Fall coloring and b-coloring of graphs

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Abstract. The b-chromatic number of G, denoted by \( \varphi(G) \), is the maximum k for which G has a b-coloring by k colors. A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex \( x_i \) having neighbors in all the other \( k-1 \) color classes. Such a vertex \( x_i \) is called a b-dominating vertex, and the set of vertices \( \{x_1, x_2 \ldots x_k\} \) is called a b-dominating system. A fall coloring of a graph G is a proper coloring, where every vertex of G is a b-dominating vertex. The fall chromatic number and the fall achromatic number of G are, respectively, the minimum and maximum integers \( \psi_f(G) \), \( \varphi_f(G) \) for which G has a fall coloring. In this paper, we obtain the results that, all these three coloring parameters are equal for the Complement graph of flower graph and sunflower graph and b-coloring for some girth graphs.

1. Introduction

The b-coloring concept was introduced by Irving and Manlove in\[7\], they prove that determining the b-chromatic number of a graph is an NP-complete problem. Kouider and Maheo \[9\] gave a useful remark that \( \chi_f(G) \leq \varphi(G) \leq \Delta(G) + 1 \), \( \Delta \) is the maximum degree of G. The b-chromatic number of G, denoted by \( \varphi(G) \), is the maximum k for which G has a b-coloring by k colors. A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex \( x_i \) having neighbors in all the other \( k-1 \) color classes. Such a vertex \( x_i \) is called a b-dominating vertex, and the set of vertices \( \{x_1, x_2 \ldots x_k\} \) is called a b-dominating system. A fall coloring of a graph G is a proper coloring, where every vertex of G is a b-dominating vertex. The fall chromatic number and the fall achromatic number of G are, respectively, the minimum and maximum integers \( \chi_f(G) \), \( \psi_f(G) \) for which G has a fall coloring. In this paper, we obtain the results that, all these three coloring parameters are equal for the Complement graph of flower graph and sunflower graph and b-coloring for some girth graphs.
Notion and terminology not mentioned here can be found in [2].

2. b-coloring and fall coloring of complement of flower graph

Theorem 2.1:

The complement graph of flower graph $Fl_n^c$, $n \geq 3$, is fall colorable and $b$-colorable with $n$ colors.

\[ i.e., \chi_f(Fl_n^c) = \varphi(Fl_n^c) = \psi_f(Fl_n^c) = n. \]

Proof:

Let $V(Fl_n) = v \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$.

By the definition of complement graph, let us connect the non-adjacent vertices of $Fl_n$. Then the vertex set of complement graph of $Fl_n$ is $V(Fl_n^c)$. The hub vertex $v$ is an isolated vertex. We consider the remaining vertices for the fall coloring and $b$-coloring.

Let $V(G) = V(Fl_n^c) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$.

Now let us assign the fall achromatic coloring to the vertex set of $V(G)$ by using the following function, $f : V(G) \rightarrow C_i : 1 \leq i \leq n$.

$f(u_i) = i, 1 \leq i \leq n$.
$f(v_i) = i, 1 \leq i \leq n$.

Therefore the vertices of $V(G)$ are colored by the above coloring process.

By the above coloring process $\exists n$ vertices of $V(G)$ with cardinality $n$ has the color class $C = \{c_i : 1 \leq i \leq n\}$.

$N(u_i) = \{u_1, u_2, \ldots, u_n\}$.

$C[N(u_i)] = \{c_i : 1 \leq i \leq n\}$.

$N(v_i) = \{v_1, v_2, \ldots, v_n\}$.

$C[N(v_i)] = \{c_i : 1 \leq i \leq n\}$.

Then $C[N(u_i)] \cup C[N(v_i)] = n$.

It gives the fall achromatic coloring. To prove it is maximum. Let us assume that, $\psi_f(Fl_n^c) > n$. Then there exists at least $n + 1$ vertices of degree $n$. The $V(G)$ has an $n$ clique formed by the set $\{u_i : 1 \leq i \leq n\}$ of degree $n - 1$. Then we can assign only $n$ colors to $V(G)$. Which is a contradiction to the fact that $\psi_f(Fl_n^c) > n$. Therefore $\psi_f(Fl_n^c) \leq n$. Hence $\psi_f(Fl_n^c) = n$. It is obvious that $\varphi(Fl_n^c) \leq \psi_f(Fl_n^c)$. Then $\varphi(Fl_n^c) = n$. The fall chromatic number is the minimum integer, where every vertex of $G$ is a $b$-dominating vertex. Since the graph $Fl_n^c$ has an $n$ clique, we should assign $n$ colors to the clique. Then the $\chi_f(Fl_n^c) = n$.

Therefore $\chi_f(Fl_n^c) = \varphi(Fl_n^c) = \psi_f(Fl_n^c) = n$. Hence the proof.

3. b-coloring and fall coloring of complement of sun flower graph

Theorem 3.1:

The complement graph of sun flower graph $SFl_n^c$, $n \geq 3$, is fall colorable and $b$-colorable with $2n$ colors.

\[ i.e., \chi_f(SFl_n^c) = \varphi(SFl_n^c) = \psi_f(SFl_n^c) = 2n. \]
Proof:

Let \( V(SFl_n) = v \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}. \)
Let us connect the non-adjacent vertices of \( SFl_n \), by using the complement graph definition. Then the vertex set of complement graph of \( SFl_n \) is \( V(SFl_n^c) \). The hub vertex \( v \) is an isolated vertex. We consider the remaining vertices for the fall coloring and b-coloring.
Let \( V(G) = V(SFl_n^c) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}. \)
Let us assign the fall achromatic coloring to the vertex set of \( V(G) \) by using the following function, \( f : V(G) \rightarrow C_i : 1 \leq i \leq 2n. \)
\[
f(u_i) = i, \quad 1 \leq i \leq n.
\]
\[
f(v_i) = 1, \quad 1 \leq i \leq n.
\]
\[
f(w_i) = n + i, \quad 1 \leq i \leq n.
\]
Therefore the vertices of \( V(G) \) are colored by the above coloring process.
By the above coloring process \( \exists 2n \) vertices of \( V(G) \) with cardinality \( 2n \) has the color class \( C = \{c_i : 1 \leq i \leq 2n\}. \)
\( N(u_i) = \{u_1, u_2, \ldots, u_n\}. \)
\( C[N(u_i)] = \{c_i : 1 \leq i \leq n\}. \)
\( N(v_i) = \{v_1, v_2, \ldots, v_n\}. \)
\( C[N(v_i)] = \{c_i : 1 \leq i \leq n\}. \)
\( N(w_i) = \{w_1, w_2, \ldots, w_n\}. \)
\( C[N(w_i)] = \{c_i : n + i \leq i \leq 2n\}. \)
Therefore \( C[N(u_i)] \cup C[N(v_i)] \cup C[N(w_i)] = 2n. \)
It gives the fall achromatic coloring. To prove it is maximum. Let us assume that, \( \psi_f(SFl_n^c) > 2n. \)
Then there exists atleast \( 2n + 1 \) vertices of degree \( 2n \). The \( V(G) \) has an \( 2n \) clique formed by the set \( \{u_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \) of degree \( 2n \). Then we can assign only \( 2n \) colors to \( V(G) \). Which is a contradiction to the fact that \( \psi_f(SFl_n^c) > 2n. \)
Then \( \psi_f(SFl_n^c) \leq 2n. \) Hence \( \psi_f(SFl_n^c) = 2n. \) It is obvious that \( \varphi(SFl_n^c) \leq \psi_f(SFl_n^c). \) Then \( \varphi(SFl_n^c) = 2n. \) The fall chromatic number is the minimum integer, where every vertex of \( G \) is a b-dominating vertex. The graph \( SFl_n^c \) has a \( 2n \) clique, we should assign \( 2n \) colors to the clique. Then the \( \chi_f(SFl_n^c) = 2n. \)
Therefore \( \chi_f(SFl_n^c) = \varphi(SFl_n^c) = \psi_f(SFl_n^c) = 2n. \) Hence the proof.

4. b-coloring of Four regular graph with girth three

Theorem 4.1:
For every four regular graph with girth three, \( n \geq 5 \) the b- chromatic number is
\[
\varphi [G_3 (n)] = \begin{cases} 
n \quad \text{ for } n = 5, \\
\left \lfloor \frac{n}{2} \right \rfloor \quad \text{ for } n = 6, 7, 8. \\
\Delta + 1 \quad \text{ for } n \geq 9. 
\end{cases}
\]
Proof:

Let \( P : 1, 2, 3, \ldots n \) be a set of \( p \) colors, which is used to assign colors to the vertices of \( G_3 (n). \)
Assign these coloring by the following cases

Case(i): For \( n = 5, \)
A Four regular graph with girth three is nothing but a \( k_5 \). We know that the chromatic number of \( K_5 \) is 5. By assigning \( n \) colors to \( G_3 (n) \) it satisfies the b-coloring.

Case(ii): For \( 6 \leq n \leq 8, \)
Assign the consecutive colors to the vertices of \( G_3 (n) \) respectively
\[
\{1, 2, 3, 4, 1, 2, 3, \text{for } n = 6\}
\]
\[
\{1, 2, 3, 4, 1, 2, 3, \text{for } n = 7\}
\]
\[
\{1, 2, 3, 4, 1, 2, 3, \text{for } n = 8\}
\]
Case(iii): For $n \geq 9$

Subcase(i): If $n$ is odd

$G_3(n)$ have the following coloring consecutively.

1,2,3,4,5,1,2,4,5, for $n = 9$.
1,2,3,4,5,1,2,3,1,4,5, otherwise

Subcase(ii): If $n$ is even

Assign the following coloring.

1,2,3,4,5,1,2,3,4,5, for $n = 5$

$i$, $i = 1,2,3,... n$

1,2,3,4,5,1,2,3,4,1,2,3,4,5 otherwise

By using the above coloring process we can say that $\varphi [G_3(n)] \geq 5$.

Suppose that we assume $\varphi [G_3(n)] = 6$.

Since the graph is four regular, $d(x_i), x_i \in V(G)$ is four.

we can possibly assign five colors as b-coloring. Therefore $\varphi [G_3(n)] \leq 5$. Hence $\varphi [G_3(n)] = 5 = \Delta + 1$. Hence the proof.

5. b-coloring of Four regular graph with girth four

Theorem 5.1:

For every four regular graph with girth four, $n \geq 8$ the b-chromatic number is

$\varphi [G_4(n)] = \left\{ \begin{array}{ll}
\lfloor n/4 \rfloor, & \text{for } n = 8,10. \\
\Delta, & \text{for } 9 \leq n \leq 13. \\
\Delta + 1, & \text{for } n \geq 14.
\end{array} \right.$

Proof:

Let $P : \{1,2,3,... n\}$ be a set of p colors, and let $f$ be a function that assigns colors to the vertices of $G_4(n)$.

i.e., $f : V (G_4(n)) \rightarrow P$.

Color the $V(G_4(n))$ by using the following coloring process.

- For $n= 8,10$, assign the colors $p_1$ and $p_2$ consecutively.
- For $n=9$, assign the colors $p_1, p_2, p_3, p_2, p_3, p_1, p_3, p_1, p_2$.
- For $11 \leq n \leq 13$, assign the colors $p_1, p_2, p_3, p_4, p_3, p_1, p_2, p_1, \ldots p_4, p_2$ respectively.
- For $n \geq 14$, if $n$ is multiplies of 5 assign the colors $p_1, p_2, p_3, p_4, p_5$ repeatedly.
  Otherwise, assign the colors $p_1, p_2, p_3, p_4, p_5, \ldots p_1, p_2, p_3, p_4, p_3, p_4, p_5$.

By using the above coloring process we can say that $\varphi [G_4(n)] \geq 5$.

Suppose that we assume $\varphi [G_4(n)] = 6$.

Since the graph is four regular, $d(x_i), x_i \in V(G)$ is four.

we can possibly assign five colors as b-coloring. Therefore $\varphi [G_4(n)] \leq 5$. Hence $\varphi [G_4(n)] = 5 = \Delta + 1$. Hence the proof.
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