On an exponential sum related to the Möbius function

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Abstract Let \( \mu(n) \) be the Möbius function and \( e(\alpha) = e^{2\pi i \alpha} \). In this paper, we study upper bounds of the classical sum

\[
S(x, \alpha) := \sum_{1 \leq n \leq x} \mu(n)e(\alpha n).
\]

We can improve some classical results of Baker and Harman [1].

Keywords exponential sums, zero density estimates, zeta functions

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1. Introduction

Let \( S(x, \alpha) = \sum_{1 \leq n \leq x} \mu(n)e(\alpha n) \). Davenport [2] proved that

\[
\max_{\alpha \in [0,1]} |S(x, \alpha)| \ll_A x(\log x)^{-A}
\]

for any \( A > 0 \). Hajela and Smith [3] gave the following conditional bounds for \( S(x, \alpha) \).

Suppose that for every Dirichlet character \( \chi \pmod{l} \), \( L(s, \chi) \) has no zeros in the half-plane \( \sigma > a \). Then

\[
\max_{\alpha \in [0,1]} |S(x, \alpha)| \ll x^{(a+2)/3+\varepsilon}
\]

for every \( \varepsilon > 0 \). Subsequently, Baker and Harman [1] sharpen the estimate to

\[
\max_{\alpha \in [0,1]} |S(x, \alpha)| \ll x^{b+\varepsilon},
\]

where

\[
b = \begin{cases} 
  a + 1/4 & \text{for } 1/2 \leq a < 11/20, \\
  4/5 & \text{for } 11/20 \leq a < 3/5, \\
  (a + 1)/2 & \text{for } 3/5 \leq a < 1. 
\end{cases}
\]

In proving the above bound, in [1], the authors established that for any rational number \( r/q \) with \( (r, q) = 1 \),

\[
S(x, \alpha) \ll x^{a+\varepsilon}q^{1/2}(1 + x|\alpha - r/q|)^{1/2}.
\]

The corresponding estimate made implicit by Hajela and Smith [3] is

\[
S(x, \alpha) \ll x^{a+\varepsilon}q^{1/2}(1 + x|\alpha - r/q|).
\]

And the improvement comes by exploiting cancellations that occur in auxiliary sums and integrals.

The aim of this paper is to give further improvement of the above bound for certain range of \( a \).
The idea is as follows: the weak Generalized Riemann Hypothesis that all Dirichlet 
L-functions \( L(s, \chi) \) have no zeros in the half-plane \( \sigma > a \) with \( a \in [1/2, 1) \) and the zero density results of Ingham \cite{4} imply that
\[
\sum_{\chi \pmod{l}} N(\sigma, T, \chi) \ll (lT)^{(3-3\sigma)/(2-\sigma)+\epsilon},
\]
for \( 1/2 \leq \sigma < a \) and 0 for \( a \leq \sigma < 1 \), where
\[
N(\sigma, T, \chi) := \sum_{\substack{L(s, \chi) = 0 \\
\sigma \leq \beta \leq 1, \ |\gamma| \leq T}} 1.
\]
Hence, by using such a basic observation, we can obtain the following result. We can improve the classical result of Baker and Harman \cite{1} for \( 1/2 < a < 4/7 \).

**Theorem 1.1.** Let \( l \) be a positive integer and \( \alpha \) be a real number. Suppose that for every Dirichlet character \( \chi \pmod{l} \), \( L(s, \chi) \) has no zeros in the half-plane \( \sigma > a \). Then with the notions above, we have
\[
\max_{\alpha \in [0,1]} |S(x, \alpha)| \ll x^{b+\epsilon},
\]
where \( a \in [1/2, 4/7] \) and
\[
b = \frac{8a - 7a^2}{4 - 2a}.
\]

**Remark 1.** For example, for \( 1/2 \leq a \leq 4/7 \), we have
\[
1/4 + a \geq \frac{8a - 7a^2}{4 - 2a}
\]
with equality happening if and only if \( a = 1/2 \), and
\[
4/5 \geq \frac{8a - 7a^2}{4 - 2a},
\]
with equality happening if and only if \( a = 4/7 \). Hence we have a much better result.

On the other hand, our idea also implies the following conditional result.

**Theorem 1.2.** Let \( q \) be a positive integer and \( \alpha \) be a real number. Suppose that for every Dirichlet character \( \chi \pmod{l} \), \( L(s, \chi) \) has no zeros in the half-plane \( \sigma > a \). Moreover, we also assume that
\[
\sum_{\chi \pmod{l}} N(\sigma, T, \chi) \ll (lT)^{2-2\sigma+\epsilon},
\]
where
\[
N(\sigma, T, \chi) := \sum_{\substack{L(s, \chi) = 0 \\
\sigma \leq \beta \leq 1, \ |\gamma| \leq T}} 1.
\]
Then with the notions above, we have
\[
\max_{\alpha \in [0,1]} |S(x, \alpha)| \ll x^{b+\epsilon},
\]
where
\[
b = \frac{1 + a}{2}.
\]
2. Proof of the main results

Before the proof we will quote the following upper bound of $S(x,\alpha)$, which is related to the zero density results and has been used to deal with the cases of almost all in $[5]$.

**Lemma 2.1** (See [5]). Let $q$ be a positive integer and $\alpha$ be a real number. Then for any rational number $r/q$ with $(r, q) = 1$ and for any fixed $\theta$ satisfying $1/2 + \delta \leq \theta \leq 1 - \delta$ (with $\delta$ being any small positive constant), we have

$$S(x,\alpha) \ll q^{-1/2}x^{1+\varepsilon} + \sum_{d|q} (q/d)^{1/2}(\phi(q/d))^{-1}(1 + x|\alpha - r/q|)$$

$$\times \max_{1 \leq T \leq (x/q)^\delta} T^{-1}(qT)^\varepsilon \int_{1/2}^1 \left(T\phi(q/d)(x/d)^{\theta+\varepsilon} + (x/d)^{\sigma+\varepsilon} \sum_{\chi \pmod{q/d}} N(\sigma, T, \chi_d)\right) d\sigma,$$

where $\phi(n)$ is the Euler function and

$$N(\sigma, T, \chi) := \sum_{\sigma \leq \beta \leq 1, \ 10 \leq |\gamma| \leq T} 1.$$

On the other hand, we also need the following result of Baker and Harman [1].

**Lemma 2.2** (See [1]). Suppose that for every Dirichlet character $\chi \pmod{l}$, $L(s, \chi)$ has no zeros in the half-plane $\sigma > a$. Then with the notions above, for any rational number $r/q$ with $(r, q) = 1$,

$$S(x,\alpha) \ll x^{a+\varepsilon}q^{1/2}(1 + x|\alpha - r/q|)^{1/2}.$$

We now apply Dirichlet’s theorem to obtain, for any real number $\alpha$, a rational number $r/q$ with $(r, q) = 1$, $1 \leq q \leq x^a$ such that

$$\left|\alpha - \frac{r}{q}\right| < \frac{1}{qx^a}.$$

By Lemma 2.2, for $q \leq x^{1-a}$, we have

$$S(x,\alpha) \ll x^{a+\varepsilon}q^{1/2}(1 + x|\alpha - r/q|)^{1/2} \ll x^{(1+a)/2+\varepsilon}.$$

Then by Lemma 2.1 with $\theta = 1/2 + \varepsilon$, the hypothesis such that for every Dirichlet character $\chi \pmod{l}$, $L(s, \chi)$ has no zeros in the half-plane $\sigma > a$, we have

$$S(x,\alpha) \ll q^{-1/2}x^{1+\varepsilon} + x^{1/2+\varepsilon}q^{1/2}(1 + x|\alpha - r/q|)$$

$$\quad + q^{-1/2}x^{a+\varepsilon}q^{3(1-a)/(2-a)}(1 + x|\alpha - r/q|).$$

It is worth pointing out that by the hypothesis, in Lemma 2.1, the range of integration for the second term inside the maximum over $T$ is reduced to $[1/2, a]$ (for the possible non-zeros in $N(\sigma, T, \chi_d)$, Ingham’s zero density result (1.1) is applied).

Then for $x^{1-a} \leq q \leq x^a$, we have

$$x|\alpha - r/q| \leq 1$$

and

$$x^{1/2+\varepsilon}q^{1/2} + x^{1+\varepsilon}q^{-1/2} \ll x^{(1+a)/2+\varepsilon}.$$
Hence, for \(a \in [1/2, 3/4]\), we can obtain
\[
S(x, \alpha) \ll x^{1/2+\varepsilon} q^{1/2} + x^{1+\varepsilon} q^{-1/2} + q^{-1/2} x^{a+\varepsilon} q^{3(1-a)/(2-a)}
\ll x^{(8a-7a^2)/(4-2a)+\varepsilon}.
\]
This completes the proof.

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