RANDOM SECTIONS OF SPHERICAL CONVEX BODIES

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Abstract. Let $K \subset S^{d-1}$ be a convex spherical body. Denote by $\Delta(K)$ the distance between two random points in $K$ and denote by $\sigma(K)$ the length of a random chord of $K$. We explicitly express the distribution of $\Delta(K)$ via the distribution of $\sigma(K)$. From this we find the density of distribution of $\Delta(K)$ when $K$ is a spherical cap.

1. Introduction

For $d \in \mathbb{N}$ fix some $k \in \{1, \ldots, d-1\}$ and denote by $G_{d,k}$ (respectively, $A_{d,k}$) the set of all linear (respectively, affine) $k$-planes in $\mathbb{R}^d$ equipped with the unique Haar measure invariant with respect to rotations (respectively, rigid motions) normalized by

$$\nu_{d,k}\left(\{L \in G_{d,k}\}\right) = 1,$$

respectively,

$$\mu_{d,k}\left(\{E \in A_{d,k}: E \cap \mathbb{B}^d \neq \emptyset\}\right) = \kappa_{d-k},$$

where is the $k$-dimensional unit ball and $\kappa_k := |\mathbb{B}^k|$. By $|\cdot|$ we denote the volume of the appropriate dimension, by which we understand the Lebesgue measure with respect to the affine hull of a set.

When $k = 1$, we deal with the set of lines. The seminal Crofton formula says that for any convex body (convex compact set with non-empty interior) $K$ we have

$$\int_{A_{d,1}} |K \cap E|^{d+1} \nu_{d,1}(dE) = \frac{d(d+1)}{2d\kappa_d} |K|^2.$$

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It has been obtained by Crofton [4] for \( d = 2 \) and later extended by Hadwiger [5] to all \( d \). What’s less well-known is the following generalization which has been independently obtained in [3, Eq. (21)] and [6, Eq. (34)]: for \( p > -d \),

\[
\int_{A_{d,1}} |K \cap E|^{d+p+1} \mu_{d,1}(dE) = \frac{(d+p)(d+p+1)}{2d\kappa_d} \int_{K^2} |x_0 - x_1|^p \, dx_0 \, dx_1.
\]

In probabilistic language, it establishes a connection between the moments of two random variables:

\[
\mathbb{E} \sigma^{d+p+1} = \frac{(d+p)(d+p+1)}{2\kappa_{d-1}} \cdot \frac{|K|^2}{|\partial K|} \mathbb{E} \Delta^p,
\]

where \( \sigma = \sigma(K) \) denotes the length of the intersection of \( K \) with the random line uniformly distributed among all lines from \( A_{d,1} \) intersected \( K \) and \( \Delta = \Delta(K) \) denotes the distance between two random points independently and uniformly chosen in \( K \). By \( |\partial K| \) we denote the surface area of the boundary of \( K \) (the \((d-1)\)-dimensional Lebesgue measure).

Since a bounded random variable is defined by its moments, we conclude that the distribution of \( \Delta \) is defined by the distribution of \( \sigma \) (which was not obvious a priori). The explicit connection was derived in [1] for \( d = 2 \) and in [7] for any \( d \):

\[
f_\Delta(t) = \frac{t^{d-1}}{|K|} \left( d\kappa_d - \kappa_{d-1} \frac{|\partial K|}{|K|} \int_0^t (1 - F_\sigma(s)) \, ds \right),
\]

where \( f_\Delta \) is the density function of the distribution of \( \Delta \) and \( F_\sigma \) is the distribution function of \( \sigma \). To obtain this result, the authors used the polar coordinates for \( d = 2 \) and the affine Blaschke–Petkantchin formula [8, Theorem 7.2.7] in general case.

The goal of this paper to obtain a spherical analogue of (1). In the next section we first introduce some basic notation and facts form the spherical integral geometry and then formulate our main result.

2. Main result

Now we turn to spherical geometry. Since we are not going to return to the Euclidean case anymore, we will keep some notation for spherical counterparts.

Denote by \( S^{d-1} \) the \((d-1)\)-dimensional unit sphere and let \( \omega_d \) denote its \((d-1)\)-dimensional Lebesgue measure: \( \omega_d := |S^{d-1}| = d\kappa_d \).
Let $K \subset \mathbb{S}^{d-1}$ be a spherical convex body, which means that it can be represented as $K = \mathbb{S}^{d-1} \cap C$, where $C$ is a line-free closed convex cone in $\mathbb{R}^d$.

Denote by $\Delta = \Delta(K)$ the spherical distance between two random points independently and uniformly chosen in $K$. Formally, $\Delta$ is defined as an angle between two lines independently and uniformly distributed among the lines from $G_{d,1}$ which intersect $K$.

Also define $\sigma = \sigma(K)$ to be the spherical length (1-dimensional Lebesgue measure) of the intersection of $K$ with the 2-plane uniformly distributed among 2-planes form $G_{d,2}$ which intersect $K$.

Our main result is the following spherical version of (1).

**Theorem 2.1.** For any spherical body $K \subset \mathbb{S}^{d-1}$, the density function of distribution of $\Delta(K)$ can be expressed in terms of the distribution function of $\sigma(K)$ as follows:

$$f_{\Delta}(t) = \frac{(\sin t)^{d-2}}{|K|} \left( w_{d-1} - \frac{w_d}{2\pi} \kappa_{d-1} \frac{|\partial K|}{|K|} \int_0^t (1 - F_\sigma(s)) \, ds \right).$$

The proof is given in the next section. As an application let us find the density of the distribution between two random points in a spherical cap.

**Corollary 2.2.** Let $K$ be a spherical cap of spherical radius $r < \frac{\pi}{2}$. Then

$$f_{\Delta}(t) = \omega_{d-1} \frac{(\sin t)^{d-2}}{|K|} \left( 1 - \frac{w_d}{2\pi} \kappa_{d-1} \frac{1}{|K|} \int_0^t \left( 1 - \frac{(\cos r)^2}{(\cos \frac{s}{2})^2} \right)^{\frac{d-2}{2}} \, ds \right).$$

**Proof.** It is straightforward to check that the spherical length of $K \cap L$ is less than $s$ if and only if $L \cap K_s = \emptyset$, where $K_s$ is a spherical cap with the same center as $K$ and with spherical radius $\arccos \left( \frac{\cos r}{\cos \frac{s}{2}} \right)$. Thus in view of (4) from below,

$$1 - F_\sigma(s) = \frac{\mu_{d,2}(L \cap K_s \neq \emptyset)}{\mu_{d,2}(L \cap K \neq \emptyset)} = \frac{|\partial K_s|}{|\partial K|},$$

and applying Theorem 2.1 together with

$$|\partial K_s| = \omega_{d-1} \left( \sin \arccos \left( \frac{\cos r}{\cos \frac{s}{2}} \right) \right)^{d-2} = \omega_{d-1} \left( 1 - \frac{(\cos \alpha)^2}{(\cos \frac{s}{2})^2} \right)^{\frac{d-2}{2}}$$

concludes the proof. $\Box$

If $d$ is even, then $f_{\Delta}$ for a spherical cap can be expressed in terms of the elementary trigonometric functions.
Corollary 2.3. If $d = 2m + 2$, then under assumptions of Corollary 2.2 we have

$$f_{\Delta}(t) = \omega_{d-1} \frac{(\sin t)^{d-2}}{|K|} - \omega_{d-1} \frac{K_{d-1}}{|K|^2} \frac{\tan t}{2}$$

$$\times \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{(2k-2)!!}{(2k-1)!!} (\cos r)^{2(m-k)} \left( 1 + \sum_{l=1}^{m-k-1} \frac{(2l-1)!!}{(2l)!!} \frac{1}{(\cos \frac{t}{2})^{2l}} \right).$$

Proof. By the binomial theorem,

$$\int_0^t \left( 1 - \frac{(\cos r)^2}{(\cos \frac{s}{2})^2} \right)^{d-2} ds = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (\cos r)^{2(m-k)} \int_0^t \frac{1}{(\cos \frac{s}{2})^{2(m-k)}} ds.$$

Denote by $I_{2k}$ the indefinite integral of $(\cos t)^{-2k}$. Applying the reduction formula we get

$$(2k-1)I_{2k}(t) = \tan t \frac{1}{(\cos t)^{2k-2}} + (2k-2)I_{2k-2}(t).$$

Using the fact that $I_2(t) = \tan t$, we can derive by induction that

$$I_{2k}(t) = \frac{(2k-2)!!}{(2k-1)!!} \tan t \left( 1 + \sum_{l=1}^{k-1} \frac{(2l-1)!!}{(2l)!!} \frac{1}{(\cos t)^{2l}} \right).$$

Therefore,

$$\int_0^t \left( 1 - \frac{(\cos r)^2}{(\cos \frac{s}{2})^2} \right)^{d-2} ds$$

$$= 2 \tan \frac{t}{2} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (\cos r)^{2(m-k)} \frac{(2k-2)!!}{(2k-1)!!} \left( 1 + \sum_{l=1}^{m-k-1} \frac{(2l-1)!!}{(2l)!!} \frac{1}{(\cos \frac{t}{2})^{2l}} \right).$$

\[ \square \]

3. Proof of Theorem 2.1

The main ingredient of the proof is the following spherical Blaschke–Petkantchin formula: for any non-negative Borel function $f : (S^{d-1})^k \to$
we have
\[\int \left( S^{d-1} \right)^{k} f(x_1, \ldots, x_k) \lambda(dx_1) \ldots \lambda(dx_k) = (k!)^{d-k} b_{d,k} \int G_{d,k} \left( E \cap S^{d-1}\right)^{k} f(x_1, \ldots, x_k) |\text{conv}(0, x_1, \ldots, x_k)|^{d-k} \times \lambda_L(dx_1) \ldots \lambda_E(dx_k) \mu_{d,k}(dL),\]

where \( \lambda, \lambda_E \) are the spherical Lebesgue measures on \( S^{d-1}, S^{d-1} \cap L \) of dimensions \( d-1, k-1 \) respectively, \( |\text{conv}(0, x_1, \ldots, x_k)| \) denotes the Euclidean volume of the convex hull of 0, \( x_1, \ldots, x_k \), and
\[b_{d,k} := \frac{\omega_{d-k+1} \cdots \omega_d}{\omega_1 \cdots \omega_k}.\]

This formula is a special case of a more general result from [2].

Denote by \( F_\Delta \) the distribution function of \( \Delta(K) \). By definition,
\[F_\sigma(t) = \frac{\int G_{d,2} \left[ L \cap K \neq \{0\} \right] 1[\alpha(K \cap L) < t] \mu_{d,2}(dL)}{\mu_{d,2}\left\{ L \in G_{d,2} \mid L \cap K \neq \emptyset \right\}},\]

\[1 - F_\Delta(t) = \frac{1}{|K|^2} \int_{(S^{d-1})^2} 1[x_1, x_2 \in K] 1[\alpha(x_1, x_2) \geq t] \lambda(dx_1) \lambda(dx_2),\]

where by \( \alpha(x_1, x_2) \) and \( \alpha(K \cap L) \) we denote the spherical distance between points \( x_1, x_2 \) and the spherical length of the chord \( K \cap L \).

First let us evaluate \( F_\Delta \). Using (2) leads to
\[\int_{(S^{d-1})^2} 1[x_1, x_2 \in K] 1[\alpha(x_1, x_2) \geq t] dx_1 dx_2 = 2^{d-2} b_{d,2} \int G_{d,2} \left( S^{d-1} \cap L \right)^2 1[x_1, x_2 \in K, \alpha(x_1, x_2) \geq t] |\text{conv}(0, x_1, x_2)|^{d-2} \times \lambda_L(dx_1) \lambda_L(dx_2) \mu_{d,2}(dL) = 2^{d-2} b_{d,2} \int_{\alpha(K \cap L) \geq t} (K \cap L)^2 1[\alpha(x_1, x_2) \geq t] |\text{conv}(0, x_1, x_2)|^{d-2} \times \lambda_L(dx_1) \lambda_L(dx_2) \mu_{d,2}(dL).\]
Using the fact that $|\text{conv}(0, x_1, x_2)| = \frac{1}{2} \sin(\alpha(x_1, x_2))$, we get

$$
\int_{(K \cap L)^2} 1[\alpha(x_1, x_2 \geq t)] |\text{conv}(0, x_1, x_2)|^{d-2} \lambda_L(dx_1) \lambda_L(dx_2)
$$

\[= \int_0^t \int_0^{\alpha(K \cap L) \phi_1 - t} 1[|\phi_1 - \phi_2| \geq t] \left( \frac{1}{2} \sin(|\phi_1 - \phi_2|) \right)^{d-2} d\phi_1 d\phi_2
\]

\[= \frac{1}{2^{d-3}} \int_t^{\alpha(K \cap L) \phi_1} \int_0^{\alpha(K \cap L) \phi_1 - t} \sin(\phi_1 - \phi_2)^{d-2} d\phi_2 d\phi_1 = \frac{1}{2^{d-3}} \int_t^{\alpha(K \cap L) \phi_1} \int_0^{\alpha(K \cap L) \phi_1} \sin(\phi_2)^{d-2} d\phi_2 d\phi_1.
\]

Hence,

$$1 - F_\Delta(t) = \frac{2bd_2}{|K|^2} \int_t^{\alpha(E) \phi_1} \int_0^{\alpha(K \cap L) \phi_1} \sin(\phi_2)^{d-2} d\phi_2 d\phi_1.$$

Let us evaluate the inner double integral. For an integer $n \geq 1$ we have

$$\sin^n t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{e^{it}}{2i}\right)^k \left(\frac{-e^{-it}}{2i}\right)^{n-k} = \frac{1}{(2i)^n} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} e^{it(2k-n)}.$$

From that we derive an indefinite integral for $\sin^{d-2} t$,

$$F(t) := \begin{cases} 
(-1)^m \frac{2m}{2m-1} & \left[ (-1)^m \frac{2m}{m} t + \sum_{k=0}^{m-1} (-1)^k \frac{2m}{k} \frac{1}{2m-2k} \sin((2m-2k)t) \right], \quad d - 2 = 2m, \\
(-1)^{m+1} \frac{2m}{2m-1} & \left[ \sum_{k=0}^{m} (-1)^k \frac{2m}{k} \frac{1}{2m+1-2k} \cos((2m+1-2k)t) \right], \quad d - 2 = 2m + 1,
\end{cases}$$

and then an indefinite integral for $F(t)$,

$$G(t) = \begin{cases} 
(-1)^m \frac{2m}{2m-1} \frac{t^2}{2} & \left[ (-1)^m \frac{2m}{m} t^2 + \sum_{k=0}^{m-1} (-1)^k \frac{2m}{k} \frac{1}{(2m-2k)^2} \cos((2m-2k)t) \right], \quad d - 2 = 2m, \\
(-1)^{m+1} \frac{2m}{2m-1} \frac{1}{(2m+1-2k)^2} & \left[ \sum_{k=0}^{m} (-1)^k \frac{2m}{k} \frac{1}{(2m+1-2k)^2} \sin((2m+1-2k)t) \right], \quad d - 2 = 2m + 1.
\end{cases}$$

It follows that

$$\int_t^{\alpha(K \cap L) \phi_1} \int_t^{\alpha(K \cap L) \phi_1} \sin(\phi_2)^{d-2} d\phi_2 d\phi_1 = \int_t^{\alpha(K \cap L)} F(\phi_1) d\phi_1 - F(t)(\alpha(K \cap L) - t)
$$

$$= G(\alpha(K \cap L)) - F(t)\alpha(K \cap L) - G(t) + tF(t).$$
and finally, we have

\[ 1 - F_\Delta(t) = \frac{2b_{d,2}}{|K|^2} \left[ \int_{\alpha(K \cap L) \geq t} (G(\alpha(K \cap L))) \mu_{d,2}(dL) - F(t) \int_{\alpha(K \cap L) \geq t} \alpha(K \cap L) \mu_{d,2}(dL) \right. \\
+ \left. (tF(t) - G(t)) \int_{\alpha(K \cap L) \geq t} \mu_{d,2}(dL) \right] = : \frac{2b_{d,2}}{|K|^2} \left[ I_1(t) - F(t)I_2(t) + (tF(t) - G(t))I_3(t) \right] \]

(3)

By spherical Crofton’s formula [8, Section 6.5],

\[ \mu_{d,2}\{L \in G_{d,2} \mid L \cap K \neq \emptyset\} = \frac{|\partial K|}{\omega_{d-1}}. \]  

(4)

Therefore it follows from the definition of \( F_\sigma \), that

\[ I_3(t) = \frac{|\partial K|}{\omega_{d-1}} (1 - F_\sigma(t)). \]  

(5)

To calculate \( I_1(t) \) and \( I_2(t) \) we will need the following statement:

**Lemma 3.1.** Let \( R : [0, \pi] \to \mathbb{R} \) be a continuous function. Then

\[ \int_{\alpha(K \cap L) < t} R(\alpha(K \cap L)) \mu_{d,2}(dL) = \frac{|\partial K|}{\omega_{d-1}} \int_0^t R(s) dF_\sigma(s). \]  

(6)

**Proof.** Consider the function \( H(t) = \int_{\alpha(K \cap L) < t} R(\alpha(K \cap L)) \mu_{d,2}(dL) \). We have

\[ \frac{H(t + \Delta t) - H(t)}{\Delta t} = \frac{1}{\Delta t} \int_{t \leq \alpha(K \cap L) < t + \Delta t} R(\alpha(K \cap L)) \mu_{d,2}(dL) \\
= R(\theta) \mu_{d,2}\{L \in G_{d,2} \mid L \cap K \neq \emptyset\} \frac{(F_\sigma(t + \Delta t) - F_\sigma(t))}{\Delta t} \]

for some \( \theta \in [t, t + \Delta t] \).

Letting \( \Delta t \) to 0 and using continuity of \( R \) and almost everywhere differentiability of \( F_\sigma \) along with (4) we obtain

\[ dH(t) = \frac{|\partial K|}{\omega_{d-1}} R(t) dF_\sigma(t), \]
and since $H(0) = 0$,

$$H(t) = \frac{\partial K}{\omega_{d-1}} \int_0^t R(s) dF_\sigma(s).$$

Substituting $x = 0$ in (3) gives

$$\int_{L \cap K \neq \emptyset} (G(\alpha(K \cap L))) \mu_{d,2}(dL) = \frac{|K|^2}{2b_{d,2}} + F(0) \int_{L \cap K \neq \emptyset} \alpha(K \cap L) \mu_{d,2}(dL)$$

$$+ G(0) \frac{|\partial K|}{\omega_{d-1}}.$$  

Again, by spherical Crofton’s formula [8, Section 6.5],

$$\int_{L \cap K \neq \emptyset} \alpha(K \cap L) \mu_{d,2}(dL) = \frac{2\pi}{\omega_d} |K|.$$

Therefore applying (3), (7) and (8) leads to

$$I_1(t) = \int_{L \cap K \neq \emptyset} (G(\alpha(K \cap L))) \mu_{d,2}(dL) - \frac{|\partial K|}{\omega_{d-1}} \int_0^t G(s) dF_\sigma(s)$$

$$= \frac{|K|^2}{2b_{d,2}} + F(0) \frac{2\pi}{\omega_d} |K| + G(0) \frac{|\partial K|}{\omega_{d-1}}$$

$$- \frac{|\partial K|}{\omega_{d-1}} \left[ G(t) F_\sigma(t) - \int_0^t F(s) F_\sigma(s) ds \right]$$

and

$$I_2(t) = \int_{L \cap K \neq \emptyset} \alpha(K \cap L) \mu_{d,2}(dL) - \frac{|\partial K|}{\omega_{d-1}} \int_0^t s dF_\sigma(s)$$

$$= \frac{2\pi}{\omega_d} |K| - \frac{|\partial K|}{\omega_{d-1}} \left( t F_\sigma(t) - \int_0^t F_\sigma(s) ds \right)$$

$$= \frac{2\pi}{\omega_d} |K| - \frac{|\partial K|}{\omega_{d-1}} \left( t F_\sigma(t) - \int_0^t F_\sigma(s) ds \right).$$
Substituting (9), (10) and (5) in (3) we get:

\[ 1 - F_\Delta(t) = \frac{2b_{d,2}}{|K|^2} \left[ |K|^2 + F(0) \frac{2\pi}{\omega_d} |K| + G(0) \frac{|\partial K|}{\omega_{d-1}} \right. \]

\[ + \left. \frac{|\partial K|}{\omega_{d-1}} \int_0^t F(s) F_\sigma(s) \, ds - \frac{|\partial K|}{\omega_{d-1}} \int_0^t F_\sigma(s) \, ds - F(t) \frac{2\pi}{\omega_d} |K| \right] \]

\[ + (tF(t) - G(t)) \frac{|\partial K|}{\omega_{d-1}} \].

Therefore,

\[ F_\Delta(t) = \frac{2b_{d,2}}{|K|^2} \left[ (F(t) - F(0)) \frac{2\pi}{\omega_d} |K| + \frac{|\partial K|}{\omega_{d-1}} \left( G(t) - tF(t) - G(0) \right. \right. \]

\[ \left. \left. + \int_0^t (F(t) - F(s)) F_\sigma(s) \, ds \right) \right]. \]

Differentiating the last equation, we arrive at

\[ f_\Delta(t) = \frac{2b_{d,2}}{|K|^2} \left[ F'(t) \frac{2\pi}{\omega_d} |K| \right. \]

\[ + \left. \frac{|\partial K|}{\omega_{d-1}} \left( G'(t) - F(t) - tF'(t) + F'(t) \int_0^t F_\sigma(s) \, ds \right) \right] \]

\[ = \frac{2b_{d,2}}{|K|^2} \left[ (\sin t)^{d-2} \frac{2\pi}{\omega_d} |K| - \frac{|\partial K|}{\omega_{d-1}} (\sin t)^{d-2} \int_0^t (1 - F_\sigma(s)) \, ds \right]. \]

To conclude the proof it remains to note that

\[ b_{d,2} = \frac{\omega_d \omega_{d-1}}{4\pi}. \]

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