Effects of nonlocal dispersal strategies and heterogeneous environment on total population *

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Abstract

In this paper, we consider the following single species model with nonlocal dispersal strategy

\[ d\mathcal{L}[\theta](x, t) + \theta(x, t)[m(x) - \theta(x, t)] = 0, \]

where \( \mathcal{L} \) denotes the nonlocal diffusion operator, and investigate how the dispersal rate of the species and the distribution of resources affect the total population. First, we show that the upper bound for the ratio between total population and total resource is \( C\sqrt{d} \). Moreover, examples are constructed to indicate that this upper bound is optimal. Secondly, for a type of simplified nonlocal diffusion operator, we prove that if \( \frac{\sup m}{\inf m} < \frac{\sqrt{5}+1}{2} \), the total population as a function of dispersal rate \( d \) admits exactly one local maximum point in \( (\inf m, \sup m) \). These results reveal essential discrepancies between local and nonlocal dispersal strategies.

Keywords total population, nonlocal dispersal, heterogeneity

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1 Introduction

Total population is an important indicator for persistence of species. If the quantity is at low level, the risk of extinction will increase, while if the quantity is at high level, it will lead to shortage of resources and intense pressure of competition, which may jeopardize the existing stability of the multi-species systems [22]. An interesting problem in spatial ecology is how the dispersal rate of the species and the distribution of resources affect the total population. The main purpose of this paper is to investigate this problem for species adopting nonlocal dispersal strategies.

Our study is motivated by a series of intriguing questions and work related to total equilibrium population in a single logistic equation with random diffusion as follows

\[
\begin{aligned}
&u_t = d \Delta u + u[m(x) - u] & x & \in \Omega, & t & > 0, \\
&\frac{\partial u}{\partial \nu} = 0 & x & \in \partial \Omega, & t & > 0,
\end{aligned}
\]  

(1.1)

where \( u \) represents the population density of a species at location \( x \in \Omega \) and at time \( t > 0 \), \( d \) is the dispersal rate of the species which is assumed to be a positive constant, the habitat \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( \nu \) denotes the unit outward normal vector. The function \( m(x) \) is the intrinsic growth rate or carrying capacity, which reflects the environmental influence on the species \( u \). Unless designated otherwise, we assume that \( m(x) \) satisfies the following condition:

\[(M) \quad m(x) \in L^\infty(\Omega), \; m(x) \geq 0 \text{ and } m \not\equiv \text{const on } \bar{\Omega}.
\]

It is known that if \( m \) satisfies the assumption \((M)\), then for every \( d > 0 \), the problem (1.1) admits a unique positive steady state, denoted by \( \theta_{d,m}(x) \), which is globally asymptotically stable (see e.g. [11]). In addition, a remarkable property concerning \( \theta_{d,m}(x) \) was first observed in [21]

\[
\int_\Omega \theta_{d,m}(x) \, dx > \int_\Omega m(x) \, dx \quad \text{for all } d > 0.
\]  

(1.2)

Biologically, this indicates that when coupled with diffusion, a heterogeneous environment can support a total population larger than the total carrying capacity of the environment, which is quite different from homogeneous environment. Simply speaking, heterogeneity of resources can benefit species. This theory is further confirmed experimentally [36].

Now, define

\[
E(m) := \sup_{d>0} \frac{\int_\Omega \theta_{d,m} \, dx}{\int_\Omega m \, dx}.
\]  

(1.3)

The ratio \( E(m) \) is a key quantity in characterizing global dynamics of two-species Lotka-Volterra competition systems [14]. According to the observation (1.2), \( E(m) > 1 \) for any \( m \) satisfying condition \((M)\). The following question was initially proposed by W.-M. Ni:
Question. Is $E(m)$ bounded above independent of $m$? If so, what is the optimal bound?

This biological question leads us to understanding how to maximize the total population under the limited total resources by redistributing the resources, and what would “optimal” distribution be, if it exists. In the one-dimensional case, i.e., when $n = 1$ and $\Omega$ is an open interval, W.-M. Ni conjectured that the supremum of $E(m)$ over all $m$’s satisfying condition (M) is 3. This conjecture is confirmed in [3]. However, for higher dimensional case, i.e., $n \geq 2$, it is proved in [16] that the supremum of $E(m)$ is unbounded. Some further studies related to this question can be found in [24, 25]. This question is also investigated in patchy environment [29].

Another interesting issue is the dependence of the total population on its dispersal rate. According to [21, Theorem 1.1], as either $d \to 0^+$ or $d \to +\infty$, the total population

$$\int_{\Omega} \theta_{d,m}(x) dx$$

always approaches $\int_{\Omega} m(x) dx$. Thus together with the observation (1.2), one sees that total population is maximized at some intermediate diffusion rate. While in [20], some examples are constructed to demonstrate that the total population, as a function of the random diffusion rate, can have at least two local maxima. It is shown in [21] that in the competition models, the invasion of exotic species in spatially heterogeneous habitats is closely related with the total population of the resident species at equilibrium. Hence as a result of the complicated dependence of the total population on its dispersal rate, the invasion of exotic species depends on the dispersal rate of the resident species in complicated manners as well [20].

The above discussion is about single species model with random diffusion, which is the most basic local dispersal strategy. However, in ecology, in many situations (e.g. [8–10, 33]), dispersal is better described as a long range process rather than as a local one, and integral operators appear as a natural choice. A commonly used form that integrates such long range dispersal is the following nonlocal diffusion operator:

$$\mathcal{L}u := \int_{\Omega} k(x, y)u(y)dy - a(x)u(x),$$

where the dispersal kernel $k(x, y) \geq 0$ describes the probability to jump from one location to another and

- either $a(x) = 1$, which corresponds to nonlocal homogeneous Dirichlet boundary condition,
- or $a(x) = \int_{\Omega} k(y, x)dy$, which corresponds to nonlocal homogeneous Neumann boundary condition.

See [2] for details. This nonlocal diffusion operator appears commonly in different types of models in ecology. See [1, 13, 17, 18, 23, 26, 28, 32] and the references therein.
Studying different types of dispersal strategies in heterogeneous environments has been one of the key approaches to understand growth and survival of individual populations and coexistence of species. In this paper, we consider the following single species model with nonlocal dispersal strategy

\[
\begin{align*}
    u_t(x, t) &= dL[u](x, t) + u(x, t)[m(x) - u(x, t)] & x \in \Omega, \ t > 0, \\
    u(x, 0) &= u_0 \geq 0, & x \in \Omega,
\end{align*}
\]  

where the nonlocal operator is defined as (1.4) and explore properties of solutions of the problem (1.5) related to total equilibrium population.

To be more specific, in the problem (1.5), we intend to investigate the same two issues discussed above for the model with random diffusion (1.1):

- properties of the upper bound of \( E(m) \) defined in (1.3), where \( \theta_{d,m} \) denotes the positive steady state to the problem (1.5) if exists;
- dependence of the total population on its dispersal rate.

Indeed, not only the joint effects of spatial heterogeneity and nonlocal dispersal strategies on total equilibrium population will be studied, but also some essential discrepancy between local and nonlocal dispersal strategies will be demonstrated.

It is worth mentioning that, similar to the above local diffusion case, the properties of solutions of single species model also play an important role in determining population dynamics of two-species Lotka-Volterra competition systems. See [4–6] and the references therein.

From now on, assume that the kernel \( k \) satisfies

\( (K) \quad k(x, y) \in C(\mathbb{R}^n \times \mathbb{R}^n) \) is nonnegative and \( k(x, x) > 0 \) in \( \mathbb{R}^n \). \( k(x, y) \) is symmetric, i.e., \( k(x, y) = k(y, x) \). Moreover, \( \int_{\mathbb{R}^n} k(x, y)dy = 1 \).

First of all, we prepare the existence and uniqueness result for the model (1.5) provided that \( m(x) \in L^\infty(\Omega) \).

**Theorem 1.1.** Assume that \( m(x) \in L^\infty(\Omega) \) is nonconstant and the kernel \( k \) satisfies (K). Define

\[
\mu_0 = \mu_0(m) = \sup_{0 \neq \psi \in L^2(\Omega)} \frac{\int_{\Omega} (dL[\psi](x)\psi(x) + m(x)\psi^2(x)) \, dx}{\int_{\Omega} \psi^2(x) \, dx},
\]

Then the problem (1.5) admits a unique positive steady state in \( L^\infty(\Omega) \) if and only if \( \mu_0 > 0 \).

When \( m \in C(\overline{\Omega}) \), the existence and uniqueness of positive steady state for the model (1.5) has been studied thoroughly. See [7] for symmetric operators in the one dimensional
case and \[5,13\] for nonsymmetric operators. The proofs of these studies rely on the properties of nonlocal eigenvalue problems, thus the condition \(m \in C(\overline{\Omega})\) is required. However, to study the questions in this paper, the condition \(m(x) \in L^\infty(\Omega)\) is necessary.

For this purpose, we employ a new approach, which depends on the application of energy functional.

Now consider
\[
\begin{align*}
    u_t(x,t) &= dL_N[u](x,t) + u(x,t)[m(x) - u(x,t)] & x \in \Omega, \ t > 0, \\
    u(x,0) &= u_0 \geq 0 & x \in \Omega,
\end{align*}
\]  

where
\[
L_N \phi := \int_\Omega k(x,y)\phi(y)dy - \int_\Omega k(y,x)\phi(x)dy.
\]

The nonlocal operator \(L_N\) defined above is nonlocal homogeneous Neumann boundary condition. For suitably rescaled kernels, the convergence between problems with nonlocal operator \(L_N\) and those with homogeneous Neumann boundary conditions is verified \[2\].

Thanks to Theorem 1.1 when \(m(x)\) satisfies the condition (M), it is easy to see that the problem (1.7) admits a unique positive steady state, still denoted by \(\theta_{d,m}\), in \(L^\infty(\Omega)\). For further discussions, we first present some basic properties about the total equilibrium population \(\int_\Omega \theta_{d,m}(x)dx\) of the model (1.7).

**Proposition 1.2.** Assume that the assumptions (K) and (M) hold for the model (1.7). Let \(\theta_{d,m}(x)\) denote the unique positive steady state to the problem (1.7). Then the following properties hold.

(i) \(\int_\Omega \theta_{d,m}(x)dx > \int_\Omega m(x)dx\) for all \(d > 0\).

(ii) \(\lim_{d \to 0^+} \theta_{d,m} = m\) in \(L^\infty\). Moreover, if assume that \(\text{ess inf}_{x \in \Omega} m > 0\), then there exist \(d_0 > 0\) such that \(\int_\Omega \theta_{d,m}(x)dx\) is increasing in \(d\) in \([0,d_0]\).

(iii) \(\lim_{d \to \infty} \theta_{d,m} = \frac{1}{|\Omega|} \int_\Omega m(x)dx\) in \(L^\infty(\Omega)\).

Proposition 1.2 shows that, similar to the model with random diffusion (1.1), the property (1.2) still holds for the model with nonlocal dispersal strategy (1.7).

The following theorems completely answer the question proposed by W.-M. Ni for the single species model with nonlocal dispersal strategies. Our first result indicates that the order of the supremum of \(E(m)\) is at most \(O(\sqrt{d})\) when the total resources are given.
Theorem 1.3. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $m(x)$ satisfies the condition (M). Then there exists $C_0 > 0$, which depends on $\int_{\Omega} m(x)dx$ only, such that for $d \geq 1$

$$\sup \{E(m) \mid m \text{ satisfies condition (M)} \} \leq C_0 \sqrt{d},$$

(1.9)

where $E(m)$ is defined in (1.3) and $\theta_{d,m}$ denotes the unique positive steady state to the problem (1.7).

Moreover, examples are constructed as follows to show that the order $O(\sqrt{d})$ is optimal under the prescribed total resources. For $\epsilon > 0$, define

$$m_\epsilon(x) = \begin{cases} 0 & x \in \Omega \setminus \Omega_{0,\epsilon}, \\ \frac{a(x_0)}{\epsilon} & x \in \Omega_{0,\epsilon}, \end{cases}$$

(1.10)

where $\Omega_{0,\epsilon}$ denotes a ball with center $x_0 \in \Omega$ and radius $\sqrt{\epsilon}$ with $\epsilon$ small enough such that $\Omega_{0,\epsilon} \subset \Omega$. Note that the total resources are independent of $\epsilon$, since

$$\int_{\Omega} m_\epsilon(x)dx = \int_{\Omega_{0,\epsilon}} \frac{a(x_0)}{\epsilon}dx = \omega_n a(x_0),$$

where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Theorem 1.4. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ and $m_\epsilon$ is defined in (1.10). Then there exists $C_1 > 0$, independent of $d$ and $m_\epsilon(x)$, such that

$$\int_{\Omega} \theta_{d,m_\epsilon}(x)dx \geq C_1 \sqrt{d},$$

provided that $\lim_{d \to +\infty} \epsilon d = \alpha \in [0,1)$, where $\theta_{d,m_\epsilon}$ denotes the unique positive steady state to the problem (1.7) with $m(x)$ replaced by $m_\epsilon(x)$.

Remark 1.1. According to Theorem 1.4, one sees that the supremum of $E(m)$ defined in (1.3) over all $m$’s satisfying condition (M) for the problem (1.7) is always unbounded. Moreover, the unboundedness of the supremum of $E(m)$ is due to the unboundedness of diffusion rate $d$ and the dimension of domains does not affect the order of the supremum of $E(m)$. This is dramatically different from the corresponding results discussed previously for the local model (1.1).

Now we study the second question, which is about the dependence of the total population on its dispersal rate. Thanks to Proposition 1.2, one sees that the total equilibrium population of the model (1.7) must be maximized at some intermediate diffusion rate for symmetric nonlocal dispersal strategies and positive resources. In particular, for a type
of simplified nonlocal dispersal operator, we provide a sufficient condition, which guarantees that the total population, as a function of the diffusion rate, have exactly one local maximum.

To be more specific, consider the following nonlocal problem

\[
\begin{aligned}
    u_t(x, t) &= d[\bar{u}(t) - u(x, t)] + u(x, t)[m(x) - u(x, t)] & x \in \Omega, \ t > 0, \\
    u(x, 0) &= u_0 \geq 0, & x \in \Omega,
\end{aligned}
\]

(1.11)

where \( \bar{f} \) represents the spatial average of any function \( f \) in \( \Omega \). From the viewpoint of biology, this simplified nonlocal dispersal operator corresponds to the case that the movement distance of the species is much larger than the diameter of the habitat. This has been illustrated in [19]. We briefly recall it here for the convenience of readers. Consider the nonlocal dispersal operator [15]

\[
L u := d \left[ \frac{1}{L} \int_{-\infty}^{\infty} k \left( \frac{x-y}{L} \right) u(y) dy - u(x) \right],
\]

(1.12)

where \( k \) is a non-negative symmetric function satisfying \( \int_{-\infty}^{\infty} k(y) dy = 1 \), and \( k(x-y) \) represents the probability of movement between \( x \) and \( y \) and the dispersal spread length \( L \) characterizes the movement distance. When \( L \) is sufficiently small, by formal expansion, the operator \( L \) can be written as

\[
L u = \frac{1}{2} d L^2 \left( \int_{-\infty}^{\infty} k(z) z^2 \ dz \right) u_{xx} + O(L^3).
\]

That is, the dispersal operator \( L \) for small spread length \( L \) can be approximated by random diffusion operator. However, for large spread length \( L \), as explained in [19], if \( u \) is a periodic function with period \( l > 0 \), then the operator \( L \) can be approximated by operator \( L_1 \), where

\[
L_1 u := d \left[ \frac{1}{l} \int_{0}^{l} u(y) dy - u(x) \right],
\]

which leads to the nonlocal dispersal operator in (1.11). In genetic models, this nonlocal dispersal operator is introduced by T. Nagylaki [30] to represent global panmixia, which is the limiting case of long-distance migration.

Again, due to Theorem 1.1, when \( m(x) \) satisfies the condition (M), the problem (1.11) admits a unique positive steady state in \( L^\infty(\Omega) \), denoted by \( \theta_d(x) \). For simplicity, set

\[
\bar{\theta}(d) = \frac{1}{|\Omega|} \int_{\Omega} \theta_d(x) dx.
\]

We have the following result.
Theorem 1.5. Suppose that $m(x)$ satisfies the condition (M) and
\[ \frac{\sup_{\Omega} m}{\inf_{\Omega} m} < \frac{\sqrt{5} + 1}{2}, \] (1.13)
then there exists $L \in (\inf_{\Omega} m, \sup_{\Omega} m)$ such that $\bar{\theta}(d)$ is non-decreasing in $d$ when $d \leq L$ and non-increasing in $d$ when $d > L$.

In the examples constructed in [20] for the model (1.1) with random diffusion, the authors choose $m(x) = 1 + \epsilon g(x)$ with $\int_{\Omega} g(x) dx = 0$ and $\epsilon$ sufficiently small, and show that the total population has at least two local maxima as diffusion rate varies. However, obviously $m(x) = 1 + \epsilon g(x)$ satisfies (1.13) when $\epsilon$ is sufficiently small. This observation demonstrates that, for certain distribution of resources, the total population, as a function of diffusion rate, could have essentially different behaviors for the local model (1.1) and the nonlocal model (1.11).

This paper is organized as follows. Theorem 1.1 is proved in Section 2. Section 3 is devoted to the proofs of Proposition 1.2, Theorems 1.3 and 1.4. The proof of Theorem 1.5 is included in Section 4. Some miscellaneous remarks are included at the end.

2 Existence and uniqueness of positive steady state

In this section, we establish the existence and uniqueness of positive steady state to the problem of (1.5) when $m(x) \in L^\infty(\Omega)$.

Proof of Theorem 1.1. First, if the problem (1.5) admits a positive steady state, denoted by $\theta$, in $L^\infty(\Omega)$, then it is easy to see that $\mu_0 > 0$ by choosing $\psi = \theta$.

The rest of the proof is devoted to proving the other direction. Assume $M := \|m\|_{L^\infty}$ and let $u$ be the solution of
\[
\begin{cases}
u_t = d\mathcal{L}[u](x,t) + u(x,t)[m(x) - u(x,t)] \quad &x \in \Omega, \ t > 0, \\
u(x,0) = M \quad &x \in \Omega.
\end{cases}
\] (2.1)

Thus, $u$ is decreasing in $t$ and there exists $\theta^* \in L^\infty(\Omega)$ such that $u(x,t) \to \theta^*(x)$ pointwisely as $t \to \infty$. Moreover, $\theta^*$ is a steady state of (2.1).

Now we show that $\theta^* \neq 0$. Suppose that it is not true, that is $u(x,t) \to 0$ pointwisely as $t \to \infty$. Since $\mu_0 > 0$, by the definition of $\mu_0$ we can choose $0 \neq \psi_0 \in L^2$ such that
\[
\int_{\Omega} (d\mathcal{L}[\psi_0](x)\psi_0(x) + m(x)\psi_0^2(x)) \, dx \geq \frac{\mu_0}{2} \int_{\Omega} \psi_0^2 \, dx > 0.
\] (2.2)

Let $\psi_i := \min\{\psi_0, i\}$, obviously $\psi_i \to \psi_0$ in $L^2(\Omega)$ as $i \to \infty$. Combined with (2.2), we can fix $i = i_0$ large enough, such that
\[
\int_{\Omega} (d\mathcal{L}[\psi_{i_0}](x)\psi_{i_0}(x) + m(x)\psi_{i_0}^2(x)) \, dx \geq \frac{\mu_0}{4} \int_{\Omega} \psi_{i_0}^2 \, dx > 0.
\]
Set \( \phi := \varepsilon_{i_0} \psi_i \), with \( \varepsilon_{i_0} = \frac{1}{i_0} \min \{ M, \frac{\mu_0}{8} \} \). It is routine to verify that
\[
\int_\Omega (d\mathcal{L}[\phi](x)\phi(x) + m(x)\phi^2(x)) \, dx - \frac{2}{3} \int_\Omega \phi^3 \, dx \geq \left[ \frac{\mu_0}{4} - \varepsilon_{i_0} i_0 \right] \int_\Omega \phi^2 \, dx > 0. \tag{2.3}
\]
Suppose that \( v \) is the solution of
\[
\begin{aligned}
  v_t &= d\mathcal{L}[v](x, t) + (m - v)v \quad x \in \Omega, \ t > 0, \\
  v(x, 0) &= \phi \quad x \in \Omega,
\end{aligned}
\]
and define
\[
E[v](t) := \frac{1}{2} \int_\Omega (d\mathcal{L}[v]v + mv^2) \, dx - \frac{1}{3} \int_\Omega v^3 \, dx.
\]
By comparison principle \( \phi \leq M \) implies that \( v \leq u \). Thus, \( v \to 0 \) in pointwisely as \( t \to \infty \), and furthermore
\[
E[v](t) \to 0 \text{ as } t \to \infty. \tag{2.4}
\]
However, since \( k(x, y) \) is symmetric, straightforward computation yields that
\[
\frac{d}{dt} E[v](t) = \int_\Omega v_t^2 \, dx \geq 0.
\]
Together with (2.3), one sees that \( E[v](t) \) is a increasing function with positive initial data, which contradicts to (2.4).

Hence \( \theta^*(x) \geq 0 \) is a nontrivial steady state of (1.5). Furthermore, denote \( A := \{ x \in \Omega \mid \theta^*(x) = 0 \} \). Due to the assumption (K), a contradiction can be derived easily by integrating both sides of the equation satisfied by \( \theta^* \) in \( A \) if \( A \) has positive measure. This yields that \( \theta^* > 0 \) a.e. in \( \Omega \).

It remains to show the uniqueness of positive steady state to the problem of (1.5) in \( L^\infty(\Omega) \). Suppose that \( \theta \in L^\infty(\Omega) \) is a positive steady state of (1.5), i.e. \( \theta \) satisfies
\[
d\mathcal{L}[\theta](x) + \theta(x)[m(x) - \theta(x)] = 0,
\]
By multiplying both sides by \( \theta^{p-1} \) and integrating over \( \Omega \), we have
\[
\begin{aligned}
  \int_\Omega \theta^{p+1} \, dx &= \int_\Omega m(x)\theta^p \, dx \\
  &= d \int_\Omega \theta^{p-1} \left( \int_\Omega k(x, y)\theta(y) \, dy - a(x)\theta(x) \right) \, dx \\
  &\leq d \int_\Omega \theta^{p-1} \left( \int_\Omega k(x, y) \, dy \right)^{\frac{p-1}{p}} \left( \int_\Omega k(x, y)\theta^p(y) \, dy \right)^{\frac{1}{p}} \, dx - d \int_\Omega a(x)\theta^p \, dx \\
  &\leq d \left( \int_\Omega \int_\Omega k(x, y) \, dy \, dx \right)^{\frac{p-1}{p}} \left( \int_\Omega \int_\Omega k(x, y)\theta^p(y) \, dy \, dx \right)^{\frac{1}{p}} - d \int_\Omega a(x)\theta^p \, dx \\
  &= d \int_\Omega \int_\Omega k(x, y) \, dy \theta^p \, dx - d \int_\Omega a(x)\theta^p \, dx \leq 0,
\end{aligned}
\]
since \( k(x, y) \) satisfies the assumption (K) and either \( a(x) = 1 \) or \( a(x) = \int_{\Omega} k(y, x) dy \). Thus it is easy to see that
\[
\| \theta \|_{L^{p+1}} \leq \| m \|_{L^{p+1}},
\]
which yields that
\[
\| \theta \|_{L^\infty} \leq \| m \|_{L^\infty} = M,
\]
since \( p \) is arbitrary. Then thanks to (2.1), it follows that \( \theta(x) \leq \theta^*(x) \). Straightforward computation gives
\[
\int_{\Omega} (\theta^* - \theta) \theta^* dx = \int_{\Omega} (m - \theta) \theta^* dx - \int_{\Omega} (m - \theta^*) \theta^* \theta dx = -d \int_{\Omega} L[\theta] \theta^* dx + d \int_{\Omega} L[\theta^*] \theta dx = 0,
\]
which implies that \( \theta \equiv \theta^* \). The proof is complete.

3 Ratio between total population and resources

This section is devoted to the proofs of Proposition 1.2, Theorems 1.3 and 1.4. Thanks to Theorem 1.1, when \( m(x) \) satisfies the condition (M), the problem (1.7) always admits a unique positive steady state, denoted by \( \theta_{d,m} \), i.e., \( \theta_{d,m} \) satisfies
\[
d \left( \int_{\Omega} k(x, y) \theta(y) dy - a(x) \theta(x) \right) + \theta(x) [m(x) - \theta(x)] = 0 \quad x \in \Omega,
\]
where
\[
a(x) = \int_{\Omega} k(y, x) dy \leq 1.
\]

Proof of Proposition 1.2. It is routine to show that since the nonlocal operator is symmetric and \( \theta_{d,m} \) is nonconstant,
\[
\int_{\Omega} \theta_{d,m}(x) dx - \int_{\Omega} m(x) dx = \frac{d}{2} \int_{\Omega} \int_{\Omega} k(x, y) \frac{(\theta_{d,m}(x) - \theta_{d,m}(y))^2}{\theta_{d,m}(x) \theta_{d,m}(y)} dy dx > 0, \quad (3.2)
\]
i.e., (i) is proved.

According to (3.1), it is routine to show that the solution \( \theta_{d,m}(x) \) can be expressed as follows
\[
\theta_{d,m}(x) = \frac{1}{2} \left[ m(x) - da(x) + \sqrt{(m(x) - da(x))^2 + 4d \int_{\Omega} k(x, y) \theta_{d,m}(y) dy} \right],
\]
and \( \|\theta_{d,m}\|_{L^\infty} \leq \|m\|_{L^\infty} \). Thus it is easy to see that
\[
\lim_{d \to 0^+} \theta_{d,m}(x) = m(x) \quad \text{in } L^\infty(\Omega). \tag{3.3}
\]
Moreover, define
\[
T_m(d) = \int_\Omega \theta_{d,m}(x) dx.
\]
It follows from (3.2) and (3.3) that
\[
T'_m(0) = \frac{1}{2} \int_\Omega \int_\Omega k(x,y) (m(x) - m(y))^2 \frac{m(x)m(y)}{a(x)} dy dx,
\]
and \( T'_m(0) > 0 \) since \( m(x) \) is nonconstant. (ii) is verified.

At the end, consider the case that \( d \to +\infty \). Due to (3.1), it is easy to see that
\[
\lim_{d \to +\infty} \theta_{d,m}(x) - \frac{1}{\Omega} \int_\Omega k(x,y) \theta_{d,n}(y) dy a(x) = 0 \quad \text{in } L^\infty(\Omega). \tag{3.4}
\]
Since \( \frac{1}{\Omega} \int_\Omega k(x,y) \theta_{d,m}(y) dy a(x) \) is uniformly bounded and equi-continuous for \( d \geq 1 \), by Arzela-Ascoli Theorem, there exist \( \ell(x) \in C(\overline{\Omega}) \) a sequence \( \{d_j\}_{j \geq 1} \) with \( d_j \to +\infty \) as \( j \to \infty \) such that
\[
\lim_{j \to +\infty} \frac{1}{\Omega} \int_\Omega k(x,y) \theta_{d_j,m}(y) dy a(x) = \ell(x) \quad \text{in } L^\infty(\Omega),
\]
which, together with (3.4), implies that
\[
\lim_{j \to +\infty} \theta_{d,j,m}(x) = \ell(x) \quad \text{in } L^\infty(\Omega), \tag{3.5}
\]
and
\[
\ell(x) - \frac{1}{\Omega} \int_\Omega k(x,y) \ell(y) dy a(x) = 0. \tag{3.6}
\]
Suppose that \( \ell \in C(\overline{\Omega}) \) is nonconstant. Denote \( B := \{x \in \overline{\Omega} \mid \ell = \max_{\overline{\Omega}} \ell\} \). Since \( B \neq \overline{\Omega} \), a contradiction can be derived at \( \partial B \cap \Omega \). Hence, the problem (3.6) only has constant solutions.

Also notice that by the problem (3.1), one has
\[
\int_\Omega \theta_{d,m}(x)[m(x) - \theta_{d,m}(x)] dx = 0.
\]
Thus it follows from (3.5) that
\[
\ell = \frac{1}{|\Omega|} \int_\Omega m(x) dx,
\]
which is a fixed number. Hence the subsequence convergence in (3.5) can be improved to
\[
\lim_{d \to +\infty} \theta_{d,m}(x) = \ell \text{ in } L^\infty(\Omega).
\]

The proof of (iii) is complete. \(\square\)

Now we study the supremum of \(E(m)\). First, we show that the order of the supremum of \(E(m)\) is at most \(O(\sqrt{d})\) when the total resources are given.

**Proof of Theorem 1.3**

Set
\[
\Omega_1 = \{x \in \Omega \mid \theta_{d,m}(x) > K_1 d\}, \quad \text{where } K_1 = 2\|k\|_{L^\infty} |\Omega|,
\]
\[
\Omega_2 = \{x \in \Omega \mid \theta_{d,m}(x) > K_2 d\}, \quad \text{where } K_2 = \frac{4 \left( \int_\Omega m(x)dx + K_1 |\Omega| \right) \|k\|_{L^\infty}}{\min_\Omega a(x)} + 2\|k\|_{L^\infty} |\Omega|.
\]

First, we establish a rough estimate for \(\int_\Omega \theta_{d,m}(x)dx\) as follows. For any \(x \in \Omega_1\),
\[
\theta_{d,m}(x) = m(x) + d \left( \int_\Omega k(x,y) \theta_{d,m}(y)dy - a(x) \theta_{d,m}(x) \right) / \theta_{d,m}(x)
\]
\[
\leq m(x) + d \|k\|_{L^\infty} \int_\Omega \theta_{d,m}(x)dx / K_1 d
\]
\[
= m(x) + \frac{1}{2|\Omega|} \int_\Omega \theta_{d,m}(x)dx,
\]
which implies that
\[
\int_{\Omega_1} \theta_{d,m}(x)dx \leq \int_\Omega m(x)dx + |\Omega_1| / 2|\Omega| \int_\Omega \theta_{d,m}(x)dx \leq \int_\Omega m(x)dx + 1 / 2 \int_\Omega \theta_{d,m}(x)dx.
\]

Then
\[
\int_\Omega \theta_{d,m}(x)dx = \int_{\Omega_1} \theta_{d,m}(x)dx + \int_{\Omega_2} \theta_{d,m}(x)dx \leq \int_\Omega m(x)dx + 1 / 2 \int_\Omega \theta_{d,m}(x)dx + K_1 d |\Omega_1^c|.
\]

Hence for \(d \geq 1\),
\[
\int_\Omega \theta_{d,m}(x)dx \leq 2 \int_\Omega m(x)dx + 2K_1 d |\Omega_1^c| \leq 2 \left( \int_\Omega m(x)dx + K_1 |\Omega| \right) d.
\]

Next, we prepare an estimate for \(|\Omega_2|\) in term of \(d\). Denote
\[
\tilde{\Omega}_2 = \left\{ x \in \Omega \mid m(x) \geq \frac{d}{2} a(x) \right\}.
\]
Obviously, 
\[ \int_{\Omega} m(x)dx \geq \int_{\tilde{\Omega}_2} m(x)dx \geq \frac{d}{2} \min_{\Omega} a(x)|\tilde{\Omega}_2|, \]
which implies that 
\[ |\tilde{\Omega}_2| \leq \frac{1}{d \min_{\Omega} a(x)} \int_{\Omega} m(x)dx. \]
We claim that \( \Omega_2 \subseteq \tilde{\Omega}_2 \). If the claim is true, then one has 
\[ |\Omega_2| \leq \frac{1}{d \min_{\Omega} a(x)} \int_{\Omega} m(x)dx. \quad (3.9) \]
To prove this claim, fix any \( x \in \tilde{\Omega}_c \), i.e., \( m(x) < \frac{d}{2} a(x) \). Based on the equation (3.1),
\[
\theta_{d,m}(x) = \frac{1}{2} \left[ m(x) - da(x) + \sqrt{(m(x) - da(x))^2 + 4d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy} \right] \\
= \frac{2d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy}{-m(x) + da(x) + \sqrt{(m(x) - da(x))^2 + 4d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy}} \\
\leq \frac{2d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy}{da(x)} \frac{2\|k\|_{L^\infty} \int_{\Omega} \theta_{d,m}(x)dx}{\min_{\Omega} a(x)} \\
\leq \frac{4}{\min_{\Omega} a(x)} \left( \int_{\Omega} m(x)dx + K_1 |\Omega| \right) \|k\|_{L^\infty} d,
\]
where the last inequality is due to (3.8). Hence \( \theta_{d,m}(x) < K_2 d \), i.e., \( x \in \Omega_2^c \). The claim is proved and thus (3.9) is valid.

Now we are ready to improve the estimate for \( \int_{\Omega} \theta_{d,m}(x)dx \). For \( x \in \Omega_2 \), since \( \Omega_2 \subseteq \Omega_1 \), the estimate (3.7) still holds, i.e.,
\[ \theta_{d,m}(x) \leq m(x) + \frac{1}{2|\Omega|} \int_{\Omega} \theta_{d,m}(x)dx. \]
Then
\[
\int_{\Omega_2} \theta_{d,m}(x)dx \leq \int_{\Omega} m(x)dx + \frac{|\Omega_2|}{2|\Omega|} \int_{\Omega} \theta_{d,m}(x)dx \\
\leq \int_{\Omega} m(x)dx + \frac{1}{2} \int_{\Omega_2} \theta_{d,m}(x)dx + \frac{|\Omega_2|}{2|\Omega|} \int_{\Omega_2} \theta_{d,m}(x)dx,
\]
which yields that
\[ \int_{\Omega_2} \theta_{d,m}(x)dx \leq 2 \int_{\Omega} m(x)dx + \frac{|\Omega_2|}{|\Omega|} \int_{\Omega_2} \theta_{d,m}(x)dx. \quad (3.10) \]
Moreover, we analyze the solution $\theta_{d,m}$ in $\Omega_2$. According to the equation (3.1), the estimates (3.9) and (3.10), one has

\[
\int_{\Omega_2} \theta_{d,m}^2(x) dx = \int_{\Omega_2} m(x) \theta_{d,m}(x) dx + d \int_{\Omega_2} \left( \int_{\Omega} k(x,y) \theta_{d,m}(y) dy - a(x) \theta_{d,m}(x) \right) dx \\
\leq K_2 d \int_{\Omega_2} m(x) dx - d \int_{\Omega_2} \left( \int_{\Omega} k(x,y) \theta_{d,m}(y) dy - a(x) \theta_{d,m}(x) \right) dx \\
\leq K_2 d \int_{\Omega} m(x) dx + d \int_{\Omega_2} \theta_{d,m}(x) dx \\
\leq \left( K_2 d + 2d + \frac{2}{\min_{\Omega} a(x)} \frac{1}{|\Omega|} \int_{\Omega_2} \theta_{d,m}(x) dx \right) \int_{\Omega} m(x) dx \\
\leq (K_2 + 2) d \int_{\Omega} m(x) dx + \frac{2}{|\Omega|} \left( \int_{\Omega} m(x) dx \right)^2 + \frac{1}{2} \int_{\Omega_2} \theta_{d,m}^2(x) dx.
\]

This indicates that for $d \geq 1$,

\[
\int_{\Omega_2} \theta_{d,m}^2(x) dx \leq 2(K_2 + 2) d \int_{\Omega} m(x) dx + \frac{4}{|\Omega|^2} \left( \int_{\Omega_2} m(x) dx \right)^2 \leq K_3 d,
\]

where

\[
K_3 = 2(K_2 + 2) \int_{\Omega} m(x) dx + \frac{4}{|\Omega|^2} \left( \frac{1}{\min_{\Omega} a(x)} \right)^2 \left( \int_{\Omega} m(x) dx \right)^2.
\]

Therefore, together with (3.10), for $d \geq 1$

\[
\int_{\Omega} \theta_{d,m}(x) dx = \int_{\Omega_2} \theta_{d,m}(x) dx + \int_{\Omega_2} \theta_{d,m}(x) dx \\
\leq 2 \int_{\Omega} m(x) dx + \frac{|\Omega_2|}{|\Omega|} \int_{\Omega_2} \theta_{d,m}(x) dx + \int_{\Omega_2} \theta_{d,m}(x) dx \\
\leq 2 \int_{\Omega} m(x) dx + 2 \left( \frac{1}{|\Omega_2|} \int_{\Omega_2} \theta_{d,m}^2(x) dx \right)^{\frac{1}{2}} \\
\leq 2 \left( \int_{\Omega} m(x) dx + \sqrt{K_3 |\Omega|} \right) \sqrt{d}.
\]

Set

\[
C_0 = 2 \left( \int_{\Omega} m(x) dx + \sqrt{K_3 |\Omega|} \right)
\]

The desired estimate follows. \qed

Next we construct examples to demonstrate that the order $O(\sqrt{d})$ is optimal under the prescribed total resources.
Proof of Theorem 1.4. W.l.o.g., always assume that $\epsilon d \leq 1$. First of all, it is routine to show that
\[ \theta_{d,m}(x) \leq \frac{a(x_0)}{\epsilon} \leq \frac{1}{\epsilon} \text{ in } \Omega, \]
and
\[ \theta_{d,m}(x) = \begin{cases} \frac{1}{2} \left[ -da(x) + \sqrt{d^2a^2(x) + 4d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy} \right] & x \in \Omega \setminus \Omega_0, \\ \frac{1}{2} \left[ \frac{a(x_0)}{\epsilon} - da(x) + \sqrt{\left( \frac{a(x_0)}{\epsilon} - da(x) \right)^2 + 4d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy} \right] & x \in \Omega_0. \end{cases} \]

Moreover, thanks to Theorem 1.3, one sees that
\[ \lim_{d \to +\infty} \sup_{\epsilon > 0} \frac{\int_{\Omega} \theta_{d,m}(x)dx}{d} = 0, \quad (3.11) \]
and
\[ \lim_{d \to +\infty} \sup_{\epsilon > 0} \frac{\int_{\Omega} k(x,y)\theta_{d,m}(y)dy}{d} = 0 \quad (3.12) \]
uniformly in $\Omega$.

For $x \in \Omega \setminus \Omega_0$, \begin{align*}
\theta_{d,m}(x) &= \frac{1}{2} \left[ -da(x) + \sqrt{d^2a^2(x) + 4d \int_{\Omega} k(x,y)\theta_{d,m}(y)dy} \right] \\
&= \frac{d}{2}a(x) \left[ -1 + \sqrt{1 + \frac{4}{a^2(x)} \int_{\Omega} k(x,y)\theta_{d,m}(y)dy} d \right] \\
&= \frac{\int_{\Omega} k(x,y)\theta_{d,m}(y)dy}{a(x)} - (1 + \xi)^{-3/2}a^{-3}(x) \frac{1}{d} \left( \int_{\Omega} k(x,y)\theta_{d,m}(y)dy \right)^2,
\end{align*}
where
\[ 0 < \xi(x) \leq \frac{4}{a^2(x)} \frac{\int_{\Omega} k(x,y)\theta_{d,m}(y)dy}{d}. \]
This yields that
\[
\int_{\Omega \setminus \Omega_0, \epsilon} (1 + \xi)^{-3/2} a^{-2}(x) \frac{1}{d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right)^2 dx
\]
\[= \int_{\Omega \setminus \Omega_0, \epsilon} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right) dx - \int_{\Omega \setminus \Omega_0, \epsilon} a(x) \theta_{d,m}(x) dx \]
\[= \int_{\Omega \setminus \Omega_0, \epsilon} \int_{\Omega} k(x, y) \theta_{d,m}(x) dx dy - \int_{\Omega \setminus \Omega_0, \epsilon} a(x) \theta_{d,m}(x) dx \]
\[= \int_{\Omega \setminus \Omega_0, \epsilon} \left( a(x) - \int_{\Omega} k(y, x) dy - \int_{\Omega \setminus \Omega_0, \epsilon} a(x) \theta_{d,m}(x) dx \right) \theta_{d,m}(x) dx \]
\[= \int_{\Omega \setminus \Omega_0, \epsilon} a(x) \theta_{d,m}(x) dx - \int_{\Omega \setminus \Omega_0, \epsilon} \left( \int_{\Omega} k(y, x) \theta_{d,m}(x) dx \right) dy \]
\[= \int_{\Omega \setminus \Omega_0, \epsilon} a(x) \theta_{d,m}(x) dx - \int_{\Omega \setminus \Omega_0, \epsilon} \frac{a(x) - a(x_0)}{\epsilon} - da(x) + \sqrt{\int_{\Omega} k(y, x) \theta_{d,m}(x) dy} \right) dx \]
\[= \int_{\Omega \setminus \Omega_0, \epsilon} \frac{a(x)}{2} \left[ \frac{a(x_0)}{\epsilon} - da(x) + \sqrt{\int_{\Omega} k(y, x) \theta_{d,m}(x) dy} \right] dx \]
\[-d \int_{\Omega \setminus \Omega_0, \epsilon} \frac{\int_{\Omega} k(x, y) \theta_{d,m}(y) dy}{d} dx.
\]

Thus, thanks to (3.12) and the assumption \( \lim_{d \to +\infty} \epsilon d = \alpha \in [0, 1) \), one has
\[\lim_{d \to +\infty} \int_{\Omega \setminus \Omega_0, \epsilon} (1 + \xi)^{-3/2} a^{-2}(x) \frac{1}{d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right)^2 dx = \omega_n (1 - \alpha) a^2(x_0), \quad (3.13)\]

where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). Notice that \( a(x) \) is strictly positive and continuous in \( \Omega \), and \( \lim_{d \to +\infty} \xi(x) = 0 \) uniformly in \( \Omega \). Hence (3.13) indicates that there exists a constant \( C > 0 \) such that for \( d \) large,
\[\int_{\Omega \setminus \Omega_0, \epsilon} \frac{1}{d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right)^2 dx \geq C. \quad (3.14)\]

Hence under the assumption \( \lim_{d \to +\infty} \epsilon d = \alpha \in [0, 1) \), for \( d \) large, one has
\[Cd \leq \int_{\Omega} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right)^2 dx \leq \| k \|^2_{L^\infty(\Omega)} \left( \int_{\Omega} \theta_{d,m}(y) dy \right)^2.\]

Therefore,
\[\int_{\Omega} \theta_{d,m}(x) dx \geq \sqrt{\frac{C}{2 |\Omega| \| k \|_{L^\infty}}} \sqrt{d}.\]

This completes the proof. \( \square \)
4 Effects of diffusion rate and source

The purpose of this section is to investigate how the total population depends on \(d\) for the model (1.11) with a type of simplified nonlocal dispersal operator.

**Proof of Theorem 1.5.** Suppose that \(\bar{\theta}(d)\) admits more than one local maxima as \(d\) varies, then \(\bar{\theta}(d)\) must have at least one local minimal point \(d_0 > 0\), i.e., \(\bar{\theta}'(d_0) = 0\) and \(\bar{\theta}''(d_0) \geq 0\). In the rest of this proof, we use the symbol \(\prime\) to denote the derivative in \(d\).

First of all, based on the equation satisfied by \(\theta_d\) as follows
\[
d \left( \bar{\theta} - \theta(x) \right) + \theta(x)[m(x) - \theta(x)] = 0 \quad x \in \Omega,
\]
it is easy to derive that
\[
\theta_d(x) = \frac{m(x) - d}{2} + \frac{1}{2} \sqrt{[m(x) - d]^2 + 4d\bar{\theta}_d(d)}, \tag{4.1}
\]
which yields that
\[
\sqrt{[m(x) - d]^2 + 4d\bar{\theta}_d(d)} = 2\theta_d(x) - m(x) + d = \theta_d(x) + \frac{d\theta_d}{\theta_d(x)}. \tag{4.2}
\]
and
\[
\theta'_d = \frac{\theta_d - \theta_d(x) + d\theta'_d}{2\theta_d(x) - m(x) + d}. \tag{4.3}
\]
Then \(4.3\) implies that
\[
\bar{\theta}'_d = \left( 1 - \frac{1}{|\Omega|} \int_{\Omega} \frac{d}{2\theta_d(x) - m(x) + d} \right)^{-1} \frac{1}{|\Omega|} \int_{\Omega} \frac{\bar{\theta}_d - \theta_d(x)}{2\theta_d(x) - m(x) + d} dx \tag{4.4}
\]
We remark that thanks to \(4.2\),
\[
1 - \frac{1}{|\Omega|} \int_{\Omega} \frac{d}{2\theta_d(x) - m(x) + d} dx = 1 - \frac{1}{|\Omega|} \int_{\Omega} d \left( \theta_d(x) + \frac{d\theta_d}{\theta_d(x)} \right)^{-1} dx > 0,
\]
thus \(\bar{\theta}'_d\) is always well defined.

To derive the expression for \(\bar{\theta}''_d\), for clarity, set
\[
f := \frac{\theta_d - \theta_d(x)}{2\theta_d(x) - m(x) + d}; \quad g := \frac{2\theta_d(x) - m(x)}{2\theta_d(x) - m(x) + d}.
\]
Thus
\[ \bar{\theta}_d = \frac{\int_{\Omega} \bar{f} \, dx}{\int_{\Omega} \bar{g} \, dx}, \quad \bar{\theta}_d' = \frac{\int_{\Omega} \bar{f}' \, dx \int_{\Omega} g \, dx - \int_{\Omega} f \, dx \int_{\Omega} g' \, dx}{\left( \int_{\Omega} g \, dx \right)^2}. \]

In particular, according to the assumption that \( \bar{\theta}_d'(d_0) = 0 \) and \( \bar{\theta}_d''(d_0) \geq 0 \), one has
\[
\int_{\Omega} f \, dx \bigg|_{d=d_0} = \int_{\Omega} \frac{\bar{\theta}_d - \theta_d(x)}{2\theta_d(x) - m(x) + d} \, dx \bigg|_{d=d_0} = 0,
\]
and
\[
\bar{\theta}_d''(d_0) = \frac{\int_{\Omega} f' \, dx \bigg|_{d=d_0}}{\int_{\Omega} g \, dx \bigg|_{d=d_0}} \geq 0,
\]
i.e., \( \int_{\Omega} f' \, dx \bigg|_{d=d_0} \geq 0. \)

Furthermore, based on (4.2), (4.3) and the assumption that \( \bar{\theta}_d'(d_0) = 0 \), it is standard to compute as follows
\[
f' \bigg|_{d=d_0} = \frac{(\bar{\theta}_d - \theta_d)'(2\theta_d - m + d) - (\bar{\theta}_d - \theta_d)(2\theta_d' + 1)}{(2\theta_d - m + d)^2} \bigg|_{d=d_0}
= \frac{-\theta_d'(2\theta_d - m + d) - 2\theta_d' \bar{\theta}_d - \theta_d - (\bar{\theta}_d - \theta_d)}{(2\theta_d - m + d)^2} \bigg|_{d=d_0}
= \frac{-\theta_d'(2\theta_d - m + d) - 2\theta_d' \bar{\theta}_d - \theta_d - \theta_d'(2\theta_d - m + d)}{(2\theta_d - m + d)^2} \bigg|_{d=d_0}
= \frac{-2\theta_d'}{(2\theta_d - m + d)^2} \left( \theta_d + \frac{d\bar{\theta}_d}{\theta_d} + \bar{\theta}_d - \theta_d \right) \bigg|_{d=d_0}
= -2\bar{\theta}_d \theta_d' \left( \theta_d + \frac{d\bar{\theta}_d}{\theta_d} \right)^{-2} \left( \frac{d}{\theta_d} + 1 \right) \bigg|_{d=d_0}.
\]

Thus
\[
\int_{\Omega} f' \, dx \bigg|_{d=d_0} = -2\bar{\theta}_d \int_{\Omega} \theta_d' \frac{d\theta_d + \theta_d^2}{(d\theta_d + \theta_d^2)^2} \, dx \bigg|_{d=d_0}
= -2\bar{\theta}_d \int_{\Omega} \theta_d' \left( \frac{d\theta_d + \theta_d^2}{(d\theta_d + \theta_d^2)^2} - \frac{1}{d\theta_d + \theta_d^2} \right) \, dx \bigg|_{d=d_0}
= -2\bar{\theta}_d \int_{\Omega} \theta_d \frac{(\bar{\theta}_d - \theta_d)}{(d\theta_d + \theta_d^2)^2} \left( \theta_d - \theta_d \right) \left( \theta_d + \bar{\theta}_d \theta_d + d\bar{\theta}_d \theta_d - d\theta_d \theta_d \right) \, dx \bigg|_{d=d_0}.
\]
Notice that according to [19, Theorem 1.1], \( \bar{\theta}_d \) is strictly increasing when \( d < \frac{1}{2}(\bar{m} + \inf_\Omega m) \) and strictly decreasing when \( d > \sup_\Omega m \). Hence \( d_0 \in (\inf_\Omega m, \sup_\Omega m) \). Moreover, by the equation satisfied by \( \theta_d \), it is easy to show that \( \theta_{d_0}(x) \in (\inf_\Omega m, \sup_\Omega m) \) in \( \Omega \). The assumption (1.13) guarantees that

\[
\frac{\theta_{d_0}(x)}{d_0} > \frac{\inf_\Omega m}{\sup_\Omega m} > \frac{\sqrt{5} - 1}{2} \quad \text{in} \quad \Omega,
\]

which implies that

\[
\theta_{d_0}^3 + \bar{\theta}_{d_0} \theta_{d_0}^2 + d_0 \bar{\theta}_{d_0} \theta_{d_0} - d_0^2 \bar{\theta}_{d_0} > \bar{\theta}_{d_0}[\theta_{d_0}^2 + d_0 \theta_{d_0} - d_0^2] > 0 \quad \text{in} \quad \Omega.
\]

This contradicts the assumption that \( d_0 \) is a local maximum point. Therefore, there exists \( L > 0 \) such that \( \bar{\theta}(d) \) is non-decreasing in \( d \) when \( d \leq L \), non-increasing in \( d \) when \( d > L \) and the property that \( L \in (\inf m, \sup m) \) easily follows from [19, Theorem 1.1].

The proof is complete.

\[ \square \]

5 Miscellaneous remarks

Logistic equation, introduced by Verhulst in 1838 [34], is one of the most fundamental models in population dynamics. In 1951, random diffusion was introduced to model dispersal behavior of a population [35]. Since then, reaction-diffusion models, which incorporate dispersal strategies, growth rates and carrying capacities, provide a good framework for studying questions in ecology. There are tremendous studies in this direction, see the books [11, 12, 31].

It is known that the logistic model with random diffusion (1.1) and that with nonlocal dispersal (1.5) share a lot of similarity in qualitative properties of solutions except for regularity. However, in this paper, for two specific issues related to total population, our results show serious discrepancies between the local model (1.1) and the nonlocal one (1.5).

First, for the local model (1.1), in the one-dimensional case, the ratio of the total population to the total resources is always less than 3 [3]. However, for the nonlocal model (1.7), the supremum of this ratio is always unbounded regardless of dimension of domains. Indeed, our results indicate that for certain distribution of resources, this ratio goes to infinity with order \( \sqrt{d} \) as the dispersal rate \( d \to \infty \).

The second issue is about the dependence of the total population on its dispersal rate. When the distribution of resources is a perturbation of a constant, the total population, as a function of \( d \), in the local model (1.1) could have multiple maximum points [20], while in the nonlocal model (1.11), which is a special case of the model (1.5), it admits exactly one maximum point.
Based on the biological and mathematical meanings of these issues discussed before, these discrepancies reflect essential differences in local and nonlocal dispersal strategies from some concrete and subtle aspects. Our exploration in this direction is quite preliminary and more related problems are waiting to be investigated. For example, in the first issue, how to maximize total population under the limited total resources in nonlocal models? In the second issue, it is still unknown whether it is possible for the total population to have multiple maximum points in the model (1.11). Moreover, the results obtained are about the model (1.11) with simplified nonlocal operators and barely anything is known for more general nonlocal operators. We will return to these problems in a future paper.

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