BOUNDS FOR SPECTRAL PROJECTORS ON THE EUCLIDEAN CYLINDER

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Abstract. We prove essentially optimal bounds for norms of spectral projectors on thin spherical shells for the Laplacian on the cylinder \((\mathbb{R}/\mathbb{Z}) \times \mathbb{R}\). In contrast to previous investigations into spectral projectors on tori, having one unbounded dimension available permits a compact self-contained proof.

1. Introduction

1.1. Spectral projectors on general manifolds and tori. Given a Riemannian manifold with Laplace-Beltrami operator \(\Delta\), consider the spectral projector \(P_{\lambda,\delta}\) on (perhaps generalized) eigenfunctions with eigenvalues within \(O(\delta)\) of \(\lambda\). It is defined through functional calculus by the formula

\[ P_{\lambda,\delta} = P_{\lambda,\delta}^\chi = \chi\left(\frac{\sqrt{-\Delta} - \lambda}{\delta}\right), \]

where \(\chi\) is a cutoff function, which is irrelevant for our purposes.

An interesting question is to determine the operator norm from \(L^2\) to \(L^p\), with \(p > 2\), of this operator. A theorem of Sogge \([5]\) gives an optimal answer for any Riemannian manifold if \(\delta = 1\)

\[ \|P_{\lambda,1}\|_{L^2 \to L^p} \lesssim \lambda^{d-1/2 - \frac{d}{p}} + \lambda^{d-1/2} \left(\frac{1}{2} - \frac{1}{p}\right), \]

While this completely answers the question if \(\delta > 1\), the case \(\delta < 1\) is still widely open. Understanding the case \(\delta < 1\) requires a global analysis on the Riemannian manifold, which makes it very delicate.

In the case of the rational torus \(\mathbb{R}^d/\mathbb{Z}^d\), \(L^p\) bounds on eigenfunctions attracted a lot of attention; this corresponds to the choice \(\delta = 1/\lambda\). The best result in this direction is due to Bourgain and Demeter \([3]\). More recently, the authors of the present paper \([4]\) considered the problem for general values of \(\lambda\) and \(\delta\), conjectured the bound for general tori

\[ \|P_{\lambda,\delta}\|_{L^2 \to L^p} \lesssim \lambda^{d-1/2 - \frac{d}{p}} + \left(\lambda\delta\right)^{\frac{(d-1)}{2}} \left(\frac{1}{2} - \frac{1}{p}\right) \quad \text{for } \delta > 1/\lambda, \]

and were able to establish this bound for a range of the parameters \(\delta, \lambda, p\).

A full proof of this conjecture seems very challenging in every dimension \(d\). Restricting to the case \(d = 2\), consider the case \((\mathbb{R}/\mathbb{Z}) \times \mathbb{R} = \mathbb{T} \times \mathbb{R}\) instead of \(\mathbb{T}^2\). The conjecture remains identical, but a short proof, relying on \(\ell^2\) decoupling, can be provided; this is the main observation of the present paper. Generalizations to higher dimensions are certainly possible.

1.2. The Euclidean cylinder. On \(\mathbb{T} \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}\), we choose coordinates \((x, y)\), with \(x \in [0, 1]\) and \(y \in \mathbb{R}\). The Laplacian operator is given by

\[ \Delta = \partial_x^2 + \partial_y^2. \]

A function \(f\) on \(\mathbb{T} \times \mathbb{R}\) can be expanded through Fourier series in \(x\) and Fourier transform in \(y\):

\[ f(x) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \widehat{f}(k, \eta) e^{2\pi i (kx + \eta y)} \, d\eta. \]
The spectral projector can then be expressed as
 \[ P_{\lambda, \delta} f(x) = \sum_{k \in \mathbb{Z}} \int \chi \left( \frac{\sqrt{k^2 + \eta^2} - \lambda}{\delta} \right) \hat{f}(k, \eta) e^{2\pi i (kx + \eta y)} \, d\eta. \]

**Theorem 1.1.** If \( \lambda > 1 \) and \( \delta < 1 \),
\[
\| P_{\lambda, \delta} \|_{L^2 \rightarrow L^p} \lesssim_{\epsilon} \lambda^{\epsilon - \epsilon'} \left[ \lambda^{\frac{1}{2} - \frac{2}{p}} \delta^{\frac{1}{2}} + (\lambda \delta)^{\frac{1}{4} - \frac{1}{2p}} \right]
\]
Furthermore, this estimate is optimal, up to the subpolynomial factor \( \lambda^{\epsilon - \epsilon'} \) and the multiplicative constant.

1.3. **Strichartz estimates.** It is interesting to draw a parallel with Strichartz estimates in dimension 2 for the Schrödinger equation, in which case the critical exponent equals 4. It was proved in the foundational paper of Bourgain [2] that
\[
\| e^{it\Delta} f \|_{L^4([0,1] \times \mathbb{T}^2)} \lesssim \| f \|_{H^{s}(\mathbb{T}^2)} \quad \text{for } s > 0.
\]
Takaoka and Tzvetkov [6] proved that the above inequality fails for \( s = 0 \), but that, on \( \mathbb{T} \times \mathbb{R} \),
\[
\| e^{it\Delta} f \|_{L^4([0,1] \times \mathbb{T} \times \mathbb{R})} \lesssim \| f \|_{L^2(\mathbb{T} \times \mathbb{R})}.
\]
Finally, Barron, Christ and Pausader [1] determined the correct global (in time) estimate, for which a further summation index is needed. These examples suggest that optimal estimates might differ by subpolynomial factors between \( \mathbb{T}^2 \) and \( \mathbb{T} \times \mathbb{R} \).

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2. **Proof of the main theorem**

**Proof.** By Plancherel’s theorem, it suffices to prove
\[
\| f \|_{L^p} \lesssim \lambda^{\epsilon} \delta^{-\epsilon'} \left[ \lambda^{\frac{1}{2} - \frac{2}{p}} \delta^{\frac{1}{2}} + (\lambda \delta)^{\frac{1}{4} - \frac{1}{2p}} \right] \| f \|_{L^2}
\]
for \( f \) a function whose Fourier transform is localized in the corona \( C_{\lambda, \delta} \) of radius \( \lambda \) and with thickness \( \delta/\lambda \):
\[
C_{\lambda, \delta} = \left\{ (k, \eta) \text{ such that } \lambda - \delta < \sqrt{k^2 + \eta^2} < \lambda + \delta \right\}.
\]
By symmetry, one can furthermore assume that \( \hat{f}(k, \eta) \) is localized in the first quadrant \( k, \eta \geq 0 \).

The function \( f \) can be split into two pieces, which will correspond to the two terms on the right-hand side of (2.1),
\[
f(x) = \sum_{|k - \lambda| \leq \frac{1}{2}} + \sum_{|k - \lambda| > \frac{1}{2}} \int_{\mathbb{R}} \hat{f}(k, \eta) e^{2\pi i (kx + \eta y)} \, d\eta = f_1(x) + f_2(x).
\]

The case \( |k - \lambda| \leq \frac{1}{2} \) The Fourier support of \( f_1 \) is made up of a collection of segments. We will see in Lemma 3.1 below that the added length of these segments can be bounded by
\[
| \text{Supp } \hat{f}_1 | \lesssim \sqrt{\lambda \delta}.
\]
Therefore, by the Cauchy-Schwartz inequality,
\[
\| f_1 \|_{L^\infty} \leq | \text{Supp } \hat{f}_1 |^{1/2} \cdot \| \hat{f}_1 \|_{L^2}^{1/2} \lesssim (\lambda \delta)^{1/4} \| f_1 \|_{L^2}.
\]
Interpolating with \( L^2 \), this gives
\[
\| f_1 \|_{L^p} \lesssim (\lambda \delta)^{\frac{1}{2} - \frac{1}{2p}} \| f \|_{L^2}.
\]
We start by choosing a function $\phi \in \mathcal{S}$ which is $> 1/2$ on $[-1,1]$, and has Fourier support in $[-1,1]$. We use periodicity in the $x$ variable to expand the range of $x$ from $x \in [0,1]$ to $x < \delta^{-1}$, so that

$$\|f_2\|_{L^p(T \times \mathbb{R})} \lesssim \delta^{1/p} \left\|\phi(x) \sum_{|k-\lambda| > \frac{\delta}{\lambda}} \int_{\mathbb{R}} \hat{f}(k, \eta) e^{2\pi i (kx + \eta y)} \, d\eta\right\|_{L^p(\mathbb{R}^2)}.$$  

We now change variables as follows: $X = \lambda x$, $Y = \lambda y$, $K = k/\lambda$, $H = \eta/\lambda$,

$$f_2(x, y) = \lambda F(X, Y), \quad F(X, Y) = \phi\left(\frac{\delta X}{\lambda}\right) \sum_{K \in \mathbb{Z}/\lambda} \int_{\mathbb{R}} \hat{f}(\lambda K, \lambda H) e^{2\pi i (KX + HY)} \, dH$$

to obtain

$$\|f_2\|_{L^p(T \times \mathbb{R})} \lesssim \delta^{1/p} \lambda^{1-\frac{2}{p}} \|F\|_{L^p(\mathbb{R}^2)}.$$  

The effect of this change of variables is that the function of $(X, Y)$ whose $L^p$ norm we want to evaluate has Fourier transform supported in the corona $C_{1,3\delta/\lambda}$ of radius $1$ and width $\delta/\lambda$, and also in the first quadrant $X, Y \geq 0$. This enables us to apply the $\ell^2$ decoupling theorem of Bourgain and Demeter \cite{BD}: for a smooth partition of unity $(\chi_\theta)$ corresponding to a suitable almost disjoint covering of $C_{1,3\delta/\lambda}$ by caps $(\theta)$ of size $\sim \frac{\delta}{\pi} \times \sqrt{\frac{\delta}{\pi}}$,

$$\|f_2\|_{L^p(T \times \mathbb{R})} \lesssim \delta^{\frac{1}{p}} \lambda^{-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{2} + \frac{3}{2p} - \epsilon} \left(\sum_{\theta} \|\chi_\theta(D)F\|_{L^2(\mathbb{R}^2)}^2\right)^{1/2},$$

where $\chi_\theta(D)$ is the Fourier multiplier with symbol $\chi_\theta$.

We now apply the inequality $\|g\|_{L^p(\mathbb{R}^d)} \lesssim \|g\|_{L^2} \text{Supp } \hat{g}^{\frac{1}{2} - \frac{1}{p}}$ (if $p \geq 2$), which follows by applying in turn the Hausdorff-Young and Hölder inequalities, and then the Plancherel theorem. Since by Lemma 3.1 below

$$|\text{Supp } \chi_\theta(D)F| = |\text{Supp } \chi_\theta \hat{F}| \lesssim \delta^{5/2} \lambda^{-3/2},$$

it follows that

$$\|f_2\|_{L^p(T \times \mathbb{R})} \lesssim \delta^{\frac{1}{p}} \lambda^{-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{2} + \frac{3}{2p} - \epsilon} \left(\sum_{\theta} \|\chi_\theta(D)F\|_{L^2(\mathbb{R}^2)}^2\right)^{1/2}.$$  

By almost orthogonality, this becomes

$$\|f_2\|_{L^p(T \times \mathbb{R})} \lesssim \delta^{\frac{1}{p}} \lambda^{-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{2} + \frac{3}{2p} - \epsilon} (\delta^{5/2} \lambda^{-3/2})^{\frac{1}{2} - \frac{1}{p}} \|F\|_{L^2(\mathbb{R}^2)}.$$  

Finally, undoing the change of variables and using once again periodicity in the $x$ variable gives

$$\|f_2\|_{L^p(T \times \mathbb{R})} \lesssim \delta^{\frac{1}{p}} \lambda^{-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{2} + \frac{3}{2p} - \epsilon} (\delta^{5/2} \lambda^{-3/2})^{\frac{1}{2} - \frac{1}{p}} \delta^{-1/2} \|f\|_{L^2} \lesssim \lambda^{\frac{1}{2} - \frac{2}{p}} \delta^{1/2}$$

Optimality. The optimality of the statement of the theorem is proved through two examples. The first one is an analog of the Knapp example: assume $\lambda \in \mathbb{N}$, and consider the function $g$ given by its Fourier transform

$$\hat{g}(k, \eta) = 1_\lambda(k) \chi\left(\frac{\eta}{\sqrt{\lambda \delta}}\right).$$
Here, $\mathbf{1}_\lambda$ is the indicator function of $\{\lambda\}$ and $\chi$ is a cutoff function with a sufficiently small support, so that $\text{Supp} \hat{g} \subset C_{\lambda, \delta}$. In physical space,

$$g(x, y) = \sqrt{\lambda \delta e^{2\pi i \lambda x} \bar{\chi}(\sqrt{\lambda \delta} y)}.$$  

It has $L^p$ norm $\sim (\lambda \delta)^{\frac{1}{2} - \frac{1}{2p}}$, so that

$$\|g\|_{L^p} \sim (\lambda \delta)^{\frac{1}{2} - \frac{1}{2p}}.$$

We now consider the function $h$ given by its Fourier transform

$$\hat{h}(k, \eta) = \mathbf{1}_{C_{\lambda, \delta}}(k, \eta) \mathbf{1}_{[0, \lambda/2]}(k);$$

here, $\mathbf{1}_{C_{\lambda, \delta}}$ is the indicator function of the annulus, and $\mathbf{1}_{[0, \lambda/2]}$ the indicator function of the interval. It is easy to check that $|\text{Supp} \hat{h}| \sim \lambda \delta$, so that $\|h\|_{L^\infty} \sim \lambda \delta$ and $\|h\|_{L^2} \sim \sqrt{\lambda \delta}$, and finally

$$\|h\|_{L^\infty} \|h\|_{L^2} \sim \sqrt{\lambda \delta}.$$

By the Bernstein inequality,

$$\|h\|_{L^p} \|h\|_{L^2} \gtrsim \lambda^{-2/p} \|h\|_{L^\infty} \|h\|_{L^2} \sim \lambda^{\frac{1}{2} - \frac{2}{p}} \delta^{1/2}.$$

The examples $g$ and $h$ show that the statement of the theorem is optimal, up to subpolynomial losses. □

3. Bounds for the Fourier support

Lemma 3.1 (Bound on the size of Fourier support). With the notations of the proof of Theorem 1.1

(i) The function $f_1$ is a function on $\mathbb{T} \times \mathbb{R}$. As such, its Fourier transform is supported on a union of lines, and has one-dimensional measure

$$|\text{Supp} \hat{f}_1| \lesssim \sqrt{\lambda \delta}.$$

(ii) The function $\chi_\theta(D)F$ is a function on $\mathbb{R}^2$. Its Fourier transform is defined on $\mathbb{R}^2$, and has two-dimensional measure

$$|\text{Supp} \hat{\chi}_\theta F| \lesssim \delta^{5/2} \lambda^{-3/2}.$$

Proof. (i) Consider $f$ as in the proof of Theorem 1.1, namely with Fourier support in $C_{\lambda, \delta}$. Since $(k, \eta)$ range in $\mathbb{Z} \times \mathbb{R}$ with $k, \eta \geq 0$, the Fourier support of $f$ is contained in $\cup_{k \in \mathbb{Z}} \{k\} \times E_k^\lambda$, where

$$E_k^\lambda = \begin{cases} \emptyset & (k \geq k_\pm), \\ (0, \sqrt{k_+^2 - k^2}) & (|k - \lambda| < \delta), \\ (\sqrt{(\lambda - \delta)^2 - k^2}, \sqrt{k_+^2 - k^2}) & (0 \leq k \leq \lambda - \delta). \end{cases}$$

Recalling that $\hat{f}_1$ is just $\hat{f}$ restricted to $|k - \lambda| \leq \frac{1}{\delta}$, one can then add up these pieces to get the bound

$$|\text{Supp} \hat{f}_1| \lesssim \sum_{\max\{0, \lambda - \frac{1}{\delta}\} \leq k < k_+} |E_k| \lesssim \sqrt{\lambda \delta} + \sum_{k \geq 0 \atop \frac{1}{\delta} \geq \lambda - k > \delta} \frac{\delta \lambda}{\sqrt{\lambda y}} \sqrt{\lambda} - x,$$

and as $y \mapsto 1/\sqrt{y}$ is decreasing this is

$$\leq 2\sqrt{\lambda \delta} + \int_{\delta}^{\min\{\lambda, \frac{1}{\delta}\}} \frac{\delta \lambda}{\sqrt{\lambda y}} dy \leq 4\sqrt{\lambda \delta}.$$
(ii) Turning to $F$, it has Fourier support in
\begin{equation}
\bigcup_{k \in \mathbb{Z} \atop |k-\lambda| > \frac{1}{\delta}} \left[ \frac{k}{\lambda} - \frac{2\delta}{\lambda} \frac{k}{\lambda} + \frac{2\delta}{\lambda} \right] \times D_k^\lambda,
\end{equation}
where $D_k^\lambda = \left\{ H, \ 1 - \frac{\delta}{\lambda} < \sqrt{\frac{k^2}{\lambda^2} + H^2} < 1 + \frac{\delta}{\lambda} \right\}$.

Consider $\chi_\theta(D)F$, for a cap $\theta$ with dimensions $\sim \frac{\delta}{\lambda} \times \sqrt{\frac{\delta}{\lambda}}$ adapted to the corona $C_{1,3\delta/\lambda}$. Given such a cap, there is $j \in \mathbb{N}$ with $2^j > \frac{1}{\delta}$ such that every point in the intersection of $\theta$ with the set (3.1) satisfies $|\lambda - k| \sim 2^j$.

There are around $\sqrt{\delta 2^j/2}$ such values of $k$ for which the vertical strip $[\frac{k}{\lambda} - \frac{2\delta}{\lambda}, \frac{k}{\lambda} + \frac{2\delta}{\lambda}]$ intersects the cap $\theta$. For each such $k$, the size of $D_k^\lambda$ is $\sim \delta \lambda^{-1/2} 2^{-j/2}$. Hence, adding up the contributions in (3.1),
\[ |\text{Supp} \chi_\theta(D)F| \lesssim \sqrt{\delta 2^j/2} \cdot \delta \lambda^{-1} \cdot \delta \lambda^{-1/2} 2^{-j/2} = \delta^{5/2} \lambda^{-3/2}. \]

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