Motivic unipotent fundamental groupoid of $\mathbb{G}_m \setminus \mu_N$ for $N = 2, 3, 4, 6, 8$ and Galois descents.

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November 19, 2014

Abstract

We study Galois descents for categories of mixed Tate motives over $\mathcal{O}_N[1/N]$, for $N \in \{2, 3, 4, 8\}$ or $\mathcal{O}_N$ for $N = 6$, with $\mathcal{O}_N$ the ring of integers of the $N^{th}$ cyclotomic field, and construct families of motivic iterated integrals with prescribed properties. In particular this gives a basis of honorary multiple zeta values (linear combinations of iterated integrals at roots of unity $\mu_N$ which are multiple zeta values). It also gives a new proof, via Goncharov’s coproduct, of Deligne’s results ([3]): the category of mixed Tate motives over $\mathcal{O}_N[1/N]$, for $N \in \{2, 3, 4, 8\}$ is spanned by the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ with an explicit basis. By applying the period map, we obtain a generating family for multiple zeta values relative to $\mu_N$.

1 INTRODUCTION

The goal of this paper is to study the Galois action on the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ for some particular values of $N$: $N \in \{2^a 3^b, a + 2b \leq 3\}$. For $N$ fixed $\in \{1, 2, 3, 4, 6, 8\}$, let $k_N = \mathbb{Q}(\xi_N)$, where $\xi_N \in \mu_N$ is a primitive $N^{th}$ root of unity, and $\mathcal{O}_N$ is the ring of integers of $k_N$. The subscript or exponent $N$ will be omitted when it is not ambiguous.

Recall that multiple zeta values relative to $\mu_N$ (periods of the corresponding motivic multiple zeta values) are given by the coefficients of a version of Drinfeld’s associator, which are explicitly:

$$(\star) \, \zeta \left( \frac{x_1, \ldots, x_p}{\epsilon_1, \ldots, \epsilon_p} \right) := \sum_{0 \leq n_1 < n_2 < \cdots < n_p} \frac{\epsilon_1^{n_1} \cdots \epsilon_p^{n_p}}{n_1 \cdots n_p}, \, \epsilon_i \in \mu_N, \, (x_p, \epsilon_p) \neq (1, 1).$$

The weight is $\omega = \sum x_i$ and the depth is $p$. Denote by $\mathbb{Z}^N$ the $\mathbb{Q}$-vector space spanned by these multiple zeta values at arguments $x_i \in \mathbb{N}, \epsilon_i \in \mu_N$. In the case $N = 2$, $\epsilon_i \in \{\pm 1\}$, we simplify the notation:

$$\zeta(z_1, \ldots, z_p) \, \text{where } z_i \in \mathbb{Z}^* \text{ corresponds to } \left( \frac{|z_i|}{\text{sign}(z_i)} \right) = \left( \frac{x_i}{\epsilon_i} \right).$$

We will consider the motivic versions of those multiple zeta values, denoted $\zeta^m$ which span the $\mathbb{Q}$-vector space of motivic multiple zetas relative to $\mu_N$, denoted $\mathcal{H}^N$. There is a surjective homomorphism called the period map (conjectured to be an isomorphism)

$$\text{per} : \mathcal{H}^N \rightarrow \mathbb{Z}^N, \, \zeta^m(\cdot) \mapsto \zeta(\cdot).$$

Furthermore, $\mathcal{H}^N$ is a Hopf comodule with an explicit coaction $\Delta$ given by Goncharov ([11]) and extended by F. Brown ([12]). And for each $N, N'$ with $N'|N$ there are Galois groups $\mathcal{G}_N$ acting on $\mathcal{H}^N$ and Galois descents determined by this coaction.
Outlines of Results

Consider the Tannakian category of mixed Tate motives over $O_N[1/N]$ (c.f. [12, 10]):

$$\mathcal{M}_{T_N} := \mathcal{M}(O_N[1/N])$$, for $N = 2, 3, 4, 8$.

In the special case $N = 6$, the category considered will be $\mathcal{M}_6 := \mathcal{M}(O_6)$, the Tannakian category of unramified mixed Tate motives over $O_6$.

Denote by $G^\mathcal{M}T = G_m \times U^\mathcal{M}T$ its Tannaka group (c.f. [16]) with respect to the canonical fiber functor (cf. below) which is defined over $Q$ and by $A^\mathcal{M}T = O(U^\mathcal{M}T)$ its fundamental Hopf algebra and by $H^\mathcal{M}T \subset O(G^\mathcal{M}T) = A^\mathcal{M}T \otimes_Q Q[t, t^{-1}]$ free $A^\mathcal{M}T$-comodule:

$$H^\mathcal{M}T = \begin{cases} A^\mathcal{M}T \otimes_Q Q[t] & \text{for } N > 2, \\ A^\mathcal{M}T \otimes_Q Q[t^2] & \text{for } N = 1, 2. \end{cases}$$

There is a notion of 

**motivic multiple zeta values** (cf. below) which form an algebra $H^N$ which embeds non canonically into $H^{\mathcal{M}T N}$ with $(2i\pi)^m \to t$. For those specific values of $N$, its is an isomorphism (by F. Brown for $N = 1$ and for $N = 2, 3, 4, 6, 8$ by Deligne [3] or by the theorems which follow). For the sake of the introduction, we will sometimes write $H$ and forget the distinction.

We define recursively on $i$ increasing motivic filtrations on $H$, called **motivic levels**, $F_i^{kN/kN', M/M'}$, stable under the action of $G^\mathcal{M}T N$, by sub-$Q$-vector spaces. The exponent $k_N/k_{N'}$ indicates the change of cyclotomic field and $M/M'$ the change of ramification. The $0^{th}$ level $F_0^{kN/kN', N/N'}$, corresponds to invariants under the group $G_N/N'$ as above and the $i^{th}$ level $F_i^{kN/kN', N/N'}$, can be seen as the $i^{th}$ space of higher ramifications corresponding to generalised Galois descents. The associated quotients will be denoted:

$$H^{2^i} := H/F_{i-1}H, \ H^{2^0} = H.$$

The exponent $k_N/k_{N'}$, $M/M'$ if not ambiguous will be omitted when we look at a specific descent.

**Example**, for $N = 2$: The action of the Lie algebra of the Galois group factors through certain operators $D_{2r+1}$ which are obtained from the formula for the coaction $\Delta$ by restricting the left hand side to weight $2r + 1$. The Galois descent between $H^{\mathcal{M}T 2}$ and $H^{\mathcal{M}T 1}$ is then measured by $D_1$, that is to say:

$$F_{-1}H^{\mathcal{M}T 2} := 0, \ F_iH^{\mathcal{M}T 2} := \{ \xi \in H^{\mathcal{M}T 2}, \ s.t. \ D_1\xi \in F_{i-1}H^{\mathcal{M}T 2}, \ \forall r > 0, D_{2r+1}\xi \in F_iH^{\mathcal{M}T 2} \}.$$

Motivic Euler sums belonging to the $0^{th}$-level of this filtration are called **honorary motivic multiple zeta values** and are in $H^{\mathcal{M}T 1}$. Some of these periods have been studied notably by D. Broadhurst (cf. [3]).

Some of our results (if we restrict to the $0^{th}$ level of the filtrations) can be illustrated by the following diagrams:

**The cases** $N = 2, 4, 8$
The cases $N = 3, 6$

\[ B^N := \left\{ \zeta^m \left( \frac{x_1, \ldots, x_{p-1}, x_p}{\epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_p \xi_N} \right) (2\pi i)^{s,m} , x_i \in \mathbb{N}^*, s \geq 0, \begin{cases} x_i \geq 1 \text{ odd, } \epsilon_i = 1 \text{ and } s \text{ even if } N = 2 \\ x_i \geq 1 , \epsilon_i = 1 \text{ if } N = 3, 4 \\ x_i > 1 , \epsilon_i = 1 \text{ if } N = 6 \\ x_i \geq 1 , \epsilon_i = \pm 1 \text{ if } N = 8 \end{cases} \right\}. \]

Denote $\mathcal{H}^N \subset \mathcal{H}$ the $\mathbb{Q}$-vector space spanned by these elements and $B_{n,p,i}$ the subset of the motivic multiple zeta values in the basis with weight $n$, depth $p$ and level $i$.

Examples:

- For $N = 2$, the basis for the motivic Euler sums:
  \[ B^2 := \left\{ \zeta^m \left( \frac{2y_1 + 1, \ldots, 2y_p + 1}{1, 1, \ldots, 1, -1} \right) \zeta^m(2)^s, y_i \geq 0, s \geq 0 \right\}. \]
  The level is defined to be the number of $y_i$s equal to 0.

- For $N = 4$, $B^4 := \left\{ \zeta^m \left( \frac{x_1, \ldots, x_p}{1, 1, \ldots, 1, \sqrt{-1}} \right), x_i > 0 \right\}$. Here, the level is the number of even $x_i$s if we focus on the descent from $\mathcal{H}^4 \to \mathcal{H}^2$, or the number of even $x_i$s plus the number of $x_i$s equal to 1 if we focus on the Galois descent from $\mathcal{H}^4 \to \mathcal{H}^1$.

- For $N = 8$ the level includes the number of $x_i$s equal to $-1$, etc.

The previous quotients $\mathcal{H}^N, \mathcal{H}_n, \mathcal{H}^N, \mathcal{H}_n, \mathcal{O}_n, \mathcal{H}^N(\mathcal{O}_n)$, $\mathcal{H}^N, \mathcal{H}_n, \mathcal{H}^N, \mathcal{H}_n, \mathcal{O}_n, \mathcal{H}^N(\mathcal{O}_n)$, will match with the sub-families restricted to the level $B_{n,p,i}$, respectively $B_{n,p,i}$. Indeed, we first prove the following theorem, for $N$ fixed in $\{2, 3, 4, 6, 8\}$:

**Theorem 1.1.** $B_{n, \geq 1}$ is a basis of the quotient $\mathcal{H}^N_{n, \geq 1}$. \footnote{The basis $B$, in the case $\{3, 4, 8\}$ is identical to P. Deligne’s in [S], and for $N = 2$ is a linear basis analogous to his algebraic basis which is formed by Lyndon words in the odd positive integers (with $\ldots 5 \leq 3 \leq 1$).}

The linear independence is obtained by computing the coaction and an argument involving 2 or 3adic properties of the coaction on this basis. The fact that $B_{n, \geq 1}$ spans $\mathcal{H}^N_{n, \geq 1}$ follows from counting dimensions. It generalizes in particular (by setting $i = 0$) a result of P. Deligne ([S]) \footnote{The basis $B$, in the case $\{3, 4, 8\}$ is identical to P. Deligne’s in [S], and for $N = 2$ is a linear basis analogous to his algebraic basis which is formed by Lyndon words in the odd positive integers (with $\ldots 5 \leq 3 \leq 1$).}
Corollary 1.2. The elements of \( B_ν \) form a basis of \( H_ν^{MT} \), the space of motivic multiple zeta values relative to \( μ_ν \).

The period map, \( \text{per} : H \to \mathbb{C} \), induces the following result for \( N = 2, 3, 4, 8 \):

Corollary 1.3. Each multiple zeta value relative to \( N^{th} \) roots of unity is a \( \mathbb{Q} \)-linear combination of multiple zeta values of the same weight of type \( B^N \).

NB: For \( N = 6 \) the result remains true if we restrict to iterated integrals relative not to all \( 6^{th} \) roots of unity but only to those relative to primitive roots.

The Theorem 1.1 gives bases for the quotients, but the next goal is to construct basis for the filtrations spaces \( F_i \) themselves.

Indeed, we have the two following exact sequences in bijection:

\[
0 \to F_i H_n \to H_n \to H_n^{gr_{i+1}} \to 0
\]

\[
0 \to \text{Vect}_\mathbb{Q}(B_{n,\leq i}) \to \text{Vect}_\mathbb{Q}(B_n) \to \text{Vect}_\mathbb{Q}(B_{n,\geq i+1}) \to 0.
\]

The second exact sequence is split; \( B_n \) is a basis of \( H_n \) and \( B_{n,\geq i+1} \) a basis of \( H_n^{gr_{i+1}} \) as a consequence of theorem 1.1. The first exact sequence is then also split so we can express a basis of \( F_i H_n \) in terms of elements \( x \in B_{n,\leq i} \), each corrected by an element denoted \( c(x) \) of \( \text{Vect}_\mathbb{Q}(B_{n,\geq i+1}) \), as the following theorem states.

More precisely, we define a map \( c_{n,p,\geq i+1} : \text{Vect}_\mathbb{Q}(B_{n,p,\leq i}) \to \text{Vect}_\mathbb{Q}(B_{n,p,\geq i+1}) \) as the composite:

\[
\text{Vect}_\mathbb{Q}(B_{n,p,\leq i}) \hookrightarrow \text{Vect}_\mathbb{Q}(B_{n,p}) \to \text{gr}_{p}^H H_n^{gr_{i+1}} \to \text{Vect}_\mathbb{Q}(B_{n,p,\geq i+1}).
\]

By iterating the process for the elements of depth smaller than \( p \), we define similarly a map:

\[ c_{n,p,\leq i+1} : \text{Vect}_\mathbb{Q}(B_{n,p,\leq i}) \to \text{Vect}_\mathbb{Q}(B_{n,p,\leq i+1}). \]

Theorem 1.4. A basis of \( F_i H_n^N \) is formed by:

\[
\cup_p \left\{ x + c_{n,p,\leq i+1}(x), x \in B_{n,p,\leq i}^N \right\}.
\]

Corollary 1.5. A basis for the space \( H_n^{MT,N'} \), of motivic multiple zeta values relative to \( μ_{N'} \), is formed by motivic multiple zeta values relative to \( μ_N \) of level 0 each corrected by a \( \mathbb{Q} \)-linear combination of motivic multiple zeta values relative to \( μ_{N'} \) of level greater than or equal to 1:

\[
\left\{ x + c_{n,\geq i+1}(x), x \in B_{n,\geq 0}^N \right\}.
\]

The period map induces a corresponding result for a generating family for MZVs relative to \( μ_{N'} \).

Example, for \( N = 2 \): A basis for honorary motivic multiple zeta values is formed by:

\[
\zeta^m(2x_1 + 1, \cdots, 2x_p + 1)\zeta^m(2) \text{ with } x_i > 0
\]

+ \( \mathbb{Q} \) linear combination of \( \zeta^m(2y_1 + 1, \cdots, 2y_q + 1)\zeta^m(2) \), \( q \leq p \) with at least one of the \( y_i \)'s equal to 0.

Remarks:

- Recall that each basis with motivic multiple zeta values at roots of unity gives a generating family for (simple) multiple zeta values at roots of unity, by the period map.

- Descent can be calculated explicitly in small depth, less than or equal to 3, as we will explain in the last section. For instance, for \( N = 2 \), the following linear combination is a honorary motivic MZV:

\[
\zeta^m(3, 3, 3) + \frac{774}{191}\zeta^m(1, 5, 3) - \frac{804}{191}\zeta^m(1, 3, 5) + \frac{450}{191}\zeta^m(1, 1, 7) - 6\zeta^m(3, 1, 5).
\]

In the general case, we could make the part of maximal depth of \( c(x) \) explicit (by inverting a matrix with binomial coefficients) but the motivic methods do not enable us to describe the other coefficients for terms of lower depth.
Contents The second section provides some generalities and definitions about motivic multiple zeta values at roots of unity, motivic iterated integrals and the infinitesimal coactions, used for our proofs. The third section details the proof of the case \( N = 2 \). The fourth section focus on the other cases \( N = 3,4 \) resp. \( N = 6 \) and \( N = 8 \), highlighting the main differences with the case \( N = 2 \), omitting the details. The last section provides some explicit examples in small depths (2 and 3) for \( N = 2,3,4,6,8 \).

Acknowledgements The author thanks Francis Brown for many discussions and corrections on this work, and Pierre Cartier for a careful reading and helpful comments. This work was supported by ERC Grant 257638.

2 Comodule of Motivic Multiple Zeta Values at Roots of Unity

2.1 Motivic periods and motivic MZVs

Recall that \( \mathcal{MT}_N \) is a Tannakian category equipped with a weight filtration \( W_r \) indexed by even integers such that \( gr^{W_r}_2(M) \) is a sum of copies of \( \mathbb{Q}(r) \) for \( M \in \mathcal{MT}_N \). This defines a canonical fiber functor:

\[
\omega: \mathcal{MT}_N \rightarrow Vec_{\mathbb{Q}}, M \mapsto \oplus \omega_r(M)
\]

\[
\omega_r(M) := Hom_{\mathcal{MT}_N(k)}(\mathbb{Q}(r), gr^{W_r}_2(M)) \cdot gr^{W_r}_2(M) = \mathbb{Q}(r) \otimes \omega_r(M).
\]

The de Rham functor \(\omega_{dR}\) here is not defined over \( \mathbb{Q} \) but on \( k_N \) and \( \omega_{dR} = \omega \otimes_{\mathbb{Q}} k_N \), so the de Rham realisation of an object \( M \) is \( M_{dR} = \omega(M) \otimes_{\mathbb{Q}} k_N \).

We also have a Betti fiber functor, which depends on the embedding \( \sigma : k_N \hookrightarrow \mathbb{C} \) (fixed here):

\[
\omega_{B,\sigma} : \mathcal{MT}_N \rightarrow Vec_{\mathbb{C}},
\]

There are canonical comparison isomorphisms between those functors denoted by \( comp_{B,dR} : \omega_{dR}(M) \otimes_{k_N} \mathbb{C} \rightarrow \omega_B(M) \otimes_{\mathbb{Q}} \mathbb{C} \) and \( comp_{dR,B} \).

The motivic Galois groups are defined by \( \mathcal{G}_B := Aut(\omega_B) \) and \( \mathcal{G} := Aut(\omega) \) and \( P_{\omega,B} := Isom(\omega_B,\omega) \) resp. \( P_{\omega} := Isom(\omega,\omega_B) \) are \( (\mathcal{G},\mathcal{G}_B) \) resp. \( (\mathcal{G}_B,\mathcal{G}) \) bitorsors.

By the Tannakian dictionary, \( \mathcal{MT}_N \) is equivalent to the category of representations of \( \mathcal{G} \), which decomposes as \( \mathcal{G}^{\mathcal{MT}} = G_m \times \mathcal{U} \), where \( \mathcal{U} \) is a pro-unipotent group scheme defined over \( \mathbb{Q} \).

Dimensions The algebraic K-theory will provide an upper bound for the dimensions of motivic periods. Indeed, it is proved (with the results of Beilinson and Borel, cf. \([10]\), and Levine, \([14]\)) that:

\[
\begin{align*}
\text{Ext}^1_{\mathcal{MT}_N}(\mathbb{Q}(0),\mathbb{Q}(1)) &= K_1(\mathcal{O}_{k\mathbb{N}}[\frac{1}{M}]) \otimes \mathbb{Q} \\
\text{Ext}^1_{\mathcal{MT}_N}(\mathbb{Q}(0),\mathbb{Q}(n)) &= K_{2n-1}(\mathcal{O}_{k\mathbb{N}}[\frac{1}{M}]) \otimes \mathbb{Q} = K_{2n-1}(k_N) \otimes \mathbb{Q} \quad \text{for } n > 1. \\
\text{Ext}^1_{\mathcal{MT}_N}(\mathbb{Q}(0),\mathbb{Q}(n)) &= 0 \quad \text{for } i > 1 \text{ or } n \leq 0.
\end{align*}
\]

For \( N = 6 \) recall that \( \mathcal{MT}_N \) refers to \( \mathcal{MT}(\mathcal{O}_6) \) hence \( Ext^1_{\mathcal{MT}_6}(\mathbb{Q}(0),\mathbb{Q}(1)) = K_1(\mathcal{O}_{k\mathbb{N}}) \).

Let \( p(N) \) denote the number of prime factors of \( N \) and \( \varphi \) Euler’s indicator function. By Borel, for \( M|N \) (cf.\([2]\)):

\[
\dim K_{2n-1}(\mathcal{O}_{k\mathbb{N}}[1/M]) \otimes \mathbb{Q} = \begin{cases} 
1 & \text{if } N = 1 \text{ or } 2, \text{ and } n \text{ odd }, (n,N) \neq (1,1). \\
0 & \text{if } N = 1 \text{ or } 2, \text{ and } n \text{ even .} \\
\frac{\varphi(N)}{2} + p(M) - 1 & \text{if } N > 2, n = 1. \\
\frac{\varphi(N)}{2} & \text{if } N > 2, n > 1.
\end{cases}
\]

Let \( \mathcal{L}^+ \) denote the completion of the pro-nilpotent graded Lie algebra of the pro-unipotent group \( \mathcal{U} \) (defined by a limit); \( \mathcal{L}^+ \) is free since \( Ext^2 = 0 \) and graded with positive degrees from the \( G_m \)-action. Furthermore:

\[
\mathcal{L}^a = \bigoplus (Ext^1_{\mathcal{MT}_N}(\mathbb{Q}(0),\mathbb{Q}(n))^\vee \text{ in degree } n) = \bigoplus (K_{2n-1}(\mathcal{O}_{k}[1/N])^\vee \text{ in degree } n). \quad (2.1)
\]
Hence:  
\[ \mathcal{A}^\text{MT} = \mathcal{O}(U) \cong (U^{\wedge}(L^{\wedge}))^\vee \cong T(\oplus_{n \geq 1} K_{2n-1}(\mathcal{O}_k[1/N]) \otimes \mathbb{Q}). \]  

**Remark:** Recall that if \( L \) is the free Lie algebra generated by \( e_0, e_\eta, \eta \in \mu_N \), the enveloping algebra \( U(L) \) is the associative free Lie algebra \( \mathbb{Q} < e_0, (e_\eta)_{\eta \in \mu_N} > \), the completed enveloping algebra of the completed Lie algebra \( U^{\wedge}(L^{\wedge}) \) is \( \mathbb{Q} < \langle e_0, (e_\eta)_{\eta \in \mu_N} > \), the set of associative formal series, and the topological dual of \( U^{\wedge}(L^{\wedge}) \) is then a tensor algebra.

Hence, for \( N > 2 \), \( \mathcal{A}^\text{MT} \) is a cofree commutative graded Hopf algebra cogenerated by \( \varphi(N) \) elements \( f^*_i \) in degree 1, and \( \varphi(N)/2 \) elements \( f^*_i \) in degree \( i > 1 \).

Define \( d^N_n := \dim \mathcal{H}^\text{MT}N \), where \( n \) indicates the weight, and the Hilbert series: \( h_N(t) := \sum_k d^N_k t^k \).

**Lemma 2.1.** For \( N > 2 \), there is a recursive formula, with \( a_N := \frac{\varphi(N)}{2} + p(N) - 1 \), and \( b_N := \frac{\varphi(N)}{2} \):

\[
d^N_n = 1 + a_N d_{n-1} + b_N \sum_{i=2}^n d_{n-i} = (a_N + 1)d_{n-1} + (b_N - a_N)d_{n-2}, \quad d_0 = 1, \quad d_1 = a_N + 1.
\]

Hence the Hilbert series for the dimensions of \( \mathcal{H}^\text{MT}N \) is:

\[
h_N(t) = \frac{1}{1 - (a_N + 1)t + (a_N - b_N)t^2}.
\]

**Examples:**

| \( N \) \( \frac{\varphi(N)}{2} \) | Dimension relation \( d^N_n \) | Hilbert series \( h_N \) |
|---|---|---|
| 1 | \( d_n = d_{n-3} + d_{n-2}, \quad d_2 = 1, \quad d_1 = 0 \) | \( h_1(t) = \frac{1}{1 - t} \) |
| 2 | \( d_n = d_{n-1} + d_{n-2}, \quad d_0 = d_1 = 1 \) | \( h_2(t) = \frac{1}{1 - t} \) |
| 3, 4 | \( d_k = 2d_{k-1} = 2^k \) | \( h_3,4(t) = \frac{1}{1 - t^2} \) |
| 8 | \( d_k = 3d_{k-1} = 3^k \) | \( h_8(t) = \frac{1}{1 - t^3} \) |
| 6, \( \mathcal{MT}(\mathcal{O}_\frac{1}{2}) \) | \( d_k = 3d_{k-1} - d_{k-2} = 3^k \) | \( h_6(t) = \frac{1}{1 - t^3 + t} \) |
| 6, \( \mathcal{MT}(\mathcal{O}_\frac{1}{2}) \) | \( d_k = 1 + \sum_{i \geq 2} d_{k-i} = d_{k-1} + d_{k-2} \) | \( h_6(t) = \frac{1}{1 - t^3 + t} \) |

**Fundamental groupoid and motivic periods**

Let \( \Pi_{0,1} := \pi^\text{dR}_1(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}, \overrightarrow{0}, \overrightarrow{-1}) \) denote the de Rham realisation of the motivic torsor of paths from 0 to 1 (with tangential basepoints given by the tangent vectors 1 at 0 and \(-1\) at 1) on \( \mathbb{P}^1 \setminus \{0, \mu_N, \infty\} \). It is the following functor:

\[
\Pi_{0,1} : R \text{ a } \mathbb{Q} - \text{ algebra} \mapsto \{ S \in R \rhd < e_0, (e_\eta)_{\eta \in \mu_N} > \mid |\Lambda S = S \otimes S| \},
\]

the set of non-commutative formal series with \( N+1 \) generators which are group-like for the completed coproduct for which \( e_i \) are primitive. It is dual to the shuffle \( \shuffle \) relation between the coefficients of the series \( S \). Its affine ring of regular functions is \( \mathcal{O}(\Pi_{0,1}) \cong \mathbb{Q} \langle e_0, (e_\eta)_{\eta \in \mu_N} \rangle \), which is a graded algebra for the shuffle product \( \shuffle \).

Denote by \( \mathcal{MT}N \) the full Tannakian subcategory of \( \mathcal{MT}N \) generated by the motivic fundamental groupoid of \( \mathbb{P}^1 \setminus \{0, \mu_N, \infty\} \), (with tangential basepoints given by the tangent vectors 1 at 0 and \(-1\) at 1). Denote also by \( \mathcal{G} = \mathcal{G}^\text{MT}N = \mathbb{G}_m \times \mathcal{U}^\text{dR} \) its Galois group defined over \( \mathbb{Q} \), \( A = \mathcal{O}(\mathcal{U}^\text{dR}) \) its fundamental Hopf algebra and \( \mathcal{L} := \mathcal{A}_{>0}/\mathcal{A}_{>0} \cdot \mathcal{A}_{=0} \) the Lie coalgebra of indecomposable elements.

We denote by \( \mathcal{H}N \subseteq \mathcal{O}(P_{B,\omega}) \) the space of **motivic multiple zeta values** relative to \( \mu_N \) as defined below.

A **motivic period** in a tannakian category of mixed Tate motives \( \mathcal{M} \) is (cf. [9]) a triplet \( [M, v, \sigma] \), where \( M \in \text{Ind}(\mathcal{M}) \), \( v \in \omega(M), \sigma \in \omega_B(M)^\vee \). Such a motivic period \( p^{\text{mot}}_{v,\sigma} M \in \mathcal{O}(P_{B,\omega}) \) can be seen as a function on \( P_{B,\omega} \):

\[
p^{\text{mot}}_{v,\sigma} M : P_{B,\omega}(\mathbb{Q}) \to \mathbb{Q}, p \mapsto < v, p(\sigma) >.
\]

Its period is obtained by evaluation \( p^{\text{mot}}_{v,\sigma} M \) on the complex point \( \text{comp}_{B,\text{dR}} \):

\[
\text{per}(p^{\text{mot}}_{v,\sigma} M) := p_{v,\sigma} = < \text{comp}_{B,\text{dR}}(v \otimes 1), \sigma > \in \mathbb{C}.
\]
In our case, $\mathcal{M} = \mathcal{M}T_N$ and $M = \mathcal{O}(\pi_1^m(\mathbb{P}^1 - \{0, \mu_N, \infty\}), \Gamma_0^{-1}, \Gamma_0^0)$.

A **motivic iterated integral** is $I^m(w) = [M, w, dch_B]$ where $dch_B$ is the image of the straight path (droit chemin) in $\omega(M)^\vee$.

Its period is $\text{per}(I^m(w)) = I(w) = \int_0^1 w = < \text{comp}_{B,dR}(w \otimes 1), dch_B > \in \mathbb{C}$.

The space $H$ is a quotient of $\mathcal{O}(\Pi_{0,1})$.

To a word $w$ in $(0, \eta_{E_{\mu}})$, we associate its image $I^m(0; w; 1) \in H$, called the motivic iterated integral with the correspondence:

$$(a_1, \cdots, a_n) \in \{0, \eta_{E_{\mu}}\}^n \leftrightarrow \omega_{a_1} \cdots \omega_{a_n} \text{ where } \omega_{\alpha}(t) = \frac{dt}{t - \alpha}$$

Let $I^a$ respectively $I^l$ be its image in $A$, resp. $L$.

**Definition 2.2.** The **motivic multiple zeta values** relative to $\mu_N$ are defined by:

$$\zeta^m\left(\frac{x_1, \cdots, x_p}{\epsilon_1, \cdots, \epsilon_p}\right) := I^m\left(0; (\epsilon_1 \cdots \epsilon_p)^{-1}, 0, \epsilon_1^{-1}, \cdots, (\epsilon_1 \cdots \epsilon_p)^{-1}, 0, \epsilon_1^{-1}, \cdots, \epsilon_p^{-1}, 0, \epsilon_1^{-1}, \cdots, \epsilon_p^{-1}; 1\right),$$

for $\epsilon_i \in \mu_N, k \geq 0, x_i > 0$ and $(x_p, \epsilon_p) \neq (1, 1)$.

More generally, we define under the same conditions:

$$\zeta^m\left(\frac{x_1, \cdots, x_p}{\epsilon_1, \cdots, \epsilon_p}\right) := I^m\left(0; 0^k, (\epsilon_1 \cdots \epsilon_p)^{-1}, 0, \epsilon_1^{-1}, \cdots, (\epsilon_1 \cdots \epsilon_p)^{-1}, 0, \epsilon_1^{-1}, \cdots, \epsilon_p^{-1}, 0, \epsilon_1^{-1}, \cdots, \epsilon_p^{-1}; 1\right).$$

NB: An overline at the end means that the corresponding $\epsilon_i$ are 1, except the last one which is $\exp(2\pi i)$. For instance, for $N = 3$: $\zeta(3, 6, 2) = \zeta\left(\frac{3, 6, 2}{1, 1, \text{exp}(2\pi i)}\right)$.

There is a surjective homomorphism called the **period map**, conjectured to be isomorphism:

$$\text{per}: H \rightarrow \mathbb{Z}, \zeta^m\left(\frac{x_1, \cdots, x_p}{\epsilon_1, \cdots, \epsilon_p}\right) \mapsto \zeta\left(\frac{x_1, \cdots, x_p}{\epsilon_1, \cdots, \epsilon_p}\right).$$

Each identity between motivic multiple zeta values at roots of unity is then true for multiple zeta values at roots of unity and in particular each result about a basis with motivic MZVs implies the corresponding result about a generating family of MZVs by application of the period map.

Conversely, we can almost lift an identity between MZVs at roots of unity to an identity between motivic ones up to one rational coefficient thanks to the coaction (via an analogue of [3], Theorem 3.3 for roots of unity).

The comodule $H^N \subseteq \mathcal{O}(P_{B, \omega})$ embeds, non canonically, into $\mathcal{H}^\mathcal{MT}_N$.

We will work in the subcategories $\mathcal{M}T_N$, which are equivalent to $\mathcal{M}T_N$ since $H = \mathcal{H}^\mathcal{MT}$ for $N = 1, 2, 3, 4, 6, 8$ ([3] for $N = 1$, proved later for $N > 1$ or cf. [3]). For each $N, N'$ with $N' \mid N$, the motivic Galois descent has a parallel for the motivic fundamental group:

$$\xymatrix{ H^N \ar@{~>}[rr]^{\sim} \ar@{~>}[dr]_{\sim} \ar@{~>}[drr]_{\sim} & & H^\mathcal{MT}_N \ar@{~>}[dr]_{\sim} \ar@{~>}[drr]_{\sim} \ar@{~>}[rr]^{\sim} \ar@{~>}[drr] & & H^\mathcal{MT}_N \ar@{~>}[rr]^{\sim} \ar@{~>}[drr] & & (H^N)^{\mathcal{G}_{N/N'}} = H^{N'}. \ar@{~>}[drr] \ar@{~>}[drr] \ar@{~>}[drr] \ar@{~>}[drr] \ar@{~>}[drr] \ar@{~>}[drr] \ar@{~>}[drr] }$$

**Coaction** The group $\mathcal{G}^\mathcal{MT}_N$ acts on the de Rham realisation $\Pi_{0,1}$ of the motivic fundamental groupoid (cf. [10], §5.12). Since $A^\mathcal{MT} = O(U^\mathcal{MT})$, the action of $U^\mathcal{MT}$ on $\Pi_{0,1}$ gives rise by duality to a coaction: $\Delta^\mathcal{MT}$. It factorizes through $A$ since $U^\mathcal{MT}'$ is the quotient of $U^\mathcal{MT}$ by the kernel of its action on $\Pi_{0,1}$ ([10]).
Then the combinatorial coaction (on words on 0, \( \eta, \eta \in \mu_N \)) induces a coaction \( \Delta \) on \( \mathcal{H} \), which is explicit (by Goncharov \cite{Goncharov} and extended by Brown); the formula being given in a following paragraph.

\[
\begin{array}{c}
\mathcal{O}(\Pi_{0,1}) \xrightarrow{\Delta^{\text{MT}}} A^{\text{MT}} \otimes_\mathbb{Q} \mathcal{O}(\Pi_{0,1}) \\
\sim \\
\mathcal{O}(\Pi_{0,1}) \xrightarrow{\Delta^c} A \otimes_\mathbb{Q} \mathcal{O}(\Pi_{0,1}) \\
\mathcal{H} \xrightarrow{\Delta} A \otimes \mathcal{H}.
\end{array}
\]

### 2.2 Properties of motivic iterated integrals

Extend the previous definition of \( I^m(a_0; a_1, \cdots a_n; a_{n+1}) \in \mathcal{H}_n \), with \( a_i \in \mu_N \cup \{0\} \) defined as above if \( a_0 = 0 \) and \( a_{n+1} = 1 \), and extend -in an unique way- by the following properties:

(i) \( I^m(a_0; a_1) = 1 \).

(ii) \( I^m(a_0; a_1, \cdots a_n; a_{n+1}) = 0 \) if \( a_0 = a_{n+1} \).

(iii) Shuffle product:

\[
\zeta_k^m \left( \begin{array}{c} x_1, \cdots, x_p \\ \epsilon_1, \cdots, \epsilon_p \end{array} \right) = \sum_{i_1 + \cdots + i_p = k} (-1)^k \left( \begin{array}{c} x_1 + i_1 - 1 \\ i_1 \\ \vdots \\ x_p + i_p - 1 \\ i_p \end{array} \right) \zeta_n^m \left( \begin{array}{c} x_1 + i_1, \cdots, x_p + i_p \\ \epsilon_1, \cdots, \epsilon_p \end{array} \right), \tag{2.3}\]

(iv) Path composition:

\[
\forall x \in \mu_N \cup \{0\}, I^m(a_0; a_1, \cdots, a_n; a_{n+1}) = \sum_{i=1}^{n} I^m(a_0; a_1, \cdots, a_i; x) I^m(x; a_{i+1}, \cdots, a_n; a_{n+1}).
\]

(v) Path reversal: \( I^m(a_0; a_1, \cdots, a_n; a_{n+1}) = (-1)^n I^m(a_{n+1}; a_n, \cdots, a_1; a_0) \)

(vi) Homothety: \( \forall a \in \mu_N, I^m(0; a \alpha_1, \cdots, a \alpha_n; a \alpha_{n+1}) = I^m(0; a_1, \cdots, a_n; a_{n+1}) \).

**Remark:** These relations, for the multiple zeta values relative to \( \mu_N \), and for the iterated integrals \( I(a_0; a_1, \cdots a_n; a_{n+1}) := \int_{\gamma} \omega_{a_1} \cdots \omega_{a_{n+1}}, \) where \( \omega(t) = \frac{dt}{t} \) and \( \gamma \) the straight path from \( a_0 \) to \( a_{n+1} \), are obviously all easily checked, following from the properties of iterated integrals.

### 2.3 Depth filtration

**Definition 2.3.** The depth of a (simple or motivic) \( \text{MZV} \) relative to \( \mu_N \) \( \zeta \left( \begin{array}{c} x_1, \cdots, x_p \\ \epsilon_1, \cdots, \epsilon_p \end{array} \right) (2\pi)^s \) is \( p \). It defines an increasing filtration by the depth, denoted \( \mathcal{F}^D_i \).

**Remark:** Beware, depth is not a graduation. For instance: \( \zeta^m(3) = \zeta^m(1, 2) \).

In depth 1, it is known for \( A \) (cf. P. Deligne and A. B. Goncharov, \cite{DeligneGoncharov} Theorem 6.8):

**Lemma 2.4.** The elements \( \zeta^a(r; \eta) \) are subject only to the following relations in \( A \):

- **Distribution relations:**
  \[
  \forall d | N, \forall \eta \in \mu_N, (\eta, r) \neq (1, 1), \zeta^a \left( \begin{array}{c} r \\ \eta \end{array} \right) = d^{r-1} \sum_{\epsilon A \eta} \zeta^a \left( \begin{array}{c} r \\ \epsilon \end{array} \right).
  \]
Conjugation relations:

\[ \zeta^a \left( \begin{array}{c} r \\ \eta \end{array} \right) = (-1)^{r-1} \zeta^a \left( \begin{array}{c} r \\ \eta^{-1} \end{array} \right). \]

Remark: The distribution relations for multiple zeta values relative to \( \mu_N \) are the following:

\[ \forall d|N, \forall \epsilon_1 \in \mu_\infty, \zeta \left( \begin{array}{c} x_1, \ldots, x_p \\ \epsilon_1, \ldots, \epsilon_p \end{array} \right) = d \sum_{\eta_1 = \epsilon_1} \cdots \sum_{\eta_p = \epsilon_p} \zeta \left( \begin{array}{c} x_1, \ldots, x_p \\ \eta_1, \ldots, \eta_p \end{array} \right). \]

It comes from the following identity:

For \( d|N, \epsilon \in \mu_\infty \), \( \sum_{\eta_\epsilon = \epsilon} \eta^n = \begin{cases} dc & \text{if } d|n \\ 0 & \text{else} \end{cases}. \)

2.4 Infinitesimal coactions

The coaction \( \Delta : H \to A \otimes Q \) is given by the combinatorial coaction \( \Delta^c : \)

\[ \Delta^c I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{k, i_0 = 0 < \cdots < i_k < i_{k+1} = n+1} \left( \prod_{p=0}^{k} I^p(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes I^m(a_0; a_1, \ldots, a_{n+1}). \]

It has a nice geometric formulation, considering the \( a_i \) as vertices on a half-circle: \( \Delta^c I^m \) corresponds to a sum on all the possible polygons with vertices \( (a_i) \) (sub-sequence of vertices among the \( a_i \)) the product of motivic iterated integrals associated to each arc (between two consecutive vertices, from \( a_{i_p} \) to \( a_{i_{p+1}} \)) \( \otimes \) the motivic iterated integrals associated to the polygon vertices.

Define for \( r \geq 1 \), the derivation operators \( D_r : H \to \mathcal{L} \otimes Q \) composite of \( \Delta^c = \Delta^c - 1 \otimes id \) with \( \pi_r \otimes id \), where \( \pi_r \) is the projection \( A \to \mathcal{L} \to \mathcal{L} \). These maps \( D_r \) are derivations:

\[ D_r(XY) = (1 \otimes X)D_r(Y) + (1 \otimes Y)D_r(X). \]

According to the previous theorem, the action of \( D_r \) on \( I^m(a_0; a_1, \ldots, a_n; a_{n+1}) \) is:

\[ D_r I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{p=0}^{n-1} I^p(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1}) \otimes I^m(a_0; a_1, \ldots, a_p, a_{p+r+1}, \ldots, a_{n+1}). \]  

(2.4)

Geometrically, it is equivalent to keep in the previous coaction only the polygon corresponding to an unique cut of interior length \( r \).

A key point is that this action respects the weight grading and the depth filtration:

\[ D_r(H_{n-r}) \subset \mathcal{L}_r \otimes Q H_{n-r} \]

\[ D_r(F^D_p H) \subset \mathcal{L}_r \otimes F^D_{p-1} H. \]

So let us consider the right-depth-graded part \( D_{r,p} \) and define maps that will be particularly useful:

Definition 2.6.

- The map \( D_{r,p} : gr^D_p H \to \mathcal{L}_r \otimes gr^D_{p-1} H \), defined as the composition

\[ (id \otimes gr^D_{p-1}) \circ D_r|_{gr^D_p H} \] maps into \( gr^D_1 \mathcal{L}_r \otimes gr^D_{p-1} H. \)

- The map \( \partial_{n,p} = \oplus_{0<r<n}(D_{r,p}) : gr^D_p H \to \oplus_{0<r<n} gr^D_1 \mathcal{L}_r \otimes gr^D_{p-1} H_{n-r}. \)

- For a suitable \( \eta \in \mu_N \), \( D^n_r \) is the map \( gr^D_p H \to gr^D_{p-1} H \) composition of \( D_{r,p} \) followed by the projection

\[ \pi^n : gr^D_1 \mathcal{L}_r \otimes gr^D_{p-1} H \to gr^D_{p-1} H : \zeta^n(r; \epsilon) \otimes X \mapsto c_{n,r} X, \]

where \( c_{n,r} \in Q \) is the coefficient of \( \zeta^n(r; \epsilon) \) in \( \zeta^n(r; \epsilon) \), well defined for some \( \eta \) cf. the remark below.
Remark Thanks to Deligne and Goncharov’s results ([10]) in depth 1, we know a basis for \( gr^1_{\mathbb{L}} A_r = gr^1_{\mathbb{L}} L_r \), so we can identify the left hand part of \( D_{r,p} \) with \( \mathbb{Q} \) or \( \mathbb{Q} \oplus \mathbb{Q} \) and the constants \( c_{\eta, r} \in \mathbb{Q} \) are well defined and explicit, once we have chosen a basis. More precisely, in the case \( N = 2, 3, 4 \), the \( D_{r}^{\eta} \) for \( \eta \in \mu_N \) are all proportional (possibly zero, setting \( D_{r}^{\eta} = 0 \) if \( \zeta^m(r; \eta) = 0 \)), whereas for \( N = 8 \) the space of derivations spanned by the "\( D \)" is 2-dimensional, generated by \( D_{r}^{\xi} \) and \( D_{r}^{-\xi} \), where \( \xi_8 = \exp(\frac{2\pi i}{8}) \).

Using (2.3), we can calculate their action on the motivic iterated integrals:

**Lemma 2.7.** Setting \( \eta_p := \epsilon_p^{-1} \), \( \eta_i := (\epsilon_1 \cdots \epsilon_p)^{-1} \), i.e. \( \epsilon_i = \eta_i^{-1} \eta_{i+1} \), with \( x_0 = 1 \) we have:

\[
D_{r,p} \left( \zeta^m \left( x_1, \cdots, x_p \right) \right) = D_{r,p} \left( I^m \left( 0; \eta_1, 0^{x_1-1}, \cdots, \eta_p, 0^{x_p-1}; 1 \right) \right) =
\]

(TYPE A) \( \sum_{\delta_{x_i \leq x_{i-1} \leq x_{i+1} \leq 1}} \delta_{x_i \leq x_{i-1} \leq x_{i+1} \leq 1} I^l(0; 0^{r-x_i} \eta_i, 0^{x_i-1}; \eta_{i+1}) \otimes I^m(0; \cdots, \eta_i, 0^{x_i-x_{i-1}-r-1}, \eta_{i+1}, \cdots; 1) \)

(TYPE B) \( \sum_{\delta_{x_i \leq x_{i+1} \leq x_{i+2} \leq 1}} \delta_{x_i \leq x_{i+1} \leq x_{i+2} \leq 1} I^l(0; 0^{x_i-1} \eta_{i+1}, 0^{r-x_{i+1}}; 0) \otimes I^m(0; \cdots, \eta_i, 0^{x_{i+1}-x_{i+2}-r-1}, \eta_{i+2}, \cdots; 1) \)

(TYPE C) \( \sum_{\delta_{x_i \leq x_{i+1} \leq x_{i+2} \leq 1}} \delta_{x_i \leq x_{i+1} \leq x_{i+2} \leq 1} I^l(0; \eta_i, 0^{x_i-1}, \eta_{i+1}) \otimes I^m(0; \cdots, \eta_i, \eta_{i+1}, \cdots; 1) \)

(TYPE D) \( \delta_{x_p \leq x_{p-1} \leq x_{p-2} \leq 1} I^l(0; \eta_p, 0^{r-x_p}; 0) \otimes I^m(0; \cdots, \eta_{p-1}, 0^{x_p-x_{p-1}-r-1}; 1) \)

(TYPE D’) \( \delta_{x_p \leq x_{p-1} \leq x_{p-2} \leq 1} I^l(0; \eta_{p-1}, 0^{x_{p-1}-r-1}, \eta_{p-1}, 0^{x_p-1}; 1) \otimes I^m(0; \cdots, \eta_{p-1}; 1) \).

Transposing for the motivic multiple zeta values and using the properties of motivic iterated integrals previously listed, it gives the following lemma:

**Lemma 2.8.**

\[
D_{r,p} \left( \zeta^m \left( x_1, \cdots, x_p \right) \right) =
\]

(TYPE A) \( \delta_{r=x_1} \zeta^l \left( r \epsilon_1 \right) \otimes \zeta^m \left( x_2, \cdots \right) \)

\[
+ \sum_{i=2}^{p-1} \delta_{x_i \leq x_{i-1} \leq x_{i+1} \leq 1} (-1)^{r-x_i} \zeta^l \left( \frac{r-1}{r-x_i} \right) \otimes \zeta^m \left( \cdots, x_i + x_{i-1} - r, \cdots, \epsilon_i-1 \epsilon_i, \cdots \right)
\]

(TYPE B) \( \sum_{i=1}^{p-1} \delta_{x_i \leq x_{i+1} \leq x_{i+2} \leq 1} (-1)^{x_i} \zeta^l \left( \frac{r-1}{r-x_i} \right) \otimes \zeta^m \left( \cdots, x_i + x_{i+1} - r, \cdots, \epsilon_{i+1} \epsilon_i, \cdots \right) \)

(TYPE C) \( \sum_{i=2}^{p-1} \delta_{x_i \leq x_{i+1} \leq x_{i+2} \leq 1} (-1)^{x_i} \zeta^l \left( \frac{r-1}{x_i-1} \right) \zeta^l \left( \frac{r-1}{x_i} \right) \otimes \zeta^m \left( \cdots, 1, \cdots \right) \)

(TYPE D) \( \delta_{x_p \leq x_{p-1} \leq x_{p-2} \leq 1} (-1)^{x_p} \zeta^l \left( \frac{r-1}{x_p} \right) \otimes \zeta^m \left( \cdots, x_{p-1} + x_p - r, \cdots, \epsilon_{p-1} \epsilon_p \right) \)

(TYPE D’) \( \delta_{x_p \leq x_{p-1} \leq x_{p-2} \leq 1} (-1)^{x_p} \zeta^l \left( \frac{r-1}{x_p-1} \right) \zeta^l \left( \frac{r-1}{x_p} \right) \otimes \zeta^m \left( \cdots, 1, \cdots \right) \).
Remark: The terms of (Type A) correspond to cuts from a 0 to a root of unity, (Type B) from a root of unity to a 0, (Type C) between two roots of unity and (Type D, D') are the cuts ending by the last 1. The terms of (Type D, D') play a particular role since they correspond to deconcatenation for the coaction, and will be our privileged terms since modulo some congruences, we will get rid of the other terms.
3 The case $N = 2$

3.1 Definitions

Define particular families of motivic multiple zeta values relative to $\mu_2$, and a notion of level and motivic level.

NB: Here we have to consider only one Galois descent, from $\mathcal{H}^2$ to $\mathcal{H}^1$ which is reflected by motivic level 0 (usual exponents can be omitted).

Definition 3.1. \( B^2 := \{ \zeta^m(2x_1 + 1, \cdots, 2x_p + 1, 2x_p + 1) \zeta(2)^{m,k}, x_i \geq 0, k \in \mathbb{N} \} \).

Here, the level is defined as the number of $x_i$ equal to zero.

- The filtration by the motivic ($\mathbb{Q}/\mathbb{Q}, 2/1$)-level,
  \[ \mathcal{F}_i \mathcal{H} := \{ \xi \in \mathcal{H}, \text{ such that } D_{i-1}^r \xi \in \mathcal{F}_{i-1} \mathcal{H}, \forall r > 0, D_{2r+1}^r \xi \in \mathcal{F}_r \mathcal{H} \}. \]

  With $\mathcal{F}_{-1} \mathcal{H} = 0$, the associated graded is denoted: $gr_\mathcal{F}$. It’s an graded Hopf algebra’s filtration:
  \[ \mathcal{F}_i \mathcal{H}, \mathcal{F}_j \mathcal{H} \subset \mathcal{F}_{i+j} \mathcal{H}, \Delta(\mathcal{F}_n \mathcal{H}) \subset \sum_{i+j=n} \mathcal{F}_i \mathcal{A} \otimes \mathcal{F}_j \mathcal{H} \]

  Remind that $\mathcal{H}^B$ denote the sub-$\mathbb{Q}$-vector space spanned by $B$ and that in $B_{n,p,i}, B_{n,p}, B_{n,i}, B_{n,p \geq i}$ the first index denotes the weight, the second the depth and the third the level, possibly indeterminate.

The filtration thus defined also commutes with the increasing depth filtration.

Set $\mathbb{Z}_{odd} = \{ \frac{a}{b}, a \in \mathbb{Z}, b \in 2\mathbb{Z} + 1 \}$, rationnals having a 2-adic valuation positive or infinite.

Lemma 3.2. The 0\textsuperscript{th} level of this filtration is isomorphic to the algebra of (honorary) motivic multiple zeta values:

\[ \mathcal{F}_0 \mathcal{H}^{MT_2} = \mathcal{F}_0 \mathcal{H}^{N=2} = \mathcal{H}^{MT_1} = \mathcal{H}^{N=1}. \]

Proof. The first equality follows from the result proved later: $\mathcal{H}^{MT_2} = \mathcal{H}^{N=2}$.

The last equality is due to a recent result of F. Brown ([4]). In addition, there is an inclusion $\mathcal{H}^{N=1} \subset \mathcal{F}_0 \mathcal{H}$. Using an equality of dimensions for fixed weight, we can conclude.

REMARKS:

- Honorary multiple zeta value have been studied notably by D. Broadhurst. In [5], there is some examples of honorary multiple zeta values. The methods of this paper enable us also to determine when a multiple zeta value is honorary modulo elements of smaller depth because the criteria given is based on a recursive calculation of $D_{2r+1}, D_{1}$ (or here depth-graded versions of it). In another paper, we provide infinite families of honorary multiple zeta values up to depth 5 with alternating patterns of odd’s or even’s.

- We could also have defined this filtration as the biggest filtration generated by the coproduct $\Delta$, such that $\mathcal{F}_0 \mathcal{H} = \mathcal{H}^{N=1}$ and $\Delta \mathcal{F}_n \mathcal{H} \subset \sum_{i+j=n} \mathcal{F}_i \mathcal{A} \otimes \mathcal{F}_j \mathcal{H}$.

- The increasing filtration by the motivic ($\mathbb{Q}/\mathbb{Q}, 2/1$)-level is the increasing filtration coming from higher ramification groups. Indeed, $\mathcal{F}_0$ is the set of the invariants under the Galois group $G^{2/1} := Gal(\mathcal{H}^2/\mathcal{H}^1)$ (such that $U_{N=2} = G^{2/1} \times U_{N=1}$), and similarly $\mathcal{F}_{i+1}$ can be seen as the set of invariants under the $i$th higher ramification group $G_i$.

- Notice that the increasing or decreasing filtration defined from the number of 1 appearing in the motivic multiple zeta values is not preserved by the coproduct, since the number of 1 can either decrease or increase (by at the most 1) and is therefore not motivic.

Consider the following quotients:

Definition 3.3.

\[ \mathcal{H}^{\geq 0} := \mathcal{H}, \mathcal{H}^{\geq i} := \mathcal{H}/\mathcal{F}_{i-1} \mathcal{H}. \]
They are coalgebras, compatible with product $(\mathcal{H}^{\geq i}, \mathcal{H}^{\geq j}) \subset \mathcal{H}^{\geq i+j}$, and we have projections:

\[ \forall j \geq i, \pi_{i,j} : \mathcal{H}^{\geq i} \rightarrow \mathcal{H}^{\geq j}. \]

Define similarly $\mathcal{H}_n^{\geq i+MT}$ and $\mathcal{H}_n^{\geq iB}$ as the quotient of $\mathcal{H}_n^B$ by the $\mathbb{Q}$-vector space spanned by $\mathcal{B}_{n,\leq i-1}$.

**Depth 1** The distribution relation in depth 1 is:

\[ \zeta^a(2r+1;1) = (2^{-2r} - 1)\zeta^a(2r+1;1). \]

**Infinitesimal coactions** The formula (3.3) in the case $N = 2$, for the elements of $\mathcal{B}, \forall r > 0$:

\[ D_{2r+1,p}(\zeta^m(2x_1+1, \cdots, 2x_p+1)) = \zeta(2r+1) \otimes [ \delta_{r=x_1, \zeta^m(2x_2+1, \cdots, 2x_p+1)} \]

(TYPE A) $\sum_{i=1}^{p-2} \delta_{x_i+1 \leq r < x_i+x_{i+1}} \left( \begin{array}{c} 2r \\ 2x_{i+1} \end{array} \right) \zeta^m(2x_1+1, \cdots, 2x_{i-1}+1, 2(x_i+x_{i+1}-r)+1, 2x_{i+2}+1, \cdots, 2x_p+1)$

(TYPE B) $\sum_{i=1}^{p-1} \delta_{x_i \leq r < x_i+x_{i+1}} \left( \begin{array}{c} 2r \\ 2x_i \end{array} \right) \zeta^m(2x_1+1, \cdots, 2x_{i-1}+1, 2(x_i+x_{i+1}-r)+1, 2x_{i+2}+1, \cdots, 2x_p+1)$

(TYPE D) $\delta_{x_p \leq r < x_p+x_{p-1}} \left( \begin{array}{c} 2r \\ 2x_p \end{array} \right) (2^{-2r} - 1)\zeta^m(2x_1+1, \cdots, 2x_{p-2}+1, 2(x_p-1+x_p-r)+1)$

(TYPE D') $\delta_{r=x_p+x_{p-1}} \left( \begin{array}{c} 2r \\ 2x_p \end{array} \right) (2^{-2r} - 2)\zeta^m(2x_1+1, \cdots, 2x_{p-2}+1)$.

As said before, the terms of (TYPE D, D') play a particular role since they correspond to deconcatenation for the coaction, and will be the terms of minimal 2-adic valuation.

$D_{1,p}$ acts as a deconcatenation on this family:

\[ D_{1,p}(\zeta^m(2x_1+1, \cdots, 2x_p+1)) = \begin{cases} 0 & \text{if } x_p \neq 0 \\ \zeta^m(2x_1+1, \cdots, 2x_{p-1}+1) & \text{if } x_p = 0. \end{cases} \]  

**3.2 Results**

Let us prove now the following result recursively on the weight, then recursively on the depth, for all $i \geq 0$ (if $i = 0$ the $i - 1$ appearing have to be replaced by $i$).

**Theorem 3.4.** 1. $\mathcal{B}_{n,p,\geq i}$ is a linearly free family of $\text{gr}_p^D \mathcal{H}_n^{\geq i}$, basis of $\text{gr}_p^D \mathcal{H}_n^{\geq i,B}$. The following map $\partial^i_{n,p}$ is bijective:

\[ \partial^i_{n,p} : \text{gr}_p^D \mathcal{H}_n^{\geq i,B} \rightarrow \text{gr}_{p-1}^D \mathcal{H}_n^{\geq i-1,B} \oplus 1 < r+1 \leq n-p+1 \text{ gr}_p^D \mathcal{H}_n^{\geq i-1,B}. \]

2. The basis $\mathcal{B}_{n,p,\geq i}$ defines a $\mathbb{Z}_{odd}$-structure of $\text{gr}_p^D \mathcal{H}_n^{\geq i}$, the projection $\pi_{0,i} : \mathcal{H}_n \rightarrow \mathcal{H}_n^{\geq i}$ is defined over $\mathbb{Z}_{odd}$ according to these structures. That is, each element $Z = (2x_1+1, \cdots, 2x_p+1) \in \mathcal{B}_{n,p}$ decomposes in a $\mathbb{Z}_{odd}$-linear combination of $\mathcal{B}_{n,p,\geq i}$ elements, denoted $cl_{n,p,\geq i}(Z)$, plus an element of $F_{i-1} \mathcal{H}_n + F_{i}^D \mathcal{H}_n$.

It defines, in an unique way, a map:

\[ cl_{n,p,\geq i} : \text{Vect}_\mathbb{Q}(\mathcal{B}_{n,p,\leq i-1}) \rightarrow \text{Vect}_\mathbb{Q}(\mathcal{B}_{n,p,\geq i}). \]

**Proof.** 1. By recursion hypothesis (weight being strictly smaller), we know that:

\[ \mathcal{B}_{n-1,p-1,\geq i-1} \oplus 1 < r+1 \leq n-p+1 \mathcal{B}_{n-2,r-1,p-1,\geq i-1} \] basis of $\text{gr}_{p-1}^D \mathcal{H}_n^{\geq i-1,B} \oplus 1 < r+1 \leq n-p+1 \text{ gr}_{p-1}^D \mathcal{H}_n^{\geq i-1,B}$. Let $M^i_{n,p}$ denote the matrix of $\left( \partial^i_{n,p}(x) \right)_{x \in \mathcal{B}_{n,p,\geq i}}$, the image of $\mathcal{B}_{n,p,\geq i}$. We easily check by a cardinal calc that $M^i_{n,p}$ is a square matrix. We need to prove invertibility of $M^i_{n,p}$.
We can cut the matrix of \( \partial_{n,p,i} \) in 4 blocks, corresponding at \( D_{1,p} \) respectively \( D_{>1,p} \) for the rows (whose image will be in \( Vect_{Q}(B_{n-1,p-1,\ge i-1}) \) respectively \( \oplus_{r \ge 1} Vect_{Q}(B_{n-r-1,p-1,\ge i}) \), and the elements of \( B_{n,p,\ge i} \) ending by 1, respectively the elements not ending by 1 for the columns.

Or, according to \( \ref{3.2} \), \( D_{1,p} \) is zero on the elements not ending by 1, and act as a deconcatenation on the others (and the deconcatenation is invertible). Therefore, \( M_{n,p}^{i} \) is lower triangular by blocks, the left-upper-block being diagonal invertible. It remains to prove the invertibility of the right-lower-block, corresponding to \( D_{>1,p} \) and the elements of \( B_{n,p,\ge i} \) not ending by 1.

Consider from then on \( \tilde{M}_{n,p}^{i} \), matrix of \( \tilde{\partial}_{n,p} \) which represents \( \oplus_{r \ge 0} 2^{2r} D_{2r+1,p} \) which comes to multiply the rows of \( M_{n,p}^{i} \) corresponding to \( D_{2r+1} \) by \( 2^{2r} \). Let us prove that it is invertible thanks to 2-adic properties of its elements.

In the formula \( \ref{3.1} \) of \( D_{2r+1,p} \), applied to an element of \( B_{n,p,\ge i} \), some terms appear with a number of 1 greater than \( i \) but also terms of \( B_{n-2r-1,p-1,\ge i} \), with \((i-1)^{n} \). We will make disappear the latter modulo 2, being 2-adically greater.

Using recursion hypothesis for the 2. of the theorem (in weight strictly smaller), we can replace these elements of \( B_{n-2r-1,p-1,\ge i} \) in \( g_{r-1}^{P-1} H_{n-2r-1}^{\le} \) by a \( Z \)-odd-linear combination of elements in \( B_{n-2r-1,p-1,\ge i} \), which do not lower the 2-adic valuation.

Once done, we can construct the matrix \( \tilde{M}_{n,p}^{i} \) and examine its entries. Order elements of \( B \) on both sides by lexicographic order of its "reversed" elements (with for the rows \( r \) that represents the last element, i.e. we order the rows first by decreasing weight) in \( M_{n,p}^{i} \), such that the diagonal corresponds to deconcatenation terms.

Referring to \( \ref{3.1} \), we see that \( \tilde{M}_{n,p}^{i} \) has all its entries of 2-adic valuation positive or equal to zero, since the coefficients in \( \ref{3.1} \) are integers or of the form \( 2^{-2r}Z \). If we look only at the terms with 2-adic valuation zero, (which comes to consider \( M_{n,p}^{i} \) modulo 2), it only remains in \( \ref{3.1} \) the terms of TYPE D and TYPE D', that is:

\[
2^{2r} D_{2r+1,p} (\zeta^{m}(2x_{1} + 1, \cdots, 2x_{p} + 1)) \equiv \delta_{r=x_{p}+x_{p-1}} \left( \frac{2r}{2x_{p}} \right) \zeta^{m}(2x_{1} + 1, \cdots, 2x_{p-2} + 1, 1) \]
\[
+ \delta_{x_{r} \le r < x_{p} + x_{p-1}} \left( \frac{2r}{2x_{p}} \zeta^{m}(2x_{1} + 1, \cdots, 2x_{p-2} + 1, 2(x_{p-1} + x_{p} - r) + 1)( \mod 2) \right).
\]

Modulo 2, with the order previously defined, it remains only an upper triangular matrix \( \delta_{x_{r} \le r} \), with 1 on the diagonal \( \delta_{x_{p}=r} \), deconcatenation terms).

Thus, \( \det \tilde{M}_{n,p}^{i} \) has a 2-adic valuation equals to zero, and in particular can not be zero, that’s why \( \tilde{M}_{n,p}^{i} \) is invertible.

2. It follows straight from 1.

Let us however specify the case where \( Z = \zeta(2x_{1} + 1, \cdots, 2x_{p} + 1) \) does not end by 1 (and has less than \( i \), \( x_{j} \) equals to 1, else the assertion is trivial). It is enough to show the existence of a \( Q \)-linear combination \( \sum_{k} \alpha_{k} e_{k} \) of elements \( e_{k} \in B_{n,p,\ge i} \) such that:

\[
\left\{ \begin{array}{l}
D_{1,p}(\sum_{k} \alpha_{k} e_{k}) = 0 = D_{1,p}(Z) \\
D_{2r+1,p}(\sum_{k} \alpha_{k} e_{k}) \equiv D_{2r+1,p}(Z) \quad \text{in } H_{\ge i}^{i-1}.
\end{array} \right.
\]

Indeed, we will have:

\[
Z - \sum_{k} \alpha_{k} e_{k} \in F_{i-1} + F_{p-1}^{D}.
\]

According to 1., the previous decomposition in upper triangular matrix by blocks of \( M_{n,p}^{i} \) following \( D_{1,p} \) and \( D_{>1,p} \), allow us to restrict to elements \( e_{k} \) not ending by 1, so that the condition on \( D_{1} \) will be automatically satisfied and the invertibility of \( \tilde{M}_{n,p}^{i} \) guarantees the
The second exact sequence is split, and \( B_{9,3} \) is diagonal by blocks following the different values of \( s \); here we restrict to the block \( s = 0 \). Indeed, \( \zeta^m(2) \) being trivial under the coaction, the matrix \( M_{9,3} \) is diagonal by blocks following the different values of \( s \) and we prove invertibility of each block separately; here we restrict to the block \( s = 0 \).

The matrix \( M_{9,3} \) considered shows the coefficients of:

\[
\zeta^m(2x + 1, 2y + 1) = 2^{2r} D_{2r+1,3}(\zeta^m(2a + 1, 2b + 1, 2c + 1)).
\]

The chosen order for the columns, i.e. the elements \( \zeta^m(2a + 1, 2b + 1, 2c + 1) \), is lexicographic order applied to triplet \((c, b, a)\). The chosen order for the rows, on \( (D_{2r+1,3}, \zeta^m(2x + 1, 2y + 1)) \) is lexicographic order on \((r, y, x)\).

Then, with this order, here is \( M_{9,3} \) modulo 2:

| \( D_r, \zeta^m \) \( \zeta^m(7, 1) \) | \( \zeta^m(5, 3) \) | \( \zeta^m(3, 3) \) | \( \zeta^m(1, 5) \) | \( \zeta^m(1, 3) \) | \( \zeta^m(1, 7) \) |
|---|---|---|---|---|---|
| \( 7, 1, 1 \) | 1 | 0 | 0 | 0 | 0 | 0 |
| \( 5, 3, 1 \) | 0 | 1 | 0 | 0 | 0 | 0 |
| \( 3, 1, 7 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 1, 7, 1 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 5, 1, 3 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 1, 5, 3 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 1, 3, 5 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 1, 1, 7 \) | 0 | 0 | 0 | 0 | 0 | 0 |

Indeed, modulo 2, it only remains the terms of Type D and Type D', that is:

\[
2^{2r} D_{2r+1,3}(\zeta^m(2a + 1, 2b + 1, 2c + 1)) \equiv \delta_{c \leq r \leq b+c} \left( \frac{2r}{2c} \right) \zeta^m(2a + 1, 2(b + c - r) + 1) \pmod{2}.
\]

Notice that the first 4 rows are exact (no need of congruences modulo 2 for \( D_1 \) because it acts as a deconcatenation on the base).

**Remark:** There are the following two exact sequences:

\[
0 \rightarrow \text{gr}^D_{\mathcal{F}_{i-1}} \mathcal{H}_n \rightarrow \text{gr}^D_{\mathcal{H}_n} \rightarrow \text{gr}^D_{\mathcal{H}_n} \rightarrow 0
\]

\[
0 \rightarrow \text{Vect}_Q(B_{n, p, \leq i-1}) \rightarrow \text{Vect}_Q(B_{n, p}) \rightarrow \text{Vect}_Q(B_{n, p, \geq i}) \rightarrow 0.
\]

The second exact sequence is split, and \( B_{n, p, \leq i} \) resp. \( B_{n, p, \geq i} \) are free families of \( \text{gr}^D_{\mathcal{H}_n} \), resp. of \( \text{gr}^D_{\mathcal{H}_n} \).

**Consequences** The corresponding result all depths mixed, whose consequences are stated above:

**Theorem 3.5.**

1. \( B_{n, \geq i} \) is a basis of \( \mathcal{H}_n^{\geq i} = \mathcal{H}_n^{\geq i, \mathcal{M}T} = \mathcal{H}_n^{\geq i, \mathcal{B}} \), on which it defines a \( \mathbb{Q} \)-structure.

2. We have the two split exact sequences in bijection:

\[
0 \rightarrow \mathcal{F}_i \mathcal{H}_n \rightarrow \mathcal{H}_n \rightarrow \mathcal{H}_n^{\geq i+1} \rightarrow 0
\]

\[
0 \rightarrow \text{Vect}_Q(B_{n, \leq i}) \rightarrow \text{Vect}_Q(B_{n}) \rightarrow \text{Vect}_Q(B_{n, \geq i+1}) \rightarrow 0.
\]
The basis $B_{n, \geq i}$ defines a $\mathbb{Z}_{\text{odd}}$ structure of $H_n^{\geq i}$, the projection $\pi_0,i : H_n \rightarrow H_n^{\geq i}$ is defined over $\mathbb{Z}_{\text{odd}}$ with regard to these structures.

We can define in an unique way, a map $cl_{n, \leq p, \geq i} : \text{Vect}_Q(B_{n, p, \leq i-1}) \rightarrow \text{Vect}_Q(B_{n, \leq p, \geq i})$, such that $x + cl_{n, \leq p, \geq i}(x) \in F_{i-1}H_n$.

3. With these notations, a basis of $F_iH_n$ is formed by:

$$\cup_p \{ x + cl_{n, \leq p, \geq i+1}(x), x \in B_{n, p, \leq i} \} .$$

Proof. 1. The elements of $B_{n, \geq i}$ are linearly independant in $H_n^{\geq i} \subset H_n^{\geq i, \text{MT}}$ according to 1. of the previous theorem. In addition, $\text{card} B_{n, \geq i} = \dim H_n^{\geq i, \text{MT}}$, hence the theorem follow, with the equality of inclusions.

2. Indeed, the second exact sequence is obviously split $B_{n, \geq i+1}$ being a subset of $B_n$. By 1., $B_n$ is a basis of $H_n$ and $B_{n, \geq i+1}$ is a basis of $H_n^{\geq i+1}$. Therefore, it gives a map $H_n \leftarrow H_n^{\geq i+1}$ and split the first exact sequence.

The construction of $cl(x)$ is the following: $x \in B_{n, \leq i-1}$ is sent on $x \in H_n^{\geq i} \cong \text{Vect}_Q(B_{n, \leq p, \geq i})$ by the projection $\pi_{0,i}$ and so $x - x \in F_{i-1}H_n$.

3. We have previously seen, that the elements of the indicated family are linearly independant, and that their cardinal is equal to the dimension of $F_iH_n^{\text{MT}}$. It gives a basis of $F_iH_n$ with elements $x \in B_{n, \leq i}$, each corrected by an element denoted $cl(x)$ of $\text{Vect}_Q(B_{n, \geq i+1})$.

Notice that the problem of making $cl(x)$ explicit boils down to the problem of describing the map $\pi_{0,i}$ in the bases $B$.

We also have the following result for the depth-graded:

**Corollary 3.6.** The elements of $B_{n, p, \geq i}$ form a basis of $\text{gr}_p H_n / F_{i-1}H_n$.

Proof. The elements are well linearly independant according to theorem 3.4, and generators, according to the previous theorem.

Theorem 3.5 generalizes in particular a result similar to P. Deligne’s one (cf. [8]), obtained here with $i = 0$:

**Corollary 3.7.** The map $\mathcal{G}^{\text{MT}} \rightarrow \mathcal{G}^{\text{MT}_2}$ is an isomorphism.

The elements of $B_n$, $\zeta^m(2x_1 + 1, \cdots, 2x_p + 1)\zeta(2)^k$ of weight $n$, form a basis of motivic Euler sums of weight $n$, $H_n^{\text{MT}}$.

The period map, $\text{per} : H \rightarrow \mathbb{C}$, induces the following result for the Euler sums:

**Corollary 3.8.** Each Euler sum is a $\mathbb{Q}$-linear combination of Euler sums $\zeta(2x_1 + 1, \cdots, 2x_p + 1)\zeta(2)^k$, $k \geq 0, x_i \geq 0$ with same weight.

We deduce also a basis of motivic level-graded:

**Corollary 3.9.** A basis of $\text{gr}_i H_n$ is, with the previous notations:

$$B_{n, p, i} := \cup_p \{ x + cl_{n, \leq p, \geq i+1}(x), x \in B_{n, p, i} \} .$$

Proof. Remark that the family:

$$\cup_{j=0}^{n-1} (\cup_p \{ x + cl_{n, \leq p, \geq i}(x), x \in B_{n, p, j} \}) \cup (\cup_p \{ x + cl_{n, \leq p, \geq i+1}(x), x \in B_{n, p, i} \})$$

is also a basis of $F_iH_n$ , and that it is formed, according to the previous theorem, by a basis of $F_{i-1}H_n$ union a basis of $\text{gr}_i H_n$.

In particular (with $i = 0$), it gives the Galois descent from $H^{\text{MT}_{(\mathbb{Z})}}$ to $H^{\text{MT}_2}$.
Corollary 3.10. A basis of motivic multiple zeta values, $\mathcal{H}_n^{\text{MT}(\mathbb{Z})}$, is formed by terms of $B_n$ with 0-level each corrected by linear combinations of elements of $B_n$ with level 1:

$$
\begin{cases}
\zeta^m(2x_1 + 1, \ldots, 2x_p + 1)\zeta^m(2) + \sum_{y_i \geq 0} \alpha_{y_i} \zeta^m(2y_1 + 1, \ldots, 2y_p + 1), \\
\sum_{\text{lower depth } q < p, z_i \geq 0} \beta_{z_i} \zeta^m(2z_1 + 1, \ldots, 2z_q + 1),
\end{cases}
$$

Comparison with Hoffman’s basis
Denote $B^{Hoff}$ Hoffman basis of $\mathcal{H}_n^{\text{MT}(\mathbb{Z})}$ according to F. Brown ([4]) formed by motivic MZV with only 2 and 3:

$$B^{Hoff} := \{\zeta^m(x_1, \ldots, x_k), \text{ where } x_i \in \{2, 3\}\}.$$ 

Here, $B^{Hoff}_{n,p}$ design the elements of $B^{Hoff}$ with weight $n$ and $p = 3^m$ among the $x_i$. Beware, in $B^{Hoff}_{n,p,0}$, $p$ still denotes the depth but in both cases, $p$ can be seen as the ‘motivic depth’. Remark that $B^{Hoff}_{n,p}$ and $B^{Hoff}_{n,p,0}$ have same cardinal.

Theorem 3.11. $B^{Hoff}_{n,p,0}$ is a basis of $gr^D\text{Vect}_Q(B^{Hoff}_{n,p})$ and defines a $\mathbb{Z}_{\text{odd}}$-structure.
I.e. each element of the Hoffman basis of weight $n$ and with $p = 3^m$ decomposes into a $\mathbb{Z}_{\text{odd}}$-linear combination of $B^{Hoff}_{n,p,0}$ elements plus terms of depth strictly less than $p$.

Proof. We prove that result recursively on $p$, and on the weight. For $p = 0$, $B^{Hoff}_{n,0}$ is empty for odd $n$, and reduced to $\{\zeta^m(2^2)\}$ else, which is rational multiple of $\zeta^m(2^2) \in B^{Hoff}_{n,0,0} = B^{Hoff}_{n,0,0}$.

For $p = 1$, F. Brown and Don Zagier (cf. [4, 19]) gave a explicit formula for $\zeta^m(2^a 3^b) \in B^{Hoff}_{n,1}$ expressing as a linear combination of elements $\zeta^m(2^k) \zeta^m(2r + 1) \in B^{Hoff}_{n,1,0} = B^{Hoff}_{n,1,0}$. Since that $\zeta^m(2r + 1) = \frac{2r}{2r}\zeta^m(2r + 1)$, the coefficients are in $\mathbb{Z}_{\text{odd}}$.

Let now $Z \in B^{Hoff}_{n,p,0}$, $p > 0$. Then $D_{2r+1}(Z) \subset \oplus_{k \leq p - 1} K_{2r+1} \otimes \text{Vect}_Q(B^{Hoff}_{n-2r-1,k})$ and by recursion hypothesis for depths $\leq p - 1$, the right part is effectively of depth $\leq p - 1$, which allows us to consider:

$$D_{2r+1}(Z) \subset L_{2r+1} \otimes gr^D_p \text{Vect}_Q(B^{Hoff}_{n-2r-1,\leq p-1}) \subseteq \mathbb{Q}\zeta^1(2r + 1) \otimes \text{Vect}_Q(B^{Hoff}_{n-2r-1,0,p-1}).$$

According to the previous results, there exists $Y \in \text{Vect}_Q(B^{Hoff}_{n,p,0})$ such that $D_{2r+1}(Z) = D_{2r+1}(Y)$ for all $r > 0$, $Y$ expressing as $Q$-linear combination of $B^{Hoff}_{n,p,0}$ elements. By injectivity of $\partial_{n,p}$, $Z = Y + X$, where $X \in F^D_{p-1}H$. 

\[\square\]
4 THE CASES \( N = 3, 4, 8 \) OR 6

4.1 Definitions

Levels and Motivic levels These cases can be handled in a rather similar way to the case \( N = 2 \), except that the number of generators is different and that several descents are possible, hence there will be several notions of level and filtrations by the motivic level, one for each descent.

For \( N = 3, 4 \) there is a generator in each degree \( \geq 1 \) and two Galois descents.

For \( N = 8 \) there is two generators in each degree \( \geq 1 \) and three Galois descents possible: with \( \mathcal{H}^4, \mathcal{H}^2 \) or \( \mathcal{H}^1 \).

For \( N = 6 \) there is one generator in each degree \( \geq 1 \) for the unramified category and there is one Galois descent with \( \mathcal{H}^1 \).

Let \( \xi = \exp(-\frac{2\pi}{\text{ord}}) \). Set \( P = 2 \) for \( N = 4, 8 \), \( P = 3 \) for \( N = 3, 6 \) and:

\[
Z_{i[P]} := \frac{Z}{1 + PZ} = \left\{ \frac{a}{1 + bP}, \ a, b \in \mathbb{Z} \right\}.
\]

Let us define particular families - future basis., notion of level on \( \mathcal{B} \) denoted \( i \) and the filtration by the motivic level corresponding denoted \( \mathcal{F}_i \):

**Definition 4.1** (For \( N = 3, 4 \)).

- **Family**: \( \mathcal{B} := \left\{ \mathfrak{c}^{m} \left( \frac{x_1, \ldots, x_p}{1, \ldots, 1, 0} \right) (2i\pi)^{s_m}, x_i \geq 1, s \geq 0 \right\} \).

- **Level**: The \((k_N/k_N, P/1)\)-level, resp. \((k_N/Q, P/P)\)-level, resp. \((k_N/Q, P/1)\)-level, is defined as the number of \( x_i \) equals to 1, resp. the number of \( x_i \) even, resp. the number of even \( x_i \) or equal to 1.

- **Filtrations by the motivic level**: With \( F_{-1} = 0 \):
  - The motivic \((k_N/k_N, P/1)\)-level:
    \[
    \mathcal{F}^{k_N/k_N, P/1}_{i} := \left\{ Z \in \mathcal{H}, \ s. \ t. \ D_{i}^{k_N/k_N, P/1}(Z) \in \mathcal{F}^{k_N/k_N, P/1}_{i-1} \mathcal{H}, \ \forall r > 0, \ D_{i}^{k_N/k_N, P/1}(Z) \in \mathcal{F}^{k_N/k_N, P/1}_{i} \mathcal{H} \right\}.
    \]
  - The motivic \((k_N/Q, P/P)\)-level:
    \[
    \mathcal{F}^{k_N/Q, P/P}_{i} := \left\{ Z \in \mathcal{H}, \ s. \ t. \ D_{2r}^{k_N/Q, P/P}(Z) \in \mathcal{F}^{k_N/Q, P/P}_{i-1} \mathcal{H}, \ \forall r \geq 0, \ D_{2r}^{k_N/Q, P/P}(Z) \in \mathcal{F}^{k_N/Q, P/P}_{i} \mathcal{H} \right\}.
    \]
  - The motivic \((k_N/Q, P/1)\)-level:
    \[
    \mathcal{F}^{k_N/Q, P/1}_{i} := \left\{ D_{2r}^{k_N/Q, P/1}(Z), D_{2r}^{k_N/Q, P/1}(Z) \right\}_{r \geq 0} \subset \mathcal{F}^{k_N/Q, P/1}_{i-1} \mathcal{H}, \ D_{2r}^{k_N/Q, P/1}(Z) \in \mathcal{F}^{k_N/Q, P/1}_{i} \mathcal{H} \right\}.
    \]

**Definition 4.2** (For \( N = 8 \)).

- **Family**: \( \mathcal{B} := \left\{ \mathfrak{c}^{m} \left( \frac{x_1, \ldots, x_p}{\epsilon_1, \ldots, \epsilon_p, 0} \right) (2i\pi)^{s_m}, x_i \geq 1, \epsilon_i \in \{\pm 1\}, s \geq 0 \right\} \).

- **Level**: The \((k_N/k_4, 2/2)\)-level, resp. \((k_N/k_4, 2/2)\)-level resp. \((k_4/Q, 2/1)\)-level, denoted \( i \), is the number of \( \epsilon_j \) equal to \(-1\), resp. plus the number of even \( x_j \) resp. plus the number of even \( x_j \) equal to 1.

- **Filtrations by the motivic level**: With \( F_{-1} = 0 \), the increasing filtrations:
  - The motivic \((k_N/k_4, 2/2)\)-level:
    \[
    \mathcal{F}^{k_N/k_4, 2/2}_{i} := \left\{ Z \in \mathcal{H}, \ s. \ t. \ \forall r > 0, \ D_{2r}^{k_N/k_4, 2/2}(Z) \in \mathcal{F}^{k_N/k_4, 2/2}_{i-1} \mathcal{H}, \ (D_{2r}^{k_N/k_4, 2/2}(Z) \in \mathcal{F}^{k_N/k_4, 2/2}_{i} \mathcal{H} \right\}.
    \]
  - The motivic \((k_N/Q, 2/2)\)-level:
    \[
    \mathcal{F}^{k_N/Q, 2/2}_{i} := \left\{ Z \in \mathcal{H}, \ s. \ t. \ \left\{ (D_{2r+1}^{k_N/Q, 2/2}-D_{2r}^{k_N/Q, 2/2})(Z) \right\}_{r \geq 0} \cup \left\{ D_{2r}^{k_N/Q, 2/2}(Z), D_{2r}^{k_N/Q, 2/2}(Z) \right\}_{r \geq 0} \subset \mathcal{F}^{k_N/Q, 2/2}_{i-1} \mathcal{H}, \ (D_{2r+1}^{k_N/Q, 2/2} + D_{2r+1}^{k_N/Q, 2/2})(Z) \in \mathcal{F}^{k_N/Q, 2/2}_{i} \mathcal{H} \right\}.
    \]

\[ \forall r \geq 0, \ (D_{2r+1}^{k_N/Q, 2/2} + D_{2r+1}^{k_N/Q, 2/2})(Z) \in \mathcal{F}^{k_N/Q, 2/2}_{i} \mathcal{H} \right\}.
\]

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The motivic \((k_N/Q,2/1)\)-level:

\[ \mathcal{F}_{i}^{k_N/Q,2/1}H := \left\{ Z \in H, \ s. t. \ (D_{2r+1}^{s} + D_{2r+1}^{-s})(Z) \in \mathcal{F}_{i}^{k_N/Q,2/1}H \right\}. \]

\[ \left\{ (D_{2r+1}^{s} - D_{2r+1}^{-s})(Z), D_{2r+1}^{s}(Z), D_{2r+1}^{-s}(Z) \right\}_{r>0} \cup \left\{ D_{i}^{s}(Z), D_{i}^{-s}(Z) \right\} \subset \mathcal{F}_{i}^{k_N/Q,2/1}H. \]

**Definition 4.3** (For \(N=6\)).

- **Family**: \(B := \left\{ \zeta_{m}^{s} \left( z_{1}, \ldots, z_{s} \right) (2i\pi)^{s}m, x_{1} > 1, s \geq 0 \right\}. \)

- **Level**: The \((k_{i}/Q,1/1)\)-level, denoted \(i\), is defined as the number of even \(x_{j}\).

- **Filtration by the motivic** \((k_{i}/Q,1/1)\)-level:

\[ \mathcal{F}_{i}^{k_{i}/Q,1/1}H := \left\{ Z \in H, \ s. t. \ \forall r > 0, \ (D_{2r}^{s})(Z) \in \mathcal{F}_{i-1}^{k_{i}/Q,1/1}H, \ (D_{2r+1}^{s})(Z) \in \mathcal{F}_{i}^{k_{i}/Q,1/1}H \right\}. \]

**NB**: In the following results, the index \(i\), in \(\mathcal{B}_{a,p,i}\) will always refer to the level corresponding to the motivic level filtration considered (fixed for each result).

**Remarks**:

1. For \(N = 3, 4, 6, 8\), we can see the increasing filtration by the motivic \((k_{N}/Q,P/1)\)-level as an increasing filtration coming from higher ramification groups of Galois group \(G_{H_{N}/H^{1}}\). Indeed, \(\mathcal{F}_{0}^{k_{N}/Q,P/1}H^{1}\) is the set of the invariants under the Galois group \(G_{N/1} : = Gal(H_{N}/H^{1})\) (such that \(U_{N} = G_{N/1} \times U_{1}\)). The case \(N = 8\) has the similar interpretation for the descents with \(H^{2}, H^{4}\).

2. Notice that the increasing -or decreasing- filtration that we could define by the number of 1 (resp. number of even) appearing in the motivic multiple zeta values is not preserved by the coproduct, since the number of 1 can either diminish or increase (at most 1), so is not motivic.

The following result gives us information on the \(0\)-th-level of these filtrations:

**Lemma 4.4**. The level 0 of these motivic level-filtrations is isomorphic to the following algebras:

\[ N = 3, 4, 6, 8: \ \mathcal{F}_{0}^{k_{N}/Q,P/1}H^{MT_{N}} = \mathcal{F}_{0}^{k_{N}/Q,P/1}H^{N} = \mathcal{H}^{MT_{1}} = H^{1}. \]

\[ N = 3: \ \mathcal{F}_{0}^{k_{3}/Q,3/3}H^{MT_{3}} = \mathcal{F}_{0}^{k_{3}/Q,3/3}H^{3} = \mathcal{H}^{MT(\mathbb{Z}[\frac{1}{3}])}. \]

\[ N = 4, 8: \ \mathcal{F}_{0}^{k_{N}/Q,2/2}H^{MT_{N}} = \mathcal{F}_{0}^{k_{N}/Q,2/2}H^{N} = \mathcal{H}^{MT_{2}} = H^{2}. \]

\[ N = 8: \ \mathcal{F}_{0}^{k_{8}/k_{4},2/2}H^{MT_{8}} = \mathcal{F}_{0}^{k_{8}/k_{4},2/2}H^{8} = \mathcal{H}^{MT_{4}} = H^{4}. \]

**Proof**. The equalities of the kind \(\mathcal{H}^{MT_{N}} = H^{N}\) will be proved later for \(N = 3, 4, 6, 8\), and have been proved in the previous section for \(N = 2\) and by F. Brown for \(N = 1\) (cf. [1]).

Moreover, we have inclusions of the kind \(\mathcal{H}^{MT_{j}} \subseteq \mathcal{F}_{0}^{k_{N}/Q,P/1}H^{MT_{N}}\) for \(j = 1, 2, \) and we deduce the equality from dimensions at fixed weight.

**Remark**: In particular it will give a basis for \(\mathcal{H}^{MT(\mathbb{Z}[\frac{1}{3}])}\).

Some others 0-level such as \(\mathcal{F}_{0}^{k_{N}/k_{N},P/1}\), \(N = 3, 4\) which should reflect the descent from \(MT(O_{N}[^{1}/1])\) to "\(MT(O_{N})" are not known to be associated to a fundamental group.

We will still use the notation \(H^{2} := H/F_{i-1}H\) for the quotients considered.

**The depth 1** Using results of Deligne and Goncharov (cf. [10]) we know the only relations in depth 1 in \(A\) are conjugation and distribution relations, and then we deduce a basis of \(gr_{1}^{D}A_{N}^{1}\).

**Lemma 4.5**. For \(N = 3, 4\): basis of \(gr_{1}^{D}A_{N}^{1}: \ \zeta_{a}(r; \xi), r > 0\).

**Relations** for \(N = 3\):

\[ \zeta_{1} \left( \frac{2r+1}{1} \right) (1 - 3^{2r}) = 2 \cdot 3^{2r} \zeta_{1} \left( \frac{2r+1}{\xi} \right); \ \zeta_{1} \left( \frac{2r}{1} \right) = 0; \ \zeta_{1} \left( \frac{r}{\xi} \right) = (-1)^{r-1} \zeta_{1} \left( \frac{r}{\xi-1} \right). \]
Relations for $N = 4$:

\[
\zeta^l \left( \frac{r}{i} \right) (1 - 2^{r-1}) = 2^{r-1} \cdot \zeta^l \left( \frac{r}{-i} \right) \quad \text{for } r \neq 1 ; \quad \zeta^l \left( \frac{1}{i} \right) = 0 = \zeta^l \left( \frac{2r}{-i} \right)
\]

\[
\zeta^l \left( \frac{2r+1}{-i} \right) = 2^{2r+1} \cdot \zeta^l \left( \frac{2r+1}{\xi} \right) ; \quad \zeta^l \left( \frac{r}{\xi} \right) = (-1)^{r-1} \cdot \zeta^l \left( \frac{r}{-\xi} \right).
\]

- For $N = 8$: basis of $gr_p^D \mathcal{A}^8_+$ : $\zeta^a \left( \frac{r}{\xi} \right) , \zeta^a \left( \frac{r}{-\xi} \right)$. Relations

\[
\zeta^l \left( \frac{r}{\xi} \right) (1 - 2^{r-1}) = 2^{r-1} \cdot \zeta^l \left( \frac{r}{-i} \right) \quad \text{for } r \neq 1 ; \quad \zeta^l \left( \frac{1}{i} \right) = 0 = \zeta^l \left( \frac{2r}{-i} \right)
\]

\[
\zeta^l \left( \frac{2r+1}{i} \right) = 2^{2r+1} \cdot \zeta^l \left( \frac{2r+1}{\xi} \right) ; \quad \zeta^l \left( \frac{r}{-i} \right) = (-1)^{r-1} \cdot \zeta^l \left( \frac{r}{-\xi} \right) ; \quad \zeta^l \left( \frac{r}{i} \right) = (-1)^{r-1} \cdot \zeta^l \left( \frac{r}{-i} \right).
\]

Infinitesimal coaction The formula (2.5) in the case $N = 3, 4$, for the elements of $\mathcal{B}$ becomes:

\[
D_{r,p} \zeta^m \left( \frac{x_1, \cdots, x_p}{1, \cdots, 1, \xi} \right) = \quad (4.1)
\]

\[
\text{(TYPE A)} \delta_{x_1} \zeta^l \left( \frac{r}{i} \right) \otimes \zeta^m \left( \frac{x_2, \cdots, x_p}{1, \cdots, 1, \xi} \right)
\]

\[
\text{(TYPE A)} \sum_{i=1}^{p-2} \delta_{x_{i+1}} (1 - 2^{r-x_{i+1}}) \zeta^l \left( \frac{r}{i} \right) \otimes \zeta^m \left( \frac{\cdots, x_{i-1}, x_i + x_{i+1} - r, r + 1}{1, \cdots, 1, \xi} \right)
\]

\[
\text{(TYPE B)} \sum_{i=1}^{p-2} \delta_{x_2} (1 - 2^{r-x_2}) \zeta^l \left( \frac{r}{i} \right) \otimes \zeta^m \left( \frac{\cdots, x_{i-1}, x_i + x_{i+1} - r, r + 1}{1, \cdots, 1, \xi} \right)
\]

(\text{TYPE D, D'}) + \delta_{x_p} x_p \leq x_p + \sum_{i=1}^{p-1} (1 - 2^{r-x_p}) \zeta^l \left( \frac{r}{i} \right) \otimes \zeta^m \left( \frac{\cdots, x_{p-2}, x_{p-1} + x_p - r}{1, \cdots, 1, \xi} \right)

The case $N = 6$ or $N = 8$ are similar; left to the reader.

4.2 Results

Proofs being fundamentally similar to previous ones (for $N = 2$), we will be deliberately more succinct. For fixed $N = 3, 4, 6, 8$ and any fixed motivic level filtration $\mathcal{F}_i$ mentioned previously and with the associated level $i$:

**Theorem 4.6.** 1. $\mathcal{B}_{n,p \geq i}$ is a linearly free family of $gr_p^D \mathcal{H}^i_{n-r}$, hence basis of $gr_p^D \mathcal{H}^i_{n-r}$.

The following maps $\partial_{n,p}$ are bijective, for the case $N = 3, 4$:

\[
\partial_{n,p}^{(N/N, P/1)} : gr_p^D \mathcal{H}^i_{n-r} \rightarrow \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^i_{n-r-1} \oplus \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r}.
\]

\[
\partial_{n,p}^{(N/N, P/1)} : gr_p^D \mathcal{H}^i_{n-r} \rightarrow \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r} \oplus \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r}.
\]

\[
\partial_{n,p}^{(N/1, P/1)} : gr_p^D \mathcal{H}^i_{n-r} \rightarrow \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r} \oplus \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r}.
\]

\[
\partial_{n,p}^{(N/1, P/1)} : gr_p^D \mathcal{H}^i_{n-r} \rightarrow \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r} \oplus \oplus_{1 < r \leq n-p+1} gr_p^D \mathcal{H}^{i-1}_{n-r}.
\]
2. The basis $B_{n,p,i}$ defines a $\mathbb{Z}_2[p]$-structure on $gr_p^D H_n^{\geq 1}$, the projection $\pi_{0,1} : H_n \to H_n^{\geq 1}$ is defined over $\mathbb{Z}_2[p]$. That is, each element $Z \in B_{n,p,i}$ is equal to a $\mathbb{Z}_2[p]$-linear combination of $B_{n,p,i}$ elements, denoted $c_{n,p,i}(Z)$, plus an element in $F_{i-1}H_n + F_{p-1}H_n$.

It defines, in an unique way, a map:

$$c_{n,p,i} : Vect_{\mathbb{Q}}(B_{n,p,i}) \to Vect_{\mathbb{Q}}(B_{n,p,i}).$$

**Proof.** 1. By recursion hypothesis, we know that:

$$\bigoplus_{r \text{ even}} B_{n-r,p-1,1} \oplus \bigoplus_{r \text{ odd}} B_{n-r,p-1,1}$$

is a basis of $gr_p^{D-1} H_n^{\geq 1}$. It follows straight from the 1.

Let $M_{n,p}$ denote the matrix of $(\partial_{n,p}(x))_{x \in B_{n,p,i}}$, the image of $B_{n,p,i}$. To prove the invertibility of $M_{n,p}$ or of $M_{n,p}$ (representing $\otimes N^{-1}D_{r,p}$), we use recursive hypothesis to replace elements of level $\leq i$ in $D_{r,p}$, $r \geq 1$ by $\mathbb{Z}[\frac{1}{r}]$-linear combinations of elements of level $\geq i$ in the quotient $gr_p^{D-1} H_n^{\geq 1}$. Then, thanks to congruences modulo $P$, only the deconcatenation terms remain:

$$N^{-1}D_{r,p} \left( \zeta \left( \begin{array}{c} x_1, \ldots, x_p \\ 1, \ldots, 1, \xi \end{array} \right) \right) =$$

$$\delta_{z_p \leq r \leq z_p + x_p - 1} (-1)^{r-z_p} \left( \begin{array}{c} r - 1 \\ x_p - 1 \end{array} \right) \zeta \left( \begin{array}{c} x_1, \ldots, x_p - 2, x_p - 1 + x_p - r \\ 1, \ldots, 1, \xi \end{array} \right) \text{ (mod } P).$$

And the matrix once ordered is triangular with 1 on the diagonal, hence invertible.

2. It follows straight from the 1. 

**Remark:**
- Beware, for $N = 8$, $D_r$ has two independent components, $D^\xi_r$ and $D^{-\xi}_r$, corresponding to $\zeta^i(r; \xi) \otimes \cdot$ respectively to $\zeta^i(r; -\xi) \otimes \cdot$. We have to distinguish them, but the statement and the proof remains rather similar.

The corresponding bijective maps are (level decreasing sometimes by 2 or 3):

- $\partial^{(8,4/2,2)}_{n,p} : gr_p^2 H_n^{\geq 1} \to gr_p^{-1} H_n^{\geq 1} \oplus gr_p^{-1} H_n^{\geq 1}$
- $\partial^{(8,4/2,2)}_{n,p} = \bigoplus_{1 \leq r \leq n-p+1} D^{-\xi}_r \oplus \bigoplus_{1 \leq r \leq n-p+1} D^\xi_r$.
- $\partial^{(8,2/2,2)}_{n,p} : gr_p^4 H_n^{\geq 1} \to \bigoplus_{1 \leq r \leq n-p+1} D^{-\xi}_r \oplus \bigoplus_{1 \leq r \leq n-p+1} D^\xi_r$.
- $\partial^{(8,1/2,1)}_{n,p} : gr_p^4 H_n^{\geq 1} \to \bigoplus_{1 \leq r \leq n-p+1} D^{-\xi}_r \oplus \bigoplus_{1 \leq r \leq n-p+1} D^\xi_r$.
- $\partial^{(8,1/2,1)}_{n,p} = \bigoplus_{1 \leq r \leq n-p+1} D^{-\xi}_r \oplus \bigoplus_{1 \leq r \leq n-p+1} D^\xi_r$.

For instance, examining the coaction on the basis elements we realize that for the deconcatenation terms (of type D) the $(k_8/k_4, 2/2)$-level decreases of at most 2 in the case of $D^\xi_r$, or remains constant else; The remaining terms, modulo 2, are still the deconcatenation ones.

- For $N = 6$, the corresponding bijective map is:

$$\partial_{n,p}^{(6,4/1,1)} : gr_p^6 H_n^{\geq 1} \to \bigoplus_{1 \leq r \leq n-p+1} D^{-\xi}_r \oplus \bigoplus_{1 \leq r \leq n-p+1} D^\xi_r.$$

The proof is rather similar than for the case $N = 2$. Notice here the terms of the Type A, B are zero for even $r$ and else have a coefficient (behind $\zeta^i(r; \xi)$):

$$c_r = \frac{2 \cdot 6^{r-1}}{(1 - 2^{r-1})(1 - 3^{r-1})}.$$

This coefficient has a 3-adic valuation greater than 0 hence, looking at valuation zero, it will remains only the deconcatenation terms.
The corresponding result all depths mixed:

**Theorem 4.7.** 1. $\mathcal{B}_{n, i}^N$ is a basis of $\mathcal{H}_n^i = \mathcal{H}_n^{2i, MT}$.

2. The $\mathbb{Z}[P]$-structure: Each element $Z \in \mathcal{B}_{n, p}$ is equal to a $\mathbb{Z}[P]$-linear combination of $\mathcal{B}_{n, \leq p, \geq i}$ elements, denoted $c_{n, \leq p, \geq i}(Z)$, plus an element in $F_i \mathcal{H}_n$.

It defines in a unique way $c_{n, \leq p, \geq i} : \text{Vect}_\mathbb{Q}(\mathcal{B}_{n, p, \leq i-1}) \to \text{Vect}_\mathbb{Q}(\mathcal{B}_{n, \leq p, \geq i})$.

3. A basis of $F_i \mathcal{H}_n$ is formed by:

$$\bigcup_p \{x + c_{n, \leq p, \geq i+1}(x), x \in \mathcal{B}_{n, p, \leq i}\}.$$

It can be illustrated by the scheme of two split exact sequences which are in bijection:

$$0 \to F_i \mathcal{H}_n \to \mathcal{H}_n^{a_{i+1}} \mathcal{H}_n^{\geq i+1} \to 0$$

$$0 \to \text{Vect}_\mathbb{Q}(\mathcal{B}_{n, \leq i}) \to \text{Vect}_\mathbb{Q}(\mathcal{B}_{n}) \to \text{Vect}_\mathbb{Q}(\mathcal{B}_{n, \geq i+1}) \to 0.$$

We then deduce for the depth-graded, which generalizes a result of P. Deligne ($i = 0$, cf. [8]):

**Corollary 4.8.** The elements of $\mathcal{B}_{n, p, \geq i}$ form a basis of $\mathcal{g}_p^{2i} \mathcal{H}_n / F_{i-1} \mathcal{H}_n$.

In particular, the map $\mathcal{G}^{MT_n} \to \mathcal{G}^{MT_{n+1}}$ is an isomorphism. The elements of $\mathcal{B}_n^N$, form a basis of motivic multiple zeta values relative to $\mu_N, \mathcal{H}_n^N$.

Beware, for $N = 6$ we only obtain a basis of the unramified $\mathcal{H}_n^{C_b}$.

A basis for the gradient for the motivic level filtration:

**Corollary 4.9.** A basis of $g_{i+1}^{k_{N/N'}} \mathcal{H}_n$ is, considering the $(k_N/k_N, P/P')$ level $i$:

$$\mathcal{B}_{n, i}^{N, (k_N/k_N, P/P')} := \bigcup_p \{x + c_{n, \leq p, \geq i+1}(x), x \in \mathcal{B}_{n, p, \leq i}\}.$$

Hence we deduce ($i = 0$) the Galois descents, corresponding to the different filtration introduced, for $N'/N \in \{3, 4, 8\}$.

**Corollary 4.10.** A basis of $\mathcal{H}_n^{N'}$ is formed by elements of $\mathcal{B}_{n, i}$ of level 0 each corrected by linear combination of elements $\mathcal{B}_{n, i}$ of level $\geq 1$. In particular, with $\xi$ primitive:

- **Galois descent** from $N' = 1$ to $N = 3, 4$: A basis of motivic multiple zeta values:

$$\mathcal{B}_{3, 4}^{1, N} := \left\{\sum_{y_i \geq 0} \alpha_{x,y} \zeta^m \left(\begin{array}{c} y_1, \ldots, y_p \\ 1, \ldots, 1, \xi \end{array}\right) \zeta^m (2)^s \right.$$  

$$+ \sum_{\text{lower depth } q < p, \text{ at least one even or } \geq 1} \zeta^m \left(\begin{array}{c} z_1, \ldots, z_q \\ 1, \ldots, 1, \xi \end{array}\right) \zeta^m (2)^s, x_i > 0, \alpha_{x,y}, \beta_{x,z} \in \mathbb{Q}\right\}.$$

**Galois descent** from $N' = 1$ to $N = 8$: Another basis of motivic multiple zeta values:

$$\mathcal{B}_{3, 8}^{1, N} := \left\{\sum_{y_i \geq 0} \alpha_{x,y} \zeta^m \left(\begin{array}{c} y_1, \ldots, y_p \\ 1, \ldots, 1, \xi \end{array}\right) \zeta^m (2)^s \right.$$  

$$+ \sum_{\text{lower depth, level } \geq 1} \zeta^m \left(\begin{array}{c} z_1, \ldots, z_q \\ \tilde{\ell}_1, \ldots, \tilde{\ell}_q \xi \end{array}\right) \zeta^m (2)^s, x_i > 0, \alpha, \beta \in \mathbb{Q}\right\}.$$

**Galois descent** from $N' = 1$ to $N = 6$ unramified ($\mathcal{H}_n^{MT_6}$):

$$\mathcal{B}_{3, 6}^{1, N} := \left\{\sum_{y_i \geq 0} \alpha_{x,y} \zeta^m \left(\begin{array}{c} y_1, \ldots, y_p \\ 1, \ldots, 1, \xi \end{array}\right) \zeta^m (2)^s \right.$$  

$$+ \sum_{\text{lower depth, level } \geq 1} \zeta^m \left(\begin{array}{c} z_1, \ldots, z_q \\ 1, \ldots, 1, \xi \end{array}\right) \zeta^m (2)^s, x_i > 0\right\}.$$
**Galois descent** from \( N' = 2 \) to \( N = 4 \): A basis of motivic Euler sums:

\[
\mathcal{B}_{2,4}^k := \left\{ \zeta^m \left( \frac{2x_1 + 1, \ldots, 2x_p + 1}{1, \ldots, 1, \xi} \right) \zeta^m(2)^s + \sum_{y_i > 0, \text{at least one even}} \alpha \zeta^m \left( \frac{y_1, \ldots, y_p}{1, \ldots, 1, \xi} \right) \zeta^m(2)^s \right\}.
\]

**Galois descent** from \( N' = 2 \) to \( N = 8 \): A basis of motivic Euler sums:

\[
\mathcal{B}_{2,8}^k := \left\{ \zeta^m \left( \frac{2x_1 + 1, \ldots, 2x_p + 1}{1, \ldots, 1, \xi} \right) \zeta^m(2)^s + \sum_{y_i \text{ at least one even or one } \epsilon_i = -1} \alpha \zeta^m \left( \frac{y_1, \ldots, y_p}{\epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_p \xi} \right) \zeta^m(2)^s \right\}.
\]

**Galois descent** from \( N' = 4 \) to \( N = 8 \): A basis of motivic multiple zeta values relative to \( \mu_4 \):

\[
\mathcal{B}_{4,8}^k := \left\{ \zeta^m \left( \frac{x_1, \ldots, x_p}{1, \ldots, 1, \xi} \right) (2i\pi)^s + \sum_{\epsilon_i = -1, \text{ at least one } \epsilon_i} \alpha \zeta^m \left( \frac{y_1, \ldots, y_p}{\epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_p \xi} \right) (2i\pi)^s \right\}.
\]

**Nota Bene:** Those basis are weight graded. The 0\(^{th}\)-level of the other filtrations may not correspond to a fundamental group as we have already pointed out. However, let us formulate the following results, for \( N = 3, 4 \):

**Corollary 4.11.** A basis of \( \mathcal{F}_0^{k N/k N, P/1} \mathcal{H}_n^{N} \) is formed by, considering the \((k N/k N, P/1)\)-level:

\[
\bigcup_p \{ x + cl_{n,\leq p \geq 1}(x), x \in \mathcal{B}_{n,0} \}.
\]

A basis of \( \mathcal{F}_0^{k N/k,3/3} \mathcal{H}_n^{N} \) is formed by, considering the \((k N/k,3/3)\)-level:

\[
\bigcup_p \{ x + cl_{n,\leq p \geq 1}(x), x \in \mathcal{B}_{n,0} \}.
\]

5 **Examples in depth 2, 3**

5.1 The case \( N = 2, \) depth 2, 3

In depth 1 all the \( \zeta^m(\mathfrak{f}) \), for all \( s > 1 \) are honorary multiple zeta values.

Let us detail the case of depth 2 and 3 as an application of the previous results.

In depth 2, coefficients are explicit:

**Lemma 5.1.** The depth 2 - part of the basis of the honorary motivic multiple zeta values:

\[
\mathcal{B}_{2,2}^k := \left\{ \zeta^m(2a + 1,2b + 1) - \frac{2(a + b)}{2b} \zeta^m(1,2(a + b) + 1), a, b > 0 \right\}.
\]
Lemma 5.2. The depth 2 part of the basis of $F_0\mathcal{H}$:

$$B_{n,2}^2 = \{ \zeta_m(2a+1, 2b+1), (a,b) \neq (0,0) \}.$$  

Proof. No need of correction ($B_{n,2 \geq 2}$ is empty for $n \neq 2$), these elements belong to $F_0\mathcal{H}$.  

Lemma 5.3. The depth 3 part of the basis of (horizontal) motivic multiple zeta values:

$$\left\{ \zeta_m(2a+1, 2b+1, 2c+1) - \sum_{k=1}^{a+b+c} \alpha_k \zeta_m(1, 2(a+b+c-k) + 1, 2k+1) - \left( \frac{2(b+c)}{2c} \right) \zeta_m(2a+1, 1, 2(b+c)+1), a, b, c > 0 \right\}.  \tag{5.1}$$
where $\alpha_{a,b,c}^{n} \in \mathbb{Z}_{odd}$ are solutions of the system $M_{3}X = A^{a,b,c}$. With $A^{a,b,c}$ such as $r^{th}$ coefficient is:

$$\delta_{b \leq r < a+b} \left( \frac{2(n-r)}{2c} \right) \left( \frac{2r}{2b} \right) - \delta_{a < r \leq a+b} \left( \frac{2(n-r)}{2c} \right) \left( \frac{2r}{2a} \right) - \delta_{b \leq r < b+c} \left( \frac{2(n-r)}{2a} \right) \left( \frac{2r}{2b} \right)$$

$$- \delta_{r \leq a} \left( \frac{2(n-r)}{2b+c} \right) \left( \frac{2r}{2c} \right) + \delta_{r < b+c} \left( \frac{2(n-r)}{2a} \right) \left( \frac{2r}{2c} \right) + \delta_{c \leq r < b+c} \left( \frac{2r}{2c} \right) \left( \frac{2(n-r)}{2a} \right) (2^{-2r} - 1).$$

$M_{3}$ the matrix whose $(r,k)^{th}$ coefficient is:

$$\delta_{r=a+b+c} (2^{-2r} - 2) \left( \frac{2n}{2k} \right) + \delta_{k \leq r \leq n} \left( \frac{2r}{2k} \right) (2^{-2r} - 1) - \delta_{r < n-k} \left( \frac{2(n-r)}{2k} \right) - \delta_{n-k \leq r < n} \left( \frac{2r}{2(n-k)} \right).$$

**Proof.** Let $\zeta^{m}(2a+1, 2b+1, 2c+1)$, $a, b, c > 0$ fixed, and substrate elements of same weight, of depth 3 until it belongs to $gr_{3}F_{0}H$.

Calculate infinitesimal coproducts refering to the formula (3.1) in the quotient $\mathcal{H}_{\mathbb{Z}^{1}}$ and use previous results for depth 2, with $n = a + b + c$:

$$D_{2r+1,3}(\zeta^{m}(2a+1, 2b+1, 2c+1)) = \zeta^{1}(2r+1) \otimes \left[ \delta_{r=b+c} \left( \frac{2(b+c)}{2c} \right) (2^{-2r} - 2) \zeta^{m}(2a+1, 1) \right]$$

$$+ \zeta^{m}(1, 2(n-r) + 1) \left( \delta_{a=r} \left( \frac{2(n-r)}{2c} \right) + \delta_{b \leq r < a+b} \left( \frac{2r}{2b} \right) \left( \frac{2(n-r)}{2c} \right) - \delta_{a \leq r \leq a+b} \left( \frac{2r}{2a} \right) \left( \frac{2(n-r)}{2c} \right) \right)$$

$$- \delta_{b \leq r < b+c} \left( \frac{2r}{2c} \right) \left( \frac{2(n-r)}{2a} \right) + \delta_{r < b+c} \left( \frac{2r}{2c} \right) \left( \frac{2(n-r)}{2a} \right) (2^{-2r} - 1).$$

At first, subract $\left( \frac{2(b+c)}{2c} \right) \zeta(2a+1, 2(b+c) + 1)$ such that the $D_{1,2}^{-1}D_{2r+1,3}$ are equal to zero, which comes to eliminate the term $\zeta^{m}(2a+1, 1)$ appearing (case $r = b+c$).

So, we are left to substract a linear combination $\sum_{k=1}^{a+b+c} \alpha_{k}^{a,b,c} \zeta^{m}(1, 2(a + b + c - k) + 1, 2k+1)$ such that the coefficients $\alpha_{k}^{a,b,c}$ are solutions of the system $M_{3}X = A^{a,b,c}$ where $A^{a,b,c} = (A_{r}^{a,b,c})_{r}$ satisfying in $\mathcal{H}_{\mathbb{Z}^{1}}$:

$$D_{2r+1,3}(\zeta^{m}(2a+1, 2b+1, 2c+1)) \left( \frac{2(b+c)}{2c} \right) \zeta^{m}(1, 2(n-r) + 1),$$

and $M_{3} = (m_{r,b})_{r,b}$ matrix such that:

$$D_{2r+1,3}(\zeta^{m}(1, 2(a + b + c - k) + 1, 2k+1)) = m_{r,b} \zeta^{1}(2r+1) \otimes \zeta^{m}(1, 2(n-r) + 1).$$

This system has solutions since (according to previous results), the matrix $M_{3}$ is invertible because modulo 2 it is an upper triangular matrix with 1 on diagonal.

Then, the following linear combination will be in $F_{0}H$:

$$\zeta^{m}(2a+1, 2b+1, 2c+1) - \sum_{k=1}^{a+b+c} \alpha_{k}^{a,b,c} \zeta^{m}(1, 2(a+b+c-k)+1, 2k+1) \left( \frac{2(b+c)}{2c} \right) \zeta(2a+1, 1, 2(b+c) + 1).$$

The coefficients $\alpha_{k}^{a,b,c}$ belong to $\mathbb{Z}_{odd}$ since coefficients are integers, and determinant of $M_{3}$ is odd.

Reffering to calculus of infinitesimal coactions, $A^{a,b,c}$ is the vector whose $r^{th}$ coefficient is:

$$\delta_{b \leq r < a+b} \left( \frac{2(n-r)}{2c} \right) \left( \frac{2r}{2b} \right) - \delta_{a < r \leq a+b} \left( \frac{2(n-r)}{2c} \right) \left( \frac{2r}{2a} \right) - \delta_{b \leq r < b+c} \left( \frac{2(n-r)}{2a} \right) \left( \frac{2r}{2b} \right)$$

$$- \delta_{r \leq a} \left( \frac{2(n-r)}{2b+c} \right) \left( \frac{2r}{2c} \right) + \delta_{r < b+c} \left( \frac{2(n-r)}{2a} \right) \left( \frac{2r}{2c} \right) + \delta_{c \leq r < b+c} \left( \frac{2r}{2c} \right) \left( \frac{2(n-r)}{2a} \right) (2^{-2r} - 1),$$

And $M_{3}$ matrix whose $(r,k)^{th}$ coefficient is:

$$\delta_{r=a+b+c} (2^{-2r} - 2) \left( \frac{2n}{2k} \right) + \delta_{k \leq r \leq n} \left( \frac{2r}{2k} \right) (2^{-2r} - 1) - \delta_{r < n-k} \left( \frac{2(n-r)}{2k} \right) - \delta_{n-k \leq r < n} \left( \frac{2r}{2(n-k)} \right).$$
EXAMPLE: By applying this lemma, with $a = b = c = 1$ we obtain the following honorary motivic multiple zeta value:

$$\zeta^m(3, 3, 3) + \frac{774}{191}\zeta^m(1, 5, 3) - \frac{804}{191}\zeta^m(1, 3, 5) + \frac{450}{191}\zeta^m(1, 1, 7) - 6\zeta^m(3, 1, 5).$$

Indeed, in this case, with the previous notations:

$$M_3 = \begin{pmatrix} \frac{63}{64} & -1 & -1 \\ \frac{1}{16} & 1 & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \end{pmatrix}, \quad A^{1,1,1} = \begin{pmatrix} \frac{51}{53} \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, we obtain the following honorary motivic multiple zeta value:

$$\zeta^m(3, 3, 3) + \frac{850920}{203117}\zeta^m(1, 7, 3) + \frac{838338}{203117}\zeta^m(1, 5, 5) - \frac{3673590}{203117}\zeta^m(1, 3, 7) + \frac{20351100}{203117}\zeta^m(1, 1, 9) - 15\zeta^m(3, 1, 7).$$

Indeed, in this case, with the previous notations:

$$M_3 = \begin{pmatrix} \frac{63}{64} & 15 & -1 \\ \frac{1}{10} & \frac{1}{16} & -6 \\ \frac{1}{16} & \frac{1}{16} & 1 \end{pmatrix}, \quad A^{1,1,2} = \begin{pmatrix} \frac{210}{387} \\ 8 \\ 0 \end{pmatrix}.$$

REMARKS: In particular, the following elements with $a, b, c \neq 0$, are honorary multiple zeta values:

$$\left(\zeta(2a + 1, 2b + 1, 2c + 1) - \sum_{k=1}^{a+b+c} \sigma_k^{a,b,c}\zeta(1, 2(a + b + c - k) + 1, 2k) - \zeta(2a + 1, 1, 2(b + c) + 1)\right),$$

We could also show that the following ones are honorary (resp. motivic) multiple zeta values:

$$\zeta(2a + 1, 2b + 2, 2c + 1), \quad c \neq 0; \quad \zeta(2a + 1, 2b + 1, 2c), \quad c \neq 0, \text{ etc.}$$

Lemma 5.4. The depth 3 part of the basis of $\mathcal{F}_1$:

$$\left\{\zeta^m(2a + 1, 2b + 1, 2c + 1) - \delta_{a=0, b=0,c=0}(-1)^{\delta_{c=0}}\left(\frac{2(a + b + c)}{2b}\right)\zeta^m(1, 1, 2(a + b + c) + 1)\right.$$  
$$\left.\quad -\delta_{c=0}\left(\frac{2(a + b)}{2b}\right)\zeta(1, 2(a + b) + 1, 1, \mathbb{T})\right\},$$  

for $c > 0$ in the quotient $\mathcal{H}^{\geq 1}$. If $c = 0$, $D_{1,1}^1\mathcal{F}_a$ is equal to zero. Else, if $c = 0$ in $\mathcal{H}^{\geq 1}$, according to the results in depth 2 for $\mathcal{F}_0$, we subtract $(2(a+b))^c\zeta(1, 2(a + b) + 1, \mathbb{T})$ because:

$$D_{1,3}(\zeta^m(2a + 1, 2b + 1, 1, \mathbb{T})) \equiv \left(\frac{2(a + b)}{2a}\right)\zeta^m(12(a + b) + 1) \equiv \left(\frac{2(a + b)}{2a}\right)D_{1,3}(\zeta^m(1, 2(a + b) + 1, 1, \mathbb{T})).$$

Besides (equalities with $\equiv$ being in the quotient $\mathcal{H}^{\geq 1}$):

$$D_{1,2}^{-1}D_{2r+1,3}(\zeta^m(2a + 1, 2b + 1, 2c + 1)) = \delta_{r=a+b+c}\left(\frac{2r}{2c}\right)(2^{-2r-2})\zeta^m(2a + 1) \equiv \delta_{r=a+b+c}\left(\frac{2(b + c)}{2c}\right)(2^{-2(b+c)-2})\zeta^m(1, \mathbb{T}).$$

$$D_{1,2}^{-1}D_{2r+1,3}(\zeta^m(1, 1, 2(a + b + c) + 1)) = \delta_{r=a+b+c}(2^{-2(a+b+c)} - 2)\zeta^m(1, \mathbb{T}).$$

Therefore, to cancel $D_{1,2}^{-1}D_{2r+1,3}$ in the case where $a = 0$ we have to subtract $(2(b+c))\zeta^m(1, 1, 2(b + c) + 1)$ and in the case where $c = 0$, to add $(2(b+c))\zeta^m(1, 1, 2(a + b) + 1).$  

□
Example in depth 4: Let us give the simplest example in depth 4 of honorary multiple zeta value obtained in the same way:

\[-\zeta^m(3,3,3) - 3678667587000 + 9187788536750 + \frac{41712466500}{4605143289541}\zeta^m(1,1,5) + \frac{9160668717750}{4605143289541}\zeta^m(1,1,7) + \frac{1186125510300}{4605143289541}\zeta^m(1,3,7) + \frac{202283196216}{4605143289541}\zeta^m(1,3,3,5) + \frac{993035356436}{4605143289541}\zeta^m(1,3,3,5) + \frac{892810656214}{4605143289541}\zeta^m(1,5,1,5) + \frac{1488017760354}{4605143289541}\zeta^m(1,5,3,3)\]

\[+ \frac{-450}{191}\zeta^m(3,1,1,7) + \frac{804}{191}\zeta^m(3,1,3,5) + \frac{-774}{191}\zeta^m(3,1,5,3) + 6\zeta^m(3,3,1,5)\]

\[+ \alpha_1\zeta^m(1,-11) + \alpha_2\zeta^m(1,-9)\zeta^m(2) + \alpha_3\zeta^m(1,-7)\zeta^m(2)^2 + \alpha_4\zeta^m(1,-5)\zeta^m(2)^3 + \alpha_5\zeta^m(1,-3)\zeta^m(2)^4,\]

with \(\alpha_i \in \mathbb{Q}\).

5.2 The case \(N = 3, 4\), depth 2

Let us detail the case of depth 2 as an application of previous results.

We omit in the proofs, to lighten the notations, the exponents \(\xi\) that indicates the projection on the second factor of the infinitesimal coaction. Let us define some coefficients that will be useful in our future examples in depth 2.

**Definition 5.5.** Set \(a_k^{a,b} \in \mathbb{Z}\) such that \(M(a_k^{a,b})_{b+1 \leq r \leq \frac{d}{2}-1} = A^{a,b}\) where \(n = 2(a + b + 1)\) and:

\[M = \left( \binom{2r-1}{2k-1} \right)_{b+1 \leq r \leq \frac{d}{2}-1}, \quad A^{a,b} = -\left( \binom{2r-1}{2b} \right)_{b+1 \leq r \leq \frac{d}{2}-1}, \quad \beta^{a,b} = \left( \binom{n-2k}{2b} \right)_{b+1 \leq r \leq \frac{d}{2}-1} + \sum_{k=b+1}^{a+b} \alpha_k \binom{n-2k}{2k-1}.
\]

For instance, \(a_k^{a,b} = (-2b+1), \quad a_k^{a,b} = 2(2b+3), \quad a_k^{a,b} = -16(2b+5), \quad a_k^{a,b} = 272(2b+7).

And \(a_k^{a,b} = (-1)^k(2b+2k-1)c_i\), where \(c_i \in \mathbb{N}\) does not depend neither on \(b\) nor on \(a\).

With the previous notations:

**Lemma 5.6.** The depth 2 part of the basis of motivic multiple zeta values, for weight \(n = 2(a+b+1)\) even:

\[\left\{ \zeta^m \left( \frac{2a+1, 2b+1}{1, \xi} \right) - \beta^{a,b} \zeta^m \left( \frac{1, n-1}{1, \xi} \right) - \sum_{k=b+1}^{a+b} \alpha_k \zeta^m \left( \frac{n-2k, 2k}{1, \xi} \right), a, b > 0 \right\}.
\]

**Proof.** Let \(Z = \zeta^m(2a + 1, 2b + 1)\) fixed, with \(a, b > 0\). First we subtract a linear combination of \(\zeta^m \left( \frac{n-2k, 2k}{1, \xi} \right)\) in order to cancel the \(D_{2r}\). It is possible since in depth 2 (because \(\zeta^l(2r; 1) = 0\)):

\[D_{2r} \left( \zeta^m (x_1, x_2) \right) = \delta_{x_2 \leq \frac{2r}{x_1} + x_2 - 1}(-1)^{x_2} \left( \binom{2r-1}{2r-1} \right) \zeta^{l} \left( \frac{2r}{x_1} \right) \otimes \zeta^m \left( x_1 + x_2 - r \right).
\]

So it is sufficient to choose the \(\alpha_k\) such that \(M a_k^{a,b} = A^{a,b}\), with \(M_{r,k} = \delta_{r \geq k}^{2(r-1)}(r, k\) going from \(b + 1\) to \(\frac{d}{2} - 1\) and \(A^{a,b} = (-\binom{2k-1}{2b})_{b+1 \leq r \leq \frac{d}{2}-1}\). It is possible because the matrix \(M\) is lower triangular (\(\delta_{r \geq k}\) with 1 on the diagonal).

We have \(D_1(\cdot) = 0\), and \(D_2(D_{2r+1}(\cdot))\), it remains to satisfy \(D_1(D_{2r+1}(\cdot)) = 0\), for \(r = n - 1\) in order to have an element of \(\mathbb{F}_0^{k, \mathbb{Q}, P}/\mathbb{H}_n\). For that, it is enough to subtract \(\beta^{a,b} \zeta^m(1, n-1; 1, \xi)\) with \(\beta^{a,b}\) such that \(\beta^{a,b} = \left( \binom{n-2}{2b} + \sum_{k=b+1}^{a+b} \alpha_k \binom{n-2}{2k-1} \right)\) according to the calculation of \(D_1(D_{2r+1}(\cdot))\) left to the reader.

The matrix \(M\) having integers as entries and determinant equal to 1, and \(A\) having integer components, the coefficients \(a_k^{a,b}\) are obviously integers. Moreover, we notice that \(a_k^{a,b} = (-2b + 1)\), the matrix \(M\) and its inverse being lower triangular with 1 on the diagonal. Similarly for the following.
Lemma 5.7. The depth 2 part of the basis of $\mathcal{F}_{1}^{k_{N}/Q,P/1}\mathcal{H}_{n}$ is for even $n$:

$$\zeta^{m}\left(\frac{2a+1+2b+1}{1,\xi}\right) - \sum_{k=b+1}^{\alpha-1}\alpha_{k}^{a,b}\zeta^{m}\left(\frac{n-2k+2k}{1,\xi}\right), a, b \geq 0, (a,b) \neq (0,0) \right\}.$$  

For odd $n$, the part in depth 2 of the basis of $\mathcal{F}_{1}^{k_{N}/Q,P/1}\mathcal{H}_{n}$ is:

$$\left\{\zeta^{m}\left(x_{1},x_{2}\right) + (-1)^{x_{2}+1}\zeta^{m}\left(n-2,1,\xi\right), x_{1}, x_{2} > 1, \text{ one even, the other odd}\right\}.$$  

Proof. For even $n$, we need to cancel $D_{2r}$ (else $D_{2n}(D_{2r}(\cdot)) \neq 0$), so we can substract the same linear combination than in the previous lemma.  

For odd $n$, we need to cancel $D_{1}(D_{2r})$. And $D_{1}(D_{2n}(Z)) = (-1)^{x_{2}}(n-2), x_{2} > 1$, so we have to substract $(-1)^{x_{2}+1}(n-2)\zeta^{m}(1,n-1)$.

Lemma 5.8. The depth 2 part of the basis of $\mathcal{F}_{1}^{k_{N}/Q,P/P}\mathcal{H}_{n}$ ($\mathcal{H}_{n}^{MT_{2}}$ if $N = 4$) is:

$$\zeta^{m}\left(\frac{2a+1+2b+1}{1,\xi}\right) - \sum_{k=b+1}^{\alpha-1}\alpha_{k}^{a,b}\zeta^{m}\left(\frac{n-2k+2k}{1,\xi}\right), a, b \geq 0 \right\}.$$  

Proof. To cancel $D_{2r}$, we substract the linear combination precised according to a previous proof.

Lemma 5.9. The depth 2 part of the basis of $\mathcal{F}_{1}^{k_{N}/Q,P/P}\mathcal{H}_{n}$ is for even $n$:

$$\zeta^{m}\left(\frac{2a+1+2b+1}{1,\xi}\right) - \sum_{k=b+1}^{\alpha-1}\alpha_{k}^{a,b}\zeta^{m}\left(\frac{n-2k+2k}{1,\xi}\right), a, b \geq 0 \right\}.$$  

And for odd $n$, the part in depth 2 of the basis of $\mathcal{F}_{1}^{k_{N}/Q,P/P}\mathcal{H}_{n}$ is:

$$\left\{\zeta^{m}\left(x_{1},x_{2}\right), x_{1}, x_{2} \geq 1, \text{ one even, the other odd}\right\}.$$  

Proof. In the case where $n$ is even, it remains to cancel the $D_{2r}$, with always the same linear combination.

In the case where $n$ is odd, there is nothing to do: $\zeta^{m}(x_{1},x_{2};1,\xi) \in \mathcal{F}_{1}^{k_{N}/Q,P/P}\mathcal{H}_{n}$.
5.3 The case $N = 8$, depth 2

Let us explicit the results for the depth 2, proofs are similar -albeit longer- than the previous ones, and are left to the reader. Same notations than the previous cases.

The levels 0, with motivic multiple zeta values relative to $\mu_5$:

**Lemma 5.10.**

- The depth 2 part of the basis of motivic multiple zeta values relative to $\mu_5$:
  \[
  \left\{ \zeta^m \left( \frac{x_1, x_2}{1, \xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, -\xi} \right) + \zeta^m \left( \frac{x_1, x_2}{1, -\xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, \xi} \right), x_i \geq 1 \right\}.
  \]

- The depth 2 part of the basis of the motivic Euler sums:
  \[
  \left\{ \zeta^m \left( \frac{2a + 1, 2b + 1}{1, \xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{-1, -\xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{1, -\xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{-1, \xi} \right) - \frac{2a - 1}{a+b+1} \right\},
  \]

- The depth 2 part of the basis of motivic multiple zeta values:
  \[
  \left\{ \zeta^m \left( \frac{2a + 1, 2b + 1}{1, \xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{-1, -\xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{1, -\xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{-1, \xi} \right) - \frac{2a - 1}{a+b+1}, a+b \geq 0 \right\}
  \]

The levels 1:

**Lemma 5.11.**

- The depth 2 part of the basis of $F_1^{k_5/k_4, 2/2}H_n$ is, for even $n$:
  \[
  \left\{ \zeta^m \left( \frac{x_1, x_2}{1, \xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, -\xi} \right) + \zeta^m \left( \frac{x_1, x_2}{1, -\xi} \right) - \zeta^m \left( \frac{x_1, x_2}{-1, \xi} \right), \zeta^m \left( \frac{x_1, x_2}{1, \xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, -\xi} \right), x_i \geq 1 \right\}.
  \]

- The depth 2 part of the basis of $F_1^{k_5/k_4, 2/2}H_n$ is, for odd $n$:
  \[
  \left\{ \zeta^m \left( \frac{x_1, x_2}{1, \xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, -\xi} \right) + \zeta^m \left( \frac{x_1, x_2}{1, -\xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, \xi} \right), \text{ exactly one even } x_i \right\}.
  \]

The depth 2 part of the basis of $F_1^{k_5/k_4, 2/2}H_n$ is for even $n$:

\[
\left\{ \zeta^m \left( \frac{2a + 1, 2b + 1}{-1, \xi} \right) + \zeta^m \left( \frac{2a + 1, 2b + 1}{-1, -\xi} \right) - \frac{2a - 1}{a+b+1} \right\},
\]

\[
- \frac{2a - 1}{a+b+1} \right\}
\]

\[
- \frac{2a - 1}{a+b+1} \right\}, a+b \geq 0 \right\}.
\]

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The depth 2 part of the basis of $\mathcal{F}_{1}^{k_{a}/Q,2/1} H_n$ is for odd $n$:

$$
\left\{ \zeta^m \left( \frac{x_1, x_2}{1, \xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, -\xi} \right) + \zeta^m \left( \frac{x_1, x_2}{-1, \xi} \right) - \gamma^{x_1, x_2} \left( \zeta^m \left( \frac{1, n-1}{1, \xi} \right) + \zeta^m \left( \frac{1, n-1}{-1, -\xi} \right) + \zeta^m \left( \frac{1, n-1}{1, -\xi} \right) + \zeta^m \left( \frac{1, n-1}{-1, \xi} \right) \right), \text{exactly one } x_i \right\}.
$$

In even weight $n$, depth 2 part of the basis of $\mathcal{F}_{1}^{k_{a}/Q,2/1} H_n$ is:

$$
\left\{ \zeta^m \left( \frac{1, n-1}{1, \xi} \right) + \zeta^m \left( \frac{1, n-1}{-1, -\xi} \right) + \zeta^m \left( \frac{1, n-1}{1, -\xi} \right) \right\}
$$

$$
\cup \left\{ \zeta^m \left( \frac{n-1, 1}{1, \xi} \right) + \zeta^m \left( \frac{n-1, 1}{-1, -\xi} \right) + \zeta^m \left( \frac{n-1, 1}{1, -\xi} \right) + \zeta^m \left( \frac{n-1, 1}{-1, \xi} \right) + \sum_{k=1}^{n-1} \alpha_k \phi_{n-1} \left( \zeta^m \left( \frac{n-2k, 2k}{1, \xi} \right) + \zeta^m \left( \frac{n-2k, 2k}{-1, -\xi} \right) + \zeta^m \left( \frac{n-2k, 2k}{1, -\xi} \right) + \zeta^m \left( \frac{n-2k, 2k}{-1, \xi} \right) \right) \right\}
$$

$$
\cup \left\{ \zeta^m \left( \frac{2a+1, 2b+1}{1, \xi} \right) + \zeta^m \left( \frac{2a+1, 2b+1}{-1, -\xi} \right) - \beta^{a,b} \left( \zeta^m \left( \frac{1, n-1}{1, \xi} \right) + \zeta^m \left( \frac{1, n-1}{-1, -\xi} \right) \right) \right\}
$$

$$
- \sum_{k=0}^{2^{r-1}-1} \alpha_k \phi_{2^{r-1}-1} \left( \zeta^m \left( \frac{n-2k, 2k}{1, \xi} \right) + \zeta^m \left( \frac{n-2k, 2k}{-1, -\xi} \right) \right), a, b > 0, \epsilon_i \in \{ \pm 1 \}, \epsilon_1 = -e_2 \right\}.
$$

Where $\gamma^{x_1, x_2} = (-1)^{x_2} (2^{r-1}-2r-x_2)$.

### 5.4 The case $N = 6$, depth 2

In depth 2, coefficients are explicit as previously and we have:

**Lemma 5.12.** The depth 2 part of the basis of motivic multiple zeta values $\mathcal{F}_{0}^{k_{a}/Q,1/1} H_n = \mathcal{H}_{n}^{MT(Z)}$, for even weight $n$:

$$
\left\{ \zeta^m \left( \frac{2a+1, 2b+1}{1, \xi} \right) - \sum_{k=0}^{2^{r-1}-1} \alpha_k \phi_{2^{r-1}-1} \zeta \left( \frac{n-2k, 2k}{1, \xi} \right), a, b > 0 \right\}.
$$

**Proof.** Left to the reader. (cf. the cases $N = 3, 4$).

### References

[1] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes).* Publie par la Societe mathematique de France, AMS dans Paris, Providence, RI, 2004.

[2] A. Borel. *Cohomologie réelle stable de groupes $S$-arithmétiques classiques.* C. R. Acad. Sci. Paris Ser. A-B 274 (1972).

[3] F. Brown. *On the decomposition of motivic multiple zeta values.* Preprint (2010), [arXiv:1102.1310](arXiv:1102.1310 NT).

[4] F. Brown. *Mixed Tate motives over $Z$.* Preprint (2010), [arXiv:1102.1312](arXiv:1102.1312 AG).
[5] J. Blümlein, D.J. Broadhurst, J.A.M. Vermaseren *The Multiple Zeta Value Data Mine*. Preprint (2009), arXiv:0907.2557v2 [math-ph].

[6] P. Cartier. *Fonctions polylogarithmes, nombres polyzêtas and groupes pro-unipotents*. Seminaire Bourbaki 2000 – 2001, exp 885, publie dans Asterisque.

[7] K. T. Chen. *Iterated path integrals*, Bull. Amer. Math. Soc. 83, (1977), 831-879.

[8] P. Deligne. *Le groupe fondamental unipotent motivique de $G_{m}\backslash\mu_N$ pour $N = 2, 3, 4, 6$ or $8$*, dans Publications Mathematiques de L’IHES, ISSN 0073-8301, Vol. 112, N°. 1, 2010 , pp. 101 to 141.

[9] P. Deligne. *Lettre a D. Zagier et F. Brown*, Moscou, janvier 2012, et Princeton, 28 avril 2012

[10] P. Deligne, A.B. Goncharov. *Groupes fondamentaux motiviques de Tate mixte*, dans Ann. Scient. Ec. Norm. Sup., 4e serie, t. 38, 2005, pp. 1 to 56.

[11] A.B. Goncharov. *Galois symmetries of fundamental groupoids and noncommutative geometry*, in Duke Math. J. 128, no. 2 (2005), pp. 209 to 284.

[12] A.B. Goncharov. *Multiple polylogarithms and mixed Tate motives*, arXiv: math.AG/0103059

[13] A.B. Goncharov. *The dihedral Lie algebra and Galois symmetries of $\pi^{(l)}_1(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$,* in Duke Math. J. 110, (2001), pp. 397-487.

[14] M. Levine. *Tate motives and the vanishing conjectures for algebraic K-theory*, in Algebraic K-theory and algebraic topology, Lake Louise,1991, in NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer, 167-188, (1993).

[15] M. Levine. *Mixed Motives*, in Mathematical Survey and Monographs, 57 American Mathematical Society, 1998.

[16] J. Milne *Algebraic Groups, Lie Groups, and their Arithmetic Subgroups* www.jmilne.org/math/ , 2011.

[17] G. Racinet. *Doubles mélanges des polylogarithmes multiples aux racines de l’unité*, Publ. Math. Inst. Hautes Etudes Sci. 95 (2002), pp. 185-231.

[18] I. Soudères. *Motivic double shuffle*, Int. J. Number Theory 6 (2010), 339-370.

[19] D. B. Zagier. *Evaluation of the multiple zeta values $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$*, preprint 2010.