Rapidity evolution of gluon TMD from low to moderate x

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Outline

• Definition of gluon TMD
• Light-cone expansion at moderate $x$
• Rapidity factorization approach at small $x$
• Generalized evolution at moderate and small $x$
• BK, DGLAP and Sudakov limits
• Gluon TMD factorization function
Transverse momentum dependent (TMD) distribution
Transverse momentum dependent (TMD) distribution

\[
\alpha_s(M_z) = 0.1185 \pm 0.0006
\]

Heavy Quarkonia (NLO)

\(e^+e^-\) jets & shapes (res. NNLO)

DIS jets (NLO)

τ decays (N^3LO)

Lattice QCD (NNLO)

QCD \(\alpha_s(Q)\)

pp \(\rightarrow\) jets (NLO)

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Transverse momentum dependent (TMD) distribution

Collinear distribution:

TMD distribution:

\[ \alpha_s(M_z) = 0.1185 \pm 0.0006 \]

Z pole fit

Heavy Quarkonia (NLO)

\( e^+ e^- \) jets & shapes (res, NNLO)

\( e^+ e^- \) jets (NLO)

\( pp \to \) jets (NLO)

\( \tau \) decays (N³LO)

Lattice QCD (NNLO)

DIS jets (NLO)

\( \tau \) decays (N³LO)

\( e^+ e^- \) jets & shapes (res, NNLO)

\( Z \) pole fit (N³LO)

\( pp \to \) jets (NLO)
Current status

Quark TMD at moderate $x$:

- J.C. Collins, D. E. Soper and G. Sterman, Nucl. Phys. B250, 199 (1985)
- X. Ji, Jian-Ping Ma, and F. Yuan, Phys. Rev. D71, 034005 (2005)
- J. C. Collins, Foundations of Perturbative QCD (CUP, 2011)

Gluon TMD:

- F. Dominguez, A.H. Mueller, S. Munier, and Bo-Wen Xiao, Phys. Lett. B705, 106 (2011)
- P. J. Mulders and J. Rodrigues, Phys. Rev. D63, 094021 (2001)
- A. H. Mueller, Bo-Wen Xiao, and F. Yuan, Phys. Rev. D88, 114010 (2013)
- Y. V. Kovchegov, M.D. Sievert, Phys. Rev. D86, 034028 (2012)
- G. A. Chirilli, Bo-Wen Xiao, and F. Yuan, Phys. Rev. D86, 054005 (2012)
- M. G. Echevarria, T. Kasemets, P. J. Mulders, and C. Pisano, hep-ph/1502.0535
From moderate to small-$x$

I. Balitsky, Nucl. Phys. B463, 99 (1996)
Yu.V. Kovchegov, Phys. Rev. D60, 034008 (1999)

Non-linear effects at small $x$
Two definitions of gluon TMD

Moderate $x$:

$$\alpha_s \mathcal{D}(x, z_\perp) \equiv \frac{2\pi^2 \alpha_s}{N_c} G(x, z_\perp) = -\frac{1}{8\pi^2 (P \cdot n)x} \int du e^{-ixu(P \cdot n)} \langle P | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) \mathcal{F}^{a\xi}(0) | P \rangle$$

Small $x$ (Weiszacker-Williams):

$$\alpha_s \mathcal{D}(x, z_\perp) \equiv \frac{2\pi^2 \alpha_s}{N_c} G(x, z_\perp) = -\frac{1}{8\pi^2 (P \cdot n)x} \int du \langle P | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) \mathcal{F}^{a\xi}(0) | P \rangle$$

$\tilde{\mathcal{F}}^a_\xi (0) \equiv [\infty n, 0]^{am} n^\mu g F^{m}_{\mu \xi}(0)$

$\tilde{\mathcal{F}}^a_\xi (z_\perp + un) \equiv n^\mu g \tilde{F}^m_{\mu \xi}(un + z_\perp)[un + z_\perp, \infty n + z_\perp]^{ma}$
Definition of gluon TMD

\[
\beta_B \mathcal{D}(\beta_B, k_\perp) = -\frac{4}{\alpha_s \langle P | P \rangle} \int d^2 x_\perp d^2 y_\perp e^{i k_\perp (x_\perp - y_\perp)} \langle P | \tilde{F}_i^a (\beta_B, x_\perp) \mathcal{F}^{ai}(\beta_B, y_\perp) | P \rangle
\]

\[
\tilde{F}_i^a (\beta_B, x_\perp) \mathcal{F}_j^a(\beta_B, y_\perp)
\]

- We consider evolution of this operator
- Distribution and fragmentation function
Definitions of gluon TMD

\[ \tilde{F}_i^a (\beta_B, x_\perp) \tilde{F}_j^a (\beta_B, y_\perp) \]

We consider evolution of this operator

\[ \frac{2}{s} \int d\tilde{z}_\ast e^{i\beta_B \tilde{z}_\ast} g \tilde{F}_{\ast i}^m (\tilde{z}_\ast, x_\perp) [\tilde{z}_\ast, \infty)_x^{ma} \]

\[ \tilde{F}_i^a (\beta_B, x_\perp) = \frac{2}{s} \int d\tilde{z}_\ast e^{-i\beta_B \tilde{z}_\ast} g \tilde{F}_{\ast i}^m (\tilde{z}_\ast, x_\perp) [\tilde{z}_\ast, \infty)_x^{ma} \]

\[ \tilde{F}_j^a (\beta_B, y_\perp) = \frac{2}{s} \int d\tilde{z}_\ast e^{i\beta_B \tilde{z}_\ast} [\infty, z_\ast)_y^{am} g \tilde{F}_{\ast j}^m (z_\ast, y_\perp) \]

\[ p = \alpha p_1 + \beta p_2 + p_\perp \]

\[ z_\ast = z_\mu p_2^\mu = \sqrt{\frac{s}{2}} z^- \]

\[ z_\bullet = z_\mu p_1^\mu = \sqrt{\frac{s}{2}} z^+ \]
Evolution of gluon TMD

Real emission

Virtual emission

Note: there are several diagrams of this kind
Evolution at small and moderate $x$

Moderate $x$ (DGLAP dynamics)

- Ordering in transverse momentum
  $p_\perp > l_\perp$
- No ordering in rapidity
  $\alpha \sim \alpha'$

Small $x$

- Ordering in rapidity
  $\alpha > \alpha'$
- No ordering in transverse momentum
  $p_\perp \sim l_\perp$
Evolution at small and moderate $x$

Moderate $x$ (DGLAP dynamics)

Small $x$

rapidity factorization approach

light-cone expansion

We consider emission in both limits and combine results
Light-cone expansion (moderate x)

Moderate x (DGLAP dynamics)

• Parameter of expansion $l_\perp/p_\perp \ll 1$
• In coordinate representation is an expansion in powers of deviation from the light-cone
• Gluon propagator is linear in $F$

\[
\lim_{k^2 \to 0} k^2 \langle A^a_\mu(k) A^b_\nu(y) \rangle = -ie^{i\frac{k^2}{\alpha s}} y^* e^{-ik_\perp y_\perp} G^{ab}_{\mu\nu}(\infty, y^*; k_\perp)|_{y_\perp}
\]

\[
G_{\mu\nu}(\infty, y^*, k_\perp)|_{y_\perp} = g_{\mu\nu}[\infty, y^*]_{y_\perp} + \int_{y_*}^{\infty} dz_* \left( -\frac{4i}{\alpha s^2} k^j (z - y)_* g_{\mu\nu}[\infty, z_*]_{y_\perp} F_{j}[z^*, y^*]_{y_\perp} + \frac{4}{\alpha s^2} (p_{2\nu} \delta_{\mu}^j - p_{2\mu} \delta_{\nu}^j)[\infty, z_*]_{y_\perp} F_{j}[z^*, y^*]_{y_\perp} \right)
\]

The structure at small x (rapidity factorization) is different
Light-cone expansion. Real emission

Lipatov vertex

\[ L_{\mu i}^{ab}(k, y_\perp, \beta_B) = i \lim_{k^2 \to 0} k^2 \langle A_{\mu}^a(k) F_{i}^b(\beta_B, y_\perp) \rangle \]

\[ \langle \tilde{F}_i^a(\beta_B, x_\perp) F_j^a(\beta_B, y_\perp) \rangle = - \int \frac{d\alpha}{2\alpha} d^2 k_\perp \tilde{L}_i^\mu(k, x_\perp, \beta_B) L_{\mu j}(k, y_\perp, \beta_B) \]
Light-cone expansion. Real and virtual emission

\[
\frac{d}{d \ln \sigma} \langle p \mid \tilde{F}_i^a(\beta_B, x_\perp) \mathcal{F}^n_j(\beta_B, y_\perp) \rangle_{|p} = \frac{g^2 N_c}{\pi} \int d^2 k_\perp \left\{ e^{i(k, x-y)} \langle p \mid \tilde{F}_k^a(\beta_B + \frac{k^2}{\sigma s}, x_\perp) \mathcal{F}^a_l(\beta_B + \frac{k^2}{\sigma s}, y_\perp) \rangle_{|p} \right\} \left( \frac{\delta_i^k \delta_j^l}{k^2_{\perp}} - \frac{2 \delta_i^k \delta_j^l}{(\beta_B s + k^2_{\perp})^2} \right) \\\n+ \frac{k^2_{\perp} \delta_i^k \delta_j^l + \delta_j^k k_i k^l + \delta_i^l k_j k^k - \delta_i^k k_j k^l - \delta_j^l k_i k^k - g^{kl} k_i k_j - g_{ij} k^k k^l}{(\beta_B s + k^2_{\perp})^2} + \frac{2 g_{ij} k^k k^l + \delta_i^k k_j k^l + \delta_j^l k_i k^k - \delta_i^l k_j k^k}{(\beta_B s + k^2_{\perp})^3} \\
- \frac{k^4_{\perp} g_{ij} k_i k^l}{(\beta_B s + k^2_{\perp})^4} \theta(1 - \beta_B - \frac{k^2_{\perp}}{\sigma s}) - \frac{\sigma \beta_B s}{k^2_{\perp} (\beta_B s + k^2_{\perp})^3} \langle p \mid \tilde{F}_i^a(\beta_B, x_\perp) \mathcal{F}^a_j(\beta_B, y_\perp) \rangle_{|p} \right\}
\]

Collinear limit \( x_\perp = y_\perp \)

\[
\mu^2 \frac{d}{d \mu^2} \langle \tilde{F}^{ni} (\beta_B, x_\perp) \mathcal{F}^n_i (\beta_B, x_\perp) \rangle = \frac{\alpha_s(\mu)}{\pi} N_c \int_{\beta_B}^{1} dz \left[ \frac{1}{z (1-z)_+} - 2 + z (1-z) \right] \tilde{F}^{ni} (\frac{\beta_B}{z}, x_\perp) \mathcal{F}^n_i (\frac{\beta_B}{z}, x_\perp)
\]

DGLAP equation

\( (k^2_{\perp} < \mu^2) \)
Rapidity factorization approach (small $x$)

Separate slow $\alpha > \sigma'$ and fast fields $\alpha' < \sigma'$.

Fast fields form the shock-wave.

To construct transition to moderate $x$ we keep a finite width of the shock-wave.

Interaction with external field is inside very thin shock-wave.
Rapidity factorization approach. Gluon propagator

Parameter of expansion \( \frac{p_{\perp}^2}{\alpha_s} \sigma_* \ll 1 \)

No ordering in transverse momentum \( p_{\perp} \sim l_{\perp} \)

The gluon propagator has tortuous structure: nonlinear in F terms and quark contribution

\[
\lim_{k^2 \to 0} k^2 \langle A^a_\mu(k)A^b_\nu(y) \rangle = -ie^{\frac{k_\perp^2}{\alpha_s} y_\perp} e^{-ik_\perp y_\perp} \mathcal{O}^{ab}_{\mu\nu}(\infty, y_*; k_\perp)|_{y_\perp}
\]

\[
\mathcal{O}^{ab}_{\mu\nu}(\infty, y_*; k_\perp) = \mathcal{G}^{ab}_{\mu\nu}(\infty, y_*; k_\perp) + \mathcal{J}^{ab}_{\mu\nu}(\infty, y_*; k_\perp) + \bar{\mathcal{J}}^{ab}_{\mu\nu}(\infty, y_*; -k_\perp)
\]

\[
\mathcal{G}^{ab}_{\mu\nu}(\infty, y_*; p_{\perp}) = g^{ab}[\infty, y_*] + \int_{y_*}^{\infty} dz_* \left( -\frac{2i}{\alpha_s^2} (z-y)_{\perp} g^{ab}_{\mu\nu} \{2p_{\perp}[\infty, z_*]F_{*j} - i[\infty, z_*]D^{i}_{\perp} F_{*j} \} + \frac{4}{\alpha_s^2} (p_{2\mu}\delta_{\mu}^{ij} - p_{2\mu}\delta_{\mu}^{ij})[\infty, z_*]F_{*j} \right)[z_*, y_*]
\]

\[
+ \frac{8}{\alpha_s^3} \int_{y_*}^{\infty} dz_* \int_{y_*}^{z_*} dz'_* \{ g^{ab}_{\mu\nu}(z'-y)_{\perp} - \frac{2}{\alpha_s} p_{2\mu} p_{2\nu} F_{*j} F_{*j}^{i} \}[z_*, y_*]
\]
Rapidity factorization approach. Real emission

Lipatov vertex

We take into account position of the shock-wave

\[ \lim_{k^2 \to 0} k^2 (A^a_k(k)F^b_i(\beta_B, y_\perp)) = 2e^{-ik \cdot y} \left( \frac{\alpha p_i}{\alpha s} - \frac{p_1 \mu}{k^2} \right) (\mathcal{F}_i(\beta_B, y_\perp) - i\partial_i U y U^\dagger) \]

\[ + \left( k \cdot g_{\mu i} \left\{ \frac{\alpha \beta s}{\alpha \beta s + p^2} - U \frac{\alpha \beta s}{\alpha \beta s + p^2} \right\} + 2\alpha p_1 \left\{ \frac{\alpha p_i}{\alpha \beta s + p^2} - U \frac{\alpha p_i}{\alpha \beta s + p^2} \right\} - \frac{2ip_1 \mu}{p^2} \partial_i U y U^\dagger \right) \]

\[ + \left\{ \frac{2i}{\alpha s} p_2 \partial_i U - \frac{2i}{\alpha \beta s} \partial_\mu U p_i + \frac{2p_2 \mu}{\beta_B \alpha^2 s^2} \partial_\perp U p_i \right\} \frac{\alpha \beta s}{\alpha \beta s + p^2} U^\dagger y = 0 \]

\[ U \equiv [\infty, -\infty] \]

We apply the same strategy to calculation of the virtual correction.

Takes into account dependence on \( \beta_B \neq 0 \)
One loop correction for the whole region of transverse momentum

One can unify result obtained for \( p_\perp \gg l_\perp \) (light-cone expansion) and \( p_\perp \sim l_\perp \) (rapidity factorization approach)

\[
L^{ab}_{\mu\nu}(k, y_\perp, \beta_B) = i \lim_{k^2 \to 0} k^2 \langle A^a_{\mu}(k) F^b_{\nu}(\beta_B, y_\perp) \rangle =
\]

\[
\frac{2e^{-ik_\perp y_\perp}}{\alpha_B s + k^2_\perp} \left\{ \frac{\alpha_B s g_{\mu\nu}}{\alpha s} \left( k^2_\perp p_{2\mu} - \alpha p_{1\mu} \right) \delta^{ij} - k_i \delta^{ij} + \frac{2\alpha k_i k^j}{\alpha_B s + k^2_\perp} \right\} \left( F^a_{j} - i \partial_j U U^{\dagger ab} \right) +
\]

\[
\left( \frac{\alpha_B s g_{\mu\nu}}{\alpha_B s + k^2_\perp} \right) \left( 2ik^j \partial_j U - \partial^2 U \right) \frac{1}{\alpha_B s + p^2_\perp} U^{\dagger} + 2i\alpha_{1\mu} \partial_i U \frac{1}{\alpha_B s + p^2_\perp} U^{\dagger} - \frac{2ip_{1\mu} \alpha}{p^2_\perp} \partial_i U y U^{\dagger}_{y_\perp}
\]

\[
\left\{ \frac{2i}{\alpha s} p_{2\mu} \partial_i U - \frac{2i}{\alpha_B s} \partial^{\perp}_{\mu} U p_{i} + \frac{2p_{2\mu}}{\alpha_B s} \partial_{\perp}^{2} U p_{i} \right\} \frac{\alpha_B s}{\alpha_B s + p^2_\perp} U^{\dagger} |y_\perp \rangle^{ab}
\]

Virtual emission:

\[
F^n_i(\beta_B, y_\perp) = \frac{2}{s} \int_{-\infty}^{\infty} dy_\ast e^{i\beta_B y_\ast} \langle [\infty, y_\ast]^{nm} g F^m_{\ast i}(y_\ast, y_\perp) \rangle
\]

\[
= -if^{\ast nkl} \int_{\sigma'} d\alpha \left\{ (y_\perp | \frac{1}{p^2_{\perp}} (\alpha_B s i \partial_i U + \partial^2_{\perp} U p_{i}) (\alpha_B s + p^2_{\perp}) \right\} \frac{1}{\alpha_B s + p^2_{\perp}} U^{\dagger} |y_\perp \rangle^{kl}
\]

\[
+ (y_\perp | \frac{\alpha_B s}{p^2_{\perp} (\alpha_B s + p^2_{\perp})} |y_\perp \rangle (F_i(\beta_B, y_\perp) - i \partial_i U y U^{\dagger}_{y_\perp})^{kl} \right\}
\]

- Valid in a full range of \( p_\perp \)
- Takes into account dependence on \( \beta_B \neq 0 \)
Evolution equation of gluon TMD

\[ \frac{d}{d \ln \sigma} \langle p|\tilde{F}_i(x_\perp, \beta_B) F_j(y_\perp, \beta_B)|p\rangle = 2\alpha_s(p_T) Tr \left\{ \left( x_\perp - \frac{1}{2} \tilde{U} p_2^2 \delta_\perp^k + 2p_1^2 p_j^k U^\dagger - \tilde{U} (U \rightarrow 1) \right) \Theta \left[ U p_2^2 g_{jk} + 2p_j p_k U^\dagger - (U \rightarrow 1) \right] \right\} \]

\[ = 2\alpha_s(p_T) Tr \left\{ \left( x_\perp - \frac{1}{2} \tilde{U} p_2^2 \delta_\perp^k + 2p_1^2 p_j^k U^\dagger - \tilde{U} (U \rightarrow 1) \right) \Theta \left[ U p_2^2 g_{jk} + 2p_j p_k U^\dagger - (U \rightarrow 1) \right] \right\} \]

\[ + \tilde{F}_k (\beta_B + \frac{p_2^2}{\sigma s}) \left[ g_{kl} p_i + \delta_\perp^l p_i^k - \delta_\perp^i p_l^k + \frac{p_1^2 \delta_\perp^l p_i^k + 2p_j p_k p_l^k p_i^k}{(\sigma \beta_B + p_1^2)^2} \right] \Theta (\beta_B + \frac{p_2^2}{\sigma s}) \]

\[ + \tilde{F}_k (\beta_B + \frac{p_2^2}{\sigma s}) \left[ \frac{p_2^2 \delta_\perp^i p_l^k + \delta_\perp^l p_i^k - \delta_\perp^i p_l^k - 2g_{ij} p_k p_l^k p_i^k}{(\sigma \beta_B + p_1^2)^2} - \frac{\delta_\perp^i p_l^k p_i^k + p_1^2}{(\sigma \beta_B + p_1^2)^2} \right] \Theta (\beta_B + \frac{p_2^2}{\sigma s}) \]

\[ - \tilde{F}_i (\beta_B + \frac{p_2^2}{\sigma s}) \left[ \frac{p_2^2 \delta_\perp^i + 2p_j p_k U^\dagger - (U \rightarrow 1)}{\sigma \beta_B + p_1^2} \right] \]

\[ - \left[ \tilde{U} p_2^2 \delta_\perp^i + 2p_j p_k U^\dagger - (U \rightarrow 1) \right] \frac{p_2^2}{p_1^2} \Theta (\beta_B + \frac{p_2^2}{\sigma s}) \]

Nonlinear equation

- Valid for the whole range of \( \sigma \) and \( \beta_B \)
- No infrared divergency
- Real emission part contains kinematical constraint \( p_1^2 < \sigma (1 - \beta_B) s \)
DGLAP limit

\[
\int d^2 z_\perp \ e^{i(k,z)_\perp} \langle p | \tilde{F}_{ij}^{an}(z_\perp,\beta_B) F_{ij}^{an}(0_\perp,\beta_B) | p + \xi p_2 \rangle \\
= 2\pi^2 \delta(\xi) \beta_B g^2 \left[ -g_{ij} D(\beta_B, k_\perp, \eta) + \left( \frac{2k_i k_j}{m^2} + g_{ij} \frac{k^2}{m^2} \right) H(\beta_B, k_\perp, \eta) \right]
\]

\[
\frac{d}{d\eta} \alpha_s D(\beta_B, z_\perp, \eta)
= \frac{\alpha_s N_c}{\pi} \int_{\beta_B}^1 \frac{dz'}{z'} \left\{ J_0(|z_\perp| \sqrt{\frac{\sigma_s \beta_B (1 - z')}{z'}}) \left[ \left( \frac{1}{1 - z'} \right) + \frac{1}{z'} - 2 + z'(1 - z') \right] \alpha_s D(\frac{\beta_B}{z'}, z_\perp, \eta) \\
+ \frac{4}{m^2} (1 - z') z'_2 J_2(|z_\perp| \sqrt{\frac{\sigma_s \beta_B (1 - z')}{z'}}) \alpha_s H''(\frac{\beta_B}{z'}, z_\perp, \eta) \right\},
\]

\[
\beta_B = x_B \sim 1 \\
k^2_\perp \sim (x - y)^{-2} \sim s \\
\sigma \lesssim 1
\]

- Evolution equation is linear
- In the collinear case reproduce DGLAP
Small x limit

\[ \beta_B = x_B \sim \frac{(x - y)^{-2}}{s} \]
\[ k_\perp^2 \sim (x - y)^{-2} \ll s \]
\[ \frac{(x - y)^{-2}}{s} \ll \sigma \ll 1 \]

\[ \frac{d}{d \ln \sigma} \langle \tilde{U}_i^a(x_\perp) U_j^a(y_\perp) \rangle = -4\alpha_s Tr \left\{ (x_\perp | \tilde{U} p_i \tilde{U}^\dagger \left( \tilde{U} \frac{p^k}{p_\perp^2} \tilde{U}^\dagger - \frac{p^k}{p_\perp^2} \right) \left( U \frac{p^k}{p_\perp^2} U^\dagger - \frac{p^k}{p_\perp^2} \right) U p_j U^\dagger | y_\perp \right\} \\
- \tilde{U}_i(x_\perp) \left[ (y_\perp | \frac{p^k}{p_\perp^2} U \frac{p_i p_j}{p_\perp^2} U^\dagger | y_\perp) - \frac{1}{2} (y_\perp | \frac{1}{p_\perp^2} | y_\perp) U_j(y_\perp) \right] - \left[ (x_\perp | \tilde{U} \frac{p_i p^k}{p_\perp^2} \tilde{U}^\dagger \frac{p_k}{p_\perp^2} | x_\perp) - \frac{1}{2} (x_\perp | \frac{1}{p_\perp^2} | x_\perp) \tilde{U}_k(x_\perp) \right] U_j(y_\perp) \right\} \]

- Nonlinear equation

\[ \frac{d}{d \ln \sigma} \langle \tilde{U}_i^a(x_\perp) U_j^a(y_\perp) \rangle = -\frac{g^2}{8\pi^3} \int d^2 z_\perp Tr \left\{ (-i \partial_\perp^2 + \tilde{U}_i^x) (\tilde{U}_x \tilde{U}_z^\dagger - 1) \frac{(x - y)^2}{(x - z)_\perp^2 (z - y)_\perp^2} \right\} \left( U_z U_y^\dagger - 1 \right) (i \partial_\perp^2 + U_j^y) \}

I. Balitsky, A.T. (2014)

\[ U_i \equiv i \partial_i U U^\dagger \]
Sudakov limit

\[ \beta_B = x_B \sim 1 \]

\[ k_{\perp}^2 \sim (x - y)_{\perp}^{-2} \sim \text{few GeV}^2 \]

\[ \sigma \lesssim 1 \]

\[ \frac{d}{d \ln \sigma} \tilde{F}_i^a(x_{\perp}, \beta_B) F_j^a(y_{\perp}, \beta_B) = - \frac{g^2 N_c}{\pi} \int \frac{d^2 \mathbf{p}_{\perp}}{\mathbf{p}_{\perp}^2} \left[ 1 - e^{i(p \cdot (x - y))_{\perp}} \right] \tilde{F}_i^a(x_{\perp}, \beta_B) F_j^a(y_{\perp}, \beta_B) \]

Real emission is restricted to the small $p_{\perp}$ region

\[ \mathcal{D}(\beta_B, k_{\perp}, \ln \sigma) \sim \exp \left\{ - \frac{\alpha_s N_c}{2\pi} \ln^2 \frac{\sigma s}{k_{\perp}^2} \right\} \mathcal{D}(\beta_B, k_{\perp}, \ln \frac{k_{\perp}^2}{s}) \]
Fragmentation function

\[ \alpha_s D_f(x_F, z_\perp) = -\frac{z_F}{8\pi^2(p \cdot n)} \int du e^{i \frac{1}{z_F} (p \cdot n) u} \sum_X \langle 0 | \tilde{F}^a_i (z_\perp + un) | X + p \rangle \langle X + p | F^{ai}(0) | 0 \rangle \]

\[ \langle 0 | U | p + X \rangle = 0 \]

For fragmentation function non-linearity is lost

\[ \frac{d}{d\eta} \alpha_s D_f(-\beta_F, z_\perp, \eta) \]

\[ = \frac{\alpha_s N_c}{\pi} \int_{z_F}^{1} dz' \left\{ J_0 \left( |z_\perp| \sqrt{\frac{\sigma_s}{z_F}(1 - z')} \right) \left[ \left( \frac{1}{1 - z'} \right)_+ + \frac{1}{z'} - 2 + z'(1 - z') \right] \alpha_s D_f \left( \frac{z'}{z_F}, z_\perp, \eta \right) \right. \]

\[ + \left. \frac{4}{m^2} \frac{1 - z'}{z'} \frac{1}{z_\perp^2} J_2 \left( |z_\perp| \sqrt{\frac{\sigma_s}{z_F}(1 - z')} \right) \alpha_s H^{ii}_f \left( \frac{z'}{z_F}, z_\perp, \eta \right) \right\}, \]

collinear case

\[ \frac{d}{d\ln \mu^2} \alpha_s(\mu) d_g^f(z_F, \ln \mu^2) = \]

\[ = \frac{\alpha_s(\mu)}{\pi} N_c \int_{z_F}^{1} \frac{dz'}{z'} \left( \left[ \frac{1}{1 - z'} + \frac{1}{z'} + z'(1 - z') - 2 \right] \alpha_s(\mu) d_g^f \left( \frac{z_F}{z'}, \ln \mu^2 \right) \right. \]
Conclusion

• There is a difference between small and moderate x even at the level of definitions
• We’ve use light-cone expansion at moderate x and rapidity factorization at small x
• We’ve constructed evolution equation which is valid in both regions
• The equation reproduces different limits (BK, DGLAP and Sudakov)
• The equation is linear at moderate and non-linear at small x
• However evolution of the gluon fragmentation function is linear