V.M. Red’kov

Dirac-Kähler field, spinor technique, and 2-potential approach to electrodynamics with two charges

B.I. Stepanov Institute of Physics
National Academy of Sciences of Belarus
68 Nezavisimosti av., Minsk, 220072 BELARUS
redkov@dragon.bas-net.by

From the 16-component Dirac-Kähler field theory, spinor equations for two types of massless vector photon fields with different parities have been derived. Their equivalent tensor equations in terms of the strength tensor $F_{ab}$ and respective 4-vector $A_b$ and 4-pseudovector $\tilde{A}_b$ depending on intrinsic photon parity are derived; they include additional sources, electric 4-vector $j_b$ and magnetic 4-pseudovector $\tilde{j}_b$. The theories of two types of photon fields are explicitly uncoupled, their linear combination through summing or subtracting results in Maxwell electrodynamics with electric and magnetic charges in 2-potential approach. So the problem of existence of magnetic charge can be understood as a super selection rule for different photon fields in intrinsic parity. The whole analysis is extended straightforwardly to a curved space-time background. In the frames of that extended Maxwell theory, the known electromagnetic duality is described as a linear transformation mixing the field variables referred to photons with different parities. That extended dual transformation concerns both strength tensors and 4-potentials $A_b, \tilde{A}_b$.

Pacs: 02.20, 03.50.De, 03.65.Fd, 03.65.Pm

Keywords: Maxwell equations, Dirac-Kähler field, spinors, intrinsic parity, magnetic charge, 2-potential approach, extended electrodynamics, duality symmetry

1 Introduction

The bibliography on Maxwell electrodynamics is enormous, the subject attracts attention up to the present time:

Lorentz [1], Poincaré [2], Einstein [3], Silberstein [4], Minkowski [5], Abraham [6], Bateman [7], Cunningham [8], Lanczos [9], Lewis [10], Marcolongo [11], Gordon [12], Tamm – Mandelstam [13], Rainich [14], Tamm [15], Uhlenbeck – Laporte [16], Juvet [17], Abraham – Becker [18],[19], Frenkel [20], Mercier [21],[24], Rumer [22], Stratton [23], Rozen [25], Gursey [26], Gupta [27], Lichnerowicz [28], Novacu [29], Borgardt [30],[31], Moses [32],[33], Panošky – Phillips [34], Post [35], Rosen [36],[38], Lipkin [37], Ellis [38], Penney [39], Zeldovich [?], Candelin [40], Strazhev – Tomil’chik [41],[43], 44], 46], 47], 53], 57], Exton – Newman – Penrose [42], Carmeli [43], Pestov [45], Landau-Lifshitz [49], Weingarten [50], Newman [51], Mignani – Recami – Baldo [52], Jackson [54], Rosen [55], Lipkin [56], Edmonds [58], Silveira [59], Venuri [60], Chow [61], Cook [62],[62], Giannetto [65], Yépez – Brito – Vargas [66], Kidd – Ardini – Anton [67], Recami [68], Krivsky – Simulik [69], Inagaki [70], Białynicki-Birula [71],[72],[73], Sipe [74], Ghose [75], Gersten [76], Esposito [76], Dvoeglazov [77], 79], Kruglov [81], Gsponer [82], Kravchenko [83], Varlamov [86], Ivecić [80], 83], [85],[82], [92],[93], [94],[96], Donev – Tashkova [89],[90],[91], Armour [88].
Inverse to (1) has the form
\[ \Psi = x \text{rang bispinor}, \text{or equivalent set of tensor fields:} \]

In the Minkowski space, the Dirac-Kähler particle is described by a 16-component wave function, a 2-rang bispinor, or equivalent set of tensor fields:

\[ U(x), \quad \text{or} \quad \{ \Phi(x), \Phi_i(x), \Phi(x), \Phi_i(x), \Phi_{mn}(x) \} ; \]

\( \Phi(x) \) is a scalar, \( \Phi_i(x) \) is a vector, \( \Phi(x) \) represents a pseudoscalar, \( \Phi_i(x) \) represents a pseudovector, \( \Psi_{mn}(x) \) is an antisymmetric tensor. To specify connection between 2-rank bispinor \( U(x) \) and the tensorial set, let us introduce parametrization of \( U(x) \) according to

\[ U(x) = \left[ -i \Phi(x) + \gamma^I \Phi_I(x) + i \sigma^{mn} \Phi_{mn}(x) + \gamma^5 \Phi(x) + i \gamma^I \gamma^5 \Phi_I(x) \right] E^{-1}, \]

\[ \gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \sigma^{ab} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a) ; \quad (1) \]

where \( E \) stands for a metrical matrix in 4-spinor space:

\[ E = \begin{vmatrix} \epsilon & 0 & \epsilon_{\alpha \beta} & 0 \\ 0 & \epsilon & \epsilon_{\dot{\alpha} \dot{\beta}} & 0 \\ -i \sigma^2 & 0 & -i \sigma^2 & 0 \\ 0 & -i \sigma^2 & 0 & +i \sigma^2 \end{vmatrix} , \]

\[ E^{-1} = \begin{vmatrix} \epsilon^{-1} & 0 & \epsilon_{\alpha \beta} & 0 \\ 0 & \epsilon & \epsilon_{\dot{\alpha} \dot{\beta}} & 0 \\ -i \sigma^2 & 0 & -i \sigma^2 & 0 \\ 0 & -i \sigma^2 & 0 & +i \sigma^2 \end{vmatrix} , \]

\[ E^2 = -I, \quad \bar{E} = -E, \quad \text{Sp} E = 0, \quad \bar{\sigma}^{ab} E = -E \sigma^{ab} . \quad (2) \]

Inverse to (1) has the form

\[ \Phi_I(x) = \frac{1}{4} \text{Sp} \ [E \gamma_I U(x)] , \quad \bar{\Phi}_I(x) = \frac{1}{4i} \text{Sp} \ [E \gamma^5 \gamma_I U(x)] , \]

\[ \Phi(x) = \frac{i}{4} \text{Sp} \ [EU(x)] , \quad \bar{\Phi}(x) = \frac{1}{4} \text{Sp} \ [E \gamma^5 U(x)] , \]

\[ \Phi_{mn}(x) = -\frac{1}{2i} \text{Sp} \ [E \sigma_{mn} U(x)] . \quad (3) \]

Below we will use the Weyl’s basis:

\[ U = \begin{vmatrix} \epsilon_{\alpha \beta} & \Delta^\alpha_{\dot{\alpha}} \\ H_{\dot{\alpha} \beta} & \eta_{\alpha \dot{\beta}} \end{vmatrix} , \quad \gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix} , \quad \sigma^a = (I, \sigma^k), \quad \bar{\sigma}^a = (I, -\sigma^k) , \]

\[ \sigma^{ab} = \frac{1}{4} \begin{vmatrix} \bar{\sigma}^a \sigma^b - \sigma^b \sigma^a & 0 \\ 0 & \sigma^a \sigma^b - \sigma^b \sigma^a \end{vmatrix} = \begin{vmatrix} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{vmatrix} , \quad \gamma^5 = \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix} . \quad (4) \]
in which decomposition (11) looks
\[
\begin{pmatrix}
\xi_{\alpha\beta} & \Delta_{\alpha}^\beta \\
H_{\alpha}^\beta & \eta_{\alpha\beta}
\end{pmatrix}
= \begin{pmatrix}
(-i \Phi - \tilde{\Phi}) + i \Sigma^{mn} \Phi_{mn} & \epsilon^{-1} \sigma^I (\Phi_t + i \tilde{\Phi}_t) \\
\sigma^I (\Phi_t - i \tilde{\Phi}_t) \epsilon^{-1} & (-i \Phi + \tilde{\Phi}) + i \Sigma^{mn} \Phi_{mn}
\end{pmatrix}.
\]

Thus, 2-spinor and tensor descriptions of the Dirac-Kähler field are determined by
\[
\Delta = (\Phi_t + i \tilde{\Phi}_t) \sigma^I \epsilon^I,
\]
\[
H = (\Phi_t - i \tilde{\Phi}_t) \epsilon^{-1},
\]
\[
\xi = (-i \Phi - \tilde{\Phi} + i \Sigma^{mn} \Phi_{mn}) \epsilon^{-1},
\]
\[
\eta = (-i \Phi + \tilde{\Phi} + i \Sigma^{mn} \Phi_{mn}) \epsilon;
\]
and inverse relations are
\[
\Phi_t + i \tilde{\Phi}_t = \frac{1}{2} sp (\epsilon^{-1} \sigma_I \Delta),
\]
\[
\Phi_t - i \tilde{\Phi}_t = \frac{1}{2} sp (\epsilon \sigma_I H),
\]
\[
-i \Phi - \tilde{\Phi} = \frac{1}{2} sp (\epsilon \xi),
\]
\[
-i \Phi + \tilde{\Phi} = \frac{1}{2} sp (\epsilon^{-1} \xi),
\]
\[
-i \Phi^{kl} + \frac{1}{2} \epsilon^{klmn} \Psi_{mn} = sp (\epsilon \Sigma^{kl} \xi),
\]
\[
-i \Phi^{kl} - \frac{1}{2} \epsilon^{klmn} \Psi_{mn} = sp (\epsilon^{-1} \Sigma^{kl} \xi).
\]

Now let us turn to the Dirac-Kähler wave equation. In spinor basis it is a Dirac-like equation for 2-rank bispinor:
\[
(i\gamma^a \frac{\partial}{\partial x^a} - m) U(x) = 0;
\]
or in 2-spinor form
\[
(A) \quad i\sigma^a \partial_a \xi(x) = m H(x), \quad (A') \quad i\sigma^a \partial_a H(x) = m \xi(x),
\]
\[
(B) \quad i\sigma^a \partial_a \eta(x) = m \Delta(x), \quad (B') \quad i\sigma^a \partial_a \Delta(x) = m \eta(x).
\]

Equations (5) - (9) are invariant under the Lorentz group. First, let us consider continuous transformations from SL(2,C). the Dirac-Kähler field transforms according to
\[
U'(x') = [S(k, k^*) \otimes S(k, k^*)] U(x);
\]
\[
\xi'(x') = B(k) \xi(x) \tilde{B}(k), \quad \Delta'(x') = B(k) \Delta(x) \tilde{B}(k^*),
\]
\[
H'(x') = B(\bar{k}^*) H(x) \bar{B}(k), \quad \eta'(x') = B(\bar{k}^*) \eta(x) \bar{B}(k^*).
\]

We should find equation satisfied by primed quantities. From eq. (A) in (9) it follows
\[
(A) \quad i\sigma^a \partial_a B^{-1}(k) \xi'(x') \tilde{B}^{-1}(k) = m B(k^*) \Delta'(x') \tilde{B}(k^*), \quad \Rightarrow
\]
\[
iB^{-1}(k^*) \sigma^a B^{-1}(k) \partial_a \xi'(x') = m H'(x')
\]
and further, taking into account identity
\[
B^{-1}(k^*) \sigma^a B^{-1}(k) = \sigma^b L^a_b(k, k^*),
\]
from eq. (11) and (13) in 2-form can be rewritten in 4-form:

\[ B(k)\bar{\sigma}^a B(k^*) = \bar{\sigma}^b L^a_b(k, k^*) \]

from whence with the use of identity

\[ B(k)\bar{\sigma}^a B(k^*) = \bar{\sigma}^b L^a_b(k, k^*) \]

we arrive at

\[ (A') \quad i\bar{\sigma}^b \partial_{\bar{b}}' \xi'(x') = m H'(x') \]

Analogously, from eq. (A') in (5.10b) it follows

\[ (A') \quad i\bar{\sigma}^a \partial_a B(k^*) H'(x') \bar{B}^{-1}(k) = m B^{-1}(k)\xi'(x') \bar{B}^{-1}(k) \quad \Rightarrow \quad i B(k)\bar{\sigma}^a B(k^*) \partial_a H'(x') = m \xi'(x') \]

In the same manner one may prove invariance of remaining equations (B) and (B') in [Q]. Note that two identities [11] and [13] in 2-form can be rewritten in 4-form:

\[ S(k, \bar{k}^*) \gamma^a S^{-1}(k, \bar{k}^*) = \gamma^b L_b^a_k(k, k^*) \]

with the help of which one can easily demonstrate invariance of the Dirac-Kähler equation in 4-spinor form. Invariance of the Dirac-Kähler equation under two discrete transformations

\[ M = i\gamma^0, \quad \begin{vmatrix} \xi' & \Delta' \\ H' & \eta' \end{vmatrix} = (M \otimes M) \begin{vmatrix} \xi & \Delta \\ H & \eta \end{vmatrix} = \begin{vmatrix} -\eta & -H \\ -\Delta & -\xi \end{vmatrix}, \]

\[ N = \gamma^0 \gamma^5, \quad \begin{vmatrix} \xi' & \Delta' \\ H' & \eta' \end{vmatrix} = (N \otimes N) \begin{vmatrix} \xi & \Delta \\ H & \eta \end{vmatrix} = \begin{vmatrix} \eta & -H \\ -\Delta & -\xi \end{vmatrix}, \]

is readily proved with the use of two identities:

\[ M \gamma^a M^{-1} = \begin{vmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{vmatrix} = \gamma^b L_b^{(P)a}, \quad N \gamma^a N^{-1} = \begin{vmatrix} 0 & -\sigma^a \\ -\sigma^a & 0 \end{vmatrix} = \gamma^b L_b^{(T)a}. \]

Now let us derive a set of tensor equations equivalent to the above spinor form. To this end, the spinor equation is to be presented as

\[ i\gamma^c \partial_c ( -i\Phi + \gamma^l \Phi_l + i\sigma^{mn} \Phi_{mn} + \gamma^5 \Phi + i\gamma^l \gamma^5 \Phi_l ) E^{-1} - m ( -i\Phi + \gamma^l \Phi_l + i\sigma^{mn} \Phi_{mn} + \gamma^5 \Phi + i\gamma^l \gamma^5 \Phi_l ) E^{-1} = 0, \]

from whence with the use of the formulas

\[ \text{Sp} \gamma^a = 0, \quad \text{Sp} \gamma^5 = \text{Sp} \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix} = 0, \quad \text{Sp} (\gamma^5 \gamma^a) = \text{Sp} \begin{vmatrix} 0 & -\sigma^a \\ +\sigma^a & 0 \end{vmatrix} = 0; \]

\[ \text{Sp} (\sigma^a \sigma^b) = \text{Sp} (\sigma^a \sigma^b) = 2 g^{ab}, \quad \text{Sp} (\gamma^a \gamma^b) = \text{Sp} \begin{vmatrix} \sigma^a \sigma^b \\ 0 \end{vmatrix} = 4 g^{ab}; \]

\[ \text{Sp} (\gamma^5 \gamma^a \gamma^b) = 0, \quad \text{Sp} (\gamma^c \gamma^a \gamma^b) = \text{Sp} \begin{vmatrix} 0 & \sigma^c \sigma^a \sigma^b \\ -\sigma^c \sigma^a \sigma^b & 0 \end{vmatrix} = 0, \quad \text{Sp} (\gamma^5 \gamma^c \gamma^a \gamma^b) = 0; \]

\[ \text{Sp} (\sigma^d \sigma^c \sigma^a \sigma^b) = 2 ( g^{dc} g^{ab} - g^{da} g^{cb} + g^{db} g^{ca} - i \epsilon^{dcab} ), \]

\[ \text{Sp} (\sigma^d \sigma^c \sigma^a \sigma^b) = 2 ( g^{dc} g^{ab} - g^{da} g^{cb} + g^{db} g^{ca} + i \epsilon^{dcab} ), \]

\[ \text{Sp} (\gamma^d \gamma^c \gamma^a \gamma^b) = \text{Sp} \begin{vmatrix} \sigma^d \sigma^c \sigma^a \sigma^b & 0 \\ 0 & \sigma^d \sigma^c \sigma^a \sigma^b \end{vmatrix} = 4 \begin{vmatrix} g^{dc} g^{ab} - g^{da} g^{cb} + g^{db} g^{ca} \end{vmatrix}, \]

\[ \text{Sp} (\gamma^5 \gamma^d \gamma^c \gamma^a \gamma^b) = \text{Sp} \begin{vmatrix} -\sigma^d \sigma^c \sigma^a \sigma^b & 0 \\ 0 & \sigma^d \sigma^c \sigma^a \sigma^b \end{vmatrix} = 4 i \epsilon^{dcab}. \]
and also

\[ \bar{\sigma}^a \sigma^b \sigma^c = \sigma^a g^{bc} - \sigma^b g^{ac} + \sigma^c g^{ab} + i \epsilon^{abcd} \sigma_d , \]

\[ \bar{\sigma}^a \sigma^b \bar{\sigma}^c = \bar{\sigma}^a g^{bc} - \bar{\sigma}^b g^{ac} + \bar{\sigma}^c g^{ab} - i \epsilon^{abcd} \bar{\sigma}_d , \]

\[ \bar{\sigma}^a \bar{\sigma}^b \sigma^c = \sigma^a g^{bc} - \sigma^b g^{ac} + \sigma^c g^{ab} + i \epsilon^{abcd} \bar{\sigma}_d , \]

we readily arrive at

\[ \partial_l \Phi^l + m \Phi = 0 , \quad \partial_l \bar{\Phi}^l + m \bar{\Phi} = 0 , \]

\[ \partial^k \Phi + \partial_l \Phi^{kl} - m \Phi^k = 0 , \]

\[ \partial^k \bar{\Phi} - \frac{1}{2} \epsilon^{kcmn} \partial_c \Phi_{mn} - m \bar{\Phi}^k = 0 , \]

\[ \partial^d \Phi^k - \partial^k \Phi^d + \epsilon^{dke} \partial_e \bar{\Phi}_l - m \Phi^{dk} = 0 . \] (19)

It should be stressed that the present context, the completely antisymmetric object Levi-Civite \( \epsilon^{abcd} \) appears as a definite and fixed symbol, \( \epsilon^{0123} = +1 \) and so on, without any presumed properties under the Lorentz transformations.

One may easily verify that in each equation in (19) are linearly combined quantities of the same behavior under the spinor covering \( \mathcal{G}_{spin} \) of the full Lorentz group \( L \); at this the space-time coordinates behave as follows

\[ (M) \quad x^M_l = (+\delta^k_l) x_k , \quad (N) \quad x^N_l = (-\delta^k_l) x_k . \] (20)

3 On equations for particles with different parities

Let us consider eqs. (19) at four different additional constraints:

\[ S = 0 \quad \bar{\Phi} = 0 , \quad \Phi_\alpha = 0 , \quad \Phi_{\alpha \beta} = 0 , \]

\[ \partial^j \Phi^l + m \Phi = 0 , \quad 0 = 0 , \]

\[ \partial_l \Phi - m \Phi_l = 0 , \quad 0 = 0 , \]

\[ \partial^d \Phi^k - \partial^k \Phi^d = 0 ; \] (21)

\[ S = \bar{\Phi} = 0 , \quad \Phi_\alpha = 0 , \quad \Phi_{\alpha \beta} = 0 ; \]

\[ 0 = 0 , \quad \partial^j \Phi^l + m \Phi = 0 , \]

\[ 0 = 0 , \quad \partial^k \Phi - m \Phi^k = 0 , \]

\[ \epsilon^{dke} \partial_e \Phi_l = 0 ; \] (22)

\[ S = 1 \quad \bar{\Phi} = 0 , \quad \Phi_\alpha = 0 , \quad \Phi_{\alpha \beta} = 0 , \]

\[ 0 = 0 , \quad \partial^j \bar{\Phi}^l + m \bar{\Phi} = 0 , \]

\[ 0 = 0 , \quad \partial^k \bar{\Phi} - m \bar{\Phi}^k = 0 , \]

\[ \frac{1}{2} \epsilon^{kcmn} \partial_c \bar{\Phi}_{mn} + m \bar{\Phi}^k = 0 , \quad \epsilon^{dke} \partial_e \bar{\Phi}_l - m \Phi^{dk} = 0 . \] (24)
For a vector particle, we will have (the symbol of \(tr\) stands for a matrix transposition)

\[
\begin{bmatrix}
\xi \\
H
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
= \begin{bmatrix}
\xi \Phi + \Phi \xi + i \Sigma^{mn} \Phi_{mn} (-i \sigma^2) \\
\sigma^l (\Phi_l - i \Phi_l) (-i \sigma^2)
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
\begin{bmatrix}
\xi \Phi + \Phi \xi + i \Sigma^{mn} \Phi_{mn} (i \sigma^2) \\
\sigma^l (\Phi_l - i \Phi_l) (i \sigma^2)
\end{bmatrix}
\]

(25)

A pseudovector case is given by

\[
\begin{bmatrix}
\xi \\
H
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
= \begin{bmatrix}
+ \Sigma^{mn} \sigma^2 \Phi_{mn} + i \sigma^l \sigma^2 \Phi_l \\
- i \sigma^l \sigma^2 \Phi_l - \Sigma^{mn} \sigma^2 \Phi_{mn}
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
\begin{bmatrix}
+ \Sigma^{mn} \sigma^2 \Phi_{mn} - i \sigma^l \sigma^2 \Phi_l \\
- i \sigma^l \sigma^2 \Phi_l - \Sigma^{mn} \sigma^2 \Phi_{mn}
\end{bmatrix}
\]

\(\Delta = + H, \tilde{\xi} = + \xi, \tilde{\eta} = + \eta.\)

(26)

\[
\begin{bmatrix}
\xi \\
H
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
= \begin{bmatrix}
+ \Sigma^{mn} \sigma^2 \Phi_{mn} + i \sigma^l \sigma^2 \Phi_l \\
- i \sigma^l \sigma^2 \Phi_l - \Sigma^{mn} \sigma^2 \Phi_{mn}
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
\begin{bmatrix}
+ \Sigma^{mn} \sigma^2 \Phi_{mn} - i \sigma^l \sigma^2 \Phi_l \\
- i \sigma^l \sigma^2 \Phi_l - \Sigma^{mn} \sigma^2 \Phi_{mn}
\end{bmatrix}
\]

\(\Delta = - H, \tilde{\xi} = + \xi, \tilde{\eta} = + \eta.\)

(27)

4 Massless vector particle, and Lorentz condition

Bearing in mind a photon field, it is better to use the usual notation:

\[
S = 1, \quad \Phi_l \mapsto A_l, \quad \Phi_{kl} \mapsto F_{kl},
\]

\[
U(x) = \left[ + \gamma^l A_l(x) + i \sigma^{mn} F_{mn}(x) \right] E^{-1};
\]

the wave equation in spinor form is

\[
\begin{align*}
(A) & \quad i \sigma^a \partial_a \xi = m H, \\
(B) & \quad i \sigma^a \partial_a \eta = m \Delta,
\end{align*}
\]

\[
\begin{align*}
(A') & \quad i \sigma^a \partial_a H = m \xi, \\
(B') & \quad i \sigma^a \partial_a \Delta = m \eta.
\end{align*}
\]

(28)

which is equivalent to the tensor system:

\[
\begin{align*}
\partial^l A_l &= 0, \\
\partial^l F_{kl} - m A_k &= 0,
\end{align*}
\]

\[
\epsilon^{kcmn} \partial_c F_{mn} = 0, \quad \partial_d A_k - \partial_k A_d - m F_{dk} = 0.
\]

(29)

Tensors and spinors are referred by

\[
\begin{bmatrix}
\xi \\
H
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\eta
\end{bmatrix}
= \begin{bmatrix}
+ \Sigma^{mn} \sigma^2 F_{mn} + i \sigma^l \sigma^2 A_l \\
- i \sigma^l \sigma^2 A_l - \Sigma^{mn} \sigma^2 F_{mn}
\end{bmatrix}
\]

or in a more detailed form

\[
\begin{align*}
\xi - \eta &= - 2i (\sigma^1 F_{23} + \sigma^2 F_{31} + \sigma^3 F_{12}) \sigma^2, \\
\xi + \eta &= 2 (\sigma^1 F_{01} + \sigma^2 F_{02} + \sigma^3 F_{03}) \sigma^2,
\end{align*}
\]

\[
\Delta = \begin{bmatrix}
(A_1 - i A_2) & (A_0 - A_3) \\
-(A_0 + A_3) & -(A_1 + i A_2)
\end{bmatrix}, \quad H = \begin{bmatrix}
(A_1 - i A_2) & -(A_0 + A_3) \\
(A_0 - A_3) & -(A_1 + i A_2)
\end{bmatrix}.
\]

(30)
Restriction to a massless vector particle is achieved as follows (see [28] and [29]):

\[
\begin{align*}
(A) & \quad i\sigma^a \partial_a \xi = 0 \quad \iff \quad \partial^l F_{kl} = 0, \quad \epsilon^{kcmn} \partial_c F_{mn} = 0; \\
(B) & \quad i\bar{\sigma}^a \partial_a \eta = 0, \\
(A') & \quad i\bar{\sigma}^a \partial_a H = \xi \quad \iff \quad \partial^l A_l = 0, \quad \partial_k A_l - \partial_l A_k = F_{kl}. \\
(B') & \quad i\sigma^a \partial_a \Delta = \eta
\end{align*}
\]

One should note that spinor equations \((A'), (B')\) results in the Lorentz condition. With the use of 3-vector notation

\[
F_{01} = -E_1 = +E^1, \quad F_{02} = -E_2 = +E^2, \quad F_{03} = -E_3 = +E^3, \\
F_{23} = B_1 = B^1, \quad F_{31} = B_2 = B^2, \quad F_{12} = B_3 = B^3,
\]
eqs. (31) take the form

\[
\begin{align*}
(A), (B) & \quad \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = 0, \quad \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0, \\
& \quad \partial_2 E_3 - \partial_3 E_2 = -\partial_0 B_1, \quad \partial_3 E_1 - \partial_1 E_3 = -\partial_0 B_2, \quad \partial_1 E_2 - \partial_2 E_1 = -\partial_0 B_3, \\
& \quad \partial_2 B_3 - \partial_3 B_2 = +\partial_0 E_1, \quad \partial_3 B_1 - \partial_1 B_3 = +\partial_0 E_2, \quad \partial_1 B_2 - \partial_2 B_1 = +\partial_0 E_3,
\end{align*}
\]
or in 3-vector notation

\[
\begin{align*}
(A), (B) & \quad \text{div } E = 0, \quad \text{div } B = 0, \\
& \quad \text{rot } E = -\frac{\partial}{\partial t} B, \quad \text{rot } B = +\frac{\partial}{\partial t} E,
\end{align*}
\]
where \(E = (E^1, E^2, E^3), B = (B^1, B^2, B^3)\). In the same manner let us consider equations \((A')\) and \((B')\) (32) – in 3-vector form \((A = (A^1, A^2, A^3))\) they are

\[
\begin{align*}
(A'), (B') & \quad 0 = 0, \quad \frac{\partial A^0}{\partial t} + \text{div } A = 0, \\
& \quad -\frac{\partial A}{\partial t} - \text{grad } A^0 = E, \quad \text{rot } A = B.
\end{align*}
\]

5 Massless pseudovector particle , and Lorentz condition

A pseudovector particle is specified by the relations:

\[
\begin{align*}
S = \mathbf{1}; & \quad \Phi_l \mapsto \hat{A}_l, \quad \Phi_{kl} \mapsto F_{kl}, \\
U(x) &= \left[ +i \sigma^{mn} F_{mn}(x) + i \gamma^i \gamma^5 \hat{A}_l(x) \right] E^{-1}; \\
(\hat{A}) & \quad i\sigma^a \partial_a \xi = m H, \\
(\hat{B}) & \quad i\bar{\sigma}^a \partial_a \eta = m \Delta, \\
0 &= 0, \quad \partial^l \hat{A}_l = 0, \quad \epsilon^{kcmn} \partial_c F_{mn} + m \hat{A}^k = 0, \quad \epsilon^{dkcl} \partial_l \hat{A}_l - m F^{dk} = 0.
\end{align*}
\]
Tensors and spinors are referred by

\[
\begin{pmatrix}
\xi & \Delta \\
H & \eta
\end{pmatrix} = \begin{pmatrix}
\Sigma_{mn} \sigma^2 F_{mn} & -\sigma^2 \tilde{A}_t \\
-\sigma^2 \tilde{A}_t & \Sigma_{mn} \sigma^2 F_{mn}
\end{pmatrix}
\]

\[
\xi = i\sigma^3 (F_{01} - i F_{23}) + (F_{02} - i F_{31}) - i\sigma^1 (F_{03} - i F_{12}) ,
\eta = i\sigma^3 (F_{01} + i F_{23}) + (F_{02} + i F_{31}) - i\sigma^1 (F_{03} + i F_{12}) ,
\]

\[
\Delta = \begin{pmatrix}
i\tilde{A}_1 + \tilde{A}_2 & +i(\tilde{A}_0 - \tilde{A}_3) \\
-i(\tilde{A}_0 + \tilde{A}_3) & -i\tilde{A}_1 + \tilde{A}_2
\end{pmatrix} ,
\]

\[
H = \begin{pmatrix}
-i\tilde{A}_1 - \tilde{A}_2 & +i(\tilde{A}_0 + \tilde{A}_3) \\
i(\tilde{A}_0 - \tilde{A}_3) & +i\tilde{A}_1 - \tilde{A}_2
\end{pmatrix} .
\]

(36)

Transition to a massless case is realized as follows:

\[
(\tilde{A}) \quad \begin{pmatrix} i\sigma^a \partial_a \xi = 0 \end{pmatrix} \quad \iff \quad \partial^4 F_{kl} = 0 , \quad \frac{1}{2} \epsilon^{kcmn} \partial_c F_{mn} = 0 ;
\]

(37)

\[
(\tilde{B}) \quad \begin{pmatrix} i\bar{\sigma}^a \partial_a H = \eta \end{pmatrix} \quad \iff \quad 0 = 0 , \quad \partial^4 \tilde{A}_t = 0 , \quad \epsilon^{dke} \partial_e \tilde{A}_l = F^{dk} .
\]

(38)

It should be noted that eqs. (37) coincide with (31), so we need investigate additionally only equations (\(\tilde{A}')\) and (\(\tilde{B}')\) – in vector form they are

\[
(\tilde{A}'),(\tilde{B}') \quad 0 = 0 , \quad \frac{\partial A^0}{\partial t} + \text{div } \tilde{A} = 0 ,
\]

\[
\text{rot } \tilde{A} = + E , \quad B = -\frac{\partial \tilde{A}}{\partial t} - \text{grad } \tilde{A}^0 .
\]

(39)

6 Comparing results for vector and pseudovector fields

Let us collect the results obtained. Tensor equations for a S = 1 field are

\[
(\alpha) \quad \partial^4 F_{kl} = 0 , \quad \epsilon^{kcmn} \partial_c F_{mn} = 0 ;
\]

(\(\beta\)) \quad \begin{pmatrix} 0 = 0 , \quad \partial^a A_a = 0 , \quad \partial_b A_c - \partial_c A_b = F_{bc} \end{pmatrix} ,

in 3-vector form

\[
(\alpha) \quad \text{div } E = 0 , \quad \text{rot } E = -\partial_t B ,
\]

\[
(\beta) \quad 0 = 0 , \quad \partial_t A^0 + \text{div } A = 0 , \quad -\partial_t A - \text{grad } A^0 = \tilde{E} ,
\]

\[
\text{rot } A = + B ;
\]

(41)

tensor equations for a S = \(\tilde{1}\) field

\[
(\tilde{\alpha}) \quad \partial^4 F_{kl}^{\tilde{1}} = 0 , \quad \epsilon^{kcmn} \partial_c F_{mn}^{\tilde{1}} = 0 ;
\]

(\(\tilde{\beta}\)) \quad \begin{pmatrix} 0 = 0 , \quad \partial^t \tilde{A}_t = 0 , \quad \epsilon_{dk}^{\tilde{e}} \partial_e \tilde{A}_l = F_{dk}^{\tilde{1}} \end{pmatrix} .
\]

(42)
in 3-vector form

\[
\begin{align*}
\text{(\(\tilde{\alpha}\))} & \quad \text{div } E^\sim = 0, \quad \text{div } B^\sim = 0, \\
& \quad \text{rot } E^\sim = - \partial_t B^\sim, \quad \text{rot } B^\sim = + \partial_t E^\sim, \\
\text{(\(\tilde{\beta}\))} & \quad 0 = 0, \quad \partial_t \tilde{A}^0 + \text{div } \tilde{A} = 0, \quad -\partial_t \tilde{A} - \text{grad } \tilde{A}^0 = B^\sim, \quad \text{rot } \tilde{A} = +E^\sim.
\end{align*}
\]

7 Maxwell equations in presence of sources, fields with different parities

For the case \(S = 1\), spinor massless equations with sources can be obtained from equation (A),(B) in (28)

\[
S = 1 \quad \begin{align*}
(A) & \quad i\sigma^a \partial_a \xi(x) = m H(x), \\
(B) & \quad i\bar{\sigma}^a \partial_a \eta(x) = m \Delta(x)
\end{align*}
\]

by formal changes

\[
\begin{align*}
 mH(x) &= -i \sigma^l \sigma^2 m A_l(x), \quad \implies -j(x) = -i \sigma^k \sigma^2 j_k(x), \\
 m\Delta(x) &= +i \bar{\sigma}^l \sigma^2 m A_l(x), \quad \implies -\bar{j}(x) = +i \sigma^k \sigma^2 j_k(x).
\end{align*}
\]

Thus, we get

\[
S = 1 \quad \begin{align*}
(A) & \quad i\sigma^a \partial_a \xi(x) = -j(x), \\
(B) & \quad i\bar{\sigma}^a \partial_a \eta(x) = -\bar{j}(x), \\
\partial^i F_{kl}(x) &= -j_k(x), \\
\epsilon^{kemn} \partial_c F_{mn}(x) &= 0; \\
\text{div } E &= +j^0, \quad \text{div } B = 0, \\
\text{rot } E &= -\partial_t B, \quad \text{rot } B = +\partial_t E + j,
\end{align*}
\]

and remaining equation (A'), (B'):

\[
S = 1 \quad \begin{align*}
(A') & \quad \left\{ \begin{array}{l}
 i\bar{\sigma}^a \partial_a H = \xi \\
 i\sigma^a \partial_a \Delta = \eta
\end{array} \right\} \quad \implies \\
\partial^i A_l &= 0, \quad \partial_k A_l - \partial_l A_k = F_{kl}, \\
0 &= 0, \quad \frac{\partial A^0}{\partial t} + \text{div } A = 0, \\
-\frac{\partial A}{\partial t} - \text{grad } A^0 &= E, \quad \text{rot } A = B.
\end{align*}
\]

In the same manner, one should consider the case \(S = \bar{1}\). Making in spinor equations

\[
S = \bar{1} : \quad \begin{align*}
(\bar{A}) & \quad i\sigma^a \partial_a \xi(x) = m H(x), \\
(\bar{B}) & \quad i\bar{\sigma}^a \partial_a \eta(x) = m \Delta(x)
\end{align*}
\]
formal changes
\[ m H(x) = -\sigma^k \sigma^2 m \tilde{A}_k(x) , \quad \implies \quad -\tilde{j}(x) = -\sigma^k \sigma^2 \tilde{\tilde{j}}_k(x) , \]
\[ m \Delta(x) = -\sigma^k \sigma^2 m \tilde{A}_k(x) , \quad \implies \quad -\tilde{\tilde{j}}(x) = -\sigma^k \sigma^2 \tilde{\tilde{j}}_k(x) \]
we arrive at
\[ (\tilde{A}) \quad i\sigma^a \partial_a \xi(x) = -\tilde{j}(x) , \]
\[ (\tilde{B}) \quad i\sigma^a \partial_a \eta(x) = -\tilde{\tilde{j}}(x) , \]
\[ (S = \tilde{1}) \quad \partial^k \tilde{F}^\sim_{kl} = 0 , \quad \frac{1}{2} \epsilon^{kcmn} \partial_c \tilde{F}^\sim_{mn} = +\tilde{j}^k , \]
\[ \text{div } \tilde{E}^\sim = 0 , \quad \text{div } \tilde{B}^\sim = -\tilde{\tilde{j}}^0^\sim , \]
\[ \text{rot } \tilde{E}^\sim = -\partial_t \tilde{B}^\sim + \tilde{j} , \quad \text{rot } \tilde{B}^\sim = +\partial_t \tilde{E}^\sim . \quad (46) \]
and remaining equations \((\tilde{A}')\), \((\tilde{B}')\):
\[ (S = \tilde{1}) \quad \partial^k \tilde{F}^\sim_{kl} = 0 , \quad \text{div } \tilde{E}^\sim = 0 , \quad \text{div } \tilde{B}^\sim = -\tilde{\tilde{j}}^0^\sim , \]
\[ \text{rot } \tilde{E}^\sim = -\partial_t \tilde{B}^\sim + \tilde{j} , \quad \text{rot } \tilde{B}^\sim = +\partial_t \tilde{E}^\sim . \quad (47) \]

One notices that a simple algebraic summing of the systems related to \((A), (B)\) and \((\tilde{A}), (\tilde{B})\) for fields of the types \(S = 1, S = \tilde{1}\) (see [44] and [46]), one arrives at equations for Maxwell electromagnetic theory with two charge, electric and magnetic:
\[ \text{E} + \text{E}^\sim = \hat{\text{E}} , \quad \text{B} + \text{B}^\sim = \hat{\text{B}} , \]
\[ \text{div } \hat{\text{E}} = j^0 , \quad \text{div } \hat{\text{B}} = -\tilde{j}^0 , \]
\[ \text{rot } \hat{\text{E}} = -\partial_t \hat{\text{B}} + \tilde{j} , \quad \text{rot } \hat{\text{B}} = +\partial_t \hat{\text{E}} + j , \quad (48) \]
4-tensor description of that summing looks
\[ (\hat{\text{E}}, c\hat{\text{B}}), \quad \tilde{F}_{ab} = F_{ab} + F^\sim_{ab} = (\partial_a A_b - \partial_b A_a) + \epsilon_{ab}^{kl} \partial_k \tilde{A}_l , \]
\[ \partial^k [ F_{kl}(x) + F^\sim_{kl} ] = -j_k(x) , \]
\[ \frac{1}{2} \epsilon^{kcmn} \partial_c [ F_{mn}(x) + F^\sim_{mn} ] = +\tilde{j}^k . \quad (49) \]

which coincides with 2-potential approach to Maxwell electrodynamics with two charges [57].

The above method of obtaining extended Maxwell equations with magnetic charges from two independent Maxwell theories with different intrinsic parities
\[ S = 1 , \quad (A, B) , \quad (A', B') ; \]
\[ S = \tilde{1} , \quad (\tilde{A}, \tilde{B}) , \quad (\tilde{A}', \tilde{B}') ; \quad (50) \]
may be sketched by the scheme
\[(S = 1) + (S = \bar{1}) : (A, B) + (\bar{A}, \bar{B}) , \ A', B', \bar{A}', \bar{B}' . \] (51)

One could introduce a symmetrical combination by a similar scheme
\[(S = 1) - (S = \bar{1}) : (A, B) + (\bar{A}, \bar{B}) A', B', \bar{A}', \bar{B}' . \] (52)

which corresponds to
\[
E - E^\sim = \hat{E} , \quad B - B^\sim = \hat{B} , \\
\text{div } \hat{E} = j^0 , \quad \text{div } \hat{B} = + j^0 , \\
\text{rot } \hat{E} = - \partial_t \hat{B} - \hat{j} , \quad \text{rot } \hat{B} = + \partial_t \hat{E} + \hat{j} ;
\] (53)

4-tensor description of that summing looks
\[
(\hat{E}, c\hat{B}) , \quad \hat{F}_{ab} = F_{ab} - F_{ab}^\sim = (\partial_a A_b - \partial_b A_a) - \epsilon_{ab}^{\ kl} \partial_k \bar{A}_l , \\
\partial^l \left[ F_{kl}(x) - F_{kl}^\sim \right] = - j_k(x) , \\
\frac{1}{2} \epsilon^{kcmn} \partial_c \left[ F_{mn}(x) - F_{mn}^\sim \right] = - j^k .
\] (54)

Evidently, the system consisting of two independent Maxwell models, \(S = 1\) and \(S = \bar{1}\), is equivalent to the pair
\[(S = 1) \pm (S = \bar{1}) : (A, B) \pm (\bar{A}, \bar{B}) , \ A', B', \bar{A}', \bar{B}' . \] (55)

Let us call the doubled model (55) an extended Maxwell theory, it includes two photon fields different in parity, and two types of sources, electrical and magnetic charges.

8 Extension to the Maxwell theory in Riemannian space-time

Let us start with a generally covariant tetrad-based Dirac-Kähler equation in 4-spinor form [99, 100]:
\[
[ i \gamma^\alpha(x) \left( \partial/\partial x^\alpha + B_\alpha(x) \right) - m ] U(x) = 0 ,
\] (56)

where \(B_\alpha(x)\) is a 2-rank bispinor connection
\[
B_\alpha(x) = \frac{1}{2} J^{ab} e^\beta_{(a)} \nabla_\alpha (e_{(b)\beta}) = \Gamma_\alpha(x) \otimes I + I \otimes \Gamma_\alpha(x) ,
\]

and \(J^{ab} = (\sigma^{ab} \otimes I + I \otimes \sigma^{ab})\) stands for generators for 2-rank bispinor under the Lorentz group. From [55] specified in Weyl spinor basis, one derives the following equations in 2-spinor form:
\[
\begin{align*}
    i \sigma^\alpha(x) \left[ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) \right] \zeta(x) &= m H(x) , \\
    i \bar{\sigma}^\alpha(x) \left[ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) \right] H(x) &= m \xi(x) , \\
    i \bar{\sigma}^\alpha(x) \left[ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) \right] \eta(x) &= m \Delta(x) , \\
    i \sigma^\alpha(x) \left[ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) \right] \Delta(x) &= m \eta(x) .
\end{align*}
\] (57)
Symbols $\Sigma_\alpha(x)$ and $\bar{\Sigma}_\alpha(x)$ stand for Infeld – van der Vaerden connections:

$$\sigma^\alpha(x) = \sigma^a e^\alpha_{(a)}(x), \quad \bar{\sigma}^\alpha(x) = \bar{\sigma}^a e^\alpha_{(a)}(x),$$

$$\Sigma_\alpha(x) = \frac{1}{2} \Sigma^{ab}_\alpha e^\alpha_{(a)}(x) \nabla_\alpha (e_{(b)} \beta), \quad \bar{\Sigma}_\alpha(x) = \frac{1}{2} \Sigma^{ab}_\alpha e^\alpha_{(a)}(x) \nabla_\alpha (e_{(b)} \beta),$$

$$\sum^a = \frac{1}{4} (\sigma^a \bar{\sigma}^b - \bar{\sigma}^b \sigma^a), \quad \bar{\sum}^a = \frac{1}{4} (\sigma^a \sigma^b - \bar{\sigma}^b \sigma^a).$$

These spinor equations are equivalent to a generally covariant tensor system

$$\nabla^\alpha \Psi_\alpha (x) + m \Psi(x) = 0, \quad \nabla^\alpha \bar{\Psi}_I(x) + m \bar{\Psi}(x) = 0,$$

$$\nabla_\alpha \Psi(x) + \nabla^\beta \Psi_{\alpha \beta}(x) - m \Psi_\alpha(x) = 0,$$

$$\nabla_\alpha \Psi_\beta(x) - \nabla_\beta \Psi_\alpha(x) + \epsilon_{\alpha \beta \rho \sigma}(x) \nabla_\rho \bar{\Psi}_\sigma(x) - m \Psi_{\alpha \beta}(x) = 0,$$

(58)

where covariant tensor field variables are connected with local tetrade tensor variables by the relations

$$\Psi_\alpha(x) = e^{(i)}_{\alpha}(x) \Psi_i(x), \quad \bar{\Psi}_\alpha(x) = e^{(i)}_{\alpha}(x) \bar{\Psi}_i(x),$$

$$\Psi_{\alpha \beta}(x) = e^{(m)}_{\alpha}(x) e^{(n)}_{\beta}(x) \Psi_{mn}(x),$$

(59)

and the Levi-Civita object is determined by

$$\epsilon^\alpha_{\beta \rho \sigma}(x) = \epsilon^{abcd}_\alpha e^{\beta}_{(a)}(x) e^{\rho}_{(c)}(x) e^{\sigma}_{(d)}(x).$$

(60)

The fields $\Psi(x), \Psi_\alpha(x), \Psi_{\alpha \beta}(x)$ are tetrade scalars, and $\bar{\Psi}(x), \bar{\Psi}_\alpha(x)$ are tetrade pseudoscalars, and the Levi-Civita object $\epsilon^\alpha_{\beta \rho \sigma}(x)$ is a generally covariant tensor and a tetrade pseudoscalar.

Now, we should obtain generally covariant equations for ordinary boson of the types $S = 0, \bar{0}, 1, \bar{1}$:

$$S = 0, \quad \bar{\Psi}, \quad \bar{\Psi}_\alpha, \quad \Psi_{\alpha \beta} = 0,$$

$$\nabla^\alpha \Psi_\alpha + m \Psi = 0, \quad \nabla_\alpha \Psi - m \Psi_\alpha = 0, \quad \nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = 0,$$

(61)

two first are the Proca equations for scalar particle, the last equation holds identically:

$$(\partial_\alpha \partial_\beta \Psi(x) - \Gamma^\mu_{\alpha \beta}(x) \partial_\mu \Psi(x)) - (\partial_\beta \partial_\alpha \Psi(x) - \Gamma^\mu_{\beta \alpha}(x) \partial_\mu \Psi(x)) = 0,$$

For a pseudoscalar field we have

$$S = \bar{0}, \quad \nabla^\alpha \bar{\Psi}_\alpha(x) + m \bar{\Psi}(x) = 0, \quad \nabla_\alpha \bar{\Psi}(x) - m \bar{\Psi}_\alpha(x) = 0, \quad \epsilon^\alpha_{\beta \rho \sigma}(x) \nabla_\rho \Psi_\sigma(x) = 0;$$

(62)

defined in the previous two equations holds identically. Now, let it be $\Psi_\alpha(x) \neq 0, \Psi_{\alpha \beta}(x) \neq 0$, then

$$S = 1, \quad \nabla^\alpha \Psi_\alpha(x) = 0, \quad \nabla_\beta \Psi_{\alpha \beta}(x) - m \Psi_\alpha(x) = 0,$$

$$-\frac{1}{2} \epsilon^\alpha_{\beta \rho \sigma}(x) \nabla_\beta \Psi_{\rho \sigma}(x) = 0, \quad \nabla_\alpha \Psi_\beta(x) - \nabla_\beta \Psi_\alpha(x) = m \Psi_{\alpha \beta}(x).$$

(63)
Here the first and third equation hold identically:
\[
\nabla^\alpha \Psi_\alpha(x) = \frac{1}{m} \nabla^\alpha \nabla^\beta \Psi_{\alpha\beta}(x) = \frac{1}{2m} \left[ \Psi_{\alpha\nu}(x) R^\nu_{\beta} \delta^\alpha_{\beta}(x) - \Psi_{\beta\nu}(x) R^\nu_{\alpha} \delta^\beta_{\alpha}(x) \right] = 0 ,
\]
\[
- \frac{1}{2m} \epsilon_{\alpha\beta\rho\sigma}(x) \nabla^\beta \left[ \nabla_\rho \Psi_\sigma(x) - \nabla_\sigma \Psi_\rho(x) \right] = - \frac{1}{4m} \epsilon_{\alpha\beta\rho\sigma}(x) \left[ \left( \nabla_\beta \nabla_\rho(x) - \nabla_\rho \nabla_\beta \right) \Psi_\sigma(x) - \left( \nabla_\beta \nabla_\sigma(x) - \nabla_\sigma \nabla_\beta \right) \Psi_\rho(x) \right] = 0.
\]

Now, let it be \( \Psi(x) = \tilde{\Psi}(x) = \Psi_\alpha(x) = 0 \), then
\[
S = 1 , \quad \nabla^\alpha \tilde{\Psi}_\alpha(x) = 0 , \quad \nabla^\beta \tilde{\Psi}_{\alpha\beta}(x) = 0 , \quad \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma}(x) \nabla_\beta \Psi_{\rho\sigma}(x) + m \tilde{\Psi}(x) = 0 , \quad \epsilon_{\alpha\beta\rho\sigma}(x) \nabla_\rho \tilde{\Psi}_\sigma(x) - m \Psi_{\alpha\beta}(x) = 0.
\]

The first and the second equations hold identically:
\[
\nabla^\alpha \tilde{\Psi}_\alpha(x) = - \frac{1}{2m} \nabla^\alpha \epsilon_{\alpha\beta\rho\sigma}(x) \nabla_\beta \Psi_{\rho\sigma}(x) = - \frac{1}{2m} \epsilon_{\alpha\beta\rho\sigma}(x) \nabla^\alpha \nabla_\beta \Psi_{\rho\sigma}(x) = \frac{1}{4m} \epsilon_{\alpha\beta\rho\sigma}(x) \left[ \Psi_{\nu\sigma}(x) R^\nu_{\beta} \delta^\alpha_{\rho\sigma}(x) + \Psi_{\rho\nu}(x) R^\rho_{\sigma} \delta^\alpha_{\nu\sigma}(x) \right] ,
\]
\[
\nabla^\beta \Psi_{\alpha\beta}(x) = \frac{1}{m} \nabla^\beta(x) \epsilon_{\alpha\beta\rho\sigma}(x) \nabla_\rho \Psi_\sigma(x) = \frac{1}{2m} \epsilon_{\alpha\beta\rho\sigma}(x) \Psi^\nu(x) R_{\nu\rho\sigma\beta}(x) .
\]

Constraints separating four boson fields are the same as in the case of Minkowskian space:
\[
S = 0 , \quad \tilde{\Delta} = +H ; \quad \tilde{\xi} = -\eta , \quad \tilde{\eta} = -\xi , \quad \tilde{\bar{\eta}} = -\bar{\eta} .
\]
\[
S = 0 , \quad \Delta = -H ; \quad \bar{\xi} = -\xi , \quad \bar{\eta} = -\eta , \quad \bar{\bar{\eta}} = -\bar{\bar{\eta}} .
\]
\[
S = 1 , \quad \Delta = +H ; \quad \bar{\xi} = +\xi , \quad \bar{\eta} = +\eta .
\]
\[
S = 1 , \quad \tilde{\Delta} = -H ; \quad \tilde{\xi} = +\xi , \quad \tilde{\eta} = +\eta .
\]

Without any additional calculation, we can write down spinor and corresponding tensor equations for Maxwell theories with opposite intrinsic parities:

for a vector model
\[
(S = 1) \quad \begin{align*}
(A) & \quad i\sigma^\alpha(x) \left[ \partial / \partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) \right] \xi(x) = - j(x) , \\
(B) & \quad i\sigma^\alpha(x) \left[ \partial / \partial x^\alpha + \bar{\Sigma}_\alpha(x) \otimes I + I \otimes \bar{\Sigma}_\alpha(x) \right] \eta(x) = - \bar{j}(x) ,
\end{align*}
\]
\[
\nabla^\beta F_{\alpha\beta}(x) = - j_\alpha(x) , \quad \epsilon_{\alpha\beta\rho\sigma} \nabla^\beta F_{\rho\sigma}(x) = 0 ;
\]
\[
(S = 1) \quad \begin{align*}
(A') & \quad i\sigma^\alpha(x) \left[ \partial / \partial x^\alpha + \bar{\Sigma}_\alpha(x) \otimes I + I \otimes \bar{\Sigma}_\alpha(x) \right] H(x) = \xi , \\
(B') & \quad i\sigma^\alpha(x) \left[ \partial / \partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) \right] \Delta(x) = \eta ,
\end{align*}
\]
\[
\nabla^\alpha A_\alpha = 0 , \quad \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = F_{\alpha\beta} .
\]
for a pseudovector model
\[
\begin{align*}
S = \tilde{1}, & \quad (\tilde{A}) \quad i\sigma^\alpha(x) [ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) ] \xi(x) = -\tilde{j}(x), \\
(S = \tilde{1}) & \quad \nabla^\beta F_{\alpha\beta} = 0, \quad \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \nabla_\beta F_{\rho\sigma} = +\tilde{j}^\alpha;
\end{align*}
\]
\[
(S = \tilde{1}), \quad (\tilde{A}') \quad i\tilde{\sigma}^\alpha(x) [ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) ] H(x) = \xi \\
(\tilde{B}') \quad i\sigma^\alpha(x) [ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) ] \Delta(x) = \eta \quad 0 = 0, \quad \nabla^\alpha \tilde{A}_\alpha = 0, \quad \epsilon^{\alpha\beta\rho\sigma} \nabla_\rho \tilde{A}_\sigma = F^{\alpha\beta}. \quad (68)
\]

9 Symmetry in extended Maxwell theory, duality transformation

Summing and subtracting respective tensor equations in (67) and (68) we arrive at
\[
\begin{align*}
\nabla^\beta [ F_{\alpha\beta}(x) + \tilde{F}_{\alpha\beta}(x) ] & = -j_\alpha(x), & \nabla^\beta [ F_{\alpha\beta}(x) - \tilde{F}_{\alpha\beta}(x) ] & = -j_\alpha(x), \\
\frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \nabla_\beta [ F_{\rho\sigma} + \tilde{F}_{\rho\sigma} ] & = +\tilde{j}^\alpha(x), & \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \nabla_\beta [ F_{\rho\sigma} - \tilde{F}_{\rho\sigma} ] & = -\tilde{j}^\alpha(x), \\
F_{\alpha\beta}(x) \pm \tilde{F}_{\alpha\beta}(x) & = \nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x) \pm \epsilon_{\alpha\beta\rho\sigma} \nabla_\rho \tilde{A}_\sigma(x), \\
\nabla^\alpha A_\alpha(x) & = 0, \quad \nabla^\alpha \tilde{A}_\alpha(x) = 0. \quad (69)
\end{align*}
\]

Here, \(F_{\alpha\beta}(x), A_\alpha(x), j_\alpha(x)\) are tetrad scalar, and \(\epsilon_{\alpha\beta\rho\sigma}(x), \tilde{A}_\alpha, \tilde{j}_\alpha\) are tetrad pseudoscalars.

Let us introduce dual variables (marked by a star symbol):
\[
\begin{align*}
F^{*\alpha\beta}(x) & = \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma}(x) F_{\rho\sigma}(x), & F_{\alpha\beta}(x) & = -\frac{1}{2} \epsilon_{\alpha\beta}^{\mu\nu}(x) F^{*\mu\nu}(x), \\
\tilde{F}^{*\alpha\beta}(x) & = \frac{1}{2} \epsilon_{\alpha\beta}^{\rho\sigma}(x) \tilde{F}_{\rho\sigma}(x), & \tilde{F}_{\alpha\beta}(x) & = -\frac{1}{2} \epsilon_{\alpha\beta}^{\mu\nu}(x) \tilde{F}^{*\mu\nu}(x). \quad (70)
\end{align*}
\]

One can easily prove an identity
\[
\frac{1}{2} \epsilon_{\rho\sigma}^{\alpha\beta} [ F_{\alpha\beta}(x) \pm \tilde{F}^{*\alpha\beta}(x) ] = \frac{1}{2} \epsilon_{\rho\sigma}^{\alpha\beta} [ \nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x) \pm \epsilon_{\alpha\beta\delta\gamma}(x) \nabla_\delta \tilde{A}_\gamma(x) ] =
\]
\[
= \epsilon_{\rho\sigma}^{\alpha\beta} \nabla_\alpha A_\beta \mp (\nabla_\rho \tilde{A}_\sigma - \nabla_\sigma \tilde{A}_\rho),
\]
that is
\[
F^{*\rho\sigma} \pm \tilde{F}^{*\rho\sigma} = \mp (\nabla_\rho \tilde{A}_\sigma - \nabla_\sigma \tilde{A}_\rho) + \epsilon_{\rho\sigma}^{\alpha\beta} \nabla_\alpha A_\beta. \quad (71)
\]

Thus, the system of extended Maxwell theory may be presented as follows:
\[
\begin{align*}
\nabla^\beta [ F_{\alpha\beta}(x) + \tilde{F}_{\alpha\beta}(x) ] & = -j_\alpha(x), & \nabla^\beta [ F_{\alpha\beta}(x) - \tilde{F}_{\alpha\beta}(x) ] & = -j_\alpha(x), \\
\nabla^\beta [ F^{*\alpha\beta}(x) + \tilde{F}^{*\alpha\beta}(x) ] & = +\tilde{j}^\alpha(x), & \nabla_\beta [ F^{*\alpha\beta} - \tilde{F}^{*\alpha\beta} ] & = -\tilde{j}^\alpha(x), \\
F_{\alpha\beta}(x) \pm \tilde{F}_{\alpha\beta}(x) & = \nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x) \pm \epsilon_{\alpha\beta\rho\sigma} \nabla_\rho \tilde{A}_\sigma(x), \\
F^{*\rho\sigma}(x) \pm \tilde{F}^{*\rho\sigma}(x) & = \mp (\nabla_\rho \tilde{A}_\sigma - \nabla_\sigma \tilde{A}_\rho) + \epsilon_{\rho\sigma}^{\alpha\beta} \nabla_\alpha A_\beta(x). \\
\nabla^\alpha A_\alpha(x) & = 0, \quad \nabla^\alpha \tilde{A}_\alpha(x) = 0. \quad (72)
\end{align*}
\]
With the notation
\[ F_{\alpha\beta}(x) + \tilde{F}_{\alpha\beta}(x) = F_{\alpha\beta}^+(x), \quad F^*_{\alpha\beta}(x) + \tilde{F}^*_{\alpha\beta}(x) = F^*_{\alpha\beta}(x), \]
\[ F_{\alpha\beta}(x) - \tilde{F}_{\alpha\beta}(x) = F_{\alpha\beta}^-(x), \quad F^*_{\alpha\beta}(x) - \tilde{F}^*_{\alpha\beta}(x) = F^*_{\alpha\beta}(x), \]
eqs. (72) look
\[ \nabla^\beta F_{\alpha\beta}^+(x) = -j_\alpha(x), \quad \nabla^\beta F_{\alpha\beta}^-(x) = -\tilde{j}_\alpha(x), \]
\[ \nabla^\beta F^+_{\alpha\beta} = +\tilde{j}^\alpha(x), \quad \nabla^\beta F^-_{\alpha\beta} = -\tilde{j}^\alpha(x), \]
\[ F_{\alpha\beta}^+(x) = \nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x) + \epsilon_{\alpha\beta}^{\rho\sigma}(x) \nabla_\rho \tilde{A}_\sigma(x), \]
\[ F^+_{\alpha\beta}(x) = \nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x) - \epsilon_{\alpha\beta}^{\rho\sigma}(x) \nabla_\rho \tilde{A}_\sigma(x), \]
\[ F_{\rho\sigma}^+(x) = -\left( \nabla_\rho \tilde{A}_\sigma(x) - \nabla_\sigma \tilde{A}_\rho(x) \right) + \epsilon_{\rho\sigma}^{\alpha\beta}(x) \nabla_\alpha A_{\beta}(x). \]
\[ \nabla^\alpha A_\alpha(x) = 0, \quad \nabla^\alpha \tilde{A}_\alpha(x) = 0. \]

The system obtained is invariant under the following linear transformation (extended duality operation)
\[ A_\alpha \mapsto -\tilde{A}_\alpha', \quad \tilde{A}_\alpha \mapsto A_\alpha', \]
\[ F^+_{\alpha\beta} \mapsto +F^+_{\alpha\beta}', \quad F^*_{\alpha\beta} \mapsto -F^*_{\alpha\beta}', \]
\[ F^-_{\alpha\beta} \mapsto -F^-_{\alpha\beta}', \quad F^*_{\alpha\beta} \mapsto +F^*_{\alpha\beta}', \]
\[ j_\alpha \mapsto -\tilde{j}_\alpha', \quad \tilde{j}_\alpha \mapsto +j_\alpha'. \]

Also, one can note another invariance transformation:
\[ A_\alpha \mapsto +\tilde{A}_\alpha', \quad \tilde{A}_\alpha \mapsto -A_\alpha', \]
\[ F^+_{\alpha\beta} \mapsto -F^+_{\alpha\beta}', \quad F^*_{\alpha\beta} \mapsto +F^*_{\alpha\beta}', \]
\[ F^-_{\alpha\beta} \mapsto +F^-_{\alpha\beta}', \quad F^*_{\alpha\beta} \mapsto -F^*_{\alpha\beta}', \]
\[ j_\alpha \mapsto +\tilde{j}_\alpha', \quad \tilde{j}_\alpha \mapsto -j_\alpha'. \]

Relations (74) and (75) are particular cases of the continuous extended dual transformation over 4-vector and 4-pseudovector:
\[ \cos \chi A_\alpha + \sin \chi \tilde{A}_\alpha = A_\beta', \quad \cos \chi A_\beta' - \sin \chi \tilde{A}_\beta' = A_\alpha, \]
\[ -\sin \chi A_\alpha + \cos \chi \tilde{A}_\alpha = \tilde{A}_\alpha', \quad \sin \chi A_\beta' + \cos \chi \tilde{A}_\beta' = \tilde{A}_\alpha, \]

which generates the following transformation over strength tensors:
\[ F^+_{\alpha\beta}(x) = \nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x) + \epsilon_{\alpha\beta}^{\rho\sigma}(x) \nabla_\rho \tilde{A}_\sigma(x) = \]
\[ = \nabla_\alpha \left( \cos \chi A_\beta' - \sin \chi \tilde{A}_\beta' \right) - \nabla_\beta \left( \cos \chi A_\alpha' - \sin \chi \tilde{A}_\alpha' \right) + \epsilon_{\alpha\beta}^{\rho\sigma}(x) \nabla_\rho \left( \sin \chi A_\sigma' + \cos \chi \tilde{A}_\sigma'(x) \right), \]

that is
\[ F^+_{\alpha\beta} = \cos \chi F^+_{\alpha\beta} + \sin \chi F^{\prime+}_{\alpha\beta}. \]

(77)
analogously

\[ F_{\alpha\beta}^{++} = -\sin \chi \ F_{\alpha\beta}^{'} + \cos \chi \ F_{\alpha\beta}^{'+*} . \]  

(78)

In the same manner one derives

\[ F_{\alpha\beta}^{-} = \cos \chi \ F_{\alpha\beta}^{'} - \sin \chi \ F_{\alpha\beta}^{'+*} , \quad F_{\alpha\beta}^{-*} = \sin \chi \ F_{\alpha\beta}^{'} + \cos \chi \ F_{\alpha\beta}^{'+*} . \]  

(79)

Continuous dual transformations over currents are

\[ \cos \chi j_{\alpha} + \sin \chi \tilde{j}_{\alpha} = j'_{\alpha} , \quad \cos \chi j'_{\alpha} - \sin \chi \tilde{j}_{\alpha} = j_{\alpha} , \]
\[ -\sin \chi j_{\alpha} + \cos \chi \tilde{j}_{\alpha} = \tilde{j}'_{\alpha} , \quad \sin \chi j'_{\alpha} + \cos \chi \tilde{j}_{\alpha} = \tilde{j}_{\alpha} . \]  

(80)

10 Acknowledgement

Let us summarize results. From the 16-component Dirac-Kähler field theory, spinor equations for two types of massless vector photon fields with different parities have been derived. Their equivalent tensor equations in terms of the strength tensor \( F_{ab} \) and respective 4-vector \( A_{b} \) and 4-pseudovector \( \tilde{A}_{b} \) depending on intrinsic photon parity are derived; they include additional sources, electric 4-vector \( j_{b} \) and magnetic 4-pseudovector \( \tilde{j}_{b} \). The theories of two types of photon fields are explicitly uncoupled, their linear combination through summing or subtracting results in Maxwell electrodynamics with electric and magnetic charges in 2-potential approach. The whole analysis is extended straightforwardly to a curved space-time background.

This work was supported by Fund for Basic Research of Belarus and JINR F06D-006.

References

[1] H.A. Lorentz. *De l’influence du mouvement de la terre sur les phénomènes lumineux.* // Archives Néerlandaises des Sciences Exactes et Naturelles. 21 103-176 (1886); *La théorie électromagnétique de Maxwell et son application aux corps mouvants.* Archives néerlandaises des sciences exactes et naturelles. 25 363-552 (1892); *Electromagnetic phenomena in a system moving with any velocity less than that of light.* Proceedings of the Section of Sciences, Koninklijke Academie van Wetenschappen te Amsterdam. 6 809-831 (1904).

[2] H. Poincaré. *Électricité et optique.* 2 Vols. Publié par J. Blondin et B. Brunhes. Paris, Georges Carré, 1890-1891; *La théorie de Lorentz et le principe de réaction.* Archives néerlandaises des sciences exactes et naturelles. 5 252-278 (1900); *La mesure du temps.* Revue de métaphysique et de morale. 6 371-384 (1898); *Électricité et optique: la lumière et les théories électrodynamiques.* Publié par J. Blondin et E. Néculcéa. Paris: Carré et Naud, 1901; *La Science et l’hypothèse.* Paris: Flammarion, 1902; *Sur la dynamique de l’électron.* C. R. Acad. Sci. Paris. 140 1504-1508 (1905); *Sur la dynamique de l’électron.* Rendiconti del circolo matematico di Palermo. 21 129-176 (1906); *La dynamique de l’électron.* Revue générale des sciences pure et appliquées. 19 386-402 (1908).

[3] A. Einstein. *Zur Elektrodynamik der bewegten Körper.* Ann. der Phys. 17 891-921 (1905); A. Einstein, J. Laub. *Über die elektromagnetischen Grundgleichungen für bewegte Körper.* Ann. der Phys. 26 532-540 (1908); *Die im elektromagnetischen Felde auf ruhende Körper ausgeübten ponderomotorischen Kräfte.* Ann. der Phys. 26 541-550 (1908).
[4] L. Silberstein. *Elektromagnetische Grundgleichungen in bivectorieller Behandlung*. Ann. der Phys. 22 579-586 (1907); *Nachtrag zur Abhandlung Über "elektromagnetische Grundgleichungen in bivectorieller Behandlung"*. Ann. der Phys. 24 783-784 (1907); L. Silberstein. *The Theory of Relativity*. London, Macmillan. 1914.

[5] H. Minkowski. *Die Grundlagen für die electromagnetischen Vorgänge in bewegten Körpern*. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse. 53-111 (1908); reprint in Math. Ann. 68 472-525 (1910); *Raum und Zeit*. Jahresbericht der deutschen Mathematiker-Vereinigung. 18 75-88 (1909); *Raum und Zeit*. Phys. Zeit. 10 104-111 (1909); *Das Relativitätsprinzip*. Annalen der Physik. 47 927-938 (1915).

[6] Von Max Abraham. *Zur electromagnetischen Mechanik*. Phys. Zeit. 10 737-741 (1909); *Die neue Mechanik*. Scientia (Rivista di Scienza). 15 8-27 (1914); *Zur Elektrodynamik bewegter Körper*. Rend. Circ. Mat. Palermo. 28 1 (1909); *Sull’elettrodinamica di Minkowski*. Rendiconti del Circolo Matematico di Palermo. 30 33-46 (1910).

[7] H. Bateman. *On the conformal transformations of the space of four dimensional and their applications to geometric optics*. Proc. London Math. Soc. 7 70-92 (1909); *The transformation of the electrodynamical equations*. Proc. London Math. Soc. Ser. 8 223-264 (1910); *The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell’s Equations*. (1915), Cambridge (reprinted by Dover, New York 1955).

[8] E. Cunningham. *The principle of relativity in electrodynamics and an extension thereof*. Proc. London Math. Soc. 8 77-98 (1909); *The Application of the Mathematical Theory of Relativity to the Electron Theory of Matter*. Proc. London Math. Soc. 10 116-127 (1911-1912); *The Principle of Relativity*. Cambridge. Cambridge University Press. 1914.

[9] Kornel Lanczos. *Die Funktionentheoretischen Beziehungen der Maxwellschen Aethergleichungen - Ein Beitrag zur Relativitätstheorie*. Verlagshandlung Josef Németh, Budapest, 1919. 80 pp.; A. Gsponer and J.P. Hurni. Lanczos’s functional theory of electrodynamics. - A commentary on Lanczos’s PhD dissertation, in W.R. Davis et al. eds. Cornelius Lanczos Collected Published Papers With Commentaries, 1 (North Carolina State University. Raleigh. 1998. P. 215-223; [arXiv:math-ph/0402012](http://arxiv.org/abs/math-ph/0402012).

[10] G.N. Lewis. *On Four Dimensional Vector Analysis and its Application in Electrical Theory*. Proc. Amer. Acad. Arts and Science. 46 165-181 (1910); G.N. Lewis. *Über viervierdimensionale Vektoranalyse und deren Anwendung auf die Elektrizitätstheorie*. Jahrbuch der Radioaktivität und Elektronik. 7 329-347 (1910).

[11] R. Marcolongo. *Les transformations de Lorentz et les équations de l’électrodynamique*. Annales de la Faculté des Sciences de Toulouse. Ser. 3. 4 429-468 (1912).

[12] W. Gordon. *Zur Lichtfortpflanzung nach der Relativitätstheorie*. Ann. Phys. (Leipzig). 72 421-456 (1923).

[13] I.E. Tamm. *Electrodynamik von anisotropen Medien in special relativity*. ZhRFXO. 56 248-262 (1924); *Crystal optics in terms of geometry of quadratic forms*. ZhRFXO. 3-4 1 (1925); Mandelstam, I.E. Tamm. *Elektrodynamik der anisotropen Medien und der spezialen Relativitätstheorie*. Math. Annalen. 95 154-160 (1925).
[14] G. Rainich. *Electrodynamics in the general relativity theory*. Trans. Am. Math. Soc. **27** 106-136 (1925).

[15] I.E. Tamm. *Electricity theory*. Moskow - Leningrad. 1929 (First Edition).

[16] G. Uhlenbeck, O. Laporte. *New Covariant Relations Following from the Dirac Equations*. Phys. Rev. **37** 1552-1554 (1931).

[17] G. Juvet, A. Schidlof. *Sur les nombres hypercomplexes de Clifford et leurs applications à l’analyse vectorielle ordinaire, à l’électromagnétisme de Minkowski et à la théorie de Dirac*. Bull. Soc. Sci. Nat. Neuchâtel. **57** 127-141 (1932).

[18] M. Abraham, R. Becker. *Theorie der Elektrizität*. Band I. Leipzig. 1932.

[19] M. Abraham, R. Becker. *Theorie der Elektrizität*. Band II. 1933.

[20] Ya.I. Frenkel. *Electrodynamics. I. General theory of electricity*. Leningrad – Moskow, 1934; II. Microscopic electrodynamics of medias. Leningrad – Moskow, 1935.

[21] A. Mercier. *Expression des équations de l'électromagnétisme au moyen des nombres de Clifford*. Thèse de l'Université de Genève. No 953. Arch. Sci. Phys. Nat. Genève. **17** 1-34 (1935).

[22] Yu.B. Rumer. *Spinor analysis*. Moscow. 1936 (in Russian).

[23] J. Stratton. *Electromagnetic Theory*. McGraw-Hill. 1941. New York.

[24] A. Mercier. *Sur les fondements de l'électrodynamique classique (méthode axiomatique)*. Arch. Sci. Phys. Nat. Genève. **2** 584-588 (1949).

[25] Nathan Rosen. *Special theories of relativity*. Am. J. Phys. **20** 161-164 (1952).

[26] F. Gürsey. *Dual invariance of Maxwell’s tensor*. Rev. Fac. Sci. Istanbul. A. **19** 154-160 (1954).

[27] S. Gupta. *Gravitation and electromagnetism*. Phys. Rev. **96** 1683-1685 (1954).

[28] A. Lichnerowicz. *Théories relativistes de la gravitation et de l'électromagnetisme*. Paris. 1955.

[29] Valer Novacu. *Introducere in electrodinamica*. 1955

[30] A.A. Borgardt. *On principle of Larmor invariance*. ZhETP. **33** 791-792 (1957).

[31] A.A. Borgardt. *Wave equations for a photon*. ZhETP. **34** 1323-1325 (1958).

[32] H.E. Moses. *A spinor representation of Maxwell equations*. Nuovo Cimento Suppl. **7** 1-18 (1958).

[33] E. Moses. *Solutions of Maxwell’s equations in terms of a spinor notation: the direct and inverse problems*. Phys. Rev. **113** 1670-1679 (1959).

[34] Wolfgang K.H. Panofsky, Melba Phillips. *Classical Electricity and Magnetics*. Addison-Wesley Publishing Company. 1962.

[35] E.J. Post. *Formal structure of electrodynamics. General covariance and electromagnetics*. Amsterdam, 1962.
[36] G. Rosen. *Symmetries of the Einstein-Maxwell wave equations*. J. Math. Phys. 3 313 (1962).
[37] D.M. Lipkin. *Existence of a new conservation law in electromagnetic theory*. J. Math. Phys. 5 696-700 (1964).
[38] J.R. Ellis. *Maxwell’s equations and theories of Maxwell form*. Ph.D. thesis. University of London. 417 pages (1964).
[39] R. Penney. *Duality invariance and Riemannian geometry*. J. Math. Phys. 5 1431-1437 (1964).
[40] D.J. Candlin. *Analysis of the new conservation law in electromagnetic field*. Nuovo Cim. 37 1390-1397 (1966).
[41] V.I. Strazhev, L.M. Tomil’chik. *To the problem of dual symmetry*. Vesti AN BSSR. Ser. Fiz.-mat. 2 102-108 (1968).
[42] A.R. Exton, E.T. Newman, R. Penrose. *Conserved quantities in the Einstein-Maxwell theory*. J. Math. Phys. 10 1566-1570 (1969).
[43] M. Carmeli. *Group analysis of Maxwell equations*. J. Math. Phys. 10 1699-1703 (1969).
[44] V.I. Strazhev. *Dual symmetry of electromagnetic field*. Vesti AN BSSR. Ser. Fiz.-mat. 5 72-78 (1971).
[45] I.B. Pestov. *relation between Dirac equation and Maxwell equations*. Preprint 2-5798. Dubna. 1971. 18 pages.
[46] V.I. Strazhev, Nguyen Vin Xuang. *On discret' symmetries in electrodynamics*. Yadernaya Fizika. 16 614-619 (1972).
[47] V.I. Strazhev. *Dual symmetry in quantum electrodynamics*. TMF. 13 200-208 (1972).
[48] Joe Rosen. *Transformation properties of electromagnetic quantities under space inversion, time reversal, and charge conjugation*. Am. J. Phys. 41 586-588 (1973).
[49] L.D. Landay, E.M. Lifshitz. Teoretical physics. II. The field theory. Moskow. 1973.
[50] D. Weingarten. *Complex symmetries of electrodynamics*. Ann. Phys. 76 510-548 (1973).
[51] E.T. Newman. *Maxwell equations and complex Minkowski space*. J. Math. Phys. 14 102-107 (1973).
[52] R. Mignani, E. Recami, M. Baldo. *About a Dirac-like equation for the photon according to Ettore Majorana*. Lett. Nuovo Cim. 11 568-572 (1974).
[53] V.I. Strazhev. *On dual symmetry in microskopoc electrodynamics*. Vesti AN BSSR. Ser. Fiz.-mat. 1 69-72 (1974).
[54] J.D. Jackson. *Classical Electrodynamics*. Wiley. New York. 1975.
[55] V.I. Fushchich. *On the additional invariance of the Dirac and Maxwell equations*. Lett. Nuovo Cim. 11 508-512 (1974).
[56] T. Frankel. *Maxwell’s equations*. Am. Math. Mon. 81 343-349 (1974).

[57] V.I. Strazhev, L.M. Tomil’chik. Electrodynamics with magnetic charge. Minsk. 1975.

[58] J.D. Edmonds, Jr. *Comment on the Dirac-like equation for the photon*. Nuov. Cim. Lett. 13 185-186 (1975).

[59] A. Da Silveira. *Invariance algebras of the Dirac and Maxwell equations*. Nouvo Cim. A. 56 385-395 (1980).

[60] G. Venuri. A geometrical formulation of electrodynamics. Nuovo Cim. A. 65, 64-76 (1981).

[61] T.L. Chow. *A Dirac-like equation for the photon*. J. Phys. A. 14 2173-2174 (1981).

[62] R.J. Cook. *Photon dynamics*. Phys. Rev. A. 25 2164-2167 (1982).

[63] R.J. Cook. *Lorentz covariance of photon dynamics*. Phys. Rev. A. 26 2754-2760 (1982).

[64] V.I. Fushchich, A.G. Nikitin. *Symmetries of Maxwell’s Equations*. Kiev, 1983; Kluwer. Dordrecht. 1987.

[65] E. Giannetto. *A Majorana-Oppenheimer formulation of quantum electrodynamics*. Lett. Nuovo Cim. 44 140-144 (1985).

[66] H.N. Niñez Yépez, A.L. Salas Brito, and C.A. Vargas. *Electric and magnetic four-vectors in classical electrodynamics*. Revista Mexicana de Fisica. 34 636 (1988).

[67] R. Kidd, J. Ardini, A. Anton. *Evolution of the modern photon*. Am. J. Phys. 57 27 (1989).

[68] E. Recami. *Possible Physical Meaning of the Photon Wave-Function, According to Ettore Majorana. in Hadronic Mechanics and Non-Potential Interactions*. Nova Sc. Pub., New York, 231-238 (1990).

[69] I.Yu. Krivsky, V.M. Simulik. *Foundations of quantum electrodynamics in field strengths terms*. Naukova Dumka. Kiev, 1992.

[70] T. Inagaki. *Quantum-mechanical approach to a free photon*. Phys. Rev. A. 49 2839-2843 (1994).

[71] I. Bialynicki-Birula. On the wave function of the photon. Acta Phys. Polon. 86, 97-116 (1994).

[72] Bialynicki-Birula. *Photon wave function. in Progress in Optics*. 36, P. 248-294, Ed. E. Wolf (North-Holland 1996); arXiv:quant-ph/050820.

[73] J.F. Sipe. *Photon wave functions*. Phys. Rev. A. 52 1875-1883 (1995).

[74] P. Ghose. *Relativistic quantum mechanics of spin-0 and spin-1 bosons*. Found. Phys. 26 1441-1455 (1996).

[75] A. Gersten. *Maxwell equations as the one-photon quantum equation*. Found. of Phys. Lett. 12 291-8 (1998); arXiv:quant-ph/9911049.

[76] Salvatore Esposito. *Covariant Majorana formulation of electrodynamics*. Found. Phys. 28 231-244 (1998); arXiv:hep-th/9704144.
[77] Valeri V. Dvoeglazov. Historical note on relativistic theories of electromagnetism. Apeiron. 5, no 1-2, 69-88.

[78] Valeri V. Dvoeglazov. Speculations on the Neutrino Theory of Light. Annales de la Fondation Louis de Broglie. 24 111-127 (1999).

[79] Valeri V. Dvoeglazov, Generalized Maxwell and Weyl equations for massless particles. Rev. Mex. Fis. 49 99-103 (2003); arXiv:math-ph/0102001

[80] T. Ivezić. True transformations relativity and electrodynamics. Found. Phys. 31 1139 (2001).

[81] S.I. Kruglov. Dirac-Läbler equation. Int. J. Theor. Phys. 41 653-687 (2002); arXiv:hep-th/0110251.

[82] A. Gsponer. On the "equivalence" of the Maxwell and Dirac equations. Int. J. Theor. Phys. 41 689-694 (2002); arXiv:math-ph/0201053.

[83] V. Kravchenko. On the relation between the Maxwell system and the Dirac equation. arXiv:math-ph/0202009.

[84] T. Ivezić. The invariant formulation of special relativity, or the "true transformations relativity", and electrodynamics. Annales de la Fondation Louis de Broglie. 27 287-302 (2002).

[85] T. Ivezić. An invariant formulation of special relativity, or the true transformations relativity and comparison with experiments. Found. Phys. Lett. 15 27 (2002); arXiv:physics/0103026

[86] V.V. Varlamov. About Algebraic Foundations of Majorana-Oppenheimer Quantum Electrodynamics and de Broglie-Jordan Neutrino Theory of Light. Ann. Fond. L. de Broglie. 27 273-286 (2003).

[87] T. Ivezić. The proof that the standard transformations of E and B are not the Lorentz transformations. Found. Phys. 33 1339 (2003).

[88] Rollin S. Armour, Jr., Spin-1/2 Maxwell fields. Found. Phys. 34 815-842 (2004); arXiv:hep-th/0305084.

[89] Stoil Donev. Complex structures in electrodynamics. arXiv:math-ph/0106008

[90] Stoil Donev. From electromagnetic duality to extended electrodynamics. Annales Fond. Broglie. 29 375-392 (2004); arXiv:hep-th/0101137.

[91] S. Donev, M. Tashkova. Extended Electrodynamics: A Brief Review. arXiv:hep-th/0403244

[92] T. Ivezić. The difference between the standard and the Lorentz transformations of the electric and magnetic fields. Application to motional EMF. Found. Phys. Lett. 18 301 (2005).

[93] T. Ivezić. The proof that Maxwell’s equations with the 3D E and B are not covariant upon the Lorentz transformations but upon the standard transformations. The new Lorentz-invariant field equations. Found. Phys. 35 1585 (2005).

[94] T. Ivezić. Axiomatic geometric formulation of electromagnetism with only one axiom: the field equation for the bivector field F with an explanation of the Trouton-Noble experiment. Found. Phys. Lett. 18 401 (2005).
[95] Iwo Bialynicki-Birula, Zofia Bialynicka-Birula. *Beams of electromagnetic radiation carrying angular momentum: The Riemann-Silberstein vector and the classical-quantum correspondence*. arXiv:quant-ph/0511011.

[96] Tomislav Ivezić. *Lorentz Invariant Majorana Formulation of the Field Equations and Dirac-like Equation for the Free Photon*. EJTP. 3 131-142 (2006).

[97] G. Esposito. *A spinorial perspective on massless photons*. arXiv:hep-th/0701013.

[98] V.I. Strachev, I.A. Satikov, V.A. Tsionenko. *Dirac-Kähler equation, classical field*. Minsk. Belarus State University, 2007 (in Russian).

[99] V.M. Red’kov. *Dirac-Kähler equation in a curved space-time, spinor and tensor formulations*. Minsk, 1989. 53 pages, deponorovani in VINITI 7.08.89. 5336 - B89.

[100] V.M. Red’kov. *On the equation for the Dirac-Kähler field and bosons with different intrinsic parities in Riemannian space*. Vesti NAS of Belarus, Ser. Fiz.-Mat. 1 90-95 (2000).

[101] N.G. Tokarevskaya, V.M. Red’kov. *On conserved quantities in the theory of charged boson fields of spin 0 and 1*. 107-117 in: Proc. of XII Ann. Sem. Nonlinear Phenomena in Complex Systems. Minsk. 2005.