HEDGING STRATEGY FOR UNIT-LINKED LIFE INSURANCE CONTRACTS WITH SELF-EXCITING JUMP CLUSTERING

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(Communicated by Hailiang Yang)

Abstract. This paper studies the hedging problem of unit-linked life insurance contracts in an incomplete market presence of self-exciting (clustering) effect, which is described by a Hawkes process. Applying the local risk-minimization method, we manage to obtain closed-form expressions of the locally risk-minimizing hedging strategies for both pure endowment and term insurance contracts. Besides, we demonstrate the existence of the minimal martingale measure and perform numerical analyses. Our numerical results indicate that jump clustering has a significant impact on the optimal hedging strategies.

1. Introduction. A unit-linked life insurance contract is a life insurance product that integrates insurance protection of traditional insurance policies with investment opportunities of financial markets. The invention of unit-linked life insurance products is one of the most significant innovations in the insurance market in the past several decades. Since then, unit-linked life insurance products have become

2020 Mathematics Subject Classification. Primary: 91B05, 91B16; Secondary: 60H30.
Key words and phrases. Local risk-minimization, unit-linked life insurance, Hawkes process, hedging strategy, minimal martingale measure.

This work was supported by the Humanity and Social Science Youth Foundation of the Ministry of Education of China (18YJC910012), the National Natural Science Foundation of China (11771147,12071114),“Shuguang Program” supported by Shanghai Education Development Foundation and Shanghai Municipal Education Commission(18SG25), the State Key Program of National Natural Science Foundation of China (71931004), the Discovery Early Career Researcher Award (DE200101266) of the Australian Research Council, the Zhejiang Provincial Natural Science Foundation of China (LY17G010003), the 111 Project(B14019), Ningbo City Natural Science Foundation (202003N4144) and the Humanity and Social Science Foundation of Ningbo University (XYPB19002).

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increasingly popular in global insurance markets. [12] shows that unit-linked life insurance products had stable and constant growth for ten years (2005-2014) among members of the European Insurance and Reinsurance Federation. Furthermore, [12] points out that there has been a common shift of market interest from fixed yield to unit-linked products since 2015. The benefit of unit-linked life insurance contracts depends on the performance of some specific stock, index or a portfolio of stocks/index in a financial market. Unit-linked life insurance contracts can provide death benefit, maturity benefit or both. Usually, a unit-linked life insurance product is a long-term insurance contract between a policyholder and an insurance company. By issuing unit-linked life insurance products, insurance companies are committed to providing insurance benefits that are related to various risky assets in financial markets. Hence, on top of existing mortality risk in the life insurance market, the issuers are also exposed to financial risk. The financial risk is caused by the evolution of the underlying asset/portfolio, whereas the mortality risk results from the uncertainty of individual death. Unlike the complete market setting in which a derivative position can be perfectly replicated by tradable securities, an incomplete market does not allow the perfect replication. Thus, how to hedge the risk exposure in unit-linked life insurance contracts is a pertinent question for the insurers, whose objectives are to protect themselves from unexpected claim payouts and financial losses and to minimize the residual risk that is unhedgeable in the incomplete market. [25] assume that the force of mortality is governed by an m-dimensional stochastic process and study the valuation and hedging of unit-linked life insurance contracts. They design a dynamic super-hedging and sub-hedging strategy for unit-linked life insurance contracts with upper and lower price bounds. [28] consider the hedging for the unit-linked life insurance contracts with minimum death guarantees under the Black-Scholes' geometric Brownian motion (GBM) framework and the Merton’s jump diffusion model. When it comes to optimal hedging of unit-linked life insurance contracts, many studies focus on risk-minimization or local risk-minimization methods. Risk-minimization is a very well-known quadratic hedging method for contingent claims in incomplete financial markets. This method is introduced in the pioneering work of [15], in which the discounted price process of the risky asset is a martingale under the original physical measure. This theory is then extended to the case of a semi-martingale by [33], where the local risk-minimization criterion is considered. [27] first applies the risk-minimization method to the risk management of unit-linked life insurance products. [37] get inspired by the work of [32] and point out that the hedging strategy in [32] is not locally risk-minimizing under the original measure. To get over it, they obtain the real locally risk-minimizing hedging strategies for unit-linked life contracts. For other relevant works, interested readers may refer to [29], [21], [6] and [7].

The aforementioned research articles consider the optimal hedging problems for unit-linked life insurance contracts primarily under the GBM model and Merton’s jump-diffusion model. However, these classical models are not adequate to describe the clustering effect and the self-exciting property of jumps in asset price dynamics observed in the market. Indeed, empirical evidence suggests that jump events are not isolated, but are clustering/self-exciting in the financial market. Jump clustering means that the occurrence of future jumps depends on the current and past jumps. For example, with the fast spread of news or report (jump event) in the market, it usually follows an ongoing process in which more information will be exposed.
The reaction of all the related players in the market would then make an impact on the prices of corresponding equity securities. Therefore, we would claim that it is a clustering jump. During a financial crisis, the clustering of jumps occurs much more frequently. According to [1], what makes a crisis disastrous is not just because of the initial jump, but rather the ripple effect afterwards. Sudden occurrence of grouped and unexpected information might trigger some serious consequences to the financial markets. This chain result is the clustering of jumps in asset values.

[14] point out that Hawkes process is an ideal model to incorporate jumps with clustering effect. Hawkes processes are a class of stochastic point processes that are introduced by [18] and [19]. Over the past decade, more and more scholars have begun to utilize Hawkes processes to solve problems of their interest. In particular, it has become very popular in financial field as some compelling empirical evidence shows that Hawkes processes represent some of the typical characteristics of a financial time series. [5] introduces a bivariate Hawkes process to describe the joint dynamics of trades and mid-price changes of the New York Stock Exchange. [36] use a Hawkes process in an insurance problem to study the asymptotic behavior of infinite and finite horizon ruin probabilities. [39] studies the limit theorems for a Cox-Ingersoll-Ross process with Hawkes jumps, an extension of the linear Hawkes processes. [2] apply multidimensional Hawkes jump diffusions to handle an optimal portfolio and consumption problem with different utility functions. [16] introduces a bivariate Hawkes process for interest rate modelling and considers the pricing for interest rate derivatives. [26] use the Hawkes jump diffusion processes to describe both the price of an underlying asset and the value of an option-writer’s assets, and present the valuation of vulnerable European options. [38] extend the model in [16] to a more general case for modelling the interest rate and obtain the pricing formulas of zero-coupon bonds. [35] study the mean-variance portfolio selection problem in contagious markets with both self-excitation and cross-excitation effects. Furthermore, more information about Hawkes processes and their applications to finance can be found in [20]. The clustering property of the Hawkes processes makes it very useful in finance and many other fields. Specifically, the Hawkes process is suitable for modelling the risky asset price dynamics. The fact that optimal dynamic hedging strategies differ among different financial models and this motivates us to consider the optimal hedging strategy in a system driving by a Hawkes jump-diffusion process.

Our specific interest is to pursue optimal hedging for unit-linked life contracts in a financial market with jump clustering. It is assumed that the stock is the only tradable asset in our model and its price process is driven by a jump-diffusion model, in which the intensity of the jump component is modeled by a Hawkes process. This feature leads to the fact that we are considering an incomplete market in which replication strategies are not available, and classical fully hedging strategies cannot be obtained. We thus resort to the locally risk-minimizing hedging method to hedge the financial and mortality risks inherent in unit-linked life insurance contracts. In comparison to other works (e.g., [27] and [37]) on risk-minimizing hedging of life insurance products, a few unique contributions that our paper makes are as follows. First, we present a Hawkes jump-diffusion process to formulate the price dynamics of the risky asset that is linked to life insurance products. In a Hawkes process, it is likely that the occurrence of a jump will accelerate the arrival of future jumps. Therefore, the Hawkes process is suitable for modeling clustered jumps. [27] uses a diffusion process to describe the price dynamics of
risky asset and the Black-Scholes model is used in numerical analysis. [37] use Lévy process to describe the price dynamics of the risky asset. Second, the financial market in [27] is complete as the Brownian motion is the only risk source in his paper. On the contrary, we prove the existence of the minimal martingale measure in an incomplete market. Besides, [37] do not provide numerical results, while we use Monte Carlo simulation methods and manage to delineate optimal hedging strategies numerically. Therefore, we are able to visualize that jump clustering has a significant impact on the risk-minimizing strategies. As for the research work about hedging for contingent claims in presence of jump clustering, [17] obtain an optimal hedging strategy for European options under one risk neutral probability measure to minimize the quadratic hedging error. In this paper, we study the hedging strategies for unit-linked life contracts from the perspective of the locally risk-minimizing criterion and obtain the minimal martingale measure. It should be pointed out that when it comes to local risk-minimization, the real-world probability measure rather than a risk-neutral probability measure should be used to minimize the quadratic hedging error. Therefore, instead of having the hedging strategies under one risk-neutral measure, our hedging strategies are under the real-world probability measure, and thus are applicable in practice directly. In addition, we run a few simulations to illustrate the significant impact of clustering risk on hedging strategies and we compare the risk-minimizing hedging strategy with the Delta hedging strategy.

The rest of paper is organized as follows. Section 2 sets up the financial market and insurance market models. Section 3 introduces definitions and studies the existence of the minimal martingale measure. Section 4 provides analytical solutions of the locally risk minimizing strategies for unit-linked pure endowment and term insurance contracts. In Section 5, we present numerical results of the hedging strategies. The final section concludes this paper.

2. The financial and insurance models.

2.1. The financial market. Let $W := \{W_t\}_{t \in [0,T]}$, with $T$ denoting a fixed and finite time horizon, be a standard Brownian motion, $N := \{N_t\}_{t \in [0,T]}$ denote a Hawkes process, and $Z := \{Z_j, j = 1, 2, 3, \ldots\}$ represent random jump sizes. They are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$, where $P$ denotes a real-world probability measure. We define the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ by $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^N \vee \mathcal{F}^Z$, where $\mathcal{F}^W = \{\mathcal{F}_t^W\}_{t \in [0,T]}$, $\mathcal{F}^N = \{\mathcal{F}_t^N\}_{t \in [0,T]}$ and $\mathcal{F}^Z = \{\mathcal{F}_t^Z\}_{t \in [0,T]}$ denote the natural filtrations of the processes $W$, $N$ and $Z$, respectively. Moreover, we assume that $\mathcal{F}$ satisfies the usual conditions. We consider a frictionless financial market consisting of a risk-free bond and a stock. The dynamics of the risk-free bond $B := \{B_t\}_{t \in [0,T]}$ is given by

$$dB_t = r_t B_t dt, \quad B_0 = 1,$$

where stochastic interest rate $r_t > 0$ denotes the instantaneous market interest rate of the bond at time $t$.

The stock price process $S := \{S_t\}_{t \in [0,T]}$ is governed by the following stochastic differential equation:

$$\frac{dS_t}{S_{t-}} = \mu(t, S_t) dt + \sigma(t, S_t) dW_t + d \sum_{j=1}^{N_t} Z_j, \quad S_0 > 0. \quad (1)$$
Here, $\mu$ and $\sigma$ are $\mathcal{R}^+$-valued measurable functions such that the equation (1) has a unique strong solution. They denote the appreciation rate and the volatility of the stock, respectively. The Hawkes process $N$ is a self-exciting point process, whose intensity $\lambda := \{\lambda_t\}_{t\in[0,T]}$ is dependent on the past realization of the process itself. To be more specific, the Hawkes jump intensity $\lambda$ is mean-reverting (refer to [2] and [22]) and is governed by

$$d\lambda_t = \alpha(\kappa - \lambda_t)dt + \gamma dN_t, \quad \lambda_0 > 0, \quad (2)$$

where $\alpha > 0$, $\kappa > 0$, $\gamma > 0$ are constant parameters, and $\lambda_0$ is the initial jump intensity. Here, $\kappa$ is the long-term mean level and $\alpha$ is the speed of reversion toward the long-term mean level $\kappa$, and $\gamma$ is the jump size of the intensity $\lambda$.

We further assume that jump sizes $\{Z_j, j = 1, 2, 3, \ldots\}$ are i.i.d random variables with a common probability density function (PDF): $\nu(dz)$. The common PDF $\nu(dz)$ has support $(-a, a)$, where $0 < a \leq 1$, and its mean and variance are $0$ and $\delta^2$, respectively. In addition, we assume that the Brownian motion $W$, the Hawkes process $N$ and the jump sizes $Z$ are mutually independent.

For convenience, we denote $J(dz, dt)$ as a random jump measure of the compound jump process such that $\sum_{j=1}^{N_t} Z_j = \int_0^t \int_{-a}^a z J(dz, ds)$. We define $\tilde{J}(dz, dt) := J(dz, dt) - \lambda_0 \nu(dz)dt$ as the corresponding compensated martingale measure. We assume that $S$ is a square-integrable semi-martingale, and present Assumption 1 in Section 3 to ensure the square integrability of $S$. The discounted price of the risky asset at time $t$ is denoted by $\tilde{S}_t := e^{-rt}S_t$. As the mean of jump sizes $Z$ is 0, we have

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \tilde{S}_u(\mu(u, S_u) - r_u)du + \int_0^t \tilde{S}_u \sigma(u, S_u)dW_u + \int_0^t \int_{-a}^a \tilde{S}_u z \tilde{J}(dz, du). \quad (3)$$

Let

$$M_t := \int_0^t \tilde{S}_u \sigma(u, S_u)dW_u + \int_0^t \int_{-a}^a \tilde{S}_u z \tilde{J}(dz, du), \quad (4)$$

and

$$A_t := \int_0^t \tilde{S}_u(\mu(u, S_u) - r_u)du. \quad (5)$$

Then, $\tilde{S}_t$ can be decomposed as

$$\tilde{S}_t = \tilde{S}_0 + M_t + A_t, \quad (6)$$

where $M$ is a square-integrable $P$-local martingale with $M_0 = 0$ and $A$ is a predictable finite variation process starting at zero. Note that if the expected stock return $\mu(t, S_t)$ is equal to the market interest rate $r_t$ for all $t \in [0, T]$, then the discounted price process of the risky asset $\tilde{S} := \{\tilde{S}_t\}_{t\in[0,T]}$ is a martingale under the measure $P$.

\footnote{Note that one standard assumption for existence and uniqueness of the solution of a stochastic differential equation is Lipschitz continuity. One can refer to [31] for detailed proof of it. Here, we assume that the drift, diffusion and jump coefficients are all Lipschitz continuous.}
2.2. The insurance market. We use \( l_x \) to denote the number of survivors at age \( x \). Let \( \{T_1, ..., T_{l_x}\} \) be i.i.d. nonnegative random variables which denote the future lifetimes of the group of \( l_x \) survivors at age \( x \). We denote \( \mathcal{H}_t \) as the natural filtration generated by \( I_{\{T_i \leq t\}} \) and \( \mathcal{H}^t := \bigvee_{i=1}^{l_x} \mathcal{H}_t \) for \( t \in [0, T] \), where \( I_\{\cdot\} \) is the indicator function and \( i \in \{1, 2, ..., l_x\} \). Let \( (\Omega_2, \mathcal{H}, \{\mathcal{H}_t\}_{t \in [0, T]}, \hat{P}) \) be a filtered complete probability space, considered to describe the insurance market, in which \( \hat{P} \) is a real-world probability measure. We use \( \mu_{x+t} \) to denote the hazard rate for a representative survivor at age \( x + t \), and the survival function of \( T_i \) is thus determined by

\[
\varphi_x := \hat{P}(T_i > t) = e^{-\int_0^t \mu_{x+u}du}.
\]  

(7)

For \( 0 \leq t \leq T \), \( D^t_i := \sum_{i=1}^{l_x} I_{\{T_i \leq t\}} \) represents the number of deaths until time \( t \). The intensity process \( \varpi = \{\varpi_t\}_{t \in [0, T]} \) of the counting process \( D^t_i \) is defined by

\[
\mathbb{E}_\hat{P}[dD^t_i | \mathcal{H}_t^-] = (l_x - D^t_i)\mu_{x+t}dt =: \varpi_t dt,
\]

where \( \mathbb{E}_\hat{P}[\cdot | \mathcal{H}_t^-] \) denotes the conditional expectation under the probability measure \( \hat{P} \) given \( \mathcal{H}_t^- \). If we let

\[
\hat{D}^t_i := D^t_i - \int_0^t \varpi_udu,
\]

then \( \hat{D}^t_i := \{\hat{D}^t_i\}_{t \in [0, T]} \) is a \( \hat{P} \)-martingale.

2.3. The combined market. In this subsection, we will introduce the combined financial-insurance model. Motivated by [27], [32], [37], [7], we assume in our work that the insurance market is independent of the financial market and that the hazard rate is a deterministic function. Under this assumption, future lifetime of an individual does not depend on the dynamics of financial market. That said, dependence between the two markets could also be assumed in others’ work, for example, [25] and [6], which is not what we consider in our work. Let \( \mathcal{G}_t := \mathcal{F}_t \lor \mathcal{H}_t \) be the natural filtration generated by the joint information of the financial market and the insurance market up to time \( t \). Furthermore, \( \mathcal{H}_t \) and \( \mathcal{F}_T \) are assumed to be independent throughout this paper. We let \( \mathcal{P} := P \times \hat{P} \) and use \( (\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \in [0, T]}, \mathcal{P}) = (\Omega_1 \times \Omega_2, \mathcal{G}, \{\mathcal{G}_t\}_{t \in [0, T]}, \mathcal{P} \times \hat{P}) \) to describe the combined market. The insurance benefits considered in our paper include pure endowment contracts and term insurance. A pure endowment benefit is paid conditional on the survival of the policyholder at the policy maturity date \( T \). The benefit is denoted as \( g(S_T) \) in which the function \( g \) is stipulated in the contract. For the insurance portfolio, the discounted value of total claims for the \( l_x \) pure endowment contracts is

\[
H = B_{T}^{-1} g(S_T) \sum_{i=1}^{l_x} I_{\{T_i > T\}} = B_{T}^{-1} g(S_T) (l_x - D^T_i).
\]

(9)

The benefit of pure endowment can be actually considered as the reward for survivalship. While for term insurance, the death benefit is payable only if the policyholder dies within the fixed term \([0, T]\). Correspondingly, the discounted dynamics
of the death benefit is given by

\[ H = \sum_{i=1}^{t} B_{T_i}^{-1} g(S_{T_i}) I_{\{T_i \leq T\}}. \]  

(10)

For convenience, we rewrite (10) as an integral with respect to the counting process \( D \):

\[ H = \int_0^T B_u^{-1} g(S_u) dD_u. \]  

(11)

Note that the condition \( \mathbb{E}_{\bar{P}}[\sup_{t \in [0,T]} g^2(S_t)] < \infty \) is assumed here so that discount value \( H \) of the unit-linked life insurance contracts is square-integrable under the probability measure \( \bar{P} \). In view of the independence between the insurance market and the financial market, any \( (\mathcal{F}, \bar{P}) \) martingale is also a \( (\mathcal{G}, \bar{P}) \) martingale. Besides, we can see that \( W \) defined in equation (1) is now a \( (\mathcal{G}, \bar{P}) \) Brownian motion by virtue of the Lévy’s Theorem (Lévy’s Characterization of Brownian Motion).

Different from traditional life insurance products, it can be seen from (9) and (11) that benefits of unit-linked life insurance products depend on mortality rate as well as the price of some specific stocks index traded in the financial market. Though unit-linked life insurance products may offer the potential of earning higher profit for policyholders, they expose insurance companies to extra investment risk in the financial market. In this article, we explore optimal hedging strategies for pure endowment and term insurance contracts from the perspective of an insurance company, and the optimality is achieved when the residual risk of the insurance portfolio is minimized. To this end, we apply the local risk-minimizing method to derive optimal hedging strategies. We first introduce a relevant concept, namely minimal martingale measure in the next section.

3. Minimal martingale measure. When the dynamics of underlying asset price follow a Hawkes jump diffusion process, the market is incomplete since the number of random sources exceeds that of tradeable risky assets. In an incomplete market, contingent claims cannot be completely replicated by the self-financing strategies. Mean-variance hedging, super-hedging and local risk-minimization are some of the popular approaches to hedge contingent claims in incomplete markets. The uncertain cost for hedging contingent claims is one of the major risks that a hedger has to bear. Local risk-minimization is a hedging method to control the risk of portfolio strategy in which local cost fluctuations are considered and the cost process is minimized. A more detailed review of local risk-minimization is provided in the Appendix for reader’s further reference.

The first step to implement the local risk-minimization method is to identify an equivalent martingale measure, referred to as the minimal martingale measure (see [15], [34], [30]). The minimal martingale measure plays an important role in finding the locally risk-minimizing hedging strategy. To proceed, we first recall a definition as follows. One may also refer to [37] and [10] for similar definitions.

**Definition 3.1.** A martingale measure \( Q^* \) equivalent to \( P \) is called the minimal martingale measure if \( Q^* \) satisfies the following two conditions:

(i) \( Q^* = P \) on \( \mathcal{F}_0 \);

(ii) if any \( P \)-local martingale \( L \) satisfying \( \langle L, M \rangle = 0 \) remains a local martingale
under $Q^*$, where $M$ is the local martingale part of $\tilde{S}$ under $P$, and $\tilde{S}$ and $M$ are given by (3) and (4), respectively.

In Definition 3.1, $\langle L, M \rangle$ denotes the predictable quadratic covariation between $L$ and $M$. Furthermore, we introduce some notation to be used throughout the paper. We denote the optional quadratic variation of $M$ by $\langle M \rangle$ and express the predictable quadratic variation of $M$ as $\langle M \rangle$.

In the following, we make an assumption to ensure the square integrability of the risky asset price $S$ and the existence of the minimal martingale measure.

**Assumption 1.** (i) The parameters of the financial market satisfy $0 \leq (\mu(u, S_u) - r_u) a < \sigma^2(u, S_u)$, $P$-a.s., for every $u \in [0, T]$;

(ii) $E_P \left[ \exp \left\{ \int_0^T \sigma^2(t, S_t)dt \right\} \right] < \infty$;

(iii) The jump intensity $\lambda$ is exponentially integrable in the sense that $E_P \left[ \exp \left\{ \delta^2 \int_0^T \lambda_u du \right\} \right] < \infty$, where $E_P[\cdot]$ denotes the expectation under $P$ and $\delta^2 = \text{Var}(Z)$ represents the variance of the jump size $Z$.

First, $\mu(u, S_u) \geq r_u$ is a very natural assumption, since otherwise no investor would be interested in purchasing the risky stock. Indeed, the excess of $\mu(u, S_u)$ over $r_u$ serves as the compensation for assuming risk in holding the stock. Furthermore, we need the assumption $(\mu(u, S_u) - r_u) a < \sigma^2(u, S_u)$ to make sure that a density process $\beta := \{\beta_u\}_{u \in [0, T]}$, to be defined in (18), is positive. We think it is a reasonable assumption. In this paper, $a$ represents the maximum range of a stock rising and falling due to a jump, it is usually not very large. In particular, the proportion of stocks rising and falling is restricted in some stock markets. For example, $a \leq 0.1$ is required in the Chinese stock market. If the volatility is set as a typical value 0.2, then $\mu(u, S_u) - r_u < 0.4$, it seems to be a reasonable assumption. On the other hand, the exponential integrability condition in (ii) and (iii) to guarantee the square integrability of the risky asset price $S$. The condition (iii) is equivalent to saying that the moment generating function of $\int_0^T \lambda_u du$ evaluated at $\delta^2$ is finite. To justify that (iii) is not an artificial assumption, we provide a lemma to shed light on certain sufficient conditions for $E_P \left[ \exp \left\{ \delta^2 \int_0^T \lambda_u du \right\} | \mathcal{F}_t \right] < \infty$, where $t \in [0, T]$.

**Remark 1.** Note that no arbitrage assumptions on the market model require the existence of a predictable process $\vartheta$ satisfying $\vartheta_t = -\frac{dA_i}{d\langle M \rangle_t}$ and $\int_0^T \vartheta_t^2 d\langle M \rangle_t < +\infty$, $P$-a.s.. This property is also called as Structure Condition (SC) and has been discussed in many references (see [34] and [10]). In our paper, Assumption 1 ensures that the Structure Condition holds. Hence, under Assumption 1, the set of all equivalent martingale measures is not empty. In particular, there are more than one equivalent martingale measure since the financial market is incomplete.

**Lemma 3.2.** Let

$$\Phi(\delta^2, t, T) := E_P \left[ \exp \left\{ \delta^2 \int_t^T \lambda_u du \right\} \mid \mathcal{F}_t \right], \quad t \in [0, T].$$

(12)

If $B(t, T)$ is a solution to the following ordinary differential equation:

$$\delta^2 + \frac{\partial B(t, T)}{\partial t} - \alpha B(t, T) + (e^{\beta(t, T)} - 1) = 0,$$

(13)
with terminal condition \( B(T, T) = 0 \), then the conditional moment-generating function of \( \int_t^T \lambda_s du \) is given by

\[
\Phi(\delta^2, t, T) := \exp \{ A(t, T) + B(t, T) \lambda_t \}, \tag{14}
\]

where \( A(t, T) = \alpha \int_t^T B(u, T) du \).

**Proof.** We adopt the similar method as the proof of Theorem 2.5 in [38]. Let

\[
G(t, \lambda_t) := \exp \{ A(t, T) + B(t, T) \lambda_t \}. \tag{15}
\]

Using Itô’s formula to \( \tilde{G}(t, \lambda_t) := e^{\delta^2 \int_t^T \lambda_s ds} G(t, \lambda_t) \), we get

\[
d\tilde{G}(t, \lambda_t) = \delta^2 \lambda_t \tilde{G}(t, \lambda_t) dt + \left( \tilde{G}(t, \lambda_t + \gamma) - \tilde{G}(t, \lambda_t) \right) dN_t
\]

\[
+ e^{\delta^2 \int_t^T \lambda_s ds} \left( \frac{\partial \tilde{G}}{\partial t} + \frac{\partial \tilde{G}}{\partial \lambda} \alpha(t, \lambda_t) \right) dt,
\]

where \( \frac{\partial \tilde{G}}{\partial t} \) and \( \frac{\partial \tilde{G}}{\partial \lambda} \) are partial derivatives of \( G \) with respect to \( t \) and \( \lambda \), and \( \tilde{N}_t := N_t - \int_0^t \lambda_s du \) is a \( P \)-martingale. Let

\[
I := \delta^2 \lambda_t G(t, \lambda_t) + \frac{\partial \tilde{G}}{\partial t} + \frac{\partial \tilde{G}}{\partial \lambda} \alpha(t, \lambda_t) + \left( G(t, \lambda_t + \gamma) - G(t, \lambda_t) \right) \lambda_t. \tag{17}
\]

Substituting (15) into (17), we find that \( I = 0 \) by (13). Furthermore, we know \( G(T, \lambda_T) = 1 \) from (15) and terminal conditions of \( A(t, T) \) and \( B(t, T) \). Hence, by \( I = 0 \) and the Feynman-Kač formula, we obtain

\[
G(t, \lambda_t) = \mathbb{E}_P \left[ e^{\delta^2 \int_t^T \lambda_s ds} G(T, \lambda_T) | \mathcal{F}_t \right] = \mathbb{E}_P \left[ e^{\delta^2 \int_t^T \lambda_s ds} | \mathcal{F}_t \right] = \Phi(\delta^2, t, T).
\]

The proof is completed.

From Lemma 3.2, we can find that if the equation (13) has a non-explosive solution over \([0, T]\), then \( \mathbb{E}_P \left[ \exp \{ \delta^2 \int_0^T \lambda_s du \} \right] = \Phi(\delta^2, 0, T) < \infty \) is satisfied in Assumption 1. However, it should be noted that Lemma 3.2 is only a sufficient condition for \( \mathbb{E}_P \left[ \exp \{ \delta^2 \int_0^T \lambda_s du \} \right] < \infty \), not a necessary one.

Next, we first construct an equivalent martingale measure via a Radon-Nikodym derivative, and then show that it is the minimal martingale measure.

**Theorem 3.3.** Given Assumption 1 and assume \( \beta_t \) follows the stochastic differential equation

\[
d\beta_t = -\beta_t \frac{\xi_t}{S_t} dM_t = -\beta_t \xi_t \left( \sigma(t, S_t) dW_t + \int_a^\alpha z \tilde{J}(dz, dt) \right),
\]

\[
\beta_0 = 1, \quad t \in [0, T], \tag{18}
\]

where

\[
\xi_t = \frac{\mu(t, S_t) - r_t}{\sigma^2(t, S_t) + \lambda_t \delta^2}.
\]

Define a probability measure \( Q^* \) equivalent to \( P \) with the Radon-Nikodym derivative \( \beta_t = \frac{dQ^*}{dP} |_{\mathcal{F}_t} \). Furthermore, if \( \mu(t, S_t) \neq r_t \) for \( t \in [0, T] \), then \( Q^* \) is the minimal martingale measure.

**Proof.** Firstly, we prove \( \beta_t > 0 \). From (18), it suffices to show \( 1 - \xi_t Z_j > 0 \). It is clear that \( \xi_t \in (0, 1) \) thanks to Assumption 1. Combining with the fact \( Z_j \in (-a, a) \subseteq (-1, 1) \) gives the desired result.
Secondly, we prove that $\beta = \{\beta_t\}_{t \in [0,T]}$ is a $P$-martingale. It follows from (18) and $\int_{a}^{\infty} z \nu (dz) = 0$ that the density process can be expressed by $\beta_t = \mathcal{E}(X)_t$, where $\mathcal{E}(X)$ denotes the stochastic exponential of $X := \{X_t\}_{t \in [0,T]}$:

$$X_t := - \int_0^t \zeta_u \sigma (u, S_u) dW_u - \int_0^t \int_a^\infty \zeta_u - z J (dz, du).$$  \hspace{1cm} (20)

Denote by $X^c_t$ and $X^d_t$ the continuous (diffusion) part and the jump part of $X_t$, i.e.,

$$X^c_t := - \int_0^t \zeta_u \sigma (u, S_u) dW_u, \quad X^d_t := - \int_0^t \int_a^\infty \zeta_u - z J (dz, du).$$  \hspace{1cm} (21)

In order to show $\beta$ is a martingale, we use a general version of Novikov’s condition (see Corollary 15.4.4 in [11]), which states that the stochastic exponential $\mathcal{E}(X)$ of a locally square integrable martingale $X$ is a uniformly integrable martingale on $[0, T]$ if $E_P \left[ \exp \left( \frac{1}{2} \langle X^c \rangle_T + \langle X^d \rangle_T \right) \right] < \infty$. From (21), we have

$$E_P \left[ \exp \left( \frac{1}{2} \langle X^c \rangle_T + \langle X^d \rangle_T \right) \right] = E_P \left[ \exp \left( \frac{1}{2} \int_0^T \zeta_u^2 \sigma^2 (u, S_u) du + \int_0^T \int_a^\infty \lambda_u \zeta_u^2 \nu (dz) du \right) \right] \leq E_P \left[ \exp \left( \frac{1}{2} \int_0^T \sigma^2 (u, S_u) du + \int_0^T \int_a^\infty \lambda_u \nu (dz) du \right) \right] < \infty,$$  \hspace{1cm} (22)

where we have used $\zeta_u \in [0, 1)$ in the second line. So, we can confirm that $\beta$ is a $P$-martingale and $Q^*$ is a probability measure.

Thirdly, we prove that the probability measure $Q^*$ is an equivalent martingale measure. By direct calculation, we have

$$\zeta_t = \frac{\mu (t, S_t) - r_t}{\sigma^2 (t, S_t) + \lambda_t \delta^2} = \frac{\mu (t, S_t) - r_t}{\sigma^2 (t, S_t) + \lambda_t \int_a^\infty z^2 \nu (dz)} = \frac{\tilde{S}_t dA_t}{d \langle M \rangle_t}.$$

An application of the stochastic product rule gives

$$d (\beta_t \tilde{S}_t) = \beta_t d \tilde{S}_t + \tilde{S}_t d \beta_t + d \left[ \beta, \tilde{S} \right]_t = \beta_t dM_t + \beta_t dA_t + \tilde{S}_t d\beta_t - \frac{\beta_t \zeta_t - \tilde{S}_t d \langle M \rangle_t}{\tilde{S}_t} = \beta_t dM_t + \tilde{S}_t d\beta_t - \frac{\beta_t \zeta_t - \tilde{S}_t d \langle M \rangle_t}{\tilde{S}_t},$$

where $\left[ \beta, \tilde{S} \right]$ is the quadratic covariation of $\beta, \tilde{S}$. All the three parts on the right-hand side of the above equation are $P$-local martingales. As the sum of the three $P$-local martingales, $\{\beta_t \tilde{S}_t\}_{t \in [0,T]}$ is a $P$-local martingale.

Moreover, for all $t \in [0, T]$, if $\mu (t, S_t) = r_t$, then $\zeta_t = 0$ and $\beta_t \equiv 1$. In this case, the measure $Q^*$ is clearly the same as the measure $P$. Finally, we prove that the measure $Q^*$ is the minimal martingale measure when $\mu (t, S_t) \neq r_t$. Let’s consider
a $P$-local martingale $L := \{L_t\}_{t \in [0,T]}$ which is orthogonal to $\{M_t\}_{t \in [0,T]}$. Then by (18), we have

$$
\langle L, \beta \rangle_t = \int_0^t -\beta_s \frac{\zeta_s}{S_s} d \langle L, M \rangle_s = 0.
$$

Therefore, $L$ remains a local martingale under the equivalent martingale measure $Q^*$. Now, we can conclude that $Q^*$ is the minimal martingale measure from Definition 3.1.

Remark 2. Note that without Assumption 1 the minimal martingale measure $Q^*$ might be a signed measure and cause problems subsequently in the local risk-minimization method. The existence of the minimal martingale measure is an interesting research problem in its own right for self-excited jump models. To avoid the minimal martingale measure being a signed measure, the common practice is to impose some restrictions on the jump sizes or model parameters. For example, [8] assumes that the jump sizes are bounded, i.e., $\Delta X_t \in [-c_1, c_2]$, and the author makes some further assumptions on model parameters so that the minimal martingale measure was not a signed measure. [23] suppose that the jump sizes take values on $(-1,1)$ and impose some technical conditions on model parameters. [3] assume that the stock price is modeled by an exponential Lévy process. Though there is no need to restrict the jump size such that the stock price is non-negative in [3], they still make two assumptions in their paper (refer to Assumption 1.1 in [3]). In our work, technical conditions in Assumption 1 are used to ensure $Q^*$ is a well-defined probability measure.

To proceed, we need the following proposition. It is a direct application of Girsanov's theorem to point processes.

**Proposition 1.** Under the minimal martingale measure $Q^*$,

$$
\hat{W}_t := W_t + \int_0^t \zeta_u \sigma(u, S_u) du,
$$

is a standard Brownian motion. Furthermore, the change of the probability measure from $P$ to $Q^*$ does not alter the dynamics of the intensity of the Hawkes process $N$, i.e., the intensity of $N$ has the following $Q^*$-dynamics:

$$
d\lambda_t = \alpha (\kappa - \lambda_t) dt + \gamma dN_t.
$$

However, the density function $v^*_t(dz)$ of jump sizes $Z_j$ becomes $(1 - \zeta_t z) \nu(dz)$ under the new probability measure $Q^*$.

**Proof.** By virtue of Girsanov’s theorem and (18), we immediately see that $\hat{W}_t$ is a standard Brownian motion under $Q^*$. On the other hand, Girsanov’s theorem for general jump-diffusion processes yields that the compensator of $J(dz, dt)$ is given by

$$
\lambda^{Q^*}_t v^*_t(dz) = \lambda_t (1 - \zeta_t z) \nu(dz) = \lambda_t \int_{-a}^a (1 - \zeta_t z) \nu(dz) \times \frac{(1 - \zeta_t z) \nu(dz)}{\int_{-a}^a (1 - \zeta_t z) \nu(dz)}.
$$

Hence, the intensity of $N_t$ under the measure $Q^*$ is given by

$$
\lambda^{Q^*}_t := \lambda_t \int_{-a}^a (1 - \zeta_t z) \nu(dz).
$$
Suppose that as a function of the first, third and fourth variables
\[ \lambda_t \] is independent of the jump size \( Z \), we have
\[ \int_{-a}^{a} (1 - \zeta_t z) \nu(dz) = \int_{-a}^{a} \nu(dz) - \zeta_t \int_{-a}^{a} z \nu(dz) = 1. \]
Hence, \( \lambda^*_t = \lambda_t \), that is, the intensity of \( N \) remains the same under the measures \( Q^* \) and \( P \). However, the density function of jump size \( Z \) under \( Q^* \) is different from that under \( P \). From (24), we obtain
\[ \nu^*_t(dz) := \frac{(1 - \zeta_t z) \nu(dz)}{\int_{-a}^{a} (1 - \zeta_t z) \nu(dz)} = (1 - \zeta_t z) \nu(dz). \] (26)
This completes the proof. \( \square \)

4. Hedging strategy for unit-linked life insurance contracts. In this section, we consider hedging strategy for unit-linked life insurance contracts. We employ the local risk-minimization method to derive an optimal hedging strategy. To that end, we borrow some definitions and symbols in some previous literature. See, for example, [34] and [37] and references therein. Let \( \overline{Q}^* := Q^* \times \hat{P} \), where \( \hat{P} \) is given in Subsection 2.2. Recall that in Subsection 2.3 the insurance market and the financial market are assumed to be independent and \( \mathcal{F} := P \times \hat{P} \). Hence, \( \overline{Q}^* \) is also the minimal martingale measure for the combined insurance and finance market. The specific proof can refer to Lemma 4.3 in [6].

4.1. Pure endowment. For a pure endowment insurance contract, a lump-sum benefit payment is made at the end of a specific term if the insured survives to the maturity date. The pure endowment contract is appealing to investors, thanks to the opportunity of earning a substantial investment return from the financial market. In this subsection, we focus on the portfolio of \( l_x \) pure endowment contracts with the discounted payoff given by (9). Let \( V^{PE}(t, S_l, \lambda_l) \) denote the expected present value of the total payoff of the portfolio with respect to the minimal martingale measure \( \overline{Q}^* \) at time \( t \), that is,
\[ V^{PE}(t, S_l, \lambda_l) := \mathbb{E}_{\overline{Q}^*} \left[ e^{-\int_t^T r_u du} g(S_T) (l_x - D_T) | \mathcal{F}_t \right]. \]
Let \( \overline{V}^{PE}(t, S_l, \lambda_l) = e^{-\int_0^t r_u du} V^{PE}(t, S_l, \lambda_l) \) be the discounted value of \( V^{PE}(t, S_l, \lambda_l) \). Clearly, it is a \( \overline{Q}^* \)-martingale. Due to the independence assumption of the financial market and the insurance market, we have
\[ \overline{V}^{PE}(t, S_l, \lambda_l) = \mathbb{E}_{\mathcal{F}_T} \left[ e^{-\int_0^T r_u du} g(S_T) | \mathcal{F}_t \right] \cdot \mathbb{E}_{\overline{Q}^*} \left[ l_x - D_T | \mathcal{F}_t \right] \]
\[ = \overline{V}(t, T, S_l, \lambda_l)(l_x - D_T) - \mathbb{E}_{\overline{Q}^*} \left[ D_T | \mathcal{F}_t \right], \] (27)
where \( \overline{V}(t, T, S_l, \lambda_l) \) denotes the discounted value of a European contingent claim under \( \overline{Q}^* \) and the payoff of a contingent claim is \( g(S_T) \) at the maturity date \( T \).

Proposition 2. Suppose that as a function of the first, third and fourth variables \( \bar{V}(\cdot, T, \cdot, \cdot) \) is in \( C^{1,2,1}([0,T] \times \mathbb{R}^+ \times \mathbb{R}^+) \). Then \( \overline{V}^{PE}(t, S_l, \lambda_l) \) is given as follows
\[ \overline{V}^{PE}(t, S_l, \lambda_l) = \overline{V}(0, S_0, \lambda_0) + \int_0^t (l_x - D_{x-}) \mathbb{E}_{\overline{Q}^*} \left[ \frac{\partial V}{\partial S} \bar{S}_{u-} \cdot \sigma d\bar{W}_u \right] \]
\[ - \int_0^t \overline{V}(u, T, S_{u-}, \lambda_{u-}) T - u \mathbb{E}_{\overline{Q}^*} \left[ D_{x+} \right] \] (28)
where \( \mathcal{J}^{Q^*}(dz, dt) := J(dz, dt) - \lambda_{u-} \nu_t^*(dz) dt \), the symbol \( V \) is a simplified notation for \( V(u, T, S_u, \lambda_u) \) and \( V(u, T, S_u, \lambda_u) = e^{J_0^t r_s du} \tilde{V}(u, T, S_u, \lambda_u) \) and

\[
\Gamma(u, T, S_u-, z, \lambda_u-) = \tilde{V}(u, T, S_u-(1 + z), \lambda_u- + \gamma) - \tilde{V}(u, T, S_u-, \lambda_u-). \tag{29}
\]

**Proof.** Applying Itô’s formula to (27) gives

\[
\tilde{V}^{PE}(t, S_t, \lambda_t)
= \tilde{V}^{PE}(0, S_0, \lambda_0) + \int_0^t \left( \frac{\partial \tilde{V}}{\partial t} + \left( \frac{\partial \tilde{V}}{\partial S} \right)_t S_t^c + \frac{1}{2} \frac{\partial^2 \tilde{V}}{\partial S^2} d(S_t^c)^2 + \frac{\partial \tilde{V}}{\partial \lambda} \alpha(\kappa - \lambda_t) dt + \int_{-a}^a \left[ \tilde{V}(t, S_{t-} + (\nu_t \gamma), \lambda_{t-} + \gamma) - \tilde{V}(t, S_{t-}, \lambda_{t-}) \right] J(dz, dt), \tag{31}
\]

where \( \tilde{S}_t \) is given in (8) and is also a \( \mathcal{Q}^* \)-martingale.

Using Itô’s formula to \( \tilde{V}(t, T, S_t, \lambda_t) \) again, we have

\[
\frac{d\tilde{S}_t}{\tilde{S}_{t-}} = \sigma(t, S_t) d\tilde{W}_t + \int_{-a}^a z \mathcal{J}^{Q^*}(dz, dt), \tag{32}
\]

where \( \tilde{W}_t \) and \( \nu_t^*(dz) \) are defined in Proposition 1. Clearly, the stock price \( S_t = e^{J_0^t r_s du} \tilde{S}_t \) satisfies

\[
\frac{dS_t}{S_{t-}} = r_t dt + \sigma(t, S_t) d\tilde{W}_t + \int_{-a}^a z \mathcal{J}^{Q^*}(dz, dt). \tag{33}
\]

Then it follows from (31) and (33) that

\[
\tilde{V}(t, T, S_t, \lambda_t)
= \tilde{V}(0, T, S_0, \lambda_0) + \int_0^t \frac{\partial \tilde{V}}{\partial u} du + \int_0^t \frac{\partial \tilde{V}}{\partial S} S_u \left( r_u - \lambda_u \int_{-a}^a z \nu_u^*(dz) \right) du
+ \int_0^t \frac{\partial \tilde{V}}{\partial S} S_u \sigma(u, S_u) d\tilde{W}_u + \frac{1}{2} \int_0^t \frac{\partial^2 \tilde{V}}{\partial S^2} \sigma^2(u, S_u) S_u^2 du + \int_0^t \frac{\partial \tilde{V}}{\partial \lambda} \alpha(\kappa - \lambda_u) du
+ \int_0^t \int_{-a}^a \lambda_u \Gamma(u, T, S_{u-}, z, \lambda_{u-}) \nu_u^*(dz) du.
\]
Note that the finite variation part in the above formula is zero since \( \tilde{V}(t, T, S_t, \lambda_t) \) is a \( \overline{Q}^T \)-martingale. Hence, given \( S_t = S \) and \( \lambda_t = \lambda \), \( \tilde{V}(t, T, S, \lambda) \) satisfies the following partial differential integral equation:

\[
\frac{\partial \tilde{V}}{\partial t} + \frac{\partial \tilde{V}}{\partial S} \left( r_t - \lambda \int_{-a}^{a} \nu_u'(dz) \right) + \frac{1}{2} \frac{\partial^2 \tilde{V}}{\partial S^2} \sigma^2(t, S_t) S^2 \\
+ \frac{\partial \tilde{V}}{\partial \lambda} (\kappa - \lambda) + \int_{-a}^{a} \lambda \Gamma(t, S, z, \lambda) \nu_u'(dz) = 0.
\]

Given that \( \frac{\partial \tilde{V}}{\partial S} S_t = \frac{\partial \tilde{V}}{\partial S} \tilde{S}_t \), we therefore get the following result

\[
d\tilde{V}(t, T, S_t, \lambda_t) = \frac{\partial \tilde{V}}{\partial S} \tilde{S}_t \sigma(t, S_t) d\tilde{W}_t + \int_{-a}^{a} \Gamma(t, S_t, z, \lambda_t) \tilde{J}^{Q^T}(dz, dt). \tag{34}
\]

Plugging (34) into (30), we get (28).

We are now ready to derive the hedging strategy for pure endowment contracts via the local risk-minimization method. From Proposition A.2, we know that if the Föllmer-Schweizer decomposition (A.6) exists, then the value process \( \tilde{V}^{PE}(t, S_t, \lambda_t) \) has a decomposition which is given by as follows

\[
\tilde{V}^{PE}(t, S_t, \lambda_t) = \tilde{V}^{PE}(0, S_0, \lambda_0) + \int_{0}^{t} \xi^{PE*}_u d\tilde{S}_u + L_t, \tag{35}
\]

where \( L = \{L_t\}_{t \in [0, T]} \) is a square-integrable \( \overline{P} \)-martingale and orthogonal to \( M = \{M_t\}_{t \in [0, T]} \), i.e., \( \langle L, M \rangle = 0 \). Then we can make use of the fact that \( L \) is orthogonal to \( M \) and carry out necessary calculations to get the pseudo-locally risk minimizing hedging strategy \( \{\xi^{PE*}_t\}_{t \in [0, T]} \). In the following, we show the existence of the decomposition.

**Proposition 3.** \( \tilde{V}^{PE}(t, S_t, \lambda_t) = e^{-\int_{0}^{t} r_u du} V^{PE}(t, S_t, \lambda_t) \) has a decomposition

\[
\tilde{V}^{PE}(t, S_t, \lambda_t) = \tilde{V}^{PE}(0, S_0, \lambda_0) + \int_{0}^{t} \xi^{PE*}_u d\tilde{S}_u + L_t,
\]

where \( L = \{L_t\}_{t \in [0, T]} \) is a square-integrable \( \overline{P} \)-martingale and orthogonal to \( M = \{M_t\}_{t \in [0, T]} \).

**Proof.** The existence of the decomposition can be obtained by adopting similar arguments of Theorem 2.9 in [24]. We can prove it in two steps. In the first step, we prove that \( L_t = \tilde{V}^{PE}(t, S_t, \lambda_t) - \tilde{V}^{PE}(0, S_0, \lambda_0) - \int_{0}^{t} \xi^{PE*}_s d\tilde{S}_s \) is a \( P \) martingale. Since \( \tilde{V}^{PE}(t, S_t, \lambda_t) \) and \( \int_{0}^{t} \xi^{PE*}_s d\tilde{S}_s \) are \( \overline{Q}^T \) local martingales, \( L_t \) is a \( \overline{Q}^T \) local martingale. Then we prove that \( L_t \) is a \( \overline{P} \) local martingale. Finally, since \( \mathbb{E}_{\overline{P}}[\langle L_t \rangle] = \mathbb{E}_{\overline{P}}[\tilde{V}^{PE}(T, S_T, \lambda_T)^2] - \mathbb{E}_{\overline{P}}[\int_{0}^{T} (\xi^{PE*}_s)^2 d\langle M \rangle_s] < \infty \), we get \( L_t \) is a \( \overline{P} \) martingale.

In the second step, we prove \( L_t = \tilde{V}^{PE}(t, S_t, \lambda_t) - \tilde{V}^{PE}(0, S_0, \lambda_0) - \int_{0}^{t} \xi^{PE*}_s d\tilde{S}_s \) is orthogonal to \( M = \{M_t\}_{t \in [0, T]} \), i.e., \( \langle L, M \rangle = 0 \). Hence, we can show the existence of the decomposition (35). We omit the details here for brevity.

**Remark 3.** In fact, we can only need to prove the existence of the Föllmer-Schweizer decomposition since it leads to the equation (35) holds. We make an
Assumption 1 in Section 3, it leads to Proposition 3.7 in [9]. Then the existence of the Föllmer-Schweizer decomposition can be obtained using Theorem 5.5 in [9].

**Theorem 4.1.** The pseudo-locally risk-minimizing hedging strategy \( \phi^{PE*} = (\xi^{PE*}, \eta^{PE*}) \) for the portfolio of pure endowment contracts is given by

\[
\xi_t^{PE*} = \frac{\partial V}{\partial S} \bar{S}_t - \sigma^2(t, S_t) + \lambda_t - \int_a^T \Gamma (t, S_t, z, \lambda_t) z \nu(dz) \frac{S_t - \sigma^2(t, S_t) + \lambda_t}{S_t},
\]

and

\[
\eta_t^{PE*} = \bar{V}^{PE}(t, S_t, \lambda_t) - \xi_t^{PE*} \bar{S}_t.
\]

**Proof.** As discussed earlier, if the decomposition (35) exists, then we can obtain the pseudo-locally risk-minimizing hedging strategy \( \phi^{PE*} = (\xi^{PE*}, \eta^{PE*}) \). Using (35), we get

\[
\langle \bar{V}^{PE}, \bar{S} \rangle_t = \int_0^t \xi_u^{PE*} d\langle \bar{S} \rangle_u + \langle L, \bar{S} \rangle_t.
\]

Given that \( L \) is orthogonal to \( M \) under the measure \( P \), it holds

\[
\langle L, \bar{S} \rangle_t = \langle L, M \rangle_t + \langle L, A \rangle_t = 0.
\]

Combining (38) and (39), \( \xi_t^{PE*} \) can be calculated under the measure \( P \) as follows

\[
\xi_t^{PE*} = \frac{d\langle \bar{V}^{PE}, \bar{S} \rangle_t}{d\langle \bar{S} \rangle_t}.
\]

Using (28) in conjunction with (3) yields

\[
\langle \bar{V}^{PE}, \bar{S} \rangle_t = \int_0^t (l_x - D_{t-} T - \theta p_x + l_x) \left( \frac{\partial V}{\partial S} \bar{S}_u - \sigma^2(u, S_u) \right) d\bar{S}_u + \lambda_u - \bar{S}_u - \int_a^T \Gamma (u, S_u, z, \lambda_u) z \nu(dz) \right) du.
\]

Because the predictable quadratic covariation process \( \langle \bar{V}^{PE}, \bar{S} \rangle_t \) is calculated under \( P \), the intensity function of the random jump size \( Z \) should thus be \( \nu(dz) \) rather than \( \nu_t^d(dz) \). Moreover, by (3), we have

\[
\langle \bar{S} \rangle_t = \int_0^t \bar{S}_u^2 - \sigma^2(u, S_u) du + \int_0^t \lambda_u - \bar{S}_u - \int_a^T z^2 \nu(dz) du
\]

\[
= \int_0^t \bar{S}_u^2 - \sigma^2(u, S_u) du + \int_0^t \lambda_u - \bar{S}_u^2 \delta^2 du.
\]

Finally, substituting (41) and (42) into (40), we obtain (36). \( \square \)

From Proposition A.1, the pseudo-locally risk-minimizing hedging strategy is the locally risk-minimizing hedging strategy if Conditions (i)-(iii) of Proposition A.1 are satisfied. Next it suffices to verify these conditions in our framework.

**Proposition 4.** The pseudo-locally risk-minimizing hedging strategy \( \phi^{PE*} = (\xi^{PE*}, \eta^{PE*}) \) for the portfolio of pure endowment contracts in Theorems 4.1 is the locally risk-minimizing hedging strategy.
Proof. We verify the three conditions sequentially.

1. Since the Brownian motion $W$ and the Hawkes jump processes $N$ are mutually independent, we have

\[
\langle M \rangle_t = \int_0^t \tilde{S}_u^2 \sigma^2(u, S_u) du + \int_0^a \int_{-a}^{a} \lambda_u \tilde{S}_u^2 z^2 \nu(dz) du
\]

\[
= \int_0^t \tilde{S}_u^2 \sigma^2(u, S_u) du + \int_0^t \lambda_u \tilde{S}_u^2 \delta^2 du.
\]

\[
(43)
\]

It follows from Eq. (43) that $\langle M \rangle_t$ is $\mathbb{P}$ almost surely strictly increasing on $[0, T]$ due to $\lambda_u > 0$. Hence, Condition (i) is satisfied.

2. Since the Lebesgue measure of the jump times of the stock price $S$ is zero, then the finite variation part

\[
A_t = \int_0^t \tilde{S}_u - (\mu(u, S_u) - r_u) du
\]

is continuous.

3. From (43) and (44), we have

\[
\vartheta_t = \frac{dA_t}{d\langle M \rangle_t} = \frac{\mu(t, S_t) - r_t}{\tilde{S}_t - (\sigma^2(t, S_t) + \lambda_t \delta^2)}.
\]

\[
(45)
\]

then

\[
\mathbb{E}_\mathbb{P}\left[ \int_0^t \vartheta_u d\langle M \rangle_u \right] = \mathbb{E}_\mathbb{P}\left[ \int_0^t \vartheta_u^2 d\langle M \rangle_u \right]
\]

\[
= \mathbb{E}_\mathbb{P}\left[ \int_0^t \left( \frac{\mu(u, S_u) - r_u}{\sigma^2(u, S_u) + \lambda_u \delta^2} \right)^2 du \right]
\]

\[
\leq \mathbb{E}_\mathbb{P}\left[ \int_0^t \left( \frac{\mu(u, S_u) - r_u}{\sigma^2(u, S_u)} \right)^2 du \right] \leq \mathbb{E}_\mathbb{P}\left[ \frac{1}{\sigma^2} \int_0^t \sigma^2(u, S_u) du \right]
\]

\[
\leq \frac{1}{\sigma^2} \mathbb{E}_\mathbb{P}\left[ \int_0^t \sigma^2(u, S_u) du \right] < \infty.
\]

(46)

Hence, Conditions (i)-(iii) all hold true. It then follows from Proposition A.1 that the pseudo-locally risk-minimizing hedging strategy in Theorem 4.1 is the locally risk-minimizing hedging strategy.

Remark 4. If the measure $P$ is a martingale measure in the first place, the trading strategy $\varphi$ obtained in Theorem 4.1 is then the risk-minimizing hedging strategy from [15].

It can be seen from (36) that the locally risk-minimizing hedging strategy is a product of two parts. The first part $(l_x - D_{l_x})_{T-t} p_{x+t}$ is the number of survivors at time $t$ multiplied by the survival probability $T-t p_{x+t}$. It reflects the expected number of survivors at time $T$ within the cohort of size $(l_x - D_{l_x})$ at age $x+t$. Hence, the insurance company should adjust the hedging portfolio strategy dynamically according to the expected number of survivors throughout the term of the policy, in which the decrement is the death of policyholders. The second part $II$ represents the locally risk-minimizing optimal hedging ratio for the contingent claim $g(S_T)$ in an incomplete financial market. The classical Delta hedging ratio $\frac{\partial V}{\partial S}$ could completely replicate the contingent claim $g(S_T)$ in a complete financial market where the stock price were modeled by a Black-Scholes model. However, the presence of jumps in
the risky asset dynamics undermines the efficiency of the delta hedging method since the market is not complete any more.

In what follows, we offer some economic explanations of the locally risk-minimizing hedging strategy (36) and compare it with the classical delta hedging strategy. We resort to the Taylor series expansion of the function \( \Gamma (t, T, S_t - z, \lambda_t - \zeta) \) up to the second-order and get the following approximation

\[
\Gamma (t, T, S_t - z, \lambda_t - \zeta) \approx \frac{\partial V}{\partial S} \tilde{S}_t - \zeta + \frac{\partial V}{\partial \lambda} \gamma + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \tilde{S}_t^2 z^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \lambda^2} \gamma^2. \tag{47}
\]

Plugging (47) into the second part \( II \) of (36), we have

\[
\frac{\partial V}{\partial S} \tilde{S}_t - \sigma^2 (t, S_t) + \lambda_t \delta^2 \int_a^\infty z \nu(dz) = \frac{\partial V}{\partial S} + \frac{\partial V}{\partial \lambda} \gamma - \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \tilde{S}_t^2 z^2 + \frac{1}{2} \frac{\partial^2 V}{\partial \lambda^2} \gamma^2 \int_a^\infty z \nu(dz). \tag{48}
\]

The last equation comes from the fact that \( \int_a^\infty z \nu(dz) = 0 \). In addition to the Delta-hedging strategy component, we find from (48) that the locally risk-minimizing hedging strategy not only incorporates the impact of the Gamma-hedging term, but also includes the impact of stock price itself, the random jump intensity and the second and third moments of jump size. If the jump size is symmetric such that \( \int_a^\infty z^3 \nu(dz) = 0 \), the second part vanishes in (48). Otherwise, both the self-excited jump intensity and the skewness of the jump size matter in the Gamma-hedging strategy. Therefore, the locally risk-minimizing hedging strategy and the Delta/Gamma-hedging strategy are very different in our financial market with jump clustering. In the following remarks, we dig into some special cases of Theorem 4.1.

**Remark 5.** If there is no jump component in the stock price model (1), i.e., \( \lambda_t = 0 \), our financial model reduces to the classic Black-Scholes model. Then the hedging strategy for pure endowment contracts is given by

\[
\xi_{PE}^* = (l^T - D_{1+T}^t) \frac{T-t}{p_{x+t}} \frac{\partial V}{\partial S}, \tag{49}
\]

and

\[
\eta_{PE}^* = \tilde{V}_{PE}(t, S_t, 0) - \xi_{PE}^* \tilde{S}_t. \tag{50}
\]

The above result is consistent with Theorem 4.2 of [27].

In the Black-Scholes framework, the delta hedging strategy is commonly used to replicate the contingent claims. Hence, the insurer only needs to know how many insureds survive in the term of insurance policies. It is reflected exactly in the first part of the hedging strategy \( \xi_{PE}^* \) in Eq. (49). Therefore, we find that the locally risk-minimizing hedging strategy \( \xi_{PE}^* \) is a product of the (conditional) expected number of surviving insureds and the Delta hedging strategy in the Black-Scholes framework.
Remark 6. If there is no insurance market, the hedging strategy for a European contingent claim \( g(S_T) \) becomes

\[
\xi_t^* = \frac{\partial V}{\partial S_t} S_t - \sigma^2(t, S_t) + \lambda_t - \int_a^T \Gamma(t, T, S_t, z, \lambda_t) z \nu(dz)
\]

\( (51) \)

and

\[
\eta_t^* = \tilde{V}(t, S_t, \lambda_t) - \xi_t^* \tilde{S}_t.
\]

Without insurance market, a hedger only faces the financial risk. Hence, the locally risk-minimizing strategy \( \xi_t^* \) of Eq. (51) reduces to be the second part of the locally risk-minimizing strategy in Eq. (36).

From (28), (32) and Definition A.2, we are now in the position to calculate the cost process \( C_t(\varphi) \) of the locally risk-minimizing hedging strategy \( \varphi^{PE*} = (\xi^{PE*}, \eta^{PE*}) \) as follows:

\[
C_t(\varphi) = \tilde{V}^{PE}(t, S_t, \lambda_t) - \int_0^t \xi_t^{PE*} d\tilde{S}_t
\]

\[
= \tilde{V}^{PE}(0, S_0, \lambda_0) + \int_0^t \left( l_x - D_{u-}^t \right) T - u p_{x+u} \frac{\partial V}{\partial S} - \xi_t^{PE*} S_u - \sigma(u, S_u) dW_u
\]

\[
- \int_0^t \tilde{V}(u, T, S_u, \lambda_u - \tilde{S}_u - z) \tilde{D}_u^t
\]

\( (53) \)

Realizing \( C_t(\varphi) \) is also a \( \mathcal{F}_t \)-martingale, then we have

\[
C_t(\varphi) = \tilde{V}^{PE}(0, S_0, \lambda_0) + \int_0^t \left( l_x - D_{u-}^t \right) T - u p_{x+u} \frac{\partial V}{\partial S} - \xi_t^{PE*} S_u - \sigma(u, S_u) dW_u
\]

\[
- \int_0^t \tilde{V}(u, T, S_u, \lambda_u - \tilde{S}_u - z) \tilde{D}_u^t
\]

\( (54) \)

Then from (A.4), (36) and \( d(\tilde{D}_u)^t = \omega_u du = (l_x - D_{u-}^t) \mu_x du \), we can represent the residual risk process as follows

\[
R_t(\varphi) = \mathbb{E}_{\mathcal{F}_t} \left[ (C_t^{\varphi} - C_t(\varphi))^2 \mid \mathcal{G}_t \right] = \mathbb{E}_{\mathcal{F}_t} \left[ \int_0^T \left( l_x - D_{u-}^t \right) T - u p_{x+u} \right]
\]

\[
\times \left( \frac{\partial V}{\partial S} - \frac{\partial V}{\partial S} S_u - \sigma^2(u, S_u) + \lambda_u - \int_a^T \Gamma(u, T, S_u, z, \lambda_u) z \nu(dz) \right)
\]

\[
S_u - \sigma^2(u, S_u) + \lambda_u - \delta^2 \right) \left( \frac{\partial V}{\partial S} - \frac{\partial V}{\partial S} S_u - \sigma^2(u, S_u) du \mid \mathcal{G}_t \right]
\]

\[
+ \mathbb{E}_{\mathcal{F}_t} \left[ \int_0^T \tilde{V}^2(u, T, S_u, \lambda_u) T - u p_{x+u} (l_x - D_{u-}^t) \mu_x du \mid \mathcal{G}_t \right]
\]
follows from (55) and Fubini’s theorem that
with
\[ \varphi \]

Let the payoff function be
\[ g \]

payoff
we present some numerical results based on the unit-linked insurance contract with
\[ K \]

Next we consider a special unit-linked insurance contract, i.e., a unit-linked in-
\[ \int \]

-\[ R \]
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\[ R_0(\varphi) \text{ is defined by the variance of the cost process } C_t(\varphi). \text{ It measures the size of the loss of hedging for a contingent claim. From (54), we find that the cost process } C_t(\varphi) \text{ consists of three uncertain components, namely, the diffusion risk of the financial asset, the mortality risk of the insurer and the jump risk of the financial asset. Comparing (54) with (56), we can see that the residual risk } R_0(\varphi) \text{ has also three components corresponding to the diffusion, mortality and jump risks.} \]

**Example 4.2.** Let the payoff function be \( g(S_T) = S_T \), that is, the insured is to be paid the value of the stock at the maturity date \( T \). In this case, \( V(t, T, S_t, \lambda_t) = \mathbb{E}_{Q_t}[e^{-\int_t^T r_s ds} S_T] = S_t \). Then we have \( \frac{\partial V}{\partial S} = 1 \) and \( \Gamma(t, T, S_t, z, \lambda_t) = \tilde{S}_t - z \). Hence, from (36), the locally risk-minimizing hedging strategy \( \varphi^{PE*} = (\xi^{PE*}, \eta^{PE*}) \) is given by

\[
\xi^{PE*}_t = \frac{(l_x - D^i_{t-}) - \frac{\partial S_t}{\partial x} \tilde{S}_t - \lambda_t \int_a^x \nu(dz)}{S_t - \lambda_x \int_a^x \nu(dz)}
\]

and

\[
\eta^{PE*}_t = \tilde{V}^{PE}_t - \xi^{PE*}_t \tilde{S}_t - \frac{(l_x - D^i_{t-}) - \frac{\partial S_t}{\partial x} \tilde{S}_t - \lambda_t \int_a^x \nu(dz)}{S_t - \lambda_x \int_a^x \nu(dz)}
\]

where the last equality in (58) is justified by \( \Delta D^i_t = D^i_t - D^i_{t-} \). Furthermore, if \( \sigma(t, S_t) \equiv \sigma, r_t \equiv r \), from (56), we have

\[
R_0(\varphi) = \int_0^T \mathbb{E}_Q \left[ \tilde{S}_t \right] T - u p_{x+u} \mu_{x+u} du
\]

\[
= \int_0^T \mathbb{E}_Q \left[ e^{\delta^2 \int_0^u \lambda_s ds} e^{2(\mu - r) u + \sigma^2 u} T - u p_{x+u} \mu_{x+u} du \right]
\]

where \( \Phi(\delta^2, 0, u) ) \) is given (14).

Different from Example 4.1 in which the payoff of the unit-linked life contracts is linked to a financial derivative, the benefit in Example 4.2 only depends on the stock price \( S \), and thereby the insurance company can purchase the stock to completely eliminate the financial risk. Therefore, in contrast to (56), the loss caused by the mortality risk appears in (59).

**4.2. Term insurance.** Term insurance makes a lump-sum benefit payment upon the death of the policyholder, as long as death occurs before the end of a predetermined term. The insurance benefit may be left to the policyholder’s dependant(s) in the event of his or her death. In this subsection, we consider the locally risk-minimizing hedging strategy \( \varphi^{TI*} = (\xi^{TI*}, \eta^{TI*}) \) for the portfolio of term insurance contracts. Let \( \tilde{V}^{TI}(t, S_t, \lambda_t) \) denote the discounted portfolio value of term insurance contracts with respect to the minimal martingale measure \( Q^* \), which is given by

\[
\tilde{V}^{TI}(t, S_t, \lambda_t) = \mathbb{E}_{Q^*} \left[ \int_0^T B_u^{-1} g(S_u) dD_u^1 \mid \mathcal{F}_t \right].
\]
Let \( \tilde{V}(t, u, S_t, \lambda_t) = \mathbb{E}_Q^{\mathcal{F}_t} \left[ B_u^{-1} g(S_u) | \mathcal{F}_t \right] \) and \( V(t, u, S_t, \lambda_t) = e^{rt} \tilde{V}(t, u, S_t, \lambda_t) \), then we have the following result.

**Theorem 4.2.** The locally risk-minimizing hedging strategy \( \varphi^{TI*} = (\xi^{TI*}, \eta^{TI*}) \) for the portfolio of term insurance contracts is

\[
\xi^{TI*}_t = \frac{(l_x - D_{t-}^l)}{S_{t-} \left( \sigma^2(t, S_t) + \lambda_{t-} \delta^2 \right)} \times \int_t^T u - t p_{x+u} \mu_{x+u} \left( \frac{\partial V}{\partial S} \tilde{S}_{t-} \sigma^2(t, S_t) + \lambda_{t-} \int_{-a}^a \Gamma \left( t, u, S_{t-}, z, \lambda_{t-} \right) z \nu(dz) \right) du
\]

and

\[
\eta^{TI*}_t = \tilde{V}(t, S_t, \lambda_t) - \xi^{TI*}_t \tilde{S}_t,
\]

where the symbol \( \tilde{V} \) is a simplified notation for \( \tilde{V}(t, u, S_t, \lambda_t) \).

**Proof.** Following (60), we derive

\[
\tilde{V}(t, S_t, \lambda_t) = \mathbb{E}_Q^{\mathcal{F}_t} \left[ \int_t^T B_u^{-1} g(S_u) dD_u^l \mid \mathcal{F}_t \right]
\]

By (34), we have

\[
d\tilde{V}(t, u, S_t, \lambda_t) = \frac{\partial V}{\partial S} \tilde{S}_{t-} \sigma(t, S_t) d\tilde{W}_t + \int_{-a}^a \Gamma \left( t, u, S_{t-}, z, \lambda_{t-} \right) \tilde{J}Q^*(dz, dt).
\]

Applying Itô’s formula to \( \tilde{V}^{TI}(t, S_t, \lambda_t) \), then by (64) we have

\[
d\tilde{V}^{TI}(t, S_t, \lambda_t)
\]

Since \( B_t^{-1} g(S_t) = \tilde{V}(t, t, S_t, \lambda_t) \), we obtain

\[
\tilde{V}^{TI}(t, S_t, \lambda_t) = \mathbb{E}_Q^{\mathcal{F}_t} \left[ \int_t^T B_u^{-1} g(S_u) dD_u^l \mid \mathcal{F}_t \right]
\]
for the portfolio of pure endowment contracts is linked to the survival probability of the policyholder to the maturity date. The difference is that the locally risk-minimizing hedging strategy is different between the portfolios of pure endowment contracts and term insurance contracts. The benefit of a pure endowment contract is conditional on the survival of the policyholder to the maturity date, while that for the portfolio of term insurance contracts also holds since it only relies on the dynamic price of the underlying asset.

Finally, using the same method as in Theorem 4.1, we get the pseudo-locally risk-minimizing hedging strategy

\[
\xi_t^{\Pi t} = \frac{d(\bar{V}^{T_t}, \bar{S}_t)}{d(S)_t} = \frac{(l_x - D_{t-})}{\bar{S}_{t-}(\sigma^2(t, S_t) + \lambda_t \delta^2)}
\]

\[
\times \int_t^T \left( \int_{u-t}^{u} \left( \frac{\partial V}{\partial S} \bar{S}_{t-} - \sigma^2(t, S_t) + \lambda_t \int_{u-t}^{u} \Gamma (t, u, S_{t-}, z, \lambda_{t-}) \sigma du \right) du \right)
\]

Eq. (62) is a direct consequence of the relation (A.1). Furthermore, Proposition 4 for the term insurance contracts also holds since it only relies on the dynamic price of the underlying asset.

Comparing (36) and (61), we find that the locally risk-minimizing hedging strategies are different between the portfolios of pure endowment contracts and term insurance contracts. The difference is that the locally risk-minimizing hedging strategy for the portfolio of pure endowment contracts is linked to the survival probability \(T-t|_{P_{x+u}}\) of policyholders, while that for the portfolio of term insurance contract contains an integral from \(t\) to \(T\) (with respect to \(u\)) in which the integrand relies on \(u-t|_{P_{x+u}}\), i.e., the conditional density of death at age \(x+u\) (resp. time \(u\)), given survival to age \(x+u\) (time \(t\)). Indeed, the difference is caused by the different designs of the two contracts. The benefit of a pure endowment contract is conditional on the survival of the policyholder to the maturity date \(T\). To hedge the mortality risk, the insurance company only needs to figure out the survival probability from \(t\) to \(T\) in the case of pure endowment contracts. For a term insurance contract, the death benefit is payable only if the policyholder dies within a fixed term. The insurance company has to integrate the instantaneous mortality rate (i.e., conditional death density) over the time window between \([t, T]\).

Similar to Subsection 4.1, we consider a unit-linked insurance contract with the payoff function \(g(s) = \max(s, K)\) in the following examples, in which we also calculate the time-0 residual risk \(R_0(\varphi)\). By (65) and Definition A.2, the cost process is given by

\[
C_t(\varphi) = \bar{V}^{T_t}(t, S_t, \lambda_t) - \int_0^t \xi_s^{T_t} d\bar{S}_s
\]

\[
= \bar{V}^{T_t}(0, S_0, \lambda_0) + \int_0^t \left( \int_s^T \mu_{x+u}(l_x - D_{s-})u-s p_{x+s} \frac{\partial V}{\partial S} du - \xi_s^{T_t} \right) \bar{S}_{s-} \sigma d\bar{W}_s
\]

\[
+ \int_0^t \left( B_s^{-1} g(S_s) - \int_s^T \mu_{x+u} \bar{V}(s, u, S_{s-}, \lambda_{s-}) u-s p_{x+s} du \right) d\bar{D}_s
\]

\[
+ \int_0^t \int_{s-a}^s \left( \int_s^T \mu_{x+u}(l_x - D_{s-})u-s p_{x+s} \Gamma (s, u, S_{s-}, z, \lambda_{s-}) du - \xi_s^{T_t} \bar{S}_{s-} \right) d\bar{J}_Q^r (dz, ds).
\]
Example 4.3. Let the payoff function be \( g(S_u) = \max(S_u, K) \). Then, the local risk-minimizing hedging strategy \((\xi^{TI*}, \eta^{TI*})\) is given by from (61) with \( V(t, u, S_t, \lambda_t) = \mathbb{E}_{Q} \left[ \max(S_u, K) \left| g_t \right. \right] \).

From (67), the residual risk at time 0 is given by
\[
R_0(\varphi) = \mathbb{E}_{\pi} \left[ \int_0^T \left( \int_s^T \mu_{x+u}(l_x - D_{x-})u - sp_{x+s} \frac{\partial V}{\partial S} du - \xi^{TI*}_t \right) \right. \left. \right] 
\]
\[
+ \mathbb{E}_{\pi} \left[ \int_0^T \left( B_s^{-1} g(S_s) - \int_s^T \mu_{x+u} \tilde{V}(s, u, S_s, \lambda_s)u - sp_{x+s} du \right) \right. \left. \right] 
\]
\[
+ \mathbb{E}_{\pi} \left[ \int_0^T \left( \int_s^T \mu_{x+u} (l_x - D_{x-})u - sp_{x+s} \Gamma(s, u, S_{x-}, z, \lambda_{x-}) du \right) \right. \left. \right] 
\]
\[
- \xi^{TI*}_t \tilde{S}_{x-} \right) \right) \left. \right] \frac{2}{\lambda_s \nu (dz) ds} \right].
\]  
(68)

Example 4.4. Let the payoff function be \( g(S_u) = S_u \), that is, the insured is to be paid the value of the stock at the maturity date \( T \). Since \( V(t, u, S_t, \lambda_t) = \mathbb{E}_{Q^*} \left[ e^{-\int_0^T r_s ds} S_u \left| g_t \right. \right] = S_t \), we have \( \frac{\partial V}{\partial S} = 1 \) and \( \Gamma(s, u, S_s, z, \lambda_s) = \tilde{S}_{x-} \). Then from (61), the local risk-minimizing hedging strategy \((\xi^{TI*}, \eta^{TI*})\) is given by
\[
\xi^{TI*}_t = (l_x - D_{x-}^t) \int_t^T u - tp_{x+t} \mu_{x+u} du,
\]  
(69)
and
\[
\eta^{TI*}_t = \frac{\tilde{V}^{TI}(t, S_t, \lambda_t)}{\xi^{TI*}_t \tilde{S}_t} = \frac{\tilde{V}^{TI}(t, S_t, \lambda_t) - (l_x - D_{x-}^t)}{\eta^{TI*}_t \tilde{S}_t} \int_t^T u - tp_{x+t} \mu_{x+u} du \tilde{S}_t.
\]  
(70)

If \( \sigma(t, S_t) = \sigma, r_t = r \), by (67) we have
\[
R_0(\varphi) = \mathbb{E}_{\pi} \left[ \int_0^T \left( \tilde{S}_s - \int_s^T \mu_{x+u} \tilde{S}_s u - sp_{x+s} du \right) \right. \left. \right] \left( l_x - D_{x-}^t \right) \mu_{x+s} ds \right]
\]
\[
= \int_0^T \left( \int_s^T \mu_{x+u} \times \right. \left. u - sp_{x+s} du \right) \right. \left. \right] \left. \right] \mathbb{E}_{\pi} \left[ \tilde{S}_s^2 \right] \mu_{x+s} ds \right]
\]
\[
= \int_0^T \left( \int_s^T \mu_{x+u} \times \right. \left. u - sp_{x+s} du \right) \right. \left. \right] \left. \right] \mathbb{E}_{\pi} \left[ \tilde{S}_s^2 \right] \mu_{x+s} ds \right]
\]
\[
= \tilde{S}_0 \int_0^T \left( \int_s^T \mu_{x+u} \times \right. \left. u - sp_{x+s} du \right) \right. \left. \right] \left. \right] \mathbb{E}_{\pi} \left[ \tilde{S}_s^2 \right] \mu_{x+s} ds \right]
\]
\[
= \tilde{S}_0 \int_0^T \left( \int_s^T \mu_{x+u} \times \right. \left. u - sp_{x+s} du \right) \right. \left. \right] \left. \right] \mathbb{E}_{\pi} \left[ \tilde{S}_s^2 \right] \mu_{x+s} ds \right]
\]  
(71)

As Example 4.1, the insurance company suffers from three different risks, namely the diffusion, mortality and jump risks, in Example 4.3. Thanks to the simple structure of the payoff function, only mortality risk needs to be considered in Example 4.4, which is the same as Example 4.2.

5. Numerical examples. In this section, we compare the risk-minimizing hedging strategies with the Delta hedging strategies in several numerical examples. In particular, we illustrate the clustering effects on the risk-minimizing hedging strategies. For convenience, we assume that the appreciate rate, volatility rate and interest rate
are constants, i.e., \( \mu(t, S_t) \equiv \mu, \sigma(t, S_t) \equiv \sigma, r_t \equiv r \). As mentioned before, the trading strategy is the risk-minimizing hedging strategy in Theorem 4.1 if \( \mu = r \) is assumed or \( P \) is a martingale measure. From Proposition 1, we know that under the minimal martingale measure \( Q^* \), the distribution of the jump size \( Z \) is time-varying and random. To bypass this technical difficulty, we assume that the growth rate of the stock is identical to the risk-free rate and consider the special case of the risk-minimizing hedging strategy. We adopt some model parameters from [27]. To be more specific, we assume that unless stated otherwise the model parameters take the following values:

\[
\begin{align*}
l_x &= 1, \ x = 45, \ T = 15, \ \mu = r = 0.06, \ \sigma = 0.25, \\
S_0 &= 1, \ B_0 = 1, \ \lambda_0 = 0.7, \ \alpha = 2, \ \kappa = 0.4, \ \gamma = 1.
\end{align*}
\]

Moreover, we assume \( a = 0.1 \) and the common distribution of random jump sizes \( Z_j \), for \( j = 1, 2, 3, \ldots \), is a uniform form distribution on \((0.1, 0.1)\). Then the mean and the variance of the jump size are 0 and \( \delta^2 = a^2/3 = \frac{1}{3}, \) respectively. In fact, the maximum jump of 10% in the stock price seems reasonable in reality. In practice, a trading halt may occur on a specific stock if there is a dramatic change, say 10%, in the stock price within a short time window. As in [27], we also use the Gompertz-Makeham hazard function to model the mortality law of the policyholder, i.e.,

\[
\mu_{x+t} = 0.0005 + 0.000075858 \times 1.09144^{x+t}, \quad t \geq 0.
\]

So, the probability of a person at age 45 surviving for at least 15 years is \( 15p_{45} = 0.8796. \)

In what follows, we only provide the numerical results of the optimal hedging strategy for the pure endowment given in Theorem 4.1. The optimal hedging strategy for the term insurance can be calculated similarly by following the same numerical method. A pure endowment with guarantee \( K \) can be viewed as a synthesis product of a European call option and a risk-free bond since \( g(S_T) = \max \{ S_T, K \} = (S_T - K)^+ + K \). The insured is assumed to be alive at time \( t \) in our numerical calculation. Otherwise, \( \xi_{PE}^* = 0 \) in (36). The strike price \( K = e^{rT} \) is used in Figs. 3-4. It is shown in Figure 2 how the hedging strategy evolves over time with respect to different strike prices.

Let \( n = 100 \) be the number of iterations per year and denote the mesh of this partition by \( \Delta t = 1/n = 1/100. \) We use the step size \( \Delta t = 1/100 \) in all the following calculations and divide 15 years into 1,500 parts. From (36), we find that \( \xi_{t}^{PE} \) depends on the intensity process \( \lambda \) and the stock price \( S \). Hence, we follow the algorithm proposed by [13] to simulate the Hawkes process and its intensity process and include a path of the intensity process \( \lambda \) of the Hawkes process in the left panel of Fig. 1. Then we use the values of \( \lambda_t \) and the Euler-Maruyama method to simulate a sample path of the stock price \( S \) and include 1501 values of \( S_t \) in the 15 years in the right panel of Fig. 1. With the values of \( \lambda_t \) and \( S_t \), we present hedging strategies \( \xi_{t}^{PE} \) in Fig. 2. As illustrated in the sample path of \( \lambda_t \), the occurrence of jump events increases the value of the intensity \( \lambda_t \). Furthermore, we also observe that the impact of a jump event fades away, which is consistent with the dynamics of \( \lambda_t \). The rate \( \alpha \) controls the exponential decay after a jump event. Over the course of time, the value of the intensity \( \lambda_t \) gradually approaches to the reversion level \( \kappa \).
Fig. 1. Paths of intensity process \( \lambda_t \) and stock price process \( S_t \).

Fig. 2 shows the value of \( \xi_t^{PE} \) corresponding to different strike prices \( K \) and describes the change of \( \xi_t^{PE} \) with respect to time \( t \). We consider three cases: \( K = 0.5e^{rT}, K = e^{rT} \) and \( K = 2e^{rT} \). Comparing Fig. 1 with Fig. 2, it is not surprising to see that the trend of \( \xi_t^{PE} \) roughly agrees with that of \( S_t \). For the case of \( K = 0.5e^{rT} = 1.2498 \), we notice that starting from the 5th year, the stock price \( S_t \) is significantly greater than \( K \), so the option is deep in-the-money. Accordingly, \( \xi_t^{PE} \) is often close to 1. But it is not equal to 1 because the survival probability \( T - t \) serves as a scaling factor in the expression of \( \xi_t^{PE} \). Though the deeply in-the-money option is almost like a stock and offers a payoff \( S_T \), only a portion of policyholders that survives to \( T \) receive the payoff. It is also interesting to observe that the values of \( \xi_t^{PE} \) have a downward trend in the 10th year, which results from the downward trend in \( S_t \) during the same time period (refer to Fig. 1). Furthermore, it should be noted that in the 10th year, the downward trend of \( \xi_t^{PE} \) is more marked in the case of \( K = 2e^{rT} \) than the case of \( K = 0.5e^{rT} \). One possible explanation is that the option is still in-the-money for the case of \( K = 0.5e^{rT} \), but would be out-of-the-money for \( K = 2e^{rT} \). A hedger is inclined to hold less stocks when the option is out-of-the-money. The value of \( \xi_t^{PE} \) thus is more likely to go down when the stock price decreases. On the other hand, it is in accordance with our intuition that as the strike price \( K \) increases, the risk-minimizing strategy \( \xi_t^{PE} \) actually decreases. In addition, it is worthwhile to mention that when the contract is about to mature at time \( t = 15 \), \( \xi_t^{PE} \) suddenly drops to 0 rather than moves upward to 1 when \( K = e^{rT} \) and \( K = 2e^{rT} \). The driving force behind this phenomenon is that the option is out-of-the-money in these two cases since \( S_t = 2.2948 < K = e^{rT} = 2.4596 \) at time \( t = 15 \).

Fig. 3 shows how the risk-minimizing hedging strategy \( \xi_0^{PE} \) and the Delta hedging strategy \( \Delta_0^{PE} \) adjust in response to the change in the initial stock price when the jump component in the stock price follows the Hawkes process. We assume the jump size \( Z_j \in U(-0.1, 0.1) \) and \( Z_j \in U(-0.5, 0.5) \) in the left panel and the right panel of Fig. 3, respectively. It can be seen from Fig. 3 that \( \xi_0^{PE} \) is less than \( \Delta_0^{PE} \). This implies that the risk-minimizing hedging is more risk-adverse than the Delta hedging. This result is consistent with [4]. Furthermore, we can see that both \( \xi_0^{PE} \) and \( \Delta_0^{PE} \) increase as \( S_0 \) increases. Indeed, as the initial stock price increases,
the insurance company is obligated to hold more shares of stocks to prepare for the future benefit payment that will be realized with higher possibility and larger amount. Moreover, by comparing the two panels, we observe that the magnitude of the Hawkes jumps has a larger impact on the risk-minimizing hedging strategy than on the Delta hedging strategy. This again verifies that the risk minimization is more risk-averse than the delta hedge. Therefore, the insurance company modifies its risk-minimizing hedging strategy more quickly if the market surges and/or plunges more widely. This may be an advantage of the risk-minimizing hedging strategy over the Delta hedging strategy.

Figure 2. Number of stock $\xi_{PE}^t$ with different strike prices $K$.

Figure 3. Effects of the jump size $Z$ on $\xi_{PE}^0$ and $\Delta_0$. We assume the jump size $Z_j \in U(-0.1, 0.1)$ and $Z_j \in U(-0.5, 0.5)$ in the left panel and the right panel of Figure 3, respectively.
Fig. 4 demonstrates the influence of different types of intensities \( \lambda \) on the locally risk-minimizing hedging strategies. The intensity of the Poisson process 0.4 is adopted in the Fig. 4. In this figure, we compare the risk-minimizing hedging strategies under a Hawkes jump diffusion process with the corresponding strategies under the classical Merton jump diffusion model with no clustering effects, i.e., \( N_t \) is a Poisson process. The intensity of the Poisson process is assumed to be 0.4 which is consistent with the mean-reversion level \( \kappa \) in the Hawkes process. The increase of the jump intensity stimulates the occurrence of sudden jumps in stock price, and vice versa. This is the so-called clustering effect. The consequence is that jumps are more frequent due to the clustering effect in the stock price model with the Hawkes jumps. The more frequently the stock jumps brings the huge price risk, the insurance company tends to hold fewer shares so as not to cause too much loss. This implies that the risk-minimizing hedging strategies in the Hawkes jump-diffusion process are less aggressive than those in the Poisson jump-diffusion process. The differences of the risk-minimizing hedging strategies between these two models are caused by the jump clustering effects.

6. Conclusion. We study optimal hedging strategy strategies for some unit-linked life insurance products when the risky dynamics follows a Hawkes jump diffusion process. This process captures many important characteristics of stock price. Particularly, the jumps in Hawkes model tend to be clustered and systematic. This paper provides the explicit representations of local risk-minimization hedging strategy for the portfolios of unit-linked pure endowment and term insurance contracts. Under certain appropriate assumptions, we show the existence of the minimal martingale measure. Moreover, our numerical examples demonstrate the impact of key parameters on the hedging strategies. Besides, we compare the risk-minimizing hedging strategy with the Delta hedging strategy in various settings and the comparison reveal the advantage of using the risk-minimizing heading strategy for the insurance company.

Appendix. In this appendix, we provide a review of local risk-minimization. In an incomplete financial market, the contingent claims cannot be perfectly duplicated by
self-financing strategies. Hence, some approaches of obtaining an optimal hedging strategy will have to be chosen. Risk-minimization is introduced in [15] where the discounted risky asset price process is a martingale under the original measure. This theory is then extended to the case of a semi-martingale, that is, local risk-minimization, by [33]. In this section, we recall some basis concepts and knowledge of local risk-minimization. Readers can refer to [34] for further details. We start with a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)\). The discounted risky asset \(\tilde{S}\) is a semi-martingale under the measure \(P\) and has the following decomposition

\[
\tilde{S} = \tilde{S}_0 + M + A,
\]

where \(M\) is a square-integrable martingale with \(M_0 = 0\) and \(A\) is a predictable process with finite variation. A trading strategy is of the form \(\varphi = (\xi, \eta)\), where \(\xi := \{\xi_t\}_{0 \leq t \leq T}\) denotes the number of risky assets held in the portfolio and \(\eta := \{\eta_t\}_{0 \leq t \leq T}\) represents the number of risk-free bonds held in the portfolio. Then the discounted value of the portfolio at time \(t\) is given by

\[
\tilde{V}(t) = \xi_t \tilde{S}_t + \eta_t. \tag{A.1}
\]

**Definition A.1.** A pair \(\varphi = (\xi, \eta)\) is called an admissible strategy if

(i) \(\xi\) is a \(\mathcal{F}\)-predictable process and \(\eta\) is a \(\mathcal{F}\)-adapted process;

(ii) \(\xi\) satisfies the following condition

\[
\mathbb{E}_P\left[\int_0^T \xi_u^2 \, d\langle M \rangle_u + \left(\int_0^T |\xi_u \, dA_u|\right)^2\right] < \infty, \tag{A.2}
\]

where \(\mathbb{E}_P[\cdot]\) is the expectation taken under \(P\);

(iii) \(\tilde{V}(t) = \xi_t \tilde{S}_t + \eta_t\) has right-continuous paths and \(\mathbb{E}_P[\tilde{V}(t)^2] < \infty\) for any \(t \in [0,T]\).

We let \(\Theta_S\) denote the space of all processes \(\xi\) satisfying condition (A.2). For the hedging problem in an incomplete market, the mean-variance hedging and the locally risk-minimizing hedging are two major quadratic approaches. Different from being self-financing in mean-variance optimal hedging strategies, the locally risk-minimizing hedging strategy is mean-self-financing. We recall self-financing strategy and mean-self-financing strategy first and introduce a partition of a finite time horizon and residual risk after that.

**Definition A.2.** The cost process of an admissible strategy \(\varphi = (\xi, \eta)\) is

\[
C_t(\varphi) := \tilde{V}(t) - \int_0^t \xi_u d\tilde{S}_u, \tag{A.3}
\]

where \(\varphi\) is said to be self-financing if \(C_t(\varphi)\) remains constant when \(t\) varies over \([0, T]\), and be mean-self-financing if \(\{C_t(\varphi)\}_{t \in [0,T]}\) is a \(P\)-martingale.

**Definition A.3.** A partition of \([0, T]\) is a finite set \(\tau = \{t_0, t_1, ..., t_k\}\) of times with \(0 = t_0 < t_1 < t_2 < ... < t_k = T\) and the mesh size of \(\tau\) is defined as

\[
|\tau| := \max_{t_i, t_{i+1} \in \tau} (t_{i+1} - t_i), \text{ where } k \text{ can be } \tau \text{ dependent.}
\]

A sequence \(\tau_n\) of partitions is increasing if \(\tau_n \subseteq \tau_{n+1}\) for all \(n\) and is approaching to the identity if \(\lim_{n \to \infty} |\tau_n| = 0\).

**Definition A.4.** For any time \(t \in [0,T]\), the residual risk is defined by

\[
R_t(\varphi) := \mathbb{E}_P[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{F}_t], \tag{A.4}
\]
over all admissible strategies.

**Definition A.5.** A small perturbation is an $L^2$-strategy $\Delta = (\varpi, \varepsilon)$ such that $\varpi$ is bounded, the variation of $\int \varpi dA$ is bounded (uniformly in $t$ and $\omega$) and $\varpi_T = \varepsilon_T = 0$. For any subinterval $(s, t]$ of $[0, T]$, we then define the small perturbation $\Delta|_{(s, t]} := (\varpi I_{(s, t]}, \varepsilon I_{(s, t]}).$

We now can introduce the concept of locally risk-minimizing hedging strategy in the following definition.

**Definition A.6.** For an $L^2$-strategy $\varphi$, a small perturbation $\Delta$ and a partition $\tau$ of $[0, T]$, we let

$$r^\tau(\varphi, \Delta) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i} (\varphi + \Delta|_{[t_i, t_{i+1}]}) - R_{t_i} (\varphi)}{\mathbb{E}_P \left[ \langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} | \mathcal{F}_{t_i} \right]} I_{(t_i, t_{i+1})},$$

(A.5)

where $\langle M \rangle$ is the predictable quadratic variation of $M$. The strategy $\varphi$ is called locally risk-minimizing hedging strategy if $\liminf_{n \to \infty} r^\tau_n(\varphi, \Delta) \geq 0$, $P$-a.e. on $\Omega \times [0, T]$ for every small perturbation $\Delta$ and every increasing sequence $(\tau_n)_n \in \mathbb{N}$ of partitions tending to the identity.

Note that the formal definition of the locally risk-minimizing hedging strategy is quite complicated. Compared with the locally risk-minimizing hedging strategy, the pseudo-locally risk-minimizing hedging strategy is easier to find. [34] presents the definition of the pseudo-locally risk-minimizing hedging strategy, and it is stated as below.

**Definition A.7.** Let $H \in L^2(\mathcal{F}_T, P)$ be the payoff of a contingent claim. An admissible strategy $\varphi = (\xi, \eta)$ such that $\tilde{V}(T) = \xi_T \tilde{S}_T + \eta_T = H$ is pseudo-locally risk-minimizing for $H$ if $\varphi$ is mean-self-financing and the martingale $\{C_t(\varphi)\}_{t \in [0, T]}$ is strongly orthogonal to $M$ under the measure $P$.

The next proposition provides an important connection between the pseudo-locally risk-minimizing hedging strategy and the locally risk-minimizing hedging strategy. It can also be found in Proposition 1 of [37] and Theorem 3.3 of [34]. We thus just present it here without proof.

**Proposition A.1.** Let $\varphi = (\xi, \eta)$ be a pseudo-locally risk-minimizing hedging strategy. If the decomposition of $\tilde{S} = M + A$ satisfies

(i) $\langle M \rangle$ is strictly increasing $P$-a.s. on the whole interval $[0, T]$;

(ii) $A$ is continuous with respect to time $t$;

(iii) $A$ is absolutely continuous with respect to $\langle M \rangle$ with a density $\vartheta$ satisfying $\mathbb{E}_P \left[ \int \vartheta dM \right] < \infty$,

then the pseudo-locally risk-minimizing hedging strategy $\varphi = (\xi, \eta)$ is the locally risk-minimizing hedging strategy.

In fact, pseudo-locally risk-minimizing hedging strategy can be obtained from by the Föllmer-Schweizer decomposition. The following result was provided in [34].

**Proposition A.2.** (Föllmer-Schweizer decomposition) A contingent claim $H \in L^2(\mathcal{F}_T, P)$ admits a pseudo-locally risk-minimizing hedging strategy $\varphi$ with $\tilde{V}(T) = H$, $P$-a.s. if and only if $H$ can be written as

$$H = H_0 + \int_0^T \xi_u d\tilde{S}_u + L_T, \ P \text{-a.s.}$$

(A.6)
with \( H_0 \in L^2(\mathcal{F}_0, P), \xi_u \in \Theta_S \) and \( L \) is a square-integrable \( P \)-Martingale orthogonal to \( M \). The cost process is \( C_t(\varphi) = H_0 + L_t \). The pseudo-locally risk-minimizing hedging strategy \( \varphi \) is given by

\[
\varphi_t = \left( \xi_u, \int_0^t \xi_u d\tilde{S}_u + L_t - \xi_t \tilde{S}_t \right), 0 \leq t \leq T; \tag{A.7}
\]

and its value process is

\[
\tilde{V}(t) = H_0 + \int_0^t \xi_u d\tilde{S}_u + L_t, 0 \leq t \leq T. \tag{A.8}
\]

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Received July 2020; 1st revision January 2021; 2nd revision February 2021.

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