Computing the unknotting numbers of certain pretzel knots

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Abstract We compute the unknotting number of two infinite families of pretzel knots, \( P(3, 1, \ldots, 1, b) \) (with \( b \) positive and odd and an odd number of 1s) and \( P(3, 3, 3c) \) (with \( c \) positive and odd). To do this, we extend a technique of Owens using Donaldson’s diagonalization theorem, and one of Traczyk using the Jones polynomial, building on work of Lickorish and Millett.

1 Introduction

The unknotting number \( u(K) \) of a knot \( K \) is the minimal number of crossing changes (whereby one strand of the knot is passed through another) required to transform \( K \) into the unknot. Any diagram of \( K \) can be used to compute such an unknotting sequence for \( K \), and thereby place an upper bound on \( u(K) \). Calculating \( u(K) \) exactly, or even computing a lower bound, is in general a hard problem.

The pretzel link \( P(a_1, \ldots, a_n) \) with \( a_i \in \mathbb{Z} \setminus 0 \) for all \( i \) is the link shown in Figure 1 with \( a_i < 0 \) representing crossings in the opposite direction to that shown. Observe that \( P(a_1, \ldots, a_n) \) is a knot when \( n \) and all of the \( a_i \) are odd, and also when exactly one of the \( a_i \) is even; every pretzel knot is of one of these two types.

We will make use of the knot signature \( \sigma(K) \), originally defined in terms of a Seifert surface (an orientable surface embedded in \( S^3 \) bounded by \( K \); see e.g. [7]) with \( \sigma(\text{unknot}) = 0 \). It is well known (see e.g. [1]) that if \( K_- \) is obtained from \( K_+ \) by changing a positive crossing then \( \sigma(K_-) - \sigma(K_+) \in \{0, 2\} \), so that, for any knot \( K \), \( u(K) \geq \frac{1}{2} |\sigma(K)| \). The method of Gordon and
Figure 1: The pretzel link $P(a_1, \ldots, a_n)$.

Litherland [4] shows easily that $\sigma(P(a_1, \ldots, a_n)) = n - 1$ whenever the $a_i$ are all positive and odd.

That $u(P(3, 1, 3)) = 2$ was established by Lickorish [6]. Owens [10] later showed that $P(3, 1, 3)$ could not be unknotted by changing one negative and any number of positive crossings, and in a separate paper [9] also showed that $u(P(3, 1, 1, 1, 3)) = 3$.

Traczyk [13] used the Jones polynomial to show that $P(3, 3, 3)$ could not be unknotted by changing one positive and one negative crossing. Owens [9] used this work and an obstruction from Heegaard Floer theory to show that $u(P(3, 3, 3)) = 3$.

We extend the techniques of Owens and Traczyk to establish the following two theorems.

**Theorem 1.** For $K = P(3, 1, \ldots, 1, b)$ with an odd number $r$ of $1$s and $b$ positive and odd, $u(K) = \frac{1}{2}(r + 3)$. More generally, $K = P(a, 1, \ldots, 1, b)$, with an odd number $r$ of $1$s and $a$ and $b$ positive and odd, cannot be unknotted by changing $\frac{1}{2}\sigma(K) = \frac{1}{2}(r + 1)$ negative and any number of positive crossings.

**Theorem 2.** We have $u(P(3, 3, 3c)) = 3$ for $c$ positive and odd. In general, for $K = P(3a, 3b, 3c)$ with $a$, $b$ and $c$ positive and odd, the unknotting number $u(K) \geq 3$.

Both of these results are special cases of the following conjecture of Jablan and Sazdanović [5].

**Conjecture 1.1.** For $n$ odd and $a_1, \ldots, a_n$ positive and odd, and with $a_1 \leq a_2 \leq \cdots \leq a_n$,

$$u(P(a_1, \ldots, a_n)) = \frac{\sum_{i=1}^{n-1} a_i}{2}.$$
2 Unknotting rational pretzel knots

In this section we consider pretzel knots of the form \( P(a,1,\ldots,1,b) \), with \( a \) and \( b \) odd and at least 3, and an odd number \( r \) of 1s; knots of this form are also part of the family known as rational or two-bridge knots. The technique in this section was established for another family of rational knots, including \( P(3,1,3) \), by Owens [10], who also established the case of \( P(3,1,1,1,3) \) by other methods in [9].

We will require the following definition.

**Definition 2.1.** A bilinear form \( q \) on some free abelian group \( M \) of rank \( 2m \) is of half-integer surgery type if it has a matrix representation
\[
Q = \begin{pmatrix}
2I & I \\
I & *
\end{pmatrix}
\]
over some basis \( \{x_1,\ldots,x_m,y_1,\ldots,y_m\} \). [10, Definition 1]

We also make use of the following corollary of [9, Theorem 3] (see also [10, Theorem 2], [11, Theorem 1]).

**Proposition 2.2.** Let \( K \) be a knot with signature \( \sigma \), and suppose that \( K \) can be unknotted by changing \( p \) positive and \( n \) negative crossings, with \( n = \frac{1}{2}\sigma \). Then the branched double cover \( Y \) of \( S^3 \) over \( K \) bounds some smooth, oriented, positive-definite 4-manifold \( X \), with intersection form of half-integer surgery type.

We will also make use of the definition of the intersection form \( q_X \).

Recall that a lattice over the integers is a free abelian group \( L \) equipped with a non-singular bilinear form \( \cdot : L \otimes L \to \mathbb{Z} \), and that given a sub-lattice \( M \) of \( L \), the orthogonal complement of \( M \) is the sublattice \( M^\perp = \{l \in L : l \cdot m = 0, \forall m \in M\} \). Let \( \mathbb{Z}^m \) denote the free abelian group on generators \( e_1,\ldots,e_m \), with the bilinear form defined by \( e_i \cdot e_j = \delta_{ij} \) (the Kronecker delta). For convenience we introduce the notation \( \Lambda_X \) for the lattice \( (H_2(X)/\operatorname{Tor} H_2(X), q_X) \).

**Theorem 1** (restated). For \( K = P(3,1,\ldots,1,b) \) with an odd number \( r \) of 1s and a positive and odd, \( u(K) = \frac{1}{2}(r + 3) \). More generally, \( K = \)
$P(a, 1, \ldots, 1, b)$, with an odd number $r$ of $1$s and $a$ and $b$ positive and odd, cannot be unknotted by changing $\frac{r}{2}\sigma(K) = \frac{1}{2}(r + 1)$ negative and any number of positive crossings.

Proof. Let $K = P(a, 1, \ldots, 1, b)$, with $a$, $b$ and $r$ as above, and let $Y$ be the twofold branched cover of $S^3$ over $K$. A well-known result [4] states that $Y$ is obtained as the boundary of a smooth 4-manifold $Z$ with intersection form $q_Z$ equal to the Goeritz form $g_K$, having matrix representation

$$Q_Z = \begin{pmatrix} A & 0 & \alpha \\ 0 & B & \beta \\ \alpha^\tau & \beta^\tau & -r - 2 \end{pmatrix},$$

where $A$ and $B$ are respectively $(a - 1) \times (a - 1)$ and $(b - 1) \times (b - 1)$ matrices of the form

$$\begin{pmatrix} -2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

and $\alpha$ and $\beta$ are column vectors of the form $(-1, 0, \ldots, 0)$. From Sylvester’s criterion, we see that $Z$ is negative-definite.

We have two 4-manifolds, $X$ and $Z$, with diffeomorphic boundaries. Consider the manifold $-Z$, with boundary $-Y$ and intersection form $q_{-Z} = -q_Z$. We can join $X$ and $-Z$ along their boundaries such that their orientations are preserved. We denote this manifold $W$. Note that in general $W$ is not unique: we have to make a choice of diffeomorphism of the boundaries; however, it doesn’t matter which one we choose for the purposes of this argument.

We have established that $W$ is a closed, smooth 4-manifold. Consider the Mayer–Vietoris sequence

$$H_2(Y) \to H_2(X) \oplus H_2(-Z) \overset{\phi}{\to} H_2(W) \to H_1(Y).$$

First, note that since the map $\phi$ is induced by the inclusion maps of $X$ and $-Z$ into $W$, it preserves intersection forms. Since $K$ is a knot, $H_2(Y)$ is trivial, so $\phi$ is a monomorphism. The cokernel $\text{coker } \phi \subseteq H_1(Y)$ is finite since $K$ is a knot [4]. We conclude from this that $W$ is positive-definite.

Donaldson’s diagonalization theorem [2] tells us that there exists a basis such that the intersection form $q_W$ has the $m \times m$ matrix representation $I_m = \text{diag}(1, \ldots, 1)$. In terms of lattices, this means that we can embed $\Lambda_{-Z}$ in $\mathbb{Z}^m \cong \Lambda_W$. Let $\mathbb{Z}^m$ be generated by $e_1, \ldots, e_m$ as above, and let $\Lambda_{-Z}$ have a basis

$$\{\xi_1, \ldots, \xi_{a-1}, \eta_1, \ldots, \eta_{b-1}, \zeta\},$$
over which $-q_Z$ has the matrix representation shown above with reversed signs. Up to changes of sign and permutations of the $e_i$, such an embedding must have the form

$$\xi_i \mapsto e_i + e_{i+1}$$

$$\eta_j \mapsto e_{a+j} + e_{a+j+1},$$

but $\zeta$ does not embed uniquely.

We know that $\Lambda_X$ must embed as a finite-index sublattice into $\Lambda_Z$. Since $\text{rk} \Lambda_Z = a + b - 1$, it follows that $\text{rk} \Lambda_Z = m - a - b + 1$. A finite-index sublattice of half-integer surgery type must therefore have $\frac{1}{2}(m - a - b + 1)$ generators $x_i$ with $x_i \cdot x_j = 2\delta_{ij}$. Any element of $\Lambda_Z$ whose expression involves a non-zero multiple of $e_i$, with $1 \leq i \leq a$, must contain some multiple of $e_1 - e_2 + e_3 - \cdots + e_a$ by the definition of the orthogonal complement. Similarly, if $e_{a+j}$, with $1 \leq j \leq b$, is involved in the expression we have to include $e_{a+1} - e_{a+2} + \cdots + e_{a+b}$. Therefore these elements with square 2 must come out of the sublattice of $\Lambda_W$ spanned by $e_{a+b+1}, \ldots, e_m$, of which there are $m - a - b - 1$; for brevity we write $g_i = e_{a+b+i}$. First let $x_1 = g_1 + g_2$. We can’t let $x_2 = g_1 - g_2$ because in that case $x_1 \cdot y_1 \equiv x_2 \cdot y_1$ modulo 2, and we need $x_1 \cdot y_1 = 1$ and $x_1 \cdot y_2 = 0$. Therefore we have to set $x_2 = g_3 + g_4, x_3 = g_5 + g_6$ and so on. Therefore the greatest number of $x_i$ we can embed in $\Lambda_Z$ is $\frac{1}{2}(m - a - b - 1) = \frac{1}{2}(m - a - b + 1) - 1$. The second part of the result follows, and since for $P(3,1,\ldots,1,b)$ we have an explicit unknotting sequence of $\frac{1}{2}(r+3)$ (negative) crossing changes (see Figure 2) we also obtain the first part.

Remark 2.3. If [11, Theorem 1] is used in place of Proposition 2.2, then we obtain versions of Theorem 1 with the 4-ball crossing number $c^*(K)$, the concordance unknotting number $u_c(K)$ or the slicing number $u_s(K)$ (see [11] for definitions of all of these) in place of the unknotting number.

3 Unknotting more pretzels

Recall that the Jones polynomial $V(L)$ is an oriented link invariant which takes values in the ring $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ of Laurent polynomials in a single indeterminate $q^{\frac{1}{2}}$ with integer coefficients. The Jones polynomial is defined by $V(O) = 1$, where $O$ denotes the unknot, and the skein relation

$$(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V(L_0) = q^{-1}V(L_+) - qV(L_-)$$
Figure 2: An unknotting sequence for $P(3, 1, \ldots, 1, b)$.

Figure 3: A skein triple.
for a skein triple of links differing only inside a 3-ball as shown in Figure 3.
A thorough treatment of the Jones polynomial, including a proof that this
relation is indeed well-defined and sufficient to compute the Jones polynomial
of any oriented link, may be found in Chapter 3 of [7].

Using the Jones polynomial in addition to the above results obtained
using Donaldson’s diagonalization theorem, it is possible to compute a lower
bound on the unknotting number of \( P(3a, 3b, 3c) \), giving an explicit value for
\( P(3, 3, 3c) \). We will require the following result of Lickorish and Millett.

**Proposition 3.1.** For any \( r \)-component link \( L \), \( V(L; \omega) = (-1)^{s+i-1}(i\sqrt{3})^d \),
where \( d \) is the nullity of the modulo-3 reduction of the symmetrized Seifert
form of \( L \) and \( \omega = e^{i\pi/3} \). [8, Theorem 3]

**Lemma 3.2.** Let \( K = P(3a, 3b) \), where \( a \) and \( b \) are positive and odd. Then
\( V(K; \omega) = -\sqrt{3} \).

**Proof.** The symmetrized Seifert form corresponding to the pretzel link \( P(3a, b) \)
has matrix representation \( \hat{S} = (3a+b) \). Denote its modulo-3 version \( \hat{S}_3 = (\beta) \).
This has nullity 1 if \( \beta = 0 \), that is if \( 3|b \), and nullity 0 otherwise, and we
apply Proposition 3.1.

Define a skein triple \((L_+, L_-, L_0)\) by \( L_+ = P(3a, 3b - 2) \), \( L_- = P(3a, 3b) \)
and \( L_0 = O \). Noting that \( L_\pm \) have two components, we have

\[
V(L_+; \omega) = (-1)^{s+i} \\
V(L_-; \omega) = -(-1)^{s-1} \sqrt{3} \\
V(L_0; \omega) = 1.
\]

We can substitute \( q = \omega \) into the Jones skein relation, although care must be
taken, as \( \omega^{1/2} \) has two possible values. Here we take \( \omega^{1/2} = e^{i\pi/6} \), but we could
equally take \( \omega^{1/2} = e^{7i\pi/6} \); the argument in the latter case is entirely parallel
to the one given here. In any case, we have

\[
1 = \frac{\omega^{1/2} - \omega^{-1/2}}{i} = \pm \omega^{-1} - \omega(-1)^{s-1}i\sqrt{3} \\
\frac{\pm \omega^{-1} - 1}{i\omega\sqrt{3}} = (-1)^{s-1}.
\]

Take the \( \pm \) sign first to be positive. This yields

\[
(-1)^{s-1} = \frac{\omega}{i\omega\sqrt{3}} = \frac{i}{\sqrt{3}}
\]

which obviously is not satisfied for any \( s_- \).
Now let the ± be negative. Here,

\[ (-1)^s = \frac{-\omega^{-\frac{1}{2}}\sqrt{3}}{i\omega\sqrt{3}} = i\omega^{-\frac{1}{2}} = 1. \]

We conclude that \( s_\equiv 0 \) modulo 2, so that \( V(P(3a, 3b); \omega) = -\sqrt{3} \) as required.

**Lemma 3.3.** With \( K = P(3a, 3b, 3c) \) with \( a, b \) and \( c \) positive and odd, \( V(K; \omega) = 3 \).

**Proof.** The symmetrized Seifert form corresponding to \( P(3a, 3b, c) \) has matrix representation

\[ \hat{S} = \begin{pmatrix} 3(a + b) & -3b \\ -3b & 3b + c \end{pmatrix} \]

with modulo-3 version

\[ \hat{S}_3 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}. \]

This has nullity 2 if \( \gamma = 0 \), that is if \( 3|c \), and nullity 1 otherwise.

Define a skein triple by \( L_+ = P(3a, 3b, 3c - 2) \), \( L_- = P(3a, 3b, 3c) \), \( L_0 = P(3a, 3b) \). We have

\[ V(L_+; \omega) = (-1)^s \cdot i\sqrt{3} \]
\[ V(L_-; \omega) = -3(-1)^s \]
\[ V(L_0; \omega) = -\sqrt{3}. \]

The skein relation gives

\[ -i\sqrt{3} = \pm i\omega^{-1}\sqrt{3} + 3\omega(-1)^s. \]

\[ -i\sqrt{3}(1 \pm \omega^{-1}) \frac{1}{3\omega} = (-1)^s. \]

Take the ± sign to be positive. Thus

\[ (-1)^s = \frac{3i\omega^{-\frac{3}{2}}}{3\omega} = -i\omega^{-\frac{3}{2}} = -1. \]

Taking the ± sign to be negative,

\[ (-1)^s = -\frac{\sqrt{3}i\omega}{3\omega} = -\frac{i}{\sqrt{3}}, \]

which again has no solutions.

Therefore \( s_\equiv 1 \) modulo 2, so \( V(P(3a, 3b, 3c); \omega) = 3 \) as required. ∎
We now use the following result of Traczyk.

**Proposition 3.4.** Let $K$ be a knot with $V(K; \omega) = (-1)^s(i\sqrt{3})^d$ that can be transformed into the unknot by changing $n$ negative and $p$ positive crossings, such that $n + p = d$. Then $p \equiv s$ modulo 2. [13, Theorem 3.1]

**Theorem 2** (restated). We have $u(P(3, 3, 3c)) = 3$ for $c$ positive and odd. In general, for $K = P(3a, 3b, 3c)$ with $a$, $b$ and $c$ positive and odd, the unknotting number $u(K) \geq 3$.

**Proof.** Since $P(3, 1, 3)$ can be transformed into $P(3a, 3b, 3c)$ by changing $3(a + b + c) - 7$ positive and no negative crossings (to see this, it is more instructive to change negative crossings in the standard diagram of $P(3a, 3b, 3c)$ in order to reach $P(3, 1, 3)$), an unknotting sequence for $P(3a, 3b, 3c)$ with $n \leq 1$ would induce an unknotting sequence for $P(3, 1, 3)$ with the same value of $n$, which is impossible by Theorem 1. If $u = 1$ then obviously $n \leq 1$. This rules out the case $u = 1$.

Assume that $u = 2$. By Lemma 3.3, $s \equiv 1$ modulo 2. Since $0 \leq p \leq 2$, Proposition 3.4 tells us that $p \geq 1$, so that $n \leq 1$. As in the previous paragraph, Theorem 1 rules out an unknotting sequence for $P(3a, 3b, 3c)$ with $n \leq 1$.

In the case where $a = b = 1$, we have an explicit unknotting sequence of three crossing changes (Figure 4), so $u = 3$. $lacksquare$

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Figure 4: An unknotting sequence for $P(3, 3, 3c)$.

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