu)_j v_{li} \right]_{|N| \geq j > i \geq 1}
\]
Thus, $u_\mu \odot v = \partial(\text{tr}((u_\mu - 1)v))v$ if and only if for all $j \prec l \in \mu$, we have $v_{li} = 0$ for all $i < j$. The trace then is
\[
\psi_N^K(u_\mu) = \sum_{\text{vec}(N)_{j \prec \mu \implies v_{li} = 0 \land i < j}} \partial(\sum_{j \prec \mu \implies v_{ij} = 0 \land i < j}) \sum_{j \prec \mu \implies v_{ij} = 0 \land i < j} \partial(v_{ij})
\]
\[
= 0,
\]
unless $|\mu \cap K| = 0$. If $|\mu \cap K| = 0$, then
\[
\psi_N^K(u_\mu) = \sum_{\text{vec}(N)_{j \prec \mu \implies v_{li} = 0 \land i < j}} 1 = |\text{UT}_N^K| q^{-\text{wt}_{N-\mu}(K)} = q^{\text{wt}_{N}(K) - \text{wt}_{N-\mu}(K)} = q^{\text{wt}_{N-\mu}(K)},
\]
as desired. \qed

**Example.** Proposition 3.4 implies that
- $V_N^\emptyset$ is the trivial module of $\text{UT}_N$,
- $V_N^N$ is isomorphic to the regular module of $\text{UT}_N$.

We can also use Proposition 3.4 to connect these modules with the restriction problem, as follows.

**Corollary 3.5.** Let $N' = N \cup \{n_+\}$ with $N < \{n_+\}$. For $K \subseteq N$,
\[
\text{Res}_{\text{UT}_N}^{\text{UT}_{N'}} \left( \bigotimes_{k \in K} V_k \right) \cong (q - 1)^{|K|} V_N^K.
\]

**Proof.** Take traces using Proposition 2.1 on the left and Proposition 3.4 on the right. \qed

**Remark.** Corollary 3.5 implies that the left module $V_N^K$ is isomorphic to the right module
\[
\mathbb{C}\text{-span}\{\text{UT}_N^K\}
\]
where $\text{UT}_N^K = \{u \in \text{UT}_N \mid (u - \text{Id}_N)_{ij} \neq 0 \text{ implies } i \in K\}$, under right multiplication.
3.3 Core modules

We can clump the $V_N^K$ based on the size of $|K|$ to obtain the following new modules. For $0 \leq k \leq |N|$, let

$$V_N^k = \bigoplus_{K \subseteq N, |K|=k} V_N^K.$$  \hfill (3.4)

Note that $V_N^0 = V_N^0$ and $V_N^{|N|} = V_N^N$, so these modules also interpolate between the trivial module and the regular module.

Remark. In the context of Section 4, it will make sense to call these modules core modules.

Proposition 3.6. For $0 \leq k \leq |N|$,

$$V_N^k \cong \bigoplus_{\lambda \in \mathcal{S}_N, |\lambda| \leq k} q^{\text{nst}_\lambda} \left[ \frac{\mathcal{P}(\lambda)}{k - |\lambda|} \right]_q V^{\lambda}.$$  

Proof. By Proposition 3.3, the multiplicity of $V^{\lambda}$ is

$$\sum_{\lambda \subseteq K \subseteq N, \frac{|K|}{|K|-|\lambda|} \leq k} q^{\text{nst}_\lambda} q^{\text{nst}_K} = \sum_{K \subseteq N, \frac{|K|}{|K|-|\lambda|} \leq k} q^{\text{nst}_K}.$$  

Since each block in $\mathcal{P}(\lambda)$ has a unique point that is not in $\lambda$, each element in $j \in N - \lambda$ uniquely determines a block $\text{bl}_j \in \mathcal{P}(\lambda)$. Furthermore, $\lambda = \lambda$ and $\lambda = \lambda$, so for each $j \in N - \lambda$,

$$\text{nst}_j = \# \{ l \in \lambda \mid j < l \} - \# \{ k \in \lambda \mid j < k \} = \# \{ l \in \lambda \mid j < l \} - \# \{ k \in \lambda \mid j < k \} = \text{wt} \mathcal{P}(\lambda) \times (\text{bl}_j).$$

Thus,

$$q^{\text{nst}_\lambda} \sum_{K \subseteq N, \frac{|K|}{|K|-|\lambda|} \leq k} q^{\text{nst}_K} = q^{\text{nst}_\lambda} \left[ \frac{\mathcal{P}(\lambda)}{k - |\lambda|} \right]_q,$$

as desired.  \hfill \square

For $0 \leq k \leq |N|$, let

$$\psi_N^k = \text{tr}(\cdot, V_N^k).$$

Corollary 3.7. For $0 \leq k \leq |N|$ and $\mu \in \mathcal{S}_N$,

$$\psi_N^k(\mu) = q^{\binom{k}{2}} \left[ \frac{|N| - |\mu|}{k} \right]_q.$$  

Proof. If $\mathcal{T}_N$ is the fixed total order on the set $N$, then by Proposition 3.4,

$$\psi_N^k(\mu) = \left[ \mathcal{T}_N^{-\mu} \right]_q = q^{\binom{k}{2}} \left[ \frac{|N| - |\mu|}{k} \right]_q.$$  

Observe that $|\mu| = |\mu|$.  \hfill \square
By Corollary 3.7, $\psi^k_\mu(u_\mu)$ only depends on the number $|\mu|$. It is tempting to believe that the corresponding partition gives a supercharacter theory of UT$_N$, but it turns out this partition has no corresponding $\mathcal{X}$ that satisfies (SC4).

However, the decomposition of pointwise products is of some interest, and we will use the base case of the following theorem in the next section.

**Theorem 3.8.** For $0 \leq j \leq k \leq |N|$, 

$$V_N^j \otimes V_N^k \cong \bigoplus_{m=0}^{j} q^{(j-m)/2} \left[ k + m \right] \left[ k + m - j, m, j - m \right] q \left. V_N^{k+m} \right.$$ 

**Remark.** The $j = 1$ case can be derived using Corollary 3.5 and Lemma 2.2 above. However, it is less involved to just check that the answer is correct (as we do in the proof below).

**Proof.** We induct on $j$. If $j = 0$, then since we are tensoring with the trivial module the result is clear. If $j = 1$, then by taking traces we use Proposition 3.7 to compare for $\mu \in \mathcal{X}_N$,

$$[k]\psi_N^k(u_\mu) + [k+1]\psi_N^{k+1}(u_\mu) = q^{k\left[ j \right]} \frac{[k][|N| - |\mu|]!}{[k]![|N| - |\mu| - k]!} + q^{k+1\left[ j \right]} \frac{[k+1][|N| - |\mu|]!}{[k+1]![|N| - |\mu| - k - 1]!}$$

$$= q^{k\left[ j \right]} \frac{[|N| - |\mu|]!}{[k]![|N| - |\mu| - k - 1]!} \left( \frac{[k]}{[k+1]![|N| - |\mu| - k - 1]} + q^{k+1\left[ j \right]} \right)$$

$$= \psi_N^{1\left[ j \right]} \frac{[|N| - |\mu|]!}{[|N| - |\mu| - k]!} \psi_N^k(u_\mu)$$

If $j > 1$, then rewrite the base case,

$$\psi_N^j \otimes \psi_N^k = \frac{1}{[j]} \left( \psi_N^j \otimes \psi_N^{j-1} \otimes \psi_N^k - [j] \psi_N^{j-1} \otimes \psi_N^k \right),$$

so that we can use induction to get

$$\psi_N^j \otimes \psi_N^k = \frac{1}{[j]} \left( \sum_{m=0}^{j-1} q^{\left( j-m \right)/2} \left[ k + m \right] \left[ k + m - j + 1, m, j - 1 - m \right] q \left( [k + m] \psi_N^{k+m} + [k + m + 1] \psi_N^{k+m+1} \right) \right.$$ 

$$- \sum_{m=0}^{j-1} q^{\left( j-m \right)/2} \left[ k + m \right] \left[ k + m - j + 1, m, j - 1 - m \right] q \left( [j-1] \psi_N^{k+m} \right)$$

$$= \frac{1}{[j]} \left( q^{\left( j \right)/2} \left[ k - j + 1, 0, j - 1 \right] q \left( [k] - [j-1] \right) \psi_N^k \right.$$ 

$$+ \sum_{m=1}^{j-1} q^{\left( j-m \right)/2} \left[ k + m - 1 \right] \left[ k + m - j, m - 1, j - m \right] q \left( [m + k] \psi_N^{m+k} \right)$$

$$+ q^{\left( j-1 \right)/2} \left[ k + j - 1 \right] \left[ k + j \right] q \left( [k + j] \psi_N^{k+j} \right).$$

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By inspection, the coefficient of $\psi^k_N$ is
\[
q^{(j-1)} \left\{ \left[ \frac{k}{j} \right] \left[ \frac{k}{[j]} \right] \frac{k}{[j]+1, 0, j-1} \right\} q = q^{j-1} \left\{ \left[ \frac{k}{j} \right] \left[ \frac{k}{[j]} \right] \frac{k}{[j]+1, 0, j-1} \right\} q = q^{(j)} \left[ \frac{k}{[j]+1} \right] q.
\]

The coefficient of $\psi^{j+k}_N$ simplifies to
\[
\left[ \frac{k+j}{[j]} \right] q = q^{(0)} \left[ \frac{k+j}{[j]} \right] q.
\]

For $1 \leq m \leq j-1$, the coefficient of $\psi^{m+j}_N$ is
\[
q^{(j-m-1)} \left\{ \left[ \frac{k+m}{[j]} \right] \frac{k+m}{[j]-1} \right\} (q^{j-m-1} + \frac{m+k}{[j]-m} + q^{j-1}) = q^{(j-m-1)} \left\{ \left[ \frac{k+m}{[j]} \right] \frac{k+m}{[j]-1} \right\} (q^{j-m-1} + \frac{m+k}{[j]-m} + q^{j-1}) = q^{(j-m)} \left[ \frac{k+m}{[j]-m} \right] q.
\]

as desired.

We get a family of $q$-binomial identities by evaluating the traces at elements $u_\mu \in UT_N$.

**Corollary 3.9.** For $0 \leq j \leq k \leq n$ and $0 \leq l \leq n-1$,
\[
q^{(l)+\binom{k}{2}} \left\{ \left[ \frac{n-l}{[j]} \right] \frac{n-l}{k} \right\} q = \sum_{m=0}^{j} q^{(k-m)+\binom{k+m}{2}} \left[ \left[ \frac{n-l}{[k+m-j, m, j-m]} \right] q \left[ \frac{n-l}{k+m} \right] q.
\]

### 3.4 The connection to restriction problems

Following [15] (up to a sign), we will let
\[
\varphi^\mu_n(q) = \prod_{j=1}^{n} (q^j - 1) = (q - 1)^n [n]!.
\]

For $0 \leq k \leq n$, let
\[
\varphi^\mu_n(q) = \frac{\varphi^\mu_n(q)}{\varphi^\mu_{n-k}(q)} = \prod_{j=0}^{k-1} (q^{n-j} - 1) = (q - 1)^k [k]! \left[ \frac{n}{k} \right] q.
\]

Armed with core modules, we can now decompose the rainbow supercharacters with relative ease. As in (3.1), let
\[
V^{n-m} \otimes V^{n-m} \otimes \cdots \otimes V^{n-m} \cong \bigotimes_{m \text{ terms}} V^{n-m}.
\]
Theorem 3.10. Let \( N' = N \cup \{n_-, n_+\} \) with \( \{n_-\} < N < \{n_+\} \). Then
\[
\text{Res}_{UT_N}^{\uparrow N} (V^\leftarrow m_{--}^{m_{--}}) \cong (q - 1)^m \bigoplus_{k=0}^{m} \varphi_k^m(q)V_N^k.
\]

**Remark.** Note that \( V_N^k = \emptyset \) when \( k > |N| \).

**Proof.** We take traces for the proof, and induct on \( m \). If \( m = 0 \), then
\[
\text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}}) = \text{Res}_{UT_N}^{\uparrow N} (\chi) = \chi = \varphi_0^0(q)\psi_0^1.
\]
For \( m = 1 \), by Lemma 2.2 (a) and Corollary 3.5,
\[
\text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}}) = (q - 1)\left( \chi + \sum_{j \in N} \text{Res}_{UT_N}^{\uparrow N \cup \{j\}} (\chi^{j_{--}^{j_{--}}}) \right)
\]
\[
= (q - 1)\left( \psi_0^0_N + \varphi_1^1(q)\psi_1^1_N \right).
\]
If \( m > 1 \), then
\[
\text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}}) = \text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}}) \otimes \text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}})
\]
\[
= (q - 1)(\psi_0^0_N + (q - 1)\psi_1^1_N) \otimes (q - 1)^{m-1} \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k,
\]
by induction. By Theorem 3.8,
\[
\text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}}) = (q - 1)^m \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k \otimes (q - 1)^{m-1} \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k
\]
\[
= (q - 1)^m \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k \otimes (q - 1)^{m-1} \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k
\]
\[
= (q - 1)^m \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k \otimes (q - 1)^{m-1} \sum_{k=0}^{m-1} \varphi_k^{m-1}(q)\psi_N^k
\]
Collect coefficients to get
\[
\text{Res}_{UT_N}^{\uparrow N} (\chi^\leftarrow m_{--}^{m_{--}}) = (q - 1)^m \sum_{k=0}^{m} \left( (q - 1)\varphi_k^{m-1}(q)\psi_N^k \right)
\]
\[
= (q - 1)^m \sum_{k=0}^{m} \varphi_k^{m-1}(q)\left( (q^k - 1 + q^k(q^{m-k} - 1))\psi_N^k \right)
\]
\[
= (q - 1)^m \sum_{k=0}^{m} \varphi_k^{m}(q)\psi_N^k,
\]
as desired. \( \square \)

Combine Theorem 3.10 with Proposition 3.6 to get the following corollary.

**Corollary 3.11.** Let \( N' = N \cup \{n_-, n_+\} \) with \( \{n_-\} < N < \{n_+\} \). Then
\[
\text{Res}_{UT_N}^{\uparrow N} (V^\leftarrow m_{--}^{m_{--}}) \cong (q - 1)^m \bigoplus_{\lambda \in \mathcal{F}_N} \left( \sum_{k=0}^{m} \varphi_k^m(q)\left( \mathcal{P}_k^m(q) \bigotimes_{k=0}^{m} \varphi_k^m(q) \right) \right) V^\lambda.
\]
4 An onion peel approach to restriction

This section studies the next generalization where \( N' = \{n_{--}, n_-, n_+, n_{++}\} \sqcup N \) satisfies

- \( N \neq \emptyset \).
- \( n_{--} \leq n_- \leq n_+ \leq n_{++} \).
- \( n_{--} < N < n_{++} \).
- \( n_- < N \) implies \( n_- = n_{--} \) and \( N < n_+ \) implies \( n_+ = n_{++} \).

We let \( N = N_\leq \cup N_\geq \cup N_\ast \) where \( \{n_{--}\} \leq N_\leq \leq \{n_-\} \leq N_\geq \leq \{n_+\} \leq N_\ast \leq \{n_{++}\} \). This allows us to also use the conventions \( N_\leq = N_\leq \cup N_\leq \) and \( N_\ast = N_\ast \cup N_\ast \). We are interested in the multisets of the form

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
N_\leq \cup \{n_-\} & N_\geq \cup \{n_+\} \\
\bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
\]

As with the rainbow case, this “double” rainbow comes from a supercharacter restriction problem. That is, we note that if \( N'' = N_{--} \cup N_- \cup N_+ \cup N_{++} \cup N \) with \( |N_{--}| = |N_{++}|, |N_-| = |N_+| \) and

\[
N_{--} < N_\leq < N_- < N_\geq < N_\ast < N_\leq < N_{++},
\]

then if \( w : N_{--} \cup N_- \rightarrow N_+ \cup N_{++} \) is the unique bijection such that \( i < j \) if and only if \( w(j) < w(i) \), then

\[
\text{Res}_{UT_N}^{UT_{N''}} \left( V^{|i-\langle w(i) \rangle|N_{--} \cup N_-} \right) \\
\cong q^{2|N_{--}|} V^{n_{--} \mid N_- \mid n_{++}} \otimes V^{n_+ \mid N_- \mid n_+}
\]

is easy to establish by taking traces and comparing the character values on \( UT_N \).

**Remark.** The generic case is where \( N_\leq, N_\geq \) and \( N_\ast \) are all nonempty; however, the notation allows for several degenerate cases. In particular, if \( N_\leq \cup N_\ast = \emptyset \) or \( N_- = \emptyset \), then we reduce to the rainbow case of Section 3.

4.1 Right-endpoint submodules

Instead of clumping together the \( V_{N}^{K} \) as in Section 3.3, we can instead decompose them further, so

\[
V_{N}^{K} = \bigoplus_{J \subseteq N} V_{N}^{K \leftarrow J},
\]
where
\[ V_N^{K\rightarrow J} = \mathbb{C}\text{-span}\{avb \mid a \in B_N, b \in B_K, \lambda \subseteq K, \lambda = J\} \]
\[ \cong \bigoplus_{\lambda \subseteq J} q^{\text{nst}_\lambda^J} V_n \]  
(4.3)

Note that \( V^{K\rightarrow J} \) directly specifies what the right endpoints will be in the set partitions appearing in the decomposition. We can additionally specify the left endpoints by considering subsets of the appropriate size of \( K \).

**Proposition 4.1.** For \( J, K \subseteq N \),
\[ V_N^{K\rightarrow J} = \bigoplus_{\lambda \subseteq J} q^{\text{wt}_\lambda(K-I) - \text{wt}_\lambda(K-I)} V_N^{I\rightarrow J} \]

**Proof.** Note that if \( \lambda \in \mathcal{S}_N \), then
\[ \text{nst}_\lambda^J = \text{wt}_\lambda^J(K-I) - \text{wt}_\lambda^J(K-I) = \text{wt}_\lambda^J(K-I) - \text{wt}_J^J(K-I). \]

Now use (4.3). \( \Box \)

In the following sections, we will make use of the “flipped” modules (as in (2.6)) coming from the action (2.5)
\[ \mathcal{V}_N^K \cong \dagger \left( \mathcal{V}_N^{w_0(K)} \right) \quad \text{and} \quad \mathcal{V}_N^{J\rightarrow K} \cong \dagger \left( \mathcal{V}_N^{w_0(K)\rightarrow w_0(J)} \right) \]

where \( w_0 : N \rightarrow N \) is the order flipping involution as in (2.1). Note that we can use Proposition 2.3 to decompose these alternate modules using the \( \dagger \)-versions of results already proved.

### 4.2 Peel modules

Let \( N' = \{ n_-^-, n_-^+, n_+^-, n_+^+ \} \cup N \) be as in (4.1). For \( 0 \leq b \leq \min\{|N_-|, |N_+|\} \) and \( 0 \leq f \leq |N_- \cup N_+| \), let
\[ V_{N-b}^{(b,f)} \cong \bigoplus_{L \subseteq N_-^-, R \subseteq N_+^+} q^{\text{wt}_L^R(F \cap N_- \cup N_+^+)} \mathcal{V}_N^{L\rightarrow (F \cap N_- \cup N_+^+)} \otimes \mathcal{V}_N^{R\rightarrow (F \cap N_- \cup N_+^+)} \]  
(4.4)

To understand what is going on with these modules, it is perhaps easiest to consider an example. Let \( N = \{1, 2, \ldots, 12\} \) and \( N_- = \{4, \ldots, 8\} \). Then we are considering a subset of the matrices of the form

\[
\begin{bmatrix}
0 & * & 0 \\
* & * & 0 \\
* & * & 0 \\
* & * & 0 \\
* & * & 0 \\
* & * & 0 \\
* & * & 0 \\
* & * & 0 \\
\end{bmatrix}
\]

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These matrices are not closed under left multiplication by $UT_N$, so we instead let $u \in UT_N$ act by

$$
\begin{bmatrix}
0 & * & 0 \\
* & 0 & 0 \\
* & * & 0 \\
* & * & * \\
* & * & * \\
* & * & 0
\end{bmatrix}
\otimes
\begin{bmatrix}
0 & 0 & 0 & * & * & * & * & * & 0 \\
0 & 0 & 0 & * & * & * & * & * & 0 \\
0 & 0 & 0 & * & * & * & * & * & 0
\end{bmatrix}
$$

In $V_{N-\subseteq N}^{(b,f)}$ we further specify

$$b = \text{rank}
\begin{bmatrix}
* & * & * \\
* & * & * \\
* & * & * \\
* & * & *
\end{bmatrix},$$

and bound the rank of the whole matrix by $f$.

**Example.** In the degenerate cases,

- If $N_\subseteq = \emptyset$, then we do not see $n_- = n_+$, but they identify a distinguished spot (which becomes important when decomposing into supercharacters).
- If $N_\subset \cup N_\supset = \emptyset$, then $b = f = 0$ and these modules are uninteresting (aka trivial).
- If $N_\subset = \emptyset$ or $N_\supset = \emptyset$ (but not both), then $b = 0$. If, for example, $N_\subset \neq \emptyset$, the module is isomorphic to

$$\bigoplus_{F \subseteq N_\subset} V_N^F.$$  \hfill (4.5)

- If $N_\subseteq = \emptyset$ or $N_\supset = \emptyset$, then $b = 0$ and we get a core module

$$V_{\emptyset \subseteq N} \cong V_N^f.$$

By construction, it is easy to decompose the peel modules into supercharacters. For $\nu \in \mathcal{X}_N$, let

$$\alpha_{\nu\beta} = \{i \sim j \in \nu \mid i \in N_\alpha, j \in N_\beta\}. \hfill (4.6)$$

Recall that $\mathcal{P}(\nu)$ is a poset on the blocks $\text{bl}(\nu)$ of $\nu$. For $K \subseteq N$ let

$$\text{bl}_{K}(\nu) = \{i_1 \sim \cdots \sim i_\ell \in \text{bl}(\nu) \mid i_\ell \in K\}$$

or the set of blocks with right-most (resp. left-most) endpoints in $K$.

Let

$$\psi_{N-\subseteq N}^{(b,f)} = \text{tr}(\cdot, V_{N-\subseteq N}^{(b,f)}).$$
Proposition 4.2. For $0 \leq b \leq \min \{|N_\prec|, |N_\succ|\}$ and $b \leq f \leq |N_\prec \cup N_\succ|$, 

$$V^{(b,f)}_{N \leq N} \cong \bigoplus_{\nu \in N, |\nu| = f} q^{|N_\prec|+|N_\succ|} \left[ f - |\nu| \leq \text{bl}_{N_\prec}(\nu) \cup \text{bl}_{N_\succ}(\nu), 0 \right] V_\nu.$$ 

Proof. Take traces and use (4.3) to get

$$\psi^{(b,f)}_{N \leq N} = \sum_{L \leq N, R \leq N} q^{\text{wt}_R(F \cap N_\prec)} \sum_{\mu \in N, |\mu| = b} q^{\text{wt}_\mu(F \cap N_\succ)} \chi_\mu \chi_\nu \sum_{\mu \in N, |\mu| = b} q^{\text{wt}_\mu(F \cap N_\succ)} \chi_\nu.$$

as desired. \qed

The main theorem of this section decomposes the double rainbow module in terms of peel modules.

Theorem 4.3. Let $N' = \{n_-, n_+, n_+, n_+\} \cup N$ be as in (4.1), and fix $m, \ell \in \mathbb{Z}_{\geq 0}$. If $\mu = \{n_- \sim_m n_+, n_- \sim_\ell n_+\} \in \mathcal{M}_{N'}$, then as a $UT_N$-module

$$V^\mu \cong q^{\text{wt}_\mu} \bigoplus_{0 \leq b \leq \min \{|N_\prec|, |N_\succ|\}} (q - 1)^{f} q^{(m-f)b} \varphi_f(q) V^{(b,f)}_{N \leq N} \otimes V^{n_\prec \sim_\ell n_+}.$$ 

Unfortunately, the proof of Theorem 4.3 lacks a certain elegance; rather, it employs the following slight generalization of Theorem 3.10 repeatedly. Essentially, we want to know how the relevant modules decompose when there are disallowed positions.

Lemma 4.4. Let $k_-, k_+ \in N$ with $k_- < k_+$ and let $\bar{K}$ be the interval in $N$ between $k_-$ and $k_+$. Let $K \subseteq \bar{K}$ and $\lambda \in \mathcal{M}_K$ with $\lambda_\bar{K} = \emptyset$ and $X = (\lambda \cap K)$. Then as a $UT_K$-module

$$V^\lambda \otimes V^{k_- \sim_\ell k_+} \cong V^\lambda \otimes q^{\ell(K-K)} (q - 1)^{\ell} \bigoplus_{J \subseteq K - X} q^{\text{wt}_J} \varphi_{[J]}(q) V^J_K.$$ 

Proof. Note that if $j \sim k$ is an arc with $j, k \in K$, then

$$\text{Res}^{UT_K}_{UT_K}(\chi^{j \sim k}) = q^{\text{wt}_{K-K}^{j \sim k}} \chi^{j \sim k}$$

follows from Proposition 2.1. Since restriction is transitive, we see that

$$\text{Res}^{UT_K}_{UT_K}(\chi^{k_- \sim_\ell k_+}) = q^{\ell(K-K)} \text{Res}^{UT_K}_{UT_K}(\chi^{k_- \sim_\ell k_+}).$$

So WLOG, we will assume that $\bar{K} = K$. 

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Induct on $\ell$, and let $L = X$. Let $\ell = 1$. If $L \cap \mu \neq \emptyset$, then $\chi ^\lambda (u_\mu) = 0$ implies the result holds in this case. WLOG, assume $L \cap \mu = \emptyset$. Then the LHS gives
\[
\chi ^\lambda (u_\mu) \chi _{<\lambda - k_+} (u_\mu)^1 = (q - 1)|\lambda|+1q^{\text{dim}(\lambda) + (|K||\mu| - |\mu|)\text{nst}_\mu^\lambda}.
\]
On the other hand, by Proposition 3.4, the RHS gives
\[
(q - 1)\chi ^\lambda (u_\mu) \left( q^{t_L|L|} \varphi _0^J (q) \psi _K^\emptyset (u_\mu) + \sum _{j \in K - L} q^{\text{wt}_L^J (j)} \varphi _1^J (q) \psi _K^J (u_\mu) \right)
= (q - 1)^{1+|\lambda|q^{\text{dim}(\lambda) - \text{nst}_\lambda}} \left( q^{t_L|L| + \sum _{j \in K - L - \mu} (q - 1)q^{\text{wt}_L^J (j)} q^{\text{wt}_L^J (j) - \text{wt}_L^J (j)} \right)
= (q - 1)^{1+|\lambda|q^{\text{dim}(\lambda) - \text{nst}_\lambda}} \left( q^{t_L|L| + (q - 1)\sum _{j \in K - L - \mu} q^{\text{wt}_L^J (j) - \text{wt}_L^J (j)} q^{\text{wt}_L^J (j)} \right)
= (q - 1)^{1+|\lambda|q^{\text{dim}(\lambda) - \text{nst}_\lambda}} \left( q^{t_L|L| + (q - 1)|K - L - \mu|q^{(|K||\mu| - |\mu| - 1) - (|K||L||\mu| - |\mu|)|+1}} \right)
= (q - 1)^{1+|\lambda|q^{\text{dim}(\lambda) - \text{nst}_\lambda}} \left( q^{t_L|L| + (q - 1)|K - L - \mu| - 1)q^{t_L|L|} \right)
= (q - 1)^{1+|\lambda|q^{\text{dim}(\lambda) - \text{nst}_\lambda}} q^{K - L - \mu}.
\]
Suppose $\ell > 1$. Then by induction and Corollary 3.5,
\[
\chi ^\lambda \otimes \chi _{<\ell - k_+} ^\lambda = \chi ^\lambda \otimes \chi _{<\ell - k_+} ^{\ell - 1} \otimes \chi _{<\ell - k_+} ^{k_+}
= (q - 1)^{\ell - 1} \sum _{j \in K - L} q^{\text{wt}_L^J (j) + (\ell - 1 - |J|)|L|} \varphi _{[|J|]}^{\ell - 1} (q) \chi ^\lambda \otimes \psi _K^J \otimes \chi _{<\ell - k_+} ^{k_+}
= (q - 1)^{\ell - 1} \sum _{j \in K - L} (q - 1)^{|J|}q^{\text{wt}_L^J (j) + (\ell - 1 - |J|)|L|}\varphi _{[|J|]}^{\ell - 1} (q) \chi ^\lambda \otimes \otimes _{j \in \ell} \chi _{<\ell - k_+} ^{i} \otimes \chi _{<\ell - k_+} ^{k_+}
\]
By adding $\{ j - k_+ | j \in J \}$ as a multiset to $\lambda$ and then applying induction,
\[
\left( \chi ^\lambda \otimes \otimes _{j \in \ell} \chi _{<\ell - k_+} ^{i} \right) \otimes \chi _{<\ell - k_+} ^{k_+}
= \left( \chi ^\lambda \otimes \otimes _{j \in \ell} \chi _{<\ell - k_+} ^{i} \right) \otimes (q - 1) \left( q^{t_L|L|} \varphi _0^J (q) \psi _K^\emptyset + \sum _{i \in K - L - J} q^{\text{wt}_L^J (i) + \text{wt}_L^J (i)} \varphi _1^J (q) \psi _K^J \right)
= \chi ^\lambda \otimes (q - 1) \left( q^{t_L|L|} q^{t_L|J|} \varphi _0^J (q) \psi _K^\emptyset + \sum _{i \in K - L - J} (q - 1)^{|J|+1} q^{\text{wt}_L^J (i) + \text{wt}_L^J (i)} \varphi _1^J (q) \psi _K^J \right).
\]
Thus, the coefficient of $\psi _K^\emptyset$ is
\[
(q - 1)^{\ell} q^{(\ell - 1)|L|+|L|} \varphi _0^{\ell - 1} (q) = (q - 1)^{\ell} q^{\text{wt}_L^J (J) \varphi _0^{\ell - 1} (q)}.
\]
On the opposite extreme, the coefficient of $\psi _K^J$ for some $\ell$-subset $J \subseteq K - L$ is
\[
(q - 1)^{\ell + 1} \sum _{i \in J} q^{\text{wt}_L^J (J - (i)) + \text{wt}_L^J (i) + \text{wt}_L^J (i)} \varphi _{-1}^{\ell - 1} (q) = (q - 1)^{\ell + 1} q^{\text{wt}_L^J (J) \varphi _{-1}^{\ell - 1} (q)} \sum _{i \in J} q^{\text{wt}_L^J (J - (i))}
= (q - 1)^{\ell} q^{\text{wt}_L^J (J) \varphi _\ell (q)}.
\]
For the intermediate cases, fix $J \subseteq K - L$ with $1 \leq |J| \leq \ell - 1$. Then the coefficient of $\psi_K^\ell$ is

\[(q - 1)^\ell \left( q^{w_{L, L}^1(J) + (\ell - |J|)|L| + |\varphi_{|J|-1}(q)} + \sum_{i \in J} (q - 1)q^{w_{L, L}^1(J) + |\varphi_{|J|-1}(q)} \right) \]
\[(q - 1)^\ell q^{w_{L, L}^1(J) + (\ell - |J|)|L|} \varphi_{|J|-1}(q) \left( q^{|J|}(q^{\ell - |J|} - 1) + (q^{|J|} - 1) \right) \]
\[(q - 1)^\ell q^{w_{L, L}^1(J) + (\ell - |J|)|L|} \varphi_{|J|}(q), \]

as desired.

As mentioned above, the proof of Theorem 4.3 uses Lemma 4.4 repeatedly. The result is somewhat technical, so we give a brief outline of the proof before we begin. The first observation is that it suffices to consider the case when $\ell = 0$. We can visualize the steps as follows. We first show that

\[
\begin{array}{c}
n_{--} \ldots n_{--} n_+ \ldots n_+ \oplus \ldots \oplus n_{++} \ldots n_{++} \approx \ldots \oplus n_{--} \ldots n_{--} n_+ \ldots n_+ \bigcirc \ldots \bigcirc n_{++} \ldots n_{++} \quad \text{m}
\end{array}
\]

where term labeled with $l$ is the sum over all $\overline{V}_N^K$ where $K \subseteq N_<$ is an $l$-subset. Then we note that

\[
\begin{array}{c}
n_{--} \ldots n_{--} n_+ \ldots n_+ \oplus \ldots \oplus n_{++} \ldots n_{++} \approx \ldots \oplus n_{--} \ldots n_{--} n_+ \ldots n_+ \bigcirc \ldots \bigcirc n_{++} \ldots n_{++} \quad \text{m-l-r}
\end{array}
\]

where the term labeled with $r$ is the sum over all $\overline{V}_N^K_{\geq}$ where $K \subseteq N_>$ is an $r$-subset. Now the terms labeled with $l$ and the $r$ form the peel modules and the left-overs go to rainbow modules.

**Proof of Theorem 4.3.** We prove the generic case where $N_<$, $N_=$ and $N_>$ are all nonempty. The general case follows by omitting the steps (4.7) and/or (4.8) as necessary.

As usual, we take traces. Note that since tensoring by $\chi^{n_--r^{n_+}}$ is straightforward, it suffices to show that

\[
\chi^{n_--m^{n_+}} = q^{m \# \{n_-, n_+\}} \sum_{b \in \min \{|N_<=|,|N_>|\}} \sum_{b \in \min \{|m_-, m_+|\}} (q - 1)^f q^{(m-f)b} \varphi_f \psi_N^{(b,f)} \bigcirc \chi^{n_--m^{n_+}},
\]

where we also note that $\text{nst}_{\mu \cup \mu}^B = m \# \{n_-, n_+\}$ in the generic case.

By Lemma 4.4 with $K = N \cup \{n_-, n_+\}$ and $\lambda = \emptyset$,

\[
\chi^{n_--m^{n_+}} = q^{m \# \{n_-, n_+\}}(q - 1)^m \sum_{|L_1| \leq N} \varphi_{|L_1|}(q) \psi_N^{L_1}.
\]

Each subset $L_1 \subseteq N$, breaks up into $L_1^< = L_1 \cap N_\leq \subseteq N_<$ and $L_1^> = L_1 \cap N_\geq \subseteq N_>$. Use this
observation and then Lemma 4.4 “in reverse” on \( N_\geq \) with \( \tilde{K} = \{ n_+ \} \cup N_\geq \) and \( \lambda = \emptyset \) to get

\[
\chi^{n_+ - n} = q^{m \# \{ n_+ \}} \sum_{L_1^\geq \subseteq N_\geq} (q - 1)^{|L_1^\geq|} \cdot q^{m \# \{ n_+ \}} \cdot \varphi_{|L_1^\geq|} \psi_{L_1^\geq}(q) \psi_{N_{L_1^\geq}}^{L_1^\geq} \circ \sum_{L_1^\geq \subseteq N_\geq} (q - 1)^{|L_1^\geq|} \cdot \varphi_{|L_1^\geq|} \psi_{L_1^\geq}(q) \psi_{N_{L_1^\geq}}^{L_1^\geq} \chi^{n_+ - n_+} \]

where the last equality is by (4.2).

We will now begin to use the action \( \circ \) in addition to \( \circ \). We will therefore use the arrows \( \psi \) and \( \psi \) to indicate which kind of the module the trace is coming from.

Let \( R_1^\geq = R_1 \cap N_\geq \). By the \( \circ \)-version of Lemma 4.4 with \( K = N_\geq \) and \( X = R_1^\geq \),

\[
\chi^{n_+ - n} = q^{m \# \{ n_+ \}} \sum_{L_1^\geq \subseteq \tilde{R}_1} (q - 1)^{|L_1^\geq|} \cdot q^{m \# \{ n_+ \}} \cdot \varphi_{|L_1^\geq|} \psi_{L_1^\geq}(q) \psi_{R_1^\geq}^{L_1^\geq} \circ \sum_{L_1^\geq \subseteq \tilde{R}_1} (q - 1)^{|L_1^\geq|} \cdot q^{m \# \{ n_+ \}} \cdot \varphi_{|L_1^\geq|} \psi_{L_1^\geq}(q) \psi_{R_1^\geq}^{L_1^\geq} \chi^{n_+ - n_+} \]

where the last equality is by (4.2).

Let \( R_1^\geq = R_1^\geq \cap N_\geq \) and \( R_1^- = R_1^- \cap N_{\leq} \). Break up each subset \( R_2 \) into \( R_2^\geq = R_2 \cap N_{\geq} \) and \( R_2^- = R_2 \cap N_{\leq} \). We again employ Lemma 4.4 “in reverse” with \( K = N_{\leq} \) and \( X = R_1^\geq \) to get

\[
\chi^{n_+ - n} = q^{m \# \{ n_+ \}} \sum_{L_1^\leq \subseteq \tilde{R}_1} (q - 1)^{|L_1^\leq|} \cdot q^{m \# \{ n_+ \}} \cdot \varphi_{|L_1^\leq|} \psi_{L_1^\leq}(q) \psi_{R_1^\leq}^{L_1^\leq} \circ \sum_{L_1^\leq \subseteq \tilde{R}_1} (q - 1)^{|L_1^\leq|} \cdot q^{m \# \{ n_+ \}} \cdot \varphi_{|L_1^\leq|} \psi_{L_1^\leq}(q) \psi_{R_1^\leq}^{L_1^\leq} \chi^{n_+ - n_+} \]
We can plug this back into the original computation to get

$$
\chi^{n- \sim m_{++}} = q^{m \# \{n_+\}} \sum_{L_k^1 \subseteq N} (q - 1)^{|L_k^1|} q^{L_k^1 \# \{n_+\}} \varphi^{m}_{|L_k^1|}(q) \psi_N^{L_k^1 \sim R_1} \\
\otimes q^{(m-|L_k^1|) \# \{n_+\}} \sum_{R_2 \subseteq N} (q - 1)^{|R_2^1|} q^{R_2^1 \# \{n_+\}} \varphi^{m}_{|R_2^1|}(q) \psi_N^{R_2^1 \sim R_1} \\
\otimes \chi^{n- m-|L_k^1|-|R_2^1| n_+},
$$

where the last equality is by (4.2). We simplify to get

$$
\chi^{n- \sim m_{++}} = q^{m \# \{n_+\}} \sum_{L_k^1 \subseteq N} (q - 1)^{|L_k^1|} q^{L_k^1 \# \{n_+\}} \varphi^{m}_{|L_k^1|}(q) \psi_N^{L_k^1 \sim R_1} \\
\otimes F \otimes \chi^{n- m-|F| n_+}.
$$

as desired. \hfill \Box

**Remark.** If $N_\prec \cup N_\succ = \emptyset$, then Theorem 4.3 reduces to the unsightful

$$
V^{n- \sim m_{++}} \cong V^{n- \sim m_{+}}.
$$

On the other hand, $N_\prec = \emptyset$ gives a family of equations

$$
V^{n- \sim m_{++}} \cong q^{m r} (q - 1)^m \bigoplus_{0 \leq b \leq m} q^{(m-f)} \varphi_f^{m_f}(q) V^{(b,f)}_{\emptyset \subseteq N},
$$

that each depend on relative order of $n_+ = n_-$ in $N'$. For example, if $N_\prec = \emptyset$ or $N_\succ = \emptyset$, then we recover Theorem 3.10. Alternatively, if $|N| = 2\ell$ and $N_\prec$ is the first $\ell$ elements of $N$, then the coefficient of the full rainbow supercharacter

\[ \chi^{n- \sim m_{++}} = \chi \]

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Then let

4.3 The multirainbow case

For a sequence of subsets \( N = N_1 \supseteq N_2 \supseteq \cdots \supseteq N_k \supseteq \emptyset \), let

\[
N' = N \cup N_\pm, \quad \text{where} \quad N_\pm = \bigcup_{j=1}^k \{ n_j^-, n_j^+ \},
\]

and for each \( 1 \leq j \leq k \), \( \{ n_j^-, n_j^+ \} \cap N_j \) is an interval in \( N' \) with \( \{ n_j^- \} < N_j < \{ n_j^+ \} \). For \( b = (b_1, \ldots, b_{k-1}, 0) \) and \( f = (f_1, \ldots, f_k) \), a corresponding onion module is given by

\[
V_{N_k \leq \cdots \leq N_1}^{(b, f)} = \left( \bigotimes_{j=1}^{k-1} V_{N_{j+1} \leq N_j}^{(b_j, f_j)} \right) \otimes V_{N_k}^{f_k}.
\]

For any sequence \( (a_1, a_2, \ldots, a_k) \), let

\[
a_{i \leq} = a_i + \cdots + a_k \quad \text{and} \quad a_{\leq i} = a_1 + \cdots + a_i.
\]

The iterated version of Theorem 4.3 is as follows.

**Corollary 4.5.** Let \( N = N_1 \supseteq \cdots \supseteq N_k \supseteq \emptyset \) be a sequence of nested subsets with \( N' \) as in (4.9). For \( m = (m_1, \ldots, m_k) \in \mathbb{Z}_{\geq 1}^k \), define the multiset

\[
\mu = \{ n_1^- \sim m_1 \sim n_2^- \sim m_2 \sim n_2^+ \sim m_2 \sim \cdots \sim n_k^- \sim m_k \sim n_k^+ \},
\]

and the set

\[
I_m = \{ (b, f) \in \mathbb{Z}_{\geq 0}^k \mid 0 \leq b_j \leq f_j \leq m_{\leq j} - f_{\leq j-1}, 1 \leq j \leq k, b_k = 0 \}.
\]

Then

\[
\text{Res}_{\text{UT}_N'}^{\text{UT}_N} (V^\mu) \cong \sum_{m_{\leq} \in \mu} q^{m_{\leq}} (q - 1)^{m_{\leq}} \bigoplus_{(b, f) \in I_m} \left( \prod_{j=1}^k q^{(m_j - f_j) b_j} \varphi_{f_j}^{m_{\leq j} - f_{\leq j-1}} (q) \right) V_{N_k \leq \cdots \leq N_1}^{(b, f)}.
\]

One basic application of this result is to decompose \( \text{UT}_N \)-module

\[
\mathbb{C}\text{-span}\{ \text{ut}_N \},
\]

under left multiplication. The decomposition into supercharacters is not obvious (unlike in the case of the regular module \( \mathbb{C}\text{-span}\{ \text{UT}_N \} \)). The following proposition implies that this module falls within the framework of Corollary 4.5.

**Proposition 4.6.** Let \( N' = N \cup \{ n_-, n_1^+, \ldots, n_{|N|}^+ \} \), where \( n_- < n_1 < n_1^+ < n_2 < \cdots < n_{|N|} < n_{|N|}^+ \). Then

\[
\mathbb{C}\text{-span}\{ \text{ut}_N \} \cong (q - 1)^{1 - |N|} \text{Res}_{\text{UT}_N'}^{\text{UT}_N} \left( \bigotimes_{j=1}^{|N|-1} V^{n_- \sim n_j^+} \right).
\]
Proof. Let $\mu \in \mathcal{S}_N$ and consider

$$\text{tr}(u_\mu, u_N) = \sum_{v \in u_N} u_\mu v = \#\{v \in u_N \mid u_\mu v = v\}.$$ 

Since,

$$(u_\mu v)_{ik} = \begin{cases} v_{ik} + v_{jk} & \text{if } i \sim j \in \mu, \\ v_{ik} & \text{otherwise}, \end{cases}$$

we may conclude

$$\text{tr}(u_\mu, u_N) = q^{\binom{|N|}{2}} \prod_{i \sim j \in \mu} q^{-\text{wt}_N(j)}.$$ 

On the other hand, by Proposition 2.1,

$$|N|^{-1} \bigotimes_{j=1}^{|N|} \chi^{n_j} v^+_N (u_\mu) = (q - 1)^{|N| - 1} q^{\binom{|N|}{2}} \prod_{i \sim j \in \mu} q^{-\text{wt}_N(j)}.$$ 

Comparing class function values gives the result. 

We can apply Corollary 4.5 to the module

$$\bigotimes_{j=1}^{|N|} V^{n_j}$$

to decompose it into more reasonable modules.

**Corollary 4.7.** For

$$\bigotimes_{j=1}^{|N|} V^{n_j} \cong q^{\binom{|N|}{2}} (q - 1)^{|N|} \bigoplus_{A \subseteq N} (q - 1)^{|A|} \prod_{a \in A} [1 + \text{wt}_N(a)] V^A_N.$$ 

**Proof.** We prove a stronger statement by induction on $|N|$: for $m \geq 1$,

$$\chi_{m \cdot |N| \rightarrow n_j^+} \bigotimes_{j=1}^{|N|-1} \chi^{n_j} v^+_N = q^{m(|N| - 1) + \binom{|N| - 1}{2}} (q - 1)^{|N| + m - 1} \bigoplus_{A \subseteq N} (q - 1)^{|A|} \prod_{a \in A} [m + \text{wt}_N(a)] v^A_N.$$ 

When $|N| = 1$, then by Theorem 3.10,

$$\chi_{m \cdot 1 \rightarrow n_j^+} = (q - 1)^{m+1} [m] v^1_N + (q - 1)^{m+0} v^0_N,$$

as desired. Assume $|N| > 1$. Then by Corollary 4.5 and (4.5),

$$\chi_{m \cdot |N| \rightarrow n_j^+} \bigotimes_{j=1}^{|N|-1} \chi^{n_j} v^+_N = q^{m+n-1} [m] v^{n_j}_N \bigoplus_{A \subseteq N} (q - 1)^{|A|} \prod_{a \in A} [m + \text{wt}_N(a)] v^A_N.$$ 

We have

$$\chi_{m \cdot |N| \rightarrow n_j^+} = q^{m} \chi_{m+1 \cdot |N| \rightarrow n_j^+} \bigoplus_{j=1}^{|N|-2} \chi_{m+1 \cdot |N|-1 \rightarrow n_j^+} \bigoplus_{j=1}^{|N|-2} \chi_{m \cdot |N| \rightarrow n_j^+} + q^{m+0} \chi_{m+1 \cdot |N| \rightarrow n_j^+} \bigoplus_{j=1}^{|N|-2} \chi_{m+1 \cdot |N|-1 \rightarrow n_j^+}.$$ 

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Apply induction to the first sum to get
\[
q^{m+n-1}(q-1)^2[m]\psi_N^{(n_n(N))} \odot \chi^{n_{-m}n^+_{n_n(N)-1}} \odot \bigotimes_{j=1}^{[N]-2} \chi^{n_{-m}n^+_j} \\
= q^{m-[N]-1}+(\frac{[N]-1}{2})(q-1)^{[N]+m-1}[m]\psi_N^{(n_n(N))} \odot \sum_{A \in N} (q-1)^{|A|+1} \prod_{a \in A} [m+w_{N-A-n_n(N)}(a)] \psi_N^A \\
= q^{m-[N]-1}+(\frac{[N]-1}{2})(q-1)^{[N]+m-1}[m] \sum_{\{n_n(N)\} \leq A \in N} (q-1)^{|A|} \prod_{a \in A} [m+w_{N-A}(a)] \psi_N^A.
\]

Apply induction to the second sum to obtain
\[
q^{m[0]}\psi_{\mathcal{N}}^{\mathcal{N}} \odot \chi^{n_{-m+1}n^+_{n_n(N)-1}} \odot \bigotimes_{j=1}^{[N]-2} \chi^{n_{-m}n^+_j} \\
= q^{m+(m+1)[(N)-2]}+(\frac{[N]-1}{2})q^{[N]+m-1} \sum_{A \in N-\{n_n(N)\}} (q-1)^{|A|} \prod_{a \in A} [m+w_{N-A}(a)] \psi_N^A \\
= q^{m-(N)-1}+(\frac{[N]-1}{2})(q-1)^{[N]+m-1} \sum_{A \in N-\{n_n(N)\}} (q-1)^{|A|} \prod_{a \in A} [m+w_{N-A}(a)] \psi_N^A.
\]

Set \(m = 1\) to get the main result.

We apply Proposition 3.3 to get a decomposition into supercharacters.

**Corollary 4.8.** If \(n'\) is the maximal element in \(N\), then
\[
\text{tr}(\cdot, \mathbb{C}\text{-span}\{\text{ut}_{\mathcal{N}}\}) = q^{(\frac{[N]-2}{2})} \sum_{\lambda \in \mathcal{P}_N} q^{\text{nst}_{\lambda}} \left( \sum_{\lambda' \in \mathcal{P}_{N-n'}{\lambda}} q^{\text{nst}_{\lambda'-\lambda}} \prod_{a \in A} [q^{w_{N-A}(a)} - 1] \right) \chi_{\lambda}.
\]

### 4.4 The further decomposition into supercharacters

This section returns to the problem of restricting to supercharacters. The final decomposition is given in Corollary 4.11, below, which essentially unpacks Theorem 4.3. However, we first need a result that takes Lemma 4.4 further into a decomposition of supercharacters. Note that if \(\nu = \emptyset\) in the following lemma, then (a) reduces to Proposition 4.1 and (b) reduces to Corollary 3.11.

**Lemma 4.9.** Let \(\{k_-, k_+\} \cup K \subset \mathcal{N}\) be an interval with \(\{k_+\} < K < \{k_+\}\). Let \(\nu \in \mathcal{M}_N\) with \(\nu_K = \emptyset\). Then
(a) Let \(X = (\nu \cap K)\). As a UT\(_K\)-module
\[
V^{\nu} \otimes V_{K}^{\nu} \cong V^{\nu} \otimes \bigoplus_{I \in K\setminus X_{<}} q^{wt_{\lambda}(I)+wt_{\lambda}(J-J')+wt_{\lambda}(J-J')} V_{K^{J'-I}}^{\nu}.
\]
(b) Suppose \(\nu \in \mathcal{S}_N\), and let \(X = (\nu \cap K)\). As a UT\(_K\)-module
\[
V^{\nu} \otimes V^{k_+_{\nu}k_+} \cong V^{\nu} \otimes (q-1)^{\ell} \bigoplus_{\lambda \in \mathcal{S}_N} q^{\text{nst}_{\lambda}} \sum_{\ell=|\lambda|} q^{\frac{(\ell-l)\chi}{2}} \varphi_{\ell}^{\nu}(q^{P(\lambda \cup \nu)} \otimes \mathcal{B}_K(\lambda \cup \nu), 0) V_{\nu} \]
Proof. (a) By Corollary 3.5,
\[(q - 1)^{|J|} \psi_K^J = \text{Res}_{\text{UT}_K}^{\text{UT}_K \cup \{k, \ldots, k + \}} \bigotimes_{j \in J} \chi^{j-k+} \].

The decomposition in Proposition 4.1 implies that it suffices to show that if \( \mu \in \mathcal{X}_K \) with \( \mu \subseteq J \) and \( \mu \subseteq K - \mathcal{X}_K \), then the coefficient of \( \chi^{\nu} \otimes \chi^{\mu} \)

\[\chi^{\nu} \otimes \text{Res}_{\text{UT}_K}^{\text{UT}_K}(\bigotimes_{j \in J} \chi^{j-k+}) \] is

\[(q - 1)^{|J|} q^{\text{nst}_{\mu}^\nu + \text{wt}_{\chi}^\nu(\mu) + \text{wt}_{\chi}^\nu(J-J - \mu) + \text{wt}_{\chi}^{\mu}(J - \mu)} q^{\text{wt}_{\chi}^{\mu}(J - \mu)}. \tag{4.10} \]

To prove (4.10) we induct on \(|J|\). If \(|J| = 1\), then we apply the \( \dagger \)-version of Lemma 4.4 to obtain

\[\chi^{\nu} \otimes \chi^{j-k+} = \chi^{\nu} \otimes (q - 1)q^{\text{wt}_{\chi}^{\nu}(j)} \chi^{j-l} + (q - 1) \sum_{j < k, \chi \in \mathcal{X}} q^{\text{wt}_{\chi}^{\nu}(l)} \chi^{j-l} \bigotimes_{j \in J, j \neq j_0} \chi^{j-k+} \]

and all the summands satisfy (4.10).

Suppose \(|J| > 1\). Let \( j_0 \in J \) be minimal. Then by first restricting \( \chi^{j_0-k+} \),

\[\chi^{\nu} \otimes \bigotimes_{j \in J} \chi^{j-k+} = \chi^{\nu} \otimes (q - 1)q^{\text{wt}_{\chi}^{\nu}(j_0)} \chi^{j-l} + \sum_{j \neq j_0} q^{\text{wt}_{\chi}^{\nu}(l)} \chi^{j-l} \bigotimes_{j \in J, j \neq j_0} \chi^{j-k+} \]

Proposition 2.2 (b) implies that the set partition \( \mu \) only appears in the decomposition of at most one of the middle terms of the sum. That is, if \( j_0 \notin \mu \) then \( \chi^{\nu} \otimes \chi^{\mu} \) only appears in the expansion of

\[\chi^{\nu} \otimes (q - 1)q^{\text{wt}_{\chi}^{\nu}(j_0)} \chi^{j-l} \bigotimes_{j \in J, j \neq j_0} \chi^{j-k+} \]

so by induction the coefficient of \( \chi^{\nu} \otimes \chi^{\mu} \) is

\[(q - 1)q^{\text{wt}_{\chi}^{\nu}(j_0)} q^{\text{wt}_{\chi}^{\nu}(\mu)} q^{\text{wt}_{\chi}^{\nu}(J-J - \mu_{J_0}) + \text{wt}_{\chi}^{\mu}(J - \mu_{J_0})}. \tag{4.10} \]

The minimality of \( j_0 \) implies \( \text{wt}_{\chi}^{\mu}(j_0) = \text{wt}_{\chi}^{\mu}(j_0) \), and so this expression simplifies to the desired term.

If, on the other hand, \( j_0 \sim l \in \mu \), then the coefficient \( \chi^{\nu} \otimes \chi^{\mu} \) only appears in the expansion of

\[\chi^{\nu} \otimes (q - 1)q^{\text{wt}_{\chi}^{\nu}(l)} \chi^{j-l} \bigotimes_{j \in J, j \neq j_0} \chi^{j-k+} \]

Reorganize slightly so that we can use induction to show that the coefficient of \( \chi^{\nu} \cup (j_0 - l) \otimes \chi^{\mu} - (j_0 - l) \) in

\[(q - 1)q^{\text{wt}_{\chi}^{\nu}(j_0 - l)} \chi^{\nu}(j_0 - l) \bigotimes_{j \in J, j \neq j_0} \chi^{j-k+} \]

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is
\[(q - 1)^q \left[ (q - 1)^{0.5} - q \right] \left[ \sum_{\ell = 0}^{x} (q - 1)^{0.5} + q \right]^{\alpha} \left[ \sum_{\ell = 0}^{x} (q - 1)^{0.5} + q \right]^{\beta}, \]
where the minimality of \( j_0 \) again gives us the desired coefficient.

(b) Since \( \nu_K = \emptyset \), we can use Lemma 4.4 with \( X = \nu \cap K \) and (a) with \( X = \nu \cap K \) to decompose
\[
\chi^\nu \otimes \chi^{k_+ - k_-} = \chi^\nu \otimes \sum_{L \subseteq K - X} (q - 1)^{\ell} (\sum_{|L| = \ell} q^{\varphi(|L|) \chi^\psi K}) = \sum_{L \subseteq K - X} (q - 1)^{\ell} (\sum_{|L| = \ell} q^{\varphi(|L|) \chi^\psi K}) \chi^\nu
\]
\[
\otimes \sum_{R \subseteq K - X} q^{\sum_{|R| = \ell} \nu(R)} \sum_{L \subseteq K - X} q^{\sum_{|L| = \ell} \nu(L)} \sum_{\lambda \in \mathcal{F}_N} ^{\nu \cup \nu \cap K} \sum_{|\lambda| = \ell} q^{\sum_{|\lambda| = \ell} \nu(\lambda)} \sum_{|\lambda| = \ell} q^{\sum_{|\lambda| = \ell} \nu(\lambda)}
\]
\[
\chi^\nu \otimes \sum_{L \subseteq K - X} q^{\sum_{|L| = \ell} \nu(\lambda)} \sum_{|\lambda| = \ell} q^{\sum_{|\lambda| = \ell} \nu(\lambda)} \sum_{|\lambda| = \ell} q^{\sum_{|\lambda| = \ell} \nu(\lambda)} \sum_{|\lambda| = \ell} q^{\sum_{|\lambda| = \ell} \nu(\lambda)}
\]
\[
(4.11)
\]
Note that
\[
\text{wt} K(\lambda) + \text{wt} \chi(\lambda) + \text{nst}_K = \text{nst}_K^{\nu' \cup \nu'}, \quad \text{where} \quad \nu' = \{ i \sim_j \nu \mid \{ i, j \} \cap K \neq \emptyset \},
\]
and since \( K \) is an interval,
\[
\text{nst}_K^{\nu' \cup \nu'} = \text{nst}_K^{\nu \cup \nu'} = |\lambda||\nu'|
\]
Fix a point \( y \in K - X - \lambda \). Then by (2.2),
\[
\text{nst}_y^{\nu \cup \nu'} = \text{nst}_y + \text{wt}_y^{\nu}(y) - \text{wt}_y^{\nu}(y).
\]
With respect to \( y \), the elements of \( i \in \nu \) with \( i < y \) fall into 4 categories:

(1) \( i \sim_k \in \nu \) with \( k < y \).

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(2) \( i \sim k \in \nu \) with \( y < k \) and \( k \in K \),
(3) \( i \sim k \in \nu \) with \( y < k \) and \( i \in K \),
(4) \( i \sim k \in \nu \) with \( y < k \) and \( i, k \notin K \).

Thus,
\[
\text{wt}_{\nu}^+(y) - \text{wt}_{\nu}^-(y) = 0 + \text{wt}_{\chi}^+(y) + \text{wt}_{\chi}^-(y) + |\nu| \cdot 1,
\]
and we obtain
\[
\sum_{L \subseteq K - \chi - \lambda, |L| + |\lambda| = l} q^{|\text{wt}_{\chi}(L) + \text{wt}_{\chi}(L) + \text{nst}_{\lambda}^+ - \text{nst}_{\lambda}^-|} = \frac{1}{q^{(|l - |\lambda|)|\nu|}} \left[ P(\lambda \cup \nu) \bigg|_{l - |\lambda| \leq \text{bl}_{K}(\lambda \cup \nu), 0} \right].
\]

Plug this last observation and (4.12) into (4.11) to get the desired result.

\[\square\]

**Remark.** There is an inherent asymmetry in the construction in Theorem 4.3, but in theory that asymmetry should disappear again when we express the module in terms of supercharacters as in Lemma 4.9. In particular, the coefficients may be simpler if one uses the \( \odot \)-version of Lemma 4.3 rather than the \( \ominus \)-version given above. Computational evidence suggests the coefficients have a slightly simpler symmetric expression.

However, from this asymmetry, we get the following corollary.

**Corollary 4.10.** Let \( K \subseteq N \) be an interval and let \( \lambda, \nu \in \mathcal{S}_N \) satisfy (a) \( \lambda \cup \nu \in \mathcal{S}_N \), (b) \( \lambda_K = \lambda \), (c) \( \nu = \{i \sim j \in \nu \mid \{i, j\} \cap K = 1\} \). Then for \( \ell \in \mathbb{Z}_{\geq |\lambda|} \),
\[
\sum_{l = |\lambda|}^{\ell} q^{(\ell - l)\nu \cup K} \varphi_{\ell}^f(q) \left[ P(\lambda \cup \nu) \bigg|_{l - |\lambda| \leq \text{bl}_{K}(\lambda \cup \nu), 0} \right] = \sum_{l = |\lambda|}^{\ell} q^{(\ell - l)\nu \cup K} \varphi_{\ell}^f(q) \left[ P(\lambda \cup \nu) \bigg|_{l - |\lambda| \leq \text{bl}_{K}(\lambda \cup \nu), 0} \right].
\]

**Example.** If, for example, \( K = \{4, \ldots, 12\} \subseteq N = \{1, 2, \ldots, 13\} \), \( \ell = 3 \),
\[
\lambda = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \quad \text{and} \quad \nu = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet.
\]
then this corollary says that
\[
(q^3 - 1) \left( q^2 + (q^2 - 1)q(q + q^2 + 3q^3 + q^4) + (q^2 - 1)(q - 1)(q^3 + 3q^4 + 4q^5 + 4q^6 + 3q^7) \right)
= (q^3 - 1) \left( q^6 + (q^2 - 1)q^3(3q^3 + q^4) + (q^2 - 1)(q - 1)(3q^6 + 3q^7) \right).
\]

Vis-a-vis the remark, the “correct” expression may be the even simpler
\[
(q^3 - 1) \left( q^{10} + (q^2 - 1)q^5(2q^3) + (q^2 - 1)(q - 1)q^6 \right).
\]

For \( \gamma \in \mathcal{S}_N \), let
\[
\gamma_\ast = \gamma_\ast \quad \text{and} \quad \gamma_\ast \neq \gamma - \gamma_\ast,
\]
and recall the notation (4.6) that selects a subset of the arcs based on where their endpoints lie.
Corollary 4.11.

\[
V^\mu \equiv q^{n \to m} \sum_{\gamma \in \mathcal{N}} q^{\gamma} \sum_{|\gamma| \leq m} q^{(m-f-l)|\gamma|} \Phi_f(q) \Phi_l^{m-f-l}(q)
\]

Proof. By Theorem 4.3, we have

\[
\chi^{n \to m} \otimes \chi^{n \to +} = q^{n \to m} \sum_{0 < f < m} (q-1)q^{(m-f)b} \Phi_f(q) \Phi_l^{m-f}(q) \chi^{n \to +}.
\]

Use the decomposition in Proposition 4.2 to write

\[
\psi^{(b,f)}_{N \in \mathcal{N}} \otimes \chi^{n \to m-f} = \sum_{v \in \mathcal{N}, |v| \leq f} \chi^{n \to m-f} \otimes \chi^{n \to m-f} \chi^v = \sum_{v \in \mathcal{N}, |v| \leq f} \chi^{n \to m-f} \otimes \chi^{n \to +}.
\]

By Lemma 4.9 (b),

\[
\chi^v \otimes \chi^{n \to m-f} = \chi^v \otimes \chi^\lambda \otimes \chi^{n \to m-f} = \chi^\lambda \otimes \chi^{n \to m-f}.
\]

Combine the two expressions by setting \( \gamma = \lambda \cup \nu \), to get

\[
\psi^{(b,f)}_{N \in \mathcal{N}} \otimes \chi^{n \to m-f} = \sum_{\lambda \in \mathcal{N}, |\lambda| \leq f, \lambda \subseteq \mathcal{N}, |\lambda| \leq m-f} \chi^{n \to m-f} \otimes \chi^{n \to m-f} \chi^\lambda.
\]

Combine with (4.13) to get

\[
\chi^{n \to m} \otimes \chi^{n \to +} = q^{n \to m} \sum_{\gamma \in \mathcal{N}} q^{\gamma} \sum_{|\gamma| \leq m} q^{(m-f-l)|\gamma|} \Phi_f(q) \Phi_l^{m-f-l}(q)
\]

as desired.
For the coefficient of the trivial character $\chi^\varnothing$, we have a more pleasing expression.

**Corollary 4.12.** The coefficient of $\chi^\varnothing$ in the decomposition of $\chi^{n_\ell n+ \otimes n_\ell n+}$ is

$$q^{2m}(q - 1)^{m+\ell} \sum_{0 \leq f \leq m} \varphi_f^m(q) \varphi_{f-m-\ell}^\ast(q) \binom{|N - N_\ast|}{f} \binom{|N_\ast|}{l}.$$ 

### 5 Combinatorial Notation Index

- $\mathcal{S}_N$. Set of set partitions of the set $N$ (Section 2.1).
- $\lambda$. Set of left endpoints of the set partition $\lambda$ (Section 2.1).
- $\underline{\lambda}$. Set of right endpoints of the set partition $\lambda$ (Section 2.1).
- $\text{bi}(\lambda)$. The blocks of the set partition $\lambda$ (Section 2.1).
- $\uparrow(\lambda)$. (2.1).
- $\text{nst}_{\lambda}^\mu$. Number of pairs of an arc of $\mu$ nested in an arc of $\lambda$ (Section 2.1).
- $\text{nst}_{\lambda}^\mu$. Number of pairs of an element of $\lambda$ nested in an arc of $\lambda$ (Section 2.1).
- $\text{crs}(\lambda)$. Number of crossings in $\lambda$ (Section 2.1).
- $\mathcal{S}_N^\ast$. Set of noncrossing set partitions of the set $N$ (Section 2.1).
- $\lambda^\ast$. The unique non-crossing set partition with left and right endpoints sets the same as $\lambda$ (Section 2.1).
- $\text{wt}_C^\ast(A) = \text{wt}_C^\ast(C)$. The number of elements in $C$ that are greater than some element of $A$ (Section 2.1).
- $\mathcal{M}_N$. The set of multisets of arcs with endpoints in $N$ (Section 2.2).
- $u \circ v$. (2.4).
- $u \circ v$. (2.5).
- $\uparrow(g)$. The transpose across the anti-diagonal (Section 2.3).
- $n_\ell \sim n_\ast$. The multi-set of $\ell$ arcs $n_\ell \sim n_\ast$ (Section 3.0).
- $[n]_q$. $q$-analogue of $n$ (Section 3.1).
- $[n]_q^\ast$. $q$-binomial coefficient (Section 3.1).
- $\text{wt}_P^\ast(A)$. Number of elements in $P$ that are greater than some element in $A$ (Section 3.1).
- $[p]_q^\ast$. Poset analogue of $q$-binomial coefficient (3.2).
- $[p]_q^\ast$.
- $\mathcal{P}(\lambda)$. Poset on the blocks of $\lambda$ given in Section 3.1.1.
- $\varphi_{\lambda}^\ast(q)$. (3.5).
- $\alpha, \beta$. (4.6).
- $\text{bl}_{K}(\nu)$. Blocks in $\nu$ with rightmost endpoint in $K$ (Section 4.2).
- $\text{bl}_{K}(\nu)$. Blocks in $\nu$ with leftmost endpoint in $K$ (Section 4.2).
- $\mathcal{P}(\lambda)$. Poset on the blocks of $\lambda$ given in Section 3.1.1.
- $\varphi_{\lambda}^\ast(q)$. (3.5).
- $\alpha, \beta$. (4.6).
- $\text{bl}_{K}(\nu)$. Blocks in $\nu$ with rightmost endpoint in $K$ (Section 4.2).
- $\text{bl}_{K}(\nu)$. Blocks in $\nu$ with leftmost endpoint in $K$ (Section 4.2).
- $\mathcal{P}(\lambda)$. Poset on the blocks of $\lambda$ given in Section 3.1.1.
- $\varphi_{\lambda}^\ast(q)$. (3.5).
- $\alpha, \beta$. (4.6).
- $\text{bl}_{K}(\nu)$. Blocks in $\nu$ with rightmost endpoint in $K$ (Section 4.2).
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