Squeezing and entanglement in quasiparticle excitations of trapped Bose-Einstein condensates

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We estimate the amount of temperature-dependent squeezing and entanglement in the collective excitations of trapped Bose-Einstein condensates. We also demonstrate an alternative method of temperature measurement for temperatures much less than the critical temperature \( T \ll T_c \).

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I. INTRODUCTION

Since the first experimental realization of Bose-Einstein condensation in trapped atomic gases, a number of papers have highlighted the parallels between processes in atom optics and nonlinear (photon) optics. Well-known examples include four-wave mixing \(^1\) and soliton dynamics \(^2\), while the squeezing of matter wave fields has also been examined \(^3\) \(^4\) \(^5\). These latter studies consider the way in which the nonlinear interactions between the atoms can generate entangled atomic beams, for example by spin-exchanging collisions of spinor Bose-Einstein condensates.

Entangled states, to which squeezed states are closely related, have been extensively studied in the last few years because of their potential application in quantum information processing and quantum computing. In this paper, we discuss squeezing and entanglement in terms of the quasiparticle modes of a trapped gas. Quasiparticle modes represent collective excitations of the Bose-Einstein condensate (BEC), and are in fact one of the most fundamental features of its dynamics. They are easily generated by applying time-dependent perturbations to the trapping potential, and were observed experimentally quite early on; they have since been studied in some detail both theoretically and experimentally.

The low temperature regime in which the majority of the atoms are in the condensate has been studied in various experiments \(^6\) \(^7\). These experiments revealed oscillations of the condensate with almost no damping, and the results were in good agreement with predictions based on the zero temperature Gross-Pitaevskii equation (GPE) \(^8\). Excitations at higher temperatures were studied in later experiments \(^9\), where large energy shifts and rapid damping rates were observed. While the standard theory of elementary excitations due to Bogoliubov \(^10\) explains most of the observed effects, it cannot account for higher-order processes, such as the Beliaev damping. And while finite temperature studies using the GPE have explained various properties of BECs observed experimentally, they cannot fully describe the evolution of collective excitations at higher temperatures \(^11\) since the spontaneous part of the Beliaev damping of the excitations is not included. Theoretical techniques have recently been developed that take the quasiparticle interactions into account \(^12\) and experiments to test the predictions are planned for the near future. The main motivation of the present paper is to analyse these experiments on the quantal aspects of quasiparticle mode evolution in a BEC.

We note that Beliaev damping may be understood in terms of a nonlinear frequency-mixing mechanism in which a Bogoliubov quasiparticle of frequency \( \omega_2 \) interacts with the ground state (condensate) and generates two quasiparticles of frequency \( \omega_1 \) that divide the initial energy equally (Fig. \(^1\)). The mechanism is actually based on a four-wave process that masquerades as a three-wave interaction analogous to optical parametric down-conversion (OPDC) in which a photon in a nonlinear medium splits into two photons of lower energy.

We calculate the amount of squeezing for a realistic system containing 10000 \(^87\)Rb atoms in a trap with spherical geometry. We also propose an alternative method for determining the temperature of a BEC well below the critical temperature by observing the variation of the envelope of the collective excitations. This should be more accurate than current techniques based on fitting a thermal Gaussian profile to the atomic cloud.

The paper is organised as follows: In Section \(^III\) we describe the higher-order Hamiltonian that accommodates processes such as Landau and Beliaev damping in the quasiparticle excitations. We derive equations of motion for a quasiparticle mode. In Section \(^IV\) we make quantitative estimates for the degree of squeezing and entanglement in quasiparticle excitations. In addition we calculate the expected “damping rate” and coupling between modes at various temperatures which can then serve as a temperature measurement calibration for ultracold atoms. A possible way of detecting squeezed states in atom optics is discussed briefly in Section \(^V\).
II. HAMILTONIAN IN THE QUASIPARTICLE BASIS

The many-body Hamiltonian for a system of bosons with pairwise interactions can be written in the usual second quantised formalism as,

\[ \hat{H} = \sum_{ij} H_{ij}^{sp} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_{ijkl} (ij|\hat{V}|kl) \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l, \]

where the matrix elements \( H_{ij}^{sp} \) are given by

\[ H_{ij}^{sp} = \int d^3r \psi_i^*(r) \hat{H}^{sp} \psi_j(r). \]

Here, \( \hat{H}^{sp} = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} \) is the single-particle Hamiltonian with a confining potential \( V_{\text{trap}} \), and the basis state wave functions are \( \psi_i(r) \). \( (ij|\hat{V}|kl) \) denotes the matrix element for the interaction potential \( \hat{V}(r) \) between atoms. The operators \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) are the creation and annihilation operators for mode i that obey the usual Bose commutation relations

\[ [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \]

The Hamiltonian (1) is written in a single-particle basis where the operator \( \hat{a}_i \) annihilates a particle from the \( \left| \right. \) For simplicity from now on we drop the explicit notation in the operators

\[ H' = \text{const} + \sum_{i \neq 0} (\epsilon_i + \Delta \epsilon_i) \beta_i^\dagger \beta_i + \left\{ \sum_{ijk \neq 0} \left[ \zeta_{ijk} \beta_i^\dagger \beta_j \beta_k \right] + h.c. \right\}, \]

in which only processes that generally conserve energy have been included. The constant term in the equation simply defines the zero of energy and \( \Delta \epsilon_i \) is the energy shift from first order perturbation theory. The coefficients \( \zeta_{ijk} \) are determined by

\[ \zeta_{ijk} = \sqrt{N_0} \sum_{mnq \neq 0} \langle q|V|m \rangle \left[ u_{iq}^* u_{jn} u_{km} + v_{iq}^* v_{jq} u_{km} + v_{im}^* u_{jn} v_{kq} \right] 
+ \langle mn|V|q \rangle \left[ u_{in}^* u_{jq} v_{km} + u_{im}^* v_{jn} u_{kq} + v_{iq}^* v_{jq} v_{kq} \right]. \]

where the indices \( i, j \) and \( k \) are labels that denote the quasiparticle energy levels. For a 3-D condensate, the index \( i \) stands for the quantum numbers \( n, l \) and \( m \). We note that the coefficients are temperature dependent through \( u_{ij} \) and \( v_{ij} \).

The Hamiltonian given by equation (1) contains terms beyond the standard Bogoliubov approximation, thus we are using a fuller description of the BEC that involves interacting quasiparticles and can take important processes such as Landau and Beliaev damping into account. Beliaev processes occur at zero temperature and are dominant in the low-temperature regime. Landau processes
on the other hand predominate at higher temperatures; these processes, in which two quasiparticles collide to form a single quasiparticle cannot occur at zero temperature because there are no excited quasiparticles.

From the Hamiltonian (4), the Heisenberg equations of motion for $\beta_p$ and $\beta_p^\dagger$ are

\begin{align*}
  i\dot{\beta}_p &= \omega_p\beta_p + \sum_{j,k\neq0} \sigma_{jk}\beta_j\beta_k + \sum_{j,k\neq0} \nu_{jk}\beta_k^\dagger\beta_j, \\
  -i\dot{\beta}_p^\dagger &= \omega_p\beta_p^\dagger + \sum_{j,k\neq0} \sigma_{jk}^\dagger\beta_j^\dagger\beta_k^\dagger + \sum_{j,k\neq0} \nu_{jk}\beta_j^\dagger\beta_k,
\end{align*}

where

\begin{align*}
  \omega_p &= (\epsilon_p + \Delta\epsilon_p)/\hbar, \\
  \sigma_{jk} &= \zeta_{pjk}/\hbar, \\
  \nu_{jk} &= (\zeta_{pjk}^* + \zeta_{jk}^*)/\hbar.
\end{align*}

It is well known that homogeneous systems have a continuous spectrum of excitations; however, for a trapped system, the spectrum of states to which the excitation can couple is discrete. Furthermore, in the case of a spherical trap there is a degeneracy in the azimuthal quantum number $m$, which in turn means that fewer resonances are present [14, 15]. This implies that the time evolution of the excitation is dominated by a strong coupling to only a few modes. On the other hand, the geometry of the trap can be changed by adjusting the frequencies in the radial and axial directions independently, and the selection of a dominant mode in this way has been demonstrated [10]. In particular, a Beliaev process has recently been observed for a scissors mode, where one mode is resonantly coupled to two modes of half the original frequency [17]; this is the kind of process that we seek to model in the present paper. The equations of motion for modes $p = 1, 2$ in a Beliaev process reduce to

\begin{align*}
  \dot{\beta}_1 &= -i\omega_1\beta_1 - i\nu_{21}\beta_2\beta_1^\dagger, \\
  \dot{\beta}_1^\dagger &= i\omega_1\beta_1^\dagger + i\nu_{21}\beta_2^\dagger\beta_1^\dagger, \\
  \dot{\beta}_2 &= -i\omega_2\beta_2 - i\nu_{21}\beta_1\beta_1^\dagger, \\
  \dot{\beta}_2^\dagger &= i\omega_2\beta_2^\dagger + i\nu_{21}\beta_1^\dagger\beta_1^\dagger,
\end{align*}

where we have used the fact that $\sigma_{11} = \nu_{21}/2$, and we have chosen the phases so that the coefficients are real.

Equations of the form (8)-(11) have been studied in quantum optics and we apply them here to study the squeezing in the quasiparticle excitations and its temperature dependence. We note that damping of quasiparticle excitations has earlier been described in terms of nonlinear mixing of quasiparticle modes [18]; however, the present work is quite distinct as the operator nature of the quasiparticle annihilation operator $\beta_i$ is retained, and the calculation is not restricted to the quadratic approximation. We are therefore extending the work of Ref. [18], so that spontaneous quantum processes are included.

To give an idea of the order of magnitude of the numbers involved, the coefficient $\nu_{21}$ equals $1.9 \times 10^{-2}$ in trap units for the following parameters: $T = 20$ nK, $N = 1000$, $\omega_{trap} = 2\pi \times 100$ Hz, for a spherical trap geometry and using $^{87}$Rb atoms for which the s-wave scattering length $a = 110a_0$, where $a_0$ is the Bohr radius. The values of $\nu_{21}$ are significantly dependent on temperature; which implies that the equations of motion for $\beta_1$ and $\beta_1^\dagger$ are altered accordingly. We can exploit this feature as a method of measuring temperature below $T_c$. Figure 2 indicates the dependence of $\nu_{21}$ on temperature, which is an effective dependence on the number of particles in the condensate. We notice that at higher temperatures the coefficient decreases swiftly, whilst for lower temperatures it is fairly constant.

III. TEMPERATURE DEPENDENT COUPLING PROCESS

A. Non-depleted regime

We consider first the case where the higher ($p = 2$) mode - the pump - has a much larger population than the lower one and where the pump depletion is ignored so that the operator $\beta_2$ can be approximated by a c-number $b_2$. Physically, this would represent a situation in which the higher mode is being continuously driven by a resonant excitation. The solutions of the equations of motion for $\beta_1$ and $\beta_1^\dagger$ in the interaction picture are then given by

\begin{align*}
  \begin{pmatrix}
    \beta_1(t) \\
    \beta_1^\dagger(t)
  \end{pmatrix} &=
  \begin{pmatrix}
    \cosh(\Omega t) & -i\sinh(\Omega t) \\
    i\sinh(\Omega t) & \cosh(\Omega t)
  \end{pmatrix}
  \begin{pmatrix}
    \beta_1(0) \\
    \beta_1^\dagger(0)
  \end{pmatrix},
\end{align*}

where $\Omega = \nu_{21}b_2$, and we have chosen the phases such that the coefficients are real. This approximation neglects the depletion of the “pump” and the solution will cease to be valid once appreciable down-conversion occurs.

Solutions (12) are formally equivalent to equations that describe degenerate parametric down conversion in quan-
tum optics: a process known to be a significant source of
squeezed states. In our case the squeezing parameter is
given by

\[ \tau = \Omega t, \]  

which is temperature dependent through \( \Omega \).

Defining the quadrature operators as

\[ X(t) = \frac{\beta(t) + \beta^\dagger(t)}{2} \text{ and } Y(t) = \frac{\beta(t) - \beta^\dagger(t)}{2i}, \]

we have the variances

\[ \langle \Delta X(t) \rangle^2 = \frac{(2N_1 + 1)}{4} \exp(-2\Omega t), \]  

\[ \langle \Delta Y(t) \rangle^2 = \frac{(2N_1 + 1)}{4} \exp(2\Omega t), \]

where \( N_1 \) is the number of particles in mode 1. In optics,
the quadrature operators are well-defined quantities cor-
responding to the amplitude and phase of the electromagnetic
(EM) oscillation. One may think of the amplitude and
(temporal) phase of oscillations in a quasiparticle ex-
citation, in a similar way, and we therefore interpret our
quadrature operators \( X \) and \( Y \) as amplitude and phase of
oscillations. Equations (14) and (15) imply that the
degree of squeezing depends on the amount of the lower-
mode oscillations. Equations (14) and (15) can, in fact, be
uncoupled by calculating the second derivative and using the fact that
the operator \( \{X, Y\} \) is a constant of motion. The
analysis is handled more easily by considering the equations of
motion for the number operators \( N_i = \beta_i^\dagger \beta_i \), which from equations (12 - 15) are

\[ \frac{dN_1}{dt} = i\nu_{21} \left( \beta_1^\dagger \beta_1 - \beta_2^\dagger \beta_2 \right), \]

\[ \frac{dN_2}{dt} = \frac{\nu_{21}}{2} \left( \beta_2^\dagger \beta_1^\dagger - \beta_2 \beta_1 \right). \]

Equations (18) and (19) can, in fact, be uncoupled by
calculating the second derivative and using the fact that
the operator \( A = N_1 + 2N_2 \) is a constant of motion. The
uncoupled equations are given by

\[ \frac{d^2N_1}{dt^2} = \nu_{21}^2 \left( -3N_1^2 + 2AN_1 + A \right), \]

\[ \frac{d^2N_2}{dt^2} = \frac{\nu_{21}^2}{2} \left( 12N_2^2 - 8AN_2 + A^2 - A \right). \]

In the case in which \( N_2 \) is strong, we can determine the
population by taking the average of equation (21). We
end up with a c-number second-order differential equation
with the following initial conditions

\[ \frac{dN_2}{dt} \bigg|_{t=0} = 0, \]

\[ N_2(0) = N_{20}; \]

whose solution is given by

\[ N_2(t) = \begin{cases} 
N_{20} + (\alpha_2 - N_{20})\text{sn}^2 \left( \frac{\theta}{\alpha_1 - \alpha_2} \sqrt{\frac{N_{20} - N_{20}}{6}}, \sqrt{\frac{N_{20} - N_{20}}{\alpha_1 - \alpha_2}} \right), & \text{for } N_{20} < \alpha_2, \\
\alpha_1 + (N_{20} - \alpha_1)\text{nd}^2 \left( \frac{\theta}{\alpha_1 - \alpha_2} \sqrt{\frac{N_{20} - \alpha_1}{\alpha_1 - \alpha_2}}, \sqrt{\frac{N_{20} - \alpha_1}{\alpha_1 - \alpha_2}} \right), & \text{for } N_{20} > \alpha_2.
\end{cases} \]

Here \( \text{sn}(u, k) \) and \( \text{nd}(u, k) \) are Jacobi elliptic functions \[20\], \( \alpha_1 \) and \( \alpha_2 \) are the roots of the quadratic poly-
The coupling strength of squeezing in the quasiparticles, as it is proportional to $\Gamma$ as an experimentally accessible indicator of the amount of squeezing in the quasiparticles, as it is proportional to $\Gamma$ as an experimentally accessible indicator of the amount of initial population. It is noted that one could also consider the changing coupling strength and the influence of the initial population. This effect is the result of having a competition between the two excited phonon states. The first moves a particle with zero momentum to momentum $k$, while the second takes the momentum from $-k$ to zero. The interference between the two routes will influence the single particle transition. Raman scattering will be strongly influenced by the presence of coherent states in the lowest mode. The precise form will depend on the conditions, pulses used, and other experimental parameters.

In summary, we have analysed a process analogous to parametric down-conversion in trapped Bose-Einstein condensates. Physically this corresponds to Beliaev damping, where two quanta are produced by one of higher frequency. This, in turn, suggests the possibility of observing temperature-dependent squeezing in the collective excitations of the condensate. The amount of squeezing gives a direct indication of the temperature and vice versa.

IV. DISCUSSION

Squeezed states of the electromagnetic field were realized some time ago and the degree of squeezing can be measured with standard techniques such as homodyne detection. In the case of squeezed quasiparticle modes of a BEC, it will be necessary to devise a method for their detection. It is noted that a related work that uses neutron scattering was suggested by Yurke.

We suggest for our system that Raman scattering between the two excited phonon states could be used to characterise the amount of squeezing present since single particle transitions will be strongly modified by the presence of a correlated pair excitation. This effect has been demonstrated for a homogeneous gas by Stamper-Kurn et al. who used a single particle transition to excite a phonon. We then have two routes for the transfer of momentum. The first moves a particle with zero momentum to momentum $k$, while the second takes the momentum from $-k$ to zero. The interference between the two routes will influence the single particle transition. Raman scattering will be strongly influenced by the presence of coherent states in the lowest mode.

The full solution, given by equation (21), has been used in a fitting routine and a plot of $\Gamma$ as a function of temperature is shown in Figure 7. In principle, $\Gamma$ has the advantage of being readily measurable. At lower temperatures the effect of the initial population is dominant, while, for higher temperatures, the coupling coefficient $\nu_{21}$ decreases rapidly so that the effect of the initial population is masked, which happens at approximately $0.6T_c$. This effect is the result of having a competition between the changing coupling strength and the influence of the initial population. It is noted that one could also consider $\Gamma$ as an experimentally accessible indicator of the amount of squeezing in the quasiparticles, as it is proportional to the coupling strength $\nu_{21}$.

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Figure Captions

1. Schematic diagram of energy levels for a trapped Bose-Einstein condensate. In the Beliaev process, analogous to optical parametric down-conversion, a quasiparticle of frequency $\omega_2$ interacts with the ground state generating two quasiparticles of frequency $\omega_1$ that divide the initial energy equally.

2. Coefficient $\nu_{21}$ vs temperature for $N = 10000$, $\omega_{\text{trap}} = 2\pi \times 100$ Hz, a spherical trap geometry and using $^{87}$Rb atoms.

3. a) Surface plot of $\Delta X$ as a function of temperature and time. b) Surface plot of $\Delta Y$ as a function of temperature and time. At $t = 0$, the behaviour is fully determined by the Bose-Einstein (BE) distribution function.

4. Correlation functions calculated within the non-depleted regime: a) The average $\langle \beta_1^\dagger \beta_1 \rangle$. The occurrence of squeezing is implied by a non-zero value of this quantity. b) Surface plot of $\langle \beta_1^\dagger \beta_1 \rangle$, which describes the behaviour of the population in mode 1.

5. Surface plot of the population of mode 2 as a function of time and temperature.

6. Plot of $N_2(t)$, the solid line is the solution for the following parameters: $N_{20} = 151.4225$, at $T = 20nK$. The dashed line represents the amplitude of oscillations of the quasiparticle mode 2.

7. The parameter $\Gamma$ plotted against temperature is obtained so that the curve $a_1 \cos(\Gamma t)$ corresponds to the envelope shown in figure 6.
\[ \nu_2(T/T_c) \]
a) $\Delta X$

b) $\Delta Y$
a) $\langle \beta_1 \beta_1 \rangle$

b) $\langle \beta_1^* \beta_1 \rangle$
$N_2$ vs. Time ($2\pi/\omega_{\text{trap}}$)
Temperature, $T/T_c$

$\Gamma(\omega_{\text{trap}})$