Mellin transforms with only critical zeros: generalized Hermite functions

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Abstract

We consider the Mellin transforms of certain generalized Hermite functions based upon certain generalized Hermite polynomials, characterized by a parameter $\mu > -1/2$. We show that the transforms have polynomial factors whose zeros lie all on the critical line. The polynomials with zeros only on the critical line are identified in terms of certain $\,\!_{2}F_{1}(2)$ hypergeometric functions, being certain scaled and shifted Meixner-Pollaczek polynomials. Other results of special function theory are presented.

Key words and phrases

Mellin transformation, generalized Hermite polynomials, hypergeometric function, critical line, zeros, functional equation, difference equation, reciprocity

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Introduction and statement of results

In a series of papers, we are considering certain Mellin transforms comprised of classical orthogonal polynomials that yield polynomial factors with zeros only on the critical line $\text{Re } s = 1/2$ or else only on the real axis. Such polynomials have many important applications to analytic number theory, in a sense extending the Riemann hypothesis. For example, using the Mellin transforms of Hermite functions, Hermite polynomials multiplied by a Gaussian factor, Bump and Ng \cite{4} were able to generalize Riemann’s second proof of the functional equation of the zeta function $\zeta(s)$, and to obtain a new representation for it. The polynomial factors turn out to be certain $\genfrac{[}{]}{0pt}{}{2}{1}(2)$ Gauss hypergeometric functions \cite{7}.

In a different setting, the polynomials $p_n(x) = \genfrac{[}{]}{0pt}{}{2}{1}(-n, -x; 1; 2) = (-1)^n \genfrac{[}{]}{0pt}{}{2}{1}(-n, x + 1; 1; 2)$ and $q_n(x) = i^n n! p_n(-1/2 - ix/2)$ were studied \cite{14}, and they directly correspond to the Bump and Ng polynomials with $s = -x$. Kirschenhofer, Pethö, and Tichy considered combinatorial properties of $p_n$, and developed diophantine properties of them. Their analytic results for $p_n$ include univariate and bivariate generating functions, and that its zeros are simple, lie on the line $x = -1/2 + it$, $t \in \mathbb{R}$, and that its zeros interlace with those of $p_{n+1}$ on this line. We may observe that these polynomials may as well be written as $p_n(x) = \genfrac{[}{]}{0pt}{}{n+x}{n} \genfrac{[}{]}{0pt}{}{2}{1}(-n, -x; -n - x; -1)$, or

$$p_n(x) = \frac{(-1)^n 2^n \Gamma(n - x)}{n! \Gamma(-x)} \genfrac{[}{]}{0pt}{}{2}{1}\left(-n, -n; x + 1 - n; \frac{1}{2}\right),$$

where $\Gamma$ is the Gamma function. In fact, combinatorial, geometrical, and coding aspects of $p_n(x)$ at integer argument had been noted in \cite{12} and \cite{19}, and Lemmas 2.2 and 2.3 of \cite{14} correspond very closely to Lemmas 2 and 3, respectively, of \cite{19}.

The Hermite polynomials being certain cases proportional to Laguerre polynomials $x^\delta L_n^{\pm 1/2}(x^2)$, $\delta = 0$ or 1, the generalization to Mellin transforms of Laguerre functions has been made \cite{3, 7} and now the polynomial factors are a family of other $\genfrac{[}{]}{0pt}{}{2}{1}(2)$.
functions. The Laguerre functions are \( L_\alpha^n(x) = x^{\alpha/2}e^{-x/2}L_n^\alpha(x) \), for \( \alpha > -1 \), and their Mellin transform is of the form \( M_\alpha^n(s) = 2^{s+\alpha/2}\Gamma(s + \alpha/2)P_\alpha^n(s) \). Mixed recursion relations are known for the polynomials \( P_\alpha^n \), as well as a generating function, and they satisfy the functional equation \( P_\alpha^n(s) = (-1)^nP_\alpha^n(1-s) \).

The generalized Mellin transform of Legendre, associated Legendre, Chebyshev, and Gegenbauer functions has been investigated very recently elsewhere [9]. These integral transforms are on the interval \([0,1]\) and lead to families of polynomials, being certain terminating \( 3F_2(1) \) functions, with zeros only on the critical line.

In this article, we study the Mellin transforms of certain generalized Hermite functions, and are able to identify the resulting polynomial factors in terms of certain hypergeometric functions \( 2F_1(2) \). The key result is that these transforms possess zeros only on the critical line.

We use standard notation. Let \( pF_q \) be the generalized hypergeometric function, \( (a)_n = \Gamma(a+n)/\Gamma(a) = (-1)^n\frac{\Gamma(1-a)}{\Gamma(1-a-n)} \) the Pochhammer symbol, and \( B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \) the Beta function.

Szegö in a problem introduced an orthogonal polynomial sequence \( \{H_\mu^n(x)\} \) [20] (p. 377). These polynomials were also studied early on by Chihara [6]. They may be written as

\[
H_\mu^{2n}(x) = (-1)^n2^{2n}n!L_{\mu}^{-1/2}(x^2), \quad (1.1a)
\]

and

\[
H_\mu^{2n+1}(x) = (-1)^n2^{2n+1}n!xL_{\mu}^{1/2}(x^2), \quad (1.1b)
\]

where \( \mu > -1/2 \) and \( L_n^\alpha \) are Laguerre polynomials. In turn, the Laguerre polynomials are expressible in terms of the confluent hypergeometric function \( 1F_1 \) as

\[
L_n^\alpha(x) = \binom{n+\alpha}{n} \, _1F_1(-n; \alpha+1; x). \quad (1.2)
\]

The polynomials \( H_\mu^n(x) \) have been written by other authors with a different normaliza-
tion. In particular, Rosenblum [17] does not include the $2^n$ factors. The polynomials are orthogonal with respect to the weight function $|x|^{2\mu}e^{-x^2}$, for $x \in (-\infty, \infty)$, such that

$$\int_{-\infty}^{\infty} H^\mu_m(x)H^\mu_n(x)|x|^{2\mu}e^{-x^2}dx = 2^{2n} \Gamma \left( \frac{n+1}{2} \right) + \mu + \frac{1}{2} \right) \delta_{mn},$$

with $[x]$ the greatest integer function.

Just as the Hermite polynomials are connected with the excited state wavefunctions and algebraic properties of the quantum mechanical simple harmonic oscillator, the generalized Hermite polynomials may be used to develop the calculus of the Bose-like oscillator [17]. Generalized Hermite functions may also be shown to be the eigenfunctions of a generalized Fourier transform, with eigenvalues $\pm 1$ and $\pm i$.

Finite sums for Laguerre polynomials and Kummer-type relations for the hypergeometric function $2F_2$ are established in [10]. Convolution sums and other results for Laguerre polynomials are given in [11]. As we briefly show in the third section of the paper, these relations may be usefully transcribed to the generalized Hermite polynomials.

The polynomials $H^\mu_n(x)$ satisfy the recurrence relation

$$H^\mu_{n+1}(x) = 2xH^\mu_n(x) - 2(n + \theta_n)H^\mu_{n-1}(x), \quad n \geq 0,$$  \hspace{1cm} (1.3)

as well as the differential equation

$$xy'' + 2(\mu - x^2)y' + (2nx - \theta_nx^{-1})y = 0,$$  \hspace{1cm} (1.4)

where $\theta_{2m} = 0$ and $\theta_{2m+1} = 2\mu$.

Using a generating function of Laguerre polynomials, one for the generalized Hermite polynomials may be developed [6],

$$(1 + 2xw + 4w^2)(1 + 4w^2)^{-\mu+3/2} \exp\left[4x^2w^2(1 + 4w^2)^{-1}\right] = \sum_{n=0}^{\infty} H^\mu_n(x)\frac{w^n}{[n/2]!}. \hspace{1cm} (1.5)$$
The polynomials $H_n^\mu(x)$ have application in Gauss-generalized-Hermite quadrature [18].

Here we consider Mellin transformations

$$(\mathcal{M}f)(s) = \int_0^\infty f(x)x^s\frac{dx}{x}.$$  

For properties of the Mellin transform, we mention [5].

We put, for $\text{Re } s > 0$,

$$M_n^\mu(s) \equiv \int_0^\infty x^{s-1}H_n^\mu(x)x^\mu e^{-x^2/2}dx. \quad (1.6)$$

We let $p_n^\mu(s)$ denote the polynomial factor of $M_n^\mu(s)$. Clearly (1.6) is the $\mu \neq 0$ extension of the Mellin transform of Hermite functions.

**Proposition 1.** The Mellin transforms (1.3) are given by

$$M_{2n}^\mu(s) = (-1)^n 2^{2n} \frac{\Gamma(\mu + s/2)}{n!} \left(\frac{2\mu + 1}{2}\right)_n \, _2F_1\left(-n, \frac{\mu + s}{2}; \frac{\mu + 1}{2}; 2\right),$$  

and

$$M_{2n+1}^\mu(s) = (-1)^n 2^{2n+1} \frac{\Gamma(\mu + s + 1/2)}{n!} \left(\frac{2\mu + 3}{2}\right)_n \, _2F_1\left(-n, \frac{\mu + s + 1}{2}; \frac{\mu + 3}{2}; 2\right), \quad (1.7a)$$

**Proposition 2.** The Mellin transforms (1.3) satisfy the recursions

$$M_{2m+1}^\mu(s) = 2M_{2m}^\mu(s + 1) - 4mM_{2m-1}^\mu(s), \quad (1.8a)$$

and

$$M_{2m+2}^\mu(s) = 2M_{2m+1}^\mu(s + 1) - 2(2m + 2\mu + 1)mM_{2m}^\mu(s). \quad (1.8b)$$

**Proposition 3.** A generating function for the transforms (1.6) is

$$\sum_{n=0}^\infty M_n^\mu(s) \frac{t^n}{n!} = 2^{(\mu + s - 3)/2} \frac{(1 + 4t^2)^{s/2 - 1}}{(1 - 4t^2)^{(\mu + s + 1)/2}} \left[ \sqrt{2(1 - 16t^4)} \Gamma\left(\frac{\mu + s}{2}\right) + 4t \Gamma\left(\frac{\mu + s + 1}{2}\right) \right].$$

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Proposition 4. (Reciprocity) For positive integers \( m \) and \( n \),

\[
\left(\mu + \frac{1}{2}\right)_m p_{2n}^\mu(-2m-\mu) = \left(\mu + \frac{1}{2}\right)_n p_{2m}^\mu(-2n-\mu),
\]

and

\[
\left(\mu + \frac{3}{2}\right)_m p_{2n+1}^\mu(-2m-1-\mu) = \left(\mu + \frac{3}{2}\right)_n p_{2m+1}^\mu(-2n-1-\mu).
\]

Theorem 1. The polynomials \( p_n^\mu(s) \), of degree \( n \), satisfy the functional equation

\[
p_n^\mu(s) = (-1)^n p_n^\mu(1-s).
\]

These polynomials have zeros only on the critical line. Further, all zeros \( \neq 1/2 \) occur in complex conjugate pairs.

The following section of the paper contains the proof of these Propositions. After that are presented various special function theory results and some discussion.

Proof of Propositions and of Theorem 1

Proposition 1. Using the definition (1.1) for \( H_{2n}^\mu \) and the series form of (1.2) for the Laguerre polynomials in (1.6) gives

\[
M_{2n}^\mu(s) = (-1)^n 2^{2n} n! \int_0^\infty x^{s+\mu-1} e^{-x^2/2} L_{n}^{\mu-1/2}(x^2) dx
\]

\[
= (-1)^n 2^{2n} \left(\mu + \frac{1}{2}\right)_n \sum_{j=0}^n \frac{(-n)_j}{(\mu + 1/2)_j j!} \Gamma \left(j + \frac{\mu + s}{2}\right) 2^{(\mu+s)/2+j-1}
\]

\[
= (-1)^n 2^{2n} \left(\mu + \frac{1}{2}\right)_n 2^{(\mu+s)/2-1} \Gamma \left(\frac{\mu + s}{2}\right) \sum_{j=0}^n \frac{(-n)_j}{(\mu + 1/2)_j j!} \left(\frac{\mu + s}{2}\right)_j j!
\]

\[
= (-1)^n 2^{2n} 2^{(\mu+s)/2-1} \left(\mu + \frac{1}{2}\right)_n \Gamma \left(\frac{\mu + s}{2}\right) \binom{\mu + s}{2} \binom{n}{\mu + 1/2} 2^{\mu+s}/2+j-1
\]

Part (b) proceeds very similarly.

Proposition 2. The recursions follow directly from (1.1) and (1.3).

Proposition 3. The generating function follows by using (1.5) in the definition (1.6).
Remark. Expanding the generating function so obtained in powers of $t$ is another way in which to find the explicit transform expressions of Proposition 1.

**Proposition 4.** This follows from the symmetry of the $2F_1$ function in its two numerator parameters.

Remark. We highly expect that the reciprocity relation and functional equation for $p_n^\mu(s)$ are connected with the properties of an Ehrhart polynomial, possibly shifted and/or scaled, corresponding to some polytope.

**Theorem 1.** By Proposition 1, the polynomials $p_n^\mu(s)$ are of degree $n$ and have real coefficients for real values of $\mu$. The functional equation follows immediately by transforming the $2F_1(2)$ function. In particular we apply (e.g., [13] p. 1043)

$$2F_1(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} 2F_1(\alpha, \gamma - \beta; \gamma; \frac{z}{z - 1}).$$

That the polynomial zeros occur only on $\text{Re } s = 1/2$ will follow from the following difference equations, which we next derive. For $n$ even,

$$[2(n+\mu)+1](s+\mu-2)p_n^\mu(s) - (s+\mu)(s+\mu-2)p_n^\mu(s+2) + [(s-2)(s-1) + (1-\mu)\mu]p_n^\mu(s-2) = 0,$$

and for $n$ odd,

$$[2(n+\mu)-1](s+\mu-1)p_n^\mu(s) - (s+\mu+1)(s+\mu-1)p_n^\mu(s+2) + [(s-2)(s-1) - (1+\mu)\mu]p_n^\mu(s-2) = 0.$$

We put $H_n^\mu(x) = x^{-\mu}e^{x^2/2}f(x)$ into the differential equation (1.4). Then for $n$ even there results

$$e^{x^2/2}x^{-\mu-1}\{[(1-\mu)\mu + x^2 + 2(n+\mu)x^2 - x^4]f(x) + x^2f''(x)\} = 0,$$

and

$$e^{x^2/2}x^{-\mu-1}\{-(1+\mu)\mu - (1+2(n+\mu))x^2 + x^4]f(x) + x^2f''(x)\} = 0$$

for $n$ odd. The quantity in curly brackets is zero, and multiplying it by $x^{s-1}$ and integrating by parts twice and shifting $s \rightarrow s - 2$, for $n$ even we find the difference
equation of the Mellin transforms,

\[ [2(n + \mu) + 1] M^n_\mu(s) - M^n_\mu(s + 2) + [(1 - \mu)\mu + (s - 2)(s - 1)] M^n_\mu(s - 2) = 0. \]

By using the \( s \)-dependent factors of the transforms (1.7a), \( M^n_\mu(s) \propto 2^{s/2} \Gamma \left( \frac{s + \mu}{2} \right) p^n_\mu(s) \), and the functional equation \( \Gamma(z + 1) = z \Gamma(z) \), we find the stated difference equation for the polynomial factors when \( n \) is even. For \( n \) odd, the steps are very similar.

We then use shifted polynomials \( q(s) = p^n_\mu(s + 1/2) \), so that \( p^n_\mu(s) = q(s - 1/2) \).

Then, with a translation \( s \rightarrow s + 1/2 \), we find

\[
[2(n + \mu) + 1] \left( s + \mu - \frac{3}{2} \right) q(s) - \left( s + \mu + \frac{1}{2} \right) \left( s + \mu - \frac{3}{2} \right) q(s + 2)
\]
\[
+ \left[ \left( s - \frac{3}{2} \right) \left( s - \frac{1}{2} \right) + (1 - \mu)\mu \right] q(s - 2) = 0,
\]

for \( n \) even, and

\[
[2(n + \mu) - 1] \left( s + \mu - \frac{1}{2} \right) q(s) - \left( s + \mu + \frac{3}{2} \right) \left( s + \mu - \frac{1}{2} \right) q(s + 2)
\]
\[
+ \left[ \left( s - \frac{3}{2} \right) \left( s - \frac{1}{2} \right) - (1 + \mu)\mu \right] q(s - 2) = 0,
\]

for \( n \) odd. It follows that if \( r_k \) is a root of \( q \), \( q(r_k) = 0 \), that

\[
\left( r_k + \mu + \frac{1}{2} \right) q(r_k + 2) = \left( r_k - \mu - \frac{1}{2} \right) q(r_k - 2),
\]

when \( n \) is even, and

\[
\left( r_k + \mu + \frac{3}{2} \right) q(r_k + 2) = \left( r_k - \mu - \frac{3}{2} \right) q(r_k - 2),
\]

when \( n \) is odd. The equality of the absolute value of both sides of these equations provides a necessary condition that \( \text{Re } r_k = 0 \). I.e., the zeros of \( q(s) \) are pure imaginary and thus the zeros of \( p^n_\mu(s) \) are on the critical line. \( \square \)
Other results

Let $\varepsilon(m) = 0$ when $m$ is even and $= 1$ when $m$ is odd. Then we may compactly write from (1.1)

$$H_m^{\mu}(x) = (-1)^{(m-\varepsilon)/2}2^m \left(\frac{m-\varepsilon}{2}\right)! x^{\varepsilon} L_{m-\varepsilon}^{\mu-1/2+\varepsilon}(x^2). \quad (2.1)$$

We let $J_n$ be the Bessel function of the first kind of order $n$. We then have the following integral representation.

**Proposition 5.** For $\mu > -1/2$,

$$H_m^{\mu}(x) = (-1)^{(m-\varepsilon)/2}2^m e^{x^2} x^{1/2-\mu} \int_0^\infty t^{m/2+\mu/2-1/4} J_{\mu-1/2+\varepsilon}(2t) e^{-t} dt. \quad (2.2)$$

**Proof.** This follows from (2.1) and the representation ([2], p. 286) for $\alpha > -1$

$$L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha/2} \int_0^\infty t^{n+\alpha/2} J_{\alpha}(2\sqrt{xy}) e^{-t} dt.$$  

\[\square\]

**Remark.** As the representation (2.2) is conditionally convergent, it does not appear to be directly useful for application to the transforms (1.6).

**Proposition 6.** For $\beta > 0$ and $\mu > -1/2$,

$$H_m^{\mu+\beta}(x) = \frac{\Gamma\left(\frac{m+\varepsilon+1}{2} + \mu + \beta\right)}{\Gamma(\beta) \Gamma\left(\frac{m+\varepsilon+1}{2} + \mu\right)} \int_0^1 t^{\mu-1/2+\varepsilon/2} (1-t)^{\beta-1} H_m^{\mu}(x \sqrt{t}) dt$$

$$= 2 \frac{\Gamma\left(\frac{m+\varepsilon+1}{2} + \mu + \beta\right)}{\Gamma(\beta) \Gamma\left(\frac{m+\varepsilon+1}{2} + \mu\right)} \int_0^1 u^{2\mu+\varepsilon} (1-u^2)^{\beta-1} H_m^{\mu}(xu) du.$$

Here $\varepsilon = 0$ when $m$ is even and $= 1$ when $m$ is odd.

**Proof.** This follows from (1.1) and Koshlyakov’s formula ([15], [16], p. 94)

$$L_n^{\alpha+\beta}(x) = \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta) \Gamma(n+\alpha+1)} \int_0^1 t^\alpha (1-t)^{\beta-1} L_n^{\alpha}(xt) dt, \quad \alpha > -1, \quad \beta > 0.$$  

\[\square\]
Remark. This result contains the very special case of \( \mu = 2n + \epsilon = 0 \) and \( \beta \to \alpha + 1/2 \) in [13], p. 836. Indeed this corresponds to Uspensky’s formula (e.g., [16], p. 94).

**Proposition 7.** For \( \mu > -1/2 \),

\[
\sum_{m=0}^{n} \frac{(-1)^m}{2^n m!} H_{2m}^{\mu+1/2}(x) = \frac{(-1)^n}{2^n n!} H_{2n}^{\mu+3/2}(x)
\]

and

\[
\sum_{m=0}^{n} \frac{(-1)^m}{2^n m!} H_{2m+1}^{\mu+1/2}(x) = \frac{(-1)^n}{2^n n!} H_{2n+1}^{\mu+1/2}(x).
\]

**Proof.** This follows from (1.1) and the relation (e.g., [13], p. 1038) \( \sum_{m=0}^{n} L_{n}^{\alpha}(x) = L_{n}^{\alpha+1}(x) \). \( \square \)

We may write the generalized Hermite polynomials in terms of the ordinary Hermite polynomials, and vice versa, as described next.

**Proposition 8.** For \( \mu > -1/2 \), (a)

\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j 4^j (-\mu)_j H_{2(n-j)}^{\mu}(x) = H_{2n}(x),
\]

(b)

\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j 4^j (\mu)_j H_{2(n-j)}^{\mu}(x) = H_{2n}^{\mu}(x),
\]

(c)

\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j 4^j (-\mu)_j H_{2(n-j)+1}^{\mu}(x) = H_{2n+1}(x),
\]

(d)

\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j 4^j (\mu)_j H_{2(n-j)+1}^{\mu}(x) = H_{2n+1}^{\mu}(x),
\]

thus (e)

\[
H_{n}^{\mu}(x) = \sum_{j=0}^{[n/2]} \binom{n/2}{j} (-1)^j 4^j (\mu)_j H_{n-2j}^{\mu}(x)
\]

and

\[
H_{n}(x) = \sum_{j=0}^{[n/2]} \binom{n/2}{j} (-1)^j 4^j (-\mu)_j H_{n-2j}^{\mu}(x).
\]
Proof. These relations follows from (1.1) and the property

\[ L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(\alpha - \beta)_j}{j!} L_{n-j}^\beta(x). \]

Thus parts (a) and (b) follow from

\[ \sum_{j=0}^{n} (\alpha - \mu)_j \binom{n}{j} (-1)^j 4^j H_{2(n-j)}^\mu(x) = H_n^\alpha(x), \]

and (c) and (d) from

\[ \sum_{j=0}^{n} (\alpha - \mu)_j \binom{n}{j} (-1)^j 4^j H_{2(n-j)+1}^\mu(x) = H_{2n+1}^\alpha(x). \]

\[ \square \]

Proposition 8 implies a summation relation for certain \( _2F_1(2) \) functions. Stated otherwise, we have the following, with \( M_n(s) = M_n^0(s) \).

Corollary. For \( \mu > -1/2 \),

\[ M_n^\mu(s) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor}{j} (-1)^j 4^j (\mu)_j M_{n-2j}(s). \]

We have the following convolution sums for generalized Hermite polynomials.

Proposition 9. (a)

\[ H_{2n}^{\alpha+\beta+1/2}(\sqrt{x^2 + y^2}) = \sum_{k=0}^{n} \binom{n}{k} H_{2(n-k)}^\alpha(x) H_{2k}^\beta(y), \]

(b)

\[ \frac{xy}{\sqrt{x^2 + y^2}} H_{2n+1}^{\alpha+\beta+3/2}(\sqrt{x^2 + y^2}) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} H_{2(n-k)+1}^\alpha(x) H_{2k+1}^\beta(y), \]

(c)

\[ \sum_{j=1}^{n-1} \binom{n-1}{j-1} H_{2(j-1)}^\alpha(x) H_{2(n-j-1)}^{k+3/2}(y) \]

\[ = -\frac{1}{4} \frac{k!}{y^{2(k+1)}} \left[ \sum_{\ell_1=0}^{k} \frac{y^{2\ell_1}}{\ell_1!} H_{2(n-1)}^\alpha(x) \right] - \sum_{\ell_2=0}^{k} \frac{y^{2\ell_2}}{\ell_2!} H_{2(n-1)}^{\alpha+\ell_2}(\sqrt{x^2 + y^2}), \]

(d)

\[ \frac{2}{xy} \sum_{j=1}^{n-1} \binom{n-1}{j-1} H_{2j-1}^\alpha(x) H_{2(n-j)-1}^{k+1/2}(y) \]
\[- \frac{k!}{y^{2(k+1)}} \left[ \frac{1}{x} \sum_{\ell_1=0}^{k} \frac{y^{2\ell_1}}{\ell_1!} H_{2n-1}^{\alpha}(x) \right] \left[ \frac{1}{\sqrt{x^2+y^2}} \sum_{\ell_2=0}^{k} \frac{y^{2\ell_2}}{\ell_2!} H_{2n-1}^{\alpha+\ell_2}(\sqrt{x^2+y^2}) \right], \]

(c)

\[ H_{2n}^{\alpha_1+\alpha_2+\ldots+\alpha_k-1/2} \left( \sqrt{x_1^2+x_2^2+\ldots+x_k^2} \right) \]

\[ = n! \sum_{i_1+i_2+\ldots+i_k=n} \frac{H_{2i_1+1/2}(x_1)H_{2i_2+1/2}(x_2)\ldots H_{2i_k+1/2}(x_k)}{i_1!i_2!\ldots i_k!}, \]

and (f)

\[ \frac{x_1 x_2 \cdots x_k}{\sqrt{x_1^2+x_2^2+\ldots+x_k^2}} H_{2n+1}^{\alpha_1+\alpha_2+\ldots+\alpha_k-3/2} \left( \sqrt{x_1^2+x_2^2+\ldots+x_k^2} \right) \]

\[ = \frac{n!}{2^{n+1}} \sum_{i_1+i_2+\ldots+i_k=n} \frac{H_{2i_1+1/2}(x_1)H_{2i_2+1/2}(x_2)\ldots H_{2i_k+1/2}(x_k)}{i_1!i_2!\ldots i_k!}. \]

In (c) and (d), \( k > 0 \) is an integer.

**Proof.** (a) and (b) follow from (1.1) and the well known relation (e.g., [13], p. 1038)

\[ L_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} L_n^{\alpha}(x) L_k^{\beta}(y). \]

(c) and (d) follow from (1.1) and Proposition 3 of [11]: For \( \alpha > -1 \) and \( k \) a positive integer,

\[ \sum_{j=1}^{n-1} \frac{L_j^{\alpha}(x)L_{n-j-1}^{k+1}(y)}{n-j} = \frac{k!}{y^{k+1}} \left[ \sum_{\ell_1=0}^{k} \frac{y^{\ell_1}}{\ell_1!} L_{n-1}^{\alpha}(x) - \sum_{\ell_2=0}^{k} \frac{y^{\ell_2}}{\ell_2!} L_{n-1}^{\alpha+\ell_2}(x+y) \right]. \]

(e) and (f) follow from (1.1) and (e.g., [13], p. 1039)

\[ L_n^{\alpha_1+\alpha_2+\ldots+\alpha_k+k-1}(x_1+x_2+\ldots+x_k) = \sum_{i_1+i_2+\ldots+i_k=n} L_{i_1}^{\alpha_1}(x_1)L_{i_2}^{\alpha_2}(x_2)\ldots L_{i_k}^{\alpha_k}(x_k). \]

\[ \blacksquare \]

**Proposition 10.** (a) For \( |t| < 1 \),

\[ \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!} H_{2m}^{\alpha-m+1/2}(x)t^m = (1+t)^{\alpha} e^{-x^2t}, \]

\[ \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!} H_{2m+1}^{\alpha-m-1/2}(x)t^m = 2x(1+t)^{\alpha} e^{-x^2t}, \]
(b) 
\[ H^{\alpha}_{2n}(\sqrt{x^2 + y^2}) = e^{y^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^{2k} H^{\alpha+k}_{2n}(x), \]
\[ H^{\alpha}_{2n+1}(\sqrt{x^2 + y^2}) = e^{y^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^{2k} H^{\alpha+k}_{2n+1}(x), \]

(c) 
\[ \sum_{m=0}^{\infty} \frac{t^m}{(\alpha + 1)_m} \frac{(-1)^m}{4^m m!} H^{\alpha+1/2}_{2m}(x) = \Gamma(\alpha + 1) \frac{e^t}{\Gamma(\alpha/2, x^\alpha)} J_\alpha(2x\sqrt{t}), \]
\[ \sum_{m=0}^{\infty} \frac{t^m}{(\alpha + 1)_m} \frac{(-1)^m}{2^{2m+1} m!} H^{\alpha+1/2}_{2m+1}(x) = \Gamma(\alpha + 1) \frac{e^t}{\Gamma(\alpha/2, x^\alpha-1)} J_\alpha(2x\sqrt{t}), \]

(d) for \( x > 0 \) and \( \gamma \) the Euler constant,
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!} H^{1/2}_{2n}(x) = -2 \ln x - \gamma, \]
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+1} n!} H^{-1/2}_{2n+1}(x) = -x(2 \ln x + \gamma), \]

(e) 
\[ \frac{(-1)^m}{4^m m!} H^{\beta+1/2}_{2m}(x) = \sum_{n=0}^{m} \frac{(-1)^n}{4^n n!} \left( \frac{\beta + n}{\beta - n} \right) \tau^{2n}(1 - \tau^2)^{m-n} H^{\beta+1/2}_{2n}(x), \]

and (f) 
\[ \frac{(-1)^m}{2^{2m+1} m!} \frac{1}{\tau} H^{-1/2}_{2m+1}(x) = \sum_{n=0}^{m} \frac{(-1)^n}{2^{2n+1} n!} \left( \frac{\beta + n}{\beta - n} \right) \tau^{2n}(1 - \tau^2)^{m-n} H^{-1/2}_{2n+1}(x). \]

(e) and (f) continue to hold for \( \tau \to 0 \), when only the \( n = 0 \) term remains on the right side.

Proof. (a) follows from (1.1) and the generating function
\[ \sum_{m=0}^{\infty} L^{\alpha-m}_m(x) t^m = (1 + t)^\alpha e^{-xt}. \]
For (b),
\[ L^{\alpha}_n(x + y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k L^{\alpha+k}_n(x) \]
is used. (c) follows from
\[ \sum_{m=0}^{\infty} \frac{t^m}{(\alpha + 1)_m} L^{\alpha}_m(x) = e^t \,_0F_1(-; \alpha + 1; -xt) = \Gamma(\alpha + 1) \frac{e^t}{(xt)^{\alpha/2}} J_\alpha(2\sqrt{xt}), \]
where $\,_{0}F_{1}$ is a confluent hypergeometric function with a single denominator parameter. (d) is based upon

$$
\sum_{n=1}^{\infty} \frac{L_n(x)}{n} = -\ln x - \gamma, \quad x > 0.
$$

(e) and (f) use (1.1) and

$$
L_{m}^{\beta}(\tau x) = \sum_{n=0}^{m} \left( \frac{\beta + m}{\beta - n} \right) \tau^n (1 - \tau)^{m-n} L_n^{\beta}(x).
$$

The ordinary Hermite polynomials have the alternative representation

$$
H_n(x) = (2x)^n \,_{2}F_{0} \left( -[n/2], -[n/2] + (-1)^n/2; -; -\frac{1}{x^2} \right),
$$

so that another way to write the generalized Hermite polynomials is

$$
H_n^{\mu}(x) = (2x)^n \,_{2}F_{0} \left( -[n/2], -[n/2] - \mu + (-1)^n/2; -; -\frac{1}{x^2} \right).
$$

In turn, the Mellin transforms (1.6) evaluate in terms of $\,_{2}F_{1}(1/2)$ functions. These functions are to be expected, as transformations of the hypergeometric factors in (1.7),

$$
\,_{2}F_{1} \left( -n, \frac{\mu + s}{2}; \mu + \frac{1}{2}; 2 \right) = (-1)^n 2^n \frac{\Gamma \left( \mu + \frac{1}{2} \right) \Gamma \left( \frac{\mu + s}{2} + n \right)}{\Gamma \left( \frac{\mu + s}{2} \right) \Gamma \left( \mu + \frac{1}{2} + n \right)} \times \,_{2}F_{1} \left( -n, \frac{1}{2} - n - \mu; 1 - n - \frac{(\mu + s)}{2}; 1 \right).
$$

$$
\,_{2}F_{1} \left( -n, \frac{\mu + s + 1}{2}; \mu + \frac{3}{2}; 2 \right) = (-1)^n 2^n \frac{\Gamma \left( \mu + \frac{3}{2} \right) \Gamma \left( \frac{\mu + s + 1}{2} + n \right)}{\Gamma \left( \frac{\mu + s + 1}{2} \right) \Gamma \left( \mu + \frac{3}{2} + n \right)} \times \,_{2}F_{1} \left( -n, \frac{1}{2} - n - \mu; 1 - n - \frac{(\mu + s + 1)}{2}; 1 \right).
$$

The Meixner-Pollaczek polynomials $P^\lambda_n$ have the hypergeometric form (e.g., [2], p. 348)

$$
P^\lambda_n(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} \,_{2}F_{1} \left( -n, \lambda + ix; 2\lambda; 1 - e^{-2i\phi} \right), \quad \lambda > 0, \quad 0 < \phi < \pi.
$$
Therefore, comparing with (1.7), the polynomial factors there are proportional to

\[ P_{n}^{(\mu+\varepsilon)/2+1/4} \left[ i \left( \frac{1}{4} - \frac{s}{2} \right); \frac{\pi}{2} \right] = \frac{(\mu + 1/2 + \varepsilon)n}{n!} i^n \frac{n}{2} \binom{\mu + s + \varepsilon}{\mu + 1/2 + \varepsilon; 2}. \]

This identification provides an alternative method to prove Theorem 1, and, in fact, to show that the polynomial zeros on the critical line are also simple. First, the Meixner-Pollaczek polynomials are orthogonal with respect to the weight

\[ |\Gamma (\lambda + ix)|^2 e^{(2\phi - \pi)x}. \]

The Plancherel relation for Mellin transforms provides none other than this weight,

\[ \left| \Gamma \left( \frac{\mu + s + \varepsilon}{2} \right) \right|^2. \]

Then the polynomial factors \( p_{n}^{\mu}(1/2 + it) \) form an orthogonal family with respect to the corresponding measure. By standard results on orthogonal polynomials [20], the zeros are simple, and the zeros of \( p_{n}^{\mu}(1/2 + it) \) and \( p_{n+1}^{\mu}(1/2 + it) \) separate each other.
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