Research Article

Dynamics of a Diffusive Multigroup SVIR Model with Nonlinear Incidence

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Received 22 August 2020; Revised 25 October 2020; Accepted 4 November 2020; Published 9 December 2020

Academic Editor: Dimitri Volchenkov

In this paper, a multigroup SVIR epidemic model with reaction-diffusion and nonlinear incidence is investigated. We first establish the well-posedness of the model. Then, the basic reproduction number $R_0$ is established and shown as a threshold: the disease-free steady state is globally asymptotically stable if $R_0 < 1$, while the disease will be persistent when $R_0 > 1$. Moreover, applying the classical method of Lyapunov and a recently developed graph-theoretic approach, we established the global stability of the endemic equilibria for a special case.

1. Introduction

As is well-known, mathematical models have played a key role in describing the dynamical evolution of infectious diseases [1–10]. For example, during the special battle for preventing the novel coronavirus (COVID-19) spreading, not only medical and biological research but also theoretical studies based on mathematical modeling may play a crucial role in analyzing and forecasting the spreading of the disease (see, for example, [4–10] and references therein; also, some useful advices for controlling the disease spreading have been given by the researchers).

The spatial heterogeneity has been regarded as an important role that affects the spatial spreading of disease. Therefore, complex models are needed to investigate the spread of diseases. In recent years, reaction-diffusion models involving environmental heterogeneity have been proposed and studied to find reliable measures to control disease spreading [11–22]. In particular, the basic reproduction number is an important threshold value for investigation of the dynamics of epidemic models. Then, Wang and Zhao [14] defined the basic reproduction number and obtained its computation formula for general reaction-diffusion epidemic models. Xu et al. [21] studied an SVIR epidemic model with reaction-diffusion and the existences of travelling waves were investigated, while Zhang and Liu [22] also studied the existence of travelling waves for an SVIR epidemic model but with nonlocal dispersal and time delay.

Notice that the essential assumption in classical compartmental epidemic models is that the individuals are homogeneously mixed, which means each individual has the same probability to get infected. However, infected probability may be different for each individual in terms of the impact of different factors, such as education levels, age, gender, and communities. To overcome this problem, multigroup models have been proposed by many researchers by dividing the individuals into different groups and most of these work focus on the global dynamics of the models (see, for example, [23–30]). Particularly, one of the earliest works of multigroup models was proposed by Lajmanovich and York [23] when studied the gonorrhea disease transmission in a nonhomogeneous population. In order to investigate the global dynamics of multigroup models, a subtle grouping method in estimating the derivatives of Lyapunov functionals guided by graph theory for multigroup models was developed in [28–30]. For example, Kuniya [25] considered a multigroup SVIR epidemic model with vaccination and the global stability of endemic equilibria were established by the
Method of Lyapunov function. However, spatial heterogeneity was not considered in the model in [25]. To the best of our knowledge, there are few results of multigroup SVIR epidemic model with reaction-diffusion. The existing results are focusing on SIS and SIR models (see [31–34]). On the contrary, incidence rate also plays an important role in modeling the epidemic models, which has been frequently used to describe the nature of certain phenomena and obtained much exact results. In addition, applying general incidence rates can obtain the unification theory by the omission of unessential detail, see, for example, [21, 35–37] and references therein. Hence, inspired by [21, 25] and the above considerations, in this paper, we consider the following multigroup SVIR epidemic model with reaction-diffusion and general incidence rate:

\[
\begin{align*}
\frac{dS_i}{dt} &= \nabla \cdot (d_{1i}(x)\nabla S_i) + \ell_i(x) - \left( \mu_i^S(x) + \xi_i(x) \right) S_i - \sum_{j=1}^{n} \beta_{ij}(x) S_i f_j(I_j), \\
\frac{dV_i}{dt} &= \nabla \cdot (d_{2i}(x)\nabla V_i) + \xi_i(x) S_i - \sum_{j=1}^{n} \beta_{ij}(x) V_i f_j(I_j) - \mu_i^V(x) V_i, \\
\frac{dI_i}{dt} &= \nabla \cdot (d_{3i}(x)\nabla I_i) + \sum_{j=1}^{n} \left( \beta_{ij}(x) S_i + \beta_{ij}(x) V_i \right) f_j(I_j) - \left( \mu_i^I(x) + \delta_i(x) + \gamma_i(x) \right) I_i, \\
\frac{dR_i}{dt} &= \nabla \cdot (d_{4i}(x)\nabla R_i) + \gamma_i(x) I_i - \mu_i^R(x) R_i,
\end{align*}
\]

with the homogeneous Neumann boundary conditions

\[
\frac{\partial S_i}{\partial n} = \frac{\partial V_i}{\partial n} = \frac{\partial I_i}{\partial n} = \frac{\partial R_i}{\partial n} = 0, \quad t \geq 0, \quad x \in \partial \Omega, \quad 1 \leq i \leq n,
\]

and the initial conditions

\[
(S_i(0, x), V_i(0, x), I_i(0, x), R_i(0, x)) = (\phi_{S_i}(x), \phi_{V_i}(x), \phi_{I_i}(x), \phi_{R_i}(x)),
\]

for \( x \in \Omega, 1 \leq i \leq n \), where \( \Omega \) is a domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( v \) is the outward normal vector to the boundary \( \partial \Omega \). Initial functions \( \phi_{S_i}(x) \) \( (k = 1, 2, 3, 4, 1 \leq i \leq n) \) are nonnegative and continuously defined on \( \Omega \). \( S_i = S_i(t, x), V_i = V_i(t, x), I_i = I_i(t, x), \) and \( R_i = R_i(t, x) \) stand for the densities of the susceptible, vaccinated, infective, and recovered individuals in the \( i \)-th group at time \( t \) and spatial location \( x \), respectively. \( \lambda_i(x) \) is the input rate of \( S_i \) in spatial location \( x \); \( \mu_i^S(x), \mu_i^V(x), \mu_i^I(x) \), and \( \mu_i^R(x) \) denote the natural death rates of \( S_i, V_i, I_i, \) and \( R_i \) in spatial location \( x \), respectively; \( \delta_i(x) \) is the death rate induced by the disease in spatial location \( x \); \( \xi_i(x) \) is the vaccination rate of \( S_i \) in spatial location \( x \); \( \gamma_i(x) \) is the rate of recovery from infection in spatial location \( x \); \( \beta_{ij}(x) \) is the infection rate of \( S_i \) infected by \( I_j \) in spatial location \( x \); \( \beta_{ij}(x) \) is the infection rate of \( V_j \) infected by \( I_j \) in spatial location \( x \); \( d_{1i}(x), d_{2i}(x), d_{3i}(x) \), and \( d_{4i}(x) \) are the diffusion rate of \( S_i, V_i, I_i, \) and \( R_i \) in spatial location \( x \), respectively. All location-dependent parameters are continuous and strictly positive defined on \( \Omega \).

It is natural to assume that \( f_j(0) = 0 \) due to the fact that disease cannot spread if there is no infection. We assume the function \( f_j(I_j) \) satisfies the following properties:

\[
f_j(0) = 0, \quad f_j(0) > 0 \text{ for } I_j > 0, \quad f_j'(I_j) > 0, \quad f_j''(I_j) \leq 0 \text{ for all } I_j > 0.
\]

It is natural to assume that \( f_j(0) = 0 \) due to the fact that disease cannot spread if there is no infection. The disease spreads heavily with the increasing number of infected individuals; thus, it is reasonable to suppose \( f_j'(I_j) > 0 \). Since the infectivity of infected individuals cannot be unbounded, it should reach a certain level when the infected individuals are heavy. Therefore, the assumption \( f_j'(I_j) \leq 0 \) implies that there exists a peak level for the infectivity of the infected individuals at some certain time. Similar to [25, 34], we also assume the \( n \)-square matrix \( (\beta_{ij}(x))_{n \times n} \) is nonnegative and irreducible.

Because the last equation of model (1) is decoupled from other equations, we indeed need to study the following subsystem of (1):

\[
(S_i(0, x), V_i(0, x), I_i(0, x), R_i(0, x)) = (\phi_{S_i}(x), \phi_{V_i}(x), \phi_{I_i}(x), \phi_{R_i}(x)),
\]

and \( f_j(I_j) \) denotes the force of infection. We assume the function \( f_j(I_j) \) satisfies the following properties:

\[
f_j(0) = 0, \quad f_j(0) > 0 \text{ for } I_j > 0, \quad f_j'(I_j) > 0, \quad f_j''(I_j) \leq 0 \text{ for all } I_j > 0.
\]
of the model. In Section 3, we define the basic reproduction number $R_0$. In Section 4, the threshold dynamics are established in terms of $R_0$. An special case is performed as a supplementary to the theoretical results in Section 5. A brief conclusion ends the paper.

2. Well-Posedness

Throughout this paper, we denote $\bar{f} = \max_{x \in \Omega} f(x)$ and $f = \min_{x \in \Omega} f(x)$ with the norm $\| \cdot \|_Y$ and $Y = C(\Omega, \mathbb{R})$. It is easy to see that $(Y, Y^*)$ is a strongly ordered Banach space. Let $X = Y \times Y$ with norm $\| \phi \|_X = \max \{ \| \phi_1 \|_Y, \| \phi_2 \|_Y, \| \phi_3 \|_Y \}$, where $\phi = (\phi_1, \phi_2, \phi_3) \in X$, $\phi_i \in Y$. Denote $Y^* = Y^* \times Y^* \times Y^*$ be the positive cone of $X$. For convenience, set $\phi_k = (\phi_{k1}, \phi_{k2}, \ldots, \phi_{k3})$ for $k = 1, 2, 3$. $S = (S_1, S_2, \ldots, S_n)$, $V = (V_1, V_2, \ldots, V_n)$, and $I = (I_1, I_2, \ldots, I_n)$. Denote

$$T_{1i}, T_{2i}, T_{3i} : C(\Omega, \mathbb{R}) \rightarrow C(\Omega, \mathbb{R})$$

for $t \geq 0$ and $x \in \Omega$, and $\phi = (\phi_1, \phi_2, \phi_3) \in X^*$. Set $u(t, \cdot, \phi) = (S(t, \cdot, \phi), V(t, \cdot, \phi), I(t, \cdot, \phi))$ be the solution of model (5) with initial function $\phi = (\phi_1, \phi_2, \phi_3) \in X^*$, and then, model (5) can be rewritten as

$$\frac{\partial S_i}{\partial t} = \nabla \cdot (d_{i1}(x)\nabla S_i) + \lambda_i(x) - \left( \mu^i_0(x) + \xi_i(x) \right) S_i - \sum_{j=1}^{n} \beta_{ij}(x) S_j f_j(I_j), \quad t \geq 0, \quad x \in \Omega, \quad 1 \leq i \leq n,$$

$$\frac{\partial V_i}{\partial t} = \nabla \cdot (d_{i2}(x)\nabla V_i) + \xi_i(x) S_i - \sum_{j=1}^{n} \beta_{ij}(x) V_j f_j(I_j) - \mu^i_1(x) V_i, \quad t \geq 0, \quad x \in \Omega, \quad 1 \leq i \leq n,$$

$$\frac{\partial I_i}{\partial t} = \nabla \cdot (d_{i3}(x)\nabla I_i) + \sum_{j=1}^{n} \left( \beta_{ij}(x) S_j + \beta_{ij}(x) V_j \right) f_j(I_j) - \left( \mu^i_1(x) + \delta_i(x) + \gamma_i(x) I_i \right), \quad t \geq 0, \quad x \in \Omega, \quad 1 \leq i \leq n,$$

be the $C_0$ semigroup associated with $\nabla \cdot (d_{i1}(x)\nabla) - (\mu^i_0(x) + \xi_i(x)), \nabla \cdot (d_{i2}(x)\nabla) - \mu^i_1(x), \nabla \cdot (d_{i3}(x)\nabla) - (\mu^i_1(x) + \delta_i(x) + \gamma_i(x))$, respectively, subject to the Neumann boundary condition. Let $I_{ki}(t, x, y)$ be the Green function associated with $\nabla \cdot (d_{i1}(x)\nabla) - (\mu^i_0(x) + \xi_i(x)), \nabla \cdot (d_{i2}(x)\nabla) - \mu^i_1(x), \nabla \cdot (d_{i3}(x)\nabla) - (\mu^i_1(x) + \delta_i(x) + \gamma_i(x))$, respectively, subject to the Neumann boundary condition. Then, for any $\phi \in \bar{C}(\Omega, \mathbb{R})$ and $t > 0$, we have

$$T_{ki}(t)\phi(x) = \int_{\Omega} T_{ki}(t, x, y) \phi(y) dy, \quad k = 1, 2, 3, \quad 1 \leq i \leq n.$$
\( \phi = (\phi_1, \phi_2, \phi_3) \in X^*. \) Furthermore, the solution semiflow
\( \Phi(t) = u(t, \cdot) : X^* \rightarrow X^* \) of model (5) defined by

\[
\Phi(t)\phi = u(t, \cdot, \phi), \quad t \geq 0,
\]
"admits a global compact attractor.

Proof. Suppose to the contrary that \( \tau_{\infty} < \infty \), then \( \|u(t, \cdot, \phi)\| \rightarrow +\infty \) as \( t \rightarrow \tau_{\infty} \) by Theorem 2 in [39]. It follows from the first equation of model (5) that

\[
\frac{dS_i}{dt} \leq \nabla \cdot (d_{1i}(x) \nabla S_i) + \lambda_i - \left( \mu_i^S + \xi_i \right) S_i, \quad t \in [0, \tau_{\infty}), \ x \in \Omega, 1 \leq i \leq n.
\]

By the comparison principle and Lemma 2 in [40], there exists a constant \( \mathcal{M}_1 > 0 \) such that \( S_i(t, x) \leq \mathcal{M}_1 (1 \leq i \leq n) \)

\[\begin{align*}
\frac{\partial \omega_i}{\partial t} &= \nabla \cdot (d_{3i}(x) \nabla \omega_i) + \sum_{j=1}^{n} \left( \beta_{ij} \mathcal{M}_1 + \bar{\beta}_{ij} \mathcal{M}_2 \right) f_j'(0) \omega_j - \left( \mu_i^j + \delta_i^j + \gamma_i^j \right) \omega_j, \quad t > 0, \ x \in \Omega, 1 \leq i \leq n, \\
\frac{\partial \omega_i}{\partial t} &= 0, \quad t > 0, \ x \in \partial \Omega.
\end{align*}\]

\[
(I_1(t, x), I_2(t, x), \ldots, I_n(t, x)) \leq c e^{\nu_0 \hat{\phi}_3(x)}, \quad t \in [0, \tau_{\infty}), \ x \in \Omega, 1 \leq i \leq n.
\]

Thus, there exists a constant \( \mathcal{M}_3 \) such that \( I_i(t, x) \leq \mathcal{M}_3, \ x \in \Omega, 1 \leq i \leq n, \)

\[
\frac{d \mathcal{Q}_i}{dt} = \int_\Omega \left( \lambda_i^1(x) - \mu_i^S(x) \right) S_i(t, x) - \mu_i^V(x) V(t, x) - \left( \mu_i^j(x) + \delta_i^j(x) + \gamma_i^j(x) \right) I_i(t, x)dx \\
\leq \int_\Omega \lambda_i^1(x)dx - \min_{x \in \Omega} \left[ \mu_i^S(x), \mu_i^V(x), \mu_i^j(x) + \delta_i^j(x) + \gamma_i^j(x) \right]  \mathcal{Q}_i, \quad t \geq 0.
\]

Thus, there exist \( t_j > 0 \) and \( \mathcal{A}_j > 0 \) such that \( \mathcal{Q}_i(t) \leq \mathcal{A}_j (1 \leq i \leq n) \) for any \( t_j \geq t_j \). Next, we denote \( t_1 \) be the eigenvalue of \( \nabla \cdot (d_{3i}(x) \nabla) - (\mu_i^l(x) + \delta_i^l(x) + \gamma_i^l(x)) \) subject to the Neumann boundary condition with eigenfunction \( \varphi_i^j(x) \), which satisfies \( t_1 > t_2 \geq t_3 \geq \cdots \). From Chapter 5 in [42], one obtains

\[
\Gamma_{3i}(t, x, y) = \sum_{j=1}^{n} \varphi_i^j(x) \varphi_j^i(y), \quad 1 \leq i \leq n.
\]

Since \( \varphi_i^j(x) \) is uniformly bounded, there exists constant \( \kappa_{3i} > 0 \) such that \( \Gamma_{3i}(t, x, y) \leq \kappa_{3i} \sum_{j=1}^{n} e^{tj} \) for \( t > 0 \). Moreover, assume \( \varpi_j \) are eigenvalues of \( \nabla \cdot (d_{3i} \nabla) - (\mu_i^l + \delta_i^l + \gamma_i^l) \)
subject to Neumann boundary condition, which satisfies $n_j^i = -(\mu_j^i + \delta_j^i + \gamma_j^i) > n_j^i \geq n_j^i \geq \cdots n_j^i \geq \cdots$. By Theorem 2.4.7 in Wang [43], one gets $n_j^i \geq n_j^i$ for all $j \in \mathbb{N}_+$. Since $n_j^i$ decreases like $-n_j^i$, there exists $\kappa_3 > 0$ such that

$$\Gamma_j \leq \kappa_3 \sum_{j \geq 1} e^{\rho_j^i t} \leq \kappa_3 e^{\left(\mu_j^i + \delta_j^i + \gamma_j^i\right) t}, \quad \forall t > 0. \quad (19)$$

Let $\hat{t} = \max\{t_1, t_2, t_3\}$. For all $t \geq \hat{t}$, by (9) and (4), we have

$$I_i(t, x) = T_{3i}(t)I_i(\hat{t}, x) + \int_0^t T_{3i}(t - s) \left[ \sum_{j=1}^n \left( \beta_{ij}(x)S_i(s, x) + \bar{\beta}_{ij}(x)V_i(s, x) \right) f_j(\hat{t}, x) \right] ds \leq A_3 e^{\alpha_3(t-i)} \|I_i(\hat{t}, x)\| + \int_0^t \int_\Omega \Gamma_{3i}(t - s, x, y) \left[ \sum_{j=1}^n \left( \beta_{ij}(y)S_i(s, y) + \bar{\beta}_{ij}(y)V_i(s, y) \right) \times f_j(0)I_j(s, y) \right] dy ds \leq A_3 e^{\alpha_3(t-i)} \|I_i(\hat{t}, x)\| + \int_0^t e^{-\left(\mu_j^i + \delta_j^i + \gamma_j^i\right)(t-s)} \sum_{j=1}^n \left( \beta_{ij} \alpha_1 + \bar{\beta}_{ij} \alpha_2 \right) f_j(0) ds \leq A_3 e^{\alpha_3(t-i)} \|I_i(\hat{t}, x)\| + \kappa_3 \alpha_3 \sum_{j=1}^n \left( \beta_{ij} \alpha_1 + \bar{\beta}_{ij} \alpha_2 \right) f_j(0) 1 - e^{-\left(\mu_j^i + \delta_j^i + \gamma_j^i\right)(t-i)} \mu_j^i + \delta_j^i + \gamma_j^i,$$

which yields that

$$\limsup_{t \to \infty} \|I_i(t, x)\| \leq \frac{\kappa_3 \alpha_3 \sum_{j=1}^n \left( \beta_{ij} \alpha_1 + \bar{\beta}_{ij} \alpha_2 \right) f_j(0)}{\mu_j^i + \delta_j^i + \gamma_j^i}, \quad 1 \leq i \leq n. \quad (21)$$

Thus, the above discussion implies that the system (5) is point dissipative. Furthermore, by Theorem 2.2.6 in [44], the solution semiflow $\Phi(t)$ is compact for any $t > 0$. Therefore, it follows from Theorem 3.4.8 in [45] that $\Phi(t)$ has a global compact attractor in $\mathbb{R}^+$. \hfill \Box

### 3. Basic Reproduction Number

For each $1 \leq i \leq n$, consider the following subsystem of model (5):

\[
\begin{align*}
\frac{\partial S_i}{\partial t} &= \nabla \cdot (d_{1i}(x)\nabla S_i) + \lambda_i(x) - \left( \mu_i^s(x) + \xi_i(x) \right) S_i, \quad t > 0, x \in \Omega, \\
\frac{\partial V_i}{\partial t} &= \nabla \cdot (d_{2i}(x)\nabla V_i) + \xi_i(x)S_i - \mu_i^v(x)V_i, \quad t > 0, x \in \Omega, \\
\frac{\partial S_i}{\partial \nu} &= 0, \quad t > 0, x \in \partial \Omega.
\end{align*}
\]

It follows from the Lemma 2.2 in [40] that the following system
Lemma 2. The eigenvalue problem (27) admits a principle eigenvalue \( \lambda_0 \) with a strictly positive eigenfunction.

Denote \( T_3 = (T_{31}, T_{32}, \ldots, T_{3n}) \). Assume that the distribution of initial infection is \( \phi_3(x) \in Y_+ \). Then, \( \mathcal{F}(x)T_3\phi_3(x) \) is the distribution of new infective part as time evolves. Therefore, we use

\[
\mathcal{L}(\phi_3)(x) = \int_0^\infty \mathcal{F}(x)T_2(t)\phi_3(x)dt
\]

(28)
to describe the total distribution of the new infective numbers produced during the infection period.

According to [14], the basic reproduction number is defined by \( \mathcal{R}_0 = r(\mathcal{L}) \), where \( r(\mathcal{L}) \) is the spectral radius of the operator \( \mathcal{L} \). Furthermore, following Theorem 3 in [14], we have the following lemma.

Lemma 3. The principle eigenvalue \( \lambda_0 \) and \( \mathcal{R}_0 - 1 \) have the same sign, and the disease-free steady state \( E_0(x) \) is locally asymptotically stable.

4. Extinction/Persistence Result

Based on the discussion above, we now investigate the extinction and persistence of the disease.

Theorem 2. If \( \mathcal{R}_0 < 1 \), then the disease-free steady state \( E_0(x) \) is globally asymptotically stable.

Proof. By Lemma 2, we have \( \lambda_0 < 0 \) when \( \mathcal{R}_0 < 1 \). Thus, there exists a small enough \( \varepsilon > 0 \) such that \( \lambda_0 + \varepsilon < 0 \). According to Theorem 1, there exists a \( t_* > 0 \) such that \( S_i(t, x) \leq S_i^0(x) + \varepsilon_0 \) and \( V_i(t, x) \leq V_i^0(x) + \varepsilon_0 (1 \leq i \leq n) \) for all \( x \in \Omega \). Thus, from the third equation of model (5) and assumption (4) that
responding to the principal eigenvalue \( \lambda_1^* < 0 \). Assume that

\[
\lim_{t \to \infty} \frac{S_i(t)}{V_i(t)} = \alpha_i, \quad \forall i, \quad 0 < \alpha_i < 1.
\]

where \( \alpha > 0 \) is a constant. With the aid of comparison principle, we can obtain

\[
\lim_{t \to \infty} \frac{S_i(t)}{V_i(t)} = \alpha_i, \quad \forall i, \quad 0 < \alpha_i < 1.
\]

This yields \( \lim_{t \to \infty} I(t, x) = 0 \) uniformly for \( x \in \Omega \). Thus, the model (5) is asymptotic to (22). Then, by Lemma 2.2 in [40] and Corollary 4.3 in [46], we have \( \lim_{t \to \infty} S(t, x) = S_i^0(x) \) and \( \lim_{t \to \infty} V(t, x) = V_i^0(x) \).

Before proving the main results on disease persistence, we first need to establish the following lemma.

\[\text{Lemma 4.} \quad \text{Let } u(\cdot, t, \phi) = (S(\cdot, t, \phi), V(\cdot, t, \phi), I(\cdot, t, \phi)) \text{ is the solution of model (5) with } u(\cdot, 0, \phi) = (\phi, \phi, \phi) \in \mathbb{R}^3.
\]

(i) For any \( \phi \in \mathbb{R}^3 \), we always have \( S_i(\cdot, t, \phi) > 0 \) and \( V_i(\cdot, t, \phi) > 0 \) for all \( t > 0 \), and there exist constants \( \rho_1 > 0, \rho_2 > 0 \) such that

\[
\liminf_{t \to \infty} S_i(\cdot, t, \phi) \geq \rho_1, \\
\liminf_{t \to \infty} V_i(\cdot, t, \phi) \geq \rho_2.
\]

(ii) If there exists \( t^* > 0 \) such that \( I_i(\cdot, t, \phi) \equiv 0 \), then we have \( I_i(\cdot, t, \phi) > 0, \forall t > t^*, 1 \leq i \leq n \).

Proof. (i) It follows from Theorem 1 that there exists \( t_0 > 0 \) and \( M > 0 \) such that

\[
I_i(\cdot, t, \phi) \leq M, \quad \forall t \geq t_0.
\]

Then, by the first equation of model (5) and the assumption (4), we have

\[
\left\{
\begin{aligned}
\frac{\partial S_i}{\partial t} &\geq \nabla \cdot (d_i \nabla S_i) + \lambda_i(x) - \left( \mu_i^S(x) + \xi_i(x) \right) S_i - \sum_{j=1}^{n} \beta_{ij}(x) M f_j^i(0) S_j, \\
\frac{\partial S_i}{\partial v} &\geq 0,
\end{aligned}
\right.
\quad x \in \Omega, \; t \geq t_0,
\]

Thus, by Lemma 2.2 in [40], it follows that the comparison system
endemic steady state uniformly for all \(x\). By the standard parabolic comparison theorem, there exists a constant \(\rho_1 > 0\) such that \(\liminf_{t \to \infty} S_i(t, x) \geq \rho_1 \) is uniformly for \(x \in \Omega\). Moreover, it follows from the second equation of model (5) that

\[
\begin{align*}
\frac{\partial v_i}{\partial t} &\geq \nabla \cdot (d_i \nabla v_i) + \xi_i(x)v_i - \nu_i f_i(x)v_i - \sum_{j=1}^{n} \beta_{ij}(x)Mf_i(0)v_j, & x \in \Omega, t \geq t_0, \\
\frac{\partial v_i}{\partial n} &\geq \nabla \cdot (d_i \nabla v_i) + \xi_i(x)v_i - \nu_i f_i(x)v_i - \sum_{j=1}^{n} \beta_{ij}(x)Mf_i(0)v_j, & x \in \partial \Omega, t \geq t_0.
\end{align*}
\]

Similarly, there exists a constant \(\rho_2\) such that \(\liminf_{t \to \infty} V_i(t, x) \geq \rho_2\) is uniformly for \(x \in \Omega\).

(ii) Assume \(I_i(t, t^*, \phi) \equiv 0\) for some \(t^* \geq 0\). According to Theorem 1, it follows from model (5) that

\[
\begin{align*}
\frac{\partial I_i}{\partial t} &\geq \nabla \cdot (d_i \nabla I_i) - \left(\mu_i^{I_i}(x) + \delta_i(x) + \gamma_i(x)\right) I_i, & x \in \Omega, t > 0, \\
\frac{\partial I_i}{\partial n} &\geq \nabla \cdot (d_i \nabla I_i) - \left(\mu_i^{I_i}(x) + \delta_i(x) + \gamma_i(x)\right) I_i, & x \in \partial \Omega, t > 0.
\end{align*}
\]

Thus, the strong maximum principle (see, e.g., [47]), Theorem 4) and Hopf boundary lemma (see, e.g., [47], Theorem 3) imply that \(I_i(t, t^*, \phi) > 0\), \(\forall t > t^*, \) and \(x \in \Omega\). Now, we are in position to state the main results of this section.

\[
\liminf_{t \to \infty} S_i(t, x) \geq 0, \liminf_{t \to \infty} V_i(t, x) \geq 0, \liminf_{t \to \infty} I_i(t, x) \geq 0, \quad 1 \leq i \leq n.
\]

uniformly for all \(x \in \Omega\). Moreover, model (5) has at least one endemic steady state \(E_*(x)\).

Proof. Define

\[
\begin{align*}
X_0 &= \{\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{K}^3: \phi_{3i} \equiv 0, \quad 1 \leq i \leq n\}, \\
\partial X_0 &= \mathbb{K}^3 \setminus X_0 = \{\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{K}^3: \phi_{3i} \equiv 0, \quad 1 \leq i \leq n\}.
\end{align*}
\]
According to Lemma 4, for any $\phi \in \mathcal{X}_0$, we have $I_i(t, \cdot, \phi) > 0$, $\forall t > 0$, and $x \in \Omega$. Thus, $\Phi(t) \mathcal{X}_0 \subseteq \mathcal{X}_0$, which implies that $\mathcal{X}_0$ is the invariant set for solution semiflow $\Phi(t)$ of model (5). Define
\[
\mathcal{M}_\beta = \{ \phi \in \mathcal{X}^+ : \Phi(t)\phi \in \partial \mathcal{X}_0, \, \forall t \geq 0 \},
\]
and $\omega(\phi)$ be the omega limit set of the orbit $\mathcal{O}^+(\phi) = \{ \Phi(t)\phi : t \geq 0 \}$. We first prove the following claim.

Claim 1. $\omega(\phi) = [E_0] = \{(S^0(x), V^0(x), 0)\}, \forall \phi \in \mathcal{M}_\beta$. Since $\phi \in \mathcal{M}_\beta$, we have $\Phi(t)\phi \in \partial \mathcal{X}_0$. Thus, $I_i(t, \cdot, \phi) = 0$ $(1 \leq i \leq n)$, $\forall t \geq 0$. Then, model (5) reduces to model (22); by Lemma 2.2 in [40] and Corollary 4.6 in [46], we have $\lim_{t \to \infty} S_i(t, \cdot, \phi) = S_i^0(x)$ and $\lim_{t \to \infty} V_i(t, \cdot, \phi) = V_i^0(x)$ uniformly for $x \in \Omega$. Hence, $\omega(\phi) = [E_0] = \{(S^0(x), V^0(x), 0)\}$, $\forall \phi \in \mathcal{M}_\beta$; i.e., the claim holds.

Since $\mathcal{R}_0 > 1$, we have $\lambda_\beta > 0$ by Lemma 3. Using the continuity of $\lambda_\beta$, there exists a sufficient small enough positive constant $\eta_0 > 0$ such that $\lambda_\beta^0 > 0$.

Claim 2. $E_0(x)$ is uniform weak repeller for $\mathcal{X}_0$ in the sense that
\[
\limsup_{t \to -\infty} \| \Phi(t)\phi - E_0(x) \| \geq \eta_0, \quad \forall x \in \mathcal{X}_0.
\]
Suppose, by contradiction, there exists a $\phi_0 \in \mathcal{X}_0$ such that
\[
\limsup_{t \to -\infty} \| \Phi(t)\phi - E_0(x) \| < \eta_0.
\]
Then, there exists an enough large $t^* > 0$ such that
\[
S_i^0(x) - \eta_0 < S_i(t, \cdot, \phi) < S_i^0(x) + \eta_0, \quad V_i^0(x) - \eta_0 < V_i(t, \cdot, \phi) < V_i^0(x) + \eta_0, \quad 0 < I_i(t, \cdot, \phi) < \eta_0, 1 \leq i \leq n, \forall t \geq t^*, x \in \Omega.
\]
Therefore, from model (5) and assumptions (4), we have
\[
\left\{ \begin{array}{l}
\frac{\partial \psi}{\partial t} \geq \nabla \cdot (d_0(x)\nabla \psi_i) + \sum_{j=1}^{n} \left[ \beta_{ij}(x)(S_j^0(x) - \eta_0) + \bar{\beta}_{ij}(x)(V_j^0(x) - \eta_0) \right] f_i(\eta_0) I_i - \left( \mu_i(x) + \delta_i(x) + \gamma_i(x) \right) I_i, \quad x \in \Omega, t \geq t^*, 1 \leq i \leq n, \\
\frac{\partial \psi}{\partial \nu} = 0, \quad x \in \partial \Omega, t \geq t^*, 1 \leq i \leq n.
\end{array} \right.
\]

Set $(\varphi_{31}(x), \varphi_{32}(x), \ldots, \varphi_{3n}(x))$ be the strongly positive eigenfunction associated with principle eigenvalue $\lambda_\beta^0 > 0$. Since $I_i(t, \cdot, \phi_0) > 0$ for all $x \in \Omega$ and $t > 0$, there exists a constant $\xi_0 > 0$ such that $I(t^*, \cdot, \phi_0) \geq \xi_0 (\varphi_{31}(x), \varphi_{32}(x), \ldots, \varphi_{3n}(x))$ for $x \in \overline{\Omega}$. It is clear that $\omega(t, x) = \xi_0 e^{\psi_{30}(t-\tau)} (\varphi_{31}(x), \varphi_{32}(x), \ldots, \varphi_{3n}(x))$ is a solution of the following linear system:
\[
\left\{ \begin{array}{l}
\frac{\partial \omega}{\partial t} = \nabla \cdot (d_0(x)\nabla \omega_i) + \sum_{j=1}^{n} \left[ \beta_{ij}(x)(S_j^0(x) - \eta_0) + \bar{\beta}_{ij}(x)(V_j^0(x) - \eta_0) \right] f_i(\eta_0) \omega_i - \left( \mu_i(x) + \delta_i(x) + \gamma_i(x) \right) \omega_i, \quad x \in \Omega, t \geq t^*, 1 \leq i \leq n, \\
\frac{\partial \omega_i}{\partial \nu} = 0, \quad x \in \partial \Omega, t \geq t^*, 1 \leq i \leq n.
\end{array} \right.
\]

Applying the comparison principle, we have
\[
I(t, \cdot, \phi_0) \geq \xi_0 e^{\psi_{30}(t-\tau)} (\varphi_{31}(x), \varphi_{32}(x), \ldots, \varphi_{3n}(x)), \quad \forall x \in \overline{\Omega}, t \geq t^*.
\]

Due to $\lambda_\beta^0 > 0$, we conclude that $I_i(t, \cdot, \phi_0)$ is unbounded. This is a contradiction.

Define a continuous function $p : \mathcal{X}^+ \to [0, +\infty)$ by
\[
p(\phi) = \min_{x \in \Omega} \{ \min_{1 \leq i \leq n} \varphi_{3i}(x) \}, \quad 1 \leq i \leq n, \forall \phi \in \mathcal{X}^+.
\]

Clearly, $p^{-1}(0, +\infty) \subseteq \mathcal{X}_0$ and has the property that if $p(\phi) > 0$ or $p(\phi) = 0$ and $\phi \in \mathcal{X}_0$, then $p(\Phi(t)\phi) > 0$ for all $t > 0$. Hence, for the semiflow $\Phi(t)$, $\mathcal{X}^+ \to \mathcal{X}^+$, $p$ is a generalized distance function [48]. From Claim 1, it follows that any forward orbit of $\Phi(t)$ in $\mathcal{M}_\beta$ converges to $E_0(x)$. Moreover, Claim 2 implies that $E_0(x)$ is isolated in $\mathcal{X}^+$ and $W^u(E_0(x)) \cap \mathcal{X}_0 = \emptyset$, where $W^u(E_0(x))$ is the stable set of $E_0(x)$. Furthermore, there is no cycle in $\mathcal{M}_\beta$ from $E_0(x)$ to $E_0(x)$. It then follows from Theorem 3 in [48] that there exists a constant $p > 0$ such that $\liminf_{t \to -\infty} p(\Phi(t)\phi) \geq p$ for all $\phi \in \mathcal{X}_0$. This implies the uniform permanence of $I(t, \cdot, \phi)$. \]
By Lemma 4 and Theorem 4.7 in [49], we know that $\Phi(t)$ has a positive steady state $E_*(x)$ of model (5). The proof is complete.

With the homogeneous Neumann boundary conditions (2) and the initial conditions (3). We mainly focus on the global stability of the endemic steady states of model (48). The proofs of the main results applying the Lyapunov functions and a subtle grouping technique guided by graph theory were developed in [28–30].

It follows from the Lemma 2 in [13] that model (48) has a disease-free steady state $E_0 = (S^0, V^0, 0)$, where $S^0 = (S^0_1, S^0_2, \ldots, S^0_\nu)$ and $V^0 = (V^0_1, V^0_2, \ldots, V^0_\nu)$ with $S^0_1 = \lambda_i/\mu^0_i + \xi_i$ and $V^0_1 = \lambda_i^0/\mu^0_i (\mu^0_i + \xi_i)$. Define matrices

$$
\mathcal{F} = \begin{pmatrix}
\theta_{11}^0 f'_1(0) & \theta_{12}^0 f'_2(0) & \cdots & \theta_{1\nu}^0 f'_\nu(0) \\
\theta_{21}^0 f'_1(0) & \theta_{22}^0 f'_2(0) & \cdots & \theta_{2\nu}^0 f'_\nu(0) \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{\nu 1}^0 f'_1(0) & \theta_{\nu 2}^0 f'_2(0) & \cdots & \theta_{\nu\nu}^0 f'_\nu(0)
\end{pmatrix}
$$

and

$$
\mathcal{V} = \begin{pmatrix}
\mu^0_i + \delta_i + \gamma_1 & 0 & \cdots & 0 \\
0 & \mu^0_1 + \delta_1 + \gamma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu^0_\nu + \delta_\nu + \gamma_\nu
\end{pmatrix}
$$

where $\theta_{ij}^0 = \beta_{ij} S^0_i + \beta_{ij} V^0_i$, then we have

$$
\mathcal{F} \mathcal{V}^{-1} = \begin{pmatrix}
\theta_{ij}^0 f'_j(0) \\
\mu^0_j + \delta_j + \gamma_j
\end{pmatrix}_{1 \leq i, j \leq \nu}.
$$

Then, it follows from [26] that the basic reproduction number is defined as the spectral radius of $\mathcal{F} \mathcal{V}^{-1}$, i.e., $\mathcal{R}_0 = r(\mathcal{F} \mathcal{V}^{-1})$. As a consequence of Theorems 2 and 3, we can obtain the following results without proofs.

**5. Global Dynamics for a Special Case**

In this section, we consider a special case with all the parameters of model (5) as constants, except for the diffusion coefficients. Then, model (5) degenerates into the following model:

$$
\begin{align*}
\frac{dS_i}{dt} &= \nabla \cdot (d_1(x)\nabla S_i) + \lambda_i - (\mu^0_i + \xi_i)S_i - \sum_{j=1}^{\nu} \beta_{ij} S_i f'_j(I_j), \\
\frac{dV_i}{dt} &= \nabla \cdot (d_2(x)\nabla V_i) + \xi_i S_i - \sum_{j=1}^{\nu} \beta_{ij} V_i f'_j(I_j) - \mu^0_i V_i, \\
\frac{dI_i}{dt} &= \nabla \cdot (d_3(x)\nabla I_i) + \sum_{j=1}^{\nu} (\beta_{ij} S_i + \beta_{ij} V_i) f'_j(I_j) - (\mu^0_i + \delta_i + \gamma_i)I_i,
\end{align*}
$$

$$
t \geq 0, \ t \in \Omega, \ 1 \leq i \leq \nu,
$$

(48)

**Theorem 4.** The disease-free steady state $E_0$ of model (48) is globally asymptotically stable when $\mathcal{R}_0 < 1$.

**Theorem 5.** If $\mathcal{R}_0 > 1$, then there exists a constant $\varrho > 0$ such that, for any $\phi = (\phi_i, \phi_2, \phi_3) \in \mathbb{R}^+$ with $\phi_3 \equiv 0 (1 \leq i \leq \nu)$, solution $u(t, \cdot, \phi) = (S(t, \cdot, \phi), V(t, \cdot, \phi), I(t, \cdot, \phi))$ of model (48) satisfies

$$
\liminf_{t \to \infty} S(t, \cdot, \phi) \geq \varrho, \quad \liminf_{t \to \infty} V(t, \cdot, \phi) \geq \varrho, \quad \liminf_{t \to \infty} I(t, \cdot, \phi) \geq \varrho,
$$

$$
1 \leq i \leq \nu,
$$

(51)

uniformly for all $x \in \tilde{\Omega}$. Moreover, model (48) has at least one endemic steady state.

Furthermore, set the endemic steady state $E_* = (S_*, V_*, I_*)$ with $S_* = (S^*_1, S^*_2, \ldots, S^*_\nu)$, $V_* = (V^*_1, V^*_2, \ldots, V^*_\nu)$, and $I_* = (I^*_1, I^*_2, \ldots, I^*_\nu)$ satisfying

$$
\begin{align*}
\lambda_i &= (\mu^* + \xi^*)_i + \sum_{j=1}^{\nu} \beta_{ij} S^*_i f'_j(I^*_j), \quad 1 \leq i \leq \nu, \\
\xi^*_i S^*_i &= \sum_{j=1}^{\nu} \beta_{ij} V^*_i f'_j(I^*_j) + \mu^*_i V^*_i, \quad i \leq n, \\
\sum_{j=1}^{\nu} (\beta_{ij} S^*_i + \beta_{ij} V^*_i) f'_j(I^*_j) &= (\mu^*_i + \delta_i + \gamma_i)I^*_i, \quad 1 \leq i \leq \nu.
\end{align*}
$$

(52)

For the global stability of the endemic steady state $E_*$, we have the following conclusion.

**Theorem 6.** If $\mathcal{R}_0 > 1$, then the endemic steady state $E_*$ is globally asymptotically stable.
Proof. Define

\[ a_{ij} = \left( \beta_{ij} S_i^* + \tilde{b}_{ij} V_i^* \right) f_j(I_j^r), \quad (1 \leq i, j \leq n), \]

\[ \Theta = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & \sum_{j=2}^{n} a_{2j} & \cdots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & \sum_{j=n}^{n} a_{nj} \end{pmatrix}, \quad (53) \]

which is a Laplacian matrix whose column sums are zero [30]. Then, \( \Theta \) is irreducible because \( (\tilde{b}_{ij})_{mn} \) is irreducible. It follows from Lemma 1 in [29] that the solution space of the linear system \( \Theta \xi = 0 \) has dimension 1 with a base \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) with \( \xi_i = c_{ii} \), where \( c_{ii} > 0 \) is the cofactor of the \( i \)-th diagonal entry of \( \Theta \).

Define

\[ \Psi(t) = \sum_{i=1}^{n} \xi_i H_i(t), \quad (54) \]

where

\[ H_i(t) = \int_{\Omega} \left[ \left( S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} \right) + \left( V_i - V_i^* - V_i^* \ln \frac{V_i}{V_i^*} \right) + \left( I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*} \right) \right] \text{dx}. \quad (55) \]

Differentiating \( H_i(t) \), we obtain

\[
\frac{dH_i(t)}{dt} = \int_{\Omega} \left\{ \left( 1 - \frac{S_i}{S_i^*} \right) \left[ \nabla \cdot (d_{ii}(x) \nabla S_i) + \lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i f_j(I_j) - (\mu_i^S + \xi_i) S_i \right] \\
+ \left( 1 - \frac{V_i}{V_i^*} \right) \left[ \nabla \cdot (d_{ii}(x) \nabla V_i) + \xi_i S_i - \sum_{j=1}^{n} \beta_{ij} V_i f_j(I_j) - \mu_i^V V_i \right] + \left( 1 - \frac{I_i}{I_i^*} \right) \right. \\
\times \left. \left[ \nabla \cdot (d_{ii}(x) \nabla I_i) + \sum_{j=1}^{n} \left( \beta_{ij} S_i + \tilde{b}_{ij} V_i \right) f_j(I_j) - (\mu_i^I + \delta_i + \gamma_i) I_i \right] \right\} \text{dx}. \quad (56)\\
\]

It follows from [50] that

\[
\begin{align*}
\int_{\Omega} \nabla \cdot d_{ii}(x) \nabla S_i \text{dx} &= 0, \\
\int_{\Omega} \frac{1}{S_i} \nabla \cdot d_{ii}(x) \nabla S_i \text{dx} &= \int_{\Omega} d_{ii}(x) \frac{\|\nabla S_i\|^2}{S_i^2} \text{dx}, \\
\int_{\Omega} \nabla \cdot d_{ii}(x) \nabla V_i \text{dx} &= 0, \\
\int_{\Omega} \frac{1}{V_i} \nabla \cdot d_{ii}(x) \nabla V_i \text{dx} &= \int_{\Omega} d_{ii}(x) \frac{\|\nabla V_i\|^2}{V_i^2} \text{dx}, \\
\int_{\Omega} \nabla \cdot d_{ii}(x) \nabla I_i \text{dx} &= 0, \\
\int_{\Omega} \frac{1}{I_i} \nabla \cdot d_{ii}(x) \nabla I_i \text{dx} &= \int_{\Omega} d_{ii}(x) \frac{\|\nabla I_i\|^2}{I_i^2} \text{dx}. \quad (57)\]
\]
Then, using steady state equations (52), we have

\[
\frac{dH_i(t)}{dt} = \int_\Omega \left\{ \mu_i^* S_i^* \left( 2 - \frac{S_i}{S^*} \right) + \mu_i V_i^* \left( 3 - \frac{V_i}{V^*} \right) - \frac{S_i}{S^*} - \frac{I_i}{I^*} + \sum_{j=1}^n \beta_{ij} S_i^* f_j(I_j^*) \left( 2 - \frac{S_i}{S^*} - \frac{I_i}{I^*} + f_j(I_j^*) - f_j(I_j^*) \right) \right\} \ dx \\
+ \sum_{j=1}^n \bar{\beta}_{ij} V_i^* f_j(I_j^*) \left( 3 - \frac{S_i}{S^*} - \frac{I_i}{I^*} + \frac{S_i V_i^*}{S^* V_i^*} - \frac{V_i I_i f_j(I_j^*)}{V_i f_j(I_j^*)} \right) - d_{si}(x) S_i^* \left\| \nabla S_i \right\|^2 - d_{2i}(x) V_i^* \left\| \nabla V_i \right\|^2 \\
- d_{si}(x) I_i^* \left\| \nabla I_i \right\|^2 \right\} \ dx \\
- \frac{dH_i(t)}{dt} \leq \int_\Omega \left[ \sum_{j=1}^n \left( \beta_{ij} S_i^* + \bar{\beta}_{ij} V_i^* \right) f_j(I_j^*) \left( f_j(I_j^*) - f_j(I_j^*) \right) \right] \ dx.
\]

where \( \varphi(x) = 1 + \ln x - x \) with global maximum value \( \varphi(x) = 0 \). From the assumption (4), it is easy to validate that

\[
\left( f_j(I_j^*) - f_j(I_j^*) \right) \left( 1 - \frac{f_j(I_j^*)}{f_j(I_j^*)} \right) \leq 0.
\]

Moreover, according to the property that arithmetic mean is not less than the associated geometric mean,

\[
\frac{V_i}{V_i^*} + \frac{S_i}{S_i^*} + \frac{S_i V_i^*}{S_i^* V_i^*} \geq 3 \quad \text{and} \quad S_i/S_i^* + S_i^*/S_i^* \geq 2.
\]

Thus, we have

\[
\frac{dH_i(t)}{dt} \leq \int_\Omega \left[ \sum_{j=1}^n \left( \beta_{ij} S_i^* + \bar{\beta}_{ij} V_i^* \right) f_j(I_j^*) \left( f_j(I_j^*) - f_j(I_j^*) \right) \right] \ dx.
\]

Consequently, we obtain

\[
\frac{d\Phi(t)}{dt} \leq \int_\Omega \sum_{j=1}^n \left[ \sum_{j=1}^n \left( \beta_{ij} S_i^* + \bar{\beta}_{ij} V_i^* \right) f_j(I_j^*) \left( f_j(I_j^*) - f_j(I_j^*) \right) \right] \ dx.
\]
It follows from the equality $\Theta \zeta = 0$ that $\sum_{j=1}^{n} a_{ij} \zeta_j = \sum_{i=1}^{n} a_{ji} \zeta_i$ which is equivalent to $\sum_{j=1}^{n} (\beta_{ij} S_i^* + \beta_{ji} V_i^*) f_j(I_j^*) \zeta_j = \sum_{i=1}^{n} (\beta_{ji} S_i^* + \beta_{ij} V_j^*) f_i(I_i^*) \zeta_i$. Then,

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ij} S_i^* + \beta_{ji} V_j^*) f_j(I_j^*) \zeta_j = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ji} S_j^* + \beta_{ij} V_i^*) f_i(I_i^*) \zeta_i
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ij} S_j^* + \beta_{ji} V_i^*) f_i(I_i^*) \zeta_j = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ji} S_j^* + \beta_{ij} V_i^*) f_j(I_j^*) \zeta_i
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ij} S_i^* + \beta_{ji} V_j^*) f_j(I_j^*) f_i(I_i^*) \zeta_i.
$$

and thus, $\sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ij} S_i^* + \beta_{ji} V_j^*) f_j(I_j^*) (f_i(I_i^*)/f_j(I_j^*) - f_i(I_i^*)/f_i(I_i^*)) = 0$ for all $I_1, I_2, \ldots, I_n > 0$. Following the graph-theoretic approach as proposed in [28–30], similar procedures as in [26] can be applied to verify that

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_{ij} S_i^* + \beta_{ji} V_j^*) f_j(I_j^*) \ln \frac{f_j(I_j^*) f_i(I_i^*)}{f_i(I_i^*) f_j(I_j^*)} = 0.
$$

Thus, we have $d\Phi(t)/dt \leq 0$. Furthermore, it can be shown that the largest invariant set in $\{(S, V, I) \in \mathbb{R}^3 \mid d\Phi(t)/dt = 0\}$ is the singleton $\{E_\ast\}$. Therefore, by the LaSalle’s invariance principle, $E_\ast$ is globally asymptotically stable.

6. Conclusions

In this paper, a multigroup SVIR model with diffusion and nonlinear incidence rate has been investigated. We defined the basic reproduction number $R_0$ for spatially heterogeneous environment, and we further proved that $R_0$ served as a threshold index which predicts the extinction and persistence of the disease. Particularly, by using comparison principle, we proved that the disease-free steady state $E_0(x)$ is globally asymptotically stable when $R_0$ is less than one. If $R_0$ is great than one, then the disease will persist. Consequently, we obtained the existence of endemic steady state $E_\ast(x)$. Furthermore, we established the criteria on the global stability of the disease-free steady state and the endemic steady state of the model in a special case.

Although the existence of the endemic steady state is established, it is necessary to point that the uniqueness and stability of the endemic steady state remains an open problem. Moreover, the impact of the latent individuals should also be considered for some diseases during its spread, such as AIDS and COVID-19 (see, for example, [4–8, 51]), in which it has been validated that the latent individuals for these two diseases may also have probability of infection. Thus, an improved model with latent compartment should be investigated. We leave these problems for future investigation.

Data Availability

There are no data in the submission.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11701445, 11701451, and 11702214), Natural Science Basic Research Plan in Shaanxi Province of China (2018JQ1057 and 2020JQ-38), and Young Talent Fund of University Association for Science and Technology in Shaanxi, China (20180504).

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