New Forms of Non-Relativistic and Relativistic Hydrodynamic Equations as Derived by the Renormalization-Group Method

Possible Functional Ansatz in the Moment Method Consistent with Chapman-Enskog Theory

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After a brief account of the derivation of the first-order relativistic hydrodynamic equation as a construction of the invariant manifold of relativistic Boltzmann equation, we give a sketch of derivation of the second-order hydrodynamic equation (extended thermodynamics) both in the nonrelativistic and relativistic cases. We show that the resultant equation suggests a novel ansatz for the functional form to be used in Grad moment method, which turns out to give the same expressions for the transport coefficients as those given in the Chapman-Enskog theory as well as the novel expressions for the relaxation times and lengths allowing natural physical interpretation.

§1. Introduction

The relativistic hydrodynamic equation is widely and successfully used in nuclear and astrophysics: The dynamical evolution of the hot and/or dense QCD matter produced in the Relativistic Heavy Ion Collider (RHIC) experiments can be well described by relativistic hydrodynamic simulations.¹,² The suggestion that the created matter may have only a tiny viscosity prompted an interest in the origin of the viscosity in the created matter to be described using the relativistic quantum field theory and also the dissipative hydrodynamic equations. The relativistic dissipative hydrodynamic equation has been also applied to various high-energy astrophysical phenomena,³ e.g., the accelerated expansion of the universe caused by the bulk viscosity of dark matter and/or dark energy.⁴ We must, however, ask if the theory of relativistic hydrodynamics for viscous fluids is fully established? The answer is, unfortunately, no.

The fundamental problems on the relativistic dissipative hydrodynamic equation may be summarized as follows: (a) ambiguities in the definition of the flow velocity,⁵,⁶ (b) unphysical instabilities of the equilibrium state,⁷ and (c) lack of causality.⁸–¹⁰

Here, we note that the non-equilibrium process may evolve through some stages of hierarchical dynamics.¹¹ In the beginning of the time-evolution of an isolated prepared state, the whole dynamical evolution of the system will be governed by Hamiltonian dynamics that is time-reversal invariant. As time goes on, the dynamics is relaxed into the kinetic regime where the time-evolution of the system is well

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described by kinetic equations which describes a coarse-grained slower dynamics: The Boltzmann equation for the one-body distribution function is one of them;\(^\text{11}\) usually the original time-reversal invariance is lost in the description by the kinetic equation through the coarse-graining. As the system is further relaxed, the time evolution will be described in terms of the hydrodynamic quantities, i.e., the flow velocity, particle-number density, and local temperature. In this sense, the hydrodynamics is the far-infrared asymptotic dynamics of the kinetic equation.

In this report, we take the relativistic Boltzmann equation (RBE)\(^\text{8}\) as a typical kinetic equation and explore the basic problems with the relativistic hydrodynamics. For obtaining the proper relativistic hydrodynamic equation, it is a legitimate and natural way to start with RBE which is Lorentz invariant and does not have stability nor causality problems.\(^\text{8}\) For analyzing the problems (a) and (b) first, we derive hydrodynamic equations\(^\text{12,13}\) for a viscous fluid from RBE on the basis of a mechanical reduction theory called the renormalization-group (RG) method\(^\text{14–16}\) and a natural ansatz on the origin of dissipation. We then proceed to the causality problem by deriving the so-called extended thermodynamics:\(^\text{17–19}\) We shall derive the mesoscopic dynamics of the RBE by constructing the invariant/attractive manifold incorporating the first excited as well as the zero modes of the linearized collision operator. We obtain the same expressions for the transport coefficients as given by the Chapman-Enskog method\(^\text{20}\) and also the relaxation times in terms of relaxation functions which have natural physical interpretation. We also show that the resultant equation suggests a novel ansatz for the functional form to be used in the moment method.\(^\text{21}\)

§2. Relativistic Boltzmann equation

The relativistic Boltzmann equation reads\(^\text{8}\)

\[
p^\mu \partial_\mu f_\mathcal{P}(x) = C[f]_\mathcal{P}(x),
\]

(2.1)

where \(f_\mathcal{P}(x)\) denotes the one-particle distribution function defined in the phase space \((x, p)\); \(p^\mu\) denotes the four-momentum of the on-shell particle, i.e., \(p^\mu p_\mu = p^2 = m^2\) and \(p^0 > 0\). The r.h.s. is the collision integral

\[
C[f]_\mathcal{P}(x) = \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1|p_2, p_3) \left( f_{p_2}(x)f_{p_3}(x) - f_\mathcal{P}(x)f_{p_1}(x) \right),
\]

(2.2)

where \(\omega(p, p_1|p_2, p_3)\) denotes the transition probability due to the microscopic two-particle interaction. Owing to the symmetry property \(\omega(p, p_1|p_2, p_3) = \omega(p_2, p_3|p, p_1) = \omega(p_1|p_3, p_2) = \omega(p_3, p_2|p_1, p)\) and the energy-momentum conservation, the collision operator satisfies the following identity for an arbitrary vector \(\varphi_\mathcal{P}(x)\),

\[
\sum_p \frac{1}{p^0} \varphi_p(x) C[f]_\mathcal{P}(x) = \frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \times \omega(p, p_1|p_2, p_3) \left( \varphi_\mathcal{P}(x) + \varphi_{p_1}(x) - \varphi_{p_2}(x) - \varphi_{p_3}(x) \right)
\]
from which one can show that the function $\varphi_p(x) = a(x) + p^\mu b_\mu(x)$ is a collision invariant satisfying the equation $\sum_p p^\mu \varphi_p(x) C[f_p(x)] = 0$, where $a(x)$ and $b_\mu(x)$ being arbitrary functions of $x$. This form is, in fact, the most general form of a collision invariant.$^8$

§3. Reduction to hydrodynamic equation

Before developing our analysis based on the RG method, we briefly summarize ad hoc aspects in the standard methods such as the Chapman-Enskog expansion and Maxwell-Grad moment methods.$^8$

In Chapman-Enskog method,$^{20}$ the zeroth-order solution is given by the local equilibrium distribution function, i.e., the Jüttner function$^{22} f_p^{(0)}(x) = f_p^{eq}(x)$. Then, one imposes a condition of fit or matching condition, which usually consists of drastic assumptions that internal energy and the particle-number density in the nonequilibrium state are the same as those in the local equilibrium state.$^8$ Indeed these constraints are equivalent to a physical assumption that there is no internal energy nor particle-number density of dissipative origin. Such an assumption, however, does not have any solid physical foundation, since the distribution function in the nonequilibrium state should be quite different from that in the local equilibrium state. Such matching conditions are also imposed in the moment method.$^8$

In the RG method that we adopt,$^{15,16}$ one needs no such matching conditions for derivation, and their correct forms are obtained as a property of the derived equation once the frame is specified by the macroscopic-frame vector.$^{12,13}$ We have shown$^{12,13}$ that the conditions of fit in the energy frame are compatible with the underlying Boltzmann equation, but those in the particle frame adopted in the literature are not and thus will never be satisfied in any physical system.

We solve RBE (2.1) in the hydrodynamic regime, and thereby derive the hydrodynamic equations governing the hydrodynamic variables which are introduced to parametrize the distribution function. To make a coarse graining with Lorentz covariance,$^{12,13}$ we introduce a timelike Lorentz vector $a^\mu$, with $a^0 > 0$. Thus, $a^\mu$ specifies the covariant and macroscopic coordinate system where the velocity field of the hydrodynamic flow is defined. We call $a^\mu$ the macroscopic-frame vector. Although $a^\mu$ could depend on the momentum $p$ as well as the space-time coordinate $x$, i.e., $a^\mu = a^\mu_p(x)$, the possible momentum dependence may not be legitimate as a macroscopic frame vector.$^\ast$ Once $a^\mu$ is given, RBE (2.1) is rewritten in terms of the new coordinates $(\tau, \sigma^\mu)$ as

\begin{equation}
\times \left( f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x) \right),
\end{equation}

\(2.3\)

\* As is shown in 12) and 13), the Eckart frame can be realized only when the macroscopic frame vector has a $p$ dependence. This implies that the macroscopic space-time in the Eckart frame is defined for respective particle state with a definite energy-momentum, which may have a difficulty in a physical interpretation.$^{23}$ In fact, it can be shown$^{23}$ that the $p$-independent timelike macroscopic frame vector with the Lorentz covariance uniquely leads to the hydrodynamics in the Landau-Lifshitz frame.
\[
\frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot a_p(\tau, \sigma)} C[f_p(\tau, \sigma)] - \varepsilon \frac{1}{p \cdot a_p(\tau, \sigma)} p \cdot \nabla f_p(\tau, \sigma), \tag{3.1}
\]

where \(\partial/\partial \tau \equiv (1/a_p^2(x)) a_p^\mu(x) \partial_{\mu}\) and \(\nabla^\mu \equiv \Delta_p^{\mu \nu}(x) \partial_{\nu}\), with \(\Delta_p^{\mu \nu}(x) \equiv g^{\mu \nu} - a_p^\mu(x) a_p^\mu(x)/a_p^2(x)\). In Eq. (3.1), the small parameter \(\varepsilon\) represents the space-nonuniformity, which may be identified with the ratio of the average particle distance over the mean free path, i.e., the Knudsen number; \(\varepsilon\) will be set back to unity in the final stage. This seemingly trivial rewrite of the equation expresses the assumption that only the spatial inhomogeneity is the origin of the dissipation. It is noteworthy that the RG method applied to the nonrelativistic Boltzmann equation with the corresponding assumption successfully leads to the Navier-Stokes equation\(^{24}\), the present approach\(^{12,13}\) is simply a covariantization of the nonrelativistic case.

In the RG method,\(^{12,13,15,16}\) we first try to obtain the perturbative solution \(\tilde{f}_p\) to Eq. (3.1) around the arbitrary initial time \(\tau = \tau_0\) with the initial value \(f_p(\tau_0, \sigma); \tilde{f}_p(\tau = \tau_0, \sigma; \tau_0) = f_p(\tau_0, \sigma)\). Here we have made explicit that the solution depends on the initial time \(\tau_0\); we suppose that the initial value is on an exact solution. The initial value as well as the solution is expanded with respect to \(\varepsilon\) as follows: \(\tilde{f}_p(\tau, \sigma; \tau_0) = \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) + \varepsilon \tilde{f}_p^{(1)}(\tau, \sigma; \tau_0) + \cdots\), and an obviously similar form for \(f_p(\tau_0, \sigma)\).

It is easy to see that the zeroth-order equation reads
\[
\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{p \cdot a_p(\sigma; \tau_0)} C[\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)]. \tag{3.2}
\]

Since we are interested in the slow motion that would be realized asymptotically as \(\tau \to \infty\), we should take the following stationary solution or the fixed point, \(\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = 0\), which is realized when \(C[\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)] = 0\), for arbitrary \(\sigma\). Thus, we find that the zero-th order solution \(\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)\) is a local equilibrium distribution function
\[
\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{(2\pi)^3} \exp \left[ \frac{\mu(\sigma; \tau_0) - p^{\mu} u^{\mu}(\sigma; \tau_0)}{T(\sigma; \tau_0)} \right] \equiv f_p^{eq}(\sigma; \tau_0), \tag{3.3}
\]
with \(u^{\mu}(\sigma; \tau_0) u^{\mu}(\sigma; \tau_0) = 1\). Note that the would-be integration constants \(T(\sigma; \tau_0), \mu(\sigma; \tau_0), \) and \(u^{\mu}(\sigma; \tau_0)\) are independent of \(\tau\) but may depend on \(\tau_0\) as well as \(\sigma\).

Given the zero-th order solution, the first-order equation reads
\[
\frac{\partial}{\partial \tau} \tilde{f}_p^{(1)}(\tau) = \sum_q A_{pq} \tilde{f}_q^{(1)}(\tau) + F_p, \tag{3.4}
\]
where \(A_{pq}\) denotes the matrix element of the linearized collision operator \(A\) given by
\[
(A)_{pq} = A_{pq} \equiv \frac{1}{p \cdot a_p} \frac{\partial}{\partial f_q} C[f_p] \bigg|_{f = f_p^{eq}} \tag{3.5}
\]
and
\[
F_p \equiv -\frac{1}{p \cdot a_p} p \cdot \nabla f_p^{eq}. \tag{3.6}
\]
Hydrodynamic Equations from RG Method

It is essential for obtaining a slow motion to clarify the spectral properties of $A$. For this purpose, it is found convenient to convert $A$ to another linear operator,

$$L \equiv (f_{eq})^{-1} A f_{eq},$$

with the diagonal matrix $f_{eq}^{pq} \equiv f_{pq} \delta_{pq}$. To characterize the properties of the linear operator $L$, let us define an inner product between arbitrary nonzero vectors $\varphi$ and $\psi$ by

$$\langle \varphi, \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot a_p) f_{eq}^{pp} \varphi_p \psi_p.$$ (3.8)

Then it can be shown\(^{12),13}\) that the linearized collision operator has a remarkable property: it is semi-negative definite and has five zero modes given by

$$\varphi_{0p}^{\alpha} \equiv \begin{cases} p^\mu & \text{for } \alpha = \mu, \\ 1 \times m & \text{for } \alpha = 4. \end{cases}$$ (3.9)

The functional subspace spanned by the five zero modes is called $P_0$ space and the projection operator to it is denoted by $P_0$; we define $Q_0$ by $Q_0 = f_{eq}^{-1}$. In the following, we also use the modified projection operators defined by $\bar{P}_0 = f_{eq} P_0 (f_{eq})^{-1}$ and $\bar{Q}_0 = f_{eq} Q_0 (f_{eq})^{-1}$.

We proceed to solve the perturbed equations up to the second order. Then summing up the perturbative solutions, we have an approximate solution around $\tau \simeq \tau_0$: $\bar{f}_p(\tau, \sigma; \tau_0) = \bar{f}_p(0)(\tau, \sigma; \tau_0) + \varepsilon \bar{f}_p(1)(\tau, \sigma; \tau_0) + \varepsilon^2 \bar{f}_p(2)(\tau, \sigma; \tau_0) + O(\varepsilon^3)$. It should be noted that this solution contains the secular terms that apparently invalidates the perturbative expansion for $\tau$ away from the initial time $\tau_0$. We can, however, utilize the secular terms to obtain an asymptotic solution valid in a global domain.\(^{15),16}\) Indeed we have a family of curves $\bar{f}_p(\tau, \sigma; \tau_0)$ parameterized with $\tau_0$: They are all on the exact solution $f_p(\sigma; \tau)$ at $\tau = \tau_0$ up to $O(\varepsilon^3)$, although only valid for $\tau$ near $\tau_0$ locally. Then, the envelope curve of the family of curves, which is in contact with each local solution at $\tau = \tau_0$, will give a global solution in our asymptotic situation, which is shown to be the case.\(^{15),16}\) According to the classical theory of envelopes, the envelope that is in contact with any curve in the family at $\tau = \tau_0$ is obtained\(^{15}\) by

$$\frac{d}{d\tau_0} \bar{f}_p(\tau, \sigma; \tau_0) \bigg|_{\tau_0 = \tau} = 0.$$ (3.10)

The derivative w.r.t. $\tau_0$ hits the hydrodynamic variables, and hence we have the evolution equation of them that is identified with the hydrodynamic equation.\(^{12),24}\) We also note that the invariant manifold which corresponds to the hydrodynamics in the functional space of the distribution function is explicitly obtained in the present method.\(^{12),13}\)

Putting back $\varepsilon = 1$, Eq. (3.10) is nicely reduced to the following form,

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[ (p \cdot a_p) \frac{\partial}{\partial \tau} + \varepsilon p \cdot \nabla \right] \left( f_{eq}^{pp} - \left[ A^{-1} \bar{Q}_0 F \right]_{pp} \right) + O(\varepsilon^3) = 0.$$ (3.11)
If one uses the identity \((p \cdot a_p) \frac{\partial}{\partial x} + p \cdot \nabla = p^\mu \partial_\mu\), Eq. (3.11) is found to have the following form,

\[
\partial_\mu T_{1st}^{\mu\nu} = 0, \quad \partial_\mu N_{1st}^{\mu} = 0, \tag{3.12}
\]
with

\[
T_{1st}^{\mu\nu} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\nu \left\{ f_{p}^{eq} - \left[ A^{-1} Q_0 F \right]_p \right\}, \tag{3.13}
\]
\[
N_{1st}^{\mu} \equiv m^{-1} \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^4 \left\{ f_{p}^{eq} - \left[ A^{-1} Q_0 F \right]_p \right\}. \tag{3.14}
\]

3.1. Example: Landau-Lifshitz frame

In this subsection, we present the hydrodynamic equation derived by the RG method.

If we take the most natural choice for the macroscopic vector as \(a^\mu(x) = u^\mu(x)\), the resultant energy-momentum tensor and particle current turn out to be12,13

\[
T_{1st}^{\mu\nu} = e u^\mu u^\nu - (p - \zeta \nabla \cdot u) \Delta^{\mu\nu} + 2 \eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \tag{3.15}
\]
\[
N_{1st}^{\mu} = n u^\mu + \frac{1}{h^2} \nabla^\mu \frac{\mu}{T}, \tag{3.16}
\]
respectively, with \(\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu\), \(\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu\), and \(\Delta^{\mu\nu\rho\sigma} \equiv 1/2 \cdot (\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - 2/3 \cdot \Delta^{\mu\nu} \Delta^{\rho\sigma})\). Here, \(n, p,\) and \(h\) denote the particle-number density, pressure, and reduced enthalpy per particle, respectively. It is clear that these formulae completely agree with those proposed by Landau and Lifshitz.\(^6\) Indeed, the respective dissipative parts \(\delta T^{\mu\nu}\) and \(\delta N^\mu\) in Eqs. (3.15) and (3.16) meet Landau and Lifshitz’s constraints

\[
\delta e \equiv u_\mu \delta T^{\mu\nu} u_\nu = 0, \quad \delta n \equiv u_\mu \delta N^\mu = 0, \quad Q_\mu \equiv \Delta^{\mu\nu} \delta T^{\nu\rho} u_\rho = 0. \tag{3.17}
\]

Thus we have derived the dissipative hydrodynamics in the Landau-Lifshitz (energy) frame in the RG method.

Moreover, since our theory is in the level of statistical mechanics, we have the microscopic expressions for the transport coefficients appearing in the hydrodynamic tensor (3.15) and current (3.16), as follows:

\[
\zeta = -\frac{1}{T} \langle \tilde{\Pi}, L^{-1} \tilde{\Pi} \rangle, \quad \lambda = \frac{1}{3 T^2} \langle \tilde{J}^\mu, L^{-1} \tilde{J}_\mu \rangle, \quad \eta = -\frac{1}{10 T} \langle \tilde{\pi}^{\mu\nu}, L^{-1} \tilde{\pi}_{\mu\nu} \rangle. \tag{3.18}
\]

Here, we have introduced the following microscopic thermal forces \((\tilde{\Pi}_p, \tilde{J}_p^\mu, \tilde{\pi}_p^{\mu\nu}) \equiv (\Pi_p, J_p^\mu, \pi_p^{\mu\nu})/(p \cdot u)\), where

\[
\Pi_p \equiv \left( \frac{4}{3} - \gamma \right) (p \cdot u)^2 + \left( (\gamma - 1) T \hat{h} - \gamma T \right) (p \cdot u) - \frac{1}{3} m^2, \tag{3.19}
\]
\[
J_p^\mu \equiv -((p \cdot u) - T \hat{h}) \Delta^{\mu\nu} p_\nu \quad \text{and} \quad \pi_p^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma} p_\rho p_\sigma. \quad \text{Here, } \gamma \text{ denotes the ratio of the heat capacities.}
Hydrodynamic Equations from RG Method

It is noteworthy that the transport coefficients can be rewritten in the Green-Kubo formula. With the use of the "time-dependent thermal force" defined by \( \tilde{\Pi}_p(s) = \sum_q [e^{sL}]_{pq} \tilde{\Pi}_q \) and so on, the relaxation functions are given by the time-correlators

\[
R_\zeta(s) \equiv \frac{1}{T} \langle \tilde{\Pi}(0), \tilde{\Pi}(s) \rangle, \tag{3.20}
\]

and so on for \( R_\lambda(s) \) and \( R_\eta(s) \) with obvious modifications to \( R_\zeta(s) \). Then the transport coefficients given in Eq. (3.18) are rewritten as follows,

\[
\zeta = \int_0^\infty ds \ R_\zeta(s), \quad \lambda = \int_0^\infty ds \ R_\lambda(s), \quad \eta = \int_0^\infty ds \ R_\eta(s). \tag{3.21}
\]

We remark that the instability problem of the local equilibrium state is analyzed in 13) and 25).

§ 4. Second-order equations and moment method

In the derivation of the first-order hydrodynamic equations like Landau-Lifshits equation, we utilized the zero modes of the linearized collision operator, which form the invariant manifold on which hydrodynamics is defined; the would-be constant zero modes acquire the time-dependence on the manifold by the RG equation. Our formalism can be extended to include excited modes as additional components of the invariant/attractive manifold. The outcome is nothing but the extended thermodynamics or Israel-Stewart type equation, with new microscopic expressions of the relaxation times and lengths. Furthermore, we find the proper ansatz for the distribution function to be used in the moment method. For the shortage of space, we here only present the results, leaving the detailed derivation in a separate paper. We emphasize that our theory gives an explicit construction of the invariant manifold corresponding to thirteen moments, which has been long sought for. 18),19)

4.1. A brief review of Grad’s thirteen-moment method and Grad-Muller equation: non-relativistic case

In Grad’s thirteen-moment method, the one-particle distribution function \( f_v(t, x) \) is expressed in terms of the equilibrium distribution function \( f_v^{eq}(t, x) \) as \( f_v(t, x) = f_v^{eq}(t, x) \left( 1 + \Phi_v(t, x) \right) \), where \( \Phi_v(t, x) \) is assumed to have the form

\[
\Phi_v(t, x) = \Phi_v^G(t, x) = \tilde{\pi}_{ij}^v(t, x) \pi_{ij}^v(t, x) + \tilde{J}_i^v(t, x) J_i^v(t, x). \tag{4.1}
\]

Here, \( \tilde{\pi}_{ij}^v(t, x) \) and \( \tilde{J}_i^v(t, x) \) are defined by \( \tilde{\pi}_{ij}^v(t, x) \equiv m (\delta v^i(t, x) \delta v^j(t, x) - \frac{1}{2} \delta^i_j |\delta v(t, x)|^2) \) and \( \tilde{J}_i^v(t, x) \equiv (m^2 |\delta v(t, x)|^2 - \frac{5}{2} T(t, x)) \delta v^i(t, x) \), with \( \delta v(t, x) \equiv v - u(t, x) \) being the peculiar velocity.

Then the evolution equation of the thirteen coefficients are determined by the equations all of which are derived from the Boltzmann equation with use of the linearized collision operator given by \( L_{vk} \equiv (f_v^{eq})^{-1} \frac{\partial}{\partial f_k} C[f]v|f - f_v^{eq} f_k^{eq} \). Thus the
Grad-Muller equation is obtained as a closed system of the equations governing $T$, $n$, $u^i$, $\pi^{ij}$, and $J^i$, which includes

$$m n \frac{\partial}{\partial t} u^i + m n u \cdot \nabla u^i = -\nabla^j (p \delta^{ij} - 2 \eta^G \pi^{ij}),$$  \hspace{1cm} (4.2)$$

$$\pi^{ij} + \tau^G_\pi \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \pi^{ij} = \bar{X}^{ij} + \text{(other terms)},$$  \hspace{1cm} (4.3)$$

with $p$ being the pressure, where we have defined the thermodynamic force given by $\bar{X}^{ij} \equiv 1/2 \cdot (\nabla^i u^j + \nabla^j u^i - 2/3 \cdot \delta^{ij} \nabla \cdot u)$. In the Grad moment theory, the transport coefficients and relaxation times are given in terms of the inner product defined by $\langle \psi, \chi \rangle_{eq} \equiv \sum_v f_v^{eq} \psi_v \chi_v$. For example,

$$\eta^G = -\frac{1}{10T} \frac{\langle \hat{\pi}^{ij}, \hat{\pi}^{ij} \rangle_{eq} \langle \hat{\pi}^{kl}, \hat{\pi}^{kl} \rangle_{eq}}{\langle \hat{\pi}^{mn}, L \hat{\pi}^{mn} \rangle_{eq}}, \quad \tau^G_\pi = -\frac{\langle \hat{\pi}^{ij}, \hat{\pi}^{ij} \rangle_{eq}}{\langle \hat{\pi}^{kl}, L \hat{\pi}^{kl} \rangle_{eq}}.$$  \hspace{1cm} (4.4)$$

These formulae are different from those given in the Chapman-Enskog expansion method; for instance, $\eta^{CE} = -1/10T \cdot \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq}$, which clearly shows that $\eta^{CE} \neq \eta^G$.

### 4.2. The mesoscopic dynamics from the RG method: nonrelativistic case

We apply the RG method to derive the extended thermodynamics or mesoscopic dynamics\textsuperscript{18),19} from the nonrelativistic Boltzmann equation, which is equivalent to construct the invariant manifold\textsuperscript{16} in the space of the distribution function. It is nice that we can read off the proper ansatz for the deviation function $\Phi_v$ from the distribution function constructed in our method. Leaving the detailed derivation of the formulae to a separate paper,\textsuperscript{26} we here give the results: Our deviation function $\Phi_v$ and the relaxation equations read

$$\Phi_v = \Phi^{TK}_v \equiv \frac{1}{T} \left[ L^{-1} \hat{\pi}^{ij} \right]_v \pi^{ij} + \frac{1}{T} \left[ L^{-1} \hat{J}^i \right]_v J^i,$$  \hspace{1cm} (4.5)$$

and

$$\sum_v \frac{1}{T} \left[ L^{-1} \hat{\pi}^{ij} \right]_v \left[ \frac{\partial}{\partial t} + v \cdot \nabla \right] f_v = \sum_v \sum_k f_v^{eq} \frac{1}{T} \left[ L^{-1} \hat{\pi}^{ij} \right]_v L_{vk} \Phi_k,$$  \hspace{1cm} (4.6)$$

$$\sum_v \frac{1}{T} \left[ L^{-1} \hat{J}^i \right]_v \left[ \frac{\partial}{\partial t} + v \cdot \nabla \right] f_v = \sum_v \sum_k f_v^{eq} \frac{1}{T} \left[ L^{-1} \hat{J}^i \right]_v L_{vk} \Phi_k,$$  \hspace{1cm} (4.7)$$

respectively. The corresponding microscopic expressions of the transport coefficients are given by

$$\eta^{TK} = -\frac{1}{10T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq}, \quad \lambda^{TK} = -\frac{1}{3T^2} \langle \hat{J}^i, L^{-1} \hat{J}^i \rangle_{eq},$$  \hspace{1cm} (4.8)$$

which perfectly agree with those by Chapman-Enskog method, $\eta^{CE}$ and $\lambda^{CE}$. Furthermore, we have the microscopic representation of the relaxation times given by

$$\tau^{TK}_{\pi} = -\frac{\langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle_{eq}}{\langle \hat{\pi}^{kl}, L^{-1} \hat{\pi}^{kl} \rangle_{eq}}, \quad \tau^{TK}_{J} = -\frac{\langle \hat{J}^i, L^{-2} \hat{J}^i \rangle_{eq}}{\langle \hat{J}^k, L^{-1} \hat{J}^k \rangle_{eq}},$$  \hspace{1cm} (4.9)$$
which are new and different from those in the naive Grad moment method.

4.3. The mesoscopic dynamics from the RG method: relativistic case

In the relativistic case, if we express the distribution function by

\[ f_p(x) = f_{p\text{eq}}(x) \times (1 + \Phi_p(x)), \]

our RG method gives the following expression of \( \Phi_p \),

\[ \Phi_p = -\frac{1}{T} \sum_q L^{-1}_{pq} (\tilde{\Pi}_q \Pi + J^\mu_q J_\mu + \tilde{\pi}^{\mu\nu} \pi_{\mu\nu}), \tag{4.10} \]

where \( \tilde{\Pi}_p, J^\mu_p, \) and \( \tilde{\pi}^{\mu\nu}_p \) are the microscopic thermal forces. This form is different from those used by Israel-Stewart\(^ {10} \) and Denicol et al.\(^ {27} \) within the moment method.

The relaxation equations derived in our RG method read

\[ \Pi = -\nabla \cdot u - \tau_\Pi D\Pi + \text{other terms involving relaxation lengths}, \]

\[ J^\mu = \frac{1}{\hbar^2} \nabla u^\mu \mu - \tau_J \Delta^\mu a DJ_a + \text{other terms involving relaxation lengths}, \]

\[ \pi^{\mu\nu} = \Delta^{\mu\rho\sigma} \nabla u^\rho - \tau_\pi \Delta^{\mu\nu\alpha\beta} D\pi_{\alpha\beta} + \left( \kappa^{(0)}_{\pi} \Delta^{\mu\nu\alpha} \Delta^{b\rho\sigma} \Delta_{\alpha\beta\gamma} \nabla u^\gamma + \kappa^{(2)}_{\pi} \Delta^{\mu\nu\alpha} \Delta^{b\rho\sigma} \omega_{\alpha\beta} \right) \pi_{\rho\sigma} \]

+ \text{other terms involving relaxation lengths}, \tag{4.11}

with the volticity \( \omega^{\mu\nu} \equiv \frac{1}{2} (\nabla \mu u^\nu - \nabla \nu u^\mu) \) and \( D \equiv u^\mu \partial_\mu \). The energy-momentum tensor and particle current read \( T^\mu_\text{2nd} = e u^\mu u^\nu - (p + \zeta \Pi) \Delta^{\mu\nu} + 2 \eta \pi^{\mu\nu} \) and \( N^\mu_\text{2nd} = n u^\mu + \lambda J^\mu \), respectively.

The RG method gives microscopic expressions for the resulting times \( \tau_\Pi, \tau_J, \) and \( \tau_\pi \) as follows:

\[ \tau_\Pi = -\frac{\langle \tilde{\Pi}, L^{-2} \tilde{\Pi} \rangle}{\langle \tilde{\Pi}, L^{-1} \tilde{\Pi} \rangle} = \frac{\int_0^\infty ds \, R_\zeta(s)}{\int_0^\infty ds \, R_\zeta(s)}, \tag{4.12} \]

and so on for \( \tau_J \) and \( \tau_\pi \) with obvious modifications. It is noteworthy that the relaxation times \( \tau_\Pi, \tau_J, \) and \( \tau_\pi \) are represented in terms of the relaxation functions \( R_\zeta(s), R_\lambda(s) \) and \( R_\eta(s) \): These formulae have a clear physical meaning of the relaxation times as the correlated times in the relaxation function. We note that it is for the first time that such representation and the interpretation of the relaxation times are given for the extended thermodynamics, as far as we are aware of. We also note that shear viscosity derived in our method does coincide with that in the Chapman-Enskog method as shown before.

§5. Brief summary

We have reported our attempts to derive first-order and second-order relativistic hydrodynamic equations from relativistic Boltzmann equation which has a manifest Lorentz invariance and does not show any pathological behavior such as the instability and acausality seen in existing hydrodynamic equations. We have given the novel extended thermodynamics both in nonrelativistic and relativistic cases through the explicit construction of attractive manifold containing the relaxation process from Boltzmann equation.
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