THE $F$-OBJECTIVE FUNCTION METHOD FOR DIFFERENTIABLE INTERVAL-VALUED VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, a differentiable vector optimization problem with the multiple interval-valued objective function and with both inequality and equality constraints is considered. The Karush-Kuhn-Tucker necessary optimality conditions are established for such a differentiable interval-valued multiobjective programming problem. Further, a new approach, called $F$-objective function method, is introduced for solving the considered differentiable vector optimization problem with the multiple interval-valued objective function. In this method, its associated vector optimization problem with the multiple interval-valued $F$-objective function is constructed. Their equivalence is established under $F$-convexity assumptions. It is shown that the introduced approach can be used to solve a nonlinear nonconvex interval-valued optimization problem. By using the introduced approximation method, it is also presented in some cases that a nonlinear nonconvex interval-valued optimization problem can be solved by the help of methods for solving linear interval-valued optimization problems.

1. Introduction. Vector optimization problems, commonly known as multiobjective programming problems or multicriteria optimization problems, gained importance because in the real world applications we encounter such extremum problems where we have to deal with more than one objective function and they should be satisfied simultaneously for optimum operation. In optimization theory, we are used to the fact that, in general, operations research problems are usually modeling as deterministic optimization problems and then their corresponding solutions are also precise. Whereas, in most real-world applications, optimization problems which are models of real-world processes contain uncertainty. So, the decisions have to be taken in changeable conditions. We shall illustrate this situation with the help of the following example of a vector optimization problem in which its objectives are not determined precisely:

Example 1. Company is specialized on making two types of tools: pliers and screwdrivers. It uses two types of material for creating these tools: plastic for handles and iron for effective part of tools. For one plier, 50 units of iron and 50
units of plastic is needed. For one screwdriver it is 30 iron units and 40 plastic units. Company disposes with 2000 units of iron and the same amount of plastic units. The expected profit from one hammer is lying between $5e - 9e$ and from one screwdriver it is lying between $3e - 5e$. Processing one plier is saving between 25 - 30 minutes, while processing one screwdriver is saving only between 12 - 15 minutes. How many pliers and how many screwdrivers should company produce to maximize profit, as well as to maximize amount of saved time?

Mathematical formulation: Note that the considered extremum problem is a multicriteria optimization problem. Firstly, it is necessary to make some notations for the considered vector optimization problem. This multiobjective programming problem has 2 objective functions (maximizing profit and amount of saved time), 2 variables (number of produced pliers and screwdrivers), and some linear constrains (amount of iron and plastic owned by company). Clearly, the individual profits and amount of saved time (per unit) of pliers and screwdrivers are not fixed and hence it is not possible to formulate the appropriate profit and amount of saved time functions using only the fixed (real) coefficients. The expected profits of pliers and screwdrivers are represented by the intervals $[5, 9]$ and $[3, 5]$, respectively, whereas, the amounts of saved time in producing of pliers and screwdrivers are represented by the intervals $[25, 30]$ and $[12, 15]$, respectively. Hence, to maximize total profit and amount of saved time, we formulate the following corresponding vector optimization problem:

$$
\text{maximize } f(x) = ([5, 9]x_1 + [3, 5]x_2, [25, 30]x_1 + [12, 15]x_2)
$$

$$
50x_1 + 50x_2 \leq 2000, \\
30x_1 + 40x_2 \leq 2000, \\
x_1, x_2 \geq 0.
$$

Note that both the profit and amount of saved time are intervals in the above formulated vector optimization problem. It is obvious that such a type of concepts of profit and amount of saved time are more realistic than the deterministic (real) concepts of profit and amount of saved time like fixed amounts, we say $5e$ and 30 minutes, etc. Why this approach is more realistic in comparison to deterministic formulation of the production factors mentioned above? This is a consequence of the fact that different factors, for example, like market value, labor efficiency, etc. differ from time to time and, therefore, we cannot predict the actual profit and the amount of saved time in prior.

Note that the multiobjective programming problem is an example of a linear vector optimization problem with uncertainty. However, many of real-world problems with uncertainty cannot be modeled easily as linear uncertain vector optimization problems. There exist various approaches to vector optimization problems with uncertainty. This follows from the fact that the methodology for solving uncertain vector optimization problems has widely applied to many research fields. Among various types of methodologies usually used to solve multiobjective optimization models with uncertainty, the interval-valued vector optimization problems have been of much interest in recent past and thus explored the extent of optimality conditions in different areas. Namely, interval-valued multiobjective programming was introduced as an effective tool to deal with the real world vector optimization problems with uncertainty data. This is a consequence of the fact that it does not require the specification or the assumption of probabilistic distributions (as in stochastic programming) or possibilistic distributions (as in fuzzy programming).
In interval-valued optimization problems, the coefficient is taken as closed intervals and they are closely related to inexact linear programming problems. Recently, there has been an increasing interest in developing optimality conditions and duality results for such optimization problems (see, for example, [1], [2], [7], [8], [15], [21], [25], [27], [34], [37], [38], [42], [44], [45]; and the references therein).

Many researchers developed different types of interval-oriented algorithms and optimization techniques for solving interval-valued optimization problems (see, for example, [4], [10], [14], [16], [17], [32], [39]). Recently, there has been an increasing interest in introducing new methods for solving interval-valued multiobjective optimization problems. Namely, Ishibuchi and Tanaka [18] considered multiobjective programming problems with interval-valued objective functions and proposed the ordering relation between two closed intervals by considering the maximization and minimization problems separately. Chanas and Kuchta [9] generalized the concept of optimality introduced by Ishihuchi and Tanaka [18] for vector optimization problems with interval-valued objective functions to the case of the linear multiobjective programming problem with interval coefficients in the objective function based on preference relations between intervals. Urli and Nadeau [41] used an interactive method to solve the linear multiobjective programming problems with interval coefficients and they also proposed a methodology to transform a nondeterministic problem into a deterministic problem. Oliveira and Antunes [33] were the first to give an overview on multiobjective linear programming problems with interval coefficients by illustrating many numerical examples. Bhurjee and Panda [7] developed a methodology to study the existence of the solutions of an interval optimization problem. Jana and Panda [20] considered a nonlinear vector optimization problem with both linear and nonlinear interval-valued functions in the objective function as well as in the constraints. They proposed a methodology to find an efficient solution, called as a preferable efficient solution. Karmakar and Bhunia [23] proposed an alternative optimization technique via multiobjective programming for constrained optimization problems with interval-valued objectives.

In recent years, considerable attention has been given to devising new methods which solve the original multiobjective mathematical programming problem by the help of some associated vector optimization problem (see, for example, [6], [11], [12], [19], [29], [36]; and others). One of such approaches is the modified objective function method introduced by Antczak [4] in the case of differentiable vector optimization problems. In this method, for the original differentiable multiobjective programming problem, its associated vector optimization problem with the modified objective function is constructed.

In the paper, we consider a differentiable vector optimization problem with multiple interval-valued objective function and with both inequality and equality constraints. For the considered differentiable interval-valued multiobjective programming problem, we prove the Karush-Kuhn-Tucker necessary optimality conditions by using the concept of a convergence vector for a set introduced by Lin [26]. However, the main purpose of this paper is to introduce a new approach for a characterization of weak LU-Pareto optimality solutions of a nonlinear nonconvex differentiable vector optimization problem with the multiple interval-valued objective function and with both inequality and equality constraints. Namely, we introduce a new approximation method, called the modified F-objective function method, for finding a weak LU-Pareto (LU-Pareto) solution of the considered differentiable nonlinear vector optimization problem with the multiple interval-valued objective
function and with both inequality and equality constraints. In this approach, an associated interval-valued vector optimization problem with the modified multiple interval-valued objective function is constructed at a given feasible solution for the original interval-valued multiobjective programming problem. Then, under the concept of $F$-convexity, it is established the equivalence between weak LU-Pareto solutions of the original interval-valued multiobjective programming problem and its associated approximated vector optimization problem with the modified multiple interval-valued objective function. The results established in the paper have been illustrated by examples of differentiable interval-valued multiobjective problems with the multiple interval-valued objective function which are solved by using the introduced approximation method. Also it is illustrated one of the important and useful in practice property of the introduced approach - in some cases, the interval-valued optimization problem with the modified multiple interval-valued objective function is constructed in this method is linear. Therefore, methods for solving linear interval-valued optimization problems existing in the literature are applicable for solving such an extremum problem. An example of such a nonlinear nonconvex differentiable interval-valued optimization problem is presented in the paper. In order to solve it, we use the approach for solving interval-valued linear optimization problems introduced by Chanas and Kuchta [9] which is based on preference relations between intervals.

2. Notations and preliminaries. Let $R^n$ be the $n$-dimensional Euclidean space and $R^n_+$ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper.

For any vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ in $R^n$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

Let $I(R)$ be a class of all closed and bounded intervals in $R$. Throughout this paper, when we say that $A$ is a closed interval, we mean that $A$ is also bounded in $R$. If $A$ is a closed interval, we use the notation $A = [a^L, a^U]$, where $a^L$ and $a^U$ mean the lower and upper bounds of $A$, respectively. In other words, if $A = [a^L, a^U] \in I(R)$, then $A = [a^L, a^U] = \{x \in R : a^L \leq x \leq a^U\}$. If $a^L = a^U = a$, then $A = [a, a] = a$ is a real number.

Let $A = [a^L, a^U]$, $B = [b^L, b^U]$, then, by definition, we have:

(a): $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$,
(b): $-A = \{-a : a \in A\} = [-a^U, -a^L]$,
(c): $A - B = A + (-B) = \{a - b : a \in A \text{ and } b \in B\} = [a^L - b^U, a^U - b^L]$,
(d): $k + A = \{k + a : a \in A\} = [k + a^L, k + a^U]$ , where $k$ is a real number,
(e): $kA = \begin{cases} 
[ka^L, ka^U] & \text{if } k > 0, \\
[ka^U, ka^L] & \text{if } k \leq 0,
\end{cases}$ where $k$ is a real number.

For the more details on the topic of interval analysis, we refer to Moore [30], Moore et al. [31] and Alefeld and Herzberger [3].

In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another.

For $A = [a^L, a^U]$ and $B = [b^L, b^U]$, we write
\( A \leq_{LU} B \) if and only if
\[
\begin{align*}
& a^L \leq b^L \\
& a^U \leq b^U.
\end{align*}
\] (1)

It means that \( A \) is inferior to \( B \), or \( B \) is superior to \( A \). It is easy to see that \( \leq_{LU} \) is a partial ordering on \( I( \mathbb{R} ) \).

Further, we can write \( A <_{LU} B \) if and only if \( A \leq_{LU} B \) and \( A \neq B \). Equivalently, \( A <_{LU} B \) if and only if
\[
\begin{align*}
& a^L < b^L \\
& a^U \leq b^U \quad \text{or} \\
& a^L \leq b^L \\
& a^U < b^U \quad \text{or} \\
& a^L < b^L \\
& a^U < b^U.
\end{align*}
\] (2)

We say that \( A = (A_1, ..., A_p) \) is an interval-valued vector if its each component \( A_i = [a_i^L, a_i^U] \) is closed interval for \( i = 1, ..., p \). Let \( A = (A_1, ..., A_p) \) and \( B = (B_1, ..., B_p) \) be two interval-valued vectors. We shall write \( A \leq_{LU} B \) if and only if \( A_i \leq_{LU} B_i \) for any \( i = 1, ..., p \), and \( A <_{LU} B \) if and only if \( A_i \leq_{LU} B_i \) for any \( i = 1, ..., p \) and \( A_i <_{LU} B_i \) for at least one \( i^* \in \{1, ..., p\} \).

Throughout this section, let \( X \) be a nonempty open subset of \( \mathbb{R}^n \).

**Definition 2.** A function \( \psi : X \to I( \mathbb{R} ) \) is called an interval-valued function if \( \psi(x) = [\psi^L(x), \psi^U(x)] \) with \( \psi^L, \psi^U : X \to \mathbb{R} \) such that \( \psi^L(x) \leq \psi^U(x) \) for each \( x \in X \).

Now, we shall consider the differentiation of an interval-valued function. Namely, we use a very straightforward concept of differentiation introduced by Wu [42].

**Definition 3.** Let \( S \) be an open set in \( \mathbb{R} \). An interval-valued function \( \psi : S \to I( \mathbb{R} ) \) with \( \psi(x) = [\psi^L(x), \psi^U(x)] \) is called weakly differentiable at \( x \) if the real-valued functions \( \psi^L \) and \( \psi^U \) are differentiable at \( x \) (in the usual sense).

Now, we recall the definition of a sublinear functional (with respect to the third component).

**Definition 4.** A functional \( F : X \times X \times \mathbb{R}^n \to \mathbb{R} \) is sublinear (with respect to the third component) if, for all \( x, u \in X \subset \mathbb{R}^n \),
\[
\begin{align*}
1) \quad & F(x, u; q_1 + q_2) \leq F(x, u; q_1) + F(x, u; q_2), \ \forall q_1, q_2 \in \mathbb{R}^n, \\
2) \quad & F(x, u; \alpha q) = \alpha F(x, u; q), \ \forall \alpha \in \mathbb{R}_+, \ \forall q \in \mathbb{R}^n.
\end{align*}
\]

The concept of the sublinear functional was given by Hanson and Mond [13] (see also Preda [35]). By 2), it is clear that
\[
F(x, u; 0) = 0. \quad (3)
\]

Several generalizations of the definition of a convex function have been introduced to optimization theory in order to weaken the assumption of convexity in proving optimality and duality results for new classes of nonconvex optimization problems, including multiobjective programming problems. One of such generalizations in the smooth case is the definition of a scalar \( F \)-convex function introduced by Hanson and Mond [13]. Now, we extend this definition to the differentiable vectorial case.

**Definition 5.** Let \( f = (f_1, ..., f_p) : X \to \mathbb{R}^p \) be a differentiable vector-valued function defined on \( X \) and \( u \in X \) be given. If there exists a sublinear function \( F : X \times X \times \mathbb{R}^n \to \mathbb{R} \) with respect to the third component such that, the following inequalities
\[
f_i(x) - f_i(u) \geq F(x, u; \nabla f_i(u)) \quad (>) \quad i = 1, ..., p,
\] (4)
hold for all \( x \in X \), then \( f \) is said to be a (vector-valued) \( F \)-convex (strictly \( F \)-convex) function at \( u \) on \( X \).

Each function \( f_i, \, i = 1, \ldots, p \), satisfying (4) is said to be \( F \)-convex (strictly \( F \)-convex) at \( u \) on \( X \).

If inequalities (4) are satisfied at any point \( u \), then \( f \) is said to be a (vector-valued) \( F \)-convex (strictly \( F \)-convex) function on \( X \).

For other properties of a class of differentiable \( F \)-convex functions, the readers are advised to consult Hanson and Mond [13] (see also Kim [24], Preda [35]).

Now, we extend the definition of \( F \)-convexity to the case of an interval-valued function.

**Definition 6.** Let \( f : X \to I(\mathbb{R}) \) be a weakly differentiable interval-valued function and \( u \in X \). If there exists a sublinear functional \( F : X \times X \times \mathbb{R}^n \to \mathbb{R} \) such that, the inequalities

\[
F^L(x) - F^L(u) \geq F(x, u; \nabla f^L(u)), \quad (>)
\]

\[
F^U(x) - F^U(u) \geq F(x, u; \nabla f^U(u)) \quad (>)
\]

hold for all \( x \in X \), then \( f \) is said to be a (strictly) interval-valued \( F \)-convex function at \( u \) on \( X \) or a (strictly) \( LU \)-\( F \)-convex interval-valued function at \( u \) on \( X \). If the inequalities (5) and (6) are satisfied at any point \( u \), then \( f \) is said to be a (strictly) \( F \)-convex interval-valued function on \( X \).

**Example 7.** Let us consider \( f : R^2 \to I(\mathbb{R}) \) which is a nonconvex weakly differentiable interval-valued function defined by \( f(x_1, x_2) = [x_1 + \sin x_2 + 2x_2, x_1 + \sin x_2 + 2x_2 + 1] \). Let \( u \) be a given arbitrary point of \( R^2 \). We show by Definition 6 that \( f \) is an interval-valued \( F \)-convex function at \( u \) on \( R^2 \). Let \( F : R^2 \times R^2 \to R \) be a functional defined by \( F(x, u; \vartheta) := (x_1 - u_1)\vartheta_1 + 2x_2 - 2u_2 + \sin x_2 - \sin u_2\vartheta_2 \). Indeed, it can be shown that the inequalities (5) and (6) are fulfilled for all \( x \in R^2 \) at any arbitrary \( u \in R^2 \), which means by Definition 6 that \( f \) is an interval-valued \( F \)-convex function on \( R^2 \).

In the paper, consider the following vector optimization problem with multiple interval-valued objective function and with both inequality and equality constraints:

\[
f(x) = (f_1(x), \ldots, f_p(x)) \to \min
\]

\[
g(x) = (g_1(x), \ldots, g_m(x)) \leq 0,
\]

\[
h(x) = (h_1(x), \ldots, h_q(x)) = 0,
\]

\( x \in X, \) (IVP)

where each \( f_i : X \to I(\mathbb{R}), \, i \in I = \{1, \ldots, p\} \) is an interval-valued function, that is, \( f_i(x) = [f^L_i(x), f^U_i(x)] \), \( i \in I \),

and, moreover, \( g : X \to R^m, \, h : X \to R^q \) and \( X \) is a nonempty open convex set of \( R^n \). We will assume, moreover, that \( f^L_i, f^U_i : X \to R, \, i \in I \), and \( g_j : X \to R, \, j \in J = \{1, \ldots, m\}, \, h_t : X \to R, \, t \in T = \{1, \ldots, q\} \), are differentiable functions on \( R^n \). For the purpose of simplifying our presentation, we will introduce the following notations \( f^L = (f^L_1, \ldots, f^L_p)^T, \quad f^U = (f^U_1, \ldots, f^U_p)^T \). Further, let us denote by \( \Omega \) the set of all feasible solutions for the considered interval-valued multiobjective optimization problem (IVP), that is, the set \( \Omega = \{x \in X : g(x) \leq 0, \, h(x) = 0\} \) and, moreover, by \( J(x) \), the set of constraint indices that are active at a feasible solution \( x \), that is, \( J(x) = \{j \in J : g_j(x) = 0\} \).
Since each of objective function values is a closed interval, we need to provide an ordering relation between any two closed intervals. The most direct way is to invoke the ordering relation \( \preceq_{LU} \) that was defined above. However, since \( \preceq_{LU} \) is a partial ordering relation, not total ordering, on \( I(R) \), we shall follow the similar concept of a nondominated solution used in a multiobjective programming problem to investigate the solution concepts for the problem (IVP).

Similar to the concept of a nondominated solution in vector optimization problems, Wu [43] has proposed solution concepts for multiobjective programming problems with interval-valued objective functions based on a partial ordering on the set of all closed intervals. Namely, optimal solutions for such vector optimization problems are defined in terms of a weak \( LU \)-Pareto solution (a \( LU \)-weakly efficient) and a \( LU \)-Pareto solution (a \( LU \)-efficient solution) in the following sense:

**Definition 8.** A feasible point \( \overline{x} \) is said to be a weak \( LU \)-Pareto solution (a weakly \( LU \)-efficient solution) for the problem (IVP) if and only if there is no other \( x \in \Omega \) such that, for each \( i \in I \),

\[
f_i(x) \preceq_{LU} f_i(\overline{x}).
\]

**Definition 9.** A feasible point \( \overline{x} \) is said to be a \( LU \)-Pareto solution (a \( LU \)-efficient solution) for the problem (IVP) if and only if there is no other \( x \in \Omega \) such that

\[
f(x) \preceq_{LU} f(\overline{x}).
\]

Now, we re-call the definition of a convergence vector for a given set introduced by Lin [26].

**Definition 10.** [26] Let \( Y \subseteq R^p \) be given. It is said that \( z \in R^p \) is a convergence vector for \( Y \) at \( \overline{y} \in Y \) if there exist a sequence \( \{y_k\} \) in \( Y \) and a sequence \( \{\alpha_k\} \) of strictly positive real numbers such that

\[
\lim_{k \to \infty} y_k = \overline{y}, \quad \lim_{k \to \infty} \alpha_k = 0, \quad \lim_{k \to \infty} \frac{y_k - \overline{y}}{\alpha_k} = z.
\]

Let us denote by \( C(\Omega, \overline{x}) \) the cone of convergence vectors for the set \( \Omega \) at \( \overline{x} \).

**Theorem 11.** Let \( f = (f_1, ..., f_p) \), where \( f_i : R^n \to I(R), \) \( i \in I = \{1, ..., p\} \) and \( \Omega \subseteq R^q \). If \( \overline{x} \in \Omega \) is a weak \( LU \)-Pareto solution for \( f \) on \( \Omega \), then no a convergence vector for \( f(\Omega) \) at \( \overline{y} = f(\overline{x}) \) is strictly negative.

**Proof.** Assume that \( \overline{x} \in \Omega \) is a weak \( LU \)-Pareto solution for \( f \) on \( \Omega \). Hence, \( \overline{y} = (\overline{y}^L, \overline{y}^U) = (f^L(\overline{x}), f^U(\overline{x})) \) is a weak \( LU \)-Pareto solution for the set \( f(\Omega) = (f^L(\Omega), f^U(\Omega)) \). Let \( d \in R^n \) be a convergence vector for the set \( \Omega \) at \( \overline{x} \). Further, let \( \{x_k\} \subset \Omega \) be the corresponding sequence converging to \( \overline{x} \). Consider a sequence \( \{y_k\} \subset f(\Omega) = (f^L(\Omega), f^U(\Omega)) \), that is, \( y_k = (y_k^L, y_k^U) \) such that \( y_k^L = f^L(x_k) \) and \( y_k^U = f^U(x_k) \) for any integer. This means that \( \{y_k^L\} \subset f^L(\Omega) \) and \( \{y_k^U\} \subset f^U(\Omega) \) for any integer. Since \( f \) is weakly differentiable at \( \overline{x} \), by Definition 3, we have that the functions \( f^L \) and \( f^U \) are differentiable at \( \overline{x} \). By the differentiability of \( f^L \) and \( f^U \) at \( \overline{x} \), it follows that \( f^L \) and \( f^U \) are continuous at this point. Thus, the sequence \( \{y_k\} \) is convergent to \( \overline{y} = f(\overline{x}) \), which means that the sequences \( \{y_k^L\} \) and \( \{y_k^U\} \) are convergent to \( \overline{y}^L = f^L(\overline{x}) \) and \( \overline{y}^U = f^U(\overline{x}) \), respectively.

By means of contradiction, suppose that there exists a convergence vector \( \overline{z} = (\overline{z}^L, \overline{z}^U) \) for \( f(\Omega) = (f^L(\Omega), f^U(\Omega)) \) at \( \overline{y} = (\overline{y}^L, \overline{y}^U) = (f^L(\overline{x}), f^U(\overline{x})) = f(\overline{x}) \), which is a strictly negative. Then, by Definition 10, it follows that, for the sequence
\{y_k\} \subset f(\Omega), \lim_{k \to \infty} y_k = \overline{y},\) there exists a sequence of strictly positive real numbers \(\{\alpha_k\}\) converging to 0 such that
\[
\lim_{k \to \infty} \frac{y_k - \overline{y}}{\alpha_k} = \overline{z}.
\] (7)

Since \(y_k = f(x_k) \subset f(\Omega)\) for any integer and \(\overline{y} = f(\overline{x}),\) (7) gives
\[
\lim_{k \to \infty} \frac{f(x_k) - f(\overline{x})}{\alpha_k} = \overline{z}.
\] (8)

Using \(f(\overline{x}) = (f^L(\overline{x}), f^U(\overline{x})), f(x_k) = (f^L(x_k), f^U(x_k))\) and \(\overline{z} = (\overline{z}^L, \overline{z}^U),\)

According to (b), (8) yields
\[
\lim_{k \to \infty} \frac{f^L(x_k) - f^U(\overline{x})}{\alpha_k} = \overline{z}^L,
\] (9)
\[
\lim_{k \to \infty} \frac{f^U(x_k) - f^L(\overline{x})}{\alpha_k} = \overline{z}^U.
\] (10)

By assumption, \(\overline{z} = (\overline{z}^L, \overline{z}^U) < 0.\) Hence, (9) and (10) imply, respectively,
\[
\lim_{k \to \infty} \frac{f^L(x_k) - f^U(\overline{x})}{\alpha_k} < 0,
\] (11)
\[
\lim_{k \to \infty} \frac{f^U(x_k) - f^L(\overline{x})}{\alpha_k} < 0.
\] (12)

Since \(\{\alpha_k\}\) is a sequence of strictly positive real numbers for any integer, for each \(i = 1, \ldots, p,\) there exist \(K^L_i\) and \(K^U_i\) such that
\[
f^L_i(x_k) < f^U_i(\overline{x}) \text{ for any } k > K^L_i,
\] (13)
\[
f^U_i(x_k) < f^L_i(\overline{x}) \text{ for any } k > K^U_i.
\] (14)

By Definition 2, it follows that
\[
f^L_i(\overline{x}) \leq f^U_i(\overline{x}), i = 1, \ldots, p,
\] (15)
\[
f^L_i(x_k) \leq f^U_i(x_k), i = 1, \ldots, p.
\] (16)

Combining (14), (15) and (16), we get
\[
f^L_i(x_k) < f^U_i(\overline{x}) \text{ for any } k > \max \{K^L_i, K^U_i\},
\] (17)
\[
f^U_i(x_k) < f^L_i(\overline{x}) \text{ for any } k > \max \{K^L_i, K^U_i\}.
\] (18)

Let \(K_{\max} = \max \{K^L_i, K^U_i : i = 1, \ldots, p\}.\) Then, (17) and (18) imply, respectively,
\[
f^L_i(x_k) < f^U_i(\overline{x}) \text{ for any } k > K_{\max}, i = 1, \ldots, p,
\]
\[
f^U_i(x_k) < f^L_i(\overline{x}) \text{ for any } k > K_{\max}, i = 1, \ldots, p.
\]

where \(x_k \in \Omega.\) This is a contradiction (for any sufficiently large \(k\)) to the assumption that \(\overline{x} \in \Omega\) is a weak \(LU\)-Pareto solution for \(f\) on \(\Omega.\) Thus, the proof of this theorem is completed.

In order to prove the Karush-Kuhn-Tucker necessary optimality conditions for a weak \(LU\)-Pareto solution in the considered interval-valued multiobjective programming problem (IVP), we give the following constraint qualification.
Definition 12. Let the constraint functions \( g = (g_1, ..., g_m) \) and \( h = (h_1, ..., h_q) \) be differentiable at \( \bar{x} \in \Omega \). It is said that the constraint qualification (CQ) is satisfied at \( \bar{x} \) for (IVP) if

\[
C(\Omega, \bar{x}) = \{ d \in \mathbb{R}^n : \nabla g_j(\bar{x}) d \leq 0, j \in J, \nabla h_j(\bar{x}) d = 0 \}.
\] (19)

Before we establish the Karush-Kuhn-Tucker necessary optimality conditions for problem (IVP), we re-call the Motzkin’s theorem of the alternative (see Mangasarian [28]).

Theorem 13. (Motzkin’s theorem of the alternative). Let \( A, C, D \) be given matrices, with \( A \) being nonvacuous (see Definition 1.25 [28]) and \( D \) being nonzero. Then either the system of inequalities

\[
Ax < 0, \quad Cx \leq 0, \quad Dx = 0
\]

has a solution \( x \), or the system

\[
A^T y_1 + C^T y_2 + D^T y_3 = 0, \quad y_1 \geq 0, \quad y_2 \geq 0
\]

has solution \( y_1, y_2 \) and \( y_3 \), but never both.

Wu [42] proved the Karush-Kuhn-Tucker necessary optimality conditions for a scalar optimization problem with the interval-valued objective function. We now extend this result for a multiobjective programming problem with the multiple interval-valued objective function and with both inequality and equality constraints.

Theorem 14. (Karush-Kuhn-Tucker necessary optimality conditions). Let \( \bar{x} \in \Omega \) be a weak \( LU \)-Pareto solution of the differentiable vector optimization problem (IVP) with the multiple interval-valued objective function and the constraint qualification (CQ) be satisfied at \( \bar{x} \). Then there exist Lagrange multipliers \( \lambda^L \in \mathbb{R}^p, \lambda^U \in \mathbb{R}^p, \mu \in \mathbb{R}^m \) and \( \xi \in \mathbb{R}^q \) such that

\[
\sum_{i=1}^p \lambda^L_i \nabla f^L_i(\bar{x}) + \sum_{i=1}^p \lambda^U_i \nabla f^U_i(\bar{x}) + \sum_{j=1}^m \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^q \xi_k \nabla h_k(\bar{x}) = 0,
\] (20)

\[
\mu_j g_j(\bar{x}) = 0, \quad j \in J,
\] (21)

\[
\left( \lambda^L, \lambda^U \right) \geq 0, \quad \mu \geq 0.
\] (22)

Proof. Assume that \( \bar{x} \in \Omega \) is a weak \( LU \)-Pareto solution of the considered interval-valued vector optimization problem (IVP) and the constraint qualification (CQ) is satisfied at \( \bar{x} \). Therefore, \( \bar{y} = (\bar{y}^L, \bar{y}^U) = (f^L(\bar{x}), f^U(\bar{x})) \) is a weak \( LU \)-Pareto solution for the set \( f(\Omega) = (f^L(\Omega), f^U(\Omega)) \). Let \( d \in \mathbb{R}^n \) be a convergence vector for the set \( \Omega \) at \( \bar{x} \). Further, let \( \{x_k\} \subset \Omega \), \( \{\alpha_k\} \) be the corresponding sequences, where \( \{\alpha_k\} \) is a sequence of strictly positive real numbers. Then, we consider the sequences \( \{y_k^L\} \subset f^L(\Omega) \) and \( \{y_k^U\} \subset f^U(\Omega) \) such that \( y_k^L = f^L(x_k) \) and \( y_k^U = f^U(x_k) \). By assumption, \( f = (f^L, f^U) \) is weakly differentiable at \( \bar{x} \). Thus, by Definition 3, it follows that \( f^L \) and \( f^U \) are differentiable at \( \bar{x} \). Hence, they are also continuous at \( \bar{x} \). Then, by the continuity of \( f^L \) and \( f^U \) at \( \bar{x} \), the sequences \( \{y_k^L\} \) and \( \{y_k^U\} \) are convergent to \( \bar{y}^L \) and \( \bar{y}^U \), respectively. Further, by the differentiability of \( f^L \) and \( f^U \) at \( \bar{x} \), we have

\[
f^L(x_k) = f^L(\bar{x}) + \nabla f^L(\bar{x})^T (x_k - \bar{x}) + \theta^L(\|x_k - \bar{x}\|),
\] (23)

\[
f^U(x_k) = f^U(\bar{x}) + \nabla f^U(\bar{x})^T (x_k - \bar{x}) + \theta^U(\|x_k - \bar{x}\|),
\] (24)
where
\[
\theta^L(\|x_k - \bar{x}\|) \rightarrow 0 \text{ when } x_k \rightarrow \bar{x}, \quad (25)
\]
\[
\theta^U(\|x_k - \bar{x}\|) \rightarrow 0 \text{ when } x_k \rightarrow \bar{x}. \quad (26)
\]

Then, we have
\[
y^L_k - y^L = \frac{f^L(x_k) - f^L(\bar{x})}{\alpha_k} = \nabla f^L(\bar{x})^T \left( \frac{x_k - \bar{x}}{\alpha_k} \right) + \frac{\theta^L(\|x_k - \bar{x}\|)}{\alpha_k} \cdot \|x_k - \bar{x}\|, \quad (27)
\]
\[
y^U_k - y^U = \frac{f^U(x_k) - f^U(\bar{x})}{\alpha_k} = \nabla f^U(\bar{x})^T \left( \frac{x_k - \bar{x}}{\alpha_k} \right) + \frac{\theta^U(\|x_k - \bar{x}\|)}{\alpha_k} \cdot \|x_k - \bar{x}\|. \quad (28)
\]

By (27), (28) and also Definition 10, we conclude that the vectors \( z^L \) and \( z^U \) defined by
\[
z^L = \lim_{k \to \infty} \frac{y^L_k - y^L}{\alpha_k} = \nabla f^L(\bar{x})^T d, \quad (29)
\]
\[
z^U = \lim_{k \to \infty} \frac{y^U_k - y^U}{\alpha_k} = \nabla f^U(\bar{x})^T d \quad (30)
\]
are convergence vectors for \( f^L(\Omega) \) and \( f^U(\Omega) \), respectively. By the constraint qualification, it follows that \( d \in R^n \) is a convergence for the set \( \Omega \) if and only if \( d \) is a solution to the following system
\[
\nabla g_{j(\bar{x})} (\bar{x})^T d \leq 0, \quad (31)
\]
\[
\nabla h(\bar{x})^T d = 0. \quad (32)
\]
Since \( \bar{x} \in \Omega \) is a weak \( LU \)-Pareto solution of (IVP), by Theorem 11, we have that there is no a strictly negative convergence vector for the set \( f(\Omega) = (f^L(\Omega), f^U(\Omega)) \) at \( \bar{y} \). Therefore, the system
\[
\nabla f^L(\bar{x})^T d < 0, \quad (33)
\]
\[
\nabla f^U(\bar{x})^T d < 0,
\]
\[
\nabla g_{j(\bar{x})} (\bar{x})^T d \leq 0, \quad (34)
\]
\[
\nabla h(\bar{x})^T d = 0
\]
has no a solution \( d \in R^n \). Therefore, by Motzkin’s theorem of the alternative (Theorem 13), we conclude that there exist \( \bar{X}^L \in R^p, \bar{X}^U \in R^p, (\bar{X}^L, \bar{X}^U) \geq 0, \bar{p} \in R^{J(\bar{x})}, \bar{n}_j \geq 0, j \in J(\bar{x}), \) and \( \bar{z} \in R^q \) such that
\[
\sum_{i=1}^{p} \bar{X}^L_i \nabla f^L_i(\bar{x}) + \sum_{i=1}^{p} \bar{X}^U_i \nabla f^U_i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{n}_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \bar{z}_k \nabla h_k(\bar{x}) = 0. \quad (35)
\]
If we set that \( \bar{n}_j = 0 \) for all \( j \in J \setminus J(\bar{x}) \), then (35) gives (20). Further, note that also (21) is satisfied. Indeed, if \( g_j(\bar{x}) < 0 \), then \( j \in J \setminus J(\bar{x}) \) and \( \bar{n}_j = 0 \). Hence, the proof of this theorem is completed. \( \square \)

Now we show that in the case when the constraint qualification (CQ) is not satisfied then the Karush-Kuhn-Tucker necessary optimality conditions established in Theorem 14 cannot be fulfilled.
Example 15. Consider the following interval-valued vector optimization problem:  
\[
f(x_1, x_2) = \left(\left[\frac{1}{2}, 1\right], (-x_1 + x_2^2) \right), \Omega \rightarrow \min
\]
\[
g_1(x_1, x_2) = x_1^3 - x_2 \leq 0, \quad \text{(IVP1)}
g_2(x_1, x_2) = x_1^3 + x_2 \leq 0.
\]
Note that \( \Omega = \{(x_1, x_2) \in R^2 : x_1^3 - x_2 \leq 0 \land x_1^3 + x_2 \leq 0\} \). Let \( \overline{\pi} = (0, 0) \) be a given feasible solution of (IVP1). It follows by Definition 2 that \( \overline{\pi} \) is a LU-Pareto solution of (IVP1). However, the constraint qualification (CQ) is not satisfied at \( \overline{\pi} \). Indeed, by Definition 12, we have \( C(\Omega, \pi) = \{(d_1, d_2) \in R^2 : d_1 \leq 0 \land d_2 = 0\} \). Further, \( \{(d_1, d_2) \in R^2 : \nabla g_1(\pi) d \leq 0 \land \nabla g_2(\pi) d \leq 0\} = \{(d_1, d_2) \in R^2 : d_2 = 0\} \). Hence, \( C(\Omega, \pi) \subset \{(d_1, d_2) \in R^2 : \nabla g_1(\pi) d \leq 0 \land \nabla g_2(\pi) d \leq 0\} \), which means that, in fact, the constraint qualification (CQ) is not satisfied at \( \overline{\pi} = (0, 0) \). Note that the Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are violated in such a case. Indeed, by the the Karush-Kuhn-Tucker necessary optimality condition (20), it follows that the following equation \( X^L - \overline{X}^L = 0 \) should be fulfilled, which implies that \( \overline{X}^L = \left(\overline{X}^L_1, \overline{X}^L_2\right) = (0, 0) \) and \( X^U = \left(X^U_1, X^U_2\right) = (0, 0) \). This is a contradiction to the Karush-Kuhn-Tucker necessary optimality condition (22).

3. The interval-valued \( F \)-objective function method for an interval-valued vector optimization problem. In this section, we introduce a new approximation approach, the so-called modified \( F \)-objective function method, for a characterization of (weak) LU-Pareto optimality for a differentiable nonlinear vector optimization problem with multiple objective function.

Let \( \overline{\pi} \) be a given feasible solution of the interval-valued vector optimization problem (IVP) and \( F : \Omega \times \Omega \times R^n \rightarrow R \) be a given sublinear functional. Then, for the considered differentiable interval-valued vector optimization problem (IVP), we define the following approximated vector optimization problem (IVP) with multiple interval-valued modified objective function as follows:

\[
\left(f_1^l(\pi) + F(x, \pi; \nabla f_1^l(\pi)), \ldots, f_1^u(\pi) + F(x, \pi; \nabla f_1^u(\pi))\right), \ldots,
\]
\[
\left(f_p^l(\pi) + F(x, \pi; \nabla f_p^l(\pi)), \ldots, f_p^u(\pi) + F(x, \pi; \nabla f_p^u(\pi))\right) \rightarrow \min
\]
\[
g(x) = (g_1(x), \ldots, g_m(x)) \leq 0, \quad \text{(IVP)}
h(x) = (h_1(x), \ldots, h_q(x)) = 0,
\]
\[x \in X,\]

We call (IVP) the vector optimization problem with the modified multiple interval-valued objective function or the vector optimization problem with the multiple interval-valued \( F \)-objective function.

Theorem 16. Let \( \overline{\pi} \in \Omega \) be a weak LU-Pareto solution (a LU-Pareto solution) of the considered original vector optimization problem (IVP) with the multiple interval-valued objective function. Further, assume that the multiple interval-valued objective function \( f \) is \( F \)-convex at \( \overline{\pi} \) on \( \Omega \). If \( F(\pi, \pi; \cdot) = 0 \), then \( \overline{\pi} \) is also a weak LU-Pareto solution (a LU-Pareto solution) of its associated vector optimization problem (IVP) with the multiple interval-valued \( F \)-objective function.

Proof. By assumption, \( \overline{\pi} \in \Omega \) is a weak LU-Pareto solution of the considered original vector optimization problem (IVP) with the multiple interval-valued objective function.
function. Hence, by Definition 8, there is no other \( x \in \Omega \) such that the system of inequalities

\[
\begin{cases}
  f_i^L(x) < f_i^L(\xi), \\
  f_i^U(x) \leq f_i^U(\xi),
\end{cases}
\]  

(35)

or

\[
\begin{cases}
  f_i^L(x) \leq f_i^L(\xi), \\
  f_i^U(x) < f_i^U(\xi),
\end{cases}
\]

or

\[
\begin{cases}
  f_i^L(x) < f_i^L(\xi), \\
  f_i^U(x) < f_i^U(\xi)
\end{cases}
\]

is fulfilled for each \( i \in \{1, \ldots, p\} \). Using \( F \)-convexity of the multiple interval-valued objective function \( f \), by Definition 6, the inequalities

\[
f^L(x) - f^L(\xi) \geq F(x, \xi; \nabla f^L(\xi)),
\]

(36)

\[
f^U(x) - f^U(\xi) \geq F(x, \xi; \nabla f^U(\xi))
\]

(37)

hold for all \( x \in \Omega \). Combining (35), (36) and (37), we get that there is no other \( x \in \Omega \) such that the inequalities

\[
\begin{cases}
  F(x, \xi; \nabla f^L_i(\xi)) < 0, \\
  F(x, \xi; \nabla f^U_i(\xi)) \leq 0,
\end{cases}
\]

(38)

or

\[
\begin{cases}
  F(x, \xi; \nabla f^L_i(\xi)) \leq 0, \\
  F(x, \xi; \nabla f^U_i(\xi)) < 0,
\end{cases}
\]

or

\[
\begin{cases}
  F(x, \xi; \nabla f^L_i(\xi)) < 0, \\
  F(x, \xi; \nabla f^U_i(\xi)) < 0
\end{cases}
\]

are satisfied for each \( i = 1, \ldots, p \). By assumption, we have that \( F(\xi, \xi; \cdot) = 0 \). Using this assumption in (38), we get, respectively, that there is no other \( x \in \Omega \) such that the inequalities

\[
\begin{cases}
  F(x, \xi; \nabla f^L_i(\xi)) < F(\xi, \xi; \nabla f^L_i(\xi)), \\
  F(x, \xi; \nabla f^U_i(\xi)) \leq F(\xi, \xi; \nabla f^U_i(\xi))
\end{cases}
\]

(39)

or

\[
\begin{cases}
  F(x, \xi; \nabla f^L_i(\xi)) \leq F(\xi, \xi; \nabla f^L_i(\xi)), \\
  F(x, \xi; \nabla f^U_i(\xi)) < F(\xi, \xi; \nabla f^U_i(\xi))
\end{cases}
\]

or

\[
\begin{cases}
  F(x, \xi; \nabla f^L_i(\xi)) < F(\xi, \xi; \nabla f^L_i(\xi)), \\
  F(x, \xi; \nabla f^U_i(\xi)) < F(\xi, \xi; \nabla f^U_i(\xi))
\end{cases}
\]

are satisfied for each \( i = 1, \ldots, p \). By means of contradiction, suppose that \( \xi \) is not a weak \( LU \)-Pareto solution of the vector optimization problem (IVP \( F(\xi) \)) with the multiple interval-valued \( F \)-objective function. Hence, by Definition 8, there exists other feasible solution \( \bar{x} \) such that the inequalities

\[
\begin{cases}
  f^L_i(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f^L_i(\bar{x})) < f^L_i(\xi) + F(\xi, \xi; \nabla f^L_i(\xi)), \\
  f^U_i(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f^U_i(\bar{x})) > f^U_i(\xi) + F(\xi, \xi; \nabla f^U_i(\xi))
\end{cases}
\]

or

\[
\begin{cases}
  f^L_i(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f^L_i(\bar{x})) \geq f^L_i(\xi) + F(\xi, \xi; \nabla f^L_i(\xi)), \\
  f^U_i(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f^U_i(\bar{x})) \leq f^U_i(\xi) + F(\xi, \xi; \nabla f^U_i(\xi))
\end{cases}
\]

are not fulfilled.
or \[
\begin{aligned}
f^L_i(x) + F(x, \bar{x}; \nabla f^L_i(x)) &< f^L_i(x) + F(x, \bar{x}; \nabla f^L_i(x)), \\
\end{aligned}
\]
are satisfied for each \(i = 1, ..., p\). This means that there exists \(x \in \Omega\) such that the system of inequalities
\[
\begin{aligned}
F(x, \bar{x}; \nabla f^L_i(x)) &< F(x, \bar{x}; \nabla f^L_i(x)), \\
F(x, \bar{x}; \nabla f^L_i(x)) &\leq F(x, \bar{x}; \nabla f^U_i(x)), \\
\end{aligned}
\]
is satisfied for each \(i = 1, ..., p\), which is a contradiction to (39). This completes the proof of this theorem. \(\Box\)

**Theorem 17.** Let \(\bar{x} \in \Omega\) be a \(LU\)-Pareto solution of the original vector optimization problem (IVP) with multiple interval-valued objective function and the constraint qualification (CQ) be satisfied at \(\bar{x}\). Further, assume that the multiple interval-valued objective function \(f\) is \(F\)-convex at \(\bar{x}\) on \(\Omega\). If \(F(\bar{x}, \bar{x}; \cdot) = 0\), then \(\bar{x}\) is also a \(LU\)-Pareto solution of its associated vector optimization problem (IVP) with the multiple interval-valued \(F\)-objective function.

**Proof.** By assumption, \(\bar{x} \in \Omega\) is a \(LU\)-Pareto solution of the considered original vector optimization problem (IVP) with multiple interval-valued objective function. Hence, by Definition 9, there is no other \(x \in \Omega\) such that
\[
f(x) <_{LU} f(\bar{x}).
\]
Hence, by the definition of the relation \(<_{LU}\), we have
\[
\begin{aligned}
f^L(x) &< f^L(\bar{x}), \\
f^U(x) &\leq f^U(\bar{x}), \\
\end{aligned}
\]
or
\[
\begin{aligned}
f^L(x) &\leq f^L(\bar{x}), \\
f^U(x) &< f^U(\bar{x}), \\
\end{aligned}
\]
or
\[
\begin{aligned}
f^L(x) &< f^L(\bar{x}), \\
f^U(x) &< f^U(\bar{x}), \\
\end{aligned}
\]
By (40), we conclude that, there is no other \(x \in \Omega\) such that
\[
\begin{aligned}
f^L_i(x) &< f^L_i(\bar{x}), \quad i \in I, \\
f^U_i(x) &\leq f^U_i(\bar{x}), \quad i \in I, \\
\end{aligned}
\]
or
\[
\begin{aligned}
f^L_i(x) &\leq f^L_i(\bar{x}), \quad i \in I, \\
f^U_i(x) &< f^U_i(\bar{x}), \quad i \in I, \\
\end{aligned}
\]
or
\[
\begin{aligned}
f^L_i(x) &< f^L_i(\bar{x}), \quad i \in I, \\
f^U_i(x) &< f^U_i(\bar{x}), \quad i \in I. \\
\end{aligned}
\]
Using \(F\)-convexity of the multiple interval-valued objective function \(f\), by Definition 6, the inequalities
\[
f^L(x) - f^L(\bar{x}) \geq F(x, \bar{x}; \nabla f^L(x)),
\]
(42)
Theorem 18. Let $\bar{\tau} \in \Omega$ be a weak $LU$-Pareto solution of its associated vector optimization problem (IVP$_F(\bar{\tau})$) with the multiple interval-valued $F$-objective function. Further, assume that each objective function $f^L_i$, $i \in I$, and each objective function $f^U_i$, $i \in I$, are $F$-convex at $\bar{\tau}$ on $\Omega$. If $F(\bar{\tau}; \cdot) = 0$, then $\bar{\tau}$ is also a weak $LU$-Pareto solution of the considered original vector optimization problem (IVP) with multiple interval-valued objective function.

Now, we prove the converse results to those ones in Theorems 16 and 17.

(43) hold for all $x \in \Omega$. Combining (41), (42) and (43), we get that there is no other $x \in \Omega$ such that

$$f^U(x) - f^U(\bar{\tau}) \geq F(x, \bar{\tau}; \nabla f^U(\bar{\tau}))$$

or

$$f^L(x) + F(\bar{\tau}; \nabla f^L(\bar{\tau})) < f^L(\bar{\tau}) + F(\bar{\tau}; \nabla f^L(\bar{\tau}))$$

or

$$f^U(x) - f^U(\bar{\tau}) \geq F(x, \bar{\tau}; \nabla f^U(\bar{\tau}))$$

or

$$f^L(x) + F(\bar{\tau}; \nabla f^L(\bar{\tau})) < f^L(\bar{\tau}) + F(\bar{\tau}; \nabla f^L(\bar{\tau}))$$

or

$$f^U(x) - f^U(\bar{\tau}) \geq F(x, \bar{\tau}; \nabla f^U(\bar{\tau}))$$

By means of contradiction, suppose that $\bar{\tau}$ is not a $LU$-Pareto solution of the vector optimization problem (IVP$_F(\bar{\tau})$) with the multiple interval-valued $F$-objective function. Hence, by Definition 9, there exists other feasible solution $\bar{x}$ such that

$$f^L_i(\bar{x}) + F(\bar{x}; \nabla f^L_i(\bar{x})) < f^L_i(\bar{\tau}) + F(\bar{\tau}; \nabla f^L_i(\bar{\tau}))$$

or

$$f^U_i(\bar{x}) + F(\bar{x}; \nabla f^U_i(\bar{x})) \leq f^U_i(\bar{\tau}) + F(\bar{\tau}; \nabla f^U_i(\bar{\tau}))$$

or

$$f^L_i(\bar{x}) + F(\bar{x}; \nabla f^L_i(\bar{x})) < f^L_i(\bar{\tau}) + F(\bar{\tau}; \nabla f^L_i(\bar{\tau}))$$

or

$$f^U_i(\bar{x}) + F(\bar{x}; \nabla f^U_i(\bar{x})) < f^U_i(\bar{\tau}) + F(\bar{\tau}; \nabla f^U_i(\bar{\tau}))$$

Thus, this means that there exists $\bar{x} \in \Omega$ such that the system of inequalities

$$(F(\bar{x}; \nabla f^L_i(\bar{x})) < 0 \land F(\bar{x}; \nabla f^L_i(\bar{x})) \leq 0, i \in I)$$

or

$$(F(\bar{x}; \nabla f^U_i(\bar{x})) \leq 0 \land F(\bar{x}; \nabla f^U_i(\bar{x})) < 0, i \in I)$$

or

$$(F(\bar{x}; \nabla f^L_i(\bar{x})) < 0 \land F(\bar{x}; \nabla f^L_i(\bar{x})) < 0, i \in I)$$

is satisfied, which contradicts (44). This completes the proof of this theorem. \[Q.E.D.\]
Proof. By means of contradiction, suppose that $\bar{x}$ is not a weak $LU$-Pareto solution of the considered interval-valued vector optimization problem (IVP). Hence, by Definition 8, there exists other feasible solution $\tilde{x}$ such that the inequalities

\[
(f_i^L(\tilde{x}) < f_i^L(\bar{x}) \land f_i^U(\tilde{x}) \leq f_i^U(\bar{x}))
\]

or

\[
(f_i^L(\tilde{x}) \leq f_i^L(\bar{x}) \land f_i^U(\tilde{x}) < f_i^U(\bar{x}))
\]

or

\[
(f_i^L(\tilde{x}) < f_i^L(\bar{x}) \land f_i^U(\tilde{x}) < f_i^U(\bar{x})).
\]

are fulfilled for each $i = 1, \ldots, p$. By assumption, each objective function $f_i^L$, $f_i^U$, $i \in I$, and each objective function $f_i^L$, $i \in I$, are $F$-convex functions at $x$ on $\Omega$. Thus, by Definition 5, it follows that the following inequalities

\[
f_i^L(x) - f_i^L(\bar{x}) \geq F(x, \bar{x}, \nabla f_i^L(\bar{x})), \quad i \in I,
\]

\[
f_i^U(x) - f_i^U(\bar{x}) \geq F(x, \bar{x}, \nabla f_i^U(\bar{x})), \quad i \in I.
\]

hold for all $x \in \Omega$. Therefore, they are also satisfied for $x = \tilde{x} \in \Omega$. Thus, the above inequalities yield, respectively,

\[
f_i^L(\tilde{x}) - f_i^L(\bar{x}) \geq F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})), \quad i \in I,
\]

\[
f_i^U(\tilde{x}) - f_i^U(\bar{x}) \geq F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})), \quad i \in I.
\]

Combining (45), (46) and (47), we get, respectively, that the inequalities

\[
\begin{align*}
F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})) &< 0, \\
F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})) &\leq 0,
\end{align*}
\]

or

\[
\begin{align*}
F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})) &\leq 0, \\
F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})) &< 0,
\end{align*}
\]

or

\[
\begin{align*}
F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})) &\leq 0, \\
F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})) &< 0.
\end{align*}
\]

are fulfilled for each $i = 1, \ldots, p$. By assumption, we have that $F(\bar{x}, \bar{x}, \cdot) = 0$. Hence, (48) implies that the system of inequalities

\[
\begin{align*}
\left\{ \begin{array}{ll}
f_i^L(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})) &< f_i^L(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})), \\
f_i^U(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})) &\leq f_i^U(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})),
\end{array} \right.
\]

or

\[
\begin{align*}
\left\{ \begin{array}{ll}
f_i^L(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})) &\leq f_i^L(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})), \\
f_i^U(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})) &< f_i^U(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})),
\end{array} \right.
\]

or

\[
\begin{align*}
\left\{ \begin{array}{ll}
f_i^L(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})) &< f_i^L(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^L(\bar{x})), \\
f_i^U(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})) &< f_i^U(\bar{x}) + F(\tilde{x}, \bar{x}, \nabla f_i^U(\bar{x})).
\end{array} \right.
\]

holds for each $i = 1, \ldots, p$. This is, by Definition 8, a contradiction to the assumption that $\bar{x} \in \Omega$ is a weak $LU$-Pareto solution of its associated vector optimization problem (IVP$_F(\bar{x})$) with the multiple interval-valued $F$-objective function. Hence, the proof of this theorem is complete. \qed

Theorem 19. Let $\bar{x} \in \Omega$ be a $LU$-Pareto solution of its associated vector optimization problem (IVP$_F(\bar{x})$) with the multiple interval-valued $F$-objective function. Further, assume that each objective function $f_i^L$ and each objective function $f_i^U$, $i \in I$, are $F$-convex functions at $\bar{x}$ on $\Omega$. If $F(\bar{x}, \bar{x}, \cdot) = 0$, then $\bar{x}$ is also a $LU$-Pareto solution of the original interval-valued vector optimization problem (IVP).
Proof. By means of contradiction, suppose that $\bar{x}$ is not a $LU$-Pareto solution of the considered interval-valued vector optimization problem (IVP). Hence, by Definition 9, there exists other feasible solution $\tilde{x}$ such that

$$\begin{cases}
    f_i^L(\tilde{x}) < f_i^L(\bar{x}), & i \in I, \\
    f_i^U(\tilde{x}) \leq f_i^U(\bar{x}), & i \in I,
\end{cases}$$

or

$$\begin{cases}
    f_i^L(\tilde{x}) \leq f_i^L(\bar{x}), & i \in I, \\
    f_i^U(\tilde{x}) < f_i^U(\bar{x}), & i \in I,
\end{cases}$$

or

$$\begin{cases}
    f_i^L(\tilde{x}) < f_i^L(\bar{x}), & i \in I, \\
    f_i^U(\tilde{x}) < f_i^U(\bar{x}), & i \in I.
\end{cases}$$

By assumption, each objective function $f_i^L$ and each objective function $f_i^U$ $i \in I$, are $F$-convex functions at $\bar{x}$ on $\Omega$. Then, by Definition 5, it follows that the following inequalities

$$f_i^L(x) - f_i^L(\bar{x}) \geq F(x, \bar{x}; \nabla f_i^L(\bar{x})), \quad i \in I,$$

$$f_i^U(x) - f_i^U(\bar{x}) \geq F(x, \bar{x}; \nabla f_i^U(\bar{x})), \quad i \in I$$

hold for all $x \in \Omega$. Therefore, they are also satisfied for $x = \tilde{x} \in \Omega$. Thus, the above inequalities yield, respectively,

$$f_i^L(\tilde{x}) - f_i^L(x) \geq F(\tilde{x}, x; \nabla f_i^L(x)), \quad i \in I,$$

$$f_i^U(\tilde{x}) - f_i^U(x) \geq F(\tilde{x}, x; \nabla f_i^U(x)), \quad i \in I.$$ 

Combining (49), (50) and (51), we get, respectively,

$$\begin{cases}
    F(\tilde{x}, \bar{x}; \nabla f_i^L(\bar{x})) < 0, & i \in I, \\
    F(\tilde{x}, \bar{x}; \nabla f_i^U(\bar{x})) \leq 0, & i \in I,
\end{cases}$$

or

$$\begin{cases}
    F(\tilde{x}, \bar{x}; \nabla f_i^L(\bar{x})) \leq 0, & i \in I, \\
    F(\tilde{x}, \bar{x}; \nabla f_i^U(\bar{x})) < 0, & i \in I,
\end{cases}$$

or

$$\begin{cases}
    F(\tilde{x}, \bar{x}; \nabla f_i^L(\bar{x})) < 0, & i \in I, \\
    F(\tilde{x}, \bar{x}; \nabla f_i^U(\bar{x})) < 0, & i \in I.
\end{cases}$$

By assumption, we have that $F(\bar{x}, \bar{x}; \cdot) = 0$. Hence, (52) implies that the system of inequalities

$$\begin{cases}
    f_i^L(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^L(\bar{x})) < f_i^L(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^L(\bar{x})), & i \in I, \\
    f_i^U(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^U(\bar{x})) \leq f_i^U(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^U(\bar{x})), & i \in I,
\end{cases}$$

or

$$\begin{cases}
    f_i^L(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^L(\bar{x})) \leq f_i^L(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^L(\bar{x})), & i \in I, \\
    f_i^U(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^U(\bar{x})) < f_i^U(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^U(\bar{x})), & i \in I,
\end{cases}$$

or

$$\begin{cases}
    f_i^L(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^L(\bar{x})) < f_i^L(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^L(\bar{x})), & i \in I, \\
    f_i^U(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^U(\bar{x})) < f_i^U(\bar{x}) + F(\bar{x}, \bar{x}; \nabla f_i^U(\bar{x})), & i \in I,
\end{cases}$$

holds. This is, by Definition 9, a contradiction to the assumption that $\bar{x} \in \Omega$ is a $LU$-Pareto solution of its associated vector optimization problem (IVP$_F(\bar{x})$) with the multiple interval-valued $F$-objective function. Hence, the proof of this theorem is complete. \qed
By Theorems 16 and 18 (Theorems 17 and 19), it follows the equivalence between the original interval-valued multiobjective optimization problem (IVP) and its associated vector optimization problem (IVP$_F(\overline{x})$) with the multiple interval-valued $F$-objective function.

**Theorem 20.** Let all hypotheses of Theorems 16 and 18 (Theorems 17 and 19) be fulfilled. Then $\overline{x} \in \Omega$ is a weak LU-Pareto solution (a LU-Pareto solution) of the original vector optimization problem (IVP) with the multiple interval-valued objective function if and only if $\overline{x}$ is a weak LU-Pareto solution (a LU-Pareto) of its associated vector optimization problem (IVP$_F(\overline{x})$) with the multiple interval-valued $F$-objective function.

**Remark 21.** Note that if we prove sufficient optimality conditions by using generalized convexity (in such a case, $F$-convexity), we have to impose on all functions constituting the considered vector optimization problem (IVP) with the multiple interval-valued objective function the appropriate $F$-convexity assumptions. Whereas if use for this purpose the $F$-objective function method introduced in this paper, we should assume that the multiple interval-valued function is $F$-convex only. Thus, we don’t need to assume appropriate $F$-convexity hypotheses of constraint functions. This is a useful property of the $F$-objective function method.

In order to illustrate the optimality results established in the paper, we consider an example of a nonconvex vector optimization problem with the multiple interval-valued objective function in which the involved functions are differentiable $F$-convex.

**Example 22.** Consider the following nonconvex differentiable vector optimization problem with the multiple interval-valued objective function:

$$f(x) = \left( \left[ -\frac{1}{8}, -\frac{1}{8} \right] ( - \arctan^2 x_1 + x_1^2 + x_2^2) + \left[ \frac{1}{2}, \frac{1}{2} \right] (x_1 + x_2) + [0, 1] , \right.$$
$$\left. [1, 1] (x_1^2 + \arctan^2 x_2 + x_2^2) + [1, 1] (x_1 + x_2) + [0, 1] \right) \rightarrow \min$$

$$g_1(x) = \arctan^2 x_1 - \arctan x_1 \leq 0,$$

$$g_2(x) = \arctan^2 x_2 - \arctan x_2 \leq 0.$$  

(IVP1)

We now re-write the considered differentiable vector optimization problem (IVP1) with interval-valued objective functions in the following form:

$$f(x) = \left( \left[ f_1^L(x), f_1^U(x) \right] , \left[ f_2^L(x), f_2^U(x) \right] \right) =$$
$$\left( \left[ \frac{1}{8} \arctan^2 x_1 - \frac{1}{8} x_1^2 - \frac{1}{8} x_2^2 + \frac{1}{2} x_1 + \frac{1}{2} x_2 , \frac{1}{8} \arctan^2 x_1 - \frac{1}{8} x_1^2 - \frac{1}{8} x_2^2 + \frac{1}{2} x_1 + \frac{1}{2} x_2 + 1 \right] , \right.$$
$$\left. \left[ x_1^2 + \arctan^2 x_2 + x_2^2 + x_1 + x_2 , x_1^2 \right] \right) \rightarrow \min$$

(IVP1)

$$g_1(x) = \arctan^2 x_1 - \arctan x_1 \leq 0,$$

$$g_2(x) = \arctan^2 x_2 - \arctan x_2 \leq 0.$$

Note that $\Omega = \{(x_1, x_2) \in R^2 : \arctan^2 x_1 - \arctan x_1 \leq 0 \land \arctan^2 x_2 - \arctan x_2 \leq 0\}$ and $\overline{x} = (0, 0)$ is a feasible point in problem (IVP1). Further, it can be shown, by Definition 9, that $\overline{x} = (0, 0)$ is a LU-Pareto solution in the problem (IVP1). Thus, the Karush-Kuhn-Tucker necessary optimality conditions (20)-(22) are satisfied at $\overline{x} = (0, 0)$. Let us define a sublinear functional $F$ as follows

$$F(x , \overline{x}, \vartheta) = \left( -\frac{1}{4} x_1^2 + x_1 + \frac{1}{4} \vartheta_1 - \overline{x}_1 \right) \vartheta_1 + \left( -\frac{1}{4} x_2^2 + x_2 + \frac{1}{4} \vartheta_2 - \overline{x}_2 \right) \vartheta_2.$$
Note that $F(\bar{x}, \bar{x}; \cdot) = 0$. Further, it can be shown by Definition 6 that the interval-valued objective functions $f$ is $F$-convex at $\bar{x} = (0, 0)$ on $\Omega$. We use the introduced modified objective function method for solving the considered interval-valued vector optimization problem (IVP1). Then, for the considered nonconvex differentiable multiobjective programming problem (IVP1) with the multiple interval-valued objective function, we construct its associated vector optimization problem (IVP1$F(\bar{x})$) with the multiple interval-valued $F$-objective function as follows

$$f(x) = ([f^U_1(x), f^L_1(x)] + [f^U_2(x), f^L_2(x)]) =$$

$$([-\frac{1}{8}x_1^2 - \frac{1}{8}x_2^2 + \frac{1}{2}x_1 + \frac{1}{2}x_2, -\frac{1}{8}x_1^2 - \frac{1}{8}x_2^2 + \frac{1}{2}x_1 + \frac{1}{2}x_2 + 1],$$

$$[-\frac{1}{4}x_1^2 + -\frac{1}{4}x_2^2 + x_1 + x_2, -\frac{1}{4}x_1^2 + -\frac{1}{4}x_2^2 + x_1 + x_2 + 1]) \rightarrow \min$$

$$g_1(x) = \arctan^2 x_1 - \arctan x_1 \leq 0,$$

$$(\text{IVP}_F(\bar{x}))$$

$$g_2(x) = \arctan^2 x_2 - \arctan x_2 \leq 0.$$  

Since all hypotheses of Theorem 19 are satisfied, therefore, $\bar{x} = (0, 0)$ is a LU-Pareto solution of the associated vector optimization problem (IVP1$F(\bar{x})$) with the multiple interval-valued $F$-objective function.

Furthermore, we consider the converse case. Note that $\bar{x} = (0, 0)$ is a LU-Pareto solution of the associated vector optimization problem (IVP1$F(\bar{x})$) with the multiple interval-valued $F$-objective function. It can be shown that the interval-valued objective functions $f_1, f_2$ are $F$-convex at $\bar{x} = (0, 0)$ on $\Omega$. Indeed, since all hypotheses of Theorem 19 are also fulfilled, this means that $\bar{x} = (0, 0)$ is also a LU-Pareto solution of the considered nonconvex vector optimization problem (IVP1) with the multiple interval-valued objective function. Hence, we have shown the equivalence between problems (IVP) and (IVP1$F(\bar{x})$) in the sense analyzed in the paper (see Theorem 20).

One of the useful properties of the introduced approach is that, in some cases, the interval-valued optimization problem (IVP1$F(\bar{x})$) constructed in this method is linear. We give an example of such an interval-valued optimization problem and, for solving it, we apply one of the existing in the literature methods for solving linear interval-valued optimization problems. Namely, we use the method introduced by Chanas and Kuchta [9] which is based on preference relations between intervals.

**Example 23.** Consider the following nonlinear nonconvex interval-valued optimization problem:

$$f(x) = [\ln^2 (1 - x_2) + x_1^2 + 2x_1 - 2x_2, \ln^2 (1 - x_1) + x_2^2 + 3x_1 + x_2] \rightarrow \min$$

$$g_1(x) = -x_1 \leq 0,$$

$$(\text{IVP2})$$

$$h_1(x) = x_1 - x_2 = 0,$$

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1 \land x_2 < 1\}.$$  

Note that $\Omega = \{(x_1, x_2) \in X : -x_1 \leq 0 \land x_1 - x_2 = 0\}$. First, we use the introduced modified objective function method for solving problem (IVP2). Let $\bar{x} = (0, 0)$ be a given feasible solution. Note that the objective functions $f^L, f^U$ are strictly $F$-convex at $\bar{x} = (0, 0)$ on $\Omega$ and both constraint functions are $F$-convex at $\bar{x} = (0, 0)$ on $\Omega$ with respect the sublinear (with respect to the third component) functional $F$ defined by

$$F(x, \bar{x}; \vartheta) = (x_1 - \bar{x}_1) \vartheta_1 + (x_2 - \bar{x}_1) \vartheta_1.$$
For the considered nonconvex interval-valued optimization problem (IVP2), we construct at $\mathbf{x} = (0, 0)$ its associated interval-valued optimization problem (IVP$_2^F(\mathbf{x})$) with the modified interval-valued objective function as follows:

$$[2, 3]x_1 + [-2, 1]x_2 \rightarrow \min$$

$$g_1(x) = -x_1 \leq 0, \quad h_1(x) = x_1 - x_2 = 0,$$

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1 \land x_2 < 1\}.$$ (ILP$_2^F(x)$)

Now, we use the approach given by Chanas and Kuchta [9] for solving the linear interval-valued optimization problem (IVP$_2^F(x)$). Namely, Chanas and Kuchta [9] considered both the concepts of cuts $t_0$, $t_1$ of an interval and parametric linear programming. Let $t_0$ and $t_1$ be any fixed numbers such that $0 \leq t_0 < t_1 \leq 1$. Then Chanas and Kuchta showed that in order to determine all $LU$-Pareto solutions of the considered linear interval-valued optimization problem (ILP$_2^F(\mathbf{x})$), it is enough to solve the following parametrical linear optimization problem (see Theorem 7 [9])

$$(2 + t)x_1 + (-2 + 3t)x_2 \rightarrow \min$$

$$t \in (t_0, t_1), \quad (ILP^F_2(\mathbf{x})(t))$$

$$x \in \Omega.$$ (ILP$_2^F(\mathbf{x})(t)$)

Note that $\mathbf{x} = (0, 0)$ is optimal in the parametrical linear optimization problem (ILP$_2^F(\mathbf{x})(t)$) for any cuts $t_0$, $t_1$, where $0 \leq t_0 < t_1 \leq 1$. Hence, by Theorem 7 [9], it is also a $LU$-optimal solution of (ILP$_2^F(\mathbf{x})$). Since all hypotheses of Theorem 19 are fulfilled, $\mathbf{x} = (0, 0)$ is also a $LU$-optimal solution of the considered interval-valued optimization problem (IVP2). Hence, we have solved a nonlinear nonconvex interval-valued optimization problem (IVP2) by using the introduced modified objective function method and one of the existing methods for finding $LU$-optimal solutions of linear interval-valued optimization problems.

4. Conclusion. In the paper, we have considered a differentiable vector optimization problem with the multiple interval-valued objective function and with both inequality and equality constraints. For such a smooth interval-valued vector optimization problem, the Karush-Kuhn-Tucker necessary optimality conditions have been established by using the concept of a convergence vector for the set of all feasible solutions. Further, a new approximation approach, called the $F$-objective function method, for finding (weakly) $LU$-efficient solutions of the considered differentiable vector optimization problem with the multiple interval-valued objective function has also been introduced in the paper. It has been presented the main idea of the presented approach, that is, the formulation of the associated vector optimization problem (IVP$_F(\mathbf{x})$) with the multiple interval-valued $F$-objective function. Further, it has been established the equivalence between both vector optimization problems with multiple interval-valued objective functions mentioned above under appropriate $F$-convexity hypotheses. This result has been illustrated by suitable examples of differentiable vector optimization problems with multiple interval-valued objective functions which have been solved by using the introduced modified objective function method. Also it has been presented an example of such a nonlinear interval-valued minimization problem for which its associated modified interval-valued optimization problem (IVP$_F(\mathbf{x})$) is linear and for solving of which one of the existing methods for solving linear interval-valued optimization problems has
been used. As it follows even from this example, by using the $F$-objective function method introduced in the paper, it is possible to use in some cases methods for solving linear interval-valued optimization problems in the case when the original interval-valued optimization problem is nonlinear.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of nonconvex vector optimization problems with multiple interval-valued objective functions and/or for other types of such extremum problems. We shall investigate these questions in the subsequent papers.

**Compliance with ethical standards.** This article does not contain any studies with human participants or animals performed by the author.

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