HISTORICAL COMMENTS ON MONGE’S ELLIPSOID AND THE CONFIGURATIONS OF LINES OF CURVATURE ON SURFACES IMMERSED IN $\mathbb{R}^3$

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Abstract. This is an essay on the historical landmarks leading to the study of principal configurations on surfaces, their structural stability and further generalizations. Here it is pointed out that in the work of Monge, 1796, are found elements of the qualitative theory of differential equations, founded by Poincaré in 1881. Two open problems are proposed.

1. Introduction

The book on differential geometry of D. Struik [29] contains key references to the classical works on principal curvature lines and their umbilic singularities due to L. Euler, G. Monge, C. Dupin, G. Darboux and A. Gullstrand, among others (see also [28] and, for additional references, also [18]). These papers—notably that of Monge, complemented with Dupin’s—can be connected with aspects of the qualitative theory of differential equations (QTDE for short) initiated by H. Poincaré [25] and consolidated with the study of the structural stability and genericity of differential equations in the plane and on surfaces, which was made systematic from 1937 to 1962 due to the seminal works of Andronov Pontrjagin and Peixoto (see [11, 23]).

I established this connection by 1970, after being prompted by the fortunate reading of Struik [29]. The essay [27], written in the Portuguese short story style, I recount how the interpretation of the historical landmarks led to inquire about the principal configurations on smooth surfaces, their structural stability and bifurcations.

This paper discusses the historical sources for the work on the structural stability of principal curvature lines and umbilic points, developed and further extended with the collaboration of C. Gutierrez [15, 16, 17] (see also the papers devoted to other differential equations of classical geometry: the asymptotic lines [7, 14], and the arithmetic, geometric and harmonic mean curvature lines [9, 10, 11]). This direction of research led me and R. Garcia to the study of general mean curvature lines in [13].

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Here it is also pointed out that in the work of Monge, [22], are found elements of the qualitative theory of differential equations, founded by Poincaré in [25].

This paper contains a reformulation of the essentials of [27], updates the references and proposes two open problems.

2. The Landmarks before Poincaré: Euler, Monge and Dupin

Leonhard Euler (1707-1783) [19], founded of the curvature theory of surfaces. He defined the normal curvature $k_n(p, L)$ on an oriented surface $S$ in a tangent direction $L$ at a point $p$ as the curvature, at $p$, of the planar curve of intersection of the surface with the plane generated by the line $L$ and the positive unit normal $N$ to the surface at $p$. The principal curvatures at $p$ are the extremal values of $k_n(p, L)$ when $L$ ranges over the tangent directions through $p$. Thus, $k_1(p) = k_n(p, L_1)$ is the minimal and $k_2(p) = k_n(p, L_2)$ is the maximal normal curvatures, attained along the principal directions: $L_1(p)$, the minimal, and $L_2(p)$, the maximal (see Fig. 1).

Euler’s formula expresses the normal curvature $k_n(\theta)$ along a direction making angle $\theta$ with the minimal principal direction $L_1$ as $k_n(\theta) = k_1 \cos^2 \theta + k_2 \sin^2 \theta$.

Euler, however, seems to have not considered the integral curves of the principal line fields $L_i : p \to L_i(p), i = 1, 2$, and overlooked the role of the umbilic points at which the principal curvatures coincide and the line fields are undefined.

![Figure 1. Principal Directions](image)

Gaspard Monge (1746-1818) coined the mathematical term umbilic point in the sense defined above and found the family of integral curves of the principal line fields $L_i, i = 1, 2$, for the case of the triaxial ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a > b > c > 0.$$ 

In doing this, by direct integration of the differential equations of the principal curvature lines, circa 1779, Monge was led to the first example of a foliation with singularities on a surface which (from now on) will be called
the principal configuration of the oriented surface. The Ellipsoid, endowed with its principal configuration, will be called Monge’s Ellipsoid (see Fig. 2).

![Monge’s Ellipsoid](image)

**Figure 2. Monge’s Ellipsoid**

The motivation found in Monge’s paper [22] is a complex interaction of aesthetic and practical considerations and of the explicit desire to apply the results of his mathematical research to real world problems. The principal configuration on the triaxial ellipsoid appears in Monge’s proposal for the dome of the Legislative Palace for the government of the French Revolution, to be built over an elliptical terrain. The lines of curvature are the guiding curves for the workers to put the stones. The umbilic points, from which were to hang the candle lights, would also be the reference points below which to put the podiums for the speakers.

The ellipsoid depicted in Fig. 2 contains some of the typical features of the qualitative theory of differential equations discussed briefly in a) to d) below:

a) **Singular Points and Separatrices.** The umbilic points play the role of singular points for the principal foliations, each of them has one separatrix for each principal foliation. This separatrix produces a connection with another umbilic point of the same type, for which it is also a separatrix, in fact an umbilic separatrix connection.

b) **Cycles.** The configuration has principal cycles. In fact, all the principal lines, except for the four umbilic connections, are periodic. The cycles fill a cylinder or annulus, for each foliation. This pattern is common to all classical examples, where no surface exhibiting an isolated cycle was known. This fact seems to be derived from the symmetry of the surfaces considered, or from the integrability that is present in the application of Dupin’s Theorem for triply orthogonal families of surfaces.

As was shown in [15], these configurations an exceptional; the generic principal cycle for a smooth surface is a hyperbolic limit cycle (see below).
c) **Structural Stability (relative to quadric surfaces).** The principal configuration remains qualitatively unchanged under small perturbations on the coefficients of the quadric polynomial that defines the surface.

d) **Bifurcations.** The drastic changes in the principal configuration exhibited by the transitions from a sphere, to an ellipsoid of revolution and to a triaxial ellipsoid (as in Fig. 2), which after a very small perturbation, is a simple form of a bifurcation phenomenon.

**Charles Dupin** (1784-1873) considered the surfaces that belong to *triply orthogonal surfaces*, thus extending considerably those whose principal configurations can be found by integration. Monge’s Ellipsoid belongs to the family of *homofocal quadrics* (see [29] and Fig. 3).

![Figure 3. Dupin’s Theorem](image)

**Theorem 1.** [27] In the space of oriented quadrics, identified with the nine-dimensional sphere, those having principal structurally stable configurations are open and dense.

**Historical Remark**  *The global study of lines of principal curvature leading to Monge’s Ellipsoid, which is analogous of the phase portrait of a differential equation, contains elements of Poincaré’s QTDE, 85 years earlier.*

This remark seems to have been overlooked by Monge’s scientific historian René Taton (1915-2004) in his remarkable book [30].

3. **Poincaré and Darboux**

The exponential role played by **Henri Poincaré** (1857-1912) for the QTDE as well as for other branches of mathematics is well known and has been discussed and analyzed in several places (see for instance [2] and [24]).
Here we are concerned with his Memoires \cite{25}, where he laid the foundations of the QTDE. In this work Poincaré determined the form of the solutions of planar analytic differential equations near their foci, nodes and saddles. He also studied also properties of the solutions around cycles and, in the case of polynomial differential equations, also the behavior at infinity.

**Gaston Darboux** (1842-1917) determined the structure of the lines of principal curvature near a generic umbilic point. In his note \cite{4}, Darboux uses the theory of singularities of Poincaré. In fact, the Darbouxian umbilics are those whose resolution by blowing up are saddles and nodes (see Figs. 4 and 5).

![Figure 4. Darbouxian Umbilics](image)

Let $p_0 \in S$ be an umbilic point. Consider a chart $(u, v) : (S, p_0) \to (\mathbb{R}^2, 0)$ around it, on which the surface has the form of the graph of a function such as

$$k \frac{1}{2}(u^2 + v^2) + \frac{a}{6} u^3 + \frac{b}{2} uv^2 + \frac{c}{6} v^3 + O[(u^2 + v^2)^2].$$

This is achieved by projecting $S$ onto $T_S(p_0)$ along $N(p_0)$ and choosing there an orthonormal chart $(u, v)$ on which the coefficient of the cubic term $u^2 v$ vanishes.

An umbilic point is called *Darbouxian* if, in the above expression, the following 2 conditions (T) and (D) hold:

- **T)** $b(b - a) \neq 0$
- **D)** either
  - $D_1 : a/b > (c/2b)^2 + 2$,
  - $D_2 : (c/2b)^2 + 2 > a/b > 1, a \neq 2b$,
  - $D_3 : a/b < 1$.

The corroboration of the pictures in Fig 4 which illustrate the principal configurations near Darbouxian umbilics, has been given in \cite{15} \cite{17}; see also \cite{3} and Fig. 5 for the Lie-Cartan resolution of a Darbouxian umbilic.

The subscripts refer to the number of *umbilic separatrices*, which are the curves, drawn with heavy lines, tending to the umbilic point and separating regions whose principal lines have different patterns of approach.
4. Principal Configurations on Smooth Surfaces

After the seminal work of Andronov-Pontrjagin [1] on structural stability of differential equations in the plane and its extension to surfaces by Peixoto [23] and in view of the discussion on Monge’s Ellipsoid formulated above, an inquiry into the characterization of the oriented surfaces in $S$ whose principal configuration are structurally stable under small $C^r$ perturbations, for $r \geq 3$, seems unavoidable.

Call $\Sigma(a, b, c, d)$ the set of smooth compact oriented surfaces $S$ which verify the following conditions.

a) All umbilic points are Darbouxian.

b) All principal cycles are hyperbolic. This means that the corresponding return map is hyperbolic; that is, its derivative is different from 1. It has been shown in [15] that hyperbolicity of a principal cycle $\gamma$ is equivalent to the requirement that

$$\int_{\gamma} \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} \neq 0,$$

where $\mathcal{H} = (k_1 + k_2)/2$ is the mean curvature and $\mathcal{K} = k_1k_2$ is the gaussian curvature.

c) The limit set of every principal line is contained in the set of umbilic points and principal cycles of $S$.

The $\alpha$-(resp. $\omega$) limit set of an oriented principal line $\gamma$, defined on its maximal interval $I = (w_-, w_+)$ where it is parametrized by arc length $s$, is the collection $\alpha(\gamma)$-(resp. $\omega(\gamma)$) of limit point sequences of the form $\gamma(s_n)$, convergent in $S$, with $s_n$ tending to the left (resp. right) extreme of $I$. The limit set of $\gamma$ is the set $\Omega = \alpha(\gamma) \cup \omega(\gamma)$. 

Figure 5. Lie-Cartan Resolution of Darbouxian Umbilics
Examples of surfaces with non trivial recurrent principal curves, which violate condition c are given in [16, 17]. There are no examples of these situations in the classical geometry literature.

d) All umbilic separatrices are separatrices of a single umbilic point.
Separatrices which violate d are called umbilic connections; an example can be seen in the ellipsoid of Fig. 2.

To make precise the formulation of the next theorems, some topological notions must be defined.

A sequence $S_n$ of surfaces converges in the $C^r$ sense to a surface $S$ provided there is a sequence of real functions $f_n$ on $S$, such that $S_n = (I + f_n N)(S)$ and $f_n$ tends to 0 in the $C^r$ sense; that is, for every chart $(u, v)$ with inverse parametrization $X$, $f_n \circ X$ converges to 0, together with the partial derivatives of order $r$, uniformly on compact parts of the domain of $X$.

A set $\Sigma$ of surfaces is said to be open in the $C^r$ sense if every sequence $S_n$ converging to $S$ in $\Sigma$ in the $C^r$ sense is, for $n$ large enough, contained in $\Sigma$.

A set $\Sigma$ of surfaces is said to be dense in the $C^r$ sense if, for every surface $S$, there is a sequence $S_n$ in $\Sigma$ converging to $S$ the $C^r$ sense.

A surface $S$ is said to be $C^r$-principal structurally stable if for every sequence $S_n$ converging to $S$ in the $C^r$ sense, there is a sequence of homeomorphisms $H_n$ from $S_n$ onto $S$, which converges to the identity of $S$, such that, for $n$ big enough, $H_n$ is a principal equivalence from $S_n$ onto $S$. That is, it maps $U_n$, the umbilic set of $S_n$, onto $U$, the umbilic set of $S$, and maps the lines of the principal foliations $F_{i,n}$, of $S_n$, onto those of $F_i$, $i = 1, 2$, principal foliations for $S$.

**Theorem 2.** (Structural Stability of Principal Configurations [15, 17]) The set of surfaces $\Sigma(a, b, c, d)$ is open in the $C^3$ sense and each of its elements is $C^3$-principal structurally stable.

**Theorem 3.** (Density of Principal Structurally Stable Surfaces, [16, 17]) The set $\Sigma(a, b, c, d)$ is dense in the $C^2$ sense.

Extensions of these results to surfaces with generic singularities, algebraic surfaces, surfaces in $\mathbb{R}^4$ and higher dimensional manifolds have been achieved recently ( see, for example, [6, 8, 21, 5]).

The bifurcations of umbilic points have been studied in [12], where references to other aspects of the bifurcations of principal configurations are found.

5. **Two Open Problems**

To conclude, two significant problems are stated.

**Problem 1.** Raise from 2 to 3 the differentiability class in the density Theorem 3.
This remains among the most intractable questions in this subject, involving difficulties of Closing Lemma type, [26], which also permeate other differential equations of classical geometry, [13].

**Problem 2.** Determine the class of principally structurally stable cubic and higher degree surfaces. Theorem 1 deals with the case of degree 2 — quadric — surfaces.

Partial results, including the behavior of lines of curvature at infinity in non-compact algebraic surfaces, have been established by Garcia and Sotomayor [6].

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