ATTRACTORS OF HOPFIELD-TYPE LATTICE MODELS WITH INCREASING NEURONAL INPUT

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Dedicated to Juan J. Nieto on the occasion of his 60th birthday

Abstract. Two Hopfield-type neural lattice models are considered, one with local \( n \)-neighborhood nonlinear interconnections among neurons and the other with global nonlinear interconnections among neurons. It is shown that both systems possess global attractors on a weighted space of bi-infinite sequences. Moreover, the attractors are shown to depend upper semi-continuously on the interconnection parameters as \( n \to \infty \).

1. Introduction. The Hopfield model \([22, 23, 28]\) is one of the most widely used mathematical models for artificial neural networks. The basic Hopfield neural network model consists of a finite system of ordinary differential equations

\[
\mu_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=1}^{n} \lambda_{i,j} f_j(u_j(t)) + g_i, \quad i = 1, \ldots, n, \tag{1}
\]

where \( u_i \) represents the voltage on the input of the \( i \)th neuron at time \( t \), \( \mu_i > 0 \) and \( \gamma_i > 0 \) represent the neuron amplifier input capacitance and resistance of the \( i \)th neuron, respectively, and \( g_i \) is the external forcing on the \( i \)th neuron. Here \( f_j \) is the neuron activation function and usually assumed to be a sigmoid type function. The parameter \( \lambda_{i,j} \) denotes the connection strength between the \( i \)th and the \( j \)th neuron and the term \( \lambda_{i,j} f_j(u_j(t)) \) represents the electric current input to the \( i \)th neuron due to the present potential of the \( j \)th neuron.

Hopfield-type neural networks have been studied extensively during the past decades (see, e.g., \([1, 6, 13, 17, 24, 25, 29, 30, 37]\)). Very recently Han, Usman & Kloeden studied Hopfield-type lattice dynamical system \([20]\):

\[
\mu_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=i-n}^{i+n} \lambda_{i,j}^{(n)} f_j(u_j(t)) + g_i, \quad i \in \mathbb{Z}, \tag{2}
\]
where each neuron is influenced by its surrounding $2n$ neurons on either side. Amongst other things it was shown that, for each fixed $n$, system (2) possesses a global attractor $A_n$ in a sequence space $\ell^2$ under appropriate assumptions.

During the past decades, lattice dynamical systems have gained considerable attention due to their wide range of applications in the applied sciences (see, e.g., [5, 9, 11, 12, 26]) as well as their interesting mathematical properties (see, e.g., [4, 7, 8, 10, 15, 32, 33, 34, 35, 38, 39, 40] and references therein). The aim of this work is to investigate what happens as the number $n$ in (2) increases without bound leading to the lattice system

$$
\mu_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{\gamma_i} + \sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j(t)) + g_i, \quad i \in \mathbb{Z}.
$$

(3)

In particular we want to show that system (3) has an attractor $A$ and that the attractors $A_n$ of (2) converge upper semi-continuously to $A$ as $n \to \infty$ where $\lambda_{i,j}^{(n)} \to \lambda_{i,j}$ as $n \to \infty$. The novelty of this work lies in the nonlinear global interconnection structure described by the term $\sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j(t))$. There have been very few studies on lattice dynamical systems with global iterations to date [3, 19, 31], and this work provides the first result on upper semi-continuity of attractors dependent on interconnection parameters in lattice neural models.

The paper is organized as follows. In Section 2 we first introduce a weighted Hilbert space of bi-infinite sequences, where systems (2) and (3) will be considered, then we study the existence and uniqueness of solutions of each system. In Section 3 we construct an absorbing set for each system (2) and (3), establish the asymptotic compactness of solutions, and prove the existence of global attractor for system (2) and (3), respectively. In Section 4 we investigate the upper semicontinuity of the global attractor.

2. Basic properties of solutions. In [20] system (2) was studied in the usual Hilbert space of bi-infinite sequences $\ell^2$. However, due to the global interconnection term $\sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j(t))$ system (3) is not well-posed in $\ell^2$ except under very restrictive assumptions. Thus in this work we consider the more inclusive weighted Hilbert space of bi-infinite real valued sequences with positive weights $\rho_i$ (see, e.g., [19, 21, 36]) defined by

$$
\ell^2_\rho := \left\{ u = (u_i)_{i \in \mathbb{Z}} : \sum_{i \in \mathbb{Z}} \rho_i u_i^2 < \infty \right\}
$$

equipped with the inner product and the corresponding norm

$$
(u, v) := \sum_{i \in \mathbb{Z}} \rho_i u_i v_i, \quad \|u\|_\rho := \sqrt{\sum_{i \in \mathbb{Z}} \rho_i u_i^2} \quad \text{for} \quad u = (u_i)_{i \in \mathbb{Z}}, \quad v = (v_i)_{i \in \mathbb{Z}} \in \ell^2_\rho.
$$

2.1. Standing assumptions. Throughout this paper the weights are assumed to satisfy

(A0) $\rho_i > 0$ for $i \in \mathbb{Z}$ and $\rho_k := \sum_{i \in \mathbb{Z}} \rho_i < \infty$.

Then it is straightforward to check that $\ell^2_\rho$ is a separable Hilbert space and $\ell^2_\rho \supset \ell^2$. In addition, the model parameters are assumed to satisfy
(A1) the neuron amplifier input capacitance and resistance of neurons are uniformly bounded, i.e., there exist positive constants \( m_\mu, M_\mu, m_\gamma, \) and \( M_\gamma, \) such that
\[
    m_\mu \leq \mu_i \leq M_\mu, \quad m_\gamma \leq \gamma_i \leq M_\gamma, \quad \forall \ i \in \mathbb{Z};
\]

(A2) the reciprocal-weighted aggregate efficacy on each neuron is finite, in the sense that there exists \( M_\lambda > 0 \) such that
\[
    \sup_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}^2}{\rho_j^2} \leq M_\lambda, \quad \text{and} \quad \sup_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left( \frac{\lambda_{i,j}^{(n)}}{\rho_j^2} \right)^2 \leq M_\lambda \text{ for all } n;
\]

and the forcing terms are assumed to satisfy

(A3) \( f_i(0) = 0 \) for all \( i \in \mathbb{Z}; \)

(A4) the functions \( f_i : \mathbb{R} \to \mathbb{R} \) are continuously differentiable with equi-bounded derivatives, i.e., there exist positive constants \( L_i \) with \( L := (L_i)_{i \in \mathbb{Z}} \in l^2_\rho \) and such that
\[
    |f_i'(s)| \leq L_i, \quad \forall \ s \in \mathbb{R}, \ i \in \mathbb{Z};
\]

(A5) \( g := (g_i)_{i \in \mathbb{Z}} \in l^2_\rho. \)

Note that assumptions (A3) and (A4) together imply that \( f_i \) is locally Lipschitz:
\[
    |f_i(s) - f_i(r)| \leq L_i |s - r|, \quad \forall \ s, r \in \mathbb{R}, \ i \in \mathbb{Z}, \quad (4)
\]
and satisfies the growth bound
\[
    |f_i(s)| \leq L_i |s|, \quad \forall \ s \in \mathbb{R}, \ i \in \mathbb{Z}. \quad (5)
\]

2.2. Reformulation on \( l^2_\rho. \) Given any \( u = (u_i)_{i \in \mathbb{Z}} \in l^2_\rho, \) define the operators \( Au = ((Au)_i)_{i \in \mathbb{Z}}, \) \( \Lambda u = ((\Lambda u)_i)_{i \in \mathbb{Z}} \) and \( \Lambda^{(n)} u = ((\Lambda^{(n)} u)_i)_{i \in \mathbb{Z}} \) componentwise by
\[
    (Au)_i = \frac{u_i}{\mu_i}, \quad (\Lambda u)_i = \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} f_j(u_j), \quad (\Lambda^{(n)} u)_i = \sum_{j = i-n}^{i+n} \frac{\lambda_{i,j}^{(n)}}{\mu_i} f_j(u_j), \quad i \in \mathbb{Z}.
\]

Lemma 2.1. Assume that assumptions (A0) – (A4) hold. Then the operators \( A, \Lambda, \Lambda^{(n)} : l^2_\rho \to l^2_\rho. \)

Proof. First by Assumption (A1),
\[
    \sum_{i \in \mathbb{Z}} \rho_i (Au_i^2) = \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} \rho_i u_i^2 \leq \frac{1}{m_\mu} \sum_{i \in \mathbb{Z}} \rho_i u_i^2 = \frac{1}{m_\mu} ||u||_\rho^2,
\]
which implies that \( Au \in l^2_\rho \) for every \( u \in l^2_\rho. \)

By assumptions (A1), the growth bound (5), Cauchy’s inequality \((\sum_i a_i b_i)^2 \leq \sum_i a_i^2 \sum_i b_i^2\) and the inequality \( \sum_i |a_i b_i| \leq \sum_i |a_i| \sum_i |b_i|, \) for every \( i \in \mathbb{Z} \)
\[
    \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} f_j(u_j) \right)^2 \leq \frac{1}{m_\mu} \left( \sum_{j \in \mathbb{Z}} |\lambda_{i,j} L_j u_j| \right)^2 \leq \frac{1}{m_\mu} \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}^2}{\rho_j^2} \right) \left( \sum_{j \in \mathbb{Z}} \rho_j L_j^2 \right) \left( \sum_{j \in \mathbb{Z}} \rho_j u_j^2 \right).
\]
Then by assumptions (A0), (A2) and (A4) we have
\[
    \sum_{i \in \mathbb{Z}} \rho_i (\Lambda u_i^2) = \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} f_j(u_j) \right)^2 \leq \frac{\rho_\gamma}{m_\mu} M_\lambda ||L||_\rho^2 ||u||_\rho^2,
\]
i.e., \( \Lambda u \in l^2_\rho \) for every \( u \in l^2_\rho. \) Similarly, \( \Lambda^{(n)} u \in l^2_\rho \) for every \( u \in l^2_\rho. \) □
Lemma 2.2. Let assumptions (A0) – (A4) hold. Then $\Lambda$ and $\Lambda^{(n)}$ are globally Lipschitz with

$$\|\Lambda u - \Lambda v\|_\rho \leq \frac{\sqrt{\rho_c M_{\Lambda}^g}}{m_\mu} \|u - v\|_\rho, \quad \|\Lambda^{(n)} u - \Lambda^{(n)} v\|_\rho \leq \frac{\sqrt{\rho_c M_{\Lambda}^g}}{m_\mu} \|u - v\|_\rho.$$  

Proof. For each $i \in \mathbb{Z}$ and any $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2_\rho$, $v = (v_i)_{i \in \mathbb{Z}} \in \ell^2_\rho$, following calculations similar to Lemma 2.1 we can get

$$\left| (\Lambda u)_i - (\Lambda v)_i \right|^2 \leq \frac{1}{m_\mu^2} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_{i,j} \right| \left| f_j(u_j) - f_j(v_j) \right| \right)^2 \leq \frac{1}{m_\mu^2} \left( \sum_{j \in \mathbb{Z}} \lambda_{i,j}^2 \right) \left( \sum_{j \in \mathbb{Z}} \rho_j L_j^2 \right) \left( \sum_{j \in \mathbb{Z}} \rho_j |u_j - v_j|^2 \right) \leq \frac{1}{m_\mu^2} M_L \|L\|_\rho^2 \|u - v\|_\rho^2,$$

and hence

$$\|\Lambda u - \Lambda v\|_\rho \leq \frac{\sqrt{\rho_c M_{\Lambda}^g}}{m_\mu} \|u - v\|_\rho.$$  

The above estimates also hold uniformly in $n$ for the finite sum operator $\Lambda^{(n)}$. \qed

Notice that due to assumptions (A1) and (A5), $\left( g_i / \mu_i \right)_{i \in \mathbb{Z}} \in \ell^2_\rho$ and the lattice systems (2) and (3) can be rewritten as the following ODEs on $\ell^2_\rho$, respectively:

$$\frac{d u(t)}{dt} = -Au + \Lambda^{(n)} u + \left( \frac{g_i}{\mu_i} \right)_{i \in \mathbb{Z}} := F^{(n)}(u), \quad u \in \ell^2_\rho, \quad (6)$$

$$\frac{d u(t)}{dt} = -Au + \Lambda u + \left( \frac{g_i}{\mu_i} \right)_{i \in \mathbb{Z}} := F(u), \quad u \in \ell^2_\rho. \quad (7)$$

2.3. Existence and uniqueness of solutions. It follows directly from Lemma 2.1 and Lemma 2.2 that $F^{(n)}(u)$ in (6) and $F(u)$ in (7) both map $\ell^2_\rho$ into $\ell^2_\rho$. Also notice that due to the linearity of the operator $A$, we have

$$\|Au - Av\|_\rho \leq \frac{1}{m_\gamma m_\mu} \|u - v\|_\rho, \quad \forall u, v \in \ell^2_\rho.$$  

Thus

$$\|F^{(n)}(u) - F^{(n)}(v)\|_\rho \leq \left( \frac{1}{m_\gamma m_\mu} + \frac{\sqrt{\rho_c M_{\Lambda}^g}}{m_\mu} \right) \|u - v\|_\rho,$$

$$\|F(u) - F(v)\|_\rho \leq \left( \frac{1}{m_\gamma m_\mu} + \frac{\sqrt{\rho_c M_{\Lambda}^g}}{m_\mu} \right) \|u - v\|_\rho,$$

i.e., $F^{(n)}$ and $F$ are globally equi-Lipschitz on $\ell^2_\rho$ with the Lipschitz constant

$$M_L := \frac{1}{m_\gamma m_\mu} + \frac{\sqrt{\rho_c M_{\Lambda}^g}}{m_\mu} \|L\|_\rho.$$  

Standard existence and uniqueness theorems for ODEs in Banach spaces (see, e.g., [14]) then give the global existence and uniqueness of solutions of equations (6) and (7) given an initial condition $u(0) = u_o \in \ell^2_\rho$. Moreover, the solutions of each system depend continuously on initial data. In fact, given any $u_o, v_o \in \ell^2_\rho$, suppose
that \( u \) and \( v \) are the solutions of of equation (7) with the initial conditions \( u_o \) and \( v_o \). Set \( w = u - v \), then \( w \) satisfies the following equation:

\[
\frac{dw(t)}{dt} = -Aw + \Lambda u - \Lambda v, \quad w \in \ell^2_{\rho}.
\]  

(9)

Multiple both sides of the \( i \)th of the equation (9) by \( \rho_i \) and summing over \( i \in \mathbb{Z} \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{\rho}^2 = -\sum_{i \in \mathbb{Z}} \rho_i w_i^2 + \sum_{i \in \mathbb{Z}} \rho_i w_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} (f_j(u_j) - f_j(v_j)) \right)
\]

\[
\leq -\sum_{i \in \mathbb{Z}} \rho_i w_i^2 + \sum_{i \in \mathbb{Z}} \rho_i w_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} L_j |w_j| \right)
\]

\[
\leq -\frac{1}{M_{\mu} M_{\gamma}} \|w\|_{\rho}^2 + \left( \sum_{i \in \mathbb{Z}} \rho_i w_i^2 \right)^{1/2} \left( \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} L_j |w_j| \right)^2 \right)^{1/2}
\]

\[
\leq -\frac{1}{M_{\mu} M_{\gamma}} \|w\|_{\rho}^2 + \left( \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}^2}{\mu_i} \right) \left( \sum_{j \in \mathbb{Z}} \rho_j L_j^2 \right) \left( \sum_{j \in \mathbb{Z}} \rho_j w_j^2 \right) \right)^{1/2}
\]

\[
\leq -\frac{1}{M_{\mu} M_{\gamma}} \|w\|_{\rho}^2 + \|w\|_{\rho} \left( \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} \right) \left( \sum_{j \in \mathbb{Z}} \rho_j L_j^2 \right) \left( \sum_{j \in \mathbb{Z}} \rho_j w_j^2 \right) \right)^{1/2}
\]

\[
\leq \left( -\frac{1}{M_{\mu} M_{\gamma}} \|L\|_{\rho} \right) \|w\|_{\rho}^2
\]

It immediately follows

\[
\|w\|_{\rho}^2 \leq e^{2(t - \frac{1}{M_{\mu} M_{\gamma}}) + \sum_{\|L\|_{\rho}} \|w\|_{\rho}^2}
\]

which implies that the solutions of (7) depend continuously on initial data. Similarly we have the same conclusion for (6).

Therefore the solution of system (6) generates a semi-group \( \{ \varphi(t) \}_{t \geq 0} \) that maps \( \ell^2_{\rho} \) to \( \ell^2_{\rho} \), defined by \( \varphi(t)u_o = u(t; u_o) \) where \( u(t; u_o) \) is the solution of (6) satisfying \( u(0) = u_o \in \ell^2_{\rho} \). Similarly the solution of system (7) generates a semi-group \( \{ \varphi(t) \}_{t \geq 0} \) that maps \( \ell^2_{\rho} \) to \( \ell^2_{\rho} \), defined by \( \varphi(t)u_o = u(t; u_o) \) where \( u(t; u_o) \) is the solution of (7) satisfying \( u(0) = u_o \in \ell^2_{\rho} \).

3. Existence of attractors. In this section we establish existence of attractors for the semi-groups \( \{ \varphi(t) \}_{t \geq 0} \) and \( \{ \varphi(t) \}_{t \geq 0} \), respectively. For the reader’s convenience we recall the following well-known result (see, e.g., \cite{2, 16, 27}).

Proposition 1. Let \( \mathcal{X} \) be a metric space and \( \{ \varphi(t) \}_{t \geq 0} \) be a semi-group of continuous operators in \( \mathcal{X} \). If \( \{ \varphi(t) \}_{t \geq 0} \) has a bounded absorbing set and is asymptotically compact, then \( \{ \varphi(t) \}_{t \geq 0} \) has a global attractor in \( \mathcal{X} \). Moreover, if \( \varphi(t)x \) is continuous from \( (0, \infty) \) to \( \mathcal{X} \) for every \( x \in \mathcal{X} \), then the global attractor is connected.

Note that additional conditions are needed to ensure the dissipativity of systems (2) and (3), and there are multiple ways to achieve this. In particular, an upper bound for \( \|L\|_{\rho} \) can be imposed to make sure that the growth of \( \Lambda u \) would not overcome the dissipativity coming from the linear term \( -\frac{\lambda}{\mu_\gamma} \). This will essentially put a restriction on the growth rates of \( f_i \)’s depending on the magnitudes of \( \mu_i \)’s.
and $\gamma_i$’s. Alternatively, an upper bound could be imposed for the magnitude of $f_i$’s themselves, then the maximum growth rates of $f_i$’s do not need to depend on the magnitudes of $\mu_i$’s and $\gamma_i$’s. Another possibility is to impose dissipativity conditions on the functions $f_i$’s, instead of assuming bounds on $f_i$’s or $f_i'$s. Since in the context of Hopfield models, $f_i$’s are usually sigmoidal functions, therefore throughout the rest of this paper we assume that $f_i$’s are bounded in the sense that (A6) there exists $B := (B_i)_{i \in \mathbb{Z}} \in \ell^2_\rho$ such that

$$|f_i(s)| \leq |B_i|, \quad \forall \ i \in \mathbb{Z}, \ s \in \mathbb{R}.$$  

3.1. Existence of absorbing sets. In this subsection we construct the absorbing set for the semi-groups $\{\varphi^{(n)}(t)\}_{t \geq 0}$ and $\{\varphi(t)\}_{t \geq 0}$, respectively.

**Lemma 3.1.** Assume (A0) – (A6) hold. Then the semi-group $\{\varphi(t)\}_{t \geq 0}$ generated by equation (3) has a bounded absorbing set in $\ell^2_\rho$

$$K := \left\{ u \in \ell^2_\rho : \|u\|_\rho \leq R_K := \frac{\sqrt{2M_\mu M_\gamma}}{m_\mu} \sqrt{\rho_\Sigma^2 M_\lambda^2 \|B\|_\rho^2 + \|g\|_\rho^2 + 1} \right\}.$$ (10)

**Proof.** Multiple both sides of the equation (3) by $\rho_i u_i(t)$ and summing over $i \in \mathbb{Z}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_\rho^2 = -\sum_{i \in \mathbb{Z}} \rho_i u_i^2 + \sum_{i \in \mathbb{Z}} \frac{\rho_i u_i}{\mu_i} \sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j) + \sum_{i \in \mathbb{Z}} \rho_i u_i g_i.$$ (11)

First by Assumption (A1), we have

$$-\sum_{i \in \mathbb{Z}} \rho_i u_i^2 \leq -\|u\|_\rho^2 \frac{\rho_\Sigma M_\mu M_\gamma}{M_\mu M_\gamma}.$$ (12)

Then notice that by assumptions (A0), (A2) and (A6)

$$\sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j) \right)^2 \leq \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}^2}{\rho_j} \right) \left( \sum_{j \in \mathbb{Z}} \rho_j \right) \left( \sum_{j \in \mathbb{Z}} \rho_j B_j^2 \right) \leq \rho_\Sigma^2 M_\lambda^2 \|B\|_\rho^2,$$ (13)

Thus by Cauchy’s inequality and Assumption (A1) we get

$$\sum_{i \in \mathbb{Z}} \frac{\rho_i u_i}{\mu_i} \sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j) \leq \frac{1}{m_\mu} \left( \sum_{i \in \mathbb{Z}} \rho_i u_i^2 \right)^{1/2} \left( \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(u_j) \right)^2 \right)^{1/2}$$

$$\leq \frac{\rho_\Sigma \sqrt{M_\lambda \|B\|_\rho}}{m_\mu} \|u\|_\rho \leq \frac{a}{2} \|u\|_\rho^2 + \frac{1}{2a} \frac{\rho_\Sigma^2 M_\lambda \|B\|_\rho^2}{m_\mu^2}$$

for some $a > 0$. Also, for some $b > 0$ we have

$$\sum_{i \in \mathbb{Z}} \frac{\rho_i u_i}{\mu_i} g_i \leq \frac{b}{2} \|u\|_\rho^2 + \frac{1}{2b} \sum_{i \in \mathbb{Z}} \frac{\rho_i g_i^2}{\mu_i^2} \leq \frac{b}{2} \|u\|_\rho^2 + \frac{1}{2b m_\mu^2} \|g\|_\rho^2.$$ (14)

Inserting estimates (12) – (14) into (11) we obtain

$$\frac{d}{dt} \|u(t)\|_\rho^2 \leq 2 \left( -\frac{1}{M_\mu M_\gamma} + \frac{a}{2} + \frac{b}{2} \right) \|u\|_\rho^2 + \frac{\rho_\Sigma^2 M_\lambda \|B\|_\rho^2}{a m_\mu^2} + \frac{1}{b m_\mu^2} \|g\|_\rho^2.$$
Letting \( a = b = \frac{1}{2M_\mu M_\gamma} \) in the above inequality leads to
\[
\frac{d \|u(t)\|_\rho^2}{dt} \leq -\frac{1}{M_\mu M_\gamma} \|u\|_\rho^2 + \frac{2M_\mu M_\gamma}{m_\rho^2} \left( \rho^2 \sum \|B\|_\rho^2 + \|g\|_\rho^2 \right),
\]
which can be integrated from 0 to \( t \) to obtain
\[
\|u(t)\|_\rho^2 \leq e^{-\frac{t}{2M_\mu M_\gamma}} \|u(0)\|_\rho^2 + \frac{2M_\mu M_\gamma}{m_\rho^2} \left( \rho^2 \sum \|B\|_\rho^2 + \|g\|_\rho^2 \right) \left( 1 - e^{-\frac{t}{2M_\mu M_\gamma}} \right).
\]
It follows immediately that the closed and bounded set \( K \) defined in (10) is an absorbing set for the semi-group \( \{\varphi(t)\}_{t \geq 0} \) on \( \ell_\rho^2 \).

Following similar computations we can obtain the following Lemma.

**Lemma 3.2.** Assume (A0) – (A6) hold. Then \( K \) defined in (10) is an absorbing set for the semi-group \( \{\varphi^{(n)}(t)\}_{t \geq 0} \) generated by equation (2) uniformly in \( n \).

### 3.2. Asymptotic compactness.

To prove the asymptotic compactness of the semi-groups \( \{\varphi(t)\}_{t \geq 0} \) and \( \{\varphi^{(n)}(t)\}_{t \geq 0} \), we first show that the solutions of systems (2) and (3) have light tails at \( |i| \) large by using a standard cut-off argument [4, 18, 19, 35, 36].

**Lemma 3.3.** Let assumptions (A0)–(A6) hold, and let \( u_o \in K \) where \( K \) be the absorbing set defined in (10). Then for every \( \varepsilon > 0 \) there exist \( T(\varepsilon) \) and \( I(\varepsilon) \) such that the solution of equation (7) with \( u(0) = u_o \) satisfies
\[
\sum_{|i| \geq 2I(\varepsilon)} \rho_i u_i(t; u_o)^2 \leq \varepsilon \quad \text{for all} \quad t \geq T(\varepsilon).
\]

**Proof.** Let \( \xi : \mathbb{R}^+ \to [0, 1] \) be a smooth function such that \( \xi(s) = 0 \) for \( 0 \leq s \leq 1 \), \( \xi(s) = 1 \) for \( 1 \leq s \leq 2 \) and \( \xi(s) = 1 \) for \( s \geq 2 \). Then there exists \( L_\xi > 0 \) such that \( |\xi'(s)| \leq L_\xi \) for all \( s \in \mathbb{R}^+ \).

Let \( M \) be a fixed (and large) natural number to be specified later, and set
\[
v_i(t) = \xi_M(|i|) u_i(t) \quad \text{with} \quad \xi_M(|i|) = \xi \left( \frac{|i|}{M} \right), \quad i \in \mathbb{Z}.
\]

Multiply both sides of system (3) by \( \rho_i v_i(t) \) and summing over \( i \in \mathbb{Z} \), we have
\[
\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) u_i^2(t) \leq -\sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) \frac{u_i^2(t)}{\mu_i} + \sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) u_i \sum_{j \in \mathbb{Z}} \frac{\lambda_{i,j}}{\mu_i} f_j(u_j(t)) \]
\[
+ \sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) \frac{u_i}{\mu_i} g_i \]
\[
:= C_1 + C_2 + C_3,
\]
in which
\[
C_1 \leq -\frac{1}{M_\mu M_\gamma} \sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) u_i^2(t);
\]
\[
C_3 \leq \frac{b}{2} \sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) u_i^2(t) + \frac{1}{2b \rho^2} \sum_{i \in \mathbb{Z}} \rho_i \xi_M(|i|) g_i^2.
\]

Using the Cauchy inequality \( (\sum a_i b_i)^2 \leq \sum a_i^2 \sum b_i^2 \) and the inequality \( xy \leq \frac{1}{2}(ax^2 + \frac{1}{b}y^2) \) (\( a \) is positive and to be determined later), by Assumption (A0), (A1),
It follows immediately from Gronwall’s lemma that

\[ C_2 \leq \frac{1}{m_\mu} \left( \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \right)^\frac{1}{2} \left( \sum_{i \in Z} \rho_i \xi_M(|i|) \left( \sum_{j \in Z} \lambda_{i,j} f_j(u_j(t)) \right)^2 \right)^\frac{1}{2} \]

\[ \leq \frac{1}{m_\mu} \left( \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \right)^\frac{1}{2} \left( \sum_{i \in Z} \rho_i \xi_M(|i|) \left( \sum_{j \in Z} \frac{\lambda^2_{i,j}}{\rho_j} \right) \left( \sum_{j \in Z} \rho_j B_j^2 \right)^\frac{1}{2} \right) \]

\[ \leq \frac{1}{m_\mu} \left( \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \right)^\frac{1}{2} \left( \sum_{i \in Z} \rho_i \xi_M(|i|) M_\lambda \rho_\varepsilon \| B \|_\rho^2 \right)^\frac{1}{2} \]

\[ \leq \frac{1}{2cm_\mu} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) + \frac{c}{2m_\mu} \sum_{i \in Z} \rho_i \xi_M(|i|) M_\lambda \rho_\varepsilon \| B \|_\rho^2, \]

and as a result

\[ \frac{1}{2} \frac{d}{dt} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \leq \left( -\frac{1}{M_\mu M_\gamma} + \frac{b}{2} + \frac{1}{2cm_\mu} \right) \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \]

\[ + \frac{c}{2m_\mu} \sum_{i \in Z} \rho_i \xi_M(|i|) M_\lambda \rho_\varepsilon \| B \|_\rho^2 + \frac{1}{2cm_\mu} \sum_{i \in Z} \rho_i \xi_M(|i|)g_2^2. \]

Here we choose \( b = \frac{1}{2m_\gamma M_\mu} \), \( c = \frac{2M_\gamma M_\mu}{m_\mu} \), then

\[ \frac{d}{dt} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \leq -\frac{1}{M_\mu M_\gamma} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) + \frac{2M_\gamma M_\mu}{m_\mu} \sum_{i \in Z} \rho_i \xi_M(|i|)g_2^2 \]

\[ + \frac{2M_\gamma M_\mu}{m_\mu} \sum_{i \in Z} \rho_i \xi_M(|i|) M_\lambda \rho_\varepsilon \| B \|_\rho^2. \]

Since \( \rho_\varepsilon := \sum_{i \in Z} \rho_i < \infty \), then for every \( \varepsilon > 0 \), there exists \( I_1(\varepsilon) > 0 \) such that

\[ \sum_{|i| \geq M} \rho_i \leq \varepsilon \quad \text{when} \quad M \geq I_1(\varepsilon). \]

In addition, since \( g := (g_i)_{i \in Z} \in \ell^2_\rho \), then for every \( \varepsilon > 0 \), there exists \( I_2(\varepsilon) > 0 \) such that

\[ \sum_{|i| \geq M} \rho_i g_i^2 \leq \varepsilon \quad \text{when} \quad M \geq I_2(\varepsilon). \]

Consequently, when \( M \geq I(\varepsilon) := \max\{I_1(\varepsilon), I_2(\varepsilon)\} \), we have

\[ \frac{d}{dt} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \leq -\frac{1}{M_\mu M_\gamma} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) + \frac{2M_\gamma M_\mu}{m_\mu} (M_\lambda \rho_\varepsilon \| B \|_\rho^2 + 1) \varepsilon. \]

It follows immediately from Gronwall’s lemma that

\[ \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \leq e^{-\frac{1}{M_\mu M_\gamma}} \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(0) + \frac{2M_\gamma M_\mu}{m_\mu} (M_\lambda \rho_\varepsilon \| B \|_\rho^2 + 1) \varepsilon \]

\[ \leq e^{-\frac{1}{M_\mu M_\gamma}} \| u_0 \|_\rho^2 + \frac{2M_\gamma M_\mu}{m_\mu} (M_\lambda \rho_\varepsilon \| B \|_\rho^2 + 1) \varepsilon \]

which implies that there exists \( T = T(\varepsilon) \) such that

\[ \sum_{|i| \geq 2M} \rho_i u_1^2(t) \leq \sum_{i \in Z} \rho_i \xi_M(|i|)u_2^2(t) \leq \frac{4M_\gamma^2 M_\mu^2}{m_\mu^2} (M_\lambda \rho_\varepsilon \| B \|_\rho^2 + 1) \varepsilon, \]
for $\forall M \geq I(\varepsilon), t \geq T(\varepsilon)$. □

We next show the semi-group $\varphi$ is asymptotically compact, that is, if $u^n_0$ is bounded in $L^2$ and $t_n \to \infty$, then every sequence $u^n(t_n) := \varphi(t_n)u^n_0$ $(n \in \mathbb{N})$ is precompact in $L^2$.

**Lemma 3.4.** Suppose that Assumptions (A0)–(A6) hold. Then the semi-group $\varphi$ is asymptotically compact.

**Proof.** For every $u^n_0 \in L^2$, by the existence of the absorbing set, there exists $N_1 > 0$, when $n > N_1$, we have
\[
\|u^n(t_n)\|_p^2 \leq R_\varphi^2.
\]
By (15) we know that there exist $v \in L^2$ and a subsequence of $\{u^n(t_n)\}$ (still denoted by $\{u^n(t_n)\}$) such that
\[
u^n(t_n) \to v \quad \text{weakly in } L^2.
\]
In what follows, we shall prove that the weak convergence above is actually strong convergence. That is, we will show that for every $\varepsilon > 0$, there exists $N_0(\varepsilon) > 0$ such that when $n \geq N_0(\varepsilon)$,
\[
\|u^n(t_n) - v\|_p^2 \leq \varepsilon.
\]
Indeed, there exists $N_2(\varepsilon) > 0$ such that $t_n \geq T(\varepsilon)$ when $n \geq N_2(\varepsilon)$ since $t_n \to \infty$ (where $T(\varepsilon)$ is the constant in Lemma 3.3) and by the tail estimate, we find that there exists $M_1(\varepsilon) > 0$ such that
\[
\sum_{|i| \geq 2M_1(\varepsilon)} \rho_i |u^n_i(t_n)|^2 \leq \frac{\varepsilon}{8} \quad \text{for all } n \geq \max\{N_2(\varepsilon), N_1(\varepsilon)\}.
\]
On the other hand, since $v \in L^2$, there exists $M_2(\varepsilon)$ such that
\[
\sum_{|i| \geq 2M_2(\varepsilon)} \rho_i v_i^2 \leq \frac{\varepsilon}{8}.
\]

Let $M_0(\varepsilon) := \max\{M_1(\varepsilon), M_2(\varepsilon)\}$, by the weak convergence (16) we infer that, there exists $N_3(\varepsilon) > N_1(\varepsilon)$ such that when $n \geq N_3(\varepsilon)$
\[
\sum_{|i| \leq 2M_0(\varepsilon)} \rho_i |u^n_i(t_n) - v_i|^2 \leq \frac{\varepsilon}{2}.
\]
Setting $N_0(\varepsilon) := \max\{N_2(\varepsilon), N_3(\varepsilon)\}$, from (17), (18) and (19), we get that, for $n \geq N_0(\varepsilon)$,
\[
\|u^n(t_n) - v\|_p^2 = \sum_{|i| \leq 2M_0(\varepsilon)} \rho_i |u^n_i(t_n) - v_i|^2 + \sum_{|i| > 2M_0(\varepsilon)} \rho_i |u^n_i(t_n) - v_i|^2 \leq \frac{\varepsilon}{2} + 2 \sum_{|i| > 2M_0(\varepsilon)} \rho_i (|u^n_i(t_n)|^2 + |v_i|^2) \leq \varepsilon
\]
Hence, we obtain $\{u^n(t_n)\}_{n \in \mathbb{N}}$ is precompact in $L^2$. □

**Theorem 3.5.** Assume that (A0) – (A6) hold. Then system (3) has a global attractor $\mathcal{A}$ and the system (2) has a global attractor $\mathcal{A}_n$ for every $n \geq 1$. Moreover the attractors $\mathcal{A}$ and $\mathcal{A}_n$ are minimal and connected.
4. Upper semi-continuity of attractors. In this section we show that the attractors for the semi-groups \( \{ \varphi^{(n)}(t) \}_{t \geq 0} \) generated by system (2) converge upper semi-continuously to the attractor for the semi-group \( \{ \varphi(t) \}_{t \geq 0} \) generated by system (3) as \( n \to \infty \). To this end we first present a series of results that are crucial for the proof of the upper semi-continuity. In particular, in Lemma 4.1 we show that the global attractors \( A_n \) and \( A \) consist of all bounded solutions of systems (2) and (3), respectively, on the whole \( \mathbb{R} \). To this end we first present a series of results that are crucial for the proof of the upper semi-continuity. In particular, in Lemma 4.1 we show that the global attractors \( A_n \) and \( A \) consist of all bounded solutions of systems (2) and (3), respectively, on the whole \( \mathbb{R} \). While in Lemma 4.2 we compare the semi-groups \( \{ \varphi^{(n)}(t) \}_{t \in \mathbb{R}} \) and \( \{ \varphi(t) \}_{t \in \mathbb{R}} \) and in Lemma 4.3 we show that any sequence of elements picked from the \( A_n \) converges to an element in \( A \) as \( n \to \infty \).

**Lemma 4.1.** The global attractors \( A_n \) and \( A \) are strictly invariant for all \( t \in \mathbb{R} \), i.e.,

\[
\varphi(t)A = A, \quad \varphi^{(n)}(t)A_n = A_n \quad \text{for every} \quad n \geq 1 \quad \forall \ t \in \mathbb{R}.
\]

**Proof.** Note that the global attractors \( A \) and \( A_n \) obtained in Theorem 3.5 are strictly invariant for \( t \geq 0 \), i.e.,

\[
\varphi(t)A = A, \quad \varphi^{(n)}(t)A_n = A_n \quad \text{for every} \quad n \geq 1 \quad \forall \ t \geq 0.
\]

Thus we only need to show the backward uniqueness of solutions to each of the systems (2) and (3). To this end, for any \( u_o \in \ell^2_{\rho} \) let \( u(t) = \varphi(t)u_o \) and \( v(t) = \varphi(t)v_o \) be two solutions of the system (3) such that \( u(T) = v(T) \) for some \( T > 0 \). We next show that \( u(t) = v(t) \) for all \( 0 \leq t \leq T \).

Define \( w(t) := u(t) - v(t) \) then by equation (3) we have

\[
\frac{d}{dt}w_i(t) = -\frac{1}{\gamma_i}w_i(t) + \sum_{j \in Z} \lambda_{i,j} (f_j(u_j) - f_j(v_j)).
\]

Multiplying the above equation by \( -\frac{\rho_i}{\mu_i}w_i \) and summing over \( i \in \mathbb{Z} \) results in

\[
- \sum_{i \in \mathbb{Z}} \rho_i \frac{d}{dt}w_i(t) = \sum_{i \in \mathbb{Z}} \frac{\rho_i}{\mu_i \gamma_i} w_i^2(t) - \sum_{i \in \mathbb{Z}} \left( \frac{\rho_i}{\mu_i} \sum_{j \in \mathbb{Z}} \lambda_{i,j} (f_j(u_j) - f_j(v_j)) \right),
\]

in which

\[
\sum_{i \in \mathbb{Z}} \frac{\rho_i}{\mu_i \gamma_i} w_i^2(t) \leq \frac{1}{m_\mu m_\gamma} \|w(t)\|_\rho^2,
\]

and

\[
\sum_{i \in \mathbb{Z}} \left( \frac{\rho_i}{\mu_i} \sum_{j \in \mathbb{Z}} \lambda_{i,j} (f_j(u_j) - f_j(v_j)) \right) \leq \frac{1}{m_\mu} \left( \sum_{i \in \mathbb{Z}} \rho_i w_i^2 \right)^{1/2} \left( \sum_{i \in \mathbb{Z}} \rho_i \sum_{j \in \mathbb{Z}} \lambda_{i,j}^2 L_j^2 \sum_{j \in \mathbb{Z}} \rho_j w_j^2 \right)^{1/2} \leq \frac{1}{m_\mu} \sqrt{\rho \lambda} \sqrt{M_\lambda} \|L\|_\rho \|w(t)\|_\rho^2.
\]

Therefore

\[
- \sum_{i \in \mathbb{Z}} \rho_i \frac{d}{dt}w_i(t) \leq \frac{1}{m_\mu m_\gamma} \|w(t)\|_\rho^2 + \frac{\sqrt{\rho \lambda}}{m_\mu} \sqrt{M_\lambda} \|L\|_\rho \|w(t)\|_\rho^2 \leq c \|w(t)\|_\rho^2, \quad (20)
\]

where

\[
c = \frac{1}{m_\gamma} \left( \frac{1}{m_\gamma} + \sqrt{\rho \lambda} M_\lambda \|L\|_\rho \right).
\]
Now suppose (for contradiction) that there exists $t_0 \in [0, T]$ such that $w(t_0) \neq 0$. Then by continuity of solutions there exists $\tau \in [t_0, T]$ such that
\[ w(\tau) = 0 \quad \text{but} \quad w(t) \neq 0 \quad \text{for all} \quad t \in [t_0, \tau]. \]
Then we have
\[ \lim_{t \to \tau^-} \frac{1}{\|w(t)\|_\rho} = \infty. \quad (21) \]
On the other hand it follows from (20) that
\[ \frac{d}{dt} \ln \frac{1}{\|w(t)\|_\rho} = -\frac{1}{2} \frac{d}{dt} \ln \|w(t)\|^2_\rho = -\sum_{i \in \mathbb{Z}} \rho_i w_i \frac{dw_i(t)}{dt} \leq c. \]
Integrating the above inequality from $t_0$ and $t \in [t_0, \tau]$ gives
\[ \ln \frac{1}{\|w(t)\|_\rho} \leq \ln \frac{1}{\|w(t_0)\|_\rho} + c(t - t_0) \leq \ln \frac{1}{\|w(t_0)\|_\rho} + cT, \]
which contradicts with (21). Therefore the solutions to system (3) is backward unique, consequently $\varphi(t)A = A$ for all $t \in \mathbb{R}$. Following a similar procedure we can show that solutions of system (2) are backward unique, and that $\varphi^{(n)}(t)A_n = A_n$ for all $t \in \mathbb{R}$.

The rest of this section requires the convergence rate from $\lambda^{(n)}_{i,j} \to \lambda_{i,j}$ to be specified. In particular it is assumed that

(A7) $\lambda^{(n)}_{i,j} \to \lambda_{i,j}$ as $n \to \infty$ in the sense that for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that
\[ \sum_{j \in \mathbb{Z}} \frac{(\lambda^{(n)}_{i,j} - \lambda_{i,j})^2}{\rho_j} \leq \varepsilon^2, \quad \forall \ n \geq N(\varepsilon), \quad i \in \mathbb{Z}. \]

Lemma 4.2. Let assumptions (A0) – (A7) hold. Then for every $u_\alpha \in \mathcal{K}$ and $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that
\[ \|\varphi^{(n)}(t)u_\alpha - \varphi(t)u_\alpha\|_\rho < \varepsilon \quad \forall \ n \geq N(\varepsilon) \quad \text{for each fixed} \ t. \]

Proof. Write $u^{(n)}(t) = (u^{(n)}_i(t))_{i \in \mathbb{Z}} = \varphi^{(n)}(t)u_\alpha$ and $v^{(n)}(t) = (v^{(n)}_i(t))_{i \in \mathbb{Z}} = \varphi(t)u_\alpha$. Then $u^{(n)}(t)$ and $v^{(n)}(t)$ satisfy systems (2) and (3) respectively. Define
\[ w^{(n)}(t) = u^{(n)}(t) - v^{(n)}(t) = (u^{(n)}_i(t) - v^{(n)}_i(t))_{i \in \mathbb{Z}} = (w^{(n)}_i(t))_{i \in \mathbb{Z}}. \]
Then $w^{(n)}(t)$ satisfies the equations
\[ \mu_i \frac{dw^{(n)}_i(t)}{dt} = -\frac{1}{\gamma_i} w^{(n)}_i(t) + \sum_{j=i-n}^{i+n} \lambda^{(n)}_{i,j} f_j(u^{(n)}_j(t)) - \sum_{j \in \mathbb{Z}} \lambda_{i,j} f_j(v^{(n)}_j). \quad (22) \]
Multiplying both sides of the above equation by $2 \rho_i w^{(n)}_i(t)/\mu_i$ and summing over $i \in \mathbb{Z}$ we obtain
\[ \frac{d}{dt} \|w^{(n)}(t)\|^2_\rho = -\sum_{i \in \mathbb{Z}} \frac{2}{\gamma_i \mu_i} \rho_i w^{(n)}_i(t)^2 - \sum_{i \in \mathbb{Z}} \left( \frac{2}{\mu_i} \rho_i w^{(n)}_i(t) \sum_{|j-i| > n} \lambda_{i,j} f_j(v^{(n)}_j) \right) \]
\[ + \sum_{i \in \mathbb{Z}} \left( \frac{2}{\mu_i} \rho_i w^{(n)}_i(t) \sum_{j=i-n}^{i+n} (\lambda^{(n)}_{i,j} f_j(u^{(n)}_j) - \lambda_{i,j} f_j(v^{(n)}_j)) \right). \quad (23) \]
To simplify notations, for each $i \in \mathbb{Z}$ define

$$P_i := \sum_{|j-i| > n} |\lambda_{i,j} f_j(v_j^{(n)})|$$

$$Q_i := \sum_{j=i-n}^{i+n} |\lambda_{i,j}^{(n)} (f_j(u_j^{(n)}(t)) - f_j(v_j^{(n)}(t)))|$$

$$R_i := \sum_{j=i-n}^{i+n} \left| (\lambda_{i,j}^{(n)} - \lambda_{i,j}) f_j(v_j^{(n)}(t)) \right|.$$

Then by Assumption (A1), Cauchy’s inequality and triangle inequality we obtain from equation (23) that

$$\frac{d}{dt} \|w^{(n)}(t)\|_\rho^2 \leq -\frac{2}{M_\mu M_\gamma} \|w^{(n)}(t)\|_\rho^2$$

$$+ \frac{2}{m_\mu} \|w^{(n)}(t)\|_\rho \left( \sum_{i \in \mathbb{Z}} \rho_i P_i^2 \right)^{1/2} + \left( \sum_{i \in \mathbb{Z}} \rho_i Q_i^2 \right)^{1/2} + \left( \sum_{i \in \mathbb{Z}} \rho_i R_i^2 \right)^{1/2}.$$ (24)

We next estimate each term on the right hand side of (24).

Since $\sum_{i \in \mathbb{Z}} \rho_i = \rho_S < \infty$, for every $\varepsilon > 0$ there exists $I(\varepsilon) > 0$ such that $\sum_{|j| > I(\varepsilon)} \rho_i < \varepsilon^2$. Pick $N_1(\varepsilon) = 2I(\varepsilon)$, then $|j| > I(\varepsilon)$ if $|j - i| > N_1(\varepsilon)$ and $|i| \leq I(\varepsilon)$, and thus

$$\sum_{|j-i| > n} \rho_j < \varepsilon^2 \quad \forall \ n \geq N_1(\varepsilon).$$

Consequently we have

$$\sum_{i \in \mathbb{Z}} \rho_i P_i^2 \leq \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{|j-i| > n} \frac{\lambda_{i,j}^2}{\rho_j} \sum_{|j-i| > n} \rho_j \sum_{|j-i| > n} \rho_j B_j^2 \right)$$

$$\leq \sum_{|i| \leq I(\varepsilon)} \rho_i M_\lambda \varepsilon^2 \|B\|_\rho^2 + \sum_{|i| > I(\varepsilon)} \rho_i M_\lambda \rho_S \|B\|_\rho^2$$

$$\leq 2M_\lambda \rho_S \|B\|_\rho^2 \varepsilon^2, \quad \forall \ n \geq N_1(\varepsilon).$$ (25)

Then by the local Lipschitz property of $f_i$’s in (4) we have

$$\sum_{i \in \mathbb{Z}} \rho_i Q_i^2 \leq \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j=i-n}^{i+n} \frac{(\lambda_{i,j}^{(n)})^2}{\rho_j^2} \sum_{j=i-n}^{i+n} \rho_j L_j^2 \sum_{j=i-n}^{i+n} \rho_j (u_j^{(n)})^2 \right)$$

$$\leq \rho_S M_\lambda \|L\|_\rho^2 \cdot \|w^{(n)}(t)\|_\rho^2,$$ (26)

Now by Assumption (A7), for every $\varepsilon > 0$ there exists $N_2(\varepsilon) > 0$ such that $\sum_{j=i-n}^{i+n} (\lambda_{i,j}^{(n)} - \lambda_{i,j})^2 / \rho_j < \varepsilon^2$ for all $n \geq N_2(\varepsilon)$. Therefore

$$\sum_{i \in \mathbb{Z}} \rho_i R_i^2 \leq \sum_{i \in \mathbb{Z}} \rho_i \left( \sum_{j=i-n}^{i+n} \frac{(\lambda_{i,j}^{(n)} - \lambda_{i,j})^2}{\rho_j^2} \sum_{j=i-n}^{i+n} \rho_j B_j^2 \right)$$

$$\leq \varepsilon^2 \rho_S \|B\|_\rho^2, \quad \forall \ n \geq N_2(\varepsilon).$$ (27)
 Lemma 4.3. It follows directly that and then by Lemma 2.1 and Lemma 3.1 we have
\[ a \|w^{(n)}(t)\|_\rho^2 + b\|w^{(n)}(t)\|_\rho. \]
Divide the above inequality by 2\|w^{(n)}(t)\|_\rho and integrate the resulting inequality from 0 to t with \( w^{(n)}(0) = 0 \) gives
\[ \|w^{(n)}(t)\|_\rho \leq \frac{b}{a} \epsilon \cdot \max\{1, \epsilon^{n(t)}\}, \quad \forall t \geq 0 \text{ fixed } n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}. \]
The desired assertion then follows immediately after minor adjustment in \( \epsilon \).

With the above lemma we can prove the following important property.

**Lemma 4.3.** Let assumptions (A0) – (A7). Then for every sequence \( \{a_n\}_{n \geq 1} \) with \( a_n \in A_n \) there exist a subsequence \( \{a_{n_k}\}_{k \geq 1} \) of \( \{a_n\}_{n \geq 1} \) and \( a \in A \) such that \( a_{n_k} \to a \) in \( \ell^2_\rho \).

**Proof.** Let \( w^{(n)}(t; a_n) = \varphi^{(n)}(t)a_n \) be the solution of system (2) with \( w^{(n)}(0) = a_n \). Let \( v^{(n)}(t) = \varphi(t)a_n \), then it follows from Lemma 4.2 above that \( w^{(n)}(t) \to v^{(n)}(t) \)

in \( \ell^2_\rho \) for every fixed \( t \in \mathbb{R} \).

Since \( a_n \in A_n \subset K \), the absorbing set defined in (10), \( v^{(n)}(t) \in K \) for all \( t \in \mathbb{R} \) with
\[ \|v^{(n)}(t)\|_\rho \leq R_K \quad \text{for all } t \in \mathbb{R} \text{ and } n = 1, 2, \ldots, \]
where \( R_K \) is the radius of the absorbing set defined in (10).

One the one hand, by equation (7) we have
\[ \left\| \frac{dv^{(n)}(t)}{dt} \right\|_\rho \leq \|Av^{(n)}(t)\|_\rho + \|Av^{(n)}(t)\|_\rho + \frac{1}{m_\mu} \|g\|_\rho, \]
and then by Lemma 2.1 and Lemma 3.1 we have
\[ \left\| \frac{dv^{(n)}(t)}{dt} \right\|_\rho \leq \frac{1}{m_\mu m_\gamma} \|w^{(n)}(t)\|_\rho + \sqrt{\rho_\Sigma M_\lambda} \|L\|_\rho \|v^{(n)}(t)\|_\rho + \frac{1}{m_\mu} \|g\|_\rho \]
\[ \leq \frac{1}{m_\mu} \left( \frac{1}{m_\gamma} + \sqrt{\rho_\Sigma M_\lambda} \|L\|_\rho \right) R_K + \frac{1}{m_\mu} \|g\|_\rho := M_v. \]
It follows directly that
\[ \left\| v^{(n)}(t) - v^{(n)}(s) \right\| \leq \left\| \frac{dv^{(n)}(t)}{dt} \right\|_\rho \left| s - t \right| \leq M_v |s - t|, \quad \forall s, t \in \mathbb{R}, \]
i.e., \( \{v^{(n)}(t)\}_{n \geq 1} \) is equicontinuous on \( C(\mathbb{R}, \ell^2_\rho) \).

On the other hand following calculations similar to those in the proof of Lemma 3.4 we can obtain that \( \{v^{(n)}(t)\}_{n \geq 1} \) is precompact in \( \ell^2_\rho \). Then due to the Arzela-Ascoli Theorem, \( \{v^{(n)}(t)\}_{n \geq 1} \) is precompact on \( C(J, \ell^2_\rho) \) for any compact interval \( J \subset \mathbb{R} \). Notice that \( v^{(n)}(t) \) are solutions of (3) which is bounded and defined on the whole \( \mathbb{R} \). Thus \( v^{(n)}(t) \in A \) for all \( t \in \mathbb{R} \). Then by the compactness of \( A \) and the Arzela-Ascoli Theorem again, there is a subsequence \( \{v^{n_k}(t)\}_{k \geq 1} \) of \( \{v^{(n)}(t)\}_{n \geq 1} \)
and \( v^*(t) \in A \) such that \( v^{n_k}(t) \to v^*(t) \) in \( C(J, \ell^2_\rho) \) for any compact interval \( J \subset \mathbb{R} \).
Summarizing above, we have \( u^{n_k}(t) \rightarrow v^*(t) \) in \( C(J, \ell^2_\rho) \) for any compact interval \( J \subset \mathbb{R} \). And it follows directly by Lemma 4.1 that
\[
a_{n_k} = u^{n_k}(0) \rightarrow v^*(0) := a \in A \quad \text{as } k \rightarrow \infty.
\]

Finally denoting by \( \text{dist}_{\ell^2_\rho} \) the Hausdorff distance between any two subsets of \( \ell^2_\rho \), we conclude this section by the following result.

**Theorem 4.4.** Assume assumptions \((A0) – (A7)\) hold. Then the global attractors for system (2) converge to the global attractor for system (3) upper semi-continuously, i.e.,
\[
\lim_{n \rightarrow \infty} \text{dist}_{\ell^2_\rho}(A_n, A) = 0.
\]

**Proof.** Suppose (for contradiction) that \( \lim_{n \rightarrow \infty} \text{dist}_{\ell^2_\rho}(A_n, A) \neq 0 \), then by the definition of \( \text{dist}_{\ell^2_\rho} \) there exist a sequence \( n_k \rightarrow \infty \), \( a_{n_k} \in A_{n_k} \) and \( \delta > 0 \) such that
\[
\text{dist}_{\ell^2_\rho}(a_{n_k}, A) \geq \delta > 0.
\]
On the contrary, by Lemma 4.3 there exists a subsequence \( \{a_{n_{k_m}}\} \) of \( \{a_{n_k}\} \) such that
\[
\text{dist}_{\ell^2_\rho}(a_{n_{k_m}}, A) \rightarrow 0 \quad \text{as } m \rightarrow \infty.
\]
The proof is complete.

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