Multi-level Coded Caching

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Abstract

Recent work has demonstrated that for content caching, joint design of storage and delivery can yield significant benefits over conventional caching approaches. This is based on storing content in the caches, so as to create coded-multicast opportunities even amongst users with different demands. Such a coded-caching scheme has been shown to be order-optimal for a caching system with single-level content, i.e., all content is uniformly popular. In this work, we consider a system with content divided into multiple levels, based on varying degrees of popularity. The main contribution of this work is the derivation of an information-theoretic outer bound for the multi-level setup, and the demonstration that under some natural regularity conditions, a memory-sharing scheme which operates each level in isolation according to a single-level coded caching scheme, is in fact order-optimal with respect to this outer bound.

I. INTRODUCTION

Content-distribution networks (CDNs) have enabled broadband delivery of content (driven by video streaming) in the wired Internet by mirroring content in several locations, in order to bring frequently requested content closer to where it is consumed; see [1], [2], [3], [4] and references therein. Cellular data traffic is also increasingly dominated by such broadband content access. Unfortunately, the CDN solution addresses the content placement aspect of the problem in isolation from the delivery scheme, offering the most gains when the local storage is large enough to store most of the popular content, and the local communication link is not a bottleneck. Therefore, in cellular networks, where the local communication link is rate-limited, there is a need to jointly design content placement with broadcast delivery.

The emerging wireless architecture (illustrated in Figure 1), consists of a dense deployment of wireless access points (APs) with small coverage and relatively large data rates, in combination with cellular base-stations (BS) with large coverage and smaller data rates. The consequence of this emerging architecture is that a user could potentially listen to the BS as well as access several wireless APs. We envisage equipping each AP with a cache, which can be used to store part of the content, without specific knowledge of future user requests. Our main goal is to design a content placement strategy at the APs, jointly with BS delivery and (multi-level) user access to the APs, so that all user requests can be satisfied while minimizing the required BS transmission rate.

Recently, [5], [6] proposed a coded caching scheme that jointly optimizes the content placement with broadcast delivery, and demonstrated a significant improvement over conventional schemes. This was enabled by content placement that created (network-coded) multicast opportunities among users across different APs, even if they have different requests. The setup that was studied consisted of single-level content, i.e., every file in the system is uniformly demanded. However, it is well understood that content demand is non-uniform in practice, with some files being more popular than others. In this paper we model this asymmetry with multi-level content, where we divide the entire content into discrete levels based on popularity. We then design a wireless content delivery scheme\(^1\) for such a multi-level content structure, based on a storage, access, and delivery

\(^1\)In [7], the approach of [5], [6] was extended to non-uniform content popularities; we compare our setup to theirs later in the section.

Fig. 1. Multi-level storage, access, and delivery architecture.
architecture that is motivated by the emerging heterogeneous wireless network architecture. We demonstrate the approximate optimality of our design with respect to information-theoretic bounds.

The core theoretical contribution of this work is to develop an information-theoretic outer bound for such a setup, and to demonstrate that a memory-sharing scheme is order-optimal with respect to this outer bound, under some natural regularity conditions. A striking aspect of this order-optimal solution is that even when AP memory is available, in some regimes, it is better to store some less popular content without completely storing the more popular content first.

Our work is closest in spirit to [7], [8]. The case of different file popularities was recently studied in [7], where a single user per AP requests a file from the content according to a certain probability distribution, and where the goal is to minimize the expected BS transmission rate. A memory-sharing scheme, similar to the one considered in our work, is considered, and its performance is compared to the information-theoretic optimal. Instead, we model the different file popularities as multiple levels of content and varying number of users per AP requesting files from each level. This deterministic setup allows us to optimize the splitting parameters for the memory-sharing scheme, and prove a stronger result demonstrating the order-optimality of the scheme with respect to information-theoretic lower bounds.

The caching architecture studied here is similar to the one proposed in [8] for heterogeneous wireless networks, with the small-cell or WiFi access points acting as helpers by storing part of the content. A content placement scheme was formulated and posed as a linear program. Our work differs from [8] in several aspects. We utilize the macro-cell base station broadcast to assist in content delivery, which helps improve the system performance significantly. Moreover, the performance of our scheme is compared against information-theoretic bounds, which do not have any restrictions on the structure of the placement and delivery schemes.

This paper is organized as follows. Section II discusses our problem formulation and provides some illustrative examples. We present our main result in Section III and provide a brief proof sketch in Section IV. The full proofs and details are left to the appendices.

II. PROBLEM SETUP AND FORMULATION

We illustrate our problem setup with a small example.

A. A small, illustrative example

![Diagram of caching architecture](image)

Fig. 2. Setup for the small example.

See Figure 2. Let the BS hold files of $L = 2$ different popularity levels: $N_1 = 2$ popular files: $W_1^1$ and $W_2^1$, and $N_2 \geq 4$ unpopular files: $W_1^2, W_2^2, \ldots, W_{N_2}^2$, with each file of size $F$ bits. There are $K = 2$ APs in the macro-cell, each equipped with a cache of size $MF$ bits whose contents are denoted by $Z_1$ and $Z_2$. There are $U = 3$ users in the system.

The system is operated in two phases. During the placement phase, each cache stores content related to the files without any prior knowledge of the user requests. Later, during the delivery phase, the users place their requests. Users 1 and 2 request popular files, say $W_1^1$ and $W_2^1$, respectively, and user 3 requests an unpopular file, say $W_3^2$. Further, users 1 and 2 access the

2The main result of this paper focuses on the case where each user connects to a single AP. For the setting where users can access multiple APs, we refer the interested reader to [9].
cache contents of AP 1 and 2 respectively, while user 3 is required to access both APs. In addition, all users can listen to a common BS broadcast $X^r$ of size $RF$ bits, where $r = (r_1, r_2, r_3)$ denotes the request vector. Using only the accessed cache contents and the BS transmission, each user attempts to recover their requested file.

The pair $(M, R)$ is said to be feasible if there exists a placement scheme with normalized cache size $M$, such that for every request vector $r$, there exists a BS broadcast transmission $X^r$ of rate no greater than $R$ such that all the requests are satisfied. For any value of the memory size $M$, denote the optimal BS transmission rate over all possible schemes by

$$R^*(M) = \inf \{ R : (M, R) \text{ is feasible} \},$$

where the minimization is over all feasible strategies.

**Theorem 1.** For the example presented above and in Figure 2, the optimal achievable rate $R^*(M)$, for every value of $M$, is given in Figure 3 as a function of $M$.

![Fig. 3. Optimal rate $R^*(M)$ for the example in Figure 2](image)

The full proof of Theorem 1 is given in Appendix F. For brevity, we will here consider the achievable scheme for the specific tuple $(M, R) = (1, \frac{3}{2})$. We split each of the popular files $W^1_1$ and $W^1_2$ into two halves: $W^1_1 = (W^1_{1a}, W^1_{1b})$ and $W^1_2 = (W^1_{2a}, W^1_{2b})$. Let the cache contents be $Z_1 = (W^1_{1a}, W^1_{2a})$ and $Z_2 = (W^1_{1b}, W^1_{2b})$, thus satisfying the memory size constraint. When the user requests are revealed, the BS sends a broadcast of two parts: $X^r = (X^r_1, X^r_2)$. The first part, $X^r_1 = W^1_{3a}$, satisfies the request of user 3 for an unpopular file. The second part, $X^r_2 = W^1_{1b} \oplus W^1_{2a}$, along with the accessed AP cache content, enables both users 1 and 2 to recover their requested files.

Additionally, we use an information-theoretic bound to prove the optimality of the above described scheme, the details of which are in Appendix F.

While, for this small example, we were able to derive the optimal scheme for every value of $M$, it is in general hard to extend this method to networks with a larger number of caches, users, or file classes. Hence, we will instead focus on identifying order-optimal schemes, for which the achievable rate $R(M)$ is within a constant multiplicative factor of the optimal rate $R^*(M)$, i.e., $R(M)/R^*(M) \leq c$, where $c$ is independent of the problem parameters. Furthermore, while our ideas extend to the multi-access case, henceforth we restrict our attention to the case where each user accesses a single AP during the delivery phase. This enables us to solely focus on the effect of multi-level content on the minimum achievable rate of the system.

In the next section, we illustrate our achievability scheme on a specific example with four levels. Later, our main result in Theorem 2 proves the order-optimality of the general scheme.

**B. Example: general achievability scheme**

Suppose the BS holds a total of $N = 62,100$ files of size $F$ bits each, divided into $L = 4$ levels with $N_1 = 900$, $N_2 = 2700$, $N_3 = 10,500$, and $N_4 = 48,000$ files in different levels. The four levels represent different file popularities, with level 1 consisting of the most popular files and level 4 including the least popular files. Suppose also that there are $K = 25$ APs, and $U_1 = 25$, $U_2 = 16$, $U_3 = 7$ and $U_4 = 1$ users accessing each AP, each requesting a file of level 1, 2, 3, and 4, respectively. Finally, suppose that each AP has a cache of size $MF = 3600F$ bits.

Given such a setup, we want to identify what should be stored in each AP cache and how the requested files should be delivered, so that the associated BS transmission rate is minimized. One natural strategy would be to allocate the entire memory of each cache to store as many popular files as possible. Since $M = N_1 + N_2$ in this example, this would mean storing all the files of levels 1 and 2 in each of the caches. As a result, the users requesting files from levels 1 or 2 can satisfy their demands by only accessing the corresponding APs. On the other hand, the $K(U_3 + U_4) = 200$ users requesting files of levels 3 and 4.

3We refer the reader to [9] for a discussion on the multi-access case.
have to rely solely on the BS broadcast. Since all these requests can be for different files, the peak BS transmission size is $RF = 200F$ bits.

Let us now consider an alternate scheme. We partition the set of file levels $\{1, 2, 3, 4\}$ into three subsets: $H$, $I$, and $J$. Each level $i$ is allotted a portion $\alpha_i M$ of each cache memory, $\alpha_i \in [0, 1]$, depending on the set to which it belongs. Levels in $H$ are assigned no cache memory ($\alpha_i M = 0$), and thus the corresponding requests are solely served by the BS transmission. Each level in the set $J$ is assigned enough memory so that every cache can hold all its constituent files, i.e., $\alpha_i M = N_i$. Finally, the remaining memory is shared amongst the levels in set $I$, so as to minimize the resulting BS transmission rate. The assignment of the levels to the subsets $H$, $I$, and $J$ is done such that the levels assigned to $J$ are more popular than the levels in $I$, which in turn dominate the levels in $H$, thus ensuring that more popular content is given a larger portion of the cache. The values of the memory-sharing parameters $\alpha_i$’s depend on the cache memory size $M$.

For our setup with $M = 3600$, let $H = \{4\}$, $I = \{3, 2\}$, $J = \{1\}$, and set $\alpha_1 = N_1/M = 0.25$, $\alpha_2 = 0.35$, $\alpha_3 = 0.4$, and $\alpha_4 = 0$. Next, we treat each level $i$ as a separate sub-system with $N_i$ files, $K$ APs, $U_i$ users per AP, and $\alpha_i M$ cache memory. Each of these sub-systems has single-level content and, as mentioned before, such a setup was studied in [5]. Using the coded caching scheme proposed in [5] for each of these sub-systems, the total required BS transmission rate is given by $\sum_{i=1}^{L} r_i \alpha_i M/N_i$, where $r_i(M/N, K, U)$ denotes the achievable rate in a single-level system with $N$ files, $K$ APs, each with memory size $M$, and $U$ users accessing every cache. Evaluating the rate expression for our choice of $\alpha_i$’s, we get a peak BS transmission size of approximately 76.4F, which is an improvement by a factor of about 2.62 over the first scheme. The scheme illustrated for the above example can be extended to the general setup by carefully choosing the subsets $(H, I, J)$ and the memory-sharing parameters $\alpha_i$’s. We briefly discuss this in Section IV-A and provide details in Appendix A.

Illustrated through examples above, our general problem statement is as follows. Consider a caching system with a BS holding $L$ levels of content, consisting of $N_1, \ldots, N_L$ files. There are $K$ APs, each equipped with a cache of normalized size $M$, and $U_1, \ldots, U_L$ users accessing each AP with requests for a file from level $1, \ldots, L$ respectively. Given such a setup, we want to characterize the minimum achievable BS transmission rate $R^*(M)$, for any cache memory size $M$.

## III. MAIN RESULTS

The main contribution of this paper is the proof of the order-optimality of the memory-sharing achievability scheme described above, under certain natural regularity conditions which we discuss below.

Without loss of generality, we label the content levels as $\{1, \ldots, L\}$ such that $i < j$ implies that the files in level $i$ are more popular than those in level $j$. Then, we make the following three assumptions regarding the levels, all of which are observed in practice.

- **a)** $N_i \leq N_j$ for $i < j$: This means that the number of files in a more popular level is at most that in a less popular level. This is found to be true in real content distribution systems. Figure 2 which shows the popularity profile of about 18,000 Netflix movies, illustrates this effect with only the first 600 files or so (about 3%) having high popularity, beyond which popularities decrease following a power law.

- **b)** $U_i/N_i \geq q \cdot U_j/N_j$ for $i < j$, $q \gg 1$: This condition implies that the average number of users requesting a particular file from a level is much higher for more popular levels as compared to less popular levels. Again, this is commonly observed since there are many more user requests for a small number of popular files.

- **c)** $N_i \geq KU_i$ for all $i$: This represents the most relevant case where the number of files $N_i$ in any level $i$ is greater than the number of users $KU_i$ simultaneously requesting them in the system.

We now present the main result of this paper.

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*Fig. 4.* Popularity profile of 17,770 Netflix movies. Popularities are based on the number of ratings.
Theorem 2. Let the regularity conditions above hold true and the number of APs $K$ be large enough. Then the BS transmission rate $R(M)$ achieved by our proposed memory-sharing scheme is within a constant multiplicative factor of the optimal rate $R^*(M)$, where the constant is independent of $K$, $\{N_i\}$, $\{U_i\}$ and $M$. In particular, there exists a constant $c > 1$ such that for all $M \in [0, N)$, we have

$$\frac{R(M)}{R^*(M)} \leq cL^3.$$  

While we have not made any attempt to optimize $c$ in our analysis, numerical results suggest that its value is quite small. For instance, in the example presented in Section II-B, the gap does not exceed 4.85 for all $M$. Moreover, we recognize that the gap depends on the number of levels $L$. However, based on empirical evidence, we strongly believe that, for typical popularity profiles, dividing the content into a small number of levels is sufficient for order-optimality. For example, Figure 5 plots the performance of the memory-sharing scheme for different choices of $L$, over the Netflix popularity profile illustrated in Figure 4. As is apparent, the achievable rate is more or less constant for $L \geq 4$. Formalizing this intuition will be part of our future work.

![Fig. 5. Performance of our scheme against the chosen number of levels $L$, for the popularity profile of 17,770 Netflix movies.](image)

Our models have so far assumed that the files are divided into a small, discrete set of popularity levels, and that the number of users accessing each level was fixed. In practical settings, files are not constrained to such a set, but rather follow a “continuous” popularity profile, which reflects, for each file, the probability that a user will request said file. Figure 4 provides such a profile for movies in the Netflix database. In these situations, we propose that the designer group the files into a small set of popularity levels (“discretizing” the popularity). Because the resulting number of levels is small, and because there is generally a large number of users, the law of large numbers will guarantee that each level will have a fixed number of users, with negligible variation. This justifies our assumption that the parameters $\{U_i\}$ are fixed and known in advance. Formalizing and strengthening the connection between the continuous and discrete popularity profiles will also be part of our future work.

The proof of Theorem 2 is discussed briefly in Section IV and is given in full in Appendix C.

IV. SKETCH OF PROOF

A. Achievable rate

As illustrated in Section II-B, our achievability scheme partitions the popularity levels into subsets $H$, $I$, and $J$, such that $H$ contains levels with no cache memory allocation, $J$ contains levels with the maximal cache memory allocation, and $I$ contains the rest of the levels that optimally share the remaining memory. We refine this partition by further splitting the set $I$ into three parts, namely $(I_0, I', I_1)$. Next, we describe how the levels are divided into the subsets $H$, $I_0$, $I'$, $I_1$, and $J$ for any given cache memory size $M$.

To choose an appropriate such partition, we first define, for any subset $A \subseteq \{1, \ldots, L\}$, the following quantities:

$$S_A = \sum_{i \in A} \sqrt{N_i U_i}; \quad T_A = \sum_{i \in A} N_i; \quad V_A = \sum_{i \in A} \frac{N_i}{K},$$

and then introduce the following definition.
Definition 1. For any memory \( M \), a partition \((H, I_0, I', I_1, J)\) of the set of levels \(\{1, \ldots, L\}\) is said to be a refined \(M\)-feasible partition if it satisfies the following inequalities:

\[
\begin{align*}
\forall h \in H & \quad \tilde{M} \leq \frac{1}{K}Q_h, \\
\forall i_0 \in I_0 & \quad \frac{1}{K}Q_{i_0} \leq \tilde{M} \leq \frac{2}{K}Q_{i_0}, \\
\forall i \in I' & \quad \frac{2}{K}Q_i \leq \tilde{M} \leq \beta Q_i, \\
\forall i_1 \in I_1 & \quad \beta Q_{i_1} \leq \tilde{M} \leq \frac{K+1}{K}Q_{i_1}, \\
\forall j \in J & \quad \frac{K+1}{K}Q_j \leq \tilde{M},
\end{align*}
\]

where \( Q_i = \sqrt{\frac{N_i}{U_i}} \) for each \( i \), \( \tilde{M} = (M - T_J + V_I)/S_T \), and \( \beta = 2/\sqrt{q} \) with \( q \) defined in regularity condition (b).

Proposition 1. Under this definition, we have, for \( h \in H \), \( i_0 \in I_0 \), \( i \in I' \), \( i_1 \in I_1 \), and \( j \in J \):

\[
\frac{N_h}{U_h} \geq \frac{N_{i_0}}{U_{i_0}} \geq \frac{N_i}{U_i} \geq \frac{N_{i_1}}{U_{i_1}} \geq \frac{N_j}{U_j},
\]

and therefore, by the regularity conditions:

\[ j < i_1 < i < i_0 < h. \]

Given \( M \), we choose a refined \( M \)-feasible partition \((H, I_0, I', I_1, J)\) and allocate cache memory to each level accordingly. As a result, we get the following individual rates corresponding to each level:

\[
\begin{align*}
\forall h \in H \cup I_0 & \quad R_h(M) \leq KU_h, \\
\forall i \in I' & \quad R_i(M) \leq 3S_I\sqrt{\frac{N_iU_i}{M - T_J + V_I}}, \\
\forall i_1 \in I_1 & \quad R_{i_1}(M) \leq \frac{4}{\beta U_i} \left( 1 - \frac{M - T_J + V_I}{K+1} Q_{i_1} S_T \right), \\
\forall j \in J & \quad R_j(M) = 0.
\end{align*}
\]

Combining the above individual rates, we get the achievable BS transmission rate for any refined \( M \)-feasible partition \((H, I_0, I', I_1, J)\). Let the popularity level \( m \) have the largest individual rate under the scheme. Since there are \( L \) levels, the BS transmission rate can be bounded as \( R(M) \leq L \cdot \hat{R}_m(M) \).

B. Order-optimality

The proof of Theorem 2 involves the derivation of information-theoretic lower bounds on the optimal BS transmission rate, and the bounding of the gap between them and the rate achieved by our proposed scheme. We utilize two lower bounds, a cut-set bound and a general bound, presented in Lemmas 1 and 2 respectively. The cut-set lower bound is similar to the one presented in [3], where it had proven sufficient to establish order-optimality in the single-level setup.

Lemma 1. For any level \( i \in \{1, \ldots, L\} \) and \( v \in \{1, \ldots, KU_i\} \), the optimal BS transmission rate must satisfy:

\[
R^*(M) \geq v - \left\lceil \frac{v/U_i}{N_i/v} \right\rceil M.
\]

Lemma 2. For any disjoint sets \( A, B \subseteq \{1, \ldots, L\} \), any \( s \in \{1, \ldots, K\} \), and any \( b \in \mathbb{N}^+ \), the optimal BS transmission rate must satisfy:

\[
R^*(M) \geq \sum_{i \in A} \min \left\{ U_i, \frac{N_i}{b} \right\} + \sum_{j \in B} \min \left\{ sU_j, \frac{N_j}{sb} \right\} - \frac{M}{b}.
\]
Proof sketch: Consider the first $s$ APs and denote their cache contents by $Z_1, \ldots, Z_s$. Also, let $X_1^{(k)}, \ldots, X_b^{(k)}$, with $k \in \{1, \ldots, s\}$, denote $s$ sets of $b$ BS broadcasts each. We choose the $sb$ broadcasts so that the following hold true:

- For every level $i \in A$ and each AP $k$, consider the $U_i$ users connected to AP $k$ and requesting files from level $i$. Then, listening to the $b$ broadcasts $X_1^{(k)}, \ldots, X_b^{(k)}$, and accessing the AP cache content $Z_k$, the $U_i$ users are able to recover a total of $t_i$ files $W_1^{(i)}, \ldots, W_t_i$, where $t_i = \min \{b \cdot U_i, N_i\}$.
- For every level $j \in B$, consider the set of $sU_j$ users connected to any one of the $k$ APs. Then, listening to all the $sb$ BS broadcasts and accessing their corresponding AP’s cache content, the $sU_j$ users are able to recover a total of $v_j$ files $W_1^{(j)}, \ldots, W_{v_j}$, where $v_j = \min \{sb \cdot sU_j, N_j\}$.

Thus, accessing $s$ caches of size $MF$ bits each and $sb$ BS transmissions of size $RF$ bits each, the users can recover $st_i$ files per level $i \in A$ and $v_j$ files per level $j \in B$. Using Fano’s inequality, this yields

$$sbR + sM \geq \sum_{i \in A} st_i + \sum_{j \in B} v_j$$

$$\Rightarrow R \geq \sum_{i \in A} \min \left\{ \frac{N_i}{b} \right\} + \sum_{j \in B} \min \left\{ sU_j, \frac{N_j}{sb} \right\} - \frac{M}{b}.$$ 

To understand the lemmas, we first need to introduce the concept of “effective memory”. The effective memory is the part of the memory that contributes to a non-zero broadcast rate. Recall that a level $j \in J$ receives a cache memory of $N_j$, and, as a result, no information about files in level $i$ is needed in the broadcasts. The part of the memory dedicated to $J$, namely $\sum_{j \in J} N_j = T_J$, is thus a part that contributes nothing to the broadcast. The effective memory is therefore what is left, namely $M - \sum_{j \in J} N_j$. The reader will notice that, in the expressions for the achievable rate (2), the cache memory always appears as the effective memory (as $M - T_J$).

In the proof, we use Lemma 1 when $J = \emptyset$, since the effective memory is exactly $M$. When $J \neq \emptyset$, then the effective memory becomes $M - \sum_{j \in J} N_j$, and this is where we need Lemma 2. Typically, we use it by choosing $A = J$ and $B = \{m\}$ (recall from the achievability that $m$ is the level that maximizes the individual achievable rate). We then carefully choose $s$ and $b$ in order to get a bound of the form:

$$R^*(M) \geq sU_m - \frac{M - \sum_{j \in J} N_j}{b},$$

an expression that relates the rate to the effective memory.

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APPENDIX A

PROOF OF THE ACHIEVABILITY

This appendix will give a detailed analysis of the achievability scheme.

As previously discussed, we assign every popularity level a fraction $\alpha_i$ of cache memory, and then treat each level as a separate system. Each level $i$ will thus have a separate broadcast with a particular rate $R_i$; we call this rate the individual rate for level $i$. The final broadcast will simply be a concatenation of the broadcast messages of all the levels, and thus the total rate is given by:

$$R(M) = \sum_{i=1}^{L} R_i(M)$$
The individual rate for a particular level is given in this next lemma, and a useful related result in the corollary that follows. The lemma is a direct generalization of the result in [5].

**Lemma 3.** For a single-level system with \( N \) files, \( K \) caches and \( U \) users per cache, let the cache memory be \( M \in \frac{N}{K} \{0, \ldots, K\} \). Then, the following rate is achievable:

\[
R \left( \frac{M}{N}, K, U \right) = \frac{KU}{} \left(1 - \frac{M}{N}\right) \leq \frac{1}{1 + \frac{K}{M} N}
\]

For all \( M \in [0, N] \), the convex envelope of the points describe above is achievable.

In particular, the extreme points \( M = 0 \) and \( M = N \) achieve a rate of \( R(0, K, U) = KU \) and \( R(1, K, U) = 0 \) respectively.

**Corollary 1.** For a single-level system with \( N \) files, \( K \) caches, and \( U \) users per cache, the achievable rate is bounded by:

\[
\frac{KU}{} \left(1 - \frac{M}{N}\right) \leq R \left( \frac{M}{N}, K, U \right) \leq \frac{NU}{} \left(1 + \frac{1}{K}\right) - U
\]

for all \( M \in [0, N] \).

For the proof of the lemma and the corollary, see Appendix D.

So for each level \( i \in \{1, \ldots, L\} \), we have a system with \( N_i \) files, \( K_i \) caches, \( U_i \) users per cache, and \( \alpha_i M \) cache memory. Thus each level has an individual rate of:

\[
R_i(M) = R \left( \frac{\alpha_i M}{N_i}, K, U_i \right)
\]

What remains is to decide what \( \alpha_i \)'s to choose. Note that we must have \( \alpha_i \in [0, 1] \) for all \( i \), and \( \sum_i \alpha_i = 1 \). Moreover, since a rate of zero can be achieved for a specific level \( i \) once its allotted memory reaches \( N_i \), we require the additional condition that \( \alpha_i M \leq N_i \) for all \( i \), otherwise the additional memory is wasted. To sum up, we impose the following constraints on the \( \alpha_i \)'s:

\[
\begin{align*}
\alpha_i &\in [0, 1] \quad \forall i \\
\alpha_i &\leq \frac{N_i}{M} \quad \forall i \\
\sum_{i=1}^{L} \alpha_i &= 1
\end{align*}
\]

(3)

Just like in the example of Section II-B, we partition the set of levels into three subsets: \( H \), \( I \), and \( J \). We assign values for \( \alpha_i \) depending on the set to which \( i \) belongs. As previously discussed, \( H \) is the set of levels that will get no cache memory, \( J \) is the set of levels that will get full cache memory, and \( I \) consists of the levels that will share the remaining memory:

\[
\begin{align*}
&i \in H \quad \implies \quad \alpha_i M = 0 \\
&i \in I \quad \implies \quad \alpha_i M = \frac{\sqrt{N_i U_i}}{S_i} (M - T_J + V_I) - \frac{N_i}{K} \\
&i \in J \quad \implies \quad \alpha_i M = N_i
\end{align*}
\]

(4)

This choice of \( \alpha_i \)'s (particularly for \( i \in I \)) is justified as the solution to a relaxed optimization problem, which minimizes an approximate value of the achievable rate. We will not discuss this here as this does not affect the end result.

However, for a particular \( M \), not all \( (H, I, J) \) partitions can be assigned the above memory allocation without violating some of the constants on \( \alpha_i \). For instance, we cannot put an arbitrary set of the levels in \( J \) unless the memory permits it. To solve this, we introduce the notion of an \( M \)-feasible partition, which is a coarser version of a refined \( M \)-feasible partition.

**Definition 2** (\( M \)-feasible partition). Given a cache memory \( M \), we define an \( M \)-feasible partition \( (H, I, J) \) of \( \{1, \ldots, L\} \) as a partition which satisfies:

\[
\begin{align*}
&\forall h \in H, \quad M \leq m_h^{1,J} \\
&\forall i \in I, \quad m_i^{1,J} \leq M \leq M_i^{1,J} \\
&\forall j \in J, \quad M_j^{1,J} \leq M,
\end{align*}
\]

∀h ∈ H, M ≤ m_h^{1,J};
∀i ∈ I, m_i^{1,J} ≤ M ≤ M_i^{1,J};
∀j ∈ J, M_j^{1,J} ≤ M,
where, for any level \(i\), we define:

\[
m_{i,j}^{I,J} = \frac{1}{K} \sqrt{\frac{N_i}{U_i} S_I + T_J - V_i}
\]

\[
M_{i,j}^{I,J} = \frac{K+1}{K} \sqrt{\frac{N_i}{U_i} S_I + T_J - V_i}
\]

An \(M\)-feasible partition is thus a refined \(M\)-feasible partition with the sets \(I_0\), \(I'\), and \(I_1\) combined into \(I = I_0 \cup I' \cup I_1\).

**Proposition 2.** For any \(M\), there exists at least one \(M\)-feasible partition \((H, I, J)\).

The above proposition is not obvious, since the constraints on \(M\) directly depend on the choice of \(I\) and \(J\). We prove it by construction, in a rather lengthy proof which we relegate to Appendix E.

**Proposition 3.** Given \(M\) and an \(M\)-feasible partition \((H, I, J)\), the choice of \(\alpha_i\)'s given in \((4)\) satisfies all the constraints in \((5)\).

*Proof:* First, note that the \(\alpha_i\)'s sum to 1:

\[
\sum_{i=1}^{L} \alpha_i = \sum_{h \in H} \alpha_h + \sum_{i \in I} \alpha_i + \sum_{j \in J} \alpha_j
\]

\[
= \sum_{h \in H} 0 + \sum_{i \in I} \left( \sqrt{\frac{N_i}{U_i}} S_I \left( 1 - \frac{T_J}{M} + \frac{V_i}{M} \right) - \frac{N_i}{K} \right) + \sum_{j \in J} \frac{N_j}{M}
\]

\[
= \sum_{i \in I} \sqrt{\frac{N_i}{U_i}} S_I \left( 1 - \frac{T_J}{M} \right) + \frac{V_i}{M} - \frac{1}{M} \sum_{i \in I} \frac{N_i}{K} + \frac{T_J}{M}
\]

\[
= \frac{S_I}{S_I} \left( 1 - \frac{T_J}{M} \right) + \frac{V_i}{M} - \frac{V_i}{M} + \frac{T_J}{M} = 1
\]

Moreover, it follows from the conditions in Definition \(\square\) that \(\frac{N_h}{U_h} \geq \frac{N_i}{U_i} \geq \frac{N_j}{U_j}\), and thus \(j < i < h\), for all \(h \in H, i \in I,\) and \(j \in J\).

Next, we show that each \(\alpha_i\) satisfies: \(\alpha_i \in [0, 1]\) and \(\alpha_i M \leq N_i\).

For \(h \in H, \alpha_h = 0\) satisfies that trivially.

For \(j \in J\), we have \(\alpha_j M = N_j \in [0, N_j]\).

Finally, for \(i \in I\), we have:

\[
\alpha_i M = \sqrt{\frac{N_i}{U_i}} (M - T_J + V_J) - \frac{N_i}{K} = \sqrt{\frac{N_i}{U_i}} M - \frac{N_i}{K} \leq \frac{K+1}{K} N_i - \frac{N_i}{K} = N_i
\]

and

\[
\alpha_i M = \sqrt{\frac{N_i}{U_i}} M - \frac{N_i}{K} \geq \frac{1}{K} N_i - \frac{N_i}{K} = 0
\]

Since all the \(\alpha_i\)'s are non-negative and they all sum to 1, then each \(\alpha_i\) is no greater than 1. Thus, the \(\alpha_i\) choice satisfies all the constraints.

We will now compute the individual rate for each level.

**Proposition 4.** Given \(M\) and a refined \(M\)-feasible partition \((H, I_0, I', I_1, J)\) of the set of levels \(\{1, \ldots, L\}\), the value of the rate is given by the following set of inequalities:

\[
i \in H \quad \implies \quad R_i(M) = KU_i
\]

\[
i \in I_0 \quad \implies \quad \frac{1}{4} KU_i \leq R_i(M) \leq KU_i
\]

\[
i \in I' \quad \implies \quad \frac{(1 - \beta) S_I \sqrt{N_i U_i}}{M - T_J + V_J} \leq R_i(M) \leq \frac{3 S_I \sqrt{N_i U_i}}{M - T_J + V_J}
\]

\[
i \in I_1 \quad \implies \quad \frac{1}{2} U_i \left( 1 - \frac{M - T_J + V_J}{M_{i,j}^{I,J} - T_J + V_J} \right) \leq R_i(M) \leq \frac{4}{\beta} U_i \left( 1 - \frac{M - T_J + V_J}{M_{i,j}^{I,J} - T_J + V_J} \right)
\]

\[
i \in J \quad \implies \quad R_i(M) = 0
\]
Before we continue, here is a recap of the conditions on \( M \), in relation to the refined partition \((H, I_0, I', I_1, J)\) from Definition 1:

\[
\forall h \in H \quad \tilde{M} \leq \frac{1}{K} \sqrt{\frac{N_h}{U_h}}
\]

\[
\forall i_0 \in I_0 \quad \frac{1}{K} \sqrt{\frac{N_{i_0}}{U_{i_0}}} \leq \tilde{M} \leq \frac{2}{K} \sqrt{\frac{N_{i_0}}{U_{i_0}}}
\]

\[
\forall i \in I' \quad \frac{2}{K} \sqrt{\frac{N_i}{U_i}} \leq \tilde{M} \leq \beta \sqrt{\frac{N_i}{U_i}}
\]

\[
\forall i_1 \in I_1 \quad \beta \sqrt{\frac{N_{i_1}}{U_{i_1}}} \leq \tilde{M} \leq \frac{K + 1}{K} \sqrt{\frac{N_{i_1}}{U_{i_1}}}
\]

\[
\forall j \in J \quad \frac{K + 1}{K} \sqrt{\frac{N_j}{U_j}} \leq \tilde{M}
\]

**Proof of Proposition 4** We will analyze each set in \((H, I_0, I', I_1, J)\) separately.

a) For levels \( h \in H \): the cache memory is zero, and thus the server has to transmit for every user their requested file. Since there are \( K U_h \) users of level \( h \), and since there are more files than users, the worst-case broadcast rate is \( R_h = K U_h \). Indeed, this would be the result if we apply Lemma 3.

b) For levels \( j \in J \): the caches can store all the files of level \( j \), and thus no broadcast is needed. Therefore, \( R_j = 0 \).

Again, this is the same as what we would get from applying Lemma 3.

Let \( M_0 = \frac{2}{K} \sqrt{\frac{N_i}{U_i}} S_I + T_J - V_I \) and \( M_1 = \beta \sqrt{\frac{N_i}{U_i}} S_I + T_J - V_I \).

c) For levels \( i \in I_0 \): we first upper-bound the rate:

\[
R_i(M) = R\left(\frac{\alpha_i M}{N_i}, K, U_i\right) \leq R(0, K, U_i) = K U_i
\]

We then lower-bound the rate:

\[
R_i(M) = R\left(\frac{\alpha_i M}{N_i}, K, U_i\right) \geq R\left(\frac{\alpha_i M_0}{N_i}, K, U_i\right) = R\left(\frac{1}{K}, K, U_i\right) = \frac{K U_i (1 - \frac{1}{K})}{1 + \frac{1}{K}} \geq \frac{1}{4} K U_i
\]

d) For levels \( i \in I' \): The upper bound is:

\[
R_i(M) = R\left(\frac{\alpha_i M}{N_i}, K, U_i\right) \leq \frac{N_i U_i \left(1 + \frac{1}{K}\right)}{\alpha_i M} - U_i = \frac{N_i U_i \left(1 + \frac{1}{K}\right)}{\sqrt{\frac{N_i U_i}{S_I}} \left(M - T_J + V_I\right) - \frac{N_i}{K}}.
\]

Since \( i \in I' \), we have \( \frac{M - T_J + V_I}{S_I} \geq \frac{2}{K} \sqrt{\frac{N_i}{U_i}} \) and hence \( \sqrt{\frac{N_i U_i}{S_I}} \left(M - T_J + V_I\right) \geq \frac{2}{K} N_i \), and so the upper bound becomes:

\[
R_i(M) \leq \frac{N_i U_i \left(1 + \frac{1}{K}\right)}{\sqrt{\frac{N_i U_i}{S_I}} \left(M - T_J + V_I\right)} \leq 3 S_I \sqrt{N_i U_i} \sqrt{\frac{N_i U_i}{M - T_J + V_I}}.
\]

Conversely, the lower bound on the rate is:

\[
R_i(M) = R\left(\frac{\alpha_i M}{N_i}, K, U_i\right) \geq K U_i \left(1 - \frac{\alpha_i M}{N_i}\right) \geq K U_i \left(1 - \frac{\sqrt{\frac{N_i U_i}{S_I}} \left(M - T_J + V_I\right) + \frac{1}{K}}{1 + K \frac{\sqrt{\frac{N_i U_i}{S_I}} \left(M - T_J + V_I\right) - 1}}\right) \geq \frac{S_I \sqrt{N_i U_i}}{M - T_J + V_I} - U_i.
\]

But we notice that:

\[
\frac{S_I \sqrt{N_i U_i}}{M - T_J + V_I} \geq \frac{S_I \sqrt{N_i U_i}}{\beta \sqrt{\frac{N_i U_i}{S_I}}} = \frac{U_i}{\beta},
\]

and thus

\[
R_i(M) \geq \frac{(1 - \beta) S_I \sqrt{N_i U_i}}{M - T_J + V_I}.
\]
e) For levels $i \in I_1$: We know that $M$ lies between:

$$M_1 = \beta \sqrt{\frac{N_i}{U_i} S_I + T_J - V_I} \leq M \leq \frac{K + 1}{K} \sqrt{\frac{N_i}{U_i} S_I + T_J - V_I} = M^{I,J}_{i}$$

By convexity of $R_i(\cdot)$, it can be bound by the linear term:

$$\frac{R_i(M) - R_i(M^{I,J}_{i})}{M^{I,J}_{i} - M} \leq \frac{R_i(M_1) - R_i(M^{I,J}_{i})}{M^{I,J}_{i} - M_1} \Rightarrow R_i(M) \leq R_i(M^{I,J}_{i}) + \frac{M^{I,J}_{i} - M}{M^{I,J}_{i} - M_1} \left(R_i(M_1) - R_i(M^{I,J}_{i})\right)$$

The individual rate evaluated at $M_1$ and $M^{I,J}_{i}$ is:

$$R_i(M^{I,J}_{i}) = R\left(\frac{\alpha_i M^{I,J}_{i}}{N_i}, K, U_i\right) = R\left(\frac{\sqrt{N_i U_i}}{N_i S_I} (M^{I,J}_{i} - T_J + V_I) - \frac{1}{K}, K, U_i\right) = R(1, K, U_i) = 0$$

and

$$R_i(M_1) = R\left(\frac{\alpha_i M_1}{N_i}, K, U_i\right) = R\left(\beta - \frac{1}{K}, K, U_i\right) \leq U_i \left(1 + \frac{1}{\beta}\right) - U_i \leq \frac{2}{\beta} U_i$$

Finally, the linear term in the rate expression above is

$$\frac{M^{I,J}_{i} - M}{M^{I,J}_{i} - M_1} = \frac{M^{I,J}_{i} - M}{(K + 1) \beta} \sqrt{\frac{N_i}{U_i} S_I} \leq 2 \left(\frac{M^{I,J}_{i} - M}{M^{I,J}_{i} - T_J + V_I}\right) = 2 \left(1 - \frac{M - T_J + V_I}{M^{I,J}_{i} - T_J + V_I}\right)$$

Therefore, the rate can be bounded by:

$$R_i(M) \leq \frac{4}{\beta} U_i \left(1 - \frac{M - T_J + V_I}{M^{I,J}_{i} - T_J + V_I}\right)$$

For the lower inequality, we have:

$$R_i(M) = R\left(\frac{\alpha_i M}{N_i}, K, U_i\right) \geq K U_i \left(1 - \frac{\alpha_i M}{N_i}\right) \geq K U_i \left(1 - \frac{\alpha_i M}{N_i}\right)$$

$$\geq \frac{1}{2} U_i \left(1 - \frac{\alpha_i M}{N_i}\right) = \frac{1}{2} U_i \left(1 - \frac{\sqrt{N_i U_i}}{N_i} (M - T_J + V_I) - \frac{N_i}{K}\right)$$

$$= \frac{1}{2} U_i \left(1 - \sqrt{\frac{N_i}{U_i} S_I}\right) \geq \frac{1}{2} U_i \left(1 - \frac{M - T_J + V_I}{M^{I,J}_{i} - T_J + V_I}\right)$$

Thus we have completed the proof.

**APPENDIX B**

**PROOF OF THE INFORMATION THEORETIC LOWER BOUNDS**

**A. Proof of Lemma 7 (cut-set bound)**

We are given a level $i \in \{1, \ldots, L\}$, and an integer $v \in \{1, \ldots, KU_i\}$. Consider $v$ users of level $i$ connected to the smallest possible number of caches $s = \lfloor v/U_i \rfloor$. Assume these caches are $Z_1, \ldots, Z_s$. Furthermore, consider $b = \lfloor N_i/s \rfloor$ iterations, where, at iteration $j$, all the $v$ users are requesting (together) the files $W^i_{(j-1)v+1}, \ldots, W^i_{jv}$, and we transmit the broadcast message $X_j$. Thus, after $b$ iterations, all the files $W^i_1, \ldots, W^i_v$ would have been recovered at the users. Formally, by Fano’s inequality:

$$H(W^i_1, \ldots, W^i_v | Z_1, \ldots, Z_s, X_1, \ldots, X_b) \leq vb \cdot F \varepsilon_F$$

where $\varepsilon_F \to 0$ as $F \to \infty$.  


Then, the following must hold for all $F$.

$$bRF + sMF \geq H(Z_1, \ldots, Z_s, X_1, \ldots, X_b)$$

$$= H(Z_1, \ldots, Z_s, X_1, \ldots, X_b | W_1^i, \ldots, W_{v_b}^i) + I(W_1^i, \ldots, W_{v_b}^i; Z_1, \ldots, Z_s, X_1, \ldots, X_b)$$

$$\geq H(W_1^i, \ldots, W_{v_b}^i) - H(W_1^i, \ldots, W_{v_b}^i | Z_1, \ldots, Z_s, X_1, \ldots, X_b)$$

$$\geq vb \cdot F - vb \cdot F \varepsilon_F$$

$$\geq vb \cdot F (1 - \varepsilon_F)$$

By taking $F \to \infty$, $\varepsilon_F$ vanishes and so:

$$R^*(M) \geq v - \frac{s}{b} M = v - \left[ \frac{v}{U_i} \right] M$$

**B. Proof of Lemma 2**

We are given two disjoints sets $A, B \in \{1, \ldots, L\}$, and two integers $s \in \{1, \ldots, K\}$ and $b \in \mathbb{N}^+$. Then, consider the first $s$ caches, $Z_1, \ldots, Z_s$, as well as all users connecting to them, and consider $s$ sets of $b$ broadcasts: $X_1^{(k)}, \ldots, X_b^{(k)}$, $k \in \{1, \ldots, s\}$. Then,

$$sbRF + sMF \geq \sum_{k=1}^{s} H(Z_i, X_1^{(k)}, \ldots, X_b^{(k)}).$$

We choose the broadcasts so that they serve the following requests:

- For every level $i \in A$, and for each $k$, the broadcast message $X_1^{(k)}, \ldots, X_b^{(k)}$, along with the cache $Z_k$, allow the $U_i$ level-$i$ users at cache $k$ to recover (all together) the files $W_1^i, \ldots, W_{u_i}^i$, where

  $$t_i = \min \{ b \cdot U_i, N_i \} .$$

- For every level $i \in B$, we require that all $sU_i$ users, using all the $sb$ broadcasts, decode files $W_1^i, \ldots, W_{u_i}^i$, where

  $$u_i = \min \{ sb \cdot sU_i, N_i \} .$$

Thus by Fano’s inequality:

$$\forall i \in A, \forall k \in \{1, \ldots, s\}, \quad H(W_1^i, \ldots, W_{u_i}^i | Z_k, X_1^{(k)}, \ldots, X_b^{(k)}) \leq t_i \cdot F \varepsilon_F$$

$$\forall i \in B, \quad H(W_1^i, \ldots, W_{u_i}^i | Z_1, \ldots, Z_s, \left\{ X_1^{(k)}, \ldots, X_b^{(k)} \right\}_{k=1}^{s}) \leq u_i \cdot F \varepsilon_F$$

where $\varepsilon_F \to 0$ as $F \to \infty$. 

Therefore, for all $F$:

\[ sbR + sMF \geq \sum_{k=1}^{s} H \left( Z_{k}, X_{1}^{(k)}, \ldots, X_{b}^{(k)} \right) \]

\[ = \sum_{k=1}^{s} \left[ H \left( Z_{k}, X_{1}^{(k)}, \ldots, X_{b}^{(k)} \middle| \{ W_{i}^{j} \}_{i \in A} \right) + I \left( \{ W_{i}^{j} \}_{i \in A} ; Z_{k}, X_{1}^{(k)}, \ldots, X_{b}^{(k)} \right) \right] \]

\[ \geq \sum_{k=1}^{s} \left[ H \left( Z_{k}, X_{1}^{(k)}, \ldots, X_{b}^{(k)} \middle| \{ W_{i}^{j} \}_{i \in A} \right) + H \left( \{ W_{i}^{j} \}_{i \in A} \right) \cdot (1 - \varepsilon_{F}) \right] \]

\[ = \sum_{k=1}^{s} H \left( Z_{1}, \ldots, Z_{s}, \left\{ X_{1}^{(k)}, \ldots, X_{b}^{(k)} \right\}_{k=1}^{s} \middle| \{ W_{i}^{j} \}_{i \in A}, \{ W_{i}^{j} \}_{i \in B} \right) + s \sum_{i \in A} t_{i} F \cdot (1 - \varepsilon_{F}) \]

\[ \geq H \left( \{ W_{i}^{j} \}_{i \in B} ; Z_{1}, \ldots, Z_{s}, \left\{ X_{1}^{(k)}, \ldots, X_{b}^{(k)} \right\}_{k=1}^{s} \right) + s \sum_{i \in A} t_{i} F \cdot (1 - \varepsilon_{F}) \]

\[ \geq H \left( \{ W_{i}^{j} \}_{i \in B} \right) \cdot (1 - \varepsilon_{F}) + s \sum_{i \in B} u_{i} F \cdot (1 - \varepsilon_{F}) \]

\[ \geq \sum_{i \in B} u_{i} F \cdot (1 - \varepsilon_{F}) + s \sum_{i \in A} t_{i} F \cdot (1 - \varepsilon_{F}) \]

\[ = \sum_{i \in B} u_{i} + s \sum_{i \in A} t_{i} \cdot F \cdot (1 - \varepsilon_{F}) \]

The inequalities marked with (*) are due to Fano’s inequality.

Taking $F$ to infinity, $\varepsilon_{F}$ decays to zero, and thus we get:

\[ sbR + sM \geq \sum_{i \in B} u_{i} + s \sum_{i \in A} t_{i} \]

\[ R \geq \sum_{i \in B} u_{i} + \sum_{i \in A} t_{i} - \frac{M}{b} \]

\[ R^{*}(M) \geq \min \left\{ sU_{i}, \frac{N_{i}}{sb} \right\} + \sum_{i \in A} \min \left\{ U_{i}, \frac{N_{i}}{b} \right\} - \frac{M}{b} \]

In the next section, we use these lower bounds to prove order-optimality of our scheme. When we are considering the bound in Lemma 2 we need to analyze the two minimizations. For the minimization over $B$, we want to analyze:

\[ s^{2}b \leq \frac{N_{i}}{U_{i}}, \quad i \in B \]  \hspace{1cm} (6)

For the minimization in the sum over $A$, we will instead analyze:

\[ b \leq \frac{N_{i}}{U_{i}}, \quad i \in A \]  \hspace{1cm} (7)

We now present the proof of Theorem 2.

### Appendix C

**Full Proof of Theorem 2**

In this section we are given a value of the memory $M$, and a refined $M$-feasible partition $(H, I_{0}, I', I_{1}, J)$. We will show that applying our achievability scheme at this value of $M$ and with the partition $(H, I_{0}, I', I_{1}, J)$ is within a constant of the information theoretic bounds.
Recall that $I = I_0 \cup I' \cup I_1$. Define the following (not necessarily distinct) levels:

$$
m = \arg\max_{i \in \{1, \ldots, L\}} R_i(M),
$$

$$
i^* = \arg\max_{i \in I} N_i U_i,
$$

$$
\hat{i} = \arg\max_{i \in I \setminus I_1} N_i U_i, \quad \text{if } I \setminus I_1 \neq \emptyset.
$$

The level $m$ maximizes the individual rate over all levels. As a result, the achievable rate can be bounded by:

$$
R(M) = \sum_{i=1}^{L} R_i(M) \leq L \cdot R_m(M).
$$

Throughout the proof, we will regularly refer to the conditions on $M$ given in (1) in Definition 1 as well as the rate expressions (5) in Proposition 4.

### A. Useful preliminary results

Before we move on to the proof of the gap, we need to present certain preliminary results.

1) **Some results pertaining to the parameters**: The following results give some relations between the $N_i U_i$ quantities of certain levels, pertaining to the refined partition.

#### Claim 1.

For any two levels $i, j \in I$, we have:

$$
K + 1 \geq \frac{\sqrt{N_i / U_i}}{\sqrt{N_j / U_j}}.
$$

**Proof**: Since $i, j \in I$, the conditions on $M$ (1) imply:

$$
\frac{1}{K} \sqrt{N_j / U_j} \leq \hat{M} \leq \frac{K+1}{K} \sqrt{N_i / U_i} \Rightarrow K + 1 \geq \frac{\sqrt{N_i / U_i}}{\sqrt{N_j / U_j}}.
$$

#### Claim 2.

Under the regularity conditions in Section III, each of the sets $I_0$ and $I_1$ cannot contain more than one level, i.e.,

$$
|I_0| \leq 1; \quad |I_1| \leq 1.
$$

**Proof**: Assume, for the sake of contradiction, that there are two distinct levels $i, j \in I_0$ with $i < j$. Then,

$$
\frac{1}{K} \sqrt{N_i / U_i} \leq \frac{1}{K} \sqrt{N_j / U_j} \leq \hat{M} \leq \frac{2}{K} \sqrt{N_i / U_i} \leq \frac{2}{K} \sqrt{N_j / U_j}.
$$

Therefore,

$$
\frac{U_i}{N_i} \leq 4 \frac{U_j}{N_j} < q \frac{U_j}{N_j},
$$

thus contradicting the regularity conditions.

Similarly, assume that there are two distinct levels $i, j \in I_1$, with $i < j$. Then,

$$
\beta \sqrt{N_i / U_i} \leq \beta \sqrt{N_j / U_j} \leq \hat{M} \leq \frac{K+1}{K} \sqrt{N_i / U_i} \leq \frac{K+1}{K} \sqrt{N_j / U_j}.
$$

Therefore,

$$
\frac{U_i}{N_i} \leq \left(\frac{K+1}{\beta K}\right)^2 \frac{U_j}{N_j} = q \left(\frac{K+1}{2K}\right)^2 \frac{U_j}{N_j} < q \frac{U_j}{N_j},
$$

thus contradicting the regularity conditions.
2) **The Dominance lemmas**: We now present three important lemmas, which we will call the Dominance lemmas. In words, all three lemmas combined say that the level that maximizes the individual rate also dominates the $S_I$ term, or the $S_{I \setminus I_i}$ term, up to a multiplicative constant. Each lemma deals with a separate case of the set to which the level $m$ belongs, where $m$ is as defined in [8]. We will prove the lemmas in Appendix C-D.

**Lemma 4** (First Dominance lemma). Assume $m \in H \cup I_0$, where $m$ is as in [8]. Then,
\[ S_{I \setminus I_i} \leq \gamma_0 L \sqrt{N_m U_m}, \]
where $\gamma_0 = \frac{4}{1 - \beta}$.

**Lemma 5** (Second Dominance lemma). Assume $m \in I'$, where $m$ is as in [8]. Then,
\[ S_{I \setminus I_i} \leq \gamma' L \sqrt{N_m U_m}, \]
where $\gamma' = 12$.

**Lemma 6** (Third Dominance lemma). Assume $m \in I_1$, where $m$ is as in [8]. Then,
\[ S_I \leq \gamma_1 L \sqrt{N_m U_m}, \]
where $\gamma_1 = \frac{8(2 - \beta)}{\beta}$.

3) Preliminary upper and lower bounds: For technical reasons, it is necessary to identify a particular upper bound on the achievable rate, and a particular lower bound on the optimal rate, that arise in the case when $I_1$ is non-empty.

**Lemma 7** (Upper Bound A). If $I_1 = \{i_1\}$ is not empty, then, for all levels $i \in H \cup I_0 \cup I'$,
\[ R_i(M) \leq \frac{3}{\beta} \sqrt{\frac{N_i U_i}{N_{i_1}}}. \]

**Lemma 8** (Lower Bound B1). Let $I_1 = \{i_1\}$, and assume $I \setminus I_1 \neq \emptyset$ so that the level $\hat{i}$ exists, as defined in [8]. Then, the optimal rate is bounded from below by:
\[ R^*(M) \geq \frac{1}{100 L} \sqrt{\frac{N_{i_1} U_{i_1}}{N_{i_i}}} \]

**Lemma 9** (Lower Bound B2). Assume $I_1 = \{i_1\}$ and $m \in H$, where $m$ is as in [8]. Then, the optimal rate is bounded from below by:
\[ R^*(M) \geq \frac{1}{100 \gamma_0 L} \sqrt{\frac{N_m U_m U_{i_1}}{N_{i_1}}}. \]

The proofs of all three of these lemmas are in Appendix C-C.

**B. Gap**

Recall the level $m$ defined in [8]. The gap analysis is subdivided into two major Regimes, depending on $m$: $m \in H \cup I_0 \cup I'$ and $m \in I_1$. (If $m \in J$, then, by combining [9] with the rate expressions in [8], we get $R(M) = 0$).

1) **Regime 1**: $m \in H \cup I_0 \cup I'$: Recall from Claim 2 that $|I_1| \leq 1$, i.e., either $I_1 = \emptyset$ or $I_1 = \{i_1\}$ for some $i_1 \in \{1, \ldots, L\}$.

We analyze each case separately.

a) **Case 1**: $I_1 = \{i_1\}$.

By Upper Bound A in Lemma 7, $R_m(M) \leq \frac{3}{\beta} \sqrt{\frac{N_m U_m U_{i_1}}{N_{i_1}}}$. If $m \in I_0 \cup I'$, then, by the definition on level $\hat{i}$ in [8], we can write $R_m(M) \leq \frac{3}{\beta} \sqrt{\frac{N_i U_{i_1}}{N_{i_1}}}$. By (9):
\[ R(M) \leq L \cdot R_m(M) \leq L \cdot \frac{3}{\beta} \sqrt{\frac{N_i U_{i_1}}{N_{i_1}}}. \]

We can now combine (10) with Lower Bound B1 (Lemma 8) to get:
\[ \frac{R(M)}{R^*(M)} \leq \frac{L \cdot \frac{3}{\beta}}{\frac{1}{100 \gamma_0 L}} = \frac{300 L^2}{\beta}. \]

If $m \in H$, then we can instead combine (10) with Lower Bound B2 (Lemma 9):
\[ \frac{R(M)}{R^*(M)} \leq \frac{L \cdot \frac{3}{\beta}}{\frac{1}{100 \gamma_0 L}} = \frac{300 \gamma_0 L^2}{\beta}. \]
b) Case 2: \( I_1 = \phi \).

We first see that \( I = I \setminus I_1 \) and thus \( S_{I \setminus I_1} = S_I \).
This case must be further divided into \( m \in H \cup I_0 \) and \( m \in I' \).

Case 2a: \( m \in H \cup I_0 \).

Here, we have, by the definition of \( m \) in (8), and by (9):

\[
R(M) \leq L \cdot KU_m. \tag{13}
\]

Furthermore, from the first Dominance lemma (Lemma 4), we know that \( S_I \leq \gamma_0 L_N \cdot U_m \).

We will now consider the two cases of \( J = \phi \) and \( J \neq \phi \), as that will affect the lower bound on the optimal rate that we will use.

Assume that \( J \neq \phi \). We use the lower bound in Lemma 2 with \( B = \{ m \} \), \( A = J \), \( s = \left\lfloor \frac{1}{5} \gamma_0 L K \right\rfloor \) and \( b = \left\lfloor \frac{25 \gamma_0 L^2 N_m}{K^2 U_m} \right\rfloor \). We analyze the minimizations in (7) and (5). For the minimization in (6):

\[
s b \leq \frac{N_m}{U_m}.
\]

For the minimization in (7), we first show that the inside of the floor in the expression of \( b \) is greater than 1. Since there exists some \( j \in J \), then:

\[
4 N_m \left( \frac{K + 1}{K} \right) \geq \gamma_0 \frac{M - T_J}{U_m},
\]

and thus \( b \geq \frac{25 \gamma_0 L^2 N_m}{2 K^2 U_m} \geq \frac{N_j}{U_j} \) for all \( j \in J \). This follows from the fact that \( |x| \geq \frac{2}{3} \) for any \( x \geq 1 \). The bound becomes:

\[
R^*(M) \geq sU_m - \frac{M - T_J}{b}
\]

\[
\geq \left( \frac{1}{5} \gamma_0 L K - 1 \right) U_m - \frac{2 M - T_J + V_I}{25 \gamma_0 L^2 N_m}
\]

\[
= \left( \frac{1}{5} \gamma_0 L K - 1 \right) U_m - \frac{4 K}{25 \gamma_0 L^2 N_m} \cdot \sqrt{U_m \cdot \gamma_0 L_N \cdot U_m}
\]

\[
\geq \left( \frac{1}{5} \gamma_0 L K - 1 \right) U_m - \frac{4 K}{25 \gamma_0 L^2 N_m} \cdot \sqrt{U_m \cdot \gamma_0 L_N \cdot U_m}
\]

\[
= KU_m \left[ \frac{1}{5} \gamma_0 L - \frac{1}{K} \right], \tag{14}
\]

where (a) follows from \( m \in H \cup I_1 \) and the conditions on \( M \) (1), and (b) follows from the first Dominance lemma (Lemma 4).

By combining (13) and (14), and for \( K \) large enough, we have:

\[
\frac{R(M)}{R^*(M)} \leq \frac{L}{25 \gamma_0 L - \frac{1}{K}}. \tag{15}
\]
Assume that $J = \phi$. We use here the cut-set bound with $i = m$, and $v = \left\lceil \frac{1}{6\gamma_0 L}K U_m \right\rceil$ users. Then,

$$R^*(M) \geq \frac{1}{6\gamma_0 L}K U_m - 1 - \left[ \frac{1}{5\gamma_0 L}K \right] \frac{M}{6\gamma_0 L N_m - K U_m}.$$  \hspace{1cm} (16)

where $(a)$ uses $m \in H \cup I_0$ and $(\ref{9})$, and $(b)$ uses both $N_m \geq K U_m$ and the first Dominance lemma (Lemma 4).

By combining $(\ref{17})$ and $(\ref{16})$, and for $K$ large enough, we get:

$$\frac{R(M)}{R^*(M)} \leq L \frac{30\gamma_0 L - 1}{30\gamma_0 L - K}.$$  \hspace{1cm} (17)

**Case 2b:** $m \in I'$. 

By the second Dominance lemma (Lemma 5), we have:

$$S_I \leq \gamma' L \sqrt{N_m U_m}.$$ 

Therefore, by using the individual rate expression of $m \in I'$ (5), as well as $(\ref{9})$, the achievable rate is:

$$R(M) \leq L \cdot R_m(M) \leq L \cdot \frac{3S_I \sqrt{N_m U_m}}{M - T_J + V_I} \leq \frac{3\gamma' L^2 N_m U_m}{M - T_J + V_I}. \hspace{1cm} (18)$$

Again, we must separate the cases $J = \phi$ and $J \neq \phi$.

**Assume first that $J \neq \phi$**. We now use the lower bound in Lemma 2 with $B = \{m\}$, $A = J$, $s = \left\lfloor \frac{1}{M - T_J + V_I} \right\rfloor$, and $b = \left\lceil \frac{9(M - T_J + V_I)^2}{N_m U_m} \right\rceil$. Note that $s \geq 1$ if we ensure that $\beta \leq \frac{1}{3L}$.

First, we will check the minimizations in $(\ref{17})$ and $(\ref{9})$. For the minimization in $(\ref{9})$:

$$s^2 b \leq \left( \frac{1}{M - T_J + V_I} \right)^2 \frac{9(M - T_J + V_I)^2}{N_m U_m} = \frac{N_m}{U_m},$$

and thus the smaller term is $sU_m$. For the minimization in $(\ref{17})$, we first note that the expression in $b$ inside the floor is greater than 1. Indeed, since $J \neq \phi$, there exists a $j \in J$, and thus:

$$\frac{9(M - T_J + V_I)^2}{N_m U_m} \geq \frac{9}{N_m U_m} \left( \frac{K + 1}{K} \sqrt{\frac{N_j S_I}{U_j}} \right)^2 \geq \frac{S_j^2}{N_m U_m} \geq \frac{N_j}{U_j} \geq 1$$
and therefore \( b \geq \frac{9(M-T_J+V_I)^2}{2N_m U_m} \), for all \( j \in J \). The lower bound hence becomes

\[
R^*(M) \geq \frac{sU_m - M - T_J}{b} \\
\geq \left( \frac{1}{2} N_m \right) U_m - M - T_J + V_I \\
= \frac{1}{3} \frac{N_m U_m}{M - T_J + V_I} - \frac{2}{9} \frac{N_m U_m}{M - T_J + V_I} \\
= \frac{N_m U_m}{M - T_J + V_I} \left[ \frac{1}{9} - \frac{M - T_J + V_I}{N_m} \right] \\
\geq \frac{N_m U_m}{M - T_J + V_I} \left[ \frac{1}{9} - \beta \sqrt{N_m U_m S_I} \right] \\
\geq \frac{N_m U_m}{M - T_J + V_I} \left[ \frac{1}{9} - \beta \gamma' L \right],
\]

where \((a)\) follows from \( m \in I' \) and \( I \), and \((b)\) follows from the second Dominance lemma (Lemma 5).

If \( \beta \geq \frac{1}{9 \gamma' L} \), then, combining (18) with (19), and for a large enough \( K \), we get:

\[
\frac{R(M)}{R^*(M)} \leq \frac{3 \gamma' L^2}{9 - 3 \beta \gamma' L}.
\]

Now assume that \( J = \phi \). Note \( T_J = 0 \). We will use the cut-set bound in Lemma \( I \) with \( i = m \), and \( v = \left\lfloor \frac{1}{3} \frac{4N_m U_m}{M + V_I} \right\rfloor \). Then,

\[
R^*(M) \geq \frac{1}{3} \frac{N_m U_m}{M + V_I} - 1 - \frac{1}{3} \frac{\sqrt{N_m U_m S_I}}{3 \beta S_I} - \frac{1}{9} \frac{N_m U_m}{M + V_I} \\
= \frac{1}{3} \frac{N_m U_m}{M + V_I} - 1 - \frac{U_m \left\lfloor \frac{1}{3} \frac{N_m}{M + V_I} \right\rfloor}{3(M + V_I) - U_m} \\
\geq \frac{1}{3} \frac{N_m U_m}{M + V_I} - 1 - \frac{U_m \left\lfloor \frac{1}{3} \frac{N_m}{M + V_I} \right\rfloor}{3(M + V_I) - U_m} (M + V_I).
\]

Since \( m \in I' \), we have:

\[
\frac{1}{3} \frac{N_m}{M + V_I} \geq \frac{1}{3} \frac{N_m}{M + V_I} = \sqrt{N_m U_m S_I} \geq \frac{1}{3 \beta S_I} \geq 1,
\]

if \( \beta \leq \frac{1}{3 \gamma' L} \), and we also have

\[
M + V_I \geq \frac{2}{K} \sqrt{N_m S_I} \geq \frac{2}{K} N_m \geq 2U_m
\]
Therefore, (21) becomes:

\[
R^*(M) \geq \frac{1}{3} N_m U_m \frac{N_m}{M + V_I} - 1 - \frac{U_m \left[ \frac{1}{4} N_m \right]}{3(M + V_I) - U_m} (M + V_I)
\]

\[
\geq \frac{1}{3} N_m U_m \frac{N_m}{M + V_I} - 1 - \frac{4 N_m U_m}{3(M + V_I) - \frac{1}{2} (M + V_I)} (M + V_I)
\]

\[
= \frac{1}{3} N_m U_m \left[ \frac{1}{15} M + V_I \right] - \frac{4 N_m U_m}{15 M + V_I}
\]

\[
(b) \quad \frac{N_m U_m}{M + V_I} \left[ \frac{1}{15} \beta \sqrt{N_m U_m S_I} \right]
\]

\[
\geq \frac{N_m U_m}{M + V_I} \left[ \frac{1}{15} - \beta \gamma L \right]
\]

\[
\geq \frac{N_m U_m}{M + V_I} \left[ \frac{1}{15} - \beta \gamma L \right],
\]

(23)

where (a) uses (22) as well as the fact that \([x] \leq 2x\) for any \(x \geq 1\), (b) uses \(m \in I'\) with (1), and (c) uses the second Dominance lemma (Lemma 5).

As long as \(\beta < \frac{1}{15} \gamma L\), then we can combine (18) with (23) to get:

\[
\frac{R(M)}{R^*(M)} \leq \frac{3 \gamma L^2}{15 - \beta \gamma L}.
\]

(24)

2) Regime 2: \(m \in I_1\): Note that this means \(I_1 = \{m\}\). Here, we analyze three cases.

a) Case 1: \(I_1 = \{m\} = I\): We can use (5) with \(m \in I_1\), together with (9), to bound the achievable rate:

\[
R(M) \leq L \cdot \frac{2}{\beta} U_m \left( 1 - \frac{M - T_J + V_I}{M^I_J - T_J + V_I} \right) \leq L \cdot \frac{2}{\beta} U_m \left( 1 - \frac{M - T_J}{M^I_J - T_J} \right).
\]

(25)

Since \(I = \{m\}\), we have:

\[
M^I_J - T_J = \frac{K + 1}{K} \sqrt{\frac{N_m}{U_m}} S_I - V_I = \frac{K + 1}{K} \sqrt{\frac{N_m}{U_m}} \sqrt{N_m U_m} - \frac{N_m}{K} = N_m.
\]

Recall that \(m \in I_1\) and (1) imply that \(M \leq M^I_J\). Therefore, the above inequality yields:

\[
M \leq N_m + T_J.
\]

(26)

We use the lower bound in Lemma 2 with \(B = \{m\}\), \(A = J\), \(s = 1\) and \(b = [N_m/U_m]\). Note that:

\[
b \geq \frac{N_m}{U_m} \geq \frac{N_j}{U_j},
\]

for all \(j \in J\). Since also \(s = 1\), this takes care of both minimizations in (7) and (6). Also note that \(b \geq 1\) (regardless of whether or not \(J = \phi\)) which means \(b \leq 2N_m/U_m\). Then,

\[
R^*(M) \geq \frac{N_m + T_J - M}{b}
\]

\[
\geq \frac{N_m - (M - T_J)}{2N_m/U_m}
\]

\[
= \frac{U_m}{2} \left( 1 - \frac{M - T_J}{N_m} \right)
\]

\[
(b) \quad \frac{1}{2} U_m \left( 1 - \frac{M - T_J}{M^I_J - T_J} \right),
\]

(27)

where (a) follows from \(b \leq 2N_m/U_m\), and (b) follows from (26).

Therefore, by combining (25) with (27), we get:

\[
\frac{R(M)}{R^*(M)} \leq \frac{4L}{\beta}.
\]

(28)
b) Case 2: $I_1 = \{m\} \subset I$ and $M \geq \frac{1}{2} N_m + T_j - V_i$: Let us first take a closer look at the memory value $Y_0 = \frac{1}{2} N_m + T_j - V_i$. For $K$ large enough, we have $Y_0 \in I$, and therefore:

$$R_m(Y_0) = R_m \left( \frac{1}{2} N_m + T_j - V_i \right)$$

$$\leq \frac{K U_m \left( 1 - \frac{a_m(\frac{1}{2} N_m + T_j - V_i)}{N_m} + \frac{1}{K} \right)}{1 + K \frac{a_m(\frac{1}{2} N_m + T_j - V_i)}{N_m} - 1}$$

$$= \frac{N_m U_m \left( 1 + \frac{1}{K} \right) - U_m}{a_m(\frac{1}{2} N_m + T_j - V_i) - U_m}$$

$$= \left( U_m + \sqrt{\frac{U_m}{N_m} S_{I \setminus \{m\}}} \right) \left( 1 + \frac{2 \gamma_1 L}{K} \right) - U_m$$

$$\leq 2 \sqrt{\frac{U_m}{N_m}} L \sqrt{N_i U_i} + \frac{2 \gamma_1 L}{K} U_m$$

$$= \sqrt{\frac{N_i U_i U_m}{N_m} \left( 2L + \frac{2 \gamma_1 L}{K} \sqrt{\frac{N_m U_m}{N_i U_i}} \right)}$$

$$\leq (2 + 4 \gamma_1 L) \sqrt{\frac{N_i U_i U_m}{N_m}}. \quad (29)$$

The annotated inequalities are explained next.

First, (a) holds for $K$ large enough, in particular $K \geq 8 \gamma_1 L$. Indeed,

$$8 \gamma_1 L \leq K \implies 4 \gamma_1^2 L^2 \leq \frac{1}{2} \gamma_1 L K \leq (\gamma_1 L - 1) K \implies (\gamma_1 L + 1) K \leq 2 \gamma_1 L (K - 2 \gamma_1 L) \implies \frac{\gamma_1 L + 1}{K} \leq \frac{2 \gamma_1 L}{K}.$$ 

Second, (b) follows from the fact that $m, \hat{i} \in I$ and Claim 1:

$$\sqrt{\frac{N_m U_m}{N_i U_i}} = \frac{N_m}{N_i} \sqrt{\frac{N_i U_i}{N_m U_m}} \leq (K + 1) \frac{N_m}{N_i} \leq 2K \frac{N_m}{N_i}.$$ 

Third, (c) follows from the fact that $m < \hat{i}$ (since $m \in I_1$ and $\hat{i} \in I_0 \cup I'$; see Proposition 1) and thus $N_m \leq N_i$ (regularity conditions, Section III).

Since $M \geq Y_0$, we have $R_m(M) \leq R_m(Y_0)$, and therefore:

$$R(M) \leq L \cdot R_m(M) \leq L \cdot R_m(Y_0) \leq (2 + 4 \gamma_1 L)^2 \sqrt{\frac{N_i U_i U_m}{N_m}}. \quad (30)$$

By combining (30) with Lower Bound B1 (Lemma 8), we get:

$$\frac{R(M)}{R^*(M)} \leq (2 + 4 \gamma_1) 100 L^3. \quad (31)$$
c) Case 3: $I_1 = \{m\} \subset I$ and $M \leq \frac{1}{2}N_m + T_J - V_I$: By the third Dominance lemma (Lemma 6), we have:

$$S_I \leq \gamma_1 L \sqrt{N_m U_m}.$$  

Let $Y_1 = \beta \sqrt{\frac{N_m}{U_m}} S_I + T_J - V_I$ and $Y_2 = N_m + T_J - V_I$. From [1], we have $Y_1 \leq M$. Thus $Y_1 \leq M \leq Y_0 \leq Y_2$. Notice that this implies that $Y_2 - Y_1 \geq \frac{1}{2}N_m$.

Using the individual rate expressions in (35), we can compute:

$$R_m(Y_1) \leq \frac{4}{\beta} U_m \left( 1 - \frac{Y_1 - T_J + V_I}{M - T_J + V_I} \right) \leq \frac{4}{\beta} U_m$$  \hspace{1cm} (32)

Furthermore, using (29) and $Y_0 \leq Y_2$, we have:

$$R_m(Y_2) \leq R_m(Y_0) \leq (2 + 4\gamma_1)L \sqrt{\frac{N_m U_m}{N_n}}$$  \hspace{1cm} (33)

By the convexity of $R_m(M)$:

$$\frac{R_m(M) - R_m(Y_2)}{Y_2 - M} \leq \frac{R_m(Y_1) - R_m(Y_2)}{Y_2 - Y_1}$$

$$R_m(M) \leq R_m(Y_2) + \frac{Y_2 - M}{Y_2 - Y_1} (R_m(Y_1) - R_m(Y_2)) \leq R_m(Y_2) + \frac{N_m + T_J - V_I - M}{\frac{1}{2}N_m} R_m(Y_1)$$

$$= R_m(Y_2) + 2R_m(Y_1) \left( 1 - \frac{M - T_J + V_I}{N_m} \right) \leq (2 + 4\gamma_1)L \sqrt{\frac{N_m U_m}{N_n}} + \frac{8}{\beta} U_m \left( 1 - \frac{M - T_J + V_I}{N_m} \right),$$  \hspace{1cm} (34)

where (a) uses the aforementioned fact that $Y_2 - Y_1 \geq \frac{1}{2}N_m$, and (b) uses (32) and (33).

Combining (34) with (9), we get:

$$R(M) \leq (2 + 4\gamma_1)L^2 \sqrt{\frac{N_m U_m}{N_n}} + \frac{8}{\beta} U_m \left( 1 - \frac{M - T_J + V_I}{N_m} \right)$$  \hspace{1cm} (35)

So the achievable rate can be expressed as the sum of two terms: a constant term (constant in $M$) and a linear term (linear in $M$). We will match the lower bound depending on which one dominates.

**If the constant term dominates**, then we combine (35) with Lower Bound B1 (Lemma 8) to get:

$$\frac{R(M)}{R^*(M)} \leq 400(1 + 2\gamma_1)L^3.$$  \hspace{1cm} (36)

**If the linear term dominates**, we use the lower bound from Lemma 2. We set $A = J, B = \{m\}$, $s = 1$, and $b = \left\lceil \frac{N_m}{U_m} \right\rceil$.

Then,

$$b \geq \frac{N_m}{U_m} \geq \frac{N_j}{U_j},$$

for all $j \in J$. Therefore, the lower bound is:

$$R^*(M) \geq \frac{N_m + T_J - M}{\left\lceil \frac{N_m}{U_m} \right\rceil}$$

$$\geq \frac{N_m + T_J - V_I - M}{\frac{2}{U_m}}$$

$$= \frac{1}{2} U_m \left( 1 - \frac{M - T_J + V_I}{N_m} \right).$$  \hspace{1cm} (37)

By combining (35) with (37), we get:

$$\frac{R(M)}{R^*(M)} \leq \frac{32L}{\beta}.$$  \hspace{1cm} (38)

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**Gap between upper and lower bounds on the optimal rate:** The equations (11), (12), (15), (17), (20), (24), (28), (31), (36), and (38), together give a bound on the ratio between the rate achieved by our scheme and the optimal rate. By combining them all, we get:

\[
\frac{R(M)}{R^*(M)} \leq c L^3,
\]

where \(c\) is the maximum out of the bounds in all the inequalities mentioned above. This concludes the proof of Theorem 2.

**C. Proof of Upper Bound A and Lower Bound B**

Recall the definitions of the levels \(m, i^*, \text{and } i\) in (8).

We will first prove the lemma pertaining to Upper Bound A.

**Proof of Lemma 7:** Let \(I_1 = \{i_1\}\). Assume, for the sake of contradiction, that there exists a level \(i \in H \cup I_0 \cup I'\) such that

\[
R_i(M) > \frac{3}{\beta} \sqrt{\frac{N_i U_i U_{i_1}}{N_{i_1}}}. \tag{39}
\]

We identify two cases.

1) **Case 1:** \(i \in H \cup I_0\): Starting with (39):

\[
R_i(M) > \frac{3}{\beta} \sqrt{\frac{N_i U_i U_{i_1}}{N_{i_1}}},
\]

\[
K U_i > \frac{3}{\beta} \sqrt{\frac{N_i U_i U_{i_1}}{N_{i_1}}},
\]

\[
\beta \sqrt{\frac{N_{i_1}}{U_{i_1}}} > \frac{3}{K} \sqrt{\frac{N_i}{U_i}}.
\]

However, this contradicts (1), which states that:

\[
\beta \sqrt{\frac{N_{i_1}}{U_{i_1}}} \leq \frac{M - T_J + V_I}{S_I} \leq \frac{2}{K} \sqrt{\frac{N_i}{U_i}},
\]

because \(i \in H \cup I_0\) and \(i_1 \in I_1\). Therefore such a case is impossible.

2) **Case 2:** \(i \in I'\): Again, going from (39):

\[
\frac{3}{\beta} \sqrt{\frac{N_i U_i U_{i_1}}{N_{i_1}}} < R_i(M),
\]

\[
\frac{3}{\beta} \sqrt{\frac{N_i U_i U_{i_1}}{N_{i_1}}} < 3 S_I \sqrt{N_i U_i}
\]

so

\[
\frac{1}{\beta} \left( M - T_J + V_I \right) < \sqrt{\frac{N_{i_1}}{U_{i_1}}} S_I,
\]

\[
M < \beta \sqrt{\frac{N_{i_1}}{U_{i_1}}} S_I + T_J - V_I.
\]

But this contradicts \(i_1 \in I_1\) with \(1\), which states:

\[
M \geq \beta \sqrt{\frac{N_{i_1}}{U_{i_1}}} S_I + T_J - V_I,
\]

and thus this case is impossible.

We will next prove the two lemmas relating to Lower Bound B.

**Proof of Lemma 8:** Consider the lower bound in Lemma 2 with \(B = \{i\}, A = J \cup I_1 = J \cup \{i_1\}, s = \left[ \frac{1}{10 L^2 \sqrt{U_i N_i}} \right],\) and \(b = \left[ 100 L^2 N_i U_i \right].\) Looking at (7) and (6), we have:

\[
s^2 b \leq N_i U_i,
\]

\[
b \geq \frac{N_{i_1}}{U_{i_1}} \geq \frac{N_j}{U_j},
\]

\[
22
\]
We now consider two cases. First, if $I_1 \subseteq I_0 \cup I'$, then $i_1 < \hat{i}$, and, due regularity condition (b) in Section III:

$$\frac{1}{10L} \sqrt{\frac{N_1 U_{i_1}}{U_{i_1} N_{i_1}}} \geq \frac{q}{10L} \geq 1,$$

if $q \geq 10L$. We can hence write $s \geq \frac{1}{20L} \sqrt{\frac{N_1 U_{i_1}}{U_{i_1} N_{i_1}}}$. Therefore:

$$R^*(M) \geq sU_i - \frac{M - T_J - N_{i_1}}{b} \geq \frac{1}{20L} \sqrt{\frac{N_1 U_{i_1} U_{i_1}}{N_{i_1}}} - \frac{M - T_J - N_{i_1}}{50L^2 \frac{N_i}{U_{i_1}}}.$$  \hspace{1cm} (40)

Taking a closer look at $M - T_J - N_{i_1}$:

$$M - T_J - N_{i_1} \leq \frac{K + 1}{K} \sqrt{\frac{N_{i_1}}{U_{i_1}}} S_I - N_{i_1} - V_I$$

$$= \frac{K + 1}{K} \sqrt{\frac{N_{i_1}}{U_{i_1}}} S_{I \setminus \{i_1\}} - V_{I \setminus \{i_1\}}$$

$$\leq 2L \sqrt{\frac{N_{i_1} U_{i_1}}{U_{i_1}}},$$

where the last step follows form the definition of $\hat{i}$ in (8). Hence, (40) becomes:

$$R^*(M) \geq \frac{1}{20L} \sqrt{\frac{N_1 U_{i_1} U_{i_1}}{N_{i_1}}} - \frac{2L \sqrt{\frac{N_1 N_{i_1} U_{i_1}}{U_{i_1} N_{i_1}}}}{50L^2 \frac{N_i}{U_{i_1}}}$$

$$= \sqrt{\frac{N_1 U_{i_1} U_{i_1}}{N_{i_1}}} \left( \frac{1}{20L} - \frac{1}{25L} \right)$$

$$= \frac{1}{100L} \sqrt{\frac{N_1 U_{i_1} U_{i_1}}{N_{i_1}}}.$$

**Proof of Lemma** \hspace{1cm} We start the proof similarly to that of Lower Bound B1. Consider the lower bound from Lemma 2 with $B = \{m\}$, $A = J \cup I_1 = J \cup \{i_1\}$, $s = \left[ \frac{1}{10\gamma_0 L} \sqrt{\frac{N_m U_{i_1}}{U_m N_{i_1}}} \right]$, and $b = \left[ \frac{100 \gamma_0^2 L^2 N_i}{U_{i_1}} \right]$. The same first steps as before (with $\beta < \frac{1}{10\gamma_0^2}$ this time) result in:

$$R^*(M) \geq \frac{1}{20\gamma_0 L} \sqrt{\frac{N_m U_m U_{i_1}}{N_{i_1}}} - \frac{M - T_J - N_{i_1}}{50\gamma_0^2 L^2 \frac{N_i}{U_{i_1}}}.$$  \hspace{1cm} (41)

Moreover, we have:

$$M - T_J - N_{i_1} \leq \frac{K + 1}{K} \sqrt{\frac{N_{i_1}}{U_{i_1}}} S_{I \setminus \{i_1\}} - V_{I \setminus \{i_1\}}.$$

We now consider two cases. First, if $I \setminus \{i_1\} = \phi$, then $S_{I \setminus \{i_1\}} = V_{I \setminus \{i_1\}} = 0$, and the above condition becomes $M - T_J - N_{i_1} \leq 0$. The lower bound (41) thus becomes:

$$R^*(M) \geq \frac{1}{20\gamma_0 L} \sqrt{\frac{N_m U_m U_{i_1}}{N_{i_1}}}.$$

If $I \setminus \{i_1\} \neq \phi$, then, using the first Dominance lemma (Lemma 4):

$$M - T_J - N_{i_1} \leq 2\gamma_0 L \sqrt{\frac{N_{i_1} N_m U_m}{U_{i_1}}}.$$
D. Proof of the Dominance lemmas

Again, recall the definitions of the levels \( m, i^*, \) and \( \hat{i} \) in [3].
For Lemmas 4 and 5, we first identify the set to which \( \hat{i} \) belongs, upper bound \( N_iU_i \) with \( N_mU_m \), and thus get a similar upper bound for \( S_{I\setminus i} \). For Lemma 6, the same is done, but with \( i^* \) instead of \( \hat{i} \), and thus \( S_I \) instead of \( S_{I\setminus i} \).

Proof of Lemma 4: Assume \( m \in H \cup I_0 \), and let \( \gamma_0 = \frac{4}{1-\beta} \).

Case 1: \( \hat{i} \in I_0 \). By the definition of \( m \), we have

\[
R_i(M) \leq R_m(M), \\
\frac{1}{4} KU_i \leq KU_m, \\
U_i \leq 4U_m.
\]

If \( m \in H \), then, by the ordering of \( H \) and \( I_0 \), we have \( \hat{i} < m \) and thus \( \sqrt{N_m/U_m} \geq \sqrt{N_i/U_i} \), which, together with the above inequality, implies \( 4\sqrt{N_mU_m} \geq \sqrt{N_iU_i} \).

If \( m \in I_0 \), then by the level spacing condition, \( I_0 = \{m\} \) (there cannot be more than one level in \( I_0 \)). But we also know that \( \hat{i} \in I_0 \), which implies \( m = i \), and thus \( \sqrt{N_mU_m} \geq \sqrt{N_iU_i} \) trivially.

Therefore, \( S_{I\setminus i} \leq 4L\sqrt{N_mU_m} \leq \gamma_0L\sqrt{N_mU_m} \).

Case 2: \( \hat{i} \in I' \). By the definition of \( m \), we have:

\[
R_i(M) \leq R_m(M), \\
\frac{1-\beta}{2} S_I \sqrt{N_iU_i} \leq KU_m, \\
\frac{1}{2} S_I \sqrt{N_iU_i} \leq KU_m(M - T_J + V_I) \leq KU_m \cdot \frac{2}{K} \sqrt{N_mU_m} S_I, \\
\sqrt{N_iU_i} \leq \frac{4}{1-\beta} \sqrt{N_mU_m}.
\]

Therefore, \( S_{I\setminus i} \leq \frac{1}{4} L\sqrt{N_mU_m} \leq \gamma_0L\sqrt{N_mU_m} \).

Proof of Lemma 5: Assume \( m \in I' \), and let \( \gamma' = 12 \).

Case 1: \( \hat{i} \in I_0 \). By the definition of \( m \):

\[
R_m(M) \geq R_i(M), \\
\frac{3}{M - T_J + V_I} \sqrt{N_mU_m} \geq \frac{1}{4} KU_i, \\
12S_I \sqrt{N_mU_m} \geq KU_i(M - T_J + V_I) \geq KU_i \cdot \frac{1}{K} \sqrt{N_iU_i} S_I, \\
12\sqrt{N_mU_m} \geq \sqrt{N_iU_i},
\]

and hence \( S_{I\setminus i} \leq 12L\sqrt{N_mU_m} \leq \gamma' L\sqrt{N_mU_m} \).

Case 2: \( \hat{i} \in I' \). Similarly:

\[
R_i(M) \leq R_m(M), \\
\frac{1-\beta}{2} S_I \sqrt{N_iU_i} \leq 3S_I \sqrt{N_mU_m} \leq \frac{3}{M - T_J + V_I} \sqrt{N_mU_m}, \\
\sqrt{N_iU_i} \leq \frac{6}{1-\beta} \sqrt{N_mU_m},
\]

\[\]
and hence $S_{I}\leq \frac{6}{\gamma} L\sqrt{\frac{N_{m}U_{m}}{L}} \leq \gamma L\sqrt{\frac{N_{m}U_{m}}{L}}$.

Proof of Lemma 6. Assume $m \in I_1$, and let $\gamma_1 = \frac{8(2-\beta)}{\beta}$.

First note that the rate $R_m(M)$ can be bounded by a simple value in this regime:

$$R_m(M) \leq R_m\left(\beta \sqrt{\frac{N_{m}}{U_{m}}}S_I + T_J - V_I\right) \leq \frac{2-\beta}{\beta} U_m.$$  

Case 1: $i^* \in I_0$. Again by the definition of $m$:

$$R_{i^*}(M) \leq R_m(M) \leq \frac{1}{4} K U_{i^*} \leq \frac{2-\beta}{\beta} U_m.$$  

However, we also have, from (1):

$$\frac{1}{K} \sqrt{\frac{N_{i^*}}{U_{i^*}}} \leq \frac{M - T_J + V_I}{S_I} \leq \frac{K + 1}{K} \sqrt{\frac{N_{m}}{U_{m}}}$$

Combining the two:

$$\sqrt{N_{i^*}U_{i^*}} \leq 4 \frac{2-\beta}{\beta} \cdot 2\sqrt{N_{m}U_{m}} \leq \frac{8(2-\beta)}{\beta} \sqrt{N_{m}U_{m}}$$

and thus $S_I \leq \frac{8(2-\beta)}{\beta} L\sqrt{\frac{N_{m}U_{m}}{L}} \leq \gamma_1 L\sqrt{\frac{N_{m}U_{m}}{L}}$.

Case 2: $i^* \in I'$. Similarly:

$$R_{i^*}(M) \leq R_m(M),$$

$$\frac{\gamma + \beta}{M - T_J + V_I} \leq \frac{2-\beta}{\beta} U_m,$$

$$S_I \sqrt{N_{i^*}U_{i^*}} \leq \frac{2(2-\beta)}{\beta(1-\beta)} U_m(M - T_J + V_I) \leq \frac{2(2-\beta)}{\beta(1-\beta)} U_m \cdot \frac{K + 1}{K} \sqrt{\frac{N_{m}}{U_{m}}} S_I,$$

$$\sqrt{N_{i^*}U_{i^*}} \leq \frac{4(2-\beta)}{\beta(1-\beta)} \sqrt{N_{m}U_{m}},$$

and thus $S_I \leq \frac{4(2-\beta)}{\beta(1-\beta)} L\sqrt{\frac{N_{m}U_{m}}{L}} \leq \gamma_1 L\sqrt{\frac{N_{m}U_{m}}{L}}$.

Case 3: $i^* \in I_1$. Since $|I_1| \leq 1$ and $m, i^* \in I_1$, we conclude that $m = i^*$ and thus $S_I \leq L\sqrt{\frac{N_{m}U_{m}}{L}} \leq \gamma_1 L\sqrt{\frac{N_{m}U_{m}}{L}}$.

\[\square\]

APPENDIX D

DISCUSSION ON SINGLE-LEVEL SCENARIOS

In the achievability scheme presented in this paper, the problem is divided into $L$ sub-systems, each dealing with one popularity class. Each sub-system is treated separately and independently of the others. As previously explained, each such sub-system is in fact an instance of the problem discussed in this section: a single-level, multi-user setup.

In this scenario, there are $N$ files of a single popularity class, $K$ caches of memory $M$, and $U$ users accessing each cache. The scheme used in this setup is a straightforward extension of the one in [5], where there is only one user accessing each cache (i.e., $U = 1$).

Let the users at cache $h \in \{1, \ldots, K\}$ be labeled as $T_i^h, \ldots, T_i^K$. Since they are all connected to the same cache, they share the same side information. Therefore, there is no benefit to coding across these users. As a result, we serve the users with $U$ separate broadcasts, by splitting them into $U$ groups, such that group $i$ consists of the $i$-th user of each cache: $T_i^1, \ldots, T_i^K$. Each group is an instance of a single-user system, such as the one in [5].

In [5], a coding-based placement scheme with network-coded broadcast transmission was proposed for such a single-user setup, whose performance is much superior to the conventional scheme with replication and unicast transmission.

The scheme proposed in [5] is as follows. For $t = \frac{MK}{N}$, each file is split into $\binom{K}{t}$ parts of equal size, one corresponding to each subset of cardinality $t$ of the $K$ AP caches in the system. Every such part is placed in each of the $t$ AP caches corresponding to its assigned subset and it is shown that this satisfies the memory size constraint. During the delivery phase, for each subset of $(t + 1)$ users, the BS broadcasts an XOR of the file parts that are requested by a user and available to everyone else through their assigned caches. Thus each user can recover their requested part, since they have access to all but one element in the XOR. The BS transmission rate achieved by this scheme is given by

$$r_0\left(\frac{M}{N}, K\right) = \frac{K (1 - \frac{K}{N})}{1 + K \frac{M}{N}},$$

and the lower convex envelope of these points for $M \in [0, N]$. The following example illustrates the scheme and its benefits.
respectively. Note that this scheme requires the BS to transmit 2\times F/2 = F bits, so that the transmission rate is 1/2.

Next, consider the conventional placement and delivery scheme in Figure 6(b). This scheme would store the same content, say \( A_1, B_1 \) in both caches. Then, during the delivery phase the BS will have to unicast \( F \) bits, \( 1/4 \leq F \leq 2/3 \) and each file is split into \( \sqrt{N_i} \) parts, let

\[
A = (A_1, A_2), \quad B = (B_1, B_2).
\]

The two caches stores \((A_1, B_1)\) and \((A_2, B_2)\) respectively. Now, suppose the two users accessing AP1 and AP2 request the files \( A, B \) respectively. Then, by broadcasting \( A_2 \oplus B_1 \), where \( \oplus \) denotes the bit-wise XOR operation, the server can satisfy both requests simultaneously. For this scheme, the BS transmits \( F/2 \) bits, so that the total rate would be the sum of the transmission rates.

By performing the placement phase exactly as for the single-user case, we can then send one broadcast for each of the \( U \) groups mentioned above, satisfying the requirements of the users in said group. Therefore, the total rate would be the sum of the rates of the broadcasts for all the groups:

\[
r_c \left( \frac{M}{N}, K, U \right) = U \cdot r_0 \left( \frac{M}{N}, K \right) = \frac{KU \left( 1 - \frac{M}{N} \right)}{1 + K \frac{M}{N}},
\]

for \( M \in \frac{N}{K} \cdot \{0, 1, \ldots, K\} \), and the lower convex envelope of these points for \( M \in [0, N] \).

### Appendix E

**Existence of an \( M \)-Feasible Partition**

In this appendix, we will prove the result of Proposition 2 which states the existence of an \( M \)-feasible partition for any \( M \). We do so by constructing a partition, and showing that it is indeed \( M \)-feasible.

Recall that an \( M \)-feasible partition \((H, I, J)\) is defined as one that satisfies the conditions in Definition 2

\[
\begin{align*}
\forall h \in H, & \quad M \leq m^{I,J}_h; \\
\forall i \in I, & \quad m^{I,J}_i \leq M \leq M^{I,J}_i; \\
\forall j \in J, & \quad M^{I,J}_j \leq M,
\end{align*}
\]

where, for any level \( i \), we have defined:

\[
m^{I,J}_i = \frac{1}{K} \sqrt{\frac{N_i}{U_i}} S_i + T_j - V_i \\
M^{I,J}_i = \frac{K + 1}{K} \sqrt{\frac{N_i}{U_i}} S_i + T_j - V_i
\]

The terms \( m^{I,J}_i \) and \( M^{I,J}_i \) depend on both the level in question, \( i \), as well as on the partition. The difficulty hence comes from the fact that the conditions that define the partition \((H, I, J)\) are themselves dependent on it.

To resolve this, we first decouple the levels from the partition (in the conditions) by defining:

\[
\tilde{m}_i = \frac{1}{K} \sqrt{\frac{N_i}{U_i}}, \\
\tilde{M}_i = \frac{K + 1}{K} \sqrt{\frac{N_i}{U_i}}.
\]
and defining the function \( f_{I,J} : \mathbb{R} \to \mathbb{R} \) as \( f_{I,J}(x) = xS_I + T_J - V_I \). Thus, \( m_{i,J} = f_{I,J}(\tilde{m}_i) \) and \( M_{i,J} = f_{I,J}(\tilde{M}_i) \).

It is worth noting that \( \tilde{m}_i < \tilde{M}_i \) for all \( i \), and, furthermore, for any \( i \) and \( j \):

\[
i < j \iff \tilde{m}_i < \tilde{m}_j \iff \tilde{M}_i < \tilde{M}_j
\]

Moreover, since \( f_{I,J}(\cdot) \) is an increasing function, it preserves the ordering of its arguments.

Algorithm 1 gives a procedure for finding an \( M \)-feasible partition for all \( M \in [0, N] \). The algorithm works on all values of \( M \) at once, and proceeds as follows. First, it divides the whole range of \( M, [0, N] \), into \( 2L + 1 \) intervals, and assigns a partition \( (H_I, I, J_I) \) for each interval \( t \in \{0, 1, \ldots, 2L\} \). Second, based on the partition of each interval, it computes the boundaries of said intervals. The boundaries will be of the form of \( m_{i,J_i} \) or \( M_{i,J_i} \). Finally, for every \( M \), it finds the interval \( t \) to which \( M \) belongs (based on the boundaries), and assigns the corresponding partition \( (H_I, I, J_I) \) as its \( M \)-feasible partition.

We will first present the algorithm and explain its steps, and then prove its correctness. Note that the algorithm terminates in finite time; in particular, its running time is \( \Theta(L^2) \).

A. Description of the algorithm

Figure 7 provides an illustration of the algorithm on a small example with five levels.

Algorithm 1 Finds an \( M \)-feasible partition for all \( M \).

1: Sort \( \{\tilde{m}_i, \tilde{M}_i\}_i \) in ascending order, and label the resulting sequence as \( (x_1, \ldots, x_{2L}) \)

2:

3: Step 1: First, determine \( (I, J) \) for each interval, symbolized by the pair \( (x_t, x_{t+1}) \):

4: \( H_0 \leftarrow \{1, \ldots, L\} \)

5: \( I_0 \leftarrow \emptyset \)

6: \( J_0 \leftarrow \emptyset \)

7: for \( t \in \{1, \ldots, 2L\} \) do

8: \quad if \( x_t = \tilde{m}_i \) then \# Move level \( i \) from \( H \) to \( I \)

9: \quad \quad \\quad \\quad H_I \leftarrow H_{I-1} \setminus \{i\}

10: \quad \quad \\quad \\quad I_I \leftarrow I_{I-1} \cup \{i\}

11: \quad \quad \\quad \\quad J_I \leftarrow J_{I-1} \cup \{i\}

12: \quad else if \( x_t = \tilde{M}_i \) then \# Move level \( i \) from \( I \) to \( J \)

13: \quad \quad \\quad \\quad H_I \leftarrow H_{I-1} \setminus \{i\}

14: \quad \quad \\quad \\quad I_I \leftarrow I_{I-1} \cup \{i\}

15: \quad \quad \\quad \\quad J_I \leftarrow J_{I-1} \cup \{i\}

16: \quad end if

17: end for

18:

19: Step 2: Now that we have \( (I, J) \) for each interval, we compute the limits of the intervals as \( (Y_t, Y_{t+1}) \):

20: for \( t \in \{1, \ldots, 2L\} \) do

21: \quad if \( x_t = \tilde{m}_i \) then

22: \quad \quad \quad \quad Y_I \leftarrow m_{i,J_i}

23: \quad else if \( x_t = \tilde{M}_i \) then

24: \quad \quad \quad \quad Y_I \leftarrow M_{i,J_i}

25: \quad end if

26: end for

27:

28: Step 3: Finally, choose an \( M \)-feasible partition for all \( M \) depending on the intervals:

29: for all \( M \) do

30: \quad Find \( t \) such that \( M \in [Y_t, Y_{t+1}) \)

31: \quad Set \( M \)-feasible partition as: \( (H_I, I, J_I) \).

32: end for

Step 0: Sort the values of \( \{\tilde{m}_i, \tilde{M}_i\}_i \) together in ascending order, and label the resulting sequence as \( (x_1, \ldots, x_{2L}) \). These \( x_t \) terms will form the “virtual boundaries” of each interval, and will be useful in finding the appropriate partitions. Computing the actual boundaries of the intervals is a later step.

Step 1: Our initial interval is the “zeroth” interval \( t = 0 \), starting at \( M = 0 \). We assign \( H_0 = \{1, \ldots, L\} \) and \( I_0 = J_0 = \emptyset \), i.e., no popularity level is given any cache memory yet. This interval will turn out to be empty, but it will act as a base case for the next intervals.
As the memory increases, we would expect levels to start migrating from $H$ to $I$ (by being assigned some memory), and then from $I$ to $J$ (when their assigned memory becomes maximal). We capture this by going over the $(x_t)_t$ sequence in order.

At this point, it is useful to note the relation between $\tilde{m}_i$ and $m_t^{I,J}$ on one hand, and $\tilde{M}_i$ and $M_t^{I,J}$ on the other hand. Since the $m_t^{I,J}$ and $M_t^{I,J}$ terms form lower and upper boundaries, respectively, on $M$ for when level $i$ is in the set $I$, then it makes intuitive sense to move $i$ from $H$ to $I$ when we arrive at a $\tilde{m}_i$, and to move it from $I$ to $J$ when we hit a $\tilde{M}_i$. This is exactly what the algorithm does in this step: for every $t \in \{1, \ldots, 2L\}$, if $x_t = \tilde{m}_i$, then we move $i$ from $H$ to $I$, and if $x_t = \tilde{M}_i$, then we move $i$ from $I$ to $J$.

**Step 2:** Now that we have determined $(H_t, I_t, J_t)$ for each interval $t$, finding the boundaries is just a matter of applying the function $f_{I_t, J_t}(\cdot)$ or $f_{I_{t-1}, J_{t-1}}(\cdot)$ on $x_t$ (both functions turn out to give the same value when evaluated at $x_t$, but the convention that we are use in the algorithm conforms with the definitions of $m_t^{I,J}$ and $M_t^{I,J}$). The resulting boundaries are labeled as $Y_t$.

**Step 3:** Finally, for every value of $M$, we need only look up the interval $[Y_t, Y_{t+1})$ in which it lies, and choose $(H_t, I_t, J_t)$ as the $M$-feasible partition.

![Table Illustration](image-url)

*Fig. 7.* Illustration of Algorithm 1 on a small example with five levels. Notice how, at each interval, a level $i$ migrates either from $H$ to $I$, or from $I$ to $J$, depending on if $x_t = \tilde{m}_i$ or $x_t = \tilde{M}_i$. The boundaries of the intervals, $Y_t$, are computed from $x_t$ and the partition of either of the intervals that $Y_t$ separates.

**B. Correctness of the algorithm**

We will first show that the algorithm behaves correctly. Most of the steps do not require such a proof, except for two things. The first is in Step 1: moving a level $i$ from one set to the other always requires that the level be present in the former and absent in the latter. The second is in Steps 2 and 3: we must show that the values $Y_t$ are ordered the same way as $x_t$, i.e., $Y_t \leq Y_{t+1}$.

1) **Moving levels between sets:** These moves occur in two forms: either we move a level $i$ from $H$ to $I$, or we move it from $I$ to $J$. The former takes place when $x_t = \tilde{m}_i$, and the latter when $x_t = \tilde{M}_i$. Recalling that $\tilde{m}_i < \tilde{M}_i$, and that all levels initially start in $H$, it follows that $i$ will go from $H$ to $I$, and then from $I$ to $J$. Thus this step is correct.

2) **Ordering of the $Y_t$’s:** To prove the correctness of this step, we must first show that:

$$\forall t \in \{1, \ldots, 2L\}: \quad f_{I_t, J_t}(x_t) = f_{I_{t-1}, J_{t-1}}(x_t)$$

*Proof:* We will analyze two cases.

First, if $x_t = \tilde{m}_i$ for some $i$, then there are sets $H$, $I$ and $J$ such that:

$$H_{t-1} = H \cup \{i\}, \quad H_t = H, \quad I_{t-1} = I, \quad I_t = I \cup \{i\}, \quad J_{t-1} = J, \quad J_t = J.$$
Therefore:

\[
f_{I_t,J_t}(x_t) = \frac{1}{K} \sqrt{\frac{N_i}{U_i}} S_{I_t\cup\{i\}} + T_J - V_{I_t\cup\{i\}}
\]

\[
= \frac{1}{K} \sqrt{\frac{N_i}{U_i}} \left( S_I + \sqrt{N_i U_i} \right) + T_J - V_I - \frac{N_i}{K}
\]

\[
= \frac{1}{K} \sqrt{\frac{N_i}{U_i}} S_I + \frac{N_i}{K} + T_J - V_I - \frac{N_i}{K}
\]

\[
= \frac{1}{K} \sqrt{\frac{N_i}{U_i}} S_I + T_J - V_I
\]

\[
= f_{I_{t-1},J_{t-1}}(x_t)
\]

Second, if \( x_t = \tilde{M}_i \) for some \( i \), then there are sets \( H, I \) and \( J \) such that:

\[
H_{t-1} = H, \quad H_t = H,
\]

\[
I_{t-1} = I \cup \{i\}, \quad I_t = I,
\]

\[
J_{t-1} = J, \quad J_t = J \cup \{i\}.
\]

Therefore:

\[
f_{I_{t-1},J_{t-1}}(x_t) = \frac{K+1}{K} \sqrt{\frac{N_i}{U_i}} S_{I_t \cup \{i\}} + T_J - V_{I_t \cup \{i\}}
\]

\[
= \frac{K+1}{K} \sqrt{\frac{N_i}{U_i}} \left( S_I + \sqrt{N_i U_i} \right) + T_J - V_I - \frac{N_i}{K}
\]

\[
= \frac{K+1}{K} \sqrt{\frac{N_i}{U_i}} S_I + \frac{K+1}{K} N_i + T_J - V_I - \frac{N_i}{K}
\]

\[
= \frac{K+1}{K} \sqrt{\frac{N_i}{U_i}} S_I + N_i + T_J - V_I
\]

\[
= \frac{K+1}{K} \sqrt{\frac{N_i}{U_i}} S_I + T_{J_t \cup \{i\}} - V_I
\]

\[
= f_{I_t,J_t}(x_t)
\]

Now we must show that \( Y_t \leq Y_{t+1} \) for all \( t \in \{1, \ldots, 2L-1\} \). Given such a \( t \), consider \( x_t \) and \( x_{t+1} \). Due to Step 0, we know that \( x_t \leq x_{t+1} \). Since the function \( f_{I_t,J_t}(\cdot) \) is, for fixed \( I \) and \( J \), nondecreasing (increasing when \( I \neq \phi \), which, as we will see, is always the case), then

\[
Y_t = f_{I_{t-1},J_{t-1}}(x_t) = f_{I_t,J_t}(x_t) \leq f_{I_t,J_t}(x_{t+1}) = f_{I_{t+1},J_{t+1}}(x_{t+1}) = Y_{t+1}.
\]

It is now a good time to state a certain proposition which can handle some technical issues.

**Proposition 5.** The \( M \)-feasible partition found in Algorithm 1 never sets \( I = \phi \).

This proposition implies that \( S_I \neq 0 \) always.

**Proof:** First we look at the extreme cases: \( t = 0 \) and \( t = 2L \), in which \( I_t = \phi \) by definition (see Figure 1).

For \( t = 0 \), we notice that \( Y_t = f_{\phi,\phi}(x_1) = 0 \), and thus the “zeroth” interval is empty.

For \( t = 2L \), we note that \( Y_{2L} = f_{\phi,\phi}(x_{2L}) = N \), and thus the very last interval is also empty.

Finally, for any interval \( t \) in between, assume that \( I_t = \phi \). The boundaries of this interval are \( Y_t \) and \( Y_{t+1} \). But we have \( Y_t = f_{I_t,J_t}(x_t) = f_{\phi,J_t}(x_t) = T_J \), and \( Y_{t+1} = f_{I_t,J_t}(x_{t+1}) = T_J = Y_t \), and thus this interval is also empty.

Since all cases where \( I_t \) is potentially an empty set occur in empty intervals, there is no value of \( M \) that will be assigned an empty \( I \) set.

Finally, we will prove that the algorithm indeed produces an \( M \)-feasible partition.

**Proof that Algorithm 1 produces an \( M \)-feasible partition:** Let \( M \in [0, N] \) be arbitrary. There exists an interval \( t \) such that \( M \in [Y_t, Y_{t+1}] \) in the algorithm, and the chosen \( M \)-feasible partition is \((H_t, I_t, J_t)\). By the design of the algorithm, we have:

\[
\forall h \in H_t, \quad x_{t+1} \leq \tilde{m}_h;
\]

\[
\forall i \in I_t, \quad \tilde{m}_i \leq x_t \leq x_{t+1} \leq \bar{M}_i;
\]

\[
\forall j \in J_t, \quad \bar{M}_j \leq x_t.
\]
By the monotonicity of $f_{I_t,J_t}(\cdot)$, and since $f_{I_t,J_t}(x_t) = Y_t$, $f_{I_t,J_t}(x_{t+1}) = Y_{t+1}$, and $M \in \{Y_t, Y_{t+1}\}$, this results in:

\[
\forall h \in H_t, \quad M \leq f_{I_t,J_t}(\tilde{m}_h) = m_{I_t,J_t}^h; \\
\forall i \in I_t, \quad m_{I_t,J_t}^i = f_{I_t,J_t}(\tilde{m}_i) \leq M \leq f_{I_t,J_t}(\tilde{M}_i) = M_{I_t,J_t}^i; \\
\forall j \in J_t, \quad M_{I_t,J_t}^j = f_{I_t,J_t}(\tilde{M}_j) \leq M.
\]

Therefore, $(H_t, I_t, J_t)$ is an $M$-feasible partition.

Thus our algorithm correctly constructs an $M$-feasible partition for all $M$, which proves the existence of such a partition for all $M$.

APPENDIX F
FULL CHARACTERIZATION FOR THE EXAMPLE IN FIGURE 2

This appendix proves Theorem 1 in two parts. The first part presents the achievability scheme, and the second part gives the information-theoretic outer bounds.

The rate-memory curve shown in Figure 3—which we are trying to prove is both achievable and optimal—can be described using the following equation:

\[
R = \max \left\{ 3 - 2M, \frac{5}{2} - M, 2 - \frac{1}{2}M, 1 - \frac{M - 2}{N_2/2} \right\}
\]

(42)

We would like to remind the reader that, in the example considered, the number of less popular files is $N_2 \geq 4$.

A. Achievability scheme

To prove the achievability of the piece-wise linear curve in Figure 2 and in (42), we need only show the achievability of the corner points, as the rest can be achieved using a memory-sharing scheme. These $(M, R)$ points are:

\[
(0, 3), \quad \left(\frac{1}{2}, 2\right), \quad \left(1, \frac{2}{3}\right), \quad (2, 1), \quad \text{and} \quad \left(2 + \frac{N_2}{2}, 0\right).
\]

1) Point $(M, R) = (0, 3)$: When $M = 0$, we do not place any content in the caches, and instead broadcast all the requested files to the users. Since these can be three different files, the peak rate is $R = 3$.

2) Point $(M, R) = (2 + \frac{N_2}{2}, 0)$: On the other extreme, when $M = 2 + \frac{N_2}{2}$, the caches are large enough to each store all the popular files, and complementary halves of the unpopular files. Thus, a user can recover any popular file by solely accessing any cache, and can recover any unpopular file by accessing both caches. Therefore, no broadcast is needed, and thus $R = 0$ is achievable.

3) Point $(M, R) = (2, 1)$: In this case, each cache can hold up to two files. Our strategy is to dedicate this memory to the two popular files. Thus, each cache contains all the popular files, but no information about the unpopular files. Therefore, users 1 and 2 can recover their respective requested files without relying on any broadcast, whereas user 3 requires a full broadcast of his requested file. As a result, the rate $R = 1$ is achievable.

4) Point $(M, R) = (1, \frac{3}{2})$: Let us split each of the popular files into two parts of equal size: $W_1^1 = (W_{1a}^1, W_{1b}^1)$ and $W_2^1 = (W_{2a}^1, W_{2b}^1)$. We place these in the caches as follows. The first cache contains the first half of each file: $Z_1 = (W_{1a}^1, W_{2a}^1)$, whereas the second cache contains the second half of each file: $Z_2 = (W_{1b}^1, W_{2b}^1)$. When the user requests are revealed, the BS sends a common message with two parts: $X^r = (X_1^r, X_2^r)$. The first part directly serves user 3 by giving him the full file he requested: $X_1^r = W_{1a}^1$. The second part will be $X_2^r = W_{1b}^1 \oplus W_{2a}^1$, which, along with the AP cache content accessed by each of users 1 and 2, allows them to recover their respective file. The BS had to transmit one full file, plus a linear combination of two half files, and, as a result, the total broadcast rate is $R = \frac{3}{2}$.

5) Point $(M, R) = (\frac{1}{2}, 2)$: This case is slightly different in that it requires storing coded content in the caches. We again split the popular files into two halves as before. However, we this time store the following: $Z_1 = W_{1a}^1 \oplus W_{2a}^1$; and $Z_2 = W_{1b}^1 \oplus W_{2b}^1$. The BS will transmit the following broadcast: $X^r = (W_{1a}^1, W_{1b}^1, W_{2a}^1)$. This will serve all the requests of the users. The broadcasts message consists of a full file and two half-files, and thus the total rate is $R = 2$.

B. Outer bounds

We will now prove that the above scheme is optimal with respect to information-theoretic bounds. We do that by showing that the rate is larger than each one of the expressions in the maximization in (42).

In all of the following, inequalities marked by $(\ast)$ are due to Fano’s inequality, and the $\varepsilon_F$ term that arises along with these inequalities is a term that decays to zero as $F \to \infty$.  

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1) First expression: Let the request vector be \( r = (1, 2, 1) \), and consider both caches along with the broadcast \( X^r \). Then, for all \( F \),

\[
RF + 2MF \geq H(Z_1, Z_2, X^r)
\]

\[
= H\left(Z_1, Z_2, X^{(1,2,1)}|W_1^1, W_2^1, W_2^2\right) + I\left(W_1^1, W_2^1, W_2^2; Z_1, Z_2, X^{(1,2,1)}\right)
\]

\[
\geq 3F(1 - \varepsilon_F)
\]

\[
R + 2M \geq 3(1 - \varepsilon_F)
\]

By taking \( F \to \infty \), we get:

\[
R \geq 3 - 2M.
\]

2) Second expression: Consider the two request vectors \( r_1 = (1, 2, 1) \) and \( r_2 = (2, 1, 2) \).

\[
2(RF + MF) \geq H(Z_1, X^{r_1}) + H(Z_2, X^{r_2})
\]

\[
= H\left(Z_1, X^{r_1}|W_1^1\right) + I\left(W_1^1; Z_1, X^{r_1}\right) + H\left(Z_2, X^{r_2}|W_1^2\right) + I\left(W_1^2; Z_2, X^{r_2}\right)
\]

\[
\geq H\left(Z_1, Z_2, X^{r_1}, X^{r_2}|W_1^1, W_1^2\right) + 2F(1 - \varepsilon_F)
\]

\[
= H\left(Z_1, Z_2, X^{r_1}, X^{r_2}|W_1^1, W_1^2, W_2^1, W_2^2\right) + I\left(W_1^1, W_2^1, W_1^2, W_2^2; Z_1, Z_2, X^{r_1}, X^{r_2}|W_1^1\right) + 2F(1 - \varepsilon_F)
\]

\[
\geq 5F(1 - \varepsilon_F)
\]

\[
2R + 2M \geq 5(1 - \varepsilon_F)
\]

By taking \( F \to \infty \), we get:

\[
R \geq \frac{5}{2} - M.
\]

3) Third expression: Consider the following four request vectors: \( r_1 = (1, 2, 1) \), \( r_2 = (2, 1, 2) \), \( r_3 = (1, 2, 3) \), and \( r_1 = (2, 1, 4) \).

\[
2(2RF + MF) \geq H(Z_1, X^{r_1}, X^{r_2}) + H(Z_2, X^{r_3}, X^{r_4})
\]

\[
= H\left(Z_1, X^{r_1}, X^{r_2}|W_1^1, W_1^2\right) + I\left(W_1^1, X^{r_1}, X^{r_2}\right)
\]

\[
+ H\left(Z_2, X^{r_3}, X^{r_4}|W_1^1, W_1^2\right) + I\left(W_1^1, Z_2, X^{r_3}, X^{r_4}\right)
\]

\[
\geq H\left(Z_1, Z_2, X^{r_1}, X^{r_2}, X^{r_3}, X^{r_4}|W_1^1, W_1^2\right) + 2F(1 - \varepsilon_F)
\]

\[
= H\left(Z_1, Z_2, X^{r_1}, X^{r_2}, X^{r_3}, X^{r_4}|W_1^1, W_1^2, W_2^1, W_2^2, W_3^2, W_4^2\right)
\]

\[
+ I\left(W_1^1, W_2^1, W_2^2, Z_1, Z_2, X^{r_1}, X^{r_2}, X^{r_3}, X^{r_4}|W_1^1, W_1^2\right) + 4F(1 - \varepsilon_F)
\]

\[
\geq 8F(1 - \varepsilon_F)
\]

\[
4R + 2M \geq 8(1 - \varepsilon_F)
\]

By taking \( F \to \infty \), we get:

\[
R \geq 2 - \frac{1}{2}M.
\]

4) Fourth expression: Consider the \( N_2 \) request vectors \( r_1, \ldots, r_{N_2} \), such that:

\[
r_k = (1, 2, k) \quad \forall k \leq \left\lfloor \frac{N_2}{2} \right\rfloor,
\]

\[
r_k = (2, 1, k) \quad \forall k \geq \left\lfloor \frac{N_2}{2} \right\rfloor + 1.
\]
Define $\mathcal{X}^e = \{X^r_k : k \text{ is even}\}$ and $\mathcal{X}^o = \{X^r_k : k \text{ is odd}\}$. Thus, each of $\mathcal{X}^e$ and $\mathcal{X}^o$ contains at least one broadcast $X^r_k$ such that $r_k = (1, 2, k)$, and one broadcast $X^r_l$ such that $r_l = (2, 1, l)$. Then,

\[
N_2RF + 2MF \geq H(Z_1, X^e) + H(Z_2, X^o) = H(Z_1, X^e | W^1_1, W^1_2) + I(W^1_1, W^1_2; Z_1, X^e) + H(Z_2, X^o | W^1_1, W^1_2) + I(W^1_1, W^1_2; Z_2, X^o)
\]

\[
\geq H(Z_1, Z_2, X^e, X^o | W^1_1, W^1_2) + 4F(1 - \varepsilon_F)
\]

\[
= H(Z_1, Z_2, X^{r_1}, \ldots, X^{r_{N_2}} | W^1_1, W^1_2, W^2_1, \ldots, W^2_{N_2}) + I(W^2_1, \ldots, W^2_{N_2}; Z_1, Z_2, X^{r_1}, \ldots, X^{r_{N_2}} | W^1_1, W^1_2) + 4F(1 - \varepsilon_F)
\]

\[
\geq (4 + N_2)F(1 - \varepsilon_F)
\]

By taking $F \to \infty$, we get:

\[
R \geq \frac{4 + N_2 - 2M}{N_2} = 1 - \frac{M - 2}{N_2/2}.
\]