LOCALIZATION BY 2-PERIODIC COMPLEXES AND VIRTUAL STRUCTURE SHEAVES

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Abstract In [12], Kim and the first author proved a result comparing the virtual fundamental classes of the moduli spaces of \( \varepsilon \)-stable quasimaps and \( \varepsilon \)-stable LG-quasimaps by studying localized Chern characters for 2-periodic complexes.

In this paper, we study a \( K \)-theoretic analogue of the localized Chern character map and show that for a Koszul 2-periodic complex it coincides with the cosection-localized Gysin map of Kiem and Li [11]. As an application, we compare the virtual structure sheaves of the moduli space of \( \varepsilon \)-stable quasimaps and \( \varepsilon \)-stable LG-quasimaps.

Keywords: virtual structure sheaves; moduli spaces

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1. Introduction

In [12], Kim and the first author studied the localized Chern character map of Polishchuk and Vaintrob [18] and showed that with respect to the tautological Koszul 2-periodic complex, the localized Chern character map coincides with the cosection Localization
map of Kiem and Li [10]. This allowed them to prove a result comparing the virtual fundamental classes of the moduli space of stable quasimaps and the moduli space of stable $LG$-quasimaps. Chiodo in [4] constructed Witten’s top Chern class avoiding the use of the bivariant intersection theory of [18] by using purely $K$-theoretic methods. In [11], Kiem and Li constructed a cosection Localization map for virtual structure sheaves. In this paper we study the $K$-theoretic analogue of [12]. We generalise the results [12] to $K$-theory, and for a 2-periodic complex of vector bundles with appropriate support, we define a Localization map (Section 2.1.2 and Definition 2.13) which plays the role of the localized Chern character map. This map is a tensor product followed by taking the alternating sum of the even and odd cohomology sheaves, which is precisely the map studied by Chiodo (see [4, Lemma 5.3.4]). We show that this map, when applied to the tautological Koszul 2-periodic complex, coincides with the cosection-localized Gysin map of Kiem and Li [11]. Now as in [12], this allows us to prove a result comparing the virtual structure sheaves of the moduli space of stable quasimaps and the moduli space of stable $LG$-quasimaps analogous, to [12, Theorem 3.2].

The main result we prove in this text is as follows, where the notations are explained in Section 4.1:

**Theorem 4.6.** In the Grothendieck group of coherent sheaves on $Q^\epsilon_X := Q^\epsilon_g(Z(s), d)$, we have

$$[O^\text{vir}_{Q^\epsilon_X}] = (-1)^{g^\text{gen}(R\pi_*(V_2^\vee))} \det R\pi_*(V_2 \otimes \omega_{\mathcal{E}})^\vee|_{Q^\epsilon_X} \otimes [O^\text{vir}_{\mathcal{E}_{LGOQ'}}.dw_{LGOQ'}],$$

(1.1)

where $g^\text{gen}(V_2^\vee)$ is the generic virtual rank of $R\pi_*(V_2^\vee)$ and $\det R\pi_*(V_2 \otimes \omega_{\mathcal{E}})^\vee$ is the determinant (see Definition 4.1).

A conceptually more intuitive explanation of Theorem 4.6 is as follows. Let $X$ be a Deligne–Mumford stack and let $E = [A \to B]$ be a perfect obstruction theory on $X$, where $A$ and $B$ are vector bundles. Following [20, §2.1], we can define the determinant of $E$ by

$$\mathcal{K}_{\text{vir}} := (\det A)^\vee \otimes B.$$ 

A square root of $\mathcal{K}_{\text{vir}}$ is a line bundle $\mathcal{L}$ (which need not exist) such that $\mathcal{L}^2 = \mathcal{K}_{\text{vir}}$. It is usually denoted by $\mathcal{K}^{1/2}_{\text{vir}}$ [20, §2.2], and is called the square root of $\det E$.

Let $LGOQ'$ be the moduli space of $LG$-quasimaps and let $Q^\epsilon_X$ be the moduli space of $\epsilon$-stable quasimaps (see Section 4.1 and Definition 4.1 for the precise notations). Let $E_{LGOQ'}$ and $E_{Q^\epsilon_X}$ denote the canonical perfect obstruction theories defined on the respective moduli spaces (see Section 4.1). Let us assume that there exists a square root of $\det E_{LGOQ'}$ denoted by $\mathcal{K}^{1/2}_{LGOQ', \text{vir}}$, and let $\mathcal{K}_{\text{vir}, Q^\epsilon_X}$ denote the determinant of $E_{Q^\epsilon_X}$. Note that

$$\mathcal{K}_{Q^\epsilon_X, \text{vir}} = (\mathcal{K}^{1/2}_{LGOQ', \text{vir}} \otimes \det R\pi_*(V_2^\vee))|_{Q^\epsilon_X}.$$ 

Thus, if $\mathcal{K}^{1/2}_{LGOQ', \text{vir}}$ exists we have a canonical choice for the square root of $\det E_{Q^\epsilon_X}$, given by

$$\mathcal{K}^{1/2}_{Q^\epsilon_X, \text{vir}} := (\mathcal{K}^{1/2}_{LGOQ', \text{vir}} \otimes \det R\pi_*(V_2^\vee))|_{Q^\epsilon_X}.$$
Now Theorem 4.6 can be restated as follows:

\[ K_{1/2}^{\epsilon} \otimes O_{X, \text{vir}} = (-1)^{\chi_{\text{gen}}(R\pi_\ast V)} \chi_{\text{gen}}(R\pi_\ast V)_{Q_{\epsilon}, \text{vir}} \otimes O_{LQG', \text{vir}, \text{dw}}. \] (1.2)

When \( \epsilon = \infty \), note that the moduli spaces under consideration are quasi-projective, and hence (1.2) is well defined if \( Q_{\infty}^{g,k}(Z(s), d) = M_{g,k}(Z(s), d) \) possesses a square root of \( \det E_{Q_X} \) (i.e., if \( K_{1/2}^{\epsilon} \) exists).

1.1. Structure of the paper and relations to other work

In [12], Kim and the first author obtained a comparison result of virtual classes of quasimap moduli spaces and \( LG \)-quasimap moduli spaces [12, Theorem 3.2] by using the localized Chern character of 2-periodic complexes of vector bundles. Using the ideas of that work, in this text we prove similar results for the virtual structure sheaves.

In Section 2, we fix the notations and assumptions that we shall follow in the rest of the paper. Several steps in the proofs and constructions used are the \( K \)-theoretic analogues of [18, Section 2]. In Definition 2.13 we define the map \( h_{Y,X}^V(E) \) that would play the role that the localized Chern character plays in [18]. This definition is motivated by the map considered by Chiodo in [4, Lemma 5.3.4]. In the rest of Section 2.3 we study the functoriality properties of the map \( h_{Y,X}^V(E) \) with respect to various functors in \( K \)-theory. As we shall require functoriality statements with respect to various derived functors, the natural setting for our statements is the derived category of matrix factorizations. We recall some preliminaries from [5] in the beginning of Section 2; in particular, in Section 2.1.2 we recall various notions of support and acyclicity for 2-periodic complexes.

The setup of Section 3 and Section 4 is similar to [12, §2, §3]. In Section 3, we recall the definition of the Koszul 2-periodic complex associated to a section and a cosection of a vector bundle. In [11], Kiem and Li constructed a cosection-localized Gysin map to define a cosection-localized virtual structure sheaf. The goal of Section 3 is to compare the map \( h_{Y,X}^V(E) \), when \( E \) is the tautological Koszul 2-periodic complex (Section 3.1) to the construction of Kiem and Li [11, Theorem 4.1]. A similar comparison for the localized Chern character with the cosection-localized Gysin map is [12, Theorem 2.6]. Building on this comparison result with the constructions by Kiem and Li, in Section 4 we prove the main result (Theorem 4.6). This section relies on the constructions of [12, §3]; to aid the reader and fix notations, we briefly summarise the results of [12] in Section 4.1 before proving Theorem 4.6 in Section 4.2.

2. Localization in \( K \)-theory

2.1. Assumptions and notations

In this article we work over a base field \( k \) of characteristic 0. In Section 4 we shall further assume \( k = \mathbb{C} \). All schemes and algebraic stacks are assumed to be separated and of finite type over \( k \). In this article we shall mainly work with Deligne-Mumford stacks, which we shall abbreviate as \( DM \)-stacks. For a \( DM \)-stack \( \mathcal{Y} \), let \( \text{Coh}(\mathcal{Y}) \) denote the abelian category of coherent sheaves and let \( D^b(\mathcal{Y}) \) denote the bounded derived category
of coherent sheaves on \( \mathcal{Y} \). \( G_0(\mathcal{Y}) \) shall denote the Grothendieck group of coherent sheaves on \( \mathcal{Y} \), that is, \( G_0(\mathcal{Y}) := K_0(\text{Coh}(\mathcal{Y})) \). Let \( \mathcal{X} \) be a closed substack of \( \mathcal{Y} \); then we denote the category of coherent sheaves on \( \mathcal{Y} \) supported on \( \mathcal{X} \) by \( \text{Coh}_X(\mathcal{Y}) \). It follows from d\'evissage that the push-forward map induces a natural isomorphism \( G_0(\mathcal{X}) \xrightarrow{\sim} K_0(\text{Coh}_X(\mathcal{Y})) \). For a coherent sheaf \( \mathcal{F} \), we denote its class in \( G_0 \) by \([\mathcal{F}]\).

### 2.1.1. 2-periodic complexes and the category of matrix factorizations.

Let \( \mathcal{Y} \) be a \( DM \)-stack. A 2-periodic complex of quasi-coherent sheaves on \( \mathcal{Y} \) is a complex \( G_* \) as follows,

\[
G_* = \left[ \begin{array}{c} \cdots \xrightarrow{d_-} G_- \xrightarrow{d_+} G_+ \xrightarrow{d_-} G_- \xrightarrow{d_+} \cdots \end{array} \right]
\]

where \( G_- \) is in odd degree, \( G_+ \) is in even degree and \( d_+ \circ d_- = d_- \circ d_+ = 0 \). \( G_* \) is said to be a 2-periodic complex of vector bundles on \( \mathcal{Y} \) if we further require that \( G_+ \) and \( G_- \) are vector bundles. In this paper, by a 2-periodic complex we shall always mean a 2-periodic complex of quasi-coherent sheaves.

For the proofs of some functoriality results, it would be advantageous to work in a suitable derived category. As noted, a 2-periodic complex is an unbounded complex, and hence some care is needed in framing the statements correctly. We briefly recall the notion of the derived category of matrix factorizations (see [5, §2] for a detailed discussion) and notions of support, where our main focus is 2-periodic complexes.

Let \( L \) be a line bundle on \( \mathcal{Y} \) and let \( w \) be a section of \( L \). One can associate to the pair \((\mathcal{Y}, w)\) the absolute derived category of factorizations \( D^{abs}[\text{Fact}(\mathcal{Y}, w)] \) and \( D(\mathcal{Y}, w) \) (see [5, Definition 2.1.4]), which is the full subcategory of \( D^{abs}[\text{Fact}(\mathcal{Y}, w)] \) consisting of those factorizations which are isomorphic to factorizations with each component coherent. For a detailed discussion, see [5, §2]. We shall be interested in the particular case when \( L = \mathcal{O}_\mathcal{Y} \) and \( w = 0 \), where it is apparent from the definition that a 2-periodic complex of coherent sheaves can be considered to be an object of \( D(\mathcal{Y}, 0) \). We further note that following [1, Definition 3.18], there is a a folding map \( \Upsilon : D^b(\mathcal{QCoh} \mathcal{Y}) \rightarrow D^{abs}[\text{Fact}(\mathcal{Y}, 0)] \) from the bounded derived category of quasi-coherent sheaves to the absolute derived category of factorizations. The functor \( \Upsilon \) restricted to the subcategory of coherent sheaves defines a functor \( \Upsilon : D^b(\mathcal{Y}) \rightarrow D(\mathcal{Y}, 0) \).

**Remark 2.1.** For a coherent sheaf \( \mathcal{F} \) on a \( DM \)-stack \( \mathcal{Y} \), by \( \Upsilon(\mathcal{F}) \) we shall denote the folding of the complex with the coherent sheaf \( \mathcal{F} \) in degree zero and all other terms zero.

**Definition 2.2** (Tensor product of 2-periodic complexes [5, Definition 2.2.1].) Let \( E_* := (E_+, E_-, d_+^E, d_-^E) \) and \( F_* := (F_+, F_-, d_+^F, d_-^F) \) be 2-periodic complexes of coherent sheaves. The tensor product denoted by \( E_* \otimes F_* \) is the 2-periodic complex defined as follows:

\[
(E_* \otimes F_*)_+ := E_+ \otimes F_+ \bigoplus E_- \otimes F_-,
\]

\[
(E_* \otimes F_*)_- := E_+ \otimes F_- \bigoplus E_- \otimes F_+,
\]
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\[ d^E \otimes F = \begin{pmatrix} 1_{E^+} \otimes d^F_+ & d^E \otimes 1_{F_-} \\ -d^E_+ \otimes 1_{F^+} & 1_{E^-} \otimes d^F_- \end{pmatrix}, \]

\[ d^- \otimes F = \begin{pmatrix} 1_{E^+} \otimes d^F_- & -d^E \otimes 1_{F^+} \\ d^E_+ \otimes 1_{F^-} & 1_{E^-} \otimes d^F_+ \end{pmatrix}. \]

**Remark 2.3.** In Definition 2.2, assume that \( E_* \) is a 2-periodic complex of vector bundles; then the derived tensor (see [5, Definition 2.2.3]) coincides with the tensor product of 2-periodic complexes. This induces a well defined functor

\[ (E_* \otimes -) : D(Y, 0) \to D(Y, 0). \]

Further, note that essentially by definition, the folding map \( \Upsilon \) is compatible with the tensor product.

### 2.1.2. Notions of support

Let us briefly outline what we want to do in this section and why these notions of support are important to us. As before, let \( Y \) be a DM-stack and let \( E_* \) be a bounded complex of vector bundles on \( Y \) supported on a closed substack \( X \). For any coherent sheaf \( F \) on \( Y \), the complex \( E_* \otimes F \) is also supported on \( X \). This observation need not be true if our complex \( E_* \) were unbounded. Note also that by support for bounded complexes we mean cohomological support. A 2-periodic complex is unbounded by definition, and hence we need to consider different notions of support and the compatibility between them for the proofs of this section. Most of the statements are elementary, and we note them down here due to the lack of a suitable reference. Let us begin by briefly recalling the construction of the absolute derived category of factorizations. If \( F_* \) and \( G_* \) are 2-periodic complexes, then a morphism \( a : F_* \to G_* \) is just a morphism of complexes such that the obvious squares commute. Translating this to the language of factorizations, this is precisely the abelian category \( Z^0\text{Fact}(Y, 0) \) [5, Remark 2.1.2]. Let \( Ch^b(Z^0\text{Fact}(Y, 0)) \) denote the category of bounded chain complexes of factorizations (i.e., chain complexes of chain complexes). For any object \( E_* \in Ch^b(Z^0\text{Fact}(Y, 0)) \), one can associate its totalization denoted by \( \text{Tot}(E_*) \in \text{Fact}(Y, 0) \) (see [5, Definition 2.1.3]). Let \( \text{Acyc}(Y, 0) \) be the smallest full saturated subcategory of \( \text{Fact}(Y, 0) \) which contains the totalizations of exact sequences in \( Ch^b(Z^0\text{Fact}(Y, 0)) \). Now the construction of the absolute derived category follows the standard recipe of looking at the underlying homotopy category and taking the Verdier quotient by \( \text{Acyc}(Y, 0) \). The derived category of factorizations is denoted by \( D^{\text{abs}}[\text{Fact}(Y, 0)] \).

For the rest of this discussion, let \( X \hookrightarrow Y \) be a closed substack and let \( U = (Y \setminus X) \to Y \) be the open complement. As we shall be working with several notions of support, we highlight the following definitions.

**Definition 2.4 ([5, Definition 2.2.4]).** \( F_* \in D^{\text{abs}}[\text{Fact}(Y, 0)] \) is called absolutely acyclic if and only if \( 0 \to F_* \) is an isomorphism in \( D^{\text{abs}}[\text{Fact}(Y, 0)] \). Further, we say \( F_* \) is absolutely supported on \( X \) if and only if \( 0 \to j^* F_* \) is an isomorphism in \( D^{\text{abs}}[\text{Fact}(U, 0)] \).
Unlike arbitrary factorizations for 2-periodic complexes, one can also talk about the cohomology sheaves, as the composition of the differentials is zero. We can thus make sense of the following definition, which in the bounded case coincides with the natural notion of support:

**Definition 2.5 (Cohomological acyclicity).** A 2-periodic complex $F_\bullet$ is called cohomologically acyclic if and only if $H_\pm(F_\bullet) = 0$. Further, we say that $F_\bullet$ is cohomologically supported on $X$ if and only if the cohomology sheaves $H_\pm(F_\bullet)$ are supported on $X$.

**Remark 2.6.** It is not true that notions of acyclicity as in Definition 2.4 and Definition 2.5 coincide for a general 2-periodic complex. However, for a folding of a bounded complex, both the notions are the same.

The next definition is the notion of locally contractible complexes, following Polischuk and Vaintrob (see paragraph just preceding [19, Definition 3.13]).

**Definition 2.7 (Locally contractible complex).** A 2-periodic complex $E_\bullet$ is said to be locally contractible if and only if there exists a smooth atlas $Y \xrightarrow{p} Y$ such that $p^*E_\bullet$ is contractible on $Y$ (i.e., the identity map is homotopic to the zero map). Further, we say that $F_\bullet$ is locally contractible off $X$ if and only if $j^*F_\bullet$ is locally contractible on $U$.

We summarise some relations between these notions of support in the following lemma:

**Lemma 2.8.** Let $\mathcal{Y}$ be a $\mathcal{D}\mathcal{M}$-stack and let $E_\bullet$ be a 2-periodic complex on $\mathcal{Y}$. Then the following hold:

(i) $E_\bullet$ is absolutely acyclic $\implies$ $E_\bullet$ is cohomologically acyclic.

(ii) $E_\bullet$ is locally contractible $\implies$ $E_\bullet$ is cohomologically acyclic.

(iii) Let $E_\bullet$ be a 2-periodic complex of vector bundles on $\mathcal{Y}$ which is locally contractible. Then $E_\bullet \otimes F_\bullet$ is locally contractible for any 2-periodic complex $F_\bullet$ on $\mathcal{Y}$.

**Proof.** (i) follows from [14, Remark 2.2] and the discussion preceding [17, Definition 3]. (ii) is obvious from the definition of being locally contractible. To show (iii), let $E_\bullet$ be a 2-periodic complex of vector bundles on $\mathcal{Y}$ which is locally contractible. As the question is local, we can assume that $\mathcal{Y}$ is a scheme and $E_\bullet$ is contractible on $\mathcal{Y}$. Let the homotopy between the identity and zero maps be given by $h_+: E_+ \to E_-$ and $h_-: E_- \to E_+$. Further, let us assume that $E_\bullet = (E_+, E_-, d_+^E, d_-^E)$ and $F_\bullet = (F_+, F_-, d_+^F, d_-^F)$. Let $h'_\pm: (E_\bullet \otimes F_\bullet)_\pm \to (E_\bullet \otimes F_\bullet)_\mp$ be the new homotopies, defined as follows:

$$h'_+ = \begin{pmatrix} 1_{E_+} \otimes d_-^F & h_- \otimes 1_{F_-} \\ -h_+ \otimes 1_{F_+} & 1_{E_-} \otimes d_+^F \end{pmatrix},$$

$$h'_- = \begin{pmatrix} 1_{E_+} \otimes d_-^F & -h_- \otimes 1_{F_-} \\ h_+ \otimes 1_{F_+} & 1_{E_-} \otimes d_+^F \end{pmatrix}.$$  

Then $h'_\pm$ is the required homotopy, which completes the proof of (iii).
Remark 2.9. We note that in Lemma 2.8 (iii), the assumption that $E_\bullet$ is a complex of vector bundles is not necessary for the proof, but it would be required for a tensor to define an exact functor (see Definition 2.13).

By restricting to the complement of the respective supports, we have the following obvious corollary:

Corollary 2.10. Let $\mathcal{Y}$ be a $\mathcal{D}M$-stack and $\mathcal{X}$ a closed substack. Let $E_\bullet$ be a 2-periodic complex on $\mathcal{Y}$. Then the following hold:

(i) $E_\bullet$ is absolutely acyclic off $\mathcal{X}$ $\implies$ $E_\bullet$ is cohomologically acyclic off $\mathcal{X}$.

(ii) $E_\bullet$ is locally contractible off $\mathcal{X}$ $\implies$ $E_\bullet$ is cohomologically acyclic off $\mathcal{X}$.

(iii) Let $E_\bullet$ be a 2-periodic complex of vector bundles on $\mathcal{Y}$ which is locally contractible off $\mathcal{X}$. Then $E_\bullet \otimes F_\bullet$ is locally contractible off $\mathcal{X}$ for any 2-periodic complex $F_\bullet$ on $\mathcal{Y}$. In particular, $E_\bullet \otimes F_\bullet$ is cohomologically acyclic off $\mathcal{X}$.

In the next remark, we briefly explain how the proper push-forward [5, Proposition 2.2.8] and the projection formula [5, Proposition 2.2.10] in the absolute derived category of factorizations can be applied in our context.

Remark 2.11 (Proper push-forward and the projection formula). Let $g : \mathcal{Y}' \to \mathcal{Y}$ be a proper representable morphism. Then $Rg_* : D(QCoh(\mathcal{Y}')) \to D(QCoh(\mathcal{Y}))$ has finite cohomological dimension. Now following [5, Definition 2.2.4] and [15, Theorem 1.17], for any quasi-coherent sheaf we can find a finite resolution by $g_*$-acyclic sheaves, which allows us to conclude that there is a well-defined functor $Rg_* : D^{abs}[\text{Fact}(\mathcal{Y}',0)] \to D^{abs}[\text{Fact}(\mathcal{Y},0)]$.

Let $E$ be a vector bundle on $\mathcal{Y}$; then the usual projection formula implies that $g^* E \otimes F$ is $g_*$-acyclic when $F$ is a $g_*$-acyclic sheaf. Arguing as in [5, Proposition 2.2.10], we conclude that for a 2-periodic complex of vector bundles $E_\bullet$ on $\mathcal{Y}$, the projection formula holds – that is, $Rg_*(g^* E_\bullet \otimes F_\bullet) \cong E_\bullet \otimes Rg_* F_\bullet$.

2.2. The map $h^Y_X(E_\bullet)$

Let $\mathcal{Y}$ be a $\mathcal{D}M$-stack and let $\mathcal{X} \to \mathcal{Y}$ be a closed substack. Let $E_\bullet$ be a 2-periodic complex of vector bundles on $\mathcal{Y}$ such that $E_\bullet$ is locally contractible off $\mathcal{X}$. In this section we would like to define a map $h^Y_X(E_\bullet) : G_0(\mathcal{Y}) \to G_0(\mathcal{X})$ (see Definition 2.13) which would be the $K$-theoretic localized Chern character map for 2-periodic complexes. We first need the following elementary lemma:

Lemma 2.12. Let

\[ 0 \to E^1_\bullet \xrightarrow{f} E^2_\bullet \xrightarrow{g} E^3_\bullet \to 0 \]

be an exact sequence of 2-periodic complexes of coherent sheaves on $\mathcal{Y}$ which are cohomologically supported on $\mathcal{X}$. Then

\[ [H_+(E^2_\bullet)] - [H_-(E^2_\bullet)] = [H_+(E^1_\bullet)] - [H_-(E^1_\bullet)] + [H_+(E^3_\bullet)] - [H_-(E^3_\bullet)] \]

in $K_0(\text{Coh}_X \mathcal{Y})$. 
Proof. The following is a simple proof suggested by the referee. Note that we have the following (periodic) long exact sequence:

$$
\cdots \to H_+(E_1^1) \to H_+(E_2^2) \to H_+(E_3^3) \to H_-(E_1^1) \to H_-(E_2^2) \to H_-(E_3^3) \to H_+(E_1^1) \to \cdots
$$

Let $K$ denote the kernel of the map $H_+(E_1^1) \to H_+(E_2^2)$, which coincides with the image of the map $H_-(E_3^3) \to H_+(E_1^1)$, as the sequence is periodic and exact. Consider the following exact sequence:

$$
0 \to K \to H_+(E_1^1) \to H_+(E_2^2) \to H_+(E_3^3) \to H_-(E_1^1) \to H_-(E_2^2) \to H_-(E_3^3) \to H_+(E_1^1) \to 0.
$$

As $K$ is a subsheaf of $H_+(E_1^1)$, it follows that $K$ is supported on $X$, and hence every term of the exact sequence is an object of $\text{Coh} X$. The claim in the lemma now follows from the definition of $K_0(\text{Coh} X Y)$.

Let

$$
0 \to F_1 \to F_2 \to F_3 \to 0
$$

be an exact sequence in $\text{Coh}(Y)$ and let $E_\ast$ be a 2-periodic complex of vector bundles on $Y$ such that $E_\ast$ is locally contractible off $X$. Then we have

$$
0 \to E_\ast \otimes \Upsilon(F_1) \to E_\ast \otimes \Upsilon(F_2) \to E_\ast \otimes \Upsilon(F_3) \to 0, \quad (2.1)
$$

which is an exact sequence of 2-periodic complexes on $Y$. Further, from Corollary 2.10 (iii) it follows that $E_\ast \otimes \Upsilon(F)$ is cohomologically supported on $X$. Now Lemma 2.12 allows us to make the following definition:

**Definition 2.13.** Let $E_\ast$ be a 2-periodic complex of vector bundles on $Y$ such that $E_\ast$ is locally contractible off $X$. We define

$$
h_X^Y(E_\ast) : G_0(Y) \to G_0(X)
$$

by sending a coherent sheaf $G$ on $Y$ to

$$
h_X^Y(E_\ast)(G) := [H_+(E_\ast \otimes \Upsilon(G))] - [H_-(E_\ast \otimes \Upsilon(G))].
$$

It follows from Lemma 2.12 that the map $h_X^Y(E_\ast)$ extends linearly to define a group homomorphism $G_0(Y) \to G_0(X)$, where we have implicitly used the isomorphism $G_0(X) \xrightarrow{\cong} K_0(\text{Coh} X Y)$.

**2.3. Some functorialities**

In the rest of this section we prove some functoriality properties of the map $h_X^Y(E_\ast)$ defined in Definition 2.13. If $E_\ast$ is a folding of a bounded complex of vector bundles, then there is nothing to prove. For our application we need these properties when $E_\ast$ is not a folding of a bounded complex, in particular for Koszul 2-periodic complexes (Definition 3.1). The strategy of proof is to reduce the arguments to the folding complexes via Construction 2.17.
and an $A^1$-homotopy invariance argument. We first make the following definition, which is a Gysin pullback for 2-periodic complexes:

**Definition 2.14** ($\lambda^!$ of a 2-periodic complex). Let $\tilde{Y}$ be a stack over $\mathbb{A}^1_k$, and let us denote by $p$ the morphism $\tilde{Y} \rightarrow \mathbb{A}^1_k$. Let $\mathcal{X} \times_k \mathbb{A}^1_k \subset \tilde{Y}$ be a closed substack such that the composition $\mathcal{X} \times_k \mathbb{A}^1_k \subset \tilde{Y} \xrightarrow{p} \mathbb{A}^1_k$ coincides with the projection. Let $\lambda : \text{Spec } k \rightarrow \mathbb{A}^1_k$ denote the regular immersion corresponding to a closed point $\lambda$. We have the natural resolution by vector bundles of $\lambda_* \mathcal{O}_{\text{Spec } k}$ given by $Q_* : 0 \rightarrow k[t] \xrightarrow{t} k[t] \rightarrow 0$. Let $E_\bullet$ be a 2-periodic complex of coherent sheaves on $\tilde{Y}$ such that $E_\bullet$ is cohomologically supported on $\mathcal{X} \times_k \mathbb{A}^1_k$. Analogous to the refined Gysin map for $K$-theory (see [13, §2]), let $P_* := \Upsilon(Q_\bullet)$; then we define $\lambda^!(E_\bullet) := E_\bullet \otimes p^* P_\bullet$. Note that $\lambda^!(E_\bullet)$ is a 2-periodic complex on $\tilde{Y}$ cohomologically supported on $\mathcal{X} \times \lambda$.

The notation we use in Definition 2.14 for the 2-periodic Gysin map might appear to be slightly confusing to the reader. $\lambda^!$ is generally reserved for the Gysin map defined at the level of $K$-theory. The reason for this abuse of notation will be apparent from Lemma 2.16. Let us briefly recall the definition of Gysin pullback in $K$-theory.

**Definition 2.15.** [13, §2] Let $g : Z' \rightarrow Z$ be a regular immersion of quasi-projective schemes. Let $Y$ be a $DM$-stack with a map $p : Y \rightarrow Z$. Consider the following cartesian diagram:

$$
\begin{array}{ccc}
Y' & \xrightarrow{f} & Y \\
\downarrow{p'} & & \downarrow{p} \\
Z' & \xrightarrow{g} & Z.
\end{array}
$$

The refined Gysin pullback $g^! : G_0(Y') \rightarrow G_0(Y)$ is defined as follows. As $Z' \rightarrow Z$ is a regular immersion, $g_* \mathcal{O}_{Z'}$ has a finite resolution by vector bundles, given by, say, $Q_\bullet \rightarrow g_* \mathcal{O}_{Z'}$. Note that $p^* Q_\bullet$ is supported on $Y'$, and hence, in particular for any coherent sheaf $\mathcal{F}$ on $Y$, $F' := \mathcal{F} \otimes p^* Q_\bullet$ is a complex in $D^b(Y')$ which is cohomologically supported on $Y'$. In particular, we define $g^!(\mathcal{F}) = [F']$ an element of $G_0(Y')$ under the natural identification $G_0(Y') \simeq K_0(\text{Coh}_{Y'} Y)$.

**Lemma 2.16.** Let $Y$ be a $DM$-stack and let $\mathcal{X} \hookrightarrow Y$ be a closed substack which naturally defines the closed substack $\mathcal{X} \times_k \mathbb{A}^1_k \hookrightarrow Y \times_k \mathbb{A}^1_k$. Let $E_\bullet$ be a 2-periodic complex of coherent sheaves on $Y \times_k \mathbb{A}^1_k$ cohomologically supported on a closed substack $\mathcal{X} \times_k \mathbb{A}^1_k$. Let $0^!(E_\bullet)$ be the 2-periodic complex defined in Definition 2.14. Then

$$
0^!(|H_+(E_\bullet)| - |H_-(E_\bullet)|) = |H_+(0^!(E_\bullet))| - |H_-(0^!(E_\bullet))|
$$

(2.2)

in $G_0(\mathcal{X} \times \text{Spec } k)$, where $0^!(|H_+(E_\bullet)| - |H_-(E_\bullet)|)$ denotes the usual Gysin pullback in $K$-theory (see [13, §2]).

**Proof.** The proof follows from the convergence of the spectral sequence defined by the following double complex:
Note that $0^1(E_\bullet)$ is a total complex of (2.3). With notations as in Definition 2.14 and Definition 2.15, it follows that $0^1([H_+(E_\bullet)])$ is defined as the class corresponding to the complex

$$(F'_\bullet) := \ldots \to 0 \to H_+(E_\bullet) \to H_+(E_\bullet) \to 0 \ldots.$$  

(2.3)

Let Ker$_+(t)$ and Coker$_+(t)$ denote the kernel and cokernel of the multiplication by $t : H_+(E_\bullet) \to H_+(E_\bullet)$ map. From the definition of the Grothendieck group of coherent sheaves, it follows that the class defined by the complex $[(F'_\bullet)_+] = [Coker_+(t)] - [Ker_+(t)]$. Now for $H_-(E_\bullet)$ one can argue similarly, define $(F'_\bullet)_-$ and conclude that $[(F'_\bullet)_-] = [Coker_-(t)] - [Ker_-(t)]$. Therefore we have the following equality in $G_0(\mathcal{X} \times \text{Spec} k)$:

$$0^1([H_+(E_\bullet)] - [H_-(E_\bullet)]) = [Coker_+(t)] - [Ker_+(t)] - [Coker_-(t)] + [Ker_-(t)].$$  

(2.4)

Now note that we can truncate (2.3) and consider the associated first-quadrant spectral sequence, which converges to the cohomology of the total complex of the double complex. Note that the totalization of the truncated complex and (2.3) do not differ in large degrees, as the double complex is periodic. Now a comparision of the $E_2$-page of the spectral sequence allows us to conclude that $[H_+(0^1(E_\bullet))] - [H_-(0^1(E_\bullet))]$ equals the right-hand side of (2.4), which proves the claim of the lemma.

Construction 2.17. Following [5, Proposition 2.5.8], we consider the following construction. As before, let $Y$ be a $DM$-stack and let $X$ be a closed substack. Let $A_\bullet$ be a 2-periodic complex of coherent sheaves on $Y$ such that $A_\bullet$ is cohomologically supported on $X$,

$$\ldots \to A_+ \xrightarrow{d_+} A_- \xrightarrow{d} A_+ \xrightarrow{d_+} A_- \to \ldots.$$  

(2.5)
Let $I_*$ be the folding of the bounded complex $0 \to \text{Img } d_- \to \text{Img } d_+ \to 0$, and similarly let $K_*$ denote the folding of the complex $0 \to \ker d_- \to \ker d_+ \to 0$. The inclusion maps $\text{Img } d_+ \hookrightarrow \ker d_\pm$ induce a map $I_*[1] \to K_*$. We thus have the following triangle in $D(\mathcal{Y}, 0)$:

$$I_*[1] \to K_* \to C_* \xrightarrow{+1} ,$$

(2.6)

where $C_*$ is the mapping cone. Note that by construction, $C_*$ is an object of $D(\mathcal{Y}, 0)$ which is cohomologically supported on $\mathcal{X}$. Let $H_*$ be the 2-periodic complex given by $H_0(A_*)$ in even degree and $H_1(A_*)$ in odd degree, where both the odd and even differentials are zero – that is, $H_* := \Upsilon(H_0(A_*) \oplus (\Upsilon(H_1(A_*))[1])$. The two exact sequences

$$0 \to \text{Img } d_+ \to \ker d_\pm \to H_\pm(A_*) \to 0$$

induce an isomorphism

$$C_* \cong H_*$$

(2.7)

in $D(\mathcal{Y}, 0)$. There is a natural map $I_*[1] \xrightarrow{i} A_*$ induced by the inclusion $I_\pm \xrightarrow{i} A_\pm$. Let $B_* := \text{Cone}(I_*[1] \xrightarrow{i} A_*)$. Let $A^1_k = \text{Spec } k[t]$, and let $p : \mathcal{Y} \times_\mathbb{A}^1 k \to \mathcal{Y}$ be the projection to $\mathcal{Y}$. Let $B[t]_* := p^*(B_*)$; in particular, we have $B[t]_+ = A_+ [t] \oplus I_+ [t]$ and $B[t]_- = A_- [t] \oplus I_- [t]$. Let us define the $f_* : B[t]_* \to I[t]_*$ as follows:

$$f_+ := d^A_+ [t] - \text{tid}_t [t] : A_+ [t] \oplus I_+ [t] \to I_+ [t],$$

(2.8)

$$f_- := d^-_A [t] - \text{tid}_t_- [t] : A_- [t] \oplus I_- [t] \to I_- [t].$$

(2.9)

Note that $f_*$ is a surjective map of complexes, and let us denote by $A_*$ the kernel of $f_*$. We have the following triangle in $D(\mathcal{Y} \times \mathbb{A}^1_k, 0)$:

$$A_* \xrightarrow{i} A_* \to \text{Cone}(t) \xrightarrow{+1} ,$$

(2.10)

where $\text{Cone}(t)$ is the cone of the map $A_* \xrightarrow{i} A_*$ which coincides with $0^1 A_*$, as in Definition 2.14. Further, by construction we have the following exact sequence of 2-periodic complexes:

$$0 \to A_* \xrightarrow{i} A_* \to A_*/tA_* \to 0.$$  

(2.11)

Combining (2.10) and (2.11), we have an isomorphism $0^1 A_* \to A/tA$ in $D(\mathcal{Y} \times \mathbb{A}^1_k, 0)$, where both the complexes are cohomologically supported on $\mathcal{X} \times 0$. Consider the following commutative diagram:
Lemma 2.18. Let $\mathcal{A}$ be a DM-stack and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two closed substacks of $\mathcal{Y}$. Let $E_1^\bullet$ and $E_2^\bullet$ be 2-periodic complexes on $\mathcal{Y}$ such that $E_1^\bullet$ is locally contractible off $\mathcal{X}_1$.

Let $\mathcal{X} = \mathcal{X}_1 \cap \mathcal{X}_2$ and let $i_1 : \mathcal{X}_1 \to \mathcal{Y}$ denote the closed immersion. Then we have

$$h^{\mathcal{Y}}(E_1^\bullet \otimes E_2^\bullet)(-) = h^\mathcal{X} \left( i_1^* E_2^\bullet \right) \left( h^{\mathcal{Y}}(E_1^\bullet)(-) \right).$$

(2.15)

Proof. Let $\mathcal{G}$ be a coherent sheaf on $\mathcal{Y}$ and let $G_\bullet := E_1^\bullet \otimes \mathcal{Y}(\mathcal{G})$. Now substituting $A_\bullet = G_\bullet$ in Construction 2.17 and keeping all other notation the same, we can construct $A_\bullet$ on $\mathcal{Y} \times_k A_k^1$ such that, $0^! A_\bullet = j_{0*} C_\bullet$ and $1^!(A_\bullet) = j_{1*} C_\bullet$, and further the natural map $C_\bullet \to H_\bullet$ is an isomorphism. Therefore, we can conclude that $E_2^\bullet \otimes C_\bullet \to E_2^\bullet \otimes H_\bullet$ is an isomorphism in $D(\mathcal{Y}, 0)$, where now both are cohomologically supported on $\mathcal{X}$. Consider the following diagram:

where in particular the bottom row is an exact sequence, as it is termwise the cokernel of the map from the first row to the second row. Let $j_0 : \mathcal{Y} \times \{0\} \to \mathcal{Y} \times_k A_k^1$ be the natural closed immersion. Now from the definition of $f_\bullet$ (see (2.8)) we note that the kernel of $f_\bullet/t$ is precisely $j_{0*} C_\bullet$. Therefore, we can conclude that $A_\bullet/t A_\bullet \cong j_{0*} C_\bullet$. As $0^! A_\bullet \cong A/t A_\bullet$, we conclude that

$$j_{0*} C_\bullet \cong 0^! A_\bullet.$$

(2.13)

Repeating the argument, but this time for the inclusion Spec $k[t]/[t-1] \to A_k^1$ corresponding to the closed point 1 and with $j_1 : \mathcal{Y} \times \{1\} \to \mathcal{Y} \times A_k^1$ we can on the other hand conclude that

$$j_{1*} A_\bullet \cong 1^! A_\bullet.$$

(2.14)
 Firstly, note that $\mathcal{Y}$ commutes with $i_{1*}$; then it follows from the projection formula \cite[Proposition 2.2.10]{5} (see Remark 2.11) that (2.16) commutes.

Now from the definition we have

$$h_{X_1}(i_1^* E^2_\bullet)(h_{X_1}(E_1^1)(\mathcal{G})) = h_{X_1}(i_1^* E^2_\bullet)((i_{1*})^{-1}([H_+(E_1^1 \otimes \mathcal{Y}(\mathcal{G}))] - [H_-(E_1^1 \otimes \mathcal{Y}(\mathcal{G}))])$$

$$= [H_+(E_1^2 \otimes K)] - [H_-(E_1^2 \otimes K)],$$

where the second equality follows from the commutativity of (2.16). Thus it is enough to show that

$$h_{X_1}(E_1^1 \otimes E_2^2)(\mathcal{G}) = [H_+(E_1^2 \otimes C)] - [H_-(E_1^2 \otimes C)].$$

Recall that we have the following diagram:

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{j_1} & \mathcal{Y} \times_k \mathbb{A}^1_k \\
\downarrow{p} & & \downarrow{p} \\
\mathcal{Y} & & \mathcal{Y}.
\end{array}$$

(2.18)

where $p$ is the projection to \mathcal{Y}. Recall from Definition 2.15 that corresponding to a closed point $1 : \text{Speck} \to \mathbb{A}^1_k$ there is a refined Gysin pullback $1^! : G_0(X \times_k \mathbb{A}^1_k) \to G_0(X)$. The natural map, $j_{1*} : \mathcal{X} \to \mathcal{Y} \times_k \mathbb{A}^1_k$, induces an isomorphism $j_{1*} : G_0(X) \to K_0(\text{Coh}_{\mathcal{X} \times_k \mathcal{Y} \times_k \mathbb{A}^1_k})$. We suppressed the isomorphism $j_{1*}$ in the definition of the refined Gysin pullback. For any coherent sheaf $\mathcal{F} \in G_0(\mathcal{X} \times_k \mathbb{A}^1_k)$, by abusing notation we can consider $1^!(\mathcal{F})$ as an element of $K_0(\text{Coh}_{\mathcal{X} \times_k \mathcal{Y} \times_k \mathbb{A}^1_k})$; and further, as $j_{1*}$ is an isomorphism, to show that the the refined Gysin pullback of $\mathcal{F}$ coincides with an element of $G_0(X)$ it is enough to compare $j_{1*}\mathcal{F}'$ with $1^!(\mathcal{F})$ (by abuse of notation) in $K_0(\text{Coh}_{\mathcal{X} \times_k \mathcal{Y} \times_k \mathbb{A}^1_k})$. This strategy is used in the arguments to follow. Consider the following:

$$j_{1*}([H_+(E_1^2 \otimes E_2^1 \otimes \mathcal{Y}(\mathcal{G}))] - [H_-(E_1^2 \otimes E_2^1 \otimes \mathcal{Y}(\mathcal{G}))])$$

$$= [H_+(j_{1*}(E_1^2 \otimes G))] - [H_-(j_{1*}(E_1^2 \otimes G))]$$

$$= [H_+(p^* E_2^1 \otimes j_{1*} G)] - [H_-(p^* E_2^1 \otimes j_{1*} G)]$$

$$= [H_+(p^* E_2^1 \otimes 1^! \mathcal{A})] - [H_-(p^* E_2^1 \otimes 1^! \mathcal{A})]$$

$$= 1^!(H_+(p^* E_2^1 \otimes \mathcal{A})) - [H_-(p^* E_2^1 \otimes \mathcal{A})].$$

(2.19)

The second equality follows from the projection formula \cite[Proposition 2.2.10]{5} (see Remark 2.11), and the third equality from (2.14). The last equality follows from Lemma 2.16. Arguing similarly, but this time applying (2.13), we have the following:
\[ j_0^*([H_+(E_2^0 \otimes C_\bullet)] - [H_+(E_2^0 \otimes C_\bullet)]) = [H_+(p^* E_2^0 \otimes j_0^* C_\bullet)] - [H_+(p^* E_2^0 \otimes j_0^* C_\bullet)] = [H_+(p^* E_2^0 \otimes 0^1 C_\bullet)] - [H_+(p^* E_2^0 \otimes 0^1 C_\bullet)] = 0'([H_+(p^* E_2^0 \otimes A_\bullet)] - [H_+(p^* E_2^0 \otimes A_\bullet)]). \tag{2.20} \]

Note that the composition of the maps \( X \xrightarrow{\bar{\pi}} X \times_k \mathbb{A}_k^1 \xrightarrow{\pi} X \) corresponds to the identity map of \( X \), where \( j_i \) are regular immersions. It follows that the Gysin pullbacks \( 0^1 \) and \( 1^1 \) correspond to the map induced by \( L_{j_i}^*: D^b(X \times_k \mathbb{A}_k^1) \rightarrow D^b(X) \) for \( i = 0 \) and \( i = 1 \), respectively (see [22, §1.3.2]), which implies that \( j_i^*: G_0(X \times_k \mathbb{A}_k^1) \rightarrow G_0(X) \). As \( G \)-theory is contravariant with respect to maps of finite Tor dimension (see [21, §3]), it follows that \( i^1 \circ p^* = Id : G_0(X) \rightarrow G_0(X) \), which in particular is independent of \( i = 0, 1 \). This in particular implies that

\[ 0^1 = 1^1 : G_0(X \times_k \mathbb{A}_k^1) \rightarrow G_0(X), \tag{2.21} \]

as \( p^* \) is an isomorphism by homotopy invariance ([21, Proposition 3.3(4)]). Now it follows from (2.19) and (2.20) that (2.17) holds, which completes the proof.

Lemma 2.19. Let \( Z \xrightarrow{g_Z} Z \) be a regular immersion of quasi-projective schemes. Let \( Y \) be a DM-stack, with \( X \xrightarrow{i_X} Y \) a closed substack. Let \( p_Y : Y \rightarrow Z \) be a morphism. Consider the following diagram, where each square is cartesian:

\[
\begin{array}{ccc}
X' & \xrightarrow{i'_{X'}} & Y' \\
\downarrow g_X & & \downarrow g_Y \\
X & \xrightarrow{i_X} & Y \\
\downarrow p_X & & \downarrow p_Y \\
& Z & \\
\end{array}
\tag{2.22}
\]

Let \( E_\bullet \) be a 2-periodic complex of vector bundles such that \( E_\bullet \) is locally contractible off \( X \); then for \( G \in G_0(Y) \),

\[ g_X^1 h_{X'}^1(E_\bullet)(G) = h_{X'}^1(g_Y^0(E_\bullet))(g_Y^0(G)), \tag{2.23} \]

where \( g_X^1 : G_0(X) \rightarrow G_0(X') \) and \( g_Y^1 : G_0(Y) \rightarrow G_0(Y') \) denote the refined Gysin pullbacks induced by the regular immersion \( Z' \rightarrow Z \).

Proof. As \( g_Z \) is a regular closed embedding and \( Z' \) is quasi-projective, there exists a resolution of \( g_Z_* \mathcal{O}_{Z'} \) by a complex of locally free sheaves, which we denote by \( P_\bullet \). Further, note that \( Y \) is a monoidal functor with respect to the tensor product; that is, in particular the following diagram commutes:
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\[ D^b(\mathcal{Y}) \xrightarrow{\otimes p_2^*P_*} D^b(\mathcal{Y},0)_{Y'} \]

\[ \gamma \]

\[ D(\mathcal{Y},0) \xrightarrow{\otimes \gamma(p_2^*P_*)} D(\mathcal{Y},0)_{Y'}. \]

(2.24)

With notations as in Lemma 2.18, let \( E_1^* = \gamma(p_2^*P_*) \) and \( E_2^* = E_* \). Then we have \( \mathcal{X}_1 = \mathcal{Y}', \mathcal{X}_2 = \mathcal{X} \) and \( \mathcal{X}_1 \cap \mathcal{X}_2 = \mathcal{X}' \). Now from Lemma 2.18 we have the following:

\[ h^Y_{\mathcal{X}'}(\gamma(p_2^*P_*) \otimes E_*)(G) = h^Y_{\mathcal{X}'}(g_2^*E_*)(h^Y_{\mathcal{X}'}(\gamma(p_2^*P_*))(G)) \]

\[ = h^Y_{\mathcal{X}'}(g_2^*E_*)(g_2^1(G)), \] (2.25)

where the last equality follows from the definition of \( h^Y_{\mathcal{X}'} \) (see Definition 2.13), the definition of the refined Gysin pullback (see Definition 2.15) and the fact that (2.24) commutes. Arguing similarly, we apply Lemma 2.18 with \( E_1^* = E_* \) and \( E_2^* = \gamma(p_2^*P_*) \). Now we have the following:

\[ h^Y_{\mathcal{X}'}(E_* \otimes \gamma(p_2^*P_*))(G) = h^Y_{\mathcal{X}'}(\gamma(i_2^*p_2^*P_*))(h^Y_{\mathcal{X}'}(E_*)(G)) \]

\[ = g_1^1 h^Y_{\mathcal{X}'}(E_*)(G). \] (2.26)

Now note that as the tensor product of 2-periodic complexes is commutative, the maps \( h^Y_{\mathcal{X}'}(\gamma(p_2^*P_*) \otimes E_*)(- \) and \( h^Y_{\mathcal{X}'}(E_* \otimes \gamma(p_2^*P_*))(- \) coincide. Now comparing (2.25) and (2.26) completes the proof of the lemma. \( \square \)

We restate Lemma 2.19 in the setup of Definition 2.14, which is the form in which we will want to apply it. We recall the notations for the convenience of the reader.

**Lemma 2.20.** Let \( A_* \) be a 2-periodic complex of vector bundles on \( \tilde{\mathcal{Y}} \) and let \( \tilde{G} \in G_0(\tilde{\mathcal{Y}}) \). Assume that the following are satisfied:

1. There is a morphism \( \tilde{\mathcal{Y}} \to \mathbb{A}^1_k \).
2. There is a closed substack \( \mathcal{X} \times_k \mathbb{A}^1_k \hookrightarrow \tilde{\mathcal{Y}} \) such that the composition \( \mathcal{X} \times_k \mathbb{A}^1_k \to \tilde{\mathcal{Y}} \to \mathbb{A}^1_k \) is a projection and \( A_* \) is locally contractible off \( \mathcal{X} \times_k \mathbb{A}^1_k \).
3. Let \( p_t : \tilde{\mathcal{Y}}|_t \to \tilde{\mathcal{Y}} \) be the natural immersion. Suppose that there exists \( t \in \mathbb{A}^1_k \setminus \{0\} \) such that \( h^Y_{\mathcal{X}'}(p_t^*A_*)(t^1\tilde{G}) = 0 \).

Then \( h^Y_{\mathcal{X}'}(p_0^*A_*)(0^1\tilde{G}) = 0 \). Further, when \( \tilde{\mathcal{Y}} = \mathcal{Y} \times_k \mathbb{A}^1_k \), then \( h^Y_{\mathcal{X}'}(p_0^*A_* = h^Y_{\mathcal{X}'}(p_0^*A_* : G_0(\mathcal{Y}) \to G_0(\mathcal{X}) \).

**Proof.** The proof of the lemma directly follows from Lemma 2.19 and the \( \mathbb{A}^1 \)-homotopy invariance \( G_0(\mathcal{X} \times_k \mathbb{A}^1_k) \cong G_0(\mathcal{X}) \) [21, Proposition 3.3(4)]. We briefly outline the proof of the first claim, as the second claim follows similarly. Consider the following cartesian
squares:
\[
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \tilde{\mathcal{Y}}_t \\
\downarrow p' & & \downarrow p_t \\
\mathcal{X} \times_k \mathbb{A}^1_k & \longrightarrow & \tilde{\mathcal{Y}} \\
\downarrow t & & \downarrow t \\
\mathbb{A}^1_k & \longrightarrow & \mathbb{A}^1_k,
\end{array}
\]
where \( t : \text{Spec} \mathbb{K} \rightarrow \mathbb{A}^1_k \) corresponds to the closed point \( t \). Then from Lemma 2.16 we have the following equality:
\[
t' h_{\mathcal{X} \times_k \mathbb{A}^1_k}^\mathcal{Y}(A_\bullet)(\tilde{G}) = h_{\mathcal{X} \times_k \mathbb{A}^1_k}^\mathcal{Y}(p^*_t A_\bullet)(t^!(\tilde{G})).
\]
From hypothesis, there exists a \( t \in \mathbb{A}^1_k \setminus \{0\} \) such that the right-hand side of (2.28) is equal to zero. We know that
\[
0! = t! : G_0(\mathcal{Y}' \setminus \mathcal{X} \times_k \mathbb{A}^1_k) \rightarrow G_0(\mathcal{X}),
\]
(see (2.21)). Consider the following:
\[
h_{\mathcal{X} \times_k \mathbb{A}^1_k}^\mathcal{Y}(p^*_t A_\bullet)(0^!(\tilde{G})) = t! h_{\mathcal{X} \times_k \mathbb{A}^1_k}^\mathcal{Y}(A_\bullet)(\tilde{G})
\]
(2.29)
where all the equalities follow from (2.28) and the hypothesis of the lemma.

Lemma 2.21. Let \( \mathcal{Y} \) and \( \mathcal{Y}' \) be DM-stacks and let \( g : \mathcal{Y}' \rightarrow \mathcal{Y} \) be a proper representable morphism. Consider the following cartesian diagram:
\[
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{Y}' \\
\downarrow g_X & & \downarrow g_Y \\
\mathcal{X} & \longrightarrow & \mathcal{Y},
\end{array}
\]
where \( \mathcal{X} \) and \( \mathcal{X}' \) are closed substacks of \( \mathcal{Y} \) and \( \mathcal{Y}' \), respectively. Let \( E_\bullet \) be a 2-periodic complex of vector bundles such that \( E_\bullet \) is locally contractible off \( \mathcal{X} \). Then for every \( G \in G_0(\mathcal{Y}') \),
\[
h_X^\mathcal{Y}(E_\bullet)(g_Y^* G) = g_X(h_X^\mathcal{Y}(g_Y^* E_\bullet)(G)) \quad \text{in} \quad G_0(\mathcal{X}).
\]
Proof. As \( g_Y \) is a representable proper morphism, it follows that \( Rg_Y^\mathcal{Y} \) has finite cohomological dimension, and hence it follows from Remark 2.11 that \( Rg_Y^\mathcal{Y} : D^{\text{abs}}[\text{Fact}(\mathcal{Y},0)] \rightarrow D^{\text{abs}}[\text{Fact}(\mathcal{Y},0)] \) is well defined. Let \( \mathcal{U} = \mathcal{Y}' \setminus \mathcal{X} \) and let us consider the following diagram:
\[
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{Y}' \\
\downarrow g_X & & \downarrow g_Y \\
\mathcal{X} & \longrightarrow & \mathcal{Y} \\
\downarrow g_X & & \downarrow g_U \\
\mathcal{X}' \setminus \mathcal{X} & \longrightarrow & \mathcal{Y} \setminus \mathcal{X},
\end{array}
\]
where each square is cartesian. As \( E_\bullet \) is locally contractible off \( \mathcal{X} \), it follows that \( g_Y^* E_\bullet \) is locally contractible off \( \mathcal{X}' \). Let \( G \) be a coherent sheaf on \( \mathcal{Y}' \) and let us consider
the 2-periodic complex $G_\bullet := g_2^\ast E_\bullet \otimes \Upsilon(G)$. From Corollary 2.10, it follows that $G_\bullet$ is cohomologically supported on $\mathcal{X}'$. Further, from the projection formula we have

$$Rg_{\mathcal{Y}*}G_\bullet \cong E_\bullet \otimes Rg_{\mathcal{Y}*}\Upsilon(G) \cong E_\bullet \otimes \Upsilon(Rg_{\mathcal{Y}*}G).$$

(2.31)

Now note that $Rg_{\mathcal{Y}*}G$ is an object of the bounded derived category of quasi-coherent sheaves with coherent cohomology – that is, it is an object of $D^b_{\text{coh}}(\text{QCoh}\mathcal{Y})$. Therefore, $\Upsilon(Rg_{\mathcal{Y}*}G)$ is an object of $D(\mathcal{Y},0)$. In particular it, follows from (2.31) that $Rg_{\mathcal{Y}*}G_\bullet$ belongs to $D(\mathcal{Y},0)$.

The strategy of the proof now is to reduce the argument to folding complexes via Construction 2.17. First note that from (2.29) the following diagram commutes:

$$\begin{array}{ccc}
G_0(\mathcal{X}') & \xrightarrow{i'_*} & K_0(\text{Coh}_{\mathcal{X}',\mathcal{Y}}') \\
g_{\mathcal{X}*} & & g_{\mathcal{Y}*} \\
G_0(\mathcal{X}) & \xrightarrow{i_*} & K_0(\text{Coh}_{\mathcal{X},\mathcal{Y}}),
\end{array}$$

(2.32)

where the top and bottom horizontal maps are isomorphisms by dévissage. Thus, to prove the lemma it is enough to show that

$$h_{\mathcal{X}}^{\mathcal{Y}}(g_{\mathcal{X}*}\mathcal{G}) = g_{\mathcal{Y}*}(h_{\mathcal{X}}^{\mathcal{Y}}(g_2^\ast E_\bullet)(\mathcal{G})) \text{ in } K_0(\text{Coh}_{\mathcal{X},\mathcal{Y}}),$$

where we ignore the identifications via dévissage induced by $i_*$ and $i'_*$. Now let us recall two basic facts. First, for an abelian category $\mathcal{C}$, there is a natural isomorphism $K_0(\mathcal{C}) \cong K_0(D^b(\mathcal{C}))$ which is induced by sending any object of $\mathcal{C} \in \mathcal{C}$ to the complex with $d$-periodic complex.

Further, note that for any $C_\bullet \in D^b(\text{Coh}\mathcal{Y})_{\mathcal{X}}$, we have $\Sigma_i(-1)^i[H^i(C_\bullet)] = [H_+(\Upsilon(C_\bullet)) - H_-(\Upsilon(C_\bullet))]$ in $K_0(\text{Coh}_{\mathcal{X},\mathcal{Y}})$. Now essentially from the definition of push-forward at the level of $G$-theory, it follows that for a complex $W_\bullet = \Upsilon(C_\bullet)$, where $C_\bullet \in D^b(\text{Coh}\mathcal{Y})_{\mathcal{X}}$, we have the following:

$$g_{\mathcal{Y}*}([H_+(W_\bullet)] - [H_-(W_\bullet)]) = [H_+(Rg_{\mathcal{Y}*}W_\bullet)] - [H_-(Rg_{\mathcal{Y}*}W_\bullet)] \text{ in } K_0(\text{Coh}_{\mathcal{X},\mathcal{Y}}).$$

(2.33)

Now with notations as in Construction 2.17, on $\mathcal{Y}'$ with $A_\bullet := g_2^\ast E_\bullet \otimes \Upsilon(G)$ and keeping the rest of the notations as in the construction, we can construct $A_\bullet$ on $\mathcal{Y}' \times A^1_k$ such that $0^!A_\bullet \cong j_0^{-1}C_\bullet$ a folding and $1^!A_\bullet \cong j_1^{-1}(A_\bullet) = j_1^*(g_2^\ast E_\bullet \otimes \Upsilon(G))$ the original complex.
Lemma 2.16. Consider the following:

\begin{align}
0 & \xrightarrow{j_i} \mathcal{Y} \\
\mathcal{Y} & \xrightarrow{j_i'} \mathcal{Y} \times_k \mathbb{A}_k^1
\end{align}

where \(j_i\) and \(j'_i\) correspond to the closed immersion along \(\mathcal{Y} \times \{ i \}\) and \(\mathcal{Y}' \times \{ i \}\), respectively. As \(R\tilde{g}_k : D(\mathcal{Y}' \times_k \mathbb{A}_k^1) \rightarrow D(\mathcal{Y} \times_k \mathbb{A}_k^1)\) is a triangulated functor, it preserves triangles; thus, applying this to (2.13) and (2.14), we have the following triangles:

\begin{align}
R\tilde{g}_s A_\bullet & \xrightarrow{t} R\tilde{g}_s A_\bullet \rightarrow R\tilde{g}_s 0^1 A_\bullet \xrightarrow{+1} \\
R\tilde{g}_s A_\bullet & \xrightarrow{(t-1)} R\tilde{g}_s A_\bullet \rightarrow R\tilde{g}_s 1^1 A_\bullet \xrightarrow{+1}
\end{align}

in \(D(\mathcal{Y} \times_k \mathbb{A}_k^1)\). Note that the cone of the first morphisms in (2.35) and (2.36) coincide with \(0^1 R\tilde{g}_s A_\bullet\) and \(1^1 R\tilde{g}_s A_\bullet\), respectively. Hence we conclude that

\begin{align}
0^1 (R\tilde{g}_s A_\bullet) & \cong R\tilde{g}_s (0^1 A_\bullet), \\
1^1 (R\tilde{g}_s A_\bullet) & \cong R\tilde{g}_s (1^1 A_\bullet).
\end{align}

Now note that \(0^1 (|H_+(R\tilde{g}_s A_\bullet)| - |H_-(R\tilde{g}_s A_\bullet)|) = 1^1 (|H_+(R\tilde{g}_s A_\bullet)| - |H_-(R\tilde{g}_s A_\bullet)|)\) in \(G_0(\text{Coh} \mathcal{X} \mathcal{Y})\).

The proof of the lemma now follows from combining these observations with Lemma 2.16. Consider the following:

\begin{align}
g_{\gamma_*} ([H_+(A_\bullet)] - [H_-(A_\bullet)]) & = g_{\gamma_*} ([H_+(C_\bullet)] - [H_-(C_\bullet)]) \\
& = [H_+(Rg_{\gamma_*} C_\bullet)] - [H_-(Rg_{\gamma_*} C_\bullet)] \\
& = 0^1 ([H_+(R\tilde{g}_s A_\bullet)] - [H_-(R\tilde{g}_s A_\bullet)]) \\
& = 0^1 (|H_+(R\tilde{g}_s A_\bullet)| - |H_-(R\tilde{g}_s A_\bullet)|) \\
& = [H_+(Rg_{\gamma_*} A_\bullet)] - [H_+(Rg_{\gamma_*} A_\bullet)],
\end{align}

where the first equality follows from (2.7) and the second equality follows from (2.33), as \(C_\bullet\) is a folding complex. The third equality follows from the commutativity of (2.35), (2.37) and Lemma 2.16. The second-to-last equality follows from Lemma 2.16 and the fact that \(G_0(\mathcal{X} \times \mathbb{A}_k^1) \cong G_0(\mathcal{X})\) (see Lemma 2.20). The last equality again follows from the commutativity of (2.36) and (2.38) and Lemma 2.16.

\[\Box\]

Remark 2.22. The reader might note that in Definition 2.13, instead of working with a 2-periodic complex of vector bundles \(E_\bullet\) on \(\mathcal{Y}\) which is locally contractible off \(\mathcal{X}\), we could have imposed only the condition that \(E_\bullet\) is absolutely acyclic off \(\mathcal{X}\). This remark also applies to the rest of the functoriality statements of Section 2.3, as the proofs go through as is.
3. Koszul complexes

In [11, Theorem 4.1], Kiem and Li defined a $K$-theoretic version of the cosection-localized Gysin map. In this section we show that when $E_\bullet$ is the Koszul $2$-periodic complex, the map $h^Y_X(E_\bullet)$ is equivalent to the cosection-localized Gysin map (Theorem 3.5). We apply the equivalence to prove a comparison of virtual structure sheaves (Theorem 4.6). The reader is advised that what follows is precisely what was done in [12, §2] for the localized Chern character, but we are interested in similar properties for the map $h^Y_X$ (Definition 2.13). Most of the statements and their proofs rely on the constructions of [12]. In an effort to make the paper reasonably self-contained, we recall the important definitions and constructions and refer the reader [12] for further details.

We recall the notion of a Koszul $2$-periodic complex.

**Definition 3.1** (Koszul $2$-periodic complex [12, §2.2]). Let $E$ be a vector bundle on a $\mathcal{DM}$-stack $Y$ and let $\alpha \in H^0(Y, E^\vee)$, $\beta \in H^0(Y, E)$ be sections such that $\langle \alpha, \beta \rangle = 0$, where the bracket $\langle , \rangle$ denotes an evaluation, or equivalently a pairing. Let $\{\alpha, \beta\}$ denote the following $2$-periodic complex of vector bundles:

$$\bigoplus_k \wedge^{2k-1} E^\vee \overset{\wedge \alpha + \beta}{\longrightarrow} \bigoplus_k \wedge^{2k} E^\vee,$$

(3.1)

where $\iota_\beta$ denotes the interior product by $\beta$. The convention is that the direct sum of even wedges sits on even degree. $\{\alpha, \beta\}$ is called the Koszul ($2$-periodic) complex associated to the cosection $\alpha$ and the section $\beta$.

**Remark 3.2.** It follows from [5, Proposition 2.3.3] that the Koszul complex $\{\alpha, \beta\}$ is locally contractible off $X = Z(\alpha, \beta) = Z(\alpha) \cap Z(\beta)$. Further, from [12, equation 2.1] we have

$$\{\alpha, \beta\} = \{\beta, \alpha\} \otimes \Upsilon(\det E^\vee [\text{rank} E]).$$

(3.2)

We next prove a $K$-theoretic analogue of the splitting principle proved in [12, §2.4]. We recall the setup and notations as in [12]. Let $E$, $\alpha$, $\beta$ and $\{\alpha, \beta\}$ be as in Definition 3.1. Let $Q$ be a vector bundle on $Y$ which is a quotient of $E$, and let us denote by $K$ the kernel of the quotient map. We thus have the following exact sequence:

$$0 \to K \overset{f}{\longrightarrow} E \to Q \to 0.$$

(3.3)

Let us assume that the cosection $\alpha$ factors as a cosection of $Q$, which we denote by $\alpha_Q$, and let $\beta_K$ be a section of $K$. Recall from [12, §2.4.1] that we can construct a vector bundle $P$ on $Y \times \text{Spec } k[t]$ with a cosection $\alpha_P$ and a section $\beta_P$ induced by $\alpha$, $\beta$ and $\beta_K$ such that $P|_{t=0} \cong Q \oplus K$ and $P|_{t=1} \cong E$. With the setup as before, we prove the $K$-theoretic splitting principle analogous to [12, Lemma 2.4].

**Lemma 3.3.** With notations as before, let us further assume that $Z(\alpha_P, \beta_P) \subset X' \times \mathbb{A}^1_k$ for some closed stack $X'$ of $Y$. Then for any $\mathcal{G} \in G_0(Y)$,

$$h^Y_{X'}(\{\alpha, \beta\})(\mathcal{G}) = h^Y_{X'}(\{\alpha_Q, 0\} \otimes \{0, \beta_K\})(\mathcal{G}).$$

(3.4)
Proof. The proof follows from Lemma 2.20, with $\widetilde{\mathcal{Y}} = \mathcal{Y} \times \mathbb{A}^1$, $\mathcal{X} = \mathcal{X}'$ and $\mathcal{A}_\bullet = \{\alpha_F, \beta_F\}$, the Koszul (2-periodic) complex associated to the cosection $\alpha_F$ and section $\beta_F$ of $P$. \hfill \square

3.1. Tautological Koszul complex

The goal of this section is to prove Theorem 3.5, which is a comparison result between the cosection-localized Gysin map constructed by Kiem and Li (see [11, Theorem 4.1]) and the map defined in Definition 2.13. For the case of the localized Chern character this is proved in [12, Theorem 2.6]; as the strategy of the proof is completely identical, we first recall the setup of [12, §2.3].

Let $M$ be a $\mathcal{D}_M$-stack and let $F$ be a vector bundle on $M$. Let $\sigma \in H^0(M, F^\vee)$ be a cosection of $F$. Let us denote the total space of $F$ by $|F|$, and by $p : |F| \to M$ the projection map. Then $\sigma$ defines a regular function $w_\sigma : |F| \to \mathbb{A}^1$. Let us denote by $t_F$ the tautological section in $H^0(|F|, p^*F)$, and further note that $\langle p^*\sigma, t_F \rangle = w_\sigma$. Now we can define a matrix factorization $\{p^*\sigma, t_F\}$ for the regular section $w_\sigma$ in a way similar to (3.1). Let $Z(w_\sigma)$ denote the zero locus of $w_\sigma$; then $\{p^*\sigma, t_F\}$ becomes a 2-periodic complex when restricted to $Z(w_\sigma)$. Here, $Z(w_\sigma)$, $p^*F|_{Z(w_\sigma)}$, $p^*\sigma$ and $t_F$ play the roles of $\mathcal{Y}, E, \alpha$ and $\beta$, respectively, in Definition 3.1, where we change notation to maintain consistency with [12]. In the particular case when $\sigma = 0$, we note that $h^F_M(\{(0, t_F)\})$ coincides with the refined Gysin pullback by the zero section of $F$, as $\{(0, t_F)\}$ is the Koszul resolution of $0_\ast \mathcal{O}_M$.

Following [10, Theorem 4.1], we consider the blowup $M'$ of $M$ along $Z(\sigma)$. We consider the notations and setup as in [12, §2.4.2]. Let $F'$ and $\sigma'$ denote the pullback of $F$ and $\sigma$ to $M'$, respectively, and let $D$ denote the exceptional divisor. The cosection $\sigma' : F' \to \mathcal{O}_{M'}$ factors as follows:

$$
\xymatrix{ F' & \mathcal{O}_{M'}(-D) \ar[r] & 0 \\
\sigma' \ar[u] \ar[r] & s_D \quad \ar[r] & 0_{M'}.
}
$$

Let $K$ be the kernel of a surjective morphism $F' \to \mathcal{O}_{M'}(-D)$. Let $b : Z(\sigma') = D \to Z(\sigma)$ and $p' : |F'| \to M'$ denote the natural projection maps. We denote by $g$ the natural map $g : |K| \hookrightarrow Z(w_\sigma) \to Z(w_\sigma)$, which is projective, hence in particular note that it is representable.

In the next lemma we summarise an intermediate construction as in the proof of [11, Theorem 4.1]. Using the Localization sequence for $G$-theory, it follows that we have the following right exact sequence:

$$G_0(|K|) \to G_0(|K|) \xrightarrow{j_\ast} G_0(|K| \setminus |F'|_{Z(\sigma')}) \to 0. \quad (3.5)$$

Further, by construction it follows that $Z(w_\sigma) \setminus |F|_{Z(\sigma')} \cong Z(w_\sigma) \setminus |F'|_{Z(\sigma')} \cong |K| \setminus |F'|_{Z(\sigma')}$, where the first isomorphism comes from the fact that $M \setminus Z(\sigma) \cong M \setminus Z(\sigma')$ and the second isomorphism comes from the fact that $Z(w_\sigma) \cong |K| \cup |F'|_{Z(\sigma')}$. For $G \in G_0(Z(w_\sigma))$, let $G_1$ denote its pullback to $G_0(Z(w_\sigma) \setminus |F|_{Z(\sigma)})$. Under the isomorphism of $Z(w_\sigma) \setminus |F|_{Z(\sigma)}$
with $|K|/|F'|_{Z(σ')}$, let $G_2$ denote its image in $G_0(|K|/|F'|_{Z(σ')})$. Let $\tilde{G} ∈ G_0(|K|)$ be such that $j^*(\tilde{G}) = G_2$, which always exists by (3.5).

Consider the Localization sequence

$$G_0(|F|_{Z(σ)}) \xrightarrow{i^*} G_0(Z(w_0)) \xrightarrow{i^*} G_0(|K|/|F'|_{Z(σ')}) → 0.$$ 

By construction, $i^*(G - g_σ \tilde{G}) = 0$. Hence there is an element (not necessarily unique) $S ∈ G_0(|F|_{Z(σ)})$ such that $i_*(S) = G - g_σ \tilde{G}$.

**Lemma 3.4.** With notations as before, we have the following:

$$h^Z_{Z(σ)}(\{p^*σ, t_F\})(G) = h^F_{Z(σ)}(\{0, t_F|_{Z(σ)}\})(S) + b_*(−[O_D(D)] \otimes h^K_{M'}(\{0, t_K\})(\tilde{G})).$$

(3.6)

In particular, the right-hand side is independent of the choice of $\tilde{G}$ and $S$.

**Proof.** Since $h^F_{Z(σ)}(\{0, t_F|_{Z(σ)}\})(S) = h^Z_{Z(σ)}(\{p^*σ, t_F\})(G - g_σ \tilde{G})$, by Lemma 2.21, it is enough to show that $h^Z_{Z(σ)}(\{p^*σ, t_F\})(g_σ \tilde{G}) = b_*(−[O_D(D)] \otimes h^K_{M'}(\{0, t_K\})(\tilde{G})).$ We obtain

$$h^Z_{Z(σ)}(\{p^*σ, t_F\})(g_σ \tilde{G}) = b_*(h^K_{M'}(\{p^*σ', t_F'\}|_{K'})(\tilde{G})) = b_*(h^K_{M'}(\{s_D, 0\} \otimes \{0, t_K\})(\tilde{G})) = b_*(−[O_D(D)] \otimes h^K_{M'}(\{0, t_K\})(\tilde{G})).$$

Here, the first equality is by Lemma 2.21; the second equality is by Lemma 3.3; the third equality comes from Lemma 2.18; and the fourth equality follows from the short exact sequence

$$0 → O_M'(-D) \xrightarrow{s_D} O_M' → O_D → 0$$

and (3.2). $\square$

### 3.2. Cosection Localization

Let notations be as in Section 3.1, with $M$ a $DM$-stack and $σ$ a cosection of $F$. In [10], Kiem and Li defined the cosection-localized Gysin map

$$0^1_{F,σ} : G_0(Z(w_0)) → G_0(Z(σ)).$$

We note here that our notation differs from theirs, where $Z(w_0)$ and $Z(σ)$ are denoted by $F(σ)$ and $M(σ)$, respectively. The following theorem is a direct consequence of Definition 2.13, Lemma 3.4 and [11, Theorem 4.1] and is the $K$-theoretic counterpart to [12, Theorem 2.6]:

**Theorem 3.5.** $0^1_{F,σ} = h^Z_{Z(σ)}(\{p^*σ, t_F\})$ as homomorphisms $G_0(Z(w_0)) → G_0(Z(σ))$.

**Proof.** The proof of the theorem follows from directly comparing Lemma 3.4 and the construction of the cosection-localized Gysin map (see [11, Theorem 4.1]). We briefly indicate the idea for the convenience of the reader. Let $0 : M → F$ denote the zero section of $F$. Recall that when $σ = 0$, the map $h^F_M(\{0, t_F\}) : G_0(|F|) → G_0(M)$ coincides with
the refined Gysin pullback defined by the zero section of \( F \), as \( \{0, t_F\} \) corresponds to the folding of the Koszul resolution of \( 0\ast O_M \). The map \( \theta^!_{F, \sigma} \) is defined in [11, Theorem 4.1], in particular see [11, Equation 4.9]. From the previous observation and Lemma 3.4, for any \( G \in G_0(Z(w_\sigma)) \) we have
\[
\text{h}^Z_{Z(\sigma)}((p^* \sigma, t_F))(G) = \text{h}^{|Z(\sigma)|}_Z((0, t_{F|Z(\sigma)}))(S) + b_*(-[O_D(D)] \otimes \text{h}^{K_M}_{M}(\{0, t_K\})(\tilde{G})), \tag{3.7}
\]
where the notations are as in Section 3.1. The claim of the theorem directly follows by comparing [11, Theorem 4.1 and equation 4.9] with (3.7).

In [11, Theorem 5.1], Kiem and Li defined the cosection-localized virtual structure sheaf and related it to the virtual structure sheaf defined by Lee in [13, §2.3]. As a direct consequence of Theorem 3.5, in Corollary 3.6 we show that we can define the cosection-localized structure sheaf using Definition 2.13. This result is a \( K \)-theoretic analogue of [12, Corollaries 2.7 and 2.8]. The assumptions are as in [12, §2.5], which we briefly recall. Let \( A \xrightarrow{d} F \) be a complex of vector bundles on \( M \) whose dual gives rise to a perfect obstruction theory relative to a pure dimensional stack \( \mathcal{M} \), and let \( C \) denote the corresponding cone in \( F \) and \( \sigma \) a cosection of \( F \) such that \( \sigma \circ d = 0 \). With respect to the cosection \( \sigma \), let \( \mathcal{O}_{M, \sigma}^{\text{vir}} \) denote the cosection-localized virtual structure sheaf (see [11, Theorem 5.1]).

**Corollary 3.6.** With notations as before, the following holds:
\[
[\mathcal{O}_{M, \sigma}^{\text{vir}}] = 0^!_{F, \sigma}[\mathcal{O}_C] = \text{h}^Z_{Z(\sigma)}((p^* \sigma, t_F))(\mathcal{O}_C).
\]

4. A comparison result

In this section, we assume that the base field is \( \mathbb{C} \). The main goal of this section is to prove the \( K \)-theoretic analogue of [12, Theorem 3.2]. Before we state the main theorem, we need to recall the setup and notations from [12]. In Section 4.1 we recall all the notations and basic theorems which will be needed to state the main theorem precisely; Section 4.2 is dedicated to the proof of Theorem 4.6. The proof closely follows the strategy in [12, §3.2] and relies on the geometric constructions proved there. We apply Lemmas 2.18 and 2.20 and Theorem 3.5 to prove Theorem 4.6. The assumptions and notations used are as in [12], so we summarise them to make the text comprehensible.

4.1. Setup

[[12, §3.1]] We briefly recall the basic setup and notations, and refer the reader to [12, §3.1] for a detailed discussion.

**Geometric side of a hybrid gauged linear sigma model** \((V_1 \oplus V_2, G, \theta, w)\): Let \( G \) be a reductive linear algebraic group and let \( V_1 := \mathbb{A}^m_C \) and \( V_2 := \mathbb{A}^n_C \) be affine varieties over \( \text{Spec} \mathbb{C} \). Let us assume that \( V_1 \) and \( V_2 \) have a linear \( G \)-action. Note that \( V_1 \) and
V_2 could also be viewed as vector spaces over $\mathbb{C}$, and when confusion does not arise we also denote the underlying vector spaces with the same notation. Let $\theta$ be a character of $G$ such that the semistable and stable points of $V_1$ with respect to $\theta$ coincide, which we denote by $V_1^{ss}(\theta)$. We denote by $E$ the quotient stack $[(V_1^{ss}(\theta) \times_{\mathbb{C}} V_2)/G]$, which is naturally the total space of a vector bundle on the stack $[V_1^{ss}(\theta)/G]$. Note that $w \in (\text{Sym}^2 V_1^\vee) \otimes V_2^\vee)^G$ induces a regular morphism $V_1 \to V_2^\vee$, which we denote by $f$; a section $[V_1^{ss}(\theta)/G] \to E^\vee$, which we denote by $s$; and a regular morphism $E \to \mathbb{A}^1_C$, which is denoted by $w$. Let $Z(dw)$ denote the critical locus of $w$ and $Z(s)$ denote the zero locus of $s$. Note that $Z(dw) \cap [V_1^{ss}(\theta)/G] = Z(s)$, where we further work under the assumption that $Z(dw)$ is a smooth closed sublocus in $[V_1^{ss}(\theta)/G]$ of codimension $n$.

Moduli spaces:

1. Let $\mathcal{M}_{g,k}(BG,d)$ denote the moduli space of principal $G$-bundles on genus $g$, $k$-pointed prestable orbi-curves $C$ with degree $d$ such that the associated classifying map $C \to BG$ is representable. $\mathcal{M}_{g,k}(BG,d)$ is a smooth Artin stack (see [3, §2.4.5], [7, §2.1]). Let $\pi : \mathcal{C} \to \mathcal{M}_{g,k}(BG,d)$ and $\mathcal{P}$ denote the universal curve and the universal $G$-bundle on $\mathcal{C}$, respectively. For notational convenience, we let $\mathfrak{B} := \mathcal{M}_{g,k}(BG,d)$.

2. Let $Q_{g,k}^\xi(Z(s),d)$ denote the moduli space of genus $g$, $k$-pointed $\xi$-stable quasimaps to $Z(s)$ of degree $d$, which is a separated $\mathcal{D}\mathcal{M}$-stack (see [3, §2]). Let $X := Z(s)$; then we let $Q_X := Q_{g,k}^\xi(Z(s),d)$ and $Q_{V_1} := Q_{g,k}^\xi(V_1/G,d)$.

3. [12, §3.1.2] Let $L G Q_{g,k}^\epsilon(E,d)'$ denote the moduli space of genus $g$, $k$-pointed, degree $d$, $\epsilon$-stable quasimaps to $V_1/G$ with $p$-fields (see [2, 5, 8]), which is a separated $\mathcal{D}\mathcal{M}$-stack. When confusion does not arise, we abbreviate this notation to $L G Q'$.

Perfect obstruction theories:

1. Consider the notations as in [12, §3.1.1]. Let $V_i := \mathcal{P} \times_G V_i$ be the bundles on $\mathcal{C}$ for $i = 1, 2$, and let us denote by $u$ the universal section. Note here that we use the same abuse of notation as in [12], where $\mathcal{C}$ and $\mathcal{P}$ are used to denote the universal curve and universal $G$-bundle on all moduli spaces, defined earlier.

2. $Q_X$ has a canonical perfect obstruction theory relative to $\mathcal{M}_{g,k}(BG,d)$ given by the dual to

$$R\pi_*(u^*(\mathcal{P} \times_G df^\vee)) : R\pi_*V_1 \to R\pi_*V_2^\vee$$

(4.1)

(see [12, §3, equation 3.2].

3. $L G Q'$ has a canonical perfect obstruction theory relative to $\mathcal{M}_{g,k}(BG,d)$ given by

$$R\pi_*(V_1 \oplus V_2 \otimes \omega_\mathcal{C})^\vee,$$

(4.2)

where $\omega_\mathcal{C}$ is a dualising sheaf on $\mathcal{C}$ relative to $L G Q'$. Further, there is cosection $d w_{L G Q'} : R^1\pi_*(V_1 \oplus V_2 \otimes \omega_\mathcal{C}) \to \mathcal{O}_{L G Q'}$ (see [12, §3.1.2]).
Definition 4.1 \((\det R\pi_* V \& \chi^{\mathrm{gen}}(R\pi_* V))\). Let \(\pi : \mathcal{C} \rightarrow Q_{g,k}(V_1//G, d)\) be the universal curve and let \(V\) be a vector bundle on \(\mathcal{C}\). As \(\pi\) is ample, \(V\) can always fit into a short exact sequence \(0 \rightarrow V \rightarrow A \rightarrow B \rightarrow 0\) for some \(\pi\)-acyclic vector bundles \(A\) and \(B\), and further, \(\pi_* A\) and \(\pi_* B\) are vector bundles on \(Q_{g,k}(V_1//G, d)\). Then we define
\[
\det R\pi_* V := \det \pi_* A \otimes (\det \pi_* B)^\vee, 
\tag{4.3}
\]
called the determinant of \(R\pi_* V\). Further note that \(R^i\pi_* V = 0\) is zero for \(i \neq 0, 1\). Let us define
\[
\chi^{\mathrm{gen}}(R\pi_* V) := \dim_{\mathrm{gen}} R^0\pi_* V - \dim_{\mathrm{gen}} R^1\pi_* V, 
\tag{4.4}
\]
where \(\dim_{\mathrm{gen}}\) denotes the generic dimension of the respective coherent sheaves. \(\chi^{\mathrm{gen}}(R\pi_* V)\) is called the generic virtual rank of \(R\pi_* V\).

Remark 4.2. Note that Definition 4.1 is independent of the choice of the \(\pi\)-acyclic resolution. Suppose our moduli spaces are quasi-projective, and \(0 \rightarrow A_1 \rightarrow B_1 \rightarrow 0\) and \(0 \rightarrow A_2 \rightarrow B_2 \rightarrow 0\) are two complexes of vector bundles on the moduli space (not on the universal curves!) which are quasi-isomorphic; then the determinants as defined in (4.3) coincide for both the complexes. In particular, note that this is the case for \(Q_X\), \(Q_{V_1}\) and \(LGQ\) when \(\varepsilon = \infty\).

In Lemmas 4.3, 4.4 and 4.5, we briefly summarise the constructions of [12, §3.2.1] that are required for the proof of the main theorem.

Lemma 4.3 (See [12, §3.2.1]).

1. On some nonempty open substack \(\mathcal{B}^e \subset \mathcal{B}\), there are chain map representations
   \[
   [A_1 \xrightarrow{d_{A_1}} B_1] \text{ of } R\pi_* V_1 \text{ and } [Q \xrightarrow{d_Q} P'\vert_1] \text{ of } R\pi_* V_2'.
   \]

2. There is an open substack \(U^{\varepsilon} \subset |A_1|\) such that \(Z(d_{A_1} t_{A_1}) \cap U^{\varepsilon} = Q_{V_1}^{\varepsilon}\).

3. There are morphisms \(\phi_{A_1} : A_1|_{U^{\varepsilon}} \rightarrow Q|_{U^{\varepsilon}}\) and \(\phi_{B_1} : B_1|_{Q_{V_1}^{\varepsilon}} \rightarrow P'|_{Q_{V_1}^{\varepsilon}}\) (defined on different underlying spaces!) such that there is a commutative diagram
   \[
   \begin{array}{ccc}
   A_1|_{Q_{V_1}^{\varepsilon}} & \xrightarrow{d_{A_1}} & B_1|_{Q_{V_1}^{\varepsilon}} \\
   \phi_{A_1}|_{Q_{V_1}^{\varepsilon}} \downarrow & & \downarrow \phi_{B_1} \\
   Q|_{Q_{V_1}^{\varepsilon}} & \xrightarrow{d_Q} & P'|_{Q_{V_1}^{\varepsilon}} \\
   \end{array}
   \]
   which represents \((4.1)|_{Q_{V_1}^{\varepsilon}}\) on \(Q_{V_1}^{\varepsilon}\).

Proof. In [12, §3.2.1], two short exact sequences
\[
0 \rightarrow V_1 \rightarrow A_i \rightarrow B_i \rightarrow 0, \ i = 1, 3,
\]
are constructed on the universal curve of \(\mathcal{B}^e\). We obtain (1) by considering \(R\pi_*\) of these exact sequences, as \(A_i\) and \(B_i\) are \(\pi_*\)-acyclic. Note that over \(\mathcal{B}^e\), \(Q_{V_1}^{\varepsilon}\) consists of sections of \(V_1\) which are described by the kernel of \(d_{A_1}\). Now (2) follows from the fact that the stability condition is an open condition. (3) follows from [12, Lemma 3.4].
Let $K$ be the kernel of $(\phi_{B_1} - d_Q)|_{Q_X^\epsilon}$. By Lemma 4.3 (3),

$$[B_1 \oplus Q \xrightarrow{\phi_{B_1} - d_Q} P^\epsilon]|_{Q_X^\epsilon}$$

is surjective and $[A_1|_{Q_X^\epsilon} \to K]$ represents the dual of the natural perfect obstruction theory (4.1) of $Q_X^\epsilon$ over $B$. Also, by Lemma 4.3 (1), $[A_1 \oplus P \xrightarrow{d_{A_1} - d_Q} B_1 \oplus Q^\epsilon]|_{LGQ'}$ represents the dual of the natural perfect obstruction theory $R\pi_*(V_1 \oplus V_2 \otimes \omega_\epsilon)$ of $LGQ'$ over $B$.

Let $U := U^\epsilon \times_{\mathcal{B}^\epsilon} |P| \times_{\mathcal{B}^\epsilon} |Q| \cong |(P \oplus Q)|_{U^\epsilon}$ be the total space of a vector bundle $P \oplus Q|_{U^\epsilon}$ over $U^\epsilon$, where $|U^\epsilon|$ denotes the pullback under the morphism $U^\epsilon \to B^\epsilon$ and $\tilde{U} := U \times_{\mathcal{C}} A_{1,C}^1$. Let $F := (B_1 \oplus Q^\epsilon \oplus Q)|_U$ be a vector bundle on $U$ and $\tilde{F} := F|_{\tilde{U}}$. Let $\tilde{p} : \tilde{F} \to \tilde{U}$ be the projection. In [12, §3.2.3], a section $\beta : \mathcal{O}_{\tilde{U}} \to \tilde{F}$ is defined as follows:

$$\beta := (d_{A_1} t_{A_1}, -\lambda d_{Q}^\epsilon t_P, \phi_{A_1} t_{A_1} - \lambda t_Q),$$

where $\lambda$ is a coordinate on $A_{1,C}^1$. Now arguing as in Lemma 4.3(2), we obtain

$$Z(d_{A_1} t_{A_1}, \phi_{A_1} t_{A_1}) \cap U^\epsilon = Q_X^\epsilon, \ Z(d_{A_1} t_{A_1}, d_{Q}^\epsilon t_P) \cap (U^\epsilon \times_{\mathcal{B}^\epsilon} |P|) = LGQ'.$$

We summarise this discussion in the following lemma, where in particular the commutativity of (4.5) and (4.6) follows from the definition of $\beta$:

**Lemma 4.4.** We obtain

$$Z(\beta) \cong \begin{cases} 
LGQ' & \text{if } \lambda \neq 0 \in A_{1,C}^1, \\
Q_X^\epsilon \times_{\mathcal{B}^\epsilon} |P| \times_{\mathcal{B}^\epsilon} |Q| & \text{if } \lambda = 0 \in A_{1,C}^1.
\end{cases}$$

Moreover, when $\lambda \neq 0$, the following is a commutative diagram:

$$
\begin{array}{ccc}
Z(\beta|_\lambda) \xrightarrow{\cong} \tilde{U}|_\lambda & \cong & U \\
\downarrow \cong & & \downarrow pr \\
LGQ'|_\lambda & \xrightarrow{\cong} & U^\epsilon \times_{\mathcal{B}^\epsilon} |P|.
\end{array}
$$

(4.5)

Here $pr$ is a projection. Similarly, when $\lambda = 0$, we have a fibre diagram

$$
\begin{array}{ccc}
Z(\beta|_0) \xrightarrow{\cong} \tilde{U}|_0 & \cong & U \\
\downarrow pr & & \downarrow pr \\
Q_X^\epsilon |_0 & \xrightarrow{\cong} & U^\epsilon.
\end{array}
$$

(4.6)

By Lemma 4.4, we have

$$C_{LGQ'}(U^\epsilon \times_{\mathcal{B}^\epsilon} |P|) \times_{\mathcal{B}^\epsilon} |Q| \cong C_{LGQ'} \times_{\mathcal{B}^\epsilon} |Q| U \cong C_{Z(\beta)} \tilde{U}|_\lambda, \ \lambda \neq 0,$$

$$C_{Q^\epsilon} U^\epsilon \times_{\mathcal{B}^\epsilon} |P| \times_{\mathcal{B}^\epsilon} |Q| \cong C_{Q^\epsilon} \times_{\mathcal{B}^\epsilon} |P| \times_{\mathcal{B}^\epsilon} |Q| U \subset C_{Z(\beta)} \tilde{U}|_0,$$

(4.7)

where the notation $C_W$ denotes the corresponding cone for a space $W$. 
The computation of $d\beta$ allows us to conclude that the restriction to $\lambda \in \mathbb{A}_r^1$ of the dual of the perfect obstruction theory $[T_{U/\mathfrak{M}^0 \times \mathbb{A}_r^1} \xrightarrow{d\beta} \tilde{F}]|_{Z(\beta)}$ of $Z(\beta)$ over $\mathfrak{M}^0 \times \mathbb{A}_r^1$ is isomorphic to

$$[A_1 \oplus P \xrightarrow{d_{A_1} \oplus -d_Q^0} B_1 \oplus Q^\vee] \oplus [Q \xrightarrow{id} Q]|_{LGQ'}$$

(4.8)

when $\lambda \neq 0$. When $\lambda = 0$, it deforms to

$$[A_1|Q_X^\beta \rightarrow K] \oplus [P \oplus Q \xrightarrow{0} Q^\vee \oplus P^\vee]|_{Q_X^\beta \times \mathfrak{M}^0[P] \times \mathbb{A}_r^1}$$

(4.9)

(see [12, §§2.4.1 and 3.2.6]). In [12, §3.2.3], a cosection $\sigma : \tilde{F}|_{Z(\beta)} \rightarrow O_{Z(\beta)}$ is defined as follows:

$$\sigma := (\phi_{B_1} - d_Q, t_P) + (id_{Q^\vee}, t_Q)Q,$$

where $\langle, \rangle_P : P^\vee|_{Z(\beta)} \times P|_{Z(\beta)} \rightarrow O_{Z(\beta)}$ and $\langle, \rangle_Q : Q^\vee|_{Z(\beta)} \times Q|_{Z(\beta)} \rightarrow O_{Z(\beta)}$ denote the natural pairings. It follows from the definition of $\sigma$ that $Z(\sigma) \subset Z(t_P) \subset Q_X^\beta \times \mathfrak{M}^0[P] \times \mathbb{A}_r^1$.

Furthermore, note that on $Q_X^\beta \times \mathfrak{M}^0[P] \times \mathbb{A}_r^1$, $\phi_{B_1} - d_Q$ is surjective, which allows us to conclude that $Z(\sigma) = Z(t_P, t_Q) \cong Q_X^\beta \times \mathbb{A}_r^1$. We therefore have the following lemma:

**Lemma 4.5.** For the cosection $\sigma : \tilde{F}|_{Z(\beta)} \rightarrow O_{Z(\beta)}$, we have $Z(\sigma) \cong Q_X^\beta \times \mathbb{A}_r^1$. Moreover, when $\lambda \neq 0$, $\sigma|_{\lambda}$ coincides with $d_{\nu}|_{\lambda} : (B \oplus Q^\vee) \oplus Q|_{Z(\beta)|_{\lambda}} \rightarrow O_{Z(\beta)|_{\lambda}}$ under (4.8). When $\lambda = 0$, $\sigma|_0$ deforms to $0 \oplus$ taut : $K \oplus (Q^\vee \oplus P^\vee)|_{Z(\beta)|_0} \rightarrow O_{Z(\beta)|_0}$ under (4.9) (see [12, §3.2.6]). Here, the cosection taut is given by the pairing which is explained in the diagram at [12, p. 17].

### 4.2. Proof of the main theorem

Now we state and prove the main theorem of this section.

**Theorem 4.6.** In the Grothendieck group of coherent sheaves on $Q_X^\beta := Q_{g,k}(Z(s), d)$, we have

$$[O^\text{vir}_{Q_X^\beta}] = (-1)^{\chi^\text{gen}(R\pi_!(V_2 \otimes \omega_e)^\vee)}|_{Q_X^\beta} \otimes [O^\text{vir}_{LGQ', dw_{LGQ'}}],$$

(4.10)

where $\chi^\text{gen}(V_2^\vee)$ is generic virtual rank of $R\pi_!(V_2 \otimes \omega_e)^\vee$ and $\det R\pi_!(V_2 \otimes \omega_e)^\vee$ is the determinant (see Definition 4.1).

In [12, Corollary 3.5], it is shown that $Z(\beta) \subset Z(\tilde{p}^*\sigma \circ t_{\tilde{F}})$. Thus, we have the Koszul complex $\{\tilde{p}^*\sigma, t_{\tilde{F}}\}$ on $C_{Z(\beta) \tilde{U}}$. By Lemma 4.5, $\{\tilde{p}^*\sigma, t_{\tilde{F}}\}$ is exact off $Q_X^\beta \times \mathbb{A}_r^1 \subset Z(\beta) \subset C_{Z(\beta) \tilde{U}}$. For $\lambda \in \mathbb{A}_r^1$, let $p_\lambda : |F|_{Z(\beta)} \mapsto Z(\beta)$ denote the projection. In the following lemma we prove an analogue of [6, Lemma 3.6] for the structure sheaf:
Lemma 4.7. Consider the following diagram, where \( \mathcal{X}, \mathcal{Y} \) are DM-stacks and all squares are cartesian:

\[
\begin{array}{ccc}
\mathcal{Y}_{\infty} & \xrightarrow{j} & \mathcal{Y}|_{\infty} \\
\downarrow & & \downarrow \\
\mathcal{X}_{\infty} & \xrightarrow{i} & \mathcal{X}|_{\infty} \\
\downarrow & & \downarrow \\
 & \mathcal{X} \rightarrow \mathbb{P}^1_{\mathbb{C}} \\
\end{array}
\]

\( \lambda = \infty \rightarrow \mathbb{P}^1_{\mathbb{C}} - \{1\}. \) (4.11)

Assume further that the following hold:

1. \( i \) and \( j \) are closed immersions.
2. The maps \( \mathcal{X} \rightarrow \mathbb{P}^1_\mathbb{C} - \{1\} \) and \( \mathcal{X} \rightarrow \mathbb{P}^1_{\mathbb{C}} \) are flat.
3. The composite map \( \mathcal{X}_{\infty} \rightarrow \mathbb{P}^1_{\mathbb{C}} \) is flat.
4. \( \mathcal{O}_{\mathcal{X}|_{\infty}} - [i_*\mathcal{O}_{\mathcal{X}_{\infty}}] \) is supported on \( \mathcal{Y}|_{\infty} \).

Then \( j_*[\mathcal{O}_{\mathcal{Y}_{\infty}}] = \infty^j[\mathcal{O}_{\mathcal{Y}}] \) in \( G_0(\mathcal{Y}|_{\infty}) \).

Proof. The proof is similar to [6, Lemma 3.6] and follows from the functoriality of the refined Gysin pullback, proper push-forward and the self-intersection formula. Consider the following:

\[
\begin{align*}
\infty^1([\mathcal{O}_{\mathcal{Y}}]) &= 1 \infty^1 v^i[\mathcal{O}_{\mathcal{X}}] \\
&= 2 v^i[\mathcal{O}_{\mathcal{X}}] \\
&= 3 v^i[\mathcal{O}_{\mathcal{X}|_{\infty}}] \\
&= 4 v^i i_*[\mathcal{O}_{\mathcal{X}_{\infty}}] \\
&= 5 j_* v^i[\mathcal{O}_{\mathcal{X}_{\infty}}] \\
&= 6 j_*[\mathcal{O}_{\mathcal{Y}_{\infty}}].
\end{align*}
\]

(4.12)

As \( \mathcal{X} \rightarrow \mathbb{P}^1_{\mathbb{C}} \) is flat, \( =^1 \) follows from the flat base change [9, Lemma 1.2]. \( =^2 \) follows from the functoriality of the refined Gysin pullback. \( =^3 \) follows from the same argument as \( =^1 \), as \( \mathcal{X} \rightarrow \mathbb{P}^1_{\mathbb{C}} - \{1\} \) is flat. Note that the normal bundle \( \mathcal{Y}|_{\infty} \rightarrow \mathcal{X}|_{\infty} \) is trivial. Hence \( =^4 \) follows from assumption (3) in Lemma 4.7 and the excess intersection formula (see [22, p. 8]). \( =^5 \) and \( =^6 \) follow from the projection formula and the flat base change, respectively. \( \square \)

Lemma 4.8. With notations as before, we have the following:

\[
\lambda^! h^\mathbb{C}_{\mathcal{Z}(\beta)}(\tilde{\mathcal{U}}) =_\lambda h^\mathbb{C}_{\mathcal{Z}(\beta)}(\mathcal{U})(\mathcal{O}_{\mathcal{C}_{\mathcal{Z}(\beta)}} \mathcal{U}). \quad (4.14)
\]

Proof. It follows that \( \lambda^! h^\mathbb{C}_{\mathcal{Z}(\beta)}(\tilde{\mathcal{U}}) =_\lambda h^\mathbb{C}_{\mathcal{Z}(\beta)}(\mathcal{U})(\mathcal{O}_{\mathcal{C}_{\mathcal{Z}(\beta)}} \mathcal{U}) \) by Lemma 2.19. Let \( j_\lambda : \mathcal{C}_{\mathcal{Z}(\beta)} \rightarrow \mathcal{C}_{\mathcal{Z}(\beta)} \mathcal{U} \) denote the natural map;
then from Lemma 4.7 and \[6, \text{Lemma 3.6}\] it follows that
\[
\text{h}^{(C_{Z(\beta)} \bar{U})}_{Q_{X}}(p^{*}\sigma, t_{F})(\lambda^{1}[O_{C_{Z(\beta)} \bar{U}}]) = \text{h}^{(C_{Z(\beta)} \bar{U})}_{Q_{X}}(p^{*}\sigma, t_{F})(\lambda^{1}[O_{C_{Z(\beta)} \bar{U}}]).
\] (4.15)

The claim now follows by applying Lemma 2.21 to the right-hand side of (4.15).

We now complete the proof of the main theorem.

**Proof of Theorem 4.6.** Let \( p' : C_{\text{LGQ}'}(U^{e} \times _{\mathbb{A}^{1}} |P|) \subset |(B_{1} \oplus Q')|_{\text{LGQ}'} \rightarrow \text{LGQ}' \) be the projection. To simplify notation, let \( U_{P} := U^{e} \times _{\mathbb{A}^{1}} |P| \) and \( F_{1} := (B_{1} \oplus Q')|_{\text{LGQ}'} \). Now from Lemma 4.8, we have the following:

\[
(4.14)|_{0} = \text{h}^{C_{\text{LGQ}'}(U_{P} \times _{\mathbb{A}^{1}} |Q|)}_{Q_{X}}((p^{*}dw_{\text{LGQ}'}, t_{F_{1}}) \boxtimes _{\mathbb{A}^{1}} (0, t_{Q}))(|O_{C_{\text{LGQ}'}(U_{P} \times _{\mathbb{A}^{1}} |Q|)})
\]

\[
= \text{h}^{C_{\text{LGQ}'}(U_{P})}_{Q_{X}}((p^{*}dw_{\text{LGQ}'}, t_{F_{1}}))(|O_{C_{\text{LGQ}'}(U_{P})})
\]

\[
= |O_{\text{LGQ}', dw_{\text{LGQ}'}}|.
\]

Here the first equality is from Lemma 4.5, the second equality is by Lemma 2.18 and the third equality follows from Corollary 3.6.

Let \( m : C_{Z(\beta_{0})} U \rightarrow |K| \) and \( p_{0} : C_{Z(\beta_{0})} U \rightarrow Z(\beta_{0}) \) be projections. On \( C_{Z(\beta_{0})} U \), we deform the complex \( (p^{*}_{0}\sigma, t_{F}) \) supported on \( Q_{X} \) to \( (0, m^{*}\tau_{K}) \otimes \{p^{*}_{0}\text{taut}, 0\} \), supported also on \( Q_{X} \) under the deformation in Lemma 4.5 (see the commutative diagram at [12, p. 17]). By application of the splitting principle (Lemma 3.3) to this deformation, (4.14)|_{0} becomes the following:

\[
\text{h}^{C_{Z(\beta_{0})} U}_{Q_{X}^{e}}((0, m^{*}\tau_{K}) \otimes \{p^{*}_{0}\text{taut}, 0\})(|O_{C_{Z(\beta_{0})} U}|) = \text{h}^{C_{Z(\beta_{0})} U}_{Q_{X}^{e}}((0, m^{*}\tau_{K}) \otimes \gamma(\Lambda^{-\bullet}(P^{\nu} \oplus Q^{\nu}) \otimes \Lambda^{n}(P \oplus Q) |n))(|O_{C_{Z(\beta_{0})} U}|) = (-1)^{\chi(V_{2})} \gamma(\Lambda^{n}(P \oplus Q) |n) = (-1)^{\chi(V_{2})} \text{det} R\pi_{*}\langle V_{2} \otimes \omega_{\mathbb{C}}\rangle|Q_{X}^{e} \otimes |O_{\text{LGQ}', dw_{\text{LGQ}'}}|,
\]

where \( \Lambda^{-\bullet}(P^{\nu} \oplus Q^{\nu}) \) is the Koszul complex and \( n \) is the rank of \( P \oplus Q \). The first equality follows from (3.2), as

\[
\{p^{*}_{0}\text{taut}, 0\} = (0, p^{*}_{0}\text{taut}) \otimes \gamma(\Lambda^{n}(P \oplus Q) |n) = \gamma(\Lambda^{-\bullet}(P^{\nu} \oplus Q^{\nu}) \otimes \Lambda^{n}(P \oplus Q) |n).
\]

The second equality follows from Lemma 2.18 and the fact that

\[
\text{h}^{C_{Z(\beta_{0})} U}_{K}(|O_{C_{Z(\beta_{0})} U}|) = |O_{C_{Z(\beta_{0})} U}|,
\]

by Corollary 3.6. Note that the sign \((-1)^{\chi(V_{2})}\) comes from the shifting by \([n]\). The third equality follows from the definitions.

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