Geometry of log-concave functions: the $L_p$ Asplund sum and the $L_p$ Minkowski problem

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Abstract
The aim of this paper is to develop a basic framework of the $L_p$ theory for the geometry of log-concave functions, which can be viewed as a functional “lifting” of the $L_p$ Brunn-Minkowski theory for convex bodies. To fulfill this goal, by combining the $L_p$ Asplund sum of log-concave functions for all $p > 1$ and the total mass, we obtain a Prékopa-Leindler type inequality and propose a definition for the first variation of the total mass in the $L_p$ setting. Based on these, we further establish an $L_p$ Minkowski type inequality related to the first variation of the total mass and derive a variational formula which motivates the definition of our $L_p$ surface area measure for log-concave functions. Consequently, the $L_p$ Minkowski problem for log-concave functions, which aims to characterize the $L_p$ surface area measure for log-concave functions, is introduced. The existence of solutions to the $L_p$ Minkowski problem for log-concave functions is obtained for $p > 1$ under some mild conditions on the pre-given Borel measures.

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1 Introduction and overview of the main results

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be log-concave if $-\log f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function. It is well-known that the log-concave functions behave in many aspects akin to convex bodies (i.e., compact convex sets with nonempty interiors, and the set of all convex bodies is denoted by $\mathcal{K}^n$), which can be considered as analytic “lifting” of geometry of convex bodies. In convex geometry, the Brunn-Minkowski theory (and its extensions) for convex bodies encompasses a large and growing range of fundamental results on the algebraic and geometric properties of convex bodies, and hence to study the parallel algebraic and geometric properties of log-concave functions is of great significance and in great demand. Recent years have witnessed that many results and notions in the Brunn-Minkowski theory (and its extensions) for convex bodies have found their functional analogues, including but not limited to the functional Blaschke-Santaló type inequality and its inverse \cite{8,12,14,15,34,35,43}, (John, Lutwak-Yang-Zhang, and Löwner) ellipsoids for log-concave functions \cite{6,32,46}, Rogers-Shephard type inequality and its reverse for log-concave functions \cite{1,2,5,27}, the affine surface areas for log-concave functions \cite{9,21–24,45}, the variation and Minkowski type problems related to log-concave functions \cite{29,30,42,62}, and (isoperimetric) inequalities related to log-concave functions \cite{3,4,7,17,20,47,59}. Other contributions include e.g., \cite{10,13,52} among others.

The celebrated Prékopa-Leindler inequality (2.13) for log-concave functions has a formula similar to the famous dimensional-free Brunn-Minkowski inequality for convex bodies (see (3.8) for $p = 1$). Note that the Prékopa-Leindler inequality indeed works for more general functions, see \cite{36,44,55–57}. The key ingredients in the Prékopa-Leindler inequality for log-concave functions are the total mass of a log-concave function $f$ given by

$$J(f) = \int_{\mathbb{R}^n} f(x) \, dx,$$

where $dx$ denotes the Lebesgue measure in $\mathbb{R}^n$ and the Asplund sum $f \oplus g$ of two log-concave functions $f$ and $g$ defined by

$$f \oplus g(x) = \sup_{x = x_1 + x_2} f(x_1)g(x_2),$$

which is again a log-concave function. Likewise, the volume and the Minkowski addition for convex bodies (which is equivalent to (2.2) for $p = 1$ for convex bodies) are the key ingredients in the Brunn-Minkowski inequality, and their combination naturally results in an elegant variational formula (i.e., (2.4) for $p = 1$). Such a variational formula defines a crucial concept in convex geometry: the surface area measure $S_K$ for convex body $K \in \mathcal{K}^n$. Notably, the Minkowski problem \cite{53,54}, aiming to characterize the surface area measure for convex bodies, can be solved by using the Euler-Lagrange equation based on the variational formula (i.e., (2.4) for $p = 1$). Lifting to the functional setting, the first variation of the total mass of log-concave functions with respect to the Asplund sum has been established by Colesanti and Fragałà in their groundbreaking work \cite{29}, and from there one can see that things become way more complicated for log-concave functions. The challenge partially comes from the facts that the total mass is defined on $\mathbb{R}^n$ instead of compact sets ($S^{n-1}$ for convex bodies) and the family of log-concave functions contains too many “unfavourable” log-concave functions (for example, those log-concave functions which are not smooth enough and/or whose domains are not nice enough). In his elegant paper \cite{60}, Rotem improved the variation formula by Colesanti and Fragałà, and claimed that the essentially continuity should be the correct, minimal and optimal conditions to make such a variation formula hold true. In a recent paper
Rotem further obtained a Riesz representation theorem for log-concave functions which characterizes the linear and increasing functionals on the class of log-concave functions. The variational formula of the total mass for log-concave functions with respect to the Asplund sum produces a measure for log-concave functions which is similar to $SK$ for $K \in \mathcal{K}^n$; such a measure was named as the surface area measure for a log-concave function in [29] (which is also known as the moment measure of convex functions in [30]). That is, for a log-concave function $f = e^{-\psi}$ with $0 < J(f) < \infty$, the moment measure of $\psi$ or the surface area measure of $f$, denoted by $\mu(f, \cdot)$, was defined as the push-forward measure of $e^{-\psi} \, dx$ under $\nabla \psi$ where $\nabla \psi$ denotes the gradient of $\psi$. That is,

$$\int_{\mathbb{R}^n} g(y) \, d\mu(f, y) = \int_{\mathbb{R}^n} g(\nabla \psi(x)) \, e^{-\psi(x)} \, dx$$

for every Borel function $g$ such that $g \in L^1(\mu(f, \cdot))$ or $g$ is non-negative. In their groundbreaking work [30], Cordero-Erausquin and Klartag studied the Minkowski type problem for log-concave functions in order to characterize the moment measure. Note that the Minkowski problem for log-concave functions in its full formulation was independently posed by Colesanti and Fragalà in [29], and finding solutions to this problem (in its full formulation) seems to be very important but highly intractable. Under certain conditions on log-concave functions (in particular, the essential continuity for the negative of their logarithms), Cordero-Erausquin and Klartag were able to solve the Minkowski problem for log-concave functions (see [30, Proposition 1] for necessity and [30, Theorem 2] for sufficiency and uniqueness). Santambrogio in [62] presented a solution to the Minkowski problem for log-concave functions by using the technique of the optimal mass transportation.

Although many concepts for log-concave functions have been studied, what brings into our attention is the missing of the “$L_p$ addition” of log-concave functions and the corresponding variational formula. These shall be analogous to the $L_p$ addition [33] and the related variational formula for convex bodies [49] which are the crucial elements in the rapidly developing $L_p$ Brunn-Minkowski theory for convex bodies. In the same manner as the Brunn-Minkowski theory for convex bodies, the $L_p$ theory for $p > 1$ was developed based on the brilliant variational formula (2.4) by Lutwak in his influential work [49]. Again, the variational formula (2.4) relies on the combination of the $L_p$ addition for convex bodies and the volume, which can be extended to all $0 \neq p \in \mathbb{R}$ (see e.g., [66]). Naturally, the $L_p$ surface area measure $h^{1-p}_K \, dS_K$ for a convex body $K \in \mathcal{K}^n$ (see Sect. 2 for notations) can be defined from (2.4), and consequently the $L_p$ Minkowski problem [49] characterizing the $L_p$ surface area measure for convex bodies can be posed (see Problem 2.1 for its precise statement). The $L_p$ Minkowski problem is a milestone in convex geometry and receives enormous attention in many areas of mathematics. Contributions to the $L_p$ Minkowski problem include [18,25,26,38–41,48–51,64,68–70] among others.

Our main goals in this paper are to establish a basic framework of the functional “lifting” of the $L_p$ Brunn-Minkowski theory for convex bodies, and thence extend the scheme for log-concave functions to an $L_p$ setting. Based on the $L_p$ Asplund sum of log-concave functions which naturally generalizes the Asplund sum, for $p > 1$, we establish a variational formula related to the total mass, define the $L_p$ surface area measure for log-concave functions, and study the $L_p$ Minkowski problem which characterizes the $L_p$ surface area measure for log-concave functions. It is our hope that these contributions provide some useful tools in the development of geometry of log-concave functions.

More specifically, Sect. 3 dedicates to study the properties for the $L_p$ Asplund sum of log-concave functions. Let $f = e^{-\varphi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$ (see Sects. 2 and 3 for notations).
For $p > 1$ and $\alpha, \beta > 0$, the $L_p$ Asplund sum $\alpha \cdot_p f \oplus_p \beta \cdot_p g$ may be formulated by

$$\alpha \cdot_p f \oplus_p \beta \cdot_p g := e^{-\phi^*} \text{ with } \phi := \left[ (\alpha \varphi^*)_p + \beta (\psi^*)_p \right]^1_p,$$  

(1.3)

where $\varphi^*$ denotes the Fenchel conjugate of $\varphi$ given in (2.5). This definition is meaningful as it can be seen in Proposition 3.2 that $\alpha \cdot_p f \oplus_p \beta \cdot_p g \in \mathcal{A}_0$ for $p > 1$. We establish the following Prékopa-Leindler type inequality in Theorem 3.3: for $f, g \in \mathcal{A}_0$, $\lambda \in (0, 1)$ and $p > 1$, it holds that

$$J((1 - \lambda) \cdot_p f \oplus_p \lambda \cdot_p g) \geq J(f)^{1-\lambda} J(g)^{\lambda}$$

with equality if and only if $f = g$ on $\mathbb{R}^n$. We also prove the log-concavity of $J(f \oplus_p t \cdot_p g)$ on $t \in (0, \infty)$ and $J((1 - t) \cdot_p f \oplus_p t \cdot_p g)$ on $t \in (0, 1)$ in Corollary 3.4.

In Sect. 4, we define $\delta J_p(f, g)$ for $p > 1$, the first variation of the total mass at $f \in \mathcal{A}_0$ along $g \in \mathcal{A}_0$ with respect to the $L_p$ Asplund sum, by

$$\delta J_p(f, g) := \lim_{t \to 0^+} \frac{J(f \oplus_p t \cdot_p g) - J(f)}{t},$$

whenever the limit exists. To find an explicit formula to describe $\delta J_p(f, g)$ for all $f, g \in \mathcal{A}_0$ seems to be intractable and even impractical. However, when $f = g \in \mathcal{A}_0$ such that $J(f) > 0$, we are able to prove in Lemma 4.3 that, for $p > 1$,

$$\delta J_p(f, f) = \frac{n}{p} \int_{\mathbb{R}^n} f \, dx + \frac{1}{p} \int_{\mathbb{R}^n} f \log f \, dx,$$

which involves the entropy of $f$ (see (4.3) for more details). Our main result in this section is a Minkowski type inequality for $\delta J_p(f, g)$ established in Theorem 4.2, namely, for $f \in \mathcal{A}_0$ and $g \in \mathcal{A}_0$ such that $J(f) > 0$ and $J(g) > 0$, one has, for $p > 1$,

$$\delta J_p(f, g) \geq \delta J_p(f, f) + J(f) \log \left( \frac{J(g)}{J(f)} \right)$$

with equality if and only if $f = g$. This Minkowski type inequality is applied to obtain a unique determination of log-concave functions in Corollary 4.7.

Although, in general, an explicit formula for $\delta J_p(f, g)$ is not easy to get, under certain conditions on $f$ and $g$ (which differs from the one for $f = g$ in Sect. 4), such a formula can be obtained. This result consists of our main contribution in Sect. 5 and is proved in Theorem 5.7. Roughly speaking, it asserts that for $p > 1$,

$$\delta J_p(f, g) = \frac{1}{p} \int_{\mathbb{R}^n} (\psi^*(y))^p (\varphi^*(y))^{1-p} \, d\mu(f, y)$$

(1.4)

holds for $f = e^{-\varphi}$ and $g = e^{-\psi}$ smooth enough such that $(\varphi^*)_p - c(\psi^*)_p$ is a convex function for some constant $c > 0$, and the function $\frac{1}{\alpha} (\psi^*)_p^\alpha$ (understood as $\log \varphi^*$ when $\alpha = 0$) is convex for some $\alpha \in (0, 1)$. Please see more precise statements in Theorem 5.7 and Corollary 5.8. The proof of Theorem 5.7 is very technical and involves a lot of rather complicated analysis on the (first and second order) differentiability and the limit of the partial derivatives of the function $- \log(f \oplus_p t \cdot_p g)$. We would like to mention that, even in the case $p = 1$ where $\phi$ defined in (1.3) is linear, such analysis (for the differentiability and related limits) already exhibits its complexity as one can see in [29]. The nonlinearity of $\phi$ defined in (1.3) for $p > 1$ does bring extra difficulty and makes our analysis even more complicated. In particular, additional conditions on $\varphi$ are needed, such as, (5.8) or those given in Corollary 5.8.
Our last contribution is the study of the \( L_p \) Minkowski problem for log-concave functions, which will be presented in Sect. 6. In fact, formula (1.4) suggests a natural way to define the \( L_p \) surface area measure for the log-concave function \( f = e^{-\varphi} \), denoted as \( \mu_p(f, \cdot) \) and defined on \( \Omega_{\varphi^*} = \{ y \in \mathbb{R}^n : 0 < \varphi^*(y) < +\infty \} \) (this set is always assumed to be nonempty). Under certain conditions on a given log-concave function \( f = e^{-\varphi} \), the measure \( \mu_p(f, \cdot) \) for \( p \in \mathbb{R} \) is given by

\[
\int_{\Omega_{\varphi^*}} g(y) \, d\mu_p(f, y) = \int_{\{ x \in \mathbb{R}^n : \nabla \varphi(x) \in \Omega_{\varphi^*} \}} g(\nabla \varphi(x))(\varphi^*(\nabla \varphi(x)))^{1-p} e^{-\varphi(x)} \, dx
\]

for every Borel function \( g \) such that \( g \in L^1(\mu_p(f, \cdot)) \) or \( g \) is non-negative. We pose the following \( L_p \) Minkowski problem for log-concave functions: \textit{for \( p \in \mathbb{R} \), find the necessary and/or sufficient conditions on a finite nonzero Borel measure \( \nu \) defined on \( \mathbb{R}^n \) so that \( d\nu = \tau \, d\mu_p(f, \cdot) \) holds for some log-concave function \( f \) and \( \tau \in \mathbb{R} \).} We provide a solution to this problem under the following mild conditions on \( \nu \): \( \nu \) is not supported in a lower-dimensional subspace of \( \mathbb{R}^n \), \( \nu(M_v \setminus L) > 0 \) holds for any bounded convex set \( L \subset \mathbb{R}^n \) where \( M_v \) is the interior of the convex hull of the support of \( \nu \), and \( \int_{\mathbb{R}^n} | x |^p \, d\nu(x) < \infty \). Note that these conditions for \( \nu \) are all natural; see Sect. 6 for more details. Our solution to the \( L_p \) Minkowski problem for log-concave functions is proved in Theorem 6.3. That is, roughly speaking, \textit{if \( \nu \) is an even measure satisfying the above conditions, for \( p > 1 \), there exists an even log-concave function \( f = e^{-\varphi} \), such that}

\[
d\nu = \frac{\int_{\Omega_{\varphi^*}} dv(y)}{\int_{\Omega_{\varphi^*}} d\mu_p(f, y)} \, d\mu_p(f, \cdot) \quad \text{on} \quad \Omega_{\varphi^*}.
\]

Finally, we would like to comment that if \( \nu \) admits a density function with respect to the Lebesgue measure, say \( d\nu(y) = h(y) \, dy \), and \( f = e^{-\varphi} \) with the convex function \( \varphi \) smooth enough, then finding a solution to the \( L_p \) Minkowski problem for log-concave functions requires to search for a (smooth enough) convex function \( \varphi \) satisfying the following Monge-Ampère equation:

\[
h(y) = \tau \varphi^*(y)^{1-p} e^{\varphi^*(y) - (y, \nabla \varphi^* y)} \det(\nabla^2 \varphi^*(y)), \quad \text{for} \quad y \in \Omega_{\varphi^*},
\]

where \( \tau \in \mathbb{R} \) is a constant and \( \det(\nabla^2 \varphi^*(y)) \) denotes the determinant of the Hessian matrix of \( \varphi^* \) at \( y \in \mathbb{R}^n \).

## 2 Preliminaries and notations

This section provides preliminaries and notations required for (log-concave) functions and convex bodies. More details can be found in [19, 58, 63]. The paper [28] by Colesanti is also an excellent source to learn various important topics related to log-concave functions.

Let \( \mathbb{N} \) be the set of natural numbers and \( n \in \mathbb{N} \) such that \( n \geq 1 \). In the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( (x, y) \) stands for the standard inner product of \( x, y \in \mathbb{R}^n \) and \( |x| \) for the Euclidean norm of \( x \in \mathbb{R}^n \). The origin of \( \mathbb{R}^n \) is denoted by \( o \). A set \( E \subset \mathbb{R}^n \) is said to be origin-symmetric if \( -x \in E \) for any \( x \in E \). Let \( B_r = \{ x \in \mathbb{R}^n : |x| \leq r \} \) be the origin-symmetric ball with radius \( r \). By \( S^{n-1} \) we mean the unit sphere, i.e., the boundary of \( B_1 \). We also use \( V(E), \text{int}(E), \overline{E} \) and \( \partial E \) to denote the volume, interior, closure and boundary of \( E \subset \mathbb{R}^n \), respectively. By \( \mathcal{H}^k \), we mean the \( k \)-dimensional Hausdorff measure.

A set \( K \subset \mathbb{R}^n \) is said to be a convex body if \( K \) is a compact convex set with nonempty interior. The family of convex bodies is denoted by \( \mathcal{K}^n \). With \( \mathcal{K}_{(o)}^n \), we mean the collection
of convex bodies containing \( o \) in their interiors. We say \( K \in \mathcal{K}^n_{(o)} \) is a dilation of \( L \in \mathcal{K}^n_{(o)} \) if there exists a \( \lambda > 0 \) such that \( K = \lambda L \). For any \( K \in \mathcal{K}^n \), the support function of \( K \), \( h_K : \mathbb{R}^n \to \mathbb{R} \), which is a sublinear function on \( \mathbb{R}^n \) and can be used to determine \( K \) uniquely, takes the following form:

\[
h_K(x) = \sup_{y \in K} (x, y) \quad \text{for } x \in \mathbb{R}^n. \tag{2.1}
\]

Let \( \alpha, \beta > 0 \) be two constants. For \( K, L \in \mathcal{K}^n_{(o)} \), define \( \alpha \cdot_p K + \beta \cdot_p L \) for \( p \geq 1 \) (see e.g., [33,49,63]) to be the convex body whose support function is given by

\[
h_{\alpha \cdot_p K + \beta \cdot_p L} = (\alpha h_K^p + \beta h_L^p)^{\frac{1}{p}}. \tag{2.2}
\]

Of course, \( \alpha \cdot_p K + \beta \cdot_p L \) in (2.2) reduces to the Minkowski addition of convex bodies when \( p = 1 \). In [33], Firey proved the following \( L_p \) Brunn-Minkowski inequality for \( p > 1 \):

\[
V(K + p \ L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}} \tag{2.3}
\]

with equality if and only if \( K \) and \( L \) are the dilation of each other. Moreover, Lutwak in [49] also established the following remarkable variational formula: for \( K, L \in \mathcal{K}^n_{(o)} \) and \( p > 1 \), one has

\[
\frac{p}{n} \cdot \frac{d}{dt} V(K + p \ t \cdot_p L) \bigg|_{t=0^+} = \frac{1}{n} \cdot \int_{S^{n-1}} h_L^p(u) h_K^{1-p}(u) dS_K(u) := V_p(K, L). \tag{2.4}
\]

Here \( S_K \) is the surface area measure of \( K \) on \( S^{n-1} \), that is, \( S_K(\eta) = \mathcal{H}_n^{n-1}(v_K^{-1}(\eta)) \) for any Borel set \( \eta \subset S^{n-1} \), where \( v_K^{-1} \) is the reverse Gauss map of \( K \). Note that both (2.3) and (2.4) work for \( p = 1 \) as well, and inequality (2.3) becomes the classical Brunn-Minkowski inequality (in this case, equality holds if and only if \( K \) and \( L \) agree up to a translation and a dilation). We refer to [63] and references therein for more details.

The elegant formula (2.4) introduces two important geometric objects: the \( L_p \) mixed volume \( V_p(K, L) \) for \( K, L \in \mathcal{K}^n_{(o)} \) and the \( L_p \) surface area measure of \( K \in \mathcal{K}^n_{(o)} \) given by

\[
dS_p(K, \cdot) = h_K^{1-p} dS_K. \tag{2.5}
\]

Associated to the \( L_p \) mixed volume is the following \( L_p \) Minkowski inequality for \( p > 1 \):

\[
V_p(K, L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}},
\]

with equality if and only if \( K \) and \( L \) are the dilation of each other [49]. Related to the \( L_p \) surface area measure is the following well-known \( L_p \) Minkowski problem [49].

**Problem 2.1** *(The \( L_p \) Minkowski problem for convex bodies)* Let \( p \in \mathbb{R} \). What are the necessary and sufficient conditions on a nonzero finite Borel measure \( \mu \) on \( S^{n-1} \) such that there exists a convex body \( K \subset \mathbb{R}^n \) satisfying \( dS_p(K, \cdot) = d\mu \)?

Throughout the paper, let \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a convex function, that is,

\[
\varphi(\lambda x + (1 - \lambda) y) \leq \lambda \varphi(x) + (1 - \lambda) \varphi(y)
\]

holds for any \( \lambda \in (0, 1) \) and \( x, y \in \text{dom}(\varphi) \), where \( \text{dom}(\varphi) = \{ x \in \mathbb{R}^n : \varphi(x) < +\infty \} \) denotes the domain of \( \varphi \). Denote by \( \mathcal{C} \) the set of all convex functions from \( \mathbb{R}^n \) to \( \mathbb{R} \cup \{+\infty\} \). Clearly, \( \text{dom}(\varphi) \) is a convex set for any \( \varphi \in \mathcal{C} \). A convex function \( \varphi \in \mathcal{C} \) is said to be proper if \( \text{dom}(\varphi) \neq \emptyset \). By \( \mathcal{C}^1(E) \) we mean the set of all functions \( \varphi \in \mathcal{C} \) such that \( \varphi \) is continuously differentiable on \( E \subseteq \text{int(\text{dom}(\varphi))} \). Similarly, by \( \varphi \in \mathcal{C}^2_+(E) \) we mean that \( \varphi \in \mathcal{C} \) is twice continuously differentiable and its Hessian matrix is positive definite on \( E \subseteq \text{int(\text{dom}(\varphi))} \).

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Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function (not necessarily a convex function) such that $\varphi(x) < \infty$ for some $x \in \mathbb{R}^n$. The Fenchel conjugate of $\varphi$ is defined by:

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - \varphi(x) \} \quad \text{for all } y \in \mathbb{R}^n. \quad (2.5)$$

For $\alpha > 0$, (2.5) yields that, for all $y \in \mathbb{R}^n$,

$$(\alpha \varphi)^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - \alpha \varphi(x) \} = \alpha \sup_{x \in \mathbb{R}^n} \{ \langle x, y/\alpha \rangle - \varphi(x) \} = \alpha \varphi^*(y/\alpha). \quad (2.6)$$

Clearly, $\varphi^{**} \leq \varphi, \varphi^* \in C$, and $\varphi^*$ is always lower semi-continuous. It can also be checked that if $\varphi \leq \psi$, then $\varphi^* \geq \psi^*$. Moreover, if $\varphi \in C$ is proper, the Fenchel-Moreau theorem (or Fenchel biconjugation theorem) [19,58] yields that $(\varphi^*)^* = \varphi$ if and only if $\varphi$ is lower semi-continuous. It is obvious by (2.5) that $\varphi^*(0) = -\inf_{x \in \mathbb{R}^n} \varphi(x)$ for $\varphi \in C$, and hence $\varphi^*$ is proper if $\inf_{x \in \mathbb{R}^n} \varphi(x) > -\infty$. Furthermore, if $\varphi \in C$ is proper, then $\varphi^*(y) > -\infty$ for all $y \in \mathbb{R}^n$. We also let $e^{-\infty} = 0$ by default (unless otherwise stated).

Denote by $\nabla \varphi(x)$ the gradient of $\varphi$ at $x \in \text{dom}(\varphi)$ if $\varphi$ is differentiable at $x$. A convex function $\varphi \in C$ may not be differentiable at each point in its domain, however it is always continuous on $\mathbb{R}^n \setminus \partial(\text{dom}(\varphi))$ and is differentiable almost everywhere in $\text{int}(\text{dom}(\varphi))$. It can be checked that if $\varphi$ is differentiable at $x$, then the supremum in (2.5) is a maximum, and it is attained at $y = \nabla \varphi(x)$. Hence,

$$\varphi^*(\nabla \varphi(x)) = \langle x, \nabla \varphi(x) \rangle - \varphi(x). \quad (2.7)$$

This relation will be used often in later context. Denote by $\nabla^2 \varphi$ the Hessian matrix of $\varphi$.

Throughout the paper, we shall consider two classes of functions $\mathcal{L}$ and $\mathcal{A}$ given by

$$\mathcal{L} = \left\{ \varphi \in C \mid \varphi \text{ is proper such that } \lim_{|x| \to +\infty} \varphi(x) = +\infty \right\},$$
$$\mathcal{A} = \left\{ f : \mathbb{R}^n \to \mathbb{R} \mid f = e^{-\varphi}, \ \varphi \in \mathcal{L} \right\}.$$

It has been proved in [29, Lemma 2.5] that if $\varphi \in \mathcal{L}$, then for all $x \in \mathbb{R}^n$,

$$\varphi(x) \geq a|x| + b, \quad (2.8)$$

where $a > 0$ and $b$ are two constants, and $\varphi^*$ is proper and satisfies $\varphi^*(y) > -\infty$ for all $y \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be log-concave if $f = e^{-\varphi}$ for some convex function $\varphi \in C$. We are interested in log-concave functions such that $0 < J(f) < \infty$, where $J(f)$ is the total mass of $f$ defined in (1.1). It can be easily checked (see e.g. [30, p. 3835]) that, for $f = e^{-\varphi}$ a log-concave function, $0 < J(f) < \infty$ is equivalent to the facts that $\text{dom}(\varphi)$ is not contained in a hyperplane and $\lim_{|x| \to +\infty} \varphi(x) = +\infty$ (in particular, $f \in \mathcal{A}$). It follows from (2.5) and (2.8) that, if $f = e^{-\varphi}$ is not degenerate (i.e., not vanishing on some sets with positive Lebesgue measure) with $\varphi \in C$, then $J(f) < \infty$ if and only if $o \in \text{int}(\text{dom}(\varphi^*))$. To this end, if $o \in \text{int}(\text{dom}(\varphi^*))$, then there exists a $\delta > 0$ such that $U_\delta(o) = \{ y \in \mathbb{R}^n : |y| < \delta \} \subseteq \text{dom}(\varphi^*)$. From (2.5), for any $x \in \mathbb{R}^n$, let $y = \frac{\delta}{2} \frac{x}{|x|}$ and then

$$\varphi(x) \geq \langle x, y \rangle - \varphi^*(y) \geq \frac{\delta}{2} |x| - \max_{|y| \leq \delta/2} \varphi^*(y).$$

Consequently, (2.8) holds with $a = \frac{\delta}{2}$ and $b = \max_{|y| \leq \delta/2} \varphi^*(y) < \infty$, and then $J(f) < \infty$. Conversely, if $J(f) < \infty$ (and hence $f$ satisfies (2.8) for some finite constants $a > 0$
and $b$), then (2.5) implies $o \in \text{dom}(\varphi^\ast)$ because $\varphi^\ast(o) = -\inf_{x \in \mathbb{R}^n} \varphi(x) \leq -b$. Note that $f$ is not degenerate and there exists $x_0 \in \text{dom}(\varphi)$. By (2.5), one gets, for $y \in U_a(o)$,

$$\varphi^\ast(y) \geq (x_0, y) - \varphi(x_0) \geq -|x_0| \cdot |y| - \varphi(x_0) \geq -a|x_0| - \varphi(x_0) > -\infty.$$ 

On the other hand, it follows from (2.5) and (2.8) that, for all $y \in U_a(o)$,

$$\varphi^\ast(y) \leq \sup_{x \in \mathbb{R}^n} \{(x, y) - a|x| - b\} = -b + \mathbf{1}_B(y) = -b + 1 < +\infty.$$ 

This concludes that $U_a(o) \subset \text{dom}(\varphi^\ast)$ and hence $o \in \text{int(\text{dom}(\varphi^\ast))}$.

The surface area measure for a log-concave function $f$ (or the moment measure of $-\log f$ as in [30]) plays important roles in applications. The following definition follows from [30, Definition 1] (see also [29, Definition 4.1]).

**Definition 2.2** Let $f = e^{-\varphi}$ be a log-concave function. The surface area measure of $f$ (or the moment measure of $\varphi$), denoted by $\mu(f, \cdot)$, is the Borel measure on $\mathbb{R}^n$ such that $\mu(f, \cdot)$ is the push-forward measure of $e^{-\varphi(x)} \, dx$ under $\nabla \varphi$, that is,

$$\int_{\mathbb{R}^n} g(y) \, d\mu(f, y) = \int_{\mathbb{R}^n} g(\nabla \varphi(x)) e^{-\varphi(x)} \, dx,$$

for every Borel function $g$ such that $g \in L^1(\mu(f, \cdot))$ or $g$ is non-negative.

It has been proved that $\varphi^\ast \in L^1(\mu(f, \cdot))$, if $f = e^{-\varphi} \in \mathcal{A}$ with $\varphi^\ast \geq 0$ in [29, Lemma 4.12] or if $0 < J(e^{-\varphi}) < \infty$ in [30, Proposition 7], i.e.,

$$-\infty < \int_{\mathbb{R}^n} \varphi^\ast(y) \, d\mu(f, y) = \int_{\mathbb{R}^n} \varphi^\ast(\nabla \varphi(x)) e^{-\varphi(x)} \, dx < +\infty. \tag{2.10}$$

Let $(\varphi \alpha)(x) = \alpha \varphi(\frac{x}{\alpha})$ for $\alpha > 0$ and $\varphi(0) = \mathbf{1}_{\{0\}}$ where $\mathbf{1}_E$ is the indicatrix function of $E$ (i.e., $\mathbf{1}_E(x) = 0$ if $x \in E$ and $\mathbf{1}_E = +\infty$ if $x \notin E$). By (2.6), one sees $(\alpha \varphi)^\ast = \varphi^\ast \alpha$ for $\alpha > 0$. It can be easily checked from (1.2) that, for real $\alpha, \beta > 0$ and two log-concave functions $f = e^{-\varphi}$ and $g = e^{-\psi}$, the Asplund sum of $\alpha \cdot f \oplus \beta \cdot g$ may also be formulated by

$$\alpha \cdot f \oplus \beta \cdot g = e^{-\varphi \Box \psi}, \tag{2.11}$$

where $\varphi \Box \psi$ is the infimal convolution of $\varphi$ and $\psi$ defined by

$$\varphi \Box \psi(x) = \inf_{y \in \mathbb{R}^n} \{\varphi(x - y) + \psi(y)\}, \text{ for any } x \in \mathbb{R}^n.$$

It can be easily checked that

$$(\varphi \alpha \Box \psi \beta)^\ast = \alpha \varphi^\ast + \beta \psi^\ast. \tag{2.12}$$

The Prékopa-Leindler inequality implies that (see, e.g., [29, Remark 3.3]), for any integrable log-concave functions $f, g$ and $0 < \lambda < 1$, one has

$$J((1 - \lambda) \cdot f \oplus \lambda \cdot g) \geq J(f)^{1-\lambda} J(g)^{\lambda}, \tag{2.13}$$

with equality if and only if there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = g(x - x_0)$ (see, e.g., [31,36]).
The $L_p$ Asplund sum of log-concave functions

This section is dedicated to the study of properties of the $L_p$ Asplund sum of log-concave functions. In particular, a Prékopa-Leindler type inequality related to the $L_p$ Asplund sum will be proved. We focus on $\mathcal{L}_0 \subset \mathcal{L}$ and $\mathcal{A}_0 \subset \mathcal{A}$, where $\mathcal{A}_0 = \{ e^{-\psi} : \psi \in \mathcal{L}_0 \}$ with $\mathcal{L}_0 = \{ \psi \in \mathcal{L} : \psi$ is non-negative and lower semi-continuous such that $\psi(o) = 0 \}$. Clearly, if $\psi \in \mathcal{L}_0$, then $\psi$ has its minimum attained at the origin $o$. From (2.5), one sees that if $\psi(o) = 0$, then

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - \varphi(x) \} = \langle o, y \rangle - \varphi(o) = 0, \quad \text{for all } y \in \mathbb{R}^n. \quad (3.1)$$

Moreover, $\varphi^*(o) = 0$ as

$$\varphi^*(o) = \sup_{x \in \mathbb{R}^n} -\varphi(x) = -\inf_{x \in \mathbb{R}^n} \varphi(x) = 0. \quad (3.2)$$

The definition for the $L_p$ Asplund sum of log-concave functions is given below, with concentration on $p > 1$.

Definition 3.1 Let $f = e^{-\psi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. For $p > 1$, define $f \oplus_p g$, the $L_p$ Asplund sum of $f$ and $g$, by $f \oplus_p g = e^{-\psi \square_p \psi}$, where

$$\varphi \square_p \psi = \left[ (\varphi^*)^p + (\psi^*)^p \right]^{1/p}. \quad (3.3)$$

For convenience, if $\alpha > 0$, let

$$(\varphi \cdot_p \alpha)(x) = (\varphi \alpha^{1/p})(x) = \alpha^{1/p} \varphi(\alpha^{-1/p} x) \quad \text{for any } x \in \mathbb{R}^n.$$

For $\alpha, \beta > 0$, $p \geq 1$, $f = e^{-\psi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$, let $\alpha \cdot_p f \oplus_p \beta \cdot_p g = e^{-\psi \cdot_p \alpha \square_p \psi} \cdot_p \beta$, where

$$\varphi \cdot_p \alpha \square_p \psi \cdot_p \beta = \left[ (\alpha(\varphi^*)^p + \beta(\psi^*)^p \right]^{1/p}. \quad (3.4)$$

By (2.12) and (3.4), $\alpha \cdot_p f \oplus_p \beta \cdot_p g$ for $p = 1$ reduces to the Asplund sum $\alpha \cdot f \oplus \beta \cdot g$.

The following result asserts that the $L_p$ Asplund sum of log-concave functions is closed in $\mathcal{A}_0$.

Proposition 3.2 Let $f = e^{-\psi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. For $p > 1$ and $\alpha, \beta > 0$, let

$$\phi = \left[ (\alpha(\varphi^*)^p + \beta(\psi^*)^p \right]^{1/p}. \quad (3.5)$$

Then $\phi = (\varphi \cdot_p \alpha \square_p \psi \cdot_p \beta)^*$ and $\alpha \cdot_p f \oplus_p \beta \cdot_p g = e^{-\phi^*} \in \mathcal{A}_0$.

Proof Let $f = e^{-\psi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$ which imply $\varphi, \psi \in \mathcal{L}_0$. By (3.1) and (3.2), $\varphi^* \geq 0$ and $\psi^* \geq 0$ with $\varphi^*(o) = \psi^*(o) = 0$. Hence, $\varphi \cdot_p \alpha \square_p \psi \cdot_p \beta$ for $p > 1$ given in (3.4) is well-defined and $\varphi \cdot_p \alpha \square_p \psi \cdot_p \beta \in \mathcal{C}$ is lower semi-continuous. Moreover, $\phi$ defined by (3.5) is non-negative and $\phi(o) = 0$ is the minimum of $\phi$. This further yields that $(\varphi \cdot_p \alpha \square_p \psi \cdot_p \beta)(o) = 0$ by (3.2), and hence is proper. Note that $\varphi^*, \psi^* \in \mathcal{C}$ are lower semi-continuous. Consequently, for $p > 1, \alpha, \beta > 0, \phi$ is lower semi-continuous. Moreover,
\( \phi \in \mathcal{C} \). To this end, for all \( \lambda \in (0, 1) \) and \( x, y \in \text{dom}(\phi) \), the Minkowski inequality for norms yields that

\[
\phi(\lambda x + (1-\lambda)y) = \left[ \alpha(\varphi(\lambda x + (1-\lambda)y))^p + \beta(\psi(\lambda x + (1-\lambda)y))^p \right]^{\frac{1}{p}} \\
\leq \left[ \alpha(\varphi^p(x) + (1-\lambda)\varphi^p(y))^p + \beta(\psi^p(x) + (1-\lambda)\psi^p(y))^p \right]^{\frac{1}{p}} \\
\leq \lambda \left[ \alpha(\varphi^p) + \beta(\psi^p) \right]^{\frac{1}{p}} (x) + (1-\lambda) \left[ \alpha(\varphi^p) + \beta(\psi^p) \right]^{\frac{1}{p}} (y) \\
= \lambda \phi(x) + (1-\lambda)\phi(y).
\]

According to (3.4) and the Fenchel-Moreau theorem [19,58], one sees that

\[
(\varphi \cdot_p \alpha \cdot_p \psi \cdot_p \beta)^* = (\varphi^*)^* = \phi \quad \text{and} \quad \alpha \cdot_p f \oplus_p \beta \cdot_p g = e^{-\phi^*}.
\]

To claim that \( \alpha \cdot_p f \oplus_p \beta \cdot_p g \in \mathcal{O} \) for \( p > 1 \), it is enough to show that \( \phi^* \in \mathcal{O} \). To this end, we only need to claim that \( \lim_{|x| \to +\infty} \phi^* (x) = +\infty \) as we already showed that \( \phi^* \) is proper and lower semi-continuous. Inequality (2.8) implies that for all \( \phi(x) \geq a|x| + b \) and \( \psi(x) \geq a'|x| + b' \) for some \( a, a' > 0 \) and \( b, b' \in \mathbb{R} \). Let \( c = \min \{a, a'\} > 0 \) and \( d = \min \{b, b'\} \). Hence \( \phi(x) \geq c|x| + d \) and \( \psi(x) \geq c|x| + d \) for all \( x \in \mathbb{R}^n \). This further yields that \( \phi^* \leq (c|x| + d)^* \) and \( \psi^* \leq (c|x| + d)^* \). Accordingly, due to \( p > 1 \) and \( \alpha, \beta > 0 \), one has

\[
\phi = \left[ \alpha(\varphi^p) + \beta(\psi^p) \right]^{\frac{1}{p}} \leq \left[ \alpha((c|x| + d)^p) + \beta((c|x| + d)^p) \right]^{\frac{1}{p}} \\
= (\alpha + \beta)^{\frac{1}{p}} (c|x| + d)^*.
\]

It follows from (2.5) that for all \( y \in \mathbb{R}^n \),

\[
\phi^* (y) \geq \left[ (\alpha + \beta)^{\frac{1}{p}} (c|x| + d)^* \right]^* (y) = c|y| + d(\alpha + \beta)^{\frac{1}{p}}.
\]

In particular, \( \lim_{|y| \to +\infty} \phi^* (y) = +\infty \). This concludes that \( \alpha \cdot_p f \oplus_p \beta \cdot_p g \in \mathcal{O} \).

\( \square \)

Let \( \chi_E \) be the characteristic function of \( E \), i.e., \( \chi_E(x) = 1 \) if \( x \in E \) and \( \chi_E(x) = 0 \) if \( x \notin E \). It is well-known that if \( K \in \mathcal{K}(o) \), then \( \chi_K \) is a log-concave function with \( I_K = -\log(\chi_K) \in \mathcal{L} \) and hence \( \chi_K \in \mathcal{O} \). By (2.1) and (2.5), one sees that \( (I_K)^* = h_K \) and \( (h_K)^* = I_K \). Together with (2.2), for two convex bodies \( K, L \in \mathcal{K}(o) \), one has, for \( p \geq 1 \),

\[
(I_K) \cdot_p \alpha \Box_p (I_L) \cdot_p \beta = \left( (\alpha((I_K)^p))^p + \beta((I_L)^p))^p \right)^{\frac{1}{p}}^* = \left( (\alpha h_K^p + \beta h_L^p))^p \right)^{\frac{1}{p}}^*. \\
= I_{\alpha \cdot_p K + \beta \cdot_p L}.
\]

Therefore, for \( \alpha, \beta > 0 \) and \( p \geq 1 \),

\[
\alpha \cdot_p \chi_K \oplus_p \beta \cdot_p \chi_L = e^{-(I_K)^p \alpha \Box_p (I_L)^p \beta} = \chi_{\alpha \cdot_p K + \beta \cdot_p L}.
\]

Moreover, for two convex bodies \( K, L \in \mathcal{K}(o) \) and for any \( \alpha, \beta > 0 \),

\[
J(\alpha \cdot_p \chi_K \oplus_p \beta \cdot_p \chi_L) = \int_{\mathbb{R}^n} \chi_{\alpha \cdot_p K + \beta \cdot_p L}(x) \, dx = V(\alpha \cdot_p K + \beta \cdot_p L). \quad (3.7)
\]
By the $L_p$ Brunn-Minkowski inequality (2.3), for $p \geq 1$, $\lambda \in (0, 1)$, and $K, L \in \mathcal{K}_n^{(o)}$, one has

$$V((1 - \lambda) \cdot_p K + p \cdot_p L) \geq \left((1 - \lambda)V(K)^\frac{p}{n} + \lambda V(L)^\frac{p}{n}\right)^\frac{n}{p} \geq V(K)^{1 - \lambda}V(L)^{\lambda}, \quad (3.8)$$

where the last inequality follows from the fact that the logarithmic function is concave (or from the inequality between the $\frac{p}{n}$-mean and the geometric mean for $p > 1$). Combining (3.7) and (3.8), one easily sees that, for $p \geq 1$, $\lambda \in (0, 1)$, and $K, L \in \mathcal{K}_n^{(o)}$,

$$J\left((1 - \lambda) \cdot_p \chi_K \oplus_p \lambda \cdot_p \chi_L\right) \geq J(\chi_K)^{1 - \lambda}J(\chi_L)^{\lambda}. \quad (3.10)$$

This is a Prékopa-Leindler type inequality and can be extended to all $f, g \in \mathcal{A}_0$. This result is proved in the following theorem, where we focus on the case $p > 1$, as the case $p = 1$ reduces to the classical Prékopa-Leindler inequality (2.13). We would like to point out that, due to lack of homogeneity in the functional setting, it is not clear whether the geometric mean on the right side of inequality (3.9) can be replaced by a larger mean.

**Theorem 3.3** Let $f = e^{-\psi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. For $\lambda \in (0, 1)$ and $p > 1$, it holds that

$$J\left((1 - \lambda) \cdot_p f \oplus_p \lambda \cdot_p g\right) \geq J(f)^{1 - \lambda}J(g)^{\lambda} \quad (3.9)$$

with equality if and only if $f = g$ on $\mathbb{R}^n$.

**Proof** Let $f = e^{-\psi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. For $\lambda \in (0, 1)$ and $p > 1$, let $(1 - \lambda) \cdot_p f \oplus_p \lambda \cdot_p g = e^{-\phi^{*}_{\lambda}}$. According to Proposition 3.2, one has $(\phi^{*}_{\lambda})^{*} = \phi_{\lambda}$ and $\phi_{\lambda} = \left[(1 - \lambda)(\psi^{*})^p + \lambda(\psi^{*})^p\right]^\frac{1}{p}$. Since the function $t^\frac{1}{p}$ is strictly concave on $t \in [0, \infty)$ when $p > 1$, one has,

$$\phi_{\lambda} = \left[(1 - \lambda)(\psi^{*})^p + \lambda(\psi^{*})^p\right]^\frac{1}{p} \geq (1 - \lambda)\psi^{*} + \lambda\psi^{*}. \quad (3.10)$$

Together with (2.12), this in turn implies that

$$\phi^{*}_{\lambda} \leq (1 - \lambda)\psi^{*} + \lambda\psi^{*} = \psi(1 - \lambda)\Box \psi. \quad (3.11)$$

By the classical Prékopa-Leindler inequality (2.13), inequality (3.9) holds:

$$J\left((1 - \lambda) \cdot_p f \oplus_p \lambda \cdot_p g\right) = \int_{\mathbb{R}^n} e^{-\phi^{*}_{\lambda}(x)} \, dx \geq \int_{\mathbb{R}^n} e^{-\left(\psi(1 - \lambda)\Box \psi\right)(x)} \, dx \geq J\left((1 - \lambda) \cdot_p f \oplus_p \lambda \cdot_p g\right)^{1 - \lambda}J\left((1 - \lambda) \cdot_p f \oplus_p \lambda \cdot_p g\right)^{\lambda}. \quad (3.12)$$

Now let us characterize the equality condition. It is obvious that equality holds if $f = g$. On the other hand, to have equality in (3.9), equalities must occur in (3.12), and consequently equalities hold for the classical Prékopa-Leindler inequality (2.13) and for (3.11) (equivalently for (3.10)). The former one implies that there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = g(x - x_0)$, which in turn yields $\psi(x) = \psi(x - x_0)$. Combining with the latter one and the fact that $t^\frac{1}{p}$ is strictly concave, one gets $\psi^{*} = \psi^{*}$ and hence $f = g$ as desired. \hfill $\square$

**Corollary 3.4** Let $f = e^{-\psi} \in \mathcal{A}_0$, $g = e^{-\psi} \in \mathcal{A}_0$ and $p > 1$. Then, $J(f \oplus_p t \cdot_p g)$ is log-concave on $t \in (0, \infty)$ and $J\left((1 - t) \cdot_p f \oplus_p t \cdot_p g\right)$ is log-concave on $t \in (0, 1)$. 

\hfill $\square$
Proof Let $f = e^{-\varphi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. For $\lambda \in (0, 1)$ and $t, s \in (0, \infty)$, let $\eta = (1 - \lambda)t + \lambda s$. According to Proposition 3.2, one gets $f \oplus_p \eta \cdot_p g = e^{-\phi_{\eta}^*}$ with $\phi_{\eta} = \left[ \left( (\varphi^*)^p + \eta (\psi^*)^p \right)^{\frac{1}{p}} \right]$. It follows from Proposition 3.2 and the fact that the function $t^{\frac{1}{p}}$ is concave on $t \in (0, \infty)$ for $p > 1$ that
\[
\phi_{\eta} = \left[ \left( (1 - \lambda)(\varphi^*)^p + t(\psi^*)^p \right) + \lambda \left( (\varphi^*)^p + s(\psi^*)^p \right) \right]^{\frac{1}{p}} \\
\geq (1 - \lambda) \left( (\varphi^*)^p + t(\psi^*)^p \right)^{\frac{1}{p}} + \lambda \left( (\varphi^*)^p + s(\psi^*)^p \right)^{\frac{1}{p}} \\
= (1 - \lambda) (\varphi_{t}^*)^p + \lambda (\varphi_{s}^*)^p.
\]
By (2.11) and (2.12), one gets $\phi_{\eta}^* \leq ((1 - \lambda)(\varphi_{t}^*)^p + \lambda (\varphi_{s}^*)^p)^* = \phi_{t}^*(1 - \lambda) \Box \varphi_{s}^* \lambda$ and hence,
\[
f \oplus_p \eta \cdot_p g = e^{-\phi_{\eta}^*} \geq (1 - \lambda) \left( f \oplus_p t \cdot_p g \right) \oplus \lambda \left( f \oplus_p s \cdot_p g \right).
\]
By the classical Prékopa-Leindler inequality (2.13), one gets the desired log-concavity for $J(f \oplus_p t \cdot_p g)$ on $t \in (0, \infty)$, that is,
\[
\log J(f \oplus_p t \cdot_p g) \geq (1 - \lambda) \log J(f \oplus_p t \cdot_p g) + \lambda \log J(f \oplus_p s \cdot_p g).
\]
The log-concavity for $J((1 - t) \cdot_p f \oplus_p t \cdot_p g)$ on $t \in (0, 1)$ is a direct result of Theorem 3.3. \hfill \Box

4 An $L_p$ Minkowski type inequality

In this section, we propose a definition for $\delta J_p(f, g)$, the first variation of the total mass at $f$ along $g$ with respect to the $L_p$ Asplund sum. A Minkowski type inequality for $\delta J_p(f, g)$ will be established. Our definition for $\delta J_p(f, g)$ is given below.

Definition 4.1 Let $f, g \in \mathcal{A}_0$. For $p > 1$, define $\delta J_p(f, g)$ by
\[
\delta J_p(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus_p t \cdot_p g) - J(f)}{t}
\]
whenever the limit exists.

Let $J(f) > 0$ and $\text{Ent}(f)$ be the entropy of $f$ which may be formulated by
\[
\text{Ent}(f) = \int_{\mathbb{R}^n} f \log f \, dx - J(f) \log J(f) \tag{4.1}
\]
According to [29, Proposition 3.11], $\text{Ent}(f)$ is finite if $f \in \mathcal{A}$ such that $J(f) > 0$. Our main result in this section is the following Minkowski type inequality for $\delta J_p(f, g)$. We only focus on $p > 1$ as inequality (4.2) and its equality condition for $p = 1$ have already been proved in [29, Theorem 5.1].

Theorem 4.2 Let $f, g \in \mathcal{A}_0$ such that $J(f) > 0$ and $J(g) > 0$. For $p > 1$, one has,
\[
\delta J_p(f, g) \geq J(f) \left[ \frac{n}{p} + \frac{1 - p}{p} \log J(f) + \log J(g) \right] + \frac{1}{p} \text{Ent}(f), \tag{4.2}
\]
with equality if and only if $f = g$. \hfill \(\Box\) Springer
In order to prove Theorem 4.2, we shall need some preparation. The following result shows how to calculate $\delta J_p(f, f)$. Again, the case for $p = 1$ has been covered in [29, Proposition 3.11] and will not be repeated in the following result. From Lemma 4.3, one sees that $\delta J_p(f, f)$ for $p > 1$ is finite if $f \in \mathcal{A}_0$ such that $J(f) > 0$.

**Lemma 4.3** Let $f \in \mathcal{A}_0$ such that $J(f) > 0$. For $p > 1$, one has

$$
\delta J_p(f, f) = \frac{n}{p} J(f) + \frac{1}{p} \int_{\mathbb{R}^n} f \log f \, dx = \frac{n + \log J(f)}{p} J(f) + \frac{1}{p} \text{Ent}(f). \quad (4.3)
$$

**Proof** Let $f = e^{-\varphi} \in \mathcal{A}_0$ such that $J(f) > 0$. It can be checked from (3.4) that, for $p > 1$ and $t > 0$, $f \oplus_p t \cdot_p f = e^{-\varphi_t}$ with

$$
\varphi_t(x) = [\varphi \cdot_p (1 + t)](x) = (1 + t)^{\frac{1}{p}} \varphi\left(\frac{x}{(1 + t)^{\frac{1}{p}}}\right).
$$

Consequently, by letting $x = (1 + t)^{\frac{1}{p}} y$, one gets

$$
\delta J_p(f, f) = \lim_{t \to 0^+} \frac{J(f \oplus_p t \cdot_p f) - J(f)}{t} = \lim_{t \to 0^+} \frac{1}{t} \left( (1 + t)^{\frac{n}{p}} \int_{\mathbb{R}^n} e^{-(1+t)^{\frac{1}{p}} \varphi(y)} \, dy - \int_{\mathbb{R}^n} e^{-\varphi(y)} \, dy \right)
$$

$$
= \lim_{t \to 0^+} \frac{(1 + t)^{\frac{n}{p}} - 1}{t} \int_{\mathbb{R}^n} e^{-\varphi(y)} \, dy + \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-(1+t)^{\frac{1}{p}} \varphi(y)} - e^{-\varphi(y)}}{t} \, dy.
$$

Note that $\varphi \in \mathcal{L}_0$ is non-negative. It follows from the monotone convergence theorem that

$$
\lim_{t \to 0^+} \frac{(1 + t)^{\frac{n}{p}} - 1}{t} \int_{\mathbb{R}^n} e^{-(1+t)^{\frac{1}{p}} \varphi(y)} \, dy = \frac{n}{p} \lim_{t \to 0^+} \int_{\mathbb{R}^n} e^{-(1+t)^{\frac{1}{p}} \varphi(y)} \, dy = \frac{n}{p} J(f).
$$

Similarly, one can also have

$$
\lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-(1+t)^{\frac{1}{p}} \varphi(y)} - e^{-\varphi(y)}}{t} \, dy = \int_{\mathbb{R}^n} \lim_{t \to 0^+} \frac{e^{-(1+t)^{\frac{1}{p}} \varphi(y)} - e^{-\varphi(y)}}{t} \, dy
$$

$$
= -\frac{1}{p} \int_{\mathbb{R}^n} \varphi(x) e^{-\varphi(x)} \, dx.
$$

By (4.1), one gets

$$
\delta J_p(f, f) = \frac{n}{p} J(f) + \frac{1}{p} \int_{\mathbb{R}^n} f \log f \, dx = -\frac{n}{p} J(f) + \frac{1}{p} \text{Ent}(f) + \frac{1}{p} J(f) \log J(f).
$$

This is exactly the desired equality (4.3). \qed

Our second lemma is to extend [29, Lemma 3.9] for $p = 1$ to all $p > 1$.

**Lemma 4.4** Let $f = e^{-\varphi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. For $p > 1$ and for $t > 0$, set

$$
\varphi_t = \varphi \circ_p (\psi \cdot_p t) \quad (4.4)
$$

and $f_t = e^{-\varphi_t}$. Then, for any $x \in \mathbb{R}^n$ and $t, s \in (0, 1]$ such that $s < t$, one has,

$$
\varphi_1(x) \leq \varphi_t(x) \leq \varphi_s(x) \leq \varphi(x) \quad \text{and} \quad f(x) \leq f_s(x) \leq f_t(x) \leq f_1(x).
$$
Corollary 3.4 and Lemma 4.4 imply that \( \log \).
Combining (4.7) and (4.8), one gets
\[
\varphi_t^* = \left( (\varphi^*)^p + t(\psi^*)^p \right) \frac{1}{p} \geq \varphi^*.
\] (4.5)
Clearly, \( \varphi_{t+\delta}^* \geq \varphi_t^* \) and \( \varphi_{t+\delta} = (\varphi_{t+\delta})^* \leq (\varphi_t^*)^* = \varphi_t \). Moreover, for any \( x \in \mathbb{R}^n \) and \( t \in [0, 1] \),
\[
\varphi_1(x) \leq \varphi_t(x) \leq \varphi(x) \quad \text{and} \quad f(x) \leq f_t(x) \leq f_1(x).
\]
This completes the proof of this lemma.

The following lemma proves that \( \delta J_p(f, g) \) indeed exists (although may be \( +\infty \)). Again we only focus on \( p > 1 \), as \( p = 1 \) has been covered in [29, Theorem 3.6].

Lemma 4.5 Let \( f, g \in \mathcal{A}_0 \) such that \( J(f) > 0 \). For \( p > 1 \), one has \( \delta J_p(f, g) \in [0, +\infty] \).

Proof Let \( f = e^{-\varphi} \in \mathcal{A}_0 \) and \( g = e^{-\psi} \in \mathcal{A}_0 \) such that \( J(f) > 0 \). Then \( \varphi, \psi \in \mathcal{L}_0 \). Let \( \varphi_t \) be as in (4.4). It follows from Lemma 4.4 that \( \varphi(x) \geq \varphi_t(x) := \lim_{t \to 0^+} \varphi_t(x) \) for every \( x \in \mathbb{R}^n \) and
\[
J(e^{-\varphi}) = \lim_{t \to 0^+} J(e^{-\varphi_t}) \geq J(e^{-\varphi}),
\]
by the monotone convergence theorem. Note that
\[
\delta J_p(f, g) = \lim_{t \to 0^+} \frac{J(e^{-\varphi_t}) - J(e^{-\varphi})}{t}.
\] (4.6)
Therefore, \( \delta J_p(f, g) = +\infty \) if \( J(e^{-\varphi}) > J(e^{-\varphi_t}) \), and \( \delta J_p(f, g) = 0 \) if \( J(e^{-\varphi_t}) = J(e^{-\varphi}) \) for some \( t_0 > 0 \) (hence \( J(e^{-\varphi}) = J(e^{-\varphi_t}) = J(e^{-\varphi}) \) for every \( t \in [0, t_0] \) due to Lemma 4.4).

Lastly, we consider the case that \( J(e^{-\varphi_t}) > J(e^{-\varphi}) \) for all \( t > 0 \) but \( J(e^{-\varphi}) = J(e^{-\varphi_t}) \). Note that \( J(f) = J(e^{-\varphi}) > 0 \). In this case, one has
\[
\frac{J(e^{-\varphi_t}) - J(e^{-\varphi})}{t} = \frac{\log J(e^{-\varphi_t}) - \log J(e^{-\varphi})}{t} \cdot \frac{J(e^{-\varphi_t}) - J(e^{-\varphi})}{\log J(e^{-\varphi_t}) - \log J(e^{-\varphi})}.
\] (4.7)
Corollary 3.4 and Lemma 4.4 imply that \( \log J(e^{-\varphi_t}) \) is an increasing and concave function on \( t \in (0, \infty) \). Thus,
\[
J(e^{-\varphi}) = \lim_{t \to 0^+} \frac{J(e^{-\varphi_t}) - J(e^{-\varphi})}{\log J(e^{-\varphi_t}) - \log J(e^{-\varphi})},
\]
and
\[
\lim_{t \to 0^+} \frac{\log J(e^{-\varphi_t}) - \log J(e^{-\varphi})}{t} \in [0, +\infty].
\] (4.8)
Combining (4.7) and (4.8), one gets \( \delta J_p(f, g) \in [0, +\infty] \), and this completes the proof.

We also need the following lemma. The case for \( p = 1 \) has been given in [29, Lemma 5.4], so we only state the result for \( p > 1 \).

Lemma 4.6 Let \( f = e^{-\varphi} \in \mathcal{A}_0 \) and \( g = e^{-\psi} \in \mathcal{A}_0 \) such that \( J(f) > 0 \). For \( p > 1 \), one has,
\[
\lim_{t \to 0^+} \frac{J((1 - t) \cdot f \oplus_p t \cdot g) - J(f)}{t} = \delta J_p(f, g) - \delta J_p(f, f).
\] (4.9)
Proof Let $f = e^{-\varphi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$ such that $J(f) > 0$. According to Proposition 3.2, for $p > 1$ and $t \in (0, 1)$, $(1-t) \cdot_p f \oplus_p t \cdot_p g = e^{-\phi_t^p}$ where

$$\phi_t = (1-t)(\varphi^p) + t(\psi^p) = (1-t)\frac{1}{p}((\varphi^p) + \frac{t}{1-t}(\psi^p))^\frac{1}{p}.$$ 

It can be checked by (2.6) that

$$\phi_t^p(x) = (1-t)\frac{1}{p}(((\varphi^p) + \frac{t}{1-t}(\psi^p))^\frac{1}{p} + (\frac{x}{(1-t)^{1/p}}).$$

By letting $x = (1-t)^\frac{1}{p} y$ and $s = \frac{t}{1-t}$, one gets $1-t = \frac{1}{1+s}$ and

$$J((1-t) \cdot_p f \oplus_p t \cdot_p g) = \int_{\mathbb{R}^n} (1-t)\frac{n}{p} e^{-(1-t)\frac{1}{p}}(((\varphi^p) + \frac{t}{1-t}(\psi^p))^\frac{1}{p} + (\frac{y}{(1-t)^{1/p}}) dy

= \int_{\mathbb{R}^n} (1+s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}}(((\varphi^p) + s(\psi^p))^\frac{1}{p} + (\frac{y}{(1+s)^{1/p}}) dy

= \int_{\mathbb{R}^n} (1+s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \psi_s(y) + (\frac{y}{(1+s)^{1/p}}) dy,

where $\psi_s$ is given by (4.4). Like in the proof of Lemma 4.5, let $\tilde{\varphi}(x) = \lim_{s \to 0^+} \psi_s(x)$ for every $x \in \mathbb{R}^n$. By Lemma 4.4 and the monotone convergence theorem, one has $\varphi \geq \tilde{\varphi}$ and $J(e^{-\tilde{\varphi}}) = \lim_{s \to 0^+} J(e^{-\psi_s}) \geq J(e^{-\varphi})$. Lemma 4.4 also implies that $(1+s)^{-\frac{1}{p}} \psi_s(y)$ is decreasing for all $y \in \mathbb{R}^n$ with $\lim_{s \to 0^+} (1+s)^{-\frac{1}{p}} \psi_s(y) = \tilde{\varphi}(y) \leq \varphi(y)$. It follows from the monotone convergence theorem that

$$\lim_{s \to 0^+} \int_{\mathbb{R}^n} (1+s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \psi_s(y) + (\frac{y}{(1+s)^{1/p}}) dy = \lim_{s \to 0^+} \int_{\mathbb{R}^n} e^{-(1+s)^{-\frac{1}{p}} \psi_s(y) + (\frac{y}{(1+s)^{1/p}}) dy = J(e^{-\tilde{\varphi}}).$$

In summary, due to $s = \frac{t}{1-t}$, one has

$$\lim_{t \to 0^+} \frac{J((1-t) \cdot_p f \oplus_p t \cdot_p g) - J(f)}{t} = \lim_{s \to 0^+} \frac{1+s}{s} \int_{\mathbb{R}^n} (1+s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \psi_s(y) + (\frac{y}{(1+s)^{1/p}}) dy

= \lim_{s \to 0^+} \int_{\mathbb{R}^n} (1+s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \psi_s(y) + (\frac{y}{(1+s)^{1/p}}) dy + \lim_{s \to 0^+} \frac{J(e^{-\tilde{\varphi}}) - J(e^{-\varphi})}{s}.$$

Clearly, the second limit in (4.10) equals to $+\infty$ if $J(e^{-\tilde{\varphi}}) > J(e^{-\varphi})$. Note that, in this case, $\delta J_p(f, g) = +\infty$ as proved in Lemma 4.5, and this proves (4.9) if $J(e^{-\tilde{\varphi}}) > J(e^{-\varphi})$.

Now let us consider the case $J(e^{-\tilde{\varphi}}) = J(e^{-\varphi})$. As $\varphi \geq \tilde{\varphi}$, one has $e^{-\varphi(x)} = e^{-\tilde{\varphi}(x)}$ (and hence $\varphi(x) = \tilde{\varphi}(x)$) for almost all $x \in \mathbb{R}^n$. According to (4.1) and (4.3), one sees

$$\delta J_p(f, f) = \delta J_p(e^{-\tilde{\varphi}}, e^{-\tilde{\varphi}}).$$

Besides, (4.6) implies that

$$\delta J_p(f, g) = \lim_{t \to 0^+} \frac{J(e^{-\varphi_t}) - J(e^{-\varphi})}{t} = \lim_{t \to 0^+} \frac{J(e^{-\varphi_t}) - J(e^{-\tilde{\varphi}})}{t}.$$

(4.12)
Replacing \( \varphi \) by \( \tilde{\varphi} \) in (4.10), one has

\[
\lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{(1 + s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)}}{s} dy = \lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{(1 + s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)}}{s} dy
\]

\[
= B_1 + B_2 + B_3.
\]

Here \( B_1 \) is defined and calculated as below:

\[
B_1 = \lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{(1 + s)^{-\frac{n}{p}} e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)}}{s} dy
\]

\[
= \left( \lim_{s \to 0^+} \frac{(1 + s)^{-\frac{n}{p}}}{s} - 1 \right) \left( \lim_{s \to 0^+} \int_{\mathbb{R}^n} e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} dy \right) = -\frac{n}{p} J(\tilde{\varphi}),
\]

where Lemma 4.4 and the monotone convergence theorem are used. It follows from (4.12) that

\[
B_2 = \lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-\varphi_s(y)} - e^{-\tilde{\varphi}(y)}}{s} dy = \delta J_p(f, g).
\]

The term \( B_3 \) is defined and calculated as follows:

\[
B_3 = \lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-\varphi_s(y)}}{s} dy = \frac{1}{p} \int_{\mathbb{R}^n} \tilde{\varphi}(y) e^{-\tilde{\varphi}(y)} dy.
\]

Indeed, (4.14) is a consequence of the dominated convergence theorem and we now provide some details to complete the argument. Clearly, the integral of the second term in (4.14) is actually over the domain of \( \varphi_s \), since \( (1 + s)^{-\frac{1}{p}} \varphi_s(y) = +\infty \) and \( \varphi_s(y) = +\infty \) if \( y \notin \text{dom}(\varphi_s) \). Note that \( 0 \leq \varphi_s(y) < \infty \) for any \( y \in \text{dom}(\varphi_s) \). Moreover, \( 1 - (1 + s)^{-\frac{1}{p}} \) is increasing on \( s \in (0, 1) \) and

\[
\lim_{s \to 0^+} \frac{1 - (1 + s)^{-\frac{1}{p}}}{s} = \frac{1}{p},
\]

which implies the existence of a finite constant \( M < \infty \) such that for all \( s \in (0, 1) \),

\[
0 < \frac{1 - (1 + s)^{-\frac{1}{p}}}{s} < M(1 - 2^{-\frac{1}{p}}).
\]

As the function \( \frac{e^x - 1}{x} \) is increasing on \( x \in (0, \infty) \), one gets, for all \( s \in (0, 1) \) and for all \( y \in \text{dom}(\varphi_s) \),

\[
\frac{e^{(1-(1+s)^{-\frac{1}{p}})\varphi_s(y)} - 1}{(1 - (1 + s)^{-\frac{1}{p}})\varphi_s(y)} \leq \frac{e^{(1-2^{-\frac{1}{p}})\varphi_s(y)} - 1}{(1 - 2^{-\frac{1}{p}})\varphi_s(y)}.
\]
By Lemma 4.4, for all \( s \in (0, 1) \) and for any \( y \in \text{dom}(\varphi_s) \),
\[
\frac{e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-\varphi_s(y)}}{s} = \left( \frac{e^{-\varphi_s(y)}(1 - (1+s)^{-\frac{1}{p}})\varphi_s(y)}{s} \right) \left( \frac{e^{(1-(1+s)^{-\frac{1}{p}})\varphi_s(y)} - 1}{(1 - (1+s)^{-\frac{1}{p}})\varphi_s(y)} \right)
\leq Me^{-\varphi_s(y)} \left( e^{(1-2^{-\frac{1}{p}})\varphi_s(y)} - 1 \right)
\leq Me^{-2^{-\frac{1}{p}}\varphi_s(y)} \leq Me^{-2^{-\frac{1}{p}}\varphi_1(y)}.
\]

It is easily checked by Proposition 3.2 that \( e^{-2^{-\frac{1}{p}}\varphi_1} \in \mathcal{A}_0 \), and hence \( \int_{\mathbb{R}^n} e^{-2^{-\frac{1}{p}}\varphi_1(y)} \, dy < +\infty \).

For convenience, let \( \tilde{\varphi}(y)e^{-\tilde{\varphi}(y)} = 0 \) if \( y \notin \text{dom}(\tilde{\varphi}) \). By Lemma 4.4, \( \text{dom}(\tilde{\varphi}) \subseteq \text{dom}(\varphi_s) \subseteq \text{dom}(\varphi_1) \) holds for \( t, s \in (0, 1) \) such that \( s < t \). Thus, if there exists \( s_0 > 0 \) such that \( y \notin \text{dom}(\varphi_{s_0}) \), then \( y \notin \text{dom}(\varphi_s) \) for all \( s \in (0, s_0] \) and \( y \notin \text{dom}(\tilde{\varphi}) \); this in turn implies that
\[
\lim_{s \to 0^+} \frac{e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-\varphi_s(y)}}{s} = 0 = \frac{1}{p} \tilde{\varphi}(y)e^{-\tilde{\varphi}(y)}.
\]

On the other hand, if \( y \in \text{dom}(\varphi_s) \) for any \( s \in (0, 1) \), then
\[
\lim_{s \to 0^+} \frac{e^{-(1+s)^{-\frac{1}{p}} \varphi_s(y)} - e^{-\varphi_s(y)}}{s} = \left( \lim_{s \to 0^+} e^{-\varphi_s(y)} \right) \left( \lim_{s \to 0^+} \frac{e^{(1-(1+s)^{-\frac{1}{p}})\varphi_s(y)} - 1}{s} \right)
= \frac{1}{p} \tilde{\varphi}(y)e^{-\tilde{\varphi}(y)}.
\]

Hence, the dominated convergence theorem can be applied to \( B_3 \) and get (4.14).

Summing up \( B_1, B_2 \) and \( B_3 \), by (4.10), (4.11), (4.13) and Lemma 4.3, one gets
\[
\lim_{t \to 0^+} \frac{J((1-t) \cdot p \ f \oplus_p t \cdot p \ g) - J(f)}{t} = B_1 + B_2 + B_3 = \delta J_p(f, g) - \delta J_p(f, f).
\]

Hence, formula (4.9) is obtained. This completes the proof of Lemma 4.6. \( \square \)

We are now ready to prove our Theorem 4.2.

**Proof of Theorem 4.2** Let \( f = e^{-\varphi} \in \mathcal{A}_0 \) and \( g = e^{-\psi} \in \mathcal{A}_0 \) such that \( J(f) > 0 \) and \( J(g) > 0 \). For \( p > 1 \), let \( F(t) = \log J((1-t) \cdot p \ f \oplus_p t \cdot p \ g) - J(f) \) for \( t \in (0, 1) \). Let
\[
F(0^+) = \lim_{t \to 0^+} F(t) \text{ and } F(1^-) = \lim_{t \to 1^-} F(t).
\]

Note that \( F(0^+) \geq \log J(f) \) and \( F(1^-) \geq \log J(g) \). In fact, by \( \varphi^* \geq 0 \) and \( \psi^* \geq 0 \), one gets \((1-t) (\varphi^*)^p + t (\psi^*)^p \) \((1-t)^{1/p} \varphi^* \) for all \( t \in (0, 1) \). It follows from \( \varphi^{**} = \varphi, (2.5) \) and (2.6) that, for all \( x \in \mathbb{R}^n \),
\[
\left[ ((1-t) (\varphi^*)^p + t (\psi^*)^p)^{1/p} \right]^*(x) \leq (1-t)^{1/p} \varphi(1-t)^{-1/p} x.
\]

This further infers that, for all \( t \in (0, 1) \),
\[
F(t) \geq \log \left( (1-t)^{n/p} \int_{\mathbb{R}^n} e^{-(1-t)^{1/p} \varphi(x)} \, dx \right).
\]
Then, by the dominated convergence theorem, one gets $F(0+) \geq \log J(f)$. A similar argument implies $F(1-) \geq \log J(g)$.

According to Corollary 3.4, $F$ is a concave function on $t \in (0, 1)$ and thus

$$F(t) \geq (1 - t)F(0+) + tF(1- \geq \log J(f) + t(\log J(g) - \log J(f))$$

holds for $t \in (0, 1)$. Consequently, if $F(0+) > \log J(f)$, then

$$\lim_{t \to 0^+} \frac{J((1 - t) \cdot_p f \oplus_p t \cdot_p g) - J(f)}{t} = +\infty > J(f) \log \left(\frac{J(g)}{J(f)}\right).$$

While if $F(0+) = \log J(f)$, then

$$\lim_{t \to 0^+} \frac{J((1 - t) \cdot_p f \oplus_p t \cdot_p g) - J(f)}{t} = \frac{dF(t)}{dt} \bigg|_{t=0^+} = e^{F(t)} \frac{dF(t)}{dt} \bigg|_{t=0^+} \geq J(f) \log \left(\frac{J(g)}{J(f)}\right).$$

(4.15)

Lemmas 4.3 and 4.6 then yield the desired inequality (4.2) as follows:

$$\delta J_p(f, g) = \delta J_p(f, f) + \lim_{t \to 0^+} \frac{J((1 - t) \cdot_p f \oplus_p t \cdot_p g) - J(f)}{t} \geq \delta J_p(f, f) + J(f) \log \left(\frac{J(g)}{J(f)}\right)$$

$$= J(f) \left[\frac{n}{p} + \frac{1 - p}{p} \log J(f) + \log J(g)\right] + \frac{1}{p} \text{Ent}(f). \tag{4.16}$$

Now let us characterize the equality for (4.2). It is obvious that (4.2) becomes equality if $f = g$ by Lemma 4.3. Conversely, assume that (4.2) holds with equality sign which happens only in the case $F(0+) = \log J(f)$; it requires equality in (4.15). In particular, as $F(0+) = \log J(f)$, one has

$$F'(0+) = \frac{dF(t)}{dt} \bigg|_{t=0^+} = \log \left(\frac{J(g)}{J(f)}\right).$$

Note that $F(t) \leq F(0+) + tF'(0+)$ for all $t \in (0, 1)$ since $F$ is concave on $(0, 1)$. This gives

$$F(t) \leq \log J(f) + t(\log J(g) - \log J(f)),$$

Consequently, equality holds in the Prékopa-Leindler type inequality (3.9) and then $f = g$.

The following corollary provides a unique determination of log-concave functions.

**Corollary 4.7** Let $f_1, f_2 \in \mathcal{A}_0$ such that $J(f_1) = J(f_2) > 0$. For $p > 1$, if

$$\delta J_p(f_1, g) = \delta J_p(f_2, g) \tag{4.17}$$

holds for any $g \in \mathcal{A}_0$ with $J(g) > 0$, then $f_1 = f_2$.

**Proof** By letting $g = f_1$ in (4.17), it follows from Theorem 4.2 [or (4.16)] and $J(f_1) = J(f_2)$ that

$$\delta J_p(f_1, f_1) = \delta J_p(f_2, f_1) \geq \delta J_p(f_2, f_2) + J(f_2) \log \left(\frac{J(f_1)}{J(f_2)}\right) = \delta J_p(f_2, f_2) \tag{4.18}$$

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with equality if and only if \( f_1 = f_2 \). Similarly,

\[
\delta J_p(f_2, f_2) = \delta J_p(f_1, f_2) \geq \delta J_p(f_1, f_1) + J(f_1) \log \left( \frac{J(f_2)}{J(f_1)} \right) = \delta J_p(f_1, f_1).
\] (4.19)

This means that (4.18) holds with equality, which in turn gives \( f_1 = f_2 \) as desired.

\[\Box\]

5 An explicit formula for \( \delta J_p(f, g) \)

Our goal in this section is to obtain an explicit integral formula for \( \delta J_p(f, g) \) for \( p > 1 \) under additional conditions on \( f \) and \( g \). Again the case \( p = 1 \) has been discussed in [29] and hence will not be covered in this section. We shall need the subclass \( \mathcal{L}'_0 \subset \mathcal{L}_0 \) where

\[
\mathcal{L}'_0 := \left\{ \varphi \in \mathcal{L}_0 : \varphi \in C^1(\mathbb{R}^n) \cap C^2_+(\mathbb{R}^n \setminus \{0\}) \text{ is strictly convex and supercoercive} \right\}.
\]

Hereafter, a function \( \varphi \) is called supercoercive if \( \lim_{|x| \to \infty} \frac{\varphi(x)}{|x|^2} = +\infty \). It is easily checked that \( (\mathbb{R}^n, \varphi) \) for \( \varphi \in \mathcal{L}'_0 \) is a pair satisfying that \( \varphi \) is differentiable and strictly convex on \( \mathbb{R}^n \), and

\[
\lim_{i \to \infty} |\nabla \varphi(x_i)| \to +\infty \text{ for each sequence } \{x_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n \text{ such that } \lim_{i \to \infty} |x_i| = +\infty.
\] (5.1)

This pair is usually called a convex function of Legendre type (see e.g. [58, Section 26] for more general definitions and properties). In general, (5.1) holds automatically for \( \varphi \) if \( \text{dom}(\varphi^*) = \mathbb{R}^n \) and \( \varphi \) is a differentiable convex function with \( \text{dom}(\varphi) = \mathbb{R}^n \), due to [58, Lemma 26.7].

We say that \( (D, \psi) \) is the Legendre conjugate of \( (C, \varphi) \) if

\[
\psi(y) = \langle x, y \rangle - \varphi(x), \quad \text{for any } y \in D \text{ and for any } x \in \nabla \varphi^{-1}(y) = \{ z \in C : y = \nabla \varphi(z) \},
\]

where \( D = \{ y \in \mathbb{R}^n : y = \nabla \varphi(x), \ x \in C \} \). Theorem 26.5 in [58] provides a nice result regarding the relation between the Legendre conjugate and Fenchel conjugate. We shall not need the full statement of [58, Theorem 26.5], and only the special cases, when both domains are \( \mathbb{R}^n \), will be stated in the following lemma.

Lemma 5.1 Let \( \phi \in C^1(\mathbb{R}^n) \) be such that \( \text{dom}(\phi^*) = \mathbb{R}^n \). Then \( (\mathbb{R}^n, \phi) \) is a convex function of Legendre type if and only if \( (\mathbb{R}^n, \phi^*) \) is. When these conditions hold, \( (\mathbb{R}^n, \phi^*) \) is the Legendre conjugate of \( (\mathbb{R}^n, \phi) \) (and vice versa). Moreover, both \( \nabla \phi : \mathbb{R}^n \to \mathbb{R}^n \) and \( \nabla \phi^* : \mathbb{R}^n \to \mathbb{R}^n \) are continuous bijections and satisfy that \( \nabla \phi^* \) is the inverse of \( \nabla \phi \) (namely, \( \nabla^{-1} \phi = \nabla \phi^* \)).

We now prove some lemmas before we state our main result in this section.

Lemma 5.2 If \( \varphi \in \mathcal{L}'_0 \), then \( \varphi^* \in \mathcal{L}'_0 \). Moreover, \( \nabla \varphi(o) = o, \ \nabla \varphi^*(o) = o \), and

\[
\{ x \in \mathbb{R}^n : \varphi(x) = 0 \} = \{ x \in \mathbb{R}^n : \varphi^*(x) = 0 \} = \{ o \}.
\]
Proof It is clear that \( \varphi^{**} = \varphi \), as \( \varphi \) is convex and \( \varphi \in C^1(\mathbb{R}^n) \). According to the Moreau-Rockafellar theorem (see e.g. [19, Proposition 3.5.4]), a proper lower semi-continuous convex function \( \varphi \) on \( \mathbb{R}^n \) is supercoercive if and only if \( \text{dom}(\varphi^*) = \mathbb{R}^n \). Consequently, \( \text{dom}(\varphi^*) = \mathbb{R}^n \) and \( \varphi^* \) is supercoercive, due to the facts that \( \varphi \) is supercoercive, and respectively \( \text{dom}(\varphi) = \mathbb{R}^n \). It is also trivial to have \( \varphi^*(o) = 0 \) and \( \varphi^*(y) \geq 0 \) for all \( y \in \mathbb{R}^n \), as \( \varphi \in \mathcal{L}_o \).

Note that the pair \((\mathbb{R}^n, \varphi)\) is a convex function of Legendre type, so is the pair \((\mathbb{R}^n, \varphi^*)\) by Lemma 5.1. In particular, \( \varphi^* \) is strictly convex on \( \mathbb{R}^n \). Lemma 5.1 also implies that \( \nabla \varphi \) and its inverse \( \nabla \varphi^* \) are both continuous on \( \mathbb{R}^n \) and thus \( \varphi^* \in C^1(\mathbb{R}^n) \). As \( \varphi \in C^2_+((\mathbb{R}^n \setminus \{o\}), \) the Hessian matrix \( \nabla^2 \varphi \) is positive definite and continuous on \( \mathbb{R}^n \setminus \{o\} \).

It follows from the inverse mapping theorem that \( \varphi^* \in C^2_+((\mathbb{R}^n \setminus \{o\}) \), which concludes \( \varphi^* \in \mathcal{L}_o' \). In particular, \( \varphi(x) + \varphi^*(y) = \langle x, y \rangle \) holds for all \( x, y \in \mathbb{R}^n \) such that \( y = \nabla \varphi(x) \) (and \( x = \nabla \varphi^*(y) \)). The strict convexity of \( \varphi \) implies that \( \varphi \) has a unique minimizer. As \( \varphi \in \mathcal{L}_o \), \( \varphi \) attains its minimum at \( o \), hence \( \nabla \varphi(o) = 0 \) and \( \{x \in \mathbb{R}^n : \varphi(x) = 0\} = \{o\} \). The same arguments clearly work for \( \varphi^* \), and this concludes the proof of Lemma 5.2. \( \square \)

Lemma 5.3 Let \( \varphi, \psi \in \mathcal{L}_o' \). For \( p > 1 \) and \( t > 0 \), set \( \varphi_t := \varphi \cdot_p (\psi \cdot_p t) \). Then \( \varphi_t \in \mathcal{L}_o' \).

Moreover, both \((\mathbb{R}^n, \varphi_t)\) and \((\mathbb{R}^n, \varphi^*_t)\) are convex functions of Legendre type.

Proof Let \( \varphi, \psi \in \mathcal{L}_o' \subset \mathcal{L}_o \) be two convex functions. According to Proposition 3.2, \( \varphi_t \in \mathcal{L}_o \) for all \( t > 0 \). In particular, \( \varphi_t \) is non-negative on \( \mathbb{R}^n \). By Lemma 4.4, \( 0 \leq \varphi_t \leq \varphi \) for all \( t > 0 \) and \( p > 1 \). This implies \( \text{dom}(\varphi_t) = \mathbb{R}^n \). This, together with the Moreau-Rockafellar theorem (see e.g. [19, Proposition 3.5.4]), immediately implies that \( \varphi_t^* \) is supercoercive. On the other hand, by Proposition 3.2, for any \( t > 0 \), \( \text{dom}(\varphi_t^*) \) is clearly equal to \( \mathbb{R}^n \) and thus \( \varphi_t^* \) is supercoercive.

Let us check the differentiability of \( \varphi_t \) and \( \varphi_t^* \). As explained in the proof of Lemma 5.2, if \( \varphi \in \mathcal{L}_o' \), then \( \nabla \varphi \) and its inverse \( \nabla \varphi^* \) are both continuous on \( \mathbb{R}^n \) and continuously differentiable on \( \mathbb{R}^n \setminus \{o\} \). Moreover, \( \nabla^2 \varphi \) and \( \nabla^2 \varphi^* \) are positive definite and continuous on \( \mathbb{R}^n \setminus \{o\} \). Similar properties hold for \( \psi \). Let \( x \neq o \). In this case, both \( \varphi^*(x) \) and \( \psi^*(x) \) are strictly positive due to Lemma 5.2. It follows from (4.5) that, for all \( t > 0 \) and \( p > 1 \), \( \varphi_t^*(x) > 0 \) and

\[
\nabla \varphi_t^*(x) = \frac{(\varphi^*(x))^{p-1} \nabla \varphi^*(x) + t(\varphi^*(x))^{p-1} \nabla \psi^*(x)}{(\varphi_t^*(x))^{p-1}}.
\]

(5.2)

Clearly \( \nabla \varphi_t^*(x) \) is continuous at \( o \neq x \in \mathbb{R}^n \). On the other hand, one can verify \( \nabla \varphi_t^*(o) = o \) according to the usual definition of differentiability:

\[
\lim_{z \to o} \frac{\varphi_t^*(z) - \varphi_t^*(o) - \langle o, z - o \rangle}{|z - o|} = \lim_{z \to o} \frac{((\varphi^*(z))^{p} + t(\varphi^*(z))^{p})^{\frac{1}{p}}}{|z|} = \lim_{z \to o} \left( \frac{((\varphi^*(z))^{p} + t(\varphi^*(z))^{p})^{\frac{1}{p}}}{|z|} \right) = 0,
\]

where the last equality follows from \( \nabla \varphi^*(o) = \nabla \psi^*(o) = o \) due to Lemma 5.2. Moreover,

\[
\lim_{x \to o} \nabla \varphi^*(x) = \nabla \varphi^*(o) = o \quad \text{and} \quad \lim_{x \to o} \nabla \psi^*(x) = \nabla \psi^*(o) = o.
\]
These conclude that, for $p > 1$, $\varphi_t^*$ is continuously differentiable on $\mathbb{R}^n$, because

$$
\lim_{x \to o} \left| \nabla \varphi_t^*(x) \right| = \lim_{x \to o} \left| \frac{(\varphi^*(x))^{p-1} \nabla \varphi^*(x) + t (\psi^*(x))^{p-1} \nabla \psi^*(x)}{(\varphi_t^*(x))^{p-1}} \right|
\leq \lim_{x \to o} \left| \nabla \varphi^*(x) \right| \left( \frac{(\varphi^*(x))^{p-1} + t \frac{1}{p} \lim_{x \to o} \left| \nabla \psi^*(x) \right|}{(\varphi_t^*(x))^{p-1}} \right)^{1-\frac{1}{p}}
\leq \lim_{x \to o} \left| \nabla \varphi^*(x) \right| + t \frac{1}{p} \lim_{x \to o} \left| \nabla \psi^*(x) \right| = 0.
$$

Now let us check that $(\mathbb{R}^n, \varphi_t^*)$ is a convex function of Legendre type. To this end, as $\varphi_t^*$ is already proved to be differentiable, we only need to verify that $\varphi_t^*$ is strictly convex on $\mathbb{R}^n$ and (5.1) holds for $\varphi_t^*$. As mentioned before, (5.1) for $\varphi_t^*$ holds automatically because dom($\varphi$) = $\mathbb{R}^n$ and $\varphi_t^*$ is differentiable, due to [58, Lemma 26.7]. The strictly convexity of $\varphi_t^*$ is easily checked as follows: for $p > 1$, $\lambda \in (0, 1)$ and $x, y \in \mathbb{R}^n$ such that $x \neq y$, by the Minkowski inequality for norms, (4.5) and Lemma 5.2 (which implies the strict convexity of $\varphi^*$ and $\psi^*$), one has

$$
\varphi_t^*(\lambda x + (1-\lambda)y) = \left( \left(\varphi^*(\lambda x + (1-\lambda)y)\right)^p + t (\psi^*(\lambda x + (1-\lambda)y)) \right)^{\frac{1}{p}}
\leq \left( \left(\lambda \varphi^*(x) + (1-\lambda)\varphi^*(y)\right)^p + t (\lambda \psi^*(x) + (1-\lambda)\psi^*(y)) \right)^{\frac{1}{p}}
\leq \lambda \varphi_t^*(x) + (1-\lambda)\varphi_t^*(y).
$$

Therefore, $(\mathbb{R}^n, \varphi_t^*)$ is a convex function of Legendre type, and so is $(\mathbb{R}^n, \varphi_t)$ due to Lemma 5.1. Moreover, both $\nabla \varphi_t^*$ and its inverse $\nabla \varphi_t$ are continuous. Thus $\varphi_t \in C^1(\mathbb{R}^n)$ and is strictly convex.

According to Lemma 5.2, $\nabla^2 \varphi^*$ and $\nabla^2 \psi^*$ are both positive definite and continuous on $\mathbb{R}^n \setminus \{o\}$. Let $x \in \mathbb{R}^n \setminus \{o\}$ and by (5.2), one has

$$
\nabla^2 \varphi_t^* = (p - 1) \frac{(\varphi^*)^{p-2} \nabla \varphi^* \otimes \nabla \varphi^* + t (\psi^*)^{p-2} \nabla \psi^* \otimes \nabla \psi^* - (\varphi_t^*)^{p-2} \nabla \varphi_t^* \otimes \nabla \varphi_t^*}{(\varphi_t^*)^{p-1}}
\quad + \frac{(\varphi^*)^{p-2} \nabla^2 \varphi^* + t (\psi^*)^{p-1} \nabla^2 \psi^*}{(\varphi_t^*)^{p-1}},
$$

(5.3)

where $y \otimes y$ is the rank 1 matrix generated by $y \in \mathbb{R}^n$. Clearly $\nabla^2 \varphi_t^*$ is continuous on $\mathbb{R}^n \setminus \{o\}$ and positive definite with determinant being strictly positive (the calculation is standard and hence will be omitted). So $\varphi_t^* \in C^2(\mathbb{R}^n \setminus \{o\})$ for all $t > 0$ and $p > 1$. Together with $\nabla \varphi_t^{-1} = \nabla \varphi_t^*$, the inverse mapping theorem gives that $\varphi_t \in C^2(\mathbb{R}^n \setminus \{o\})$ for all $t > 0$ and $p > 1$. Moreover, for any $o \neq x \in \mathbb{R}^n$, $\nabla^2 \varphi_t(x)$ equal the inverse of $\nabla^2 \varphi_t^*(y)$ with $y = \nabla \varphi_t(x)$. This concludes that $\varphi_t \in \mathcal{L}_t^\prime$. \qed

We will also need the following lemma. The function $\varphi_t$ is as in (4.4).

**Lemma 5.4** Let $\varphi, \psi \in \mathcal{L}_t^\prime$. For $p > 1$ and $t > 0$, one has,

i) $\lim_{t \to 0^+} \varphi_t(x) = \varphi(x)$ for all $x \in \mathbb{R}^n$;

ii) for every closed bounded subset $E \subset \mathbb{R}^n$, $\lim_{t \to 0^+} \nabla \varphi_t(x) = \nabla \varphi(x)$ uniformly on $E$.

**Proof** (i) Let $\varphi, \psi \in \mathcal{L}_t^\prime$. By Lemma 4.4, for any $x \in \mathbb{R}^n$, $\varphi_t(x)$ is decreasing on $t \in (0, 1]$, and thus $\lim \sup_{t \to 0^+} \varphi_t(x) \leq \varphi(x)$. The desired argument in i) follows immediately once
the following is checked: for any \( x \in \mathbb{R}^n \),
\[
\liminf_{t \to 0^+} \varphi_t(x) \geq \varphi(x). \tag{5.4}
\]

To this end, let \( x \in \mathbb{R}^n \) be fixed and \( r > |\nabla \varphi(x)| \). Let \( B_r \) be the Euclidean ball with center at the origin and radius \( r \). For \( p > 1 \), it can be checked, by \( \varphi^*, \psi^* \geq 0 \), that for all \( t > 0 \),
\[
\varphi_t^* = \left( (\varphi^*)_p + t(\psi^*)_p \right) \frac{1}{p} \leq \varphi^* + t \frac{1}{p} \psi^*. \tag{5.5}
\]

It follows from (5.5) that, for \( t \in (0, 1] \),
\[
\varphi_t(x) = \sup_{y \in \mathbb{R}^n} \left\{ \langle x, y \rangle - \varphi_t^*(y) \right\} \geq \sup_{y \in B_r} \left\{ \langle x, y \rangle - \varphi_t^*(y) \right\}
\geq \sup_{y \in B_r} \left\{ \langle x, y \rangle - \varphi^*(y) - t \frac{1}{p} \psi^*(y) \right\}.
\]

Define the finite constant \( c \) to be \( c = \max\{\psi^*(y) : y \in B_r\} \). The fact that \( r > |\nabla \varphi(x)| \) implies \( \nabla \varphi(x) \in B_r \), and hence, for \( t \in (0, 1] \),
\[
\varphi_t(x) \geq \sup_{y \in B_r} \left\{ \langle x, y \rangle - \varphi^*(y) - t \frac{1}{p} \psi^*(y) \right\}
\geq \langle x, \nabla \varphi(x) \rangle - \varphi^*(\nabla \varphi(x)) - t \frac{1}{p} c = \varphi(x) - t \frac{1}{p} c.
\]

The desired inequality (5.4) follows by letting \( t \to 0^+ \). This completes the proof for part i). ii) This is a direct consequence of [58, Theorem 25.7]; in a slight different form, it reads: if \( \{\varphi_t\}_{t \in \mathbb{N} \cup \{0\}} \) is a sequence of finite and differentiable convex functions on an open convex set \( E \) such that \( \varphi_t \to \varphi_0 \) pointwisely on \( E \), then \( \nabla \varphi_t \to \nabla \varphi \) pointwisely on \( E \) and uniformly on every closed bounded subset of \( E \).

The following lemma provides the derivative of \( \varphi_t \) with respect to \( t \).

**Lemma 5.5** Let \( \varphi, \psi \in \mathcal{L}_0' \). For \( p > 1 \) and for \( t > 0 \), set \( \varphi_t = \varphi \Box_p (\psi \cdot t \varphi_t) \). Then,
\[
\frac{d}{dt} \varphi_t(x) = \begin{cases} 
-\frac{1}{p} \left( \psi^*(\nabla \varphi_t(x)) \right)^p \left( \varphi_t^*(\nabla \varphi_t(x)) \right)^{1-p}, & \text{if } o \neq x \in \mathbb{R}^n; \\
0, & \text{if } x = o.
\end{cases} \tag{5.6}
\]

In particular, one has \( \frac{d}{dt} \varphi_t(o) \big|_{t=0^+} = 0 \), and for all \( o \neq x \in \mathbb{R}^n \),
\[
\frac{d}{dt} \varphi_t(x) \big|_{t=0^+} = -\frac{1}{p} \left( \psi^*(\nabla \varphi(x)) \right)^p \left( \varphi^*(\nabla \varphi(x)) \right)^{1-p}. \tag{5.7}
\]

**Proof** Let \( \varphi, \psi \in \mathcal{L}_0' \) and \( p > 1 \). Formula (5.6) for \( x = o \) and \( \frac{d}{dt} \varphi_t(o) \big|_{t=0^+} = 0 \) clearly hold, following from Lemmas 5.2 and 5.3 (the latter one gives \( \varphi_t \in \mathcal{L}_0 \) and hence \( \nabla \varphi_t(o) = o \)).

Let \( x \neq o \). By Lemma 5.3 (and its proof), one sees that the mapping \( F \) defined by \( F(x, y, t) = \nabla \varphi_t^*(y) - x \) is continuously differentiable on \((\mathbb{R}^n \setminus \{o\}) \times (\mathbb{R}^n \setminus \{o\}) \times (0, +\infty)\). Note that \( \frac{\partial F}{\partial y} = \nabla^2 \varphi_t^*(y) \) is nonsingular for every \( y \in \mathbb{R}^n \setminus \{o\} \) by (5.3). The implicit function theorem yields (locally) the existence of a unique continuously differentiable mapping \( y = y(x, t) \) for \((x, t) \in (\mathbb{R}^n \setminus \{o\}) \times (0, +\infty)\) such that \( F(x, y(x, t), t) = o \). That is, \( x = \nabla \varphi_t^*(y(x, t)) \). According to Lemma 5.3, \( \nabla \varphi_t = \nabla^{-1} \varphi_t^* \). Thus, \( y(x, t) = \nabla \varphi_t(x) \) and \( x = \nabla \varphi_t(x) \)
\[ \nabla \varphi_t^* (\nabla \varphi_t(x)) \] for \( x \neq o \). Moreover, for \( x = o \), one has \( \varphi_t(x) = \langle x, \nabla \varphi_t(x) \rangle - \varphi_t^* (\nabla \varphi_t(x)) \). Taking derivative from both sides, one gets, for any \( t \in (0, \infty) \) and (fixed) \( x \in \mathbb{R}^n \setminus \{ o \} \),

\[
\frac{d}{dt} \varphi_t(x) = \langle x, \frac{d}{dt} \nabla \varphi_t(x) \rangle - \frac{1}{p} \left( \psi^* (\nabla \varphi_t(x)) \right)^p \left( \varphi_t^* (\nabla \varphi_t(x)) \right)^{1-p} - (\nabla \varphi_t^* (\nabla \varphi_t(x)), \frac{d}{dt} \nabla \varphi_t(x)) = - \frac{1}{p} \left( \psi^* (\nabla \varphi_t(x)) \right)^p \left( \varphi_t^* (\nabla \varphi_t(x)) \right)^{1-p}.
\]

This concludes the proof of (5.6). Consequently, (5.7) follows from part ii) of Lemma 5.4 and by letting \( t \to 0^+ \) in (5.6).

In order to obtain an explicit formula for \( \delta J_p(f, g) \), we need to define the notion of admissible \( p \)-perturbation. See [29] for the case for \( p = 1 \).

**Definition 5.6** Let \( f = e^{-\psi} \in \mathcal{A}_0 \) and \( p > 1 \). The function \( g = e^{-\psi} \in \mathcal{A}_0 \) is said to be an admissible \( p \)-perturbation for \( f \), if there exists a constant \( c > 0 \) such that \( (\varphi^*)^p - c(\psi^*)^p \) is a convex function.

Our main result in this section is the following integral formula for \( \delta J_p(f, g) \). Again, we only focus on \( p > 1 \) and the case \( p = 1 \) has been covered in [29, Theorem 4.5].

**Theorem 5.7** Let \( f = e^{-\psi} \in \mathcal{A}_0' \) and \( g = e^{-\psi} \in \mathcal{A}_0' \). For \( p > 1 \), assume that \( g \) is an admissible \( p \)-perturbation for \( f \). In addition, suppose that there exists a constant \( k > 0 \) such that

\[
\det \left( \nabla^2 (\varphi^*)(y) \right) \leq k(\varphi^*)(y)^{n(p-1)} \det \left( \nabla^2 \varphi^*(y) \right) \tag{5.8}
\]

holds for all \( y \in \mathbb{R}^n \setminus \{ o \} \). Then

\[
\delta J_p(f, g) = \frac{1}{p} \int_{\mathbb{R}^n} (\psi^* (\nabla \varphi(x)))^p (\varphi^* (\nabla \varphi(x)))^{1-p} e^{-\varphi(x)} dx = \frac{1}{p} \int_{\mathbb{R}^n} (\psi^*(y))^p (\varphi^*(y))^{1-p} d\mu(f, y). \tag{5.9}
\]

**Proof** Let \( f = e^{-\psi} \in \mathcal{A}_0' \) and \( g = e^{-\psi} \in \mathcal{A}_0' \). Let \( t > 0 \) be fixed. According to (4.4), Proposition 3.2, and Definition 4.1, one sees that

\[
\delta J_p(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus_p t \cdot g) - J(f)}{t} = \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-\varphi_t(x)} - e^{-\varphi(x)}}{t} dx. \tag{5.10}
\]

By Lemma 5.5 (in particular, (5.7)), it holds that, for \( x \in \mathbb{R}^n \),

\[
\lim_{t \to 0^+} \frac{e^{-\varphi_t(x)} - e^{-\varphi(x)}}{t} = \frac{1}{p} \left( \psi^* (\nabla \varphi(x)) \right)^p \left( \varphi^* (\nabla \varphi(x)) \right)^{1-p} e^{-\varphi(x)}. \tag{5.11}
\]

Consequently, the first formula in (5.9) follows immediately from (5.10) and (5.11) once the assumptions of the dominated convergence theorem are verified. The second formula in (5.9) follows directly from Definition 2.2.

Now let us verify that the dominated convergence theorem can be applied for (5.10). For any fixed \( x \in \mathbb{R}^n \), Lemma 5.4 implies that \( \varphi_t(x) \) is continuous at \( t = 0 \) and Lemma 5.5 yields the differentiability of \( \varphi_t(x) \) on \( t \in (0, \infty) \). Together with (5.6), the Lagrange mean
The function on the right, for any $s \in [0, t]$, is integrable over $\mathbb{R}^n$, namely,
\[
\Psi(s) = \frac{1}{p} \int_{\mathbb{R}^n} \left( \psi^*(\nabla \varphi_s(x)) \right)^p \left( \varphi_s^*(\nabla \varphi_s(x)) \right)^{1-p} e^{-\varphi_s(x)} \, dx < \infty. \tag{5.13}
\]
Indeed, (4.5) yields $s(\psi^*)^p \leq (\varphi_s^*)^p$ for any $s > 0$. Together with (2.10), one has
\[
s \Psi(s) = \frac{1}{p} \int_{\mathbb{R}^n} s \left( \psi^*(\nabla \varphi_s(x)) \right)^p \left( \varphi_s^*(\nabla \varphi_s(x)) \right)^{1-p} e^{-\varphi_s(x)} \, dx \\
\leq \frac{1}{p} \int_{\mathbb{R}^n} \left( \varphi_s^*(\nabla \varphi_s(x)) \right)^p \left( \varphi_s^*(\nabla \varphi_s(x)) \right)^{1-p} e^{-\varphi_s(x)} \, dx \\
= \frac{1}{p} \int_{\mathbb{R}^n} \varphi_s^*(\nabla \varphi_s(x)) e^{-\varphi_s(x)} \, dx < +\infty.
\]
For $s = 0$, the assumption that $g$ is an admissible $p$-perturbation of $f$ is needed. Recall that $\varphi^*(o) = \psi^*(o) = 0$ if $f = e^{-\varphi} \in \mathcal{A}_0$ and $g = e^{-\psi} \in \mathcal{A}_0$. According to Definition 5.6, there exists a constant $c > 0$, such that $(\varphi^*)^p - c(\psi^*)^p$ is a convex function. By Lemma 5.2, one has $(\varphi^*(o))^p - c(\psi^*(o))^p = 0$ and $\nabla((\varphi^*)^p - c(\psi^*)^p)(o) = 0$, which in turn yields that, for any $y \in \mathbb{R}^n$,
\[
((\varphi^*)^p - c(\psi^*)^p)(y) \geq ((\varphi^*)^p - c(\psi^*)^p)(o) + \langle y, \nabla((\varphi^*)^p - c(\psi^*)^p)(o) \rangle = 0. \tag{5.14}
\]
Consequently, by (2.10), the following holds:
\[
c \Psi(0) = \int_{\mathbb{R}^n} c \left( \psi^*(\nabla \varphi(x)) \right)^p \left( \varphi^*(\nabla \varphi(x)) \right)^{1-p} e^{-\varphi(x)} \, dx \\
\leq \int_{\mathbb{R}^n} \varphi^*(\nabla \varphi(x)) e^{-\varphi(x)} \, dx < +\infty.
\]
By (5.10), (5.12) and (5.13), we obtain that, for all $s \in (0, t)$,
\[
\delta J_p(f, g) = \frac{1}{p} \lim_{s \to 0^+} \int_{\mathbb{R}^n} \left( \psi^*(\nabla \varphi_s(x)) \right)^p \left( \varphi_s^*(\nabla \varphi_s(x)) \right)^{1-p} e^{-\varphi_s(x)} \, dx. \tag{5.15}
\]
Let $y = \nabla \varphi_s(x)$. From Lemmas 5.2 and 5.3 (in particular, its proof), (5.15) can be rewritten as
\[
\delta J_p(f, g) = \frac{1}{p} \lim_{s \to 0^+} \int_{\mathbb{R}^n \setminus \{0\}} \left( \psi^*(\nabla \varphi_s(x)) \right)^p \left( \varphi_s^*(\nabla \varphi_s(x)) \right)^{1-p} e^{-\varphi_s(x)} \, dx \\
= \frac{1}{p} \lim_{s \to 0^+} \int_{\mathbb{R}^n \setminus \{0\}} \left( \psi^*(y) \right)^p \left( \varphi_s^*(y) \right)^{1-p} e^{-\varphi_s(y)} \det \left( \nabla^2 \varphi_s^*(y) \right) \, dy. \tag{5.16}
\]
The desired formula (5.9) follows once the dominated convergence theorem is verified for (5.16).
According to Lemma 5.3, one has $\phi_t^* \in C^2_t(\mathbb{R}^n \setminus \{o\})$. For $y \neq o$, (5.3) can be rewritten as
\[
\nabla^2 ((\phi_t^*)^p)(y) = \nabla^2 ((\phi^*)^p)(y) + \lambda \nabla^2 ((\phi^*)^p)(y)
\]
\[
= \left( p(p - 1)(\phi_t^*)^{p-2} \nabla \phi_t^* \otimes \nabla \phi_t^* + p(\phi_t^*)^{p-1} \nabla^2 \phi_t^* \right)(y).
\]  
(5.17)

For a positive definite matrix $A$ and a positive semi-definite matrix $B$, the following holds:
\[
\det(A + B) \geq \det A + \det B,
\]  
(5.18)

where $\det A$ denotes the determinant of $A$. This inequality can be applied to (5.17) to get
\[
p^n(\phi_t^*(y))^{n(p-1)} \det(\nabla^2 \phi_t^*(y)) \leq \det(\nabla^2 (\phi^*)^p(y)) = \det(\nabla^2 (\phi^*)^p(y) + t \nabla^2 (\psi^*)^p(y)).
\]  
(5.19)

As $g = e^{-\psi} \in \mathcal{M}_0'$ is an admissible $p$-perturbation for $f$, there exists a constant $c > 0$ such that $\phi = (\phi^*)^p - c(\psi^*)^p$ is convex. Hence $\nabla^2 (\psi^*)^p = \frac{1}{c} \nabla^2 (\phi^*)^p - \frac{1}{c} \nabla^2 \phi$. It follows from (5.8), (5.18) and (5.19) that, for $y \neq o$ and $p > 1$,
\[
\det(\nabla^2 \phi_t^*(y)) \leq p^{-n}(\phi_t^*(y))^{n(1-p)} \det(\nabla^2 (\phi^*)^p(y) + t \nabla^2 (\psi^*)^p(y))
\]
\[
\leq p^{-n}(\phi_t^*(y))^{n(1-p)} \det(\nabla^2 (\phi^*)^p(y) + \frac{t}{c} \nabla^2 (\psi^*)^p(y))
\]
\[
\leq \left( \frac{t + c}{cp} \right)^n (\phi_t^*(y))^{n(1-p)} \det(\nabla^2 (\phi^*)^p(y))
\]
\[
\leq k \left( \frac{t + c}{cp} \right)^n \det(\nabla^2 \phi^*(y)),
\]

where the constant $k > 0$ is given by (5.8). Let $k_0 = k \left( \frac{1+c}{cp} \right)^n$. Hence, if $t \in (0, 1)$, one gets
\[
\det(\nabla^2 \phi_t^*(y)) \leq k_0 \det(\nabla^2 \phi^*(y)).
\]  
(5.20)

For any $o \neq y \in \mathbb{R}^n$, by (2.7), (5.2) and Lemma 5.2, one gets
\[
\frac{d}{dt} \left( (\phi_t^*(y))^p - \langle y, (\phi_t^*(y))^{p-1} \nabla \phi_t^*(y) \rangle \right) = (\psi^*(y))^p - \langle y, (\psi^*(y))^{p-1} \nabla \psi^*(y) \rangle
\]
\[
= (\psi^*(y))^{p-1} \left( \psi^*(y) - \langle y, \nabla \psi^*(y) \rangle \right)
\]
\[
= -(\psi^*(y))^{p-1} \psi(\nabla \psi^*(y)) \leq 0.
\]

Together with (4.5) and (5.14), one gets that, for $p > 1$, $o \neq y \in \mathbb{R}^n$, and $t \in (0, 1)$,
\[
\phi_t^*(y) - \langle y, \nabla \phi_t^*(y) \rangle \leq (\phi_t^*(y))^{1-p} \left( (\phi^*(y))^p - \langle y, (\phi^*(y))^{p-1} \nabla \phi^*(y) \rangle \right)
\]
\[
= \left( \frac{\phi_t^*(y)}{\phi_t^*(y)} \right)^{p-1} (\phi^*(y) - \langle y, \nabla \phi^*(y) \rangle)
\]
\[
\leq \left( \frac{c}{1 + c} \right)^{p-1} (\phi^*(y) - \langle y, \nabla \phi^*(y) \rangle) \leq 0.
\]  
(5.21)

Formulas (4.5) and (5.14) also yield that, for any $t > 0$,
\[
(\psi^*)^p \leq c^{-1} \phi_t^*(\phi_t^*)^{p-1}.
\]  
(5.22)
Now we are ready to check the interchange of orders of limit and integration in (5.16). Combining (2.7), (5.20) (5.21), (5.22), one has, for \( s \in (0, 1) \) and \( o \neq y \in \mathbb{R}^n \),

\[
h(y) = k_0 c^{-1} \varphi^*(y) \exp \left( \left( \frac{c}{1 + c} \right)^{p-1} \varphi^*(y) - (y, \nabla \varphi^*(y)) \right) \det (\nabla^2 \varphi^*(y))
\geq (\varphi^*(y))^p \left( \varphi_s^*(y) \right)^{1-p} e^{-\varphi_s(\nabla \varphi^*(y))} \det (\nabla^2 \varphi_s^*(y)).
\]

The function \( h \) is integrable by (2.6) and (2.10). Indeed, a calculation similar to (5.16) leads to

\[
\int_{\mathbb{R}^n \setminus \{o\}} h(y) \, dy = k_0 c^{-1} \int_{\mathbb{R}^n \setminus \{o\}} \varphi^*(y) \exp \left( \left( \frac{c}{1 + c} \right)^{p-1} \varphi^*(y) - (y, \nabla \varphi^*(y)) \right) \times \det (\nabla^2 \varphi^*(y)) \, dy
\]

\[
= k_0 c^{-1} \int_{\mathbb{R}^n} \varphi^*(\nabla \varphi(x)) \exp \left( - \left( \frac{c}{1 + c} \right)^{p-1} \varphi(x) \right) \, dx
\]

\[
= k_0 c^{-1} \left( \frac{c}{1 + c} \right)^{1-p} \int_{\mathbb{R}^n} \tilde{\varphi}^*(\nabla \tilde{\varphi}(x)) e^{-\tilde{\varphi}(x)} \, dx \in (-\infty, \infty),
\]

where \( \tilde{\varphi} = \left( \frac{c}{1 + c} \right)^{p-1} \varphi \). Therefore, the dominated convergence theorem can be applied to (5.16) and the desired formula (5.9) holds. \( \square \)

The condition (5.8) is indeed natural and many widely used functions do satisfy this condition, for example, the Gaussian function \( e^{-|x|^2/2} \). The following results give some convenient ways to check condition (5.8).

**Corollary 5.8** Assume that \( \varphi \in C^2_+(\mathbb{R}^n \setminus \{o\}) \). Then (5.8) holds if any one of the following holds:

(i) for some \( \alpha \in [0, 1) \), the function \( \frac{1}{\alpha} (\varphi^*)^\alpha \) (understood as \( \log \varphi^* \) when \( \alpha = 0 \)) is convex;

(ii) there exists a constant \( k_1 \), such that, for any \( y \in \mathbb{R}^n \setminus \{o\} \),

\[
\left( \nabla \varphi^*(y), (\nabla^2 \varphi^*(y))^{-1} \nabla \varphi^*(y) \right) \leq k_1 \varphi^*(y). \tag{5.23}
\]

**Proof** For \( p > 1 \), condition (5.8) is equivalent to, for any \( y \neq o \),

\[
H(y) = \det \left( \nabla^2 \varphi^*(y) + (p-1)(\varphi^*(y))^{-1} \nabla \varphi^*(y) \otimes \nabla \varphi^*(y) \right) \leq kp^{-n} \det (\nabla^2 \varphi^*(y)). \tag{5.24}
\]

\[i\) Let \( \alpha \in [0, 1) \) and the function \( \frac{1}{\alpha} (\varphi^*)^\alpha \) be convex. For any \( o \neq y \in \mathbb{R}^n \),

\[
A(y) = \nabla^2 \varphi^*(y) + (\alpha - 1)(\varphi^*(y))^{-1} \nabla \varphi^*(y) \otimes \nabla \varphi^*(y)
\]

is a positive semi-definite matrix. Therefore, (5.18) yields

\[
H(y) = \det \left( 1 + \frac{p-1}{1-\alpha} \right) \nabla^2 \varphi^*(y) + \frac{p-1}{\alpha-1} A(y) \leq \left( 1 + \frac{p-1}{1-\alpha} \right)^n \det (\nabla^2 \varphi^*(y)).
\]

Hence, (5.24) and condition (5.8) hold true.
ii) It can be calculated that $I_n + z \otimes z$ for any $z \in \mathbb{R}^n$ has its determinant to be $1 + |z|^2$, where $I_n$ denotes the identity matrix of order $n$. Hence, for any $o \neq y \in \mathbb{R}^n$, (5.23) yields

$$H(y) = \det \left( \nabla^2 \varphi^*(y) \right) \left( 1 + (p - 1)(\varphi^*(y))^{-1} \left( \nabla \varphi^*(y), (\nabla^2 \varphi^*(y))^{-1} \nabla \varphi^*(y) \right) \right)$$

$$\leq \det \left( \nabla^2 \varphi^*(y) \right) \left( 1 + k_1(p - 1) \right).$$

This implies (5.24) and hence condition (5.8) holds true. □

6 The $L_p$ Minkowski problem for log-concave functions

This section aims to investigate the $L_p$ Minkowski problem for log-concave functions. Actually Theorem 5.7 suggests a new measure for log-concave functions. Let $\mathcal{L} = \{ \varphi \in \mathcal{L} : \varphi \geq 0 \}$, $\Omega_{\varphi^*} = \{ y \in \mathbb{R}^n : 0 < \varphi^*(y) < +\infty \}$ and $\Omega = \{ y \in \mathbb{R}^n : \varphi^*(y) = 0 \}$. The set $\Omega_{\varphi^*}$ is always assumed to be nonempty. The subscript $\varphi^*$ is often omitted if there is no confusion.

The $L_p$ surface area measure of $f$ can be defined as follows.

Definition 6.1 Let $f = e^{-\varphi}$ be a log-concave function with $\varphi \in \mathcal{L}$ such that $\varphi^* \in \mathcal{L}^+$ and $\Omega$ is nonempty. For $p \in \mathbb{R}$, the $L_p$ surface area measure of $f$, denoted by $\mu_p(f, \cdot)$, is the Borel measure on $\Omega$ such that

$$\int_{\Omega} g(y) d\mu_p(f, y) = \int_{\{ x \in \text{dom}(\varphi) : \nabla \varphi(x) \in \Omega \}} g(\nabla \varphi(x))(\varphi^*(\nabla \varphi(x)))^{1-p} e^{-\varphi(x)} d\nu (6.1)$$

holds for every Borel function $g$ such that $g \in L^1(\mu_p(f, \cdot))$ or $g$ is non-negative.

In general, $d\mu_p(f, \cdot) = (\varphi^*)^{1-p} d\mu_1(f, \cdot)$ on $\Omega$. The $L_p$ surface area measure of $f$ for $p = 1$ in Definition 6.1 is the restriction of the surface area measure $f$ given in (2.9) on $\Omega$, i.e., $\mu_1(f, \cdot) = \mu(f, \cdot)|_{\Omega}$. If the Lebesgue measure of $\Omega$ is zero, $\mu_1(f, \cdot)$ can be extended to $\mathbb{R}^n$ (more precisely to the interior of $\text{dom}(\varphi^*)$), and reduces to $\mu(f, \cdot)$. The measure $\mu_1(f, \cdot)$ is always finite for $f \in \mathcal{A}$. As $\varphi^* = 0$ on $\Omega$, one can even extend $\mu_1(f, \cdot)$ for $p < 1$ to $\text{int}(\text{dom}(\varphi^*))$. When $p = 0$, one has the $L_0$ (or the logarithmic) surface area measure of $f$ which is again a finite measure for $f \in \mathcal{A}$ based on (2.10).

The problem to characterize the $L_p$ surface area measure of a log-concave function $f$ can be naturally formulated as follows.

Problem 6.2 (The $L_p$ Minkowski problem for log-concave functions) Let $v$ be a finite nonzero Borel measure on $\mathbb{R}^n$ and $p \in \mathbb{R}$. Find the necessary and/or sufficient conditions on $v$, so that,

$$d\nu = \tau d\mu_p(f, \cdot) \quad \text{or} \quad (\varphi^*)^{p-1} d\nu = \tau d\mu_1(f, \cdot)$$

hold for some log-concave function $f = e^{-\varphi}$ and $\tau \in \mathbb{R}$.

When $p = 1$ and the Lebesgue measure of $\widehat{\Omega}$ is zero, Problem 6.2 for $p = 1$ reduces to the Minkowski problem for moment measures investigated in [30] by Cordero-Erausquin and Klartag. See [29, Section 7] for a full formulation to this problem. In this case, a solution has been provided in [30], including [30, Proposition 1] for necessity and [30, Theorem 2] for sufficiency and uniqueness.
When \( f = e^{-\varphi} \) is smooth enough so that \( \nabla \varphi : \text{int}(\text{dom}(\varphi)) \to \text{int}(\text{dom}(\varphi^*)) \) is smooth and bijective, then formula (2.7) and Definition 6.1 deduce that, by letting \( y = \nabla \varphi(x) \),

\[
\int_{\Omega} g(y) \, d\mu_p(f, y) = \int_{\{x \in \mathbb{R}^n : \nabla \varphi(x) \in \Omega\}} g(\nabla \varphi(x))(\varphi^*(\nabla \varphi(x)))^{1-p} e^{-\varphi(x)} \, dx
\]

\[
= \int_{\Omega} g(y) \varphi^*(y)^{1-p} \det(\nabla^2 \varphi^*(y)) e^{-\varphi^*(y)-\langle y, \nabla \varphi^*(y) \rangle} \, dy,
\]

holds for every Borel function \( g \) such that \( g \in L^1(\mu_p(f, \cdot)) \) or \( g \) is non-negative. That is, \( \mu_p(f, \cdot) \) for \( p \in \mathbb{R} \) is absolutely continuous with respect to the Lebesgue measure and satisfies

\[
\frac{d\mu_p(f, y)}{dy} = \varphi^*(y)^{1-p} e^{\varphi^*(y)-\langle y, \nabla \varphi^*(y) \rangle} \det(\nabla^2 \varphi^*(y)) \quad \text{for} \quad y \in \Omega.
\]

Consequently, if \( \nu \) admits a density function with respect to the Lebesgue measure, say \( d\nu(y) = h(y) \, dy \), then finding a solution to the \( L_p \) Minkowski problem for log-concave functions requires to obtain a (smooth enough) convex function \( \varphi \) satisfying the following Monge-Ampère equation:

\[
h(y) = \tau \varphi^*(y)^{1-p} e^{\varphi^*(y)-\langle y, \nabla \varphi^*(y) \rangle} \det(\nabla^2 \varphi^*(y)) \quad \text{for} \quad y \in \Omega,
\]

where \( \tau \in \mathbb{R} \) is a constant.

Our main goal in this section is to provide a solution to Problem 6.2 for \( p > 1 \). Let \( M_\nu \) be the interior of the convex hull of the support of the Borel measure \( \nu \). For convenience, denote by \( \mathscr{M} \) the set of all even finite nonzero Borel measures on \( \mathbb{R}^n \), such that, if \( \nu \in \mathscr{M} \), then \( \nu(\mathbb{R}^n) < \infty \), \( \nu \) is not supported in a lower-dimensional subspace of \( \mathbb{R}^n \), \( \nu(M_\nu \setminus L) > 0 \) holds for any bounded convex set \( L \subset \mathbb{R}^n \), and

\[
\int_{\mathbb{R}^n} |x|^p \, d\nu(x) < \infty. \tag{6.2}
\]

Note that the conditions for \( \nu \in \mathscr{M} \) are all natural. Indeed, the assumption that \( \nu \) is not supported in a lower-dimensional subspace of \( \mathbb{R}^n \) is essential in the solution to the Minkowski problem for moment measures in [30]. The requirement for (6.2) is to guarantee that the optimization problem (6.3) is not taken over an empty set. Finally, the condition that \( \nu(M_\nu \setminus L) > 0 \) holds for any bounded convex set \( L \subset \mathbb{R}^n \) is to guarantee that \( \nu(M_\nu \setminus \tilde{\Omega}_\varphi) > 0 \) for any \( \varphi \) and thus avoid \( \Omega_\varphi \) being either empty or a null set. We notice that, complementary to our solutions of the \( L_p \) Minkowski problems for log-concave functions for \( p > 1 \) and even data, the case for \( 0 < p < 1 \) was solved by Rotem in his paper [60] (which was uploaded to arXiv on the same day as the present paper).

**Theorem 6.3** Let \( \nu \in \mathscr{M} \). For \( p > 1 \), there exists an even log-concave function \( f = e^{-\varphi} \), such that, \( \varphi \in \mathcal{C} \) is even and lower semi-continuous, \( \varphi^* \in \mathcal{L}^+ \), and

\[
d\nu = \tau \varphi^*(y)^{1-p} \, d\mu_1(f, \cdot) = \tau \, d\mu_p(f, \cdot) \quad \text{on} \quad \Omega,
\]

where the constant \( \tau \) takes the following form:

\[
\tau = \frac{\int_{\Omega} \varphi^*(y)^p \, d\nu(y)}{\int_{\Omega} \varphi^*(y) \, d\nu(y)} = \frac{\int_{\Omega} d\nu(y)}{\int_{\Omega} d\mu_p(f, y)}.
\]
Before we prove Theorem 6.3, we need some preparation. Let $p > 1$ and $v \in \mathcal{M}$. Denote by $L_{p,e}(v)$ the set of even non-negative functions on $\mathbb{R}^n$ which have finite $L^p$ norms with respect to the measure $v$ and whose function values are 0 at $o$. For $\phi \in L_{p,e}(v)$, let
\[
\Phi_{p,v}(\phi) = \frac{1}{p} \int_{\mathbb{R}^n} \phi(x)^p \, dv(x) - \log J(e^{-\phi^*}).
\]

To find a solution to Problem 6.2, one needs to search for a solution to
\[
\Theta = \inf \left\{ \Phi_{p,v}(\phi) : \phi \in L_{p,e}(v) \; \text{and} \; 0 < J(e^{-\phi^*}) < \infty \right\}. \tag{6.3}
\]

The following lemma shall be needed to solve (6.3). Similar results for $p = 1$ can be found in [30, Lemmas 14 and 15].

**Lemma 6.4** Let $p > 1$ and $v \in \mathcal{M}$. Then there exists a constant $c_v > 0$, such that, for any $\phi \in L_{p,e}(v) \cap C$ satisfying $0 < J(e^{-\phi^*}) < \infty$, the following holds:
\[
\int_{\mathbb{R}^n} \phi(x)^p \, dv(x) \geq c_v \left( J(e^{-\phi^*}) \right)^{\frac{1}{p}} - \int_{\mathbb{R}^n} \, dv(x). \tag{6.4}
\]

**Proof** Let $p > 1$ and $v \in \mathcal{M}$. Assume that $0 < J(e^{-\phi^*}) < \infty$. For any $\phi \in L_{p,e}(v) \cap C$, $e^{-\phi}$ must be an even convex function. It is well-known that there exist constants $c_1, c_2 > 0$ such that for any $n \in \mathbb{N}$ and any even log-concave function $f = e^{-\phi}$ with $J(e^{-\phi}) \in (0, \infty)$, the following holds:
\[
c^n_1 \leq J(e^{-\phi}) J(e^{-\phi^*}) \leq c^n_2. \tag{6.5}
\]

We refer the readers to [43, Theorem 1.1] for more details on inequality (6.5). The sharp constant for $c_2$ is $2\pi$ and the upper bound is indeed the Blaschke-Santaló inequalities for (even) log-concave functions, see e.g., [8, 14, 34, 35]. Applying inequality (6.5) to the log-concave function $e^{-\phi^*}$, one gets that $0 < J(e^{-\phi^*}) < \infty$.

For $\phi \in L_{p,e}(v) \cap C$ satisfying $0 < J(e^{-\phi^*}) < \infty$, let $K_\phi = \{ x \in \mathbb{R}^n : \phi(x) \leq 1 \}$. Clearly, $K_\phi$ is an origin-symmetric bounded convex set containing the origin $o$ in its interior (due to $\phi(o) = 0$). As $0 < J(e^{-\phi}) < \infty$, one has $V(K_\phi) < \infty$. Denote by $\text{vrad}(\phi)$ the volume radius of $K_\phi$, that is,
\[
\text{vrad}(\phi) = \left( \frac{V(K_\phi)}{V(B_1)} \right)^{1/n} \in (0, \infty).
\]

Note that $K_\phi$ cannot contain an Euclidean ball whose radius is greater than $\text{vrad}(\phi)$. Consequently, one may find a vector $\theta_0 \in S^{n-1}$ such that
\[
\sup_{x \in K_\phi} |\langle x, \theta_0 \rangle| = \sup_{x \in K_\phi} \langle x, \theta_0 \rangle \leq \text{vrad}(\phi). \tag{6.6}
\]

The fact that $\phi$ is convex yields that, for $p > 1$ and for any $x \in \mathbb{R}^n$ with $|\langle x, \theta_0 \rangle| \geq \text{vrad}(\phi)$,
\[
\phi^p \left( \frac{\text{vrad}(\phi)}{|\langle x, \theta_0 \rangle|} \right) x \leq \frac{\text{vrad}(\phi)}{|\langle x, \theta_0 \rangle|} \phi^p(x). \tag{6.7}
\]

According to (6.6) and (6.7), if $x \in \mathbb{R}^n$ is such that $|\langle x, \theta_0 \rangle| \geq \text{vrad}(\phi)$, then $\phi(x) \geq 1$ and
\[
\phi^p(x) \geq \frac{|\langle x, \theta_0 \rangle|}{\text{vrad}(\phi)} \phi^p \left( \frac{\text{vrad}(\phi)}{|\langle x, \theta_0 \rangle|} \right) x \geq \frac{|\langle x, \theta_0 \rangle|}{\text{vrad}(\phi)} \geq \frac{|\langle x, \theta_0 \rangle|}{\text{vrad}(\phi)} - 1. \tag{6.8}
\]
Indeed, (6.8) holds for all \( x \in \mathbb{R}^n \) as it is trivial to have (6.8) for those \( x \in \mathbb{R}^n \) such that \( |\langle x, \theta_0 \rangle| < \text{vrad}(\phi) \), due to \( \phi(x) \geq 0 \). Consequently,

\[
\int_{\mathbb{R}^n} \phi^p(x) d\nu(x) \geq \frac{1}{\text{vrad}(\phi)} \int_{\mathbb{R}^n} |\langle x, \theta_0 \rangle| d\nu(x) - \int_{\mathbb{R}^n} d\nu(x) \geq \frac{m_\nu}{\text{vrad}(\phi)} - \int_{\mathbb{R}^n} d\nu(x),
\]

where \( m_\nu = \inf_{\theta \in S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta \rangle| d\nu(x) \). Note that \( m_\nu \in (0, \infty) \) is a direct consequence of the conditions on \( \nu \in \mathcal{M} \), in particular, the function \( \theta \mapsto \int_{\mathbb{R}^n} |\langle x, \theta \rangle| d\nu(x) \) is positive and continuous on \( \theta \) due to the dominated convergence theorem. On the other hand,

\[
\int_{\mathbb{R}^n} e^{-\phi(x)} dx \geq \int_{K_\phi} e^{-\phi(x)} dx \geq \frac{V(K_\phi)}{e} = \frac{V(B_1)}{e} e^{-\text{vrad}(\phi)n}.
\]

Therefore, an application of inequality (6.5) with \( c_2 = 2\pi \) immediately yields

\[
\int_{\mathbb{R}^n} \phi^p(x) d\nu(x) \geq \frac{(V(B_1))^{1/n} m_\nu}{e^{1/n}} \left( \int_{\mathbb{R}^n} e^{-\phi(x)} dx \right)^{-\frac{1}{n}} - \int_{\mathbb{R}^n} d\nu(x)
\]

\[
\geq c_\nu \left( \int_{\mathbb{R}^n} e^{-\phi^*(x)} dx \right)^{\frac{1}{n}} - \int_{\mathbb{R}^n} d\nu(x),
\]

by letting \( c_\nu = \frac{(V(B_1))^{1/n} m_\nu}{2\pi e^{1/n}} \). This concludes the desired formula (6.4).

It has been proved in [30, Lemma 16] that, if \( \nu \) is a nonzero finite Borel measure on \( \mathbb{R}^n \) that is not supported in a lower-dimensional subspace of \( \mathbb{R}^n \), for \( x_0 \in M_\nu \), then there exists a constant \( C_{\nu,x_0} > 0 \) with the following property:

\[
\phi(x_0) \leq C_{\nu,x_0} \int_{\mathbb{R}^n} \phi(x) d\nu(x)
\]

holds for any \( \nu \)-integrable convex function \( \phi : \mathbb{R}^n \rightarrow [0, \infty] \). This can be applied to \( \nu \in \mathcal{M} \) and \( p > 1 \) to get that, for \( x_0 \in M_\nu \), then there exists a constant \( C_{\nu,x_0} > 0 \), such that,

\[
\phi^p(x_0) \leq C_{\nu,x_0} \int_{\mathbb{R}^n} \phi^p(x) d\nu(x)
\]

(6.9)

holds for any \( \phi \in C \cap L_{p,\nu}(\nu) \).

We shall need the following lemma given by [58, Theorem 10.9].

**Lemma 6.5** Let \( C \subset \mathbb{R}^n \) be a relatively open convex set, and let \( \phi_1, \phi_2, \ldots, \) be a sequence of finite convex functions on \( C \). Suppose that the real numbers \( \phi_1(x), \phi_2(x), \ldots, \) are bounded for each \( x \in C \). It is then possible to select a subsequence of \( \phi_1(x), \phi_2(x), \ldots, \) which converges uniformly on closed bounded subsets of \( C \) to some finite convex function \( \phi \).

The following result was proved in [60, Proposition 2.1], and it is needed in our later context. Similar results can be found in [11,16].

**Lemma 6.6** Let \( \phi, g : \mathbb{R}^n \rightarrow (-\infty, \infty] \) be lower semi-continuous functions. Assume that \( g \) is bounded from below, \( g(o) < \infty \) and \( \phi(o) < \infty \). Then, at every point \( x_0 \in \mathbb{R}^n \) where \( \phi^* \) is differentiable, we have

\[
\frac{d}{dt} \left. (\phi + tg)^* (x_0) \right|_{t=0^+} = -g \left( \nabla \phi^*(x_0) \right).
\]

We are now ready to prove the following lemma. The case \( p = 1 \) has been discussed in [30, Lemma 17].

\( \square \) Springer
Lemma 6.7 Let $p > 1$ and $v \in \mathcal{M}$. Assume that $\phi \in C \cap L_{p,e}(v)$ for any $l \in \mathbb{N}$ satisfy
\[
\sup_{l \in \mathbb{N}} \int_{\mathbb{R}^n} \phi_l(x)^p \, d\nu(x) < +\infty.
\] (6.10)
Then there exists a subsequence $\{\phi_{j_l}\}_{l \in \mathbb{N}}$ of $\{\phi_l\}_{l \in \mathbb{N}}$ and a non-negative convex function $\phi \in C \cap L_{p,e}(v)$, such that,
\[
\int_{\mathbb{R}^n} \phi(x)^p \, d\nu(x) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \phi_{j_l}(x)^p \, d\nu(x)
\]
and
\[
\int_{\mathbb{R}^n} e^{-\phi^*(x)} \, dx \geq \limsup_{j \to \infty} \int_{\mathbb{R}^n} e^{-\phi_{j_l}^*(x)} \, dx.
\] (6.11)

Proof We prove this lemma following the ideas of the proof of [30, Lemma 17], with emphasis on the difference and modification.

As $v \in \mathcal{M}$ is an even measure which is not supported in a lower-dimensional subspace of $\mathbb{R}^n$, the open set $M_v$ is nonempty and origin-symmetric with $o \in M_v$. By (6.9) and (6.10), for any $x \in M_v$, one has $\sup_{l \in \mathbb{N}} \phi_l(x)^p < +\infty$. We would like to mention that, if $\phi \in L_{p,e}(v)$, then $\phi$ is finite near the origin and $\text{dom}(\phi) \supseteq M_v$. Lemma 6.5 can be applied to $\phi_{j_l}$ and $C = M_v$ to obtain the existence of a subsequence $\{\phi_{j_l}\}_{l \in \mathbb{N}}$ of $\{\phi_l\}_{l \in \mathbb{N}}$, which converges to an even convex function $\phi : M_v \to \mathbb{R}$ pointwisely on $M_v$ and also uniformly on any closed bounded subset of $M_v$. The finiteness of $\phi$ on $M_v$ implies the continuity of $\phi$ on $M_v$. Moreover, $\phi$ is non-negative in $M_v$ and achieves its minimum at the origin with $\phi(o) = 0$ (as $\phi_l(o) = 0$ for each $l \in \mathbb{N}$). The function $\phi : M_v \to \mathbb{R}$ can be extended on $\mathbb{R}^n$, and the new function will still be denoted by $\phi$. That is, let $\phi(x) = +\infty$ for $x \notin M_v$; while $\phi(x) = \lim_{\lambda \to 1^-} \phi(\lambda x)$ if $x \in \partial M_v$. The limit in the latter case always exists, although it may be $+\infty$, due to the fact that the function $\lambda \mapsto \phi(\lambda x)$ is increasing on $\lambda \in (0, 1)$ following from the convexity of $\phi$ and $\phi(o) = 0$. Moreover, for any $x \in M_v$, $\phi(\lambda x)$ is increasing to $\phi(x)$ as $\lambda$ is increasing to $1$. It is evident that $\phi : \mathbb{R}^n \to \mathbb{R}$ is an even, non-negative, and lower semi-continuous convex function with $\phi(o) = 0$.

Note that the support of $\nu$ is a subset of $\overline{M_v}$. It follows from Fatou’s lemma and $\phi_{j_l} \to \phi$ pointwisely in $M_v$ that, for any given $\lambda \in (0, 1)$, one has
\[
\int_{\mathbb{R}^n} \phi(\lambda x)^p \, d\nu(x) = \int_{M_v} \phi(\lambda x)^p \, d\nu(x) \leq \liminf_{j \to \infty} \int_{M_v} \phi_{j_l}(\lambda x)^p \, d\nu(x)
\]
\[
\leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \phi_{j_l}(x)^p \, d\nu(x) < +\infty,
\]
where the second inequality again follows from the monotonicity of the function $\lambda \mapsto \phi_{j_l}(\lambda x)$. Similarly, by the monotone convergence theorem, one can also obtain that
\[
\int_{\mathbb{R}^n} \phi(x)^p \, d\nu(x) = \lim_{\lambda \to 1^-} \int_{\mathbb{R}^n} \phi(\lambda x)^p \, d\nu(x) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \phi_{j_l}(x)^p \, d\nu(x) < +\infty.
\]
This completes the proof of the first inequality in (6.11). In particular, $\phi \in L_{p,e}(v) \cap C$, $o \in M_v \subseteq \text{dom}(\phi)$ and hence $J(e^{-\phi^*}) < \infty$.

The second inequality in (6.11) indeed follows immediately from the proof of [30, Lemma 17]. For completeness, a brief explanation extracted from [30, p. 3861-3862] is provided here. First of all, for any $y \in \mathbb{R}^n$, one has $\phi^*(y) = \sup_{l \in \mathbb{N}} \{\langle x_i, y \rangle - \phi(x_i)\}$ where $\{x_i\}_{l \in \mathbb{N}}$ is a dense sequence in $M_v$. When $j$ is large enough, the origin lies in the interior of the convex hull of $\{x_1, \cdots, x_j\}$, and this in turn implies that $\exp(-h_j)$ with $h_j(y) = \sup_{1 \leq i \leq j} \{\langle x_i, y \rangle - \phi(x_i)\}$
is integrable. Clearly, $h_j$ is increasing to $\phi^*$ as $j$ is increasing to $\infty$. By the monotone convergence theorem, one gets
\[ \int_{\mathbb{R}^n} e^{-\phi^*(y)} \, dy = \lim_{j \to \infty} \int_{\mathbb{R}^n} e^{-h_j(y)} \, dy. \]
Let $\varepsilon > 0$. An integer $j_0$ (depending only on $\varepsilon$) can be found to have
\[ 0 \leq \int_{\mathbb{R}^n} e^{-h_{j_0}(y)} \, dy - \int_{\mathbb{R}^n} e^{-\phi^*(y)} \, dy < \varepsilon. \]  
(6.12)
The pointwise convergence of $\phi_{j'} \to \phi$ (on $\{x_1, \ldots, x_{j_0}\}$) yields that, for sufficiently large $j$, $\phi_{j'}^*(x) \geq h_{j_0}(x) - \varepsilon$ holds for all $x \in \mathbb{R}^n$. Together with (6.12), the following holds:
\[ \int_{\mathbb{R}^n} e^{-\phi^*(y)} \, dy > \int_{\mathbb{R}^n} e^{-h_{j_0}(y)} \, dy - \varepsilon \geq e^{-\varepsilon} \limsup_{j \to \infty} \int_{\mathbb{R}^n} e^{-\phi_{j'}^*(y)} \, dy - \varepsilon. \]
The second argument in (6.11) then follows by letting $\varepsilon \to 0$. \hfill \Box

Now we are ready to prove our main result, i.e., Theorem 6.3. The $L_p$ surface area measure for log-concave functions in general does not have homogeneity, hence solving the related Minkowski problem (i.e., Problem 6.2) usually requires more delicate analysis. Most of the times, such problems shall require to solve constrained optimization problems, and the method of Lagrange multipliers as in [37,65,67] should be used; this method should work here for a proof of Theorem 6.3. However, we find that the ideas in the proof of [30, Theorem 2] work well in our case for $p > 1$. Therefore, we decide to adopt the ideas in [30, Theorem 2] in the proof of Theorem 6.3. Unfortunately, due to the lack of homogeneity for the $L_p$ surface area measure for log-concave functions for $p > 1$, it is unlikely to have the uniqueness of solutions to Problem 6.2.

**Proof** We will search a solution for the following optimization problem (6.3):
\[ \Theta = \inf \left\{ \Phi_{p,v}(\phi) : \phi \in L_{p,e}(v) \text{ and } 0 < J(e^{-\phi^*}) < \infty \right\}. \]
According to $(\phi^*)^* \leq \phi$ and $((\phi^*)^*)^* = \phi^*$ for any $\phi \in L_{p,e}(v)$, it can be checked that
\[ \Phi_{p,v}(\phi) = \frac{1}{p} \int_{\mathbb{R}^n} \phi(x)^p \, dv(x) - \log J(e^{-\phi^*}) \geq \frac{1}{p} \int_{\mathbb{R}^n} ((\phi^*)^p)(x) \, dv(x) - \log J(e^{-\phi^*}) = \Phi_{p,v}((\phi^*)^*). \]
Consequently, to find a solution to the optimization problem (6.3), it is enough to focus on the class of functions $\phi \in L_{p,e}(v) \cap C$ which are also lower semi-continuous.

We now claim that the optimization problem (6.3) is well-defined. First of all, (6.2) implies that $|x| \in L_{p,e}(v) \cap C$. Note that $|x|$ is also lower semi-continuous. It is well-known that $|x|^* = I_{B_1}$. Therefore, $J(e^{-|x|^*}) = V(B_1)$ is finite. This in turn yields that the infimum of (6.3) is not taken over an empty set and hence $\Theta < \infty$. On the other hand, $\Theta$ is bounded from below. To see this, by (6.4), for any lower semi-continuous function $\phi \in L_{p,e}(v) \cap C$ with $J(e^{-\phi^*}) \in (0, \infty)$, one has
\[ \Phi_{p,v}(\phi) = \int_{\mathbb{R}^n} \phi(x)^p \, dv(x) - \log J(e^{-\phi^*}) \geq H(J(e^{-\phi^*})) - \int_{\mathbb{R}^n} d\nu(x), \]  
(6.13)
where \( H(t) = c_v t^{1/n} - \log t \) for \( t \in (0, \infty) \). It can be easily checked that \( H(\cdot) \) achieves its minimum at \( t_0 = (nc_v^{-1})^n \) and \( H(t) \geq H(t_0) > -\infty \). Thus, \( \Theta > -\infty \) and the optimization problem (6.3) is well-defined. We also would like to mention that

\[
\lim_{t \to 0^+} H(t) = \lim_{t \to +\infty} H(t) = +\infty.
\]

(6.14)

Let \( \{\phi_l\}_{l \in \mathbb{N}} \subset L_{p,v} \cap C \) be a minimizing sequence of lower semi-continuous functions such that \( J(e^{-\phi_l^p}) \in (0, \infty) \) for each \( l \in \mathbb{N} \) and

\[
\Theta = \lim_{l \to \infty} \Phi_{p,v}(\phi_l) \leq \Phi_{p,v}(|x|) < +\infty.
\]

Without loss of generality, we can always assume that

\[
\sup_{l \in \mathbb{N}} \Phi_{p,v}(\phi_l) \leq \Phi_{p,v}(|x|) + 1 < +\infty.
\]

(6.15)

Together with (6.13) and (6.14), one sees that

\[
0 < \inf_{l \in \mathbb{N}} J(e^{-\phi_l^p}) \leq \sup_{l \in \mathbb{N}} J(e^{-\phi_l^p}) < +\infty.
\]

(6.16)

Combining with (6.15), the following holds:

\[
\sup_{l \in \mathbb{N}} \int_{\mathbb{R}^n} \phi_l(x)^p \, dv(x) < +\infty.
\]

This is exactly the condition (6.10). Therefore, Lemma 6.7 can be applied to get a subsequence \( \{\phi_{l_j}\}_{j \in \mathbb{N}} \) of \( \{\phi_l\}_{l \in \mathbb{N}} \) and a non-negative convex function \( \phi_0 \in C \cap L_{p,v} \), such that (6.11) holds, namely,

\[
\int_{\mathbb{R}^n} \phi_0(x)^p \, dv(x) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \phi_{l_j}(x)^p \, dv(x)
\]

and

\[
\int_{\mathbb{R}^n} e^{-\phi_0^p(x)} \, dx \geq \limsup_{j \to \infty} \int_{\mathbb{R}^n} e^{-\phi_{l_j}^p(x)} \, dx.
\]

According to (6.11), one immediately has

\[
\Phi_{p,v}(\phi_0) \leq \liminf_{l \to \infty} \Phi_{p,v}(\phi_l) = \lim_{l \to \infty} \Phi_{p,v}(\phi_l) = \Theta \leq \Phi_{p,v}(\phi_0).
\]

(6.17)

Hence, \( \phi_0 \) solves the optimization problem (6.3). Moreover, \( 0 < J(e^{-\phi_0^p}) < \infty \) following from (6.14). Inequality (6.5) yields that \( 0 < J(e^{-\phi_0}) < \infty \) as well. In particular,

\[
\lim_{|x| \to +\infty} \phi_0(x) = +\infty.
\]

(6.18)

Let \( \bar{\Omega}_{\phi_0} = \{y \in \mathbb{R}^n : \phi_0(y) = 0\} \) and \( \Omega_{\phi_0} = \{y \in \mathbb{R}^n : 0 < \phi_0(y) < \infty\} \). By the facts that \( \text{dom}(\phi_0) \supseteq M_v \) and \( \phi_0 \) is even, one gets that \( \phi_0 \) is continuous on \( M_v \) and both \( \bar{\Omega}_{\phi_0} \) and \( \Omega_{\phi_0} \) are origin-symmetric. Moreover, (6.18) yields that \( \Omega_{\phi_0} \) is a bounded closed (due to the lower semi-continuity of \( \phi_0 \)) convex set. In particular, \( M_v \setminus \bar{\Omega}_{\phi_0} \subseteq \Omega_{\phi_0} \) as \( M_v \subseteq \text{dom}(\phi_0) \). Recall that if \( v \in \mathcal{M} \), then \( v(M_v \setminus L) > 0 \) holds for any bounded convex set \( L \subset \mathbb{R}^n \). Consequently, \( v(M_v \setminus \bar{\Omega}_{\phi_0}) = v(M_v \cap \Omega_{\phi_0}) > 0 \) and

\[
\int_{\mathbb{R}^n} \phi_0^p(y) \, dv(y) = \int_{\bar{\Omega}_{\phi_0}} \phi_0^p(y) \, dv(y) \geq \int_{M_v \setminus \bar{\Omega}_{\phi_0}} \phi_0^p(y) \, dv(y) > 0.
\]

Thus, \( \Omega_{\phi_0} \) is a nonempty open set whose Lebesgue measure is strictly positive.
Let \( g : \mathbb{R}^n \to \mathbb{R} \) be any even compactly supported continuous function such that the support of \( g \), denoted by \( \text{supp}(g) \), is a proper subset of \( \Omega_{\phi_0} \). Moreover, the compact set \( \text{supp}(g) \) is contained in \( \Omega_{\phi_0} \). Let \( \phi_t = \phi_0 + tg \) and \( \phi_t \) is a continuous function on \( \Omega_{\phi_0} \). Note that \( \phi_0 > 0 \) on \( \text{supp}(g) \) and hence min\( x \in \text{supp}(g) \phi_0(x) > 0 \). This further yields the existence of \( t_0 > 0 \) such that \( \phi_t \) is non-negative on \( \Omega_{\phi_0} \) for all \( t \in [-t_0, t_0] \). It is easily checked that for all \( t \in [-t_0, t_0] \), \( x \in \mathbb{R}^n \) and \( p > 1 \), one has \( \phi_t^p(x) \leq 2^p(\phi_0^p(x) + |t|g(x)|^p) \) and hence

\[
\int_{\mathbb{R}^n} \phi_t^p(x) \, d\nu(x) \leq 2^p \int_{\mathbb{R}^n} \phi_0^p(x) \, d\nu(x) + 2^p |t|^p \int_{\mathbb{R}^n} |g(x)|^p \, d\nu(x) < \infty.
\]

This means that \( \phi_t \in L_{p,e}(\nu) \) for all \( t \in [-t_0, t_0] \). Also note that

\[
\phi_0(x) - |t| \max_{x \in \mathbb{R}^n} |g(x)| \leq \phi_t(x) \leq \phi_0(x) + |t| \max_{x \in \mathbb{R}^n} |g(x)|.
\]

By (2.5), one obtains that, for all \( t \in [-t_0, t_0] \) and all \( y \in \text{dom}(\phi_t^*) \),

\[
\phi_t^*(y) - |t| \max_{x \in \mathbb{R}^n} |g(x)| \leq \phi_t^*(y) \leq \phi_0^*(y) + |t| \max_{x \in \mathbb{R}^n} |g(x)|.
\]

(6.19)

This concludes that \( J(e^{-\phi_t^*}) \in (0, \infty) \) for all \( t \in [-t_0, t_0] \) as

\[
e^{-t_0 \max_{x \in \mathbb{R}^n} |g(x)|} J(e^{-\phi_t^*}) \leq J(e^{-\phi_t^*}) \leq e^{t_0 \max_{x \in \mathbb{R}^n} |g(x)|} J(e^{-\phi_0^*}).
\]

Consequently, for any even and compactly supported continuous function \( g : \mathbb{R}^n \to \mathbb{R} \) with \( \text{supp}(g) \subseteq \Omega_{\phi_0} \), there exists \( t_0 > 0 \) such that \( \Phi_{p,v}(\phi_0) \leq \Phi_{p,v}(\phi_t) \) holds for all \( t \in [-t_0, t_0] \). Thus, \( \phi_0 \) satisfies that

\[
\frac{d}{dt} \Phi_{p,v}(\phi_t) \bigg|_{t=0} = \frac{d}{dt} \left( \frac{1}{p} \int_{\mathbb{R}^n} \phi_t(x)^p \, d\nu(x) \right) \bigg|_{t=0} - \frac{d}{dt} \left( \log J(e^{-\phi_t^*}) \right) \bigg|_{t=0} = 0.
\]

(6.20)

By the dominated convergence theorem, one can easily get that

\[
\frac{d}{dt} \left( \frac{1}{p} \int_{\mathbb{R}^n} \phi_t(x)^p \, d\nu(x) \right) \bigg|_{t=0} = \frac{d}{dt} \left( \frac{1}{p} \int_{\mathbb{R}^n} (\phi_0(x) + tg(x))^p \, d\nu(x) \right) \bigg|_{t=0} = \int_{\mathbb{R}^n} g(x)\phi_0(x)^{p-1} \, d\nu(x).
\]

(6.21)

Applying Lemma 6.6 to \( \phi_0 \) and \( g \), and to \( \phi_0 \) and \( -g \), one gets, for all \( x \in \mathbb{R}^n \) where \( \phi_0^* \) is differentiable,

\[
\frac{d}{dt} \phi_t^*(x) \bigg|_{t=0^+} = -g(\nabla\phi_t^*(x)) \quad \text{and} \quad \frac{d}{dt} \phi_t^*(x) \bigg|_{t=0^-} = -g(\nabla\phi_0^*(x)).
\]

Consequently, at \( x \in \mathbb{R}^n \) where \( \phi_0^* \) is differentiable, one gets

\[
\frac{d}{dt} \phi_t^*(x) \bigg|_{t=0} = -g(\nabla\phi_0^*(x)).
\]

(6.22)

This further yields that formula (6.22) holds for almost all \( x \) in the interior of \( \text{dom}(\phi_0^*) \). On the other hand, it follows from (6.19) that, for all \( t \in [-t_0, t_0] \) and all \( y \in \text{dom}(\phi_0^*) \),

\[
0 \leq \frac{e^{-\phi_t^*(y)} - e^{-\phi_0^*(y)}}{t} \leq \max \left\{ \frac{e^{|t| \max_{x \in \mathbb{R}^n} |g(x)|} - 1}{|t|}, \frac{1 - e^{-|t| \max_{x \in \mathbb{R}^n} |g(x)|}}{|t|} \right\} \leq D \max_{x \in \mathbb{R}^n} |g(x)| < +\infty.
\]
for some positive constant $D$. The dominated convergence theorem then gives
\[
\frac{d}{dt} \left( \log J(e^{-\varphi_t}) \right) \bigg|_{t=0} = \frac{1}{J(e^{-\varphi_0})} \int_{\mathbb{R}^n} g(\nabla \varphi_0^*(x)) e^{-\varphi_0^*(x)} \, dx. \quad (6.23)
\]
Together with (2.9) for $\varphi = \varphi_0^*$, (6.20), (6.21), and (6.23), one has
\[
\int_{\Omega_{\varphi_0}} g(y) \varphi_0^*(y)^{p-1} \, d\nu(y) = \int_{\mathbb{R}^n} g(y) \varphi_0(y)^{p-1} \, d\nu(y)
= \frac{1}{J(e^{-\varphi_0})} \int_{\mathbb{R}^n} g(\nabla \varphi_0^*(x)) e^{-\varphi_0^*(x)} \, dx
= \frac{1}{J(e^{-\varphi_0})} \int_{\mathbb{R}^n} g(y) \, d\mu(e^{-\varphi_0^*}, y)
= \frac{1}{J(e^{-\varphi_0})} \int_{\Omega_{\varphi_0}} g(y) \, d\mu(e^{-\varphi_0^*}, y)
\]
for any even and compactly supported continuous function $g : \mathbb{R}^n \to \mathbb{R}$ with $\text{supp}(g) \subseteq \Omega_{\varphi_0}$. This concludes that, on $\Omega_{\varphi_0}$,
\[
\varphi_0^* \mu^{-1} \, d\nu = \frac{1}{J(e^{-\varphi_0})} \, d\mu(e^{-\varphi_0^*}, \cdot). \quad (6.24)
\]
Let $\varphi = \varphi_0^*$. Then $\varphi \in C$ is an even lower semi-continuous convex function, $\varphi^* \in \mathcal{L}^+$, and
\[
d\nu = \frac{1}{J(e^{-\varphi})} (\varphi^*)^{1-p} \, d\mu_1(e^{-\varphi}, \cdot) = \frac{1}{J(e^{-\varphi})} \, d\mu_p(e^{-\varphi}, \cdot) \quad \text{on } \Omega_{\varphi^*}. \quad (6.25)
\]
By taking the integration from both sides of (6.24) or (6.25) on $\Omega_{\varphi^*}$, one sees that
\[
\frac{1}{J(e^{-\varphi})} = \frac{\int_{\Omega} (\varphi^*(y))^{p-1} \, d\nu(y)}{\int_{\Omega} \, d\mu_1(e^{-\varphi}, y)} = \frac{\int_{\Omega} \, d\nu(y)}{\int_{\Omega} \, d\mu_p(e^{-\varphi}, y)}.
\]
This completes the proof of Theorem 6.3. \hfill \Box

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