The Crossed Product by a Partial Endomorphism and the Covariance Algebra

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Abstract

Given a local homeomorphism $\sigma : U \to X$ where $U \subseteq X$ is clopen and $X$ is a compact and Hausdorff topological space, we obtain the possible transfer operators $L_\rho$ which may occur for $\alpha : C(X) \to C(U)$ given by $\alpha(f) = f \circ \sigma$. We obtain examples of partial dynamical systems $(X_A, \sigma_A)$ such that the construction of the covariance algebra $C^*(X_A, \sigma_A)$ and the crossed product by partial endomorphism $O(X_A, \alpha, L)$ associated to this system are not equivalent, in the sense that there does not exists invertible function $\rho \in C(U)$ such that $O(X_A, \alpha, L_\rho) = C^*(X_A, \sigma)$.

1 Introduction

We start with a summary of the construction of the crossed product by a partial endomorphism. Details may be seen in [3]. A partial $C^*$-dynamical system $(A, \alpha, L)$ consists of a (closed) ideal $I$ of a $C^*$-algebra $A$, an idempotent self-adjoint ideal $J$ of $I$ (not necessarily closed), a *-homomorphism $\alpha : A \to M(I)$ and a linear positive map (which preserves *) $L : J \to A$ such that $L(aa(b)) = L(a)b$ for each $a \in J$ and $b \in A$. The map $L$ is called transfer operator. Define in $J$ an inner product (which may be degenerated) by $(x, y) = L(x^*y)$. Then we obtain an inner product $\langle , \rangle$ in the quotient $J_0 = J/\{x \in J : L(x^*x) = 0\}$ defined by $\langle \tilde{x}, \tilde{y} \rangle = L(x^*y)$, which induces a norm $\| \|$. Define $M = \overline{J_0}$, which is a right Hilbert $A$-module and also a left $A$-module, where the left multiplication is defined by the *-homomorphism $\varphi : A \to L(M)$ (the adjointable operators in $M$), where $\varphi(a)(\tilde{x}) = \tilde{ax}$ for each $x \in J$. The Toeplitz algebra associated to $(A, \alpha, L)$ is the universal $C^*$-algebra $T(A, \alpha, L)$ generated by $A \cup M$ with the relations of $A$, of $M$, the bi-module products and $m^*n = \langle m, n \rangle$.

\[\text{Supported by Cnpq}\]
A redundancy in $T(A, \alpha, L_L)$ is a pair $(a, k) \in A \times \hat{K}_1$, ($\hat{K}_1 = \text{span}\{mn^*, m, n \in M\}$), such that $am = km$ for every $m \in M$. The Crossed Product by a Partial Endomorphism $O(A, \alpha, L)$ is the quotient of $T(A, \alpha, L)$ by the ideal generated by all the elements $a - k$ where $(a, k)$ is a redundancy and $a \in \ker(\varphi)^{-1} \cap \varphi^{-1}(K(M))$.

In [3] it was defined the algebra $O(X, \alpha, L)$. This algebra is constructed from a partial $C^*$-dynamical system $(C(X), \alpha, L)$ induced by a local homeomorphism $\sigma : U \to X$, where $U$ is an open subset of a compact topological Hausdorff space $X$. More specifically,

$$\alpha : C(X) \to C^b(U)$$

$$f \mapsto f \circ \sigma$$

where $C^b(U)$ is the space of all continuous bounded functions in $U$ and $L : C_c(U) \to C(X)$ ($C_c(U)$ is the set of the continuous functions with compact support in $U$) is defined by

$$L(f)(x) = \begin{cases} 
\sum_{y \in \sigma^{-1}(x)} f(y) & \text{if } x \in \sigma(U) \\
0 & \text{otherwise}
\end{cases}$$

for every $f \in C_c(U)$ and $x \in X$.

In [3] it was defined the algebra $C^*(X, \alpha)$, called covariance algebra. This algebra is also constructed from a partial dynamical system, that is, a continuous map $\sigma : U \to X$ where $X$ is a topological compact Hausdorff space, $U$ is a clopen subset of $X$ and $\sigma(U)$ is open.

If we suppose that $\sigma : U \to X$ is an local homeomorphism, $U$ clopen (and so $\sigma(U)$ is always open) then $(X, \sigma)$ gives rise to two $C^*$-algebras, the covariance algebra $C^*(X, \sigma)$ and the crossed product by a partial endomorphism $O(X, \sigma, L)$.

In this paper we identify the transfer operators $L_\rho$ which may occur for $\alpha$. Moreover we show that the constructions of covariance algebra and crossed product by partial endomorphism are not equivalent, in the following sense: we obtain examples of partial dynamical systems $(X_A, \sigma_A)$ such that there does not exists invertible function $\rho$ such that $O(X_A, \alpha, L_\rho) = C^*(X, \alpha)$.

Given $C^*$-algebras $A$ and $B$, if we write $A = B$, we will say that $A$ and $B$ are *-isomorphic.

Acknowledgements
2 Transfer operators of $X$ for $\alpha$

Let $\sigma : U \to X$ be a local homeomorphism and $U$ an open subset of the compact Hausdorff space $X$. This local homeomorphism induces the $*$-homomorphism

$$\alpha : C(X) \to C^b(U)$$

$$f \mapsto f \circ \sigma .$$

Given a positive function $\rho \in C(U)$, for all $f \in C_c(U)$ we may define

$$L_\rho(f)(x) = \begin{cases} 
\sum_{y \in \sigma^{-1}(x)} \rho(y)f(y) & \text{if } x \in \sigma(U) \\
0 & \text{otherwise}
\end{cases}$$

for each $x \in X$. Note that $L_\rho(f) = L(\rho f)$, and since $\rho f \in C_c(U)$ and $L(\rho f) \in C(X)$ (see [3]) then $L_\rho(f)$ in fact is an element of $C(X)$. In this way we may define the map $L_\rho : C_c(U) \to C(X)$, which is linear and positive (by the fact that $\rho$ is positive). It is easy to see that $L_\rho(f \circ (g)) = L_\rho(f)g$ for each $f \in C_c(U)$ and $g \in C(X)$. The following proposition shows that if $U$ is clopen in $X$ then every transfer operator for $\alpha$ is of the form $L_\rho$ for some $\rho \in C(U)$.

**Proposition 2.1** Let $L : C_c(U) \to C(X)$ (U clopen in X) a transfer operator for $\alpha$, that is, $L$ is linear, positive and $L(g \circ (f)) = L(g)f$ for each $f \in C(X)$ and $g \in C_c(U)$. Then there exists $\rho \in C(U)$ such that $L = L_\rho$.

**Proof.**

Let $\{V_i\}_{i=1}^n$ be an open cover of $U$ such that $\sigma_{|V_i}$ is a homeomorphism. (such cover exists because $U$ is compact and $\sigma$ is a local homeomorphism). For each $i$ take an open subset $U_i \subseteq V_i$ such that $\overline{U_i} \subseteq V_i$ and $\{U_i\}_i$ is also a cover for $U$. Consider the partition of unity $\{\varphi_i\}$ subordinated to $\{U_i\}_i$ and define $\xi_i = \sqrt{\varphi_i}$. Since $\xi_i$ is positive for each $i$ then $L(\xi_i)$ is a positive function. Define $\rho = \sum_{i=1}^n \alpha(L(\xi_i))\xi_i$ which is also positive. Given $f \in C_c(U)$ define for each $i$, 

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Claim 1: \( g_i \in C(X) \) for all \( i \)

Let \( x_j \to x \). Suppose \( x \in \sigma(V_i) \). Since \( \sigma(V_i) \) is open we may suppose that \( x_j \in \sigma(V_i) \) for each \( j \). Since \( \sigma|_{V_i} \) is a homeomorphism then \( \sigma^{-1}(x_j) \to \sigma^{-1}(x) \) in \( V_i \) and so \( g_i(x_j) = (\xi_i f)(\sigma^{-1}(x_j)) \to (\xi_i f)(\sigma^{-1}(x)) = g_i(x) \). If \( x \notin \sigma(V_i) \) then \( x \notin \sigma(\overline{V_i}) \), which is closed. Therefore we may suppose that \( x_j \notin \sigma(\overline{U_i}) \) and so \( g_i(x_j) = 0 = g_i(x) \).

Claim 2: \( \xi_i \alpha(g_i) = \varphi_i f \)

If \( x \notin U_i \) then \( (\xi_i \alpha(g_i))(x) = 0 = (\varphi_i f)(x) \). If \( x \in U_i \) then \( \alpha(g_i)(x) = g_i(\sigma(x)) = \xi_i(x)f(x) \) and so \( \xi_i(x)\alpha(g_i)(x) = \xi^2(x)f(x) = \varphi(x)f(x) \).

Since \( \varphi \) is partition of unity then \( f = \sum_{i=1}^{n} \varphi_i f = \sum_{i=1}^{n} \xi_i \alpha(g_i) \), where the last equality follows by claim 2. Then

\[
L(f) = \sum_{i=1}^{n} L(\xi_i \alpha(g_i)) = \sum_{i=1}^{n} L(\xi_i) g_i.
\]

We show that \( L = L_\rho \). If \( x \notin \sigma(U) \) then \( L_\rho(f)(x) = 0 = L(f)(x) \) by definition.

Given \( x \in \sigma(U) \),

\[
L_\rho(f)(x) = \sum_{y \in \sigma^{-1}(x)} \rho(y)f(y) = \sum_{y \in \sigma^{-1}(x)} \sum_{i=1}^{n} \alpha(L(\xi_i))(y)\xi_i(y)f(y) = \sum_{y \in \sigma^{-1}(x)} \sum_{i:y \in U_i} L(\xi_i)(x)\xi_i(y)f(y).
\]

On the other hand,

\[
L(f)(x) = \sum_{i=1}^{n} L(\xi_i)(x)g_i(x) = \sum_{i:x \in \sigma(U_i)} L(\xi_i)(x)\xi_i(\sigma^{-1}(x))f(\sigma^{-1}(x)).
\]

To see that those two expressions are equal note that the summands are the same.
Denote by $M_\rho$ the Hilbert bi-module generated by $C_c(U)$ with the inner product given by $L_\rho$ and by $\tilde{K}_\rho$ the algebra generated by $nm^*$ in $T(X,\alpha,L_\rho)$. Moreover, denote by $\varphi_\rho : C(X) \to L(M_\rho)$ the $^*$-homomorphism given by the left product of $A$ by $M_\rho$.

**Lemma 2.2** Let $\rho, \rho' \in C(U)$ positive functions. If $\ker(\rho) = \ker(\rho')$ then $\ker(\varphi_\rho) = \ker(\varphi_{\rho'})$.

**Proof.**

Let $f \in C(X)$. Then $f \in \ker(\varphi_\rho) \iff fm = 0$ for each $m \in M_\rho \iff \tilde{f}g = \tilde{f}\tilde{g} = 0$ for each $g \in C_c(U)$. It is easy to check that $\tilde{f}g = 0$ in $M_\rho$ if and only if $\rho fg = 0$. Then $f \in \ker(\varphi_\rho)$ if and only if $\rho fg = 0$ for each $g \in C_c(U)$. In the same way, $f \in \ker(\varphi_{\rho'})$ if and only if $\rho' fg = 0$ for each $g \in C_c(U)$. Since $\ker(\rho) = \ker(\rho')$ then $\rho fg = 0$ if and only if $\rho' fg = 0$ for each $g \in C_c(U)$.

\[\square\]

**Proposition 2.3** If $\rho$ and $\rho'$ are elements of $C(U)$ such that there exists $r \in C(U)$ such that $r(x) \neq 0$ for each $x \in U$ and $\rho = rp'$ then $\mathcal{O}(X,\alpha,L_\rho)$ and $\mathcal{O}(X,\alpha,L_{\rho'})$ are $^*$-isomorphic.

**Proof.**

Let us define a $^*$-homomorphism from $\mathcal{O}(X,\alpha,L_\rho)$ to $\mathcal{O}(X,\alpha,L_{\rho'})$. Define

$$\psi_1 : C(X) \to T(X,\alpha,L_{\rho'})$$

$$f \mapsto f$$

Let $\xi = \sqrt{r}$, and note that for each $g \in C_c(U)$,

$$\|\tilde{g}\|_\rho^2 = \|L_\rho(g^*g)\| = \|L(\rho g^*g)\| = \|L(rp'g^*g)\| = \|L_{\rho'}((\xi g)^*\xi g)\| = \|\tilde{\xi}g\|_{\rho'}^2.$$ 

So we may define $\psi_2 : M_\rho \to T(X,\alpha,L_{\rho'})$ by $\psi_2(\tilde{g}) = \tilde{\xi}g$. Let $\psi_3 = \psi_1 \cup \psi_2$. We show that $\psi_3$ extends to $T(X,\alpha,L_\rho)$. For each $f \in C(X)$ and $g \in C_c(U)$ we have

$$\psi_3(f)\psi_3(\tilde{g}) = f\tilde{\xi}g = \tilde{\xi}fg = \psi_3(\tilde{fg})$$

and

$$\psi_3(\tilde{g})\psi_3(f) = \tilde{\xi}gf = \tilde{\xi}g\alpha(f) = \psi_3(g\alpha f).$$
Moreover, if \( h \in C_c(U) \) then
\[
\psi_3(\xi g^*)^* \xi h = L_{\rho'}((\xi g)^* \xi h) = L_{\rho'}(g^* h) = L_{\rho}(g^* h) = \psi_3(L_{\rho}(g^* h)).
\]

So \( \psi_3 \) extends to \( \mathcal{T}(X, \alpha, L_{\rho}) \). Let \((f, k) \in C(X) \times \hat{K}_{1, \rho} \) a redundancy with \( f \in \ker(\varphi_{\rho})^\perp \cap \varphi_{\rho}^{-1}(K(M_{\rho})) \). Since \( \psi_3(M_{\rho}) \subseteq M_{\rho'} \) it follows that \( \psi_3(k) \in \hat{K}_{1, \rho'} \) and so \((\psi_3(f), \psi_3(k)) \in C(X) \times \hat{K}_{1, \rho'} \). Moreover, given \( g \in C_c(U) \) and \( \xi^{-1} g \) from where \( \psi_3(M_{\rho}) \) is dense in \( M_{\rho'} \), and so, since \( fm = km \) for each \( m \in M_{\rho} \) then \( \psi_3(f)n = \psi_3(k)n \) for every \( n \in M_{\rho'} \). Therefore \((\psi_3(f), \psi_3(k)) \) is a redundancy. Since \( \xi^{-1} g \) is dense in \( M_{\rho'} \), and so, since \( f \in \ker(\varphi_{\rho})^\perp \), by the previous lemma, \( \psi_3(f) \in \ker(\varphi_{\rho'})^\perp \). Then, since \((\psi_3(f), \psi_3(k)) \) is a redundancy of \( \mathcal{T}(X, \alpha, L) \) then by \([3:2.6]\), \( \psi_3(f) \in \varphi^{-1}(K(M_{\rho'})) \). So \( \psi_3(f) \in \ker(\varphi_{\rho'})^\perp \cap \varphi_{\rho'}^{-1}(K(M_{\rho})) \). This shows that if \( \phi \) is the quotient *-homomorphism from \( \mathcal{T}(X, \alpha, L) \) in \( \mathcal{O}(X, \alpha, L) \) then \( \phi \circ \psi_3 : \mathcal{T}(X, \alpha, L_{\rho}) \to \mathcal{O}(X, \alpha, L) \) is a homomorphism which vanishes on all the elements of the form \((a - k)\) where \((a, k)\) is a redundancy and \( a \in \varphi_{\rho}^{-1}(K(M_{\rho})) \cap \ker(\varphi_{\rho})^\perp \). So we obtain a *-homomorphism
\[
\psi : \mathcal{O}(X, \alpha, L_{\rho}) \to \mathcal{O}(X, \alpha, L_{\rho'})
\]
\[
f \mapsto f
\]
\[
\tilde{g} \mapsto \tilde{\xi}g
\]

In the same way we may define the *-homomorphism
\[
\psi_0 : \mathcal{O}(X, \alpha, L_{\rho'}) \to \mathcal{O}(X, \alpha, L_{\rho})
\]
\[
f \mapsto f
\]
\[
\tilde{g} \mapsto \tilde{\xi}^{-1}g
\]

Note that \( \psi_0 \) is the inverse of \( \psi \), showing that the algebras are *-isomorphic.

\( \square \)

**Corollary 2.4** If \( \rho \in C(U) \) is a positive function such that \( \rho(x) \neq 0 \) for all \( x \in U \) then \( \mathcal{O}(X, \alpha, L_{\rho}) \) is *-isomorphic to \( \mathcal{O}(X, \alpha, L) \).
Proof.

Note that the transfer operator $L$ associated to the algebra $O(X, \alpha, L)$ is the operator $L_{1_U}$.

Since $\rho = 1_U$ is invertible, taking $r = \rho^{-1}$, by the previous proposition follows the corollary.

□

3 Relationship between the Covariance Algebra and the Crossed Product by Partial Endomorphism

We show here that given a partial dynamical system $\sigma : U \to X$, where $U$ is clopen, there exists an other partial dynamical system $\tilde{\sigma} : \tilde{U} \to \tilde{X}$ (called in [4] the $\sigma$-extension of $X$) such that $C^*(\sigma, X) = O(\tilde{X}, \alpha, L)$. Moreover, if $\sigma$ is injective then $C^*(\sigma, X) = O(X, \alpha, L)$.

3.1 The Covariance Algebra as an Crossed Product by a Partial Endomorphism

Let us start with a summary of the construction of the covariance algebra. Let $\sigma : U \to X$ a continuous map, $U \subseteq X$ clopen, $X$ compact Hausdorff and $\sigma(U)$ open. Denote $\sigma(U) = U_{-1}$.

Consider the space $X \cup \{0\}$, where $\{0\}$ is a symbol, which we define to be clopen. So $X \cup \{0\}$ is a compact and Hausdorff space.

Define $\tilde{X} \subset \prod_{i=0}^{\infty} X \cup \{0\}$,

$$\tilde{X} = \bigcup_{N=0}^{\infty} X_N \cup X_\infty$$

onde

$$X_N = \{(x_0, x_1, ..., x_N, 0, 0, ...) : \sigma(x_i) = x_{i-1} \text{ e } x_N \notin U_{-1}\}$$

and

$$X_\infty = \{(x_0, x_1, ...) : \sigma(x_i) = x_{i-1}\}.$$

In $\tilde{X}$ we consider the product topology induced from $\prod X \cup \{0\}$.
By [4: 2.2] \( \widetilde{X} \) is compact. Define

\[
\Phi : \quad \widetilde{X} \to X
\]

\[
(x_0, x_1, ...) \mapsto x_0
\]

which is continuous and surjective. Consider the clopen subsets \( \widetilde{U} = \Phi^{-1}(U) \) and \( \widetilde{U}_{-1} = \Phi^{-1}(U_{-1}) \) and the continuous map

\[
\tilde{\sigma} : \quad \widetilde{U} \to \widetilde{U}_{-1}
\]

\[
(x_0, x_1, ...) \mapsto (\sigma(x_0), x_0, x_1, ...)
\]

Those maps satisfies the relation

\[
\Phi(\tilde{\sigma}(\tilde{x})) = \sigma(\Phi(\tilde{x})).
\]

Note that \( \tilde{\sigma} \) is in fact an homeomorphism. This homeomorphism induces the *-isomorphism

\[
\theta : C(\widetilde{U}_{-1}) \to C(\widetilde{U})
\]

\[
f \mapsto f \circ \tilde{\sigma}
\]

So we may consider the partial crossed product \( C(\widetilde{X}) \rtimes_\theta \mathbb{Z} \) (see [1]).

**Definition 3.1 (4: 4.2)** The covariance algebra associated to the partial system \((X, \sigma)\) is the algebra \( C(\widetilde{X}) \rtimes_\theta \mathbb{Z} \) and will be denoted \( C^*(X, \sigma) \).

**Lemma 3.2** If \( \sigma : U \to X \) is injective, \( U \) clopen and \( U_{-1} \) open then \( C(X) \rtimes_\theta \mathbb{Z} = \mathcal{O}(X, \alpha, L) \), where \( \theta : C(U_{-1}) \to C(U) \) is given by \( \theta(f) = f \circ \sigma \).

**Proof.**

Define \( \psi_1 : C(X) \cup M \to C(X) \rtimes_\theta \mathbb{Z} \) by \( \psi_1(f) = f \delta_0 \) and \( \psi_1(1_{U}) = 1_U \delta_1 \). It is easy to check that \( \psi_1 \) extends to \( T(X, \alpha, L) \). We show that \( \Psi_1 \) vanishes on the redundancies.

Let \((f, k)\) redundancy with \( f \in \ker(\varphi)^{\perp} \cap \varphi^{-1}(\mathbb{K}(M)) \). By [3 2.6], \( f \in C(U) \). Then \( \psi_1(f) \psi_1(1_U) = f \delta_1 1_{U_{-1}} \delta_{-1} = \theta(\theta^{-1}(f) 1_{U_{-1}}) \delta_0 = \psi_1(f) \). Take \((k_n) \in \mathbb{K}_1 \), \( k_n = \sum_{i} m_{ni} l_{ni} \).
where \( m_{ni}, l_{ni} \in M \). Then

\[
(\psi_1(f) - \psi(k))(\psi_1(f) - \psi(k))^* = (\psi_1(f) - \psi_1(k))\psi_1(f - k) = \psi_1(f - k)(\psi_1(\tilde{1}_U) - \psi_1(k))^* = \\
= \psi(f - k)(\tilde{1}_U \tilde{f}^* - k) = \lim_{n \to \infty} \psi(f - k)(\tilde{1}_U \tilde{f}^* - k_n) = 0.
\]

The last equality follows by the fact that \((f - k)m = 0\) for each \( m \in M \). So, by passage to the quotient we may consider \( \psi : \mathcal{O}(X, \alpha, L) \to C(X) \rtimes_{\theta} \mathbb{Z} \). By the other hand, define

\[
\psi_0 : C(X) \to \mathcal{O}(X, \alpha, L) \quad f \mapsto f
\]

which is a *-homomorphism. Note that for each \( f \in C(U_{-1}) \),

\[
\tilde{1}_U \psi_0(f)\tilde{1}_U^* = 1_U \alpha(f) \tilde{1}_U^* = 1_U \alpha(f) = \theta(f) = \psi_0(\theta(f))
\]

and moreover \( \tilde{1}_U \) is a partial isometry such that \( \tilde{1}_U \tilde{1}_U^* = 1_U \) and \( \tilde{1}_U^* \tilde{1}_U = 1_{U_{-1}} \). Then, since \((\psi_0, \tilde{1}_U)\) is a covariant representation of \( C(X) \) in \( \mathcal{O}(X, \alpha, L) \), there exists a *-homomorphism \( \psi' : C(X) \rtimes_{\theta} \mathbb{Z} \to \mathcal{O}(X, \alpha, L) \) such that \( \psi'(f \delta_n) = f \tilde{1}_U^n \) (see [1, 5]). The *-homomorphisms \( \psi \) and \( \psi' \) are inverses of each other, and so the algebras are *-isomorphic.

\[\square\]

**Corollary 3.3** \( C^*(X, \sigma) = \mathcal{O}(\tilde{X}, \alpha, L) \)

**Proof.**

Follows by the definition of covariance algebras and by the previous lemma.

\[\square\]

By the following proposition, if \( \sigma \) is injective then the constructions of covariance algebra and crossed product by partial endomorphism are equivalent.

**Proposition 3.4** If \( \sigma : U \to X \) is injective then \( C^*(X, \sigma) = \mathcal{O}(X, \alpha, L) \).

**Proof.**
By [4: 2.3] the map
\[ \Phi : \tilde{X} \rightarrow X \]
\[(x_0, x_1, ...) \mapsto x_0 \]
is a homeomorphism. Moreover, since \( \Phi \circ \tilde{\sigma} = \sigma \circ \Phi \) then \( C(\tilde{x}) \times_{\theta} \mathbb{Z} = C(x) \times_{\theta} \mathbb{Z} \). By the previous lemma \( C(x) \times_{\theta} \mathbb{Z} = \mathcal{O}(X, \alpha, L) \).

\[ \square \]

### 3.2 Cuntz-Krieger algebras

We show examples of partial dynamical system \( \sigma_A : U \rightarrow X_A \) such that there does not exist an invertible function \( \rho \in C(U) \) such that \( \mathcal{O}(X, \alpha, L_\rho) \) and \( C^*(X, \alpha) \) are *-isomorphic.

The examples are based on the Cuntz-Krieger algebras.

Let \( A \) be a \( n \times n \) matrix with \( a_{i,j} \in \{0, 1\} \). Denote by \( Gr(A) \) the directed graph of \( A \), that is, the vertex set is \( \{1, ..., n\} \) and \( A(i,j) \) is the number of oriented edges from \( i \) to \( j \). A path is a sequence \( x_1, ..., x_m \) such that \( A(x_i, x_{i+1}) = 1 \) for each \( i \). The graph \( Gr(A) \) is transitive if for each \( i \) and \( j \) there exists a path from \( i \) to \( j \), that is, a path \( x_1, ..., x_m \) such that \( x_1 = i \) and \( x_m = j \). The graph is a cycle if for each \( i \) there exists only one \( j \) such that \( A(i,j) = 1 \).

Let

\[ X_A = \{ x = (x_1, x_2, ..) \in \{1, ..., n\}^\mathbb{N} : A(x_i, x_{i+1}) = 1 \forall i \} \subseteq \{1, ..., n\}^\mathbb{N} \]

and

\[ \sigma_A : X_A \rightarrow X_A \]
\[(x_0, x_1, ...) \mapsto (x_1, x_2, ...). \]

Consider the set

\[ \overline{X_A} = \{(x_i)_{i \in \mathbb{Z}} \in \{1, ..., n\}^\mathbb{Z} : A(x_i, x_{i+1}) = 1 \forall i \} \subseteq \{1, ..., n\}^\mathbb{Z} \]

and the map \( \overline{\sigma_A} : \overline{X_A} \rightarrow \overline{X_A} \) defined by \( \overline{\sigma_A}(i)_{i \in \mathbb{Z}} = (x_{i+1})_{i \in \mathbb{Z}} \). It is showed in [4: 2.8] that there exists a homeomorphism \( \Phi : \overline{X_A} \rightarrow \overline{X_A} \) such that \( \Phi \circ \overline{\sigma_A} = \overline{\sigma_A} \circ \Phi \). Therefore \( \mathcal{O}(\overline{X_A}, \alpha, L) = \mathcal{O}(\overline{X_A}, \alpha, L) \) and so \( C^*(X_A, \sigma_A) = \mathcal{O}(\overline{X_A}, \alpha, L) \). So we may analyze the ideal structure of \( C^*(X_A, \sigma_A) \) by using the theory developed for \( \mathcal{O}(\overline{X_A}, \alpha, L) \) in [3]. This theory
is based on the $\sigma_A,\sigma_A^{-1}$ invariant open subsets of $X_A$. (In a system $\sigma: U \to X$, a subset $V \subseteq X$ is $\sigma,\sigma^{-1}$ invariant if $\sigma(U \cap V) \subseteq V$ and $\sigma^{-1}(V) \subseteq V$).

**Proposition 3.5** If $Gr(A)$ is transitive and is not a cicle then there exists at least one open non trivial $\sigma_A,\sigma_A^{-1}$ invariant subset of $X_A$.

**Proof.**

Let $r = x_1, x_2, \ldots, x_n$ an admissible word (that is, $A(x_i, x_{i+1}) = 1$ for each $i$). Let $V_r = \{x \in X_A : r \in x\}$. Note that $V_r$ is open and $\sigma_A, \sigma_A^{-1}$ invariant. We show that there exists a such non trivial $V_r$. Take $x_1 \in \{1, \ldots, n\}$. Consider an admissible word $x_1, \ldots, x_m$ where $x_j \neq x_1$ for each $j > 1$ and $A(x_m, x_1) = 1$. Such word exists because $Gr(A)$ is transitive. Let $r = x_1, \ldots, x_m, x_1$. Then

$$y = (\ldots, x_m, x_1, x_2, \ldots, x_m, x_1, x_2, \ldots) \in V_r$$

where $x_1^\circ$ means $y_0 = x_1$.

We conclude the proof by showing that $V_r \neq X_A$. Suppose that there exists $y_0 \in \{1, \ldots, n\}$ with $y_0 \notin \{x_1, \ldots, x_m\}$. Let $x_1, y_1, \ldots, y_l, y_0, s_1, \ldots, s_l$ an admissible word such that $y_j \neq x_1$ and $s_j \neq x_1$ for each $j$ and $A(s_l, x_1) = 1$. Then

$$(\ldots, s_l^\circ, x_1, y_1, \ldots, y_l, y_0, s_1, \ldots, s_l, x_1, \ldots) \notin V_r.$$ 

If $\{x_1, \ldots, x_m\} = \{1, \ldots, n\}$, since $Gr(A)$ is not a cicle, for some $x_i$ there exists $x_t$ such that $A(x_i, x_t) = 1$ and $x_t \neq x_{i+1}$. (if $i = m$ consider $x_{i+1} = x_1$). If $x_t = x_1$ (and so $i \neq m$) consider an admissible word $x_1, \ldots, x_i, x_1$ and note that

$$(\ldots, x_i^\circ, x_1, x_2, \ldots, x_i, x_1, \ldots) \notin V_r.$$ 

If $x_t \neq x_1$ consider an admissible word $x_1, x_2, \ldots, x_i, x_t, y_1, \ldots, y_l$ such that $y_j \neq x_1$ and $A(y_l, x_1) = 1$ (if there does not exists $y_1 \neq x_1$ such that $A(x_t, y_1) = 1$ then $y_1, \ldots, y_l$ is the empty word) and so

$$(\ldots, y_l^\circ, x_1, x_2, \ldots x_i, x_t, y_1, \ldots, y_l, y_1, \ldots) \notin V_r.$$ 

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So $V_t \neq X_A$.

Now we analyse the $\sigma_A, \sigma_A^{-1}$ invariant subsets of $X_A$.

**Proposition 3.6** If $Gr(A)$ is transitive and is not a cicle the the unique open $\sigma_A$-invariant subset of $X_A$ are $\emptyset$ and $X_A$.

**Proof.**
Let $V \subseteq X_A$ an open nonempty $\sigma_A$ invariant susbet of $X_A$. Let $x \in V$ and $V_m$ a open neighbourhood of $x$, $V_m \subseteq V$,

$$V_m = \{ y \in X_A : x_i = y_i \text{ for each } 1 \leq i \leq m \}.$$ 

Given $z \in X_A$ take $r = r_1, ..., r_t$ a path from $x_m$ to $z_1$. Then

$$s = (x_1, ..., x_m, r_2, ..., r_{t-1}, z_1, z_2, ...) \in V_m$$

and since $V$ is $\sigma_A$ invariant then $z = \sigma_A^{m+t-2}(s) \in V$. So $V = X_A$.

□

According [3] a partial dynamical system $\sigma : U \to X$ is topologically free if the closure of $V^{i,j} = \{ x \in U : \sigma^i(x) = \sigma^j(x) \}$ has empty interior for each $i, j \in \mathbb{N}, i \neq j$.

**Proposition 3.7** If $Gr(A)$ is transitive and is not a cicle then $(X_A, \sigma_A)$ is topologically free.

**Proof.**
Suppose that $\overline{V^{i,j}}$ has nonempty interior and $i < j$, $j = i + k$. Let $x'$ be an interior point of $\overline{V^{i,j}}$ and $V_{x'} \subseteq \overline{V^{i,j}}$ open neighbourhood of $x'$. Take $x \in V^{i,j} \cap V_{x'}$. Since $\sigma_A^i(x) = \sigma_A^j(x)$ then $z_i + t = z_{j + t}$ for each $t \in \mathbb{N}$ and since $j = i + k$ then $x = (x_1, ..., x_{i-1}, r, r, ...)$ where $r = x_i x_{i+1} ... x_{i+k-1}$. Consider the open subset

$$V_m = \{ z \in X_A : z_i = x_i, 1 \leq i \leq m \}$$

where $m$ is such that $m \geq i + k$ and $V_m \subseteq V_{x'}$. Then, if $y \in V_m$ with $y \in V^{i,j}$ then $y = x$.  

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Therefore $V_m = \{x\}$. We show that there exists $z \in V_m$ with $z \neq x$, and that will be a contradiction. Suppose $y_0 \in \{1, ..., n\}$ and $y_0 \notin \{x_i, ..., x_{i+k-1}\}$. Take a path $s = s_1, ..., s_t$ from $x_i$ to $x_{i+k-1}$ such that $s_j = y_0$ for some $j$. Then $z = (x_1, ..., x_{i-1}, r, r, ..., r, s, s, ...) \in V_m$ (where $r$ is repeated $m$ times) but $z \neq x$. Suppose $\{1, ..., n\} = \{x_i, ..., x_{i+k-1}\}$. Since $\text{Gr}(A)$ is not a cycle then for some $x_j$ there exists $x_t \neq x_{j+1}$ (consider $x_{j+1} = x_i$ if $j = i+k-1$) such that $A(x_j, x_t) = 1$. Let $s$ be a path from $x_t$ to $x_{i+k-1}$ and define $p = x_i, ..., x_j, x_t$. Then

$$z = (x_1, ..., x_{i-1}, r, r, ..., r, x_i, ..., x_{j}, x_t, p, p, p, ...,) \in V_m$$

(where $p$ is repeated $m$ times) but $z \neq x$. So, it is showed that there exists $z \in V_m, z \neq x$. Therefore, $V^i_j$ has empty interior for each $i, j$. 

\[\square\]

**Theorem 3.8** If $\text{Gr}(A)$ is transitive and is not a cycle then $C^*(X_A, \sigma_A)$ and $O(X_A, \alpha, L)$ are not *-isomorphic $C^*$-algebras.

**Proof.**

By [3.2] $C^*(X_A, \sigma_A) = O(\overline{X_A}, \alpha, L)$ and since $O(\overline{X_A}, \alpha, L) = O(\overline{X_A}, \alpha, L)$ then $C^*(X_A, \sigma_A) = O(\overline{X_A}, \alpha, L)$. By [3.5] $X_A$ has at least one non trivial open $\sigma_A, \sigma_A^{-1}$ invariant subset and by [3.9] $O(\overline{X_A}, \alpha, L)$ has at least on non trivial ideal. On the other hand, by [3.6] $(X_A, \sigma_A)$ has no open $\sigma_A, \sigma_A^{-1}$ invariant subsets and by [3.7] $(X_A, \sigma_A)$ is topologically free. By [3.8] $O(X_A, \alpha, L)$ is simple. So $C^*(X_A, \sigma_A)$ and $O(X_A, \alpha, L)$ are not *-isomorphic.

\[\square\]

**Corollary 3.9** If $\text{Gr}(A)$ is transitive and is not a cycle then there does not exists transfer operator $L_{\rho}$, with $\rho(x) \neq 0$ for each $x \in U$ such that $C^*(X_A, \sigma_A)$ and $O(X_A, \alpha, L)$ are *-isomorphic $C^*$-algebras.

**Proof.**

Follows by the previous theorem an by [2.4] 

\[\square\]
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