Monopoles and string tension in SU(2) QCD

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Abstract

Monopole and photon contributions to abelian Wilson loops are calculated using Monte-Carlo simulations of SU(2) QCD in the maximally abelian gauge. The string tension is well reproduced only by monopole contributions, whereas photons alone are responsible for the Coulomb coefficient of the abelian static potential.

I. INTRODUCTION

QCD confines color, but color confinement mechanism is not yet clarified. A promising picture is the dual Meissner effect due to condensation of some color magnetic quantity [1,2]. This picture is realized in the confinement phase of lattice compact QED [3–5]. Especially interesting are the following facts:

1) A dual transformation can be done, leading us to an action describing a monopole Coulomb gas [4,6–8]. Monopole condensation is shown to occur in the confinement phase from energy-entropy balance.

2) The monopole contribution alone can reproduce the full value of the string tension [9].

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In the case of QCD, there is a difficult problem. We have to find a color magnetic quantity in QCD. The 'tHooft idea of abelian projection of QCD [10] is very interesting in this respect. The abelian projection of QCD is to extract an abelian theory performing a partial gauge-fixing such that the maximal abelian torus group remains unbroken. After the abelian projection, $SU(3)$ QCD can be regarded as a $U(1) \times U(1)$ abelian gauge theory with magnetic monopoles and electric charges. 't Hooft conjectured that the condensation of the abelian monopoles is the confinement mechanism in QCD [10].

There are, however, infinite ways of extracting such an abelian theory out of $SU(3)$ QCD. It seems important to find a good gauge in which the conjecture is seen clearly to be realized. A gauge called maximally abelian (MA) gauge has been shown to be very interesting [11–13]. In the MA gauge, there are phenomena which may be called abelian dominance [12,14]. Especially the string tension in $SU(2)$ QCD is well reproduced by Wilson loops composed of abelian link fields alone in the MA gauge. Moreover the monopole current $k_\mu(s)$ can be defined similarly as in compact QED [3]. It is shown in the MA gauge that the abelian monopoles are dense and dynamical in the confinement phase, whereas they are dilute and static in the deconfinement phase [13].

Recently we have derived an effective $U(1)$ monopole action in the MA gauge and in $SU(2)$ QCD [13,16]. Entropy dominance over energy of the monopole loops, i.e., condensation of the monopole loops seems to occur in the confinement phase if extended monopoles [17] are considered [15,16]. After the abelian projection in the MA gauge, infrared behaviors of $SU(2)$ QCD may be described by a compact-QED like $U(1)$ theory with the running coupling constant instead of the bare one and with the monopole mass on a dual lattice.

If the monopoles alone are responsible for the confinement mechanism, the string tension which is a key quantity of confinement must be explained by monopole contributions alone. This is realized in compact QED [3]. The aim of this note is to show that the same thing happens also in $SU(2)$ QCD by means of evaluating monopole and photon contributions to abelian Wilson loops. Preliminary results are reported by the present authors [15] and other group [18].
II. ABELIAN PROJECTION AND GAUGE INVARIANCE

We adopt the usual SU(2) Wilson action. The maximally abelian gauge is given \[11\] by performing a local gauge transformation

\[
V(s) \rightarrow \tilde{V}(s, \hat{\mu}) = V(s)U(s, \hat{\mu})V^{-1}(s + \hat{\mu}).
\]

is maximized. Here

\[
\tilde{U}(s, \hat{\mu}) = V(s)U(s, \hat{\mu})V^{-1}(s + \hat{\mu}).
\]

After the gauge fixing is over, there still remains a $U(1)$ symmetry. We can extract an abelian link gauge variable from the $SU(2)$ ones as follows;

\[
\tilde{U}(s, \hat{\mu}) = A(s, \hat{\mu})u(s, \hat{\mu}),
\]

where $u(s, \hat{\mu})$ is diagonal and $A(s, \hat{\mu})$ has off-diagonal components. It is easy to show that the above fields $u(s, \hat{\mu})$ and $A(s, \hat{\mu})$ behave under the residual $U(1)$ transformation $d(s)$ as an abelian gauge field and charged matters, respectively:

\[
u(s, \hat{\mu}) \rightarrow u'(s, \hat{\mu}) = d(s)u(s, \hat{\mu})d^*(s + \hat{\mu}),
\]

\[
A(s, \hat{\mu}) \rightarrow A'(s, \hat{\mu}) = d(s)A(s, \hat{\mu})d^*(s).
\]

Let us repeat that a $U(1)$ invariant quantity written in terms of the abelian link variables $u(s, \hat{\mu})$ after an abelian projection can be rewritten in a $SU(2)$ invariant (but complicated) form using the original link variables $U(s, \hat{\mu})$ \[14\]. The gauge function $V(s)$ which maximizes $R$ is a functional of $U(s, \hat{\mu})$. First study the transformation property of $V(s)$ under any $SU(2)$ transformation $W(s)$, fixing the $U(1)$ ambiguity of $V(s)$ in some way. Considering that a gauge-fixed quantity does not transform, we see

\[
V(s) \rightarrow V^W(s) = d(s)V(s)W^{-1}(s),
\]

where $d(s) \in U(1)$ ensures the form invariance of $V^W(s)$ and $V(s)$ and is determined uniquely by $V(s)$ and $W(s)$. Using the definition \[11\] and the transformation property \[5\], we get
\[(\tilde{U}(s, \hat{\mu}))^W = d(s)\tilde{U}(s, \hat{\mu})d^\dagger(s + \hat{\mu})\]  

(6)

for any \(SU(2)\) transformation \(W(s)\). Hence all \(U(1)\) invariant quantities composed of \(\tilde{U}(s, \hat{\mu})\) (and \(u(s, \hat{\mu})\)) are automatically \(SU(2)\) invariant after the abelian projection.

As an example, consider a \(U(1)\)-invariant \(1 \times 1\) plaquette variable \(u_P(s, \mu, \nu)\) composed of \(u(s, \hat{\mu})\) alone. Using (1) and (2), we get

\[u_P(s, \mu, \nu) = \frac{1}{2}\text{Tr}(u(s, \hat{\mu})u(s + \hat{\mu}, \hat{\nu})u^\dagger(s + \hat{\nu}, \hat{\mu})u^\dagger(s, \hat{\nu}))\]

(7)

\[= \frac{1}{2}\text{Tr}(U'(s, \hat{\mu})U'(s + \hat{\mu}, \hat{\nu})U'^\dagger(s + \hat{\nu}, \hat{\mu})U'^\dagger(s, \hat{\nu}))\],

(8)

where \(U'(s, \hat{\mu})\) is a modified link field defined by \(U'(s, \hat{\mu}) = B(s, \hat{\mu})U(s, \hat{\mu})\) and \(B(s, \hat{\mu}) = V^\dagger(s)A^\dagger(s, \hat{\mu})V(s)\). Under an arbitrary \(SU(2)\) transformation \(W(s)\), we see from (4), (5) and (6)

\[B(s, \hat{\mu}) \rightarrow W(s)B(s, \hat{\mu})W^\dagger(s),\]

(9)

\[U'(s, \hat{\mu}) \rightarrow W(s)U'(s, \hat{\mu})W^\dagger(s + \hat{\mu}).\]

(10)

Namely, \(U'(s, \hat{\mu})\) transforms like the corresponding original link field \(U(s, \hat{\mu})\). The \(SU(2)\) invariance of \(u_P(s, \mu, \nu)\) is seen also from this property. Moreover, it is seen how different the abelian plaquette variable is from the full one. \(u_P(s, \mu, \nu)\) in a different abelian projection corresponds to a differently modified full plaquette variable composed of \(U'(s, \hat{\mu})\).

III. MONOPOLE AND PHOTON CONTRIBUTIONS TO ABELIAN WILSON LOOPS

We show an abelian Wilson loop operator written in terms of \(u(s, \hat{\mu})\) alone after the abelian projection is given by a product of monopole and photon contributions. Here we take into account only a simple Wilson loop, say, of size \(I \times J\). Then such an abelian Wilson loop operator is expressed as

\[W = \exp\{i \sum J_\mu(s)\theta_\mu(s)\},\]

(11)
where $J_\mu(s)$ is an external current taking ±1 along the Wilson loop and $\theta_\mu(s)$ is an angle variable defined from $u(s, \hat{\mu})$ as follows:

$$u(s, \hat{\mu}) = \begin{pmatrix} e^{i\theta_\mu(s)} & 0 \\ 0 & e^{-i\theta_\mu(s)} \end{pmatrix}.$$  \hfill (12)

Since $J_\mu(s)$ is conserved, it is rewritten for such a simple Wilson loop in terms of an antisymmetric variable $M_{\mu\nu}(s)$ as $J_\nu(s) = \partial'_\mu M_{\mu\nu}(s)$, where $\partial'$ is a backward derivative on a lattice. $M_{\mu\nu}(s)$ takes ±1 on a surface with the Wilson loop boundary. Although we can choose any surface of such a type, we adopt a minimal surface here. We get

$$W = \exp\{-i \sum M_{\mu\nu}(s) f_{\mu\nu}(s)\},$$  \hfill (13)

where $f_{\mu\nu}(s) = \partial_\mu \theta_\nu(s) - \partial_\nu \theta_\mu(s)$ and $\partial_\mu$ is a forward derivative on a lattice. The gauge plaquette variable can be decomposed into $f_{\mu\nu}(s) = \bar{f}_{\mu\nu}(s) + 2\pi n_{\mu\nu}(s)$ where $\bar{f}_{\mu\nu}(s) \in [-\pi, \pi]$ corresponds to a field strength and $n_{\mu\nu}(s)$ is an integer-valued plaquette variable denoting the Dirac string. Since $M_{\mu\nu}(s)$ and $n_{\mu\nu}(s)$ are integers, the latter does not contribute to Eq. (13). Hence $f_{\mu\nu}(s)$ in Eq. (13) is replaced by $\bar{f}_{\mu\nu}(s)$. Using a decomposition rule

$$M_{\mu\nu}(s) = -\sum D(s - s')\left[\partial'_{\alpha}(\partial_\mu M_{\alpha\nu} - \partial_\nu M_{\alpha\mu})(s') \right.$$  

$$\left. \quad + \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \epsilon_{\alpha'\beta'\rho\sigma} \partial'_{\alpha} \partial'_{\alpha'} M_{\rho\sigma}(s') \right],$$

we get

$$W = W_1 \cdot W_2$$  \hfill (14)

$$W_1 = \exp\{-i \sum \partial'_\mu \bar{f}_{\mu\nu}(s) D(s - s') J_\nu(s')\}$$  

$$W_2 = \exp\{2\pi i \sum k_\beta(s) D(s - s') \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} \partial_\alpha M_{\rho\sigma}(s')\},$$

where a monopole current $k_\mu(s)$ is defined as $k_\mu(s) = (1/4\pi) \epsilon_{\mu\alpha\beta\gamma} \partial_\alpha \bar{f}_{\beta\gamma}(s)$ following DeGrand-Toussaint. $D(s)$ is the lattice Coulomb propagator. Since $\bar{f}_{\mu\nu}(s)$ corresponds

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1. This condition $\bar{f}_{\mu\nu}(s) \in [-\pi, \pi]$ is applicable only in the case of defining a smallest $1^3$ monopole.
to the field strength of the photon field, \( W_1(W_2) \) is the photon (the monopole) contribution to the Wilson loop. To study the features of both contributions, we evaluate the expectation values \( \langle W_1 \rangle \) and \( \langle W_2 \rangle \) separately and compare them with those of \( W \).

**IV. SIMULATIONS AND EXTENDED MONOPOLES**

The Monte-Carlo simulations were done on \( 24^4 \) lattice from \( \beta = 2.4 \) to \( \beta = 2.7 \). All measurements were done every 30 sweeps after a thermalization of 1500 sweeps. We took 50 configurations totally for measurements. The gauge-fixing criterion is the same as done in Ref. \[19\]. Using gauge-fixed configurations, we evaluated monopole currents and obtained the ensemble of monopole currents.

As shown in the previous note \[15,16\], type-2 extended monopole loops with \( b > b_c \sim 5.2 \times 10^{-3}(\Lambda_L)^{-1} \) seem to condense, where \( b = na(\beta) \) for \( n^3 \) extended monopoles and \( a(\beta) \) is the lattice constant. So we measure \( 2^3 \) \((3^3)\) extended monopoles with \( b = 2a(\beta)(b = 3a(\beta)) \) of the type-2 \[17\] as well as usual smallest ones. The extended monopole of the type-2 is defined on an extended cube as the sum of the smallest ones included in the cube as shown in Fig. 1. Note that the definition of the type-2 extended monopoles corresponds to making a block spin transformation of the monopole currents with the scale factor \( n \). Hence the effective lattice volume is reduced. We call the effective lattice as a renormalized lattice. We evaluate the averages of \( W \) using abelian link variables (called abelian) on the original lattice, of \( W_1 \) (photon part), and \( W_2 \) (monopole part) on each renormalized lattice, separately.

**V. RESULTS**

The results are shown in the following:

1. The monopole contributions to Wilson loops are obtained with relatively small errors. Surprisingly enough, the Creutz ratios \( \chi(I,J) \) of the monopole contributions having
small errors are almost independent of the loop size for example as shown in Fig. 2. This means that the monopole contributions are composed only of an area, a perimeter, and a constant terms without a Coulomb term.

2. Assuming the static potential is given by a linear + Coulomb + constant terms, we try to determine the potential using the least square fit. There are various ways, but we adopt a method similar to that [20] using the Creutz ratios. The assumption of the form of the static potential leads us to the Creutz ratio

\[ \chi(I, J) = \chi_0 - \chi_1 \left( \frac{1}{I(I - 1)} + \frac{1}{J(J - 1)} \right) + \chi_2 \left( \frac{1}{I(I - 1)J(J - 1)} \right), \]

where \( \chi_0 \) is the string tension and \( \chi_1 \) corresponds to the Coulomb coefficient of the static potential. Using the fitted values of \( \chi_0 \) and \( \chi_1 \), we can reproduce each static potential.

We plot their data in Fig. 3 (at \( \beta = 2.5 \)) and in Fig. 4 (at \( \beta = 2.6 \)). We find the monopole contributions are responsible for the linear-rising behaviors. When the 2\(^3\) extended monopoles are used, we obtain almost the same results. The photon part contributes only to the short-ranged region. There seems to exist a small discrepancy between the abelian and the monopole + photon parts for \( R/a = 12 \), but finite-size effects are expected there. Moreover, the assumption of the form of the static potential looks too simple. Similar data are obtained for \( \beta = 2.4 \) and 2.7.

3. This is seen more clearly from the data of the string tensions which are determined from the static potentials. They are shown in Fig. 3. The full and the abelian string tensions have large errors but they are seen to be well reproduced by the monopoles alone for \( \beta \leq 2.7 \) and the photon part has almost vanishing string tensions.

4. The string tensions from the monopoles and the photons of various sizes are plotted in Fig. 3. It is interesting to see that they are almost independent of the extendedness contrary to our preliminary data on a smaller lattice [21], although the monopole
actions determined in [15,16] depend on the extendedness. Note that the extended
monopoles of the type-2 are composed of the sum of the smallest monopoles. Hence
even when the extended monopoles alone look to condense, the smallest monopoles
also have some information of the condensation.

5. We have derived also Coulomb coefficients from the static potentials as shown in Fig.
7. The monopole part has almost vanishing Coulomb coefficients which is in agreement
with the constant behaviors of the Creutz rations of the monopole part as shown above.
The $1^3$ photon part has large coefficients and they reproduce well the coefficients of
the abelian static potentials.

6. We also measured the Coulomb coefficients of the photon parts on each renormalized
lattice. The values depend on the extendedness. But the following fact is interesting.
The photon parts are evaluated on a renormalized lattice with $b = na(\beta)$. Hence they
have different values of $b$ for different extendedness. The Coulomb coefficients of the
photon parts are well reproduced by the $SU(2)$ running coupling constants $g(b)$ with
$b = na(\beta)$, i.e., $-g(b)^2/16\pi$, where

$$
g^{-2}(b) = \frac{11}{24\pi^2} \ln\left(\frac{1}{b^2 \Lambda^2}\right) + \frac{17}{44\pi^2} \ln \ln \left(\frac{1}{b^2 \Lambda^2}\right).
$$

The scale parameter $\Lambda$ determined is $\Lambda \sim 48\Lambda_L$ which is quite near to the value
$\Lambda \sim 42\Lambda_L$ fixed from the monopole action [15].

In conclusion, our analyses done here strongly suggest that abelian monopoles are respon-
sible for confinement in $SU(2)$ QCD and condensation of the monopoles is the confinement
mechanism if the abelian projection is done in the MA gauge. To extend our method to
a finite-temperature system and also to $SU(3)$ with or without dynamical quarks is very
interesting. These studies are also in progress.

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FIGURES

FIG. 1. An extended cube on which an \( 2^3 \) extended monopole is defined as the sum of eight \((=2^3)\) smallest monopoles.

FIG. 2. Creutz ratios \( \chi(I,J) \) from abelian and \( 2^3 \) monopole Wilson loops at \( \beta = 2.6 \) versus \( I \times J \). The monopole Creutz ratio values are divided by 4, being adjusted to those in unit \( a(\beta) \).

FIG. 3. Static potentials \( aV(R) \) versus \( R/a \) at \( \beta = 2.5 \). The values are shifted by a constant.

FIG. 4. Static potentials \( aV(R) \) versus \( R/a \) at \( \beta = 2.6 \). The values are shifted by a constant.

FIG. 5. String tensions at \( \beta = 2.4, 2.5, 2.6, \) and 2.7.

FIG. 6. String tensions on each renormalized lattice.

FIG. 7. Coulomb coefficients.

FIG. 8. Coulomb coefficients on each renormalized lattice.
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CREUTZ RATIOS

beta = 2.6

abelian

$2^3$ monopole
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\text{abelian} \quad \beta = 2.6

- \text{1}^3 \text{ monopole}
- \text{1}^3 \text{ photon}
- \text{1}^3 \text{ monopole+photon}
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string tensions

\[ \frac{\sigma}{(\Lambda)^2} \]

- full SU(2)
- abelian
- \(1^3\) monopole
- \(1^3\) photon
string tensions

- 1$^3$ monopole / photon
- 2$^3$ monopole / photon
- 3$^3$ monopole / photon
Coulomb coefficients

Lambda = 48(Lambda)_L

abelian
1^3 photon
1^3 monopole
abelian

1 \textsuperscript{3} photon

2 \textsuperscript{3} photon

3 \textsuperscript{3} photon

\Lambda = 48(\Lambda_L)

Coulomb coefficients

\begin{align*}
\text{BETA} & : \quad 2.4 \quad 2.5 \quad 2.6 \quad 2.7 \\
\end{align*}