CHARACTERIZATION OF FUNCTIONS VIA COMMUTATORS OF BILINEAR FRACTIONAL INTEGRALS ON MORREY SPACES

SUZHEN MAO AND HUOXIONG WU

ABSTRACT. For $b \in L^{1}_{\text{loc}}(\mathbb{R}^n)$, let $I_\alpha$ be the bilinear fractional integral operator, and $[b, I_\alpha]$ be the commutator of $I_\alpha$ with pointwise multiplication $b$ ($i = 1, 2$). This paper shows that if the commutator $[b, I_\alpha]$ for $i = 1$ or 2 is bounded from the product Morrey spaces $L^{p_1, \lambda_1}(\mathbb{R}^n) \times L^{p_2, \lambda_2}(\mathbb{R}^n)$ to the Morrey space $L^{q, \lambda}(\mathbb{R}^n)$ for some suitable indexes $\lambda, \lambda_1, \lambda_2$ and $p_1, p_2, q$, then $b \in BMO(\mathbb{R}^n)$, as well as that the compactness of $[b, I_\alpha]$ for $i = 1$ or 2 from $L^{p_1, \lambda_1}(\mathbb{R}^n) \times L^{p_2, \lambda_2}(\mathbb{R}^n)$ to $L^{q, \lambda}(\mathbb{R}^n)$ implies that $b \in CMO(\mathbb{R}^n)$ (the closure in $BMO(\mathbb{R}^n)$ of the space of $C^\infty_c(\mathbb{R}^n)$ functions with compact support). These results together with some previous ones give a new characterization of $BMO(\mathbb{R}^n)$ functions or $CMO(\mathbb{R}^n)$ functions in essential ways.

1. Introduction

Let $\mathbb{R}^n$ be the Euclidean space with $n \geq 2$, and $BMO(\mathbb{R}^n)$ denote the space of functions with bounded mean oscillation, which consists of all locally integrable functions $b$, such that

$$
\|b\|_* := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx < \infty,
$$

where $Q$ is a cube with sides parallel to the axes, and $b_Q$ is the average of $b$ over $Q$. Also, let $CMO(\mathbb{R}^n)$ be the closure in the $BMO(\mathbb{R}^n)$ norm of $C^\infty_c(\mathbb{R}^n)$, which represents the space of infinitely differentiable functions with compact support.

Received July 1, 2015.
2010 Mathematics Subject Classification. Primary 42B20, 47B07; Secondary 42B25, 42B99.

Key words and phrases. bilinear fractional integrals, commutators, Morrey spaces, $BMO(\mathbb{R}^n)$, $CMO(\mathbb{R}^n)$, boundedness, compactness.

This work is financially supported by the NNSF of China (Grant Nos. 11371295, 11471041) and the NSF of Fujian Province of China (Grant No. 2015J01025).

©2016 Korean Mathematical Society
For $0 < \alpha < 2n$, let us consider the bilinear fractional integral operator $I_\alpha$ defined originally for $f, g \in C_0^\infty(\mathbb{R}^n)$ by

$$I_\alpha(f,g)(x) := \int_{\mathbb{R}^{2n}} \frac{1}{(|x-y| + |x-z|)^{2n-\alpha}} f(y)g(z)dydz,$$

and its commutators with symbol $b$ given by

1. \[ [b, I_\alpha]_1 (f,g) := I_\alpha(bf,g) - bI_\alpha(f,g), \]
2. \[ [b, I_\alpha]_2 (f,g) := I_\alpha(f,bg) - bI_\alpha(f,g). \]

The boundedness and compactness of $[b, I_\alpha]$ on variant function spaces have been the topic in many articles recently, see \cite{1, 2, 3, 5, 6, 12, 13, 17, 18, 19, 26}, among numerous references. One of the interesting questions on $[b, I_\alpha]$ is whether it can be used to characterize $BMO(\mathbb{R}^n)$ by boundedness, or $CMO(\mathbb{R}^n)$ by compactness, as those in the linear setting (see \cite{7, 9, 27} etc).

Recently, Chaffee \cite{5} established the following result.

**Theorem A.** For $b \in L^1_{loc}(\mathbb{R}^n), 0 < \alpha < 2n$ and $1 < p_1, p_2$, and $q$ satisfying

$$0 < \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q} < 1,$$

we have, for $i = 1$ or $2$,

$[b, I_\alpha]_i : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a bounded operator $\iff b \in BMO(\mathbb{R}^n)$.

Subsequently, Chaffee and Torres \cite{6} obtained the following characterization by compactness in Lebesgue spaces.

**Theorem B.** For $b \in L^1_{loc}(\mathbb{R}^n), 0 < \alpha < 2n$ and $1 < p_1, p_2$, and $q$ satisfying

$$0 < \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q} < 1,$$

we have, for $i = 1$ or $2$,

$[b, I_\alpha]_i : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a compact operator $\iff b \in CMO(\mathbb{R}^n)$.

In this paper, we aim to extend the above results to Morrey spaces, which is defined as follows.

**Definition.** For $0 < \lambda < n$, $1 \leq p < \infty$, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{ f \in L^p_{loc} : \| f \|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \},$$

where

$$\| f \|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{t \in \mathbb{R}^n, R > 0} \left( \frac{1}{R^\lambda} \int_{B(t, R)} |f(x)|^p dx \right)^{1/p},$$

and $B(t, R)$ is the ball centered at $t$ and with radius $R > 0$. 

The space $L^{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space with norm $\| \cdot \|_{L^{p,\lambda}(\mathbb{R}^n)}$. Moreover, for $1 \leq p < \infty$, then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and $L^{p,n}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$ isometrically. If $\lambda > n$, then $L^{p,\lambda}(\mathbb{R}^n) = \{0\}$.

It is well known that the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ was originally introduced by Morrey [20] to study certain problems in elliptic equations and was subsequently found to have many important applications to partial differential equations, such as elliptic equations, Navier-Stokes equations and Schrödinger equations, see [4, 15, 21, 23, 24] et al. and references therein. Also, the boundedness for the classical operators in harmonic analysis and the compactness of the commutators for such classical operators in $L^{p,\lambda}(\mathbb{R}^n)$ were extensively studied, for examples see [8, 9, 10, 11, 14, 17, 28] and references therein. In particular, Ding and Mei [17] established the following boundedness and compactness of $[b, \mathcal{I}_\alpha]$, for $i = 1, 2$ in Morrey spaces.

**Theorem C.** For $0 < \alpha < 2n$, $0 < \lambda$, $\lambda_1, \lambda_2 < n$. Suppose that $1/2 < p < \infty$, $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, $1 < q < \infty$ with $1/q = 1/p - \alpha/(n - \lambda)$. Then

(i) for $b \in BMO(\mathbb{R}^n)$, there exists a constant $C > 0$ such that for $i = 1, 2$,

$$\| [b, \mathcal{I}_\alpha]_i (f, g) \|_{L^{\alpha,\lambda}(\mathbb{R}^n)} \leq C \| b \| \| f \|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \| g \|_{L^{p_2,\lambda_2}(\mathbb{R}^n)};$$

(ii) for $b \in CMO(\mathbb{R}^n)$, $[b, \mathcal{I}_\alpha]_i$ is a compact operator from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{\alpha,\lambda}(\mathbb{R}^n)$, $i = 1, 2$.

Compared Theorem C with Theorems A and B, it is natural to ask whether the boundedness or compactness of the commutator $[b, \mathcal{I}_\alpha]_i$, for $i = 1$ or 2, from the product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to the Morrey space $L^{\alpha,\lambda}(\mathbb{R}^n)$, can imply that $b \in BMO(\mathbb{R}^n)$ or $b \in CMO(\mathbb{R}^n)$. The main purpose of this paper is to address the question above. Our results can be formulated as follows.

**Theorem 1.1.** For $0 < \alpha < 2n$, $0 < \lambda$, $\lambda_1, \lambda_2 < n$, suppose that $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, $1 < q < \infty$ with $1/q = 1/p - \alpha/(n - \lambda)$. If the commutator $[b, \mathcal{I}_\alpha]_i$, for $i = 1$ or 2, is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{\alpha,\lambda}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.

**Theorem 1.2.** For $0 < \alpha < 2n$, $0 < \lambda_1, \lambda_2, \lambda < n$, suppose that $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, and $1 < q < \infty$ with $1/q = 1/p - \alpha/(n - \lambda)$. If the commutator $[b, \mathcal{I}_\alpha]_i$, for $i = 1$ or 2, is a compact operator from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{\alpha,\lambda}(\mathbb{R}^n)$, then $b \in CMO(\mathbb{R}^n)$.

Moreover, combining Theorems 1.1 and 1.2 with Theorem C, we have the following equivalent characterizations.

**Theorem 1.3.** Under the assumptions of Theorem 1.1 or 1.2, for $i = 1$ or 2, we have
which we can express (\(|b, I_\alpha|_i : L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n) \to L^{q,\lambda}(\mathbb{R}^n)\) is bounded \(\iff\) \(b \in BMO(\mathbb{R}^n)\));
(ii) \([b, I_\alpha]_i : L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n) \to L^{q,\lambda}(\mathbb{R}^n)\) is compact \(\iff\) \(b \in CMO(\mathbb{R}^n)\).

**Remark.** Obviously, Theorems A and B can be regarded as the extreme case of Theorem 1.3 in \(\lambda = \lambda_1 = \lambda_2 = 0\) since \(L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)\) for any \(1 \leq p < \infty\). Therefore, our results essentially extend the corresponding ones in [5, 6]. In addition, Theorem 1.3 can also be regarded as the generalization of the corresponding result in [9] from the linear setting to the multilinear setting.

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1.1 and the proof of Theorem 1.2 will be given in Section 3. We remark that our ideas are greatly motivated by [5, 6, 9, 16].

We shall use the following conventions:

- \(C\) always denotes a positive constant that is independent of main parameters involved but whose value may differ from line to line.
- For a set \(E \subset \mathbb{R}^n\), \(\chi_E\) denotes its characteristic function.
- For \(p \in [1, \infty)\), we use \(p'\) to denote the dual exponent of \(p\), namely \(p' = p/(p - 1)\).
- For a ball \(B \subset \mathbb{R}^n\) and \(c > 0\), we use \(cB\) to denote the ball concentric with \(B\) whose radius is \(c\) times of \(B\)'s.

**2. Proof of Theorem 1.1**

This section is devoted to proving Theorem 1.1. The techniques in our arguments are taken from [5, 16], which originate from [7].

*Proof of Theorem 1.1.* By symmetry of the kernel of \([b, I_\alpha]|_i\), we will give our arguments to \([b, I_\alpha]|_i\). For \(\delta > 0\), let \(B((y_0, z_0), \delta \sqrt{n}) \subset \mathbb{R}^{2n}\) be the ball for which we can express \((|y| + |z|)^{2n-\alpha}\) as an absolutely convergent Fourier series of the form

\[ (|y| + |z|)^{2n-\alpha} = \sum_{j=0}^{\infty} a_j e^{i\xi_j(y,z)}, \]

which \(\sum_{j=0}^{\infty} |a_j| < \infty\) in the neighborhood \(|y - y_0| + |z - z_0| \leq 2\delta \sqrt{n}\).

The specific vectors \(\xi_j\) will not play a role in the proof. We will express them as \(\xi_j = (\xi_j^1, \xi_j^2) \in \mathbb{R}^n \times \mathbb{R}^n\). Note that due to the homogeneity of \((|y| + |z|)^{2n-\alpha}\), we can take \((y_0, z_0)\) such that \(|(y_0, z_0)| > 2\sqrt{n}\) and \(\delta < 1\) such that \(B \cap \{0\} = \emptyset\).

Set \(y_1 = y_0 \delta^{-1}\) and \(z_1 = z_0 \delta^{-1}\), and note that

\[ |y - y_1| + |z - z_1| < 2\sqrt{n} \Rightarrow |\delta y - y_0| + |\delta z - z_0| \leq 2\delta \sqrt{n}, \]

and so for all \((y, z)\) satisfying the inequality on the left. We have

\[ (|y| + |z|)^{2n-\alpha} = (|\delta y| + |\delta z|)^{2n-\alpha} \delta^{-2n+\alpha}. \]
Let \( Q = Q(x_0, r_0) \) be an arbitrary cube in \( \mathbb{R}^n \). Set \( \tilde{y} = x_0 - r_0y_1 \), \( \tilde{z} = x_0 - r_0z_1 \) and take \( Q' = Q(\tilde{y}, r_0) \subset \mathbb{R}^n \) and \( Q'' = Q(\tilde{z}, r_0) \subset \mathbb{R}^n \). Then for any \( x \in Q \) and \( y \in Q' \), we have

\[
\left| \frac{x - y}{r_0} - y_1 \right| = \left| \frac{x - x_0}{r_0} - \frac{x_0 - \tilde{y}}{r_0} \right| \leq \left| \frac{x - x_0}{r_0} \right| + \left| \frac{\tilde{y} - y}{r_0} \right| \leq \sqrt{n}.
\]

The same estimate holds for \( x \in Q \) and \( z \in Q'' \), and so we have

\[
\left| \frac{x - y}{r_0} - y_1 \right| + \left| \frac{x - z}{r_0} - z_1 \right| \leq 2\sqrt{n}.
\]

Let \( \sigma(x) = \text{sgn} (b(x) - b_{Q'}) \). Then

\[
\int_Q |b(x) - b_{Q'}| dx \\
= \int_Q (b(x) - b_{Q'}) \sigma(x) dx \\
= \frac{1}{|Q'|} \int_{Q'} \int_Q (b(x) - b(y)) dy \sigma(x) dx \\
= \frac{1}{|Q''| |Q'|} \int_{Q''} \int_{Q'} (b(x) - b(y)) \sigma(x) dz dy dx \\
= r_0^{-2n} \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y| + |x - z|} |x - z|^{2n-\alpha} \left( \left| \frac{x - y}{r_0} \right| + \left| \frac{x - z}{r_0} \right| \right)^{2n-\alpha} \\
\times \sigma(x) \chi_Q(x) \chi_{Q'}(y) \chi_{Q''}(z) dz dy dx \\
= r_0^{-\alpha} \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y| + |x - z|} \delta^{\alpha-2n+\alpha} \sum_{j=0}^{\infty} a_j e^{i\delta y_j} (\frac{x - y}{r_0}, \frac{x - z}{r_0}) \\
\times \sigma(x) \chi_Q(x) \chi_{Q'}(y) \chi_{Q''}(z) dz dy dx.
\]

Let \( f_j(y) = e^{-\frac{i}{\delta} y_j^* y} \chi_{Q'}(y) \), \( g_j(z) = e^{-\frac{i}{\delta} z_j^* z} \chi_{Q''}(z) \) and

\( h_j(x) = e^{\frac{i}{\delta} y_j^* (x,y)} \sigma(x) \chi_Q(x) \).

Then \( f_j \in L^{p_1, \lambda_1}(\mathbb{R}^n) \), \( g_j \in L^{p_2, \lambda_2}(\mathbb{R}^n) \) and by Hölder’s inequality and the assumption the boundedness of \([b, T_\alpha]\) from \( L^{p_1, \lambda_1}(\mathbb{R}^n) \times L^{p_2, \lambda_2}(\mathbb{R}^n) \) to \( L^{q, \lambda}(\mathbb{R}^n) \), we have

\[
\int_Q |b(x) - b_{Q'}| dx \\
= r_0^{-\alpha} \delta^{-2n+\alpha} \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y| + |x - z|} \delta^{\alpha-2n+\alpha} \sum_{j=0}^{\infty} a_j f_j(y) g_j(z) h_j(x) dz dy dx
\]
\[ \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \leq C |Q|^{-1} \int_Q |b(x) - c| dx \text{ for any } c, \]

which implies that \( b \in BMO(\mathbb{R}^n) \) and completes the proof of Theorem 1.1. \( \Box \)
3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. The following characterization of $CMO(\mathbb{R}^n)$ will play key role in our arguments.

Lemma 3.1 (cf. [25]). A function $b \in BMO(\mathbb{R}^n)$ is in $CMO(\mathbb{R}^n)$, if and only if $b$ satisfies the following three conditions:

1. \( \lim_{a \to 0} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0, \)
2. \( \lim_{a \to \infty} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0, \)
3. \( \lim_{|y| \to \infty} \frac{1}{|Q|} \int_Q |b(x + y) - b_Q| dx = 0 \) for each $Q$.

Proof of Theorem 1.2. Note that a compact operator is bounded, by Theorem 1.1, we know that the symbol $b$ of a compact operator must be at least in $BMO(\mathbb{R}^n)$. In what follows, we will prove that $b \in CMO(\mathbb{R}^n)$.

Employing the ideas of [6, 9], our approach is as follows: Under the assumption of that $[b, \mathcal{I}_a]_i$ is a compact operator from $L^{p_i, \lambda_i}(\mathbb{R}^n) \times L^{p_i, \lambda_i}(\mathbb{R}^n)$ to $L^{q, \lambda}(\mathbb{R}^n)$ for $i = 1$ or 2, we will show that if $b$ fails to satisfy one of the conditions (1)-(3) in Lemma 3.1, then one can construct sequences of functions $\{f_j\}_{j=1}^\infty$ uniformly bounded on $L^{p_i, \lambda_i}(\mathbb{R}^n)$ and $\{g_j\}_{j=1}^\infty$ uniformly bounded on $L^{p_i, \lambda_i}(\mathbb{R}^n)$, such that $\{[b, \mathcal{I}_a](f_j, g_j)\}_{j=1}^\infty$ has no convergent subsequence in $L^{q, \lambda}(\mathbb{R}^n)$, which contradicts the compactness assumption. It then follows that if $[b, \mathcal{I}_a]_i$ is compact, the symbol $b$ must satisfy all three conditions and hence be an element of $CMO(\mathbb{R}^n)$.

By the symmetry of the kernel of $[b, \mathcal{I}_a]$, again, we will give the arguments only to $[b, \mathcal{I}_a]_i$. Before constructing the sequence, we make some preliminaries.

Assume that $b \in BMO(\mathbb{R}^n)$ with $\|b\|_\infty = 1$. Then there exist $\epsilon > 0$ and a sequence of cubes $\{Q_j(y_j, d_j)\}_{j}^\infty$ such that for every $j$,

\[ \frac{1}{|Q_j|} \int_{Q_j} |b(y) - b_{Q_j}| dy > \epsilon. \]

We define

\[ f_j(y) = |Q_j|^{-(n-\lambda_1)/(np_1)} \left( \text{sgn} (b(y) - b_{Q_j}) - c_0 \right) \chi_{Q_j}(y), \]

where $c_0 = |Q_j|^{-1} \int_{Q_j} \text{sgn} (b(y) - b_{Q_j}) dy$. It is easy to check that $|c_0| < 1$ and $\{f_j\}$ has the following properties

1. $\text{supp} f_j \subset Q_j$, 
2. $f_j(y)(b(y) - b_{Q_j}) \geq 0$, 
3. $\int_{\mathbb{R}^n} f_j(y) dy = 0$, 
4. $\int_{\mathbb{R}^n} f_j(y)(b(y) - b_{Q_j}) dy = |Q_j|^{-(n-\lambda_1)/(np_1)} \int_{Q_j} |b(y) - b_{Q_j}| dy$. 

We define $\mathcal{I}_a = \sum_{Q_i} \frac{1}{|Q_i|} \int_{Q_i} |b(y) - b_{Q_i}| dy$.
(11) \[ |f_j(y)| \leq 2|Q_j|^{-(n-\lambda_1)/(np_1)}. \]

(11) gives us that \( \{ ||f_j||_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \}_{j=1}^{\infty} \) is bounded uniformly. In fact, for any \( t \in \mathbb{R}^n \),

\[
\left( \frac{1}{R^{\lambda_1}} \int_{B(t,R)} |f_j(y)|^{p_1} dy \right)^{1/p_1} \leq \begin{cases} C_1 \left( \frac{R}{T} \right)^{(n-\lambda_1)/p_1} & 0 < R \leq d_j; \\
C_1 \left( \frac{d_j}{T} \right)^{\lambda_1/p_1} & R > d_j > 0,
\end{cases}
\]

where \( C_1 \) is independent of \( j, R, t \).

For the other functions, we will simply define

\[
g_j = \frac{\lambda Q_j}{|Q_j|^{(n-\lambda_2)/(p_2 n)}},
\]

which satisfies

\[
\left( \frac{1}{R^{\lambda_2}} \int_{B(t,R)} |g_j(y)|^{p_2} dy \right)^{1/p_2} \leq \begin{cases} C_2 \left( \frac{R}{T} \right)^{(n-\lambda_2)/p_2} & 0 < R \leq d_j; \\
C_2 \left( \frac{d_j}{T} \right)^{\lambda_2/p_2} & R > d_j > 0,
\end{cases}
\]

where \( C_2 \) is independent of \( j, R, t \). Thus the sequence \( \{ [b, \mathcal{I}_\alpha] f_j, g_j \}_{j=1}^{\infty} \) is also bounded in \( L^{q,\lambda}(\mathbb{R}^n) \).

Next we establish several technical estimates. For a cube \( Q_j \) with centered \( y_j \) and satisfying (5) for some \( \epsilon > 0 \), \( f_j, g_j \) as above, and all \( x \in (2\sqrt{n}Q_j)^c \) the following point-wise estimates hold:

(13) \[ |\mathcal{I}_\alpha((b - b_{Q_j})f_j, g_j)(x)| \leq C|Q_j|^{2-(n-\lambda)/(np)|x-y_j|^{-2n+\alpha}}, \]

(14) \[ |\mathcal{I}_\alpha((b - b_{Q_j})f_j, g_j)(x)| \geq C|Q_j|^{2-(n-\lambda)/(np)|x-y_j|^{-2n+\alpha+\epsilon}}, \]

(15) \[ |\mathcal{I}_\alpha(f_j, g_j)(x)| \leq C|Q_j|^{2-(n-\lambda)/(np)+1/n|x-y_j|^{-2n+\alpha-1}}, \]

where the constants involved are independent of \( b, f_j, g_j \) and \( \epsilon \).

To prove (13), we use that \(|x-y_j| \approx |x-y|\) for all \( y \in Q_j \), and that \( ||b||_* = 1 \) to obtain

\[
|\mathcal{I}_\alpha((b - b_{Q_j})f_j, g_j)(x)| \\
= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y) - b_{Q_j})f_j(y)g_j(z)}{|x-y| + |x-z|}^{2n-\alpha} \, dy \, dz \right| \\
= \left| \int_{Q_j} \int_{Q_j} \frac{(b(y) - b_{Q_j})f_j(y)g_j(z)}{|x-y| + |x-z|}^{2n-\alpha} \, dy \, dz \right| \\
\leq |Q_j|^{-2(\lambda_1)/(np_1)+2(\lambda_1-\lambda_2)/(np_2)}|x-y_j|^{-2n+\alpha} \int_{Q_j} \int_{Q_j} |b(y) - b_{Q_j}| \, dy \, dz \\
\leq C|Q_j|^{2-(n-\lambda)/(np)|x-y_j|^{-2n+\alpha}||b||_*}.
\]
\[ C|Q_j|^{2-(n-\lambda)/(np)}|x - y_j|^{-2n+\alpha}. \]

By (5), (8) and (10), we have

\[ |I_\alpha((b - bQ_j)f_j, g_j)(x)| \]
\[ = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(b(y) - bQ_j)f_j(y)g_j(z)}{|x - y| + |x - z|^{2n-\alpha}} \, dy \, dz \right| \]
\[ \geq C|Q_j|^{1-(n-\lambda_2)/(np_2)}|x - y_j|^{-2n+\alpha} \int_{Q_j} (b(y) - bQ_j)f_j(y) \, dy \]
\[ = C|Q_j|^{1-(n-\lambda_3)/(np_3)-2n+\alpha/(np-\lambda_2)} \int_{Q_j} |b(y) - bQ_j| \, dy \]
\[ \geq C|Q_j|^{2-(n-\lambda)/(np)|x - y_j|^{-2n+\alpha}} \epsilon, \]

which gives (14). Finally, using that \( f_j \) has mean zero we obtain (15) in the following way,

\[ |I_\alpha(f_j, g_j)(x)| \]
\[ = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{f_j(y)g_j(z)}{|x - y| + |x - z|^{2n-\alpha}} \right) \, dy \, dz \right| \]
\[ \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_j(y)g_j(z)}{|x - y| + |x - z|^{2n-\alpha+1}} \, dy \, dz \]
\[ \leq C \frac{|Q_j|^{1/n}}{|x - y_j|^{2n-\alpha+1}} \int_{Q_j} \int_{Q_j} |f_j(y)g_j(z)| \, dy \, dz \]
\[ \leq C|Q_j|^{2-(n-\lambda)/(np)+1/n} |x - y_j|^{-2n+\alpha-1}. \]

Following [25] and [9], we now use the above point-wise estimates (13)–(15) to prove some \( L^{q,\lambda}(\mathbb{R}^n) \) inequalities for \( [b, I_\alpha]_1(f_j, g_j) \). We may get the following claims.

**Claim 1.** There exist constants \( \gamma_2 > \gamma_1 > 2 \) and \( \gamma_3 > 0 \), which are depending only on \( p_1, p_2, n, \epsilon, \lambda \) and \( b \), such that

\[ \left( \int_{\gamma_1 d_j < |x - y_j| < \gamma_2 d_j} |[b, I_\alpha]_1(f_j, g_j)(x)|^q \, dx \right)^{1/q} \geq \gamma_3 |Q_j|^{\lambda/(nq)}, \]

and

\[ \left( \int_{|x - y_j| > \gamma_2 d_j} |[b, I_\alpha]_1(f_j, g_j)(x)|^q \, dx \right)^{1/q} \leq \frac{\gamma_4}{4} |Q_j|^{\lambda/(nq)}. \]

**Claim 2.** There exists a constant \( 0 < \beta < \gamma_2 \) depending only on \( p_1, p_2, n, \epsilon, \lambda \) and \( b \), such that

\[ \left( \int_E |[b, I_\alpha]_1(f_j, g_j)(x)|^q \, dx \right)^{1/q} \leq \frac{\gamma_4}{4} |Q_j|^{\lambda/(nq)} \]
holds for every measurable set $E$ satisfying

$$E \subset \{ x : \gamma_1 d_j < |x - y_j| < \gamma_2 d_j \} \text{ and } \frac{|E|}{|Q_j|} < \beta^n.$$ 

Now we prove (16)-(18). Starting $\tilde{\gamma}_1 > \max\{16, n\}$, using (15) and the fact that $2n - \alpha - n/q > 0$, $1/p_1 + 1/p_2 < 2$. Since $|b_2 Q - b| \leq C \|b\|_* = C$, by $\|b\|_* = 1$ we have

$$\left( \int_{2^s d_j < |x - y_j| < 2^{s+1} d_j} |b(x) - b_{Q_j}|^q dx \right)^{1/q} \leq C s 2^{sn/q} |Q_j|^{1/q}.$$ 

Then

$$\left( \int_{|x - y_j| > \tilde{\gamma}_1 d_j} |(b(x) - b_{Q_j}) I_\alpha(f_j, g_j)(x)|^q dx \right)^{1/q}$$

$$\leq C |Q_j|^{2-(n-\lambda)/(np)+1/n} \sum_{s = \log_2 \tilde{\gamma}_1}^{\infty} \left( \int_{2^s d_j < |x - y_j| < 2^{s+1} d_j} |b(x) - b_{Q_j}|^q \frac{|x - y_j|^q}{|x - y_j|^{q+2n-\alpha}} dx \right)^{1/q}$$

$$\leq C |Q_j|^{2-(n-\lambda)/(np)+1/n} \sum_{s = \log_2 \tilde{\gamma}_1}^{\infty} 2^{-s(2n-\alpha+1)} Q_j^{-2+\alpha/n-1/n}$$

$$\times \left( \int_{2^s d_j < |x - y_j| < 2^{s+1} d_j} |b(x) - b_{Q_j}|^q dx \right)^{1/q}$$

$$\leq C |Q_j|^{\lambda/(mq)} \sum_{s = \log_2 \tilde{\gamma}_1}^{\infty} 2^{-s(2n-\alpha-n/q+1/2)},$$

where we have used that $s \leq 2^{s/2}$ for $4 \leq \log_2 \tilde{\gamma}_1 \leq s$. Thus we obtain that

$$\left( \int_{|x - y_j| > \tilde{\gamma}_1 d_j} |(b(x) - b_{Q_j}) I_\alpha(f_j, g_j)(x)|^q dx \right)^{1/q}$$

$$\leq C |Q_j|^{\lambda/(mq)^2} \gamma_1^{-2n+\alpha+n/q-1/2}. $$

Next, for $\gamma_2 > \tilde{\gamma}_1$, using (14) and (19), we have the following

$$\left( \int_{\tilde{\gamma}_1 d_j < |x - y_j| < \gamma_2 d_j} |b I_\alpha(f, g_j)(x)|^q dx \right)^{1/q}$$

$$\geq \left( \int_{\tilde{\gamma}_1 d_j < |x - y_j| < \gamma_2 d_j} |I_\alpha((b - b_{Q_j}) f_j, g_j)(x)|^q dx \right)^{1/q}$$

$$- \left( \int_{\tilde{\gamma}_1 d_j < |x - y_j| < \gamma_2 d_j} |(b(x) - b_{Q_j}) I_\alpha(f_j, g_j)(x)|^q dx \right)^{1/q}$$
there exists some positive constant $c$

On the other hand, by the same arguments as in [9, p. 309], we can show that

$$
(22)
$$

Similarly, from (13) and (19), we have

$$
(21)
$$

Using (20) and (21) (since $1 < p < n/\alpha$, $1/q = 1/p - \alpha/(n - \lambda)$, $n/q + \alpha < n$), we can select $\gamma_1, \gamma_2$ in place $\tilde{\gamma}_1, \tilde{\gamma}_2$, with $\gamma_2 > \gamma_1$, such that (16) and (17) are verified for some $\gamma_3 > 0$.

We now verify (18). Let $E \subset \{ x : \gamma_1 d_j < |x - y_j| < \gamma_2 d_j \}$ be an arbitrary measurable set. Then by (13) and (15), we get

$$
(22)
$$

On the other hand, by the same arguments as in [9, p. 309], we can show that there exists some positive constant $c_1$ depending on $\gamma_1$ and $\gamma_2$ and $b$ such that

$$
\frac{1}{|Q_j|} \int_E |b(x) - b_{Q_j}|^q dx \leq C \frac{|E|}{|Q_j|} \left( 1 + \log \left( \frac{c_1 |Q_j|}{|E|} \right) \right)^{q-1}.
$$
This together with (22), if we take \(0 < \beta < \min(c_1^{1/\alpha}, \gamma_2)\), implies that (18) holds.

We are left with constructing the sequences to lead to a contradiction. The arguments are again borrowed from [6, 9].

**Case 1.** If \(b\) does not satisfy (1), then there exist some \(\epsilon > 0\) and sequence \(\{Q_j\}\) with \(\lim_{j \to \infty} |Q_j| = 0\) such that for every \(j\), (5) holds.

By \(\lim_{j \to \infty} d_j = 0\), we may choose a subsequence \(\{Q_{j_k}^{(1)} = Q(y_{j_k}, q_{j_k}^{(1)})\}\), such that their radius \(\{d_{j_k}^{(1)}\}\) satisfying

\[
\frac{d_{j_k}^{(1)}}{d_{j_k}^{(1)}} < \frac{\beta}{\gamma_2}.
\]

We also let \(\{f_{j_k}\}\) and \(\{g_{j_k}\}\) be the subsequence associated to the selected cubes \(\{Q_{j_k}^{(1)}\}\) as defined earlier on. For fixed \(k\) and \(m\), we define the following sets

\[
G = \{x : \gamma_1 d_{j_k}^{(1)} < |x - y_{j_k}| < \gamma_2 d_{j_k}^{(1)}\},
\]

\[
G_1 = G \setminus \{x : |x - y_{j_k+m}| \leq \gamma_2 d_{j_k+m}^{(1)}\},
\]

and

\[
G_2 = \{x : |x - y_{j_k+m}| > \gamma_2 d_{j_k+m}^{(1)}\},
\]

where \(\gamma_1\) and \(\gamma_2\) are defined as before. Note that

\[
G_1 \subset B(y_{j_k}, \gamma_2 d_{j_k}^{(1)}) \cap G_2,
\]

and

\[
G_1 = G \setminus (G_2 \cap G).
\]

Also, by construction and our choice of \(Q_{j_k}^{(1)}\), one can easily see that

\[
\frac{|G_2 \cap G|}{|Q_{j_k}|} \leq \frac{(\gamma_2 d_{j_k+m}^{(1)})^n}{(d_{j_k}^{(1)})^n} \leq \gamma_2^n \left(\frac{\beta^n}{\gamma_2^n}\right)^m < \beta^n.
\]

It follows that

\[
\left(\int_{B(y_{j_k}, \gamma_2 d_{j_k}^{(1)})} |[b, T_{\alpha}]_1 (f_{j_k}, g_{j_k})(x) - [b, T_{\alpha}]_1 (f_{j_k+m}, g_{j_k+m})(x)|^q dx\right)^{1/q}
\]

\[
\geq \left(\int_{G_1} |[b, T_{\alpha}]_1 (f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q} - \left(\int_{G_2} |[b, T_{\alpha}]_1 (f_{j_k+m}, g_{j_k+m})(x)|^q dx\right)^{1/q}
\]

\[
\geq \left(\int_{G} |[b, T_{\alpha}]_1 (f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q} - \left(\int_{G_2 \setminus G} |[b, T_{\alpha}]_1 (f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q}
\]

\[
- \frac{\gamma_2}{4} Q_{j_k+m}^{1/\lambda(nq)}
\]

\[
\geq \left(\frac{\gamma_2^n}{4} Q_{j_k}^{1/\lambda(nq)} \int_{G_2 \setminus G} |[b, T_{\alpha}]_1 (f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q} - \frac{\gamma_2}{4} Q_{j_k+m}^{1/\lambda(nq)}.
\]
By (26) and applying (18) with $E := G_2 \cap G$, we have
\[
(27) \quad \int_{G_2 \cap G} \| [b, I_{\alpha}]_1 (f_{j_k}, g_{j_k})(x) \|^q dx \leq \left( \frac{3}{4} \right)^q |Q_{j_k}|^{\lambda/n}.
\]
This together with (26) and the fact that $|Q_{j_{k+m}}^{(1)}| < |Q_{j_k}^{(1)}|$ for any $m \in \mathbb{N}$, yields that there exists $\delta_0 = \delta_0(\gamma_3, q, n) > 0$ such that
\[
\left( \int_{B(y_{j_k}, \gamma_2 d_{j_k}^{(1)})} \| [b, I_{\alpha}]_1 (f_{j_k}, g_{j_k})(x) - [b, I_{\alpha}]_1 (f_{j_{k+m}}, g_{j_{k+m}})(x) \|^q dx \right)^{1/q} \geq \delta_0 |Q_{j_k}^{(1)}|^{\lambda/(nm)}.
\]
Thus
\[
\left( \frac{1}{d_{j_k}^{(1)}} \int_{B(y_{j_k}, \gamma_2 d_{j_k}^{(1)})} \| [b, I_{\alpha}]_1 (f_{j_k}, g_{j_k})(x) - [b, I_{\alpha}]_1 (f_{j_{k+m}}, g_{j_{k+m}})(x) \|^q dx \right)^{1/q} \geq \delta,
\]
where $\delta = \delta(\delta_0, n, q, \lambda)$ and $\delta$ is independent on $m$.

Hence, $[b, I_{\alpha}]_1$ is not a compact operator from $L^{p_1, \lambda_1}(\mathbb{R}^n) \times L^{p_2, \lambda_2}(\mathbb{R}^n)$ to $L^{2, \lambda}(\mathbb{R}^n)$. This contradiction shows that $b$ must satisfy the condition (1) of Lemma 3.1.

Case 2. If $b$ does not satisfy (2), then there also exist $\epsilon$ and sequence of cubes $\{Q_j\}$, this time with $|Q_j| \to \infty$ as $j \to \infty$ such that (3.1) holds, too. This time we choose the subsequence $\{Q_{j_i}^{(2)} = Q(y_{j_i}, d_{j_i}^{(2)})\}$ so that
\[
\frac{d_{j_i}^{(2)}}{d_{j_{i+1}}^{(2)}} < \frac{\beta}{\gamma_2}.
\]
We can use a similar method as in the previous case, but the diameters are increasing, so the sets of definition in a reversed order. Thus, for fixed $m, i$, we have
\[
\tilde{G} = \{ x : \gamma_1 d_{j_{i+m}}^{(2)} < |x - y_{j_{i+m}}| < \gamma_2 d_{j_{i+m}}^{(2)} \},
\]
\[
\tilde{G}_1 = \tilde{G} \setminus \{ x : |x - y_{j_i}| \leq \gamma_2 d_{j_i}^{(2)} \},
\]
\[
\tilde{G}_2 = \{ x : |x - y_{j_i}| > \gamma_2 d_{j_i}^{(2)} \}.
\]
As before, (24) and (25) hold, and from this, the calculations are identical to those in Case 1.

Case 3. It remains to show that $b$ must satisfying (3). In fact, in this case, if $b$ does not satisfy (3), then there exists a cube $Q$ with its diameter $d$ and a sequence $\{y_j\}$ with $\lim_{j \to \infty} y_j = \infty$, such that (5) holds for the sequence $\{Q_j := Q + y_j\}$. Thus, if we take the sequences $\{f_j\}$ and $\{g_j\}$ defined by (6) and (12), respectively. Then (16) and (17) hold still. Now we denote
\[
B_j = \{ x \in \mathbb{R}^n : |x - y_j| < \gamma_2 d \}
and choose a subsequence \( \{B_{j_k} = B(y_{j_k}, \gamma_2 d)\} \) such that \( B_{j_k} \cap B_{j_l} = \emptyset \) for \( l \neq k \). For selected \( j_k \), let \( f_{j_k} \) and \( g_{j_k} \) be the functions associated with \( Q_{j_k} \). We also define

\[
\tilde{G} = \{ x : \gamma_1 d < |x - y_{j_k}| < \gamma_2 d \},
\]

\[
\tilde{G}_1 = \tilde{G} \setminus \{ x : |x - y_{j_k+m}| \leq \gamma_2 d \},
\]

\[
\tilde{G}_2 = \{ x : |x - y_{j_k+m}| > \gamma_2 d \};
\]

We see that \( \tilde{G}_1 = \tilde{G} - \tilde{G}_2 = \tilde{G} \), by \( B_{j_k} \cap B_{j_k+m} = \emptyset \). Thus, for any \( k, m \in \mathbb{N} \), by (16) and (17) we get

\[
\left( \int_{B_{j_k}} |[b, I_{\alpha}]_1 (f_{j_k}, g_{j_k}) (x) - [b, I_{\alpha}]_1 (f_{j_k+m}, g_{j_k+m}) (x)|^q dx \right)^{1/q} \\
\geq \left( \int_{\tilde{G}} |[b, I_{\alpha}]_1 (f_{j_k}, g_{j_k}) (x)|^q dx \right)^{1/q} - \left( \int_{\tilde{G}_2} |[b, I_{\alpha}]_1 (f_{j_k+m}, g_{j_k+m}) (x)|^q dx \right)^{1/q} \\
\geq \gamma_3 |Q|^{1/nq} - \frac{\gamma_3}{4} |Q|^{1/nq} \\
> \frac{\gamma_3}{2} |Q|^{1/nq}.
\]

Hence,

\[
\left( \frac{1}{d^2} \int_{B_{j_k}} |[b, I_{\alpha}]_1 (f_{j_k}, g_{j_k}) (x) - [b, I_{\alpha}]_1 (f_{j_k+m}, g_{j_k+m}) (x)|^q dx \right)^{1/q} \geq C\gamma_3.
\]

This contradicts the compactness of \([b, I_{\alpha}]_1 \) from \( L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n) \) to \( L^{q,\lambda}(\mathbb{R}^n) \). So \( b \) must also satisfy the condition (3) in Lemma 3.1. Theorem 1.2 is proved. \( \square \)

References

[1] Á. Bényi, W. Damián, K. Moen, and R. Torres, Compactness properties of commutators of bilinear fractional integrals, Math. Z. 280 (2015), no. 1-2, 569–582.
[2] Á. Bényi, Compact bilinear commutators: the weighted case, Michigan Math. J. 64 (2015), no. 1, 39–51.
[3] Á. Bényi and R. Torres, Compact bilinear operators and commutators, Proc. Amer. Math. Soc. 141 (2013), no. 10, 3609–3621.
[4] L. Caffarelli, Elliptic second order equations, Rend. Sem. Mat. Fis. Milano 58 (1988), 253–284.
[5] L. Chaffee, Characterizations of \( BMO \) through commutators of bilinear singular integral operator, arXiv:1410.4587v3.
[6] L. Chaffee and R. Torres, Characterizations of compactness of the commutators of bilinear fractional integral operators, Potential Anal. 43 (2015), no. 3, 481–494.
[7] S. Chanillo, A note on commutators, Indiana Univ. Math. J. 31 (1982), no. 1, 7–16.
[8] Y. Chen and Y. Ding, Compactness of the commutators of parabolic singular integrals, Sci. China Math. 53 (2010), no. 10, 2633–2648.
[9] Y. Chen, Y. Ding, and X. Wang, Compactness of commutators of Riesz potential on Morrey spaces, Potential Anal. 30 (2009), no. 4, 301–313.
[10] _, Compactness for commutators of Marcinkiewicz integral in Morrey spaces, Taiwanese J. Math. 15 (2011), no. 2, 633–658.
[11] _, Compactness for commutators for singular integrals on Morrey spaces, Canad. J. Math. 64 (2012), no. 2, 257–281.
[12] S. Chen and H. Wu, Multiple weighted estimates for commutators of multilinear fractional integral operators, Sci. China Math. 56 (2013), no. 9, 1879–1894.
[13] X. Chen and Q. Xue, Weighted estimates for a class of multilinear fractional type operators, J. Math. Anal. Appl. 362 (2010), no. 2, 355–373.
[14] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Math. Appl. 7 (1987), no. 3-4, 273–279.
[15] F. Chiarenza, M. Frasca, and P. Longo, Interior $W^{2,p}$-estimates for nontdivergence elliptic equations with discontinuous coefficients, Ricerche. Mat. 40 (1991), no. 1, 149–168.
[16] Y. Ding, A characterization of BMO via commutators for some operators, Northeast. Math. J. 13 (1997), no. 4, 422–432.
[17] Y. Ding and T. Mei, Boundedness and compactness for the commutators of bilinear operators on Morrey spaces, Potential Anal. 42 (2015), no. 3, 717–748.
[18] J. Lian, B. Ma, and H. Wu, Commutators of multilinear fractional integrals with weighted Lipschitz functions, Acta Math. Sci. 33A (2013), no. 3, 494–509.
[19] J. Lian and H. Wu, A class of commutators for multilinear fractional integrals in nonhomogeneous spaces, J. Inequal. Appl. 2008 (2008), Article ID 373050, 17 pages.
[20] C. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1938), no. 1, 126–166.
[21] D. Palagachev and L. Softova, Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's, Potential Anal. 20 (2004), no. 3, 237–263.
[22] A. Ruiz and L. Vega, Unique continuation for Schrödinger operators with potential in Morrey spaces, Publ. Mat. 35 (1991), no. 1, 291–298.
[23] Z. Shen, The periodic Schrödinger operators with potentials in the Morrey class, J. Funct. Anal. 193 (2002), no. 2, 314–345.
[24] M. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, Comm. Partial Differential Equations 17 (1992), no. 9-10, 1407–1456.
[25] A. Uchiyama, On the compactness of operators of Hankel type, Tôhoku Math. J. 30 (1978), no. 1, 163–171.
[26] H. Wang and W. Yi, Multilinear singular and fractional integral operators on weighted Morrey spaces, J. Funct. Spaces Appl. 2013 (2013), Art. ID 735795, 11 pages.
[27] S. Wang, The compactness of the commutator of fractional integral operator, China Ann. Math. Ser. A 4 (1987), no. 3, 475–482.
[28] J. Zhang and H. Wu, Oscillation and variation inequalities for singular integrals and commutators on weighted Morrey spaces, Front. Math. China, DOI 10.1007/s11464-014-0186-5 (in press).

Suzhen Mao  
School of Mathematical Sciences  
Xiamen University  
Xiamen Fujian 361005, P. R. China  
E-mail address: suzhen.8605060163.com

Huoxiong Wu  
School of Mathematical Sciences  
Xiamen University  
Xiamen Fujian 361005, P. R. China  
E-mail address: huoxwu@xmu.edu.cn