The Kurdyka-Łojasiewicz-Simon inequality and stabilisation in nonsmooth infinite-dimensional gradient systems

Ralph Chill and Sebastian Mildner

Abstract. We state and prove a stabilisation result for solutions of abstract gradient systems associated with nonsmooth energy functions on infinite dimensional Hilbert spaces. One feature is that in this general setting the assumption on the range of the solution can be considerably relaxed, which considerably simplifies the applicability of the stabilisation result even in the case of smooth energies.

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1. Introduction

The Łojasiewicz gradient inequality for real analytic functions on $\mathbb{R}^N$ [9] [10] and its generalisations to functions definable in o-minimal structures [8] or to smooth functions on infinite dimensional Hilbert spaces [11] [7] [4] have proved to be major tools in the study of asymptotic behaviour of gradient and gradient-like systems. The Łojasiewicz-Simon inequality for smooth energy functions on infinite-dimensional Hilbert spaces has been applied in order to prove stabilisation of bounded solutions of many parabolic equations such as diffusion equations, Cahn-Hilliard type equations for describing phase separation phenomena, or geometric evolution equations, but also to hyperbolic equations such as damped wave equations; the literature being vast, we merely refer to the monographs by Haraux & Jendoubi [5], Huang [6] and the references therein.

In this article, we consider nonsmooth gradient systems in infinite dimensional Hilbert spaces, associated with semiconvex, lower semicontinuous energy functions and their subgradients. We show that the Kurdyka-Łojasiewicz-Simon
inequality may also be applied in this general setting in order to prove stabilisation
of solutions of gradient systems. The point of this article is, however, not only this
generalisation. Unlike in the situation of smooth energy functions, which are at
least continuously differentiable functions defined on (open subsets of) a Banach
space, a natural energy space is not present in the case of energy functions defined
on a Hilbert space and taking values in the extended real line. The role of energy
space is taken over by the effective domain which, however, carries in general no
linear structure.

This article starts with a small but useful observation. The effective domain
of a function $E$ on a metric space $M$ always carries a natural topology $\tau_E$, so that
$(\text{dom}\ E, \tau_E)$ is continuously embedded into $(M, d)$ and so that $E$ is continuous on
$(\text{dom}\ E, \tau_E)$. Actually, we take the coarsest topology with these two properties. We
show that in the case of the classical Dirichlet energy of the Neumann-Laplacian on
$L^2(\Omega)$, but also for semilinear perturbations of this energy, this natural topology
coincides with the norm topology on the Sobolev space $H^1(\Omega)$.

This small observation is used in the second part where we state and prove
the stabilisation result for global, bounded solutions of associated gradient sys-
tems. This result uses the Kurdyka-Lojasiewicz-Simon inequality, named after the
Kurdyka-Lojasiewicz inequality for functions definable in o-minimal structures and
after the Lojasiewicz-Simon inequality for functions defined on Hilbert spaces. A
new feature is that the usual assumption of relative compactness of the global
solution in the energy space (or in the effective domain equipped with the topol-
ogy mentioned above) can be considerably weakened to the assumption of relative
compactness of the solution in the ambient Hilbert space. In many applications
of the Lojasiewicz-Simon inequality the verification of the relative compactness of
the range of the solution in the energy space required a lot of efforts and advanced
techniques, while the relative compactness of the range of the solution in the ambi-
ent Hilbert space often follows from a standard application of Rellich-Kondrachov.
Our result thus seems to be of interest even in the case of smooth energies with
effective domains having a linear structure.

2. Topology and metric induced by the energy

Let $(M, d)$ be a metric space and let $E : M \to \mathbb{R} \cup \{+\infty\}$ be an energy function
with values in the extended real line. We suppose that $E$ is proper in the sense
that the effective domain $\text{dom}\ E := \{E < +\infty\}$ is nonempty. We equip $\text{dom}\ E$ with
a topology $\tau_E$, namely the coarsest topology for which the natural embedding
$\text{dom}\ E \to M$ and the mapping $E : \text{dom}\ E \to \mathbb{R}$ are continuous. A net $(u_\alpha)$ in
$\text{dom}\ E$ thus converges to $u \in \text{dom}\ E$ with respect to the topology $\tau_E$ if and only if
$\lim\alpha \ d(u_\alpha, u) = 0$ and $\lim\alpha \ E(u_\alpha) = E(u)$. As a consequence of the simple structure
of the topology $\tau_E$, we have the following lemma.
Lemma 2.1. The topology \( \tau_{E} \) is metrizable. For example, the topology \( \tau_{E} \) is induced by the metric \( d_{E} : \text{dom } E \times \text{dom } E \to \mathbb{R} \) given by

\[
d_{E}(u, v) := d(u, v) + |E(u) - E(v)| \quad (u, v \in \text{dom } E).
\]

Example 2.2. On the Hilbert space \( H = L^{2}(\Omega) \) (\( \Omega \subseteq \mathbb{R}^{N} \) open) we consider the function \( \mathcal{E}_{1} : L^{2}(\Omega) \to \mathbb{R} \cup \{+\infty\} \) given by \( \mathcal{E}_{1}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^{2} \) with effective domain \( \text{dom } \mathcal{E}_{1} = H^{1}(\Omega) \). A sequence \( (u_{n}) \) in \( H^{1}(\Omega) \) converges with respect to \( \tau_{\mathcal{E}_{1}} \) to some element \( u \in H^{1}(\Omega) \) if and only if \( \lim_{n} u_{n} = u \) in \( L^{2}(\Omega) \) and \( \lim_{n} \mathcal{E}_{1}(u_{n}) = \mathcal{E}_{1}(u) \) in \( \mathbb{R} \). As a consequence, if a sequence \( (u_{n}) \) converges to \( u \in H^{1}(\Omega) \) with respect to \( \tau_{\mathcal{E}_{1}} \), then necessarily \( (u_{n}) \) is bounded in \( (H^{1}(\Omega), \| \cdot \|_{H^{1}}) \). By reflexivity of \( H^{1}(\Omega) \) and by continuity of the embedding \( H^{1}(\Omega) \to L^{2}(\Omega) \), the sequence \( (u_{n}) \) thus converges weakly to \( u \in H^{1}(\Omega) \). However, the convergence in \( \tau_{\mathcal{E}_{1}} \) implies in addition that \( \lim_{n} \| u_{n} \|_{H^{1}} = \| u \|_{H^{1}} \), and hence \( (u_{n}) \) converges to \( u \) in the norm topology of \( H^{1}(\Omega) \). Obviously, the converse implication – saying that convergence in the norm topology implies convergence in \( \tau_{\mathcal{E}_{1}} \) – is true, too, and hence, using also Lemma 2.1, both topologies coincide.

Lemma 2.3. Let \((M, d)\) be a metric space. Let \( \mathcal{E}_{1}, \mathcal{E}_{2} : M \to \mathbb{R} \cup \{+\infty\} \) be two functions, and let \( \mathcal{E} := \mathcal{E}_{1} + \mathcal{E}_{2} \). Then:

(a) If \( \mathcal{E}_{2} \) is continuous with respect to the topology \( \tau_{\mathcal{E}_{1}} \), then \( \tau_{\mathcal{E}} \) is coarser than \( \tau_{\mathcal{E}_{1}} \).

(b) If \( \mathcal{E}_{2} \) is continuous with respect to the topology in \( M \), then \( \tau_{\mathcal{E}} = \tau_{\mathcal{E}_{1}} \).

Proof. (a) By assumption and by definition of \( \tau_{\mathcal{E}_{1}} \), both \( \mathcal{E}_{1} \) and \( \mathcal{E}_{2} \) are continuous with respect to the topology \( \tau_{\mathcal{E}_{1}} \), and hence \( \mathcal{E} \) is continuous with respect to this topology. By definition again, the topology \( \tau_{\mathcal{E}} \) must be coarser than the topology \( \tau_{\mathcal{E}_{1}} \).

(b) This follows by symmetry \( (\mathcal{E}_{1} = \mathcal{E} - \mathcal{E}_{2}) \) and by applying (a). \( \square \)

Example 2.4. On the Hilbert space \( H = L^{2}(\Omega) \) we consider the function \( \mathcal{E} \) given by

\[
\mathcal{E}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^{2} + \int_{\Omega} F(u) = \mathcal{E}_{1}(u) + \mathcal{E}_{2}(u),
\]

where \( \mathcal{E}_{1} \) is as in Example 2.2 and \( \mathcal{E}_{2}(u) = \int_{\Omega} F(u) \) for some function \( F \in C^{1}(\mathbb{R}) \) with globally Lipschitz continuous derivative \( F' \). The function \( \mathcal{E}_{2} \) is continuous with respect to the norm topology in \( L^{2}(\Omega) \). By Example 2.2 and Lemma 2.3 \( \tau_{\mathcal{E}} \) coincides with the norm topology in \( H^{1}(\Omega) \).

Example 2.5. More generally, if \( \mathcal{E} \) is a function on a Hilbert space \( H \), if the effective domain \( V := \text{dom } E \) is a subspace of \( H \), equipped with a seminorm \( \| \cdot \|_{V} \) such that \( \| H \|_{V} := \| \cdot \|_{V} + \| \cdot \|_{H} \) is a complete norm and \((V, \| \cdot \|_{V})\) is a dual Banach space, and if \( \mathcal{E} \) is a function of this seminorm (that is, \( \mathcal{E} = f \circ | \cdot |_{V} \) for some continuous \( f : \mathbb{R} \to \mathbb{R} \)), then \( \tau_{\mathcal{E}} \) is in general coarser than the norm topology of \( V \). For
example, consider the choice $H = L^2(\Omega)$, $V = L^2(\Omega) \cap BV(\Omega)$ and $E(u) = |u|_{TV}$ (the total variation seminorm).

**Example 2.6.** Let $E : M \to \mathbb{R} \cup \{+\infty\}$ be a function on a metric space $(M, d)$. Given a subset $C \subseteq M$, we define the characteristic function $1_C : M \to \mathbb{R} \cup \{+\infty\}$ by

$$1_C(u) := \begin{cases} 
0 & \text{if } u \in C, \\
+\infty & \text{else}, 
\end{cases}$$

and we let $E_C := E + 1_C$. Then $E_C$ is proper if $\text{dom } E_C = \text{dom } E \cap C \neq \emptyset$. The topology $\tau_{E_C}$ is the topology induced by $\tau_E$ on $\text{dom } E_C$. Indeed, the metrics $d_E$ and $d_{E_C}$ (compare with Lemma 2.1) coincide on $\text{dom } E_C$.

## 3. Stabilisation of global solutions of nonsmooth gradient systems

Let $H$ be a Hilbert space and let $E : H \to \mathbb{R} \cup \{+\infty\}$. We say that $E$ is semiconvex, if there exists $\omega \in \mathbb{R}$ such that $u \mapsto E(u) + \frac{\omega}{2} \|u\|_H^2$ is convex. The subgradient of $E$ is the relation

$$\partial E := \{(u, f) \in H \times H : u \in \text{dom } E \text{ and for every } v \in H \}
\liminf_{\lambda \to 0^+} \frac{E(u + \lambda v) - E(u)}{\lambda} \geq \langle f, v \rangle_H \}.$$ 

For semiconvex $E$ and $\omega \in \mathbb{R}$ large enough,

$$\partial E = \{(u, f) \in H \times H : u \in \text{dom } E \text{ and for every } v \in H \}
E(v) - E(u) + \frac{\omega}{2} \|v - u\|_H^2 \geq \langle f, v - u \rangle_H \}.$$ 

For every $u \in H$ we set $\partial E(u) := \{f \in H : (u, f) \in \partial E\}$, which is a closed and convex set. Furthermore, we define the slope $|\partial E(u)| := \inf \{\|f\|_H : f \in \partial E(u)\}$, with the convention $\inf \emptyset = \infty$. If $\partial E(u)$ is nonempty, then $|\partial E(u)| = \|P_{\partial E(u)}0\|_H$, where $P_{\partial E(u)}$ denotes the orthogonal projection onto $\partial E(u)$.

**Lemma 3.1.** Let $E : H \to \mathbb{R} \cup \{+\infty\}$ be proper, semiconvex and lower semicontinuous. Let $((u_n, f_n))$ be a sequence in $\partial E$ and $(u, f) \in H \times H$ such that

$$\lim_{n \to \infty} u_n = u \text{ and weak-}\lim_{n \to \infty} f_n = f.$$ 

Then

$$(u, f) \in \partial E \text{ and } \lim_{n \to \infty} E(u_n) = E(u).$$

**Proof.** By the characterisation of the subgradient of semiconvex functions, for some $\omega \in \mathbb{R}$ large enough, and for every $v \in H$ and every $n \in \mathbb{N}$,

$$E(v) \geq E(u_n) + \langle f_n, v - u_n \rangle - \frac{\omega}{2} \|v - u_n\|_H^2.$$ (3.1)
By taking the limit inferior on the right-hand side of this inequality, as \( n \to \infty \), and by using the lower semicontinuity of \( \mathcal{E} \),
\[
\mathcal{E}(v) \geq \mathcal{E}(u) + \langle f, v - u \rangle - \frac{\omega}{2} \| v - u \|_H^2 \quad \text{for every } v \in H.
\]
This inequality implies first (choose \( v \in \text{dom} \mathcal{E} \)) that \( u \in \text{dom} \mathcal{E} \), and second that \( (u, f) \in \partial \mathcal{E} \). Choosing now \( v = u \) in (3.1), and taking the limit superior on the right-hand side of that inequality, one obtains \( \mathcal{E}(u) = \lim_{n \to \infty} \mathcal{E}(u_n) \).

If \( \mathcal{E} \) is a proper, semiconvex, lower semicontinuous function on \( H \), then the gradient system
\[
\dot{u} + \partial \mathcal{E}(u) \ni f
\]
(3.2)
adopts for every \( u_0 \in \text{dom} \mathcal{E} \) and every \( f \in L^2(\mathbb{R}_+; H) \) a unique strong solution \( u \in H^1_{\text{loc}}(\mathbb{R}_+; H) \) satisfying the initial condition \( u(0) = u_0 \) [3, Théorème 3.6, p.72], [2, Theorem 4.11]. Strong solution means that \( \langle u(t), f(t) - \dot{u}(t) \rangle \in \partial \mathcal{E} \) for almost every \( t \in \mathbb{R}_+ \). For every strong solution \( u \) the composition \( \mathcal{E}(u) \) is absolutely continuous, and for almost every \( t \in \mathbb{R}_+ \) the energy equality
\[
\frac{d}{dt} \mathcal{E}(u) = -\frac{1}{2} \| \dot{u} \|_H^2 - \frac{1}{2} P_{\partial \mathcal{E}(u)} f_\|_H^2 + \frac{1}{2} \| f \|_H^2
\]
(3.3)
holds (use [2, Lemma 4.4] or compare with [3, Théorème 3.6, p.72], [1, Theorem 2.3.3]). In particular, the function \( \mathcal{H} : \mathbb{R}_+ \to \mathbb{R}_+ \), defined by
\[
\mathcal{H}(t) = \mathcal{E}(u(t)) + \frac{1}{2} \int_t^\infty \| f(s) \|_H^2 \, ds,
\]
(3.4)
is absolutely continuous and decreasing.

**Lemma 3.2.** Let \( \mathcal{E} : H \to \mathbb{R} \cup \{+\infty\} \) be proper, semiconvex and lower semicontinuous and \( f \in L^2(\mathbb{R}_+; H) \). Let \( u \in H^1_{\text{loc}}(\mathbb{R}_+; H) \) be a solution of the gradient system (3.2), and consider its \( \omega \)-limit set
\[
\omega(u) := \{ \varphi \in H : \exists (t_n) \nearrow \infty \text{ s.t. } \lim_{n \to \infty} u(t_n) = \varphi \text{ in } H \}.
\]

Then:
(a) For every \( \varphi \in \omega(u) \) one has \( \lim_{t \to \infty} \mathcal{E}(u(t)) = \mathcal{E}(\varphi) \).
(b) The function \( \mathcal{E} \) is constant on \( \omega(u) \).
(c) One has
\[
\omega(u) = \{ \varphi \in H : \exists (t_n) \nearrow \infty \text{ s.t. } \lim_{n \to \infty} u(t_n) = \varphi \text{ w.r.t. } \tau_E \}.
\]

**Proof.** Let \( \varphi \in \omega(u) \), and let \( (t_n) \) be a sequence in \( \mathbb{R}_+ \) such that \( \lim_{n \to \infty} t_n = \infty \) and \( \lim_{n \to \infty} u(t_n) = \varphi \) in \( H \). Let \( \mathcal{H} \) be the function defined in (3.3). By lower semicontinuity of \( \mathcal{E} \),
\[
\liminf_{n \to \infty} \mathcal{H}(t_n) = \liminf_{n \to \infty} \mathcal{E}(u(t_n)) \geq \mathcal{E}(\varphi),
\]
so that \( \mathcal{H} \) is bounded from below. Since \( \mathcal{H} \) is also decreasing,
\[
\lim_{t \to \infty} \mathcal{H}(t) = \lim_{t \to \infty} \mathcal{E}(u(t)) \exists.
\]
(3.5)
Moreover, by the energy equality, \( \dot{u} \in L^2(\mathbb{R}_+; H) \). From here we deduce, for every \( s \in [0, 1] \),
\[
\limsup_{n \to \infty} \|u(t_n + s) - \varphi\|_H \leq \limsup_{n \to \infty} (\|u(t_n + s) - u(t_n)\|_H + \|u(t_n) - \varphi\|_H)
\leq \limsup_{n \to \infty} \left( \int_0^1 \|\dot{u}(t_n + r)\|_H \, dr + \|u(t_n) - \varphi\|_H \right) = 0.
\]
Since \( \dot{u}, f \in L^2(\mathbb{R}_+; H) \) and \((u(t), f(t) - \dot{u}(t)) \in \partial \mathcal{E}\) for almost every \( t \), we thus find a sequence \((s_n) \in [0, 1]\) (depending on the representatives of the measurable functions \( f \) and \( \dot{u} \)) such that \((u(t_n + s_n), f(t_n + s_n) - \dot{u}(t_n + s_n)) \in \partial \mathcal{E}\),
\[
\lim_{n \to \infty} u(t_n + s_n) = \varphi \quad \text{and} \quad \lim_{n \to \infty} (f(t_n + s_n) - \dot{u}(t_n + s_n)) = 0.
\]
By Lemma 3.4, this implies \((\varphi, 0) \in \partial \mathcal{E}\) and
\[
\lim_{n \to \infty} \mathcal{E}(u(t_n + s_n)) = \mathcal{E}(\varphi).
\]
From here and the convergence of \( \mathcal{E}(u) \) (see 3.3) follows (a). Assertions (b) and (c) are direct consequences of (a). \( \square \)

We say that a function \( \mathcal{E} : H \to \mathbb{R} \cup \{+\infty\} \) satisfies the Kurdyka-Łojasiewicz-Simon inequality on a set \( U \subseteq H \) if there exists a strictly increasing \( \Theta \in W^{1,1}_{loc}(\mathbb{R}) \) such that \( |\partial(\Theta \circ \mathcal{E})(v)| \geq 1 \) for every \( v \in U \) with \( 0 \not\in \partial \mathcal{E}(v) \).

**Theorem 3.3.** Let \( H \) be a Hilbert space and let \( \mathcal{E} : H \to \mathbb{R} \cup \{+\infty\} \) be proper, semicontinuous and lower semicontinuous. Let \( u \in H^1_{loc}(\mathbb{R}_+; H) \) be a global strong solution of the gradient system (3.2) with \( f = 0 \). Assume that there exists \( \varphi \in \omega(u) \) such that \( \mathcal{E} \) satisfies the Kurdyka-Łojasiewicz-Simon inequality in a \( \tau_\mathcal{E} \)-neighbourhood of \( \varphi \). Then \( u \) has finite length in \( H \) and \( \lim_{t \to \infty} u(t) = \varphi \) in \( \tau_\mathcal{E} \).

For the proof of Theorem 3.3 we need the following chain rule.

**Lemma 3.4.** Let \( \mathcal{E} : H \to \mathbb{R} \cup \{+\infty\} \) be proper, \( u \in \text{dom} \mathcal{E} \) and let \( \Theta : \mathbb{R} \to \mathbb{R} \) be continuous, strictly increasing and differentiable at \( \mathcal{E}(u) \). Then \( \Theta'(\mathcal{E}(u)) \partial \mathcal{E}(u) \subseteq \partial(\Theta \circ \mathcal{E})(u) \). Moreover, if \( \Theta'(\mathcal{E}(u)) \neq 0 \), then \( \Theta'(\mathcal{E}(u)) \partial \mathcal{E}(u) = \partial(\Theta \circ \mathcal{E})(u) \).

**Proof.** Let \( f \in \partial \mathcal{E}(u) \) and \( v \in H \). Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that
\[
\inf_{\lambda \in (0, \delta)} \frac{\mathcal{E}(u + \lambda v) - \mathcal{E}(u)}{\lambda} \geq (f, v) - \varepsilon,
\]
that is,
\[
\mathcal{E}(u + \lambda v) \geq \mathcal{E}(u) + \lambda ((f, v) - \varepsilon) \quad \text{for every} \quad \lambda \in (0, \delta).
\]
Due to the monotonicity of \( \Theta \), we obtain
\[
\frac{(\Theta \circ \mathcal{E})(u + \lambda v) - (\Theta \circ \mathcal{E})(u)}{\lambda} \geq \Theta(\mathcal{E}(u) + \lambda ((f, v) - \varepsilon)) - \Theta(\mathcal{E}(u)) \rightarrow \Theta'(\mathcal{E}(u)) ((f, v) - \varepsilon) \quad \text{as} \quad \lambda \to 0+.
\]
Therefore, since this inequality holds for every $\varepsilon > 0$,
\[
\liminf_{\lambda \to 0^+} \frac{(\Theta \circ \mathcal{E})(u + \lambda v) - (\Theta \circ \mathcal{E})(u)}{\lambda} \geq \langle \Theta'(\mathcal{E}(u)) f, v \rangle.
\]

As a consequence $\Theta'(\mathcal{E}(u)) f \in \partial(\Theta \circ \mathcal{E})(u)$.

If $\Theta'(\mathcal{E}(u)) \neq 0$, we may repeat the argument above with the inverse function $\Theta^{-1}$, which is continuous, strictly increasing, and differentiable at $(\Theta \circ \mathcal{E})(u)$, and we obtain the converse inclusion. □

**Proof of Theorem 3.3.** Let $\varphi$ be as in the assumption, and let $U$ be a $\tau_\mathcal{E}$-neighbourhood of $\varphi$ such that $\mathcal{E}$ satisfies the Kurdyka-Lojasiewicz-Simon inequality in $U$. This means that there exists a strictly increasing $\Theta \in W^{1,1}_{loc}(\mathbb{R})$ such that $|\partial(\Theta \circ \mathcal{E})(v)| \geq 1$ for every $v \in U$ with $0 \notin \partial\mathcal{E}(v)$.

Since the energy is decreasing along the solution $u$, $\mathcal{E}(u(t)) \geq \mathcal{E}(\varphi)$ for every $t \in \mathbb{R}_+$. If $\mathcal{E}(u(t)) = \mathcal{E}(\varphi)$ for some $t \in \mathbb{R}_+$, then the energy is eventually constant along $u$, which implies that $u$ is eventually constant. In this case, there remains nothing to prove.

Hence, we may assume that $\mathcal{E}(u(t)) > \mathcal{E}(\varphi)$ for every $t \in \mathbb{R}_+$. In this case, $\mathcal{E}(u)$ is strictly decreasing, and $\dot{u}(t) \neq 0$ for almost every $t \in \mathbb{R}_+$. By assumption and Lemma 3.2(c), there exists a sequence $(t_n)$ in $\mathbb{R}_+$ such that $\lim_{n \to \infty} t_n = \infty$ and $\lim_{n \to \infty} u(t_n) = \varphi$ in $\tau_\mathcal{E}$. Without loss of generality, we may assume that $u(t_n) \in U$ for every $n$. For every $n$ we set
\[s_n := \sup\{s \in [t_n, \infty) : u(t) \in U \text{ for every } t \in [t_n, s]\}.
\]

Since $u$ is continuous with values in $(\text{dom } \mathcal{E}, \tau_\mathcal{E})$ and since $U$ is open in this space, $s_n > t_n$. For almost every $t \in [t_n, s_n)$, by the chain rule, the energy equality and Lemma 3.4
\[
-\frac{d}{dt}(\Theta \circ \mathcal{E})(u(t)) = -\Theta'(\mathcal{E}(u(t))) \frac{d}{dt}\mathcal{E}(u(t))
\]
\[= \frac{1}{2} \Theta'(\mathcal{E}(u(t))) (\|\dot{u}(t)\|^2_H + |\partial\mathcal{E}(u(t))|^2)
\]
\[\geq \Theta'(\mathcal{E}(u(t))) \|\dot{u}(t)\|_H |\partial\mathcal{E}(u(t))|
\]
\[\geq \|\dot{u}(t)\|_H |\partial(\Theta \circ \mathcal{E})(u(t))|
\]
\[\geq \|\dot{u}(t)\|_H. \tag{3.6}
\]

Integrating both sides, we obtain
\[
\|u(t) - u(t_n)\|_H \leq \int_{t_n}^t \|\dot{u}(s)\|_H \, ds
\]
\[\leq (\Theta \circ \mathcal{E})(u(t_n)) - (\Theta \circ \mathcal{E})(u(t))
\]
\[\leq (\Theta \circ \mathcal{E})(u(t_n)) - (\Theta \circ \mathcal{E})(\varphi).
\]
Assume now that all $s_n$ are finite. Then, by continuity, the preceding inequality remains true for $t$ replaced by $s_n$, and thus
\[
\|u(s_n) - \varphi\|_H \leq \|u(s_n) - u(t_n)\|_H + \|u(t_n) - \varphi\|_H \\
\leq (\Theta \circ \mathcal{E})(u(t_n)) - (\Theta \circ \mathcal{E})(\varphi) + \|u(t_n) - \varphi\|_H.
\]
The convergence of $(u(t_n))$ to $\varphi$ in $\tau_E$ and the continuity of $\Theta$ then imply that the right-hand side of this inequality converges to 0 as $n \to \infty$. As a consequence,
\[
\lim_{n \to \infty} u(s_n) = \varphi \text{ in the norm topology of } H.
\]

This and Lemma 3.2 (a) yield
\[
\lim_{n \to \infty} u(s_n) = \varphi \text{ in the topology } \tau_E,
\]
which is, however, a contradiction since $u(s_n) \not\in U$ for every $n$. Hence, the assumption that all $s_n$ are finite was false. There thus exists $n$ such that $s_n = \infty$. In this case, the estimate (3.6) implies $\dot{u} \in L^1([t_n, \infty); H)$, so that $u$ has finite length in $H$. By Cauchy’s criterion, combined with Lemma 3.2 (a), we deduce $\lim_{t \to \infty} u(t) = \varphi$ in $\tau_E$.

Remark 3.5. We emphasize that the $\omega$-limit set of the solution $u$ in Theorem 3.3 is taken with respect to the norm topology in the ambient Hilbert space $H$. A condition for the nonemptiness of the $\omega$-limit is the condition that the range of $u$ is relatively compact in the norm topology of $H$. In many applications, this follows from mere boundedness of the solution in $H$, from the boundedness of the energy along $u$, and from standard compact embedding theorems.

Many articles on applications of the Lojasiewicz-Simon inequality in the context of smooth gradient systems required in addition nonemptiness of the $\omega$-limit set in a finer topology. In the context of Example 2.4, this would be the norm topology of the Sobolev space $H^1(\Omega)$. This was usually verified by showing that the solution has relatively compact range in the underlying energy space, sometimes with considerable effort. Note that in Example 2.4, the norm topology in $H^1(\Omega)$ and the topology $\tau_E$ coincide. Moreover, by Lemma 3.2 (c), the $\omega$-limit set with respect to the norm topology in $H$ and the $\omega$-limit set with respect to the topology $\tau_E$ coincide.

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Ralph Chill
Technische Universität Dresden, Institut für Analysis, 01062 Dresden, Germany
e-mail: ralph.chill@tu-dresden.de

Sebastian Mildner
Technische Universität Dresden, Institut für Analysis, 01062 Dresden, Germany
e-mail: sebastian.mildner@tu-dresden.de