Functional equations for transfer-matrix operators in open Hecke chain models

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Abstract. We consider integrable open chain models formulated in terms of generators of affine Hecke algebras. The hierarchy of commutative elements (which are analogs of the commutative transfer-matrices) are constructed by using the fusion procedure. These elements satisfy a set of functional relations which generalize functional relations among a family of transfer-matrices in solvable spin chain models of $U_q(gl(n|m))$ type.

1 Introduction

In our recent paper [1] the non-polynomial baxterized solutions of reflection equations associated to the affine Hecke and affine Birman-Murakami-Wenzl algebras have been found. From these solutions one can produce (see [1] for details) polynomial solutions proposed in [2], [3] for the Hecke algebra case. Relations to integrable chain models with nontrivial boundary conditions were also discussed in [1].

In this paper we concentrate to the investigation of chain models formulated in terms of generators of the affine Hecke algebra. Let braid elements $\{\sigma_1, \ldots, \sigma_M\}$ and affine operator $y_1$ be generators of the affine Hecke algebra $\hat{H}_{M+1}(q)$ (for the definition of $\hat{H}_{M+1}(q)$ see next Section). In [1] we have suggested to consider an open chain model with the Hamiltonian

$$\mathcal{H}_M = \sum_{k=1}^M \sigma_k + (q - q^{-1}) \frac{\xi}{y_1 - \xi} \in \hat{H}_{M+1}(q), \quad (1.1)$$

where $\xi$ is a parameter of the model. This parameter fixes the boundary conditions for the chain and results from the non-polynomial solution [1]

$$y_1(x) = \frac{y_1 - \xi x}{y_1 - \xi x^{-1}},$$

of the reflection equation (see eq. (2.10) below; $x$ is a spectral parameter) associated to the affine Hecke algebra. The model (1.1) generalizes the XXZ open spin chain model and, as it was shown in [1], this model describes universally a set of integrable systems for any representation of the algebra $\hat{H}_{M+1}(q)$. The most interesting representations of $\hat{H}_{M+1}(q)$ (from the point of view of applications) are the so-called $R$-matrix representations. It is known that the Hecke algebra can be realized via

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$R$-matrices in the fundamental representation for $U_q(gl(n))$ [4], [5] and $U_q(gl(n|m))$ [6]. For $U_q(gl(n|m))$ case this realization is

$$\rho(\sigma_i) = I^\otimes(i-1) \otimes \hat{R} \otimes I^\otimes(M-i) =: \hat{R}_{ii+1},$$

where $I \in \text{Mat}(n+m)$ is the identity matrix and $\hat{R}$ is the braid form of fundamental $R$-matrix for $U_q(gl(n|m))$ (the explicit form of $\hat{R}$ is presented in Conclusion). For the affine element we have

$$\rho(y_1) = L \otimes I^\otimes M =: L_1,$$

where elements of $L \in \text{Mat}(n+m)$ are generators of $U_q(gl(n|m))$-type reflection equation algebra $\hat{R}_{12}L_1\hat{R}_{12}L_1 = L_1\hat{R}_{12}L_1\hat{R}_{12}$. All these $R$-matrix representations $\rho$ lead to integrable chain systems with the Hamiltonians $\rho(H_M)$, where $H_M$ is defined in (1.1). Considering the special case of the $U_q(gl(2))$-type representation $\rho$ of the algebra $\hat{H}_{M+1}$ we reproduce from (1.1) a Hamiltonian for the $XXZ$ open spin chain model with the general boundary condition for one side of the chain. In view of this we call the model formulated in terms of the generators of $\hat{H}_{M+1}$ with the universal Hamiltonian (1.1) as the open Hecke chain model.

In this paper the hierarchy of commutative elements (which are analogs of the transfer-matrices) are constructed for open Hecke chain model by using the algebraic version of the fusion procedure [7], [8], [9]. The transfer-matrix type elements satisfy functional relations which generalize functional relations for transfer-matrices in solvable open chain models (see, e.g., [10], [11] and refs. therein).

The paper is organized as follows. In Sect. 2 we give the definition of the affine Hecke algebra and recall the algebraic formulation [1] of the system of open Hecke chain. In particular we investigate commuting transfer-matrix type elements which are generating functions for the higher Hamiltonians of the model. In Sect. 3 the algebraic description of the fusion procedure for the solutions of the reflection equation and for the transfer-matrix type elements are presented. In this Section we deduce the functional relations for transfer-matrix type elements of the open Hecke chain. In Sections 4 and 5 we discuss the Temperley-Lieb "quotient" of the Hecke model for which the closed functional equations for the transfer-matrix type elements can be deduced. We note that this analysis is closely related to the approach proposed in [12] for the investigation of spin chain systems constructed on the base of the Temperley-Lieb algebra of special $R$-matrices. In Conclusion we discuss briefly the interrelations between the algebraic approach of this paper and standard reflection equation technique [13] developed for the investigation of the open chain systems.

2 Open Hecke chains

A braid group $\mathcal{B}_{M+1}$ is generated by Artin elements $\sigma_i$ ($i = 1, \ldots M$) subject to relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1. \tag{2.1}$$
An $A$-Type Hecke algebra $H_{M+1}(q)$ is a quotient of the group algebra of the braid group $\mathcal{B}_{M+1}$ by additional Hecke relations

$$\sigma_i^2 - 1 = \lambda \sigma_i, \quad (i = 1, \ldots, M),$$

(2.2)

where $\lambda := (q - q^{-1})$ and $q \in \mathbb{C}\setminus \{0\}$ is a parameter. Let $x \in \mathbb{C}$ be a spectral parameter. We introduce baxterized elements [4] (see also [6], [15])

$$\sigma_n(x) := \sigma_n - x \sigma_n^{-1} \in H_{M+1}(q),$$

(2.3)

which solve the Yang-Baxter equation

$$\sigma_n(x) \sigma_{n-1}(xy) \sigma_n(y) = \sigma_{n-1}(y) \sigma_n(xy) \sigma_n(x),$$

(2.4)

and satisfy $\sigma_n(x) \sigma_n(y) = \lambda \sigma_n(xy) + (1 - x)(1 - y)$. The normalized elements [1]

$$e_n^-(x) := \frac{\sigma_n - x \sigma_n^{-1}}{q x - q^{-1}}, \quad e_n^+(x) := \frac{\sigma_n - x \sigma_n^{-1}}{q - q^{-1} x},$$

(2.5)

obey the unitarity condition $e_n^+(x) e_n^+(x^{-1}) = 1$. These baxterized elements are useful for the definition of symmetrizers $A_{1\to n}^+$ and antisymmetrizers $A_{1\to n}^-$ ($n = 1, 2, \ldots, M + 1$) in the Hecke algebra $H_{M+1} := H_{M+1}(q)$. The operators $A_{1\to n}^\pm$ can be defined inductively by using recurrent relations [4] (see also [15] and references therein)

$$A_{1\to n+1}^\pm = A_{1\to n}^\pm e_n^+(q^{\mp 2n}) A_{1\to n}^\pm, \quad A_{1\to 1}^\pm := 1,$$

(2.6)

and obey

$$\sigma_i A_{1\to n}^\pm = A_{1\to n}^\pm \sigma_i = \pm q^\pm A_{1\to n}^\pm (i = 1, \ldots, n - 1),$$

$$A_{1\to n}^\pm A_{1\to m}^\pm = A_{1\to m}^\pm A_{1\to n}^\pm (\forall n \geq m), \quad A_{1\to n}^\pm A_{1\to m}^\pm = 0,$$

(2.7)

$$e_i^\pm(x) A_{1\to n}^\pm = A_{1\to n}^\pm e_i^\pm(x) = A_{1\to n}^\pm (i = 1, \ldots, n - 1).$$

An affine Hecke algebra $\hat{H}_{M+1}$ (see, e.g., [14], Chapter 12.3) is an extension of the Hecke algebra $H_{M+1}$. The algebra $\hat{H}_{M+1}$ is generated by elements $\sigma_i$ ($i = 1, \ldots, M$) of $H_{M+1}$ and affine generators $y_k$ ($k = 1, \ldots, M + 1$) which satisfy:

$$y_{k+1} = \sigma_k y_k \sigma_k, \quad y_k y_j = y_j y_k, \quad y_j \sigma_i = \sigma_i y_j \quad (j \neq i, i + 1).$$

(2.8)

The elements $\{y_k\}$ form a commutative subalgebra in $\hat{H}_{M+1}$, while symmetric functions of $y_k$ form a center in $\hat{H}_{M+1}$.

One can prove by induction that the element

$$y_n(x) = \sigma_{n-1}(\frac{x}{\xi_n - 1}) \cdots \sigma_2(\frac{x}{\xi_2 - 1}) \sigma_1(\frac{x}{\xi_1 - 1}) y_1(x) \sigma_1(x \xi_1) \sigma_2(x \xi_2) \cdots \sigma_{n-1}(x \xi_{n-1}),$$

(2.9)

solves $(\forall \xi_1, \ldots, \xi_{n-1} \in \mathbb{C}\setminus \{0\})$ the reflection equation

$$\sigma_n(x z^{-1}) y_n(x) \sigma_n(x z) y_n(z) = y_n(z) \sigma_n(x z) y_n(x) \sigma_n(x z^{-1}),$$

(2.10)
where \( y_1(x) \in \hat{H}_{M+1} \) is any local (i.e., \([y_1(x), \sigma_k] = 0 \ \forall k > 1\)) solution of (2.10) for \( n = 1 \). In [1] we have found such local solution which is rational in \( y_1 \):

\[
y_1(x) = \frac{y_1 - \xi x}{y_1 - \xi x^{-1}},
\]

(2.11)

where \( y_1 \) is the affine generator of the algebra \( \hat{H}_{M+1} \) and \( \xi \in \mathbb{C} \) is a parameter. The operator \( y_1(x) \) (2.11) is regular \( y_1(1) = 1 \), and obeys the unitarity condition: \( y_1(x)y_1(x^{-1}) = 1 \). Below we consider the special form of the element (2.9) (when all \( \xi_k = 1 \))

\[
y_n(x) = \sigma_{n-1}(x) \cdots \sigma_2(x)\sigma_1(x)y_1(x)\sigma_1(x)\sigma_2(x) \cdots \sigma_{n-1}(x),
\]

(2.12)

which is a Hecke algebra analog of the Sklyanin’s monodromy matrix [13]. The elements (2.12) has been used in [1] for the construction of the integrable chain systems.

Consider the following inclusions of the subalgebras \( \hat{H}_1 \subset \hat{H}_2 \subset \ldots \subset \hat{H}_{M+1} \):

\[
\{y_1; \sigma_1, \ldots, \sigma_{n-1}\} = \hat{H}_n \subset \hat{H}_{n+1} = \{y_1; \sigma_1, \ldots, \sigma_{n-1}, \sigma_n\}.
\]

Then, following [1] we equip the algebra \( \hat{H}_{M+1} \) by linear mappings

\[
Tr_{D(n+1)} : \hat{H}_{n+1} \rightarrow \hat{H}_n,
\]

from the algebras \( \hat{H}_{n+1} \) to its subalgebras \( \hat{H}_n \), such that for all \( X, X' \in \hat{H}_n \) and \( Y \in \hat{H}_{n+1} \) we have

\[
Tr_{D(n+1)}(X) = D^{(0)}X, \quad Tr_{D(n+1)}(XYX') = XTr_{D(n+1)}(Y)X',
\]

\[
Tr_{D(n+1)}(\sigma_{n}^{\pm 1}X\sigma_{n}^{\mp 1}) = Tr_{D(n)}(X), \quad Tr_{D(n+1)}(X\sigma_nX') = X X',
\]

\[
Tr_{D(1)}(y_1^k) = D^{(k)}, \quad Tr_{D(n)}Tr_{D(n+1)}(\sigma_nY) = Tr_{D(n)}Tr_{D(n+1)}(Y\sigma_n),
\]

(2.13)

where \( D^{(k)} \in \mathbb{C}\setminus\{0\} \) (\( k \in \mathbb{Z} \)) are constants. We stress that \( D^{(k)} \) could be considered as additional generators of an abelian subalgebra \( \hat{H}_0 \) which extends \( \hat{H}_{M+1} \) and be central in \( \hat{H}_{M+1} \), but for us it is enough to put \( D^{(k)} \) to constants.

Using the maps \( Tr_{D(n+1)} \) one can show that the baxterized elements \( \sigma_n(x) \) (2.3) obey the following identity (\( \forall X \in \hat{H}_n, \ \forall x, z \)):

\[
Tr_{D(n+1)}(\sigma_n(x)X\sigma_n(z)) = (1-x)(1-z)Tr_{D(n)}(X) + \lambda \left( 1 - \frac{xz}{b} \right) X,
\]

(2.14)

where

\[
b := \frac{1}{1 - \lambda D^{(0)}}.
\]

Applying this identity to the reflection eq. (2.10) we deduce

\[
[\tau_{n-1}(x), y_n(z)] = \frac{\lambda(xb^{-1} - x^{-1})}{((x + x^{-1}) - (z + z^{-1}))} [y_n(x), y_n(z)],
\]

where operators

\[
\tau_n(x) = Tr_{D(n+1)}(y_{n+1}(x)),
\]

(2.15)
form a commutative family \[ \tau_n(x), \tau_n(z) \] = 0 (\forall x, z) (see [1]).

Now we formulate the open Hecke chains using the operator analogs of Sklyanin’s transfer-matrices [13]. In our case these transfer-matrix operators \( \tau_n(x) \) are the elements of \( \hat{H}_n \) and represented by (2.15) where the solution \( y_{n+1}(x) \) of the reflection equation is taken in the special form (2.12):

\[
\tau_n(x) = Tr_{D(n+1)} \left( y_{n+1}(x) \right) = Tr_{D(n+1)} \left( \sigma_n(x) \cdots \sigma_1(x) y_1(x) \sigma_1(x) \cdots \sigma_n(x) \right),
\]

where \( y_1(x) \) is any local and regular solution of (2.10) for \( n = 1 \) (e.g. one can take the operator (2.11)). The local Hamiltonian of the open Hecke chains is

\[
\mathcal{H}_n = \sum_{m=1}^{n-1} \sigma_m - \frac{\lambda}{2} y'_1(1).
\]

This Hamiltonian (up to a normalization factor and additional constant) can be obtained by differentiating \( \tau_n(x) \) with respect to \( x \) at the point \( x = 1 \). The Hamiltonian (2.17) describes the open chain model with nontrivial boundary condition on the first site (given by the second term in (2.17)) and free boundary condition on the last site of the chain. The Hamiltonian (1.1) is obtained by the substitution of (2.11) in (2.17).

It follows from (2.14) that the transfer-matrix operator \( \tau_n(x) \) (2.16) satisfies recurrent equation

\[
\tau_n(x) = \lambda (1 - \frac{x^2}{b}) y_n(x) + (1 - x)^2 \tau_{n-1}(x),
\]

which can be solved as

\[
\tau_n(x) = \lambda \left( 1 - \frac{x^2}{b} \right) \left( \sum_{k=0}^{n-1} (1 - x)^{2k} y_{n-k}(x) \right) + (1 - x)^2 Tr_{D(1)} \left( y_1(x) \right),
\]

where monodromy elements \( y_k(x) \) (2.12) are the solutions of the reflection equation (2.10) for \( n = k \).

Consider the open chain model with free boundary condition for both sides of the chain, i.e. \( y_1(x) = 1 \). In this case we deduce from (2.19)

\[
\tau_n(x) \big|_{y_1(x)=1} = Tr_{D(n+1)} \left( \sigma_n(x) \cdots \sigma_2(x) \sigma_1(x) \cdot \cdot \cdot \sigma_n(x) \right) = \\
= \lambda \left( 1 - \frac{x^2}{b} \right) J_n(x) + (1 - x)^2 D(0) =: T^{(1)}_n(x),
\]

where \( J_n(x) \) are special polynomials in spectral parameter \( x \) of the order \( 2n - 2 \):

\[
J_n(x) = \left( \sum_{k=0}^{n-2} (1 - x)^{2k} \sigma_{n-k-1}(x) \cdots \sigma_1^2(x) \cdots \sigma_{n-k-1}(x) \right) + (1 - x)^{2(n-1)} = \\
= \lambda \left( 1 + x \right) \left( \sum_{a=1}^{2n-3} (1 - x)^a \left( \lambda x \right)^{2n-3-a} j_a \right) + f_n(x),
\]
and \( f_n(x) \) are scalar polynomial functions of \( x \):
\[
f_n(x) = \sum_{k=0}^{n-2} (1-x)^{2k} \left( (1-x)^2 + \lambda^2 x^2 \right)^{n-k-1} + (1-x)^{2(n-1)}.
\]
The representation (2.21) follows from equations
\[
\sigma_k(x) = (1-x)\sigma_x + \lambda x, \quad \sigma_k^2(x) = \lambda(1+x)(1-x)\sigma_k + [(1-x)^2 + \lambda^2 x^2].
\]
Since the transfer - matrix type elements \( \tau_n(x) \) form a commutative family \([\tau_n(x), \tau_n(z)] = 0 \), we have from eqs. (2.20), (2.21) a set \( \{j_1, j_2, \ldots, j_{2n-3}\} \) of \( (2n-3) \) commuting elements of the Hecke algebra \( H_n \). It is clear that
\[
J_n(0) = \sigma_{n-1} \cdots \sigma_1^2 \cdots \sigma_{n-1} + \sigma_{n-2} \cdots \sigma_1^2 \cdots \sigma_{n-2} + \ldots + \sigma_1^2 + 1 = \\
\lambda \sum_{m=1}^{n-1} \sum_{k=1}^{m} (\sigma_k \cdots \sigma_{m-1} \sigma_m \sigma_{m-1} \cdots \sigma_k) + n = \lambda j_{2n-3} + f_n(0),
\]
and the operator \( j_{2n-3} \) (as well as \( J_n(0) \)) is a central element in \( H_n \). On the other hand we have
\[
J_n'(1) = -2\lambda^{2n-3} \left( \sum_{m=1}^{n-1} \sigma_m \right) + 2\lambda^{2n-2}(n-1) = -2\lambda^{2n-3} j_1 + f_n'(1),
\]
and according to (2.17) we obtain that
\[
j_1 = H_n^{(\text{free})} = \sum_{m=1}^{n-1} \sigma_m, \tag{2.22}
\]

is the Hamiltonian for the open Hecke chain with free ends.

For the first nontrivial case \( n = 3 \) we have the set of commuting elements
\[
j_1 = \sigma_1 + \sigma_2, \quad j_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_1, \quad j_3 = \sigma_2 \sigma_1 \sigma_2 + \sigma_1 + \sigma_2. \tag{2.23}
\]
The element \( j_1 \) is the Hamiltonian \( H_3^{(\text{free})} \) for the open Hecke chain and \( j_3 \) is a central element in \( H_3 \). The characteristic identity for the Hamiltonian \( j_1 = H_3^{(\text{free})} \) is calculated directly:
\[
(j_1 + 2q^{-1})(j_1 - 2q)(j_1 - \lambda - 1)(j_1 - \lambda + 1) = 0.
\]
and it means that \( \text{Spec}(H_3^{(\text{free})}) = (2q, -2q^{-1}, \lambda \pm 1) \). The first two eigenvalues (\( \pm 2q^{\pm 1} \)) correspond to the one dimensional representations \( \sigma_i = \pm q^{\pm 1} \) \( (i = 1, 2) \) of \( H_3 \). The doublet \( (\lambda \pm 1) \) corresponds to the 2-dimensional irrep of \( H_3(q) \) (for the representation theory of Hecke algebras see [15], [16], [17] and refs. therein).

For the next case \( n = 4 \) we obtain the following set of commuting elements
\[
j_1 = \sum_{i=1}^{3} \sigma_i, \quad j_2 = \{\sigma_1, \sigma_2\} + \{\sigma_2, \sigma_3\} + 2\sigma_3 \sigma_1, \\
j_3 = \{\sigma_1 \sigma_3, \sigma_2\} + \{\sigma_1 + \sigma_3\} \sigma_2 (\sigma_1 + \sigma_3) + \lambda \sigma_3 \sigma_1 + 2 \sum_{i=1}^{3} \sigma_i, \\
j_4 = \{\sigma_2 \sigma_3 \sigma_2, \sigma_1\} + \{\sigma_2 \sigma_1 \sigma_2, \sigma_3\} + \{\sigma_2, \sigma_3\} + \{\sigma_2, \sigma_1\}, \\
j_5 = \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_1 + \sigma_2 \sigma_3 \sigma_2 + \sigma_1 \sigma_2 \sigma_1 + \sigma_1 + \sigma_2 + \sigma_3,
\]
The coefficient $j_5$ is a central element in $H_4$. We note that the longest element in $H_4$: $j = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 = (j_5 - j_1)(j_1 - \lambda) - j_4$ commutes with the Hamiltonian $j_1 = H_4^{(\text{free})}$. The spectrum of $H_4^{(\text{free})}$ consists of the eigenvalues: $(\pm 3q^{\pm 1})$ for two 1-dimensional irreps, $(2q - q^{-1}, 2q - q^{-1} \pm \sqrt{2})$ and $(-2q^{-1} + q, -2q^{-1} + q \pm \sqrt{2})$ for two dual 3-dimensional irreps, and $(\frac{3}{2}\lambda \pm \frac{1}{2}\sqrt{q^{-2} + 10 + q^2})$ for 2-dimensional irrep of $H_4$.

**Remark.** Consider the element $j$ of the braid group $B_{M+1}$

$$j := (\sigma_1 \cdots \sigma_M)(\sigma_1 \cdots \sigma_{M-1}) \cdots (\sigma_1\sigma_2) \cdot \sigma_1.$$  

This element commutes with the element $(\sum_{i=1}^{M} \sigma_i)$ of the group algebra of $B_{M+1}$, since $\sigma_i j = j \sigma_{M+1-i}$. Thus, for the Hecke quotient $H_{M+1}$ of group algebra of $B_{M+1}$, the element $j \in H_{M+1}$ is a conservation charge for the model of the chain with the Hamiltonian $H_{M+1}^{(\text{free})}$ (2.22). Using the longest element $j \in H_n$ one can show that operators (2.20), (2.21) are invariant under the mirror transformation $\sigma_k \leftrightarrow \sigma_{n-k}$. It demonstrates the mirror symmetry of the open Hecke chain described by the Hamiltonian (2.22) with respect to the exchange of the first and last sites of the chain.

### 3 Fusion for baxterized solutions of reflection equation

The fusion procedure for the solutions of the Yang-Baxter equation has been proposed in [7], [8]. The fusion for the solutions of the reflection equation was considered in [9], where it was applied to the investigation of the XXZ open spin chain model. In this Section we formulate the fusion for the solutions of the reflection equations in terms of the affine Hecke algebra generators. Below we use concise notations

$$e_n^x := e_1^+(x), \quad y_n^x := y_1(x)$$

for baxterized elements $e_1^+(x)$ (2.5) and solutions $y_1(x)$ of the reflection equation (2.10).

The reflection equation (2.10) can be graphically represented in the form

![Fig. 1](image)

Considering the sequence of the pictures
etc., one can easily construct fusions for solutions of the reflection equation. Indeed, the algebraic expressions for these diagrams are

\[ Y_{1 \to 2}(x) = e_1^{q-2} \cdot [y_1^e e_1^{xq^2} y_1^{xq^2}] = e_1^{q-2} \cdot y_{1 \to 2}(x) , \]

\[ Y_{1 \to 3}(x) = [e_1^{q-2} e_2^{q-4} e_1^{q-2}] \cdot [y_1^e e_1^{xq^2} y_1^{xq^2} e_2^{xq^2} e_1^{xq^6} y_1^{xq^6}] = [e_1^{q-2} e_2^{q-4} e_1^{q-2}] \cdot y_{1 \to 3}(x) , \]

where we have used the concise notations \( e_1^x, y_1^e \) (3.24). The formula which generalizes (3.25) is

\[ Y_{1 \to k}(x) = A_{1 \to k}^+ \cdot y_{1 \to k}(x) = \bar{Y}_{1 \to k}(x) \cdot A_{1 \to k}^+ , \quad (3.26) \]

where

\[
y_{1 \to k}(x) := y_1^e \left[ e_1^{xq^2} y_1^{xq^2} \right] \left[ e_2^{xq^2} e_1^{xq^2} y_1^{xq^2} \right] \left[ e_3^{xq^6} e_2^{xq^6} e_1^{xq^{10}} y_1^{xq^6} \right] \cdots \\
\cdots \left[ e_1^{xq^{2(2k-1)}} e_2^{xq^{2k}} e_1^{xq^{2(2k-1)-1}} \right] .
\]

and \( \bar{Y}_{1 \to k}(x) \) is obtained from \( y_{1 \to k}(x) \) by writing it from right to left (the second equality in (3.26) is obtained with the help of the Yang-Baxter (2.4) and reflection equations (2.10) and can be justified immediately from the consideration of its graphical representation analogous to Fig. 2). The corresponding fusion for the baxterized elements is given by the expression

\[ \sigma_{1 \to k+1 \to 2k}(x) = A_{1 \to k}^+ \cdot A_{k+1 \to 2k}^+ \cdot \sigma_{k+1 \to 2k+2}^{xq^2} \cdots \sigma_{2k \to k}^{xq^{2(k-1)}} \equiv \]

\[ \equiv A_{1 \to k}^+ \cdot A_{k+1 \to 2k}^+ \cdot \sigma_{k \to 2k}^{xq^{-2(k-1)}} \cdots \sigma_{2 \to 1}^{xq^{-2}} \sigma_{1 \to k}^x , \quad (3.27) \]

where

\[ \sigma_{k+1 \to 1}^x = e_1^{xq^{-2(k-1)}} \cdots e_2^{xq^{-2}} e_1^x , \quad e_{k \to k+1}^x = e_1^{xq^2} e_2^{xq^2} \cdots e_k^{xq^{2(k-1)}} . \]

The elements (3.27) solve the highest Yang-Baxter equations. One can check directly (by using the graphical representation of the type Fig. 2) that (3.26) and (3.27) satisfy the equation

\[ \sigma_{1 \to k, k+1 \to 2k}(x z^{-1}) Y_{1 \to k}(x) \sigma_{1 \to k, k+1 \to 2k}(x z) Y_{1 \to k}(z) = \]

\[ = Y_{1 \to k}(z) \sigma_{1 \to k, k+1 \to 2k}(x z) Y_{1 \to k}(x) \sigma_{1 \to k, k+1 \to 2k}(x z^{-1}) . \]

which is a fusion version of the reflection equation (2.10).
The highest transfer-matrix type element \( \tau^{(k)}(x) \) which corresponds to the fusion solution (3.26) of the reflection equation (3.28) is
\[
\tau^{(k)}(x) = Tr_{D(1 \rightarrow k)} (Y_{1 \rightarrow k}(x)) = Tr_{D(1 \rightarrow k)} \left( A_{1 \rightarrow k}^+ \cdot y_{1 \rightarrow k}(x) \right). \tag{3.29}
\]
In particular we have for \( k = 1, 2 \)
\[
\tau^{(1)}(x) = Tr_{D(1)} (y_1^x), \quad \tau^{(2)}(x) = Tr_{D(1 \rightarrow 2)} \left( e_1^q x^{-2} y_1^x \left[ e_1^q x^2 y_1^{xq} \right]^2 \right). \tag{3.30}
\]

Below we need the following Lemma.

**Lemma 1.** In the Hecke algebra the following identity holds \( (\forall x) \)
\[
A_{1 \rightarrow k}^+ \left[ \sigma_k^x \sigma_k^{xq} \cdots \sigma_1^{xq^{2(k-1)}} \right] = \left[ \sigma_k^{xq^{2(k-1)}} \sigma_k^{xq^{2(k-2)}} \cdots \sigma_2^{xq^2} \sigma_1^x \right] A_{2 \rightarrow k+1}^+. \tag{3.31}
\]
where \( A_{1 \rightarrow k}^+ \) are symmetrizers (2.6).

**Proof.** We prove eq. (3.31) by induction. Let (3.31) is correct for some fixed \( k \). Consider the left hand side of (3.31) for the case \( k \rightarrow k + 1 \)
\[
A_{1 \rightarrow k+1}^+ \left[ \sigma_k^x \sigma_k^{xq^2} \cdots \sigma_1^{xq^{2k}} \right] = \left[ e_1^{q^{k-2}} \cdots e_k^{q^{2(k-1)}} \right] A_{1 \rightarrow k}^+ \sigma_{k+1}^x \left[ \sigma_k^{xq^2} \cdots \sigma_1^{xq^{2k}} \right] =
\]
\[
= (3.31) = \left[ e_1^{q^{k-2}} \cdots e_k^{q^{2(k-1)}} \right] \sigma_{k+1}^x \left[ \sigma_k^{xq^2} \cdots \sigma_1^{xq^{2k}} \right] A_{2 \rightarrow k+1}^+ =
\]
\[
= (2.4) = \left[ \sigma_k^{xq^{2k}} \cdots \sigma_2^{xq^2} \sigma_1^x \right] \sigma_{k+1}^x \left[ e_1^{q^{k-2}} \cdots e_k^{q^{2(k-1)}} \right] A_{2 \rightarrow k+1}^+ =
\]
\[
= \left[ \sigma_k^{xq^{2k}} \cdots \sigma_2^{xq^2} \sigma_1^x \right] A_{2 \rightarrow k+2}.
\]
This equation is equivalent to (3.31) for \( k \rightarrow k + 1 \). \( \square \)

**Proposition 1.** The transfer-matrix type elements \( \tau^{(k)}(x) \) (3.26) satisfy the following identity
\[
\tau^{(k)}(x) \tau^{(1)}(xq^{2k}) = \phi_k'(x) \tau^{(k+1)}(x) + \phi_k''(x) \tau^{(k,1)}(x), \tag{3.32}
\]
where coefficient functions are
\[
\phi_k'(x) = \frac{\left( 1 - x^2 q^{4k} b^{-1} \right) \left( 1 - x^2 q^{2(k-1)} \right)}{\left( 1 - x^2 q^{2k} b^{-1} \right) \left( 1 - x^2 q^{4k-2} \right)},
\]
\[
\phi_k''(x) = \frac{q^k (1 - q^{-2k})}{(1 - q^{-2(k+1)})(1 - q^2)} \frac{\left( 1 - x^2 q^{2(k-1)} b^{-1} \right) \left( 1 - x^2 q^{2(k-1)} \right)}{\left( 1 - x^2 q^{2k} b^{-1} \right) \left( 1 - x^2 q^{4k-2} \right)},
\]
and
\[
\tau^{(k,1)}(x) = Tr_{D(1 \rightarrow k+1)} \left( \left[ \sigma_k^x \sigma_2^{xq^2} \cdots \sigma_1^{xq^{2k}} A_{1 \rightarrow k+1}^+ \right] \cdot y_{1 \rightarrow k+1}(x) \right),
\]
is a new transfer-matrix type element.
Proof. Using identity (2.14) for $z = b/x$ we deduce

$$
\eta_k(x) \tau^{(1)}(xq^{2k}) = \eta_k(x) T_{R_D(1)}(y_1 x^{q^{2k}}) =
$$

$$
= T_{R_D(k+1)} \left( e_k^{x^2 q^{2k}} e_k^{-x^2 q^{2(k+1)}} \cdots e_1^{x^2 q^{2(2k-1)}} y_1 x^{q^{2k}} \frac{b}{e_1^{x^2 q^{2k}} \cdots e_k^{x^2 q^{2k}}} \right),
$$

where

$$
\eta_k(x) = \frac{(1 - x^2 q^{2(k-1)}) (1 - x^2 q^{2k} b^{-1})}{(1 - x^2 q^{2(k-1)}) (1 - x^2 q^{4k} b^{-1})}.
$$

Thus, we have

$$
\eta_k(x) \tau^{(k)}(x) \tau^{(1)}(xq^{2k}) = T_{R_D(1 \rightarrow k+1)} \left( A_{1 \rightarrow k}^+ \cdot y_{1 \rightarrow k+1}(x) e_1^{x_2 q^{4k-2}} \cdots e_k^{x_2 q^{2k}} \right) (3.33)
$$

One can represent the identity element as linear combination of two baxterized elements

$$
1 = e_k(q^{-2k}) + \frac{(1 - q^{-2k})}{(q^2 - q^{-2k})} \sigma_k(q^2), \quad e_k(q^{-2k}) := \frac{\sigma_k(q^{-2k})}{q - q^{-2k-1}}. (3.34)
$$

Taking into account eqs. (3.34), (3.31) we write eq. (3.33) in the form

$$
\eta_k(x) \tau^{(k)}(x) \tau^{(1)}(xq^{2k}) = T_{R_D(1 \rightarrow k+1)} \left( (A_{1 \rightarrow k+1}^+ + \frac{(1 - q^{-2k})}{(q^2 - q^{-2k})} \sigma_k q^2 A_{1 \rightarrow k}^+) \cdot y_{1 \rightarrow k}(x).$$

$$
\cdot [e_k^{x^2 q^{2k}} e_k^{-x^2 q^{2(k+1)}} \cdots e_1^{x^2 q^{2k}} y_1 x^{q^{2k}}] e_1^{x_2 q^{4k-2}} \cdots e_k^{x_2 q^{2k}} \right) =
$$

$$
= (3.31) = T_{R_D(1 \rightarrow k+1)} \left( (A_{1 \rightarrow k+1}^+ y_{1 \rightarrow k+1}(x) e_1^{x_2 q^{4k-2}} \cdots e_k^{x_2 q^{2k}}) + \frac{(1 - q^{-2k})}{(q^2 - q^{-2k})} \sigma_k^2 \right).
$$

Since $\sigma_k^2 = 0$ one can deduce the equation

$$
\sigma_{k+1}(q^{-2}) e_k(x) \sigma_{k+1}(q^2) = \frac{(1 - x)}{(1 - q^2)(q - q^{-1} x)} \sigma_{k+1}(q^{-2}) \sigma_k(q^2) \sigma_{k+1}(q^2), \quad (3.36)
$$

which leads to

$$
A_{2 \rightarrow k+1}^+ \left[ e_1^{x_2 q^{4k-2}} \cdots e_1^{x_2 q^{2k}} e_k^{x_2 q^{2k}} \sigma_k^2 \right] = \frac{(-q^{-k-1})(1 - x^2 q^{2(k-1)})}{(1 - q^2) x^{-1} (1 - \frac{b}{x^2 q^{2k}})} A_{2 \rightarrow k+1}^+ \left[ \sigma_1^2 \cdots \sigma_k^2 \right]. 
$$

$$
\text{We have also the following identity (see last eqs. in (2.7))}

$$
A_{1 \rightarrow k+1}^+ \left[ e_1^{x_2 q^{4k-2}} \cdots e_k^{x_2 q^{2k}} \right] = A_{1 \rightarrow k+1}^+. \quad (3.38)
$$

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Finally using (3.37) and (3.38) we obtain for (3.35)

\[ \eta_k(x) \tau(k)(x)\tau^{(1)}(xq^{2k}) = \tau^{(k+1)}(x) + \]

\[
+ \frac{q^{-k-2}(1-q^{-2k})}{(1-q^{-2k})^2} \left( \frac{1-b}{1-x^2q^{2k+1}(1-q^{-2k})^2} \right) \cdot 
\]

\[ T_r_{D(i\rightarrow k+1)} \left( A^+_{1\rightarrow k} \cdot y_{1\rightarrow k+1}(x) \{ \sigma_i^2 \cdots \sigma_i^2 \sigma_k \} \right), \]

which is equivalent to (3.32).

\[ \square \]

4 Temperley-Lieb quotient for the Hecke chain model.

Eqs. (3.32) are not closed with respect to the set of the transfer-matrix type elements \( \tau(k)(x) \) (3.29). To obtain closed set of relations we need to consider special quotients of the Hecke algebra \( H_{M+1} \). In this Section we consider a Temperley-Lieb quotient \( H_{M+1}^{(TL)} \) which is defined by imposing the additional constraints on the generators \( \sigma_i \in H_{M+1} \) \( (i = 1, \ldots, M) \):

\[ \sigma_i(q^2)\sigma_{i\pm 1}(q^4)\sigma_i(q^2) = 0. \] (4.39)

These constraints are equivalent to the vanishing of the third rank antisymmetrizers \( A_{i\rightarrow i+2} = 0 \) (2.6). Moreover, they fix the parameter \( D(0) \) (or \( b \)):

\[ 0 = T_r_{D(i\rightarrow i+1)}(\sigma_{i-1}(q^2)\sigma_i(q^4)\sigma_{i-1}(q^2)) = \sigma_{i-1}(q^2) \left( T_r_{D(i\rightarrow i+1)}(\sigma_i(q^4)) \right) \sigma_{i-1}(q^2) = \]

\[ = \sigma_{i-1}^2(q^2)(1-q^4(1-\lambda D(0))) = \sigma_{i-1}^2(q^2)(1-q^4/b) \Rightarrow \]

\[ \Rightarrow b = q^4, \quad D(0) = \frac{1-q^{-4}}{\lambda}. \] (4.40)

A Temperley-Lieb (TL) quotient of the affine Hecke algebra, in addition to (4.39), is defined by imposing the constraint on the affine generator of \( \hat{H}_{M+1} \):

\[ y_1 \sigma_1 y_1 \sigma_1^2 = \sigma_1^2 y_1 \sigma_1 y_1 = \Delta \sigma_1^2, \] (4.41)

where \( \Delta = \frac{\lambda \eta}{(1-q^{-4})(D(2) - q(D(1)))} \) is a central element in \( \hat{H}_{M+1} \). In the presence of the map (2.13), eq. (4.41) requires that the affine generator \( y_1 \) obeys quadratic characteristic identity:

\[ y_1^2 = q D(1) y_1 = q^{-1} \Delta, \]

and after that (4.41) is written in the form \( \sigma_1^2 y_1 \sigma_1^2 = q^2(q^2 - 1)D(1)\sigma_1^2 \). Thus, this factor of the affine Hecke algebra coincides with the one-boundary TL, or blob, algebra \([18], [19]\).

The constraint (4.41) also leads to the constraints

\[ y_1^2 e_1^2 q^2 y_1^2 \sigma_1^2 = \sigma_1^2 y_1^2 e_1^2 q^2 y_1^2 q^2 \Rightarrow \Delta (z) \sigma_1^2, \] (4.42)
where \( y_1^* \in \hat{H}_{M+1} \) is any local solution of the reflection equation (2.10) for \( n = 1 \) and \( \Delta^{(0)}(z) \) is a central element in \( \hat{H}_{M+1} \).

**Proposition 2.** In the case of the TL quotient for the open Hecke chain, the functional relations (3.32) are closed with respect to the family of the transfer-matrix type elements \( \tau^{(k)}(x) \):

\[
\tau^{(k)}(x)\tau^{(1)}(xq^{2k}) = \phi'_k(x)\tau^{(k+1)}(x) + \phi''_k(x)\Delta^{(0)}(xq^{2k})\tau^{(k-1)}(x)
\]  

(4.43)

where \( \tau^{(0)}(x) := 1 \) and

\[
\phi'_k(x) := \frac{1 - x^2q^{2(k-1)}}{1 - x^2q^{2k-4}} \left( 1 - x^2q^{4k-4} \right),
\]

\[
\phi''_k(x) := \frac{1}{(-q^2)} \frac{1 - x^2q^{2k-6}}{1 - x^2q^{2k-4}} \left( 1 - x^2q^{4k-4} \right).
\]

**Proof.** We write (3.32) in the form

\[
\tau^{(k)}(x)\tau^{(1)}(xq^{2k}) = \phi'_k(x)\tau^{(k+1)}(x) + \phi''_k(x)\Delta^{(0)}(xq^{2k}).
\]

\[
Tr_{D(1\rightarrow k+1)}(A_{1\rightarrow k} \cdot y_{1\rightarrow k-1}(x) \cdot [e_{k-1}^{2(x^{2(1-k)})} \cdots e_1^{2q^{2(k-1)}}] [e_k^{2q^{2k}} \cdots e_2^{2q^{4(k-1)}}] [\sigma_1^{q^2} \cdots \sigma_k^{q^2}] = \ldots = \ldots
\]

Constraints (4.39), which defines the TL quotient for the Hecke algebra, lead to the following identities

\[
\sigma_i(q^2)\sigma_{i+1}(x)\sigma_i(q^2) = \lambda q^3(1 - q^{-4}x)\sigma_i(q^2),
\]

\[
e_i(x)e_{i+1}(xq^2)\sigma_i(q^2) = -\xi(x)\sigma_{i+1}(q^2)e_i(q^2) \quad (\forall x),
\]

where \( \xi(x) = \frac{(1-q^2)}{q^2(1-x)} \). We use these identities to simplify (4.44) by means of the following relation

\[
[e_{k-1}^{2q^{2k-7}} \cdots e_2^{q^{4k-8}}] [e_k^{2q^{2k}} \cdots e_2^{q^{4k-6}}] [\sigma_1^{2} \cdots \sigma_{k-1}^{2} \sigma_k^{2}] = \ldots = \ldots
\]

\[
\Pi_{n=1}^{k-1}(-\xi(x^2q^{4k-4-2n})) [\sigma_k^{q^2} \cdots \sigma_{k-1}^{q^2}] = (\lambda q)^{2(k-1)} \Pi_{n=1}^{k-1}(-\xi(x^2q^{4k-4-2n})) [\sigma_k^{q^2} \cdots \sigma_{k-1}^{q^2}] = \ldots = \ldots
\]

Taking into account this relation and identities

\[
Tr_{D(k+1)}(\sigma_k^{q^2}) = 1 - q^2/b, \quad Tr_{D(k)} A_{1\rightarrow k}^+ = \frac{1}{q - q^{-2k+1}} A_{1\rightarrow k-1}^+
\]

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which follow from the definitions (2.6), (2.13), we write (4.44) for $b = q^4$ in the form (4.43).

Eqs. (4.43) are closed with respect to the set of commuting elements $\tau^{(k)}(x)$. We note that eqs. (4.43) can be used for definition of elements $\tau^{(k)}(x)$ in the case when integers $k < 0$.

5 $T-Q$ Baxter equation for open chains of TL type

Now we make a change of variables $z = xq^{2k}$ in (4.43) and put

$$Q^{(k)}(z) := (1 - z^2 q^{-2(2k+2)}) \tau^{(k)}(z q^{-2k}) .$$

As a result we write (4.43) in the form

$$\frac{(1 - z^2 q^{-2})}{(1 - z^2 q^{-4})} Q^{(k)}(z) \tau^{(1)}(z) = Q^{(k+1)}(z q^2) - \frac{\Delta^{(0)}(z)}{q^2} Q^{(k-1)}(z q^{-2}) ,$$

where $\tau^{(1)}(z)$ and $\Delta^{(0)}(z)$ are defined in (3.30) and (4.42), respectively. Introduce the generating function $Q(z, w)$ of the elements $Q^{(k)}(z)$

$$Q(z, w) := \sum_{k=-\infty}^{\infty} w^k Q^{(k)}(z) .$$

Then eq. (5.45) is represented in the form of the $T-Q$ Baxter equation

$$\frac{(1 - z^2 q^{-2})}{(1 - z^2 q^{-4})} Q(z, w) \tau^{(1)}(z) = \frac{1}{w} Q(z q^2, w) - \frac{\Delta^{(0)}(z) w}{q^2} Q(z q^{-2}, w) .$$

We remind that our aim is to find the spectrum of the transfer-matrix type operator $\tau^{(1)}(z)$ for some fixed determinant function $\Delta^{(0)}(z)$. We fix the explicit expression for the determinant $\Delta^{(0)}(z) =: \Delta_N^{(0)}(z)$ is

$$\Delta_N^{(0)}(z) \sigma^{2z^2}_{N+1} = y_N^{q-2} e_N^{q-2} y^{N+1}_N \sigma^{2}_{N+1}$$

where monodromy element $y_N^{q-1}$ is a normalized version of (2.12) and given by

$$y^{\hat{x}}_{N+1} = e_N^{+}(z) y_N e_N^{+}(z) = \ldots = e_N^{+}(z) \ldots e_1^{+}(z) y_1 e_1^{+}(z) \ldots e_N^{+}(z) .$$

Let $y_1^{+}$ be a polynomial in $z$ up to a multiplication by some inessential scalar function of $z$. Such solutions $y_1^{+}$ of the reflection equation have been proposed in [2], [3]. We note that for the case when $y_1^{+}$ is a polynomial in $z$ of order $K$ the transfer-matrix type element is represented in the form

$$\tau^{(1)}_N(z) = Tr_{D(N+1)}(y^{\hat{x}}_{N+1}) = \frac{1}{(q - zq^{-1})^{2N}} \tau_N(z) ,$$

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where the element $\tau_N(z)$ (2.19) is a polynomial in $z$ of order $2N + K$. Then, from (4.7) we have

$$\Delta_N^{(0)}(z) \sigma_N^{q^2} = (e_N^{zq^{-2}} y_N^{zq^{-2}} e_N^{zq^{-2}}) e_N^{zq^{-2}} (e_N^z y_N^z e_N^z) \sigma_N^{q^2} =$$

$$= e_N^{zq^{-2}} e_N^{zq^{-2}} y_N e_N^{zq^{-2}} (e_N^{zq^{-2}} y_N e_N^{zq^{-2}}) \sigma_N^{q^2} =$$

$$= -\xi(z/q^2) e_N^{zq^{-2}} e_N^{zq^{-2}} y_N (\xi(z/q^2))^2 \sigma_N^{q^2} =$$

$$= \cdots = (\xi(zq^{-2}))^{2N} \sigma_N^{q^2} = (\xi(zq^{-2}))^{2N} \sigma_N^{q^2}$$

or

$$\Delta_N^{(0)}(z) = \Delta_N^{(0)}(z) \left(\xi(zq^{-2})\lambda q\right)^{2N} = \Delta_N^{(0)}(z) \left(\frac{1 - z}{q - zq^{-1}}\right)^{2N}. \quad (5.50)$$

We choose the trivial boundary condition $y_1(z) = 1$ which corresponds to the model of the open chain with free ends described by the Hamiltonian $\mathcal{H}_N^{(\text{free})}$ (2.22). In this case the initial determinant $\Delta_N^{(0)}(z)$ is fixed by

$$\Delta_N^{(0)}(z) \sigma_N^{q^2} = e_N^{zq^{-2}} \sigma_N^{q^2} = -q^{-2} \frac{(1 - z^2)}{(1 - z^2 q^{-1})} \sigma_N^{q^2}.$$

Thus, the functional equations (5.45) are represented as

$$\frac{(1 - z^2 q^{-2})}{(1 - z^2 q^{-1})} Q^{(k)}(z) T_N^{(1)}(z) = Q^{(k+1)}(z q^2) + q^{-4} \frac{(1 - z^2)}{(1 - z^2 q^{-1})} \left(\frac{1 - z}{q - zq^{-1}}\right)^{2N} Q^{(k-1)}(zq^{-2}) \quad (5.51)$$

where we substitute $T_N^{(1)}(z) = \frac{T_N^{(1)}(z)}{(q - zq^{-1})^{2N}}$ and $T_N^{(1)}(z)$ is the polynomial in $z$ of order $2N$ (2.20) which has been considered in Sect. 2.

Eqs. (5.46) and (5.51) could be used (by applying the standard procedure) for derivation of analytical Bethe ansatz equations which contains the information about the spectrum of the transfer-matrix type operators $\tau^{(1)}(z)$ and, in particular, about the spectrum of the Hamiltonians (2.17).

6 Conclusion

In Conclusion we briefly discuss the interrelations between the algebraic approach presented above and standard technique developed for the investigation of the open spin chain systems and based on the matrix reflection equation [13].

Consider the $R$-matrix representation $\rho_R$ of the affine Hecke algebra $\hat{H}_{M+1}$:

$$\rho_R(\sigma_1) = \hat{R}_{ii+1} := I \otimes (i-1) \otimes \hat{R} \otimes I \otimes (M-i),$$

$$\rho_R(y_1) = S(L^-) \ L_1^+ := (1/L^-) \ L^- \otimes I \otimes M. \quad (6.52)$$
Here $I$ is $(n + m) \times (n + m)$ identity matrix; $\hat{R} \in \text{End}(V_{n+m}^{\otimes 2})$ is the fundamental $R$-matrix for $U_q(gl(n|m))$ written in the braid form [6], [15] (for the standard form see [20]):

$$\hat{R} = \sum_i (-1)^{|i|} q^{1-2|\nu_i|} e_{ii} \otimes e_{ii} + \sum_{i \neq j} (-1)^{|i||j|} e_{ij} \otimes e_{ji} + \lambda \sum_{j > 1} e_{ii} \otimes e_{jj},$$

$$\hat{R}^2 = \lambda \hat{R} + 1, \quad (-1)^{|12|} \hat{R}_{12} = \hat{R}_{12} (-1)^{|12|}, \quad \lim_{q \to 1} (\hat{R}_{12}) = \mathcal{P}_{12},$$

where $\lambda = q - q^{-1}$, $e_{ij}$ are matrix units, $|i| = 0, 1 (mod 2)$ denotes the parity of the components of supervectors in $V_{n+m}$, $\mathcal{P}_{12} := (-1)^{|12|} P_{12}$ is a superpermutation matrix ($P_{12}$ is a permutation matrix) and we have used concise matrix notations, e.g., $((-1)^{|12|})_{1j2} = (-1)^{|i||j|} \delta_{j1}^2 \delta_{j1}^2$. The operator-valued matrices $L^\pm (L^-) \in \text{Mat}(n + m)$ are invertible, upper (lower) triangular, and satisfy:

$$\hat{R}_{12} L^\pm_2 (-1)^{|12|} L^\pm_1 = L^\pm_2 (-1)^{|12|} L^\pm_1 \hat{R}_{12},$$

$$\hat{R}_{12} L^-_2 (-1)^{|12|} L^-_1 = L^-_2 (-1)^{|12|} L^-_1 \hat{R}_{12}.$$  \hspace{1cm} (6.53)

We prescribe to the elements $(L^\pm)_j$ the grading $([i] + [j])$. According to the approach of [5], $L^\pm$ are matrices of Cartan type generators of $U_q(gl(n|m))$. Note that, in the representation $\rho_R$ (6.52), the solution (2.11) has the form

$$\rho_R(y_1(x)) = \frac{1}{(L^+_1 - \xi x^{-1} L^-_1) (L^+_1 - \xi x L^-_1)} =: K_1(x),$$  \hspace{1cm} (6.54)

and one can immediately check that (6.54) solves the reflection equation (2.10) written in the $R$-matrix form

$$\hat{R}_{12}(x/z) K_1(x) \hat{R}_{12}(x z) K_1(z) = K_1(z) \hat{R}_{12}(x z) K_1(x) \hat{R}_{12}(x/z),$$  \hspace{1cm} (6.55)

where

$$\hat{R}_{nn+1}(x) := \hat{R}_{nn+1} - x \hat{R}_{nn+1}^{-1} = \rho_R(\sigma_n),$$

is the $R$-matrix image of the baxterized element (2.3). To prove (6.55) one needs only the fact that $K(x)$ (6.54) is represented in the factorized form $L^{-1}(\xi/x)L(\xi x)$, where the operator-valued matrix

$$L(x) = (L^+ - x L^-),$$  \hspace{1cm} (6.56)

defines the evaluation representation [14] of the affine superalgebra $U_q(\hat{gl}(n|m))$ and satisfies the intertwining relations

$$\hat{R}_{12}(x) L_2(xy) (-1)^{|12|} L_1(y) = L_2(y) (-1)^{|12|} L_1(xy) \hat{R}_{12}(x).$$  \hspace{1cm} (6.57)

We note that all grading factors $(-1)^{|12|}$ have disappeared in (6.55). The comultiplications for the quantum superalgebras with defining relations (6.53) and (6.57) have the usual form $\Delta L^\pm = L^\pm \otimes L^\pm$ and $\Delta L(x) = L(x) \otimes L(x)$, where $\otimes$ is a graded tensor product, i.e., $(a \otimes b)(c \otimes d) = (-1)^{|b||d|}(ac \otimes bd)$. 

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We note that reflection equation solutions of the type (6.54) were called in [21] as solutions which admit a “regular factorization”. Given a matrix \( L(x) \) which satisfies (6.57) one can construct by means of the standard quantum inverse scattering method an operator-valued monodromy matrix of the chain with \((N + 1)\) sites

\[
T_a(\xi, \xi_1, \ldots, \xi_N|x) = L_a(\xi x) \otimes L_a(\xi_1 x) \otimes \ldots \otimes L_a(\xi_N x),
\]

(6.58)

where \( a \) denotes the number of auxiliary matrix space and \( \otimes \) is a graded tensor product of the operator spaces. The monodromy matrix (6.58) satisfies the intertwining relations (6.57)

\[
\hat{R}_{12}(x) T_a(\xi_1, \ldots |xy) (-1)^{[1][2]} T_1(\xi_1, \ldots |y) = T_a(\xi, \ldots |y) (-1)^{[1][2]} T_1(\xi, \ldots |xy) \hat{R}_{12}(x).
\]

Then, we construct the two-row Sklyanin’s monodromy matrix

\[
K_a(\xi, \xi_1, \ldots, \xi_N|x) = T_a^{-1}(\xi, \xi_1, \ldots, \xi_N|x^{-1}) \cdot T_a(\xi, \xi_1, \ldots, \xi_N|x),
\]

(6.59)

which obviously solves the reflection equation (6.55). The corresponding Sklyanin’s transfer matrix is given by the quantum supertrace

\[
\tau_N(\xi, \xi_1, \ldots, \xi_N|x) = Tr_a(D_a K_a(\xi, \xi_1, \ldots, \xi_N|x)),
\]

(6.60)

where \( D^i_j = (-1)^i q^{2m^i+(-1)^i(2i-2m-1)} \delta^i_j [15] \) is the matrix of quantum supertrace (for the case of quantum superalgebra \( U_q(gl(n|m)) \)) and one can check that this matrix is a constant solution of the conjugated reflection equation. Taking into account the fact that the monodromy matrix (6.58) satisfies the intertwining relations (6.57) one can use a matrix representation in which the image of (6.58) is

\[
\hat{R}_{12}(x) T_a(\xi_1, \ldots |xy) (-1)^{[1][2]} T_1(\xi_1, \ldots |y) = \hat{R}_{12}(x) .
\]

Then, we construct the two-row Sklyanin’s monodromy matrix

\[
\hat{R}_{12}(x) T_a(\xi_1, \ldots |xy) (-1)^{[1][2]} T_1(\xi_1, \ldots |y) = \hat{R}_{12}(x).
\]

(6.61)

where we have applied the matrix homomorphism to all factors in (6.58) except the first one and \(1, 2, \ldots, N\) denote the numbers of matrix spaces which replace the operator spaces. Then, for the Sklyanin’s monodromy matrix (6.59) we obtain the representation

\[
K_a(\xi, \xi_1, \ldots, \xi_N|x) \rightarrow \hat{R}_{N-1, N}(\xi_N|x) \hat{R}_{N-1, N}(\xi_{N-1}|x) \ldots \hat{R}_{12}(\xi_1 x) \hat{R}_{12}(\xi_1 x) \ldots \hat{R}_{N-1, N}(\xi_{N-1}|x) \hat{R}_{N-1, N}(\xi_{N-1}|x) = \rho R \left( e^+_N(\xi_N|x) e^+_N(\xi_{N-1}|x) \ldots e^+_1(\xi_1|x) e^+_1(\xi_1|x) \ldots e^+_1(\xi_{N-1}|x) e^+_1(\xi_{N-1}|x) \right)
\]

(6.61)

where we have taken into account the identity \( \hat{R}(x^{-1}) \hat{R}(x) = (q - x q^{-1})(q - x^{-1} q^{-1}) \), put \( a = (N + 1) \) and used the notations (2.5), (6.52). It is clear that (6.61) is the \( \hat{R} \)-matrix image of (5.48) for \( x = z, \xi_i = 1 \) and \( y_1 = y_{1, \xi} - \xi/z \). The quantum trace
(6.60) of (6.61) coincides with the $R$-matrix image $\rho_R$ of the transfer matrix type element (5.49).

Thus, we have demonstrated that the integrable models based on the consideration of the monodromy matrices (6.59) and models based on (5.48) in general are different and coincides only on their $R$ matrix projections. This identification is clarified by the formula (6.61).

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