PROFILE DECOMPOSITIONS
OF FRACTIONAL SCHröDINGER EQUATIONS
WITH ANGULARLY REGULAR DATA

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ABSTRACT. We study the fractional Schrödinger equations in $\mathbb{R}^{1+d}$, $d \geq 3$ of order $d/(d-1) < \alpha < 2$. Under the angular regularity assumption we prove linear and nonlinear profile decompositions which extend the previous results [9] to data without radial assumption. As applications we show blowup phenomena of solutions to mass-critical fractional Hartree equations.

1. Introduction

We continue the study of the fractional Schrödinger equations with Hartree type nonlinearity which was carried out in our previous work [9] under the assumption that the initial data are radial. Let us consider the following equation with initial data of angular regularity:

$$(1.1) \begin{aligned}
   iu_t + (-\Delta)^{\alpha/2} u &= \lambda (|x|^{-\alpha} * |u|^2) u, \quad (t,x) \in \mathbb{R}^{1+d}, \quad d \geq 3, \\
u(0,x) &= f(x) \in L^2_{\rho} H_\sigma^\gamma,
\end{aligned}$$

where $\lambda = \pm 1$, $(-\Delta)^{\alpha/2} = F^{-1}|\xi|^\alpha F$. Here, the Sobolev space $L^2_{\rho} H_\sigma^\gamma$ is defined by the norm

$$\|f\|^2_{L^2_{\rho} H_\sigma^\gamma} = \int_0^\infty \int_{S^{n-1}} |D_\sigma f(\rho \sigma)|^2 d\sigma \rho^{n-1} d\rho$$

and the operator $D_\sigma^\gamma$ is given by $(1 - \Delta_\sigma)^{\gamma/2}$ while $\Delta_\sigma$ is the Laplace-Beltrami operator defined on the unit sphere. In dimension 3, $\Delta_\sigma$ is the square of angular momentum operator. So, the norm $\|f\|_{L^2_{\rho} H_\sigma^\gamma}$ can be referred as a quantity associated with mass and initial angular momentum. The index $\alpha \in (0,2)$ is the fractional order of equation which is known for Lévy stability index. In [22] Laskin introduced the fractional quantum mechanics in which he generalized the Brownian-like quantum mechanical path, in the Feynman path integral approach to quantum mechanics, to the $\alpha$-stable Lévy-like quantum mechanical path. The equation (1.1) of other types of nonlinearities also appears in astrophysics or water waves, particularly with $\alpha = 1/2$ or $3/2$. See [14, 17] and references therein.

The solutions to equation (1.1) have the conservation laws for the mass and the energy:

$$M(u) = \int |u|^2 \, dx, \quad E(u) = \frac{1}{2} \int \nabla |\nabla|^{\alpha} u \, dx - \frac{\lambda}{4} \int \pi(|x|^{-\alpha} * |u|^2) u \, dx.$$
We say that (1.1) is focusing if $\lambda = 1$, and defocusing if $\lambda = -1$. The equation (1.1) is mass-critical, as $M(u)$ is invariant under scaling $u(t, x) \rightarrow u_\rho(t, \rho x)$, $\rho > 0$, and $u_\rho$ is again a solution to (1.1) with initial datum $\rho^{-d/2}u(0, x/\rho)$. The Cauchy problem (1.1) is locally well-posed in $L^2_0 H^\gamma_0$ if $\gamma \geq \gamma_0$ for a certain $\gamma_0$. See Appendix A. There are well-posedness results with initial data in different Sobolev spaces. See [7, 17] for results with critical or noncritical nonlinearity.

In the previous works [8, 9] it was intended to extend the theory of critical nonlinear Schrödinger equations to the fractional order equations. For the focusing case, the authors [8] used a virial argument to show the finite time blowup with radial data provided that the energy $E(u)$ is negative.

In [9] the linear profile decomposition for the radial $L^2$ data was established. In this paper, we extend the profile decomposition to general data while assuming an extra angular regularity and apply it to show blowup phenomena of solutions to mass-critical fractional Hartree equations.

Related to nonlinear dispersive equations with the critical nonlinearity, the profile decompositions have been intensively studied and led to various recent developments. For example, see [10]. Profile decompositions for the Schrödinger equations with $L^2$ data were obtained by Merle and Vega [23] when $d = 2$, Carles and Keraani [5], $d = 1$, and Bégout and Vargas [2], $d \geq 3$. (Also see [11, 25] for results on the wave equation and [26, 21] on general dispersive equations.) These results are based on refinements of Strichartz estimates (see [24, 3]). There is a different approach which makes use of the Sobolev imbedding [13] but such approach is not applicable especially when the equation is $L^2$-critical.

Our approach here also relies on a refinement of Strichartz estimate which strengthens the usual estimate. However, when $\alpha < 2$, due to insufficient dispersion, we do not have a proper linear estimate for general data which matches with the natural scaling $u \rightarrow u_\rho$. In order to get around this one may consider the Strichartz estimates which are accompanied by a loss of derivative or integrability. For instance, when one assumes the data is radial or more regular in angular direction, Strichartz estimates have wider admissible range. (See [16] and [11].) More precisely we make use of the estimate (2.1). Thanks to the extended admissible range of (2.1) it is relatively simpler to obtain the refinement (see Proposition 2.3 which is used for the proof of profile decomposition) but the estimate (2.1) suffers from large loss of angular regularity which makes it difficult to use (2.1) directly. Hence we need smoothing in angular variables to compensate the loss. This is done in Lemma 2.4 by making use of a bilinear $L^2$ estimate.

We now denote by $U(t)f$ the solution of the linear equation $iu_t + (-\Delta)^\frac{\alpha}{2} u = 0$ with initial datum $f$. Then it is formally given by

$$U(t)f = e^{it(-\Delta)^\frac{\alpha}{2}} f := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x\cdot \xi + |\xi|^\alpha)} \hat{f}(\xi) d\xi.$$ 

Here $\hat{f}$ denotes the Fourier transform of $f$ such that $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot \xi} f(x) dx$.

The following is our main result.

**Theorem 1.1.** Let $d \geq 3$, $\frac{d}{d+1} < \alpha < 2$, and $2 < q, r < \infty$ satisfy $\frac{\alpha}{q} + \frac{d}{r} = \frac{d}{2}$. Suppose that $(u_n)_{n \geq 1}$ is a sequence of complex-valued functions satisfying $\|u_n\|_{L^r_0 H^\gamma_0} \leq 1$ for some $\gamma \geq 0$. Then up to a
subsequence, for any \( l \geq 1 \), there exist a sequence of functions \((\phi^j)_{1 \leq j \leq l} \subset L^2_{\rho}H^s_{\gamma} \), \( \omega^j_n \subset L^2_{\rho}H^s_{\gamma} \) and a family of parameters \((h_n^j, t_n^j)_{1 \leq j \leq l, n \geq 1} \) such that

\[
u_n(x) = \sum_{1 \leq j \leq l} U(t_n^j)(h_n^j)^{-d/2}\phi^j(\cdot/h_n^j)(x) + \omega_n^j(x)
\]

and the following properties are satisfied:

1. If \( \hat{\gamma} < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r} \), then

\[
\lim_{l \to \infty} \limsup_{n \to \infty} \|U(\cdot)\omega^l_n\|_{L^q_tL^r_x H^\gamma_{\alpha}(\mathbb{R} \times \mathbb{R}^d)} = 0.
\]

2. For \( j \neq k \), \((h_n^j, t_n^j)_{n \geq 1} \) and \((h_n^k, t_n^k)_{n \geq 1} \) are asymptotically orthogonal in the sense that

\[
\limsup_{n \to \infty} \frac{h_n^j h_n^k + |t_n^j - t_n^k| + |t_n^j - t_n^k|}{(h_n^j)^\alpha (h_n^k)^\alpha} = \infty.
\]

3. For each \( l \geq 0 \),

\[
\lim_{n \to \infty} \left[ \|\nu_n\|_{L^2_{\rho}H^s_{\gamma}}^2 - \left( \sum_{1 \leq j \leq l} \|\phi^j\|_{L^2_{\rho}H^s_{\gamma}}^2 + \|\omega_n^j\|_{L^2_{\rho}H^s_{\gamma}}^2 \right) \right] = 0.
\]

It should be noted that \( \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r} \) is positive. So we can take a positive \( \hat{\gamma} \) for \((1.2)\), which is important for proof of nonlinear profile decomposition. The regularity requirement \( \hat{\gamma} < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r} \) is far from being optimal. Concerning the parameters which appear in Theorem \(1.1\) one notices that the space translation and modulation (=frequency translation) are absent. It is not surprising in that they are not noncompact symmetries of the linear estimate \(2.1\). For instance, testing with a translating sequence \( f_n(\cdot) = f(\cdot - nx_0) \) for \( x_0 \neq 0 \), one can observe that \( \|f_n\|_{L^2} = c \) but \( \|D^\gamma_\rho U(\cdot)f_n\|_{L^q_tL^r_x H^\gamma_{\alpha}} \to 0 \).

Once we obtain the linear profile decomposition, we can apply it to the nonlinear problem \((1.1)\). The procedure is now well established and rather standard. Especially, the equation \((1.1)\) in the angularly regular case is similar to the radial case \(9\) (For nonlinear Schrödinger equation, see \(20\)). Hence, we mostly omit its proof. But for the sake of completeness we provide statements of results regarding the blowup problem (see Section 4).

Using a perturbation argument and global well-posedness for small data (see Appendix \(A\)), we can extend the linear profile decomposition to the nonlinear profile decomposition. As in the linear profile decomposition, parameters in the nonlinear profile decomposition also have the asymptotic orthogonality. Then it is easy to show the existence of blowup solutions of minimal quantity associated with angular regularity, and the mass concentration phenomena of finite time blowup solutions. See Section 4 for detail.

The rest of the paper is organized as follows: In Section 2, we will show the refined Strichartz estimate. Section 3 will be devoted to proving the main theorem, establishing the linear profile decomposition. In Section 4, we discuss applications to nonlinear profile decomposition and blowup

\[1\] In view of Sobolev embedding, the derivative \(D^\gamma_\rho \) is not sufficient to recover the loss of integrability in \(L^2_\rho\), when compared the usual Strichartz estimate.
profile. In Appendix A, we provide the proof of small data global well-posedness for the Cauchy problem (1.1), as the initial data is not usual Sobolev data. The proofs of theorems in Section 4 rely on Propositions A.1, A.3.

2. Refined Strichartz Estimates

For the fractional Schrödinger equation with \( \alpha < 2 \), it is known that the Strichartz estimate for \( L^2 \)-data has a loss of regularity. However, if one imposes an angular regularity on data, one can recover some of loss of regularity. Recently, almost optimal range of admissible pairs was established in [16] and the range was further extended in [11, 18] to include the remaining endpoint cases.

For \( \frac{d}{d-1} < \alpha < 2 \) and \( 2 \leq q, r \leq \infty \), let us set

\[
\beta(\alpha, q, r) = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}.
\]

We now recall from [10] the estimate

\[
\| \nabla |^{-\beta(\alpha, q, r)} D_{\sigma}^{\frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}} U(\cdot) f \|_{L^q_t L^p_x L^\infty_r} \lesssim \| f \|_{L^2}
\]

for \( \frac{2(d-2)}{d-1} < r < \infty \). Then by interpolating with mass conservation (the case \( q = \infty \), \( r = 2 \)) we have

\[
(2.1) \quad \| \nabla |^{-\beta(\alpha, q, r)} D_{\sigma}^{\frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}} U(\cdot) f \|_{L^q_t L^p_x L^\infty_r} \lesssim \| f \|_{L^2}
\]

holds for \( \frac{\hat{\gamma}}{2} < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r} \) whenever \( q, r \geq 2 \), \( r \neq \infty \) and \( \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \leq \frac{1}{q} < (d-1) \left( \frac{1}{2} - \frac{1}{r} \right) \). Using frequency decomposition, we rewrite (2.1). Let \( P_k, k \in \mathbb{Z} \), denote the Littlewood-Paley operator with symbol \( \chi(\xi/2^k) \in C^\infty_0 \) which is radial and supported in the annulus \( A_k = \{ 2^{k-1} < |\xi| < 2^{k+1} \} \) such that \( \sum_{k \in \mathbb{Z}} P_k = id \).

**Lemma 2.1.** Let \( \frac{d}{d-1} < \alpha < 2 \), \( q, r \geq 2 \), and \( r \neq \infty \), and let \( \beta(\alpha, q, r) = \frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q} \). If \( \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \leq \frac{1}{q} < (d-1) \left( \frac{1}{2} - \frac{1}{r} \right) \), then for \( \frac{\hat{\gamma}}{2} < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r} \)

\[
\| D_{\sigma}^{\frac{d}{2}} U(\cdot) f \|_{L^q_t L^p_x L^\infty_r} \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{2k\beta(\alpha, q, r)} \| P_k f \|_{L^2}^2 \right)^{1/2}.
\]

When initial data is localized in frequency, it is possible to improve angular regularity to wider range (for example, see proof of Lemma 4.1 [11] and [15]). However, it is not clear that such estimates can be used to get estimates without frequency localization.

If we consider interaction of two linear waves of different frequencies, then we can obtain an improved form of bilinear Strichartz estimate, which enjoys extra smoothing.

**Lemma 2.2.** Let \( l \in \mathbb{N} \). Then for \( \frac{\hat{\gamma}}{2} < (d-2)/4 \) there exists an \( \epsilon > 0 \) such that

\[
\left\| D_{\sigma}^{\frac{d}{2}} [U(\cdot) P_t f] \right\|_{L^q_t} \left\| D_{\sigma}^{\frac{d}{2}} [U(\cdot) P_0 g] \right\|_{L^p_t} \left\| L^2 dt |x|^{d-1} dl |x| \right\| \lesssim 2^{\varepsilon(\beta(\alpha, q, r) \pm \epsilon)} \| f \|_2 \| g \|_2,
\]

where
Proof. Let \((Y^m_n)\) be the orthonormal spherical harmonic functions of order \(n\). Using the spherical harmonic expansion of \(\tilde{f}\) and \(\tilde{g}\),

\[
\tilde{f}(\rho \sigma) = \sum_{n,m} a^n_m(\rho)Y^m_n(\sigma), \quad \tilde{g}(\rho \sigma) = \sum_{n',m'} b^{m'}_{n'}(\rho)Y^{m'}_{n'}(\sigma),
\]

we rewrite \(U(t)P_\ell f\) and \(U(t)P_0 g\) as

\[
U(t)P_\ell f(x) = r^{-\frac{d-2}{2}} \sum_{n,m} c_n \mathcal{T}_n^\ell(a^n_m)(t,r) Y^m_n(\sigma),
\]

\[
U(t)P_0 g(x) = r^{-\frac{d-2}{2}} \sum_{n',m'} c_{n'} \mathcal{T}_n^{0}(b^{m'}_{n'})(t,r) Y^{m'}_{n'}(\sigma),
\]

where

\[
\mathcal{T}_n^\ell(a^n_m)(t,r) = \int_0^\infty e^{int} J_\nu(\sqrt{\rho} \psi)(\rho/2^\ell) \frac{\partial}{\partial \rho} a^n_m(\rho) d\rho,
\]

and \(x = r \sigma, |c_n| = c\) independent of \(n, m, \nu(\nu) = n + (d - 2)/2\). See \([27]\) for detail.

By the spectral theory we have

\[
\langle D^\sigma_n Y^m_n, D^\sigma_n Y^{m'}_{n'} \rangle = (1 + n(n + d - 2))^{-1} \delta_{n,n'} \delta_{m,m'},
\]

where \(\langle u, v \rangle_\sigma = \int_{\mathbb{R}^{d-1}} \overline{u} v d\sigma\) and \(\delta_{n,n'}\) is the Kronecker delta. By this it follows that for \(j = 0, \ell\)

\[
\|U(t)P_\ell f\|_{L^2_\sigma}^2 = c^2 r^{-(d-2)} \sum_{n,m} |\mathcal{T}_n^\ell(a^n_m)(t,r)|^2,
\]

\[
\|D^\sigma_n U(t)P_\ell f\|_{L^2_\sigma}^2 = c^2 r^{-(d-2)} \sum_{n,m} (1 + n(n + d - 2)) \frac{d-2}{2} |\mathcal{T}_n^\ell(a^n_m)(t,r)|^2.
\]

And by the change of variables \(\rho \mapsto \rho^{\frac{1}{2}}\), we obtain

\[
\|D^\sigma_n U(t)P_\ell f\|_{L^2_\sigma}^2 \|U(t)P_0 g\|_{L^2_\sigma}^2 = c^4 r^{-(d-2)} \sum_{n,n',m,m'} (1 + n(n + d - 2)) \frac{d-2}{2} |\mathcal{T}_n^\ell(a^n_m) \overline{\mathcal{T}_{n'}^0(b^{m'}_{n'})}|^2
\]

\[
\lesssim r^{-(d-2)} \sum_{n,n',m,m'} (1 + n(n + d - 2)) \frac{d-2}{2} \left|\int_0^\infty \int_0^\infty e^{itn} \overline{F_n^m(\rho) C_n^{m'}(\rho')} d\rho d\rho' \right|^2
\]

\[
\lesssim r^{-(d-2)} \sum_{n,n',m,m'} (1 + n(n + d - 2)) \frac{d-2}{2} \left|\overline{F_n^m(\rho)} C_n^{m'}(\rho')\right|^2,
\]

where \(\overline{F_n^m(\rho)} = F_n^m(-\rho)\), and

\[
F_n^m(\rho) = \chi_{(0,\infty)}(\rho) J_\nu(\sqrt{\rho} \psi)(\rho) \frac{d-2(n+1)}{2^{\ell-1}} \psi(\rho^{\frac{1}{d-2}}) a^n_m(\rho^{\frac{1}{d-2}}),
\]

\[
G_n^m(\rho') = \chi_{(0,\infty)}(\rho') J_\nu(\sqrt{\rho} \psi)(\rho') \frac{d-2(n+1)}{2^{\ell-1}} \psi(\rho^{\frac{1}{d-2}}) b^n_{m'}(\rho'^{\frac{1}{d-2}}).
\]

Taking \(L^2\)-norm with respect to \(t\)-variable, from Plancherel’s theorem and Young’s convolution inequality, we get

\[
\left\|D^\sigma_n U(t)P_\ell f\right\|_{L^2_\sigma} \left\|U(t)P_0 g\right\|_{L^2_\sigma} \lesssim r^{-(d-2)} \left( \sum_{n,n',m,m'} (1 + n(n + d - 2)) \frac{d-2}{2} \left|\overline{F_n^m(\rho)} \right|_{L^2_\rho} \left|C_n^{m'}(\rho')\right|_{L^2_\rho'} \right)^{\frac{1}{2}}.
\]
We then take $L^2(r^{d-1}dr)$ on both sides of (2.3) to obtain
\[
\left\|D_{\sigma}^{d-2}U(t)P_t f \right\|_2^2 \left\|U(t)P_0 g \right\|_2^2 \leq \left( 1 + n \right)^{d-2} \sum_{n,n',m,m'} \int_0^\infty r^{-(d-2)} \left\| F_n^m \right\|_{L^2(d\rho)}^2 \left( \sup_r \left\| G_{n'}^m \right\|_{L^1(d\rho')}^2 \right) \cdot \left( \sup_r \left\| G_{n'}^m \right\|_{L^1(d\rho')}^2 \right)^{\frac{1}{2}},
\]
where
\[
A_n^m = \left( \int_0^\infty r^{-(d-2)} \left\| F_n^m \right\|_{L^2(d\rho)}^2 \left( \sup_r \left\| G_{n'}^m \right\|_{L^1(d\rho')}^2 \right) \right)^{\frac{1}{2}}, \quad B_{n'}^m = \left( \sup_r \left\| G_{n'}^m \right\|_{L^1(d\rho')}^2 \right)^{\frac{1}{2}}.
\]
Making the change of variables $\rho \mapsto \rho^\alpha$, we have for $A_n^m$ that
\[
[A_n^m]^2 = \int_0^\infty \left( \int_0^\infty \left| J_{\nu(n)}(r\rho) \right|^2 r^{-(d-2)} \left( \rho^{d-(\alpha-1)} \left| \psi(\rho/2^\ell) \right| \right) \left| a_n^m(\rho) \right|^2 \left( \sup_r \left\| G_{n'}^m \right\|_{L^1(d\rho')}^2 \right) \right) \left( \rho^{d-(\alpha-1)} \left( \psi(\rho/2^\ell) \right) \right)^{\frac{1}{2}} d\rho.
\]
From the Bessel function estimate (see p. 403 of [28]) and the Stirling’s formula it follows that
\[
\int_0^\infty |J_{\nu(n)}(r\rho)|^2 r^{-(d-2)} dr = \left( \frac{\rho/2}{\Gamma(d-2)} \right)^{\frac{1}{2}} \frac{1}{2 \Gamma(d-2)} \leq \rho^{d-(1+n)-(d-2)}.
\]
Thus we have
\[
[A_n^m]^2 \leq (1 + n)^{-(d-2)} 2^\ell (d-\alpha-1) \int |a_n^m(\rho)|^2 \rho^{d-1} d\rho.
\]
For $B_{n'}^m$, after change of variables, we estimate
\[
[B_{n'}^m]^2 \leq \sup_r \left( \int |J_{\nu(n')} (r\rho') \right|^2 r^{d-1} \left( \sup_r \left| \psi(\rho') \right| \left| b_{n'}^{m'}(\rho') \right|^2 \right) \left( \sup_r \left| \psi(\rho') \right| \left| b_{n'}^{m'}(\rho') \right|^2 \right)^{\frac{1}{2}} \left( \sup_r \left| \psi(\rho') \right| \left| b_{n'}^{m'}(\rho') \right|^2 \right)^{\frac{1}{2}} d\rho' \leq \sup_r \int_{\rho_0}^{\infty} \left| J_{\nu(n')} (r\rho') \right|^2 r^{d-1} d\rho' \leq \sup_r \int_{\rho_0}^{\infty} \left| J_{\nu(n')} (r\rho') \right|^2 r^{d-1} d\rho'.
\]
Since $\sup_{\rho'} \left( \int_{\rho_0}^{\infty} |J_{\nu(n)}(\rho')|^2 d\rho' \right) \leq 1$ (see (3.4) of [11]), we have
\[
[B_{n'}^m]^2 \leq \int |b_{n'}^{m'}(\rho')|^2 \rho^{d-1} d\rho'.
\]
Plugging (2.5), (2.6) into (2.4), we obtain
\[
\left\|D_{\sigma}^{d-2}U(t)P_t f \right\|_2^2 \left\|U(t)P_0 g \right\|_2^2 \leq 2^{\ell(d-\alpha-1)/2} \left( \sum_{n,n'} |a_n^m|^2 \rho^{d-1} d\rho \right)^{\frac{1}{2}} \left( \sum_{n',m'} |b_{n'}^{m'}|^2 \rho^{d-1} d\rho' \right)^{\frac{1}{2}} \leq 2^{\ell(d-\alpha-1)/2} \left\|f\right\|_2 \left\|g\right\|_2.
\]
On the other hand, repeating the above argument we have

\[ \|U(t)P_t f\|_{L^2_p}^2 \|D^{d-2}_s U(t)P_0 g\|_{L^2}^2 \leq r^{-(d-2)} \sum_{n,n',m,m'} (1 + n'(n' + d - 2)) \frac{d-2}{2} \|F_n^{m*} G_{n'}^{m'}(t)\|^2. \]

Hence by using Young’s convolution inequality again as in \((2.3)\), we get

\[ \left\| \|U(t)P_t f\|_{L^2_p} \|D^{d-2}_s U(t)P_0 g\|_{L^2} \right\|_{L^2(dt)} \leq r^{-(d-2)} \left( \sum_{n,n',m,m'} (1 + n'(n' + d - 2)) \frac{d-2}{2} \|F_n^{m*} \|_{L^1(dp)} \|G_{n'}^{m'}\|_{L^2(dp')} \right)^{\frac{1}{2}}. \]

And we also have

\[ \left\| \|U(t)P_t f\|_{L^2_p} \|D^{d-2}_s U(t)P_0 g\|_{L^2} \right\|_{L^2(v^{d-1}drdt)} \leq \sum_{n,n',m,m'} (1 + n)^{d-2} [\tilde{A}_n^m]^2 [\tilde{B}_{n'}^{m'}]^2, \]

where \(\tilde{A}_n^m = (\sup_r (r \|F_n^{m*}\|_{L^1(dp)}))^\frac{1}{2}\) and \(\tilde{B}_{n'}^{m'} = (\int_0^\infty r^{-(d-2)} \|G_{n'}^{m'}\|_{L^2(dp')} \, dr)^\frac{1}{2}\). Changing the role of \(F_n^{m*}\) and \(G_{n'}^{m'}\) in \((2.9)\) and \((2.10)\), one can easily show that

\[ [\tilde{A}_n^m]^2 \leq 2^f \int |a_n^m(\rho)|^2 \rho^{d-1} d\rho, \quad [\tilde{B}_{n'}^{m'}]^2 \leq (1 + n')^{-(d-2)} \int |b_{n'}^{m'}(\rho)|^2 \rho^{d-1} d\rho, \]

which implies

\[ \left\| \|U(t)P_t f\|_{L^2_p} \|D^{d-2}_s U(t)P_0 g\|_{L^2} \right\|_{L^2(v^{d-1}drdt)} \leq 2^{f/2} \|f\|_2 \|g\|_2. \]

Finally, interpolation between \((2.7)\) and \((2.9)\) gives the desired estimate. \(\square\)

In \(\[9\]\) a refined Strichartz estimate is shown for radial function. We extend it to the functions with angular regularity. This extension will play a crucial role in proving profile decomposition of angularly regular data. Here we combine the argument in \(\[9\] [10]\) with the spherical harmonic expansion.

For \(\alpha < 2\), we say the pair \((q,r)\) \(\alpha\)-admissible, if \(\frac{\alpha}{q} + \frac{d}{r} = \frac{d}{2}\) for \(2 \leq q, r \leq \infty\).

**Proposition 2.3.** Let \(\frac{d}{q-1} < \alpha < 2\), \(q > 2\) and \(r \neq \infty\). Then for each \(\alpha\)-admissible pair \((q,r)\) there exist \(\theta,p\) with \(\theta \in (0,1), p \in [1,2)\) such that for any \(\gamma < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r}\),

\[ \|D^\gamma U(\cdot)f\|_{L^q_t L^p_x L^2} \lesssim (\sup_k 2^{kd(\frac{1}{2} - \frac{1}{r}))} \|P_k f\|_p)^\theta \|f\|_2^{1-\theta}. \]

In order to show Proposition \(2.3\) we need the following lemma.

**Lemma 2.4.** Let \(d \geq 3\). Then for \(\gamma < (d-2)/4\)

\[ \|D^\gamma U(\cdot)f\|_{L^q_t L^p_x L^2} \lesssim \left( \sum_k (2^{k\beta(d,\gamma)} \|P_k f\|_2)^4 \right)^{\frac{1}{4}}. \]
Proof. By Littlewood-Paley decomposition we write

\[
\langle D^\delta_\sigma U(t) f, D^\delta_\sigma U(t) f \rangle = \sum_{j=-\infty}^{\infty} \sum_k \langle D^\delta_\sigma U(t) P_k f, D^\delta_\sigma U(t) P_{j+k} f \rangle.
\]

For (2.11) it is sufficient to show that for some \( \epsilon > 0 \)

\[
(2.12) \quad \| \sum_k \langle D^\delta_\sigma U(\cdot) P_k f, D^\delta_\sigma U(\cdot) P_{j+k} f \rangle \|_{L^2_t L^2_\sigma} \lesssim 2^{-|j|} \epsilon \left( \sum_k \| D^\delta_\sigma U(t) P_k f \|_{L^2_\sigma} \right)^4.
\]

We show the cases \(|j| \leq 3\) and \(|j| > 3\), separately. Let us first consider the case \(|j| \leq 3\). By the Hölder’s and the Cauchy-Schwarz inequalities, we have

\[
\left| \sum_k \langle D^\delta_\sigma U(t) P_k f, D^\delta_\sigma U(t) P_{j+k} f \rangle \right|^2 \lesssim \left( \sum_k \| D^\delta_\sigma U(t) P_k f \|_{L^2_\sigma} \right)^2 \left( \sum_k \| D^\delta_\sigma U(t) P_{j+k} f \|_{L^2_\sigma} \right)^2
\]

\[
\lesssim \left( \sum_k \| D^\delta_\sigma U(t) P_k f \|_{L^2_\sigma} \| D^\delta_\sigma U(t) P_{j+k} f \|_{L^2_\sigma} \right)^2 \lesssim \sum_{l=-\infty}^{\infty} \sum_k \| D^\delta_\sigma U(t) P_k f \|_{L^2_\sigma} \| D^\delta_\sigma U(t) P_{l+i} f \|_{L^2_\sigma}^2.
\]

Hence we get

\[
\text{LHS of (2.12)} \lesssim \sum_{l=-\infty}^{\infty} \sum_k \| D^\delta_\sigma U(t) P_k f \|_{L^2_\sigma} \| D^\delta_\sigma U(t) P_{l+i} f \|_{L^2_\sigma}^2.
\]

Lemma 2.2 and the Cauchy-Schwarz inequality imply (2.12) when \(|j| \leq 3\).

We now consider the case \(|j| > 3\). As previously, using the spherical harmonic expansion such that \( \hat{f} = \sum a_n^m Y_n^m \) and \( \hat{g} = \sum b_m^m Y_n^m \) and making the change of variables \( \rho \mapsto \rho' \), we get

\[
\langle D^\delta_\sigma [U(t) P_k f], D^\delta_\sigma [U(t) P_{j+k} g] \rangle = c^2 \alpha^{-2} 2^{-(d-2)} \sum_{n,m} \left( 1 + n(n+d-2) \right) \int_{\mathbb{R}^d} e^{it \rho' - \rho''} G_n^m (\rho') F_n^m (\rho') d\rho' d\rho''
\]

\[
= c^2 \alpha^{-2} 2^{-(d-2)} \sum_{n,m} \left( 1 + n(n+d-2) \right) \int \tilde{F}_n^m (t),
\]

where

\[
F_n^m (\rho) = \chi_{(0,\infty)} (\rho) J_{\nu(n)} (r \rho^{1/2}) \rho^{d-2(\alpha-1)/2} \psi (\rho^{1/2} / 2^k) a_n^m (\rho^{1/2}),
\]

\[
G_n^m (\rho') = \chi_{(0,\infty)} (\rho') J_{\nu(n)} (r' \rho'^{1/2}) (\rho')^{d-2(\alpha-1)/2} \psi (\rho'^{1/2} / 2^{j+k}) b_n^m (\rho'^{1/2}).
\]

Hence the Fourier support of \( \langle D^\delta_\sigma [U(t) P_k f], D^\delta_\sigma [U(t) P_{j+k} g] \rangle \) with respect to \( t \) is \( \{ \tau : 2^{\alpha(j+k-2)} \leq \tau \leq 2^{\alpha(j+k+2)} \} \). So the Fourier supports of \( \langle D^\delta_\sigma U(t) P_k f, D^\delta_\sigma U(t) P_{j+k} f \rangle \) with respect to \( t \) are boundedly overlapping. Then Plancherel’s theorem in \( t \) gives

\[
\text{LHS of (2.12)} = \| \sum_k \langle D^\delta_\sigma U(\cdot) P_k f, D^\delta_\sigma U(\cdot) P_{j+k} f \rangle \|_{L^2_t L^2_\sigma}^2 \lesssim \sum_k \| \langle D^\delta_\sigma U(t) P_k f, D^\delta_\sigma U(t) P_{l+i} f \rangle \|_{L^2_t L^2_\sigma}^2.
\]
Hence using Cauchy-Schwarz and Lemma 2.2 we see that
\[
\text{LHS of (2.12)} \lesssim \sum_k \left\| D^\sigma_\theta U(t) P_k f \right\|_{L^q_t L^r_x L^\alpha_{\nu}}^2 \left\| D^\sigma_\theta U(t) P_j+k f \right\|_{L^q_t L^r_x L^\alpha_{\nu}}^2 \\
\lesssim \sum_k 2^{-|j|c_2k^{\beta(\alpha,4,4)} \| P_k f \|_2^2 2^{2(j+k)\beta(\alpha,4,4)} \| P_j+k f \|_2^2} \lesssim 2^{-|j|c_2 k^{\beta(\alpha,4,4)} \| P_k f \|_2}.
\]
For the last inequality we used the Cauchy-Schwarz inequality. This completes the proof. □

Now we are ready to prove Proposition 2.3.

**Proof.** To begin with we note that $\beta(\alpha, q, r) = 0$ because $(q, r)$ is $\alpha$-admissible. From Lemma 2.1 we have
\[
\left\| D^\sigma_\theta U(\cdot) f \right\|_{L^q_t L^r_x L^\alpha_{\nu}} \lesssim \left( \sum_k \left\| \hat{P}_k f \right\|_2^{2/q} \right)^{1/2}
\]
for any $\alpha$-admissible pair $(q, r)$. Then (2.10) follows from interpolation of (2.13) and the following two estimates: for some $p_*, q_*$ with $p_* < 2 < q_*$,
\[
\left\| D^\sigma_\theta U(\cdot) f \right\|_{L^q_t L^r_x L^\alpha_{\nu}} \lesssim \left( \sum_k \left\| \hat{P}_k f \right\|_2^{q_*/q_0} \right)^{1/q_0},
\]
\[
\left\| D^\sigma_\theta U(\cdot) f \right\|_{L^q_t L^r_x L^\alpha_{\nu}} \lesssim \sum_k 2^{kd(\frac{1}{2} - \frac{1}{r_0})} \left\| \hat{P}_k f \right\|_{p_*}.
\]
In fact, the interpolation among (2.13), (2.14) and (2.15) gives
\[
\left\| D^\sigma_\theta U(\cdot) f \right\|_{L^q_t L^r_x L^\alpha_{\nu}} \lesssim \left( \sum_k \left( 2^{kd(\frac{1}{2} - \frac{1}{r_0})} \left\| \hat{P}_k f \right\|_{p_0} \right)^{q_0} \right)^{1/q_0}
\]
for $(1/q_0, 1/p_0)$ contained in the triangle with the vertices $(1/2, 1/2)$, $(1/p_*$, 1) and $(1/2, 1/q_*)$. So, there exist $q_0, p_0, p_* < 2 < q_0$, for which (2.16) holds. Hence,
\[
\left\| D^\sigma_\theta U(\cdot) f \right\|_{L^q_t L^r_x L^\alpha_{\nu}} \lesssim \left( \left( \sup_k 2^{kd(\frac{1}{2} - \frac{1}{r_0})} \left\| \hat{P}_k f \right\|_{p_0} \right)^{q_0} \sum_k \left( 2^{kd(\frac{1}{2} - \frac{1}{r_0})} \left\| \hat{P}_k f \right\|_{p_0} \right)^{q_0} \right)^{1/q_0}
\]
\[
\lesssim \left( \left( \sup_k 2^{kd(\frac{1}{2} - \frac{1}{r_0})} \left\| \hat{P}_k f \right\|_{p_0} \right)^{(q_0-2)/q_0} \left( \sum_k \left\| \hat{P}_k f \right\|_2^{2/q_0} \right)^{1/q_0}
\]
\[
\lesssim \left( \left( \sup_k 2^{kd(\frac{1}{2} - \frac{1}{r_0})} \left\| \hat{P}_k f \right\|_{p_0} \right)^{(q_0-2)/q_0} \left\| \hat{P}_k f \right\|_2^{2/q_0} \right).
\]
We need only to set $p = p_0$ and $\theta = 1 - 2/q_0$ to get (2.10).

Now we show (2.14) and (2.15). First we consider the inequality (2.14). Let $(q, r)$ be an $\alpha$-admissible pair with $2 < q \ll \infty$. Since $\hat{\gamma} < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r}$, $\hat{\gamma}$ is less than $\frac{d-1}{2} - \frac{1}{q_0} - \frac{d-1}{r_0}$ on a small neighborhood of $q, r$. One can easily check that $\frac{d}{2} - \frac{d}{6} - \frac{\alpha}{q_0} > 0$. Hence there exist $2 < q_0, r_0 < \infty$ such that $\frac{d}{2} - \frac{d}{6} - \frac{\alpha}{q_0} < 0, \frac{1}{q_0} \leq (d-1)(\frac{1}{2} - \frac{1}{r_0})$, $\hat{\gamma} < \frac{d-1}{2} - \frac{1}{q_0} - \frac{d-1}{r_0}$ and $(\frac{1}{q}, \frac{1}{r}) = (\frac{\theta}{q_0}, \frac{\theta}{r_0}) + (1-\theta)(\frac{1}{q_0}, \frac{1}{r_0})$, $0 < \theta < 1$. For this pair $(q_0, r_0)$ we have $\left\| D^\sigma_\theta U(\cdot) P_0 f \right\|_{L^{q_0}_{\nu} L^{r_0}_x L^\alpha_{\nu}} \lesssim \left\| \hat{P}_0 f \right\|_2$ from (2.1). Then interpolating this with the estimate
\[
\left\| D^\sigma_\theta U(\cdot) P_0 f \right\|_{L^{q_0}_{\nu} L^{r_0}_x L^\alpha_{\nu}} \lesssim \left\| D^\sigma_0 U(\cdot) P_0 f \right\|_{L^{\sigma}_{\nu} L^{\sigma}_x L^\alpha_{\nu}} \lesssim \left\| \hat{P}_0 f \right\|_2,
\]
which follows from \( \|D_\sigma^\frac{d-2q}{4} P_0 f\|_{L_t^q L_x^r} \lesssim \|P_0 f\|_2 \) and the trivial estimate \( \|U(\cdot)P_0 f\|_{L_t^\infty L_x^\infty} \lesssim \|P_0 f\|_1 \), implies
\[
\|D_\sigma^\frac{d-2q}{4} U(\cdot) P_0 f\|_{L_t^q L_x^r, L_\sigma^2} \lesssim \|P_0 f\|_{p_*}
\]
for some \( p_* \) with \( 1 < p_* < 2 \). By rescaling it follows \( \|D_\sigma^\frac{d-2q}{4} U(\cdot) P_k f\|_{L_t^q L_x^r, L_\sigma^2} \lesssim 2^{kd(\frac{1}{2} - \frac{1}{p_*})} \|P_k f\|_{p_*} \). Now Minkowski’s inequalities give
\[
\|D_\sigma^\frac{d-2q}{4} U(\cdot) f\|_{L_t^q L_x^r, L_\sigma^2} \leq \sum_k \|D_\sigma^\frac{d-2q}{4} U(\cdot) f\|_{L_t^q L_x^r, L_\sigma^2} \lesssim \sum_k 2^{kd(\frac{1}{2} - \frac{1}{p_*})} \|P_k f\|_{p_*}.
\]

Finally, the proof of (2.14) is easy. It can be done by interpolating (2.11) with the estimates in Lemma 2.3 for \((q, r)\) with \( \frac{d}{\alpha} \left( \frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q} \leq (d - 1) \left( \frac{1}{2} - \frac{1}{r} \right), \ 2 \leq q, r \leq \infty \). This completes the proof. \( \square \)

### 3. Linear profile decomposition

In this section we prove Theorem 1.1. Throughout this section we assume that \( \frac{d}{d-1} < \alpha < 2 \), the pair \((q, r)\) is \( \alpha \)-admissible with \( \frac{d}{\alpha} \left( \frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q} < (d - 1) \left( \frac{1}{2} - \frac{1}{r} \right) \), and \( \hat{\gamma} < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{r} \).

#### 3.1. Preliminary decomposition

Thanks to the refined Strichartz estimate (2.11), we can extract frequencies and scaling parameters to get a preliminary decomposition.

**Proposition 3.1.** Let \( (u_n)_{n \geq 1} \) be a sequence of complex valued functions satisfying \( \|u_n\|_{L_t^\gamma H_x^2} \leq 1 \) for some \( \gamma \geq 0 \). Then for any \( \delta > 0 \), there exists \( N = N(\delta), \rho_n^0 \in (0, \infty) \) and \( (f_n^j)_{1 \leq j \leq N, n \geq 1} \subset L_t^2 H_x^2 \) such that
\[
u_n = \sum_{j=1}^N f_n^j + q_n^N
\]
and the following properties hold:

1. There exists a compact set \( K = K(N) \subset \{ \xi : R_1 < |\xi| < R_2 \} \) satisfying that
\[
(\rho_n^j)^{d/2} |D_\sigma^\frac{d-2q}{4} f_n^j(\rho_n^j, \xi)| \leq C_{\delta, \chi, K}(\xi) \text{ for every } 1 \leq j \leq N,
\]

2. \( \lim_{n \to \infty} (\frac{\rho_n^j}{\rho_n^{j+1}})^{d/2} (\rho_n^j)^{d/2} = 1 \), if \( j \neq k \),

3. \( \lim_{n \to \infty} \|U(\cdot)q_n^N\|_{L_t^q L_x^r, L_\sigma^\gamma} \leq \delta \) for any \( N \geq 1 \),

4. \( \lim_{n \to \infty} (\|u_n\|_{L_t^\gamma H_x^2} - (\sum_{j=1}^N \|f_n^j\|_{L_t^\gamma H_x^2} + \|q_n^N\|_{L_t^\gamma H_x^2})) = 0 \).

**Proof.** The argument here is similar to the one for the radial case [9]. To begin with, let us set
\[
\nu_n := D_\sigma^\frac{d-2q}{4} u_n.
\]

We may assume that \( \|U(\cdot)\nu_n\|_{L_t^\gamma L_x^\alpha, L_\sigma^\gamma} > \delta \) for all \( n \geq 1 \), otherwise there is nothing to prove. For \( \rho_n^1 > 0 \) let us set \( A_n^1 = \{ \xi : \frac{\rho_n^1}{\rho_n^j} < |\xi| < \rho_n^1 \} \) and \( \nu_n^1 = \nu_n \chi_{A_n^1} \). By the refined Strichartz estimates (Proposition 2.3), there exists \( \rho_n^1 \) such that
\[
(3.1) \quad c_1 (\rho_n^1)^{d(\frac{1}{p} - \frac{1}{2})} \|\nu_n^1\|_p \leq \|\nu_n^1\|_p.
\]
for some positive constant $c_1$, $p \in (1, 2)$, $\theta \in (0, 1)$ as stated in Proposition 2.3. And since $\|v_n^1\|_{L^2} \leq \|v_n\|_{L^2} \leq 1$, $\int_{\{|v^1_n| > \lambda\}} |v^1_n|^p d\xi = \int_{|v_n^1| > \lambda} (\lambda^{2-p} |v^1_n|^p) \lambda^{p-2} d\xi \leq \lambda^{p-2}$ for any $\lambda > 0$. Thus we have

$$\left( \int_{\{|v^1_n| > \lambda\}} |v^1_n|^p d\xi \right)^{\frac{1}{p}} \leq \lambda^{1 - \frac{2}{p}}.$$

Now let us set $\lambda = \left( c_1 / 2^p \right)^{\frac{1}{p-2}} (\rho_n^1)^{-\frac{d}{p-2}} \delta^\frac{1}{p-2}$. Then (3.1) gives

$$\left( \int_{\{|v^1_n| < \lambda\}} |v^1_n|^p d\xi \right)^{\frac{1}{p}} \leq \lambda^{2 - \frac{p}{p-2}} \left( \int_{\{|v^1_n| < \lambda\}} |v^1_n|^2 d\xi \right)^{\frac{1}{2}},$$

where $\omega_d$ is the measure of unit sphere. This implies

$$\left( \int_{\{|v^1_n| < \lambda\}} |v^1_n|^2 d\xi \right)^{\frac{1}{2}} \leq \left( \int_{\{|v^1_n| < \lambda\}} |v^1_n|^p d\xi \right)^{\frac{1}{p}} \left( c_1 \omega_d^{\frac{1}{2} - \frac{1}{p}} \right).$$

Now define $G^1_n(\psi)(\xi)$ by $(\rho^1_n)^{d/2} \psi(\rho_1 n \xi)$ for measurable function $\psi$. Then by letting $\tilde{w}_n^1 = \tilde{w}_n^1 \chi_{\{|v^1_n| < \lambda\}}$, we get $\|w^1_n\|_2 \geq \frac{1}{2} c_1 \delta^\frac{1}{p-2}$ and $|G^1_n(\tilde{w}_n^1)(\xi)| = |(\rho^1_n)^{\frac{d}{p}} \tilde{w}_n^1(\rho^1 n \xi)| \leq C_0 \chi_{A_{1/2, 1}}(\xi)$. Here $A_{R_1, R_2}$ is the annulus $\{\xi : R_1 < |\xi| < R_2\}$. We can repeat the above process with $v_n - w^1_n$ replacing $v_n$. After $N' = \tilde{N}'(\delta)$ steps, we get $(w^1_n)_{1 \leq j \leq N'}$ and $(\rho^1_n)$ such that

$$v_n = \sum_{j=1}^{N'} w^1_j + q_n^{N'}, \quad \|v_n\|_{L^2} \leq \sum_{j=1}^{N'} \|w^1_j\|_{L^2} + \|q_n^{N'}\|_{L^2}, \quad \|U(t)q_n^{N'}\|_{L^2} \leq \delta.$$

The second identity follows from disjointness of Fourier supports of $w^1_j$ and $q_n^{N'}$. In the sequel, we say $\rho^1_n$ is orthogonal to $\rho^k_n$ if $\limsup_{n \to \infty} (\rho^k_n / \rho^1_n) = \infty$. Define $f^1_n$ to be the sum of those $w_n^j$ whose $\rho^1_n$ are not orthogonal to $\rho^1_n$. Take least $j_0 \in [2, N']$ such that $\rho^1_{j_0}$ is orthogonal to $\rho^1_n$ and define $\tilde{f}^1_n$ to be the sum of $w^1_j$ whose $\rho^1_n$ are orthogonal to $\rho^1_n$ but not to $\rho^0_{j_0}$. After $N$ step for $N \leq N'$, we have $(\tilde{f}^1_n)_{1 \leq j \leq N}$. Let us set

$$f^1_n = D^{-\gamma} f^1_n, \quad q_n^{N'} = D^{-\gamma} q_n^{N'}.$$

Then $\{f^1_n\}$ satisfy the properties 2) and 4) of Proposition 3.1 because the disjointness of the Fourier supports are disjoint and the third inequality of (3.2) gives Property 3).

Now it remains to check Property 1). We only consider $f^1_n$, as other cases can be treated similarly. Since $w^1_n$ collected in $f^1_n$ has $\rho^1_n$ which is not orthogonal to $\rho^1_n$, we have

$$\limsup_{n \to \infty} \left( \frac{\rho^1_n}{\rho^1_n} + \frac{\rho^1_n}{\rho^1_n} \right) < \infty.$$

Moreover, by construction, we have $|G^1_n(\tilde{w}_n^1)| \leq C_0 \chi_{A_{1/2, 1}}$. Here $G^1_n(\psi)(\xi) = (\rho^1_n)^{\frac{d}{2}} \psi(\rho^1_n \xi)$. From $G^1_n(\tilde{w}_n^1) = G^1_n(G^1_n)^{-1} G^1_n(\tilde{w}_n^1)$ and $G^1_n(G^1_n)^{-1} \psi(\xi) = \left( \frac{\rho^1_n}{\rho^1_n} \right)^{\frac{d}{2}} \psi \left( \frac{\rho^1_n}{\rho^1_n} \xi \right)$, and the non-orthogonality [3.3], there exist $R_1$ and $R_2$ with $0 < R_1 < R_2$ such that $|G^1_n(\tilde{w}_n^1)| \leq C_0 \chi_{A_{R_1, R_2}}$ for all $w^1_n$ collected in $f^1_n$. This completes the proof of Proposition 3.1. □
In the next step, we further decompose \( \{f_\ell^2\} \) into time translated profiles.

**Proposition 3.2.** Suppose that \( (f_n) \subset L_p^2 H^\gamma_\alpha \) for some \( \gamma \geq 0 \) with \( (\rho_n)^{d/2} |D^\gamma \tilde{F}_n| |\rho_n \xi| \leq \tilde{F}'(\xi) \), where \( \tilde{F} \in L^\infty(K) \) for some compact set \( K \subset A = \{ \xi : 0 < R_1 < |\xi| < R_2 \} \). Then there exist a family \( (\rho_n^\ell)_{\ell \geq 1} \subset R \) and a sequence \( (\phi^\ell)_{\ell \geq 1} \subset L_p^2 H^\gamma_\alpha \) satisfying the following:

1) For \( \ell \neq \ell' \)

\[
\limsup_{n \to \infty} |s_n^\ell - s_n^{\ell'}| = \infty.
\]

2) For every \( M \geq 1 \), there exists \( (e_n^M) \subset L_p^2 H^\gamma_\alpha \) such that

\[
f_n(x) = \sum_{\ell = 1}^{M} (\rho_n)^{d/2} (U(s_n^\ell)\phi^\ell)(\rho_n x) + e_n^M(x), \quad \limsup_{n \to \infty} \|U(\cdot) e_n^M\|_{L_p^2 H^\gamma_\alpha} = 0.
\]

3) For any \( M \geq 1 \),

\[
\limsup_{n \to \infty} \left( \|f_n\|_{L_p^2 H^\gamma_\alpha}^2 - \sum_{\ell = 1}^{M} \|\phi^\ell\|_{L_p^2 H^\gamma_\alpha}^2 + \|e_n^M\|_{L_p^2 H^\gamma_\alpha}^2 \right) = 0.
\]

**Proof.** Denote by \( \mathcal{F} \) the collection of functions \( (F_n)_{n \geq 1} \) which are given by \( \tilde{F}_n(\xi) = (\rho_n)^{d/2} f_n(\rho_n \xi) \), and define

\[
\mathcal{W}(\mathcal{F}) = \{ \text{weak-limit } U(-s_n^1)F_n(x) \text{ in } L_p^2 H^\gamma_\alpha : s_n^1 \in R \}, \quad \mu(\mathcal{F}) = \sup_{\phi \in \mathcal{W}(\mathcal{F})} \|\phi\|_{L_p^2 H^\gamma_\alpha}.
\]

Here by the weak limit \( \psi_n \) in \( L_p^2 H^\gamma_\alpha \) we mean that \( \langle \psi_n - \psi, \phi \rangle_{L_p^2 H^\gamma_\alpha} \equiv \langle D_\gamma^\phi(\psi_n - \psi), D_\gamma^\phi \phi \rangle_{L^2} \to 0 \) as \( n \to \infty \) for any \( \phi \in L_p^2 H^\gamma_\alpha \). Then \( \mu(\mathcal{F}) \leq \limsup_{n \to \infty} \|F_n\|_{L_p^2 H^\gamma_\alpha} \).

We may assume that \( \mu(\mathcal{F}) > 0 \), otherwise we are done by using a forthcoming inequality \([3.4]\). Let us choose subsequences \( (F_n)_{n=1}^\infty \), \( (s_n^1) \) and \( \phi^1 \) such that \( U(-s_n^1)F_n(x) \to \phi^1 \) as \( n \to \infty \) and \( \|\phi^1\|_{L_p^2 H^\gamma_\alpha} \geq \frac{1}{2} \mu(\mathcal{F}) \). Let \( F_n^1 = F_n - U(s_n^1)\phi^1(x) \) and \( \mathcal{F}^1 = \{ F_n^1 \}_{n=1}^\infty \). Then

\[
\limsup_{n \to \infty} \|F_n^1\|_{L_p^2 H^\gamma_\alpha}^2 = \limsup_{n \to \infty} \|F_n - U(s_n^1)\phi^1(x)\|_{L_p^2 H^\gamma_\alpha}^2
\]

\[
= \limsup_{n \to \infty} \|U(-s_n^1)F_n(x) - \phi^1\|_{L_p^2 H^\gamma_\alpha}^2
\]

\[
= \limsup_{n \to \infty} \left( \|F_n - (F_n^1)\|_{L_p^2 H^\gamma_\alpha}^2 - \langle F_n, F_n^1 \rangle_{L_p^2 H^\gamma_\alpha} \right)
\]

\[
= \limsup_{n \to \infty} \|F_n\|_{L_p^2 H^\gamma_\alpha}^2 - \|\phi^1\|_{L_p^2 H^\gamma_\alpha}^2.
\]

Repeat the process with \( F_n^1 \) to get \( s_n^2, \phi^2, F_n^2 \) and so on. By taking a diagonal sequence we may write

\[
F_n(x) = \sum_{\ell = 1}^{M} U(s_n^\ell)\phi^\ell(x) + F_n^M,
\]

and we have

\[
\limsup_{n \to \infty} \|F_n\|_{L_p^2 H^\gamma_\alpha}^2 = \sum_{\ell = 1}^{M} \|\phi^\ell\|_{L_p^2 H^\gamma_\alpha}^2 + \limsup_{n \to \infty} \|F_n^M\|_{L_p^2 H^\gamma_\alpha}^2.
\]

Thus \( \sum_{\ell = 1}^{M} \|\phi^\ell\|_{L_p^2 H^\gamma_\alpha}^2 \) converges. Hence we obtain \( \limsup_{\ell \to \infty} \|\phi^\ell\|_{L_p^2 H^\gamma_\alpha} = 0 \) and by \( \mu(\mathcal{F}^M) \leq 2\|\phi^{M+1}\|_{L_p^2 H^\gamma_\alpha} \) we get \( \limsup_{M \to \infty} \mu(\mathcal{F}^M) = 0 \).
Now we define $e_n^M$ by setting 

$$\rho_n^{\frac{d}{2}} e_n^M(\rho_n \xi) = \hat{F}_n^M.$$ 

Then we are left to show

$$\limsup_{n \to \infty} \|U(\cdot)e_n^M\|_{L_t^q L_x^p H^\gamma_\rho} \lesssim \mu(F^M)$$ 

for some $\theta$ with $0 < \theta < 1$. By construction, we may assume that $D_\theta^\gamma 1_{t \leq M}$ has common compact support $K$. Invoking that the pair $(q, r)$ is admissible, we get

$$\|U(\cdot)e_n^M\|_{L_t^q L_x^p H^\gamma_\rho} \leq (\|U(\cdot)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}} + d - 2) \frac{t}{q(6-d)} \frac{t}{(6-q) \frac{d}{3} - d - 2} \frac{1}{\frac{q(6-q)}{q(6-d)}}$$

$$\lesssim \|U(\cdot)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}} + d - 2 \frac{1}{\frac{q(6-q)}{q(6-d)}}$$

for some $(\tilde{q}, \tilde{r})$ satisfying $\frac{2}{\tilde{q}} - \frac{2}{\tilde{r}} - \frac{2}{q} > 0$, $\frac{1}{\tilde{q}} \leq (d - 1)(\frac{1}{q} - \frac{1}{r})$ and $(\frac{1}{\tilde{q}}, \frac{1}{\tilde{r}}) = (\bar{\theta}(\frac{1}{q}, \frac{1}{r}) + (1 - \bar{\theta})(\frac{1}{q}, \frac{1}{r}))$ for some $0 < \bar{\theta} < 1$. Concerning the first factor, from Lemma 2.1, we have

$$\|U(\cdot)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}} \lesssim R_1 \frac{d}{q(2d-a)} \frac{d}{q(6-d)} \frac{1}{\frac{q(6-q)}{q(6-d)}} \{F_n^M\} \|H^\gamma_\rho \lesssim R_1 \frac{d}{q(2d-a)} \frac{1}{\frac{q(6-q)}{q(6-d)}} ,$$

and Lemma 2.4 gives

$$\|U(\cdot)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}} \lesssim R_2^{\alpha(4,4)} \{F_n^M\} \|H^\gamma_\rho \lesssim R_2^{\alpha(4,4)} .$$

Thus for (3.4) it suffices to show $\limsup_{n \to \infty} \|U(t)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}_\rho} \lesssim \mu(F^M)$. For this we may assume that there exists $\delta > 0$ such that

$$\limsup_{M \to \infty} \|U(t)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}_\rho} \geq \delta .$$

Let $(s_n^M, \rho_n^M)$ be a pair such that such that $\frac{1}{2} \|U(t)F_n^M\|_{L_t^q L_x^p H^{\gamma_\rho}_\rho} \leq \|U(s_n^M)F_n^M(\rho_n^M)\|_{H^{\gamma_\rho}_\rho}$. Then we show that $(\rho_n^M)$ is uniformly bounded. Let us first observe that for any $\rho_1, \rho_2 \in \mathbb{R}$

$$\|D_{\rho}^\gamma U(t)(F_n^M(\rho_2)) - D_{\rho}^\gamma U(t)(F_n^M(\rho_1))\|_{L_t^q L_x^p H^{\gamma_\rho}_\rho} \leq \|D_{\rho}^\gamma U(t)(F_n^M(\rho_1))\|_{L_t^q L_x^p H^{\gamma_\rho}_\rho} \lesssim \sup_x |\nabla(D_{\rho}^\gamma U(t)(F_n^M))(x)|\|\rho_2 - \rho_1\| \lesssim \int |\xi| |\xi|^{\alpha(4,4)} D_{\rho}^\gamma F_n^M(\xi) d\xi |\rho_2 - \rho_1| \lesssim |\rho_2 - \rho_1| .$$

From this, we deduce that $\|U(s_n^M)F_n^M(\rho)\|_{H^{\gamma_\rho}_\rho} \geq \frac{\delta}{2}$ if $|\rho - \rho_n^M| \leq c \frac{\delta}{2}$ for some small constant $c > 0$. Taking $L^2_\rho$-norm on the set $\{ |\rho_n^M| - c \frac{\delta}{2} < |\rho| < |\rho_n^M| + c \frac{\delta}{2} \}$, we have $\frac{\delta}{2} |\rho_n^M|^{-\frac{1}{2}} \leq \|F_n^M\|_2 \leq 1$, which implies the uniform boundedness of $(\rho_n^M)$.

Since $(\rho_n^M)$ is uniformly bounded, there exists $\rho_0^M$ such that $\rho_n^M \to \rho_0^M$ as $n \to \infty$, after taking a subsequence if necessary. Then for large $n$, we have

$$\|U(s_n^M)(F_n^M(\rho_0^M))\|_{H^{\gamma_\rho}_\rho} \geq \frac{1}{2} \|U(s_n^M)(F_n^M(\rho_n^M))\|_{H^{\gamma_\rho}_\rho} .$$

Let us choose $\sigma_n^M \in S^{d-1}$ such that $|D_{\sigma}^\gamma U(s_n^M)(F_n^M(\rho_0^M \sigma_n^M))| \geq \frac{3}{4} \|D_{\sigma}^\gamma U(s_n^M)(F_n^M(\rho_0^M \sigma_n^M))\|_{L_t^q L_x^p H^{\gamma_\rho}_\rho}$. Since $S^{d-1}$ is compact, $\sigma_n^M \to \sigma_0^M$ as $n \to \infty$ for some $\sigma_0^M \in S^{d-1}$. Then for large $n$, we have

$$\|U(s_n^M)(F_n^M(\rho_0^M \sigma_0^M))\|_{H^{\gamma_\rho}_\rho} \geq \frac{1}{2} \|U(s_n^M)(F_n^M(\rho_0^M \sigma_0^M))\|_{H^{\gamma_\rho}_\rho} .$$
\[ |D^2 U(s_n^M)(F_n^M)(\rho_0^M \sigma_0^M)| \geq \frac{1}{2} |D^2 U(s_n^M)(F_n^M)(\rho_0^M \sigma_0^M)|, \]

Set \( \psi \in C_0^\infty(\mathbb{R}^d) \) be such that \( \psi = 1 \)
on \( K \) and \( \psi^M \) be a Schwartz function such that \( \widehat{D^2 \psi^M} = \psi \delta_0 \), where \( \delta_0 \) is Dirac-delta measure. Then we have
\[
\limsup_{n \to \infty} \|U(t)F_n^M\|_{L^\infty_t L^\infty_x H^s_d} \lesssim \limsup_{n \to \infty} |D^2 U(s_n^M)(F_n^M)(\rho_0^M \sigma_0^M)| \\
\lesssim \limsup_{n \to \infty} \|\langle U(s_n^M)(F_n^M)(y), \psi^M \rangle\|_{L^2_t H^s_d} \leq \mu(F^M) \|\psi^M\|_{L^2_t H^s_d} \lesssim \mu(F^M).
\]

This completes the proof of Proposition 3.2.

Now we are ready to prove Theorem 1.1.

3.2. Proof of Theorem 1.1. We begin with a preliminary decomposition. From Propositions 3.1 and 3.2, we have
\[
(3.5) \quad u_n = \sum_{j=1}^N \sum_{\ell=1}^{M_j} \Phi_{n,j}^{\ell,j} + \omega_{n,M_1,\ldots,M_N},
\]
where
\[
\Phi_{n,j}^{\ell,j} = U(t\ell,j)(\rho_{n,j}^{-d/2} \phi^{\ell,j} (\cdot/h_n^j)),
\]
\[
(h_n^j, t_n^j) = ((\rho_{n,j}^{-1}), (\rho_{n,j}^{-\alpha}s_n^j), \quad \omega_{n,M_1,\ldots,M_N} = \sum_{j=1}^N c_n^{j,M_j} + q_n^N.
\]
Then we have

1. the orthogonality of parameter family \((h_n^j, t_n^j)\) (the property (2) in Theorem 1.1),
2. the asymptotic orthogonality, i.e.
\[
\|u_n\|_{L^2_t H^s_d}^2 = \sum_{j=1}^N \sum_{\ell=1}^{M_j} \|\phi^{\ell,j}\|_{L^2_t H^s_d}^2 + \|\omega_{n,M_1,\ldots,M_N}\|_{L^2_t H^s_d}^2 + o_n(1)
\]
and
\[
\|\omega_{n,M_1,\ldots,M_N}\|_{L^2_t H^s_d}^2 = \sum_{j=1}^N \|c_n^{j,M_j}\|_{L^2_t H^s_d}^2 + \|q_n^N\|_{L^2_t H^s_d}^2.
\]

We will show that \( U(t) \omega_{n,M_1,\ldots,M_N} \) converges to zero in a Strichartz norm, i.e.
\[
(3.6) \quad \limsup_{n \to \infty} \|U(t) \omega_{n,M_1,\ldots,M_N}\|_{L^q_t L^r_x H^s_d} \to 0 \quad \text{as} \quad \min\{N, M_1, \ldots, M_N\} \to \infty,
\]
where \((q, r)\) is an \( \alpha \)-admissible pair for \( \frac{d}{d-1} < \alpha < 2 \). We enumerate the pair \((\ell, j)\) by an order function \( n \) satisfying
\[
n(\ell, j) < n(\ell', k) \quad \text{if} \quad \ell + j < \ell' + k \quad \text{or} \quad \ell + j = \ell' + k \quad \text{and} \quad j < k.
\]
After relabeling, we get
\[
u_n = \sum_{1 \leq j \leq l} U(t_j^l)(\rho_j^l)^{-d/2} \phi^l (\cdot/h_n^l)(x) + \omega_l^l
\]
where \( \omega_l^l = \omega_{M_1,\ldots,M_N} \) with \( l = \sum_{j=1}^N M_j \). Then the proof is completed by (3.6).

Now let us prove (3.5). Given \( \varepsilon > 0 \), we take a positive number \( \Lambda \) such that for \( N \geq \Lambda \),
\[
\limsup_{n \to \infty} \|U(t) q_n^N\|_{L^q_t L^r_x H^s_d} \leq \varepsilon/3.
\]
Then for \( N \geq \Lambda \), we can find \( \Lambda_N \) such that whenever
\(M_j \geq \Lambda N, \limsup_{n \to \infty} \|U(t) \epsilon_n^{j,M_j} \|_{L^q_t L^r_x H^\gamma + \frac{\gamma}{2}} \leq \frac{\varepsilon}{3N}\) for \(1 \leq j \leq N\). Now we rewrite \(\omega^{N,M_1,\ldots,M_N}_n\) by
\[
\omega^{N,M_1,\ldots,M_N}_n = q_n^M + \sum_{1 \leq j \leq N} \epsilon_n^{j,M_j \vee \Lambda N} + R_n^{N,M_1,\ldots,M_n},
\]
where \(M_j \vee \Lambda N\) denotes \(\max\{M_j, \Lambda N\}\) and
\[
R_n^{N,M_1,\ldots,M_N} = \sum_{1 \leq j \leq N} (\epsilon_n^{j,M_j} - \epsilon_n^{j,\Lambda N}) = \sum_{1 \leq j \leq N} \sum_{M_j < \ell < \Lambda N} \Phi_{\ell,j}^n.
\]
Then we have
\[
\lim_{n \to \infty} \|U(t) \omega^{N,M_1,\ldots,M_N}_n \|_{L^q_t L^r_x H^\gamma + \frac{\gamma}{2}} \leq \frac{2\varepsilon}{3} + \lim_{n \to \infty} \|U(t) R_n^{N,M_1,\ldots,M_N} \|_{L^q_t L^r_x H^\gamma + \frac{\gamma}{2}}.
\]
In order to handle the last term, we need the following lemma.

**Lemma 3.3.** For every \(N, M_1, \ldots, M_N\), we have
\[
\limsup_{n \to \infty} \bigg\| \sum_{j=1}^N \sum_{\ell=1}^{M_j} U(t) \Phi_{n}^{\ell,j} \bigg\|_{L^q_t L^r_x H^\gamma + \frac{\gamma}{2}}^2 \leq \sum_{j=1}^N \sum_{\ell=1}^{M_j} \limsup_{n \to \infty} \bigg\| U(t) \Phi_{n}^{\ell,j} \bigg\|_{L^q_t L^r_x H^\gamma + \frac{\gamma}{2}}^2.
\]

**Proof of Lemma 3.3** It suffices to show that for \((j, \ell) \neq (k, \ell')\),
\[
\lim_{n \to \infty} \|D_{\sigma}^{\gamma + \frac{\gamma}{2}} U(t) \Phi_{n}^{\ell,j} D_{\sigma}^{\gamma + \frac{\gamma}{2}} U(t) \Phi_{n}^{\ell',k} \|_{L^q_t L^r_x L^2_x} = 0.
\]

When \((j, \ell) \neq (k, \ell')\), there are two possibilities:
1. \(\limsup_{n \to \infty} \left( \frac{h_k^n}{h_n^n} + \frac{h_{\ell'}^n}{h_n^n} \right) = \infty\),
2. \((h_k^n) = (h_{\ell'}^n)\) and \(\limsup_{n \to \infty} \frac{|t_n^{\ell,j} - t_n^{\ell',k}|}{(h_n^n)^a} = \infty\).

More generally, we will prove that if \(D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_1, D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_2 \in L^q_t L^r_x L^2_x\), then
\[
\limsup_{n \to \infty} \bigg\| \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_1 \left( \frac{t-t_n^{\ell,j}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_2 \left( \frac{t-t_n^{\ell',k}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \bigg\|_{L^q_t L^r_x L^2_x} = 0.
\]

By density argument, it suffices to show this for \(D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_1, D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_2 \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)\). Using the Hölder inequality and scaling in spatial variables,
\[
A_n := \left\| \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_1 \left( \frac{t-t_n^{\ell,j}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_2 \left( \frac{t-t_n^{\ell',k}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \right\|_{L^q_t L^r_x L^2_x}
\]
\[
\leq \left\| \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_1 \left( \frac{t-t_n^{\ell,j}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \right\|_{L^q_t L^r_x L^2_x} \left\| \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_2 \left( \frac{t-t_n^{\ell',k}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \right\|_{L^q_t L^r_x L^2_x}
\]
\[
\leq \left\| \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_1 \left( \frac{t-t_n^{\ell,j}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \right\|_{L^q_t L^r_x L^2_x} \left\| \frac{1}{(h_n^n)^a} D_{\sigma}^{\gamma + \frac{\gamma}{2}} \Psi_2 \left( \frac{t-t_n^{\ell',k}}{(h_n^n)^a}, x, \frac{x}{h_n^n} \right) \right\|_{L^q_t L^r_x L^2_x}.
\]

Note that \(D_{\sigma}\) commutes dilation.
Then by time translation and scaling in time, we estimate
\[
A_n = \left\| \frac{1}{(h_n^j)^{\frac{d}{2} + \frac{d}{r} - \frac{2\alpha}{q}}} D^\sigma_{\gamma+5} \Psi_1(t, x) \right\|_{L^q_t L^\gamma(x)} + \frac{1}{(h_n^j)^{\frac{d}{2} - \frac{d}{r}}} \left\| D^\sigma_{\gamma+5} \Psi_2((\frac{h_n^j}{h_n^k})^\alpha t - \frac{t^{e,k} - t^{e,j}}{(h_n^k)^\alpha}, x) \right\|_{L^q_t L^\gamma(x)}
\]
\[
\leq \left\| \left(\frac{h_n^j}{h_n^k}\right)^{\frac{d}{2} + \frac{d}{r} - \frac{2\alpha}{q}} D^\sigma_{\gamma+5} \Psi_1(t, x) \right\|_{L^q_t L^\gamma(x)} + D^\sigma_{\gamma+5} \Psi_2((\frac{h_n^j}{h_n^k})^\alpha t - \frac{t^{e,k} - t^{e,j}}{(h_n^k)^\alpha}, x) \right\|_{L^q_t L^\gamma(x)}
\]
Since the support in time of \( D^\sigma_{\gamma+5} \Psi_1(t, \cdot) \) is compact, from the above condition (1) or (2) it follows that \( \limsup_{n \to \infty} A_n = 0 \). This completes the proof of Lemma 3.3 \( \square \)

By Lemma 3.3 and the Strichartz estimates (Lemma 2.1) it follows that
\[
\limsup_{n \to \infty} \left\| U(t) P_n^{N,M_1, \ldots, M_N} \right\|^2_{L^q_t L^\gamma(x)} \leq \sum_{1 \leq j \leq N} \sum_{M_j < \Lambda_N} \limsup_{n \to \infty} \left\| U(t) \Phi_{\gamma+5}^{f,j} \right\|^2_{L^q_t L^\gamma(x)}
\]
\[
\lesssim \sum_{1 \leq j \leq N} \sum_{\ell > M_j} \| \phi^{f,j} \|^2_{L^2_t L^\gamma(x)}
\]
Since \( \sum_{j, \ell} \| \phi^{f,j} \|^2_{L^2_t L^\gamma(x)} \) is convergent, we have
\[
\limsup_{n \to \infty} \left( \sum_{j=1}^{N} \sum_{\ell > M_j} \left\| U(t) \Phi_{\gamma+5}^{f,j} \right\|^2_{L^q_t L^\gamma(x)} \right)^{\frac{1}{2}} \lesssim \frac{\varepsilon}{3},
\]
provided that \( \min(N, M_1, \ldots, M_N) \) is sufficiently large. This completes the proof of Theorem 1.1

4. Application: Blowup phenomena

In this section, we present applications of linear profile decomposition to the mass-critical Hartree equations (1.1). Almost all parts of the proofs of results are very similar to the radial case. So, we omit them and refer readers to [9] for them. Firstly, we get nonlinear profile decompositions of the solutions to (1.1).

4.1. Nonlinear profile decomposition. Let us set
\[
(q_0, r_0) = \left(3, \frac{6d}{3d - 2\alpha} \right), \quad \gamma_1 = \frac{d^2 - \alpha d + \alpha}{4d}, \quad \gamma_2 = \frac{d - 1 + \gamma_1}{3}, \quad \gamma = \gamma_2 - \gamma_1 + \gamma.
\]
As it will be shown in Appendix A by the usual fixed point argument and the Strichartz estimate in Lemma 2.1, the local well-posedness theory can be based on the estimate of space-time norm \( \left\| u \right\|_{L^q_t L^r(x; L^\alpha(H^\gamma_x ; \mathbb{R}^d))} \). For a given sequence of angularly regular data \( (u_n^i) \subset L^2_t L^\gamma_x \), using the linear profile decomposition (Theorem 1.1), we have sequences \( (\phi^i)_{1 \leq j < l} \in L^2_t L^\gamma_x \), \( \omega^i_n \in L^2_t L^\gamma_x \), \( (h_n^i, t_n^i)_{1 \leq j < l, n \geq 1} \) which satisfy (1)–(3) in Theorem 1.1. Then by taking subsequence, if necessary, we may assume that \( t^j \in \{-\infty, 0, \infty\} \). Here we denote \( t^j = \lim_{n \to \infty} t_n^i \). Using the local well-posedness theorem with initial data at \( t = 0 \) or \( t = \pm \infty \) (see Proposition A.3 below), we define the nonlinear profile by the maximal nonlinear solution for each linear profile.
Definition 4.1. Let \((h_n, t_n)\) be a family of parameters and \((t_n)\) have a limit in \([-\infty, \infty]\). Given a linear profile \(\phi \in L^2_\rho H^2_\delta\) with \((h_n, t_n)\), we define the nonlinear profile associated with them to be the maximal solution \(\psi\) to (1.1) which is in \(C_t L^2_\rho H^2_\delta((-T_{\min}, T_{\max}) \times \mathbb{R}^d)\) satisfying that
\[
\lim_{n \to \infty} \|U(t_n)\phi - \psi(t_n)\|_{L^2_\rho H^2_\delta} = 0.
\]
Here \((-T_{\min}, T_{\max})\) is the maximal existence time interval.

Then, the linear profile decomposition yields the nonlinear profile decomposition. It is the key tool for proving blowup phenomena in what follows.

Theorem 4.2. Let \((u^0_n) \subset L^2_\rho H^2_\delta\) be a bounded sequence. Suppose that \((\phi^j)_{1 \leq j \leq l} \subset L^2_\rho H^2_\delta\), \(\omega^j_n \in L^2_\rho H^2_\delta\), and \((h^j_n, t^j_n)_{1 \leq j \leq l, n \geq 1}\) are sequences obtained from Theorem 4.1. Let \(u_n \in C_t L^2_\rho H^2_\delta(J_n \times \mathbb{R}^d)\) be the maximal solution of (1.1) with initial data \(u_n(0) = u^0_n\). For each \(j \geq 1\), suppose \((\psi^j)_{1 \leq j \leq l} \subset C_t L^2_\rho H^2_\delta((-T^j_{\min}, T^j_{\max}) \times \mathbb{R}^d)\) is the maximal nonlinear profile associated with \((\phi^j)_{1 \leq j \leq l}\) and \((h^j_n, t^j_n)_{1 \leq j \leq l, n \geq 1}\). Let \((I_n)\) be a family of nondecreasing time intervals containing 0. Then, the following two are equivalent;

1. \(\limsup_{n \to \infty} \|\Gamma_n^j \psi^j\|_{L^\infty_t L^2_\rho H^2_\delta(I_n \times \mathbb{R}^d)} < \infty\), \(j \geq 1\),
2. \(\limsup_{n \to \infty} \|u_n\|_{L^\infty_t L^2_\rho H^2_\delta(I_n \times \mathbb{R}^d)} < \infty\).

Here \(\Gamma_n^j \psi^j = \frac{1}{(h^j_n)^{d/2}} \psi^j(\frac{\cdot - t^j_n}{h^j_n})\). Moreover, if (1) or (2) holds true, we have a decomposition
\[
u_n = \sum_{j=1}^{l} \Gamma_n^j \psi^j + U(\cdot)\omega_n^j + e^j_n
\]
with \(\lim_{l \to \infty} \limsup_{n \to \infty} (\|U(\cdot)\omega_n^j\|_{L^\infty_t L^2_\rho H^2_\delta(I_n \times \mathbb{R}^d)} + \|e^j_n\|_{L^\infty_t L^2_\rho H^2_\delta(I_n \times \mathbb{R}^d)}) = 0\).

4.2. Applications. We consider blowup phenomena of solutions to (1.1). If the solution fails to persist, then the space-time norm blows up. The blowup solution is defined as follows.

Definition 4.3. A solution \(u \in C_t L^2_\rho H^2_\delta((-T_{\min}, T_{\max}) \times \mathbb{R}^d)\) to (1.1) is said to blow up if \(\|u\|_{L^\infty_t L^2_\rho H^2_\delta((-T_{\min}, T_{\max}) \times \mathbb{R}^d)} = \infty\). Here \((-T_{\min}, T_{\max}) \in [-\infty, \infty]\) denotes the maximal time interval of existence of the solution.

Since \(T_{\max}\) or \(T_{\min}\) may be \(\infty\), we regard non-scattering global solutions as blowup solutions at infinite time. We also define a minimal quantity of solutions from which a solution may ignore to blow up.

Definition 4.4. Define
\[
\delta_0 = \sup \{ A : \text{for any } u_0 \text{ with } \|u_0\|_{L^2_\rho H^2_\delta} \leq A, \text{ (1.1) is globally well-posed on } \mathbb{R} \text{ satisfying } \|u\|_{L^\infty_t L^2_\rho H^2_\delta(\mathbb{R} \times \mathbb{R}^d)} < \infty \}.
\]

By the small data global existence (see Section 3A below), we have \(\delta_0 > 0\). Moreover, for any \(\delta > \delta_0\) there exists a blowup solution \(u\) with \(\delta_0 \leq \|u_0\|_{L^2_\rho H^2_\delta} \leq \delta\). Such a solutions satisfies that \(\|u(t)\|_{L^2_\rho H^2_\delta} \geq \delta_0\) for all \(t \in (-T_{\min}, T_{\max})\). As opposed to \(L^2_\rho\)-norm, \(L^2_\rho H^2_\delta\) is not conserved in time.
We only have a lower bound from the mass conservation law. One may compare $\delta_0$ with minimal mass of radial blowup solutions \[9\]. If we set

$$\delta_{0,rad} = \sup\{A : \text{for any radial data } u_0 \text{ with } \|u_0\|_{L^2} \leq A, (1.1) \text{ is globally well-posed on } \mathbb{R}$$

satisfying \(\|u\|_{L^\infty_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)} < \infty\),

then, it is clear that \(\delta_0 \leq \delta_{0,rad}\).

Following the same lines of arguments which were used for the radial case (Theorem 1.6, \[9\]), we obtain the existence of minimal blowup solutions with initial data in \(L^2_\rho H^\gamma_{\alpha}\).

**Theorem 4.5.** Assume \(\delta_0 < \infty\). Then, there exists a blowup solution \(u\) to (1.1) with initial data \(u_0 \in L^2_\rho H^\gamma_{\alpha}\) such that \(\|u_0\|_{L^2_\rho H^\gamma_{\alpha}} = \delta_0\).

Note that the minimal blowup solution obtained above may not be radial, and so \(\|u(t)\|_{L^2}\) may be smaller than \(\delta_0\).

In view of the local theory in Appendix A there are two possible blowup scenarios:

1. \(\sup_{t \in (-T_{\min},T_{\max})} \|u(t)\|_{L^2_\rho H^\gamma_{\alpha}} = \infty\),
2. \(\sup_{t \in (-T_{\min},T_{\max})} \|u(t)\|_{L^2_\rho H^\gamma_{\alpha}} < \infty\) and \(\|u\|_{L^\infty_t L^\infty_x((-T_{\min},T_{\max}) \times \mathbb{R}^d)} = \infty\).

We focus on the second scenario. In the case of (2), we deduce from the linear and nonlinear profile decompositions a compactness of the trajectory of solution \(u(t)\) as in the radial case. If especially \(\sup_{t \in (-T_{\min},T_{\max})} \|u(t)\|_{L^2_\rho H^\gamma_{\alpha}}^2 < 2\delta_0^2\), then the blowup solution does not form more than one blowup profile. Thus, this gives a weaker form of compactness property of the blowup solutions.

**Theorem 4.6.** Let \(u\) be finite time blowup solution of (1.1) at \(T^*\) with \(\sup_{0 < t < T^*} \|u(t,\cdot)\|_{L^2_\rho H^\gamma_{\alpha}} < \sqrt{2}\delta_0\) and let \(t_n \nrightarrow T^*\). Then there exist \(\phi \in L^2_\rho H^\gamma_{\alpha}\) and \((h_n)_{n=1}^\infty\) satisfying \(h_n^{d/2} u(t_n, h_n x) \rightharpoonup \phi\) (weakly) in \(L^2_\rho H^\gamma_{\alpha}\) and if solution of (1.1) with initial data \(\phi\) blows up at \(T^{**}\), then

$$\lim_{n \to \infty} \frac{h_n}{(T^* - t_n)^{1/\alpha}} \leq \frac{1}{(T^{**})^{1/\alpha}}$$

up to subsequence.

When a blowup occurs, only one profile blows up by shrinking in scale. See \[9\] for the radial case and \[9\] for related results when \(\alpha > 2\). As a byproduct of Theorem 4.6, we obtain a concentration property of blowup solution as follows.

**Theorem 4.7.** Let \(u\) be a finite time blowup solution at \(T^*\) with \(\sup_{0 < t < T^*} \|u(t,\cdot)\|_{L^2_\rho H^\gamma_{\alpha}} < \sqrt{2}\delta_0\) and let \(t_n \nrightarrow T^*\). Then for \(\lambda(t_n)\) satisfying \(\frac{(T^* - t_n)^{1/\alpha}}{\lambda(t_n)} \to 0\)

$$\limsup_{n \to \infty} \int_{|x| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \|\phi\|_{L^2}^2,$$

where \(\phi\) is defined as in Theorem 4.6
Appendix A. Local wellposedness and global well-posedness with small data

In this section we show local well-posedness and global well-posedness with small data by the standard fixed point argument. Local well-posedness and global well-posedness with small data. Let \( q_0, r_0, \gamma_1, \gamma_2, \) and \( \tilde{\gamma} \) be given by \((1.1)\).

**Proposition A.1.** For any \( u_0 \in L^2_\rho H^\gamma_\sigma \), there exists a unique solution \( u \) to \((1.1)\) such that \( u \in C_t L^2_\rho H^\gamma_\sigma ((-T_{\text{min}}, T_{\text{max}}) \times \mathbb{R}^d) \cap L^2_{t,\text{loc}} L^r_\rho H^\gamma_\sigma ((-T_{\text{min}}, T_{\text{max}}) \times \mathbb{R}^d) \) whenever

\[
\gamma \geq \frac{d^2 - \alpha d + \alpha}{4d}.
\]

Moreover, if \( \|u_0\|_{L^2_\rho H^\gamma_\sigma} \) is sufficiently small, then \( T_{\text{min}} = T_{\text{max}} = \infty \) and \( u \in L^q_{t} L^r_\rho H^\gamma_\sigma (\mathbb{R}^{1+d}) \).

**Proof.** The proof is based on the standard fixed point argument. Let us consider the integral equation

\[
(A.1) \quad u(t) = U(t)u_0 + i\lambda \int_0^t U(t-s)(|x|^{-\alpha} |u|^2)u(s)ds
\]

on Banach space \( X = X_{T, \mu} \) given by

\[
X := \{ v \in L^q_{t} L^r_\rho H^\gamma_\sigma([-T, T] \times \mathbb{R}^d) : \|v\|_{L^q_{t} L^r_\rho H^\gamma_\sigma([-T, T] \times \mathbb{R}^d)} \leq \mu \}.
\]

For simplicity we denote \( L^q_{t} L^r_\rho H^\gamma_\sigma([-T, T] \times \mathbb{R}^d) \) by \( L^q_{t} L^r_\rho H^\gamma_\sigma \). Then \( X \) is obviously a complete metric space with metric \( d(u, v) = \|u - v\|_{L^q_{t} L^r_\rho H^\gamma_\sigma} \). To proceed we consider nonlinear mapping \( \mathcal{N} \) defined by

\[
\mathcal{N}(v)(t) := U(t)u_0 + i\lambda \int_0^t U(t-s)(|x|^{-\alpha} |v(s)|^2)(v(s))ds
\]

and show that it is self mapping on \( X \).

In fact, by Strichartz estimates (Lemma 2.1), we have

\[
\|\mathcal{N}(u)\|_{L^q_{t} L^r_\rho H^\gamma_\sigma} \lesssim \|U(\cdot)u_0\|_{L^q_{t} L^r_\rho H^\gamma_\sigma} + \|D^\gamma_\sigma((|x|^{-\alpha} |u|^2)u)\|_{L^q_{t} L^r_\rho H^\gamma_\sigma}.
\]

By Leibniz rule on the unit sphere and the Hölder’s inequality, one obtain

\[
\|D^\gamma_\sigma((|x|^{-\alpha} |u|^2)u)\|_{L^q_{t} L^r_\rho H^\gamma_\sigma} \lesssim \||x|^{-\alpha} |u|^2\|_{L^q_{t} L^r_\rho H^\gamma_\sigma} \|D^\gamma_\sigma u\|_{L^q_{t} L^r_\rho L^p_{t} L^p_\rho} + \||x|^{-\alpha} (D^\gamma_\sigma |u|^2)\|_{L^q_{t} L^r_\rho H^\gamma_\sigma} \|u\|_{L^q_{t} L^r_\rho L^p_{t} L^p_\rho}.
\]

where

\[
\frac{1}{q_1} = \frac{2}{3}, \quad \frac{1}{q_2} = \frac{1}{3}, \quad \frac{1}{r_1} = \frac{2}{r_0} - \frac{d - \alpha}{d}, \quad \frac{1}{r_2} = \frac{1}{r_0}, \quad \frac{1}{p_1} = 2\left(\frac{1}{2} - \frac{\gamma_2}{d - 1}\right),
\]

\[
\frac{1}{p_2} = 2\left(\frac{\gamma_2 - \gamma_1}{d - 1}\right), \quad \frac{1}{p_3} = 1 - \frac{\gamma_2 - \gamma_1}{d - 1} - \frac{\gamma_2 - \gamma_1}{d - 1}, \quad \frac{1}{p_4} = \frac{1}{2} - \frac{\gamma_2}{d - 1}.
\]

To treat the convolution term we use the following lemma about fractional integration in the space \( L^p_\rho L^p_\rho \).

**Lemma A.2.** Let \( 1 < r, r' < \infty, 0 < \beta < n, \) and \( 1 \leq p \leq \infty \). If \( \frac{1}{r} = \frac{1}{r'} - \frac{n - \beta}{n} \), then

\[
\|\|x|^{-\beta} * f\|_{L^p_\rho L^p_\rho} \lesssim \|f\|_{L^p_\rho L^p_\rho}.
\]
Proof of Lemma A.2. We use the following pointwise estimate, which is shown in p.15 of [12]:

\[ \int_{S^{n-1}} |x|^{-\beta} |f| (\rho \sigma) \, d\sigma \leq (|x|^{-\beta} F)(\rho), \]

where \( F(\rho) = \int_{S^{n-1}} |f(\rho \sigma)| \, d\sigma \). By taking \( L^\infty_\rho \) on both sides of (A.2), from Hardy-Littlewood-Sobolev inequality we get

\[ \| |x|^{-\beta} f \|_{L^p_\rho L^q_\rho} \lesssim \| |x|^{-\beta} F \|_{L^p_\rho} \lesssim \| f \|_{L^p_\rho L^q_\rho}. \]

On the other hand, we have

\[ \langle |x|^{-\beta} f, g \rangle = \langle f, |x|^{-\beta} g \rangle \lesssim \| f \|_{L^p_\rho L^q_\rho} \| |x|^{-\beta} g \|_{L^p_\rho L^q_\rho} \lesssim \| f \|_{L^p_\rho L^q_\rho} \| g \|_{L^p_\rho L^q_\rho}. \]

Interpolation between these two estimates gives the desired estimates. \( \square \)

Then by Lemma A.2, we get

\[ \| |x|^{-\alpha} |u|^2 \|_{L^p_\rho L^q_\rho} \lesssim \| D_{\rho}^\gamma u \|_{L^p_\rho L^q_\rho} + \| D_{\rho}^{\gamma+1} u \|_{L^p_\rho L^q_\rho} \lesssim \| D_{\rho}^\gamma u \|_{L^p_\rho L^q_\rho} + \| D_{\rho}^{\gamma+1} u \|_{L^p_\rho L^q_\rho} \]

Finally, the Leibniz rule and Sobolev embedding on the unit sphere gives

\[ \| u \|_{L^p_\rho L^q_\rho} L^p_\rho L^q_\rho \lesssim \| D_{\rho}^\gamma u \|_{L^p_\rho L^q_\rho} + \| D_{\rho}^{\gamma+1} u \|_{L^p_\rho L^q_\rho} \lesssim \| D_{\rho}^\gamma u \|_{L^p_\rho L^q_\rho} L^p_\rho L^q_\rho \]

Since \( \| U(\cdot) u_0 \|_{L^p_\rho L^q_\rho} \lesssim \| u_0 \|_{L^p_\rho H^\gamma_\rho} \), for suitable \( T \) and \( \mu \) we have

\[ \| N(u) \|_{L^p_\rho L^q_\rho} \lesssim C(T) \| U(\cdot) u_0 \|_{L^p_\rho L^q_\rho} + \mu^3 \lesssim \mu. \]

Similarly one can easily show \( u \rightarrow N(u) \) is a contraction map on \( X \) for suitable \( T \) and \( \mu \). This implies that there exists a unique solution \( u \in L^q_T L^\infty_\rho H^\gamma_\rho \) to (A.1). Now using Lemma 2.1 again, we get

\[ \| N(u) \|_{L^p_\rho L^q_\rho} \lesssim \| u_0 \|_{L^p_\rho H^\gamma_\rho} + C \mu^3. \]

Thus \( u \in L^p_T L^q_\rho H^\gamma_\rho \cap L^q_T L^\infty_\rho H^\gamma_\rho \) and from the uniqueness and the formula (A.1) the well-posedness is straightforward.

On the other hand, if \( \| u_0 \|_{L^p_\rho H^\gamma_\rho} \) and \( \mu \) are sufficiently small, then the functional \( N \) is shown to be a contraction map on complete metric space \( Y \) given by

\[ Y := \{ v \in (C L^2_\rho H^\gamma_\rho \cap L^\infty_\rho L^1_\rho H^\gamma_\rho) (\mathbb{R}^{1+d}) : \| v \|_{L^p_\rho L^q_\rho L^\infty_\rho L^1_\rho H^\gamma_\rho (\mathbb{R}^{1+d})} \leq \mu \}. \]

We omit the details. \( \square \)

The well-posedness for a given asymptotic state is also similar and fairly standard. We provide its proof for completeness.
Proposition A.3. Given \( u_\infty \in L^2_p H^\gamma_0 (\mathbb{R}^d) \), there exists a positive \( T \) and a unique solution \( u \) to (1.1) such that \( u \in C(T, \infty) \times \mathbb{R}^d \) and \( L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d \) and \( \| u(t) - U(t)u_\infty \|_{L^2_p H^\gamma_0} \to 0 \) as \( t \to \infty \).

Proof. We define a nonlinear mapping \( N \) by

\[
N(v)(t) := i\lambda \int_t^\infty U(t - s)(|x|^{-\alpha} * |U(s)u_\infty + v(s)|^2)(U(s)u_\infty + v(s))ds
\]

for \( v \) in Banach space \( Z = Z_{T, \mu} \) given by

\[
Z := \{ v \in C(T, \infty) \times \mathbb{R}^d \cap L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d) : \|v\|_{L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d} \leq \mu \}.
\]

Similarly to proof of Proposition A.1, one can get

\[
\|N(v)\|_{L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d} \lesssim \|U(\cdot)u_\infty\|^2_{L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d} + \|v\|^2_{L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d}.
\]

Since \( \|U(\cdot)u_\infty\|_{L_t^q L^\gamma_0 H^\gamma (T, \infty) \times \mathbb{R}^d} \lesssim \|u_\infty\|_{L^2_p H^\gamma_0} \) by Lemma 2.1, \( N \) becomes a self-mapping on \( Z \) for sufficiently large \( T \). Similarly one can easily show that \( N \) is a contraction mapping on \( Z \). Let \( v \) be the fixed point of \( N \) in \( Z \). Then by continuity we get \( \|v(t)\|_{L^2_p H^\gamma_0} \to 0 \) as \( t \to \infty \).

Now we write \( u(t) \) as \( u(t) = U(t)u_\infty + v(t) \). Then it follows that \( \|u(t) - U(t)u_\infty\|_{L^2_p H^\gamma_0} \to 0 \) as \( t \to \infty \). It remains to show that

\[
(A.3) \quad u(t) = U(t - t)u(t) - i\lambda \int_t^\tau U(t - s)((|x|^{-\alpha} * |u|^2)u)(s)ds.
\]

In fact, since \( v(\tau) = N(v)(\tau) \), one can show that

\[
v(\tau) = U(\tau - t)v(t) - i\lambda \int_t^\tau U(\tau - s)((|x|^{-\alpha} * |u|^2)u)(s)ds.
\]

Thus

\[
u(\tau) = U(\tau - t)u_\infty + v(\tau) = U(\tau - t)(U(t)u_\infty + v(t)) - i\lambda \int_t^\tau U(\tau - s)((|x|^{-\alpha} * |u|^2)u)(s)ds,
\]

which yields (A.3).

\[\Box\]

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