Thermalization in Weakly Coupled Nonabelian Plasmas

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Abstract: We investigate how relativistic, nonabelian plasmas approach equilibrium in a general context. Our treatment is entirely parametric and for small Yang-Mills coupling $\alpha$. First we study isotropic systems with an initially nonequilibrium momentum distribution. We consider both the case of initially very high occupancy and initially very low occupancy. Then we consider systems which are anisotropic. We consider both weak anisotropy and large anisotropy, and allow the occupancy to be parametrically large or small. Writing the typical momentum of an initial excitation as $Q$ and the final temperature as $T_{\text{final}}$, full equilibration occurs in a time $t_{\text{eq}} \sim \alpha^{-2} T_{\text{final}}^{-1}$ for $T_{\text{final}} > Q$, and $t_{\text{eq}} \sim \alpha^{-2} Q^{\frac{5}{2}} T_{\text{final}}^{\frac{-1}{2}}$ for $T_{\text{final}} < Q$, unless the initial system is sufficiently anisotropic and $T_{\text{final}} > \alpha^{\frac{3}{2}} Q$, in which case equilibration occurs somewhat faster, $t_{\text{eq}} \sim \max(\alpha^{-2} T^{-1}, \alpha^{\frac{3}{2}} Q^{\frac{5}{2}} T_{\text{final}}^{\frac{-1}{2}})$.

Keywords: nonabelian gauge theory, equilibration, thermal field theory, QCD, weak coupling
1 Introduction and summary of results

Nonabelian plasmas out of equilibrium are a rather generic feature of early universe cosmology. For instance, they can arise in the process of reheating after inflation, whether by perturbative decay of inflatons [1], by resonant decay of an inflaton into Standard Model...
fields [2], or due to other late-decaying relics. In some cases the nonabelian plasmas created
in these processes can be locally very anisotropic; for instance, Asaka et al [3] show that
late decays of inflatons can produce jet-shaped regions of heated plasma which may be
important for electroweak baryogenesis; the interiors of these jets are generically highly
anisotropic. Nonequilibrium and anisotropic plasmas can also arise in cosmological phase
transitions, such as the electroweak phase transition [4]. In all of these cases, due to
asymptotic freedom in QCD and the weakly coupled nature of weak isospin, the relevant
nonabelian coupling is relatively small.

The early stages of highly relativistic heavy ion collisions also probably produce a
nonabelian (QCD) plasma out of equilibrium. Probably in real-world applications the
coupling is not small and a perturbative approach is suspect; but at least in the theoretically
clean limit of extremely high energy collisions the coupling is also weak in this context.

In any case we think it is a well motivated theoretical question to ask, how in general
does a nonabelian plasma approach equilibrium? For the case of a plasma which is relatively
close to equilibrium, we believe that this is well understood. The dominant processes in this
case are elastic scattering and inelastic, number-changing (splitting and joining) processes.
These are well described by an effective kinetic theory [5]. There is a natural energy scale
$T$, and the time for the system to approach equilibrium is parametrically $t_{eq} \sim \alpha^{-2}T^{-1}$ up
to logarithmic factors which we will systematically ignore.

However, it is less clear what physics is most important for a system which starts out
very far from equilibrium. This is particularly the case if the system is also anisotropic.
In this case, there can be plasma instabilities [6–11] which may dominate the dynamics in
some cases. To our knowledge, there has been no comprehensive study of this case. The
literature is also somewhat fragmentary for the case of a system which is isotropic, but
in which the modes which dominate the system’s energy have typical occupancies much
larger or much smaller than 1.

In this paper we will give a parametric treatment of each of these cases; systems
which are far from equilibrium but isotropic, and systems which are far from isotropic.
By a parametric treatment we mean that we will identify the most important physics
in each case, as determined by counting powers of the gauge coupling $\alpha$, and estimate
relevant momentum and time scales in the process of equilibration as powers of $\alpha$. We
will not attempt to keep track of factors of the logarithm of the coupling. Nor will we
attempt to estimate any order-1 coefficients, or determine the range of $\alpha$ values over which
our parametric estimates are reliable. If a more quantitative estimate is needed in any
particular situation, then we have at least identified what the relevant physics is in making
such an estimate.

We will not present estimates of the equilibration of systems which are initially spatially
nonuniform (inhomogeneous). This presents too general a class of situations to allow
any comprehensive study. However there is a general strategy for using our results to
study these systems as well. Namely, one can consider the way in which spatial nonuniformity
leads to local momentum-space nonuniformity through propagation, and then use
our results for the evolution of momentum-space anisotropic systems to determine the local
physics which this causes. We will use the results of this paper to study one particularly
interesting case, that of a system under 1-dimensional “Bjorken” expansion, in a future publication.\footnote{We cannot help giving away one central result of our future publication. Assuming the energy density scales as $\varepsilon \sim \alpha^{-1}Q^4T^{-1}$ before equilibration, and using the estimate for the final equilibration time given in the abstract, one finds $\tau_{eq} \sim \alpha^{-\frac{3}{2}}Q^{-3}$, slightly different than the conclusions of Baier et. al. \cite{12}.}

In the main body of the text we will go through each case in detail, introducing the physics relevant to each situation as the need arises. However, for the convenience of the reader we will begin with a summary of our results.

First we consider isotropic plasmas. For simplicity we only consider plasmas where all physics is initially dominated by excitations which have a single characteristic momentum scale $Q$, with a typical occupancy $f(Q)$ which we write parametrically as $f(Q) \sim \alpha^{-c}$. Cases with $c > 0$ represent initially overoccupied systems; those with $c < 0$ represent initially underoccupied systems.

We find three cases. For $c > 1$ (extreme overoccupancy) the Nielsen-Olesen instability \cite{13} rapidly converts the system into one with a larger $Q$ and $c = 1$.

For $0 < c < 1$ (overoccupancy), the dominant processes are elastic scattering and number-changing effective 2 $\to$ 1 “merging” processes. Excitations quickly arrange into an $f(p) \propto T_s/p$ distribution with a cutoff scale $p_{\text{max}}$. Initially $p_{\text{max}} \sim Q$ and $T_s \sim \alpha^{-c}Q$, but both scales evolve with time, see eq. (2.30):

$$p_{\text{max}} \sim \alpha^{\frac{c}{2+c}}Q^\frac{2}{3}t^\frac{2}{3}, \quad T_s \sim \alpha^{-\frac{3}{2+c}}Q^\frac{1}{3}t^{-\frac{1}{3}}, \quad (t > \alpha^{-2+2c}Q^{-1}) . \quad (1.1)$$

Equilibration occurs when $p_{\text{max}} \sim T_s \sim \alpha^{-c/4}Q$, at $t_{\text{eq}} \sim \alpha^{-2+\frac{c}{2}}Q^{-1} \sim \alpha^{-2}T^{-1}$. The final temperature is determined by the initial energy density; and $t_{\text{eq}}$ is the characteristic equilibration time for a bath of temperature $T$.

For $c < 0$ (underoccupancy), the physics is a little more involved (though a basically correct exposition was given in Ref. \cite{12}, in a slightly different context). The most important physics is the formation of a bath of low-momentum “daughter” particles, which thermalize and begin to dominate the system’s dynamics at a time scale $t \sim \alpha^{-2+\frac{c}{2}}Q^{-1}$. The soft thermal bath then catalyzes the breakup of the hard excitations, absorbing their energy into the thermal bath. Both processes are dominated by number-changing effective $1 \to 2$ processes, and it is essential to include the modification of the number-changing rate due to the Landau-Pomeranchuk-Migdal (LPM) effect \cite{14, 15}. The temperature of the thermal bath rises with time as $T \sim \alpha^{4-c}Q^{2}t^2$. The hard excitations are consumed and thermalization completes when $T = T_{\text{final}} \sim \alpha^{-\frac{c}{2}}Q$ at $t_{\text{eq}} \sim \alpha^{-2+\frac{c}{2}}Q^{-1}$. The temperature is again dictated by the initial energy density; the equilibration time is $t_{\text{eq}} \sim \alpha^{3/8}\alpha^{-2}T_{\text{final}}^{1}$, longer by $\alpha^{c/8}$ than the characteristic equilibration time of a thermal bath at temperature $T_{\text{final}}$. But $t_{\text{eq}}$ is much shorter than the large-angle elastic scattering time scale for the initial distribution of excitations. Therefore, large angle change through elastic scattering between the initially present excitations actually plays no role in the system’s thermalization.

There is a simple way to understand the equilibration time for an underoccupied system. Because of LPM modified bremsstrahlung emission, a hard $p \gg T$ excitation
in a thermal bath loses energy at a rate \( dE/dt \sim \alpha^2 T^2 \sqrt{E/T} \) [15]. Therefore \( t_{\text{eq}} \) is the characteristic time scale for an excitation of energy \( Q \), traversing a thermal bath of temperature \( T_{\text{final}} \), to radiate away its energy. Equilibration could not proceed faster than this.

Having treated isotropic systems, we then turn to anisotropic ones. Again we assume that the system starts with excitations possessing a single characteristic momentum scale \( Q \). But the distribution of particles is now characterized by two quantities: a typical occupancy, \( f(p) \sim \alpha^{-c} \), and a measure of anisotropy characterized by a strength \( \alpha^{-d} \). For weakly anisotropic systems with \( d < 0 \), we define \( \epsilon \equiv \alpha^{-d} \) as the relative variation of the occupancy with direction (so if \( \epsilon = 0.1 \) then the occupancy in some directions is 10% larger than in other directions). For strongly anisotropic systems with \( d > 0 \), we define \( \delta \equiv \alpha^{-d} \), as the angular range within which most of the excitations reside. We concentrate on the case of an oblate distribution, with most excitations’ momenta \( p \) lying within an angle \( \delta \) of the \( xy \) plane. In this case, if \( \delta = 0.1 \) then the occupancy is \( f(p) \sim \alpha^{-c} \) if \( |p_z| < 0.1|p| \), but \( f(p) \) is small outside this range. We also consider prolate distributions with most excitations’ momenta within an angle \( \delta \) of the \( z \) axis.

Our findings for oblate anisotropic plasmas are summarized in figure 1 and table 1. Depending on the values of \( c \) and \( d \) (occupancy and anisotropy), the system’s evolution can take many different forms. In almost all regions plasma instabilities play a key role in the dynamics; but after some time scale new physics comes to dominate and the behavior of the system changes. We have only tried to track the evolution until the first significant change in the dominant physics. In summary, the regions are:

1. Plasma instabilities induce joining processes which suppress the anisotropy.

| Region | Boundaries | timescale for new physics | New physics which occurs |
|--------|------------|---------------------------|-------------------------|
| 1      | \( c > 0, 3d - c + 1 > 0, 7d - c + 1 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | joining reduces anisotropy |
| 2      | \( c < 0, 3d + c + 1 > 0, 5d + c + 1 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | new plasma instabilities |
| 3a     | \( c < 0, 5d + c + 1 > 0, d < 0, 7d + 3c + 1 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | new plasma instabilities |
| 3b     | \( d > 0, 8d + 3c + 3 > 0, 9d + 3c + 1 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | new plasma instabilities |
| 4a     | \( 7d - c + 1 > 0, 7d + 3c + 1 > 0, d < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | angle randomization |
| 4b     | \( d > 0, 23d + 9c + 3 > 0, 3d + c - 1 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | angle broadening |
| 5      | \( 3d + c + 1 > 0, 8d + 3c + 3 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | new plasma instabilities? |
| 6      | \( 9d + 3c + 1 > 0, 23d + 9c + 3 < 0, 9d + 3c - 1 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | new, weak instabilities |
| 7      | \( 3d + c - 1 > 0, d - c + 1 > 0 \) | \( \alpha^{-d/2} Q^{-1} \) | angle broadening |
| 8      | \( c > 1, d - c + 1 < 0 \) | less than \( Q^{-1} \) | Nielsen-Olesen instabilities |
| 9      | \( 3d + c + 1 < 0, c < 0 \) | as per isotropic | as per isotropic |
| 10     | \( 9d + 3c - 1 > 0, 37d + 15c + 7 < 0 \) | \( \alpha^{-d/2} Q^{-1} \) | thermal bath elastic scatt |

Table 1. Main regions in the occupancy\( (c) \)–anisotropy\( (d) \) plane, indicating the time scale on which the physics changes and the nature of the new physics.
2. Plasma instabilities induce splitting processes, split daughters are anisotropic and generate new plasma instabilities which come to dominate the dynamics. The most important daughters have occupancies limited by re-absorption.

3. Same as 2. except re-absorption is irrelevant; the most important momentum scale is set by the angle randomization of split daughters.

4. Plasma instabilities randomize the directions of hard excitations.

5. Radiated daughters induce new plasma instabilities before the original plasma instabilities finish growing to their “saturation” size. This region is not fully understood.

6. Radiated daughters form a nearly-thermal bath, but residual anisotropy of the bath generates new plasma instabilities.

7. Plasma instabilities broaden the momentum distribution before the unstable modes’ growth can saturate. The hard-loop approximation does not apply in this region.

8. Nielsen-Olesen instabilities rapidly lower the maximum occupancy.

9. Plasma instabilities play no important role. Equilibration proceeds as in an isotropic system.

10. Plasma instabilities cause splitting which generates a soft thermal bath. The bath comes to dominate the dynamics through elastic scattering. (This region lies off the edge of Fig. 1.)

Note that in regions 1, 4, 7, and 8 the processes which redistribute particle momentum (\(\dot{q}\)) become less efficient with time; whereas on the contrary, in regions 2, 3, 5, 6, and 10 the new physics which emerges actually makes the redistribution of particle momentum more efficient with time.

Figure 1. Main regions in the occupancy-anisotropy plane. Descriptions of each region are in the text, and in Fig. 13.
Though we have not followed the evolution of anisotropic systems through to final thermalization, we have an estimate for the final thermalization time. First one determines $T_{\text{final}} \sim \varepsilon_{\pm}^{-\frac{4}{3}}$; for $d > 0$ it is $T_{\text{final}} \sim \alpha_{\pm}^{-\frac{4}{3}} Q$. If $T_{\text{final}} \geq Q$ then $t_{\text{eq}} \sim \alpha_{\pm}^{-2} T_{\text{final}}^{-1}$. If $T_{\text{final}} \ll Q$ then $t_{\text{eq}} \sim \alpha_{\pm}^{-\frac{13}{7}} Q^{-\frac{5}{7}} T_{\text{final}}^{-\frac{12}{7}}$ if $T_{\text{final}} > \alpha_{\pm}^{-2} Q^{\frac{1}{2}} T_{\text{final}}^{-\frac{3}{2}}$ if $T_{\text{final}} < \alpha_{\pm}^{-2} Q$. Both estimates are the time it takes for an excitation of momentum $Q$ to lose its energy while traversing a thermal bath of temperature $\sim T_{\text{final}}$; the difference is that for $T_{\text{final}} > \alpha_{\pm}^{-\frac{4}{3}} Q$, the thermal bath is somewhat anisotropic and plasma instabilities, not elastic scattering, determine how the $p \sim Q$ momentum excitation loses energy. If the resulting $t_{\text{eq}}$ is shorter than $\alpha_{\pm}^{-2} T_{\text{final}}^{-1}$, then the hard $p \sim Q$ excitations break up before the remaining thermal bath can equilibrate with itself. But it still takes time $\sim \alpha_{\pm}^{-2} T_{\text{final}}^{-1}$ for the bath to fully self-equilibrate, so this remains a lower bound on the total thermalization time.

Having summarized our findings, we will now present the details leading to these results. Since we presented the main conclusions here, we will not end with a discussion or conclusions section.

2 Isotropic distributions

In this section we will consider systems which are both homogeneous and isotropic. Nevertheless they can be very far from equilibrium. Such systems might arise rather generically during preheating after inflation. We will also re-encounter many of the physical processes relevant to this case when we consider systems which are not isotropic. We begin by reviewing the relevant physical processes, then we will consider their application to nonequilibrium systems.

2.1 Elastic and inelastic scattering

We begin by reviewing the basic physical processes relevant for equilibration in a plasma. All of the physics discussed here is well known, and is presented in more complete detail in Ref. [5]. At weak coupling and at sufficiently long length and time scales, the dynamics of the plasma can be described by that of long-lived weakly interacting quasiparticle excitations. The perpetual interaction with the medium affects the quasiparticle’s dispersion relation, and to leading order, the correction looks like an effective mass

$$E(p) \sim \sqrt{p^2 + m^2}, \quad (2.1)$$

which introduces a new scale, the screening scale $m$, which is parametrically

$$m^2 \sim \alpha \int_p f(p) \frac{1}{p}, \quad (2.2)$$

where $\int_p$ is a shorthand for $\int d^3p$.

Under certain conditions, the non-equilibrium dynamics of the plasma can be described by the Boltzmann equations that schematically read

$$(\partial_t + v(p) \cdot \nabla_x) f(x, p, t) = -C[f], \quad (2.3)$$
where \( f(x, p, t) \) is the phase space density of quasiparticles in the plasma, and \( v(p) \) is the velocity of a quasiparticle with momentum \( p \). In the following, we will restrict ourselves to spatially homogeneous systems so that \( f \) is only a function of \( p \) and \( t \). In the presence of several particle species (gluons, up- and down-quarks, etc.) the distribution function is a multicomponent vector with a component for each species. In the absence of external forces, the quasiparticles change their momentum states by mutual interactions; \( C[f] \) is a spatially local collision term that represents the rate at which particles get scattered out of state \( p \) minus the rate they get scattered into this state.

The conditions under which the non-equilibrium dynamics of a theory can be described by the Boltzmann equations of eq. (2.3) are

- that the typical size of the wave packets of quasiparticles is smaller than the mean free path of quasiparticles, and
- the quantum mechanical formation time of scatterings is small compared to the mean free time.

Generally these conditions will be met for those degrees of freedom in a plasma with typical occupancy \( f(p) \ll \alpha^{-1} \). An exception is that certain inelastic processes can have formation times which exceed the mean free time between scatterings for the excitations involved. This case requires special treatment (the LPM effect), which we will return to at length below.

There are two qualitatively different processes that contribute in leading order to the collision term, elastic scattering and near-collinear splitting processes. The collision term for elastic scattering reads

\[
C^{2+2}[f] = \frac{1}{2} \int_{k, p', k'} |M(p, k, p', k')|^2 (2\pi)^4 \delta^4(P + K - P' - K') \times \left\{ f(p) f(k) \left[ 1 \pm f(p') \right] \left[ 1 \pm f(k') \right] - f(p') f(k') \left[ 1 \pm f(p) \right] \left[ 1 \pm f(k) \right] \right\}. \tag{2.4}
\]

Here \( P, P', K, \) and \( K' \) denote on-shell four-vectors. The \( [1 \pm f] \)-factors arise from final state Bose stimulation or Pauli blocking. \( M \) is the elastic scattering amplitude in non-relativistic normalization, related to the usual relativistically normalized matrix element \( \mathcal{M} \) by

\[
|M(p, k, p', k')|^2 \sim \frac{|\mathcal{M}(p, k, p', k')|^2}{(2p_0)(2k_0)(2p'_0)(2k'_0)}. \tag{2.6}
\]
In the leading order for gauge bosons, the $\mathcal{M}$ originates from diagrams in Fig. 2 (from now on we will only consider gauge bosons, which due to the stimulated interaction rates dominate the equilibration). In vacuum the matrix element reads

$$|\mathcal{M}|_{\text{vacuum}}^2 \sim \alpha^2 \left(3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2}\right),$$

where $s$, $t$, and $u$ are the usual Mandelstam variables. The process is dominated by small momentum exchange, and in this limit the matrix element becomes

$$|\mathcal{M}|_{\text{vacuum}}^2 \sim \frac{\alpha^2}{(q_\perp^2)^2},$$

where $q_\perp$ is the momentum transfer in the elastic scattering. In vacuum the total cross-section $\sim \int d^2 q_\perp |\mathcal{M}|^2$ is infrared divergent, but this divergence is regulated by including the medium dependent self-energy to the exchange line in the diagram of Fig. 2. Then the amplitude becomes $[16]^3$

$$|\mathcal{M}|^2 \sim \frac{\alpha^2}{q_\perp^2(q_\perp^2 + m^2)},$$

A particle traveling through the plasma undergoes successive uncorrelated elastic scatterings that diffuse the momentum of the traveling particle. The momentum of the particle evolves as a random walk in momentum space, characterized by the momentum diffusion constant $\hat{q}_{\text{elastic}}$; the average squared momentum transfer $\Delta p^2$ grows linearly in time,

$$\Delta p^2 \sim \hat{q}_{\text{elastic}} t.$$  

The rate of elastic scatterings is

$$\frac{d\Gamma_{\text{el}}}{d^2 q_\perp} \sim \int dq_z \int_{p'} |\mathcal{M}|^2 f(p')[1+f(p'-q)],$$

which for $q_\perp < |p|$ is

$$\frac{d\Gamma_{\text{el}}}{d^2 q_\perp} \sim \frac{\alpha^2}{q_\perp^2(q_\perp^2 + m^2)} \int_{p'} f(p')[1+f(p')],$$

and the mean squared momentum transfer per unit time is

$$\hat{q}_{\text{elastic}} \sim \int d^2 q_\perp \frac{d\Gamma_{\text{el}}}{d^2 q_\perp} q_\perp^2.$$ 

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2If the typical initial occupancy is large, we must be speaking of gauge bosons. If the initial occupancy is small and the system is composed primarily of fermions, then the radiated “daughter” particles are primarily gauge bosons. The rate of radiation differs only by an order-1 group Casimir factor and all parametric estimates are identical to the case with primary gauge bosons. All of these remarks also apply to the anisotropic case.

3The screening does not affect nearly static magnetic fields, which is why the matrix element scales as $1/q_\perp^2$ for very small $q_\perp$. Because of this behavior the total cross-section actually remains logarithmically divergent, but the divergence is too weak to influence the rate of angle or particle number change.
Collisions with small $q_{\perp} \sim m$ are the most frequent but change momentum of the propagating particle the least, while scatterings with large $q_{\perp}$ are rare but produce larger angle deflections. Inserting eq. (2.12) into eq. (2.13), one sees that all available scales of momentum transfer contribute equally to the momentum diffusion, such that $\hat{q}_{\text{elastic}}$ reads\footnote{That all the available momentum transfer scales contribute to the momentum diffusion gives rise to a logarithmic enhancement, which is however neglected in our power counting.}

$$\hat{q}_{\text{elastic}} \sim \alpha^2 \int_p f(p)[1+f(p)].$$

Due to the peculiarities of soft and collinear enhancements in QCD, there is another class of processes which are also leading-order. These are the inelastic, number changing, collisions $[5, 17]$. In the vacuum, $1 \leftrightarrow 2$ processes are kinematically disallowed for massless particles, but in the presence of the medium, soft elastic scatterings can take the particles slightly $\sim m$ off-shell allowing for subsequent “$1 \leftrightarrow 2$” nearly collinear splitting processes, depicted in Fig. 3. A particle propagating through a plasma experiences soft elastic scatterings with rate

$$\Gamma_{\text{el}} \sim \frac{\alpha^2}{m^2} \int_{p'} f(p')[1+f(p')] \sim \frac{\hat{q}_{\text{elastic}}}{m^2}.$$  

In vacuum, each individual uncorrelated scattering event induces a collinear radiation with probability $\alpha$ per logarithmic range in angle, per logarithmic range in energy. The same applies in-medium (modulo Bose stimulation factors) provided that the formation time of the emission process is shorter than the mean time scale between scattering events. When the formation time is longer than the time between scatterings, there is destructive interference between emission processes, the LPM effect, which we will discuss in due course. Assuming the formation time is shorter, and neglecting the log over emission angles, the particle splitting rate is parametrically

$$\frac{d\Gamma_{\text{split}}^{\text{BH}}}{dp/p} \sim \alpha \Gamma_{\text{el}},$$

where $p$ is the energy of the emitted particle. (The superscript BH stands for Bethe-Heitler, since the regime where coherence effects are negligible corresponds to the original...
calculation of Bremsstrahlung radiation in QED by Bethe and Heitler [18]. Including the statistical factors, the effective collision term for \( 1 \leftrightarrow 2 \) processes reads

\[
C^{-1} \leftrightarrow 2^f \sim \int \frac{d\Gamma_{\text{BH}}^{\text{split}}}{dk} (f(p)[1+f(p-k)][1+f(k)] - f(p-k)f(k)[1+f(p)]),
\]

where energy and momentum conservation are implied.

2.2 \( 0 < c < 1 \): Cascade by elastic scattering

We start by considering isotropic and homogeneous systems whose initial distribution at \( t = 0 \) has occupancies \( \sim \alpha^{-c} \) up to a cutoff scale \( Q \)

\[
f(p < Q) \sim \alpha^{-c}, \quad f(p > Q) < (Q/p)^4
\]

so that the high momentum particles \( (p > Q) \) do not dominate energy density or anything else and can be neglected in the discussion. For \( c > 0 \), the soft modes are overpopulated compared to the thermal ensemble with the same energy density. We expect thermalization to proceed by energy cascading to higher \( p \) modes with lower occupancy, until the typical occupancy reaches 1 and quantum mechanics cuts off further cascading. We will see that if \( c < 1 \), both the elastic scattering and inverse collinear splitting are of comparable effectiveness in transferring energy to higher momenta. For \( c > 1 \), the occupancies are so large, that even in the weak coupling limit, the thermalization is driven by non-perturbative physics. This will be discussed separately in the next subsection.

The energy density of the system is

\[
\varepsilon \sim \int d^3p \, p \, f(p) \sim \alpha^{-c} Q^4,
\]

and the final temperature is \( T_{\text{final}} \sim \varepsilon^{1/4} \sim \alpha^{-c/4} Q \), so the equilibration cannot occur faster than in time \( t_{\text{eq}} \sim \alpha^{-2} T_{\text{final}}^{-1} \sim \alpha^{-2+c/4} Q^{-1} \), which is the equilibration time for a small deviation from thermal equilibrium in a system at temperature \( T_{\text{final}} \). On the other hand systems with high occupancies have Bose-stimulated interaction rates so we also expect it should not take longer than this time scale for the system to thermalize.

Let us now estimate the time scale it takes for a particle with \( p \sim Q \) to appreciably change its angle or momentum. For the initial conditions of eq. (2.18), the momentum diffusion due to elastic scattering is dominated by modes with \( p \sim Q \) and is parametrically of order

\[
\dot{q}_{\text{elastic}} \sim \alpha^2 \int p \, f(p)[1+f(p)] \sim \alpha^{2-2c} Q^3,
\]

and the time for large-angle change is

\[
t_{\text{large angle}} \sim Q^2 / \dot{q} \sim \alpha^{-2+2c} Q^{-1}.
\]
in the sense that the “gain” and “loss” terms in the Boltzmann equation cancel. Such a soft tail does not dominate either screening, elastic scattering, or the energy density of the system, so it plays no role in directing the dynamics of the typical excitations. Such $f(p) \propto 1/p$ tails will be a common feature in the sections which follow, so we introduce some notation to describe them. In our case, at and after the time $t_{\text{large angle}}$ the particles have had time to organize themselves into such a $1/p$ tail. The distribution is characterized by two scales; the maximum momentum $p_{\text{max}}$ below which the occupancy behaves as $\propto 1/p$, and an “effective temperature” determining the occupancy below $p_{\text{max}}$:

$$f(p) \sim \frac{T_*}{p} \Theta(p_{\text{max}} - p).$$

Of course the Heaviside function is an oversimplification of the transition from $\propto T_*/p$ behavior to rapid falloff; but for parametric estimates it will be sufficient. The parameter $T_*$ is solved from energy conservation

$$\varepsilon \sim T_* p_{\text{max}}^3.$$  

The subsequent equilibration is then controlled by the evolution of the scale $p_{\text{max}}$. Before thermalization $p_{\text{max}} < T_{\text{final}} < T_*$, and when $p_{\text{max}}$ increases $T_*$ decreases. When $p_{\text{max}} \sim T_{\text{final}} \sim T_*$, the typical occupancies at scale $p_{\text{max}}$ reach 1, and all scales below $T_{\text{final}}$ have thermalized. After that, the cascade will continue (without Bose stimulation) to higher momentum scales. However, in thermal ensemble the energy density, screening, and elastic scattering are dominated by scale $T_{\text{final}}$, so that even though the ultraviolet tail of the distribution has not completely adjusted to the Maxwell-Boltzmann form, the system can be considered thermalized.

The scale $p_{\text{max}}$ can grow either by elastic scatterings or inverse splitting processes. The rate for large angle elastic collisions is

$$\dot{q}_{\text{elastic}} \sim \alpha^2 T_*^2 p_{\text{max}},$$

$$\Gamma_{\text{large angle}} \sim \frac{\dot{q}_{\text{elastic}}}{p_{\text{max}}} \sim \alpha^2 T_* \left( \frac{T_*}{p_{\text{max}}} \right),$$

and is enhanced compared to the equilibrium one $\sim \alpha^2 T$.

The rate for a hard joining process, in which two particles with $p \sim p_{\text{max}}$ merge, so that the momentum of the resulting particle has changed by order 1, is given by the eq. (2.17) with $p \sim k \sim (p-k) \sim p_{\text{max}}$ so that

$$\partial_t f(p_{\text{max}}) \sim \Gamma_{\text{hard merge}} f(p_{\text{max}}) \sim \alpha \Gamma_{\text{el}} f(p_{\text{max}})[1+f(p_{\text{max}})],$$

and inserting the elastic scattering rate,

$$\Gamma_{\text{hard merge}} \sim \alpha^2 T_* \left( \frac{T_*}{p_{\text{max}}} \right).$$

The rates for large momentum transfer either from elastic scattering or hard joining are the same (up to logarithms) so both the two processes compete, but all parametric estimates made using either one will give the same results.
Figure 4. Evolution of the momentum scales in a log-log plot during the cascade with initial distribution of eq. (2.18) with $c = 1$.

During the evolution of $p_{\text{max}}$ the total particle number changes, and we have assumed that the number changing processes can keep up with the change of $p_{\text{max}}$ and hence $T_\ast$. We expect that this is true because number-changing $2 \leftrightarrow 1$ splitting processes are parametrically as efficient as elastic scattering. But even if they are numerically less efficient, particle number is still efficiently changed, because the particle number changing rate is large in the infrared. Specifically, the total particle number changing rate is of order

$$\Gamma \sim \alpha^{-1} \sim \int \frac{dk}{d\epsilon} \frac{\Gamma_{\text{BH \, \text{split}}}}{d\epsilon} \sim \alpha \Gamma_{\text{el \, \text{f}}} f(m) \sim (2.28)$$

which is greater than the hard merging or the large angle elastic scattering rate by a power of $(p_{\text{max}}/m) \sim \sqrt{p_{\text{max}}/\alpha T_\ast}$ [19].

The time evolution of $p_{\text{max}}$ and $T_\ast$ can be determined by self-consistently solving

$$p_{\text{max}}^2 \sim \hat{q}_{\text{elastic}} t, \quad \text{and} \quad T_\ast \sim \alpha^{-1} \sim \frac{\hat{q}_{\text{elastic}}}{\hat{q}_{\text{elastic}}},$$

(2.29)

These expressions hold for all times $t$ such that $p_{\text{max}} > Q$, that is, $t > \alpha^{-c} Q^{-1}$, until equilibration is complete, which occurs when $p_{\text{max}} \sim T_\ast$, at time

$$t_{\text{eq}} \sim \alpha^{-2+c} Q^{-1} \sim \alpha^{-2} T_\ast \sim \alpha^{-2} T_{\text{final}}.$$  

(2.31)

These results are summarized in Fig. 4.

2.3 Nielsen-Olesen Instability

In the last subsection we considered the case $f(p \sim Q) \sim \alpha^{-c}$ with $0 < c < 1$. Now we briefly comment on what happens in the case $c > 1$, that is, the case of extremely high initial occupancy. The physics in this regime is nonperturbative; large occupation numbers of order $\gtrsim \alpha^{-1}$ can cancel the suppression of interactions coming from the small coupling
constant. Specifically, the nonperturbative physics of the Nielsen-Olesen instability \(^{13}\) will dominate in this regime.

Very large occupancy fields can be viewed as effectively classical. In our case, we have classical color-magnetic fields of coherence length \(l_{\text{coh}} \sim Q^{-1}\) and of field strength \(B^2 \sim \alpha^{-c} Q^4 \rightarrow B \sim \alpha^{-c/2} Q^2\). The Lamor radius of a particle propagating in such a field is \(L \sim \alpha^{-1/4} B^{-1/2} \sim \alpha^{(c-1)/4} Q^{-1}\), which is shorter than the coherence length of the field. Therefore we can neglect the spatial variation of the magnetic field. In an intense and uniform magnetic field, relativistic excitations split up into Landau levels with energy spacing \(\sim L^{-1}\). But as pointed out by Nielsen and Olesen \(^{13}\), the way the spin-magnetic interaction splits the Landau levels means that, for spin-1 particles, one spin state of the lowest Landau level is actually exponentially unstable, with a growth time \(\gamma \sim L^{-1} \sim \alpha^{1/4} Q\). In a time scale\(^5\) \(t \sim L^{-1}\) these unstable modes grow to absorb the energy density of the magnetic fields, transferring it to modes of typical wave number \(p \sim L^{-1} \sim \alpha^{1/4} Q \gg Q\). So the instability transfers energy towards ultraviolet and works towards thermalizing the distribution. Using energy conservation, \(\alpha^{-c} Q^4 \sim f_{\text{new}}(p)p^4\), we find the final occupancy of these modes to be \(f(p) \sim \alpha^{-1}\). Therefore the final state has

\[
Q_{\text{new}} \sim \alpha^{1/4} Q, \quad f_{\text{new}} \sim \alpha^{-1}.
\]

So the end product of the Nielsen-Olesen instability is a distribution with \(c = 1\). The equilibration of this “final” state will then proceed as described in the previous subsection.

### 2.4 Landau-Pomeranchuk-Migdal suppression

We will turn next to dilute systems, \(c < 0\) or \(f(p) \ll 1\). In this case the screening scale \(m^2\) gets smaller, but surprisingly the elastic scattering rate does not; while there are fewer scattering “targets,” this is compensated by the weaker screening, so the explicit factor of \(f(p)\) in eq. (2.15) is canceled by the implicit factor in \(1/m^2\). But the individual elastic scatterings are by smaller and smaller angles. If the scattering is by a sufficiently small angle, any emitted radiation stays “on top” of the emitter for so long that it can still interfere with an emission from the next scattering, which is called the Landau-Pomeranchuk-Migdal (LPM) effect \(^{14}\), after the scientists who understood the effect in the context of QED. This effect was first understood correctly in the context of QCD by Baier et al ("BDMPS") \(^{15}\). These interference effects will prove to play an important role in the case of a dilute (low initial occupancy) system, so we will review the physics of the LPM effect before we discuss the low-occupancy case.

Consider the quantum mechanical formation time \(t_{\text{form}}\) of a nearly collinear splitting process, defined as the time it takes to separate the wave packet of the emitted daughter particle from the emitting mother particle in the transverse direction. Take \(k\) to be the momentum of the daughter particle and \(k_{\perp}\) its transverse momentum with respect to the mother particle. The transverse size of the wave packet is then \(\Delta x_{\perp} \sim 1/k_{\perp}\), and

\(^{5}\) The timescale also depends on the size of the initial seed fluctuations in \(p \sim L^{-1}\) modes. But at minimum there are vacuum fluctuations, which behave as \(f(p) \sim 1/2\). So the time for the fluctuations to grow from \(f \sim 1\) to \(f \sim \alpha^{-1}\) is \(\sim L^{-1} \ln(\alpha^{-1})\), longer than \(L^{-1}\) by a logarithm.
its transverse velocity $v_{\perp} \sim k_{\perp}/k$, so that the daughter’s wave packet overlaps with the mother particle until the time

$$t_{\text{form}} \sim \frac{\Delta x_{\perp}}{v_{\perp}} \sim \frac{k}{k_{\perp}^2}.$$  

(2.33)

The daughter particle acquires transverse momentum by random elastic scatterings with the other particles in the medium. If $t_{\text{form}}$ is longer than the mean time between such scatterings, we can replace individual scatterings with transverse momentum diffusion and estimate that $k_{\perp}^2 \sim \hat{q}_{\text{elastic}} t_{\text{form}}$, giving the formation time$^6$

$$t_{\text{form}}(k) \sim \sqrt{\frac{k}{\hat{q}_{\text{elastic}}}}.$$  

(2.34)

If the formation time is longer than the time between elastic collisions $\Gamma_{\text{el}}^{-1}$, the kinetic theory treatment of eq. (2.3) is no longer adequate as the diagrams depicted in Fig. 5 interfere coherently, an effect not taken into account in the kinetic treatment. Physically, the emitting particle can no longer resolve individual kicks originating from individual collisions, and effectively sees only the net deflection from all collisions during the formation time; the particle’s trajectory bends smoothly. Then the rate for splitting processes reads$^7$

$$\Gamma_{\text{split}}^{\text{LPM}}(k) \sim \alpha t_{\text{form}}^{-1}(k),$$  

(2.35)

$$\Gamma_{\text{split}}(k) \sim \min[\Gamma_{\text{BH}}^{\text{split}}(k), \Gamma_{\text{split}}^{\text{LPM}}(k)].$$  

(2.36)

Equivalently, the spectrum of split particles changes from Bethe-Heitler log($k$) to LPM type $k^{-1/2}$ spectrum at the momentum scale $k_{\text{LPM}}$ where the formation time is of the same

$^6$In nonabelian theories, both the mother and daughter particles are subject to elastic scattering as the gauge bosons are charged. In an abelian theory the photon is not charged and the formation time depends on the momentum of the mother particle instead of that of the daughter. The analogous estimate in the abelian case is $t_{\text{form}}(p) \sim \sqrt{|p^2|/k_{\text{elastic}}}$, where $p, k$ are the momenta of the mother and daughter respectively.

$^7$Note that the formation time of an inverse splitting process in the previous subsection (overoccupied case) is $t_{\text{form}} \sim (\alpha T_\gamma)^{-1}$. This is of the same order of magnitude as the time between elastic collisions $\Gamma_{\text{el}}^{-1} \sim (\alpha T_\gamma)^{-1}$, so the LPM suppression does not affect the parametric estimates.
order of magnitude as the time between elastic scatterings:

$$t^2_{\text{form}}(k_{\text{LPM}}) \sim \Gamma_{\text{el}}^{-2}$$

$$k_{\text{LPM}} \frac{m^2}{q_{\text{elastic}}} \sim \left(\frac{m^2}{q_{\text{elastic}}}\right)^2$$

$$k_{\text{LPM}} \sim \frac{m^4}{q_{\text{elastic}}}.$$

(2.37)

2.5 \(c < 0\): Low occupancy and “Bottom-up” thermalization

Next we consider the equilibration of an isotropic homogeneous system, whose initial distribution is given again by eq. (2.18) with typical occupancies \(\alpha^{-c} < 1\) \((c < 0)\), so that the initial distribution consists of underoccupied hard particles at a typical scale \(Q > T_{\text{final}}\). The thermalization proceeds in close analogy to the “bottom-up” scenario described in Ref. [12]: First, a population of soft modes with momenta of order \(\sim m\) is generated by soft splitting, induced by small angle elastic collisions between the hard particles. The soft sector subsequently thermalizes in a time \(Qt \sim \alpha^{-2-c/3}\). By this time, the soft sector carries only a small portion of the total energy of the system, but it draws energy from the hard modes by hard LPM suppressed splitting. The system becomes fully thermalized when the hard sector has lost all its energy to the soft sector at \(Qt \sim \alpha^{-2+c/8}\).

2.5.1 \(Qt < \alpha^{-2-c}\)

For the underoccupied initial condition, the screening scale, momentum diffusion constant due to elastic scattering, and small angle elastic scattering rates read

$$\hat{q}_{\text{elastic}} \sim \alpha^2 n_h \sim \alpha^{2-c} Q^3,$$

(2.38)

$$m_h^2 \sim \alpha n_h Q \sim \alpha^{1-c} Q^2,$$

(2.39)

$$\Gamma_{\text{el}} \sim \hat{q}_{\text{elastic}} \sim \alpha Q,$$

(2.40)

where \(n_h \sim \alpha^{-c} Q^3\) is the number of initial hard particles. The index \(h\) indicates that the screening is due to the original hard modes. If the equilibration would proceed via elastic scatterings as in the overoccupied case, the estimate for the equilibration time would be dictated by the large angle collision time for the hardest modes \(t_{eq} \sim Q^2/\hat{q}_{\text{elastic}} \sim \alpha^{2+c} Q^{-1}\). However, as we will see, inelastic scatterings provide a faster route to the thermal distribution.

The hard particles radiate soft gluons creating a new population of particles at small momentum. The production rate of gluons with soft momentum \(k\) by hard particles with momentum \(p\) is Bose stimulated. The production rate of soft particles of momentum \(k\), from eq. (2.17), is

$$\Gamma_{\text{prod}}(k) \sim \int_p \Gamma_{\text{split}}(k)f(p)[1+f_s(k)][1+f(p)] \sim \int_p \Gamma_{\text{split}}(k)f(p)[1+f_s(k)],$$

(2.41)

where \(f_s\) is the distribution of the new soft modes. But the stimulation factor \([1+f_s(k)]\) is cancelled by the inverse process where the soft gluon merges onto a hard particle

$$\Gamma_{\text{absorb}} \sim \int_p \Gamma_{\text{split}}(k)f_s(k)f(p)[1+f(p)] \sim \int_p \Gamma_{\text{split}}(k)f_s(k)f(p),$$

(2.42)
so that the total production rate of soft particles becomes

$$\Gamma_{\text{prod}} - \Gamma_{\text{absorb}} \sim \int p \Gamma_{\text{split}}(k) f(p).$$  \hfill (2.43)

The spectrum of the emitted particles is then

$$f_s(p) \sim n_h \Gamma_{\text{split}}(p) t / p^3,$$  \hfill (2.44)

where $p^{-3}$ arises from the momentum space element. Using eq. (2.16) and eq. (2.35),

$$f_s(p) \sim \alpha n_h \Gamma_{\text{split}}(p) t / p^3,$$  \hfill (2.45)
$$f_s(p) \sim \alpha n_h \sqrt{q_{\text{elastic}} t} / p^{7/2},$$  \hfill (2.46)

with (see eq. (2.37), eq. (2.38) and eq. (2.39))

$$k_{\text{LPM}} \sim \alpha^{-c} Q.$$  \hfill (2.47)

As long as $c > -1$ so that $k_{\text{LPM}} > m_h$, the production of the softest modes is described by the Bethe-Heitler rate.\(^8\) However, interactions can change the actual spectrum from this production rate. In particular, an $f(p) \sim p^{-3}$ spectrum rises more steeply than a thermal spectrum. Below some scale $p_{\text{max}}$ the spectrum will collapse into a thermal-like form,

$$f_s(p) \sim T_\ast / p, \quad \text{for } p < p_{\text{max}}$$  \hfill (2.48)

in close analogy to the overoccupied case discussed in subsection 2.2. The soft sector’s contribution to the screening and elastic scattering are dominated by the scale $p_{\text{max}}$, as they were in the overoccupied case. The major difference is, however, that the growth of the soft sector is driven by soft splittings from the hard particles. In addition to interacting among themselves, the soft particles can scatter with the hard particles.

Let us now determine what physics dominates the evolution of $p_{\text{max}}$. There are four mechanisms which compete in redistributing the soft sector:

- Elastic scattering with hard particles, described by $\hat{q}_{\text{elastic}}$: The scale which has had enough time to experience order 1 momentum changes is

$$p_{\text{max}}^{\text{hard scat}} \sim (\hat{q}_{\text{elastic}} t)^{1/2} \sim \alpha^{1-c/2} Q(Q t)^{1/2}.$$  \hfill (2.49)

All scales below $p_{\text{max}}^{\text{hard scat}}$ have had time to redistribute, via elastic scattering, into an $f(p) \propto 1/p$ thermal-like tail.

- Elastic scattering among soft particles: That the integral in eq. (2.13) diverges for $\propto 1/p^3$ already shows the inconsistency of such a soft distribution. While the total number of soft particles is initially smaller than that of the hard particles, their

\(^8\)For extremely dilute initial conditions $c < -1$, the production of even the softest modes is LPM suppressed. In this case the first soft gluons emerge at the time $t \sim t_{\text{form}}(m_h) \sim \alpha^{-3/4+c/4} Q^{-1}$, after which the evolution will follow the normal $c > -1$ case discussed in the next sub-subsection.
interaction rates are Bose stimulated, so it could be possible that elastic scattering is dominated by the soft sector.

Assuming that the evolution of $p_{\text{max}}$ is dominated by the elastic scatterings between soft particles, we can estimate the momentum diffusion constant arising from the soft sector $\hat{q}_{s,\text{elastic}} \sim \alpha^2 T^2 p_{\text{max}}^{\text{soft scat}}$. This momentum diffusion coefficient rearranges the soft particles out to $p_{\text{max}} \sim (\hat{q}_{s,\text{elastic}} t)^{1/2}$. The energy density of the soft sector is dominated by the particles emitted at scale $p_{\text{max}}$, so the energy density of the soft sector is $\varepsilon_s \sim n_b \Gamma_{\text{split}}(p_{\text{max}}^{\text{soft scat}}) t \sim T_s p_{\text{max}}^3$. Combining and solving self-consistently, we get

$$f_{\text{max}}^{\text{soft scat}} \sim \alpha^{6/5 - 2c/5} Q(Q t)^{3/5}. \quad (2.50)$$

- Merging at the $p_{\text{max}}$ scale: The particle number in the overoccupied regime can decrease by merging pairs of particles to create fewer particles at higher momentum, analogously to the hard merging in the overoccupied case. All modes up to a limiting scale $p_{\text{hard merge}}$ have had time to undergo repeated merging to push the extra particle number to the scale $p_{\text{hard merge}}$.

To estimate the scale $p_{\text{hard merge}}$ we find the hardest modes which have had time to have an order 1 probability of merging

$$\Gamma_{\text{merge}}(p_{\text{max}}^{\text{hard merge}}) t \sim 1 \quad (2.51)$$

with the Bose stimulated rate

$$\Gamma_{\text{merge}}(p) \sim \Gamma_{\text{split}}(p) [1 + f(p)], \quad (2.52)$$

so that

$$p_{\text{hard merge}} \sim \alpha^{4/3 - c/3} Q(Q t)^{2/3}. \quad (2.53)$$

- Saturation: For a given soft momentum $k \ll Q$, emissions will cause $f_s(k)$ to grow until it reaches a size where absorption (joining) processes shut off the production. Let us find the occupancy where this occurs. The production and absorption rates per hard particle $f(p)$ are

$$f(p) (\Gamma_{\text{prod}} - \Gamma_{\text{absorb}}) \sim \Gamma_{\text{split}}(k) \left[ f(p) [1 + f(p - k)] [1 + f_s(k)] - f(p - k) f_s(k) [1 + f(p)] \right]$$

$$\sim \Gamma_{\text{split}}(k) \left[ k f_s(k) \frac{df}{dp} + f(p) [1 + f(p)] \right]. \quad (2.54)$$

This rate, integrated over $p$, goes to zero when

$$f_s(k) \sim \frac{\int_p f(p) [1 + f(p)]}{\int_p -k df/dp}. \quad (2.55)$$

Since the phase space opens up as $p^2 dp$, it must be that $df(p)/dp$ is on average negative. So we can estimate $df/dp \sim -f(p)/p$. Using this estimate and using that $p \sim Q$ is the dominant scale, we find

$$f_s(k) \sim \frac{p}{k} [1 + f(p)] \sim \frac{Q}{k} (1 + \alpha^{-c}). \quad (2.56)$$
(The $1 + \alpha^{-c}$ term is included so the formula can also be applied for overoccupancy.)

Whenever the occupancy of soft modes reaches this value, soft modes are absorbed as quickly as they are emitted. The production mechanism shuts off and the occupancy grows no further. If this shutoff is responsible for deciding where the momentum tail goes from $k^{-3}$ to $k^{-1}$ behavior, then the scale $p_{\text{max}}$ is determined by the condition that the emitted spectrum, $f_s(p_{\text{max}})$ from eq. (2.44), has $f_s(p_{\text{max}}) \sim Q/p_{\text{max}}$. Inserting eq. (2.45), we find

$$p_{\text{saturation}}^{\text{max}} \sim \alpha^{1-c/2}Q(Qt)^{1/2},$$

which turns out to be the same as the scale set by elastic scatterings with hard particles.

At early times the hierarchy of these scales is

$$p_{\text{max}}^{\text{hard scat}} \sim p_{\text{saturation}}^{\text{max}} > p_{\text{max}}^{\text{hard merge}} > p_{\text{max}}^{\text{soft scat}}$$

so that the evolution of $p_{\text{max}}$ is set by elastic scatterings with the hard particles (as well as saturation of soft particle production):

$$p_{\text{max}} \sim p_{\text{max}}^{\text{hard scat}} \sim \alpha^{1-c/2}Q(Qt)^{1/2}.$$

The occupancies in the infrared tail can be solved again by estimating how much energy density the soft sector has had time to eat. The energy density in the infrared tail is dominated by the scale $p_{\text{max}}$ so that

$$\varepsilon_s \sim T_s p_{\text{max}}^3 \sim p_{\text{max}} n_s(p_{\text{max}}),$$

$$n_s(p_{\text{max}}) \sim \Gamma_{\text{split}}^{\text{BH}}(p_{\text{max}}) t_h,$$

$$T_s \sim \Gamma_{\text{split}}^{\text{BH}}(p_{\text{max}}) t_h / p_{\text{max}}^2 \sim Q,$$

showing that $T_s$ is a constant scale.

There are a number of assumptions we have made whose breaking may lead to a revision of the time evolution of $p_{\text{max}}$ and $T_s$. First, we assumed that the hard sector dominates elastic scattering, screening and energy density. Of these, screening is most sensitive to soft modes. The contribution of the soft sector to screening is dominated by the scale $p_{\text{max}}$,

$$m_s^2 \sim \int_p f_s(p) p \sim \alpha T_s p_{\text{max}} \sim \alpha^{2-c/2}Q^2(Qt)^{1/2},$$

which becomes comparable to hard screening at the time (compare eq. (2.62) to eq. (2.39))

$$Qt \sim \alpha^{-2-c}.$$ 

Second, we also made the assumption of the hierarchy given in eq. (2.58), which also breaks down at the same time $Qt \sim \alpha^{-2-c}$, when all four scales reach

$$p_{\text{max}}^{\text{hard scat}} \sim p_{\text{max}}^{\text{saturation}} \sim p_{\text{max}}^{\text{hard merge}} \sim p_{\text{max}}^{\text{soft scat}} \sim k_{\text{LPM}}.$$
2.5.2 $\alpha^{-2-c} < Q t < \alpha^{-2+c/3}$

In the second stage, the scale $p_{\text{max}}$ continues its growth, but as it enters the LPM suppressed regime, the increase of energy to the soft sector is no longer enough to keep $T_s$ a constant scale. At the same time, the soft sector grows large enough to dominate screening, so that the rate of elastic scattering reduces and the new LPM scale follows $p_{\text{max}}$.

Assuming that the evolution of $p_{\text{max}}$ is still set by the elastic scattering off the hard particles, we get the same estimate as before,

$$p_{\text{max}} \sim (\hat{q}_{\text{elastic}} t)^{1/2} \sim \alpha^{1-c/2} Q (Q_t)^{1/2}. \quad (2.65)$$

Then, from the energy density that the infrared tail has had time to eat we get

$$T_s \sim T_{\text{split}}(p_{\text{max}}) t n_h / p_{\text{max}}^2 \sim \alpha^{-1/2-c/4} (Q t)^{-1/4} Q. \quad (2.66)$$

The contribution of the soft screening and elastic scattering from soft particles are then

$$m_s^2 \sim \alpha^{3/2-3c/4} Q (Q t)^{1/4}, \quad (2.67)$$
$$\hat{q}_{s,\text{elastic}} \sim \alpha^{2-c/3} Q^3, \quad (2.68)$$

so that the soft and hard sectors give comparable contributions to elastic scattering but the soft sector dominates screening. Hence, the assumption that $p_{\text{max}}$ can be estimated from hard elastic scattering is selfconsistent. Furthermore, as the dynamics of the soft sector is set by the soft sector itself, we know already from the overoccupied case that the hard merging scale will equal $p_{\text{max}}$.

The growth of $m_s$ decreases the elastic scattering rate, and so the LPM scale starts to grow. The new LPM scale is then

$$k_{\text{LPM}} \sim m_s^4 / \hat{q}_{\text{elastic}} \sim p_{\text{max}}, \quad (2.69)$$

so that using either the BH rate or the LPM rate will give the correct estimate for the energy density of the infrared tail.

When $p_{\text{max}} \sim T_s$, the occupancies at the scale $p_{\text{max}}$ become of order 1 and the soft sector thermalizes. This happens at $Q t \sim \alpha^{-2+c/3}$, and at later times, the estimates again need to be revised.

2.5.3 $\alpha^{-2+c/3} < Q t < \alpha^{-2+3c/8}$

In the final stage of the equilibration the soft particles dominate screening, particle number and $\hat{q}$ but the hard sector still carries most of the energy. The soft sector is thermalized and is described by a single scale, its temperature $T$, with $k_{\text{LPM}} \sim T$, $n_s \sim T^3$, $\varepsilon_s \sim T^4$, and

$$\hat{q}_{\text{elastic}} \sim \alpha^2 T^3. \quad (2.70)$$

The hard sector loses its energy to the soft sector by splitting off daughters, which split further, cascading down in momentum to join the thermal bath. There is a subtlety, first
identified by Baier et al [12]. As discussed, the spectrum of excitations emitted by a hard \( p \sim Q \) particle scales as \( k^{-1/2} dk/k \). In terms of energy lost, the spectrum is dominated by the largest \( k \); in terms of number of excitations, it is dominated by small \( k \). But in a given amount of time \( t \), a sufficiently high-energy daughter will not actually give its energy to the thermal bath, but will remain intact. In terms of energy gain by the thermal bath, the most important emitted daughters are therefore the highest-energy daughters which can break down completely and yield their energy to the thermal bath within time \( t \). These are precisely those excitations which can hard-split in less than time \( t \). The rate for a particle of momentum \( k \) to hard-split is of order the same as the rate for a much harder \( p \sim Q \) particle to emit a particle of momentum \( k \). Therefore the most important daughter momentum scale \( k \) is that, such that each hard particle emits \( O(1) \) such daughters in time \( t \). Call this momentum scale \( k_{\text{split}} \); parametrically (see eq. (2.34), eq. (2.35) and eq. (2.70))

\[
\Gamma^{\text{LPM}}_{\text{split}}(k_{\text{split}}) t \sim 1, \quad k_{\text{split}} \sim \alpha^4 T^3 t^2. \quad (2.71)
\]

The particles with momentum \( k_{\text{split}} \) then undergo multiple splittings, in a time scale which is much shorter than the formation time of the initial emission, depositing all their energy to the soft thermal bath. The energy density and the temperature of the soft sector are then

\[
\varepsilon_s \sim n_h k_{\text{split}}, \quad T \sim \varepsilon_s^{\frac{1}{4}}, \quad (2.72)
\]

and solving self consistently we get for the temperature and for the splitting scale

\[
T \sim \alpha^4 n_h t^2 \sim \alpha^{4-c} Q^3 t^2, \quad (2.73)
\]

\[
k_{\text{split}} \sim \alpha^{16-3c} Q(Q t)^8. \quad (2.74)
\]

By the time the splitting scale reaches \( Q \), all of the hard modes have had time to undergo hard splitting and join the thermal bath, and the system thermalizes when

\[
Qt \sim \alpha^{-2+3c/8}, \quad T_{\text{final}} \sim \alpha^{-c/4} Q. \quad (2.75)
\]

The final equilibration time should be compared to the characteristic time scale \( t \sim \alpha^{-2T^{-1}} \sim \alpha^{-2+c/4} Q \) for the thermal bath to self-equilibrate. The equilibration time is longer but only by a factor of \( \sqrt{Q/T} \sim \alpha^{c/8} \sim f_{\text{initial}}^{-\frac{2}{5}} \). In particular, equilibration is much faster than the naive large-angle scattering time for the original hard excitations, \( t_{\text{eq}} \sim \alpha^{-2+c} Q^{-1} \). Physically, the factor \( \sqrt{Q/T} \) arises because the time it takes a particle of momentum \( Q \gg T \) to radiate away its energy in a thermal bath of temperature \( T \) is \( t \sim (\alpha^2 T)^{-1} \sqrt{Q/T} \). So the final equilibration time is just the time for the hard excitations to break up in a thermal bath at the final temperature.

3 Weakly anisotropic systems

In some applications we expect a nonabelian plasma to form, in which, rather than being evenly distributed in direction, particle occupancy is larger in some directions than in
others. In some cases the excess occupation may be modest. For instance, a nonabelian plasma which is experiencing shear flow develops an anisotropic particle distribution. This has been discussed by Asakawa et al. [20], who argue that in a range of shear flow strengths, the plasma remains nearly isotropic but only because of the physics of plasma instabilities. Weakly anisotropic plasmas should also be rather generic in the presence of phase interfaces during early Universe phase transitions, such as the electroweak phase transition (if it exists).

After an overview of the physics of plasma instabilities, we will first treat this case of weakly anisotropic plasmas, and then consider plasmas with a very high degree of anisotropy in the next section. Since anisotropic plasmas are rather complex and because the “parameter space” describing them is larger than for the isotropic case, we will not attempt to follow the dynamics through every stage to equilibrium. Instead we will identify the relevant dynamics initially, and then determine the time scale before the dynamics change appreciably and what the new relevant dynamics becomes.

3.1 Plasma instabilities

So far we have only considered systems which are statistically locally isotropic. A new complication enters when we consider systems which are locally anisotropic. Namely, certain long-wavelength magnetic gauge field excitations are then generically unstable to exponential growth, in a process called the Weibel instability (see [6] for the electromagnetic case and [7–10] for an overview of the nonabelian case and [21–23] for some nonabelian numerical studies). At weak coupling, in many circumstances these fields are expected to grow large enough that they come to dominate much of the dynamics of the system. In particular, the “hard” excitations which dominate the energy density of the system may predominantly change direction by deflection in these magnetic fields. These deflections
may then dominate the splitting or joining processes these excitations undergo. The key physics involved in plasma instabilities is the physics of screening. So far we have only needed a very crude description of plasma screening, but plasma instabilities involve finer details of the screening process. Therefore we begin with a more careful discussion of the physics of screening.

There are already nice, physical and quantitative discussions of plasma instabilities in the literature, see for instance section II of Ref. [9] and Refs. [8, 10]. Nevertheless we will give a quick qualitative overview to capture certain salient points for our discussion. We start by explaining how to think about the screening scale \( m \) we previously introduced.

Consider the case where many particle-like excitations\(^9\) all move along the \( \pm x \) axis, in the presence of a magnetic field with wave-vector \( \mathbf{k} \) in the \( y \) direction and \( \mathbf{B} \) field in the \( z \) direction, as illustrated in Fig. 7. The force on a particle is

\[
F = q^a v \cdot B^a. \tag{3.1}
\]

If the particle trajectories are undeflected at time \( t = 0 \), then by time \( t \) their velocities and positions have changed, to linear order in \( B \), by

\[
v_y(t) = \frac{q^a B^a t}{p}, \quad \delta y(t) = \frac{q^a B^a t^2}{2p}. \tag{3.2}
\]

This particle motion concentrates particles in some regions and depletes them in others. The concentration fraction is \( k \delta y(t) \), and the net current associated with this concentration is

\[
J^a \sim \int d^3 p f(p) q^a k \delta y(t) \sim \int d^3 p f(p) \frac{q^a q^b}{2p} kB^b t^2. \tag{3.3}
\]

Here \( \int d^3 p f(p) \) is just counting the density of particles. We recognize the combination

\[
\int d^3 p f(p) \frac{q^a q^b}{p} = \delta_{ab} g^2 T_r \int \frac{d^3 p f(p)}{p} \sim \delta_{ab} m^2
\]

so the induced current is

\[
J^a \sim kB^a m^2 t^2. \tag{3.5}
\]

This current enters the equations of motion for the magnetic field, where it competes with terms of form \( \nabla \times B \sim kB \). Therefore the current term becomes important in the evolution of a magnetic field whenever \( m^2 t^2 \sim 1 \). So the correct physical interpretation of the thermal mass \( m \) is that it is the inverse of the time scale for particle deflections to add up to enough that they influence the gauge field dynamics. That is, if particles remain coherently in a region of electric or magnetic field of one sign for time scales of order \( 1/m \) or longer, then...

---

\(^9\)By particle-like excitations we mean, excitations which can be described in terms of wave packets which are of smaller extent than the coherence length of the classical fields in which they move; and yet are built of a narrow enough distribution of momenta that they move nearly non-dispersively. If the typical particle momentum is \( p \), the range of momenta particles carry (and therefore the range of momenta out of which wave packets are built) is \( \Delta p \), and the characteristic wave number of the background fields they move in is \( k \), this requires \( p \gg \Delta p \gg k \). We will encounter a case where these criteria cannot be met in subsection 4.8, but in most of our discussion the scale ordering will be true parametrically.
the induced currents are large enough to significantly influence the field evolution. In the case illustrated, since the particles enhance the magnetic field which deflects them, they lead to an exponential growth in the strength of the magnetic field $B(t) \sim B(t=0) \exp(\gamma t)$, with a growth rate $\gamma \sim m$.

For the case considered in Fig. 7 the currents from the particle excitations strengthen the magnetic field, which destabilizes the plasma towards magnetic field growth. Particles flying in certain other directions contribute currents which oppose the magnetic field. For an isotropic distribution of particle momenta, a magnetic field will not induce any current. That is because a magnetic field induces an overall rotation of the distribution of particles; but if the particle distribution is isotropic then a rotation does not change it. Therefore an isotropic distribution is neutral in the sense that it neither stabilizes nor destabilizes magnetic fields. For an anisotropic distribution of particles, we can instead ask about the average over directions and polarizations of magnetic fields. This is equivalent to averaging over the particles’ momentum distribution. Hence, for anisotropic systems, the plasma’s impact on magnetic fields, averaged over directions and polarizations of magnetic fields, is neutral. That means that there will be some directions and polarizations where the particles stabilize the magnetic fields (generate currents which oppose the magnetic field), while for other directions and polarizations they will destabilize the magnetic fields (generate currents which enhance the magnetic field). Hence, in an anisotropic plasma, magnetic instabilities will always be present in some field directions and polarizations.

In practice we will be interested in magnetic fields which change with time, which always means there are also electric fields present. Electric fields are stabilized by an isotropic distribution of particles, so time-changing fields tend to be more stable than static magnetic ones. Also, if the electromagnetic fields vary on too short of a length scale, particles will not have trajectories which stay in a coherent region of field strength for the requisite $t \gtrsim 1/m$ needed for currents to dominate the Yang-Mills equations of motion. To make somewhat more precise statements about which modes are unstable under which circumstances, we will sketch a diagrammatic evaluation.
At the linearized level, an instability will occur if the retarded propagator has a pole for some wave number $k$ and pure imaginary frequency $\omega = i\gamma$, corresponding to exponential growth. The transverse part of the inverse retarded propagator is

$$G_{ij}^{-1}(k, \gamma) = (k^2 + \gamma^2)(\delta_{ij} - \hat{k}_i\hat{k}_j) - \Pi_{ij}(k, \gamma).$$

(3.6)

If the self-energy loop momentum $p$ is large compared to the external momentum $k$, then the self-energy behavior becomes spin independent and we can treat the simplest case, which is a scalar in the loop. For this case,

$$\Pi_{\mu\nu}(k) \sim g^2 \int d^4p \left[ -2g^{\mu\nu}\delta(p^2)f(p) + (2p + k)^\mu(2p + k)^\nu \frac{\delta(p^2)f(p)}{(p + k)^2} \right. + \left. (2p - k)^\mu(2p - k)^\nu \frac{\delta(p^2)f(p)}{(p - k)^2} \right].$$

(3.7)

Assuming $p \cdot k \gg k^2$, we can expand in $2p \cdot k \gg k^2$. Together with the spin-independent form for the self-energy, this constitutes the hard-loop approximation for the self-energy. In this approximation, the self-energy is

$$\Pi_{ij} = -g^2 Tr \int \frac{f(p)}{p} \left[ \delta_{ij} - \frac{k_i v_j + k_j v_i}{v \cdot k - i\gamma} + \frac{(k^2 + \gamma^2)v_i v_j}{(v \cdot k - i\gamma)^2} \right].$$

(3.8)

Positive-sign contributions inside the square brackets contribute positively to eq. (3.6), stabilizing the field. Negative contributions destabilize the field, and if they are large enough – if the RHS of eq. (3.8) equals or is larger than $2(k^2 + \gamma^2)$ – then they allow an exponential instability. We see that those particles in the angular range $v \cdot k < \gamma$ are the ones contributing to the instability; all others stabilize.

The above discussion is based on linearizing in the size of the magnetic field, and possibly in other interactions. It breaks down when higher loop contributions to the self-energy become as large as the one-loop self-energy discussed above. Most of the time, the dynamics controlling the equilibration of the full system take place on a time scale long compared to $1/\gamma$ the growth rate of the instabilities. In this case, the instabilities will grow until some nonlinear physics limits their size. After this, the relevant magnetic fields will remain large, evolving but in a “quasi-stationary” fashion so that statistical properties of their distribution evolve only on the time scale of the system’s approach to equilibrium. The saturation of instabilities and quasi-stationary evolution have been verified in numerical studies [11, 24, 25].

### 3.2 Instabilities for weak anisotropy

Consider a system along the lines of what we discussed in the previous sections; that is, a distribution of “particles” with a typical momentum scale $p \sim Q$, sufficiently fast falloff in $f(p)$ above this scale, and typical occupancies $f(p) \sim \alpha^{-c}$ for $p \sim Q$. We will consider both the possibility $c > 0$ (overoccupancy) and $c < 0$ (underoccupancy). But now we will
consider the case where \( f(p) \) is also a function of the direction \( \hat{p} \). In this section we will take this angular dependence to be weak,

\[
f(p) = f(p) \times (1 + \epsilon F(\hat{p}))
\]

(3.9)

with \( F(\hat{p}) \) some order-1 combination of spherical harmonics of number 2 and higher, but dominated by low numbers, and \( \epsilon \) characterizing how small the non-spherical component of \( f(p) \) is. We also define \( d \) as \( \epsilon \equiv \alpha^{-d} \) (note that \( d \) is negative) to make it easier to mix powers of \( \alpha \) and of \( \epsilon \).

Inevitably the distribution we consider here is not the most general we could consider. While there is no need to allow \( Y_{1m}(\hat{p}) \) content in \( F(\hat{p}) \) we could have considered cases where \( f(p) \) does not factorize into a radial and an angular function, that is, where the typical particle energy is larger in some directions than in others.

First we compute the screening scale and elastic scattering \( \hat{q} \),

\[
m^2 \sim \alpha Q^2 f(Q) \sim \alpha^{1-c} Q^2, \quad \hat{q}_{\text{elastic}} \sim \alpha^2 Q^3 f[1+f] \sim \left\{ \begin{array}{ll} \alpha^{2-2c} Q^3 & c > 0, \\ \alpha^{2-c} Q^3 & c < 0. \end{array} \right. \tag{3.10}
\]

To determine whether plasma instabilities play any role in this system, we need to determine what gauge field modes become plasma-unstable, how large the amplitudes of the associated magnetic fields grow, and how much deflection these magnetic fields cause in the hard particles. That is, we need to compute \( \hat{q}_{\text{inst}} \) arising from soft-unstable magnetic fields and compare it to \( \hat{q}_{\text{elastic}} \) to determine whether the instabilities play an important role.

First let us see how many modes are unstable and how fast they grow. As discussed in the last section, a mode is unstable and grows if the inverse retarded propagator, eq. (3.6), vanishes for an imaginary frequency \( \omega = i\gamma \). \(-\Pi_{ij}\) gets more positive as \( \gamma \) is increased, so the range of \( k \) which are unstable is determined by considering \( \gamma = 0^+ \). The isotropic part of the integral vanishes for \( \gamma = 0^+ \) and gives \( \pi \gamma / k \) for small \( \gamma \). The anisotropic part averages to zero over \( \hat{k} \) directions, but is positive in some directions and negative in other directions, \( \epsilon \) in unstable directions

\[
- \Pi_{ij} \sim \frac{m^2}{2} \left( \frac{\pi \gamma}{k} - \epsilon \right). \tag{3.11}
\]

Applying this we find instabilities in about half of directions, with maximum \( k \) of

\[
k_{\text{inst}}^2 \sim \epsilon m^2. \tag{3.12}
\]

A simpler way to get this scale would be to compute \( m^2 \) just from the anisotropic piece of the particle distribution, which gives \( m_{\text{inst}}^2 \sim \epsilon m^2 \). The unstable modes are those with \( k_{\text{inst}}^2 \sim m_{\text{inst}}^2 \). So long as \( \epsilon \leq 1 \) and \( c \leq 1 \), the scale \( k_{\text{inst}} \ll Q \) and so the “hard-loop” approximation used to establish these estimates is self-consistent.

The maximal growth rate is roughly \( \gamma \) such that the \( m^2 \pi \gamma / 2k \) term cancels the negative term, giving

\[
\gamma \sim \epsilon k_{\text{inst}} \sim \epsilon^4 m. \tag{3.13}
\]

\[\text{[That is, the } Y_{00}(\hat{p}) \text{ component defines } f(p). \text{ Any } Y_{1m}(\hat{p}) \text{ component can be removed by a choice of rest frame.]}\]
Figure 8. A cartoon illustrating how color rotation could inhibit plasma instabilities. A blue-anti-red magnetic field deflects blue and red color charges. However where color rotation occurs, the induced current will be in another color direction (such as green). This does not amplify the inducing magnetic field because it is in the wrong color direction; hence plasma instabilities are inhibited.

This is not the answer we would get by just considering the contribution of the unstable modes. That is because a time-varying gauge field always has an electric component, and an isotropic distribution of particles exerts a stabilizing effect on electric fields. So while the isotropic part of the particle distribution does not change which modes are unstable, it can slow down the growth of the unstable modes. (This is an example of the general phenomenon that scales $k \ll m$ have slow dynamics [26]).

This analysis of which modes are unstable is perturbative. The mode amplitudes grow until the perturbative treatment breaks down, which will occur when some nonabelian term becomes important somewhere in the above calculation. The evaluation of the hard loop assumes particles can propagate a distance $1/k_{\text{inst}}$ without color rotating. Color rotation destroys the growth of plasma instabilities, as illustrated in figure 8: when particles color-rotate, the current they induce is in a different color direction than the magnetic field which generated the current. Hence the current does not enhance the generating magnetic field. Color rotation is important when a Wilson line $U = \text{Pexp} ig \int dl \cdot A$ of length $l \sim k_{\text{inst}}^{-1}$ generically contains an order-1 phase factor. A gauge-invariant criterion, for such phase factors to be large, is that a Wilson loop of size $1/k_{\text{inst}}$ on a side have an order-1 phase factor, see Fig. 9. The phase factor of such a Wilson loop is $g^{-1}B/k_{\text{inst}}^2$. So color randomization is important if $B^2 \sim \alpha^{-1}k_{\text{inst}}^4$. This translates to an occupancy for the unstable modes of $O(\alpha^{-1})$. In other words, the growth of plasma instabilities is shut off when plasma-unstable modes have nonperturbatively large occupancies. We would arrive at the same conclusions by investigating where mutual interactions between such modes significantly influence their evolution. We would also reach the same conclusions by asking where two-loop self-energy corrections grow to be as large as the one-loop self-energy correction we studied.
Figure 9. Setup for the Weibel instability as in Fig. 7, but with a Wilson loop indicated. When the phase around the Wilson loop is order-1, then in any gauge one or both of the indicated particle trajectories suffers a large color rotation. Choosing the Wilson loop to be of length $\sim 1/m$ and width $\sim 1/k$, an order-1 phase in the Wilson loop is sufficient to ensure that the type of color rotation illustrated in Fig. 8 will occur over the length scale relevant for the development of the instability.

3.3 Momentum diffusion from the instability

Next we compute the mean squared rate at which these unstable modes deflect the excitations of momentum $\sim Q$. Such an excitation feels a time varying force of order $F \sim gB$. This force remains coherently in the same direction for the characteristic coherence length of the magnetic field, $t_{coh} \sim k_{inst}^{-1}$. (This is also the typical time scale it takes the particle to rotate in color, which would also randomize the direction of the force.) Provided $c < 1$ and $\epsilon < 1$, the momentum accumulated in one coherence length,

$$\Delta p \sim t_{coh} gB \sim k_{inst}^{-1} g \frac{k_{inst}^2}{g} \sim k_{inst} \sim \epsilon^\frac{1}{2} \alpha^\frac{1}{2} Q^3$$  \hspace{1cm} (3.14)$$

is small compared to the typical particle momentum $Q$. Therefore the change in particle momentum will accumulate as many small incoherent “kicks” and can be well characterized by a momentum diffusion coefficient $\hat{q}_{inst}$,

$$\hat{q}_{inst} \sim F^2 t_{coh} \sim \alpha B^2 / k_{inst}.$$  \hspace{1cm} (3.15)$$

We already saw that the magnetic field saturates at an amplitude $B^2 \sim \alpha^{-1} k_{inst}^4$. Therefore

$$\hat{q}_{inst} \sim \alpha B^2 t_{coh} \sim \frac{\alpha B^2}{k_{inst}} \sim k_{inst}^3 \sim \epsilon^\frac{3}{2} m^3 \sim \epsilon^\frac{3}{2} \alpha^\frac{3}{2} (1-c) Q^3.$$  \hspace{1cm} (3.16)$$

The instability is important if this $\hat{q}$ dominates the scattering from elastic processes, but we expect it to be irrelevant if it is smaller than the scattering already occurring due to normal elastic processes. So the domain where the instability is important is

$$\hat{q}_{inst} > \hat{q}_{elastic}, \hspace{1cm} \epsilon^\frac{3}{2} \alpha^\frac{3}{2} (1-c) Q^3 > \alpha^{2-(2.1) c} Q^3$$  \hspace{1cm} (3.17)$$
leading to

\[
\text{Instability important if } \begin{cases} 
\epsilon > \alpha \frac{1+c}{1-2} c < 0, \\
\epsilon > \alpha \frac{3+c}{1-2} c > 0.
\end{cases}
\] (3.18)

When the instability is important, it can change two things. One thing it can do is to enhance angle change, erasing \(\epsilon\). The other thing it can do is for this angle change to enhance splitting/joining processes. This can cause \(c\) and \(\epsilon\) to evolve, and in the case of low occupancy it can create a “cloud” of soft excitations which can dominate the anisotropy of screening even if they do not replace the hard particles numerically. It cannot lead to an evolution of \(c\) through elastic scattering because soft unstable fields are primarily magnetic, and only electric fields change particle energy. To estimate the electric field subdominance, note that \(E \sim \partial_t A\) while \(B \sim \partial_x A\), so \(E \sim (\gamma/k_{\text{inst}})B \sim \epsilon B\). Therefore the diffusion of particle energy \(\sim E^2 t_{\text{coh}}\) is slow compared to the diffusion of particle direction. The time scale for particle energy to change is correspondingly longer than the time scale for \(\epsilon\) to be erased, ending the instability.

The rate for \(\epsilon\) to change appreciably due to scattering is the rate for large angle change, which is

\[
t_{\text{large angle}}^{-1} \sim \frac{\hat{\gamma}_{\text{inst}}}{Q^2} \sim \epsilon^{\frac{3}{2}} \alpha^{\frac{7}{4}} Q^{\frac{3}{2}} (1-c)^{\frac{3}{4}} Q^{\frac{1}{2}}. 
\] (3.19)

This time scale is larger (rate smaller) than the growth rate of the unstable modes, \(\gamma \sim \epsilon^{\frac{3}{2}} \alpha^{\frac{7}{4}} Q\), so the treatment is self-consistent in that the unstable modes have time to grow before they significantly modify \(\epsilon\).

To find the rate at which splitting/joining processes occur, we need to determine the formation time of emitted particles. Moving through the soft-unstable fields, an emitted particle picks up a kick of \(k_{\text{inst}}^2\) every length \(1/k_{\text{inst}}\), which is the effective inter-scattering distance. The formation time for radiation of a particle of energy \(k\) is therefore

\[
t_{\text{form}}(k) \sim \frac{k}{k_{\perp}} \sim \frac{k}{\hat{q} t_{\text{form}}} \Rightarrow t_{\text{form}}(k) \sim k^{-1} k_{\text{inst}}^{-\frac{3}{2}} \sim k^{-1} \epsilon^{-\frac{3}{2}} \alpha^{-\frac{7}{4}} (1-c)^{\frac{3}{4}} Q^{-\frac{3}{2}}
\] (3.20)

which is always longer than \(1/k_{\text{inst}}\). Therefore split/join processes are always in the LPM regime. The rate of these processes is a factor of \(\alpha[1+f(k)]\) over the formation time;

\[
\Gamma_{\text{split/join}}(k) \sim \alpha[1+f(k)] t_{\text{form}}^{-1} \sim k^{-\frac{3}{2}} \epsilon^{-\frac{3}{4}} \alpha^{-\frac{7}{4}} Q^{\frac{3}{2}} [1+f(k)]. 
\] (3.21)

To determine whether this rate is physically important, we have to consider the physics of the particles generated.

### 3.4 Joining and \(\epsilon\) suppression for overoccupancy

For the case \(c > 0\) \((f(Q) > 1)\) we need not worry about splitting processes; we saw in section 2.2 that \(f(p < Q)\) scales at most as \(1/p\), which is too soft to dominate screening. We also see, combining eq. (3.21), eq. (3.19) and eq. (3.18), that \(\Gamma_{\text{split/join}} t_{\text{large angle}} \ll 1\). Therefore the particles randomize direction, and \(\epsilon\) is degraded, before particle number can change appreciably.

Nevertheless it is possible that \(\epsilon\) is significantly affected by joining processes. The rate of this process is \(\Gamma_{\text{split/join}}(Q) \sim \epsilon^{\frac{3}{2}} \alpha^{\frac{7}{4}} (1-c) Q\) \((\text{eq. (3.21)}\) with \(k \sim Q\) and \(f \sim \alpha^{-c}\)). But what
this expression disguises is that this rate is anisotropic in direction, at the order one level. The reason is that \( \hat{q} \) arising from plasma instabilities is in general strongly anisotropic, and peaked for those particles which are overoccupied. So what can happen is the following. Only an \( \mathcal{O}(\epsilon) \) fraction of particles are anisotropic; but they generate instabilities, which cause an order-one anisotropic \( \hat{q} \). This makes the rate of joining processes faster, by an order-one amount, in the directions which have an excess of particles. Joining reduces the number of particles moving in those directions, which is precisely a reduction in \( \epsilon \).

The anisotropy is erased by the joining when an \( \epsilon \) fraction of particles in the overoccupied direction have had time to join, creating an \( \epsilon \) change in \( f(p) \). This takes place at the time

\[
t_{\epsilon} \sim \epsilon \Gamma_{\text{split/join}}^{-1} \sim \epsilon^{\frac{1}{4}} \alpha^{-\frac{3}{2}+\frac{7}{4}} Q^{-1}.
\] (3.22)

This must be compared to the rate at which \( \epsilon \) is degraded by angular deflection; the anisotropy is also erased in a time it takes to redistribute the directions of particles \( t_{\text{large angle}} \). Whichever of these processes is faster dominates, so that the dominant mechanism bringing down \( \epsilon \) is particle joining if

\[
t_{\epsilon} < t_{\text{large angle}} \sim \epsilon^{\frac{1}{4}} \alpha^{-\frac{3}{2}+\frac{7}{4}} Q^{-1},
\]

\[
\epsilon < \alpha^{\frac{1}{2}}(1-c).
\] (3.23)

Otherwise anisotropy is mostly removed by angle change due to \( \hat{q} \). The time scale for isotropization is

\[
t_{\text{change}} \sim \begin{cases} 
  t_{\epsilon} \sim \epsilon^{\frac{1}{4}} \alpha^{-\frac{3}{2}+\frac{7}{4}} Q^{-1}, & \epsilon < \alpha^{\frac{1}{2}}(1-c), \\
t_{\text{large angle}} \sim \epsilon^{\frac{1}{4}} \alpha^{\frac{3}{2}(1-c)} Q^{-1}, & \epsilon > \alpha^{\frac{1}{2}}(1-c).
\end{cases}
\] (3.24)

In either case, after time \( t_{\text{change}} \) the plasma has become nearly isotropic, and equilibration proceeds as described in subsection 2.2. Note that, in every case, \( t_{\text{change}} \) is short compared to \( \alpha^{-2}T_{\text{final}}^{-1} \), the full equilibration time found in subsection 2.2.

### 3.5 Soft emissions and \( \epsilon \) for underoccupancy

Next consider the case \( c < 0 \), so \( f(Q) < 1 \) (underoccupancy). In this case the rate of hard particles to split off softer particles is larger than the rate for them to join. Soft particles are more efficient at screening than hard particles. The splitting rate is also highly anisotropic, because \( \hat{q} \) is. Therefore, more soft particles get generated in the directions where there is already a particle number excess. This tends to increase the anisotropy of screening, potentially enhancing the strength of plasma instabilities. In fact we will see that this happens over most, but not all, of the parameter range of interest.

The soft particle production is in the LPM regime and, analogously to the isotropic case, the spectrum of the soft particles is given by eq. (2.46), eq. (2.48) (replacing \( \hat{q}_{\text{elastic}} \) with the dominant \( \hat{q}_{\text{inst}} \)),

\[
f_s(p) \sim \alpha n_h \sqrt{\hat{q}_{\text{inst}} t/p}^{7/2}, \quad \text{for } p > p_{\text{max}},
\] (3.25)

\[
f_s(p) \sim T_s/p, \quad \text{for } p < p_{\text{max}}.
\] (3.26)
In this case the evolution of $p_{\text{max}}$ is dominated by saturation, eq. (2.56), so that $f_s(p)$ is limited from above by $f_s(p) \lesssim Q/p$ and $T_\ast \sim Q$. We find $p_{\text{max}}$ by equating eq. (3.25), eq. (3.26) and using eq. (3.16):

$$T_\ast \sim Q, \quad p_{\text{max}} \sim \hat{q}_{\text{inst}}^{1/5} (\alpha n_h/Q)^{2/5} (\frac{2}{3})^{2/5} \sim \epsilon \frac{3}{4} \alpha^{\frac{3}{4}} (1-c) Q Q t^{2/5}. \quad (3.27)$$

As the $\hat{q}_{\text{inst}}$ is $\mathcal{O}(1)$ anisotropic, the soft particles are created with an $\mathcal{O}(1)$ anisotropy. The modes are then driven towards isotropy by momentum diffusion dominated by $\hat{q}_{\text{inst}}$. The anisotropy of the soft modes is described by $\epsilon_s(k)$, and how anisotropic they are depends on the momenta of the soft particles and how large $t$ is. The momentum scale which is just becoming isotropic is

$$k_{\text{iso}} \sim \sqrt{q_{\text{inst}} t} \sim \epsilon \frac{3}{4} \alpha^{\frac{3}{4}} (1-c) Q (Q t t)^{1/2}. \quad (3.29)$$

At early times $p_{\text{max}}$ is larger than $k_{\text{iso}}$, while for large $t$ the larger of the two scales is $k_{\text{iso}}$. In particular, the time when $k_{\text{iso}}$ catches up with $p_{\text{max}}$ is

$$t_{\text{iso}}(p_{\text{max}}) \sim \frac{m_h^8}{q_{\text{inst}}} \sim \epsilon \alpha^{-\frac{5}{2}(1-c)} Q^{-1}. \quad (3.30)$$

For $k > k_{\text{iso}}$, the soft particles have the order 1 anisotropy they where created with, $\epsilon_s(k > k_{\text{iso}}) \sim 1$. Below $k_{\text{iso}}$, the particles that were created less than $t_{\text{iso}}(k) \sim k^2/\hat{q}_{\text{inst}}$ ago are still anisotropic; particles created longer ago have had time to undergo large angle change and isotropize. The fraction of anisotropic particles, and hence $\epsilon_s$, at the scale $k < k_{\text{iso}}$ is then

$$\epsilon_s(k) \sim t_{\text{iso}}(k)/t \sim k^2/k_{\text{iso}}^2 \quad \text{for} \quad k < k_{\text{iso}}, \quad (3.31)$$

$$\epsilon_s(k) \sim 1 \quad \text{for} \quad k > k_{\text{iso}}. \quad (3.32)$$

The new anisotropic population of soft particles can subsequently cause new plasma instabilities which contribute to $\hat{q}_{\text{inst}}$, and may eventually dominate it. The contribution to $\hat{q}_{s,\text{inst}}$ from the new soft particles at the scale $k$ is

$$\hat{q}_{s,\text{inst}}(k) \sim \epsilon_s^{3/2} (k) m_s^3(k), \quad (3.33)$$

where $m_s(k)$ is the screening from particles at scale $k$. The scale $p_{\text{max}}$ always dominates soft-particle contributions to screening: for $k < p_{\text{max}}$, the distribution only grows as $1/k$ and screening depends on $\int kdkf(k)$. On the other hand for $k > p_{\text{max}}$ the soft distribution falls as $k^{-\frac{3}{2}}dk$, so that screening falls as $\int k^{-\frac{3}{2}}dk$, and gets its dominant contribution again from the scale $p_{\text{max}}$. Then, if $p_{\text{max}} > k_{\text{iso}}$ so that the particles which dominate screening are order-1 anisotropic, then also $\hat{q}_{s,\text{inst}}$ is dominated by the particles at the scale $p_{\text{max}}$. In this case, the new $\hat{q}_{s,\text{inst}}$ is

$$\hat{q}_{s,\text{inst}} \sim m_s^3(p_{\text{max}}) \sim \alpha^{3/2} p_{\text{max}}^{3/2} f_s^{3/2} \sim \epsilon \frac{3}{4} \alpha^{\frac{3}{4}} (1-c) Q^3 (Q t)^{3/5}, \quad \text{for} \quad p_{\text{max}} > k_{\text{iso}}. \quad (3.34)$$
On the other hand, if $k_{\text{iso}} > p_{\text{max}}$, then the scale $p_{\text{max}}$ may dominate screening but it will not dominate soft contributions to $\hat{q}$ as $\epsilon_s(k) \propto k^2$ decreases too fast with decreasing $k$. So the contribution to $\hat{q}$ scales as $\int^{k_{\text{iso}}} k^{-\frac{5}{2}} k^2 dk \sim \int^{k_{\text{iso}}} k^{-\frac{1}{2}} dk$ which is dominated by $k_{\text{iso}}$, and

$$
\hat{q}_{\text{s,inst}}(t) = m_s^3(k_{\text{iso}}) \sim \left[ \alpha k_{\text{iso}}^2 f_s(k_{\text{iso}}) \right]^{3/2} \sim \alpha^3 n_h \hat{q}_{\text{inst}}^{3/8} t^{3/8}
$$

$$
\sim \epsilon^{-\frac{9}{16}} \frac{\alpha^{\frac{11}{8}}}{\alpha^3} Q^2(Qt)^{\frac{1}{8}},
$$

for $p_{\text{max}} < k_{\text{iso}}$. (3.36)

As the soft sector grows, the new $\hat{q}_{\text{s,inst}}$ increases and it may start to dominate over the $\hat{q}_{\text{inst}}$ from the hard particles; we define $t_{\text{change}}$ as the time when

$$
\hat{q}_{\text{s,inst}}(t_{\text{change}}) \sim \hat{q}_{\text{inst}}.
$$

We then have three possibilities:

- The soft $\hat{q}_{\text{s,inst}}$ starts to dominate $\hat{q}$ at early times when particles at $p_{\text{max}}$ are still fully anisotropic, $p_{\text{max}}(t_{\text{change}}) > k_{\text{iso}}(t_{\text{change}})$. In this case the soft sector takes over $\hat{q}_{\text{inst}}$ at time

$$
t_{\text{change}} \sim \epsilon^{-\frac{7}{4}} \alpha^{\frac{7}{8}} \frac{Q}{Q}.
$$

This is consistent with the assumption for (see eq. (3.30), eq. (3.38))

$$
t_{\text{change}} < t_{\text{iso}}(p_{\text{max}}) \quad \Rightarrow \quad \epsilon < \alpha^{\frac{1}{8}}.
$$

This occurs if

$$
t_{\text{iso}}(p_{\text{max}}) < t_{\text{change}} < t_{\text{large angle}} \quad \Rightarrow \quad \alpha^{\frac{11}{8}} \epsilon^{\frac{1}{8}} \alpha^{\frac{1}{8}} \sim \epsilon^{\frac{11}{4}} \alpha^{\frac{1}{8}} - \frac{9}{2} \epsilon Q^{-1}.
$$

This is the Region 2 in Fig. 10.

- The new plasma instabilities start to dominate after the scale $p_{\text{max}}$ has isotropized, but before time $t_{\text{large angle}}$ which is when $k_{\text{iso}} \sim Q$ and momentum diffusion has isotropized all modes. In this case the soft sector begins to dominate $\hat{q}_{\text{inst}}$ by the time (see eq. (3.35))

$$
\hat{q}_{\text{inst}}(t_{\text{change}}) \sim \hat{q}_{\text{s,inst}} \quad \Rightarrow \quad t_{\text{change}} \sim \alpha^{-8} Q^{1/4} Q^{1/4} \sim \epsilon^{-\frac{11}{4}} \alpha^{\frac{1}{8}} - \frac{3}{2} \epsilon Q^{-1}.
$$

This occurs if

$$
t_{\text{iso}}(p_{\text{max}}) < t_{\text{change}} < t_{\text{large angle}} \quad \Rightarrow \quad \alpha^{\frac{11}{8}} \epsilon^{\frac{1}{8}} \alpha^{\frac{1}{8}} - \frac{9}{2} \epsilon Q^{-1}.
$$

This is the Region 3 in Fig. 10.

In this regime the evolution of $p_{\text{max}}$ may cease to be controlled by saturation before $t_{\text{change}}$. In particular if $\epsilon > \alpha^{\frac{11}{8}}$, elastic scattering among the soft particles will start to push $p_{\text{max}}$ to higher scales (reducing $T_*$ as in the isotropic case) before the new instabilities take over momentum diffusion. However, $\hat{q}$ remains dominated by particles at the scale $k_{\text{iso}}$, so this can have no effect on $t_{\text{change}}$.

- The third possibility is that soft particles never grow to dominate $\hat{q}_{\text{inst}}$ and the large angle changes erase the anisotropy, which occurs if

$$
\epsilon > \alpha^{\frac{11}{8}}.
$$

(3.42)
Figure 10. For weak anisotropy, the $d$, $c$ plane (representing the level of anisotropy and of over- or underoccupancy) divides into four regions. In region 1, plasma instabilities induce particle joining processes which erase the anisotropy. In regions 2 and 3, particle splitting creates soft particles which are anisotropic and generate new plasma instabilities. In region 2 this occurs because of particles which have saturated occupancies. In region 3 it occurs later, when the saturated particles have become isotropic and higher energy, still-anisotropic particles are responsible for the new anisotropy. In region 4, the particles isotropize due to instability-induced angle change. Below the blue triangle, angle change is dominated by ordinary elastic scattering rather than by plasma instabilities.

This is the Region 4 in Fig. 10. In this case the anisotropy shrinks appreciably at $t_{\text{change}} \sim t_{\text{large angle}}$ given in eq. (3.19). When $\epsilon$ falls below $\alpha^{-\frac{1+3}{3}}$ the system will enter region 3.

These conclusions are displayed in Fig. 10.

All we have computed above is the time scale before some new physics significantly changes the dynamics; we have not computed the final equilibration time $t_{\text{eq}}$. We will postpone doing so until the end of the next section.

As a final aside, note that the total screening of soft particles actually dominates the screening arising from hard particles before $t_{\text{large angle}}$ if there are enough particles at the scale $p_{\max}$ at that time. The total screening from such particles is $m_2^2 \sim \alpha p_{\max} T_\star \sim \alpha p_{\max} Q$. This is to be compared with $m_2^2 \sim \alpha^{1-c} Q^2$ from hard particles. Using eq. (3.28) and eq. (3.19), this occurs if $\epsilon < \alpha^{-\frac{1+3}{3}}$. However this effect is not important because even if soft particles dominate screening, if they are isotropic they have no influence on which modes are unstable to plasma instabilities and how large the plasma instabilities grow. Therefore this large contribution to $m_2^2$ does not actually influence the physics of plasma instabilities and can be ignored.

4 Large anisotropy

A nonabelian plasma with a violent creation mechanism, or under strong anisotropic expansion or compression, can be very far from equilibrium in a strongly anisotropic fashion.
In particular, we expect very strong anisotropy at early times in heavy ion collisions, at least in the theoretically clean case of extremely high energy nuclei (so that the energy density after the collision represents a scale large compared to $\Lambda_{QCD}$ and asymptotic freedom ensures perturbative couplings) \cite{12}. Therefore we will consider very anisotropic systems, which may also possess typical occupancies very far from 1.

Since we cannot study everything, we will concentrate on two of the simplest possibilities; an initial momentum distribution with a single characteristic scale $Q$ and a strongly oblate (planar) momentum distribution, and a similar distribution but with a very prolate (linear) momentum distribution. We will try to determine the evolution of such a system, starting with the growth of the plasma instabilities, and continuing until some other piece of physics takes over as the dominant mechanism driving thermalization. We will not follow the system’s evolution all the way to thermalization for every case we introduce, but at the end we will give an estimate of the final equilibration time scale $t_{eq}$. 

4.1 Instabilities of an oblate distribution

Consider first a distribution of excitations with a characteristic momentum $p \sim Q$, but with a very oblate angular distribution. Namely, the occupancy is $f(p) \sim \alpha^{-c}$ if $p \lesssim Q$ and $p_z \lesssim \delta Q$, but $f(p)$ is small outside this range. That is, the excitations are concentrated in an angular range of width $\sim \delta$ about the $p_z = 0$ plane (so in spherical coordinates, $|\cos(\theta)| \lesssim \delta$ for most of the excitations). We take the angular range $\delta$ to be small, characterizing it when convenient as $\delta \sim \alpha d$. Thus, $d < 0$ (last section) represents distributions which are nearly isotropic, while $d > 0$ represents distributions which are very far from isotropic.

The energy density of this system is

$$\varepsilon \sim \int d^3p \ p f(p) \sim Q^4 \delta \alpha^{-c}$$

(4.1)

where the factor $\delta$ is because the phase space volume which is occupied is only $Q^3 \delta$, not $Q^3$. If the energy density is larger than $\alpha^{-1} Q^4$ then there will be Nielsen-Olesen instabilities, see subsection 2.3. This occurs if $\delta \alpha^{-c} > \alpha^{-1}$ or $c - d > 1$. We have already discussed the Nielsen-Olesen instability so we will not consider this region further.

The screening scale is

$$m^2 \sim \alpha \int \frac{d^3p}{p} f(p) \sim \alpha^{1-c} \delta Q^2 \sim \alpha^{1-c+d} Q^2$$

(4.2)

and magnetic modes are unstable even in a larger range of $k_z$. According to Ref. \cite{25}, the range of wave numbers which exhibit plasma instabilities is $k$ such that $k_x, k_y \lesssim m$ and $k_z \lesssim \delta^{-1} m$, and the growth rate is $\gamma \sim m$. We can quickly re-derive these results by turning again to eq. (3.6) and eq. (3.8). If $k_z / k_z \lesssim \delta$ and $k_y / k_z \lesssim \delta$ then $v \cdot k \lesssim \delta k$ for most of the excitations in the plasma. If $\gamma > v \cdot k$ over most of this range then the contribution from the second term in eq. (3.8) is negative and dominant, and $\Pi_{ij} \sim m^2 k^2 / \gamma^2$. The largest this can be, given $\gamma > v \cdot k \sim \delta k$, is $\Pi_{ij} \sim m^2 \delta^{-2}$. Substituting into eq. (3.6), the largest value of $k$ which can be unstable is $k^2 \sim \Pi \sim \delta^{-2} m^2$ or $k \sim \delta^{-1} m$. Since we required
Let us develop a clearer understanding of this result. We saw previously that the physical meaning of the scale $m$ is that excitations must remain coherently within a region of magnetic field of stable sign and magnitude for a time scale $1/m$ if the current is to grow big enough to modify field evolution. It is immediately clear that $\gamma \lesssim m$, since $\gamma$ is directly an inverse time scale for the magnetic field to change. Now $k$ determines the spatial nonuniformity of the magnetic field, and $v \cdot k$ tells how fast a particle explores this nonuniformity. If the field varies only in the $z$ direction but a particle moves purely in the $xy$ plane, it will remain in a region of coherent magnetic field forever. Since particles propagate almost purely in the $xy$ plane, the $k_x$ and $k_y$ components must be $\lesssim m$ so that in-plane propagation remains in a region of coherent magnetic field. Since the $z$ component of an excitation’s propagation is small, the field can vary faster in the $z$ direction without the particle feeling different signs of $B$ field. This is illustrated in figure 11.

There is an assumption we have built into the above discussion. We have assumed that the excitations which give rise to these plasma instabilities can be treated as point-like entities. This only makes sense if they can be described using wave packets with spatial extent $\Delta z < 1/k_z$, requiring momenta $\Delta p_z > k_z$. Clearly $\Delta p_z$ cannot be larger than $\delta Q$, the width of the $p_z$ distribution of our excitations. Therefore our treatment is only self-consistent if $\delta^{-1} m < \delta Q$. Using eq. (4.2), this requires

$$m < \delta^2 Q \quad \text{and} \quad m \sim \delta^4 \alpha^{1-c} Q \quad \Rightarrow \quad \delta^3 > \alpha^{1-c} \quad \text{or} \quad c < 1 - 3d. \quad (4.3)$$

Above this line, we must start from eq. (3.7) and re-derive what modes are unstable and how fast they grow. We will return to the case $c > 1 - 3d$ at the end of this section. In the remainder of this section we will assume $c < 1 - 3d$, so the hard-loop treatment is valid.

As in the previous section, plasma instabilities give rise to large infrared magnetic fields, which can dominate the dynamics by deflecting the hard excitations, with a characteristic momentum diffusion strength $\hat{q}_{\text{inst}}$. We want to determine $\hat{q}_{\text{inst}}$, which requires understanding how large the instabilities grow. The plasma instabilities will grow until either

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{Illustration of how a magnetic field can vary rapidly in the $z$ direction, and yet particles will have long free paths in same-sign regions of magnetic field because they propagate in a narrow range of angles.}
\end{figure}
1. they cause some physics which changes the distribution of excitations or otherwise changes the dynamics in an essential way, or

2. they get large enough that nonlinear effects limit their growth.

Case 2 occurs first over almost all the parameter space we explore, so we will consider it in more detail. There will be a narrow region where case 1 actually occurs first, and we will handle it when it arises.

The growth of instabilities will be cut off when any mechanism either interferes with the process which causes growth, or provides an efficient mechanism to remove energy from the unstable modes. Color randomization is a mechanism to cut off the growth mechanism, just as in the weak-anisotropy case. Refer again to Fig. 9. When the Wilson line of a propagating particle randomizes its phase over a length scale \( \frac{1}{v \cdot k} \sim \frac{1}{m} \), then color randomization occurs and instability growth is shut off. This occurs if the typical amplitude for a gauge field (associated with the plasma-unstable B fields) is \( A \sim \frac{m}{g} \).

The magnetic field is \( B \sim \nabla \times A \sim k_{\text{inst}} A \sim m^2 \delta^{-1} g^{-1} \), corresponding to

\[
B^2 \sim \alpha^{-1} \delta^{-2} m^4. \tag{4.4}
\]

Equivalently, we need a Wilson loop of size \( \frac{1}{m} \times \frac{1}{k} \) to contain an order-1 phase, requiring \( B \sim \frac{km}{g} \), returning the same estimate. Other estimates for the limiting size of the magnetic fields, such as the scale where nonlinear interactions between soft B fields become important, give the same estimate. This estimate is also supported by lattice studies [25].

The limiting size of the instabilities again corresponds to occupancies \( f(k) \sim \alpha^{-1} \) for the unstable modes. To see this, note that the energy density in soft magnetic fields is

\[
\varepsilon \sim B^2 \sim \int_{k_{\text{inst}}}^{k_{\text{coh}}} d k_\perp d^2 k_\perp k f(k) \sim k_{\text{inst}}^2 m^2 f(k) \sim m^4 \delta^{-2} f(k) \tag{4.5}
\]

requiring the typical unstable field occupancy to be \( f(k) \sim \alpha^{-1} \).

These magnetic fields give rise to momentum diffusion, but now the momentum diffusion coefficient \( \hat{q} \) is strongly direction dependent. For a particle moving at an angle \( \theta > \delta \) with respect to the \( xy \) plane (that is, with \( p_z/p = \sin \theta \)), the magnetic field changes orientation on a length scale \( k_{\text{inst}}/\theta \), and so

\[
\hat{q}(\theta) \sim F^2 t_{\text{coh}} \sim \alpha B^2 k_{\text{inst}}^2 \theta \sim \delta^{-2} m^4 \frac{\theta}{\delta m} \sim \frac{m^3}{\delta \theta}. \tag{4.6}
\]

For generic angles this is \( m^3/\delta \), while for very planar directions it is \( m^3/\delta^2 \). The latter case is relevant for the typical excitations with \( p \sim Q, p_z \sim \delta Q \). So for the typical excitations we have

\[
\hat{q}_{\text{inst}}(\delta) \sim \delta^{-2} m^3 \sim \delta^{-2} \alpha^{\frac{1}{2}(1-c)} Q^3. \tag{4.7}
\]

The situation becomes still more complicated if we consider relatively low-momentum excitations, with \( p \lesssim m/\delta^2 \). Specifically, for excitations with \( p_z < m/\delta \), the extent of the wave packet is larger than the inhomogeneity scale of the magnetic field and the particle-approximation (WKB) treatment of propagation, implicit above, breaks down. Therefore we must consider \( p \theta < m/\delta \) as a special case.
The key to understanding the response of such a low-momentum particle is to remember that the momentum transfer has to be a multiple of the wave-number of the nonabelian field. But if a particle of momentum \( p \ll m/\delta^2 \) picks up a single kick of momentum \( k_z \sim m/\delta \), then its angle will instantly change to be \( \theta \sim k_z/p \sim m/(p\delta) \gg \delta \). In this case, one should estimate the rate of momentum accumulation using this larger value of \( \theta \). Using this estimate, we find \( \dot{\theta} \sim m^3/\delta \theta \) but \( \theta \sim m/p\delta \), so \( \dot{\theta} \sim pm^2 \). This estimate is valid for \( p > m/\delta \). For \( p < m/\delta \), there are \( k \)-values which the momentum-\( p \) particle simply cannot pick up, since the magnetic field is nearly static and does not change the particle energy. Hence the largest \( k \) which can be picked up is \( k \sim p \). Again, such a \( k \) would cause large-angle change to the particle, so the coherence time is only \( 1/k \). This again gives the estimate

\[
\dot{\theta} \sim pm^2, \quad \text{when } pm^2 < \frac{m^3}{\delta \theta}.
\]

### 4.2 Momentum broadening

Now we are ready to determine the range of momenta where plasma instabilities are more important than ordinary elastic scattering. The rate of ordinary elastic scattering is

\[
\dot{\theta}_{\text{elastic}} \sim \alpha^2 \int d^3k f[1+f] \sim \alpha^2 \delta Q^3[1+f] \sim \left\{ \begin{array}{ll}
\alpha^2-2\epsilon \delta Q^3 & c > 0, \\
\alpha^2-\epsilon \delta Q^3 & c < 0.
\end{array} \right. \quad (4.9)
\]

Later we will often find it convenient to work only in terms of the variables \( \alpha, \delta, m, Q \). Using that \( m^2 \sim \alpha \int f(p) d^3p \sim \alpha f \delta Q^3 \), we can eliminate the factors of \( f \) in favor of factors of \( m^2 \) and write

\[
\dot{\theta}_{\text{elastic}}(c < 0) \sim \alpha m^2 Q. \quad (4.10)
\]

Now we need to compare \( \dot{\theta}_{\text{elastic}} \) with \( \dot{\theta}_{\text{inst}} \) to determine where plasma instabilities dominate the dynamics. Within the range we are considering, \( d > 0 \) and \( c < 1 - 3d \), we find \( \dot{\theta}_{\text{inst}} > \dot{\theta}_{\text{elastic}} \) whenever \( c > 0 \), while for \( c < 0 \) the instability dominates provided that

\[
\frac{\dot{\theta}_{\text{inst}}}{\dot{\theta}_{\text{elastic}}} > \alpha \frac{2d-c}{3} - 2d Q^3 > \alpha^{2-c-1} d Q^3
\]

or equivalently \( d > -\frac{1-c}{3} \). Outside this region, elastic scattering is more efficient than plasma instabilities. Note that the region where elastic scattering wins out corresponds to systems with extremely low occupancy, \( c < -1 \) or \( f(p) < \alpha \). We will not try to determine the physics in this region.\(^{11}\) Also note that, for excitations at generic angles \( \theta \sim 1 \), the plasma instabilities give a smaller \( \dot{\theta} \), \( \dot{\theta} \sim m^3/\delta \) from eq. (4.6). For such particles, plasma instabilities dominate in the narrower region \( c > -1 - d \). But this fact will turn out not to play a role in our discussion below.

\(^{11}\)One might expect that the physics here will be analogous to anisotropic, low occupancy systems studied in section 2.5. This may be true but it is not obvious, since low-momentum radiated daughters may be anisotropic and may generate important plasma instabilities. We leave this problem as an exercise to the reader.
Having identified the region where plasma instabilities dominate $\hat{q}$ and having found the relevant $\hat{q}$, it remains to determine what physics first leads to a significant change to the dynamics. We will take as our criterion that some new physics causes $\hat{q}$ for our typical excitations to change appreciably. The simplest possibility is that the $\hat{q}$ we just found causes the typical excitations’ angular distribution to get broader, increasing the value of $\delta$ and reducing the value of $c$. Since the mean squared change in $p_z$ is

$$\Delta p_z^2(t) \sim \hat{q} t,$$

this occurs when $\Delta p_z^2(t) > p_z^2 \sim \delta^2 Q^2$, which requires a time

$$t_{\text{broaden}} \sim \frac{\delta^2 Q^2}{\hat{q}_{\text{inst}}} \sim \frac{\delta^4 Q^2}{m^3} \sim \delta^2 \alpha^{-1} \frac{1}{(1-c)Q-1}.$$

Self-consistency requires $t > m^{-1}$. We see that this is satisfied but is marginal for the boundary case $c = 1 - 3d$, see eq. (4.3).

### 4.3 Splitting in highly anisotropic plasmas

Just as for isotropic and weakly anisotropic systems, we also expect joining (for $c > 0$) and splitting (for $c < 0$) may be important. As for weak anisotropy, these processes will generically be in the LPM regime, so an analysis in terms of $\hat{q}$ and formation times is relevant. And the split daughters can again dominate the dynamics, either through elastic scattering or through new plasma instabilities.$^{12}$

Assuming $\hat{q}_{\text{inst}}$ is the dominant momentum exchange process, and estimating the rate of LPM modified splitting or joining in the same way we did in previous sections, the formation time to split or join to the momentum scale $p < Q$ is

$$t_{\text{form}}^2(p) \sim \frac{p}{\hat{q}_{\text{inst}}} \sim \begin{cases} \frac{p \delta^2}{m^2}, & p > \delta^{-2} m, \\ \frac{1}{m}, & p < \delta^{-2} m. \end{cases}$$

The different value when $p < \delta^{-2} m$ happens both because the momentum broadening rate is smaller for such soft momenta, and because these emissions are not actually LPM suppressed; $1/m$ is the mean time between scattering events so the emission rate will be the same as if we assumed each scattering [every time $1/m$] provides the opportunity for an emission. The rate for a hard excitation to emit an excitation of momentum $p$ is

$$\frac{d\Gamma}{dtdp/p} \sim \alpha t_{\text{form}}^{-1}[1+f(Q)] \equiv \Gamma_{\text{split}}(p).$$

Here $[1+f(Q)]$ is a Bose stimulation factor for the case that the “hard” $p \sim Q$ excitations are at high occupancy. In this case, the final-state hard particle is Bose enhanced.$^{12}$

$^{12}$In [24, 27] the authors argued based on numerical simulations of plasma instabilities that $k \gtrsim m$ excitations are also produced by re-scattering of the unstable fields (the “nonabelian cascade”). While true, we find that this production method is less efficient than bremsstrahlung emission, a process which the lattice techniques employed in [24] do not incorporate. Therefore we will neglect the nonabelian cascade in the remainder of this paper.
The time scale on which a typical hard excitation undergoes hard splitting or joining is

$$t_{\text{split}}(Q) \sim \alpha^{-1} \delta^{-1/2} t_{\text{form}}(Q) \sim \alpha^{-1+0.1\epsilon} Q^{2/3} \delta^{-1/2} \sim \alpha^{2/3} Q^{-1}.$$  \hspace{1cm} (4.16)

Here $(0, c)$ refers to the low-occupancy or high-occupancy case respectively. This time scale is shorter than $t_{\text{broaden}}$ if

$$\delta^{2/3} \alpha^{-2/3} Q^{2/3} < \delta^{2/3} \alpha^{-2/3}(1-c)$$

If $1 > c > 0$, in which case we use the $-c/4$ case, this expression is never satisfied and broadening always happens before hard joining. Splitting will also play no important role when $c > 0$; just as in subsection 3.4, soft occupancies will become an $f(p) \sim \alpha^{-c} Q/p$ tail, which however never dominates screening, scattering, or plasma instabilities. Therefore if $c > 0$ then the first new physics is always momentum broadening at time $t_{\text{broaden}}$.

However if $c < 0$ then in the region $\delta > \alpha^{-1-3c}/9$, or $c < -3d-1/3$, hard splitting will occur before the plasma instabilities can broaden the hard particle distribution. Therefore splitting will play a role in a region at least this large. The actual region will be larger, because soft $p \ll Q$ split-off daughters can dominate the dynamics before hard splitting becomes important.

### 4.4 Scales $k_{\text{iso}}$ and $k_{\text{split}}$

The analysis of what physics takes over from the “ordinary” instabilities will be a little intricate. In particular, one complication is that, as we have seen, the momentum broadening rate $\dot{q}_{\text{inst}}(\theta)$ is angle-dependent. Since we will often be concerned with whether or not the radiated daughters are isotropic, this is a complicating effect. We summarize how the daughters fill in small momentum scales, including this complication but neglecting splitting of daughters, in Appendix A.

However, in the end we are always interested in what is happening “at the last minute” when some other piece of physics comes to dominate $\dot{q}(\theta \sim \delta)$. In every case we will find that this new piece of physics has a $\dot{q}$ which is not parametrically anisotropic. That is, $\dot{q}$ will be anisotropic by an $O(1)$ amount, but not by a power of $\delta$ or $\alpha$. Therefore, at the time scale when a new $\dot{q}$ is coming to dominate, we may treat $\dot{q}$ as approximately isotropic. Similarly, in eq. (4.14) we can ignore the $p < \delta^{-2}m$ case, which occurred because of the anisotropy of $\dot{q}$, and always take $t_{\text{form}}$ to comply to the first expression. It is most important to get this last time scale correct. If we also treat $\dot{q}$ as isotropic at all times, while we will mis-represent physics at early times, we always get the final stage, the time scale and physical mechanism which replace the original plasma instabilities correct.

To understand how radiated daughters can come to dominate $\dot{q}$, we need to study the time development of the population of such daughters. We will distinguish two important scales. The first is the scale $k_{\text{iso}}(t)$, introduced in eq. (3.29). This is the time-dependent
momentum scale such that particles at this scale have had just enough time for transverse momentum diffusion to drive them to an isotropic distribution at the order-1 level. It is determined by

\[ k_{iso}^2 \sim \hat{q}_{\text{inst}} t \quad \Rightarrow \quad k_{iso} \sim \delta^{-1} m^3 t^{\frac{1}{2}} \sim \alpha^{-\frac{3-3\epsilon-d}{4}} t^\frac{d}{2} Q^\frac{d}{2}. \] (4.18)

Excitations at the scale \( k_{iso} \) can grow to dominate \( \hat{q} \) by providing a new source for plasma instabilities. Unlike in the case for weak anisotropy, the scale \( p_{\text{max}} \) lies below \( k_{iso}(t) \) for all \( t > m^{-1} \). The scale \( p_{\text{max}} \) can only play a role by dominating \( \hat{q} \) through elastic scattering. However, if elastic scattering ever provides \( \hat{q}_{\text{elastic}} \sim \hat{q}_{\text{inst}} \), then all excitations with \( k \gtrsim k_{iso} \) thermalize and \( \hat{q}_{\text{elastic}} \) is controlled by the scale \( k_{iso} \). Later we will check whether elastic scattering from the scale \( k_{iso} \) can dominate, and we find that this is not the case in the regime of interest.

The other important scale is \( k_{\text{split}}(t) \), defined in eq. (2.71): \( \Gamma_{\text{split}}(k_{\text{split}}) t \sim 1 \). As discussed in subsection 2.5.3, a daughter with \( k \lesssim k_{\text{split}} \) in turn has time to re-split and fragment completely. Therefore, all particles with \( k \lesssim k_{\text{split}} \) redistribute into a thermal bath or thermal tail. Applying eq. (4.14) and remembering \( c < 0 \) so \( 1+f(Q) \simeq 1 \), we find

\[ t \sim \Gamma_{\text{split}}^{-1} \sim \alpha^{-1} t_{\text{form}} \sim \alpha^{-1} \frac{k_{\text{split}}^\frac{1}{2} \delta}{m^2} \quad \Rightarrow \quad k_{\text{split}} \sim \delta^{-2} \alpha^{-2} m^3 t^2 \sim \alpha^{-\frac{d-3\epsilon}{2}} Q^\frac{d}{2} t^2. \] (4.19)

Both \( k_{iso} \) and \( k_{\text{split}} \) increase with time; but \( k_{\text{split}} \) rises faster, as \( t^2 \) rather than as \( t^{\frac{3}{2}} \). The scales cross at a momentum \( k_{\text{isosplit}} \) and time \( t_{\text{isosplit}} \) such that

\[
\begin{align*}
    k_{\text{split}}(t_{\text{isosplit}}) &= k_{\text{iso}}(t_{\text{isosplit}}) \equiv k_{\text{isosplit}} \\
    \delta^{-2} \alpha^{-2} m^3 t_{\text{isosplit}}^2 &\sim \delta^{-1} m^3 t_{\text{isosplit}}^2 \\
    t_{\text{isosplit}}^2 &\sim \delta^2 \alpha^{-\frac{4}{3}} m^{-1} \sim \alpha^{-\frac{12}{11} + \frac{6}{11} \epsilon + \frac{1}{11} Q^{-1}} \\
    k_{\text{isosplit}} &\sim \delta^{-2} \alpha^{-2} m^3 t_{\text{isosplit}}^2 \sim \delta^{-\frac{4}{3}} \alpha^{-\frac{2}{3}} m \sim \alpha^{-\frac{1-3\epsilon-d}{6}} Q.
\end{align*}
\] (4.20) (4.21)

At all times, modes with \( k > k_{\text{isosplit}} \) split before \( \hat{q} \) drives them to isotropy, while modes with \( k < k_{\text{isosplit}} \) become isotropic before splitting.

### 4.5 \( t < t_{\text{isosplit}} \): New Instabilities

Before the time \( t_{\text{isosplit}} \), \( k_{iso} > k_{\text{split}} \), and we can effectively ignore re-splittings of daughters. The daughter particles are generically anisotropic and can give rise to “new” plasma instabilities. We will now show that the most important scale for any “new” plasma instabilities is the scale \( k_{iso} \). Due to the nonabelian LPM effect, the number of particles varies with scale as \( f(k) d^3 k \sim k^{-1/2} dk/k \) which is IR dominated; the contribution to screening scales as \( f(k) d^3 k/k \sim k^{-3/2} dk/k \) and is more IR dominated. Define the contribution to screening from momenta around \( k \) to be \( m^2(k) \); we see \( m^2(k) \propto k^{-3/2} \). The level of anisotropy also varies with scale. For \( k > k_{iso} \) the particles are strongly anisotropic but are few in number. Since the mean squared transverse momentum they can accumulate is \( k_{iso}^2 \), they will lie in
an angular range \( \theta(k) \sim k_{\text{iso}}/k \). These daughters will give rise to new instabilities with a \( \dot{q} \) of order

\[
\dot{q}(k > k_{\text{iso}}) \sim \theta^{-2}m^3(k) \propto (k/k_{\text{iso}})^2k^{-9/4} \propto k^{-1/4}
\]

which is dominated by small \( k \). Meanwhile, for \( k < k_{\text{iso}} \), the excitations are nearly isotropic. Only the fraction which were introduced too recently to isotropize are anisotropic, and this fraction is of order \((k/k_{\text{iso}})^2\). Therefore, for \( k < k_{\text{iso}} \), we have weak anisotropy with \( \epsilon \sim (k/k_{\text{iso}})^2 \). Then we find

\[
\dot{q}(k < k_{\text{iso}}) \sim \epsilon^{\frac{3}{2}}m^3(k) \sim (k/k_{\text{iso}})^3k^{-\frac{9}{4}} \propto k^{\frac{1}{4}}
\]

which is large \( k \) dominated. Hence the most important scale is \( k_{\text{iso}} \), where the anisotropy is order-1.

To determine the value of \( \dot{q}(k_{\text{iso}}(t)) \equiv \dot{q}_{s,\text{inst}} \) due to plasma instabilities at the scale \( k_{\text{iso}}(t) \), we need to determine \( m^3(k_{\text{iso}}) \). We have that \( t^2_{\text{form}}(k_{\text{iso}}) \sim k_{\text{iso}}^{-1} \). But \( \dot{q} \sim k_{\text{iso}}^{-1} \) so \( t^2_{\text{form}} \sim t/k_{\text{iso}} \sim \delta t \frac{\theta}{m} \). The number of daughters produced is \( n(k_{\text{iso}}) \sim n_{\text{hard}} \alpha t/t_{\text{form}} \). And \( m^2 \sim \alpha n_{\text{hard}}/Q \) so \( \alpha n_{\text{hard}} \sim m^2 Q \). Therefore

\[
n(k_{\text{iso}}) \sim \alpha n_{\text{hard}} t/t_{\text{form}}(k_{\text{iso}}) \sim m^2 Q t/t_{\text{iso}}^2 \sim m^2 k_{\text{iso}}^{\frac{1}{2}} Q t^{\frac{1}{2}}
\]

and \( m^2(k_{\text{iso}}) \sim \alpha n(k_{\text{iso}})/k_{\text{iso}} \), so

\[
m^2(k_{\text{iso}}) \sim \alpha m^2 k_{\text{iso}}^{\frac{1}{2}} Q t^{\frac{1}{2}} \sim \alpha \delta m^2 Q t^{\frac{1}{2}} \Rightarrow \dot{q}_{s,\text{inst}} \sim m^3(k_{\text{iso}}) \sim \alpha \delta m^2 Q t^{\frac{1}{2}}.\]

The strength of plasma instabilities from daughters grows with time as \( t^{\frac{3}{2}} \).
These “new” plasma instabilities will catch up with $\hat{q}_{\text{inst}} \sim \delta^{-2} m^3$ at the time scale

$$t_{s,\text{inst}} \sim \frac{\alpha^3}{\delta^2} m^\frac{15}{8} Q^\frac{3}{2} t_s^\frac{3}{8} \sim \frac{\alpha^3}{\delta^2} m^\frac{9}{8} Q^{-\frac{3}{2}} \sim \frac{\alpha^3}{\delta^2} m^\frac{9}{8} Q^{-\frac{3}{2}} \sim \frac{\alpha^3}{\delta^2} m^3 Q^{-\frac{3}{2}} t_s^\frac{3}{8} \sim \frac{\alpha^3}{\delta^2} m^3 Q^{-\frac{3}{2}} t_s^\frac{3}{8}.$$  (4.26)

Of course, we assumed $t_{s,\text{inst}} < t_{\text{isosplit}}$, so this answer is only self-consistent if

$$t_{s,\text{inst}} < \frac{1}{m} \quad \Rightarrow \quad \alpha^{-2} \frac{\delta^{-\frac{22}{5}} m^3 Q^{-4}}{\alpha^{-\frac{4}{3}} + \frac{\beta}{3} + \frac{\gamma}{3}} > 1 \quad \Rightarrow \quad c < -3d - \frac{1}{3}, \quad (4.27)$$

Below this line in the $c, d$ plane, new instabilities from split daughters dominate $\hat{q}$ before $p \sim Q$ particle angle-change becomes important. This happens to be the same as the line we found earlier, when we determined where hard splitting happens before $t_{\text{broaden}}$.

We should also check that the time scale we found in eq. (4.26) is longer than the time scale $1/m$. After all, the instabilities grow on this time scale, and it is the shortest formation time of any daughter induced by scattering in the instabilities. Therefore, if we find $t_{s,\text{inst}} \leq 1/m$, all of our calculations are inconsistent. The condition is

$$t_{s,\text{inst}} > m^{-1} \quad \Rightarrow \quad \alpha^{-2} \frac{\delta^{-\frac{22}{5}} m^3 Q^{-4}}{\alpha^{-\frac{4}{3}} + \frac{\beta}{3} + \frac{\gamma}{3}} > 1 \quad \Rightarrow \quad d > \frac{-3(1 + c)}{8}, \quad (4.28)$$

which is a line starting at $d = 0, c = -1$ and rising with slope $-3/8$. Between this line and the line $d = -(1 + c)/3$, daughters become important and control $\hat{q}$ via plasma instabilities in a time scale $\sim 1/m$, possibly before the plasma instabilities from the original hard particles have finished growing. We have not completely worked out the physics in this region, and we suspect that the physics in the region immediately below the $d = -(1 + c)/3$ line may also be nontrivial. However we will be lazy and leave the complete understanding of this region for future work.

4.6 Elastic scattering versus new instabilities

Let us return to the region with $d < -(1 + 3c)/9$ but $d > -3(1 + c)/8$. We just found that new plasma instabilities, associated with the scale $k_{\text{iso}}$, dominate $\hat{q}$ before $t_{\text{isosplit}}$ in this region. However there is one more thing we should check. The modes at and below the scale $k_{\text{iso}}$ could in principle also dominate $\hat{q}$ via elastic scattering. Before continuing, we should pause to make sure that this does not occur.

Suppose to the contrary that $\hat{q}_{\text{elastic}} \gtrsim \hat{q}(k_{\text{iso}})$ which we just computed. Then $k_{\text{iso}}$ due to elastic scattering will be at least as large as that from plasma instabilities. But as discussed in subsection 2.5, elastic scattering will fully thermalize all $k < k_{\text{iso}}$, either into a thermal bath or an $f(p) \propto T_s/p$ type tail. We will consider separately the case $f(k_{\text{iso}}) \gg 1$, in which case we get a $T_s/p$ type tail, and the case $f(k_{\text{iso}}) \ll 1$, in which case we get a thermal bath.
We want to determine whether split daughters can change this at times a thermal distribution. The temperature is set by the energy density, $k$
Meanwhile, particles with $\hat{q}\sim m^3$ing eq. (4.8), we find $\hat{q}_{\text{elastic}}(t)\sim t^{-\frac{4}{3}}$ and a quick calculation shows $f(k_{\text{iso}})\sim k_{\text{iso}}^{-\frac{2}{3}} t \sim t^{-\frac{4}{3}}$, and therefore $\hat{q}_{\text{elastic}} \sim t^0$.

On the other hand, if $f(k_{\text{iso}}, t_{s,\text{inst}}) < 1$ and again assuming that $\hat{q}_{\text{elastic}} \gtrsim \hat{q}_{s,\text{inst}}$, then elastic scattering re-organizes the momentum distribution of $k < k_{\text{iso}}$ particles into a thermal distribution. The temperature is set by the energy density, $T^4 \sim f(k_{\text{iso}})k_{\text{iso}}^4$. Meanwhile, particles with $k$ just above $k_{\text{iso}}$ have not “collapsed” and still provide $\hat{q} \sim \hat{q}_{s,\text{inst}}$ via plasma instabilities. Making the comparison again, we find

$$\hat{q}_{\text{elastic}} \sim \alpha^2 T^3 \quad \text{whereas} \quad \hat{q}_{s,\text{inst}} \sim m^3$$

$$\hat{q}_{\text{elastic}} \sim \alpha^2 (fk_{\text{iso}}^4)^{\frac{2}{3}} \quad \text{whereas} \quad \hat{q}_{s,\text{inst}} \sim (\alpha f k_{\text{iso}}^2)^{\frac{2}{3}}$$

$$\hat{q}_{\text{elastic}} \sim \alpha^2 f^2 k_{\text{iso}}^3 \quad \text{whereas} \quad \hat{q}_{s,\text{inst}} \sim \alpha^2 f^2 k_{\text{iso}}^3. \quad (4.29)$$

In this case, $\hat{q}_{s,\text{inst}}$ is larger provided $f(k_{\text{iso}}(t_{s,\text{inst}})) > \alpha^2$. In this case $T$ grows with time, and so $\hat{q}_{\text{elastic}}$ was smaller at all earlier times. Combining the two cases, $\hat{q}_{s,\text{inst}}$ dominates provided that $\alpha^{-1} \gg f \gg \alpha^2$.

Now let us check the actual value of $f(k_{\text{iso}}, t_{s,\text{inst}})$. Using $f \sim n(k_{\text{iso}})k_{\text{iso}}^3$ and combining eq. (4.18), eq. (4.24), and eq. (4.26), we find

$$f(k_{\text{iso}}, t_{s,\text{inst}}) \sim \frac{n(k_{\text{iso}})}{k_{\text{iso}}^3} \sim m^2 k_{\text{iso}}^{-\frac{2}{3}} Q t_{s,\text{inst}}^{\frac{1}{2}} \sim \delta^2 m^{-\frac{2}{3}} Q t_{s,\text{inst}}^{-\frac{3}{7}} \sim \alpha^3 \delta^8 m^{-1} Q^4 \sim \alpha^{1+2c+6d}. \quad (4.31)$$

This is to be applied in the region $-(1+c)/3 < d < -(1+3c)/9$. In this region, $f$ varies from $\alpha^{-1}$, at $d = -(1+c)/3$, to $\alpha^2$, at $d = -(1+3c)/9$. Therefore we self-consistently find that new instabilities dominate elastic scattering when they come to dominate overall $\hat{q}$, throughout this region.

**4.7 \ t > t_{\text{isosplit}}: new thermal bath**

Now consider the region $(1-c)/3 > d > -(1+3c)/9$. In this case, at the time $t_{\text{isosplit}}$, the total $\hat{q}$ is still dominated by $\hat{q}_{\text{inst}}$ arising from instabilities induced by the $p \sim Q$ excitations. We want to determine whether split daughters can change this at times $t_{\text{isosplit}} < t < t_{\text{broaden}}$. This range of momenta only exists if $t_{\text{isosplit}} < t_{\text{broaden}}$, which, using eq. (4.13) and eq. (4.20), requires $d < -(1+3c)/7$.

Below the scale $k_{\text{split}}$, excitations undergo split-join processes on a time scale shorter than the age of the system. This allows the excitations with $k < k_{\text{split}}$ to equilibrate into a nearly-thermal bath with temperature $T$, in analogy with what we found in Sub-subsection 2.5.3. Let us find $T(t)$ as a function of $t$. The thermal bath is made by breaking down all
daughters produced with \( k \lesssim k_{\text{split}} \). As in subsection 2.5, most of the energy comes from the scale \( k_{\text{split}} \); the number of daughters is \( \sim n_{\text{hard}} \), so

\[
\varepsilon \sim k_{\text{split}} n_{\text{hard}} \sim \alpha^{-1} k_{\text{split}} m^{2} Q, \quad T \sim \varepsilon^{\frac{1}{2}} \sim \alpha^{-\frac{1}{2}} k_{\text{split}}^{\frac{3}{2}} m^{\frac{3}{2}} Q^{\frac{1}{2}}. \tag{4.32}
\]

Substituting in eq. (4.19),

\[
T \sim \alpha^{\frac{1}{2}} \delta^{-\frac{1}{2}} m^{\frac{5}{2}} Q^{\frac{1}{2}} t^{\frac{3}{2}}. \tag{4.33}
\]

This grows more slowly with time than \( k_{\text{split}} \). We can also explicitly check that at the earliest time we are considering, namely \( t_{\text{isosplit}} \),

\[
T(t_{\text{isosplit}}) \sim \alpha^{-\frac{5}{2}} \delta^{-\frac{1}{2}} m^{\frac{3}{2}} Q^{\frac{1}{2}} \sim \alpha^{-\frac{1}{2}} m^{\frac{3}{2}} \frac{Q}{t},
\]

whereas, from eq. (4.21),

\[
k_{\text{isosplit}} \sim \alpha^{-\frac{3}{2}} \delta^{-\frac{1}{2}} m \sim \alpha^{-\frac{4}{24} - \frac{3}{4}} Q. \tag{4.34}
\]

Since we are considering a region where \( 3c > -1 - 9d \), we find \( T(t_{\text{isosplit}}) \ll k_{\text{isosplit}} \). Hence \( T \ll k_{\text{split}} \) at all subsequent times. We also find that \( T \ll k_{\text{iso}} \) at all times, since \( k_{\text{iso}} = k_{\text{split}} \) at time \( t_{\text{isosplit}} \), and \( T \) and \( k_{\text{iso}} \) scale with time in the same way, as \( t^{\frac{3}{2}} \).

Most of the particles in the system are in the thermal bath. To see this, note that each particle at energy \( k_{\text{split}} \) turns into \( (k_{\text{split}}/T) \gg 1 \) thermal bath particles. Since \( \sim n_{\text{hard}} \) particles of energy \( k_{\text{split}} \) get made, the bath particles outnumber hard particles, \( k \sim k_{\text{split}} \) particles, and (as we will soon see) particles with \( k_{\text{split}} \gg k \gg T \). Hence we expect that elastic scattering will be dominated by the thermal bath. The value of \( \dot{q} \) due to elastic scattering with thermal bath particles is

\[
\dot{q}_{\text{elast}} \sim \alpha^{2} T^{3} \sim \alpha^{\frac{1}{2}} \delta^{-\frac{1}{2}} m^{\frac{15}{4}} Q^{\frac{3}{2}} t^{\frac{3}{2}}. \tag{4.36}
\]

In addition, the thermal bath is generally not fully isotropic, and neither are any of the excitations at scales \( > T \). Therefore there can also be plasma instabilities due to the anisotropy of excitations at any scale between \( T \) and \( k_{\text{split}} \). First we need to estimate occupancies in this range. We know \( n(k_{\text{split}}) \sim n_{\text{hard}} \). What about scales below \( k_{\text{split}} \)? Below the scale \( k_{\text{split}} \), the flux of energy density through a logarithmic “bin” of momentum \( k \) is \( \sim k_{\text{split}} n_{\text{hard}} / t \). The energy density equals this energy “flux” times the mean lifetime of a particle to undergo democratic splitting,\(^{13}\) which is \( t_{\text{split}}(k) \propto k^{\frac{3}{2}} \) (see eq. (4.14), eq. (4.15)). Since the energy density is \( kn(k) \), we find \( n(k) \propto k^{-\frac{3}{2}} \), specifically \( n(k) \sim n_{\text{hard}}(k_{\text{split}}/k)^{\frac{3}{2}} \).

The strength of screening from these particles is \( m^{2}(k) \sim \alpha^{-1} n(k) / k \propto k^{-\frac{1}{2}} \).

Recall that, for \( k > k_{\text{isosplit}} \), particles split before their directions are randomized. Since \( k_{\text{split}} > k_{\text{isosplit}} \), the largest \( k \) values in the cascade will be highly anisotropic. Specifically, in the time \( t_{\text{split}}(k) \) during which a particle resides at momentum \( k \) before splitting further, it picks up a mean-squared momentum \( \dot{q}_{\text{split}}(k) \). This induces an angular range of

\[
\theta^{2}(k) \sim \frac{\Delta k^{2}}{k^{2}} \sim \frac{\dot{q}_{\text{split}}(k)}{k^{2}} \propto k^{-\frac{1}{2}}. \tag{4.37}
\]

\(^{13}\)By democratic splitting we mean a splitting process where the daughters have comparable energy. This is the main process which removes particles from some \( k \)-bin.
There is also a contribution to $\theta^2$ from the angular distribution of the parent, but since we find $\theta^2$ increases with smaller $k$, the parent-distribution is subdominant and we can use the above estimate. The strength of plasma instabilities from a scale $k \gg k_{\text{isosplit}}$ is

$$\hat{q}_{s,\text{inst}}(k) \sim \theta^{-2}(k)m^3(k) \propto k^{\frac{1}{2}}k^{-\frac{3}{2}} \sim k^{-\frac{3}{4}}$$

which is small $k$ dominated.

If $T < k_{\text{isosplit}}$, then there are also nearly-isotropic scales between $k_{\text{split}}$ and $T$. The time scale for a particle of momentum $k < k_{\text{isosplit}}$ to be direction-randomized is $t_{\text{iso}}(k) \sim \frac{k^2}{\hat{q}_{\text{split}}(k)}$. A fraction $t_{\text{iso}}(k)/t_{\text{split}}(k)$ of particles have not had time to randomize in direction, giving rise to a residual (weak) anisotropy of size

$$\epsilon(k) \sim \frac{k^2}{\hat{q}_{\text{split}}(k)} \propto k^{\frac{3}{4}}.$$ (4.39)

The plasma instabilities arising from such excitations are of order

$$\hat{q}_{s,\text{inst}}(k < k_{\text{isosplit}}) \sim (\epsilon m^2)^{\frac{3}{2}} \propto k^0,$$ (4.40)

essentially flat in $k$. Therefore it is fair to say that, up to logs, the size of new plasma instabilities is given by the contribution from the softest scale in the game, which is $T$ itself.\footnote{If $T < k_{\text{isosplit}}$, the argument is less clear. In this case, the excitations may isotropize as they cascade from the scale $k_{\text{isosplit}}$ to the scale $T$, “landing” on the thermal bath already isotropic. However, in this case the $\hat{q}$ of plasma instabilities from the $O(1)$ anisotropy at the scale $k_{\text{isosplit}}$ turns out to be the same as the $\hat{q}$ we find for the thermal bath below, so our final answer remains correct.}

At the scale $T$, the time it takes for an excitation to randomize its direction is $t_{\text{iso}}(T) \sim T^2/\hat{q}$. Any excitation which arrived longer than $t_{\text{iso}}(T)$ ago has had its direction randomized; recent additions to the bath have not and are anisotropic. Therefore the degree of anisotropy of the bath is

$$\epsilon(t) \sim \frac{t_{\text{iso}}(T)}{t} \sim \frac{T^2}{\hat{q}t} \sim \frac{\alpha^{\frac{3}{2}}\delta^{-1}m^3Q^2}{\delta^{-2}m^3t} \sim \alpha^{\frac{1}{2}}\delta^{1}m^{-\frac{1}{2}}Q^2 \sim \alpha^{\frac{3}{4}+3d}.$$ (4.41)

which turns out to be $t$ independent.

If we consider the thermal bath as a weakly-anisotropic system, in the spirit of section 3, then it has $c = 0$. In this case, we saw there that instabilities dominate $\hat{q}$ from the thermal bath if $\epsilon > \alpha^{\frac{3}{4}}$. Otherwise elastic scatterings dominate. Therefore plasma instabilities are more important provided

$$\epsilon > \alpha^{\frac{1}{4}} \quad \Rightarrow \quad \alpha^{\frac{1+\epsilon+3d}{4}} > \alpha^{\frac{1}{4}} \quad \Rightarrow \quad c < \frac{1-9d}{3}.$$ (4.42)

In this region, the value of $\hat{q}$ from $T$-bath plasma instabilities is

$$\hat{q}_{s,\text{inst}}(t) \sim \frac{3}{2}m^3 \sim \epsilon^{\frac{3}{2}}\alpha^{\frac{3}{4}}T^3 \sim \alpha^3m^3Q^2t^{\frac{3}{2}}.$$ (4.43)
Now we just need to check whether \( \hat{q} \) from eq. (4.36) or eq. (4.43) can exceed \( \hat{q}_{\text{inst}} \sim \delta^{-2}m^3 \) before the time \( t = t_{\text{broaden}} \). New instabilities catch up with \( \hat{q}_{\text{inst}} \) at time \( t_{s,\text{inst}} \), defined as

\[
\hat{q}_{s,\text{inst}}(t_{s,\text{inst}}) \sim \hat{q}_{\text{inst}} \\
\alpha^3 m^3 Q^3 t_{s,\text{inst}}^2 \sim \delta^{-2}m^3 \\
t_{s,\text{inst}} \sim \alpha^{-2} \delta^{-4} \pi Q^{-1}.
\] (4.44)

Comparing with \( t_{\text{broaden}} \) from eq. (4.13), we find

\[
t_{s,\text{inst}} \ll t_{\text{broaden}} \quad \text{if} \\
\alpha^{-2} \delta^{-4} \pi Q^{-1} \ll \delta^4 m^{-3} Q^2 \\
\alpha^{-2} \delta^{-4} \pi Q^{-1} \ll \alpha^{-3+3c+5d} Q^{-1} \\
0 > 3 + 9c + 23d
\] (4.45)
a line starting at \( c = -1/3 \) and with slope \(-9/23\). Therefore, new plasma instabilities from the scale \( T_s \) come to dominate the dynamics in the region with \( 9d > -1 - 3c \), \( 23d < -3 - 9c \), and \( 9d < 1 - 3c \).

For \( 9d > 1 - 3c \), the thermal bath is too isotropic to have important plasma instabilities but \( \hat{q}_{s,\text{elastic}} \) can grow to dominate. This happens at time

\[
\hat{q}_{s,\text{elastic}}(t_{s,\text{elast}}) \sim \hat{q}_{\text{inst}} \\
\alpha^{11/12} \delta^{-2} \frac{4}{3} m^{15/4} Q^{3/4} t_{s,\text{elast}}^2 \sim \delta^{-2}m^3 \\
t_{s,\text{elast}} \sim \alpha^{11/12} \delta^{-1/3} m^{1/2} Q^{7/12}.
\] (4.46)

Again comparing with \( t_{\text{broaden}} \), we find

\[
t_{s,\text{elast}} \ll t_{\text{broaden}} \quad \text{if} \\
\alpha^{11/12} \delta^{-1/3} m^{1/2} Q^{7/12} \ll \delta^4 m^{-3} Q^2 \\
\alpha^{25+3c+7d} Q^{-1} \ll \alpha^{18+18c+30d} Q^{-1} \\
0 > 7 + 15c + 37d
\] (4.47)

Therefore elastic scattering from a thermal bath dominates in a region satisfying \( 37d < -7 - 15c \) but \( 9d > 1 - 3c \). This is a triangle in the \( c, d \) plane with its apex at \( c = -25/6, d = 3/2 \).

This completes our study of the region where plasma instabilities are important and are well described by hard loops. The regions we have found and their properties are summarized in figure 13. The last region studied, where elastic scattering from a thermal bath comes to dominate, lies off the edge of the figure.

### 4.8 Highly anisotropic oblate distribution

Now we return to the case \( c > 1 - 3d \). Recall that, in this region, the hard-loop approximation returned a range of unstable modes which is broader than \( \delta Q \). This indicated
an inconsistency in the hard-loop approximation. We also found for the marginal case 
$c = 1 - 3d$ that the time scale $t_{\text{broaden}}$ for the momentum distribution to broaden was the 
same as the $1/m$ time scale for the instabilities to grow. Let us try to understand the 
physics in this regime.

We claim that the dominant physics remains a Weibel-type instability, with large 
magnetic fields deflecting the “hard” $p \sim Q$ excitations and broadening the $p_\perp$ distribution 
of their momenta. However, the range of plasma-unstable modes will be different than 
what we found using the hard-loop approximation; and they will grow large enough to 
broaden the $p \sim Q$ excitations’ angular distribution before they grow large enough for 
nonlinear (nonabelian) interactions to cut off the instability growth. That is, the growth of 
instabilities will be regulated by the angular broadening of the hard excitation distribution, 
rather than nonabelian interactions between unstable modes.

We will guess that the unstable modes still have $k_z \gg k_\perp \sim m$ but $k_z \ll Q$. But we 
now assume $k_z > p_z \sim \theta Q$ in contrast to the previous section. The one-loop self-energy is 
still given by eq. (3.7), but now $p \cdot k \sim Qk^0 + Qk_\perp + p_z k_z$ need not be large compared to 
k^2 \sim k_z^2. In order to find the range of unstable $k_z$ available, take $k^0$ and $k_\perp$ to zero; then 
$p \cdot k \sim p_z k_z$. But we expect $k_z > p_z$, so $k^2 \sim k_z^2$ is larger than $p \cdot k$. So to find the range 
of unstable $k_z$, drop the $p \cdot k$ terms in favor of $k^2$ terms. In this approximation, eq. (3.7) 
simplifies to

$$\Pi^{\mu\nu}(k) \sim g^2 \int d^4p \frac{p^2}{k_z^2} \delta(p^2) f(p) \sim \frac{Q^2}{k_z^2} g^2 \int d^4p \delta(p^2) f(p) \sim \frac{Q^2}{k_z^2} m^2. \quad (4.48)$$

The propagator equation has instabilities when $k_z^2 \sim \Pi(k)$, which we see requires

$$k_z^2 \sim \frac{Q^2}{k_z^2} m^2 \quad \Rightarrow \quad k_z \sim \sqrt{mQ}. \quad (4.49)$$

This is the range of unstable momenta. In the parametric regime considered, it is wider 
than $p_z \sim \delta Q$ (so the treatment is self-consistent) but narrower than $\delta^{-1} m$.

What is the range of $k_\perp$ and the maximum growth rate $\gamma$? They are determined as 
the values which significantly reduce the value of $\Pi^{\mu\nu}$ estimated above. This occurs when 
$p \cdot k \sim k_z^2$. Since $p^0 \sim Q \sim p_\perp$, this occurs whenever $\gamma k_\perp \sim m$. So the transverse 
momentum range and growth rate of the unstable modes remain $\sim m$.

To what amplitude do the plasma instabilities grow? As before, they grow until either 
nonlinear physics interferes with the growth mechanism, or the unstable fields modify the 
dynamics in some other way such that the calculation of the instability growth rate gets 
amended. In this region we will find that the latter occurs. Our previous estimate for the 
magnetic field strength where nonlinearities such as color randomization shut off the 
instability can be extended in a straightforward manner. The fields would still saturate 
with occupancy $f(k) \sim \alpha^{-1}$, with a magnetic field strength of $B^2 \sim \alpha^{-1} k_z^2 m^2 \sim \alpha^{-1} m^3 Q$.

The maximum value of $\hat{q}$ would then be $m^2 Q$.

The unstable fields grow exponentially, so the momentum randomization will explore 
all values smaller than the above $\hat{q}$ before achieving this value of $\hat{q}$. However, as soon as 
$\hat{q}_{\text{inst}} \gtrsim \delta^2 mQ^2$, then the hard excitations undergo significant broadening, $\Delta p_\perp > p_\perp \sim \delta Q$, 

\[ -46 - \]
Figure 13. Cartoon of $d,c$ plane indicating the regimes found for oblate distributions. Plasma instabilities are important in regions 1 to 7. In region 1, instabilities induce splitting which reduces anisotropy. In 2, instabilities induce splitting, generating anisotropic, occupancy-saturated particles which cause stronger plasma instabilities. Region 3 is similar but the new particles are not occupancy-saturated. In regions 4 and 7, plasma instabilities directly lead to the broadening of the primary-particle distribution. In 4 they do this after amplitude-saturation, in region 7 they do so before their amplitude saturates and they are not well described by hard-loops. In region 5, splitting creates new anisotropic particles on a time scale of order the instability growth rate. In region 6, splitting generates a thermal bath, but it is incompletely isotropic and its plasma instabilities come to dominate dynamics. Region 8 is Nielsen-Olesen unstable. In region 9, elastic scatterings play the dominant role. Not shown is region 10, between regions 4,6 at very negative $c$ and large $d$, where elastic scattering from a new thermal bath comes to dominate dynamics. The lower boundaries of regions 2,5 have not been completely studied and may require revision.

In a $1/m$ time scale. In the region we are considering, $m > \delta^2 Q$ and so $\delta^2 m Q^2 < m^2 Q$. Therefore the unstable modes reach a strength where they broaden the hard excitation distribution before they finish growing to become nonlinear.

This broadening of the hard excitation distribution changes which modes are unstable, how large they can grow, and so forth. In fact, it moves where the system lies in the $c,d$ plane. Therefore it constitutes an important change to the properties of the system.

Hence we conclude that the physics of the region $c > 1 - 3d$ but $c < 1 + d$ is, that plasma instabilities grow in a time scale $m \sim \alpha^{1-3d}$ to become large enough that they broaden the distribution of hard excitations. The values of $c,d$ then move along a line of fixed energy density, which is fixed $c - d$, towards smaller $c,d$, reaching $c = 1 - 3d$ in a time scale $t \sim 1/m$ up to logs. Thereafter they enter the $c < 1 - 3d$ regime we have treated above.

4.9 Final equilibration times

In the previous subsections we have shown that plasma instabilities generally dominate the dynamics at early times, and we have determined what new phenomenon takes over from
them and after what time scale. The 10 regions we found can be broken into four broad categories:

1. In region 9, elastic scattering dominates and the discussion of section 2 should be used.

2. In region 8, N-O instabilities almost instantly change the system to be one lying just off the boundary of region 8. The equilibration then proceeds as it would in that other region.

3. In regions 1, 4, and 7 plasma instabilities cause the momentum distribution to become more isotropic. For $d > 0$ this means $c, d$ each decline with time; for $d < 0$ only $d$ declines with time. Therefore after some amount of time the system enters region 3, 6, 9, or 10.

4. In regions 2, 3, 5, 6, and 10, a bath of soft excitations develops and comes to dominate the dynamics.

Since we already solved the first case, and since the second and third cases turn into either the first or fourth after some amount of time, we only need to address what happens in the last case, when a bath of soft excitations emerges to dominate the physics. This bath may start out nearly thermal (regions 6, 10) or far from thermal (regions 2, 3, 5). But in every case we have that the energy density $\varepsilon \sim \alpha^{-c}Q^4$ (if $d < 0$) or $\sim \alpha^{-c+d}Q^4$ (if $d > 0$) is small compared to $Q^4$, so the final temperature $T_{\text{final}} \sim \varepsilon^\frac{1}{2} \ll Q$.

In every case we expect the late evolution to be controlled by the physics of a bath of soft $p \lesssim T_{\text{final}}$ excitations, which steal energy from the hard $p \sim Q$ modes and eventually become a thermal bath. Since the thermalization time of soft excitations is generally shorter than for hard excitations, one expects that the soft bath becomes nearly thermal before it finishes breaking down the $p \sim Q$ excitations. It is safe to assume that, as in the isotropic case, the most time consuming stage is the final breakdown of the hard excitations when the bath temperature approaches $T \sim T_{\text{final}}$.

There are two possibilities. Near the end, the thermal bath can either be somewhat anisotropic, with $\varepsilon > \alpha^{\frac{1}{4}}$; or it can be more isotropic, $\varepsilon < \alpha^{\frac{1}{4}}$. In the former case the dominant physics will be plasma instabilities; in the latter case it will be elastic scattering. In either case, what we need to compute is the time scale at which the hard $p \sim Q$ excitations fragment completely, that is, $k_{\text{split}}(t_{\text{eq}}) \sim Q$. Using our previous results for $k_{\text{split}},$

$$t_{\text{eq}} \sim \alpha^{-1} \hat{q}^{-\frac{1}{2}} Q^{\frac{1}{2}}. \quad (4.50)$$

It remains to self-consistently determine $\varepsilon$ and $\hat{q}$.

First assume $\varepsilon > \alpha^{\frac{1}{4}}$ so plasma instabilities matter. Then $\hat{q} \sim \varepsilon^{\frac{2}{3}\alpha^{\frac{8}{3}} T_{\text{final}}^3}$. We also have that $k_{\text{isosplit}} \sim \alpha^{-1} \varepsilon^{\frac{1}{2}} T_{\text{final}}$ which is $\gg T_{\text{final}}$; so the fragmenting daughters are nearly isotropic just above the scale $T_{\text{final}}$. However, because $\hat{q}$ is caused by plasma instabilities, it is order-1 anisotropic, and so the particles fragmenting into the thermal bath are order-1
The level of anisotropy of the thermal bath $\epsilon$ is then set by the ratio of the time scale for direction randomization, $t_{iso} \sim T_{final}^2 \hat{q}$, and $t_{eq}$ -- since a fraction $\sim t_{iso}/t_{eq}$ of the thermal bath is made up of excitations which landed less than $t_{iso}$ ago and have not isotropized. That is,

$$\epsilon \sim \frac{t_{iso}}{t_{eq}} \sim \frac{T_{final}^2 \hat{q}^{-1}}{\alpha^{-1} \hat{q}^{-\frac{2}{3}} Q^2} \sim \alpha^{\frac{1}{4}} \epsilon \frac{3}{2} T_{final}^{\frac{1}{2}} Q^{-\frac{1}{2}}$$

and hence

$$\epsilon \sim \alpha^{\frac{1}{4}} (T_{final}/Q)^{\frac{1}{4}}, \quad t_{eq} \sim \alpha^{-3} Q^2 T_{final}^{\frac{2}{2}}. \quad (4.52)$$

This result only makes sense if $\epsilon > \alpha^{\frac{1}{4}}$, which we see is $T_{final} > \alpha^{\frac{4}{3}} Q$. This is the region $0 > c > -8/3$ for $d < 0$ and $0 > c - d > -8/3$ for $d > 0$. It also only makes sense if the resulting $t_{eq} \geq \alpha^{-2} T_{final}^{-1}$. Otherwise, the $p \sim Q$ excitations break down before the bath can completely thermalize with itself, and this final thermalization is the slowest process.

For $T_{final} < \alpha^{\frac{4}{3}} Q$, thermalization takes so long that the thermal bath is very nearly isotropic and elastic scattering dominates. In this case $\hat{q} \sim \alpha^2 T_{final}^3$, and we find

$$t_{eq} \sim \alpha^{-2} Q^2 T_{final}^3. \quad (4.53)$$

This case is the same as the isotropic case from subsection 2.5.

There are two weaknesses in this analysis. The first is that we assumed that there is no long time-scale “hang-up” on the way to the final thermalization we discuss. This seems safe; we checked explicitly that no such “hang-up” happens in isotropic systems, and anisotropic systems should generally have faster dynamics due to plasma instabilities. Also, all the $t_{change}$ time scales in Regions 1–7, 10 in table 1 are parametrically shorter than $t_{eq}$. The second weakness is that we assumed that, for $T_{final} > \alpha^{\frac{4}{3}} Q$, the thermal bath will be anisotropic. Once it is anisotropic, the anisotropic arrival of daughters maintains this anisotropy, but it has to get that way to start with. However, in regions 2, 3, 5, and 6 the soft bath does start out anisotropic; and region 10 (where it does not) is purely in the region where $T_{final} < \alpha^{\frac{4}{3}} Q$. So this seems consistent.

It is possible that, even if a system starts out with $\hat{q}_{elastic}$ somewhat larger than $\hat{q}_{inst}$, the thermal bath will still manage to become anisotropic. Therefore the boundaries between regions 2 and 9, and between regions 5 and 9, are somewhat in doubt. We will not try to resolve this issue here but leave it for future work, if physical systems which lie in this regime are found.

### 4.10 Prolate distribution

Now we will repeat the exercise of the last subsections, but for the case of a highly prolate anisotropic plasma, that is, one where $f(p) \sim \alpha^{-c}$ provided $p_z \lesssim Q$ and $p_x, p_y \lesssim \delta Q$ with $\delta \sim \alpha^d \ll 1$.

\footnote{This sounds like a contradiction; the particles are nearly isotropic as they fragment, but they land on the thermal bath in an anisotropic fashion. The reason this can be true is that, at every stage, the fragmentation happens in an anisotropic fashion dominated by particles near the $p_x, p_y$ plane, but they then isotropize before the next step in the fragmentation.}
The number density of typical excitations is \( n \sim f\delta^2 Q^3 \), now with two powers of \( \delta \) because momentum on two axes is more constrained. The energy density is

\[
\varepsilon \sim nQ \sim \alpha^{-c}\delta^2 Q^4 \tag{4.54}
\]

and we will assume \( \alpha^{-c}\delta^2 \ll \alpha^{-1} \), or \( c < 1 + 2d \) as otherwise there are Nielsen-Olesen instabilities.

The screening scale is

\[
m^2 \sim \alpha n_{\text{hard}} Q^{-1} \sim \delta^2 \alpha^{1-c} Q^2 \sim \alpha^{1-c+2d} Q^2. \tag{4.55}
\]

Since the “hard” \( p \sim Q \) modes all travel nearly in the \( z \) direction, they remain coherently in a region of constant magnetic field provided that \( k_z \ll m \) for the magnetic field; but both \( k_x \) and \( k_y \) may be larger by a factor of \( \delta^{-1} \) and the particles will remain in a region of coherent field. Hence we find that the range of unstable momenta and the growth rate are

\[
k_z \sim m, \quad k_\perp \sim m\delta^{-1}, \quad \gamma \sim m. \tag{4.56}
\]

All unstable modes have polarization vectors in the \( z \) direction. The hard-loop approximation breaks down if \( k_\perp \) for unstable modes exceeds the typical transverse momentum of a “hard” excitation, \( k_\perp > \delta Q \), which is if \( m > \delta^2 Q \) or \( c > 1 - 2d \). The region \( 1 + 2d > c > 1 - 2d \) is analogous to the case of subsection 4.8; in this region the growth of plasma instabilities is cut off before amplitude saturation because the hard momentum distribution gets broader.

The unstable mode amplitudes saturate when, in traversing a distance \( 1/m \) along the \( z \) axis, a particle’s color is rotated by an \( \mathcal{O}(1) \) angle, as previously discussed. This requires

\[
A_z \sim m/g, \quad B \sim kA \sim m^2\delta^{-1}\alpha^{-7/2}. \tag{4.57}
\]

Unlike the oblate or weak-anisotropy cases, saturation now occurs before the typical occupancy of an unstable mode is \( \sim \alpha^{-1} \), \( B^2 \sim f k_z^3 \) so \( f \sim \alpha^{-1}\delta \) for the unstable modes.

The functional form of \( \hat{q}_{\text{elastic}} \) is the same, when expressed in terms of \( m, \delta, \alpha, Q \), as for the oblate case, eq. (4.7). The same will be true, for \( c < 0 \), of \( \hat{q}_{\text{elastic}} \), as we now show. The momentum diffusion due to ordinary scatterings is

\[
\hat{q}_{\text{elastic}} \sim \alpha^2 \int d^3pf[1+f] \sim \begin{cases} \alpha^{2-2c}\delta^2 Q^3 & c > 0, \\ \alpha m^2 Q & c < 0. \end{cases} \tag{4.58}
\]

For \( c > 0 \) instabilities always dominate \( \hat{q} \) and we can ignore \( \hat{q}_{\text{elastic}} \). For \( c < 0 \), eq. (4.58) has the same form as eq. (4.10). The region where plasma instabilities dominate is

\[
\frac{\hat{q}_{\text{inst}}}{\hat{q}_{\text{elastic}}} > \alpha^{2(1-c)} \delta Q^3 \sim \alpha^{2-c\delta^2 Q^3} \quad \delta^{-1} > \alpha^{\frac{1+d}{2}}. \tag{4.59}
\]
or $2d > -1 - c$.

We now go quickly through the prolate versions of the arguments we gave in the previous sections for oblate systems. We will find that, when expressed in terms of $m^2, \delta, \alpha, \text{and } Q$, the results from the oblate case carry over to the prolate case. Expressing in terms of $c, d$, the difference will just be the power of $d$ appearing in $m^2$, eq. (4.55). Therefore the distinct regions in the $c, d$ plane are the same as before but the boundaries are found by replacing factors of $d - c$ in the oblate case with $2d - c$ in the prolate case (since $m^2 \propto \alpha^{1+d-c}$ in eq. (4.2) but $m^2 \propto \alpha^{1+2d-c}$ in eq. (4.55)).

In particular, we find that eq. (4.13) for $t_{\text{broaden}}$, eq. (4.14) for $t_{\text{form}}$ and eq. (4.15) for $t_{\text{split}}$, eq. (4.18) for $k_{\text{iso}}$, eq. (4.19) for $k_{\text{split}}$, and eq. (4.20) and eq. (4.21) for $t_{\text{isosplit}}$ and $k_{\text{isosplit}}$, all still hold if one reads the versions expressed in terms of $m^2, \alpha, \delta, Q$. So do our arguments regarding the most important scales for new daughters. The temperature of the thermal bath of daughters, $T_\ast$, is also still given by eq. (4.33), and the criteria for its dominance are determined by the same expressions when we use the $m^2, \alpha, \delta, Q$ variables.

Therefore we find the same regions as for an oblate distribution. In terms of $c, d$ the regions’ boundaries are shifted, but in terms of $d$ and the energy density (which depends on $c - d$ for oblate and $c - 2d$ for prolate distributions) they are the same. This is summarized in figure 14. Similarly we expect no parametric difference in the final equilibration times.

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A Daughters neglecting re-splitting

In most of section 4, when treating the split-off daughters of “hard” $p \sim Q$ excitations, we made the simplifying approximation that $\hat{q}(\theta)$ is uniform in angle and nearly independent of an excitation’s energy. We justified this approximation by arguing that we were really most interested in any new physics which comes to supplant $\hat{q}_{\text{inst}}$ as the dominant source of $\hat{q}$. And in every case we found that the new $\hat{q}$ has at most order-1 anisotropy. Therefore, at the moment when a new mechanism takes over controlling $\hat{q}$, one is free to treat $\hat{q}$ as isotropic. But what happens before this moment?

We will present at least a partial answer to this question, by looking at the evolution of particle occupancy and angular distribution, as a function of momentum $p$ and time $t$ for $m < p < Q$. We will treat $\hat{q}_{\text{broaden}} > t > 1/m$, and we will also only try to understand $p > k_{\text{split}}(t)$. We will only consider the case of an oblate momentum distribution.

Further, we will neglect elastic scattering. We saw previously that this is a good approximation for narrow angles and large momenta provided that $\hat{q}_{\text{inst}} \sim 0$, which requires $1 + c + 3d > 0$. For generic angles it requires $\hat{q}_{\text{inst}}(\theta \sim 1) \sim \delta^{-1}m^3 > \alpha m Q$, which requires $1 + c - d > 0$. For the lowest-momentum excitations, $p \sim m$, it requires $m^3 \gg \alpha m^2 Q$, which is true if $1 + c - d > 0$. These restrictions, and the restriction $p > k_{\text{split}}(t)$, must be taken into account when applying the results of this appendix.

Our results are presented in figure 15. It is most convenient to write $p = m\delta^{-a}$ and $t = \delta^{-b}/m$, that is, to use $m$ as the characteristic scale and powers of $\delta^{-1}$ to distinguish how much larger than $m$, $1/m$ the momentum $p$ and time $t$ are. In these units, $Q \sim \delta^{-K}m$ with $K = (1 - c + d)/(2d)$. Lines of equal $K$ are lines of constant slope starting at $c = 1$, $d = 0$; the line $c = 1 - 3d$ corresponds to $Q \sim \delta^{-2}m$ and the $d = 0$ axis is $K = \infty$.

First, we determine the range of angles which split-daughters will take, as a function of $p$ and $t$. The key facts we need are the following. Radiated daughters are naturally born in the narrow angular range of the hard parent population, and they then undergo transverse momentum diffusion to gain mean-squared momentum $\Delta p^2 \sim \hat{q}t$. The range of angles will be

$$\theta^2 \sim \begin{cases} \frac{\hat{q}t}{p^2} & \text{if less than 1} \\ \sim 1 & \text{if } \frac{\hat{q}t}{p^2} > 1 \end{cases} \quad (A.1)$$

The value of $\hat{q}$ is dependent on both $p$ and $\theta$ (see eq. (4.6) and eq. (4.8)):

$$\hat{q}(p, \theta) \sim \min \left( \frac{m^3}{\partial \theta}, m^2 p \right). \quad (A.2)$$

First consider excitations with $p < m/\delta$; then we always have $\hat{q} \sim m^2p$. At early times the angular range is $\theta^2 \sim \hat{q}t/p^2 \sim m^2t/p$. These particles become isotropic on a time scale

$$t \sim \frac{p^2}{\hat{q}} \sim \frac{p^2}{pm^2} \sim \frac{p}{m}m^{-1} \sim \frac{p}{m}m^{-1}. \quad (A.3)$$
Next consider $m/\delta^2 > p > m/\delta$. Initially $\hat{q} \sim m^2p$ and again $\theta \sim m\frac{1}{p}p^{\frac{1}{2}}$; but this changes when $m^2p \sim m^3/\delta\theta$, which is at $t \sim \delta^{-2} p^{-1}$. After this, $\hat{q} \sim m^3/\delta\theta$, and we find

$$\theta^2 \sim \frac{\hat{q}t}{p^2} \sim \frac{m^3t}{p^2\delta\theta} \Rightarrow \theta \sim \delta^{-\frac{1}{2}} mp^{-\frac{1}{2}} t^{\frac{3}{2}} \quad (A.4)$$

until $\theta \sim 1$ at the time scale $t \sim \delta^{-1} p^2 m^{-3}$.

Finally, if $p > \delta^{-2} m$, then initially $\theta \sim \delta$ the angular range inherited from the radiating parent. Enough momentum broadening has accumulated to overcome this value when eq. (A.4) returns $\theta \sim \delta$, which is at $t \sim \delta^{-4} p^2 m^{-3}$. After this the value of $\theta$ is given by eq. (A.4) until $t \sim \delta^{-1} p^2 m^{-3}$, when it is again order-1. These time and momentum scales, $\hat{q}$ values and angular ranges are summarized in figure 15.

Regarding the occupancy as a function of $p$ and $t$, we already saw that the formation time for a radiated daughter is $t_{\text{form}}^2 \sim p/\hat{q}$, which is $t_{\text{form}} \sim m^{-1}$ for $p < \delta^{-2} m$ and is $t_{\text{form}} \sim \delta p m^{-\frac{3}{2}}$ for $p > \delta^{-2} m$. The rate of particle emission by a hard parent is $\Gamma \sim \alpha[1+f(Q)]/t_{\text{form}}(dtdp/p)$ and the density of hard parents is $n_{\text{hard}} \sim \alpha^{-1} m^2 Q$. The powers of $\alpha$ cancel, so the number of soft excitations $n_p \equiv dn/d\ln p$ is

$$n_p \sim \begin{cases} m^3 Q t[1+f(Q)] & p \ll \delta^{-2} m, \\ \delta^{-1} p^{-\frac{1}{2}} m^2 Q t[1+f(Q)] & p \gg \delta^{-2} m. \end{cases} \quad (A.5)$$

Here $[1+f(Q)]$ is a stimulation factor which is important when the hard particles are at large occupancy.
To convert these into an occupancy at the scale $p$, we use $n_p \sim p^3 \theta f(p)$. The factor of $\theta$ is because not all of the angular range is filled. When the resulting occupancy is larger than $Q[1+f(Q)]/p$, then joining (re-absorption) will compete with splitting and the occupancy will saturate. This occurs for $t > p^2 m^{-3}$ for $p < \delta^{-2} m$ and for $t > \delta^2 p^2 m^{-\frac{7}{2}}$ for $p > \delta^{-2} m$.

Therefore, in terms of the nine regions marked in figure 15, the typical occupancy will be:

$$f(p) \sim \begin{cases}
m^2 p^{-\frac{3}{2}} Q^{\frac{3}{2}} & \text{region 1},
m^3 p^{-3} Q t & \text{region 2},
Q/p & \text{region 3},
Q/p & \text{region 4},
m^3 p^{-3} Q t & \text{region 5},
\delta^{-1} m^2 p^{-\frac{7}{2}} Q t & \text{region 6},
\delta^4 m^2 p^{-\frac{7}{2}} Q t & \text{region 7},
\delta^{-\frac{2}{3}} m^2 p^{-\frac{7}{2}} Q t & \text{region 8},
\delta^{-2} m^2 p^{-\frac{7}{2}} Q t & \text{region 9}.
\end{cases} \quad (A.6)$$

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