Comments on long distance contribution to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$

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Abstract

We study the long distance contribution to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ by using chiral perturbation theory. We find that the tree level $O(P^2)$ contribution vanishes identically without assuming the large $N_c$ limit. The leading contribution arises from the one-loop amplitude, which is $O(P^4)$ in the chiral power counting. The branching ratio of the long distance contribution is found to be of order $10^{-7}$ smaller compared with that of short distance one and hence is completely negligible.
Introduction: The rare decay mode $K^+ \rightarrow \pi^+\nu\bar{\nu}$ is suppressed by the GIM mechanism, the leading contribution starts from the one loop level for the short distance effect. It also receives contribution from the long distance effect. The heavy quark effect manifests itself in very different fashions for these two kinds of contributions. For the long distance effect, it realizes itself as the coefficients of the low energy effective lagrangian which contains no explicit heavy quark mass dependence. While for the short distance effect, it appears explicitly in the propagators of one loop amplitude for the process of interest. Due to the explicit dependence of heavy quark mass, in particular $m_t$, the short distance effect dominates the total amplitude. Since the short distance contribution contains explicit dependence on $m_t$ and $V_{td}$, it offers an opportunity for experimentalists to measure the standard model parameters. Because the amplitude is GIM suppressed, it also leaves a window for physics beyond the standard model.

The short distance contribution has been calculated by Inami and Lim [1] and the long distance one has been calculated by Rein and Sehgal [2], by Hagelin and Littenberg [3] and more recently by Lu and Wise [4]. It is the purpose of the present work to reanalyze $K^+ \rightarrow \pi^+\nu\bar{\nu}$ within the same framework, namely chiral perturbation theory, as in [4]. With different identification of left-handed and right-handed currents, we find that the tree level amplitude vanishes identically without assuming the large $N_c$ limit, as opposed to the result in [4]. The leading contribution arises from one loop amplitude and the branching ratio of the long distance contribution is about $10^{-7}$ smaller than that of the short distance one [5,6]. So the $K^+ \rightarrow \pi^+\nu\bar{\nu}$ mode is virtually a pure short distance effect. This leads to the conclusion that the accuracy of the determination of the standard model parameters by extracting from the experimental data is as good as the experimental measurement.

Chiral Lagrangian: The chiral lagrangian incorporates the external fields into the covariant derivatives according to the handness of the external fields. In the standard model, the $Z^0$ acquires both the left-handed and right-handed components due to the mixing with the hypercharge. In the three flavour space, the matrix representing the $Z^0$ particle contains a singlet component $I$, which is not included in the $SU(3)\times SU(3)$ chiral symmetry. In order to
incorporate the singlet piece into the chiral lagrangian, we first assume nonet symmetry and then use a nonet symmetry breaking parameter $\xi$ to indicate the degree of nonet symmetry breaking, $\xi = 1$ for exact nonet symmetry. The covariant derivative is then read

$$D_\mu U = \partial_\mu U - i r_\mu U + iU l_\mu = \partial_\mu U + \frac{i g}{\cos \theta_W} (UQ - \frac{\xi}{6} U - \sin^2 \theta_W [U, Q]) Z^0_\mu, \quad (1)$$

where $Q$ is the quark charge matrix, $Q = \text{diag}(2/3, -1/3, -1/3)$, and $U$ is the nonlinear realization of meson octet

$$U = \exp(i\Phi/f_\pi), \quad (2)$$

in which

$$\Phi = \phi^a \lambda^a = \sqrt{2} \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & K^0 & -2\eta/\sqrt{6} \end{pmatrix} \quad (3)$$

and $f_\pi = 93 MeV$ is the pion decay constant. Note that this identification of covariant derivative is different from that in Ref. [4]. The covariant derivative in Eq. (16) of Ref. [4] has a wrong identification of left-handed and right-handed currents, namely the currents have been placed in wrong positions.

The chiral lagrangians, in terms of the covariant derivative so constructed, are given by

$$\mathcal{L}_2 = \frac{f_\pi^2}{4} Tr \left[ D_\mu U^\dagger D^\mu U + 2B_0 M(U + U^\dagger) \right] \quad (4)$$

for strong interaction, and

$$\mathcal{L}_2^{\Delta S=1} = G_8 f_\pi^4 Tr \lambda_6 D_\mu U^\dagger D^\mu U \quad (5)$$

for weak interaction. The coefficient $G_8$ is related to the Fermi constant and the CKM matrix elements, the numerical value is given by $G_8 = 9.1 \times 10^{-6} GeV^{-2}$. There are two

\footnote{Our covariant derivative in Eq. (1) is the same as that, for example, in Eq. (3.11) of Ref. [7].}
ways to calculate the tree level amplitude of $K^+ \rightarrow \pi^+\nu\bar{\nu}$, using the conventional basis in Eq. (3) or using the diagonalized basis

$$\pi^+ \rightarrow \pi^+ - \frac{2m_K^2 f_{\pi}^2 G_8}{m_K^2 - m_{\pi}^2} K^+$$

(6)

$$K^+ \rightarrow K^+ + \frac{2m_K^2 f_{\pi}^2 G^*_8}{m_K^2 - m_{\pi}^2} \pi^+$$

(7)

$$\pi^0 \rightarrow \pi^0 + \frac{\sqrt{2} m_K^2 f_{\pi}^2}{m_K^2 - m_{\pi}^2} (G_8 K^0 + G^*_8 \bar{K}^0)$$

(8)

$$K^0 \rightarrow K^0 - \frac{\sqrt{2} m_K^2 f_{\pi}^2 G^*_8}{m_K^2 - m_{\pi}^2} \pi^0 + \sqrt{2} \frac{m_{\pi}^2 f_{\pi}^2 G^*_8}{3 m_{\eta}^2 - m_K^2} \eta$$

(9)

$$\eta \rightarrow \eta - \sqrt{2} \frac{m_{\pi}^2 f_{\pi}^2}{3 m_{\eta}^2 - m_K^2} (G_8 K^0 + G^*_8 \bar{K}^0)$$

(10)

which eliminates the $K^+ - \pi^+$ mixing. There are three Feynman diagrams, Fig. 1(a), 1(b) and 1(c), which contribute to the tree level amplitude in the conventional basis while, in the diagonalized basis, the amplitude could only receive a contribution from Fig. 1(a) since the vertex $K^+ - \pi^+$ is removed.

We now evaluate the contribution arising from the terms proportional to $I$, $Q$ and the $[Q, U]$ separately in the conventional basis. The singlet part, proportional to $\xi$, gives no contribution to $K^+ K^+ Z^0$, $\pi^+ \pi^+ Z^0$ and $K^+ \pi^+ Z^0$ vertices and thus it does not appear in the amplitudes. The rest two parts have nonvanishing contributions to the vertices and, explicitly, they lead to the following amplitudes

$$A^{(a)} = -\frac{2i g G_8 f_{\pi}^2 (1 - 2 \sin^2 \theta_W)}{\cos \theta_W} p_K \cdot \epsilon$$

$$A^{(b)} = -\frac{2i g G_8 f_{\pi}^2 (1 - 2 \sin^2 \theta_W)}{\cos \theta_W} \frac{m_{\pi}^2}{m_K^2 - m_{\pi}^2} p_K \cdot \epsilon$$

$$A^{(c)} = \frac{2i g G_8 f_{\pi}^2 (1 - 2 \sin^2 \theta_W)}{\cos \theta_W} \frac{m_{K}^2}{m_K^2 - m_{\pi}^2} p_K \cdot \epsilon.$$  

(11)

However, the sum of the above three amplitudes in Eq. (11) is zero, i.e.,
\begin{equation}
A^{(a)} + A^{(b)} + A^{(c)} = 0.
\end{equation}

In the diagonalized basis, it is easy to show that the vertex $K^+ \pi^+ Z^0$ arising from the only possibly diagram Fig. (a) also vanishes, resulting in a vanishing amplitude. This result differs from that in Ref. \cite{4} significantly.

One-loop Amplitude: The amplitude receives contribution starting from $O(P^4)$ in the chiral power counting. In the $O(P^4)$ chiral lagrangian of weak interaction, the number of counter terms is so large \cite{9} that fitting the coefficients from data is beyond the reach of current experiments. As a standard practice, we only calculate the finite part of the one loop amplitude, the so-called chiral logarithmic piece, to estimate the order of magnitude of the decay rate. The amplitude of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ is found to be

\begin{equation}
A(K^+ \rightarrow \pi^+ \nu \bar{\nu}) = -\frac{i \alpha G_8 (1 - 2 \sin^2 \theta_W)}{64 \pi M_Z^2 \sin^2 \theta_W \cos^2 \theta_W} J(m_{K}^2)(P_K + P_\pi)^\mu \bar{\nu} \gamma_\mu (1 - \gamma_5) \nu,
\end{equation}

where the loop function is defined as

\begin{equation}
J(m^2) = \frac{1}{i \pi^2} \int d^n q \frac{1}{q^2 - m^2} = m^2(\Delta - \ln \frac{m^2}{4 \pi^2 f_\pi^2}).
\end{equation}

The divergent part is given by

\begin{equation}
\Delta = \frac{2}{\epsilon} - \gamma - \ln \pi + 1,
\end{equation}

where $\gamma$ is the Euler number and $\epsilon = 4 - n$. The decay rate can be evaluated analytically \cite{10} and it reads as

\begin{equation}
\Gamma(K^+ \rightarrow \pi^+ \nu \bar{\nu}) = \frac{\alpha^2 G_8^2 m_K^5 (1 - 2 \sin^2 \theta_W)^2}{2^{19} \pi^5 M_Z^2 \sin^4 \theta_W \cos^4 \theta_W} (1 - 8 r_\pi + 8 r_\pi^3 - r_\pi^4 - 12 r_\pi^2 \ln r_\pi)|A(m_{K}^2)|^2,
\end{equation}

where

\begin{equation}
r_\pi = \frac{m_\pi^2}{m_K^2}.
\end{equation}

The long distance contribution gives arise to the branching ratio

\begin{equation}
\frac{\Gamma_{long}}{\Gamma_{total}} = \frac{1}{1 + \frac{m_\pi^2}{m_K^2} \ln \frac{m_\pi^2}{m_K^2}}.
\end{equation}
\[ Br(K^+ \to \pi^+ \nu \bar{\nu})_{L.D.} = 7.71 \times 10^{-18} \] (18)

which is roughly of order $10^{-7}$ smaller than that of the short distance contribution [11].

In summary, the $K^+ \to \pi^+ \nu \bar{\nu}$ amplitude is shown to be much smaller than what has been calculated before, the effect starts to appear only at $O(P^4)$. This is a general feature shared by many other GIM suppressed processes, which will be elaborated fully elsewhere. With the result of the present work, the uncertainty of the determination of the standard model parameters from experiments is further restricted and it makes the proposed experiments more interesting.

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FIGURES

**Fig 1:** Feynman diagrams which contribute to the tree level $K^+ \to \pi^+ \nu \bar{\nu}$ amplitude. The cross stands for the weak interaction. Only diagram (a) could in principle contribute in the diagonalized basis.