Silting Theory in triangulated categories with coproducts

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Abstract

We introduce the notion of noncompact (partial) silting and (partial) tilting sets and objects in any triangulated category $D$ with arbitrary (set-indexed) coproducts. We show that equivalence classes of partial silting sets are in bijection with t-structures generated by their co-heart whose heart has a generator, and in case $D$ is compactly generated, this restricts to: i) a bijection between equivalence classes of self-small partial silting objects and left nondegenerate t-structures in $D$ whose heart is a module category and whose associated cohomological functor preserves products; ii) a bijection between equivalence classes of classical silting objects and nondegenerate smashing and co-smashing t-structures whose heart is a module category.

We describe the objects in the aisle of the t-structure associated to a partial silting set $T$ as the Milnor (or homotopy) colimit of sequences of morphisms with successive cones in $\text{Sum}(T)[n]$. We use this fact to develop a theory of tilting objects in very general AB3 abelian categories, a setting and its dual on which we show the validity of several well-known results of tilting and cotilting theory of modules. Finally, we show that if $T$ is a bounded tilting set in a compactly generated algebraic triangulated category $D$ and $H$ is the heart of the associated t-structure, then the inclusion $H \hookrightarrow D$ extends to a triangulated equivalence $D(H) \xrightarrow{\sim} D$ which restricts to bounded levels.

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1 Introduction

Silting sets and objects in triangulated categories were introduced by Aihara and Iyama [1], as a way of overcoming a problem inherent to tilting objects, namely, that mutations are sometimes impossible to define. By extending the class of tilting objects to the wider class of silting objects, they were able to define a concept of silting mutation that always worked. In the initial definition of silting object, a strong generation property was required, in the sense that the ambient triangulated category had to be the thick subcategory generated by the object. As a consequence, the study of silting objects was mainly concentrated on 'categories of compact objects', specially in the perfect derived category of an algebra.

From contributions of several authors (see [1, 39, 27, 30, ...]) it soon became clear that silting complexes were connected with several concepts existing in the literature. For instance, with co-t-structures (equivalently, weight structures), as defined in [12 and 47], with t-structures as defined in [11 and, in the context of Representation theory, also with the so-called simple minded collections (see [30]). As the final point of this route, K"onig and Yang [33] gave, for a finite dimensional algebra \( \Lambda \), a precise bijection between equivalence classes of silting complexes in \( \text{per}(\Lambda) \cong \mathcal{K}^b(\Lambda) \), bounded co-t-structures in \( \mathcal{K}^b(\Lambda) \), bounded t-structures in \( \mathcal{D}^b(\text{mod} - \Lambda) \) whose heart is a length category and equivalence classes of simple minded collections in \( \mathcal{D}^b(\text{mod} - \Lambda) \). In addition, they showed that these bijections were compatible with the concepts of mutation defined in each set. Similar results, also presented in several meetings, were independently obtained in [31] for homologically smooth and homologically nonpositive dg algebras with finite dimensional homology.

In a route similar to the one followed by tilting modules and, more generally, tilting complexes, a few authors (see [62 and 41]) extended the notion of silting object to the unbounded derived category \( \mathcal{D}(\mathcal{R}) \) of a ring \( \mathcal{R} \). The strong generation condition had necessarily to be dropped, but the newly defined concept of 'big' silting complex allowed them to extend K"onig-Yang bijection, except for the simple-minded collections, to the unbounded setting (see [41]). A further step in this direction is done independently in [50] and in this paper. Here we shall introduce a notion of partial silting object in any triangulated category with coproducts, which will still allow a sort of K"onig-Yang bijection. In fact any such partial silting object defines a t-structure in the triangulated category whose heart has a projective generator. This leads naturally to the question of whether this is the way of obtaining all t-structures whose heart has a projective generator. Even more specifically, whether this is the way of obtaining all t-structures whose heart is the module category over a small \( K \)-category or over an ordinary algebra. Our results in the paper give partial answers to these questions and, using the dual concept of partial cosilting object, we can also address the question of which t-structures and, using the dual concept of partial cosilting object, we can also address the question of which t-structures and, using the dual concept of partial cosilting object, we can also address the question of which t-structures have a heart which is a Grothendieck category.

If one follows the development of Tilting theory for modules and complexes of modules, one sees that the so-called classical (i.e. compact) tilting complexes give equivalences of categories. Indeed, as shown by Rickard and Keller work (see [51, 52 and 28]), if \( T \) is a classical tilting complex in \( \mathcal{D}(\mathcal{A}) \) and \( B = \text{End}_{\mathcal{D}(\mathcal{A})}(T) \), then there is an equivalence of triangulated categories \( \mathcal{D}(\mathcal{A}) \sim \mathcal{D}(\mathcal{B}) \) which takes \( T \) to \( B \). When replacing such a classical (=compact) tilting complex by a noncompact one, we do not have an equivalence of triangulated categories but, after replacement of \( T \) by some power \( T^{(1)} \), one actually has a recollement of triangulated categories (see [9 and 13 Section 7]). One of the common features is that a tilting complex \( T \), be it compact or noncompact, defines the t-structure \( \tau_T = (T^{\perp > 0}, T^{\perp < 0}) \) in \( \mathcal{D}(\mathcal{A}) \). In the classical (=compact) case the associated heart \( \mathcal{H}_T \) turns out to be equivalent to \( \text{Mod} - \text{End}_{\mathcal{D}(\mathcal{A})}(B) \) and the inclusion functor \( \mathcal{H}_T \hookrightarrow \mathcal{D}(\mathcal{A}) \) extends to an equivalence of categories \( \mathcal{D}(\mathcal{H}_T) \sim \mathcal{D}(\mathcal{A}) \) (see [19] for the case of a classical tilting object in an abelian category). Surprisingly, with the tilting theory for AB3 abelian categories developed in Section 4 one has that this phenomenon is still true when \( T \) is an infinitely generated \( n \)-tilting module and, even more generally, for any \( (n-) \)tilting object in such an AB3 abelian category (see [17 and 18]). So it is very natural to ask if a similar phenomenon holds for (nonclassical) tilting objects in any triangulated category with coproducts. Note that the phenomenon is discarded if, more generally, one deals just with silting objects (see [50 Corollary 5.2]).

In this paper we define the co-heart of a t-structure in a triangulated category (see the first paragraph
of Section 3) and, when such a category has coproducts, we introduce the notions of (partial) tilting and (partial) silting sets of objects, calling them classical when they consist of compact objects (see Definitions 2 and 3). Our first main result and two of its consequences are the following, all stated for any triangulated category \( D \) with coproducts:

- (Part of Theorem 1) A t-structure \( \tau \) in \( D \) is generated by a partial silting set if and only if it is generated by its co-heart and its heart has a generator. When, in addition, \( D \) is compactly generated, this is equivalent to saying that \( \tau \) is left nondegenerate, its heart has a projective generator and the cohomological functor \( \tilde{H} : D \to \mathcal{H} \) preserves products.

- (Part of Corollary 4) When \( D \) is compactly generated, there is a bijection between equivalence classes of self-small (resp. classical) partial silting sets and left nondegenerate (resp. left nondegenerate smashing) t-structures in \( D \) whose heart is the module category over a small k-category and whose associated cohomological functor \( \tilde{H} \) preserves products. This bijection restricts to another one obtained by replacing 'set' by 'object' and 'small k-category' by 'ordinary algebra'.

- (Part of Proposition 2) When \( D \) is compactly generated, there are: i) a bijection between equivalence classes of classical silting sets (resp. objects) and nondegenerate smashing and co-smashing t-structures whose heart is a module category over a small k-category (resp. ordinary algebra); ii) a bijection between equivalence classes of cosilting pure-injective objects \( Q \) such that \( \perp_{<0} Q \) is closed under taking products and smashing and co-smashing t-structures whose heart is a Grothendieck category.

The concept of partial silting set in our general setting has the problem that it is sometimes difficult to check its defining conditions for a given set of objects. On the other hand, even in the case of a silting object \( T \), where the aisle is \( T^{<0} \), it is not clear how the objects of this aisle can be defined in terms of \( T \). Our second main result partially solves these problems:

- (Part of Theorem 3) When \( T \) is a strongly nonpositive set in \( D \) (see Definition 7), it is partial silting if and only if there is a t-structure \( (\mathcal{V}, \mathcal{V}^\perp) \) such that \( T \subset \mathcal{V} \) and, for some \( q \in \mathbb{Z} \), the functor \( \text{Hom}_D(T, x) \) vanishes on \( \mathcal{V}[q] \) for all \( T \in \mathcal{T} \). Moreover, each object in the aisle of the associated t-structure is a Milnor (or homotopy) colimit of a sequence

\[
X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \ldots \xrightarrow{x_n} X_n \xrightarrow{x_{n+1}} \ldots,
\]

where \( X_0 \in \text{Sum}(\mathcal{T}) \) and \( \text{cone}(x_n) \in \text{Sum}(\mathcal{T})[n] \), for all \( n \geq 0 \).

This description of the aisle has an important (not straightforward) consequence, when \( D = D(A) \) is the derived category of an abelian category \( A \) and \( T \) is an object of \( A \) which is partial silting in \( D(A) \). In this case, the objects in the aisle are precisely those chain complexes which are isomorphic in \( D(A) \) to complexes \( \ldots \rightarrow T^{-n} \rightarrow \ldots \rightarrow T^{-1} \rightarrow T^0 \rightarrow 0 \ldots \), with all the \( T^{-k} \) in \( \text{Sum}(\mathcal{T}) \) (see Proposition 3). This led us to think that it might be possible to extend the well-established theory of tilting modules (see, e.g., 14, 25 and 40 for the classical part, and 3 and 16 for the infinitely generated part) to any abelian category \( A \) whose derived category has Hom sets and arbitrary coproducts. This is indeed the case and Section 3 is devoted to developing such a theory. Definition 8 introduces the concept of tilting object in such an abelian category and the main result of the section, Theorem 5, shows that several known characterizations of tilting modules also work in this general setting. The advantage of the new theory is that it is apt to dualization. In this way tilting and cotilting theory are two sides of a unique theory. This has already been exploited in 18.

The final one of the main results and its consequences provide a partial answer to the question of whether the inclusion from the heart can be extended to a triangulated equivalence.

- (Theorem 7 and Corollaries 7 and 8) Let \( D \) be any compactly generated algebraic triangulated category and let \( T \) be a bounded tilting set in \( D \) (see Definition 9). If \( H = H_T \) is the heart of the associated t-structure in \( D \), then the inclusion \( H \hookrightarrow D \) extends to a triangulated equivalence \( \Psi : D(H) \simto D^* \) which restricts to equivalences \( D^*(H) \simto D^* \), for \( * \in \{+, -1, b\} \), with an appropriate definition of \( D^* \) which is the classical one when \( D = D(A) \), for a dg algebra \( A \).

The organization of the paper goes as follows. Section 2 is of preliminaries, and there we introduce most of the needed terminology. In Section 3 we introduce the co-heart of a t-structure and study its properties. In Section 4 partial silting and partial tilting sets and objects in a triangulated category with coproducts \( D \) are introduced and the mentioned Theorem 1 is proved together with some corollaries which give a König-Yang-like bijection between equivalence classes of partial silting objects and some t-structures in \( D \). In Section 5 we prove Theorem 2. Then Section 6 is devoted to developing a tilting
2 Preliminaries and terminology

All throughout this paper, we shall work over a commutative ring $k$, fixed from now on. All categories will be $k$-categories. That is, the morphisms between two objects form a $k$-module and the composition of morphisms will be $k$-bilinear. All of them are assumed to have Hom sets and, in case of doubts as for the derived category of an abelian category, this Hom set hypothesis will be required. Unless explicitly said otherwise categories will be also additive and all subcategories will be full and closed under taking isomorphisms. Coproducts and products will be always small (i.e. set-indexed). The expression $'$A has coproducts (resp. products)' will then mean that $A$ has arbitrary set-indexed coproducts (resp. products).

When $S \subseteq \text{Ob}(A)$ is any class of objects, we shall denote by $\sum_A(S)$ (resp. $\sum_A(S)$) the subcategory of objects which are finite (resp. arbitrary) coproducts of objects in $S$. Then $\text{add}_A(S)$ (resp. $\text{Add}_A(S)$) will denote the subcategory of objects which are direct summands of objects in $\sum_A(S)$ (resp. $\sum_A(A)$). Also, we will denote by $\text{Prod}_A(S)$ the class of objects which are direct summands of arbitrary products of objects in $S$. When $S = \{V\}$, for some object $V$, we will simply write $\sum_A(V)$ (resp. $\sum_A(V)$) $\text{add}_A(V)$ (resp. $\text{Add}_A(V)$) and $\text{Prod}_A(V)$.

If $S$ is a class (resp. set) as above, we will say that it is a class (resp. set) of generators when, given any nonzero morphism $f : X \rightarrow Y$ in $A$, there is a morphism $g : S \rightarrow X$, for some $S \in S$, such that $f \circ g \neq 0$. Note that when $S$ is a set, this is equivalent to saying that the functor $\prod_{S \in S} \text{Hom}_A(S, ?) : A \rightarrow \text{Ab}$ is faithful. An object $G$ is a generator of $A$, when $\{G\}$ is a set of generators. When $A$ is abelian and $P$ is a class (resp. set) of projective objects, then $P$ is a class (resp. set) of generators if and only if, given any $0 \neq X \in \text{Ob}(A)$, there is nonzero morphism $P \rightarrow X$, for some $P \in P$. When $A$ is $\text{AB3}$ abelian (see definition below) and $S$ is a set, it is a set of generators exactly when each object $X$ of $A$ is an epimorphic image of a coproduct of objects of $S$ (see [57, Proposition IV.6.2]). The concepts of class (resp. set) of cogenerators and of cogenerator are defined dually. Sometimes, in the case of an abelian category $A$ we will employ a stronger version of these concepts. Namely, a class $S \subseteq \text{Ob}(A)$ will be called a generating (resp. cogenerating) class of $A$ when, for each object $X$ of $A$, there is an epimorphism $S \rightarrow X$ (resp. monomorphism $X \rightarrow S$), for some $S \in S$.

We shall say that idempotents split in $A$, when given any object $X$ of $A$ and any idempotent endomorphism $e = e^2 \in \text{End}_A(X)$, there is an isomorphism $f : X \xrightarrow{\sim} X \coprod X$, for some $X, X \in \text{Ob}(A)$, such that $e$ is the composition

$$X \xrightarrow{f} X \coprod X \xrightarrow{(1 \ 0)} X \xrightarrow{(0 \ 1)} X \xrightarrow{f^{-1}} X.$$  

When $A$ is abelian, idempotents split in it.

When $A$ has coproducts, we shall say that an object $X$ is a compact (or small) object when the functor $\text{Hom}_A(X, ?) : A \rightarrow \text{Ab}$ preserves coproducts. That is, when the canonical map $\prod_{i \in I} \text{Hom}_A(T, X_i) \rightarrow \text{Hom}_A(T, \coprod_{i \in I} X_i)$ is bijective, for each family $(X_i)_{i \in I}$ of objects of $A$. More generally, we will say that a set $S$ of objects is self-small, when the canonical map $\prod_{i \in I} \text{Hom}_A(S, S_i) \rightarrow \text{Hom}_A(S, \coprod_{i \in I} S_i)$ is bijective, for each $S \in S$ and each family $(S_i)_{i \in I}$ of objects of $S$. An object $X$ will be called self-small when $\{X\}$ is a self-small set.

If $X$ is a subcategory and $M$ is an object of $A$, a morphism $f : X_M \rightarrow M$ is an $X$-precover (or right $X$-approximation) of $M$ when $X_M$ is in $X$ and, for every morphism $g : X \rightarrow M$ with $X \in X$, there is a morphism $v : X \rightarrow X_M$ such that $f \circ v = g$. When $A$ has coproducts and $S$ is a set of objects, every object $M$ admits a morphism which is both an $\text{Add}(S)$-precover and a $\text{Sum}(S)$-precover, namely the canonical morphism $\epsilon_M : \coprod_{S \subseteq S} \text{S}(\text{Hom}_A(S, M)) \rightarrow M$. If, for each $S \subseteq S$ and each $f \in \text{Hom}_A(S, M)$, we denote by $\iota_{(S, f)} : S \rightarrow \coprod_{S \subseteq S} \text{S}(\text{Hom}_A(S, M))$ the corresponding injection into the coproduct, then $\epsilon_M$ is the unique morphism such that $\epsilon_M \circ \iota_{(S, f)} = f$, for all $S \subseteq S$ and $f \in \text{Hom}_A(S, M)$.

We will frequently use the following 'hierarchy' among abelian categories introduced by Grothendieck (22). Let $A$ be an abelian category.

...
- \( \mathcal{A} \) is \( AB3 \) (resp. \( AB3^* \)) when it has coproducts (resp. products);

- \( \mathcal{A} \) is \( AB4 \) (resp. \( AB4^* \)) when it is \( AB3 \) (resp. \( AB3^* \)) and the coproduct functor \( \coprod : [I, \mathcal{A}] \to \mathcal{A} \) (resp. product functor \( \prod : [I, \mathcal{A}] \to \mathcal{A} \)) is exact, for each set \( I \);

- \( \mathcal{A} \) is \( AB5 \) (resp. \( AB5^* \)) when it is \( AB3 \) (resp. \( AB3^* \)) and the direct limit functor \( \lim \) : \([I^{op}, \mathcal{A}] \to \mathcal{A}\) is exact, for each directed set \( I \).

Note that the \( AB3 \) (resp. \( AB3^* \)) condition is equivalent to the fact that \( \mathcal{A} \) is cocomplete (resp. complete). An \( AB5 \) abelian category \( \mathcal{G} \) having a set of generators (equivalently, a generator), is called a Grothendieck category. A classical example of a Grothendieck category is the category of right modules \( \text{Mod}_k \) over a field \( k \).

In particular, \( \mathcal{G} \) is an \( AB3 \) abelian category, \( \mathcal{S} \subset \text{Ob}(\mathcal{G}) \) is any class of objects, and \( n \) is a natural number, we will denote by \( \text{Pres}^n(\mathcal{S}) \) the subcategory of objects \( X \in \text{Ob}(\mathcal{G}) \) which admit an exact sequence \( \Sigma^{-n} \to \cdots \to \Sigma^{-1} \to \Sigma^0 \to X \to 0 \), with the \( \Sigma^{-k} \) in \( \text{Sum}(\mathcal{S}) \) for all \( k = 0, 1, \ldots, n \).

We refer the reader to [11] for the precise definition of triangulated category, but, divesting from the terminology in that book, for a given triangulated category \( \mathcal{D} \), we will denote by \( \ell[1] : \mathcal{D} \to \mathcal{D} \) its suspension functor. We will then put \( \ell[0] = 1_{\mathcal{D}} \) and \( \ell[k] \) will denote the \( k \)-th power of \( \ell[1] \), for each integer \( k \). (Distinguished) triangles in \( \mathcal{D} \) will be denoted \( X \to Y \to Z \to \ell[1]X \), or also \( X \to Y \to Z \to \ell[n]X \) when the connecting morphism \( w \) needs to be emphasized. A triangulated category between triangulated categories is one which preserves triangles.

Given any additive category \( \mathcal{A} \) and any class \( \mathcal{S} \) of objects in it, we shall denote by \( \mathcal{S}^\perp \) (resp. \( \perp \mathcal{S} \)) the subcategory of objects \( X \in \text{Ob}(\mathcal{A}) \) such that \( \text{Hom}(\mathcal{A}(X, S)) = 0 \) (resp. \( \text{Hom}(\mathcal{A}(S, X)) = 0 \)), for all \( S \in \mathcal{S} \). In the particular case when \( \mathcal{A} = \mathcal{D} \) is a triangulated category and \( n \in \mathbb{Z} \) is an integer, we will denote by \( \mathcal{S}^\perp \mathcal{D} \) (resp. \( \perp \mathcal{S} \mathcal{D} \)) the subcategory of \( \mathcal{D} \) consisting of the objects \( Y \) such that \( \text{Hom}\mathcal{D}(S, Y[k]) = 0 \), for all \( S \in \mathcal{S} \) and all integers \( k \geq n \) (resp. \( k \leq n \)). Symmetrically, the subcategory \( \perp \mathcal{S}^\perp \mathcal{D} \) (resp. \( \mathcal{S}^\perp \mathcal{D} \)) will be the one whose objects \( X \) satisfy \( \text{Hom}\mathcal{D}(X, S[k]) = 0 \), for all \( S \in \mathcal{S} \) and all \( k \geq n \) (resp. \( k \leq n \)). By analogous recipe, one defines \( \mathcal{S}^\perp_{\geq n} \), \( \mathcal{S}^\perp_{< n} \), \( \perp \mathcal{S}^\perp_{\geq n} \) and \( \perp \mathcal{S}^\perp_{< n} \). We will use also the symbol \( \mathcal{S}^\perp_{k \in \mathbb{Z}} \) (resp. \( \perp \mathcal{S}^\perp_{k \in \mathbb{Z}} \)) to denote the subcategory of those objects \( X \) such that \( \text{Hom}\mathcal{D}(S, X[k]) = 0 \) (resp. \( \text{Hom}\mathcal{D}(X, S[k]) = 0 \)), for all \( k \in \mathbb{Z} \).

Unlike the terminology used in the general setting of additive categories, in the specific context of triangulated categories a weaker version of the term ‘class (resp. set) of generators’ is commonly used. Namely, a class (resp. set) \( \mathcal{S} \subset \text{Ob}(\mathcal{D}) \) is called a class (resp. set, resp. set of generators) of \( \mathcal{D} \) when \( \mathcal{S}^\perp \mathcal{D} = \{0\} \). Dually \( \mathcal{C} \) is a class (resp. set) of cogenerators of \( \mathcal{D} \) when \( \perp \mathcal{C} \mathcal{D} = \{0\} \). In case \( \mathcal{D} \) has coproducts, we will say that \( \mathcal{D} \) is compactly generated when it has a set of compact generators. A triangulated category is called algebraic when it is equivalent to the stable category of a Frobenius exact category (see [23], [28]).

Recall that if \( \mathcal{D} \) and \( \mathcal{A} \) are a triangulated and an abelian category, respectively, then an additive functor \( H : \mathcal{D} \to \mathcal{A} \) is a cohomological functor when, given any triangle \( X \to Y \to Z \to \ell[1]X \), one gets an induced long exact sequence in \( \mathcal{A} \):

\[
\cdots \to H^{n-1}(Z) \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots,
\]
where \( H^n := H \circ (\cdot^n) \), for each \( n \in \mathbb{Z} \). Each representable functor \( \text{Hom}_D(X, ?) : D \to k \text{-Mod} \) is cohomological. We will say that \( D \) satisfies Brown representability theorem when \( D \) has coproducts and each cohomological functor \( H : D^{op} \to k \text{-Mod} \) that preserves products (i.e. that, as a contravariant functor \( D \to k \text{-Mod} \), it takes coproducts to products) is representable. We will say that \( D \) satisfies Brown representability theorem for the dual when \( D^{op} \) satisfies Brown representability theorem. Each compactly generated triangulated category satisfies both Brown representability theorem and its dual (\cite{5} Theorem B and subsequent remark).

Given a triangulated category \( D \), a subcategory \( E \) will be called a suspended subcategory when it is closed under taking extensions and \( E[1] \subseteq E \). If, in addition, we have \( E = E[1] \), we will say that \( E \) is a triangulated subcategory. A triangulated subcategory closed under taking direct summands is called a thick subcategory. When the ambient triangulated category \( D \) has coproducts, a triangulated subcategory closed under taking arbitrary coproducts is called a localizing subcategory. Note that such a subcategory is always thick (see the proof of \cite{11} Proposition 1.6.8), which also shows that idempotents split in any triangulated category with coproducts. Clearly, there are dual concepts of cosuspended subcategory and colocalizing subcategories, while those of triangulated and thick subcategory are self-dual. Given any class \( S \) of objects of \( D \), we will denote by \( \text{Sus}^D(S) \) (resp. \( \text{Tri}(S) \), resp. \( \text{Thick}(S) \)) the smallest suspended (resp. triangulated, resp. thick) subcategory of \( D \) containing \( S \). When \( D \) has coproducts, we will let \( \text{Sus}^D(S) \) and \( \text{Loc}_D(S) \) be the smallest suspended subcategory closed under taking coproducts and the smallest localizing subcategory containing \( S \), respectively.

Given an additive category \( A \), we will denote by \( C(A) \) and \( K(A) \) the category of chain complexes of objects of \( A \) and the homotopy category of \( A \). Diverting from the classical notation, we will write superindices for chains, cycles and boundaries in ascending order. We will denote by \( C^-(A) \) (resp. \( K^-(A) \)), \( C^+(A) \) (resp. \( K^+(A) \)) and \( C^b(A) \) (resp. \( K^b(A) \)) the full subcategories of \( C(A) \) (resp. \( K(A) \)) consisting of those objects isomorphic to upper bounded, lower bounded and (upper and lower) bounded complexes, respectively. Note that \( K(A) \) is always a triangulated category of which \( K^-(A) \), \( K^+(A) \) and \( K^b(A) \) are triangulated subcategories. Furthermore, when \( A \) has coproducts, \( C(A) \) and \( K(A) \) also have coproducts, which are calculated pointwise. When \( A \) is an abelian category, we will denote by \( D(A) \) its derived category, which is the one obtained from \( C(A) \) by formally inverting the quasi-isomorphisms (see \cite{61} for the details). Note that, in principle, the morphisms in \( D(A) \) between two objects do not form a set, but a proper class. Therefore, in several parts of this paper, we will require that \( D(A) \) has Hom sets. We shall denote by \( D^-(A) \) (resp. \( D^+(A) \), resp. \( D^b(A) \)) the full subcategory of \( D(A) \) consisting of those complexes \( X^* \) such that \( H^k(X^*) = 0 \), for all \( k \gg 0 \) (resp. \( k \ll 0 \), resp. \( |k| \gg 0 \)), where \( H^k : D(A) \to A \) denotes the \( k \)-th cohomology functor, for each \( k \in \mathbb{Z} \).

When \( D \) is a triangulated category with coproducts, we will use the term Milnor colimit of a sequence of morphisms \( X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_2} \ldots \xrightarrow{x_{n-1}} X_n \) what in \cite{41} is called homotopy colimit. It will be denoted \( \text{Mcolim}(X_n) \), without reference to the \( x_n \). However, for the dual concept in a triangulated category with products we will retain the term homotopy limit, denoted \( \text{Holim}(X_n) \).

Given two subcategories \( X \) and \( Y \) of the triangulated category \( D \), we will denote by \( X \star Y \) the subcategory of \( D \) consisting of the objects \( M \) which fit into a triangle \( X \to M \to Y \). Due to the octahedral axiom, the operation \( \star \) is associative, so that \( X_1 \star X_2 \star \ldots \star X_n \) is well-defined, for each family of subcategories \( (X_i)_{1 \leq i \leq n} \).

A t-structure in \( D \) (see \cite{11} Section 1) is a pair \( (U, W) \) of full subcategories, closed under taking direct summands in \( D \), which satisfy the following properties:

\begin{enumerate}
  \item \( \text{Hom}_D(U, W[-1]) = 0 \), for all \( U \in U \) and \( W \in W \);
  \item \( U[1] \subseteq U \);
  \item For each \( X \in \text{Ob}(D) \), there is a triangle \( U \to X \to V \xrightarrow{\tau} \) in \( D \), where \( U \in U \) and \( V \in W[-1] \) (equivalently, we have \( D = U \star (W[-1]) \)).
\end{enumerate}

It is easy to see that in such case \( W = U^{1}[1] \) and \( U = \perp(W[-1]) = \perp(U^{1}) \). For this reason, we will write a t-structure as \( \tau = (U, U^{1}) \). We will call \( U \) and \( U^{1} \) the aisle and the co-aisle of the t-structure, which are, respectively, a suspended and a cosuspended subcategory of \( D \). The objects \( U \) and \( V \) in the above triangle are uniquely determined by \( X \), up to isomorphism, and define functors \( \tau_U : D \to U \) and \( \tau_U^{1} : D \to U^{1} \) which are right and left adjoints to the respective inclusion functors. We call them the
left and right truncation functors with respect to the given t-structure. Note that \((U[k], U^\perp[k + 1])\) is also a t-structure in \(D\), for each \(k \in \mathbb{Z}\). The full subcategory \(\mathcal{H} = U \cap W = U \cap U^\perp[1]\) is called the heart of the t-structure and it is an abelian category, where the short exact sequences ‘are’ the triangles in \(D\) with its three terms in \(\mathcal{H}\). Moreover, with the obvious abuse of notation, the assignments \(X \rightsquigarrow (\tau_U \circ \tau_U^{-1}[1])(X)\) and \(X \rightarrow (\tau_U^{-1} \circ \tau_U)(X)\) define naturally isomorphic functors \(D \rightarrow \mathcal{H}\) which are cohomological (see \([11]\)). The t-structure \(\tau = (U, U^\perp[1])\) will be called left (resp. right) nondegenerate when \(\bigcap_{k \in \mathbb{Z}} U[k] = 0\) (resp. \(\bigcap_{k \in \mathbb{Z}} U^\perp[k] = 0\)). It will be called nondegenerate when it is left and right nondegenerate. We shall say that \(\tau\) is a semi-orthogonal decomposition when \(U = U[1]\) (equivalently, \(U^\perp = U^\perp[1]\)). In this case \(\tau = (U, U^\perp)\) and both \(U\) and \(U^\perp\) are thick subcategories of \(D\).

When in the last paragraph \(D\) has coproducts, the aisle \(U\) is closed under coproducts, but the coaisle \(U^\perp\) need not be so. When this is also the case or, equivalently, when the truncation functor \(\tau_U : D \rightarrow U\) preserves coproducts, we shall say that \(\tau\) is a smashing t-structure. Dually, when \(D\) has products, \(\tau\) is said to be a co-smashing t-structure when \(U\) is closed under taking products. Assuming that \(D\) has coproducts, if \(S \subset U\) is any class (or set) of objects, we shall say that the t-structure \(\tau\) is generated by \(S\) or that \(S\) is a class (resp. set) of generators of \(\tau\) when \(U^\perp = S^{\perp \leq 0}\). We shall say that \(\tau\) is compactly generated when there is a set of compact objects which generates \(\tau\). Note that such a t-structure is always smashing. Generalizing a bit the classical definition, the following phenomenon will be called infinite dévissage (see \([30]\) Theorem 12.1)).

**Lemma 1.** Let \(D\) be a triangulated category with coproducts, let \(\tau = (U, U^\perp[1])\) be a t-structure (resp. semi-orthogonal decomposition) in \(D\) and let \(S \subset U\) be a set of objects which are compact in \(D\) and such that \(S^{\perp \leq 0} = U^\perp\) (resp. \(S^{\perp \leq 0} = U^\perp\)). If \(V \subset U\) is a closed under coproduct supended (resp. localizing) subcategory of \(D\) such that \(S \subset V\), then we have \(V = U\).

**Example 1.** The following examples of t-structures are relevant for us:

1. (see \([11]\) Example 1.3.2]) Let \(A\) be an abelian category and, for each \(k \in \mathbb{Z}\), denote by \(D^{\leq k}(A)\) (resp. \(D^{\geq k}(A)\)) the subcategory of \(D(A)\) consisting of the complexes \(X^\bullet\) such that \(H^j(X^\bullet) = 0\), for all \(j > k\) (resp. \(j < k\)). The pair \((D^{\leq k}(A), D^{\geq k}(A))\) is a t-structure in \(D(A)\) whose heart is equivalent to \(A\). Its left and right truncation functors will be denoted by \(\tau^{\leq k} : D(A) \rightarrow D^{\leq k}(A)\) and \(\tau^{\geq k} : D(A) \rightarrow D^{\geq k}(A) := D^{\geq k}(A)[-1]\). For \(k = 0\), the t-structure is known as the canonical t-structure in \(D(A)\).

2. Let \(D\) be a triangulated category with coproducts and let \(T\) be a classical tilting object (see Definition \([5]\)). It is well-known, and will be a particular case of our results in Section \([4]\) that the pair \((T^{\perp \geq 0}, T^{\perp \leq 0})\) is a t-structure in \(D\) generated by \(T\). Its heart \(\mathcal{H}_T\) is equivalent to \(\text{Mod} - E\), where \(E = \text{End}_D(T)\).

### 3 The co-heart of a t-structure

In this section \(D\) will be a triangulated category and \(\tau := (U, U^\perp[1])\) will be a t-structure in \(D\). Apart from its heart \(\mathcal{H} = U \cap U^\perp[1]\), we will also consider its co-heart \(\mathcal{C} := U \cap U^\perp[1]\). Note that we do not assume the existence of any co-t-structure in \(D\) with \(C\) as its co-heart. We will denote by \(\tilde{H} : D \rightarrow H\) the associated cohomological functor.

The objects of the co-heart were called ‘Ext-projectives with respect to \(U\)’ in \([5]\). The proof of assertion 1 of the next lemma is essentially that of \([5]\) Lemma 1.3]. We reproduce it here since it will be frequently used throughout the paper.

**Lemma 2.** The following assertions hold:

1. The restriction to the co-heart \(\tilde{H}|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{H}\) is a fully faithful functor and its image consists of projective objects.

2. Suppose now that \(D\) has coproducts and let \(C \in \mathcal{C}\) be any object of the co-heart. The following statements hold true:

   (a) The functor \(\tilde{H}|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{H}\) preserves coproducts;
(b) $\tilde{H}(C)$ is compact in $\mathcal{H}$ if and only if $C$ is compact in $\mathcal{U}$. In particular, when $\tau$ is smashing, $\tilde{H}(C)$ is compact in $\mathcal{H}$ if and only if $C$ is compact in $\mathcal{D}$.

Proof. For each $U \in \mathcal{U}$, we have a canonical truncation triangle

$$\tau_{U[1]}U \xrightarrow{\eta_U} U \xrightarrow{p_U} \tilde{H}(U) \xrightarrow{\tau_{U[1]}1} \tau_{U[1]}U[1].$$

1) Let $C, C' \in \mathcal{C}$ be any two objects and consider the induced map $\text{Hom}_\mathcal{C}(C, C') \to \text{Hom}_\mathcal{H}(\tilde{H}(C), \tilde{H}(C'))$. If $f : C \to C'$ is in the kernel of the latter map, then $p_{C'} \circ f = 0$, which implies that $f$ factors in the form $f : C \xrightarrow{\pi} \tau_{U[1]}C' \xrightarrow{\eta} C'$. But $g = 0$ since $C \in \perp \mathcal{U}[1]$, and hence $\tilde{H}_C$ is faithful.

Suppose now that $h : \tilde{H}(C) \to \tilde{H}(C')$ is a morphism in $\mathcal{H}$. Then the composition $w_{C'} \circ h \circ p_C$ is zero, since $\text{Hom}_\mathcal{D}(C, ?)$ vanishes on $\mathcal{U}[2]$. We then get a morphism $f : C \to C'$ such that $p_{C'} \circ f = h \circ p_C$.

Note that, by definition of $\tilde{H}$, we also have that $p_{C'} \circ f = \tilde{H}(f) \circ p_C$. It follows that $(h - \tilde{H}(f)) \circ p_C = 0$, which implies that $h - \tilde{H}(f)$ factors in the form $\tilde{H}(C) \xrightarrow{\eta_C} (\tau_{U[1]}C)[1] \xrightarrow{\pi} \tilde{H}(C')$. But the second arrow in this composition is zero since it has domain in $\mathcal{U}[2]$ and codomain in $\perp[1]$. Therefore $\tilde{H}_C$ is also faithful.

Note now that if $C \in \mathcal{C}$ and $M \in \mathcal{H}$, then adjunction gives an isomorphism, functorial on both variables,

$$\text{Hom}_\mathcal{H}(\tilde{H}(C), M) \cong \text{Hom}_{\mathcal{U}[1]}(\tau^{U[1]}C, M).$$

If now $\pi : M \to N$ is an epimorphism in $\mathcal{H}$ and we put $K := \text{Ker}_\mathcal{H}(\pi)$, then we get a triangle $M \xrightarrow{\pi} N \to K[1] \xrightarrow{+}$. Since $\text{Hom}_\mathcal{D}(C, K[1]) = 0$, for all $C \in \mathcal{C}$, we conclude that the induced map

$$\pi_* : \text{Hom}_\mathcal{H}(\tilde{H}(C), M) \cong \text{Hom}_\mathcal{D}(C, M) \to \text{Hom}_\mathcal{D}(C, N) \cong \text{Hom}_\mathcal{H}(\tilde{H}(C), N)$$

is surjective, which shows that $\tilde{H}(C)$ is a projective object of $\mathcal{H}$.

2) All throughout the proof of this assertion, we will use that $\mathcal{H}$ is $\text{AB3}$ (see [43] Proposition 3.2).

a) Note also that if $(M_i)_{i \in I}$ is a family of objects of $\mathcal{H}$, then its coproduct in this category is precisely $\tilde{H}(\coprod_{i \in I} M_i)$. This is a direct consequence of the fact that $\tilde{H}_{\perp} : \mathcal{U} \to \mathcal{H}$ is left adjoint to the inclusion $\mathcal{H} \to \mathcal{U}$ (see [46] Lemma 3.1) and the inclusion $\mathcal{U} \to \mathcal{D}$ preserves coproducts. On the other hand, if $(C_i)_{i \in I}$ is a family of objects of $C$, by [41] Remark 1.2.2 we have a triangle

$$\coprod_{i \in I} \tau_{U[1]}C_i \to \coprod_{i \in I} C_i \to \prod_{i \in I} \tilde{H}(C_i) \xrightarrow{+}.$$ .

It immediately follows that $\tilde{H}(\coprod_{i \in I} C_i) \cong \tilde{H}(\coprod_{i \in I} \tilde{H}(C_i))$, and the second member of this isomorphism is precisely the coproduct of the $\tilde{H}(C_i)$ in $\mathcal{H}$.

b) Fix $C \in \mathcal{C}$. By the first triangle of this proof and the fact that $\text{Hom}_\mathcal{D}(C, ?)$ vanishes on $\mathcal{U}[1]$, we have a functorial isomorphism $(p_U)_* : \text{Hom}_\mathcal{D}(C, U) \xrightarrow{\sim} \text{Hom}_\mathcal{D}(C, \tilde{H}(U))$, for each $U \in \mathcal{U}$. Let now $(M_i)_{i \in I}$ be any family of objects of $\mathcal{H}$. Considering that $\tilde{H}_{\perp} : \mathcal{U} \to \mathcal{H}$ is left adjoint to the inclusion functor, we then have a chain of morphisms:

$$\prod_{i \in I} \text{Hom}_\mathcal{D}(C, M_i) = \prod_{i \in I} \text{Hom}_\mathcal{U}(C, M_i) \cong \prod_{i \in I} \text{Hom}_\mathcal{H}(\tilde{H}(C), M_i) \xrightarrow{\text{can}} \text{Hom}_\mathcal{H}(\tilde{H}(C), \tilde{H}(\prod_{i \in I} M_i)) = \text{Hom}_\mathcal{H}(\tilde{H}(C), \tilde{H}(\coprod_{i \in I} M_i)) \cong \text{Hom}_\mathcal{D}(C, \tilde{H}(\coprod_{i \in I} M_i)) \cong \text{Hom}_\mathcal{D}(C, M_i),$$

where $\prod_{i \in I} M_i$ denotes the coproduct in $\mathcal{H}$. It follows that $\tilde{H}(C)$ is compact in $\mathcal{H}$ if and only if $\text{Hom}_\mathcal{D}(C, ?)$ preserves coproducts (in $\mathcal{D}$) of objects of $\mathcal{H}$.

But if $(U_i)_{i \in I}$ is any family of objects in $\mathcal{U}$, then we get a triangle

$$\coprod_{i \in I} \tau_{U[1]}U_i \to \coprod_{i \in I} U_i \to \prod_{i \in I} \tilde{H}(U_i) \xrightarrow{+}$$

(see [41] Remark 1.2.2), which gives an isomorphism $\text{Hom}_\mathcal{D}(C, \coprod_{i \in I} U_i) \xrightarrow{\sim} \text{Hom}_\mathcal{D}(C, \prod_{i \in I} \tilde{H}(U_i))$ since $\text{Hom}_\mathcal{D}(C, ?)$ vanishes on $\mathcal{U}[1]$. It follows immediately that $\text{Hom}_\mathcal{D}(C, ?)$ preserves coproducts (in $\mathcal{D}$) of objects of $\mathcal{H}$ if and only if it preserves coproducts of objects of $\mathcal{U}$. This proves that $\tilde{H}(C)$ is compact in $\mathcal{H}$ if and only if $C$ is compact in $\mathcal{U}$. 


Assume now that $\tau$ is smashing. Proving that $C$ is compact in $\mathcal{D}$ whenever $\operatorname{Hom}_\mathcal{D}(C, ?)$ preserves coproducts of objects in $\mathcal{U}$ is a standard argument. Let $(X_i)_{i \in I}$ be any family of objects of $\mathcal{D}$ and consider the adjoint pair $(\iota_U, \tau_U)$, where $\iota_U : \mathcal{U} \hookrightarrow \mathcal{D}$ is the inclusion functor. Bearing in mind that, due to the smashing condition of $\tau$, the functor $\tau_U : \mathcal{D} \to \mathcal{U}$ preserves coproducts, we then get a chain of isomorphisms

\[
\prod_{i \in I} \operatorname{Hom}_\mathcal{D}(C, X_i) = \prod_{i \in I} \operatorname{Hom}_\mathcal{D}((\iota_U(C), X_i)) \cong \prod_{i \in I} \operatorname{Hom}_U(\iota_U(C), \tau_U(X_i)) = \operatorname{Hom}_\mathcal{D}(C, \prod\iota_U(X_i)) \cong \operatorname{Hom}_\mathcal{D}(C, \prod \tau_U(X_i)) = \operatorname{Hom}_\mathcal{D}(C, \prod X_i) = \operatorname{Hom}_\mathcal{D}(C, \prod X_i).
\]

The following result is a generalization of [11, Proposition 1.3.7]:

**Lemma 3.** Let $X$ be an object of $\mathcal{D}$. The following assertions are equivalent:

1. $\tilde{H}^j(X) = 0$, for all $j > 0$ (resp. $j \leq 0$).

2. $\tau^{U^\perp} X$ is in $\bigcap_{n \in \mathbb{Z}} U^{\perp}[n]$ (resp. $\tau_U X$ is in $\bigcap_{n \in \mathbb{Z}} U[n]$).

**Proof.** We will prove the ‘not in-between brackets’ assertion, the other one following by the duality principle.

Note that if $U \in \mathcal{U}$ then $\tilde{H}^j(U) = \tilde{H}(U[j]) = \tau^{U^\perp[1]}(U[j]) = 0$, for all $j > 0$. If follows from this that if $U \in \bigcap_{n \in \mathbb{Z}} U[n]$, then we have that $\tilde{H}^j(U) = \tilde{H}^j(U[j - 1]) = 0$, for all $j \in \mathbb{Z}$, because $U[j - 1] \in \mathcal{U}$. By the duality principle, we first get that $\tilde{H}^j$ vanishes on $U^\perp$, for all $j \leq 0$, and we also get that if $V \in \bigcap_{n \in \mathbb{Z}} U^{\perp}[n]$, then $\tilde{H}^j(V) = 0$, for all $j \in \mathbb{Z}$.

In the rest of the proof we put $U = \tau_U X$ and $V = \tau^{U^\perp} X$, and consider the corresponding truncation triangle $U \to X \to V \to 0$. The long exact sequence associated to $\tilde{H}$ gives an exact sequence

\[0 = \tilde{H}^j(U) \to \tilde{H}^j(X) \to \tilde{H}^j(V) \to \tilde{H}^{j+1}(U) = 0\]

in $\mathcal{H}$, for all $j > 0$. It follows that assertion 1 holds if and only if $\tilde{H}^j(V) = 0$, for all $j > 0$. But this in turn is equivalent to say that $\tilde{H}^j(V) = 0$, for all $j \in \mathbb{Z}$, due to the previous paragraph. The implication 2) $\Rightarrow$ 1) is then clear.

On the other hand, the implication 1) $\Rightarrow$ 2) reduces to prove that if $V \in U^\perp$ and $\tilde{H}^j(V) = 0$, for all $j \in \mathbb{Z}$, then $V \in \bigcap_{n \in \mathbb{Z}} U^{\perp}[n]$. Suppose that this is not the case, so that there exists an integer $n > 0$ such that $V \in U^{\perp}[n - 1] \setminus U^{\perp}[n]$. That is, we have $V[n - 1] \in U^\perp$, but $V[n] \notin U^\perp$. But then we have $0 = \tilde{H}^n(V) = \tilde{H}(V[n]) = \tau_U(V[n])$, since $V[n] \in U^{\perp}[1]$. It follows that $V[n] \in U^\perp$, which is a contradiction. \qed

**Lemma 4.** The following assertions are equivalent:

1. $\tau$ is generated by $\mathcal{C}$.

2. $\tau$ is a left non-degenerated $t$-structure and, for each $M \in \mathcal{H} \setminus \{0\}$, there is a nonzero morphism $f : C \to M$, where $C \in \mathcal{C}$.

In this case $\tilde{H}(\mathcal{C})$ is a class of projective generators of $\mathcal{H}$.

**Proof.** 1) $\Rightarrow$ 2) Suppose that $M \in \mathcal{H}$ and $\operatorname{Hom}_\mathcal{D}(?, M)$ vanishes on $\mathcal{C}[k]$, for each $k \geq 0$, because $M \in U^{\perp}[1]$. Assertion 1 then says that $M \in U^\perp$, so that $M \in \mathcal{U} \cap U^{\perp} = 0$.

Let us take $U \in \bigcap_{n \in \mathbb{Z}} U[n]$. By Lemma 3 we get that $\tilde{H}^j(U) = 0$, for all $j \leq 0$ (in fact, it even follows that $\tilde{H}^j(U) = 0$, for all $j \in \mathbb{Z}$). Bearing in mind that $\mathcal{C} \subset \mathcal{U}$, for each $j \leq 0$ and each $C \in \mathcal{C}$, we have that

\[0 = \operatorname{Hom}_\mathcal{D}(C, \tilde{H}^j(U)) = \operatorname{Hom}_\mathcal{D}(C, \tau_U \tau^{U^{\perp}[1]}(U[j])) \cong \operatorname{Hom}_\mathcal{D}(C, \tau^{U^{\perp}[1]}(U[j]))\]

But, by the fact that $\operatorname{Hom}_\mathcal{D}(C, ?)$ vanishes on $U[1]$, we also have an isomorphism $\operatorname{Hom}_\mathcal{D}(C, U[j]) \cong \operatorname{Hom}_\mathcal{D}(C, \tau^{U^{\perp}[1]}(U[j]))$. It follows that $\operatorname{Hom}_\mathcal{D}(C, U[j]) = 0$, for all $C \in \mathcal{C}$ and all $j \leq 0$, so that $U \in C^{\perp \leq 0} = U^\perp$ and hence $U \in \mathcal{U} \cap U^\perp = 0$. \qed
2) \implies 1) Since we have \( C[k] \subseteq U \), for all \( k \geq 0 \), the inclusion \( U^\perp \subseteq C^{\perp \subseteq} \) is obvious. Conversely, let us consider \( Y \in C^{\perp \subseteq} \). Since we have an isomorphism \( \Hom_D(C[k], Y) \cong \Hom_D(C[k], Y) \), for all \( C \in C \) and all \( k \geq 0 \), we can assume without loss of generality that \( Y \in U \), and the goal is shifted to prove that \( Y = 0 \). Considering the triangle in the first paragraph of the proof of Lemma 2 with \( U = Y \), we see that \( \Hom_D(C[Y], Y) \) is an isomorphism, for each \( C \in C \). We then have that \( \Hom_D(Y, \tilde{H}(Y)) \) vanishes on \( C \), which, by assertion 2, means that \( \tilde{H}(Y) = 0 \). That is, we have that \( Y \in D[1] \). But then \( Y = Y'[1] \), where \( Y' \in U \) and \( \Hom_D(C[k], Y') = \Hom_D(C[k+1], Y) = 0 \), for all \( k \geq 0 \). It follows that \( Y' \in D[1] \), so that \( Y \in D[2] \). By iterating the process, we conclude that \( Y \in \bigcap_{n \geq 0} U[n] = 0 \).

Suppose now that assertions 1 and 2 hold. For each \( M \in \mathcal{H} \), we have an object \( C \in C \) and a nonzero morphism \( C \to M \) which, by the proof of Lemma 2, factors in the form \( C \to \tilde{H}(C) \to M \). Since \( \tilde{H}(C) \) consists of projective objects (see Lemma 2), we get that it is a class of projective generators of \( \mathcal{H} \).

\[
4 \quad \text{A silting bijection at the unbounded level}
\]

The following concept will be important for us.

**Definition 1.** Let \( \mathcal{D} \) be a triangulated category. A class (or set) \( \mathcal{T} \) of objects in such a category will be called nonpositive (resp. exceptional) when \( \Hom_D(T, T'[k]) = 0 \), for all \( T, T' \in \mathcal{T} \) and all integers \( k \geq 0 \) (resp. \( k \neq 0 \)). When \( \mathcal{D} \) has coproducts, we will say that \( \mathcal{T} \) is strongly nonpositive (resp. strongly exceptional) when \( \text{Sum}(\mathcal{T}) \) (or, equivalently, \( \text{Add}(\mathcal{T}) \)) is a nonpositive (resp. exceptional) class.

All throughout the rest of the section, we assume that \( \mathcal{D} \) is a triangulated category with coproducts.

**Definition 2.** Let \( \mathcal{T} \) be a set of objects of \( \mathcal{D} \). We shall say that \( \mathcal{T} \) is partial silting when the following conditions hold:

1. The pair \( (\mathcal{U}_T, \mathcal{U}_T^\perp[1]) := (\perp(\mathcal{T}^{\subseteq \subseteq}), \mathcal{T}^{\subseteq \subseteq}) \) is a t-structure in \( \mathcal{D} \);
2. \( \Hom_D(T, ?) \) vanishes on \( \mathcal{U}_T[1] \), for all \( T \in \mathcal{T} \).

Note that if \( \mathcal{T} \) is a partial silting set in \( \mathcal{D} \), then \( \mathcal{T} \subseteq \mathcal{U}_T \), and hence \( \mathcal{T} \) is strongly nonpositive. A strongly exceptional partial silting set will be called partial tilting. When \( \mathcal{T} = \{T\} \) is a partial silting (resp. partial tilting) set, we will say that \( T \) is a partial silting (resp. partial tilting) object of \( \mathcal{D} \).

A partial silting (resp. partial tilting) set (resp. object) will be called a silting (resp. tilting) set (resp. object) when it generates \( \mathcal{D} \) as a triangulated category.

**Remark 1.** The explicit definition of the dual notions of (partial) cosilting and (partial) cotilting sets and objects, which make sense in any triangulated category with arbitrary products, are left to the reader. We also leave to her/him the dualization of all the results obtained in this paper.

**Remark 2.** For many 'well-behaved' triangulated categories with coproducts, condition 1 of Definition 2 is automatic. For instance, if \( \mathcal{E} \) is a Frobenius exact category of Grothendieck type as defined in \([53]\), then it was proved in \([56]\) Theorem 2.13 and Proposition 3.9] that, for each set \( \mathcal{T} \) of objects of \( \mathcal{D} := \mathcal{E} \), the pair \( (\perp(\mathcal{T}^{\subseteq \subseteq}), \mathcal{T}^{\subseteq \subseteq}) \) is a t-structure in \( \mathcal{D} \) (see also \([2]\)). Actually, \( \perp(\mathcal{T}^{\subseteq \subseteq}) \) consists of those objects which are isomorphic in \( \mathcal{D} \) to direct summands of objects that admit a continuous transfinite filtration in \( \mathcal{E} \) with successive factors in \( \bigcup_{k \geq 0} \mathcal{T}[k] \).

**Remark 3.** In an independent recent work, Psaroudakis and Vitoria (see \([56]\) Definition 4.1]) call an object \( T \) of \( \mathcal{D} \) silting when \( (\mathcal{T}^{\subseteq \subseteq}, \mathcal{T}^{\subseteq \subseteq}) \) is a t-structure such that \( T \in \mathcal{T}^{\perp \subseteq \subseteq} \). From Theorem 4 below one can easily derive that their definition is equivalent to our definition of silting object.

We immediately get some examples. An extension of the second one will be given in later sections.

**Example 2.** 1. Let \( \mathcal{D} \) be compactly generated (e.g. \( \mathcal{D} = \mathcal{D}(A) \), where \( A \) is a dg algebra). Any set \( \mathcal{T} \) of compact objects such that \( \Hom_D(T, T'[k]) = 0 \), for all \( T, T' \in \mathcal{T} \) and all \( k > 0 \) (resp. \( k \neq 0 \)), is a partial silting (resp. partial tilting) set. In such case, it is a silting (resp. tilting) set if and only if \( \mathcal{S} \subseteq \text{thick}_D(T) \), for some (resp. every) set \( \mathcal{S} \) of compact generators of \( \mathcal{D} \) (e.g. \( \mathcal{S} = \{A\} \) when \( \mathcal{D} = \mathcal{D}(A) \)).
2. When $\mathcal{D} = \mathcal{D}(A)$ is the derived category of an ordinary algebra $A$ and $\mathcal{T} = \{T\}$, where $T$ is a big silting complex (see [2], also called big semi-tilting complex in [62]). That is, when $\text{thick}_{\mathcal{D}(R)}(\text{Add}(T)) = \text{thick}_{\mathcal{D}(R)}(\text{Add}(R)) = \mathcal{K}^b(\text{Proj} - R)$ and $\text{Hom}_\mathcal{D}(T, T[k]) = 0$, for all sets $I$ and integers $k > 0$.

Proof. 1) We will prove the statement for the partial silting case. The corresponding statement when replacing ‘silting’ by ‘tilting’ is clear.

By [30] Theorems 12.1 and 12.2, we know that each $X \in \mathcal{U}_\mathcal{T}[1]$ is the Milnor colimit of a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} \ldots,$$

where $X_0$ and all cones of the $f_n$ are in $\text{Sum}(\coprod_{T \in \mathcal{T}, k > 0} T[k])$. It follows by induction that $\text{Hom}_\mathcal{D}(T, X_n) = 0$, for all $T \in \mathcal{T}$ and all $n \in \mathbb{N}$, from which we get that $\text{Hom}_\mathcal{D}(T, X) \cong \lim_{\to} \text{Hom}_\mathcal{D}(T, X_n) = 0$ since each $T \in \mathcal{T}$ is a compact object.

In this situation $\mathcal{T}$ is a silting set if and only if $\mathcal{T}$ is a set of compact generators of $\mathcal{D}$. By [25] Theorem 5.3, this is equivalent to the condition mentioned in the statement.

2) It follows from the work in [62] and [4].

Historically, silting and tilting objects or complexes were assumed to be compact. The terminology that we use in this paper, in particular Definition 2, is reminiscent of the one used for tilting modules and this justifies the following.

**Definition 3.** A classical (partial) silting (resp. classical (partial) tilting) set of $\mathcal{D}$ will be a (partial) silting (resp. (partial) tilting) set consisting of compact objects. An object $T$ will be called classical (partial) silting object (resp. classical (partial) tilting object) when the set $\{T\}$ is so.

**Example 3.**

1. Let $\mathcal{A}$ be an abelian category such that $\mathcal{D}(\mathcal{A})$ has Hom sets and arbitrary coproducts (see Setup [11] below), and let $\mathcal{P}$ be a set of projective generators. Then $\mathcal{P}$ is a tilting (and hence silting) set of $\mathcal{D}(\mathcal{A})$.

2. Let $\mathcal{A}$ be an abelian category such that $\mathcal{D}(\mathcal{A})$ has Hom sets and arbitrary products, and let $\mathcal{I}$ be a set of injective cogenerators. Then $\mathcal{I}$ is a tilting (and hence silting) set in $\mathcal{D}(\mathcal{A})^{op} \cong \mathcal{D}(\mathcal{A}^{op})$.

Proof. Example 1 is [50] Example 4.2(ii). Example 2 is dual.

**Lemma 5.** Let $\mathcal{T}$ be a partial silting set in $\mathcal{D}$, let $\tau = (\mathcal{U}, \mathcal{U}^\perp[1])$ be its associated t-structure. The following assertions hold:

1. $\mathcal{T}^{\perp > 0}$ consists of the objects $X \in \mathcal{D}$ such that $\tau^\mathcal{U}X \in \mathcal{T}^{\perp \leq 0}$

2. Let $f : T' \to U$ be a morphism, where $T' \in \text{Sum}(\mathcal{T})$ and $U \in \mathcal{U}$. The following statements are equivalent:

   (a) $f$ is a Sum(\mathcal{T})-precover.

   (b) If $T' \xrightarrow{f} U \to Z \xrightarrow{+} \text{ is a triangle in } \mathcal{D}$, then $Z \in \mathcal{U}[1]$.

   (c) In particular, if $(U_i)_{i \in I}$ is a family of objects of $\mathcal{U}$ and $(f_i : T_i \to U_i)_{i \in I}$ is a family of Sum(\mathcal{T})-precovers, then the morphism $\coprod f_i : \coprod_{i \in I} T_i \to \coprod_{i \in I} U_i$ is also a Sum(\mathcal{T})-precover.

Proof. 1) Due to the definition of partial silting set, we have $\mathcal{U} \subseteq \mathcal{T}^{\perp > 0}$, and then the inclusion $\mathcal{U} \ast \mathcal{T}^{\perp \leq 0} \subseteq \mathcal{T}^{\perp > 0}$ is clear since $\mathcal{T}^{\perp > 0}$ is closed under extensions. Therefore all objects $X$ such that $\tau^\mathcal{U}X \in \mathcal{T}^{\perp \leq 0}$ are in $\mathcal{T}^{\perp > 0}$.

Conversely, let $X \in \mathcal{T}^{\perp > 0}$ and consider the triangle

$$X \to \tau^\mathcal{U}X \to (\tau_\mathcal{U}X)[1] \to \ldots$$

given by truncating with respect to $\tau$. Its outer vertices are in $\mathcal{T}^{\perp > 0}$ and, hence, its three vertices are in this subcategory. In particular, $Y := \tau^\mathcal{U}X$ satisfies that $\text{Hom}_\mathcal{D}(T, Y[k]) = 0$, for all $k > 0$. But we also have that $Y \in \mathcal{U}^\perp = \mathcal{T}^{\perp \leq 0}$. It follows that $Y \in \mathcal{T}^{\perp \leq 0}$. 

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2) We only need to prove the implication $a) \implies b)$ for the reverse one is obvious since $\text{Hom}_D(T, \cdot)$ vanishes on $U[1]$. Note first that $Z \in \mathcal{U}$. Moreover, by applying the long exact sequence associated to $\text{Hom}_D(T, \cdot)$ and the surjectivity of $f_\tau = \text{Hom}_D(T, f) : \text{Hom}_D(T, T') \to \text{Hom}_D(T, U)$, one also gets that $\text{Hom}_D(T, Z) = 0$. We then get that $\text{Hom}_D(T, Z[j]) = 0$, for all $j \geq 0$ and all $T \in \mathcal{T}$. By assertion 1, we then have $Z \in \mathcal{T}_{\leq 0} = \mathcal{T}_{\leq 0}[1] = \mathcal{U}[1] \ast \mathcal{T}_{\leq 0}$, so that $\tau_0 U[1]Z \in \mathcal{T}_{\leq 0}$. But then the canonical morphism $Z \to \tau_0 U[1]Z$ is the zero one. Indeed, we have $U[1] = \mathcal{T}_{\leq 0} \supset \mathcal{T}_{\leq 0}$, which implies that $U = \mathcal{A}(\mathcal{T}_{\leq 0}) \subset \mathcal{A}(\mathcal{T}_{\leq 0})$. We then get that the canonical morphism $\tau_0 U[1]Z \to Z$ is a retraction, so that $Z \in U[1]$. \hfill \square

**Lemma 6.** Let $\mathcal{T}$ be a partial silting set in $\mathcal{D}$ and $(\mathcal{U}, \mathcal{U}^\perp[1]) := (\mathcal{A}(\mathcal{T}^\perp), \mathcal{A}(\mathcal{T}^\perp))$ be its associated $t$-structure. Then we have $\mathcal{C} = \text{Add}(\mathcal{T})$, where $\mathcal{C} = \mathcal{U} \cap \mathcal{U}^\perp[1]$ is the co-heart of the $t$-structure.

**Proof.** Let $C \in \mathcal{C}$ be any object. We claim that the canonical $\text{Sum}(T)$-precover $f : T' := \bigsqcup_{T \in \mathcal{T}} \text{Hom}_D(T, C) \to C$ is a retraction. Indeed, if we complete to a triangle $T' \xrightarrow{f} C \xrightarrow{\theta} Z \to$, then the previous lemma says that $Z \in U[1]$, which implies that $g = 0$ due to the definition of the coheart. Therefore $f$ is a retraction, as claimed. \hfill \square

Recall that an abelian category is **locally small** when the subobjects of any object form a set. We are now ready to prove the main result of this section.

**Theorem 1.** Let $\mathcal{D}$ be a triangulated category with coproducts and let $\tau = (\mathcal{U}, \mathcal{U}^\perp[1])$ be a $t$-structure in $\mathcal{D}$, with heart $\mathcal{H}$ and co-heart $\mathcal{C}$. The following assertions are equivalent:

1. $\tau$ is generated by a partial silting set.
2. $\tau$ is generated by a set $\mathcal{T}$ such that $\mathcal{T}^\perp = \mathcal{U} \ast \mathcal{T}_{\leq 0}$.
3. $\tau$ is generated by $\mathcal{C}$ and $\mathcal{H}$ has a set of (projective) generators.
4. $\tau$ is generated by a set, also generated by $\mathcal{C}$ and $\mathcal{H}$ is locally small.

When in addition $\mathcal{D}$ satisfies Brown representability theorem for the dual (e.g. when $\mathcal{D}$ is compactly generated), they are also equivalent to:

5. $\tau$ is left nondegenerate, $\mathcal{H}$ has a projective generator and the cohomological functor $\hat{H} : \mathcal{D} \to \mathcal{H}$ preserves products.

**Proof.** 1) $\implies$ 2) is a direct consequence of Lemma 5 when taking as $\mathcal{T}$ any partial silting set which generates $\tau$.

2) $\implies$ 1) The inclusion $\mathcal{U} \subseteq \mathcal{T}^\perp$ gives condition 2) of the definition of partial silting set for $\mathcal{T}$. Condition 1) of that definition is automatic since, by hypothesis, $\tau$ is generated by $\mathcal{T}$.

1) $\implies$ 3) Let $\mathcal{T}$ be a partial silting set which generates $\tau$. By Lemma 6 we get that $\tau$ is generated by $\mathcal{C}$. Moreover, this same lemma together with Lemma 2 give that $\hat{H}(\mathcal{C}) = \text{Add}(\hat{H}(\mathcal{T}))$. Therefore $\hat{H}(\mathcal{T})$ is a set of projective generators of $\mathcal{H}$ due to Lemma 4.

3) $\implies$ 4) By [1] Proposition IV.6.6], we know that $\mathcal{H}$ is locally small. It remains to check that $\tau$ is generated by a set. Let $G$ be a generator of $\mathcal{H}$. The images in $\mathcal{H}$ of morphisms $\hat{H}(C) \to G$, with $C \in \mathcal{C}$, form a set, denoted by $\mathcal{Y}$ in the sequel, because $\mathcal{H}$ is locally small. We put $t(G) = \sum_{Y \in \mathcal{Y}} Y$. This sum exists because $\mathcal{H}$ is AB3 (see [6] Proposition 3.2]). If we had $G/t(G) \neq 0$, then we would have a nonzero morphism $f : \hat{H}(C') \to G/t(G)$, for some $C' \in \mathcal{C}$ (see Lemma 4 and its proof). Due to the projective condition of $\hat{H}(C')$ in $\mathcal{H}$, this morphism would factor in the form $f : \hat{H}(C') \xrightarrow{\pi} G \xrightarrow{\pi} G/t(G)$, where $\pi$ is the projection. But then $\text{Im}(\hat{H}(g)) \subseteq t(G)$, which would imply that $f = \pi \circ g = 0$, and thus a contradiction. Therefore we have $t(G) = G$. For each $Y \in \mathcal{Y}$, fix now a morphism $f_Y : \hat{H}(C_Y) \to G$, with $C_Y \in \mathcal{C}$, such that $\text{Im}(f_Y) = Y$. It immediately follows that if we put $\mathcal{T} := \{C_Y : Y \in \mathcal{Y}\}$, then $\hat{H}(\mathcal{T})$ is a set of projective generators of $\mathcal{H}$. But, due to Lemma 4 we then have $\hat{H}(\mathcal{C}) = \text{Add}(\hat{H}(\mathcal{T}))$, and also $\mathcal{C} = \text{Add}(\mathcal{T})$. Then $\mathcal{S} := \mathcal{T}$ is a set which generates $\tau$ since $\tau$ is generated by $\mathcal{C}$.

4) $\implies$ 1) Let $0 \neq M \in \mathcal{H}$ be any object. By Lemma 4 we have a nonzero morphism $g : C \to M$, for some $C \in \mathcal{C}$. By the proof of Lemma 2, we know that it factors through $\hat{H}(C)$, so that we have a nonzero morphism $\hat{H}(C) \to M$, for some $C \in \mathcal{C}$. Then, due to the locally small condition of $\mathcal{H}$ and the fact that,
by Lemma\cite{4} $\tilde{H}(C)$ is a class of projective generators of $\mathcal{H}$, we get that each object of $\mathcal{H}$ is an epimorphic image of a coproduct of objects of $\tilde{H}(C)$.

Let now $S$ be a set of generators of $\tau$. Note that if $0 \neq M \in \mathcal{H}$, then there is a nonzero morphism $f : S \to M$, for some $S \in S$, for otherwise we would have $\text{Hom}(S[k], M) = 0$, for all $S \in S$ and $k \geq 0$, because $\mathcal{H} \subseteq \mathcal{U}^{-1}[1]$. That is, we would have that $M \in S^{-1}\mathcal{U} = \mathcal{U}^{-1}$, and hence $M \in \mathcal{U} \cap \mathcal{U}^{-1} = 0$, which is a contradiction.

Note also that, by the proof of Lemma\cite{2} $f$ factors through $\tilde{H}(S)$, so that, for each $M \in \mathcal{H}$, we have a nonzero morphism $\tilde{H}(S) \to M$, with $S \in S$.

Fixing now an epimorphism $\pi_S : \tilde{H}(C_S) \to \tilde{H}(S)$ in $\mathcal{H}$, where $C_S \in C$, for each $S \in S$, we get that $\mathfrak{T} : \{C_S : S \in S\}$ then $\tilde{H}(T)$ is a set of projective generators of $\mathcal{H}$, because, for each $M \in \mathcal{H}$, there is a nonzero morphism $\tilde{H}(C_S) \to M$, for some $S \in S$. In particular, we get that $\tilde{H}(C) = \text{Add}(\tilde{H}(T))$ and, by Lemma\cite{2} we conclude that $C = \text{Add}(\mathfrak{T})$. Then $\mathfrak{T}$ is the desired partial silting set which generates $\tau$.

1) $= 3) \implies 5)$ For this implication we do not need the full strength of the dual Brown representability theorem. It is enough for $D$ to have products. The left nondegeneracy of $\tau$ follows from Lemma\cite{3} By Proposition 3.2, we know that $\mathcal{H}$ is $\text{AB3 and AB3}^*$. On the other hand, the proof of implication 1) $\implies 3)$ shows that $\tilde{H}(\mathfrak{T})$ is a set of projective generators of $\mathcal{H}$, so that this category is also $\text{AB4}^*$ (use the dual of \cite{46} Corollary 3.2.9]). Put now $T_0 := \prod_{T \in \mathfrak{T}} T$. Then $\tilde{H}(T_0)$ a projective generator of $\mathcal{H}$. We claim that the composition of functors

$$D \xrightarrow{\tilde{H}} \mathcal{H} \xrightarrow{\text{Hom}_H(\tilde{H}(T_0), ?)} \text{Ab}$$

is naturally isomorphic to the functor $\text{Hom}_D(T_0, ?) : D \to \text{Ab}$. Indeed, by the proof of Lemma\cite{2} we know that if $X \in D$ and we put $M = \tilde{H}(X)$, $U = \gamma \mu X$ and $C = T_0$ in that proof, we get an isomorphism

$$\text{Hom}_H(\tilde{H}(T_0), \tilde{H}(X)) \cong \text{Hom}_D(T_0, \tilde{H}(X)) \cong \text{Hom}_D(T_0, \gamma \mu X)$$

which is functorial on $X$. But the adjoint pair $(\gamma \mu : U \leftarrow D, \gamma \mu : D \to U)$ and the fact that $T_0 \in U$ give another functorial isomorphism

$$\text{Hom}_D(T_0, \gamma \mu X) = \text{Hom}_U(T_0, \gamma \mu X) \cong \text{Hom}_D(T_0, X).$$

We now prove that $\tilde{H} : D \to \mathcal{H}$ preserves products. Let $(X_i)_{i \in I}$ be a family of objects of $D$ and consider the canonical morphism $\psi : \tilde{H}(\prod_{i \in I} X_i) \to \prod_{i \in I} \tilde{H}(X_i)$, where $\prod^*$ stands for the product in $\mathcal{H}$. Due to the fact that $\tilde{H}(T_0)$ is a projective generator of $\mathcal{H}$, in order to prove that $\psi$ is an isomorphism it is enough to prove that the induced map

$$\psi_* : \text{Hom}_H(\tilde{H}(T_0), \tilde{H}(\prod_{i \in I} X_i)) \to \text{Hom}_H(\tilde{H}(T_0), \prod_{i \in I} \tilde{H}(X_i)) \cong \prod_{i \in I} \text{Hom}_H(\tilde{H}(T_0), \tilde{H}(X_i))$$

is an isomorphism. But this is a direct consequence of the previous paragraph and the fact that the functor $\text{Hom}_D(T_0, ?) : D \to \text{Ab}$ preserves products.

5) $\implies$ 1) In the rest of the proof we assume that $D$ satisfies Brown representability theorem for the dual (BRT* in the sequel). Fix now a projective generator $P$ of $H$ and consider the composition functor

$$D \xrightarrow{\tilde{H}} \mathcal{H} \xrightarrow{\text{Hom}_H(P, ?)} \text{Ab}.$$
By [49 Theorem 7.2], we know that if \( D \) is well-generated in the sense of Neeman (see [41 Definition 8.1.6 and Remark 8.1.7] and [55]) and it is algebraic, then there exists a dg category \( \mathcal{A} \) and a set \( S \) of objects in \( D(\mathcal{A}) \) such that \( D \cong \text{Loc}_{D(\mathcal{A})}(S) \). In particular \( D \) has products in that case and, by [41 Proposition 8.4.2], we also know that \( D \) satisfies Brown representability theorem. Note that the derived category of a Grothendieck category is an example of well-generated algebraic triangulated category.

**Corollary 1.** Let \( \tau \) be a t-structure in \( D \). The following assertions hold:

1. If \( D \) satisfies Brown representability theorem for the dual, and \( \tau \) is left nondegenerate co-smashing and its heart has a projective generator, then \( \tau = (\perp(T^{\perp \leq 0}), T^{\perp \leq 0}) \) for some partial silting \( T \) in \( D \).
2. If \( D \) has products and satisfies Brown representability theorem (e.g. if \( D \) is well-generated algebraic), then the following statements are equivalent:
   
   (a) \( \tau \) is right nondegenerate, its heart \( H \) has an injective cogenerator and the cohomological functor \( \tilde{H} : D \rightarrow H \) preserves coproducts.
   
   (b) \( \tau = (\perp < 0), (\perp < 0)^\perp) \), for some partial cosilting set \( Q \) in \( D \) (see Remark 7).

**Proof.**
1) Note that if \( \tau \) is co-smashing, then \( H \) is closed under taking products in \( D \), so that products in \( H \) are calculated as in \( D \). On the other hand, if \( (M_i)_{i \in I} \) is a family of objects of \( D \), then the associated truncation triangle with respect to \( \tau = (\mathcal{U}, \mathcal{U}^{\perp [1]}) \) is

\[
\prod_{i \in I} \tau_\mathcal{U} M_i \rightarrow \prod_{i \in I} M_i \rightarrow \prod_{i \in I} \tau_\mathcal{H} M_i \rightarrow.
\]

Then both truncation functors \( \tau_\mathcal{U} : D \rightarrow \mathcal{U} \) and \( \tau_\mathcal{H} : D \rightarrow \mathcal{H} \) preserve products, and the same can be said about the 'shifted' truncations functors \( \tau_\mathcal{U}[1] \) and \( \tau_\mathcal{H}[1] \). In particular, the cohomological functor \( \tilde{H} = \tau_\mathcal{H}[1] \circ \tau_\mathcal{U} : D \rightarrow H \) preserves products, and assertion 1 follows from Theorem 1.

2) The equivalence of assertions 2.a and 2.b is the dual version of the equivalence of assertions 1 and 5 in Theorem 1.

**Corollary 2.** Let \( T \) be a partial silting set in \( D \), let \( \tau \) be the associated t-structure and let \( \tilde{H} : D \rightarrow H \) the induced cohomological functor. The following assertions are equivalent:

1. \( \tilde{H}(T) \) is a set of compact projective generators of \( H \);
2. \( T \) is a self-small set in \( D \).

**Proof.** By the proof of implication 1) \( \Rightarrow \) 3) in last theorem, we know that \( \tilde{H}(T) \) is a set of projective generators of \( H \).

1) \( \Rightarrow \) 2) By Lemma 2, we know that all objects of \( T \) are compact in \( \mathcal{U} = \perp(T^{\leq 0}) \). Since \( T \subseteq \mathcal{U} \) the self-smallness of \( T \) is clear.

2) \( \Rightarrow \) 1) We will prove that each \( T \in T \) is compact in \( \mathcal{U} \), which, by Lemma 2, will end the proof. Let \( (U_i)_{i \in I} \) be a family in \( \mathcal{U} \) and fix a Sum(\( T \))-precover \( f_i : T_i \rightarrow U_i \) and complete to a corresponding triangle \( T_i \rightarrow U_i \rightarrow Z_i \rightarrow \), for each \( i \in I \). Then \( f = \prod f_i : \prod_{i \in I} T_i \rightarrow \prod_{i \in I} U_i \) is also a Sum(\( T \))-precover (see Lemma 3). By this same lemma, we have that \( \text{Hom}_D(T, ?) \) vanishes on each \( Z_i \) and on \( \prod_{i \in I} Z_i \), for all \( T \in T \). We then have the following commutative square, where the horizontal arrows are epimorphisms:

\[
\begin{array}{ccc}
\prod_{i \in I} \text{Hom}_D(T, T_i) & \rightarrow & \prod_{i \in I} \text{Hom}_D(T, U_i) \\
\downarrow \sim & & \downarrow \\
\text{Hom}_D(T, \prod_{i \in I} T_i) & \rightarrow & \text{Hom}_D(T, \prod_{i \in I} U_i)
\end{array}
\]

for every \( T \in T \). Moreover, the left vertical arrow is an isomorphism because \( T \) is self-small. It follows that the right vertical arrow is an epimorphism. But it is always a monomorphism. Therefore \( T \) is compact in \( \mathcal{U} \), as desired. \( \square \)
The following definition is very helpful.

**Definition 4.** Two strongly nonpositive sets $\mathcal{T}$ and $\mathcal{T}'$ in $\mathcal{D}$ will be called equivalent when $\text{Add}(\mathcal{T}) = \text{Add}(\mathcal{T}')$. In particular, two partial silting objects $T$ and $T'$ will be equivalent when $\text{Add}(T) = \text{Add}(T')$.

**Corollary 3.** Let $\mathcal{D}$ be a triangulated category with coproducts. The assignment $\mathcal{T} \mapsto \tau_{\mathcal{T}} = (\mathcal{T}^{>0}, \mathcal{T}^{<0})$ gives a one-to-one correspondence between equivalence classes of partial silting sets (with just one object) and $t$-structures in $\mathcal{D}$ generated by their co-heart whose heart has a generator. This restricts to:

1. A bijection between equivalence classes of silting objects and (right) nondegenerate $t$-structures in $\mathcal{D}$ generated by their co-heart whose heart has a generator.

2. A bijection between equivalence classes of self-small (resp. classical) partial silting sets and $t$-structures (resp. smashing $t$-structures) in $\mathcal{D}$ generated by their co-heart whose heart is the module category over a small $k$-category.

3. A bijection between equivalence classes of self-small (resp. classical) partial silting objects and $t$-structures (resp. smashing $t$-structures) in $\mathcal{D}$ generated by their co-heart whose heart is the module category over an ordinary algebra.

4. A bijection between equivalence classes of self-small (resp. classical) silting objects and (right) nondegenerate $t$-structures (resp. (right) nondegenerate smashing $t$-structures) in $\mathcal{D}$ generated by their co-heart whose heart is the module category over an ordinary algebra.

**Proof.** The general bijection is a consequence of Theorem 1 and Lemma 6. Note that a set $\mathcal{T}$ is partial silting if and only if $\hat{T} \mathcal{D} \text{Add}(\mathcal{T}) = \text{Add}(\mathcal{T}')$. Therefore we have $\tau_{\mathcal{T}} = (\mathcal{T}^{>0}, \mathcal{T}^{<0})$.

Let now check that this bijection restricts to the indicated bijections:

1) Given a partial silting set $\mathcal{T}$ of $\mathcal{D}$ and putting $\hat{T}$ as above, we have a chain of double implications $\mathcal{T} \mathcal{D} \tau_{\mathcal{T}}$ generates $\mathcal{D} \tau_{\mathcal{T}}$.

2) The general part of this bijection, concerning self-small partial silting sets, follows directly from Corollary 2 and Proposition 1. If in this bijection the set $\mathcal{T}$ consists of compact objects, then $\tau_{\mathcal{T}}$ is smashing. Conversely, suppose that $\tau$ is a smashing $t$-structure. We select then a set $\mathcal{P}$ of compact projective generators of $\mathcal{H}$. Since also $\hat{\mathcal{H}}(\mathcal{C})$ is a class of projective generators of $\mathcal{H}$, each $P \in \mathcal{P}$ is a direct summand of an object $\hat{\mathcal{H}}(C)$, with $C \in \mathcal{C}$. It follows from lemma 2 that $P \cong \hat{\mathcal{H}}(C_P)$, for some $C_P \in \mathcal{C} \cap \mathcal{D}^c$. Then $\mathcal{T} = \{C_P: P \in \mathcal{P}\}$ is a classical partial silting set. Moreover, we then have that $\hat{\mathcal{H}}(\mathcal{C}) = \text{Add}(\mathcal{P}) = \text{Add}(\hat{\mathcal{H}}(\mathcal{T}))$, which implies that $\mathcal{C} = \text{Add}(\mathcal{T})$. Therefore we have $\tau = \tau_{\mathcal{T}}$.

3) The bijection here is an obvious restriction of the bijection in 2.

4) This bijection follows from the bijection in 1 and from Corollary 2 for the self-small case, and that this bijection restricts to the one for classical (=compact) silting objects follows as the bijection in 2) or 3).

In the particular case when $\mathcal{D}$ satisfies BRT*, we can use assertion 5 of Theorem 1 to obtain the following bijection, whose proof goes along the lines of the previous corollary and is left to the reader.

**Corollary 4.** Suppose that $\mathcal{D}$ satisfies Brown representability theorem for the dual (e.g. when $\mathcal{D}$ is compactly generated). The assignment $\mathcal{T} \mapsto \tau_{\mathcal{T}} = (\mathcal{T}^{\leq 0}, \mathcal{T}^{>0})$ gives a bijection between equivalence classes of partial silting (resp. silting) sets (with just one object) and left (resp. left and right) nondegenerate $t$-structures in $\mathcal{D}$ whose heart has a projective generator and whose associated cohomological functor preserves products. This bijection restricts to:
1. A bijection between equivalence classes of self-small (resp. classical) partial silting sets and left nondegenerate (resp. left nondegenerate smashing) $t$-structures in $D$ whose heart is the module category over a small $k$-category and whose associated cohomological functor preserves products.

2. A bijection between equivalence classes of self-small (resp. classical) partial silting objects and left nondegenerate (resp. left nondegenerate smashing) $t$-structures in $D$ whose heart is the module category over an ordinary algebra and whose associated cohomological functor preserves products.

**Definition 5.** Let $D$ have arbitrary coproducts and products. An object $Y$ of $D$ will be called pure-injective when, given any set $I$ and the canonical coproduct $\lambda: Y^{(I)} \to Y^I$, the transpose map $\lambda^*: \text{Hom}_D(Y^I, Y) \to \text{Hom}_D(Y^{(I)}, Y)$ is surjective.

Note that if $D$ is compactly generated, then $\lambda: Y^{(I)} \to Y^I$ is a pure monomorphism in the terminology of [34]. In particular, by [34] Theorem 1.8 our notion of pure-injectivity agrees in that case with the classical one. Recall also that if $T$ is a compact object of $D$ and $E$ is the minimal injective cogenerator of $\text{Mod} - k$, then $\text{Hom}_k(\text{Hom}_D(T, ?), E): \mathcal{D} \to \text{Mod} - k$ is a contravariant cohomological functor which takes coproducts to products. When $D$ satisfies Brown representability theorem, this functor is represented by an object $D(T)$, usually called the Brown-Comenetz dual of $T$, uniquely determined up to isomorphism.

**Example 4.** Let $\mathcal{D}$ satisfy Brown representability theorem and let $\mathcal{T}$ be a set of compact objects in $\mathcal{D}$. Then $D(\mathcal{T}) := \prod_{T \in \mathcal{T}} D(T)$ is a pure-injective object of $\mathcal{D}$.

**Proof.** Put $Y := D(\mathcal{T})$. The map $\lambda^* : \text{Hom}_D(Y^I, Y) \to \text{Hom}_D(Y^{(I)}, Y)$ is surjective if and only if $\lambda^* : \text{Hom}_D(Y^I, D(T)) \to \text{Hom}_D(Y^{(I)}, D(T))$ is surjective, for each $T \in \mathcal{T}$. By definition of $D(\mathcal{T})$, this is equivalent to saying that $\lambda : \text{Hom}_D(T, Y^{(I)}) \to \text{Hom}_D(T, Y^I)$ is injective, for all $T \in \mathcal{T}$. This is clear due to the compactness of all $T \in \mathcal{T}$. \qed

For our next result, we warn the reader that the dual notion of equivalence of (partial) silting objects, that of equivalence of (partial) cosilting objects, is defined by the fact that two (partial) cosilting objects $Q$ and $Q'$ are equivalent exactly when $\text{Prod}(Q) = \text{Prod}(Q')$.

**Proposition 2.** Suppose that both $\mathcal{D}$ and $\mathcal{D}^{op}$ satisfy Brown representability theorem and consider the classes $\mathcal{S}_i$ ($i = 1, \ldots, 4$) whose elements are the following:

1. The equivalence classes of classical silting sets (resp. objects) in $\mathcal{D}$;

2. The smashing and co-smashing nondegenerate $t$-structures in $\mathcal{D}$ whose heart is a module category over some small $k$-category (resp. ordinary algebra);

3. The smashing and co-smashing nondegenerate $t$-structures in $\mathcal{D}$ whose heart is a Grothendieck category;

4. The equivalence classes of pure-injective cosilting objects $Q$ in $\mathcal{D}$ such that $\perp^c Q$ is closed under taking products in $\mathcal{D}$.

There are bijections and injections

$$\mathcal{S}_1 \xleftrightarrow{\sim} \mathcal{S}_2 \xrightarrow{\sim} \mathcal{S}_3 \hookrightarrow \mathcal{S}_4.$$  

The composed map $\mathcal{S}_1 \to \mathcal{S}_4$ takes $\mathcal{T}$ to the equivalence class of $D(\mathcal{T}) := \prod_{T \in \mathcal{T}} D(T)$, where $D(\mathcal{T})$ is the Brown-Comenetz dual of $T$. Moreover, when $\mathcal{D}$ is compactly generated, the map $\mathcal{S}_3 \to \mathcal{S}_4$ is bijective.

**Proof.** The bijection of Corollary [12] clearly restricts to a bijection between $\mathcal{S}_1$ and the class of nondegenerate smashing $t$-structures $\tau$ in $\mathcal{D}$ whose heart $\mathcal{H} = \mathcal{H}_\tau$ is a module category over some small $k$-category and whose associated cohomological functor $\tilde{H} : \mathcal{D} \to \mathcal{H}$ preserves products. We will prove that this class of $t$-structures is precisely $\mathcal{S}_2$, which will give the bijection $\mathcal{S}_1 \xleftarrow{\sim} \mathcal{S}_2$. When $\tau = (\mathcal{U}, \mathcal{U}^{[-1]})$ is smashing and co-smashing, both $\mathcal{U}$ and $\mathcal{U}^{[-1]}$ are closed under coproducts and products, which easily implies that the inclusion $\mathcal{H} \hookrightarrow \mathcal{D}$ preserves coproducts and products. This in turn implies that $\tilde{H}$ preserves coproducts and products. Conversely, suppose that $\tau$ is nondegenerate smashing and that $\tilde{H}$ preserves products. If $(U_i)_{i \in I}$ is a family of objects of $\mathcal{U}$, then

$$\tilde{H}(\bigoplus U_i) = \tilde{H}(\bigoplus U_i[k]) \cong \bigoplus \tilde{H}(U_i[k]) \cong \prod \tilde{H}(U_i) = 0,$$
for all $k > 0$, where $\prod^*$ denotes the product in $\mathcal{H}$. By Lemma 2 and the right nondegeneracy of $\tau$, we get that $\tau^{\oplus_{i=1}^k}(\prod U_i) = 0$ and so $\prod U_i \in \mathcal{U}$. That is, $\tau$ is co-smashing.

Let us assume now that $\tau$ is a t-structure as in 3. As proved in the previous paragraph, such a t-structure satisfies condition 2.a of Corollary 1. Therefore, we have a partial cosilting set $\mathcal{Q}$, uniquely determined up to equivalence, such that $\tau = (\{<0 \mathcal{Q}, \{<0 \mathcal{Q}\} \})$. Putting $\mathcal{Q} = \prod_{Q \in \mathcal{Q}} Q'$, we can assume that $\mathcal{Q} = \{Q\}$. But the left nondegeneracy of $\tau$ implies that $\mathcal{Q}$ cogeners $\mathcal{D}$, so that $\mathcal{Q}$ is actually a cosilting set. In particular, by the dual of Theorem 1 when taken for the silting situation, we have that $\tau = (\{<0 \mathcal{Q}, \{<0 \mathcal{Q}\} \})$.

In order to have a (clearly injective) map $S_3 \rightarrow S_4$, we just need to check that $\mathcal{Q}$ is a pure-injective object. We go to a more general situation and assume that $\mathcal{Q}$ is isomorphic to $\mathcal{Q}$, which is the case when $\mathcal{Q}$ is closed under finite coproducts which is abelian and such that the inclusion functor $\mathcal{Q} \hookrightarrow \mathcal{D}$ is pure-injective. In order to have a (clearly injective) map $S_3 \rightarrow S_4$, we just need to check that $\mathcal{Q}$ is a pure-injective object. We go to a more general situation and assume that $\mathcal{Q}$ is closed under finite coproducts which is abelian and such that the inclusion functor $\mathcal{Q} \hookrightarrow \mathcal{D}$ is pure-injective. In order to have a (clearly injective) map $S_3 \rightarrow S_4$, we just need to check that $\mathcal{Q}$ is a pure-injective object. We go to a more general situation and assume that $\mathcal{Q}$ is closed under finite coproducts which is abelian and such that the inclusion functor $\mathcal{Q} \hookrightarrow \mathcal{D}$ is pure-injective.

Note that the composed map $\mathcal{Q} \hookrightarrow \mathcal{Q}$ takes the equivalence class of $\tau$ to the equivalence class of $\mathcal{Q}$ if and only if it is pure-injective. That is, $\mathcal{Q}$ is pure-injective in $\mathcal{D}$ if and only if so is $\mathcal{Q}$, as it was claimed. If now $\tau = (\{<0 \mathcal{Q}, \{<0 \mathcal{Q}\} \})$ is as in 3, then, due to the fact that $\mathcal{H}(\mathcal{Q})$ is an injective cogenerator of $\mathcal{H}$, the right vertical arrow of the last diagram is an epimorphism. Then $\mathcal{Q}$ is a pure-injective object of $\mathcal{D}$.

Note that the composed map $S_3 \rightarrow S_4$ takes the equivalence class of $\tau$ to the equivalence class of $\mathcal{Q}$. But, by construction of the Brown-Comenetz dual, we have that $\tau = (\{<0 \mathcal{Q}, \{<0 \mathcal{Q}\} \})$. Moreover, the equality $\mathcal{H}(\mathcal{Q}) = 0$ gives the equality $\mathcal{H}(\mathcal{Q}) = 0$, for all $\tau \in \tau$ and $j > 0$, which implies that $\mathcal{D}(\mathcal{T}) = \tau \mathcal{T} \mathcal{T}$. This together with the fact that $(\{<0 \mathcal{D}(\mathcal{T}), \{<0 \mathcal{D}(\mathcal{T})\} \})$ is a t-structure implies that $\mathcal{D}(\mathcal{T})$ is a cosilting object clearly equivalent to $\mathcal{Q}$.

It remains to see that, when $\mathcal{D}$ is compactly generated, the map $S_3 \rightarrow S_4$ is surjective, for which we just need to prove that the heart $\mathcal{H}$ is a Grothendieck category. But note that, as seen above, the fact that $\mathcal{Q}$ is pure-injective in $\mathcal{D}$ implies that $\mathcal{H}(\mathcal{Q})$ is also pure-injective in $\mathcal{D}$. Then the result is a consequence of the next lemma.

Recall from [31 Définition 1.2.5] that an admissible abelian subcategory $\mathcal{A}$ of $\mathcal{D}$ is a full subcategory closed under finite coproducts which is abelian and such that the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{D}$ takes short exact sequences to triangles. The following result was communicated to us by Stovicek [60].

**Lemma 7.** Let $\mathcal{D}$ be a compactly generated triangulated category and let $\mathcal{A}$ be an $AB3^*$ admissible abelian subcategory such that the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{D}$ preserves products. If $\mathcal{A}$ admits an injective cogenerator $\mathcal{Y}$ which is a pure-injective object of $\mathcal{D}$, then $\mathcal{A}$ is a Grothendieck category.

**Remark 4.** Under sufficiently general assumptions on $\mathcal{D}$, as those of Remark 2, one has that every orthogonal pair in $\mathcal{D}$ generated by a set gives triangles (see [50 Proposition 3.3]). Then, in similarity with König-Yang bijection, one can include co-t-structures in the last bijections. For instance, there is a bijection between equivalence classes of partial silting sets and co-t-structures in $\mathcal{D}$ generated by their co-heart and such that this co-heart has an additive generator (i.e. there is an object $\tau \in \mathcal{C} := \mathcal{U} \cap \mathcal{D}[1]$
such that $C = \text{Add}(T)$). We leave to the reader the statement of the corresponding restricted bijections in the line of Corollary [3].

One can easily give examples of nonclassical self-small partial silting objects:

**Example 5.** If $f : A \rightarrow B$ is a homological epimorphism of ordinary algebras (i.e. the multiplication map $B \otimes_A B \rightarrow B$ is an isomorphism and $\text{Tor}^A_i(B, B) = 0$, for all $i > 0$), then the restriction of scalars $f_* : \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ preserves partial silting (resp. tilting) objects and preserves self-smallness. In particular, $B$ is a self-small partial tilting object of $\mathcal{D}(A)$ which need not be compact.

**Proof.** The proof is a direct consequence of the fact that $f_* : \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ is fully faithful and preserves coproducts (see [21 Theorem 4.4] and [13 Lemma 4]). Taking the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, we see that $B$ need not be compact in $\mathcal{D}(A)$.

However, the following seems to be a more delicate question. As a byproduct of our Sections 6 and 7 [18 Corollary 2.5] and our Corollary [10] give partial affirmative answers.

**Question 2.** Let $\mathcal{D}$ be a triangulated category with coproducts (even compactly generated). Is any self-small silting object necessarily compact (=classical)?

## 5 The aisle of a partial silting t-structure

The main result of this section, Theorem [2] gives a handy criterion to identify those strongly nonpositive objects in a triangulated category with coproducts which are partial silting objects. In such case, it also gives a precise description of the objects in the aisle of the associated t-structure. We first need a preliminary lemma.

**Lemma 8.** Let $\mathcal{D}$ be a triangulated category and let $\mathcal{E}, \mathcal{F}$ be extension-closed subcategories of $\mathcal{D}$ such that $\text{Hom}(E, F[1]) = 0$, for all $E \in \mathcal{E}$ and $F \in \mathcal{F}$. Then $\mathcal{E} \ast \mathcal{F}$ is closed under extensions in $\mathcal{D}$. In particular, if $\mathcal{D}$ has coproducts and $T$ is a strongly nonpositive class in $\mathcal{D}$, then we have an equality:

$$\text{thick}(\text{Sum}(T)) = \text{thick}(\text{Add}(T)) = \bigcup \{ \text{Sum}(T)[r] \ast \text{Sum}(T)[r+1] \ast \ldots \ast \text{Sum}(T)[s] \} = \bigcup \{ \text{Add}(T)[r] \ast \text{Add}(T)[r+1] \ast \ldots \ast \text{Add}(T)[s] \},$$

where the unions range over all pairs of integers $(r, s)$ such that $r \leq s$.

**Proof.** Let $X, X' \in \mathcal{E} \ast \mathcal{F}$ be any objects and consider triangles in $\mathcal{D}$:

$$E \xrightarrow{u} X \xrightarrow{v} F \xrightarrow{+}$$
$$E' \xrightarrow{u'} X' \xrightarrow{v'} F' \xrightarrow{+}$$
$$X \xrightarrow{f} M \xrightarrow{g} X' \xrightarrow{h} X[1],$$

where $E, E' \in \mathcal{E}$ and $F, F' \in \mathcal{F}$. The goal is to prove that $M \in \mathcal{E} \ast \mathcal{F}$. Note that $v \circ h[-1] \circ u'[-1] \in \text{Hom}(E'[1], F) = 0$. This gives a (non-unique) morphism $h_E[-1] : E'[1] \rightarrow E$ such that $u \circ h_E[-1] = h[-1] \circ u'[-1]$. Verdier’s $3 \times 3$ Lemma (see [38 Lemma 1.7]) says that we can form a commutative diagram, with all rows and columns being triangles:

$$E'[1] \xrightarrow{u'[-1]} X'[-1] \xrightarrow{h_E[-1]} F'[-1]$$

$$E \xrightarrow{u} X \xrightarrow{v} F \xrightarrow{h} X[1]$$

Clearly $\tilde{E} \in \mathcal{E}$ and $\tilde{F} \in \mathcal{F}$, which proves that $M \in \mathcal{E} \ast \mathcal{F}$. 

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The key point for the final equalities is that \( \text{Add}(T)[k] \) and \( \text{Sum}(T)[k] \) are both closed under taking extensions, whenever \( T \) is a strongly nonpositive class of objects and \( k \in \mathbb{Z} \). Then the chain of equalities will follow automatically from the first statement, once we check that \( \bigcup \text{Sum}(T)[r] \ast \text{Sum}(T)[r+1] \ast \ldots \ast \text{Sum}(T)[s] \) is closed under taking direct summands. Indeed, if \( X \) is in \( \text{Sum}(T)[r] \ast \text{Sum}(T)[r+1] \ast \ldots \ast \text{Sum}(T)[s] \), for some integers \( r \leq s \), then the same is true for \( X^{(n)} \) since coproducts of triangles in \( \mathcal{D} \) are again triangles (see \cite[Proposition 1.2.1 and Remark 1.2.2]{11}). If now \( Y \) is a direct summand of \( X \) then, by the proof of \cite[Proposition 1.6.8]{11}, we know that there is a triangle \( X^{(n)} \rightarrow X^{(n)} \rightarrow Y \rightarrow \text{Sum}(T)[s] \), for all integers \( n \). It follows that \( Y \in (\text{Sum}(T)[r] \ast \text{Sum}(T)[r+1] \ast \ldots \ast \text{Sum}(T)[s+1]) \).

Throughout the rest of the section \( \mathcal{D} \) will be a triangulated category with coproducts. Apart from the usual equivalence of partial silting sets (see Definition \ref{def:weak equivalence}), we will use the following weaker version for strongly nonpositive sets.

**Definition 6.** Let \( \mathcal{T} \) and \( \mathcal{T}' \) be two strongly nonpositive sets of \( \mathcal{D} \). We will say that they are weakly equivalent when \( \text{thick}_\mathcal{D}(\text{Sum}(\mathcal{T})) = \text{thick}_\mathcal{D}(\text{Sum}(\mathcal{T}')) \).

**Theorem 2.** Let \( \mathcal{D} \) be a triangulated category with coproducts and let \( \mathcal{T} \) be a strongly nonpositive set of objects of \( \mathcal{D} \). The following assertions are equivalent:

1. \( \mathcal{T} \) is a partial silting set.
2. \( \mathcal{T} \) is weakly equivalent to a partial silting set.
3. There is a t-structure \( (\mathcal{V},\mathcal{V}^{\perp}[1]) \) in \( \mathcal{D} \) such that
   
   (a) \( \mathcal{T} \subset \mathcal{V} \);
   
   (b) There is an integer \( q \) such that \( \text{Hom}_\mathcal{D}(\mathcal{T},?) \) vanishes on \( \mathcal{V}[q] \), for all \( T \in \mathcal{T} \).

In such case, if \( \tau_\mathcal{T} = \{ \mathcal{T}^{\perp \leq 0}, \mathcal{T}^{\perp \leq 0} \} \) is the associated t-structure, then \( \mathcal{U}_\mathcal{T} := \{ \mathcal{T}^{\perp \leq 0} \} \) consists of the objects \( X \) in \( \mathcal{D} \) which are the Milnor colimit of some sequence

\[
X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \ldots \xrightarrow{x_n} X_n \xrightarrow{x_{n+1}} \ldots
\]

such that \( X_0 \in \text{Sum}(\mathcal{T}) \) and \( \text{cone}(x_n) \in \text{Sum}(\mathcal{T})[n] \), for each \( n > 0 \).

**Proof.** 1) \( \Rightarrow \) 2) is clear.

2) \( \Rightarrow \) 3) Let \( \mathcal{S} \) be a partial silting set weakly equivalent to \( \mathcal{T} \). We will prove that, up to shift, the associated t-structure \( \tau_\mathcal{S} = (\mathcal{U}_\mathcal{S},\mathcal{U}_\mathcal{S}^{\perp}[1]) \) satisfies the requirements. Indeed, put \( T = \coprod_{T \in \mathcal{T}} T \). Then, by the hypothesis and Lemma \ref{lem:equivalence}, we have that \( T \in \text{Add}(\mathcal{S})[p] \ast \text{Add}(\mathcal{S})[p+1] \ast \ldots \ast \text{Add}(\mathcal{S})[p+t] \), for some \( p \in \mathbb{Z} \) and \( t \in \mathbb{N} \). Replacing \( \mathcal{S} \) by \( \mathcal{S}[p] \), we can assume without loss of generality that \( \text{Sum}(\mathcal{T}) \subset \text{Add}(\mathcal{S}) \ast \text{Add}(\mathcal{S})[1] \ast \ldots \ast \text{Add}(\mathcal{S})[t] \) (\(^*\)). Then condition 3.a clearly holds with \( \mathcal{V} = \mathcal{U}_\mathcal{S} \). On the other hand, due to the partial silting condition of \( \mathcal{S} \), we know that \( \text{Hom}_\mathcal{D}(\mathcal{S},?) \) vanishes on \( \mathcal{U}_\mathcal{S}[1] \), for all \( S \in \mathcal{S} \). But then the inclusion \( (*) \) gives that \( \text{Hom}_\mathcal{D}(\mathcal{T},?) \) vanishes on \( \mathcal{U}_\mathcal{S}[t+1] \), for all \( T \in \mathcal{T} \).

3) \( \Rightarrow \) 1) Consider a sequence

\[
X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \ldots \xrightarrow{x_n} X_n \xrightarrow{x_{n+1}} \ldots
\]

as in the statement and put \( X_\infty = \text{Mcolim} \ X_n \).

First step: The canonical morphism \( \lim_\mathcal{D} \text{Hom}_\mathcal{D}(\mathcal{T}[k],X_n) \rightarrow \text{Hom}_\mathcal{D}(\mathcal{T}[k],X_\infty) \) is an epimorphism, for each \( k \in \mathbb{Z} \), and hence \( \text{Hom}_\mathcal{D}(\mathcal{T},X_\infty) = 0 \).

Let us fix \( k \) and consider \( q \) as in condition 3.b. Due to the inclusion \( \mathcal{T} \subset \mathcal{V} \), we know that \( \text{cone}(x_n) \in \mathcal{V}[q+k+1] \), for all \( n > q+k \). We put \( m(k) = q+k \) in the sequel, and we also put \( X'_n = X_n \), for \( n \leq m(k) \), and \( X'_n = X_{m(k)} \), for all \( n > m(k) \). Note that we get a new sequence

\[
X'_0 \xrightarrow{x'_1} X'_1 \xrightarrow{x'_2} \ldots \xrightarrow{x'_n} X'_n \xrightarrow{x'_{n+1}} \ldots
\]

where \( x'_n \) is the identity map, for each \( n > m(k) \). In particular, by \cite[Lemma 1.6.6]{11}, we know that \( X'_\infty := \text{Mcolim} \ X'_n \) is isomorphic to \( X'_m(k) \), with the canonical composition map \( \mu_{m(k)} : X'_m(k) \rightarrow \coprod_{n \in \mathbb{N}} X'_n \rightarrow X'_\infty \), \( X'_\infty \) being an isomorphism. Moreover, we clearly have a ‘morphism of sequences’ \( (X'_n,x'_n) \rightarrow (X_n,x_n) \).
Fix any object \( T \in \mathcal{T} \). For each \( n \in \mathbb{N} \), we have a triangle \( X'_n \longrightarrow X_n \longrightarrow X''_n \), where \( X''_n \in \mathcal{V}[q+k+1] \) (note that \( X''_n = 0 \) for \( n \leq m(k) \)). As a consequence, we get that \( \prod_{n \in \mathbb{N}} X''_n \in \mathcal{V}[q+k+1] \), so that, by the previous paragraph, \( \text{Hom}_D(T[k], \prod_{n \in \mathbb{N}} X'_n) = 0 \), for all \( j \geq -1 \). In particular, the canonical morphism \( \text{Hom}_D(T[k], \prod_{n \in \mathbb{N}} X'_n) \longrightarrow \text{Hom}_D(T[k], \prod_{n \in \mathbb{N}} X_n) \) is an isomorphism. By applying the 3x3 lemma (see [38 Lemma 1.7]), we can form the following commutative diagram whose rows and columns are triangles:

\[
\begin{array}{ccc}
\prod X'_n & \longrightarrow & \prod X'_n \\
\downarrow & & \downarrow \\
X_n & \longrightarrow & X_n \\
\downarrow & \downarrow & \downarrow \\
\prod X''_n & \longrightarrow & \prod X''_n
\end{array}
\]

We then have that \( \text{Hom}_D(T[k], Z[j]) = 0 \), for all \( j \leq -1 \), which in turn implies that the morphism \( h_* : \text{Hom}_D(T[k], X'_n[j]) \longrightarrow \text{Hom}_D(T[k], X_n[j]) \) is an isomorphism, for all \( j \geq 0 \), and an epimorphism for \( j = -1 \). Denote by \( i'_m(k) : X'_m(k) = M \longrightarrow \prod_{n \in \mathbb{N}} X'_n \) and \( i_m(k) : X_m(k) \longrightarrow \prod_{n \in \mathbb{N}} X_n \) the injections into the respective coproducts and put \( \mu'_m(k) = p' \circ i'_m(k) \) and \( \mu_m(k) = p \circ i_m(k) \). By the last diagram, we know that \( h \circ \mu'_m(k) = \mu_m(k) \). And, as mentioned above, we know that \( \mu'_m(k) \) is an isomorphism. It then follows that \( \mu_m(k)[j] \ast : \text{Hom}_D(T[k], X_m(k)[j]) \longrightarrow \text{Hom}_D(T[k], X_n[j]) \) is an isomorphism, for all \( j \geq 0 \), and an epimorphism for \( j = -1 \). But \( (\mu_m(k)[j]) \ast \) factors in the form

\[
\text{Hom}_D(T[k], X_m(k)[j]) \longrightarrow \lim \text{Hom}_D(T[k], X_n[j]) \longrightarrow \text{Hom}_D(T[k], X_n[j]).
\]

Then the canonical map \( \lim \text{Hom}_D(T[k], X_n[j]) \longrightarrow \text{Hom}_D(T[k], X_n[j]) \) is an epimorphism, for all \( j \geq 0 \).

Second step: \( \mathcal{U}_T := \mathcal{F}(T^{\perp} M) \) is an aisle of \( \mathcal{D} \) and \( \text{Hom}_D(T, \mathcal{?}) \) vanishes on \( \mathcal{U}_T \). The proof is inspired by that of [30] Theorem 12.2. Let \( M \in \mathcal{D} \) be any object. We construct a direct system of triangles \((X_n \xrightarrow{f_n} M \xrightarrow{p_n} Y_n \xrightarrow{+} M)_{n \in \mathbb{N}}\), with the property that \( \text{Hom}_D(T[k], Y_n) = 0 \), for all \( 0 \leq k \leq n \). For \( n = 0 \), the map \( f_0 : X_0 \longrightarrow M \) is a \( \text{Add}(T) \)-precover of \( M \) and \( g_0 \) and \( Y_0 \) are obtained by choosing a (fixed) completion to a triangle \( X_0 \xrightarrow{f_0} M \xrightarrow{p_0} Y_0 \xrightarrow{+} M \). If \( n > 0 \) and we already have defined the direct system up to step \( n-1 \), then we choose a \( \text{Add}(T)[n] \)-precover \( p_n : T[n] \longrightarrow Y_{n-1} \). Note that \( p_n \) is also a \( \text{Add}(T)[n] \)-precover since \( \text{Hom}_D(T[k], Y_{n-1}) = 0 \), for \( 0 \leq k \leq n-1 \). When completing to a triangle \( T[n] \xrightarrow{p_n} Y_{n-1} \xrightarrow{g_n} Y_n \xrightarrow{+} M \), from the precovering condition of \( p_n \) and the fact that \( \text{Hom}_D(T[k], Y_{n-1}) = 0 \) for \( j > 0 \), we easily deduce that \( \text{Hom}_D(T[k], Y_n) = 0 \), for all \( 0 \leq k \leq n \). Now, applying the octahedral axiom, we get the following commutative diagram, which gives the triangle \( X_n \xrightarrow{f_n} M \xrightarrow{p_n} Y_n \xrightarrow{+} M \) at step \( n \), together with the connecting morphisms \( x_n : X_{n-1} \longrightarrow X_n \) and \( y_n : Y_{n-1} \longrightarrow Y_n \):

\[
\begin{array}{ccc}
X_{n-1} & \xrightarrow{x_n} & X_n \\
\downarrow & & \downarrow \\
X_{n-1} & \xrightarrow{f_{n-1}} & M \\
\downarrow & \downarrow & \downarrow \\
Y_{n-1} & & Y_n \\
\downarrow & \downarrow & \downarrow \\
Y_{n-1} & \xrightarrow{y_n} & Y_n \\
\downarrow & \downarrow & \downarrow \\
T[n] & \xrightarrow{p_n} & Y_n \\
\end{array}
\]

Denote by \( \mu_r \) the composition \( X_r \xrightarrow{\iota_r} \prod_{n \in \mathbb{N}} X_n \xrightarrow{p} \text{Mcolim}X_n = X_\infty \). If \( f : \prod_{n \in \mathbb{N}} X_n \longrightarrow M \) is the unique morphism such that \( f \circ \iota_r = f_r \), for all \( r \in \mathbb{N} \), then the composition \( \prod_{n \in \mathbb{N}} X_n \xrightarrow{1} \prod_{n \in \mathbb{N}} X_n \xrightarrow{f} M \) is the zero morphism. We then get a morphism \( f : X_\infty \longrightarrow M \) such that \( f \circ p = \hat{f} \), and hence \( f \circ \mu_r = f_r \), for each \( r \in \mathbb{N} \), together with a triangle \( X_\infty \xrightarrow{\hat{f}} M \xrightarrow{g} Y \xrightarrow{+} M \). It is clear that \( X_\infty \in \mathcal{U}_T \).

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We will prove that $Y \in T_{\leq 0}$, so that $\tau T = (\perp T_{\leq 0}, T_{\leq 0})$ will be a t-structure in $\mathcal{D}$ with $\mathcal{U}_T$ as aisle. Note that then the description of the objects of $\mathcal{U}_T = (\perp T_{\leq 0})$ as Milnor colimits of a sequence as in the statement will be automatic, and the fact that $\text{Hom}_D(T, ?)$ vanishes on $\mathcal{U}_T[1]$ will follow from Step 1.

Fix $k \in \mathbb{N}$. The map $(f_r)_* : \text{Hom}_D(T[k], X_r) \to \text{Hom}_D(T[k], M)$ is an epimorphism for $r > k$ and $T \in T$, because $\text{Hom}_D(T[k], Y_r) = 0$ for $r > k$. Since we have $(f_r)_* = f_r \circ (\mu_r)_*$, we conclude that $f_r$ is an epimorphism, for all $k \in \mathbb{N}$. This implies in particular that $\text{Hom}_D(T, Y) = 0$, because $\text{Hom}_D(T, X_{\infty}[1]) = 0$.

We now prove that $f_* : \text{Hom}_D(T[k], X_n) \to \text{Hom}_D(T[k], M)$ is a monomorphism (and hence an isomorphism), for all $k \geq 0$ and all $T \in T$. Take any $\varphi \in \text{Hom}_D(T[k], M)$. Since, by Step 1, the canonical morphism $\lim \text{Hom}_D(T[k], X_n) \to \text{Hom}_D(T[k], X_\infty)$ is surjective, we get that $\text{Hom}_D(T[k], X_\infty)$ is the union of all the images of the maps $(\mu_r)_* : \text{Hom}_D(T[k], X_r) \to \text{Hom}_D(T[k], X_\infty)$. In particular, we have that $\varphi = (\mu_r)_*(\psi)$, for some $r \in \mathbb{N}$ and some $\psi \in \text{Hom}_D(T[k], X_r)$. We can assume without loss of generality that $r > k + 1$. We then have

$$0 = f_*(\varphi) = (f_* \circ (\mu_r)_*)(\psi) = f_\bullet \circ \psi = f_\bullet \circ \psi.$$

Using now the triangle $Y_r[-1] \xrightarrow{w} X_r \xrightarrow{f_r} X_{\infty}$, we conclude that $\psi$ factors in the form $\psi : T[k] \to Y_r[-1] \xrightarrow{w} X_r$. But $\text{Hom}_D(T[k], Y_r[-1]) \cong \text{Hom}_D(T[k + 1], Y_r)$ is zero, because $r > k + 1$. We then get $\psi = 0$, and also $\varphi = 0$.

The fact that $f_*$ is an isomorphism, for all $k \geq 0$, and that $\text{Hom}_D(T, X_{\infty}[1]) = 0$ imply that $\text{Hom}_D(T[k], Y) = 0$, for all $k \geq 0$ and all $T \in T$, so that $Y \in T_{\leq 0}$ as desired.

We will frequently use the following two auxiliary results. The first one is a slight improvement of [50] Lemma 4.10 (i,ii).

**Lemma 9.** Let $A$ be any abelian category such that $D(A)$ has Hom sets, and let $G$ be a class of generators of $A$. If $k \in \mathbb{Z}$ and $X \in D(A)$ are such that $\text{Hom}_{D(A)} (?, X[k])$ vanishes on $G$, then $H^k(X) = 0$. In particular, we have an equality $(\perp G_{\leq 0}, G_{\leq 0}) = (D_{\leq 0}(A), D_{\geq 0}(A))$.

**Proof.** If we assume that $H^k(X) \neq 0$ and $p : Z^k(X) \to H^k(X)$ is the epimorphism from $k$-cycles to $k$-homology, then there is a morphism $\alpha : G \to Z^k(X)$, for some $G \in G$, such that $p \circ \alpha \neq 0$. If now $s : X \to X$ is any quasi-isomorphism, then $p$ factors in the form $Z^k(X) \to Z^k(X) \to H^k(X)$, which implies that the induced chain map $\tilde{\alpha} : G \to X[k]$ has the property that $s[k] \circ \tilde{\alpha}$ is a non-zero morphism in $K(A)$, for all quasi-isomorphisms $s$ with domain $X$. This implies that $\tilde{\alpha}$ is a non-zero morphism in $D(A)$, which contradicts the fact that $\text{Hom}_{D(A)} (?, X[k])$ vanishes on $G$.

**Lemma 10.** Let $A$ be any abelian category such that $D(A)$ has Hom sets, let $M$ be an object of $A$ and let $n$ be a natural number. The following assertions are equivalent:

1. $\text{Ext}_A^n(M, N) = 0$, for all integers $k > n$ and all objects $N$ of $A$.
2. The functor $\text{Hom}_{D(A)}(M, ?) : D(A) \to \text{Ab}$ vanishes on $D^{< - n}(A)$.
3. There exists an exact sequence $0 \to P^{-n} \to \ldots \to P^{-1} \to P^0 \to M \to 0$, where all the $P^{-k}$ are projective objects of $A$.

**Proof.** The equivalence of assertions 1 and 3 when $A$ has enough projectives is standard, and the implication 2) $\implies$ 1) is clear. As for the implication 1) $\implies$ 2), consider a complex $Y^\bullet \in D^{< - n}(A)$ and suppose that $\text{Hom}_{D(A)}(M, Y^\bullet) \neq 0$. Then, up to replacement of $Y^\bullet$ by a quasi-isomorphic complex, we can assume that we have a chain map $f : M \to Y^\bullet$ which represents a nonzero morphism in $D(A)$. Then this map factors in the form $f : M \xrightarrow{\hat{f}} \sigma_{\geq - n - 1}X^\bullet \xrightarrow{\text{can}} Y^\bullet$, where $\sigma_{\geq - n - 1}$ denotes the stupid truncation at $-n - 1$. But $\sigma_{\geq - n - 1}Y^\bullet$ has homology concentrated in degree $-n - 1$ since $Y^\bullet \in D^{< - n}(A)$. It follows that $\sigma_{\geq - n - 1}Y^\bullet$ is isomorphic in $D(A)$ to a stalk complex $N[n + 1]$, which implies that $\hat{f} = 0$ in $D(A)$ since $\text{Ext}_A^{n+1}(M, N) = 0$. Therefore we have $f = 0$ in $D(A)$, which is a contradiction.
Definition 7. An object $M$ of the abelian category $\mathcal{A}$ will be said to have projective dimension $\leq n$, written $\text{pd}_\mathcal{A}(M) \leq n$, when it satisfies condition 1 of Lemma 17. The concept of injective dimension $\leq n$, written $\text{id}_\mathcal{A}(M) \leq n$, is the dual. The category $\mathcal{A}$ is said to have global dimension $\leq n$, written $\text{gldim}(\mathcal{A}) \leq n$ when each object has projective (equivalently, injective) dimension $\leq n$. We will say that $\mathcal{A}$ has finite global dimension when there is a $n \in \mathbb{N}$ such that $\text{gldim}(\mathcal{A}) \leq n$.

In the rest of the section, we consider the following situation.

Setup 1. $\mathcal{A}$ is an abelian category with the property that its derived category $\mathcal{D}(\mathcal{A})$ has Hom sets and arbitrary coproducts.

Lemma 11. Let $\mathcal{A}$ be an abelian category as in Setup 7 let $(X^*_j)_{j \in J}$ be a family of objects in $\mathcal{D}^{\leq 0}(\mathcal{A})$ and let $i_j : X^*_j \hookrightarrow \prod_{i \in I} X^*_i$ be the $j$-th injection into the coproduct, for each $j \in J$. Then $H^0(\prod_{i \in I} X^*_i)$ together with the morphisms $H^0(i_j) : H^0(X^*_j) \to H^0(\prod_{i \in I} X^*_i)$ ($j \in I$), is a coproduct of the $H^0(X^*_i)$ in $\mathcal{A}$. In particular $\mathcal{A}$ is AB3.

Proof. The functor $H^0 : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ is naturally identified with the cohomological functor afforded by the canonical t-structure. By [16, Lemma 3.1], we know that $H^0_{|\mathcal{D}^{\leq 0}(\mathcal{A})} : \mathcal{D}^{\leq 0}(\mathcal{A}) \to \mathcal{A}$ is left adjoint to the canonical 'inclusion' $\mathcal{A} \to \mathcal{D}^{\leq 0}(\mathcal{A}) (M \leadsto M[0])$. The result is then an easy exercise of Category Theory. \hfill \Box

Example 6. Each AB4 abelian category with enough projectives, each Grothendieck category and the dual category of any Grothendieck category are abelian categories as in Setup 7.

Proof. If $\mathcal{A}$ is AB4, then $\mathcal{D}(\mathcal{A})$ has coproducts and they are calculated 'pointwise', i.e., as in $\mathcal{C}(\mathcal{A})$ (see [42]). In such case, when either $\mathcal{A}$ has enough projectives (see [51, Theorem 1]) or $\mathcal{A}$ is a Grothendieck category, we know that $\mathcal{D}(\mathcal{A})$ has Hom sets.

Suppose finally $\mathcal{A} = \mathcal{G}^{\text{op}}$, where $\mathcal{G}$ is a Grothendieck category. We have an induced equivalence of categories $\mathcal{D}(\mathcal{A}) \cong \mathcal{D}(\mathcal{G})^{\text{op}}$. The result in this case is just a consequence of the fact that $\mathcal{D}(\mathcal{G})$ has products. \hfill \Box

Theorem 2 has now the following (nondirect) consequence.

Proposition 3. Let $\mathcal{A}$ be as in Setup 7 and let $T \in \mathcal{A}$ be an object satisfying both of the following conditions:

i) The coproduct of $I$ copies of $T$, denoted $T^{(I)}$, is the same in $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$, for all sets $I$;

ii) $\text{Ext}^k_{\mathcal{A}}(T, T^{(I)}) = 0$, for all integers $k > 0$ and all sets $I$.

The following assertions hold:

1. $\text{thick}_{\mathcal{D}(\mathcal{A})}(\text{Sum}(T))$ consists of the complexes isomorphic in $\mathcal{D}(\mathcal{A})$ to bounded complexes of objects in $\text{Sum}(T)$ (or $\text{Add}(T)$).

2. If $m \leq n$ are integers, then the subcategory $\text{Add}(T)[m] \star \text{Add}(T)[m+1] \star ... \star \text{Add}(T)[n]$ consists of those complexes isomorphic in $\mathcal{D}(\mathcal{A})$ to complexes of $\mathcal{K}^{[-n,-m]}(\text{Add}(T))$.

3. If $T$ is partial silting and $\tau_T = (\perp T^{\perp \leq 0}, T^{\perp > 0})$ is the associated t-structure in $\mathcal{D}(\mathcal{A})$, then $\perp (T^{\perp \leq 0})$ consists of the complexes isomorphic in $\mathcal{D}(\mathcal{A})$ to complexes in $\mathcal{K}^{\leq 0}(\text{Sum}(T)) = \mathcal{K}^{\leq 0}(\text{Add}(T))$.

Proof. 1) Consider the composition $F : \mathcal{K}^{b}(\text{Add}(T)) \xrightarrow{\iota} \mathcal{K}(\mathcal{A}) \xrightarrow{q} \mathcal{D}(\mathcal{A})$, where $\iota$ and $q$ are the inclusion and the localization functors, respectively. We will prove that $F$ is fully faithful by slightly modifying the proof of [44, Proposition 7.3]. First, it is clear that $\text{thick}_{\mathcal{K}(\mathcal{A})}(\text{Sum}(T)) = \mathcal{K}^{b}(\text{Add}(T))$. Consider the subcategory $\mathcal{C}$ of $\mathcal{K}^{b}(\text{Add}(T))$ consisting of the $M$ such that the map

$$\text{Hom}_{\mathcal{K}(\mathcal{A})}(M, T^{(I)}[p]) \to \text{Hom}_{\mathcal{D}(\mathcal{A})}(F(M), F(T^{(I)}[p]),$$

induced by $F$, is bijective, for all $p \in \mathbb{Z}$ and all sets $I$. We clearly have that $T^{(J)} \in \mathcal{C}$, for all sets $J$, and that $\mathcal{C}$ is closed under extensions, all shifts and direct summands. That is, $\mathcal{C}$ is a thick subcategory of $\mathcal{K}^{b}(\text{Add}(T))$ containing Sum(T). It follows that $\mathcal{C} = \mathcal{K}^{b}(\text{Add}(T))$. 

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Fix now $M \in \mathcal{K}_b(\mathrm{Add}(T))$ and consider the subcategory $\mathcal{C}_M'$ of $\mathcal{K}_b(\mathrm{Add}(T))$ consisting of those $N$ such the induced map
\[
\mathrm{Hom}_{\mathcal{K}_b(A)}(M, N[p]) \to \mathrm{Hom}_{\mathcal{D}(A)}(F(M), F(N)[p])
\]
is bijective, for all $p \in \mathbb{Z}$. Again, we have that $\mathcal{C}_M'$ is a thick subcategory of $\mathcal{K}_b(\mathrm{Add}(T))$ which, due to the previous paragraph, contains $\mathrm{Sum}(T)$. It follows that $\mathcal{C}_M' = \mathcal{K}_b(\mathrm{Add}(T))$, for each $M \in \mathcal{K}_b(\mathrm{Add}(T))$. Therefore $F$ is a fully faithful functor.

Due to the fully faithful condition of $F$, we know that $F(\mathcal{K}_b(\mathrm{Add}(T)))$ is a thick subcategory of $\mathcal{D}(A)$ such that $F(\mathcal{K}_b(\mathrm{Add}(T))) = F(\mathrm{thick}_{\mathcal{A}}(\mathrm{Sum}(T))) \subseteq \mathrm{thick}_{\mathcal{D}(A)}(\mathrm{Sum}(T))$. But the reverse inclusion also holds, because $T^{(f)} = F(T^{(f)})$, for each set $I$.

2) The proof of assertion 1 gives that the functor $F$ induces an equivalence of triangulated categories $\mathcal{K}_b(\mathrm{Add}(T)) \xrightarrow{\sim} \mathrm{thick}_{\mathcal{D}(A)}(\mathrm{Sum}(T))$. Assertion 2 is then a consequence of the fact that, as a subcategory of $\mathcal{K}_b(\mathrm{Add}(T))$, the category $\mathrm{Add}(T)[m] \ast \mathrm{Add}(T)[m+1] \ast \ldots \mathrm{Add}(T)[n]$ consists precisely of the complexes in $\mathcal{K}_{n-m}[\mathrm{Add}(T)]$.

3) Put $\mathcal{U}_T := (T^{\perp_{\leq 0}})$ in the sequel. By Theorem [2] each object $X \in \mathcal{U}_T$ is the Milnor colimit of a sequence
\[
X_0 \xrightarrow{x_1} X_1 \xrightarrow{x_2} \ldots \xrightarrow{x_{n-1}} X_{n-1} \xrightarrow{x_n} X_n \xrightarrow{x_{n+1}} \ldots,
\]
where $X_0 = T_0 \in \mathrm{Sum}(T)$ and $\mathrm{cone}(x_n) = T_n[n]$, for some $T_n \in \mathrm{Sum}(T)$. The proof of assertion 1 tells us that each morphism $x_n$ is a chain map and that the induced triangle $X_{n-1} \xrightarrow{x_n} X_n \xrightarrow{T_n[n]} +$ may be viewed as a triangle in $\mathcal{K}_b(\mathcal{A})$, for each $n > 0$. This will allow us to construct a complex $T^\bullet : \ldots \to T_n \to T_{n-1} \to \ldots \to T_1 \to T_0 \to 0 \to \ldots$ which is isomorphic to $X$ in $\mathcal{D}(A)$. We will construct $T^\bullet$ inductively. Namely, for each $n > 0$, we will give a morphism $f_n : T_n \to T_{n-1}$ in $\mathcal{A}$ satisfying the following properties:

- a) The composition of two consecutive maps in the sequence $T_n \xrightarrow{f_n} T_{n-1} \to \ldots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0$ is the zero map;
- b) The complex $\ldots \to 0 \to T_n \xrightarrow{f_n} T_{n-1} \to \ldots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \to 0 \to \ldots$
  is isomorphic to $X_n$ in $\mathcal{D}(A)$;
- c) Under the isomorphism of b) (and the ones from the preceding steps), the morphism $x_n : X_{n-1} \to X_n$ is identified with the chain map given by the vertical arrows of the following diagram:

\[
\begin{array}{cccccccccc}
\cdots & 0 & \to & T_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & T_1 & \xrightarrow{f_1} & T_0 & \to & 0 & \to & \cdots \\
\cdots & 0 & \to & T_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_2} & T_1 & \xrightarrow{f_1} & T_0 & \to & 0 & \to & \cdots \\
\end{array}
\]

Once we will have proved this, the complex
\[
T^\bullet : \ldots \xrightarrow{f_{n-1}} T_n \xrightarrow{f_n} \ldots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \to 0 \to \ldots
\]
will be the desired one and the proof will be finished. Indeed, by taking stupid truncations, we have that $\sigma_{\geq -n}T^\bullet$ is the complex of condition b) above, and the canonical morphism $\sigma_{\geq -n+1}T^\bullet \to \sigma_{\geq -n}T^\bullet$ is precisely the map of condition c) above. Therefore we will have an isomorphism $T^\bullet \cong \mathrm{Mcolim}(\sigma_{\geq -n}T^\bullet) \cong \mathrm{Mcolim}X_n = X$ in $\mathcal{D}(A)$.

By definition of the sequence $(X_n, x_n)$, we have a triangle $T_1[0] \to X_0 = T_0[0] \xrightarrow{x_1} X_1 \xrightarrow{\pm}$, and the morphism $T_1[0] \to T_0[0]$ is of the form $f_1[0]$, for some morphism $f_1 : T_1 \to T_0$ in $\mathcal{A}$. Suppose now that $n > 1$. The induction hypothesis gives a complex $T_{n-1}^\bullet : \ldots \to T_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} T_0 \to 0 \to \ldots$, which is isomorphic to $X_{n-1}$ in $\mathcal{D}(A)$. Since we have a triangle $X_{n-1} \xrightarrow{x_n} X_n \xrightarrow{T_n[n]} +$, we also get a triangle $T_n[n-1] \xrightarrow{\alpha_n} T_{n-1} \xrightarrow{\beta_n} X_n \xrightarrow{\pm}$, where $\beta_n$ is identified with $x_n$ using the isomorphism $T_{n-1}^\bullet \cong X_{n-1}$. 

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But assertion 1 tells us that $\alpha_n$ 'is' a chain map. It is then given by the vertical arrows of the following commutative diagram, for some morphism $f_n : T_n \rightarrow T_{n-1}$ in $\mathcal{G}$ such that $f_{n-1} \circ f_n = 0$:

$$
\cdots \rightarrow 0 \rightarrow T_n \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \\
\cdots \rightarrow 0 \rightarrow T_{n-1} \rightarrow T_n-1 \rightarrow \cdots \rightarrow f_2 T_1 \rightarrow f_1 T_0 \rightarrow 0 \rightarrow \cdots
$$

Note that the cone of the last mentioned chain map is isomorphic to the complex $T^*_n : \ldots \rightarrow 0 \rightarrow T_n \xrightarrow{f_n} T_{n-1} \rightarrow \cdots \rightarrow f_2 T_1 \rightarrow f_1 T_0 \rightarrow 0 \rightarrow \ldots$. Then all needed conditions a)-c) are satisfied.

$\Box$

6 A tilting theory for objects in $\text{AB}3$ abelian categories

The goal of this section is to show that the results in the previous section allow to extend the well-established theory of infinitely generated $n$-tilting modules, for $n \in \mathbb{N}$, to any abelian category as in Setup 1 (see [15] for a similar attempt, when $n = 1$ and $\mathcal{A}$ is a Grothendieck category).

**Definition 8.** An object $T$ of $\mathcal{A}$ will be called partial $n$-tilting when the following conditions hold:

- $T0$ The coproduct of $I$ copies of $T$, denoted by $T^{(I)}$, is the same in $\mathcal{A}$ and $D(\mathcal{A})$, for each set $I$;

- $T1$ $\text{Ext}^k_{\mathcal{A}}(T, T^{(I)}) = 0$, for all integers $k > 0$ and all sets $I$;

- $T2$ The projective dimension of $T$ is $\leq n$;

We will say that $T$ is a $n$-tilting object if it is partial $n$-tilting and, in addition, the following condition holds:

- $T3$ There is a generating class $\mathcal{G}$ of $\mathcal{A}$ such that, for each $G \in \mathcal{G}$, there is an exact sequence $0 \rightarrow G \rightarrow T^0 \rightarrow T^1 \rightarrow \ldots \rightarrow T^n \rightarrow 0$, where all the $T^k$ are in $\text{Add}(T)$.

We will say that $T$ is a (partial) tilting object of $\mathcal{A}$ when it is (partial) $n$-tilting, for some $n \in \mathbb{N}$. Finally, a classical (partial) tilting object of $\mathcal{A}$ will be a (partial) tilting object which is compact as an object of $D(\mathcal{A})$.

**Remarks 1.**

1. We will see in the proof of Theorem 8 that we could have chosen any $m \in \mathbb{N}$ in condition $T3$. That is, 'n-tilting object' is synonymous of 'tilting object of projective dimension $\leq n$'.

2. The reader will have noted a strange condition $T0$ in Definition 8. This condition is always satisfied when $\mathcal{A}$ is $\text{AB}4$ (e.g. a Grothendieck category). We could have chosen to define a notion of partial tilting object in $\mathcal{A}$ by replacing conditions 0 and 1 by the condition that $\text{Hom}_{D(\mathcal{A})}(T, T^{(I)}[k]) = 0$, for all integers $k > 0$ and all sets $I$, where $T^{(I)}$ denotes the coproduct of $I$ copies of $T$ in $D(\mathcal{A})$. But some of the nice properties disappear. For instance, the description of the aisle of the associated t-structure given in Corollary 2 below need not be true.

3. The reader is invited to define the dual notions of (partial) $n$-cotilting object and (partial) cotilting object, which make sense in any abelian category $\mathcal{A}$ such that $D(\mathcal{A})$ has $\text{Hom}$ sets and arbitrary products. We also leave to him/her the statements of the results dual to those which will be proved in the rest of the section for (partial) tilting objects.

Recall that a Grothendieck category $\mathcal{A}$ is locally noetherian when it has a set $\mathcal{S}$ of noetherian generators, i.e. all the objects in $\mathcal{S}$ satisfy ACC on subobjects.

**Example 7.** Let $\mathcal{A}$ be a locally noetherian Grothendieck category of finite global dimension (e.g. $\mathcal{A} = \text{Qcoh}(\mathcal{X})$, where $\mathcal{X}$ is a smooth algebraic variety or $\mathcal{A} = \text{Mod} - R$ for a right noetherian ring $R$ of finite global dimension). Let $G$ be a generator of $\mathcal{A}$ and let $0 \rightarrow G \rightarrow E^0 \rightarrow E^1 \rightarrow \ldots \rightarrow E^m \rightarrow 0$ be its minimal injective resolution. Then $T = \oplus_{0 \leq i \leq m} E^i$ is a tilting object of $\mathcal{A}$.
Proof. We check the conditions of Definition 8. Since $\mathcal{A}$ is AB4 condition $T_0$ holds. By the locally noetherian condition, we know that each coproduct of injective objects is injective (see [57] Proposition V.4.3), so that $T(I)$ is an injective object of $\mathcal{A}$, for each set $I$. This gives condition $T_1$, while condition $T_2$ holds for some $n \in \mathbb{N}$ due to the finite global dimension of $\mathcal{A}$. Finally, taking as generating class $\mathcal{G} = \text{Sum}(G)$, we immediately get condition $T_3$ using the exactness of coproducts and the fact that coproducts of injective objects are injective (see Remark [11]).

Corollary 5. Each partial tilting object $T$ of $\mathcal{A}$ is a partial silting object of $D(\mathcal{A})$ whose associated t-structure is $(\mathcal{K}^{\leq 0}(\text{Sum}(T)), T^{\perp, < 0})$.

Proof. The set $\mathcal{T} = \{T\}$ of $D(\mathcal{A})$ satisfies assertion 3 of Theorem 2 by taking $(\mathcal{V}, \mathcal{V}^{\perp}[1]) = (D^{\leq 0}(\mathcal{A}), D^{\geq 0}(\mathcal{A}))$ (see Lemma 10). That the associated t-structure is as indicated follows from Proposition 3.

Lemma 12. Let $T$ be an object of $\mathcal{A}$ satisfying properties $T_0$ and $T_1$ of Definition 8 and let $\mathcal{Y} := \bigcap_{k>0} \text{Ker}(\text{Ext}_k^b(T, ?))$. If there is an $m \in \mathbb{N}$ such that $\text{Pres}^m(T) = \mathcal{Y}$, then $\text{Pres}^m(\mathcal{Y}) \subseteq \mathcal{Y}$.

Proof. Let $Y^0 \xrightarrow{f_1} Y^1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} Y^m$ be an exact sequence in $\mathcal{A}$, with all the $Y^k$ in $\mathcal{Y}$, and let us prove that $\text{Coker}(f^{m-1}) \in \mathcal{Y}$. We put $B^k = \text{Im}(f^{k-1})$, for $k = 1, \ldots, m$ and put $B^{m+1} = \text{Coker}(f^{m-1})$. We shall prove by induction on $k = 1, \ldots, m+1$ that $B^k \in \text{Pres}^{k-1}(T)$. For $k = m+1$, this will give that $B^{m+1} \in \text{Pres}^m(T) = \mathcal{Y}$ and will end the proof.

For $k = 1$ is clear. Let us take $k > 1$ (and $k \leq m+1$). We consider the induced exact sequence $0 \to B^{k-1} \xrightarrow{u} Y^{k-1} \xrightarrow{f^{k-1}} B^k \to 0$ and fix a $\text{Sum}(T)$-precover $p : T' \to Y^{k-1}$, which is necessarily an epimorphism. Note that, by applying the exact sequence of $\text{Ext}$ to the sequence $0 \to \text{Ker}(p) \to T' \xrightarrow{p} Y^{k-1}$, we readily see that $\text{Ker}(p) \in \mathcal{Y}$. By taking the pullback of $p$ and $u$ and using properties of pullbacks, we then get an exact sequence $0 \to \text{Ker}(p) \to \text{Ker}(f^{k-1} \circ p) \to B^{k-1} \to 0$. Now the proof of Lemma 3.8 is valid on any AB3* abelian category, and its dual applies to our case. It follows that $\text{Ker}(f^{k-1} \circ p) \in \text{Pres}^{k-1}(T)$. By considering the exact sequence $0 \to \text{Ker}(f^{k-1} \circ p) \to T' \xrightarrow{f^{k-1} \circ p} B^k \to 0$, we conclude that $B^k \in \text{Pres}^{k-1}(T)$.

The following lemma was pointed out to us by Luisa Fiorot. We here reproduce the essential idea of her proof and thank her for the help.

Lemma 13. Let $T$ be a $n$-tilting object in $\mathcal{A}$ and let $\mathcal{H}_T = T^{\perp, > 0} \cap T^{\perp, < 0}$ be the heart of the associated t-structure in $D(\mathcal{A})$. Then the class $\mathcal{Y} := \bigcap_{k\geq 0} \text{Ker}(\text{Ext}_k^b(T, ?))$ coincides with $\mathcal{A} \cap \mathcal{H}_T$ and is a cogenerating class in $\mathcal{A}$. Moreover, any object $A \in \mathcal{A}$ admits an exact sequence $0 \to A \to Y \to T^1 \to \cdots \to T^n \to 0$, where $Y \in \mathcal{Y}$ and $T^k \in \text{Add}(T)$ for any $k = 1, \ldots, n$.

Proof. The aisle of the associated t-structure in $D(\mathcal{A})$ is $\mathcal{U}_T = T^{\perp, > 0} = \mathcal{K}^{\leq 0}(\text{Add}(T))$ (see Corollary 5). We then have an equality of subcategories $\mathcal{Y} = \mathcal{A} \cap T^{\perp, > 0} = \mathcal{A} \cap T^{\perp, > 0} \cap T^{\perp, < 0} = \mathcal{A} \cap \mathcal{H}_T$ since there are no negative extensions between objects of $\mathcal{A}$. Moreover, by Lemma 10 we have that $D^{\leq -n}(\mathcal{A}) \subseteq T^{\perp, > 0}$, and so $\mathcal{A} \subseteq \mathcal{U}_T[\leq -n] = \mathcal{K}^{\leq n}(\text{Add}(T))$.

Given any object $A \in \mathcal{A}$, we then have a complex

$$T^* : \cdots \to T^{-k} \xrightarrow{d^{-k}} T^{-k+1} \xrightarrow{d^{-k+1}} \cdots \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{n-1}} T^n \to 0 \to \cdots,$$

with all the $T^j$ in $\text{Add}(T)$, which is isomorphic to $A[0]$ in $D(\mathcal{A})$. Then $Y := \text{Coker}(d^{-1})$ is in $\mathcal{Y}$ because, using stupid truncations, we have an isomorphism $\sigma_{\leq 0} T^* \cong Y[0]$ in $D(\mathcal{A})$ and $\sigma_{\leq 0} T^* \in \mathcal{K}^{\leq 0}(\text{Add}(T)) = T^{\perp, > 0}$. But we have a monomorphism $A \cong H^0(T^*) \to Y$ and, hence, $\mathcal{Y}$ is a cogenerating class of $\mathcal{A}$. On the other hand, using intelligent truncation, we have an isomorphism $A[0] \cong \tau_{\geq 0} T^*$ in $D(\mathcal{A})$, and $\tau_{\geq 0} T^*$ is identified with the induced complex

$$\cdots \to 0 \to Y \to T^1 \to \cdots \to T^m \to 0 \to \cdots$$

The following main result of the section shows that several common characterizations of tilting modules pass naturally to our general setting.
Theorem 3. Let $\mathcal{A}$ be an abelian category such that $\mathcal{D}(\mathcal{A})$ has Hom sets and arbitrary coproducts, and let $T$ be an object of $\mathcal{A}$ such that the coproduct $T^{(1)}$ is the same in $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$. The following assertions are equivalent:

1. $T$ is a tilting object of $\mathcal{A}$.
2. $T$ has finite projective dimension and is a silting (or tilting) object of $\mathcal{D}(\mathcal{A})$.
3. $T$ is a (partial) silting object of $\mathcal{D}(\mathcal{A})$ such that, for some generating class $G$ of $\mathcal{A}$, there is an $n \in \mathbb{N}$ such that the inclusions $G \subset \text{Add}(T)[-n] \star \text{Add}(T)[-n + 1] \star \ldots \star \text{Add}(T)[0] \subseteq \text{thick}_{\mathcal{D}(\mathcal{A})}(G)$ hold.
4. $\text{Ext}^k_{\mathcal{A}}(T, T^{(1)}) = 0$, for all integers $k > 0$ and all sets $I$, and there is a generating class $G$ of $\mathcal{A}$ such that:
   
   (a) There is a common finite upper bound on the projective dimensions of the objects of $G$;
   
   (b) $\text{thick}_{\mathcal{D}(\mathcal{A})}(G) = \text{thick}_{\mathcal{D}(\mathcal{A})}(\text{Sum}(T))$.
5. If $Y = \bigcap_{k > 0} \ker(\text{Ext}^k_{\mathcal{A}}(T, ?))$, then:
   
   (a) $Y = \text{Pres}^m(T)$, for some integer $m \in \mathbb{N}$;
   
   (b) $Y$ is a cogenerating class of $\mathcal{A}$.

If any of the equivalent conditions hold, the associated $t$-structure in $\mathcal{D}(\mathcal{A})$ is $\tau_T = (K^{\leq 0}(\text{Sum}(T)), T^{\perp < 0}) = (T^{> 0}, T^{< 0})$.

Proof. 1) $\Rightarrow$ 2) By Corollary $\mathbf{5}$ we know that $T$ is a partial silting object of $\mathcal{D}(\mathcal{A})$. On the other hand, by property T3 in the definition of tilting object and Lemma $\mathbf{9}$ we easily get that $T$ is a generator of $\mathcal{D}(\mathcal{A})$.

2) $\Rightarrow$ 1) It immediately follows that $T$ is partial tilting object, and only condition T3 in Definition $\mathbf{8}$ needs to be checked. Let us put $\text{pd}_{\mathcal{A}}(T) = n$. If $M \in \mathcal{A}$ is any object, then we have that $M[n] \in T^{> 0} = \mathcal{U}_T = K^{\geq 0}(\text{Add}(T))$, due to Lemma $\mathbf{10}$. Then we have a complex

\[ T_M^\bullet : \ldots \rightarrow T_M^{-n-1} \rightarrow T_M^{-n} \rightarrow \ldots \rightarrow T_M^0 \rightarrow 0 \rightarrow \ldots \]

with homology concentrated in degree $-n$ and $H^{-n}(T_M^\bullet) \cong M$, such that $T_M^{-k} \in \text{Add}(T)$ for all $k \in \mathbb{N}$. Taking coboundaries and cocycles in degree $-n$, we get an exact sequence $0 \rightarrow B_M^n \rightarrow Z_M^{-n} \rightarrow M \rightarrow 0$. Putting $G = \{Z_M^{-n} : M \in \mathcal{A}\}$, we get a generating class of $\mathcal{A}$ satisfying the mentioned property T3.

1) $\Rightarrow$ 3) By Corollary $\mathbf{5}$ we know that $T$ is a partial silting object of $\mathcal{D}(\mathcal{A})$ whose associated $t$-structure is the one of the final statement of this theorem. Let $n \in \mathbb{N}$ be such that $T$ is $n$-tilting. By assertion 2 of Proposition $\mathbf{3}$ and condition T3, we get that $G \subset \text{Add}(T)[-n] \star \text{Add}(T)[-n + 1] \star \ldots \star \text{Add}(T)[0]$.

Consider now a $G$-resolution of $T$:

\[ \ldots \rightarrow G^{-m} \rightarrow G^{-m+1} \rightarrow \ldots \rightarrow G^{-1} \rightarrow G^0 \rightarrow T \rightarrow 0 \]

and denote by $Z^{-m}$ the kernel of the differential $G^{-m} \rightarrow G^{-m+1}$, for each $m \in \mathbb{N}$. We then have an induced complex

\[ X^\bullet : \ldots \rightarrow 0 \rightarrow Z^{-n} \rightarrow G^{-n} \rightarrow G^{-n+1} \rightarrow \ldots \rightarrow G^1 \rightarrow G^0 \rightarrow 0 \rightarrow \ldots \]

which is quasi-isomorphic to the stalk complex $T[0]$. If we consider the truncated complex

\[ G^\bullet : \ldots 0 \rightarrow G^{-n} \rightarrow G^{-n+1} \rightarrow \ldots \rightarrow G^1 \rightarrow G^0 \rightarrow 0, \ldots, \]

then $X^\bullet$ is just the cone in $\mathcal{K}(\mathcal{A})$ of the chain map $Z^{-n}[n] \rightarrow G^\bullet$ determined by the inclusion $Z^{-n} \hookrightarrow G^{-n}$. We then get a triangle

\[ Z^{-n}[n] \rightarrow G^\bullet \rightarrow T[0] \rightarrow Z^{-n}[n + 1] \]

in $\mathcal{D}(\mathcal{A})$, which splits since $\text{Ext}^1_{\mathcal{A}}(T, Z^{-n}) = 0$. It follows that $T[0]$ is isomorphic to a direct summand of $G^\bullet$ in $\mathcal{D}(\mathcal{A})$, and we clearly have $G^\bullet \in \text{thick}_{\mathcal{D}(\mathcal{A})}(G)$.
3) \implies 4) is clear, except for condition 4.a. By assertion 2 of Proposition 3, we know that each \( G \in \mathcal{G} \) admits an exact sequence sequence \( 0 \rightarrow G \rightarrow T^0 \rightarrow T^1 \rightarrow \cdots \rightarrow T^n \rightarrow 0 \) (*), with all the \( T^k \) in \( \text{Add}(T) \). But, by hypothesis, we know that \( T \) is a partially silting object of \( \mathcal{D}(\mathcal{A}) \), so that \( \mathcal{U}_T = \mathcal{U} = \mathcal{U}_T^{\perp} = \mathcal{U}_{T^0}^{\perp} \) is \( \tau \)-injective. Then the inclusion \( \mathcal{G} \subseteq \text{Add}(\mathcal{T}) \) together with Lemma 3 imply that \( \mathcal{D}^{\leq 0}(\mathcal{A}) = \mathcal{U}_T^{\perp} \subseteq \mathcal{U}_T^{\perp} \mathcal{U}_T^{\perp} \subseteq \mathcal{U}_T^{\perp} \), so that \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, ?) \) vanishes on \( \mathcal{D}^{\leq 0}(\mathcal{A}) \) for all \( G \in \mathcal{G} \), using a classical argument.

To end the proof of this implication, we only need to check that we can choose \( p = n \). If \( p < n \) is clear: we simply put \( T^k = 0 \) for \( k = p + 1, \ldots, n \). So we assume that \( p > n \). Let us consider the induced exact sequence

\[
0 \rightarrow K \rightarrow T^{p-n} \rightarrow T^{p-n} \rightarrow \cdots \rightarrow T^p \rightarrow 0,
\]

which is an \( \text{Add}(T) \)-coresolution of \( K \). Since \( \text{Ext}^k_{\mathcal{A}}(T, ?) \) vanishes on all the \( T^k \), \( \text{Ext}^j_{\mathcal{A}}(T, K) \) is the \( j \)-th cohomology group of the induced complex of abelian groups

\[
\cdots \rightarrow 0 \rightarrow \text{Hom}_{\mathcal{A}}(T, T^{p-n-1}) \rightarrow \text{Hom}_{\mathcal{A}}(T, T^{p-n}) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{A}}(T, T^p) \rightarrow 0 \rightarrow \cdots,
\]

where \( \text{Hom}_{\mathcal{A}}(T, T^k) \) is in degree \( k + n + 1 - p \) for each \( k = p - n - 1, p - n, \ldots, p \). In particular, we have that \( 0 = \text{Ext}^{n+1}_{\mathcal{A}}(T, K) \) is the cokernel of the map \( \text{Hom}_{\mathcal{A}}(T, T^{p-1}) \rightarrow \text{Hom}_{\mathcal{A}}(T, T^p) \), so that this map is surjective. It follows that \( d^{n-1} : T^{p-1} \rightarrow T^p \) is a retraction and, hence, that \( \text{Ker}(d^{n-1}) \) is also in \( \text{Add}(T) \). That is, if there exists a sequence as (***) of length \( p \), then there also exists one of length \( p - 1 \).

By iterating the process, we arrive at an exact sequence like (***) of length exactly \( n \).

1) \( \implies 5) \) Due to Lemma 13, we know that condition 5.b holds. Let us put \( n = \text{pd}_A(T) \). We have already seen in the proof of that lemma that \( \mathcal{Y} = \mathcal{U}_T \cap \mathcal{A} \) and, by the already proved last statement of the proposition, we get that \( \mathcal{Y} = \mathcal{K}^{\leq 0}(\mathcal{A}) \cap \mathcal{A} \subseteq \text{Pres}^{n-1}(\mathcal{A}) \).

We will also prove the converse inclusion. Let \( Y \in \text{Pres}^{n-1}(\mathcal{A}) \) be any object and consider an exact sequence in \( \mathcal{A} \)

\[
\begin{array}{ccccccccc}
T^{n+1} & d^{n+1} & T^n & \cdots & T^1 & T^0 & Y & 0.
\end{array}
\]

\text{Let us put } \text{Ker}(d^{n+1}), \text{consider a } \mathcal{G}-\text{resolution of } \text{Ker}(d^{n+1}) \text{and patch it with the sequence (***)}. \text{We then obtain a complex}

\[
X^\bullet : \cdots G^{-n-1} \rightarrow G^{-n} \rightarrow T^{-n-1} \rightarrow \cdots \rightarrow T^{-1} \rightarrow T^0 \rightarrow 0.
\]

which is quasi-isomorphic to the stalk complex \( Y[0] \). By taking the stupid truncation at \( -n + 1 \), we get a triangle \( \sigma_{-n+1}X^\bullet \rightarrow X^\bullet \rightarrow \sigma_{\leq -n}X^\bullet \) in \( \mathcal{K}(\mathcal{A}) \). But \( \sigma_{-n+1}X^\bullet \) is a complex of objects in \( \text{Add}(T) \) concentrated in degrees \( -n + 1, \ldots, -1, 0 \). It follows that \( \sigma_{-n+1}X^\bullet \in \text{Add}(T)[0] \) and \( \sigma_{n}X^\bullet \in \mathcal{D}^{\leq -n}(\mathcal{A}) \). Therefore the functor \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, ?[k]) : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{A} \) vanishes both on \( \sigma_{-n+1}X^\bullet \) and \( \sigma_{n}X^\bullet \), for all \( k > 0 \), due to the silting condition of \( T \) and the fact that \( \text{pd}_A(T) \leq n \). We then get that

\[
\text{Ext}^k_{\mathcal{A}}(T, Y) \cong \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, Y[k]) \cong \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, X^\bullet[k]) = 0,
\]

for all \( k > 0 \).
5) $\implies$ 1) We have $T^{(I)} \in \text{Pres}^m(T)$, so that $\text{Ext}_A^k(T, T^{(I)}) = 0$, for all sets $I$. Moreover, the cogenerating condition of $\mathcal{Y}$ together with Lemma 13 imply that each $M \in \text{Ob}(\mathcal{A})$ admits an exact sequence $0 \to M \to Y^0 \to Y^1 \to \ldots \to Y^m \to Y^{m+1} \to 0$, where the $Y^i$ are in $\mathcal{Y}$. Consider now the complex

$$Y^\bullet : \ldots 0 \to Y^0 \to Y^1 \to \ldots \to Y^m \to Y^{m+1} \to 0 \ldots$$

Since $\text{Ext}_A^k(T, ?)$ vanishes on all $Y^j$ for all $k > 0$, we get that $\text{Ext}_A^k(T, M)$ is the $j$-th cohomology group of the complex $\text{Hom}_{\mathcal{A}}(T, Y^\bullet)$. It follows that $\text{Ext}_A^k(T, M) = 0$, for all $k > m + 1$, and so $\text{pd}_A(T) \leq m + 1$. Therefore $T$ is a partial $(m + 1)$-tilting module.

In order to check property T3 of Definition 3 note that we can apply the dual of [6 Theorem 1.1], with $X$, $\omega$, $\check{X}$ and $\check{\omega}$ replaced by $\mathcal{Y}$, $\text{Add}(T)$, $\mathcal{A}$ and $\text{Add}(T)^\vee$, respectively. Here $\text{Add}(T)^\vee$ denotes the subcategory consisting of those objects $M$ which admit an exact sequence $0 \to M \to T^0 \to T^1 \to \ldots \to T^p \to 0$, for some $p \in \mathbb{N}$, with all the $T^k$ in $\text{Add}(T)$. It follows that $\mathcal{G} := \text{Add}(T)^\vee$ is a generating class. It remains to see that if $G \in \mathcal{G}$ and we fix an exact sequence $0 \to G \to T^0 \to T^1 \to \ldots \to T^p \to 0$, with the $T^k$ in $\text{Add}(T)$, then we can choose one such $\text{Add}(T)$-coresolution with $p = \text{pd}_A(T) = (m + 1)$. This has been done in the proof of 4) $\implies$ 1).

Note that if $\mathcal{A}$ has enough projectives, then $\text{Proj} \mathcal{A} \subset \text{add}(\mathcal{G})$, for any generating class $\mathcal{G}$. Furthermore, one has $\text{thick}_{\mathcal{D}(\mathcal{A})}(\mathcal{G}) = \text{thick}_{\mathcal{D}(\mathcal{A})}(\text{Sum}(\mathcal{P})) = K^b(\text{Proj} \mathcal{A})$ whenever $\mathcal{P}$ is a class of projective generators and there is a common finite upper bound in the projective dimensions of the objects of $\mathcal{G}$. Therefore we immediately get from Theorem 3 the following result, which is well-known for module categories (see [8 Theorem 3.11], [2 Corollary 3.7]).

**Corollary 6.** Let $\mathcal{A}$ as in Setup 1 have enough projectives and let $T$ be an object of $\mathcal{A}$ such that the coproduct $T^{(I)}$ is the same in $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$, for every set $I$. The following assertions are equivalent:

1. $T$ is a tilting object of $\mathcal{A}$.

2. The following assertions hold:

   (a) $\text{Ext}_A^k(T, T^{(I)}) = 0$, for all integers $k > 0$ and all sets $I$.

   (b) There exists an exact sequence $0 \to P^{-n} \to \ldots \to P^{-1} \to P^0 \to T \to 0$, where all the $P^{-k}$ are projective objects;

   (c) For some (resp. every) class of projective generators $\mathcal{P}$ of $\mathcal{A}$, there is $m \in \mathbb{N}$ such that each object $P \in \mathcal{P}$ admits an exact sequence $0 \to P \to T^0 \to T^1 \to \ldots \to T^m \to 0$, where all the $T^k$ are in $\text{Add}(T)$.

3. $\text{Ext}_A^k(T, T^{(I)}) = 0$, for all integers $k > 0$ and all sets $I$, and $\text{thick}_{\mathcal{D}(\mathcal{A})}(\text{Add}(T)) = K^b(\text{Proj} \mathcal{A})$.

4. If $\mathcal{Y} := \bigcap_{k > 0} \text{Ker}(\text{Ext}_A^k(T, ?))$, then $\mathcal{Y} = \text{Pres}^m(T)$, for some $m \in \mathbb{N}$, and $\mathcal{Y}$ is a cogenerating class of $\mathcal{A}$.

If $T$ satisfies any of these equivalent conditions, then $\mathcal{A}$ has a projective generator and $\tau_T = (T^{\perp > 0}, T^{\perp < 0}) = (K^{\leq 0}(\text{Sum}(T)), T^{\perp < 0})$ is a $t$-structure in $\mathcal{D}(\mathcal{A})$.

**Proof.** By our previous comments, the only thing that does not follow immediately from Theorem 3 is the fact that $\mathcal{A}$ has a projective generator. To see this, consider condition 2.b and take $P = \bigoplus_{0 \leq k \leq n} P^{-k}$. Since $T$ is a silting object, whence a generator, of $\mathcal{D}(\mathcal{A})$ we get that also $P$ is a generator of $\mathcal{D}(\mathcal{A})$. Therefore, for each $0 \neq M \in \mathcal{A}$, we have that $\text{Hom}_{\mathcal{A}}(P, M) = 0$ since $\text{Hom}_{\mathcal{D}(\mathcal{A})}(P[j], M) = 0$ for $j \neq 0$. Then $P$ is a generator of $\mathcal{A}$.

**Remarks 2.** 1. It is evident from the proof of Theorem 3 that the $n$ of $n$-tilting is the same natural number that appears in assertion 2 of the theorem and equals $m + 1$, for the $m$ of assertion 4 in the theorem. This and its dual generalize [8 Theorem 3.11].

2. From the dual of Theorem 3 one derives that [5, Theorem 4.5] is valid for any cotilting object in an abelian category $\mathcal{A}$ such that $\mathcal{D}(\mathcal{A})$ has products and Hom sets (e.g. for any cotilting object in a Grothendieck category).
7 Triangulated equivalences induced by tilting objects

Given a triangulated category with coproducts \( \mathcal{D} \) and a \( \tau \)-structure \( \tau \) in it with heart \( \mathcal{H}_\tau \), one can ask naturally the following:

**Question 3.** Under which conditions the inclusion \( \mathcal{H}_\tau \to \mathcal{D} \) extends to a triangulated equivalence \( \mathcal{D}(\mathcal{H}_\tau) \to \mathcal{D} \)? At least, when does it extend to a fully faithful functor \( \mathcal{D}(\mathcal{H}_\tau) \to \mathcal{D} \)?

Very recently, Psaroudakis and Vitoria (see [51 Section 5]) have used filtered categories and the realisation functor introduced in [11] to study in depth the second part of the question, and have obtained precise answers in case \( \tau \) is induced by a tilting object of \( \mathcal{D} \), for some choices of \( \mathcal{D} \). Also very recently, Fiorot, Mattiello and the second named author have used the tilting theory of last section to show in a very simple way that if \( A \) is an abelian category as in Setup 1 \( T \) is a tilting object of \( A \) and \( \mathcal{H}_T \) denotes the heart of the associated \( \tau \)-structure in \( \mathcal{D}(A) \), then the inclusion \( \mathcal{H}_T \to \mathcal{D}(A) \) extends to a triangulated equivalence \( \mathcal{D}(\mathcal{H}_T) \to \mathcal{D}(A) \) (see [18 Corollary 2.5]). The main result of this section, Theorem 4 will show that the analogous of the latter result holds when we replace \( \mathcal{D}(A) \) by any compactly generated algebraic triangulated category and \( T \) by any bounded tilting object in \( \mathcal{D} \) (see definition below).

The sets of objects with which we shall be dealing here are the following.

**Definition 9.** Let \( \mathcal{D} \) be a triangulated category with arbitrary coproducts and let \( \mathcal{T} \) be a set of objects of \( \mathcal{D} \). We will say that \( \mathcal{T} \) is a bounded (partial) tilting set when it is (partial) tilting (see Definition 2) and weakly equivalent to a classical (partial) silting set (see Definitions 4 and 3). Of course the same adjectives are applied to an object \( T \) when the set \( \{T\} \) is so.

A \( \tau \)-structure \( \tau = (\mathcal{U}, \mathcal{U}^\perp [1]) \) will be called a (bounded, resp. classical) (partial) tilting \( \tau \)-structure when there is a (bounded, resp. classical) (partial) tilting set \( \mathcal{T} \) of \( \mathcal{D} \) such that \( \tau = (\mathcal{U}(\mathcal{T})^{\perp_{\leq 0}}, \mathcal{U}^{\perp_{< 0}}) \).

**Remark 5.** Although, we won’t use it in this paper, one can define bounded (partial) silting set and bounded (partial) silting \( \tau \)-structure just replacing ‘tilting’ by ‘silting’ last definition. When \( A \) is an ordinary algebra, one can easily see that a bounded silting object of \( \mathcal{D}(A) \) is exactly what is called a semi-tilting complex in [22] and a big silting complex in [4].

In the proof of our next result and throughout the rest of this section, we will assume that the reader is acquainted with (small) dg categories and their derived categories. Our terminology is mainly taken from [29], but we will use several results from [28]. However, we will still keep the notation \( K(A) \) (instead of \( \mathcal{H}A \) as in Keller’s papers) for the homotopy category of \( A \).

**Proposition 4.** Let \( \mathcal{D} \) be a compactly generated algebraic triangulated category and let \( \tau \) be a bounded tilting \( \tau \)-structure in \( \mathcal{D} \), whose heart is denoted by \( \mathcal{H} \). There is a bounded tilting object \( T \) in \( \mathcal{D} \) satisfying the following properties:

1. \( \tau = (\mathcal{T}^{\perp_{> 0}}, \mathcal{T}^{\perp_{< 0}}) \) is the \( \tau \)-structure associate to \( T \);

2. If \( E := \text{End}_\mathcal{D}(T) \), then the functor \( \text{Hom}_\mathcal{H}(T, ?) : \mathcal{H} \to \text{Mod} - E \) is fully faithful, exact and has a left adjoint which preserves products.

**Proof.** Let us assume that \( \mathcal{T} \) is a tilting set such that \( \tau = (\mathcal{T}^{\perp_{> 0}}, \mathcal{T}^{\perp_{< 0}}) \). By taking \( \bar{\mathcal{T}} := \bigsqcup_{T \in \mathcal{T}} T \), we can and shall assume that \( \mathcal{T} = \{\bar{T}\} \). By definition, \( \mathcal{D} \) has a silting set \( \mathcal{S} \) of compact objects which is weakly equivalent to \( \mathcal{T} \). By [28 Theorem 4.3], we have a dg category \( A \), which can be chosen to be \( k \)-flat, and a triangulated equivalence \( \mathcal{D} \xrightarrow{\sim} \mathcal{D}(A) \) which takes \( \mathcal{S} \) onto the set \( \{\mathcal{A}^\wedge : A \in A\} \) of representable \( A \)-modules. These form then a silting set of compact generators of \( \mathcal{D}(A) \), which implies that \( H^n(A, A') = 0 \), for all \( n > 0 \) and all \( A, A' \in A \). So in the rest of the proof, we assume that \( \mathcal{D} = \mathcal{D}(A) \), where \( A \) is a \( k \)-flat dg category with this homology upper boundedness condition, and that \( \text{thick}_{\mathcal{D}(A)}(\text{Sum}(\mathcal{T})) = \text{thick}_{\mathcal{D}(A)}(\text{Sum}(\mathcal{A}^\wedge : A \in A)) \). It follows that there is a set \( I \) such that \( \mathcal{A}^\wedge \in \text{thick}_{\mathcal{D}(A)}(\overline{\mathcal{T}(I)}) \), for all \( A \in A \). We put \( T = \overline{\mathcal{T}(I)} \) in the sequel, and will check that it satisfies the requirements.

Property 1 is straightforward. We can assume without loss of generality that \( T \) is a homotopically projective dg \( A \)-module (i.e. that \( \text{Hom}_{K(A)}(T, ?) \) vanishes on acyclic complexes) and put \( B = \text{End}_{\mathcal{D}(A)}(T) \). Then \( B \) is a dg algebra such that \( H^k(B) \cong \text{Hom}_{K(A)}(T, T[k]) \cong \text{Hom}_{\mathcal{D}(A)}(T, T[k]) \), for all \( k \in \mathbb{Z} \), so that \( B \) has homology concentrated in degree 0 and \( H^0(B) \cong \text{End}_{\mathcal{D}(A)}(T) =: E \). Note that the classical truncation
at zero of $B$ gives a dg subalgebra $\tau^{\leq 0} B$. The corresponding inclusion $j : \tau^{\leq 0} B \hookrightarrow B$ of dg algebras is a quasi-isomorphism, which implies that the restriction of scalars $j_* : D(B) \to D(\tau^{\leq 0} B)$ is a triangulated equivalence. Without loss of generality, we can replace $B$ by $\tau^{\leq 0} B$ and assume that $B = \oplus_{n \geq 0} B^{-n}$ is concentrated in degrees $\leq 0$. Note that $T$ is canonically a dg $B - A$-bimodule and we have now a canonical projection $p : B \to H^0(B) = E$ which is also a quasi-isomorphism of dg algebras. This implies that if $p^* = E \otimes_B ? : C_dg(B^{op}) \to C_dg(E^{op})$ is the extension of scalars, then its left derived functor $Lp^* : D(B^{op}) \to D(E^{op})$ is a triangulated equivalence. If necessary, we replace $T$ by a quasi-isomorphic homotopically projective $B - A$-bimodule. The $k$-flatness of $A$ implies that $BT$ is homotopically flat (just adapt the proof of [44, Lemma 3.6]) to the case when $A$ is a dg category instead of dg algebra). This implies that $p \otimes_B 1_T : T = B \otimes_B T \to E \otimes_B T$ is a quasi-isomorphism of $B - A$-modules and that $Lp^*(T) = E \otimes_B T$. As a final step of reduction, we replace $B$ by $E$ and $T$ by $E \otimes_B T$, thus assuming in the rest of the proof that $T$ is an $E - A$-bimodule.

Now assertion 4 of [44, Theorem 6.4] holds (with $E$ instead of $B$), so that the classical derived functor $R\Hom_A(T, ?) : D(A) \to D(E)$ is fully faithful and its left adjoint $? \otimes^L_E T : D(E) \to D(A)$ has itself a (fully faithful) left adjoint $\Lambda : D(A) \to D(E)$. We now put $G := \Hom_H(T, ?) : \mathcal{H} \to \Mod - E$. We claim that the functor $F : \Mod - E \to \mathcal{H}$ defined as the following composition

$$\Mod - E \xrightarrow{can} D(E) \xrightarrow{\otimes^{L_E}_E T} D(A) \xrightarrow{\tilde{H}} \mathcal{H}$$

is a left adjoint to $G$, where $\tilde{H}$ is the cofunctor associated to the $t$-structure $\tau$. Note that if $M$ is an $E$-module, then $M[0] \in D^{\leq 0}(E)$, so that $M[0]$ is isomorphic in $D(E)$ to the Milnor colimit of a sequence $M_0 \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} M_n \xrightarrow{\cdots}$, where $M_0 \in \operatorname{Sum}(E)[0]$ and $\operatorname{cone}(\alpha_n) \in \operatorname{Sum}(E)[n]$, for all $n \geq 0$. We then have $M[0] \otimes^L_E T \cong \operatorname{Molim}(M_n \otimes^L_E T)$, which is an object of the aile $\mathcal{U} = T^{1>0}$ (see Theorem [2]). Bearing in mind that $\tilde{H}(\mathcal{U}) : \mathcal{H} \to \mathcal{H}$ is left adjoint to the inclusion functor $\mathcal{H} \hookrightarrow \mathcal{U}$ (see [31, Lemma 3.1]), we get a sequence of isomorphisms, for $M \in \Mod - E$ and $Y \in \mathcal{H}$

$$\Hom_\mathcal{H}(F(M), Y) = \Hom_\mathcal{H}(\tilde{H}(M[0] \otimes^L_E T), Y) \cong \Hom_\mathcal{H}(M[0] \otimes^L_E T, Y) = \Hom_{D(A)}(M[0] \otimes^L_E T, Y) \cong \Hom_{D(E)}(M[0], R\Hom_A(T, Y)).$$

But we have that $Y \in \mathcal{H} = T^{1>0} \cap T^{1<0}$, which implies that $H^k(R\Hom_A(T, Y)) = \Hom_{D(A)}(T, Y[k]) = 0$, for all $k \neq 0$. It follows that we have an isomorphism $R\Hom_A(T, Y) \cong \Hom_{D(A)}(T, Y[0]) = G(Y)[0]$ in $D(E)$. We then have an isomorphism $\Hom_\mathcal{H}(F(M), Y) \cong \Hom_\mathcal{H}(M[0], G(Y)[0]) \cong \Hom_{E}(M, G(Y))$. It is routine to check that this isomorphism is natural on $M$ and $Y$, so that $(F, G)$ is an adjoint pair.

In order to prove that $G$ is fully faithful, we will check that the counit $\epsilon : F \circ G \to 1_\mathcal{H}$ is a natural isomorphism. Due to the fact that $(? \otimes^L_E T, R\Hom_A(T, ?))$ is an adjoint pair with fully faithful right component, the counit of this adjunction $\delta : (? \otimes^L_E T) \circ R\Hom_A(T, ?) \to 1_{D(A)}$ is a natural isomorphism. Then $\tilde{H}(\delta) : \tilde{H}(R\Hom_A(T, Y) \otimes^L_E T) \to \tilde{H}(Y) = Y$ is an isomorphism in $\mathcal{H}$, for all $Y \in \mathcal{H}$. We have already seen that there is an isomorphism $\operatorname{RHom}_A(T, Y) \cong G(Y)[0]$ in $D(E)$, which allows us to view $\tilde{H}(\delta)$ as a morphism $(F \circ G)(Y) = \tilde{H}(G(Y)[0] \otimes^L_E T) \to Y$. This morphism is easily identified with the counit morphism $\epsilon_Y$.

The functor $? \otimes^L_E T : D(E) \to D(A)$ preserves products since it has a left adjoint. Moreover, by Theorem [1] we know that $\tilde{H} : D(A) \to \mathcal{H}$ preserves products. It follows that the three functors in the composition (*) which defines $F$ preserve products. Therefore $F$ preserves products.

Recall the following definition (see [36, Subsection 1.2.1] and [42, Section 1]).

**Definition 10.** Let $\mathcal{H}$ be an abelian category. We say that its derived category $D(\mathcal{H})$ is left complete when the induced (non-canonical) morphism $M \to \operatorname{Holim}^{\leq n} M$ is an isomorphism, for each $M \in D(\mathcal{H})$.

The following result is a direct consequence of [36, Proposition 1.2.1.19] since any $\text{AB3}^*$ abelian category with a projective generator is $\text{AB4}^*$ (see the dual of [48, Corollary 3.2.9]). See also [50, Lemma 6.4] for a related result.

**Lemma 14.** If $\mathcal{H}$ is an $\text{AB3}^*$ abelian category with a projective generator, then $D(\mathcal{H})$ is left complete.

In the rest of the section, we will assume the following situation, as derived from the proof of Proposition [4].
Setup 2. A will be a nonpositively graded k-flat dg category, E will be an ordinary k-algebra and \( T \) will be a dg \( E - A \)-bimodule satisfying the following conditions:

1. \( T \) is a tilting object of \( D(A) \) such that \( \mathcal{A}^\circ \in \text{thick}_{D(A)}(T) \), for all \( A \in \mathcal{A} \);
2. The canonical map \( E \to \text{End}_{D(A)}(T) \) is an isomorphism of algebras.

By [44, Theorem 6.4 and Proposition 6.9], we know that we have adjoint pairs \( (\otimes^L_E T, R\text{Hom}_A(T, ?)) \) and \( (\Lambda, \otimes^R_E T) \) of triangulated functors, where \( R\text{Hom}_A(T, ?) \) and \( \Lambda : D(A) \to D(E) \) are fully faithful and preserve compact objects.

Lemma 15. The functor \( G = \text{Hom}_H(T, ?) : H \to \text{Mod} - E \) induces a triangulated functor \( G = R\text{G} : D(H) \to D(E) \) whose essential image is contained in the essential image of \( R\text{Hom}_A(T, ?) : D(A) \to D(E) \). Moreover, if \( LF : D(E) \to D(H) \) is the left derived functor of \( F \), then \( (LF, G) \) is an adjoint pair.

Proof. Since \( G : H \to \text{Mod} - E \) is an exact functor, we have \( G = R\text{G} \). That is, its right derived functor \( R\text{G} : D(H) \to D(E) \) exists and takes the complex

\[
Y^\bullet : \cdots Y^{k-1} \to Y^k \to Y^{k+1} \to \cdots
\]

to the complex

\[
G(Y^\bullet) : \cdots G(Y^{k-1}) \to G(Y^k) \to G(Y^{k+1}) \to \cdots
\]

The last sentence of the lemma is then a standard fact about derived functors.

We now consider the full subcategory \( C \) of \( D(H) \) consisting of those complexes \( Y^\bullet \in D(H) \) such that \( G(Y^\bullet) \in \text{Im}(R\text{Hom}_A(T, ?)) =: \mathcal{Z} \). The goal is to prove that \( C = D(H) \). Note that \( \mathcal{Z} \) is a full triangulated subcategory of \( D(E) \) closed under taking products and, hence, it is also closed under taking homotopy limits. Moreover, the category \( \mathcal{H} \) is \( \text{AB3}^* \) (see [46, Proposition 3.2]) and has \( T \) as a projective generator. It then follows that \( \mathcal{H} \) is \( \text{AB4}^* \) and, hence, products in \( D(H) \) are calculated pointwise. In particular \( G = R\text{G} \) preserves products and, hence, also homotopy limits. We then have that \( \mathcal{C} \) is closed under taking homotopy limits in \( D(H) \), and this is a left complete triangulated category by Lemma 14. Our task is then reduced to prove that \( D^+(H) \subseteq \mathcal{C} \).

Let now \( Y^\bullet \in D^+(H) \) be any bounded below complex of objects in \( \mathcal{H} \) which, without loss of generality, we assume to be concentrated in degrees \( \geq 0 \). By taking stupid truncations, we have that \( Y^\bullet = \text{Holim}_{s \leq n} Y^\bullet \) and, since the functor \( G = R\text{G} \) acts pointwise, we get that \( G(Y^\bullet) = \text{Holim}G(\sigma_{\leq n}Y^\bullet) = \text{Holim}\sigma_{\leq n}G(Y^\bullet) \). In this way, the proof is further reduced to prove that \( D^+(H) \subseteq \mathcal{C} \). But each object \( Y^\bullet \) of \( D^+(H) \) is a finite iterated extension of the stalk complexes \( H^k(Y^\bullet)[-k] \). This finally reduces the proof to check that if \( Y \in \mathcal{H} \), then each stalk complex \( G(Y)[k] \) is in \( \mathcal{Z} \). But this is clear since we have seen in the proof of Proposition 3 that \( G(Y)[0] \cong \text{RHom}_A(T, Y) \).

\[\square\]

Lemma 16. Let

\[
C^\bullet : \cdots \to C^{k-1} \to C^k \to C^{k+1} \to \cdots
\]

be a complex of \( E \)-modules such that \( C^k[0] \otimes^L_E T = 0 \), for all \( k \in \mathbb{Z} \). Then we also have that \( C^\bullet \otimes^L_E T = 0 \).

Proof. By taking stupid truncations, we have an isomorphism \( C^\bullet \cong \text{Mcolim}(\sigma_{\geq -n}C^\bullet) \) in \( \mathcal{K}(E) \) (and, hence, also in \( D(E) \)). We then get an isomorphism \( C^\bullet \otimes^L_E T \cong \text{Mcolim}(\sigma_{\geq -n}C^\bullet \otimes^L_E T) \) in \( D(A) \). This reduces the proof to the case when \( C^\bullet \in D^+(E) \). Without loss of generality, we assume that \( C^\bullet \in D^+(E) \). Then we have an isomorphism \( C^\bullet = \text{Holim}(\sigma_{\leq n}C^\bullet) \) in \( D(E) \). But the functor \( \otimes^L_E T : D(E) \to D(A) \) preserves products, and hence homotopy limits, because it has a left adjoint. It follows that \( C^\bullet \otimes^L_E T \cong \text{Holim}(\sigma_{\leq n}C^\bullet \otimes^L_E T) \), so that the proof is reduced to the case when \( C^\bullet \in C^b(E) \). But in this case \( C^\bullet \) is a finite iterated extension of the stalk complexes \( C^k[-k] \) and we have \( C^k[-k] \otimes^L_E T = (C^k[0] \otimes^L_E T)[-k] = 0 \), for each \( k \in \mathbb{Z} \).

\[\square\]

In the proof of the following lemma, we will follow the terminology and notation of [54] to which we refer for all definitions. We will denote by \( \mathcal{K}(\mathcal{H}) \) the homotopy category of homological projective multicomplexes over \( \mathcal{H} \) and will consider the functor \( \kappa : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}) \) which takes any \( X^{**} \in \mathcal{K}(\mathcal{H}) \) to its totalization complex \( \text{Tot}(X^{**}) \). Note that then \( \text{Im}(\kappa) \subseteq \mathcal{K}({\text{Proj}}\mathcal{H}) = \mathcal{K}({\text{Sum}}(T)) \). Recall that if \( q : \mathcal{K}(\mathcal{H}) \to D(\mathcal{H}) \) is the usual localization functor, then the composition \( q \circ \kappa : \mathcal{K}(\mathcal{H}) \to D(\mathcal{H}) \) is triangulated equivalence (see [54 Theorem 1]).
Lemma 17. Let \((X_i^\bullet)_{i\in I}\) be a family of objects of \(D(\mathcal{H})\) which has a coproduct in this category and consider the induced triangle in \(D(E)\)

\[
\prod G(X_i^\bullet) \xrightarrow{i} G(\prod X_i^\bullet) \xrightarrow{\alpha} C \xrightarrow{\beta} N
\]

Then we have \(C \otimes^L_{E} T = 0\) in \(D(A)\).

Proof. All throughout the proof, we will see the action of the canonical functor \(q : K(\mathcal{H}) \rightarrow D(\mathcal{H})\) as the identity. That is, abusing of notation, we will put \(X = q(X)\) and \(f = q(f)\), for any object \(X\) and morphism \(f\) in \(K(\mathcal{H})\). Note that the restriction functor \(q_{\text{Im}(\kappa)} : \text{Im}(\kappa) \rightarrow D(\mathcal{H})\) is full and dense since \(q \circ \kappa\) is an equivalence of categories. This allows us to assume all throughout the proof that \(X_i^\bullet \in \text{Im}(\kappa) \subset K(\text{Proj}\mathcal{H})\), for each \(i \in I\).

Consider now the coproducts \(\prod X_i^\bullet\) and \(\prod X_i^\bullet\) of the \(X_i^\bullet\) in \(K(\mathcal{H})\) and \(D(\mathcal{H})\), respectively. Note that the first one is calculated as in \(\mathcal{C}(\mathcal{H})\) (i.e. pointwise). Let us denote by \(\lambda_j : X_j^\bullet \rightarrow \prod X_i^\bullet\) and \(\iota_j : X_j^\bullet \rightarrow \prod X_i^\bullet\) the respective injections into the coproduct, in \(D(\mathcal{H})\) and \(K(\mathcal{H})\), respectively. By the universal property of the coproduct \(\prod X_i^\bullet\), we have a unique morphism \(f : \prod X_i^\bullet \rightarrow \prod X_i^\bullet\) in \(D(\mathcal{H})\) such that \(f \circ \lambda_j = \iota_j\), for all \(j \in I\). It follows that, for each \(Y^\bullet \in \text{Im}(\kappa)\), we have the following commutative diagram, where the horizontal arrows are the canonical isomorphisms coming from the universal property of coproducts:

\[
\begin{array}{ccc}
\text{Hom}_{K(\mathcal{H})}(\prod X_i^\bullet, Y^\bullet) & \xrightarrow{\sim} & \prod_{i \in I} \text{Hom}_{K(\mathcal{H})}(X_i^\bullet, Y^\bullet) \\
\downarrow q & & \downarrow q \\
\text{Hom}_{D(\mathcal{H})}(\prod X_i^\bullet, Y^\bullet) & \xrightarrow{\sim} & \prod_{i \in I} \text{Hom}_{D(\mathcal{H})}(X_i^\bullet, Y^\bullet)
\end{array}
\]

Due to the fullness of \(q_{\text{Im}(\kappa)}\), the right vertical arrow of last diagram is an epimorphism, which implies that \(f^*\) is an epimorphism, for each \(Y^\bullet \in \text{Im}(\kappa)\).

But each object of \(D(\mathcal{H})\), in particular \(\prod X_i^\bullet\), is in the image of \(q_{\text{Im}(\kappa)}\). This means that we can (and shall) assume that, as an object of \(K(\mathcal{H})\), one has \(\prod X_i^\bullet \in \text{Im}(\kappa)\). Putting then \(Y = \prod X_i^\bullet\) in the last paragraph, we get that the map \(f^* : \text{Hom}_{D(\mathcal{H})}(\prod X_i^\bullet, \prod X_i^\bullet) \rightarrow \text{Hom}_{D(\mathcal{H})}(\prod X_i^\bullet, \prod X_i^\bullet)\) is an epimorphism. Therefore \(f\) is a section in \(D(\mathcal{H})\).

The octahedral axiom gives the following commutative diagram in \(D(E)\), where the morphism \(\prod G(X_i^\bullet) \rightarrow G(\prod X_i^\bullet)\) (resp. \(\prod G(X_i^\bullet) \rightarrow G(\prod X_i^\bullet)\)) is the canonical one in \(D(E)\) (resp. \(K(E)\)) induced by the universal property of the coproduct (here we use that the coproduct of complexes in \(D(E)\) can be calculated pointwise and, hence, \(\text{‘coincides’ with the coproduct in } K(E)\)):

\[
\begin{array}{ccc}
\prod G(X_i^\bullet) & \xrightarrow{G(\alpha)} & G(\prod X_i^\bullet) \\
\uparrow & & \uparrow \beta \\
\prod G(X_i^\bullet) & \xrightarrow{\beta} & C' \\
\uparrow_{\gamma} & & \uparrow_{\gamma} \\
N & \xrightarrow{\gamma} & N
\end{array}
\]

The fact that \(G(\alpha)\) is a section implies that \(\beta\) is a retraction, so that \(\alpha\) is also a retraction in \(D(E)\) and the dotted triangle splits. Since \(D(E)\) has arbitrary coproducts, we know that idempotents split and, hence, we have \(C' \cong C \oplus N\).

The proof is whence reduced to check that \(C' \otimes^L_{E} T = 0\). But the canonical morphism \(u : \prod G(X_i^\bullet) \rightarrow G(\prod X_i^\bullet)\) is the image of the \(\text{‘same’}\) morphism in \(\mathcal{C}(E)\), which is a monomorphism in this category since \(G : \mathcal{H} \rightarrow \text{Mod} - E\) is exact and preserves finite coproducts. Therefore we can assume that we have an exact sequence \(0 \rightarrow \prod G(X_i^\bullet) \xrightarrow{u} G(\prod X_i^\bullet) \rightarrow C' \rightarrow 0\) in the abelian category \(\mathcal{C}(E)\) whose image by
the usual functor $C(E) \longrightarrow \mathcal{D}(E)$ is the second horizontal triangle of the diagram above. We then have an exact sequence

\[ 0 \to \bigoplus_{i \in I} G(X^i_k) \to G\bigl(\bigoplus_{i \in I} X^i_k\bigl) \to C^{ik} \to 0 \quad (\ast) \]

in $\text{Mod} - E$, for each $k \in \mathbb{Z}$. But, due to the exactness of $G : \mathcal{H} \longrightarrow \text{Mod} - E$, we know that $G(Y^\bullet)^k = G(Y^k)$, for each $Y^\bullet \in \mathcal{C}(\mathcal{H})$ and each $k \in \mathbb{Z}$. This together with the fact that coproducts in $\mathcal{K}(\mathcal{H})$ are calculated pointwise allows us to rewrite the sequence $(\ast)$ as

\[ 0 \to \bigoplus_{i \in I} G(X^i_k) \to G\bigl(\bigoplus_{i \in I} X^i_k\bigl) \to C^{ik} \to 0 \quad (\ast) \]

We have seen in the proof of Proposition\textsuperscript{4} that $G(Y)[0] \cong \text{RHom}_\mathcal{A}(T, Y)$, for each $Y \in \mathcal{H}$. The last exact sequence then gives a triangle in $\mathcal{D}(E)$

\[ \bigoplus_{i \in I} \text{RHom}_\mathcal{A}(T, X^i_k[0]) \longrightarrow \text{RHom}_\mathcal{A}(T, \bigl(\bigoplus_{i \in I} X^i_k[0]\bigl)) \longrightarrow C^{ik}[0] \overset{+}{\longrightarrow}. \]

In principle, the coproduct of the $X^i_k$ has been taken in $\mathcal{H}$. But, by the choice of the complexes $X^i$, we know that each $X^i_k$ is a projective object of $\mathcal{H}$, so that the coproduct $\bigoplus_{i \in I} X^i_k$ is the same in $\mathcal{H}$ and in $\mathcal{D}(\mathcal{H})$. Applying now the functor $? \otimes_E^L T : \mathcal{D}(E) \longrightarrow \mathcal{D}(\mathcal{A})$ to the last triangle and bearing in mind that the counit (? $\otimes_E^L T \otimes \text{RHom}_\mathcal{A}(T, ?)) \longrightarrow 1_{\mathcal{D}(\mathcal{A})}$ is a natural isomorphism, we easily conclude that $C^{ik}[0] \otimes_E^L T = 0$, for all $k \in \mathbb{Z}$. Then $C^i \otimes_E^L T = 0$ in $\mathcal{D}(\mathcal{A})$, by Lemma\textsuperscript{16}

\[ \square \]

We are now ready for the main result of this section. We refer the reader to [13] for a simple proof of the corresponding result when $\mathcal{D} = \mathcal{D}(\mathcal{A})$ is the derived category of an abelian category $\mathcal{A}$ as in Setup\textsuperscript{11} and $T$ is a tilting object of $\mathcal{A}$.

**Theorem 4.** Let $k$ be a commutative ring, let $\mathcal{D}$ be a compactly generated algebraic triangulated $k$-category, let $\mathcal{T}$ be a bounded tilting set of $\mathcal{D}$ and let $\mathcal{H}$ be the heart of the associated t-structure. The inclusion $\mathcal{H} \hookrightarrow \mathcal{D}$ extends to a triangulated equivalence $\Psi : \mathcal{D}(\mathcal{H}) \overset{\sim}{\longrightarrow} \mathcal{D}$.

**Proof.** As shown in the proof Proposition\textsuperscript{4} we can assume that $\mathcal{T} = \{T\}$ and that we are in the situation of Setup\textsuperscript{2} with $\mathcal{D} = \mathcal{D}(\mathcal{A})$. We can then use the functors $G$ and $F$ given by such proposition. Using the adjoint pairs ($\text{LF}, G$) and ($\Lambda, ? \otimes_E^L T$), we see that we have two compositions of triangulated functors

\[ \Phi : \mathcal{D} = \mathcal{D}(\mathcal{A}) \overset{\Lambda}{\longrightarrow} \mathcal{D}(E) \overset{\text{LF}}{\longrightarrow} \mathcal{D}(\mathcal{H}) \]

\[ \Psi : \mathcal{D}(\mathcal{H}) \overset{G}{\longrightarrow} \mathcal{D}(E) \overset{\gamma \otimes_E^L T}{\longrightarrow} \mathcal{D}(\mathcal{A}) = \mathcal{D}, \]

such that $(\Phi, \Psi)$ is an adjoint pair of triangulated functors.

Put $\mathcal{T}' := \text{thick}_{\mathcal{D}(\mathcal{H})}(T)$ in the sequel. We claim that the restriction $\Psi|_{\mathcal{T}'} : \mathcal{T}' \longrightarrow \mathcal{D}(\mathcal{A})$ is fully faithful. It is enough to check that the induced map $\text{Hom}_{\mathcal{D}(\mathcal{H})}(T, T[k]) \longrightarrow \text{Hom}_{\mathcal{D}(\mathcal{A})}(\Psi(T), \Psi(T)[k])$ is an isomorphism, for all $k \in \mathbb{Z}$. But $\Psi(T) = E \otimes_E^L T = T$ and the last map gets identified with the composition

\[ \text{Hom}_{\mathcal{D}(\mathcal{H})}(T, T[k]) \overset{G}{\longrightarrow} \text{Hom}_{\mathcal{D}(E)}(E, E[k]) \overset{\gamma \otimes_E^L T}{\longrightarrow} \text{Hom}_{\mathcal{D}(\mathcal{A})}(T, T[k]). \]

This map is bijective since the three appearing $k$-modules are zero when $k \neq 0$.

By Lemma\textsuperscript{15} we know that $\text{Im}(G) \subseteq \text{Im}(\text{RHom}_\mathcal{A}(T, ?))$. Moreover, the restriction of $? \otimes_E^L T$ induces an equivalence of triangulated categories $\text{Im}(\text{RHom}_\mathcal{A}(T, ?)) \overset{\sim}{\longrightarrow} \mathcal{D}(\mathcal{A})$. The already proved fully faithful condition of $\Psi|_{\mathcal{T}'} = (? \otimes_E^L T \otimes \text{G}|_{\mathcal{T}'})$ then implies that also $\text{G}|_{\mathcal{T}'} : \mathcal{T}' \longrightarrow \mathcal{D}(E)$ is fully faithful. Moreover, since each object of $\text{per}(E) = \mathcal{D}^c(E)$ is a direct summand of a finite iterated extension of stalk complexes $E[k] = G(T[k])$, we readily see that $G$ induces an equivalence of categories $\mathcal{T}' = \text{thick}_{\mathcal{D}(\mathcal{H})}(T) \overset{\sim}{\longrightarrow} \text{thick}_{\mathcal{D}(\mathcal{H})}(T) = \text{per}(E)$ whose inverse is necessarily $\text{LF}|_{\text{per}(E)} : \text{per}(E) \longrightarrow \text{thick}_{\mathcal{D}(\mathcal{H})}(T)$. Therefore $\text{LF}$ is fully faithful when restricted to $\text{per}(E)$.

We will now prove that $\Phi : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{H})$ is fully faithful. To do it we follow a standard method (see the proof of [28, Lemma 4.2]). Consider the full subcategory $\mathcal{Z}$ of $\mathcal{D}(\mathcal{A})$ consisting of the $\text{dg} \mathcal{A}$-modules $N$ such that the canonical map $\Phi : \text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{A}^\wedge, N[k]) \longrightarrow \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(\mathcal{A}^\wedge), \Phi(N)[k])$ is an isomorphism, for
all $A \in \mathcal{A}$ and all integers $k \in \mathbb{Z}$. It is clearly a triangulated subcategory and it contains all representable $A$-modules $B^A$. Indeed, by [32] Theorem 6.4 and Proposition 6.9, we know that $A$ preserves compact objects and we have already seen that the restriction of $\mathcal{L}F$ to $\mathcal{P}(E)$ is fully faithful. We then get that the restriction of $\Phi$ to $\mathcal{P}(A) = \mathcal{D}^c(A)$ is fully faithful, so that $B^A \in \mathcal{Z}$, for all objects $B \in \mathcal{A}$. We claim that $\mathcal{Z}$ is also closed under taking coproducts, which, by dévissage, will imply that $\mathcal{Z} = \mathcal{D}(A)$.

For this it is enough to prove that, for each $A \in \mathcal{A}$, the object $\Phi(A^\wedge)$ is a compact in $\mathcal{D}(\mathcal{H})$, in the sense that if a family of objects $(Y^*_i)_{i \in I}$ of $\mathcal{D}(\mathcal{H})$ has a coproduct in this category, then the canonical map $\prod_{i \in I} \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), Y^*_i) \to \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \prod_{i \in I} Y^*_i)$ is an isomorphism. Indeed, since $\Phi$ is a composition of left adjoint functors, it preserves coproducts. That is, if $(N_i)_{i \in I}$ is a family of objects of $\mathcal{D}(A)$, then the coproduct in $\mathcal{D}(\mathcal{H})$ of the $\Phi(N_i)$ exists and is isomorphic to $\Phi(\prod_{i \in I} N_i)$. If the compactness condition of $\Phi(A^\wedge)$ is checked, then we will get an isomorphism

$$\prod_{i \in I} \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \Phi(N_i)) \cong \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \prod_{i \in I} \Phi(N_i)) \cong \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \Phi(\prod_{i \in I} N_i)).$$

When the $N_i$ are in $\mathcal{Z}$, this isomorphism can be then inserted in a commutative diagram, where the two horizontal arrows and the left vertical one are isomorphisms:

$$\begin{array}{ccc}
\prod \text{Hom}_{\mathcal{D}(\mathcal{A})}(A^\wedge, N_i) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{D}(\mathcal{A})}(A^\wedge, \prod N_i) \\
\downarrow & & \downarrow \phi \\
\prod \text{Hom}_{\mathcal{D}(\mathcal{H})}(\phi(A^\wedge), \phi(N_i)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{D}(\mathcal{H})}(\phi(A^\wedge), \phi(\prod N_i))
\end{array}$$

It will follow that also the right vertical arrow is an isomorphism, which in turn implies that $\prod_{i \in I} N_i \in \mathcal{Z}$, so that $\mathcal{Z}$ will be closed under taking coproducts in $\mathcal{D}(A)$ as desired.

Let us prove then that $\Phi(A^\wedge)$ is compact in $\mathcal{D}(\mathcal{H})$, for each $A \in \mathcal{A}$. Using successively the adjunctions $(\mathcal{L}F, G)$ and $(A, ? \otimes^L_E T)$, given a family $(Y^*_i)$ of objects in $\mathcal{D}(\mathcal{H})$ that has a coproduct in this category and an object $A \in \mathcal{A}$, we have a chain of isomorphisms

$$\text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \prod Y^*_i) \cong \text{Hom}_{\mathcal{D}(E)}(A(A^\wedge), G(\prod Y^*_i)) \cong \text{Hom}_{\mathcal{D}(A)}(A^\wedge, G(\prod Y^*_i) \otimes^L_E T).$$

But, by Lemma [17] we have an isomorphism $(\prod G(Y^*_i)) \otimes^L_E T \cong G(\prod Y^*_i) \otimes^L_E T$. This fact, the isomorphism (**), and the compactness of $A^\wedge$ in $\mathcal{D}(A)$ tell us that we have an isomorphism

$$\text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \prod Y^*_i) \cong \text{Hom}_{\mathcal{D}(A)}(A^\wedge, \prod G(Y^*_i) \otimes^L_E T) \cong \prod \text{Hom}_{\mathcal{D}(\mathcal{A})}(A^\wedge, G(Y^*_i) \otimes^L_E T) \cong \prod \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), Y^*_i),$$

which is easily seen to be the inverse of the canonical morphism $\prod \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), Y^*_i) \to \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge), \prod Y^*_i)$.

We now finish the proof of the fully faithful condition of $\Phi$. For any $N \in \mathcal{D}(A)$ fixed, we consider the full subcategory $\mathcal{C}^N$ of $\mathcal{D}(A)$ consisting of those dg $A$-modules $M$ such that $\Phi : \text{Hom}_{\mathcal{D}(A)}(M[k], N) \to \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(M)[k], \Phi(N))$ is an isomorphism, for all $k \in \mathbb{Z}$. This is a triangulated subcategory of $\mathcal{D}(A)$ clearly closed under taking coproducts. The previous paragraphs of this proof show that $\mathcal{C}^N$ contains all representable $A$-modules $A^\wedge$, with $A \in \mathcal{A}$. By dévissage, we conclude that $\mathcal{C}^N = \mathcal{D}(A)$, and hence that $\Phi$ is fully faithful.

Let us put $\mathcal{X} = \text{Im}(\Phi)$, which is then a full triangulated subcategory of $\mathcal{D}(\mathcal{H})$ closed under taking coproducts, when they exist. The fact that $\Phi$ is fully faithful and has a right adjoint implies that the inclusion functor $\mathcal{X} \hookrightarrow \mathcal{D}(\mathcal{H})$ has a right adjoint. By [32] Proposition 1, we know that $(\mathcal{X}, \mathcal{X}^\perp)$ is a semi-orthogonal decomposition of $\mathcal{D}(\mathcal{H})$. We will prove that $\mathcal{X}^\perp = 0$, which will imply that $\mathcal{X} = \mathcal{D}(\mathcal{H})$ and, hence, that $\Phi$ is an equivalence of categories. Then its quasi-inverse $\Psi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(A)$ will be the desired functor. Indeed, if $X \in \mathcal{H}$ then, as seen in the proof of Proposition [32] we have $\Psi(X) = G(X)[0] \otimes^L_E T \cong X$, for each $X \in \mathcal{H}$, from which it is easily seen that $\Psi|_{\mathcal{H}} : \mathcal{H} \to \mathcal{D}$ is naturally isomorphic to the inclusion functor.

Let $Y^*$ be an object of $\mathcal{X}^\perp$. Then, for each $A \in \mathcal{A}$ and $k \in \mathbb{Z}$, we have that $0 = \text{Hom}_{\mathcal{D}(\mathcal{H})}(\Phi(A^\wedge)[k], Y^*) \cong \text{Hom}_{\mathcal{D}(A)}(A^\wedge[k], \Psi(Y^*))$. This implies that $0 = \Psi(Y^*) = (\otimes^L_E T) \circ G(Y^*)$. 

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By Lemma [15] we know that $G(Y^\bullet) \cong \text{RHom}_A(T, M)$, for some $M \in \mathcal{D}(A)$ fixed in the sequel. Moreover, since the counit $\delta : (\otimes^B E) \circ \text{RHom}_A(T, ?) \to 1_{\mathcal{D}(A)}$ is a natural isomorphism we get an isomorphism

$$0 = \Phi(Y^\bullet) = \text{RHom}_A(T, M) \otimes^B T \xrightarrow{\sim} M.$$  

This implies that $G(Y^\bullet) = 0$. In other words, if $Y^\bullet$ is the complex in $\mathcal{C}(\mathcal{H})$

$$\ldots \to Y^{k-1} \to Y^k \to Y^{k+1} \to \ldots,$$

then the complex in $\mathcal{C}(E)$

$$G(Y^\bullet) : \ldots \to G(Y^{k-1}) \to G(Y^k) \to G(Y^{k+1}) \to \ldots$$

is acyclic. But due to the exactness of $G$, we have that $0 = H^k(G(Y^\bullet)) = G(H^k(Y^\bullet))$, so that we have $\text{Hom}_\mathcal{H}(T, H^k(Y^\bullet)) = 0$, for all $k \in \mathbb{Z}$. This in turn implies that $H^k(Y^\bullet) = 0$, for all $k \in \mathbb{Z}$, because $T$ is a projective generator of $\mathcal{H}$. That is, $Y^\bullet$ is acyclic and hence $Y^\bullet = 0$ in $\mathcal{D}(\mathcal{H})$. \qed

A few corollaries follow now (see [50] for related results when $\mathcal{D} = \mathcal{D}(A)$ is the derived category of an abelian category as in Setup [1].

**Corollary 7.** Let $\mathcal{D}$ be a compactly generated algebraic triangulated category, let $\mathcal{T}$ be a bounded tilting set in $\mathcal{D}$ and let $\mathcal{S}$ be a silting set of compact objects of $\mathcal{D}$ which is weakly equivalent to $\mathcal{T}$. If $\mathcal{H}$ is the heart of the $t$-structure in $\mathcal{D}$ associated to $\mathcal{T}$ and $\Psi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}$ is any triangulated equivalence which extends the inclusion functor $\mathcal{H} \hookrightarrow \mathcal{D}$, then the following assertions hold:

1. $\Psi(\mathcal{D}^b(\mathcal{H}))$ consists of the objects $M$ of $\mathcal{D}$ such that, for some natural number $n = n(M)$, one has $\text{Hom}_\mathcal{D}(S, M[k]) = 0$ whenever $|k| > n$ and $S \in \mathcal{S}$.

2. $\Psi(\mathcal{D}^-(\mathcal{H}))$ (resp. $\Psi(\mathcal{D}^+(\mathcal{H}))$) consists of the objects $M$ of $\mathcal{D}$ such that, for some natural number $n = n(M)$, one has $\text{Hom}_\mathcal{D}(S, M[k]) = 0$ whenever $k > n$ (resp. $k < -n$) and $S \in \mathcal{S}$.

**Proof.** Without loss of generality, we assume that $\mathcal{T} = \{T\}$. Let $X^\bullet \in \mathcal{D}(\mathcal{H})$ be any object. Using the fact that $T$ is a projective generator of $\mathcal{D}(\mathcal{H})$ and that $\Psi(T) = T$, we get a chain of double implications:

$$X^\bullet \in \mathcal{D}^-(\mathcal{H}) \iff \text{Hom}_\mathcal{D}(\mathcal{H})(T, X^\bullet[k]) = 0, \text{ for } k > 0 \iff \text{Hom}_\mathcal{D}(T, \Psi(X^\bullet)[k]) = 0, \text{ for } k > 0.$$  

The fact that $\text{thick}_\mathcal{D}(\text{Sum}(T)) = \text{thick}_\mathcal{D}(\text{Sum}(\mathcal{S}))$ gives then that $X^\bullet \in \mathcal{D}^-(\mathcal{H})$ if and only if there exists $a$ such that $\text{Hom}_\mathcal{D}(S, \Psi(X^\bullet)[k]) = 0$, for all $S \in \mathcal{S}$ and all integers $k > n$. Then the part of assertion 2 concerning the description of $\Psi(\mathcal{D}^-(\mathcal{H}))$ is clear. A symmetric argument proves the corresponding assertion for $\Psi(\mathcal{D}^+(\mathcal{H}))$, which in turn proves also assertion 1. \qed

**Corollary 8.** Let $A$ be a dg algebra (e.g. an ordinary algebra), let $\mathcal{T}$ be a bounded tilting set in $\mathcal{D}(A)$ and let $\mathcal{H}$ be the heart of the $t$-structure $(\mathcal{T}^{(+)} \mathcal{T}^{(-)})$ in $\mathcal{D}(A)$. If $\Psi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(A)$ is any triangulated equivalence which extends the inclusion functor $\mathcal{H} \hookrightarrow \mathcal{D}(A)$, then it induces by restriction triangulated equivalences $\mathcal{D}^*(\mathcal{H}) \to \mathcal{D}^*(A)$, for $* \in \{b, +, -\}$.

**Proof.** As usual, we assume that $\mathcal{T} = \{T\}$. For each classical silting set $\mathcal{S}$ of $\mathcal{D}(A)$, one has $\text{thick}_{\mathcal{D}(A)}(\mathcal{S}) = \text{per}(A) = \text{thick}_{\mathcal{D}(A)}(A)$ (*) (see [28] Theorem 5.3). Note that then we have a finite subset $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $A \in \text{thick}_{\mathcal{D}(A)}(\mathcal{S}_0)$. Then $\mathcal{S}_0$ is also a classical silting set, because it is classical partial silting and generates $\mathcal{D}(A)$. By [11] Theorem 2.18, we conclude that $\mathcal{S} = \mathcal{S}_0$, so that $\mathcal{S}$ is finite. But the equality $\text{thick}_{\mathcal{D}(A)}(\text{Sum}(\mathcal{S})) = \text{thick}_{\mathcal{D}(A)}(\text{Sum}(T))$ gives that, up to shift, we have an inclusion $\mathcal{S} \hookrightarrow \text{Add}(T) \ast \text{Add}(T)[1] \ast \ldots \ast \text{Add}(T)[n]$, for some $n \in \mathbb{N}$. Due to the tilting condition of $T$, we know that $\text{Hom}_{\mathcal{D}(A)}(X, Y[k]) = 0$, for $|k| > 2n$, whenever $X, Y \in \text{Add}(T) \ast \text{Add}(T)[1] \ast \ldots \ast \text{Add}(T)[n]$. Therefore we have that $\text{Hom}_{\mathcal{D}(A)}(S, S'[k]) = 0$, for $|k| > n$, whenever $S, S' \in \mathcal{S}$.  

Bearing in mind the equality (*) above, we get an $p \in \mathbb{N}$ such that $H^k(A) \cong \text{Hom}_{\mathcal{D}(A)}(A, A[k]) = 0$, for $|k| > p$. That is, $A$ is homologically bounded. Moreover, the finiteness of $\mathcal{S}$ implies that we actually have $\text{thick}_{\mathcal{D}(A)}(\text{Sum}(\mathcal{S})) = \text{thick}_{\mathcal{D}(A)}(\text{Sum}(A))$. It then follows in a straightforward way that a dg $A$-module $Y$ is in $\mathcal{D}^-(A)$ if and only if there is a $m = m(Y) \in \mathbb{N}$ such that $\text{Hom}_{\mathcal{D}(A)}(S, Y[k]) = 0$ for $k > m$. A similar argument works when we take $+$ or $b$ instead of $-$. Now apply Corollary [7]. \qed
Theorem 4 allows us to say something when we replace 'tilting' by 'partial tilting'. We refer to [11, Section 1.4] for the definition of recollement of triangulated categories.

**Corollary 9.** Let $D$ be a compactly generated algebraic triangulated category, let $T$ be a bounded partial tilting set of $D$, let $\tau = (^\perp (T_\geq 0), T_\leq 0)$ be its associated $t$-structure and let $H$ be its heart. The inclusion $H \hookrightarrow D$ extends to a fully faithful functor $j_1 : D(H) \to D$ which fits in a recollement

\[
\begin{array}{ccc}
D' & \xrightarrow{i_*} & D \\
\downarrow{i^*} & & \downarrow{j^*} \\
\downarrow{j_*} & & \downarrow{D(H)} \\
\end{array}
\]

**Proof.** Let $S$ be a classical partial silting set in $D$ which is weakly equivalent to $T$. Consider the associated localizing subcategory $D'' := \text{Loc}_D(T) = \text{Loc}_D(S)$. Since it is compactly generated it is smashing and we have a recollement

\[
\begin{array}{ccc}
D' & \xrightarrow{i_*} & D \\
\downarrow{i^*} & & \downarrow{j^*} \\
\downarrow{j_*} & & \downarrow{D''} \\
\end{array}
\]

(see [13, Theorem 5]), where the upper arrow $D \leftarrow D''$ is the inclusion functor. Moreover, $D''$ is also an algebraic triangulated category and $T$ is a bounded tilting set of $D''$. By Theorem 1 and its proof, we have that $T_{\geq 0} = U_T * T_{\leq 0} \in Z$, which implies that $T_{\geq 0} (D'') := T_{\geq 0} \cap D'' = U_T = T_{\geq 0} (T \leq 0)$. Then the heart of the $t$-structure in $D''$ associated to $T$ is

\[
H' = T_{\geq 0} (D'') \cap T_{\leq 0} (D'') = U_T \cap T_{\leq 0} \cap D'' = U_T \cap T_{\leq 0} = H
\]

since $U_T \subset D''$. Theorem 4 gives then a triangulated equivalence $\Psi : D(H) \simto D''$. The desired functor $j_1$ is the composition $D(H) \xrightarrow{\Psi} D'' \xrightarrow{incl} D$.

We end by giving a partial affirmative answer to Question 2 (see the end of Section 4).

**Corollary 10.** Let $D$ be a compactly generated algebraic triangulated category and let $T$ be an object of $D$. Then $T$ is a self-small bounded tilting object if and only if it is a classical tilting object.

**Proof.** Just adapt the proof of [18, Corollary 2.3], using our Theorem 4 instead of [18, Theorem 1.5].

8 Relation with exceptional sequences

Starting with Rudakov’s seminar (see [53]) the concepts of exceptional and strongly exceptional sequence of coherent sheaves have played a fundamental role in Algebraic Geometry and it still an open problem to identify those algebraic varieties $X$ for which there exists an exceptional sequence in the category $\text{coh}(X)$ of coherent sheaves (see the introduction of [26]). In this final section we want to stress the relationship between these classical concepts and those of partial silting (or tilting) sets studied in earlier sections. In a typical pattern of this paper, we shall pass from finite (strongly) exceptional sequences to infinite ones and from 'coherent' objects to arbitrary ones.

**Definition 11.** Let $G$ be any Grothendieck category and let $(T_n)_{n \in \mathbb{Z}}$ be a sequence of (some possibly zero) objects of $G$. We will say that $(T_n)_{n \in \mathbb{Z}}$ is

1. Exceptional when $\text{Ext}^k_G(T_n, T_p(I)) = 0$, for all integers $n, p, k$ such that $n \geq p$ and $k > 0$ and all sets $I$,
2. Strongly exceptional when it is exceptional and $\text{Ext}^k_G(T_n, T_p(I))$, for all $k > 0$, all $n, p \in \mathbb{Z}$ and all sets $I$.
3. Superexceptional when $\text{Ext}^k_G(T_n, T_p(I)) = 0$, for all $p, n \in \mathbb{Z}$, all sets $I$ and all natural numbers $k > p - n$.

All these sequences will be called complete when they generate $D(G)$. 

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Note that 'strongly exceptional' implies 'superexceptional', which in turn implies 'exceptional'.

Recall that an abelian category $\mathcal{A}$ is hereditary when $\text{gldim}(\mathcal{A}) \leq 1$ (see Definition 4). The category of quasi-coherent sheaves on a weighted projective line (see 20) or the module category over the path algebra of a possibly infinite quiver are examples of hereditary Grothendieck categories. Note that if $\mathcal{G}$ is a hereditary Grothendieck category, then a sequence $(T_n)_{n \in \mathbb{Z}}$ in $\mathcal{G}$ is superexceptional if and only if it is exceptional.

Recall that a Grothendieck category $\mathcal{G}$ is locally coherent when the objects $X$ such that $\text{Hom}_\mathcal{G}(X,?) : \mathcal{G} \to \text{Ab}$ preserve direct limits, usually called finitely presented objects, form a skeletally small class $fp(\mathcal{G})$ of generators of $\mathcal{G}$ which is closed under taking kernels.

We are ready for the main result of the section.

**Proposition 5.** Let $\mathcal{G}$ be any Grothendieck category with $\text{gldim}(\mathcal{G}) \leq d < \infty$, and let $(T_n)_{n \in \mathbb{Z}}$ be a sequence of objects in $\mathcal{G}$. The following assertions hold:

1. $(T_n)_{n \in \mathbb{Z}}$ is a superexceptional sequence if and only if $\mathcal{T} = \{T_n[n] : n \in \mathbb{Z}\}$ is a strongly nonpositive set in $\mathcal{D}(\mathcal{G})$.

2. $(T_n)_{n \in \mathbb{Z}}$ is exceptional if and only if $\mathcal{T} = \{T_n[nd] : n \in \mathbb{Z}\}$ is a strongly nonpositive set in $\mathcal{D}(\mathcal{G})$.

Moreover, when $\mathcal{G}$ is hereditary, each strongly nonpositive set in $\mathcal{D}(\mathcal{G})$ is equivalent to a set $\mathcal{T}$ as above (see Definition 4).

For general $d > 0$, under conditions 1 or 2 above, if either one of the conditions below holds then $\mathcal{T}$ is partial silting, and it is even partial tilting in case $(T_n)_{n \in \mathbb{Z}}$ is strongly exceptional.

a) The family $(T_n)_{n \in \mathbb{N}}$ is finite, i.e. $T_n = 0$ for almost all $n \in \mathbb{Z}$;

b) $\mathcal{G}$ is locally coherent and all the $T_n$ are finitely presented objects of $\mathcal{G}$.

**Proof.** For the first part of the proposition, we just need to prove assertion 1 since, as it is easily verified and left to the reader, the sequence $(T_n)_{n \in \mathbb{Z}}$ is exceptional if and only if the sequence $(T'_n)_{n \in \mathbb{Z}}$ is superexceptional, where $T'_n = 0$ for $n \notin d\mathbb{Z}$ and $T'_n = T_{Nn}$ for $n \in d\mathbb{Z}$.

We then prove assertion 1. The set $\mathcal{T}$ is strongly nonpositive if and only if $\text{Hom}_{\mathcal{D}(\mathcal{G})}(T_n[n], \bigoplus_{p \in \mathbb{Z}} T_p^{(lp)}[p + k]) = 0$ (1), for all $n, p \in \mathbb{Z}$, for all integers $k > 0$ and for each family $(l_p)_{p \in \mathbb{Z}}$ of sets. Let us fix $n \in \mathbb{Z}$ and $k > 0$ and decompose $X := \bigoplus_{p \in \mathbb{Z}} T_p^{(lp)}[p + k]$ as $X = Y \bigoplus Z \bigoplus W$, where $Z = \bigoplus_{p \geq n - k + d + 1} T_p^{(lp)}[p + k]$, $Y = \bigoplus_{p \leq n - k} T_p^{(lp)}[p + k]$ and $W = \bigoplus_{p \geq n - k} T_p^{(lp)}[p + k]$. Then we have that $Y \in \mathcal{D}^{\leq -n - d - 1}(\mathcal{G})$ while $W \in \mathcal{D}^{> -n}(\mathcal{G})$. Then $\text{Hom}_{\mathcal{D}(\mathcal{G})}(T_n[n], W) = 0$ and, by Lemma 6 and the fact that each object of $\mathcal{G}$ has projective dimension $\leq d$, we also have that $\text{Hom}_{\mathcal{D}(\mathcal{G})}(T_n[n], Y) = 0$. Therefore equality (1) holds for $n$ and $k$ if and only if $\text{Hom}_{\mathcal{D}(\mathcal{G})}(T_n[n], Z) = 0$, which is in turn equivalent to saying that $\text{Ext}^j_G(T_n, T_n^{(l_k+j)}) = 0$, for all all $j = 0, 1, ..., d$. Bearing in mind that $\text{Ext}^j_G(?, ?) : \mathcal{G}^d \times \mathcal{G} \to \text{Ab}$ vanishes for $j \in \mathbb{N} \setminus \{0, 1, ..., d\}$, we deduce that the equality (1) holds if and only if $\text{Ext}^j_G(T_n, T_n^{(l)}) = 0$, for all sets $I$, whenever $j$ is a natural number such that $j > p - n$. That is, if and only if $(T_n)_{n \in \mathbb{Z}}$ is a superexceptional sequence in $\mathcal{G}$.

Suppose now that $\mathcal{G}$ is hereditary and that $\mathcal{T}'$ is a strongly nonpositive set of objects in $\mathcal{D}(\mathcal{G})$. It is well-known that in this case each $X \in \mathcal{D}(\mathcal{G})$ is isomorphic to $\bigoplus_{n \in \mathbb{Z}} H^{-n}(X)[n]$. This implies that $\mathcal{T}' = \{H^{-n}(T') : n \in \mathbb{Z}, T' \in \mathcal{T}'\}$ is also strongly nonpositive. Taking now $T_n = \bigoplus_{T' \in \mathcal{T}'} H^{-n}(T')$, we get a set $\{T_n[n] : n \in \mathbb{Z}\}$ which is strongly nonpositive in $\mathcal{D}(\mathcal{G})$ and satisfies that $\text{Add}(\mathcal{T}') = \text{Add}(\mathcal{T})$. By the previous paragraph, we know that $(T_n)_{n \in \mathbb{Z}}$ is a (super)exceptional sequence in $\mathcal{G}$.

We pass to prove the partial silting condition. When $(T_n)_{n \in \mathbb{Z}}$ is finite, due to the fact that $\text{gldim}(\mathcal{G}) < \infty$, we know that there are integers $r < s$ such that $\mathcal{T} \subset \mathcal{D}^{\leq s}(\mathcal{G})$ and $\text{Hom}_{\mathcal{D}(\mathcal{G})}(T_n[?, n])$ vanishes on $\mathcal{D}^{\geq r}(\mathcal{G}) = \mathcal{D}^{\leq s}(\mathcal{G})[s - r]$, for all $n \in \mathbb{Z}$. Then Theorem 2 applies, with $\mathcal{V} = \mathcal{D}^{\leq s}(\mathcal{G})$ and $q = s - r$.

Suppose now that $\mathcal{G}$ is locally coherent. We claim if $X$ is a finitely presented object of $\mathcal{G}$, then $X = X[0]$ is a compact object of $\mathcal{D}(\mathcal{G})$. Once this is proved the partial silting condition of $\mathcal{T}$ under assumption 2 will follow directly from Example 1). We need to prove that if $(M_i)_{i \in I}$ is any family of objects of $\mathcal{D}(\mathcal{G})$, then the canonical map $\prod_{i \in I} \text{Hom}_{\mathcal{D}(\mathcal{G})}(X, M_i) \to \text{Hom}_{\mathcal{D}(\mathcal{G})}(X, \prod_{i \in I} M_i)$ is an isomorphism.

By taking successively the canonical truncation triangles $\tau^{\leq -d - 1} M_i \to M_i \to \tau^{> -d - 1} M_i \to$ and...
and that it generates $D \rightarrow R$ in $\mathcal{D}$.

The fact that, under hypotheses a) or b), the set $i_d$ has a fully faithful right adjoint $M \rightarrow KQ$ per) exceptional sequence in the hereditary category $\mathcal{D} = \bigoplus_{\kappa \in \mathbb{Z}} \mathcal{D} \bigoplus_{\kappa \in \mathbb{Z}} \mathcal{D} \bigoplus_{\kappa \in \mathbb{Z}} \mathcal{D}$, so that each $\mathcal{D}$ is a strongly exceptional sequence of not necessarily coherent sheaves.

We now give an example of an infinite (super)exceptional sequences of ‘coherent’ objects and a finite one consisting of ‘noncoherent’ objects.

Example 8.

1. (See also \[50, Example 4.18\]) Consider an acyclic quiver $Q$ admitting an infinite path $i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_n \rightarrow \ldots$. Putting $T_n = 0$ for $n < 0$, and $T_n = KQe_i$ for $n \geq 0$, we get a (super)exceptional sequence in the hereditary category $K Q - \text{Mod}$.

2. Let $\mathbb{X} = \mathbb{P}^n(k)$ be the projective $n$-space over the algebraically closed field $k$. Let $0 \rightarrow O_{\mathbb{X}} \rightarrow E^0 \rightarrow E^1 \rightarrow \ldots \rightarrow E^n \rightarrow 0$ be the minimal injective resolution in $\text{Qcoh}(\mathbb{X})$ of the structural sheaf. If $\mathcal{E} = \{ E_1, \ldots, E_n \}$ is a complete strongly exceptional sequence in $\text{Qcoh}(\mathbb{X})$.

Proof. Example 1 is clear. As for example 2, we claim that $\text{Hom}_{\text{Qcoh}(\mathbb{X})}(E^r(i), E^n(j)) = 0$ whenever $r > s$ and $i, j \in \{ 0, 1, \ldots, n \}$. To see this, we use Serre’s theorem and identify $\text{Qcoh}(\mathbb{X})$ with the ‘category of tails’ $R - \text{Gr}/R - \text{Tor}$, where $R = K[x_0, x_1, \ldots, x_n]$, $R - \text{Gr}$ is the category of graded $R$-modules and $R - \text{Tor}$ consists of the graded $R$-modules whose elements are annihilated by powers of the ideal $R^+ = (x_0, \ldots, x_1)$. Adapting to the graded situation the argument for the ungraded case (see [37, Theorem 18.8]) we get that if $0 \rightarrow R \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots \rightarrow I^n \rightarrow I^{n+1} \rightarrow 0$ is the minimal injective resolution of $R$ in $R - \text{Gr}$, then $I^k$ is the direct sum of the $E_{gr}(R/p)$, where $E_{gr}(?)$ denotes the injective envelope in $R - \text{Gr}$ and $p$ ranges over all graded prime ideals of $R$ with graded height equal to $k$ (with the obvious definition). This means that $I^k$ is torsionfree for $k = 0, 1, \ldots, n$ while $I^{n+1}$ is in $R - \text{Tor}$. Then if $q : R - \text{Gr} \rightarrow R - \text{Gr}/R - \text{Tor} \cong \text{Qcoh}(\mathbb{X})$ is the quotient functor, we get that the minimal injective resolution of $q(R) \cong O_{\mathbb{X}}$ in $R - \text{Gr}/R - \text{Tor} \cong \text{Qcoh}(\mathbb{X})$ is $0 \rightarrow q(I^0) \rightarrow q(I^1) \rightarrow \ldots \rightarrow q(I^n) \rightarrow 0$. Recall that $q$ has a fully faithful right adjoint $j : R - \text{Gr}/R - \text{Tor} \rightarrow R - \text{Gr}$ whose essential image consists of those $Y \in R - \text{Gr}$ such that $\text{Ext}_{R - \text{Gr}}(T, Y) = 0$, for $i = 0, 1$ and all $T \in R - \text{Tor}$. Then the unit map $I^k \rightarrow (j \circ q)(I^k)$ is an isomorphism, for all $k = 0, 1, \ldots, n$, and we have

$$
\text{Hom}_{\text{Qcoh}(\mathbb{X})}(E^r(i), E^n(j)) = \text{Hom}_{R - \text{Gr}/R - \text{Tor}}(q(I^r)(i), q(I^n)(j)) \cong \text{Hom}_{R - \text{Gr}}((j \circ q)(I^r)(i), (j \circ q)(I^n)(j)) \cong \text{Hom}_{R - \text{Gr}}(I^r(i), I^n(j)).
$$

But if $\text{Ext}_{R - \text{Gr}}(T, Y)$ were a nonzero morphism in $R - \text{Gr}$, then $\text{Im}(f)$ would be a nonzero graded submodule of $I^n(j)$ whose graded support consists of graded prime ideals of graded height $\geq r$ and whose associated graded prime ideals are of graded height $s$. This is a contradiction, and our claim is settled. This also implies that $\mathcal{E} = \{ \mathbb{E}^0, \mathbb{E}^1, \ldots, \mathbb{E}^n \}$ is a strongly exceptional sequence due to the injective condition of
all coproducts of the $\hat{E}^k$ in $\text{Qcoh}(X)$. On the other hand, we know that $\mathcal{T} := \{O_X, O_X(1), \ldots, O_X(n)\}$ is a classical tilting set of $\text{Qcoh}(X)$ (see [10] Lemma 2), so that it generates $\mathcal{D}(X)$ as a triangulated category. Since we clearly have that $\text{thick}_{\mathcal{D}(X)}(\mathcal{T}) \subseteq \text{thick}_{\mathcal{D}(X)}(\mathcal{E})$ we conclude that $\mathcal{E}$ generates $\mathcal{D}(X)$ and, hence, $\mathcal{E}$ is complete.

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