Longitudinal Force on a Moving Potential

Jian-Ming Tang and D. J. Thouless
Department of Physics, University of Washington
Box 351560, Seattle WA 98195-1560
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We show a formal result of the longitudinal force acting on a moving potential. The potential can be velocity-dependent, which appears in various interesting physical systems, such as electrons in the presence of a magnetic flux-line, or phonons scattering off a moving vortex. By using the phase-shift analysis, we are able to show the equivalence between the adiabatic perturbation theory and the kinetic theory for the longitudinal force in the dilute gas limit.

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I. INTRODUCTION

In recent work on the theory of the transverse force on vortex moving relative to a superfluid[1],[2] we have used a technique in which the force is calculated by applying a pinning potential which is made to move, and the reaction on the pinning potential is calculated perturbatively. A number of people have objected privately that this method does not give the longitudinal force correctly. We do not agree with this objection, since we think that it can be used to determine the longitudinal force. For simple problems our method gives results that are in agreement with those obtained by more familiar methods. This paper examines this question for some special cases. We conclude that any differences from other perturbative methods which may occur are due to uncertainties in limiting procedures.

For the component of force transverse to the vortex velocity on a vortex in a superfluid[3],[4] we have used a technique which is made to move, and the reaction on the pinning potential is calculated perturbatively. A number of people have objected privately that this method does not give the longitudinal force correctly. We do not agree with this objection, since we think that it can be used to determine the longitudinal force. For simple problems our method gives results that are in agreement with those obtained by more familiar methods. This paper examines this question for some special cases. We conclude that any differences from other perturbative methods which may occur are due to uncertainties in limiting procedures.

In Sec. II we consider the effect of a moving short-ranged spherically symmetric potential on a system of noninteracting particles, and show that the longitudinal force obtained from the method of TAN is, to lowest order in the velocity, identical to that which is obtained from kinetic theory. In Sec. III we consider the long-ranged Aharonov–Bohm type potential, and again find no difference in the longitudinal force. The transverse force seems to be more sensitive to limiting procedures.

II. THE LONGITUDINAL FORCE

The Hamiltonian describing a system of noninteracting particles in a system which is uniform apart from a spherically symmetric scattering potential whose center is time-dependent has the form

\begin{equation}
H = \sum_{i=1}^{N} \left\{ -\frac{\nabla_i^2}{2m_0} + V[\mathbf{r}_i - \mathbf{r}_0(t)] \right\}.
\end{equation}

It is convenient to work with a velocity which is switched adiabatically, so that \( \mathbf{r}_0 = \mathbf{v}_V \ e^{\gamma t} \). Since we will only be considering effects linear in \( \mathbf{v}_V \), the frequency dependence of the response can be obtained by analytical continuation of the \( \gamma \) dependence into the complex plane. The method used by TAN involves calculating the expectation value of the force on the potential to first order in \( \mathbf{v}_V \), using the instantaneous eigenstates of \( H \) as a basis. This gives

\begin{equation}
\mathbf{F} = -\sum_{\alpha} f_\alpha \langle \Psi_\alpha | (\nabla_0 H)|\Psi_\alpha \rangle + \sum_{\alpha} f_\alpha \left\langle \Psi_\alpha | (\nabla_0 H) \frac{iP_\alpha}{E_\alpha - H + i\gamma} \mathbf{v}_V \cdot \nabla_0 + \text{h.c.} |\Psi_\alpha \rangle, \right.
\end{equation}

where \( |\Psi_\alpha \rangle \) is an instantaneous eigenvalue of \( H \) with eigenvalue \( E_\alpha \). \( f_\alpha \) is the equilibrium probability that the state is occupied, and \( P_\alpha \) is the projection operator off the state \( \alpha \). The first term is not proportional to the vortex velocity \( \mathbf{v}_V \), and vanishes if the system has spherical symmetry. We can simplify Eq. (2) further by substituting \( \nabla_0 |\Psi_\alpha \rangle \) with \( \nabla_0 H \), which is \( \nabla_0 V \), because \( \nabla_0 H \) is the commutator of \( \nabla_0 \) and \( H \),

\begin{equation}

\nabla_0 |\Psi_\alpha \rangle = \frac{1}{E_\alpha - H} (\nabla_0 V)|\Psi_\alpha \rangle.
\end{equation}
This gives the resultant longitudinal force as
\[
\mathbf{v}_V \cdot \mathbf{F} = \sum_{\alpha \neq \beta} \frac{f_\alpha - f_\beta}{E_\alpha - E_\beta} \left| \langle \Psi_\beta | \mathbf{v}_V \cdot (\nabla_0 V) | \Psi_\alpha \rangle \right|^2
\]
\[
\gamma \frac{1}{(E_\alpha - E_\beta)^2 + \gamma^2}.
\]
(4)

In the limit that \( \gamma \) goes to zero, which is the case that we consider in the rest of this paper, the last term gives
\[
f_\gamma
\]

For noninteracting particles the sum over states can be replaced by a sum over single-particle wave functions, with the \( f_\alpha \) replaced by fermionic or bosonic occupation probabilities. For a central potential the energy-conserving matrix elements in Eq. (4) can be expressed in terms of phase shifts, using the relation between the plane wave and spherical wave expansions,
\[
\langle r | \psi_k \rangle = \frac{4\pi}{\sqrt{V}} \sum_{lm} i^l A_l(kr) Y_{lm}(\hat{r}) Y_{lm}^* (\hat{k}),
\]
(5)

where \( V \) is the system volume, and the radial wave functions at large distances are
\[
A_l(kr) = e^{i\delta_l} \left[ \cos \delta_l k r - \sin \delta_l m l (kr) \right].
\]
(6)
The matrix element which needs to be evaluated for the longitudinal force given by Eq. (4) is
\[
\langle \psi_\beta | \frac{\partial V}{\partial z} | \psi_\alpha \rangle = \frac{4\pi}{V} \sum_{lm} \sum_{l'm'} i^{-l'-l} Y_{l'm'}^*(\hat{\beta}) Y_{lm}^*(\hat{\alpha})
\]
\[
\int dr r^2 A_{l'}^*(kr) \frac{dV}{dr} A_l(kr) \int d\Omega \cos \theta Y_{l'm'}(\hat{r}) Y_{lm}(\hat{r}),
\]
(7)

if \( r_0 \) is taken to be in the \( \hat{z} \) direction. The last angular integral can be carried out by using the following recurrence relation,
\[
\cos \theta Y_{lm}(\theta, \phi) = \sqrt{\frac{(l + m + 1)(l - m + 1)}{(2l + 3)(2l + 1)}} Y_{l+1,m}(\theta, \phi)
\]
\[
+ \sqrt{\frac{(l + m)(l - m)}{(2l + 1)(2l - 1)}} Y_{l-1,m}(\theta, \phi),
\]
(8)

where the orbital angular momenta are changed by one because the gradient is a tensor operator of rank one. Taking the square of its modulus and integrating \( \hat{k}_\alpha \) and \( \hat{k}_\beta \) over all possible directions will get rid of the remaining spherical harmonics, and end up with a summation over orbital angular momenta,
\[
\int d\Omega_{k\alpha} d\Omega_{k\beta} \left| \langle \psi_\beta | \frac{\partial V}{\partial z} | \psi_\alpha \rangle \right|^2
\]
\[
= \frac{(4\pi)^4}{V^2} \sum_l \frac{2}{3} \left( l + 1 \right) \int dr r^2 A_{l+1}^*(kr) \frac{dV}{dr} A_l(kr). \]
(9)

The summation over magnetic quantum numbers can be easily carried out because there is only one independent direction left after the integration. The radial integral gives the difference between phase shifts (see Appendix A),
\[
2m_0 \int dr r^2 A_{l+1}^*(kr) \frac{dV}{dr} A_l(kr)
\]
\[
= -e^{i(\delta_{l+1} - \delta_l)} \sin(\delta_{l+1} - \delta_l). \]
(10)

There is no contribution to the force from bound states because their wave functions vanish at large distances, and resonance states may potentially play an important role here. So the longitudinal force can be written as an integral over all scattering states,
\[
\frac{F}{Vv} = N \int dk \frac{\partial f_k}{\partial k} \frac{2k^2}{3\pi} \sum_l (l + 1) \sin^2(\delta_{l+1} - \delta_l). \]
(11)

This result was obtained in a similar way by Bönig and Schönhammer.

There is a very similar calculation for scattering by an axially symmetric potential, for which the wave functions are
\[
\langle x | \psi_k \rangle = \frac{1}{\sqrt{V}} \sum_m i^m A_m(k \rho) e^{im(x - \phi_k)}. \]
(12)

The calculation for the matrix element is almost the same except for a different normalization constant,
\[
\left| \langle \psi_\beta | \frac{\partial V}{\partial x} | \psi_\alpha \rangle \right|^2
\]
\[
= \frac{i\pi}{\sqrt{V}} \sum_m e^{-im\phi_\alpha} \int d\rho \frac{dV}{d\rho} A_m
\]
\[
\left( -e^{i(m+1)\phi_\beta} A^*_m + e^{im\phi_\beta} A^*_m \right).
\]
(13)

After integrating all incident directions, only the radial integral remains,
\[
\int d\phi_k \frac{d\phi_k}{dV} \left| \langle \psi_\beta | \frac{\partial V}{\partial x} | \psi_\alpha \rangle \right|^2
\]
\[
= \frac{(2\pi)^2 \pi}{V^2} \sum_m \left| \int d\rho A^*_m \frac{dV}{d\rho} A_m \right|^2,
\]
(14)

and this gives the connection to the phase shifts in the same fashion,
\[
2m_0 \int d\rho A^*_m \frac{dV}{d\rho} A_m
\]
\[
= \frac{2k}{\pi} e^{i(\delta_m - \delta_{m+1})} \sin(\delta_{m+1} - \delta_m). \]
(15)

This gives the longitudinal force as
\[
\frac{F}{Lvv} = N \int dk \frac{\partial f_k}{\partial k} \frac{k^2}{2\pi} \sum_m \sin^2(\delta_{m+1} - \delta_m),
\]
(16)
where $L$ is the length of the system in the axial direction.

Equations (11) and (16) are identical to the equations obtained from the standard kinetic theory argument. In kinetic theory the usual procedure is to keep the scattering potential fixed and to shift the equilibrium distribution so that particles have an average momentum at large distances given by an undetermined multiplier $v_V$. The force is the momentum transfer per unit time from particles to the scattering center. In these dimensions this is related to the differential scattering cross section $\sigma(k, \theta)$ by

$$F = \dot{v}_V \cdot \int \frac{d^d k}{(2\pi)^d} \langle k - m_0 v_V \rangle N \frac{f_k}{m_0} |k - m_0 v_V|$$

(17)

$$\int d\Omega (1 - \cos \theta) \sigma(k - m_0 v_V, \theta) \simeq N_v \int dk k^{d+1} \frac{\delta f_k}{\delta k} \int d\Omega (1 - \cos \theta) \sigma(k, \theta) ,$$

(18)

where, on the right side of the equation, only the first order term in the velocity has been kept. The angular integral over all incident directions can be carried out independently because of the isotropy, and gives

$$\int d\Omega_k \cos^2 \theta_k = \frac{2\pi^{d/2}}{\Gamma(d/2)^d} .$$

(19)

The last scattering cross section integral is often called the transport cross section, $\sigma_{tr}$. For a spherically symmetric potential in three dimensions the scattering amplitude can be expanded in Legendre polynomials as

$$\mathcal{F}(\theta) = \frac{1}{k} \sum_l (2l + 1) \frac{e^{i\delta_l}}{l+l-1} P_l(\cos \theta) ,$$

(20)

where $\delta_l$ is the phase shift for the $l$-th partial wave. The differential cross section is just the modulus squared of the scattering amplitude, and the transport cross section is

$$\sigma_{tr} = \int d\Omega (1 - \cos \theta) \sigma(\theta) = \frac{4\pi}{k^2} \sum_l (l+1) \sin^2(\delta_{l+1} - \delta_l) .$$

(21)

This is in agreement with Eq. (11).

In two dimensions, the scattering amplitude is expanded in Fourier series,

$$\mathcal{F}(\phi) = \sum_m \frac{2i}{\pi k} \sin \delta_m e^{-im\phi} ,$$

(22)

and then transport cross section follows similarly,

$$\sigma_{tr} = \int d\phi (1 - \cos \phi) \sigma(\phi) = \frac{2}{k} \sum_m \sin^2(\delta_{m+1} - \delta_m) ,$$

(23)

in agreement with Eq. (16).

III. LONG-RANGED POTENTIALS

For problems such as the scattering of phonons by a quantized vortex or the scattering of charged particles by an Aharonov–Bohm flux line we need to consider potentials which fall off with distance like $1/r^2$ and for which the phase shifts have a nonzero limit for large angular momentum. A potential of the form

$$2m_0 V(\rho, k_\rho) = \frac{1}{\rho^2} \left( -2i\alpha \frac{\partial}{\partial \phi} + \beta \right) .$$

(24)

This gives an approximate form for scattering of phonons from a quantized vortex with $\alpha = k \kappa_0 / 2\pi c$, and for scattering of charged particles from an Aharonov–Bohm flux line for $\beta = \alpha^2$. The radial wave function is the Bessel function

$$A_m(k\rho) = e^{i\delta_m} J_{\nu(m)}(k\rho) ,$$

(25)

where

$$\nu(m) = \sqrt{m^2 + 2am + \beta} ,$$

(26)

so that the phase shifts have the form

$$\delta_m = \left[ |m| - \nu(m) \right] \pi/2 .$$

(27)

In place of Eq. (14) we have

$$\int d\phi_k \alpha \beta \left| \psi_{\beta} \right|^2 = \left( \frac{2\pi}{m_0} \right)^{\frac{3}{2}} \sum_m \int d\rho A^*_m A_m .$$

(28)

The integral on the right side of this equation can be evaluated in terms of standard integrals of Bessel functions and is equal to

$$(k/\pi) \cos \left( |m| \pi/2 \right) e^{i(\delta_m - \delta_{m+1})} \times$$

$$\frac{1}{|m| \pi/2 - \nu(m)^2 - 1 - 4(a \nu(m)^2 + \beta)} .$$

(29)

Substitution of Eq. (23) into the fraction shows that it is equal to unity, while Eq. (27) shows that the cosine is equal to $\sin(\delta_{m+1} - \delta_m)$ so that the force is given in exactly the same form as in Eqs. (11) and (23).

For this type of potential the forward scattering amplitude diverges like $1/\phi$, but for the transport cross section the factor of $1 - \cos \phi$ cancels the divergence. For the transverse cross section the situation is much more delicate, as is shown in the recent discussion by Wexler and Thouless.
IV. SUMMARY

We have shown that, both for a short-ranged central potential, and for a long-ranged potential of the sort given by a flux line or vortex, calculation of the longitudinal force on a moving potential in a noninteracting background by kinetic theory and by time-dependent perturbation theory give the same results to lowest order in the velocity.

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APPENDIX A: RADIAL SCHröDINGER EQUATION

Here is the key identity for establishing the equivalence between the two theories. Since we did not specify a particular pinning potential, any general property of the matrix element could be derived from the radial wave equation which in three dimensions is

$$
\frac{1}{r} \frac{d}{dr} \left[ r A_l(kr) \right] + \left[ k^2 - 2m_0 V(r) - \frac{l(l+1)}{r^2} \right] A_l(kr) = 0 .
$$

(A1)

A general relation between different \( l \)'s can be derived as following,

$$
r^n \left[ A_l \frac{dA_l}{dr} - \frac{dA_{l'}}{dr} A_l \right] \bigg|_0 = (k'^2 - k^2) \int drr^n A_{l'} A_l
$$

+ \( (n-2) \int drr^{n-1} \left[ A_{l'} \frac{dA_l}{dr} - \frac{dA_{l'}}{dr} A_l \right] \]

- \( l'(l'+1) - l(l+1) \) \int drr^{n-2} A_{l'} A_l . \quad (A2)

To evaluate the radial part of the matrix element in Eq. (3), we first integrate it by parts, replace \( V(r)A_l(kr) \) using the radial wave equation, and then use Eq. (A2) with \( n = 1 \),

$$
2m_0 \int drr^2 A_{l'} \frac{dV}{dr} A_l = \left[ r^2 \frac{dA_{l'}}{dr} - \frac{2l(l+1)}{l'(l'+1) - l(l+1)} rA_{l'} A_l \right] \bigg|_0
$$

+ \( (k'^2 - k^2) \int dr \left[ r^2 A_{l'} \frac{dA_l}{dr} - 2l'(l'+1) - l(l+1) \right] rA_{l'} A_l \]

- \( l'(l'+1) - l(l+1) - 2l'(l'+1) + l(l+1) \) \int dr A_{l'} \frac{dA_l}{dr} . \quad (A3)

Using the asymptotic expansion for the Bessel functions, the boundary term gives the difference between two phase shifts. Similar relation could be established from the wave equation in two dimensions, or could be done by directly substituting \( A_l(x) \rightarrow \sqrt{\frac{x}{2\pi}} A_m(x) \) with \( l = m - 1/2 \),

$$
2m_0 \int d\rho A_{m'} \frac{dV}{d\rho} A_m = \left[ \rho \frac{dA_{m'}}{d\rho} - \frac{\rho k^2 A_{m'}}{A_m} \right] \bigg|_0 \infty
$$

+ \( (k'^2 - k^2) \int d\rho \left[ \rho A_{m'} \frac{dA_m}{d\rho} - m'^2 + 3m^2 - 1 \right] \]

- \( m'^2 - m^2 - \frac{2m'^2 + 2m^2 - 1}{m'^2 - m^2} \) \int d\rho \left[ A_{m'} \frac{dA_m}{d\rho} - \frac{A_m A_{m'}}{2\rho^2} \right] . \quad (A4)

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