THE POP- SWITCH PLANAR ALGEBRA AND THE JONES- WENZL IDEMPOTENTS

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ABSTRACT. The Jones-Wenzl idempotents are elements of the Temperley-Lieb planar algebra that are important, but complicated to write down. We will present a new planar algebra, the pop-switch planar algebra, which contains the Temperley-Lieb planar algebra. It is motivated by Jones’ idea of the graph planar algebra of type $A_n$. In the tensor category of idempotents of the pop-switch planar algebra, the $n$th Jones-Wenzl idempotent is isomorphic to a direct sum of $n + 1$ diagrams consisting of only vertical strands.

1. Introduction

The Temperley-Lieb algebras were first introduced by Temperley and Lieb [LT71] in their work on transfer matrices in statistical mechanics. Vaughn F. R. Jones independently rediscovered Temperley-Lieb algebras in his work on von Neumann algebras [Jon99]. He assembled these algebras together to form the Temperley-Lieb planar algebra, the simplest example of a subfactor planar algebra.

The Jones-Wenzl idempotents, first introduced in [Wen87], are elements of the Temperley-Lieb algebras. One way they arise naturally is in representation theory. The Temperley-Lieb algebras encode the category of representations of $U_q(sl_2)$, and the Jones-Wenzl idempotents represent the irreducible representations. Chapters in books have been devoted to them [KL94]. They have been categorified by [CK10] and [FSS10], and generalized [OY97].

While important, the Jones-Wenzl idempotents are difficult to write down explicitly. The $n$th Jones-Wenzl idempotent is a linear combination of every diagram with $n$ non-intersecting strands. The number of these diagrams is the $n$th Catalan number. To find the coefficient of a given diagram requires a complicated algorithm originally given by Frankel and Khovanov [FK97] and later written down by Morrison [Mor].

In this paper, we define the pop-switch planar algebra, a new planar algebra that contains the Temperley-Lieb planar algebra. Our original motivation was a diagrammatic treatment of the graph planar algebra introduced by Jones [Jon00]. The pop-switch planar algebra captures with simple diagrams the complicated calculations involved in working with objects in the graph planar algebra.

The main theorem of this paper shows that each Jones-Wenzl idempotent is isomorphic to a direct sum of diagrams with only vertical strands. It is to be hoped that this makes them easier to work with, and gives a new approach to some open problems.
2. Background

For convenience, we work over the field \( \mathbb{C} \) and let \( q \) be a nonzero complex number that is not a root of unity. Many of the results hold over other fields, but if \( q \) is a root of unity the proofs fail due to division by zero.

**Definition 2.1.** The \( n \)th quantum number is defined as
\[
[n] = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}
\]
and the quantum binomial is defined as
\[
\binom{n}{k} = \frac{[n][n-1] \cdots [n-k+1]}{[k][k-1] \cdots [1]}
\]
where \( 0 \leq k \leq n \) are natural numbers.

We have the following identities.

**Lemma 2.2.** \( [k+l] = [k][l+1] - [k-1][l] \).

**Proof.** This follows from the definition and a simple computation. \( \Box \)

**Corollary 2.3.** \( \binom{k+l}{l} = [l+1] \binom{k+l-1}{l} - [k-1] \binom{k+l-1}{l-1} \).

**Proof.** After taking a common denominator and cancelling common terms, this reduces to the previous lemma. \( \Box \)

2.1. **Planar algebras.** We won’t define planar algebras in great detail. See Jones’ original paper [Jon99] for a formal definition. See [MPS08] for a helpful introduction.

We will use what are sometimes called vanilla planar algebras. These lack any of the optional extra features or properties that are often included in the definition.

A planar tangle \( T \) consists of:
- a disk \( D \) called the output disk,
- a finite set of disjoint disks \( D_i \) called the input disks in the interior of \( D \),
- a point called a basepoint of \( \partial D \) and of each \( \partial D_i \), and
- a collection of disjoint curves called strands in \( D \).

The strands can be closed curves, or can have endpoints on \( \partial D \) or \( \partial D_i \) or both. Apart from the endpoints, the strands lie in the interior of \( D \) and do not intersect \( D_i \). The basepoints do not coincide with endpoints of strands. Planar tangles are considered up to isotopy in the plane.

It is sometimes possible to insert a planar tangle \( T_1 \) into one of the input disks of another planar tangle \( T_2 \) to obtain a new planar tangle. Specifically, this is possible if the number of endpoints on the output disk of \( T_1 \) is the same as the number of endpoints on the chosen input disk of \( T_2 \). Then we can use an isotopy to make the endpoints match up. This still leaves an ambiguity of how to rotate \( T_1 \). The basepoints remove this ambiguity: we require the basepoint of the output disk of \( T_1 \) to coincide with the basepoint of the chosen input disk of \( T_2 \).

The planar tangles, together with this operation of inserting one planar tangle into an input disk of another, form a rather general type of algebraic gadget called an operad. Briefly, a planar algebra is a representation of the operad of planar tangles.
More concretely, a planar algebra $\mathcal{P}$ is a sequence of vector spaces $\mathcal{P}_i$ for $i \geq 0$. Suppose $T$ is a planar tangle with input disks $D_1, \ldots, D_n$. Let $d_i$ be the number of endpoints on $\partial D_i$ and let $d$ be the number of endpoints on $\partial D$. Suppose $v_i \in \mathcal{P}_{d_i}$ for all $i$. Then there is an action of $T$

$$T(v_1, \ldots, v_n) \in \mathcal{P}_d.$$ 

The action of planar tangles must be multilinear, and it must be compatible with the operad structure in a natural sense.

The definition of a planar algebra may seem complicated. However it formalizes a fairly simple idea, familiar to knot theorists, of tangle-like diagrams that can be glued together in arbitrary planar ways. Perhaps the main novelty is that we allow formal linear combinations of diagrams, which glue together in a multilinear way.

An example might help.

2.2. The Temperley-Lieb planar algebra. The simplest planar algebra is the Temperley-Lieb planar algebra $\mathcal{T}\mathcal{L}$. The vector space $\mathcal{T}\mathcal{L}_i$ is spanned by tangle diagrams that have no input disks and $i$ endpoints on the output disk.

There is one relation. A closed loop in a diagram may be deleted at the expense of multiplying by the scalar $q + q^{-1}$. We call this the bubble-bursting relation.

If $i$ is odd then $T_i$ is zero. A basis for $T_{2n}$ is given by tangle diagrams that have $n$ strands and no closed loops.

In practice, most planar algebras can be thought of as formal linear combinations of diagrams that are similar to Temperley-Lieb diagrams, but with optional extra features, like crossings, orientations, colors, or vertices.

2.3. The category corresponding to a planar algebra. Suppose $\mathcal{P}$ is a planar algebra. We now describe how $\mathcal{P}$ can be thought of as a category. In this context, the input and output disks in the definition of $\mathcal{P}$ should be thought of as rectangles instead of round disks.

The category $\mathcal{C}$ corresponding to $\mathcal{P}$ is as follows.

- The objects are the non-negative integers.
- The morphisms from $i$ to $j$ are the elements of $\mathcal{P}_{i+j}$, thought of as having $i$ endpoints on the bottom of the rectangle and $j$ on the top.
- The composition $f \circ g$ is given by stacking $f$ on top of $g$.

Let $\mathcal{P}_i^j$ denote $\mathcal{P}_{i+j}$ with the elements treated as morphisms from $i$ to $j$. An idempotent is an element $p$ of $\mathcal{P}_n^n$ such that $p^2 = p$.

We can expand the objects in the category by a construction known as the Karoubi envelope. This new category $\mathcal{C}'$ is defined as follows.

- The objects of $\mathcal{C}'$ are the idempotents of $\mathcal{C}$.
- The morphisms from $p$ to $q$ are morphisms in $\mathcal{C}$ of the form $qxp$.

Next, note that $\mathcal{C}$ and $\mathcal{C}'$ are also tensor categories, where $x \otimes y$ is obtained by placing $x$ to the left of $y$.

Finally, we can define a matrix category of $\mathcal{C}'$. The objects are formal direct sums of objects of $\mathcal{C}'$ and the morphisms are formal matrices. Instead of this abstract definition, all we need is the following lemma.

Lemma 2.4. Suppose $p$ and $q_1, \ldots, q_n$ are idempotents such that

$$p = q_1 + \cdots + q_n,$$
and \( q_i q_j = 0 \) whenever \( i \neq j \). Then

\[ p \simeq q_1 \oplus \cdots \oplus q_n. \]

2.4. Jones-Wenzl idempotents. The Jones-Wenzl idempotent \( p_n \) is the unique element of \( \mathcal{T}_n \) such that

- \( p_n \neq 0 \)
- \( p_n^2 = p_n \)
- \( a p_n = 0 \) if \( a \) is any diagram that includes a strand with both endpoints at the bottom of the rectangle.
- \( p_n b = 0 \) if \( b \) is any diagram that includes a strand with both endpoints at the top of the rectangle.

Because of these last two properties, the Jones-Wenzl idempotents are sometimes referred to as “uncappable.” If \( q \) is a root of unity, the Jones-Wenzl idempotents do not exist for all \( n \).

3. The pop-switch planar algebra

3.1. The pop-switch planar algebra.

Definition 3.1. Let the pop-switch planar algebra \( \mathcal{PSA} \) be the planar algebra generated by oriented strands modulo the following relations.

- The pop-switch relations

\[
\begin{align*}
\circ & \Rightarrow \bigcirc, \\
\bigcirc & \Rightarrow \bigcirc
\end{align*}
\]

- The bubble-bursting relation

\[
\bigcirc + \bigcirc = (q + q^{-1})\epsilon,
\]

where \( \epsilon \) denotes the empty diagram.

This contains the Temperley-Lieb planar algebra; a non-oriented strand is the sum of each orientation.

We need some tools to move the diagrams around.

Denote \( n \) parallel strands oriented in the same direction by a single oriented strand labelled \( n \).

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\]

If \( n \) is a negative integer, \( \modulon {\downarrow} = \modulon {\downarrow} \)

Let \( \iota_n \) denote \( n \) vertical strands oriented up. Let \( \beta_n \) denote \( n \) parallel strands that form a bubble oriented counterclockwise. Let \( \alpha_n \) denote a \( \beta_{-n} \) inside a \( \beta_n \).

\[
\iota_n = \modulon {\downarrow} \beta_n = \modulon {\downarrow} \alpha_n = \modulon {\downarrow} \modulon {\downarrow} \modulon {\downarrow} \modulon {\downarrow} \modulon {\downarrow} \modulon {\downarrow} .
\]

Lemma 3.2. Suppose \( x \in \mathcal{PSA}_0 \) and \( y \) is a sequence of \( 2n \) vertical strands such that \( n \) are oriented up and \( n \) are oriented down. Then \( x \otimes y = y \otimes x \).
Proof. Use the pop-switch relation repeatedly to create a gap and pass $x$ through. Then use the pop-switch relation repeatedly to restore the original $2n$ vertical strands.

Lemma 3.3. The multi-pop-switch relations The pop-switch relations hold for multiple strands.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=3cm]{pop-switch.png}
\end{array}
\end{array}
\]

Proof. Without loss of generality, consider the first equality. Induct on $n$. The case $n = 1$ is the pop-switch relations. For the case $n = k + 1$, move the innermost $\beta_k$ across two strands using the previous lemma. Then use the case $n = k$, and finally the case $n = 1$.

Corollary 3.4. $\iota_k \otimes \alpha_n = \iota_k$ and $\iota_{-k} \otimes \alpha_{-n} = \iota_{-k}$ for $k \geq n \geq 0$.

Proof. Consider $\iota_k \otimes \alpha_n$. Use the multi-pop-switch relation by popping the innermost $\beta_n$ of the $\alpha_n$. Then straighten out the $\iota_n$. The other case is similar.

Corollary 3.5.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=3cm]{pop-switch.png}
\end{array}
\end{array}
\]

Proof. Start with the left side of the first equality. Use a multi-pop-switch relation on the $n-1$ strands, as shown below.

By Lemma 3.2 we can move the $\beta_{n+1}$ into the $\alpha_1$ to achieve the result.

The second identity is proved similarly.

Lemma 3.6. $\iota_n = \beta_{-n} \otimes \iota_n \otimes \beta_n$.

Proof. This follows from the multi-pop switch relations.

Now we give some relations involving the Jones-Wenzl idempotents $p_n$. First, we need some notation for them. We will use a rectangle to represent $p_n$. It should always be assumed that $p_n \in P^n$ even if the strands are not drawn.

Notation 3.7. $p_n = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=3cm]{pop-switch.png}
\end{array}
\end{array}$
We can make use of the fact that they are uncappable.

**Lemma 3.8.** \[\begin{tikzpicture}[baseline=-0.5ex]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0.5,0) -- (1,0);
\draw[thick,->] (1,0) -- (1.5,0);
\draw[thick,->] (1.5,0) -- (2,0);
\draw[thick,->] (2,0) -- (2.5,0);
\filldraw[black] (0,0) circle (2pt);
\filldraw[black] (2,0) circle (2pt);
\end{tikzpicture} = (-1)^{n+1} \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0.5,0) -- (1,0);
\draw[thick,->] (1,0) -- (1.5,0);
\draw[thick,->] (1.5,0) -- (2,0);
\draw[thick,->] (2,0) -- (2.5,0);
\filldraw[black] (0,0) circle (2pt);
\filldraw[black] (2,0) circle (2pt);
\end{tikzpicture} \] This relation remains true if all arrows are reversed.

**Proof.** For the case \( n = 0 \), use the fact that an unoriented cap gives zero. For the case \( n = 1 \), use the case \( n = 0 \) and the pop-switch relation.

For the general case, use induction on \( n \). Start by using the case \( n = k \) as follows:

\[ \begin{array}{c}
\begin{tikzpicture}[baseline=-0.5ex]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0.5,0) -- (1,0);
\draw[thick,->] (1,0) -- (1.5,0);
\draw[thick,->] (1.5,0) -- (2,0);
\draw[thick,->] (2,0) -- (2.5,0);
\filldraw[black] (0,0) circle (2pt);
\filldraw[black] (2,0) circle (2pt);
\end{tikzpicture}
\end{array}
= (-1)^{k+1} \begin{array}{c}
\begin{tikzpicture}[baseline=-0.5ex]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0.5,0) -- (1,0);
\draw[thick,->] (1,0) -- (1.5,0);
\draw[thick,->] (1.5,0) -- (2,0);
\draw[thick,->] (2,0) -- (2.5,0);
\filldraw[black] (0,0) circle (2pt);
\filldraw[black] (2,0) circle (2pt);
\end{tikzpicture}
\end{array}
\] Next use the case \( n = 1 \), followed by Lemma 3.2, to achieve the result.

\[ (-1)^{k+2} \begin{array}{c}
\begin{tikzpicture}[baseline=-0.5ex]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0.5,0) -- (1,0);
\draw[thick,->] (1,0) -- (1.5,0);
\draw[thick,->] (1.5,0) -- (2,0);
\draw[thick,->] (2,0) -- (2.5,0);
\filldraw[black] (0,0) circle (2pt);
\filldraw[black] (2,0) circle (2pt);
\end{tikzpicture}
\end{array}
\] □

4. PROOF OF THE MAIN THEOREM

The aim of this section is to prove the following.

**Theorem 4.1.** The \( n \)th Jones-Wenzl idempotent is isomorphic to a direct sum of \( n + 1 \) diagrams:

\[ p_n \simeq \bigoplus_{i=0}^{n} t_{-i} \otimes t_n \otimes t_{-i} \]

\[ = \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{n} \bigoplus_{j=1}^{n} \bigoplus_{j=n}^{n} \bigoplus_{j=n}^{n} \]

**Proof.** Since \( p_n \) is an idempotent, \( p_n = p_n^2 = p_n \text{id}_n p_n \), where \( \text{id}_n \) is \( n \) nonoriented parallel strands, the multiplicative identity in \( \text{T} \text{L}^n_n \). Now write \( \text{id}_n \) as a sum of \( 2^n \) different ways of orienting \( n \) vertical strands. Break this sum into \( n + 1 \) sums depending on how many strands are oriented up.

**Definition 4.2.** Let \( p_{n-k}^k \) denote the sum of \( \binom{n}{k} \) diagrams obtained from \( p_n \text{id}_n p_n \) by orienting \( k \) strands up and \( n - k \) strands down in the \( \text{id}_n \).

Then \( p_n = p_n^0 + p_{n-1}^1 + \cdots + p_0^n \). If \( k_1 \neq k_2 \), then \( p_{n-k_1}^{k_1} p_{n-k_2}^{k_2} = 0 \). Thus, by Lemma 2.4

\[ p_n \simeq p_n^0 \oplus p_{n-1}^1 \oplus \cdots \oplus p_0^n. \]

It remains only to show

\[ p_l^k \simeq t_{-l} \otimes t_{k+l} \otimes t_{-l}. \]

This is done in Lemma 4.9 □

To prove Lemma 4.9, we first define \( X_l^k \), which we will show is equal a scalar times \( p_l^k \) in Lemma 4.8.
Definition 4.3.

$$X^k_l =$$

Lemmas 4.4 and 4.5 are similar and begin the inductive step of the proof of Lemma 4.8.

Lemma 4.4. \( p_{k+l}(X^{k-1}_{l} \otimes \iota_1)p_{k+l} = (-1)^l X^k_l \otimes \beta_{-1}. \)

Proof.

By a pop-switch relation we have the following.

Then by Lemma 3.8 we can move the arc across the \( l - 1 \) strands creating a \( \beta_{l-1} \) on the right. Next we use Lemma 3.6 to replace the arc with \( \beta_{-1} \otimes \iota_1 \otimes \beta_1. \)

Move the innermost \( \beta_1 \) from the \( \beta_l \) to the far right across \( l - 1 \) strands in both directions by Lemma 3.2.
Then remove the two bubbles on the bottom right of the diagram by Lemma 3.6.

Lastly, by Lemma 3.2 move the $\beta_{l-1}$ into the $\beta_1$ and the $\beta_{-1}$ into the $\beta_{-(l-1)}$.

Lemma 4.5. $p_{k+l}(X_{l-1}^k \otimes \iota_{-1})p_{k+l} = (-1)^k X_l^k \otimes \beta_k$.

Proof.

By a pop-switch relation we have the following.

Then by Lemma 3.8 we can move the arc across the $k-1$ strands creating a $\beta_{-(k-1)}$ on the right. Next we use Lemma 3.2 to move the $\beta_1$ across the $l-1$ strands in both directions into the $\beta_{l-1}$.
Replace the arc with $\beta_{-1} \otimes \iota_1 \otimes \beta_1$.

$$= (-1)^{k+1} \cdot \iota_{k+1} \otimes \beta_{k+1}$$

Lastly, by Lemma 3.2 move the $\beta_1$ across the $k - 1$ strands in both directions into the $\beta_{k-1}$. By the same lemma, move the $\beta_{-1}$ into the $\beta_{k-1}$.

$$= (-1)^k \cdot \iota_k \otimes \beta_k$$

Lemma 4.6 is the key to proving Lemma 4.7, which is required to complete the proof of Lemma 4.8. It is worth noting that in Lemma 4.7 the $X_k^k$ merely acts as a catalyst to provide enough strands to use 4.6. All that is necessary is the presence of $\iota_{-1}$ and $\iota_k$ on the left as specified in Lemma 4.6 for the purpose of implementing Corollary 3.4.

Lemma 4.6. For $k \geq n - 1$,

$$\iota_k \otimes \beta_n = [n] \iota_k \otimes \beta_1 - [n - 1] \iota_k$$
and

$$\iota_{-k} \otimes \beta_{-n} = [n] \iota_{-k} \otimes \beta_{-1} - [n - 1] \iota_{-k}$$

Proof. We prove the first identity, since the second is similar. Consider the case $n = 2$ with $k \geq 1$. Use the bubble-bursting relation on the innermost loop of $\beta_2$. Corollary 3.4 then gives the result.

$$\iota_k \otimes \beta_2 = [2] \iota_k \otimes \beta_1 - [2] \iota_k \otimes \alpha_1 = [2] \iota_k \otimes \beta_1 - [2] \iota_k$$

Now assume $k \geq n - 1$. Use the bubble-bursting relation on the innermost loop of $\beta_n$. Corollary 3.5 and induction give

$$\iota_k \otimes \beta_n = [2] \iota_k \otimes \beta_{n-1} - [2] \iota_k \otimes \beta_{n-2}$$

$$= [2] ([n-1] \iota_k \otimes \beta_1 - [n-2] \iota_k) - ([n-2] \iota_k \otimes \beta_1 - [n-3] \iota_k)$$

$$= ([2] [n-1] - [n-2]) \iota_k \otimes \beta_1 - ([2] [n-1] - [n-3]) \iota_k$$

$$= [n] \iota_k \otimes \beta_1 - [n-1] \iota_k$$

$\square$
Lemma 4.7. If \( k + l = n \) then
\[
\left[ \binom{n-1}{l} \right] X^k \otimes \beta_{-l} + \left[ \binom{n-1}{k} \right] X^k \otimes \beta_k = \left[ \binom{n}{k} \right] X^k.
\]

Proof. Note that every term in the equation contains \( X^k \). However, the result will hold so long as there are both a \( \iota_{-l+1} \) and \( \iota_{k-1} \) on the left of each diagram in order to use Lemma 4.6. Thus it suffices to prove
\[
\left[ \binom{n-1}{l} \right] (\lceil l \rceil \beta_{-l} - \lfloor l - 1 \rfloor) + \left[ \binom{n-1}{k} \right] (\lceil k \rceil \beta_{1} - \lfloor k - 1 \rfloor) = \left[ \binom{n}{k} \right].
\]

Use the identity
\[
\left[ \binom{n-1}{l} \right] = \left[ \binom{n-1}{k} \right],
\]
and the bubble bursting relation \( \beta_{-l} + \beta_{1} = [2] \) to eliminate \( \beta_{-l} \) and \( \beta_{1} \) from the left side. Then simplify further using the identity \[2\]\( l \rceil - \lfloor l - 1 \rfloor = \lceil l + 1 \rfloor \). We obtain
\[
\left[ \binom{n-1}{l} \right] \lceil l + 1 \rceil - \left[ \binom{n-1}{k} \right] \lfloor k - 1 \rfloor.
\]

By Corollary 2.3, this is equal to \( \left[ \binom{n}{k} \right] \), as desired. \( \Box \)

Lemma 4.8. \( p^k_l = (-1)^{kl} \left[ \binom{k+l}{l} \right] X^k_l \)

Proof. Induct on \( n = k + l \). Notice \( p^l_1 = \iota_1 = X^l_0 \) and \( p^0_1 = \iota_{-1} = X^0_1 \). Assume \( k > 0 \) and \( l > 0 \). Then
\[
p^k_l = p_{k+l}(p^{k-1}_l \otimes \iota_1)p_{k+l} + p_{k+l}(p^{k-1}_{l-1} \otimes \iota_{-1})p_{k+l}
\]

By Lemma 4.4 and Lemma 4.5
\[
= (-1)^{kl} \left[ \binom{k+l-1}{l} \right] X^k_l \otimes \beta_{-l} + (-1)^{kl} \left[ \binom{k+l-1}{k} \right] X^k_l \otimes \beta_k
\]

By Lemma 4.7
\[
= (-1)^{kl} \left[ \binom{k+l}{k} \right] X^k_l \]

\( \Box \)

Lemma 4.9. \( p^k_l \cong \iota_{-l} \otimes \iota_{l+k} \otimes \iota_{-l} \).

Proof. The explicit isomorphisms are:
\[
f = (-1)^{kl} \left[ \binom{k+l}{k} \right] \quad g = \iota_{-l} \otimes \iota_{l+k} \otimes \iota_{-l}.
\]

Then \( f \circ g = (-1)^{kl} \left[ \binom{k+l}{k} \right] X^k_l = p^k_l \) by Lemma 4.8. Thus \( f \circ g \) is the identity morphism from \( p^k_l \) to \( p^k_l \).

On the other hand, \( g \circ f = \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}, \) the identity morphism from \( \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l} \) to \( \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l} \).
The second equality holds by performing two multi-pop-switch relations: one on the $\beta_{-l}$ at the top with the $l$ strands to the left and the $l$ strands on the bottom left, and the other on the $\beta_l$ and the $l$ strands on the right and top right. Now expand the Jones-Wenzl idempotent. The only non-zero term comes from one of the following Temperley-Lieb diagrams.

Thus the result of $g \circ f$ must be a scalar times $l_{-l} \otimes l_{k+l} \otimes l_{-l}$. Since $f \circ g$ is the identity and $g \circ f$ is a scalar times the identity, that scalar must be 1. $\square$

5. Graph planar algebra and the Temperley-Lieb planar algebra

This section is motivation for the definition of the pop-switch planar algebra. We start with a summary of the definition of the graph planar algebra, first defined in [Jon00].

Throughout this section, fix a simple graph $\Gamma$. For Jones, all planar algebras are shaded, and $\Gamma$ is required to be bipartite. We will ignore this issue.

Let $\mu$ be a function from the vertices of $\Gamma$ to $\mathbb{R}_{>0}$. We will define the graph planar algebra $\mathcal{P}$ corresponding to $(\Gamma, \mu)$.

For each $k > 0$, let $\mathcal{P}_{2k}$ be the vector space of complex valued functions on the set of loops of length $2k$ on $\Gamma$.

Suppose $T$ is a tangle. For each input disk of $T$, let $v_b$ be a corresponding input vector. We must define a corresponding output vector $v$. Thus we must define $v(\gamma)$ for every loop $\gamma$ in $\Gamma$ that has length equal to the number of endpoints on the outer boundary of $T$.

A state $\sigma$ of $T$ is a function from the set of regions of $T$ to the set of vertices of $\Gamma$ such that adjacent regions are sent to adjacent vertices.

Suppose $r$ is a region of $T$. This is a planar surface with boundary that may include some right-angled corners. The Euler measure $e(r)$ is defined in a similar way to the Euler characteristic, using the usual formula $V - E + F$ for a triangulation of $r$. The difference is, every corner must be a vertex and only counts as $\frac{1}{4}$, any other vertex on a boundary only counts as $\frac{1}{2}$, and every edge on a boundary only counts as $\frac{1}{2}$. 

We are finally ready to define the image vector \( v \) of the vectors \( v_b \) under the action of the tangle \( T \).

\[
v(\gamma) = \sum_{\sigma} \left( \prod_r \mu(\sigma(r))^{c(r)} \right) \left( \prod_b v_b(\sigma|\partial b) \right).
\]

The sum is over all states \( \sigma \) that are compatible with \( \gamma \). The first product is over all regions \( r \) of \( T \). The second product is over all input disks \( b \) of \( T \).

The Temperley-Lieb planar algebra is a subfactor planar algebra of type \( A_{\infty} \). It can be found inside the graph planar algebra associated to \( \Gamma = A_{\infty} \), which is the ray with vertices indexed by positive integers. The function \( \mu \) assigns the quantum integer \([n]\) to the \( n \)th vertex. (Note we are still assuming \( q \) is not a root of unity. If \( q \) is a primitive \((n + 1)\)th root of unity then we should use the graph \( A_n \).)

Suppose \( T \) is an oriented tangle. Define a state of \( T \) to be a function from the set of regions of \( T \) to the set of vertices of \( A_\infty \) such that, for any strand of \( T \), if the region to its right is sent to vertex \( n \) then the region to its left is sent to vertex \( n + 1 \). Thus, a state is determined by the vertex associated to a single region. In a sense, the orientation on the strands removes the ambiguity in the state of a Temperley-Lieb diagram.

Now suppose \( T \) and \( T' \) differ by a pop-switch relation. There is an obvious correspondence between states of \( T \) and states off \( T' \). Furthermore, the total Euler measure of the region associated to any given vertex is the same. We therefore have a well-defined embedding of the pop-switch planar algebra in the graph planar algebra of the graph \( A_\infty \).

One can think of the pop-switch planar algebra as a diagrammatic way to keep track of computations inside the graph planar algebra of \( A_\infty \).

References

[CK10] B. Cooper and V. Krushkal. Categorification of the Jones-Wenzl projectors. ArXiv e-prints, May 2010.

[FK97] I. Frenkel and M. Khovanov. Canonical bases in tensor products and graphical calculus for \( u_q(sl_2) \). Duke Math. J., 87(3):409–480, 04 1997.

[FSS10] I. Frenkel, C. Stroppel, and J. Sussan. Categorifying fractional euler characteristics, Jones-Wenzl projector and \( 3j \)-symbols. ArXiv e-prints, July 2010.

[Jon99] V. F. R. Jones. Planar algebras, I. ArXiv Mathematics e-prints, September 1999.

[Jon00] V. F. R. Jones. The planar algebra of a bipartite graph. Knots in Hellas ’98, 24:94–117, 2000.

[KL94] L. Kauffman and S. Lins. Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds. Princeton University Press, 1994.

[LT71] H. N. V. Lieb and E. H. Temperley. Relations between the ‘Percolation’ and ‘Colouring’ Problem and other Graph-Theoretical Problems Associated with Regular Planar Lattices: Some Exact Results for the ‘Percolation’ Problem. Proceedings of the Royal Society of London., pages 251–280, April 1971.

[Mor] S. Morrison. A formula for the Jones-Wenzl projections. Unpublished. Available at https://tqft.net/math/JonesWenzlProjections.pdf.

[MPS08] S. Morrison, E. Peters, and N. Snyder. Skein theory for the \( d_{2n} \) planar algebras. ArXiv e-prints, August 2008.

[OY97] T. Ohtsuki and S. Yamada. Quantum \( su(3) \) invariant of 3-manifolds via linear skein theory. Journal of Knot Theory and Its Ramifications, 06(03):373–404, 1997.

[Wen87] H. Wenzl. On sequences of projections. C. R. Math. Rep. Acad. Sci. Canada, 9(1):5–9, 1987.
