A Privacy-Preserving Distributed Control of Optimal Power Flow

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Abstract—We consider a distributed optimal power flow formulated as an optimization problem that maximizes a nondifferentiable concave function. Solving such a problem by the existing distributed algorithms can lead to data privacy issues because the solution information exchanged within the algorithms can be utilized by an adversary to infer the data. To preserve data privacy, in this paper we propose a differentially private projected subgradient (DP-PS) algorithm that includes a solution encryption step. We show that a sequence generated by DP-PS converges in expectation, in probability, and with probability 1. Moreover, we show that the rate of convergence in expectation is affected by a target privacy level of DP-PS chosen by the user. We conduct numerical experiments that demonstrate the convergence and data privacy preservation of DP-PS.

Index Terms—Differential privacy, projected subgradient algorithm, optimal power flow, dual decomposition.

I. INTRODUCTION

Optimal power flow (OPF) is an important problem in reliably and economically operating electric grids. Currently, the problem is solved by independent system operators in a centralized manner. Recently, however, distributed OPF has been spotlighted as a result of the introduction of microgrids with energy storage [1] and increasing penetrations of distributed energy resources [2]. Distributed OPF consists of (i) a set of OPF subproblems defined for each zone of the power grid and (ii) consensus constraints that link the subproblems. Distributed OPF can be solved by the existing distributed algorithms (e.g., [3], [4], [5], [6]), which do not require sharing private data information (e.g., demand data from each zone) but send the local solutions to the central machine. Unfortunately, an adversary may be able to estimate the data based on the solutions (e.g., reverse engineering [7]), thus motivating the need for solution encryption.

Differential privacy (DP) is a randomization technique that guarantees the existence of multiple datasets with similar probabilities of resulting in the encrypted solution, thus preserving data privacy [8]. A differentially private algorithm is an algorithm that incorporates differential privacy for preserving data privacy during the algorithmic process [9]. Several DP algorithms have been proposed to solve various distributed optimization problems. For example, (i) a DP alternating direction method of multipliers (ADMM) was proposed for solving a distributed empirical risk minimization problem [10], [11], [12] and a distributed DC OPF [13], and (ii) a DP stochastic gradient descent (SGD) method was proposed for solving a classification problem [14], a resource allocation problem [15], and deep neural networks [16].

We define target privacy level (TPL) as a user parameter for DP algorithms to control data privacy. While guaranteeing stronger privacy, increasing TPL of the DP algorithms may affect the convergence. For example, DP-ADMM with higher TPL is shown to find suboptimal solutions, implying the need for a trade-off between data privacy and solution quality [11], [13]. On the other hand, the numerical results in [14] show that the solution accuracy from DP-SGD may be close to that of non-private SGD. Also, Huang et al. [12] report numerical experiments that DP-SGD has good noise resilience compared with that of DP-ADMM, but it converges slowly. While numerical evidence has been demonstrated for the trade-off between the convergence of DP algorithms and TPL, only a few studies (e.g., DP-ADMM in [12]) develop the theoretical links.

In this paper we present a DP projected subgradient (PS) algorithm for solving a distributed OPF while preserving data privacy, and we study how TPL affects the convergence of DP-PS theoretically and numerically. We first formulate the distributed second-order conic (SOC) and alternating current (AC) OPF (see, e.g., [17], [18], [19]) based on dual decomposition by taking the Lagrangian relaxation with respect to the consensus constraints, where supergradients are computed by solving the OPF subproblems in parallel. Moreover, in order to guarantee data privacy, the supergradients exchanged within the algorithm are systematically randomized by adding random noise extracted from a Laplace distribution. Under three rules of specifying search direction and step size, we show that a sequence generated by DP-PS converges in expectation, in probability, and with probability 1. In particular, we show that the convergence
complexity is affected by a constant factor only as TPL increases. In summary, this paper answers the following questions:

- How can we preserve the privacy of the data communicated in the distributed OPF setting?
- Can DP guarantee data privacy?
- What are the implications of adding the DP technique to the distributed OPF setting?
- Can our technical development be numerically demonstrated?

The remainder of the paper is organized as follows. In Section II, we present a distributed OPF problem. In Section III, we describe a differentially private control with the proposed DP-PS. In Section IV, we study the convergence of DP-PS. We conduct case studies in Section V and summarize our conclusions in Section VI. We denote by \( \mathbb{N} \) a set of natural numbers. For \( A \in \mathbb{N} \), we define \([A] := \{1, \ldots, A\}\). We use \((\cdot, \cdot)\) and \(\|\cdot\|\) to denote the scalar product and the Euclidean norm.

## II. DISTRIBUTED OPTIMAL POWER FLOW

We depict a power network by a graph \((\mathcal{N}, \mathcal{L})\), where \( \mathcal{N} \) is a set of buses and \( \mathcal{L} \) is a set of lines. For every line \( \ell_{ij} \in \mathcal{L} \), where \( i \) is a from bus and \( j \) is a to bus of line \( \ell \), we are given line parameters, including bounds \([\theta_{ij}, \pi_{ij}]\) on voltage angle difference, thermal limit \(\pi_{ij} \), resistance \(r_{ij} \), reactance \(x_{ij} \), impedance \(z_{ij} := r_{ij} + \mathbf{i}x_{ij} \), line charging susceptance \(b_{ij} \), tap ratio \(\tau_{ij} \), phase shift angle \(\theta_{ij} \), and admittance matrix \(Y_{ij}\), namely,

\[
Y_{ij} := \begin{bmatrix} Y_{ij}^{R} & Y_{ij}^{P} \\ Y_{ij}^{P} & Y_{ij}^{I} \end{bmatrix} = \begin{bmatrix} (\tau_{ij}^{-1}+\frac{1}{\theta_{ij}}) \frac{1}{\xi_{ij}} \mathbf{1}x - z_{ij}^{-1} \frac{1}{\tau_{ij} x} \mathbf{1}y \mathbf{1}x \mathbf{1}y \\ -z_{ij}^{-1} \frac{1}{\tau_{ij} x} \mathbf{1}y \mathbf{1}x \mathbf{1}y \end{bmatrix},
\]

\(G_{ij} := \Re(Y_{ij}^{R})\), \(B_{ij} := \Im(Y_{ij}^{P})\), \(G_{ii} := \Im(Y_{ii}^{R})\), \(B_{ii} := \Im(Y_{ii}^{P})\), \(G_{jj} := \Re(Y_{jj}^{R})\), \(B_{jj} := \Re(Y_{jj}^{P})\), and \(B_{ij} := \Im(Y_{ij}^{P})\). For every bus \( i \in \mathcal{N} \), we are given bus parameters, including bounds \([\omega_{i}, \bar{\omega}_{i}] \) on voltage magnitude, active (resp., reactive) power demand \(p_{i}^{d} \) (resp., \(q_{i}^{d}\)), shunt conductance \(g_{i}^{s} \), and shunt susceptance \(b_{i}^{s} \). Furthermore, for every \( i \in \mathcal{N} \), we define subsets \(L_{i}^{c} := \{\ell_{ij} : j \in \mathcal{N}, \ell_{ij} \in \mathcal{L}\} \) and \(L_{i}^{g} := \{\ell_{ij} : j \in \mathcal{N}, \ell_{ij} \in \mathcal{L}\} \) of a set of lines and a set of generators \(G_{i}\). For every generator \( g \in G_{i} \), we are given generator parameters, including bounds \([g_{i}^{c}, \bar{g}_{i}^{c}] \) (resp., \([g_{i}^{q}, \bar{g}_{i}^{q}]\)) on the amounts of active (resp., reactive) power generation and coefficients \((c_{1,g}, c_{2,g})\) of the quadratic generation cost function.

Next we present decision variables. For every line \( \ell_{ij} \in \mathcal{L} \), we denote active (resp., reactive) power flow along line \( \ell \) by \( p_{ij}^{l}, q_{ij}^{l} \) (resp., \( g_{ij}^{l}, f_{ij}^{l} \)). For every \( i \in \mathcal{N} \), we denote the complex voltage by \( V_{i} = v_{i}^{g} + \mathbf{i}v_{i}^{l} \), and we introduce the following auxiliary variables:

\[
w_{ij}^{R} = v_{i}^{g} v_{j}^{g}, \quad w_{ij}^{I} = v_{i}^{g} v_{j}^{l} + v_{i}^{l} v_{j}^{g}, \quad w_{ij}^{R} = v_{i}^{l} v_{j}^{l}, \quad \forall j \in \mathcal{N}. \tag{1}
\]

For every generator \( g \in G_{i} \), we denote the amounts of active (resp., reactive) power generation by \( p_{g}^{d} \) (resp., \( q_{g}^{d} \)). In the following, we present a SOC OPF formulation:

\[
\min \sum_{i \in \mathcal{N}} \sum_{g \in G_{i}} \left( c_{1,g} p_{g}^{d} + c_{2,g} (p_{g}^{d})^{2} \right) \tag{2a}
\]

subject to

\[
\forall \ell_{ij} \in \mathcal{L} : \quad p_{ij}^{l} = G_{ij} (w_{ij}^{R} + w_{ij}^{I}) + B_{ij} (w_{ij}^{I} + w_{ij}^{R}) + G_{ij} (w_{ij}^{R} + w_{ij}^{I}) \tag{2b}
\]

\[
q_{ij}^{l} = G_{ij} (w_{ij}^{I} + w_{ij}^{R}) - B_{ij} (w_{ij}^{I} + w_{ij}^{R}) - G_{ij} (w_{ij}^{R} + w_{ij}^{I}) \tag{2c}
\]

\[
\forall i \in \mathcal{N} : \quad \sum_{\ell_{ij} \in L_{i}^{c}} p_{ij}^{l} + \sum_{\ell_{ij} \in L_{i}^{g}} p_{ij}^{l} = \sum_{g \in G_{i}} p_{g}^{d} - p_{g}^{d} - G_{ij} (w_{ij}^{R} + w_{ij}^{I}) \tag{2h}
\]

\[
\forall i \in \mathcal{N}, \forall g \in G_{i} : \quad p_{ij}^{l} = p_{ij}^{d} \tag{2i}
\]

\[
\forall i \in \mathcal{N} : \quad \sum_{\ell_{ij} \in L_{i}^{c}} q_{ij}^{l} + \sum_{\ell_{ij} \in L_{i}^{g}} q_{ij}^{l} = \sum_{g \in G_{i}} q_{g}^{d} - q_{g}^{d} + B_{ij} (w_{ij}^{R} + w_{ij}^{I}) \tag{2j}
\]

\[
\forall i \in \mathcal{N} : \quad \sum_{\ell_{ij} \in L_{i}^{c}} B_{ij} (w_{ij}^{I} + w_{ij}^{R}) + \sum_{\ell_{ij} \in L_{i}^{g}} B_{ij} (w_{ij}^{I} + w_{ij}^{R}) = \sum_{g \in G_{i}} B_{ij} (w_{ij}^{I} + w_{ij}^{R}) \tag{2k}
\]

\[
\forall i \in \mathcal{N} : \quad (w_{ij}^{R} + w_{ij}^{I})^{2} + (w_{ij}^{I} + w_{ij}^{R})^{2} \leq \left( \frac{w_{ij}^{R} + w_{ij}^{I} + w_{ij}^{I} + w_{ij}^{R}}{2} \right)^{2} \tag{2l}
\]

where (2a) is to minimize the generation cost, (2b)–(2e) represent power flow, (2f) represent line thermal limit, (2g) represent bounds on voltage angle differences, (2h)–(2i) represent power balance, (2j) represent bounds on voltage magnitudes, (2k) represent bounds on power generation, and (2l) represent SOC constraints that ensure linking between auxiliary variables.

We remark that this paper uses the SOC OPF formulation for example. The technical development and results should remain true with any convex relaxation of the OPF problem. For example, one can introduce SOCP strengthening techniques [19], [20], semidefinite programming relaxation, or quadratic convex relaxation [17]. In this work, however, we focus on solving one of the convex relaxation techniques, SOC OPF (2) in a distributed and privacy-preserving manner.

We decompose the network into several zones indexed by \( Z := \{1, \ldots, Z\} \). Specifically, we split a set \( \mathcal{N} \) of buses into subsets \( \{\mathcal{N}_{z}\}_{z \in Z} \) such that \( \mathcal{N} = \cup_{z \in Z} \mathcal{N}_{z} \) and \( \mathcal{N}_{z} \cap \mathcal{N}_{z'} = \emptyset \) for \( z, z' \in Z \), \( z \neq z' \). For each zone \( z \in Z \) we define a line set \( L_{z} := \{\ell_{ij} \in \mathcal{L} : \ell_{ij} \in \mathcal{L}_{z}\} \); an extended node set \( V_{z} := \{i \in \mathcal{N}_{z} : i \not\in \mathcal{N}_{z} \} \), where \( \mathcal{A}_{i} \) is a set of adjacent buses of \( i \); and a set of cuts \( C_{z} := \{z \in \mathcal{N}_{z} : z \neq \mathcal{N}_{z} \} \). Note that \( \{\mathcal{N}_{z}\}_{z \in Z} \) is a collection of disjoint sets, while \( \{\mathcal{L}_{z}\}_{z \in Z} \) and \( \{V_{z}\}_{z \in Z} \) are not. Using these notations, we rewrite problem (2) as

\[
\min \sum_{z \in Z} \sum_{i \in \mathcal{N}} f_{z} \sum_{x_{z}} \tag{3a}
\]
\[ (x_z, y_z) \in F_z(\bar{D}_z), \ \forall z \in Z, \quad (3b) \]
\[ \phi_i = x_{zi}, \ \forall z \in Z, \forall i \in C(z), \quad (3c) \]
\[ \phi_i \in R, \ \forall i \in C, \ \text{where} \]
\[ x_z \leftarrow \{ p_{\overline{D}, q_{\overline{D}}, \overline{q}_{\overline{D}}, \overline{q}_{\overline{D}}}, u_{\overline{D}}, w_{\overline{D}} \} \cup \{ p_{\delta}, q_{\delta}, \overline{q}_{\delta}, w_{\overline{D}}, \overline{w}_{\overline{D}} \} \cup \{ \ell_{ij} \} \in \mathcal{C}_z \]
\[ \phi \leftarrow \{ p_{\overline{D}, q_{\overline{D}}, \overline{q}_{\overline{D}}, \overline{q}_{\overline{D}}}, u_{\overline{D}}, w_{\overline{D}} \} \cup \{ \ell_{ij} \} \in \mathcal{C}_z \]
\[ C(z) \text{ is an index set that indicates each element of } y_z, \ C := \{ x_{zi} : (2b) - (2g), \ \forall i \in C(z) \} \]
\[ \text{Note that the constraints consist of redundant but numerically beneficial. By introducing a dual vector } \lambda := \{ x_{zi} \} \in \mathcal{C}_z \text{ with associated constraints (3c), one can construct a } \]
\[ \max \left\{ H(\lambda) := \sum_{z \in Z} h_z(\lambda_z) \right\} , \quad (4a) \]
\[ \text{subject to } f_z(x_z) + \sum_{i \in C(z)} \lambda_{zi} y_{zi} \leq 0, \quad (4b) \]
\[ \text{We emphasize that, for a given } \lambda, \text{ evaluating } H(\lambda) \text{ can be done by solving the subproblem (4b) in parallel. Let } \lambda^* \text{ be a maximizer of the nondifferentiable concave function } H(\lambda). \text{ Then } H(\lambda^*) \text{ is the optimal value of (3) by the strong duality from the convexity of } F_z(\bar{D}_z). \]

**Remark 1.** If (2l) are replaced with (1), then (2) is a rectangular formulation of AC OPF. In this case, (4) may not provide a solution that satisfies (3c) because of the nonconvexity of \( F_z(\bar{D}_z) \).

**Remark 2.** There exists \( y^*_{zi}, y^*_i \in R \) such that \( y_{zi} \in [y^*_{zi}, y^*_i], \ \forall i \in C, \forall z \in F(i) \).

### III. DIFFERENTIALLY PRIVATE CONTROL

The Lagrangian dual problem (4) can be solved by any nonsmooth convex optimization algorithms. In this paper we consider the PS algorithm, 
\[ \lambda^{k+1} = \text{Proj}_A(\lambda^k + \alpha_k \gamma_k^*), \ \forall k \in [K], \quad (5) \]
where \( \text{Proj}_A(\cdot) \) represents the orthogonal projection onto \( A, \alpha_k \) is a step size, \( \gamma_k^* \) is a search direction, and \( K \) is the total number of iterations.

1) **Motivating Example (Data Leakage):** Throughout the paper, we consider a hypothetical strong adversary that can access every but private load data of a control zone in a power system and tries to infer the data by intercepting the communication among the control zones. In this example, we demonstrate that the existing distributed algorithms are susceptible to inference attacks (e.g., [7]). Specifically, the adversary can intercept the communication data \( \{ y^k_{zi} \}_{k=1}^K \) of the PS algorithm (5) and try to infer private demand data \( D_z \) in (4b). Such inference attack can be easily conducted by solving an adversary problem as in [13]. The adversary problem is described as follows.

Let \( K \) be a set of PS iterations observed by an adversary who aims to infer a demand data at node \( l \) in zone \( z \), namely \( D_{zl} \). We assume that the strong adversary knows (i) all the demand information except \( D_{zl} \), (ii) all the topological information of zone \( z \), and (iii) the exchanged supergradient \( \{ \bar{y}^k_{zi} \}_{k \in K} \) and the local solution \( \{ x^k_\bar{z} \}_{k \in K} \). Our assumption is justified as to give the most advantages to the adversary, which can be considered as the worst-case data leakage scenario to the privacy-preserving control.

\[ \min_{D_{zl}, x^0_{z\bar{l}}} \sum_{k \in K} f_k(x^k_{\bar{z}}) + \sum_{k \in K} \| x^k_{z\bar{l}} - \bar{x}^k_{z\bar{l}} \|^2 + \| y^k_{z\bar{l}} - \bar{y}^k_{z\bar{l}} \|^2 \] 
\[ \text{subject to } \forall k \in K:\]
\[ (2b) - (2g), \forall l \in L; (2h), \forall i \in N_z; (2j), \forall i \in V_z; (2k), \forall i \in N_z, y^k_{g} \in G_i; (2l), \forall l, \forall j \in V_z; \]
\[ \sum_{i} p_{k}^{\delta_{i}} + \sum_{i} p_{k}^{\delta_{i}} = \sum_{g} p_{k}^{\delta_{g}} - D_{zl} - g_i^k[w_{k\bar{z}}^{\text{RR}} + w_{k\bar{z}}^{\text{RL}}]; \]
\[ \sum_{i} p_{k}^{\delta_{i}} + \sum_{i} p_{k}^{\delta_{i}} = \sum_{g} p_{k}^{\delta_{g}} - g_i^k[w_{k\bar{z}}^{\text{RR}} + w_{k\bar{z}}^{\text{RL}}], \]
\[ \forall i \in N_z \setminus \{ l \}, \]
\[ \text{where } \Gamma > 0 \text{ is a penalty parameter. Note that } D_{zl} \text{ is a decision variable as well as } \{ x^k_{\bar{z}}, y^k_{z\bar{l}} \}_{k \in K}. \text{ By solving (6), the adversary aims to obtain the unknown demand } D_{zl} \text{ that minimizes the distance between } \{ x^k_{\bar{z}}, y^k_{z\bar{l}} \}_{k \in K} \text{ and the solutions } \{ x^k_{\bar{z}}, y^k_{z\bar{l}} \}_{k \in K} \text{ obtained from the PS algorithm. With a sufficiently large } \Gamma, \text{ the adversary can find the demand data at node } l \text{ that produces } \{ x^k_{\bar{z}}, y^k_{z\bar{l}} \}_{k \in K}, \text{ thus identifying } D_{zl}. \text{ As the cardinality of } K \text{ increases, moreover, the accuracy of the demand estimated by (6) increases while sacrificing computation. We denote by } K \text{ a collection of various } K \text{ and by } D_{zl}(K) \text{ a demand estimated by (6) with } K \in K. \]

We demonstrate the effectiveness of adversary problem (6) by using an instance “case 14” from Matpower.
with the decomposition into 3 zones (see Table II). We solve the distributed OPF of (4) by using PS. At each iteration $k$, an approximation error is measured as below and reported in Figure 1:

$$AE_k = 100|Z^* - Z_k|/Z^*$, $\forall k \in [K],$$

where $Z^*$ is the optimal objective value and $Z_k$ is the objective values computed at the $k$th iteration of PS, respectively.

We consider the adversary who aims to estimate the demand at node $l = 4$ in zone $\hat{z} = 1$, namely, $D_{l4} = 47.8$ MW. For every trial $K \in \hat{K}$, we solve (6) and report in Figure 1 a demand estimation error:

$$DE(K) := 100|\hat{D}_{l4} - D_{l4}(K)|/\hat{D}_{l4}, \text{ } \forall K \in \hat{K},$$

where $\hat{K} \leftarrow \{\{1\}, \ldots, \{K\}\}$ (various $\hat{K}$ will be discussed in Section V). Figure 1 shows that the adversary is highly likely to estimate $D_{l4}$, and hence this situation motivates the need for the solution encryption to preserve data privacy.

2) Differential Privacy in PS: The motivating example suggests that PS for solving (4) might be vulnerable to data leakage. To preserve data privacy, we introduce differential privacy (see [9] for more details).

**Definition 1. ($\epsilon$-differential privacy) A randomized function $R$ that maps data $D$ to some random numbers gives $\epsilon$-differential privacy if**

$$\ln\left(\frac{\mathbb{P}(R(D') \in S)}{\mathbb{P}(R(D'') \in S)}\right) \leq \epsilon, \forall (D', D'') \in \mathcal{D}_\beta, \forall S \subseteq \text{Range}(R),$$

where $\epsilon > 0$, the probability taken is over the coin tosses of $R$, and $\mathcal{D}_\beta$ is a collection of two datasets $(D', D'')$ differing in one element by $\beta \in \mathbb{R}_+$. For small $\epsilon \approx \ln(1 + \bar{\epsilon})$, we have $\mathbb{P}(R(D') \in S) \in [1 - \epsilon, 1 + \bar{\epsilon}]$, which implies that distinguishing $D'$ from $D''$ based on $S$ becomes more difficult as $\epsilon$ decreases. To construct $R(D)$ that ensures $\epsilon$-differential privacy on data $D$, one can utilize a Laplace mechanism [8]. More specifically, a query function $Q : D \rightarrow \mathbb{R}$ mapping data to true answer is perturbed by adding Laplacian noise described in Definition 2.

**Definition 2. (Laplacian noise) Laplacian noise $\tilde{\xi} \in \mathbb{R}$ is a random variable following the Laplace distribution whose probability density function is $L(\xi|b) = \frac{1}{2b} \exp\left(-\frac{\xi}{b}\right)$ for $b > 0$. The randomized function $R(D) := Q(D) + \tilde{\xi}$ provides $\epsilon$-differential privacy if $\tilde{\xi}$ is drawn from the Laplace distribution with $b = \max\{(D, D'') \in \mathcal{D}_\beta | Q(D') - Q(D'')|/\epsilon\}$. The main idea of DP-PS is to perturb $y^k$ with the noise $\Delta^k$ such that**

$$\tilde{y}^k \leftarrow y^k + \Delta^k, \forall z \in \mathcal{Z}, \forall i \in C(z),$$

for every iteration $k$ of PS. We describe the algorithmic steps of DP-PS in Algorithm 1. In line 3, we find a supergradient $\dot{y}^k$ of the concave function $H$ at $\lambda^k$. In lines 6–8, we generate the Laplacian noise $\Delta^k$ and the noisy supergradient $\tilde{y}^k$. In line 9, we update dual variables based on the step size $\alpha_k$ and the search direction $s^k(\tilde{y}^k)$ determined in advance (see Section IV).

**Algorithm 1 DP Projected Subgradient Algorithm**

1. Set $k \leftarrow 1$ and $\lambda^1 \leftarrow 0$.  
2. for $k \in \{1, \ldots, K\}$ do  
3. Given $\lambda^k$, find $\dot{y}^k$ by solving (4b) in parallel. 
4. Store $H_{\text{best}}(\lambda^k) \leftarrow \max_{x \in \mathcal{X}}\{H(x)\}$.  
5. # Perturbation of $y^k$  
6. Solve (11) to find $\{\Delta^k\}_{z, i \in \mathcal{Z}, i \in C(z)}$.  
7. Extract $\xi^k$ from $L(\xi^k|\Delta^k)/\epsilon$ in Definition 2.  
8. Compute $\tilde{y}^k$ by (9).  
9. # Update dual variables  
10. $\lambda^{k+1} \leftarrow \text{Proj}_{\Lambda}(\lambda^k + \alpha_k s^k(\tilde{y}^k))$.  

Now we describe how to generate the noise $\xi^k_{zi}$ in (9) so that the $\epsilon$-differential privacy in Definition 1 on $D$ is ensured. First, we define a query function as follows:

$$Q^k_{zi} : D_z \rightarrow y^k_{zi}, \forall z \in \mathcal{Z}, \forall i \in C(z),$$

where $y^k_{zi}$ is obtained by solving (4b) for given $\lambda^k$ and $D_z \in \mathbb{R}^{|\mathcal{X} |}$. Second, we draw $\xi^k_{zi}$ in (9) from the Laplace distribution in Definition 2 with $b = \Delta^k_{zi}/\epsilon$ and

$$\Delta^k_{zi} = \max_{D_z \in \mathcal{D}_\beta(D_z)} |Q^k_{zi}(D_z) - Q^k_{zi}(\hat{D}_z)|,$$

where $\hat{D}_z$ is a given demand and $\hat{D}_\beta(D_z)$ is a collection of $D'_z$ differing in one element from $D_z$ by $\beta$, namely

$$\hat{D}_\beta(D_z) := \bigcup_{j \in \mathcal{X}} \left\{D'_z \in \mathbb{R}^{|\mathcal{X}|} : D'_{zj} = D_{zj}, \forall j \in \mathcal{X} \setminus \{i\}, D'_{zi} \in [D_{zi}(1 - \beta), D_{zi}(1 + \beta)]\right\}.$$
\( R^k_z (D_z) := Q^k_z (D_z) + \xi^k_z \), \( \forall z \in Z, \forall i \in C(z) \), \( k \in [\bar{K}] \), where \( Q^k_z \) is defined in (10), \( \xi^k_z \) is extracted from \( L(\xi^k_z | \Delta^k_z) \) in Definition 2, and \( \Delta^k_z \) is from (11). For all \( z \in Z, \forall i \in C(z) \), we have
\[
\ln \left( \frac{P( R^k_z (D_z) \in S_k)}{P( R^k_z (D_z) \in S_{k-1})} \right) \leq \xi^k_z, \quad \forall d \in D^k_z \), \( \forall S_k \in \text{Range}(R^k_z) \),
\]
where \( R^k_z (D_z) \) is equal to \( \bar{y}^k_z \) in (9). This implies that \( \xi^k_z \) is a \( \bar{y}^k_z \) in (9). This implies that \( \xi^k_z \) is extracted from \( L(\xi^k_z | \Delta^k_z) \).

\[\text{Proof. See Appendix A.}\]

| Rule | Step size | Search direction |
|------|-----------|------------------|
| 1    | \( \alpha_k = a/k \), where \( a > 0 \) | \( s^k (y^k) := y^k \) |
| 2    | \( \alpha_k := \frac{\lambda (k+1)}{\lambda k} \) | \( s^k (y^k) := y^k + \alpha_k (y^k - y^k \alpha_k) \) |
| 3    | \( \alpha_k := \frac{\lambda (k+1)}{\lambda k} \) | \( s^k (y^k) := y^k + \alpha_k (y^k - y^k \alpha_k) \) |

Assumption 1. \( A \) is compact and \( \lambda^* \in \Lambda \) maximizes \( \lambda \).

Remark 4. For the Laplacian noise \( \xi^k_z \) in (9), we notice that there exists \( \xi^k_z (\bar{e}) \in \mathbb{R}^+, \) which increases as \( \bar{e} \) decreases, such that \( \xi^k_z (\bar{e}) \in [\xi^k_z (\bar{e}), \xi^k_z (\bar{e})] \) for all \( k \in [\bar{K}] \), where \( K \) is the total number of iterations.

Lemma 1. For all \( k \in [\bar{K}] \), (i) \( ||y^k||^2 \in [G^k, G^k (\bar{e})] \), where \( G^k \) is a small positive number, and
\[ G^k (\bar{e}) := \sum_{z \in Z} \sum_{i \in C(z)} \left\{ \left[ \max_{y \in Z} (|y^k|, |y^k|) \right]^2 + \xi^k_z (\bar{e})^2 + 2 \xi^k_z (\bar{e}) \right\}.
\] and (ii) have the following basic inequality:
\[ ||\lambda^k - \lambda^*|| \leq ||\lambda^k - \lambda^*||^2 + 2 \left( \frac{\bar{e}^2}{\bar{e}^{\bar{e}}} \right) \left( H(\lambda^k) - H(\lambda^*) \right) + 2 \alpha_k \left( H(\lambda^k) - H(\lambda^*) \right), \forall k \in \mathbb{N}. \]

Proof. (14) holds from (9), and Remarks 2 and 4. (15) holds because of the nonexpansion property of the projection and the supergradient inequality, namely, \( H(\lambda) - H(\lambda^k) \leq \langle y^k, \lambda - \lambda^k \rangle \) for all \( \lambda \in \Lambda \). We emphasize that \( G^k(\bar{e}) \) increases as \( \bar{e} \) decreases.

1) Rule 1: Under Rule 1 it follows from (15) that
\[ E[||\lambda^k - \lambda^*||^2 | \lambda^k] \leq ||\lambda^k - \lambda^*||^2 + 2 \alpha_k \left( H(\lambda^k) - H(\lambda^*) \right), \forall k \in \mathbb{K}. \]

By taking the conditional expectation on (16), one can derive the following inequality:
\[ E[||\lambda^k - \lambda^*||^2 | \lambda^k] \leq ||\lambda^k - \lambda^*||^2 + 2 \alpha_k \left( H(\lambda^k) - H(\lambda^*) \right), \forall k \in \mathbb{K}. \]

Since \( \exists L \) : \( \lambda^* \geq ||\lambda^k - \lambda^*||^2 \) by Assumption 1 and \( E[||\lambda^k - \lambda^*||^2] \geq 0 \), (19) can be expressed as
\[ \lambda^* + G^k (\bar{e}) \sum_{k=1}^{K} \alpha_k \geq 2 \sum_{k=1}^{K} \alpha_k \left( H(\lambda^*) - E[H(\lambda^*)] \right) \geq \left( \sum_{k=1}^{K} \alpha_k \right) \left( H(\lambda^*) - E[H(\lambda^*)] \right), \]
where the last inequality holds due to Jensen’s inequality. By substituting \( \alpha_k = a/k \) in (20), we obtain
\[ H(\lambda^*) - E[H_{\text{best}}(\lambda^k)] \leq \lambda^* + G^k (\bar{e}) \sum_{k=1}^{K} \alpha_k, \]
where \( H_{\text{best}}(\lambda^k) := \max_{k \in \mathbb{K}} H(\lambda^k) \).
Theorem 2. Algorithm 1 with Rule 1 provides a sequence that converges in expectation and probability, namely,
\begin{align}
\lim_{K \to \infty} \mathbb{E}[H_{best}(\lambda^K)] &= H(\lambda^*), \quad (22a) \\
\lim_{K \to \infty} \mathbb{P}\{H(\lambda^*) - H_{best}(\lambda^K) \geq \epsilon\} &= 0, \quad (22b)
\end{align}
for any \( \epsilon > 0 \). Furthermore, the rate of convergence in expectation is \( O(\log(K)) \), where \( G(\bar{c}) \) increases as \( \bar{c} \) decreases.

Proof. See Appendix B.

To show that Algorithm 1 provides a sequence that converges with probability 1, we introduce the notion of the stochastic quasi-Feyer sequence in Definition 3.

Definition 3. (Stochastic quasi-Feyer sequence [24]) A sequence of random vectors \( \{z_k\}_{k=1}^{\infty} \) is a stochastic quasi-Feyer sequence for a set \( Z \subset \mathbb{R}^n \) if \( \mathbb{E}[\|z_k\|^2] < \infty \), and for any \( z \in Z \),
\[ \mathbb{E}[\|z - z^{k+1}\|^2 \mid z^1, \ldots, z^k] \leq \|z - z^k\|^2 + d_k, \quad \forall k \in \mathbb{N}, \]
\[ d_k \geq 0, \quad \forall k \in \mathbb{N}, \quad \sum_{k=1}^{\infty} \mathbb{E}[d_k] < \infty. \]

Theorem 3. Algorithm 1 with Rule 1 provides a sequence that converges with probability 1, namely,
\[ \mathbb{P}\{\lim_{K \to \infty} H_{best}(\lambda^K) = H(\lambda^*)\} = 1. \quad (23) \]

Proof. See Appendix C.

2) Rule 2: Under Rule 2 it follows from (15) that
\[ \|\lambda^{k+1} - \lambda^*\|^2 \leq \|\lambda^k - \lambda^*\|^2 - \frac{(H(\lambda^*) - H(\lambda^k))^2}{\|y^k\|^2} \]
\[ + 2 \frac{H(\lambda^*) - H(\lambda^k)}{\|y^k\|^2} \|y^k\| \|\lambda^k - \lambda^*\| 
\leq \|\lambda^k - \lambda^*\|^2 - \frac{(H(\lambda^*) - H(\lambda^k))^2}{G(\bar{c})} + M(\bar{c}), \quad (24) \]
where the first inequality holds due to the Cauchy-Schwarz inequality and the last inequality holds due to the existence of \( M(\bar{c}) \in (0, \infty) \) based on Remark 4, Lemma 1, and Assumption 1. By taking the expectation and applying Jensen’s inequality, we have
\[ \mathbb{E}[\|\lambda^{k+1} - \lambda^*\|^2] \leq \mathbb{E}[\|\lambda^k - \lambda^*\|^2] - 
\frac{(H(\lambda^*) - H(\lambda^k))^2}{G(\bar{c})} + M(\bar{c}). \]

Following the similar derivation from Rule 1, we obtain
\[ H(\lambda^*) - \mathbb{E}[H_{best}(\lambda^K)] \leq \sqrt{\frac{(\lambda^k + KM(\bar{c}))G(\bar{c})}{K}}. \quad (26) \]
Based on (26), we state the following proposition.

Proposition 1. Algorithm 1 with Rule 2 produces a sequence that converges in expectation to a point within \( \sqrt{M(\bar{c})G(\bar{c})} \) of the optimal value. Since \( M(\bar{c})G(\bar{c}) \) increases as \( \bar{c} \) decreases, it implies that there exists a trade-off between TPL and solution accuracy.

We show, however, that the trade-off vanishes under the following assumption.

Assumption 2. (Adapted from Assumption 3.1 in [25]) There exists \( \mu > 0 \) such that
\begin{align}
\|\lambda - \lambda^*\| &\leq H(\lambda^*) - H(\lambda), \quad \forall \lambda \in \Lambda, \quad (27a) \\
\|s^k(\hat{y}^k) - y^k\| &\leq \mu/2, \quad \forall k \in \mathbb{N}, \quad (27b)
\end{align}
where the first inequality indicates that the function \( H \) has a sharp set of maxima over a convex set \( \Lambda \) and the second inequality indicates that the distance between the search direction and the supergradient is bounded.

Assumption 2 is mild since the function \( H \) is polyhedral for our case with a reasonable choice of TPL.

Theorem 4. Under Assumption 2, Algorithm 1 with Rule 2 provides a sequence that converges in expectation, in probability, and with probability 1. The rate of convergence in expectation is \( O(G(\bar{c})/\sqrt{K}) \), where \( G(\bar{c}) \) increases as \( \bar{c} \) decreases.

Proof. See Appendix D.

3) Rule 3: Under Rule 3 the search direction \( s^k(\hat{y}^k) \) is a linear combination of \( \{\hat{y}^k\}_{k \in \mathbb{N}} \).

Lemma 2. Under Rule 3 we have
\[ \|s^k(\hat{y}^k)\|^2 = \|\hat{y}^k + \zeta_k s^k(\hat{y}^k)\|^2 \leq \|\hat{y}^k\|^2, \quad \forall k \in \mathbb{N}. \]

Proof. If \( \zeta_k = 0 \), then \( s^k(\hat{y}^k) = \hat{y}^k \). If \( \zeta_k > 0 \), then
\[ \|\hat{y}^k + \zeta_k s^k(\hat{y}^k)\|^2 - \|\hat{y}^k\|^2 = \zeta_k^2 \|s^k(\hat{y}^k)\|^2 + 2 \zeta_k \langle s^k(\hat{y}^k), \hat{y}^k \rangle 
= \chi_k^2 \|s^k(\hat{y}^k)\|^2 - 2 \chi_k \|s^k(\hat{y}^k)\| \|\hat{y}^k\| 
\leq 0, \]
where the last inequality holds since \( \chi_k \in [0, 2] \) as defined in Table 1.

From Lemma 2, similar results from Rule 2 can be derived. Under Rule 3 it follows from (15) that
\begin{align}
\|\lambda^{k+1} - \lambda^*\|^2 &\leq \|\lambda^k - \lambda^*\|^2 - \frac{(H(\lambda^*) - H(\lambda^k))^2}{\|s^k(\hat{y}^k)\|^2} \]
\[ + 2 \frac{H(\lambda^*) - H(\lambda^k)}{\|s^k(\hat{y}^k)\|^2} \|s^k(\hat{y}^k)\| \cdot \|\lambda^k - \lambda^*\| 
\leq \|\lambda^k - \lambda^*\|^2 - \frac{(H(\lambda^*) - H(\lambda^k))^2}{G(\bar{c})} + R(\bar{c}), \quad (28) \]
where the last inequality holds due to Lemma 2 and the existence of \( R(\bar{c}) \in (0, \infty) \) based on the boundness of \( s^k(\hat{y}^k) \) by its construction, Lemma 1, and Assumption 1. We emphasize that (28) is similar to (24). Thus one can derive results similar to (26), Proposition 1, and Theorem 4 under Rule 3.

Remark 5. We remark that all the results related to the convergence of DP-PS also hold when solving AC OPF.
described in Remark 1. However, the strong duality does not hold for AC OPF, so the consensus constraints may not be satisfied at termination.

V. NUMERICAL EXPERIMENTS

To support our findings from Sections III and IV, we showcase that increasing TPL of DP-PS does not affect the solution accuracy, although it does affect computation. In all the experiments, we solve optimization models by IPOPT [26] via Julia 1.5.0 on a personal laptop with an Intel Core i9 CPU and 64 GB of RAM.

1) Experimental Settings: For the power network instances, we consider case 14 and case 118 from Matpower [21]. The optimal objective values of SOC OPF (2) are obtained by utilizing IPOPT: $Z^* = 8075.1$ for case 14 and $Z^* = 129341.9$ for case 118. The networks are decomposed as described in Table II.

| Zone | Buses |
|------|-------|
| Zone 1 | 1–5 |
| Zone 2 | 7–10 |
| Zone 3 | 6, 11–14 |

We consider an adversary who aims to estimate $\bar{D}_{14} = 47.8$ MW for case 14 and $\bar{D}_{13} = 39$ MW for case 118, respectively, by solving the adversarial problem (6) for $|\hat{K}(T)|$ times, where $T$ is any integer number less than total iterations $K$ of DP-PS and

$$\hat{K}(T) := \bigcup_{t=1}^{K/T} \{ (t-1)T + 1, \ldots, tT \}. \tag{29}$$

Recall that $\hat{K}$ in Section III-1 is $\hat{K}(1)$.

On the other hand, we aim to protect demand data from the adversary by using the proposed DP-PS. First, we compute $\Delta_{\beta}$ in (11) with $\beta = 5\%$. Second, we consider various $\bar{\epsilon} \in \{0.01, 0.05, 0.1, 1, 10, \infty\}$ of DP-PS, where $\bar{\epsilon} = \infty$ represents a non-private PS and smaller $\bar{\epsilon}$ ensures stronger data privacy. Note that DP-PS with $K$ total iterations and $1/\bar{\epsilon}$ TPL provides $\bar{\epsilon}$-DP (resp., $K\bar{\epsilon}$-DP) against an adversary with $T = 1$ (resp., $T = K$) in (29). We use Rule 3 depicted in Section IV for our experiments.

2) Comparison with DP-ADMM: We compare the proposed DP-PS with the existing DP-ADMM [13] (see Appendix E for details on DP-ADMM).

In Figure 2 we report the objective values resulting by DP-PS and DP-ADMM for solving the case-14 and case-118 instances. When $\bar{\epsilon} = 0.01$ (i.e., larger noises are introduced for stronger data privacy), the objective value of DP-ADMM significantly fluctuates and is even larger than the optimal objective value $Z^*$ of the SOC OPF model. This implies that the sequence provided by DP-ADMM does not converge especially when stronger data privacy is required. In contrast, the proposed DP-PS always provides a lower bound on $Z^*$ and the sequence converges to $Z^*$.

3) Convergence of DP-PS: We demonstrate the numerical support for Theorem 4 that increasing TPL does not affect the solution accuracy of DP-PS, although it does affect computation.

Solution Accuracy: In Figure 3 we report the optimality gap at each iteration $k$ of DP-PS. The results show that the sequence generated by DP-PS converges regardless of the $\bar{\epsilon}$ value. We also discuss the impact of the number of zones on the convergence of DP-PS in Section V-6.

Computation: Figure 3 demonstrates that DP-PS with smaller $\bar{\epsilon}$ requires more iterations to converge. In Figure 4 we report the total number of iterations required for DP-PS to converge to a solution within $1\%$ of the optimality gap. The results show the decreasing trends of total iterations as $\bar{\epsilon}$ increases. This implies that there exists a trade-off between TPL and computation.

4) Data Privacy Preservation: We numerically show that increasing TPL provides higher data privacy.

First, we consider various $\hat{K}(T)$ when constructing the adversarial problem (6). As $T$ increases, theoretically, the accuracy of the demand estimated by solving (6) with $K \in \hat{K}(T)$ increases. We report in
Figure 5 an average demand estimation error (DEE): $\sum_{\hat{K} \in \hat{K}(T)} \text{DE}(\hat{K})/|\hat{K}(T)|$, where $\text{DE}(\hat{K})$ is defined in (8). The results show (i) decreasing trends of the average DEE as $\hat{\epsilon}$ increases for fixed $T$ and (ii) decreasing trends of the average DEE as $T$ increases for fixed $\hat{\epsilon}$. Moreover, the average DEE for fixed $\hat{\epsilon}$ seems to converge to a point as $T$ increases. The results imply that increasing TPL produces stronger data privacy (e.g., see $\hat{K}(100)$ in Figure 5 (right) when $\hat{\epsilon} = 0.01$.

5) Summary: We report in Figure 6 the optimality gap at the termination of DP-PS and the adversarial’s chance of success (CoS) defined as follow:

$$\text{CoS}(\mathcal{G}) = 100 \times \sum_{T \in \mathcal{T}} \sum_{\hat{K} \in \hat{K}(T)} \mathcal{I}(\text{DE}(\hat{K})) \leq \mathcal{G}/|\hat{K}(T)|,$$

where $\mathcal{G}$ is a prespecified value (e.g., $\mathcal{G} = 1\%$), $\mathcal{T}$ is a collection of various $T$, $\hat{K}(T)$ is in (29), $\text{DE}(\hat{K})$ is in (8), $\mathcal{I}(\text{DE}(\hat{K}) \leq \mathcal{G}) = 1$ if $\text{DE}(\hat{K}) \leq \mathcal{G}$ and $\mathcal{I}(\text{DE}(\hat{K}) \leq \mathcal{G}) = 0$ otherwise. The results demonstrate that as $\hat{\epsilon}$ decreases, the adversarial’s chance of successful demand estimation decreases while the optimality gap still remains the same.

6) The impact of the number of zones on the convergence: As the number of zones increases, the convergence of DP-PS can become slower. To see this, we use a partitioning algorithm [28] to generate different zones of the power systems. This algorithm is available at Metis.jl. In Figure 7, we report the optimality gap produced by DP-PS with $\hat{\epsilon} = 0.1$ for case 14 and case 118 instances decomposed by $|Z| \in \{3, 5, 7\}$ and $|Z| \in \{3, 10, 50\}$, respectively. Note that there are about 2 buses for each zone of case 14 and case 118 systems when $|Z| = 7$ and $|Z| = 50$, respectively. We observe that the optimality gap slightly increases as the number of zones increases, but not significantly.

7) AC OPF: In this section we show that the convergence and the data privacy preservation of DP-PS are also achieved when solving AC OPF (see Remarks 1 and 5). To this end we present Figures 8 and 9, which are counterparts of Figures 3 and 5, respectively. When $\hat{\epsilon} = 0.01$, more iterations may be required for the convergence. Again, the sequence produced by DP-PS may converge to an infeasible point for AC OPF.

VI. CONCLUSION

We studied a privacy-preserving distributed OPF and proposed a differentially private projected subgradient (DP-PS) algorithm that includes a solution encryption step. In this algorithm Laplacian noise is introduced to encrypt solution exchanged within the algorithm, which leads to $\hat{\epsilon}$-differential privacy on data. The target privacy
level of DP-PS is chosen by users, which affects not only the data privacy but also the convergence of the algorithm. We showed that a sequence provided by DP-PS converges to an optimal solution regardless of the $\bar{\epsilon}$ value, but more iterations are required for the convergence as $\bar{\epsilon}$ decreases. Also, we demonstrated that, as $\bar{\epsilon}$ decreases, the adversarial’s chance of successful demand data inference decreases while the optimality gap remains the same. Finally, as stated in Remarks 1 and 5, the proposed DP-PS can lead to an infeasible solution with respect to the AC OPF model, calling for privacy-preserving algorithms for the model.

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APPENDIX A

Proof of Theorem 1

First, in the kth iteration of DP-PS, for all \( z \in \mathbb{Z} \) and \( i \in C(z) \), we denote by \( P_{R_z^k(D)}(\tilde{y}_z^k) \) the probability density at any \( \tilde{y}_z^k \in S_k \), where \( R_z^k \) is defined in (12) and \( S_k \) is a subset of \( \text{Range}(R_z^k) \). Then we have

\[
P(R_z^k(D) \in S_k) = \int_{S_k} P_{R_z^k(D)}(\tilde{y}_z^k)d\tilde{y}_z^k.
\]

Now consider the following ratio:

\[
\frac{P_{R_z^k(D)}(\tilde{y}_z^k)}{P_{R_z^k(D)}(\tilde{y}_z^k)} = \frac{L(\tilde{y}_z^k - Q_z^k(D_z))/\epsilon}{L(\tilde{y}_z^k - Q_z^k(D_z))/\epsilon} = \exp \left( \left( \epsilon / \Delta_z^k \right) \left( |\tilde{y}_z^k - Q_z^k(D_z)| - |\tilde{y}_z^k - Q_z^k(D_z)| \right) \right) \leq \exp \left( \epsilon / \Delta_z^k \right) \left( Q_z^k(D_z) - Q_z^k(D_z) \right)
\]

where \( L \) is from Definition 2, the first inequality holds due to the reverse triangle inequality, namely, \( |a| - |b| \leq |a - b| \) and the last inequality holds since \( \Delta_z^k \geq |Q_z^k(D_z) - Q_z^k(D_z)| \) for all \( D_z \in \delta(D_z) \) from (11). Similarly, one can obtain a lower bound as follows:

\[
\exp \left( \left( \epsilon / \Delta_z^k \right) \left( |\tilde{y}_z^k - Q_z^k(D_z)| - |\tilde{y}_z^k - Q_z^k(D_z)| \right) \right) \geq \exp \left( -\epsilon / \Delta_z^k \right) \left( Q_z^k(D_z) - Q_z^k(D_z) \right)
\]

where the first inequality holds due to the reverse triangle inequality, namely, \( |a| - |b| \geq |a - b| \). Therefore, we have

\[
\exp(-\epsilon) \leq \frac{P_{R_z^k(D)}(\tilde{y}_z^k)}{P_{R_z^k(D)}(\tilde{y}_z^k)} \leq \exp(\epsilon), \; \forall D_z \in \delta(D_z),
\]

and integrating \( \tilde{y}_z^k \) over \( S_k \) yields (13). This proves that \( \epsilon \)-differential privacy on data is guaranteed for each iteration \( k \) of DP-PS.

Second, for all \( z \in \mathbb{Z} \) and \( i \in C(z) \), we denote by \( R_z^k \) a randomized function that maps the dataset \( D_z \in [N_z] \) to \( \tilde{y}_z^k := \left\{ \hat{y}_z^k \right\}_{k=1}^K, \) where \( K \) is the total number of iterations consumed by DP-PS. It suffices to show that

\[
\left| \ln \left( \frac{P(R_z^k(D_z) \in S)}{P(R_z^k(D_z) \in S)} \right) \right| \leq \epsilon, \; \forall D_z \in \delta(D_z), \; \forall S \subseteq \text{Range}(R_z^k).
\]

We denote by \( P_{R_z^k(D_z)}(\tilde{y}_z^k) \) the joint density at any \( \tilde{y}_z^k \in S_{D_z} \). Then we have

\[
P(R_z^k(D_z) \in S) = \int_S P_{R_z^k(D_z)}(\tilde{y}_z^k)d\tilde{y}_z^k.
\]

The joint density function can be expressed by the conditional density functions:

\[
P_{R_z^k(D_z)}(\tilde{y}_z^k) = P_{R_z^k(D_z)}(\tilde{y}_z^k \mid R_z^k(D_z)) = \prod_{k=1}^K P(\tilde{y}_z^k \mid R_z^k(D_z))
\]

where \( R_z^k(D_z) \) is defined in (12). This proves (13).

APPENDIX B

Proof of Theorem 2

Since \( H(\lambda^k) - H(\lambda^k) \geq 0 \) and the right-hand side of (21) goes to zero as \( K \to \infty \), (22a) holds. Also, (22b) holds due to Markov’s inequality, namely, for \( \epsilon > 0 \),

\[
P \left( H(\lambda^k) - H(\lambda^k) \geq \epsilon \right) \leq \epsilon \left| \delta(\lambda^k) - \delta(\lambda^k) \right| / \epsilon,
\]

where the right-hand side of (31) goes to zero as \( K \to \infty \). From the right-hand side of (21), the rate of convergence in expectation is \( O(G^i(\bar{\epsilon})/\ln(K)) \). This completes the proof.

APPENDIX C

Proof of Theorem 3

By taking a conditional expectation on (16), we obtain

\[
\mathbb{E}[\|\lambda^{k+1} - \lambda^*\|^2 \mid \lambda^1, \ldots, \lambda^k] \leq \|\lambda^k - \lambda^*\|^2 + \alpha_k^2 G^i(\bar{\epsilon}),
\]

where the inequality holds since \( \alpha_k^2 H(\lambda^k) - H(\lambda^k) \leq 0 \) and \( \mathbb{E}[\|\lambda^k - \lambda^*\|^2] = 0, \forall z \in Z, \forall i \in C(z) \). Since \( \lambda^1 \) is bounded by Assumption 2, \( \alpha_k^2 G^i(\bar{\epsilon}) \geq 0 \), and \( G^i(\bar{\epsilon}) \sum_{i=1}^\infty \alpha_k^2 < \infty \), the sequence \( \{\lambda^k\} \) generated by Algorithm 1 with Rule 1 is a stochastic quasi-Feyer
sequence for a set $A^*$ of maximizers. Based on Theorem 6.1 in [24] and the existence of a subsequence $\{\lambda^{k_s}\}$ such that $H_{\text{best}}(\lambda^{k_s})$ converges to $H(\lambda^*)$ with probability 1 due to (22b), one can conclude that the sequence $\{\lambda^k\}$ converges to a point in $A^*$. For more details, we refer the reader to the proof of Theorem 6.2 in [24].

APPENDIX D
PROOF OF THEOREM 4

Under Assumption 2, it follows from (24) that

$$
\begin{align*}
&\|\lambda^{k+1} - \lambda^*\|^2 \leq \|\lambda^k - \lambda^*\|^2 - \frac{(H(\lambda^*) - H(\lambda^k))^2}{\|\hat{y}\|^2} + \\
&2 \frac{H(\lambda^*) - H(\lambda^k)}{\|\hat{y}\|^2} \|s(\hat{y}^k) - s(k)\| \cdot \|\lambda^k - \lambda^*\|
\end{align*}
$$

(32)

where the first inequality holds since $\hat{\xi}^{k} = s(\hat{y}^k) - s(k)$, the second inequality holds due to Assumption 2, and the last inequality holds due to $(1-2\|s^k(\hat{y}^k) - s(k)\|/\mu) \in (0, 1)$ from Assumption 2 and Lemma 1. By taking similar steps in Section IV-2, we obtain

$$
0 \leq H(\lambda^*) - E[\max_{k \in [K]} H(\lambda^k)] \leq \sqrt{\frac{\lambda^* G^0(\hat{\epsilon})}{G - \hat{K}}}.
$$

(T33)

Taking similar steps in the proof of Theorem 2, we conclude from (33) that the sequence produced by DP-PS with Rule 2 under Assumption 2 converges in expectation and in probability. Also, the rate of convergence in expectation is $O(G^0(\hat{\epsilon})/K)$. It follows from (32) that $\|\lambda^{k+1} - \lambda^*\|^2 \leq \|\lambda^k - \lambda^*\|^2$. By taking a conditional expectation, we obtain

$$
E[\|\lambda^{k+1} - \lambda^*\|^2 \mid \lambda^1, \ldots, \lambda^k] \leq \|\lambda^k - \lambda^*\|^2.
$$

Thus, the sequence $\{\lambda^k\}$ generated by DP-PS with Rule 2 under Assumption 2 is a stochastic quasi-Feyer sequence for a set $A^*$ of maximizers. As discussed in the proof of Theorem 3, it proves the convergence with probability 1. This completes the proof.

APPENDIX E

DP-ADMM

We present the augmented Lagrangian dual problem given by

$$
\begin{align*}
\max \min_{\lambda^k} & \sum_{x \in Z} \left\{ f_x(x) + \sum_{i \in C(z)} \left( \lambda_{zi} (\phi_i - y_{zi}) + \frac{\rho}{2} (\phi_i - y_{zi})^2 \right) \right\} \\
\text{s.t.} & \quad (x, y) \in \mathcal{P}_z(D_x), \forall z \in Z, \\
& \quad \phi_i \in R, \forall i \in C,
\end{align*}
$$

where $\lambda$ is a dual vector associated with constraint (3c).

For every iteration $k$ of the ADMM algorithm, it updates $(y^k, \phi^k, \lambda^k) \rightarrow (y^{k+1}, \phi^{k+1}, \lambda^{k+1})$ by solving a sequence of the following subproblems:

$$
\begin{align*}
&y_{zi}^{k+1} \leftarrow \arg \min_{(x, y) \in \mathcal{P}_z(D_x)} f_x(x) + \sum_{i \in C(z)} \left( \lambda_{zi} (\phi_i - y_{zi}) + \frac{\rho}{2} (\phi_i - y_{zi})^2 \right), \forall z \in Z, \quad \text{(34a)} \\
&\phi_i^{k+1} \leftarrow \arg \min_{\phi_i} \sum_{z \in \mathcal{Z}} \lambda_{zi} (\phi_i - y_{zi})^2, \forall i \in C, \quad \text{(34b)} \\
&\lambda_{zi}^{k+1} = \lambda_{zi}^k + \rho (\phi_i^{k+1} - y_{zi}^{k+1}), \forall z \in Z, \forall i \in C(z). \quad \text{(34c)}
\end{align*}
$$

In Algorithm 2, we describe DP-ADMM. In line 3, we solve the subproblem (34a) to find $y^{k+1}$, which is perturbed by adding the Laplacian noise $\xi^k$ in line 7. In line 9, we solve the subproblem (34b) with $\tilde{y}^{k+1}$ to find $\phi^{k+1}$. In line 10, the dual variable is updated from $\lambda^k$ to $\lambda^{k+1}$.

Algorithm 2 DP-ADMM

1: Set $k \leftarrow 1$, $\lambda^1 \leftarrow 0$, and $\phi^1 \leftarrow 0$.
2: for $k \in \{1, \ldots, K\}$ do
3: Given $\lambda^k$ and $\phi^k$, find $y^{k+1}$ by solving (34a).
4: # Perturbation of $y^{k+1}$
5: Solve (11) to find $\{\Delta_{zi}\}_{z \in Z, i \in C(z)}$.
6: Extract $\tilde{\xi}^k$ from $\{\Delta_{zi}\}$ in Definition 2.
7: $y_{zi}^{k+1} \leftarrow y_{zi}^{k+1} + \tilde{\xi}_{zi}, \forall z \in Z, \forall i \in C(z)$.
8: # Solve the second-block problem
9: Solve (34b) with $\tilde{y}^{k+1}$.
10: # Update dual variables
11: $\lambda_{zi}^{k+1} = \lambda_{zi}^k + \rho (\phi_i^{k+1} - y_{zi}^{k+1}), \forall z \in Z, \forall i \in C(z)$.
end for