Virial coefficients in $(\tilde{\mu}, q)$-Bose gas model related to compositeness of particles and their interaction: temperature-dependence problem

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We establish the relation of the second virial coefficient of certain $(\tilde{\mu}, q)$-deformed Bose gas model, recently proposed by the authors in [Ukr. J. Phys., 2013], to the interaction and compositeness parameters when either of these factors is taken into account separately. When the interaction is dealt with, the deformation parameter becomes linked directly to the scattering length, and the effective radius of interaction (in general, to scattering phases). The additionally arising temperature dependence is a new feature absent in the deformed Bose gas model within adopted interpretation of the deformation parameters $\tilde{\mu}$ and $q$. Here the problem of the temperature dependence is analyzed in detail and its possible solution is proposed.

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I. INTRODUCTION

Nonlinear physical system involving either of the nonideality/nonlinearity factors are often effectively described by means of some deformed (algebraic, phenomenological or other) models as counterpart to their corresponding “ideal” prototype. Deformed Bose gas models (DBGMs) along with deformed oscillators, deformed quantum mechanics and some other extensions, being evolved since the end of 1980s till now, belong to such models of an effective description. Generally speaking the DBGMs may or may not be directly linked with deformed oscillators, which in many cases are taken as the structural object underlying the former. Soon after introducing deformed oscillators or deformed bosons [1–3], their use for elaborating respective deformed analogs of Bose gas model became popular. Already the early instances of DBGM, to list a few [4–9], had witnessed the appearance of very important new direction with a long-term perspective and with good potential for useful (and realistic) applications. The latter range from $^4$He system [10] to e.g. such high-energy physics objects as two- and three-particle correlations of pions generated in relativistic collisions of nuclei [11]. These applications yielding a good effective description stimulated the study of deeper reasons of the applicability of DBGMs to real physical systems. Helpful, from this viewpoint, appears the idea that deformation of ideal Bose gas model could and should provide an efficient effective description of the properties of more realistic (i.e. non-ideal) gases of Bose like particles. Moreover, the deviation from strict ideality may originate for several reasons (“nonideality” factors). It was demonstrated [12] that non-pointlike form of particles may serve as the first and most obvious such a factor, and it is possible to link the parameter $q$ of deformation directly with the ratio of excluded volume (the sum of nonzero proper volumes of the particles) to the whole volume. The next factor of non-ideality is the interaction between the particles, and as shown in [13] it can be naturally taken into account by a version of deformation.

More recently it became clear that the possibility of realization (of operator algebra) of composite bosons by deformed bosons proven in [14] is naturally promoted to the elaboration of DBGM able to effectively account for compositeness of Bose particles (the compositeness makes them quasi-bosons, differing from strict bosons). At last, let us mention recent work [15] which shows how to incorporate simultaneously two different factors of non-ideality: the compositeness of particles and their interaction. That work motivated our present study.

Let us mention some versions of DBGMs and their applications to physical systems in different contexts. Thermodynamics of the $q$-DBGMs was studied e.g. in [16, 17]; for the Bose condensation of the deformed gases see [18]. The DBGMs and many-body systems of $q$-bosons were applied to phonon gas in $^4$He [10], to excitons in [19], to a study of pairing correlations in nuclei [20]. Some of DBGMs were applied when studying two-particle correlation functions [11, 21–24]. The extent or strength of deformation of the mentioned models usually is characterized by one or more deformation parameters. Till now, the question about the relation between the deformation parameters and the microscopic nonideality factors and their parameters remained opened, and the microscopic analysis of the correspondence between a physical system and its deformed counterpart was still absent.

In this work, similarly to [14] where the deformation parameter for the realization of bifermionic composites (quasibosons) by deformed bosons was related to the wavefunctions of bifermionic states being realized, we establish the relation between the deformation of a special
is constructed so that the thermodynamics/statistical physics is given through the partition function $i.e. N = \sum_{n=0}^{\infty} \frac{N^n}{n!} = \exp(N^{(\tilde{\mu},q)}(z, V, T))$. This structure function determines quantitatively how the virial coefficients within the DBGMs studied therein depend only on the deformation parameter(s) which in our interpretation are interrelated with nonideality factors and thus should not depend on the temperature. On the other hand, the virial coefficients for a gas with interaction manifest temperature dependence [27]. Just this problem is in the focus of the present work.

II. RELATION OF DEFORMATION PARAMETERS TO THE INTERACTION BETWEEN QUASIBOSONS AND TO THEIR COMPOSITENESS

We start with the recently obtained [15] deformed virial expansion for the $\tilde{\mu},q$-deformed Bose gas whose thermodynamics/statistical physics is given through the structure function $\varphi_{\tilde{\mu},q}(N)$ (denote $[N]_q \equiv 1 - q^N$),

$$\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}([N]_q) = (1 + \tilde{\mu})[N]_q - \tilde{\mu}[N]_q^2.$$  
(1)

The structure function determines quantitatively how the thermodynamics/statistical physics is “deformed” for that or another system. Namely, in [15] the $\tilde{\mu},q$-DBGM is constructed so that $\varphi_{\tilde{\mu},q}(z \frac{d}{dz})$ replaces the derivative $z \frac{d}{dz}$ in the known relation for the total number of particles given through partition function i.e. $N = z \frac{d}{dz} \ln Z$, yielding the definition for the deformed total number of particles in terms of the nondeformed partition function:

$$\tilde{N} \equiv N^{(\tilde{\mu},q)}(z, V, T) \equiv \varphi_{\tilde{\mu},q}(z \frac{d}{dz}) \ln Z.$$  
(2)

All other deformed physical quantities are recovered using the (non-deformed) version of the relations of ideal quantum Bose gas. For instance, for the second virial coefficient, which is of interest for us herein, within the $\tilde{\mu},q$-DBGM we have obtained [15]

$$V_2^{(\tilde{\mu},q)} = \frac{\varphi_{\tilde{\mu},q}(2)}{2^{7/2}} = \frac{(1 + q)(1 - \tilde{\mu}q)}{2^{7/2}}.$$  
(3)

In our treatment, the parameter $q$ of $\varphi_{\tilde{\mu},q}(N)$ corresponds to effective taking the interparticle interaction into account, and $\tilde{\mu}$ – to composite-structure effects. Somewhat earlier, the Arik-Coon structure function $[N]_q$ was used to effectively incorporate [13] the interaction between the particles of a gas of elementary bosons.

Note that if, in addition to deformed thermodynamic relations, the structure function $\varphi_{\tilde{\mu},q}(N)$ describes some deformed boson algebra related to the $\tilde{\mu},q$-DBGM studied herein, certain ranges of admissible $\tilde{\mu}$ and $q$ hold. These can be deduced from the condition $\varphi_{\tilde{\mu},q}(n) \geq 0$, $n = 1, N_{\text{max}}$ which corresponds to non-negativity of the norm of deformed boson Fock states ($N_{\text{max}}$ is maximum occupation number). In particular the non-negativity of $\varphi_{\tilde{\mu},q}(2)$ yields $\tilde{\mu}q \leq 1$ and $q \geq -1$. However we do not appeal to the relation with a deformed boson algebra.

Besides $\varphi_{\tilde{\mu},q}(N)$, one can take yet another versions of combining the two structure functions $\varphi_{\tilde{\mu}}(N)$ and $[N]_q$, e.g. in the form $\varphi_{\tilde{\mu},q}(N) = [\varphi_{\tilde{\mu}}(N)]_q$, or as the family with one more parameter: $t\varphi_{\tilde{\mu},q}(N) + (1-t)\varphi_{\tilde{\mu},q}$, for $0 \leq t \leq 1$. Remark that the treatment below can be extended to the case of even more general structure function $\varphi(N)$ when some of the deformation parameters are responsible for interparticle interaction, and the others – for the composite structure of particles in the effective description.

a. Effective account for the particle-particle interaction to $(\lambda^3/\nu)^2$-terms. As known, the deviation (from the ideal or non-interacting case) of the second virial coefficient $V_2$ due to the two-particle interaction is expressed through the partial wave phaseshifts $\delta_l(k)$ and the bound state (if any) energies $\epsilon_B$ as follows [27]

$$V_2 - V_2^{(0)} = -\frac{8^{1/2}}{\pi} \sum_{l=1}^{\infty} \int_0^{\infty} e^{-\frac{\beta}{k^2}l_m} \frac{\partial \delta_l(k)}{\partial k} dk.$$  
(4)

Here $B$ runs over bound states, $l$ is the angular momentum quantum number and the summation is performed over even $l$ in bosonic case, and over odd $l$ in the fermionic case. In low-energy approximation we retain in (4) only the $l = 0$ summand ($s$-wave approximation). The corresponding phaseshift $\delta_0(k)$ generally can be determined by solving Schrodinger equation for a specified interaction potential. However, in the low-energy limit (when $l = 1$ effects are negligible) the following expansion known as effective range approximation holds [28–30]:

$$k \cot \delta_0 - \frac{1}{a} + \frac{1}{2} r_0 k^2 + ..., \quad r_0 = 2 \int dr \left[ \left( \frac{1}{r} - \frac{1}{a} \right)^2 - \chi_0^2(r) \right].$$  
(5)

where $a$ is the scattering length, $r_0$ – effective range (radius), and $\chi_0(r)$ being the radial wavefunction of the lowest state multiplied by $r$. Since for some typical potentials $r_0$ depends only on the range and depth of the potential, this expansion is sometimes called as “shape-independent approximation”. For the shape-independent approximation we find $\frac{\partial \delta_0}{\partial k} = -a + (a - 3 r_0/2) a^2 k^2 + \ldots$
Putting this derivative in (4) and performing integration, within the s-wave approximation we obtain

\[ V_2 - V_2^{(0)} = -8^{1/2} \sum_B \frac{e^{-\beta \varepsilon_B} + 2a}{\lambda_T} - 2\pi^2 \left( 1 - \frac{3}{2} \frac{\alpha}{\lambda_T} \right)^3 + O((a/\lambda_T)^5), \]  

(6)

where \( \lambda_T \equiv \hbar / \sqrt{2\pi mk_B T} \) is the thermal wavelength.

Below we give the explicit expressions for \( V_2 - V_2^{(0)} \), or for the pair \( a \) and \( r_0 \) through which it is expressed in (6), for a number of potentials (their definitions and some details are relegated to appendix A):

- **Hard spheres interaction potential** (A1). We have [27]

\[ V_2 - V_2^{(0)} = 2 \frac{\lambda}{\lambda_T} + \frac{10\pi^2}{3} \left( \frac{\lambda}{\lambda_T} \right)^5 + \ldots \quad (l = 0, 2). \]  

(7)

- **Constant repulsive potential** (A3). For this and subsequent potentials the corresponding quantities are given using [29]. So, we have

\[ a = R \left( 1 - \frac{\text{th} K_0}{K_0 R} \right), \quad r_0 = 0. \]  

(8)

- **Square-well potential** (A5). For this,

\[ a = -R \left( \frac{\text{tg} K_0}{K_0 R} - 1 \right), \quad r_0 = R \left( 1 - \frac{1}{K_0^2} \frac{R^2}{3a^2} \right). \]  

(9)

- **Anomalous scattering potential** (A7). For this,

\[ a = R - \frac{\text{th} K_0 (R - r_1) + \frac{\pi}{\lambda_T} \text{tg} K r_1}{K_0 (1 + \frac{\pi}{\lambda_T} \text{tg} K r_1) \text{th} K_0 (R - r_1)}. \]  

Somewhat awkward expression for \( r_0 \) is omitted.

- **Scattering resonances** (A9). For this,

\[ a = \frac{\Omega}{\Omega + 1} R, \quad r_0 = \frac{2}{3} \frac{\Omega}{\Omega + 1} R. \]  

(11)

- **Modified Pöschl-Teller potential** (A11). At integer \( \lambda \),

\[ a = \frac{1}{\alpha} \sum_{n=1}^{\lambda-1} \frac{1}{n}, \quad r_0 = \frac{2}{3\alpha} \left( \frac{\sum_{n=1}^{\lambda-1} n^{-1}}{n} \right)^3 - \frac{\sum_{n=1}^{\lambda-1} n^{-3}}{\left( \sum_{n=1}^{\lambda-1} n^{-1} \right)^2}. \]  

(12)

- **Inverse power repulsive potential** (A13). For this,

\[ a = r_0 \frac{\Gamma \left( 1 - \frac{2}{\eta} \right)}{\Gamma \left( 1 + \frac{2}{\eta} \right)} \left( \frac{\eta}{2\eta} \right)^{1/2}, \quad \eta = \frac{n - 2}{2}, \]  

(13)

and \( r_0 \) can be found from (5) using (A15).

With the data given above, we have the deviation of the second virial coefficient in eq. (6) from that of ideal Bose gas, for each of the considered potentials (A1), (A3), ..., (A13).

On the other hand, within \( \varphi_{\tilde{\mu}, a} \)-deformed Bose gas model we have [15] (see also eq. (3)):

\[ V_2^{(\tilde{\mu}, a)} - V_2^{(0)} \big| _{\tilde{\mu}=0} = \frac{2 - \varphi_{\tilde{\mu}, a}(2)}{2^{7/2}} \big| _{\tilde{\mu}=0} = \frac{1 - q}{2^{7/2}}. \]  

(14)

By juxtaposing this with (6), we obtain

\[ q = q(a, r_0, T) = 1 - 2^{9/2} \frac{a}{\lambda_T} + 2^{13/2} \frac{a^2}{(1 + \frac{2}{\lambda_T})^3} + \ldots + 2^{13/2} \sum_B e^{-\beta \varepsilon_B}, \]  

(15)

which constitutes one of our main results. Of course, this formula should be appended with \( a \) and \( r_0 \) taken e.g. for the chosen cases from (8)-(13), or for any other desired case.

The temperature dependence of the deformation parameter in (15) appears somewhat unexpected since, in our interpretation, the deformation parameter characterizes the nonideality of deformed Bose gas model as a whole, and \( T \) is its internal parameter. One of the approaches to resolve this issue consists in a modification of the very deformation in deformed Bose gas model. For instance, we can use the extended deformed derivative (here \( z = e^{\beta \mu} \) is the fugacity, \( \mu \) the chemical potential)

\[ z \frac{\partial}{\partial z} \rightarrow z \tilde{D}_z \equiv \varphi \left( z \frac{\partial}{\partial z} \right) + \chi \left( z \frac{\partial}{\partial z} \right) \frac{\partial}{\partial \beta} + g(\beta) \rho \left( z \frac{\partial}{\partial z} \right), \]  

(16)

with structure functions \( \varphi, \chi, \rho \), in the relation

\[ \tilde{N} = z \tilde{D}_z \ln Z^{(0)} = \varphi(\frac{\partial}{\partial z}) \ln Z^{(0)} - \chi(\frac{\partial}{\partial z}) U^{(0)} - \beta g(\beta) \rho(\frac{\partial}{\partial z}) \tilde{\Phi}_G^{(0)}. \]  

Here \( Z^{(0)}, U^{(0)} \) and \( \tilde{\Phi}_G^{(0)} \) are nondeformed partition function, internal energy and Gibbs thermodynamic potential respectively; \( z, V, T \) serve as independent variables. Thus, on the thermodynamics level, \( \chi \) and \( \rho \) reflect the effect on the total number of particles of the internal energy and Gibbs thermodynamic potential which now appear on the same footing as the logarithm of grand partition function. The corresponding analysis will be carried out in sec. III below.

*Remark.* It is worth to estimate the relative magnitude of the terms \(-8^{1/2} \sum_B e^{-\beta \varepsilon_B} \) and \( 2 \frac{\alpha}{\lambda_T} \) in (6) at low-energy scattering when bound states do exist. According to [29] we have the following estimate for the binding energy in terms of scattering data:

\[ \varepsilon_B \simeq -\frac{\hbar^2}{2ma^2} (1 + \frac{r_0}{a}). \]  

Using this we come to

\[ -8^{1/2} e^{-\beta \varepsilon_B} + 2 \frac{a}{\lambda_T} \simeq -8^{1/2} e^{\frac{a^2}{2ma^2} (1 + \frac{r_0}{a})} + 2 \frac{a}{\lambda_T} = \]  

\[ = -8^{1/2} e^{\frac{a}{2m} (1 + \frac{r_0}{a})} + 2 \frac{a}{\lambda_T} < 0 \quad \text{for} \quad a/\lambda_T < 1. \]  

(17)

Thus, the binding energy term in (6) (if a bound state exists) is dominating over \( 2 a/\lambda_T \) for small \( a/\lambda_T \).
b. Effective account for the compositeness of particles up to \((\lambda^3/\nu)^2\)-terms. Let us now evaluate the second virial coefficient in the absence of explicit interaction between quasibosons (composite bosons). Note that the partition function from which the second virial coefficient can be extracted, for the system of composite bosons within a general framework was considered in [31]. Within our approach (which is both effective and efficient), however, the task of obtaining the virial coefficient(s) is completely tractable leading for the deformed Bose gas to exact results.

Two-component quasibosons concerned here have the following creation/annihilation operators [12, 14]
\begin{equation}
A_\alpha^\dagger = \sum_{\mu \nu} \Phi_{\alpha \mu}^\dagger a_\mu b_\nu, \quad A_\alpha = \sum_{\mu \nu} \Phi_{\alpha \mu} b_\mu a_\nu,
\end{equation}

where \(a_\mu^\dagger, b_\nu^\dagger\), and \(a_\mu, b_\nu\) are the creation and annihilation operators for the constituent fermions, and the set of matrices \(\Phi_{\alpha \mu}\) determine the quasiboson wavefunction. As a starting point we take the known general expression for 2nd virial coefficient [27]
\begin{equation}
V_2 = \frac{1}{2V}[(\text{Tr}_1 e^{-\beta H_1})^2 - \text{Tr}_2 e^{-\beta H_2}].
\end{equation}

Here \(\text{Tr}_1\) denotes the trace over one-quasiboson states and \(\text{Tr}_2\) over the states of two quasibosons; \(H_1\) and \(H_2\) are respectively one- and two-quasibosonic Hamiltonians. The distinction between the second virial coefficients for the ideal Bose- and ideal Fermi gases is caused by the nilpotency of the fermionic creation operators, and consequently by the nullifying of the respective terms in \(\text{Tr}_2 e^{-\beta H_2}\) from (19). Analogously, in the case of bi-fermionic quasibosons the nonzero summands from \(\text{Tr}_2 e^{-\beta H_2}\) are determined by the condition 
\begin{equation}
|\langle A_\alpha^\dagger A_\beta^\dagger \rangle|^2 \neq 0.
\end{equation}

Let us calculate \(|\langle A_\alpha^\dagger A_\alpha^\dagger \rangle|^2\):
\begin{equation}
|\langle A_\alpha^\dagger A_\alpha^\dagger \rangle|^2 = \sum_{\mu_1 \mu_2 \nu_1 \nu_2} \langle 0| b_\nu^\dagger a_{\nu_2}^\dagger a_{\mu_2}^\dagger \Phi_{\alpha \mu_2}^\dagger \Phi_{\alpha \mu_1}^\dagger \Phi_{\alpha \nu_2}^\dagger b_{\nu_1} a_{\nu_1} a_{\mu_1} 0 \rangle.
\end{equation}

The traces in (19) are calculated as follows:
\begin{equation}
\text{Tr}_1 e^{-\beta H_1} = \sum_{k_1 n_1} \langle k_1 n_1| e^{-\beta H_1} A_\alpha^\dagger A_\alpha^\dagger |k_1 n_1\rangle = \sum_{k_1 n_1} e^{-\beta \varepsilon_{k_1 n_1}},
\end{equation}

\begin{equation}
\text{Tr}_2 e^{-\beta H_2} = 1/2 \sum_{(k_1 n_1) \neq (k_2 n_2)} \langle k_2 n_2| e^{-\beta H_2} A_\alpha^\dagger A_\alpha^\dagger |k_1 n_1\rangle + \frac{1}{2} \langle A_\alpha^\dagger A_\alpha^\dagger |A_\alpha^\dagger A_\alpha^\dagger \rangle = 
\end{equation}

\begin{equation}
= \frac{1}{2} \left( \sum_{k_1 n_1} e^{-\beta \varepsilon_{k_1 n_1}} \right)^2 - \frac{1}{2} \sum_{k_1 n_1} e^{-2\beta \varepsilon_{k_1 n_1}} + \sum_{k_1 n_1} e^{-2\beta \varepsilon_{k_1 n_1}}.
\end{equation}

Here \(k_{1,2}\) is the momentum quantum number, \(n_{1,2}\) contains all the other quasibosonic quantum numbers, \(\varepsilon_{k_{1,2}}\) is the energy of quasiboson in the state \(|k_{1,2}\rangle\) and the prime in \(\sum\) implies the summation over all the modes \((k,n)\) for which \(\langle A_{k,n}^\dagger A_{k,n}^\dagger \rangle \neq 0\). Substituting (21) and (22) in (19) and splitting \(\varepsilon_{k_{1,2}}\) into kinetic energy \(\frac{\hbar^2 k_{1,2}^2}{2m}\) and internal energy \(\varepsilon_{n_{1,2}}\) as \(\varepsilon_{k_{1,2}} = \frac{\hbar^2 k_{1,2}^2}{2m} + \varepsilon_{n_{1,2}}\) we obtain
\begin{equation}
V_2(T) = \frac{1}{2\beta^2} \sum_{k_{1,2}} e^{-2\beta \varepsilon_{n_{1,2}}} - \frac{\lambda_2^2}{2V} \sum_{k_{1,2}} e^{-2\beta \varepsilon_{n_{1,2}}} + \varepsilon_{n_{1,2}}.
\end{equation}

If for all the \((k,n)\)-modes \(\langle A_{k,n}^\dagger A_{k,n}^\dagger \rangle \neq 0\), then performing the summation over \(k\) according to \(\sum_k e^{-2\beta \frac{\hbar^2 k^2}{2m}} = 2^{-3/2}/V \lambda_2^2\) we obtain
\begin{equation}
V_2(T) - V_2^{(0)} = -\frac{1}{2\beta^2} \left( \sum_{k_{1,2}} e^{-2\beta \varepsilon_{n_{1,2}}} - 1 \right).
\end{equation}

On the other hand, in the deformed case we have the (exact) result [15] (see also eq. (3)), i.e.
\begin{equation}
V_2^{(\tilde{\mu},q)} - V_2^{(0)} |_{q=1} = 2 - \frac{\varphi_{\tilde{\mu},q}(2)}{2^{3/2}} |_{q=1} = \frac{\tilde{\mu}}{2^{5/2}}
\end{equation}

from which after juxtaposing, according to our interpretation, with (24) we arrive at
\begin{equation}
\tilde{\mu} = \mu(\varepsilon_{n_{1,2}}, \Phi_{\alpha \mu}, T) = 1 - \sum_{n_{1,2}} e^{-2\beta \varepsilon_{n_{1,2}}}
\end{equation}

(the dependence on \(\Phi_{\alpha \mu}\) is retained for general case). As now is seen, the obtained difference (24) is mainly related with the internal energy of a quasiboson, not with its (nonbosonic) commutation relations.

The structure function \(\varphi_{\tilde{\mu},q}(N)\) with \(q = q(a, r_0, T)\), \(\mu = \mu(\varepsilon_{n_{1,2}}, \Phi_{\alpha \mu}, T)\) is chosen for the goal of the effective account (in certain approximation) for the factors of interaction and of composite structure of particles of a gas. Let us emphasize that the direct microscopic treatment may lead to quite different relation between the second virial coefficient incorporating the both factors (interaction and compositeness) and the virial coefficients involving only one nontrivial factor. The functional composition as in (1) may not already hold, nevertheless, the linear part of the Taylor expansion of \(V_2^{(\tilde{\mu},q)}\) in small \(\epsilon = q - 1\) and \(\tilde{\mu}\) may coincide with the corresponding part found from the microscopic treatment.

It is clear that the modification of deformation according to (16) may lead to quite different dependence of deformation parameters on the characteristics of interaction and compositeness.

Let us note that the major deformation structure function \(\varphi\) in (16) is a general one. The choice \(\varphi\left(\frac{z}{\sqrt{\pi}}\right) = \varphi_{\mu,q}\left(\frac{z}{\sqrt{\pi}}\right)\) results in the formulas (14) and (25) for the virial coefficient \(V_2\). Clearly, other choices for \(\varphi\) in (16) will result in other form of respective virial coefficient \(V_2\) and the respective temperature dependence.
III. MODIFICATION OF DERIVATIVE $z\frac{\partial}{\partial z}$
AIMED TO YIELD TEMPERATURE DEPENDENT VIRIAL COEFFICIENTS

As already mentioned, we can obtain temperature dependent deformed (i.e. within the deformation-based approach) virial coefficients say by performing the extension of the deformed derivative, see (16). The functions $\varphi$, $\chi$, $\rho$ should not depend on the temperature. The term $g(\beta)\rho\left(z\frac{\partial}{\partial z}\right)$ is introduced in order to reflect the ambiguity in the (left or right) position of $\partial/\partial\beta$, i.e. to cover the terms like $\frac{\partial}{\partial\beta}\chi\left(z\frac{\partial}{\partial z}\right)$. This can be verified by means of the commutation relation

$$\left[\partial/\partial\beta, f(z\partial/\partial z)\right] = -\beta^{-1}(z\partial/\partial z)\cdot f'(z\partial/\partial z).$$  \hspace{1cm} (27)

The noncommutativity of derivatives $\partial/\partial\beta$ and $z\partial/\partial z$ is observed after presenting $z\partial/\partial z$ as $\beta^{-1}\partial/\partial\mu$, where $\mu$ is chemical potential (recall that $z = e^{\beta\mu}$).

Applying deformed derivative (16) to the known expansion for the partition function

$$\ln Z^{(0)}(z, V, T) = \frac{V}{\lambda_T^3} \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}}$$

we obtain the following series for the deformed (that is why we use tilde) total number of particles

$$\tilde{N} = z\tilde{D}_2 \ln Z^{(0)}(z, V, T) = \frac{V}{\lambda_T^3} \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z - 3/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z^n}{n^{5/2}}. $$ \hspace{1cm} (28)

Deformed partition function is then recovered as

$$\ln \tilde{Z} = (d/dz)^{-1}\tilde{N} = \frac{V}{\lambda_T^3} \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z - 5/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z^n}{n^{7/2}}. $$ \hspace{1cm} (29)

Expanding fugacity as $z = z_0 + z_1\frac{\lambda_T^3}{\tilde{v}} + z_2(\frac{\lambda_T^3}{\tilde{v}})^2 + ...$, denoting by $\tilde{v} = \frac{\lambda_T^3}{\tilde{v}}$ the deformed specific volume, and remembering that $z_i = z_i(T)$, $i = 0, 1, ...$, after substituting the resulting expansion into (28) we obtain the relation

$$\frac{\lambda_T^3}{\tilde{v}} = \sum_{n=0}^{\infty} R_n(T; \varphi, \chi, \rho) \left(\frac{\lambda_T^3}{\tilde{v}}\right)^n$$ \hspace{1cm} (30)

where the coefficients at the same powers of $\frac{\lambda_T^3}{\tilde{v}}$ in the l.h.s. and r.h.s. should be

$$R_0 \equiv \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z_0 - 3/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z_0^n}{n^{5/2}} = 0,$$ \hspace{1cm} (31)

$$R_1 \equiv \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z_0 - 1/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z_0^{n-1}}{n^{1/2}} = 1,$$ \hspace{1cm} (32)

$$R_2 \equiv \left(\frac{z_0}{z_1} - \frac{1}{2}\frac{z_1^2}{z_0}\right) \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z_0 + 1/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z_0^{n-1}}{n^{1/2}} = 0,$$ \hspace{1cm} (33)

Similarly, for the deformed equation of state $\frac{\tilde{P}}{k_BT} = \ln \tilde{Z}$, using (29) we find the following virial $\lambda_T^3/\tilde{v}$-expansion

$$\frac{\tilde{P}}{k_BT} = \frac{1}{\lambda_T^3} \tilde{V}_0(T; \varphi, \chi, \rho) + \tilde{v}^{-1} \sum_{n=1}^{\infty} \tilde{V}_n(T; \varphi, \chi, \rho) \left(\frac{\lambda_T^3}{\tilde{v}}\right)^{n-1}$$ \hspace{1cm} (34)

with virial coefficients

$$\tilde{V}_0 \equiv \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z_0 - 5/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z_0^n}{n^{1/2}} = 0,$$ \hspace{1cm} (35)

$$\tilde{V}_1 \equiv \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z_0 - 3/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z_0^{n-1}}{n^{1/2}} = 1,$$ \hspace{1cm} (36)

$$\tilde{V}_2 \equiv \left(\frac{z_0}{z_1} - \frac{1}{2}\frac{z_1^2}{z_0}\right) \tilde{V}_1 + \frac{1}{2}\frac{z_1^2}{z_0} \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z_0 - 1/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z_0^{n-1}}{n^{1/2}},$$  \hspace{1cm} (37)

The equalities in (35), (36) are imposed in order that virial expansion (34) reproduces the corresponding limit of classical ideal gas. One of the solutions of (31) and (35) is $z_0 = 0$. Let us dwell on this case. Deformed equation of state $\tilde{P} = \tilde{P}(\lambda_T^3/\tilde{v})$, see (34), can be written in the implicit parametric form (see (28), (29)),

$$\frac{\lambda_T^3}{\tilde{v}} = \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z - 3/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z^n}{n^{5/2}} +$$ \hspace{1cm} (38)

$$\frac{\tilde{P}}{k_BT} = \frac{1}{\lambda_T^3} \sum_{n=1}^{\infty} \left[\varphi(n) + \beta^{-1}(n\chi(n)\ln z - 5/2\chi(n) + n\chi'(n)) + g(\beta)\rho(n)\right] \frac{z^n}{n^{1/2}},$$ \hspace{1cm} (39)

Value $z = z_0 = 0$ corresponds to $\lambda_T^3/\tilde{v}|_{z=0} = 0$, as $z \ln z \to 0$ at $z \to 0$ in (38). Consider the first derivative of (39) by $\lambda_T^3/\tilde{v}$ namely
\[ \frac{\partial (\tilde{P}/(k_BT))}{\partial (\lambda T^2/\tilde{v})} = \frac{\partial (\tilde{P}/(k_BT))/\partial z}{\partial (\lambda T^2/\tilde{v})/\partial z} = \frac{1}{\lambda T^2} \]

\[ \sum_{n=1}^{\infty} \left[ \varphi(n) + \beta^{-1}(n \ln z - \frac{1}{2})\chi(n) + n\chi'(n) \right] \frac{z^n}{n^{3/2}} = \sum_{n=1}^{\infty} \left[ \varphi(n) + \beta^{-1}(n \ln z - \frac{1}{2})\chi(n) + n\chi'(n) \right] \frac{z^n}{n^{3/2}} \]

For the second derivative we obtain

\[ \frac{\partial^2 (\tilde{P}/(k_BT))}{\partial (\lambda T^2/\tilde{v})^2} = \frac{1}{\lambda T^2} \left[ \varphi(1) + \beta^{-1}(1)\chi(1)(\ln z - 1/2) + \chi'(1) \right] \sum_{n=1}^{\infty} \left[ \varphi(n) + \beta^{-1}(n \ln z - \frac{1}{2})\chi(n) + n\chi'(n) \right] \frac{z^n}{n^{3/2}} \]

\[ \sum_{n=1}^{\infty} \left[ \varphi(n) + \beta^{-1}(n \ln z - \frac{1}{2})\chi(n) + n\chi'(n) \right] g(\beta) \rho(1) \]

\[ \to \frac{1}{z^2(\lambda T)^2} \left[ \varphi(1) + \beta^{-1}(1)\chi(1)(\ln z - \frac{1}{2}) + \chi'(1) \right] g(\beta) \rho(1) \to \frac{1}{z^2(\lambda T)^2} \]

This completes the derivation for the second derivative of \( \chi(2) \) in terms of \( \varphi(2) \).

Likewise, the requirement of finiteness leads to \( \chi(2) = 0 \) and hence to

\[ \tilde{V}_2 = \frac{\beta^2}{2\tilde{v}^2} \frac{\partial^2 (\tilde{P}/(k_BT))}{\partial (\lambda T^2/\tilde{v})^2} = \frac{1}{\lambda T^2} \left[ \varphi(1) + \beta^{-1}(1)\chi(1) + \beta^{-1}(1)\chi'(1) \right] g(\beta) \rho(1) \]

Using the last expression we can compare the result of the microscopic treatment with the action of the deformation. In the r.h.s. of (42) we have exactly the r.h.s.

of (28). Taking there \( \chi(n) = 0, n = 1, 2, \ldots \) (as the simplest variant to exclude singularity at \( z \to z_0 = 0 \)) and comparing the first two terms with the corresponding ones in the l.h.s. of (42) we come to the relations

\[ \varphi(1) + g(\beta)\rho(1) = 1, \]

\[ \varphi(2) + g(\beta)\rho(2) = 2(1 - 27/2a/\lambda T). \]

From these we find

\[ g(\beta) = 2^{9/2}(a/\lambda T_0 - a/\lambda T)^{-1/2}, \]

\[ \varphi(1) = 1, \quad \rho(1) = 0, \]

where \( T_0 \) is defined from \( \varphi(2) = 2(1 - 27/2a/\lambda T_0) \).

The first example of the respective deformed derivative is

\[ \left[ \frac{z d}{dz} \right] + (\lambda T_0/\lambda T - 1)(q - 1) \left( \frac{z d}{dz} - 1 \right) \]

where

\[ q = 1 - 2^{9/2}a/\lambda T_0. \]

Note, the form of the latter is very natural: it shows that the extent (magnitude) \( 1 - q \) of deformation is just proportional to the scattering length \( a \) divided by \( \lambda T_0 \).

More general case is the \( \mu, q \)-deformed one. For it,

\[ z\tilde{D}_2 = \frac{\varphi(2)}{\lambda T} \left( \frac{z d}{dz} + (\lambda T/\lambda T - 1)(q - 1) \right) \left( \frac{z d}{dz} - 1 \right) - \frac{1}{1 - e^{-2\beta_0^2}} \sum_n \frac{e^{-2\beta_0^2 e^{2n\varphi-q}}} {\mu \left( \frac{z d}{dz} - 1 \right)} \]

\[ q = 1 - 2^{9/2}a/\lambda T_0, \quad \mu = 1 - \sum_n e^{-2\beta_0^2 e^{2n\varphi-q}}. \]

The comparison of eqs. (47) and (49) shows the difference between the two situations. In the former, only interaction is effectively taken into account, while in the latter, more general, case the both factors – interaction and compositeness – are involved.

Remark that besides modified deformed derivative (16), its further extensions may be considered, e.g.

\[ z\tilde{D}_2 \equiv \frac{\varphi(k)}{\lambda T} \left( \frac{z d}{dz} + \chi \left( \frac{z d}{dz} \right) + g(\beta) \rho \left( \frac{z d}{dz} \right) + \ldots \right) \]

For the commutator \([h(\partial/\partial \beta), f(z\partial/\partial z)]\) we obtain

\[ \left[ h \left( \frac{\partial}{\partial \beta} \right), f \left( z \frac{\partial}{\partial z} \right) \right] = \sum_{k=1}^{\infty} \frac{\beta^{-k}}{k!} Q_k \left( \frac{z d}{dz} \right) h^{(k)} \left( \frac{\partial}{\partial \beta} \right) \]

where \( Q_k(x) \equiv (-1)^k x^{-k} \left( \frac{z d}{dz} \right)^k f(x) \). So, since the ambiguity in the position of \( h(\partial/\partial \beta) \) is still present, the terms with higher derivatives \( h^{(i)} \left( \frac{\partial}{\partial \beta} \right) \) may enter the “...” in (50).
IV. CONCLUDING REMARKS

In the analysis of the second virial coefficient of non-ideal Bose gas from the viewpoint of the role of such important factors of non-ideality as interaction of particles and their compositeness, we have found explicit expression for $V_2 - V_2^{(0)}$ given through the scattering length $a$ and the effective radius $r_0$ of interaction. The latter result, when compared to virial coefficient (14) of deformed Bose gas, has led us to one of our main formulas, eq. (15). Though the dependence of deformation parameter on $a$ and $r_0$ is rather expected, the encountered $T$-dependence has become a kind of surprise. With the goal to find reasonable explanation, and to justify the appearance of temperature dependence in the effective deformation of a Bose gas with temperature dependent interaction are present simultaneously. Note that the coefficient when the both factors of the compositeness and interaction are present simultaneously. Note that the corresponding dependence $V_2 = V_2(a, r_0, \varepsilon^{int}, \ldots, T)$ may be different from that obtained above, and thus lead to some other structure functions of deformation.

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Appendix A: Examples of the interaction potential

Here we present a number of examples of the interaction potentials and respective phaseshifts or scattering length/effective radius through which the second virial coefficient in (4) or (6) is expressed.

a. **Hard spheres interaction potential.** It is given by

$$U(r) = \begin{cases} +\infty, & r < D; \\ 0, & r > D. \end{cases}$$ (A1)

Direct calculation using (4) yields (see e.g. [27])

$$V_2 - V_2^{(0)} = \begin{cases} \frac{2D}{\lambda_T} + \frac{10\pi^2}{3} \left( \frac{D}{\lambda_T} \right)^5 + \ldots & \text{(Bose case, } l = 0, 2); \\ 6\pi \left( \frac{D}{\lambda_T} \right)^3 - 18\pi^2 \left( \frac{D}{\lambda_T} \right)^5 + \ldots & \text{(Fermi case, } l = 1). \end{cases}$$ (A2)

b. **Constant repulsive potential.** It is defined by

$$U(r) = \begin{cases} U_0 > 0, & r < R; \\ 0, & r > R. \end{cases}$$ (A3)

Respective $l = 0$ phaseshift and scattering length are then given as (for this and further examples see e.g. [29])

$$\delta_0 = kR \left( \frac{\text{th} K_0 R}{K_0 R} - 1 \right), \quad a = R \left( 1 - \frac{\text{th} K_0 R}{K_0 R} \right)$$ (A4)

where $K_0^2 = \frac{2mU_0}{kR^2}$, and $r_0 = 0$ (see also eq. (8)). The difference $V_2 - V_2^{(0)}$ can be calculated using (6) like in the previous case (this concerns also the rest of examples).

c. **Square-well potential.** It has the definition:

$$U(r) = \begin{cases} -U_0 < 0, & r < R; \\ 0, & r > R. \end{cases}$$ (A5)

The scattering length and effective radius equal to

$$a = -R \left( \frac{\text{tg} K_0 R}{K_0 R} - 1 \right), \quad r_0 = R \left( 1 - \frac{1}{K_0^2 R a} - \frac{R^2}{3a^2} \right)$$ (A6)

with $K_0$ defined as in the previous example.

d. **Anomalous scattering potential.** In this case

$$\frac{2m}{kR^2} U(r) = \begin{cases} -K_1^2, & 0 < r < r_1; \\ +K_0^2, & r_1 < r < R; \\ 0, & R \leq r. \end{cases}$$ (A7)

For $l = 0$ phaseshift we have

$$\delta_0 = -kR + \arctg \left\{ \frac{kR}{\kappa R} \cdot \frac{\text{th} \kappa (R-r_1)}{1 + \kappa R \text{th} \kappa (R-r_1)} \right\}$$ (A8)

where $\kappa^2 = K_0^2 - k^2$, $K_0^2 = K_1^2 + k^2$.

e. **Scattering resonances.** The corresponding potential is

$$U(r) = \frac{\Omega}{2m R} \delta(r - R).$$ (A9)

Phaseshift $\delta_0$ is given as

$$\text{tg}(kR + \delta_0) = -\frac{\text{tg} kR}{1 + \Omega \text{tg} kR}.$$ (A10)
f. Modified Pöschl-Teller potential. The potential is
\[
U(r) = -\frac{\hbar^2 \alpha^2 \lambda (\lambda - 1)}{2m \chi^2 \alpha r}.
\] (A11)

The respective phaseshift \(\delta_0\) reads:
\[
\delta_0 = \arctg \frac{2\tilde{k}}{\lambda} - \arctg (\frac{\pi \lambda}{2} \frac{th \pi \tilde{k}}{n}) + \sum_{n=1}^{\infty} \{ \arctg \frac{2\tilde{k}}{\lambda + n} - \arctg \frac{2\tilde{k}}{n} \}, \quad \tilde{k} = \frac{k}{2 \alpha}.
\] (A12)

g. Inverse power repulsive potential, that is
\[
U(r) = \frac{\hbar^2 g^2 (r_0 / r)^n}{2m r_0}.
\] (A13)

For it, the scattering length and the wavefunction of the lowest state (through which the effective radius is expressed) are respectively given as
\[
a = r_0 \frac{\Gamma(1 - \frac{1}{2n})}{\Gamma(1 + \frac{1}{2n})} \left( \frac{g}{2n} \right)^{\frac{1}{n}}, \quad \eta = \frac{n - 2}{2},
\] (A14)

\[
\chi_0 = C \sqrt{\frac{r}{r_0}} K_{\eta} \left( \frac{g}{\eta} (r/r_0)^{-\eta} \right),
\] (A15)

\(K_{\nu}(z)\) being the modified Hankel function.

h. First Born approximation. Finally, let us present the expression for the \(l\)th phaseshift in the first Born approximation:
\[
\delta_l \simeq \frac{-2mk}{\hbar^2} \int_0^\infty U(r) (j_l(kr))^2 r^2 dr
\] (A16)

where \(j_l\) is the spherical Bessel function. However its applicability is quite restricted, and the respective validity conditions reduce to the ones on \(U(r)\) of the first Born approximation.

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