Stability and global attractors for thermoelastic Bresse system

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In this article, we consider the stability properties for thermoelastic Bresse system which describes the motion of a linear planar, shearable thermoelastic beam. The system consists of three wave equations and two heat equations coupled in certain pattern. The two wave equations about the longitudinal displacement and shear angle displacement are effectively damped by the dissipation from the two heat equations. We use the multiplier techniques to prove the exponential stability result when the wave speed of the vertical displacement coincides with the wave speed of longitudinal or of the shear angle displacement. Moreover, the existence of the global attractor is first achieved.

Keywords: exponential decay; Bresse beam; thermoelasticity; attractor

AMS Subject Classifications: 34D45; 35B40; 74K10

1. Introduction

In this article, we will consider the following system:

\( \rho w_{1tt} = \left( E h(w_{1x} - kw_3) - \alpha \theta_1 \right)_x - kGh(\phi_2 + w_{3x} + kw_1), \)  
(1.1)

\( \rho w_{3tt} = Gh(\phi_2 + w_{3x} + kw_1)_x + kEh(w_{1x} - kw_3) - k\alpha \theta_1, \)  
(1.2)

\( \rho I \phi_{2tt} = EI\phi_{2xx} - G h(\phi_2 + w_{3x} + kw_1) - \alpha \theta_{3xx}, \)  
(1.3)

\( \rho c \theta_{1tt} = \theta_{1xx} - \alpha T_0(w_{1x} - kw_3), \)  
(1.4)

\( \rho c \theta_{3tt} = \theta_{3xx} - \alpha T_0 \phi_{2xx}, \)  
(1.5)

together with initial conditions

\( w_1(x, 0) = u_0(x), \quad w_{1t}(x, 0) = v_0(x), \quad \phi_2(x, 0) = \phi_0(x), \)  
\( \phi_{2t}(x, 0) = \psi_0(x), \quad w_3(x, 0) = w_0(x), \quad w_{3t}(x, 0) = \varphi_0(x), \)  
\( \theta_1(x, 0) = \theta_0(x), \quad \theta_2(x, 0) = \xi_0(x), \quad \theta_{3t}(x, 0) = \eta_0(x) \)  
(1.6)

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and boundary conditions
\[ w_1(x, t) = w_3(x, t) = \phi_2(x, t) = \theta_1(x, t) = \theta_3(x, t) = 0 \quad \text{for } x = 0, 1, \quad (1.7) \]
where \( w_1, w_3, \phi_2 \) are the longitudinal, vertical and shear angle displacement, \( \theta_1, \theta_3 \) are the temperature deviations from the \( T_0 \) along the longitudinal and vertical directions, \( E, \ G, \ \rho, \ I, \ m, \ k, \ h, \ c \) are positive constants for the elastic and thermal material properties.

From this seemingly complicated system, very interesting special cases can be obtained. In particular, the isothermal system is exactly the system obtained by Bresse [1] in 1856. The Bresse system, Equations (1.1)–(1.3) with \( \theta_1, \theta_3 \) removed, is more general than the well-known Timoshenko system where the longitudinal displacement \( w_1 \) is not considered. If both \( \theta_1 \) and \( w_1 \) are neglected, the Bresse thermoelastic system simplifies to the following Timoshenko thermoelastic system:

\[
\begin{align*}
\rho hw_{3tt} &= Gh(\phi_2 + w_3)_x, \\
\rho I \phi_{2tt} &= E I \phi_{2xx} - Gh(\phi_2 + w_3) - \alpha \theta_{3tt}, \\
\rho c \theta_{3tt} &= \theta_{3xx} + \theta_{3xx} - \alpha T_0 \phi_{2xx},
\end{align*}
\]

which was studied by Messaoudi and Sid-Houari [2]. For the boundary conditions:
\[ w_3(x, t) = \phi_2(x, t) = \theta_3(x, t) = 0, \quad x = 0, 1, \quad (1.11) \]
they obtained exponential stability for the thermoelastic Timoshenko system (1.8)–(1.10) when \( E = G \). We refer the reader to [3–9] for the Timoshenko system with other kinds of damping mechanisms such as viscous damping, viscoelastic damping of Boltzmann type acting on the motion equation of \( w_3 \) or \( \phi_2 \). Recently, Liu and Rao [10] considered a similar system; they used semigroup method to show that the exponentially decay rate is preserved when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, only a polynomial type decay rate can be obtained; their main tools are the frequency domain characterization of exponential decay obtained by Prüss [11] and Huang [12], and of polynomial decay obtained recently by Liu and Rao [13]. For the attractors, we refer to [14–19].

In the present work, we consider system (1.1)–(1.5), That is, we use the multiplier techniques to prove the exponential stability result only for an equal wave speed case.

2. Equal wave speeds case: \( E = G \)

Here we state and prove a decay result in the case of equal wave-speeds propagation.

Define the state spaces
\[ \mathcal{H} = H_0^1 \times H_*^1 \times H_0^1 \times H_0^1 \times (L^2)^5, \]
where
\[ H_*^1 = \left\{ f \in H^1(0, 1) \mid \int_0^1 f(x) = 0 \right\}. \]
The associated energy term is given by
\[
E(t) = \frac{1}{2} \int_0^1 \left( [ Eh(w_{1x} - kw_3)^2 + Gh(\phi_2 + w_{3x} + kw_1)^2 + E\phi_{2x}^2 ] + [ \rho h(w_{1x}^2 + w_{3x}^2) + \rho I\phi_{2x}^2 ] + \frac{\rho c}{T_0} (\theta_1^2 + \theta_2^2 + \theta_3^2) \right) dx. \tag{2.1}
\]

By a straightforward calculation, we have
\[
\frac{dE(t)}{dt} = -\frac{1}{T_0} \left( \|\theta_1\|^2 + \|\theta_3\|^2 \right). \tag{2.2}
\]

From the semigroup theory [6,7], we have the following existence and regularity result, for the explicit proofs, we refer the reader to [12].

**Lemma 2.1** Let \( u_0(x), w_0(x), \phi_0(x), \xi_0(x), \psi_0(x), \theta_0(x), \eta_0(x) \in \mathcal{H} \) be given, then problem (1.1)–(1.5) has a unique global weak solution \((u, \psi, \theta)\) verifying the
\[
w_3(x, t) \in C(R^+, H^1(0, 1)) \cap C^1(R^+, L^2(0, 1)), \quad (w_1(x, t), \phi_2(x, t), \theta_1(x, t), \theta_3(x, t)) \in C(R^+, H^1(0, 1)) \cap C^1(R^+, L^2(0, 1)).
\]

We are now ready to state our main stability result.

**Theorem 2.1** Suppose that \( E = G \) and \( u_0(x), w_0(x), \phi_0(x), \xi_0(x), \psi_0(x), \theta_0(x), \eta_0(x) \in \mathcal{H} \), then the energy \( E(t) \) decays exponentially as time tends to infinity; that is, there exist two positive constants \( \epsilon \) and \( \mu \) independent of the initial data and \( t \), such that
\[
E(t) \leq CE(0)e^{-\mu t}, \quad t > 0. \tag{2.3}
\]

The proof of this result will be established through several lemmas.

**Lemma 2.2** Let \( w_1, w_3, \phi_2, \theta_1, \theta_3 \) be a solution of (1.1)–(1.5), then we have, \( \forall \epsilon > 0, \)
\[
\frac{dI_1(t)}{dt} \leq -\frac{E}{2} \left( \|\phi_{2x}\|^2 + \rho I \|\phi_{2x}\|^2 + \epsilon_1 \left( \|w_{3x}\|^2 + \| (w_{1x} - kw_3) \|^2 \right) \right) + C(\epsilon_1) \left( \|\theta_{3x}\|^2 + \|\theta_{1x}\|^2 + \|\phi_2\|^2 \right). \tag{2.6}
\]

**Proof**
\[
\frac{dI_1}{dt} = -EI\|\phi_{2x}\|^2 + \rho I \|\phi_{2x}\|^2 - \int_0^1 \sigma_{3x} \phi_2 \, dx \\
- kEh \int_0^1 (w_{1x} - kw_3) f \, dx - k\sigma \int_0^1 \theta_1 f \, dx + \rho \int_0^1 w_{3x} f \, dx. \tag{2.7}
\]
By using the inequalities
\[
\int_0^1 f_x^2 \, dx \leq \int_0^1 \phi_x^2 \, dx \leq \int_0^1 \phi_x^2 \, dx,
\]
\[
\int_0^1 f_t^2 \, dx \leq \int_0^1 f_{tx}^2 \, dx \leq \int_0^1 \phi_t^2 \, dx
\]
and Young’s inequality, the assertion of the lemma follows.

Let
\[
I_2 = \rho c \phi h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) w_{1t} \, dx.
\]

**Lemma 2.3** Let \( w_1, w_3, \phi_2, \theta_1, \theta_3 \) be a solution of (1.1)–(1.5), then we have, \( \forall \epsilon_2 > 0, \)
\[
\frac{dI_2(t)}{dt} \leq -\frac{\alpha \rho T_0}{2} \int_0^1 w_{1t}^2 \, dx + C(\epsilon_2) \left( \|\theta_{1x}\|^2 + \|w_{3t}\|^2 \right)
+ \epsilon_2 \left( \|w_{1x} - \epsilon_3 w_3\|^2 + \|\phi_2 + w_{3x} + k w_1\|^2 \right).
\]

**Proof** Using Equations (1.4) and (1.1), we get
\[
\frac{dI_2(t)}{dt} = \rho c \phi h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) w_{1t} \, dx + \rho c \phi h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) w_{1t} \, dx
\]
\[
= \rho h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) \left( w_{1t} - \epsilon_3 (w_{1x} - \epsilon_3 w_3) \right) w_{1t} \, dx
\]
\[
+ \rho h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) \left( (Eh w_{1x} - \epsilon_3 w_3) - \alpha \theta_{1x} \right) - KGH(\phi_2 + w_{3x} + k w_1) \right) \, dx
\]
\[
= \rho h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) w_{1t} \, dx - \rho h \alpha T_0 \int_0^1 w_{1t}^2 \, dx + \rho h \int_0^1 \left( \int_0^x \theta_3 \, dy \right) w_{1t} \, dx
\]
\[
+ \rho h Eh \int_0^1 \left( \int_0^x \theta_1, w_{1t} + k \theta_1 w_3 + \alpha \theta_1^2 \right) \, dx
\]
\[
- \rho c \phi h \int_0^1 \left( \int_0^x \theta_1 \, dy \right) (\phi_2 + w_{3x} + k w_1) \, dx.
\]
The assertion of the Lemma then follows, using Young’s and Poincaré’s inequalities.

Let
\[
I_3 = \rho c \phi l \int_0^1 \left( \int_0^x \theta_3 \, dy \right) \phi_2 \, dx.
\]

**Lemma 2.4** Let \( w_1, w_3, \phi_2, \theta_1, \theta_3 \) be a solution of (1.1)–(1.5), then we have, \( \forall \epsilon_3 > 0, \)
\[
\frac{dI_3}{dt} \leq -\frac{\alpha \rho IT_0}{2} \|\phi_2\|^2 + C(\epsilon_3) \|\theta_{3x}\|^2 + \|\theta_{3x}\|^2
+ \epsilon_3 \|\phi_{2x}\|^2 + \epsilon_3 \|\phi_2 + w_{3x} + k w_1\|^2.
\]
Using Equations (1.3) and (1.5), we have

\[
\frac{dI_3}{dt} = \rho c \rho I \int_0^1 \left( \int_0^x \theta_{3_{tt}} \, dy \right) \phi_{2_t} \, dx + \rho c \rho I \int_0^1 \left( \int_0^x \theta_{3_t} \, dy \right) \phi_{2_{tt}} \, dx
\]

\[
= \rho I \int_0^1 \left( \int_0^x \left( \theta_{3_{xxt}} + \theta_{3_{xx}} - \alpha T_0 \theta_{2x_t} \right) \, dy \right) \phi_{2_t} \, dx
\]

\[
+ \rho c \int_0^1 \left( \int_0^x \theta_{3_t} \, dy \right) \left( EI \theta_{2x_{xx}} - Gh \left( \phi_2 + w_{3x} + \kappa w_1 \right) - \alpha \theta_{3_x} \right) \, dx
\]

\[
= \rho I \int_0^1 \left( \theta_{3_{xx}} \phi_{2_t} + \theta_{3_x} \phi_{2_{t}} \right) \, dx + \int_0^1 \phi_{2_t}^2 \, dx + \rho c EI \int_0^1 \theta_{3x} \phi_{2x} \, dx
\]

Then using Young’s and Poincaré’s inequalities, we can obtain the assertion.

Next, we set

\[
I_4 = h \rho I \int_0^1 \phi_{2x} (\phi_2 + w_{3x} + \kappa w_1) \, dx + h \rho I \int_0^1 \phi_{2x} w_{3t} \, dx.
\]

**Lemma 2.5** Let \( w_1, w_3, \phi_2, \theta_1, \theta_3 \) be a solution of (1.1)–(1.5), then we have, \( \forall \varepsilon_4 > 0, \)

\[
\frac{dI_4}{dt} \leq - \frac{Gh^2}{2} \int_0^1 \left( \phi_2 + w_{3x} + \kappa w_1 \right)^2 \, dx + C(\varepsilon_4) \left( \| \theta_{3_{xx}} \|^2 + \| \theta_{1x} \|^2 \right)
\]

\[+ \frac{1}{2} h \rho I \left( \| \phi_{2x} \|^2 + \| w_{3t} \|^2 \right) + C(\varepsilon_4) \| \phi_{2x} \|^2 + \varepsilon_4 \| w_{1x} - \kappa w_3 \|^2. \tag{2.13}
\]

**Proof.** Let (1) = \( I \int_0^1 \phi_{2t} (\phi_2 + w_{3x} + \kappa w_1) \, dx, \) (2) = \( h \rho I \int_0^1 \phi_{2x} w_{3t} \, dx, \) then using Equations (1.3), (1.5), we have

\[
(1)’ = \rho I \int_0^1 \phi_{2tt} (\phi_2 + w_{3x} + \kappa w_1) \, dx + h \rho I \int_0^1 \phi_{2t} (\phi_2 + w_{3x} + \kappa w_1) \, dx
\]

\[= h EI \int_0^1 \phi_{2xx} (\phi_2 + w_{3x} + \kappa w_1) \, dx - \frac{Gh^2}{2} \int_0^1 (\phi_2 + w_{3x} + \kappa w_1)^2 \, dx
\]

\[+ \alpha h \int_0^1 \theta_{3xx} (\phi_2 + w_{3x} + \kappa w_1) \, dx + h \rho I \int_0^1 \phi_{2t}^2 \, dx + h \rho I \int_0^1 \phi_{2t} (w_{3x} + \kappa w_1) \, dx,
\]

\[
(2)’ = I \rho h \int_0^1 \phi_{2xt} w_{3t} \, dx + I \rho h \int_0^1 \phi_{2x} w_{3tt} \, dx
\]

\[= - I \rho h \int_0^1 \phi_{2t} w_{3x} \, dx + I Gh \int_0^1 \phi_{2x} (\phi_2 + w_{3x} + \kappa w_1) \, dx
\]

\[+ I k Eh \int_0^1 \phi_{2x} (w_{1x} - \kappa w_3) \, dx - \alpha I k \int_0^1 \phi_{2x} \theta_1 \, dx.
\]
Notice that $E = G$, then

$$I_4 = (1') + (2')$$

$$= -Gh^2 \int_0^1 (\phi_2 + w_{3x} + k w_1)^2 \, dx - \alpha h \int_0^1 \theta_{3x}(\phi_2 + w_{3x} + k w_1) \, dx + h \rho I \int_0^1 \phi_2^2 \, dx$$

$$+ kh I \rho \int_0^1 \phi_2 w_1 \, dx + Ik E h \int_0^1 \phi_{2x}(w_{1x} - k w_3) \, dx - \alpha k \int_0^1 \phi_{2x} \theta_1 \, dx,$$

then use Young’s inequality, we can obtain the assertion.

We set

$$I_5 = -h \rho \int_0^1 w_{3x}(w_{1x} - k w_3) \, dx - h \rho \int_0^1 w_1(\phi_2 + w_{3x} + k w_1) \, dx. \tag{2.14}$$

**Lemma 2.6** Let $w_1, w_3, \phi_2, \theta_1, \theta_3$ be a solution of (1.1)–(1.5), then we have, $\forall \varepsilon > 0$,

$$\frac{dI_5}{dt} \leq - \frac{kh}{2} \|(w_{1x} - k w_3)\|^2 - \frac{\rho}{2} \|w_{1x}\|^2 + k \rho \|w_{3x}\|^2$$

$$+ \frac{\rho}{2} \|\phi_{2x}\|^2 + C(\varepsilon) \|\theta_{1x}\|^2 + (kGh + \varepsilon) \|(\phi_2 + w_{3x} + k w_1)\|^2. \tag{2.15}$$

**Proof** Let 

$$(1') = -h \rho \int_0^1 w_{3x}(w_{1x} - k w_3) \, dx, \quad (2') = \alpha h \int_0^1 \theta_{3x}(\phi_2 + w_{3x} + k w_1) \, dx,$$

then using Equations (1.1), (1.2), we have

$$(1') = -Gh \int_0^1 (\phi_2 + w_{3x} + k w_1)(w_{1x} - k w_3) \, dx - kh \int_0^1 (w_{1x} - k w_3)^2 \, dx$$

$$+ \alpha h \int_0^1 \theta_{3x}(w_{1x} - k w_3) \, dx + k \rho \int_0^1 w_{3x}^2 \, dx - \rho \int_0^1 w_{3x} w_{1x} \, dx,$$

$$(2') = -kh \int_0^1 \phi_2 w_1 \, dx + kh \int_0^1 \phi_{2x}(w_{1x} - k w_3) \, dx - \alpha k \int_0^1 \phi_{2x} \theta_1 \, dx$$

$$+ \rho \int_0^1 w_{1x} w_{3x} \, dx.$$

Then, note that $E = G$ again, from the above two equalities and Young’s inequality, we can obtain the assertion.

Next, we set

$$I_6 = -\rho h \int_0^1 w_3 w_3 \, dx - \rho h \int_0^1 w_{1x} w_1 \, dx. \tag{2.16}$$

**Lemma 2.7** Let $w_1, w_3, \phi_2, \theta_1, \theta_3$ be a solution of (1.1)–(1.5), then we have,

$$\frac{dI_6}{dt} \leq - \rho h \left( \|w_3\|^2 + \|w_1\|^2 \right) + C \|\theta_{1x}\|^2 + C \|\phi_{2x}\|^2. \tag{2.17}$$
Proof Using Equations (1.1), (1.2), we have
\[
I_6 = -\rho h \int_0^1 w_3^2 \, dx - \rho h \int_0^1 w_1^2 \, dx + Eh \int_0^1 (w_{1x} - kw_3)^2 \, dx \\
+ Gh \int_0^1 (\phi_2 + w_{3x} + kw_1)(w_{3x} + kw_1) \, dx - \alpha \int_0^1 \theta_1(w_{1x} - kw_3) \, dx.
\] (2.18)
From (2.1) and (2.2), we have that, \(\exists C > 0\), satisfy
\[
-\alpha \int_0^1 \theta_1(w_{1x} - kw_3) \, dx \leq C\|\theta_{1x}\|^2 - Eh\|w_{1x} - kw_3\|^2,
\] (2.19)
similarly,
\[
Gh \int_0^1 (\phi_2 + w_{3x} + kw_1)(w_{3x} + kw_1) \, dx \\
= Gh\|\phi_2 + w_{3x} + kw_1\|^2 - Gh \int_0^1 (\phi_2 + w_{3x} + kw_1) \phi_2 \, dx \\
\leq C\|\phi_{2x}\|^2,
\] (2.20)
then , insert (2.19) and (2.20) into (2.18), the assertion of the lemma follows.
Now, we set
\[
I_7 = \alpha \int_0^1 \theta_2 \, dx + \frac{1}{2}\|\theta_3\|^2 + \alpha T_0 \int_0^1 \phi_2 \theta_3 \, dx.
\] (2.21)
Lemma 2.8 Let \(w, w_3, w_{3x}, \theta_1, \ldots\) be a solution of (1.1)–(1.5), then we have, \(\forall \varepsilon_7 > 0\),
\[
\frac{d}{dt} \|\theta_{3x}\|^2 \leq -\|\theta_{3x}\|^2 + C(\varepsilon_7)\|\theta_{3x}\|^2 + \varepsilon_7\|\phi_{2x}\|^2.
\] (2.22)
Proof Using Equations (1.5), we have
\[
\frac{dI_7}{dt} = -\|\theta_{3x}\|^2 - \alpha T_0 \int_0^1 \phi_2 \theta_{3x} \, dx.
\]
Then using Young’s and Poincaré’s inequalities, we can obtain the assertion.
Now, let \(N, N_1, N_2, N_3, N_4, N_5, N_6, N_7 > 0\), we define the Lyapunov functional \(\mathcal{F}\) as follows
\[
\mathcal{F} = NE + N_1 I_1 + N_2 I_2 + N_3 I_3 + N_4 I_4 + N_5 I_5 + N_6 I_6 + N_7 I_7.
\] (2.23)
By using (2.2), (2.6), (2.9), (2.11), (2.13), (2.15), (2.17), (2.22), we have
\[
\frac{d\mathcal{F}}{dt} \leq \gamma_1\|\theta_{1x}\|^2 + \gamma_2\|\theta_{3x}\|^2 + \gamma_3\|\phi_{2x}\|^2 + \gamma_4\|w_{1x}\|^2 + \gamma_5\|\phi_{2x}\|^2 + \gamma_6\|\phi_2 + w_{3x} + kw_1\|^2 + \gamma_7\|w_{1x} - kw_3\|^2 + \gamma_8\|w_{3x}\|^2 + \gamma_9\|\theta_{3x}\|^2,
\] (2.24)
where

\[ \begin{align*}
\Upsilon_1 &= -\frac{N}{T_0} + C(\varepsilon_1)N_1 + N_2C(\varepsilon_2) + N_4C(\varepsilon_4) + N_5C(\varepsilon_5) + C, \\
\Upsilon_2 &= -\frac{N}{T_0} + C(\varepsilon_1)N_1 + N_3C(\varepsilon_3) + N_4C(\varepsilon_4) + N_7C(\varepsilon_7), \\
\Upsilon_3 &= -\frac{N_1 EI}{2} + \varepsilon_3 N_3 + C(\varepsilon_4)N_4 + CN_6, \\
\Upsilon_4 &= -\frac{\alpha_0 N_2}{2} + \frac{k h N_4}{2} - \frac{\rho h N_5}{2} - \rho h N_6, \\
\Upsilon_5 &= -\frac{\alpha_0 N_3}{2} + \frac{k h N_4}{2} + N_1 \rho I + N_1 C(\varepsilon_1) + \frac{\rho h N_5}{2}, \\
\Upsilon_6 &= -\frac{G h^2 N_4}{2} + k G h N_5 + N_5 \varepsilon_5 + N_3 \varepsilon_3 + N_2 \varepsilon_2, \\
\Upsilon_7 &= -\frac{k E h N_5}{2} + N_4 \varepsilon_4 + N_1 \varepsilon_1 + N_2 \varepsilon_2, \\
\Upsilon_8 &= -N_6 \rho h + N_5 k \rho h + C(\varepsilon_2)N_2 + N_1 \varepsilon_1, \\
\Upsilon_9 &= -N_7 + N_5 C(\varepsilon_3).
\end{align*} \]

At this point, we chose our constants very carefully and properly so that the existing \( \omega > 0 \), (2.24) takes the form

\[ \frac{d\mathcal{F}}{dt} \leq -\omega \left( \|\phi_{1x}\|^2 + \|\phi_{2x}\|^2 + \|\theta_{1x}\|^2 + \|\phi_{2}\|^2 + \|\phi_{3}\|^2 + \|w_1\|^2 \right) + \|\phi_{2}\|^2 + \|\phi_{3}\|^2 + \|w_{1x} - k w_3\|^2 + \|w_{3x}\|^2, \quad (2.25) \]

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1** First, from the definition of \( \mathcal{F} \), we have

\[ \mathcal{F} \sim E(t), \quad (2.26) \]

Then from (2.25) and (2.26), it leads to

\[ \frac{d\mathcal{F}}{dt} \leq -\mu \mathcal{F}, \quad (2.27) \]

integrating (2.27) over \((0, t)\), and use of (2.26) leads to (2.3), this completes the proof of Theorem 2.1.

### 3. Global attractors

In this section, we establish the existence of the global attractor for system (1.1)–(1.5).

Setting \( v = w_{1t}, \ \varphi = w_{3t}, \ \psi = \phi_{2t}, \ \eta = \theta_{3t} \). Then, Equations (1.1)–(1.5) can be transformed into the system

\[ w_{1t} = v, \quad (3.1) \]
We consider the problem in the following Hilbert space

\[ \mathcal{H} = H^1_0 \times H^1_0 \times H^1_0 \times (L^2)^5 \]

Recall that the global attractor of \( S(t) \) acting on \( \mathcal{H} \) is a compact set \( \mathcal{A} \) enjoying the following properties:

1. \( \mathcal{A} \) is fully invariant for \( S(t) \), that is, \( S(t)\mathcal{A} = \mathcal{A} \) for every \( t \geq 0 \).
2. \( \mathcal{A} \) is an attracting set, namely, for any bounded set \( \mathcal{R} \subset \mathcal{H} \),

\[
\lim_{t \to \infty} \delta_\mathcal{H}(S(t)\mathcal{R}, \mathcal{A}) = 0,
\]

where \( \delta_\mathcal{H} \) denotes the Hausdorff semi-distance on \( \mathcal{H} \).

More details on the subject can be found in the books [17,20,21].

**Remark 3.1** The uniform energy estimate (2.3) implies the existence of a bounded absorbing set \( \mathcal{R}^* \subset \mathcal{H} \) for the \( C_0 \) semigroup \( S(t) \). Indeed, if \( \mathcal{R}^* \) is any ball of \( \mathcal{H} \), then for any bounded set \( \mathcal{R} \subset \mathcal{H} \) it is immediate to see that there exists \( t(\mathcal{R}) \geq 0 \) such that

\[ S(t)\mathcal{R} \subset \mathcal{R}^* \]

for every \( t \geq t(\mathcal{R}) \).

Moreover, if we define

\[ \mathcal{R}_0 = \bigcup_{t \geq 0} S(t)\mathcal{R}^*, \]

it is clear that \( \mathcal{R}_0 \) is still a bounded absorbing set which is also invariant for \( S(t) \), that is, \( S(t)\mathcal{R}_0 \subset \mathcal{R}_0 \) for every \( t \geq 0 \).

In the sequel, we define the operator \( A \) as \( Af = -f_{xx} \) with Dirichlet boundary conditions. It is well known that \( A \) is a positive operator on \( L^2 \) with domain \( D(A) = H^2 \cap H^1_0 \). Moreover, we can define the powers \( A^s \) of \( A \) for \( s \in \mathbb{R} \). The space \( V_{2s} = D(A^s) \) turns out to be a Hilbert space with the inner product

\[ \langle u, v \rangle_{V_{2s}} = \langle A^s u, A^s v \rangle, \]

where \( \langle \cdot \rangle \) stands for \( L^2 \)-inner product on \( L^2 \).
In particular, $V_{-1} = H^{-1}$, $V_{0} = L^{2}$, $V_{1} = H_{0}^{1}$. The injection $V_{s_{1}} \hookrightarrow V_{s_{2}}$ is compact whenever $s_{1} > s_{2}$. For further convenience, for $s \in R$, introduce the Hilbert space

$$H_{s} = V_{1+s} \times V_{1+s} \times V_{1+s} \times V_{1+s} \times (V_{s})^{5}.$$ 

Clearly, $H_{0} = H$.

Now, let $z_{0} = (u_{0}, w_{0}, \phi_{0}, \xi_{0}, v_{0}, \psi_{0}, \theta_{0}, \eta_{0})$, where $R_{0}$ is the invariant, bounded absorbing set of $S(t)$ given by Remark 3.1, take the inner product in $H_{0}$ of (3.1)–(3.9) and $(A^\alpha w_{1}, A^\alpha w_{3}, A^\alpha \phi_{2}, A^\alpha \theta_{3}, A^\alpha \psi, A^\alpha \psi, A^\alpha \theta_{1}, A^\alpha \eta)$ to get

$$\frac{d}{dt} \left( Eh \left\| (w_{1x} - kw_{3}) \right\|_{\sigma}^{2} + Gh \left\| (\phi_{2} + w_{3x} + kw_{1}) \right\|_{\sigma}^{2} + EI \left\| \phi_{2} \right\|_{1+\sigma}^{2} \right. \\
+ \left. \rho h \left( \left\| w_{1} \right\|_{\sigma}^{2} + \left\| w_{3} \right\|_{\sigma}^{2} \right) + \rho I \left\| \phi_{2} \right\|_{\sigma}^{2} + \frac{\rho c}{T_{0}} \left( \left\| \theta_{1} \right\|_{\sigma}^{2} + \left\| \theta_{3} \right\|_{\sigma}^{2} + \left\| \theta_{3} \right\|_{1+\sigma}^{2} \right) \right) \\
= - \frac{2}{T_{0}} \left( \left\| \theta_{1} \right\|_{1+\sigma}^{2} + \left\| \theta_{3} \right\|_{1+\sigma}^{2} \right). \quad (3.10)$$

Here, the boundary term of integration by parts is neglected since we are working with more regular functions. We denote

$$E_{2}(t) = Eh \left\| (w_{1x} - kw_{3}) \right\|_{\sigma}^{2} + Gh \left\| (\phi_{2} + w_{3x} + kw_{1}) \right\|_{\sigma}^{2} + EI \left\| \phi_{2} \right\|_{1+\sigma}^{2} \right. \\
+ \rho h \left( \left\| w_{1} \right\|_{\sigma}^{2} + \left\| w_{3} \right\|_{\sigma}^{2} \right) + \rho I \left\| \phi_{2} \right\|_{\sigma}^{2} + \frac{\rho c}{T_{0}} \left( \left\| \theta_{1} \right\|_{\sigma}^{2} + \left\| \theta_{3} \right\|_{\sigma}^{2} + \left\| \theta_{3} \right\|_{1+\sigma}^{2} \right).$$

Then, introduce the functional

$$J(t) = NE_{2}(t) + N_{1}F_{1} + N_{2}F_{2} + N_{3}F_{3} + N_{4}F_{4} + N_{5}F_{5} + N_{6}F_{6} + N_{7}F_{7} \quad (3.11)$$

by repeating a similar argument as in the proofs of Lemma 2.2–2.8 and (3.10), choosing our constants very carefully and properly, we get

$$\frac{d}{dt} J(t) + c E_{2}(t) \leq 0.$$
On the other hand, 
\[ J(t) \sim E_2(t), \]
so that
\[ \frac{d}{dt} J(t) + c_1 J(t) \leq 0, \]
which gives
\[ E_2(t) \sim J(t) \leq c_2 e^{-c_1 t}, \quad \|z(t)\|_{\mathcal{H}_u} \leq c_2 e^{-c_1 t}. \] (3.12)

Let \( \mathcal{R}(t) \) be the ball of \( V_{3/2} \times V_{3/2} \times V_{3/2} \times V_{3/2} \times (V_{1/2})^5 \), from the compact embedding \( V_{3/2} \times V_{3/2} \times V_{3/2} \times V_{3/2} \times (V_{1/2})^5 \hookrightarrow H_0^1 \times H_0^1 \times H_0^1 \times H_0^1 \times (L^2)^5 \), \( \mathcal{R}(t) \) is compact in \( \mathcal{H} \). Then, due to the compactness of \( \mathcal{R}(t) \), for every fixed \( t \geq 0 \) and every \( d > c_2 e^{-c_1 t} \), there exist finitely many balls of \( \mathcal{H} \) of radius \( d \) such that \( z(t) \) belongs to the union of such balls, for every \( z_0 \in \mathcal{R}_0 \). This implies that
\[ \alpha_{\mathcal{H}}(S(t)\mathcal{R}_0) \leq c_2 e^{-c_1 t} \quad \forall t \geq 0, \] (3.13)
where \( \alpha_{\mathcal{H}} \) is the Kuratowski measure of non-compactness, defined by
\[ \alpha_{\mathcal{H}}(\mathcal{R}) = \inf \{d : \mathcal{R} \text{ has a finite cover of balls of } \mathcal{H} \text{ of diameter less than } d \}. \]

Since the invariant, connected, bounded absorbing set \( \mathcal{R}_0 \) fulfills (3.13), exploiting a classical result of the theory of attractors of semigroups (see, e.g. [22]), we conclude that the \( \omega \)-limit set of \( \mathcal{R}_0 \), that is,
\[ \omega(\mathcal{R}_0) = \bigcup_{t \geq 0} S(t)\mathcal{R}_0, \]
is a connected and compact global attractor of \( S(t) \). Therefore we have proved the following result.

**Theorem 3.1.** Under assumption of \((H_1)\)–\((H_2)\), problem (3.1)–(3.9) possesses a unique global attractor \( A \).

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