A LIOUVILLE-TYPE THEOREM FOR COOPERATIVE PARABOLIC SYSTEMS

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ABSTRACT. We prove Liouville-type theorem for semilinear parabolic system of the form

\[ \begin{align*}
    u_t - \Delta u &= a_{11} u^p + a_{12} u^r v^{s+1}, & (x,t) \in \Omega \times I, \\
    v_t - \Delta v &= a_{21} u^r v^s + a_{22} v^p, & (x,t) \in \Omega \times I,
\end{align*} \]

where \( r, s > 0, 
 p = r + s + 1 \). The real matrix \( A = (a_{ij}) \) satisfies conditions
\( a_{12}, a_{21} \geq 0 \) and \( a_{11}, a_{22} > 0 \). This paper is a continuation of Phan-Souplet \( (\text{Math. Ann.}, 366, 1561-1585, 2016) \) where the authors considered the special case \( s = r \) for the system of \( m \) components. Our tool for the proof of Liouville-type theorem is a refinement of Phan-Souplet, which is based on Gidas-Spruck \( (\text{Commun. Pure Appl.Math.} 34, 525–598 1981) \) and Bidaut-Véron \( (\text{Équations aux dérivées partielles et applications. Elsevier, Paris, pp 189–198, 1998}) \).

1. Introduction. In this article, we consider the semilinear parabolic system of the form

\[ \begin{align*}
    u_t - \Delta u &= a_{11} u^p + a_{12} u^r v^{s+1}, & (x,t) \in \Omega \times I, \\
    v_t - \Delta v &= a_{21} u^r v^s + a_{22} v^p, & (x,t) \in \Omega \times I,
\end{align*} \]

where \( r, s > 0, 
 p = r + s + 1 \), \( \Omega \) is a domain of \( \mathbb{R}^N \), and \( I \) is an interval of \( \mathbb{R} \). Throughout this paper, the real matrix \( A = (a_{ij}) \) is assumed to satisfy the following conditions
\( a_{12}, a_{21} \geq 0 \) and \( a_{11}, a_{22} > 0 \).

System \( (1) \) arises in different mathematical models in physics, chemistry and biology. It has been used to describe heat propagations in a two-component combustible mixture \([3, 9]\). In this case, \( u \) and \( v \) stand for the temperatures of the interacting components. In dynamics of biological groups \([6, 14]\), the system \( (1) \) models the interaction of two biological groups where the speed of the diffusion is slow. Furthermore, it can be used to describe some models of Bose-Einstein condensation \([7]\), or of chemical processes \([12]\).

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So far, system (1) has attracted much attention in various mathematical directions. For instance, the local and global existence was obtained via theory of abstract evolution equations (see e.g. [1]). The results on the regularity, symmetry property, or blow-up phenomena were considered in e.g. [7, 10, 9]. Recently, the singularity estimates and the Liouville-type theorems have been studied in [15, 18, 16].

The aim of this paper is to prove the nonexistence of nontrivial solution of problem (1) in the entire space, such a result is called the Liouville-type theorem. This is a continuation of [16] where the author prove the Liouville-type theorem in the special case \( s = r \) for the system of \( m \) components

\[
\partial_t u_i - \Delta u_i = \sum_{j=1}^{m} a_{ij} u^r v^{s+1}, \quad i = 1, 2, \ldots, m
\]

under the range of \( p < p_B \). Here, \( p_B \) is the Bidaut-Véron exponent

\[
p_B := \begin{cases} \frac{N(N+2)}{N-2} & \text{if } N \geq 2, \\ \infty & \text{if } N = 1. \end{cases}
\]

We propose to study the system of two components (1) and look for the difficulty arising on the nonexistence of solutions when the exponents \( r \neq s \). Before stating the main result, let us first recall the elliptic counterpart

\[
\begin{cases} -\Delta u = a_{11} u^p + a_{12} u^r v^{s+1}, & x \in \mathbb{R}^N, \\ -\Delta v = a_{21} u^{r+1} v^s + a_{22} v^p, & x \in \mathbb{R}^N, \end{cases}
\]

which arises in mathematical models of physical phenomena, such as nonlinear optics and Bose-Einstein condensation (see e.g. [2, 8, 21]). It is well known that the Liouville-type result for (3) plays an important role in the study of both elliptic and parabolic problems. The optimal Liouville-type theorem for problem (3) was completely proved in [20] (see also [13]) via moving sphere techniques under the optimal range \( p < p_S \), where \( p_S \) is Sobolev exponent

\[
p_S := \begin{cases} \frac{N+2}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2. \end{cases}
\]

For the corresponding parabolic problem (1), the Liouville property is less understood. First, by adding up the two equations and using Young’s inequality, one can easily reduce to a scalar parabolic inequality and deduce the Fujita-type result of problem (1), which asserts that, there is no positive solution in \( \mathbb{R}^N \times \mathbb{R}_+ \) if \( 1 < p \leq 1 + \frac{2}{N} \). The Liouville-type theorem for problem (1) under the restrictions \( a_{11} = a_{22} = 1, a_{12} = a_{21}, r = s \) has been proved in [15] for the class of radial solutions in any dimension. Recently, Quittner [18] has provided a new important technique to prove the Liouville-type theorem for parabolic systems with gradient structure, under the condition \( (N-2)p < N \). More precisely, the result of Quittner [18, Theorem 3] is formulated for parabolic system of the form \( U_t - \Delta U = F(U) \), where \( U = (u_1, u_2, \ldots, u_m) \) and \( F \) satisfies the conditions:

- \( F = \nabla G \) with \( G \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^m, \mathbb{R}) \) for some \( \alpha > 0 \),
- \( G(0) = 0, G(U) > 0 \) if \( U \neq 0 \),
- \( F(\lambda U) = \lambda^p F(U) \) for \( U \in \mathbb{R}^m, \lambda > 0 \),
- there exists \( \xi \in (0, \infty)^m \) such that \( \xi \cdot F(U) > 0 \) for \( U \neq 0 \).
**Theorem A** [Quittner [18]]. Assume either \( N \leq 2 \), or \( p < N/(N - 2) \) if \( N \geq 3 \). Then the system \( U_t - \Delta U = F(U) \) does not possess any nontrivial nonnegative classical solution in \( \mathbb{R}^N \times \mathbb{R} \).

In particular, the Liouville-type result of [18] is optimal in dimensions \( N \leq 2 \), and there is an additional condition \( p < \frac{N}{N - 2} \) in dimension \( N \geq 3 \). The main tools in [18] are scaling argument and energy estimates. We note that, by a simple scaling, one can reduce the system (1) to a parabolic system with gradient structure as in Theorem A. More recently, Phan and Souplet [16] have used a different approach to establish a Liouville-type theorem for problem (1) in a larger range of \( p \) in dimension \( N \geq 3 \), under the restrictions \( r = s \) and \( a_{12} = a_{21} \).

In this paper, we establish a Liouville-type theorem for problem (1) in general case which allows \( r \neq s \) and/or \( a_{12} \neq a_{21} \). Our main result is as follows.

**Theorem 1.1.** Assume that \( p < p_B \) if \( N \geq 3 \). Then, under the assumption (2), the system (1) does not possess any nontrivial nonnegative classical solution in \( \mathbb{R}^N \times \mathbb{R} \).

Due to the fact that \( \frac{N}{N - 2} < p_B \) when \( N \geq 3 \), our result is a partial improvement of that in Quittner [18] in high dimensions (as seen for system of two components). Our tool for the proof of Theorem 1.1 is a refinement of [16], which is essentially based on Gidas-Spruck technique [11] developed by Bidaut-Véron [4] (see also [5] for elliptic system). This technique consists of nonlinear integral estimates and Bochner formula. We remark that, for the Liouville-type theorem, the condition \( p < p_B \) is the best up-to-date one for parabolic problem in dimension \( N \geq 3 \), even for the simplest model \( u_t - \Delta u = u^p \).

We stress that the proof of Theorem 1.1 is not straightforward in comparison with that in [16]. The only main difficulty arising in this paper is the presence of different exponents \( r \) and \( s \). This makes the Gidas-Spruck and Bidaut-Véron techniques become more delicate. In addition, it requires a suitable combination of nonlinear integral estimates on each component, see the Lemma 2.1. The rest of the proof is similar to that in [16].

From Theorems 1.1, one can deduce the singularity estimates by rescaling and doubling arguments. We just give here the result without proof since it is totally similar to that in [17, Theorem 3.1].

**Proposition 1.** Assume that \( p < p_B \) if \( N \geq 3 \), \( \Omega \) is a domain of \( \mathbb{R}^N \), and (2). Let \((u, v)\) be a nonnegative solution of (1) in \( \Omega \times (0, T) \). Then for all \((x, t) \in \Omega \times (0, T)\), there holds

\[
(u + v)(x, t) \leq C(N, r, s, A) \left( t^{-1/(p-1)} + (T - t)^{-1/(p-1)} + \text{dist}^{-2/(p-1)}(x, \partial \Omega) \right). 
\]

(4)

We note that the condition \( p < p_B \) in Proposition 1 can be removed if the solution is radial and \( \Omega \) is a symmetric domain (see [15, Section 2]). By a symmetric domain, we mean either the whole space \( \mathbb{R}^N \), a ball \( B_R = B(0, R) \), an annulus \( \{ x \in \mathbb{R}^N : R_1 < |x| < R_2 \} \), or an exterior domain \( \{ x \in \mathbb{R}^N : |x| > R \} \).

The rest of this paper is devoted to the proof of Theorem 1.1.

2. **Proof of Liouville-type theorem.** For the sake of simplicity, we denote by \( \int \) the integral \( \int_{B_1 \times (-1, 1)} \) dxdt.
Lemma 2.1. Assume that $N \geq 3$, $p < p_B$ and (2). Let $0 \leq \varphi \in \mathcal{D}(B_1 \times (-1,1))$ and $(u,v)$ be a positive classical solution of (1) in $B_1 \times (-1,1)$. Denote

$$I_1 = \int \varphi u^{-2}|\nabla u|^4, \quad I_2 = \int \varphi v^{-2}|\nabla v|^4, \quad I = (r + 1)a_{21}I_1 + (s + 1)a_{12}I_2,$$

$$L_1 = \int \varphi (a_{11}u^p + a_{12}uv^{s+1})^2, \quad L_2 = \int \varphi (a_{21}u^{r+1}v^s + a_{22}v^p)^2,$$

$$L = (r + 1)a_{21}L_1 + (s + 1)a_{12}L_2.$$

Then there holds

$$I + L \leq C \int \varphi (|u_1|u^{-1}|
abla u|^2 + |v_1|v^{-1}|
abla v|^2)$$

$$+ C \int |\nabla \varphi \cdot \nabla u| (u^p + u^r v^{s+1} + |u| + u^{-1}|
abla u|^2)$$

$$+ C \int |\nabla \varphi \cdot \nabla v| (v^p + u^{r+1}v^s + |v| + v^{-1}|
abla v|^2)$$

$$+ C \int |\Delta \varphi| (|\nabla u|^2 + |\nabla v|^2) + C \int |u_1| (u^{p+1} + v^{p+1}) + C \int \varphi (u_1^2 + v_1^2),$$

where $C = C(N,r,s,A)$.

Proof. Denoting

$$J_1 = \int \varphi u^{-1}|
abla u|^2 \Delta u, \quad J_2 = \int \varphi v^{-1}|
abla v|^2 \Delta v,$$

$$J = (r + 1)a_{21}J_1 + (s + 1)a_{12}J_2,$$

$$K_1 = \int \varphi (\Delta u)^2, \quad K_2 = \int \varphi (\Delta v)^2, \quad K = (r + 1)a_{21}K_1 + (s + 1)a_{12}K_2.$$

Applying [19, Lemma 8.9] with $q = 0$, $-1 \neq k < 0$, we have two following inequalities

$$- \left( \frac{N-1}{N}k + 1 \right) kI_1 + \frac{N+2}{N}kJ_1 - \frac{N-1}{N}K_1$$

$$\leq \frac{1}{2} \int |\nabla u|^2 \Delta \varphi + \int (\Delta u - ku^{-1}|
abla u|^2) \nabla u \cdot \nabla \varphi,$$

$$- \left( \frac{N-1}{N}k + 1 \right) kI_2 + \frac{N+2}{N}kJ_2 - \frac{N-1}{N}K_2$$

$$\leq \frac{1}{2} \int |\nabla v|^2 \Delta \varphi + \int (\Delta v - kv^{-1}|
abla v|^2) \nabla v \cdot \nabla \varphi.$$

Hence,

$$- \left( \frac{N-1}{N}k + 1 \right) kI + \frac{N+2}{N}kJ - \frac{N-1}{N}K$$

$$\leq C \int (|\nabla u|^2 + |\nabla v|^2) |\Delta \varphi| + C \int (|\Delta u - ku^{-1}|
abla u|^2) \nabla u \cdot \nabla \varphi$$

$$+ C \int (|\Delta v - kv^{-1}|
abla v|^2) \nabla v \cdot \nabla \varphi.$$

(6)
Next, the integration by parts gives

\[
\int \varphi |\nabla u|^2 u^{r-1} v^{s+1} = \int \varphi v^{s+1} \nabla u \cdot \nabla \left( \frac{u^r}{r} \right)
= -\frac{1}{r} \int \varphi u^r v^{s+1} \Delta u + \frac{s+1}{r} \int u^r v^s \nabla u \cdot \nabla v
- \frac{1}{r} \int u^r v^{s+1} \nabla \varphi \cdot \nabla v, \tag{7}
\]

\[
\int \varphi |\nabla v|^2 u^{r+1} v^{s-1} = \int \varphi u^{r+1} v^s \nabla v \cdot \nabla \left( \frac{v}{s} \right)
= -\frac{1}{s} \int \varphi u^{r+1} v^s \Delta v + \frac{r+1}{s} \int u^r v^s \nabla u \cdot \nabla v
- \frac{1}{s} \int u^{r+1} v^s \nabla \varphi \cdot \nabla v. \tag{8}
\]

Set \( \lambda = \frac{(s+1)s}{(r+1)r} \), it follows from (7)-(8) that

\[
\int \varphi |\nabla u|^2 u^{r-1} v^{s+1} + \lambda \int \varphi |\nabla v|^2 u^{r+1} v^{s-1}
= -\frac{1}{r} \int \varphi u^r v^{s+1} \Delta u - \frac{\lambda}{s} \int \varphi u^{r+1} v^s \Delta v
- \frac{2(s+1)}{r} \int u^r v^s \nabla u \cdot \nabla v - \frac{1}{r} \int u^r v^{s+1} \nabla \varphi \cdot \nabla u
- \frac{1}{s} \int u^{r+1} v^s \nabla \varphi \cdot \nabla v. \tag{9}
\]

Using the Young inequality \( 2u^r v^s \nabla u \cdot \nabla v \leq |\nabla u|^2 u^{r-1} v^{s+1} + |\nabla v|^2 u^{r+1} v^{s-1} \), we deduce from (9) that,

\[
\left(1 + \frac{s+1}{r}\right) \int \varphi |\nabla u|^2 u^{r-1} v^{s+1} + \left(\lambda + \frac{s+1}{r}\right) \int \varphi |\nabla v|^2 u^{r+1} v^{s-1}
\geq -\frac{1}{r} \int \varphi u^r v^{s+1} \Delta u - \frac{\lambda}{s} \int \varphi u^{r+1} v^s \Delta v
- \frac{1}{r} \int u^r v^{s+1} \nabla \varphi \cdot \nabla u
- \frac{1}{s} \int u^{r+1} v^s \nabla \varphi \cdot \nabla v.
\]

Or,

\[
(r+1) \int \varphi |\nabla u|^2 u^{r-1} v^{s+1} + (s+1) \int \varphi |\nabla v|^2 u^{r+1} v^{s-1}
\geq \frac{-r+1}{p} \int \varphi u^r v^{s+1} \Delta u - \frac{s+1}{p} \int \varphi u^{r+1} v^s \Delta v
- \frac{r+1}{p} \int u^r v^{s+1} \nabla \varphi \cdot \nabla u
- \frac{s+1}{p} \int u^{r+1} v^s \nabla \varphi \cdot \nabla v. \tag{10}
\]

Next, the integration by parts gives

\[
\int \varphi u^{p-1} |\nabla u|^2 = \int \varphi u^{p-1} \nabla u \cdot \nabla \left( \frac{u^p}{p} \right)
= -\frac{1}{p} \int \varphi u^p \Delta u - \frac{1}{p} \int u^p \nabla \varphi \cdot \nabla u, \tag{11}
\]

\[
\int \varphi v^{p-1} |\nabla v|^2 = \int \varphi v^{p-1} \nabla v \cdot \nabla \left( \frac{v^p}{p} \right)
= -\frac{1}{p} \int \varphi v^p \Delta v - \frac{1}{p} \int v^p \nabla \varphi \cdot \nabla v. \tag{12}
\]
Using $-\Delta u = a_{11}u^p + a_{12}u^r v^{s+1} - u_t$, $-\Delta v = a_{21}u^{r+1}v^s + a_{22}v^p - v_t$, we compute $J$ as follows

$$
J = (r+1)a_{21} \int \varphi u^{-1} \nabla u^2 (-\Delta u) + (s+1)a_{12} \int \varphi v^{-1} \nabla v^2 (-\Delta v) \\
= (r+1)a_{21} \int \varphi a_{11}u^{p-1} \nabla u^2 + (s+1)a_{12} \int \varphi a_{22}v^{p-1} \nabla v^2 \\
+ a_{12}a_{21} \left[(r+1) \int \varphi |\nabla u|^2 u^{r-1}v^{s+1} + (s+1) \int \varphi |\nabla v|^2 u^{r+1}v^{s-1}\right] \\
- (r+1)a_{21} \int \varphi u^{-1} |\nabla u|^2 u_t - (s+1)a_{12} \int \varphi v^{-1} |\nabla v|^2 v_t. \tag{13}
$$

Taking into account the estimates (10)-(12), we deduce from (13) that,

$$
J \geq \frac{1}{p}(r+1)a_{21} \int \varphi (a_{11}u^p + a_{12}u^r v^{s+1})(-\Delta u) \\
+ \frac{1}{p}(s+1)a_{12} \int \varphi (a_{21}u^{r+1}v^s + a_{22}v^p)(-\Delta v) \\
- C \int \left( |\nabla u^{-1}| \nabla u^2 + |\nabla v^{-1}| \nabla v^2 \right) \\
- C \int \left( (u^p + u^r v^{s+1}) |\nabla \varphi \cdot \nabla u| + (v^p + u^{r+1} v^s) |\nabla \varphi \cdot \nabla v| \right). \tag{14}
$$

Again, substituting $-\Delta u = a_{11}u^p + a_{12}u^r v^{s+1} - u_t$, $-\Delta v = a_{21}u^{r+1}v^s + a_{22}v^p - v_t$ into (14) and integrating by parts in $t$, we have

$$
J \geq \frac{1}{p}(r+1)a_{21} \int \varphi (a_{11}u^p + a_{12}u^r v^{s+1})^2 \\
+ \frac{1}{p}(s+1)a_{12} \int \varphi (a_{21}u^{r+1}v^s + a_{22}v^p)^2 \\
- C \int \left( |\nabla u^{-1}| \nabla u^2 + |\nabla v^{-1}| \nabla v^2 \right) \\
- C \int \left( (u^p + u^r v^{s+1}) |\nabla \varphi \cdot \nabla u| + (v^p + u^{r+1} v^s) |\nabla \varphi \cdot \nabla v| \right) \\
- C \int |\varphi_t|(u^{p+1} + v^{p+1} + u^{r+1}v^{s+1}).
$$

Using the Young inequality $u^{r+1}v^{s+1} \leq C(u^{p+1} + v^{p+1})$ for the last term in the above estimate, we get

$$
J \geq \frac{1}{p}L - C \int \left( |\nabla u^{-1}| \nabla u^2 + |\nabla v^{-1}| \nabla v^2 \right) \\
- C \int \left( (u^p + u^r v^{s+1}) |\nabla \varphi \cdot \nabla u| + (v^p + u^{r+1} v^s) |\nabla \varphi \cdot \nabla v| \right) \\
- C \int |\varphi_t|(u^{p+1} + v^{p+1}). \tag{15}
$$

Next, we compute $K$ as

$$
K = (r+1)a_{21} \int \varphi (a_{11}u^p + a_{12}u^r v^{s+1} - u_t)^2 \\
+ (s+1)a_{12} \int \varphi (a_{21}u^{r+1}v^s + a_{22}v^p - v_t)^2
$$
where we have used again the Young inequality (17).

Lemma follows from (6) and (17).

\[
\phi \text{ the test-function }
\]

We follow the argument as in proof of [19, Proposition 21.5]. One can choose \(\varepsilon > 0\) by the following estimates.

\[
\varepsilon \lambda > 0, \quad \text{using (19) and the Young inequality, for any } p < p_B, \text{ we can take } k > -N/(N - 1) \text{ close to } -N/(N - 1) \text{ such that}
\]

\[
\left(\frac{N - 1}{N} \cdot k + 1\right)(-k) > 0 \quad \text{and} \quad \frac{N + 2}{pN}(-k) - \frac{N - 1}{N} > 0.
\]

Lemma follows from (6) and (17). \(\Box\)

**Lemma 2.2.** Assume that \(N \geq 3, \quad p < p_B \) and (2). Assume in addition that \(a_{12}, a_{21} > 0\). Let \((u, v)\) be a positive classical solution of (1) in \(B_1 \times (-1, 1)\), then

\[
\int_{B_{1/2}} \int_{-1/2}^{1/2} (u^{2p} + v^{2p}) dx dt \leq C(N, r, s, A). \tag{18}
\]

**Proof.** We follow the argument as in proof of [19, Proposition 21.5]. One can choose the test-function \(\varphi \in \mathcal{D}(B_1 \times (-1, 1))\) such that \(\varphi = 1\) in \(B_{1/2} \times (-1/2, 1/2), \quad 0 \leq \varphi \leq 1\) and

\[
\begin{align*}
|\nabla \varphi| & \leq C\varphi^{(3p+1)/4p}, \\
|\varphi_t| & \leq C\varphi^{(p+1)/2p},
\end{align*}
\]

Recall the notation \(\int = \int_{B_1} \int_{-1}^{1} dx dt\). By the proof of [19, Proposition 21.5] (see the formula (21.10)), for any constant \(\eta > 0\) and any positive function \(w \in \mathcal{C}^{2,1}(B_1 \times (-1, 1))\), we have

\[
\int |\nabla w|^2 (|\Delta \varphi| + \varphi^{-1} |\nabla \varphi|^2 + |\varphi_t|) \leq \eta \int \varphi(w^{-2}|\nabla w|^4 + w^{2p}) + C(\eta). \tag{19}
\]

Using (19) and the Young inequality, for any \(\varepsilon > 0\), we can control the RHS of (5) by the following estimates.
\[
\int \varphi |u_t|^2 |\nabla u|^2 \leq \varepsilon \int \varphi u^{-2} |\nabla u|^4 + \frac{1}{4\varepsilon} \int \varphi \, |u_t|^2,
\]
\[
\int |\nabla \varphi \cdot \nabla u| (u^p + u^r v^{s+1}) \leq \varepsilon \int \varphi (u^p + u^r v^{s+1})^2 + \frac{1}{4\varepsilon} \int \varphi^{-1} |\nabla \varphi|^2 |\nabla u|^2
\]
\[
\leq \varepsilon \int \varphi (u^p + u^r v^{s+1})^2 + \varepsilon \int \varphi (u^{-2} |\nabla u|^4 + u^{2p}) + C(\varepsilon),
\]
\[
\int |\nabla \varphi \cdot \nabla u||u_t|^2 \leq \varepsilon \int \varphi u^{-2} |\nabla u|^4 + \frac{1}{4\varepsilon} \int \varphi^{-1} |\nabla \varphi|^2 |\nabla u|^2
\]
\[
\leq \varepsilon \int \varphi u^{-2} |\nabla u|^4 + \varepsilon \int \varphi (u^{-2} |\nabla u|^4 + u^{2p}) + C(\varepsilon),
\]
\[
\int |\Delta \varphi| |\nabla u|^2 \leq \varepsilon \int \varphi (u^{-2} |\nabla u|^4 + u^{2p}) + C(\varepsilon),
\]
\[
\int \varphi |u_t|^{p+1} \leq \varepsilon \int \varphi u^{2p} + C(\varepsilon) \int \varphi^{-1} |\nabla \varphi|^{2p/(p-1)}
\]
\[
\leq \varepsilon \int \varphi u^{2p} + C(\varepsilon). \quad (20)
\]

Similarly for the estimates corresponding to \( v \), we then deduce from the Lemma 2.1 that
\[
I + L \leq C.\varepsilon \int \varphi \left( (u^p + u^r v^{s+1})^2 + (v^p + u^r v^{s+1} v^s)^2 + u^{-2} |\nabla u|^4 + v^{-2} |\nabla v|^4 \right)
\]
\[
+ C(\varepsilon) + C\left( 1 + \frac{1}{\varepsilon} \right) \int \varphi (u_t^2 + v_t^2).
\]

Since \( a_{ij} > 0 \) for all \( i, j = 1, 2 \), it follows that
\[
I + L \leq C.\varepsilon (I + L) + C(\varepsilon) + C\left( 1 + \frac{1}{\varepsilon} \right) \int \varphi (u_t^2 + v_t^2). \quad (21)
\]

On the other hand,
\[
\int \varphi \left( (r+1)a_{21}|u_t|^2 + (s+1)a_{12}|v_t|^2 \right)
\]
\[
= \int \varphi (r+1)a_{21}u_t(\Delta u + a_{11}u^p + a_{12}u^r v^{s+1})
\]
\[
+ \int \varphi (s+1)a_{12}v_t(\Delta v + a_{21}u^r v^{s+1} + a_{22}v^p)
\]
\[
= - \int \varphi \partial_t \left( \frac{(r+1)a_{21} |\nabla u|^2 + (s+1)a_{12} |\nabla v|^2}{2} \right)
\]
\[
+ \int \varphi \partial_t \left( \frac{(r+1)a_{21} a_{11} u^{p+1} + (s+1)a_{12} a_{22} v^{p+1}}{p+1} + a_{12} a_{21} u^r v^{s+1} \right)
\]
\[
- \int \left( (r+1)a_{21} u_t \nabla \varphi \cdot \nabla u + (s+1)a_{12} v_t \nabla \varphi \cdot \nabla v \right).
\]
Integrating by parts in $t$, we have
\[
\int \varphi \left( (r + 1)a_{21} |u_t|^2 + (s + 1)a_{12} |v_t|^2 \right)
\]
\[= \int \varphi_t \left( (r + 1)a_{21} |\nabla u|^2 + (s + 1)a_{12} |\nabla v|^2 \right)
\]
\[- \int \varphi_t (r + 1)a_{21} a_{11} u^{p+1} + (s + 1)a_{12} a_{22} v^{p+1}
\]
\[- \int \varphi_t a_{12} a_{21} u^{r+1} v^{s+1} - \int \left( (r + 1)a_{21} u_t \nabla \varphi \cdot \nabla u + (s + 1)a_{12} v_t \nabla \varphi \cdot \nabla v \right)
\]
\[\leq C \int |\varphi_t| \left( |\nabla u|^2 + |\nabla v|^2 + u^{p+1} + v^{p+1} \right)
\]
\[+ \frac{1}{2} \int \varphi \left( (r + 1)a_{21} |u_t|^2 + (s + 1)a_{12} |v_t|^2 \right) + C \int \varphi^{-1} |\nabla \varphi|^2 (|\nabla u|^2 + |\nabla v|^2).
\]
Hence,
\[
\int \varphi (r + 1)a_{21} |u_t|^2 + (s + 1)a_{12} |v_t|^2 \leq C \int |\varphi_t| \left( |\nabla u|^2 + |\nabla v|^2 + u^{p+1} + v^{p+1} \right)
\]
\[+ C \int \varphi^{-1} |\nabla \varphi|^2 (|\nabla u|^2 + |\nabla v|^2).
\]
(22)

Using $a_{12}, a_{21} > 0$, for any $\eta > 0$, we deduce from (19), (20) and (22) that,
\[
\int \varphi (|u_t|^2 + |v_t|^2) \leq C \eta (I + L) + C(\eta).
\]
(23)

Therefore, it follows from (21) and (23) that
\[
I + L \leq C \varepsilon (I + L) + C(\varepsilon) + C \left( 1 + \frac{1}{\varepsilon} \right) \left( C \eta (I + L) + C(\eta) \right).
\]

By taking $\eta = \varepsilon^2$ and choosing $\varepsilon$ sufficiently small, we obtain $I, L \leq C$ and the Lemma follows.

Proof of Theorem 1.1. We recall that, by the same argument of Quittner [18], Theorem is true for $N \leq 2$. Moreover, Theorem is straightforward if $a_{12} = 0$ or $a_{21} = 0$ since it is reduced to scalar equation. We then assume that $a_{12} > 0$, $a_{21} > 0$ and $N \geq 3$.

We first consider the case of bounded solutions. Assume for contradiction that $(u, v)$ is a nontrivial, bounded, nonnegative solution of (1) in $\mathbb{R}^N \times \mathbb{R}$. Since the components $u$ and $v$ are supersolutions of the heat equation, it follows from the strong maximum principle that either $(u, v)$ is positive in $\mathbb{R}^N \times \mathbb{R}$, or there exists $t_0 \in \mathbb{R}$ such that $u = v = 0$ in $\mathbb{R}^N \times (\infty, t_0]$. In the later case, since $(u, v)$ is bounded, we have $\partial_{tt} u - \Delta u \leq C(u + v)$, $\partial_{tt} v - \Delta v \leq C(u + v)$ for some constant $C > 0$. The maximum principle (see e.g. [16, Proposition 6.1]) then guarantees that $u = v = 0$ in $\mathbb{R}^N \times (t_0, \infty)$, hence $u \equiv v \equiv 0$. Consequently, we may assume without loss of generality that $(u, v)$ is positive in $\mathbb{R}^N \times \mathbb{R}$.

For any $R > 0$, we rescale
\[
u_R(x, t) = R^{2/(p-1)} \nu(Rx, R^2 t), \quad v_R(x, t) = R^{2/(p-1)} \nu(Rx, R^2 t).
\]
Then \((u_R, v_R)\) is also a solution to (1). By Lemma 2.2, we have
\[
\int_{|y|<R/2} \int_{|s|<R^2/2} (u^{2p} + v^{2p})(y, s) \, dy \, ds
= R^{N+2-4p/(p-1)} \int_{|x|<1/2} \int_{|t|<1/2} (u_{R}^{2p} + v_{R}^{2p})(x, t) \, dx \, dt
\leq CR^{N+2-4p/(p-1)}.
\]
Letting \(R \to \infty\) and noting \(p < p_B \leq p_S\), we deduce that \(u \equiv v \equiv 0\). This is a contradiction.

Finally, for the general case, we recall that the Liouville-type property of Theorem 1.1 for bounded solutions is sufficient for the proof of Proposition 1. But after a time shift, formula (4) in Proposition 1 guarantees that any solution of (1) in \(\mathbb{R}^N \times (-T, T)\) always satisfies
\[u(x, t) + v(x, t) \leq CT^{-1/(p-1)} \text{ in } \mathbb{R}^N \times (-T/2, T/2).\]
The conclusion then follows by letting \(T \to \infty\).

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REFERENCES

[1] H. Amann, Global existence for semilinear parabolic systems, J. Reine Angew. Math., 360 (1985), 47–83.
[2] T. Bartsch, N. Dancer and Z.-Q. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, Calc. Var. Partial Differential Equations, 37 (2010), 345–361.
[3] J. Bebernes and D. Eberly, Mathematical Problems from Combustion Theory, vol. 83 of Applied Mathematical Sciences, Springer-Verlag, New York, 1989.
[4] M.-F. Bidaut-Véron, Initial blow-up for the solutions of a semilinear parabolic equation with source term, in Équations aux dérivées partielles et applications, Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998, 189–198.
[5] M.-F. Bidaut-Véron and T. Raoux, Asymptotics of solutions of some nonlinear elliptic systems, Comm. Partial Differential Equations, 21 (1996), 1035–1086.
[6] R. S. Cantrell, C. Cosner and V. Hutson, Permanence in ecological systems with spatial heterogeneity, Proc. Roy. Soc. Edinburgh Sect. A, 123 (1993), 533–559.
[7] E. N. Dancer, K. Wang and Z. Zhang, Uniform Hölder estimate for singularly perturbed parabolic systems of Bose-Einstein condensates and competing species, J. Differential Equations, 251 (2011), 2737–2769.
[8] E. N. Dancer, J. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 953–969.
[9] M. Escobedo and M. A. Herrero, Boundedness and blow up for a semilinear reaction-diffusion system, J. Differential Equations, 89 (1991), 176–202.
[10] J. Földes and P. Poláčik, On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry, Discrete Contin. Dyn. Syst., 25 (2009), 133–157.
[11] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., 34 (1981), 525–598.
[12] P. Glandsdorf and I. Prigogine, Thermodynamic Theory of Structure Stability and Fluctuations, 1971.
[13] Y. Guo and J. Liu, Liouville type theorems for positive solutions of elliptic system in \(\mathbb{R}^N\), Comm. Partial Differential Equations, 33 (2008), 263–284.
[14] H. Meinhardt, Models of Biological Pattern Formation, vol. 6, Academic Press London, 1982.
[15] Q. H. Phan, Optimal Liouville-type theorems for a parabolic system, Discrete Contin. Dyn. Syst., 35 (2015), 399–409.
[16] Q. H. Phan and P. Souplet, A Liouville-type theorem for the 3-dimensional parabolic Gross–Pitaevskii and related systems, *Math. Ann.*, 366 (2016), 1561–1585.

[17] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations, *Indiana Univ. Math. J.*, 56 (2007), 879–908.

[18] P. Quittner, Liouville theorems for scaling invariant superlinear parabolic problems with gradient structure, *Math. Ann.*, 364 (2016), 269–292.

[19] P. Quittner and P. Souplet, *Superlinear Parabolic Problems*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2007, Blow-up, global existence and steady states.

[20] W. Reichel and H. Zou, Non-existence results for semilinear cooperative elliptic systems via moving spheres, *J. Differential Equations*, 161 (2000), 219–243.

[21] J. Wei and T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Ration. Mech. Anal.*, 190 (2008), 83–106.

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