The Chern class mapping on abelian surfaces

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Abstract

We examine the Chern class mapping $c_1 : \text{NS}(S)/p\text{NS}(S) \to H^1(S, \Omega^1_S)$ for an abelian surface $S$ in characteristic $p \geq 3$, and give a basis of the kernel $c_1$ for the superspecial abelian surface.

1 Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$, and $S$ be a nonsingular complete algebraic surface over $k$. We denote by $H^2_{dR}(S)$ the second de Rham cohomology group of $S$, and by $\text{NS}(S)$ the Néron-Severi group of $S$. $\text{NS}(S)$ is a finitely generated abelian group, and the rank $\rho(S)$ of $\text{NS}(S)$ is called the Picard number. We have the Chern class mapping $\text{NS}(S)/p\text{NS}(S) \to H^2_{dR}(S)$ and this map is injective if the Hodge-to-de Rham spectral sequence of $S$ degenerates at $E_1$-term (cf. Ogus [8]). We also have the Chern class mapping $c_1 : \text{NS}(S)/p\text{NS}(S) \to H^1(S, \Omega^1_S)$. This map is, whereas, not necessarily injective even if the Hodge-to-de Rham spectral sequence of $S$ degenerates at $E_1$-term (cf. Ogus [8]).

In this paper, we examine this map $c_1$ in the case of abelian surfaces. For abelian surfaces, the Chern class mapping $c_1$ is injective if and only if the abelian surface is not superspecial (for the definition, see Section 2). This fact was implicitly proved in Ogus [8] by using the notion of K3 crystal. We give here a down-to-earth proof of this fact and determine a basis of the kernel of the Chern class mapping $c_1$ for the superspecial abelian surface. To calculate a basis of $\text{Ker } c_1$, in Section 2 we examine the structure of the Néron-Severi group of the superspecial abelian surface. Using the theory of quaternion algebra, problems on divisors on superspecial abelian surfaces are replaced by problems on matrices with coefficients in quaternion division algebra. Main part of Section 2 will be known to specialists. But, since we cannot find a suitable reference which includes explicit results, we give here a complete proof. As an example, we give an explicit description on our theory in the case of characteristic 3. Finally, we examine the Chern class mapping of Kummer surfaces and show similar results to the case of abelian surfaces.

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2 The Néron-Severi group

Let $k$ be an algebraically closed field of characteristic $p > 0$. An abelian surface is said to be superspecial if it is isomorphic to a product of two supersingular elliptic curves. Note that a superspecial abelian surface is unique up to isomorphism (cf. Shioda [11]). In this section, we examine the structure of the Néron-Severi group of the superspecial abelian surface.

Let $E$ be a supersingular elliptic curve defined over $k$, and we consider the superspecial abelian surface $A = E_1 \times E_2$ with $E_1 = E_2 = E$. We denote by $O_E$ the zero point of $E$. We take a divisor $X = E_1 \times \{O_{E_2}\} + \{O_{E_1}\} \times E_2$, which gives a principal polarization on $A$. We also denote $E_1 \times \{O_{E_2}\}$ (resp. $\{O_{E_1}\} \times E_2$) by $E_1$ (resp. by $E_2$) for the sake of simplicity. We set $\mathcal{O} = \text{End}(E)$ and $B = \text{End}^0(E) = \text{End}(E) \otimes \mathbb{Q}$. Then, $B$ is a quaternion division algebra over the rational number field $\mathbb{Q}$ with discriminant $p$, and $\mathcal{O}$ is a maximal order of $B$. For an element $a \in B$, we denote by $\bar{a}$ the canonical involution. For a divisor $L$, we have a homomorphism

$$\varphi_L : A \longrightarrow \text{Pic}^0(A) \quad x \mapsto T_x^*L - L,$$

where $T_x$ is the translation by $x \in A$ (cf Mumford [7]). We set

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\} \mid \alpha, \delta \in \mathbb{Z}, \gamma, \beta \in \mathcal{O}, \gamma = \bar{\beta}.$$

Main part of the following theorem will be known to specialists (cf. Ibukiyama, Katsura and Oort [5]).

**Theorem 2.1** The homomorphism

$$j : \text{NS}(A) \longrightarrow H \quad L \mapsto \varphi^{-1}_X \circ \varphi_L$$

is bijective. By this correspondence, we have

$$j(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad j(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $L_1, L_2 \in \text{NS}(A)$ such that

$$j(L_1) = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \quad j(L_2) = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},$$

...
the intersection number \((L_1, L_2)\) is given by
\[
(L_1, L_2) = \alpha_2 \delta_1 + \alpha_1 \delta_2 - \gamma_1 \beta_2 - \gamma_2 \beta_1.
\]
In particular, for \(L \in \text{NS}(A)\) such that \(j(L) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\) we have
\[
L^2 = 2 \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]
\((L, E_1) = \alpha, (L, E_2) = \delta.
\]
We have also \(j(nD) = nj(D)\) for an integer \(n\).

The first statement of this theorem is given in Mumford [7]. The final statement follows from the definition of \(\varphi_L\). To prove the others, we need some lemmas.

**Lemma 2.2** The restriction homomorphism
\[
\begin{array}{ccc}
\text{Res} : \text{Pic}^0(A) & \rightarrow & \text{Pic}^0(E_1) \times \text{Pic}^0(E_2) \\
L & \mapsto & (L|_{E_1}, L|_{E_2})
\end{array}
\]
is an isomorphism, and the following diagram commutes:
\[
\begin{array}{ccc}
A & \xrightarrow{\varphi_X} & \text{Pic}^0(A) \ni L \\
\| & & \| \\
E_1 \times E_2 & \xrightarrow{\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}} & \text{Pic}^0(E_1) \times \text{Pic}^0(E_2) \ni (L|_{E_1}, L|_{E_2})
\end{array}
\]

**Proof** The first statement is well-known (cf. Mumford [7]). For \(x = (x_1, x_2) \in A\), we have
\[
\text{Res} \circ \varphi_X(x) = \text{Res}(T_x^*X - X) = (T_{x_1}^*O_{E_1} - O_{E_1}, T_{x_2}^*O_{E_2} - O_{E_2}) = (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})(x)
\]

We now examine the canonical involution of \(B\). Since we have \(B = \text{End}(E) \otimes \text{Q}\), it suffices to define it for the elements of \(\text{End}(E)\). For \(g \in \text{End}(E)\), we define the canonical involution by
\[
\tilde{g} = \varphi_{O_E}^{-1} \circ g^* \circ \varphi_{O_E}.
\]

For the elliptic curve \(E\), we have
\[
\tilde{g} \circ g = \varphi_{O_E}^{-1} \circ g^* \circ \varphi_{O_E} \circ g = \deg g.
\]

**Lemma 2.3** The Rosatti involution \(g'\) of \(g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(B)\) is given by
\[
g' = \begin{pmatrix} \frac{\alpha}{\delta} & \frac{-\beta}{\delta} \\ \frac{\gamma}{\delta} & \frac{-\delta}{\delta} \end{pmatrix}
\]

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**Proof** We denote by $\hat{g}$ the dual morphism of $g$. As the action to divisors, we have $\hat{g} = g^*$. The Rosatti involution is given by $g' = \varphi_X^{-1} \circ \hat{g} \circ \varphi_X$. We calculate the right-hand-side term concretely. We have a commutative diagram

\[
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}} & \text{Pic}^0(E_1) \times \text{Pic}^0(E_2) \\
\downarrow \varphi_X & & \uparrow \\
\text{Pic}^0(E_1 \times E_2) & \xrightarrow{\text{Res}} & \text{Pic}^0(E_1) \times \text{Pic}^0(E_2) \\
\uparrow \hat{g} & & \uparrow \text{Res} \circ \hat{g} \circ \text{Res}^{-1} \\
\text{Pic}^0(E_1 \times E_2) & \xrightarrow{\text{Res}} & \text{Pic}^0(E_1) \times \text{Pic}^0(E_2) \\
\downarrow \varphi_X & & \uparrow \\
E_1 \times E_2 & \xrightarrow{\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}} & \text{Pic}^0(E_1) \times \text{Pic}^0(E_2).
\end{array}
\]

Using this diagram, for the point $\left( \frac{x_1}{x_2} \right) \in E_1 \times E_2$ we have

\[
g' \left( \frac{x_1}{x_2} \right) = \varphi_X^{-1} \circ \hat{g} \circ \varphi_X \left( \frac{x_1}{x_2} \right) = (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \text{Res} \circ \hat{g} \circ \text{Res}^{-1} \circ (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}) \left( \frac{x_1}{x_2} \right)
\]

\[
= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \text{Res} \circ \hat{g} \circ \text{Res}^{-1} \left( \varphi_{O_{E_1}}(x_1) \text{Res}(p_1 \circ g)^* \varphi_{O_{E_1}}(x_1) + (p_2 \circ g)^* \varphi_{O_{E_2}}(x_2) \right)
\]

We denote by $m_i$ the addition of $E_i$ ($i = 1, 2$). Then, we have

\[
p_1 \circ g = m_1 \circ (\alpha \times \beta), \quad p_2 \circ g = m_2 \circ (\gamma \times \delta).
\]

We denote by $q_i$ ($i = 1, 2$) the i-th projection $E_1 \times E_1 \to E_1$. Then by Mumford [7], for $L \in \text{Pic}^0(E_1)$ we have

\[
m_1^* L \sim q_1^* L + q_2^* L \text{ (linearly equivalent)}.
\]

Therefore we have

\[
(p_1 \circ g)^* \varphi_{O_{E_1}}(x_1) = (m_1 \circ (\alpha \times \beta))^* \varphi_{O_{E_1}}(x_1) = (\alpha \times \beta)^* m_1^* \varphi_{O_{E_1}}(x_1)
\]

\[
= (\alpha \times \beta)^* (q_1^* \varphi_{O_{E_1}}(x_1) + q_2^* \varphi_{O_{E_1}}(x_1))
\]

\[
= \{q_1 \circ (\alpha \times \beta)\}^* \varphi_{O_{E_1}}(x_1) + \{q_2 \circ (\alpha \times \beta)\}^* \varphi_{O_{E_1}}(x_1).
\]

Since we have commutative diagrams

\[
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{\alpha \times \beta} & E_1 \times E_1 \\
\downarrow p_1 & & \downarrow q_1 \\
E_1 & \xrightarrow{\alpha} & E_1,
\end{array}
\]

\[
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{\alpha \times \beta} & E_1 \times E_1 \\
\downarrow p_2 & & \downarrow q_2 \\
E_1 & \xrightarrow{\beta} & E_1.
\end{array}
\]

\[
4
\]
Since $E\alpha x$

Now, we examine as

we restrict the divisor $L$

In a similar way, we have

Therefore, we have

Therefore, we have

Therefore, we have

Since $E_1 = E_2 = E$ and $\varphi_{O_1} = \varphi_{O_2}$, we conclude

Lemma 2.4 For a divisor $L \in \text{Pic}(E_1 \times E_2)$ with $j(L) = g = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)$, we have

Proof Since $\alpha$ is an integer and

Now, we examine $\alpha x$.

We restrict the divisor $L$ to $E_1$ and denote it by $e$. Then, the divisor is expressed as

$$e \sim \sum_{i=1}^{\lambda} n_i P_i$$
with integers \( n_i \) and points \( P_i \) on \( E_1 \) \((i = 1, 2, \cdots, \lambda)\). We have

\[
(L, E_1) = \deg e = \sum_{i=1}^{\lambda} n_i.
\]

We set \( n = \sum_{i=1}^{\lambda} n_i \). Using these notation, we have the following form:

\[
g\left(\frac{x}{O_{E_2}}\right) = (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \left( T_x^* e - e \right).
\]

We denote by \( \oplus \) the addition of \( E_1 \), and by \( \ominus \) the subtraction of \( E_1 \). Then, we have

\[
T_x^* e \sim \sum_{i=1}^{\lambda} n_i (P_i \ominus x).
\]

By Abel’s theorem, we see that

\[
\varphi_{O_{E_1}}^-((-n) - O_{E_1}) = nx, \quad \text{and}
\]

\[
g\left(\frac{x}{O_{E_2}}\right) = \left(\begin{array}{c} nx \\ \ast \end{array}\right).
\]

Hence, comparing (1) and (2), we have \( \alpha = n = (L, E_1) \). In a similar way, we have \( \beta = (L, E_2) \). \( \blacksquare \)

**Lemma 2.5** \( j(E_1) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right) \) and \( j(E_2) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \)

**Proof** For a point \((x_1, x_2) \in E_1 \times E_2\), we have

\[
\varphi_X^{-1} \circ \varphi_{E_1} \left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = \varphi_X^{-1} \{T_{(x_1, x_2)}^* E_1 - E_1\}
\]

\[
= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ Res\{T_{(x_1, x_2)}^* E_1 - E_1\}
\]

\[
= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \left(\begin{array}{c} O_{E_1} - O_{E_1} \\ O_{E_1} - O_{E_2} \end{array}\right) = \left(\begin{array}{c} O_{E_1} \\ x_2 \end{array}\right).
\]

Therefore, we have

\[
j(E_1) = \varphi_X^{-1} \circ \varphi_{E_1} = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right).
\]

In a similar way, we have the latter assersion. \( \blacksquare \)
Lemma 2.6 For $L \in \text{NS}(E_1 \times E_2)$, we set $j(L) = g = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \delta \end{pmatrix}$. Then,

$$L^2 = 2 \det g.$$  

Proof Since $\alpha, \delta \in \mathbb{Z}$, we have

$$\varphi^{-1}_X \circ \varphi_{(L-\alpha E_2-\delta E_1)} = \varphi^{-1}_X \circ \varphi_L - \alpha \varphi^{-1}_X \circ \varphi_{E_2} - \delta \varphi^{-1}_X \circ \varphi_{E_1} = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \delta \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & \bar{\gamma} \\ \gamma & 0 \end{pmatrix}.$$  

Since the right hand-side is contained in $H$, there exists a divisor $Z$ such that

$$\varphi^{-1}_X \circ \varphi_Z = \begin{pmatrix} 0 & \bar{\gamma} \\ \gamma & 0 \end{pmatrix}.$$  

Since $\varphi_X$ is an isomorphism, by the Riemann-Roch theorem, we have

$$\deg(\varphi^{-1}_X \circ \varphi_Z) = \deg \varphi_Z = (Z^2/2)^2.$$  

On the other hand,

$$\deg(\varphi^{-1}_X \circ \varphi_Z) = \deg \gamma \cdot \deg \bar{\gamma} = (\deg \gamma)^2 = (\gamma \bar{\gamma})^2.$$  

By Lemma 2.4, we have

$$(Z, E_1) = (Z, E_2) = 0.$$  

Therefore, we have $(Z, E_1 + E_2) = 0$. Since $(E_1 + E_2)^2 = 2 > 0$, by the Hodge index theorem we see $Z^2 < 0$. Therefore, we have, $Z^2/2 = -\gamma \bar{\gamma}$.

On the other hand, since $\varphi_X$ is an isomorphism and $\varphi^{-1}_X \circ \varphi_{L-\alpha E_2-\delta E_1-Z} = 0$, we have $\varphi_{(L-\alpha E_2-\delta E_1-Z)} = 0$. Therefore, we have

$$0 \equiv L - \alpha E_2 - \delta E_1 - Z.$$  

Hence, we have

$$L^2 = 2\alpha \delta + Z^2 = 2(\alpha \delta - \gamma \bar{\gamma}) = 2 \det g.$$  

Lemma 2.7 Let $L_1$ and $L_2$ be two divisors with $j(L_1) = g_1$ and $j(L_2) = g_2$. Let $g$ be an automorphism of $A$. Then, $g^* L_1 \equiv L_2$ if and only if $^t \bar{g} g_1 g = g_2$. 

\[ \boxed{ } \]
Proof We have
\[ g^*L_1 \equiv L_2 \iff \varphi g^*L_1 = \varphi L_2 \]
\[ \iff \hat{g} \circ \varphi L_1 \circ g = \varphi L_2 \]
\[ \iff \varphi_X^{-1} \circ \hat{g} \circ \varphi_X \circ (\varphi_X^{-1} \circ \varphi L_1) \circ g = \varphi_X^{-1} \circ \varphi L_2 \]
\[ \iff g' \circ g \circ g = g. \]

Let \( m : E \times E \rightarrow E \) be the addition of \( E \), and we set
\[ \Delta = \text{Ker } m. \]

We have \( \Delta = \{(P, -P) \mid P \in E\} \). Note that this \( \Delta \) is different from the usual diagonal. For two endomorphisms \( a_1, a_2 \in \text{End}(E) \), we set
\[ \Delta_{a_1, a_2} = (a_1 \times a_2)^* \Delta. \]

Using this notation, we have \( \Delta = \Delta_{1,1} \). We have the following theorem (cf. [6]).

**Theorem 2.8**

\[ j(\Delta_{a_1, a_2}) = \left( \bar{a}_1 a_1 \bar{a}_2 a_2 \right). \]

In particular, we have
\[ j(\Delta) = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right). \]

**Proof** We set
\[ \varphi_X^{-1} \circ \varphi \Delta = \left( \begin{array}{cc} \alpha & \bar{\gamma} \\ \gamma & \delta \end{array} \right). \]

Then, by \( (E_1, \Delta) = (E_2, \Delta) = 1 \), we have \( \alpha = \delta = 1 \). Since we have
\[ \varphi_X^{-1} \circ \varphi \Delta \left( \begin{array}{c} x \\ -x \end{array} \right) = \varphi_X^{-1} \{ T^*_{(x, -x)} \Delta - \Delta \} = \varphi_X^{-1}(0) = \left( \begin{array}{c} O_1 \\ O_2 \end{array} \right), \]

we have \( \gamma(x) = x \) for any \( x \in E \). Therefore, we have \( \gamma = 1 \).

By definition, we have
\[ \Delta_{a_1, a_2} = (a_1 \times a_2)^* \Delta. \]

Therefore, we have
\[ j(\Delta_{a_1, a_2}) = t \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right) = \left( \begin{array}{cc} \bar{a}_1 a_1 & \bar{a}_2 a_2 \\ \bar{a}_2 a_1 & \bar{a}_2 a_2 \end{array} \right) \]

\[ \square \]
3 Non-superspecial cases

In this section, we examine the injectivity of the Chern class mapping of abelian surfaces. The following theorem follows essentially from the results in Ogus [8], but we give here a down-to-earth proof.

**Theorem 3.1** Let $X$ be an abelian surface defined over $k$. Then, the Chern map

$$c_1 : \text{NS}(X)/p\text{NS}(X) \to H^1(X, \Omega^1_X)$$

is injective if and only if $X$ is not superspecial.

**Proof** The only-if-part will be proved in Theorem 4.4. We prove here the if-part. We denote by $r(X)$ the $p$-rank of the group of $p$-torsion points of $X$, and by $a(X)$ the a-number of $X$ defined in Oort [9], which relates to the local-local group scheme $\alpha^p$ in $X$. By Oort [10], $X$ is superspecial if and only if $a(X) = 2$. Therefore, we assume $a(X) \neq 2$. Take an affine open covering $\{U_i\}$ of $X$, and suppose that there is a divisor $D = \{f_{ij}\}$ which is not zero in $\text{NS}(X)/p\text{NS}(X)$, such that $c_1(D) = \{df_{ij}/f_{ij}\} \sim 0$ in $H^1(X, \Omega^1_X)$. Then, there exists $\omega_i \in H^0(U_i, \Omega^1_X)$ such that

$$df_{ij}/f_{ij} = \omega_j - \omega_i.$$

Suppose $d\omega_i = 0$. Then, operating the Cartier operator $C$, we have

$$df_{ij}/f_{ij} = C(\omega_j) - C(\omega_i).$$

Therefore, we have

$$C(\omega_j) - \omega_j = C(\omega_i) - \omega_i \quad \text{on } U_i \cap U_j,$$

and we have a regular 1-form $\omega'$ on $X$ which is defined by

$$C(\omega_i) - \omega_i \quad \text{on } U_i.$$

Since $C - \text{id} : H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$ is surjective, there exists a regular 1-form $\omega \in H^0(X, \Omega^1_X)$ such that $(C - \text{id})(\omega) = \omega'$. Therefore, we have

$$C(\omega_i - \omega) = \omega_i - \omega.$$

By the property of the Cartier operator, there exists an regular function $f_i$ on $U_i$ such that

$$\omega_i - \omega = df_i/f_i,$$

and we have

$$df_{ij}/f_{ij} = df_j/f_j - df_i/f_i.$$

This means $d(f_{ij}f_i/f_j) = 0$ and we conclude $D \in p\text{NS}(X)$, which contradicts $D \neq 0$ in $\text{NS}(X)/p\text{NS}(X)$. 
Now, we assume $d\omega_i \neq 0$. Then, we have $d\omega_i = d\omega_j$ on $U_i \cap U_j$ and we get a non-zero regular 2-form on $X$. Since this regular 2-form is $d$-exact and is a basis of $H^0(X, \Omega_X^2)$, the Cartier operator acts on $H^0(X, \Omega_X^2)$ as the zero map. Therefore, $X$ is not ordinary, that is, $r(X) \neq 2$. Therefore, we have either $r(X) = 1$ and $a(X) = 1$, or $r(X) = 0$ and $a(X) = 1$.

Now, we consider the absolute Frobenius $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$. Since $a(X) = 1$ in both cases, there exists a non-zero element $\beta = \{g_{ij}\}$ in $H^1(X, \mathcal{O}_X)$ such that $\beta^p = 0$. This means that there exists a regular function $g_i$ on $U_i$ such that $g_{ij}^p = g_j - g_i$. Since $dg_i = dg_j$ on $U_i \cap U_j$, we have a non-zero regular 1-form $\eta$ on $X$ given by $dg_i$ on $U_i$. Since $\dim H^0(X, \Omega_X^1) = 2$, in both cases there exists a non-zero regular 1-form $\eta'$ such that $\langle \eta, \eta' \rangle$ gives a basis of $H^0(X, \Omega_X^1)$ with $C(\eta') \neq 0$. In fact, we can take $\eta'$ with $C(\eta') = \eta$ if $r(X) = 0$ and $a(X) = 1$, and we can take $\eta'$ with $C(\eta') = \eta'$ if $r(X) = 1$ and $a(X) = 1$. Since we have $H^0(X, \Omega_X^2) = \wedge^2 H^0(X, \Omega_X^1)$, $\eta \wedge \eta'$ gives a basis of $H^0(X, \Omega_X^2)$. Therefore, there exists a non-zero element $a \in k$ such that

$$d\omega_i = a\eta \wedge \eta' = a(d(g_i, \eta')).$$

We set $\theta_i = \omega_i - ag_i \eta'$. Then, $\theta_i$ is $d$-closed and we have

$$df_{ij}/f_{ij} = ag_i \eta' - ag_j \eta' + \theta_j - \theta_i = ag_{ij} \eta' + \theta_j - \theta_i.$$  

Operating the Cartier operator, we have

$$df_{ij}/f_{ij} = a^{1/p} g_{ij} C(\eta') + C(\theta_j) - C(\theta_i).$$

This means that

$$c_1(D) \sim a^{1/p} \beta \otimes C(\eta') \in H^1(X, \mathcal{O}_X) \otimes H^0(X, \Omega_X^1) \cong H^1(X, \Omega_X^1)$$

Since $\beta \neq 0$ in $H^1(X, \mathcal{O}_X)$ and $C(\eta') \neq 0$ in $H^0(X, \Omega_X^1)$, we see $\beta \otimes C(\eta') \neq 0$ in $H^1(X, \Omega_X^1)$. A contradiction. Hence, if $a(X) \neq 2$, we conclude that $c_1$ is injective.

4 Superspecial cases

Let $k$ be an algebraically closed field of characteristic $p \geq 3$. For an elliptic curve $E$ over $k$, we examine the action of endomorphisms of $E$ on $H^0(E, \Omega_E^1)$ and $H^1(E, \mathcal{O}_E)$.

**Lemma 4.1** Let $E$ be an elliptic curve and $\alpha \in \text{End}(E)$. Assume $\alpha$ acts on $H^1(E, \mathcal{O}_E)$ as the multiplication by $\beta \in k$ ($\beta \neq 0$). Then, $\alpha$ acts on $H^0(E, \Omega_E^1)$ as the multiplication by $\frac{\text{deg} \alpha}{\beta}$. 

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Proof By the endomorphism $\alpha : E \to E$, we have a commutative diagram

$$
\begin{array}{ccc}
\text{NS}(E)/p\text{NS}(E) & \xrightarrow{\alpha^*} & \text{NS}(E)/p\text{NS}(E) \\
\downarrow c_1 & & \downarrow c_1 \\
\text{H}^1(E, \Omega^1_E) & \xrightarrow{\alpha^*} & \text{H}^1(E, \Omega^1_E) \\
\downarrow & & \downarrow \\
\text{H}^1(E, \mathcal{O}_E) \otimes \text{H}^0(E, \Omega^1_E) & \xrightarrow{\alpha^* \otimes \alpha^*} & \text{H}^1(E, \mathcal{O}_E) \otimes \text{H}^0(E, \Omega^1_E).
\end{array}
$$

Take a point $Q \in E$, and bases $\omega \in \text{H}^0(E, \Omega^1_E)$, $\eta \in \text{H}^1(E, \mathcal{O}_E)$. Then, we have $\alpha^*(Q) = (\deg \alpha)Q$, and $(\alpha^* \otimes \alpha^*)(\omega \otimes \eta) = (\beta \omega) \otimes \alpha^* \eta$. The result follows from the diagram. $\blacksquare$

For an integer $n$, we have an endomorphism $[n]_E : E \to E$ given by $P \mapsto nP$ ($P \in E$).

Lemma 4.2 The induced homomorphism

$$
[n]_E^* : \text{H}^0(E, \Omega^1_E) \to \text{H}^0(E, \Omega^1_E)
$$

is the multiplication by $n$, i.e., $[n]_E^* \omega = n \omega$ for $\omega \in \text{H}^0(E, \Omega^1_E)$.

Proof This follows from the fact that $[n]_E^*$ is given by the multiplication $n$ on the tangent space at the origin (Mumford [7]). $\blacksquare$

Assume $p \neq 2$. Take a prime number $q$ such that $-q \equiv 5$ (mod 8) and $(\frac{-q}{p}) = -1$, and take an integer $a$ such that $a^2 \equiv -p$ (mod $q$). Here, $(\frac{-q}{p})$ is the Legendre symbol. Note that $p$ and $q$ are prime to each other. By Ibukiyama (cf. [4]), the quaternion division algebra $B$ over $\mathbb{Q}$ with discriminant $p$ and a maximal order $\mathcal{O}$ of $B$ are given by

$$
B = \mathbb{Q} \oplus \mathbb{Q}F \oplus \mathbb{Q}a \oplus \mathbb{Q}Fa
$$

with $F^2 = -p$, $a^2 = -q$, $F = -aF$.

$$
\mathcal{O} = \mathbb{Z} + Z(\frac{1+a}{2}) + Z(\frac{F(1+a)}{2}) + Z(\frac{(a+F)a}{q}).
$$

Then, we know that there exists a supersingular elliptic curve $E$ over $k$ with $\text{End}(E) = \mathcal{O}$ and $\text{End}^0(E) = B$ (cf. Deuring [2]). We need the following well-known lemma.

Lemma 4.3 For a non-singular complete algebraic curve $X$ and with an affine open covering $\{U_i\}$ of $X$

$$
\begin{align*}
\text{Pic}(X)/p\text{Pic}(X) & \to \text{H}^1(X, \Omega^1_X) \\
\{f_{ij}\} & \mapsto \{df_{ij}/f_{ij}\}
\end{align*}
$$

is injective.
Proof Suppose \( \{df_{ij}/f_{ij}\} \sim 0 \). Then, there exists \( \omega_i \in \Omega^1_X(U_i) \) such that

\[
\frac{df_{ij}}{f_{ij}} = \omega_j - \omega_i.
\]

Since \( X \) is one-dimensional, \( \omega_i \)'s are d-closed. By the Cartier operator \( C \), we have

\[
\frac{df_{ij}}{f_{ij}} = C(\omega_j) - C(\omega_i).
\]

Therefore, we have

\[
C(\omega_i) - \omega_i = C(\omega_j) - \omega_j.
\]

Hence, \( C(\omega_i) - \omega_i \) on \( U_i \) gives a global regular 1-form \( \omega \in H^0(X, \Omega^1_X) \). Since \( C - \text{id}_X \) is surjective on \( H^0(X, \Omega^1_X) \), there exists \( \{\tilde{\omega}\} \) such that \( (C - \text{id}_X)(\tilde{\omega}) = \omega \). Replace \( \omega_i \) by \( \omega_i - \tilde{\omega} \), we may assume \( C(\omega_i) = \omega_i \). Hence, there exists \( f_i \) such that \( \omega_i = \frac{df_i}{f_i} \). The result follows from this fact (cf. the proof of Theorem 3.1).

Let \( E_1 = E_2 = E \), and put \( A = E_1 \times E_2 \). We have the natural isomorphism

\[
H^1(A, \Omega^1_A) \cong H^1(A, \mathcal{O}_A) \otimes H^0(A, \Omega^1_A)
\]

with

\[
H^1(A, \mathcal{O}_A) \cong H^1(E_1, \mathcal{O}_{E_1}) \oplus H^1(E_2, \mathcal{O}_{E_2}),
\]

\[
H^0(A, \Omega^1_A) \cong H^0(E_1, \Omega^1_{E_1}) \oplus H^0(E_2, \Omega^1_{E_2}).
\]

Therefore, we have a decomposition

\[
(*) \quad H^1(A, \Omega^1_A) \cong H^1(E_1, \mathcal{O}_{E_1}) \otimes H^0(E_1, \Omega^1_{E_1}) \oplus H^1(E_1, \mathcal{O}_{E_1}) \otimes H^0(E_2, \Omega^1_{E_2})
\]

\[
\oplus H^1(E_2, \mathcal{O}_{E_2}) \otimes H^0(E_1, \Omega^1_{E_1}) \oplus H^1(E_2, \mathcal{O}_{E_2}) \otimes H^0(E_2, \Omega^1_{E_2}).
\]

We have projections

\[
pr_i : A \rightarrow E_i \quad (i = 1, 2).
\]

Then, we have injective homomorphisms

\[
pr_i^* : H^1(E_i, \Omega^1_{E_i}) \hookrightarrow H^1(A, \Omega^1_A).
\]

Note that

\[
H^1(E_1, \Omega^1_{E_1}) \cong H^1(E_1, \mathcal{O}_{E_1}) \otimes H^0(E_1, \Omega^1_{E_1})
\]

\[
H^1(E_2, \Omega^1_{E_2}) \cong H^1(E_2, \mathcal{O}_{E_2}) \otimes H^0(E_2, \Omega^1_{E_2}),
\]

and we have the following commutative diagram

\[
(***) \quad \begin{align*}
\text{NS}(A)/p\text{NS}(A) & \overset{\phi}{\twoheadrightarrow} H^1(A, \Omega^1_A) \\
\uparrow & \uparrow pr_i^* \\
\text{Pic}(E_i)/p\text{Pic}(E_i) & \overset{\phi}{\twoheadrightarrow} H^1(E_i, \Omega^1_{E_i})
\end{align*}
\]

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The image of the homomorphism \( pr_i^* \) is the one dimensional subspace \( H^1(E_i, \mathcal{O}_{E_i}) \otimes H^0(E_i, \Omega^1_{E_i}) \) \((i = 1, 2)\) in \( H^1(A, \Omega^1_A) \).

Now, we consider the Chern class mapping

\[
\text{NS}(A)/p\text{NS}(A) \cong \text{Pic}(A)/p\text{Pic}(A) \xrightarrow{c_1} H^1(A, \Omega^1_A).
\]

For the divisors \( E_2 \) (resp. \( E_1 \)) on \( A \), we set \( \Omega_1 = c_1(E_2) \) (resp. \( \Omega_4 = c_1(E_1) \)). Then, by the diagram (**) \( \Omega_1 \) (resp. \( \Omega_4 \)) is a basis of \( H^1(A, \Omega^1_A) \otimes H^0(E_1, \mathcal{O}_{E_1}) \), (resp. \( H^1(E_2, \mathcal{O}_{E_2}) \)).

We set

\[
\Delta_a = \Delta_{\text{id}, a}.
\]

Here, \( \text{id} \) is the identity endomorphism of \( E \). Then we have

\[
j(\Delta_a) = \begin{pmatrix} 1 & \bar{a} \\ a & \bar{aa} \end{pmatrix}
\]

Since \( \{ \text{id}, \frac{1+ \alpha}{2}, F \frac{1+ \alpha}{2}, \frac{(a+F)\alpha}{q} \} \) is a basis of \( \mathcal{O} = \text{End}(E) \), we see that

\[
E_1, E_2, \Delta = \Delta_{\text{id}, \Delta_{\frac{1+\alpha}{2}}, \Delta_{\frac{(a+F)\alpha}{q}}, \Delta_{\frac{(a+F)\alpha}{q}}}
\]

is a basis of \( \text{NS}(A) \). Since \( \alpha^2 = -q \), we see that \( \alpha \) acts on \( H^0(E, \Omega^1_E) \) as the multiplication by \( \pm \sqrt{-q} \). We can choose \( \alpha \) such that the action \( \alpha \) on \( H^0(E, \Omega^1_E) \) is the multiplication by \( \sqrt{-q} \). \( F \) acts on \( H^0(E, \Omega^1_E) \) as the zero-map. Therefore, \( \frac{1+ \alpha}{2}, \frac{1+ \alpha}{2} \) and \( \frac{(a+F)\alpha}{q} \) act on \( H^0(E, \Omega^1_E) \) respectively as

\[
\frac{1 + \sqrt{-q}}{2}, 0, \frac{a \sqrt{-q}}{q}.
\]

Since \( H^1(E, \mathcal{O}_E) \) is dual to \( H^0(E, \Omega^1_E) \), by Lemma 4.1 the actions of \( \frac{1+ \alpha}{2}, F \frac{1+ \alpha}{2} \) and \( \frac{(a+F)\alpha}{q} \) on \( H^1(E, \mathcal{O}_E) \) are respectively given by

\[
\frac{1 - \sqrt{-q}}{2}, 0, -\frac{a \sqrt{-q}}{q}.
\]

Therefore, on the decomposition (*) of the space \( H^1(A, \Omega^1_A) \) the endomorphisms \( \text{id} \times \frac{1+ \alpha}{2} \), \( \text{id} \times F \frac{1+ \alpha}{2} \), \( \text{id} \times \frac{(a+F)\alpha}{q} \) of \( A \) act respectively as

\[
(1, \frac{1 + \sqrt{-q}}{2}, \frac{1 - \sqrt{-q}}{2}, \frac{1 + q}{4}), \quad (1, 0, 0, 0), \quad (1, \frac{\sqrt{-q}}{q}, -\frac{\sqrt{-q}}{q}, \frac{a^2}{q})
\]

on each direct summand.

We consider the automorphism \( \tau \) of \( A \) defined by

\[
\tau : \quad A = E_1 \times E_2 \quad \rightarrow \quad A = E_1 \times E_2 \\
(P_1, P_2) \quad \mapsto \quad (P_2, P_1).
\]
We denote by $\Omega_2$ a basis of $H^1(E_1, O_{E_1}) \otimes H^0(E_2, \Omega^1_{E_2})$. We set $\Omega_3 = \tau^* \Omega_2$. Then, $\Omega_3$ is a basis of $H^0(E_1, \Omega^1_{E_1}) \otimes H^1(E_2, O_{E_2})$. We set

$$c_1(\Delta) = c_1(\Delta_{\sigma}) = \alpha_1 \Omega_1 + \alpha_2 \Omega_2 + \alpha_3 \Omega_3 + \alpha_4 \Omega_4$$

with $\alpha_i \in \mathbb{k} (i = 1, 2, 3, 4)$. We consider the inclusions

$$\epsilon_1 : E_1 \to E_1 \times E_2 = A \quad \epsilon_2 : E_2 \to E_1 \times E_2 = A$$

$P \mapsto (P, O_{E_2}) \quad P \mapsto (O_{E_1}, P)$

Then, we have the following diagram induced by $\epsilon_i$.

$$\begin{array}{ccc}
\text{Pic}(E_1)/p\text{Pic}(E_1) & \xrightarrow{c_1} & H^1(E_1, \Omega^1_{E_1}) \\
\uparrow & & \uparrow \\
\text{NS}(A)/p\text{NS}(A) & \xrightarrow{c_1} & H^1(A, \Omega^1_A).
\end{array}$$

Using this diagram, by $(\Delta, E_1) = 1$ and $(\Delta, E_2) = 1$ we see $\alpha_1 = \alpha_4 = 1$. Since $\tau^* \Delta = \Delta$, we also have $\alpha_2 = \alpha_3$, which we denote it by $\alpha$.

We consider the natural inclusion $\phi : \Delta \hookrightarrow A = E_1 \times E_2$ and the diagram

$$\begin{array}{ccc}
\text{Pic}(\Delta)/p\text{Pic}(\Delta) & \xrightarrow{c_1} & H^1(\Delta, \Omega^1_\Delta) \\
\uparrow & & \uparrow \\
\text{NS}(A)/p\text{NS}(A) & \xrightarrow{c_1} & H^1(\Delta, \Omega^1_\Delta).
\end{array}$$

Since $(\Delta, \Delta) = 0$, we have $\phi^* c_1(\Delta) = 0$. On the other hand, since $(\Delta, E_1) = (\Delta, E_2) = 1$, we have $\phi^* c_1(E_1) = \phi^* c_1(E_2) \neq 0$. Therefore, we see $\alpha \neq 0$. Replacing $\Omega_2$ by $\alpha \Omega_2$, we may assume $\alpha = 1$.

Summarizing these results, we have

$$c_1(E_1) = \Omega_4, \quad c_1(E_2) = \Omega_1,$$

$$c_1(\Delta) = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4,$$

$$c_1(\Delta_{\frac{\Delta}{p}}) = \Omega_1 + \frac{1 + 4q}{q} \Omega_2 + \frac{1 + 4q}{q} \Omega_3 + \frac{1 + 4q}{q} \Omega_4,$$

$$c_1(\Delta_{\frac{\Delta}{p} + \frac{2a}{q}}) = \Omega_1 + \frac{a\sqrt{-q}}{q} \Omega_2 - \frac{a\sqrt{-q}}{q} \Omega_3 + \frac{a^2}{q} \Omega_4.$$ 

Since $2q$ is prime to $p$, there exists an integer $\ell$ such that $\ell \equiv \frac{a}{2q} \pmod{p}$. Using these notation, we have the following theorem.

**Theorem 4.4** The kernel $\text{Ker } c_1$ is two-dimensional over $\mathbb{F}_p$, and a basis of $\text{Ker } c_1$ is given by divisors $\Delta_{\frac{\Delta}{p} + \frac{2a}{q}} - E_2$ and $\Delta_{\frac{\Delta}{p} + \frac{2a}{q}} - \ell \Delta_{\frac{\Delta}{p}} + 2\ell \Delta - (\ell + 1)E_2 - (1 - q + 2a)\ell E_1$.

**Proof** With respect to the basis $\langle \Omega_1, \Omega_2, \Omega_3, \Omega_4 \rangle$, the Chern classes $c_1(E_1)$, $c_1(E_2)$, $c_1(\Delta)$, $c_1(\Delta_{\frac{\Delta}{p}})$, $c_1(\Delta_{\frac{\Delta}{p} + \frac{2a}{q}})$, $c_1(\Delta_{\frac{\Delta}{p} + \frac{2a}{q}})$ are respectively represented
as the following vectors:

\[ u_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \]

\[ u_4 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{-q}}{2} \\ \frac{1-\sqrt{-q}}{2} \\ \frac{1}{4} \end{pmatrix}, \quad u_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_6 = \begin{pmatrix} 1 \\ \frac{a\sqrt{-q}}{q} \\ -\frac{a\sqrt{-q}}{q} \\ \frac{a^2}{q} \end{pmatrix}. \]

Since \( u_1, u_2, u_3, u_4 \) are linearly independent over \( \mathbb{F}_p \) and we have

\[ u_5 = u_2, \quad u_6 = \frac{2a}{q} u_4 - \frac{a}{q} u_3 + \left( \frac{a}{2q} + 1 \right) u_2 + \left( \frac{a}{2q} - \frac{a}{2} + \frac{a^2}{q} \right) u_1, \]

we see \( \dim_{\mathbb{F}_p} \text{Im} \ c_1 = 4 \). Since \( \dim_{\mathbb{F}_p} \text{NS}(A)/p\text{NS}(A) = 6 \), we have \( \dim_{\mathbb{F}_p} \ker c_1 = 2 \). Since \( \langle E_1, E_2, \Delta, \Delta_{F+\alpha}, \Delta_{F+\alpha} \rangle \) is a basis of \( \text{NS}(A)/p\text{NS}(A) \), the latter part follows from our construction.

Using this theorem, we have the following well-known corollary (cf. van der Geer and Katsura [3], for instance).

**Corollary 4.5** Let \( A \) be a superspecial abelian surface. Then, \( H^1(A, \Omega^1_A) \) is generated by algebraic cycles.

**Proof** This follows from the fact that \( u_1, u_2, u_3, u_4 \) are linearly independent also over \( k \).

5 **Example**

We give here one concrete example. Assume characteristic \( p = 3 \). Then, there exists up to isomorphism only one supersingular elliptic curve and the supersingular elliptic curve is defined by

\[ E : y^2 = x^3 - x \]

We consider two automorphisms defined by

\[ \sigma : x \mapsto x + 1, \ y \mapsto y, \]
\[ \tau : x \mapsto -x, \ y \mapsto \sqrt{-1}y \]

We have a morphism defined by

\[ \pi : E \rightarrow \mathbb{P}^1, \quad (x, y) \mapsto x \]
By the result of Ibukiyama ([4]), we have
\[ \text{End}(E) = Z + Z\tau + Z\iota \circ \tau + Z\tau \circ \iota \circ \sigma. \]

Here, \( \iota \) is the inversion of \( E \).

Let \( P \) be the point on \( \mathbb{P}^1 \) given by the local equation \( x = 0 \), and \( \tilde{P} \) a point on \( E \) such that \( \pi(\tilde{P}) = P \). We consider an affine open covering \( \{U_0, U_1\} \) of \( \mathbb{P}^1 \) which is given by
\[ U_0 = \{x \in \mathbb{P}^1 \mid x \neq \infty\}, U_1 = \{x \in \mathbb{P}^1 \mid x \neq 0\}. \]
The divisor \( P \) is given by the functions \( x \) on \( U_0 \), \( 1 \) on \( U_1 \).

Under the notation, we have the following diagram.
\[ \begin{array}{ccc}
2\tilde{P} & \in & \text{Pic}(E)/3\text{Pic}(E) \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
2\tilde{P} & \in & \text{Pic}(\mathbb{P}^1)/3\text{Pic}(\mathbb{P}^1)
\end{array} \quad \begin{array}{rll}
\text{c}_1 : & & H^1(E, \Omega_E^1) \cong k \\
\text{c}_1 : & & H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \cong k
\end{array} \]

In this diagram, we have \( c_1(P) = \{dx\} \), and \( c_1(\tilde{P}) = \{\frac{dx}{x}\} \).

We set \( A = E_1 \times E_2 \) with \( E_1 = E_2 = E \). We consider the Chern map
\[ c_1 : \text{NS}(A)/3\text{NS}(A) \to H^1(A, \Omega_A^1) \cong H^1(A, \mathcal{O}_A) \otimes H^0(A, \Omega_A^1) \]

We also consider the natural inclusion defined by
\[ \varphi : E \to E_1 \times E_2 = A \]
\[ P \mapsto (P, O_{E_2}) \]

We have a commutative diagram
\[ \begin{array}{ccc}
\text{NS}(A)/3\text{NS}(A) & \xrightarrow{\varphi} & \text{NS}(E)/3\text{NS}(E) \\
\downarrow c_1 & & \downarrow c_1 \\
H^1(A, \Omega_A^1) & \xrightarrow{\varphi^*} & H^1(E, \Omega_E^1)
\end{array} \]

Then, we have
\[ \varphi^*(c_1((\Delta))) = c_1(\varphi^*(\Delta)) = c_1(O_E) = \{\frac{dx}{2x}\} \neq 0. \]

We determine the action of \( \text{End}(E) \) on \( H^0(E, \Omega_E^1) \). A basis of \( H^0(E, \Omega_E^1) \) is given by \( \frac{dx}{y} \) and we have
\[ (\iota \circ \sigma) \frac{dx}{y} = -\frac{dx}{y}, \quad \tau \frac{dx}{y} = -\sqrt{-1} \frac{dx}{y}, \quad (\tau \circ \iota \circ \sigma) \frac{dx}{y} = \sqrt{-1} \frac{dx}{y}. \]
Since $H^1(E, \Omega_E^1)$ is dual to $H^0(E, \Omega_E^{1'})$, the actions of $\iota \circ \sigma$, $\tau$ and $\tau \circ \iota \circ \sigma$ are respectively given by the multiplications

$$-1, \sqrt{-1}, -\sqrt{-1}$$

by Lemma 4.1. Since we have

$$H^1(A, \Omega_A^1) \cong H^1(A, \mathcal{O}_A) \otimes H^0(A, \Omega_A^1)$$

$$\cong H^1(E_1, \mathcal{O}_{E_1}) \otimes H^0(E_1, \mathcal{O}_{E_1}) \oplus H^1(E_1, \mathcal{O}_{E_1}) \otimes H^0(E_2, \Omega_{E_2}^1)$$

$$\oplus H^1(E_2, \mathcal{O}_{E_2}) \otimes H^0(E_1, \Omega_{E_1}^1) \oplus H^1(E_2, \mathcal{O}_{E_2}) \otimes H^0(E_2, \Omega_{E_2}^1),$$

the actions $\text{id} \times \iota \circ \sigma$, $\text{id} \times \tau$ and $\text{id} \times \tau \circ \iota \circ \sigma$ are respectively given by the multiplication on each summand by

$$(1, -1, -1, 1)$$

$$(1, \sqrt{-1}, -\sqrt{-1}, 1)$$

$$(1, -\sqrt{-1}, \sqrt{-1}, 1).$$

By our general theory,

$$E_1, E_2, \Delta, \Delta_{\iota \circ \sigma}, \Delta, \Delta_{\tau \circ \iota \circ \sigma}$$

gives a basis of $\text{NS}(A)$ over $\mathbb{Z}$. Therefore, $\Delta + \Delta_{\iota \circ \sigma} + E_1 + E_2$ and $\Delta + \Delta_{\tau \circ \iota \circ \sigma} + E_1 + E_2$ are linearly independent divisors in $\text{NS}(A)/3\text{NS}(A)$ over $\mathbb{F}_3$. Moreover, considering the actions of the endomorphisms $\text{id} \times \iota \circ \sigma$, $\text{id} \times \tau$ and $\text{id} \times \tau \circ \iota \circ \sigma$ on $H^1(A, \Omega_A^1)$ and the commutative diagram

$$\begin{array}{ccc}
\text{NS}(A) & \xrightarrow{f} & \text{NS}(A) \\
\downarrow c_1 & & \downarrow c_1 \\
H^1(A, \Omega_A^1) & \xrightarrow{f} & H^1(A, \Omega_A^1)
\end{array}$$

we conclude that the Chern classes of these two divisors are zero. Therefore, we see that

$$\Delta + \Delta_{\iota \circ \sigma} + E_1 + E_2, \Delta, \Delta_{\tau \circ \iota \circ \sigma} + E_1 + E_2$$

gives a basis of $\text{Ker} c_1$ over $\mathbb{F}_3$.

6 An application to Kummer surfaces

Let $A$ be an abelian surface defined over $k$, and $\iota$ be the inversion. We denote by $\tilde{A}$ the surface made of 16 blowings-up at 16 two-torsion points on $A$. Then, $\iota$ induces the action $\tilde{i}$ on $\tilde{A}$ and $\text{Km}(A) = \tilde{A}/\tilde{i}$ is the Kummer surface. We denote by $\pi : \tilde{A} \to \text{Km}(A)$ the projection. A K3 surface $X$ is called supersingular if the Picard number $\rho(X)$ is equal to the second Betti number $b_2(X) = 22$. For a supersingular K3 surface, the discriminant of $\text{NS}(X)$ is equal to the form $-p^{2^g}$.
and \( \sigma \) is called an Artin invariant. We know \( 1 \leq \sigma \leq 10 \) (cf. Artin [1]). A supersingular K3 surface with Artin invariant 1 is said to be superspecial. Such a K3 surface is unique up to isomorphism and is isomorphic to the Kummer surface \( \text{Km}(A) \) such that \( A \) is superspecial (cf. Ogus [8]). We also know that a supersingular K3 surface with \( \sigma = 2 \) is isomorphic to a Kummer surface \( \text{Km}(A) \) such that \( A \) is supersingular and non-superspecial (cf. Ogus [8]).

We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{NS}(\text{Km}(A))/p\text{NS}(\text{Km}(A)) & \xrightarrow{c_1} & H^1(\text{Km}(A), \Omega^1_{\text{Km}(A)}) \\
\downarrow & & \downarrow \\
\text{NS}(\tilde{A})/p\text{NS}(\tilde{A}) & \xrightarrow{c_1} & H^1(\tilde{A}, \Omega^1_{\tilde{A}}).
\end{array}
\]

Since we have \( 2\text{NS}(\tilde{A}) \subset \pi^*\text{NS}(\text{Km}(A)) \subset \text{NS}(\tilde{A}) \) by Shioda [11] and \( p \neq 2 \), we see \( \text{NS}(\text{Km}(A))/p\text{NS}(\text{Km}(A)) \cong \text{NS}(\tilde{A})/p\text{NS}(\tilde{A}) \). Since \( \iota \) acts on \( H^1(A, \mathcal{O}_A) \) and \( H^0(A, \Omega^1_A) \) as the multiplication by \(-1\). Since \( H^1(A, \Omega^1_A) \cong H^1(A, \mathcal{O}_A) \otimes H^0(A, \Omega^1_A) \), we see that \( \iota \) acts identically on \( H^1(A, \Omega^1_A) \). Therefore, \( \iota \) acts identically on \( H^1(A, \Omega^1_A) \). Hence, we have \( H^1(\text{Km}(A), \Omega^1_{\text{Km}(A)}) \cong H^1(A, \Omega^1_A) \). Summarizing these results, by Theorems 3.1 and [?] we have the following theorem.

**Theorem 6.1** For a Kummer surface \( \text{Km}(A) \), the Chern class mapping \( c_1 : \text{NS}(\text{Km}(A))/p\text{NS}(\text{Km}(A)) \longrightarrow H^1(\text{Km}(A), \Omega^1_{\text{Km}(A)}) \) is injective if and only if \( \text{Km}(A) \) is not superspecial. If \( \text{Km}(A) \) is superspecial, \( \dim_F \ker c_1 = 2 \) holds.

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