An Interesting Fitting Formula of Quark Masses

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Abstract

We show an interesting empirical formula of quark masses here, which is found by implementing a least squares fit. In this formula the measured QCD coupling is almost a "best fitting coupling".

There are many papers working on the masses of quarks, in theory or in experiment. For theory, the difficulty is, masses of quarks cannot been determined by first principle. The masses of known six quarks correspond to six free parameters in standard model (SM). On the other hand, there appears a mystery problem in the SM, that is, comparing to $t$ quarks, the other five quarks can be considered as ones with vanishing mass. However, there are empirical formulae to illuminate this hierarchy problem, such as reference [1]. We show another interesting mass fitting for the known six quarks in this note, which can be compared with the results in [1]. Although it is empirical, this formula is significant agreement with experiment values. We expect this formula can help us to discover the correct theory of flavor. A byproduct of this formula is, that the best fitting coupling is just the experiment one.

We first list the $\overline{MS}$ quark masses in table 1. These data are from reference [2] in different scale. Because pole masses of light quarks are physical meaningless and pole masses of heavy quarks also have evil definitions [3], we do not use the data of pole mass here.

| $n_g$ | $n_s$ | Mass (MeV) |
|-------|-------|------------|
|       | 1     | 3.25 ± 1.75 MeV (2 GeV) | 1.2 ± 0.2 GeV ($m_c$) | 174 ± 5 GeV$^*$ |
|       | 2     | 7.0 ± 2.0 MeV (2 GeV)     | 115 ± 35 MeV (2 GeV)   | 4.25 ± 0.2 GeV ($m_b$) |

Table 1: Masses of quarks in $\overline{MS}$ scheme from [2]. $n_g$ is the number of generator and $n_s$ is the number of "isospin": $n_s = 3(1 - |Q|)$, where $Q$ is the charge of quark. Since reference [2] only gives mass ranges of different quarks, we take its midpoint as our input. For instance, the mass of $u$ quark 3.25 ± 1.75 MeV (2 GeV) just corresponds to the range 1.5 MeV ≤ $m_u$ (2 GeV) ≤ 5 MeV in reference [2]. The quantities in bracket are scales where we obtained quark masses.

$^*$: This is the pole mass of top quark, which should be converted into $\overline{MS}$ mass using Eqn. [1] [2].

We then renormalize different quark masses to the mass of Z particle $m_Z = 91.2$ GeV using renormalization group equation. Relevant formulae have been shown in reference [4]. We take here $\alpha_s(m_Z) = 0.117 [2]$, which corresponds $\Lambda^{(5)} = 197$ MeV, $\Lambda^{(4)} = 274$ MeV and
\( \Lambda^{(3)} = 310 MeV \). The results have been shown in Table 2. As argued in Tab. 1, one should convert the pole mass of \( t \) quark into \( \overline{MS} \) mass. Since \( m_t \gg m_q \), where \( q = u, d, s, c, b \), the mass is converted by a simpler relation,

\[
m^{pole} = m(m)[1 + \frac{4\alpha_s(m)}{3\pi} + (13.44 - 1.04 \times 5)(\frac{\alpha_s(m)}{\pi})^2]. \tag{1}
\]

| \( n_s = 1 \) | \( n_g = 1 \) | \( n_g = 2 \) | \( n_g = 3 \) |
|---|---|---|---|
| \( n_g = 1 \) | 0.00019(10) | 0.597(100) | 173(5) |
| \( n_g = 2 \) | 0.00041(12) | 0.067(21) | 2.94(17) |

Table 2: The \( \overline{MS} \) masses (Gev) at scale \( m_Z \).

In QCD, the contributions of quark mass to high energy parameters, such as anomalous dimension functions, decay behaviors beyond tree level, coupling, and even more, the evolutions of masses themselves, have the form \( \ln \frac{m}{m_0} \), where \( m_0 \) is some subtraction mass. Thus, we consider here the relations of the logarithm of quark mass, for instance, \( \ln m/m_0 \), where we choose \( m_0 = 1 Gev \). The results are listed in Table 3.

| \( n_s = 1 \) | \( n_g = 1 \) | \( n_g = 2 \) | \( n_g = 3 \) |
|---|---|---|---|
| \( n_g = 1 \) | -6.26(54) | -0.515(167) | 5.155(29) |
| \( n_g = 2 \) | -5.49(29) | -2.696(304) | 1.079(59) |

Table 3: \( y(n_g, n_s) = \ln m/m_0 \) at scale \( m_Z \), where \( m_0 = 1 Gev \).

We find that, for definite \( n_s \), there is an approximate linear relation among different \( n_g \). One can use formula \( a_1n_sn_g + a_2n_g + a_3n_s + a_4 \) to fit quark masses. We use the least squares method with weight to obtain \( a_i \). That is, find parameters \( a_i \) to minimize function

\[
f = \sum_{n_g=1,2,3; n_s=1,2} (y(n_g, n_s) - (a_1n_s n_g + a_2n_g + a_3n_s + a_4))^2 w(n_s, n_g), \tag{2}
\]

where \( w(n_s, n_g) \) is the weight: \( w(n_s, n_g) = dy^{-2}(n_s, n_g) \), where \( dy \) is the error of \( y = \ln m/m_0 \). The coefficients are \( a_1 = -2.31(20) \), \( a_2 = 7.99(32) \), \( a_3 = 2.85(58) \), and \( a_4 = -14.73(94) \).

Notice that

\[
a_1n_s n_g + a_2n_g + a_3n_s + a_4 = (a_1n_g + a_3)(n_s + \frac{a_2}{a_1}) + (a_4 - \frac{a_2a_3}{a_1}), \tag{3}
\]

a more convenient approach is to redefine \( m_0 \) and then fit mass using \( (c_1n_g + c_2)(n_s + c_3) \). From Eqn. 3 we let \( m_0 \) satisfy \( \ln \frac{m_0}{1 Gev} = a_4 - \frac{a_2a_3}{a_1} = -4.886 \), or \( m_0 (m_Z) = 7.55 Mev \). Data of \( \ln m/m_0 \) have been shown in Table 4.

| \( n_g = 1 \) | \( n_g = 2 \) | \( n_g = 3 \) |
|---|---|---|
| \( n_g = 1 \) | -1.375(538) | 4.372(167) | 10.041(30) |
| \( n_g = 2 \) | -0.608(286) | 2.191(304) | 5.965(59) |

Table 4: \( y(n_g, n_s) = \ln m/m_0 \) at scale \( m_Z \), where \( m_0 = 0.00755 Gev \).
Using the least squares method, that is, minimizing the function

$$\sum_{n_g=1,2,3} (y(n_g, n_s) - (c_1n_g + c_2)(n_s + c_3))^2w(n_s, n_g),$$  \hspace{1cm} (4)

one obtains $c_1 = -2.311(25)$, $c_2 = 2.848(28)$ and $c_3 = -3.459(35)$. In fact, one can use a more symmetrical form, $c_1(n_g + c'_2)(n_s + c_3)$, where $c'_2 = c_2/c_1$, to fit $y(n_g, n_s)$. Notice $c_3 \simeq 3c_2/c_1$, one can furthermore reduce parameters $c_1, c_2, c_3$ into two parameters $c_1, c_2$, if he performs constraint $c_3 = 3c_2/c_1 = 3c'_2$. We did not do it here. Quark masses and the fitting formula are plotted in Fig. 1. This empirical formula can be compared with the results introduced in ref. \[1\].

To discuss the quality of fitting in Eqn. \[4\], we study modified coefficient of determination for statistics:

$$R^2 = 1 - \frac{SSE/N_E}{SST/N_T},$$ \hspace{1cm} (5)

In Eqn. \[5\] $N_E = 6 - 3 - 1 = 2$ is the degree of sum squared error (SSE), $SSE = \sum_{n_s,n_g} (y(n_g, n_s) - \hat{y}(n_g, n_s))^2$, where $\hat{y}$ is the fitting value of $y$, $N_T = 6 - 1 - 1 = 4$ is the degree of total sum of squares (SST), $SST = \sum_{n_s,n_g} (y(n_g, n_s) - \bar{y})^2$, where $\bar{y}$ is the average of $y$: $\bar{y} = \frac{1}{n_sn_g} \sum_{n_g} y(n_g, n_s)$. The additional subtraction of 1 is due to subtraction constance $m_0$.

In other literature $R^2$ is written as $\bar{R}^2$.

Generally, $0 \leq R^2 \leq 1$. One obtains here that $R^2 = 0.99584$, which is very close to 1. This roughly means that, about 99.6 percent of the mass statistics can be interpreted by this empirical fitting. Or, the part which can not been interpreted by the fitting is no more than 1%. We conclude that this fitting is a quite good empirical formula.

All the calculations given above depend on the coupling $\alpha_s(m_Z)$. But, $\alpha_s(m_Z)$ itself is determined by experiment and it also has error. Therefore, it is interesting to study the fitting behavior at different $\alpha_s(m_Z)$. Due to the experiment interesting, we vary $\alpha_s(m_Z)$ from 0.09 to 0.13 here. We repeat all the calculations, taking Tab. 1 as input, and then study the behavior of modified coefficient of determination $R^2$ for the fitting formula \[4\], \[5\]. The results are shown in Fig 2. One just finds that $R^2$ approaches its maximum at the range $0.115 < \alpha_s(m_Z) < 0.12$. We call this coupling as best fitting coupling $\alpha_s^b$, where the fitting formula \[4\] works best. We use $d\alpha_s^b = \sqrt{\sum_{n_s,n_g} (\partial \alpha_s^b/\partial m(n_s, n_g))^2 dm(n_s, n_g)}$, where $m(n_s, n_g)$ is the data in Tab. 1 and $\partial \alpha_s^b/\partial m(n_s, n_g)$ can be extracted by a small shifting of $m(n_s, n_g)$, to estimate error of $\alpha_s^b$, $d\alpha_s^b$. The result is $\alpha_s^b = 0.118(31)$. The big error is mainly due to the errors of $u, d, s$ quarks masses.

We see that the measured coupling is very close to the best fitting coupling. Here is a possible interpretation. As we know, the masses in Tab. 1 are mean ones, which are extracted by connecting various theories and experiments. In this sense, we say that the data in Tab. 1 is unprejudiced estimation of the true masses of quarks, as long as we have performed enough estimations. Suppose QCD and renormalization theory are both right theories and remember that the mean measured coupling $\alpha_s(m_Z) = 0.117$ is just the estimation of the true coupling $\alpha_s^t$. We conclude that, if fitting equation \[4\] is a right or an approximate right behavior, $\alpha_s^b$ should also be a estimation to $\alpha_s^t$, although $\alpha_s^t$ itself depends on the level of loops calculations. On one hand, if the behavior of $y(n_g, n_s)$ is complete random or is not linear at

\[1\] Here, for simplification, we just make an assumption that the data in Tab. 1 is irrelevant with $\alpha_s(m_Z)$. 

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all, or in other words, the fitting is not a correct one, one should obtain two bad results, one is that $R^2$ is not so close to 1, the other is that, generally, $\alpha_s^b \neq \alpha_t^s$, unless by chance. Since $R^2$ is very close to unitary, we expect the equation is a right or an approximate right behavior of quarks. On the other hand, when one says that the fitting equation is a right or an approximate right behavior, he always implies that this statement is obtained at correct coupling, $\alpha_t^s$. This means that, when the coupling deviates away $\alpha_t^s$, the fitting will go to bad. Or, in other words, the $\alpha_t^s$ should be equal to the best coupling, $\alpha_b^s$, provided one takes correct mass input in Tab. 1. Therefore, $\alpha_b^s$ is also an estimation of $\alpha_t^s$. It is understood that $\alpha_b^s \simeq \alpha_s$.

From Fig. 1 and $R^2$ check, the linear fitting is quite good agreement with the experiment data. $s$ quark lies a little below the fitting line, (or on the contrary, $d$ quark lies a little above the fitting line,) which may be considered as the correction due to QED and statistic error. In fact, if one uses linear fitting to fit the masses of leptons, which have $|Q| = 1$, the experiment of $\mu$ lepton should lies above the fitting line (Since the lepton does not enjoy strong interaction, we do not discuss the fitting for lepton masses in detail here).

If the logarithm of quark masses does have linear behavior, there are some interesting inferences immediately.

For instance, one can use this formula to extract masses of four generator quarks. For the heavier quark $t'$, we get $m_{t'}(m_Z) = 51(17)TeV$, which is beyond our experiment capability. But the searching of lighter quark $b'$ is not beyond our experiment capability. In fact, using extraction of the linear fitting, we obtain $m_{b'}(m_Z) = 85(23)Gev$, which corresponds to the pole mass $m_{b'}^{pole} = 91(25)Gev$. In other words, $m_{b'}^{pole} \approx m_Z$ dramatically.

It is also a puzzle that whether the fourth-generate quarks exist or not. Until nowadays, we did not find the fourth-generate quarks. But since $m_{b'}^{pole} \approx m_Z$, one should check the data at the vicinity around $m_Z$ more carefully. According to reference, the mass gap between the fourth-generate quarks is so large that possibly there is no fourth-generate quark at all.

It is hard to understand why top quark has so large mass in SM. In some seesaw mechanism, for instance, the source of top quark is significant different with that of other quarks and therefore the formula of top quark is also different with that of other quarks. But we see here that the mass of top quark extracted from the linear behavior of the logarithm of quark mass agrees well with the measured ones. This implies that the top quark is also a "common" quark and the masses of top quark and all the other known quarks share the same source, although $m_t >> m_Z$. We expect this is helpful to understand the source of particle masses and correct theory of flavor.

In some references, the vanishing of up quark mass is used to solve strong CP puzzle in QCD. However, if the fitting equation is right or approximate right, the mass of up quark is never vanishing. This means that the solution of CP broken in QCD is not the vanishing of up quark mass. It is possibly from other mechanism, for instance, U(1) symmetry.

At last, the subtract mass is $m_0(m_Z) = 7.55MeV$, which corresponds $m_0(m_0) = 428MeV$, or roughly equals to constituent light quarks mass.

In summary, the significant agreement of equation shows that the masses of known quarks are never random. Therefore, equation should be included in the full theory. We expect this equation should give some clue of full theory, such as the source of masses or flavor physics.

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Figure 1: $\ln\left(\frac{m_{nq}}{m_0}\right)$ vs. $n_q$, where $n_q$ stands for the generator of quarks while $m_0$ is pointed out in Tab. 4. The dashing line is $\ln\left(\frac{m_z}{m_0}\right)$.

Figure 2: $R^2$ vs. $\alpha_s(m_Z)$. 