On the ruin problem in the renewal risk processes perturbed by diffusion

Min Song

Abstract

In this paper, we consider the perturbed renewal risk process. Systems of integro-differential equations for the Gerber-Shiu functions at ruin caused by a claim and oscillation are established, respectively. The explicit Laplace transforms of Gerber-Shiu functions are obtained, while the closed form expressions for the Gerber-Shiu functions are derived when the claim amount distribution is from the rational family. Finally, we present numerical examples intended to illustrate the main results.

Keywords: Diffusion process; Gerber-Shiu discounted penalty function; Renewal risk process
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* Corresponding author: School of Mathematical Sciences, Nankai University, Tianjin, China
E-mail: nksongmin@yahoo.com.cn (M. Song)
1. Introduction

Consider a continuous time renewal risk process perturbed by diffusion

\[ U(t) = u + ct - \sum_{i=1}^{N(t)} Z_i + \sigma B(t), \quad t \geq 0, \tag{1.1} \]

where \( u \geq 0 \) is the initial capital. \( c > 0 \) is the constant rate of premium. The ordinary renewal process \( \{N(t), t \geq 0\} \) denotes the number of claims up to time \( t \), with \( N(t) = \max\{n \geq 1 : V_1 + V_2 + \cdots + V_n \leq t\} \). Then \( V_i, i = 1, 2, \ldots \) are the interclaim random variables. They are independent and assumed to have common distribution function \( K \), density function \( k \), and Laplace transforms \( \hat{k}(s) = \int_{0}^{\infty} e^{-sx}k(x)dx \). \( \{Z_i, i \geq 1\} \) are independent claim-size random variables with common distribution \( P \) (such that \( P(0) = 0 \)) and density \( p \). \( \{B(t), t \geq 0\} \) is a standard Brownian motion with \( B(0) = 0 \). It is assumed that \( \{N(t)\}, \{B(t)\} \) and \( \{Z_i\} \) are mutually independent and that \( cE(V_i) > E(Z_i) \) providing a positive safety loading factor.

The perturbed risk model of form (1.1) was firstly introduced by Gerber (1970) and has been studied by many authors. See, for example, Dufre^se and Gerber (1991), Furre and Schmidli (1994), Schmidli (1995), Gerber and Landry (1998), Wang and Wu (2000), Tsai and Willmot (2002a, b), Zhang and Wang (2003), Li and Garrido (2005) and the references therein.

Let \( T = \inf\{t : U(t) \leq 0\} \) (with \( \inf\{\emptyset\} = \infty \)) be the time of ruin for risk process (1.1). Two important nonnegative random variables in connection with the time of ruin \( T \) are \( |U(T)| \), the deficit at the time of ruin, and \( U(T-) \), the surplus immediately before the time of ruin. Consider a penalty scheme which is defined by a constant \( w_0 \) if ruin occurs by oscillation and \( w(U(T-), |U(T)|) \) if ruin is caused by a jump. Then,
the Gerber-Shiu discounted penalty function at ruin $\phi(u)$ is defined by

$$\phi(u) = \phi_w(u) + w_0\phi_d(u),$$

where for $\delta \geq 0$,

$$\phi_w(u) = E\left[e^{-\delta T}I(T < \infty, U(T) < 0)|U(0) = u\right],$$

(with $\phi_w(0) = 0$) is the expected discounted penalty function at ruin caused by a claim, and

$$\phi_d(u) = E[e^{-\delta T}I(T < \infty, U(T) = 0)|U(0) = u],$$

(with $\phi_d(0) = 1$) is the Laplace transform of the ruin time $T$ due to oscillation. Many ruin-related quantities can be analyzed by appropriately choosing special penalty function $w$, for example, let $\delta = 0$ and $w = 1$, then $\phi_w(u) \triangleq \psi_w(u)$ gives the probability of ruin due to a claim and $\phi_d(u) \triangleq \psi_d(u)$ is the probability of ruin that is caused by oscillation.

The evaluation of the Gerber-Shiu discounted penalty function, first introduced in Gerber and Shiu(1998), is now one of the main research problems in ruin theory. See, for example, Gerber and Landry (1998), Tsai and Willmot (2002a,b) for the classical surplus process perturbed by diffusion, Li and Garrido (2005) for the generalized Erlang \((n)\) risk process perturbed by diffusion, Albrecher and Boxma (2005) for the semi-Markov model, Willmot (2007) and Landriault and Willmot (2007) for the renewal risk model, Lu and Tsai (2007) for the Markov-Modulated process perturbed by diffusion.

The rest paper is organized as follows. In Section 2 we derive systems of integro-differential equations for Gerber-Shiu functions. Section 3 discusses a generalized Lundberg’s equation and its roots. And the Gerber-Shiu functions for the model are fully
analyzed in Section 4. Section 5 contains several numerical examples intended to illustrate the main results.

2. Integro-differential equations

More recently, Ren (2007) considered the risk process with phase-type interclaim times, i.e., the distribution of the interclaim time $K$ is phase-type with representation $(\alpha, B, b)$, where $\alpha$ and $b$ are row vectors of length $n$ and $B$ is a $n \times n$ matrix. That is, each of random variables $V_k, k = 1, 2, \ldots$ corresponds to the time to absorption in a terminating continuous-time Markov Chain $J^{(k)}_k$ with $n$ transient states $\{E_1, E_2, \ldots E_n\}$ and are absorbing state $E_0$. Let $e$ denote a row vector of length $n$ with all elements being one. Then $b^T = -Be^T$. Following Asmussen (2000),

$$K(t) = 1 - \alpha e^t B e^T, \quad t \geq 0,$$

$$k(t) = \alpha e^t b^T, \quad t \geq 0,$$

and

$$\hat{k}(s) = \int_0^\infty e^{-st} k(t) dt = \alpha(sI - B)^{-1} b^T. \quad (2.1)$$

For $i = 1, 2, \ldots, n$, let $\phi(u; i)$ denote the Gerber-Shiu function given $U(0) = u$ and $J^{(1)}_0 = E_i$, that is,

$$\phi(u; i) = E[e^{-\delta T} I(T < \infty)w(U(T^-), |U(T)|)|U(0) = u, J^{(1)}_0 = E_i], \quad i = 1, 2, \ldots, n.$$ 

Then the Gerber-Shiu function may be computed by

$$\phi(u) = \alpha \phi(u)$$

where $\phi(u) = (\phi(u; 1), \ldots, \phi(u; n))^T$ is a column vector of functions. Similarly, we write $\phi_w(u; i)$ and $\phi_d(u; i)$ for the Gerber-Shiu functions at ruin caused by a claim
and oscillation respectively, given \( U(0) = u \) and \( J_0^{(1)} = \mathcal{E}_i \). Denoted by \( \phi_w(u) \triangleq (\phi_w(u; 1), \ldots, \phi_w(u; n))^\top \) and \( \phi_d(u) \triangleq (\phi_d(u; 1), \ldots, \phi_d(u; n))^\top \).

Our first result gives integro-differential equations for Gerber-Shiu functions.

**Theorem 2.1.** Let \( u > 0 \). Then, \( \phi_w(u) \) satisfies the following equation

\[
\frac{\sigma^2}{2} \phi''_w(u) + c \phi'_w(u) + (B - \delta I) \phi_w(u) + \left[ \int_0^u \alpha \phi_w(u - x)p(x)dx + \omega(u) \right] \mathbf{b}^\top = 0 \quad (2.2)
\]

where \( \omega(u) = \int_u^\infty w(u, x - u)p(x)dx \), \( I = \text{diag}(1, 1, \ldots, 1) \), \( \mathbf{0} \) denotes a column vector of length \( n \) with all elements being 0, and \( \phi_w(0) = \mathbf{0} \), while \( \phi_d(u) \) satisfies

\[
\frac{\sigma^2}{2} \phi''_d(u) + c \phi'_d(u) - \delta \phi_d(u) + B \phi_d(u) + \left[ \int_0^u \alpha \phi_d(u - x)p(x)dx \right] \mathbf{b}^\top = 0, \quad (2.3)
\]

with \( \phi_d(0) = e^\top \).

**Proof.** Define \( \{J(t), t \geq 0\} \) by piecing the \( \{J_t^{(k)}\} \) together,

\[
J(t) = \{J_t^{(1)}\}, \quad 0 \leq t < V_1, \quad J(t) = \{J_{t-V_1}^{(2)}\}, \quad V_1 \leq t < V_1 + V_2, \ldots
\]

Then \( \{J(t)\} \) is Markov. Jacobson (2005) showed that the joint process \( \{(U(t), J(t)), t \geq 0\} \) is a Markovian additive process and thus \( \{(U(t), J(t)), t \geq 0\} \) is a homogeneous Markov process. Then we shall use the technique developed in Grandell (1991, P.84) and Wu and Wei (2004) to derive the integro-differential equations for \( \phi_w(u; i) \) and \( \phi_d(u; i) \) for \( i = 1, 2, \ldots, n \). Let us consider \( \phi_w(u; i) \) first. Consider a short time interval \([0, h]\). By noting the fact that claim occurs at the time when the state for \( \{J(t), t \geq 0\} \) changes, we separate the three possible cases:

1. no change of state occurs in \([0, h]\) for the underlying Markov chain \( \{J(t), t \geq 0\} \) and denoted by \( J[0, h] \equiv \mathcal{E}_i \) in this case;
2. the state for \( \{J_t, t \geq 0\} \) changes in \([0, h]\) but no claim occurs;
3. at least one claim occur in \([0, h]\),
In the case (1). Denoted by $E^{(u,i)}$ the conditional expectation given the initial $(U(0), J(0)) = (u, E_i)$ and $V(h) \triangleright u + ch + \sigma B(h)$. Let $\mathcal{F}^J$ and $\mathcal{F}^{(U,J)}$ denote the natural filtration of processes $\{J(t)\}$ and $\{(U(t), J(t))\}$ respectively. For $t \geq 0$, let $\theta_t$ be the shift operators (see, Revuz and Yor (1991, P.34)). It follows from the Markov property of both the underlying process $\{J(t)\}$ and the vector process $\{(U(t), J(t))\}$ that

$$
E^{(u,i)}[e^{-\delta T}I(T < \infty, U(T) < 0)w(U(T), |U(T)|)I(J[h] \equiv E_i)]
$$

= $E\left\{E^{(u,i)}[w(U(T), |U(T)|)e^{-\delta T}I(T < \infty, U(T) < 0)I(J[h] \equiv E_i) | \mathcal{F}^{(U,J)}_h] | \mathcal{F}^J_{\infty}\right\}$

= $E\left\{E^{(u,i)}[e^{-\delta h}I(J[h] \equiv E_i)]

E^{(u,i)}[(w(U(T), |U(T)|)e^{-\delta T}I(T < \infty, U(T) < 0)) \circ \theta_{\delta h} | \mathcal{F}^{(U,J)}_h] | \mathcal{F}^J_{\infty}\right\}$

= $e^{-\delta h}E\left\{E[I(J[h] \equiv E_i) | \mathcal{F}^{(U,h,J(h))}_h] | \mathcal{F}^J_{\infty}\right\}$

= $e^{-\delta h}e^{b_{ii}h}E[\phi_w(V(h); i)], \hspace{1cm} (2.4)$

where $b_{ij}$ is the $(i, j)$th entry of matrix $B$.

For case (2) and (3), by the similar argument to that of case (1), we have

$$
E^{(u,i)}[e^{-\delta T}I(T < \infty, U(T) < 0)w(U(T), |U(T)|)I(J[h] \neq E_i, N(h) = 0)]
$$

= $e^{-\delta h}(1 - e^{b_{ii}h}) \sum_{j=1, j \neq i}^n \left(\frac{b_{ij}}{-b_{ii}}\right)E[\phi_w(V(h); j)] + o(h), \hspace{1cm} (2.5)$
where $o(h)/h \to 0$ as $h \to 0$, and

$$E^{(u,i)} \left[ e^{-\delta T} I(T < \infty, U(T) < 0) w(U(T) - |U(T)|) I(N(h) \geq 1) \right]$$

$$= e^{-\delta h} (1 - e^{b_i h}) \left( \frac{b_i}{b_{ii}} \right) \sum_{j=1}^{n} \alpha_j E \left[ \int_{0}^{\nu(h)} \phi_w(\nu(h) - x; j)p(x)dx + \int_{\nu(h)}^{\infty} w(\nu(h), x - \nu(h))p(x)dx \right] + o(h), \quad (2.6)$$

where $b_i$ denotes the $i$th entry of vector $b$.

Summarizing the above analysis, it follows from (2.4), (2.5) and (2.6) that

$$\phi_w(u; i) = (1 - \delta h + b_{ii} h) E[\phi_w(\nu(h); i)] + h \sum_{j=1, j\neq i}^{n} b_{ij} E[\phi_w(\nu(h); j)] + h b_i \sum_{j=1}^{n} \alpha_j E \left[ \int_{0}^{\nu(h)} \phi_w(\nu(h) - x; j)p(x)dx + \int_{\nu(h)}^{\infty} w(\nu(h), x - \nu(h))p(x)dx \right] + o(h). \quad (2.7)$$

Apply Itô’s lemma for jump-diffusion processes (McDonald (2006), Section 20.8.) to $\phi_w(\nu(h); k)$, for $k = 1, 2, \ldots, n$ we have

$$E[\phi_w(\nu(h); k)] = \phi_w(u; k) + [c\phi_w'(u; k) + \frac{\sigma^2}{2}\phi_w''(u; k)]h + o(h).$$

Substituting the above expressions into (2.7), canceling $\phi_w(u; i)$ from both sides, dividing $h$ and letting $h \to 0$ yields a system of integro-differential equations for $\phi_w(u; i)$ given the initial surplus $u$ and the initial state of the phase-type distribution $E_i$:

$$\frac{\sigma^2}{2}\phi_w''(u; i) + c\phi_w'(u; i) - \delta \phi_w(u; i) + \sum_{j=1}^{n} b_{ij} \phi_w(u; j) + b_i \sum_{j=1}^{n} \alpha_j \left( \int_{0}^{u} \phi_w(u - x; j)p(x)dx + \omega(u) \right) = 0, \quad \text{for} \quad i = 1, 2, \ldots, n.$$ 

Writing the above equations in matrix form we get (2.2) and note that $\phi_w(0; i) = 0$ for $i = 1, 2, \ldots, n$ since $P(T < \infty, U(T) < 0|U(0) = 0) = 0$. Using arguments similar to those used in deriving (2.2), it is not difficult to get (2.3) and $\phi_d(0; i) = 1$ for $i = 1, 2, \ldots, n$ since $P(T < \infty, U(T) = 0|U(0) = 0) = 1$. □
Remark 2.1. When the distribution of the interclaim time is a generalized Erlang (n) distribution,

$$\alpha = (1, 0, \ldots, 0), \quad B = \begin{pmatrix}
-\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\
0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & -\lambda_n
\end{pmatrix}, \quad b^T = \begin{pmatrix} 0 \\ \vdots \\ \lambda_n \end{pmatrix}. \quad (2.8)$$

For $i = 1, 2, \ldots, n - 1$, from (2.2) that

$$\lambda_i \phi_w(u; i + 1) = (\lambda_i + \delta) \phi_w(u, i) - c \phi'_w(u; i) - \frac{\sigma^2}{2} \phi''_w(u; i)$$

and

$$(\lambda_n + \delta) \phi_w(u; n) - c \phi'_w(u; n) - \frac{\sigma^2}{2} \phi''_w(u; n) = \lambda_n \left[ \int_0^u \phi_w(u - x; 1)p(x)dx + \omega(u) \right],$$

which are formulae (7) and (8), respectively, in Li and Garrido (2005).

Assume that $\lim_{u \to \infty} e^{-su} \phi_w(u) = 0$ and $\lim_{u \to \infty} e^{-su} \phi'_w(u) = 0$ hold for $\Re(s) > 0$. Taking Laplace transforms on both sides of equation (2.2) and noting that $\phi_w(0) = 0$, we have

$$L(s) \hat{\phi}_w(s) = \frac{\sigma^2}{2} \phi'_w(0) - \hat{\omega}(s)b^T, \quad s \in \mathbb{C},$$

where

$$L(s) = \left( \frac{\sigma^2}{2}s^2 + cs - \delta \right)I + B + b^T \alpha \hat{\rho}(s),$$

and $\hat{\omega}(s) = \int_0^\infty e^{-su} \omega(u)du$. Denoted by $Q_w(s) \triangleq \frac{\sigma^2}{2} \phi'_w(0) - \hat{\omega}(s)b^T$. Then the vector Laplace transforms $\hat{\phi}_w(s)$ can be solved as

$$\hat{\phi}_w(s) = [L(s)]^{-1}Q_w(s).$$
Thus
\[
\hat{\phi}_w(s) = \frac{1}{\det[L(s)]} L^*(s) Q_w(s), \quad s \in \mathbb{C},
\] (2.9)
where \(L^*(s)\) is the adjoint of matrix \(L(s)\).

Similarly, assume that \(\lim_{u \to \infty} e^{-su} \phi_d(u) = 0\) and \(\lim_{u \to \infty} e^{-su} \phi'_d(u) = 0\) hold for \(\Re(s) > 0\). We have
\[
\hat{\phi}_d(s) = \frac{1}{\det[L(s)]} L^*(s) Q_d(s), \quad s \in \mathbb{C},
\] (2.10)
where \(Q_d(s) = (\frac{\sigma^2}{2} \phi'_d(0;1) + \frac{\sigma^2}{2} s + c, \frac{\sigma^2}{2} \phi'_d(0;2) + \frac{\sigma^2}{2} s + c, \ldots, \frac{\sigma^2}{2} \phi'_d(0;n) + \frac{\sigma^2}{2} s + c)^T\).

It is observed from (2.9) and (2.10) that the explicit expressions for the Gerber-Shiu functions are closely related to the roots of equation: \(\det[L(s)] = 0\). This is discussed in the next section.

3. Solutions of Lundberg’s fundamental equation

Let \(a(s) = \frac{\sigma^2}{2} s^2 + cs - \delta\). For values of \(s\) such that the matrix \(a(s)I + B\) is invertible, using the same arguments as in Ren (2007), we have
\[
\det[L(s)] = \det[a(s)I + B]\det[I + (a(s)I + B)^{-1}b^T \hat{\alpha} \hat{p}(s)] = \det[a(s)I + B][1 + \hat{\alpha}(a(s)I + B)^{-1}b^T \hat{p}(s)] = \det[a(s)I + B][1 - \hat{k}(-a(s)) \hat{p}(s)],
\] (3.1)
where we utilize (2.1) in the last step. Since the matrix \(a(s)I + B\) is assumed to be invertible, (3.1) indicates that the solutions for \(\det[L(s)] = 0\) and the solutions for Lundberg’s fundamental equation
\[
\hat{k}(\delta - cs - \frac{\sigma^2 s^2}{2}) \hat{p}(s) = 1, \quad s \in \mathbb{C},
\] (3.2)
as defined in Gerber and Shiu (2005a) and Li and Garrido (2005) are identical.

**Theorem 3.1.** For $\delta > 0$, Lundberg’s fundamental equation in (3.2) has exactly $n$ roots, say $\rho_1(\delta, \sigma), \rho_2(\delta, \sigma), \ldots, \rho_n(\delta, \sigma)$ with a positive real part $\Re(\rho_i(\delta, \sigma)) > 0$.

**Proof.** The idea of the proof comes from Gerber and Shiu (2005b). Let $\gamma(s) = 1/\hat{k}(-a(s))$. Then, as pointed out by Gerber and Shiu (2005b), its zero occurs at $-a(s) = \xi$, where $\xi$ ranges over all $n$ eigenvalues of $B$. However, $B$ is an intensity matrix of all transient states in a continuous time Markov chain, it is nonsingular matrix with all its eigenvalues having negative real parts (see Corollary 8.2.1 in Rolski et al. (1999)). Therefore, we see that $\gamma(s)$ has exactly $n$ positive zeros and $n$ negative zeros.

Consider a domain that is a half disk centered at 0, lying in the right half of the complex plane, and with a sufficiently large radius. For $\Re(s) \geq 0$, obviously that $|\hat{p}(s)| \leq 1$. Because $\gamma(s)$ has exactly $n$ positive zeros. The theorem follows from Rouché’s theorem if we can show that $|\gamma(s)| > 1$ on the boundary of such a half disk.

Since phase-type distribution belongs to the rational family distributions (see Section 4.2., below), $\gamma(s)$ has the form of the ratios of two polynomials, in which the degree of the denominator is less than the degree of nominator. Then $|\gamma(s)| > 1$ for $|s|$ sufficiently large. Now, for $s$ on the imaginary axis, $\Re(s) = 0$, we have $|\gamma(s)| > 1/|\hat{k}(-a(s))| > 1$ also. This ends the proof. \(\square\)

**Remark 3.1.** If $\delta \to 0+$ then $\rho_i(\delta, \sigma) \to \rho_i(0, \sigma)$ for $1 \leq i \leq n$. and Eq. (3.2) becomes

$$\hat{k}(-cs - \frac{\sigma^2 s^2}{2})\hat{p}(s) = 1, \quad s \in \mathbb{C}.$$

Evidently 0 is one of the roots.

In the rest of paper, $\rho_i(\delta, \sigma)$ are simply denoted by $\rho_i$ for $1 \leq i \leq n$ and $\delta \geq 0$. 


4. Main results

4.1. Explicit expressions for $\hat{\phi}$

Here, we recall the concept of dividend differences (see Gerber and Shiu (2005a)). For a function $L(s)$, its dividend differences, with respects to distinct numbers $\rho_1, \rho_2, \ldots$, can be defined recursively as:

$$L(s) = L(\rho_1) + (s - \rho_1)L[\rho_1, s], \quad L[\rho_1, s] = L[\rho_1, \rho_2] + (s - \rho_2)L[\rho_1, \rho_2, s], \ldots$$

The definition of the dividend differences obviously can be extended to any vector or matrix that is a function of a single variable. For example, for matrix $L(s)$,

$$L[\rho_1, s] = \frac{L(s) - L(\rho_1)}{s - \rho_1},$$

$$L[\rho_1, \rho_2, s] = \frac{L[\rho_1, s] - L[\rho_1, \rho_2]}{s - \rho_2},$$

and so on.

We assume that the roots $\rho_1, \rho_2, \ldots, \rho_n$ are distinct in the sequel. Since $\hat{\phi}_w(s)$ is finite for $\Re(s) \geq 0$. From (2.9) that

$$\frac{\sigma^2}{2} L^*(\rho_i) \phi_w'(0) = L^*(\rho_i) \hat{\omega}(\rho_i) b^T, \quad \text{for } i = 1, 2, \ldots, n.$$ 

Therefore

$$\frac{\sigma^2}{2} L^*[\rho_1, \rho_2] \phi_w'(0) = \left[ L^*(\rho_1) \hat{\omega}[\rho_1, \rho_2] + L^*[\rho_1, \rho_2] \hat{\omega}(\rho_2) \right] b^T.$$ 

and recursively

$$\frac{\sigma^2}{2} L^*[\rho_1, \rho_2, \ldots, \rho_n] \phi_w'(0) = \left[ \sum_{i=1}^{n} L^*[\rho_1, \rho_2, \ldots, \rho_i] \hat{\omega}[\rho_1, \ldots, \rho_n] \right] b^T.$$ 

Then we obtain the following results for $\phi'_w(0)$ and $\phi'_d(0)$:
Theorem 4.1. The differential of Gerber-Shiu functions at zero can be given by

\[
\phi'_w(0) = \frac{1}{\sigma^2/2} \left( \sum_{i=1}^{n} L^*[\rho_1, \rho_2, \ldots, \rho_i] \hat{\omega}[\rho_i, \ldots, \rho_n] \right) b^T, \tag{4.1}
\]

\[
\phi'_d(0) = -\frac{1}{\sigma^2/2} \left( \sum_{i=1}^{n} L^*[\rho_1, \rho_2, \ldots, \rho_i] e^T (\frac{\sigma^2}{2} \rho_n + c) + \frac{\sigma^2}{2} L^*[\rho_1, \rho_2, \ldots, \rho_{n-1}] e^T \right). \tag{4.2}
\]

Applying the dividend differences repeatedly to the numerator of Eq. (2.9) and (2.10), we obtain the following explicit expressions for the Laplace transform of Gerber-Shiu functions:

Theorem 4.2. The Laplace transform of Gerber-Shiu functions are given by

\[
\hat{\phi}_w(s) = \prod_{i=1}^{n} \frac{(s - \rho_i)}{\det[L(s)]} \left\{ L^*[\rho_1, \rho_2, \ldots, \rho_n] \left( \frac{\sigma^2}{2} \phi'_w(0) - \hat{\omega}(\rho_n) b^T \right) - L^*[\rho_1, \ldots, \rho_{n-1}, s] b^T \hat{\omega}[\rho_n, s] \right\}, \tag{4.3}
\]

and

\[
\hat{\phi}_d(s) = \prod_{i=1}^{n} \frac{(s - \rho_i)}{\det[L(s)]} \left\{ L^*[\rho_1, \rho_2, \ldots, \rho_n] \left( \frac{\sigma^2}{2} \phi'_d(0) + \left( \frac{\sigma^2}{2} \rho_n + c \right) e^T \right) + \frac{\sigma^2}{2} L^*[\rho_1, \rho_2, \ldots, \rho_{n-1}, s] e^T \right\}. \tag{4.4}
\]

Proof. By the fact that \( s = \rho_1 \) is a root of the numerator in (2.9), we get

\[
L^*[s]Q_w(s) = L^*[s]Q_w(s) - L^*[\rho_1]Q(\rho_1)
\]

\[
= [L^*[s] - L^*(\rho_1)] Q_w(s) + L^*(\rho_1) [Q_w(s) - Q_w(\rho_1)]
\]

\[
= (s - \rho_1) \left\{ L^*[\rho_1, s] Q_w(s) + L^*(\rho_1) Q_w[\rho_1, s] \right\}
\]

\[
= (s - \rho_1) \left\{ L^*[\rho_1, s] Q_w(s) - L^*(\rho_1) b^T \hat{\omega}[\rho_1, s] \right\}. \tag{4.5}
\]
Further note that \( s = \rho_2 \) is also a root of numerator in (2.9), implying that \( s = \rho_2 \) is a zero of the expression within the braces in (4.5), Then

\[
\begin{align*}
\mathbf{L}^*(s)Q_w(s) &= (s - \rho_1) \left\{ \left[ \mathbf{L}^*[\rho_1, s]Q_w(s) - \mathbf{L}^*(\rho_1)b^T\hat{\omega}[\rho_1, s] \right] \\
&\quad - \left[ \mathbf{L}^*[\rho_1, \rho_2]Q_w(\rho_2) - \mathbf{L}^*(\rho_1)b^T\hat{\omega}[\rho_1, \rho_2] \right] \right\} \\
&= (s - \rho_1)(s - \rho_2) \left\{ \mathbf{L}^*[\rho_1, \rho_2, s]Q_w(s) - \right. \\
&\quad \left. \sum_{i=1}^{2} \mathbf{L}^*[\rho_1, \rho_i]b^T\hat{\omega}[\rho_i, \rho_2] \right\},
\end{align*}
\]

recursively from the fact \( s = \rho_3, \ldots, s = \rho_{n-1} \) are roots of the numerator in (2.9) we obtain

\[
\begin{align*}
\mathbf{L}^*(s)Q_w(s) &= \prod_{i=1}^{n-1}(s - \rho_i) \left\{ \mathbf{L}^*[\rho_1, \ldots, \rho_{n-1}, s]Q_w(s) \\
&\quad - \sum_{i=1}^{n-1} \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_i]b^T\hat{\omega}[\rho_i, \ldots, \rho_{n-1}, s] \right\}. \quad (4.6)
\end{align*}
\]

Further note that \( s = \rho_n \) is a root of the numerator in (2.9), from (4.6) that

\[
\begin{align*}
\mathbf{L}^*(s)Q_w(s) &= \prod_{i=1}^{n-1}(s - \rho_i) \left\{ \mathbf{L}^*[\rho_1, \ldots, \rho_{n-1}, s](Q_w(s) - Q_w(\rho_n)) + \\
&\quad (\mathbf{L}^*[\rho_1, \ldots, \rho_{n-1}, s] - \mathbf{L}^*[\rho_1, \ldots, \rho_{n-1}, \rho_n])Q_w(\rho_n) - \\
&\quad \sum_{i=1}^{n-1} \mathbf{L}^*[\rho_1, \ldots, \rho_i]b^T(\hat{\omega}[\rho_i, \ldots, \rho_{n-1}, s] - \hat{\omega}[\rho_i, \ldots, \rho_{n-1}, \rho_n]) \right\} \\
&= \prod_{i=1}^{n-1}(s - \rho_i) \left\{ \mathbf{L}^*[\rho_1, \ldots, \rho_{n-1}, s]Q_w(\rho_n) + \mathbf{L}^*[\rho_1, \ldots, \rho_n, s]Q_w(\rho_n) - \\
&\quad \sum_{i=1}^{n-1} \mathbf{L}^*[\rho_1, \ldots, \rho_i]b^T\hat{\omega}[\rho_i, \ldots, \rho_n, s] \right\},
\end{align*}
\]

thus formula (4.3) is derived. Formula (4.4) can be proofed in the same way. \( \square \)

**4.2. Closed form expressions of \( \phi \) for rational family claim-size distribution**
In some cases the functions $\phi_w(u)$ and $\phi_d(u)$ can be explicitly and analytically determined by inversion of (4.3) and (4.4), respectively. Consider the case where the claim-size distribution $P$ belongs to the rational family, i.e., its density Laplace transform is of the form

$$\hat{p}(s) = \frac{r_{m-1}(s)}{r_m(s)}, \quad m \in \mathbb{N}^+,$$

where $r_{m-1}(s)$ is a polynomial of degrees $m - 1$ or less, while $r_m(s)$ is a polynomial of degrees $m$ with only negative roots, all have leading coefficient 1 and satisfy $r_{m-1}(0) = r_m(0)$. This wide class of distributions includes the Erlang, Coxian and phase-type distributions, and also the mixtures of these (see Cohen (1982); Tijms (1984)).

Multiply both numerator and denominator of Eq. (4.3) by $r_m(s)$, yielding

$$\hat{\phi}_w(s) = \frac{\prod_{i=1}^n (s - \rho_i)}{r_m(s) \det[L(s)]} \left( r_m(s) L^*[\rho_1, \rho_2, \ldots, \rho_n, s] (\frac{\sigma^2}{2} \phi_w'(0) - \hat{\omega}(\rho_n) b^T) - r_m(s) \right) \left( L^*[\rho_1, \ldots, \rho_{n-1}, s] b^T \hat{\omega}[\rho_n, s] - r_m(s) \sum_{i=1}^{n-1} L^*[\rho_1, \rho_2, \ldots, \rho_i] b^T \hat{\omega}[\rho_i, \ldots, \rho_n, s] \right) \quad (4.7)$$

Clearly that $r_m(s) \det[L(s)]$ is a polynomial of degrees $m+2n$ with leading coefficient $(\frac{\sigma^2}{2})^n$. So the equation $r_m(s) \det[L(s)] = 0$ has $m + 2n$ roots on the complex plane. By Theorem 3.1. and the definition of the rational distribution, $r_m(s) \det[L(s)] = 0$ has $n$ positive real roots $\rho_1, \rho_2, \ldots, \rho_n$, and $m + n$ negative real roots. Thus we can express $r_m(s) \det[L(s)]$ by all its roots, i.e.

$$r_m(s) \det[L(s)] = (\frac{\sigma^2}{2})^n \prod_{i=1}^n (s - \rho_i) \prod_{i=1}^{m+n} (s + R_i),$$

where all $R_i$’s have positive real parts. For simplicity we assume that these $R_i$’s are distinct. Cancel the term $\prod_{i=1}^n (s - \rho_i)$ from both numerator and denominator of (4.7).
Consequently Eq. (4.7) can be rewritten as

\[
\hat{\phi}_w(s) = \frac{1}{(\sigma^2 n) \prod_{i=1}^{m+n} (s + R_i)} \left\{ \frac{r_m(s) \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_n, s]}{\frac{\sigma^2}{2} \phi'_w(0) - \hat{\omega}(\rho_n) \mathbf{b}^T} - r_m(s) \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_n, s] \right\} (4.8)
\]

It is not difficult to see that the elements in matrix \( r_m(s) \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_n, s] \) or \( r_m(s) \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_{n-1}, s] \) are polynomials of degrees which are less than \( m + n \). and all \( \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_i] \) for \( i = 1, 2, \ldots, n \) are constants. By decomposing the rational expressions in Eq. (4.8) into partial fractions, we get

\[
\frac{r_m(s) \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_j, s]}{\prod_{i=1}^{m+n} (s + R_i)} = \sum_{i=1}^{m+n} \frac{M_i^{(j)}}{s + R_i}, \quad \text{for } j = n - 1, n,
\]

where \( M_i^{(j)} \) for \( i = 1, 2, \ldots, m + n \) are coefficient matrices with

\[
M_i^{(j)} = \frac{r_m(-R_i) \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_j, -R_i]}{\prod_{\ell=1, \ell \neq i}^{m+n} (R_\ell - R_i)}, \quad \text{for } j = n - 1, n, (4.9)
\]

and

\[
\frac{r_m(s)}{\prod_{i=1}^{m+n} (s + R_i)} = \sum_{i=1}^{m+n} \frac{G_i}{s + R_i},
\]

where \( G_i \) for \( i = 1, 2, \ldots, m + n \) are coefficients given by

\[
G_i = \frac{r_m(-R_i)}{\prod_{\ell=1, \ell \neq i}^{m+n} (R_\ell - R_i)}. (4.10)
\]

Thus, by partial fraction Eq. (4.8) can be expressed as

\[
\hat{\phi}_w(s) = \frac{1}{(\sigma^2/2)^n} \sum_{i=1}^{m+n} \frac{1}{s + R_i} \left\{ M_i^{(n)} \mathbf{Q}_w(\rho_n) - M_i^{(n-1)} \mathbf{b}^T \hat{\omega}[\rho_n, s] \right. \\
\left. - G_i \sum_{\ell=1}^{n-1} \mathbf{L}^*[\rho_1, \rho_2, \ldots, \rho_\ell] \mathbf{b}^T \hat{\omega}[\rho_\ell, \ldots, \rho_n, s] \right\} (4.11)
\]

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and similarly,

$$
\hat{\Phi}_d(s) = \frac{1}{(\sigma^2/2)^n} \sum_{i=1}^{m+n} \frac{1}{s + R_i} \left\{ M_i^{(n)} Q_d(\rho_n) + \frac{\sigma^2}{2} M_i^{(n-1)} e^T \right\}. 
$$

(4.12)

In order to determine the explicit Laplace inverse of \( \hat{\phi}_w(s) \) and \( \hat{\phi}_d(s) \), we define the same operator \( T_x \) as in Dickson and Hipp (2001), i.e., for an integrable real valued function \( f \) with respect to a complex number \( x \) \((\Re(x) \geq 0)\):

$$
T_x f(x) = \int_x^\infty e^{-x(y-x)} f(y) dy, \quad x \geq 0.
$$

It is clear that the Laplace transform of \( f \), \( \hat{f}(s) \), can be expressed as \( T_s f(0) \), and for distinct \( x_1, x_2 \in \mathbb{C} \)

$$
T_{x_1} T_{x_2} f(x) = T_{x_2} T_{x_1} f(x) = \frac{T_{x_1} f(x) - T_{x_2} f(x)}{x_2 - x_1}, \quad x \geq 0.
$$

Properties of this operator can be found in Li and Garrido (2004). Gerber and Shiu (2005a) also presented the following useful result on the relationship between the operator \( T_x \) and the corresponding dividend difference:

$$
\left[ \prod_{i=1}^n T_{x_i} f \right](0) = (-1)^{n+1} \hat{f}[x_1, x_2, \ldots, x_n].
$$

(4.13)

By (4.13) and \( T_s f(0) = \hat{f}(s) \), Reserving the Laplace transform of Eq. (4.11) and Eq. (4.12), we get the following theorem.

**Theorem 4.3.** If the claim-size distribution belongs to the rational family, the Gerber-Shiu functions are given by:

$$
\Phi_w(u) = \frac{1}{(\sigma^2/2)^n} \sum_{i=1}^{m+n} \left\{ M_i^{(n)} Q_w(\rho_n) e^{-R_i u} + e^{-R_i u} \right\} \left[ M_i^{(n-1)} b^T T_{\rho_n} \omega(u) + G_i \sum_{\ell=1}^{n-1} (-1)^{n-\ell} L^* \left[ \rho_1, \rho_2, \ldots, \rho_{\ell} \right] b^T \left( \prod_{j=\ell}^{n} T_{\rho_j} \right) \omega(u) \right\},
$$

(4.14)
where \( * \) is the convolution operator, the constants \( M_i^{(j)} (j = n-1, n) \), \( G_i \) and \( Q_w(\rho_n) = \frac{\sigma^2}{2} \phi_w'(0) - \hat{\omega}(\rho_n)b^T \) are obtained by Eq. (4.9), (4.10) and (4.1) respectively. And

\[
\phi_d(u) = \frac{1}{(\sigma^2/2)^n} \sum_{i=1}^{m+n} e^{-R_iu} \left\{ M_i^{(n)} Q_d(\rho_n) + \frac{\sigma^2}{2} M_i^{(n-1)} e^T \right\},
\]

where \( Q_d(\rho_n) = \frac{\sigma^2}{2} \phi_d'(0) + (\frac{\sigma^2}{2} \rho_n + c)e^T \) can be calculated from (4.2).

4.3. Ruin probability

This subsection illustrates the application of the previous results in a special case that \( \delta = 0 \), \( w_0 = 1 \), \( w(x,y) = 1 \), \( p(x) = \beta e^{-\beta x} \) for \( x > 0 \) and the interclaim times follow (2.8) with \( n=2 \). Then (4.14) and (4.15) become the ruin probabilities caused by a claim and oscillation respectively. We now have \( \hat{p}(s) = \beta/(s + \beta) \), \( \omega(u) = e^{\beta u} \) for \( u \geq 0 \). The matrix

\[
L(s) = \begin{pmatrix}
\frac{\sigma^2}{2} s^2 + cs - \lambda_1 & \lambda_1 \\
\frac{\beta \lambda_2}{s + \beta} & \frac{\sigma^2}{2} s^2 + cs - \lambda_2
\end{pmatrix}
\]

has exactly two positive real roots \( \rho_1 = 0 \), \( \rho_2 \) and three negative real roots \( -R_1, -R_2 \) and \( -R_3 \). It follows form (4.1) and (4.2) that

\[
Q_w(\rho_2) = \frac{\sigma^2}{2} \phi_w'(0) - \hat{\omega}(\rho_2)b^T
\]

\[
= \frac{\lambda_1 \lambda_2}{\beta(\beta + \rho_2)(\frac{\sigma^2}{2} \rho_2 + c)^2} \left( \frac{\sigma^2}{2} \rho_2 + c \quad \frac{\sigma^2}{2} \rho_2 + c - \frac{\lambda_2}{\beta + \rho_2} \right)^T,
\]

\[
Q_d(\rho_2) = \frac{\sigma^2}{2} \phi_d'(0) + (\frac{\sigma^2}{2} \rho_2 + c)b^T
\]

\[
= \frac{\lambda_1 + \lambda_2}{\rho_2 + 2c/\sigma^2} \left( 1 \quad 1 - \frac{\lambda_2}{(\beta + \rho_2)/\sigma^2} \right)^T.
\]

Moreover, noting that \( \psi_w(u) = \alpha \psi_w(u) = \psi_w(u; 1) \) and \( \psi_d(u) = \psi_d(u; 1) \), together
with (4.14) and (4.15) give the following formulae for ruin probabilities

\[
\psi_w(u) = \frac{1}{(\sigma^2/2)^2} \sum_{i=1}^{3} \frac{\lambda_1 \lambda_2}{\beta (\beta + \rho_2)} \prod_{\ell \neq i} (R_\ell - R_i) \left\{ (1 + \frac{\sigma^2}{2}(\beta - R_i)) e^{-R_i u} - e^{-\beta u} \right\},
\]

(4.18)

\[
\psi_d(u) = \sum_{i=1}^{3} \frac{\beta - R_i}{\prod_{\ell \neq i} (R_\ell - R_i)} \left\{ (\lambda_1 + \lambda_2 - R_i + \frac{2c}{\sigma^2}) e^{-R_i u} \right\}.
\]

(4.19)

5. Numerical Examples

In this section, we will present some numerical examples. In all calculations \( c = 1, \sigma = 1, w_0 = 1, p(x) = e^{-x} \) for \( x \geq 0 \) are fixed. Let the interclaim times be distributed with phase-type representation

\[
\alpha = (1, 0), \quad B = \begin{pmatrix}
-1 & \frac{1}{2} \\
0 & -4
\end{pmatrix}, \quad b^T = \begin{pmatrix}
\frac{1}{2} \\
4
\end{pmatrix}.
\]

Then the mean \( E[V_1] = 9/8 \), which indicates the relative safety loading is \( \frac{1}{8} \).

Example 5.1. (Ruin probability) Let \( \delta = 0, w(x, y) = 1 \) for \( x \geq 0, y \geq 0 \). In this case, we have

\[
L(s) = \begin{pmatrix}
\frac{s^2}{2} + s - 1 + \frac{1}{2(s+1)} & \frac{1}{2} \\
4 & \frac{s^2}{2} + s - 4
\end{pmatrix},
\]

the Lundberg’s equation \( \text{det}[L(s)] = 0 \) has roots:

\[
\rho_1 = 0, \quad \rho_2 = 2.06412,
\]

and \( R_1 = 3.90909, \quad R_2 = 3.0744, \quad R_3 = 0.0806231. \)
Then from (4.14) and (4.15) the different ruin probability components become:

\[ \psi_w(u) = 0.27603e^{-3.90909u} - 0.8912e^{-3.0744u} + 0.6151e^{-0.0806231u}, \]

\[ \psi_d(u) = -0.2675e^{-3.90909u} + 0.9368e^{-3.0744u} + 0.33066e^{-0.0806231u}, \]

\[ \psi(u) = 0.00853e^{-3.90909u} + 0.0456e^{-3.0744u} + 0.9458e^{-0.0806231u}. \]

Fig. 1. Decomposition of the ruin probability.

With the help of Matlab, we get Fig. 1. for these ruin probabilities for different values of u, as well as their decomposition into the ruin probabilities due to claims and those due to oscillations. From the graph it can be observed that the ruin probability due to oscillations is a strictly decreasing function (from 1 to 0) of the initial surplus u. Moreover, when u is small, it decreases sharply, while it decreases slowly when u is large. By contrast, the ruin probability due to claims increases quickly at first but
then deceases slowly after that.

**Example 5.2.** (The Laplace transform of the ruin time) When \( w = 1 \), we give Fig. 2. of \( E[e^{-\delta T}I(T < \infty)|U(0) = u] \) in the case \( \delta = 0.1 \) and \( \delta = 0.2 \).

![Fig. 2. The Laplace transform of T.](image)

As expected, the Laplace transform of the ruin time is high for low \( \delta \), the force of interest.

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