Random skew plane partitions and the Pearcey process

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Abstract
We study random skew 3D partitions weighted by $q^{\text{vol}}$ and, specifically, the $q \to 1$ asymptotics of local correlations near various points of the limit shape. We obtain sine-kernel asymptotics for correlations in the bulk of the disordered region, Airy kernel asymptotics near a general point of the frozen boundary, and a Pearcey kernel asymptotics near a cusp of the frozen boundary.

1 Introduction
A plane partition $\pi = (\pi_{ij})$ is an array of nonnegative numbers indexed by $(i,j) \in \mathbb{N}^2$ that is monotone, that is,

$$\pi_{ij} \geq \pi_{i+r,j+s}, \quad r,s \geq 0$$

and finite in the sense that $\pi_{ij} = 0$ when $i + j \gg 0$. Plane partitions have a obvious generalization which we call skew plane partitions. A skew plane partition is again a monotone array $(\pi_{ij})$ which is now indexed by points $(i,j)$ of a skew shape $\lambda/\mu$, where $\mu \subset \lambda$ is a pair of ordinary partitions. We call $\mu$ and $\lambda$ the inner and outer shape of $\pi$, respectively. In fact, in this paper we will only consider the case when the outer shape $\lambda$ is a $a \times b$ rectangle. Here is an example with $\mu = (1,1)$ and $a = b = 5$.

$$\begin{array}{cccccc}
7 & 4 & 2 & 1 \\
5 & 3 & 2 & 0 \\
\pi = & 6 & 3 & 1 & 1 & 0 \\
4 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
\end{array}$$

(1)

Placing $\pi_{ij}$ cubes over the $(i,j)$ square in $\lambda/\mu$ gives a three-dimensional object which we will call a skew 3D partition and denote by the same letter $\pi$. Its volume is $|\pi| = \sum \pi_{ij}$. For $\pi$ as in (1), it shown in Figure 1.

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Given a parameter $0 < q < 1$, define a probability measure on the set of all skew plane partitions with given inner and outer shapes by setting

$$\text{Prob}(\pi) \propto q^{\ell(\pi)}. \quad (2)$$

The corresponding random skew 3D partition model has a natural random growth interpretation, the parameter $q$ being the fugacity. Also, a simple bijection, which should be clear from Figure 1, and is recalled below, relates this model to a random tiling problem.

We are interested in the thermodynamic limit in which $q \to 1$ and both inner and outer shapes are rescaled by $1/r$ where

$$r = -\ln q \to +0.$$  

The results of [6] imply the following form of the law of large numbers: scaled by $r$ in all directions, the surface of our random skew 3D partition converges to a nonrandom surface — the limit shape. This limit shape will be easy to see in the exact formulas discussed below. A simulation showing the formation of the limit shape is presented in Figure 2.

An important qualitative feature of limit shape is the presence of both ordered and disordered regions, separated by the frozen boundary. Furthermore, the frozen boundary has various special points, namely, it has cusps (there is one forming in Figure 2) it can be seen more clearly in Figures 16 and 17) and also turning points where the limit shape is not smooth. In Figure 15–17, the turning points are the points of tangency to any of the lines in the same figure.

One expects that the microscopic properties of the random surface, in particular, the correlation functions of local operators, are universal in the sense that they are determined by the macroscopic behavior of the limit shape at that point. More specifically, one expects that:
1. in the bulk of the disordered region, the correlation are given by the incomplete beta kernel \[12, 15\] with the parameters determined by slope of the limit shape (a special case of this is the discrete sine kernel);

2. at a general point of the frozen boundary, suitably scaled, the correlation are given by the extended Airy kernel \[17, 9, 19\];

3. at a cusp of the frozen boundary, correlation, suitably scaled, are given by the extended Pearcey kernel, discussed below and in \[20\].

In this paper we prove all these statements for the model at hand. The required techniques were developed in our paper \[15\], of which this one is a continuation. Namely, as will be reviewed below, our random skew plane partition model is a special case of Schur process. This yields an exact contour integral formulas for correlation functions. The asymptotics is then extracted by a direct albeit laborious saddle point analysis. The striking resemblance of the above list to classification of singularities is not accidental for, as we will see, these three situation correspond precisely to the saddle point being a simple, double, or triple critical point.

We will also see that the frozen boundary is essentially an algebraic curve and that it has precisely one cusp per each exterior corner of the inner shape \(\mu\).

We expect that near a turning point the correlations behave like eigenvalues of a \(k \times k\) corner of a GUE random \(N \times N\) matrix, where \(N \gg 0\) and \(k\) plays the role of time. We hope to return to this question in a future paper.
The results presented here were obtained in 2002-03 and were reported by us at several conferences. The period between then and now saw many further developments in the field. Most notably, the Pearcey process, which we found describes the behavior near a cusp of the frozen boundary arose in the random matrix context in the work of Tracy and Widom [20]. Pearcey asymptotics for equal time correlations of eigenvalues were obtained earlier by Brezin and Hikami [3, 4] and also by Aptekarev, Bleher, and Kuijlaars [2]. We enjoyed and benefited from the correspondence with C. Tracy on subject.

In [8], Ferrari and Spohn derived from the exact formulas of [15] the Airy process asymptotics in the case of unrestricted 3D partitions (the \( \mu = \emptyset, a = b = \infty \) case in our notation). In a related but technically more involved context, the Airy process asymptotics was found by K. Johansson in [10].

In [16], the partition function of the random surface model studied here was related to the topological vertex of [1] and, thus, to the Gromov-Witten theory of toric Calabi-Yau threefolds. They were many subsequent developments, some of which are reviewed in [14]. The papers [7, 18] may the closest to the material presented here. Also, much more general results on algebraicity of the frozen boundary are now available [13]. We would like to thank R. Kenyon and C. Vafa for numerous discussions.

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2 Preliminaries

2.1 Skew 3D partition as a sequence of its slices

We associate to \( \pi \) the sequence \( \{\lambda(t)\} \) of its diagonal slices, that is, the sequence of partitions

\[
\lambda(t) = (\pi_{i,t+i}), \quad i \geq \max(0,-t), \quad t \in \mathbb{Z}.
\]

Throughout this paper, we assume that the outer shape of our skew partition is an \( a \times b \) box and, in particular, we will use the letter \( \lambda \) to denote diagonal slices, not the outer shape.

Notation \( \lambda \succ \nu \) as usual means that \( \lambda \) and \( \nu \) interlace, that is,

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \ldots .
\]

It is easy to see that the sequence \( \{\lambda(t)\} \) corresponds to a skew plane partition if and only if it satisfies the following conditions:

- if the slice \( \lambda(t_0) \) is passing through an inner corner of the skew plane partition then

\[
\cdots < \lambda(t_0 - 2) < \lambda(t_0 - 1) < \lambda(t_0) \succ \lambda(t_0 + 1) \succ \lambda(t_0 + 2) \succ \ldots ;
\]

4
• if the slice $\lambda(t_0)$ is passing through an outer corner of the skew plane partition then

$$\cdots \succ \lambda(t_0 - 2) \succ \lambda(t_0 - 1) \succ \lambda(t_0) \prec \lambda(t_0 + 1) \prec \lambda(t_0 + 2) \prec \ldots .$$  \hspace{0.5cm} (5)$$

For example, the configuration $\{\lambda(t)\}$ corresponding to the partition $\{1\}$ is

$$(2) \prec (4) \prec (6,1) \succ (3,1) \prec (5,1) \prec (7,3,1) \succ (4,2) \succ (2) \succ (1).$$

We will denote the sequence of inner and outer corners of the inner shape by $\{v_i\}_{1 \leq i \leq N}$ and $\{u_i\}_{1 \leq i \leq N-1}$, respectively. We assume that they are numbered so that

$$v_1 < u_1 < v_2 < u_2 < \ldots u_{N-1} < v_N.$$

We also assume that the point $t = 0$ is chosen so that

$$\sum_{1 \leq i \leq N} v_i = \sum_{1 \leq i \leq N-1} u_i.$$ \hspace{0.5cm} (6)$$

2.2 Connection to tilings

There is a well-known mapping of 3D diagrams to tilings of the plane by rhombi. Namely, the tiles are the images of faces of the 3D diagram under the projection

$$(x, y, z) \mapsto (t, h) = (y - x, z - (x + y)/2).$$ \hspace{0.5cm} (7)$$

This mapping is a bijection between 3D diagrams and tilings with appropriate boundary conditions. The horizontal tiles of the tiling corresponding to the diagram in Figure 1 are shown in Figure 3.

It is clear that the positions of horizontal tiles uniquely determine both the tiling and the partition $\pi$. The set

$$\sigma(\pi) = \{(i - j, \pi_{ij} - (i + j - 1)/2)\} \subset \mathbb{Z} \times \frac{1}{2} \mathbb{Z}$$ \hspace{0.5cm} (8)$$

is precisely the set of the centers of the horizontal tiles. Notice that if $(h, t)$ is a center of a tile $h + t/2 + 1/2$ is always an integer.

Define

$$B(t) = \frac{1}{2} \sum_{i=1}^{N} |t - v_i| - \frac{1}{2} \sum_{i=1}^{N-1} |t - u_i| .$$

The image of the inner boundary of our skew plane partitions in the $(h, t)$-plane is the curve

$$h = -B(t),$$

see an example of this curve in Figure 4. In particular, the highest layer of horizontal rhombi for an empty plane partitions is the set of points with coordinates $h = -B(t) - 1/2$. 

5
Figure 3: Horizontal tiles of the tiling corresponding to the partition in Figure 1 in \((t,h)\)-coordinates.

Figure 4: Coordinates of corners.
2.3 Partition function and correlation functions

Generalizing (2), introduce a probability measure on skew plane partitions by

\[
\text{Prob}(\{\lambda(t)\}) \propto \prod_{t \in \mathbb{Z}} q_t^{\lambda(t)},
\]

where \(0 \leq q_t < 1\) are parameters.

We assume that \(q_t = 0, t < u_0 = -a\) or \(t > u_N = b\) and so the plane partition is confined to a \(a \times b\) outer box. The homogeneous case when all nonzero \(q_t\) are equal corresponds to (2).

For fixed inner shape \(\mu\), the partition function is defined by

\[
Z(\{q_t\}, \mu) = \sum_{\{\lambda(t)\} \in \mathbb{Z}} \prod_{t \in \mathbb{Z}} q_t^{\lambda(t)} = \sum_{\pi} \prod_{t \in \mathbb{Z}} q_t^{\pi_t}.
\]

where \(|\pi_t| = \sum_i \pi_{i,t+i}i\).

The correspondence \(\pi \mapsto \sigma(\pi)\) defined in (8) makes a random skew partition a random subset of \(\mathbb{Z} \times (\mathbb{Z} + \frac{1}{2})\), that is, a random point field on a lattice. This motivates the following

**Definition 1.** Given a subset \(U \subset \mathbb{Z} \times (\mathbb{Z} + \frac{1}{2})\), define the corresponding correlation function by

\[
\rho(U) = \text{Prob}(U \subset \sigma(\pi)) = \frac{1}{Z} \sum_{\pi, U \subset \sigma(\pi)} \prod_{t \in \mathbb{Z}} q_t^{\pi_t}.
\]

These correlation functions depend on parameters \(q_t\) and on the fixed inner shape \(\mu\) of skew plane partitions.

Consider the following “local” functions on skew plane partitions:

\[
\rho_{h,t}(\pi) = \begin{cases} 
1 & \text{if } (h, t) \in \sigma(\pi), \\
0 & \text{otherwise}.
\end{cases}
\]

If \(U = \{(h_1, t_1), \ldots, (h_n, t_n)\}\) with \(t_1 \geq \cdots \geq t_n\) and \((h_i, t_i) \neq (h_j, t_j)\), the correlation function (10) can be written as:

\[
\rho(U) = \langle \rho_{h_1,t_1} \cdots \rho_{h_n,t_n} \rangle = \frac{1}{Z} \sum_{\pi} \rho_{h_1,t_1}(\pi) \cdots \rho_{h_n,t_n}(\pi) \prod_{t \in \mathbb{Z}} q_t^{\pi_t}.
\]

3 Schur processes

3.1 General Schur processes

Schur process, introduced in [15] is a probability measure on sequences of partitions.
Parameters of the Schur process are sequences of pairs of functions \( \{\phi^\pm_t(z)\}_{t \in \mathbb{Z}} \) such that \( \phi^+_t(z) \) is analytic at \( z = 0 \) and \( \phi^-_t(z) \) is analytic at \( z = \infty \). For such pair of functions \( \phi_t(z) \pm \) consider skew Schur functions

\[
s_{\lambda/\mu}[\phi^+](\lambda, \mu) = \det(\phi^+_{\lambda_i - \mu_j - i + j}) ,
\]

and

\[
s_{\lambda/\mu}[\phi^-](\lambda, \mu) = \det(\phi^-_{\lambda_i + \mu_j + i - j}) .
\]

Some basic notions about Schur functions are recalled in the Appendix A.

Define the transition weight by the formula

\[
S_\phi(\lambda, \mu) = \sum_\nu s_{\lambda/\nu}[\phi^+] s_{\mu/\nu}[\phi^-] .
\]

**Definition 2.** The probabilities of the Schur process are given by

\[
\text{Prob}(\{\lambda(t)\}) = \frac{1}{Z} \prod_{m \in \mathbb{Z} + 1/2} S_{\phi_m}(\lambda(m - \frac{1}{2}), \lambda(m + \frac{1}{2})) ,
\]

where the transition weight \( S_\phi \) is defined above, by \( \phi_t \) we denoted the pair of functions \( \phi_t(z) \pm \) and \( Z \) is the normalizing factor (partition function)

\[
Z = \sum_{\{\lambda(t)\}} \prod_{m \in \mathbb{Z} + 1/2} S_{\phi[m]}(\lambda(m - \frac{1}{2}), \lambda(m + \frac{1}{2})) .
\]

If instead of infinite sequences \( \{\lambda(t)\} \) we have finite sequences of length \( N \) we will say that the Schur process is of length \( N \).

### 3.2 Polynomial Schur processes and height distributions on skew plane partitions

We will say that the Schur process is *polynomial* if functions \( \phi^\pm_t(z) \) are polynomials in \( z^\pm \). Let us show that the measure \( \Phi \) is closely related to a polynomial Schur process.

We will parametrize the inner shape as before by assuming that \( v_i, 1 \leq i \leq N, v_i \in \mathbb{Z} \) are positions of inner corners of the inner shape of the plane partition (see Fig. 4) and \( u_i \in (v_i, v_{i+1}) \), \( 1 \leq i \leq N - 1 \) are positions of outer corners.

**Theorem 1.** The restriction of the measure \( \Phi \) to random variables supported on subsequences \( \{\lambda(m)\}, m < v_1, \{\lambda(v_i)\}, 1 \leq i \leq N, \{\lambda(m)\}, m > v_N \) coincides with the polynomial Schur process with parameters

\[
\phi^-_m(z) = (z - x^-_m), \quad \phi^+_m(z) = 1 , \quad (12)
\]

\[
\phi^+_i(z) = \prod_{v_i < m < u_i, m \in \mathbb{Z} + \frac{1}{2}} (z - x^+_m) , \quad (13)
\]

\[
\phi^-_i(z) = \prod_{u_i < m < v_{i+1}, m \in \mathbb{Z} + \frac{1}{2}} (z - x^-_m) , \quad (14)
\]

\[
\phi^+_m(z) = (z - x^+_m), \quad \phi^-_m(z) = 1 , \quad (15)
\]
where parameters $x_i^\pm$ and $q_t$ are related as follows:

\[
\begin{align*}
\frac{x_{m+1}^+}{x_m^+} &= q_{m+1/2}, \quad v_i < m < u_i - 1, \text{ or } m > v_N \quad (16) \\
x_{u_i - 1/2}^+ x_{u_i + 1/2}^- &= q_{u_i}^{-1}, \quad (17) \\
x_{v_i - 1/2}^- x_{v_i + 1/2}^+ &= q_{v_i}, \quad (18) \\
\frac{x_m^-}{x_{m+1}^-} &= q_{m+1/2}, \quad u_i < m < v_{i+1} - 1, \text{ or } m < v_1 \quad (19)
\end{align*}
\]

**Proof.** Let us restrict the process (9) to the subsequence $\{\lambda_{v_i}\}$. It is easy to see that transition probability from $v_i+1$ to $v_i$ in such subprocess are

\[
S_i(\lambda(v_i), \lambda(v_{i+1})) = \sum_{s_{\lambda(v_i)/\lambda(u_i)}} (x_{v_i+1/2}^+, x_{v_i+3/2}^+, \ldots, x_{u_i-1/2}^-) (20)
\]

where $x_m^\pm$ are related to $q_t$ as in (16). Thus this process (9) is a polynomial Schur process with $\phi^\pm_i$ given by (12). Conversely, it is clear, that due to the identity (78) any polynomial Schur process can be extended to a probability measure (9) on sequences of interlacing partitions with parameters $\{q_t\}$ defined as in (12)(16).

**4 Fermionic representation for correlation functions**

**4.1**

For $m \in \mathbb{Z} + \frac{1}{2}$ define $\varepsilon(m) = +$ if $v_i < m < u_i$ and $1 \leq i \leq N$, and $\varepsilon(m) = -$ if $u_i < m < v_{i+1}$ and $0 \leq i \leq N - 1$. This is shown on Fig.3 Define $D^+ = \{m | \varepsilon(m) = +\}$ and $D^- = \{m | \varepsilon(m) = -\}$.

Let $x_m^\pm$ be positive numbers related to $q_t$ as in (16). Notice that for given $q_t$ the numbers $x_m^\pm$ are defined up to a transformation $x_m^\pm \to x_m^\pm a^{\pm 1}$.

**Theorem 2.** 1. The partition function for the height distribution on skew plane partitions can be represented as the matrix element of the product of vertex operator described in the Appendix B.2 as follows

\[
\begin{align*}
Z &= \left( \prod_{u_N > m > v_N} \Gamma_-(x_m^+) \cdots \prod_{u_i < m < u_{i+1}} \Gamma_+(x_m^-) \prod_{v_i < m < u_i} \Gamma_-(x_m^+) \right) \\
&\quad \prod_{u_1 > m > u_0} \Gamma_+(x_m^-) v^{(m)}_0, v^{(m)}_0 \right) = \left( \prod_{m \in \mathbb{Z} + \frac{1}{2}, u_0 < m < u_N} \Gamma_-(x_m^-(m)) v^{(m)}_0, v^{(m)}_0 \right) \quad (22)
\end{align*}
\]
and
\[ Z = \prod_{m_1 < m_2, m_1 \in D^-, m_2 \in D^+} (1 - x_{m_1}^+ x_{m_2}^-)^{-1}, \quad m_i \in \mathbb{Z} + \frac{1}{2} \]

2. Assume \( t_1 > \cdots > t_n \), then
\[
\langle \rho_{h_1,t_1} \cdots \rho_{h_n,t_n} \rangle = \frac{1}{Z} \left( \prod_{m \in \mathbb{Z} + \frac{1}{2}, u_N > m > t_1} \Gamma_{-\varepsilon(m)}(x_m^{\varepsilon(m)}) \psi_{j_1} \psi_{j_1}^\ast \cdots \right. \\
\prod_{t_i < m < t_{i-1}} \Gamma_{-\varepsilon(m)}(x_m^{\varepsilon(m)}) \psi_{j_i} \psi_{j_i}^\ast \prod_{t_{i+1} < m < t_i} \Gamma_{-\varepsilon(m)}(x_m^{\varepsilon(m)}) \\
\left. \cdots \psi_{j_n} \psi_{j_n}^\ast \prod_{t_n > m > u_0} \Gamma_{-\varepsilon(m)}(x_m^{\varepsilon(m)}) \psi_{j_0} \psi_{j_0}^\ast \right) 
\]
(23)

Here and below \( j_i = h_i - B(t_i) + 1/2 \).

3. Correlation functions (11) are determinants:
\[
\langle \rho_{h_1,t_1} \cdots \rho_{h_n,t_n} \rangle = \det(K((t_i, h_i), (t_k, h_k)))_{1 \leq i, k \leq n} 
\]
(24)

where
\[
K((t_1, h_1), (t_2, h_2)) = \\
\frac{1}{(2\pi)^2} \int_{|z| < R(t_1)} \int_{R^*(t_2) < |w|} \frac{\Phi_-(z, t_1) \Phi_+(w, t_2)}{\Phi_+(z, t_1) \Phi_-(w, t_2)} \\
\frac{\sqrt{zw} - e^{-h_1 + B(t_1) - 1/2} w B(t_2) + 1/2}{z \bar{w}} 
\]
(25)

Here \(|w| < |z|\) for \( t_1 \geq t_2 \), \(|w| > |z|\) for \( t_1 < t_2 \), \( R(t) = \min_{m \geq t} (|x_m^+|^{-1}) \) and \( R^*(t) = \max_{m < t} (|x_m^-|) \). Functions \( \Phi_\pm(z, t) \) are:
\[
\Phi_+(z, t) = \prod_{m > t, m \in D^+, m \in \mathbb{Z} + \frac{1}{2}} (1 - z x_m^+) 
\]
(26)
\[
\Phi_-(z, t) = \prod_{m < t, m \in D^-, m \in \mathbb{Z} + \frac{1}{2}} (1 - z^{-1} x_m^-) 
\]
(27)

Proof. The fact that the partition function and correlation functions for the height distribution of plane partitions the matrix element of the product of vertex operators as above follows form the formula (91) for matrix elements of products of vertex operators \( \Gamma_\pm(x) \).

Using the commutation relations (86), and the fact that \( \Gamma_-(x) v(0) = 0 \) we obtain the product formula for the partition function.

The operators \( \psi_j \psi_j^\ast \) act on the vector \( v_\lambda^{(0)} \) as follows:
\[
\psi_j \psi_j^\ast v_\lambda^{(0)} = v_\lambda^{(0)} 
\]
if $j = \lambda_i - i + 1/2$ for some $i = 1, 2, \ldots$ and

$$\psi_j \psi_j^* v_\lambda^{(0)} = 0$$

otherwise. Using this fact and the formula for the matrix elements of $\Gamma_\pm(x)$ we obtain the formula for the correlation functions of densities.

Moving operators $\Gamma_-$ to the right and $\Gamma_+$ to the left we obtain the following formula for the correlation functions

$$\langle \rho_{h_1, t_1} \cdots \rho_{h_n, t_n} \rangle = (\psi_{j_1}(t_1) \psi_{j_1}^*(t_1) \cdots \psi_{j_i}(t_i) \psi_{j_i}^*(t_i) \cdots \psi_{j_n}(t_n) \psi_{j_n}^*(t_n) v_0^{(0)} , v_0^{(0)})$$

where

$$\psi_j(t) = \prod_{m > t, m \in D^+} \Gamma_-(x_m^+) \prod_{m < t, m \in D^-} \Gamma_+(x_m^-)^{-1} \psi_j$$

and

$$\psi_j^*(t) = \prod_{m > t, m \in D^+} \Gamma_-(x_m^+) \prod_{m < t, m \in D^-} \Gamma_+(x_m^-)^{-1} \psi_j^*$$

Here the operators on the right are given by power series. Commuting formal power series gives the following identities:

$$\Gamma_+(x)^{-1} \psi_k \Gamma_+(x) = \psi_k - x \psi_{k+1}$$

$$\Gamma_-(x) \psi_k \Gamma_-(x)^{-1} = \sum_{n \geq 0} x^n \psi_{k-n}$$

$$\Gamma_+(x)^{-1} \psi_k^* \Gamma_+(x) = \sum_{n \geq 0} x^n \psi_{k-n}^*$$

$$\Gamma_-(x) \psi_k^* \Gamma_-(x)^{-1} = \psi_k^* - x \psi_{k+1}^*$$

Applying these identities to the formal Fourier transform of $\psi_j(t)$ and $\psi_j^*(t)$ we obtain:

$$\psi(z, t) = \prod_{m > t, m \in D^+} (1 - zx_m^+) \prod_{m < t, m \in D^-} (1 - z^{-1}x_m^-) \psi(z), \quad (30)$$

$$\psi^*(z, t) = \prod_{m > t, m \in D^+} (1 - zx_m^+) \prod_{m < t, m \in D^-} (1 - z^{-1}x_m^-) \psi^*(z), \quad (31)$$
where both sides are power series in \( x_m^\pm \) and are formal Laurent power series in \( z \). If \( v \in F \), then \( \psi(z)v = z^{1/2}F[z^{-1}, z] \) and \( \psi^*(z)v = z^{1/2}F[z^{-1}, z] \).

For the inverse Fourier transform of \( \psi(z,t)v \) and \( \psi(z,t) \) we obtain the following integral representations:

\[
\psi_k(t)v = \frac{1}{2\pi i} \int_{|z|<R(t)} \frac{\Phi_+(z,t)}{\Phi_-(z,t)} z^k \psi(z)v dz \quad (32)
\]

\[
\psi^*_k(t)v = \frac{1}{2\pi i} \int_{R^*(t)<|w|<1} \frac{\Phi_+(w,t)}{\Phi_-(w,t)} w^k \psi^*(w)v dw \quad (33)
\]

Here \( v \in F \), both sides are vectors in \( F[[x_m^\pm]] \) and these power series converge for sufficiently small \( x \)’s.

The contour of integration for \( z \) is chosen in such a way that none of the poles of \( \Phi_+(z,t) \) will be inside of it, this gives \( |z| < R(t) = \min_{m>\ell}(|x_m^+|^{-1}). \)

The contour of integration for \( w \) is such that none of the poles of \( \Phi_-(w,t) \) are outside the contour. This gives \( |w| > R^*(t) = \max_{m<\ell}(|x_m^-|) \).

As it follows from the Wick’s lemma \[13\] that correlation functions \[11\] are determinants of matrices of correlation functions of two Clifford operators \[28\][29].

\[
\langle \rho_{h_1,t_1} \cdots \rho_{h_n,t_n} \rangle = \det(K_{a,b})_{1 \leq a,b \leq n}
\]

where

\[
K_{a,b} = K((t_a,h_a),(t_b,h_b)) = \begin{cases} 
(\psi_j(t_a) \psi^*_j(t_b) v_0, v_0), & a \leq b, \\
-(\psi^*_j(t_b) \psi_j(t_a) v_0, v_0), & a > b.
\end{cases}
\]

Notice that \( a \leq b \) iff \( t_a \geq t_b \) and \( a > b \) iff \( t_a < t_b \). Now, substitute \[22\] and \[32\] into \( K_{a,b} \) and take into account \[32\]. This proves the formula for correlation functions.

\[ \square \]

4.2

Notice that operators \( \psi(z,t) \) and \( \psi^*(z,t) \) satisfy the difference equations:

\[
\psi(z,t+1) = (1-zx_{t+1/2}^+)\psi(z,t), \quad t \in D_+
\]

\[
\psi(z,t+1) = (1-z^{-1}x_{t+1/2}^-)\psi(z,t), \quad t \in D_-
\]

\[
\psi^*(z,t-1) = (1-zx_{t-1/2}^-)\psi^*(z,t), \quad t \in D_+
\]

\[
\psi^*(z,t-1) = (1-z^{-1}x_{t-1/2}^+)\psi(z,t), \quad t \in D_-
\]

These difference equations give the following difference equations for correlation functions:

\[
K((t_1,h_1),(t_2,h_2)) - K((t_1-1,h_1+1/2),(t_2,h_2)) + x_{t_1-1/2}^+ K((t_1-1,h_1-1/2),(t_2,h_2)) = \delta_{t_1,t_2} \delta_{h_1,h_2}, \quad t_1 \in D_+ , \quad (34)
\]
\[ K((t_1, h_1), (t_2, h_2)) - K((t_1 - 1, h_1 - 1/2), (t_2, h_2)) \
+ x_{t_1-1/2}^- K((t_1 - 1, h_1 + 1/2), (t_2, h_2)) = \delta_{t_1, t_2} \delta_{h_1, h_2}, \ t_1 \in D_- . \quad (35) \]

Using these equations and similar difference equations in \( t_2 \) one can express all correlation functions in terms of equal time correlation functions.

**Remark 1.** The equations (34) and (35) together with appropriate boundary conditions are the equations for the inverse Kasteleyn matrix for the corresponding dimer model.

### 4.3 The homogeneous restricted case

In the homogeneous restricted case \( 0 < q_l = q < 1 \) for \( u_0 \leq t \leq u_N \) and \( q_l = 0 \) otherwise.

In the homogeneous case \( x_m^\pm = a^{\pm 1} q^\pm m \). The partition function does not depend on \( a \). The functions \( \Phi_\pm(z, t) \) are:

\[
\Phi_+(z, t) = \prod_{m > t, m \in D^+} (1 - z q^m a), \quad (36)
\]
\[
\Phi_-(z, t) = \prod_{m < t, m \in D^-} (1 - z^{-1} q^{-m} a^{-1}) \quad (37)
\]

In this case we have

\[
R(t) = \begin{cases} 
  a^{-1} q^{-v_i}, & u_i < t < v_i \\
  a^{-1} q^{-t}, & v_{i-1} < t < u_i 
\end{cases}
\]

\[
R^*(t) = \begin{cases} 
  a^{-1} q^{-t}, & u_i < t < v_i \\
  a^{-1} q^{-v_{i-1}}, & v_{i-1} < t < u_i 
\end{cases}
\]

Notice, that when \( q \to 0 \) the density of horizontal tiles converges to

\[
\rho(h, t) = \frac{1}{(2\pi i)^2} \int_{|z|=1+\epsilon} \int_{|w|=1+\epsilon} \frac{1}{z - w} \frac{dz \, dw}{z^{h+B(t)+\frac{1}{2}} (z^{h-B(t)+\frac{1}{2}} + \epsilon).}
\]

This integral is 1 when \((h, t)\) is on a “floor” and is 0 when it is on the “wall”.

### 5 The thermodynamic limit

Here we will study the limit \( q \to 1 \) of the homogeneous Gauss distribution on restricted skew plane partitions when the number of corners in the inner shape of diagrams remain finite.

We assume that \( q = \exp(-r) \), \( r \to +0 \) and \( U_i = r u_i \), \( V_i = r v_i \) and \( N \) remain fixed and that \( U_0 < V_1 < U_1 < \cdots < V_N < U_N \).
5.1 Asymptotics of the partition function

It is easy to compute the free energy of the system in this limit:

\[ F = -\log Z = - \sum_{m<n, m \in D^-, n \in D^+} \log(1 - q^{n-m}) = \]

\[ -\frac{1}{r^2} \left( \sum_{1 \leq i \leq j \leq N} \int_{U_{i-1} < \mu < V_i} \int_{V_j < \nu < U_j} \log(1 - e^{\mu - \nu}) d\mu d\nu \right) + o\left(\frac{1}{r^2}\right) \]

Similarly, one can compute the asymptotic of the average volume of a 3D partition:

\[ \langle |\pi| \rangle = q \frac{\partial}{\partial q} Z = \sum_{m<n, m \in D^-, n \in D^+} \frac{n-m}{1 - q^{n-m}} = \]

\[ \frac{1}{r^3} \left( \sum_{1 \leq i \leq j \leq N} \int_{U_{i-1} < \mu < V_i} \int_{V_j < \nu < U_j} \frac{\nu - \mu}{1 - e^{\mu - \nu}} d\mu d\nu \right) + o\left(\frac{1}{r^3}\right) \]

The first formula reflects essentially two dimensional nature of the problem. The second formula implies that \( r^{-1} \) is the characteristic length of the system when \( r \to 0 \).

5.2 The function \( S(z) \)

Now let us analyze the correlation functions in the limit \( r \to +0 \). Since \( r^{-1} \) is a characteristic scale of the system in this limit we assume \( \tau = t_i r, \chi = h_i r \) remain finite. Depending on the value of \((\chi, \tau)\) we will either keep the differences \( t_i - t_j \) and \( h_i - h_j \) finite, or we will scale them as appropriate powers of \( r \).

When \( r \to +0 \) the functions in the integral defining correlation functions behave as

\[ \frac{\Phi_-(z, t)}{\Phi_+(z, t)} = e^{S(z)} F(z)(1 + O(r)) \]

where

\[ S(z) = \int_{\mu < \tau, \mu \in D_-} \log(1 - z^{-1} e^{\mu}) d\mu - \int_{\mu > \tau, \mu \in D_+} \log(1 - z e^{-\mu}) d\mu - (\chi + B(\tau)) \ln(z) \]

and \( F(z) \) can be computed explicitly.

In this limit the integral becomes

\[ K(h_1, t_1, h_2, t_2) = \frac{1}{(2\pi i)^2} \times \]

\[ \int_{C_z} \int_{C_w} e^{S(z; \chi_1, \tau_1) - S(w; \chi_2, \tau_2)} \frac{F(z; \chi_1, \tau_1)}{F(w; \chi_2, \tau_2)} \frac{\sqrt{z} \, dz}{z} \frac{\sqrt{w} \, dw}{w} (1 + o(1)) . \]
The integration contours are described in the previous section. For example, if $N = 2$ and $U_1 < \tau_1, \tau_2 < V_2$ the contours are:

$$C_z \times C_w = \begin{cases} e^{V_2} > |z|, |w| > |z|, |w| > e^\tau & \text{if } \tau_1 > \tau_2 \\ e^{V_2} > |z| > |w| > e^\tau & \text{if } \tau_1 \geq \tau_2 \end{cases}$$

The integral can be computed by the steepest descent method. In order to do this one should first analyze critical points of $S(z)$ and then deform contours of integration accordingly.

6 Critical points of $S(z)$ and the deformation of contours

The function $S(z)$ can be written as a sum of dilogarithms:

$$S(z) = -(\chi + B(\tau)) \ln(z) + \sum_{i=0}^{N} \mathbb{L}_2(ze^{-U_i}) - \sum_{i=1}^{N} \mathbb{L}_2(ze^{-V_i}) - \mathbb{L}_2(ze^{-\tau}).$$

where

$$\mathbb{L}_2(z) = \int_{0}^{z} \frac{\ln(1-x)}{x} dx$$

Critical points of $S(z)$ are zeros of

$$z \frac{\partial}{\partial z} S(z) = -(\chi + B(\tau)) + \int_{\mu > \tau, \mu \in D_+} \frac{ze^{-\mu}}{1 - ze^{-\mu}} d\mu + \int_{\mu < \tau, \mu \in D_-} \frac{z^{-1}e^\mu}{1 - ze^{-\mu}} d\mu$$

This is equivalent to the following equations.

$$-\chi - B(\tau) - \sum_{0 \leq j < i} \log \left( \frac{1 - z e^{-V_j+1}}{1 - z e^{-U_j}} \right) + \log \left( \frac{1 - ze^{-U_i}}{1 - ze^{-\tau}} \right) + \sum_{N \geq j > i} \log \left( \frac{1 - ze^{-U_j}}{1 - ze^{-V_j}} \right) = 0$$

when $V_i < \tau < U_i$ and

$$-\chi - B(\tau) - \sum_{0 \leq j < i} \log \left( \frac{1 - z^{-1}e^{V_j+1}}{1 - z^{-1}e^{U_j}} \right) - \log \left( \frac{1 - z e^{-\tau}}{1 - z e^{-U_i}} \right) + \sum_{N \geq j > i} \log \left( \frac{1 - z e^{-U_j}}{1 - z e^{-V_j}} \right) = 0$$

when $U_i < \tau < V_i+1$.

Exponentiating these equations we obtain

$$\exp(-\chi - B(\tau) - L(\tau)) \prod_{0 \leq j \leq N} (1 - ze^{-U_j}) = (1 - ze^{-\tau}) \prod_{1 \leq j \leq N} (1 - ze^{-V_j}) \quad \text{(39)}$$

where

$$L(\tau) = \begin{cases} \sum_{0 \leq j < i} (V_{j+1} - U_j) & \text{if } V_i < \tau < U_i \\ \sum_{0 \leq j < i} (V_{j+1} - U_j) + \tau - U_i & \text{if } U_i < \tau < V_{i+1} \end{cases}$$
The number $L(\tau)$ is the total length of $(-)$ intervals which are to the left of $\tau$.

It is easy to see that

$$L(t) + B(t) = t/2 - u_0.$$ 

Therefore the equation (39) can be written as

$$e^{\chi-\tau/2}z - e^{\chi+\tau/2} = f(z) \quad (40)$$

where

$$f(z) = e^{u_0} \prod_{i}^{N} (ze^{-U_i} - 1) \prod_{i}^{N} (ze^{-V_i} - 1).$$

### 6.1 The number of roots

**Theorem 3.** The equation (40) has either $N$ real solutions or $N-2$ real solutions and two complex conjugate.

**Proof.** The function $f(z)$ has simple poles at $z = v_i$ and $z = \infty$ and simple zeros at $z = u_i$. A sample graph of the function $f(z)$ is plotted in Figure 5.

![Figure 5: The graph of $f(z)$ for $\{U_0, V_1, U_1, V_2, U_2\} = \{-2, -0.5, 1.5, 2, 3\}$](image-url)
Solutions to the equation (40) are intersection points of the line $e^x - \frac{\tau}{z} - e^x + \frac{\tau}{2}$ with the graph of the function $f(z)$. Any straight line of non-infinite slope obviously intersects the graph of $f(z)$ in at least $N - 2$ points (and in at most $N$ points, since the degree of $f$ equals $N$). Hence among the roots of (40) there is at most one complex conjugate pair.

6.2 The $N = 1$ case

First, consider the equation (39). When $N = 1$ the normalization of $U$'s and $V$'s given by the equation (6) implies $V_1 = 0$. Introduce variables

$$X = e^{x + \frac{\tau}{2}}, \quad T = e^{-\tau}, \quad \alpha = e^{U_0}, \quad \beta = e^{-U_1},$$

The equation for critical points of $S(z)$ is quadratic:

$$(\beta - XT)z^2 + (X + XT - (1 + \alpha \beta))z + (\alpha - X) = 0 \quad (41)$$

The discriminant of this equation is:

$$\Delta = (X - XT - \alpha + \beta)^2 - (X + XT)2(1 - \alpha)(1 - \beta) + (1 - \alpha^2)(1 - \beta^2). \quad (42)$$

The structure of solutions depend on the value of the discriminant $\Delta$.

- When $\Delta < 0$ there are two complex conjugated critical points
- When $\Delta > 0$ these critical points are real.
- When $\Delta = 0$ the two simple critical points degenerate into one double critical point.

6.3 The $N = 2$ case

This is the smallest value of $N$ when the function $S(z)$ can have a triple critical point.

The function $S(z)$ for the case when $N = 2$, and $V_1 + V_2 < \tau < V_2$ is

$$S(z) = \int_{U_0}^{V_1} \ln(1 - z^{-1}e^\mu) d\mu + \int_{V_1 + V_2}^{\tau} \ln(1 - z^{-1}e^\mu) d\mu - \int_{V_2}^{U_2} \ln(1 - z^{-1}e^\mu) d\mu - (\chi + B(\tau)) \ln z$$

We choose branches of logarithms such that the derivative of $S(z)$ has branch cuts $[e^{U_0}, e^{V_1}]$, $[e^{V_1 + V_2}, e^\tau]$, and $[e^{V_2}, e^{U_2}]$.

The function $S(z)$ has three critical points. They are either all real or there is a complex conjugate pair of simple complex critical points. Geometrically these critical points correspond to intersection points of the line $e^{x - \tau/2}z - e^{x + \tau/2}$ with the graph of the function $f(z)$.

When the line $e^{x - \tau/2}z - e^{x + \tau/2}$ intersects the graph of the function $f(z)$ transversally, the intersection points are simple critical points of $S(z)$.

At double critical points the line $e^{x - \tau/2}z - e^{x + \tau/2}$ is tangent to $f(z)$ but it does not bisect the graph of $f(z)$ at the point where it is tangent to the graph.
At a triple critical point the line is tangent to the graph of \( f(z) \) and bisects it.

By the definition, a double critical point \( z \) of the function \( S(z) \) satisfies the two equations \( S'(z) = 0 \) and \( S''(z) = 0 \). We have

\[
\frac{z \partial}{\partial z} S(z) = -\chi - \ln(ze^{-\tau} - e^{2\tau}) + \ln f(z)
\]

\[
(z \frac{\partial}{\partial z})^2 S(z) = -\frac{ze^{-\tau}}{ze^{-\tau} - e^{2\tau}} + z \frac{f'(z)}{f(z)}
\]

This gives the system of equations for double critical points:

\[
e^{\chi - \tau/2} - e^{\chi + \tau/2} = f(z) \quad (43)
\]

\[
e^{\chi_0 - \tau_0/2} = f'(z) \quad (44)
\]

These equations define the curve in the \((\chi, \tau)\) plane. We will say the point \((\chi, \tau)\) on this curve is generic if it is not a triple critical point, i.e. if \( S'''(z) \neq 0 \) where \( z \) is the corresponding double critical point of \( S(z) \) (a solution to (43).

Denote by \( z_0 \) the triple critical point of \( S(z) \) and by \((\chi_0, \tau_0)\) the corresponding values of \( \chi \) and \( \tau \). By definition \( S'(z_0) = S''(z_0) = S'''(z_0) = 0 \), which gives one more equation in addition to (43):

\[
f''(z_0) = 0
\]

It is clear from the shape graph of \( f(z) \) that for each value of \( \tau \) there are either 2 or 4 double critical points of \( S(z) \). They satisfy the following inequalities:

1. \(-\infty < \tau < V_1\), then \( z_1 < 0 \) and \( 0 < z_2 < e^{V_1} \)
2. \( V_1 < \tau < U_1\), then \( z_1 < 0 \) and \( e^{V_1} < z_2 < e^\tau \)
3. \( U_1 < \tau < \tau_0\), then \( z_1 < 0, e^{V_1} < z_2 < z_0, z_0 < z_3 < e^{U_1} \), and \( e^\tau < z_4 < e^{V_2} \).
4. \( \tau_0 < \tau < V_2\), then \( z_1 < 0 \) and \( e^\tau < z_2 < e^{V_2} \).
5. \( V_2 < \tau < U_2\), then \( z_1 < 0 \) and \( e^{V_2} < z_2 \).

It is also clear that if \( \tau_0 > U_1 \), then \( e^{V_1} < z_0 < e^{U_1} \).

### 6.4 Deformation of integration contours

We want to deform contours of integration in (38) to position them in the way the steepest descent methods requires. If \( z_c \) is a critical point of \( S(z) \), and if it does not lie on a branch cut of the function \( S(z) \), the integration contour should be deformed to a contour which lies on the curve \( \Im(S(z)) = \Im(S(z_c)) \).
The function $S(z)$ has branch cuts along the real line. However, only $\exp\left(\frac{S(z)}{r}\right)F(z)$, which is the leading term of the asymptotic of $\frac{\Phi_-(z,t)}{\Phi_+(z,t)}z^{-\frac{h-B(t)}{2}}$ appear in the integral.

Zeros of $\Phi_+(z)$ are accumulating along $(e^{V_2}, e^{U_2})$. Therefore we can not deform $C_z$ through this segment but we can deform through any other part of the real line.

Similarly, zeros of $\Phi_-(w)$ are accumulating in the segments $(e^{U_0}, e^{V_1})$ and $(e^{U_1}, e^\tau)$. Thus, the contour $C_w$ can not be deformed through these segments but can be deformed through any other segment of the real line.

Therefore we have to deform contours $C_z$ and $C_w$ to the union of appropriate branches of curves $\Im(S(z)) = \Im(S(z_c + i0))$ and $\Im(S(z)) = \Im(S(z_c - i0))$. Figures 6, 7, and 8 show these curves for simple critical points, double critical points and the triple critical point, respectively.

Figure 6: Level curves of $\Im(S)$ when all critical points are simple

Figure 7: Level curves of $\Im(S)$ when there is a double critical point
Deformed contours of integration $C_z$ and $C_w$ are shown in Figures 13, 14, 9, 10, 11, 12 for simple, double, and triple critical points respectively.

Figure 9: Deformation of the integration contour $C_z$ for a double critical point

While deforming contours $C_z$ and $C_w$ one should keep track on the residues at the pole $z = w$. We will discuss this later.

7 Simple critical points and bulk limit

7.1 Bulk limit of correlation functions

Here we will compute the asymptotic of the integral (38) when there is a pair of complex conjugate critical points and when $\Delta h = h_1 - h_2$ and $\Delta t = t_1 - t_2$ are fixed in the limit $r \to 0$. 

20
Figure 10: Deformation of the integration contour $C_w$ for a double critical point

Figure 11: Deformation of the integration contour $C_z$ for a triple critical point
Figure 12: Deformation of the integration contour $C_w$ for a triple critical point

Figure 13: Deformation of the integration contour for a simple critical point when $t_1 \geq t_2$
Figure 14: Deformation of the integration contours for a simple critical point when \( t_1 < t_2 \)

Deform contours \( C_z \) and \( C_w \) to \( C_1 \) and \( C_2 \) respectively as it is shown on Fig. 13 for \( t_1 \geq t_2 \) and as it is shown on Fig. 14 for \( t_1 < t_2 \).

Taking into account the residue at \( z = w \) we have the identity:

\[
\int_{C_z} \int_{C_w} \cdots = \int_{C_1} \int_{C_2} \cdots + \int_{\gamma_{\pm}} \cdots
\]

(45)

Here we take + for \( \Delta t \geq 0 \) and − otherwise. If \( U_i < \tau < V_{i+1} \) then \( \gamma_+ \) is a simple curve connecting \( z_c \) and \( \bar{z}_c \) and passing through positive part of the real line with \( e^{V_{i+1}} < z \). Similarly, \( \gamma_- \) is a simple curve connecting \( z_c \) and \( \bar{z}_c \) and passing through the negative part of the real line. If \( V_i < \tau < U_i \) then \( \gamma_{\pm} \) are again simple curves connecting \( z_c \) and \( \bar{z}_c \) in such a way that \( \gamma_- \) intersects the negative part of the real line and \( \gamma_+ \) intersects the positive part of the real line at \( e^{\tau} < z \).

As \( r \to 0 \) only the second term in the right hand side of (45) will have finite limit. The first term will vanish. The computations are similar to [15]. The integrals along \( \gamma_{\pm} \) converge as \( r \to 0 \) to

\[
\kappa(\Delta h, \Delta t) = \begin{cases} 
B^{(+)}_{\gamma_+}(\Delta t, \Delta h + \varepsilon(\tau)\Delta t/2), & \Delta t \geq 0 \\
B^{(-)}_{\gamma_-}(\Delta t, \Delta h + \varepsilon(\tau)\Delta t/2), & \Delta t < 0
\end{cases}
\]

(46)

where

\[
B^{(\pm)}_{\gamma}(k, l) = \frac{1}{2\pi i} \int_{\gamma} (1 - e^{\mp \tau} w^{\pm 1})^k w^{-l-1} dw,
\]

Here \( \varepsilon(\tau) = 1 \) when \( V_i < \tau < U_i \) and \( \varepsilon(\tau) = -1 \) when \( U_i < \tau < V_{i+1} \). The contours \( \gamma_{\pm} \) are as above.

For the limit of correlation functions we obtain the following answer:

\[
\lim_{r \to 0} \langle \rho_{h_1, t_1} \cdots \rho_{h_n, t_n} \rangle = \det(\kappa_{a, b})_{1 \leq a, b \leq n}
\]

23
where
\[ \kappa_{a,b} = \kappa((h_a - h_b), (t_a - t_b)) = \lim_{r \to +0} K((t_a, h_a), (t_b, h_b)) \]

Correlation functions for finite \( q \) satisfy recurrence equations (34) and (35). In the limit these recurrence relations turn into difference equations for correlation functions (7.1):

\[ \kappa(h, t) - \kappa(h + \varepsilon(\tau)1/2, t - 1) + e^{-\varepsilon(\tau)\tau} \kappa(h - \varepsilon(\tau)1/2, t - 1) = \delta_{h,0}\delta_{t,0} \]

where \( \varepsilon(\tau) = +1 \) when \( \tau \in D^+ \), \( \varepsilon(\tau) = -1 \) when \( \tau \in D^- \). These equations can also be directly deduced from the integral representation of correlation functions.

**Remark 2.** The difference operator in these equations is the Kasteleyn matrix on the infinite hexagonal lattice. The pairwise correlation function (7.1) can be regarded as the inverse for this matrix with the boundary conditions determined by \( (\tau, \chi) \).

For one-time correlation functions we have:

\[ \kappa(h, 0) = |z_c|^{-h} \frac{\sin(\theta h)}{\pi h} \]

where \( \theta \) is the argument of \( z_c \), that is, \( z_c = |z_c|e^{i\theta} \).

Notice that the factor \( |z_c|^h \) does not contribute to the equal-time density correlation functions and we have:

\[ \lim_{r \to 0} \langle \rho_{h_1,t} \cdots \rho_{h_n,t} \rangle = \det \left( \frac{\sin(\theta (h_a - h_b))}{\pi (h_a - h_b)} \right) \]

Here we assume that \( h_1 > h_2 > \ldots h_n \).

### 7.2 Limit shape

The form of the one-point correlation function implies that the density of horizontal tiles is

\[ \rho(\tau, \chi) = \frac{\theta}{\pi} \]

where \( \theta \) is the argument of \( z_c \). The limit shape can reconstructed from this density by integration:

\[ z(\tau, \chi) = \int_{-b(\tau)}^{\chi} (1 - \rho(\tau, s)) \, ds . \quad (47) \]

\[ x(\tau, \chi) = z(\tau, \chi) - \chi - \frac{\tau}{2}, \quad y(\tau, \chi) = z(\tau, \chi) - \chi + \frac{\tau}{2} . \quad (48) \]

Here \( b(\tau) = \max(-\tau/2 + U_0, \tau/2 + U_N) \). Thus, the information about the limit shape is in the structure of critical points of the function \( S(z) \).

When the point \( (\tau, \chi) \) is such that all critical points of \( S(z) \) are real the limit of correlation functions (7.1) is either 1 or 0. It is zero if the maximum of \( S(z) \)
is inside of the cycle of integration and it is 1 if it is outside. The corresponding point \((x, y, z)\) lies on a facet (flat part of the limit shape).

If \((\tau, \chi)\) is such that there is a pair of complex conjugate simple critical points, the point \((x, y, z)\) lies in the disordered region (curved part of the limit shape).

The frozen boundary (that is, the boundary between the disordered region and the facets) limit shape correspond to \((\chi, \tau)\) for which there exists a real zero of \(S(z)\) of the multiplicity at least 2. Cusps of the frozen boundary correspond to triple critical points.

We plotted the frozen boundary in the \((\tau, \chi)\)-plane on Fig. 15 for \(N = 1\) and on Fig. 16 for \(N = 1\). The vicinity of the cusp is magnified on Fig. 17. It is instructive to compare these curves with the result of numeric simulation in Figure 2.

![Figure 15: An example of frozen boundary for N = 1](image1)

![Figure 16: An example of frozen boundary for N = 2](image2)
8 Double critical points and the scaling limit near the boundary

8.1

Now, let us assume that \((\chi_0, \tau_0)\) are such that \(z_0\) is a double critical point of \(S(z)\). Consider the vicinity of \((z_0, \chi_0, \tau_0)\) with coordinates 
\[
z = z_0 \exp(\xi), \chi = \chi_0 + \delta \chi, \tau = \tau_0 + \delta \tau.
\]

Lowest degree terms in the Taylor expansion around \((z_0, \chi_0, \tau_0)\) are:

\[
S(z, \chi, \tau) = S(z_0, \chi_0, \tau_0) + \frac{A}{3!} \xi^3 - (\delta \chi - B \delta \tau) \xi + \frac{1}{2} C \delta \tau \xi^2 + \frac{1}{2} D \delta \tau^2 \xi + \frac{1}{3!} E \delta \tau^3 + H \delta \tau + \frac{1}{2} G \delta \tau^2 - \ln(z_0) \delta \chi + \ldots
\]

where

\[
A = (\frac{\partial}{\partial z})^3 S(z)|_{z_0, \tau_0} = \frac{z_0^2 f''(z_0)}{f(z_0)}
\]

\[
B = z \frac{\partial^2}{\partial \tau \partial z} S(z)|_{z_0, \tau_0} = \frac{z_0 e^{-\tau} + e^{\tau}}{2(z_0 e^{-\tau} - e^{\tau})}
\]

\[
C = (z \frac{\partial}{\partial z})^2 \frac{\partial}{\partial \tau} S(z)|_{z_0, \tau_0} = -\frac{z_0}{(z_0 e^{-\tau} - e^{\tau})^2}
\]

\[
D = z \frac{\partial}{\partial z} \frac{\partial^2}{\partial \tau^2} S(z)|_{z_0, \tau_0} = \frac{z_0}{(z_0 e^{-\tau} - e^{\tau})^2}
\]

\[
E = \frac{\partial^3}{\partial \tau^3} S(z)|_{z_0, \tau_0} = -\frac{z_0}{(z_0 e^{-\tau} - e^{\tau})^2}
\]

\[
H = \frac{\partial}{\partial \tau} S(z)|_{z_0, \tau_0} = \begin{cases} 
-\frac{1}{2} \ln(z_0) + \ln(z_0 - e^{\tau_0}), & \tau_0 \in D_- \\
-\frac{1}{2} \ln(z_0) + \ln(e^{\tau} - z_0) - \tau_0, & \tau_0 \in D_+
\end{cases}
\]
\[ G = \frac{\partial^2}{\partial \tau^2} S(z) \big|_{z_0, \tau_0} = \begin{cases} \frac{-e^{\tau_0}}{z_0 - e^{-\tau_0}}, & \tau_0 \in D_- \\ \frac{e^\tau}{e^{\tau_0} - z_0}, & \tau_0 \in D_+ \end{cases} \]  

(55)

Notice that \( C = -D, E = -D, D > 0, \frac{A}{B} > 0 \). The sign of \( A \) depends on the nature of the interface. It is positive if the frozen region is above the melted region and it is negative otherwise. It changes signs at the points \( z_0 = \exp(U_i) \) and at the triple critical points where \( f''(z_0) = 0 \).

Rescaling local coordinates \( \xi, \delta \chi \) and \( \delta \tau \) as

\[ \xi = r^{\frac{4}{3}} \sigma, \quad \delta \chi - B \delta \tau = r^{\frac{4}{3}} x, \quad \delta \tau = r^{\frac{4}{3}} y. \]

we have

\[
\frac{S(z, \chi, \tau) - S(z_0, \chi_0, \tau_0)}{r} = \frac{A}{3!} \sigma^3 - x \sigma + \frac{1}{2} C y \sigma^2 + \frac{1}{2} D y^2 \sigma + \frac{1}{3!} E y^3 + o(1)
\]

8.2

As \( r \to 0 \) the leading asymptotic of the integral \( K \) is determined by the leading asymptotic of the integral:

\[
\text{K}(h_1, t_1), (h_2, t_2)) = \frac{1}{(2\pi i)^2} \int_{C_z} \int_{C_w} e^{\frac{S(z, \chi, \tau) - S(z_0, \chi_0, \tau_0)}{r}} \sqrt{\frac{2w}{z-w}} \frac{dz}{z} \frac{dw}{w} (1+O(r^{1/3}))
\]

(56)

If \( V_1 < \tau < V_2 \), the contour of integration is

\[ C_z \times C_w = \begin{cases} 1 > |z| > |w| > e^{V_1} & \text{if} \ V_1 < \tau < U_1 \\ e^{V_1} > |z| > |w| > e^\tau & \text{if} \ U_1 < \tau < V_2 \end{cases} \]

for \( t_1 \geq t_2 \). When \( t_1 < t_2 \) the only difference is that \( |z| < |w| \).

Assume that coordinates \((h_i, t_i)\) are in the vicinity of the point \((\chi_0, \tau_0)\), and that as \( r \to 0 \) they scale in the following way:

\[ \chi_i = r h_i = \chi_0 + r^{\frac{4}{3}} x_i - B r^{\frac{4}{3}} y_i, \quad \tau_i = r t_i = \tau_0 + r^{\frac{4}{3}} y_i. \]

(57)

In this limit condition \( t_1 \geq t_2, \ t_2 > t_1 \) translate to \( y_1 \geq y_2 \) and \( y_2 > y_1 \) respectively.

Deform contours of integration in such a way that they will pass through this critical point and will follow the branches of curves \( \Im(S(z)) = \Im(S(z_0)) \) and \( \Im(S(w)) = \Im(S(z_0)) \) where \( \Re(S(z)) \) and \( -\Re(S(w)) \) have local maximum. Deformed contours are described in the section 6.3.

The leading contribution to the asymptotic comes the vicinity of the point \( z_0 \) where we will use local coordinates \( z = z_0 (1 + r^{\frac{4}{3}} \sigma) \) and \( w = z_0 (1 + r^{\frac{4}{3}} \kappa) \). The saddle point integration contours for \( \sigma \) and \( \kappa \) in the limit \( r \to 0 \) are shown on Fig. 18 and Fig. 19 for \( A > 0 \).
Figure 18: Integration contour for $\sigma$

Figure 19: Integration contour for $\kappa$
The leading term of the asymptotic of the integral (56) is given by

\[ K((h_1, t_1), (h_2, t_2)) = \exp\left(\frac{H}{r^{2/3}}(y_1 - y_2) + \frac{G}{r^{1/3}}(y_1^2 - y_2^2) - \frac{\ln(z_0)}{r^{1/3}}(x_1 - x_2)ight) \]

\[ - \frac{B \ln(z_0)}{r^{2/3}}(y_1 - y_2) + \frac{1}{3!} E(y_1^3 - y_2^3) K_A((x_1 - \frac{D}{2}y_1^2, y_1), (x_2 - \frac{D}{2}y_2^2, y_2))(1+O(r^{1/3})) \]

(58)

For \( y_1 \geq y_2 \) the function \( K_A \) is

\[ K_A((a_1, b_1), (a_2, b_2)) = \frac{1}{(2\pi i)^2} \int_{C_+} \int_{C_-} \exp\left(\frac{A}{3!}(\sigma^3 - \kappa^3) - \frac{D}{2}(b_1 \sigma^2 - b_2 \kappa^2) - a_1 \sigma + a_2 \kappa\right) \frac{d\sigma d\kappa}{\sigma - \kappa} \] \hspace{1cm} (59)

For \( y_1 < y_2 \) it has an extra term which comes from the residue at \( \sigma = \tau \):

\[ K_A((a_1, b_1), (a_2, b_2)) = \frac{1}{(2\pi i)^2} \int_{C_+} \int_{C_-} \exp\left(\frac{A}{3!}(\sigma^3 - \kappa^3) - \frac{D}{2}(b_1 \sigma^2 - b_2 \kappa^2) - a_1 \sigma + a_2 \kappa\right) \frac{d\sigma d\kappa}{\sigma - \kappa} + \frac{1}{2\pi i} \int_{C_+} \exp\left(-\frac{D}{2}(b_1 - b_2)\sigma^2 - (a_1 - a_2)\sigma\right) d\sigma \] \hspace{1cm} (60)

The function \( K_A((x_1, y_1), (x_2, y_2)) \) is discontinuous at \( y_1 = y_2 \). The discontinuity is determined by the second term. It is not difficult to see that it is equal to

\[ \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{(x_1 - x_2)^2}{2D|y_1 - y_2|}\right)} \]

Notice that since correlation functions (23) are given by determinants, the exponential factors in front of the integral in (59) will be canceled and we have:

\[ \lim_{r \to 0} \langle \rho_{h_1, t_1} \ldots \rho_{h_n, t_n} \rangle = \det(K_A((x_a - \frac{D}{2}y_a^2, y_a), (x_b - \frac{D}{2}y_b^2, y_b))) \]

Here we assume that \((h_i, t_i)\) are scaled as in (57).

It is easy to verify that

\[ K_A((x_1, 0), (x_2, 0)) = \frac{\text{Ai}(x_1) \text{Ai}'(x_2) - \text{Ai}(x_2) \text{Ai}'(x_1)}{\lambda(x_1 - x_2)} \]

where \( \text{Ai}(x) \) is the Airy function:

\[ \text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(\frac{i}{3} t^3 + ixt\right) dt, \]

and \( \lambda = \left(\frac{2A}{3}\right)^{1/3} \).
For the scaling limit of the density of horizontal tiles we have:

\[
\rho_A(x, y) = -\frac{i}{(2\pi)^2} \lambda \int_{\mathbb{R}} \int_{\mathbb{R}} \exp \left( -i \frac{s^3 - t^3}{3!} - i \frac{C^2 y^2}{2A} - x \right) (s - t) \frac{ds dt}{s - t - i0}.
\]

It can also be written as

\[
\rho_A(x, y) = \int_{-\infty}^{\infty} A_i(v) dv
\]

In this form it is clear that the density function is positive.

## 9 Triple critical points and the scaling limit near the cusp

### 9.1 The singularity of the limit curve

Then the following holds if \( z_0 \) is a triple critical point

\[
(z \frac{\partial}{\partial z})^4 S(z) \big|_{z_0} = z_0^3 \frac{f^{(3)}(z_0)}{f(z_0)}
\]

The proof is straightforward. First, one computes the fourth derivative:

\[
(z \frac{\partial}{\partial z})^4 S(z) = \frac{2z^2 e^{-\frac{\tau}{2}}}{(ze^{-\frac{\tau}{2}} - e^{\frac{\tau}{2}})^3} + (zf' + 3z^2 \frac{f'}{f} + z^3 (\frac{f''}{f}'))
\]

This expression reduces to \( \frac{2z^2 e^{-\frac{\tau}{2}}}{(ze^{-\frac{\tau}{2}} - e^{\frac{\tau}{2}})^3} \) after taking into account equations for \( z_0 \). The second derivative of \( f(z) \) vanishes at \( z_0 \) and, as it follows from the graph of \( f(z) \), it is negative for \( z < z_0 \) and positive for \( z > z_0 \). This implies the positivity of \( f^{(3)}(z_0) \). This means that the sign of \( A \) is determined by the sign of \( f(z_0) \). The later is negative if \( z_0 < \exp(U_1) \) (when the cusp on the limit shape is turned to the right) and is positive otherwise (i.e. when the cusp is turned to the left).

Now, let us find the behavior of the boundary curve near the cusp. Assume that \( z_0 \) is a triple critical point corresponding to the cusp with singularity at \( (\chi_0, \tau_0) \), i.e. \( z_0, a_0 \) and \( b_0 \) satisfy the system:

\[
e^{\chi_0 - \tau_0/2} = f'(z_0), \quad (63)
\]

\[
e^{\chi_0 + \tau_0/2} = z_0 f''(z_0) - f(z_0), \quad (64)
\]

\[
0 = f''(z_0). \quad (65)
\]

If a point \( (\chi, \tau) \) is at the boundary curve, it satisfies the equations:

\[
e^{\chi - \tau/2} = f'(z), \quad e^{\chi + \tau/2} = zf'(z) - f(z), \quad (66)
\]
Let \((z, \chi, \tau)\) be a double critical point in a small vicinity of \((z_0, \chi_0, \tau_0)\):

\[
\chi = \chi_0 + \delta \chi, \quad \tau = \tau_0 + \delta \tau, \quad z = z_0 + \varepsilon
\]

Then from the equations (66,68) we obtain the following asymptotic of the boundary curve near the cusp:

\[
\exp(\chi_0 - \tau_0/2)(\delta \chi - \delta \tau/2) = \frac{1}{2} f^{(3)}(z_0)\varepsilon^2 + \frac{1}{6} f^{(4)}(z_0)\varepsilon^3 + O(\varepsilon^4)
\]

\[
\exp(\chi_0 + \tau_0/2)(\delta \chi + \delta \tau/2) = \frac{1}{2} z_0 f^{(3)}(z_0)\varepsilon^2 + \frac{1}{6} z_0 f^{(4)}(z_0) + 2 f^{(3)}(z_0)\varepsilon^3 + O(\varepsilon^4)
\]

when \(\varepsilon \to 0\). From here we have:

\[
\delta \chi = \frac{1}{4} \exp(-\chi_0 - \frac{\tau_0}{2})((z_0 + \exp(\tau_0)) f^{(3)}(z_0)\varepsilon^2 + \frac{1}{3} ((z_0 + \exp(\tau_0)) f^{(4)}(z_0) + 2 f^{(3)}(z_0)\varepsilon^3 + ...)
\]

(67)

\[
\delta \tau = \frac{1}{2} \exp(-\chi_0 - \frac{\tau_0}{2})((z_0 - \exp(\tau_0)) f^{(3)}(z_0)\varepsilon^2 + \frac{1}{3} ((z_0 - \exp(\tau_0)) f^{(4)}(z_0) - 2 f^{(3)}(z_0)\varepsilon^3 + ...)
\]

(68)

After reparametrization

\[
\varepsilon = \sigma - \frac{(z_0 - e^{\tau_0}) f^{(4)}(z_0) - 2 f^{(3)}(z_0)}{6 f^{(3)}(z_0)(z_0 - e^{\tau_0})}
\]

we have:

\[
\delta \tau = \frac{1}{2} \exp(-\chi_0 - \frac{\tau_0}{2})((z_0 - \exp(\tau_0)) f^{(3)}(z_0)\sigma^2 + O(\sigma^3),
\]

\[
\delta \chi = \frac{1}{4} \exp(-\chi_0 - \frac{\tau_0}{2})((z_0+\exp(\tau_0)) f^{(3)}(z_0)\sigma^2 - \frac{4}{3} \frac{\exp(\tau_0)}{z_0 - \exp(\tau_0)} f^{(3)}(z_0)\sigma^3) + O(\sigma^4).
\]

This is a parametrization of a cusp singularity in the boundary of the limit shape.

### 9.2 The asymptotic of correlation functions

Expanding the function \(S(z, \chi, \tau)\) near the triple critical point \(z_0\) we obtain the following lowest degree terms of the Taylor expansion:

\[
S(z, \chi_0 + \delta \chi, \tau_0 + \delta \tau) = S(z_0, \chi_0, \tau_0) + \frac{A}{4!} \xi^4 - \delta \chi \xi + B \delta \tau \xi + \frac{1}{2} C \delta \tau \xi^2 +
\]

\[
+ \frac{1}{2} D \delta \tau^2 - \ln(z_0)\delta \chi + H \delta \tau + o(1)
\]
where

\[ A = z_0^3 f^{(3)}(z_0) \]

and \( B \) and \( C \) are as before, given by (49) (51). Recall that \( A < 0 \) is the cusp in the limit shape is turned right and \( A > 0 \) if it is turned left.

Scaling local coordinates as

\[ \xi = r^{1/4} \sigma, \quad \delta \chi = r^{1/2} x, \quad \delta \tau = r^{1/2} y \]

we obtain the following asymptotic for \( S \):

\[
S(z, \chi, \tau) - S(z_0, \chi_0, \tau_0) \approx \frac{A}{4!} \sigma^4 - \frac{x}{y} \sigma + \frac{C}{2} \sigma^2 y^2 + \frac{D}{2} y^2 + \\
\frac{H}{r^{1/2} y} - \frac{\ln(z_0)}{r^{1/4} x} x + \frac{\ln(z_0)}{r^{1/4} y} y + o(1)
\]

Now let us find the asymptotic of the integral (56) as \( r \to 0 \) assuming that coordinates are scaled as

\[ rh_i = \chi_0 + r^{3/4} x_i - Br^{3/4} y_i, \quad rt_i = \tau_0 + r^{3/4} y_i. \]  

(69)

where \((\chi_0, \tau_0)\) are coordinates of the tip of the cusp.

The asymptotic of the integral (56) as \( r \to 0 \) can be evaluated by the steepest descent method. It is determined by the contribution from the triple critical point \( z_0 \) and after deforming contours of integration as it described in section 6.4 the leading term of the asymptotic is given by the integral

\[
K((h_1, t_1)(h_2, t_2)) = \exp(-\frac{\ln(z_0)}{r^{1/4}}(x_1 - x_2) + \frac{H}{r^{1/2}}(y_1 - y_2) + \frac{B \ln(z_0)}{r^{1/4}}(y_1 - y_2) + \\
\frac{D}{2}(y_1^2 - y_2^2))K_P((x_1, y_1), (x_2, y_2))(1 + O(r^{1/4})
\]

(70)

where for \( y_1 > y_2 \) we have

\[
K_P((x_1, y_1), (x_2, y_2)) = \\
\frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \exp\left(\frac{A}{4!}(\sigma^4 - \kappa^4) + \frac{C}{2}(y_1 \sigma^2 - y_2 \kappa^2) - x_1 \sigma + x_2 \kappa \right) \frac{d\sigma d\kappa}{\sigma - \kappa} + \\
\sqrt{\frac{1}{2\pi D|y_1 - y_2|}} \exp\left(-\frac{(x_1 - x_2)^2}{2D(y_1 - y_2)}\right).
\]

(71)

For \( y_1 < y_2 \) there is an extra term coming from the residue at \( \sigma = \tau \):
\[ K_P((x_1, y_1), (x_2, y_2)) = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \exp\left(\frac{A}{4!} (\sigma^4 - \kappa^4) + C \left( y_1 \sigma^2 - y_2 \kappa^2 \right) - x_1 \sigma + x_2 \kappa \right) \frac{d\sigma d\kappa}{\sigma - \kappa}. \quad (72) \]

Integration contours for \( A > 0 \) are shown on Fig. 21 and 20.

Notice that the exponential factors cancels in the limit of correlation functions and we have:

\[ \lim_{r \to 0} \langle \rho_{h_1, t_1} \ldots \rho_{h_n, t_n} \rangle = \det(K_P((x_a, y_a), (x_b, y_b))) \]

Here we assume \((h_i, t_i)\) scale as in [33].
10 Some properties of Pearcey kernel

Recall that the asymptotic of correlation functions is determined by the following integral, which we will call Pearcey kernel

\[ K_P((x_1, y_1), (x_2, y_2)) = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \exp\left(\frac{A}{4!}(\sigma^4 - \kappa^4) + \frac{C}{2}(y_1\sigma^2 - y_2\kappa^2) - x_1\sigma + x_2\kappa\right) \frac{d\sigma d\kappa}{\sigma - \kappa}. \quad (73) \]

After the appropriate scaling of variables we can set \( A = 6 \) and \( C = 1 \). Below we will review some of its properties.

10.1

The function \( K_P((x_1, y_1), (x_2, y_2)) \) (with \( A = 6 \) and \( C = 1 \)) satisfies the following differential equations:

\[ (\frac{\partial}{\partial y_i} - \frac{1}{2} \frac{\partial^2}{\partial x_i^2}) K_P = 0 \]

for \( i = 1, 2 \) and

\[ (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) K_P = -P_+(x_1, y_1)P_-(x_2, y_2) \]

where

\[ P_{\pm}(x, y) = \frac{1}{2\pi i} \int_{C_{\pm}} \exp(\pm(\frac{\sigma^4}{4} + \frac{1}{2}y\sigma^2 - x\sigma)) d\sigma \]

For functions \( P_{\pm} \) we have:

\[ (\frac{\partial^3}{\partial x^3} \pm x + \frac{\partial}{\partial x}) P_{\pm} = 0 \]

\[ (\pm \frac{\partial}{\partial y} - \frac{1}{2} \frac{\partial^2}{\partial x^2}) P_{\pm} = 0 \]

10.2

The function \( P_+(x, y) \) can be written as

\[ P_+(x, y) = \frac{1}{2\pi i}\left(-\int_{\omega\mathbb{R}^+} + \int_{\omega-1\mathbb{R}^+} - \int_{\omega-3\mathbb{R}^+} + \int_{\omega^3\mathbb{R}^+}\right) \]

or, as

\[ P_+(x, y) = \frac{1}{\pi} \Im(\omega) \int_0^\infty \exp(-\frac{s^4}{4} - \frac{i}{2}s^2)(e^{x\omega s} - e^{-x\omega s}) ds \]

34
From this integral representation one can find a power series formula for the integral:

\[
P_+ (x, y) = \frac{2}{\pi} \sum_{n \geq 0, m \geq 0, n + m = 0(2)} (-1)^{n+m} 2^{n-1} \Gamma \left( \frac{n+m+1}{2} \right) x^{2n+1} (-y)^m \frac{1}{(2n+1)! m!} \]

or,

\[
P_+ (x, y) = \frac{1}{\pi} \sum_{k, l \geq 0} (-1)^{k+l/2} 2^{k+l+1} \frac{\Gamma (k+l+1/2)}{(4k+1)(2l)!} x^{4k+1} y^{2l} + \]

\[
+ \frac{1}{\pi} \sum_{k, l \geq 0} (-1)^{k+l/2} 2^{k+l+1} \frac{\Gamma (k+l+3/2)}{(4k+3)(2l+1)!} x^{4k+3} y^{2l+1} \]

Now, using the identity \( \Gamma (k+1/2) = 2^{-k+1/2} \sqrt{\pi} \Gamma (k) \), we arrive to the formula

\[
P_+ (x, y) = \frac{2}{\sqrt{\pi}} \sum_{k, l \geq 0} (-1)^{k+l/2} 2^{k+l+1} \frac{\Gamma (k+l+1/2)}{(4k+1)(2l)!} x^{4k+1} y^{2l} +
\]

\[
+ \frac{2}{\sqrt{\pi}} \sum_{k, l \geq 0} (-1)^{k+l/2} 2^{-k+l+2} \frac{\Gamma (k+l+2)}{(4k+3)(2l+1)!} x^{4k+3} y^{2l+1} \]

10.3

The integrals

\[
P_- (x, y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left( -\frac{t^4}{4} + \frac{1}{2} yt^2 + ixt \right) dt
\]

can be easily expanded into a power series in \( x \) and \( y \):

\[
P_- (x, y) = \frac{1}{\pi} \sum_{n, m \geq 0} (-1)^n 2^{n-3/2} \frac{\Gamma (n+m+1/4)}{(2n)! m!} x^{2n} y^m,
\]

10.4

One-time correlation functions can be expressed in terms of \( P_\pm \) as follows:

\[
K(x_1, x_2) = \frac{P''_+ (x_1, y) P_- (x_2, y) - P'_+ (x_1, y) P'_- (x_2, y) + P'_+ (x_1, y) P''_- (x_2, y) + y P'_+ (x_1, y) P_- (x_2, y)}{x_1 - x_2}
\]

(74)

Here \( P'(x, y) = \frac{\partial}{\partial x} P(x, y) \). This identity follows from

\[
\int_{C_1} \int_{C_2} \left( \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \kappa} \right) \exp \left( \frac{A}{4!} (\sigma^4 - \kappa^4) + \frac{C}{2} (y_1 \sigma^2 - y_2 \kappa^2) - x_1 \sigma + x_2 \kappa \right) \frac{d\sigma d\kappa}{\sigma - \kappa} = 0.
\]

(75)
Taking the limit $x_1 \to x_2 = x$ in (74) we obtain the following expression for the scaling limit of the density of tiles near the cusp:

$$
\rho(x,y) = P'_+(x,y)P''_+(x,y) - P'_+(x,y)P'_-(x,y) - xP_+(x,y)P_-(x,y)
$$

This function is plotted on Fig 22.

Figure 22: The scaling limit of the density of horizontal tiles near the cups

A Schur functions

Recall that a partition is a sequence of integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$$

A diagram of a partition $\lambda$ (also known as its Young diagram) has $\lambda_1$ boxes in the upper row, $\lambda_2$ boxes in the next row etc., see Figure 23. Unless this leads to a confusion, we identify partitions with their diagrams. The sum

$$|\lambda| = \sum_{i \geq 1} \lambda_i$$

36
is the area of the diagram $\lambda$.

Figure 23: A Young diagram

A skew diagram $\lambda/\mu$ is a pair of two partitions $\lambda$ and $\mu$ such that $\lambda_1 \geq \mu_1, \lambda_2 \geq \mu_2, \ldots$. Graphically, it is obtained from $\lambda$ by removing $\mu_1$ first boxes from the first row, $\mu_2$ first boxes from the second row etc. The size of the skew diagram $\lambda/\mu$ is

$$|\lambda/\mu| = |\lambda| - |\mu|$$

A semistandard tableau of the shape $\lambda/\mu$ with entries $1, 2, 3, \ldots$ is the result of writing numbers $1, 2, 3, \ldots$ in boxes of the diagram, one in each box, in such a way that the numbers weakly decrease along the rows and strictly decrease along the columns, see an example in Figure 24.

Figure 24: A semi-standard tableau

The Schur function corresponding to the skew tableau $\lambda/\mu$ is a symmetric function of variables $x_1, x_2, \ldots$ which can be defined as the sum over all semi-standard tableaux of shape $\lambda$ of monomials in $x_i$:

$$s_{\lambda/\mu}(x_1, x_2, \ldots) = \sum_{T_{\lambda/\mu}} x_1^{m_1} x_2^{m_2} \ldots$$

where $m_i$ is the number of occurrences of $i$ in the tableau.

A semi-standard tableaux of shape $\lambda/\mu$ can be identified with sequences of skew diagrams in the following way.

Let us say that $\lambda \succeq \mu$ if $\lambda$ and $\mu$ interlace, that is,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \ldots$$

Let us call the sequence of diagrams $\lambda(1), \lambda(2), \ldots, \lambda(n)$ interlacing if $\lambda(1) \succeq \lambda(2) \succ \cdots \succ \lambda(n)$. It is clear that if we will associate with such sequence the semi-standard tableau with $|\lambda(1)| - |\lambda(2)|$ entrees of $n$, $|\lambda(2)| - |\lambda(3)|$ entrees of $n-1$, etc.
etc. It is also clear that this correspondence gives a bijection between semi-
standard tableaux of shape $\lambda(1)/\lambda(n)$ and interlacing sequences of diagrams
which start with $\lambda(1)$ and ends with $\lambda(n)$.

For skew Schur functions this bijection gives the following formula

$$s_{\lambda/\mu}(x_1, \ldots, x_n, 0, 0, \ldots) = \sum_{\lambda \succ \lambda(1) \succ \cdots \succ \lambda(n-1) \succ \mu} x_1^{[\lambda|-\lambda(1)|-|\lambda(2)|} x_2^{[\lambda(1)|-|\lambda(3)|} \cdots x_n^{[\lambda(n-1)|-|\mu|}$$  \hspace{1cm} (76)

from now on we will write $s_{\lambda/\mu(x_1, x_2, \ldots, x_n)}$ for $s_{\lambda}(x_1, \ldots, x_n, 0, 0, \ldots)$.

Notice that

$$s_{\lambda/\mu}(x_1, 0, \ldots) = \begin{cases} x_1^{[\lambda|-|\mu|}, & \mu \prec \lambda, \\ 0, & \mu \not\prec \lambda. \end{cases}$$  \hspace{1cm} (77)

and therefore we can write

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{\lambda \succ \lambda(1) \succ \cdots \succ \lambda(n-1) \succ \mu} s_{\lambda/\lambda(1)}(x_1) s_{\lambda(1)/\lambda(2)}(x_2) \cdots s_{\lambda(n-1)/\mu}(x_n)$$  \hspace{1cm} (78)

The function $s_{\lambda}(x_1, x_2, \ldots, x_n)$ is the character of the irreducible representation of $GL_n$ with the highest weight $\lambda$ computed on the diagonal element with entries $x_1, x_2, \ldots, x_n$. The formula (76) is the result of the computation this character in the Gelfand-Tsetlin basis.

B Semiinfinite forms and vertex operators

B.1 Semiinfinite forms

Let the space $V$ be spanned by $k \in \mathbb{Z} + \frac{1}{2}$. The space $F = \Lambda \hat{V}$ is, by
definition, spanned by vectors

$$v_S = s_1 \wedge s_2 \wedge s_3 \wedge \ldots,$$

where $S = \{s_1 > s_2 > \ldots\} \subset \mathbb{Z} + \frac{1}{2}$ is such a subset that both sets

$$S_+ = S \setminus (\mathbb{Z}_{\leq 0} - \frac{1}{2}), \quad S_- = (\mathbb{Z}_{\leq 0} - \frac{1}{2}) \setminus S$$

are finite. We equip $\Lambda \hat{V}$ with the inner product $(.,.)$ in which the basis $\{v_S\}$ is
orthonormal. The space $F$ is also called the fermionic Fock space.

The infinite Clifford algebra $Cl(V)$ is generated by elements $\psi_k, \psi_k^*$, $k \in \mathbb{Z} + \frac{1}{2}$ with defining relations

$$\psi_k \psi_l + \psi_l \psi_k = 0, \quad \psi_k^* \psi_l^* + \psi_l^* \psi_k^* = 0, \quad \psi_k \psi_l^* + \psi_l^* \psi_k = \delta_{k,l},$$

It acts on the Fock space $F$ as
\[ \psi_k (s_1 \wedge s_2 \wedge s_3 \wedge \ldots) = k \wedge s_1 \wedge s_2 \wedge s_3 \wedge \ldots, \]

\[ \psi_k^* (s_1 \wedge s_2 \wedge \ldots s_l \wedge k \wedge s_{l+1} \wedge \ldots) = (-1)^l s_1 \wedge s_2 \wedge \ldots s_l \wedge s_{l+1} \wedge \ldots, \]

\[ \psi_k^* (s_1 \wedge s_2 \wedge s_3 \wedge \ldots) = 0, k \in \mathbb{Z} \setminus S \]

The space \( F \) is an irreducible representation of \( Cl(V) \). Notice that the operator representing \( \psi_k^* \) is conjugate to the operator representing \( \psi_k \) with the respect to the scalar product in \( F \).

The Lie algebra \( \mathfrak{gl}_\infty \) of \( \mathbb{Z} \times \mathbb{Z} \)-matrices with finitely many entries acts naturally on \( V \) and therefore acts diagonally on semi-infinite wedge space \( F \). This action extends to the action of \( a_\infty \) [11] and is reducible. Irreducible components \( F(m) \) are eigenspaces of the operator

\[ C = \sum_{k<0} \psi_k^* \psi_k - \sum_{k<0} \psi_k \psi_k^* \]

The Fock space decomposes into the direct sum

\[ F = \bigoplus_{m \in \mathbb{Z}} F(m) \]

of irreducible representations of \( a_\infty \) [11].

The subspace \( F(m) \) is spanned by vectors \( s_1 \wedge s_2 \wedge s_3 \wedge \ldots \) with \( s_i = m - i + 1 \) for sufficiently large \( i \). It is generated by the action of \( a_\infty \) on the vacuum vector (the \( a_\infty \) highest weight vector in \( F(m) \)):

\[ v_0^{(m)} = m - 1/2 \wedge m - 3/2 \wedge m - 5/2 \wedge \ldots \]

The operators \( \psi_k \) and \( \psi_k^* \) act on \( v_0^{(m)} \) as

\[ \psi_k v_0^{(m)} = 0, \quad k \leq m - 1/2, \]

\[ \psi_k^* v_0^{(m)} = 0, \quad k > m - 1/2, \]

They can be regarded as \( a_\infty \)-intertwining operators \( \psi : V \otimes F^{(m)} \to F^{(m+1)} \), where \( V = \mathbb{C}^{2+\frac{3}{2}} \) is the vector representation of \( \mathfrak{gl}_\infty \).

Vectors in the space \( F^{(m)} \) can be parameterized by partitions. For a partition \( \lambda \) define

\[ v_\lambda^{(m)} = \lambda_1 + m - 1/2 \wedge \lambda_2 + m - 3/2 \wedge \ldots \]

It is clear that these vectors span \( F^{(m)} \).
B.2 Clifford algebra and vertex operators

Consider elements
\[ \alpha_n = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_{k+n} \psi_k^*, \quad n = \pm 1, \pm 2, \ldots, \]

They satisfy commutation relations
\[
[\alpha_n, \alpha_m] = -n \delta_{n,-m}, \\
[\alpha_n, \psi_k] = \psi_{k+n}, \\
[\alpha_n, \psi_k^*] = -\psi_{k-n}^*
\]

(81)

It is clear that
\[ \alpha_n v_0^{(m)} = 0 \]
for \( n < 0 \) and \( m \in \mathbb{Z} \). Vertex operators are the formal power series
\[
\Gamma_+(x) = \exp(\sum_{n \geq 1} x^n \alpha_n), \quad \Gamma_-(x) = \exp(\sum_{n \geq 1} x^n \alpha_{-n})
\]

The operator \( \Gamma_-(x) \) acts finitely in the space \( F \) (applied to any vector of \( F \) it acts as a polynomial in \( x \)). In particular
\[
\Gamma_-(x) v_0^{(m)} = v_0^{(m)}
\]

The operator \( \Gamma_+(x) \) is conjugate to \( \Gamma_-(x) \):
\[
(\Gamma_-(x)v, w) = (v, \Gamma_+(x)w)
\]

(84)

and since the scalar product is symmetric
\[
(\Gamma_-(x)v, w) = (\Gamma_+(x)w, v)
\]

Notice that its action is defined in \( F \) not only as a formal power series in \( x \). In a weak sense operators \( \Gamma_{\pm} \) are operator-valued functions which are analytic at \( x = 0 \).

Define the formal Fourier transform of \( \psi_k, \psi_k^* \) as power series
\[
\psi(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} z^k \psi_k, \quad \psi^*(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} z^{-k} \psi_k^*.
\]

(85)

These operators and vertex operators satisfy the following commutation relations:
\[
\Gamma_+(x) \Gamma_-(y) = (1 - xy) \Gamma_-(y) \Gamma_+(x), \\
\Gamma_+(x) \psi(z) = (1 - z^{-1}x)^{-1} \psi(z) \Gamma_+(x), \\
\Gamma_+(x) \psi^*(z) = (1 - xz)^{-1} \psi^*(z) \Gamma_+(x), \\
\Gamma_-(x) \psi^*(z) = (1 - x^{-1}z) \psi^*(z) \Gamma_+(x), \\
\Gamma_-(x) \psi(z) = (1 - xz) \psi(z) \Gamma_-(x)
\]

(86)

Here left and right sides are corresponding formal power series.

The following is a well known statement.

40
Theorem 4. The following identity holds:

\[
(\Gamma_-(x_1)\Gamma_-(x_2)\ldots\Gamma_-(x_n)v^{(m)}_{\lambda},v^{(m)}_{\mu}) = s_{\lambda/\mu}(x_1,\ldots,x_n)
\]  

(91)

This is a well known statement, for a proof see for example [11]. We will give a proof of it here. The key step is to show that the identity (91) holds for \( n = 1 \) which can be easily derived from the fact that

\[
v^{(m)}_{\lambda} = \psi_{j_1}\ldots\psi_{j_n}v^{(m-n)}_0
\]

where \( j_1 = \lambda_1 + m - 1/2, j_2 = \lambda_2 + m - 3/2, \ldots \) and from the identity

\[
\Gamma_-(x)\psi(z_1)\ldots\psi(z_n)v^{(m-n)}_0 = \prod_{i=1}^n(1-xz_i)^{-1}\psi(z_1)\ldots\psi(z_n)v^{(m-n)}_0
\]

for generating functions.

For matrix elements of generating functions \( \psi(z) \) and \( \psi^*(w) \) we have:

\[
(\psi^*(w)\psi(z)v^{(m)}_0,v^{(m)}_0) = (\frac{z}{w})^{m+1/2}\frac{1}{1-z/w}, \quad |z| < |w|
\]

\[
(\psi(z)\psi^*(w)v^{(m)}_0,v^{(m)}_0) = (\frac{z}{w})^{m-1/2}\frac{1}{1-w/z}, \quad |z| > |w|.
\]

(92)

These identities follow from the summation of a geometric series.

Let \( A^a_i = \sum_j A^a_{ij}\psi_j \) and \( B^a_i = \sum_j B^a_{ij}\psi^*_j \) for \( a = 1,\ldots,n \). The following identity is known as a Wick’s lemma:

\[
(A^1_{i_1}B^1_{j_1}A^2_{i_2}B^2_{j_2}\ldots A^n_{i_n}B^n_{j_n}v^{(m)}_0,v^{(m)}_0) = \det(K_{ab})_{1\leq a,b\leq n}
\]

where

\[
K_{ab} = \begin{cases} 
(A^a_{i_a}B^b_{j_b},v^{(m)}_0,v^{(m)}_0), & a \leq b \\
-(B^b_{j_b}A^a_{i_a},v^{(m)}_0,v^{(m)}_0), & a > b
\end{cases}
\]

(93)

C Some asymptotic for limit shapes

C.1

The frozen boundary is singular at \( \tau = V_j \). When \( \tau \to V_j \) the singular branch of the boundary curve behave as

\[
\chi(\tau) = 2\ln\left(\frac{\tau - V_j}{2}\right) + |V_j|/2 + \sum_{i=0}^{j-1}(V_{i+1} - U_i) + \\
\ln(\prod_{k\neq j,1\leq k\leq N}|1 - e^{-U_k + U_j}|/\prod_{1\leq k\leq N}|1 - e^{-U_k + V_j}|) + O(|\tau - V_j|).
\]

41
C.2

The limit curve is tangent to the lines $\tau = U_0$ and $\tau = U_N$ at the points $(\chi_L, U_0)$ and $(\chi_R, U_N)$ where

$$
\chi_L = \sum_{j=1}^{N} \ln \left( \frac{1 - e^{U_0 - V_j}}{1 - e^{U_0 - U_j}} \right) + \frac{U_0}{2}
$$

$$
\chi_R = \sum_{j=1}^{N} \ln \left( \frac{1 - e^{U_N - V_j}}{1 - e^{U_N - U_j}} \right) + U_0 - \frac{U_N}{2}
$$

If $\tau = U_0 + \epsilon$ or $\tau = U_N - \epsilon$ and $\epsilon \to +0$ the asymptotic of two double critical points $z^{(\pm)}(\tau)$ is

$$
z^{(\pm)}(\tau) = \begin{cases} 
    e^{U_0} \left( 1 \pm \sqrt{\epsilon C L + O(\epsilon)} \right) & \text{if } \tau = U_0 + \epsilon \\
    e^{U_N} \left( 1 \pm \sqrt{\epsilon C R + O(\epsilon)} \right) & \text{if } \tau = U_N - \epsilon
\end{cases}
$$

where $C_L$ and $C_R$ are some constants.

The boundary curve near this point behave as:

$$
\chi(\tau) = \begin{cases} 
    \chi_L(1 + \sqrt{\tau - U_0} D_L + O(|U_0 - \tau|)) & \text{if } \tau \to U_0 + 0 \\
    \chi_L(1 + \sqrt{U_N - \tau} D_R + O(|U_N - \tau|)) & \text{if } \tau \to U_N - 0
\end{cases}
$$

where, again, $D_L$ and $D_R$ are some constants.

Similar solution exists near each point $\tau = U_j$, $j = 1, \ldots, N - 1$.

C.3

Let us find the asymptotic of the density function near the left boundary of the limit shape. Assume

$$
\chi = \chi_L + \delta \chi, \quad \tau = U_0 + \delta \tau
$$

for some positive $\delta \tau \to 0$.

Solutions to $[40]$ have the asymptotic $z = \exp(U_0)(1 + \delta z)$. Let us find $\delta z$ as a function of $\delta \chi$ and $\delta \tau$. We have the following asymptotical expansions:

$$
\prod_{j=1}^{N} \frac{1 - ze^{-V_j}}{1 - ze^{-U_j}} = \prod_{j=1}^{N} \frac{1 - e^{U_0 - V_j}}{1 - e^{U_0 - U_j}} \left( 1 - C \delta z + O(\delta z^2) \right)
$$

where

$$
C = \sum_{i=1}^{N} \left( \frac{e^{U_0 - V_j}}{1 - e^{U_0 - U_j}} - \frac{e^{U_0 - U_j}}{1 - e^{U_0 - V_j}} \right) > 0
$$

Keeping leading orders in $\delta \chi$, $\delta \tau$, and $\delta z$ in $[40]$ we obtain the equation for $\delta z$:

$$
C \delta z^2 + \delta z((C + 1/2) \delta \tau - \delta \chi) + \delta \tau = 0
$$
In the region $\delta \tau \propto (\delta \chi)^2$ we have two asymptotical solutions

$$\delta z_{1,2} \simeq \frac{\delta \chi}{2} \pm \sqrt{\frac{(\delta \chi)^2}{2} - \frac{\delta \tau}{C}}$$

From here we obtain the asymptotic of the density function in this region:

$$\rho = \frac{\theta}{\pi} \simeq \arctan\left(\sqrt{\frac{4\delta \tau}{C(\delta \chi)^2} - 1}\right)$$

Here $\theta$ is the argument of $\delta z_1$. Notice that $\rho \to 1/2 - 0$ as $\delta \chi \to +0$.

### D The symmetry of correlation functions

Change variables in the integral

$$K((t_1, h_1), (t_2, h_2)) = \frac{1}{(2\pi)^2} \int_{|z|<\min\{1, R(t_1)\}} \int_{R^*(t_2)<|w|<1} \frac{\Phi_-(z, t_1)\Phi_+(w, t_2) \sqrt{zw}}{\Phi_+(z, t_1)\Phi_-(w, t_2)} \frac{dzdw}{zw}$$

from $z$ to $w^{-1}$ and from $w$ to $z^{-1}$. It becomes

$$K((t_1, h_1), (t_2, h_2)) = \frac{1}{(2\pi)^2} \int_{|w|^{-1}<\min\{1, R(t_1)\}} \int_{R^*(t_2)<|z|^{-1}<1} \frac{\Phi_-(z, -t_2)\Phi_+(w, t_1) \sqrt{zw}}{\Phi_+(z, t_2)\Phi_-(w, t_1)} \frac{dzdw}{zw}$$

where

$$\Phi_+(z, t) = \prod_m (1 - z \hat{x}_m^+)$$

$$\Phi_-(z, t) = \prod_m (1 - z\hat{x}_m^-)$$

and $\hat{x}_m^\pm = x_m^\mp$.

Thus, we have the following “reflection” symmetry of correlation functions:

$$K((j_1, t_1), (j_2, t_2)) = K((j_2, -t_1)(j_1, -t_1))$$

This symmetry is obvious on the “microscopical level”. It corresponds to the reflection of the tiling in $t$-direction.
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