Quantum correlations require multipartite information principles

Rodrigo Gallego, Lars Erik Würflinger, Antonio Acín, and Miguel Navascués

ICFO-Institut de Ciències Fotòniques, E-08860 Castelldefels, Barcelona, Spain
ICREA-Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, 08010 Barcelona, Spain
Department of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.

Identifying which correlations among distant observers are possible within our current description of Nature, based on quantum mechanics, is a fundamental problem in Physics. Recently, information concepts have been proposed as the key ingredient to characterize the set of quantum correlations. Novel information principles, such as information causality or non-trivial communication complexity, have been introduced in this context and successfully applied to some concrete scenarios. We show in this work a fundamental limitation of this approach: no principle based on bipartite information concepts is able to single out the set of quantum correlations for an arbitrary number of parties. Our results reflect the intricate structure of quantum correlations and imply that new and intrinsically multipartite information concepts are needed for their full understanding.

Introduction An ubiquitous problem in Physics is to understand which correlations can be observed among different events. In fact, any theoretical model aims at predicting the experimental results of measurements, or actions, performed at different space-time locations. Naively, one could argue that any kind of correlations are in principle possible within a general physical theory, and that only the details of the devices used for establishing the correlations imply limitations on them. Interestingly, this intuition is not correct: general physical principles impose non-trivial constraints on the allowed correlations among distant observers, independently of any assumption on the internal working of the devices. It is then a crucial question to identify which correlations among distant observers are compatible with our current description of Nature based on Quantum Physics. In particular, it would be desirable to understand why some correlations cannot be realized by quantum means, even if they do not allow any faster-than-light communication.

Recently, information concepts have been advocated as the key missing ingredient needed to single-out the set of quantum correlations. The main idea is to identify ‘natural’ information principles, formulated in terms only of correlations, which are satisfied by quantum correlations and proven to be violated by supra-quantum correlations. The existence of these supra-quantum correlations, then, would have implausible consequences from an information point of view. These information principles would provide a natural explanation of why the correlations observed in Nature have the quantum form. Celebrated examples of these principles are information causality or non-trivial communication complexity. While the use of these information concepts has been successfully applied to some specific scenarios, proving, or disproving, the validity of a principle for quantum correlations is extremely challenging. On the one hand, it is rather difficult to derive the Hilbert space structure needed for quantum correlations from information quantities. On the other hand, proving that some supra-quantum correlations are fully compatible with an information principle seems out of reach, as one needs to consider all possible protocols using these correlations and show that none of them leads to a violation of the principle. Thus, it is still open whether this approach is able to fully determine the set of quantum correlations.

In this work, we consider a general scenario consisting of an arbitrary number of observers and show a fundamental limitation of this information-based program: no information principle based on bipartite concepts is able to determine the set of quantum correlations. Our results imply that determining the set of quantum correlations for an arbitrary number of observers, requires principles of an intrinsically multipartite structure.

Non-signaling, local and quantum correlations The analyzed scenario consists of $n$ distant observers that can perform $m$ possible measurements of $d$ possible results on their systems. The observed correlations are described by the joint probability distribution $P(a_1, \ldots, a_n|x_1, \ldots, x_n)$, where $x_i = 0, \ldots, m-1$ denotes the measurement performed by party $i = 1, \ldots, n$; and $a_i = 0, \ldots, d-1$, the corresponding result. Each system is just seen as a black box producing the output $a_i$ given the input $x_i$.

Consider first the situation in which the measurements by the observers define space-like separated events. Then, the laws of special relativity guarantee that no signal has been able to propagate among the different observers. Under these conditions, the statistics seen by the observers are independent of any measurement performed by the other $n-k$ observers. Indeed, if this were not the case, the $n-k$ observers could signal to the remaining $k$ ones, even if they were causally disconnected. Mathematically, the impossibility of faster-than-light communication is imposed on the set of probabilities by requiring that

$$P(a_1 \ldots a_k|x_1 \ldots x_k) = \sum_{a_{k+1} \ldots a_n} P(a_1 \ldots a_n|x_1 \ldots x_n) \quad (1)$$

$$N_{\text{max}}(X) = \max_{X \in \mathcal{C}} P(X)$$
be independent of \(x_{k+1}, \ldots, x_n\). Similar relations hold for any partition of the \(n\) parties in two groups. These linear constraints define the set of non-signaling correlations.

A subset of the non-signaling correlations is the set of correlations having a local hidden variable model,

\[
P_L(a_1 \ldots a_n|x_1 \ldots x_n) = \sum_{\lambda} p_{\lambda} P_1(a_1|x_1, \lambda) \ldots P_n(a_n|x_n, \lambda). \tag{2}
\]

These correlations are also called local or classical and have a clear operational meaning: they can be established among the observers when each of them produces locally the outcome \(a_i\) using the input \(x_i\) and some pre-established classical instructions, denoted by \(\lambda\). As first shown by Bell, they satisfy some non-trivial linear constraints, known as Bell inequalities \(\dagger\). It can also be shown that some correlations are local if, and only if, they are compatible with the no-signaling principle and determinism \(\ddagger\). Indeed, they can always be decomposed as mixtures of points where the result for each measurement is assigned in a deterministic manner.

Quantum correlations correspond to those that can be obtained by performing local measurements on an \(n\)-partite state. Formally, one has

\[
P_Q(a_1 \ldots a_n|x_1 \ldots x_n) = \text{tr}(\rho M^{(1)}_{a_1,x_1} \otimes \ldots \otimes M^{(n)}_{a_n,x_n}), \tag{3}
\]

where \(\rho\) is the \(n\)-partite quantum state and \(M_{a_i,x_i}^{(i)}\) the measurement operator by party \(i\) yielding outcome \(a_i\) given measurement choice \(x_i\). Quantum correlations are known to lie between the set of classical and general non-signaling correlations as there exist quantum correlations which violate a Bell inequality and therefore have no classical analog \(\dagger\), and non-signaling correlations which are supra-quantum \(\ddagger\), i.e., they cannot be written in the form \(\S\). Despite having a clear mathematical definition \(\S\), the set of quantum correlations lacks a nice interpretation in terms of general principles, contrary to the classical and non-signaling counterparts. As said, it has been suggested that information concepts could provide the missing principles for quantum correlations.

It is worth mentioning before proceeding with the proof of the results that most of the existing examples of information principles have been formulated in the bipartite scenario. For example, information causality considers a scenario in which a first party, Alice, has a string of \(n_A\) bits. Alice is then allowed to send \(m\) classical bits to a second party, Bob. Information causality bounds the information Bob can gain on the \(n_A\) bits held by Alice whichever protocol they implement making use of the pre-established bipartite correlations and the message of \(m\) bits. Alice and Bob can violate this principle when they have access to some supra-quantum correlations \(\dagger\). In the case \(m = 0\), information causality implies that in absence of a message, pre-established correlations do not allow Bob to gain any information about any of the bits held by Alice, which is nothing but the no-signaling principle. The multipartite version of the no-signaling principle consists in the application of its bipartite version to all possible partitions of the \(n\) parties into two groups, see \(\S\). This suggests the following generalization of information causality to an arbitrary number of parties: given some correlations \(P(a_1, \ldots, a_n|x_1, \ldots, x_n)\), they are said to be compatible with information causality whenever all bipartite correlations constructed from them satisfy this principle. This generalization ensures the correspondence between no-signaling and information causality when \(m = 0\) for an arbitrary number of parties. This generalization of information causality has recently been applied to the study of extremal tripartite non-signaling correlations \(\ddagger\).

Regarding non trivial communication complexity, it studies how much communication is needed between two distant parties to compute probabilistically a function of some inputs in a distributed manner. It can also be interpreted as a generalization of the no-signaling principle, as it imposes constraints on correlations when a finite amount of communication is allowed between parties. Different multipartite generalizations of the principle have been studied, see \(\S\). However, as for information causality, one can always consider the straightforward generalization in which the principle is applied to every partition of the \(n\) parties in two groups.

**Supra-quantum correlations fulfilling information principles** In this work, we show that any physical principle that, similarly to no-signaling, is applied to every bipartition in the multipartite scenario is not sufficient to characterize the set of quantum correlations. We show this by finding tripartite correlations that, on one hand, fulfill any information principle based on bipartite concepts and, on the other hand, are supra-quantum.

To grant that our distributions are compatible with any bipartite information principle, we will restrict our search to a set of tripartite correlations which behave classically under any system bipartition. Let \(P(a_1a_2a_3|x_1x_2x_3)\) be a non-signaling tripartite distribution. We say that \(P(a_1a_2a_3|x_1x_2x_3)\) admits a time-ordered bi-local (TOBL) model if it can be written as

\[
P(a_1a_2a_3|x_1x_2x_3) = \sum_{\lambda} p^i_j k P(a_i|x_1, \lambda)P_{j-k}(a_ja_k|x_jx_k, \lambda) \tag{4}
\]

for \((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)\), with the distribu-
tions $P_{j \rightarrow k}$ and $P_{j \leftarrow k}$ obeying the conditions

$$P_{j \rightarrow k}(a_j|x_j, \lambda) = \sum_{a_k} P_{j \rightarrow k}(a_j, a_k|x_j x_k, \lambda), \quad (5)$$

$$P_{j \leftarrow k}(a_k|x_k, \lambda) = \sum_{a_j} P_{j \leftarrow k}(a_j, a_k|x_j x_k, \lambda). \quad (6)$$

The notion of TOBL correlations first appeared in [15] (see [16] for a proper introduction and further motivation for such a models). As can be seen from relations (5) and (6) we impose the distributions $P_{j \rightarrow k}$ and $P_{j \leftarrow k}$ to allow for signaling at most in one direction, indicated by the arrow, see Table I.

| $x_2$ | $x_3$ | $a_2$ | $a_3$ |
|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     |
| 0     | 1     | 0     | 1     |
| 1     | 0     | 1     | 1     |
| 1     | 1     | 0     | 1     |

TABLE I. Different examples of deterministic bipartite probability distributions $P_{2,3}(a_2 a_3|x_2 x_3, \lambda)$ characterized by output assignments to the four possible combination of measurements. Left: inputs and outputs corresponding to a point $P_{2 \leftarrow 3}(a_2 a_3|x_2 x_3, \lambda)$ in the decomposition (4). Center: inputs and outputs corresponding to a point $P_{3 \rightarrow 2}(a_2 a_3|x_2 x_3, \lambda)$ in (4). Right: inputs and outputs corresponding to a distribution which allows signaling in the two directions.

To understand the operational meaning of these models, consider the bipartition 1|23 for which systems 2 and 3 act together. In this situation, $P(a_1 a_2 a_3|x_1 x_2 x_3)$ can be simulated if a classical random variable $\lambda$ with probability distribution $p^{1|23}_{\lambda}$ is shared by parties 1 and the composite system 2−3, and they implement the following protocol: given $\lambda$, 1 generates its output according to the distribution $P(a_1|x_1, \lambda)$; on the other side, and depending on which of the parties 2 and 3 measures first, 2−3 uses either $P_{2 \leftarrow 3}(a_2 a_3|x_2 x_3, \lambda)$ or $P_{2 \rightarrow 3}(a_2 a_3|x_2 x_3, \lambda)$ to produce the two measurement outcomes. Likewise, any other bipartition of systems 1,2,3 admits a classical simulation.

By construction, the set of tripartite TOBL models is convex and includes all tripartite probability distributions of the form (4). Moreover, it becomes classical under postselection: indeed, suppose that we are given a tripartite distribution $P(a_1 a_2 a_3|x_1 x_2 x_3)$ satisfying condition (4), and a postselection is made on the outcome $a_3$ of measurement $\tilde{x}_3$ by party 3. Then one has

$$P(a_1 a_2|x_1 x_2 \tilde{x}_3 a_3) = \sum_{\lambda} p^{1}_{\lambda} P(a_1|x_1, \lambda) P^i(a_2|x_2, \lambda), \quad (7)$$

with $\lambda = \frac{p^{1|23}_{\lambda}}{P_{3}(a_3|x_3, \lambda)} P_{2 \leftarrow 3}(a_3|x_3, \lambda), \quad P^i(a_2|x_2, \lambda) = P_{2 \rightarrow 3}(a_2|x_2 \tilde{x}_3 a_3, \lambda). \quad (8)$

Postselected tripartite TOBL boxes can thus be regarded as elements of the TOBL set with trivial outcomes for one of the parties.

As mentioned in the introduction, to demonstrate that a set of correlations is compatible with an information principle one needs to consider all possible protocols using these correlations and ensure that the correlations obtained this way are in accordance with the principle. The most general protocol consists in distributing an arbitrary number of boxes described by $P^i_1, P^i_2, \ldots, P^i_N$ among three parties which are split into two groups, $A$ and $B$. Both groups can process the classical information provided by their share of the $N$ boxes. For instance, outputs generated by some of the boxes can be used as inputs for other boxes, see figure 1. This local processing of classical information is usually referred to as wirings [17]. Thus, in order to prove our result in full generality, we should consider all possible wirings of tripartite boxes. We show next that if $P^i_1, P^i_2, \ldots, P^i_N$ are in TOBL, then the resulting correlations $P_{\text{fin}}$ obtained after any wiring protocol have a local decomposition with respect to the bipartition $A/B$, and therefore fulfill any bipartite information principle.

For simplicity, we illustrate our procedure for the wiring shown in figure 1 where boxes $P^i_1, P^i_2, P^i_3$ are distributed between two parties $A$ and $B$, and party $A$ only holds one subsystem of each box. The construction is nevertheless general: it applies to any wiring and also covers situations where for some TOBL boxes party $A$ holds two subsystems instead of just one (or even the whole box).

From (4) we have

$$P^i(a_1 a_2 a_3|x_1 x_2 x_3) = \sum_{\lambda} p^{1}_{\lambda} P^i(a_1|x_1, \lambda) P_{2 \rightarrow 3}(a_2 a_3|x_2 x_3, \lambda), \quad (9)$$

$$P^i(a_1|x_1, \lambda) P_{2 \rightarrow 3}(a_2 a_3|x_2 x_3, \lambda), \quad (9)$$

for $i = 1, 2, 3$. Consider the first box that receives an input, in our case subsystem 2 of $P^1$. The first outcome $a_1$ can be generated by the probability distribution $P_{2 \rightarrow 3}(a_2 a_3|x_2 x_3, \lambda)$ encoded in the hidden variable $\lambda$. That models these first correlations. This is possible because for this decomposition $a_2$ is defined independently of $x_3$, the input in subsystem 3. Then, the next input $x_3$, which is equal to $a_1$, generates the output $a_3$ according to the probability distribution $P_{2 \rightarrow 3}(a_2 a_3|x_2 x_3, \lambda)$ encoded in $\lambda$. The subsequent outcomes $a_2$ and $a_3$ are
generated in a similar way. The general idea is that outputs are generated sequentially using the local models according to the structure of the wiring on $2-3$. Finally, subsystem $1$ can generate its outputs $a^i$ by using the probability distribution $P^i_1(a^i|x^i,\lambda^i)$. This probability distribution is independent of the order in which parties 2 and 3 make their measurement choices for any of the boxes. Averaging over all hidden variables one obtains $P_{\text{fin}}$. This construction provides the desired local model for the final probability distribution.

To show the absence of a quantum realization for some elements of the TOBL set of correlations, we use the Bell inequality known as ‘Guess Your Neighbor’s Input’ \cite{19}. Formally we have

$$B_{\text{max}} = \max \mathbf{B}(P)$$

subject to

$$P(a_1a_2a_3|x_1x_2x_3) \in \text{TOBL}.$$  

The maximization yields a value of $B_{\text{max}} = \frac{7}{2}$, implying the existence of supra-quantum correlations in TOBL. Details of this probability distribution attaining the maximum of $7/6$ and its TOBL decomposition can be found in the Supplemental Material of this article \cite{18}.

**Conclusion**  To summarize, we have shown that there exist tripartite non-signaling correlations that fulfill the principles of information causality and non-trivial communication complexity although they do not belong to the set of quantum correlations. The presented reasoning also applies to every other principle applied to the bipartitions of a multiparty system. This result provides a helpful insight for the formulation of a future principle aiming at distinguishing between quantum and supra-quantum correlations. In contrast to the no-signaling principle, such a forthcoming principle will need to be an intrinsically multipartite concept. This suggests that future research should be devoted to the development of information concepts of genuinely multipartite character.

More specifically, one could investigate which multiparticle generalizations of non trivial communication complexity can be considered intrinsically multipartite, and furthermore, how to generalize information causality for the case of multipartite communication protocols.

**Note added** After completion of this work, an extremal point of the tripartite non-signaling polytope which is supra-quantum and in TOBL was reported in \cite{13}.

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APPENDIX

This appendix presents a tripartite non-signaling probability distribution that attains the maximum of 7/6 for the ‘Guess Your Neighbor’s Input’ inequality, as well as its TOBL decomposition. To simplify notation, let us switch from \((a_1a_2a_3)\) to \((abc)\); and from \((x_1x_2x_3)\), to \((xyz)\). Now, consider the no-signaling tripartite probability distribution \(P(a, b, c|x, y, z)\) given by the probabilities shown in Table II.

| \(a\) | \(b\) | \(c\) | \(d\) | \(e\) | \(f\) | \(g\) | \(h\) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |

TABLE II. Tripartite probability distribution \(P(abc|xyz)\) attaining the maximum of 7/6 for the ‘Guess Your Neighbor’s Input’ inequality, where the rows correspond to the inputs xyz and the columns to the outputs abc.

The value of the ‘Guess Your Neighbor’s Input’ inequality for \(P(a, b, c|x, y, z)\) equals

\[
B(P) = \frac{2}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{7}{6} \neq 1,
\]

and thus \(P(a, b, c|x, y, z)\) cannot be approximated by any quantum system. Next we will prove that \(P(a, b, c|x, y, z)\) belongs to the TOBL set of correlations, and so it is compatible with any bipartite information principle.

First, notice that \(P(a, b, c|x, y, z)\) is invariant under permutations of the three parties. It is therefore enough to show that it admits a decomposition of the form [4] for the partition \(A|BC\). Along this bipartition, probability distributions appearing in the decomposition [4] are such that the outcome \(a\) only depends on the measurement choice \(x\) for every given \(\lambda\); let \(a_x\) denote this outcome for \(x = 0, 1\). Conditions [5] and [6] tell us that for every \(\lambda\) the marginal \(P_{B\rightarrow C}(b|y, \lambda)\) is independent of \(z\), and the marginal \(P_{B\rightarrow C}(c|z, \lambda)\) is independent of \(y\). Thus, for \(B \rightarrow C\) we have that \(b\) depends on \(y\) and \(c\) depends on both \(z\) and \(y\). The possible outcomes will then be denoted \(b_y, c_{yz}\). Similarly, for \(B \leftarrow C\), the possible outcomes are \(b_{yz}, c_z\). Tables III and IV contain the output assignments corresponding to deterministic probability distributions together with the weights \(p_{\lambda}\) for \(A|B \rightarrow C\) and \(A|B \leftarrow C\), respectively. Note that, in agreement with [4], the outcome assignments for \(A\) and the weights \(p_{\lambda}\) are the same for both decompositions.

| \(\lambda\) | \(p_{\lambda}\) | \(a_0\) | \(a_1\) | \(b_0\) | \(b_1\) | \(c_0\) | \(c_1\) |
|---|---|---|---|---|---|---|---|
| 1 | 1/12 | 0 | 0 | 0 | 1 | 0 | 1 |
| 2 | 1/12 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 1/12 | 0 | 0 | 0 | 0 | 0 | 1 |
| 4 | 1/12 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1/12 | 0 | 1 | 0 | 1 | 0 | 0 |
| 6 | 1/12 | 0 | 1 | 0 | 0 | 1 | 0 |
| 7 | 1/12 | 0 | 1 | 0 | 0 | 0 | 0 |
| 8 | 1/12 | 0 | 1 | 0 | 1 | 0 | 0 |
| 9 | 1/12 | 0 | 1 | 1 | 1 | 1 | 0 |

TABLE III. TOBL decomposition into deterministic probability distributions characterized by outcome assignments for the bipartition \(A|BC\) in the case \(A|B \rightarrow C\). For every \(\lambda\) the outcome \(a\) only depends on \(x\), and \(b\) only depends on \(y\).

| \(\lambda\) | \(p_{\lambda}\) | \(a_0\) | \(a_1\) | \(b_0\) | \(b_1\) | \(c_0\) | \(c_1\) |
|---|---|---|---|---|---|---|---|
| 1 | 1/12 | 0 | 0 | 0 | 0 | 1 | 0 |
| 2 | 1/12 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3 | 1/12 | 0 | 0 | 0 | 0 | 1 | 0 |
| 4 | 1/12 | 0 | 0 | 0 | 0 | 1 | 0 |
| 5 | 1/12 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 1/12 | 0 | 1 | 0 | 0 | 0 | 0 |
| 7 | 1/12 | 0 | 1 | 0 | 0 | 0 | 0 |
| 8 | 1/12 | 0 | 1 | 0 | 0 | 0 | 0 |
| 9 | 1/12 | 0 | 1 | 1 | 1 | 1 | 0 |
| 10 | 1/12 | 0 | 1 | 1 | 1 | 0 | 1 |

TABLE IV. TOBL decomposition into deterministic probability distributions characterized by outcome assignments for the bipartition \(A|BC\) in the case \(A|B \leftarrow C\). For every \(\lambda\) the outcome \(a\) only depends on \(x\), and \(c\) only depends on \(z\).

It is trivial to see that both tables indeed reproduce \(P(a, b, c|x, y, z)\), and hence such a distribution belongs to the TOBL set.