Periodic solutions for a 1D-model with nonlocal velocity via mass transport

Lucas C. F. Ferreira *
Universidade Estadual de Campinas, Departamento de Matemática,
CEP 13083-859, Campinas-SP, Brazil.
E-mail:lcff@ime.unicamp.br

Julio C. Valencia-Guevara
Universidade Estadual de Campinas, Departamento de Matemática,
CEP 13083-859, Campinas-SP, Brazil.
E-mail:ra099814@ime.unicamp.br

Abstract

This paper concerns periodic solutions for a 1D-model with nonlocal velocity given by the periodic Hilbert transform. There is a rich literature showing that this model presents singular behavior of solutions via numerics and mathematical approaches. For instance, they can blow up by forming mass-concentration. We develop a global well-posedness theory for periodic measure initial data that allows, in particular, to analyze how the model evolves from those singularities. Our results are based on periodic mass transport theory and the abstract gradient flow theory in metric spaces developed by Ambrosio et al. [2]. A viscous version of the model is also analyzed and inviscid limit properties are obtained.

AMS MSC2010: 35Q35; 76B03; 35L67; 35A15; 35K15
Keywords: Nonlocal fluxes; Periodic solutions; Gradient flows; Optimal transport; Inviscid limit

*L. Ferreira was supported by FAPESP and CNPQ, Brazil. (corresponding author)
1 Introduction

We consider the following one-dimensional model

\[ u_t + (H(u)u)_x = 0 \]  

(1.1)

with the initial condition \( u(x, 0) = u_0 \), where \( H \) stands for the periodic Hilbert transform

\[ H(u)(x) = \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \cot\left(\frac{x - y}{2}\right) u(y) dy, \]  

(1.2)

and the unknown \( u : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) is 2\( \pi \)-periodic in the spatial variables. The viscous version of (1.1) is also studied. In the non-periodic case, (1.2) should be changed to the continuous Hilbert transform

\[ H_c(u)(x) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(y)}{x - y} dy. \]  

(1.3)

The model (1.1) is a continuity equation with non-local velocity field and has a strong analogy with some physical models; for instance, 2D inviscid quasi-geostrophic equation (see [11]) and 2D vortex sheet problems (see [4]). It also appears in modeling of dislocation dynamics in crystals where \( u \geq 0 \) stands for the density of defects in the material (see [16],[26],[5]). Applying the Hilbert transform over (1.1), and computing the \( x \)-derivative, the resulting equation can be used, in a first approximation, to study the dynamics of the interface between two fluids; one governed by Stokes equations and other by Euler equations (see [10, Appendix A]). For other examples of 1D-equations appearing as models for PDEs defined in higher dimensions, we refer the reader to [12],[13],[22],[25] and their references.

In particular, the non-conservative variant of (1.1)

\[ u_t + H(u)u_x = 0 \]  

(1.4)

and its viscous versions have been studied by several authors via mathematical fluid mechanics arguments, see [13],[14],[19],[17] and their references.

Beyond that, (1.1) has a mathematical interest of its own due to its non-local structure and singular behavior with respect to existence of global solutions. For instance, in comparison with (1.4), a difficulty of handling (1.1) is the lack of maximum principle for the \( L^\infty \)-norm. In fact, in [11], the authors showed there is no global periodic solutions of (1.1) in \( C^1([\pi, \pi] \times [0, \infty)) \) for \( u_0 \in C^1([\pi, \pi]) \) with \( \int_{-\pi}^{\pi} u_0(x) dx = 0 \) and \( u_0 \not\equiv 0 \). If, instead, one assumes

\[ \int_{-\pi}^{\pi} u_0(x) dx \geq 0 \text{ and } \min_x u_0(x) < 0 \]  

(1.5)

then the \( C^1 \)-breakdown still holds true. The authors of [10] considered the non-periodic version of (1.1) and showed local well-posedness of nonnegative \( H^2(\mathbb{R}) \)-solutions. These develop a finite time singularity provided that there is a \( x_0 \in \mathbb{R} \) such that \( u_0(x_0) = 0 \). For \( 0 < \delta < 1 \) and strictly positive \( u_0 \in L^2(\mathbb{R}) \cap C^{1,\delta}(\mathbb{R}) \) vanishing at infinity, they showed there
is a unique global solution $u \in C([0, \infty); L^2 \cap C^{1,\delta})$ for (1.1) and a version of it with viscous term $-\nu \mathcal{H}_c(u_x)$. Considering the fractional viscosity $(-u_{xx})^{\alpha/2}$, the paper [18] extended some results in [10] by showing blow up of smooth solutions for $0 \leq \alpha \leq 2$ and smooth positive initial data. Moreover, we prove that solutions of (1.1) with an opposite sign but solutions correspond easily by changing $\pi \int_0^T \rho(x, t) dt = 1$.

In fact, they showed that $u$ converges in $D'([0, T^*)$) as $t \to T^*$ to the periodic measure

$$
\mu_a = a + \sum_{n \in \mathbb{Z}} \delta_0(x - 2\pi n),
$$

where $a$ is a constant and $\delta_0$ is the Dirac delta distribution. In other words, by standard periodic identification, $u(\cdot, t) \to a + \delta_0([-\pi, \pi])$ in $D'([0, T^*)$ as $t \to T^*$. For $\nu = 0$ and a positive data $u_0 \in C^\infty(\mathbb{T})$, blow up of $L^\infty(\mathbb{T})$-norm was proved in [4] which also indicates a concentration of mass due to sign-preservation and mass-conservation for solutions of (1.1). Taking data in the form

$$
u u_0(x) = a_0 + a_1 \cos(x)
$$

where $|a_1| > \nu \geq 0$ and $a_0 \neq 0$, solutions with $L^2(\mathbb{T})$-norm blowing up at a finite time were obtained in [30, p.157]. These solutions can (or not) be nonnegative according to the choice of the parameters $a_1$ and $a_0$.

The above results corroborate the singular feature of (1.1) and, in particular, show that solutions can exhibit mass concentration. So, it is natural to wonder about a framework in which solutions could continue after the blow-up time and how the PDE evolves from singular data. We consider (1.1) with $\nu = 0$ and $\nu > 0$ both for $u_0$ belonging to the set of periodic probability measures $\mathcal{P}(S^1)$ endowed with the periodic Wasserstein metric (see [3],[24],[15]). For all initial data $u_0 \in \mathcal{P}(S^1)$, solutions converge towards a stationary state as $t \to \infty$, which is the unique minimum for the associated energy functional. Moreover, we prove that solutions of (1.6) with $\nu > 0$ converge in $\mathcal{P}(S^1)$ to those of (1.1) when $\nu \to 0^+$ (inviscid limit). In view of the mass-conservation property, notice that (by making a normalization) the constraint $\int_{-\pi}^\pi u_0 dx = 1$ is not an essential one.

We also point out that the evolution of (1.1) from initial measures may be of interest due to its connection with some problems involving 2D vortex sheet which is in a layer of vorticity distributed as a delta function on a curve. In [11], an explicit formula for solutions of (1.6) with $\nu > 0$ was obtained by using the Hopf-Cole transform and complex Burgers equation. To do this, it is necessary to have $\mathcal{H}(u_0)$ at least belonging to $L^1_{loc}$ what is not verified for a general $u_0 \in \mathcal{P}(S^1)$; for instance, $\mathcal{H}(\delta_0) \notin L^1_{loc}(-\delta, \delta)$, for all $0 < \delta < \pi$.

Formally, the PDEs (1.1) and (1.6) can be rewrite as a continuity equation

$$
\partial_t u(x, t) = \nabla \cdot (\nu(x, t) u(x, t))
$$

(1.9)
with velocity field \( \mathbf{v} = \nabla \delta E / \delta \mathbf{u} \) given by the gradient of the variational derivative of the corresponding free energy functional (see Section 2 for details). Equations in this form have the so-called gradient-flow structure (see [2]) and their solutions can be obtained by means of an interactive variational scheme based on optimal transport theory and properties of \( E \), what goes back to the seminal work [27] for the linear Fokker-Planck equation. Roughly speaking, the basic idea is to construct solutions that follows the direction of steepest descent of the energy functional in a probability measure space endowed with a suitable metric. In the non-periodic setting, an appropriate space is \( \mathcal{P}_2(\mathbb{R}^d) \) (probability measures with finite second moments) endowed with the 2-Wasserstein distance.

In fact, the above approach has been applied in \( \mathcal{P}_2(\mathbb{R}^d) \) to several equations (see [1],[8],[31]) and an abstract theory has been developed to a general class of continuity equation in \( \mathbb{R}^d \) with energy functional

\[
E[u] := \int_{\mathbb{R}^d} U(u(x)) \, dx + \int_{\mathbb{R}^d} u(x) V(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) u(x) u(y) \, dx \, dy, \quad (1.10)
\]

where the terms \( U : \mathbb{R}^+ \to \mathbb{R}, V : \mathbb{R}^d \to \mathbb{R} \) and \( W : \mathbb{R}^d \to \mathbb{R} \) are a density of internal energy, a confinement potential and an interaction potential, respectively (see e.g. [2],[7]). The concept of displacement convexity for \( E \) introduced in [28] (more generally, \( \lambda \)-convexity) plays a core role in the theory, as well as lower semicontinuity and coercivity properties of \( E \). More precisely, in [2, Chapter 4], a gradient-flow theory is developed for (1.9) in general metric spaces by assuming these properties for an abstract functional \( E \). An important example is the Wasserstein metric space \( \mathcal{P}_2(X) \) where \( X \) is a separable Hilbert space. This general theory was successfully used in [2, Chapter 11, p. 298-303] to study some PDEs in \( X = \mathbb{R}^d \) with functionals having the concrete form (1.10) and satisfying certain smooth and growing conditions on \( U, V, W \).

In [6], the authors analyzed (1.1) and (1.6) in the non-periodic case where the interaction potential \( W(x) = -1/\pi \log |x| \) is singular at origin. There, by employing the results from [2, Chapter 11], a gradient-flow solution in \( \mathcal{P}_2(\mathbb{R}^d) \) was obtained after making a self-similar change of variables in the equations and proving the needed properties for \( E \). This change of variables generates a confinement term \( V(x) = \frac{|x|^2}{2} \) that helps to control the lack of boundedness from below of the interaction part of \( E \). The asymptotic behavior of solutions in [6] is described by a self-similar one while here the dynamics is attracted to a unique stationary solution.

Let us also comment about motivation from a general point of view. Periodic solutions are widely studied in PDE-theory and appear naturally in several physical phenomena, specially in fluid mechanics. So, it is important that different approaches get to deal with this kind of solution. In this direction, our results seem to be the first construction of space-periodic gradient flows in the context of fluid mechanics. In fact, there are a few works dealing with existence of periodic solutions for PDEs via optimal mass transport. For instance, we would like to mention the papers [9], [3] and [24]. In the former, the authors analyzed the family of
first-order displacement-convex functionals in $\mathcal{P}(S^1)$

$$E(\rho) = \int_0^1 \left[ \left( \frac{1}{\rho^\beta} \right)_x \right]^2 \, dx, \text{ for } \beta \in [1, \frac{3}{2}], \quad (1.11)$$

whose associated gradient flows are periodic weak-solutions of a class of fourth-order degenerate parabolic equations. In [3], existence of Eulerian distribution solutions for semi-geostrophic equations was obtained by using regularity and stability properties found in [20, 21] for Alexandrov solutions of the Monge-Ampere equation and optimal mass transport in $T^2$. The authors of [24] developed a weak KAM theory in $\mathcal{P}(T^d)$ with $d \geq 1$ and obtained existence and asymptotic behavior of solutions for the nonlinear Vlasov system.

Another motivation is that objects like (1.7) and (1.8) do not belong to $\mathcal{P}_2(\mathbb{R})$ and then they are not covered by the results in [6]. Moreover, it is worthy to mention that periodic conditions prevent the use of the self-similar change employed by [6]. So, we need to handle the singular interaction potential of (1.1) and (1.6) in original variables in order to obtain the key properties for $E$ and carry out in $\mathcal{P}(S^1)$ the general theory in metric spaces of [2, Chapter 4].

In what follows, we comment on some technical difficulties. Unlike when $X$ is Hilbert, the sphere $S^1$ is not a convex set and then a displacement interpolation curve like $((1 - t)H_1 + tH_2)_{\#}\mu$ with $H_i : S^1 \to S^1$ could not be well-defined in $\mathcal{P}(S^1)$. In Section 3.3, we work with a concept of generalized geodesic in $\mathcal{P}(S^1)$ as curves of equivalence classes (see Definition 3.2 and Remark 3.3). By using this, we define and show a type of convexity for functionals. In particular, in spite of the cost $d^2_{\text{per}}(x, \cdot)$ in (2.1) is not convex, we show the 2-convexity of the square of the periodic Wasserstein distance $d^2_{\text{per}}(\mu, \cdot)$ (see Lemma 3.9), which is essential for the convergence of the steepest descent scheme (4.2)-(4.3). This is obtained by employing the equivalent representations $\mathcal{P}(S^1)$ and $\mathcal{P}_2(\mathbb{R})/\sim$ (see (2.1) and (2.2)) and the identity (2.5). Also, in Lemma 2.4, we prove a certain invariance property for $\int_{\mathbb{R}^d} U(u(x)) \, dx$ (where $U(0) = 0$) with respect to the equivalence relation (2.2). This is key in the proof of the convexity of the entropy functional $U[\mu] = \int_{[-\pi, \pi]} u \log u \, dx$ insofar as it assures the invariance of the integral (3.3) for elements of an equivalent class in $\mathcal{P}_2(\mathbb{R})/\sim$ supported in some interval of the type $[a, a + 2\pi)$. Similarly, in order to obtain convexity of the interaction functional $\int \int_{[-\pi, \pi]^2} W(x - y)du(x)du(y)$, we need an invariance property that is stated in Remark 2.3.

Furthermore, to our knowledge, there is no stability result in general metric spaces for solutions obtained via the theory of gradient flows found in [2, Chapter 4]. The authors of [6] employed a stability result of [2, Chapter 11] in the space $\mathcal{P}_2(X)$ where $X$ is Hilbert. Since this is not the case of $S^1$, it is necessary to obtain a version of such results for the periodic setting (see Theorem 4.3). In fact, the periodic condition allows to perform a proof more direct than that in [2, Chapter 11] and could be extended to study stability of gradient flows in $\mathcal{P}(S^1)$ generated by a family of general functionals $\{E_\alpha\}$ under relatively simpler conditions.

The plan of this paper is as follows. In Section 2, we summarize some facts about optimal mass transport in $S^1$ and present the gradient-flow structure of (1.1) and (1.6) in a more
detailed way. Section 3 is devoted to prove key properties of the free energy functional $E$. Finally, global well-posedness of gradient-flow solutions in $\mathcal{P}(\mathbb{S}^1)$ and inviscid limit are proved in Section 4.

2 Mass transport and Gradient-Flow Structure

2.1 Mass Transport in $\mathbb{S}^1$

In this section, we resume the theory of optimal transport relevant for our purposes. The circle $\mathbb{S}^1$ is considered as the quotient space $\mathbb{R}/2\pi\mathbb{Z}$ and functions on $\mathbb{S}^1$ are considered as $2\pi$-periodic functions in $\mathbb{R}$. The space $C^r(\mathbb{S}^1)$ stands for the set of $2\pi$-periodic functions of class $C^r$, for $r \geq 0$. In the case $r = 0$, we denote $C^0(\mathbb{S}^1)$ by $C(\mathbb{S}^1)$. Also, $\mathbb{S}^1$ will be identified with the interval $[-\pi, \pi]$ whenever convenient.

We denote by $\mathcal{P}(\mathbb{S}^1)$ the space of periodic probability measures endowed with the periodic 2-Wasserstein distance

$$d^2_{\text{per}}(\mu, \rho) = \inf \left\{ \int_{\mathbb{S}^1 \times \mathbb{S}^1} d^2_{\text{per}}(x, y) \, d\gamma(x, y) : \gamma \in \Gamma(\mu, \rho) \right\},$$

where $\Gamma(\mu, \rho)$ stands for the set of probability measures with marginals $\mu$ and $\rho$, and $d_{\text{per}}$ denotes the geodesic distance in $\mathbb{S}^1$. The subspace of absolutely continuous measures in $\mathcal{P}(\mathbb{S}^1)$ is denoted by $\mathcal{P}_{\text{ac}}(\mathbb{S}^1)$. From [29], for $\mu, \rho \in \mathcal{P}(\mathbb{S}^1)$ with $\mu \in \mathcal{P}_{\text{ac}}(\mathbb{S}^1)$, there exists an optimal transport map $t^\mu : \mathbb{S}^1 \to \mathbb{S}^1$ for the Monge problem with quadratic distance cost. Indeed it is possible to show that $t^\mu$ exists if $\mu$ does not give mass on points (see the argument below), i.e., $\mu$ has no atoms.

Following [9], we can consider the application $t^\nu$ from $[-\pi, \pi]$ to $\mathbb{S}^1$. Define the application $t^\nu_\tilde{\mu} : [-\pi, \pi] \to [-\pi, 3\pi]$ given by: $t^\nu_\tilde{\mu}(x)$ is the smallest element in the equivalence class $t^\nu_\mu(x)$ such that $|t^\nu_\tilde{\mu}(x) - x| \leq \pi$. In this case, the geodesic distance of $x$ and $t^\nu_\tilde{\mu}(x)$ coincides with the Euclidean one between $x$ and $t^\nu_\tilde{\mu}(x)$. So, considering Euclidean quadratic cost, if $\tilde{\rho} := t^\nu_\tilde{\mu} \# \mu$ then $t^\nu_\mu$ is the optimal transport map between $\mu$ and $\tilde{\rho}$. Thus, $t^\nu_\mu$ is monotone and

$$t^\nu_\mu(\pi) = 2\pi + t^\nu_\mu(-\pi) =: 2\pi + a.$$

It follows from monotonicity that $t^\nu_\mu([\pi, \pi]) \subset [a, a + 2\pi]$. In short, we can think in optimal transports as maps from $[-\pi, \pi]$ to $[a, a + 2\pi]$.

Let us also comment on another way to see the space $\mathcal{P}(\mathbb{S}^1)$. Recently, in analogy with the construction of the torus as a quotient space, the authors of [24] defined an equivalence relation in $\mathcal{P}_2(\mathbb{R})$ by setting

$$\mu \sim \rho \iff \int_\mathbb{R} \zeta d\mu = \int_\mathbb{R} \zeta d\rho, \forall \zeta \in C^0(\mathbb{S}^1).$$
Given $\mu \in \mathcal{P}_2(\mathbb{R})$, there exists a $\hat{\mu}$ equivalent to $\mu$ concentrated in $[-\pi, \pi)$. For that, just take the pushforward of $\mu$ by the map that sends $x \in \mathbb{R}$ to the (unique) element of $[x] \cap [-\pi, \pi)$ (here $[x]$ denotes the equivalence class in $\mathbb{R}/2\pi\mathbb{Z}$). Indeed $\hat{\mu}$ is the unique representative of $[\mu]$ with these properties. In the sequel, they showed a relation between the metrics of $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})/\sim$ similar to that between $\mathbb{R}$ and $\mathbb{S}^1$. Indeed, if $d_2$ is the Wasserstein metric in $\mathcal{P}_2(\mathbb{R})$ then

$$d^2_{\text{per}}(\mu, \rho) = \min\{d^2_2(\mu, \rho^*) : \rho \sim \rho^*\}$$

(2.3)$$= \min\{d^2_2(\mu^*, \rho^*) : \mu \sim \mu^* \text{ and } \rho \sim \rho^*\}. \quad (2.4)$$

We observe that the minimum in (2.3) is reached with the map $\tilde{\mu}^\rho$ (built above), when $\mu$ is supported in $[-\pi, \pi)$. In fact, one can check that $\tilde{\rho} \sim \rho$ and

$$d^2_2(\mu, \tilde{\rho}) \leq \int_{[-\pi, \pi]} |x - \tilde{\mu}^\rho(x)|^2 \, d\mu(x)$$

$$= \int_{[-\pi, \pi]} d^2_{\text{per}}(x, \tilde{t}_\mu^\rho(x)) \, d\mu(x)$$

$$= d^2_{\text{per}}(\mu, \rho),$$

as desired. By (2.3), there exist $\gamma \in \mathcal{P}_2(\mathbb{R}^2)$ and $\rho^* \in \mathcal{P}_2(\mathbb{R})$ such that

$$d^2_{\text{per}}(\mu, \rho) = \int_{\mathbb{R}^2} |x - y|^2 \, d\gamma(x, y), \gamma \in \Gamma(\mu, \rho^*) \text{ and } \rho^* \sim \rho. \quad (2.5)$$

Finally, since $\mathbb{S}^1$ is compact, let us remark that the weak topology (narrow) in $\mathcal{P}(\mathbb{S}^1)$ coincides with that induced by the $p$-Wasserstein metric. In particular, by Prokhorov lemma, $\mathcal{P}(\mathbb{S}^1)$ is a compact metric space. The above two ways of seeing the space $\mathcal{P}(\mathbb{S}^1)$ will be exploited by us throughout the paper.

### 2.2 Geodesics with Constant Velocity in $\mathcal{P}(\mathbb{S}^1)$

In this section, we consider the elements of $\mathbb{S}^1$ as equivalence classes $[x]$. In what follows, we recall the definition of a type of geodesic in metric spaces.

**Definition 2.1.** Let $(X, d)$ be a metric space. A curve $\varphi : [0, 1] \to X$ is called a geodesic with constant velocity if

$$d(\varphi(s), \varphi(t)) = |t - s|d(\varphi(0), \varphi(1)) \quad \forall s, t \in [0, 1].$$

Here we present an explicit construction of a geodesic with constant velocity connecting two arbitrary measures in $\mathcal{P}(\mathbb{S}^1)$. Define the multivalued map $g_t : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ given by $g_t([x], [y]) = [(1 - t)\bar{x} + t\bar{y}]$, where

$$(\bar{x}, \bar{y}) \in \text{Argmin}\{|\bar{x} - \bar{y}| : \bar{x} \in [x], \bar{y} \in [y]\}.$$
Note that $g_t$ is a function well defined in the set
$$\{(x, y) \in S^1 \times S^1 : \text{d}_{\text{per}}([x], [y]) < \pi\}.$$ If $d_{\text{per}}([x], [y]) = \pi$ then we can redefine $g_t$ by requiring that $\hat{y} < \hat{x}$. It follows that
$$d_{\text{per}}(g_t([x], [y]), g_s([x], [y])) = |t - s|d_{\text{per}}([x], [y]), \text{ for all } s, t \in [0, 1].$$

Given $\mu_0, \mu_1 \in \mathcal{P}(S^1)$ and $\gamma \in \Gamma(\mu_0, \mu_1)$ an optimal plan for $d_{\text{per}}(\mu_0, \mu_1)$, we define
$$\mu_t := g_t \# \gamma, \quad t \in [0, 1].$$

(2.6)

**Proposition 2.2.** Let $\mu_0, \mu_1 \in \mathcal{P}(S^1)$. Then, the curve defined in (2.6) is a geodesic with constant velocity with respect to the Wasserstein metric in $\mathcal{P}(S^1)$.

**Proof.** Define $\gamma_{t,s} = (g_t, g_s) \# \gamma \in \Gamma(\mu_t, \mu_s)$. It follows that
$$d^2_{\text{per}}(\mu_t, \mu_s) \leq \int_{S^1 \times S^1} d^2_{\text{per}}(g_t([x], [y]), g_s([x], [y])) \, d\gamma$$
$$= (t - s)^2 \int_{S^1 \times S^1} d^2_{\text{per}}([x], [y]) \, d\gamma,$$
and then $d_{\text{per}}(\mu_t, \mu_s) \leq |t - s|d_{\text{per}}(\mu_0, \mu_1)$. If there exist $s, t \in [0, 1]$ such that $s < t$ and the inequality in (2.7) is strict, then
$$d_{\text{per}}(\mu_0, \mu_1) < s d_{\text{per}}(\mu_0, \mu_1) + |t - s|d_{\text{per}}(\mu_0, \mu_1) + (1 - t)d_{\text{per}}(\mu_0, \mu_1)$$
$$= d_{\text{per}}(\mu_0, \mu_1),$$
which gives a contradiction. Therefore, we have indeed an equality in (2.7), as required. 

2.3 Gradient-Flow Structure

Formally, we can write (1.1) and (1.6) as
$$w_t = \left[ u \left( \nu \frac{u_x}{u} - \mathcal{H}(u) \right) \right]_x$$
$$= \left[ u \left( \nu \log u - \frac{1}{\pi} \log |\sin(x/2)| \ast u \right) \right]_x, \quad \text{for } \nu \geq 0.$$ 

(2.8)

Since we are looking for solutions in $\mathcal{P}(S^1)$, equation (2.8) suggests to define the interaction kernel as
$$W(x) = \begin{cases} -\frac{1}{\pi} \log |\sin(x/2)| & \text{if } x \in [-\pi, \pi), \ x \neq 0; \\ \infty & \text{if } x = 0; \\ W(x + 2\pi) = W(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

(2.9)
Now consider the free energy functional $\mathcal{F}_\nu : \mathcal{P}(S^1) \to (-\infty, \infty]$ defined in the following way:

$$\mathcal{F}_\nu[\mu] = \nu \int_{[-\pi, \pi)} \mu \log \mu dx + \int \int \left[-\pi, \pi\right)^2 W(x-y) d\mu(x) d\mu(y)$$

$$=: \nu \mathcal{U}[\mu] + \mathcal{F}_0[\mu], \text{ for all } \mu \in \mathcal{P}_{ac}(S^1) \text{ and } \nu > 0. \quad (2.10)$$

Here we are identifying an absolutely continuous measure with your density with respect to Lebesgue measure. For $\mu \in \mathcal{P}(S^1) \setminus \mathcal{P}_{ac}(S^1)$ and $\nu > 0$, $\mathcal{F}_\nu[\mu] = \mathcal{U}[\mu] = \infty$. In the case $\nu = 0$ we simply define the functional as

$$\mathcal{F}_{\nu=0}[\mu] = \mathcal{F}_0[\mu], \text{ for all } \mu \in \mathcal{P}(S^1). \quad (2.11)$$

We recall that the domain of the functional, denoted by $D(\mathcal{F}_\nu)$, is defined as the set

$$D(\mathcal{F}_\nu) = \{ \mu \in \mathcal{P}(S^1); \mathcal{F}_\nu[\mu] < \infty \}. \quad (2.12)$$

So, we can write (2.8) in the form (1.9) with $\nu = \frac{d\mathcal{F}_\nu}{du}$, that is

$$u_t = \left[ u \left( \frac{\delta \mathcal{F}_\nu}{\delta u} \right) \right]_x \quad (2.13)$$

which has the structure of gradient flow in $\mathcal{P}(S^1)$ corresponding to the energy functional $\mathcal{F}_\nu$.

**Remark 2.3.** Let us remark that if we adopt the interpretation of $\mathcal{P}(S^1)$ as $\mathcal{P}_2(\mathbb{R})/\sim$, the definition of $\mathcal{F}_\nu$ is as follows:

(i) For the entropy part of $\mathcal{F}_\nu$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, we consider the unique $\mu^* \sim \mu$ such that $\mu^*$ is supported in $[-\pi, \pi)$ and define $\mathcal{U}[\mu] := \mathcal{U}[\mu^*]$ when $\mu \in \mathcal{P}_{ac}(S^1)$, and $\mathcal{U}[\mu] = \infty$ otherwise.

(ii) For the interaction part, first observe that

$$\int \int_{\mathbb{R}^2} W(x-y) d\mu(x) d\mu(y) = \int \int_{\mathbb{R}^2} W(x-y) d\mu^*(x) d\mu^*(y),$$

for any $\mu \sim \mu^*$ in $\mathcal{P}_2(\mathbb{R})$. In fact, this equality follows by approximating the kernel $W$ monotonically from below by periodic continuous functions and then applying the monotone convergence theorem. Therefore, we define the interaction functional by

$$\mathcal{F}_0[\mu] = \int \int_{\mathbb{R}^2} W(x-y) d\mu^*(x) d\mu^*(y).$$

Connected to item (i) in the previous remark, let us show some kind of invariance for the entropy functional $\mathcal{U}[\mu]$. 

9
Lemma 2.4. Let \( U : [0, \infty) \rightarrow \mathbb{R} \) with \( U(0) = 0, \mu, \rho \in \mathcal{P}_{ac}(\mathbb{R}) \) such that \( \mu \sim \rho, \mu \) is supported in \([a, a + 2\pi)\) and \( \rho \) is supported in \([b, b + 2\pi)\). Then, if \( f \) and \( h \) are the densities of \( \mu \) and \( \rho \) respectively, we have \( U \circ f \in L^1(\mathbb{R}, dx) \) if and only if \( U \circ h \in L^1(\mathbb{R}, dx) \) and
\[
\int_{[a,a+2\pi)} U \circ f \, dx = \int_{(b,b+2\pi)} U \circ h \, dx.
\]

Proof. Without loss of generality, we can assume that \( \mu \) is supported in \([0, 2\pi)\). Define the function \( T : [b, b + 2\pi) \rightarrow [0, 2\pi) \) by
\[
T(x) = \begin{cases} 
  x - 2\pi \left\lfloor \frac{b}{2\pi} \right\rfloor & \text{if } b \leq x < 2\pi \left(1 + \left\lfloor \frac{b}{2\pi} \right\rfloor \right); \\
  x - 2\pi \left(1 + \left\lfloor \frac{b}{2\pi} \right\rfloor \right) & \text{if } 2\pi \left(1 + \left\lfloor \frac{b}{2\pi} \right\rfloor \right) \leq x < b + 2\pi,
\end{cases}
\]
where \( \lfloor \cdot \rfloor \) stands for the greatest integer function. Then, clearly \( T \) is bijective and it is straightforward to check that \( T^* \rho \sim \rho \) and \( T^* \rho \) is supported in \([0, 2\pi)\). Since there exists a unique representative equivalent to \( \rho \) supported in \([0, 2\pi)\) (see Section 2.1), we conclude that \( T^* \rho = \mu \). Let \( f \) and \( h \) be the densities of \( \mu \) and \( \rho \), respectively. Then, for any \( \zeta \in C(S^1) \), we have
\[
\int_{[0,2\pi]} \zeta(x) f(T^{-1}(x)) \, dx = \int_{[0,2\pi]} \zeta(x) f(x) \, dx.
\]
It follows that \( h \circ T^{-1} = f \) a.e. in \([0, 2\pi)\) and a change of variables completes the proof. 

Remark 2.5. A natural question is to know how much large in \( \mathcal{P}(S^1) \) the set \((2.12)\) is when \( \nu = 0 \). In fact, \( \mathcal{F}_0[\mu] = +\infty \) when \( \mu \in \mathcal{P}(S^1) \backslash \mathcal{P}_{ac}(S^1) \) gives mass on points. On the other hand, it is interesting to note that \( \mathcal{F}_0 \) may be finite on singular measures (with respect to Lebesgue one) that concentrate mass on sets with positive Hausdorff dimension (see proposition below).

Proposition 2.6. For \( s = \log(2)/\log(3) \), let \( \mathcal{H}^s \) denote the \( s \)-dimensional Hausdorff measure in \( \mathbb{R} \) and let \( C \) denote the Cantor ternary set in \([-\pi, \pi]\). Let \( \mu^s \in \mathcal{P}(S^1) \) be defined by
\[
\mu^s(A) = \frac{1}{2\pi} \mathcal{H}^s(A \cap C), \tag{2.14}
\]
for all Lebesgue measurable set \( A \). Then, \( \mu^s \) is singular with respect to the Lebesgue measure and \( \mathcal{F}_0[\mu^s] < \infty \).

Proof. The Hausdorff dimension of \( C \) is \( s := \log(2)/\log(3) \) and \( \mathcal{H}^s(C) = 2\pi \) (see [23, p. 34]). Define the probability measure \( \mu^s(A) = \frac{1}{2\pi} \mathcal{H}^s(A \cap C) \) in \( S^1 \equiv [-\pi, \pi) \). Now notice that
\[
- \log \left| \sin \left( \frac{x}{2} \right) \right| \leq - \log \left| \frac{x}{4} \right|,
\]
where \( x \in [0, 2\pi) \).
for $|x| \leq x_0$ and some $x_0 > 0$. Thus

$$F_0[\mu^s] \leq \frac{-1}{\pi} \int \int_{|x-y| \leq x_0 \cap [-\pi, \pi]^2} \log \left| \frac{x-y}{4} \right| d\mu^s(y)d\mu^s(x) + \frac{-1}{\pi} \int \int_{|x-y| > x_0 \cap [-\pi, \pi]^2} \log \left| \sin \left( \frac{x-y}{2} \right) \right| d\mu^s(y)d\mu^s(x).$$

(2.15)

The second integral in (2.15) is obviously finite, while the first can be bounded after showing

$$\int_{[-\pi, \pi]} -\log \left| \frac{x-y}{2\pi} \right| d\mu(y) \leq C, \quad (2.16)$$

for some universal constant $C > 0$. The estimate (2.16) can be showed by approximating monotonically from below the function $-\log \left| \frac{x-y}{2\pi} \right|$ by functions based on the construction of the Cantor set and then by applying the monotone convergence theorem.

Remark 2.7. There is an optimal transport map $t^\nu_\rho$ between $\mu$ and $\rho$, when $\mu$ does not give mass to points (see [32, p.75]). Then, the subject of Remark 2.5 is not a restriction to the existence of an optimal transport map between elements of $D(F_\nu)$.

3 Properties of the Functionals

The aim of this section is to obtain lower semicontinuity, coercivity and convexity properties for the free energy functional $F_\nu$.

3.1 Lower semicontinuity

In the next lemma we adapt some ideas of [28] for our context.

Lemma 3.1. The functional $F_\nu$ is lower semicontinuous with respect to the weak topology that coincides with the topology induced by the Wasserstein metric.

Proof. In view of (2.10), it is sufficient to prove the lower semicontinuity of $U$ and $F_0$.

Step 1 (Semicontinuity of $U$): Denote by $U(t) = t\log t$, $t \geq 0$. We start with the lower semicontinuity of the functional $U$. Let $\mu_k \rightarrow \mu$ be a sequence weakly convergent in $\mathcal{P}(S^1)$ and assume that $\mu \in \mathcal{P}_{ac}(S^1)$. We can assume that $\liminf_{k \rightarrow \infty} U(\mu_k) < \infty$ and then $\mu_k \in \mathcal{P}_{ac}(S^1)$ with $d\mu_k = \rho_k dx$. Let $\eta$ be a nonnegative smooth compactly supported mollifier with mass equal to 1 in $S^1$. Let $\eta_\delta(x) = \frac{1}{\delta} \eta \left( \frac{x}{\delta} \right)$, for $\delta > 0$. Then, by Jensen inequality, we have

$$\int_{-\pi}^{\pi} \rho_k(y) \log \rho_k(y) \eta_\delta(x-y) dy \geq U \left( \int_{-\pi}^{\pi} \rho_k(y) \eta_\delta(x-y) dy \right). \quad (3.1)$$
It follows by integrating in $x$ that
\[
\liminf_{k \to \infty} U(\mu_k) \geq \liminf_{k \to \infty} U(\rho_k \ast \eta_\delta) 
\geq \int_{-\pi}^{\pi} \liminf_{k \to \infty} U(\rho_k \ast \eta_\delta(x)) \, dx 
= \int_{-\pi}^{\pi} U(\rho \ast \eta_\delta(x)) \, dx,
\]
where we have used Fatou lemma in the second inequality. The equality follows from the weak convergence of $\mu_k$ and continuity of the function $U$. Since the support of $\eta_\delta$ is shrinking to a point, as $\delta \to 0$, then we obtain from Lebesgue differentiation theorem that
\[
\lim_{\delta \to 0} \rho \ast \eta_\delta(x) = \rho(x) \text{ a.e. in } S^1.
\]
The semicontinuity follows by applying again Fatou lemma and continuity of $U$. Let us assume now that $U(\mu) = \infty$. Without loss of generality, we can assume that $U(\mu_k) < \infty$, for $k$ large enough. Let $d\mu = d\mu_{\text{sing}} + \rho \, dx$ a Lebesgue decomposition in singular and absolutely continuous parts. By regularity of $\mu_{\text{sing}}$, there exist a compact $K \subset S^1$ with null Lebesgue measure and a number $m > 0$ such that $\mu_{\text{sing}}(K) \geq m > 0$. Thus, there exists an open set $O$ containing $K$ with Lebesgue measure arbitrarily small. Recall that weak convergence implies $\liminf_{k \to \infty} \mu_k(O) \geq \mu(O)$. We denote by $|O|$ the Lebesgue measure of $O$ and $c_0 = \inf_{t \geq 0} U(t)$. By Jensen inequality, we have
\[
U[\mu_k] - 2\pi c_0 = \int_{-\pi}^{\pi} (U(\rho_k(x)) - c_0) \, dx 
\geq |O| \left[ U \left( \int_{O} \frac{\rho_k}{|O|} \, dx \right) - c_0 \right] 
= |O| \left[ U \left( \frac{\mu_k(O)}{|O|} \right) - c_0 \right].
\]
Since $U(t)/t$ is increasing and $U(t)/t \to +\infty$ as $t \to \infty$, we obtain
\[
U[\mu_k] - 2\pi c_0 \geq \mu_k(O) \frac{|O|}{m} U \left( \frac{m}{|O|} \right) - c_0 |O| 
\geq |O| U \left( \frac{m}{|O|} \right) - c_0 |O|,
\]
for $k$ large enough. We conclude by letting $|O| \to 0$.

**Step 2** (Semicontinuity of $\mathcal{F}_0$): Note that $W$ can be approximated monotonically from below by periodic bounded continuous functions $W_l$. Thus, if $\mu_k \to \mu$ weakly then
\[
\int \int_{[-\pi,\pi]^2} W_l(y - x) \, d\mu(y) \, d\mu(x) = \lim_{k \to \infty} \int \int_{[-\pi,\pi]^2} W_l(y - x) \, d\mu_k(y) \, d\mu_k(x) 
\leq \liminf_{k \to \infty} \int \int_{[-\pi,\pi]^2} W(y - x) \, d\mu_k(y) \, d\mu_k(x),
\]
12
because the weak convergence of $\mu_k$ implies weak convergence of $\mu_k \times \mu_k \rightarrow \mu \times \mu$. Now, an application of the monotone convergence theorem finishes the proof.

3.2 Existence of Minimizer

At this level, we can show the existence of a minimizer for $F_\nu$. In fact, it is bounded from below, because

$$\nu \mathcal{U}[\mu] \geq 2\pi \nu \inf_{t > 0} t \log(t) = -2\pi \nu^{-1},$$

and

$$W[\mu] \geq 0.$$

Choose a minimizer sequence $\mu_k$ for $\inf F_\nu$. In view of the weak compactness of $P(S^1)$, we can assume (up to a subsequence) that $\mu_k \rightarrow \mu_0$ in $P(S^1)$. It follows from the lower semicontinuity that

$$\inf F_\nu \leq F_\nu[\mu_0] \leq \lim \inf_{k \rightarrow \infty} F_\nu[\mu_k] = \inf F_\nu,$$

and so $\mu_0$ is a minimizer of $F_\nu$.

3.3 Convexity in $P(S^1)$

The minimizer obtained in the previous section is unique since we show some kind of convexity for $F_\nu$.

Given $\mu_0, \mu_1, \omega \in P(S^1)$, we know that there exist $\mu_0^* \sim \mu_0, \mu_1^* \sim \mu_1$ and plans $\gamma_0 \in \Gamma(\omega, \mu_0^*), \gamma_1 \in \Gamma(\omega, \mu_1^*)$ such that (2.5) is true. In the case when $\omega$ is supported in $[-\pi, \pi)$ and does not give mass to points, we know that it is possible to choose $\mu_i^* = t_{\omega}^{\mu_i} \# \omega$ and $\gamma_i = (I, t_{\omega}^{\mu_i} \# \omega), i = 0, 1$, where $t_{\omega}^{\mu_i}$ is the map built in Section 2.1. We can consider it as $t_{\omega}^{\mu_i} : [-\pi, \pi) \rightarrow [a_i, a_i + 2\pi), i = 0, 1$. Thus, the map

$$(1 - t)t_{\omega}^{\mu_0} + tt_{\omega}^{\mu_1} : [-\pi, \pi) \rightarrow [(1 - t)a_0 + ta_1, (1 - t)a_0 + ta_1 + 2\pi)$$

can be seen as a map from $\mathbb{S}^1$ to $\mathbb{S}^1$.

In what follows, following the terminology of [2], we define the concepts of generalized geodesic and convexity along generalized geodesics in $P(S^1)$.

**Definition 3.2.** Given $\mu_0, \mu_1, \omega \in P(S^1)$, choose pairs $(\mu_0^*, \gamma_0)$ and $(\mu_1^*, \gamma_1)$ such that $\gamma_0 \in \Gamma(\omega, \mu_0^*)$ and $\gamma_1 \in \Gamma(\omega, \mu_1^*)$ and (2.5) is valid. A generalized geodesic connecting $\mu_0$ to $\mu_1$ in $P(S^1)$, with base point in $\omega$ and induced by $\gamma$, is a curve of equivalence classes of the type $\mu_t^g := ((1 - t)P_2 + tP_3) \# \gamma$, $t \in [0, 1]$, where $P_2$ and $P_3$ are the second and third projections from $\mathbb{R}^3$ onto $\mathbb{R}$, and $\gamma \in \Gamma(\omega, \mu_0^*, \mu_1^*)$ is such that $(P_1, P_2) \# \gamma = \gamma_0$ and $(P_1, P_3) \# \gamma = \gamma_1$. 

13
Remark 3.3. Notice that the convex combination \((1 - t) P_2 + tP_3\) can be outside of \(\mathbb{S}^1\). Then, \(\mu^\varphi\) should be understood as a curve of equivalence classes, according to the identification between \(\mathcal{P}(\mathbb{S}^1)\) and \(\mathcal{P}(\mathbb{R})/\sim\).

Remark 3.4. When \(\omega, \mu_0\) and \(\mu_1\) are supported in \([-\pi, \pi]\) and \(\omega\) does not give mass to points, we can take \(\gamma = (I, \tilde{t}^0_\omega, \tilde{t}^\mu_\omega)_\# \omega\). Therefore, a generalized geodesic in \(\mathcal{P}(\mathbb{S}^1)\) is given by the equivalence class of \(\mu^\varphi = ((1 - t)\tilde{t}^0_\omega + t\tilde{t}^\mu_\omega)_\# \omega\).

Definition 3.5. We say that a functional \(\mathcal{F}: \mathcal{P}(\mathbb{S}^1) \to (-\infty, \infty]\) is \(\lambda\)-convex along generalized geodesics, for some \(\lambda \in \mathbb{R}\), if given \(\omega, \mu_0, \mu_1 \in D(\mathcal{F})\) (the domain of \(\mathcal{F}\)) there exists a generalized geodesic \(\mu^\varphi\) connecting \(\mu_0\) to \(\mu_1\), based in \(\omega\) and induced by \(\gamma\), such that

\[
\mathcal{F}[\mu^\varphi] \leq (1 - t)\mathcal{F}[\mu_0] + t\mathcal{F}[\mu_1] - \frac{\lambda}{2} t (1 - t) d^2_\gamma(\mu_0^\varphi, \mu_1^\varphi),
\]

where \(\mu_i^\varphi \sim \mu_i\) and \(d^2_\gamma(\mu_0^\varphi, \mu_1^\varphi) = \int_{\mathbb{R}^3} |x_2 - x_3|^2 \, d\gamma(x_1, x_2, x_3)\).

Remark 3.6. In Definition 3.5, from (2.4) notice that \(d^2_\gamma(\mu_0^\varphi, \mu_1^\varphi) \geq d^2_2(\mu_0^\varphi, \mu_1^\varphi) \geq d^2_{\text{per}}(\mu_0, \mu_1)\), for all \(\mu_0, \mu_1 \in \mathcal{P}(\mathbb{S}^1)\).

The next lemma shows that \(\mathcal{F}_\nu\) is convex along generalized geodesics.

Lemma 3.7. The energy functional \(\mathcal{F}_\nu\) is strictly \(0\)-convex along generalized geodesics. Thus, the minimizer of \(\mathcal{F}_\nu\) obtained in Section 3.2 is unique.

Proof. Let \(\omega, \mu_0, \mu_1 \in D(\mathcal{F}_\nu)\) be supported in \([-\pi, \pi]\). Since \(\omega, \mu_0\) and \(\mu_1\) have no atoms, we can choose the generalized geodesic with representative \(\mu^\varphi = ((1 - t)\tilde{t}^0_\omega + t\tilde{t}^\mu_\omega)_\# \omega\) (see Remark 3.4).

For the entropy functional \(\mathcal{U}\), we know that \(\omega, \mu_0, \mu_1 \in \mathcal{P}_{ac}(\mathbb{S}^1)\) and have that \(\mu^\varphi \in \mathcal{P}_{ac}(\mathbb{S}^1)\) with support contained in \([(1 - t)a_0 + ta_1, (1 - t)a_0 + ta_1 + 2\pi]\). Denoting by \(f_t\) the density of \(\mu^\varphi\) and using Lemma 2.4, we have that the entropy is given by

\[
\mathcal{U}[\mu^\varphi] = \int f_t \log(f_t) \, dx. \tag{3.3}
\]

Now the convexity follows by arguing as in [2, Proposition 9.3.9] (see also [28] for the case of displacement interpolation curves).

On the other hand, recall that \(W(x) = -\frac{1}{\pi} \log |\sin \left(\frac{x}{2}\right)|\) (for \(x \neq 0\)) is the kernel of \(\mathcal{F}_0\) and we have

\[
W''(x) = \frac{1}{4\pi} \csc^2 \left(\frac{x}{2}\right) > 0,
\]

for \(-2\pi < x < 2\pi\) with \(x \neq 0\). Thus, \(W\) is convex on the segments \((-2\pi, 0)\) and \((0, 2\pi)\). Now, a modification of an argument in [6] can be used in order to show that \(\mathcal{F}_0\) is convex along generalized geodesics. In fact, by Remark 2.3 (ii), we have that

\[
\mathcal{F}_0[\mu^\varphi] = \int \int_{[-\pi, \pi]^2} W((1 - t)(\tilde{t}^0_\omega(x) - \tilde{t}^0_\omega(y)) + t(\tilde{t}^\mu_\omega(x) - \tilde{t}^\mu_\omega(y))) \, d(\omega \times \omega).
\]
Note that \((x, y) \in [-\pi, \pi]^2\) implies
\[
-2\pi < \tilde{t}_\omega^\mu (x) - \tilde{t}_\omega^\mu (y) < 2\pi, \quad i = 0, 1,
\]
and monotonicity of Euclidean transport maps implies that \(\tilde{t}_\omega^\mu (x) - \tilde{t}_\omega^\mu (y) \geq 0\) if and only if \(\tilde{t}_\omega^\mu (x) - \tilde{t}_\omega^\mu (y) \geq 0\). Using the convexity of \(W\) on \((-2\pi, 0)\) and \((0, 2\pi)\) separately, we are done.

\[\square\]

**Remark 3.8.** For the functionals above, it is possible to show convexity along geodesics with constant velocity (see Section 2.2) instead of generalized geodesics.

The next lemma contains an essential property for the convergence of the Euler scheme.

**Lemma 3.9.** For each fixed \(\rho \in \mathcal{P}(S^1)\), the functional \(\rho \rightarrow d_{\text{per}}^2(\mu, \rho)\) is 2-convex along generalized geodesics.

**Proof.** Let \(\omega, \mu_0, \mu_0^*, \mu_1, \mu_1^*\) and \(\gamma\) be as in Definition 3.2 such that \(\mu_0^*\) and \(\mu_1^*\) are minimum points in (2.3) for \(d_{\text{per}}^2(\omega, \mu_0)\) and \(d_{\text{per}}^2(\omega, \mu_1)\), respectively. Using the 2-convexity of the 2-Wasserstein metric in \(\mathcal{P}_2(\mathbb{R})\) and (2.3), we obtain
\[
d_{\text{per}}^2(\omega, \mu_1^*) \leq d_2^2(\omega, \mu_1^*) = (1 - t)d_2^2(\omega, \mu_0^*) + t d_2^2(\omega, \mu_1^*) - t(1 - t) d_\gamma^2(\mu_0^*, \mu_1^*),
\]
as desired.

\[\square\]

## 4 Global well-posedness and inviscid limit

In the previous sections we have obtained key properties for the functional \(F_{\nu}\) in order to construct gradient-flow solutions via an abstract Euler scheme. In fact, this scheme can be carried out in general metric spaces since the corresponding functional satisfies certain conditions (see [2, Chapter 4]).

### 4.1 Gradient-Flow Solutions

Here we consider the Euler discrete approximation scheme for gradient flows in \(\mathcal{P}(S^1)\). Let \(\mu \in \mathcal{P}(S^1)\) and let \(\tau > 0\) be the time step. Consider the functional \(\Psi_\nu(\tau, \mu; \cdot) : \mathcal{P}(S^1) \rightarrow (-\infty, \infty]\) as
\[
\Psi_\nu(\tau, \mu; \rho) := \frac{1}{2\tau} d_{\text{per}}^2(\mu, \rho) + F_{\nu}[\rho],
\]

\[4.1\]
where $d_{\text{per}}$ stands for the 2-Wasserstein distance in $\mathcal{P}(S^1)$ (see (2.1)) and $\mathcal{F}_\nu$ is the functional associated to (1.6), for $\nu \geq 0$.

For $\mu_0 \in \mathcal{P}(S^1)$, define the interactive sequence $(\mu_{\nu,\tau}^k)_{k=0}^\infty$ by
\[
\begin{align*}
\mu_{\nu,\tau}^0 &= \mu_0; \\
\mu_{\nu,\tau}^k &= \text{Argmin}\Psi_{\nu}(\tau, \mu_{\nu,\tau}^{k-1}; \cdot).
\end{align*}
\] (4.2)

Again, the minimizer above exists by the compactness of $\mathcal{P}(S^1)$ and is unique by convexity properties of $\mathcal{F}_\nu$ and $d_{\text{per}}$ (see Section 3.3). This allows to define the approximate discrete solution of the gradient flow equation (see [27] and [2])
\[
\frac{\partial \mu}{\partial t} = -\text{grad}_{d_{\text{per}}} (\mathcal{F}_\nu)
\]
\[= \partial_x \cdot (\mu \partial_x (\delta \mathcal{F}_\nu/\delta \mu))
\]
by setting
\[
\mu_{\nu,\tau}(t) = \mu_{\nu,\tau}^k \text{ if } t \in [k\tau, (k+1)\tau).
\] (4.3)

In the following we obtain the well-posedness in $\mathcal{P}(S^1)$ and some properties of the gradient flow of $\mathcal{F}_\nu$.

**Theorem 4.1.** Let $\nu \geq 0$, $\mu_0 \in \mathcal{P}(S^1)$ and $\mathcal{F}_\nu$ be the functional defined in (2.10)-(2.11).

(i) The discrete solution $\mu_{\nu}(t)$ defined in (4.3) converges locally uniformly to a locally Lipschitz curve $\mu(t) = S[\mu_0](t)$ in $\mathcal{P}(S^1)$ which is the unique gradient flow of $\mathcal{F}_\nu$ with $\mu(0) = \mu_0$.

(ii) The map $t \to S[\mu_0](t)$ is a 0-contracting semigroup in $\mathcal{P}(S^1)$, i.e.
\[
d_{\text{per}}(S[\mu_0](t), S[\rho_0](t)) \leq d_{\text{per}}(\mu_0, \rho_0), \text{ for } \mu_0, \rho_0 \in \mathcal{P}(S^1).
\]

(iii) If $\nu > 0$ then $\mu(t) \in \mathcal{P}_{\text{ac}}(S^1)$, for all $t > 0$. In the case $\nu = 0$, the measure solution $\mu(\cdot, t) = \mu(t)$ concentrates mass at most on sets of positive Hausdorff dimension.

(iv) Let $\bar{\mu}$ be the unique minimum of $\mathcal{F}_\nu$. Then, the map $t \to d_{\text{per}}(\mu(t), \bar{\mu})$ is not increasing and
\[
\mathcal{F}_\nu(\mu(t)) - \mathcal{F}_\nu(\bar{\mu}) \leq \frac{d^2_{\text{per}}(\mu_0, \bar{\mu})}{2t}, \text{ for all } t > 0.
\] (4.4)

(v) The minimum $\bar{\mu}$ is a stationary gradient flow solution, i.e., $\bar{\mu} = S[\bar{\mu}](t)$. Moreover, for all $\mu_0 \in \mathcal{P}(S^1)$, $\mu(t) \to \bar{\mu}$ in $\mathcal{P}(S^1)$, as $t \to \infty$.

(vi) If $\mu_0 \in D(\mathcal{F}_\nu)$, then we have the following error estimate
\[
d^2_{\text{per}}(\mu_{\nu}(t), \mu(t)) \leq \tau \left( \mathcal{F}_\nu(\mu_0) + 2\pi \nu e^{-1} \right).
\] (4.5)
Proof. We have showed that the energy functional $F_\nu$ is lower bounded (and so coercive), lower semicontinuous (Lemma 3.1) and 0-convex along generalized geodesics (Lemma 3.7). On the other hand, Lemmas 3.7 and 3.9 imply that the functional defined in (4.1) is convex along generalized geodesics. Finally we recall that $L^\infty(\mathbb{S}^1) \cap P(\mathbb{S}^1) \subset D(F_\nu)$ is dense in $P(\mathbb{S}^1)$, and therefore we can take $\mu_0 \in P(\mathbb{S}^1)$. Now, items (i), (ii), (iv) and (vi) follow from the abstract theory of [2, Theorems 4.0.4 and 4.0.7] in general metric spaces. For (iii), we know that $S[\mu_0](t) \in D(F_\nu) \subset P_{ac}(\mathbb{S}^1)$ for $\nu > 0$. In the case $\nu = 0$, we have that $S[\mu_0](t) \in D(F_0)$ and, by Proposition 2.6, $D(F_0)$ contains singular measures but without atoms. For item (v), notice first that $\bar{\mu}$ is also a minimizer of $\Psi \nu(\tau, \mu; \cdot)$. It follows that $\mu^k_{\nu, \tau} = \bar{\mu}$ in (4.2), for all $k$, and so $S[\bar{\mu}](t) = \mu_{\nu, \tau}(t) = \bar{\mu}$, for all $t > 0$ and $\tau > 0$. Since $F_\nu(\mu(t)) \to F_\nu(\bar{\mu})$ (by (4.4)), as $t \to \infty$, the convergence $\mu(t) \to \bar{\mu}$ follows from the compactness of $P(\mathbb{S}^1)$ and the lower semicontinuity of $F_\nu$.

Remark 4.2. By item (iii), we can not exclude the possibility that in the case $\nu = 0$ the gradient flow of (1.1) is singular with respect to the Lebesgue measure, although atoms in the solution are not allowed. In the viscous case, the entropy part of the functional prevents the existence of singular measures in the flow at $t > 0$.

Theorem 4.3. Let $\nu \geq 0$ and, for $0 < \nu < \epsilon_0$, let us denote by $\mu^\nu(t)$ and $\mu_\nu(t)$ the gradient flows associated to the energy functionals $F_\nu$ and $F_0$, respectively, with the same initial data $\mu_0 \in P(\mathbb{S}^1)$. Then, $\mu^\nu(t) \to \mu(t)$ in $P(\mathbb{S}^1)$, locally uniformly in $[0, \infty)$, as $\nu \to 0^+$.

Proof. By simplicity we can assume that $\mu_0 \in D(F_{l_0})$. The general case follows by using the same argument together with a discrete version of item (ii) of Theorem 4.1.

Step 1. Let $\mu^\nu(t) \to \mu$ as $\nu \to 0^+$ and let $\mu^\nu_\nu(t)$ be the minimizer of $\Psi_\nu(\tau, \mu^\nu; \cdot)$. Then, for each fixed $\tau > 0$, $\mu^\nu_\nu(t) \to \mu_\nu(t)$ as $\nu \to 0^+$, where $\mu_\nu(t)$ is the minimizer of $\Psi_\nu(\tau, \mu; \cdot)$.

In fact, by the compactness of $P(\mathbb{S}^1)$, we can extract a convergent subsequence $(\mu^\nu_\nu^i)_{i=1}^\infty$ such that $\nu_i \to 0^+$ and $\rho = \lim_{i \to \infty} \mu^\nu_\nu^i$. Given $\omega \in D(F_0)$, we have

$$\Psi_{\nu_i}(\tau, \mu_{\nu_i}; \mu^\nu_\nu^i) \leq \Psi_{\nu_i}(\tau, \mu^\nu_\nu^i; \omega).$$

(4.6)

It is straightforward to check that $\lim_{i \to \infty} \Psi_{\nu_i}(\tau, \mu_{\nu_i}; \omega) = \Psi_0(\tau, \mu; \omega)$. Also, by lower semicontinuity, it follows that

$$\liminf_{i \to \infty} F_{\nu_i}[\mu^\nu_\nu^i] \geq F_0[\rho]$$

and

$$\liminf_{i \to \infty} d^2_{per}(\mu^\nu_\nu^i; \mu^\nu_\nu^i) \geq d^2_{per}(\mu, \rho).$$

Then, we can conclude from (4.6) that $\rho$ is a minimizer of $\Psi_0(\tau, \mu; \cdot)$ and so $\rho = \mu_\tau$ (by uniqueness).

Step 2. Given $T > 0$ finite and $\tau > 0$, let $\mu_{\nu, \tau}(t)$ and $\mu_\tau(t)$ be the discrete solution defined in (4.3). We have that

$$\lim_{\nu \to 0^+} d_{per}(\mu_{\nu, \tau}(t), \mu_\tau(t)) = 0,$$

(4.7)
uniformly in $[0, T]$, for each $\tau > 0$.

Recalling the definition of $\mu^{\nu,k}_\tau$ in (4.2), notice that Step 1 and an induction argument yield
\[
\lim_{\nu \to 0^+} d_{\text{per}}(\mu^{\nu,k}_\tau, \mu^k_\tau) = 0, \text{ for all } k \in \{0\} \cup \mathbb{N}. \quad (4.8)
\]
Since $\mu_{\nu,\tau}(t)$ and $\mu_\tau(t)$ are step functions with $\mathcal{P}(S^1)$-values based on $\mu^{\nu,k}_\tau$ and $\mu^k_\tau$ (see (4.3)), respectively, the uniform convergence (4.7) in $[0, T]$ follows from (4.8).

**Step 3.** Finally, we can use Theorem 4.1 (vi) in order to get
\[
\begin{align*}
\frac{d}{dt} \int_{-\pi}^{\pi} \phi(x) d\mu_t &\leq \tau^{1/2} \left( \mathcal{F}_\nu(\mu_0) + 2\pi \nu e^{-1} \right)^{1/2} + d_{\text{per}}(\mu_{\nu,\tau}(t), \mu_\tau(t)) \\
&\quad + \tau^{1/2} F_0(\mu_0)^{1/2}, \text{ for all } t \in [0, T]. 
\end{align*}
\] (4.9)

Since $\tau > 0$ is arbitrary, we obtain the desired convergence by making $\nu \to 0^+$ in (4.9) and using (4.7).

### 4.2 Solutions in distributional sense

We already know that the semigroup $\mu(t) = S[\mu_0](t)$ is an absolutely continuous curve in $\mathcal{P}(S^1)$. The characterization given in [2] for such curves provides that $\mu(t)$ satisfies the continuity equation in the sense of distributions. More precisely, for the periodic setting, the analogous one is given in [24, Theorem 3.2]. In this case, the corresponding vector field is the minimal selection in the sub-differential $\partial \mathcal{F}_\nu(\mu(t))$ and its $L^2$-norm with respect to $\mu(t)$ is a $L^1_{\text{loc}}$ function in $(0, \infty)$.

We say that a curve $\mu_t$ is a solution of the equation (1.1), with initial data $\mu_0$, if, for each $\phi \in C^\infty(S^1)$, the equation
\[
\frac{d}{dt} \int_{-\pi}^{\pi} \phi(x) d\mu_t = \nu \int_{-\pi}^{\pi} \phi''(x) d\mu_t \\
- \int \int_{[-\pi,\pi]^2} \cot \left( \frac{x-y}{2} \right) (\phi'(x) - \phi'(y)) d(\mu_t \times \mu_t) 
\] (4.10)

is verified in the distributional sense in $(0, \infty)$ and $\mu_t \rightharpoonup \mu_0$ weakly-asterisk as measures, as $t \to 0^+$.

For $\mu \in \mathcal{P}(S^1)$, we define $L_\mu : C^\infty(S^1) \to \mathbb{R}$ by
\[
L_\mu(\phi) = \int \int_{[-\pi,\pi]^2} \cot \left( \frac{x-y}{2} \right) (\phi(x) - \phi(y)) d(\mu \times \mu). \quad (4.11)
\]

The functional (4.11) is continuous in $C^1(S^1)$. In fact, since $\sin(0)/0 = 1$ (by definition),
we can choose $\delta > 0$ such that $|t| < \delta$ implies that $\sin(t)/t \geq 1/2$. Thus

$$
|L_\mu(\phi)| \leq \frac{1}{2\pi} \int \int_{[-\pi,\pi)^2 \cap |x-y|<2\delta} \left| \cot \left( \frac{x-y}{2} \right) \right| (\phi(x) - \phi(y)) \, d(\mu \times \mu)
+ \frac{1}{2\pi} \int \int_{[-\pi,\pi)^2 \cap |x-y|\geq2\delta} \left| \cot \left( \frac{x-y}{2} \right) \right| (\phi(x) - \phi(y)) \, d(\mu \times \mu)
\leq \frac{1}{\pi} [1 + 2\pi \cot(\delta)] \|\phi\|_{L^\infty}.
$$

Indeed, we can refine the above estimate by considering $\sin(t)/t \geq \sin(\delta)/\delta$ for $|t| < \delta$. This shows the continuity of $L_\mu : C^\infty(S^1) \to \mathbb{R}$. Now, arguing as in [6] we have that there exists a Radon measure $\xi_\mu \in \mathcal{M}(S^1)$ such that the following representation for (4.11) is valid

$$
L_\mu(\phi) = \int_{[-\pi,\pi]} \phi' d\xi_\mu, \text{ for all } \phi \in C^1(S^1). \quad (4.12)
$$

**Lemma 4.4.** For $\nu \geq 0$, let $\mathcal{F}_\nu$ be the functional defined in (2.10)-(2.11) and $\mu \in D(\mathcal{F}_\nu)$. If $\mu \in D(|\partial \mathcal{F}_\nu|)$ then there exists $\theta \in L^2(S^1, d\mu)$ such that

$$
\theta \mu = \frac{d}{dx} (\nu \mu + \xi_\mu), \quad (4.13)
$$

where $\xi_\mu$ is as in (4.12). The vector $\theta$ is the minimal selection in the sub-differential $\partial \mathcal{F}_\nu(\mu)$. Moreover, for $\nu > 0$ we have $\nu \mu + \xi_\mu \in W^{1,1}(S^1, dx)$ where $dx$ is the Lebesgue measure on $[-\pi, \pi) \equiv S^1$.

**Proof.** We start by calculating the directional derivatives of $\mathcal{F}_0$.

Let $\phi \in C^\infty(S^1)$, and define $r_\varepsilon(x) = x + \varepsilon \phi(x)$. Note that $r_\varepsilon(\pi) = 2\pi + r_\varepsilon(-\pi)$, and so we can consider $r_\varepsilon$ as a function from $S^1$ to itself. Given $\mu \in D(\mathcal{F}_0)$, we can consider $r_\varepsilon \# \mu$ and, for $\varepsilon < \text{Lip}_\phi^{-1}$, $r_\varepsilon(x) - r_\varepsilon(y) > 0$ provided that $x - y > 0$. Analogously if $x - y < 0$ then $r_\varepsilon(x) - r_\varepsilon(y) < 0$.

Therefore, since $\frac{-1}{\pi} \log |\sin \left( \frac{x}{\pi} \right)|$ is convex in each one of the segments $(0, 2\pi]$ and $[-2\pi, 0)$, we can use monotonicity properties of convex functions in order to obtain

$$
\frac{1}{\varepsilon} [\mathcal{F}_0(r_\varepsilon \# \mu) - \mathcal{F}_0(\mu)]
= -\frac{1}{\varepsilon \pi} \int \int_{[-\pi,\pi)^2} \left( \log \left| \sin \left( \frac{r_\varepsilon(x) - r_\varepsilon(y)}{2} \right) \right| - \log \left| \sin \left( \frac{x-y}{2} \right) \right| \right) d(\mu \times \mu). \quad (4.14)
$$

Splitting the integral (4.14) into the subsets $\{x > y\}$ and $\{x < y\}$, we observe that each resulting integrand is nondecreasing in $\varepsilon$ (by convexity). By applying the monotone convergence theorem, we obtain

$$
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [\mathcal{F}_0(r_\varepsilon \# \mu) - \mathcal{F}_0(\mu)] = -\frac{1}{2\pi} \int \int_{[-\pi,\pi)^2} \cot \left( \frac{x-y}{2} \right) (\phi(x) - \phi(y)) d(\mu \times \mu)
= -L_\mu(\phi).
$$

19
Next we deal with the derivative of $U$. Since $r_{\epsilon \# \mu}$ is supported in $[a, a + 2\pi)$, for some $a \in \mathbb{R}$, we can use Lemma 2.4 to obtain

$$U[r_{\epsilon \# \mu}] = \int_{\mathbb{R}} f_{\epsilon}(x) \log(f_{\epsilon}(x)) \, dx,$$

where $f_{\epsilon}$ is the density of $r_{\epsilon \# \mu}$. Now, the computation of the directional derivative of $U$ follows similarly to that of the entropy functional in the whole space $\mathbb{R}$ (see [32]). It follows that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} [U(r_{\epsilon \# \mu}) - U(\mu)] = -\nu \int_{-\pi}^{\pi} \phi' d\mu. \quad (4.15)$$

We turn to the complete functional $F_{\nu}$. First recall that $d_{\text{per}}(r_{\epsilon \# \mu}, \mu) \leq \|\phi\|_{L^2(S^1, d\mu)}$. Since $\mu \in D(|\partial F_{\nu}|)$, we have that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} [F_{\nu}[r_{\epsilon \# \mu}] - F_{\nu}[\mu]] = \nu \int_{-\pi}^{\pi} \phi' d\mu - L_{\mu}(\phi) \geq -|\partial F_{\nu}|(\mu)\|\phi\|_{L^2(S^1, d\mu)}.$$

By changing $\phi$ by $-\phi$, it follows that

$$\left| \nu \int_{-\pi}^{\pi} \phi' d\mu - L_{\mu}(\phi) \right| \leq |\partial F_{\nu}|(\mu)\|\phi\|_{L^2(S^1, d\mu)}.$$

Therefore, by Riesz representation theorem, there exists $\theta \in L^2(S^1, d\mu)$ such that

$$-\nu \int_{-\pi}^{\pi} \phi' d\mu - \int_{[-\pi, \pi]} \phi' d\xi_{\mu} = \int_{-\pi}^{\pi} \theta \phi d\mu, \text{ for all } \phi \in C^\infty(S^1). \quad (4.16)$$

From (4.16), we have $\theta \mu = \frac{d}{dx}(\nu \mu + \xi_{\mu})$ in the distributional sense and then $\theta$ is the minimal selection of $\partial F_{\nu}(\mu)$. Moreover, for $\nu > 0$, $\mu$ is absolutely continuous with respect to the Lebesgue measure. Thus, by definition of $U$ and Holder inequality, it follows that $\theta \mu \in L^1(S^1, dx)$, i.e. $\frac{d}{dx}(\nu \mu + \xi_{\mu}) \in L^1(S^1, dx)$, and thereby $\nu \mu + \xi_{\mu} \in W^{1,1}(S^1, dx)$. In particular, this implies that $\xi_{\mu}$ is absolutely continuous with respect to the Lebesgue measure.

Now we are in position to show that our gradient flows are solutions for (1.1) and (1.6) in distributional sense.

**Theorem 4.5.** Let $\nu \geq 0$ and let $F_{\nu}$ be the functional defined in (2.10)-(2.11). Let $\mu_t$ be the gradient flow for $F_{\nu}$ with initial data $\mu_0 \in \mathcal{P}(S^1)$ and let $\xi_{\mu,t}$ be the Radon measure associate
to functional $L_\mu$ according to (4.12). Then, $\mu_t := \mu(x,t)$ is a weak solution of (1.6) in the sense of (4.10). Moreover, for $\nu > 0$ we have that

$$\mu(x,t) \in L^1_{loc}((0, \infty); W^{1,1}(S^1, dx))$$

(4.17)

$$\nu \mu_t + \xi_{\mu_t} \in W^{1,1}(S^1, dx), \text{ for all } t > 0,$$

(4.18)

where $dx$ is the Lebesgue measure on $[-\pi, \pi) \equiv S^1$.

**Proof.** We start by showing that $\mu_t$ is a distributional solution of (1.6). In fact, by definition of gradient flow, we know that the continuity equation

$$\partial_t \mu_t + \partial_x (\theta_t \mu_t) = 0$$

is satisfied in the sense of distributions, where $\theta_t \mu_t = \frac{d}{dx}(\nu \mu_t + \xi_{\mu_t})$. For $\phi \in C^\infty(S^1)$ and $\eta(t) \in C^\infty_c((0, \infty))$, we have that

$$\int_0^\infty \int_{[-\pi, \pi)} [\eta'(t) \phi(\xi) + \eta(t) \phi'(\xi) \theta_t(\xi)] \mu_t(\xi) d\xi dt = 0$$

and then

$$\int_0^\infty \int_{[-\pi, \pi)} \eta'(t) \phi(\xi) \mu_t(\xi) d\xi dt$$

$$= \int_0^\infty \eta(t) \int_{[-\pi, \pi)} \phi''(\xi) d(\nu \mu_t + \xi_{\mu_t}) dt$$

$$= \nu \int_0^\infty \eta(t) \int_{[-\pi, \pi)} \phi''(\xi) d\mu_t dt + \int_0^\infty \eta(t) L_{\mu_t}(\phi') dt$$

$$= \nu \int_0^\infty \eta(t) \int_{[-\pi, \pi)} \phi''(\xi) d\mu_t dt - \nu \int_0^\infty \eta(t) \int_{[-\pi, \pi)^2} \cot(\frac{x-y}{2}) (\phi'(x) - \phi'(y)) d(\mu_t \times \mu_t) dt,$$

which gives (4.10).

Next, for $\nu > 0$, notice that

$$\| \partial_x \mu(\cdot, t) \|_{L^1(S^1, dx)} = \| \frac{\partial_x \mu(x, t)}{\mu(x, t)} \|_{L^1(S^1, d\mu_t)} \leq \| \frac{\partial_x \mu(x, t)}{\mu(x, t)} \|_{L^2(S^1, d\mu_t)}.$$

The properties (4.17)-(4.18) follows from Lemma 4.4 and the fact that $\frac{\partial_x \mu(x, t)}{\mu(x, t)}$ is the minimal selection for the functional $\mathcal{U}$ in $\mu_t$, and then $\| \frac{\partial_x \mu(x, t)}{\mu(x, t)} \|_{L^2(S^1, d\mu_t)} \in L^1_{loc}((0, \infty))$.

**References**

[1] Agueh, M., *Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory*, Adv. Differential Equations 10 (3) (2005), 309–360.
[2] Ambrosio, L., Gigli, N., Savaré, G., *Gradient flows: in metric spaces and in the space of probability measures*, Birkhäuser, (2005).

[3] Ambrosio, L., Colombo, M., De Philippis, G., Figalli, A., *Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case*, Comm. Partial Differential Equations 37 (12) (2012), 2209–2227.

[4] Baker, G.R., Li, X., Morlet, A.C., *Analytic structure of two 1D-transport equations with nonlocal fluxes*, Phys. D 91 (4) (1996), 349–375.

[5] Biler, P., Karch, G., Monneau, R., *Nonlinear diffusion of dislocation density and self-similar solutions*, Comm. Math. Phys. 294 (2010), 145–168.

[6] Carrillo, J., Ferreira, L.C.F., Precioso, J., *A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity*, Adv. Math. 231 (1) (2012), 306–327.

[7] Carrillo, J. A., McCann, R. J., Villani, C., *Contractions in the 2-Wasserstein length space and thermalization of granular media*, Arch. Ration. Mech. Anal. 179 (2) (2006), 217–263.

[8] Carrillo, J. A., Di Francesco, M., Figalli, A., Laurent, T., Slepčev, D., *Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations*, Duke Math. J. 156 (2011), 229–271.

[9] Carrillo, J. A., Slepčev, D., *Example of a first order displacement convex functional*, Calculus of Variations and PDEs 36 (2009), 547–564.

[10] Castro, A., Córdoba, D., Global existence, singularities and ill-posedness for a nonlocal flux, Adv. Math. 219 (6) (2008), 1916–1936.

[11] Chae, D., Córdoba, A., Córdoba, D., Fontelos, M. A., *Finite time singularities in a 1D model of the quasi-geostrophic equation*, Adv. Math. 194 (1) (2005), 203–223.

[12] Constantin, P., Lax, P., Majda, A., *A simple one-dimensional model for the three dimensional vorticity*, Comm. Pure Appl. Math. 38 (1985), 715–724.

[13] Córdoba, A., Córdoba, D., Fontelos, M. A, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. 162 (3) (2005), 1377–1389.

[14] Córdoba, A., Córdoba, D., Fontelos, M. A., *Integral inequalities for the Hilbert transform applied to a nonlocal transport equation*, J. Math. Pures Appl. (9) (6) 86 (2006), 529–540.

[15] Cordero-Erausquin, D., *Sur le transport de mesures périodiques*, C. R. Acad. Sci. Paris Sér. I Math., 329 (1999), 199–202.
[16] Deslippe, J., Tedstrom, R., Daw, M.S., Chrzan, D., Neeraj, T., Mills, M., *Dynamics scaling in a simple one-dimensional model of dislocation activity*, Phil. Mag. 84 (2004), 2445–2454.

[17] Li, D., Rodrigo, J. L., *Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation*, Adv. Math. 217 (6) (2008), 2563–2568.

[18] Li, D., Rodrigo, J. L., *On a one-dimensional nonlocal flux with fractional dissipation*, SIAM J. Math. Anal. 43 (1) (2011), 507–526.

[19] Dong, H., *Well-posedness for a transport equation with nonlocal velocity*, J. Funct. Anal. 255 (11) (2008), 3070–3097.

[20] De Philippis, G., Figalli, A., *$W^{2,1}$ regularity for solutions of the Monge-Ampère equation*, Invent. Math. 192 (1) (2013), 55–69.

[21] De Philippis, G., Figalli, A., *Second order stability for the Monge-Ampère equation and strong Sobolev convergence of optimal transport maps*, Anal. PDE 6 (4) (2013), 993–1000.

[22] Escudero, C., *On one-dimensional models for hydrodynamics*, Physica D 217 (2006), 58–63.

[23] Falconer, K., *Fractal geometry. Mathematical foundations and applications*, Second edition, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[24] Gangbo, W., Tudorascu, A., *Weak KAM theory on the Wasserstein torus with multidimensional underlying space*, Comm. Pure Appl. Math. 67 (3) (2014), 408–463.

[25] De Gregorio, S., *On a one-dimensional model for the three-dimensional vorticity equation*, J. Statist. Phys. 59 (1990), 1251–1263.

[26] Head, A. K., *Dislocation group dynamics III. Similarity solutions of the continuum approximation*, Phil. Mag. 26 (1972), 65–72.

[27] Jordan, R., Kinderlehrer, D., Otto, F., *The variational formulation of the Fokker-Plank Equation*, SIAM J. Math Anal. 29 (1) (1998), 1–17.

[28] McCann, R. J., *A convexity principle for interacting gases*, Adv. Math. 128 (1) (1997), 153–179.

[29] McCann, R. J., *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. 11 (2001), 589-608.

[30] Morlet, A. C., *Further properties of a continuum of model equations with globally defined flux*, J. Math. Anal. Appl. 221 (1) (1998), 132–160.
[31] Otto, F., *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations 26 (1-2) (2001), 101–174.

[32] Villani, C., *Topics in Optimal Transportation*, Graduate Studies in Mathematics 58, American Mathematical Society, Providence, RI, 2003.