Topological Transcendental Fields †

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Abstract: This article initiates the study of topological transcendental fields \( F \) which are subfields of the topological field \( C \) of all complex numbers such that \( F \) only consists of rational numbers and a nonempty set of transcendental numbers. \( F \), with the topology it inherits as a subspace of \( C \), is a topological field. Each topological transcendental field is a separable metrizable zero-dimensional space and algebraically is \( \mathbb{Q}(T) \), the extension of the field of rational numbers by a set \( T \) of transcendental numbers. It is proven that there exist precisely \( 2^{\aleph_0} \) countably infinite topological transcendental fields and each is homeomorphic to the space \( \mathbb{Q} \) of rational numbers with its usual topology. It is also shown that there is a class of \( 2^{\aleph_0} \) of topological transcendental fields of the form \( \mathbb{Q}(T) \) with \( T \) a set of Liouville numbers, no two of which are homeomorphic.

Keywords: topological field; transcendental number; algebraic; countably infinite; homeomorphic; extension field; subfield

1. Preliminaries

We begin by setting out our notation and making some simple preliminary observations.

Remark 1. We shall discuss four fields: \( C \), the field of all complex numbers; \( \mathbb{R} \), the field of all real numbers; \( \mathbb{A} \), the field of all algebraic numbers; and \( \mathbb{Q} \), the field of all rational numbers. Observe the following easily verified facts:

(i) Fields \( C \) and \( \mathbb{R} \) have cardinality \( \epsilon \), the cardinality of the continuum;
(ii) Fields \( \mathbb{A} \) and \( \mathbb{Q} \) have cardinality \( \aleph_0 \);
(iii) \( C \) with its Euclidean topology is homeomorphic to \( \mathbb{R} \times \mathbb{R} \), where \( \mathbb{R} \) has its Euclidean topology;
(iv) Each of these four fields has a natural topology; \( C \) and \( \mathbb{R} \) have Euclidean topologies, while \( \mathbb{A} \) and \( \mathbb{Q} \) inherit a natural topology as a subspace of \( C \);
(v) Field \( \mathbb{Q} \) is a dense subfield of the topological field \( \mathbb{R} \) (that is, the closure, in the topological sense, of \( \mathbb{Q} \) is \( \mathbb{R} \));
(vi) Topological field \( \mathbb{A} \) is a dense subfield of the topological field \( \mathbb{C} \);
(vii) \( C \supseteq \mathbb{A} \supseteq \mathbb{A} \cap \mathbb{R} \supseteq \mathbb{Q} \), but \( \mathbb{A} \) is not a subset of \( \mathbb{R} \);
(viii) Field \( C \) is a vector space of dimension \( \epsilon \) over \( \mathbb{A} \) and it is also a vector space of dimension \( \epsilon \) over \( \mathbb{Q} \);
(ix) \( \mathbb{A} \) is a vector space of countably infinite dimension over \( \mathbb{Q} \);
(x) \( \mathbb{N} \) denotes the set of positive integers and \( \mathbb{Z} \) denotes the set of all integers, each with the discrete topology;
(xi) \( \mathbb{T} \) is the topological space of all transcendental numbers, where \( \mathbb{T} = C \setminus \mathbb{A} \) and has a natural topology as a subspace of \( C \). The topology of \( \mathbb{T} \) is separable, metrizable, and zero-dimensional. Furthermore, the cardinality of \( \mathbb{T} \) is \( \epsilon \) and \( \mathbb{T} \) is dense in \( C \).
Remark 2. Now, we mention some not so easily verified known results:

(i) $\mathcal{T}$ is homeomorphic to the space $\mathbb{P}$ of all irrational real numbers. $\mathbb{P}$ is also homeomorphic to the countably infinite product $\mathbb{N}^\mathbb{N}$. (see ([11], §1.9));

(ii) $\mathcal{T} \mathbb{Q}$ denotes the set $\mathcal{T} \cup \mathbb{Q}$. It is also homeomorphic to $\mathbb{P}$;

(iii) In 1932, Kurt Mahler classified the set of all transcendental numbers $\mathcal{T}$ into three disjoint classes: $S$, $T$, and $U$. For a discussion of this important classification, see ([2], Chapter 8). It has been proven that each of these sets has cardinality $c$. Furthermore, the Lebesgue measure of $T$ and $U$ are each zero. Thus, $S$ has full measure, that is its complement has a measure of zero;

(iv) We introduce the classes $\mathcal{S} \cup \mathbb{Q} = \mathcal{S}$, $\mathcal{T} \cup \mathbb{Q} = \mathcal{T}$, $\mathcal{U} \cup \mathbb{Q} = \mathcal{U}$. Clearly $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ each have cardinality $c$, $\mathcal{T}$, and $\mathcal{U}$ have measure zero, and $\mathcal{S}$ has full measure;

(v) In 1844, Joseph Liouville showed that all members of a certain class of numbers, now known as the Liouville numbers, are transcendental. A real number $x$ is said to be a Liouville number if, for every positive integer $n$, there exists a pair $(p, q)$ of integers with $q > 1$, such that $0 < |x - \frac{p}{q}| < \frac{1}{q^n}$ (see [3]). The Liouville numbers are a subset of the Mahler class $U$. We denote the set of Liouville numbers by $\mathcal{L}$ and the set $\mathcal{L} \cup \mathbb{Q}$ by $\mathcal{L} \cup \mathbb{Q}$.

Recall the following definitions from [4]. While Weintraub stated the definitions and propositions using countably infinite sets, there is no problem to state these using the sets of any cardinality.

Definition 1. Let $\mathbb{E}$ be an extension field of $\mathbb{F}$. Then, $\alpha \in \mathbb{E}$ is said to be transcendental over $\mathbb{F}$ if $\alpha$ is not a root of any nonzero polynomial $p(X) \in \mathbb{F}[X]$, the ring of polynomials over $\mathbb{F}$ in the variable $X$ with coefficients in $\mathbb{F}$. The quantity $\alpha \in \mathbb{E} \setminus \mathbb{F}$ is said to be algebraic over $\mathbb{F}$ if it is not transcendental over $\mathbb{F}$.

Definition 2. An extension field $\mathbb{E}$ of a field $\mathbb{F}$ is said to be a completely transcendental extension of $\mathbb{F}$ if $\alpha$ is transcendental over $\mathbb{F}$, for every $\alpha \in \mathbb{E} \setminus \mathbb{F}$.

Definition 3. Let $\mathbb{E}$ be an extension field of a field $\mathbb{F}$. Then, $\mathbb{E}$ is a purely transcendental extension of $\mathbb{F}$ if $\mathbb{E}$ is isomorphic to the field of rational functions $\mathbb{Q}(\{X_i : i \in I\})$ of variables $\{X_i : i \in I\}$, where $I$ is a finite or infinite index set.

Definition 4. Let field $\mathbb{E}$ be an extension of the field $\mathbb{F}$. If $I$ is any index set, the subset $S = \{s_i : i \in I\}$ of $\mathbb{E}$ is said to be algebraically independent over $\mathbb{F}$ if for all finite subsets $\{i_1, \ldots, i_n\}$ of $I$, all nonzero polynomials $p \in \mathbb{F}[X_{i_1}, \ldots, X_{i_n}]$, $p(s_{i_1}, \ldots, s_{i_n}) \neq 0$. By convention, if $S = \emptyset$, then $S$ is said to be algebraically independent over $\mathbb{F}$.

Remark 3. Observe that, if a set $S$ is algebraically independent over $\mathbb{Q}$, then it is algebraically independent over $\mathbb{A}$. Furthermore, algebraic independence implies linear independence.

Remark 4. Central to their definition of the classes $S$, $T$, and $U$, was the feature that Mahler wanted, namely that any two algebraically dependent transcendental numbers lie in the same class $S$, $T$, or $U$.

We shall use ([4], Lemmas 6.1.5 and 6.1.8) which are stated here as Proposition 2 and Proposition 1. In this context, it is useful to recall the classical result of Jacob Lüroth, proven in 1876, that every field that lies between any field $\mathbb{F}$ and an extension field $\mathbb{F}(\alpha)$ is itself an extension field of $\mathbb{F}$ by a single element of the field $\mathbb{F}(\alpha)$.

Proposition 1. Let $\mathbb{E}$ be a purely transcendental extension of a field $\mathbb{F}$. Then, $\mathbb{E}$ is a completely transcendental extension of $\mathbb{F}$.
Proposition 2. Let $E$ be an extension field of the field $F$ and let $S = \{s_i : i \in I\}$ be algebraically independent over $F$, where $I$ is an index set. Then, the extension field $F(S)$ is a purely transcendental extension.

Recall the following definition from, for example, [5,6]:

Definition 5. A field $F$ with a topology $\tau$ is said to be a topological field if the field operations:

(i) $(x, y) \rightarrow x + y$ from $F \times F$ to $F$;
(ii) $x \rightarrow -x$ from $F \setminus \{0\}$ to $F \setminus \{0\}$;
(iii) $(x, y) \rightarrow xy$ from $F \times F$ to $F$; and
(iv) $x \rightarrow x^{-1}$ from $F$ to $F$ are all continuous.

The standard examples of topological fields of characteristic 0 are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ with the usual Euclidean topologies. Indeed, by ([7], §27, Theorem 22), the only connected locally compact Hausdorff fields are $\mathbb{R}$ and $\mathbb{C}$. However, Shakhmatov in [8] proved the following beautiful result:

Theorem 1. On every field $F$ of infinite cardinality $\aleph_0$, there exist precisely $2^{2^{\aleph_0}}$ distinct topologies which make $F$ a topological field.

Motivated by the definition of a transcendental group introduced in [9], we define here the notion of a topological transcendental field.

Definition 6. The topological field $F$ is said to be a topological transcendental field if algebraically it is a subfield of $\mathbb{C}$, is a subset of $\mathbb{Q} \cup T$, and has the topology it inherits as a subspace of $\mathbb{C}$.

Remark 5. Of course, the underlying field of a topological transcendental field is a completely transcendental extension of $\mathbb{Q}$.

2. Countably Infinite Transcendental Fields

Proposition 3. If $t$ is any transcendental number, then $\mathbb{Q}(t)$ is a topological transcendental field.

Proof. This proposition is an immediate consequence of Propositions 1 and 2. \(\square\)

Remark 6. Of course it is not true that if $t_1$ and $t_2$ are transcendental, then $\mathbb{Q}(t_1, t_2)$ is necessarily a transcendental field. For example, if $t_1 = \pi$ and $t_2 = \pi + \sqrt{2}$, then $\mathbb{Q}(t_1, t_2)$ is not a topological transcendental field as $\sqrt{2} \in \mathbb{Q}(t_1, t_2)$. In fact, Paul Erdős [10] proved that for every real number $r$ there exist Liouville numbers $t_3, t_4, t_5, t_6$ such that $t_3 \cdot t_4 = r$ and $t_5 + t_6 = r$. Indeed, he proved that for each real number $r$, there are uncountably many Liouville numbers $t_3, t_4$ and $t_5, t_6$ with these properties. As a consequence, we see that if $L$ is the set of all Liouville numbers, then $\mathbb{Q}(L)$ is not a topological transcendental field.

Having established the existence of countably infinite topological transcendental fields, we now describe a very concrete example. However, first we state a well-known theorem on transcendental numbers—please see Theorem 1.4 and the comments following it, in [2].

Theorem 2. (Lindemann–Weierstrass Theorem) For any $m \in \mathbb{N}$ and any algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ which are linearly independent over $\mathbb{Q}$, the numbers $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_m}$ are algebraically independent.
Theorem 3. Let \( S = \{a_1, a_2, \ldots, a_n, \ldots \} \) be a countably infinite set of algebraic numbers which are linearly independent over \( \mathbb{Q} \). If \( T = \{e^{a_1}, e^{a_2}, \ldots, e^{a_n}, \ldots \} \), then \( \mathbb{Q}(T) \), is a topological transcendental field.

Proof. By Propositions 1 and 2, \( \mathbb{Q}(T) \) is a topological transcendental field. \( \square \)

Theorem 4. There exist precisely \( 2^{\aleph_0} \) countably infinite topological transcendental fields, each of which is homeomorphic to \( \mathbb{Q} \).

Proof. Using the notation of Theorem 3, there are \( 2^{\aleph_0} \) subsets of \( T \) and, due to algebraic independence, any two such subsets \( V, W, V \neq W \), are such that \( \mathbb{Q}(V) \neq \mathbb{Q}(W) \).

Furthermore, there are only \( 2^{\aleph_0} \) countably infinite subsets of \( \mathbb{C} \). Thus, there exist precisely \( 2^{\aleph_0} \) countably infinite topological transcendental groups.

By ([1], Theorem 1.9.6), the space \( \mathbb{Q} \) of all rational numbers up to homeomorphism is the unique non-empty countably infinite separable metrizable space without isolated points. In a topological field (indeed in a topological group), there are isolated points if and only if the topological field has a discrete topology. However, by ([11], Theorem 6), the only discrete subgroups of \( \mathbb{C} \) are isomorphic to \( \mathbb{Z} \) and \( \mathbb{Z} \times \mathbb{Z} \), neither of which has the algebraic structure of a field. Thus, every countably infinite topological transcendental field is homeomorphic to \( \mathbb{Q} \). \( \square \)

3. Topological Transcendental Fields of Continuum Cardinality

Theorem 5. Let \( \mathbb{K} \) be a topological transcendental field of cardinality \( \text{card}(\mathbb{K}) \). Then, the extension field \( \mathbb{K}(t) \) is a topological transcendental field for all \( t \in \mathbb{C} \).

Proof. The extension field \( \mathbb{K}(t) \) consists of elements \( z \) of the form

\[
z = \frac{c_0 + c_1 t + c_2 t^2 + \ldots + c_m t^m}{d_0 + d_1 t + d_2 t^2 + \ldots + d_m t^m}
\]

for \( c_0, c_1, \ldots, c_n, d_0, d_1, \ldots, d_m \in \mathbb{K} \), \( n, m \in \mathbb{N} \). If \( z \) is an algebraic number \( a \), then

\[
c_0 + c_1 t + c_2 t^2 + \ldots + c_m t^m - a d_0 - a d_1 t - a d_2 t^2 - \cdots - a d_m t^m = 0. \tag{*}
\]

For any given \( n, m \in \mathbb{N} \), given \( c_0, c_1, \ldots, c_n, d_0, d_1, \ldots, d_m \in \mathbb{K} \), and given \( a \in \mathbb{A} \), the Fundamental Theorem of Algebra says that there at most \( \max(n, m) \) algebraic number solutions of \((*)\) for \( t \). As there are only a countably infinite number of algebraic numbers \( a \), we see that for given \( c_0, c_1, \ldots, c_n, d_0, d_1, \ldots, d_m \in \mathbb{K} \), there are a countable number of solutions of \((*)\) for \( t \). Noting that the number of choices of \( c_0, c_1, \ldots, c_n, d_0, d_1, \ldots, d_m \in \mathbb{K} \) is \( \text{card}(\mathbb{K}) \), for each \( n, m \in \mathbb{N} \), we obtain that \( z \) is a transcendental number except for at most \( \aleph_0 \times \text{card}(\mathbb{K}) = \text{card}(\mathbb{K}) \) values of \( t \), which proves the theorem. \( \square \)

Noting our Remark 6, Corollary 1 is of interest.

Corollary 1. If \( t_1, t_2 \) are transcendental numbers, then \( \mathbb{Q}(t_1, t_2) \) is a topological transcendental field for all but a countably infinite number of pairs \( (t_1, t_2) \). Indeed, if \( W \) is a countable set of transcendental fields, then \( \mathbb{Q}(W) \) is a topological transcendental field for all but a countably infinite number of sets \( W \). \( \square \)

Corollary 2. Let \( \mathbb{K} \) be a topological transcendental field of cardinality \( \aleph_0 < 2^{\aleph_0} \). Then, there exists a \( t \in \mathbb{C} \) such that \( \mathbb{K}(t) \) is a topological transcendental field which properly contains \( \mathbb{K} \). \( \square \)

Theorem 6. Let \( E \) be any set of cardinality \( \aleph \) of transcendental numbers. Then, there exists a topological transcendental field \( \mathbb{Q}(T) \) of cardinality \( \aleph \), where \( T \subseteq E \). Further, \( \mathbb{Q}(T) \) has \( 2^\aleph \) distinct topological transcendental subfields.
Proof. Consider the set $\mathcal{F}$ of all topological transcendental fields $\mathbb{Q}(F)$, where $F$ is a subset of $E$, with the property that for each pair $W, V \subset F$ such that $W \neq V$, $\mathbb{Q}(V) \neq \mathbb{Q}(W)$.

By Corollary 1 and the fact that $E$ is uncountable, there exist $s, t \in E, t \notin \mathbb{Q}(s), s \notin \mathbb{Q}(t)$, and $\mathbb{Q}(s,t)$ is a topological transcendental field. Then, $\mathbb{Q}(s,t) \in \mathcal{F}$.

Put a partial order on the members of $\mathcal{F}$ by set theory containment. Consider any totally ordered subset $\mathcal{S}$ of members of $\mathcal{F}$. Let $\mathbb{K}$ be the union of members of $\mathcal{S}$. Clearly it is a member of $\mathcal{F}$ and is an upper bound of $\mathcal{S}$. Therefore, by Zorn’s Lemma, $\mathcal{F}$ has a maximal member $\mathbb{Q}(T)$, where $T \subset E$.

Suppose that $T$ has cardinality $\aleph < \epsilon$, then by the proof of Theorem 4, there exists an element $\epsilon \in \mathbb{E}$, such that $\mathbb{Q}(T)(\epsilon) = \mathbb{Q}(T, \{\epsilon\})$ is a topological transcendental field which is easily seen to be a member of $\mathcal{F}$. This contradicts the maximality of $\mathbb{Q}(T)$. Thus, $T$ has cardinality $\epsilon$.

Furthermore, by the definition of $\mathcal{F}$, $\mathbb{Q}(T)$ has $2^\epsilon$ distinct topological transcendental subfields. $\square$

Theorem 7. Let $E$ be a set of transcendental numbers of cardinality $\epsilon$. Then, there exist $2^\epsilon$ distinct topological transcendental fields $\mathbb{Q}(T)$, where $T \subset E$, no two of which are homeomorphic.

Proof. By the Laverentieff Theorem, Theorem A8.5 of [1], there are at most $\epsilon$ subspaces of $\mathbb{C}$ which are homeomorphic. Thus, from Theorem 6 there are $2^\epsilon$ topological transcendental fields, no two of which are homeomorphic. $\square$

Corollary 3. Let $E$ be the set $L$ of Liouville numbers or the Mahler set $U$ or the Mahler set $T$ or the Mahler set $S$. Then, there exist $2^\epsilon$ distinct topological transcendental fields $\mathbb{Q}(T)$, where $T \subset E$, no two of which are homeomorphic. $\square$

As noted in Remark 3, the Mahler sets $T$ and $U$ and the set of Liouville numbers, being a subset of $U$, have Lebesgue measure zero, while the Mahler set $S$ has full measure; we thus conclude by asking whether there are any topological transcendental fields of non-zero Lebesgue measure.

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