Strong Approximate Consensus Halving
and the Borsuk-Ulam Theorem

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Abstract

In the consensus halving problem we are given \(n\) agents with valuations over the interval \([0,1]\). The goal is to divide the interval into at most \(n+1\) pieces (by placing at most \(n\) cuts), which may be combined to give a partition of \([0,1]\) into two sets valued equally by all agents. The existence of a solution may be established by the Borsuk-Ulam theorem. We consider the task of computing an approximation of an exact solution of the consensus halving problem, where the valuations are given by distribution functions computed by algebraic circuits. Here approximation refers to computing a point that \(\varepsilon\)-close to an exact solution, also called strong approximation. We show that this task is polynomial time equivalent to computing an approximation to an exact solution of the Borsuk-Ulam search problem defined by a continuous function that is computed by an algebraic circuit.

The Borsuk-Ulam search problem is the defining problem of the complexity class \(\text{BU}\). We introduce a new complexity class \(\text{BBU}\) to also capture an alternative formulation of the Borsuk-Ulam theorem from a computational point of view. We investigate their relationship and prove several structural results for these classes as well as for the complexity class \(\text{FIXP}\).

1 Introduction

Many computational problems, e.g. linear and semidefinite programming, are most naturally expressed using real numbers. When the model of computation is discrete, these problems must be recast as discrete problems. In the case of linear programming this causes no problems. Namely, when the input is given as rational numbers and an optimal solution exists, a rational valued optimal solution exists and may be computed in polynomial time. For semidefinite programming however, it may be the case that all optimal solutions are irrational. For dealing with such cases we may instead consider the weak optimization problem as defined by Grötschel, Lovász and Schrijver [GLS88]: Given \(\varepsilon > 0\), the task is to compute a rational-valued vector \(x\) that is \(\varepsilon\)-close to the set of feasible solutions and has objective value \(\varepsilon\)-close to optimal. Assuming we are also given, as an additional input, a strictly feasible solution and a bound on the magnitude of the coordinates of an optimal solution, the weak optimization problem may be solved in polynomial time using the ellipsoid algorithm [GLS88]. Let us note however that without additional assumptions, even the complexity of the basic existence problem of semidefinite feasibility is unknown. In fact, the problem is likely to be computationally very hard [TV08]. More precisely, it is hard for the problem PosSLP, which
is the fundamental problem of deciding whether an integer given by a division free arithmetic circuit is positive [ABKM09].

In this paper we consider real valued search problems, where existence of a solution is guaranteed by topological existence theorems such as the Brouwer fixed point theorem and the Borsuk-Ulam theorem. This means that the search problems are total, thereby fundamentally differentiating them from general search problems where, as described above, even the existence problem may be computational hard. We are mainly interested in the approximation problem: given \( \epsilon > 0 \), the task is to compute a rational-valued vector \( x \) that is \( \epsilon \)-close to the set of solutions.

Recall that the Brouwer fixed point theorem states every continuous function \( f : B^n \to B^n \), where \( B^n \) is the unit \( n \)-ball, has a fixed point, i.e. there is \( x \in B^n \) such that \( f(x) = x \) [Bro11]. The Borsuk-Ulam theorem states that every continuous function \( f : S^n \to \mathbb{R}^n \), where \( S^n \) is the unit \( n \)-sphere in \( \mathbb{R}^{n+1} \), maps a pair of antipodal points of \( S^n \) to the same point in \( \mathbb{R}^n \), i.e. there is \( x \in S^n \) such that \( f(x) = f(-x) \) [Bor33]. The Brouwer fixed point theorem is of course not restricted to apply to the domain \( B^n \), but applies to any domain that is homeomorphic to \( B^n \). Similarly the Borsuk-Ulam theorem applies to any domain homeomorphic to \( S^n \) by an antipode-preserving homeomorphism. It is well-known that the Borsuk-Ulam theorem generalizes the Brouwer fixed point theorem, in the sense that the Brouwer fixed point theorem is easy to prove using the Borsuk-Ulam theorem [Su97; Vol08].

The Brouwer fixed point theorem and the Borsuk-Ulam theorem naturally define corresponding real valued search problems, and thereby also corresponding approximation problems. In addition, the statements of the theorems naturally leads to another notion of approximation. For the case of the Brouwer fixed point theorem we may look for an almost fixed point, i.e. \( x \in B^n \) such that \( f(x) \) is \( \epsilon \)-close to \( x \), and for the case of the Borsuk-Ulam theorem we look for a pair of antipodal points that almost map to the same point, i.e. \( x \in S^n \) such that \( f(x) \) and \( f(-x) \) are \( \epsilon \)-close. Following [EY10], we shall refer to this notion of approximation as weak approximation and to make the distinction clear we refer to the former (and general) notion of approximation as strong approximation. In the setting of weak approximation in relation to the Borsuk-Ulam theorem we assume that \( f \) has co-domain \( B^n \).

In their seminal work, Etessami and Yannakakis [EY10] introduced the complexity class FIXP to capture the computational complexity of the real-valued search problems associated with the Brouwer fixed point theorem, and proved that the problem of finding a Nash equilibrium in a given 3-player game in strategic form is FIXP-complete. In order to have a notion of completeness, the class FIXP is defined to be closed under reductions. The type of reductions chosen by Etessami and Yannakakis, SL-reductions, consists of mapping between sets of solutions by a composition of a projection reduction followed by individual affine transformation applied to each coordinate.

Etessami and Yannakakis consider different ways to cast real valued search problems as discrete search problems. In addition to the approximation problem, these are the partial computation problem where the task is to compute a solution to a given number of bits of precision and decision problems, where the task is to evaluate a sign condition of the set of solutions given the promise that either all solutions satisfy the condition or none of them do. Of these we shall only consider the approximation problem. The class FIXP\(_a\) denotes the class of discrete search problems corresponding to strong approximation of Brouwer fixed points and is defined to be closed under polynomial time reductions. Etessami and Yannakakis also prove that the problem PosSLP reduce to the problem of approximating a Nash equilibrium, thereby showing that FIXP\(_a\) likely contains search problems that are computationally very hard.

While the notion of SL-reductions is very restricted, it is sufficient for proving completeness of the problem of finding Nash equilibrium. Likewise, SL-reductions are sufficient for showing that FIXP is robust with respect to the choice of domain for the Brouwer function.

Another important reason for using SL-reductions is that they immediately imply polynomial time reductions between the corresponding decision and approximation problems (the partial computation problem is more fragile and requires additional assumptions, cf. [EY10]). As we are mainly interested in the approx-
imtion problem more expressive notions of reducibility can be considered, while maintaining the property that reducibility implies polynomial time reducibility between the corresponding approximation problems. A sufficient condition for this is that the mapping of solutions is polynomially continuous and polynomial time computable.

1.1 The Borsuk-Ulam Theorem

Deligkas, Fearnley, Melissourgos, and Spirakis [DFMS21] recently introduced a complexity class BU to capture, in an analogy to FXP, the computational complexity of the real-valued search problems associated with the Borsuk-Ulam theorem.

The Borsuk-Ulam theorem has a number of equivalent statements that are also easy to derive from each other. A function \( f \) defined on the unit sphere \( S^n \) is odd if \( f(x) = -f(-x) \) for all \( x \in S^n \). Note that the boundary \( \partial B^n \) of the unit \( n \)-ball \( B^n \) is identical to \( S^{n-1} \). We thus say that a function \( f \) defined on \( B^n \) is odd on \( \partial B^n \) if \( f \) is odd when restricted to \( S^{n-1} \). We present the simple proof of the known fact that the different formulations can be derived from each other, for the purpose of discussing equivalence from a computational point of view.

**Theorem 1** (Borsuk-Ulam). The following statements hold:

1. If \( f : S^n \to \mathbb{R}^n \) is continuous there exists \( x \in S^n \) such that \( f(x) = f(-x) \).
2. If \( g : S^n \to \mathbb{R}^n \) is continuous and odd there exists \( x \in S^n \) such that \( g(x) = 0 \).
3. If \( h : B^n \to \mathbb{R}^n \) is continuous and odd on \( \partial B^n \) there exists \( x \in B^n \) such that \( h(x) = 0 \).

**Proof of equivalence.** Given \( f \) we may define \( g(x) = f(x) - f(-x) \). Clearly \( g \) is odd and we have \( g(x) = 0 \) if and only if \( f(x) = f(-x) \), which shows that (2) implies (1). Conversely, given \( g \) we simply let \( f = g \). If \( f(x) = f(-x) \), then since \( g \) is odd we have \( f(x) = g(x) = -g(-x) = -f(-x) = -f(x) \) and hence \( g(x) = f(x) = f(x) = 0 \), which therefore shows (1) implies (2).

We may view \( S^n \) as two hemispheres, each homeomorphic to \( B^n \), which are glued together along their equators. Let \( \pi : S^n \to B^n \) be the orthogonal projection defined by \( \pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n) \). Then given \( h \) we may define

\[
g(x) = \begin{cases} 
  h(\pi(x)) & \text{if } x_{n+1} \geq 0 \\
  -h(-\pi(x)) & \text{if } x_{n+1} \leq 0 
\end{cases}.
\]

The assumption that \( h \) is odd on \( \partial B^n \) makes \( g \) a well-defined continuous odd function. We have \( g(x) = 0 \) if and only if \( h(x) = 0 \), which shows that (2) implies (3). Conversely, given \( g \) we define \( h \) by \( h(x) = g\left(x, (1 - ||x||_2^2)\frac{1}{2}\right) \). Then \( h \) is continuous and odd on \( \partial B^n \), since \( x \in \partial B^n \) if and only if \( ||x||_2^2 = 1 \). Clearly if \( h(x) = 0 \) we may let \( y = (x, (1 - ||x||_2^2)\frac{1}{2}) \) and have \( g(y) = 0 \). On the other hand, when \( g(y) = 0 \) we may define \( x = (y_1, \ldots, y_n) \) if \( y_{n+1} \geq 0 \) and \( x = (-y_1, \ldots, -y_n) \) if \( y_{n+1} < 0 \), and we have \( h(x) = 0 \). Together this shows that (3) implies (2).

The class \( BU \) defined in [DFMS21] corresponds to first formulation of the above theorem. We may clearly consider the second formulation equivalent to the first also from a computational point of view. In particular, when translating between formulations, the set of solutions is unchanged. Note that this set of solutions has the property that all solutions come in pairs: when \( x \) is a solution then \( -x \) is a solution as well. For the third formulation of the theorem this property only holds for solutions on the boundary \( \partial B^n \).

In contrast, while the mapping of solutions of the third formulation to the second (and first) formulation given above is continuous this is not the case in the other direction. More precisely, consider \( y \in S^n \) such that
Namely we may consider the function \( F \) on an interval has length \( n \) and is precisely a division into \( n \) cuts, such that unions of these intervals form another partition \( A = A^+ \cup A^- \) of \( A \) satisfying \( \mu_i(A^+) = \mu_i(A^-) \) for every \( i \). We may think of the intervals being assigned a label from the set \{\(+, -\)\}, and \( A^+ \) is precisely the union of the intervals labeled by \(+\). Such a partition is also known as a consensus halving. Using the Borsuk-Ulam theorem, Simmons and Su [SS03] proved that a consensus halving using at most \( n \) cuts always exists. Simmons and Su represent a division of \( A \) as a point \( x \) on the unit \( n \)-sphere \( S^n_1 \) with respect to the \( \ell_1 \)-norm. The point \( x \) is viewed as representing a division into precisely \( n + 1 \) intervals, where some intervals are possibly empty. More precisely, the \( i \)-th interval has length \( |x_i| \), and intervals of length 0 may simply be discarded. The intervals of positive length are then labeled according to \( \text{sgn}(x_i) \). Note that for any \( x \), the antipode \(-x\) represent the division where the sets \( A^+ \) and \( A^- \) are exchanged. This naturally leads to a formulation using the Borsuk-Ulam theorem [SS03]. Namely we may consider the function \( F : S^n_1 \to \mathbb{R}^n \) given by \( F(x)_i = \mu_i(A^+) \), and note that any \( x \in S^n_1 \) for which \( F(x) = F(-x) \) represent a consensus halving.

We are interesting in the simple setting of additive measures, where we have corresponding density functions \( f_1, \ldots, f_n \) such that \( \mu_i(B) = \int_B f_i(x) \, dx \). To cast the consensus halving problem as a real valued search problem we follow [DFMS21] and assume that the measures \( \mu_1, \ldots, \mu_n \) are given by the distribution functions \( F_1, \ldots, F_n \) defined by \( \int_0^1 f_i(x) \, dx \). An instance of the consensus halving problem is then given as a list of algebraic circuits computing these distribution functions.

Corresponding to the different formulations of the Borsuk-Ulam theorem as a real valued search problem with domain \( S^n \) or \( B^n \) we get two different formulations of the consensus halving problem. We denote these by CH and BCH respectively. Deligkas et al. proved membership of CH in BU following the proof of Simmons and Su, and proved hardness of CH for FIXP. Combining these, it follows that FIXP \( \subseteq \) BU.

### 1.2 Consensus Halving

The Consensus halving problem is a classical problem of fair division [SS03]. We are given a set of \( n \) bounded and continuous measures \( \mu_1, \ldots, \mu_n \) defined on the interval \( A = [0, 1] \). The goal is to partition the interval \( A \) into at most \( n + 1 \) intervals, i.e. by placing at most \( n \) cuts, such that unions of these intervals form another partition \( A = A^+ \cup A^- \) of \( A \) satisfying \( \mu_i(A^+) = \mu_i(A^-) \) for every \( i \). We may think of the intervals being assigned a label from the set \{\(+, -\)\}, and \( A^+ \) is precisely the union of the intervals labeled by \(+\). Such a partition is also known as a consensus halving. Using the Borsuk-Ulam theorem, Simmons and Su [SS03] proved that a consensus halving using at most \( n \) cuts always exists. Simmons and Su represent a division of \( A \) as a point \( x \) on the unit \( n \)-sphere \( S^n_1 \) with respect to the \( \ell_1 \)-norm. The point \( x \) is viewed as representing a division into precisely \( n + 1 \) intervals, where some intervals are possibly empty. More precisely, the \( i \)-th interval has length \( |x_i| \), and intervals of length 0 may simply be discarded. The intervals of positive length are then labeled according to \( \text{sgn}(x_i) \). Note that for any \( x \), the antipode \(-x\) represent the division where the sets \( A^+ \) and \( A^- \) are exchanged. This naturally leads to a formulation using the Borsuk-Ulam theorem [SS03]. Namely we may consider the function \( F : S^n_1 \to \mathbb{R}^n \) given by \( F(x)_i = \mu_i(A^+) \), and note that any \( x \in S^n_1 \) for which \( F(x) = F(-x) \) represent a consensus halving.

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### 1.3 Strong versus Weak Approximation

The difference between weak and strong approximation was studied in detail in the general context of the Brouwer fixed point theorem by Etessami and Yannakakis. A central example is the problem of finding a Nash equilibrium (NE). An important notion of approximation of a NE is the notion of an \( \varepsilon \)-NE. Computing an \( \varepsilon \)-NE of a given strategic form game \( \Gamma \) is polynomial time equivalent to computing a weak
\(\varepsilon\)-approximation to a fixed point the Nash’s Brouwer function \(F_{\Gamma}\) associated to \(\Gamma\) [EY10, Proposition 2.3]. In turn, computing a weak \(\varepsilon\)-approximation to a fixed point of \(F_{\Gamma}\) polynomial time reduces to computing a strong \(\varepsilon''\)-approximation to a fixed point of \(F_{\Gamma}\) [EY10, Proposition 2.2], since the function \(F_{\Gamma}\) is polynomially continuous and polynomial time computable. In general however an \(\varepsilon\)-NE might be far from any actual NE, unless \(\varepsilon\) is inverse doubly exponentially small as a function of the size of the game [EY10, Corollary 3.8].

For the problem of consensus halving we can illustrate the difference between weak and strong approximation by a simple example. We shall refer to a weak \(\varepsilon\)-approximation of a consensus halving as simply an \(\varepsilon\)-consensus halving. Consider a single agent whose measure \(\mu\) is on the interval \([0, 1]\) is given by the following density

\[
f(x) = \begin{cases} 
(1 + \varepsilon)/\varepsilon & \text{if } 0 \leq x < \varepsilon/2 \\
0 & \text{if } \varepsilon/2 \leq x < 1 - \varepsilon/2 \\
(1 - \varepsilon)/\varepsilon & \text{if } 1 - \varepsilon/2 \leq x \leq 1
\end{cases}
\]

We have \(\mu([0, 1]) = 1\) and since \(\mu\) is a step function, the corresponding distribution function \(F\) is piece-wise linear. The unique consensus halving is obtained by placing a cut at the point \(\varepsilon/2 - \varepsilon^2/(2 + 2\varepsilon)\). Placing a cut at any point \(t \in [\varepsilon/2 - \varepsilon^2/(1 + \varepsilon), 1 - \varepsilon/2]\) results in an \(\varepsilon\)-consensus halving, i.e. such that \(|\mu([0, t]) - \mu([t, 1])| \leq \varepsilon\). Thus an \(\varepsilon\)-consensus halving might be very far from an actual consensus halving. Note also that placing a cut at any point \(t \in [0, 3\varepsilon/2 - \varepsilon^2/(2 + 2\varepsilon)]\) is a strong \(\varepsilon\)-approximation, which illustrates that a strong approximation is not necessarily a weak approximation. On the other hand, a strong \((\varepsilon^2/2)\)-approximation is also an \(\varepsilon\)-consensus halving.

The Brouwer fixed point theorem and the Borsuk-Ulam theorem can both be proved starting from combinatorial analogues of the two theorems, namely from Sperner’s lemma and Tucker’s lemma, respectively. The proofs of these two lemmas are constructive, but using them to derive the Brouwer fixed point theorem and the Borsuk-Ulam theorem involve a nonconstructive limit argument. Let us in passing note that while Sperner’s lemma, like the Borsuk-Ulam theorem, has several different formulations, it is usually formulated as the combinatorial analogue of the third formulation of Theorem 1.

Sperner’s and Tucker’s lemma give rise to total NP search problems. These turn out to be complete for the complexity classes PPAD and PPA introduced in seminal work by Papadimitriou [Pap94]. Papadimitriou proved PPAD-completeness of the problem given by Sperner’s lemma as well as membership in PPA of the problem given by Tucker’s lemma, while PPA-completeness of the latter problem was proved recently by Aisenberg, Bonet, and Buss [ABB20]. These results also imply that the classes PPAD and PPA corresponds to the problems of computing weak approximations to Brouwer fixed points and to Borsuk-Ulam points.

The computational complexity of the problems of computing an \(\varepsilon\)-NE and of computing an \(\varepsilon\)-consensus halving was settled in breakthroughs of two lines of research. Computing an \(\varepsilon\)-NE was shown to be PPAD-complete by Daskalakis, Goldberg and Papadimitriou [DGP09] and Cheng and Deng [CD06]. Computing an \(\varepsilon\)-consensus halving was shown to be PPA-complete by Filos-Ratsikas and Goldberg [FG18; FG19].

### 1.4 Our Results

Our main result is that the problem of strong approximation of consensus halving is equivalent to strong approximation of the Borsuk-Ulam theorem.

**Theorem 2.** The strong approximation problem for CH is \(\text{BU}_a\)-complete.

As described we view the consensus halving problem as the real valued search problem with its domain being either the unit sphere or the unit ball with respect to the \(\ell_1\)-norm. The theorem is proved by reduction from the real valued search problem associated with the Borsuk-Ulam theorem on the domain being the unit ball with respect to the \(\ell_\infty\)-norm, i.e. from a defining problem of the class BBU.
It is of general interest to study the relationship between search problems given by the Borsuk-Ulam theorem on different domains from a computational point of view. The reduction establishing the proof of Theorem 2 gives additional motivation for this. The domains we consider are unit spheres $S^n_p$ and unit balls $B^n_p$ with respect to the $\ell_p$-norm for $p \geq 1$ or $p = \infty$. It is of course straightforward to construct homeomorphisms between unit spheres or unit balls with respect to different norms, and these could be used to define reductions between the different problems. We would however like that the mapping of solutions is simple, and in particular we would like to avoid divisions and root operations. We prove that one may in fact reduce between domains using SL-reductions.

Deligkas et al. gave a reduction from the FIXP-complete problem of finding a Nash equilibrium to CH. Combined with membership of CH in BU, this gives the inclusion $\text{FIXP} \subseteq \text{BU}$. We observe that a proof due to Volovikov [Vol08] of the Brouwer fixed point theorem from the Borsuk-Ulam theorem may be adapted to give a simple proof of the inclusion $\text{FIXP} \subseteq \text{BU}$.

For the class FIXP we prove two interesting structural properties that do not appear to have been observed earlier. While FIXP is defined using SL-reductions, we show that FIXP is closed under polynomial time reductions where the mapping of solutions is expressed by general algebraic circuits. This in particular supports that one may reasonably define the classes BU and BBU using less restrictive notions of reductions than SL-reductions. We propose to have the mapping of solutions be computed by algebraic circuits involving the operations of addition, multiplication by scalars, as well as maximization. This means that the mapping of solutions is a piecewise linear function, and we refer to these as PL-reductions. The second structural result for FIXP is a characterization of the class by very simple Brouwer functions. These are defined on the unit-hypercube domain $[0, 1]^n$ and each coordinate function is simply one of the operations $\{+, -, \ast, \max, \min\}$, modified to be have the output truncated to the interval $[0, 1]$.

For the classes BU and BBU we prove that they are also closed under reductions where the mapping of solutions is computed by general algebraic circuits, but with the additional requirement that this function must be odd.

For the class FIXP, an interesting consequence of the proof that finding a Nash equilibrium is complete, is that the class may be characterized by Brouwer functions computed by algebraic circuits without the division operation. The proof also shows that the class FIXP is unchanged even when allowing root operations as basic operations. We prove by a simple transformation that the classes BU and BBU may be characterized using algebraic circuits without the division operation. Furthermore, as a consequence of Theorem 2 the class of strong approximation problems $\text{BU}_a = \text{BBU}_a$ is unchanged even when allowing root operations as basic operations.

1.5 Comparison to previous work

As a precursor to the proof of PP A-completeness of computing an $\varepsilon$-consensus halving, Filos-Ratsikas, Frederiksen, Goldberg and Zhang [FFGZ18] proved the problem to be PPAD-hard. Deligkas et al. [DFMS21] uses ideas from this proof together with additional new ideas to obtain their proof of FIXP-hardness for computing an exact consensus halving.

While PPAD $\subseteq$ PPA, the PPAD-hardness result of [FFGZ18] is not implied by the recent proofs of PPA-completeness. In particular, the work [FFGZ18] proves PPAD-hardness even for constant $\varepsilon$, while the work of [FG19] only proves PPA-hardness for $\varepsilon$ being inverse polynomially small. In the same way, while FIXP $\subseteq$ BU, FIXP-hardness of computing an exact consensus halving is not implied by our reduction, since Theorem 2 establishes $\text{BU}_a$-hardness rather than BU-hardness. Recently a considerably simpler proof of PPA-hardness for computing an $\varepsilon$-consensus halving was given by Filos-Ratsikas, Hollender, Sotiraki and Zampetakis [FHSZ20], and our reduction is inspired by this work.

All reductions described above are similar in the sense that one or more evaluations of a circuit are expressed in the consensus halving instance. The full interval $A$ is partitioned into subintervals, cuts within
these subintervals encode values in various ways, and agents implement the gates of the circuit by placing cuts. A main difference between the reductions establishing PPAD-hardness and FIXP-hardness to those establishing PPA-hardness is that in the former reductions, all cuts are constrained to be placed in distinct subintervals. This reason this is possible is that the objective is to find a fixed point of the circuit, which means that inputs and outputs may be identified.

In the setting of PPA and BBU the objective is to find a “zero” of the circuit. More precisely, for the setting of PPA the objective is to find two adjacent points of a given Tucker labeling that receive complementary labels, i.e. labels of different sign but same absolute value. For the setting of BBU the objective is to find an actual zero point of the circuit. All of the reductions establishing PPA-hardness of computing an $\epsilon$-consensus halving have the property that cuts encoding the input of the circuit are free cuts, meaning that they can in principle be placed anywhere, and as a result also interfere with the evaluations of the circuit. This is also the case for our reduction, and this invariably limits its applicability to the approximation problem.

In the reduction of [FHSZ20], the interval $A$ is structured into different regions, a coordinate-encoding region, a constant-creation region, several circuit-simulation regions, and finally a feedback region. Our reduction also has a coordinate-encoding region and several circuit simulation regions, but the functions performed by the constant-creation region and feedback regions perform in [FHSZ20] is our reduction integrated in the individual circuit simulation regions and done differently.

A novelty of the reduction of [FHSZ20] compared to previous reductions is in how values are encoded by cuts in subintervals. In previous reductions, values are encoded by what we will call position encoding. Here it is required that there is exactly one cut in the subinterval, and the value encoded is determined by the distance between the cut position and the left endpoint of the interval. In [FHSZ20] values are encoded by what we will call label encoding. Here there is no requirement on the number of cuts in the subinterval, and the value encoded is simply the difference between the Lebesgue measures of the subsets of the interval receiving label $+$ and label $-$. We shall employ a hybrid approach where the coordinate-encoding region uses label encoding while the circuit-simulation regions uses position encoding. The first step performed in a circuit-simulation region is thus to copy the input from the coordinate-encoding region. Switching to position encoding allows us in particular to implement a multiplication gate, similarly to [DFMS21]. Here the multiplication $xy$ is computed via the identity $xy = ((x+y)^2 - x^2 - y^2)/2$. In [DFMS21] where values range over $[0, 1]$, the squaring operation may be implemented directly by agents. In our case values range over the interval $[-1, 1]$, and the squaring operation is decomposed further, having agents compute it separately over the intervals $[-1,0]$ and $[0,1]$.

In analogy to [FHSZ20] we have feedback agents that ensures that the circuit evaluates to 0 on the encoded input. The criteria that the agents check is however different, and for our purposes it is crucial that we have the same sign pattern in the position encoding of the output of the circuit as the copy of the input made by the circuit-simulation region. The actual detection of an output of 0 is performed by using approximations of the Dirac delta function. For computing the distribution functions of the feedback agents, we make use of the fact that these are computed by algebraic circuits, which enable us to make a strong approximation of the Dirac delta function via repeated squaring.

### 1.6 Organization of Paper

In Section 2 we introduce necessary terminology and we give a detailed account of real valued search problems and reducibility between these. Our structural results for FIXP are given in Section 3 and our structural results for BU and BBU are given in Section 4. Section 4 also includes the simple proof of the inclusion $\text{FIXP} \subseteq \text{BU}$. We present our main result, Theorem 2, in Section 6.
2 Preliminaries

2.1 Algebraic Circuits

Let \( B \) be a finite set of real valued functions, for example \( B = \{+, -, \times, \div, \max, \min\} \). An algebraic circuit \( C \) with \( n \) inputs and \( m \) outputs over the basis \( B \) is given by an acyclic graph \( G = (V, A) \) as follows. The size of \( C \) is equal to the number of nodes of \( G \), which are also referred to as gates. The depth of \( C \) is equal to the length of the longest path of \( G \). Every node of indegree 0 is either an input gate labeled by a variable from the set \( \{x_1, \ldots, x_n\} \) or a constant gate labeled by a real valued constant. Every other node is labeled by an element of \( B \) called the gate function. If a node \( v \) is labeled by a gate function \( g : A \to \mathbb{R} \) with \( A \subseteq \mathbb{R}^k \) we require that \( g \) has exactly \( k \) ingoing arcs with a linear order specifying the order of arguments to \( g \). The output of \( C \) is specified by an ordered list of \( m \) (not necessarily distinct) nodes of \( G \). The computation of \( C \) on a given input \( x \in \mathbb{R}^n \) is defined in the natural way. Computation may fail in case a gate of \( C \) labeled by a function \( g : A \to \mathbb{R} \) receives an input outside \( A \), and in this case the output of \( C \) is undefined. Otherwise we say that the output is well defined and denote its value by \( C(x) \). If \( D \subseteq \mathbb{R}^n \) we say that \( C \) computes a function \( f : D \to \mathbb{R}^m \) if \( C(x) \) is well defined for all \( x \in D \).

We shall in this paper just consider algebraic circuits where the basis consists only of continuous functions. This means in particular that any algebraic circuits computes a continuous function as well. We shall also consider constant gates labeled with rational numbers. In this case we are also interested in the bitsize of the encoding of the constants, which is the maximum bitsize of the numerator or denominator. An important special class of algebraic circuits are those over the basis \( \{+,-,\times,\div, \max, \min\} \) and using just the constant 1. We refer to these as arithmetic circuits. An arithmetic circuit with no division gates is called division-free. Note that any integer of bitsize \( \tau \) may be computed by a division-free arithmetic circuit of size \( O(\tau) \).

By using multiplication with the constant \(-1\), the functions \(-\) and \(\min\) may be simulated using \(+\) and \(\max\), respectively. In this way we may convert a circuit over the full basis \( \{+,-,\times,\div, \max, \min\} \) into an equivalent \( \{+,-,\times,\div, \max\} \)-circuit. We shall also consider circuits where use of the multiplication operator \(*\) is restricted to having one of the arguments being a constant gate. We denote this by the symbol \(*\zeta\) and use it in particular for defining \( \{+,-,\zeta,\max\} \)-circuits.

At times it will conveninent to consider gate functions with their output range truncated to stay within a given interval. If \( g : A \to \mathbb{R} \) is a gate function and \( a \leq b \) defines a real interval \([a,b]\) we denote by \( g_{T[a,b]} \) the gate function defined by \( g_{T[a,b]}(x) = a \) if \( g(x) < a \), \( g_{T[a,b]}(x) = b \) if \( g(x) > b \), and \( g_{T[a,b]}(x) = g(x) \) otherwise. Note that \( g_{T[a,b]} \) is continuous whenever \( g \) is continuous.

While we shall not consider circuits with the discontinuous sign function \( \text{sgn} \), in the context of approximating functions, it is sometimes sufficient to use an approximation of \( \text{sgn} \) instead. A typical use of \( \text{sgn}(z) \) is to perform a selection between two values \( x \) and \( y \). We define the \( \delta \)-approximate selection function to be the function that based on \( \text{sgn}(z) \) outputs values \( x \) or \( y \) except in the interval of length \( \delta \) centered around 0 where it instead linearly interpolates between \( x \) and \( y \).

**Definition 1.** For given \( \delta > 0 \), the (two-sided) \( \delta \)-approximate selection function \( \text{Sel}_\delta \) is defined by

\[
\text{Sel}_\delta(x,y,z) = \begin{cases} 
  x & \text{if } z \leq -\delta/2 \\
  (y-x)z/\delta + (y+x)/2 & \text{if } -\delta/2 \leq z \leq \delta/2 \\
  y & \text{if } \delta/2 \leq z 
\end{cases}
\]

We note that \( \text{Sel}_\delta \) may be computed as \( \text{Sel}_\delta(x,y,z) = (1-t)/2 \cdot x + (1+t)/2 \cdot y \), where \( t \) defined by \( t = \max(\min(z,\delta/2),-\delta/2)/(\delta/2) \) is the \( \delta \)-approximation of \( \text{sgn}(z) \). In particular is \( \text{Sel}_\delta(x,y,z) \) computed by a \( \{+,\times,\max\} \)-circuit (or a \( \{+,\times,\div,\max\} \)-circuit if we also view \( \delta \) as a variable).
2.2 Search problems

A general search problem $\Pi$ is defined by specifying to each input instance $I$ a search space (or domain) $D_I$ and a set $\text{Sol}(I) \subseteq D_I$ of solutions. We distinguish between discrete and real-valued search problems. For discrete search problems we assume that $D_I \subseteq \{0,1\}^{d_I}$ for an integer $d_I$ depending on $I$. Analogously, for real-valued search problems we assume that $D_I \subseteq \mathbb{R}^{d_I}$ for an integer depending on $I$. One could likewise distinguish between search problems with discrete input and real-valued input. We are however mostly interested in problems where the input is discrete, that is we assume that instances $I$ are encoded as strings over a given finite alphabet $\Sigma$ (e.g. $\Sigma = \{0,1\}$).

A very important class of discrete search problems arise from decision problems given as languages in NP, thereby forming the class of NP search problems. More precisely, these are the discrete search problems where we assume there are polynomial time algorithms that (i) given $I$ compute $d_I$ whose magnitude is polynomial in $|I|$, (ii) given $I$ and $x \in \{0,1\}^{d_I}$ checks whether $x \in D_I$, and lastly, (iii) given $I$ and $x \in D_I$ checks whether $x \in \text{Sol}(I)$. The corresponding language in NP is then $L = \{I \mid \text{Sol}(I) \neq \emptyset\}$. The class of all NP search problems is denoted by FNP. The subclass TFNP of FNP consists of the NP search problems for which $\text{Sol}(I) \neq \emptyset$ for every input $I$. An NP search problem $\Pi$ is said to be solvable in polynomial time if there is a Turing machine running in polynomial time that on input $I$ gives as output some member $y$ of $\text{Sol}(I)$ in case $\text{Sol}(I) \neq \emptyset$ and rejects otherwise. The subclass of FNP consisting of the search problems solvable in polynomial time is denoted by FP, and it holds that $\text{FP} = \text{FNP}$ if and only if $\text{P} = \text{NP}$.

Many natural search problems are however defined with a continuous search space. Not all of these may adequately be recast as discrete search problems, but are more naturally viewed as real-valued search problems. One approach for studying such problems would be to switch to the Blum-Shub-Smale model of computation [BSS89]. A BSS machine resembles a Turing machine, but operates with real numbers instead of symbols from a finite alphabet. In particular is the input real-valued, and input instances are therefore encoded as real-valued vectors. All basic arithmetic operations and comparisons are unit-cost operations. One may then define real-valued analogues of Turing machine based classes. In particular, Blum, Shub and Smale defined and studied the real-valued analogues $\text{P}_R$ and $\text{NP}_R$ of $	ext{P}$ and $	ext{NP}$. A BSS machine may in general make use of real-valued machine constants. If a BSS machine only uses rational valued machine constants we shall call it constant-free. Real-valued analogues of the classes FP, FNP, and TFNP for the BSS machine model do not appear to be defined in the literature, but can be defined in a straight-forward way. Let us just note that the proof that $\text{P} = \text{NP}$ implies $\text{FP} = \text{FNP}$ does not generalize to the setting of BSS machines, since it crucially depends on the search space being discrete.

For the classes $\text{P}_R$ and $\text{NP}_R$, if we simply restrict the input to be discrete and consider only constant-free BSS machines this results in complexity classes, denoted by $\text{BP}(\text{P}_R^0)$ and $\text{BP}(\text{NP}_R^0)$, that may directly be compared to Turing machine based complexity classes. Indeed, it was proved by Allender, Bürgisser, Kjeldgaard-Pedersen and Miltersen [ABKM09, Proposition 1.1] that $\text{BP}(\text{P}_R^0) = \text{pPosSLP}$, where PosSLP is the problem of deciding whether an integer given by a division free arithmetic circuit is positive. While the precise complexity of PosSLP is not known, Allender et al. proved that it is contained in the counting hierarchy CH (not to be confused with the consensus halving problem whose abbreviation coincides).

The class $\text{BP}(\text{NP}_R^0)$ is equal to the class $\exists \mathbb{R}$ that was defined by Schaefer and Štefankovič [SS17] to capture the complexity of the existential theory of the reals ETR. It is known that $\text{NP} \subseteq \exists \mathbb{R} \subseteq \text{PSPACE}$, where the latter inclusion follows from the decision procedure for ETR due to Canny [Can88]. Schaefer and Štefankovič also prove $\exists \mathbb{R}$-completeness for deciding existence of a probability-constrained Nash equilibrium in a given 3-player game in strategic form; later works have extended this to $\exists \mathbb{R}$-completeness for many other decision problems about existence of Nash equilibria satisfying different properties in 3-player games in strategic form [GMVY18; BM16; BM17; BH19]. The proofs of $\exists \mathbb{R}$-hardness makes critical use of the fact that the input is discrete and it is not known if these problems are also complete for $\text{NP}_R$.

We define the class of $\exists \mathbb{R}$ search problems as the following subclass of all real valued search problems.
Instaces $I$ are encoded as string over a given finite alphabet $\Sigma$ and we assume there is a polynomial time algorithm that given $I$ computes $d_I$, where $D_I \subseteq \mathbb{R}^{d_I}$. We next assume that there are polynomial time constant free BSS machines that given $I$ and $x \in \mathbb{R}^{d_I}$ checks whether $x \in D_I$, and given $I$ and $x \in D_I$ checks whether $x \in \text{Sol}(I)$. The corresponding language in $\exists \mathbb{R}$ is then $L = \{I \mid \text{Sol}(I) \neq \emptyset\}$.

### 2.3 Solving real-valued search problems

Let $\Pi$ be a $\exists \mathbb{R}$ search problem. In analogy with the case of NP search problems, one could consider the task of solving $\Pi$ to be that of giving as output some member $y$ of $\text{Sol}(I)$ in case $\text{Sol}(I) \neq \emptyset$. In general each member of $\text{Sol}(I)$ may be irrational valued which precludes a Turing machine to compute a solution explicitly. This is in general also the case for a BSS machine, even when allowing machine constants. Regardless, we shall restrict our attention to Turing machines below.

On the other hand, when $\text{Sol}(I) \neq \emptyset$ a solution is guaranteed to exist with coordinates being algebraic numbers, since a member of $\text{Sol}(I)$ may be defined by an existential first-order formula over the reals with only rational-valued coefficients. This means that one could instead compute an indirect description of the coordinates of a solution, for instance by describing isolated roots of univariate polynomials. If such a description could be computed in polynomial time in $|I|$ we could consider that to be a polynomial time solution of $\Pi$.

Etessami and Yannakakis [EY10] suggest several other computational problems one may alternatively consider in place of solving a search problems $\Pi$ explicitly or exactly. Our main interest is in the problem of approximation. We shall assume for simplicity that $D_I \subseteq [-1,1]^{d_I}$. Together with an instance $I$ of $\Pi$ we are now given as an auxiliary input a rational number $\epsilon > 0$, and the task is to compute $x \in \mathbb{Q}^{d_I}$ such that there exist $x' \in \text{Sol}(I)$ with $\|x' - x\|_\infty \leq \epsilon$. We shall turn this into a discrete search problem by encoding the coordinates of $x$ as binary strings. More precisely, to $\Pi$ we shall associate a discrete search problem $\Pi_\alpha$, where instances are of the form $(I,k)$, where $I$ is an instance of $\Pi$ and $k$ is a positive integer. We define $\epsilon = 2^{-k}$ and let the domain of $(I,k)$ be $D_{I,k} = \{0,1\}^{d_I(k+3)}$, thereby allowing the specification of a point $x \in D_I$ with coordinates of the form $x_i = a_i2^{-k+1}$, where $a_i \in \{-2^{k+1}, \ldots, 2^{k+1}\}$. The solution set $\text{Sol}(I,k)$ is defined from $\text{Sol}(I)$ by approximating each coordinate. That is, we define $\text{Sol}(I,k) = \{x \in D_{I,k} \mid \exists x' \in \text{Sol}(I) : \|x' - x\|_\infty \leq \epsilon\}$. Note that if we had defined $\text{Sol}(I,k)$ by instead truncating the coordinates of solutions $x' \in \text{Sol}(I)$ to $k$ bits of precision, we would have obtained the possibly harder problem of partial computation which was also considered by Etessami and Yannakakis [EY10].

We say that $\Pi$ can be approximated in polynomial time if the approximation problem $\Pi_\alpha$ can be solved in time polynomial in $|I|$ and $k$.

### 2.4 Reductions between search problems

Let $\Pi$ and $\Gamma$ be search problems. A many-one reduction from $\Pi$ to $\Gamma$ consists of a pair of functions $(f,g)$. The function $f$ is called the instance mapping and the function $g$ the solution mapping. The instance mapping $f$ maps any instance $I$ of $\Pi$ to an instance $f(I)$ of $\Gamma$ and for any solution $y \in \text{Sol}(f(I))$ of $\Gamma$ the solution mapping $g$ maps the pair $(I,y)$ to a solution $x = g(I,y) \in \text{Sol}(I)$ of $\Pi$. It is required that $\text{Sol}(f(I)) \neq \emptyset$ whenever $\text{Sol}(I) \neq \emptyset$. We will only consider many-one reductions, and will refer to these simply as reductions.

If $\Pi_1$ and $\Pi_2$ are discrete search problems a reduction $(f,g)$ between $\Pi_1$ and $\Pi_2$ is a polynomial time reduction if both functions $f$ and $g$ are computable in polynomial time. If $\Pi_1$ and $\Pi_2$ are real-valued search problems it is less obvious which notion of reduction is most appropriate and we shall consider several different types of reductions. For all these we assume that $f$ is computable in polynomial time. The reduction $(f,g)$ is a real polynomial time reduction if $g$ is computable in polynomial time by a constant free BSS machine. We shall generally consider this notion of reduction too powerful. In particular the definition

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does not guaranteed that the function $g$ is a continuous function in its second argument $y$. For this reason we instead consider reductions defined by algebraic circuits over a given basis $B$ of real-valued basis functions.

We say that the reduction $(f, g)$ is a \textit{polynomial time $B$-circuit reduction} if there is a function computable in polynomial time that maps an instance $I$ to a $B$-circuit $C_I$ in such a way that $C_I$ computes a function $C_I : D(I) \rightarrow D_I$ where $g(I, y) = C_I(y)$ for all $y \in \text{Sol}(f(I))$. Note in particular that the size of $C_I$ and the bitsize of all constant gates are bounded by a polynomial in $|I|$. If in addition there exists a constant $h$ such that the depth of $C_I$ is bounded by $h$ for all $I$ we say that the reduction $(f, g)$ is a \textit{polynomial time constant depth $B$-circuit reduction}. Etessami and Yannakakis [EY10] defined the even weaker notion where the solution set is of the form $\text{Sol}(f(I))$, i.e. all $y \in \text{Sol}(f(I))$.

The class $\text{FIXP}$ consists of all total $\exists \mathbb{R}$ search problems that are $\exists \mathbb{R}$ reducible to a basic $\exists \mathbb{R}$ search problem for which each domain $D_I$ is a convex polytope described by a set of linear inequalities with rational coefficients and the function $f_I$ is defined by a $\{+, -, \times, \div, \text{max}, \text{min}\}$-circuit $C_I$.

2.5 Total real-valued search problems

Like in the case of TFNP where interesting classes of total NP search problems may be defined in terms of existence theorems for finite structures [Pap94; GP18], we may define classes of total real valued $\exists \mathbb{R}$ search problems based on existence theorems concerning domains $D_I \subseteq \mathbb{R}^n$. Typical examples of such domains $D_I$ are spheres and balls. Suppose $p$ is either a real number $p \geq 1$ or $p = \infty$. By $S^n_p$ and $B^n_p$ we denote the unit $n$-sphere and unit $n$-ball with respect to the $\ell_p$-norm defined as $S^n_p = \{x \in \mathbb{R}^{n+1} \mid \|x\|_p = 1\}$ and $B^n_p = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$, respectively. If $p$ is not specified, we simply assume $p = 2$.

2.5.1 The Brouwer fixed point theorem and $\exists \mathbb{R}$ search problems

We recall here the definition of the class $\exists \mathbb{R}$ search problem by Etessami and Yannakakis [EY10]. The class $\exists \mathbb{R}$ search is defined by starting with $\exists \mathbb{R}$ search problems given by the Brouwer fixed point theorem, and afterwards closing the class with respect to $\exists \mathbb{R}$-reductions. We shall refer to these defining problems as basic $\exists \mathbb{R}$ search problems.

Definition 2. An $\exists \mathbb{R}$ search problem $\Pi$ is a basic $\exists \mathbb{R}$ search problem if every instance $I$ describes a nonempty compact convex domain $D_I$ and a continuous function $F_I : D_I \rightarrow D_I$, computed by an algebraic circuit $C_I$, and these descriptions must be computable in polynomial time. The solution set is $\text{Sol}(I) = \{x \in D_I \mid F_I(x) = x\}$.

The Brouwer fixed point theorem guarantees that every basic $\exists \mathbb{R}$ search problem is a total $\exists \mathbb{R}$ search problem. To define the class $\exists \mathbb{R}$ search, Etessami and Yannakakis restrict attention to a concrete class of basic $\exists \mathbb{R}$ search problems.

Definition 3. The class $\exists \mathbb{R}$ search consists of all total $\exists \mathbb{R}$ search problems that are $\exists \mathbb{R}$-reducible to a basic $\exists \mathbb{R}$ search problem for which each domain $D_I$ is a convex polytope described by a set of linear inequalities with rational coefficients and the function $f_I$ is defined by a $\{+, -, \times, \div, \text{max}, \text{min}\}$-circuit $C_I$. 

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The class \( \text{FIXP}_a \) is the class of strong approximation problems corresponding to \( \text{FIXP} \). More precisely, \( \text{FIXP}_a \) consist of all discrete search problems polynomial time reducible to the problem \( \Pi_a \) for \( \Pi \in \text{FIXP} \).

The definition of \( \text{FIXP} \) is quite robust with respect to the choice of domain and set of basis functions allowed by circuits in the basic \( \text{FIXP} \) problems. Etessami and Yannakakis proved that basic \( \text{FIXP} \) problems defined by \( \{+, -*, \div, \max, \min, \sqrt[n]{\cdot}\} \)-circuits are still in the class \( \text{FIXP} \). Likewise, basic \( \text{FIXP} \)-problems where \( D_I \) is a ball with rational-valued center and diameter, or more generally an ellipsoid given by a rational center-point and a positive-definite matrix with rational entries, are still in the class \( \text{FIXP} \) \( [\text{EY10, Lemma 4.1}] \). The same argument allows for using as domain the ball \( B^d_1 \) with respect to the \( \ell_p \) norm for any rational \( p \geq 1 \) or \( p = \infty \), with the coordinates possibly transformed by individual affine functions.

On the other hand, Etessami and Yannakakis also proved that one may greatly restrict the class of basic \( \text{FIXP} \) problems used to define \( \text{FIXP} \) without changing the class. The domains may be restricted to be unit hypercubes \([0, 1]^d \) and the circuits may be restricted to \( \{+, *, \max\} \)-circuits. Both restrictions may in fact be imposed at the same time. The restriction to \( \{+, *, \max\} \)-circuits is a consequence of first proving that the problem of finding a Nash equilibrium in a given finite game in strategic form is hard for \( \text{FIXP} \) with respect to \( \text{SL} \)-reductions and then proving \( \text{FIXP} \)-membership of this problem using \( \{+, *, \max\} \)-circuits.

Another way to restrict circuits is by limiting their depth. The function of Nash for expressing Nash equilibrium as Brouwer fixed points involve divisions but as noted by Etessami and Yannakakis it may be viewed as a constant depth circuit, if one allows for addition gates of arbitrary fanin. Thus in the definition of \( \text{FIXP} \) one may restrict circuits to be constant depth \( \{+, *, \max\} \)-circuits, where the addition gates are allowed to have unbounded fanin.

We show in Proposition 3 of Section 3 that one may in fact take this much further and completely flatten the circuits of defining problems for \( \text{FIXP} \) to be depth 1 circuits of fanin at most 2, additionally also without requiring division. In other words, each coordinate function becomes just a simple function of at most 2 coordinates of the input. We also show in Proposition 1 that \( \text{FIXP} \) is closed under much more powerful reductions than just the basic \( \text{SL} \)-reductions used to define the class \( \text{FIXP} \).

### 2.5.2 The Borsuk-Ulam theorem and \( \text{BU} \)

A new class \( \text{BU} \) of total \( \exists \mathbb{R} \) search problems based on the Borsuk Ulam theorem was recently introduced by Deligkas et al. [DFMS21]. The definition of \( \text{BU} \) is meant to capture the Borsuk-Ulam theorem as stated in formulation (1) of Theorem 1. Following the definition of \( \text{FIXP} \) by Etessami and Yannakakis, Deligkas et al. first consider a set of basic search problems and then close the class under reductions.

**Definition 4.** An \( \exists \mathbb{R} \) search problem \( \Pi \) is a basic \( \text{BU} \) problem if every instance \( I \) describes a domain \( D_I \subset \mathbb{R}^d \) which is homeomorphic to \( S^{d-1} \) by an antipode preserving homeomorphism and a continuous function \( F_I : D_I \rightarrow \mathbb{R}^{d-1} \), computed by an algebraic circuit \( C_I \), and these descriptions must be computable in polynomial time. The solution set is \( \text{Sol}(I) = \{x \in D_I \mid F_I(x) = F_I(-x)\} \).

In defining the class Deligkas et al. restrict their attention to spheres with respect to the \( \ell_1 \)-norm as domains and functions computed by \( \{+, -*, \max, \min\} \)-circuits. Compared to the definition of \( \text{FIXP} \), division gates are thus excluded. However we show later in Section 4 that division gates can always be eliminated. Having thus fixed the set of basic \( \text{BU} \) search problems what remains in order to define \( \text{BU} \) is to settle on a notion of reductions. In their journal paper, Deligkas et al. [DFMS21] suggest using reductions computable by general algebraic circuits including non-continuous comparison gates, whereas in the preceding conference paper [DFMS19] they did not precisely define a choice of reductions. We shall revisit the question of choice of reduction in Section 4 before proposing our definition of \( \text{BU} \).

### 2.6 Consensus Halving

We give here a formal definition of consensus halving with additive measures as real valued search problems.
where \( t_0 = 0 \) and \( t_j = \sum_{k \leq j} \lvert x_k \rvert, \) for \( j = 1, \ldots, n + 1. \)

Given \( \{+, -, \ast, \div, \max, \min\} \)-circuits computing the distribution functions \( F_i \), the function \( F \) computing the left-hand-side of equation (1) may clearly be computed by \( \{+, -, \ast, \div, \max, \min\} \)-circuits as well. The result of Deligkas et al. that \( \text{CH} \) is contained in \( \text{BU} \) follows.

The existence proof of a consensus halving by Simmons and Su as well the formulation of a \( \exists \mathbb{R} \) search problem by Deligkas et al. match the Borsuk-Ulam theorem as stated in formulation (1) of Theorem 1. We now assume that the statement \( \Phi(\epsilon, \delta) \) is simply a Boolean formula whose atoms involve univariate polynomial equalities and inequalities. The bounds given by Basu, Pollack and Roy for the result of quantifier elimination imply that the degree of the univariate polynomials are bounded by \( d^{O(k_1)\ldots O(k_m)} \) with coefficients of bitsize at most \( \max(k, \tau)d^{O(k_1)\ldots O(k_m)} \). We may now appeal to Theorem 13.17 of [BPR16] to conclude that \( \Psi(\delta) \), and hence also \( \Phi(\epsilon, \delta) \) is true, for some \( \delta \geq 2^{-\max(k, \tau)d^{O(k_1)\ldots O(k_m)}} \).

### 2.7 Tools from Real Algebraic Geometry

For obtaining our results concerning strong approximation we need concrete bounds on \( \delta > 0 \) as a function of \( \epsilon > 0 \) witnessing the truth of “epsilon-delta” statements. When such a statement is expressible in the first-order theory of the reals, such bounds can be obtained in a generic way using the general machinery of real algebraic geometry [BPR16]. This approach has been used several times previously for establishing \( \text{FIXP}_o \) membership of the problem of strong approximation of Nash equilibrium refinements [EHMS14; Ete20; HL18].

Concretely, suppose that \( \Phi(\epsilon, \delta) \) is a formula with free variables \( \epsilon \) and \( \delta \) of the form

\[
\Phi(\epsilon, \delta) = (Q_1 x_1 \in \mathbb{R}^{n_1}) \ldots (Q_\omega x_\omega \in \mathbb{R}^{k_\omega}) F(x_1, \ldots, x_\omega, \epsilon, \delta),
\]

where \( Q_i \in \{\forall, \exists\} \), and \( F \) is a Boolean formula whose atoms are polynomial equalities and inequalities involving polynomials of degree at most \( d \) and having integer coefficients of bitsize at most \( \tau \).

We now assume that the statement \( (\forall \epsilon > 0)(\exists \delta > 0)\Phi(\epsilon, \delta) \) is true, and fix \( \epsilon = 2^{-k} \), for a positive integer \( k \), resulting in the formula \( (\delta > 0) \land \Phi(\epsilon, \delta) \), with \( \delta \) as the only variable. We may now perform quantifier elimination [BPR16, Algorithm 14.21] on this to obtain an equivalent quantifier free formula \( \Psi(\delta) \). The formula \( \Psi(\delta) \) is simply a Boolean formula whose atoms involve univariate polynomial equalities and inequalities. The bounds given by Basu, Pollack and Roy for the result of quantifier elimination imply that the degree of the univariate polynomials are bounded by \( d^{O(k_1)\ldots O(k_m)} \) with coefficients of bitsize at most \( \max(k, \tau)d^{O(k_1)\ldots O(k_m)} \). We may now appeal to Theorem 13.17 of [BPR16] to conclude that \( \Psi(\delta) \), and hence also \( \Phi(\epsilon, \delta) \) is true, for some \( \delta \geq 2^{-\max(k, \tau)d^{O(k_1)\ldots O(k_m)}} \).
In our applications, the formula $\Phi$ is defined from a given instance $I$. Both $\tau$ and $d$ will be bounded by fixed polynomials of $|I|$. The number of blocks $\omega$ of quantified variables will be a fixed constant, and $k_i$ for $1 \leq i \leq \omega$ are bounded by fixed polynomials of $|I|$ as well. In other words there will be a fixed polynomial $q$ such that the formula $\Phi(\epsilon, \delta)$ is true for some $\delta \geq (1/\epsilon)^{2q(|I|)}$.

The first-order formulas we consider are expressed using also the evaluation of functions computable by algebraic circuits as a primitive. We may in a generic way transform such formulas to having only polynomial inequalities and equalities that each gate is computed correctly, and the variables corresponding to the output gates may then be used instead in place of the function. As long as the number of evaluations of functions is constant, this leaves the number of blocks of quantified variables constant.

3 Structural Properties of FIXP

Recall that FIXP is defined to be the closure of all basic FIXP problems with respect to the very simple notion of SL-reductions. We first show that FIXP is in fact closed under general circuit reductions.

**Proposition 1.** Suppose that $\Pi$ is a $\exists R$ search problem defined with unit hypercube domains and reduces to $\Gamma \in$ FIXP by a polynomial time $\{+,-,\times,\div,\max,\min,\sqrt[n]{\cdot}\}$-circuit reduction. Then $\Pi$ belongs to FIXP as well.

**Proof.** We may without loss of generality assume the domain of $\Gamma$ is also the unit hypercube. Let $(f, g)$ be the assumed reduction from $\Pi$ to $\Gamma$. Let $I$ be an instance of $\Pi$. By assumption $D_I = [0,1]^m$ and $D_{f(I)} = [0,1]^n$, where $m = d_I$ and $n = d_{f(I)}$. From the definition of $(f, g)$ we may given $I$ in polynomial time compute $f(I)$ as well as the circuit $C_I$ that defines a function $G: [0,1]^n \rightarrow [0,1]^m$ such that $g(I, x) = G(x)$ for all $x \in \text{Sol}(f(I))$. By assumption on $\Gamma$ we may in polynomial time compute another circuit $C_{f(I)}$ that defines a function $F: [0,1]^m \rightarrow [0,1]^n$ such that $\text{Sol}(f(I))$ are the fixed points of $F$.

We now define the function $H: [0,1]^{n+m} \rightarrow [0,1]^{n+m}$ by $H(x, y) = (F(x), G(x))$. Clearly the set of fixed points of $H$ is equal to $\{(x, G(x)) \mid x \in \text{Sol}(f(I))\}$, and since $H$ is computable by a $\{+,-,\times,\div,\max,\min,\sqrt[n]{\cdot}\}$-circuit this defines a $\exists R$ search problem $\Lambda$ in FIXP with the same set of instances as $\Pi$. We note that the projection of a fixed point of $H$ to the last $m$ coordinates gives a solution to $\Pi$ from which it follows that $\Pi$ in particular SL-reduces to $\Lambda$. Therefore $\Pi$ belongs to FIXP as well.

Our next basic result is based on properties of the basic FIXP problem used by Etessami and Yannakakis to show that the division operation is not necessary to express all of FIXP. We give a brief review of their construction. An instance $I$ describes a $d$-player game in strategic form. Player $i$ has a set $S_i$ of $n_i = |S_i|$ pure strategies and a utility function $u_i: S_1 \times \cdots \times S_d \rightarrow \mathbb{R}$. Let $n = n_1 + \cdots + n_d$ be the total number of strategies. The domain is given as $D_I = \Delta_{n_1-1} \times \cdots \times \Delta_{n_d-1}$, where the $(n_i-1)$-dimensional unit simplex $\Delta_{n_i-1}$ is identified with the set of probability distributions on $S_i$, for $i = 1, \ldots, d$. The domain $D_I$ may be viewed as a subset of $\mathbb{R}^n$ in the natural way. The utility functions define the function $v: D_I \rightarrow \mathbb{R}^n$ given by

$$v(x)_{ia_i} = \sum_{a_{-i} \in S_{-i}} u_i(a_1, \ldots, a_d) \prod_{j \neq i} x_{ja_j},$$

where $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \cdots \times S_d$ and $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) \in S_{-i}$. Define further the function $h: D_I \rightarrow \mathbb{R}^n$ by $h(x) = x + v(x)$ and finally let $G_I: D_I \rightarrow D_I$ be defined by letting $G_I(x)$ be the projection of $h(x)$ onto $D_I$. For all $i = 1, \ldots, d$, it holds that $G_I(x)_{ia_i} = \max(h_{ia_i} - t_i, 0)$, where $t_i$ is the unique value satisfying $\sum_{a_{-i} \in S_{-i}} \max(h_{ia_i} - t_i, 0) = 1$. The fixed points of $G_I$ are exactly the Nash equilibria of the game described by $I$ [EY10, Lemma 4.5], and the search problem is therefore FIXP-complete [EY10, Theorem 4.3].
The definitions of the functions \( v, h, \) and \( G_I \) allows us to extend their domain from \( D_I \) to the \( n \)-dimensional unit cube \([0,1]^n\). By definition of \( G_I \) this does not change the set of fixed points of \( G_I \). Likewise, applying the same affine transformation to \( u_I(x) \), for \( i = 1,\ldots,d \), does not change the set of fixed points of \( G_I \). We may thus assume that \( u_I \) has codomain \([0,1]\). Making use of a sorting network, Etessami and Yannakakis show that \( G_I \) may be computed by a polynomial size \( \{+,\,-,\,\cdot\,\,\text{min},\text{max}\}\)-circuit \( C_I \) [EY10, Lemma 4.6].

It is furthermore straightforward to ensure that all constants used in \( C_I \) as well as values computed by gate functions of \( C_I \) belong to the interval \([0,1]\) for any input \( x \in [0,1]^n \) (cf. [DFMS21]). We summarize these observations below.

**Proposition 2.** There is a basic \( \text{FIXP} \) problem \( \Pi_{\text{NE}} \), complete for \( \text{FIXP} \) under SL-reductions, such that for any instance \( I \) it holds that \( D_I = [0,1]^{d_I} \) and such that \( C_I \) is a \( \{+,\,-,\,\cdot\,\,\text{min},\text{max}\}\)-circuit that satisfies that all gate functions of \( C_I \) compute values in \([0,1]\) given input \( x \in D_I \).

From here we may derive a characterization of \( \text{FIXP} \) in terms of depth 1 circuits, where the addition and subtraction operators (necessarily) are truncated to the interval \([0,1]\). This is simply done by a Tseitin-style transformation. One may note that a Tseitin-style transformation is already used in the proof that \( \Pi_{\text{NE}} \) is \( \text{FIXP} \)-hard. This means such a transformation is applied twice at different points of the proof to yield the statement below.

**Proposition 3.** There is a basic \( \text{FIXP} \) problem \( \Pi \), complete for \( \text{FIXP} \) under SL-reductions, such that for any instance \( I \) it holds that \( D_I = [0,1]^{d_I} \) and such that \( C_I \) is a depth 1 \( \{+,\,-,\,\cdot\,\,\text{max},\text{min}\}\)-circuit, using only constants from the interval \([0,1]\).

**Proof.** We reduce from the problem \( \Pi_{\text{NE}} \) of Proposition 2. The instances of \( \Pi \) are the same instances of \( \Pi_{\text{NE}} \). Let \( I \) be an instance of \( \Pi_{\text{NE}} \) and let \( D = [0,1]^{d_I} \) and \( C_I \) be the corresponding domain and \( \{+,\,-,\,\cdot\,\,\text{min},\text{max}\}\)-circuit as given by Proposition 2. Suppose that \( C_I \) has \( m_I \) gates \( g_1,\ldots,g_{m_I} \). We define the new domain \( D_I' \) for \( \Pi \) simply by \( D_I' = [0,1]^{d_{I'}} \), where \( d_{I'} = d_I + m_I \). We next define the gates of \( C_I' \) which all are output gates of \( C_I \). We may consider the input as pairs \((x,y)\in [0,1]^d \times [0,1]^m \) and we may think of the output gates as variables, similarly grouped as \((z,w)\) and ranging over \([0,1]^{d_I} \times [0,1]^{m_I} \). If \( g_j \) is an input gate labeled by \( x_j \), we let \( w_j = x_j \), and if \( g_j \) is a constant gate labeled by \( c \in [0,1] \) we let \( w_j = c \). If \( g_j \) is an addition gate taking as input gates \( g_k \) and \( g_l \) we let \( w_j = (y_k + y_l) \in [0,1] \), i.e. the addition of \( g_k \) and \( g_l \) is simulated by a truncated addition of \( y_k \) and \( y_l \). The case of subtraction is analogous. If \( g_j \) is a multiplication gate taking as input \( g_k \) and \( g_l \) we let \( w_j = y_k \cdot y_l \). The case of maximum and minimum gates are analogous. Finally if \( g_j \) is the \( i \)th output gate of \( C_I \) we let \( z_i = y_j \). By construction \( C_I' \) computes a function \( F_I' : D_I' \rightarrow D_I' \) and \( F_I'(x,y) = (x,y) \) if and only if \( g_j \) computes the value \( y_j \) on input \( x \) for all \( I \) and \( C_I(x) = x \). We thus obtain \( x \) such that \( C_I(x) = x \) as the projection of \((x,y)\) to the first \( d_I \) coordinates.

In case we prefer to construct a normal \( \{+,\,-,\,\cdot\,\,\text{max},\text{min}\}\)-circuit without truncated operations we can clearly simulate the truncated addition and subtraction operations by depth 3 circuits. We can also easily convert the circuits to constant depth \( \{+,\,\cdot\,\,\text{max}\}\) circuits by considering the the domain \( B^{d_I}_{p} = [-1,1]^{d_I} \) instead of \([0,1]^{d_I} \).

### 4 Definition and Structural Properties of BU and BBU

In this section we define two classes of \( \exists \mathbb{R} \) search problems \( \text{BU} \) and \( \text{BBU} \) based on the Borsuk-Ulam theorem corresponding to formulations (1) and (3) of Theorem 1. We start by defining basic \( \text{BU} \) and basic \( \text{BBU} \) problems. We shall restrict our attention to the unit \( n \)-sphere and unit \( n \)-ball, but with regards to any \( \ell_p \) norm for \( p \geq 1 \) or \( p = \infty \). For the case of \( \text{BU} \) this amounts to specializing Definition 4.

**Definition 7.** A basic \( \text{BU} \) problem is a basic \( \ell_p \)-\( \text{BU} \) problem if for every instance \( I \) we have \( D_I = S^{d_I}_{p} \).
Similarly we define the set of basic BBU problems with respect to the $\ell_p$-norm.

**Definition 8.** An $\exists \mathbb{R}$ search problem $\Pi$ is a basic $\ell_p$-BBU problem if for every instance $I$ we have $D_1 = \mathbb{B}_p^n$ and $I$ describes a continuous function $F_I : D_1 \to \mathbb{R}^{d_1}$, which is odd on the boundary $\partial \mathbb{B}_p^n$. The function $F_I$ must be computed by an algebraic circuit $C_I$ whose description is computable in polynomial time. The solution set is $\text{Sol}(I) = \{x \in D_I \mid F_I(x) = 0\}$.

The condition that the function $F_I$ is odd on $\partial \mathbb{B}_p^n$ is a semantic condition. However, typically the function $F_I$ would be defined from a basic $\ell_p$-BU problem by a transformation done in a similar way as in the proof of Theorem 1, and thereby $F_I$ would satisfy the condition automatically.

To define the classes BU and BBU, we restrict our attention to domains with respect to the $\ell_\infty$-norm.

**Definition 9.** The class BU (respectively, BBU) consists of all total $\exists \mathbb{R}$ search problems that are PL-reducible to a basic $\ell_\infty$-BU problem (respectively, basic $\ell_\infty$-BBU problem) for which the function $F_I$ is defined by a $\{+, -, \times, \div, \max, \min\}$-circuit $C_I$.

While the definition of BU in [DFMS21] was using as domain the unit sphere with respect to the $\ell_1$-norm and not allowing for division gates, we show in this section these changes do not change the class. We propose choosing PL-reductions for closing the class under reductions. PL-reductions are sufficient for obtaining all of our results and they are polynomially continuous. Another reason for this choice is that if we restrict the circuits defining the classes FIXP and BU to also be piecewise linear, i.e. be $\{+, \times, \max\}$-circuits, we obtain the classes LinearFIXP and LinearBU, that when closed under polynomial-time reductions are equal to PPAD and PPA, respectively [EY10; DFMS21].

### 4.1 Elimination of Division Gates

In this section, we show how to eliminate division gates from circuits defining an instance of the BU or BBU problems. Let therefore $C$ denote an algebraic circuit defined over the basis $\{+, -, \times, \div, \max, \min, \sqrt{\cdot}\}$.

**Moving Divisions to the Top.** In the paper [EY10], it is shown how to move all division gates to the top of the circuit by keeping track of the numerator and denominator of every gate. For sake of completeness we describe this transformation. Every gate $g_i$ is replaced by two gates $g_i'$ and $g_i''$ keeping track of the numerator and denominator, that is the value of $g_i$ in the original circuit will be equal to the value of $g_i'/g_i''$ in the transformed circuit. Firstly, if $g_i$ is an input gate or a constant-$c$ gate we put $g_i' = x_j$ for appropriate $j$ (respectively $g_i' = c$) and $g_i'' = 1$. In order to maintain the equality $g_i = g_i'/g_i''$, we proceed as follows: if $g_i = g_j \pm g_k$ is an addition/subtraction gate in the original circuit, then we update the numerator and denominator to $g_i' = g_j' \cdot g_k' \pm g_j'' \cdot g_k''$ and $g_i'' = g_j'' \cdot g_k''$; if $g_i = g_j \cdot g_k$, then $g_i' = g_j' \cdot g_k'$ and $g_i'' = g_j'' \cdot g_k''$; if $g_i = g_j \div g_k$, then $g_i' = g_j' \cdot g_k''$ and $g_i'' = g_j'' \cdot g_k'$. For root gates, we note that if $g_j = g_j'$ is input to a $\sqrt{\cdot}$-gate $g_i$ for $k$ even, then $g_j \geq 0$, from which it follows that $\text{sgn}(g_j') = \text{sgn}(g_j'')$. With this in mind, we see that we may maintain the numerator and denominator of $g_i$ by putting $g_i' = \sqrt{g_j' g_k'}$ and $g_i'' = \sqrt{g_j'' g_k''}$. Finally, for the max-gate we note that $\max(c a, c b) = c \max(a, b)$ for $c \geq 0$. Using this we see that if $g_i = \max(g_j, g_k)$, then we may maintain the numerator and denominator via the formulas $g_i' = \max(g_j', g_k') \cdot (g_j'' \cdot g_k'' \cdot (g_j')^2)$ and $g_i'' = (g_j'')^2 \cdot (g_k'')^2$. We note that all this can be done only blowing up the size of the circuit by a constant factor. In the aforementioned paper, the authors then have division gates at the top outputting $\text{out}_i = \text{out}_i'/\text{out}_i''$. However, for our application this is unnecessary and we may completely remove division gates.
Removing Division Gates for BBU. Suppose that $\Pi$ is a BBU problem. Let $I$ be an instance of $\Pi$ and denote by $C_I$ an algebraic circuit computing a continuous function $F_I: B^{d_I} \rightarrow \mathbb{R}^{d_I}$ that is odd on $S^{d_I-1}$ such that $\text{Sol}(I) = \{x \in B^{d_I} \mid F_I(x) = 0\}$. As described above, we may transform the circuit $C_I$ to a circuit $C_I^+$ that maintains the numerator and denominator of every gate. In the same way we define a circuit $C_I^-$ that is exactly like $C_I^+$, except it multiplies the input by $-1$ at the very beginning. Let $\text{out}_i^{+}, \text{out}_i^{-}$ denote the gates in $C_I^+$ representing the numerators and denominators of the output gates of $C_I$. We now define a circuit $C_I^*$ that on input $x$ feeds this into $C_I^+$ and $C_I^-$ and then outputs the values $\text{out}_i^{+} \cdot \text{out}_i^{-}$ for $i = 1, \ldots, d_I$. If we denote by $F_i = F_i^+/F_i^-$ the coordinate functions of $F_I$, then $C_I^*$ is a circuit computing the function $F_i^*$ with coordinate functions $F_i^*(x) = F_i^+(x)/F_i^-(x)$. Now, if $x \in S^{d_I}$ then $F_i^*(x) = F_i^+(x)/F_i^-(x)$, so $F_i^*(x)F_i^-(x) = F_i^+(x)F_i^-(x)$, meaning that $F_i^*$ is odd on the boundary. In this way we have defined a BBU problem $\Gamma$ with the same instances as $\Pi$. Furthermore, given an instance $I$ of $\Pi$ one may in polynomial time compute an instance $f(I)$ of $\Gamma$ by computing $C_I^*$. We note that for any $x \in B^{d_I}$ it holds that $F_I(x) = 0$ if and only if $F_i^*(x) = 0$. We conclude that $\Pi$ SL-reduces to the division-free BBU–problem $\Gamma$.

Removing Division Gates for BU. Now let $I$ be an instance of a BU–problem $\Pi$ and denote by $C_I$ an algebraic circuit computing a continuous function $F_I: S^{d_I} \rightarrow \mathbb{R}^{d_I}$ such that $\text{Sol}(I) = \{x \in S^{d_I} \mid F_I(x) = F_I(-x)\}$. We make the same reduction as for BBU defining a circuit $C_I^*$ that computes a function $F_i^* : S^{d_I} \rightarrow \mathbb{R}^{d_I}$ whose coordinate functions are given by $F_i^*(x)F_i^-(x)$ where $F_i^+(x)/F_i^-(x)$ is the $i$th coordinate function of $F_i$. By definition, $x$ is a BU-point of $F_I$ if and only if $F_i^*(x)/F_i^-(x) = F_i^+(x)/F_i^-(x)$ for all $i$. This happens if and only if $F_i^+(x)F_i^-(x) = F_i^-(x)F_i^+(x)$ for all $i$, meaning that $x$ is a BU-point of $F_i^*$. Again, we conclude that $\Pi$ SL-reduces to a division-free BU–problem.

In the previous two paragraphs, we have shown the following result.

**Proposition 4.** The classes BU and BBU remain the same even if the circuits are restricted to not using division gates.

### 4.2 Relationship with FIXP

As a consequence of their results about consensus halving, Deligkas et al. proved that $\text{FIXP} \subset \text{BU}$. We observe here that the direct proof that the Bosu-Valum theorem implies the Brouwer fixed point theorem due to Volovikov [Vol08] gives a much simpler way to derive this relationship. For completeness we present the construction and proof of Volovikov.

**Proposition 5** (Volovikov). Let $f: B^{d}_\infty \rightarrow B^{d}_\infty$ be a continuous function. Define the continuous function $g: S^{d}_\infty \rightarrow \mathbb{R}^d$ by $g(x,t) = (1 + t)(f(x) - x)$. If $g(x,t) = g(-x,-t)$ then $|t| = 1$ and $f(tx) = tx$.

**Proof.** Note first that

$$g(x,t) - g(-x,-t) = t \left[(1 + t)f(x) + (1 - t)f(-x)\right] - 2x .$$

It follows that $g(x,t) = g(-x,-t)$ if and only if $k(x,t) = x$, where

$$k(x,t) = \frac{t}{2} \left[(1 + t)f(x) + (1 - t)f(-x)\right] .$$

If $(x,t) \in S^{d}_\infty$ and $|t| < 1$ it holds that $\|x\|_\infty = 1$. Then since

$$\|k(x,t)\|_\infty \leq \frac{|t|}{2} \left[(1 + t)\|f(x)\|_\infty + (1 - t)\|f(-x)\|_\infty\right]$$

$$\leq \frac{|t|}{2} \left[(1 + t) + (1 - t)\right] = |t| < 1 ,$$

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we have \(k(x,t) \neq x\). Thus \(g(x,t) = g(-x,-t)\) implies that \(|t| = 1\). When \(|t| = 1\) we clearly have \(k(x,t) = tf(tx)\). In conclusion, \(g(x,t) = g(-x,-t)\) implies \(tf(tx) = x\), or equivalently that \(f(tx) = tx\).

The above construction immediately give a simple reduction from any basic FIXP problem with domains \(B^d_{\infty}\) to a basic \(\ell_{\infty}\)-BU problem. The solution mapping of the reduction must map solutions \((x,t)\) to \(tx\). This may be done by simply using multiplication gates. But since any solution \((x,t)\) has \(|t| = 1\) the multiplication \(tx\) may also be expressed as \(Sel_2(-x_i,x_i,t)\), which means the solution mapping can also be computed by constant depth \(\{+,*\}\)-circuits.

**Proposition 6.** Any \(\Pi \in \text{FIXP}\) reduces to a basic \(\ell_{\infty}\)-BU problem with \(\{+,-,\cdot,\max,\min\}\)-circuit by polynomial time constant depth B-circuit reductions, for both \(B = \{+,-,\cdot\}\) and \(B = \{+,-,\cdot,\zeta,\max,\min\}\).

**Proof.** Any \(\Pi \in \text{FIXP}\) SL-reduces to a basic FIXP problem \(\Gamma\) with domains \(D_I = B^d_{\infty}\) and \(\{+,*\}\)-circuits \(C_I\). From that, the instance mapping as described by Proposition 5 produces a \(\{+,*\}\)-circuit and domain \(S^d_{\infty}\). The composition of the SL-reduction and the reduction described above then yields the claimed types of reductions.

### 4.3 Change of Domains for BU and BBU

In this section we show reduce between different domains for the BBU and BU problems.

**Proposition 7.** Let \(B\) be a set of gates that contains \(\{+,-,\cdot,\max,\min\}\). Suppose that \(\Pi\) is an \(\exists \mathbb{R}\) search problem whose domains are contained in hypercubes that reduces to a basic \(\ell_p\)-BBU problem \(\Gamma\) by a polynomial time \(B\)-circuit reduction \((f,g)\). Furthermore, suppose that for any instance \(I\) of \(\Pi\) the function \(g(I,\cdot)\) mapping solutions of \(f(I)\) to solutions of \(I\) is odd and assume that \(C_{f(I)}\) is also a \(B\)-circuit. (i) If \(p = \infty\) then \(\Pi\) SL-reduces to a basic \(\ell_{\infty}\)-BBU problem using gates in \(B\). (ii) If \(1 \leq p < \infty\) then \(\Pi\) SL-reduces to a basic \(\ell_p\)-BBU problem using \(B \cup \{\sqrt{\cdot}\}\)-circuits.

**Proof.** (i) First assume that the domains of \(\Gamma\) are unit hypercubes. Let \(I\) denote an instance of \(\Pi\). By assumption \(D_I \subseteq [-1,1]^m\) and \(D_{f(I)} = [-1,1]^n\) where \(m = d_I\) and \(n = d_{f(I)}\). From the definition of \((f,g)\) we may given \(I\) in polynomial time compute \(f(I)\) and a circuit \(C_I\) computing a function \(G: [-1,1]^n \rightarrow [-1,1]^m\) such that \(G(x) = g(I,x) \in \text{Sol}(I)\) for every \(x \in \text{Sol}(f(I))\). By assumption of \(\Gamma\) we may in polynomial time compute another circuit \(C_{f(I)}\) that defines a function \(F: [-1,1]^n \rightarrow \mathbb{R}^n\) that is odd on the boundary such that \(\text{Sol}(f(I))\) are the zeroes of \(F\).

Define \(H: [-1,1]^{n+m} \rightarrow \mathbb{R}^{n+m}\) by \(H(x,y) = ((1 - ||y||_\infty)F(x),y - \frac{1}{2}G(x))\). As \(G\) is odd and \(F\) is odd on the boundary, one may verify that \(H\) is odd on the boundary of \([-1,1]^{n+m}\). As \(H\) is polynomial-time computable by a \(B\)-circuit, it defines an \(\ell_{\infty}\)-BBU problem \(\Lambda\) with the same instances as \(\Pi\). Furthermore, if \((x,y)\) is a zero of \(H\), then \(y = G(x)/2\), so \(||y||_\infty < 1\). The equality \((1 - ||y||_\infty)F(x) = 0\) from the first component then implies \(F(x) = 0\). Therefore, the zeroes of \(H\) are contained in \(\{(x,G(x)/2) \mid x \in \text{Sol}(f(I))\}\). Given a zero of \(H\) one may recover a solution to \(\Pi\) by projecting onto the last \(m\) coordinates and multiplying by 2. In particular, \(\Pi\) SL-reduces to \(\Lambda\).

(ii) Now, suppose that the domains of \(\Gamma\) are \(p\)-balls, where \(1 \leq p < \infty\). Again by assumption we have that \(D_I \subseteq [-1,1]^m\) and \(D_{f(I)} = B^n_p\) where \(m = d_I\) and \(n = d_{f(I)}\), and we may given an instance \(I\) of \(\Pi\) in polynomial time compute a circuit \(C_I\) defining a function \(G: B^n_p \rightarrow [-1,1]^m\) such that \(G(x) = g(I,x) \in \text{Sol}(I)\) for every \(x \in \text{Sol}(f(I))\). Furthermore, we may in polynomial time compute another circuit \(C_{f(I)}\) computing a function \(F: B^n_p \rightarrow \mathbb{R}^n\) that is odd on \(S_{p}^{n-1}\) such that \(\text{Sol}(f(I))\) is the zeroes of \(F\).

Now define an odd function \(h: B^n_p \rightarrow \mathbb{R}^n\) by \(h(x) = x/\max(1/2,||x||_p)\), which may be computed by a circuit using also \(\sqrt{\cdot}\) gates, and define \(H: B^{n+m}_p \rightarrow \mathbb{R}^{n+m}\) by

\[
H(x,y) = (\max(0,\frac{1}{2} - ||y||_p^2)F(h(x)),y - \frac{1}{2}G(h(x)))
\]
First we remark that $H$ is odd on the boundary of $B^p_{m+1}$. Clearly, the second coordinate is always odd, and the first coordinate evaluates to 0 if $|y|^p_p > 1/2$. If $(x, y) \in S^p_{m+1}$ and $|y|^p_p < 1/2$, then $|x|^p_p > 1/2$ which implies that $|x|^p_p > 1/2$. This then implies that $h(x) = x/|x|^p_p$ and so $F(h(x)) = -F(-h(x)) = -F(h(-x))$, because $h$ is odd and $F$ is odd on $S^p_{m+1}$.

Now, if $(x, y)$ is a zero of $H$, then $y = \sum y G(h(x))$ and so $|y|^p_p \leq (n|y|_\infty)^p \leq \frac{1}{p} < \frac{1}{2}$. From the first first coordinate equality $\max(0, 1/2 - |y|^p_p) F(h(x)) = 0$ one then obtains that $F(h(x)) = 0$ so $h(x) \in \text{Sol}(f(I))$. Thus, the set of zeroes of $H$ are contained in $\{x, \frac{1}{2n} G(h(x)) \mid h(x) \in \text{Sol}(f(I))\}$. Furthermore, $H$ can be computed by circuit over $B \cup \{\sqrt n\}$, so this defines a basic $\ell_p - BBU$ problem $\Lambda$ with $(B \cup \{\sqrt n\})$-circuits and the same instances as $\Pi$. From a zero of $H$ we may again recover a solution to $\Pi$ by projecting onto the last $m$ coordinates and multiplying the result by $3n$. We conclude that $\Pi$ SL-reduces to $\Lambda$.

**Proposition 8.** Any basic $\ell_\infty - BBU$ problem SL-reduces to a basic $\ell_p - BBU$ problem using gates in $\{+, -, *, \div, \max, \min, \sqrt n\}$.

**Proof.** Let $\Pi$ be a basic $\ell_\infty - BBU$ problem. By the previous proposition it suffices to argue that $\Pi$ polynomial time $\{+, -, *, \div, \max, \min\}$-reduces to a a basic $\ell_p - BBU$ problem. Given an instance $I$ of $\Pi$, compute in polynomial time a circuit $C_I$ defining a function $F : B^m_0 \rightarrow \mathbb{R}^n$ that is odd on $S^p_{m-1}$ such that $\text{Sol}(I)$ are the zeroes of $F$. Also, define the map $\pi : B^m_0 \rightarrow B^0_0$ by $\pi(x) = x/\max(1/2, |x|_\infty)$. Now we may in polynomial time compute $\{+, -, *, \div, \max, \min\}$-circuit computing the function $G : B^m_p \rightarrow \mathbb{R}^n$ given by $G(x) = F(\pi(x))$. If $|x|^p_p = 1$, then $|x|_\infty \geq 1/(2n)$, so $|\pi(x)|_\infty = 1$, implying that $G(x) = G(-x)$. Thus we have defined a map $f$ taking instances $I$ of $\Pi$ to instances $f(I)$ of a basic $\ell_p - BBU$ problem $\Gamma$. We note that $f$ is computable in polynomial time. Furthermore, from $x \in \text{Sol}(f(I))$ one may recover a solution by $g(I, x) = \pi(x)$ to $I$. As the function $g(I, \cdot)$ is odd and computable by a $\{+, -, *, \div, \max, \min\}$-circuit we conclude that $(f, g)$ satisfies the requirements of the previous proposition. We conclude that $\Pi$ SL-reduces to a $\ell_p - BBU$ using gates from $\{+, -, *, \div, \max, \min, \sqrt n\}$.

**Proposition 9.** Any basic $\ell_p - BBU$ problem $\Pi$ SL-reduces to a basic $\ell_\infty - BBU$ problem where the circuits are allowed to use $\sqrt n$ gates.

**Proof.** Let $\Pi$ be a basic $\ell_p - BBU$ problem and let $I$ denote an instance of $\Pi$. We may compute a circuit $C_I$ defining a function $F : B^m_0 \rightarrow \mathbb{R}^n$ that is odd on the boundary of $B^m_p$ such that $\text{Sol}(I)$ is the set of zeroes of $F$. Now, define a function $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $h(x) = x/\max(1/2, |x|_p)$. We may now in polynomial time compute $a \{+, -, *, \div, \max, \min, \sqrt n\}$-circuit computing the function $H : B^m_0 \rightarrow \mathbb{R}^n$ given by $H(x) = F(h(x))$.

If $x \in \partial B^m_0$, then $|x|^p_p \geq |x|_\infty = 1$ which shows that $h(x) = x/|x|^p_p$ so $|h(x)|_p = 1$. As $h$ is odd and $F$ is odd on the boundary, it follows that $H(x) = F(h(x)) = F(-h(-x)) = -F(h(-x)) = -H(-x)$ showing that $H$ is odd on the boundary of $B^m_0$, so it defines an instance of an $\ell_\infty - BBU$ problem $\Gamma$. Mapping back solutions amounts to computing $h(x)$ which can be done by a $\{+, -, *, \div, \max, \min, \sqrt n\}$-circuit. The result now follows from part (i) of Proposition 7.

Now we proceed with showing the reductions between basic $\ell_p - BU$ problems.

**Proposition 10.** Suppose that $\Pi$ is an $\exists \mathbb{R}$ search problem whose domains are contained in hypercubes that reduces to a basic $\ell_p - BU$ problem by a $B-$circuit reduction $(f, g)$, where $\{+, -, *, \div, \max, \min\} \subseteq B$. Assume also that for every instance $I$ of $\Pi$, and that $g(I, \cdot)$ is an odd map $\mathbb{R}^{|d(I)|} \rightarrow [-1, 1]^{|d(I)|}$. (i) If $p = \infty$ then $\Pi$ SL-reduces to a basic $\ell_\infty - BU$ problem with circuits over $B$. (ii) If $1 \leq p < \infty$ then $\Pi$ SL-reduces to a basic $\ell_p - BU$ problem with circuits over $B \cup \{\sqrt n\}$.

**Proof.** (i) Let $I$ denote an instance of $\Pi$ and let $m = d_I$. By assumption of $(f, g)$ we may in polynomial time compute a circuit defining a function $F : S^m_0 \rightarrow \mathbb{R}^n$ such that $\text{Sol}(f(I))$ consists of the $x \in S^m_0$ such
that $F(x) = F(-x)$. By the result in Section 4.1 we may assume that the circuit computing $F$ is division-free, and so we may extend the domain of $F$ to be $\mathbb{R}^{n+1}$. Also, we may in polynomial time compute a circuit defining a function $G : \mathbb{R}^{n+1} \rightarrow [-1,1]^m$ mapping $\text{Sol}(f(I))$ to $\text{Sol}(I)$. Define $H : S^{n+m}_o \rightarrow \mathbb{R}^{m+n}$ by $H(x,y) = (F(x),y - \frac{1}{2}G(x))$. We note that $H$ may be computed by a circuit over $B$, so it defines a basic $\ell_\infty$-BU problem $\Lambda$ with $B$-circuits and the same instances as $\Pi$. If $(x,y) \in S^{n+m}_o$ has $H(x,y) = H(-x,-y)$ then $y - \frac{1}{2}G(x) = -y + \frac{1}{2}G(x)$, as $G$ is odd. Therefore $\|y\|_\infty = \|G(x)/2\|_\infty < 1$ and so $\|x\|_\infty = 1$. Also, the first coordinate shows that $F(x) = F(-x)$. This says that $x \in \text{Sol}(f(I))$, and so $G(x) \in \text{Sol}(I)$. As $2y = G(x)$ we see that $\Pi$ SL-reduces to $\Lambda$.

(ii) Again let $I$ denote an instance of $\Pi$ with $m = d_I$. From $f(I)$ we may in polynomial time compute a circuit computing a map $F : S^m_p \rightarrow \mathbb{R}^n$ such that $\text{Sol}(f(I)) = \{x \in S^m_p \mid F(x) = F(-x)\}$ and a $B$-circuit computing a map $G : \mathbb{R}^{n+1} \rightarrow [-1,1]^m$ sending $\text{Sol}(f(I))$ to $\text{Sol}(I)$. Again, we may extend the domain of $F$. Define a map $h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ by $h(x) = x/\max(1/2,\|x\|_p)$ and $H : S^{m+n}_o \rightarrow \mathbb{R}^{m+n}$ by

$$H(x,y) = (F(h(x)),y - \frac{1}{2}G(h(x)))$$

In this way, we have defined a basic $\ell_p$-BU problem $\Lambda$ with $B \cup \{\sqrt{}\}$-circuits and the same instances as $\Pi$. If $(x,y) \in S^{n+m}_o$ has $H(x,y) = H(-x,-y)$ we find that $y = \frac{1}{2}G(h(x))$ so $\|y\|_p \leq 1/2$. This implies that $\|x\|_p \geq 1/2$, and so $h(x) = x/\|x\|_p \in S^m_p$. Also, the first component shows that $F(h(x)) = F(-h(x))$. We use that $h$ is odd. Therefore, $h(x) \in \text{Sol}(f(I))$, and so $G(h(x)) \in \text{Sol}(I)$. As $2y = G(h(x))$, we conclude that $\Pi$ SL-reduces to $\Lambda$.

**Proposition 11.** Let $B = \{+,-,\ast,\div,\max,\min\}$. (i) A basic $\ell_p$-BU problem $\Pi$ with $B$-circuits SL-reduces to a basic $\ell_\infty$-BU problem with $B \cup \{\sqrt{}\}$-circuits. (ii) A basic $\ell_\infty$-BU problem $\Pi$ with $B$-circuits SL-reduces to a basic $\ell_p$-BU problem using $B$-circuits.

**Proof.** (i) Let $\Pi$ denote a basic $\ell_p$-BU problem. Suppose an instance $I$ is defined by some continuous function $F : S^m_p \rightarrow \mathbb{R}^n$. By Section 4.1 we may assume that the circuit computing $F$ is division-free and so extend $F$ to be defined in all of $\mathbb{R}^{n+1}$. Define a function $g : \mathbb{R}^{n+1} \rightarrow [-1,1]^{n+1}$ by $g(x) = x/\max(1/2,\|x\|_p)$ and $H : S^m_o \rightarrow \mathbb{R}^n$ by $H(x) = F(g(x))$. Let $f$ denote the map sending the instance $I$ to the instance $f(I)$ given by $H$ of a basic $\ell_\infty$-BU problem $\Gamma$. One may verify that $(f,g)$ is a reduction satisfying the properties of Proposition 10, so by part (i) of Proposition 10 we have that $\Pi$ SL-reduces to the basic $\ell_\infty$-BU.

(ii) Let $\Pi$ denote a basic $\ell_p$-BU problem. Suppose an instance $I$ is defined by some continuous function $F : S^m_o \rightarrow \mathbb{R}^n$. Again, we may extend $F$. Similarly to the case above, we define a function $g : \mathbb{R}^{n+1} \rightarrow [-1,1]^{n+1}$ by $g(x) = x/\max(1/(n+1),\|x\|_p)$ and $H : S^m_p \rightarrow \mathbb{R}^n$ by $H(x) = F(g(x))$. Let $f$ denote the map sending the instance $I$ to the instance $f(I)$ given by $H$ of a basic $\ell_\infty$-BU problem $\Gamma$.

First, the map $g$ satisfies the condition of Proposition 10. If $x \in \text{Sol}(f(I))$, then it holds that $x \in S^m_p$, and so $1 = \|x\|_p \leq (n+1)\|x\|_\infty$, implying that $\|x\|_\infty \geq 1/(n+1)$. From this it follows that $g(x) = x/\|x\|_\infty$ by definition. Furthermore, using that $g$ is odd we find that $F(g(x)) = H(x) = H(-x) = F(g(-x)) = F(-g(x))$. We conclude that $g(x) \in \text{Sol}(I)$. In conclusion, $(f,g)$ is a reduction from $\Pi$ to $\Gamma$ satisfying the properties of Proposition 10. By part (ii) of Proposition 10 we conclude that $\Pi$ SL-reduces to a basic $\ell_\infty$-BU problem.

**5 Relation between $\ell_p$-BU and $\ell_p$-BBU**

Let $B$ be some finite set of gates containing $\{+,-,\ast,\div,\max,\min\}$. In this section we study reductions between $\ell_p$-BU problems and $\ell_p$-BBU problems. Suppose we are given a basic $\ell_p$-BU problem $\Pi$ with circuits defined over $B$. In order to show that $\Pi$ reduces to a basic $\ell_p$-BBU problem we follow the proof of Theorem 1. Given an instance of $\Pi$ we may in polynomial time compute the dimension $n = d_I$ and a
circuit over $B$ defining a map $F_I: S_p^0 \rightarrow \mathbb{R}^n$ such that $\text{Sol}(I) = \{ x \in S_p^0 \mid F_I(x) = F_I(-x) \}$. Define also the map $\pi: B_p^0 \rightarrow S_p^0$ by

$$
\pi(x) = \begin{cases} 
(x, (1 - ||x||_p^p)^{1/p}) & \text{if } 1 \leq p < \infty \\
(x/t, 2 \cdot (1 - t)) & \text{if } p = \infty
\end{cases}
$$

where $t = \max(1/2, ||x||_\infty)$. Define a map $H: B_p^0 \rightarrow \mathbb{R}^n$ by $H(x) = F_I(\pi(x)) - F_I(-\pi(x)).$ If $x \in S_p^{n-1}$ then the last coordinate of $\pi(x)$ vanishes and so $\pi(x) = -\pi(-x)$ implying that $H(x) = -H(-x)$, so $H$ is odd on the boundary. As $H$ is computable by a $(B \cup \{ \frac{\pi}{\gamma} \})$-circuit if $p < \infty$ (and $B$-circuit if $p = \infty$) this defines an $\ell_p - \text{BBU}$ problem $\Gamma$ with the same instances as $\Pi$. Furthermore, the set of BU-points of $H$ is exactly $\{ x \in B_p^0 \mid F_I(\pi(x)) = F_I(-\pi(x)) \}$, so mapping solutions $x$ of $\Gamma$ to solutions of $\Pi$ amounts to computing $\pi(x)$ which can be done by a circuit over $B \cup \{ \frac{\pi}{\gamma} \}$ if $p < \infty$ (and over $B$ if $p = \infty$).

However, when $p \neq 1$ these reductions make use of -$\gamma$-gates for $p < \infty$ or division gates for $p = \infty$. We can remedy this by applying Propositions 8, 9, and 11 which give that we may go back and forth between different domains for BBU and BU by SL-reductions. Specifically, for any $\ell_p - \text{BU}$ problem we may SL-reduce to a $\ell_1 - \text{BU}$ problem (that also uses $\gamma$-gates if $p < \infty$). Then we may apply the above $\{+, \ast, \zeta\}$-reduction from $\ell_1 - \text{BU}$ to $\ell_1 - \text{BBU}$. And from there we may again SL-reduce to an $\ell_p - \text{BBU}$ problem. In conclusion we obtain the following result.

**Proposition 12.** Any basic $\ell_p - \text{BU}$ problem $\{+, \ast, \zeta\}$-reduces to a basic $\ell_p - \text{BBU}$ problem.

Note that the reductions of this proposition are a special case of PL-reductions. Because these are polynomially continuous, we automatically also get the following result.

**Proposition 13.** Any basic $\ell_p - \text{BU}_a$ problem polynomial time reduces to a basic $\ell_p - \text{BBU}_a$ problem.

For reductions in the other direction, consider an instance $H: B_p^0 \rightarrow \mathbb{R}^n$ of a basic $\ell_p - \text{BBU}$ problem. Given this instance we define an instance of a basic $\ell_\infty - \text{BU}$ problem given by $F: S_\infty^0 \rightarrow \mathbb{R}^n$ where

$$
F(x) = \text{Sel}_2(-H(-\pi(x)), H(\pi(x)), x_{n+1})
$$

and $\pi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the projection $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n).$ Now suppose that $x \in S_\infty^0$ satisfies $F(x) = F(-x).$ If $x_{n+1} = 1$ this implies that

$$
H(\pi(x)) = F(x) = F(-x) = -H(-\pi(-x)) = -H(\pi(x))
$$

showing that $H(\pi(x)) = 0$, so $\pi(x)$ is a solution to the original problem. Similarly, if $x_{n+1} = -1$, then $H(-\pi(x)) = 0$, so $-\pi(x)$ is a solution to the original problem. In the case where $|x_{n+1}| < 1$ we have that $||\pi(x)||_\infty = 1$ and so $H(\pi(x)) = -H(-\pi(x))$, because $H$ is odd on the boundary. By definition of the selection-function Sel this implies that

$$
F(x) = \text{Sel}_2(-H(-\pi(x)), H(\pi(x)), x_{n+1}) = \text{Sel}_2(H(\pi(x)), H(\pi(x)), x_{n+1}) = H(\pi(x))
$$

and similarly $F(-x) = -H(\pi(x)).$ The equality $F(x) = F(-x)$ then implies that $H(\pi(x)) = H(-\pi(x)) = 0$, so both $\pi(x)$ and $-\pi(x)$ is a solution to the original instance in this case. In conclusion, if we could recover the sign of $x_{n+1}$ then we could define a solution map sending $x$ to $\text{sgn}(x_{n+1})\pi(x)$, but we do not allow this. However, in the approximate version, we may do this.

**Proposition 14.** Any basic $\ell_p - \text{BBU}_a$ problem polynomial time reduces to a basic basic $\ell_p - \text{BU}_a$ problem.

---

1We are grateful to Alexandros Hollender for noting that the reduction is possible without introducing approximation error.
Proof. After changing domain we may assume that \( p = \infty \). Given an instance \((H, \varepsilon)\) of a basic \( \ell_\infty - \text{BBU}_a \) problem we apply the above construction and the map \( f \) outputs the instance \((F, \varepsilon')\) of a basic \( \ell_\infty - \text{BBU}_a \) problem where \( \varepsilon' = \min(\varepsilon, 1/2) \). Now suppose that \( x \) is a solution to the problem \((F, \varepsilon')\). This means there exists some \( x^* \) with \(|x - x^*|_\infty \leq \varepsilon'\) and \( F(x^*) = F(-x^*) \). We now claim that we may map back the solution \( x \) of \((F, \varepsilon')\) to a solution of \((H, \varepsilon)\) by the map \( g(x) = \text{sgn}(x_{n+1}) \pi(x) \).

If \(|x_{n+1}| \geq 1/2\) then we have that \( \text{sgn}(x_{n+1}) = \text{sgn}(x_{n+1}^*) \) as \( \varepsilon' \leq 1/2 \). Therefore

\[
||\pi(x) - \pi(x^*)||_\infty = ||\pi(x) - \pi(x^*)||_\infty \leq ||x - x^*||_\infty \leq \varepsilon' \leq \varepsilon
\]

Also \( \text{sgn}(x^*)\pi(x^*) \) is a zero of \( H \) by the discussion above the proposition. In the case where \(|x_{n+1}| < 1/2\) we have that \(|x_{n+1}^*| < 1\) and so both of \( \pm \pi(x^*) \) is a zero of \( H \). As \( \text{sgn}(x) \pi(x) \) is \( \varepsilon \)-close to \( \pm \pi(x^*) \) or \( \pi(x^*) \), we conclude that \( g(x) \) is a solution to the problem \((H, \varepsilon)\).

Combining Proposition 13 and Proposition 14 we obtain the following result.

**Theorem 3.** \( \text{BU}_a = \text{BBU}_a \)

6 Consensus Halving

In this section we present the proof of our main result Theorem 2. This result enables an additional structural result, given in Section 6.5 about the class of strong approximation problems \( \text{BU}_a = \text{BBU}_a \), showing that the class is unchanged even when allowing root operations as basic operations.

Suppose we are given a basic \( \ell_\infty - \text{BBU}_a \) problem \( \Pi_a \) with circuits over the basis \(+, -, \ast, \max, \min\). Let \((I, k)\) denote an instance of \( \Pi_a \) and put \( \varepsilon = 2^{-k} \). We may in polynomial time compute a circuit \( C \) defining a function \( F : B^a_\infty \to \mathbb{R}^a \) that is odd on the boundary \( S^a_\infty \) such that \( \text{Sol}(I) = \{ x \in B^a_\infty \mid F(x) = 0 \} \). We now provide a reduction from \( \Pi_a \) to a CH\(_a\)-problem. In the reduction we will make use of the "almost implies near" paradigm.

**Lemma 1.** Let \( F : B^a_\infty \to \mathbb{R}^a \) be a continuous map. For any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \(|F(x)|_\infty \leq \delta\) then there is an \( x^* \in B^a_\infty \) such that \(|x - x^*|_\infty \leq \varepsilon \) and \( F(x^*) = 0 \).

**Proof.** Let \( F \) and \( \varepsilon > 0 \) be given. Suppose the claim is false. Then for any \( n \in \mathbb{N} \) there is an \( x_n \) such that \(|F(x)|_n \leq 1/n\) and if \( x^* \in B^a_\infty \) has \(|x_n - x^*|_\infty \leq \varepsilon \) then \( F(x^*) \neq 0 \). By compactness the Bolzano-Weierstrass theorem implies the existence of a subsequence \( \{x_{n_i}\} \) converging to some \( x^* \in B^a_\infty \). By continuity of \( F \) and \(|\cdot|_\infty \) we get that \(|F(x^*)|_\infty = \lim_{i \to \infty} |F(x_{n_i})|_\infty = 0 \), showing that \( F(x^*) = 0 \). However, for sufficiently large \( i \in \mathbb{N} \) it holds that \(|x_{n_i} - x^*|_\infty \leq \varepsilon \) contradicting the choice of the \( x_{n_i} \).

This lemma says that for any \( \varepsilon > 0 \), if \(|F(x)|_\infty \) is sufficiently close to being zero, then \( x \) is \( \varepsilon \)-close to a real zero of \( F \). When \( F \) is computed by an algebraic circuit of polynomial size, it follows by the results in Section 2.7 that there exists some fixed polynomial \( q \) with integer coefficients such that the above lemma holds true for some \( \delta \geq (\varepsilon) 2^{2(|I|)} \). The lemma then holds true for \( \delta = (\varepsilon) 2^{2(|I|)} \), and we may construct this number using a circuit of polynomial size by repeatedly squaring the number \( \varepsilon \) exactly \( q(|I|) \) times. This number will be used by the feedback agents in our CH\(_a\) instance in order to ensure that any solution gives a solution to the \( \ell_\infty - \text{BBU}_a \) instance.

6.1 Overview of the Reduction

**Overview.** As in previous works, we describe a consensus halving instance on an interval \( A = [0, M] \), where \( M \) is bounded by a polynomial in \(|I|\), rather than the interval \([0, 1]\). This instance may then be translated to an instance on the interval \([0, 1]\) by simple scaling. Like [FHSZ20], in the leftmost end of the instance
we place the Coordinate-Encoding region consisting of $n$ intervals. In a solution $S$, these intervals will encode a value $x \in [-1, 1]^n$. A circuit simulator $C$ will simulate the circuit of $F$ on this value $x$. The circuit simulators will consist of a number of agents each implementing one gate of the circuit. However, such a circuit simulator may fail in simulating $C$ properly, so we will use a polynomial number of circuit simulators $C_1, \ldots, C_{p(n)}$. Each of these circuit simulators will output $n$ values $[C_j(x)]_1, \ldots, [C_j(x)]_n$ into intervals $I_{1j}, \ldots, I_{nj}$ immediately after the simulation. Finally, we introduce the so-called feedback agents $f_1, \ldots, f_n$. The agent $f_i$ will have some very thin Dirac blocks centered in each of the intervals $I_{ij}$ where $j \in [p(n)]$. These agents will ensure that if $z$ is an exact solution to the CH instance, then the encoded value $x$ satisfies that $||F(x)||_\infty$ is sufficiently small that we may conclude that $x$ is $\varepsilon$-close to a zero $x^*$ of $F$.

**Label Encoding.** For a unit interval $I$ we let $I^\pm$ denote the subsets of $I$ assigned the corresponding label. We define the label encoding of $I$ to be a value in $[-1, 1]$ given by the formula $v_I(I) := \lambda(I^+) - \lambda(I^-)$, where $\lambda$ denotes the Lebesgue measure on the real line $\mathbb{R}$. This makes sense as $I^\pm$ is measurable, because they are the union of a finite number of intervals.

**Coordinate-Encoding Region.** The interval $[0,n]$ is called the Coordinate-Encoding region. For every $i \in [n]$, the subinterval $[i-1,i]$ of the Coordinate-Encoding region encodes a value $v_i := v_I([i-1,i])$ via the label encoding.

**Position Encoding.** For an an interval $I$ which contains only a single cut, thus dividing $I$ into two subintervals $I = I_a \cup I_b$, we define the position encoding of $I$ to be the value $v_p(I) := \lambda(I_1) - \hat{\lambda}(I_2)$. We note that $v_p(I) = v_I(I)$ if the labeling sequence is $-/+$, and $v_p(I) = -v_I(I)$ in the case the labeling sequence is $+/-$.

**From Label to Position.** Before a circuit simulator there is a sign detection interval $I_\varepsilon$ which detects the labeling sequence. Unless it contains a stray cut, this interval will encode a sign $s = \pm 1$ (to be precise $1$ if the label is $+$ and $-1$ is the label is $-$). By placing agents that flip the label as indicated below, we may now obtain position encodings of the values $s x_1, \ldots, s x_n$. These values will be read-in as inputs to the subsequent circuit simulator.

\[\begin{array}{cccccccc}
  & x_1 & x_2 & \cdots & x_n & s & s x_1 & s x_2 & \cdots \\
\end{array}\]

**Circuit Simulators.** As mentioned above, the circuit simulator $C_j$ will read-in the values $s_j x_1, \ldots, s_j x_n$ and simulate the circuit computing $F$ on this input. They then output their values into $n$ intervals immediately after the simulation.

**Feedback Agents.** By the discussion after the proof of Lemma 1 we may by repeated squaring construct a circuit of polynomial size in $|I|$ computing a tiny number $\delta > 0$ such that if $||F(x)||_\infty < \delta$ then $x$ is $(\varepsilon/2)$-close to a zero of $F$. Now fix $i \in [n]$ and let $c_{ij}$ denote the centre of the feedback interval $I_{ij}$ outputs the value $[C_j(s_j \cdot x)]_i$. We then define the $i$th feedback agent to have constant density $1/\delta$ in the intervals $[c_{ij} - \delta/2, c_{ij} + \delta/2]$.

The reason for having the feedback agents have these very narrow Dirac blocks is that if $F_i(x) > \delta$ for some $i$, then in any of the "uncorrupted" circuits (i.e. circuits outputting the correct values) all the density of the $i$th agent will contribute to the same label. Moreover, we will show using the boundary condition of $F$ that the contribution is to the same label in all the uncorrupted circuit simulators. This will contradict that the feedback agents should value $I^+$ and $I^-$ equally. That is the feedback agents ensure that $||F(x)||_\infty \leq \delta$ if $x$ is the value encoded by an exact solution to the consensus halving instance we construct.
Stray Cuts. Any of the agents implementing one of the gates in a circuit simulator will force a cut to be placed in an interval in that same circuit simulator. The only agents whose cuts we have no control over are the \( n \) feedback agents. The expectation is that these agents should make cuts in the Coordinate-Encoding region that flip the label. If they do not do this we will call it a stray cut. If a circuit simulator contains a stray cut, we will say nothing about its value.

Observation 1. If it is not the case that every unit interval encoding a coordinate \( x_i \) in the Coordinate-Encoding region contains a cut that flips the label, then the encoded point \( x \in B^n_\infty \) will lie on the boundary \( S^n_\infty \). With this in mind we may ensure that \( x \in S^n_\infty \) or \( s_1 = s_2 = \cdots = s_{p(n)} = \pm 1 \) where the sign is the same as the label of the first interval. This can be done by, if necessary, placing one single-block agent after the Coordinate-Encoding region and each of the circuit simulators (if placing such an agent is necessary depends on, respectively, the number of variables \( n \) and the size of the circuits).

6.2 Construction of Gates

In this section we describe how to construct Consensus-Halving agents implementing the required gates \( \{+, -, *, \max, \min\} \). First, we show that we may transform the circuit such that all gates only take values in the interval \([-1, 1]\).

Transforming the Circuit. By propagating every gate to the top of the circuit we may assume that the circuit is layered. Let \( C' \) denote the resulting circuit. By repeated squaring we may maintain a gate with value \( 1/2^{2^d} \) in the \( d \)th layer. Suppose \( g = \alpha(g_1, g_2) \) is a gate with inputs \( g_1, g_2 \) in layer \( d \). We modify the gates as follows: if \( \alpha \in \{+, -, \max, \min\} \) then we multiply \( g_i \) by \( 1/2^{2^d} \) before applying \( \alpha \); if \( \alpha = * \), then we multiply the input by 1 before applying \( \alpha \). Finally, we transform \( C' \) into the circuit \( C'' \) as follows: on input \( x \), the circuit \( C'' \) multiplies the input by \( 1/2 \) and then evaluates \( C' \) on input \( x/2 \). Inductively, one may show that if \( g \) is a gate in layer \( d \) in the circuit \( C' \), then the corresponding gate in in the circuit \( C'' \) has value \( g/2^{2^d} \). As all the gates are among \( \{+, -, *, \max, \min\} \), this ensures that all the gates in \( C'' \) take values in \([-1, 1]\).

Addition Gate \([G_+]\). We may construct an addition gate using two agents. The first agent has two unit input intervals that we assume contain one cut each. This then forces a cut in the long output interval that has length 3. The second agent then truncates this value.

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\[\text{\includegraphics[width=0.5\textwidth]{addition_gate}}\]
```

Constant Gate \([G_\zeta]\). Let \( \zeta \in [-1, 1] \cap \mathbb{Q} \) be a rational constant. The agent will have a block of unit height in the sign interval and a block of width \( \zeta/2 \) and height \( 2/\zeta \) centered in another interval.

```
\[\text{\includegraphics[width=0.5\textwidth]{constant_gate}}\]
```

Before proceeding with the remaining gates, we construct a general function gate, an agent that implements any decreasing function.
Function Gate $[G_h]$. Let $-1 \leq a < b \leq 1$ and $-1 \leq c < d \leq 1$ be rational numbers and consider a continuously differentiable map $h: [a, b] \rightarrow [c, d]$ satisfying $h(a) = d$ and $h(c) = c$. Let $h_1$ denote the extension of $h$ that is constant on $[-1, a]$ and $[b, 1]$. We now construct an agent with input interval $I$ and output interval $O$ computing this map, that is the agent should force a cut in the output interval such that $h_1(v_p(I)) = v_p(O)$.

The agent that we construct has a block of height 2 in the output interval and density $f$ is positive in this interval as $h$ is assumed to be a decreasing map, so it makes sense for the agent to have density $f$. One may verify that the agent values the input interval and output interval equally. We further add two rectangles to the output interval colored blue and red in the sketch below. These will ensure that if the cut in the input interval is placed at $z \leq (a + 1)/2$ such that $v_p(I) \leq a$, then the cut in the output interval must be placed at $z^* = (d + 1)/2$, meaning that $v_p(O) = d$. Similarly, if $v_p(I) \geq b$ then $v_p(O) = c$.

Suppose cuts are placed in $z$ in the input interval and in $z^*$ in the output interval. As the agent must value the parts with positive and negative label equally, we get the equality

$$1 = \int_{(a+1)/2}^{z} \frac{-2h'(2t-1)}{d-c} dt + \left( z^* - \frac{c+1}{2} \right)^2 \frac{2}{d-c}$$

From this we obtain that

$$d - c = -\int_{a}^{2z-1} h'(u) du + 2z^* - c - 1$$

$$= -h(2z-1) + d + 2z^* - c - 1$$

where we use that $h(a) = d$ by assumption. We conclude that $h(2z-1) = 2z^* - 1$, that is we obtain the equality $h(v_p(I)) = v_p(O)$.

Using this general function gate, we may now build up the remaining gates required by the circuit.

Multiplication By -1 Gate $[G_{-1}]$. In order to realise this gate, we consider the function $h: [-1, 1] \rightarrow [-1, 1]$ given by $x \mapsto -x$. The agent’s density function in the input interval is then given by $f(z) = 1$.

Subtraction Gate $[G_-]$. We may build this using the gates $G_{-1}$ and $G_+$.
Multiplication by $\zeta \in [-1, 1] [G_{\zeta}]$. If $\zeta < 0$ we may construct $G_{\zeta}$ as a function gate using the function $h: [-1, 1] \to [\zeta, -\zeta]$. If $\zeta > 0$ we construct using $-\zeta$ and a minus gate, i.e. $G_{\zeta} = -G_{(-\zeta)}$.

Maximum Gate $[G_{\text{max}}]$. First we show how to construct a gate computing the absolute value of the input. We may construct gates $G_1, G_2$ such that $G_1(x) = -\max(x, 0)$ and $G_2(x) = \max(-x, 0)$ as function gates by using the functions $h_1 : [0, 1] \to [-1, 0]$ given by $x \mapsto -x$ and $h_2 : [-1, 0] \to [0, 1]$ given by $x \mapsto -x$. Now, we may construct the absolute value gate as $G_{|\cdot|} = -G_1 + G_2$. We may now construct $G_{\text{max}}$ by using the formula $\max(x, y) = (x + y + |x - y|)/2$.

Minimum Gate $[G_{\text{min}}]$. We may build this using $\min(x, y) = x + y - \max(x, y)$.

Multiplication Gate $[G_{\times}]$. We start off by constructing a gate squaring the input. First we construct $G_1$ and $G_2$ as function gates with respect to $h_1 : [-1, 0] \to [0, 1]$ given by $x \mapsto x^2$ and $h_2 : [0, 1] \to [-1, 0]$ given by $x \mapsto -x^2$. Then we may construct the squaring gate as $G_{(\cdot)^2} = G_1 - G_2$. Now we may use the previously constructed gates to make a multiplication gate via the identity $xy = ((x + y)^2 - x^2 - y^2)/2$.

6.3 Describing valuation functions as circuits.

In the description above, we described the valuations of the agents by providing formulas for their densities. However, an instance of CH actually consists of a list of algebraic circuits computing the distribution functions of the agents. In order to construct gates, it is sufficient for agents to have densities that are piece-wise polynomial. Therefore, consider an agent with polynomial densities $f_i$ in the intervals $[a_i, b_i)$ for $i = 1, \ldots, s$, and let $F_i$ denote the indefinite integral of $f_i$. We note that $F_i$ is a polynomial so it may be computed by an algebraic circuit. Now we claim that the distribution function of this agent may be computed by an algebraic circuit via the formula

$$F(x) = \sum_{i=1}^{s} [F_i(\max(a_i, \min(x, b_i))) - F_i(a_i)]$$

This is the case, because the summands will be equal to $F_i(a_i) - F_i(a_i) = 0$ if $x < a_i$, to $F_i(x) - F_i(a)$ if $a_i \leq x \leq b_i$ and to $F_i(b_i) - F_i(a_i)$ if $x > b_i$, meaning that this formula does indeed calculate the valuation of the agent in the interval $[0, x]$.

6.4 Reduction and Correctness

Recall that we are given an instance $(F, \varepsilon)$ of the BBUn problem and that we have to construct an instance of the CHb problem. The reduction now outputs an instance of the CHb problem where the consensus halving instance is constructed as above with $p(n) = 2n + 1$ circuit simulators and the approximation parameter is given by $\varepsilon' = \varepsilon/(4n)$. Let $z$ denote a solution to this CHb instance. By definition, there exists an exact solution $z^*$ to the consensus-halving problem such that $\|z - z^*\|_{\infty} \leq \varepsilon'$. Let $x$ and $x^*$ denote the values encoded by respectively $z$ and $z^*$ in the Coordinate-Encoding region. Suppose, generally, we are given an interval $I$ with a number of cut points $t_1, \ldots, t_s$. Moving a cut point by a distance $\leq \varepsilon'$ we create a new interval $I'$. This changes the label encoding by at most $2\varepsilon'$, that is $|v_I(I) - v_I(I')| \leq 2\varepsilon'$. Successively, if we move all the cuts by a distance $\leq \varepsilon'$, then we get an interval $I''$ such that $|v_I(I) - v_I(I'')| \leq 2s\varepsilon'$. As $\|z - z^*\|_{\infty} \leq \varepsilon'$ and any of the subintervals in the Coordinate-encoding region can contain at most $n$ cuts, we conclude that $\|x - x^*\|_{\infty} \leq 2n\varepsilon' = 2n(\varepsilon/(4n)) = \varepsilon/2$. In order to show that $x$ is $\varepsilon$-close to a zero of $F$, it now suffices by the triangle inequality to show that $x^*$ is $(\varepsilon/2)$-close to a zero of $F$. This will follow from the two following lemmas.
Lemma 2. If there are no stray cuts in the exact solution $z^*$, then the associated value $x^*$ encoded in the Coordinate-encoding region satisfies $F(x^*) = 0$.

Proof. We recall that if the solution $z^*$ contain no stray cuts, then the signs of all the circuit simulators are equal $s_1 = \cdots = s_{2n+1} = s$ where $s = \pm 1$. Furthermore, all the circuit simulators will output the same values $F_1(sx^*), \ldots, F_n(sx^*)$ into the feedback intervals. Thus, there can be no cancellation, so in order for the feedback agents to value the positive and negative part equally it must be the case that $F(sx^*) = 0$. □

Lemma 3. If there is a stray cut in the exact solution $z^*$, then the associated value $x^*$ encoded in the Encoding-region satisfies the inequality $\|F(x^*)\|_\infty \leq \delta$.

Proof. Suppose toward contradiction that $|F(x)_i| > \delta$ for some $i$. Without loss of generality we assume that $F(x)_i > \delta$. As there is a stray cut, the Coordinate-Encoding region can contain at most $n-1$ cuts. Thus, at least one of the coordinates $x^*_i$ must be $\pm 1$ showing that $x^* \in S^{n-1}$. From this and the boundary condition we conclude that $F(x^*) = -F(-x^*)$. Furthermore, there is at most $n$ stray cuts, so at most $n$ circuit simulators can become corrupted. This means that $n + 1$ circuit simulators work correctly. Now suppose that the circuit simulator $C_j$ is uncorrupted. If the label is $s_j = +1$, then $C_j$ will output $F(x)$ into the feedback region and the labeling sequence will be $+/-$; if the label is $s_j = -1$ then $C_j$ will output $F(-x) = -F(x)$ into the feedback region and the labeling sequence will be $-/+$. This is indicated below:

\[
\begin{array}{ccc}
+ & - & + \\
F(x)_i \geq \delta & F(-x)_i = -F(x)_i < -\delta \\
\end{array}
\]

From this we conclude that the $n + 1$ uncorrupted circuit simulators altogether contribute $(n + 1)\delta$ to the part with negative label. However, the $n$ corrupted circuit simulators can contribute at most $n\delta$ to the part with positive label. This implies that $f_j$ cannot value the negative and positive part equally. This contradicts the assumption that $z^*$ is an exact consensus-halving. We conclude that $\|F(x^*)\|_\infty \leq \delta$. □

By the two lemmas above, it follows that the value $x^*$ encoded by the exact consensus-halving $z^*$ satisfies the inequality $\|F(x^*)\|_\infty \leq \delta$. By choice of $\delta$, this implies that there exists some $x^{**}$ such that $\|x^* - x^{**}\|_\infty \leq \epsilon/2$ and $F(x^{**}) = 0$. From the discussion before the two lemmas, it follows that $x$ is $\epsilon$-close to a zero of $F$ and is thus a solution to the BBU$_a$ instance $(F, \epsilon)$.

Mapping back a Solution. What remains is to show that we may recover a solution $x$ to the BBU$_a$ instance from the solution $z$ to the CH$_a$ instance. Recall that in a solution $z = (z_1, \ldots, z_N)$ to the consensus-halving problem $|z_i|$ and $\text{sgn}(z_i)$ represents the length and label of the $i$th interval. For $i \leq n$ and $j \leq n + 1$ we introduce

\[
t_j = \sum_{k=1}^{j-1} |z_k| \\
x_{ij}^+ = \max(0, \min(t_{j-1} + z_j, i) - \max(t_{j-1}, i - 1)) \\
x_{ij}^- = \max(0, \min(t_{j-1} - z_j, i) - \max(t_{j-1}, i - 1))
\]

These numbers may be computed efficiently by a circuit over $\{+, -\}, \max, \min\$. We notice that if $z_j > 0$ then $x_{ij}^- = 0$ and if $z_j < 0$ then $x_{ij}^+ = 0$. Furthermore, by checking a couple of cases, one finds that if $z_j > 0$ (respectively $z_j < 0$) then $x_{ij}^+$ (respectively $x_{ij}^-$) is the length of the $j$th interval that is contained in $[i - 1, i]$. 

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As the coordinate-encoding region can contain at most \( n \) cuts (corresponding to at most \( n + 1 \) intervals), we deduce from the above that the values encoded can be computed as

\[
x_i = \sum_{j=1}^{n+1} x_{ij}^+ - x_{ij}^-
\]

for every \( i \leq n \). If there is a stray cut then both \( x \) and \( -x \) are valid solutions by the boundary condition of \( F \). If there is no stray cut, then \( s_1 = s_2 = \cdots = s_{p(n)} = s = \text{sgn}(z_1) \) by Observation 1 and in this case we may recover a solution as \( sx \).

6.5 Removing Root Gates.

In this subsection, we argue by going through \( CH_a \) that the strong approximation problems \( BU_a = BBU_a \) do not change even if we allow the circuits to use root-operations as basic operations.

**Proposition 15.** The class \( \ell_\infty - BBU_a \) remains unchanged even if we allow the circuits to use root-gates.

**Proof.** Let \( \Pi_a \) be a basic \( \ell_\infty - BBU_a \) problem where the circuits are allowed to use gates from the basis \( \{+, -, *, \text{max}, \text{min}, \sqrt{}\} \). In the previous section, we constructed a polynomial time reduction from \( \Pi_a \) to a \( CH_a \) problem \( \Gamma_a \) in such a way that the circuits computing the distribution functions of the agents are defined over \( \{+, -, *, \text{max}, \text{min}\} \). Namely, the root gates can be implemented by first noting that the power-gate \( (\cdot)^k \) can be implemented by an agent with polynomial densities by using the general function gate construction. Then, in order to construct an agent implementing the root gate we simply interchange the input interval and output interval of the power-gate. By the proof of the result of Deligkas et al. that \( CH \) is contained in \( BU \), the problem \( \Gamma_a \) polynomial time reduces to a \( \ell_1 - BU_a \) problem \( \Lambda \) that only uses gates from \( \{+, -, *, \text{max}, \text{min}\} \). By Proposition 13, \( \Lambda \) reduces to a basic \( \ell_1 - BBU_a \) problem \( \Xi \) which again uses only gates from \( \{+, -, *, \text{max}, \text{min}\} \). Finally, by Proposition 9, \( \Xi \) reduces to a basic \( \ell_\infty - BBU_a \) problem, again using only gates from \( \{+, -, *, \text{max}, \text{min}\} \). Altogether, we see that \( \Pi_a \) polynomial time reduces to a \( \ell_\infty - BBU_a \) without root-gates. \( \Box \)

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