APPLICATIONS OF SIEGEL’S LEMMA TO A SYSTEM OF LINEAR FORMS AND ITS MINIMAL POINTS

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ABSTRACT. Consider a real matrix Θ consisting of rows \((\theta_{i,1}, \ldots, \theta_{i,n})\), for \(1 \leq i \leq m\). The problem of making the system linear forms \(x_1\theta_{i,1} + \cdots + x_n\theta_{i,n} - y_i\) for integers \(x_j, y_i\) small naturally induces an ordinary and a uniform exponent of approximation, denoted by \(w(\Theta)\) and \(\hat{w}(\Theta)\) respectively. For \(m = 1\), a sharp lower bound for the ratio \(w(\Theta)/\hat{w}(\Theta)\) was recently established by Marnat and Moshchevitin. We give a short, new proof of this result upon a hypothesis on the best approximations integer vectors associated to \(\Theta\). Our bound applies to general \(m > 1\), but is probably not optimal in this case. Thereby we also complement a similar conditional result of Moshchevitin, who imposed a different assumption on the best approximations. Our hypothesis is satisfied in particular for \(m = 1, n = 2\) and unconditionally confirms a previous observation of Jarník. We formulate our results in a very general context of approximation of subspaces of Euclidean spaces by lattices. We further establish criteria upon which a given number \(\ell\) of consecutive best approximation vectors are linearly independent. Our method is based on Siegel’s Lemma.

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1. A SYSTEM OF LINEAR FORMS

1.1. Exponents of approximation and minimal points. A standard problem in Diophantine approximation is, for \(mn\) given real numbers \(\theta_{i,j}\), \(1 \leq i \leq m, 1 \leq j \leq n\), to study simultaneously small absolute values of \(m\) linear forms

\[
\theta_{i,1} x_1 + \cdots + \theta_{i,n} x_n + y_i, \quad 1 \leq i \leq m,
\]

with integers \(x_i, y_i\) not all 0. Let \(\Theta \in \mathbb{R}^{m \times n}\) be the corresponding matrix and \(\vec{\theta}_i = (\theta_{i,1}, \ldots, \theta_{i,n})\) for \(1 \leq i \leq m\) its rows. Define the extended matrix \(\Theta^E \in \mathbb{R}^{m \times (m+n)}\) by gluing the \(m \times m\) identity matrix to the right of \(\Theta\), i.e. the \(i\)-th row \(\vec{\theta}^E_i\) of \(\Theta^E\) equals

\[
\vec{\theta}^E_i = (\vec{\theta}_i, e_i) \in \mathbb{R}^{n+m},
\]

for \(e_i\) the \(i\)-th canonical base vector in \(\mathbb{R}^m\). Further writing \(\vec{x} = (x_1, \ldots, x_n)\) and \(\vec{y} = (y_1, \ldots, y_m)\) and \(\vec{z} = (\vec{x}, \vec{y})\) which lies in \(\mathbb{Z}^{n+m}\), our system can be equivalently written as

\[
\Theta^E \cdot \vec{z}.
\]

In this paper we consider Euclidean spaces \(\mathbb{R}^N\) equipped with the maximum norm \(|\xi| = \max_{1 \leq i \leq N} |\xi_i|\) for a vector \(\xi = (\xi_1, \ldots, \xi_N)\) in it, for simplicity we make no notational

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reference of the involved dimension $N$. However, all results remain true when considering any other norm. All vectors below are considered column vectors for convenience.

First we recall the notion of classical exponents of approximation associated to $\Theta$. According to \cite{11}, we denote by $w(\Theta)$ the supremum of $w$ such that
\[
\|\Theta^E z\| < \|z\|^{-w}
\]
holds for certain integer vectors $z$ of arbitrarily large norm $\|z\|$. We further define the uniform exponent $\hat{w}(\Theta)$ as the supremum of real parameters $\hat{w}$ so that the estimate
\[
\|z\| \leq X, \quad \|\Theta^E z\| < X^{-\hat{w}}
\]
has a solution $z \in \mathbb{Z}^{n+m} \setminus \{0\}$ for all large $X$. These exponents satisfy the relations
\[
\infty \geq w(\Theta) \geq \hat{w}(\Theta) \geq \frac{n}{m}
\]
by a pigeon hole principle argument as in Dirichlet’s Theorem (or Minkowski’s Convex Body Theorem). More precisely, for every parameter $X > 1$ there exists some $z \in \mathbb{Z}^{n+m} \setminus \{0\}$ of norm $\|z\| \leq X$ for which the vector (11) has norm $\ll_{m,n} X^{-n/m}$. The notation $A \ll B$ always means that there is a constant $c = c(.)$ depending only on the index variables so that $A \leq cB$. Moreover $A \asymp B$ means $A \ll B \ll A$. We write $\ll, \gg, \asymp$ without index if the implied constants are absolute. We remark that $\hat{w}(\Theta) = 1$ if $m = n = 1$ and $\Theta \notin \mathbb{Q}$, hence $\hat{w}(\Theta) \leq 1$ for $n = 1$ and any $m$ and vectors $\Theta \notin \mathbb{Q}^m$, whereas $\hat{w}(\Theta) = \infty$ occurs for certain matrices with algebraically independent entries as soon as $n > 1$. As customary we call $\Theta$ very well approximable if $w(\Theta) > n/m$, and recall that $\Theta$ is singular in the sense of Diophantine approximation if $\hat{w}(\Theta) > n/m$ (the definition of singularity uses a slightly weaker condition though).

The exponents $w, \hat{w}$ are closely related to best approximations that we discuss now. Assume the columns of $\Theta^E$ are $\mathbb{Q}$-linearly independent, that is $\Theta$ is non-degenerate in the sense of Jarník \cite{13}. Then $\Theta$ induces a sequence of points $z = (x, y)$ in $\mathbb{Z}^{n+m}$ of increasing norms, which we denote by $(\tilde{z}_k)_{k\geq 1}$, with the property that $\|\Theta^E \tilde{z}_k\| > 0$ minimizes $\|\Theta^E z\|$ upon all choices of $z \in \mathbb{Z}^{n+m} \setminus \{0\}$ with $\|z\| < \|\tilde{z}_{k+1}\|$ (note that this sequence depends on the chosen norm). Even for non-degenerate $\Theta$, it may still happen that $\theta_{i,j}$ are $\mathbb{Q}$-linearly dependent together with $\{1\}$ and then the sequence $(\tilde{z}_k)_{k\geq 1}$ may not be unique (up to sign). For simplicity we also want to exclude this case, however remark that some of our results might extend to non-degenerate matrices upon choosing any appropriate sequence in case of ambiguity. If the sequence is well-defined, we say $\Theta$ is “good”, the terminology originates in \cite{22}. So in the sequel we always assume $\Theta$ is good, i.e. induces a uniquely determined (up to sign) sequence $(\tilde{z}_k)_{k\geq 1}$, and shall call these best approximations or minimal points associated to $\Theta$. The sequence of minimal points obviously satisfies
\[
\|\tilde{z}_1\| < \|\tilde{z}_2\| < \cdots, \quad \|\Theta^E \tilde{z}_1\| > \|\Theta^E \tilde{z}_2\| > \cdots .
\]
It is easy to see that we can choose the vectors $z$ realizing the exponents $w(\Theta), \hat{w}(\Theta)$ among the sequence $(\tilde{z}_k)_{k\geq 1}$. Regarding the latter, more precisely for any $\epsilon > 0$ we have
\[
0 < \|\Theta^E \tilde{z}_k\| \leq \|\tilde{z}_{k+1}\|^{-\hat{w}(\Theta)+\epsilon} < \|\tilde{z}_k\|^{-\hat{w}(\Theta)+\epsilon}, \quad k \geq k_0(\epsilon),
\]
complementary to the well-known estimates

\[ 0 < \| \Theta^E z_k \| \ll_{m,n} \| z_{k+1} \|^{-n/m} < \| z_k \|^{-n/m}, \quad k \geq 1, \]

that are slightly stronger than (4) if \( \hat{w}(\Theta) = n/m \).

The sequence of minimal points has already been investigated by Jarník, see for example [12]. Important special case are \( m = 1 \) and \( n = 1 \). For one linear form, i.e. \( m = 1 \), this sequence was studied by Davenport and Schmidt [8], [9] when studying approximation to a real number by algebraic integers. In the same paper they also dealt with the analogous sequence with respect to the dual setting of simultaneous approximation, corresponding to \( n = 1 \) in our notation. Investigation of the latter was emphasized with contributions by several authors, including a series of papers by Lagarias starting from [15] in 1979 and later Moshchevitin. We also refer to the more recent paper by Chevallier [7] for an introduction to the simultaneous approximation setting, including a wealth of references. The general case of arbitrary \( m, n \) has been studied for example in [17], [22] and several other papers by Moshchevitin that we will recall later.

1.2. Outline of the paper. One purpose of this paper is to study lower bounds for the quotient \( w(\Theta)/\hat{w}(\Theta) \). This topic already goes back to Jarník [12,13] and has attracted interest lately, it gave rise to a series of papers within the last decade including [23], [30], [31], [10], [18], [25], particularly on the cases \( m = 1 \) or \( n = 1 \). We survey known results in Section 2, in particular Marnat and Moshchevitin [18] and Moshchevitin [24]. Thereby we encompass results on linear independence of minimal points. In Section 3 we will establish a new complementary conditional bound and compare it with [18], [24]. In certain cases our result is unconditional and implies observations of Jarník [12], that in turn is the special case \( n = 2 \) of Theorem 2.1 below that originates in [18].

The second purpose is to use a very similar method to derive conditions under which a given number \( \ell \) of consecutive minimal points are linearly independent. For \( \ell = n + m \), this question results in studying the regularity of the quadratic matrices whose columns are these best approximations, a classical topic. In Section 4 we want to treat the case \( \ell < n + m \), with emphasis on \( \ell = 3 \).

2. Known lower bounds for \( w(\Theta)/\hat{w}(\Theta) \)

2.1. Cases \( m = 1 \) or \( n = 1 \). In these cases \( m = 1 \) or \( n = 1 \), the minimum ratio \( w(\Theta)/\hat{w}(\Theta) \) was established by Marnat and Moshchevitin [18] (see also Rivard-Cooke’s PhD-thesis [25] for a different proof). We only state their result for \( m = 1 \).

**Theorem 2.1** (Marnat, Moshchevitin). Let \( n \geq 2 \). If \( m = 1 \), for any good \( \Theta \in \mathbb{R}^{m \times n} \) we have

\[ \frac{w(\Theta)}{\hat{w}(\Theta)} \geq G_{1,n}, \]

where \( G_{1,n} = G_{1,n}(\hat{w}(\Theta)) \) is the unique positive real root of \( P_{1,n}(x) = 1 - \hat{w}(\Theta) + \sum_{j=1}^{n-1} x^j \). Equality is attained for certain \( \Theta \), thus the bound is optimal.
2.2. Linear independence of minimal points. In this section, we prepare some notation and further survey some more facts on minimal points. The topic of linear independence of subsets of best approximations \(z_j\) is related to the exponents \(w(\Theta), \hat{w}(\Theta)\) and has been investigated in Diophantine approximation. It is easy to see that any two consecutive best approximations \(z_k, z_{k+1}\) are linearly independent, a short argument in fact shows that any such pair spans (as a \(\mathbb{Z}\)-module) the lattice obtained from intersecting their real span real with \(\mathbb{Z}^{n+m}\), see [7, Lemma 4] (there the case of simultaneous approximation \(n = 1\) is treated, but for any system of linear forms an analogous argument applies). On the other hand, it may happen that all large best approximations lie in a fixed space of dimension only 2, see Theorem 2.2 below. The next definition deals in more detail with the dimension of the sublattice of \(\mathbb{Z}^{n+m}\) spanned by minimal points. Recall the notion of a good matrix from Section 1.1.

**Definition 1.** Let \(n \geq 1, m \geq 1\) integers and consider good matrices \(\Theta \in \mathbb{R}^{m \times n}\) as above.

Let

\[ R(\Theta) = \min \{ h : \exists k_0 \text{ such that the vectors } (z_k)_{k \geq k_0} \text{ span a space of dimension } h \}. \]

We denote this real subspace by \(\mathcal{S}_\Theta \subseteq \mathbb{R}^{n+m}\). For \(1 \leq h \leq m+n\), define

\[ \mathcal{G}_h = \mathcal{G}_h^{m,n} = \{ \Theta : R(\Theta) = h \}, \quad \mathcal{H}_h = \mathcal{H}_h^{m,n} = \{ \Theta : R(\Theta) \leq h \}. \]

The notation \(R(\Theta)\) was introduced in [22]. Obviously \(\mathcal{G}_h\) are disjoint in \(h\) and

\[ \emptyset = \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots \subseteq \mathcal{H}_{n+m} = \mathbb{R}^{m \times n}, \quad m, n \geq 1. \]

Clearly, the generic case is \(\Theta \in \mathcal{G}_{m+n}\). As observed above, \(R(\Theta) \geq 2\) for any \(\Theta\) and equality occurs in some cases. Combining claims from [22, Section 2.2] we get the following almost complete classification on \(R(\Theta) = 2\).

**Theorem 2.2** (Moshchevitin). If \(2 \leq n < m\), the set of good matrices in \(\mathcal{G}_2\) as a subset of \(\mathbb{R}^{m\times n}\) has Hausdorff dimension at least \(m(n-1)\), in particular is not empty. If \(n > m\) then \(\mathcal{G}_2 = \emptyset\). If the entries of \(\Theta\) are algebraically independent, then \(\Theta \notin \mathcal{G}_2\).

See also [21, Section 1.3] for a sketch of the proof when \(m = 1\). The same is true if the elements of \(\Theta\) are algebraically independent [22, Corollary 2]. There are also partial results on the open case \(m = n\) in [22]. The next theorem comprises two more results by Moshchevitin [22, Theorem 14]; the first deals with the case of best approximations ultimately lying in a 3-dimensional sublattice of \(\mathbb{Z}^{n+m}\) and is similarly surprising.

**Theorem 2.3** (Moshchevitin). For every \(m \geq 1, n \geq 3\), there exist uncountably many good \(\Theta \in \mathcal{G}_3\). On the other hand, for \(n = 1\) and \(m \geq 1\), the set \(\mathcal{H}_{m+n-1} = \mathcal{H}_m\) is empty.

Even in the case \(m = 1\) also available in [20, 21], where we can obviously take \(n = 2\) as well, the first result is very surprising. Reverse to the latter claim of the theorem, for \(n = 1\) the corresponding determinants formed by \(m+1\) consecutive minimal points may be 0 for \(k \geq k_0\), see again Moshchevitin [21].
2.3. A conditional bound by Moshchevitin. For \( \min\{m, n\} > 1 \), the optimal bound for the ratio \( w(\Theta)/\hat{w}(\Theta) \) is unknown. Some unconditional estimates, as well as counterexamples for reasonably sharper estimates, due to Jarník are recalled in [24, Section 3.1]. Here we just explicitly want to state

\[
(5) \quad w(\Theta) \geq \frac{\hat{w}(\Theta)}{n(\hat{w}(\Theta))^n} - \frac{3\hat{w}(\Theta)}{n-1} - \frac{(5n^2)n}{\hat{w}(\Theta)}, \quad \text{if } n \geq 3 \text{ and } \hat{w}(\Theta) \geq \frac{(5n^2)n}{\hat{w}(\Theta)}. 
\]

See also Moshchevitin [23] when \( m \geq 3, n = 2 \) and \( \hat{w}(\Theta) \geq 1 \).

Recall \( \mathcal{G}_h \) and \( \mathcal{S}_\Theta \) from Definition 1. To formulate a result indicated in [24], we consider the following subsets of matrices \( \Theta \in \mathcal{G}_h \) obtained from some (rather strong) linear independence property.

**Definition 2.** Let \( 1 \leq h \leq m + n \) be an integer. Let \( \mathbf{L}_h \subseteq \mathcal{G}_h \) be the set of matrices within \( \mathcal{G}_h \) with the property that for infinitely many integers \( t \geq 1 \), the \( h \) consecutive best approximations \( z_t, z_{t+1}, \ldots, z_{t+h-1} \) are linearly independent (thus span \( \mathcal{S}_\Theta \)). If the property holds for all large \( t \), we call the corresponding smaller set \( \mathbf{S}_h \subseteq \mathcal{G}_h \).

We will omit the dependence on \( m, n \) in the notation. We want to point out that results quoted in Section 2.2 imply for small \( h \) the identities

\[
\mathcal{G}_2 = \mathbf{L}_2 = \mathbf{S}_2, \quad \mathcal{G}_3 = \mathbf{L}_3 = \mathbf{S}_3.
\]

However \( \mathbf{S}_3 \subseteq \mathbf{S}_3 \). Under the assumption \( \Theta \in \mathbf{L}_h \), as pointed out to the author by the referee, there is a short argument giving a strong bound for the ratio \( w(\Theta)/\hat{w}(\Theta) \). For \( a, b \) positive integers and \( \Theta \in \mathbb{R}^{m \times n} \), denote by \( G_{a,b} \) the positive root of

\[
P_{a,b}(x) = -\sum_{j=1}^{a-1} \frac{\hat{w}(\Theta)}{x^j} + 1 - \hat{w}(\Theta) + \sum_{j=1}^{b-1} x^j = 0.
\]

When \( m = 1 \), the definition agrees with \( G_{1,n} \) defined in Theorem 2.1. For \( \Theta \in \mathbf{L}_h \) recall the \( h \)-dimensional subspace \( \mathcal{S}_\Theta \subseteq \mathbb{R}^{m+n} \) from Definition 1 and associate to \( \Theta \) another subspace in \( \mathbb{R}^{m+n} \) given as

\[
\mathcal{L}_\Theta = \{(\hat{z}, \Theta \hat{z}) : \hat{z} \in \mathbb{R}^n\} \subseteq \mathbb{R}^{m+n},
\]

and derive

\[
n' = \dim(\mathcal{S}_\Theta \cap \mathcal{L}_\Theta), \quad m' = h - n'.
\]

Notice that in the generic situation \( h = m + n \) we just have \( \mathcal{S}_\Theta = \mathbb{R}^{m+n} \), and \( m = m', n = n' \). With this notation, the following bounds for \( \Theta \in \mathbf{L}_h \) hold.

**Theorem 2.4 (Moshchevitin (essentially)).** Let \( 1 \leq h \leq m + n \). For \( \Theta \in \mathbf{L}_h \) the estimate

\[
(6) \quad \frac{w(\Theta)}{\hat{w}(\Theta)} \geq G_{m',n'}
\]

holds. In particular, in the generic case \( h = m + n \), we have

\[
(7) \quad \frac{w(\Theta)}{\hat{w}(\Theta)} \geq G_{m,n}.
\]
As pointed out, claim (7) is already stated, but without proof, in [24, Section 3.5]. If \( m = 1 \) this simplifies to the sharp (unconditional) bound in Theorem 2.1. The bound is likely to be optimal, possibly even without assumption \( \Theta \in L_h \), for general \( m, n \). We provide a proof of Theorem 2.4 reported to the author by the referee in the Appendix in Section 6.

3. New conditional lower bounds for \( w(\Theta)/\hat{w}(\Theta) \)

3.1. General \( h \). From \((z_k)_{k \geq 1}\) the sequence of best approximation associated to \( \Theta \in \mathbb{R}^{m \times n} \), derive

\[
\tau = \limsup_{k \to \infty} \frac{\log \|z_{k+1}\|}{\log \|z_k\|} \geq 1, \quad \tilde{\tau} = \liminf_{k \to \infty} \frac{\log \|z_{k+1}\|}{\log \|z_k\|} \geq 1.
\]

We introduce simplifying (strong) short vector hypotheses on \( \Theta \) that are slightly weaker than \( L_h \) resp. \( SL_h \), for given \( h \). Assume \( m, n \) are fixed in the sequel.

Definition 3. Let \( 1 \leq h \leq m+n \) be an integer. Let \( V_h = V_h^{m,n} \) be the set of good matrices \( \Theta \in \mathbb{R}^{m \times n} \) with the property that for infinitely many integers \( t \geq 1 \) the lattice

\[
\langle z_t, \ldots, z_{t+h-1} \rangle \cap \mathbb{Z}^{m+n}
\]

obtained by intersecting the real span of the \( h \) consecutive best approximations with the integer lattice, contains a short integer vector \( v = v_t \in \mathbb{Z}^{m+n} \) of norm \( \|v_t\| \ll \|z_t\|^{o(1)} \) as \( t \to \infty \). If we assume the property for all large \( t \), we denote the induced smaller set by \( SV_h = SV_h^{m,n} \).

We stress that we do not need to restrict \( \Theta \) to \( H_h \) here. For readability we will again omit upper case indices \( m, n \) in \( V_h, SV_h \) below. The conditions \( \Theta \in V_h \) or \( \Theta \in SV_h \) become less stringent the larger \( h \) is. A generic matrix \( \Theta \in \mathbb{R}^{m \times n} \) lies in \( SV_h \), but in no \( SV_b \) with \( b < m+n \). It is not hard to see that our conditions are indeed, at least formally, less restrictive than those in Definition 2.

Proposition 1. For every \( 1 \leq h \leq m+n \), we have

\[
L_h \subseteq V_h, \quad SL_h \subseteq SV_h.
\]

Proof. We check the first inclusion only. Let \( \Theta \in L_h \). For \( k_0 \) as in Definition 1 and any \( t \) that satisfies the hypothesis in Definition 2, take the constant vector \( v_t = z_{k_0} \). Since by assumption \( \Theta \in G_h \), this vector \( v_t \) lies in the space \( J_\Theta \) spanned by \( z_t, z_{t+1}, \ldots, z_{t+h-1} \), moreover it has absolutely bounded norm \( \|v_t\| = \|z_{k_0}\| = O(1) = \|z_{k_0}\|^{o(1)} \). Thus indeed \( \Theta \in V_h \).

We believe that the difference sets \( V_h \setminus L_h \) and \( SV_h \setminus SL_h \) are non-empty, however we do not have examples at hand. Our main result of this section below admits a short proof with Siegel’s Lemma. We interpret \( \infty / \infty = \infty \).
Theorem 3.1. Let \( h \geq 2 \) and \( m \geq 1, n \geq 1 \) integers. Assume \( \Theta \in \mathbb{R}^{m \times n} \) lies in \( \mathbf{V}_h \).

Then
\[
\hat{w}(\Theta) \leq 1 + \frac{w(\Theta)}{\hat{w}(\Theta)} + \left( \frac{w(\Theta)}{\hat{w}(\Theta)} \right)^2 + \cdots + \left( \frac{w(\Theta)}{\hat{w}(\Theta)} \right)^{h-2}.
\]

More precisely we have
\[
\hat{w}(\Theta) \leq 1 + \frac{1}{\tau} + \frac{1}{\tau^2} + \cdots + \frac{1}{\tau^{h-2}} = \frac{\tau^{h-1} - 1}{\tau - 1}.
\]

If \( \Theta \in \mathbf{S} \mathbf{V}_h \), then
\[
\hat{w}(\Theta) \leq 1 + \frac{1}{\tau}(1 + \frac{1}{\tau} + \frac{1}{\tau^2} + \cdots + \frac{1}{\tau^{h-3}}) = 1 + \frac{\tau^{h-2} - 1}{\tau - 1}.
\]

In particular, if \( \Theta \in \mathbf{S} \mathbf{V}_h \) and there is equality in (9), then
\[
\lim_{k \to \infty} \frac{\log \| z_{k+1} \|}{\log \| z_k \|} = \frac{w(\Theta)}{\hat{w}(\Theta)}.
\]

Remark 1. When we relax the condition in Definition 3 to \( \| v_t \| \ll \| z_t \|^{\Delta + o(1)} \) for \( \Delta \geq 0 \) a parameter, the argument of the proof with minor adaptations implies the bound
\[
\hat{w}(\Theta) \leq 1 + \frac{w(\Theta)}{\hat{w}(\Theta)} + \left( \frac{w(\Theta)}{\hat{w}(\Theta)} \right)^2 + \cdots + \left( \frac{w(\Theta)}{\hat{w}(\Theta)} \right)^{h-2} + \frac{2\Delta w(\Theta)}{\tau} \leq 1 + \frac{w(\Theta)}{\hat{w}(\Theta)} + \left( \frac{w(\Theta)}{\hat{w}(\Theta)} \right)^2 + \cdots + \left( \frac{w(\Theta)}{\hat{w}(\Theta)} \right)^{h-2} + 2\Delta w(\Theta)
\]
in terms of \( \Delta \), essentially since then \( \varepsilon \) from the proof can be replaced by \( \Delta w(\Theta) \). For \( \Delta = 0 \) it naturally coincides with (9), for \( \Delta \geq 1/2 \) it becomes always trivial.

We see from \( \tau \leq w(\Theta)/\hat{w}(\Theta) \), the claim of Lemma 1 below, that (10) directly implies (9). We compare the claim with Theorem 2.1. For one linear form \( m = 1 \) and \( h = m + n = n + 1 \), we recognize (9) again as equivalent to (7), thus it again yields the sharp bound in Theorem 2.1. If \( m > 1 \), our estimate is weaker than (7), however upon a more moderate assumption. We should remark that refinements in the spirit of (10), (11) can be obtained in Theorem 2.4 as well. At this point we also want to refer again to Moshchevitin [23] for a stronger bound than (9) when \( m \geq 3, n = 2 \) and \( \hat{w}(\Theta) \geq 1 \).

A natural problem on the gap between Definition 3 and Definition 2 arises.

Problem 1. Do we have the stronger estimate (6) for any \( \Theta \in \mathbf{V}_h \)?

A consequence of Theorem 3.3 is that \( \mathbf{V}_h \) is small if \( n/m \) is large compared to \( h \).

Corollary 1. Assume \( m, n \) satisfy
\[
n > (h - 1)m.
\]

Then any \( \Theta \in \mathbf{V}_h \) is very well approximable, i.e. \( w(\Theta) > n/m \). Hence, upon (13), the set \( \mathbf{V}_h \subseteq \mathbb{R}^{mn} \) has \( mn \)-dimensional Hausdorff measure 0 (in fact Hausdorff dimension smaller than \( mn \)), and contains no matrix with only algebraic entries.
Proof. We readily check that (2), (9) and (13) implies \( w(\Theta) > n/m \). The metric implication is then a well-known generalization of a result of Jarník [11], see Beresnevich and Velani [2] for reasonably stronger versions. The claim for \( \mathbb{Q} \)-linearly independent algebraic matrices follows as they satisfy \( w(\Theta) = n/m \) by a direct consequence of Schmidt’s Subspace Theorem (see [4, Theorem 2.8, 2.9]). □

We wonder if we can relax the assumption to \( h < m + n \). We also include a speculation on the uniform exponent motivated by the construction in [21].

**Problem 2.** Let \( m \geq 1, n \geq 1 \) and \( h < m + n \). Are all \( \Theta \in \mathcal{V}_h \) (if any exist) very well approximable, i.e. \( w(\Theta) > n/m \)? Does the set \( \mathcal{V}_h \) have \( mn \)-dimensional Lebesgue-measure 0 (Hausdorff dimension smaller than \( mn \)) and not contain algebraic matrices? Is the stronger conclusion \( \hat{w}(\Theta) > n/m \) true for any \( \Theta \in \mathcal{V}_h \) (at least for large \( n/m \))?

For \( m = 1 \) and \( \Theta \in \mathcal{H}_h \) in place of \( \Theta \in \mathcal{V}_h \), a positive answer concerning the ordinary exponent can be inferred from Theorem [21] with a similar deduction as Corollary [1] from Theorem [3,1] (see also the appendix), as pointed out to the author in private correspondence by N. Moshchevitin. Moreover, for any \( m, n \) it is true for \( \Theta \in \mathcal{L}_h \) as well by (6), see also the last paragraph of [12, Section 8]. On the other hand, for \( m > 1 \) and general matrices in \( \mathcal{H}_h \) the problem seems open, as for \( \Theta \in \mathcal{V}_h \) in Problem 2.

We finally remark that similar, unconditional, quantitative claims as (10), (11), (12), relating best approximations with classical exponents, were recently established for simultaneous approximation (i.e. \( n = 1 \)) by Nguyen, Poels and Roy [25].

### 3.2. Special case \( \Theta \in \mathcal{V}_2 \).

From Theorem 3.1 we get

**Theorem 3.2.** Let \( \Theta \in \mathcal{V}_2 \). Then

\[
\hat{w}(\Theta) \leq 1.
\]

By (3) in particular \( n \leq m \).

If we restrict to \( \Theta \in \mathcal{G}_2 = \mathcal{L}_2 \), we can obtain a slightly stronger estimate already observed by Moshchevitin [22, Theorem 8], with a new proof. If \((\bar{z}_k)_{k \geq 1}\) is the sequence of best approximations associated to \( \Theta \in \mathcal{G}_2 \), we have

\[
\|\Theta^E\bar{z}_t\| > c: \|\bar{z}_{t+1}\|^{-1}, \quad t \geq 1,
\]

for some \( c = c(\Theta) > 0 \). That is a partial claim of Theorem 3.1 above. The stronger version follows from our proof of Theorem 3.1 below, upon using the stronger assumption \( \|z\| = O(1) \) compared to Definition 3 valid for \( \Theta \in \mathcal{G}_2 = \mathcal{L}_2 \), see the proof of Proposition 1.

### 3.3. Special case \( \Theta \in \mathcal{V}_3 \).

We derive a new proof a result of Jarník.

**Theorem 3.3.** Let \( m \geq 1, n \geq 1 \) integers and assume \( \Theta \in \mathcal{V}_3 \). Then

\[
w(\Theta) \geq \hat{w}(\Theta)^2 - \hat{w}(\Theta).
\]

In fact we have

\[
\hat{w}(\Theta) \leq \tau + 1.
\]
If \( \Theta \in SV_3 \), then
\[
\hat{w}(\Theta) \leq 2 + 1,
\]
in particular then equality in (16) implies
\[
\lim_{k \to \infty} \frac{\log \|z_{k+1}\|}{\log \|z_k\|} = \hat{w}(\Theta).
\]

If \( m = 1, n = 2 \), formula (16) is unconditional and already occurs in Jarník [12, Theorem 2] with a different proof. Jarník’s result is the special case \( n = 2 \) in the linear form result of Theorem 2.1. As thankfully pointed out to the author by N. Moshchevitin, Jarník’s proof can be extended to the more general situation \( \Theta \in G_3 = L_3 \) for any \( m, n \), which however for general \( m, n \) is still slightly weaker than Theorem 3.3 where we assume \( \Theta \in V_3 \), at least formally.

We briefly discuss the consequence of Corollary 1 for \( h = 3 \). If \( m = 1 \), any vector \( \Theta \in V_3 \) for \( n \geq 3 \) is very well approximable. For \( \Theta \in G_3 = L_3 \subseteq V_3 \) this follows from Jarník [12] already. On the other hand, any matrix in \( V_2 \) induces the upper bound \( \hat{w}(\Theta) \leq 1 \) independent of \( m, n \), see Theorem 3.2 above. This clearly does not exclude that a matrix in \( V_2 \) is very well approximable.

For sake of completeness, we state a related result on simultaneous approximation \( n = 1 \) where our hypothesis (13) fails. Lagarias [16, Theorem 5.2] showed that for \( m = 2 \) and a badly approximable vector \( \Theta = (\theta_1, \theta_2)^t \in \mathbb{R}^2 \) there is an absolute upper bound on the number of consecutive triples of linearly dependent minimal points \( z_k, z_{k+1}, z_{k+2} \). In the same paper he shows that the claim is not true if the restriction to badly approximable vectors is dropped, see also [19] for a generalization.

4. Criteria for linear independence of consecutive minimal points

Let \( \ell \geq 3 \) be a given integer. We study under which assumptions on a good matrix \( \Theta \in \mathbb{R}^{m \times n} \) we can deduce that \( \ell \) consecutive minimal points \( z_k, \ldots, z_{k+\ell-1} \) are linearly independent, for all large \( k \) or certain arbitrarily large \( k \). Our assumptions will involve bounds for the logarithmic quotients of consecutive linear form evaluations and norms of best approximations, more precisely we employ the quantities
\[
\sigma_k := \frac{\log \|\Theta^E z_{k+1}\|}{\log \|\Theta^E z_k\|}, \quad \tau_k := \frac{\log \|z_{k+1}\|}{\log \|z_k\|}, \quad \nu_k := -\frac{\log \|\Theta^E z_k\|}{\log \|z_k\|}.
\]
For \( m = 1 \) or \( n = 1 \), similar quantities regarding quotients consecutive minimal point norms on one hand and consecutive approximation qualities on the other hand, but without taking logarithms, have recently been studied by Akhunzhanov and Moshchevitin [1]. Some consequences of their work are briefly sketched in Example 1 below.

We have \( \sigma_k > 1, \tau_k > 1 \) by (3), and \( \nu_k > \hat{w}(\Theta) - o(1) \geq n/m - o(1) \) as \( k \to \infty \) by (4). Moreover, the upper limit of \( \nu_k \) as \( k \to \infty \) coincides with \( w(\Theta) \). Furthermore
\[
\sigma_k = \frac{\tau_k \nu_{k+1}}{\nu_k}, \quad \tau_k \nu_{k+1} > \nu_k.
\]
where the right claim follows from (3). We give more details on these quantities below Corollary 2. We will further use the derived values

\[ \underline{\sigma} := \liminf_{k \to \infty} \sigma_k, \quad \overline{\sigma} := \limsup_{k \to \infty} \sigma_k, \]

that complement (8) and are again bounded from below by 1 and may attain the formal value +∞. Let

\[ \Gamma(\Theta) = 1 + \frac{\log(\hat{w}(\Theta)(\tau - 1)(\underline{\sigma} - 1) + 1)}{\log \tau}, \]

and

\[ \tilde{\Gamma}(\Theta) = 1 + \frac{\log ((\tau - 1)\left(\hat{w}(\Theta)(\underline{\sigma} - 1) + 1 - \frac{1}{\tau}\right) + 1)}{\log \tau}, \]

where here and below we take the right limit if \( \tau = 1. \) Since \( \tau \geq 1, \) for any \( \Theta \) we have

\[ \Gamma(\Theta) \leq \tilde{\Gamma}(\Theta). \]

Our first result is the following

**Theorem 4.1.** Let \( m, n \geq 1 \) and \( \Theta \in \mathbb{R}^{m \times n} \) a good matrix with associated minimal point sequence \((z_k)_{k \geq 1}. \) If we let \( \epsilon > 0 \) and \( k \geq k_0(\epsilon) \), then the assumption

(21) \quad \hat{w}(\Theta) > \frac{\tau_{k+1} + \tau_{k}^{-1}}{\sigma_k - 1} + \epsilon

implies that \( z_k, z_{k+1}, z_{k+2} \) are linearly independent. Now assume that

\[ \sigma > 1, \quad \tau < \infty. \]

If the integer \( \ell \geq 1 \) satisfies

(22) \quad \ell < \tilde{\Gamma}(\Theta),

which is in particular true if \( \ell < \Gamma(\Theta), \) then for all large indices \( k \) the vectors

(23) \quad \tilde{z}_k, \tilde{z}_{k+1}, \ldots, \tilde{z}_{k+\ell-1}

are linearly independent. If we assume that either of the slightly weaker conditions

(24) \quad \ell < 1 + \min \left\{ \tilde{\Gamma}(\Theta), \frac{\log ((\tau - 1)\left(\hat{w}(\Theta)(\underline{\sigma} - 1) + 1 - \frac{1}{\tau}\right) + 1)}{\log \tau} \right\},

or

(25) \quad \ell < 1 + \min \left\{ \tilde{\Gamma}(\Theta), \frac{\log ((\tau - 1)\left(\hat{w}(\Theta)(\underline{\sigma} - 1) + 1 - \tau^{-1}\right) + 1)}{\log \tau} \right\}

holds, then (23) are linearly independent for infinitely many \( k. \)

**Remark 2.** In view of (2), we can relax the conditions in all (21)-(25) by replacing \( \hat{w}(\Theta) \) by \( n/m. \)
Notice that the right bounds in the minima in (24) and (25) are obtained by replacing \( \sigma \) by \( \overline{\sigma} \) and \( \tau \) by \( \overline{\tau} \) respectively in \( \tilde{\Gamma}(\Theta) \), and subtracting 1. This indeed relaxes (22) in both cases. A weakened version of the right bound in the minimum in (24) with simpler bound expression is obtained via replacing \( \tau \) by 1, likewise as \( \Gamma(\Theta) \) arises from \( \tilde{\Gamma}(\Theta) \).

We deduce a corollary.

**Corollary 2.** For any good matrix \( \Theta \in \mathbb{R}^{m \times n} \) we have

\[
\hat{w}(\Theta) \leq \frac{\overline{\tau}^{n+m} - 1 - (\overline{\tau} - 1)(1 - \frac{1}{\overline{\sigma}})}{(\overline{\tau} - 1)(\overline{\sigma} - 1)}.
\]

In particular

\[
\frac{n}{m} \leq \frac{\overline{\tau}^{n+m} - 1}{(\overline{\tau} - 1)(\overline{\sigma} - 1)},
\]

and \( \hat{w}(\Theta) = \infty \) implies that either \( \overline{\sigma} = 1 \) or \( \overline{\tau} = \infty \).

The last assertion applies in particular to the very singular vectors belonging to \( V_3 \) constructed by Moshchevitin [21] for \( m = 1 \). Probably the latter claim \( \overline{\tau} = \infty \) is true. Jarník’s [11] estimate (5) implies the ratio \( w(\Theta)/\hat{w}(\Theta) \) tends to infinity with \( \hat{w}(\Theta) \), however for \( \overline{\tau} \) this seems not quite clear. Lemma 1 below contains reverse estimates.

**Proof.** For \( \ell = m + n + 1 \), the vectors (23) are clearly linearly dependent, hence the estimate (22) must be false. This is equivalent to the first claim. The weaker second claim then follows from (2) and \( \tau \geq 1 \). \(\square\)

Roughly speaking, Theorem 4.1 and Corollary 2 tell us that if the approximation qualities induced by any two consecutive best approximations differ significantly, then the norms of certain two consecutive best approximations must also increase at some minimum rate. This relation gets even stronger if \( \hat{w}(\Theta) \) exceeds \( n/m \) significantly.

To give some flavor of the strength of the bounds, if \( \overline{\sigma} > 1, \overline{\tau} > 1 \) and the ratio \( n/m =: c \) are all fixed, then by (2) we satisfy (22) for an \( \ell \geq \log c + d - o(1) \) for some \( d \), independent of \( n \). For \( c, d \) not too small this may be of interest. If for all large (resp. infinitely many) \( k \) we can improve the trivial lower bound \( \overline{\sigma} \) and/or upper bound \( \overline{\tau} \) for \( \ell - 1 \) consecutive values \( \sigma_k, \ldots, \sigma_{k+\ell-2} \) and/or \( \tau_k, \ldots, \tau_{k+\ell-2} \), the conditions (22) (resp. (24) or (25)) of Theorem 4.1 can be relaxed. See also Theorem 4.2 below. A sharp upper estimate for both in terms of exponents of approximation is provided in the following lemma.

**Lemma 1.** Let \( \Theta \in \mathbb{R}^{m \times n} \) a good matrix and assume \( \hat{w}(\Theta) < \infty \). Then we have

\[
1 \leq \max\{\overline{\sigma}, \overline{\tau}\} \leq \max\{\overline{\sigma}, \overline{\tau}\} \leq \frac{w(\Theta)}{\hat{w}(\Theta)} \leq \frac{m}{n} \cdot w(\Theta).
\]

Unfortunately we require a non-trivial lower estimate for \( \overline{\sigma} \) in our applications. A generic \( \Theta \) satisfies \( w(\Theta) = \hat{w}(\Theta) = n/m \) and hence induces \( \overline{\sigma} = \overline{\tau} = \overline{\tau} = \overline{\tau} = 1 \). On the other hand, for a "typical" \( \Theta \) satisfying \( w(\Theta) > n/m \), we expect \( \overline{\sigma} > 1 \) (and \( \overline{\tau} > 1 \)) or at least \( \overline{\sigma} > 1 \) (and \( \overline{\tau} > 1 \)), see Theorem 2.1 or Theorem 3.1. However, the relation between the exponents \( w(\Theta), \hat{w}(\Theta) \) and the values \( \overline{\sigma}, \overline{\tau}, \overline{\tau}, \overline{\tau} \) can be complicated as the next example demonstrates.
Example 1. Let \( \theta \in \mathbb{R} \) an extremal number as defined by Roy [27]. If \( m = 2, n = 1 \) and \( \Theta = (\theta, \theta^2) \), then Roy’s results in that paper (in particular [27, Theorem 5.1] and its proof) imply

\[
\tau = \sigma = \frac{w(\Theta)}{\hat{w}(\Theta)} = \frac{1}{\sqrt{\tau^* - 1}} = \sqrt{\frac{5 + 1}{2}}.
\]

Possibly also \( \tau = \sigma = \frac{(\sqrt{5} + 1)/2}{\sqrt{\tau^* - 1}} \), however this seems not clear from [27, Theorem 5.1]. Similarly for \( m = 1, n = 2 \) and \( \Theta = (\theta, \theta^2) \). Regardless if this is true, there is identity at least in the third inequality in (26). On the other hand, for \( m = 1, n = 3 \) and \( \Theta = (\theta, \theta^2, \theta^3) \), the description of the associated parametric graph in [28] shows that actually \( \tau = 1 \) and \( \sigma = 1 \), even though \( w(\Theta) = \sqrt{5 + 2} > 3 = \hat{w}(\Theta) \). However, the construction suggests that \( \sigma = \frac{w(\Theta)}{\hat{w}(\Theta)} = \frac{(\sqrt{5} + 1)/2}{\sqrt{\tau^* - 1}} \), however this seems not clear from [27, Theorem 5.1].

The method in [18] shows that for \( m = 1 \), the assumption \( \hat{w}(\Theta) > n \) implies \( \tau \geq G^* > 1 \), and similarly if \( n = 1 \) then \( \tau \geq G > 1 \), with \( G = G(\hat{w}(\Theta), m) \) and \( G^* = G^*(\hat{w}(\Theta), n) \) defined as in [18, Theorem 1]. The latter \( G^*(\hat{w}(\Theta), n) \) we denoted by \( G_{1,n}(\hat{w}(\Theta)) \) in the rephrased Theorem 2.1 above.

We continue with a variant of Theorem 4.1 where we impose a bound on the logarithmic quotients of the largest by the smallest vector norm of a set of consecutive best approximations instead.

**Theorem 4.2.** Let \( m, n, \Theta, (\tilde{z}_k)_{k \geq 1} \) as above and \( \ell \geq 3 \) and integer. Let \( \sigma' > 1 \) and \( \tau' \geq 1, \tau^* \geq 1 \) be real numbers and \( k \) be a large integer. Assume for \( \sigma_j, \tau_j \) defined in (19) we have

\[
\sigma_j \geq \sigma', \quad \tau_j \geq \tau', \qquad k \leq j \leq k + \ell - 3,
\]

and

\[
\tau_k \tau_{k+1} \cdots \tau_{k+\ell-2} \leq \frac{\log \| \tilde{z}_{k+\ell-1} \|}{\log \| \tilde{z}_k \|} \leq \tau^*.
\]

Then with

\[
\Lambda := 1 + \tau'^{-1} + \tau'^{-2} + \cdots + \tau'^{-(\ell-2)} = \frac{1 - \tau'^{1-\ell}}{1 - \tau'^{-1}} \leq \ell - 1,
\]

if we have

\[
\hat{w}(\Theta) > \frac{\tau^* \Lambda - \tau' + 1}{\tau'(\sigma' - 1)},
\]

then the best approximations \( \tilde{z}_k, \tilde{z}_{k+1}, \ldots, \tilde{z}_{k+\ell-1} \) are linearly independent. By rearrangements, the same conclusion holds if

\[
\ell < 1 - \frac{\log \left( \frac{1 - \tau'^{-1}}{\tau'^{-1}}, \frac{\hat{w}(\Theta)\tau'(\sigma' - 1) + \tau'^{-1}}{\tau'^{-1}} \right)}{\log \tau'}.
\]
Dealing with consecutive minimal points is not too crucial in Theorem 4.2, it can be generalized in a straightforward way to any increasingly ordered minimal points satisfying similar relations. We notice that
\[ \tau_{k-1} - \epsilon \leq \tau^* \leq \tau_k + \epsilon \]
for large \( k \geq k_0(\epsilon) \). Again by (2) we can replace the factor \( \hat{w}(\Theta) \) by the possibly smaller value \( n/m \), to obtain a weaker result that avoids exponents. We want to state two weaker but simpler conditions in a corollary.

**Corollary 3.** With the notation of Theorem 4.2, if
\[ \ell < \frac{\sigma' - 1}{\tau^*} \cdot (\hat{w}(\Theta) + \tau' - 1) + 1, \]
which is in particular true if
\[ \ell < \frac{\sigma' - 1}{\tau^*} \cdot \hat{w}(\Theta) + 1, \]
then the best approximations \( z_k, z_{k+1}, \ldots, z_{k+\ell-1} \) are linearly independent. Thus for any good \( \Theta \in \mathbb{R}^{m \times n} \) we have
\[ \hat{w}(\Theta) \leq \frac{(m + n)\tau^*}{\sigma' - 1}. \]

**Proof.** Implication from (31) follows from (29) when estimating \( \Lambda \leq \ell - 1 \), then using \( \tau' \geq 1 \) gives the weaker condition (32). Finally for \( \ell = m + n + 1 \) the linear independence conclusion fails, hence the reverse inequality of (32) must hold, giving the claim (33). \( \square \)

The last claim (33) holds for general \( \Theta \in \mathcal{G}_h \) with the factor \( m + n \) replaced by \( h \). For example any \( \Theta = \theta \in \mathcal{H}_{3,n} \) satisfies \( \hat{w}(\Theta) \leq 3\tau^*/(\sigma' - 1) \). Weaker claims by replacing \( \tau^* \) by \( \tau_{k-1} \) can be stated, for \( \ell = 3 \) this is implied by (21). We provide another linear independence criterion for \( \ell = 3 \) complementary to (21), where we make hypotheses on two consecutive approximation qualities, reflected by \( \nu_k, \nu_{k+1} \).

**Theorem 4.3.** Keep the notation of Theorem 4.2 and let \( \epsilon > 0 \). Assume that \( k \geq k_0(\epsilon) \) is large and as in (19) let
\[ \nu_k = -\frac{\log \| \Theta^E z_k \|}{\log \| z_k \|}, \quad \nu_{k+1} = -\frac{\log \| \Theta^E z_{k+1} \|}{\log \| z_{k+1} \|}, \quad \tau_k = \frac{\log \| z_{k+1} \|}{\log \| z_k \|}, \quad \tau_{k+1} = \frac{\log \| z_{k+2} \|}{\log \| z_{k+1} \|}. \]

If at least one of the three conditions
\[ \nu_k - \epsilon > \frac{\tau_k \tau_{k+1} + 1}{\sigma_k - 1} \]
or
\[ \nu_k + 1 + \epsilon < \tau_k (\nu_{k+1} - \tau_{k+1}) \]
or
\[ \tau_k (\tau_k \nu_{k+1} - \nu_k) \hat{w}(\Theta)^2 - \nu_k \hat{w}(\Theta) - \tau_k \nu_k \nu_{k+1} > \epsilon, \]
holds, then \( z_k, z_{k+1}, z_{k+2} \) are linearly independent.
The second condition (35) is just slightly stronger than
\[ \nu_{k+1} - \tau_{k+1} > \frac{\nu_k}{\tau_k} - \varepsilon. \]
Up to subtraction of (the possibly large) \( \tau_{k+1} \) this resembles (20). Observe that the bracket expression in (35) is positive by (20). Again we can write \( n/m \) in place of \( \hat{w}(\Theta) \) in (36), and also \( \nu_i \geq n/m - o(1) \) as \( i \to \infty \) by Dirichlet’s Theorem (2) and \( \tau_i > 1 \) by (3). The third hypothesis (36) holds in particular if \( \nu_{k+1} \) is sufficiently large and \( \nu_k/\tau_k < \hat{w}(\Theta)^2 \), the latter being true if \( \nu_k/\tau_k < (n/m)^2 \). Theorem 4.3 should be viewed as a "local result", the fact that three consecutive minimal points are linearly independent for infinitely many \( k \) often follows without further assumption, as recalled in Section 2.2.

We finish this section by remarking that some considerations concerning the simultaneous approximation case \( n = 1 \) can be extracted from Davenport and Schmidt [8], see in particular Lemma 5 in that paper. We believe that the underlying arguments can be adapted to get more insight. For \( n = 1, m = 2 \) recall Lagarias’ result from [15] quoted in Section 3.3. See also Section 5.2 below.

4.1. The Veronese curve. We now consider \( n = 1 \) and more specifically that \( \Theta = (\theta, 1) \in \mathbb{R}^{m \times 1} \) consists of successive powers of a number, that is \( (\theta, 1) \) lies on the twisted Veronese curve \( V_n := \{ (\theta^n, \ldots, \theta^2, \theta, 1) : \theta \in \mathbb{R} \} \) with coordinates in reverse order. We will sporadically identify the vector \( \Theta \) with its first coordinate \( \theta \in \mathbb{R} \) in the sequel. Then the scalar product of the minimal points \( z_k \) with \( \Theta_E = (\theta, 1) \in \mathbb{R}^{n+1} \) can be interpreted as an integer polynomial of degree at most \( n \) evaluated at \( \theta \). We denote by \( P_k \) this polynomial that realizes \( \Theta_E z_k = P_k(\theta) \), call \( P_k \) best approximation polynomial associated to the pair \( \theta, n \) and write \( H(P_k) \) for \( \|z_k\| \) and call it height of \( P_k \). According to (3), the sequence \( (P_k)_{k \geq 1} \) satisfies
\[
H(P_1) < H(P_2) < \cdots, \quad |P_1(\theta)| > |P_2(\theta)| > \cdots.
\]
The classical notation for the linear form exponents of approximation in this case is
\[
w(\Theta) = w_n(\theta), \quad \hat{w}(\Theta) = \hat{w}_n(\theta).
\]
The claims of previous sections clearly apply to the special case of the Veronese curve. We first highlight a consequence of Theorem 2.1 when combined with a result of Sprindžuk [32].

Definition 4. Let \( G_{h,n} \subseteq G_h^{1,n} \) be the points in \( G_h^{1,n} \) of the form \( (\theta^n, \theta^{n-1}, \ldots, \theta) \).

Corollary 4. Let \( n > h \geq 3 \) be integers. Then the set \( G_{h,n} \) has 1-dimensional Lebesgue measure 0 (Hausdorff dimension less than 1) and contains no vector with algebraic \( \theta \).

The first metric claim is valid for the much larger class of so-called extremal curves, including any smooth curve that is properly curved. We only want to refer here to a very general result by Kleinbock and Margulis [14]. Concrete bounds for the Hausdorff dimensions of \( G_{h,n} \) for \( h < n \) can be derived from combining (9) with the metric result of Bernik [3], for \( h = 3 \) we get that \( G_{3,n}^{1} \) has dimension at most \( (n+1)/(n^2 - n + 1) = O(n^{-1}) \) for \( n \geq 2 \), smaller than 1 if \( n > 2 \).

Our proof of the next result requires the Veronese curve setting. We adapt the notation concerning \( \sigma, \tau, \nu \) from Section 4.
Theorem 4.4. Let $n \geq 1$ and a real number $\theta$ not algebraic of degree $\leq n$ be given and consider the best approximation polynomials $(P_k)_{k \geq 1}$ associated to $\theta, n$. Assume for any large $k$ the polynomials $P_k, P_{k+1}$ have no common factor and we have

\begin{equation}
\nu := \liminf_{k \to \infty} -\frac{\log |P_k(\theta)|}{\log H(P_k)} > 2n - 1.
\end{equation}

Then

\begin{equation}
\sigma := \liminf_{k \to \infty} \frac{\log |P_{k+1}(\theta)|}{\log |P_k(\theta)|} \geq \frac{\nu - n + 1}{n}, \quad \tau := \limsup_{k \to \infty} \frac{\log H(P_{k+1})}{\log H(P_k)} \leq \frac{w_n(\theta)}{\hat{w}_n(\theta)},
\end{equation}

and hence if the integer $\ell \geq 1$ satisfies

\begin{equation}
\ell < \frac{\log \left(\frac{(\nu - 2n + 1)(w_n(\theta) - \hat{w}_n(\theta))}{n} + 1\right)}{\log(w_n(\theta)/\hat{w}_n(\theta))} + 1 \leq \frac{\log \left(\frac{(\nu - 2n + 1)(w_n(\theta) - n)}{n} + 1\right)}{\log(w_n(\theta)/n)} + 1,
\end{equation}

then for every large $k$ the polynomials $P_k, P_{k+1}, \ldots, P_{k+\ell-1}$ are linearly independent.

The condition (39) can be slightly relaxed, see the connection between $\Gamma$ and $\tilde{\Gamma}$ in Section 4. Moreover variants with relaxed conditions and conclusions for infinitely many $k$ only can be readily derived. We state some other remarks.

Remark 3. We may also state a stronger version than (38) involving the accordingly defined quantity $\tau$, analogously to (22). Note that $\nu$ and $w_n(\theta)$ are related by \[ \nu = \liminf_{k \to \infty} -\frac{\log |P_k(\theta)|}{\log H(P_k)} \leq \limsup_{k \to \infty} -\frac{\log |P_k(\theta)|}{\log H(P_k)} = w_n(\theta). \]

It may be true that $n = 2$ and $\theta$ any extremal number $[27]$ provide a non-trivial equality case, compare this with Example [1] above. Unfortunately, it is not clear how to link $\nu$ with $\hat{w}_n(\theta)$.

Remark 4. The coprimality condition is satisfied as soon as $w_{n-1}(\theta) < \nu$, as then the polynomials $P_k$ are irreducible of degree precisely $n$ for every large $k$, so in particular if $w_{n-1}(\theta) \leq 2n - 1$. In case of $w_n(\theta) > w_{n-1}(\theta)$ and $\hat{w}_n(\theta) > n$, due to Lemma [1] and [6] Theorem 2.2] we can estimate

\[ \max\{\sigma, \tau\} \leq \max\{\sigma, \tau\} \leq \frac{w_n(\theta)}{\hat{w}_n(\theta)} \leq \frac{n - 1}{\hat{w}_n(\theta) - n}. \]

If $w_n(\theta) \geq \delta n$ for $\delta > 2$, then we may choose $\ell \gg \log n$ again with an implied constant independent from $n$. The condition (38) of the theorem states that all best approximation polynomials induce very small evaluations at $\theta$, with the natural exponent $n$ replaced by some value $> 2n - 1$. We could similarly derive variants of Theorem 4.4 in the spirit of Theorem 4.2 for the Veronese curve under assumption of (37). We only want to state an improvement of Theorem 4.3 in the Veronese curve case.
Theorem 4.5. Let \( \theta \) be a transcendental real number and \( n \geq 2 \) be an integer and denote by \( (P_j)_{j \geq 1} \) the sequence of best approximation polynomials associated to \( \theta, n \). Let \( \epsilon > 0 \). Assume \( k \) is a large index and that \( P_k \) and \( P_{k+1} \) are coprime. As in (19) let

\[
\nu_k = -\frac{\log |P_k(\theta)|}{\log H(P_k)}, \quad \nu_{k+1} = -\frac{\log |P_{k+1}(\theta)|}{\log H(P_{k+1})}, \quad \tau_{k+1} = \frac{\log H(P_{k+1})}{\log H(P_{k+1})}.
\]

Assume that \( \nu_k > 2n - 1 \) and

- either the relation

\[
(\chi_k^2\nu_{k+1} - \chi_k\nu_k)\hat{w}_n(\theta)^2 - \nu_k\hat{w}_n(\theta) - \chi_k\nu_k\nu_{k+1} > 0, \quad \chi_k = \frac{\nu_k - n + 1}{n},
\]

- or

\[
(\nu_{k+1} - \tau_{k+1})\frac{\nu_k - n + 1}{\nu_k} > n.
\]

holds. Then \( P_k, P_{k+1}, P_{k+2} \) are linearly independent.

It can be verified that upon \( \nu_k > 2n - 1 \) the condition (40) relaxes (36) and (41) relaxes (34), when we trivially estimate \( \tau_k \) by 1 in (36) resp. (34). Finally we want to generalize Theorems 4.1, 4.2 to certain sets of polynomials derived from consecutive best approximation polynomials by multiplication with integer polynomials of small degree \((\leq d)\). Sets of this type have been of interest in [29], where it was shown that certain mild linear independence conditions imply good upper bounds on the classical exponent \( \hat{w}_n(\theta) \). The main obstacle for our method in this setting is that for \( d > 0 \) the new polynomials may have small evaluations at \( \theta \) as well. For this reason the quantity \( w_d(\theta) \) will occur. We agree on the notation \( w_0(\theta) = 0 \).

Theorem 4.6. Let \( n \geq 1 \) be an integer and \( \theta \) be a real number and let \( (P_k)_{k \geq 1} \) be the best approximation polynomial sequence associated to \( n, \theta \). Define \( \sigma \geq 1, \tau \geq 1 \) as in (38) and let \( \ell \geq 3, d \geq 0 \) be other integers satisfying \((d + 1)\ell \leq n + d + 1 \). Assume the equivalent conditions

\[
\ell < \frac{\hat{w}_n(\theta)(\sigma - 1)\tau}{d + 1} + 1 \iff \hat{w}_n(\theta) > \frac{[(d + 1)\ell - 1](w_d(\theta) + 1)\tau^{\ell - 1}}{(\sigma - 1)\tau},
\]

hold. Define the sets of polynomials

\[
\mathcal{A}_j(T) = \{P_j(T), TP_j(T), \ldots, T^dP_j(T)\}, \quad j \geq 1.
\]

Then for all large indices \( k \), the set \( \mathcal{B}_k := \mathcal{A}_k \cup \mathcal{A}_{k+1} \cup \cdots \cup \mathcal{A}_{k+\ell - 1} \) consisting of \((d + 1)\ell \) polynomials of degree at most \( n + d \), is linearly independent.

As before we may replace \( \hat{w}_n(\theta) \) by \( n \) in (42) to get weaker claims. The choice \( d = 0 \) leads to criterion (32) of Corollary 3 in the special case of the Veronese curve upon identifying \( \tau^* \) with \( \tau^{\ell - 1} \), see also the remarks below Theorem 4.2. Some improvements in the spirit of Theorem 4.2 can be obtained upon certain refinements in the proof, we do not state them explicitly. We see that if \( \sigma > 1, \tau \) are fixed and \( w_d(\theta) \ll d \) then for large \( n \) again we have that \( \mathcal{B}_k \) in the theorem is linearly independent for \( \ell \) up to some value \( \gg \log n - 2 \log d \). If \( d \) is fixed as well and \( w_d(\theta) < \infty \), again for large \( n \) the claim is true for \( \ell \) up to \( \gg \log n \).
5. Proofs

5.1. Siegel’s Lemma. A crucial ingredient of our proofs is Siegel’s Lemma. The most effective variant for our purposes is reproduced below. See also Davenport and Schmidt [9, Theorem 3] proved in Section 11 of their paper.

**Lemma 2** (Siegel’s Lemma). Consider a system of linear equations

\[ Bx = 0 \]

where \( B \in \mathbb{Z}^{m \times u} \) is a matrix with \( m \) rows and \( u \) columns, and \( u > m \). Assume the rows are linearly independent, i.e. the matrix has rank \( m \). Then there is a solution \( x = (x_1, \ldots, x_u)^t \in \mathbb{Z}^u \setminus \{0\} \) of norm \( \|x\| \leq (u - m)V_{1/u-m}^{\text{max}} \), for \( V \) the maximum modulus of the \( m \times m \)-subdeterminants of the matrices formed by \( m \) columns of \( B \).

We point out that the occurring determinants can be estimated up to a factor \( \ll m^1 \) by the product of the column norms by Hadamard’s inequality. Moreover the standard version of Siegel’s Lemma with \( \|x\| \ll m \max_{i,j} |b_{ij}|^{m/(u-m)} \), where \( b_{ij} \) are the entries of \( B \), follows directly. We will apply the following modified version.

**Corollary 5.** Let \( B' \) be any integer \( m \times u \)-matrix of rank \( s < u \) (possibly with \( m > u \)). Then the system \( B'x = 0 \) has a solution \( x \in \mathbb{Z}^u \setminus \{0\} \) with \( \|x\| \ll m V'_{1/(u-s)} \leq V' \) where again \( V' \) is the maximum absolute value of the \( s \times s \)-subdeterminants of \( B' \).

**Proof.** We form a new auxiliary matrix \( B \) by taking any \( s \) linearly independent rows from \( B' \) and define \( V \) for \( B \) as above. We can apply Siegel’s Lemma in the above version to \( B \) and obtain that \( Bx = 0' \) has a solution \( x \in \mathbb{Z}^u \setminus \{0\} \) of norm \( \|x\| \leq V'_{1/(u-s)} \leq V \). However, since the potential other \( m - s \) lines of \( B' \) are each a linear combination of the \( s \) linearly independent lines of \( B \) (since \( B' \) has rank \( s \)), clearly \( x \) is also a solution to the original system \( B'x = 0 \). Finally, since every \( s \times s \) submatrix of \( B \) is also a submatrix of \( B' \), clearly \( V \leq V' \). \( \square \)

5.2. Outline of proofs. The proofs of all main results of the paper below basically follow the same line. We assume a putative linear dependence equation

\[ a_1\xi_1 + a_2\xi_2 + \cdots + a_v\xi_v = 0, \]

for \( \xi_j = z_j \), certain best approximations, mostly consecutive, associated to \( \Theta \) and suitable \( v \). From Siegel’s Lemma in the form of Corollary [5] and Hadamard’s estimate we derive upper bounds for \( \|a\| = \max |a_j| \) in terms of the norms \( \|\xi_v\| \). The above identity implies

\[ a_1\Theta^E_1 + \cdots + a_v\Theta^E_v = \Theta^E(a_1\xi_1 + a_2\xi_2 + \cdots + a_v\Theta^E_v) = \Theta^E \cdot 0 = 0. \]

Now if the maximum of the terms, say \( \|\Theta^E_1\| \), is reasonably larger than all other expressions \( \|\Theta^E_i\|, i \neq 1 \), using the bounds for the coefficients we get a contradiction by triangular inequality, unless \( a_1 = 0 \) which must be considered separately. We finish this short section with the proof of the auxiliary lemma. Observe there is a typographical difference between different quantities \( \epsilon \) and \( \varepsilon \).
\textbf{Proof of Lemma 1.} Let \( \epsilon > 0 \). Let \( \tilde{z}_k \) be a best approximation of large index \( k \). Then by definition of \( w(\Theta) \) we have
\[
\nu_k = -\frac{\log \|\Theta E \tilde{z}_k\|}{\log \|\tilde{z}_k\|} \leq w(\Theta) + \epsilon.
\]
Now let \( \epsilon = 2 \tilde{w}(\Theta) \epsilon > \tilde{w}(\Theta) \epsilon > 0 \) and
\[
X := \|\tilde{z}_k\|^{w(\Theta)/\tilde{w}(\Theta) - \epsilon}.
\]
By definition of \( \tilde{w}(\Theta) \) the system
\[
\|\tilde{z}\| \leq X, \quad \|\Theta E \tilde{z}\| \leq X^{\tilde{w}(\Theta) + \epsilon}
\]
has a solution \( \tilde{z} \in \mathbb{Z}^{n+m} \setminus \{0\} \) if \( k \) was chosen large enough. Note that the right estimate is not satisfied for \( \tilde{z} = z_k \) by choice of \( \epsilon \). Thus by definition of best approximations \( \tilde{w}(\Theta) \) we infer \( X \geq \|z_{k+1}\| \), showing the estimate for \( \tau \) as \( \epsilon \) and thus \( \epsilon \) can be chosen arbitrarily small.

For the estimate for \( \sigma \) again start with any large \( k \) and observe that a slight modification of the proof of the estimate for \( \tau \) above (writing \( \nu_k \) in place of \( w(\Theta) \)) shows that
\[
\tau_k = \frac{\log \|\tilde{z}_{k+1}\|}{\log \|\tilde{z}_k\|} \leq \frac{\nu_k}{\tilde{w}(\Theta)} + \epsilon.
\]
See the proof of Theorem 4.3 below for a concise justification. Observe further that
\[
\nu_{k+1} = \frac{\log \|\Theta E \tilde{z}_{k+1}\|}{\log \|\tilde{z}_{k+1}\|} \leq w(\Theta) + \epsilon
\]
holds. Combining these properties yields
\[
\log \|\Theta E \tilde{z}_{k+1}\| \leq \log \|\tilde{z}_{k+1}\| \cdot \frac{\log \|\tilde{z}_{k+1}\|}{\log \|\tilde{z}_k\|} \cdot \frac{\log \|\tilde{z}_k\|}{\log \|\Theta E \tilde{z}_k\|} \leq (w(\Theta) + \epsilon)(\frac{\nu_k}{\tilde{w}(\Theta)} + \epsilon)\nu_k^{-1}.
\]
The claim follows as \( \epsilon, \epsilon \to 0 \). The most right inequality in (26) now comes from (2).

5.3. \textbf{Proof of Theorem 3.1.} In the proofs below any appearing \( \epsilon_i \) will be positive but arbitrarily small. We first observe the following easy, auxiliary result. Notice again the typographical difference between \( \epsilon \) and \( \epsilon \) in the proof.

\textbf{Proposition 2.} Assume \( w(\Theta) < \infty \). Then if for every \( t \) we choose any \( \tilde{v}_t \in \mathbb{Z}^{m+n} \) with \( \|\tilde{v}_t\| \ll \|\tilde{z}_t\|^{o(1)} \) as \( t \to \infty \), we have
\[
\|\Theta E \tilde{v}_t\| \geq \|\tilde{z}_t\|^{-o(1)}.
\]

\textbf{Proof.} We may clearly assume \( \|\tilde{v}_t\| \) tends to infinity with \( t \). Then by definition of \( w(\Theta) \) for large \( t \geq t_0 \) we have
\[
\|\Theta E \tilde{v}_t\| \geq \|\tilde{v}_t\|^{-2w(\Theta)}.
\]
By assumption, for any \( \epsilon > 0 \) and large \( t \geq t_1(\epsilon) \), we have \( \|\tilde{v}_t\| \leq \|\tilde{z}_t\|^\epsilon \). For given \( \epsilon > 0 \), with \( \epsilon := \epsilon/(2w(\Theta)) \), we conclude
\[
\|\Theta E \tilde{v}_t\| \geq \|\tilde{z}_t\|^{-2w(\Theta)} \geq \|\tilde{z}_t\|^{-\epsilon}, \quad t \geq \max\{t_0, t_1\}.
\]
As \( \epsilon \) can be arbitrarily small, the claim follows.
Let \( m \geq 1, n \geq 1 \) and \( 1 \leq h \leq m + n \) be fixed and \( \Theta \in \mathbb{R}^{m \times n} \) belong to \( \mathbf{V}_h \). Consider sets of consecutive minimal vectors \( \tilde{z}_t, \tilde{z}_{t+1}, \ldots, \tilde{z}_{t+h-1} \) for large \( t \) as in the definition of \( \mathbf{V}_h \). To shorten notation, let

\[
\mathcal{F}_t = \langle \tilde{z}_t, \tilde{z}_{t+1}, \ldots, \tilde{z}_{t+h-1} \rangle \subseteq \mathbb{R}^{m+n}, \quad t \geq 1,
\]

be the vector spanned by the \( \tilde{z}_t \). By assumption some integer vector \( \mathbf{v}_t \) of norm \( \| \mathbf{v}_t \| \ll \| \tilde{z}_t \|^\Theta \| \) as \( t \to \infty \) lies in the lattice \( \mathcal{F}_t \cap \mathbb{Z}^{m+n} \). For each \( t \geq 1 \) let

\[
\{ \mathbf{v}_1, \ldots, \mathbf{v}_w \} \subseteq \{ \tilde{z}_t, \tilde{z}_{t+1}, \ldots, \tilde{z}_{t+h-1} \}, \quad 1 \leq w \leq h,
\]

where \( w = w(t) \) and the \( \mathbf{v}_i = \mathbf{v}_i(t) \) depend on \( t \) as well, be a linearly independent set spanning the same space

\[
\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_w \rangle \subseteq \mathcal{F}_t,
\]

in other words a vector space basis of \( \mathcal{F}_t \). Assume the norms \( \| \mathbf{v}_i \| \) are naturally increasingly ordered. Let \( Y := \| \tilde{z}_t \| \in \mathbb{N} \). Now by linear independence of the \( \mathbf{v}_i \) and since \( Y \cdot \mathbf{v}_i \) obviously lies in the lattice \( \mathcal{F}_t \cap \mathbb{Z}^{m+n} \) as well, we have a one-dimensional solution space to the identity

\[(44) \quad a_0 Y \mathbf{v}_i + a_1 y_{i,1} + a_2 y_{i,2} + \cdots + a_w y_{i,w} = 0, \]

in \( \mathbf{a} = (a_0, \ldots, a_w) \), and any non-zero solution has \( a_0 \neq 0 \). Since we deal with integer vectors, the integral solutions to \( (44) \) form a one-dimensional lattice in \( \mathbb{Z}^{w+1} \). In other words, we have a unique generator solution \( \mathbf{a} = (a_0, \ldots, a_w) \) that is a primitive (i.e. largest common divisor equals 1) integer vector with \( a_0 \neq 0 \), and all other integer solutions to \( (44) \) are integer multiples of it. In particular our \( \mathbf{a} \) minimizes the norm among all non-zero integer solutions. Fix this \( \mathbf{a} \) in the sequel, for simplicity we do not invent new notation for it. If \( w(\Theta) = \infty \) the claim is trivial, so we can assume \( w(\Theta) < \infty \). Then for given \( \varepsilon > 0 \), by Proposition 2 we have \( \| \Theta^E \mathbf{v}_i \| \geq \| \tilde{z}_t \|^\Theta \varepsilon = Y^{-\varepsilon} \) for large \( t \). Since \( a_0 \neq 0 \), thus

\[
\| a_0 \Theta^E Y \mathbf{v}_i \| = |a_0| \cdot Y \cdot \| \Theta^E \mathbf{v}_i \| \geq Y \cdot \| \Theta^E \mathbf{v}_i \| \geq Y^{1-\varepsilon}, \quad t \geq t_0.
\]

On the other hand, by \( (44) \) we have

\[
\begin{align*}
a_0 \Theta^E (Y \mathbf{v}_i) + a_1 \Theta^E y_{i,1} + a_2 \Theta^E y_{i,2} + \cdots + a_w \Theta^E y_{i,w} \\
= \Theta^E (a_0 Y \mathbf{v}_i + a_1 y_{i,1} + a_2 y_{i,2} + \cdots + a_w y_{i,w}) = 0,
\end{align*}
\]

so for \( t \geq t_0 \) we infer

\[(45) \quad S := \| a_1 \Theta^E y_{i,1} + a_2 \Theta^E y_{i,2} + \cdots + a_w \Theta^E y_{i,w} \| = \| a_0 Y \Theta^E \mathbf{v}_i \| \geq Y^{1-\varepsilon}.
\]

Let

\[
\alpha_i = \alpha_{i,t} = \frac{\log \| \tilde{z}_{i+1} \|}{\log \| \tilde{z}_i \|} > 1, \quad 1 \leq i \leq h-1,
\]

so that \( \| \tilde{z}_{i+1} \| = Y^{\alpha_i} \). Now equation \( (44) \) can be written \( B \mathbf{a} = \mathbf{0} \) for \( B \) the integer matrix whose \( w + 1 \) columns consist of the vectors \( Y \mathbf{v}_i, y_{i,1}, y_{i,2}, \ldots, y_{i,w} \) respectively and \( \mathbf{a} = (a_0, a_1, a_2, \ldots, a_w) \). By assumption \( B \) has rank \( w \). Any \( w \times w \) subdeterminant of \( B \) can by Hadamard’s inequality be estimated up to a factor \( \ll_{m,n} 1 \) by the product of the
column norms. Since $\|Yw\| = Y \cdot \|w\| \leq Y^{1+\varepsilon}$, Siegel’s Lemma in form of Corollary 5 and the minimality of $\|a\|$ thus imply that our generator solution $a$ to (14) satisfies
\begin{equation}
\|a\| = \max \{\|\alpha\| \leq m, n \max \{\|y_1\|, Y^{1+\varepsilon} \cdot \|y_2\|, \ldots, \|y_w\|\} \leq Y^{1+\varepsilon} \cdot \|z_t+1\| \cdots \|z_t+h-1\| = Y^{1+\alpha_1+\cdots+\alpha_{h-1}+\varepsilon},
\end{equation}
no matter whether $z_t = z_t$ or not. Now $\|z_t\| \geq \|z_t\|$ and (5), (11) imply
\begin{equation}
\max_{1 \leq i \leq w} \|\Theta^t y_i\| \leq \|\Theta^t z_t\| \leq \|z_t+1\|^{\alpha_1} \cdots \|z_t+h-1\|^{\alpha_h} \leq \|z_t\|^{\alpha_1} \cdots \|z_t\|^{\alpha_h} = Y^{-\alpha_1 \theta + \varepsilon}.
\end{equation}
Since the moduli of the scalar products are decreasing according to (3), combining (46), (17) yields that the sum in (15) can be estimated from above by
\begin{equation}
S \leq w \|a\| \cdot \|\Theta^t z_t\| \leq h \|a\| \cdot \|\Theta^t z_t\| \leq \|\Theta^t z_t\| + 1 Y^{1+\alpha_1+\cdots+\alpha_{h-1}+\alpha_{h} \theta + \varepsilon}.
\end{equation}
Since $Y \to \infty$ as $t \to \infty$ and $\varepsilon$ can be arbitrarily small, making up for the multiplicative factor with arbitrarily small quantity and combining with the lower estimate (15) yields
\begin{equation}
\alpha_1 (1 - \theta) + \alpha_2 + \cdots + \alpha_{h-1} \geq -\varepsilon_4.
\end{equation}
Hence, no matter if $\hat{\theta}(\Theta) > 1$ or not, as $t \to \infty$ we conclude
\begin{equation}
\hat{\theta}(\Theta) \leq 1 + \frac{\alpha_2 + \cdots + \alpha_{h-1}}{\alpha_1} + \varepsilon_5.
\end{equation}
By Lemma 11 we see that $\alpha_{i+1}/\alpha_i = \tau_{i+1} \leq \theta(\Theta)/\hat{\theta}(\Theta) + \varepsilon_6$ for all $i$ under consideration. Hence the right hand side can be estimated via
\begin{equation}
\hat{\theta}(\Theta) \leq 1 + \frac{\theta(\Theta)}{\hat{\theta}(\Theta)} + \left(\frac{\theta(\Theta)}{\hat{\theta}(\Theta)}\right)^2 + \cdots + \left(\frac{\theta(\Theta)}{\hat{\theta}(\Theta)}\right)^{h-2} + \varepsilon_7
\end{equation}
the claim (9) follows as $\varepsilon_7$ will be arbitrarily small. The claim (10) is clear from (48) as well. Finally upon $\Theta \in SV_k$, we can choose $t$ so that the quotient $\alpha_2/\alpha_1 = \tau_{i+1}$ is arbitrarily close to $\tau_t$ and (11) follows. The proof is finished.

5.4. Proofs of Section 4 Similar ideas are employed to prove the results of Section 4. The notation $[x]$ indicates the largest integer smaller than or equal to $x \in \mathbb{R}$.

Proof of Theorem 4.4. Let $m, n, \Theta$ as in the theorem. We first show the claim involving (22). Hence let $\ell \geq 3$ be another fixed integer to be specified later. Assume the opposite, that is $\tilde{z}_{k}, \ldots, \tilde{z}_{k+\ell-1} \in \mathbb{Z}^{n+m}$ are linearly dependent for some large $k$, which we consider fixed in the sequel. Derive $y_1, \ldots, y_w$, with $w = w(t)$, a subset that forms a basis of $F_k = \langle \tilde{z}_1, \ldots, \tilde{z}_{k+\ell-1} \rangle$, labeled with increasing norms, very similar to the proof of Theorem 3.1. Notice $w \leq \ell - 1$ here. First assume we may choose the $y_j$ so that $\tilde{z}_k \notin \{y_1, \ldots, y_w\}$. Let $y_0 = \tilde{z}_k$. For simplicity, let
\begin{equation}
Y_j := \|y_j\|, \quad 0 \leq j \leq w.
\end{equation}
Put $Z := |Y_1/Y_0| \geq 1$ so that $\|Zy_0\| \gg \|y_0\|$. Let $B$ be the $(m+n) \times (w+1)$-matrix with first column $Zy_0$ and $j$-th column $y_{j-1}$ for $2 \leq j \leq w+1$. Consider the system
\begin{equation}
Ba = a_0 Z y_0 + a_1 y_1 + \cdots + a_w y_w = 0.
\end{equation}
for \(a = (a_0, \ldots, a_w)\). By the same argument as in Theorem 3.1 there is a unique primitive integer vector \(a\) with \(a_0 > 0\) that generates the one-dimensional lattice of all integer solutions. Recall \(\tau_j\) from (19), that clearly satisfy \(\tau_j > 1\) for all \(j\) by (3). Since \(\|Zy\| = ZY_0 \ll Y_1\) and \(B\) has rank \(w\) and by the minimality of \(\|a\|\), Siegel’s Lemma in form of Corollary 5 implies that our primitive integer solution vector has entries

\[
\|a\| = \max_{0 \leq j \leq w} |a_j| \leq Y_1 Y_2 \cdots Y_w \leq \|z_{k+1}\| \cdot \|z_{k+2}\| \cdots \|z_{k+\ell-1}\| \leq Y_0^R,
\]

where we have put \(R := \tau_k + \tau_k \tau_{k+1} + \cdots + \tau_k \tau_{k+\ell-2}\).

Let \(\varepsilon > 0\). We may assume \(k\) was chosen large enough that

\[
\sigma_k = \frac{\log \|\Theta^E z_{k+1}\|}{\log \|\Theta^E z_k\|} > \sqrt{R} - \varepsilon.
\]

From \(a_0 \neq 0\) we further infer

\[
\|\Theta^E a_0 Zy\| = |a_0| Z \cdot \|\Theta^E y\| \geq Z\|\Theta^E y\| > 0.
\]

Since in view of (19) we have

\[
\Theta^E a_0 Zy_0 + \Theta^E a_1 y_1 + \cdots + \Theta^E a_w y_w = \Theta^E \cdot 0 = 0,
\]

we infer

\[
S := \|\Theta^E a_1 z_1 + \cdots + \Theta^E a_w z_w\| = \|a_0 Z\Theta^E y_0\| \geq Z\|\Theta^E y_0\|.
\]

On the other hand, by (3) and (51) we have

\[
\max_{1 \leq j \leq w} \|\Theta^E y_j\| = \|\Theta^E y_1\| \leq \|\Theta^E z_{k+1}\| \leq \|\Theta^E z_k\| \zeta^{-\varepsilon} = \|\Theta^E y_0\| \zeta^{-\varepsilon}.
\]

Consequently we can estimate

\[
|S| \leq \ell \|a\| \cdot \|\Theta^E y_0\| \ll_{m,n} Y_0\|\Theta^E y_0\| \zeta^{-\varepsilon} \text{ and hence}
\]

\[
Y_0^R \|\Theta^E y_0\| \zeta^{-\varepsilon} \gg_{m,n} Z \|\Theta^E y_0\|,
\]

or equivalently

\[
\|\Theta^E y_0\| \geq Z^{1/(\zeta^{-1}) + \epsilon_1} Y_0^{-R/(\zeta^{-1}) + \epsilon_2} = Y_0^{(\tau_k - 1)/(\zeta^{-1}) + \epsilon_1} Y_0^{-R/(\zeta^{-1}) + \epsilon_2},
\]

for \(\epsilon_1 > 0, \epsilon_2 > 0\) small variations of \(\epsilon\). On the other hand, since \(y_0\) is a minimal point, by (1) we infer

\[
\|\Theta^E y_0\| \leq Y_0^{1 - \tau_k \hat{w}(\Theta) + \epsilon_3}.
\]

Combining with (53) yields

\[
\tau_k \hat{w}(\Theta) \leq \frac{R - \tau_k + 1}{\zeta - 1} + \epsilon_4.
\]

We may assume \(k\) is large enough that

\[
\tau - \epsilon \leq \tau_i \leq \tau + \epsilon, \quad i \geq k - 1.
\]

The value \(R\) can be bounded

\[
R \leq \tau_k (1 + \tau + \tau^2 + \cdots + \tau^{\ell-2}) + \epsilon_5 = \tau_k \frac{\tau^{\ell-1} - 1}{\tau - 1} + \epsilon_5.
\]
Here and below we always take the limit if \( \tau = 1 \). From (54) we infer
\[
\hat{w}(\Theta)\tau_k(\sigma - 1) + \tau_k - 1 \leq \tau_k\frac{\tau^{\ell-1} - 1}{\tau - 1} + \epsilon_6.
\]
Estimating \( \tau_k \geq \tau - \epsilon \) we get
\[
\tau^{\ell-1} - 1 \geq (\tau - 1) \left( (\hat{w}(\Theta)(\sigma - 1) + 1 - \frac{1}{\tau}) + \epsilon_7. \right.
\]
Solving for \( \ell \), we see that
\[
\ell > 1 + \frac{\log \left( (\tau - 1) \left( (\hat{w}(\Theta)(\sigma - 1) + 1 - \frac{1}{\tau}) + 1 \right) \right)}{\log \tau} + \epsilon_8 = \tilde{\Gamma}(\Theta) + \epsilon_8.
\]
Taking the contrapositive yields the claim involving (22) of the theorem. The specialization (21) follows since if \( \ell = 3 \), then we have \( R = \tau_k(1 + \tau_{k+1}) \) and also can take \( \sigma_k \) instead of \( \sigma - \epsilon \) because the two consecutive minimal points \( \tilde{z}_{k+1}, \tilde{z}_{k+2} \) are always linearly independent. Then a short calculation indeed verifies (21).

Now assume otherwise that we cannot choose the \( y_i \) so that \( \tilde{z}_k \notin \{ y_1, \ldots, y_w \} \). This means that \( (\tilde{z}_{k+1}, \ldots, \tilde{z}_{k+\ell-1})_R \) span a proper subspace of \( \mathcal{F}_k = (\tilde{z}_k, \tilde{z}_{k+1}, \ldots, \tilde{z}_{k+\ell-1})_R \). Hence the set \( \{ \tilde{z}_{k+1}, \ldots, \tilde{z}_{k+\ell-1} \} \) is linearly dependent as well. Thus upon index shift \( k+1 \) becoming \( k \), we have reduced the problem from \( \ell \) to \( \ell - 1 \). By an inductive argument, upon accordingly redefining \( k \), we must end up at some point where can assume the property \( \tilde{z}_k \notin \{ y_1, \ldots, y_w \} \) is satisfied. Thus we infer (55) for some \( \ell' \leq \ell - 1 \) replacing \( \ell \) in the right hand side, and with the same arguments finally end up at the stronger condition
\[
\ell \geq \ell' + 1 > 1 + \tilde{\Gamma}(\Theta) + \epsilon_8
\]
for linear dependence. Again taking the contrapositive, we conclude that condition (22) suffices in any case for linear independence.

Finally we prove the last claims. We start with large \( k \) that satisfy \( \sigma_k \geq \tau - \epsilon \) instead of (51) and assume \( \tilde{z}_k, \tilde{z}_{k+1}, \ldots, \tilde{z}_{k+\ell-1} \) are linearly dependent. Then proceeding as above and again distinguishing the two cases \( \tilde{z}_k \notin \{ y_1, \ldots, y_w \} \) and \( \tilde{z}_k \in \{ y_1, \ldots, y_w \} \), we get the reverse estimates as in the right and left bound in (24) for \( \ell \), respectively. Hereby we use (57) for the latter case. Again taking the contrapositive yields the claim. Similarly, we can assume \( \tau_k \geq \tau - \epsilon \) by \( \tau \) for certain arbitrarily large \( k \). This leads to a replacement of \( \tau \) by \( \tilde{\tau} \) in (55), and the analogous arguments yield the sufficient condition (25).

**Remark 5.** Assume the space \( \mathcal{F}_k \) has dimension \( w < \ell - 1 \) strictly for all large \( k \). Then we can readily refine the bound \( \tilde{\Gamma}(\Theta) \) in (56) for \( \ell \), as we may take the smaller value
\[
\tilde{R}(w) = \tau_k\tau_{k+1} \cdots \tau_{k+\ell-1-w} + \tau_k\tau_{k+1} \cdots \tau_{k+\ell-w} + \cdots + \tau_k\tau_{k+1} \cdots \tau_{k+\ell-2}
\]
in place of \( R \). Similarly if we assume the property for infinitely many \( k \). This applies in particular to \( \Theta \in \mathcal{V}_h \) when we identify \( h = w \).

The proof of Theorem 4.2 works very similarly, we just estimate the coefficients in (49) with Siegel’s Lemma in a slightly different way. Again it is understood that \( \epsilon_i \) will all be positive but arbitrarily small as the initial \( \epsilon > 0 \) tends to 0.
Proof of Theorem 4.3. For $\ell > 0$ an integer to be fixed later and large $k$ again assume the opposite that $z_k, \ldots, z_{k+\ell-1} \in \mathbb{Z}^{n+m}$ are linearly dependent. Define $y_i$ and $Y_i$, $0 \leq i \leq w$, for $w = w(t) \leq \ell - 1$, as in Theorem 4.1. Then for the same reasons, again (39) induces a primitive integer vector $a = (a_0, \ldots, a_w)$ with $a_0 > 0$ that generates the lattice of all integer solutions. Again first assume $z_k \notin \{y_1, \ldots, y_w\}$ and take $Z$ as in Theorem 4.1. For simplicity put $X := Y_w = \|y_w\|$. Write (49) again as a system $B \cdot a = 0$ with $a = (a_0, \ldots, a_w)^t$ and $B$ the $(m+n) \times (w+1)$ integer matrix of deficient rank $w$ whose columns are the vectors $Zy_0, y_1, \ldots, y_w$. By (3) and Siegel's Lemma and Hadamard's estimate, bounding the column norms via the assumption on $\tau'$, since $w \leq \ell - 1$ we can estimate

\begin{equation}
\|a\| \ll_{m,n} Y_w Y_{w-1} \cdots Y_1 \ll X^\Lambda,
\end{equation}

with

\[ \Lambda = 1 + \tau' - 1 + \tau' - 2 + \cdots + \tau' - (\ell - 2) = \frac{1 - \tau' - (\ell - 1)}{1 - \tau' - 1}, \]

as in (28). By assumption we have

\begin{equation}
\sigma_k = \frac{\log \|\Theta^E z_{k+1}\|}{\log \|\Theta^E z_k\|} \geq \sigma'.
\end{equation}

Now since $a_0 \neq 0$ again we have $\|a_0 Z \Theta^E y_0\| = |a_0| Z \cdot \|\Theta^E y_0\| \geq Z \|\Theta^E y_0\|$. As in the proof of Theorem 4.1 in view of (49) we infer

\begin{equation}
\|a_1 \Theta^E y_1 + \cdots + a_w \Theta^E y_w\| = \|a_0 Z \Theta^E y_0\| \geq Z \|\Theta^E y_0\|.
\end{equation}

On the other hand, by (59) we infer

\[ \max_{j \geq 1} \|\Theta^E y_j\| = \|\Theta^E y_1\| \leq \|\Theta^E y_0\|^{\sigma' - \epsilon}, \]

so by (58) the left hand side in (60) is at most $\ell \|a\| \cdot \|\Theta^E y_0\|^{\sigma' - \epsilon} \ll_{m,n} X^\Lambda \|\Theta^E y_0\|^{\sigma' - \epsilon}$. Combining gives

\[ X^{\Lambda} \|\Theta^E y_0\|^{\sigma' - \epsilon} \gg_{m,n} Z \|\Theta^E y_0\|, \]

or

\[ X^{\Lambda} \gg_{m,n} Z \|\Theta^E y_0\|^{- (\sigma' - 1 - \epsilon)}. \]

Now by assumption $\tau_k \geq \tau' \geq 1$, thus $Z \gg Y_0^{\tau' - 1}$ and

\[ \|\Theta^E y_0\| \geq Y_0^{\frac{\tau' - 1}{\tau' - 1 + \epsilon_1}} \cdot X^{- \frac{\Lambda}{\tau' - 1 + \epsilon_1}}. \]

On the other hand by assumption

\[ X = Y_w \leq \|z_{k+\ell-1}\| \leq \|z_k\|^{\tau'} = Y_0^{\tau'}, \]

inserting gives

\begin{equation}
\|\Theta^E y_0\| \geq Y_0^{\frac{\tau' - 1}{\tau' - 1 + \epsilon_1}} \cdot \frac{\Lambda}{\tau' - 1 + \epsilon_2}.
\end{equation}

On the other hand, from Dirichlet's Theorem (4) we infer

\[ \|\Theta^E y_0\| \ll Y_0^{\tau' \hat{g}(\Theta) + \epsilon_3}. \]
As all $e_i$ can be made arbitrarily small, combining with (61) yields
\[
\hat{w}(\Theta) \leq \frac{\tau^*\Lambda - \tau' + 1}{\tau'(\sigma' - 1)} + \epsilon_4.
\]
Taking the contrapositive and as $\epsilon_4$ can be arbitrarily small shows that (29) indeed implies the linear independence of $\hat{z}_k, \ldots, \hat{z}_{k+\ell-1}$. Inserting for $\Lambda$ from (28), condition (29) can be rearranged to (30). Finally, if the assumption $\hat{z}_k \notin \{y_1, \ldots, y_{\nu}\}$ does not hold, we reduce it to this case precisely as in the last paragraph of the proof of Theorem 3.1. \[\square\]

**Proof of Theorem 4.3.** Assume otherwise $\hat{z}_k, \hat{z}_{k+1}, \hat{z}_{k+2}$ are linearly dependent so that we have an identity
\[a_k \hat{z}_k + a_{k+1} \hat{z}_{k+1} + a_{k+2} \hat{z}_{k+2} = 0,
\]
with integers $a_k, a_{k+1}, a_{k+2}$ not all 0. We have $a_k \neq 0$ since $\hat{z}_{k+1}, \hat{z}_{k+2}$ are linearly independent for every $k$, see Section 1.1. Upon the first condition we proceed as in the proof of Theorem 4.1. Note that with $Y_0 := \|\hat{z}_k\|$ we have $\|\hat{z}_{k+1}\| \cdot \|\hat{z}_{k+2}\| = Y_0^{\tau_k + \tau_{k+1}}$. On the one hand with $Z = \|\hat{z}_{k+1}\|/\|\hat{z}_k\| > Y_0^{\tau_k - 1}$ the linear dependence of $Z\hat{z}_k, \hat{z}_{k+1}, \hat{z}_{k+2}$ very similarly as in (53) yields
\[
\|\Theta^E \hat{z}_k\| \geq Z^{1/(\sigma_k - 1)} Y_0^{-(\tau_k + \tau_{k+1})/(\sigma_k - 1)} \geq Y_0^{-\mu}, \quad \mu = \frac{\tau_k \tau_{k+1} + 1}{\sigma_k - 1} > 0.
\]
On the other hand by definition
\[
\|\Theta^E \hat{z}_k\| = Y_0^{-\nu_k}.
\]
Combining yields $\nu_k \leq (\tau_k \tau_{k+1} + 1)/(\sigma_k - 1) + o(1)$ as $k \to \infty$. Thus assuming the reverse inequality (31), we cannot have linear dependence for large $k$. The second condition (35) is equivalent to (34) via identity (20), which reads
\[
\sigma_k = \frac{\log \|\Theta^E \hat{z}_{k+1}\|}{\log \|\Theta^E \hat{z}_k\|} = \frac{\tau_k \nu_{k+1}}{\nu_k} > 1,
\]
after a short rearrangement (upon modifying $\epsilon$).

For the conclusion from the third hypothesis we verify the sufficient condition (21) i.e.\[
\hat{w}(\Theta) > \frac{\tau_k + 1}{\sigma_k - 1} + \epsilon.
\]
We bound $\tau_{k+1}$ from above. We claim for any $\epsilon_1 > 0$ we have
\[
\tau_{k+1} \leq \frac{\nu_{k+1}}{\hat{w}(\Theta)} + \epsilon_1, \quad k \geq k_0(\epsilon_1).
\]
Inserting for $\tau_{k+1}$ and $\sigma_k$ from (64) and (62) in (63), we derive the third criterion (56) after a short rearrangement. Hereby we use $\tau_k \nu_{k+1} - \nu_k > 0$ by (62). We are left to verify (64), which we may do for index $k$ instead of $k+1$ for simplicity. However, this estimate follows again from Dirichlet’s Theorem, similar to (4). Assume (64) fails (for index $k$ instead of $k+1$). Then for suitable small $\epsilon_2 > 0$ (in dependence of $\epsilon_1$ above) that we can let tend to 0 as $\epsilon_1 \to 0$ and the parameter $X = \|\hat{z}_k\|^{\tau_k - \epsilon_2}$, we have
\[
X < \|\hat{z}_{k+1}\|, \quad \|\Theta^E \hat{z}_k\| = \|\hat{z}_k\|^{-\nu_k} > X^{-\hat{w}(\Theta) + \epsilon_3},
\]
with some fixed small modification $\epsilon_3 > 0$ of $\epsilon_2$. Hence, since $\mathcal{Z}_k, \mathcal{Z}_{k+1}$ are consecutive minimal points, the system \(|z| \leq X\) and \(\| \Theta^F z \| < X^{-\bar{\omega}(\Theta)+\epsilon_3}\) would have no solution in an integer vector $z \in \mathbb{Z}^{n+m} \setminus \{0\}$. This obviously contradicts the definition of $\bar{\omega}(\Theta)$ as $k \to \infty$ and thereby $X \to \infty$. \hfill \Box

5.5. Proofs for the Veronese curve. The improvements for the Veronese curve rely on the following estimate based on a variation of Liouville’s inequality from [9].

Lemma 3. Let $n \geq 1$ be an integer and $\theta$ real and not algebraic of degree at most $n$. Let $(P_k)_{k \geq 1}$ be the associated best approximation polynomial sequence. Let $\epsilon > 0$. Assume for some $k \geq k_0(\epsilon)$ we have that $P_k, P_{k+1}$ have no common factor and that $|P_k(\theta)| = H(P_k)^{-\nu_k}$ for some $\nu_k > 2n - 1$. Then we have

$$
\tau_k = \frac{\log H(P_{k+1})}{\log H(P_k)} \geq \frac{\nu_k - n + 1}{n} - \epsilon.
$$

Proof. As a direct consequence of [9, Lemma 3.1], if $\theta$ is any real number and $P, Q$ are coprime polynomials of degree at most $n$ and $H(Q) > H(P)$, we have

$$
\max\{|P(\theta)|, |Q(\theta)|\} \gg_n H(P)^{-n+1}H(Q)^{-n}.
$$

Application to best approximation polynomials $P = P_k$ and $Q = P_{k+1}$ yields

$$
H(P_k)^{-\nu_k} = |P_k(\theta)| = \max\{|P_k(\theta)|, |P_{k+1}(\theta)|\} \gg_n H(P_k)^{-n+1}H(P_{k+1})^{-n}.
$$

The claim follows after minor rearrangements. \hfill \Box

We remark that the estimate (65) is known to be sharp if $n = 2$ and $\theta$ is a Sturmian continued fraction (see [5, Theorem 3.1]) or any extremal number [27].

Proof of Theorem 4.4. We need to show (65), the bound on $\ell$ then follows essentially as a special case of condition (22) from Theorem 4.1. For the left inequality we use Lemma 3. Let $\epsilon > 0$ and $k$ be large. Write

$$
\sigma_k = \frac{\log |P_{k+1}(\theta)|}{\log |P_k(\theta)|} = \frac{\tau_k \nu_{k+1}}{\nu_k}
$$

where

$$
\nu_k = -\frac{\log |P_k(\theta)|}{\log H(P_k)}, \quad \nu_{k+1} = -\frac{\log |P_{k+1}(\theta)|}{\log H(P_{k+1})}, \quad \tau_k = \frac{\log H(P_{k+1})}{\log H(P_k)}.
$$

By assumption $\nu_{k+1} \geq \nu - \epsilon$. Moreover $\tau_k$ can be bounded in terms of $\nu_k$ by Lemma 3 via

$$
\tau_k = \frac{\log H(P_{k+1})}{\log H(P_k)} \geq \frac{\nu_k - n + 1}{n} - \epsilon.
$$

Combining yields that

$$
\sigma_k \geq \frac{\nu (\nu_k - n + 1)}{n \nu_k} - \epsilon.
$$

Now by assumption $\nu_k \geq \nu - \epsilon$ as well, and letting $\epsilon \to 0$ we see that the expression is minimized if $\nu_k = \nu$ which gives the lower bound $(\nu - n + 1)/n$ of the theorem for $\sigma$. The right estimate for $\sigma$ is just (26). \hfill \Box
Proof of Theorem 4.5. For the first condition we combine criterion (36) from Theorem 4.3 with (65) from Lemma 3. Observe that when expanding the expression in (36) as a quadratic function in \( \tau_k \), since it has positive leading coefficient and negative constant term, it has a positive and a negative real root. Thus, if (36) holds for some value of \( \tau_k > 1 > 0 \), then also for any larger value. Hence in view of Lemma 3 it suffices to have (36) for \( \tau_k = \chi_k = (\nu_k - n + 1)/n \). Inserting and expanding, we derive the sufficient hypothesis (40). Similarly, we combine (35) with (65) to obtain the criterion (41), hereby using \( \nu_k+1 - \tau_k > 0 \) as a consequence of (61) and \( \hat{w}_n(\theta) \geq n \geq 2, \nu_k+1 > 1 \) by (2).

For the proof of Theorem 4.6 we essentially proceed as in the proof of Theorem 4.2.

Proof of Theorem 4.6. Assume the opposite that \( \mathcal{B}_k \) is linearly dependent. Then we have a polynomial identity
\[
P_k U_k + \cdots + P_{k+\ell-1} U_{k+\ell-1} \equiv 0,
\]
with \( U_k \) integer polynomials of degree at most \( d \), not all identically 0. The identity can be written in coordinates in form of a linear equation system \( BA = 0 \) with \( B \) a matrix with \( n + d + 1 \) rows and \( (d+1)\ell \) columns whose entries are coefficients of the polynomials \( P_j \), and \( a \in \mathbb{Z}^{(d+1)\ell} \) the vector consisting of the coefficients of all \( U_j \). By considering if necessary a maximum linearly independent subset of the columns, and distinguishing the cases where some of the remaining columns originates from \( P_k U_k \) and where this is not the case, we can assume that \( B \) above has corank 1 and the first polynomial \( U_k \) does not vanish in any such non-trivial solution. This indeed works very similar to the proof of Theorem 4.1. Then again we obtain a one-dimensional solution lattice for \( a \).

Since there is a non-trivial solution the matrix \( B \) has rank less than \( (d+1)\ell \) and application of Siegel’s Lemma when trivially estimating the subdeterminants by \( X^{(d+1)\ell-1} \) now gives that each \( U_j \) has height \( H(U_j) \ll_n X^{(d+1)\ell-1} \), where again \( X := H(P_{k+\ell-1}) \). Then since \( U_k \) does not vanish identically we can estimate
\[
|U_k(\theta)P_k(\theta)| = |U_k(\theta)| \cdot |P_k(\theta)| \gg H(U_k)^{-w_d(\theta)+\varepsilon} \cdot |P_k(\theta)| \gg X^{-(d+1)\ell-1} \cdot |w_d(\theta)+\varepsilon| \cdot |P_k(\theta)|.
\]
In view of
\[
|P_k(\theta)U_k(\theta)| = |P_{k+1}(\theta)U_{k+1}(\theta) + \cdots + P_{k+\ell-1}(\theta)U_{k+\ell-1}(\theta)| \ll_n |P_{k+1}(\theta)| \max H(U_j),
\]
similar to Theorem 4.2 we obtain the relation
\[
X^{(d+1)\ell-1} |P_k(\theta)|^{-\varepsilon} \gg_n |P_k(\theta)| \cdot X^{-[(d+1)\ell-1] \cdot |w_d(\theta)+\varepsilon|}.
\]
By \( X \ll H(P_k)^{\varepsilon^{-1}+\varepsilon} \) we conclude
\[
|P_k(\theta)| \geq H(P_k)^{-\mu+\varepsilon}, \quad \mu = \frac{[(d+1)\ell-1] \cdot (w_d(\theta)+\varepsilon) - \varepsilon}{\sigma - 1} > 0,
\]
for \( \varepsilon > 0 \) some modification of \( \varepsilon \). On the other hand since \( P_k \) is a best approximation polynomial, by Dirichlet’s Theorem (31) we have \( |P_k(\theta)| \ll_n H(P_k)^{-2\hat{w}_n(\theta)+\varepsilon} \). Combining
yields
\[ n \leq \hat{w}_n(\theta) \leq \frac{[(d + 1)\ell - 1](w_d(\theta) + 1)\tau^{\ell - 1}}{(\tau - 1)\tau} + \varepsilon. \]
Hence again assuming the reverse inequality and letting \( \varepsilon \to 0 \) and thus \( \varepsilon \to 0 \) we cannot have the assumed linear dependence relation. \( \square \)

6. ANNEX: A PROOF OF THEOREM 2.4

The following proof of the claim (6) stated in the unpublished online resource [24] was pointed out to the author by the referee.

First consider the generic case \( h = m+n \) only. Denote \( \tilde{z}_k = (x_{1,k}, \ldots, x_{n,k}, y_{1,k}, \ldots, y_{m,k}) \) for \( k \geq 1 \) the \( k \)-th minimal point associated to \( \Theta \). If \( m+n \) consecutive minimal points \( \tilde{z}_t, \ldots, \tilde{z}_{t+m+n-1} \) are linearly independent, their determinant
\[
\Delta = \begin{vmatrix}
  x_{1,t} & \cdots & x_{n,t} & y_{1,t} & \cdots & y_{m,t} \\
  x_{1,t+1} & \cdots & x_{n,t+1} & y_{1,t+1} & \cdots & y_{m,t+1} \\
  \vdots & \cdots & \vdots & \cdots & \cdots & \vdots \\
  x_{1,t+m+n-1} & \cdots & x_{n,t+m+n-1} & y_{1,t+m+n-1} & \cdots & y_{m,t+m+n-1}
\end{vmatrix}
\]
is non-zero. By taking linear combinations of columns we can make any \( y_{j,k} \) smaller than \( \| \Theta_E \tilde{z}_k \| \), for \( 1 \leq k \leq m \) and \( t \leq j \leq t+m+n-1 \), and we estimate
\[
1 \leq |\Delta| \ll \|\tilde{z}_{t+m+n-1}\| \cdot \|\tilde{z}_{t+m+n-2}\| \cdots \|\tilde{z}_{t+n}\| \cdot \|\Theta_E \tilde{z}_{t+n-1}\| \cdot \|\Theta_E \tilde{z}_{t+n-2}\| \cdots \|\Theta_E \tilde{z}_t\|.
\]
By definition of \( \hat{w}(\Theta) \) for \( \varepsilon > 0 \) and \( t \geq t_0(\varepsilon) \) it gives
\[
1 \ll \|\tilde{z}_{t+m+n-1}\| \cdot \|\tilde{z}_{t+m+n-2}\| \cdots \|\tilde{z}_{t+n}\| \cdot \|\hat{w}(\Theta) + \varepsilon\| \cdot \|\tilde{z}_{t+n-1}\| \cdot \|\tilde{z}_{t+n-2}\| \cdots \|\hat{w}(\Theta) + \varepsilon\|.
\]
One checks that this implies for some \( j \in \{t+1, \ldots, t+n-2\} \) the estimate
\[
\|\tilde{z}_{j+1}\| \geq \|\tilde{z}_j\| G_{m,n}^{-\varepsilon}
\]
with \( G_{m,n} \) as in the theorem and \( \varepsilon > 0 \) that tends to 0 as \( \varepsilon \) does. Thus \( \tau_j \geq G_{m,n} - \varepsilon \) in our notation from Section 3. From \( L_{m+n} \) it follows there are arbitrarily large such \( t \), and since \( \varepsilon > 0 \) can be arbitrarily small Lemma 3 implies \( w(\Theta) < \hat{w}(\Theta) \geq \tau \geq G_{m,n} \).

For general \( h \), one can reduce the problem to \( m' \times n' \) matrices by considering the subspace \( L_{\Theta} \cap R_{\Theta} \) with \( L_{\Theta} \) and \( R_{\Theta} \) as in the theorem.

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