ON SEMISTABILITY OF $CAT(0)$ GROUPS

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Abstract. Does every one-ended $CAT(0)$ group have semistable fundamental group at infinity? As we write, this is an open question. Let $G$ be such a group acting geometrically on the proper $CAT(0)$ space $X$. In this paper we show that in order to establish a positive answer to the question it is only necessary to check that any two geodesic rays in $X$ are properly homotopic. We then show that if the answer to the question is negative, with $(G, X)$ a counter-example, then the boundary of $X$, $\partial X$ with the cone topology, must have a weak cut point. This is of interest because a theorem of Papadoglou and the second-named author [PS09] has established that there cannot be an example of $(G, X)$ where $\partial X$ has a cut point. Thus, the search for a negative answer comes down to the difference between cut points and weak cut points. We also show that the Tits ball of radius $\frac{\pi}{2}$ about that weak cut point is a “cut set” in the sense that it separates $\partial X$. Finally, we observe that if a negative example $(G, X)$ exists then $G$ is rank 1.

1. Semistability at infinity

Let $X$ be a proper $CAT(0)$ space having one end, and let $\partial X$ denote its compactifying boundary. A point of $\partial X$ is, by definition, an equivalence class of geodesic rays in $X$ any two of which are boundedly close. A simple model for $\partial X$ is obtained by choosing a base point $b \in X$: then the space of all geodesic rays starting at $b$, with the compact-open topology, is homeomorphic to $\partial X$. Since $X$ has one end $\partial X$ is connected. Following classical terminology we refer to a connected compact Hausdorff space as a continuum; all continua in this paper will be metrizable.

One says that $X$ has semistable fundamental group at infinity (or has one strong end) if any two proper rays in $X$ are properly homotopic. We recall that the gap between the words and the definition is explained as follows: An inverse sequence of groups $\{H_n, f^m_n\}$ is said to be semistable (or Mittag-Leffler) if, for each $n$, the images of the bonding homomorphisms $f^m_n : H_m \to H_n$ are the same for all but finitely many values of $m > n$. The relevance to our $X$ is this: Let $\{K_n\}$ be an exhausting sequence of compact subsets of $X$ such that, for all $n$, $K_n$ is a subset of the interior of $K_{n+1}$. Then $\{U_n := X - K_n\}$ is a basic system of neighborhoods of infinity. Choosing a suitably parametrized geodesic base ray $\omega$ in $X$, we consider the inverse sequence of groups $\{\pi_1(U_n, \omega), (\text{inclusion})_\#\}$. It is a fact that this sequence of groups is semistable if and only if any two proper rays in $X$ are properly homotopic; see [Geo08], Section 16.1 for details.

Extending our simple model of $\partial X$ as the space of geodesic rays beginning at $b$, we define $\hat{X}$ to be the space of all maps $[0, \infty) \to X$ which are either geodesic rays starting at $b$, or geodesic segments on a finite interval $[0, \rho]$, mapping 0 to $b$, and are constant on
\([\rho, \infty)\). Understood here is the compact-open topology, so, by the Arzela-Ascoli Theorem, \(\hat{X}\) is compact. It is only a minor abuse of language to say that \(X\) is open and dense in \(\hat{X}\). It is obvious that \(\partial X\) is a \(Z\)-set\(^2\) in \(\hat{X}\). We write \(\hat{U}_n := U_n \cup \partial X\). Because \(\partial X\) is a \(Z\)-set we obtain an inverse sequence of open subsets of \(\hat{X}\) with the following property:

\[
\text{The “inclusion map” } \{\pi_1(U_n, \omega), (\text{inclusion})_\#\} \to \{\pi_1(\hat{U}_n, \omega(\infty)), (\text{inclusion})_\#\} \text{ induces an isomorphism of pro-groups.}
\]

Thus, semistability at infinity of \(X\) can be read off from either inverse sequence.

We will be dealing with the relationship between \(X\) having semistable fundamental group at infinity and shape properties of \(\partial X\). There is a delicate issue here which makes necessary a brief discussion of shape theory.

2. Remarks on shape theory

For our purposes, shape theory of continua can be presented in the following limited way. Suppose given a compact absolute retract (AR) \(\hat{W}\) and a subcontinuum \(Y\) which is a \(Z\)-set in \(\hat{W}\). We choose a base point \(y \in Y\). The dense open set \(W := \hat{W} - Y\) is a locally compact AR having one end. If \(\{V_n\}\) is a basic system of neighborhoods of infinity in \(W\) then \(\hat{V}_n := V_n \cup Y\) is a basic system of neighborhoods of \(Y\) in \(\hat{W}\). Moreover:

1. If each \(V_n\) is closed in \(W\) then \(\{\hat{V}_n\}\) is a basic system of closed neighborhoods of \(Y\) in \(\hat{W}\);
2. If each \(V_n\) is an absolute neighborhood retract (ANR), then \(\{\hat{V}_n\}\) is a basic system of ANR neighborhoods of \(Y\) in \(\hat{W}\) (because “adding” a \(Z\)-set does not destroy the ANR property).

One really wants both (1) and (2) to hold because then one has a basic sequence of compact ANR neighborhoods of \(Y\) in the AR \(\hat{W}\), in which case any pro-homotopy\(^3\) invariant of the inverse sequence \(\{\hat{V}_n, y\}\) is an intrinsic shape theoretic invariant of \((Y, y)\). In particular, this means that if \(\{\hat{V}_n\}\) is a sequence of compact ANR neighborhoods of a \(Z\)-set copy of \(Y\) in a (possibly different) AR \(W'\), then the same (pointed) pro-homotopy properties will hold there. One such property is semistability of the inverse sequence of fundamental groups. If any such inverse sequence of pointed compact ANR’s has semistable fundamental pro-group then \((Y, y)\) is said to have \emph{semistable fundamental pro-group}\(^4\). It is a fact that (because \(Y\) is connected) this property is independent of the choice of base point.

\textbf{Example:} It is well-known that every proper \(CAT(0)\) space is an AR. Since the passage from \(X\) to \(\hat{X}\) is achieved by “adding” a \(Z\)-set, namely \(\partial X\), the compact space \(\hat{X}\) is also an...

\(^2\)This means that for any open set \(U\) in \(\hat{X}\), the inclusion map \(U \cap X \to U\) is a homotopy equivalence. Equivalently, for any positive \(\epsilon\), any map of a polyhedron \(P\) into \(\hat{X}\) can be \(\epsilon\)-homotoped off \(\partial X\), holding fixed an arbitrarily large closed subset of \(P\) lying in the pre-image of \(X\).

\(^3\)Because of our concern with fundamental group we are really interested in pointed shape invariance — i.e. invariance under pro-homotopy where all maps and homotopies respect base points. It is cumbersome to keep inserting the word “pointed”, and we shall often omit it.

\(^4\)Or, in the language of shape theory, \(Y\) is \emph{pointed 1-movable}. 

2
AR. Thus $(\hat{X}, \partial X)$ is an example of $(\hat{W}, Y)$ of the previous paragraphs. We apply the above discussion to the neighborhoods $U_n$ of Section II. In order to equate “$X$ having semistable fundamental group at infinity” with “$\partial X$ having semistable fundamental pro-group” we would like to be able to choose the $U_n$ (or a cofinal sequence of neighborhoods $U'_n$) to be closed ANR subsets of $X$. i.e. to satisfy (1) and (2). When $X$ is triangulable (for example, a $CAT(0)$ cubical complex) this is easy to achieve. But we do not know how to find such closed ANR neighborhoods of infinity in an arbitrary proper $CAT(0)$ space $X$. Nevertheless, we assert:

**Proposition 2.1.** A proper $CAT(0)$ space $X$ has semistable fundamental group at infinity if and only if its boundary $\partial X$ has semistable fundamental pro-group.

Because the proof of Proposition 2.1 for the non-triangulable case involves ideas outside the scope of this paper we defer it to the Appendix.

3. **Geodesic rays**

Since geodesic rays are proper, one may ask if geodesic rays alone determine whether or not $X$ has semistable fundamental group at infinity.

**Theorem 3.1.** For a one-ended proper $CAT(0)$ space $X$ the following are equivalent:

(i) $X$ has semistable fundamental group at infinity;
(ii) any two geodesic rays in $X$ are properly homotopic;
(iii) $\partial X$ has semistable fundamental pro-group.

**Remark 3.2.** Any two geodesic rays in $X$ are properly homotopic through geodesic rays (i.e. every level of the homotopy a geodesic ray) if and only if $\partial X$ is path connected. Theorem 3.1 describes something weaker than path connectedness.

**Remark 3.3.** It has long been known (see Krasinkiewicz [Kra77]) that if a continuum is locally connected then it has semistable fundamental pro-group, so Theorem 3.1 is mainly of interest when $\partial X$ is not known to be locally connected.

**Remark 3.4.** That theorem of Krasinkiewicz [Kra77] says more: A continuum has semistable fundamental pro-group if and only if it is shape equivalent to a locally connected continuum.

We need some shape theoretic terminology.

(1) A strong shape component of $\partial X$ is a proper homotopy class of proper rays in $X$.

(2) The proper ray $c : [0, \infty) \to X$ ends at the point $p \in \partial X$ if the map $c$ extends continuously to $\dot{c} : [0, \infty] \to \hat{X}$ by mapping the point $\infty$ to $p$.

(3) Two points $p$ and $q$ of $\partial X$ lie in the same component of joinability if there are proper rays in $X$ ending at $p$ and at $q$ which are properly homotopic.

**Remark 3.5.** For strong shape theory see, for example, Section 17.7 of [Geo08]. Components of joinability were introduced in [KM79].

**Lemma 3.6.** If two proper rays end at the same point of $\partial X$ then they are properly homotopic.
Proof. Let the proper rays $c$ and $c'$ end at the point $p$. AR’s are locally simply connected, so we can choose a basic system of neighborhoods of $p$ in $\hat{X}$

$$\cdots \subseteq W_n \subseteq V_n \subseteq U_n = W_{n-1} \subseteq V_{n-1} \subseteq U_{n-1} \subseteq \cdots$$

such that

(i) $[c(n), \infty) \cup [c'(n), \infty) \subseteq W_n$;
(ii) any two points in $W_n$ can be joined by a path in $V_n$;
(iii) any loop in $V_n$ is homotopically trivial in $U_n$.

For each $n$ choose a path $\omega_n$ in $V_n$ joining $c(n)$ to $c'(n)$. Choose a trivializing homotopy in $U_{n-1}$ of the loop formed by the segments $[c'(n-1), c'(n)], [c(n-1), c(n)], \omega_{n-1}$ and $\omega_n$. Together, these give a homotopy $[0, \infty) \times [0, 1] \to \hat{X}$ between $c$ and $c'$. Because $\partial X$ is a $Z$-set, this homotopy can be pulled off $\partial X$ skeleton by skeleton with the required amount of control to give a proper homotopy in $X$ between $c$ and $c'$.

We denote by $[c]$ the strong shape component of $\partial X$ defined by the proper ray $c$. If some $c'$ in $[c]$ ends at a point of $\partial X$ we say that the strong shape component $[c]$ of $\partial X$ is non-empty. In general there may also be empty strong shape components, meaning examples of $[c]$ containing no such $c'$. The existence of these is explored in [GK91] where, in particular the following is proved ([GK91] Corollary 3.6 and Theorem 5.1):

**Theorem 3.7.** If $\partial X$ has an empty strong shape component then it has uncountably many such, and also uncountably many components of joinability.

**Remark 3.8.** The proof in [GK91] is for metrizable continua in general, not specifically for boundaries of $CAT(0)$ spaces.

**Proof of Theorem 3.7.** The equivalence of (i) and (iii) is Proposition 2.1. The direction “(i) implies (ii)” is trivial. Assume that any two geodesic rays in $X$ are properly homotopic. Then $\partial X$ has only one component of joinability. By Theorem 3.7 $\partial X$ has no empty strong shape components. So any strong shape component $[c]$ contains a proper ray $c'$ which ends at a point $p$ of $\partial X$. There is also a geodesic ray ending at $p$, and, by Lemma 3.6 it is properly homotopic to $c'$. It follows that $X$ has semistable fundamental group at infinity. □

4. Relevance to group theory

A finitely presented one-ended group $G$ has semistable fundamental group at infinity if the universal cover of some (equivalently, any) finite 2-complex whose fundamental group is isomorphic to $G$ has semistable fundamental group at infinity in the sense defined above for $X$. This property of finitely presented groups is a quasi-isometry invariant; see, for example, Section 18.2 of [Geo08]. It is unknown if every finitely presented group has this property. All one-ended hyperbolic groups have it, as do many other classes of groups.

When the discrete group $G$ acts geometrically (i.e. properly discontinuously and cocompactly) on the one-ended proper $CAT(0)$ space $X$ then $G$ is quasi-isometric to $X$, so $G$ has one end and is finitely presented. Moreover, $G$ has semistable fundamental group at infinity if and only if the same is true of $X$. The relevance to group theory is that in order to show that a $CAT(0)$ group $G$ has semistable fundamental group at infinity, one need only check
the condition on geodesic rays given in Theorem 3.1. And it need only be checked on one proper $CAT(0)$ space on which $G$ acts geometrically.

This is of interest because of the following still-unsolved problem:

**Question 4.1.** Is it true that every one-ended $CAT(0)$ group has semistable fundamental group at infinity?

An immediate corollary of Theorem 3.1 is:

**Corollary 4.2.** If $G$ is not rank 1 then $G$ has semistable fundamental group at infinity.

**Proof.** If $G$ is not rank 1 then, by a theorem of Ballmann and Buyalo [BB08], the Tits diameter of $\partial X$ is finite, so $\partial X$ is Tits path connected. The obvious function from the Tits boundary to $\partial X$ is continuous so $\partial X$ is path connected. Remark 3.2 applies. □

The theorem of Krasinkiewicz [Kra77] mentioned in Remark 3.4 implies that if there exists a $CAT(0)$ group $G$ which does not have semistable fundamental group at infinity then $\partial X$ is not shape equivalent to a locally connected continuum.

5. Cut points and semistability

We continue to explore properties which $\partial X$ would have to possess if its fundamental pro-group were not semistable.

A point $c$ of a continuum $Y$ is a **cut point** if $Y - \{c\}$ is not connected. The point $c$ is a **weak cut point** if there are two other points $a$ and $b$ of $Y$ such that any subcontinuum containing $a$ and $b$ must also contain $c$. In that case we say that $c$ weakly separates $a$ from $b$.

**Example 5.1.** The limit segment $I$ of the Topologist’s Sine Curve $S$ provides an example of the difference: $c$ is the mid point of $I$, $a$ and $b$ are its two end points. This $c$ is a weak cut point in $S$ but it is not a cut point.

Let $X$ be as before, and let $G$ act geometrically on $X$. It is proved in [PS09] that the existence of such a group $G$ implies that $\partial X$ cannot contain a cut point. In this section we show that if $\partial X$ does not have semistable fundamental pro-group then $\partial X$ contains a weak cut point. Thus in the search for a non-semistable $CAT(0)$ example we are down to the difference between cut points and weak cut points.

A proper $CAT(0)$ space $X$ is **almost geodesically complete** if there is a number $\lambda$ such that for any points $p$ and $q$ of $X$ there is a geodesic ray starting at $p$ and passing within distance $\lambda$ of $q$. It is proved in [Ont05] that when (as here) there is a group $G$ acting geometrically on $X$ by isometries, then $X$ is almost geodesically complete. *From now on this number $\lambda$ will be part of the data coming with $X$.*

**Lemma 5.2.** Let $p \in X$. Suppose that for every $\rho > 0$ there exists $\sigma > \rho$ with the following property: Whenever $\tau \geq \sigma$ and $\alpha, \beta : [0, \infty) \to X$ are unit speed geodesic rays starting at $p$ with $d(\alpha(\tau), \beta(\tau)) < 2\lambda + 1$ then the geodesic segment $[\alpha(\tau), \beta(\tau)]$ can be homotoped, relative to $\alpha$ and $\beta$, out of the closed ball $B(p, \tau + \lambda + 1)$ by a homotopy missing $B(p, \rho)$. Then $X$ has semistable fundamental group at infinity. □
Proof. Let $\omega$ be a geodesic base ray starting at $p$, and let $\rho > 0$ be given. Choose $\sigma$ as in the hypothesis, and let $E$ be any compact set containing $D := B(p, \sigma + 2\lambda + 1)$. Let $\gamma : [0, 1] \to X$ be a loop in $X - D$ based at $\omega$ and let $\delta := d(p, \gamma(0))$. By very small homotopies, we may assume that $\gamma$ is piecewise geodesic. Choose a partition $0 = x_0, x_1, \ldots, x_n = 1$ with the property that the length of $\gamma |: [x_{i-1}, x_i] \to X$ is at most 1. For each $1 \leq i < n$ choose a unit speed geodesic ray $\omega_i$ starting at $p$ with $d(\omega_i(y_i), \gamma(x_i)) \leq \lambda$ for some $y_i$ (necessarily, $y_i \geq \sigma$), setting $\omega_0 = \omega_n = \omega$, and $y_0 = d = y_n$.

The loop formed from $\gamma([x_{i-1}, x_i])$ and the geodesic segments $[\gamma(x_i), \omega_i(y_i)], [\gamma(x_{i-1}), \omega_{i-1}(y_{i-1})]$ and $[\omega_{i-1}(y_{i-1}), \omega_i(y_i)]$ has length bounded by $4\lambda + 2$. The straight line homotopy of this loop to the point $\gamma(x_i)$ misses $B(p, \rho)$. Thus we may homotope $\gamma$ to the piecewise geodesic loop with vertices $\omega_0(y_0), \omega_1(y_1), \ldots, \omega_{n-1}(y_{n-1})$. Each segment has length at most $2\lambda + 1$, and $y_i \geq \sigma + \lambda + 1$. This new loop is homotopic to the piecewise geodesic segment with vertices $\omega_0(\sigma + \lambda + 1), \omega_1(\sigma + \lambda + 1), \ldots, \omega_{n-1}(\sigma + \lambda + 1)$; again, the homotopy misses $B(p, \rho)$. By hypothesis, we can homotope this, relative to $\omega_0, \ldots, \omega_{n-1}$, to a loop $\eta$ supported in $X - B(p, \sigma + 2\lambda + 2)$, again missing $B(p, \rho)$. Now consider the subpath $\eta_i$ of $\eta$ from $\omega_i$ to $\omega_{i+1}$ (mod $n$) Repeating the above process we can homotope each path $\eta_i$ outside of $B(p, \sigma + 2\lambda + 3)$ relative to $\omega_i$ and $\omega_{i+1}$ missing $B(p, \rho)$. Gluing these homotopies we get a homotopy of $\gamma$, supported outside of $B(p, \sigma + 2\lambda + 3)$. Iterating the process, for any $n$ we can homotope $\gamma$ out of $B(p, \sigma + 2\lambda + n)$ by a homotopy missing $B(p, \rho)$. Thus we can move $\gamma$ off the compact set $E$. □

For $Z \subseteq \partial X$ and $x \in X$ the cone with base $Z$ and vertex $x$ is the subset $C_xZ$ of $\hat{X}$ consisting of $Z$ and the images in $X$ of all the geodesic rays $[0, \infty) \to X$ starting at $x$ and ending in $Z$.

**Lemma 5.3.** When $Z$ is compact and $c \in \partial X - Z$ is defined by the geodesic ray $\gamma$ starting at $x$ then, given $\rho > 0$, there is $\sigma > 0$ such that for all $\tau > \sigma$ the distance in $X$ from $\gamma(\tau)$ to $C_xZ$ is greater than $\rho$.

**Remark 5.4.** In fact it can be proved that the cone is compact when $Z$ is compact, but we will not need this.

**Proof.** (of Lemma 5.3) Suppose not. Then there exists $\rho > 0$, a sequence $(\xi_n)$ in $Z$, and sequences of numbers $(\tau_n) \to \infty$ and $(\nu_n) \to \infty$ such that for all $n$ we have $d(\gamma(\tau_n), \xi_n(\nu_n)) \leq \rho$. Since $Z$ is compact we may assume $(\xi_n)$ converges to some $\xi \in Z$. By a “comparison triangles argument” $\gamma$ and $\xi$ are asymptotic geodesic rays, and since they both start at $p$ they are identical. This contradicts the fact that $c \notin Z$. □

**Theorem 5.5.** If $X$ does not have semistable fundamental group at infinity then there are points $a, b, c \in \partial X$ such that $c$ weakly separates $a$ from $b$.

**Proof.** By Lemma 5.2 we know that for some $p \in X$ there exists $\rho > 0$ such that for any $n \in \mathbb{N}$ there exists $\sigma_n > n$ and unit speed geodesic rays $\alpha_n, \beta_n : [0, \infty) \to X$ starting at $p$ with $d(\alpha_n(\sigma_n), \beta_n(\sigma_n)) \leq 2\lambda + 1$, but the segment $[\alpha_n(\sigma_n), \beta_n(\sigma_n)]$ cannot be homotoped relative to $\alpha_n, \beta_n$ out of $B(p, \sigma_n + \lambda + 1)$ by a homotopy missing $B(p, \rho)$. Choose a compact set $K$ whose $G$-translates cover $X$, and for each $n$ choose an isometry $g_n$ with $g_n(\alpha_n(\sigma_n)) \in K$. Passing to a subsequence we may assume:
(i) \(\{g_n p\}\) converges to a point \(c \in \partial X\);
(ii) \(\{g_n(\alpha_n(\sigma_n))\}\) converges to \(x\) and \(\{g_n(\beta_n(\sigma_n))\}\) converges to \(y\);
(iii) \(\{g_n(\alpha_n)\}\) converges to \(\alpha\) and \(\{g_n(\beta_n)\}\) converges to \(\beta\), where \(\alpha\) and \(\beta\) are geodesic lines with \(\alpha(-\infty) = \beta(-\infty) = c\). Here, convergence is uniform on compact subsets.

We parametrize \(\alpha, \beta : \mathbb{R} \to X\) with unit speed so that \(\alpha(0) = x\) and \(\beta(0) = y\). We note that \(d(x, y) \leq 2\lambda + 1\). Let \(a = \alpha(\infty)\) and let \(b = \beta(\infty)\).

Suppose there is a continuum \(Z \subseteq \partial X\) containing \(a\) and \(b\) but not \(c\). By Lemma \[5.3\] since \(c \notin Z\) we may choose \(n\) large enough that the cone \(C_x Z\) is disjoint from \(B(g_n p, p + 2\lambda + 2)\).

For \(d \in Z\) let \(\gamma_d : [0, \infty) \to X\) be the geodesic ray (“cone line”) starting at \(x\) and ending at \(d\). By compactness of \(Z\) there is \(\kappa\) with \(\gamma_d(\tau) \notin B(g_n p, \sigma_n + \lambda + 1)\) for all \(\tau \geq \kappa\) and all \(d \in Z\). Let \(\tilde{Z} = \{\gamma_d(\kappa) \mid d \in Z\}\). This is a connected subset of \(X\) since it is a continuous image of \(Z\).

For small \(\epsilon\), there is a piecwise geodesic path \(\theta\) from \(\alpha(\kappa)\) (i.e. \(\alpha(\kappa)\)) to \(\beta(\kappa)\), where \(\theta\) lies uniformly within \(\epsilon\) of \(\tilde{Z}\). Certainly, \(\epsilon\) can be chosen so that \(\theta\) misses \(B(g_n p, q_n + \lambda + 1)\). The end-point of \(\theta\) is \(\gamma_\kappa(\kappa)\). There is a geodesic segment \(\sigma\) joining \(\gamma_\kappa(\kappa)\) to \(\beta(\kappa)\) whose length is at most \(2\lambda + 1\). The combined path \(\theta \ast \sigma\) joins \(\alpha(\kappa)\) to \(\beta(\kappa)\).

The straight line unit-speed contraction of \(\theta\) to the point \(x\) misses \(B(g_n p, s + 2\lambda + 2)\), and during that contraction the point \(\gamma_\kappa(\kappa)\) traverses the segment \([\gamma_\kappa(\kappa), x]\). That segment lies within \(2\lambda + 1\) of the segment \([\beta(\kappa), y]\). Consider the geodesic quadrilateral \([\beta(\kappa), y] \cup [y, x] \cup [x, \gamma_\kappa(\kappa)] \cup [\gamma_\kappa(\kappa), \beta(\kappa)]\). The geodesic segment from each point on \(\sigma\) to the corresponding point on \([x, y]\) defines a homotopy between \(\sigma\) and \([x, y]\) which stays within \(2\lambda + 1\) of \([x, \gamma_\kappa(\kappa)]\). This can be matched with the previously defined contraction. The result is a homotopy of \(\theta \ast \sigma\) to \([x, y]\) relative to \(\alpha, \beta\), which misses \(B(g_n p, p + 1)\) (see II.2.2 of [BH99]).

By small homotopies this gives a homotopy from the path \([\alpha(0), \beta(0)]\) to the path \([\alpha_n(\kappa), \beta_n(\kappa)]\) relative to \(\alpha_n, \beta_n\) missing \(B(g_n p, p)\). Translating this back by \(g_n^{-1}\) gives a contradiction. \(\square\)

6. Separation in the Tits metric topology

Under the hypotheses of Theorem \[5.5\] it is clear that any small open neighborhood of \(c\) in the cone topology separates \(a\) from \(b\). In this section we improve that statement:

**Theorem 6.1.** If \(X\) does not have semistable fundamental group at infinity then there are points \(a, b, c \in \partial X\) such that the Tits ball \(B := B_T(c, \frac{\pi}{2})\) separates \(a\) from \(b\).

**Proof.** The points \(a, b, c\) are those found in the proof of Theorem \[5.5\] and we continue to use notation from that proof. As \(n \to \infty\) the balls \(B(g_n p, \sigma_n + \lambda + 1)\) converge to a horoball \(H\) about \(c\), and by Lemma 1 of [Swe13] the boundary of \(H\) in \(\partial X\) is \(B\). The points \(a\) and \(b\) do not lie in \(B\) because the lines \(\alpha\) and \(\beta\) leave \(H\).

Suppose \(a\) and \(b\) were in the same component of \(\partial X - B\). Then for any cover \(\mathcal{G}\) of \(\partial X - B\) by sets open in \(\tilde{X} - B\) there would be sets \(W_1, \ldots, W_j\) in \(\mathcal{G}\) such that each intersection \(W_i \cap W_{i+1}\) is non-empty, \(a \in W_i\) and \(b \in W_j\). We choose \(\mathcal{G}\) to consist of sets \(W\) such that any two geodesics starting at \(x\) and ending in \(W\) fellow-travel for a long time after they leave
Using these sets $W_i$ we construct a path $\omega$ in $X - H$ joining a point on $\alpha$ to a point on $\beta$. The loop formed by $\omega, [x, y]$ and the obvious segments of $\alpha$ and $\beta$ contracts in a compact subset of $X$. This contraction misses $B(g_n, \rho)$ when $n >> 0$. As in the previous proof this gives a contradiction.

7. Appendix: Proof of Proposition 2.1

Recall that the space $X$, and hence also the space $\hat{X}$ (obtained by adding a Z-set to $X$), are AR’s. Let $Q$ denote the Hilbert Cube, equipped with the metric it inherits as a subspace of Hilbert Space. Then $X \times Q$, with the usual product metric, is also a proper $CAT(0)$ space. The boundary of $X \times Q$ is again $\partial X$. If $\{U_n := X - K_n\}$ of Section 1 is a basic sequence of neighborhoods of infinity in $X$ then $\{U_n \times Q\}$ plays the same role in $X \times Q$.

By theorems of West [Wes77] and Chapman [Cha74], $X \times Q$ is a $Q$-manifold and hence is homeomorphic to $P \times Q$ for some locally compact polyhedron $P$. Clearly the polyhedral structure gives a sequence $\{V_n\}$ of closed ANR neighborhoods of infinity in $P \times Q$ (which become compact ANR’s when compactified by the boundary). So the shape theory of $\partial X$ can be read off from the pro-homotopy of the sequence $\{V_n\}$. The image, under the homeomorphism, of the $V_n$’s in $X \times Q$ gives a sequence of closed ANR neighborhoods of infinity in $X \times Q$. This image is cofinal in $\{U_n \times Q\}$. So one has semistable fundamental group if and only if the other has.

In summary, semistability of $\{V_n\}$ is equivalent to $\partial X$ having semistable fundamental pro-group, and is also equivalent to semistability of the inverse sequence of fundamental groups of the $U_n \times Q$’s, hence also to semistability of the inverse sequence of fundamental groups of the $U_n$’s.

Remark 7.1. It might be objected that the use of the deepest theorems of infinite-dimensional topology to prove the “almost obvious” Proposition 2.1 is overkill. Other proofs are possible. One could follow a method used in [Ont05] where the $CAT(0)$ structure is used to construct a good open cover of $X$ whose nerve is proper homotopy equivalent to $X$. One would then proceed, much as in our proof, to find closed ANR neighborhoods of infinity in that nerve. Another approach would be to use more general shape theory as described in the monograph [MsS82] (see especially pages 18-19) to prove that the sequence $\{U_n\}$ is an “expansion” of $\partial X$. Open subsets of $X$ are ANR’s and hence are homotopy equivalent to CW complexes, so this is a “polyhedral expansion” in their sense. This would establish that the shape theory of $\partial X$ can be read off from $\{U_n\}$.

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