Tooth Surface Modeling and Stress Analysis of 3-DOF Spherical Gear

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Abstract. A new type of spherical gear is proposed, which is formed by combining the tooth profile of the plane involute annular tooth surface and spherical involute spherical cone tooth surface. 3-DOF of pitch, deflection and rotation can be realized without principle error transmission. Based on the forming principle of plane involute and spherical involute, the equations of annular tooth surface and spherical cone tooth surface are derived by coordinate transformation theory; Combined with MATLAB and Pro/E, the 3D accurate model of 3-DOF spherical gear was obtained, and the contact stress of tooth surface was analyzed based on the finite element method. The simulation results show that the stress concentration always occurs at the tooth root and the tooth top of the spherical and conical tooth surface under the condition of constant spherical distance and the same torque but different coaxial intersection angle, which can provide reference for the tooth surface modification and optimization of spherical gear.

1. Introduction

Traditional gear has only one degree of freedom, which cannot meet the requirements of multi-drive degrees of freedom in emerging fields. The development of modern transmission field urgently needs the new type gear with multi-degree of freedom.

In the early 1980s, Ole. Monlang[1] invented a Trallfa flexible wrist based on spherical crown gear. However, its disadvantages include the inability to achieve accurate spherical motion with fixed transmission ratio, and its difficulty in processing and manufacturing and high cost. Shyue-cheng Yang et al[2] proposed to change the discrete conical teeth of Trallfa spherical gear into arc-shaped teeth, and transfer the motion through the engagement of arc-shaped convex teeth and arc-shaped concave teeth. Then they proposed a spherical gear with a discrete ring involute tooth profile[3]. Yang, Hsueh Cheng et al[4] designed a spherical gear pair with continuous involute tooth profile, which was shaped as convex and concave drum. Pan cunyun[5] proposed the involute ring spherical gear, which is formed by the rotation of the plane involute around the axis. This kind of continuous gear and tooth mechanism fundamentally overcomes the transmission error defect of discrete Trallfa gear and realizes continuous meshing transmission and is easy to manufacture.

In this paper, a new spherical gear structure with 3-DOF is proposed, which combines a planar involute ring tooth surface with a spherical involute cone tooth surface. Based on MATLAB and Pro/E, the precise 3d model of spherical gear was established. Besides, the Workbench was used for static analysis of spherical gear to obtain the contact stress of tooth surface, which provided basis and reference for the optimization design of spherical gear.
2. Analysis of transmission principle of spherical gear
As shown in figure 1, the meshing motion of the spherical gear can be regarded as pure rolling between two spheres, which always maintain tangential contact, and the transmission ratio is 1:1. The rotation of the spherical gear 1 around the $x_1$, $y_1$, and $z_1$ axis of the coordinate system with the center of the spherical as the origin and the axis can respectively realize the pitch, deflection and rotation motion of the gear pair. The weft of the pitch sphere is the pitch line of the partial pendulum meshing motion, and the longitude of the pitch sphere is the pitch line of the pitch meshing motion.

![Figure 1. Schematic diagram of spherical gears movement](image)

3. Mathematical model and 3D modeling of spherical gear

3.1 Single tooth surface formation of spherical gear
As shown in figure 2, a coordinate system $O-xyz$ is established at the center of the base sphere, and $O_1-x_1y_1z_1$ is established at a point on the upper surface. The annular tooth surface of the plane involute is formed by rotating the plane involute around the z axis. The starting point $Q_r$，$Q_l$ is on different weft lines of the base sphere, and the radius of the base circle $r_b$ is the radius of the base sphere. The tooth surface of spherical involute spherical cone is composed of a family of spherical involute curves with a single parameter. The starting point $Q$ is on the longitude line of the base sphere, the included angle between $oQ$ and z axis is $\beta$, and the apex of the base cone is always rejoined with the z axis. The planar involute annular tooth surface and spherical involute spherical cone tooth surface are combined to form a 3-DOF spherical gear single tooth surface.

![Figure 2. Schematic diagram of tooth formation of single tooth of spherical gear.](image)

![Figure 3. Involute distribution of middle convex tooth.](image)

3.2 Equation of annular tooth surface
Figure 3 is the section diagram of figure 2 in the $xoz$ plane. The point of the left involute rotates clockwise around the origin and the coordinates of $\beta$ are...
\[ \begin{bmatrix} x_w \\ z_w \end{bmatrix} = \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} r_b \sin(u + \beta) - r_b \cos(u + \beta) \\ r_b \cos(u + \beta) + r_b \sin(u + \beta) \end{bmatrix} \]

where \( r_b \) is the radius of the base circle, and \( u \) is the involute expansion angle.

The right involute is the point on the left involute that rotates \( \Theta_v \) clockwise around the origin, and its coordinates are

\[ \begin{bmatrix} x_v \\ z_v \end{bmatrix} = \begin{bmatrix} r_b \sin(u + (\beta + \Theta_v)) - r_b u \cos(u + (\beta + \Theta_v)) \\ r_b \cos(u + (\beta + \Theta_v)) + r_b u \sin(u + (\beta + \Theta_v)) \end{bmatrix} \]

where \( \Theta_v \) is the included angle between left and right involute symmetric points, \( \alpha_p \) is the node pressure angle, and \( z \) is the number of teeth.

As shown in figure 3, for the middle convex tooth, the left and right involutes are symmetric about the \( z \) axis, and the rotation angle \( \beta_1(k) \) of the left involute is

\[ \beta_1(k) = -\frac{\Theta_{10}}{2} + \frac{2k\pi}{z_1} \]

where \( \Theta_{10} = \pi / \alpha_2 + 2\sin(\alpha_p) \), \( k = 0, \pm 1, \pm 2, \ldots \), \( \mid k \mid \leq z_1 / 4 - 1 / 2 \).

For the middle concave tooth, the left involute rotation angle \( \beta_2(k) \) is

\[ \beta_2(k) = \pi / \alpha_2 + \beta_1(k) \]

The tooth profile of the involute ring tooth surface can be obtained, where the equation of the left involute tooth surface is

\[ r_{id} = \begin{bmatrix} r_b [\sin(u + \beta(k)) - u \cos(u + \beta(k))] \cos v \\ r_b [\sin(u + \beta(k)) - u \cos(u + \beta(k))] \sin v \\ r_b [\cos(u + \beta(k)) + u \sin(u + \beta(k))] \end{bmatrix} \]

3.3 Equation of spherical cone tooth surface
In figure 4, the circular plane \( \pi \) rolls along the surface of the base cone, and the trajectory formed by point \( Q \) is spherical involute. The fixed coordinate system \( O_2 - x_2, y_2, z_2 \) is established with the cone tip as the origin, and the length of the base cone is \( r_0 \), so the projection length of \( O_2 P \) on the \( O_2 x_2, y_2 \) plane is \( r_0 \sin \gamma \), and the projection length of \( O_2 P \) on the \( z_2 \) axis is \( r_0 \cos \gamma \).
Similar to the involute of the plane, the left involute rotates angle $\theta_s$ around the $o_2o_3$ axis to obtain the right spherical involute, then the equation of the right spherical involute is

$$
\mathbf{r}_s = \begin{bmatrix}
    r_0 \sin \gamma \sin (\psi + \theta_s) \\
    r_0 \sin \gamma \cos (\psi + \theta_s) \\
    r_0 \cos \gamma \\
    1
\end{bmatrix}
$$  \hspace{1cm} (7)

where $\theta_s$ is the center angle corresponding to the tooth thickness, $\theta_s = \frac{\pi}{N} - 2(\psi - \psi_p)$, the cone angle $\gamma$ is the included angle between $o_2P$ and axis $o_2o_3$, and the polar angle $\psi$ is the included angle between $o_3Q$ and $o_3P$, and then

$$
\psi = \frac{1}{\sin \gamma_b} \cos^{-1} \left( \frac{\cos \gamma}{\cos \gamma_b} \right) - \cos^{-1} \left( \frac{\tan \gamma_b}{\tan \gamma} \right)
$$  \hspace{1cm} (8)

Figure 5 shows the section of figure 2 in the xoz plane. The shaded part is the projection of spherical involute spherical cone tooth surface in the xoz plane. The curve $c_1$ is the top curve of the spherical involute tooth, and point $a$ is the end point of the plane involute. When $\gamma_b = \gamma_b'$, the starting point $Q$ of the spherical involute moves on the base circle to the point $\bar{Q}$, and the vertex $o_2$ of the base cone moves along the z axis to $o_2'$. The curved surface formed by the trajectory of spherical involute at different starting points is called spherical involute spherical cone tooth surface.

According to the law of sine and cosine of triangles, we can obtain

$$
r_0 = r_1 \sin \beta / \sin \gamma_b
$$  \hspace{1cm} (9)

$$
L = \sqrt{r_0^2 + r_2^2 + 2r_0r_2 \cos(\gamma_b + \beta)}
$$  \hspace{1cm} (10)

where $\beta$ is the included angle between $\overrightarrow{oQ}$ and z axis, $L$ is the length of $\overrightarrow{o_2o_2}$, and $\gamma_b$ is the base cone angle.

The coordinates of the base cone vertex $o_2$ on $\sigma$ are $(0, 0, L)^T$. The transformation matrix from frame $\sigma_2(x_2, y_2, z_2)$ to $\sigma(x, y, z)$ is

$$
\mathbf{M}_{02} = \begin{bmatrix}
    0 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 \\
    0 & 0 & -1 & L \\
    0 & 0 & 0 & 1
\end{bmatrix}
$$  \hspace{1cm} (11)

The equation of spherical cone tooth surface of spherical involute on the left side in $\sigma(x, y, z)$ coordinates is

$$
\mathbf{r}_{sl} = \mathbf{M}_{02} \begin{bmatrix}
    r_0 \sin \gamma \sin \psi \\
    r_0 \sin \gamma \cos \psi \\
    r_0 \cos \gamma \\
    1
\end{bmatrix} = \begin{bmatrix}
    r_0 \sin \gamma \cos \psi \\
    r_0 \sin \gamma \sin \psi \\
    -r_0 \cos \gamma + L \\
    1
\end{bmatrix}
$$  \hspace{1cm} (12)

When $\gamma_b$ changes, $\beta$ also changes with it, and the polar angle $\psi$ also changes with the cone angle $\gamma$. The spherical involute of the spherical cone tooth surface is the involute of the variable base cone angle, and the starting point $Q$ is on the base circle of the plane involute in figure 3. At this point, the spherical involute tooth surfaces on the left and right sides of the spherical cone formed do not have symmetry. The center angle corresponding to the base tooth thickness is
\[ \theta_{sb} (\beta) = \pi / 2 + 2 \sin (\gamma_0 (\beta)) \psi_r (\gamma_0 (\beta)) \]  

Then the rotation angle of the involute with variable cone angle is 

\[ \Delta \theta_s = \frac{\theta_{sb} (\beta) - \theta_{sb} (0)}{2} \]  

The equation of the left involute of the tooth surface of the spherical cone with variable cone angle in the coordinate system \((x, y, z)\) is 

\[
\begin{bmatrix}
\cos \Delta \theta_s & -\sin \Delta \theta_s & 0 & 0 \\
\sin \Delta \theta_s & \cos \Delta \theta_s & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{r}_{slv}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{r}_0 \sin \gamma \cos (\Delta \theta_s - \psi) \\
-\mathbf{r}_0 \sin \gamma \sin (\Delta \theta_s - \psi) \\
\mathbf{L}(\beta) - \mathbf{r}_0 \cos \gamma \\
1
\end{bmatrix}
\]  

### 3.4 3D model of spherical gear

With the help of software Pro/E and MATLAB, spherical gear can be quickly modeled. The data processing function of MATLAB is used to export the data of tooth surface point cloud, and import it into Pro/E, so as to obtain the tooth surface model of single ring tooth and spherical cone tooth of spherical gear in figure 6, and then obtain the 3D model of spherical gear. Figure 7 shows the meshing model of the spherical gear under the condition that the spherical center distance is \(D = 30.5 \text{mm}\) and the axis intersection angle is 90°.

![Figure 6. Schematic diagram of ring tooth and spherical cone tooth single tooth.](image_url)

![Figure 7. The model of spherical gear with an angle of 90°.](image_url)

### 4. Tooth surface contact stress analysis

Set the spherical gear material as structural steel, the contact form is friction contact, friction factor is 0.2. Considering the complexity of internal working conditions and the requirements of actual working conditions when the spherical gear pair meshes, the driving wheel concave gear exerts a driving torque of \(M = 1000 \text{N} \cdot \text{mm}\) and limits all degrees of freedom of the slave wheel convex gear.

This paper focuses on the analysis of the change of the contact stress of the tooth surface under the meshing state of the two gears with the axis intersection angle of 45°, 90° and 180°. The simulation results are shown in figure 8. According to the simulation results, the maximum stress is shown in table 1.

![Simulation results](image_url)
Figure 8. Stress distribution diagram of concave and convex gear when angle is 45°, 90° and 180°.

Table 1 The maximum stress at different coaxial angles

| Torque (N·m) | Distance of centre of sphere (mm) | Angle (°) | Maximum stress value (MPa) |
|--------------|----------------------------------|-----------|---------------------------|
| 1000         | 30.5                             | 45        | 46                        |
| 180          | 61                               | 90        | 24                        |

As can be seen from the above, with the increase of the intersection angle of the two spherical gears, the contact area of gear teeth increases first and then decreases, while the maximum stress value of gear teeth decreases first and then increases. It meets the law that spherical gear teeth from low latitude to high latitude gradually reduced. When the axial intersection angle is 90°, the maximum stress of gear teeth reaches the minimum of 24MPa. When the axial intersection angle is 180°, the maximum stress of gear teeth is 61MPa. The stress concentration occurs at the tooth root and the tooth top of the spherical gear under different meshing conditions, which can provide guidance for the tooth surface modification and optimization design of spherical gears.

5. Conclusion

According to the generating principle of involute, the equation of spherical and ring tooth surface in the direction of longitude and weft is deduced. Based on the finite element method, the contact stress of the spherical gear pair was analyzed statically, and the contact stress of the contact position of the tooth surface was extracted. It was found that the stress concentration always occurred at the tooth root and the tooth top of the spherical and ring tooth surface of the gear, and the contact stress decreased first and then increased with the increase of the axial intersection angle.

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