Obtaining Bounds on the Sum of Divergent Series in Physics

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Abstract

Under certain circumstances, some of which are made explicit here, one can deduce bounds on the full sum of a perturbation series of a physical quantity by using a variational Borel map on the partial series. The method is illustrated by applying it to various examples, physical and mathematical.

1 Introduction

As there are hardly any problems in physically relevant quantum field theories that are exactly solvable, one must resort to approximate calculations. The only analytical tool that in principle allows for such approximate computations to be systematically improved, and for the errors to be estimated, is perturbation theory. Unfortunately in many cases the perturbation parameter is not small so that the series, of which in practice only a few terms are known, diverges or gives a poor representation of the physical quantity. Consequently, several techniques have been used to estimate the full sum

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of an infinite series from the small number of given terms, see for example \cite{1, 4, 5} and references therein.

A variational method which was introduced in Ref.\cite{4} has the novelty of apparently giving a monotonically converging sequence of approximations and thus possible bounds on the sum of an infinite series. That method has been applied to the physically important case of the free energy of gauge theories at high temperature in \cite{4, 5} and the purpose of this paper is to provide a detailed technical justification for the methodology used there. In essence this paper is a condensed and slightly modified version of the unpublished report \cite{6}. The exposition here is not meant to satisfy a mathematical physicist but is rather suggestive and is illustrated with several examples to make some of the technical assumptions plausible. It is hoped that some readers will find the method sufficiently interesting to re-examine it more rigorously in the future.

In the following section, I describe in a detailed but abstract manner how the technique of \cite{4} can be used to obtain lower or upper bounds for some series, and also in many cases to obtain accurate estimates of the exact sum. Concrete examples are given in Sec.(3). A conclusion and summary of open problems appears in Sect.(4). Finally, the Appendix contains an approximate but important analysis of the relevant equations which explains some of the empirically observed trends.

2 The Method

Consider the expansion

$$\hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n ,$$

(1)

with $\lambda$ the perturbation parameter. It will be assumed in this paper that the relevant expansions are infinitely long, but that only a finite number of terms are known. Since perturbation expansions in quantum field theories have the generic behaviour $f_n \sim n!$ for $n$ large \cite{1}, it is natural to consider the Borel transform

$$B_N(z) = \sum_{n=0}^{N} \frac{f_n}{n!} z^n .$$

(2)

The series (1) can then be recovered from the Borel integral,

$$\hat{S}_N(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-z/\lambda} \ B_N(z) \ dz .$$

(3)
If the exact function $B(z) \equiv B_\infty(z)$ is known, Eq.(3) can be taken as defining the exact sum, $S = \hat{S}_\infty$, of the full perturbation series if the Borel integral is well-defined. The poor convergence of (1) can be attributed to the expansion of $B(z)$ in a power series as in (2) and the subsequent use of that series beyond its radius of convergence in (3).

Suppose initially that $B(z)$ has only one singularity in the complex $z$-plane (the Borel plane), at $z = -1/p$, with $p$ real and positive. Then the radius of convergence of the Borel series (2) is $1/p$. Therefore in order to construct an approximation to the exact sum, one must extend the domain of convergence of the partial series in Eq.(3). The method of Loeffel, Le-Guillou and Zinn-Justin \[7, 8\] is to use a conformal map

$$w(z) = \frac{\sqrt{1 + zp} - 1}{\sqrt{1 + zp} + 1},$$ (4)

which maps the $z$-plane into a unit circle in the $w$-plane, with the singularity at $z = -1/p$ mapped to $w = -1$. The inverse of this map is

$$z(w) = \frac{4w}{p} \frac{1}{(1 - w)^2}.$$ (5)

Making the change of variables (5) in (3) one obtains,

$$\hat{S}_N(\lambda) = \frac{1}{\lambda} \int_0^1 dw \frac{dz}{dw} e^{-z(w)/\lambda} B_N(z(w)),$$ (6)

where $B_N(z(w))$ is understood as a power series in $w$ obtained by an expansion of (2). In terms of the variable $w$, the series $B_N(z(w))$ converges for $|w| < 1$, so that the potential problem in (3) is only at the upper limit of integration. Now, the difference between $B_N(z(w))$ and $B_{N+1}(z(w))$ begins at order $w^{N+1}$. If one knows the coefficients $f_n$ only up to $n = N$, then it is consistent to keep only terms up to order $w^N$ in $B_N(z(w))$. Performing this truncation in (6) and riverting back to the $z$ variable, one obtains

$$S_N(\lambda) \equiv \frac{1}{\lambda} \sum_{n=0}^{n=N} \frac{f_n}{n!} \left(\frac{4}{p}\right)^n \sum_{k=0}^{N-n} \frac{(2n + k - 1)!}{k!(2n - 1)!} \int_0^\infty dz \, e^{-z/\lambda} w(z)^{(k+n)}.$$ (7)

$S_N(\lambda)$ is a non-trivial resummation of the original series $\hat{S}_N(\lambda)$. In Ref.\[8, 9\], Eq.(7) has been used to resum long perturbation expansions for critical
exponents, with \( z = -1/p \) the location of the instanton singularity of \( \phi^4_3 \) field theory. Note that after a scaling, (7) may be written as

\[
S_N(\lambda) \equiv \sum_{n=0}^{n=N} \frac{f_n}{n!} \left( \frac{4}{p} \right)^n \sum_{k=0}^{N-n} \frac{(2n+k-1)!}{k!(2n-1)!} \int_0^\infty dz \ e^{-z} \ w(\lambda z)^{(k+n)}.
\] (8)

Since \( w(z) \) is a bounded and slowly varying function, this shows that the resummed expression \( S_N(\lambda) \) is a much slower varying function of the coupling than the original series \( \hat{S}_N(\lambda) \).

The resummation (7) has also been applied to quantum chromodynamics (QCD), with \( z = -1/p \) the location of the first ultraviolet renormalon pole \([10, 11]\). Actually, QCD is Borel-nonsummable \([12]\), which means that \( B(z) \) also has poles for \( z > 0 \), rendering the Borel integral ambiguous. In such a case, one can still use (3) to define the sum of the perturbation series once the integral is made definite through some prescription such as the principal value.

In all the applications of (7) in \([8, 9, 10, 11]\), \( p \) is a known and fixed constant which determines the location of the singularity of \( B(z) \) closest to the origin. In Ref.\([4]\), the expression (7) was used as the starting point for a new technique which can be used even in cases when the singularity structure of \( B(z) \) is unknown. Indeed, as seen above, the conventional transformation of a series

\[
\hat{S}_N(\lambda) = \sum_{n=0}^N f_n \lambda^n.
\] (9)

by the Borel-conformal map method results in the reorganised expression

\[
S_N(\lambda, p) \equiv \frac{1}{\lambda} \sum_{n=0}^{n=N} \frac{f_n}{n!} \left( \frac{4}{p} \right)^n \sum_{k=0}^{N-n} \frac{(2n+k-1)!}{k!(2n-1)!} \int_0^\infty dz \ e^{-z/\lambda} \ w(z)^{(k+n)}.
\] (10)

with \( p \) a fixed constant. However, notice that \( p \) does not figure in (9) but enters (10) only through the conformal map (4). Thus instead of treating \( p \) as some fixed constant as in Ref.\([8, 9, 10, 11]\), one may consider \( p > 0 \) a free parameter which defines the conformal map (4) and Eq.(10) then represents a continuous family of resummations, one for each value of \( p \). Thus, from now on, \( p \) will not refer to the location of some singularity in \( B(z) \). In fact, except for the regularity of \( B(z) \) at the origin, no other knowledge about the singularity structure of \( B(z) \) is required for the method elaborated below.
Of course having liberated ourselves from the usual interpretation of $p$ in (10), we also lose definiteness in our resummation. Therefore some new condition must be imposed to fix the value of $p$ and hence of the expression (10). For a start, for each $N$, choose $p > 0$ to be the location of an extremum of $S_N(\lambda, p)$. Since (10) depends on $\lambda$, the value of $p$ in principle will also depend on $\lambda$. However the procedure then becomes too unwieldy, and to simplify it $p$ is determined at some reference value $\lambda = \lambda_0$, say at the midpoint of the range of interest:

$$\frac{\partial S_N(\lambda_0, p)}{\partial p} = 0.$$  \hspace{1cm} (11)

As mentioned after (8), $S(\lambda, p)$ is a relatively slowly varying function of $\lambda$, hence determining $p$ at some fixed $\lambda = \lambda_0$, and then using the same $p$ in (10) for various $\lambda$ is sufficient for practical purposes. Indeed, in many applications, one requires the sum of the series at one particular value of the coupling, or in a very narrow range, so the simplification made in (11), is both useful and sufficient.

In some cases, (11) will not have any solutions. For example, if all the $f_n$ are of the same sign, $S_N(\lambda, p)$ will be a monotonic function of $p$. Thus for the procedure to work, at least some of the $f_n$ must be of a different sign. This will be assumed to be the case from now on. Suppose next that $f_1 \neq 0$. Then for $p$ large, and since the $p$ dependence of $w(z)$ is mild,

$$S_N(p) \sim f_0 + \frac{f_1}{p}.$$  \hspace{1cm} (12)

Therefore for $f_1 < 0$, $S_N$ will first decrease as $1/p$ increases and then increase when the next term $f_m/p^m > 0$ dominates the sum. If the last non-zero term $f_N/p^N$ is positive then one expects $S_N(p)$ to have a global minimum at some $p > 0$. However if $f_N/p^N < 0$ then $S_N(p)$ can become very small as $p \to 0^+$ and so the minimum would likely be only a local extremum. From this heuristic argument one concludes that if the first non-zero coefficient $f_n, (n > 0)$ is negative, Eq. (11) will have global minima solutions for some $N$.

For conciseness, unless otherwise stated, I will discuss from now on only the case when a global minimum exists, as the arguments for the global maximum case are then obvious. For a series $S_N(\lambda, p)$, define $p(N)$ to be the position of the global minima for the values of $N$ when they exist, and for other values of $N$ let $p(N)$ denote the location of the local minima. Also, let
\[ S_N \equiv S_N(\lambda, p(N)). \] (13)

Obviously the definition of \( p(N) \) has been chosen because it is useful. If \( p(N) \) is a location of a global minimum, then \( S_N \) certainly is a lower bound on \( S_N(\lambda, p) \). However this does not imply that \( S_N \) is a lower bound on the exact sum \( S \) of the perturbation series because for finite \( N \), \( S \) might lie outside of the space of resummations labelled by \( p \). For each \( N \), let \( p^*(N) \) be the value of \( p \) that is optimal, that is, it is the value which when used in (10) gives the best estimate of \( S \). Define, \( S_N^* = S_N(\lambda, p^*(N)) \). Then for a global minima one has

\[ S_N \leq S_N^* \] (14)

Presumably \( S_N^* \) converges to \( S \) as \( N \to \infty \). Then for those \( N \) when global minima exist,

\[ S_{N \to \infty} \leq S. \] (15)

(This implicitly assumes that the sub-sequence of global minima is infinite: That is, given any positive integer \( N_0 \), there is some \( n > N_0 \) for which \( f_n \) is positive.)

Though the inclusion of global minima in the definition of \( p(N) \) is quite clear, the inclusion of local minima requires some explanation. As mentioned earlier, if \( S_N(\lambda, p) \) has a global minimum, then \( S_{N+1}(\lambda, p) \) will probably not have a global minimum if \( f_{N+1} \) is negative because \( S_{N+1}(\lambda, p) \) might become very small as \( p \to 0^+ \). However \( S_{N+1}(\lambda, p) \) will still have a local minimum (and hence also a local maximum). The local minimum will occur for moderate values of \( p \) close to \( p(N) \), the global minimum position of \( S_N(\lambda, p) \). Therefore one might expect that the local minimum for \( S_{N+1}(\lambda, p) \) still gives a bound on \( S_{N+1}^* \). Indeed, suppose that the inequality

\[ S_N \leq S_{N+1} \] (16)

holds for all \( N \) up to \( N = \infty \), then clearly

\[ S_N \leq S \] (17)

and one concludes that \( S_N(\lambda, p(N)) \) is a lower bound on the sum of the full perturbation series. Of course proving (16) is equivalent to knowing \( f_n \) explicitly for all \( n \), which is not the situation in reality. As a practical matter, it is sufficient to draw a conclusion after observing the trend (16) (or lack
of) for the available terms of the series. Arguments given in the Appendix indicate that the trend, once started, will continue.

An interesting question is whether the inequality (15) is saturated. In most of the examples studied this has been found to be the case. In fact the convergence is so rapid that one can conclude with a high degree of confidence, not only that $S_N$ is a lower bound, but that it is close to the exact value. However the example in Sec.(3.5) shows that the series $S_N$, though forming a bound on the actual value $S$, might not converge to $S$. In that example the exact result $S(\lambda)$ has a first derivative $dS(\lambda)/d\lambda$ that varies rapidly with $\lambda$. An explanation of why in such cases the bounds $S_N$ do not converge to the exact value is given in the Appendix. However, as described below, such situations can also be taken care of by the resummation procedure (10-11).

Recall that when a local minima occurs for some $N$, one expects also a local maximum. Define $\bar{p}_N$ to be the position of those local maxima and $\bar{S}_N$ the corresponding value of the resummed series. It turns out that $\bar{S}_N$ also satisfies an inequality like (10) (see Sec.(3.5)). However since for the sequence $\bar{S}_N$ of only local extrema one does not seem to have a statement like (14-15), it is not a priori obvious that they form bounds to the exact result. Of course if one believes that the $p^\star(N)$ as defined above always take moderate values, then a statement like (14) might also be made for the sequence $\bar{S}_N$.

Furthermore, given a series which has global minima solutions to (11), then by ‘removing’ the first nontrivial coefficient of that series one forms an auxiliary series, $S'_N(\lambda, p)$, which has global maxima solutions. In this way one can form alternative bounds to the sum of the original series complementing those obtained from $S_N$. This is illustrated in Sec.(3.5).

Using the sequences $S_N$, $\bar{S}_N$, and $S'_N$, one can obtain constraints on $S$ and, with some physical input, an estimate of $S$ itself. A question now arises for practical problems where exact results are not known. Is it possible to determine whether the unknown exact result lies closer to the upper or lower bound? Empirically, it appears that the series $S_N$ converges to the exact result $S(\lambda)$ if $\partial^2 S(\lambda)/\partial \lambda^2$ is very small in magnitude in the entire range of interest, $0 < \lambda < \lambda_0$. Otherwise the exact result seems to be better approximated by the series $\bar{S}_N$ or by the auxiliary series method discussed above. An explanation of why the bounds $S_N$ have slowly varying first derivatives, and why the bounds $\bar{S}_N$ and those from the auxiliary series have faster varying first derivatives is given in the Appendix.

Although the main discussion in this paper will be for the $p(N)$ (or $\bar{p}(N)$
and $p'(N)$ as defined above, in some cases one finds (by inspection) solutions $p_0(N)$ to (11) which have the property that as $N \to \infty$, $p_0(N) \to p_0$, a constant. In such a case one is tempted to speculate that the fixed point $p_0$ indicates the existence of a singularity at $z = -1/p$ in $B(z)$. This is indeed found to be the case in the examples studied, though apparently the converse is not necessarily true: the singularities of $B(z)$ need not show up as solutions of (11). Furthermore, the convergence of the series $S_{(0)N}$ is not expected to be monotonic since in general the $p_0(N)$ refer to both maxima and minima.

Though not manifest at first sight, the analysis in the Appendix suggests that even the $p(N)$ defined above Eq. (13) actually will converge to a fixed value as $N \to \infty$. This trend can be observed in the examples studied.

3 Examples

In this section a number of examples with different properties are studied to illustrate the resummation technique of Eqs. (10-11) and to show that the various technical assumptions made there are not vacuous. Except for the last example, all define Borel-summable series, that is, $B(z)$ has no singularities at real, positive $z$. A Borel-nonsummable example will be considered in Sec.(3.3). All of the examples here have global minima as solutions to the extremum condition (11) for some $N$. (From any such series $S_N$ one can trivially construct $C - S_N$, with $C$ a constant, which then gives an example of a series with global maxima solutions to (11). The reader is reminded that the existence of global minima does not by itself imply that one has obtained a lower bound on the sum of the series. The additional condition (16) must be satisfied for a lower bound. The Euler-Heisenberg series below shows how one can end up with an upper bound from global minima! (See also the Appendix).

It is also re-emphasized that although in these examples the exact singularity structure of $B(z)$ is known, that information will not be used in the resummation. The resummation of the partial series

$$\hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n$$

will proceed using Eqs. (10-11). The exact $B(z)$ will only be used to compare the resummed results with the exact sum of the full series given by the Borel integral.
\[ S(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-z/\lambda} B(z) dz . \] (19)

In addition to the examples below, the interested reader may find others in [6].

### 3.1 A Prototype

Consider a model defined by the Borel function,

\[ B(z) = \frac{1}{1 + z} . \] (20)

Expanding \( B(z) \) to \( N \)-th order in \( z \) and using the result in (19) gives the truncated series

\[ \hat{S}_N = \sum_{n=0}^{N} (-\lambda)^n n! . \] (21)

The divergent nature of this series is displayed in Fig.(1a). Using \( f_n = (-1)^n n! \) in (10) and solving Eq.(11) at the reference value \( \lambda_0 = 1 \) gives the following solutions (minima): \( p(2) = 2.65 \), \( p(3) = 5.1 \) and \( p(4) = 8.4 \). As expected from the arguments of Sec.(2), there is no solution for \( N = 1 \), the solutions for \( N = 2 \) and \( N = 4 \) are global minima while that for \( N = 3 \) is a local minimum. The resummed series is shown in Fig.(1b). The convergence of the \( S_N \) is monotonic and satisfies the condition (16). Hence the approximants \( S_N \) can be argued to form lower bounds to the exact result. That this is indeed the case can be seen by inspection of the exact result in Fig.(1b) as obtained from (20) and (19). Furthermore, it is clear that the lower bounds converge to the exact value, and so may be used to estimate it. At \( \lambda = 0.5 \), the exact value is 0.722657, while the approximants are \( S_2 = 0.704 \), \( S_3 = 0.709 \), \( S_4 = 0.711 \).

For this model there are other solutions, for each \( N \), to the extremum condition (17) in addition to the minima. By inspection one picks out the sequence of values, \( p_0(3) = 1.6 \) (local maximum), \( p_0(4) = 1.3 \) (local minimum) and \( p_0(5) = 1.15 \) (local maximum) as plausibly approaching a fixed point. Indeed we already know that in this model the exact singularity of \( B(z) \) is a pole at \( z = -1/p = -1 \), so the fixed point probably refers to the location of this singularity. It should come as no surprise that if the approximants (10) are evaluated at the values \( p_0(N) \), the convergence to the exact
value will be much faster, and this is indicated in Fig.(1c). At $\lambda = 0.5$ the values of the approximants are $S_{(0)3} = 0.726$, $S_{(0)4} = 0.7219$, $S_{(0)5} = 0.7228$. Notice however that in this case the convergence is not monotonic (and was not expected to be).

### 3.2 Beta function of a Two-Dimensional Field Theory

In [13] the exact beta function in a two-dimensional field theory was obtained and the result checked explicitly up to three-loop order. After a scaling, that beta function is essentially of the form

$$S(\lambda) = \frac{1}{1 + \lambda}. \quad (22)$$

In this case a power expansion of $S(\lambda)$ is actually convergent for $|\lambda| < 1$. However close to $\lambda = 1$, the convergence is very slow and one requires a large number of terms of the series to obtain accurate results. The truncated perturbation series corresponding to (22) is

$$\hat{S}_N = \sum_{0}^{N} (-1)^n \lambda^n. \quad (23)$$

Notice that there is no factorial growth of the coefficients in (23). The slow convergence of this series for $\lambda$ close to 1 is displayed in Fig.(2a).

The utility of the resummation procedure in this case is now demonstrated. Using $f_n = (-1)^n$ in (10) and solving Eq.(11) at the reference value $\lambda_0 = 1$, gives the following solutions (minima): $p(2) = 1.2$, $p(3) = 2.9$, $p(4) = 5$. The resummed series is shown in Fig.(2b) together with the exact result. The convergence is rapid, monotonic, satisfies the condition (16), and the approximants $S_N$ form lower bounds to the exact result. With only four terms, the resummed series already shows a dramatically improved convergence compared to the original series (23), even for couplings as large as $\lambda = 0.8$: The exact sum at that coupling is 0.556, while the resummed values are $S_2 = 0.492$, $S_3 = 0.507$, $S_4 = 0.512$.

This example illustrates that the resummation procedure (10-11) is useful even for a convergent series, especially when one is close to the radius of convergence and when only a few terms of the series are available (which is often the situation in practice). Furthermore this example emphasizes that the parameter $p$ has no obvious relation to the singularities of $B(z)$. Indeed
the function (22) corresponds to the Borel transform

\[ B(z) = e^{-z}. \]  

(24)

which is regular everywhere in the Borel plane.

### 3.3 A Borel-Nonsummable Model

For most of the physical quantities calculable from the Standard Model of particle physics, the perturbation expansion is not expected to be Borel summable. In simple terms, this means that the function \( B(z) \) has poles on the positive semi-axis of the Borel plane thus rendering the Borel integral (19) ambiguous. If one choses an \( i\epsilon \) prescription then the resulting ambiguity is in the imaginary part. Sometimes these imaginary parts are of direct physical relevance [14, 3]. More generally they indicate that the perturbation series does not give the full answer, but must be supplemented with some non-perturbative contributions [10, 12]. Since the imaginary parts will be of the form \( e^{-1/\lambda} \), the additional real non-perturbative terms are expected to take the same form. These expectations have been confirmed in some lower dimensional field theories (see [12] and references therein).

One can also construct mathematical models to illustrate the arguments of the last paragraph. Suppose, for simplicity, that the only singularity of \( B(z) \) for \( z > 0 \) is a single pole at \( z = q \). Then the sum of the perturbation series can be defined by the principal value prescription,

\[ S_{\text{pert}} \equiv \frac{1}{\lambda} \mathcal{P} \int_{0}^{\infty} dz \ e^{-z/\lambda} B(z). \]  

(25)

With this definition, one focuses only on the real part of the physical quantity. Suppose furthermore, again for simplicity, that \( B(z) \) is integrable at infinity. Then the Eq.(25) may be rewritten as

\[ S_{\text{pert}} = \frac{1}{\lambda} \int_{0}^{\infty} dz \ (e^{-z/\lambda} - e^{-q/\lambda}) B(z) + \frac{e^{-q/\lambda}}{\lambda} \mathcal{P} \int_{0}^{\infty} \ dz \ B(z). \]  

(26)

The first term on the right-hand-side is finite and unambiguous. Call it \( S_{\text{exact}} \), and denote the second term on the right-hand-side as \( S_{np} \). Thus we have

\[ S_{\text{exact}}(\lambda) = S_{\text{pert}}(\lambda) - S_{np}(\lambda). \]  

(27)
In this way, the exact result $S_{\text{exact}}$, has been broken into two components, $S_{\text{pert}}$ which is purely perturbative, and $S_{\text{np}}$ which is purely non-perturbative. However while $S_{\text{exact}}$ is well-defined, both $S_{\text{pert}}$ and $S_{\text{np}}$ are only defined through the principal value prescription. Though highly simplified, this model plausibly represents the situation, for example, in Quantum Chromodynamics (QCD).

In order to illustrate explicitly the resummation technique (10-11) in the Borel-nonsummable case, set

$$B(z) = \frac{1}{(1+z)(5-z)}. \quad (28)$$

Then,

$$S_{\text{pert}}(\lambda) = \frac{1}{\lambda} \mathcal{P} \int_0^\infty \frac{e^{-z/\lambda}}{(1+z)(5-z)}. \quad (29)$$

The truncated perturbation series corresponding to (29) is

$$\hat{S}_N = \sum_{n=0}^{N} f_n \lambda^n, \quad (30)$$

with

$$f_n = ((-1)^n + 5^{-(n+1)}) \frac{n!}{6}. \quad (31)$$

The divergent series is oscillatory, analogous to that Fig.(1a). The solution of Eq.(11) at the reference value $\lambda_0 = 1$, gives (minima): $p(2) = 2.8$, $p(3) = 5.4$, $p(4) = 9$, $p(5) = 13.5$. The resummed partial series is shown in Fig.(3a) together with the exact sum of the full perturbation series given by (29). The convergence is rapid, monotonic, satisfies the condition (16), and the approximants $S_N$ do indeed form lower bounds. Also, the convergence is so fast that, except for a small interval at intermediate coupling, $S_5$ is not only a lower bound, but also a very good approximation to $S_{\text{pert}}$.

Now, corresponding to the Borel function (28), one has from the definitions before Eq.(27),

$$S_{\text{np}}(\lambda) = \frac{\ln 5}{6\lambda} e^{-5/\lambda} \quad (32)$$

and

$$S_{\text{exact}}(\lambda) = \frac{1}{\lambda} \int_0^\infty \frac{e^{-z/\lambda} - e^{-5/\lambda}}{(1+z)(5-z)}. \quad (33)$$
In Fig.(3b), the curves for $S_{\text{exact}}$ and $S_5$ are plotted for a bigger range of $\lambda$. This shows that except for a small range of couplings, the resummed perturbative approximant $S_5$ lies above and deviates significantly from the exact result although, as seen above, $S_5$ does form a converging lower bound to the perturbative component $S_{\text{pert}}$ of the exact result. The difference is of course the non-perturbative piece $S_{\text{np}}$. In the same figure there is a plot of $(S_5 - S_{\text{np}})$ which, according to the definition (27) should approximate $S_{\text{exact}}$. Indeed the agreement is very good and, amusingly, it improves at large coupling: At $\lambda = 5$, $S_{\text{exact}} = 0.0533$ and $S_5 - S_{\text{np}} = 0.0457$, while at $\lambda = 10$, $S_{\text{exact}} = 0.0219$ and $S_5 - S_{\text{np}} = 0.0193$.

This toy model discussion illustrates concretely the arguments given in Ref.[4] for the free-energy density of thermal SU(3) gauge theory. There it was found that the exact result, given by lattice data, differed significantly from the resummed perturbative result, and it was argued that the difference was caused by the Borel-nonsummability of the theory. Assuming a non-perturbative component of the form $A e^{-q/\lambda}$, the constants $A$ and $q$ were determined from the difference between $S_{\text{exact}}$ and $S_5$. A more detailed discussion of the results in [4] and their extension to full QCD and other non-Abelian gauge theories is presented in [4].

### 3.4 The Euler-Heisenberg Series

Schwinger’s [15] effective Lagrangian (density) for QED in a uniform magnetic field $B$ is,

$$L(\lambda) = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left( \coth(s) - \frac{1}{s} - \frac{s}{3} \right) e^{-s/\sqrt{\lambda}} \tag{34}$$

where $\lambda \equiv \frac{e^2 B^2}{m^4}$, $e_r$ is the renormalized electron coupling, and $m$ the electron mass.

Define the dimensionless quantity

$$S(\lambda) = 100 \int_0^\infty \frac{ds}{s^2} \left( \coth(s) - \frac{1}{s} - \frac{s}{3} \right) e^{-s/\sqrt{\lambda}} \tag{35}$$

which is related in an obvious way to (34). An expansion of (35) is given by

$$\hat{S}(\lambda) = 1600 \sum_{n=1}^\infty f_n \lambda^n \tag{36}$$
with

\[ f_n = \frac{4^{n-1} B_{2n+2}}{2n (2n+1) (2n+2)}, \]  

(37)

and where \( B_{2n} \) are the Bernoulli numbers which alternate in sign, and diverge factorially for large \( n \). Equation (36) is the Euler-Heisenberg [10] expansion which is equivalent to a sum of an infinite number of Feynman diagrams consisting of one closed fermion loop with an even number of external photons. The Euler-Heisenberg series diverges for large values of \( \lambda \) and displays oscillatory behaviour analogous to that in Fig.(1a). Consider now a resummation of the divergent series using (10-11). At the reference value \( \lambda = 10 \), the solutions to (11) are (global minima) \( p(2) = 0.77, p(3) = 1.4, p(6) = 2.25. \) No solutions were found for \( N = 3, 5, 7.\)

The exact result of Schwinger, given by (35), is plotted in Fig.(4), together with the approximants \( S_2, S_4, S_6. \) The approximants form upper bounds although the \( p(N) \) are positions of global minima. See again the caution prior to Eq.(14) and the Appendix. The convergence of the approximants \( S_N \) to the exact expression is manifest. For example, at \( \lambda = 10 \), where the Euler-Heisenberg series is badly divergent, the Schwinger’s exact value is \( S(10) = -8.056 \) while the estimates are, \( S_2 = -5.9, S_4 = -6.9, S_6 = -7.3. \)

The Euler-Heisenberg series (36) is Borel summable [14], and therefore provides an example of a Borel-summable series in a theory (QED) which is generally considered Borel-nonsummable. However it should be noted that the Euler-Heisenberg series (36) represents a one-fermion-loop result whereas the ”renormalon” singularities [12] which signal Borel-nonsummability are expected to arise when multi-fermion loop corrections to (34) are taken into account.

### 3.5 Critical Exponents in Three Dimensions

The three-dimensional \( O(N) \) symmetric \( \phi^4 \) field theory can be used as an effective theory (see [8, 9] and references therein) to describe the critical behaviour of many physical systems near a second-order phase transition. For this purpose, the renormalization group functions of this theory have been calculated to very high order. A detailed list of references can be found in [8, 17]. For example, the beta function for the polymer case, \( N = 0, \) is
given by

$$
\beta(\lambda) = -\lambda + \lambda^2 - 0.439815\lambda^3 + 0.389923\lambda^4 - 0.447316\lambda^5 + 0.633855\lambda^6 - 1.03493\lambda^7,
$$

where \(\lambda\) is a dimensionless coupling (usually denoted as \(g\) or \(\tilde{g}\) in the literature \cite{9}). The objective is to find the non-trivial infrared fixed point, \(\lambda^\star\), of the theory, which is given by the zero of the beta function,

$$
\beta(\lambda^\star) = 0, \quad \frac{\partial \beta}{\partial \lambda}_{\lambda^\star} \equiv \omega > 0.
$$

The expression (38) is divergent for large \(\lambda \sim 1\) where a nontrivial zero is expected. Using the resummation (10) with \(S_N\) denoting approximants to the beta function, and choosing the reference point \(\lambda_0 = 1\), one gets as solutions to Eq.(11) (minima), \(p(2) = 1.3, \ p(3) = 3.2, \ p(4) = 5.6, \ p(5) = 8.6, \ p(6) = 12.25, \ p(7) = 16.5\). The curves are shown in Fig.(5a). They form rapidly converging lower bounds but intercept the \(\lambda\)-axis only at the origin, thus giving only a trivial zero to the beta function. The situation is similar to the \(\sin(\pi \lambda)\) toy model studied earlier.

Now, since for \(N = 3, 5, 7\) the minima are only local, one expects local maxima to exist also. Indeed the local maxima are located at, \(\bar{p}(3) = 0.18, \ \bar{p}(5) = 0.21, \ \bar{p}(7) = 0.19\). The curves for \(\bar{S}_N\) are shown in Fig(5b). Though these are local maxima, they obey the inequality \(\bar{S}_{N+1} > \bar{S}_N\) and so appear to form lower bounds! Since the lower bounds due to \(\bar{S}_N\) are higher than the lower bounds due to \(S_N\) (Fig.(5a)), one may argue that those due to \(\bar{S}_N\) provide more accurate information. The curves in Fig.(5b) do in fact have a non-trivial zero. For \(N = 7\), the zero is near \(\lambda = 1.425\). It is reasonable to suppose that as both the approximants \(S_N\) and \(\bar{S}(N)\) provide lower bounds, it is simply a matter of choosing the highest lower bound to estimate the sum of the series.

Given the physical importance of this example, it is useful to perform further checks. So let us re-analyse the problem using the auxiliary series method discussed in Sect.(2). That is, let \(S'_N(\lambda, p) = S_N(\lambda, p)/\lambda\). The auxiliary series will then obviously have maxima as solutions to (11). The solutions at the reference point \(\lambda_0 = 1\) are \(p'(2) = 0.6, \ p'(3) = 0.875, \ p'(4) = 0.33, \ p'(5) = 0.4, \ p'(6) = 0.2525\). (Actually, \(p'(5)\) is a point of inflexion). The curves for \(\lambda S'_N\) are shown in Fig.(5c). Again, though determined by positions
of maxima, the curves appear to form rapidly converging lower bounds which have a non-trivial zero.

In order to compare with the results in the literature, more precise numbers are now quoted. Conjecturing that the curves in Fig.(5c) are indeed lower bounds to the exact result, the $N = 6$ curve, shown magnified near its nontrivial zero in Fig.(5d), gives an upper bound of 1.4193 to the nontrivial zero of the beta function. Re-optimizing the $N = 6$ equation (11) at $\lambda_0 = 1.419$ does not change the curve for $\lambda S_N^e$ significantly to modify that bound at the level of accuracy quoted. The slope of the beta function at the non-trivial zero can also be determined from the $N = 6$ curve in Fig.(5d). It is $\omega = 0.7955$. Since the curves appear to get steeper as $N$ increases, this value of $\omega$ is a lower bound. These values can now be compared with those of Ref.[9]: there it was found that $\lambda^* = 1.413 \pm 0.006$ and $\omega = 0.812 \pm 0.016$. The agreement with the bounds found here is excellent. (Comparison of results obtained by other methods and other authors may be found in [9]).

It is important to note the difference between the methodology used in this paper and that employed in Ref.[9]. In [9] the authors used a Borel-Leroy transformation with a variable parameter $b$ but they used the conventional conformal map [8] with the value of $p$ fixed at the precise location of the instanton singularity, $p^* = 0.166246$ (the value for $N = 0$). The parameter $b$, together with some other parameters introduced in [9] were used to check the convergence of the series and to estimate their errors. Here instead the usual Borel transform (2) is used but the conformal map has a single variational parameter $p$ determined according to the condition (11). From the numbers quoted above, it is clear that the values $p(N)$ used here are not the same as the value $p^*$. Furthermore, in the approach of this paper, the only assumptions made about the analyticity structure of the Borel transform are (i) $B(z)$ should be regular at the origin, and (ii) in order for the resummed perturbation series to faithfully represent the physical quantity, the series should be Borel summable. Borel summability of the $\phi^4_3$ theory has been established in [18].

From this example it is clear that the novel resummation presented here, with the parameter $p$ determined from (11), can be used to complement the analysis done in [9].
4 Conclusion

The method used in Ref.[5] to resum the perturbative free energy of gauge theories at high temperature has been detailed. It can be used more generally to obtain bounds on the full perturbation expression, $S(\lambda)$, of a physical quantity if in addition to the given partial perturbation series

$$\hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n,$$

some additional requirements are met and some assumptions made.

The first requirement is that not all the $f_n$ be of the same sign. Then solutions exist to the extremum condition (11). If the first nontrivial coefficient $f_n(n > 0)$ is negative, then the solutions to (11) will be global or local minima, depending on the sign of $f_N$. The approximants $S_N$ as defined through (13) then form lower bounds to $S(\lambda)$ if the inequality (16) is satisfied for all $N$ and if the assumptions leading to (15) are admitted. The inequality (16) must first be explicitly tested for the available terms of the series, then the arguments given in the Appendix suggest that the trend will continue for those cases where $c(N) > 1$ at low orders. Therefore even with partial information as in (14), one can apparently deduce lower bounds to the exact value $S(\lambda)$ if the technical assumptions mentioned above are valid. Under similar conditions, additional bounds may be obtained by using the auxiliary series method described in Sec.(2).

Sometimes one finds minima solutions to (11) but for which the $S_N$ obey an inequality opposite to (15). In such cases, even though the convergence is monotonic, it is not a priori obvious that the $S_N$ are actually upper bounds to the exact value. For problems where the exact result is unknown, additional input from theory or physics is required before a definitive statement can be made. However in all the examples encountered of this type, the $S_N$ did actually bound the exact result. A similar caveat concerns the approximants $\bar{S}$ defined near the end of Sec.(2).

The observed rapid convergence of the bounds has been explained in the Appendix. The obvious question is whether the bounds converge to the exact value itself. Empirically, it is found that the bounds formed by $S_N$ actually converge to the exact value if $S(\lambda)$ has a slowly varying first derivative, $\partial S(\lambda)/\partial \lambda$. Otherwise the complementary bounds formed by $\bar{S}(\lambda)$ or the auxiliary series method give better approximations to the exact result. The reason for this phenomena has been given in the Appendix.
Although most of the observed features of the resummation method developed here have been explained in the Appendix, at least in a semi-quantitative way, more patterns were detected than could be explained. For example, (i) Is it true that in all cases, as $N \to \infty$, $c(N) \equiv p(N+1)/p(N) \to 1$ (Similarly for the $\bar{p}(N)$)? (ii) Is it true that in all cases the approximants $S_N$ and $\bar{S}_N$ form bounds to the full perturbative result? (iii) Is it true that the bounds $S_N$ always approximate the exact result $S$ well if $|\partial^2 S / \partial \lambda^2 / \partial S| / \partial \lambda|$ is small throughout the range, $0 < \lambda < \lambda_0$, of interest, and otherwise the alternative bounds $\bar{S}_N$ or those from the auxiliary series give better approximations?

Clearly it would be desirable to have a better understanding of the conditions under which some of the technical assumptions made above hold rigorously. Until that is achieved, and in particular for those physical cases where little is known about the exact series anyway, the method can nevertheless be used as another independent estimator of the sum of a perturbation series, complementing other techniques in the literature. I point out that the resummation method discussed here can be viewed as an example of an "order-dependent mapping" introduced in [19] by Seznac and Zinn-Justin, but applied here to the Borel series rather than the original perturbation expansion. In addition, the extremum condition (11) can be thought of as an example of Stevensen’s Principal of Minimum Sensitivity [20].

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Appendix A1

The resummed perturbation series with a variable parameter $p$ is given by

$$S_N(\lambda, p) \equiv \sum_{n=0}^{N} \frac{f_n}{n!} \left(\frac{4}{p}\right)^n \sum_{k=0}^{N-n} \frac{(2n+k-1)!}{k!(2n-1)!} \int_0^\infty dz \ e^{-z} \ w(z\lambda)^{(k+n)}. \quad (42)$$

Write the equation above in the compact form

$$S_N(\lambda, p) = \sum_{n=0}^{N} \frac{A_n(N)}{p^n}, \quad (43)$$

where $A_n(N)$ is a $p$ and $\lambda$ dependent constant that can be read off from (42). In particular note that the sign of $A_n(N)$ is the same as the sign of $f_n$. The extremum condition (11) applied to (43) results in

$$\lambda \frac{\partial S_N}{\partial \lambda} \mid_{\lambda_0} = \sum_{n=1}^{N} \frac{nA_n(N)}{p(N)^n}, \quad (44)$$

where use has been made of the form $w(z\lambda)$ in transforming the $p$ derivative of $A_n(N)$ into a $\lambda$ derivative. At the solution $p = p(N)$ of (11), one has

$$S_N \equiv \sum_{n=0}^{N} \frac{A_n(N)}{p(N)^n} \quad (45)$$

Since for a given $N$ there is in general more than one solution to the extremum equation (11), here $p(N+1)$ and $p(N)$ will refer to solutions at consecutive orders which are both positions of minima or both positions of maxima. Now define

$$p(N+1) = c(N) \ p(N) \quad (46)$$

where $c(N)$ is some function of $N$. Then from equation (14) and the corresponding one at order $N+1$, one easily deduces

$$\lambda \frac{\partial (S_{N+1} - S_N)}{\partial \lambda} \mid_{\lambda_0} = \sum_{n=1}^{N} \frac{n}{p(N)^n} \left( \frac{A_n(N+1)}{c(N)^n} - A_n(N) \right) + \frac{(N+1)A_{N+1}(N+1)}{c(N)^{N+1}p(N)^{N+1}}. \quad (47)$$

Now for large $N$, $A_n(N) \sim A_n(N+1)$, and write this simply as $A_n$. Also define, $\Delta S_N \equiv S_{N+1} - S_N$. Assume now that $p(N)$ is large and $c(N) > 1$. 

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Also assume that all the coefficients \( A_n \) are generically of the same order, or at least do not increase rapidly with \( n \) (the factorial growth of \( f_n \) has already been taken care of by the Borel transform). Then keeping terms needed to solve for \( 1/c(N) \) at leading order, (47) simplifies to

\[
\lambda \frac{\partial \Delta S_N}{\partial \lambda} |_{\lambda_0} = \frac{A_1}{p(N)} \left( \frac{1}{c(N)} - 1 \right) - \frac{2A_2}{p(N)^2}.
\]  

(48)

Similarly, from eq.(45) and the corresponding one at order \( N + 1 \) one deduces at large \( N \) and to leading order in \( 1/p(N) \),

\[
\Delta S_N = \frac{A_1}{p(N)} \left( \frac{1}{c(N)} - 1 \right),
\]  

(49)

where it is implicit in this equation that \( \lambda = \lambda_0 \). Now, if \( \lambda_0 \) is varied, then the leading change in (49) comes from \( A_1 \). Thus using (49), Eq.(48) can be simplified to

\[
\frac{\alpha}{p(N)} \left( \frac{1}{c(N)} - 1 \right) \approx \frac{2A_2}{p(N)^2}.
\]  

(50)

where \( \alpha = (A_1 - \lambda \frac{\partial A_1}{\partial \lambda})|_{\lambda_0} \). Thus,

\[
\frac{1}{c(N)} \approx 1 + \frac{2A_2}{\alpha p(N)}.
\]  

(51)

This solution will be self-consistent with the initial assumption \( c(N) > 1 \) only if \( A_2/\alpha \) is negative, which then requires that that \( f_2 \) and \( f_1 \) should be of opposite signs (since \( A_1 \) is larger than \( \frac{\partial A_1}{\partial \lambda} \)). Note that for \( N = 2 \) the solution \( p(2) \) exists in the first place only if \( f_2 \) and \( f_1 \) are of opposite signs. Hence one concludes that \( c(2) > 1 \) is generically expected. Comparing (51) with the corresponding equation at the next order \( N + 1 \) one obtains at large \( N \) the recursive relation

\[
\frac{1}{c(N + 1)} = 1 - \frac{1}{c(N)} + \frac{1}{c(N)^2},
\]  

(52)

which can also be written as

\[
\frac{1}{c(N + 1)} - \frac{1}{c(N)} = \left( 1 - \frac{1}{c(N)} \right)^2.
\]  

(53)
showing that
\[ c(N + 1) < c(N). \] (54)

Indeed the large \( N \) solution of (52) is
\[ \frac{1}{c(N)} \approx 1 + \frac{1}{N + K}. \] (55)

with \( K \) a constant. Therefore \( c(N \to \infty) \to 1^+ \), that is, though \( p(N) \) will increase with \( N \), it will approach a constant as \( N \to \infty \). Now, since \( p(N) \) increases with \( N \), this means that the various approximations that led from (52) to (53-54) will become increasingly accurate. What is remarkable is that in the examples studied with \( c(N) > 1 \), (54) is already satisfied at low \( N \) and furthermore the relation (52) too is a reasonable approximation.

The above relations (52-54) were obtained under the assumption \( c(N) > 1 \). If \( c(N) < 1 \) then there are apparently no simple relations. For the auxiliary series of the beta function in Sec.(3.5), \( c(N) \) alternated between being slightly larger than one and much smaller than one.

Consider now the Eq.(49) which is valid at large \( N \) and large \( p(N) \),
\[ \Delta S_N = \frac{A_1}{p(N)} \left( \frac{1}{c(N)} - 1 \right), \] (56)

Remarkably, this simple equation summarizes most of the observed trends. When \( c(N) > 1 \), (56) implies that for \( A_1 < 0 \), which corresponds to \( f_1 < 0 \) and minima solutions to (54), \( \Delta S_N > 0 \), which is indeed observed in the examples studied and leads to lower bounds. Similarly if \( A_1 > 0 \), which corresponds to \( f_1 > 0 \) and maxima solutions, \( \Delta S_N < 0 \), implying upper bounds. The only exception observed so far is the Euler-Heisenberg series in Sec.(3.4) which had \( c(N) > 1 \), \( A_1 < 0 \), and yet gave upper bounds. Presumably for that example the approximation (56) is not appropriate.

Now suppose that \( c(N) < 1 \), as happens roughly for the auxiliary series in Sec.(3.5). Then Eq.(56) implies that minima, corresponding to \( f_1 < 0 \) and hence \( A_1 < 0 \), give \( \Delta S < 0 \), which explains some of the oddities observed in (8).

The equation (56) also explains the rapid convergence of the bounds. Indeed one sees that convergence can be achieved simply by one of two ways. Firstly, if \( c(N) \to 1 \) as \( N \to \infty \), that is \( p(N) \to p_0 \). This type of behaviour, is manifested, for example, by the toy-model in Sec.(3.5) of Ref.8. The second way is for \( c(N) > 1 \) for all \( N \), which leads to \( p(N) \) increasing with \( N \).
This is what has been generically observed. Actually, as shown above, the second behaviour also gives $c(N) \to 1^+$, and hence the convergence is doubly fast. Explicitly, from Eqs. (51-53, 56) one deduces for the $c(N) > 1$ case,

$$\Delta S \approx \frac{\alpha A_1}{2A_2} \left( \frac{1}{c(N)} - 1 \right)^2$$

$$= \frac{\alpha A_1}{2A_2} \frac{1}{(N+K)^2}.$$  \hspace{1cm} (57)  

(58)

In summary, for practical problems, the strongest statements can be made for the case when $c(N) > 1$ is observed for the given terms of a partial perturbative series, that is when $p(N)$ increases with $N$. Then the various approximations leading to the above equations become increasingly accurate at large $N$, allowing one to make assertions about all $N$. Firstly, one deduces $c(N \to \infty) \to 1^+$. Secondly, for $S_N$ which are global (and local) minima, if $S_N < S_{N+1}$ for the given terms of the series, the trend will continue for larger $N$, the convergence of the $S_N$ will be rapid, and they will form lower bounds to the exact result. However if $S_N > S_{N+1}$ is observed for the global (and local) minima, (as in Sec. (3.4)), then though the trend will continue and though the convergence of the $S_N$ will be rapid, it is not \textit{a priori} obvious that they will be upper bounds to the exact result, because then key pieces (14-15) of the argument are missing. (The same loophole occurs for the approximants $\tilde{S}_N$ formed from the $\tilde{p}(N)$’s.)

If it is observed that $c(N) < 1$ for the given terms of a series, then the various equations above are not necessarily accurate at higher $N$. In that case one can only conjecture that the observed monotonic and rapid convergence will continue at higher orders.

A2

Consider now the slope of the approximants $S_N(\lambda)$. From (12), as $\lambda \to 0$, $S_N(\lambda) \to f_0 + f_1 \lambda$, so that

$$\frac{\partial S_N}{\partial \lambda}|_{\lambda=0} = f_1.$$  \hspace{1cm} (59)

Thus all the bounds approach the origin with the same value ($f_0$) and slope ($f_1$) independent of $p$ and $N$. This fact can be seen in all the figures. Consider
next Eq. (44) for large $p(N)$,

$$\frac{\partial S_N}{\partial \lambda}|_{\lambda_0} \sim \frac{1}{\lambda_0 p(N)} A_1. \quad (60)$$

Since the sign of $A_1$ is the same as the sign of $f_1$, this shows that the curves $S_N(\lambda)$ are not expected to change direction as $\lambda$ varies. This is indeed observed, and explains why the bounds $S_N(\lambda)$ to $S(\lambda)$ are also good estimates of $S(\lambda)$ itself only when the latter has a slowly varying first derivative.

As was observed in Sec.(3.5) (see also [4]), the approximants $\bar{S}(\lambda)$ formed from the local extrema $\bar{p}(N)$ had a more varying slope. In order to understand this, set for simplicity $f_0 = 0$ so that $\bar{S}_N(0) = 0$ and let us demand that

$$\bar{S}_N(\lambda_0) = 0, \quad (61)$$

so that $\bar{S}_N(\lambda)$ curves back to its value at the origin, and so can better approximate functions like $\sin(\lambda \pi)$. The condition (61) then leads from (43) to

$$0 = \sum_{n=1}^N A_n \bar{p}(N)^n. \quad (62)$$

Note that since the $\bar{p}(N)$ also have to satisfy the extremum condition (11), eqns. (11.62) are actually two coupled equations for $\lambda_0$ and $\bar{p}(N)$ which we would like to analyse for consistency. Firstly, (62) can be used to eliminate the leading term in the 'barred' version of (44), so that now

$$\lambda \frac{\partial \bar{S}_N}{\partial \lambda}|_{\lambda_0} = \sum_{n=2}^N \frac{(n-1)A_n}{p(N)^n} \quad (63)$$

which at large $\bar{p}(N)$ gives the slope

$$\frac{\partial \bar{S}_N}{\partial \lambda}|_{\lambda_0} \sim \frac{A_2}{\lambda_0 p(N)^2}. \quad (64)$$

If $f_2$ is opposite in sign to $f_1$ then by comparing (64) with (59) one sees that indeed the approximants $\bar{S}$ can change direction. Of course this just shows that the approximate large $\bar{p}(N)$ analysis above is self-consistent. Now, in Sec.(2), for $f_1 < 0, f_2 > 0$, the $\bar{p}(N)$ have been defined as positions of local maxima when the $p(N)$ are positions of local minima. We can compare the
relative magnitudes of the two values as follows. For large $\bar{p}(N)$, (62) has the approximate solution

$$\bar{p}(N) \sim -\frac{A_2}{A_1}. \quad (65)$$

By contrast the $p(N)$ are approximate solutions of (44) with the left-hand-side deleted, which is equivalent to ignoring the mild $p$ dependence of the $A_n$’s,

$$p(N) \sim -\frac{2A_2}{A_1}. \quad (66)$$

Thus the values of $\bar{p}(N)$ are expected to be smaller than those of $p(N)$, and the interested reader may verify from the given examples that this is indeed the case. Reversing the logic of the argument above, one concludes as follows. For that $N$ when $p(N)$ is the position of a local minimum, one expects a local maximum at $\bar{p}(N)$. While the $S_N$ curve is expected to be monotonic in $\lambda$, that of $\bar{S}(\lambda)$ will not be monotonic if the value of $\bar{p}(N)$ is smaller than that of $p(N)$.

A similar discussion can be carried out for the curves formed from an auxiliary series $S'_N$. Again set for simplicity $f_0 = 0$, and define

$$\hat{S}'_N \equiv \frac{\hat{S}'_N}{\lambda}. \quad (67)$$

The extremization (11) in $p$ is done with respect to the resummed auxiliary series $S'_N$, and one deduces for large $p'(N)$, from an equation analogous to (44), that

$$\text{sign} \left( \lambda \frac{\partial S'_N}{\partial \lambda} \right)_{\lambda_0} = \text{sign}(f_2). \quad (68)$$

Now demanding

$$S_N(\lambda_0) = 0, \quad (69)$$

for the reconstructed resummed series $S_N \equiv \lambda S'_N$ gives,

$$\frac{\partial S_N}{\partial \lambda}|_{\lambda_0} = \lambda_0 \left( \frac{\partial S'_N}{\partial \lambda} \right)_{\lambda_0}, \quad (70)$$

which when combined with (68) shows that for $f_1$ and $f_2$ of opposite signs, the slope of the reconstructed $S_N$ at $\lambda = \lambda_0$ is opposite in sign to its slope at the origin which is given by (59).
Thus if $f_1$ and $f_2$ are of opposite signs, then the auxiliary series may be expected to give a reconstructed $S_N$ with a slope that has variation in sign, compared with the slope of the approximant $S_N$ which is obtained by direct means, if the two conditions (11) and (69) for $S'$ have a consistent solution $p'(N)$ for some $\lambda_0$. 
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Figure Captions

Figure (1a): Plots of the divergent series $\hat{S}_N(\lambda)$ for the model in Sec.(3.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_5, \hat{S}_3, S, \hat{S}_2, \hat{S}_4$.

Figure (1b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_2, S_3, S_4, S$.

Figure (1c): Plots of the resummed series $S_{(0)}(\lambda)$ for the model in Sec.(3.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_4, S, S_5, S_3$. Compared to Fig.(1b), the convergence is faster but not monotonic.

Figure (2a): Plots of the series $\hat{S}_N(\lambda)$ for the model in Sec.(3.2), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_3, S, \hat{S}_4, \hat{S}_2$.

Figure (2b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.2), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_2, S_3, S_4, S$.

Figure (3a): Plots of the resummed perturbation series $S_N(\lambda)$ for the model in Sec.(3.3), together with the full perturbative result $S_{\text{pert}}(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_2, S_3, S_4, S_5, S_{\text{pert}}$.

Figure (3b): Plots of $S_{\text{exact}}(\lambda), S_5(\lambda)$ and $S_5(\lambda) - S_{\text{np}}(\lambda)$ as defined in Sec.(3.3). At $\lambda = 10$, the lowest curve is $S_5 - S_{\text{np}}$, and the highest one is $S_5$.

Figure (4): Plots of the resummed Euler-Heisenberg series $S_N(\lambda)$, together with Schwinger’s exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S, S_6, S_4, S_2$. The approximants approach the exact result from above.

Figure (5a): Plots of the resummed beta function $S_N(\lambda)$ of Sec.(3.5). Starting from the lowest curve and moving upwards, one has $S_7, S_6, S_5, S_4, S_3, S_2$.

Figure (5b): Plots of the resummed series $\bar{S}_N(\lambda)$ for beta function in
Sec.(3.5). Starting from the lowest curve and moving upwards, one has $\bar{S}_3$, $\bar{S}_5$, $\bar{S}_7$. The approximants appear to form upper bounds.

Figure (5c): Plots of the resummed beta function $S_N(\lambda)$ of Sec.(3.5), obtained through the auxiliary series. Starting from the lowest curve and moving upwards, one has $S_2$, $S_3$, $S_4$, $S_5$, $S_6$. The approximants appear to form upper bounds. The curves for $N = 2$ and $N = 3$ are indistinguishable, and similarly, those for $N = 4$ and $N = 5$ are very close.

Figure (5d): Magnification of the $N = 6$ curve of Fig.(3.5) near its non-trivial zero.
