THE ANALOGUE OF THE BRZ EXACT SEQUENCE FOR TATE-SHAFAREVICH GROUPS

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Abstract. We find an exact sequence in term of Tate-Shafarevich groups (assuming being finite) III(E/K) and III(E/L) of elliptic curve E over a finite Galois extension L/K of number fields. This is the analogue of the “BRZ exact sequence” [12] for the relative Pólya group Po(L/K) [3] and the Ostrowski quotient Ost(L/K) [12].

Keywords: Tate-Shafarevich group, BRZ exact sequence, relative Pólya group, Ostrowski quotient, transgression map.

1. Introduction

Notations. The following notations will be used throughout this article:

For a number field K, the notations I(K), P(K), Cl(K), O_K, h_K, U_K, denote the group of fractional ideals, group of principal fractional ideals, ideal class group, ring of integers, class number, and group of units of K, respectively. For a finite extension L/K, ε_L/K : Cl(K) → Cl(L) denotes the capitulation map.

Let E be an elliptic curve defined over a number field K. The “Tate-Shafarevich group” of E/K, denoted by III(E/K), is the group of everywhere locally trivial homogeneous spaces (modulo equivalence) for E/K [14, Chapter X]. Using a cohomological analogy between the Mordell-Weil group E(K) and the unit group U_K, one may define the Tate-Shafarevich group III(K) of K which is isomorphic to the ideal class group of K, see Proposition (4.2). In particular, III(K) is finite, whereas finiteness of III(E/K) is an widely open conjecture stated by Tate and Shafarevich. Likewise, for L/K a finite Galois extension of number fields with Galois group G, the “locally trivial cohomology group” H^1_{lt}(G, U) is defined to be the group of everywhere locally trivial elements of H^1(G, U_L) and one can show that H^1_{lt}(G, U) is canonically isomorphic to the capitulation kernel Ker(ε_L/K), see Proposition (1.4).

Using the cohomological methods of Brumer-Rosen [1] and Zantema [16], one can find the BRZ exact sequence (2.2), which relates Ker(ε_L/K) to the relative Pólya group Po(L/K) [3, 8], where Po(L/K) is the subgroup of Cl(L) generated by all classes of the ambiguous ideals in L/K, see Proposition (2.2). It can be seen that Po(L/K) also coincides with Cl(L)^G_{trans}, where

Cl(L)^G_{trans} := Ker (Cl(L)^G → H^1(G, P(L))),

denotes the group of transgressive ambiguous classes, see [3, §2]. More precisely, the exact sequence given by Gnozále-Avilés in [5, Theorem 2.4] will be equivalent to the BRZ exact sequence (2.2).

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In this paper, using the aforementioned results along with some results of Yu [15], we find an exact sequence which is an analogue of the BRZ sequence (2.2), and the one in [3.2] due to Gnozalez-Avilés, in term of Tate-Shafarevich groups of elliptic curves over finite Galois extension of number fields, see Theorem (5.6). As a consequence, we find a generalization of [15, Corollary 7] and [4, Main Theorem].

2. Relative Pólya Group and Ostrowski Quotient

The notion of Pólya group [2, §11.4] has been recently generalized to the Relative Pólya group in the following sense (this paper is not concerning Pólya fields and Pólya groups, and the reader is referred to [2, 3, 7, 8, 16] for some results in these subjects):

**Definition 2.1.** [3, 8] Let $L/K$ be a finite extension of number fields. The relative Pólya group of $L$ over $K$, denoted by $\text{Po}(L/K)$, is the subgroup of $\text{Cl}(L)$ generated by the classes of the relative Ostrowski ideals

\[ \Pi_{P^f}(L/K) := \prod_{M \in \text{Max}(\mathcal{O}_L)} \mathcal{M}, \tag{2.1} \]

where $P$ is a prime ideal of $K$, $f$ is a positive integer, and $N_{L/K}(\mathcal{M}) = P^f$. By the convention, if $L$ has no ideal with relative norm $P^f$ (over $K$), then $\Pi_{P^f}(L/K) = \mathcal{O}_L$. In particular, $\text{Po}(L/Q) = \text{Po}(L)$ and $\text{Po}(L/L) = \text{Cl}(L)$.

If $L/K$ is a Galois extension with Galois group $G$, the ideals $\Pi_{P^f}(K/P)$ freely generate the ambiguous ideals $I(L)^G$. Hence in terms of the action of the Galois group on the class group, $\text{Po}(L/K)$ coincides with the group of strongly ambiguous ideal classes of $L$. Using some results of Brumer-Rosen [1, §2], one may find the following result which is also obtained by Zantema [16, §3] for $K = Q$:

**Proposition 2.2.** [8 Theorem 2.2] Let $L/K$ be a finite Galois extension of number fields with Galois group $G$. Then the following sequence is exact:

\[ \{0\} \to \text{Ker}(\epsilon_{L/K}) \to H^1(G, U_L) \to \bigoplus_{P} \frac{\mathbb{Z}}{e_{P^f}(L/K)\mathbb{Z}} \to \frac{\text{Po}(L/K)}{\epsilon_{L/K}(\text{Cl}(K))} \to \{0\}, \tag{2.2} \]

where $P$ is a (finite) prime of $K$ and $e_{P^f}(L/K)$ denotes its ramification index in $L/K$.

We refer to exact sequence (2.2) as the “Brumer-Rosen-Zantema” exact sequence or shortly “BRZ” [13, Remark 1.4]. Some interesting consequences of the BRZ sequence, especially generalizations of some results for Pólya fields, have been found in [8, §2], see Proposition (2.4) below for the selected ones. Moreover, using the BRZ exact sequence one may find simple proofs for some classical well known results for the capitulation problem [13 §2.2.1]. However, as a modification of the relative Pólya group, the notion of “Ostrowski quotient” has been recently introduced as follows:

**Definition 2.3.** [12] For a finite extension $L/K$ of number fields, the Ostrowski quotient $\text{Ost}(L/K)$ is defined as

\[ \text{Ost}(L/K) := \frac{\text{Po}(L/K)}{\text{Po}(L/K) \cap \epsilon_{L/K}(\text{Cl}(K))}. \tag{2.3} \]
In particular, $\text{Ost}(L/Q) = \text{Po}(L/Q) = \text{Po}(L)$ and $\text{Ost}(L/L) = \{0\}$. The extension $L/K$ is called “Ostrowski” (or $L$ is called $K$-Ostrowski) if $\text{Ost}(L/K)$ is trivial.

Note that if $L/K$ is a Galois extension, then $\epsilon_{L/K}(\text{Cl}(K)) \subseteq \text{Po}(L/K)$ [8, § 2], hence $\text{Ost}(L/K) = \frac{\text{Po}(L/K)}{\epsilon_{L/K}(\text{Cl}(K))}$. We summarize some results obtaining from the BRZ exact sequence (2.2) which we need for the rest of the paper:

**Proposition 2.4.** [8][13] For a finite Galois extension $L/K$ of number fields with Galois group $G$, the following assertions hold:

(i) \[
\# \text{Po}(L/K) = \frac{h_K \cdot \prod\ EP(L/K)}{\# H^1(G, U_L)}, \quad \# \text{Ost}(L/K) = \frac{\# \text{Ker}(\epsilon_{L/K}) \cdot \prod\ EP(L/K)}{\# H^1(G, U_L)}
\]

(ii) If $\text{gcd}(h_K, [L : K]) = 1$, then the following sequence is exact:

\[
\{0\} \to H^1(G, U_L) \to \bigoplus_{p \mid \text{disc}(L/K)} \mathbb{Z} / \text{EP}(L/K) \mathbb{Z} \to \frac{\text{Po}(L/K)}{\text{Cl}(K)} \to \{0\}.
\]

(iii) If either $\text{gcd}(h_K, [L : K]) = 1$ or all finite places of $K$ are unramified in $L$, then $\text{Po}(L/K) = \epsilon_{L/K}(\text{Cl}(K))$, i.e. $L/K$ is Ostrowski. In particular, the extensions $H(K)/K$ and $\Gamma(K)/K$ are Ostrowski, where $H(K)$ and $\Gamma(K)$ denote the Hilbert class field and genus field of $K$, respectively.

**3. Localization and transgression maps**

In the rest of the paper, for $L/K$ a finite Galois extension of number fields, we fix the following notations:

- $\overline{K}$: the algebraic closure of $K$
- $G_K := \text{Gal}(\overline{K}/K)$
- $M_K$: a complete set of places of $K$
- $M_K^\infty$: the set of all archimedean places of $K$
- $K_v$: the completion of $K$ at a place $v \in M_K$
- $v_L$: a fixed place of $L$ lying above $v \in M_K$
- $U_{v_L}$: the group of local units in $L_{v_L}$
- $G_v := \text{Gal}(L_{v_L}/K_v)$ for a place $v \in M_K$
- $\overline{K}_v$: the algebraic closure of $K_v$
- $G_{K_v} := \text{Gal}(\overline{K}_v/K_v)$

Let $S$ be a finite set of primes of $K$ containing $M_K^\infty$. In [5], González-Avilés found some interesting results on the kernel and cokernel of the $S$-capitulation map. Assuming $S = M_K^\infty$, we summarize a few results of González-Avilés which are closely related to our purpose:

**Proposition 3.1.** [5, § 2] For $L/K$ a finite Galois extension of number fields with Galois group $G$, the following assertions hold:

(i) $\text{Cl}(L)^G_{\text{trans}} \simeq I(L)^G / P(L)^G$, where

\[
\text{Cl}(L)^G_{\text{trans}} := \text{Ker} \left( \text{Cl}(L)^G \xrightarrow{\alpha} H^1(G, L^*/U_L) \right).
\]

(ii) The following sequence is exact

\[
\{0\} \to \text{Ker}(\epsilon_{L/K}) \to H^1(G, U_L) \xrightarrow{\Delta} \bigoplus_{v \mid \text{disc}(L/K)} H^1(G_{v_L}, U_{v_L}) \to \frac{\text{Cl}(L)^G_{\text{trans}}}{\epsilon_{L/K}(\text{Cl}(K))} \to \{0\}.
\]
**Remark 3.2.** Compositing the embedding $H^1(G, L^*/U_L) \to H^2(G, U_L)$, with the map $\alpha$ in the part (i) of Proposition (5.1), we get the transgression map

$$\text{trans}_{L/K} : \text{Cl}(L)^G \to H^2(G, U_L),$$

whose kernel, i.e. $\text{Cl}(L)^{\text{trans}}_G$, is called “the group of transgressive ambiguous classes” [5 §2].

**Corollary 3.3.** For $L/K$ a finite Galois extension of number fields with Galois group $G$, we have

$$\text{Po}(L/K) \simeq \text{Cl}(L)^{\text{trans}}_G \quad \text{and} \quad \text{Ost}(L/K) \simeq \text{Coker}(\lambda),$$

where

$$\lambda : H^1(G, U_L) \to \bigoplus_{v \in M_K} H^1(G_{vL}, U_{vL}).$$

denotes the “localization map” as in the exact sequence (3.2).

**Proof.** Since $I(L)^G$ would be generated by the all relative Ostrowski ideals $\Pi_{P_f}(L/K)$ defined as in (2.1), see [1, proof of Proposition 2.2], $I(L)^G/P(L)^G$ coincides with the relative Pólya group $\text{Po}(L/K)$. Using Propositions (2.2) and (3.1), we obtain the desired isomorphisms.

As mentioned before, we aim to find the analogue of the exact sequences (2.2) and (3.2), as well as their consequences, for Tate-Shafarevich group of elliptic curves over finite Galois extensions of number fields.

### 4. On Tate-Shafarevich Groups

Let $E/K$ be an elliptic curve defined over a number field $K$. By the Mordell-Weil Theorem the group of $K$-rational points of $E$, denoted by $E(K)$, is finitely generated. An important tool in study of the Mordell-Weil group $E(K)$, is the “Tate-Shafarevich group” $\text{III}(E/K)$, which measures the failure of the Hasse local-global principle for curves that are principle homogeneous spaces for $E/K$. More formally, $\text{III}(E/K)$ is defined to be the group of everywhere locally trivial elements of $H^1(G_K, E(K))$ [14 Chapter X]:

$$\text{III}(E/K) := \text{Ker} \left( H^1(G_K, E(K)) \to \prod_{v \in M_K} H^1(G_{K_v}, E(K_v)) \right).$$

There is a famous conjecture, due to Tate and Shafarevich, which states that $\text{III}(E/K)$ is finite [14 Conjecture 4.13]. Though it has been shown that if the analytic rank of $E$ is smaller that 2, then $\text{III}(E/K)$ is finite [10 Theorem 5.12], the general case has not yet been proved. An important fact that makes the Tate-Shafarevich conjecture seemingly be true, is concerning the similar situation for number fields (see Proposition (4.2) below):

There is a $G_K$-action on $U_K$ whose fixed points are precisely the unit group $U_K$. By Dirichlet Unit Theorem, $U_K$ is also a finitely generated group. Using the cohomological analogy between the Mordell-Weil group $E(K)$ and the unit group $U_K$, one can define the analogue of the Tate-Shafarevich group for number fields in the following sense:
**Definition 4.1.** [11 §1] The “Tate-Shafarevich group $\Sha(K)$” of $K$ is defined as

$$
(4.2) \quad \Sha(K) := \ker \left( H^1(G_K, U_K) \to \prod_{v \in M_K} H^1(G_{K_v}, U_{K_v}) \right),
$$

where for non-archimedean $v$ (resp. archimedean $v$), $U_{K_v}$ denotes the valuation ring in $K_v$ (resp. $U_{K_v} = K_v$). Note that by Hilbert’s Theorem 90, $\Sha(K)$ can be defined just in terms of the non-archimedean places.

Despite the Tate-Shafarevich conjecture for elliptic curves is still open, in the case of number fields we have:

**Proposition 4.2.** [11 Proposition 1] $\Sha(K)$ is canonically isomorphic to the ideal class group of $K$. In particular, $\Sha(K)$ is finite.

Similar to $\Sha(K)$, replacing $K$ with a finite Galois extension of $K$, yields the following notion:

**Definition 4.3.** [11 §1] Let $L/K$ be a finite Galois extension of number fields with Galois group $G$. The locally trivial cohomology group $H^1_{lt}(G, U_L)$ is defined as

$$
(4.3) \quad H^1_{lt}(G, U_L) := \ker \left( H^1(G, U_L) \to \prod_{v \in M_K} H^1(G_{vL}, U_{vL}) \right).
$$

**Proposition 4.4.** [11 Corollary of Proposition 1] For a finite Galois extension $L/K$ of number fields with Galois group $G$, one has

$$
(4.4) \quad \ker(\epsilon_{L/K}) \cong H^1_{lt}(G, U_L).
$$

**Remark 4.5.** Note that $H^1_{lt}(G, U_L)$ is the kernel of the “localization map” $\lambda$ as described by González-Avilés in [5, page 80]. Indeed the exact sequences (3.2) and the BRZ (2.2), using Corollary (3.3), imply the above isomorphism, too. In order to find the analogue of the BRZ exact sequence (2.2) for Tate-Shafarevich groups, we shall obtain the analogue of the isomorphism (4.4), see Lemma (5.4) below.

Using the BRZ exact sequence (2.2) and Proposition (4.4), we restate some results in [13 §2.1] for the “capitulation problem” in terms of the locally trivial cohomology groups:

**Corollary 4.6.** For $L/K$ a finite Galois extension of number fields with Galois group $G$, the following assertions hold:

(i) [13 Proposition 2.9] For $H(K)$ the Hilbert class field of $K$, one has

$$
\Sha(K) \cong H^1_{lt}(\text{Gal}(H(K)/K), U_K) \cong \text{Gal}(H(K)/K).
$$

(ii) [13 Proposition 2.10] If all finite places of $K$ are unramified in $L$, then

$$
H^1_{lt}(G, U_L) = H^1(G, U_L).
$$

(iii) [13 Theorem 2.12, part (i)] If $L/K$ is cyclic and unramified at all (finite and infinite) places of $K$, then

$$
\#H^1_{lt}(G, U_L) = (U_K : \text{Norm}_{L/K}(U_L)) [L : K].
$$
5. The BRZ sequence for Tate-Shafarevich groups

In [15] Yu found some interesting results concerning Tate-Shafarevich group of abelian varieties over finite Galois extensions of number fields. In this paper, we are just interested in dimension one abelian varieties, namely the elliptic curves, for which we restate some needed results of Yu in their terms. However, our results in this section may be thought for arbitrary abelian varieties, as well.

**Convention.** We assume that the Tate-Shafarevich groups \( X(E/K) \) and \( X(E/L) \), appearing throughout this section, are finite. Also we will use the same notations as introduced in Section (3).

**Proposition 5.1.** [15, Theorem 1] For an alliptic curve \( E/K \) over \( K \), the following sequence is exact:

\[
\{0\} \to \Sha(E/K) \to H^1(G_K, E) \xrightarrow{\Omega} \bigoplus_{v \in M_K} H^1(G_{K_v}, E) \to \widehat{E}(K)^* \to \{0\},
\]

where \( \widehat{E}(K) \) denotes the completion of \( E(K) \) with respect to the topology defined by the subgroups of finite index, and \( E(K)^* \) denotes the group of continuous characters of finite order of \( E(K) \), i.e. \( E(K)^* = \text{Hom}_{cts}(E(K), \mathbb{Q}/\mathbb{Z}) \).

**Remark 5.2.** The last term in the original form of the exact sequence (5.1) appears based on the dual of abelian varieties [15, Theorem 1], whereas in the exceptional case the dual of an elliptic curve \( E \) will be isomorphic to \( E \) itself. More precisely, in the characteristic 0, the dual of an abelian variety \( A \) is isomorphic to the “degree zero divisor group \( \text{Pic}^0(A) \)” of \( A \) [9, §8], and for an elliptic curves \( E \) one has \( \text{Pic}^0(E) \simeq E \) [14, Chapter III, Proposition 3.4].

For \( L/K \) a finite Galois extension of number fields with Galois group \( G \), recall the transgression map \( \text{trans}_{L/K} : \text{Cl}(L)^G \to H^2(G, U_L) \) as mentioned in Remark (5.2). Using Yu’s paper [15, §4] and the analogy between \( \text{Cl}(L) \) and \( \Sha(E/L) \), see Section (4), one may find the analogue of the transgression map in term of Tate-Shafarevich groups:

For \( E/K \) an elliptic curve over \( K \), let

\[
\text{res}^E_{L/K} : H^1(G_K, E) \to H^1(G_L, E)^G
\]

be the restriction map, and

\[
\text{trans}^E_{L/K} : H^1(G_L, E)^G \to H^2(G, E(L))
\]

be the transgression map (for the full detailed definition, see [15, §4]).

**Lemma 5.3.** [6, Theorem 2] With the above notations, one has

\[
\text{Image} \left( \text{res}^E_{L/K} \right) = \text{Ker} \left( \text{trans}^E_{L/K} \right).
\]

Denote by \( \widetilde{\text{res}}^E_{L/K} \) the map induced by the restriction map \( \text{res}^E_{L/K} \) whose domain is the Tate-Shafarevich group \( \Sha(E/K) \), i.e.

\[
\widetilde{\text{res}}^E_{L/K} := \text{res}^E_{L/K} |_{\Sha(E/K)}.
\]

Then by Lemma (5.3), we have

\[
\text{Image}(\widetilde{\text{res}}^E_{L/K}) \subseteq \text{Ker}(\text{trans}^E_{L/K}).
\]
Lemma 5.4. Let $L/K$ be a finite Galois extension of number fields with Galois group $G$, $E/K$ be an elliptic curve over $K$, $\tilde{\res}^E_{L/K} = \res^E_{L/K} |_{\Sha(E/L)}$ and $\mathcal{F} : H^1(G, E(L)) \rightarrow \bigoplus_{v \in \mathcal{M}_K} H^1(G_v, E(L_v))$ be the localization map. Then

\begin{equation}
\text{Image}(\tilde{\res}^E_{L/K}) \subseteq \Sha(E/L)^G,
\end{equation}

and

\begin{equation}
\text{Ker}(\tilde{\res}^E_{L/K}) \simeq \ker(\mathcal{F}).
\end{equation}

Proof. Applying the snake lemma on the commutative diagram

\begin{equation}
\begin{array}{cccccc}
\{0\} & \rightarrow & H^1(G, E(L)) & \rightarrow & H^1(G_K, E) & \stackrel{\res^E_{L/K}}{\rightarrow} & \text{Image}(\res^E_{L/K}) & \rightarrow & \{0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \rightarrow & \bigoplus_{v \in \mathcal{M}_K} H^1(G_v, E(L_v)) & \rightarrow & \bigoplus_{v \in \mathcal{M}_K} H^1(G_v, E) & \stackrel{\sigma}{\rightarrow} & \bigoplus_{v \in \mathcal{M}_K} H^1(L_v, E).
\end{array}
\end{equation}

we find the following exact sequence

\begin{equation}
\{0\} \rightarrow \text{Ker}(\mathcal{F}) \overset{i}{\rightarrow} \Sha(E/K) \overset{\tilde{\res}^E_{L/K}}{\rightarrow} \text{Ker}(\mathcal{H}) \rightarrow \text{Coker}(\mathcal{F}) \rightarrow \mathcal{I}(\text{Coker}(\mathcal{F})) \rightarrow \{0\},
\end{equation}

where $\tilde{i} := i|_\mathcal{F}$ and

\begin{equation}
\mathcal{I} : \text{Coker}(\mathcal{F}) \rightarrow \text{Coker}(\mathcal{G})
\end{equation}

is obtained from the above diagram, see [15, §2]. Using exact sequence (5.9) and the equality

\begin{equation}
\text{Ker}(\mathcal{H}) = \Sha(E/L)^G \cap \text{Ker}(\trans^E_{L/K}),
\end{equation}

see [15, page 213], we obtain the containment (5.6). Exact sequence (5.9) also implies that

$$\text{Ker}(\mathcal{F}) \simeq \tilde{i}(\text{Ker}(\mathcal{F})) = \text{Ker}(\tilde{\res}^E_{L/K}).$$

□

Remark 5.5. By Lemma (5.4), the map $\tilde{\res}^E_{L/K}$ can be written as

$$\tilde{\res}^E_{L/K} : \Sha(E/K) \rightarrow \Sha(E/L)^G,$$

which is the analogue of the capitulation map $\epsilon_{L/K} : \text{Cl}(K) \rightarrow \text{Cl}(L)^G$ as in [5, §2]. Moreover, the isomorphism (5.7) in the above lemma, may be thought as the analogue of the Schoof-Washington result in Proposition (4.4) for Tate-Shafarevich groups.

Similar to the transgression map $\trans : \text{Cl}(L)^G \rightarrow H^2(G, U_L)$ [5, §2], let

\begin{equation}
\hearttrans_{L/K}^E := \trans_{L/K}^E |_{\Sha(E/L)^G} : \Sha(E/L)^G \rightarrow H^2(G, E(L)).
\end{equation}

By equation (5.11), we have

\begin{equation}
\text{Ker}(\mathcal{H}) = \text{Ker}(\hearttrans_{L/K}^E),
\end{equation}

and exact sequence (5.9) yields

\begin{equation}
\text{Image}(\tilde{\res}^E_{L/K}) \subseteq \text{Ker}(\hearttrans_{L/K}^E).
\end{equation}
Theorem 5.6. Let \( L/K \) be a finite Galois extension of number fields with Galois group \( G \), and \( E/K \) be an elliptic curve. Then the sequence
\[
\{0\} \to \text{Ker}(\overline{\text{res}}_{L/K}^E) \to H^1(G, E(L)) \xrightarrow{\mathcal{F}} \bigoplus_{v \in M_K} H^1(G_{vL}, E(L_{vL})) \to \text{Ker}(\overline{\text{trans}}_{L/K}^E) \to \{0\},
\]
is exact if and only if \( \mathcal{I} : \text{Coker}(\mathcal{F}) \to \text{Coker}(\mathcal{G}) \), as in (5.10), is the zero map. In particular, if \( \widehat{E}(K)^* \), introduced in Proposition 5.1, is trivial then the above sequence is exact.

Proof. By Lemma (5.4), we have
\[
\text{Ker}(\overline{\text{res}}_{L/K}^E) \simeq \ker(\mathcal{F}).
\]
Hence sequence (5.15) is exact if and only if
\[
\frac{\text{Ker}(\overline{\text{trans}}_{L/K}^E)}{\overline{\text{res}}_{L/K}^E (\text{III}(E/K))} \simeq \text{Coker}(\mathcal{F}).
\]
By exact sequence (5.9) and equality (5.13), the last isomorphism holds if and only if \( \mathcal{I}(\text{Coker}(\mathcal{F})) = \{0\} \). Also by Proposition 5.1, one has
\[
\text{Image}(\mathcal{I}) \subseteq \text{Coker}(\mathcal{G}) \simeq \widehat{E}(K)^*.
\]

Remark 5.7. Let
\[
\mathcal{F}_0' : \widehat{H}^0(G, E(L)) \to \prod_{v \in M_K} \widehat{H}^0(G_{vL}, E(L_{vL})).
\]
Then one can show that \( \# \text{Image}(\mathcal{I}) = \# \text{Image}(\mathcal{F}_0') \), see [15 Lemma 5]. Hence by Theorem (5.6), sequence (5.15) is exact if and only if \( \mathcal{F}_0' \) is the zero map.

Using Corollary 3.3 and comparing exact sequence (5.15) with the BRZ sequence (2.2), one may consider \( \text{Ker}(\overline{\text{trans}}_{L/K}^E) \) and \( \frac{\text{Ker}(\overline{\text{trans}}_{L/K}^E)}{\overline{\text{res}}_{L/K}^E (\text{III}(E/K))} \) as the analogous notions of \( \text{Po}(L/K) \) and \( \text{Ost}(L/K) \), respectively. Based on this viewpoint, the following corollary correspond naturally to Proposition 2.4:

Corollary 5.8. For \( L/K \) a finite Galois extension of number fields with Galois group \( G \), and an elliptic curve \( E/K \), if \( \mathcal{I} : \text{Coker}(\mathcal{F}) \to \text{Coker}(\mathcal{G}) \), as in (5.10), is the zero map, then the following assertions hold:

(i)
\[
\frac{\# \text{Ker}(\overline{\text{trans}}_{L/K}^E)}{\# \text{III}(E/K)} = \frac{\prod_{v \in M_K} \# H^1(G_{vL}, E(L_{vL}))}{\# H^1(G, E(L))}.
\]

Further, \( \overline{\text{trans}}_{L/K}^E \) is the zero map if and only if
\[
\frac{\# \text{III}(E/L)^G}{\# \text{III}(E/K)} = \frac{\prod_{v \in M_K} \# H^1(G_{vL}, E(L_{vL}))}{\# H^1(G, E(L))}.
\]
(ii) If \( \gcd(\#\text{III}(E/K), [L : K]) = 1 \), then the sequence

\[
0 \to H^1(G, E(L)) \xrightarrow{\varphi} \bigoplus_{v \in M_K} H^1(G_v L, E(L_v L)) \to \ker\left(\widetilde{\text{trans}}_{L/K}^E\right)_{\text{III}(E/K)} \to 0,
\]

is exact.

(iii) If \( \gcd(\#\text{III}(E/L), [L : K]) = 1 \), then \( \ker(\widetilde{\text{res}}_{L/K}^E(\text{III}(E/K))) = 0 \).

Proof. Immediately follows from Theorem (5.6). \( \square \)

As a generalization of the main theorem in [4], Yu proved:

**Proposition 5.9.** \([15, \text{Corollary 7}]\) Let \( L/K \) be a finite Galois extension of number fields with Galois group \( G \), and \( E/K \) be an elliptic curve. If \( \widetilde{\text{H}}^0(G, E'(L)) = H^2(G, E(L)) = \{0\} \),

then

\[
\frac{\#\text{III}(E/L)^G}{\#\text{III}(E/K)} = \prod_{v \in M_K} \frac{\#H^1(G_v L, E(L_v L))}{\#H^1((G, E(L)))}.
\]

**Remark 5.10.** Note that if \( \widetilde{\text{H}}^0(G, E'(L)) = \{0\} \), then the map \( \varphi \) is zero, see Remark (5.7). Also by the definition (5.12), triviality of \( H^2(G, E(L)) \), implies that \( \widetilde{\text{trans}}_{L/K}^E \) is the zero map. Whereas, the converse of these statements may not hold in general. Therefore, the part (i) of Corollary (5.8) can be seen as a generalization of Proposition (5.9) and [3, Main Theorem], as well.

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