ON BOUNDEDNESS OF CALDERÓN-TOEPLITZ OPERATORS

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Abstract. We study the boundedness of Toeplitz-type operators defined in the context of the Calderón reproducing formula considering the specific wavelets whose Fourier transforms are related to Laguerre polynomials. Some sufficient conditions for simultaneous boundedness of these Calderón-Toeplitz operators on each wavelet subspace for unbounded symbols are given, where investigating the behavior of certain sequence of iterated integrals of symbols is helpful. A number of examples and counterexamples is given.

1 Introduction

Calderón-Toeplitz operators are integral operators which arise in the context of wavelet analysis in connection with the Calderón reproducing formula, cf. [2]. This formula gives rise to a class of Hilbert spaces with reproducing kernels, the so-called spaces of Calderón (or, wavelet) transforms. These operators were formally defined in [14] as a wavelet counterpart of Toeplitz operators defined on Hilbert spaces of holomorphic functions. Therefore the name “Calderón-Toeplitz” reflects the close relationship with the Calderón reproducing formula on one side and, on the other side, it emphasizes the fact that this operator is unitarily equivalent to the Toeplitz-type operator

\[ P_\psi M_a : W_\psi (L_2(\mathbb{R})) \to W_\psi (L_2(\mathbb{R})), \]

where \( M_a \) is the multiplication operator by \( a \) and \( P_\psi \) is the orthogonal projection from \( L_2(\mathbb{R} \times \mathbb{R}_+, v^{-2} du dv) \) onto the space of wavelet transforms

\[ W_\psi (L_2(\mathbb{R})) = \left\{ W_\psi f(u,v) = \langle f, \psi_{u,v} \rangle ; f \in L_2(\mathbb{R}) \right\}, \]

where \( \psi \in L_2(\mathbb{R}) \) is an admissible wavelet and \( \psi_{u,v}, (u,v) \in \mathbb{R} \times \mathbb{R}_+ \), are the shifted and scaled versions of \( \psi \), see Section 2. These operators are also a useful localization tool which enables to localize a signal both in time and frequency. For further details about localization operators and wavelets transforms we refer to the Wong book [19].

In [6] we have described the structure of the space of Calderón transforms \( W_\psi (L_2(\mathbb{R})) \) inside \( L_2(\mathbb{R} \times \mathbb{R}_+, v^{-2} du dv) \). This representation was further used to study Calderón-Toeplitz operators acting on spaces of Calderón transforms for general (admissible) wavelets in [7]. Considering certain specific wavelets in [8] we were able to simplify these general results and discover the interesting connection between the spaces of Calderón transforms and poly-analytic

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This connection was further investigated in [17] where also some interesting applications were given. The specificity of this choice of wavelets $\psi^{(k)}$, whose Fourier transforms are related to Laguerre polynomials, has enabled us to study in [9] in more detail the family of Calderón-Toeplitz operators $T^{(k)}_a$ acting on wavelet subspaces $A^{(k)}$ (with parameter $k = 0, 1, 2, \ldots$ being the degree of Laguerre polynomial $L_k$) with symbols depending only on vertical variable in the upper half-plane $\Pi$. In this specific case of wavelets the corresponding Calderón-Toeplitz operators generalize classical Toeplitz operators acting on the Bergman space in an interesting way which differs from the case of Toeplitz operators acting on the weighted Bergman spaces studied in [5]. On the other hand, the classical Toeplitz operators and Calderón-Toeplitz operators share many features in common.

In this paper we continue the detailed study of Calderón-Toeplitz operators $T^{(k)}_a$ acting on wavelet subspace $A^{(k)}$, which was initiated in [9]. For a bounded symbol $a$ on $G$ the Calderón-Toeplitz operator $T^{(k)}_a$ is clearly bounded on $A^{(k)}$. However, an interesting and important feature of these operators on wavelet subspaces is that they can be bounded for symbols that are unbounded near the boundary. Therefore the aim of this paper is to study in detail the boundedness properties of Calderón-Toeplitz operators with such unbounded symbols and to give sufficient conditions for their simultaneous boundedness on all wavelet subspaces. The main tool in our study is the result, which we have shown in [9], that the Calderón-Toeplitz operator $T^{(k)}_a$ acting on $A^{(k)}$ with a symbol $a = a(3\zeta)$, $\zeta = (u, v)$, is unitarily equivalent to the multiplication operator $\gamma_{a,k}I$ acting on $L^2(\mathbb{R}_+)$ with

$$\gamma_{a,k}(\xi) = \chi_+(\xi) \int_{\mathbb{R}_+} a\left(\frac{v}{2\xi}\right) \ell^2_k(v) \, dv,$$

where the functions $\ell_k(x) = e^{-x^2/2}L_k(x)$ forms an orthonormal basis in $L^2(\mathbb{R}_+)$ with $L_k(x)$ being the Laguerre polynomial of order $k = 0, 1, \ldots$. Thus, the boundedness of function $\gamma_{a,k}$ is responsible also for the boundedness of operator $T^{(k)}_a$. We will also show that for unbounded symbols $a = a(v)$, $v \in \mathbb{R}_+$, the behavior of iterated means

$$C^{(1)}_a(v) = \int_0^v a(t) \, dt,$$

$$C^{(m)}_a(v) = \int_0^v C^{(m-1)}_a(t) \, dt, \quad m = 2, 3, \ldots,$$

rather than the behavior of symbol $a$ itself plays a crucial role in the boundedness properties. Contrary to the case of Toeplitz operators on weighted Bergman spaces studied in [5] these means do not depend on a weighted parameter $k$. In our case $k = 0, 1, \ldots$ may be viewed as a level of consideration of time-scale analysis of a signal. We present a number of examples and construct wide families of unbounded symbols for which the Calderón-Toeplitz operator is not only bounded, but also belongs to the algebra of bounded Calderón-Toeplitz operators generated by bounded symbols on $\mathbb{R}_+$ having limits at the endpoints of $[0, +\infty)$. This extends results for bounded symbols from [9] to certain unbounded ones.
In the last Section [4] we show how Calderón-Toeplitz operators with unbounded symbols can appear as uniform limits of Calderón-Toeplitz operators with bounded symbols.

2 Preliminaries

Here we briefly recall some necessary notations and results from our previous works, mainly from [9]. As usual, \( \mathbb{R} (\mathbb{C}, \mathbb{N}) \) is the set of all real (complex, natural) numbers, \( \mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\} \) is the two-point compactification of \( \mathbb{R} \), and \( \mathbb{R}_+ \) is the positive half-line with \( \chi_+ \) being its characteristic function. Let \( L_2(G, dv) \) be the space of all square-integrable functions on \( G \) with respect to measure \( dv \), where \( G = \{ \zeta = (u, v); u \in \mathbb{R}, v > 0 \} \) is the locally compact “ax + b”-group with the left invariant Haar measure \( dv(\zeta) = v^{-2} du dv \). In what follows we identify the group \( G \) with the upper half-plane \( \Pi = \{ \zeta = u + iv; u \in \mathbb{R}, v > 0 \} \) in the complex plane \( \mathbb{C} \). Then the square-integrable representation \( \rho \) of \( G \) on \( L_2(\mathbb{R}) \) is given by

\[
(\rho \xi f)(x) = f(\xi(x)) = \frac{1}{\sqrt{v}} f \left( \frac{x - u}{v} \right), \quad f \in L_2(\mathbb{R}),
\]

with \( \zeta = (u, v) \in G \). The function \( \psi \in L_2(\mathbb{R}) \) is called an admissible wavelet if it satisfies the so-called admissibility condition

\[
\int_{\mathbb{R}_+} |\hat{\psi}(x\xi)|^2 \frac{d\xi}{\xi} = 1
\]

for almost every \( x \in \mathbb{R} \), where \( \hat{\psi} \) stands for the unitary Fourier transform \( \mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) given by

\[
\mathcal{F}\{g\}(\xi) = \hat{g}(\xi) = \int_{\mathbb{R}} g(x)e^{-2\pi i x \xi} \, dx.
\]

The Laguerre polynomials \( L_n^{(\alpha)}(x) \) of degree \( n \in \mathbb{N} \cup \{0\} \), and type \( \alpha \) are given by

\[
L_n^{(\alpha)}(y) = \frac{y^{-\alpha} e^y}{n!} \frac{d^n}{dy^n} (e^{-y} y^{n+\alpha}) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-y)^k}{k!}, \quad y \in \mathbb{R}_+,
\]

cf. [3, formula 8.970.1]. For \( \alpha = 0 \) we simply write \( L_n(y) \). Recall that the system of functions

\[
\ell_n(y) = e^{-y/2} L_n(y), \quad y \in \mathbb{R}_+, \quad n \in \mathbb{Z}_+,
\]

forms an orthonormal basis in the space \( L_2(\mathbb{R}_+, dy) \). For appropriate parameters introduce the (non-negative) function

\[
\Lambda_{p,m,n}^{(\alpha,\beta)}(x) = x^p e^{-x} |L_m^{(\alpha)}(x) L_n^{(\beta)}(x)|, \quad x \in \mathbb{R}_+.
\]

Then for each \( \alpha, \beta \geq -\frac{1}{2}, \Re p > -1, x \in \mathbb{R}_+ \), and \( m, n \in \mathbb{Z}_+ \) we have

\[
\Lambda_{p,m,n}^{(\alpha,\beta)}(x) \leq \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{(\alpha + 1)_{m-i} (\beta + 1)_{n-j}}{(m-i)! (n-j)!} x^{p+i} e^{-x}, \quad (1)
\]
where \( \mathcal{R} \) respectively. Let \( f \) \( \in \mathcal{H}_p \), \( p \in (0, 1) \), with respect to wavelets \( \varphi \), resp. \( \bar{\varphi} \), on \( \mathbb{R} \), whose Fourier transforms are given by

\[
\hat{\varphi}\left(\xi\right) = \chi_{+}\left(\xi\right)\sqrt{2\pi}\ell_{k}\left(2\xi\right)\quad \text{and}\quad \hat{\bar{\varphi}}\left(\xi\right) = \hat{\varphi}\left(-\xi\right),
\]

respectively. Let \( A^{(k)} \), resp. \( \bar{A}^{(k)} \), be the spaces of wavelet transforms of functions \( f \in H^{+}_{2}\left(\mathbb{R}\right) \), resp. \( f \in H^{-}_{2}\left(\mathbb{R}\right) \), with respect to wavelets \( \psi^{(k)} \), resp. \( \bar{\psi}^{(k)} \), where

\[
H^{+}_{2}\left(\mathbb{R}\right) = \left\{ f \in L_{2}\left(\mathbb{R}\right); \ \text{supp} \hat{f} \subseteq [0, +\infty) \right\},
\]

\[
H^{-}_{2}\left(\mathbb{R}\right) = \left\{ f \in L_{2}\left(\mathbb{R}\right); \ \text{supp} \hat{f} \subseteq (-\infty, 0] \right\},
\]

are the Hardy spaces, respectively.

Consider the unitary operators

\[
U_{1} = \left(\mathcal{F} \otimes I\right) : L_{2}\left(\mathbb{G}, d\nu\left(\zeta\right)\right) \rightarrow L_{2}\left(\mathbb{R}, d\nu\right) \otimes L_{2}\left(\mathbb{R}^{+}, v^{-2}d\nu\right)
\]

with \( \zeta = (u, v) \in \mathbb{G} \), and

\[
U_{2} : L_{2}\left(\mathbb{R}, d\nu\right) \otimes L_{2}\left(\mathbb{R}^{+}, v^{-2}d\nu\right) \rightarrow L_{2}\left(\mathbb{R}, dx\right) \otimes L_{2}\left(\mathbb{R}^{+}, dy\right)
\]

given by

\[
U_{2} : F\left(u, v\right) \mapsto \sqrt{\frac{2|x|}{y}} F\left(x, \frac{y}{2|x|}\right).
\]

In [5] we have proved the following important result describing the structure of wavelet subspaces \( A^{(k)} \) inside \( L_{2}\left(\mathbb{G}, d\nu\right) \).

**Theorem 2.1** The unitary operator \( U = U_{2}U_{1} \) gives an isometrical isomorphism of the space \( L_{2}\left(\mathbb{G}, d\nu\right) \) onto \( L_{2}\left(\mathbb{R}, dx\right) \otimes L_{2}\left(\mathbb{R}^{+}, dy\right) \), under which the wavelet subspace \( A^{(k)} \) is mapped onto \( L_{2}\left(\mathbb{R}^{+}\right) \otimes L_{k} \), where \( L_{k} \) is the rank-one space generated by function \( \ell_{k}\left(y\right) = e^{-y^{2}/2}\ell_{k}\left(y\right) \).
This result is a ”wavelet” analog of results obtained for the Bergman and poly-Bergman spaces, cf. [16], and enables to study an interesting connection between wavelet spaces related to Laguerre polynomials and poly-Bergman spaces in more detail, which is done in paper [1].

Following the general scheme presented in [12], let us introduce the isometric imbedding

\[ Q_k : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+) \]

given by

\[ (Q_k f)(x, y) = \chi_+(x)f(x)\ell_k(y), \]

here the function \( f \) is extended to an element of \( L^2(\mathbb{R}) \) by setting \( f(x) \equiv 0 \) for \( x < 0 \). The adjoint operator

\[ Q_k^* : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+) \]

is given by

\[ (Q_k^* F)(x) = \chi_+(x) \int_{\mathbb{R}_+} F(x, \tau)\ell_k(\tau) d\tau, \]

and we have

**Theorem 2.2** The operator \( R_k : A^{(k)} \rightarrow L^2(\mathbb{R}_+) \), where

\[ (R_k F)(\xi) = \chi_+(\xi)\sqrt{2\xi} \int_{\mathbb{R}\times\mathbb{R}_+} F(u, v)\ell_k(2v\xi)e^{-2\pi i u\sqrt{\xi}} \frac{dudv}{v}, \tag{4} \]

is an isometrical isomorphism admitting decomposition \( R_k = Q_k^* U_2(\mathcal{F} \otimes I) \).

**Corollary 2.3** The inverse isomorphism \( R_k^* : L^2(\mathbb{R}_+) \rightarrow A^{(k)} \) given by

\[ (R_k^* f)(u, v) = \sqrt{2\xi} \int_{\mathbb{R}_+} f(\xi)\ell_k(2\xi v)e^{2\pi i u\sqrt{\xi}} \frac{d\xi}{\sqrt{\xi}} \tag{5} \]

admits the decomposition \( R_k^* = (\mathcal{F}^{-1} \otimes I)U_2^{-1} Q_k \) with \( \mathcal{F}^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) being the inverse Fourier transform.

The above representation of wavelet subspaces is especially important in the study of Toeplitz-type operators related to wavelets whose symbols depend only on vertical variable \( v = \Im \zeta \) in the upper half-plane \( \Pi \) of the complex plane \( \mathbb{C} \). For a given \( L^\infty(G, dv) \)-function \( a(\zeta) = a(v) \) depending only on \( v = \Im \zeta, \zeta \in G \), define the Calderón-Toeplitz operator \( T_a^{(k)} : A^{(k)} \rightarrow A^{(k)} \) with symbol \( a \) as

\[ T_a^{(k)} = P^{(k)} M_a, \]

where \( M_a \) is the operator of pointwise multiplication by \( a \) and \( P^{(k)} \) is the orthogonal projection from \( L^2(G, dv) \) to the wavelet subspace \( A^{(k)} \). Formally, these operators were introduced in [14] and further studied in general e.g. in papers [7], [10], [11] and [15]. The following important result, which enables to reduce Calderón-Toeplitz operator to a certain multiplication operator, was proved in [9]. In fact, it is the main tool of our study.
Theorem 2.4 Let \( a = a(v), \ v \in \mathbb{R}_+ \), be a measurable symbol on \( G \). Then the Calderón-Toeplitz operator \( T_a^{(k)} \) acting on \( A^{(k)} \) is unitarily equivalent to the multiplication operator
\[
\gamma_{a,k} I = R_k T_a^{(k)} R_k^* 
\]
acting on \( L_2(\mathbb{R}_+) \), where \( R_k \) and \( R_k^* \) are given by (4) and (5), respectively. The function \( \gamma_{a,k} \) is given by
\[
\gamma_{a,k}(\xi) = \int_{\mathbb{R}_+} a \left( \frac{v}{2\xi} \right) \ell^2_k(v) \, dv, \quad \xi \in \mathbb{R}_+. 
\]

Note that all the above results and definitions may be stated analogously for the space \( \overline{A}^{(k)} \). In what follows we restrict our attention only to wavelet subspaces \( A^{(k)} \) and operators \( T_a^{(k)} \) acting on them. For more results on properties of Calderón-Toeplitz operators \( T_a^{(k)} \) with symbols \( a = a(v) \) and properties of the corresponding function \( \gamma_{a,k}(\xi) \) responsible for many interesting features of these operators and their algebras, see the recent paper [9].

3 Boundedness of Calderón-Toeplitz operator

Clearly, if \( a = a(v) \) is a bounded symbol on \( G \), then the operator \( T_a^{(k)} \) is bounded on \( A^{(k)} \), and for its operator norm holds
\[
\| T_a^{(k)} \| \leq \text{ess-sup} |a(v)|.
\]
Thus all spaces \( A^{(k)}, \ k \in \mathbb{Z}_+, \) are naturally appropriate for Calderón-Toeplitz operators with bounded symbols. However, we may observe that the result of Theorem 2.4 suggests considering not only \( L_\infty(G, d\nu) \)-symbols, but also unbounded ones. In this case, we obviously have

Corollary 3.1 Calderón-Toeplitz operator \( T_a^{(k)} \) with a measurable symbol \( a = a(v), \ v \in \mathbb{R}_+ \), is bounded on \( A^{(k)} \) if and only if the function \( \gamma_{a,k}(\xi) \) is bounded on \( \mathbb{R}_+ \), and
\[
\| T_a^{(k)} \| = \sup_{\xi \in \mathbb{R}_+} |\gamma_{a,k}(\xi)|.
\]

From this result, we immediately have that the Calderón-Toeplitz operator \( T_a^{(1)} \) with unbounded symbol
\[
a(v) = \frac{1}{\sqrt{v}} \sin \frac{1}{v}, \quad v \in \mathbb{R}_+, 
\]
is bounded on \( A^{(1)} \) because the corresponding function
\[
\gamma_{a,1}(\xi) = \frac{\sqrt{2\pi}}{4} e^{-2\sqrt{\xi}} \left[ (2\sqrt{\xi} - 8\xi) \frac{\cos 2\sqrt{\xi}}{2\sqrt{\xi}} + (3 - 2\sqrt{\xi}) \frac{\sin 2\sqrt{\xi}}{2\sqrt{\xi}} \right], \quad \xi \in \mathbb{R}_+,
\]
is bounded, see [9, Example 4.4]. However, due to computational limitations (to find an explicit form of the function \( \gamma_{a,k}(\xi) \)) we can not say anything about the boundedness of \( T_a^{(k)} \) for arbitrary \( k \). Fortunately, according to Theorem 5.5 we will be able to show much more for a more general class of unbounded symbols including that symbol given by (7).
Example 3.2 For oscillating symbol $a(v) = e^{2\pi i v}$ (with $i^2 = -1$) we have

$$
\gamma_{a,k}(\xi) = \frac{(-1)^k}{(\xi - 1)^{2k+1}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{2j+1}, \quad \xi \in \mathbb{R},
$$

see [9, Example 4.5]. Thus, the Calderón-Toeplitz operator $T_a^{(k)}$ acting on $A^{(k)}$ is bounded for each $k \in \mathbb{Z}_+$, and moreover $\gamma_{a,k}(\xi) \in C[0, +\infty]$.

In both the above mentioned examples (more precisely, in the first one just for the case $k = 1$, but we will show later that also for all $k \in \mathbb{Z}_+$) we are in situation that the Calderón-Toeplitz operator $T_a^{(k)}$ belongs to the C*-algebra $T_k \left( L^{(0, +\infty)}(\mathbb{R}_+) \right)$ generated by (bounded) Calderón-Toeplitz operators with $L_\infty(\mathbb{R}_+)$-symbols having limits at the points 0 and $+\infty$, cf. [9, Section 4]. Recall that the Calderón-Toeplitz operator $T_a^{(k)}$ with a symbol $a = a(v)$ belongs to the algebra $T_k \left( L^{(0, +\infty)}(\mathbb{R}_+) \right)$ if and only if the corresponding function $\gamma_{a,k}(\xi)$ belongs to $C[0, +\infty]$. This means that the algebra $T_k \left( L^{(0, +\infty)}(\mathbb{R}_+) \right)$ contains many more Calderón-Toeplitz operators than was described in [9], because it also contains (bounded) Calderón-Toeplitz operators whose (generally unbounded) symbols $a(v)$ need not have limits at the endpoints 0 and $+\infty$.

Example 3.3 An easy example of unbounded symbol, for which the Calderón-Toeplitz operator is unbounded for each $k \in \mathbb{Z}_+$, is the function $a = a(v) = v^p$ with $\Re p > -1$ and $\Re p \neq 0$. The explicit formula for $\gamma_{a,k}$ has the form

$$
\gamma_{a,k}(\xi) = \int_{\mathbb{R}_+} a \left( \frac{v}{2\xi} \right) \ell_k^2(v) dv = \frac{1}{(2\xi)^p} \int_{\mathbb{R}_+} A^{(0,0)}_{p,k,k}(v) dv, \quad \xi \in \mathbb{R}.
$$

Since by the formula (3) the last integral is a constant, the function $\gamma_{a,k}(\xi)$ is clearly unbounded on $\mathbb{R}_+$. Thus the operator $T_a^{(k)}$ is not bounded on $A^{(k)}$ and does not belong to the algebra $T_k \left( L^{(0, +\infty)}(\mathbb{R}_+) \right)$. This case of symbols is subsumed in Theorem 3.11.

As we have shown in [9, Theorem 4.2], the behavior of a bounded function $a(v)$ near the point 0, or $+\infty$ determines the behavior of function $\gamma_{a,k}(\xi)$ near the point $+\infty$, or 0, respectively. The existence of limits of $a(v)$ in these endpoints guarantees the continuity of $\gamma_{a,k}(\xi)$ on the whole $\mathbb{R}_+$, however this condition is not necessary even for bounded symbols, see [9, Remark 4.3]. Continuity of function $\gamma_{a,k}$ on the whole $\mathbb{R}_+$ then guarantees its boundedness, and therefore by Corollary 3.1 the boundedness of the corresponding Calderón-Toeplitz operator $T_a^{(k)}$ on wavelet subspace $A^{(k)}$.

However, as example of symbol (7) shows (we will do it exactly and more generally in Example 3.8) the Calderón-Toeplitz operator $T_a^{(k)}$ can be bounded and belong to the algebra $T_k \left( L^{(0, +\infty)}(\mathbb{R}_+) \right)$ for each $k \in \mathbb{Z}_+$ even for unbounded symbols $a = a(v)$. Thus, in what follows we study this phenomena in more detail and will be interested in unbounded symbols to have a sufficiently large class of them common to all admissible $k$. For this purpose denote by $L_1(\mathbb{R}_+, 0)$ the class of functions $a = a(v)$ such that

$$
a(v)e^{-\varepsilon v} \in L_1(\mathbb{R}_+), \quad \text{for any } \varepsilon > 0.
$$
We give some conditions on the behavior of $L_1(\mathbb{R}_+,0)$-symbols (in fact on the behavior of certain means of these symbols) which guarantees the boundedness of function $\gamma_{a,k}(\xi)$.

**Remark 3.4** Using several formulas, more precisely [4, formula 8.976.3], [13, formula (5), p. 209] and [4, formula 8.976.1], we may rewrite the function $\gamma_{a,k}(\xi)$ as follows

$$\gamma_{a,k}(\xi) = \frac{1}{2^{2k+1}} \sum_{i=0}^{k} \sum_{j=0}^{2k} \sum_{r=0}^{j} \left( \binom{2k-2i}{k-i} \binom{2i}{j} \binom{2j}{i} \frac{(-1)^r}{r!} (1-4\xi)^{2r-j} (4\xi)^{j+1} I_r(\xi), \right.$$} \[ \text{where} \]

$$I_r(\xi) = \int_{\mathbb{R}_+} a(v) v^r e^{-2v\xi} dv.$$ 

Then the last integral is, in fact, the integral in the formula of function $\gamma_{a,\lambda}(x)$ for Toeplitz operators on the upper half-plane (the so-called parabolic case), see [17, formula (13.1.1), p. 329], or [5, formula (2.6)]. Therefore it would be natural to consider certain means of symbols depending on parameter $k$ as it is done therein, but we will not do it this way and our means of symbols will not depend on weight parameter.

For any $L_1(\mathbb{R}_+,0)$-symbol $a(v)$ define the following averaging functions

$$C_a^{(1)}(v) = \int_0^v a(t) dt,$$

$$C_a^{(m)}(v) = \int_0^v C_a^{(m-1)}(t) dt, \quad m = 2, 3, \ldots$$

The functions $C_a^{(m)}$ constitute a "sequence of iterated integrals" of symbol $a$.

**Theorem 3.5** Let $a = a(v) \in L_1(\mathbb{R}_+,0)$ and for any $m \in \mathbb{N}$ suppose that the function $C_a^{(m)}$ has the following asymptotic behavior

$$C_a^{(m)}(v) = O(v^m), \quad \text{as} \quad v \to 0, \quad (8)$$

and

$$C_a^{(m)}(v) = O(v^m), \quad \text{as} \quad v \to +\infty. \quad (9)$$

Then for each $k \in \mathbb{Z}_+$ we have

$$\sup_{\xi \in \mathbb{R}_+} |\gamma_{a,k}(\xi)| < +\infty.$$ 

Consequently, the corresponding Calderón-Toeplitz operator $T_a^{(k)}$ is bounded on $A^{(k)}$ for every $k \in \mathbb{Z}_+$.

**Proof.** Let $m \geq 1$. The condition (8) together with the condition (9) imply that for all $v \in \mathbb{R}_+$ the estimate

$$|C_a^{(m)}(v)| \leq \text{const} \, v^m \quad (10)$$

8
holds, where "const" does not depend on \( v \in \mathbb{R}_+ \). Integrating by parts \( m \)-times we have for all \( \xi \in \mathbb{R}_+ \) that

\[
\gamma_{a,k}(\xi) = 2\xi \int_{\mathbb{R}_+} e^{2\xi v} e^{-2\xi v} |C_a^{(m)}(v)| e^{-2\xi v} |L_{k-i-j}^{(i-j)}(2\xi v) L_{k-j}^{(j)}(2\xi v)| \, dv
\]

\[
= (-1)^m 2\xi \int_{\mathbb{R}_+} \frac{d^m}{dv^m} e^{2\xi v} e^{-2\xi v} |C_a^{(m+1)}(v)| \, dv.
\]

Using the estimates (10) and (2) we then get

\[
|\gamma_{a,k}(\xi)| \leq (2\xi)^{m+1} \sum_{i=0}^m \sum_{j=0}^i \left( \binom{m}{i} \binom{i}{j} \right) \int_{\mathbb{R}_+} |C_a^{(m)}(v)| e^{-2\xi v} |L_{k-i-j}^{(i-j)}(2\xi v) L_{k-j}^{(j)}(2\xi v)| \, dv
\]

\[
\leq \text{const} \sum_{i=0}^m \sum_{j=0}^i \left( \binom{m}{i} \binom{i}{j} \right) \int_{\mathbb{R}_+} (2\xi v)^m e^{-2\xi v} |L_{k-i-j}^{(i-j)}(2\xi v) L_{k-j}^{(j)}(2\xi v)| 2\xi \, dv
\]

\[
= \text{const} \sum_{i=0}^m \sum_{j=0}^i \left( \binom{m}{i} \binom{i}{j} \right) \int_{\mathbb{R}_+} x^{(i-j)}(x) \, dx
\]

\[
\leq \text{const} \sum_{i=0}^m \sum_{j=0}^i \left( \binom{m}{i} \binom{i}{j} \right) \text{const} \leq \text{const},
\]

thus the function \( \gamma_{a,k} \) is bounded for each \( k \in \mathbb{Z}_+ \), and by Corollary 3.4 the Calderón-Toeplitz operator \( T_a^{(b)} \) is bounded as well. \( \square \)

**Remark 3.6** The condition (8) guarantees the boundedness of the function \( \gamma_{a,k}(\xi) \) at a neighborhood of \( \xi = +\infty \), while the condition (9) guarantees its boundedness at a neighborhood of \( \xi = 0 \). Observe that if the conditions (8) and (9) hold for some \( m = m_0 \), then according to (10) hold also for \( m = m_0 + 1 \). Indeed,

\[
|C_a^{(m+1)}(v)| \leq \int_0^v |C_a^{(m)}(t)| \, dt \leq \text{const} \int_0^v t^{m_0} \, dt \leq \text{const} v^{m_0+1}.
\]

The main advantage of Theorem 3.5 is that we need not have an explicit form of the corresponding function \( \gamma_{a,k} \) for an unbounded symbol \( a = a(v) \) to decide about its boundedness, as it is e.g. in Example 3.2.

**Theorem 3.7** Let \( a = a(v) \in L_1(\mathbb{R}_+, 0) \). If for any \( m, n \in \mathbb{N} \), any \( \lambda_1 \in \mathbb{R}_+ \) and any \( \lambda_2 \in (0, n + 1) \) holds

\[
C_a^{(m)}(v) = O \left( v^{m+\lambda_1} \right), \quad v \to 0,
\]
and
\[ C_a^{(n)}(v) = \mathcal{O}(v^{n-\lambda_2}), \quad \text{as} \quad v \to +\infty, \]  \hspace{1cm} (12)

then for each \( k \in \mathbb{Z}_+ \) we have
\[ \lim_{\xi \to +\infty} \gamma_{a,k}(\xi) = 0 = \lim_{\xi \to 0} \gamma_{a,k}(\xi). \]

**Proof.** Analogously as in the proof of Theorem 3.5 using the condition (11) we have
\[ |\gamma_{a,k}(\xi)| \leq \frac{1}{(2\xi)^{m+1}} \text{const} \sum_{i=0}^{m} \sum_{j=0}^{i} \binom{m}{i} \binom{i}{j} \int_{\mathbb{R}_+} \Lambda_{m+\lambda_1,k-i+j,k-j}(x) \, dx, \]
where "const" does not depend on \( v \in \mathbb{R}_+ \). Thus, \( \lim_{\xi \to +\infty} \gamma_{a,k}(\xi) = 0 \).

Similarly, integrating by parts \( n \)-times and using the condition (12) we get
\[ |\gamma_{a,k}(\xi)| \leq (2\xi)^{\lambda_2} \text{const} \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} \int_{\mathbb{R}_+} \Lambda_{n-\lambda_2,k-i+j,k-j}(x) \, dx, \]
where (the different) "const" does not depend on \( v \in \mathbb{R}_+ \). Letting \( \xi \to 0 \) we have again the desired result. \( \square \)

In other words, Theorem 3.7 gives the condition on the behavior of \( L_1(\mathbb{R}_+,0) \)-symbols such that the function \( \gamma_{a,k}(\xi) \in C[0, +\infty] \), and thus the corresponding Calderón-Toeplitz operator \( T_a^{(k)} \) belongs to the algebra \( \mathcal{T}_k \left( L^{(0,+)\infty}(\mathbb{R}_+) \right) \). In the next example we present a wide class of oscillating symbols \( a = a(v) \) for which the Calderón-Toeplitz operator \( T_a^{(k)} \) belongs to \( \mathcal{T}_k \left( L^{(0,+)\infty}(\mathbb{R}_+) \right) \) for each \( k \in \mathbb{Z}_+ \).

**Example 3.8** For \( \alpha > 0 \) and \( \beta \in (0, 1) \) consider the unbounded symbol
\[ a(v) = v^{-\beta} \sin v^{-\alpha}, \quad v \in \mathbb{R}_+. \]

However, the function \( a(v) \) is continuous at \( v = +\infty \) for all admissible values of parameters, and therefore \( \gamma_{a,k}(0) = a(+\infty) = 0 \). On the other side, it is difficult to verify the behavior of function \( \gamma_{a,k}(\xi) \) at the endpoint \( +\infty \) by a direct computation. According to [17, Example 13.1.4] we have
\[ C_a^{(1)}(v) = \frac{v^{\alpha-\beta+1}}{\alpha} \cos v^{-\alpha} + \mathcal{O}(v^{2\alpha-\beta+1}), \quad \text{as} \quad v \to 0. \]

From it follows that for \( \alpha > \beta \) the first condition in (11) holds for \( m = 1 \) and \( \lambda_1 = \alpha - \beta \). By Theorem 3.7 the function \( \gamma_{a,k}(\xi) \) is bounded, and therefore the corresponding Calderón-Toeplitz operator \( T_a^{(k)} \) is bounded for each \( k \in \mathbb{Z}_+ \). Observe that this is in accordance with the obtained result of a special case for the symbol (7) and \( k = 1 \). Here we have extended it for much more general class of unbounded symbols and the whole range of parameters \( k \).

If \( \alpha \leq \beta \), then
\[ C_a^{(m)}(v) = \mathcal{O}(v^{m\alpha-\beta+m}), \quad \text{as} \quad v \to 0. \]
Thus for each \( \alpha \leq \beta \) there exists \( m_0 \in \mathbb{N} \) such that \( m_0 \alpha > \beta \), and therefore the first condition in (11) holds for \( m = m_0 \) and \( \lambda_1 = m_0 \alpha - \beta \), which guarantees that \( \gamma_{a,k}(\xi) \) is continuous at \( \xi = 0 \). Since for all parameters \( \alpha > 0 \) and \( \beta \in (0,1) \) the function \( \gamma_{a,k}(\xi) \in C[0, +\infty) \), then each Calderón-Toeplitz operator \( T^{(k)}_a \) belongs to the algebra \( \mathcal{T}_k \left( L^{(0, +\infty)}_{\infty}(\mathbb{R}_+) \right) \) for each \( k \in \mathbb{Z}_+ \).

In fact, Theorem 3.10 partially extends the result [9, Theorem 4.2] stated for bounded symbols to certain unbounded ones. In Example 3.2 and Example 3.8 we have provided such oscillating symbols \( a = a(v) \) for which the Calderón-Toeplitz operator \( T^{(k)}_a \) belongs to the algebra \( \mathcal{T}_k \left( L^{(0, +\infty)}_{\infty}(\mathbb{R}_+) \right) \) for the whole range of parameters \( k \). Now we give an example of a bounded oscillating symbol such that the bounded operator \( T^{(k)}_a \) does not belong to the algebra \( \mathcal{T}_k \left( L^{(0, +\infty)}_{\infty}(\mathbb{R}_+) \right) \).

Example 3.9 The function

\[
a(v) = v^i = e^{i \ln v}, \quad v \in \mathbb{R}_+,
\]

is oscillating near the endpoints 0 and +\( \infty \), but it is bounded on \( \mathbb{R}_+ \), and therefore the Calderón-Toeplitz operator \( T^{(k)}_a \) is bounded for each \( k \in \mathbb{R}_+ \).

Changing the variable \( x = 2v \xi \) yields

\[
\gamma_{a,k}(\xi) = 2\xi \int_{\mathbb{R}_+} v^i v^i_k(2v \xi) dv = (2\xi)^{-1} \int_{\mathbb{R}_+} x^i v^i_k(x) dx
\]

\[
= (2\xi)^{-1} \int_{\mathbb{R}_+} A_{i,k,k}(x) dx.
\]

Since by the formula (3) the last integral is a constant depending on \( k \), the function \( \gamma_{a,k}(\xi) \) oscillates and has no limit when \( \xi \to 0 \) as well as when \( \xi \to +\infty \). Thus the bounded Calderón-Toeplitz operator \( T^{(k)}_a \) does not belong to the algebra \( \mathcal{T}_k \left( L^{(0, +\infty)}_{\infty}(\mathbb{R}_+) \right) \). Hence not all oscillating symbols (even bounded and continuous) generate an operator from \( \mathcal{T}_k \left( L^{(0, +\infty)}_{\infty}(\mathbb{R}_+) \right) \).

In the following theorem we show that the boundedness of a Toeplitz operator on the Bergman space with non-negativity of symbol or its means guarantees the boundedness of Calderón-Toeplitz operator on each wavelet subspace as well.

Theorem 3.10 (i) Let \( a = a(v) \in L_1(\mathbb{R}_+, 0) \) be non-negative almost everywhere. If \( T^{(0)}_a \) is bounded on \( A^{(0)} \), then the operator \( T^{(k)}_a \) is bounded on \( A^{(k)} \) for each \( k \in \mathbb{Z}_+ \).

(ii) Let \( C^{(m)}_a \) be non-negative almost everywhere for a certain \( m = m_0 \). If \( T^{(0)}_a \) is bounded on \( A^{(0)} \), then the operator \( T^{(k)}_a \) is bounded on \( A^{(k)} \) for each \( k \in \mathbb{Z}_+ \).

Proof. (i) From assumptions we have

\[
\gamma_{a,0}(\xi) = 2\xi \int_{\mathbb{R}_+} a(v)e^{-2v \xi} dv \geq 2\xi \int_0^{(2\xi)^{-1}} a(v)e^{-2v \xi} dv \geq \frac{2\xi}{e} C^{(1)}_a((2\xi)^{-1}).
\]

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Putting $(2\xi)^{-1} = v$ we get
\[
C_a^{(1)}(v) \leq \left( e \sup_{\xi \in \mathbb{R}_+} |\gamma_{a,0}(\xi)| \right) v = \text{const},
\]
which by Theorem 3.5 means that $T_a^{(k)}$ is bounded on $A^{(k)}$ for each $k \in \mathbb{Z}_+$.

(ii) Integrating by parts $m_0$-times we obtain
\[
\gamma_{a,0}(\xi) = (-1)^{m_0} 2\xi \int_{\mathbb{R}_+} C_a^{(m_0)}(v) \frac{d^{m_0}v}{dv^{m_0}} e^{-2v\xi} dv = (2\xi)^{m_0+1} \int_{\mathbb{R}_+} C_a^{(m_0)}(v)e^{-2v\xi} dv \geq (2\xi)^{m_0+1} e^{-1} C_a^{(m_0+1)}((2\xi)^{-1}).
\]
Again putting $(2\xi)^{-1} = v$ we have
\[
C_a^{(m_0+1)}(v) \leq \left( e \sup_{\xi \in \mathbb{R}_+} |\gamma_{a,0}(\xi)| \right) v^{m_0+1}
\]
and by Theorem 3.5 the boundedness of $T_a^{(k)}$ on $A^{(k)}$ for each $k \in \mathbb{Z}_+$ follows. □

According to the presented examples an unbounded symbol must have a sufficiently sophisticated oscillating behavior at neighborhoods of the points 0 and $+\infty$ to generate a bounded Calderón-Toeplitz operator. In what follows we show that infinitely growing positive symbols (as in the case of identity, or its powers) cannot generate bounded Calderón-Toeplitz operators in general. For this purpose for a non-negative function $a = a(v)$ put
\[
\theta_a(v) = \inf_{t \in (0,v)} a(t) \quad \text{and} \quad \Theta_a(v) = \inf_{t \in (v/2,v)} a(t).
\]

Theorem 3.11 For a given non-negative symbol $a = a(v)$ if either
\[
\lim_{v \to 0} \theta_a(v) = +\infty \quad (13)
\]
or
\[
\lim_{v \to +\infty} \Theta_a(v) = +\infty, \quad (14)
\]
then the Calderón-Toeplitz operator $T_a^{(k)}$ is unbounded on each $A^{(k)}$, $k \in \mathbb{Z}_+$.

Proof. If the condition (13) holds, then
\[
C_a^{(1)}(v) = \int_0^v a(t) dt \geq v \theta_a(v),
\]
which yields $v^{-1} C_a^{(1)}(v) \to +\infty$, as $v \to 0$. 

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If the condition (14) holds, then
\[
v^{-1}C_a^{(1)}(v) > v^{-1} \int_{v/2}^{v} a(t) \, dt \geq \frac{1}{2} \Theta_a(v),
\]
which again yields \(v^{-1}C_a^{(1)}(v) \to +\infty\), as \(v \to +\infty\).

Example 3.12 For the family of non-negative symbols on \(\mathbb{R}_+\) in the form
\[
a(v) = v^{-\beta} \ln^2 v^{-\alpha}, \quad \beta \in [0, 1], \quad \alpha > 0,
\]
we have that for all admissible parameters holds \(\lim_{v \to 0} \theta_a(v) = +\infty\), and thus by Theorem 3.11 the Calderón-Toeplitz operator \(T_a^{(k)}\) is unbounded on \(A^{(k)}\) for each \(k \in \mathbb{Z}_+\).

In the following example we use the result of Theorem 3.11 to study boundedness of a Calderón-Toeplitz operator with unbounded symbol as a product of two symbols for which the corresponding Calderón-Toeplitz operators are bounded on each wavelet subspace.

Example 3.13 Let us consider two symbols on \(\mathbb{R}_+\) in the form
\[
a(v) = v^{-\beta} \sin v^{-\alpha}, \quad \beta \in (0, 1), \quad \alpha \geq \beta,
\]
and
\[
b(v) = v^\tau \sin v^{-\alpha}, \quad \tau \in (0, \beta).
\]
According to Example 3.8 for the unbounded symbol \(a(v)\) the Calderón-Toeplitz operator \(T_a^{(k)}\) is bounded for each \(k \in \mathbb{Z}_+\). Since the symbol \(b(v) \in C[0, +\infty]\), then the Calderón-Toeplitz operator \(T_b^{(k)}\) is bounded for each \(k \in \mathbb{Z}_+\) as well. Put
\[
c(v) = a(v)b(v) = v^{-\delta} - \frac{v^{-\delta}}{2} \cos 2v^{-\alpha} = c_1(v) + c_2(v),
\]
where \(\delta = \beta - \tau \in (0, 1)\). Clearly, \(c(v)\) is an unbounded symbol. However, \(T_{c_2}^{(k)}\) is bounded for each \(k \in \mathbb{Z}_+\) (analogously as in Example 3.8 replacing sin by cos). Since
\[
\theta_{c_1}(v) = \inf_{t \in (0, v)} \frac{1}{2t} = \frac{1}{2v^\delta} \to +\infty, \quad {v \to 0},
\]
then by Theorem 3.11 the Calderón-Toeplitz operator \(T_{c_1}^{(k)}\) is unbounded for each \(k \in \mathbb{Z}_+\). Thus, the Calderón-Toeplitz operator \(T_{ab}^{(k)}\) is unbounded on \(A^{(k)}\) for each \(k \in \mathbb{Z}_+\). Moreover, this result shows that the semi-commutator
\[
\left[ T_a^{(k)}, T_b^{(k)} \right] = T_a^{(k)}T_b^{(k)} - T_{ab}^{(k)}
\]
is not compact. This interesting feature and the algebras of Calderón-Toeplitz operators will be considered elsewhere.
4 Calderón-Toeplitz operators with unbounded symbols as uniform limits of Calderón-Toeplitz operators with bounded symbols

In connection with the obtained results we now show how Calderón-Toeplitz operators with unbounded symbols given in Example 3.8 can appear as uniform limits of Calderón-Toeplitz operators with bounded symbols.

Theorem 4.1 The Calderón-Toeplitz operator $T^{(k)}_a$ with symbol

$$a(v) = v^{-\beta} \sin v^{-\alpha}, \quad v \in \mathbb{R}_+,$$

where $\beta \in (0,1)$ and $\alpha > \beta$, belongs to the $C^*$-algebra generated by Calderón-Toeplitz operators with smooth bounded symbols on each wavelet subspace $A^{(k)}$.

Proof. Consider the sequence $\vartheta_n = (\pi n)^{-1/\alpha}$, $n \in \mathbb{N}$, of zeros of function $a = a(v)$ and define the sequence $a_n(v) = \begin{cases} a(v), & v \in [\vartheta_n, +\infty), \\ 0, & v \in [0, \vartheta_n). \end{cases}$

Each symbol $a_n(v)$ is bounded and continuous. Further each $a_n(v)$ can be uniformly approximated by smooth symbols, and thus belongs to the $C^*$-algebra generated by Calderón-Toeplitz operators with smooth bounded symbols. According to Theorem 2.4 the Calderón-Toeplitz operator $T^{(k)}_a$ acting on $A^{(k)}$ is unitarily equivalent to the multiplication operator $\gamma_{a,k} I$ acting on $L^2(\mathbb{R}_+)$, where the function $\gamma_{a,k}$ is given by (6). Thus,

$$\|T^{(k)}_a - T^{(k)}_{a_n}\| = \|T^{(k)}_{a_n} - T^{(k)}_{a - a_n}\| = \sup_{\xi \in \mathbb{R}_+} |\gamma_{(a - a_n),k}(\xi)|$$

$$= \sup_{\xi \in \mathbb{R}_+} \left| 2\xi \int_0^{\vartheta_n} a(v) \ell_k^2(2v\xi) \, dv \right|$$

$$= \sup_{\xi \in \mathbb{R}_+} \left| 2\xi C_a^{(1)}(\vartheta_n) \ell_k^2(2\vartheta_n\xi) + 4\xi^2 \int_0^{\vartheta_n} C_a^{(1)}(v) \ell_k^2(2v\xi) \, dv \right. + 8\xi^2 \int_0^{\vartheta_n} C_a^{(1)}(v) e^{-2v\xi} L_k(2v\xi) L_k^{(1)}(2v\xi) \, dv \right|,$$

where integration by parts has been used. Since

$$C_a^{(1)}(v) = \int_0^v a(t) \, dt = \frac{v^{\alpha - \beta + 1}}{\alpha} \cos v^{-\alpha} + \mathcal{O}(v^{2\alpha - \beta + 1}), \quad \text{as} \quad v \to 0,$$

see Example 3.8 then

$$|C_a^{(1)}(v)| \leq \text{const} \, v^{\alpha - \beta + 1},$$

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where "const" does not depend on \( v \in (0, 1) \), and thus the Calderón-Toeplitz operator \( T_a^{(k)} \) is bounded on \( A^{(k)} \) for each \( k \in \mathbb{Z}_+ \). Then

\[
\|T_{a-n}^{(k)}\| \leq \text{const} \sup_{\xi \in \mathbb{R}_+} \left( 2\xi |C_a^{(1)}(\vartheta_n)|\ell_2^2(2\vartheta_n) + 4\xi^2 \int_0^{\vartheta_n} |C_a^{(1)}(v)|\ell_2^2(2v) \, dv \right) \\
+ 8\xi^2 \int_0^{\vartheta_n} |C_a^{(1)}(v)|e^{-2v\xi}|L_k(2v\xi)L_{k-1}^{(1)}(2v\xi)| \, dv \\
\leq \text{const} \sup_{\xi \in \mathbb{R}_+} \vartheta_n^{\alpha-\beta}(2\vartheta_n)\ell_2^2(2\vartheta_n) \\
+ \text{const} \sup_{\xi \in \mathbb{R}_+} 4\xi^2 \int_0^{\vartheta_n} v^{\alpha-\beta+1}\ell_2^2(2v\xi) \, dv \\
+ \text{const} \sup_{\xi \in \mathbb{R}_+} 8\xi^2 \int_0^{\vartheta_n} v^{\alpha-\beta+1}e^{-2v\xi}|L_k(2v\xi)L_{k-1}^{(1)}(2v\xi)| \, dv \\
= I_1 + I_2 + I_3.
\]

Since \( \sup_{\xi \in \mathbb{R}_+} (2\vartheta_n)\ell_2^2(2\vartheta_n) < +\infty \), then \( I_1 \leq q_1(k)\vartheta_n^{\alpha-\beta} \). To evaluate \( I_2 \) we use the estimate (2), and we have

\[
I_2 = \text{const} \sup_{\xi \in \mathbb{R}_+} \int_0^{\vartheta_n} v^{\alpha-\beta}(2v\xi)\ell_2^2(2v\xi) \, dv \\
\leq \text{const} \vartheta_n^{\alpha-\beta} \sup_{\xi \in \mathbb{R}_+} \int_0^{\vartheta_n} (2v\xi)\ell_2^2(2v\xi) \, dv \\
= \text{const} \vartheta_n^{\alpha-\beta} \sup_{\xi \in \mathbb{R}_+} \int_0^{2\vartheta_n\xi} x\ell_2^2(x) \, dx \\
\leq \text{const} \vartheta_n^{\alpha-\beta} \int_{\mathbb{R}_+} \Lambda_{0,0}^{(0)}(x) \, dx \\
\leq q_2(k)\vartheta_n^{\alpha-\beta}.
\]

Similarly for \( I_3 \) we get

\[
I_3 = 2\text{const} \sup_{\xi \in \mathbb{R}_+} \int_0^{\vartheta_n} v^{\alpha-\beta}(2v\xi)e^{-2v\xi}|L_k(2v\xi)L_{k-1}^{(1)}(2v\xi)| \, dv \\
\leq 2\text{const} \vartheta_n^{\alpha-\beta} \sup_{\xi \in \mathbb{R}_+} \int_0^{\vartheta_n} (2v\xi)e^{-2v\xi}|L_k(2v\xi)L_{k-1}^{(1)}(2v\xi)| \, dv \\
= 2\text{const} \vartheta_n^{\alpha-\beta} \sup_{\xi \in \mathbb{R}_+} \int_0^{2\vartheta_n\xi} xe^{-x}|L_k(x)L_{k-1}^{(1)}(x)| \, dx \\
\leq 2\text{const} \vartheta_n^{\alpha-\beta} \int_{\mathbb{R}_+} \Lambda_{0,1}^{(0)}(x) \, dx \\
\leq q_3(k)\vartheta_n^{\alpha-\beta}.
\]

Thus,

\[
\|T_{a-n}^{(k)} - T_{a}^{(k)}\| \leq q(k)\vartheta_n^{\alpha-\beta},
\]

where the constant \( q(k) \) depends on \( k \), but does not depend on \( n \), and \( \vartheta_n \to 0 \) whenever \( n \to +\infty \). \( \Box \)
It seems to be natural to ask whether the boundedness of Calderón-Toeplitz operator (and by Corollary 3.1 the boundedness of corresponding function $\gamma$) is equivalent to the boundedness of its Wick symbol. According to the result of Nowak [10] it is true for non-negative symbols $a$ and sufficiently smooth wavelets. Thus, we immediately have the following result.

**Corollary 4.2** For a non-negative symbol $a = a(v)$ the following statements are equivalent:

(i) operator $T_{a}^{(k)}$ is bounded;

(ii) the function $\gamma_{a,k}$ is bounded;

(iii) the Wick symbol $\tilde{a}_{k}$ of $T_{a}^{(k)}$ is bounded.

In connection with it we also mention another Nowak’s result, cf. [10], for compactness of Calderón-Toeplitz operator: for a non-negative symbol the Calderón-Toeplitz operator is compact if and only if its Wick symbol tends to 0 at infinity. In our case of symbol $a = a(v)$ we are in a different situation because according to Theorem 2.4 the operator $T_{a}^{(k)}$ is unitarily equivalent to a multiplication operator, and thus never compact.

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