Autocorrelation Function for Dispersion-Free Fiber Channels with Distributed Amplification

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Abstract—The autocorrelation function of the output signal given the input signal is derived in closed form for dispersion-free fiber channels with distributed optical amplification (OA). The autocorrelation function is used to upper bound the output power of bandlimited or time-resolution limited receivers, and thereby to bound spectral broadening and the capacity of receivers with thermal noise. The output power scales at most as the square-root of the launch power, and thus capacity scales at most as one-half the logarithm of the launch power. The propagating signal bandwidth scales at least as the square-root of the launch power. However, in practice the OA bandwidth should exceed the signal bandwidth to compensate attenuation. Hence, there is a launch power threshold beyond which the fiber model loses practical relevance. Nevertheless, for the mathematical model an upper bound on capacity is developed when the OA bandwidth scales as the square-root of the launch power, in which case capacity scales at most as the inverse fourth root of the launch power.

Index Terms—Autocorrelation, Channel capacity, Dispersion, Kerr effect, Noise

I. INTRODUCTION

One obstacle to understand the capacity of optical fiber is that nonlinearity and distributed optical amplification (OA) cause spectral broadening that is difficult to characterize. One approach to make progress is to study simplified models that retain the essential features of spectral broadening. The model studied here is dispersion-free fiber with distributed OA.

There are two existing approaches to proceed. The first is by Mecozzi [1] who derived the per-sample statistics of the channel, including the channel conditional probability distribution. Turitsyn et al. [2], and Yousefi and Kschischang [3], redervive this distribution with other methods. They further argue that, for large launch power $P$, the per-sample capacity is the same as the capacity of an additive white Gaussian noise (AWGN) channel with a direct detection receiver, i.e., capacity is closely approached with intensity modulation, and grows as $\frac{1}{2} \log P$ for large $P$. Refined results were recently posted in [4, 5].

A second approach considers the entire received waveform. Tang [6, 7] studied the auto- and crosscorrelation functions of the channel input and output signals when the input signals are Gaussian and stationary, and in particular when the input signals are sinc pulses with complex and circularly symmetric Gaussian modulation. The autocorrelation function defines the signal power spectral density (PSD) that lets one study spectral broadening. Tang used the PSD to evaluate Pinsker’s capacity lower bound [8] for wavelength division multiplexing (WDM) and per-channel receivers without cooperation.

A. Limitations of the Per-Sample Model

The per-sample model is attractive because one has closed-form expressions for the statistics. However, the model has several limitations. First, the per-sample statistics do not capture spectral broadening, and this tempts one to consider only the launch signal bandwidth rather than the propagating signal bandwidth. The propagating signal bandwidth $W$ grows with the launch power $P$ and a practical requirement is that the OA bandwidth $B$ exceed $W$ to compensate attenuation, i.e., one requires $B \geq W$. However, we show that there is a $P$ beyond which $B$ does not exceed $W$ and the model loses practical relevance. The growth of $W$ is due to signal-noise mixing that cannot be controlled by waveform design.

Second, a per-sample receiver has infinite bandwidth while practical receivers are bandlimited. In other words, a per-sample analysis takes limits in a particular order: first the receiver bandwidth is made infinite and then $P$ is made large. However, for a given system the receiver bandwidth is fixed, and changing the order of limits (first $P$ is made large) can change the capacity scaling.

Third, the per-sample model ignores correlations in the received waveform, and this can lead to suboptimal receivers. In fact, we show that a three-sample receiver achieves infinite capacity for any $P$ for the model studied in [1, 2, 3]. The per-sample rate $\frac{1}{2} \log P$ thus underestimates capacity. This issue will also appear for the nonlinear Schrödinger equation (NLSE) with dispersion, nonlinearity, and distributed noise.

The result may be understood as follows: the noise in the model of [1] has limited bandwidth, and by sending signal energy in the noise-free spectrum one achieves large rate, cf. [10] Thm. 5. The reader may expect that an obvious fix is to add white (thermal or electronic) noise to the channel or receiver models. However, the per-sample capacity is then zero. This conundrum shows that reasonable and precise noise models, device models, and spectral analyses are needed when analyzing capacity, e.g., see [11] Sec. IX-A-B.

1There are many reasonable definitions for bandwidth. We use a common one, namely the length of the frequency range centered at the carrier frequency that contains a specified fraction of the signal power.

2The short article [9] also argues that the model of [1] may be impractical for large $P$. The arguments are based on qualitative and empirical observations concerning spectral broadening and signal-noise mixing.

3The potential for capacity increase was noted in [3] Sec. VIII) but without recognizing the extent of the effect, i.e., that the noise model is unreasonable. Hence, the main conclusions in [2] Sec. VIII should be treated with caution, namely that the capacity of dispersion-free fiber grows as $\frac{1}{2} \log P$, and that a potential peak of spectral efficiency curves is due to deterministic effects only, and not due to signal-noise mixing.
Based on these observations, we conclude that one must study the waveform model, and not only the per-sample model. We proceed by studying two-sample statistics for the waveform channel. We additionally place practical constraints on the OAs, transmitter, and receiver. First, we model the receiver as performing projections with white noise, e.g., due to thermal noise. Second, the receiver has either finite bandwidth or finite time resolution. Finally, we study OA bandwidth that grows with the propagating signal bandwidth.

B. Organization

This paper is organized as follows. Section II describes notation, second order statistics, AWGN channels and their capacity, and certain hyperbolic functions. Section III describes the fiber models under study. Section IV points out limitations notation, second order statistics, AWGN channels and their capacity. Section V states our main result: the autocorrelation function in closed form for dispersion-free optical fiber with OA. Section VI studies the autocorrelation function for rectangular pulses. Section VII develops upper bounds on the output power and energy of the receivers, as well as lower bounds on the propagating signal bandwidth. Section VIII uses the power bounds to develop capacity upper bounds. Section IX concludes the paper. The appendices provide supporting material, including a review of theory from [1].

II. Preliminaries

A. Notation

We study signals \( u(z,t) \) where \( z \) is a spatial variable and \( t \) is a time variable. The position \( z = 0 \) is where the information-bearing signal \( u(0,\cdot) \) is launched. To simplify notation, we often write \( u_z(t) = u(z,t) \), and even drop \( z \) if the position is clear from the context. For example, we often write \( u(t) \) for \( u_z(t) \). We also often drop the time indices for convenience, e.g., we write \( u_0 = u_0(t) \) and \( u_0' = u_0(t') \).

We write random variables with uppercase letters and realizations of random variables with the corresponding lowercase letters. For example, we follow [11] and study the statistics of the random variables \( U(z,t) \) for different \( t \) when conditioned on the launch signal \( u_0(\cdot) \). The expectation of \( X \) is denoted by \( E[X] \), and the conditional expectation based on the event \( Y = y \) is denoted by \( E[X|Y = y] \). The mutual information of two random variables \( X \) and \( Y \) is written as \( I(X;Y) \). The differential entropy of \( Y \) without and with conditioning on \( X \) is denoted by \( h(Y) \) and \( h(Y|X) \), respectively.

The notation \( y^* \) refers to the complex conjugate of \( y \). \( \Re(y) \) and \( \Im(y) \) are the real part and imaginary part of \( y \). The function \( 1(\cdot) \) is the indicator function that takes on the value 1 if its argument is true, and is otherwise 0. The function \( \delta(\cdot) \) is the Dirac-\( \delta \) operator, and we write \( \sin c(\pi y)/(\pi y) \) with \( \sin c(0) = 1 \). The functions \( I_0(\cdot) \) and \( I_1(\cdot) \) are the modified Bessel functions of the first kind of orders 0 and 1, respectively. We write \( Q \lesssim P^x \) if \( \lim_{P \to \infty} \log Q/\log P \leq x \), and similarly for \( Q \gtrsim P^x \).

B. Autocorrelation Functions and Power Spectral Density

We study the conditional and unconditional autocorrelation functions

\[
A_z(t,t') = E[ U_z(t) U_z(t')^* ] \quad \text{if } u_0(\cdot) = u_0(\cdot) \\
\bar{A}_z(t,t') = E[ A_z(t,t') ] = E[ U_z(t) U_z(t')^* ]
\]

(1)

(2)

where the bar above \( A_z(t,t') \) specifies that we have taken the expectation with respect to the launch signal \( U_0(\cdot) \). As described above, we often drop the subscript \( z \) for convenience, e.g., we write \( A(t,t') \) for \( A_z(t,t') \). A basic property of the autocorrelation function is \( A(t,t') = A(t',t)^* \).

The power spectral density (PSD) is defined as

\[
\bar{P}(f) = \lim_{T \to \infty} \bar{P}(f,T) \quad \text{assuming the limit exists, where}
\]

\[
\bar{P}(f,T) = \frac{1}{T} E \left[ \left( \int_{-T/2}^{T/2} U(t)e^{-j2\pi ft} dt \right)^2 \right] = \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \bar{A}(t,t')e^{-j2\pi f(t-t')} dt' dt.
\]

(3)

(4)

C. Pulse Amplitude Modulation

We will sometimes consider pulse-amplitude modulation (PAM) with time period \( T_s \) for which the launch signals are

\[
u_0(t) = \sum_k x_k g(t - kT_s)
\]

(5)

where the \( x_k \) are complex-valued modulation symbols and \( g(\cdot) \) is a pulse shape with unit energy. The signals \( u_0 \) are cyclostationary [12, p. 70], so for all integers \( \ell \), we have

\[
\bar{A}(t - \ell T_s, t' - \ell T_s) = \bar{A}(t, t').
\]

We may thus focus on the time-averaged autocorrelation function

\[
\bar{A}(\tau) = \frac{1}{T_s} \int_0^{T_s} \bar{A}(t,t - \tau) dt
\]

(6)

and we have

\[
\bar{P}(f) = \int_{-\infty}^{\infty} \bar{A}(\tau)e^{-j2\pi f\tau} d\tau.
\]

(7)

D. Additive White Gaussian Noise Channels

The classic way of dealing with noise for linear channels is to use the AWGN model

\[
u(t) = u_0(t) + n_E(t)
\]

(8)

where \( n_E(\cdot) \) is a realization of the complex, circularly symmetric, white, Gaussian process \( N_E(\cdot) \) with a one-sided PSD of \( 0 \) W/Hz across all frequencies. We will consider thermal noise with \( N_0 = k_B T_e \) where \( k_B \approx 1.381 \times 10^{-23} \) Joules/Kelvin is Boltzmann’s constant, and where \( T_e \) is the temperature in Kelvin.

The model (8) is artificial because \( n_E(\cdot) \) has infinite bandwidth and infinite power.\(^4\) Of course, noise encountered in

\(^4\)The per-sample capacity of the AWGN channel is therefore zero.
practice has finite bandwidth and power, but the idea is that the noise PSD is flat for frequencies much larger than those of the processing capabilities of the transmitter or receiver. An optimal receiver projects its input signal onto the linear subspace spanned by the transmit signals, see Sec. IV-B.

Consider next the bandlimited AWGN channel

\[ u(t) = (u_0(t) + n_E(t)) \ast W \text{sinc}(Wt) \]  

where \( \ast \) denotes convolution. One can convert this channel into a discrete-time channel by sampling \( u(\cdot) \) at the Nyquist rate \( W \) Hz. The capacity under the average power constraint

\[ \bar{P}_T = \frac{1}{T} \int_{-T/2}^{T/2} E \left[ |U_0(t)|^2 \right] dt \leq P \]  

for large \( T \) is achieved by using PAM, sinc pulses, and Gaussian modulation, and is given by \([13, \text{Sec. 25}]\)

\[ C(W) = W \log_2 \left( 1 + \frac{P}{WN_0} \right) \text{ bits/s}. \]  

The value \( C \) increases with \( W \), and we have

\[ \lim_{W \to \infty} C(W) = \frac{P}{N_0} \log_2(e) \text{ bits/s}. \]  

In other words, capacity scales logarithmically with the signal-to-noise ratio (SNR) \( P/(WN_0) \) with a bandwidth limitation, and linearly with \( P/N_0 \) without a bandwidth limitation.

The spectral efficiency is defined as

\[ \eta(W) = \frac{C(W)}{W} = \log_2 \left( 1 + \frac{P}{WN_0} \right) \text{ bits/s/Hz} \]  

and we have \( \eta(W) \to 0 \) in the limit of large \( W \). However, one usually studies \( P = E/T_s \), where \( E \) is the average energy of PAM with sinc pulses that are offset by \( T_s = 1/W \) seconds. We thus have

\[ \eta(W) = \frac{C(W)}{W} = \log_2 \left( 1 + \frac{E}{N_0} \right) \text{ bits/s/Hz} \]  

which is independent of \( W \). Note that this approach has a transmit power \( P = EW \) that grows with \( W \).

**Remark 1:** The constraint \([10]\) permits peaky or flash signals with arbitrarily large amplitudes if \( T \to \infty \). In practice, however, the input amplitude is limited, i.e., we require \( |u_0(t)| \leq A_{\text{max}} \) for all \( t \) and for some positive \( A_{\text{max}} \). The capacity under an input amplitude constraint was studied in \([13, \text{Sec. 26}]\), for example.

**Remark 2:** Suppose \( U_0(\cdot) \) has the PSD \( \bar{P}_0(\cdot) \) so that the power at the output of the channel \([9]\) is

\[ \bar{P}_r(W) = WN_0 + \int_{-W/2}^{W/2} \bar{P}_0(f) df. \]  

Suppose further that, instead of the launch constraint \([10]\), the receiver signal \( U(\cdot) \) must satisfy \( \bar{P}_r(W) \leq P + WN_0 \). An upper bound on the capacity of the channel \([9]\) is then (see \([13, \text{Sec. 29}]\) and \([14]\))

\[ C(W) \leq W \log_2 \left( 1 + \frac{P}{WN_0} \right) \text{ bits/s}. \]  

**E. Hyperbolic Functions**

We use the following functions with complex arguments:

\[ S(c) = \text{sech} \left( \sqrt{2c} z \right) \]  

\[ T(c) = \tanh \left( \sqrt{2c} z \right) \]  

where the dependence on \( z \) on the left-hand side (LHS) of \([17]\) and \([18]\) was suppressed for notational simplicity. As further simplification, we write \( S_R(c) = \Re(S(c)), S_I(c) = \Im(S(c)), T_R(c) = \Re(T(c)), T_I(c) = \Im(T(c)) \).

Consider \( c = -jx/z^2 \) where \( x \) is real and non-negative.

The following bounds are valid numerically, see Figs. 1 and 2:

\[ |S(c)| \leq 1, \quad |T(c)| \leq z \]  

\[ -0.136 \leq S_R(c) \leq 1, \quad 0 \leq T_R(c) \leq z \]  

\[ -0.028 \leq S_I(c) \leq x, \quad 0 \leq T_I(c) \leq (2/3)xz. \]  

For small \( x \), we have

\[ |S(c)| \geq d, \quad |T(c)| \geq d \cdot z \]  

\[ S_R(c) \geq d, \quad T_R(c) \geq d \cdot z \]  

\[ S_I(c) \geq d \cdot x, \quad T_I(c) \geq d \cdot (2/3)xz \]  

for a constant \( d \) that approaches 1 as \( x \) approaches 0. We further have the following bounds, see Figs. 1 and 2:

\[ |S(c)| \leq \sqrt{3} e^{-\sqrt{3}} \]  

\[ T_I(c) \geq \frac{2}{3} \min \left( x, \frac{1}{\sqrt{x}} \right) \]  

\[ \frac{S_I(c)^2}{T_I(c)} \leq \frac{3x}{2z}. \]  

**III. Fiber and Distributed Amplification**

**A. Nonlinear Schrödinger Equation**

Consider the slowly varying component \( a(z, T) \) of a single-mode, linearly-polarized, electromagnetic wave \( a(z, T)e^{j2\pi f_0 T} \) along an optical fiber, where \( z \) is the position, \( T \) is time, and \( f_0 \) is the carrier frequency. The propagation
of $a(z,T)$ is governed by the stochastic NLSE (see \cite{15} Eq. (2.3.27))

$$\frac{\partial a}{\partial z} + \frac{\alpha}{2} a + \beta_1 \frac{\partial a}{\partial T} + j \frac{\beta_2}{2} \frac{\partial^2 a}{\partial T^2} - j\gamma |a|^2 a - n = 0 \quad (24)$$

where $\alpha$ is the loss coefficient, $\beta_1$ is the group velocity dispersion (GVD) parameter, and $\gamma$ is the Kerr coefficient. All of these parameters are frequency dependent in general. The variables $n(z, T)$ are realizations of noise random variables $N(z, T)$ whose characteristics we discuss below in Sec. III-B.

Remark 3: The wave propagation is sometimes defined using the complex conjugate of $a(z, T)$. For example, this approach is used in \cite{1}.

The NLSE is usually expressed using the retarded time $t = T - \beta_1 z$ and the amplified signal $u(z, t) = e^{\alpha z^2/2} a(z, t)$. Inserting these modifications into (24), we have the simplified equation

$$\frac{\partial u}{\partial z} = -j \frac{\beta_2}{2} \frac{\partial^2 u}{\partial T^2} + j\gamma |u|^2 u e^{-\alpha z} + n e^{-\alpha z/2} \quad (25)$$

where we abuse notation and write $n(z, t)$ for $n(z, t + \beta_1 z)$. A commonly studied version of (25) has $\alpha = 0$ so that

$$\frac{\partial u}{\partial z} = -j \frac{\beta_2}{2} \frac{\partial^2 u}{\partial T^2} + j\gamma |u|^2 u + n. \quad (26)$$

The model (26) has many interesting features. For example, if $\beta_2 < 0$ and there is no noise, i.e., $n(t) = 0$ for all $t$, then the fiber supports bright solitons, see \cite{15} Ch. 5). However, the general model seems to have no closed-form solution and, to gain insight, one often studies channels without nonlinearity ($\gamma = 0$) or without dispersion ($\beta_2 = 0$) or without noise.

B. Optical Amplifier Noise Model

The signal $n(z, \cdot)$ in (23)-(26) represents OA noise, which is usually modeled as a bandlimited Gaussian process with the same bandwidth $B$ as the OA bandwidth. Two common choices for OA are erbium-doped fiber amplifiers (EDFAs) at specific positions along the fiber and distributed Raman amplification \cite{11} Sec. IX.B.

We consider distributed Raman amplification for which the noise power distance density (PDD) is $K W/m$ that is proportional to $B$, i.e., we have $K = N_A B W/m$ where $N_A$ is the OA noise power spectral-distance density (PSDD) in W/Hz/m. For distributed amplification, the noise is assumed independent across positions $z$ so that we have the spatiotemporal autocorrelation function:

$$E[N(z, t) N(z', t')] = K \delta(z - z') \text{sinc}(B(t - t')). \quad (27)$$

More precisely, the accumulated noise at distance $z$ is modeled as a spatial Wiener process $W(z, t)$ that has the form $W(z, t) = (W_R(z, t) + jW_I(z, t))/\sqrt{2}$ where $W_R(z, t)$ and $W_I(z, t)$ are independent standard real Wiener processes. A useful model is that $W(z, t)$ is the limit for large $\ell$ of the processes

$$W_\ell(z, t) = \sum_{i=1}^{\lfloor \ell z \rfloor} \frac{1}{\sqrt{\ell}} N_i(t) \quad (28)$$

where the time processes in the sequence $\{N_i(\cdot)\}_{i=1}^\infty$ are independent and identically distributed (i.i.d.), complex, circularly-symmetric, bandlimited, Gaussian random processes with mean 0 and autocorrelation function

$$E[N_i(t) N_i(t')] = \text{sinc}(B(t - t')). \quad (29)$$

Thus, $W_\ell(z, t)$ and $W_\ell(z, t')$ are correlated in general, since from (28)-(29) we have the correlation coefficient

$$\rho_\ell(z, t, t') = \frac{E[W_\ell(z, t) W_\ell(z, t')]}{\sqrt{E[|W_\ell(z, t)|^2] E[|W_\ell(z, t')|^2]}} = \text{sinc}(B(t - t')). \quad (30)$$

Observe that (30) is independent of $\ell$, $z$, and the absolute time $t$, so we use the notation $\rho(t - t')$ instead of $\rho_\ell(z, t, t')$.

C. Lossless and Linear Fiber Model

Consider (26) but with $\gamma = 0$. Taking Fourier transforms, the propagation equation is

$$\frac{\partial \tilde{u}}{\partial z} = \frac{j \beta_2}{2} (2\pi f)^2 \tilde{u} + \tilde{n} \quad (31)$$

where $\tilde{u}(z, \cdot)$ and $\tilde{n}(z, \cdot)$ are the respective Fourier transforms of $u(z, \cdot)$ and $n(z, \cdot)$. The solution of (31) is

$$\tilde{u}(f, z) = \tilde{u}(0, f) e^{j \beta_2 (2\pi f)^2 z} + \sqrt{K} \tilde{w}(f, z) \quad (32)$$

where $\tilde{w}(z, \cdot)$ is the Fourier transform of $w(z, \cdot)$, i.e., $\tilde{w}(\cdot)$ is a realization of a spatial Wiener process if $|f| < B/2$, and is zero otherwise. The channel filter is therefore an all-pass filter with phase shifts proportional to $f^2$; the frequency-dependence of the phase is called chromatic dispersion. Furthermore, the channel is noise-free outside the band $|f| < B/2$. This property is problematic when considering information theoretic limits of communication, see Sec. IV-A below.

\footnote{The transformation $t = T - \beta_1 z$ does not change this equation.}
D. Lossless and Dispersion-Free Fiber Model

Consider (34) but without dispersion. We have
\[
\frac{\partial u}{\partial z} = j\gamma |u|^2 u + n. \tag{33}
\]

The solution of (33) is (see [1] eq. (13))
\[
u(z, t) = \left[u(0, t) + \sqrt{K} w(z, t)\right] 
\exp \left[j \gamma \int_0^z |u(0, t) + \sqrt{K} w(z', t)|^2 dz' \right]. \tag{34}
\]

If we expand the quadratic term of the exponential in (34), then we obtain three parts: a self-phase modulation term \(|u(0, t)|^2\), a signal-noise mixed term \(2\sqrt{K} \Re\{u(0, t) w(z', t)^*\}\), and a noise term \(K|w(z, t)|^2\). If \(|u(0, t)|\) is large, then the signal-noise mixed term will cause large and random phase variations that result in uncontrolled spectral broadening. Understanding this effect seems key to understanding the nonlinear Shannon limit of optical fiber, i.e., the limitation of the capacity [11].

IV. NOISE AND RECEIVER MODELS

A. Large Capacity

The capacity of the channel (32) is well-understood. For example, if the launch signal \(\tilde{u}(0, \cdot)\) can have energy outside the noise band, then the capacity is infinity for any launch power [10, Thm. 5].

The nonlinear model (34) can also have infinite capacity. To see this, consider \(T \leq 1/B \leq T_s\) and the launch signal
\[
u(0, t) = x g(t - T/2) - 2x g(t - 3T/2) \tag{35}
\]
where \(x\) is an information symbol and \(g(\cdot)\) is a rectangular pulse of unit norm in the interval \([0, T_s]\). We claim that, for small \(T\), the launch bandwidth is described mainly by \(T_s\) and not \(T\). We have
\[
|\tilde{u}(0, f)| = |x| \cdot |\tilde{g}(f)| \cdot \sqrt{5 - 4 \cos(2\pi f T)} \tag{36}
\]
where \(\tilde{g}(\cdot)\) is the Fourier transform of \(g(\cdot)\). Choosing small \(T\) thus does not increase the launch bandwidth, e.g., if one uses a measure such as the bandwidth having 99% of the power.

We now choose \(T \ll 1/B\) so that the noise variables at \(t = 0, T, 2T\) are approximately the same, i.e., we have
\[
w(z, 0) \approx w(z, T) \approx w(z, 2T).
\]

We can make the approximations as accurate as desired by choosing sufficiently small \(T\). Let \(w(z, 0) = w_R + j w_I\) and suppose \(x\) is real. We choose a 3-sample receiver that outputs
\[
y_0 = |u_z(0)|^2 = K (w_R^2 + w_I^2) \tag{37}
\]
\[
y_1 = |u_z(T)|^2 \approx (x/\sqrt{T_s} + \sqrt{K} w_R)^2 + K w_I^2 \tag{38}
\]
\[
y_2 = |u_z(2T)|^2 \approx (-x/\sqrt{T_s} + \sqrt{K} w_R)^2 + K w_I^2 \tag{39}
\]
and compute
\[
x^2 \approx T_s \left(y_1 + y_2 - y_0\right) \tag{40}
\]
to any desired accuracy by choosing sufficiently small \(T\). This means that, for the launch energy \(x^2(1 + 4T/T_s) \approx x^2\), we can achieve any rate by choosing sufficiently small \(T\). In other words, the capacity is unbounded for any launch power.

Remark 4: The reader might consider this example unsatisfactory because it requires rapid signaling. In fact, the example does not work with sinc pulses, yet it seems artificial to limit attention to such pulses. The main purpose of the example is to show that bandlimited noise has pitfalls when dealing with capacity, and that care is needed in treating bandwidth [16].

Remark 5: The above observations apply also to the full NLSE model in Sec. III-A.

B. Receiver Noise

A natural approach to circumvent the problems with bandlimited noise is to add AWGN to the nonlinear model (34). In other words, the new model has two noise processes: the bandlimited distributed OA noise and the receiver AWGN. More precisely, consider an electronic receiver that operates on a noisy signal
\[
u_E(t) = u(z, t) + n_E(t) \tag{41}
\]
where \(n_E(\cdot)\) is the same as in [8].

Now consider the set \(L^2[0, T]\) of continuous and finite-energy signals in the time interval \([0, T]\). This set has a complete orthonormal basis \(\{\phi_m(\cdot)\}_{m=1}^{\infty}\) and one usually has receivers that put out a finite number of projection values
\[
Y_m = \int_0^T u_E(z, t) \phi_m(t)^* dt = Z_m + \int_0^T U(z, t) \phi_m(t)^* dt \tag{42}
\]
for \(m = 1, 2, \ldots, M\) where
\[
Z_m = \int_0^T N_E(t) \phi_m(t)^* dt. \tag{43}
\]
In other words, \(Z_1, Z_2, \ldots, Z_M\) is a string of statistically independent, complex, circularly-symmetric, Gaussian random variables with variance \(N_0\). The set \(\{Y_m\}_{m=1}^{M}\) of measurements forms a set of sufficient statistics if every possible signal \(u(z, \cdot)\) lies in the subspace spanned by the signals \(\{\phi_m(\cdot)\}_{m=1}^{M}\). Otherwise, one must let \(M \rightarrow \infty\) in general.

Remark 6: An alternative to introducing receiver thermal noise is to assume that \(u(z, \cdot)\) has spectral components inside the OA bandwidth only. However, this approach prevents considering finite-time pulses such as rectangular pulses. Furthermore, spectral broadening prevents \(u(z, \cdot)\) from remaining strictly bandlimited even if \(u(0, \cdot)\) is strictly bandlimited.

Remark 7: A second alternative is to study OA with infinite bandwidth but with a finite PDD of \(K \text{ W/m}\). Another way to think of this is that the noise PSDD is \(N_A = K/B \text{ W/Hz/m}\) and one considers the limit of increasing \(B\). This is effectively what was done in [11, Sec. IV], and we develop results for this model in Appendix [9]. The model is artificial but it has two useful features: the analysis greatly simplifies and the model

\footnote{One may also consider the set of continuous and finite-energy signals in a frequency interval.}
gives insight into systems where the optical noise process has much larger bandwidth than the signals propagating along the fiber. Related studies on models with white phase noise can be found in [17], [13], [19].

Remark 8: We show in Appendix [r] that the nonlinearity can increase capacity. The idea is to use the nonlinearity to convert amplitude-shift keying (ASK) to orthogonal frequency-shift keying (FSK). We remark that orthogonal FSK achieves capacity for large bandwidth $W$ [12], p. 207] and that capacity grows linearly in launch power for large $W$, see [12].

C. Bandlimited Receiver

We will consider two receivers that are related. The first is bandlimited to $W$ Hz, i.e., the receiver collects energy in the frequency band $f \in [-W/2, W/2]$ only. The average receiver power after filtering is (see [15])

$$ P_r(W) = \int_{-W/2}^{W/2} \mathcal{P}(f) df. \quad (44) $$

Note that we have not included the noise $N_E(\cdot)$ in $P_r(W)$; this noise will contribute an additional $W N_0$ Watts.

For convenience, we define (see [4])

$$ \tilde{P}_r(W, T) = \int_{-W/2}^{W/2} \tilde{\mathcal{P}}(f, T) df. \quad (45) $$

To later help us bound $\tilde{P}_r(W, T)$, we upper bound the PSD of unit height in the frequency interval $[-W/2, W/2]$ by

$$ \tilde{b}(f) = \begin{cases} 2 \left(1 - \frac{|f|}{W}\right), & |f| \leq W \\ 0, & \text{else.} \end{cases} \quad (46) $$

The motivation for this step is to ensure that the absolute value of the corresponding time signal

$$ b(t) = 2W \operatorname{sinc} (Wt)^2 $$

integrates to a finite value for $t \geq 0$, namely the value

$$ \int_0^\infty |b(t)| dt = 1. \quad (48) $$

Inserting (46) into (45), we have

$$ \tilde{P}_r(W, T) \leq \int_{-W/2}^{W/2} \tilde{\mathcal{P}}(f, T) \tilde{b}(f) df $$

$$ = \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \tilde{A}(t, t') b(t - t') dt' dt. \quad (49) $$

For cyclostationary signals, we obtain

$$ \tilde{P}_r(W) \leq \int_{-\infty}^{\infty} \tilde{A}(\tau) b(\tau) d\tau. \quad (50) $$

D. Time-Resolution Limited Receiver

The second receiver is time-resolution limited to $T_r$ seconds where $T_r \leq T_s$. More precisely, we consider a normalized integrate-and-dump filter over $T_r$ seconds. The energy output by the receiver is

$$ E_m(T_r) = E \left[ \left| \int_{mT_r}^{(m+1)T_r} \frac{1}{\sqrt{T_r}} U(t) dt \right|^2 \right] U_0(\cdot) = u_0(\cdot) $$

$$ = \frac{1}{T_r} \int_{mT_r}^{(m+1)T_r} \int_{mT_r}^{(m+1)T_r} \tilde{A}(t, t') dt' dt. \quad (51) $$

Thus, the average energy is

$$ \tilde{E}_m(T_r) = \frac{1}{T_r} \int_{mT_r}^{(m+1)T_r} \int_{mT_r}^{(m+1)T_r} \tilde{A}(t, t') dt' dt. \quad (52) $$

The value $\tilde{E}_m(T_r)$ is closely related to the RHS of (49).

Remark 9: One can build a receiver with time-resolution $T_r/2$ seconds with two receivers with time resolution $T_r$ seconds by offsetting their integration times by $T_r/2$ seconds. Thus, it might make more sense to state that our receivers have limited time precision. Similarly, one can build a bandwidth $2W$ receiver with two bandwidth $W$ receivers whose center frequencies are offset by $W$ Hz. Of course, these approaches increase complexity and cost, and in practice one is limited by the available receivers.

V. AUTOCORRELATION FUNCTION

The following theorem is our main analytic result. Consider

$$ c = -j\gamma (K/2) \sqrt{1 - \rho^2} \quad (53) $$

so that in Sec. II-E we have $c = -jx/z^2$ where

$$ x = (K/2)z^2 \sqrt{1 - \rho^2}. \quad (54) $$

Theorem 1: The conditional autocorrelation function $[1]$ for the signal (34) is

$$ A(t, t') = |S(c)|^2 \left[ T_R(c)K \rho + \left( S_R(c) u_0 + j \frac{S_I(c)}{\sqrt{1 - \rho^2}} (u_0 - \rho u_0') \right) \left( S_R(c) u_0' + j \frac{S_I(c)}{\sqrt{1 - \rho^2}} (u_0' - \rho u_0) \right)^* \right. $$

$$ \exp \left( j\gamma T_R(c) \left| u_0 \right|^2 - \left| u_0' \right|^2 \right) $$

$$ \exp \left( -\gamma \frac{T_R(c)}{\sqrt{1 - \rho^2}} \left| u_0 \right|^2 + \left| u_0' \right|^2 - 2\rho \Re \{ u_0 u_0' \} \right) \right]. \quad (55) $$

Proof: See Appendix [D]. The proof builds on the development in [1] that is reviewed in Appendices [A, C].

Remark 10: The first exponential of (55) describes self-phase modulation (SPM) and the second exponential is due to signal-noise mixing. The argument of the latter exponential is real, non-positive, and decreases with the instantaneous powers $|u_0|^2$ and $|u_0'|^2$ of the launch signal unless $t = t'$. 

Example 1: Consider $\gamma = 0$ so the channel is linear. We have $c = 0$, $S(c) = 1$, $T(c) = z$, and therefore

$$ A(t, t') = K z \rho(t - t') + u_0(t) u_0(t')^*. \quad (56) $$
Example 2: Consider $t = t'$ so that $u_0 = u'_0$, $\rho = 1$, $c = 0$, $S(c) = 1$, $T(c) = z$, and therefore

$$A(t, t) = E \left[ |U_z(t)|^2 \right] U_0(t) = u_0(t)$$

$$= K z + |u_0(t)|^2. \quad (57)$$

The instantaneous power is thus preserved, see (54).

Example 3: Consider $u_0(t) = 0$ for all $t$. The noise autocorrelation function is

$$A(t, t') = |S(c)|^2 T_R(c) K \rho(t - t') \quad (58)$$

and for $t \neq t'$ the value $|A(t, t')|$ first increases but eventually decreases as $z$ grows. This means that the noise PSD eventually broadens with $z$. However, one usually operates in regimes where $|c|$ is small so that $A(t, t') \approx K \rho z$.

Example 4: Consider $u_0(t) = \sqrt{P} e^{i\phi(t)}$ for all $t$, i.e., the launch signal has constant envelope. We compute

$$A(t, t') = |S|^2 \left[ T_R K \rho + P e^{i\phi} \right] \left[ (S_R^2 + \frac{S^2}{1 - \rho^2}) (1 - \rho e^{-j\phi_{\Delta}})^2 \right] \exp \left( -\gamma T_i \frac{1 - \rho^2}{1 - \rho^2} \right) \quad (59)$$

where $\phi_{\Delta}(t, t') = \phi(t) - \phi(t')$, and where we have suppressed the dependence on $c$ and $t$ for notational convenience.

A. Low Noise-Nonlinearity-Distance Product

A commonly studied regime is where the noise and/or the Kerr coefficient are small with respect to the distance. More precisely, we consider $\sqrt{\gamma K z^2}$ small enough so that

$$S(c) \approx 1 - 2c(z^2/2) = 1 + j \gamma K \sqrt{1 - \rho^2} (z^2/2) \quad (60)$$

$$T(c) \approx z - 2c(z^3/3) = z + j \gamma K \sqrt{1 - \rho^2} (z^3/3) \quad (61)$$

are accurate approximations for the various constants in (55). The autocorrelation function is thus approximately

$$A(t, t') = \left( K \rho z + u_0 u'_0 + j \gamma K \frac{z^2}{2} \left[ |u_0|^2 - |u'_0|^2 \right] \right) \exp \left( \frac{-\gamma T_i}{1 - \rho^2} \right) \exp \left( -\frac{K}{2} [ |u_0|^2 + |u'_0|^2 - 2\rho \Re \{u_0 u'_0^* \} ] \right) \quad (62)$$

where

$$\kappa = 2\gamma^2 K z^2/3. \quad (63)$$

and where we have kept up to second order terms in $\sqrt{\gamma K z^2}$.

B. Bounds on the Autocorrelation Amplitude

Consider the argument of the last exponential in (55) for which we have

$$|u_0|^2 + |u'_0|^2 - 2\rho \Re \{u_0 u'_0^* \}$$

$$= |u'_0 - \rho u_0|^2 + |u_0|^2 (1 - \rho^2). \quad (64)$$

Now suppose that $|u_0| \geq |u'_0|$ so that

$$|u_0 - \rho u_0| \leq 2 |u_0|$$

$$|u'_0 - \rho u_0| \leq 2 |u_0|. \quad (65)$$

We may use (64) and (65) with (19) to bound

$$|A(t, t')| \leq \left[ K z + |u_0|^2 \left( 1 + \frac{1 + 2 |S_I(c)|^2}{\sqrt{1 - \rho^2}} \right) \right] \exp \left( -\gamma T_i \frac{|u_0|^2}{1 - \rho^2} \right) \leq \left[ K z + |u_0|^2 \left( 1 + \gamma K z^2 \right) \right] \exp \left( -\gamma T_i \frac{|u_0|^2}{1 - \rho^2} \right). \quad (66)$$

We further have $\gamma K z^2 \approx 0.017$ for certain parameter ranges that we are interested in, see Table I.

Remark 11: The amplitude $|A(t, t')|$ captures the influence of signal-noise mixing, but it removes the SPM exponential in (55). The reason for focussing on signal-noise mixing is because this effect cannot be controlled, as opposed to the deterministic effects of SPM and cross-phase modulation (XPM). However, in a network environment, the XPM cannot necessarily be controlled either, and interference can be the main limitation on capacity (11).

The bound (66) is useful when $B$ and $K = BN_A$ are fixed. However, we will also be interested in scaling $B$ with the launch power. To treat such cases, we keep the $|S(c)|^2$ term from (55). We further use (64) and its symmetric counterpart to write

$$|u_0|^2 + |u'_0|^2 - 2\rho \Re \{u_0 u'_0^* \}$$

$$= \frac{|u_0 - \rho u_0|^2 + |u'_0 - \rho u'_0|^2}{2} \leq |u_0|^2 + |u'_0|^2 \left( 1 - \rho^2 \right). \quad (67)$$

We now use (19) to bound

$$|A(t, t')| \leq |S(c)|^2 \left[ K z + \left( \frac{|u_0| + |S_I(c)|}{\sqrt{1 - \rho^2}} \left| u_0 - \rho u'_0 \right| e^{-\frac{\gamma T_i}{1 - \rho^2} \frac{|u_0 - \rho u'_0|^2}{2}}} \right]$$

$$\exp \left( -\gamma T_i \frac{|u_0|^2 + |u'_0|^2}{2} \frac{|u_0 - \rho u'_0|^2}{1 - \rho^2} \right) \quad (68)$$

Applying (236) in Appendix I we have

$$|A(t, t')| \leq |S(c)|^2 \left[ K z + \left( |u_0| + \delta \right) \left| u'_0 + \delta \right| \right]$$

$$\exp \left( -\gamma T_i \frac{|u_0|^2 + |u'_0|^2}{2} \frac{|u_0 - \rho u'_0|^2}{1 - \rho^2} \right) \quad (69)$$

where

$$\delta = \sqrt{\frac{S_I(c)}{e \gamma T_i(c) \sqrt{1 - \rho^2}}} \quad (70)$$

Using (23) with (54), we have

$$\delta \leq \sqrt{\frac{3K z}{4e}} \approx 1.4 \times 10^{-3} \quad (71)$$

where the approximation is for the parameters of Table I. We here have $|u_0| + \delta \approx |u_0|$ for “large” signal powers such as $|u_0|^2 \geq 1 \times 10^{-3}$ W (or 0 dBm).
VI. RECTANGULAR PULSES

Consider rectangular pulses for which the SPM term

\[ \exp(j \gamma T_R(c) [\|u_0\|^2 - |u'_0|^2]) \]

in (55) is unity. Rectangular pulses are thus convenient for studying the spectral broadening characterized by the signal-noise mixing exponential in (55).

Consider the fiber parameters shown in Table I (see Tables I-III). We have \( \sqrt{\gamma K z^2} \approx 0.130 \) so that the approximations (60)-(62) are accurate. We further have \( \kappa \approx 28.6 \) so that the second exponential of (62) is small (less than 1/e) for power levels beyond \( |u_0|^2 = |u'_0|^2 = 1/\kappa \approx 0.035 \) W (or 15.4 dBm) if \( \rho = 0 \).

A. Isolated Rectangular Pulse

Consider an isolated rectangular pulse

\[ u_0(t) = \begin{cases} \sqrt{P}, & |t| \leq T_s/2 \\ 0, & \text{else} \end{cases} \]

(72)

that has energy \( PT_s \) Joules. The approximation (62) is

\[
A(t, t') =
\begin{cases}
(K \rho z + P) e^{-\kappa P (1-\rho)}, & |t| \leq T_s/2, \ |t'| \leq T_s/2 \\
(K \rho z + j \gamma K \rho z^2 P) e^{j \gamma z P} e^{-\kappa P}, & |t| \leq T_s/2, \ |t'| > T_s/2 \\
(K \rho z - j \gamma K \rho z^2 P) e^{-j \gamma z P} e^{-\kappa P}, & |t| > T_s/2, \ |t'| \leq T_s/2 \\
K \rho z, & |t| > T_s/2, \ |t'| > T_s/2.
\end{cases}
\]

(73)

For example, for a linear channel we have \( \gamma = 0 \) and

\[
A(t, t') = \begin{cases} K \rho z + P, & |t| \leq T_s/2, \ |t'| \leq T_s/2 \\
K \rho z, & \text{else}
\end{cases}
\]

(74)

We choose \( T_s = 10 \) ps to correspond to the symbol rate of 100 Gbaud.

Fig. 3 shows \( |A(t, t')| \) in dB for the pulse (72), \( t' = 0 \) and \( t' = 5.1 \) ps, and for \( P = 10, 100, 200, 400 \) mW. The plot of the amplitude for the pulse (72) and \( A(5, 5.1) \) is close to \( K \rho z \). However, for \( P = 10 \) mW the function \( |A(t, 0)| \) already has a small bulge at \( t = 0 \). We have thus entered the nonlinear regime where signal-noise mixing causes spectral broadening. As the power increases further, \( 10 \log_{10} |A(t, 0)| \) develops a sinc pulse shape in the range \( t = [-5, 5] \) ps due to the exponential factor \( e^{-\kappa P (1-\rho)} \). The narrow autocorrelation function for \( P = 400 \) mW implies that the spectrum has broadened considerably.

B. PAM with Rectangular Pulses and Ring Modulation

Consider PAM in (5) with rectangular pulses that are time-limited to \([0, T_s]\). We study a constant amplitude \( \sqrt{P} \), and a phase that is uniformly distributed over \([-\pi, \pi] \), i.e., ring modulation or phase-shift keying (PSK). Using (62) with \( u_0(t) = \sqrt{P} e^{j \Phi(t)} \) and \( u_0'(t) = \sqrt{P} e^{j \Phi'(t)} \) we have

\[
A(t, t') = (K \rho z + P e^{j \Phi}) e^{-\kappa P (1-\rho) \cos \Phi_{\Delta}}.
\]

(75)

where \( \Phi_{\Delta} = \Phi - \Phi' \). If \( t \) and \( t' \) are in the same symbol interval, then we have \( \Phi_{\Delta} = 0 \); otherwise \( \Phi \) and \( \Phi' \) are independent and \( \Phi_{\Delta} \) is uniform. The average autocorrelation function is
therefore
\[ \tilde{A}(t,t') = \begin{cases} (K\rho z + P)e^{-\kappa P(1-\rho)}, & \Phi_{\Delta} = 0 \\ [K\rho z I_0 (\kappa P \rho) + P I_1 (\kappa P \rho)]e^{-\kappa P}, & \text{else.} \end{cases} \] (76)
Note that \( \tilde{A}(t,t') \) is real-valued. We further compute
\[ \tilde{A}(\tau) = \left( 1 - \frac{|\tau|}{T_s} \right) (K\rho z + P)e^{-\kappa P(1-\rho)} \]
\[ + \frac{|\tau|}{T_s} [K\rho z I_0 (\kappa P \rho) + P I_1 (\kappa P \rho)] e^{-\kappa P} \] (77)
for \( |\tau| < T_s \), and otherwise
\[ \tilde{A}(\tau) = [K\rho z I_0 (\kappa P \rho) + P I_1 (\kappa P \rho)] e^{-\kappa P}. \] (78)
Note that \( \rho \) is a function of \( \tau \), and that \( \tilde{A}(\tau) \) is a real-valued and even function of \( \tau \).

A plot of \( |A(\tau)| \) is shown in Fig. 5 along with the amplitude of the average autocorrelation function from 10^4 Monte Carlo simulations of the random signals and noise. Observe that \( |A(\tau)| \) accurately matches the simulations. The dash-dotted curve is for \( \gamma = 0 \), and it shows that the nonlinearity has caused substantial narrowing of the autocorrelation function, i.e., there is substantial spectral broadening. This phenomenon is clearly apparent in Fig. 6 that plots the PSDs \( \tilde{P}(f) \) for \( P = 10, 50, 100, 500, 1000 \) mW as well as the PSD when \( P = 10 \) mW and \( \gamma = 0 \). Recall that the OA bandwidth is \( B = 500 \) GHz, and observe that \( \tilde{P}(f) \) is large well-beyond this frequency already for \( P = 100 \) mW. The model is therefore inaccurate at this launch power since the OA compensates attenuation only for frequencies up to \( B \).

Remark 12: The received signal energy is \( \tilde{A}(0) = \bar{A}(0) = Kz + P \), and as \( \gamma \to 0 \) we have
\[ \tilde{A}(\tau) \to \begin{cases} K\rho z + \left( 1 - \frac{|\tau|}{T_s} \right) P, & |\tau| \leq T_s \\ K\rho z, & \text{else.} \end{cases} \] (79)
The same limiting expression (79) is valid for \( \bar{A}(\tau) \).

VII. POWER, BANDWIDTH, AND ENERGY BOUNDS
This section studies the power and energy at the output of bandlimited and time-resolution limited receivers. The analysis ultimately lets us bound the propagating signal bandwidth as a function of the launch power. A simple but useful bound for low launch power is
\[ \tilde{P}_r(W,T) \leq \tilde{P}_r(\infty,T) = Kz + P_T \leq Kz + P \] (80)
where the second inequality is from (10).

A. Bandlimited Receiver
Consider (49) and split the double integral into three parts: one where \( |u_0| > |u_0'| \), one where \( |u_0| < |u_0'| \), and one where \( |u_0| = |u_0'| \). The first two double integrals are identical due to the symmetry in the arguments of the integrand, i.e., for every pair \((t,t') = (t_1, t_2)\) where \( |u_0| > |u_0'| \) there is a pair \((t,t') = (t_2, t_1)\) for which \(|u_0| < |u_0'| \) and
\[ |\tilde{A}(t_1, t_2) b(t_1 - t_2)| = |\tilde{A}(t_2, t_1) b(t_2 - t_1)|. \]
In other words, using (49) we have
\[ \tilde{P}_r(W,T) \leq \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} |\tilde{A}(t,t') b(t - t')| dt' dt \]
\[ = \frac{2}{T} \int_{\mathcal{I}_1} |\tilde{A}(t,t') b(t - t')| dt' dt + \frac{1}{T} \int_{\mathcal{I}_2} |\tilde{A}(t,t') b(t - t')| dt' dt \] (81)
where
\[ \mathcal{I}_1 = \{(t,t') : |u_0| > |u_0'|, |t| \leq T/2, |t'| \leq T/2\} \] (82)
\[ \mathcal{I}_2 = \{(t,t') : |u_0| = |u_0'|, |t| \leq T/2, |t'| \leq T/2\}. \] (83)
We further have \( |\tilde{A}(t,t')| = |E[\tilde{A}(t,t')]| \leq E[|\tilde{A}(t,t')|] \).
Thus, using (69) and \(|u_0| \geq |u_0'| \) for all \((t,t') \in \mathcal{I}_1 \cup \mathcal{I}_2\), we have
\[ \tilde{P}_r(W,T) \leq \frac{1}{T} \int_{-T/2}^{T/2} E[|\tilde{P}_r(W,T)|] dt \] (84)
where
\[ P_r(W, t) = 2 \left[ Kz + \left( \sqrt{P_t} + \delta \right)^2 \right] \]
\[ \int_{-T/2}^{T/2} |S(c)|^2 \exp \left( -\gamma T_s(c) \frac{P_t}{2} \sqrt{1 - \rho^2} \right) |b(t - t')| \, dt' \]
and we have defined \( P_t = |u_0(t)|^2 \).

**B. Bounds on the Instantaneous Received Power**

Recall that \( W \) is the receiver bandwidth and \( B \) is the OA bandwidth. In Appendix G we prove the following lemmas that bound the instantaneous received power \( P_r(W, t, t) \). We distinguish cases where \( x \geq 1 \) cannot or can occur.

- **Lemma 2** applies to the usual case with \( W \leq B \) and low-noise nonlinearity-distance product, i.e., \( \gamma (K/2) z^2 \leq 1 \).
- **Lemma 3** is for \( W \leq B \) and \( \gamma (K/2) z^2 \geq 1 \).
- **Lemma 4** is for \( W \geq B \) and \( \gamma (K/2) z^2 \leq 1 \).

We do not consider the case \( W \geq B \) and \( \gamma (K/2) z^2 \geq 1 \) because this case is slightly more complicated than the others, and because we are mainly interested in \( W \leq B \).

**Lemma 2:** If \( W \leq B \) and \( \gamma (K/2) z^2 \leq 1 \), then we have
\[ P_r(W, t, t) \leq 2 \left[ Kz + \left( \sqrt{P_t} + \delta \right)^2 \right] \frac{2 W/B}{\sqrt{(\kappa/8)P_t}} \left( e^{-\gamma K z^2} - (\kappa/9)P_t \right) . \]

**Lemma 3:** If \( W \leq B \) and \( \gamma (K/2) z^2 \geq 1 \), then we have
\[ P_r(W, t, t) \leq 4 \left[ Kz + \left( \sqrt{P_t} + \delta \right)^2 \right] \frac{2 W/B}{\sqrt{(\kappa/8)P_t}} \left( e^{-\gamma K z^2} + 5e^{-\gamma K z^2 - \sqrt{\pi/3} P_t} \right) . \]

**Lemma 4:** If \( W \geq B \) and \( \gamma (K/2) z^2 \leq 1 \), then we have
\[ P_r(W, t, t) \leq 4 \left[ Kz + \left( \sqrt{P_t} + \delta \right)^2 \right] \frac{2 W/B}{\sqrt{(\kappa/8)P_t}} \left( e^{-(\kappa/8)P_t(B/W)^2} + 5e^{-\gamma K z^2 - (\kappa/9)P_t} \right) . \]

**Remark 13:** The above bounds are valid for any launch signal. The bounds may be very loose, e.g., for launch signals with bandwidth larger than \( B \).

**Remark 14:** We have
\[ (\sqrt{\pi}/2) \text{erf}(y) \approx y, \text{ small } |y| \]
\[ \text{erf}(y) \approx 1, \text{ large } y. \]

Thus, for large \( P_t \), the received power scales at most as \( \sqrt{P_t} \) for all the regimes considered in Lemmas 2-4. The power loss factor of \( 1/\sqrt{P_t} \) is due to signal-noise mixing, and the square-root character of the power loss is due to the quadratic behavior of \( \rho(t - t') = \text{sinc}(B(t - t')) \) near \( t = t' \). The shape of the OA noise PSD thus directly affects the power scaling.

**Remark 15:** For small \( P_t \) or small \( \gamma \), we know that the instantaneous received power \( P_r(W, t, t) \) can be \( Kz + P_t \), as we expect for a memoryless, noisy, linear channel. For example, small \( P_t \) the RHS of (86) approaches
\[ 4 \left[ Kz + \left( \sqrt{P_t} + \delta \right)^2 \right] \left( \frac{2 W}{B} + 5e^{-\gamma K z^2} \right) . \]

Note that \( W/B \) can be small but the term \( 5e^{-\gamma K z^2} \) is larger than one if \( \gamma (K/2) z^2 \leq 1 \). However, for \( \gamma (K/2) z^2 \geq 1 \) and small \( P_t \), the RHS of (87) approaches
\[ 4 \left[ Kz + \left( \sqrt{P_t} + \delta \right)^2 \right] \left( \frac{29 W}{\gamma K z^2} + 5e^{-\gamma K z^2} \right) . \]

Now the receiver may put out less power than \( Kz + P_t \) due to the large noise nonlinearity-distance product.

**Remark 16:** If \( B \) is very large, then the \( W/B \) terms in (86)-(88) are small. We may thus encounter \( P_r(W, t, t) \) with an exponential behavior in \( P_t \), see Remark 7 and Appendix E.

**Remark 17:** The case \( W > B \) has the receiver measuring signals in bands where there is no attenuation yet the noise is small, as discussed in the introduction and Sec. IV-A. Moreover, for fixed \( P_t \), fixed \( B \), and large \( W \) we approach the regime of the per-sample receiver where (88) becomes
\[ P_r(\infty, T, t) \leq Kz + \left( \sqrt{P_t} + \delta \right)^2 \left[ 9 + 20e^{-\gamma K z^2} - (\kappa/9)P_t \right] . \]

The correct answer on the RHS of (92) is \( Kz + P_t^{\prime} \); the extra factors are due to loose bounding steps that were designed for large \( P_t \).

**C. Bounds on the Average Received Power**

We continue to study the case \( W \leq B \) as the range of practical and theoretical interest. We would next like to develop a bound on the average received power \( P_{\text{av}}(W, T) \) as a function of the maximum average launch power \( P \), see (10). For this purpose, define the function
\[ f(s, P) = \left[ Kz + \left( \sqrt{P} + \delta \right)^2 \right] \frac{\sqrt{\pi} \text{erf}(\sqrt{sP})}{2sP} \]
and an offset power \( P_0 = 3(Kz + \delta^2) \). In Appendix G we prove the following lemmas.

**Lemma 5:** If \( W \leq B \) and \( \gamma (K/2) z^2 \leq 1 \), then we have
\[ P_{\text{av}}(W, T) \leq c_1 + \frac{8W}{B} f \left( \frac{\kappa}{8} P + P_0 \right) \]
where
\[ c_1 = \left( Kz + \delta^2 + \sqrt{\frac{18}{\kappa e}} \delta + 9 \frac{\kappa e}{K} \right) e^{-\gamma K z^2} . \]

Furthermore, the RHS of (94) is non-decreasing and concave in \( P \).

**Lemma 6:** If \( W \leq B \) and \( \gamma (K/2) z^2 \geq 1 \), then we have
\[ P_{\text{av}}(W, T) \leq c_2 + \frac{16(W/B)}{\gamma K z^2} f \left( \frac{1}{3Kz}, P + P_0 \right) \]
(96)
where
\[
\begin{align*}
c_2 &= \frac{100}{\gamma K z^2} \left[ K z + \delta^2 + \sqrt{\frac{6K z}{e} \delta + \frac{\sqrt{18K z}}{e}} \right] \quad (97) \\
c_3 &= 20 \left[ K z + \delta^2 + \left( \frac{80K z}{\gamma e^2} \right)^{1/4} \delta + \sqrt{\frac{20K}{\gamma e^2}} \right] e^{-\gamma K z^2} \quad (97).
\end{align*}
\]

Furthermore, the RHS of (96) is non-decreasing and concave in \(P\).

Remark 18: The values \(c_1\), \(c_2\), and \(c_3\) are independent of \(P\) and \(B\), but they depend on \(B\), \(\gamma\), and \(z\).

Remark 19: The RHSs of (94) and (96) scale as \(\sqrt{P}\) for large \(P\).

D. Propagating Signal Bandwidth

We proceed to develop a bound on the propagating signal bandwidth, which we also write as \(W\) (in the previous sections, the parameter \(W\) represented the receiver filter bandwidth). We are particularly interested in large \(P\) where spectral broadening occurs. We interpret the regime \(W \leq B\) as being “practically relevant” and \(W > B\) as being “impractical”.

The average total received power for a linear channel is \(Kz + P\). Suppose we require that 99% of this power is inside the band \(f \in [-W/2, W/2]\), i.e., we require
\[
\bar{P}_r(W, T) \geq 0.99(Kz + P). \quad (98)
\]

We remark that the value 99% is not crucial; the results below remain valid for any other choice near 100%.

Consider first \(W \leq B\) and \(\gamma(K/2)z^2 \leq 1\). Combining (94) and (98), and using \(\text{erf}(y) \leq 1\), we have (see (102))
\[
\frac{W}{B} \geq \frac{0.99(Kz + P) - c_1}{8f(\kappa/8, P + P_0)} \geq \frac{0.99(Kz + P) - c_1}{8 \left[ Kz + (\sqrt{P + P_0 + \delta})^2 \right]} \quad (99).
\]

Thus, for fixed \(B\) and large \(P\), we find that \(W\) scales at least as a constant times \(\sqrt{P}\). This means that there is some power threshold for which \(W > B\). We conclude that the model loses practical relevance beyond some launch power threshold.

An upper bound on the threshold follows by computing the \(P\) for which the RHS of (99) is one. For example, for the parameters in Table I, we compute \(P \leq 18.6\) W. However, the power 18.6 W seems unrealistically large, which suggests that our bounds are very loose. Fig[1] also suggests that the bound is loose, since there is substantial spectral broadening already at \(P = 50\) mW. However, recall that the bound is valid for any launch signal, and not only PAM with rectangular pulses and ring modulation.

\[\text{This definition does not always make sense, e.g., for very noisy signals where the useful part of the signal has small bandwidth.}\]

Consider next \(W \leq B\) and \(\gamma(K/2)z^2 \geq 1\). Combining (96) and (98), and using \(\text{erf}(y) \leq 1\), we have
\[
\frac{W}{B} \geq \frac{0.99(Kz + P) - c_1}{\frac{16\sqrt{3\pi}}{\gamma K z^2}f(\frac{1}{\sqrt{3\pi}}, P + P_0)} \geq \frac{0.99(Kz + P) - c_1}{c_2\sqrt{P + P_0} + \frac{16\sqrt{3\pi}}{\gamma K z^2} \left[ Kz + (\sqrt{P + P_0 + \delta})^2 \right]} \quad (100)
\]

Thus, for fixed \(B\) and large \(P\), we again find that \(W\) scales at least as a constant times \(\sqrt{P}\). We again conclude that the model loses practical relevance beyond some launch power threshold.

E. Distributed Amplification Bandwidth

The bounds (99)-(102) let us study whether we can increase the range of practically relevant \(P\) by increasing \(B\). We show that this is not possible in general. In fact, as \(B\) increases we must limit ourselves to progressively smaller \(P\), while at the same time dealing with more noise power \(Kz = N_A Bz\).

We study the following problem. Suppose the OA bandwidth scales as \(B = P^\beta\) for some non-negative constant \(\beta\). For large \(P\), we thus study the case \(\gamma(K/2)z^2 \geq 1\) where the relevant bounds are (96) and (101)-(102). Note that \(K\), \(\kappa\), and \(P_0\) are proportional to \(B\), while \(\delta\) is proportional to \(\sqrt{B}\). Thus, \(c_2\) remains a constant and \(c_3\) vanishes for large \(B\). Inserting \(B = P^\beta\) into (96), the scaling behavior of \(P_r(W, T)\) for large \(P\) is bounded as
\[
\bar{P}_r(W, T) \sim \begin{cases} P^{(1-3\beta)/2}, & 0 \leq \beta \leq 1 \\ P^{-\beta}, & \beta \geq 1. \end{cases} \quad (103)
\]

The average receiver power thus decreases with the average launch power if \(\beta > 1/3\) and \(P\) is sufficiently large.

Next, inserting \(B = P^\beta\) into (101), the scaling behavior of \(W\) is bounded as
\[
W \gtrsim \begin{cases} P^{(1+3\beta)/2}, & 0 \leq \beta \leq 1 \\ P^{2\beta}, & \beta \geq 1. \end{cases} \quad (104)
\]

The condition \(W \leq B\) for large \(P\) requires \(P^{(1+3\beta)/2} \lesssim P^\beta\) for \(0 \leq \beta \leq 1\), or \(P^{2\beta} \lesssim P^\beta\) for \(\beta \geq 1\), neither of which is possible. We conclude that there is no scaling of \(B\) through which we can make the model practically relevant for large launch power.

F. Time-Resolution Limited Receiver

Bounds for the time-resolution limited receiver can be developed using the same steps as those for the bandlimited receiver. For instance, using the same steps as in (81) but with (52) rather than (49), we have the analog of (84)-(85), namely
\[
\bar{E}_m(T_r) \geq \frac{1}{T_r} \int_{mT_r}^{(m+1)T_r} \mathbb{E}[E_m(T_r, t)] \, dt \quad (105)
\]

where
\[
E_m(T_r, t) = 2 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] \int_{mT_r}^{(m+1)T_r} |S(c)|^2 \text{exp} \left( -\gamma T_f(c) \frac{P_t}{2} \sqrt{1 - \bar{P}} \right) \, dt'. \quad (106)
\]
Next, by following similar steps as (195)–(200) that were used to derive (86), for $T_r \geq 1/B$ we have

$$E_m(T_r, t) \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right]$$

$$\frac{1}{B} \sqrt{\frac{4}{\pi}} \frac{\sqrt{\pi}}{2} \text{erf} \left( \sqrt{\frac{(Kz^2)}{P_t}} \right) + 5 \left( T_r - \frac{1}{B} \right) e^{-\sqrt{\gamma Kz^2 - (\kappa/9)P_t}}. \quad (107)$$

Note that there is no extra factor of two in front of the $\text{erf}(\cdot)$ term, cf. (86), because we do not need to use the filter (46).

For $T_r < 1/B$, we have

$$E_m(T_r, t) \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right]$$

$$\frac{1}{B} \sqrt{\frac{4}{\pi}} \frac{\sqrt{\pi}}{2} \text{erf} \left( \sqrt{\frac{(Kz^2)}{P_t}} \right) B T_r \quad (108)$$

which is simpler than (87) because there is only one integration interval, rather than four as in Appendix C; see (206). As before, for large $P_t$ the energy $E_m(T_r, t)$ scales at most as $\sqrt{P_t}$. The same claim is valid for the average energy $E_{\bar{m}}(T_r)$ by using the concavity steps in Appendix C; see (216)–(220).

Remark 20: The time resolution $T_r$ must scale to zero at least as fast as $1/\sqrt{P_t}$ to have the RHS of (108) scale as $P_t$.

Remark 21: Consider PAM and fixed $P_t$. As $T_r$ decreases to zero, the RHS of (108) becomes

$$4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] T_r \quad (109)$$

which decreases to zero. However, there are $T_s/T_r$ samples per transmitted symbol, so the energy collected per symbol is proportional to $T_s$.

Remark 22: If $\gamma \to 0$ then the RHS of (108) becomes

$$4 (Kz + P_t) T_r. \quad (109)$$

This is loose by a factor of four: a factor of two is from the step corresponding to (81), and another factor of two is from the step corresponding to (200) where the interval $T_1$ was enlarged. We show in Appendix D how to improve these steps to obtain the expected $(Kz + P_t) T_r$ for PAM with rectangular pulses, ring modulation, and $T_r = T_s = 1/B$.

VIII. CAPACITY UPPER BOUNDS

Recall from (16) that the capacity satisfies

$$C(W) \leq W \log_2 \left( 1 + \frac{\bar{P}_r(W)}{W N_0} \right) \text{bits/s}. \quad (110)$$

We may thus use the bounds (80), (94) and (96) to upper bound (110).

Consider the fiber parameters in Table I and the receiver bandwidth $W = B = 500$ GHz. We study both the normalized capacity $C(W)/W$ and the spectral efficiency

$$\eta = \frac{C(W)}{\max(W, W_{\text{min}})} \text{bits/s/Hz} \quad (111)$$

where $W_{\text{min}}$ is the smallest received signal bandwidth that satisfies (99). Both expressions are measured in bits/s/Hz.

Fig. 7 shows the resulting bounds as the curves labeled “Upper bound” and “\eta bound.” We also plot a lower bound from (111) Fig. 36, curve (1)]. This bound was computed for 5 WDM signals, each of bandwidth 100 GHz, but with dispersion and optical filtering (OF). The upper and lower bounds are thus not directly comparable at high launch power. However, at low launch power both channels are basically linear and have the same capacity. We remark that we have shifted the lower bound by $10 \log_{10}(5) \approx 7$ dB to the right, since the $P_{\text{in}}$ in [11] Fig. 36 is the power per WDM channel.

We comment on the behavior of the curves.

- The upper bound increases with $P$.
- The model is no longer practically relevant according to (99) for $P > 18.6$ W, or 42.7 dBm. This bound is shown as the vertical dashed line in Fig. 7. The real threshold for practical relevance is much lower.
- The upper bound has two parts. The part on the left (small to large $P$) up to the vertical dashed line is based on the known bound (80). The part on the right (very large $P$) is new and is based on (94).
- The upper bound is far above the lower bound from (111) Fig. 36). This suggests that the upper bound is very loose.
- The upper bound seems extremely loose for small $P$. To understand why, observe that for small $P$ the RHS of (110) is

$$\log_2 \left( 1 + \frac{Kz}{W N_0} \right) \approx 11.7 \text{bits/s/Hz}. \quad (112)$$

In fact, we expect that $Kz$ should appear in the denominator of the SNR in (110) and (112), and not in the numerator. This issue is discussed in Sec. VIII-A below.

- Beyond the threshold, $\bar{P}_r(W)$ scales as $\sqrt{P}$. The slope of the bound thus changes from approximately 3 dB per bit to 6 dB per bit. However, we expect that the signal phase cannot be used to transmit information at large $P$, cf. [3] Sec. VI.A]. If this is true, then $C(W)/W$ eventually scales at most as $1/2 \log P$, and the slope of the upper bound becomes 12 dB per bit.
• The upper bound on \((\ref{eq:upper_bound})\) decreases rapidly beyond the power threshold because of spectral broadening.

A. Rates with OA Noise

One might expect that a \(Kz\) term should appear in the denominator of the SNR in \((\ref{eq:SNR})\). However, we have so far been unable to prove this for the model \((\ref{eq:NLSE})\). The difficulty is related to the signal-noise mixing, the bandlimited nature of the OA noise, and to the discussion in Sec. IV-A.

However, suppose the propagating signal remains inside the OA band, as required by the inequality \(W \leq B\). Suppose further that the propagating signal is accurately characterized by considering only frequencies within the band \(f \in [-W/2, W/2]\) for all \(z\). We can then apply the theory in \((\ref{eq:theory})\), \((\ref{eq:theory2})\) to improve \((\ref{eq:SNR})\) to

\[
C(W) \leq W \log_2 \left( \frac{\bar{P}_r(W) + W N_0}{K z + W N_0} \right) \text{ bits/s. (113)}
\]

Consider again the fiber parameters in Table I and \(W = B = 500\) GHz. Fig. 7 shows the resulting bound on \(C(W)/W\) as the curve labeled “Upper bound 2”. We comment on the behavior of the curve.

• The upper bound now seems reasonable for small \(P\).
• We do not plot the upper bound or spectral efficiency beyond the threshold \(P = 42.7\) dBm because the signal no longer remains inside the OA band, and hence the theory of \((\ref{eq:theory})\), \((\ref{eq:theory2})\) does not apply. In fact, substantial spectral broadening occurs at much smaller launch power, so this theory is more limited than suggested by Fig. 7.

B. OA Bandwidth Scales with the Launch Power

Although the models are impractical for large launch power, we can nevertheless follow \((\ref{eq:models})\), \((\ref{eq:models2})\) and study the capacities of the mathematical models for large \(P\). For example, suppose \(B\) scales as \(\sqrt{P}\), which is a lower bound on the spectral broadening rate, see \((\ref{eq:theory})\). We may use the bounds \((\ref{eq:bounds})\) and \((\ref{eq:bound2})\) to upper bound the receiver power, and we can apply the capacity bounds \((\ref{eq:upper_bound})\) and \((\ref{eq:upper_bound2})\).

Consider again the fiber parameters in Table I and the receiver bandwidth \(W = 500\) GHz. However, we now scale the OA bandwidth as \(B = W \max \left( 1, \sqrt{P} \right)\). Fig. 8 shows the normalized capacity bounds, which are similar to Fig. 7. The main change is that, at high power, both \(\bar{P}_r(W)\) and \(C(W)/W\) scale at most as \(P^{-1/4}\), as predicted by \((\ref{eq:theory})\) with \(\beta = 1/2\).

Observe also that the slopes of the upper bounds change earlier than the threshold \(P = 42.7\) dBm. This is an artifact of multiplying \(\sqrt{P}\) by \(W\) rather than some other number for large \(P\). For example, a natural choice based on \((\ref{eq:theory})\) and \((\ref{eq:theory2})\) is to choose \(B = W \max \left( 1, \sqrt{\kappa P/512} \right)\).

The reader might expect that the rates in Fig. 8 should not decrease with \(P\). However, note that the capacities are normalized, and that the figure is for a system where the OA bandwidth \(B\) changes with \(P\). In fact, we expect that the real (normalized) capacities at large \(P\) will be much smaller than the upper bounds shown in Fig. 7 or Fig. 8.

\[\text{In other words, as } P \text{ increases, } C(W)/W \text{ first grows as } \log(1 + P/(WN_0)), \text{ but is then upper bounded by } k_1 - \frac{1}{2} \log P \text{ for some constant } k_1, \text{ and finally is upper bounded by } k_2 P^{-1/4} \text{ for some constant } k_2.\]

Fig. 8. Normalized capacity bounds for dispersion-free fiber with \(W = 500\) GHz and where \(B\) scales as \(W \sqrt{P}\) for large \(P\). The vertical dashed line is the same one as in Fig. 7.

IX. Conclusions

Our main result is a closed-form expression for the autocorrelation function of the output signal given the input signal of dispersion-free fiber channels with distributed OA. The expression gives a bound on the output power of bandlimited and time-resolution limited receivers. The theory shows that there is a launch power beyond which the OA bandwidth \(B\) can no longer dominate the propagating signal bandwidth \(W\), and the model loses practical relevance. The growth of \(W\) due to signal-noise mixing that cannot be controlled by waveform design.

The receiver power bounds can be converted to capacity bounds. However, the latter bounds are far above the true capacity, and an interesting problem is to improve them. For example, one can improve the following steps.

• Treat the noise term in \((\ref{eq:NLSE})\) separately. An upper bound on the received noise power is \(K z \cdot \min(W/B, 1)\).
• Replace \((\ref{eq:bound})\) with a PSD more like the PSD of unit height in the frequency interval \([-W/2, W/2]\).
• For small \(\sqrt{\gamma K z^2}\), replace \((\ref{eq:bound})\) with a bound similar to \((2/3)xz\).
• For small \(\sqrt{\gamma K z^2}\), use \((\ref{eq:bound})\) rather than \((\ref{eq:bound})\), since \((\ref{eq:bound})\) does not have the factor 1/2 inside the exponential. We chose \((\ref{eq:bound})\) in order to treat large \(B\).
• Replace \((\ref{eq:bound})\) with tighter bounds.
• Use the SPM exponential in \((\ref{eq:NLSE})\).

Furthermore, one can improve the bounds for special choices of launch signals, e.g., bandlimited signals or PAM with rectangular pulses, cf. Sec. VI-B and Appendix E.

Although the bounds are loose, we suspect that they give reasonable guidance on the capacity behavior of NLSE-based fiber models. A challenging open problem is to develop the autocorrelation function for NLSE models with noise, nonlinearity, and dispersion. There is some hope for progress. For example, if the dispersion is due to an all-pass filter, then this filter does not change the PSD.
APPENDIX A
CAMERON-MARTIN THEORY
Consider the Cameron-Martin paper \[22\]. The space \( C \) is
the set of real-valued functions \( x(t) \) that are continuous on \([0,1] \)
and have \( x(0) = 0 \). The Wiener measure is defined as (see \[22\], p. 73)
\[
\frac{1}{(\pi^n t_1 (t_2 - t_1) \ldots (t_n - t_{n-1}))^{1/2}} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \exp \left( \frac{-s_1^2}{t_1} - \frac{(s_2-s_1)^2}{t_2-t_1} - \cdots - \frac{(s_n-s_{n-1})^2}{t_n-t_{n-1}} \right) ds_1 \cdots ds_n. \tag{114}
\]
The Wiener integral over the space \( C \) of a functional \( F[x] \) is
defined using this measure, and the integral is written as
\[
\int_C^W F[x] d_w x. \tag{115}
\]
Note that (115) corresponds to an expectation with respect to a
Wiener process that is multiplied by \(1/\sqrt{2} \).
Let \( p(t) \) be real-valued, continuous, and positive on \(0 \leq t \leq 1 \)
and consider a complex number \( \lambda \). Let \( \lambda_0 \) be the least
characteristic value of the differential equation
\[
f''(t) + \lambda p(t) f(t) = 0 \tag{116}
\]
subject to the boundary conditions \( f(0) = f'(1) = 0 \). Let \( f_\lambda(t) \) be
any non-trivial solution of (116) satisfying \( f_\lambda(1) = 0 \).
We have the following lemma by using Theorem 2 in \[22\] for
real \( \lambda \) (see especially (3.2) and (3.3) of \[22\]), and by using the
same argument as in \[23\] pp. 218-219] to extend to complex
\( \lambda \) and \( g(\cdot) \).

Lemma 7 (See \[22\] and \[23\]): If \( \Re(\lambda) < \lambda_0 \) then we have
\[
\int_C^W \exp \left( \lambda \int_0^1 \left[ p(t)x^2(t) + 2g(t)x(t) \right] dt \right) d_w x
\]
\[= \left( \frac{f_\lambda(1)}{f_\lambda(0)} \right)^{1/2} \exp \left( \lambda^2 \beta^2 \right) \tag{117}
\]
where
\[
\beta^2 = \int_0^1 \left[ \frac{1}{f_\lambda(t)} \int_1^t g(s)f_\lambda(s)ds \right]^2 dt. \tag{118}
\]
Example 5 (Example 1 in \[22\]): If \( p(t) = 1 \) for all \( t \),
then we have \( \lambda_0 = \pi^2/4 \). For \( \Re(\lambda) < \pi^2/4 \), we thus have
\[
f_\lambda(t) = \cos \left( \lambda^{1/2}(t-1) \right). \tag{119}
\]

APPENDIX B
MECOZZI’S IDENTITY
We prove a (slightly corrected) result from \[1\] eq. (18)].

Lemma 8 (See \[1\] eq. (18)):: Consider a standard real
Wiener process \( W(\cdot) \). For complex numbers \( a \) and \( b \),
and imaginary-valued \( c \), we have
\[
E \left[ \exp \left( aW(z) + bW(z')dz' - c \int_0^z W(z')^2dz' \right) \right]
= \sqrt{S(c)} \exp \left( \lambda^2 \beta^2 \right) \tag{120}
\]
where
\[
\lambda^2 \beta^2 = \left( \frac{a^2}{2} - \frac{b^2}{4c} \right) T(c) + \frac{ab}{2c} (1 - S(c) + \frac{b^2}{4c}). \tag{121}
\]

Proof: Consider the change of variables \( y = z'/z \), and write
(120) as
\[
E \left[ \exp \left( aW(z) + bW(z^2)dz - cz \int_0^1 W(y) dy \right) \right]
= E \left[ \exp \left( a\sqrt{2}z^{3/2} \int_0^1 W(y) dy \right) \right]
- 2e^{-z^2} \int_0^1 W(y) dy \right). \tag{122}
\]
Now consider the function
\[
g(s) = \frac{1}{2s} \left( a\sqrt{2}z^{3/2} \left[ \frac{1}{1 - s} - \frac{1}{s} \right] + b\sqrt{2}z^{3/2} \right) \tag{123}
\]
where \( \epsilon \) is a small positive number. The idea of including the
function \( 1(\cdot) \) is to avoid a Dirac-\( \delta \) function, and so that we
can write
\[
W(1) = \lim_{\epsilon \to 0} \int_0^1 \left[ \frac{1}{1 - s} - \frac{1}{s} \right] W(s) ds. \tag{124}
\]
We apply (117), (119) with \( \lambda = -2c \) and compute
\[
\lambda^2 \beta^2 = \int_0^1 \left[ - \frac{a^2}{2} \cos \left( \frac{1}{\lambda^{1/2} \epsilon} \right) \sin \left( \frac{1}{\lambda^{1/2} \epsilon} \right) \right] \lambda^{1/2} \epsilon \tag{125}
\]
For vanishing \( \epsilon \), the sine ratio becomes \( 1 \), and we have
\[
\lambda^2 \beta^2 = \left[ - \frac{a^2}{2} \tan \left( \frac{\lambda^{1/2}}{2} \right) \right] \lambda^{1/2} \epsilon^{-1/2}
+ \frac{b^2z^3}{2\lambda^{3/2}} \tan \left( \frac{\lambda^{1/2} \epsilon^{-1/2}}{2} \right) \tag{126}
\]
We obtain (120) and (121) by using
\[
\tan(\lambda x) = \lambda \tan x, \cos(\lambda x) = \cosh x. \]

Example 6: If \( c \to 0 \) then we have
\[
\lambda^2 \beta^2 \to \frac{z}{2} \left( a^2 + abz + \frac{b^2}{3}z^2 \right) \tag{127}
\]
which follows by using the Taylor series expansions
\[
\tan x \approx x - x^3/3, \sech x \approx 1 - x^2/2. \]
Alternatively, one can prove (127) without using (117) by
observing that, for \( c = 0 \), the term inside the exponential in
(120) is a zero-mean Gaussian random variable.

APPENDIX C
ONE-SAMPLE STATISTICS
We review the sample moments computed in \[1\] eq. (17)]. As
we consider only one time instant \( t \), we drop the time variables
for convenience of notation.
Recall that \( u_0 = u_{0R} + j u_{0I} \). Consider the conditional moments

\[
\mu_{m,n} = E\left[U^m (U^*)^n | U_0 = u_0\right]
\]

and the moment generating function

\[
M_{m,n}(s_1, s_2) = E \left[ \exp \left( s_1 [u_0 + \sqrt{K} W(z)] + s_2 [u_0 + \sqrt{K} W(z)]^* \right) + j \gamma (m - n) \int_0^z |u_0 + \sqrt{K} W(z')|^2 dz' \right]
\]

so that

\[
\mu_{m,n} = \frac{\partial^{m+n}}{\partial s_1^m \partial s_2^n} M_{m,n}(s_1, s_2) \mid_{s_1=s_2=0}.
\]

We compute

\[
M_{m,n}(s_1, s_2) = e^{d} \cdot E \left[ e^{Z} \right]
\]

where

\[
d = s_1 u_0 + s_2 u_0^* + j \gamma (m - n) |u_0|^2 z
\]

\[
Z = \sqrt{K} [s_1 W(z) + s_2 W(z)^*] + j \gamma (m - n) \sqrt{K} \int_0^z 2 \Re \{u_0 W(z')^*\} dz' + j \gamma (m - n) K \int_0^z |W(z')|^2 dz'.
\]

Recall that \( W(z) = (W_R(z) + j W_I(z))/\sqrt{2} \), where \( W_R(\cdot) \) and \( W_I(\cdot) \) are independent, standard, real, Wiener processes of unit variance. We may thus simplify \( E \left[ e^{Z} \right] = E \left[ e^{A+B} \right] = E \left[ e^{A} \right] E \left[ e^{B} \right] \) where

\[
A = \sqrt{\frac{K}{2}} (s_1 + s_2) W_R(z)
+ j \gamma (m - n) \sqrt{2 K} u_{0R} \int_0^z W_R(z') dz'
+ j \gamma (m - n) \frac{K}{2} \int_0^z W_R(z')^2 dz'.
\]

\[
B = j \sqrt{\frac{K}{2}} (s_1 - s_2) W_I(z)
+ j \gamma (m - n) \sqrt{2 K} u_{0I} \int_0^z W_I(z') dz'
+ j \gamma (m - n) \frac{K}{2} \int_0^z W_I(z')^2 dz'.
\]

Now define the values

\[
a_1 = \sqrt{\frac{K}{2}} (s_1 + s_2), \quad a_2 = j \sqrt{\frac{K}{2}} (s_1 - s_2),
\]

\[
b_1 = j \gamma (m - n) \sqrt{2 K} u_{0R}, \quad b_2 = j \gamma (m - n) \sqrt{2 K} u_{0I},
\]

\[
c_1 = c_2 = c = -j \gamma (m - n) K/2
\]

so that

\[
\frac{a_1^2 + a_2^2}{2} = K s_1 s_2,
\]

\[
\frac{b_1^2 + b_2^2}{4c} = j \gamma (n - m) |u_0|^2
\]

\[
\frac{a_1 b_1 + a_2 b_2}{2c} = -(s_1 u_0 + s_2 u_0^*)..
\]

We have the following expression using (120):

\[
E \left[ e^{A} \right] E \left[ e^{B} \right] = S(c) \exp \left[ \left( K s_1 s_2 + j \gamma (m - n) |u_0|^2 \right) T(c) \right]
- (s_1 u_0 + s_2 u_0^*) (1 - S(c))
- j \gamma (m - n) |u_0|^2 z'.
\]

This gives a result corresponding to [11] eq. (19):

\[
M_{m,n}(s_1, s_2) = S(c) \exp \left[ (s_1 u_0 + s_2 u_0^*) S(c) \right]
+ (K s_1 s_2 + j \gamma (m - n) |u_0|^2) T(c)
\]

where \( c \) is given in (136). For example, for \( m = n \) we have \( c = 0 \) from (136), and therefore \( S(c) = 1 \) and \( T(c) = z \). If we further have \( m = n = 1 \), then (130) gives

\[
\mu_{1,1} = \frac{\partial}{\partial s_1 \partial s_2} M_{1,1}(s_1, s_2) \mid_{s_1=s_2=0}
= K z + |u_0|^2
\]

as expected from (57).

A. First Moment

The \( m \)th moment is

\[
E \left[ U^m \right] = \mu_{m,0} = u_0^m E_m(c) S(c)^{m+1}.
\]

where \( c = -j \gamma m K/2 \) and

\[
E_m(c) = \exp \left( j \gamma m |u_0|^2 T(c) \right).
\]

For example, the first moment has \( c = -j \gamma K/2 \) and

\[
E \left[ U \right] = u_0 E_1(c) S(c)^2.
\]

Now consider small \( \sqrt{K z^2} \) for which (see (60)–(61))

\[
T(c) \approx z + j \gamma K z^3/3
\]

\( S(c)^2 \approx 1. \)

We thus have

\[
E \left[ U \right] \approx u_0 \exp \left( \left( |u_0|^2 \right) \right)
= u_0 \exp \left( \left( |u_0|^2 \right) \right)
= u_0 \exp \left( \left( |u_0|^2 \right) \right)
\]

where we have used \( \gamma = 2 \gamma K z^3/3 \) as in (63). The first moment thus experiences a power reduction of

\[
f(z) = |E_1(c) S(c)^2|^2 \approx \exp \left( -\gamma |u_0|^2 \right).
\]

This matches Meccozzi’s equations (29) and (30) from [11].

Remark 23: The value of the first moment may seem curious from the following perspective. The first moment is

\[
E \left[ \left[ u_0 + \sqrt{K} W(z) \right] \exp \left( j \gamma \int_0^z |u_0 + \sqrt{K} W(z')|^2 dz' \right) \right]
\]

where \( n = 1 \). One might guess that (147) simplifies to

\[
u_0 M_{1,0}(0,0) = u_0 E \left[ \left( j \gamma \int_0^z |u_0 + \sqrt{K} W(z')|^2 dz' \right) \right]
\]
Since the term with $\sqrt{K}W(z)$ seems to evaluate to zero. However, the first moment would then be

$$u_0 M_{1,0}(0,0) = u_0 E_1(c)S(c).$$

Note that the $S(c)$ term is not squared. In fact, we have

$$E \left[ \sqrt{K}W(z) \exp \left( j\gamma \int_0^z |u_0 + \sqrt{K}W(z')|^2 dz' \right) \right] = u_0 |S(c)| E_1(c)S(c)$$

which gives the desired result.

**APPENDIX D**

**TWO-SAMPLE STATISTICS**

We write $u_0 = u_0(t)$, $u_0' = u_0(t')$, and similarly for $u(t) = u_z(t)$. Consider the conditional moments

$$\mu_{mnkt} = E \left[ U^m(U^*)^n(U'T^*)^t \bigg| U_0(\cdot) = u_0(\cdot) \right]$$

and the moment generating function

$$M_{mnkt}(s) = E \left[ \exp \left( s_1[u_0 + \sqrt{K}W(z,t)] + s_2[u_0 + \sqrt{K}W(z,t)]^* + s_3[u_0' + \sqrt{K}W(z,t')] + s_4[u_0' + \sqrt{K}W(z,t')]^* + j\gamma \int_0^z (m-n) |u_0 + \sqrt{K}W(z,t)|^2 dz' \right) \right]$$

where $s = [s_1, s_2, s_3, s_4]$ so that

$$\mu_{mnkt} = \frac{\partial^{m+n+k+l} M_{mnkt}(s)}{\partial s_1 \partial s_2 \partial s_3 \partial s_4} \bigg|_{s=0}$$

**A. Autocorrelation Function**

For the autocorrelation function, we choose $mnkt = 1001$. For simplicity of notation, we replace $s_4$ with $s_2$ and write

$$M_{1001}(s_1, s_2) = e^d \cdot E \left[ e^{2z} \right]$$

where

$$d = s_1 u_0 + s_2 u_0'^* + j\gamma z \left[ |u_0|^2 - |u_0'|^2 \right]$$

$$Z = \sqrt{K} \left[ s_1 W(z,t) + s_2 W(z,t)^* \right]$$

$$+ j\gamma \sqrt{K} \int_0^z \left[ 2\Re \left\{ |u_0 W(z',t)^* - u_0' W(z',t')^* \right\} dz' \right]$$

$$+ j\gamma K \int_0^z |W(z',t)|^2 - |W(z',t')|^2 dz'.$$

**B. Noise**

Observe that $W(\cdot,t)$ and $W(\cdot,t')$ are correlated complex Wiener processes. Since $W(\cdot,t)$ and $W(\cdot,t')$ are circularly symmetric, we may write

$$W(z,t) = \rho^* W(z,t) + \sqrt{1 - |\rho|^2} W(z,t')$$

where the correlation coefficient is

$$\rho = E \left[ W(z,t)W(z,t)^* \right]$$

and $\tilde{W}(z,t') = (\tilde{W}(z,t') + j\tilde{W}_1(z,t'))/\sqrt{2}$ where $\tilde{W}(\cdot,t')$ and $\tilde{W}_1(\cdot,t')$ are independent, standard, real, Wiener processes that are jointly independent of $W(\cdot,t)$. Using $\rho = \rho_R + j\rho_I$, we have

$$W(z,t') = (W_R(z,t') + jW_I(z,t'))/\sqrt{2}$$

$$W_R(z,t') = \rho_R W_R(z,t) + \rho_I W_I(z,t) + \sqrt{1 - |\rho|^2} \tilde{W}_1(z,t'),$$

$$W_I(z,t') = \rho_R W_I(z,t) - \rho_I W_R(z,t) + \sqrt{1 - |\rho|^2} \tilde{W}_1(z,t').$$

(158)

We thus have $E[W_R(z,t')W_I(z,t')] = 0$ as required.

We are particularly interested in real $\rho$, see (30). In this case, we have

$$W_R(z,t') = \rho R W_R(z,t) + \sqrt{1 - \rho^2} \tilde{W}_1(z,t')$$

$$W_I(z,t') = \rho I W_R(z,t) + \sqrt{1 - \rho^2} \tilde{W}_1(z,t').$$

(159)

Thus, the real and imaginary processes are independent, i.e., $W_R(z,\cdot)$ is independent of $W_I(z,\cdot)$.

**C. Analysis of Z**

Inserting (159) into (155), we have

$$Z = \sqrt{K} \left[ s_1 W(z,t) + s_2 \rho W(z,t)^* + s_2 \sqrt{1 - \rho^2} \tilde{W}_1(z,t') \right]$$

$$+ j\gamma \sqrt{K} \int_0^z \left[ 2\Re \left\{ |u_0(t) - u_0(t')| \rho W(z',t)^* - |u_0(t')| \rho W(z',t)^* \right\} dz' \right]$$

$$+ j\gamma K \int_0^z \left( 1 - \rho^2 \right) \left( |W(z',t)|^2 - |W(z',t')|^2 \right) \right]$$

$$- 2\Re \left\{ \rho \sqrt{1 - \rho^2} W(z,t) \tilde{W}_1(z,t') \right\} dz'.$$

(160)

The quadratic form in the last integral is $W^T Q W$, where

$$W = [W'(z,t) \ W'(z,t')]^T$$

$$Q = \begin{bmatrix} 1 - \rho^2 & -\rho \sqrt{1 - \rho^2} \\ -\rho \sqrt{1 - \rho^2} & -(1 - \rho^2) \end{bmatrix}.$$ (162)

The eigenvalue decomposition is $Q = SAS^T$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \sqrt{1 - \rho^2} & 0 \\ 0 & -\sqrt{1 - \rho^2} \end{bmatrix}$$

$$S = [e_1 \ e_2] = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

with

$$a = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \rho^2}}$$

$$b = \frac{1}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \rho^2}} \cdot \text{sgn}(\rho).$$

(164)

Note that $ST S = I$, $a^2 + b^2 = 1$, $a^2 - b^2 = \sqrt{1 - \rho^2}$, and $ab = \rho / 2$. The quadratic form of interest is $W^T (SAS^T) W$. We thus define

$$V = S^T W$$

where $V = [V_1(z') V_2(z')]^T$ with

$$V_1(z') = (V_1 R(z') + jV_1 I(z'))/\sqrt{2}$$

$$V_2(z') = (V_2 R(z') + jV_2 I(z'))/\sqrt{2}. $$

(167)
Since the columns of $S$ are orthonormal, the random processes $V_{1R}(\cdot)$, $V_{1I}(\cdot)$, $V_{2R}(\cdot)$, and $V_{2I}(\cdot)$ are jointly independent standard Wiener. We further have

$$W = SV = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} V \quad (168)$$

$$W^\dagger Q W = V^\dagger \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V. \quad (169)$$

We expand $Z$ as

$$Z = (a_{1R}V_{1R} + a_{1I}V_{1I}) + (a_{2R}V_{2R} + a_{2I}V_{2I})$$

$$+ \int_0^z (b_{1R}V_{1R} + b_{1I}V_{1I}) + (b_{2R}V_{2R} + b_{2I}V_{2I}) \, dz'$$

$$- \int_0^z (c_{1R}V_{1R}^2 + c_{1I}V_{1I}^2) + (c_{2R}V_{2R}^2 + c_{2I}V_{2I}^2) \, dz' \quad (170)$$

where

$$a_{1R} = \sqrt{\frac{K}{2}} (a(s_1 + s_2 \rho) - b s_2 \sqrt{1 - \rho^2})$$

$$a_{1I} = j \sqrt{\frac{K}{2}} (a(s_1 - s_2 \rho) + b s_2 \sqrt{1 - \rho^2})$$

$$a_{2R} = \sqrt{\frac{K}{2}} (b(s_1 + s_2 \rho) + a s_2 \sqrt{1 - \rho^2})$$

$$a_{2I} = j \sqrt{\frac{K}{2}} (b(s_1 - s_2 \rho) - a s_2 \sqrt{1 - \rho^2})$$

$$b_{1R} = j \gamma \sqrt{2K} (a(u_{0R} - \rho u'_0) + b u'_0 \sqrt{1 - \rho^2})$$

$$b_{1I} = j \gamma \sqrt{2K} (a(u_{0I} - \rho u'_0) + b u'_0 \sqrt{1 - \rho^2})$$

$$b_{2R} = j \gamma \sqrt{2K} (b(u_{0R} - \rho u'_0) - a u'_0 \sqrt{1 - \rho^2})$$

$$b_{2I} = j \gamma \sqrt{2K} (b(u_{0I} - \rho u'_0) - a u'_0 \sqrt{1 - \rho^2})$$

$$c = c_{1R} = c_{1I} = -c_{2R} = -c_{2I} = -j \gamma \sqrt{\frac{K}{2}} \sqrt{1 - \rho^2}$$

and where

$$u_{0R} = u_{0R}(t), \quad u_{0I} = u_{0I}(t), \quad u'_0 = u_{0R}(t)'.$$

$$u'_0 = u_{0I}(t).$$

D. Moment Generating Function

We use (17), (18), the identities

$$\text{sech} \left( \sqrt{-jx} \right) = \text{sech} \left( \sqrt{jx} \right)$$

$$\tanh \left( \sqrt{-jx} \right) / \sqrt{-jx} = \left( \tanh \left( \sqrt{jx} \right) / \sqrt{jx} \right)^*$$

for real $x$, and Lemma [8] to calculate

$$M_{1001}(s_1, s_2) = e^{st} |S(c)|^2$$

$$\cdot \exp \left[ \frac{a_{1R}^2 + a_{1I}^2}{2} - \frac{b_{1R}^2 + b_{1I}^2}{4c} \right] T(c) +$$

$$\frac{a_{1R} b_{1R} + a_{1I} b_{1I}}{2c} \left( 1 - S(c) \right) +$$

$$\frac{b_{1R}^2 + b_{1I}^2}{4c} \right]. \quad (171)$$

We rewrite (171) as

$$M_{1001}(s_1, s_2) = e^{st} |S(c)|^2 \exp \left[ T(c) \right.$$

$$\cdot \left( \frac{a_{1R}^2 + a_{1I}^2 + a_{2R}^2 + a_{2I}^2}{2} - \frac{b_{1R}^2 + b_{1I}^2 + b_{2R}^2 + b_{2I}^2}{4c} \right.$$

$$+ j T_I(c) \right.$$

$$\cdot \left( \frac{a_{1R}^2 + a_{1I}^2 - a_{2R}^2 - a_{2I}^2}{2} - \frac{b_{1R}^2 + b_{1I}^2 + b_{2R}^2 + b_{2I}^2}{4c} \right.$$

$$+ (1 - S_R(c)) \frac{a_{1R} b_{1R} + a_{1I} b_{1I} - a_{2R} b_{2R} - a_{2I} b_{2I}}{2c}$$

$$- j S_I(c) \frac{a_{1R} b_{1R} + a_{1I} b_{1I} + a_{2R} b_{2R} + a_{2I} b_{2I}}{2c}$$

$$+ \frac{b_{1R}^2 + b_{1I}^2 + b_{2R}^2 + b_{2I}^2}{4c} \right]. \quad (172)$$

The sums in this expression are

$$\frac{a_{1R}^2 + a_{1I}^2 + a_{2R}^2 + a_{2I}^2}{2} = K s_1 s_2 \rho$$

$$\frac{a_{1R}^2 + a_{1I}^2 - a_{2R}^2 - a_{2I}^2}{2} = 0$$

$$\frac{b_{1R}^2 + b_{1I}^2 + b_{2R}^2 + b_{2I}^2}{4c}$$

$$= \frac{-j \gamma \sqrt{2K} \sqrt{1 - \rho^2}}{4c} \left[ |u_0|^2 + |u'_0|^2 - 2 \rho \Re \{u_0 u'_0 \} \right]$$

$$\frac{b_{1R}^2 + b_{1I}^2 - b_{2R}^2 - b_{2I}^2}{2c} = -j \gamma (|u_0|^2 - |u'_0|^2)$$

$$\frac{a_{1R} b_{1R} + a_{1I} b_{1I} + a_{2R} b_{2R} + a_{2I} b_{2I}}{2c}$$

$$= \frac{1}{\sqrt{1 - \rho^2}} \left[ s_1 (\rho u'_0 - u_0) + s_2 (u'_0 - \rho u_0) \right]$$

$$\frac{a_{1R} b_{1R} + a_{1I} b_{1I} - a_{2R} b_{2R} - a_{2I} b_{2I}}{2c}$$

$$= -s_1 u_0 - s_2 u'_0.$$

We thus have

$$M_{1001}(s_1, s_2) = |S(c)|^2$$

$$\cdot \exp \left[ T(c) \cdot \left( K s_1 s_2 \rho + j \gamma (|u_0|^2 - |u'_0|^2) \right) \right.$$\n
$$- j \gamma \frac{T_I(c)}{\sqrt{1 - \rho^2}} \left[ |u_0|^2 + |u'_0|^2 - 2 \rho \Re \{u_0 u'_0 \} \right]$$

$$+ S_R(c) \{ s_1 u_0 + s_2 u'_0 \}$$

$$- j \frac{S_I(c)}{\sqrt{1 - \rho^2}} \left[ s_1 (\rho u'_0 - u_0) + s_2 (u'_0 - \rho u_0) \right] \right]. \quad (173)$$

Taking derivatives and setting $s_1 = s_2 = 0$ we obtain the autocorrelation function [33].

APPENDIX E

INFINITE BANDWIDTH AND FINITE POWER NOISE

Consider large $B$ but fixed $K$ as in Remark [4] i.e., we have a noise PDD of $K \, W/m$ that is independent of $B$. Such infinite bandwidth noise is relatively easy to treat because, conditioned on the input, any two samples $U(t)$ and $U(t')$ with $t' \neq t$ are statistically independent.
A. Autocorrelation and PSD

We have \( \rho = 1(t = t') \) and use (34) to compute
\[
A(t, t') = \begin{cases} \frac{K_x + |u_0(t)|^2}{2}, & t = t' \\ \frac{v(t) v(t')^*}{2}, & t \neq t' \end{cases}
\]
where (cf. (141) and (142))
\[v(t) = u_0(t) E_1(c, t) S(c)^2\] (175)
\[E_1(c, t) = \exp \left( j \gamma |u_0(t)|^2 T(c) \right)\] (176)
and \( c = -j \gamma K/2 \). The PSD is therefore
\[
\bar{P}(f, T) = E \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{T}} V(t) e^{-j 2\pi ft} dt \right]^2
\]
(177)
where \( V(\cdot) \) is the random signal with realization \( v(\cdot) \), and where the expectation is over the random launch signal \( U_0(\cdot) \).

B. A Discrete-Time Model

We develop a discrete-time model. Let \( \{\phi_m(\cdot)\}_{m=1}^{\infty} \) be a complete orthonormal basis for \( L^2[0, T] \). Consider the projection output
\[
Y_m = \int_0^T U(t) \phi_m(t)^* dt
\]
(178)
and collect these values in the sequence \( Y = \{Y_m\}_{m=1}^{\infty} \) with energy \( ||Y||^2 = \sum_{m=1}^{\infty} |Y_m|^2 \). We can use (149) and the same steps as in [18], [19] (see also [17], Sec. IV.C-D) to show that if \( U_0(\cdot) = u_0(\cdot) \), then we have
\[
Y_m = \int_0^T E \left[ \left| U(t) \right| \right] \phi_m(t)^* dt = \int_0^T E_1(c, t) S(c)^2 u_0(t) \phi_m(t)^* dt
\]
(179)
where \( E_1(c, t) \) is given by (16). In other words, the channel effectively modulates \( u_0(t) \) by the factor \( E_1(c, t) S(c)^2 \). This means there is no phase noise since \( E_1(c, t) \) is a function of \( u_0(t) \). Instead, the signal loses energy since \( |E_1(c, t) S(c)|^2 < 1 \) if \( |u_0(t)| > 0 \), \( \gamma > 0 \), and \( K > 0 \).

The result (179) suggests that we study the complex-alphabet and continuous-time model
\[
Y(t) = E_1(c, t) S(c)^2 u_0(t) + N_E(t)
\]
(180)
where \( N_E(\cdot) \) is an AWGN process with a one-sided PSD of \( N_0 \) W/Hz. The model (180) has several interesting features. First, an optimal receiver\(^{10} \) may use matched filtering for signals of the form \( u_0(t) E_1(c, t) \). Moreover, suppose \( \sqrt{\gamma K} z^2 \) is small so that we have (see (176) and (143))
\[
E_1(c, t) S(c)^2 \approx \exp \left( j \gamma z - \kappa/2 \right) |u_0(t)|^2
\]
(181)
where \( \kappa = 2 \gamma K z^2 / 3 \) as in (63). We see that the receiver modulates the phase and amplitude of the received signal as a function of \( |u_0(t)|^2 \). In particular, if we use PAM with rectangular pulses that are time-limited to \( [0, T_s] \), then the standard matched filter is optimal but the receiver a-posteriori probability calculation should account for the channel’s symbol-dependent attenuation and phase shift.

C. Capacity Bounds

Consider the channel (180) with an amplitude constraint \( |u_0(t)| \leq A_{\text{max}} = \sqrt{PT_s} \). For rectangular pulses, we have
\[
C = \max_{X:|X|^2 \leq P} \frac{1}{T_s} \left[ h(Y) - \log_2 (\pi e N_0) \right]
\]
\[
\leq \frac{1}{T_s} \log_2 \left( 1 + \frac{\max_{X:|X|^2 \leq P} E \left[ T_s |X|^2 e^{-\kappa |X|^2} \right]}{N_0} \right)
\]
\[
= \left\{ \begin{array}{ll} \frac{1}{T_s} \log_2 \left( 1 + \frac{P}{N_0} \right), & \text{if } P < 1/\kappa \\
\frac{1}{T_s} \log_2 \left( 1 + \frac{T_s}{\kappa e N_0} \right), & \text{else} \end{array} \right.
\]
(182)
The smallest \( P \) that achieves the maximum upper bound is \( P = 1/\kappa \). In fact, the bound on the RHS of (182) can be approached if \( A_{\text{max}} \to 0 \), cf. [24]. Furthermore, for fixed \( P \), we can maximize the RHS of (182) over \( T_s \) to obtain \( T_s \to 0 \) and therefore
\[
\lim_{T_s \to 0} C \leq \frac{1}{\kappa e N_0} \log_2(e) \text{ bits/s.} \]
(183)
The optimal signaling thus uses very fast pulses, and the capacity \( C \) decreases inversely proportional to \( \gamma^2, K \), and \( z^3 \).

APPENDIX F

NONLINEARITY CAN INCREASE CAPACITY

We show that nonlinearity can increase capacity even with receiver noise. In the absence of OA noise, the model (41) is defined by (34) is
\[
u_E(t) = u_0(t)e^{\gamma |u_0(t)|^2 + N_E(t)}
\]
(184)
Suppose the transmitter uses PAM with square-root pulses
\[
g(t) = \left\{ \begin{array}{ll} \sqrt{\frac{t - T_s + 1}{T_s / (1 - T_s/2)}}, & t \in [0, T_s) \\
0, & \text{else} \end{array} \right.
\]
(185)
where \( T_s \leq 1 \), see (5). For \( \gamma = 0 \) the capacity is
\[
C = \frac{1}{T_s} \log_2 \left( 1 + \frac{PT_s}{N_0} \right) \text{ bits/s.} \]
(186)
This capacity can be achieved by scaling and shaping quadrature amplitude modulation (QAM) symbols \( x_k = x_{R,k} + jx_{I,k} \) where the \( x_{R,k} \) and \( x_{I,k} \) take on values in \( \{ \pm 1, \pm 3, \ldots \} \). Note that the capacity scales as \( \log_2 P \) for large \( P \).

Suppose now that \( \gamma > 0 \). The main observation is that the nonlinearity in (183) converts the pulse (185) to a tone whose frequency and power is proportional to \( |x_k|^2 \). More precisely, the noise-free output signals have the form
\[
u(t) = x_k \sqrt{\frac{t - T_s + 1}{T_s (1 - T_s/2)}} \exp \left( j 2\pi h |x_k|^2 (t - T_s + 1) \right)
\]
(187)
for \( t \in [kT_s, (k+1)T_s) \), where
\[
h = \frac{\gamma}{2\pi T_s (1 - T_s/2)}
\]
(188)
A modulation index \([12], \text{p. 118}\). Suppose we use intensity modulation where we choose the \( M \) symbols
\[
x_k \in \{(2i - 1)\Delta : i = M + 1, M + 2, \ldots, 2M\}
\]
each with probability 1/M. The average energy is then $E = (28M^2 - 1)\Delta^2/3$. We further choose $\gamma$ so that $ht_s$ is a positive integer, e.g., $\gamma = 2\pi(1 - T_s/2)$ so that $ht_s = 1$. The pulses $I^{\gamma_{ii}}_1$ are then mutually orthogonal: for $x_i \neq x_m$ we have

$$\int_0^{T_s} t - T_s + 1 \left( t - T_s / 2 \right) e^{i2\pi h(|x_i|^2 - |x_m|^2)(t - T_s / 2)} dt = 0. \quad (189)$$

The channel has thus converted the ASK signals to orthogonal FSK signals for which the frequency grows with the power.

Next, a standard upper bound on the error probability of signal sets is the union bound [12 p. 185]

$$P_e \leq (M - 1) Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right) \quad (190)$$

where $d_{\text{min}}$ is the minimum Euclidean distance between different pulses. Since our FSK signals are mutually orthogonal, the minimum distance corresponds to the signals with $i = M + 1$ and $i = M + 2$, i.e., we have

$$d_{\text{min}} = \sqrt{(2M + 1)^2 + (2M + 3)^2} \cdot \Delta \geq \sqrt{8M} \Delta$$

and therefore

$$P_e \leq (M - 1) Q \left( \frac{2M\Delta}{\sqrt{2N_0}} \right). \quad (191)$$

We use $R = \log_2 M$ bits/symbol and $Q(x) \leq e^{-x^2/2}$ for positive $x$ to write

$$P_e < \exp \left( R \ln 2 - \frac{6}{28} \frac{E}{N_0} \right). \quad (192)$$

This bound shows that, for any choice of target error probability $P_e$, we can choose the rate as

$$\frac{R}{T_s} = \frac{6}{28} \frac{P}{N_0} \log_2 e + \frac{1}{T_s} \ln P_e \text{ bits/s.} \quad (193)$$

The capacity thus scales linearly with $P$ rather than logarithmically as for $\gamma = 0$.

**Remark 24:** The reason for the capacity gain is because the channel has spread the spectrum of the PAM signal. The gain is thus at the expense of using more frequency resources.

**Remark 25:** The above example shows that intensity modulation can achieve a capacity that grows linearly with $P$ for large $P$. The per-sample rate $\frac{1}{2} \log P$ from [2], [3] thus underestimates capacity even with AWGN at the receiver.

**Remark 26:** The channel is artificial because we have assumed the channel is lossless without amplification.

**APPENDIX G**

**PROOFS OF LEMMAS**

We repeat (85) here for convenience:

$$P_e(W, T, t) = 2 \left[ Kz + (\sqrt{P_t + \delta})^2 \right] \int_{T/2}^{T/2} |S(c)|^2 \exp \left( -\gamma T_f(c) \frac{P_t}{2} \sqrt{1 - \rho^2} \right) |b(t - t')| dt'. \quad (194)$$

**Proof of Lemma 2**

Consider $W \leq B$ and $\gamma(K/2)z^2 \leq 1$. We have $x \leq 1$ (see (54)) and the bound (22) gives

$$\gamma T_f(c) \geq \gamma \frac{z x}{3} = \left( \frac{k}{4} \right) \left( \sqrt{1 - \rho^2} \right). \quad (195)$$

We further use the crude bounds (233) in Appendix I to upper bound the exponential of (194) as

$$\exp \left( -\gamma T_f(c) \frac{P_t}{2} \sqrt{1 - \rho^2} \right) \leq \left\{ \begin{array}{ll} \exp \left( -\gamma \frac{Z}{8} P_t B^2 (t - t')^2 \right), & B |t - t'| < 1 \\
\exp \left( -\gamma \frac{Z}{9} P_t \right), & B |t - t'| > 1. \end{array} \right. \quad (196)$$

We now define $\sigma = \sqrt{(\gamma/8)P_t} B$, $y = \sigma (t - t')$ and use the time intervals

$$I_1 = \{ t' : |t - t'| \leq 1/B, |t'| \leq T/2 \} \quad (197)$$

$$I_2 = \{ t' : |t - t'| > 1/B, |t'| \leq T/2 \} \quad (198)$$

to bound (194) as

$$P_e(W, T, t) \leq 2 \left[ Kz + (\sqrt{P_t + \delta})^2 \right] \left[ \int_{I_1} 2W e^{-\sigma^2(t - t')^2} dt' + \int_{I_2} |S(c)|^2 e^{-\gamma \frac{Z}{8} P_t} B^2 (t - t') \right]$$

$$\leq 4 \left[ Kz + (\sqrt{P_t + \delta})^2 \right] \left[ \int_0^{\sigma B} 2W e^{-y^2} dy \right]$$

$$+ \int_0^\infty 5e^{-\gamma \frac{Z}{8} P_t} e^{-\gamma \frac{Z}{8} P_t} B |b(t)| d\tau. \quad (200)$$

Step (a) in (200) follows by using (196), $|b(t - t')| \leq 2W$, and $|S(c)|^2 \leq 1$; step (b) follows by using $\tau = t - t'$, inserting (21), and applying the second inequality in (233) to bound $x \leq \gamma(K/2)z^2$. Evaluating the integrals and using (48) gives

$$P_e(W, T, t) \leq 4 \left[ Kz + (\sqrt{P_t + \delta})^2 \right]$$

$$\left[ \frac{2W/B}{\sqrt{(\gamma/8)P_t}} \frac{\sqrt{\pi}}{2} \text{erf} \left( (\sqrt{\gamma/8})P_t \right) + 5e^{-\gamma \frac{Z}{8} P_t} e^{-\gamma \frac{Z}{8} P_t} \right]. \quad (201)$$

**Proof of Lemma 3**

Consider $W \leq B$ and $\gamma(K/2)z^2 > 1$. We now have the situation that $x > 1$ can occur, so that we need both bounds in (22) depending on the value of $\tau = t - t'$. We further need both bounds of (233) in Appendix I depending on whether $|\tau|$ is smaller or larger than $1/B$. This leads to four integration regions in general, as described below.

We begin with (233) to write

$$B |\tau| \leq \sqrt{1 - \sin^2(\pi B \tau)} \leq 2B |\tau| \quad (202)$$

for $0 \leq B |\tau| \leq 1$. Using (54), we thus have

$$\gamma(K/2)z^2 B |\tau| \leq x \leq \gamma K z^2 B |\tau|. \quad (203)$$
Defining
\[ \tau^* = \frac{1}{(\gamma K z^2 B)} \]  (204)
we have \( 2\tau^* \leq 1/B \) by hypothesis, and (203) gives
\[ |\tau| \leq \tau^* \Rightarrow x \leq 1 \]
\[ |\tau| \geq 2\tau^* \Rightarrow x \geq 1 . \]  (205)

We proceed to upper bound \( P_r(W, T, t) \) by splitting the integral (194) into four parts with \( |\tau| \leq \tau^* , \tau^* \leq |\tau| \leq 2\tau^* , \) \( 2\tau^* \leq |\tau| \leq 1/B , \) and \( |\tau| > 1/B . \) Using (22) and (233), we upper bound the signal-noise exponential as
\[
\exp \left( -\gamma T_1(c) \frac{P_t}{2} \sqrt{1 - \rho^2} \right) \leq \begin{cases} 
\exp \left( -(\kappa/8)P_t B^2 \tau^2 \right) , & |\tau| \leq \tau^* \\
\exp \left( -\frac{B}{18K} P_t \sqrt{\gamma} \right) , & 2\tau^* \leq |\tau| \leq 1/B \\
\exp \left( -\frac{\sqrt{2}}{20K} P_t \right) , & |\tau| > 1/B . 
\end{cases}
\]

For the regime \( \tau^* \leq |\tau| \leq 2\tau^* , \) we use upper bound the sum of the first and second terms on the RHS of (206).

Inserting (206) into (194) and following similar steps as in (197)-(200), we have
\[ P_r(W, T, t) \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] \left[ \int_0^{2\tau^*} \frac{2W}{\sigma} e^{-y^2} dy + \int_{\tau^*}^{1/B} 10W e^{-a\sqrt{\tau}} d\tau + \int_{1/B}^{\infty} 5e^{-\sqrt{\gamma K z^2 - \sqrt{\pi} a\rho^2}} P_t |b(\tau)| d\tau \right] \]
\[ \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] \left[ \frac{2W/B}{\sqrt{(\kappa/8)P_t}} \frac{\sqrt{\pi}}{2} \text{erf} \left( \sqrt{(\kappa/8)P_t} B \tau^* \right) + \frac{20W}{a} \left( \tau^* + \frac{1}{a} \right) e^{-a\sqrt{\tau^*}} + 5e^{-\sqrt{\gamma K z^2 - 20Kz}} P_t \right] \]
(207)

where for step (b) we have defined
\[ a = \sqrt{\frac{\gamma B}{18K}} P_t + \sqrt{2\gamma K z^2 B} . \]  (208)

Step (a) in (207) used the first inequality in (233) to bound \(|S(c)|^2\) for the second integral. For step (b), we applied
\[ \int_{\tau^*}^{1/B} e^{-a\sqrt{\tau}} d\tau = \int_{\tau^*}^{1/B} e^{-at} 2t dt \]
\[ \leq \frac{2}{a} \left( \sqrt{\tau^* + \frac{1}{a}} \right) e^{-a\sqrt{\tau^*}} . \]  (209)

Finally, we use
\[ a \geq \max \left( \sqrt{\frac{\gamma B}{18K}} P_t, \sqrt{2\gamma K z^2 B} \right) \]
\[ \sqrt{(\kappa/8)P_t} 2B \tau^* = \sqrt{\frac{P_t}{3Kz}} \]
(210)
(211)
to simplify (207) and obtain
\[ P_r(W, T, t) \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] \left[ \frac{2W/B}{\sqrt{(\kappa/8)P_t}} \frac{\sqrt{\pi}}{2} \text{erf} \left( \sqrt{\frac{P_t}{3Kz}} \right) + \frac{25W/B}{\gamma K z^2} e^{-\frac{\sqrt{\gamma K z^2 - \sqrt{20Kz}}}{\sqrt{\pi} a\rho^2}} P_t \right] . \]  (212)

Proof of Lemma 7

Consider the case \( W \geq B \) and \( \gamma(K/2)z^2 \leq 1 . \) We again use the bound (195) and the parameters (197) to write
\[ P_r(W, T, t) \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] \left[ \int_0^{\sigma/W} \frac{2W}{\sigma} e^{-y^2} dy + \int_{1/W}^{1/B} e^{-a\tau^2} |b(\tau)| d\tau + 5e^{-\sqrt{\gamma K z^2 - (\kappa/9) P_t}} \right] . \]
(213)

We now use \( |b(t)| \leq 2(W(\pi t^2)^{-1} \) to upper bound the second integral in (213) as follows:
\[ \int_{1/W}^{1/B} e^{-a\tau^2} \frac{2}{\pi^2 W \tau^2} d\tau \leq e^{-(\kappa/8)P_t(B/W)^2} W - B \]
(214)

Inserting into (213), we have
\[ P_r(W, T, t) \leq 4 \left[ Kz + (\sqrt{P_t} + \delta)^2 \right] \left[ \frac{2W/B}{\sqrt{(\kappa/8)P_t}} \frac{\sqrt{\pi}}{2} \text{erf} \left( \sqrt{(\kappa/8)P_t} B \right) + \frac{1}{4} \left( 1 - \frac{B}{W} \right) e^{-(\kappa/8)P_t(B/W)^2} + 5e^{-\sqrt{\gamma K z^2 - (\kappa/9) P_t}} \right] . \]
(215)

Proof of Lemma 5

Observe that the RHS of (201) includes the form
\[ f(P_t) = (a + b\sqrt{P_t} + cP_t) \frac{\sqrt{\pi}}{2} \text{erf}(sP_t) \]
(216)

where \( a = Kz + \delta^2 , b = 2\delta , c = 1 , \) and \( s = \kappa/8 . \) The results (238)-(239) derived in Appendix I state that (216) is non-decreasing and concave in \( P_t \) if \( P_t \geq 3(Kz + \delta^2) . \) Thus, if we replace \( P_t \) with \( P_t + P_o \), where the offset power is \( P_o = 3(Kz + \delta^2) \), then (216) is non-decreasing and concave for \( P_t \geq 0 . \)

Next, by using (235)-(236) in Appendix I we have
\[ 20 \left( Kz + (\sqrt{P_t} + \delta)^2 \right) e^{-\sqrt{\gamma K z^2 - (\kappa/9)P_t}} \leq c_1 \]
(217)

where
\[ c_1 = 20 \left[ Kz + \delta^2 + \sqrt{\frac{18}{\kappa e}} \delta + \frac{9}{\kappa e} \right] e^{-\sqrt{\gamma K z^2}} . \]
(218)

Observe that \( c_1 \) is independent of \( P_t \) and \( W \).

We now loosen (201) to
\[ P_r(W, T, t) \leq c_1 + 8(W/B) f(P_t + P_o) \]
(219)
where the RHS is non-decreasing and concave in \( P_t \). Jensen’s inequality applied to the RHS of (84) thus gives the bound
\[
\bar{P}_r(W, T) \leq c_1 + 8(W/B)f(P_T + P_o)
\]  
(220)
where we have replaced \( P_t \) with \( \bar{P}_r \), see \((10)\). Furthermore, we require \( \bar{P}_T \leq P \), so we have
\[
\bar{P}_r(W, T) \leq c_1 + 8(W/B)f(P + P_o).
\]  
(221)

We may simplify the bound further without changing the scaling behavior that we are interested in. We use \( \text{erf}(y) \leq 1 \) and loosen (221) to
\[
\bar{P}_r(W, T) \leq c_1 + 8W^2 K z + (\sqrt{P + P_o + \delta})^2 e^{-\gamma K z^2} \leq c_2
\]  
(222)
For example, the RHS of (222) scales as \( \sqrt{P} \) for large \( P \).

**Proof of Lemma 6**

We repeat the above steps (216)-(222) for (212). We now have \( s = 1/(3K z) \) in (216), and we use (235)-(236) to compute
\[
\frac{100}{\gamma K z^2} [K z + (\sqrt{P + \delta})^2] e^{-\gamma K z^2} = c_2
\]  
(223)
where
\[
c_2 = \frac{100}{\gamma K z^2} [K z + \delta^2 + \sqrt{6K z \delta} + \sqrt{18K z \delta}] e^{-\gamma K z^2}
\]  
(224)
\[
c_3 = 20 \left[ K z + \delta^2 + \frac{80K \delta}{\gamma e^2} \right] e^{-\gamma K z^2}.
\]  
(225)
Observe that \( c_2 \) and \( c_3 \) are independent of \( P_t \) and \( W \). We loosen (212) and use Jensen’s inequality to write
\[
\bar{P}_r(W, T) \leq \frac{W}{B} c_2 + c_3 + \frac{16(W/B)\gamma K z^2}{\gamma K z^2 - f} (P_T + P_o)
\]  
(226)
which is the analog of (220). The RHS of (225) is increasing in \( \bar{P}_T \), so we have
\[
\bar{P}_r(W, T) \leq \frac{W}{B} c_2 + c_3 + \frac{16(W/B)\gamma K z^2}{\gamma K z^2 - f} (P + P_o).
\]  
(227)

To study large \( P \), we may simplify (226) by again using \( \text{erf}(y) \leq 1 \). We arrive at a similar bound as (222), and \( \bar{P}_r(W, T) \) again scales at most as \( \sqrt{P} \) for large \( P \).

**APPENDIX H**

**ENERGY BOUND FOR PAM WITH RECTANGULAR PULSES AND RING MODULATION**

PAM with rectangular pulses and ring modulation has a constant envelope, so we can apply (59). Suppose \( T_r = T_s \), so that for \( mT_r \leq t, t' < (m + 1)T_r \), we have \( \phi(t, t') = 0 \) and
\[
\bar{A}(t, t') = A(t, t')
\]  
(228)
where
\[
\bar{P}_r(W, T) = c_1 + 8(W/B)f(P + P_o).
\]  
(229)

Note that (227) is real-valued. Using (52) rather than (49), we thus have (cf. (66))
\[
E_m(T_r) \leq \frac{1}{T_r^2} \left[ K z + P \left( 1 + \gamma^2 K^2 z^4 \right) \right]
\]  
(230)
\[
\int_0^{T_r} \int_0^{T_r} \exp \left( -\frac{T}{\gamma K z^2} \right) dt' dt.
\]  
(231)
Consider \( T_r = 1/B \) and \( \gamma (K/2) z^2 \leq 1 \). We use (195) and (233) to upper bound the exponential of (228) with
\[
\exp \left( -(\gamma/2) P B^2 (t - t')^2 \right)
\]  
(232)
for the range of interest with \( B|t - t'| \leq 1 \). We thus have
\[
E_m(T_r) \leq \frac{1}{T_r^2} \left[ K z + P \left( 1 + \gamma^2 K^2 z^4 \right) \right]
\]  
(233)
\[
\int_0^{T_r} \int_0^{T_r} \exp \left( -\frac{T}{\gamma K z^2} \right) dt' dt.
\]  
(234)
where \( \sigma = \sqrt{(\gamma/2) P B} \). Evaluating the integral gives
\[
E_m(T_r) \leq \left[ K z + P \left( 1 + \gamma^2 K^2 z^4 \right) \right] \frac{2T_r}{\sqrt{(\gamma/2) P}} \sqrt{\pi} \left[ \text{erf} \left( \sqrt{\gamma/2} P \right) - \frac{1}{\sqrt{(\gamma/2) P}} \Gamma \left( 1 - e^{-(\gamma/2) P} \right) \right].
\]  
(235)
As \( \gamma \to 0 \) we have \( \kappa \to 0 \), and we find that the RHS of (233) becomes \( (K z + P) T_r \), as expected for a linear channel.

**APPENDIX I**

**VARIOUS BOUNDS**

**Sinc Function**

We have the following bounds, see Fig. 9
\[
|\text{sinc}(y)| = \left| \frac{\sin(\pi y)}{\pi y} \right| \leq \begin{cases} 1 - y^2, & |y| \leq 1 \\ 1/4, & |y| > 1 \end{cases}
\]  
(236)
\[
|\text{sinc}(y)|^2 \leq \begin{cases} 1 - y^2, & |y| \leq 1 \\ 1/20, & |y| > 1 \end{cases}
\]  
(237)
\[
|\text{sinc}(y)|^2 \geq 1 - 4y^2.
\]  
(238)

**Exponential Function**

We use two bounds on the exponential function with \( a > 0 \):
\[
y e^{-ay} \leq \frac{1}{ae} \quad \text{with equality if } y = \frac{1}{a}
\]  
(239)
\[
y e^{-ay}^2 \leq \frac{1}{2ae} \quad \text{with equality if } y = \frac{1}{\sqrt{2a}}.
\]  
(240)
From (238), we see that $f$ is concave if $P \geq a/c$. Thus, $f(P + 3a/c)$ is both non-decreasing and concave if $P \geq 0$.

Fig. 9. Simple bounds on $|\text{sinc}(x)|$ and $\text{sinc}(x)^2$.

Concavity of a Special Function
Consider the function

$$f(P) = (a + b\sqrt{P} + cP) \sqrt{\frac{\pi}{sP}} \text{erf} \left( \frac{\sqrt{sP}}{2} \right)$$

(237)

where $a$, $b$, and $c$ are non-negative constants and $s$ is a positive constant. We compute

$$\frac{df}{dP} = \frac{e^{-sP}}{2sP^{3/2}} \left[ (cP + a)s\sqrt{P} + bsP + \sqrt{s}(cP - a)e^{sP}\sqrt{\frac{\pi}{2}} \text{erf} \left( \frac{\sqrt{sP}}{2} \right) \right]$$

(238)

$$\frac{d^2f}{dP^2} = -\frac{e^{-sP}}{4P^2\sqrt{sP}} \left[ 2s\sqrt{P}(cP + a) + 2bsP + b \right]$$

$$+(cP - 3a) \left( e^{sP}\sqrt{\frac{\pi}{2}} \text{erf} \left( \frac{\sqrt{sP}}{2} \right) - \text{sinc}(P) \right).$$

(239)

From (238), we see that $f(P)$ is non-decreasing if $P \geq a/c$. Similarly, for (239) we use [25, 8.253.1] to bound

$$e^{y^2} \cdot \frac{\sqrt{\pi}}{2} \text{erf}(y) \geq y$$

(240)

for $y \geq 0$ and find that $f(P)$ is concave if $P \geq 3a/c$. Thus, $f(P + 3a/c)$ is both non-decreasing and concave if $P \geq 0$.

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