FLAT PLUMBING BASKET AND CONTACT STRUCTURE

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Abstract. A flat plumbing basket is a Seifert surface consisting of a disk and bands contained in distinct pages of the disk open book decomposition of the 3-sphere. We generalize a notion of flat plumbing basket in general open books and examine close connection between flat plumbing baskets and the contact structure supported by the open book. As an application we give lower bounds for the flat plumbing basket numbers and determine the flat plumbing basket numbers for various knots and links, including the torus links.

1. Introduction

In [7] Furihata-Hirasawa-Kobayashi showed that every oriented link in $S^3$ bounds a certain special Seifert surface which they called a flat plumbing basket. A surface $F$ in $S^3$ is a flat plumbing basket if it is obtained from a disk by plumbing unknotted and untwisted annuli so that the gluing regions are contained in the disk. We call a flat plumbing basket $F$ whose boundary is $L$ a flat plumbing basket presentation of $L$.

Equivalently, a flat plumbing basket can be seen as a surface which consists of a page of the disk open book decomposition of $S^3$ and finitely many bands contained in distinct pages of the open book. In this point of view, it is quite natural to expect a connection between flat plumbing baskets and contact geometry since an open book decomposition is a decomposition of contact 3-manifolds.

In this paper, we discuss a connection between flat plumbing basket presentations and contact geometry. We show that a flat plumbing basket presentation can be seen as Legendrian and transverse link representatives in several natural ways. Actually, we will generalize a notion of a flat plumbing basket for general open books, and we show that a flat plumbing basket $F$ is a convex surface that respects the structure of band-plumbing decomposition of $F$.

As an application of contact geometry point of view, we give various lower bounds for the flat plumbing basket number $fpbk(L)$, the minimal number of bands which is needed to obtain a flat plumbing basket presentation of the link $L$ [11]. Let $\overline{L}$ be the mirror image of $L$. Then we obtained the following lower bounds for $fpbk(L)$.

**Theorem 1.1.** Let $L$ be a non-trivial oriented link in $S^3$. Then $fpbk(L) \geq \max\{-sl(L), -sl(\overline{L})\} - 1$.

**Theorem 1.2.** Let $L$ be a non-trivial oriented link in $S^3$. Then $2fbpk(L) \geq \max\{-tb(L), -tb(\overline{L})\} + b(L)$.

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Here $\text{sl}(L)$ and $\text{tb}(L)$ are the maximal self-linking number and the maximal Thurston-Bennequin number of $L$, and $b(L)$ denotes the braid index of $L$. It turns out these lower bounds are quite useful. We determine the flat plumbing basket number for various knots, including the $(p, q)$-torus link $T_{p,q}$ (Example 3.8).

This paper is organized as follows: In Section 2, we recall the definition of flat plumbing basket presentation of a link in $S^3$.

In Section 3, we show that a flat plumbing basket presentation can be viewed as a front projection of a Legendrian link $L_F$. By considering the transverse push-off of $L_F$ we prove Theorem 1.1 (Theorem 3.2). We also point out that a flat plumbing basket presentation can be seen as a closed braid, which gives a simple and direct alternative proof of Theorem 1.1. We then study when the lower bounds in Theorem 1.1 is sharp, and determine the flat plumbing basket number for various knots.

In Section 4, we discuss another way to relate a flat plumbing basket presentation and a Legendrian link. We explain how to view a flat plumbing basket presentation as an arc presentation and how to get the corresponding grid diagram. This gives a different legendrian link $L_G$, which leads to Theorem 1.2.

A reader who is only interested in knots and links in $S^3$ and the disk open book can skip Section 5. In Section 5 we generalize a notion of flat plumbing basket for a general open book decomposition of general (contact) 3-manifolds. Then we show that not only its boundary $L = \partial F$, a flat plumbing basket $F$ itself is closely related to the contact geometry: we show that $F$ is a convex surface whose dividing set divides $F$ into a page and bands in an obvious way. This point view allows us to generalize Theorem 1.1 for null-homologous link general open books (Theorem 5.4).

Section 6 contains tables of the flat plumbing basket numbers for prime knots with less than or equal to 9 crossings, improving a previous table [9, Table 1]. We determine the flat plumbing basket numbers for these knots, with 6 exceptions.

This paper is an expanded and improved version of second author’s preprint [18].

2. Flat plumbing basket in $S^3$

Let $U$ be the oriented unknot in $S^3$ and $\pi: S^3 \setminus U \to S^1 = [0, 2\pi]/(0 \sim 2\pi)$ be the disk fibration of its complement. We call the pair $(U, \pi)$ the disk open book decomposition of $S^3$. The closure a fiber $D_\theta := \pi^{-1}(D_\theta)$ is called a page, and $U$ is called the binding. (see Figure 1).

A Seifert surface $F$ is a flat plumbing basket if there are finitely many disjoint bands $B_1, \ldots, B_n$ and $0 < \theta_1 < \cdots < \theta_n < 2\pi$ such that $F = D_{\theta_1} \cup B_1 \cup \cdots \cup B_n$, each band $B_i$ is contained in $D_{\theta_i}$ and $B_i \cap U$ consists of two arcs, where $n = b_1(F)$.

We call a flat plumbing basket $F$ whose boundary is an oriented link $L$ a flat plumbing basket presentation of $L$ (see Figure 2).

The flat plumbing basket number $fpbk(L)$ of an oriented link $L$ is the minimal number of bands among all flat plumbing basket presentation of $L$. Namely,

$$fpbk(L) := \min \{ b_1(F) \mid F \text{ is a flat plumbing basket presentation of } L \},$$

where $b_1(F)$ is the first betti number of $L$. We remark that $fpbk(L) \in 2\mathbb{Z}_{\geq 0} + |L| - 1$, where $|L|$ is the number of the components of $L$, and $fpbk$ is preserved under taking mirror image.
FIGURE 1. Open book decomposition

FIGURE 2. An example of a flat plumbing basket \( F \). It is a flat plumbing basket presentation of the negative trefoil.

3. Flat plumbing basket as front projection

Let \( F \) be a flat plumbing basket with bands \( B_1, \ldots, B_n \). We draw \( F \) as a union of bands and a rotated rectangle, where the bands are attached to the rectangle at the top left edge (see the middle picture in Figure 3). Moreover, we draw each band so that

- it is sufficiently thin and
- the core of the band consists of three line segments: one is parallel to the top left edge of the rectangle and the other two are vertical to the top left edge.

Let \( h_i \) be the distance between the top left edge of the rectangle and the line segment of the core of \( B_i \) parallel to the edge. We deform each band so that \( h_i > h_j \) if \( i > j \). Then, we obtain a diagram of \( L = \partial F \) which satisfies that the strand with small slope is in front at each crossing. After replacing the local maxima and local minima in the horizontal coordinate with cusps, and smoothing the local maxima and local minima in the vertical coordinate, we obtain a front projection of a Legendrian link in the standard contact structure \( \xi_{std} \) on \( S^3 \) (see the right picture in Figure 3 – This picture answers [18, Question 6.2]).

We denote the Legendrian link by \( L_F \) and call it the Legendrian link associated with \( F \).
Figure 3. How to draw a front projection from a flat plumbing basket: For a flat plumbing basket $F$, we draw $F$ as the middle picture, so that $h_i > h_j$ holds if $i > j$ (the middle figure). With suitable modification such a diagram can be naturally regarded as a front projection of a Legendrian link $\mathcal{L}_F$ (the right picture).

One can immediately read the Thurston-Bennequin number and the rotation number of $\mathcal{L}_F$ from the front projection.

**Lemma 3.1.** Let $F$ be a flat plumbing basket with $b_1(F) > 0$ and $\mathcal{L}_F$ be the Legendrian link associated with $F$. Then we obtain

\begin{align*}
(3.1) & \quad tb(\mathcal{L}_F) = -2b_1(F), \\
(3.2) & \quad rot(\mathcal{L}_F) = -b_1(F) + 1.
\end{align*}

**Proof.** By construction of the front projection, the front projection of $\mathcal{L}_F$ has $2b_1(F)$ right cusps and the sum of signs of crossings is zero. Hence, we have $tb(\mathcal{L}_F) = 0 - 2b_1(F) = -2b_1(F)$.

The front projection of $\mathcal{L}_F$ has $2b_1(F) - 1$ up right cusps and $1$ down right cusp, and the numbers of up and down left cusps are equal so $rot(\mathcal{L}_F) = \frac{1}{2}(1 - (2b_1(F) - 1)) = -b_1(F) + 1$. \hfill \square

Let $\mathcal{T}_F$ be the transverse push-off of $\mathcal{L}_F$. By Lemma 3.1 we have

\begin{equation}
(3.3) \quad sl(\mathcal{T}_F) = tb(\mathcal{L}_F) - rot(\mathcal{L}_F) = -b_1(F) - 1.
\end{equation}

Let $\overline{sl}(L)$ is the maximal self-linking number of topological link $L$, that is,

$$
\overline{sl}(L) = \max\{sl(T) \mid T \text{ is a transverse link topologically isotopic to } L\}.
$$

For a flat plumbing basket presentation $F$ of $L$, we have obvious inequalities $\overline{sl}(L) \geq sl(\mathcal{T}_F)$ and $fpbk(L) \leq b_1(F)$. Also, $fpbk(L) = fpbk(\overline{L})$ where $\overline{L}$ denotes the mirror image of $L$. Hence we obtain the following.
Theorem 3.2 (Theorem 1.1). Let $L$ be a non-trivial oriented link in $S^3$. Then we have

$$fpbk(L) \geq \max\{-sl(L), -\overline{sl}(L)\} - 1.$$

Here is an alternative, direct proof of Theorem 3.2 which is interesting in its own right. For a flat plumbing basket $F$, we view $L = \partial F$ as a closed $(b_1(F)+1)$-braid by viewing each band as a union of disk and two twisted bands with opposite twisting (see Figure 4). This closed braid provides a transverse link $T_{br}$ with $sl(T_{br}) = -b_1(F) - 1$.

![Figure 4. Flat plumbing basket, viewed as closed $(b_1(F)+1)$-braid](image)

Remark 3.3. The transverse link $T_{br}$ is transverse isotopic to $T_F$, the transverse push-off of $L_F$. This can be checked, for example, by converting the arc presentation from the front projection $L_F$ into a closed braid representative, as we discuss in the next section.

Corollary 3.4. Let $L$ be an oriented link. Then, we have

$$\maxdeg_{v} P_{L}(v, z) \leq fpbk(L).$$

Proof. The HOMFLYPT bound on the self-linking number (the Morton-Frank Williams inequality [6, 14]) implies

$$\maxdeg_{v} P_{L}(v, z) \leq -sl(L) - 1.$$

Hence by Theorem 3.2, we finish the proof. \(\square\)

3.1. Applications and discussions. It turns out Theorem 3.2 is quite useful, especially in the case the link $L$ is a closure of a positive braid.

Let $b(L)$ the braid index of $L$, the minimum number of strands needed to represent $L$ as a closed braid. Let $\beta$ be a $b(L)$-braid whose closure is $L$. By [4],[13] a closed braid representative of the minimum braid index always attains the maximal self-linking number so we have

$$\overline{sl}(L) = -b(L) + e(\beta)$$

and

$$\overline{sl}(\overline{L}) = -b(L) - e(\beta),$$

where $e(\beta)$ denotes the exponent sum of the braid $\beta$. Consequently, we have

$$-\overline{sl}(L) - \overline{sl}(\overline{L}) = 2b(L).$$

On the other hand, by Bennequin’s inequality we have

$$\overline{sl}(L) \leq -\chi(L),$$
where \( \chi(L) \) be the maximum Euler characteristic of Seifert surface of \( L \). Therefore we have the following.

**Corollary 3.5.** Let \( L \) be a non-trivial link such that the Bennequin inequality (3.5) is sharp, namely, \( \overline{sl}(L) = -\chi(L) \) holds. Then \( f_{pbk}(L) \geq -\chi(L) + 2b(L) - 1 \).

It is interesting to compare this bounds with other lower bounds of \( f_{pbk}(K) \). First of all, a flat plumbing basket \( F \) is always compressible with \( \chi(F) = -b_1(F) + 1 \) so we have a lower bound

\[
(3.6) \quad f_{pbk}(L) \geq -\chi(L) + 1.
\]

which we call the trivial genus bound. Hirose-Nakashima [9, Theorem 1.3] proved \( (3.7) \)

\[
\chi(L) = \deg \Delta_L(t) + 4, \quad \text{where } \Delta_L(t) \text{ denotes the Alexander polynomial of } L. \quad \text{We call (3.7) Hirose-Nakashima bound.}
\]

Since \( \deg \Delta_L(t) + 4 \leq -\chi(L) + 3 \) and \( b(L) \geq 2 \) for any non-trivial link \( L \), Corollary 3.5 gives a better estimate than the trivial genus bound (3.6) and Hirose-Nakashima bound (3.7) when the Bennequin inequality (3.5) is sharp.

An \( n \)-braid \( \beta \) is strongly quasipositive if \( \beta \) is a product of so-called the band generators

\[
a_{i,j} = (\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i)^{-1}\sigma_j(\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i) \quad (1 \leq i < j \leq n).
\]

A band generator can be viewed as a positively twisted band connecting the \( i \)-th and the \( j \)-th strands of the braid. Consequently, when \( L \) is a closure of a strongly quasipositive braid, we have a Seifert surface \( F \) of \( L \) with \( \overline{sl}(L) = -\chi(F) \) so Bennequin’s inequality (3.5) is sharp.

Hence Theorem 1.1 will be effective for strongly quasipositive links. In Tables 2 and 3, for prime knots \( K \) with less than or equal to 9 crossings we will see that \( -\overline{sl}(K) - 1 = f_{pbk}(K) \) if \( K \) is represented by the closure of a strongly quasipositive braid. This leads to the following question:

**Question 3.6.** If a link \( L \) is represented by the closure of a strongly quasipositive braid, is the inequality in Theorem 1.1 (equivalently, the inequality in Corollary 3.5) equality? Namely, for such \( L \), is it true that

\[
\overline{sl}(L) - 1 = -\chi(L) + 2b(L) - 1?
\]

As a partial answer, we show the following.

**Theorem 3.7.** If \( L \) is presented by the closure of a positive \( b(L) \)-braid, then

\[
f_{pbk}(L) = -\overline{sl}(L) - 1 = -\chi(L) - 1 + 2b(L).
\]

**Proof.** By Corollary 3.5 we have \( f_{pbk}(L) \geq -\overline{sl}(L) - 1 = -\chi(L) - 1 + 2b(L) \) so we check the converse inequality.

Let \( \beta \) be a positive \( b(L) \)-braid whose closure is \( L \) and \( \overline{\beta} \) be the mirror image of the braid \( \beta \). Then \( \overline{\beta} \) is a negative \( b(L) \)-braid whose closure is \( \overline{L} \). Therefore \( \overline{sl}(L) = -b(L) + e(\overline{\beta}) = -b(L) - e(\beta) \), where \( e(\beta) \) denotes the exponent sum of \( \beta \). Since \( \overline{\beta} \) is a negative braid, it is a product of \( e(\beta) \) standard negative generators \( \{\sigma_{1}^{-1}, \ldots, \sigma_{b(L)-1}^{-1}\} \). Thus the mirror image \( \overline{L} \) is a closure of a braid of the form

\[
(\sigma_{b(L)-1}\cdots\sigma_{1})(\sigma_{1}^{-1}\cdots\sigma_{b(L)-1}^{-1})\overline{\beta}
= (\sigma_{b(L)-1}\cdots\sigma_{1})(\text{braid word consisting of } e(\beta) + (b(L) - 1) \text{ negative generators}).
\]
By [7, Theorem 2.4], from such a closed braid representative we construct a flat plumbing basket presentation $F$ of $L$ with $b_1(F) = e(\beta) + b(L) - 1 = -\overline{sl}(T) - 1$. □

Although this do not fully answer Question 3.6, it can be used to determine the flat plumbing basket for the torus links.

**Example 3.8** (Torus links). For any $p \geq q > 1$, let $T_{(p,q)}$ be the $(p,q)$ torus knot. Since $b(T_{(p,q)}) = q$ and $T_{(p,q)}$ is the closure of $q$ braid $(\sigma_1 \cdots \sigma_{q-1})^{-p}$, by Theorem 3.7

$$fpbk(T_{(p,q)}) = fpbk(T_{(p,-q)}) = -\overline{sl}(T_{(p,-q)}) - 1 = pq - p + q - 1.$$  

**Example 3.9** (Twist knots). Let $K_m$ be the $m$-twist knot (Figure 5). By [5, Theorem 1.2] and Theorem 3.2

$$2k \leq fpbk(K_{2k}) \leq 2k + 4 \leq fpbk(K_{2k+1}).$$

Mikami Hirasawa showed that $fpbk(K_{2k+1}) \leq 2k + 4$ for $k \geq 0$, and $fpbk(K_{2k}) \leq 2k$ for $k \geq 3$ in his forthcoming paper. Hence

- $fpbk(K_{2k+1}) = 2k + 4$ for $k \geq 0$,
- $fpbk(K_{2k}) = 2k$ for $k \geq 3$,
- $fpbk(K_2) = 4$ and $fpbk(K_4) = 6$.

![Figure 5. The m-twist knot $K_m$](image)

The flat plumbing basket number is subadditive under the connected sum of knots but it is not additive in general. For example, Hirose-Nakashima [9, Remark 1.4(b)] proved that $fpbk(3_1) = fpbk(\overline{3_1}) = 4$ but $fpbk(3_1 \overline{3_1}) = 6$. Nao Kobayashi (Imoto) proved that $fpbk(6_1) = fpbk(\overline{6_1}) = 6$ but $fpbk(6_1 \overline{6_1}) = 8$ in [12, Proposition 5.4].

Theorem 3.2 gives some sufficient conditions for $fpbk$ is additive.

**Example 3.10.** For knots $K$ and $K'$, $\overline{sl}(K \sharp K') = \overline{sl}(K) + \overline{sl}(K') + 1$ holds. This shows that if $K_1, \ldots, K_n$ satisfies $fpbk(K_i) = -\overline{sl}(K_i) - 1$ then the following holds:

$$fpbk(K_1 \sharp K_2 \sharp \cdots \sharp K_n) = fpbk(K_1) + fpbk(K_2) + \cdots + fpbk(K_n).$$

For example, $fpbk(3_1 \sharp 3_1) = fpbk(3_1) + fpbk(3_1) = 8$.

4. Flat plumbing basket and arc presentation

In this section, we give another natural way to relate flat plumbing baskets and Legendrian links. For basics of arc presentation we refer [3]. See [16] for a relation to Legendrian and transverse links and arc presentation.

As the definition already suggests, a flat plumbing basket presentation can be naturally seen as an arc presentation as follows. We view a flat plumbing basket $F$ as a disk $D^2$ with $n = b_1(F)$ bands $B_1, \ldots, B_n$. By collapsing each connected
component of $\partial(F \setminus \bigcup_{i=1}^{n} B_i)$ we get an arc presentation $A_F$ with arc number $2n$ (See Figure 6). This immediately leads to the following.

**Theorem 4.1.** Let $\alpha(L)$ denotes the arc index of $L$. Then

$$2f_{pbk}(L) \geq \alpha(L).$$

Since $\alpha(L) = c(L) + 2$ for non-split alternating link $L$, where $c(L)$ denotes the crossing number of $L$, we have

**Corollary 4.2.** Let $L$ be a non-split alternating link. Then

$$2f_{pbk}(L) \geq c(L) + 2.$$

It seems likely that this inequality holds for general non-split links.

Unfortunately, this lower bound is not so good. To extract more detailed information, let us consider the grid diagram representative $G_F$ of the arc presentation $A_F$, which is obtained as follows. A part of $L = \partial F$ that comes from a boundary of a band $B_i$ appears as oppositely oriented vertical line segments in $G_F$ (Figure 7(i)). By connecting the endpoints of these vertical lines by horizontal lines, we get a grid diagram on the annulus (binding $U$) $\times [0, 1]$. To get a usual grid diagram on rectangle, finally we flip one vertical line so that it is contained in the rectangle (Figure 7(ii), and see also the middle and the rightmost picture of Figure 6).

Let $\mathcal{L}_G$ and $\mathcal{T}_G$ be the Legendrian and the transverse links represented by the Grid diagram $G_F$. The Thurston-Bennequin number and the rotation number of $\mathcal{L}_G$ are given by

$$tb(\mathcal{L}_G) = \text{writhe} - \frac{1}{2} \#(\vec{r}, \vec{\lambda}), \quad rot(\mathcal{L}_G) = \frac{1}{2} \left( \#(\vec{r}, \vec{\lambda}) - \#(\vec{r}, \vec{\mu}) \right)$$
where \( \#(\text{\scriptsize{□}}, \text{\scriptsize{□}}) \), \( \#(\text{\scriptsize{□}}, \text{\scriptsize{□}}) \), and \( \#(\text{\scriptsize{□}}, \text{\scriptsize{□}}) \), denotes the number of corners of the indicated shapes in the grid diagram \( G_F \).

For each band \( B_i \), there are \( 2^4 = 16 \) possibility of the orientations of horizontal lines from the endpoints of the pair of vertical segments. We classify the bands of \( F \) into the following five types \( A, B, C, D \) and \( E \), according to its contribution to the Thurston-Bennequin number and the rotation number, as we show in Table 1.

| Type | local picture | \( tb \) | rot |
|------|---------------|--------|-----|
| A    | ![A](image1)  | 0      | 1   |
| B    | ![B](image2)  | \(-\frac{3}{2}\) | \(-\frac{1}{2}\) |
| C    | ![C](image3)  | \(-\frac{3}{2}\) | \(-\frac{1}{2}\) |
| D    | ![D](image4)  | \(-\frac{3}{2}\) | \(-\frac{1}{2}\) |
| E    | ![E](image5)  | \(-2\)  | \(-1\) |

Table 1. Five types of local picture of grid diagram near the bands
Let \( A, B, C, D, E \) be the number of bands of type \( A, B, C, D, E \) respectively. Then \( A + B + C + D + E = b_1(F) \) holds. Since vertical segment of the grid diagram \( G_F \) appears in pair with opposite orientation, the other crossing does not affect the writhe. Therefore we have

\[
\begin{align*}
(4.1) \quad & \text{tb}(L_G) = -\frac{1}{2} B - C - \frac{3}{2} D - 2E, \\
(4.2) \quad & \text{rot}(L_G) = A + \frac{1}{2} B - \frac{1}{2} D - E + 1, \\
(4.3) \quad & \text{sl}(T_G) = \text{tb}(L_G) - \text{rot}(L_G) = -(A + B + C + D + E) - 1 = -b_1(F) - 1.
\end{align*}
\]

Here the last +1 in the rotation number comes from a final modification (Figure 7(ii)) to get a Grid diagram. We remark that we can also prove Theorem 1.1 by the equation (4.3).

Let \( H \) be the number of horizontal lines in the grid diagram \( G_D \) which are oriented from right to left. Then we get a closed \( H \)-braid representing \( T_G \) by flipping the horizontal lines oriented from right to left (see Figure 8). By definition of type \( A, B, C, D, E \), we have

\[
(4.4) \quad H = \frac{1}{2}(4A + 3B + 2C + D) = b_1(F) + \text{rot}(L_G) - 1.
\]

Thus we have \( \text{tb}(L_G) = H - 2b_1(F) \). Let \( \overline{tb}(L) \) be the maximum Thurston-Bennequin number. Since \( H \geq b(L) \) where \( b(L) \) denotes the braid index of \( L \), we get the following estimate:

**Theorem 4.3.** For any oriented link \( L \), we obtain

\[
2\overline{fbpk}(L) \geq \max\{-\overline{tb}(L), -\overline{tb}(L^\prime)\} + b(L).
\]

It turns out Theorem 4.3 is quite useful, although it rarely attains the equality compared with the lower bound in Theorem 3.2. As we will see in Section 6, Theorem 4.3 determines the flat plumbing basket number for many knots. A key trick is that for a knot \( K \), \( fpk(K) \in 2\mathbb{Z} \) hence a weaker inequality like \( 2\overline{fbpk}(K) \geq 9 \) actually implies a stronger one \( \overline{fbpk}(K) \geq 6 \). It is interesting to find a condition for the inequality in Theorem 4.3 is an equality, or, to find a refinement of Theorem 4.3 that yields the equality more often.
5. Flat plumbing basket in general open book and convex surface

5.1. Flat plumbing basket in general open book. In this section we extend a notion of flat plumbing basket in a general open book.

An open book decomposition $(B, \pi)$ of an oriented closed 3-manifold $M$ is a pair consisting of a fibered link $B$ in $M$ and its fibration $\pi : M \setminus L \to S^1$. We call the link $B$ is called the binding. For $t \in S^1 = [0, 2\pi]/0 \sim 2\pi$, we denote the closure of the fiber $\pi^{-1}(t)$ by $S_t$ and we call it a page.

Let $(B, \pi)$ be an open book decomposition of $M$ and $L$ be an oriented null-homologous link in $M$.

Definition 5.1. A Seifert surface $F$ of $L$ is a flat plumbing basket with respect to an open book $(B, \pi)$ if $F$ is obtained from the page $S_0$ by attaching finitely many bands $B_1, \ldots, B_n$ so that each band lies in a page $S_{\theta_i}$ with $0 < \theta_1 < \cdots < \theta_n < 2\pi$. Then, we call $F$ a flat plumbing basket presentation of $L$ with respect to $(B, \pi)$ (f.p.b-presentation of $L$ w.r.t. $(B, \pi)$, for short).

An orientation-preserving diffeomorphism $f : M \to M$ is said to be fiber-preserving (with respect to the open book $(B, \pi)$) if $f(B) = B$ and $f(S_t) = S_{\Theta(t)}$ holds, where $\Theta : S^1 = [0, 2\pi]/0 \sim 2\pi \to [0, 2\pi]/0 \sim 2\pi = S^1$ is an orientation preserving diffeomorphism.

Two flat plumbing baskets $F$ and $F'$ with respect to the same open book $(B, \pi)$ are equivalent if there is a fiber-preserving diffeomorphism (with respect to $(B, \pi)$) $f : M \to M$ that sends $F$ to $F'$.

We call the subscript $i$ of $B_i$ the band $B_i$ label of the band. The core $\gamma_i$ of a band $B_i$ is a properly embedded arc in $S_{\theta_i}$. We denote by $[\gamma_i]$ the isotopy class of this arc and call it the isotopy label of the band.

Assume that the binding $B$ is an $\ell$-component link, and let us write its connected components by $C_1, \ldots, C_\ell$. By recording the labels of the bands and as one travel along each $C_i$, one obtain a cyclic word $W_F(i)$ over $\{1, \ldots, n\}$ where $n$ is the number of the bands of $F$. We call the pair $(W_F, H_F)$ consisting of the set of cyclic words $W_F = \{W_F(1), \ldots, W_F(\ell)\}$ and the sequence of isotopy labels $\{[\gamma_1], \ldots, [\gamma_n]\}$ the (extended) flat basket code for $F$.

By definition, the equivalence class of a flat plumbing basket is determined by its (extended) flat basket code.

Example 5.2. In a case open book is the disk open book decomposition of $(U, \pi)$ of $S^3$, the isotopy labellings are irrelevant so the cyclic word $W_F$ determines the equivalence class of a flat plumbing basket. The cyclic word $W_F$ is called a flat basket code for $F$ in [7].

5.2. Flat plumbing basket is a convex surface. Throughout this section we fix an open book decomposition $(B, \pi)$ of $M$, and we regard $M$ as a contact 3-manifold $(M, \xi)$ where $\xi$ is a contact structure supported by $(B, \pi)$. For basics of (3-dimensional) contact geometry we refer to [17].

A crucial property which we use is that we can take a contact structure $\xi$ supported by the open book $(B, \pi)$ so that each page $S_0$ is convex with dividing set equal to $B = \partial S_0$ (see [8, 19]).

Let $F$ be a flat plumbing basket presentation of $L$ with $n$ bands $B_1, \ldots, B_n$. We assume that $B_i \subset S_{\theta_i}$. We slightly perturb each page $S_{\theta_i}$ near the binding so that $S_0 \cup S_{\theta_i}$ forms a smooth and convex surface, with dividing set equal to $B$. After this modification $L = \partial F$ is smooth.
By definition, the boundary \( \partial F = L \) consists of two types of arcs – the boundary of bands \( B_i \) contained in a page and the subarc of the binding \( B \). We push each subarc of the binding \( \gamma \subset L \) into the page \( S_0 \) so that \( L = \partial F \) transverse to the binding. By applying Legendrian realization principle [10, Theorem 3.7] we realize each subarc of \( L \) contained in a page as a Legendrian arc, with respect to the contact structure \( \xi \). Since we have first arranged page \( S_{\theta_i} \) so that \( S_0 \cup S_{\theta_i} \) is smooth, these Legendrian arcs are connected smoothly hence we actually realize \( L = \partial F \) as a Legendrian link \( \mathcal{L}_F \) in a contact 3-manifold \((M, \xi)\).

**Remark 5.3.** In the case \((B, \pi)\) is the disk open book of the standard contact \( S^3 \), the Legendrian link \( \mathcal{L}_F \) is equal to the Legendrian link given by the front projection in Section 3.

Actually we achieved more: \( F \) is now a convex surface with Legendrian boundary, and its dividing set is identified with the attached regions of the bands. From this convex Seifert surface \( F \) one immediately reads the Thurston-Bennequin number and the rotation number of \( \mathcal{L}_F = \partial F \): we have

\[
\text{tb}(\mathcal{L}_F) = \frac{1}{2} \#(\partial F \cup \Gamma) = -2n
\]

and

\[
\text{rot}(\mathcal{L}_F, F) = \chi(F_+) - \chi(F_-) = \chi(S_0) - n,
\]

where \( F_+ \) (resp. \( F_- \)) is a union of component of \( F \setminus \Gamma \) such that the orientation of a contact vector field \( v \) and the normal orientation of \( F \) coincide (resp. opposite). In our settings, \( F_+ = S_0 \) and \( F_- = B_1 \cup \cdots \cup B_n \). Consequently, the self-linking number of the transverse push-off of \( \mathcal{L}_F \) (with respect to the surface \( F \)) is given by

\[
\text{sl}(\mathcal{T}_F) = \text{tb}(\mathcal{L}_F) - \text{rot}(\mathcal{L}_F) = -n - \chi(S_0).
\]

Let \( L \) be a null-homologous transverse link \( L \). We define maximal self-linking number by

\[
\overline{\text{sl}}(L) = \max \{ \text{sl}(L, \Sigma) \mid \Sigma \text{ is a Seifert surface of } L \}.
\]
Here when the maximum do not exists (say, \( L \) is a loose link in overtwisted contact 3-manifold) we define \( s\ell(L) = +\infty \). Also, we define the flat plumbing basket number of \( L \) (with respect to open book \((B, \pi)\)) by
\[
fpbk_{(B, \pi)}(L) = \min \{ \# \text{bands of } F \mid F \text{ is a f.p.b-presentation of } L \text{ w.r.t. } (B, \pi) \}.
\]
Here when \( L \) do not admit a flat plumbing basket presentation, we define \( fpbk_{(B, \pi)} = +\infty \).

Then we have the following inequality that generalizes Theorem 1.1.

**Theorem 5.4.** Let \( M \) be an oriented closed 3-manifold. Let \((B, \pi)\) be an open book decomposition of \( M \), \( S_0 \) be a page of \((B, \pi)\) and \( L \) be an oriented null-homologous link in \( M \). Then, we obtain
\[
fpbk_{(B, \pi)}(L) \geq -s\ell(L) - \chi(S_0).
\]

**Remark 5.5.** We remark that for any open book decomposition \((B, \pi)\) of \( S^3 \) and any oriented link \( L \) in \( S^3 \), we obtain
\[
fpbk(L) + 1 - \chi(S_0) \geq fpbk_{(B, \pi)}(L).
\]

We can prove this fact as follows: Draw the page \( S_0 \) of \((B, \pi)\) as a union of a rectangle and \( 1 - \chi(S_0) \) 1-handles as the left part of Figure 10. Give the 1-handles an order. Let \( B_i' \) be a band on \( S_0 \) which is properly embedded and whose core is the cocore of the \( i \)-th 1-handle of \( S_0 \). After pushing off each \( B_i' \) to distinct pages, \( S_0 \cup B_1' \cup \cdots \cup B_n' \) gives a flat plumbing basket presentation of the unknot with respect to \((B, \pi)\). Let \( D_0 \cup B_1 \cup \cdots \cup B_n \) be a flat plumbing basket presentation of \( L \) with respect to the disk open book (by [7], it always exists). Then,
\[
F = S_0 \#_0 D_0 \cup B_1' \cup \cdots \cup B_n' \cup B_1 \cup \cdots \cup B_n
\]
is a flat plumbing basket presentation of \( L \) with respect to \((B, \pi)\), where \( S_0 \#_0 D_0 \) is a boundary sum of \( S_0 \) and \( D_0 \) (see Figure 10). By taking \( n = fpbk(L) \), we have
\[
fpbk(L) + 1 - \chi(S_0) \geq fpbk_{(B, \pi)}(L).
\]

\[\text{Figure 10. A flat plumbing basket presentation of the negative trefoil knot with respect to a general open book}\]

6. Tabulation

By using Theorem 1.1 and Theorem 1.2 we improve the table of \( fpbk(K) \) given in [9, Table 1] as Tables 2 and 3.

In the table,
- The asterisks * are improved points.
• The double asterisks $\ast\ast$ are given by [2] and Mikami Hirasawa. In fact, Mikami Hirasawa taught the second author that $8_{13}$ and $9_{44}$ has the flat plumbing basket number 6.
• $g(K)$ is the genus.
• The check mark in the row “Not monic?” means that its Alexander polynomial is not monic. Note that for knots with less than or equal to nine crossings $\deg\Delta_K(t) = 2g(K)$ holds hence it implies that a better bound $f_{pbk}(K) \geq 2g(K) + 4$ than the trivial genus bound.
• $\alpha(K)$ is the arc index.
• $b(K)$ is the braid index.
• $TB(K)$ denotes $\max\{-tb(K), -\overline{tb}(K)\}$.
• $SL(K)$ denotes $\max\{-sl(K), -\overline{sl}(K)\}$.
• The daggers $\dag$ in the row $g(K)$, “Not monic?”, $\alpha(K)$, $b(K)$ and $SL(K)$ represent that the trivial genus bound (3.6), Hirose-Nakashima bound (3.7), the arc index bound (Theorem 4.1), TB+braid index bound in Theorem 1.2, and the self-linking number bound in Theorem 1.1 determine the $f_{pbk}$, respectively.

We refer to [15, Proposition 1.6] for $\overline{sl}(K)$ and [1] for the genus $g(K)$, arc index $\alpha(K)$, $b(K)$, $\overline{tb}(K)$ and $\Delta_K(t)$.

Here we list some of simple observations.
• The lower bound $2f_{pbk}(K) \geq TB(K) + b(K)$ in Theorem 1.2 quite often determine the $f_{pbk}(K)$.
• The knots $8_{13}$ and $9_{49}$ give an example where the lower bound $f_{pbk}(K) \geq SL(K) - 1$ in Theorem 1.1 gives a better bound than the lower bound $2f_{pbk}(K) \geq TB(K) + b(K)$ in Theorem 1.2. Thus the lower bounds in Theorem 1.1 and Theorem 1.2 are independent.
• In general, the arc index bound Theorem 4.1 rarely detects the flat plumbing basket number. Nevertheless, as the knot $7_7$ shows, it can detect the flat plumbing basket number even if the lower bounds from Theorem 1.1 and Theorem 1.2 cannot.
• As the knot $6_3$ (resp. $8_4$) demonstrates, sometimes the trivial genus bound (resp. Hirose-Nakashima bound) determines $f_{pbk}(K)$ whereas other lower bounds Theorem 4.1, Theorem 1.1 and Theorem 1.2 cannot.

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Table 2. Table of flat plumbing basket numbers \(fpbk(K)\) for prime knots \(K\) with up to 9 crossings. For the notations, see Section 6.
| $K$ | $g(K)$ | Not monic? | $\alpha(K)$ | $b(K)$ | $TB(K)$ | $SL(K)$ | $fpbk(K)$ |
|-----|--------|------------|------------|--------|---------|---------|-----------|
| 9_4 | 2      | ✓          | 11         | 5†     | 8       | 7       | 8         |
| 9_9 | 3      | ✓          | 11         | 3†     | 16      | 11†     | 10        |
| 9_10| 2      | ✓          | 11         | 4†     | 14      | 11†     | 10*       |
| 9_13| 3      |             | 11         | 4†     | 12      | 9†      | 8         |
| 9_12| 2      | ✓          | 11         | 5†     | 10      | 9†      | 8         |
| 9_19| 2      | ✓          | 11         | 4†     | 14      | 11†     | 10*       |
| 9_14| 2      | ✓          | 11         | 5       | 7       | 7       | 8         |
| 9_15| 2      | ✓          | 11         | 5†     | 10      | 9†      | 8         |
| 9_16| 3      | ✓          | 11         | 3†     | 16      | 11†     | 10        |
| 9_17| 3      |             | 11         | 4       | 8       | 5       | 8         |
| 9_18| 2      | ✓          | 11         | 4†     | 14      | 11†     | 10*       |
| 9_19| 2      | ✓          | 11         | 5       | 6       | 5       | 8         |
| 9_20| 3      |             | 11         | 4†     | 12      | 9†      | 8         |
| 9_21| 2      | ✓          | 11         | 5†     | 10      | 9†      | 8         |
| 9_22| 3      |             | 11         | 4       | 8       | 5       | 8         |
| 9_23| 2      | ✓          | 11         | 4†     | 14      | 11†     | 10*       |
| 9_24| 3      |             | 11         | 4       | 6       | 5       | 8         |
| 9_25| 2      | ✓          | 11         | 5       | 10      | 9       | 8–10      |
| 9_26| 3      |             | 11         | 4†     | 9       | 7       | 8         |
| 9_27| 3      |             | 11         | 4       | 6       | 5       | 8         |
| 9_28| 3      |             | 11         | 4†     | 9       | 7       | 8         |
| 9_29| 3      |             | 11         | 4       | 8       | 5       | 8         |
| 9_30| 3      |             | 11         | 4       | 6       | 5       | 8         |
| 9_31| 3      |             | 11         | 4       | 9       | 7       | 8         |
| 9_32| 3      |             | 11         | 4†     | 9       | 7       | 8         |
| 9_33| 3      |             | 11         | 4       | 6       | 5       | 8         |
| 9_34| 3      |             | 11         | 4       | 6       | 5       | 8–12      |
| 9_35| 1      | ✓          | 11         | 5†     | 12      | 11†     | 10*       |
| 9_36| 3      |             | 11         | 4†     | 12      | 9†      | 8         |
| 9_37| 2      | ✓          | 11         | 5       | 6       | 5       | 8         |
| 9_38| 2      | ✓          | 11         | 4†     | 14      | 11†     | 10*       |
| 9_39| 2      | ✓          | 11         | 5       | 10      | 9       | 8–10      |
| 9_40| 3      |             | 11         | 4       | 9       | 7       | 8–12      |
| 9_41| 2      | ✓          | 11         | 5       | 7       | 7       | 8–10      |
| 9_42| 2      |             | 8          | 4†     | 5       | 5       | 6         |
| 9_43| 3      |             | 9          | 4       | 10      | 9       | 8–10      |
| 9_44| 2      |             | 9†         | 4†     | 6       | 5       | 6**       |
| 9_45| 2      |             | 9          | 4†     | 10      | 9†      | 8*        |
| 9_46| 1      | ✓          | 8          | 4†     | 7       | 7†      | 6         |
| 9_47| 3      |             | 9          | 4       | 7       | 7       | 8         |
| 9_48| 2      |             | 9†         | 4†     | 8       | 7†      | 6         |
| 9_49| 2      | ✓          | 9          | 4       | 12      | 11†     | 10*       |

Table 3. Table of flat plumbing basket numbers $fpbk(K)$ for prime knots $K$ with up to 9 crossings (continuation of Table 2).
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