Penrose quasi-local energy and Kerr–Schild metrics

Mahdi Godazgar and André Kaderli

Institut für Theoretische Physik, Eidgenössische Technische Hochschule Zürich, Wolfgang-Pauli-Strasse 27, 8093 Zürich, Switzerland

E-mail: godazgar@phys.ethz.ch and kaderlia@student.ethz.ch

Received 3 August 2018, revised 17 September 2018
Accepted for publication 26 September 2018
Published 19 October 2018

Abstract
Specialising in the case of Kerr–Schild spacetimes, which include the Kerr black hole and gravitational wave solutions, we propose a modification of the Penrose quasi-local energy. The modification relies on the existence of a natural Minkowski background for Kerr–Schild spacetimes. We find that the modified surface integral reduces to a volume integral of the Einstein tensor, which has been proposed previously as an appropriate definition for quasi-local energy for Kerr–Schild backgrounds. Furthermore, in the special case that the Kerr–Schild null vector is normal to the surface of interest, we construct a 1–1 map between the 2-surface twistors in the Kerr–Schild background and Minkowski twistors projected onto the surface.

Keywords: quasi-local energy, Kerr–Schild metrics, twistors

1. Introduction

A reasonable notion of gravitational energy\(^1\) remains an unresolved issue in the context of general relativity more than a century after its discovery\(^2\). The main difficulty is the lack of a background geometry; a difficulty shared with another important outstanding problem in general relativity: that of quantisation. This means that the well-established Noether–Belinfante–Rosenfeld procedure for the energy–momentum density of matter fields cannot be generalised to include the gravitational field. Indeed, due to the principle of equivalence any tensorial gravitational energy–momentum is expected to vanish locally. Different coordinate-dependent approaches have been suggested, such as energy–momentum pseudotensors and bimetric formulations, see e.g. [2–4]. Alternatively, it is reasonable to look for so-called quasi-local quantities associated to 3-volumes \(V\) and their boundary 2-surfaces \(S\), see e.g. [5–8]. In particular, in a recent interesting proposal, Wang and Yau attempt to formulate a reasonable

\(^1\) It should be emphasised that amenability to relative ease of computation is an important aspect of reasonability.

\(^2\) See [1] for a review of (quasi-local) energy–momentum constructions in GR.
definition of quasi-local energy by considering isometric embeddings in flat spacetime, hence overcoming the problem of background independence [9, 10].

In this paper, we consider the issue of a background geometry and its seeming necessity for a reasonable definition of quasi-local energy by focusing on Penrose’s proposal for quasi-local energy in the context of Kerr–Schild spacetimes. Kerr–Schild spacetimes, which include the Kerr(–Newman) black hole and gravitational wave solutions, are a simple, yet non-trivial, class of metrics with a natural background spacetime to work on.

The Penrose charge integral [6, 11] (see also [12]) is an energy–momentum construction based on the form of the corresponding charge integral in the weak-field limit. Penrose showed that in flat spacetime, one can write the energy–momentum in some volume in terms of a surface integral involving twistors. The simplest way of understanding twistors is that they act as potentials for Killing vectors. He then generalises this construction to curved spacetime. However, because twistors do not generally exist in curved spacetime [13], the notion of twistors is replaced by the notion of a 2-surface twistor, which is a projection of the twistor equation transverse to the surface. The 2-surface twistor equation is generally hard to solve and the Penrose mass has only been calculated for certain special surfaces, such as spherically symmetric surfaces in spherically symmetric spacetimes [14, 15].

We exploit the existence of a natural Minkowski background for Kerr–Schild spacetimes as well as the relation of the Penrose charge integral to the Nester–Witten 2-form [16] to modify the original Penrose construction. In particular, we extend the Penrose construction from the spinor bundle to the bundle of linear frames. The modified Penrose quasi-local energy is then given by a basis transformation from the induced null tetrad to the Cartesian coordinate basis, which is simply the orthonormal basis of the Minkowski background associated with the Kerr–Schild spacetime. The modified Penrose energy is then equal to the integral of the Nester–Witten 2-form, evaluated on the linear frame bundle using the Cartesian coordinate basis over the surface bounding the volume of interest:

$$P_{KS}^{a} = \frac{1}{8\pi} \int_{S} W_{a}[^{g}a].$$

(1.1)

Using Stokes’ theorem, the above expression can be rewritten as a volume integral, which, using properties of the Kerr–Schild ansatz, reduces to

$$P_{KS}^{a} = \frac{1}{8\pi} \int_{V} G_{a}^{b} S_{b},$$

(1.2)

i.e. a volume integral of the Einstein tensor. The above charge integral has been proposed before for Kerr–Schild spacetimes [17–19]. In particular, it coincides with the Einstein and Landau–Lifshitz pseudotensors. However, there are also other inequivalent proposals for the Kerr–Schild quasi-local energy [20]. Furthermore, from a physical point of view, considering a Kerr–Schild spacetime as an exact perturbation on Minkowski spacetime, it would make sense that the Einstein tensor represent the gravitational energy–momentum tensor, since the Einstein tensor vanishes for Minkowski spacetime.

In addition, given the close relation between Kerr–Schild and Minkowski spacetimes, one would expect that there is a relation between solutions of the 2-surface twistor equation on a Kerr–Schild background and twistors in Minkowski spacetime. We investigate also this possibility and find that indeed for the case where the Kerr–Schild null vector is orthogonal to the surface and shear-free, there is a one-to-one correspondence between the 2-surface twistors on the Kerr–Schild background and those on Minkowski spacetime.

3 A brief description of the conventions is provided at the end of the introduction.

4 The Goldberg–Sachs theorem [21] guarantees this to be the case for vacuum Einstein solutions.
While the results in this paper are restricted to the setting of Kerr–Schild spacetimes, which have the advantage of admitting a natural background, and it is hard to see how one can generalise the construction in this paper, the result is significant in that it not only emphasises the importance of a background geometry, but also, perhaps more importantly the significance of the bundle on which the quasi-local energy is defined. The fact that the equivalence principle precludes a tensorial energy–momentum quantity makes pseudotensors, viewed as sections of a particular bundle, the only real viable option.

We begin in sections 2 and 3 by giving a brief review of Kerr–Schild metrics and the Penrose quasi-local energy, respectively. In section 4, we describe how the Penrose construction can be modified for the class of Kerr–Schild metrics, leading to a volume integral of the Einstein tensor. In section 5, we construct a 1–1 map between 2-surface twistors in Kerr–Schild backgrounds and those in Minkowski spacetime for the special case where the Kerr–Schild null vector is normal to the surface of interest. We finish in section 6 by showing that our proposed modification reduces to the ADM energy in the global case. The appendix provides a brief review of spinors and the GHP formalism, knowledge of which will be assumed in the main text of the paper.

1.1. Conventions and notation

We work with geometrised units, the metric signature is $-2$, the Riemann (curvature) tensor $R_{abc}^\ d$ is defined by
\[
(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = R_{abc}^\ d \omega_d
\]
for any derivative operator $\nabla_a$ and dual vector field $\omega_d$. The symmetrisation of a tensor $T_{ab}$ is denoted by
\[
T_{(ab)} = \frac{1}{2!} (T_{ab} + T_{ba})
\]
and the antisymmetrisation by
\[
T_{[ab]} = \frac{1}{2!} (T_{ab} - T_{ba}),
\]
with the corresponding generalisations for tensors and spinors of arbitrary valence. We use the abstract index notation [22] and denote discrete indices with bold Latin letters ($\bf{a}, \bf{b}, \ldots$) or bold Greek letters ($\alpha, \beta, \ldots$) for orthonormal frames, for example a vector basis is denoted by ($\delta^a_\ a$), where $a = 0, \ldots, 3$. Latin letters ($a, b, \ldots$) denote tangent space indices and capital Latin letters ($A, B, \ldots$) denote spinor indices. Furthermore, in sections 3 and 4, we use Greek letters ($\alpha, \beta, \ldots$) also to denote sections of the twistor bundle. It should be clear from the context what the Greek letter indices denote.

2. Kerr–Schild metrics

The class of Kerr–Schild metrics can be thought of as exact linear perturbations on flat Minkowski spacetime$^5$. Working in Cartesian coordinates $(t, x, y, z)$, the Kerr–Schild class is defined by the following line element
\[
g_{ab} = \gamma_{ab} - 2Sk_ak_b, \tag{2.1}
\]
where $\gamma_{ab}$ is the standard Minkowski line element in Cartesian coordinates and $S$ and $k$ are a scalar function and a null vector defined on the background Minkowski spacetime, respectively.

$^5$ See [23] for an introduction.
We generally focus on Kerr–Schild solutions of the Einstein equation where the matter energy–momentum tensor satisfies

\[ T_{ab} k^a k^b = 0. \]  

(2.2)

This includes vacuum Kerr–Schild spacetimes, which are sufficient for gravitational energy constructions.

Assuming that equation (2.2) is satisfied, we list some important properties of Kerr–Schild metrics [23]:

\[ k^a \nabla_a k^b = 0, \quad k^a k^c C_{abcd} = (D^2 S - R/6) k_b k_d, \]  

(2.3)

where \( D \equiv k^a \partial_a \). The first equation gives that \( k \) is an affinely parametrised geodesic, while the second equation implies that \( k \) is a multiple principal null direction of the Weyl tensor \( C_{abcd} \), and hence, the spacetime is algebraically special. Moreover, the Einstein tensor is simply given by

\[ G^a_b = 2 \eta_{ac} \partial_d \left( \partial^e \partial^d \left( S k^b k^d - S k^b k^d \right) \right). \]  

(2.4)

An important example of such a metric is that corresponding to the Kerr–Newman black hole given by

\[ S = 2MR^3 - Q^2 R^2, \]  

(2.5)

\[ (k_0, k_1, k_2, k_3) = \left( 1, \frac{Rx + ay}{R^2 + a^2}, \frac{Ry - ax}{R^2 + a^2}, \frac{z}{R} \right) \]  

with

\[ x^2 + y^2 + z^2 = R^2 + a^2 \left( 1 - \frac{z^2}{R^2} \right). \]  

(2.6)

### 3. Penrose quasi-local energy

The Penrose charge integral associated to generic 2-surfaces \( S \) of spherical topology is given by

\[ A_S[\omega^A, \lambda^A] \equiv \frac{i}{8\pi} \int_S \omega^A \lambda^B R^D_{ABCD} d\tau^A \wedge d\tau^D, \]  

(3.1)

where \( \omega^A \) and \( \lambda^B \) solve the 2-surface twistor equation

\[ \bar{\partial} \omega^0 = \sigma \omega^0, \quad \bar{\partial} \omega^1 = \sigma' \omega^1 \]  

(3.2)

for the components of the spinor fields \( \omega^A \) and \( \lambda^A \) with respect to a GHP spin frame \((\alpha^A, \iota^A)\) and

\[ R^A_{BCD} = \frac{1}{2} \tilde{R}^A_{BCD} \]  

(3.3)

with \( R_{abcd} \) the Riemann curvature tensor.

\[ ^6 \text{The construction is introduced in [6] and is thoroughly discussed in chapter 9 of [11]. For an introduction to 2-surface twistors and the compacted spin-coefficient formalism, see [11, 22].} \]
Penrose’s charge integral and the corresponding propagation law for the spinor fields can be derived by considering the weak-field limit of general relativity on Minkowski spacetime, as we will now briefly review following [1]. As for global energy constructions in the weak-field approximation and the gravitational mass in Newtonian gravity, the quasi-local energy of the gravitational field is expected to be a surface integral of the curvature associated to an extended, but finite 2-surface $S$. It should be equal to the charge integral of the non-gravitational fields acting as a source for the curvature. Thus, we expect that

$$\int_V K_a T^{ab} \Sigma_b = \frac{1}{8\pi} \int_S \omega^{AB} R_{ABcd} dx^c \wedge dx^d,$$

where $K_a$ is a Killing vector, $T^{ab}$ the (linearised) matter energy–momentum tensor, $\omega^{AB}$ an arbitrary symmetric $(2, 0)$-spinor and $\Sigma_b$ the volume 3-form. By means of the Einstein equation, the integrand on the left-hand side equals the exterior derivative of the integrand on the right-hand side iff $\omega^{AB}$ solves the valence 2 twistor equation and acts as a potential for the Killing vector $K_a$.

The above integral $A_2$ can be considered to constitute a bilinear map in $\omega^A$ and $\lambda^B$, and defines by its value the so-called angular momentum twistor $A_{\alpha\beta}$

$$A_{\alpha\beta} Z^\alpha Z^\beta \equiv i A_2[\omega^A, \lambda^B]$$

for 2-surface twistors $Z^\alpha = (\omega^A, i \Delta_{A/A} \omega^A)$ and $Z^\alpha = (\lambda^A, i \Delta_{A/A} \lambda^A)$ with $\Delta_{AA'}$ the 2-dimensional Sen operator [1, 24]. The 2-dimensional Sen operator $\Delta_a$ is simply the projection of the Levi-Civita connection to the tangent space of the 2-surface $S$,

$$\Delta_a = \Pi^b a \nabla_b,$$  

(3.6)

where $\Pi^b a$ is the projection operator onto the surface. More precisely, given an orthonormal set of timelike and spacelike vectors $(t^a, v^a)$, respectively, normal to the surface $S$,

$$\Pi^b a = \delta^b_a - t^b t^a + v^b v^a.$$  

(3.7)

The energy–momentum defined by the Penrose charge integral is an off-diagonal spinor part of the angular momentum twistor $A_{\alpha\beta}$, which can be extracted in terms of the following contraction

$$P^{\alpha \gamma} \pi_\alpha \tilde{\pi}_{\gamma} = -A_{\alpha\beta} Z^\alpha I^{\beta\gamma} Z^\gamma = A_2[i \Pi^b a \nabla_{BB'} \omega^A \wedge dx^b],$$

where

$$W[\pi_A] \equiv i \Pi^b a \nabla_{BB'} \pi_A dx^b \wedge dx^b, \quad \tilde{\pi}_{A'} = i \nabla_{AA'} \omega^A$$

(3.8)

is the Nester–Witten 2-form on the bundle of spin frames and $I^{\beta\gamma}$ is the infinity twistor, which ensures that the overall contraction with the angular momentum twistor yields the left-hand side of equation (3.8)\(^7\). The last equality in (3.8) follows from the 2-surface twistor equation [16]. The Nester–Witten 2-form can be extended to the bundle of linear frames

\(^7\) See the beginning of section 4 for a discussion of the problems associated with this construction.
by a vierbein $\theta^\alpha_a$ and the transformation matrix $A^a_\alpha$ relating the induced null basis to the Minkowski basis according to

$$
\epsilon^A_\alpha \otimes n^a_\alpha \rightarrow \theta^\alpha_a \rightarrow \delta^a_\alpha,
$$

(3.10)

where $(\epsilon^A_\alpha)$ is a spin frame, $(n^a_\alpha)$ the null basis obtained by the tensor product of $(\epsilon^A_\alpha)$ with itself, $(\theta^\alpha_a)$ the canonical Minkowski basis given by a constant linear combination of the induced null basis and $(\delta^a_\alpha)$ the Cartesian coordinate basis. Thus, considering the spin frame $\epsilon^A_\alpha$ with $\epsilon^A_\alpha = \pi_A$, the 00-component of the energy–momentum is given by the Nester–Witten integral

$$
P^{00} = P^{\Lambda^\alpha A_\alpha} \pi_A \bar{\pi} A_\alpha'
$$

$$
= \frac{1}{8\pi} A^\alpha_\alpha \int_S \theta^\mu A_\alpha W_a[\delta^\nu_a],
$$

where

$$
W_a[\delta^\nu_a] = \frac{1}{2} \epsilon_{abcd} d^d b \wedge \omega^{cd}
$$

(3.11)

is the extension of the Nester–Witten 2-form $W[\pi_A]$ to the bundle of linear frames with $\omega^{cd} = \omega^c_\beta d \omega^b_\beta$ and $\omega^\mu_\nu$ being the spin connections defined according to $\nabla^\mu \theta^\nu = \omega^\mu_\nu \theta^\nu$. We denote the inverse transformations of $A^a_\alpha$ and $\theta^\alpha_a$ by $A^\alpha_a$ and $\theta^a_\alpha$, respectively.

4. Modifying the Penrose energy

The contraction $P^{\Lambda^\alpha A_\alpha} \pi_A \bar{\pi} A_\alpha'$ is expected to be constant on the surface $S$ as can be seen from its definition in terms of the original Penrose charge integral, see equation (3.8). This constancy as well as other issues such as the existence of an infinity twistor and hence, the well-definedness of the overall contraction $P^{\Lambda^\alpha A_\alpha} \pi_A \bar{\pi} A_\alpha'$ may neither be constant on $S$ nor extractable from $A^a_\alpha \theta^\alpha_a$ at all. Therefore, he proposes to consider the Nester–Witten 2-form integral directly as the defining value of the contraction

$$
P^{\Lambda^\alpha A_\alpha} \pi_A \bar{\pi} A_\alpha' \equiv \frac{1}{4\pi} \int_S W[\pi_A].
$$

(4.1)

For the following discussion, we assume that the original Penrose construction is well-defined for Kerr–Schild spacetimes or, alternatively, adopt the suggestion of Helfer and consider the Nester–Witten 2-form integral evaluated on the bundle of spin frames as the defining value of the contraction $P^{\Lambda^\alpha A_\alpha} \pi_A \bar{\pi} A_\alpha'$. We discuss 2-surface twistors on Kerr–Schild spacetimes in section 5.

By the assumed constancy, we can write $P^{00}$ as

$$
P^{00} = \frac{1}{\text{area}(S)} \int_S dS P^{\Lambda^\alpha A_\alpha} n^a_\alpha
$$

$$
= \frac{1}{8\pi} A^\alpha_\alpha \int_S \theta^\mu A_\alpha W_a[\delta^\nu_a]
$$

(4.2)

8 We denote the discrete index of the induced Minkowski basis by Greek letters in order to distinguish the membership of the different indices to the different bases.
and interpret the integrand as the 0-component of the energy–momentum surface density with respect to the induced null tetrad with \( n^a_0 = \pi_A \pi_{\bar{A}} \). Therefore, the quasi-local energy–momentum with respect to the Cartesian coordinate basis is given by

\[
P_{\pi a}^{KS} \equiv \frac{1}{\text{area}(S)} \int_S dS g_{ab} P^{\pi A'} \delta^b_a. \tag{4.3}
\]

This integral can be related to the above \( P^0 \) as follows. The 2-surface twistor equation has generically exactly four linearly independent solutions for surfaces of spherical topology [26] (see, however, [27]). These four solutions define four spin frames with \( \epsilon^A_0 = \pi_A \) and four induced null frames \( n^a_i \). Thus, we have four integrals of the type (4.2), denoted by

\[
P^0_i \equiv \frac{1}{\text{area}(S)} \int_S dS P^{\pi i A'} n^a_0 \tag{4.4}
\]

and obtain a new basis by combining the 0th basis vectors of these null frames \( n^a_i \). These vectors are indeed linearly independent over \( \mathbb{R} \), since two of them are linearly independent over \( \mathbb{C} \). The coordinate basis can be rewritten in terms of this new basis via a basis transformation

\[
\delta^a_\alpha = \sum_i D^a_i n^0_i, \tag{4.5}
\]

which provides a relation between \( P_{\pi a}^{KS} \) and the integrals \( P^0 \). First, note that if we expand the above basis transformation in terms of the induced Minkowski tetrads and the corresponding vierbein followed by the transformation to the coordinate basis, we obtain

\[
\delta^a_\alpha = \sum_i D^a_i n^0_i = \sum_i D^a_i A^\alpha_3 g_{bc} \theta^a_i W^b_c[\delta^\alpha_\alpha].
\]

A comparison of the coefficients of the above expansions yields

\[
\sum_i D^a_i A^\alpha_3 \theta^b_i = g^{ab}. \tag{4.6}
\]

Therefore, from equations (4.3)–(4.6) we obtain

\[
P_{\pi a}^{KS} = \frac{1}{8\pi} \int_S g_{ac} \sum_i D^c_i A^\alpha_3 \theta^b_i W^b_c[\delta^\alpha_\alpha] = \frac{1}{8\pi} \int_S W^a_c[\delta^\alpha_\alpha]. \tag{4.7}
\]

Stokes’ theorem allows us to rewrite the above surface integral in terms of a volume integral

\[
P_{\pi a}^{KS} = \frac{1}{8\pi} \int_V dW^a_c[\delta^\alpha_\alpha] \tag{4.8}
\]

and as we find below this turns out to be a particularly interesting quantity.

Note that the Sparling equation [28–30] gives a beautiful relation between the exterior derivative of the Nester–Witten 2-form and the Einstein tensor.
\[ dW_\alpha[\theta^\alpha_a] = -G_\alpha^\beta \Sigma_\beta - S_\alpha[\theta^\alpha_a], \]  

(4.9)

where \( S_\alpha[\theta^\alpha_a] \) is the Sparling 3-form and

\[
\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma = \epsilon^{\alpha\beta\gamma\delta} \Sigma_\delta.
\]

(4.10)

However, the above equation (4.9) is clearly only valid on the bundle of orthonormal frames.

In what follows, we calculate the Nester–Witten 2-form and its exterior derivative for the Kerr–Schild class on the bundle of linear frames. We express the vierbein relating the orthonormal frame \( (\theta^\alpha_a) \) induced by the 2-surface twistor equation to the coordinate basis \( (\delta^a_a) \), such that \( g_{ab} = \theta^\alpha_a \theta^\beta_b \eta_{\alpha\beta} \), as (see [31])

\[
\theta^\alpha_a = e^\alpha_a - S k^\alpha_a,
\]

(4.11)

where \( k^\alpha = e^\alpha_k a^k \) and \( e^\alpha_a \) is the vierbein connecting the induced orthonormal frame to the background Minkowski basis. Expanding the torsion-free, metric compatible covariant derivative of an arbitrary vector with respect to the orthonormal and the coordinate basis yields the following relation between the spin connections and the vierbein

\[
\omega^\alpha_a \beta = \theta^\beta_b \partial^b (2 \theta^\alpha_a \theta^\beta_b) + \theta^\alpha_a \partial^b \theta^\beta_b \gamma.
\]

(4.12)

The evaluation of equation (4.12) for the vierbein given by (4.11) leads to

\[
\omega^\alpha_a \beta = 2 \epsilon_{a}^{b} \epsilon_{b}^{\alpha} \partial^b (S k^b \delta^\alpha)
\]

(4.13)

for the spin connection of Kerr–Schild metrics. This can be used to evaluate the Nester–Witten 2-form given by (3.11). The final expression is

\[
W_a[\delta^a_a] = \frac{1}{4} \Sigma_{ab} \partial_b (g^{ce} g^{bd} - g^{ed} g^{bc}) \Sigma_{cd},
\]

(4.14)

where \( \Sigma_{ab} = \frac{1}{2} \epsilon_{abcd} dx^c \wedge dx^d \). The exterior derivative of (4.14) turns out to be the Einstein tensor (see equation (2.4)) contracted with the volume 3-form \( \Sigma_a = \frac{1}{2} \epsilon_{abcd} dx^b \wedge dx^c \wedge dx^d \) for Kerr–Schild solutions of the Einstein equation where the matter energy–momentum tensor satisfies \( T_{ab} \delta^a \delta^b = 0 \), i.e.

\[
dW_a[\delta^a_a] = G_a^b \Sigma^b_b.
\]

(4.15)

Substituting the above equation into equation (4.8) gives that the quasi-local energy–momentum of Kerr–Schild spacetimes given by (4.7) is the Einstein tensor integral

\[
p^{KS}_a = \frac{1}{8 \pi} \int_V G_a^b \Sigma^b_b.
\]

(4.16)

While the original Penrose charge integral was only defined for the generic surfaces of spherical topology allowing for a 4 complex-dimensional 2-surface twistor space, the Einstein tensor integral is defined for arbitrary volumes in Kerr–Schild spacetimes. Thus, our construction may be evaluated for arbitrary surfaces.

---

9 In [30], the Sparling equation has been used directly on the bundle of linear frames leading them to conclude that the Sparling form vanishes for Kerr–Schild metrics. When computed on the bundle of orthonormal frames, it is clear that the Sparling form is non-zero.
5. 2-surface twistors on Kerr–Schild spacetimes

In this section, we would like to address some questions associated with the 2-surface twistor space for Kerr–Schild spacetimes.

For simplicity, we consider 2-surfaces $S$ in Kerr–Schild spacetimes for which the geodesic null vector $k$ is a normal vector. In this setting, the spin frame $(\kappa^A, \iota^A)$ with $k^a = \kappa^A_k\bar{\kappa}_a^A$ is a GHP frame on $S$ and induces a null tetrad of the form $(k^a, n^a, m^a, \bar{m}^a)$. Thus, the Kerr–Schild line element can be expressed as

$$g_{ab} = \eta_{ab} - 2Sk_a k_b = 2k_a(n_b - 2m_a\bar{m}_b).$$

Rearranging the above equation then gives a null tetrad for the Minkowski background

$$\eta_{ab} = g_{ab} + 2Sk_a k_b = 2k_a(n_b + Sk_b) - 2m_a\bar{m}_b.$$

Hence, the tetrad $(k^a, n^a - Sk^a, m^a, \bar{m}^a)$ is a null tetrad with respect to the Minkowski background with $k$ normal to $S$. The spin coefficients appearing in the 2-surface twistor equation with respect to the GHP frame $(\kappa^A, \iota^A)$ are

$$\sigma = m^a\delta k_a, \quad \sigma' = \bar{m}^a\delta' n_a, \quad \beta = \frac{1}{2}(n^a\delta k_a + m^a\delta\bar{m}_a), \quad \beta' = \frac{1}{2}(k^a\delta' n_a + \bar{m}^a\delta' m_a),$$

where $\delta = m^a\nabla_a$ and $\delta' = \bar{m}^a\nabla_a$, see e.g. [11, 22]. Due to the geodesic and null properties of $k$, the only change in the above spin coefficients due to a change of basis from $(k^a, n^a, m^a, \bar{m}^a)$ to $(k^a, n^a - Sk^a, m^a, \bar{m}^a)$ occurs in $\sigma'$, which transforms as

$$\sigma' \rightarrow \sigma' + S\bar{\sigma}.$$

From section 2 and equation (2.3) in particular, we know that aligned Kerr–Schild spacetimes are algebraically special. Therefore, the Goldberg–Sachs theorem guarantees that for vacuum Kerr–Schild spacetimes, the null vector $k$ is shear-free and hence, $\sigma = 0$. More generally, if the congruence of $k$ is indeed shear-free, the 2-surface twistor equation for the components of the spinor solution with respect to $(\kappa^A, \iota^A)$ are the same as the 2-surface twistor equation for the components on the Minkowski background with respect to the spin frame with induced null tetrad $(k^a, n^a - Sk^a, m^a, \bar{m}^a)$. Since the latter are simply tangential projections of the twistor equation on the Minkowski background, we find a one-to-one correspondence between the 2-surface twistors for surfaces with normal $k$ and the background Minkowski twistors for Kerr–Schild spacetimes. For the case where the congruence is shearing, we know that the solution is of Petrov type N, see section 32.4.2 of [23].

6. Relation to pseudotensors and ADM energy

The notion that the Einstein tensor acts as an energy–momentum density for Kerr–Schild spacetimes is supported by other energy–momentum constructions.

The energy–momentum pseudotensors of Einstein, Landau and Lifshitz, Tolman, Papapetrou, and Weinberg are proportional to the Einstein tensor with respect to the Cartesian coordinates of Kerr–Schild spacetimes [19].
Furthermore, the quantity $P_{KS}^0$ reproduces the ADM energy for asymptotically flat Kerr–Schild spacetimes in the appropriate limit. The ADM energy is the numerical value of the Hamiltonian surface integral of linearised gravity [32] and is given by

$$E_{ADM} = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S^2_r} d^2S n_i (\partial_j h_{ij} - \partial_i h_{ij}),$$

(6.1)

where $h_{ij}$ is the linear perturbation from the flat (spatial) metric in terms of asymptotically flat Euclidean coordinates, the summation over the spatial indices $i$ and $j$ runs from 1 to 3 and the vector $n^i$ is the unit outward normal of the 2-sphere $S^2_r$ with asymptotically Euclidean radius $r$. Considering asymptotically flat Kerr–Schild spacetimes, where the components of the Kerr–Schild deviation in terms of the Cartesian coordinate basis fall off sufficiently fast at large radii $r = \sqrt{x^i x^i}$ and a potential time dependence at spatial infinity does not contribute to the final integral, equations (2.4) and (4.16) lead to

$$P_{KS}^0 = -\frac{1}{8\pi} \int_{S^2} d^2S n_i (\partial_j (Sk_{ij} - \partial_i (Sk_{ij}))).$$

(6.2)

The deviation from the induced spatial metric of the asymptotically flat Kerr–Schild spacetime is $h_{ij} = -2Sk_{ij}$, thus from equations (6.1) and (6.2) we find

$$P_{KS}^0 = E_{ADM}|h_{ij} = -2Sk_{ij}.$$  

(6.3)

Therefore, the Einstein tensor integral satisfies the correct large sphere behaviour at spatial infinity for asymptotically flat Kerr–Schild spacetimes by reproducing the ADM energy.

**Acknowledgments**

MG is partially supported by grant no. 615203 from the European Research Council under the FP7.

**Appendix. Spinors and the Geroch–Held–Penrose (GHP) formalism**

Spinors are defined as sections of the spinor bundle whose fibre at a point is a 2-dimensional complex vector space, called the spin space. There exists a unique inner product on this space given by

$$\{\kappa, \omega\} = \epsilon_{AB} \kappa^A \omega^B,$$

(A.1)

where $\epsilon_{AB}$ is an antisymmetric spinor such that

$$\epsilon_{AC} e^B = \delta_A^B.$$

(A.2)

Equivalently, $\epsilon_{AB}$ and $\epsilon_{B}^A$ can be used to lower and raise spinor indices, respectively. Thus\textsuperscript{12},

$$\kappa_A = \kappa^B \epsilon_{BA}, \quad \kappa^A = e^B \kappa_B.$$  

(A.3)

We may choose a GHP [33, 34] spin frame $(\sigma^A, \epsilon^A)$, i.e. a basis with the spinors normalised such that

$$\sigma_A \epsilon^A = 1.$$  

(A.4)

\textsuperscript{12}This convention for the product of spinors is known as the ‘North East–South West’ convention.
The spinor map, or the isomorphism between the proper, orthochronous Lorentz group $\text{SO}^+(1,3)$ and $\text{SL}(2,\mathbb{C})$, allows one to write vectors as bilinears of spinors. Hence, we identify tangent bundle indices with spinor indices
\[ a = AA', \ldots \] (A.5)

The metric
\[ g_{ab} = \epsilon_{AB} \epsilon_{A'B'} \] (A.6)

and we construct a complex null frame $(\ell^a, n^a, m^a, \bar{m}^a)$ via the following identification
\[ \ell^a = \theta^A \partial^A, \quad n^a = \iota^A \partial^A, \quad m^a = \bar{\theta}^A \partial^A, \quad \bar{m}^a = \bar{\iota}^A \partial^A, \] (A.7)

where, for example, $\bar{\theta}^A$ is the complex conjugate of $\theta^A$. Equations (A.4) and (A.6) then imply that the only non-trivial inner products between the null frame vectors are
\[ \ell^a n_a = 1, \quad m^a \bar{m}_a = -1 \] (A.8)

and that
\[ g_{ab} = \ell_a n_b + n_a \ell_b - m_a \bar{m}_b - \bar{m}_a m_b. \] (A.9)

Using such a Newman–Penrose [33] or GHP [34] frame, one can construct scalars by taking the components of tensors along the null frame directions. For example, for a covector $V^a$,
\[ V^0 \equiv \ell^a \ell_a, \quad V^1 \equiv n^a \ell_a, \quad V^m \equiv m^a \ell_a, \quad V_{\bar{m}} \equiv \bar{m}^a \ell_a. \] (A.10)

Taking the components of the covariant derivatives of the frame vectors along the null frame gives twelve independent complex spin coefficients. For example, for some components of the derivative of $\ell_a$, we have
\[ \kappa = m^b \nabla_b \ell_a, \quad \sigma = m^b m^d \nabla_b \ell_d, \quad \rho = m^b \bar{m}_b \nabla_a \ell^b, \quad \tau = m^b n^d \nabla_a \ell_d. \] (A.11)

Physically, the above scalars parametrise the geodesity, shear, expansion and twist, respectively, of the null congruence of $\ell^a$. Defining a prime operation so that
\[ \ell^a \rightarrow \lambda \ell^a, \quad n^a \rightarrow \lambda^{-1} n^a, \quad m^a \rightarrow \lambda m^a, \quad \bar{m}^a \rightarrow \lambda^{-1} \bar{m}^a, \] (A.12)

\[ \kappa', \sigma', \rho' \text{ and } \tau' \text{ parametrise the geodesity, shear, expansion and twist, respectively, of the null congruence of } n^a. \]

All the eight spin coefficients defined above are GHP scalars in the following sense: a GHP scalar $\eta$ of spin weight $\{p, q\}$ is defined as a scalar such that under the following transformation
\[ \theta^A \rightarrow \lambda \theta^A, \quad \ell^A \rightarrow \lambda^{-1} \ell^A, \] (A.13)

for some arbitrary complex scalar $\lambda$,
\[ \eta \rightarrow \lambda^p \bar{\lambda}^q \eta. \] (A.14)

For example, $\kappa$ is a GHP scalar of weight $\{3, 1\}$, as can be easily verified from its definition in equation (A.11).

Of the twelve independent complex spin coefficients, the remaining four
\[ \beta = \frac{1}{2} (n^b m^d \nabla_b \ell_d + m^b m^d \nabla_b \ell_d), \quad \epsilon = \frac{1}{2} (n^b \ell^d \nabla_b \ell_d + m^b \ell^d \nabla_b \ell_d). \] (A.15)

Equivalently,
\[ (\theta^A)' = i \theta^A, \quad (\ell^A)' = i \ell^A. \] (11)
as well as $\beta'$ and $\epsilon'$ are not GHP scalars. These spin coefficients can be used to define ‘GHP covariant’ derivative operators acting on a GHP scalar $\eta$ of spin weight $\{p,q\}$ as follows:

$$\nabla_{\alpha} \eta = (\ell^a \nabla_a - p\epsilon - q\bar{\epsilon}) \eta, \quad \nabla^{\alpha} \eta = (m^a \nabla_a + p\epsilon' + q\bar{\epsilon}') \eta,$$

$$\partial \eta = (m^a \nabla_a - p\bar{\beta} + q\bar{\beta}') \eta, \quad \partial^{\alpha} \eta = (m^a \nabla_a + p\beta' - q\beta) \eta.$$  \hfill (A.16)

The spin weights of the derivative operators are

$$\nabla : \{1,1\}, \quad \nabla^{\alpha} : \{-1,-1\}, \quad \partial : \{1,-1\}, \quad \partial^{\alpha} : \{-1,1\}. \quad (A.17)$$

**ORCID iDs**

Mahdi Godazgar \(https://orcid.org/0000-0001-8926-7745\)

**References**

[1] Szabados L B 2004 Quasi-local energy-momentum and angular momentum in GR: a review article Living Rev. Relativ. **7** 4
[2] Einstein A 1916 Die Grundlage der allgemeinen relativitätstheorie Ann. Phys. **49** 769–822
[3] Landau L D and Lifschitz E M (ed) 1975 The gravitational field equations The Classical Theory of Fields (Course of Theoretical Physics vol 2) 4th edn (Amsterdam: Pergamon) ch 11, pp 259–94
[4] Rosen N 1985 Localization of gravitational energy Found. Phys. **15** 997–1008
[5] Hawking S 1968 Gravitational radiation in an expanding universe J. Math. Phys. **9** 598–604
[6] Penrose R 1982 Quasilocal mass and angular momentum in general relativity Proc. R. Soc. A **381** 53–63
[7] Bartnik R 1989 New definition of quasilocal mass Phys. Rev. Lett. **62** 2346–8
[8] Brown J D and York J W Jr 1993 Quasilocal energy and conserved charges derived from the gravitational action Phys. Rev. D **47** 1407–19
[9] Wang M-T and Yau S-T 2009 Quasilocal mass in general relativity Phys. Rev. Lett. **102** 021101
[10] Wang M-T and Yau S-T 2009 Isometric embeddings into the Minkowski space and new quasi-local mass Commun. Math. Phys. **288** 919–42
[11] Penrose R and Rindler W 1988 Spinors, Space-Time. Vol 2: Spinor and Twistor Methods in Space-Time Geometry (Cambridge: Cambridge University Press)
[12] Tod K 1990 Penrose’s quasi-local mass Twistors in Mathematics and Physics (London Mathematical Society Lecture Note Series (Book 156)) ed T Bailey and R Baston (Cambridge: Cambridge University Press) pp 164–88
[13] Lewandowski J 1991 Twistor equation in a curved space-time Class. Quantum Grav. **8** L11–8
[14] Tod K P 1983 Some examples of Penrose’s quasi-local mass construction Proc. R. Soc. A **388** 457–77
[15] Tod K P 1986 More on Penrose’s quasi-local mass Class. Quantum Grav. **3** 1169
[16] Szabados L B 1994 Two-dimensional Sen connections and quasilocal energy momentum Class. Quantum Grav. **11** 1847–66
[17] Gurses M and Feza G 1975 Lorentz covariant treatment of the Kerr–Schild metric J. Math. Phys. **16** 2385
[18] Virbhadra K S 1990 Energy associated with a Kerr–Newman black hole Phys. Rev. D **41** 1086–90
[19] Aguirregabiria J M, Chamorro A and Virbhadra K S 1996 Energy and angular momentum of charged rotating black holes Gen. Relativ. Gravit. **28** 1393–400
[20] Xulu S S 2000 Moller energy for the Kerr–Newman metric Mod. Phys. Lett. **A15** 1511–7
[21] Goldberg J and Sachs R 2009 Gen. Relativ. Gravit. **41** 43 (republication of below)
Goldberg J and Sachs R 1962 A theorem on Petrov types Acta Phys. Pol. **22** 13
[22] Penrose R and Rindler W 1986 Spinors, Space-Time. Vol 1: Two-Spinor Calculus and Relativistic Fields (Cambridge: Cambridge University Press)
[23] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)

[24] Szabados L B 1994 Two-dimensional Sen connections in general relativity Class. Quantum Grav. 11 1833–46

[25] Helfer A D 1992 Difficulties with quasilocal momentum and angular momentum Class. Quantum Grav. 9 1001

[26] Baston R 1984 The index of the 2-twistor equations Twistor Newsl. 17 31–2

[27] Jeffryes B P 1986 ‘Extra’ solutions to the 2-surface twistor equations Class. Quantum Grav. 3 L9

[28] Sparling G 1983 Einstein’s vacuum equations Univ. Pittsburg

[29] Sparling G 1984 Twistor theory and the characterization of Fefferman’s conformal structures Univ. Pittsburg

[30] Mason L and Frauendiener J 1990 The Sparling 3-form, Ashtekar variables and quasi-local mass Math. Soc. Lect. Notes 156 189

[31] Godazgar M 2017 Entanglement entropy, the Einstein equation and the Sparling construction J. High Energy Phys. JHEP09(2017)108

[32] Arnowitt R L, Deser S and Misner C W 2008 The dynamics of general relativity Gen. Relativ. Gravit. 40 1997–2027

[33] Newman E and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients J. Math. Phys. 3 566–78

[34] Geroch R P, Held A and Penrose R 1973 A space-time calculus based on pairs of null directions J. Math. Phys. 14 874–81