Fusion rules in N=1
superconformal minimal models

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Abstract

The generalization to N=1 superconformal minimal models of the relation between the modular transformation matrix and the fusion rules in rational conformal field theories, the Verlinde theorem, is shown to provide complete information about the fusion rules, including their fermionic parity. The results for the superconformal Tri-critical Ising and Ashkin-Teller models agree with the known rational conformal formulation. The Coulomb gas description of correlation functions in the Ramond sector of N=1 minimal models is also discussed and a previous formulation is completed.
1 Introduction.

One of the most interesting results in 2D conformal field theory is the Verlinde theorem [1][2][3]. It gives the number of conformal blocks of a RCFT on a punctured Riemann surface in terms of elements of the modular matrix. The result arises as a consequence of the well established, but nevertheless still surprising, fact that the modular matrix $S$ implementing the modular transformation $\tau \rightarrow -1/\tau$ on the space of genus one conformal blocks, diagonalizes the fusion rules. The proof of the theorem relies on the technical assumption that both left and right extended chiral algebras consist only of generators with integral conformal weight, the so called rational conformal field theories (RCFT’s), and thus evidently excludes the superconformal case.

A generalized Verlinde formula which describes the fusion rules in all sectors of $\mathcal{N} = 1$ superconformal theories was obtained in [4]: by considering some examples of correlation functions of primary fields and employing certain bases of conformal blocks, it was argued that the Verlinde conjecture extends to the $\mathcal{N} = 1$ superconformal unitary series. In this letter we show that the generalized Verlinde formula contains information about the fermionic parity of the fusion rules.

In Section 2, we complete the formulation of the Coulomb gas description [5] of correlation functions in the $R$ sector performed in [6] and [4], by considering the parity of the $R$ fields. In Section 3, we discuss the generalized Verlinde formula and compute the fusion rules of the Tricritical Ising model (TIM) and the critical Ashkin-Teller model (AT). These models are known to have both RCFT as well as superconformal descriptions, so they can be regarded as a check of our results. Since the superconformal Coulomb gas method remains at the level of a prescription, the correlation functions have to pass several checks such as null state decoupling and correct behavior under degeneration of the torus to the plane as well as in the factorization limit. Consistency with the conformal fusion rules can therefore be considered another successful check on the superconformal blocks and on the extension of the Verlinde formula.
2 Two-Point Conformal Blocks.

Contour integral representations of the conformal blocks can be computed with the Dotsenko-Fateev Coulomb gas technique \(^\text{[5]}\) by introducing Feigin-Fuks screening operators to make correlation functions background charge neutral. Correlation functions of \(N = 1\) superconformal primary fields on the torus were considered in \(^\text{[6]}\)\(^\text{[4]}\) where one contour examples, corresponding to Ramond primary fields, and double contour integrals, necessary when considering NS primary fields, were studied. In this section we rely heavily on notation, results and the discussion contained in these references.

The \(N=1\) superconformal minimal models are characterized by the following discrete series of central charges \(c\) and allowed conformal weights of the primary fields \(\Delta_{r,s}\) \(^\text{[7]}\).

\[
\begin{align*}
  c &= \frac{3}{2} - \frac{12}{p(p+2)} \quad (p = 3, 4, ...) \\
  \Delta_{r,s} &= \frac{[(p+2)r - sp]^2 - 4}{8p(p+2)} + \frac{1}{32}[1 - (-1)^{r-s}] \quad (1)
\end{align*}
\]

where \(1 \leq s \leq r \leq p - 1\) with \(r - s\) even in the NS sector and \(1 \leq s \leq r - 1\) for \(1 \leq r \leq \left\lfloor \frac{p-1}{2} \right\rfloor\) or \(1 \leq s \leq r + 1\) for \(\left\lceil \frac{p+1}{2} \right\rceil \leq r \leq p - 1\) with \(r - s\) odd in the R sector.

In order to compute correlation functions, the fields are represented by vertex operators of the form \(^\text{[8]}\)

\[
N_B^{\alpha}(z) = e^{i\alpha \phi(z)} \quad N_F^{\alpha}(z) = \psi(z)e^{i\alpha \phi(z)} \quad R^{\pm}_{\alpha}(z) = \sigma^{\pm}(z)e^{i\alpha \phi(z)} \quad (2)
\]

where \(\phi\) and \(\psi\) are a free boson and a Majorana fermion, \(N_B^{\alpha}\) and \(N_F^{\alpha}\) are the bosonic and fermionic components of a \(NS\) field, \(\sigma^{\pm}\) are the two spin fields with conformal weight \(\frac{1}{16}\) that are to be identified respectively with the spin field and the disorder field of the Ising model, and \(R^{\pm}_{\alpha}\) are the two \(R\) fields whose conformal weights correspond to the charge \(\alpha\) (only the ground states of conformal weight \(c = \frac{1}{24}\) are not degenerate). They obey the following OPE

\[
\psi(z)\sigma^{\pm}(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\sigma^{\mp}(w) + ...
\]
\[
\sigma^\pm(z)\sigma^\pm(w) \sim \frac{1}{(z-w)^{\frac{3}{2}}} + (z-w)^{\frac{3}{2}}\psi(w) + ...
\]
\[
\sigma^\pm(z)\sigma^\mp(w) \sim (z-w)^{\frac{3}{2}}\psi(w) + ...
\]

(3)

Let us compute the conformal blocks corresponding to the correlator \(\langle \phi_{1,2}\phi_{1,2} \rangle\) using the Coulomb gas formalism and completing the results in [6] and [4] by taking into account the fermionic parities of the fields, i.e., considering the three possible two-point functions \(\langle \phi_{1,2}^+\phi_{1,2}^+ \rangle, \langle \phi_{1,2}^-\phi_{1,2}^- \rangle\) and \(\langle \phi_{1,2}^+\phi_{1,2}^- \rangle\). The conformal blocks are

\[
G_i^{(\text{sign1, sign2});\nu}(z_1, z_2) = \int_{C_i} dz \left\langle \sigma^{\text{sign1}}(z_1)\sigma^{\text{sign2}}(z_2)\psi(z) \right\rangle_\nu \left\langle e^{-\frac{i}{2}\alpha^+\phi(z_1)}e^{-\frac{i}{2}\alpha^-\phi(z_2)}e^{i\alpha^+\phi(z)} \right\rangle_\nu
\]

(4)

where \(\nu = 1, 2, 3, 4\) label the spin structure and

\[
\alpha^+ = \sqrt{\frac{p+2}{2p}} \quad \alpha^- = -\sqrt{\frac{p}{2(p+2)}}
\]

(5)

The closed contour basis \(\{C_1, C_2\}\) has been specified in [6, 4], and \(\text{sign1, sign2} \in \{+, -\}\).

The correlation function of \(2n\) spin fields and \(2m\) disorder fields of the Ising model on the torus is given by

\[
\langle \sigma^+(w_1, \bar{w}_1)...\sigma^+(w_{2n}, \bar{w}_{2n})\sigma^-(u_1, \bar{u}_1)...\sigma^-(u_{2m}, \bar{u}_{2m}) \rangle_\nu^2 =

= \sum_{\sum_{i=\pm1} \sum_{i'k=\pm1}} \left| \Theta_\nu \left( \frac{\sum_{i} w_i + \sum_{i'} u_k}{2} \right) \right|^2 \prod_k \frac{\Theta_1(w_i - w_j)}{\Theta_1'(0)} \frac{\epsilon_i \epsilon'_j}{2} \prod_{i,k} \frac{\Theta_1(u_i - u_k)}{\Theta_1'(0)} \frac{\epsilon_i \epsilon'_k}{2}
\]

(6)

where \(\Theta_\nu\) are the usual Jacobi \(\Theta\)-functions.
We take the square root on both sides of this equality and keep the holomorphic part only (obtaining in this way the so-called ‘holomorphic square root’). We then specialize to \( n = 2, m = 0; n = 0, m = 2 \) and \( n = m = 1 \) and take the limits \( w_3 \to w_4, u_3 \to u_4 \) and \( w_2 \to u_3 \) respectively. In all cases (because of (3)) the residue of the \( \frac{3}{8} \) order pole is kept. We thus obtain

\[
\langle \sigma^+(z_1)\sigma^+(z_2)\psi(z) \rangle_\nu = \langle \sigma^-(z_1)\sigma^-(z_2)\psi(z) \rangle_\nu \\
= \left[ \frac{\Theta_1(z_1 - z_2)}{\Theta_1'(0)} \right]^{\frac{3}{8}} \left[ \frac{\Theta_1(z_1 - z)}{\Theta_1'(0)} \right]^{-\frac{1}{2}} \times \left[ \frac{\Theta_1(z_2 - z)}{\Theta_1'(0)} \right]^{-\frac{1}{2}} \left[ \frac{\Theta_\nu \left( \frac{1}{2}(z_1 - z_2) \right)}{\Theta_\nu(0)} \right]^{-\frac{1}{2}} \times \frac{\Theta_\nu \left( \frac{1}{2}(z_1 + z_2 - 2z) \right)}{\Theta_\nu(0)} \tag{7}
\]

\[
\langle \sigma^+(z_1)\sigma^-(z_2)\psi(z) \rangle_\nu = \left[ \frac{\Theta_1(z_1 - z_2)}{\Theta_1'(0)} \right]^{-\frac{1}{2}} \left[ \frac{\Theta_1(z_1 - z)}{\Theta_1'(0)} \right]^{\frac{1}{2}} \left[ \frac{\Theta_1(z_2 - z)}{\Theta_1'(0)} \right]^{\frac{1}{2}} \times \left[ \frac{\Theta_1(z_1 - z)\Theta_1(z_2 - z)\Theta_\nu \left( \frac{1}{2}(z_1 - z_2) \right)}{\Theta_1'(0)^2} \right]^{-\frac{1}{2}} \times \frac{\Theta_1(z_1 - z)\Theta_1(z_2 - z)\Theta_\nu \left( \frac{1}{2}(z_1 - z_2) \right)}{\Theta_1'(0)^2} \tag{8}
\]

By calculating the lattice contribution in (4) we get

\[
G_i^{(\text{sign1},\text{sign2})\nu}(r, s) = \iint_{C_i} \exp \left[ \gamma^{\nu}(\lambda - (-1)^{r+1+s}(r+s+1)^{\delta_{1\nu}} + r(\text{sign}1)\delta_{4\nu}) \right] \left[ \frac{\Theta_1(z_1 - z_2)}{\Theta_1'(0)} \right]^{\frac{1}{2}} \left[ \frac{\Theta_1(z_1 - z)\Theta_1(z_2 - z)}{\Theta_1'(0)^2} \right]^{-\alpha^2} \times \left[ \gamma^{\nu}(\lambda) - (-1)^{r+1+s}(r+s+1)^{\delta_{1\nu}} + r(\text{sign}1)\delta_{4\nu} \right] \tag{9}
\]
where

\[
\Gamma^\nu(\lambda) = \frac{\Theta^\nu(0)^{\frac{1}{2}}}{\eta^2} e \frac{i \pi}{6} \left( \frac{\lambda^2 - 1}{12} \right) \delta_{4\nu} \sum_{n \in \mathbb{Z}} (-1)^{np(\delta_{1\nu} + \delta_{4\nu})} \times q^{\frac{(\lambda + \eta N)^2}{8N}} e^{i \pi \frac{1}{6N} \alpha - (2z_1 - z_2)}
\]

(10)

\[
\lambda = r(p + 2) - sp \quad \tilde{\lambda} = -r(p + 2) - sp \quad N = 2p(p + 2)
\]

(11)

Notice that only the fermionic correlator (7) was considered in [4].

Clearly from eq. (7) \( G^\nu(\lambda, \lambda) = G^\nu(\tilde{\lambda}, \tilde{\lambda}) \). Nevertheless, it is possible to see that both \( \langle \sigma^+(z_1)\sigma^+(z_2)\psi(z) \rangle^\nu \) and \( \langle \sigma^+(z_1)\sigma^-(z_2)\psi(z) \rangle^\nu \) have the same monodromy and modular properties as well as degeneration and factorization limits. Indeed, in the degeneration limit, \( q \to 0 \), the four-point functions on the sphere are recovered, namely

\[
\langle V^\alpha_1(0)R_{1,2}^{\text{sign}1}(x_1)R_{1,2}^{\text{sign}2}(x_2)V_{2\alpha_2-\alpha}(\infty) \rangle
\]

(12)

where the conjugate vertices \( V^\alpha_1 \) and \( V_{2\alpha_2-\alpha} \) are either \( N^B_\Delta, N^F_\Delta \) or \( R^\pm_\Delta \) and the contours \( C_1 \) and \( C_2 \) on the torus degenerate to Pochhammer contours on the extended complex plane.

The properties of the conformal blocks under monodromy transformations are not altered. Recall that \( \phi(a) \) and \( \phi(b) \) are the Verlinde operators which implement these transformations as the point \( z_1 \) (or \( z_2 \)) is transported once around either an \( a \) or \( b \) cycle on the torus. These operators are not modified when the correlation functions contain spin fields of different parity.

Another check on the conformal blocks is provided by the factorization limit \( z_1 \to z_2 \), where the intermediate states are precisely those dictated by the fusion rules. The problematic feature appearing for even \( p \) however, remains. In fact, the \( N = 1 \) superconformal partition function can always be obtained by factorizing the modular and monodromy invariant two point correlation function on the identity intermediate state. However in the \( p = 4 \) case or AT model at criticality, the \( C_1 \) contour in the factorization limit reproduces all the terms of the corresponding superconformal partition function [10], except for the contribution \( \text{Tr}_{[R]}(-1)^F \) of the \( \nu = 1 \) sector. This can
be understood since the intermediate state in the \( \nu = 1 \) sector is a fermion, so we do not expect to obtain the partition function but the fermion expectation value. Actually when \( \nu = 1 \) the residue is zero, reflecting the fact that the fermion is a null state.

Therefore taking into account the parities of the \( R \) fields does not modify the checks on the conformal blocks that have been performed previously in [6].

Let us now consider the modular transformation matrix \( S \) on the basis of conformal blocks \( G_{i=1}^{(\text{sign}1,\text{sign}2)}(r, s) \). It is possible to see that \( \langle \sigma^+(z_1)\sigma^+(z_2)\psi(z) \rangle_\nu \) and \( \langle \sigma^+(z_1)\sigma^-(z_2)\psi(z) \rangle_\nu \) have the same \( S \) transformation properties, so the form of the \( S \) matrix does not depend on the parity of the fields. Notice that the \( S_{\nu'=1,2,3,4}^{-1} \) matrices are just those given in [4]. We list them here (up to phase factors) for completeness.

\[
(S_{\nu'=3,4})_{r,s}^{r',s'} = -\frac{4}{\sqrt{p(p+2)}} \sin \pi \left( rr'\alpha_2^2 - \frac{rs'}{2} \right) \sin \pi \left( ss'\alpha_2^2 - \frac{sr'}{2} \right) \tag{13}
\]

\[
(S_{\nu'=2})_{r,s}^{r',s'} = -\frac{4\gamma_{r',s'}}{\sqrt{p(p+2)}} \sin \pi \left( rr'\alpha_2^2 - \frac{sr'}{2} \right) \sin \pi \left( ss'\alpha_2^2 - \frac{rs'}{2} \right) \tag{14}
\]

\[
(S_{\nu'=1})_{r,s}^{r',s'} = -\frac{4}{\sqrt{p(p+2)}} \sin \pi \left( rr'\alpha_2^2 - \frac{sr'}{2} + \frac{r'}{2} \right) \cos \pi \left( ss'\alpha_2^2 - \frac{rs'}{2} - \frac{r'}{2} \right) \tag{15}
\]

with

\[
\gamma_{r',s'} = 1 - \frac{1}{2} \delta_{r',2p} \delta_{r',2} \delta_{s',p+2} \delta_{s',p+2} \tag{16}
\]

Recall that the physical origin of the factor \( \gamma_{r',s'} \) is the double degeneracy of the \( R \) states, apart from the vacuum \( (r', s') = (p^2, p^2) \).

These equations are essentially the modular transformations of superconformal characters [11] except for \( S^{R\rightarrow \bar{R}} \) or \( S_{\nu'=1}^{-1} \) which is not defined except for the \( (r, s) = \left( \frac{p^2}{2}, \frac{p^2}{2} \right) \) state where it is the identity.
3 The Verlinde theorem in N=1 superconformal models.

Even though the proof of the Verlinde theorem \([1],[2],[3]\) requires that left and right extended chiral algebras consist only of generators with integral conformal weight, it was shown in \([4]\) that it is possible to construct a complete Verlinde basis in N=1 superconformal minimal models, namely

\[
G = (E_+ E_- O_+ O_-)
\]

with

\[
E_\pm(r, s) = \frac{1}{2}\left[G_1^{\nu=3}(r, s) \pm e^{-i\pi/12} G_1^{\nu=4}(r, s)\right],
\]

\[
O_\pm(r, s) = \frac{1}{2}\left[G_1^{\nu=2}(r, s) \pm e^{-i\pi/4} G_1^{\nu=1}(r, s)\right]
\]

which is an eigenstate of \(\phi(a)\) and with respect to which \(\phi(b)\) yields the fusion rule coefficients. In this basis the descendants in a \(q\) expansion are always at integer level spacing above the highest weight state (regarding superdescendants as Virasoro primaries). Moreover, the proof of the Verlinde theorem relies only on conformal and duality properties under certain manipulations of conformal blocks in their degeneration and factorization limits. Since we have checked that these limits are indeed consistent with the fusion rules, this suggests that by working in the appropriate basis the proof can always be carried through. Even though we have no general proof, we will show in this section that the generalized Verlinde formula gives the correct fusion coefficients in the TIM and AT models, including the fermionic parity of the fields.

Let us summarize the arguments leading to the generalization of the Verlinde formula. The fusion coefficients \(N^K_{IJ}\) are defined as

\[
\varphi_I \times \varphi_J = N^K_{IJ} \varphi_K
\]

where \(\varphi_I\) is one of the operators \(N^B_\alpha, N^F_\alpha\) or \(R^\pm_\alpha\), and the upper case indices denote both \(r, s\) and spin structure sector \(\nu\) (or, rather, the appropriate combinations of spin structures discussed above). As shown in \([4]\), the \(b\) cycle monodromy operator in the \(t\) channel, \(\phi^t(b)\), yields the expected superconformal fusion rules. Since we are interested in the action of \(\phi_I(b)\) on characters
rather than two-point blocks, we may factorize the $t$ channel blocks on the identity intermediate state. In general one has an equation of the form

$$\phi_t(b)\chi_J = \sum_K N^K_{IJ} \chi_K$$

(20)

where $\chi_I$ denotes a particular character (or combination of characters) and the normalization condition is $N^K_{I0} = \delta^K_I$, where $J = 0$ denotes the identity or NS vacuum character. Furthermore, with respect to the same basis of conformal blocks, one has under $\phi_t(a)$

$$\phi_t(a)\chi_J = \lambda^{(j)}_I \chi_J$$

(21)

One may now proceed as in [1], using the conjugation relation between $a$ and $b$ cycle monodromy, $\phi_t(a) = S^{-1} \phi_t(b) S$, to express the fusion coefficients in terms of the modular matrices and the eigenvalue of $\phi_t(a)$:

$$N^K_{IJ} = \sum_L S^L_J \lambda^{(L)}_I (S^{-1})^K_L$$

(22)

The modular matrix $S^K_I$ acts on the Verlinde basis (17) as

$$G(r,s|\tau) = \sum_{r',s'} (S)_{r,s}^{r',s'} G(r',s'| -1 \tau)$$

(23)

and is given by

$$S = \begin{pmatrix}
S^{-1}_{\nu'=3} & S^{-1}_{\nu'=3} & S^{-1}_{\nu'=2} & S^{-1}_{\nu'=2} \\
S^{-1}_{\nu'=3} & -S^{-1}_{\nu'=2} & S^{-1}_{\nu'=2} & -S^{-1}_{\nu'=2} \\
S^{-1}_{\nu'=4} & -S^{-1}_{\nu'=1} & -S^{-1}_{\nu'=1} & S^{-1}_{\nu'=1} \\
S^{-1}_{\nu'=4} & -S^{-1}_{\nu'=1} & S^{-1}_{\nu'=1} & -S^{-1}_{\nu'=1}
\end{pmatrix}$$

(24)

Using the normalization condition allows the eigenvalue to be expressed as

$$\lambda^{(L)}_I = \frac{S^L_I}{S^L_{I=0}}$$

(25)

whence

$$N^K_{IJ} = \sum_L \frac{S^L_J S^L_{IJ} (S^{-1})^K_L}{S^L_{I=0,L}}$$

(26)
Since \( S^2 = 1 \), multiplying by \( S^2 \) on the right yields the result for the number of couplings between three operators labeled by \( I, J, K \):

\[
N_{IJK} = \sum_L \frac{S_{I,L}S_{J,L}S_{L,K}}{S_{I=0,L}} \quad (27)
\]

Notice that the \( S \) matrix \((24)\) is symmetric and unitary for \( p \) odd. This is in accord with the well known result that for \( p = 3 \) in particular, there are two equivalent representations of the TIM- either as a \( p = 4 \) minimal conformal model or as the \( N = 1 \) superconformal model that we are discussing. In fact, for \( p = 3 \) the combinations of blocks \((17)\) in the factorization limit are the Virasoro characters

\[
\begin{align*}
E_+(1, 1) & \sim \chi_0^{Vir} \\
E_+(1, 3) & \sim \chi_{\frac{1}{10}}^{Vir} \\
O_\pm(2, 3) & \sim \chi_{\text{Vir}}^{\frac{1}{50}}
\end{align*}
\]

\[
E_-(1, 1) \sim \chi_{\frac{3}{2}}^{Vir} \\
E_-(1, 3) \sim \chi_{\frac{7}{16}}^{Vir} \\
O_\pm(2, 1) \sim \chi_{\text{Vir}}^{\frac{7}{16}}
\quad (28)
\]

However for \( p \) even the factor \( \gamma_{r,s} \) in the matrix \( S_{p'=2}^{-1} \), associated with the Ramond vacuum state, and the vanishing of \( S_{p'=1}^{-1} \) for \((r, s) = \left(\frac{p}{2}, \frac{p+2}{2}\right)\), appear to prevent \( S \) from being either symmetric or unitary, unlike in the conformal case.

Let us compute the fusion coefficients in the first two cases of the \( N = 1 \) superconformal minimal series, \( p = 3 \) and 4. Notice that in equation \((27)\) \( I = 0 \) corresponds to \( E_+(1, 1) \) and \( L = E_+(r, s), E_-(r, s), O_+(r, s), O_-(r, s) \). We have to consider the three possibilities \( NS \times NS \sim NS, R \times R \sim NS \) and \( NS \times R \sim R \). In the first case we find, for \( p = 3 \), i.e. the TIM,

\[
\begin{align*}
N_{(1,3,\pm)(1,3,\pm)(1,1,+)} & = N_{(1,3,\pm)(1,3,\pm)(1,3,-)} = 1 \\
N_{(1,3,+)(1,3,-)(1,1,-)} & = N_{(1,3,+)(1,3,-)(1,3,+)} = 1 \\
N_{(1,1,-)(1,1,-)(1,1,+)} & = N_{(1,3,\pm)(1,1,-)(1,3,\mp)} = 1
\end{align*}
\]

\quad (29)

where we have written only the non-trivial nonvanishing coefficients. They correspond respectively to the conformal fusion rules \([12, 13]\)

\[
\begin{align*}
\frac{1}{10} \times \frac{1}{10} & = 0 + \frac{3}{5}, & \frac{3}{5} \times \frac{3}{5} & = 0 + \frac{3}{5}, & \frac{1}{10} \times \frac{3}{5} & = \frac{3}{2} + \frac{1}{10} \\
\frac{1}{10} \times \frac{3}{2} & = \frac{3}{5}, & \frac{3}{5} \times \frac{3}{2} & = \frac{1}{10}, & \frac{3}{2} \times \frac{3}{2} & = 0
\end{align*}
\]

\quad (30)
and contain the information about the fermionic parity of the fields in accord with previous results [14].

Let us now compute the fusions $R \times R \sim NS$. The nonvanishing coefficients are

\[
\begin{align*}
N_{(2,3,\pm)(2,3,\pm)(1,1,\pm)} &= N_{(2,3,\pm)(2,3,\pm)(1,3,-)} = 1 \\
N_{(2,3,\pm)(2,3,\mp)(1,1,\pm)} &= N_{(2,3,\pm)(2,3,\mp)(1,3,+) = 1} \\
N_{(2,1,\pm)(2,3,\pm)(1,3,-)} &= N_{(2,1,\pm)(2,3,\mp)(1,3,+) = 1} \\
N_{(2,1,\pm)(2,1,\pm)(1,1,\pm)} &= N_{(2,1,\pm)(2,1,\pm)(1,1,-)} = 1
\end{align*}
\]

These should be compared to the conformal fusion rules [12, 13]

\[
\begin{align*}
\frac{3}{80} \times \frac{3}{80} &= 0 + \frac{3}{2} + \frac{1}{10} + \frac{3}{5} \\
\frac{7}{16} \times \frac{7}{16} &= 0 + \frac{3}{2}, \quad \frac{7}{16} \times \frac{3}{80} = \frac{1}{10} + \frac{3}{5}
\end{align*}
\]

From the first two lines in eq. (31) we get

\[
[(2,3,\mp) + (2,3,\pm)] \times [(2,3,\pm) + (2,3,\mp)] = 2 \left( (1,1,\mp) + (1,1,\pm) + (1,3,\mp) + (1,3,\pm) \right)
\]

and the first equality in eq. (32) is reproduced by making the identification

\[
\frac{1}{\sqrt{2}} [(2,3,\pm) + (2,3,\mp)] = \left( \frac{3}{80} \right)
\]

where a $\frac{1}{\sqrt{2}}$ factor has been introduced since the double degeneracy of the $R$ states requires the identification of two $R$ states with only one state in the conformal case (see eq. (28)). The analogous identification for $O_{\pm}(2,1)$ allows to reproduce the remaining fusions (32).

Similarly we obtain for the mixed fusion rules $R \times NS \sim R$

\[
\begin{align*}
N_{(2,1,\pm)(1,3,\pm)(2,3,\pm)} &= N_{(2,1,\pm)(1,3,\mp)(2,3,\pm)} = 1 \\
N_{(2,3,\pm)(1,3,\pm)(2,3,-)} &= N_{(2,3,\pm)(1,3,\mp)(2,3,-)} = 1 \\
N_{(2,3,\pm)(1,3,\pm)(2,1,+)} &= N_{(2,3,\pm)(1,3,\mp)(2,1,-)} = 1 \\
N_{(2,3,\pm)(1,1,-)(2,3,\mp)} &= N_{(2,1,\pm)(1,1,-)(2,1,\mp)} = 1
\end{align*}
\]
which correspond to

\[
\begin{align*}
\frac{7}{16} \times \frac{1}{10} &= \frac{7}{16} \times \frac{3}{5} = \frac{3}{80} \\
\frac{3}{80} \times \frac{1}{10} &= \frac{3}{80} \times \frac{3}{5} = \frac{3}{80} + \frac{7}{16} \\
\frac{3}{80} \times \frac{3}{2} &= \frac{3}{80}, \quad \frac{7}{16} \times \frac{3}{2} = \frac{7}{16}
\end{align*}
\]

(36)

For example, the first line in eq. (35) corresponds to

\[
[(2, 1, +) + (2, 1, -)] \times \frac{1}{10} = [(2, 3, +) + (2, 3, -)] \\
[(2, 1, +) + (2, 1, -)] \times \frac{3}{5} = [(2, 3, +) + (2, 3, -)]
\]

(37)

in accord with the first line of eq. (36), and analogously it is possible to reproduce the remaining fusions.

In the \( p = 4 \) case the formalism is completely analogous. The identification among the superconformal \([10]\) and the Virasoro \([15]\) characters is as follows

\[
\begin{align*}
\chi_{NS}^0 &= \chi_{V_{ir}} + \chi_{V_{ir,(1)}} \quad \chi_{NS}^0 = e^{-i\pi/72} \left( \chi_{V_{ir}} - \chi_{V_{ir,(1)}} \right) \\
\chi_{NS}^1 &= \chi_{V_{ir}} + \chi_{V_{ir,(2)}} \quad \chi_{NS}^1 = e^{-i\pi/72} \left( \chi_{V_{ir}} - \chi_{V_{ir,(2)}} \right) \\
\chi_{NS}^2 &= \chi_{V_{ir}} + \chi_{V_{ir}} \quad \chi_{NS}^2 = e^{i\pi/4} \left( \chi_{V_{ir}} + \chi_{V_{ir}} \right) \\
\chi_{NS}^3 &= \chi_{V_{ir,(1)}} + \chi_{V_{ir,(1)}} \quad \chi_{NS}^3 = e^{i\pi/4} \left( \chi_{V_{ir,(1)}} - \chi_{V_{ir,(1)}} \right) \\
\chi_{4/24}^1 &= \chi_{V_{ir}} + \chi_{V_{ir}} \quad \chi_{4/24}^1 = e^{i\alpha} \left( \chi_{V_{ir}} - \chi_{V_{ir}} \right) \\
\chi_{4/38}^2 &= \chi_{V_{ir}} \quad \chi_{4/38}^2 = \chi_{V_{ir,(1)}} \\
\chi_{9/16}^3 &= \chi_{V_{ir,(2)}} \quad \chi_{9/16}^3 = \chi_{V_{ir,(2)}}
\end{align*}
\]

(38)

where \( \alpha \in \mathbf{R} \), the \( \chi_{4/24}^1 \) character is an arbitrary phase, and we have taken into account that in the RCFT description as a \( Z_2 \) orbifold of a Gaussian model with rational radius, there are two representations of \( \frac{3}{2}, \frac{1}{16} \) and \( \frac{9}{16} \).
As in the \( p = 3 \) situation, all the doubly degenerate states of the \( R \) sector must include a \( \frac{1}{\sqrt{2}} \) factor to be compared with the corresponding states in the conformal case. For the \( \frac{1}{24} \) state we must take into account that in the \( \nu = 1 \) sector our correlation functions cannot reproduce the corresponding character in the factorization limit because the intermediate state is a fermion instead of the identity. So we have to introduce a \( \frac{1}{\sqrt{2}} \) factor here again to take into account the contribution of the state \( \frac{1}{24} \) in both the \( \nu = 1 \) and \( \nu = 2 \) sectors. This corresponds to the states \( \frac{1}{24} \) and \( \frac{25}{24} \) in the Gaussian model case. Following the same procedure as in the \( p = 3 \) case it is possible to see that the fusion rules of the conformal case, as calculated in [15], are reproduced. Let’s consider, as an example, the \( \frac{1}{24} \times \frac{1}{24} \) fusion rule. We obtain

\[
[(2, 3, +) + (2, 3, -)] \times [(2, 3, +) + (2, 3, -)] = 4 ((1, 1, +) + (1, 1, -) + (3, 1, +) + (3, 1, -) + (1, 3, +) + (1, 3, -)) \quad (39)
\]

in complete agreement, after rescaling properly, with the addition of the conformal fusion rules [15]

\[
\frac{1}{24} \times \frac{1}{24} = \frac{25}{24} \times \frac{25}{24} = 0 + \frac{1}{6} \quad \frac{1}{24} \times \frac{25}{24} = \frac{25}{24} \times \frac{1}{24} = \frac{2}{3} + 2 \left( \frac{3}{2} \right) \quad (40)
\]

Notice that these fusion rules disagree with those in App. E of ref. [4] (where the set of eqs. (5.9) of that work were used).

In conclusion we have provided strong evidence that by working in an appropriate basis the Verlinde theorem carries through to the \( N=1 \) superconformal minimal models. We have also shown that the generalized Verlinde formula contains information about the fermionic parity of the fields.

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