A TALE OF TWO SHUFFLE ALGEBRAS

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Abstract. As a quantum affinization, the quantum toroidal algebra $U_q,\varpi(\hat{\mathfrak{gl}}_n)$ is defined in terms of its “left” and “right” halves, which both admit shuffle algebra presentations (\cite{6}, \cite{10}). In the present paper, we take an orthogonal viewpoint, and give shuffle algebra presentations for the “top” and “bottom” halves instead, starting from the evaluation representation $U_q(\hat{\mathfrak{gl}}_n) \rtimes C^n(z)$ and its usual $R$-matrix $R(z) \in \text{End}(C^n \otimes C^n)(z)$ (see \cite{7}). An upshot of this construction is a new topological coproduct on $U_q,\varpi(\hat{\mathfrak{gl}}_n)$ which extends the Drinfeld-Jimbo coproduct on the horizontal subalgebra $U_q(\hat{\mathfrak{gl}}_n) \subset U_q,\varpi(\hat{\mathfrak{gl}}_n)$.

1. Introduction

1.1. The affine quantum group $U_q(\hat{\mathfrak{sl}}_n) = U_q(\hat{\mathfrak{sl}}_n)$ (hats will be replaced by points in the present paper) has the following two presentations:

- as the quantum affinization of $U_q(\mathfrak{sl}_n)$
- as the Drinfeld-Jimbo quantum group whose Dynkin diagram is an $n$-cycle

However, the two presentations above yield different bialgebra structures on $U_q(\hat{\mathfrak{sl}}_n)$, which is evidenced by the fact that the coproduct in the first bullet is only topological (i.e. $\Delta$ is an infinite sum, which only makes sense in a certain completion). Moreover, the two bullets above yield different triangular decompositions of $U_q(\hat{\mathfrak{sl}}_n)$ into positive, Cartan, and negative halves:

\begin{align*}
(1.1) \quad U_q(\hat{\mathfrak{sl}}_n) &\cong U_q^+(\hat{\mathfrak{sl}}_n) \otimes (\text{Cartan subalgebra}) \otimes U_q^-(\hat{\mathfrak{sl}}_n) \\
(1.2) \quad U_q(\hat{\mathfrak{sl}}_n) &\cong U_q^+(\hat{\mathfrak{sl}}_n) \otimes (\text{Cartan subalgebra}) \otimes U_q^-(\hat{\mathfrak{sl}}_n)
\end{align*}

The two decompositions above are quite different: the positive subalgebra $U_q^+(\hat{\mathfrak{sl}}_n)$ of (1.1) is generated by Drinfeld’s elements $e_{i,k}$ over all $1 \leq i < n$ and $k \in \mathbb{Z}$, while the positive subalgebra $U_q^+(\hat{\mathfrak{sl}}_n)$ of (1.2) is generated by the Drinfeld-Jimbo elements $\{e_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$. The connection between these two presentations was given in \cite{1}.

1.2. The main purpose of the present paper is to extend the description above to the quantum toroidal algebra $U_q,\varpi(\hat{\mathfrak{gl}}_n)$, which is defined as in the first bullet:

\[ U_q,\varpi(\hat{\mathfrak{gl}}_n) := \text{affinization of } U_q(\hat{\mathfrak{gl}}_n) \]

This construction naturally comes with a triangular decomposition (see Subsection 3.12 for an overview of the quantum toroidal algebra, as well as of our conventions):

\begin{align*}
(1.3) \quad U_q,\varpi(\hat{\mathfrak{gl}}_n) &\cong \bar{U}_q,\varpi(\hat{\mathfrak{gl}}_n) \otimes \bar{U}_q,\varpi(\hat{\mathfrak{gl}}_n)
\end{align*}
Our $\tilde{U}_{q,\mathfrak{g}}(\mathfrak{g}_n)$ and $\check{U}_{q,\mathfrak{g}}(\mathfrak{g}_n)$ are the Borel subalgebras of the quantum toroidal algebra, and they explicitly arise as a unipotent part tensored with a Cartan part:

\begin{align}
\tilde{U}_{q,\mathfrak{g}}(\mathfrak{g}_n) & \cong U_{q,\mathfrak{g}}(\mathfrak{g}_n) \otimes U_q^\geq(\mathfrak{g}_1)^n \\
\check{U}_{q,\mathfrak{g}}(\mathfrak{g}_n) & \cong U_{q,\mathfrak{g}}(\mathfrak{g}_n) \otimes U_q^\leq(\mathfrak{g}_1)^n
\end{align}

There is a well-known topological coproduct of $U_{q,\mathfrak{g}}(\mathfrak{g}_n)$, which preserves the subalgebras (1.4) and (1.5), and extends the (almost) cocommutative coproduct on the “vertical” subalgebra:

\begin{align}
U_q^\geq(\mathfrak{g}_1)^n \otimes U_q^\leq(\mathfrak{g}_1)^n & = U_q(\mathfrak{g}_1)^n \subset U_{q,\mathfrak{g}}(\mathfrak{g}_n)
\end{align}

The main goal of this paper is to define another decomposition into subalgebras:

\begin{align}
U_{q,\mathfrak{g}}(\mathfrak{g}_n) & \cong \tilde{U}_{q,\mathfrak{g}}(\mathfrak{g}_n) \otimes \check{U}_{q,\mathfrak{g}}(\mathfrak{g}_n)
\end{align}

(see Corollary 3.30). We will explicitly construct the tensor factors of (1.7) as:

\begin{align}
\tilde{U}_{q,\mathfrak{g}}(\mathfrak{g}_n) & \cong U_{q,\mathfrak{g}}(\mathfrak{g}_n) \otimes U_q^\geq(\mathfrak{g}_1)^n \\
\check{U}_{q,\mathfrak{g}}(\mathfrak{g}_n) & \cong U_{q,\mathfrak{g}}(\mathfrak{g}_n) \otimes U_q^\leq(\mathfrak{g}_1)^n
\end{align}

where the “horizontal” subalgebra:

\begin{align}
U_q^\geq(\mathfrak{g}_1)^n \otimes U_q^\leq(\mathfrak{g}_1)^n & = U_q(\mathfrak{g}_1)^n \subset U_{q,\mathfrak{g}}(\mathfrak{g}_n)
\end{align}

will be the quantum group in the RTT presentation \cite{7}. Moreover, we endow $U_{q,\mathfrak{g}}(\mathfrak{g}_n)$ with a new topological coproduct which preserves the subalgebras (1.8), (1.9), and extends the usual (Drinfeld-Jimbo) coproduct on $U_q(\mathfrak{g}_1)^n \subset U_{q,\mathfrak{g}}(\mathfrak{g}_n)$.

1.3. To represent the aforementioned decompositions pictorially, we will recall that the quantum toroidal algebra is graded by $\mathbb{Z}^n \times \mathbb{Z}$, where $\mathbb{Z}^n$ is the root lattice of $U_q(\mathfrak{sl}_n)$ and $\mathbb{Z}$ is the affinization direction. Then the following picture indicates the various subalgebras of $U_{q,\mathfrak{g}}(\mathfrak{g}_n)$, by displaying which degrees they live in:

\begin{figure}
\centering
\includegraphics{grading.pdf}
\caption{The grading of $U_{q,\mathfrak{g}}(\mathfrak{g}_n)$ and its various subalgebras}
\end{figure}
In the particular case $n = 1$, the quantum toroidal algebra is isomorphic to the well-known Ding-Iohara-Miki ([4], [13]) a.k.a. elliptic Hall algebra ([2], [20]), which has an action of $SL_2(\mathbb{Z})$ by automorphisms. In particular, the decomposition (1.7) is obtained from the decomposition (1.3) by applying the automorphism corresponding to rotation by 90 degrees. However, in the general $n$ case, the algebras featuring in the two decompositions are not isomorphic to each other, which is sensible given the fact that the grading axes $\mathbb{Z}^n$ and $\mathbb{Z}$ are quite different.

1.4. To describe $U^+_{q,\bar{q}}(\mathfrak{gl}_n)$ and $U^-_{q,\bar{q}}(\mathfrak{gl}_n)$ of (1.3), let us consider the vector space:

\[(1.11) \quad \mathcal{S}^+ \subset \bigoplus_{(d_1, \ldots, d_n) \in \mathbb{N}^n} \mathbb{Q}(q, \bar{q}^{1/2})(z_{111}, \ldots, z_{1d_11}, \ldots, z_{n11}, \ldots, z_{nd_n1})^{\text{Sym}}\]

of rational functions which satisfy the wheel conditions (as in [8], [10]): namely that such rational functions have at most simple poles at $z_{ia}q^2 - z_{i+1,b}$ (for all $i, a, b$) and that the residue at such a pole is divisible by $z_{ia} - z_{i+1,b}$ and $z_{ia} - z_{i+1,b'}$ for all $a' \neq a$ and $b' \neq b$. The vector space (1.11) is called a shuffle algebra, akin to the classical construction of Feigin and Odesskii concerning certain elliptic algebras ([10]). Explicitly, the product on (1.11) is constructed using the rational function (3.40), see Definition 3.16. An algebra homomorphism was constructed in [6]:

\[U^+_{q,\bar{q}}(\mathfrak{gl}_n) \rightarrow \mathcal{S}^+\]

and it was shown to be an isomorphism in [16]. Similarly, $U^-_{q,\bar{q}}(\mathfrak{gl}_n) \cong \mathcal{S}^- = (\mathcal{S}^+)^{\text{op}}$.

To describe the subalgebras $U^+_{q,\bar{q}}(\mathfrak{gl}_n)$ and $U^-_{q,\bar{q}}(\mathfrak{gl}_n)$ which appear in (1.7), we will introduce a new kind of shuffle algebra (let $V$ be an $n$–dimensional vector space):

\[(1.12) \quad \mathcal{A}^+ \subset \bigoplus_{k=0}^{\infty} \text{End}_{\mathbb{Q}(q, \bar{q}^{1/2})} (V \otimes \ldots \otimes V)(z_1, \ldots, z_k)\]

and the algebra structure on the RHS is constructed using the $R$–matrix (4.1), see Propositions 4.5 and 4.6. By definition, the subspace (1.12) precisely consists of End($V^{\otimes k}$)–valued rational functions which have at most simple poles at $z_{a}q^2 - z_{b}$ (for all $a, b$) and whose residue at such a pole satisfies the conditions outlined in Definition 4.8. The subalgebra $\mathcal{A}^-$ is defined similarly, but with $\bar{q}^{-1}q^{-n}$ instead of $\bar{q}$.

**Theorem 1.5.** There exist injective algebra homomorphisms:

\[\mathcal{A}^+, \mathcal{A}^-_{\text{op}} \hookrightarrow U_{q,\bar{q}}(\mathfrak{gl}_n)\]

Denoting the images of these maps by $U^+_{q,\bar{q}}(\mathfrak{gl}_n)$ and $U^-_{q,\bar{q}}(\mathfrak{gl}_n)$ yields the decomposition (1.7). Moreover, there exist topological coproducts on the subalgebras:

\[(1.13) \quad \bar{\mathcal{A}}^+ = \mathcal{A}^+ \otimes U^\geq_{q}(\mathfrak{gl}_n) \hookrightarrow U_{q,\bar{q}}(\mathfrak{gl}_n)\]

\[(1.14) \quad \bar{\mathcal{A}}^-_{\text{op}} = (\mathcal{A}^- \otimes U^\leq_{q}(\mathfrak{gl}_n))^{\text{op}} \hookrightarrow U_{q,\bar{q}}(\mathfrak{gl}_n)\]

which extend the Drinfeld-Jimbo coproduct on the horizontal subalgebra (1.10), and realize $U_{q,\bar{q}}(\mathfrak{gl}_n)$ as the Drinfeld double of its subalgebras (1.13) and (1.14).
In [11], the authors claim that $U_q(\mathfrak{gl}_n)$ admits a Drinfeld double structure via generators and relations, but there seem to be fundamental differences between the subalgebra called $B$ in loc. cit. and our $U^+_{q,\pi}(\mathfrak{gl}_n)$. For example, in degree $\mathbb{Z}^n \times \{1\}$, the former algebra has elements parametrized by the positive roots of $U_q(\mathfrak{sl}_n)$, while the latter has elements parametrized by all roots of $U_q(\mathfrak{sl}_n)$. Therefore, we do not make any claims concerning the connection of our work with loc. cit.

We emphasize the fact that $U^+_{q,\pi}(\mathfrak{gl}_n)$ is not the same as the “vertical subalgebra” that was studied in [9] and numerous other works. The latter construction has to do with $U_q(\mathfrak{sl}_n)$ presented as the affinization of $U_q(\mathfrak{sl}_n)$ and thus implicitly breaks the symmetry among the vertices of the cyclic quiver. Meanwhile, our construction takes the “horizontal subalgebra” $U_q(\mathfrak{gl}_n)$ and its evaluation representation $V = \mathbb{C}^n(z)$ as an input, and outputs half of the quantum toroidal algebra.

More generally, starting from a quantum group $U_q(\mathfrak{g})$ and a representation $V$ endowed with a unitary $R$–matrix, one may ask if the double shuffle algebra:

\begin{equation}
\mathcal{D}\left(\text{an appropriate subalgebra of } \bigoplus_{k=0}^{\infty} \text{End}(V \otimes^k)\right)
\end{equation}

(defined as in Section 2) is related to the quantum group $U_q(\mathfrak{g})$. Theorem 1.5 deals with the case $\mathfrak{g} = \mathfrak{gl}_n$ and $V = \mathbb{C}^n(z)$. If something along these lines is true in affine types other than $A$, we venture to speculate that the algebra (1.15) might be related to the extended Yangians of [21], appropriately $q$–deformed and doubled. Such a realization of quantum affinizations is to be expected from the work [12], [19], who have realized Yangians and their $q$–deformations inside endomorphism rings of tensor products of certain geometrically defined representations of $\mathfrak{g}$.

1.6. The structure of the present paper is the following:

- In Section 2 we construct a shuffle algebra $A^+$ starting from a vector space $V$ and a unitary $R$–matrix $\in \text{End}(V \otimes^2)$ (see also [14]). By adding certain elements, we construct the extended shuffle algebra $\tilde{A}^+$, which admits a coproduct. From two such extended shuffle algebras, we construct their Drinfeld double $A$.

- In Section 3 we recall the quantum group $U_q(\mathfrak{gl}_n)$ and its PBW presentation from [17]. This will allow us to construct the decomposition (1.7) as algebras.

- In Section 4 we construct a version of the shuffle algebra of Section 2 that corresponds to the $R$–matrix with spectral parameter (4.1), thus yielding (1.12).

- In Section 5 we construct the extended version of the shuffle algebra of Section 4 and endow it with a topological coproduct, and construct a PBW basis of it.

- In Section 6 we construct a bialgebra pairing between two copies of the extended shuffle algebras of Section 5. The corresponding Drinfeld double will precisely match $U_q(\mathfrak{gl}_n)$, thus completing the proof of Theorem 1.5.
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1.7. Given a finite-dimensional vector space $V$, we will often write elements $X \in \text{End}(V^\otimes k)$ as $X_{i_1,...,i_k}$ in order to point out the set of indices of $X$. If $V = \mathbb{C}^n$, then:

$$X = \sum_{i_1,...,i_k,j_1,...,j_k} \text{coefficient} \cdot E_{i_1j_1} \otimes ... \otimes E_{i_kj_k}$$

for certain coefficients, where $E_{ij} \in \text{End}(V)$ denotes the matrix with entry 1 on row $i$ and column $j$, and 0 everywhere else. For any permutation $\sigma \in S(k)$, we write:

$$\sigma X \sigma^{-1} = X_{\sigma(1),...,\sigma(k)}$$

(1.17)

where $\sigma \curvearrowright V^\otimes k$ by permuting the factors (therefore, the effect of conjugating (1.16) by $\sigma$ is to replace the indices $i_1,...,j_k$ by $i_{\sigma(1)},...,j_{\sigma(k)}$). Moreover, we will write:

$$X_{i_1,...,i_k} = X_{i_1} \otimes ... \otimes X_{i_k} \in \text{End}(V^\otimes N) \otimes \text{End}(V^\otimes k-N) \cong \text{End}(V^\otimes k)$$

(1.18)

if we wish to set apart the first $i$ tensor factors from the last $k-i$ tensor factors of $X$. There is an implicit summation in the right-hand side of (1.18) which we will not write down, much alike Sweedler notation. For any $a \in \mathbb{N}$, we will write:

$$E_{ij}^{(a)} = 1 \otimes ... \otimes E_{ij} \otimes ... \otimes 1 \in \text{End}(V^\otimes k)$$

with $a$-th position

(2.1)

2. Shuffle algebras and $R$–matrices

2.1. The main goal of the present Section is to study shuffle algebras associated to the data contained in the 4 bullets below:

- a vector space $V$, assumed finite-dimensional for simplicity

- an element ($R$–matrix) $R \in \text{Aut}(V^\otimes 2)$ satisfying the Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

(2.1)

- an element $\tilde{R} \in \text{Aut}(V^\otimes 2)$ satisfying:

$$\tilde{R}_{21}\tilde{R}_{31}R_{23} = R_{23}\tilde{R}_{31}\tilde{R}_{21}$$

(2.2)

$$R_{12}\tilde{R}_{31}\tilde{R}_{32} = \tilde{R}_{32}\tilde{R}_{31}R_{12}$$

(2.3)
• a scalar $f$ so that:

\begin{equation}
R_{12}R_{21} = f \cdot \text{Id}_{V \otimes V} = R_{21}R_{12}
\end{equation}

The present Section will be concerned with generalities in the context above, while Section 4 deals with a particular setting, namely that of:

\begin{equation}
R(x) = \text{RHS of (3.87)} \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)(x)
\end{equation}

and $\tilde{R}(x) = R_{21}(x^{-1}\bar{q}^{-2})$, for a parameter $\bar{q}$. Many Propositions in the current Section have counterparts in Section 4 and we will only prove such statements once.

2.2. We will represent the tensor product $V \otimes^k$ as $k$ labeled dots on a vertical line, and certain elements of $\text{End}(V \otimes^k)$ will be represented as braids between two such collections of $k$ labeled dots situated on parallel vertical lines. Specifically, the crossings below represent either the automorphisms $R$ or $\tilde{R}$, with indices given by the labels of the strands (which are inherited from the labels of their endpoints):

![Various crossings](image1.png)

**Figure 1.** Various crossings

The strands are represented either as straight or squiggly, because we wish to indicate whether the picture in question refers to either $R$ or $\tilde{R}$. Compositions are always read left-to-right, for example the following equivalence of braids underlies the Yang-Baxter relations (2.1):

![Reidemeister III move - version 1](image2.png)

**Figure 2.** Reidemeister III move - version 1

while the following equivalences underlie equations (2.2) and (2.3), respectively:
We will equivalate braids connected by the Reidemeister III type moves above.

2.3. We will now recall the construction of Section 5.2 of [14] (itself a dual version of the construction of [7]) and present it in the language of shuffle algebras. We will then construct an extended shuffle algebra which admits a coproduct and bialgebra pairing, and then define the corresponding Drinfeld double.

**Proposition 2.4.** For $V, R, \tilde{R}$ as in Subsection 2.1, the assignment:

$$A_{1\ldots k} * B_{1\ldots l} = \sum_{\{1, \ldots, k+l\} = \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_l\}} \sum_{\text{only if } a_i < b_j} R_{a_kb_l} \ldots R_{a_1b_1} \left[ A_{a_2 \ldots a_k} \tilde{R}_{a_1b_1} \ldots \tilde{R}_{a_kb_l} B_{b_2 \ldots b_l} \right]$$

(2.6)

yields an associative algebra structure on the vector space:

$$\bigoplus_{k=0}^{\infty} \text{End}(V^\otimes k)$$

with unit $1 \in \text{End}(V^\otimes 0)$. We will call (2.6) the “shuffle product”.

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1 The meaning of the indexing sets in the three products of $R$’s or $\tilde{R}$’s is that the factors in:

$$\tilde{R}_{a_1b_1} \ldots \tilde{R}_{a_kb_l}$$

are taken in any order such that $(a_i, b_j)$ is to the left of $(a_i', b_j')$ if $i < i'$ and to the right of $(a_i, b_j')$ if $j < j'$. The text “only if $a_i < b_j$” or “only if $a_i > b_j$” under such a product means that only those pairs of indices $(a_i, b_j)$ satisfying the respective inequalities occur in the product.
We note that the second line of (2.6) can be represented by the following braid, depicted here for $k = 2$, $l = 2$ and $(a_1, a_2) = (1, 3)$, $(b_1, b_2) = (2, 4)$:

![Braid Diagram](image)

**Figure 5.** $A \ast B$ as a braid

The proof of Proposition 2.4, namely that the multiplication defined above is associative, is a straightforward consequence of the following equivalence of braids:

![Braid Diagram](image)

**Figure 6.** $(A \ast B) \ast C$

![Braid Diagram](image)

**Figure 7.** $A \ast (B \ast C)$

Indeed, in the top picture, one can pull the straight red strands to the left of the blue-green crossings, and the squiggly red strands below the blue-green crossings.
This procedure is simply a succession of the Reidemeister III moves of Figures 2, 3 and 4, which in the end produces the bottom picture.

2.5. The symmetrization of a tensor $X \in \text{End}(V^\otimes k)$ is defined as:

\begin{equation}
\text{Sym } X = \sum_{\sigma \in S(k)} R_\sigma \cdot (\sigma X \sigma^{-1}) \cdot R^{-1}_\sigma
\end{equation}

where $\sigma X \sigma^{-1}$ refers to the permutation of the tensor factors of $X$ in accordance with $\sigma$, and $R_\sigma \in \text{End}(V^\otimes k)$ is any braid connecting the $i$–th endpoint on the right with the $\sigma(i)$–th endpoint on the left.

![Figure 8. A braid representation of $R_\sigma \cdot (\sigma X \sigma^{-1}) \cdot R^{-1}_\sigma$](image)

Choosing one braid lift of $\sigma$ over another is just the ambiguity of choosing $R_{ab}$ over $R_{ba}$ for any crossing between the strands labeled $a$ and $b$. Since (2.4) says that these two endomorphisms differ by a scalar, the ambiguity does not affect (2.8).

A tensor $Y \in \text{End}(V^\otimes k)$ is called symmetric if:

\begin{equation}
Y = R_\sigma \cdot (\sigma Y \sigma^{-1}) \cdot R^{-1}_\sigma
\end{equation}

$\forall \sigma \in S(k)$. It is easy to see that any symmetrization (2.8) is a symmetric tensor.

**Proposition 2.6.** The shuffle product of Proposition 2.4 preserves the vector space:

\begin{equation}
\mathcal{A}^+ \subset \bigoplus_{k=0}^{\infty} \text{End}(V^\otimes k)
\end{equation}

consisting of symmetric tensors. We will call $\mathcal{A}^+$ the shuffle algebra.

**Proof.** Let $A_{1...k} \in \text{End}(V^\otimes k)$ and $B_{1...l} \in \text{End}(V^\otimes l)$ be any symmetric tensors. A permutation $\mu \in S(k+l)$ is called a $(k,l)$–shuffle if:

\begin{align}
a_1 := \mu(1) < \ldots < a_k := \mu(k) \\
b_1 := \mu(k+1) < \ldots < b_l := \mu(k+l)
\end{align}

Because of the diagram (depicted for $k = 2, l = 2, (a_1, a_2) = (1, 3), (b_1, b_2) = (2, 4)$):
Figure 9. $R_{\mu} \cdot (\mu \Phi^{-1}) \cdot R_{\mu}^{-1}$

it is easy to see that the definition (2.6) can be restated as:

(2.12) \[ A * B = \sum_{(k,l)\text{-shuffles}} R_{\mu} \cdot (\mu \Phi^{-1}) \cdot R_{\mu}^{-1} \]

where:

(2.13) \[ \Phi = \left[ R_{k,k+1} \ldots R_{1,k+1} \right] A_{1 \ldots k} \left[ \tilde{R}_{1,k+1} \ldots \tilde{R}_{k,k+1} \right] B_{k+1 \ldots k+l} \]

For any $\tau \in S(k) \times S(l) \subset S(k+l)$, we have:

$R_{\tau} \cdot (\tau \Phi \tau^{-1}) \cdot R_{\tau}^{-1} = \Phi$

which can be seen from the fact that the braid below:

Figure 10. $R_{\tau} \cdot (\tau \Phi \tau^{-1}) \cdot R_{\tau}^{-1}$

is equivalent to $\Phi$ (since we can cancel the braids representing $R_{\tau}$ and $R_{\tau}^{-1}$ by pulling them through the symmetric tensors $A$ and $B$). Then (2.12) implies:

$$A * B = \frac{1}{k!!} \sum_{(k,l)\text{-shuffles}} \sum_{\tau \in S(k) \times S(l)} R_{\mu} \cdot \mu \left( R_{\tau} \cdot \tau \Phi \tau^{-1} \cdot R_{\tau}^{-1} \right) \mu^{-1} \cdot R_{\mu}^{-1}$$

Since any $\sigma \in S(k+l)$ can be written uniquely as $\mu \circ \tau$, where $\mu$ is a $(k,l)$–shuffle and $\tau \in S(k) \times S(l) \subset S(k+l)$, the formula above yields:

(2.14) \[ A * B = \frac{1}{k!!l!!} \sum_{\sigma \in S(k+l)} R_{\sigma} \cdot (\sigma \Phi \sigma^{-1}) \cdot R_{\sigma}^{-1} = \frac{1}{k!!l!!} \cdot \text{Sym} \ \Phi \]
(we have used the fact that $R_{\mu \tau} = R_{\mu} \cdot \mu R_{\tau} \mu^{-1}$, times a product of scalars (2.4)). Since the symmetrization of any tensor is symmetric, this concludes the proof.

By analogy with formula (2.14), one has the following:

$$A * B = \frac{1}{k!!} \cdot \text{Sym} \Psi$$

where $\Psi = A_{l+1...l+k} \left[ \tilde{R}_{l+1,l}...\tilde{R}_{l+k,1} \right] B_{1...l} \left[ R_{l+k,1}...R_{l+1,l} \right]$.

2.7. Let us fix a basis $v_1, ..., v_n$ of $V$ and write $E_{ij}$ for the elementary symmetric matrix with a single 1 at the intersection of row $i$ and column $j$, and 0 otherwise.

**Definition 2.8.** Consider the extended shuffle algebra:

$$\tilde{A}^+ = \left\langle A^+, s_{ij}, t_{ij} \right\rangle \big/ \text{relations (2.16)–(2.20)}$$

In order to concisely state the relations, it makes sense to package the new generators $s_{ij}, t_{ij}$ into generating functions:

$$S = \sum_{i,j=1}^n s_{ij} \otimes E_{ij} \in \tilde{A}^+ \otimes \text{End}(V)$$

$$T = \sum_{i,j=1}^n t_{ij} \otimes E_{ij} \in \tilde{A}^+ \otimes \text{End}(V)$$

and the required relations take the form:

(2.16) $RS_1S_2 = S_2S_1R$

(2.17) $T_1T_2R = RT_2T_1$

(2.18) $T_1\tilde{R}S_2 = S_2\tilde{RT}_1$

as well as:

(2.19) $X_{1...k} \cdot S_0 = S_0 \cdot \frac{R_{k0}...R_{l0}}{f_{k0}...f_{l0}} X_{1...k} \tilde{R}_{l0}...\tilde{R}_{k0}$

(2.20) $T_0 \cdot X_{1...k} = \tilde{R}_{0k}...\tilde{R}_{01} X_{1...k} \frac{R_{01}...R_{0k}}{f_{01}...f_{0k}} \cdot T_0$

$\forall X_{1...k} \in \text{End}(V^{\otimes k})$. The latter two formulas should be interpreted as identities in $\tilde{A}^+ \otimes \text{End}(V)$, where the latter copy of $V$ is the one represented by index 0.

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2We write $f_{ij} = f_{ji}$ for the scalar $f \in \text{Aut}(V \otimes V)$, interpreted as an element of $\text{Aut}(V^{\otimes k})$ via the inclusion of the $i$–th and $j$–th tensor factors. This notation will come in handy in Section 5, when $f_{ij}$ will be a scalar-valued rational function in the variable $z_i/z_j$. 

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Proposition 2.9. The following assignments make $\tilde{A}^+$ into a bialgebra:

$$\Delta(S) = (1 \otimes S)(S \otimes 1), \quad \varepsilon(S) = \text{Id}$$  \hspace{1cm} (2.21) $$

$$\Delta(T) = (T \otimes 1)(1 \otimes T), \quad \varepsilon(T) = \text{Id}$$  \hspace{1cm} (2.22) $$

while for all $X = X_{1...k} \in \mathcal{A}^+ \subset \tilde{A}^+$ for $k \geq 1$, we set $\varepsilon(X) = 0$ and:

$$\Delta(X) = \sum_{i=0}^{k} (S_{k...S_{i+1}} \otimes 1) \left( \prod_{1 \leq u \leq i < v \leq k} f_{uv}^{-1} \right) (T_{i+1...T_k} \otimes 1)$$  \hspace{1cm} (2.23) $$

where the notation $X_{1...k} = X_{1...i} \otimes X_{i+1...k}$ is explained in (1.18).

Proof. The facts that the counit extends to an algebra homomorphism, and that it interacts appropriately with the coproduct, are easy to see. In order to show that the coproduct extends to an algebra homomorphism:

$$\tilde{A}^+ \rightarrow \tilde{A}^+ \otimes \tilde{A}^+$$

we must show that (2.21), (2.22), (2.23) respect relations (2.16)–(2.20), as well as formula (2.6) for the shuffle product. To this end, note that:

$$\Delta(RS_1S_2) = (1 \otimes S_1)(S_1 \otimes 1)(1 \otimes S_2)(S_2 \otimes 1) = (1 \otimes S_2)(S_2 \otimes 1)(1 \otimes S_1)(S_1 \otimes 1)R = \Delta(S_2S_1R)$$

An analogous argument shows that $\Delta(T_1T_2R) = \Delta(RT_2T_1)$. As for (2.18):

$$\Delta(T_1RS_2) = (T_1 \otimes 1)(1 \otimes T_1)\tilde{R}(1 \otimes S_2)(S_2 \otimes 1) = (1 \otimes S_2)(1 \otimes T_1)\tilde{R}(T_1 \otimes S_2)(S_2 \otimes 1) = \Delta(S_2\tilde{R}T_1)$$

Let us now apply the coproduct to the left-hand side of (2.19):

$$\Delta(X_{1...k}S_0) = \sum_{i=0}^{k} (S_{i...S_{i+1}} \otimes 1)(X_{1...i} \otimes X_{i+1...k})(T_{i+1...T_k} \otimes 1)(1 \otimes S_0)(S_0 \otimes 1)$$

$$\sum_{i=0}^{k} (1 \otimes S_0) \frac{R_{k0}...R_{i+1,0}}{f_{k0}...f_{i+1,0}} (S_{k...S_{i+1}} \otimes 1)(X_{1...i} \otimes X_{i+1...k})(T_{i+1...T_k} \otimes 1)\tilde{R}_{i+1,0}...\tilde{R}_{k0}(S_0 \otimes 1)$$

$$\sum_{i=0}^{k} (1 \otimes S_0) \frac{R_{k0}...R_{i+1,0}}{f_{k0}...f_{i+1,0}} (S_{k...S_{i+1}} \otimes 1)(X_{1...i} \otimes X_{i+1...k})(S_0 \otimes 1)\tilde{R}_{i+1,0}...\tilde{R}_{k0}(T_{i+1...T_k} \otimes 1)$$

$$\sum_{i=0}^{k} (1 \otimes S_0) \frac{R_{k0}...R_{i+1,0}}{f_{k0}...f_{i+1,0}} (S_{k...S_{i+1}}S_0 \otimes 1) \frac{R_{i0}...R_{01}}{f_{i0}...f_{01}} (X_{1...i} \otimes X_{i+1...k})\tilde{R}_{i0}...\tilde{R}_{k0}(T_{i+1...T_k} \otimes 1)$$

$$\sum_{i=0}^{k} (1 \otimes S_0)(S_0 \otimes 1)(S_{k...S_{i+1}} \otimes 1) \frac{R_{k0}...R_{i0}}{f_{k0}...f_{i0}} (X_{1...i} \otimes X_{i+1...k})\tilde{R}_{i0}...\tilde{R}_{k0}(T_{i+1...T_k} \otimes 1)$$
The last line above equals $\Delta(\text{RHS of (2.19))}, so we are done. The computation showing that $\Delta$ respects relation (2.20) is analogous, and therefore left to the interested reader. As for the shuffle product itself, we must show that $\Delta(A_{1\ldots k} \ast B_{1\ldots l}) = \Delta(A_{1\ldots k}) \ast \Delta(B_{1\ldots l})$. Applying formulas (2.6) and (2.23) implies:

$$
\Delta(A_{1\ldots k}) \ast \Delta(B_{1\ldots l}) = \sum_{d=0}^{k} \sum_{e=0}^{l} (S_{k\ldots e} \otimes S_{d+1}) \left( \frac{A_{1\ldots d} \otimes A_{d+1\ldots k}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{uv}} \right) (T_{d+1\ldots T_{k} \otimes 1}^1) (B_{1\ldots e} \otimes B_{e+1\ldots l}) \left( \frac{B_{1\ldots e} \otimes B_{e+1\ldots l}}{\prod_{1 \leq u \leq e \leq v \leq l} f_{uv}} \right) (T_{e+1\ldots T_{l} \otimes 1}) \left( \frac{T_{e+1\ldots T_{l} \otimes 1}}{\prod_{1 \leq u \leq e \leq v \leq l} f_{uv}} \right)
$$

In the next-to-last row of the expression above, we may apply (2.18) in order to move the $T's$ to the right and the $S's$ to the left. Afterwards, we apply (2.19) and (2.20) to move the $S's$ to the left of $A_{a_1\ldots a_d}$ and the $T's$ to the right of $B_{b_1\ldots b_e}$:

$$
\Delta(A_{1\ldots k}) \ast \Delta(B_{1\ldots l}) = \sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a_1 < \ldots < a_d} \sum_{b_1 < \ldots < b_e} \sum_{\{1, \ldots, d+e\} = \{a_1, \ldots, a_d\} \cup \{b_1, \ldots, b_e\}} \sum_{\{d+e+1, \ldots, k+l\} = \{a_{d+1}, \ldots, a_k\} \cup \{b_{e+1}, \ldots, b_l\}} \left[ \frac{R_{a_1 b_1} \cdots R_{a_d b_d}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{R_{a_{d+1} b_1} \cdots R_{a_k b_d}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{B_{b_1} \cdots B_{b_e}}{\prod_{1 \leq u \leq e \leq l} f_{b_u b_v}} \right] \left[ \frac{T_{a_1} \cdots T_{a_d}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{T_{b_1} \cdots T_{b_e}}{\prod_{1 \leq u \leq e \leq l} f_{b_u b_v}} \right] \left[ \frac{T_{a_1} \cdots T_{a_d} \otimes T_{b_1} \otimes \ldots \otimes T_{b_e}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{T_{a_1} \cdots T_{a_d} \otimes T_{b_1} \otimes \ldots \otimes T_{b_e}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right]
$$

Finally, we may use (2.16) and (2.17) to move the outermost products of $R_{a_i b_j}$ past the $S's$ and the $T's$, at the cost of re-ordering the latter:

$$
\Delta(A_{1\ldots k}) \ast \Delta(B_{1\ldots l}) = \sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a_1 < \ldots < a_d} \sum_{b_1 < \ldots < b_e} \sum_{\{1, \ldots, d+e\} = \{a_1, \ldots, a_d\} \cup \{b_1, \ldots, b_e\}} \sum_{\{d+e+1, \ldots, k+l\} = \{a_{d+1}, \ldots, a_k\} \cup \{b_{e+1}, \ldots, b_l\}} \left[ \prod_{x \in \{a_{d+1}, \ldots, a_k, b_{e+1}, \ldots, b_l\}} S_x \otimes 1 \right] \left[ \frac{R_{a_1 b_1} \cdots R_{a_d b_d}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{R_{a_{d+1} b_1} \cdots R_{a_k b_d}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{R_{a_1 b_1} \cdots R_{a_d b_d}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right] \left[ \frac{B_{b_1} \cdots B_{b_e}}{\prod_{1 \leq u \leq e \leq l} f_{b_u b_v}} \right] \left[ \frac{B_{b_1} \cdots B_{b_e}}{\prod_{1 \leq u \leq e \leq l} f_{b_u b_v}} \right] \left[ \frac{B_{b_1} \cdots B_{b_e}}{\prod_{1 \leq u \leq e \leq l} f_{b_u b_v}} \right] \left[ \frac{A_{a_1 \ldots a_d} \otimes A_{a_{d+1} \ldots a_k}}{\prod_{1 \leq u \leq d \leq v \leq k} f_{a_u b_v}} \right]
$$
\[ [R_{a,k_1} \cdots R_{a,k_n}] \begin{cases} R_{a,b_1} \cdots R_{a,b_{k-1}} & \text{if } a_i > b_j \\ R_{a,b_{k-1}} & \text{if } a_i < b_j \end{cases} \begin{cases} R_{a,b_1} \cdots R_{a,b_{k-1}} & \text{if } a_i > b_j \\ R_{a,b_{k-1}} & \text{if } a_i < b_j \end{cases} \left( \prod_{x \in \{a_{d+1}, \ldots, a_k, b_{e+1}, \ldots, b_l\}} T_x \otimes 1 \right) \]

The right-hand side is simply $\Delta$ applied to the RHS of (2.6), as we needed to prove. \( \square \)

2.10. Let us consider two copies of the extended shuffle algebra, denoted $\bar{A}^+, \bar{A}^-$, defined as in the previous Subsections with respect to the same $R$, but with:

\begin{align*}
(2.24) & \quad \bar{R}^+ = \bar{R}, \\
(2.25) & \quad \bar{R}^- = (\bar{R}^{11})^{-1}^{21}
\end{align*}

where $\text{End}(V \otimes k) \overset{1s}{\leftrightarrow} \text{End}(V \otimes k)$ denotes the transposition of the $s$-tensor factor:

\begin{equation}
(2.26) \quad (E_{i_1 j_1} \otimes \ldots \otimes E_{i_k j_k})^{1s} = E_{i_1 j_1} \otimes \ldots \otimes E_{i_k j_k}
\end{equation}

It is an elementary exercise to show that if properties (2.2)–(2.3) hold for $R$, then they also hold for $\bar{R}$ given by formula (2.25). We will now define a pairing:

\begin{equation}
(2.27) \quad \bar{A}^+ \otimes \bar{A}^- \rightarrow \text{ground field}
\end{equation}

which respects the bialgebra structure in the following sense:

\begin{align*}
(2.28) & \quad \langle ab, c \rangle = \langle b \otimes a, \Delta(c) \rangle, & \forall a, b \in \bar{A}^+, \ c \in \bar{A}^- \\
(2.29) & \quad \langle a, bc \rangle = \langle \Delta(a), b \otimes c \rangle, & \forall a \in \bar{A}^+, \ b, c \in \bar{A}^-
\end{align*}

We will henceforth write $\langle X \rangle$ for the copy of $X \in \text{End}(V \otimes k)$ in $\bar{A}^\pm$. The analogous notation will apply to $S^\pm, T^\pm \in \bar{A}^\pm \otimes \text{End}(V)$.

**Proposition 2.11.** The assignments:

\begin{align*}
(2.30) & \quad \langle S_2^+, S_1^+ \rangle = \bar{R}^+, \quad \langle T_2^+, T_1^- \rangle = \bar{R}^- \\
(2.31) & \quad \langle S_2^+, T_1^- \rangle = \frac{R}{f}, \quad \langle T_2^+, S_1^- \rangle = \bar{R}^-
\end{align*}

and for all $X, Y \in \text{End}(V \otimes k)$:

\begin{equation}
(2.32) \quad \langle X^+, Y^- \rangle = \frac{1}{k!} \text{Tr}_{V \otimes k} \left( XY \prod_{1 \leq i < j \leq k} \frac{1}{f_{ij}} \right)
\end{equation}

(the pairings between $X^+, Y^-$ on one side and $S^+, T^+, S^-, T^-$ on the other side are defined to be 0) generate a bialgebra pairing (2.27) satisfying (2.28)–(2.29).

**Proof.** The data provided are sufficient to completely define the pairing, in virtue of (2.28) and (2.29). The thing that we need to check is that the defining relations of the extended shuffle algebras, namely (2.16)–(2.20) and the definition of the shuffle product in (2.6), are preserved by the pairing. For (2.16), we have:

\begin{align*}
\langle R_{a,b} S_1^+ S_2^+, S_3^- \rangle & \overset{2.28}{=} R_{12}(S_1^+ \otimes S_2^+, S_3^- \otimes S_3^-) = R_{12} \bar{R}_{31} \bar{R}_{32} \overset{2.23}{=} \\
&= \bar{R}_{32} \bar{R}_{31} R_{12} = \langle S_2^+ \otimes S_1^+, S_3^- \otimes S_3^- \rangle R_{12} \overset{2.28}{=} \langle S_2^+ S_1^+ R_{12}, S_3^- \rangle
\end{align*}
and:

\[
\langle R_{12}S_3^+ S_2^+, T_3^- \rangle = \frac{2.28}{R_{12}} \langle S_2^+ \otimes S_1^+, T_3^- \otimes T_3^- \rangle = \frac{R_{12}R_{32}R_{31}}{f_{32}f_{31}} (2.1)
\]

\[
= \frac{R_{31}R_{32}R_{12}}{f_{32}f_{31}} = \langle S_1^+ \otimes S_2^+, T_3^- \otimes T_3^- \rangle R_{12} \frac{2.28}{(S_2^+ S_1^+) R_{12}, T_3^-}
\]

We leave the analogous formulas when (2.16) is replaced by (2.17), or when the roles of \( S \) and \( T \) are switched, or when the roles of the two arguments of the pairing are switched, as exercises to the interested reader. As for (2.18), we have:

\[
\langle T_1^+ \tilde{R}_{12} S_2^+, S_3^- \rangle = \frac{2.28}{(T_1^+, S_3^-) \tilde{R}_{12}(S_2^+, S_3^-) = R_{31} \tilde{R}_{12} \bar{R}_{32}}
\]

The analogous formulas when \( S_3^- \) is replaced by \( T_3^- \), or when the roles of the arguments of the pairing are switched, are left as exercises to the interested reader. To prove that (2.19) pairs correctly with elements of \( A^- \), note that (2.23) implies:

\[
\Delta(Y_{1...k}^-) = Y_{1...k}^- \otimes 1 + (S_k^- \ldots S_1^- \otimes 1)(1 \otimes Y_{1...k}^-)(T_1^- \ldots T_k^- \otimes 1) + \ldots
\]

where \( ... \) stands for summands in which \( Y_{1...k}^- \) has a non-zero number of indices on either side of the \( \otimes \) sign. Then we have:

\[
\langle X_{1...k}^+ S_0^+, Y_{1...k}^- \rangle = \frac{2.28}{(S_0^+ \otimes X_{1...k}^+, (S_k^- \ldots S_1^- \otimes 1)(1 \otimes Y_{1...k}^-)(T_1^- \ldots T_k^- \otimes 1) =}
\]

\[
(2.33) = \frac{1}{k!} \text{Tr}_{V^{\otimes k}} \left[ X_{1...k} \left( \tilde{R}_{k0}^{10} \ldots \tilde{R}_{10}^{10} \right) \frac{R_{10}^{10} \ldots R_{k0}}{f_{10} \ldots f_{k0}} \prod_{1 \leq i < j \leq k} \frac{1}{f_{ij}} \right]
\]

where \( \text{Tr}_{V^{\otimes k}} \) denotes trace with respect to the indices \( 1, \ldots, k \) only (therefore the expression above is valued in \( \text{End}(V) \), corresponding to the index 0). The equality between the two rows of (2.33) is proved as follows: because both sides are bilinear in the tensors \( X_{1...k} \) and \( Y_{1...k} \), it suffices to prove that they are equal for:

\[
X = E_{i_1 j_1} \otimes \ldots \otimes E_{i_k j_k}, \quad Y = E_{i_1' j_1'} \otimes \ldots \otimes E_{i_n' j_n'}
\]

for arbitrary \( i_a, j_a, i'_a, j'_a \in \{1, \ldots, n\} \). In this case, the equality between the two rows of (2.33) is a straightforward exercise, which is performed by expanding \( S_3^+ \) and \( T_3^- \) in terms of the elementary matrices \( E_{ij} \), and using (2.29), (2.30), (2.31). Similarly, one can show that:

\[
\langle S_0 R_{k0} \ldots R_{10} f_{k0} \ldots f_{10} X_{1...k} \tilde{R}_{10} \ldots \tilde{R}_{k0}, Y_{1...k} \rangle =
\]

\[
= \frac{1}{k!} \text{Tr}_{V^{\otimes k}} \left[ \frac{R_{k0} \ldots R_{10} f_{k0} \ldots f_{10} X_{1...k} \tilde{R}_{10} \ldots \tilde{R}_{k0} Y_{1...k} \prod_{1 \leq i < j \leq n} \frac{1}{f_{ij}} \right]
\]

because of \( \varepsilon(S_0) = \text{Id} \). Because trace is invariant under cyclic permutations, the right-hand side of the expression above equals the right-hand side of (2.33), thus showing that relation (2.19) is preserved by the pairing. The proof that (2.20) is preserved by the pairing is analogous, and left as an exercise to the interested reader.
Before proving that the pairing (2.32) intertwines the shuffle product with the coproduct, let us show that the trace pairing is symmetric, in the sense that:

\[ \frac{1}{k!} \text{Tr} \left( [\text{Sym } A] \cdot Y \right) = \text{Tr} (A \cdot Y) \]

(2.34)

\[ \frac{1}{k!} \text{Tr} \left( X \cdot [\text{Sym } B] \right) = \text{Tr} (X \cdot B) \]

(2.35)

for all symmetric tensors \( X, Y \in \text{End}(V^\otimes k) \) and all tensors \( A, B \in \text{End}(V^\otimes k) \). Indeed, (2.34) follows from the fact that \( \forall \sigma \in S(k) \), we have:

\[ \text{Tr} \left( R_\sigma (\sigma A \sigma^{-1}) R_\sigma^{-1} Y \right) = \text{Tr} (A Y) \]

where the latter equality is the conjugation invariance of trace. Property (2.35) is proved likewise. As a consequence of (2.15) and (2.34), proving formula (2.28) for \( a = A^+ \), \( b = B^+ \) and \( c = Y^- \) boils down to the following equality:

\[ \frac{1}{k!!} \text{Tr} \left( \prod_{1 \leq i < j \leq k+1} \frac{1}{f_{ij}} \right) \]

(2.36)

\[ = \left( B^+_{1...l} \otimes A^+_{i+1...l+k+1} \right) \left( S^-_{k+1...l} \otimes Y^-_{1...k} \right) \frac{1}{\prod_{1 \leq u < v < k+1} f_{uv}} \left( T^-_{k+1...l} \right) \]

which we will now prove. We have:

\[ \Delta (B^+_{1...l}) = B^+_{1...l} \otimes 1 + (S^+_{1...l} \otimes 1)(1 \otimes B^+_{1...l})(T^+_{1...l} \otimes 1) + ... \]

where the ellipsis denotes summands whose second tensor factor has a non-zero number of indices on either side of the \( \otimes \) sign. Because of this, formula (2.29) when one of \( b \) and \( c \) is either \( S^- \) or \( T^- \) (which we have already checked yields a consistent bialgebra pairing) implies that:

\[ \langle B^+_{1...l}, S^-_{k+1...l} \rangle Y^-_{1...k} = \left( \Delta (B^+_{1...l}), S^-_{k+1...l} \otimes Y^-_{1...k} \right) \]

\[ = \text{Tr} \left( [\tilde{R}^+_{1...l}, R^+_{k+1...l}] B_{1...l}[R^+_{1...l}, R_{k+1...l}] Y_{1...l} \right) \]

Therefore, the RHS of (2.36) is precisely equal to the LHS of (2.36), as required. Similarly, proving (2.29) for \( a = A^+ \), \( b = A^- \), \( c = B^- \) boils down to the equality:

\[ \frac{1}{k!!} \text{Tr} \left( \prod_{1 \leq i < j \leq k+1} \frac{1}{f_{ij}} \right) \]

(2.37)

\[ = \left( S^+_{1...l} \otimes A^+_{1...l+k+1} \right) X^+_{k+1...l} \frac{1}{\prod_{1 \leq u < k+1} f_{uv}} \left( T^+_{k+1...l} \right) \]

The equality (2.37) is proved by analogy with (2.36), and we skip the details.
2.12. Proposition 2.11 allows us to define the Drinfeld double:
\begin{equation}
A = \tilde{A}^+ \otimes \tilde{A}^{-,op,coop}
\end{equation}

such that \( \tilde{A}^+ \cong \tilde{A}^+ \otimes 1 \) and \( \tilde{A}^{-,op,coop} \cong 1 \otimes \tilde{A}^{-,op,coop} \) are sub-bialgebras of \( A \), and the commutation of elements coming from the two factors is governed by:
\begin{equation}
(a, b_1)a_2b_2 = b_1a_1(a_2, b_2)
\end{equation}
for all \( a \in \tilde{A}^+ \) and \( b \in \tilde{A}^{-,op,coop} \). Let us now spell out formula (2.39) when \( a \) and \( b \) are among the generators of the double algebra (2.38). The quadratic relations (2.16) - (2.20) hold as stated between the + generators, and hold with the opposite multiplication between the - generators. As for the relations that involve one of the + generators and one of the - generators, we have:

**Proposition 2.13.** We have the following formulas in \( A \):
\begin{equation}
S_2^+ R S_1^- = S_1^- R S_2^+, \quad RS_1^+ T_2^- = T_2^- S_1^+ R
\end{equation}
\begin{equation}
T_1^+ R T_2^- = T_2^- R T_1^+, \quad RT_2^+ S_1^- = S_1^- T_2^+ R
\end{equation}
as well as:
\begin{equation}
S_0^+ \cdot X_{1...k}^\pm = \bar{R}_0^\pm \cdots \bar{R}_{01}^\pm X_{1...k}^\pm R_{01} \cdots R_{0k} \cdot \pm S_0^\mp
\end{equation}
\begin{equation}
X_{1...k}^\pm \cdot T_0^\pm = T_0^\pm \cdots R_{k0} \cdots R_{10} X_{1...k}^\pm \tilde{R}_0^\pm \cdots \tilde{R}_{k0}
\end{equation}
where \( \cdot + = \cdot \) and \( \cdot - = \cdot^{op} \) (the opposite multiplication in \( A \)). Finally, we have:
\begin{equation}
[E_{i,j}, E_{i',j'}] = s_{j,i}^+ t_{j',i}^+ - t_{j,i}^- s_{j',i}^-
\end{equation}
\( \forall i, j, i', j' \in \{1, ..., n\} \), where \( E_{i,j} \) are elements in the \( k = 1 \) summand of (2.7).\n
**Proof.** Let us now prove the first formula in (2.40) and leave the second one and (2.41) as exercises for the interested reader. Since:
\begin{equation}
\Delta(S^+) = (1 \otimes S^+)(S^+ \otimes 1) \quad \text{in} \quad \tilde{A}^+
\end{equation}
\begin{equation}
\Delta(S^-) = (S^- \otimes 1)(1 \otimes S^-) \quad \text{in} \quad \tilde{A}^{-,op,coop}
\end{equation}
formula (2.39) for \( a = S_2^+ \) and \( b = S_1^- \) implies:
\begin{equation}
S_2^+ (S_2^+, S_1^-) S_1^- = S_1^- (S_2^+, S_1^-) S_2^+
\end{equation}
Using (2.30) to evaluate the pairing implies precisely the first formula in (2.40).

Let us now prove (2.42) and leave the analogous formula (2.43) as an exercise. We will do so in the case \( \pm = + \), as \( \pm = - \) just involves the opposite of all relations.
\begin{equation}
\Delta(X_{1...k}^+) = X_{1...k}^+ \otimes 1 + (S_k^+ \cdots S_1^+ \otimes 1)(1 \otimes X_{1...k}^+)(T_k^+ \cdots T_1^+ \otimes 1) + ...\n\end{equation}
where the rightmost ellipsis stands for terms which have a non-zero number of indices on either side of the \( \otimes \) sign, so they pair trivially with \( S^- \). Meanwhile, \( \Delta(S^-) \) is given by (2.46). Applying (2.39) for \( a = X_{1...k}^+ \) and \( b_0 = S^- \) yields:
\begin{equation}
(S_k^+ \cdots S_1^+, S_0^-) X_{1...k}^+ (T_k^+ \cdots T_1^+, S_0^-) S_0^- = S_0^- X_{1...k}^+
\end{equation}
Formulas (2.28), (2.30) and (2.31) transform the formula above precisely into (2.42).
As for (2.44), consider relation (2.23) for \( k = 1 \) and \( X = E_{ij}^+ \):

\[
\Delta(E_{ij}^+) = E_{ij}^+ \otimes 1 + \sum_{x,y=1}^{n} s_{xj}^t t_{xy}^+ \otimes E_{xy}^+
\]
as well as the (op, coop) version of the above equality that holds in \( \tilde{A}^{-\text{op,coop}} \):

\[
\Delta(E_{ij}^-) = \sum_{x',y'=1}^{n} E_{x'y'}^- \otimes t_{jy}^+ s_{x'i}^t + 1 \otimes E_{ij}^-.
\]

Then (2.44) follows by applying (2.39) for \( a = E_{ij}^+ \) and \( b = E_{ij}^- \).

\( \square \)

3. Quantum toroidal \( \mathfrak{g}l_n \)

3.1. Fix \( n > 1 \), and let us start with a few notational remarks. The symbol \( \delta \) will refer to several different notions throughout the present paper. Specifically, the main usage of \( \delta \) is the following “mod \( n \)” variant of the Kronecker symbol:

\[
\delta_i^j = \begin{cases} 
1 & \text{if } i \equiv j \mod n \\
0 & \text{otherwise}
\end{cases}
\]

We will write \( \delta_i^j \mod g \) if we need the notion above for congruences modulo another number \( g \) instead of \( n \). Moreover, if \( i,j,i',j' \in \mathbb{Z} \), we will write:

\[
\delta_{(i,j)}^{(i',j')} = \begin{cases} 
1 & \text{if } (i,j) \equiv (i',j') \mod (n,n) \\
0 & \text{otherwise}
\end{cases}
\]

The ordinary Kronecker symbol will be denoted by \( \tilde{\delta} \), and it takes the value 1 if and only if \( i = j \) as integers, and 0 otherwise. Finally, we will use the notation:

\[
\delta(z) = \sum_{k \in \mathbb{Z}} z^k
\]

for the formal \( \delta \) series.

3.2. Let \( \hat{\mathfrak{sl}}_n \) be the Kac-Moody Lie algebra of type \( \hat{A}_n \). The corresponding Drinfeld-Jimbo quantum group\(^3\) is defined to be the associative algebra:

\[
U_q(\hat{\mathfrak{sl}}_n) = \mathbb{Q}(q)\langle x_i^\pm, \psi_i^\pm, c \rangle_{i \in \mathbb{Z}/n\mathbb{Z}, s \in \{1,\ldots,n\}}
\]

modulo the fact that \( c \) is central, as well as the following relations:

\[
\psi_s \psi_{s'} = \psi_{s'} \psi_s
\]

\[
\psi_s x_i^\pm = q^{\pm(\delta_i^{s+1} - \delta_i)} x_i^\pm \psi_s
\]

\[
[x_i^\pm, x_j^\pm] = 0 \quad \text{if } j \notin \{i - 1, i + 1\}
\]

\[
(x_i^\pm)^2 x_j^\pm - (q + q^{-1}) x_i^\pm x_j^\pm x_i^\pm + x_j^\pm (x_i^\pm)^2 = 0 \quad \text{if } j \in \{i - 1, i + 1\}
\]

\( ^3\)We note that the algebra defined below is slightly larger than the usual quantum group, since the Cartan part of the latter is generated by the ratios \( \psi_i^{\pm1} / \psi_j \), instead of \( \psi_i^{\pm1} \), ..., \( \psi_n^{\pm1} \) themselves.
for all $i, j \in \mathbb{Z}/n\mathbb{Z}$ and $s, s' \in \{1, \ldots, n\}$. We will extend the notation $\psi_s$ to all $s \in \mathbb{Z}$ by setting $\psi_{s+n} = c \psi_s$. We also consider the $q$–deformed Heisenberg algebra:

\begin{equation}
U_q(\mathfrak{gl}_1) = \mathbb{Q}(q) \langle p_{\pm k}, c \rangle_{k \in \mathbb{N}}
\end{equation}

where $c$ is central, and the $p_{\pm k}$ satisfy the commutation relation:

\begin{equation}
[p_k, p_l] = k \delta^0_{k+l} \cdot c^k - c^{-k} / q^k - q^{-k}
\end{equation}

Then we will consider the algebra:

\begin{equation}
U_q(\mathfrak{gl}_n) = U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{gl}_1) / (c \otimes 1 - 1 \otimes c)
\end{equation}

which serves as an affine $q$–version of the Lie algebra isomorphism $\mathfrak{gl}_n \cong \mathfrak{sl}_n \oplus \mathfrak{gl}_1$.

### 3.3. We can make $U_q(\mathfrak{sl}_n)$ into a bialgebra by using the counit $\varepsilon(c) = 1$, $\varepsilon(\psi_s) = 1$, $\varepsilon(x^\pm_i) = 0$, $\varepsilon(p_k) = 0$ and the coproduct given by $\Delta(c) = c \otimes c$ and:

\begin{equation}
\Delta(\psi_s) = \psi_s \otimes \psi_s
\end{equation}

\begin{equation}
\Delta(x^+_i) = \psi_i \otimes x^+_i + x^+_i \otimes 1
\end{equation}

\begin{equation}
\Delta(x^-_i) = 1 \otimes x^-_i + x^-_i \otimes \psi_i
\end{equation}

\begin{equation}
\Delta(p_k) = 1 / c^k \otimes p_k + p_k \otimes 1
\end{equation}

\begin{equation}
\Delta(p_{-k}) = 1 \otimes p_{-k} + p_{-k} \otimes c^k
\end{equation}

Moreover, the sub-bialgebras:

\begin{equation}
U^>_q(\mathfrak{sl}_n) = \mathbb{Q}(q) \langle x^+_i, p_k, \psi_s^{\pm 1}, c^{\pm 1} \rangle_{i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{N}, s \in \{1, \ldots, n\}} \subset U_q(\mathfrak{sl}_n)
\end{equation}

\begin{equation}
U^<_q(\mathfrak{sl}_n) = \mathbb{Q}(q) \langle x^-_i, p_{-k}, \psi_s^{\pm 1}, c^{\pm 1} \rangle_{i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{N}, s \in \{1, \ldots, n\}} \subset U_q(\mathfrak{sl}_n)
\end{equation}

are endowed with a bialgebra pairing:

\begin{equation}
\langle \psi_s, \psi_{s'} \rangle = q^{-\delta^s_{s'}}, \quad \langle x^+_i, x^-_j \rangle = \frac{\delta^i_j}{q - q^{-1}}, \quad \langle p_k, p_{-l} \rangle = \frac{k \delta^k_l}{q^k - q^{-k}}
\end{equation}

and all other pairings between generators are 0. It is well-known that $U_q(\mathfrak{sl}_n)$ is the Drinfeld double corresponding to the data (3.17) (modulo the identification of the symbols $\psi_s$ in the two factors of (3.17)). The algebra $U_q(\mathfrak{sl}_n)$ is $\mathbb{Z}^n$–graded:

\begin{align*}
\text{deg } c &= 0, \quad \text{deg } \psi_s = 0, \quad \text{deg } x^\pm_i = \pm \varsigma^i, \quad \text{deg } p_{\pm k} = \pm k \delta
\end{align*}
where \( \varsigma^i = (0, ..., 0, 1, 0, ..., 0) \) and \( \delta = (1, ..., 1) \).

**Remark 3.4.** The elements \( x^i_\pm \) of \( U_q(\mathfrak{gl}_n) \) are called simple (root) generators, while the elements \( p_{\pm k} \) are called imaginary (root) generators. Up to constant multiples, these are all the primitive elements of \( U_q(\mathfrak{gl}_n) \) (see Definition 3.8).

3.5. We will now give a different incarnation of the bialgebra (3.10), which is obtained by combining the results of [3] and [5]. We will use the notation of [16].

**Definition 3.6.** Consider the algebra:

\[
E = \mathbb{Q}(q) \left\langle f_{\pm[i;j]}, \psi_s^{\pm 1}, c^{\pm 1} \right\rangle_{(i,j) \in \frac{\mathbb{Z}}{n \mathbb{Z}}} \quad \text{subject to relations (3.96)–(3.97)}
\]

where \( c \) is central, and the quadratic relations (3.96)–(3.97) will be explained later.

The algebra \( E \) is a bialgebra with respect to the coproduct \( \Delta(c) = c \otimes c \) and:

\[
\Delta(\psi_s) = \psi_s \otimes \psi_s \\
\Delta(f_{[i;j]}) = \sum_{s=i}^j f_{[s;j]} \psi_s \otimes f_{[i;s]} \\
\Delta(f_{-[i;j]}) = \sum_{s=i}^j f_{-[i;s]} \otimes f_{-[s;j]} \frac{\psi_s}{\psi_i}
\]

where the notation \( \psi_s \) is extended to all \( s \in \mathbb{Z} \) by \( \psi_{s+n} = c \psi_s \). The sub-bialgebras:

\[
E^\geq = \mathbb{Q}(q) \left\langle f_{+[i;j]}, \psi_s^{\pm 1}, c^{\pm 1} \right\rangle_{(i,j) \in \frac{\mathbb{Z}}{n \mathbb{Z}}} \subset E \\
E^\leq = \mathbb{Q}(q) \left\langle f_{-[i;j]}, \psi_s^{\pm 1}, c^{\pm 1} \right\rangle_{(i,j) \in \frac{\mathbb{Z}}{n \mathbb{Z}}} \subset E
\]

are endowed with a bialgebra pairing:

\[
E^\geq \otimes E^\leq \xrightarrow{\text{op, coop}} \mathbb{Q}(q)
\]

generated by properties (2.28), (2.29) and:

\[
\langle \psi_s, \psi_{s'} \rangle = q^{-\delta_{s,s'}}, \quad \langle f_{[i;j]}, f_{-[i';j']} \rangle = \delta_{[i,j],[i',j']}(1 - q^{-2})
\]

where the right-most delta symbol is defined in (3.2). It is well-known that \( E \) is the Drinfeld double corresponding to the datum (3.25). The algebra \( E \) is \( \mathbb{Z}^n \)-graded:

\[
\deg c = 0, \quad \deg \psi_s = 0, \quad \deg f_{\pm[i;j]} = \pm[i;j]
\]

where \([i;j] = \varsigma^i + ... + \varsigma^{j-1}\) (we write \( \varsigma^k = \varsigma^k \mod n \)).
3.7. The subalgebras:
\[ \mathcal{E} \supset \mathcal{E}^\pm = \mathbb{Q}(q) \left\langle f_{\pm[i;j]} \right\rangle_{(i,j) \in \mathbb{Z}^2/(n,n)} \]
are graded by \( \pm \mathbb{N}^n \), and we will write \( \mathcal{E}_{\pm d} \) for their graded pieces, for all \( d \in \mathbb{N}^n \).

The dimensions of these graded pieces are given by the following formula:
\[ \dim \mathcal{E}_{\pm d} = \# \text{ partitions } \left\{ d = [i_1; j_1] + \ldots + [i_u; j_u] \right\}_{(i_a < j_a) \in \mathbb{Z}^2/(n,n)}^{u \in \mathbb{N}} \]
In fact, ordered products of the various \( f_{\pm[i;j]} \) give rise to a PBW basis of \( \mathcal{E}^\pm \).

**Definition 3.8.** An element \( x \in \mathcal{E}_{\pm d} \) is called primitive if:
\[ \Delta(x) \in \langle \psi^{\pm 1}_s \rangle_{s \in \mathbb{Z}} \otimes x + x \otimes \langle \psi^{\pm 1}_s \rangle_{s \in \mathbb{Z}} \]
We will write \( \mathcal{E}_{\pm d}^{\text{prim}} \subset \mathcal{E}_{\pm d} \) for the vector subspace of primitive elements.

As shown in [16], we have:
\[ \dim \mathcal{E}_{\pm d}^{\text{prim}} = \begin{cases} 1 & \text{if } d \text{ is either } [i; i + 1) \text{ or } k \delta \\ 0 & \text{otherwise} \end{cases} \]
for various \( i \in \{1, \ldots, n\} \) and \( k \in \mathbb{N} \). Therefore, up to scalar multiples, there is a unique choice of primitive elements:
\[ x^+_{i} \in \mathcal{E}_{\pm [i; i+1)}, \quad p^\pm k \in \mathcal{E}_{\pm k \delta} \]
which will be called simple and imaginary (respectively) primitive generators of \( \mathcal{E} \).

Comparing this with Remark 3.4 allows us to obtain the following:

**Theorem 3.9.** ([16]) Any choice of simple and imaginary primitive elements (3.29) of \( \mathcal{E} \) gives rise to an isomorphism of \( \mathbb{Z}^n \)-graded bialgebras \( U_q(\mathfrak{gl}_n) \cong \mathcal{E} \).

3.10. As is clear from the Theorem above, understanding the bialgebra structure of \( \mathcal{E} \) boils down to controlling the up-to-scalar ambiguity in choosing the primitive elements. To do this, we consider a formal parameter \( \bar{q} \) and define linear functionals:
\[ \alpha_{\pm[i;j]} : \mathcal{E}_{\pm[i;j]} \to \mathbb{Q}(q, \bar{q}^{\pm}) \]
for all \( (i < j) \in \mathbb{Z}^2/(n,n) \), satisfying the following properties:
\[ \alpha_{\pm[i;j]}(r \cdot r') = \begin{cases} \alpha_{\pm[s;j]}(r)\alpha_{\pm[i;s]}(r') & \text{if } \exists s \text{ s.t. } r \in \mathcal{E}_{\pm[s;j]}, r' \in \mathcal{E}_{\pm[i;s]} \\ 0 & \text{otherwise} \end{cases} \]
and (let \( \bar{q}^+_+ = \bar{q} \) and \( \bar{q}^- = (q^n \bar{q})^{-1} \)):
\[ \alpha_{\pm[i;j]}(f_{\pm[i';j']}) = \delta_{(i,j)_{(i',j')}} (1 - q^2) \bar{q}^{i-j} \]
We henceforth fix the elements (3.29) by making the choice of [17], namely:
\[ \alpha_{\pm[i; i+1]}(x^+_{i}) = \pm 1 \]
\[ \alpha_{\pm[s;s+nk]}(p^k) = \pm 1 \]
\( \forall k > 0, \ i, s \in \mathbb{Z}/n\mathbb{Z}. \) The fact that the right-hand side of (3.34) does not depend on \( s \) is a consequence of the \( \mathbb{Z}/n\mathbb{Z} \)-invariance of the elements \( p_{\pm k} \), see [16].

3.11. It is easy to note that the bialgebra \( U_q(\mathfrak{gl}_n) \cong \mathcal{E} \) possesses an antipode: \(^4\)
\[
A(\psi_s) = \psi_s^{-1}, \quad A(x_i^+) = -\frac{\psi_{i+1}}{\psi_i} x_i^+, \quad A(x_i^-) = -x_i^- \frac{\psi_i}{\psi_{i+1}}, \quad A(p_{\pm k}) = -e^{\pm k} p_{\pm k}
\]
In terms of the root generators, we may write:
\[
A^{\pm 1}(f_{\pm [i;j]}) = \frac{\psi_i^{\pm 1}}{\psi_{i+1}^{\pm 1}} f_{\pm [i;j]} q_{\mp}^{2(i-j)} - q^{(i,j)}
\]
where the elements \( \tilde{f}_{\pm [i;j]} \) are inductively defined in terms of \( f_{\pm [i;j]} \) by the formulas:
\[
\sum_{s=i}^{j} \tilde{f}_{\pm [s;j]} f_{\pm [s;i]} q_{\mp}^{2(j-s)} = \sum_{s=i}^{j} f_{\pm [s;j]} \tilde{f}_{\pm [s;i]} q_{\mp}^{2(j-s)} = 0
\]
Alternatively, the elements \( \tilde{f}_{\pm [i;j]} \) are completely determined by their coproduct:
\[
\Delta(\tilde{f}_{[i;j]}) = \sum_{s=i}^{j} \frac{\psi_s}{\psi_{s+1}} \tilde{f}_{[s;i]} \otimes \tilde{f}_{[s;j]}
\]
\[
\Delta(\tilde{f}_{-[i;j]}) = \sum_{s=i}^{j} \tilde{f}_{-[s;j]} \otimes \frac{\psi_j}{\psi_{j+1}} \tilde{f}_{-[s;i]}
\]
and by their values under the linear maps (3.30):
\[
\alpha_{[i;j]}(\tilde{f}_{[i';j']}) = \delta_{[i';j']}^{[i;j]} (1-q^{-2}) q_{\pm}^{i-j}
\]
Since \( \mathcal{E} \cong U_q(\mathfrak{gl}_n) \), we will use the notation \( f_{\pm [i;j]} \) (respectively \( \tilde{f}_{\pm [i;j]} \)) for the elements of either algebra. These elements will be called root generators of either algebra \( \mathcal{E} \cong U_q(\mathfrak{gl}_n) \), because \([i;j]\) are positive roots of the affine \( A_n \) root system.

3.12. Affinizations of quantum groups are defined by replacing each generator \( x_i^\pm \) as in Subsection 3.2 by an infinite family of generators \( \{x_{i,k}^\pm\}_{k \in \mathbb{Z}}. \) To define affinizations explicitly, let us consider variables \( z \) as being colored by an integer \( i \), denoted by “col \( z \)”. Then we may define the following color-dependent rational function:
\[
\zeta\left(\frac{z}{w}\right) = \begin{cases} 
\frac{z q^{2k} - w q^{-1}}{z q^{2k} - w} & \text{if col } i - \text{col } j = nk \\
1 & \text{if col } i - \text{col } j \notin \{ -1, 0 \} \mod n \\
\frac{z q^{2k} - w}{z q^{2k} - w q^{-1}} & \text{if col } i - \text{col } j = nk - 1
\end{cases}
\]
for any variables \( z, w \) of colors \( i \) and \( j \) respectively.

\(^4\)The antipode is a bialgebra anti-automorphism \( A : \mathcal{E} \rightarrow \mathcal{E} \) satisfying certain compatibility properties with the product, coproduct and pairing. We choose to write \( A(x) \) instead of the more common \( S(x) \) so as to not confuse the antipode with the series \( S(x) \) of Subsection 3.32.
Definition 3.13. The quantum toroidal algebra is:

\[ U_{q, \tau} (\tilde{\mathfrak{g}}_n) = \mathbb{Q}(q, \tau^\pm) \langle x_{i,k}^\pm, \varphi_i^\pm, \psi_s^{\pm 1}, c^{\pm 1}, \overline{c}^{\pm 1} \rangle_{k \in \mathbb{Z}, k' \in \mathbb{N}, i \in \mathbb{Z}/n\mathbb{Z}, s \in \mathbb{Z}} / \text{relations } \begin{array}{c}
(3.41) - (3.47)
\end{array} \]

Consider the series \( x_i^\pm (z) = \sum_{k \in \mathbb{Z}} \frac{x_{i,k}^\pm}{z^k} \) and \( \varphi_i^\pm (w) = \frac{\psi_i^{\pm 1}}{w^{\pm 1}} + \sum_{k=1}^\infty \frac{\varphi_k^\pm}{w^{k\pm}}, \) and set:

\[ c, \overline{c} \text{ central, } \psi_{s+n} = \psi_s c, \forall s \in \mathbb{Z} \]

(3.42) \( \psi_s \) commutes with \( \psi \)'s and \( \varphi \)'s, and satisfies (3.3) with \( x_i^\pm \mapsto x_i^\pm (z) \)

(3.43) \[ \varphi_i^\pm (z) \varphi_j^\pm (w) \left( \frac{w}{z} \right)^{\pm 1} = \varphi_j^\pm (w) \varphi_i^\pm (z) \left( \frac{z}{w} \right)^{\pm 1} \]

(3.44) \[ x_i^\pm (z) x_j^\pm (w) \left( \frac{w}{z} \right)^{\pm 1} = x_j^\pm (w) x_i^\pm (z) \left( \frac{z}{w} \right)^{\pm 1} \]

(3.45) \[ [x_i^+ (z), x_i^- (w)] = \frac{\delta_i^j}{q - q^{-1}} \left[ \delta \left( \frac{z}{w} \right) \varphi_i^+ (z) - \delta \left( \frac{w}{z} \right) \varphi_i^- (w) \right] \]

and, for \( i \in \{i-1, i+1\} \):

\[ x_i^\pm (z) x_j^\pm (w) - (q + q^{-1}) x_i^\pm (z) x_j^\pm (w) x_i^\pm (z_2) + x_j^\pm (w) x_i^\pm (z_1) x_i^\pm (z_2) + \]

(3.46) same expression with \( z_1 \) and \( z_2 \) switched = 0

for all choices of \( \pm, \pm', i, j \), where the variables \( z \) and \( w \) have color \( i \) and \( j \), respectively, for the purpose of defining the rational function \( \zeta \). Note that we extend the index \( i \) to arbitrary integers, by applying the convention:

\[ x_{i+n,k}^\pm = x_{i,k}^\pm \overline{q}^{-2k}, \quad \varphi_{i+n,k}^\pm = \varphi_{i,k}^\pm \overline{q}^{-2k} \]

We consider the subalgebras \( U_{q, \tau}^\pm (\tilde{\mathfrak{g}}_n) \subset U_{q, \tau} (\tilde{\mathfrak{g}}_n) \) generated by \( \{x_{i,k}^\pm \in \mathbb{Z}/n\mathbb{Z} \} \).

Remark 3.14. In the notation of Subsection 1.3 we have \( U_{q, \tau}^+ (\tilde{\mathfrak{g}}_n) = U_{q, \tau} (\tilde{\mathfrak{g}}_n) \) and \( U_{q, \tau}^- (\tilde{\mathfrak{g}}_n) = U_{q, \tau} (\tilde{\mathfrak{g}}_n) \), but we will henceforth use the \( \pm \) notation.

3.15. Let us now recall the classical shuffle algebra realization of \( U_{q, \tau}^\pm (\tilde{\mathfrak{g}}_n) \). Consider variables \( z_{ia} \) of color \( i \), for various \( i \in \{1, ..., n\} \) and \( a \in \mathbb{N} \). We call a function \( R(z_{11}, ..., z_{1d_1}, ..., z_{n1}, ..., z_{nd_n}) \) color-symmetric if it is symmetric in \( z_{i1}, ..., z_{id_i} \) for all \( i \in \{1, ..., n\} \). Depending on the context, the symbol “Sym” will refer either to either color-symmetric functions in variables \( z_{ia} \), or to the symmetrization operation:

\[ \text{Sym } F(\ldots, z_{i1}, \ldots, z_{id_i}, \ldots) = \sum_{(\sigma^1, \ldots, \sigma^n) \in S(d_1) \times \cdots \times S(d_n)} F(\ldots, z_{i, \sigma^1(1)}, \ldots, z_{i, \sigma^1(d_1)}, \ldots) \]

Let \( d! = d_1! \cdots d_n! \). The following construction arose in the context of quantum groups in [3], by analogy to the work of Feigin-Odesskii on certain elliptic algebras.
Consider the vector space:
\[
\bigoplus_{d=(d_1,...,d_n)\in \mathbb{N}^n} \mathbb{Q}(q, \tilde{q}^\pm)(..., z_{i_1}, ..., z_{i_d}, ...)^{\text{Sym}}
\]
and endow it with an associative algebra structure, by setting \( R \ast R' \) equal to:
\[
\text{Sym} \left[ \frac{R(..., z_{i_1}, ..., z_{i_d}, ...)}{d!} R'(..., z_{i_1}, ..., z_{i_d}, ...) \prod_{i,d' \geq 1}^{n} \prod_{d_i < a \leq d_i + d_i'} z_{i_a} \left( \frac{z_{i_{a'}}}{z_{i_{a'}}} \right) \right]
\]
Let \( S^+ \) the subalgebra of (3.48) generated by \( \{ z_{k_i} \}_{k \in \mathbb{Z}} \) and we let \( S^- = (S^+)^{\text{op}} \).

The algebras \( S^\pm \) are graded by \( \pm \mathbb{N} \times \mathbb{Z} \), where \( \deg R(..., z_{i_1}, ..., z_{i_d}, ...) = (\pm d, k) \) if \( d = (d_1, ..., d_n) \), while \( k \) denotes the homogeneous degree of \( R \). We will write \( S_{\pm a} \) for the graded pieces of \( S^\pm \) with respect to the \( \pm \mathbb{N} \) direction only.

**Theorem 3.17.** ([14]) The subalgebras \( S^\pm \) coincide with the \( \mathbb{Q}(q, \tilde{q}^\pm) \)-vector subspaces of (3.48) consisting of rational functions \( R(..., z_{i_a}, ...) \) that satisfy:
\[
R(..., z_{i_a}, ...) = \prod_{i=1}^{n} \prod_{a'=i}^{n} (z_{i_a}q - z_{i_{a'}+1}q^{-1})
\]
where \( r \) is an arbitrary Laurent polynomial which vanishes at the specializations:
\[
(z_{i_1}, z_{i_2}, z_{i_{-1}}) \mapsto (w, wq^2, w), \quad (z_{i_1}, z_{i_2}, z_{i_{+1}}) \mapsto (w, wq^{-2}, w)
\]
for any \( i \in \{1, ..., n\} \). This vanishing property is the natural analogue of the wheel conditions studied in [10]. By convention, we set \( z_{n+1,a} = z_{n+1}q^{-2} = z_{na}q^2 \).

It is a well-known fact (see [3]) that \( S^\pm \cong U_{q, \tilde{q}}(\mathfrak{gl}_n) \). In order to obtain the entire quantum toroidal algebra and not just its halves, define the double shuffle algebra:
\[
S = S^+ \otimes S^0 \otimes S^- \big/ \text{relations akin to (3.44), (3.46)}
\]
where:
\[
S^0 = \frac{\mathbb{Q}(q, \tilde{q}^\pm)(\varphi_{i,k}, \psi_{i,k}^{\pm 1}, c_{i,k}^{\pm 1}, \overline{c}_{i,k}^{\pm 1})_{i,k \in \mathbb{N}, i \in \mathbb{Z}/n\mathbb{Z}}}{\text{relations (3.41)-(3.43)}}
\]
Therefore, there is an isomorphism:
\[
S \cong U_{q, \tilde{q}}(\mathfrak{gl}_n)
\]
given by sending \( (z_{k_i})^{\pm} \mapsto x_{i,k}, \varphi_{i,k}^{\pm}, \psi_{i,k}^{\pm}, \varphi_{i,k}, \psi_s, \psi_s \).

### 3.18. Let us consider the following halves of \( S \):
\[
S^\geq = S^+ \otimes \frac{\mathbb{Q}(q, \tilde{q}^\pm)(\varphi_{i,k}^{\pm}, \psi_{s}^{\pm 1}, c_{\pm 1}, \overline{c}_{\pm 1})_{i,k \in \mathbb{N}, i \in \mathbb{Z}/n\mathbb{Z}}}{\text{relations (3.41)-(3.43)}} \big/ \text{relations modeled after (3.44)}
\]
\[
S^\leq = S^- \otimes \frac{\mathbb{Q}(q, \tilde{q}^\pm)(\varphi_{i,k}^{\pm}, \psi_{s}^{\pm 1}, c_{\pm 1}, \overline{c}_{\pm 1})_{i,k \in \mathbb{N}, i \in \mathbb{Z}/n\mathbb{Z}}}{\text{relations (3.41)-(3.43)}} \big/ \text{relations modeled after (3.44)}
\]
Moreover, the algebras (3.51) and (3.52) are endowed with topological coproducts:

\[
\Delta(R^+) = \sum_{e \leq d} \left[ \prod_{i \leq e} \varphi_{i,e_i}^+ \right] R^+ \left( \prod_{i \leq e} \varphi_{i,e_i}^+ \right) \prod_{i > e} \zeta \left( \frac{\zeta_{i,a} \tilde{c}_1}{z_{i,a}} \right) \prod_{i \leq e} \varphi_{i,e_i}^+ \zeta \left( \frac{\zeta_{i,a} \tilde{c}_1}{z_{i,a}} \right)
\]

and there exists a pairing between the two halves given by:

\[
\langle R^+, R^- \rangle = \frac{1 - q^{-2} |d|}{d!} \int \prod_{i,j=1}^n \prod_{a \leq d_i, b \leq d_j} \zeta \left( \frac{\zeta_{i,a} \tilde{c}_2}{z_{i,a}} \right) \zeta \left( \frac{\zeta_{i,a} \tilde{c}_1}{z_{i,a}} \right) \prod_{i \leq e} \varphi_{i,e_i}^+ \prod_{i \leq e} \varphi_{i,e_i}^+ \frac{dz_{i,a}}{2\pi iz_{i,a}}
\]

for any \(R^+ \in S_d\) and \(R^- \in S_{-d}\) (we refer the reader to [10] or [17] for details).

**Remark 3.19.** To think of (3.55) as a tensor, we expand the right-hand side in non-negative powers of \(z_{i,a}/z_{i,a}'\) for \(a \leq e_i, e_i' < a'\), thus obtaining an infinite sum of monomials. In each of these monomials, we put the symbols \(\varphi_{i,d}^+\) to the very left of the expression, then all powers of \(z_{i,a}\) with \(a \leq e_i\), then the \(\otimes\) sign, and finally all powers of \(z_{i,a}\) with \(a > e_i\). The powers of the central element \(\tilde{c}_1 = \tilde{c} \otimes 1\) are placed in the first tensor factor. The resulting expression will be a power series, and therefore lies in a completion of \(S^2 \otimes S^2\). The same argument applies to (3.54).

**Remark 3.20.** In formula (3.55), the integral is defined in such a way that the variable \(z_{i,a}\) traces a contour which surrounds \(z_{i,b}q^2, z_{i-1,b}\) and \(z_{i+1,b}q^{-2}\) for all \(i \in \{1, \ldots, n\}\) and all \(a, b\) (a particular choice of contours which achieves this aim is explained in Proposition 3.8 of [10]).

3.21. In [16], [17], we constructed a PBW basis of \(S^\pm \cong U_{q, q^{-1}}(\mathfrak{g}_n)\). More precisely we construct particular elements of \(S^\pm\) called “PBW generators”, indexed by a totally ordered set, and claim that a linear basis of \(S^\pm\) is given by ordered products of the PBW generators. In the case of the algebra \(E^\pm \cong U_{q, q^{-1}}(\mathfrak{gl}_n)\), we have already seen in Subsection 3.7 that the PBW generators of \(E^\pm\) are indexed by:

\[
(i < j) \in \frac{Z^2}{(n, n)Z}
\]

It should come as no surprise that the PBW generators of \(S^\pm\) are indexed by:

\[
\left( (i < j), k \right) \in \frac{Z^2}{(n, n)Z} \times Z
\]

If we write \(\mu = \frac{i - j}{k}\), we will find it more useful to index the PBW generators by:

\[
\left( (i < j), \mu \right) \in \frac{Z^2}{(n, n)Z} \times \mathbb{Q}
\]
such that $\frac{z_i}{\mu} \in \mathbb{Z}$. For any choice of $i < j$ and $\mu$ as above, we define:

\[ A^\mu_{\pm[i;j]} = \text{Sym} \left[ \prod_{a = 2}^{j-1} \left( z_{a|q|}^{2a} \right) \left( \frac{n - a - 1}{z_a} \right) \prod_{i \leq a < b < j} \zeta \left( \frac{z_b}{z_a} \right) \right] \in S^a \]

\[ B^\mu_{\pm[i;j]} = \text{Sym} \left[ \prod_{a = 2}^{j-1} \left( z_{a|q|}^{2a} \right) \left( \frac{n - a - 1}{z_a} \right) \prod_{i \leq a < b < j} \zeta \left( \frac{z_a}{z_b} \right) \right] \in S^a \]

In order to think of the RHS of (3.56) and (3.57) as shuffle elements, we relabel the variables $z_i$, $z_j$, $z_{j-1}$ according to the following rule $\forall a \in \{i, \ldots, i + n - 1\}$:

\[ z_a, z_{a+n}, z_{a+2n}, \ldots \rightarrow z_{a1}, z_{a2}q^{-2}, z_{a3}q^{-4}, \ldots \]

and thus the RHS of (3.56) and (3.57) lie in the vector space (3.48). If $\frac{z_i}{\mu} \notin \mathbb{Z}$, the LHS of (3.56) and (3.57) are defined to be 0. We will occasionally write:

\[ A^{(\pm k)}_{\pm[i;j]} = A^\mu_{\pm[i;j]} \quad \text{and} \quad B^{(\pm k)}_{\pm[i;j]} = B^\mu_{\pm[i;j]} \]

where $k = \frac{z_i}{\mu}$, in order to indicate the fact that $\deg A^{(\pm k)}_{\pm[i;j]} B^{(\pm k)}_{\pm[i;j]} = \pm((i; j), k)$.

3.22. We will now define an algebra $D$ that is isomorphic to $S \cong U_{q, \tau}(\mathfrak{gl}_n)$, much like the algebra $E$ was isomorphic to $U_q(\mathfrak{gl}_n)$. The first step is to define an infinite family of the latter algebras. Specifically, for any coprime $(a, b) \in \mathbb{Z} \times \mathbb{N}$, define:

\[ E_{\frac{a}{\tau}} = U_q(\mathfrak{gl}_{\frac{a}{\tau}})^{\otimes g} \]

where $g = \gcd(n, a)$. The root generators of (3.58) are parametrized by:

triples $(u, v, r)$ where $(u, v) \in \mathbb{Z}^2 \left( \frac{n}{g}, \frac{n}{g} \right) \mathbb{Z}$ and $r \in \{1, \ldots, g\}$

However, we choose to replace this triple by:

\[ (i, j) = (r + au, r + av) \in \mathbb{Z}^2 \left( \frac{n}{g}, \frac{n}{g} \right) \mathbb{Z} \]

and therefore we will use the following notation for the root generators of (3.58):

\[ f^{(k)}_{i;j} = 1^{\otimes r-1} \otimes f_{u,v} \otimes 1^{\otimes g-r} \]

\[ f^{(k)}_{i;j} = 1^{\otimes r-1} \otimes f_{u,v} \otimes 1^{\otimes g-r} \]

for any indices $i, j$, which are connected to $u, v, r$ by (3.59). Note that we can have $i > j$ in formula (3.59), via the convention:

\[ [i; j] = -[j; i] \quad \text{if} \quad i > j \]

Moreover, formula (3.59) implies that $k := (j - i)\frac{b}{a}$ is an integer, so we will write:

\[ f^{(k)}_{i;j} = f^{(k)}_{i;j} \quad \text{and} \quad f^{(k)}_{i;j} = f^{(k)}_{i;j} \]

and set (3.62) equal to 0 if $k = (j - i)\frac{b}{a} \notin \mathbb{Z}$.
3.23. For any \((i, j) \in \mathbb{Z}^2/(n, n)\mathbb{Z}\) and \(k \in \mathbb{N}\), let us write \(\mu = \frac{i - j}{k}\). The assignment:

\[\deg f^{(k)}_{[i; j]} = \deg \bar{f}^{(k)}_{[i; j]} = ([i; j], k)\]

makes \(E_{\mu}\) into a \(\mathbb{Z}^n \times \mathbb{Z}\) graded algebra. For all \(d \in \mathbb{Z}^n\), we will write \(E_{\mu;d}\) for its degree \(d \times \mathbb{Z}\) graded piece. Consider two invertible central elements \(c, \bar{c}\), and recall the Cartan elements \(\psi_1, \ldots, \psi_n\) of \(E_0 = E\). As \(\mu\) ranges over \(\mathbb{Q} \sqcup \mathbb{Q}\), we will identify the central elements of the algebras \(E_{\mu}\) according to the rule:

\[
\left(\text{central element of } E_{\frac{i}{k}}\right) = c^\frac{i}{k} \bar{c}^{\frac{n-i}{k}}
\]

and identify the Cartan elements of the algebras \(E_{\mu}\) according to the rule:

\[
\left(\psi_s \text{ on } \mu\text{-th factor of } U_q(g\mathfrak{gl}_n) \otimes g = E_{\frac{i}{k}}\right) = \psi_{p+s} c^{bs}
\]

where \(g = \gcd(n, a)\). Hence we have the following relation:

\[
(3.63) \quad \psi_s X = q^{-(s, d)} X \psi_s \quad \forall X \in E_{\mu;\mathcal{Q}}, \forall s \in \{1, \ldots, n\}
\]

where \((\cdot, \cdot)\) is the bilinear form on \(\mathbb{Z}^n\) given by \((\xi, \eta) = \delta_i^j - \delta_i^{j-1}\).

We henceforth extend the scalars in \(E_{\mu}\) from \(\mathbb{Q}(q)\) to \(\mathbb{Q}(q, \bar{q})\). The following is an obvious consequence of the structure defined in Subsections 3.5 and 3.11.

**Proposition 3.24.** For any \(\mu\), the algebra \(E_{\mu}\) has a coproduct \(\Delta_{\mu}\), for which:

\[
\Delta_{\mu}(f_{[i; j]}^{(\mu)}) = \sum_{s \in \{1, \ldots, j\}} f^{(\mu)}_{[s; j]} \frac{\psi_s}{\psi_s} c^{\frac{i-s}{k}} \otimes f^{(\mu)}_{[i; s]}
\]

\[
\Delta_{\mu}(\bar{f}_{[i; j]}^{(\mu)}) = \sum_{s \in \{1, \ldots, j\}} \frac{\psi_s}{\psi_s} \bar{f}^{(\mu)}_{[s; j]} c^{\frac{j-s}{k}} \otimes \bar{f}^{(\mu)}_{[i; s]}
\]

\[
\Delta_{\mu}(f_{[-i; j]}^{(\mu)}) = \sum_{s \in \{1, \ldots, j\}} f^{(\mu)}_{[-s; j]} \otimes f^{(\mu)}_{[i; s]} \psi_s c^{\frac{i-s}{k}}
\]

\[
\Delta_{\mu}(\bar{f}_{[-i; j]}^{(\mu)}) = \sum_{s \in \{1, \ldots, j\}} \bar{f}^{(\mu)}_{[-s; j]} \otimes \bar{f}^{(\mu)}_{[i; s]} \psi_s c^{\frac{j-s}{k}}
\]

\[\forall (i, j) \in \mathbb{Z}^2/(n, n)\mathbb{Z} \text{ such that } \frac{i-j}{k} \in \mathbb{N}. \text{ For all such } i, j, \text{ we have linear maps:}
\]

\[
\alpha_{\pm[i; j]} : E_{\mu;\pm[i; j]} \rightarrow \mathbb{Q}(q, \bar{q})
\]

that satisfy property (3.31) and:

\[
\alpha_{\pm[i; j]}(f_{\pm[i'; j']}^{(k)}) = \delta_{\pm[i'; j']}^{(i, j)} (1 - q^2) \frac{\gcd(j-i, k)}{n}
\]

\[
\alpha_{\pm[i; j]}(\bar{f}_{\pm[i'; j']}^{(k)}) = \delta_{\pm[i'; j']}^{(i, j)} (1 - q^{-2}) \frac{\gcd(j-i, k)}{n}
\]

\[5\text{The formulas below differ from (3.21), (3.22), (3.37), and (3.38) by powers of } \bar{c}. \text{ This is achieved by rescaling the } f \text{ and } \bar{f} \text{ generators by certain powers of } \bar{c}, \text{ which we will tacitly do.}
\]

\[6\text{The notation } s \in \{1, \ldots, j\} \text{ means "s runs between } i \text{ and } j', \text{ for either } i < j \text{ or } i > j.\]
3.25. Let us consider the subalgebras $E_\mu^\pm \subset E_\mu$ generated by those elements \((3.62)\) where the sign of $k$ is equal to $\pm$. As in \((3.29)\), we obtain the following elements:

\begin{align}
\{ p_{\pm [i;i+a]}^\mu \}_{i \in \mathbb{Z}/n\mathbb{Z}} & \subset E_\mu^\pm \quad \text{are simple generators, if } n \notdivides a \\
\{ p_{\pm [i;8],r}^\mu \}_{i \in \mathbb{Z}/n\mathbb{Z}} & \subset E_\mu^\pm \quad \text{are imaginary generators}
\end{align}

\(\forall \mu = \frac{\xi}{n} \in \mathbb{Q} \cap \mathbb{R}\). These elements are all primitive for the coproduct $\Delta_\mu$ and satisfy:

\begin{align}
\alpha_{\pm[u,v]}(p_{\pm [i;i+a]}^\mu) &= \pm \delta_{(u,v)}^{(i,i+a)} \\
\alpha_{\pm[\kappa,\kappa;+l]}(p_{\pm [i;8],r}^\mu) &= \pm \delta_s \mod g
\end{align}

for any $u, v, s$. We may use the notation:

\(p_{\pm [i;i+a]}^{(\pm b)} = p_{\pm [i;i+a]}^\mu \quad \text{and} \quad p_{\pm [i;8],r}^{(\pm b)} = p_{\pm [i;8],r}^\mu\)

to emphasize the fact that $\deg p_{\pm}^{(k)} = (d, k) \in \mathbb{Z}^n \times \mathbb{Z}$. Let us write:

\((3.71)\) \[ \bar{i} = i - n \left\lfloor \frac{i-1}{n} \right\rfloor \]

for all $i \in \mathbb{Z}$, and recall that $\delta_{\bar{i}}^j = 1$ if $i \equiv j \mod n$, and 0 otherwise.

**Definition 3.26.** Consider the algebras:

\(3.72\) \[ \mathcal{D}^\pm = \bigotimes_{\mu \in \mathbb{Q}} \frac{E_\mu^\pm}{(3.73) \div (3.74)} \]

whose generators, by the discussion above, are denoted by:

\[ \left\{ p_{\pm [i;j],r}^{(\pm k)} \right\}_{(i,j) \in \mathbb{Z}/n\mathbb{Z}^2, r \in \mathbb{Z}/g\mathbb{Z}} \]

Whenever $d := \det \begin{pmatrix} k & k' \\ j \quad \text{mod} \quad n \end{pmatrix}$ satisfies $|d| = \gcd(k', nl)$, we impose the relation:

\(3.73\) \[ p_{\pm [i;j],r}^{(\pm k)} p_{\pm [i;j],r}^{(\pm k')} = \pm p_{\pm [i;j],r}^{(\pm k \pm k')} \left( \delta_i^r \mod g \frac{q^d}{q^d - 1} - \delta_j^r \mod g \frac{q^d}{q^d - 1} \right) \]

and whenever $\det \begin{pmatrix} k & k' \\ j \quad \text{mod} \quad n \end{pmatrix} = \gcd(k + k', j + j' - i - i')$, we impose:

\(3.74\) \[ p_{\pm [i;j],r}^{(\pm k)} p_{\pm [i;j],r}^{(\pm k')} q^{\delta_i^r - \delta_j^r} = p_{\pm [i;j],r}^{(\pm k')} p_{\pm [i;j],r}^{(\pm k)} q^{\delta_j^l - \delta_i^l} = \]

\[ \sum_{l \in \mathbb{Z}/n\mathbb{Z}} f_{\pm [t,j],l}^{\mu} f_{\pm [i,s]}^{\mu} \left( \delta_j^l q^{\bar{i} - j} - \delta_i^l q^{\bar{i} - j'} \right) \]

\(3.74\) \[ \mu = \frac{i+j'-i'-j}{k+k'} \].
3.27. Let us write \( D^0 = E_\infty \), and use the notation \( p^{(0)}_{\pm; i+j+1} \) and \( p^{(0)}_{\pm; i;j+1} \) for its simple and imaginary generators, respectively, as defined in Subsection 3.25.

**Definition 3.28.** Let us define the double of the algebras \([3.72]\) as:

\[
D = D^+ \otimes D^0 \otimes D^- \bigg/ \text{relations } \begin{array}{l}
(3.76) \quad [p^{(1)}_{\pm; i;j}, p^{(0)}_{\pm; i+1,j}] = \pm p^{(1)}_{\pm; i;j+1} \left( \frac{\eta_{\pm}^i}{\eta_{\pm}^j} - \frac{\eta_{\pm}^j}{\eta_{\pm}^i} \right) \\
(3.77) \quad [p^{(1)}_{\pm; i;j}, p^{(0)}_{\pm; i;j+1}] = \pm p^{(1)}_{\pm; i;j+1} \left( \frac{\eta_{\pm}^i}{\eta_{\pm}^j} - \frac{\eta_{\pm}^j}{\eta_{\pm}^i} \right) \\
(3.78) \quad p^{(1)}_{\pm; i;j} p^{(0)}_{\pm; i;j+1} q^{\delta_{i+1}-\delta_i} - p^{(1)}_{\pm; i;j+1} p^{(1)}_{\pm; i;j} q^{\delta_{i+1}-\delta_i} = \\
= \pm \left( \delta_{i+1} \frac{\eta_{\pm}^i}{\eta_{\pm}^{i+1}} - \delta_i \frac{\eta_{\pm}^i}{\eta_{\pm}^j} \right) p^{(1)}_{\pm; i;j+1} \\
(3.79) \quad \left[ p^{(1)}_{\pm; i;j}, p^{(0)}_{\pm; i;j+1} \right] = \pm \left( \delta_{i+1} \frac{\eta_{\pm}^i}{\eta_{\pm}^{i+1}} p^{(1)}_{\pm; i;j+1} - \delta_i \frac{\eta_{\pm}^i}{\eta_{\pm}^j} p^{(1)}_{\pm; i;j} \right)
\end{array}
\]

and:

\[
(3.80) \quad \left[ p^{(1)}_{\pm; i;j}, p^{(-1)}_{\pm; i;j'} \right] = \frac{1}{q^{-1} - q}
\]

\[
\left( \sum_{\left[ i'-n \right] \leq k \leq \left[ i'-m \right]} f^{(0)}_{i',k} \frac{\psi_{i',j'}^{(0)}}{\psi_{i',k} c^{(0)}_{i',j'}} - \sum_{\left[ j'-n \right] \leq k \leq \left[ j'-m \right]} f^{(0)}_{j',k} \frac{\psi_{j',i}^{(0)}}{\psi_{j',k} c^{(0)}_{j',i'}} \right)
\]

Relations \([3.77]-[3.80]\) are “sufficient” to describe all the relations between the three tensor factors of \([3.75]\), because the algebras \( D^\pm \) are generated by:

\[
(3.81) \quad \left\{ p^{(1)}_{\pm; i;j} \right\}_{(i,j) \in \frac{Z^2}{(n,m)Z}}
\]

(we will prove this in Proposition 3.34). Therefore, relations \([3.77]-[3.80]\) allow us to “straighten” any product of elements from the subalgebras \( D^+, D^0, D^- \), i.e. to write said product as a sum of products of elements from \( D^+, D^0, D^- \), in this order.

**Theorem 3.29.** \( \{17\} \) There is an isomorphism \( D \cong S \).

*Proof. (sketch, see Proposition 5.15 of loc. cit. for details)* The subalgebra:

\[
S \supseteq T_\mu = \left\{ \begin{array}{ll}
A^\mu_{i; j}, B^\mu_{i; j} \bigg/ \frac{\eta_{\pm}^i}{\eta_{\pm}^j} & \text{if } \mu > 0 \text{ or } \mu = \infty \\
A^-_{i; j}, B^-_{i; j} \bigg/ \frac{\eta_{\pm}^i}{\eta_{\pm}^j} & \text{if } \mu < 0
\end{array} \right\}
\]

\( \mu \in \mathbb{Z} \) is actually \( \frac{1}{\mu} \) in our notation.

\[\text{Note that the slope } \mu \text{ in loc. cit. is actually } \frac{1}{\mu} \text{ in our notation.} \]
is isomorphic to $\mathcal{E}_\mu$ of (3.58) for all $\mu \in \mathbb{Q} \sqcup \{\infty\}$, by sending:

$$f_{\mu}^{ij} \mapsto \frac{1}{\psi_\mu} A_{\mu}^{ij} \psi_\mu e^{i \frac{\xi_{\mu}}{\mu} \cdot (q_{\mu}^+)^{\mu}}$$

$$f_{-\mu}^{ij} \mapsto \frac{1}{\psi_\mu} B_{-\mu}^{ij} \psi_\mu e^{i \frac{\xi_{\mu}}{\mu} \cdot (q_{\mu}^+)^{\mu}}$$

if $\mu > 0$ or $\mu = \infty$, while:

$$f_{\mu}^{ij} \mapsto B_{\mu}^{ij} \cdot q^{j-i}$$

$$f_{-\mu}^{ij} \mapsto A_{-\mu}^{ij} \cdot q^{j-i}$$

if $\mu < 0$. Similarly, $T_0 := \mathcal{S}^0$ is isomorphic to a tensor product of $n$ Heisenberg algebras. As in Subsection 3.25, this allows us to construct the images of the simple and imaginary generators:

$$p^{(\pm k)}_{\pm ij} \mapsto X^{(\pm k)}_{\pm ij} \in \mathcal{S}, \quad p^{(\pm k')}_{\pm \\delta, r} \mapsto X^{(\pm k')}_{\pm \delta, r} \in \mathcal{S}$$

In loc. cit., we showed that the simple and imaginary generators $X_{-} \in \mathcal{S}$ defined above satisfy relations (3.73–3.74) and (3.76–3.80) with $p$’s replaced by $X$’s, hence we obtain an algebra homomorphism:

$$\Phi : \mathcal{D} \to \mathcal{S}$$

This map is an isomorphism because ordered products of the elements $f_{\mu}^{ij}$ (respectively their images under the assignments (3.82–3.85)) in increasing order of $\mu$ were shown in [17] (respectively [16]) to form a linear basis of $\mathcal{D}$ (respectively $\mathcal{S}$).

**Corollary 3.30.** If we combine (3.50) with (3.86), we obtain an isomorphism:

$$\Psi : \mathcal{D} \cong U_q(\mathfrak{gl}_n)$$

If we write:

$$\Psi(\mathcal{D}^+) = U_{q,q}(\mathfrak{gl}_n), \quad \Psi(\mathcal{D}^-) = U^q_{q,q}(\mathfrak{gl}_n)$$

and consider the usual triangular decomposition:

$$\Psi(\mathcal{D}^0) = U_q(\mathfrak{gl}_n) = U^g_q(\mathfrak{gl}_n) \otimes U^\leq_q(\mathfrak{gl}_n)$$

then we obtain the decomposition (1.7) as algebras.

3.31. The purpose of the remainder of the present paper is to make:

$$\mathcal{D}^+ \otimes U^\geq_q(\mathfrak{gl}_n) \quad \text{and} \quad \mathcal{D}^- \otimes U^\leq_q(\mathfrak{gl}_n)$$

into bialgebras, in such a way that $\mathcal{D}$ becomes their Drinfeld double. This will be done by recasting $\mathcal{D}^\pm$ as a new type of shuffle algebra, as described in Subsection 1.4. With this in mind, we will prove the following more explicit version of Theorem 1.5.

**Theorem 1.5 (explicit):** If $\mathcal{A}^+$ and $\mathcal{A}^-$ are the shuffle algebras that will be introduced in Definitions 4.8 and 6.2 respectively, then we have algebra isomorphisms:

$$\mathcal{D}^+ \cong \mathcal{A}^+, \quad \mathcal{D}^- \cong \mathcal{A}^{-, \text{op}}$$

Moreover, the extended algebras $\mathcal{\tilde{A}}^\pm$ defined in (6.6) have bialgebra structures and a bialgebra pairing between them, such that the Drinfeld double:

$$\mathcal{A} = \mathcal{\tilde{A}}^+ \otimes \mathcal{\tilde{A}}^{-, \text{op, coop}}$$
is isomorphic to \( \mathcal{D} \) (and hence also with \( \mathcal{S} \) and \( U_q(\hat{\mathfrak{sl}}_n) \)) as an algebra.

3.32. In the remainder of this Section, we will study the algebra \( \mathcal{E} \cong U_q(\hat{\mathfrak{sl}}_n) \) in more detail, and fill the gaps left in the discussion above. Let us consider the following matrix-valued rational function called an \( R \)-matrix:

\[
R(x) = \sum_{1 \leq i,j \leq n} E_{ii} \otimes E_{jj} \left( q - q^{-1}\frac{x}{1-x} \right) + (q - q^{-1}) \sum_{1 \leq i \neq j \leq n} E_{ij} \otimes E_{ji} \frac{x^{\delta_{i,j}}}{1-x}
\]

where \( E_{ij} \) denotes the \( n \times n \) matrix with a single 1 at the intersection of row \( i \) and column \( j \), and zeroes everywhere else. We will now give an alternate version of Definition 3.33, and afterwards show how to match notations:

**Definition 3.33.** Consider the algebra:

\[
\mathcal{E} := \mathbb{Q}(q) \left\langle s_{(i,j)}; t_{(i,j)}; c^{\pm 1} \right\rangle_{i \leq j \leq n} / \text{relations (3.89) - (3.92)}
\]

where:

\[
s_{(i,j)} t_{(i,j)} = 1
\]

\[
R \left( \frac{x}{y} \right) S_1(x) S_2(y) = S_2(y) S_1(x) R \left( \frac{x}{y} \right)
\]

\[
R \left( \frac{x}{y} \right) T_1(x) T_2(y) = T_2(y) T_1(x) R \left( \frac{x}{y} \right)
\]

\[
R \left( \frac{xc}{y} \right) S_1(x) T_2(y) = T_2(y) S_1(x) R \left( \frac{x}{yc} \right)
\]

where \( Z_1 = Z \otimes \text{Id} \) and \( Z_2 = \text{Id} \otimes Z \) for any symbol \( Z \), and:

\[
S(x) = \sum_{1 \leq i,j \leq n, d \geq 0} s_{(i,j+n)d} \cdot E_{ij} x^{-d}
\]

\[
T(x) = \sum_{1 \leq i,j \leq n, d \geq 0} t_{(i,j+n)d} \cdot E_{ji} x^d
\]

The series \( S(x) \), \( T(y) \) are the transposes of the series denoted \( T^-(x) \), \( T^+(y) \) in \( \text{[16]} \), which explains the discrepancy between our conventions and those of \( \text{loc. cit.} \).

3.34. We will write \( \psi_k = s_{(k;k)}^{-1} = t_{(k;k)} \in \mathcal{E} \) for all \( 1 \leq k \leq n \), and set:

\[
f_{(i,j)} = s_{(i,j)} \psi_i, \quad f_{-(i,j)} = t_{(i,j)} \psi_i^{-1}
\]

\( \forall 1 \leq i \leq n \) and \( i \leq j \in \mathbb{Z} \). We will extend our notation to all integers by setting:

\[
f_{\pm(i+n,j+n)} = f_{\pm(i,j)}, \quad \psi_{k+n} = c^n \psi_k
\]

\( \forall i \leq j, k \). It is elementary to see that relations (3.90)-(3.92) can be rewritten as:

\[
\psi_k f_{\pm(i,j)} = q^{i\delta_{i,j}^}\delta_{i,j} f_{\pm(i,j)} \psi_k
\]

---

\footnote{Relations (3.90) and (3.91) can be made explicit by expanding in either positive or negative powers of \( x/y \), but (3.92) must be expanded in negative powers of \( x/y \).}
and:

\[
\sum_{\pm[i;j] \pm'[i';j'] = d} \text{coefficient} \cdot f_{\pm[i;j]} f_{\pm'[i';j']} = 0
\]

for all \( \pm, \pm' \in \{+, -\} \) and \( d \in \mathbb{Z}^n \). We will not need to spell out the coefficients in (3.97) explicitly, but they can easily be obtained by expanding (3.90)–(3.92) as power series in \( x/y \) and equating matrix coefficients of every \( E_{ij} \otimes E_{i'j'} \).

The bialgebra (and Drinfeld double) structure on \( \mathcal{E} = \mathcal{E}^\geq \otimes \mathcal{E}^\leq \) from (3.20)–(3.22) can be presented in terms of the matrix-valued power series (3.93)–(3.94) as:

\[
\Delta(\mathcal{S}(x)) = (1 \otimes \mathcal{S}(xc_1)) \cdot (\mathcal{S}(x) \otimes 1) \quad (3.98)
\]

\[
\Delta(\mathcal{T}(x)) = (1 \otimes \mathcal{T}(x)) \cdot (\mathcal{T}(xc_2) \otimes 1) \quad (3.99)
\]

where \( \cdot \) denotes matrix multiplication (i.e. the formula \( E_{ij} \cdot E_{i'j'} = \delta_{ij}^1 E_{ij} \)), and \( c_1 = c \otimes 1, c_2 = 1 \otimes c \). It is straightforward to check that these coproducts respect relations (3.90)–(3.92), i.e. extend to well-defined coproducts on the algebra \( \mathcal{E} \).

3.35. The pairing (3.26) takes the form:

\[
\langle S_1(x), T_2(y) \rangle = R \left( \frac{x}{y} \right)^{-1} \quad \Leftrightarrow \quad \langle S_1(x), T_2(y)^{-1} \rangle = R \left( \frac{x}{y} \right)
\]

It is elementary to show that (3.100) generates a bialgebra pairing, i.e. it intertwines the product with the coproduct on \( \mathcal{E} \), and that \( \mathcal{E} \) is the Drinfeld double of its halves with respect to this pairing. Moreover, the linear maps (3.30) which were used to normalize primitive elements of \( \mathcal{E} \) can be easily seen to come from the assignment:

\[
\alpha^{\pm}_{\pm[i;j]} : \mathcal{E}_{\pm[i;j]} \rightarrow \mathbb{Q}(q, q^{\pm})
\]

as the coefficients of the maps \( \alpha^{\pm} \), appropriately renormalized as follows:

\[
\alpha^{+}(r) = \sum_{(i \leq j) \in \mathbb{Z}_{\geq 0}^2} \alpha^{+}_{(i;j)}(r) \cdot E_{ij}x^{\lfloor \frac{i-1}{n} \rfloor - \lfloor \frac{j-1}{n} \rfloor} q^{\frac{i}{n}}
\]

\[
\alpha^{-}(r) = \sum_{(i \leq j) \in \mathbb{Z}_{\geq 0}^2} \alpha^{-}_{(i;j)}(r) \cdot E_{ij}x^{\lfloor \frac{i-1}{n} \rfloor - \lfloor \frac{j-1}{n} \rfloor} q^{-\frac{i}{n}}
\]

Then it is elementary to observe that the bialgebra property (2.28)–(2.29) of the pairing, together with definition (3.101), imply the multiplicativity property (3.31).
3.36. We will henceforth write $S^+(x) = S(x)$ and $T^-(x) = T(x)$, so we have:

$$S^+(x) = \begin{cases} \sum_{1 \leq i,j \leq n,d \geq 0} f_{[i;j+n;d]} \psi^1_i \cdot E_{ij}x^{-d} & \text{if } d=0 \text{ then } i\leq j \\ \sum_{1 \leq i,j \leq n,d \geq 0} f_{-[i;j+n;d]} \psi^1_i \cdot E_{ji}x^d & \text{if } d=0 \text{ then } i<j \end{cases}$$

$$T^-(x) = \begin{cases} \sum_{1 \leq i,j \leq n,d \geq 0} f_{[i;j+n;d]} \psi^1_i \cdot E_{ij}x^{-d} & \text{if } d=0 \text{ then } i\leq j \\ \sum_{1 \leq i,j \leq n,d \geq 0} f_{-[i;j+n;d]} \psi^1_i \cdot E_{ji}x^d & \text{if } d=0 \text{ then } i<j \end{cases}$$

as elements of $\mathcal{E} \otimes \operatorname{End}(\mathbb{C}^n)[[x^{\pm 1}]]$. We define series $S^-(x)$ and $T^+(x)$ by:

$$S^-(x)T^-(xq^2) = 1 \tag{3.105}$$

$$D^{-1}S^+(xq^{2n}q^2)^\dagger DT^+(x)^\dagger = 1 \tag{3.106}$$

where $D = \operatorname{diag}(q^2, \ldots, q^{2n})$. The coefficients of these series will be denoted by:

$$T^+(x) = \begin{cases} \sum_{1 \leq i,j \leq n,d \geq 0} \psi_j f_{[i;j+n;d]}q^{-\frac{2(i-1)}{n}} c^{-d} \cdot E_{ij}x^{-d} & \text{if } d=0 \text{ then } i\leq j \\ \sum_{1 \leq i,j \leq n,d \geq 0} \psi_j f_{-[i;j+n;d]}q^{-\frac{2(i-1)}{n}} c^d \cdot E_{ji}x^d & \text{if } d=0 \text{ then } i<j \end{cases}$$

$$S^-(x) = \begin{cases} \sum_{1 \leq i,j \leq n,d \geq 0} \psi_j f_{[i;j+n;d]}q^{-\frac{2(i-1)}{n}} c^d \cdot E_{ij}x^d & \text{if } d=0 \text{ then } i\leq j \\ \sum_{1 \leq i,j \leq n,d \geq 0} \psi_j f_{-[i;j+n;d]}q^{-\frac{2(i-1)}{n}} c^{-d} \cdot E_{ji}x^{-d} & \text{if } d=0 \text{ then } i<j \end{cases}$$

It is straightforward to show that formulas (3.105) and (3.106) imply (3.36). The following identities are easy to prove, as consequences of (3.90), (3.91), (3.92):

$$T_1^+(x)R_{21} \left( \frac{y}{xq} \right) S_2^+(y) = S_2^+(y)R_{21} \left( \frac{y}{xq} \right) T_1^+(x) \tag{3.107}$$

$$S_1^-(x)R_{21} \left( \frac{y}{xq^2} \right) T_2^-(y) = T_2^-(y)R_{21} \left( \frac{y}{xq^2} \right) S_1^-(x) \tag{3.108}$$

$$R \left( \frac{xc}{y} \right) S_2^+(x)T_1^-(y) = T_2^-(y)S_1^+(x)R \left( \frac{xc}{yq} \right) \tag{3.109}$$

$$R \left( \frac{xc}{y} \right) T_2^+(y)S_1^-(x) = S_2^+(y)T_2^+(y)R \left( \frac{xc}{yq} \right) \tag{3.110}$$

$$S_1^+(x)R \left( \frac{x}{yq^2} \right) S_2^-(y) = S_2^-(y)R \left( \frac{xc}{yq^2} \right) S_1^+(x) \tag{3.111}$$

$$T_2^+(y)R \left( \frac{x}{yq^2} \right) T_1^-(x) = T_1^-(x)R \left( \frac{xc}{yq^2} \right) T_2^+(y) \tag{3.112}$$
3.37. In the present Subsection, we will show how to rewrite the relations between the generators of the algebras $D^+, D^0, D^-$ (specifically relations (3.76)–(3.78) and (3.77)–(3.79)) in terms of the series $S^\pm(x)$ and $T^\pm(x)$.

**Proposition 3.38.** Under the substitution:

\[
p^{(1)}_{[ij, n, x]} \sim E_{ij} z^d \frac{q^{2i-1}}{1 - q^2}
\]

for all $1 \leq i, j \leq n$ and $d \in \mathbb{Z}$, the following relations hold in $D$:

\[
X_i^+(z) \cdot S^+_2(w) = S^+_2(w) \cdot R_{12} \left( \frac{z}{w} \right) X_i^+(\frac{w}{z})
\]

\[
T_i^+(w) \cdot X_i^+(z) = R_{12} \left( \frac{z}{w} \right) X_i^+(\frac{w}{z}) \cdot T_i^+(w)
\]

\[
S_i^-(w) \cdot X_i^+(z) = R_{12} \left( \frac{z}{w} \right) X_i^+(\frac{w}{z}) \cdot S_i^-(w)
\]

\[
X_i^+(z) \cdot T_i^-(w) = T_i^-(w) \cdot R_{12} \left( \frac{z}{w} \right) X_i^+(\frac{w}{z})
\]

for any $X_i^+(z) \in \text{End}(\mathbb{C}^n)[z^\pm 1]$, where $f(x) \in \mathbb{Q}(q, x)$ is defined in (4.7).

To understand the meaning of relations (3.114)–(3.117), let us spell out the first of these. Letting $S^+_2(w) = \sum_{u, v}^k s^+_2(\frac{w}{u}, \frac{v}{w})$ and $X(z) = \sum_{x}^\infty$, formula (3.114) reads:

\[
\sum_{k \geq 0} E_{ij} \frac{w^k}{u^d} s^+_2(\frac{w}{u}, \frac{v}{w}) = \sum_{k \geq 0} E_{ij} \frac{w^k}{u^d} z^k
\]

where $r$ and $r'$ are the coefficients of the power series expansions:

\[
R_{12} \left( \frac{z}{w} \right) \cdot f \left( \frac{z}{w} \right)^{-1} = \sum_{i,j,u,v}^k r_{ij,k} \cdot E_{ij} \otimes E_{uw} \frac{z^k}{w^k}
\]

\[
R_{21} \left( \frac{w}{z} \right) = \sum_{i,j,u,v}^k r_{ij,k} \cdot E_{ij} \otimes E_{uw} \frac{z^k}{w^k}
\]

Equating the coefficients of $\cdots \otimes \frac{w^d}{u^d}$ in the two sides of (3.118) yields the identity:

\[
E_{ij} \frac{w^d}{u^d} s^+_2(\frac{w}{u}, \frac{v}{w}) = \sum_{a, b \geq 0}^k r_{ij,a} r_{uv}^b \cdot s^+_2(\frac{w}{u}, \frac{v}{w})^b
\]

which is a relation in the algebra $D$, once we perform the substitution (3.113).

**Proof.** We will only prove relation (3.114), and leave the analogous formulas (3.115)–(3.117) as exercises to the interested reader. Let us rewrite (3.114) as:

\[
S^+_2(w)^{-1} X_1^+(z) S^+_2(w) = R_{21} \left( \frac{w}{z} \right)^{-1} X_1^+(z) R_{21} \left( \frac{w}{z} \right)
\]
If we let $A : \mathcal{E} \to \mathcal{E}$ denote the antipode, then we have $A^{-1}(S(w)) = S(wc)^{-1}$ as a consequence of \((3.98)\). With this in mind, \((3.119)\) reads:

\[(3.120) \quad X^+_1(z) \heartsuit S^+_2(w) = X^+_1(z) \heartsuit S^+_2(w)\]

where for any $e \in U_q^{>\ge}(\mathfrak{g}l_n)$, we write:

\[(3.121) \quad X^+_1(z) \heartsuit e = A^{-1}(e_2)X^+_1(z)e_1\]

\[(3.122) \quad X^+_1(z) \clubsuit e = \left\langle e_2, T_1(z)X^+_1(z)\right\rangle e_1, T_1(z\theta^2)^{-1}\]

Indeed, when $e = S_2(w)$, the right-hand sides of \((3.121)\) and \((3.122)\) match the LHS and RHS of \((3.119)\), respectively, due to \((3.100)\). It is easy to see that the operations $\heartsuit$ and $\clubsuit$ are additive in $e$, and moreover:

\[X^+_1(z) \heartsuit (ee') = (X^+_1(z) \heartsuit e) \heartsuit e'\]

\[X^+_1(z) \clubsuit (ee') = (X^+_1(z) \clubsuit e) \clubsuit e'\]

Since $\mathcal{E}^+$ is generated by the $p_{(l,1)}^{(0)}$’s and $p_{(s,s+1)}^{(0)}$’s, formula \((3.120)\) is equivalent to:

\[(3.123) \quad X^+_1(z) \heartsuit e = X^+_1(z) \heartsuit e \quad \forall e \in \left\{p_{(l,1)}^{(0)}, p_{(s,s+1)}^{(0)}\right\}_{l \in \mathbb{N}}\]

When $e = p_{(l,1)}^{(0)}$ (whose coproduct is $\Delta(e) = e \otimes 1 + e^{-1} \otimes e$), relation \((3.123)\) reads:

\[X^+_1(z)p_{(l,1)}^{(0)} - p_{(l,1)}^{(0)}X^+_1(z) = X^+_1(z)\left\langle p_{(l,1)}^{(0)}, T_1(z\theta^2)^{-1}\right\rangle + \left\langle p_{(l,1)}^{(0)}, T_1(z)\right\rangle X^+_1(z)\]

Formulas \((3.69)\) and \((3.101)\) imply that the pairings in the right-hand side of the formula above are equal to $\theta'$ and $-\theta'^{-1}$, respectively, hence we obtain:

\[\left[X^+_1(z), p_{(l,1)}^{(0)}\right] = z'X^+_1(z)(\theta' - \theta'^{-1})\]

If we plug $X^+_1(z) = E_{ji}z^d\theta^{\frac{1}{2}i}$ into the equation above and use the correspondence \((3.113)\), the formula above reduces to \((3.76)\) for $\pm = +$. Similarly, if we plug $e = p_{(s,s+1)}^{(0)}$ into \((3.123)\), then the resulting formula reduces to \((3.78)\) for $\pm = +$. \(\square\)

By analogy with Proposition \((3.38)\), we have the following result (whose proof is quite close to the one above, hence left as an exercise to the interested reader):

**Proposition 3.39.** Under the substitution:

\[(3.124) \quad p_{(l,i+jnd)}^{(-1)} \sim E_{ji}z^{d} \cdot \frac{\theta^{1-2i}}{q - q^{-1}}\]
the following relations hold in \(D\) (recall that \(\cdot \) denotes the opposite product):

\[
X_1^-(z) \cdot_\text{op} S_2^+(w) = S_2^-(w) \cdot_\text{op} R_{12} \left(\frac{\hat{z}}{\hat{w}}\right) X_1^-(z) D_2 \cdot \frac{q^n q^{2n}}{z} D_2^{-1}
\]

\[
T_2^-(w) \cdot_\text{op} X_1^-(z) = D_1 R_{12} \left(\frac{q^n q^{2n}}{w c}\right) D_1^{-1} X_1^-(z) \cdot_\text{op} T_2^+(w)
\]

\[
S_2^+(w) \cdot_\text{op} X_1^-(z) = D_1 R_{12} \left(\frac{q^n q^{2n}}{w c}\right) D_1^{-1} X_1^-(z) R_{21} \left(\frac{w}{z}\right) \cdot_\text{op} S_2^+(w)
\]

Finally, let us perform both substitutions (3.113) and (3.124) simultaneously:

\[
p^{(\pm)}_{i,j} \Rightarrow (E_{ji} z^d)^\pm \cdot \frac{q^n q^{2n}}{1 - q^2}
\]

\[
l^{(\pm)}_{i,j} \Rightarrow (E_{ji} z^d)^- \cdot \frac{q^n q^{2n}}{q - q^{-1}}
\]

(129) \[
\left[\left(\frac{E_{ij}}{z^d}\right)^+, \left(\frac{E_{ij'}}{z^d}\right)^-\right] = (q^2 - 1) \sum_{k \in \mathbb{Z}} \left(s^+_j s^+_{i+n+k} t^+_{i';i+n+d^d-k} - t^-_{i';j+n+k} s^-_{i';j+n(d^d-k)} c^{-d^d}\right)
\]

as an immediate consequence of formula (3.80) (we set \(s^\pm_{i,j} = t^\pm_{i,j} = 0\) if \(i > j\)).

3.40. We will now prove a useful Lemma about the structure of the algebra \(\mathcal{E}^+\) of Subsection 3.3. Let us write \(\text{LHS}_d\) for the quantity in the left-hand side of (3.37), when the signs are \(\pm = \pm' = +\). Then we have:

\[
\mathcal{E}^+ = \mathbb{Q}(q) \left\langle f_{[i;j]} \right\rangle_{(i < j)} \bigg/ \text{LHS}_d d \in \mathbb{N}^n
\]

In Section 5 we will find ourselves in the situation of having an algebra \(\mathcal{B}^+\) and wanting to construct an algebra isomorphism \(\Upsilon : \mathcal{E}^+ \cong \mathcal{B}^+\). Of course, the straightforward way to do this is to construct elements:

\[
f'_{[i;j]} \in \mathcal{B}^+\] declare that \(\Upsilon(f_{[i;j]}) = f'_{[i;j]}\)

and directly check that \(\Upsilon(\text{LHS}_d) = 0\), \(\forall d \in \mathbb{N}^n\). However, such a check is not handy in our situation, and we will instead rely on some additional structure:

Lemma 3.41. Assume \(\mathcal{B}^+\) is a \(\mathbb{Z}^n\)-graded \(\mathbb{Q}(q)\)-algebra, such that:

\[
\mathcal{B}^+ = \left\langle \mathcal{B}^+^+, \psi_{a}^{\pm}, c^{\pm} \right\rangle_{a \in \mathbb{Z}}
\]

\[
\left\langle \psi_{x} - q^{-(\deg x \cdot c^d)} x \psi_{x}, \psi_{s+n} - c \psi_{a}, c \text{ central} \right\rangle_{x \in \mathcal{B}^+, s \in \mathbb{Z}}
\]

\[9\] is a bialgebra. Assume there exist elements \(0 \neq f'_{(i;j)} \in \mathcal{B}_{(i;j)}\) and linear maps:
\[\alpha'_{(i;j)} : \mathcal{B}_{(i;j)} \to \mathbb{Q}(q) \quad \forall i \in \{1, \ldots, n\} \text{ and } j > i\]
such that the analogues of (3.21), (3.22), (3.31), and (3.32) hold. If:
\[(3.130) \quad \left\{ x \text{ primitive and } \alpha'_{(i;j)}(x) = 0, \forall i, j \right\} \Rightarrow x = 0\]
then the map \(E^+ \overset{\Upsilon}{\to} B^+\), \(\Upsilon(f_{(i;j)}) = f'_{(i;j)}\) is an injective algebra homomorphism.

**Proof.** Let us consider the left-hand side of (3.97) with \(f'_{(i;j)}\) instead of the \(f_{(i;j)}\):
\[LHS'_d = \sum_{|i;j|+|j';j''|=d} \text{coefficient} \cdot f'_{(i;j)}f'_{(j';j''')} \in \mathcal{B}_d\]
To show that \(\Upsilon\) is an algebra homomorphism, we would need to show that \(LHS'_d = 0\), which we will prove by induction on \(d \in \mathbb{N}^n\). The base case \(d = 0\) is trivial, so we will only prove the induction step. We have:
\[\Delta(LHS'_d) \in \langle \psi_s^{\pm 1} \rangle_{s \in \mathbb{Z}} \otimes LHS'_d + \ldots + LHS'_d \otimes \langle \psi_s^{\pm 1} \rangle_{s \in \mathbb{Z}}\]
where the middle terms denoted by the ellipsis are equal to \(\Upsilon \otimes \Upsilon\) applied to the middle terms of \(\Delta(LHS_d)\). Since the latter are 0 (as \(LHS_d = 0\) in \(E^+\)), we conclude that \(LHS'_d\) is primitive. Moreover, the analogues of (3.31) and (3.32) imply that:
\[\alpha'_{(i;j)}(LHS'_d) = \alpha_{(i;j)}(LHS_d) = 0 \quad \forall d, i < j\]
Therefore, the assumption (3.130) implies that \(LHS'_d = 0\) for all \(d\), thus establishing the fact that \(\Upsilon\) is a well-defined algebra homomorphism. To show that \(\Upsilon\) is injective, assume that its kernel is non-empty. Since \(\Upsilon\) preserves degrees, we may choose \(0 \neq x \in E^+\) of minimal degree \(d \in \mathbb{N}^n\) such that \(\Upsilon(x) = 0\). Since \(\Upsilon\) preserves the coproduct and is injective in degrees \(< d\) (by the minimality of \(d\)), we conclude that \(x\) is primitive. However, since \(\Upsilon\) intertwines the linear maps \(\alpha_{(i;j)}\) with \(\alpha'_{(i;j)}\), we conclude that \(x\) is also annihilated by the linear maps \(\alpha'_{(i;j)}\), hence \(x = 0\).

Since an injective linear map of finite-dimensional vector spaces \(\Phi : V_1 \to V_2\) is an isomorphism if \(\dim V_2 \leq \dim V_1\) (as well as the similar statement in the graded case, if \(V_1\) and \(V_2\) have finite-dimensional graded pieces which are preserved by \(\Phi\)) we obtain the following:

**Corollary 3.42.** If the assumption of Lemma 3.41 holds, and moreover, if \(\dim \mathcal{B}_d \leq \text{RHS of (3.27)}\) for all \(d \in \mathbb{N}^n\), then \(\Upsilon : E^+ \cong B^+\) is an isomorphism.

**Proposition 3.43.** The \(\mathbb{Q}(q, \bar{q}^{\pm 1})\)-algebra \(D^\pm\) is generated by the elements:
\[(3.131) \quad \left\{ P_{\pm |i;j|}^{(\pm 1)} \right\}_{(i,j) \in \mathbb{Z}^2 / \mathbb{Z}^n} \]

---

9In the formula above, \(\langle \cdot, \cdot \rangle\) is the bilinear form on \(\mathbb{Z}^n\) given by \(\langle \varsigma^i, \varsigma^j \rangle = \delta^i_1 \delta^j_1 - 1\)
Proof. Without loss of generality, let $\pm = +$. We will prove that $p^{(k)}_{(i;j)}$ (resp. $p^{(k')}_{(\delta,r)}$) lies in the subalgebra generated by the elements (3.131) for all choices of indices $i, j, l, r$, by induction on $k$ (respectively $k'$). To this end, let us choose a lattice triangle of minimal size with the vector $(j-i, k)$ (respectively $(nl, k')$) as an edge:

![Diagram of lattice triangle](image)

In the case of the picture on the left, namely that of the element $p^{(k)}_{(i;j)}$, we have:

$$
\text{det} \begin{pmatrix} b & b-k \cr a & j-i+a \end{pmatrix} = 1
$$

(3.132)

Recall that $i \not\equiv j \mod n$. If $a \equiv j - i$ modulo $n$, then relation (3.73) gives us:

$$
\left[ p^{(b)}_{(i+a)} p^{(k-b)}_{(i+a;j)} \right] = p^{(k)}_{(i;j)} \left( q^k - q^{-\frac{1}{a}} \right)
$$

while if $a \not\equiv j - i$ modulo $n$, then relation (3.74) gives us:

$$
p^{(b)}_{(i+a)} p^{(k-b)}_{(i+a;j)} q^{-\frac{1}{a}} = p^{(k)}_{(i;j)} - q^{-\frac{1}{a}} q^{-\frac{1}{n}} p^{(k)}_{(i;j)}
$$

In either of the two formulas above, the induction hypothesis implies that the left-hand side lies in the algebra generated by the elements (3.131). Therefore, so does the right-hand side, and the induction step is complete.

The case of the picture on the right, namely that of the element $p^{(k')}_{(\delta,r)}$, is proved analogously. We will therefore only sketch the main idea, and leave the details to the interested reader. As shown in Lemma 4.4 of [17], relation (3.74) implies:

$$
p^{(b)}_{(u+a)} p^{(k-b)}_{(u+a;u+n)} q^{-\frac{1}{a}} - p^{(k-b)}_{(u+a;u+n)} p^{(b)}_{(u+u+a)} q^{-\frac{1}{a}} = \sum_{l_1 + l_2 = l} \tilde{j}^{(\mu)}_{(u+a+l_1;u+a)} \tilde{j}^{(\mu)}_{(u+a+l_2)}
$$

where $\mu = k'/nl$, and $p^{(a)}_{(u;u+x)} = \sum_{v' \in \mathbb{Z}/n} p^{(a)}_{(u;v')} q^{-\frac{1}{a}} \tilde{j}^{(\mu)}_{(u;u+v')}$.

By the induction hypothesis, the left-hand side of the expression above lies in the subalgebra generated by the elements (3.131), hence so does the right-hand side, which we henceforth denote RHS. Clearly, RHS $\in B^+_{\mu}$, and it can therefore be expressed as a sum of products of the simple and imaginary generators:

$$
\text{RHS} = \sum \alpha p^{\nu}_{(\delta,r)} + \text{sum of products of more than one generator of } B^+_{\mu}
$$

By the induction hypothesis, all products of more than one simple or imaginary generator lie in the subalgebra generated by the elements (3.131). We conclude that
\[ \sum_{r} \alpha^u_r \cdot p^u_{\delta_r} \] also does, for all \( u \in \{1, ..., n\} \), hence so does each \( p^u_{\delta_r} \) (the matrix \( \alpha^u_r \) has full rank, as shown in the last paragraph of the proof of Theorem 4.5 of [17]).

\[ \square \]

4. The shuffle algebra with spectral parameters

4.1. We will now generalize the construction of Section 2 by allowing the coefficients of matrices in \( \text{End}(V \otimes k) \) to be rational functions. We will recycle all the notations from Section 2, so fix a basis of an \( n \)-dimensional vector space \( V \), and let:

\[ (4.1) \quad R(x) \in \text{End}_{\mathbb{Q}(q)}(V \otimes V)(x) \]

be given by formula (3.87). For a parameter \( q \), we define:

\[ (4.2) \quad \tilde{R}(x) = R_{21} \left( \frac{1}{xq^2} \right) \in \text{End}_{\mathbb{Q}(q, q)}(V \otimes V)(x) \]

It is well-known that \( R(x) \) satisfies the Yang-Baxter equation with parameter:

\[ (4.3) \quad R_{12} \left( \frac{z_1}{z_2} \right) R_{13} \left( \frac{z_1}{z_3} \right) R_{23} \left( \frac{z_2}{z_3} \right) = R_{23} \left( \frac{z_2}{z_3} \right) R_{13} \left( \frac{z_1}{z_3} \right) R_{12} \left( \frac{z_1}{z_2} \right) \]

and it is easy to show that \( \tilde{R}(z) \) satisfies the following analogue of (2.2)–(2.3):

\[ (4.4) \quad \tilde{R}_{21} \left( \frac{z_2}{z_1} \right) \tilde{R}_{31} \left( \frac{z_3}{z_1} \right) R_{23} \left( \frac{z_2}{z_3} \right) = R_{23} \left( \frac{z_2}{z_3} \right) \tilde{R}_{31} \left( \frac{z_3}{z_1} \right) \tilde{R}_{21} \left( \frac{z_2}{z_1} \right) \]

\[ (4.5) \quad R_{12} \left( \frac{z_1}{z_2} \right) \tilde{R}_{31} \left( \frac{z_3}{z_1} \right) \tilde{R}_{32} \left( \frac{z_3}{z_2} \right) = \tilde{R}_{32} \left( \frac{z_3}{z_2} \right) \tilde{R}_{31} \left( \frac{z_3}{z_1} \right) R_{12} \left( \frac{z_1}{z_2} \right) \]

Finally, we note that the \( R \)-matrix (4.1) is (almost) unitary, in the sense that:

\[ (4.6) \quad R_{12}(x)R_{21} \left( \frac{1}{x} \right) = f(x) \cdot \text{Id}_{V \otimes V} \]

where:

\[ (4.7) \quad f(x) = \frac{(1-xq^2)(1-xq^{-2})}{(1-x)^2} \]

It is easy to see that “half” of the rational function \( f(x) \) could have been absorbed in the definition of \( R(x) \), but we will prefer our current conventions.

4.2. We will represent elements of \( \text{End}(V \otimes k)(z_1, ..., z_k) \) as braids on \( k \) strands. The only difference between the present setup and that of Section 2 is that each strand carries not only a label \( i \in \{1, ..., k\} \) but also a variable \( z_i \). With this in mind, we make the convention that the endomorphism corresponding to a positive crossing of strands labeled \( i \) and \( j \), endowed with variables \( z_i \) and \( z_j \) respectively, is:

\[ R_{ij} \left( \frac{z_i}{z_j} \right) \]

Because of (4.2), we can represent both \( R \) and \( \tilde{R} \) as crossings of braids of the same kind (i.e. we do not need the dichotomy of straight strands versus squiggly strands, of Subsection 2.2), if we remember to change the variable on one of our strands. We will always write the variable next to every braid. For example, the braids:
Figure 11. Braids decorated with variables represent the following compositions $\in \text{End}(V^\otimes 2)(z_1, z_2)$:

$$R_{12} \left( \frac{z_1}{z_2} \right) A_1(z_1) \tilde{R}_{12} \left( \frac{z_1}{z_2} \right) B_2(z_2) \quad \text{and} \quad A_2(z_2) \tilde{R}_{21} \left( \frac{z_2}{z_1} \right) B_1(z_1) \tilde{R}_{21} \left( \frac{z_2}{z_1} \right)$$

respectively. The variable does not change along a strand, except at a box.

4.3. We make an unusual convention on residues of rational functions, by stipulating that $(\alpha - x)^{-1}$ has residue 1 at $x = \alpha$ (instead of the more usual $-1$). With this in mind, note that $R(x)$ has a pole at $x = 1$, with residue $(q - q^{-1}) \cdot (12)$. Thus:

$$\text{Res}_{x=\frac{1}{q}} R(x) = (q^{-1} - q) \cdot (12) \in \text{End}_{Q(q, \bar{q})}(V \otimes V)$$

where $(12)$ denotes the permutation operator of the two factors. Pictorially, the endomorphism $(4.8)$ will be represented by two black dots indicating a color change (recall that the color encodes the index $\in \{1, ..., k\}$ of a strand) between two strands:

Figure 12. Black dots can slide past arbitrary strands

The equality of braids depicted in Figure 12 means that one can move the black dots as far left or as far right as we wish, no matter how many other strands we pass over or under. Explicitly, the equality depicted above reads:

$$(q^{-1} - q)(12) \cdot R_{i2} \left( \frac{z_i}{y_1} \right) R_{i1}^{-1} \left( \frac{y_2}{z_j} \right) = R_{i1} \left( \frac{z_i}{y_1} \right) R_{2j} \left( \frac{y_2}{z_j} \right) \cdot (q^{-1} - q)(12)$$

which is a true identity. Finally, we note that due to formula (4.6), we can always change a crossing in a braid, at the cost of multiplying with the function (4.7):

Figure 13. Changing a crossing
4.4. The following result is proved just like Proposition 2.4.

Proposition 4.5. Let $A = A_{1...k}(z_1, ..., z_k)$, $B = B_{1...l}(z_1, ..., z_l)$. The assignment:

$$A \ast B = \sum_{\{1, ..., k+l\} = \{a_1, ..., a_k\} \cup \{b_1, ..., b_l\}} \left[ \prod_{i=k}^{l} \prod_{j=1}^{i} R_{a_ib_j} \left( \frac{z_{a_i}}{z_{b_j}} \right) \right]$$

(4.9)

$$A_{a_1...a_k}(z_{a_1}, ..., z_{a_k}) \left[ \prod_{i=1}^{k} \prod_{j=1}^{l} \tilde{R}_{a_ib_j} \left( \frac{z_{a_i}}{z_{b_j}} \right) \right] B_{b_1...b_l}(z_{b_1}, ..., z_{b_l})$$

yields an associative algebra structure on the vector space:

$$\bigoplus_{k=0}^{\infty} \text{End}_{Q(q^\frac{1}{k})}(V^\otimes k)(z_1, ..., z_k)$$

with unit $1 \in \text{End}_{Q(q^\frac{1}{k})}(V^\otimes 0)$. We call (4.9) the “shuffle product”.

Proposition 4.6. The shuffle product above preserves the vector space:

$$A_{a_1...a_k}(z_{a_1}, ..., z_{a_k}) \left[ \prod_{i=1}^{k} \prod_{j=1}^{l} \tilde{R}_{a_ib_j} \left( \frac{z_{a_i}}{z_{b_j}} \right) \right] B_{b_1...b_l}(z_{b_1}, ..., z_{b_l})$$

consisting of tensors $X = X_{1...k}(z_1, ..., z_k)$ which simultaneously satisfy:

- $X = \frac{x(z_1, ..., z_k)}{\prod_{1 \leq i \neq j \leq k}(z_i - z_j q^2)}$ for some $x \in \text{End}_{Q(q^\frac{1}{k})}(V^\otimes k)[z_1^{\pm 1}, ..., z_k^{\pm 1}]$.

- $X$ is symmetric, in the sense that:

$$X = R_\sigma \cdot (\sigma X \sigma^{-1}) \cdot R_\sigma^{-1}$$

(4.10)

$$\forall \sigma \in S(k), \text{ where } R_\sigma = R_\sigma(z_1, ..., z_k) \text{ is any braid lift of the permutation } \sigma, \text{ and:}$$

$$\sigma X \sigma^{-1} = X_{\sigma(1), ..., \sigma(k)}(z_{\sigma(1)}, ..., z_{\sigma(k)})$$

Proof. The fact that the shuffle product (4.9) preserves the vector space of symmetric tensors is proved word-for-word like Proposition 2.6. Therefore, it remains to prove that if $A$ and $B$ only have simple poles at $z_i = z_j q^2$, then $A \ast B$ has the same property. Since $R_{ab}(z)$ has a simple pole at $z = 1$, a priori $A \ast B$ could have simple poles at $z_i = z_j$, so it remains to show that the residues at these poles vanish.

Without loss of generality, we will prove the vanishing of the residue at $z_1 = z_k$. We will show that any symmetric tensor $X = X_{1...k}(z_1, ..., z_k)$ with at most a simple pole at $z_1 = z_k$ is actually regular there. Since only the indices/variables 1 and $k$ will play an important role in the following, we will use ellipses $\ldots$ for the indices/variables 2, ..., $k-1$. Let us consider (4.10) in the particular case $\sigma = (1k)$:

(4.11) 

$$-Y_{12...k-1,k}(z_1, z_2, ..., z_{k-1}, z_k) \cdot R_\sigma = R_\sigma \cdot Y_{k2...k-1,1}(z_k, z_2, ..., z_{k-1}, z_1)$$
where \( Y(z_1, \ldots, z_k) = X(z_1, \ldots, z_k) \cdot (z_1 - z_k) \) is regular at \( z_1 = z_k \). We may choose:

\[
R_{\sigma} = R_{12} \left( \frac{z_1}{z_2} \right) \ldots R_{1,k-1} \left( \frac{z_1}{z_{k-1}} \right) R_{1,k} \left( \frac{z_1}{z_k} \right) R_{k,k-1}^{-1} \left( \frac{z_k}{z_{k-1}} \right) \ldots R_{k,k}^{-1} \left( \frac{z_k}{z_2} \right)
\]

Since the residue of \( R_{1,k} \left( \frac{z_1}{z_k} \right) \) at \( z_1 = z_k \) is \((q - q^{-1}) \cdot (1k)\), the residue of (4.11) is:

\[
-Y_1(x, \ldots, x) \cdot R_{12} \left( \frac{x}{z_2} \right) \ldots R_{1,k-1} \left( \frac{x}{z_{k-1}} \right) : (1k) \cdot R_{k,k-1}^{-1} \left( \frac{x}{z_{k-1}} \right) \ldots R_{k,k}^{-1} \left( \frac{x}{z_2} \right) = \]

\[
R_{12} \left( \frac{x}{z_2} \right) \ldots R_{1,k-1} \left( \frac{x}{z_{k-1}} \right) R_{1,k}^{-1} \left( \frac{x}{z_k} \right) \ldots R_{12}^{-1} \left( \frac{x}{z_2} \right) \cdot (1k) = \]

\[
R_{12} \left( \frac{x}{z_2} \right) \ldots R_{1,k-1} \left( \frac{x}{z_{k-1}} \right) R_{1,k}^{-1} \left( \frac{x}{z_k} \right) \ldots R_{12}^{-1} \left( \frac{x}{z_2} \right) Y_{1,k}(x, \ldots, x) \cdot (1k)
\]

After canceling all the \( R \) factors and the permutation operators \((1k)\), we are left with \( Y_{1,k}(x, \ldots, x) = 0 \), which implies that \( X \) was regular at \( z_1 = z_k \) to begin with.

\( \square \)

4.7. For a rational function \( X(z_1, \ldots, z_k) \) with at most simple poles, we let:

\[
\text{Res}_{\{z_i = y_i, z_{i+1} = y_i q^2, \ldots, z_k = y_k q^{2(1-\ell)}\}} X\]

be the rational function in \( y, z_{i+1}, \ldots, z_k \) obtained by successively taking the residue at \( z_2 = z_1 q^2 \), then at \( z_3 = z_1 q^4 \), ..., then at \( z_k = z_1 q^{2(1-\ell)} \) and finally relabeling the variable \( z_1 \mapsto y \). More generally, for any collection of natural numbers:

\[
1 = c_1 < c_2 < \ldots < c_u < c_{u+1} = k + 1
\]

we will write:

\[
\text{Res}_{\{z_x = y_x, z_{x+1} = y_x q^2, \ldots, z_{x+1-1} = y_x q^{2(1-s_x)}\}_{x \in \{1, \ldots, u\}}} X
\]

for the rational function in \( y_1, \ldots, y_u \) obtained by applying the iterated residue (4.11) construction for the groups of variables indexed by \( \{c_1, c_1+1, \ldots, c_2-1\}, \ldots, \{c_u, c_u + 1, \ldots, c_{u+1} - 1\} \). Moreover, we set:

\[
\prod_{i=1}^{k} x_i = \prod_{1 \leq i \leq k} x_i = x_1 x_2 \ldots x_k \quad \prod_{i=k}^{1} x_i = \prod_{k \geq i \geq 1} x_i = x_k x_{k-1} \ldots x_1
\]

for any collection of potentially non-commuting symbols \( x_1, \ldots, x_k \).

**Definition 4.8.** Let \( \mathcal{A}^+ \subset \mathcal{A}^+_{\text{big}} \) be the vector subspace of elements \( X \) such that for any composition \( k = \lambda_1 + \ldots + \lambda_u \) we have (let \( \lambda_s = c_{s+1} - c_s \) for all \( s \)):

\[
\text{Res}_{\{z_x = y_x, z_{x+1} = y_x q^2, \ldots, z_{x+1-1} = y_x q^{2(1-s_x)}\}_{x \in \{1, \ldots, u\}}} X =
\]

\[
(q^{-1} - q)^{k-u} \prod_{(s,d) \neq (t,e)} \text{unordered pairs} \quad \prod_{1 \leq s,t \leq u, 1 \leq d < \lambda_s, 1 \leq e < \lambda_t} f \left( \frac{y_x q^{2d}}{y_t q^{2e}} \right)
\]
\[ \prod_{u \geq s \geq 1 \geq t \leq u} R_{c_t+c_s} \left( \frac{y_t}{y_s q^{2\lambda_t}} \right) \cdot X^{(\lambda_1,...,\lambda_u)}(y_1,...,y_u), \]
\[
\prod_{1 \leq s \leq u} \prod_{u \geq t > s} R_{c_t+c_s} \left( \frac{y_t q^{2\lambda_t}}{y_s q^{2\lambda_s}} \right) \prod_{s=1}^{u} \begin{pmatrix} c_s & \ldots & c_{s+1} - 2 & c_{s+1} - 1 & c_s \\ c_s + 1 & \ldots & c_{s+1} - 1 & c_s \end{pmatrix}
\]

for some \( X^{(\lambda_1,...,\lambda_u)} \in \text{End}(V^\otimes u)(y_1,...,y_u) \).

Pictorially, the RHS of (4.13) may be represented as follows:

Figure 14. The RHS of (4.13) for \( u = 2, \lambda_1 = 4, \lambda_2 = 3 \)

Note the symbol “blue over blue” to the right of Figure 14. Given two colors \( \gamma_1 \) and \( \gamma_2 \), placing \( \gamma_1 \) over \( \gamma_2 \) is a prescription that indicates that the braid in question be multiplied by the product of \( f(y/y') \), where \( y \) (respectively \( y' \)) goes over all variables on strands whose left endpoint has color \( \gamma_1 \) (respectively \( \gamma_2 \)), and the leftmost endpoint with variable \( y \) is above the leftmost endpoint with variable \( y' \).

Remark 4.9. We will call (4.13) the wheel conditions in the current matrix-valued setting, because the \( E_{11} \otimes \ldots \otimes E_{11} \) coefficient of any \( X \) satisfying (4.13) is a symmetric rational function in \( z_1,\ldots,z_k \) that satisfies the wheel conditions of [8].

Remark 4.10. Note that when \( k = 1 \), the wheel condition (4.13) is vacuous, but it is already non-trivial for \( k = 2 \) (as opposed from the \( n = 1 \) case of loc. cit.)

Proposition 4.11. The vector subspace \( A^+ \) of Definition 4.8 is preserved by the shuffle product (and will henceforth be called the “shuffle algebra”).

Proof. Assume that two matrix-valued rational functions \( A \) and \( B \) in \( k \) and \( l \) variables, respectively, satisfy the wheel condition (4.13). To prove that their shuffle product \( A \ast B \) also satisfies the wheel condition, we must take the iterated residue of the right-hand side of (4.9) at:

\[ z_{cs} = y_s, z_{cs+1} = y_s q^2, \ldots, z_{cs+1-1} = y_s q^{2(\lambda_s-1)} \]
for any composition \(k + l = \lambda_1 + \ldots + \lambda_u\). We will show that, at such a specialization, each summand in the RHS of (4.9) has the form predicated in the RHS of (4.13), so we henceforth fix a shuffle \(a_1 < \ldots < a_k, b_1 < \ldots < b_l\). Because \(\tilde{R}(z)\) has a simple pole at \(z = q^{-2}\), the only way such a shuffle can have a non-zero residue is if:

\[
\{a_1, \ldots, a_k\} = \bigcup_{s=1}^{u} \{c_s, c_s + 1, \ldots, r_s, r_s - 1\}
\]

\[
\{b_1, \ldots, b_l\} = \bigcup_{s=1}^{u} \{r_s, r_s + 1, \ldots, c_s+1, c_s+1 - 1\}
\]

for some choice of \(r_s \in \{c_s, \ldots, c_s+1\} - 1\) for all \(s \in \{1, \ldots, u\}\). We will indicate this choice by using the following colors for the strands of our braids:

- **red** for \(c_s\),
- **blue** for \(c_s + 1, \ldots, r_s - 1\)
- **purple** for \(r_s\),
- **green** for \(r_s + 1, \ldots, c_s+1 - 1\)

With this in mind, the summand of (4.9) corresponding to our chosen shuffle is represented by the following braid (to keep the pictures reasonable, we will only depict the case \(u = 2\), but the modifications that lead to the general case are straightforward; although we only depict a single blue and green strand in each of the \(u\) groups, the reader may obtain the general case by replacing each of them with any number of parallel blue and green strands, respectively):

**Figure 15.**

The black dots in the middle of the braid appear because the variables on the braids in question are set equal to each other in the iterated residue. By sliding the black dots as far to the right as possible (which is allowed, due to Figure 12), we obtain:

**Figure 16.**
One readily notices that certain pairs of braids are twisted twice around each other, and these twists can be canceled up to a factor of \( f(y/y') \) (due to the identity in Figure 13), where \( y \) and \( y' \) are the variables on the braids in question. Keeping in mind that the variables on the red strands are modified to the right of the red boxes, this yields the following braid:

\[
\begin{array}{cccc}
1 & \ldots & k - 1 & k \\
2 & \ldots & k & 1
\end{array}
= (12)(23) \ldots (k - 1, k) = (1k)(1 - k) \ldots (12)
\]

in the symmetric group \( S(k) \). Therefore, the braid in Figure 17 is precisely of the form predicated in the right-hand side of (4.13), which concludes our proof.

\[ \square \]

**Proposition 4.12.** For any \( X \in A^+ \) and any composition \( k = \lambda_1 + \ldots + \lambda_u \), the tensor \( Y = X^{(\lambda_1, \ldots, \lambda_u)} \) that appears in (4.13) has at most simple poles at:

\[
y_s q^{2d} - y_t q^{2e} \quad \forall \quad 1 \leq s < t \leq u, \quad 0 \leq d < \lambda_s, \quad 0 \leq e < \lambda_t
\]

for all \( 1 \leq s < t \leq u \) and any \( 0 \leq d < \lambda_s \). Moreover, if \( \lambda_s = \lambda_t \) then:

\[
Y_{...,s,...,t,...}(\ldots, y_s, \ldots, y_t, \ldots) = R_{(st)} \cdot Y_{...,t,...,s,...}(\ldots, y_t, \ldots, y_s, \ldots) \cdot R_{(st)}^{-1}
\]

for any braid lift \( R_{(st)} = R_{(st)}(y_1, \ldots, y_u) \) of the transposition \( (st) \).

**Proof.** Let us first prove the statement about the poles of \( Y \). In the course of this proof, all poles will be counted with multiplicities, in the sense that whenever we refer to a “set of poles”, the reader should assume this means “multiset of poles”. Because of the first bullet of Proposition 4.6 which determines the allowable poles of \( X \in A^+ \), the left-hand side of (4.13) has a simple pole at:

\[
y_s q^{2d} - y_t q^{2e} \quad \forall \quad 1 \leq s < t \leq u, \quad 0 \leq d < \lambda_s, \quad 0 \leq e < \lambda_t
\]

On the other hand, the right-hand side of (4.13) has a double pole at:

\[
y_s q^{2d} - y_t q^{2e} \quad \forall \quad 1 \leq s < t \leq u, \quad 1 \leq d < \lambda_s, \quad 1 \leq e < \lambda_t
\]

because of the \( f \) factors, and a simple pole at:

\[
\begin{cases}
y_s - y_t q^{2e}, \quad \text{and} \\
y_s q^{2\lambda_s} - y_t q^{2e}
\end{cases} \quad \forall \quad 1 \leq s < t \leq u, \quad 1 \leq e < \lambda_t
\]
because of the simple pole of $R(z)$ at $z = 1$. Eliminating the multiset of poles in (4.17) and (4.18) from the multiset of poles in (4.16) yields the allowable poles of $Y(y_1, ..., y_u)$, and it is elementary to see that they are precisely of the form (4.14).

As for (4.15), we will prove it pictorially. To keep the pictures legible, we will only show the case $s = 1$, $t = u = 2$, but the interested reader may easily generalize the argument. Because of property (4.10), the tensor $X(y_1, y_1 q_2^2, ..., y_2, y_2 q_2^2, ...)$ (which is represented by a braid akin to Figure 14) is also equal to the following braid:

![Figure 18.](image)

(we ignore the scalar-valued rational functions $f$ in the diagrams above, as they commute with all the braids involved). The braid called $R_\sigma$ interchanges the two collections of $\lambda_s = \lambda_t$ braids corresponding to the variables $y_s q_2^2$ and $y_t q_2^2$. Although we could choose the crossings of $R_\sigma$ arbitrarily, the choice we make above is that the two red strands cross above all other ones, then the two blue strands next to the red strands cross above all remaining ones, then the two blue strands next to the previous blue strands cross etc. In virtue of Figure 12, we may move the black dots to the very right of the picture above, obtaining the braid below:

![Figure 19.](image)

Then we pull the red strands as far up as possible, and notice that the blue strands are all unlinked, thus yielding the braid in Figure 20 below.
The red strands in Figure 20 correspond to the endomorphism:

\[ R(st) \cdot (st)Y(st) \cdot R_{st}^{-1} \in \text{End}(V^{\otimes 2})(y_s, y_t) \]

which we may equate with \( Y \) due to the braid equivalences described above. \( \square \)

4.13. The shuffle algebra \( \mathcal{A}^+ \) has a “vertical” and a “horizontal” grading:

\[ N \ni v \text{deg } f(z_1, ..., z_k)E_{i_1j_1} \otimes ... \otimes E_{i_kj_k} = k \]

(4.19)  \[ Z^n \ni h \text{deg } f(z_1, ..., z_k)E_{i_1j_1} \otimes ... \otimes E_{i_kj_k} = (\text{hom } \text{deg } f)\delta + \sum_{a=1}^{k} \text{deg } E_{i_a j_a} \]

where \( \delta = (1, ..., 1) \) and the grading on \( \text{End}(V) \) is defined by:

(4.20)  \[ \text{deg } E_{ij} = -[i; j] \]

We will find it convenient to extend the notation \( E_{ij} \) to all \( i, j \in \mathbb{Z} \), according to:

(4.21)  \[ E_{ij} = E_{\bar{i} \bar{j}} z_{\lfloor \frac{i-1}{n} \rfloor}^{-1} \bar{z}_{\lfloor \frac{j-1}{n} \rfloor}^{-1} \in \text{End}(V)[z^\pm 1] \]

where \( \bar{i} \) denotes the residue class of \( i \) in the set \( \{1, ..., n\} \). Then the grading (4.20) makes sense for arbitrary integer indices \( E_{i_a j_a} \), and formula (4.21) also makes sense for all integers \( i, j \). We will denote the graded pieces of the shuffle algebra by:

\[ \mathcal{A}^+ = \bigoplus_{k=0}^{\infty} \mathcal{A}_k, \quad \mathcal{A}_k = \bigoplus_{d \in \mathbb{Z}^n} \mathcal{A}_{d,k} \]

and refer to \( (d, k) \) as the degree of homogeneous elements. Finally, we write:

\[ |d| = d_1 + ... + d_n \]

for any \( d = (d_1, ..., d_n) \in \mathbb{Z} \) and refer to the number:

(4.24)  \[ \mu = \left| (\text{hom } \text{deg } f)\delta + \sum_{a=1}^{k} \text{deg } E_{i_a j_a} \right| \in \mathbb{Q} \]

10More generally, we will extend the notation above to a \( k \)-fold tensor, by the rule:

(4.23)  \[ E_{i_1 j_1} \otimes ... \otimes E_{i_k j_k} = E_{\bar{i}_1 \bar{j}_1} \otimes ... \otimes E_{\bar{i}_k \bar{j}_k} z_{\lfloor \frac{i_1-1}{n} \rfloor}^{-1} \bar{z}_{\lfloor \frac{j_1-1}{n} \rfloor}^{-1} ... z_{\lfloor \frac{i_k-1}{n} \rfloor}^{-1} \bar{z}_{\lfloor \frac{j_k-1}{n} \rfloor}^{-1} \]

as elements of \( \text{End}(V^{\otimes k})[z_{\pm 1}^1, ..., z_{\pm 1}^k] \)
as the slope of the matrix-valued rational function \( f(z_1, \ldots, z_k)E_{i_1j_1} \otimes \ldots \otimes E_{i_kj_k} \).

We will consider the partial ordering on \( \mathbb{Z}^n \) given by:

\[
(d_1, \ldots, d_n) \leq (d'_1, \ldots, d'_n) \text{ if } \begin{cases} 
  d_n < d'_n \\
  d_n = d'_n \text{ and } \sum_{i=1}^{n-1} d_i \leq \sum_{i=1}^{n-1} d'_i
\end{cases}
\]

5. THE EXTENDED SHUFFLE ALGEBRA WITH SPECTRAL PARAMETER

5.1. We will now replicate the construction of Subsection 2.7 in the situation of the \( R \)-matrix with spectral parameter (3.87).

**Definition 5.2.** Consider the extended shuffle algebra:

\[
A^+ = \left( A^+, s_{[i;j]} \right)_{1 \leq i \leq n}^{i \leq j} / \text{relations (5.1) and (5.2)}
\]

In order to concisely state the relations, it makes sense to package the new generators \( s_{[i;j]} \) into the following matrix-valued generating function:

\[
S(x) = \sum_{1 \leq i,j \leq n, \ d \geq 0} s_{[i;j]+nd} \otimes \frac{E_{ij}}{x^d} \in \bar{A}^+ \otimes \text{End}(V)[[x^{-1}]]
\]

We impose the following analogues of relations (2.17) and (2.19):

\[
(5.1) \quad R \left( \frac{x}{y} \right) S_1(x)S_2(y) = S_2(y)S_1(x)R \left( \frac{x}{y} \right)
\]

\[
(5.2) \quad X \cdot S_0(y) = S_0(y) \cdot \frac{R_{k0} \left( \frac{z_k}{y} \right) \ldots R_{10} \left( \frac{z_1}{y} \right)}{f \left( \frac{z_k}{y} \right) \ldots f \left( \frac{z_1}{y} \right)} X \bar{R}_{10} \left( \frac{z_1}{y} \right) \ldots \bar{R}_{k0} \left( \frac{z_k}{y} \right)
\]

for any \( X = X_{1\ldots k}(z_1, \ldots, z_k) \in A^+ \subset \bar{A}^+ \).

Note that the grading on \( A^+ \) extends to one on \( \bar{A}^+ \), by setting:

\[
\deg s_{[i;j]} = ([i;j], 0) \in \mathbb{Z}^n \times \mathbb{N}
\]

5.3. We may also consider the elements \( t_{[i;j]} \in \bar{A}^+ \) defined by (3.106), where:

\[
(5.3) \quad T(x) = \sum_{1 \leq i,j \leq n, \ d \geq 0} t_{[i;j]+nd} \otimes \frac{E_{ij}}{x^d}
\]

Then it is a straightforward computation (which we leave as an exercise to the interested reader) to see that (5.1), (5.2) imply analogues of (2.17), (2.18), (2.20):

\[
(5.4) \quad T_1(x)T_2(y)R \left( \frac{x}{y} \right) = R \left( \frac{x}{y} \right) T_2(y)T_1(x)
\]

\[
(5.5) \quad T_1(x)\bar{R} \left( \frac{x}{y} \right) S_2(y) = S_2(y)\bar{R} \left( \frac{x}{y} \right) T_1(x)
\]
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\[(5.6) \quad T_0(y) \cdot X = \tilde{R}_{0k} \left( \frac{y}{z_k} \right) \ldots \tilde{R}_{01} \left( \frac{y}{z_1} \right) X \frac{R_{01} \left( \frac{w}{z_1} \right) \ldots R_{0k} \left( \frac{w}{z_k} \right)}{f \left( \frac{w}{z_1} \right) \ldots f \left( \frac{w}{z_k} \right)} \cdot T_0(y)\]

Therefore, we conclude that the series \( S(x) \) and \( T(x) \) satisfy the same relations as in Definition 2.8 (modified in order to account for the variables \( z_i \)), even though the \( s \)'s and the \( t \)'s are not independent of each other anymore. We will write:

\[s_{i;i}^{-1} = \psi_i = t_{i;i}\]

and note that formulas (5.1) imply that:

\[(5.7) \quad \psi_i \psi_j = \psi_j \psi_i\]

\[(5.8) \quad \psi_s X = q^{-\langle \text{deg } X, \varsigma^* \rangle} X \psi_s\]

\[\forall X \in \tilde{A}^+, \text{where } \langle \cdot, \cdot \rangle \text{ is the bilinear form on } \mathbb{Z}^n \text{ given by } \langle \varsigma^i, \varsigma^j \rangle = \delta^j_i - \delta^{j-1}_i.\]

5.4. Consider the following topological coproduct on the algebra \( \tilde{A}^+ \), which is the natural analogue of the coproduct studied in Proposition 2.9

\[(5.9) \quad \Delta(S(x)) = (1 \otimes S(x)) \cdot (S(x) \otimes 1)\]

(hence \( \Delta(T(x)) = (T(x) \otimes 1) \cdot (1 \otimes T(x)) \)) by \( (3.106) \)) and:

\[(5.10) \quad \Delta(X_{1...k}) = \sum_{i=0}^{k} (S_i(z_k) \ldots S_{i+1}(z_{i+1}) \otimes 1) \cdot \left( X_{1...i}(z_1, ..., z_i) \otimes X_{i+1...k}(z_{i+1}, ..., z_k) \right) \cdot \left( T_{i+1}(z_{i+1}) \ldots T_k(z_k) \otimes 1 \right) \cdot \frac{\prod_{1 \leq u \leq i \leq v \leq k} f \left( \frac{z_u}{z_v} \right)}{\prod_{1 \leq u \leq i \leq v \leq k} f \left( \frac{z_u}{z_v} \right)}\]

for all \( X_{1...k}(z_1, ..., z_k) \in A^+ \subset \tilde{A}^+ \). The fact that \( \Delta \) defined above gives rise to a coassociative coalgebra structure which respects the algebra structure is proved by analogy with Proposition 2.9 and we leave the details to the interested reader.

Remark 5.5. Because \( S(x) \) is a power series in \( x \), the coproduct defined above takes values in a completion of \( \tilde{A}^+ \otimes \tilde{A}^+ \). Specifically, to make sense of the second line of (5.10), we must expand the rational function:

\[\frac{X_{1...k}(z_1, ..., z_k)}{\prod_{1 \leq u \leq i \leq v \leq k} f \left( \frac{z_u}{z_v} \right)}\]

for \( z_1, ..., z_i \ll z_{i+1}, ..., z_k \), then collect all tensors of the form \( X_{1...i}(z_1, ..., z_i) \) to the left of \( \otimes \), and all tensors of the form \( X_{i+1,...,k}(z_{i+1}, ..., z_k) \) to the right of \( \otimes \).
5.6. Given \( X \in \mathcal{A} \), we will represent its degree \( \text{deg} X = (d, k) \) on a 2 dimensional lattice, via the projection \((d, k) \rightarrow ([d], k)\), and hence assign to \( X \) the lattice point \(([d], k)\). Similarly, we will assign to the tensor \( X_1 \otimes X_2 \) the two-segment path:

![Diagram of tensor with two-segment path]

The intersection of the arrows, namely \(([d_2], k_2)\), is called the hinge of \( X_1 \otimes X_2 \).

**Definition 5.7.** Let \( \mu \in \mathbb{Q} \). We let \( \mathcal{A}_{\leq \mu}^+ \subset \mathcal{A}^+ \) be the set of those \( X \) such that:

\[
\Delta(X) = \Delta_{\mu}(X) + (\text{anything}) \otimes (\text{slope} < \mu)
\]

where \( \Delta_{\mu}(X) \) consists only of summands \( X_1 \otimes X_2 \) with slope \( X_2 = \mu \), as in (4.24). In terms of the pictorial definitions of hinges above, \( X \in \mathcal{A}_{\leq \mu}^+ \) if and only if all summands in \( \Delta(X) \) have hinge at slope \( |d|/k \leq \mu \) as measured from the origin.

It is easy to see that \( \mathcal{A}_{\leq \mu}^+ \) is a vector space. Let us define its graded pieces:

\[
\mathcal{A}_{\leq \mu|d,k} = \mathcal{A}_{\leq \mu}^+ \cap \mathcal{A}_{d,k}, \quad \mathcal{A}_{\leq \mu|d,k} = \mathcal{A}_{\leq \mu}^+ \cap \mathcal{A}_{d,k}
\]

and note that \( \mathcal{A}_{\leq \mu|d,k} \neq 0 \) only if \( |d| \leq k\mu \).

**Proposition 5.8.** For any \( \mu \in \mathbb{Q} \), the subspace \( \mathcal{A}_{\leq \mu}^+ \) is a subalgebra of \( \mathcal{A}^+ \), and:

\[
\Delta_{\mu}(X \ast Y) = \Delta_{\mu}(X) \ast \Delta_{\mu}(Y)
\]

for all \( X, Y \in \mathcal{A}_{\leq \mu}^+ \). We call \( \mathcal{A}_{\leq \mu}^+ \) a **slope subalgebra**.

**Proof.** Note that degree is multiplicative, i.e. (assume the LHS is non-zero):

\[
\text{deg} \left( f(z_1, \ldots, z_k)E_{i_1j_1} \otimes \ldots \otimes E_{i_kj_k} \right) \left( f'(z_1, \ldots, z_k)E_{i_1'j_1'} \otimes \ldots \otimes E_{i_k'j_k'} \right) = \\
\text{deg} f(z_1, \ldots, z_k)E_{i_1j_1} \otimes \ldots \otimes E_{i_kj_k} + \text{deg} f'(z_1, \ldots, z_k)E_{i_1'j_1'} \otimes \ldots \otimes E_{i_k'j_k'}
\]

Therefore, if \( \Delta(X) = X_1 \otimes X_2 \) and \( \Delta(Y) = Y_1 \otimes Y_2 \) with slope \( X_2 \), slope \( Y_2 \leq \mu \), then slope \( X_2Y_2 \leq \mu \). Since \( \Delta(XY) = X_1Y_1 \otimes X_2Y_2 \), this implies the conclusion.

5.9. Our reason for introducing the slope subalgebras is that \( \{ \mathcal{A}_{\leq \mu|d,k} \}_{\mu \in \mathbb{Q}} \) yield a filtration of \( \mathcal{A}_{d,k} \) by finite-dimensional vector spaces.
Lemma 5.10. The dimension of $A_{\leq \mu; d,k}$ as a vector space over $\mathbb{Q}(q, q^{1/2})$ is at most the number of unordered collections:

\[
(i_1, j_1, \lambda_1), \ldots, (i_u, j_u, \lambda_u)
\]

where:

- $\lambda_s \in \mathbb{N}$ with $\sum_{s=1}^{u} \lambda_s = k$
- $(i_s, j_s) \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}$ with $\sum_{s=1}^{u}(i_s; j_s) = d$
- $j_s - i_s \leq \mu \lambda_s$ for all $s \in \{1, \ldots, u\}$

In [5.13], we will show that the dimension of $A_{\leq \mu; d,k}$ is in fact equal to the number of unordered collections (5.13). The argument below follows that of [8, 15, 16].

Proof. To any partition $\lambda = (\lambda_1 \leq \ldots \leq \lambda_u)$ of $k \in \mathbb{N}$, we associate the linear map:

\[
A_{\leq \mu; d,k} \xrightarrow{\varphi_\lambda} \text{End}(V^{\otimes u})(y_1, \ldots, y_u)
\]

\[
X \mapsto X^{(\lambda_1, \ldots, \lambda_u)}
\]

Consider the dominance ordering $\lambda' > \lambda$ on partitions, and define:

\[
A_{\leq \mu; d,k}^\lambda = \bigcap_{\lambda' > \lambda} \ker \varphi_{\lambda'}
\]

Since $A_{\leq \mu; d,k}^\lambda = A_{\leq \mu; d,k}$, then the desired bound on $\dim A_{\leq \mu; d,k}$ would follow from:

\[
\dim \varphi_\lambda\left(A_{\leq \mu; d,k}^\lambda\right) \leq \# \left\{ \text{unordered } (i_1, j_1), \ldots, (i_u, j_u) \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}, \right. \quad \left. \text{such that } \sum_{s=1}^{u}(i_s; j_s) = d \text{ and } j_s - i_s \leq \mu \lambda_s \text{ for all } s \in \{1, \ldots, u\} \right\}
\]

[11] for any $\lambda = (\lambda_1 \leq \ldots \leq \lambda_u)$. By Proposition 4.12 any $Y \in \text{Im } \varphi_\lambda$ is of the form:

\[
Y(y_1, \ldots, y_u) = \frac{\text{End}(V^{\otimes u})(y_1^{\pm 1}, \ldots, y_u^{\pm 1})}{\prod_{1 \leq s < t \leq u} \prod_{d=0}^{\lambda_s - 1} (y_s q^{2d} - y_t q^{-2})(y_s q^{2d} - y_t q^{2\lambda_t})} X
\]

However, if $Y = \varphi_\lambda(X)$ for some $X \in A_{\leq \mu; d,k}^\lambda$, we claim that $Y$ is a Laurent polynomial. Indeed, let us show that $Y$ is regular at $y_s q^{2d} - y_t q^{-2}$. We have:

\[
\text{Res}_{y_s q^{2d} = y_t q^{-2}} \left( \frac{\text{Res}_{z_{r'} = y_{r}, \ldots, z_{r'+1} = y_{r} q^{2(\lambda_{r'} - 1)}} X}{z_{r'} = y_{r}, \ldots, z_{r'+1} = y_{r} q^{2(\lambda_{r'} - 1)}} \right)_{r' \in \{1, \ldots, u\}} = \text{Res}_{y_s q^{2d} = y_t q^{-2}} \left( \frac{\text{Res}_{z_{r'} = y_{r}, \ldots, z_{r'+1} = y_{r} q^{2(\lambda_{r'} - 1)}} X}{z_{r'} = y_{r}, \ldots, z_{r'+1} = y_{r} q^{2(\lambda_{r'} - 1)}} \right)_{r' \in \{1, \ldots, u\}}
\]

where $\lambda'$ is obtained from $\lambda$ by replacing $\lambda_s$ and $\lambda_t$ by $\lambda_s - d - 1$ and $\lambda_t + d + 1$. The right-hand side of the expression above vanishes because $X \in A_{\leq \mu; d,k}^\lambda$ and $\lambda' > \lambda$.

[11] Above, the word “unordered” means that we identify pairs $(i_a, j_a) = (i_b, j_b)$ iff $\lambda_a = \lambda_b$. 
Similarly, one can show that the residue of \( Y \) vanishes at \( y_s q^{2d} - y_t q^{2\lambda_s} \) for any \( s < t \) and \( 0 \leq d < \lambda_s \) and this precisely implies that \( Y \) is a Laurent polynomial:

\[
Y(y_1, \ldots, y_n) = \sum_{1 \leq \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \leq n} \text{coefficient} \cdot y_1^{\alpha_1} \ldots y_n^{\alpha_n} E_{\alpha_1\beta_1} \otimes \ldots \otimes E_{\alpha_n\beta_n}
\]

Since the matrices \( R \) and \( \tilde{R} \) have total degree 0, the horizontal degree of \( Y \) is equal to that of \( X \), namely \( \mathbf{d} \), so we conclude that the only summands with non-zero coefficient in \((5.16)\) satisfy:

\[
(h_1 + \ldots + h_n)\delta + \deg E_{\alpha_1\beta_1} + \ldots + \deg E_{\alpha_n\beta_n} = \mathbf{d}
\]

Finally, the slope condition on \( X \) implies an analogous slope condition on \( Y \); in each variable \( y_s \), we have:

\[
 nh_s + \alpha_s - \beta_s \leq \mu \lambda_s
\]

Therefore, the number of coefficients that one gets to choose in \((5.16)\) is at most the number of collections \((j_s = \alpha_s + nh_s, i_s = \beta_s)\) satisfying the conditions in the right-hand side of \((5.14)\). The reason why we need to take unordered collections is the symmetry property of \( Y \) proved in Proposition \([4.12]\).

\[
\square
\]

5.11. The defining property \((4.13)\) of the shuffle algebra allows us to associate to \( X = X_{1\ldots\lambda}(z_1, \ldots, z_k) \in A^+ \) the matrix-valued power series \( X^{(k)}(y) \in \text{End}(V)[y^{\pm 1}] \).

\textbf{Proposition 5.12.} For any \( A \in A_{d,k} \) and \( B \in A_{e,t} \), we have:

\[
(A \ast B)^{(k+l)}(y) = A^{(k)}(y)B^{(l)}(y)q^{2ke_n}
\]

where \( e_n \) is the last component of the vector \( \mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n \).

\textbf{Proof.} The proof is precisely the \( u = 1 \) case of the proof of Proposition \([4.11]\) since the equality of braids therein indicates the fact that:

\[
(A \ast B)^{(k+l)}(y) = A^{(k)}(y)B^{(l)}(y)q^{2ke_k}
\]

The fact that \( B \in A_{e,t} \) implies the homogeneity property:

\[
B(z_1 \xi, \ldots, z_k \xi) = \xi^{e_n} B(z_1, \ldots, z_k)
\]

Since \( R \)-matrices are invariant under rescaling variables, the rational function \( B^{(l)}(y) \) of \((4.13)\) also satisfies \( B^{(l)}(y\xi) = \xi^{e_n} B^{(l)}(y) \). Then \((5.18)\) implies \((5.17)\).

\[
\square
\]

For all \((i, j) \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}\), define the linear maps:

\[
\bigoplus_{k=0}^{\infty} A_{(i,j),k} \xrightarrow{\alpha_{(i,j)}} \mathbb{Q}(q, q^{\frac{1}{2}})
\]

\[
X_{1\ldots k}(z_1, \ldots, z_k) \xrightarrow{\alpha_{(i,j)}} \text{coefficient of } E_{ji} \text{ in } X^{(k)}(y)(1 - q^2)^{k}q^{\frac{h_{(i,j)} + (i-j) + k - 2k\delta}{n}}
\]

(recall that \( E_{ij} = \frac{E_{ij}}{y^{\frac{i-j}{n}}[\frac{n}{2}]} \)). As a consequence of Proposition \([5.12]\) we have:
Corollary 5.13. For any $k, l \in \mathbb{N}$ and $(i, j) \in \frac{\mathbb{Z}^2}{(m, n)^2}$, we have:
\begin{equation}
(5.20) \quad a_{[i,j]}(A \ast B) = a_{[i,j]}(A) a_{[i,j]}(B) \cdot q^{\frac{(e-s) - (i-j)}{m}}
\end{equation}
whenever $\deg A = ([s; j), k)$ and $\deg B = ([i; s), l)$ for some $s$ between $i$ and $j$. If such an $s$ does not exist, then the RHS of (5.20) is set equal to 0, by convention.

Proof. The corollary is an immediate consequence of (5.17) and (5.19). The only thing we need to check is that the power of $q$ is the same in the left as in the right-hand sides of (5.20), which happens due to the elementary identity:
\[ q^{(k+1)(l+1) - 2(k+l+1)} = q^{\frac{(e-s)-2i}{n}} \cdot q^{\frac{(e-s)-2j}{n}} - 2k([\frac{e-s+1}{n}] - [\frac{i-j+1}{n}]) q^{\frac{(e-s) - (i-j)}{n}} \]
since if $e = [i; s)$, then $e_n = \lfloor \frac{e-1}{n} \rfloor - \lfloor \frac{i-j+1}{n} \rfloor$.

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□
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5.14. Let $B_{\mu|d} = A_{\mu|d, \frac{d}{n}}$. Particular importance will be given to the subalgebra:
\begin{equation}
(5.21) \quad B_{\mu}^+ = \bigoplus_{d \in \mathbb{Z}^n} B_{\mu|d}
\end{equation}
As a consequence of Proposition 5.8, the leading order term $\Delta_{\mu}$ of (5.11) restricts to a coproduct on the enhanced subalgebra:
\begin{equation}
(5.22) \quad B_{\mu}^+ = \left\langle B_{\mu}^+ , \psi_{\frac{d}{n}}^{\pm 1} \right\rangle_{s \in \{1, \ldots, n\}} / \text{relations (5.7), (5.8)}
\end{equation}

Lemma 5.15. If $X \in B_{\mu}^+$ is primitive with respect to the coproduct $\Delta_{\mu}$, and:
\begin{equation}
(5.23) \quad a_{[i,j]}(X) = 0
\end{equation}
for all $(i, j) \in \frac{\mathbb{Z}^2}{(m, n)^2}$ such that $\deg X \in [i; j) \times \mathbb{N}$, then $X = 0$.

Proof. The assumption $X \in B_{\mu}^+$ implies that $\deg X = (d, k)$ with:
\begin{equation}
(5.24) \quad |d| = \mu k
\end{equation}
As we observed in the proof of Lemma 5.10, it suffices to show that $\varphi_{\lambda}(X) = 0$ for all partitions $\lambda$, which we will do in reverse dominance order of the partition $\lambda$. The base case is when $\lambda = (k)$, which is satisfied because $\varphi_{(k)}(X) = 0$ is precisely the context of the assumption (5.23). For a general partition $\lambda \neq (k)$, we may invoke the induction hypothesis to conclude that $\varphi_{\lambda'}(X) = 0$ for all partitions $\lambda' > \lambda$, and in this case $\varphi_{\lambda}(X)$ takes the form of (5.16). However, the fact that $X$ is a primitive element requires every summand appearing in the RHS of (5.16) to satisfy:
\[ nh_s + \alpha_s - \beta_s < \mu \lambda_s \]
for all $1 \leq s \leq u$ (we have $u = l(\lambda) > 1$, since we are dealing with the case $\lambda \neq (k)$). However, relation (5.24) forces the following identity:
\[ \sum_{s=1}^{u} (nh_s + \alpha_s - \beta_s) = \mu k = \mu \sum_{s=1}^{u} \lambda_s \]
This yields a contradiction, hence $\varphi_{\lambda}(X) = 0$, thus completing the induction step.

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□
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5.16. We will now construct particular elements of $B_\mu^+$, which together with Lemma 5.10 will yield a PBW basis of the shuffle algebra, leading to the proof of Theorem 1.5. Consider the following notion of symmetrization, analogous to (2.8):

(5.25) $\text{Sym} \ X = \sum_{\sigma \in S(k)} R_\sigma \cdot X_{\sigma(1)...\sigma(k)}(z_{\sigma(1)}, \ldots, z_{\sigma(k)}) \cdot R_\sigma^{-1}$

where $R_\sigma$ is the product of $R_{ij}(\frac{z_i}{z_j})$ associated to any braid lift of $\sigma$. For instance:

(5.26) $R_{\omega_k}(z_1, \ldots, z_k) = \prod_{i=1}^{k-1} \prod_{j=i+1}^k R_{ij}(\frac{z_i}{z_j})$

lifts the longest element $\omega_k \in S(k)$. Consider the matrix-valued rational functions:

(5.27) $Q(x) = q^{-1} \sum_{1 \leq i,j \leq n} (xq^2)^{\delta_{i<j}} \frac{E_{ij}}{1 - xq^2} \otimes E_{ji}$

(5.28) $\bar{Q}(x) = -q \sum_{1 \leq i,j \leq n} (xq^2)^{\delta_{i<j}} \frac{E_{ij}}{1 - xq^2} \otimes E_{ji}$

which have, up to scalar, the same simple pole and residue as $\bar{R}(x)$:

(5.29) $q \cdot \text{Res}_{x=q^{-2}} Q(x) = -q^{-1} \cdot \text{Res}_{x=q^{-2}} \bar{Q}(x) = (q^{-1} - q)^{-1} \cdot \text{Res}_{x=q^{-2}} \bar{R}(x) = (12)$

(see formula (4.8)). Moreover, it is easy to check the following identity:

(5.30) $Q(x) + \bar{Q}(x) = \bar{R}(x) - \sum_{1 \leq i \neq j \leq n} E_{ii} \otimes E_{jj}$

**Proposition 5.17.** For any $(i,j) \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}$ and $\mu \in \mathbb{Q}$ such that $k = \frac{i-j}{\mu} \in \mathbb{N}$, set:

(5.31) $F^\mu_{(i,j)} = F^{(k)}_{|i;j|} := \text{Sym} \ R_{\omega_k}(z_1, \ldots, z_k)$

\[
\prod_{a=1}^k \left[ \bar{R}_{1a} \left( \frac{z_1}{z_a} \right) \ldots \bar{R}_{a-2,a} \left( \frac{z_a-2}{z_a} \right) Q_{a-1,a} \left( \frac{z_a-1}{z_a} \right) E_{s_a-1,s_a} \left( \frac{z}{q^{n-a}} \right) \right]
\]

(5.32) $\bar{F}^\mu_{(i,j)} = \bar{F}^{(k)}_{|i;j|} := (-q^2 q^2)^{-k} \cdot \text{Sym} \ R_{\omega_k}(z_1, \ldots, z_k)$

\[
\prod_{a=1}^k \left[ \bar{R}_{1a} \left( \frac{z_1}{z_a} \right) \ldots \bar{R}_{a-2,a} \left( \frac{z_a-2}{z_a} \right) Q_{a-1,a} \left( \frac{z_a-1}{z_a} \right) E_{s_a-1,s_a} \left( \frac{z}{q^{n-a}} \right) \right]
\]

(recall (4.22)) where $s_a = j - \lfloor \mu a \rfloor$, $s'_a = j - \lfloor \mu a \rfloor$. Then $F^\mu_{(i,j)}, \bar{F}^\mu_{(i,j)} \in B_\mu^+$, and:

(5.33) $\Delta_{\mu} \left( F^\mu_{(i,j)} \right) = \sum_{s \in \{i, \ldots, j\}} F^\mu_{[s,i]} \otimes F^\mu_{[s,j]}$

(5.34) $\Delta_{\mu} \left( \bar{F}^\mu_{(i,j)} \right) = \sum_{s \in \{i, \ldots, j\}} \bar{F}^\mu_{[s,i]} \otimes \bar{F}^\mu_{[s,j]}$

where we set $F^\mu_{(i,j)} = \bar{F}^\mu_{(i,j)} = 0$ if $\frac{i-j}{\mu} \notin \mathbb{N}$.
Proof. Note that if \( Q, \bar{Q} \) were replaced by \( \tilde{R} \) in formulas (5.31), (5.32), then the right-hand sides of the aforementioned formulas would precisely equal:

\[
E_{s_0 \delta_1 \frac{\sigma}{\bar{\sigma}}} \bar{\sigma} \tau_{\bar{\sigma}} \cdots \bar{E}_{s_k-1 \epsilon_k \frac{\sigma}{\bar{\sigma}}} \tau_{\bar{\sigma}} \quad \text{and} \quad E_{s_0' \delta_1' \frac{\sigma}{\bar{\sigma}}} \bar{\sigma} \tau_{\bar{\sigma}} \cdots \bar{E}_{s_k' - 1 \epsilon_k' \frac{\sigma}{\bar{\sigma}}} \tau_{\bar{\sigma}}
\]

The fact that the shuffle elements (5.35) satisfy the wheel conditions is simply a consequence of iterating Proposition 4.11 a number of \( k-1 \) times. As far as the elements \( F_{\mu_{(i,j)}}^\nu, F_{\nu_{(i,j)}}^\mu \) are concerned, the fact that they satisfy the wheel conditions is proved similarly with the fact that (5.35) satisfies the wheel conditions: this is because Proposition 4.11 uses the fact that \( \text{Res}_{x=\cdots} \bar{R}(x) \) is a multiple of the permutation matrix, and we have already seen in (5.29) that the residues of \( Q(x), \bar{Q}(x), \bar{R}(x) \) are the same up to scalar. We leave the details to the interested reader.

Let us prove that the shuffle elements (5.31) and (5.32) lie in \( B_n^+ \). We will only prove the former case, since the latter case is analogous. We have:

\[
F := F_{\mu_{(i,j)}}^\nu = \text{Sym} R_{\omega_k} (z_1, \ldots, z_k) X_{1 \ldots k} (z_1, \ldots, z_k)
\]

where \( X \) is the expression on the second line of (5.31). For any \( l \in \{0, \ldots, k\} \) we need to look at the first \( l \) tensor factors of \( \text{Sym} R_{\omega_k} X \) and isolate the terms of minimal \( \text{hdeg} \). If \( Y \) is a \( k \)-tensor, we will henceforth use the phrase “initial degree of \( Y \)” instead of “total \( \text{hdeg} \) of the first \( l \) factors of \( Y \)”. Because:

\[
\lim_{x \to 0} R_{ab}(x) = \sum_{i,j=1}^n q^{-1} E_{ii}^a \otimes E_{jj}^b + \sum_{i>j} (q-q^{-1}) E_{ij}^a \otimes E_{ji}^b
\]

\[
\lim_{x \to \infty} R_{ab}(x) = \sum_{i,j=1}^n q^{-1} E_{ii}^a \otimes E_{jj}^b - \sum_{i<j} (q-q^{-1}) E_{ij}^a \otimes E_{ji}^b
\]

for all indices \( a \) and \( b \), we obtain the following easy (but very useful) fact:

Claim 5.18. For any \( 1 \leq a \neq b \leq k \), multiplying a \( k \)-tensor \( Y \) by:

\[
R_{ab} \left( \frac{z_a}{z_b} \right) \quad \text{or} \quad \bar{R}_{ab} \left( \frac{z_a}{z_b} \right) \quad \text{or} \quad Q_{ab} \left( \frac{z_a}{z_b} \right) \quad \text{or} \quad \bar{Q}_{ab} \left( \frac{z_a}{z_b} \right)
\]

(either on the left or on the right) cannot decrease the minimal initial degree of \( Y \).

Therefore, it suffices to compute the minimal initial degree of:

\[
X_{\sigma(1) \ldots \sigma(k)} (z_{\sigma(1)}, \ldots, z_{\sigma(k)}) = \prod_{a=1}^k \bar{R}_{\sigma(1) \sigma(a)} (\frac{z_{\sigma(1)}}{z_{\sigma(a)}}) \ldots \bar{R}_{\sigma(a-1) \sigma(a)} (\frac{z_{\sigma(a-1)}}{z_{\sigma(a)}}) Q_{\sigma(a-1) \sigma(a)} (\frac{z_{\sigma(a-1)}}{z_{\sigma(a)}}) E_{\sigma(a)} (\frac{z_{\sigma(a)}}{z_{\sigma(a)}}) \bar{\sigma} \tau_{\bar{\sigma}}
\]

for any permutation \( \sigma \) of \( \{1, \ldots, k\} \). Claim 5.18 implies that the minimal initial degree (henceforth denoted “m.i.d.”) comes from the various \( E_{s_{a-1} s_a} \) factors:

\[
\text{m.i.d. of (5.39)} = \sum_{a \in A} (s_{a-1} - s_a) + \#
\]
where $A = \{\sigma^{-1}(1), \ldots, \sigma^{-1}(l)\}$ and the number $\#$ counts those $a \in A$ such that $a - 1 \not\in A$. This number must be added to the minimal initial degree because $Q(\infty)$ has 0 on the diagonal, by definition. It is elementary to show that:

$$\text{RHS of (5.40)} = \sum_{a \in A} \left( [\mu a] - [\mu (a-1)] \right) + \# \geq \sum_{s=1}^{t} \left( [\mu \beta_s] - [\mu \alpha_s] \right)$$

if $A$ splits up into consecutive blocks of integers:

$$A = \{\alpha_1 + 1, \ldots, \beta_1, \alpha_2 + 1, \ldots, \beta_2, \ldots, \alpha_t + 1, \ldots, \beta_t\}$$

with $0 \leq \alpha_1$, while $\beta_s < \alpha_{s+1}$ for all $s$, and $\beta_t \leq k$. Since:

$$\text{RHS of (5.41)} \geq \mu \sum_{s=1}^{t} (\beta_s - \alpha_s) = \mu \cdot \# A = \mu \# A$$

we conclude that $F \in \mathcal{A}_{\mu}^+$. Because $\text{hdeg } F = j-i$ and $\text{vdeg } F = k$, then $F \in \mathcal{B}_{\mu}^+$.

Moreover, the terms of minimal initial degree in $F$ correspond to those situations where we have equality in all the inequalities above, and these require $\mu l \in \mathbb{Z}$ and:

$$A = \{1, \ldots, l\}$$

Let us now compute the summands which achieve the minimal initial degree in:

$$R_{\omega k} X = R_{\omega l}(z_1, \ldots, z_l) \left[ R_{1, l+1} \left( \frac{z_1}{z_{l+1}} \right) \ldots R_{l, k} \left( \frac{z_l}{z_k} \right) \right]$$

$$R_{\omega_{k-l}}(z_{l+1}, \ldots, z_k) E_{s_{l|\ldots|s_k}}^{(1, \ldots, l)} \left[ \tilde{R}_{1, l+1} \left( \frac{z_1}{z_{l+1}} \right) \ldots \tilde{R}_{l, k} \left( \frac{z_l}{z_k} \right) \right] E_{s_{l|\ldots|s_k}}^{(l+1, \ldots, k)}$$

where the notation $E_{u_{c_{a-1}} c_{a-1}}^{(u_{c_{a-1}})}$ is shorthand for:

$$\prod_{a=1}^{u} \left[ \tilde{R}_{a, a} \left( \frac{z_a}{z_a} \right) \ldots \tilde{R}_{a-2, a} \left( \frac{z_{a-2}}{z_a} \right) Q_{a-1, a} \left( \frac{z_{a-1}}{z_a} \right) E_{c_{a-1} c_a}^{(a)} \right]$$

If the terms with the squiggly red underline were not present in (5.44), then we would conclude that the terms of minimal initial degree would be precisely:

$$\text{m.i.d. } R_{\omega k} X = R_{\omega l} E_{s_{l|\ldots|s_k}}^{(1, \ldots, l)} \otimes R_{\omega_{k-l}} E_{s_{l|\ldots|s_k}}^{(l+1, \ldots, k)}$$

and upon symmetrization, this would almost imply (5.33). However, we must deal with the contribution of the terms with the squiggly red underline. As we have seen in the discussion above (specifically (5.37) and (5.38)), these factors only contribute a diagonal matrix to the terms of minimal initial degree. Specifically, if:

$$E_{s_{l|\ldots|s_k}}^{(1, \ldots, l)} = \sum_{x_{a, y_a=1}} \text{coefficient} \cdot E_{x_1 y_1} \otimes \ldots \otimes E_{x_l y_l} \otimes 1^{\otimes k-l}$$

$$E_{s_{l|\ldots|s_k}}^{(l+1, \ldots, k)} = \sum_{x_{a, y_a=1}} \text{coefficient} \cdot 1^{\otimes l} \otimes E_{x_{l+1} y_{l+1}} \otimes \ldots \otimes E_{x_k y_k} \otimes 1^{\otimes k-l}$$
For any Proposition 5.21. under the linear functionals (5.19). Let us therefore compute the latter:

determined by the coproduct relations (5.33) and (5.34), together with their value (5.48)

Claim 5.19. The quantity (5.45) is a sum of tensors $E_{x_1 y_1} \otimes \cdots \otimes E_{x_l y_l}$, where:

$$\#\{x_u \equiv r \mod n\} - \#\{y_u \equiv r \mod n\} = \delta^r_{cu} - \delta^r_{cv}$$

for any $r \in \mathbb{Z}/n\mathbb{Z}$.

we may rewrite (5.47) as:

$$\Delta_\mu(R_{\omega_k} X) = \sum_{x_u, y_u = 1}^n \text{coefficient} \cdot \psi_{y_1} \cdots \psi_{y_k} R_{\omega_k - 1}(1 \otimes \cdots \otimes E_{x_1 y_1 + 1} \otimes \cdots \otimes E_{x_k y_k})$$

As a consequence of the following straightforward claim:

$$\Delta_\mu(R_{\omega_k} X) = \sum_{x_u, y_u = 1}^n \text{coefficient} \cdot R_{\omega_l}(E_{x_1 y_1} \otimes \cdots \otimes E_{x_l y_l} \otimes 1 \otimes \cdots \otimes 1)$$

Upon symmetrization with respect to those permutations $\sigma \in S(k)$ which preserve the set $\{1, \ldots, l\}$, this yields precisely (5.33). \hfill \Box

5.20. According to Lemma 5.15, the elements $F_{\mu}^{i,j, l}$ are completely determined by the coproduct relations (5.33) and (5.34), together with their value under the linear functionals (5.19). Let us therefore compute the latter:

**Proposition 5.21.** For any $(i, j) \in \mathbb{Z}^2/(n, n)\mathbb{Z}$ and $\mu \in \mathbb{Q}$ such that $\frac{j-1}{\mu} \in \mathbb{N}$, we have:

$$\alpha_{[u, v]} \left( F^{\mu}_{[i,j]} \right) = \delta^{(i,j)}_{(u, v)} (1 - q^2)q^{-\gcd(k, j-1)}$$

$$\alpha_{[u, v]} \left( F^{\mu}_{[i,j]} \right) = \delta^{(i,j)}_{(u, v)} (1 - q^{-2})q^{-\gcd(k, j-1)}$$

for any $(u, v) \in \mathbb{Z}^2/(n, n)\mathbb{Z}$ such that $[u; v] = [i; j]$.

**Proof.** Recall that $F = F^{\mu}_{[i,j]}$ is given by the symmetrization (5.36), namely:

$$F = \sum_{\sigma \in S(k)} R_\sigma \cdot \sigma \left( R_{\omega_k} \cdot \text{second line of (5.31)} \right) \sigma^{-1} \cdot R_\sigma^{-1}$$
where $R_\sigma$ is an arbitrary braid which lifts the permutation $\sigma$. Of the $k!$ summands in the right-hand side, only the one corresponding to the identity permutation is involved in the iterated residue of $F$ at $z_k = z_{k-1} \varpi^2, ..., z_2 = z_1 \varpi^2$, hence we obtain:

\begin{equation}
(5.50) \quad \text{Res}_{\{z_1=y, z_2=y\varpi^2, ..., z_k=y\varpi^{2k-2}\}} F = R_{\omega_k}(y, y\varpi^2, ..., y\varpi^{2k-2})
\end{equation}

\[
\prod_{a=1}^{k} \left[ R_{1a} \left( \varpi^{2-2a} \right) \cdots R_{a-2,a} \left( \varpi^{-4} \right) \cdot \left( \text{Res}_{x=\varpi^{-2}} Q_{a-1,a}(x) \right) \cdot E_{s_a-1,s_a}^{(a)}(x) \right]_{z_a=y\varpi^{2a-2}}
\]

(as $E_{ij}^{(a)} = E_{ij}^{(a)} \left[ \frac{l_{a,i}}{a} \right] - \left[ \frac{l_{a,i}}{a} \right]$, we must specialize $z_a = y\varpi^{2a-2}$ in (5.50)). Note that:

\[
R_{\omega_k}(z_1, ..., z_k) = \prod_{a=1}^{k-1} \prod_{b=a+1}^{k} R_{ab} \left( \varpi^{2a-2b} \right) = R_{12} \left( \varpi^{-2} \right) \cdots R_{1k} \left( \varpi^{2k-2} \right) \prod_{b=k}^{2} R_{ab} \left( \varpi^{2a-2b} \right)
\]

due to (4.3). Since $\bar{R}$ is given by (4.2) and $\text{Res}_{x=\varpi^{-2}} Q(x) = q^{-1} \cdot (12)$, we have:

LHS of (5.50) = $q^{1-k} \prod_{b=1}^{k} R_{b,a+1} \left( \varpi^{2b-2a} \right) \cdot \prod_{b=1}^{k} E_{s_a-1,s_a}^{(1)}(x) \cdot \left( \frac{1}{2} \cdots \frac{k}{1} \right)$

If we move the permutations ($b - 1, b$) all the way to the right, then we obtain:

LHS of (5.50) = $q^{1-k} \prod_{b=3}^{k} R_{b,a+1} \left( \varpi^{2b-2a} \right) \cdot \prod_{b=1}^{k} E_{s_a-1,s_a}^{(1)}(x) \cdot \left( \frac{1}{2} \cdots \frac{k}{1} \right)$

\[
= q^{1-k} \prod_{2 \leq a < b \leq k} f \left( \varpi^{2a-2b} \right) \cdot R_{12} \left( \varpi^{-2} \right) \cdots R_{1k} \left( \varpi^{-2k} \right) \cdot (E_{ij} \otimes 1 \otimes ... \otimes 1) \sum_{b=1}^{k} \left( \varpi^{2b-2k} \right) \left( \frac{2b-2k}{n} - \left[ \frac{2b-2k}{n} \right] + \frac{\varpi}{n} \right) \cdot \left( \frac{1}{2} \cdots \frac{k}{1} \right)
\]

With this in mind, (4.13) implies that:

\[
F^{(k)} = E_{ji} \cdot q^{1-k} \prod_{b=1}^{k} \left( \frac{1}{2} \cdots \frac{k}{1} \right) \sum_{b=1}^{k} \left( \varpi^{2b-2k} \right) \left( \frac{2b-2k}{n} - \left[ \frac{2b-2k}{n} \right] + \frac{\varpi}{n} \right) \cdot \left( \frac{1}{2} \cdots \frac{k}{1} \right)
\]

Given formula (5.19) and the elementary identity:

\[
\sum_{a=1}^{k} \left\lfloor \frac{ad}{k} \right\rfloor = \frac{dk + d + k - \text{gcd}(d,k)}{2}
\]

we conclude (5.48). Formula (5.49) is proved analogously. □
5.22. Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ be coprime, set $g = \gcd(n, a)$ and:

\[
\mu = \frac{a}{b}
\]

Recall the discussion in Subsection 3.22, in which the algebra:

\[
E_\mu = U_q(\mathfrak{gl}_n)^{\otimes g}
\]

was made to be $\mathbb{Z}^n \times \mathbb{Z}$ graded, and its root vectors were indexed as:

\[
f_\mu[i; j], \quad \forall (i, j) \in \mathbb{Z}^2/(n, n)\mathbb{Z}
\]

Comparing (3.21) and (3.32) with (5.33) and (5.48), respectively, Lemma 3.41 implies that there exists a bialgebra homomorphism:

\[
E_\mu^+ \xrightarrow{\Upsilon_\mu} B_\mu^+, \quad f_\mu[i; j] \mapsto F_\mu[i; j]
\]

Lemma 5.10 for $|d| = k\mu$ implies that:

\[
\dim B_\mu|d = \# \{\text{partitions } d = [i_1; j_1] + \ldots + [i_u; j_u] \text{ s.t. } j_a - i_a \in \mu \mathbb{N}, \forall a \in \{1, \ldots, u\}\}
\]

The right-hand side above is precisely equal to the dimension of the algebra $E_\mu^+$ in degree $d$, so Corollary 3.42 implies:

**Proposition 5.23.** For any $\mu \in \mathbb{Q}$, the map $\Upsilon_\mu : E_\mu^+ \rightarrow B_\mu^+$ is an isomorphism.

5.24. Since $B_\mu^+$ is isomorphic to the algebra $E_\mu^+$, we may present it instead in terms of simple and imaginary generators (see Subsection 3.25 for the notation):

\[
P_{[i; j]}^\mu = \Upsilon_\mu\left(p_{[i; j]}^\mu\right)
\]

\[
P_{l; \delta, r}^\mu = \Upsilon_\mu\left(p_{l; \delta, r}^\mu\right)
\]

Since $\Upsilon_\mu$ preserves the maps $\alpha_{[u; v]}$, we have:

\[
\alpha_{[u; v]}\left(p_{[i; j]}^\mu\right) = \delta_{(u, v)}^{(i, j)}
\]

\[
\alpha_{[u; u+n]}\left(p_{l; \delta, r}^\mu\right) = \delta_{u \mod g}^r
\]

for all $(u, v) \in \frac{\mathbb{Z}^2}{(n, n)\mathbb{Z}}$. By (5.48) and (5.49), the simple generators are given by:

\[
P_{[i; j]}^\mu = \frac{F_{[i; j]}^\mu}{q^{\frac{a}{b}}(1 - q^2)} = \frac{\tilde{F}_{[i; j]}^\mu}{q^{-\frac{a}{b}}(1 - q^{-2})}
\]

if $\gcd(j - i, \mu(j - i)) = 1$, but we do not know a closed formula for the imaginary generators (5.56). We will sometimes use the notation:

\[
P_{[i; j]}^{(k)} = P_{[i; j]}^\mu, \quad P_{l; \delta, r}^{(k)} = P_{l; \delta, r}^\mu
\]

if $j - i = \mu k$, $nl = \mu k'$, in order to emphasize the fact that $\deg P_{d}^{(k)} = (d, k)$. Set:

\[
P_{[i; j]}^{(k)} = P_{l; \delta, r}^{(k)}
\]

if $j = i + nl$, $\gcd(k, j - i) = 1$ and $r \equiv i \mod n$. 
Theorem 5.25. We have an algebra isomorphism:
\[ D^+ \overset{\chi^+}{\cong} A^+, \quad P_{[i;j]}^{(k)} \mapsto P_{[i;j]}^{(k)}, \quad P_{[i;j]}^{(k')} \mapsto P_{[i;j]}^{(k')} \]
where \( D^+ \) is the explicit algebra of Definition 3.26.

5.26. To prove Theorem 5.25, we need to show that the simple and imaginary generators of \( B^+_{\mu} \) satisfy the analogues of relations (3.73) and (3.74).

Proposition 5.27. Assume \( \gcd(j - i, k) = 1 \), and consider the lattice triangle \( T \):

uniquely determined as the triangle of maximal area situated completely to the right of the vector \((j - i, k)\), which does not contain any lattice points inside. Let \( \mu \) denote the slope of one of the edges of \( T \), as indicated in the pictures above. Then:
\[
\Delta \left( P_{[i;j]}^{(k)} \right) = P_{[i;j]}^{(k)} \otimes 1 + \frac{\psi_k}{\psi_j} \otimes P_{[i;j]}^{(k)} + \left( \text{tensors with hinge strictly right of } T \right) + \\
\sum_{1 \leq u < v \leq \lambda} \begin{cases} 
F_{[i;j]}^{(u)} \frac{\psi_k}{\psi_v} \otimes \bar{F}_{[i;j]}^{(u)} F_{[i;j]}^{(v)} 
& \text{for the picture on the left} \\
\psi_k \otimes P_{[u;j]}^{(v)} \frac{\psi_k}{\psi_u} \otimes \bar{F}_{[i;j]}^{(u)} F_{[i;j]}^{(v)} 
& \text{for the picture on the right} 
\end{cases}
\]
where \( \bullet = k - \frac{j - i + u - v}{\mu} \).

Proof. Due to (5.57), we may replace all \( P \)'s by \( F \)'s in formula (5.59). We will refine the proof of Proposition 5.17 so we will freely adapt the notations therein: let \( F \) and \( X \) be given by (5.36). The summands of \( \Delta(F) \) with leftmost possible hinge correspond to those tensors of minimal initial degree, which is:
\[
\text{minimal initial degree of } F \geq x := \sum_{s=1}^{t} \left( \left( \frac{(j - i)\beta_s}{k} \right) - \left( \frac{(j - i)\alpha_s}{k} \right) \right) 
\]
(as in (5.41)) If we let \( y := \sum_{s=1}^{t} \beta_s - \alpha_s = \#A \), where \( A = \{ \sigma^{-1}(1), \ldots, \sigma^{-1}(l) \} \), then it is straightforward to see that the lattice point \((x, y)\) lies on the boundary or to the right of the lattice triangle \( T \). This implies that all hinges of summands of \( \Delta(F) \) lie on the boundary or to the right of the triangle \( T \). The boundary cases correspond to equality in (5.60), and they explicitly are:
for the picture on the left in Figure 22: \( A = \{1, \ldots, \beta\} \cup \{\alpha + 1, \ldots, k\} \) where \( \beta < \alpha \) have the property that \( \mu \alpha, \mu \beta \in \mathbb{Z} \)

for the picture on the right in Figure 22: \( A = \{\alpha + 1, \ldots, \beta\} \) where \( \alpha < \beta \) have the property that \( \mu \alpha, \mu \beta \in \mathbb{Z} \)

We will only treat the situation in the first bullet (i.e. in the picture on the left), because that of the second bullet is analogous and will not be used in the present paper. Recall that m.i.d. \( Y \) stands for the terms of minimal initial degree of a \( k \)-tensor \( Y \), i.e. the smallest value of the total \( \text{hdeg} \) of the first \( l \) tensor factors of \( Y \) (for fixed \( l \leq k \)). We will now show that the terms of minimal initial degree give rise precisely to the tensors on the second line of (5.59). Explicitly, we have:

\[
R_{\omega_k} X = R_{\omega_1} (z_1, \ldots, z_\beta) \left[ R_{1,\beta+1} \left( \frac{z_1}{z_{\beta+1}} \right) \ldots R_{\beta,\alpha} \left( \frac{z_\beta}{z_\alpha} \right) \right] R_{\omega_{\alpha-\beta}} (z_{\beta+1}, \ldots, z_{\alpha})
\]

\[
\begin{bmatrix}
R_{1,\alpha+1} \left( \frac{z_1}{z_{\alpha+1}} \right) \ldots R_{\beta,k} \left( \frac{z_\beta}{z_k} \right) \\
R_{\beta+1,\alpha+1} \left( \frac{z_{\beta+1}}{z_{\alpha+1}} \right) \ldots R_{\alpha,k} \left( \frac{z_\alpha}{z_k} \right)
\end{bmatrix}
\]

\[
E_{\alpha+1} (\omega_1, \ldots, \omega_{\alpha+1}) \cdot \sum_{x_1, y_1=1}^{n} \text{coefficient} \cdot E_{x_1 y_1} \otimes \ldots \otimes E_{x_{\beta+1} y_{\beta+1}} \otimes 1^{\otimes \alpha - \beta} \otimes 1^{\otimes k - \alpha}
\]

where the symbol \( E_{c_1, \ldots, c_{\alpha+1}} \) is defined in (5.45). As we have seen in the latter part of Proposition 5.17 the terms with the squiggly red underline contribute certain powers of \( q \) to the minimal initial degree of \( R_{\omega_k} X \). Specifically, let:

\[
\begin{align*}
E_{s_\alpha, \ldots, s_\beta}^{(1, \ldots, \beta)} &= \sum_{x_1, y_1=1}^{n} \text{coefficient} \cdot E_{x_1 y_1} \otimes \ldots \otimes E_{x_\beta y_\beta} \otimes 1^{\otimes \alpha - \beta} \otimes 1^{\otimes k - \alpha} \\
E_{s_\alpha, \ldots, s_\beta}^{(\beta+1, \ldots, \beta+1)} &= \sum_{x_1, y_1=1}^{n} \text{coefficient} \cdot 1^{\otimes \beta} \otimes E_{x_{\beta+1} y_{\beta+1}} \otimes \ldots \otimes E_{x_{\alpha+1} y_{\alpha+1}} \otimes 1^{\otimes k - \alpha} \\
E_{s_{\alpha+1}, \ldots, s_k}^{(\alpha+1, \ldots, k)} &= \sum_{x_1, y_1=1}^{n} \text{coefficient} \cdot 1^{\otimes \beta} \otimes 1^{\otimes k - \alpha} \otimes E_{x_{\alpha+1} y_{\alpha+1}} \otimes \ldots \otimes E_{x_k y_k}
\end{align*}
\]

(we note that \( x_1 = j, y_\beta = s_\beta = x_{\beta+1}, y_\alpha = s_\alpha = x_{\alpha+1}, y_k = i \) in all summands above that have non-zero coefficient). Then m.i.d. \( \Delta(R_{\omega_k} X) \), which differs by certain powers of \( \psi_s^{1} \) from m.i.d. \( R_{\omega_k} X \), is given by:

\[
(5.61) \text{ m.i.d. } \Delta(R_{\omega_k} X) = \sum_{x_1, y_1=1}^{n} \text{coefficient} \cdot \psi_{x_{\beta+1}}^{-1} \ldots \psi_{x_\alpha}^{-1} R_{\omega_1} \left[ \prod_{1 \leq a \leq \beta} R_{a} \left( \frac{z_a}{z_c} \right) \right]
\]

\[
R_{\omega_{\alpha-\beta}} (E_{x_1 y_1} \otimes \ldots \otimes E_{x_\beta y_\beta} \otimes 1^{\otimes k - \beta}) \left[ \prod_{1 \leq a \leq \beta} R_{a} \left( \frac{z_a}{z_c} \right) \right] (1^{\otimes \alpha} \otimes E_{x_{\alpha+1} y_{\alpha+1}} \otimes \ldots \otimes E_{x_k y_k})
\]

\[
\psi_{y_{\beta+1}} \ldots \psi_{y_{\alpha+1}} R_{\omega_{\alpha-\beta}} (1^{\otimes \beta} \otimes E_{x_{\beta+1} y_{\beta+1}} \otimes \ldots \otimes E_{x_{\alpha+1} y_{\alpha+1}} \otimes 1^{k - \alpha})
\]
Symmetrizing the expression above with respect to all permutations  

\[ (-q^{-2})^\frac{1}{2} \sum_{\beta < \alpha \leq \delta} q^{-1} - \sum_{\beta < \alpha \leq \delta} q^{-1} E_{\alpha, \beta}^{(5.63)} q^{-1} - \sum_{\beta < \alpha \leq \delta} q^{-1} E_{\alpha, \beta}^{(5.65)} q^{-1} - \sum_{\beta < \alpha \leq \delta} q^{-1} E_{\alpha, \beta}^{(5.67)} q^{-1} - \sum_{\beta < \alpha \leq \delta} q^{-1} E_{\alpha, \beta}^{(5.69)} q^{-1} \]

\[ \text{first squiggle} \quad \text{second squiggle} \quad \text{third squiggle} \quad \text{fourth squiggle} \quad \text{conjugation} \]

Before we move on, we must explain three issues concerning the expression above: the power of \( q \) labeled “conjugation”, why \( y_0 = x_{\alpha+1} = s_\alpha \) were increased by 1, and why the factor \( -q^{-2} \) arose. The first issue, namely the power of \( q \), appeared from the diagonal terms of arbitrary conjugation matrices \( R_\alpha \) and \( R_\sigma^{-1} \) as in (5.25), where \( \sigma \) is any permutation which switches the variables \( \{\beta + 1, \ldots, \alpha\} \) and \( \{\alpha + 1, \ldots, k\} \) (this is because in the definition of the coproduct, the variables to the left of the \( \otimes \) sign must all have smaller indices than the variables to the right of the \( \otimes \) sign).

The latter two issues happened because of the presence of \( Q_{\alpha, \alpha+1}(\infty) \) in the fourth squiggle. Because \( Q_{\alpha, \alpha+1}(\infty) \) has 0 on the diagonal, its contribution of minimal initial degree comes from the immediately off-diagonal terms, which are:

\[ -\frac{q^{-2\delta s_\alpha}}{q} \sum_t \ldots \otimes 1 \otimes E_{t,t+1} \otimes E_{t+1,t} \otimes 1 \otimes \cdots \]

Therefore, for any indices \( u \) and \( v \), we have:

\[ E_{u,v}^{(\alpha)} \cdot \left( -\frac{q^{-2\delta s_\alpha}}{q} \sum_t E_{t,t+1}^{(\alpha)} \right) \cdot E_{u,v}^{(\alpha+1)} = (-q^{-2}q^{2\delta s_\alpha})^{(\alpha)} \cdot q^{\delta s_\alpha} E_{u,s_{\alpha+1}}^{(\alpha)} E_{s_{\alpha+1},v}^{(\alpha+1)} \]

Using (5.8), we may move certain \( \psi \) factors around in (5.61), in order to cancel the powers of \( q \) with underbraces beneath:

\[
\text{m.i.d. } \Delta(R_{\omega k} X) = (-q^2 q^{2\alpha})^{-1} \sum_{x, y_{\alpha} = 1}^n \text{coefficient} \cdot R_{\omega_{\beta}} \left[ \prod_{1 \leq \alpha \leq \beta} R_{\alpha c} \left( \frac{z_a}{z_c} \right) \right] R_{\omega_{k-\alpha}}
\]

\[
(E_{1} \otimes \cdots \otimes E_{k})^{(\alpha)} \cdot \psi_{x_{\alpha}+1} \otimes \psi_{y_{\alpha}} \psi_{x_{\alpha}} \prod_{1 \leq \alpha \leq \beta} \tilde{R}_{\alpha c} \left( \frac{z_a}{z_c} \right)
\]

As a consequence of Claim 5.19, we may write the expression above as:

\[
\text{m.i.d. } \Delta(R_{\omega k} X) = (-q^2 q^{2\alpha})^{-1} \sum_{x, y_{\alpha} = 1}^n \text{coefficient} \cdot R_{\omega_{\beta}} \left[ \prod_{1 \leq \alpha \leq \beta} R_{\alpha c} \left( \frac{z_a}{z_c} \right) \right] R_{\omega_{k-\alpha}}
\]

\[
E_{s_{\alpha+1} \cdots |s_{\beta}}^{(\alpha+1 \cdot k)} \tilde{R}_{s_{\alpha+1} \cdots |s_{\beta}} \left( \frac{z_a}{z_c} \right) E_{s_{\alpha+1} \cdots |s_{\beta}}^{(\alpha+1 \cdot k)} \otimes R_{\omega_{\alpha-\beta}} E_{s_{\alpha+1} \cdots |s_{\beta}}^{(\alpha+1 \cdot k)}
\]

Symmetrizing the expression above with respect to all permutations \( \sigma \in S(k) \) which fix the set \( A \) gives rise to m.i.d. \( \Delta(F) \). To obtain the expression on the second line of (5.59), it remains to establish the formulas below (let \( u = s_\alpha + 1 \) and \( v = s_\beta \)):

\[
F_{[u,v]}^{(\alpha)} = \text{Sym } R_{\omega_{\beta}} E_{s_{\alpha+1} \cdots |s_{\beta}}^{(\alpha+1 \cdot k)}
\]

\[
F_{[u,v]}^{(\bullet)} = \text{Sym } R_{\omega_{\alpha-\beta}} E_{s_{\alpha+1} \cdots |s_{\beta}}^{(\alpha+1 \cdot k)}
\]

To prove the formulas above, we start with an easy computation:
Claim 5.28. We have the identity:

\[(5.66) \quad E^{(1...k)}_{c_1|c_2|...|c_{k-1}|c_k} = F^{(1...k)}_{c_1|c_2|...|c_{k-1}|c_k} \cdot (-q^2 \eta^2)^{1-k} \]

where \(F^{(u...v)}_{c_{u-1}|...|c_v} = \prod_{a=u}^{v} \left[ \bar{R}_{aa} \left( \frac{z_{a-1}}{z_a} \right) ... \bar{R}_{a-2,a} \left( \frac{z_{a-2}}{z_{a-1}} \right) \bar{Q}_{a-1,a} \left( \frac{z_{a-1}}{z_a} \right) E^{(a)}_{c_{u-1}|...|c_v} \eta^\frac{2u}{n} \right]. \)

Let us first show how the Claim allows us to complete the proof of the Proposition. Because of (5.66), formula (5.65) is equivalent to:

\[(5.67) \quad \bar{F}^u_{(i|u)} = (-q^2 \eta^2)^{\alpha-k} \cdot \text{Sym} \bar{R}_{\omega_{k-a}} F^{(1...k-\alpha)}_{s_{a+1}|...|s_{k-1}|s_k} \]

Then (5.63), (5.64), (5.67) follow from (5.31), (5.32) and the formulas below:

\[
\begin{align*}
    j - \left( \frac{(j - i)t}{k} \right) & = j - \left[ \mu t \right] \quad \forall t \in \{1, ..., \beta \} \\
    j - \left( \frac{(j - i)t}{k} \right) + \delta^a_t & = v - \left( \frac{(v - u)(t - \beta)}{\alpha - \beta} \right) \quad \forall t \in \{ \beta + 1, ..., \alpha \} \\
    j - \left( \frac{(j - i)t}{k} \right) + 1 & = u - \left[ \mu (t - \alpha) \right] \quad \forall t \in \{ \alpha + 1, ..., k \}
\end{align*}
\]

which are all straightforward consequences of our assumption on the triangle \(T\).

This completes the proof of formula (5.59) for the picture on the left in Figure 22 (as we said, the case of the picture on the right is analogous, and left to the interested reader). As for Claim 5.28, we start with the following identity:

\[(5.68) \quad (E_{j,t} \otimes 1)Q \left( \frac{z_1}{z_2} \right) (1 \otimes E_{t,i}) \eta^\frac{2u}{n} = \]

\[
= (E_{j,t+1} \otimes 1)Q \left( \frac{z_1}{z_2} \right) (1 \otimes E_{t+1,i}) \eta^\frac{2u}{n} \cdot (-q^2 \eta^2)^{-1}
\]

for all \(i, j, t \in \mathbb{Z}\). Indeed, by plugging in (5.27)–(5.28), formula (5.68) reads:

\[
\begin{align*}
q^{-1} \sum_{u=1}^{n} \frac{n}{z_1} \left[ \frac{\frac{z_1}{z_2}}{1 - \frac{z_1^2}{z_2}} \right] \left( \frac{z_1^2}{z_2} \right)^{\eta^u} (E_{j,u} \otimes E_{u,i}) \eta^\frac{2u}{n} = \]

\[
= (-q) \sum_{u=1}^{n} \frac{n}{z_1} \left[ \frac{\frac{z_1}{z_2}}{1 - \frac{z_1^2}{z_2}} \right] \left( \frac{z_1^2}{z_2} \right)^{\eta^u} (E_{j,u} \otimes E_{u,i}) \eta^\frac{2u}{n} \cdot (-q^2 \eta^2)^{-1}
\]

which is elementary. Identity (5.66) follows by \(k - 1\) applications of (5.68).

\[\square\]

Proof. of Theorem 5.25. We have already seen that, for fixed \(\mu\), the assignment:

\[f^\mu_{(i|j)} \text{ of (3.60) } \leftrightarrow F^\mu_{(i|j)} \text{ of (5.31)} \]

yields an algebra homomorphism \(E^\mu_{(i|j)} \rightarrow A^+.\) To extend this to a homomorphism:

\[(5.69) \quad \Upsilon^+ : D^+ \rightarrow A^+ \]

we need to prove that formulas (3.73) and (3.74) hold with \(p, f\) replaced by \(P, F\). In order to show that (3.73) holds in this setup, let us first show that the linear maps \(\alpha_{(u,v)}\) take the same value on both sides of the equation. By (5.20), we have:

\[
\alpha_{(u,v)}(\text{LHS of (3.73)}) = \alpha_{(u,v)} \left( P^{(k)}_{(i|j)} P^{(k)}_{l|z|} \right) - \alpha_{(u,v)} \left( P^{(k)}_{(i|j)} P^{(k)}_{l|z|} \right) =
\]

\[\]

\[\]
\[
\alpha_{[u;nl,v]} \left( P_{[ij]}^{(k)} \right) \alpha_{[u;nl]} \left( P_{[ij]}^{(k')} \right) \frac{p^2}{3} = \alpha_{[v;nl,v]} \left( P_{[ij]}^{(k')} \right) \alpha_{[u;v;nl]} \left( P_{[ij]}^{(k)} \right) \frac{p^2}{3} = \\
\delta_{[u;nl,v]}^{(i,j)} \delta^{(i,j)}_{[u;nl,v]} \frac{p^2}{3} - \delta_{[u;v;nl]}^{(i,j)} \delta^{(i,j)}_{[u;v;nl]} \frac{p^2}{3} = \alpha_{[u;v]} (\text{RHS of } (3.73))
\]

∀u, v. Therefore, Lemma 3.15 implies that the equality (3.73) would follow from:

\[\text{LHS of (3.73)} \in B^{+}_{\frac{1}{n+l+1}}\]

We may depict the degree vectors of the elements \( P_{[ij]}^{(k)} \), \( P_{[ij]}^{(k')} \), \( P_{[ij+n]}^{(k+k')} \) as:

\[
\begin{array}{c}
\ldots \\
\ldots \\
(j-i+k+k') \\
\ldots \\
\ldots \\
(0,0)
\end{array}
\]

We need to show that all the hinges of summands of \( \Delta(\text{LHS of (3.73)}) \) are to the right the vector \((j-i+nl, k+k')\). Since coproduct is multiplicative, the hinges of \( \Delta(XY) \) are all among the sums of hinges of \( \Delta(X) \) and \( \Delta(Y) \), as vectors in \( \mathbb{Z}^2 \).

By definition, the hinges of \( \Delta(P_{[ij]}^{(k)}) \) and \( \Delta(P_{[ij]}^{(k')}) \) lie to the right of the vectors \((j-i,k)\) and \((nl,k')\), respectively. The sum of any two such hinges lies to the right of the parallelogram in the picture, except for the sum of the two hinges below:

\[
\Delta(P_{[ij]}^{(k)}) = \ldots + P_{[ij]}^{(k)} \otimes 1 + \ldots \text{ has a hinge at } (j-i,k)
\]

\[
\Delta(P_{[ij]}^{(k')}) = \ldots + 1 \otimes P_{[ij]}^{(k')} + \ldots \text{ has a hinge at } (0,0)
\]

Therefore, \( \Delta(P_{[ij]}^{(k)} P_{[ij]}^{(k')}) \) and \( \Delta(P_{[ij]}^{(k')} P_{[ij]}^{(k)}) \) both have a hinge at the point \((j-i, k)\), but the corresponding summand in both coproducts is:

\[
P_{[ij]}^{(k)} \otimes P_{[ij]}^{(k')}
\]

We conclude that this summand vanishes in \( \Delta(\text{LHS of (3.73)}) \), which therefore has all the hinges to the right of \((j-i+nl, k+k')\). This completes the proof of (3.73).

Let us now prove (3.74) by induction on \( k+k' \) (the base case \( k+k' = 1 \) is trivial). Recall that \( \mu = \frac{j+i'-i'j'}{k+k'} \), and let us represent the degrees of \( P_{[ij]}^{(k)} \) and \( P_{[i'j']}^{(k') \prime} \) as:
We have the following formulas, courtesy of Proposition 5.27

\[ \Delta \left( P^{(k)}_{[i;j]} \right) = P^{(k)}_{[i;j]} \otimes 1 + \frac{\psi_{i}}{\psi_{j}} \otimes P^{(k)}_{[i;j]} + \sum_{i \leq u < v \leq j} F^{\mu}_{[u;j]} \frac{\psi_{u}}{\psi_{v}} \tilde{F}^{\mu}_{[u;u]} \otimes P^{(\bullet)}_{[u;u]} + \ldots \]

\[ \Delta \left( P'^{(k)}_{[i';j']} \right) = P'^{(k)}_{[i';j']} \otimes 1 + \frac{\psi'_{i'}}{\psi'_{j'}} \otimes P'^{(k)}_{[i';j']} + \ldots \]

where the ellipsis denotes terms whose hinges lie to the right of the line of slope \( \mu \) (this convention will remain in force for the remainder of this proof), and \( \bullet \) denotes the natural number which makes the two sides of the expressions above have the same vertical degree. Letting LHS denote the left-hand side of (3.74), we have:

\[ \Delta(\text{LHS}) = \text{LHS} \otimes 1 + \frac{\psi_{i}}{\psi_{j}} \frac{\psi'_{i'}}{\psi'_{j'}} \otimes \text{LHS} + \sum_{i \leq u < v \leq j} q^{\delta_{ij}'} \psi_{u} \psi_{v} \widetilde{F}^{\mu}_{[u;j]} \frac{\psi_{u}}{\psi_{v}} \frac{\psi'_{u}}{\psi'_{v}} \otimes P^{(k)}_{[i;j]} \widetilde{P}^{(k)}_{[i';j']} - q^{\delta_{ij}'} \psi'_{u} \psi'_{v} \widetilde{F}^{\mu}_{[u;j]} \frac{\psi'_{u}}{\psi'_{v}} \frac{\psi'_{u}}{\psi'_{v}} \otimes P^{(k)}_{[i;j]} \widetilde{P}^{(k)}_{[i';j']} + \ldots \]

Relation 5.8 allows us to move \( \psi' \)'s around, and show that the expression with the squiggly underline vanishes, while the expression on the second line yields:

\[ \Delta(\text{LHS}) = \text{LHS} \otimes 1 + \frac{\psi_{i}}{\psi_{j}} \frac{\psi'_{i'}}{\psi'_{j'}} \otimes \text{LHS} + \sum_{i \leq u < v \leq j} F^{\mu}_{[u;j]} \frac{\psi_{i}}{\psi_{j}} \psi_{v} \frac{\psi'_{i'}}{\psi'_{j'}} \frac{\psi'_{v}}{\psi'_{v}} \otimes \left[ q^{\delta_{ij}'} \frac{\psi_{u}}{\psi_{v}} \widetilde{P}^{(k)}_{[u;v]} P^{(k)}_{[i';j']} - q^{\delta_{ij}'} \frac{\psi'_{u}}{\psi'_{v}} \widetilde{P}^{(k)}_{[u;v]} P^{(k)}_{[i';j']} + \ldots \right] \]

If we are only interested in the leading term \( \Delta_{\mu} \) of the coproduct, we may neglect the terms represented by the ellipsis. The induction hypothesis allows us to replace the term in square brackets by the RHS of (3.74) for \((i, j, k) \mapsto (u, v, \bullet)\), hence:

\[ \Delta_{\mu}(\text{LHS}) = \text{LHS} \otimes 1 + \frac{\psi_{i}}{\psi_{j}} \frac{\psi'_{i'}}{\psi'_{j'}} \otimes \text{LHS} + \sum_{i \leq u < v \leq j} \left[ F^{\mu}_{[u;j]} \frac{\psi_{i}}{\psi_{j}} \psi_{v} \frac{\psi'_{i'}}{\psi'_{j'}} \frac{\psi'_{v}}{\psi'_{v}} \otimes \left( \delta_{ij}' q^{\frac{-\delta_{ij}'}{q-1}} - \delta_{ij}' \frac{(q^{\frac{1}{h}})^{n-2d+1} - (q^{\frac{1}{h}})^{n-2d+1}}{q^{n-1} - q^n} \right) \right] \]

Let \( \gamma_{i,j,k}(s) \in \mathbb{Q}(q, q^{\frac{1}{h}}) \) denote the constant in the round brackets on the second line above. By (5.33) and (5.34), notice that the formula above matches \( \Delta_{\mu}(\text{RHS}) \). By Lemma 5.14, to prove that \( \text{LHS} = \text{RHS} \), it suffices to show that the two sides of equation (3.74) take the same values under the maps (5.19). To this end:

\[ \alpha_{[u;v]}(\text{LHS}) = \sum_{i \leq u < v \leq j} \delta_{(u,v)}(i,j,i') q^{\delta_{ij}'} \delta_{u,v} \delta_{i,j}' \frac{q^{\frac{1}{h}}}{q^{n-1} - q^n} \gamma_{i,j,k}(s) \]

where \( d = \text{gcd}(k + k', j + j' - i - i') \), while:

\[ \alpha_{[u;v]}(\text{RHS}) = \sum_{i \leq u < v \leq j} \delta_{(u,v)}(i,j,i') \frac{q^{\text{gcd}(i,j,i') - i'}}{q^{n-1} - q^n} \gamma_{i,j,k}(s) \]
The equality between the right-hand sides of (5.70) and (5.71) was established in Claim 4.9 of [17]. This concludes the proof of (3.74), so there exists an algebra homomorphism $\Upsilon^+$ as in (5.69). In the remainder of this proof, we need to show that $\Upsilon^+$ is an isomorphism. As explained in [17] (following the similar argument of [2]), one may use relations (3.73) and (3.74) to express an arbitrary product of the generators (5.55) and (5.56) as a linear combinations of products of the form:

$$\prod_{\mu \in \mathbb{Q}^+} x_{\mu}^{(i)} \quad \text{all but finitely many of the } x_{\mu}^{(i)} \text{ are 1}$$

where $\{x_{\mu}^{(i)}\}$ go over any linear basis of $B_{\mu}^+$, and $\prod_{\mu} \rightarrow$ is taken in increasing order of $\mu$.

**Claim 5.29.** The elements (5.72) are all linearly independent in $A^+$.

Let us first show that the Claim completes the proof of the Theorem. The linear independence of the elements (5.72) implies that:

$$\dim A_{\leq \mu |d,k} \geq \# \{ \text{unordered collections (5.13)} \}$$

for all $\mu, k, d$ (indeed, formula (3.27) implies that the number of products (5.72) with $\mu$ bounded above is precisely equal to the number in the RHS of (5.73)). Combining (5.73) with Lemma 5.10 we conclude that the products (5.72) actually form a linear basis of $A^+$, and therefore $A^+$ is generated by the elements (5.55) and (5.56). This implies that $\Upsilon^+$ is an isomorphism, since we showed in [17] that (5.72) also form a linear basis of $D^+$.

Let us now prove Claim 5.29. We assume that the basis vectors $x_{\mu}^{(i)}$ of $B_{\mu}^+$ are ordered in non-decreasing order of $|\hdeg|$, i.e.:

$$i \geq i' \Rightarrow |\hdeg x_{\mu}^{(i)}| \geq |\hdeg x_{\mu}^{(i')}|$$

Now suppose we have a non-trivial linear relation among the various products (5.72). We may rewrite this hypothetical relation as:

$$x_{\mu}^{(i)} \prod_{\nu > \mu} x_{\nu}^{(j')} = \sum \text{coefficient} \cdot x_{\mu}^{(i')} \prod_{\nu > \mu} x_{\nu}^{(j)}$$

where all terms in the RHS have $i' < i$. Since the coproduct is multiplicative, then all the hinges of $\Delta(XY)$ are sums of hinges of $\Delta(X)$ and hinges of $\Delta(Y)$, as vectors in $\mathbb{Z}^2$. Therefore, $\Delta(\text{LHS of (5.75)})$ has a single summand with hinge at the lattice point:

$$\left( |\hdeg x_{\mu}^{(i)}, \vdeg x_{\mu}^{(i)}| \right)$$

and the corresponding summand is precisely:

$$\Delta(\text{LHS of (5.75)}) = ... + \psi \prod_{\nu > \mu} x_{\nu}^{(j')} \otimes x_{\mu}^{(i)} + ...$$

where $\psi$ stands for a certain (unimportant) product of $\psi_{a,b}^{i,i'}$s. Meanwhile, the coproduct of the RHS of (5.75) can only have a hinge at a lattice point (5.76) if
equality is achieved in (5.74). The corresponding summand in the coproduct is:

$$\Delta(\text{RHS of (5.75)}) = \ldots + \text{coefficient} \times x^{(i')}_{\nu} \otimes x^{(i')}_{\mu} + \ldots$$

Since $x^{(i)}_{\mu}$ cannot be expressed as a linear combination of $x^{(i')}_{\nu'}$ with $i' < i$, the right-hand sides of expressions (5.77) and (5.78) cannot be equal. This contradiction implies that there can be no relation (5.75), which proves Claim 5.29. □

Corollary 5.30. The algebra $A^+$ is generated by the $v\deg = 1$ elements:

$$\{ E_{ij} \}_{(i,j) \in \mathbb{Z}_2^{(n,n)}}$$

The corollary is an immediate consequence of Proposition 3.43 and Theorem 5.25.

6. The double shuffle algebra with spectral parameters

In the previous Section, we constructed the extended shuffle algebra corresponding to the $R$–matrix with spectral parameters (3.87). We will now take two such extended shuffle algebras and construct their double, as was done in Subsections 2.10 and 2.12 for $R$–matrices without spectral parameters. This will conclude the proof of Theorem 1.5.

6.1. Let $\bar{q}_+ = \bar{q}$ and $\bar{q}_- = q^{-n}\bar{q}^{-1}$. If $\bar{R}^+(x) = \bar{R}(x)$ is given by (4.2), then:

$$\bar{R}^-(x) = \left[\bar{R}^{11} \left( \frac{1}{x} \right)^{-1} \right]_{21}^{11} \in \text{End}(V \otimes V)(x)$$

given by:

$$\bar{R}^-(x) = \sum_{1 \leq i,j \leq n} E_{ii} \otimes E_{jj} \left( \frac{q^{-1} - qxq_+^{-2}}{1 - xq_+^{-2}} \right)^{\delta_{11}} - (q^{-1} - q^{-1}) \sum_{1 \leq i \neq j \leq n} E_{ij} \otimes E_{ji} q^{2(j-i)} \frac{xq_+^{-2}}{1 - xq_+^{-2}}$$

Note that we have the equality:

$$\bar{R}^-(x) = D_2 \bar{R}^+(x) D_2^{-1} \mid_{\bar{q}_+ \mapsto \bar{q}_-}$$

where $D = \text{diag}(q^2, \ldots, q^{2n}) \in \text{End}(V)$.

Definition 6.2. The shuffle algebra $A^-$ is defined just like in Definition 4.8, using $\bar{q}_-$ instead of $\bar{q}$, and the multiplication (4.9) uses $\bar{R}^-$ instead of $\bar{R}$.

Because of (6.2), the map:

$$A^+ \xrightarrow{\Phi} A^-,$$

$$X_{1\ldots k}(z_1, \ldots, z_k) \mapsto D_1 \ldots D_k X_{1\ldots k}(z_1, \ldots, z_k) \mid_{\bar{q}_+ \mapsto \bar{q}_-}$$

is a $\mathbb{Q}(q)$–linear algebra isomorphism. The following elements of $A^-$ are the images of the elements (5.31)–(5.32) under $\Phi$, times a factor of $\bar{q}_+^{2k-2n}$:
(6.4) \[ F^{(-k)}_{\{i;j\}} = \text{Sym} \ R_{\omega_k}(z_1, \ldots, z_k) \]
\[ \prod_{a=1}^{k} [ \tilde{R}_{1a} (z_1) \cdots \tilde{R}_{a-2,a} (z_2) Q_{a-1,a} (z_{a-1}) E_{s_{a-1}^a} (\frac{z_{a-1}}{z_a}) \] \[ - \bar{Q}_{a-1,a} (z_{a-1}) E_{s_{a-1}^a} (\frac{z_{a-1}}{z_a}) ] \]

(6.5) \[ \bar{F}^{(-k)}_{\{i;j\}} = (-\bar{\eta}^2)^k \cdot \text{Sym} \ R_{\omega_k}(z_1, \ldots, z_k) \]
\[ \prod_{a=1}^{k} [ \tilde{R}_{1a} (z_1) \cdots \tilde{R}_{a-2,a} (z_2) Q_{a-1,a} (z_{a-1}) E_{s_{a-1}^a} (\frac{z_{a-1}}{z_a}) \] \[ - \bar{Q}_{a-1,a} (z_{a-1}) E_{s_{a-1}^a} (\frac{z_{a-1}}{z_a}) ] \]
where \( s_a = j - [\mu a], \ s'_a = j - [\mu a], \ Q^- = D_2QD_2^{-1}\mid_{\eta=\bar{\eta}}, \ \bar{Q}^- = D_2\bar{Q}D_2^{-1}\mid_{\eta=\bar{\eta}}. \)

6.3. In Definition 5.2, we defined the extended shuffle algebra by introducing new generators. We will now add two more central elements \( c \) and \( \bar{c} \), and define instead:

(6.6) \[ \bar{A}^\pm = \bigoplus_{c, \ bar{c} \text{ central and relations (6.7) and (6.8)}}^\{i \leq j\}^\{1 \leq i \leq n\} \]

where:

(6.7) \[ R \left( \frac{x}{y} \right) S^\pm_1(x) S^\pm_2(y) = S^\pm_2(y) S^\pm_1(x) R \left( \frac{x}{y} \right) \]

(6.8) \[ X^\pm \cdot S^\pm_0(y) = S^\pm_0(y) \cdot \frac{R_{k_0} (z_k) \cdots R_{10} (z_1)}{f (z_k) \cdots f (z_1)} X^\pm \tilde{R}_{k_0}^\pm \left( \frac{z_1}{y} \right) \cdots \tilde{R}_{10}^\pm \left( \frac{z_k}{y} \right) \]

for any \( X^\pm = X^\pm_{1..k}(z_1, \ldots, z_k) \in \bar{A}^\pm \subset \bar{A}^\pm \), where:

\[ S^\pm(x) = \sum_{d=0}^{d=0} s^\pm_{\{i;j\}+nd} \cdot \left\{ \begin{array}{ll} E_{ij} x^{-d} & \text{if } \pm = + \\
- E_{ij} x^d & \text{if } \pm = - \end{array} \right. \]

If we define the series \( T^\pm(x) \) by (3.105) and (3.106), then (6.8) is equivalent to:

(6.9) \[ T^\pm_0(y) \cdot X^\pm = \tilde{T}^\pm_{0k} \left( \frac{y}{z_k} \right) \cdots \tilde{T}^\pm_{01} \left( \frac{y}{z_1} \right) X^\pm \frac{R_{k_0} (\frac{y}{z_k}) \cdots R_{10} (\frac{y}{z_1})}{f (\frac{y}{z_k}) \cdots f (\frac{y}{z_1})} \cdot T^\pm_0(y) \]

6.4. The algebras \( \bar{A}^\pm \) are graded by \( \mathbb{Z}^n \times \mathbb{Z} \), with:
\[ \deg X^\pm_{1..k}(z_1, \ldots, z_k) = (d, \pm k), \quad \forall X^\pm \in \bar{A}^\pm \]
\[ \deg s^\pm_{\{i;j\}} = (\pm (i, j), 0), \quad \forall i \leq j \]
where \( d \in \mathbb{Z}^n \) is defined in (4.20). We write \( \deg X = (\text{vdeg} \ X, \text{vdeg} \ X) \) to specify the components of the degree vector in \( \mathbb{Z}^n \) and \( \mathbb{Z} \), respectively. The reason why we introduced central elements \( c \) and \( \bar{c} \) to the algebras (6.6) is to twist the coproduct. Specifically, let \( \Delta_{\text{old}} \) be the coproduct of (5.9) - (5.10), and define:

(6.10) \[ \Delta : \bar{A}^\pm \rightarrow \bar{A}^\pm \otimes \bar{A}^\pm \]
Explicitly, the product in $R$ braid lifting the permutation $\sigma$ because they both lift the permutation by the formulas

\[ \Delta(\sigma) = \sum_{\sigma^{-1}(i) > \sigma^{-1}(j)} R_{ij} \left( \frac{z_i}{z_j} \right), \quad R_{ij} = \sum_{\sigma^{-1}(i) > \sigma^{-1}(j)} R_{ij} \left( \frac{z_j}{z_i} \right) \]

for all $i,j \leq k$.

Proposition 6.6. There is a pairing (of vector spaces):

\[ \mathcal{A}^+ \otimes \mathcal{A}^- \rightarrow \mathbb{Q}(q, q^\pm) \]

given by:

\[ \langle I_1^+ \cdots I_k^+ , X_{1\ldots k}^- (z_1, \ldots, z_k) \rangle = (q^2 - 1)^k \int_{|z_1| \ll \cdots \ll |z_k|} \text{Tr} \left( R_{\omega_k} \prod_{a=1}^k \left[ I_a^{(a)}(z_a) \prod_{b=a+1}^k \tilde{R}_{ab}^+ \left( \frac{z_a}{z_b} \right) \right] X_{1\ldots k}(z_1, \ldots, z_k) \right) \]

and

\[ \langle X_{1\ldots k}^+(z_1, \ldots, z_k), J_1^- \cdots J_k^- \rangle = (q^2 - 1)^k \int_{|z_1| \ll \cdots \ll |z_k|} \text{Tr} \left( R_{\omega_k} \prod_{a=1}^k \left[ J_a^{(a)}(z_a) \prod_{b=a+1}^k \tilde{R}_{ab}^- \left( \frac{z_a}{z_b} \right) \right] X_{1\ldots k}(z_1, \ldots, z_k) \right) \]
for all $I_a, J_a \in \text{End}(V)[z^\pm 1]$ and all $X^\pm \in \mathcal{A}^\pm$. The notation:

$$\int_{|z_1| \ll \ldots \ll |z_k|} F(z_1, \ldots, z_k)$$

refers to the iterated residue of $F$ at 0: first in $z_1$, then in $z_2$, ..., finally in $z_k$.

Note that the pairing $\langle X^+, Y^- \rangle$ is only non-zero for pairs of elements of opposite degrees, i.e. $X^+ \in \mathcal{A}_{d,k}$ and $Y^- \in \mathcal{A}_{-d,-k}$ for various $(d, k) \in \mathbb{Z}^n \times \mathbb{N}$.

**Proof.** Formula (6.14) is well-defined as a linear functional in the second argument, while (6.15) is well-defined as a linear functional in the first argument. Therefore, to show that (6.13) is well-defined as a linear functional in both arguments, we only need to show that (6.14) and (6.15) produce the same result when $X^\pm$ is of the form (6.12) (this statement implicitly uses Corollary 5.30 which states that any element in $\mathcal{A}^\pm$ is a linear combination of the elements (6.12)). To this end, we have:

$$\frac{1}{(q^2 - 1)^k} \left( I_1^+ \ast * I_k^+, J_1^- \ast * J_k^- \right) \text{ according to } (6.14) = \int_{|z_1| \ll \ldots \ll |z_k|} \sum_{\sigma \in S(k)} \sum_{a \in A} R_{\sigma \omega_k} \prod_{i=1}^{k} R_{\sigma(a)}(z_{\sigma(a)})$$

(6.16) $\text{Tr} \left( \prod_{1 \leq i < j \leq k} \frac{1}{f(z_i)} \cdot R_{\omega_k} \prod_{i=1}^{k} R_{\sigma(a)}(z_{\sigma(a)}) \right)$

and:

$$\frac{1}{(q^2 - 1)^k} \left( I_1^+ \ast * I_k^+, J_1^- \ast * J_k^- \right) \text{ according to } (6.15) = \int_{|z_1| \ll \ldots \ll |z_k|} \sum_{\sigma \in S(k)} \sum_{a \in A} R_{\sigma \omega_k} \prod_{i=1}^{k} R_{\sigma(a)}(z_{\sigma(a)})$$

(6.17) $\text{Tr} \left( \prod_{1 \leq i < j \leq k} \frac{1}{f(z_i)} \cdot R_{\omega_k} \prod_{i=1}^{k} R_{\sigma(a)}(z_{\sigma(a)}) \right)$

By using the cyclic property of the trace and the straightforward identity:

$$R_{\sigma \omega_k} = \sigma R_{\sigma^{-1} \omega_k} \sigma^{-1} \prod_{1 \leq i < j \leq k} \frac{1}{f(z_i)} \frac{1}{f(z_j)}$$

(6.18) (which uses (4.6)) we may rewrite the formulas above as:

$$\text{RHS of (6.16)} = \sum_{\sigma \in S(k)} \int_{|z_1| \ll \ldots \ll |z_k|} \prod_{1 \leq i < j \leq k} \frac{1}{f(z_i)} \frac{1}{f(z_j)} \text{Tr} \left( \prod_{i=1}^{k} I_{a(i)}(z_i) \prod_{i=1}^{k} R_{\sigma(a)}(z_{\sigma(a)}) \right)$$

(6.19)
and:

\[
\text{RHS of (6.17)} = \sum_{\sigma \in S(k)} \int_{|z_1| \leq \ldots \leq |z_k|} \prod_{1 \leq i < j \leq k} \frac{1}{f(z_j)} \cdot \text{Tr} \left( \prod_{a=1}^{k} I_a(z_a) \prod_{b=a+1}^{k} \tilde{R}_{ab}^+ \left( \frac{z_a}{z_b} \right) \right) \text{R}_{\sigma^{-1}(i)} \text{Tr} \left( \prod_{a=1}^{k} J_a(z_a) \prod_{b=a+1}^{k} \tilde{R}_{ab}^- \left( \frac{z_a}{z_b} \right) \right) \text{R}_{\sigma^{-1}(i)}^{-1} \right) \]

If we replace \( \sigma \mapsto \sigma^{-1} \) and replace \( z_a \mapsto z_{\sigma(a)} \) in the latter formula, then it precisely matches (6.19) (this uses the fact that \( \text{Tr}(Y) = \text{Tr}(\sigma Y \sigma^{-1}) \) for any tensor \( Y \)), up to the fact that the order of the contours changes:

\[
\text{RHS of (6.17)} = \sum_{\sigma \in S(k)} \int_{|z_{\sigma(1)}| \leq \ldots \leq |z_{\sigma(k)}|} \prod_{1 \leq i < j \leq k} \frac{1}{f(z_j)} \cdot \text{Tr} \left( \prod_{a=1}^{k} I_a(z_a) \prod_{b=a+1}^{k} \tilde{R}_{ab}^- \left( \frac{z_a}{z_b} \right) \right) \text{R}_{\sigma^{-1}(i)} \text{Tr} \left( \prod_{a=1}^{k} J_a(z_a) \prod_{b=a+1}^{k} \tilde{R}_{ab}^+ \left( \frac{z_a}{z_b} \right) \right) \text{R}_{\sigma^{-1}(i)}^{-1} \right) \]

Therefore, we may conclude that (6.16) equals (6.17) (which is what we need to prove), once we show that we may change the contours of the integral (6.19):

\[
\text{RHS of (6.19)} = \sum_{\sigma \in S(k)} \int_{|z_1| \leq \ldots \leq |z_k|} \prod_{1 \leq i < j \leq k} \frac{1}{f(z_j)} \cdot \text{Tr} \left( \prod_{a=1}^{k} I_a(z_a) \prod_{b=a+1}^{k} \tilde{R}_{ab}^- \left( \frac{z_a}{z_b} \right) \right) \text{R}_{\sigma^{-1}(i)} \text{Tr} \left( \prod_{a=1}^{k} J_a(z_a) \prod_{b=a+1}^{k} \tilde{R}_{ab}^+ \left( \frac{z_a}{z_b} \right) \right) \text{R}_{\sigma^{-1}(i)}^{-1} \right) \]

The integrand of (6.19) has three kinds of poles:

- \( z_i = z_j q^2 \) if \( i < j \) and \( \sigma^{-1}(i) \neq \sigma^{-1}(j) \), which arise from the zeroes of \( f(x) \)
- \( z_i q^{-2} = z_j \) if \( i < j \), which arise from the poles of \( \tilde{R}^+(x) \)
- \( z_i q^{-2} = z_j \) if \( \sigma^{-1}(i) < \sigma^{-1}(j) \), which arise from the poles of \( \tilde{R}^-(x) \)

As we move the contours as in (6.21), the only poles encountered involve \( z_i \) and \( z_j \) for \( i < j \) and \( \sigma^{-1}(i) > \sigma^{-1}(j) \), so already the poles in the first bullet do not come up. Meanwhile, the poles in the second bullet may come up, and in the remainder of this proof, we will show that the corresponding residue is 0 (the situation of the poles in the third bullet is analogous, so we skip it). To this end, recall that:

\[
\tilde{R}^{+,+1}_{21}(z) \tilde{R}^{-1}_{12} \left( \frac{1}{z} \right) = \text{Id}_{V \otimes V}
\]

by the definition of \( \tilde{R}^\pm \) in (6.1). This implies the identity:

\[
\text{Tr}_{V \otimes V} \left( \tilde{R}^{+}_{21}(z) A_2 \tilde{R}^{-}_{12} \left( \frac{1}{z} \right) B_1 \right) = \text{Tr}(A) \text{Tr}(B)
\]

for any \( A, B \in \text{End}(V) \). Taking the residue at \( z = q^{-2} \), we obtain:

\[
\text{Tr}_{V \otimes V} \left( (12) A_2 \cdot \tilde{R}^{-}_{12}(q^2) \cdot B_1 \right) = 0
\]

We may generalize the formula above to:

\[
\text{Tr}_{V \otimes k} \left( (ij) A_{1\ldots,i\ldots,k} \cdot \tilde{R}^{-}_{ij}(q^2) \cdot B_{1\ldots,j\ldots,k} \right) = 0
\]
\[ \forall i \neq j \text{ and } A, B \in \text{End}(V^\otimes k^{-1}). \text{ Formula (6.22) implies (6.23) because none the} \]

indices, other than the \(i\)-th and \(j\)-th, play any role in the vanishing of the trace.
We will use (6.23) to prove that the residue of (6.19) at \(z_iq^2_i = z_j\) vanishes. Let us consider the expression on the second line of (6.19) for \(\sigma = \omega_k\) and take its residue at \(z_iq^2_i = z_j\). The corresponding quantity is precisely represented by Braid 1 on the previous page (we draw braids top-to-bottom instead of left-to-right, for better legibility), if we make the convention that the variable on each strand is multiplied by \(q^2\) and \(q_2\) (respectively) as soon as it reaches the first and the second (respectively) box on that strand. Braids 1, 2, 3 and 4 are equivalent due to Reidemeister moves and the move in Figure 12. Braids 4 and 5 do not represent equal endomorphisms of \(V^{\otimes k}\), but they are equal upon taking the trace (since \(\text{Tr}(AB) = \text{Tr}(BA)\)). Finally, Braids 5 and 6 are equal due to Reidemeister moves. Because of the identity (6.23), Braid 6 has zero trace, thus yielding the conclusion.

Strictly speaking, the argument just given covers the case \(i = 2, j = 4\) and \(k = 5\), but it is obvious that we may replace the strands labeled 1, 3, 5 by any number of parallel strands, thus yielding the situation of arbitrary \(i, j, k\). More crucial is the fact that we have only shown the vanishing of the residue at \(z_iq^2_i = z_j\) of the \(\sigma = \omega_k\) summand of (6.19). The case of general \(\sigma\) would require one to insert the positive braid representing the permutation \(\sigma\omega_k\) at the middle of the braids above and the positive braid representing the permutation \(\sigma^{-1}\omega_k\) at the bottom of the braids above. The braid moves involved are analogous to the ones just performed, with the idea being to move the crossing between the blue and green strands to the very left of all other crossings. We leave the visual depiction of this fact to the interested reader, but we stress the fact that we only need to check the vanishing of the residue for those \(i < j\) such that \(\sigma^{-1}(i) > \sigma^{-1}(j)\). This implies that the green and blue strands do not cross except at the two points already depicted in the braids above, and this is what allows the argument to carry through.

\[\Box\]

**Proposition 6.7.** There exists a bialgebra pairing:

\[(6.24) \quad \tilde{A^+} \otimes \tilde{A^-} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Q}(q, q^n)\]

generated by \[6.13\] and:

\[(6.25) \quad \langle S^+_1(y), S^-_1(x) \rangle = R^+ \left(\frac{x}{y}\right) \quad \langle T^+_2(y), T^-_1(x) \rangle = R^- \left(\frac{x}{y}\right)\]

\[(6.26) \quad \langle S^+_2(y), T^-_1(x) \rangle = R \left(\frac{x}{y}\right) f^{-1} \left(\frac{x}{y}\right) \quad \langle T^+_2(y), S^-_1(x) \rangle = R \left(\frac{x}{y}\right)\]

(all rational functions above are expanded in the region \(|x| \ll |y|\)).

**Proof.** The proof follows that of Proposition \[2.11\] very closely, so we will only sketch the main ideas and leave the details to the interested reader. Take any:

- \(a, b \in \{X^+, S^+(x), T^+(x), \text{ for } X \in A^+\}\)
- \(c \in \{X^-, S^-(x), T^-(x), \text{ for } X \in A^-\}\)

and define \(\langle ab, c \rangle\) to be the RHS of \[2.28\]. Then if \(\sum_i a_i b_i = 0\) holds in \(\tilde{A^+}\), we must show that the pairing:

\[\left\langle \sum_i a_i b_i, c \right\rangle\]
thus defined is 0. If at least one of $a, b, c$ is a coefficient of either $S^\pm(x)$ or $T^\pm(x)$, then the statement in question is proved just like in Proposition 2.11, if one is careful to expand $x$ around $\infty^{\pm 1}$ (since (6.14) also involve integrals, we need to stipulate that $x$ should be closer to $\infty^{-\pm 1}$ than any of the variables $z_1, \ldots, z_k$).

The remaining case is when $a, b, c$ are all in $A^\pm$, and we must prove that:

\[(6.27) \quad \left\langle A^+ \ast B^+, Y^- \right\rangle = \left\langle B^+ \otimes A^+, \Delta(Y^-) \right\rangle\]

for all $A_{1\ldots k}(z_1, \ldots, z_k), B_{1\ldots l}(z_1, \ldots, z_l) \in A^+$ and $Y_{1\ldots k+l}(z_1, \ldots, z_k+l) \in A^-$. By Corollary 5.30, it suffices to consider $A = I_1 \ast \cdots \ast I_k$ and $B = I_{k+1} \ast \cdots \ast I_{k+l}$ for various $I_a \in \text{End}(V^+)[z_{\pm 1}]$. In this case, we may rewrite (6.14) as:

\[(6.28) \quad \left\langle I_1^+ \ast \cdots \ast I_k^+, X_{1\ldots k}(z_1, \ldots, z_k) \right\rangle = (q^2 - 1)^k \int_{|z_1| \gg \cdots \gg |z_k|} \text{Tr} \left( \prod_{b=k}^{k+l} I_{b+k-l}^+(z_b) \prod_{a=b-1}^{a=k} \tilde{R}_ba^+ \left( \frac{z_b}{z_a} \right) \right) \tilde{R}_{\omega k} \cdot \frac{X_{1\ldots k}(z_1, \ldots, z_k)}{\prod_{1 \leq i < j \leq k} f \left( \frac{z_i}{z_j} \right)} \]

by reversing the order of the tensor factors of $V^{\otimes k}$ and relabeling the variables $z_a \mapsto z_{k+1-a}$, as well as using the symmetry property (4.10) of $X$. Then we have:

\[(6.29) \quad \text{LHS of (6.27)} = (q^2 - 1)^{k+l} \int_{|z_1| \gg \cdots \gg |z_{k+l}|} \text{Tr} \left( \prod_{b=k+l}^{b=k+l} I_{b+k-l}^+(z_b) \prod_{a=b-1}^{a=k} \tilde{R}_ba^+ \left( \frac{z_b}{z_a} \right) \right) \tilde{R}_{\omega k} \cdot \frac{Y_{1\ldots k+l}(z_1, \ldots, z_{k+l})}{\prod_{1 \leq i < j \leq k+l} f \left( \frac{z_i}{z_j} \right)} \]

Meanwhile, the right-hand side of (6.27) is computed just like the right-hand side of (2.36), with the specification that $\Delta(Y)$ is expanded in the region when the first $k$ tensor factors are much smaller than the last $l$ tensor factors, one obtains:

\[
\text{RHS of (6.27)} = (q^2 - 1)^{k+l} \int_{|z_1| \gg \cdots \gg |z_{k+l}|} \text{Tr} \left( \frac{Y_{1\ldots k+l}(z_1, \ldots, z_{k+l})}{\prod_{1 \leq i < j \leq k+l} f \left( \frac{z_i}{z_j} \right)} \right) \prod_{b=k+l}^{l} I_{b+k-l}^+(z_b) \prod_{a=b-1}^{a=k} \tilde{R}_ba^+ \left( \frac{z_b}{z_a} \right) \tilde{R}_{\omega j} \left( \frac{z_{j+1}, \ldots, z_{l+k}}{z_1} \right) \frac{\tilde{R}_{l+1,l} \left( \frac{z_{l+1}}{z_1} \right) \cdots \tilde{R}_{l+1,k+1} \left( \frac{z_{l+k}}{z_1} \right)}{\prod_{1 \leq i < j \leq k+l} f \left( \frac{z_i}{z_j} \right)} \]

We may move $\tilde{R}_{\omega j}$ to the very right of the expression above, and obtain precisely (6.29). This completes the proof of the fact that the pairing (6.24) respects (2.28). The situation of (2.29) is analogous, so we leave it as an exercise to the reader. \(\square\)
6.8. Having proved Proposition 6.7, we may construct the Drinfeld double:

\[ \mathcal{A} = \tilde{A}^+ \otimes \tilde{A}^{-,\text{op,coop}} / \left( s^{+}_{[i;i]} s^{-}_{[i;i]} - 1 \right) \]

We will often use the notation \( \psi_i = (s^{+}_{[i;i]})^{-1} \).

**Proposition 6.9.** We have the following commutation relations in the algebra \( \mathcal{A} \):

\[ S^+_0(w) \cdot X^\pm = \left( \tilde{R}^{\pm}_{ik}(w) \right)_{\pm} \]

\[ = \tilde{R}^{\pm}_{0k} \left( \frac{wc}{z_k} \right) \cdots \tilde{R}^{\pm}_{01} \left( \frac{wc}{z_1} \right) X^\pm \tilde{R}^{\pm}_{k0} \left( \frac{wc}{z_k} \right) \cdots \tilde{R}^{\pm}_{0k} \left( \frac{wc}{z_k} \right) \cdot S^+_0(w) \]

\[ X^\pm \cdot T^+_0(w) = \left( T^+_0(w) \right) \cdot X^\pm \]

\[ = T^+_0(w) \cdot R_{k0} \left( \frac{z_k}{wc} \right) \cdots R_{10} \left( \frac{z_1}{wc} \right) X^\pm R_{10} \left( \frac{z_1}{wc} \right) \cdots R_{k0} \left( \frac{z_k}{wc} \right) \]

if \( X^\pm = X^\pm_{[z_1, \ldots, z_k]} \in \mathcal{A}^\pm \), where we recall that \(-_+ = \cdot\) and \(-_- = \cdot^{\text{op}}\). Finally:

\[ (q^2 - 1) \sum_{k \in \mathbb{Z}} \left( s^+_{[i;i+n]} t^+_{[i;j+n]} s^-_{[j;j+n]} \right) c^{-d^d c^{d-1}} - t^-_{[i;j+n]} s^-_{[j;j+n(d+d-k)]} c^{-d^d c} \]

(we set \( s^+_{[i;j]} = t^+_{[i;j]} = 0 \) if \( i > j \)).

**Proof.** Formulas (6.31) and (6.32) are proved just like (2.42) and (2.43) (the presence of \( c \) and \( \tilde{c} \) are due to the twist in the coproduct (6.10)), and so we leave them as exercises to the interested reader. As far as (6.33) is concerned, we note the definition (6.11) of the coproduct implies the following analogue of formula (2.47):

\[ \Delta \left( E_{ij}^{\pm} \right) = E_{ij}^{\pm} \otimes 1 + \sum_{a,b \geq 0} s_{[x;i+n]}^{\pm} t_{[y;j+n]}^{\pm} c^{a+b+c-1} \otimes E_{xy}^{\pm} \]

in \( \tilde{A}^+ \), as well as the following analogue of (2.48):

\[ \Delta \left( E_{ij}^{\pm} \right) = \sum_{a,b \geq 0} E_{xy}^{\pm} \otimes t_{[y;j+n]}^{\pm} s_{[x;i+n]}^{\pm} c^{d^d a - b} + 1 \otimes E_{ij}^{\pm} \]

in \( \tilde{A}^{-,\text{op,coop}} \). Then (6.33) is simply an application of (2.39).

**Proof.** of Theorem 1.5 (in the formulation of Subsection 3.31): Let us write:

\[ \mathcal{A}^0 \subset \mathcal{A} \]

for the subalgebra generated by the coefficients of the series \( S^\pm(x) \) and \( T^\pm(x) \). Then \( \mathcal{A}^0 \) is isomorphic to the subalgebra \( D^0 \) of Subsection 3.27 because they are both isomorphic to the algebra \( \mathcal{E} \) of Definition 3.33. Moreover, Theorem 5.25 (and its analogue when \( \mathcal{A}^+ \) is replaced by \( \mathcal{A}^- \)) give rise to algebra isomorphisms:

\[ \Upsilon^\pm : D^\pm \rightarrow \mathcal{A}^\pm \]
Putting the preceding remarks together yields an isomorphism of vector spaces:

$$D \cong D^+ \otimes D^0 \otimes D^- \xrightarrow{\Upsilon} A^+ \otimes A^0 \otimes A^- \cong A$$

where the first isomorphism holds by definition, and the last isomorphism follows from \((6.30)\). To show that \(\Upsilon\) is an algebra isomorphism, one needs to show that:

$$\Upsilon(ab) = \Upsilon(a) \Upsilon(b)$$

for all \(a, b \in D\). By Proposition \((3.43)\) we may assume that:

$$a = x_1...x_k \alpha y_1...y_l \quad \text{and} \quad b = x'_1...x'_{k'} \beta y'_1...y'_l'$$

for various \(x_i, x'_i \in D^+\) of \(vdeg = 1\), \(y_i, y'_i \in D^-\) of \(vdeg = -1\), and \(\alpha, \beta \in D^0\). To compute the left-hand side of \((6.37)\), one takes:

$$ab = x_1...x_k \alpha y_1...y_l x'_1...x'_{k'} \beta y'_1...y'_l'$$

and uses relations \((3.114)-(3.117), (3.125)-(3.128), (3.129)\) to write it as:

$$ab = \sum \text{coefficient} \cdot x'_1...x'_{u} \gamma y'_1...y'_l'$$

for various \(x'_i \in D^+\) of \(vdeg = 1\), \(y'_i \in D^-\) of \(vdeg = -1\), and \(\gamma \in D^0\). Similarly:

$$\Upsilon(a) \Upsilon(b) = \Upsilon(x_1)...\Upsilon(x_k) \Upsilon(\alpha) \Upsilon(y_1)...\Upsilon(y_l) \Upsilon(x'_1)...\Upsilon(x'_{k'}) \Upsilon(\beta) \Upsilon(y'_1)...\Upsilon(y'_l')$$

can be expressed using relations \((6.8)-(6.9), (6.11)-(6.32)\) and \((6.33)\) as:

$$\Upsilon(a) \Upsilon(b) = \sum \text{coefficient} \cdot \Upsilon(x'_1)...\Upsilon(x'_{u}) \Upsilon(\gamma) \Upsilon(y'_1)...\Upsilon(y'_l')$$

where the coefficients and the various \(x'_i, y'_i, \gamma\) are the same ones as in \((6.38)\). The reason for the latter fact is that the squiggly underlined relations above match each other pairwise. It is clear that \(\Upsilon\) applied to the right-hand side of \((6.38)\) is precisely the right hand side of \((6.39)\), thus completing the proof.

\[\square\]

6.10. We will now study the bialgebra pairing between \(\tilde{A}^+\) and \(\tilde{A}^-\) in more detail, with the goal of proving certain formulas that will be used in \([18]\). Let us consider the restriction of the pairing \((6.24)\) to the following subalgebras:

$$\mathcal{B}^-_{\mu} \otimes \mathcal{B}^0_{\mu} \xrightarrow{(\cdot, \cdot)} \mathbb{Q}(q, q^\pm)$$

for all \(\mu \in \mathbb{Q} \cup \infty\).

**Proposition 6.11.** For all \(\mu \in \mathbb{Q} \cup \infty\), the pairing \((6.40)\) is a bialgebra pairing, i.e. it intertwines the product with the coproduct \(\Delta_{\mu}\), in the sense of \((2.28)-(2.29)\).

**Proof.** Let us prove \((2.28)\), and leave \((2.29)\) as an exercise to the interested reader. Take formula \((6.27)\), which we already proved in the course of Proposition 6.7

$$\left\langle A^+ * B^+, Y^- \right\rangle = \left\langle B^+ \otimes A^+, \Delta(Y^-) \right\rangle$$

If we let \(A^+, B^+ \in \mathcal{B}^+_{\mu}\) and \(Y^- \in \mathcal{B}^-_{\mu}\), then our task is equivalent to showing that the formula above holds with \(\Delta\) replaced by \(\Delta_{\mu}\). Since \(Y^- \in \mathcal{B}^-_{\mu}\), we have:

$$\Delta(Y^-) = \Delta_{\mu}(Y^-) + (\text{slope } < \mu) \otimes (\text{slope } > \mu)$$

(the reason why the formula above differs from \((5.11)\) is that slope lines are reflected across the horizontal line when going from \(A^+\) to \(A^-\)). All the summands in the
right-hand side of (6.42) other than $\Delta_{\mu}(Y^{-})$ pair trivially with $B^{+} \otimes A^{+}$, for degree reasons. This implies that (6.41) holds with $\Delta$ replaced by $\Delta_{\mu}$.

As a consequence of Proposition 6.11 the Drinfeld double:
\[
\mathcal{B}_{\mu} = \mathcal{B}_{\mu}^{\oplus} \otimes \mathcal{B}_{\mu}^{\leq \text{op,coop}} \left( (\psi_{i} \otimes 1)(1 \otimes \psi_{i}^{-1}) - 1 \otimes 1 \right)
\]
defined with respect to the coproduct $\Delta_{\mu}$, will be a subalgebra of the algebra $\mathcal{A}$ of (6.30). Moreover, the following diagram commutes, for all $\mu \in \mathbb{Q} \sqcup \infty$:
\[
\begin{array}{ccc}
\mathcal{E}_{\mu} & \longrightarrow & \mathcal{D} \\
\sim & & \sim \\
\mathcal{B}_{\mu} & \longrightarrow & \mathcal{A}
\end{array}
\]
where the vertical map on the left is the natural double of the isomorphism $\Upsilon_{\mu}$ of Proposition 5.23 and the vertical map on the right is the isomorphism $\Upsilon$ of (6.36).

6.12. Let us consider any $\mu \in \mathbb{Q} \sqcup \infty$. As we have seen in Subsection 3.22 the subalgebra $\mathcal{E}_{\mu} \subset \mathcal{D}$ is generated by the elements:
\[
\left\{ f^{(\pm k)}_{\pm[i,j]}, f^{(\pm k)}_{\pm[i,j]} \right\}_{i,j} \in \mathbb{Z} \otimes \mathbb{N}, k \in \mathbb{N}
\]
which satisfy the coproduct relations of Proposition 3.24 as well as formulas (3.64)–(3.65) for their images under the maps $\alpha_{\pm[i,j]}$. When we pass the elements above under the vertical isomorphisms of diagram (6.43), we obtain:
\[
\left\{ F^{(\pm k)}_{\pm[i,j]}, F^{(\pm k)}_{\pm[i,j]} \right\}_{i,j} \in \mathbb{Z} \otimes \mathbb{N}, k \in \mathbb{N}
\]
defined in (5.31)–(5.32) when the sign is $+$, and in (6.4)–(6.5) when the sign is $-$. As proved for the case $\pm = +$ in Proposition 5.17 (the case $\pm = -$ is analogous, and we leave it as an exercise to the interested reader), the elements $F, F$ satisfy the analogous coproduct relations as $f, f$. Similarly, we have the following formulas:
\[
\alpha_{\pm[i,j]}(F^{(\pm k)}_{\pm[i',j']}) = \delta^{(i,j)}_{(i',j')}(1 - q^{2})q_{\pm}^{\frac{\gcd(j-i,k)}{n}}
\]
\[
\alpha_{\pm[i,j]}(\bar{F}^{(k)}_{\pm[i',j']}) = \delta^{(i,j)}_{(i',j')}(1 - q^{-2})\bar{q}_{\pm}^{-\frac{\gcd(j-i,k)}{n}}
\]
where the linear maps:
\[
\bigoplus_{k=0}^{\infty} A_{\pm[i,j], \pm}^{\alpha_{\pm[i,j]}} \longrightarrow \mathbb{Q}(q, q^{-\frac{1}{n}})
\]
are given by:
\[
X_{1,...,k}(z_{1},...,z_{k}) \xrightarrow{\alpha_{\pm[i,j]}} \text{coefficient of } E_{ij} \text{ in } X^{(k)}(y)(1 - q^{2})^{k}q_{+}^{\frac{k(1-\pm)(j-i)+k-2k\pm}{n}}
\]
\[
Y_{1,...,k}(z_{1},...,z_{k}) \xrightarrow{\alpha_{\pm[i,j]}} \text{coefficient of } E_{ij} \text{ in } Y^{(k)}(y)(1 - q^{2})^{k}\bar{q}_{-}^{\frac{k(1-\pm)(j-i)+k-2(k-1)+2i}{n}}
\]
Recall that $X^{(k)}(y) \in \text{End}(V)[y^{\pm 1}]$ was defined by Figure 14, for any $X \in A_{k}$. Meanwhile, for any $Y \in A_{-k}$, the symbol $Y^{(k)}(y)$ is defined analogously, but with $\bar{q}$ replaced by $\bar{q}_{-}$, and the operators $D_{1}...D_{k}$ placed in front of the braid in Figure
14. Formulas (6.44)–(6.45) were proved when \( \pm = + \) in Proposition 5.17, and the case when \( \pm = - \) is an analogous exercise that we leave to the interested reader.

6.13. As we have seen in Subsection 3.25, the subalgebra \( \mathcal{E}_\mu \) is also generated by the primitive elements:

\[
\begin{align*}
\{ p^{(\pm b)}_{\pm[i;i+a]}, p^{(\pm i+a)}_{\pm[i+a]} \}_{i \in \mathbb{Z}/g\mathbb{Z}} & \quad \text{of (3.70)}
\end{align*}
\]

where \( \mu = \frac{a}{b} \) with \( \gcd(a,b) = 1, \ b \geq 0 \) and \( g = \gcd(n,a) \). These primitive elements satisfy formulas (3.68)–(3.69). When we pass the elements above under the vertical isomorphisms of diagram (6.43), we obtain:

\[
\begin{align*}
\{ P^{(\pm b)}_{\pm[i;i+a]}, P^{(\pm i+a)}_{\pm[i+a]} \}_{i \in \mathbb{Z}/n\mathbb{Z}} & \quad \in \mathcal{B}_\mu
\end{align*}
\]

The elements (6.47) are primitive for the coproduct \( \Delta_\mu \), and moreover satisfy the following analogues of formulas (3.68) and (3.69):

\[
\begin{align*}
\alpha_{\pm[i;v]} \left( P^\mu_{\pm[i;i+a]} \right) & = \pm 1 \\
\alpha_{\pm[s+tn]} \left( P^\mu_{\pm[i;i+a]} \right) & = \pm s \mod g
\end{align*}
\]

6.14. Proposition 3.24 gives us formulas for the coproducts of the elements (3.62). Meanwhile, the elements (3.70) are primitive, so there are no intermediate terms in their coproduct. Therefore, we may apply relation (2.39) between the two halves of the bialgebra \( \mathcal{B}_\mu \), and obtain the following formulas:

\[
\begin{align*}
\left[ P^\mu_{\pm[i;j]}, F^\mu_{\mp[i';j']} \right] & = \left< P^\mu_{\pm[i;j]}, F^\mu_{\mp[i';j']} \right> \left[ \left( \frac{\psi_{i,j}}{\psi_{j'}} \right)^{\pm 1} - \left( \frac{\psi_{i,j}}{\psi_{j'}} \right)^{\mp 1} \right] \\
\left[ P^\mu_{\pm[i;j]}, \bar{F}^\mu_{\mp[i';j']} \right] & = \left< P^\mu_{\pm[i;j]}, \bar{F}^\mu_{\mp[i';j']} \right> \left[ \left( \frac{\psi_{i,j}}{\psi_{j'}} \right)^{\pm 1} - \left( \frac{\psi_{i,j}}{\psi_{j'}} \right)^{\mp 1} \right]
\end{align*}
\]

for all \((i, j), (i', j') \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}\) such that \( j - i = j' - i' \in \mathbb{N}_\mu \). Similarly, we have:

\[
\begin{align*}
\left[ P^\mu_{\pm[l;e]}, F^\mu_{\mp[e';j']} \right] & = \left< P^\mu_{\pm[l;e]}, F^\mu_{\mp[e';j']} \right> \left[ c^{\mp l} e^{\pm n} - c^{\mp l} e^{\mp n} \right] \\
\left[ P^\mu_{\pm[l;e]}, \bar{F}^\mu_{\mp[e';j']} \right] & = \left< P^\mu_{\pm[l;e]}, \bar{F}^\mu_{\mp[e';j']} \right> \left[ c^{\mp l} e^{\pm n} - c^{\mp l} e^{\mp n} \right]
\end{align*}
\]

for all \( l \in \mathbb{Z} \) and \((i', j') \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}\) such that \( nl = j' - i' \in \mathbb{N}_\mu \), and \( r \in \mathbb{Z}/g\mathbb{Z} \) where \( g = \gcd(n, \text{numerator } \mu) \). Similar formulas were worked out in [17] between the \( p \) and \( f, \bar{f} \) generators in \( \mathcal{E}_\mu \), but with explicit numbers instead of the pairings in the right-hand side. Therefore, the fact that \( \Upsilon_\mu \) is an isomorphism implies the following
explicit formulas for the pairings above:

\begin{align}
\langle P_\pm^{(i,j)} F_\pm^{(i',j')} \rangle &= \pm \delta_{(i,j)}^{(i',j')} \cdot \frac{\gcd(k,j-i)}{n} \\
\langle P_\pm^{(i,j)} F_\pm^{(i',j')} \rangle &= \pm \delta_{(i,j)}^{(i',j')} \cdot \frac{\gcd(k,j-i)}{n} \\
\langle P_\pm^{\delta,r} F_\pm^{[i',j']} \rangle &= \pm \delta_r \mod g \cdot \frac{\gcd(k,n)}{n} \\
\langle P_\pm^{\delta,r} F_\pm^{[i',j']} \rangle &= \pm \delta_r \mod g \cdot \frac{\gcd(k,n)}{n}
\end{align}

for all applicable indices. Comparing the formulas above with \((6.48)-(6.49)\) yields:

\begin{proposition}
For any \((i,j) \in \frac{\mathbb{Z}^2}{(n,n)\mathbb{Z}}\) and \(k \in \mathbb{N}\) such that \(\mu = \frac{i-j}{k}\), we have:

\begin{align}
\langle X, F^{(-k)}_{i,j} \rangle &= \alpha_{[i,j]}(X) \cdot \frac{\gcd(j-i,k)}{n} \\
\langle F^{(k)}_{i,j}, Y \rangle &= \alpha_{-[i,j]}(Y) \cdot \frac{\gcd(j-i,k)}{n}
\end{align}

for all \(X \in B^+_{\mu} \) and \(Y \in B^-_{\mu} \).

\end{proposition}

\begin{proof}
We will only prove \((6.55)\) as it will be used in [18], and leave the analogous formula \((6.54)\) as an exercise to the interested reader. Comparing formulas \((6.48)-(6.49)\) with \((6.51)-(6.53)\) shows us that formula \((6.55)\) holds when \(Y\) is one of the primitive generators of \(B^+_{\mu} \). Therefore, all that remains to show is that if \((6.55)\) holds for \(Y, Y' \in B^-_{\mu} \), then it also holds for \(Y \ast Y' \). This happens by comparing:

\begin{align}
\langle F^{\mu}_{i,j}, Y \ast Y' \rangle &= \begin{cases} \langle F^{\mu}_{i,j}, Y \rangle \langle F^{\mu}_{i,j}, Y' \rangle & \text{if } \exists s \text{ s.t. } \text{hdeg } Y = -[s;j], \text{hdeg } Y' = -[i;s] \\
0 & \text{otherwise} \end{cases} \\
\end{align}

with \((3.31)\).
\end{proof}

\begin{thebibliography}{9}
\bibitem{Beck} Beck J., \textit{Braid group action and quantum affine algebras}, Commun. Math. Physics vol. 165 (1994), 555-568
\bibitem{Burban} Burban I., Schiffmann O., \textit{On the Hall algebra of an elliptic curve}, I, Duke Math. J. 161 (2012), no. 7, 1171–1231
\bibitem{Ding} Ding J., Frenkel I., \textit{Isomorphism of two realizations of quantum affine algebra \(U_q(\widehat{\mathfrak{g}_n})\)}, Comm. Math. Phys. 156 (1993), no. 2, 277–300
\bibitem{Ding2} Ding J., Iohara K., \textit{Generalization of Drinfeld quantum affine algebras}, Lett. Math. Phys., 41 (1997), no. 2, 181–193
\bibitem{Drinfeld} Drinfeld V. G., \textit{A new realization of Yangians and quantized affine algebras}, Soviet Math. Dokl. 36 (1988), 212–216
\bibitem{Fulco} Enriquez B., \textit{On correlation functions of Drinfeld currents and shuffle algebras}, Transform. Groups 5 (2000), no. 2, 111 - 120
\bibitem{Fad} Faddeev L., Reshetikhin N., Takhtajan L., \textit{Quantization of Lie groups and Lie algebras}, Leningrad Math. J. 1 (1990) 193–226
\bibitem{Feigin} Feigin B., Hashizume K., Hoshino A., Shiraishi J., Yanagida S., \textit{A commutative algebra on degenerate \(CP^1\) and MacDonald polynomials}, J. Math. Phys. 50 (2009), no. 9
\bibitem{Feigin2} Feigin B., Jimbo M., Miwa T., Mukhin E., \textit{Representations of quantum toroidal \(g\mathfrak{sl}_n\)}, Journal of Algebra vol. 380 (2013), 78–108
\end{thebibliography}
[10] Feigin B., Odesskii A., *Vector bundles on elliptic curve and Sklyanin algebras*, Topics in Quantum Groups and Finite-Type Invariants, Amer. Math. Soc. Transl. Ser. 2, 185 (1998), Amer. Math. Soc., 65–84

[11] Jing N., Zhang H., *Hopf algebraic structures of quantum toroidal algebras*, arXiv:1604.05416

[12] Maulik D., Okounkov A., *Quantum groups and quantum cohomology*, Astérisque, Volume 408 (2019) 212 pp

[13] Miki K., *A \((q, \gamma)\) analog of the \(W_{1+\infty}\) algebra*, J. Math. Phys., 48 (2007), no. 12

[14] Mudrov A. I., *Reflection equation and twisted Yangians*, Journal of Mathematical Physics 48, 093501 (2007)

[15] Negut A., *Shuffle algebra revisited*, Int. Math. Res. Not., Volume 2014, Issue 22, 2014, 6242–6275

[16] Negut A., *Quantum toroidal and shuffle algebras*, Adv. Math., Volume 372 (2020), 107288

[17] Negut A., *PBW basis for \(U_q(\widehat{\mathfrak{gl}_n})\)*, arXiv:1905.06277

[18] Negut A., *Deformed \(W\)–algebras in type \(A\) for rectangular nilpotent*, arXiv:2004.02737

[19] Okounkov A., Smirnov A., *Quantum difference equation for Nakajima varieties*, arXiv:1602.09007

[20] Schiffmann O., *Drinfeld realization of the elliptic Hall algebra*, Journal of Algebraic Combinatorics, vol 35 (2012), no 2, 237–262

[21] Wendlandt C., *The \(R\)-Matrix Presentation for the Yangian of a Simple Lie Algebra*, Comm. Math. Phys., October 2018, Volume 363, Issue 1, 289–332

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