HEAT ASYMPTOTICS WITH SPECTRAL BOUNDARY CONDITIONS

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Abstract. Let $P$ be an operator of Dirac type on a compact Riemannian manifold with smooth boundary. We impose spectral boundary conditions and study the asymptotics of the heat trace of the associated operator of Laplace type.

1. Introduction

Let $M$ be a compact Riemannian manifold of dimension $m \geq 3$ with smooth boundary $\partial M$. Let $E_i$ be unitary bundles over $M$ and let

\[ P : C^\infty(E_1) \to C^\infty(E_2) \]

where $P$ is first order partial differential operator. Let $P^*$ be the formal adjoint of $P$. We say display (1) is an elliptic complex of Dirac type if the associated second order operators

\[ D_1 := P^*P \text{ and } D_2 := PP^* \]

on $C^\infty(E_1)$ and on $C^\infty(E_2)$ are of Laplace type - i.e. if these operators have scalar leading symbol given by the metric tensor. If $E_1 = E_2$ and if $P = P^*$, then $P$ is said to be an operator of Dirac type; it is convenient, however, to work in this slightly more general context.

We impose spectral boundary conditions; these were first introduced by Atiyah, Patodi, and Singer [3] in their study of the index theorem for manifolds with boundary. Let $\gamma$ be the leading symbol of the operator $P$. Then $\gamma + \gamma^*$ defines a unitary Clifford module structure on $E_1 \oplus E_2$. We choose a unitary connection $\nabla = \nabla_1 \oplus \nabla_2$ on $E_1 \oplus E_2$ so that

\[ \nabla(\gamma + \gamma^*) = 0 \text{ and } (\nabla s, \bar{s}) + (s, \nabla \bar{s}) = d(s, \bar{s}); \]

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such a connection always exists [3] and is said to be a compatible unitary connection. We note that equation (2) does not determine $\nabla$ uniquely; there are many compatible unitary connections.

Use the inward geodesic flow to identify a neighborhood of the boundary with the collar $\partial M \times [0, \epsilon)$ for some $\epsilon > 0$; if $y \in \partial M$, then the curves $y(t) = (y, t)$ are unit speed geodesics perpendicular to the boundary. If $y = (y^1, \ldots, y^{m-1})$ are local coordinates on $\partial M$, let $x = (y, x^m)$ be local coordinates on the collar where $x^m$ is the geodesic distance to the boundary. Let $\partial \mu := \frac{\partial}{\partial x^\mu}$; $\partial_m$ is the inward geodesic normal vector field on the collar. Let $\nabla_\mu$ be covariant differentiation with respect to $\partial_\mu$. We may decompose

$$P = \sum_{1 \leq \mu \leq m} \gamma_\mu \nabla_\mu + \psi$$

where $\psi$ is a 0th order operator. Since we do not assume that the structures are product near the boundary, the connection 1 form of $\nabla$, the leading symbol $\gamma$, and the endomorphism $\psi$ can depend on the normal variable. Relative to a local frame on the collar which is parallel along the normal geodesic rays which are perpendicular to the boundary, we have $\nabla_m = \partial_m$. We set $x^m = 0$ to define a tangential operator

$$B_0(y) := \gamma^m(y, 0)^{-1} \left( \sum_{\alpha < m} \gamma_\alpha(y, 0) \nabla_\alpha + \psi(y, 0) \right)$$

on $C^\infty \left( E_1|_{\partial M} \right)$. Let $\Theta$ be an auxiliary self-adjoint endomorphism of $E_1|_{\partial M}$. Let

$$A := \frac{1}{2} \left( B_0 + B_0^* \right) + \Theta$$

on $C^\infty \left( E_1|_{\partial M} \right)$. Here, the adjoint of $B_0$ is taken with respect to the structures on the boundary. The operator $A$ is a self-adjoint operator of Dirac type on $C^\infty \left( E_1|_{\partial M} \right)$. Let the boundary operator $B$, which we will use to define the boundary conditions for the operator $P$, be orthogonal projection on the span of the eigenspaces for the non-negative spectrum of $A$. Denote the realization of $P$ and the associated self-adjoint operator of Laplace type by

$$P_B$$

and $D_{1,B} := (P_B)^* P_B$.

Let $F \in C^\infty(E_1)$ be an auxiliary function we use to localize the problem; $F$ is called the smearing function. Results of Grubb and Seeley [13, 14, 15] show that there is an asymptotic series as $t \downarrow 0$ of the form:

$$\text{Tr}_{L^2} \left( F e^{-tD_1, B} \right) \sim \sum_{0 \leq k \leq m-1} a_k(F, B) t^{(k-m)/2} + O(t^{-1/8}).$$

(There is a complete asymptotic series with log terms, but we shall not need this fact as we shall only be interested in the first few terms in the series).
The coefficients $a_k$ in equation (3) are locally computable. We shall determine $a_0$, $a_1$, and $a_2$ in this paper. Our purpose is at least partly expository, so we will present several different techniques which yield information about these asymptotic coefficients.

We adopt the following notational conventions. Roman indices $i$ and $j$ will range from 1 to $m$ and index a local orthonormal frame for $TM$; Greek indices will index a local coordinate frame. Near the boundary, we choose the frame so that $e_m$ is the inward unit geodesic normal vector; we let indices $a$ and $b$ range from 1 through $m-1$ and index the corresponding frame for the tangent bundle of the boundary. We adopt the Einstein convention and sum over repeated indices. We let $';$ denote multiple covariant differentiation of the tensors involved. Decompose

\[ D_1 = -(g^{\mu\nu}\partial_\mu \partial_\nu + a^\mu \partial_\mu + b). \]

Let $\Gamma$ be the Christoffel symbols of the Levi-Civita connection on $M$. There is a canonical connection $D\nabla$ on the bundle $E_1$ and there is a canonical endomorphism $E$ of the bundle $E_1$ so that $D_1 = -\{\text{Tr} (D\nabla^2) + E\}$; we refer to [14] for details. If $E_1 = E_2$ and if $P = P^*$, then $D\nabla$ is a unitary connection; however $D\nabla \gamma$ need not vanish in general so $D\nabla$ is not in general a compatible connection. Let $\omega$ be the connection 1 form of $D\nabla$. We have

\[
\omega_\delta := \frac{1}{2} g_{\nu\delta} (a^\nu + g^{\mu\sigma} \Gamma^\nu_{\mu\sigma}) \quad \text{and} \quad 
E := b - g^{\mu\nu} (\partial_\nu \omega_\mu + \omega_\nu \omega_\mu - \omega_\sigma \Gamma^\nu_{\nu\mu} \omega^\sigma).
\]

Decompose $P = \gamma_i \nabla_i + \psi$ and let $\hat{\psi} := \gamma_m^{-1} \psi$. Let $\mathcal{R}$ be the scalar curvature of the metric on $M$. The main result of this paper is the following:

**Theorem 1.1.** We have

1. $a_0(F, D_1, \mathcal{B}) = (4\pi)^{-m/2} \int_M \text{Tr} (F)$. 
2. $a_1(F, D_1, \mathcal{B}) = \frac{1}{2} \left[ \frac{\Gamma(m)}{\Gamma(1/2)\Gamma(3/2)} - 1 \right] (4\pi)^{-(m-1)/2} \int_{\partial M} \text{Tr} (F)$. 
3. $a_2(F, D_1, \mathcal{B}) = (4\pi)^{-m/2} \int_M \frac{1}{8} \text{Tr} \left\{ F(\mathcal{R} + 6E) \right\}$

\[ + (4\pi)^{-m/2} \int_{\partial M} \text{Tr} \left\{ \frac{1}{2} [\hat{\psi} + \hat{\psi}^*] F + \frac{1}{3} \left[ 1 - \frac{3\Gamma(1/2)\Gamma(3/2)}{4\Gamma(5/2)} \right] L_{aa} F \right\} \left[ 1 - \frac{\Gamma(1/2)\Gamma(3/2)}{2\Gamma(5/2)} \right] F_m. \]
If \( k < m \), then there exist locally computable invariants \( a^M_k \) and \( a^\partial M_k \) so that
\[
a_k(F, D_1, \mathcal{B}) = (4\pi)^{-m/2} \int_M \text{Tr} (a^M_k(F, D_1, x))
+ (4\pi)^{-m/2} \int_{\partial M} \text{Tr} (a^\partial M_k(F, D_1, \mathcal{B}, y)).
\]

We have included a normalizing factor of \((4\pi)^{-m/2}\) to simplify the formulas for the local invariants \( a^M_k \) and \( a^\partial M_k \). We use dimensional analysis to see that the invariants \( a^M_k \) are homogeneous of weight \( k \) and the invariants \( a^\partial M_k \) are homogeneous of weight \( k - 1 \) in the jets of the symbols of \( P \) and \( P^* \). The invariants \( a^M_{2j+1} \) vanish. We use Theorem 4.1.6 [12] to see that:

\[
a_0^M(F, D_1, x) = \text{Tr} \{ F \} \quad \text{and} \quad a_2^M(F, D_1, x) = \text{Tr} \{ F(\hat{\psi} + \hat{\psi}^*) \}.
\]

The bundles \( E_1 \) and \( E_2 \) are distinct; we must use \( \gamma_m \) to identify \( E_1 \) and \( E_2 \) near the boundary. This observation reduces the number of invariants which are homogeneous of weight 1; for example, \( \text{Tr}(\psi) \) is not invariantly defined. There are universal constants so that
\[
a_1^\partial M(F, D_1, \mathcal{B}) = b_1(m) \text{Tr}(F)
+ c_0(m)F(\hat{\psi} + \hat{\psi}^*) + c_1(m)F(\hat{\psi} - \hat{\psi}^*)
+ c_2(m)F\Theta + c_3(m)FL_{aa} + c_4(m)F_{;m}.
\]

In contrast to the situation when the boundary operator \( \mathcal{B} \) is local, the constants exhibit non-trivial dependence upon the dimension.

We will use three different methodologies to compute the unknown coefficients. In Section 2, we use results of Grubb and Seeley [18] for structures that are product near the boundary to compute the constants \( b_1(m) \) and \( c_0(m) \); see Lemma 2.1 for details. In Section 3, we use functorial properties of these invariants to determine the coefficients \( c_0(m) \), \( c_1(m) \), and \( c_2(m) \); see Lemma 3.1 for details. In Section 4, we use computations on the ball to determine the coefficients \( b_1(m) \), \( c_3(m) \), and \( c_4(m) \); see Lemma 4.1 for details. As a check on our methods, we give two different derivations of the relation
\[
(m - 2)c_4(m) + (m - 1)c_3(m) = -\frac{m - 1}{6}
\]
in Sections 3 and 4. Our purpose in this paper is partly pedagogical; we wish to illustrate different methodologies which can be used to study these invariants.

2. Product Formulas

We use results of Grubb and Seeley [18] to show:
Lemma 2.1.

1. We have \( b_1(m) = \frac{1}{4} \left[ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma(\frac{m}{2})} - 1 \right] (4\pi)^{-(m-1)/2} \).

2. We have \( c_0(m) = \frac{1}{2} \).

Proof of Lemma 2.1: Suppose that \( P = \gamma_m(\nabla_m + B) \) where \( B \) is a self-adjoint tangential operator of Dirac type with coefficients which are independent of the normal variable; we take \( \Theta = 0 \) so \( A = B \). In this setting, we say the structures are product near the boundary. Let \( \tilde{D}_1 \) be the associated operator of Laplace type on the double. We ignore the effect of the 0 spectrum and define:

\[
\eta(s, B) := \text{Tr}_{L^2}(B(B^2)^{-s-1}), \quad \zeta(2s, B) := \text{Tr}_{L^2}((B^2)^{-s})
\]

\[
\zeta(s, \tilde{D}_1) := \text{Tr}_{L^2}(\tilde{D}_1^{-s}), \quad \zeta(s, D_1) := \text{Tr}_{L^2}(D_1^{-s}).
\]

We refer to Theorem 2.1 \cite{18} for the proof that:

\[
\Gamma(s)\zeta(s, D_1) = R(s) + \Gamma(s) \left\{ \frac{1}{2} \zeta(s, \tilde{D}_1) + \frac{1}{4} \left( \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{1}{2})} \right) - 1 \right\} \zeta(s, B^2)
\]

\[
-\frac{1}{4} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{1}{2})} \eta(2s, B)
\]

(7)

where the remainder \( R \) is regular away from \( s = 0 \). Expand

\[
\text{Tr}_{L^2}(Be^{-tB^2}) \sim \sum_k a_k^\eta(B) t^{(-m-1)/2}.\]

The invariants \( a_{2j}^\eta \) vanish. Formulas of Branson and Gilkey \cite{8} show that

\[
a_1^\eta(B) = -(4\pi)^{-(m-1)/2}(m - 2) \int_{\partial M} \text{Tr}(\hat{\psi}).
\]

(8)

One can use the Mellin transformation to relate the asymptotics of the heat equation to the pole structure of the eta and zeta functions. Let \( N \) be a manifold of dimension \( n \), let \( D_N \) be an operator of Laplace type on \( N \), and let \( Q_N \) be an operator of Dirac type on \( N \). If the boundary of \( N \) is non-empty, impose spectral boundary conditions. We then have, see for example Theorem 1.12.2 \cite{23},

\[
a_k(D_N) = \text{Res}_{s=-k} \Gamma(s)\zeta(s, D_N) \quad \text{and}
\]

\[
a_k^\eta(Q_N) = \text{Res}_{s=-k-1} \Gamma(s)\eta(2s-1, Q_N).
\]

(9)
The following identities now follow from equations (7) and (9):

\[ a_n(1, D, B) = \frac{1}{2} a_{n-1}(1, B) \quad \text{if } n \equiv 0 \mod 2, \]

\[ a_n(1, \tilde{D}, B) = -\frac{1}{2} (\eta - 1) a_{n-1}(1, B) \quad \text{if } n \equiv 1 \mod 2. \]

We prove the first assertion of Lemma 2.1 by using equations (5) and (11) to compute:

\[ (4\pi)^{-m/2} a_1^\partial M(1, D, B, y) = \frac{1}{4} \left( \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m+n+1}{2})} - 1 \right) (4\pi)^{-(m-1)/2} a_0(1, B, y) \]

\[ = (4\pi)^{-(m-1)/2} \frac{1}{4} \left( \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m+n+1}{2})} - 1 \right) \text{Tr}(1). \]

We prove the second assertion of Lemma 2.1 by using equations (8) and (10) to compute:

\[ (4\pi)^{-m/2} a_2^\partial M(1, D, B, y) = -\frac{1}{2(m-2)\Gamma(\frac{m}{2})} (4\pi)^{-(m-1)/2} a_1^\eta(1, B, y) \]

\[ = (4\pi)^{-m/2} \text{Tr}(\hat{\psi}). \]

3. Functorial Method

The invariants \( a_k \) have many functorial properties. We use these properties to establish the following result.

Lemma 3.1.

1. We have \( c_2(m) = 0 \).
2. We have \( c_0(m) = \frac{1}{2} \).
3. We have \( c_1(m) = 0 \).
4. We have \( (m-2)c_4(m) + (m-1)c_3(m) = -\frac{m-1}{6} \).

Proof of Lemma 3.1: We take \( E_1 = E_2 \) and let \( P \) be a formally self-adjoint operator of Dirac type on \( C^\infty(E_1) \). Let \( \Theta(\varepsilon) := \Theta + \varepsilon \) where \( \varepsilon \) is a real parameter. For generic values of \( \varepsilon \), \( \ker(A(\varepsilon)) \) is trivial. For such a value, the boundary conditions determined by the boundary operator \( B(\varepsilon) \) are locally constant and thus \( a_2 \) is independent of \( \varepsilon \). The first assertion of Lemma 3.1
now follows; this implies that \( a_2(\cdot) \) is not sensitive to the particular boundary condition chosen among the family we are considering.

For the remainder of the proof of Lemma 3.1, let \( T := S^1 \times \ldots \times S^1 \) be the \( m-1 \) dimensional torus and let \( M := T \times [0,1] \); we give \( T \) and \( M \) the canonical product flat metrics and let \((x_1,\ldots,x_m)\) be the usual parameters. We identify \( T \) with \( T \times \{0\} \) in \( M \). Let \( L \) be the line bundle defined by a non-trivial \( \mathbb{Z}_2 \) valued representation of the fundamental group of \( T \) and let \( V \) be the trivial bundle of dimension \( 2^m \) with coefficients in \( L \). Let \( \gamma_i \) be real skew-adjoint matrices acting on \( V \) which satisfy the Clifford commutation relations

\[
\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}.
\]

The twisting defined by \( L \) ensures that the kernel of the associated tangential operator \( B \) is trivial. Let \( \nabla_i := \partial_i \) define a compatible unitary connection on \( C^\infty(V) \). Let \( \Delta_0 := (\gamma_i \partial_i)^2 = -\partial_i^2 \) be the associated operator of Laplace type on \( C^\infty(V) \). Let \( f = f(x_m) \) be a smooth real valued function on \( M \) which vanishes identically near \( x_m = 1 \).

We use the index theorem to evaluate the coefficient \( c_0(m) \). Let

\[
Q := \gamma_i \partial_i + f \gamma_m, \quad Q^* := \gamma_i \partial_i - f \gamma_m, \\
D_1 := Q^* Q = \Delta - 2\gamma_m \gamma_a f \partial_a + f^2 - f_m, \\
D_2 := QQ^* = \Delta + 2\gamma_m \gamma_a f \partial_a + f^2 + f_m.
\]

We use equation (4) to compute:

\[
\omega_{1,a} = \gamma_m \gamma_a f, \quad \omega_{1,m} = 0, \quad E_1 = f_m + (m - 2)f^2, \\
\omega_{2,a} = -\gamma_m \gamma_a f, \quad \omega_{2,m} = 0, \quad E_2 = -f_m + (m - 2)f^2.
\]

We have \( \hat{\psi}_1 = f \) and \( \hat{\psi}_2 = -f \). The local formula for the index shows that the super trace vanishes for \( n \neq m \). Since \( m \geq 3 \), we have

\[
0 = a_2(1,D_1,B) - a_2(1,D_2,B) \\
= \text{dim}(V) \left\{ 2 \int_M f_m + 4c_0(m) \int_T f \right\} \\
= \text{dim}(V) (-2 + 4c_0(m)) \int_T f.
\]

This shows that \( c_0(m) = \frac{1}{2} \) and proves the second assertion of Lemma 3.1. Note that this value agrees with the result obtained previously in the proof of Lemma 2.1.

Next we study the coefficient \( c_1 \). Let

\[
P_f := \gamma_i \partial_i + \sqrt{-1} f \gamma_m;
\]
this operator is formally self-adjoint. Note that $A_f = -\gamma_m \gamma_a \nabla_a$ so the boundary operator $\mathcal{B}$ is independent of the function $f$. We expand

$$
P^2_f = \Delta_0 - 2\sqrt{-1}f \partial_m - \sqrt{-1}f_m + f^2.$$

Since the $\gamma_i$ were real, complex conjugation preserves the boundary conditions and intertwines $P^2_f$ with $P^2_{-f}$. Thus $P^2_f$ and $P^2_{-f}$ are isospectral so

$$
a_2(1, P^2_f, \mathcal{B}) = a_2(1, P^2_{-f}, \mathcal{B}).$$

We use equation (14) to see the interior integrand vanishes by computing:

$$
\omega_m(f) = \sqrt{-1}f \quad \text{and} \quad E(f) = 0.
$$

Since $\hat{\psi}_f = \sqrt{-1}f$, we may use equation (14) to show $c_1 = 0$ by computing:

$$
0 = a_2(1, P^2_f, \mathcal{B}) - a_2(1, P^2_{-f}, \mathcal{B}) = 4\sqrt{-1}c_1(m) \int_T f.
$$

We can also show $c_1(m) = 0$ using gauge invariance. Let $\hat{P} = e^{-\sqrt{-1}f}P_0e^{\sqrt{-1}f}$ be defined by a global unitary change of gauge. We have

$$
\psi_f = \psi + \sqrt{-1}f_i \gamma_i \quad \text{so} \quad \hat{\psi}_f = \hat{\psi} + \sqrt{-1}f_i \gamma_i^{-1} \gamma_i.
$$

Thus $\text{Tr} \left( \hat{\psi}_f - \hat{\psi}^*_f \right) = -2\sqrt{-1}f_m \text{dim}(V)$. As $a_2$ is gauge invariant, $c_1(m) = 0$.

To prove the final assertion of Lemma 3.1, we shall need the following technical result; we postpone the proof until the end of this section.

**Sublemma 3.2.** Let $ds^2(\varepsilon) = e^{2\varepsilon f} ds^2$ and let $P(\varepsilon) := e^{-\varepsilon f} P$. There exists a compatible family of unitary connections $\nabla(\varepsilon)$ so that

$$
\psi(\varepsilon) = e^{-\varepsilon f} \left\{ \psi(0) - \frac{m-1}{2\varepsilon} \right\} f_i \gamma_i.
$$

We use Sublemma 3.2 to complete the proof of Lemma 3.1. Let $P_0 := \gamma_i \partial_i$ and let

$$
\begin{align*}
ds^2(\varepsilon) &= e^{2\varepsilon f} ds^2, & P(\varepsilon) := e^{-\alpha(m) \varepsilon f} P_0 e^{-\beta(m) \varepsilon f}, \\
d\text{vol}(\varepsilon) &= e^{m \varepsilon f} d\text{vol}, & P(\varepsilon)^* = e^{(-\beta(m)-m) \varepsilon f} P_0^* e^{(m-\alpha(m)) \varepsilon f}.
\end{align*}
$$

We fix the inner product on $V$. Let $\alpha(m) = \frac{1+m}{2}$ and let $\beta(m) := \frac{1-m}{2}$. Then $\alpha(m) + \beta(m) = 1$ and $\alpha(m) = \beta(m) + m$ so the metric determined by the leading symbol of $P(\varepsilon)$ is $ds^2(\varepsilon)$ and $P(\varepsilon)$ is formally self-adjoint. We assume that $f = f(x_m)$ vanishes on the boundary of $M$ and that $f$ vanishes identically near $x_m = 1$. Thus the leading symbol of $A(\varepsilon)$ is independent of $\varepsilon$ so by defining $\Theta(\varepsilon)$ appropriately, we can assume the boundary operator $\mathcal{B}(\varepsilon)$
is independent of \( \varepsilon \). Let \( D(\varepsilon) := P(\varepsilon)^2 \). Let \( \delta := \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \) We compute the variation:

\[
\delta \text{Tr}_{L^2}(e^{-tD(\varepsilon)}) = -t \text{Tr}_{L^2}(\{\delta D(\varepsilon)\}e^{-t\Delta_0}) \\
= -2t \text{Tr}_{L^2}(\{\delta P(\varepsilon)\}P_0e^{-t\Delta_0}) = 2t \text{Tr}_{L^2}(f\Delta_0e^{-t\Delta_0}) \\
= -2t \partial_t \text{Tr}_{L^2}(f e^{-t\Delta_0}).
\]

We equate coefficients in the asymptotic expansions to see

\[
\delta a_2(1, D(\varepsilon), \mathcal{B}) = (m - 2)a_2(f, \Delta_0, \mathcal{B}).
\]

If \( \varepsilon = 0 \), then \( D = \Delta_0 \) and \( a_2 = 0 \). We are interested in the coefficient of \( \varepsilon f_{,mm} \). Since \( \omega(0) = 0 \), since \( \Gamma(0) = 0 \), and since \( E(0) = 0 \), we may compute

\[
b(D(\varepsilon)) = -\beta(m)\varepsilon f_{,mm} + O(\varepsilon^2)
\]

\[
a^n(D(\varepsilon)) = (m - 2\alpha(m) - 2\beta(m))\varepsilon f_m
\]

\[
\omega_m(D(\varepsilon)) = \frac{m - 2\alpha(m) - 2\beta(m)}{2}\varepsilon f_m + \omega_m(e^{-2\varepsilon f}\Delta_0),
\]

\[
E(D(\varepsilon)) = b(\varepsilon) - \partial_m \omega_m(\varepsilon) + E(e^{-2\varepsilon f}\Delta_0) + O(\varepsilon^2)
\]

\[
= -\frac{m - 2\alpha(m)}{2}\varepsilon f_{,mm} + E(e^{-2\varepsilon f}\Delta_0) + O(\varepsilon^2).
\]

We use results of [7] to see that

\[
\delta E(e^{-2\varepsilon f}\Delta_0) = -2fE + \frac{1}{2}(m - 2)f_{,ii},
\]

\[
\delta \mathcal{R} = -2f\mathcal{R} - 2(m - 1)f_{,ii}, \text{ and}
\]

\[
\delta L_{aa} = -fL_{aa} - (m - 1)f_{,m}.
\]

This permits us to compute the variation of the interior integral:

\[
\delta \int_M a_2^M(D(\varepsilon)) = \left\{ \frac{-2m - 2}{6} + \frac{m - 2}{2} - \frac{m - 2\alpha(m)}{2} \right\} \int_M f_{,mm}
\]

\[
= \left\{ -\alpha(m) + \frac{m + 2}{3} \right\} \int_T f_m.
\]

We have \( P(\varepsilon) = e^{-\varepsilon f}(P_0 - \beta(m)\varepsilon \gamma_{(m)f_{,m}}) \). Since \( \psi(P_0) = 0 \), Lemma 3.2 (2) implies that \( \psi(e^{-\varepsilon f}P_0) = -\frac{m - 1}{2}\varepsilon e^{-\varepsilon f}f_{,i}g_{ji} \). As \( f \) vanishes on \( \partial M \),

\[
\delta \dot{\psi}(P(\varepsilon)) = -\left\{ \beta(m) + \frac{m - 1}{2} \right\} f_m \text{ and}
\]

\[
\delta \int_T a_2^M(D(\varepsilon), \mathcal{B}) = \left\{ -(m - 1)c_3(m) - \beta(m) - \frac{m - 1}{2} \right\} \int_T f_m.
\]

Recall that \( \alpha(m) + \beta(m) = 1 \) and that \( a_2(\Delta_0) = 0 \). We use equations (13), (14), and (15). The final assertion of Lemma 3.1 follows from the following
Proof of Sublemma 3.2: We follow the argument given in [12]. Let $M$ be an arbitrary Riemannian manifold and let $\gamma$ be a Clifford module structure on a vector bundle over $M$. Let $\partial_\mu$ and $dx^\mu$ be local coordinate frames for the tangent and cotangent bundles. We define:

\[
\gamma^\mu(\varepsilon) := e^{-f_\varepsilon} \gamma^\mu \quad \gamma_\mu(\varepsilon) := e^{f_\varepsilon} \gamma_\mu \quad \theta_\mu := \frac{\varepsilon}{4}\left\{ 2f_{\nu\lambda} \gamma^\nu \gamma_\mu + cf_{\mu\nu} \right\}.
\]

Let $\nabla(\varepsilon) := \nabla + \theta$. Since the original connection is compatible, we have

\[
0 = \nabla_\mu(\gamma) (dx^\nu) = \partial_\mu(\gamma^\nu) + [\omega_\mu, \gamma^\nu] + \Gamma^\nu_{\sigma\kappa}(\varepsilon)(0).
\]

Let $E_\mu(\varepsilon) := \nabla_\mu(\varepsilon) \gamma(\varepsilon)$ and let $E^\nu_\mu := E_\mu(dx^\nu)$. We then have

\[
E^\nu_\mu(\varepsilon) = -\varepsilon f_{\mu\nu} \gamma^\kappa(\varepsilon) + [\theta_\mu, \gamma^\nu](\varepsilon) + (\Gamma^\nu_{\sigma\kappa}(\varepsilon) - \Gamma^\nu_{\kappa\sigma}(\varepsilon)) \gamma^\sigma(\varepsilon).
\]

Fix $x_0 \in M$ and choose the local coordinates near $x_0$ so $g_{\mu\nu}(x_0) = \delta_{\mu\nu}$. Then

\[
\Gamma^\nu_{\mu\sigma}(x_0) = \frac{1}{2} g^{\sigma\tau}(\partial_\nu g_{\mu\tau} + \partial_\tau g_{\mu\nu} - \partial_\sigma g_{\mu\nu})(x_0)
\]

\[
(\Gamma^\nu_{\mu\sigma}(\varepsilon) - \Gamma^\nu_{\mu\sigma}(0))(x_0) = \varepsilon(\delta_{\mu\nu} f_{\sigma} + \delta_{\nu\sigma} f_{\mu} - \delta_{\mu\sigma} f_{\nu})(x_0)
\]

\[
\theta_\mu(x_0) = \frac{\varepsilon}{4}\left\{ 2f_{\sigma\nu} \gamma_\mu + cf_{\mu\nu} \right\}(x_0).
\]

We must show $E^\nu_\mu(x_0) = 0$. If $\mu \neq \nu$, we compute:

\[
E^\nu_\mu(x_0) = e^{-f_\varepsilon}\left\{ -\varepsilon f_{\mu\nu} \gamma_\kappa + \frac{1}{2} \varepsilon f_{\sigma}(\gamma_\sigma \gamma_\mu, \gamma_\nu) + \varepsilon(- f_{\nu\gamma} \gamma_\mu + f_{\mu\gamma_\nu})(x_0)
\]

\[
= e^{-f_\varepsilon}\left\{ -\varepsilon f_{\mu\nu} \gamma_\kappa + \varepsilon(f_{\sigma} \gamma_\sigma) \right\}(x_0) = 0.
\]

If $\mu = \nu$, then (don’t sum over $\mu$):

\[
E^\nu_\mu(x_0) = e^{-f_\varepsilon}\left\{ -\varepsilon f_{\mu\nu} \gamma_\mu + [\theta_\mu, \gamma^\mu] + \sum_\sigma (\Gamma^\mu_{\mu\sigma}(\varepsilon) - \Gamma^\mu_{\mu\sigma}(0)) \gamma_\sigma \right\}(x_0)
\]

\[
= -e^{-f_\varepsilon}\left\{ \varepsilon f_{\mu\gamma_\mu} - \varepsilon \sum_{\sigma \neq \mu} f_{\sigma} \gamma_\sigma + \varepsilon \sum_\sigma f_{\sigma} \gamma_\sigma \right\}(x_0) = 0.
\]

This shows that $\nabla(\varepsilon)$ is compatible. Let $c = 2$. Then $\theta_\mu + \theta^*_\mu = 0$ and $\nabla(\varepsilon)$ is unitary. We complete the proof of Sublemma 3.2 by computing

\[
\psi(\varepsilon) = e^{-f_\varepsilon}\left\{ \psi - \frac{1}{4} \varepsilon (2f_{\mu\nu} \gamma^\mu \gamma_\nu + cf_{\mu\nu} \gamma^\mu) \right\}
\]

\[
= e^{-f_\varepsilon}\left\{ \psi - \frac{1}{4} \varepsilon f_{\mu\nu} \gamma^\mu (2m - 4 + c) \right\}. \quad \Box
\]
4. Computations on the Disk

We perform computations on the disk to prove:

**Lemma 4.1.**

1. We have $b_1(m) = \frac{1}{4} \left[ \frac{\Gamma(m/2)}{\Gamma(1/2)} - 1 \right] (4\pi)^{-(m-1)/2}$.

2. We have $c_3(m) = \frac{1}{6} \left[ 1 - \frac{3\Gamma(1/2)\Gamma(m/2)}{4\Gamma(1/2)} \right]$.

3. We have $c_4(m) = -\frac{m-1}{2(m-2)} \left[ 1 - \frac{1}{2} \Gamma(1/2) \frac{\Gamma(m/2)}{\Gamma(1/2)} \right]$.

4. We have $(m-2)c_4(m) + (m-1)c_3(m) = -\frac{m-1}{6}$.

**Proof:** Let $M$ be the unit ball in $\mathbb{R}^m$ with the usual metric. If $r \in [0, 1]$ is the radial normal coordinate and if $dS^2$ is the usual metric on the unit sphere $S^{m-1}$, then $ds^2 = dr^2 + r^2 dS^2$. The inward unit normal on the boundary is $-\partial_r$. The only nonvanishing components of the Christoffel symbols are

$$\Gamma_{abc} = \frac{1}{r} \tilde{\Gamma}_{abc}$$

and $\Gamma_{abm} = \frac{1}{r} \delta_{ab}$;

the second fundamental form is given by $\Gamma_{abm} = L_{ab}$. Here $\tilde{\Gamma}_{abc}$ are the Christoffel symbols associated with the metric $dS^2$ on the sphere $S^{m-1}$ and tilde will always refer to this metric.

The spin representation $\gamma$ is an irreducible representation of the Clifford algebra; we refer to [2] for details. Let $P = \gamma^\nu \partial_\nu$ be the Dirac operator on the ball; we take the flat connection $\nabla$ and set $\psi = 0$. We suppose $m$ even (there is a corresponding decomposition for $m$ odd) to find a local decomposition:

$$\gamma^a_{(m)} = \begin{pmatrix} 0 & \sqrt{-1} \cdot \gamma^a_{(m-1)} \\ -\sqrt{-1} \cdot \gamma^a_{(m-1)} & 0 \end{pmatrix}$$

and

$$\gamma^m_{(m)} = \begin{pmatrix} 0 & \sqrt{-1} \cdot 1_{m-1} \\ -\sqrt{-1} \cdot 1_{m-1} & 0 \end{pmatrix}.$$

We stress that $\gamma^j_{(m)}$ are the $\gamma$-matrices projected along some vielbein system. Decompose $\nabla_j = e_j + \omega_j$ where $\omega_j = \frac{1}{4} \Gamma_{jkl} \gamma^k \gamma^l$ is the connection 1 form of the spin connection. Note that

$$\nabla_a = \frac{1}{r} \left( \begin{pmatrix} \tilde{\nabla}_a & 0 \\ 0 & \tilde{\nabla}_a \end{pmatrix} + \frac{1}{2} \delta_{ab}(-\gamma^m_{(m)} \gamma^b_{(m)}) \right).$$
Let $\tilde{P}$ the Dirac operator on the sphere. We have:

$$P = \left( \frac{\partial}{\partial x_m} - \frac{m-1}{2r} \right) \gamma^m_{(m)} + \frac{1}{r} \begin{pmatrix} 0 & \sqrt{-1} \tilde{P} \\ -\sqrt{-1} \tilde{P} & 0 \end{pmatrix}. $$

Let $d_s$ be the dimension of the spin bundle on the disk; $d_s = \frac{2m}{2}$ if $m$ is even. The spinor modes $Z^{(n)}_{\pm}$ on the sphere are discussed in [9]. We have

$$\tilde{P} Z^{(n)}_{\pm}(\Omega) = \pm \left(n + \frac{m-1}{2}\right) Z^{(n)}_{\pm}(\Omega) \text{ for } n = 0, 1, \ldots;$$

$$d_n(m) := \dim Z^{(n)}_{\pm} = \frac{1}{2} d_s \left( \frac{m+n-2}{n} \right).$$

Let $J_\nu(z)$ be the Bessel functions. These satisfy the differential equation and functional relations [15]:

$$\frac{d^2 J_\nu(z)}{dz^2} + \frac{1}{z} \frac{d J_\nu(z)}{dz} + \left( 1 - \frac{\nu^2}{z^2} \right) J_\nu(z) = 0,$$

$$z \frac{d}{dz} J_\nu(z) + \nu J_\nu(z) = z J_{\nu-1}(z), \text{ and}$$

$$z \frac{d}{dz} J_\nu(z) - \nu J_\nu(z) = -z J_{\nu+1}(z).$$

Let $P \varphi_{\pm} = \pm \lambda \varphi_{\pm}$ be an eigen function of $P$. Modulo a suitable radial normalizing constant $C$, we may express:

$$(16) \quad \varphi_{\pm}^{(+)} = \frac{C}{r^{(d-1)/2}} \begin{pmatrix} i J_{n+m/2}(kr) Z^{(n)}_{\pm}(\Omega) \\ \pm J_{n+m/2-1}(kr) Z^{(n)}_{\pm}(\Omega) \end{pmatrix}, \text{ and}$$

$$(17) \quad \varphi_{\pm}^{(-)} = \frac{C}{r^{(d-1)/2}} \begin{pmatrix} \pm J_{n+m/2-1}(kr) Z^{(n)}_{-}(\Omega) \\ i J_{n+m/2}(kr) Z^{(n)}_{-}(\Omega) \end{pmatrix}. $$

Let $T_{\gamma^a_{(m)}} := -\gamma^m_{(m)} \gamma^a_{(m)}$ and let $T \nabla_a := \nabla_a - \frac{1}{2} L_{ab} T_{\gamma^b_{(m)}}$. Then $T \nabla$ is a compatible unitary connection for the induced Clifford modules structure $T \gamma$; see [13] for details. We may express the tangential operator $B$ in the form:

$$B = -\gamma^m_{(m)} \gamma^a_{(m)} \nabla_a = T_{\gamma^a_{(m)}} \nabla_a = T_{\gamma^a_{(m)}} \left( T \nabla_a + \frac{1}{2} L_{ab} T_{\gamma^b_{(m)}} \right)$$

$$= \left( -\tilde{P} - \frac{m-1}{2} \begin{array}{cc} 0 & 1 \\ 0 & \tilde{P} - \frac{m-1}{2} \end{array} \right).$$

Thus in particular $B = B^\ast$. We take $\Theta = \frac{m-1}{2} 1_m$. We then have:

$$A = \left( -\tilde{P} \begin{array}{cc} 0 \\ 0 \end{array} \tilde{P} \right).$$
The eigenstates and eigenvalues of $A$ then are given by:

$$A \left( \frac{Z^{(n)}_+}{Z^{(n)}_-} \right) = - \left( n + \frac{m-1}{2} \right) \left( \frac{Z^{(n)}_+}{Z^{(n)}_-} \right)$$

for $n = 0, 1, ...$

The boundary condition suppresses the non-negative spectrum of $A$. We use equation (17) to see that the non-negative modes of $A$ are associated with the radial factor $J_{n + \frac{m-1}{2}}(\lambda r)$. Hence the implicit eigenvalue equation is

$$J_p(\lambda) = 0 \text{ where } p = n + \frac{m}{2} - 1.$$  

The first and third authors developed a method for calculating the associated heat-kernel coefficients for smearing function $F = 1$ in [5, 6, 10]; they generalized this method to deal with $F = F(r)$ in [11]. We summarize the essential results from these papers briefly; in principal one could calculate any number of coefficients. We first suppose that $F = 1$. Instead of looking directly at the heat-kernel we will consider the zeta-function $\zeta(s)$ of the operator $P^2$ and use the relationship provided by equation (9) between the pole structure of the zeta function and the asymptotics of the heat equation:

$$a_k = \text{Res}_{s = \frac{m-k}{2}} \Gamma(s) \zeta(s).$$

Thus to compute $a_0$, $a_1$, and $a_2$, we must determine the residues of the zeta-function $\zeta(s)$ at the values $s = \frac{m}{2}$, $s = \frac{m-1}{2}$, and $s = \frac{m}{2} - 1$. We use the eigenvalue equation (18) to express

$$\zeta(s) = 4 \sum_{n=0}^{\infty} d_n(m) \int_C \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln J_p(k),$$

where the contour $C$ runs counterclockwise and encloses all the solutions of (18) which lie on the positive real axis. The factor of four comes from the four types of solutions in (16) and (17). As it stands, equation (20) is well defined only for $\Re s > m/2$, so the first task is to construct the analytical continuation to the left. We define a modified zeta function

$$\zeta^{(n)}(s) = \int_C \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln k^{-p} J_p(k);$$

the additional factor $k^{-p}$ has been introduced to avoid contributions coming from the origin. Since no additional pole is enclosed, the integral is unchanged.

The behaviour of $\zeta^{(n)}(s)$ as $n \to \infty$ controls the convergence of the sum over $n$; different orders in $n$ can be studied by shifting the contour to the imaginary axis and by using the uniform asymptotic expansion of the resulting Bessel function $I_p(k)$. To ensure that the resulting expression converges for some
range of $s$ when shifting the contour to the imaginary axis, we add a small positive constant to the eigenvalues. For $s$ in the strip $1/2 < \Re s < 1$, we have:

\[
\zeta^{(n)}(s) = \frac{\sin(\pi s)}{\pi} \int_{\epsilon}^{\infty} dk (k^2 - \epsilon^2)^{-s} \frac{\partial}{\partial k} \ln k^{-p} I_p(k).
\]

We introduce some additional notation dealing with the uniform asymptotic expansion of the Bessel function. For $p \to \infty$ with $z = k/p$ fixed, we use results of [1] to see that:

\[
(21) \quad I_p(zp) \sim \frac{1}{\sqrt{2\pi p}} \frac{e^{pn}}{(1 + z^2)^{1/4}} \left[ 1 + \sum_{l=1}^{\infty} \frac{u_l(t)}{p^l} \right]
\]

where

\[
t = 1/\sqrt{1 + z^2} \quad \text{and} \quad \eta = \sqrt{1 + z^2 + \ln[z/(1 + \sqrt{1 + z^2})]}.
\]

Let $u_0(t) = 1$. We use the recursion relationship given in [1] to determine the polynomials $u_l(t)$ which appear in equation (21):

\[
u_{l+1}(t) = \frac{1}{2} t^2 (1 - t^2) u_l'(t) + \frac{1}{8} \int_0^t d\tau (1 - 5 \tau^2) u_l(\tau).
\]

We also need the coefficients $D_m(t)$ defined by the cumulant expansion:

\[
(22) \quad \ln \left[ 1 + \sum_{l=1}^{\infty} \frac{u_l(t)}{p^l} \right] \sim \sum_{q=1}^{\infty} \frac{D_q(t)}{p^q}.
\]

The eigenvalue multiplicities $d_n(m)$ are $O(n^{m-2})$ as $n \to \infty$. Consequently, the leading behaviour of every term is on the order of $p^{-2s-q+m-2}$; thus on the half plane $\Re s > (m - 3)/2$, only the value $q = 1$ contributes to the residues of the zeta-function. We have $D_1(t) = \frac{1}{8} t^2 - \frac{5}{24} t^3$. We use equation (21) to decompose

\[
\zeta^{(n)}(s) = A^{(n)}_{-1}(s) + A^{(n)}_0(s) + A^{(n)}_1(s) + R^{(n)}(s), \quad \text{where}
\]

\[
A^{(n)}_{-1}(s) = \frac{\sin \pi s}{\pi} \int_{\epsilon/p}^{\infty} dz [(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} \ln (z^{-p} e^{pn}),
\]

\[
A^{(n)}_0(s) = \frac{\sin \pi s}{\pi} \int_{\epsilon/p}^{\infty} dz [(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} \ln (1 + z^2)^{-1/4},
\]

\[
A^{(n)}_1(s) = \frac{\sin \pi s}{\pi} \int_{\epsilon/p}^{\infty} dz [(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} \left( \frac{D_1(t)}{p} \right).
\]

The remainder $R^{(n)}(s)$ is such that $\sum_{n=0}^{\infty} d_n(m) R^{(n)}(s)$ is analytic on the half plane $\Re s > (m - 3)/2$. 

Let \(2F_1\) be the hypergeometric function. We have
\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt t^{b-1}(1 - t)^{c-b-1}(1 - t z)^{-a}, \quad \text{and}
\]
\[
\int_{\epsilon/p}^\infty dz \left[(zp)^2 - \epsilon^2\right]^{-s} \frac{\partial}{\partial z} t^i = -\frac{l}{2} \frac{\Gamma(s + \frac{l}{2})\Gamma(1 - s)}{\Gamma(1 + \frac{l}{2})} p^i [\epsilon^2 + p^2]^{-s-1/2}.
\]

We use the first identity to study \(A_{-1}^{(n)}(s)\) and \(A_0^{(n)}(s)\); we use the second identity to study \(A_1^{(n)}(s)\). This shows that
\[
A_{-1}^{(n)}(s) = \frac{\epsilon^{-2s+1}}{2\Gamma(\frac{1}{2})} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} 2F_1(-\frac{1}{2}, s - \frac{1}{2}; -\frac{p^2}{\epsilon^2} - \frac{p^2}{2}\epsilon^{-2s},
\]
\[
A_0^{(n)}(s) = -\frac{1}{4}(p^2 + \epsilon^2)^{-s},
\]
\[
A_1^{(n)}(s) = \frac{1}{8\Gamma(s)} \left[ -\frac{\Gamma(s + \frac{3}{2})}{\Gamma(\frac{1}{2})} (p^2 + \epsilon^2)^{-s-\frac{3}{2}} \right] - \frac{5}{24\Gamma(s)} \left[ -2\frac{\Gamma(s + \frac{3}{2})}{\Gamma(\frac{1}{2})} \Gamma(s + 1) \Gamma(s + \frac{1}{2}) \right].
\]

We take the limit as \(\epsilon \to 0\); the resulting zeta-function which appears is connected to the spectrum on the sphere. We define the base zeta-function \(\zeta_{S^4}\)
and the Barnes zeta-function \(\zeta_{B}(s, a) = \sum_{n=0}^{\infty} d_n(m)(n + a)^{-s}\).

We then have the relation \(\zeta_{S^4}(s) = 2d_s\zeta_{B}(2s, \frac{m}{2} - 1)\). For \(i = -1, i = 0,\) and \(i = 1,\) let \(A_i(s) = 4\sum_{n=0}^{\infty} d_n(m) A_i^{(n)}(s)\). We take the limit as \(\epsilon \to 0\) to see that
\[
A_{-1}(s) = \frac{1}{4\Gamma(\frac{1}{2})} \frac{\Gamma(s + 1)}{\Gamma(s + \frac{1}{2})} \zeta_{S^4}(s - \frac{1}{2}),
\]
\[
A_0(s) = -\frac{1}{4}\zeta_{S^4}(s), \quad \text{and}
\]
\[
A_1(s) = -\frac{1}{2\Gamma(s)} \left[ \frac{1}{8\Gamma(\frac{1}{2})} \Gamma(s + \frac{1}{2}) - \frac{5}{12\Gamma(\frac{1}{2})} \Gamma(s + \frac{3}{2}) \right].
\]

We use the Mellin-Barnes integral representation of the hypergeometric functions \(2F_1\) to calculate \(A_{-1}(s)\):
\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_C dt \frac{\Gamma(a + t)\Gamma(b + t)\Gamma(-t)}{\Gamma(c + t)} (-z)^t.
\]

We choose the contour of integration so that the poles of \(\Gamma(a+t)\Gamma(b+t)/\Gamma(c+t)\) lie to the left of the contour and so that the poles of \(\Gamma(-t)\) lie to the right of
the contour. We stress that before interchanging the sum and the integral, we
must shift the contour $C$ over the pole at $t = 1/2$ to the left; this cancels the
term $-\frac{p}{2} e^{-2s}$ appearing in the expression for $A_{-1}$ appearing above.
This reduces the analysis of the zeta function on the ball to analysis of a zeta
function on the boundary. We compute the residues of $\zeta(s)$ from the residues
of $\zeta_B(s,a)$. Let $d := m - 1$. To compute these residues, we first express
$\zeta_B(s,a)$ as a contour integral. Let $C$ be the Hankel contour.

$$\zeta_B(s,a) = \sum_{n=0}^{\infty} \binom{d+n-1}{n} (n+a)^{-s} = \sum_{\vec{m} \in \mathbb{N}_0^d} (a + m_1 + \ldots + m_d)^{-s}$$

The residues of $\zeta_B(s,a)$ are intimately connected with the generalized Bernoulli
polynomials \[19\],

$$e^{-at} \frac{(1-e^{-t})^d}{(1-e^{-t})^d} = (-1)^d \sum_{n=0}^{\infty} \frac{(-t)^{n-d}}{n!} B_n^{(d)}(a).$$

We use the residue theorem to see that

$$\text{Res}_{s=z} \zeta_B(s,a) = \frac{(-1)^d+z}{(z-1)!(d-z)!} B_{d-z}^{(d)}(a),$$

for $z = 1, \ldots, d$. The needed leading poles are

$$\text{Res}_{s=d} \zeta_B(s,a) = \frac{1}{(d-1)!},$$
$$\text{Res}_{s=d-1} \zeta_B(s,a) = \frac{d-2a}{2(d-2)!},$$
$$\text{Res}_{s=d-2} \zeta_B(s,a) = \frac{12a^2 - d - 12ad + 3d^2}{24(d-3)!}.$$

We may now determine the residues of $\zeta(s)$. At $s = \frac{m}{2}$, only $A_{-1}(s)$ contributes.
We use equation (23) to see

$$\text{Res}_{s=\frac{m}{2}} \zeta(s) = d_s \frac{1}{4\sqrt{\pi}} \Gamma \left( \frac{m-1}{2} \right) \frac{1}{\Gamma \left( \frac{m}{2} + 1 \right) \Gamma(m-1)}.$$}

We use the ‘doubling formula’ $\frac{\Gamma(z)}{\Gamma(2z)} = \frac{\sqrt{\pi} z!}{\Gamma(z+1/2)}$ for the $\Gamma$ function, we use
equation (13), and we use the observation $\text{Tr}(1) = d_s$ to check that we have
derived the correct formula for $a_0$:

$$a_0 = \frac{d_s}{2^m \Gamma \left( \frac{m}{2} + 1 \right)} = (4\pi)^{-m/2} \int_{B^m} \text{Tr}(1).$$
Next, we study the pole at \( s = \frac{m-1}{2} \). This time, \( A_{-1}(s) \) and \( A_0(s) \) contribute. We compute
\[
\text{Res}_{s=\frac{m-1}{2}} A_{-1}(s) = \frac{1}{2^m \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{m-1}{2}\right)},
\]
\[
\text{Res}_{s=\frac{m-1}{2}} A_0(s) = -\frac{1}{4 \Gamma(m-1)}.
\]
This implies that
\[
a_1 = \frac{1}{2^m \Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{2^m \Gamma\left(\frac{m}{2}\right)} = \frac{1}{4} \left[ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{m+1}{2}\right)} - 1 \right] (4\pi)^{-\frac{m-1}{2}} \int_{S_{m-1}} \text{Tr} \ (1).
\]
This determines \( b_1(m) \) and proves the first assertion of Lemma \([\text{I.1}]\). Next we compute \( a_2 \) and thereby determine \( c_3(m) \):
\[
\text{Res}_{s=\frac{m-2}{2}} A_{-1}(s) = \frac{1}{6} \frac{4 - m}{2^m \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m-2}{2}\right)},
\]
\[
\text{Res}_{s=\frac{m-2}{2}} A_0(s) = -\frac{1}{8 \Gamma(m-2)},
\]
\[
\text{Res}_{s=\frac{m-2}{2}} A_1(s) = -\frac{1}{6} \frac{8 - 5m}{2^m \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m-2}{2}\right)}.
\]
We can now prove the second assertion of Lemma \([\text{I.1}]\) by determining \( c_3(m) \). We compute:
\[
a_2 = \frac{2}{3} \left( \frac{(m-1)}{2^m \Gamma\left(\frac{m}{2}\right)} \right) - \frac{1}{2} \left( \frac{(m-1) \Gamma\left(\frac{1}{2}\right)}{2^m \Gamma\left(\frac{m+1}{2}\right)} \right)
\]
\[
= \frac{1}{3} \left( 1 - \frac{3 \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{m}{2}\right)}{4 \Gamma\left(\frac{m+1}{2}\right)} \right) (4\pi)^{-m/2} \int_{S_{m-1}} \text{Tr} \ (L_{aa}).
\]
The pattern of the universal constants involving a dimensionless constant and a combination of \( \Gamma \)-functions is evident. This pattern holds for the higher coefficients \([\text{I.1}]\).

We introduce the weighting (or smearing) function \( F = 1 - r^2 \) to determine \( c_4(m) \); since \( E \) and \( R \) vanish and since \( F \) vanishes on the boundary, only the term involving \( F_{:m} = 2 \) survives. (Note: we checked these computations by studying a more general function of the form \( F(r) = f_0 + f_1 r^2 + f_2 r^4 \) but omit details in the interests of brevity). We note that the radial normalization constant is given by \( C = 1/J_{p+1}(\lambda) \). We denote the normalized Bessel function by
\[
\bar{J}_k(\lambda r) := J_k(\lambda r)/J_{p+1}(\lambda).
\]
Instead of the zeta function we consider now the smeared analogue:

\[ \zeta(F; s) = \sum_{\lambda} \int_{B^n} F(x) \varphi^*(x) \varphi(x) \frac{1}{\lambda^{2s}}. \]  

(28)

Since \( F \) depends only on the normal variable, the integral in equation (28) over the sphere \( S^{n-1} \) behaves as in the case \( F = 1 \) so that

\[ \zeta(F; s) = 4 \sum_{n=0}^{\infty} d_n(m) \int_{C} \frac{dk}{2\pi i} k^{-2s} \]

\[ \cdot \int_{0}^{1} dr F(r) r (\bar{J}_{p+1}^2(kr) + \bar{J}_p^2(kr)) \frac{\partial}{\partial k} \ln J_p(k). \]

We set \( F(r) = 1 - r^2 \) and compute the radial integrals:

\[ \int_{0}^{1} r^3 [\bar{J}_p^2(\lambda r) + \bar{J}^2_{p+1}(\lambda r)] = \frac{2p^2 + 3p + 1}{3\lambda^2} + \frac{1}{3} \text{ so } \]

\[ \zeta(r^2; s) = 4 \sum_{n=0}^{\infty} d_n(m) \int_{C} \frac{dk}{2\pi i} k^{-2s} \left[ \frac{2p^2 + 3p + 1}{3k^2} + \frac{1}{3} \right] \frac{\partial}{\partial k} \ln J_p(k). \]

We use equation (24) to evaluate this expression; the second term is given by equation (20) and simple substitutions suffice to evaluate the remaining parts. The factor \( 1/k^2 \) is absorbed by using \( s + 1 \) instead of \( s \) in equations (23), (24), and (25). The powers of \( p \) lower the argument of the base zeta-function by 1, by \( \frac{1}{2} \) and by 0. It is now a straightforward matter to compute:

\[ A_{-1}(r^2; s) = \frac{1}{4} \frac{\Gamma(s+1/2)}{\Gamma(s+1)} \zeta_{Sd}(s+1/2) \left[ \frac{1}{3} + \frac{2s-1}{3s+1} \right] \]

\[ + \frac{1}{4} \frac{\Gamma(s+1/2)}{\Gamma(s+2)} \left[ \zeta_{Sd}(s) + \frac{1}{3} \zeta_{Sd}(s+1/2) \right], \]

\[ A_0(r^2; s) = -\frac{1}{4} \zeta_{Sd}(s) - \frac{1}{4} \zeta_{Sd}(s+1/2) + ... \]

\[ A_1(r^2; s) = -\frac{2}{3\Gamma(s+1)} \zeta_{Sd}(s+1/2) \left[ \frac{1}{8\Gamma(1/2)} \Gamma(s+3/2) - \frac{5}{12\Gamma(1/2)} \Gamma(s+5/2) \right] \]

\[ -\frac{1}{3\Gamma(s)} \zeta_{Sd}(s+1/2) \left[ \frac{1}{8\Gamma(1/2)} \Gamma(s+3/2) - \frac{5}{12\Gamma(1/2)} \Gamma(s+3/2) \right] + ... \]

The coefficient \( a_0 \) gives the leading term in the expansion of the heat trace; this is correctly reproduced by the first term in \( A_{-1}(r^2; s) \). We also confirm the invariant form of the next coefficient \( a_1 \). We determine the value of \( c_4(m) \)
by considering the pole at the value $s = \frac{m-2}{2}$. Note that $F;m = 2$. We compute

$$\text{Res}_{s=\frac{m-2}{2}} \Gamma(s) A_{-1}(1 - r^2; s) = -\frac{1}{12} \frac{m}{m-2} \frac{5m-6}{(4\pi)^{-m/2}} \int_{S_{m-1}} \text{Tr} (2)$$

$$\text{Res}_{s=\frac{m-2}{2}} \Gamma(s) A_{0}(1 - r^2; s) = \frac{1}{4} \frac{\Gamma\left(\frac{1}{2}\right) m-1}{m-2} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma((m+1)/2)} (4\pi)^{-m/2} \int_{S_{m-1}} \text{Tr} (2)$$

$$\text{Res}_{s=\frac{m-2}{2}} \Gamma(s) A_{1}(1 - r^2; s) = -\frac{1}{12} \frac{5m-6}{m-2} (4\pi)^{-m/2} \int_{S_{m-1}} \text{Tr} (2).$$

We sum the contributions to prove the third assertion of Lemma \[11\] by evaluating $c_4$. The final assertion of Lemma \[11\] is now immediate. \(\square\)

5. Conclusion

In Section \[2\], we used results of Grubb and Seeley to study the setting when the structures are product near the boundary. In this setting, $\hat{\psi}$ was self-adjoint and only the terms $\text{Tr} (1)$ and $\text{Tr} (\hat{\psi})$ were non-trivial. We used these results to determine the coefficients $b_1(m)$ and $c_0(m)$.

In Section \[3\], we used functorial methods to determine the coefficients $c_0(m)$, $c_1(m)$, and $c_2(m)$; the value we computed for $c_0(m)$ agreed with that determined in Section \[2\]. We used variational methods to obtain a non-trivial linear relationship between the coefficients $c_3(m)$ and $c_4(m)$.

In Section \[4\], we determined $b_1(m)$, $c_3(m)$ and $c_4(m)$ by studying the ball in Euclidean space. The value of $b_1(m)$ computed in this fashion agreed with the value which was determined in Section \[2\]. The values for the coefficients $c_3(m)$ and $c_4(m)$ satisfied the linear relationship derived in Section \[3\].

In Section \[5\], we chose $c = 2$ to ensure that the 1 parameter family $\nabla(\varepsilon)$ of connections was unitary; the second author originally missed this point and his error in choosing a different value of $c$ led to a seeming contradiction, which has now been cleared up, between the methods of Section \[5\] and the methods of Section \[3\]. We find it interesting that each of the three methods we have discussed gives some, but not all, of the coefficients and that none of the results of the three sections is a proper subset of the other.

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