Gradient estimates for the heat semigroup on forms in a complete Riemannian manifold.

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Abstract

We study the heat equation \( \frac{\partial u}{\partial t} - \Delta u = 0 \), \( u(x, 0) = \omega(x) \), where \( \Delta := dd^* + d^*d \) is the Hodge laplacian and \( u(\cdot, t) \) and \( \omega \) are \( p \)-differential forms in the complete Riemannian manifold \((M, g)\). Under weak bounded geometrical assumptions we get estimates on its semigroup of the form:

Acting on \( p \)-forms with \( p \geq 1 \) and \( k \geq 0 \):
\[ \forall t \geq 1, \left\| \nabla^k e^{-t\Delta} \right\|_{L^r(M) \to L^r(M)} \leq c(n, r, k). \]

Acting on functions, i.e. with \( p = 0 \), we get a better result:
\[ \forall k \geq 1, \forall t \geq 1, \left\| \nabla^k e^{-t\Delta} \right\|_{L^r(M) \to L^r(M)} \leq c(n, r, k)t^{-1/2}. \]

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1 Introduction.

In physics the heat equation is a partial differential equation that describes how the distribution of heat evolves over time in a solid medium.

For instance in $\mathbb{R}^n$ the heat equation is the following: $\frac{\partial u}{\partial t} - \Delta u = 0$, $u(x, 0) = \omega(x)$, where $\Delta$ is the laplacian in $\mathbb{R}^n$. In this case we have a solution given by the kernel:

$$\Phi(x, t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, \ t > 0 \\ 0 & x \in \mathbb{R}^n, \ t \leq 0 \end{cases}$$

and the solution is: $u(x,t) := \int_{\mathbb{R}^n} \omega(y) \Phi(x - y, t) dy$.

Clearly working in $\mathbb{R}^n$ is not enough: take the case of a shuttle coming back through the atmosphere. This can be modeled by a Riemannian manifold of dimension two, so we are naturally leaded to study the heat equation on Riemannian manifolds. And now the geometry enters because we need to deal with differential forms and the heat equation with them. In $\mathbb{R}^n$ the Laplacian operates diagonally on $p$-forms so we have to deal only with functions. This is not the case on Riemannian manifolds and the treatment has to be more delicate.

In the following $M := (M, g)$ will be a $C^\infty$ smooth connected complete Riemannian manifold without boundary unless otherwise stated. We shall just say "Riemannian manifold" to mean it.

The study of $L^r$ estimates for the solutions of the heat equation in a Riemannian manifold started long time ago. A basic work was done by R.S. Strichartz [14]. In particular he proved that the heat kernel is a contraction on the space of functions in $L^r(M)$ for $1 \leq r \leq \infty$.

The study of general parabolic equations in $\mathbb{R}^n$ is also well advanced, see for instance [7] and the references therein. In the case of parabolic equations in Riemannian manifold we can see for instance [12] and the references therein.

In [2] we also study parabolic equations in vector bundles on Riemannian manifold with mixed time-space Lebesgue or Sobolev norm. Here we get pointwise in time estimates, and we use essentially the same philosophy as in [2] to pass from local to global by use of the "admissible balls".

So our aim in this work is to get estimates on the covariant derivatives of any order of solutions of the heat equation $\frac{\partial u}{\partial t} - \Delta u = 0$, $u(x, 0) = \omega(x)$, where $\Delta := dd^* + d^*d$ is the Hodge laplacian and $u(x, t)$ and $\omega(x)$ are $p$-differential forms in the Riemannian manifold $M$. We shall denote $L^r(U)$ the space of forms, or derivatives of them, in the Lebesgue space $L^r(U)$ for a measurable set $U \subset M$, with the same notations as for functions.

We introduce $(m, \epsilon)$-admissible balls $B_{m, \epsilon}(x)$ in $(M, g)$ as in [2]. These balls are the ones defined in the work of Hebey and Herzlich [9] but without asking for the harmonicity of the local coordinates.

**Definition 1.1.** Let $M$ be a Riemannian manifold and $x \in M$. We shall say that the geodesic ball $B(x, R)$ is $(0, \epsilon)$-admissible if there is a chart $(B(x, R), \varphi)$ such that:

1) $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$ in $B(x, R)$ as bilinear forms, and it will be $(m, \epsilon)$-admissible for $m \geq 1$, if, moreover:

2) $\sum_{1 \leq |\beta| \leq m} R^{|\beta|} \sup_{i,j=1,\ldots,n, \ y \in B_\epsilon(R)} |\partial^{\beta} g_{ij}(y)| \leq \epsilon.$
We shall denote \( \mathcal{A}_m(\epsilon) \) the set of \((m, \epsilon)\)-admissible balls.

**Definition 1.2.** Let \( x \in M \), we set \( R'(x) = \sup \{ R > 0 : B(x, R) \in \mathcal{A}(\epsilon) \} \). We shall say that \( R_\epsilon(x) := \min \{ 1, R'(x)/2 \} \) is the \( \epsilon \)-admissible radius at \( x \).

We shall follow a natural path to proceed: first we use known result in \( \mathbb{R}^n \) via the Duhamel formula to get precise local estimates on \( M \), then we globalise them.

Let \( x \in M \), \( B := B(x, R) \) be an \( \epsilon \)-admissible ball.

- Using Duhamel formula we first get local estimates for any solutions \( u \) of
  \[
  \frac{\partial u}{\partial t} - \Delta u = 0, \quad u(x, 0) = \omega(x).
  \]
- We suppose now that \( \omega \in L^2(M) \cap L^p(M) \) and, because there is a global solution \( u(\cdot, t) \in L^2(M) \) such that \( \frac{\partial u}{\partial t} - \Delta u = 0, \quad u(x, 0) = \omega(x) \), this global solution verifies also the local estimates.
- Using Vitali type covering, plus a weight \( w(x) \) coming from the \( \epsilon \)-admissible radius \( R_\epsilon \), we globalise the result.

For \( p \geq 0 \) let \( \Lambda^p(M) \) be the set of \( C^\infty \) smooth \( p \)-forms in \( M \). We know that

\[
\nabla^k : \Lambda^p \to \Lambda^p \otimes T^* M \otimes \cdots \otimes T^* M,
\]

for the case of general vector bundle with a metric connection instead of just the bundle of \( p \)-forms, see for instance [3] or [2] Section 2.3, p. 6 to have the weights added. On this tensor product we have a pointwise modulus which allows us to define, with a weight \( w \):

\[
\forall u \in \Lambda^p(M), \quad \| \nabla^k u \|^r_{L^r(M, w)} := \int_M \| \nabla^k u \|^r \, w dv.
\]

For instance in the case of a function \( u \), then \( \nabla u \) can be seen as the 1-form \( du \), or as the usual gradient vector. We compute \( \nabla^k u \) locally in Section 3.1, formula (3.4).

We shall weakened the usual definition of bounded geometry to suit our purpose.

**Definition 1.3.** A Riemannian manifold \( M \) has \( k \)-order weak bounded geometry if:

- the injectivity radius \( r_{\text{inj}}(x) \) at \( x \in M \) is bounded below by some constant \( i > 0 \) for any \( x \in M \);
- for \( 0 \leq j \leq k \), the covariant derivatives \( \nabla^j R \) of the Ricci curvature tensor are bounded in \( L^\infty(M) \) norm.

Now we can state our main theorem.

**Theorem 1.4.** Let \( M \) be a Riemannian manifold. Let \( r \in [1, \infty] \). For any \( \delta > 0 \), there is an \( \epsilon(\delta) > 0 \) such that for any \( \epsilon \leq \epsilon(\delta) \), for any \( k \geq 0 \) and any \( p \)-form \( \omega \in L^r(M) \cap L^2(M) \), we have, with \( u = e^{\Delta_\omega} \), the canonical solution of the heat equation:

\[
\forall t \in (\delta, 1), \quad \| \nabla^k u(\cdot, t) \|^r_{L^r(M, w)} \leq c(n, r) (\frac{\delta}{t - \delta^{3/2}}) \| \omega \|_{L^r(M)}.
\]

For \( \omega \) any \( p \)-form with \( p \geq 1 \) and any \( k \geq 0 \):

\[
\forall t \geq 1, \quad \| \nabla^k u(\cdot, t) \|^r_{L^r(M, w)} \leq c(n, r) \| \omega \|_{L^r(M)}.
\]

and for any function \( \omega \), we have the better result for \( k \geq 1 \):

\[
\forall t \geq 1, \quad \| \nabla^k u(\cdot, t) \|^r_{L^r(M, w)} \leq c(n, r) t^{-1/2} \| \omega \|_{L^r(M)}.
\]

With the weight \( w(x) := R_\epsilon(x)^{kr + mr} \).

If \( k \leq 1 \) we have \( R_\epsilon(x) := R_{m, \epsilon}(x) \) is the admissible radius for the \((m, \epsilon)\)-admissible balls, with \( m = 1 \) if \( p = 0 \) and \( m = 2 \) if \( p > 1 \). If \( k \geq 2 \), then \( R_\epsilon(x) \) is the admissible radius for the \((k, \epsilon)\)-admissible balls for \( p \geq 1 \) and for the \((k - 1, \epsilon)\)-admissible balls for \( p = 0 \).
To get "classical estimates", i.e. estimates without weights, we use [9, Corollary, p. 7] and we prove:

**Theorem 1.5.** Let M be a Riemannian manifold. Let $r \in [1, \infty]$ and $\omega \in L^r(M) \cap L^2(M)$. For $k = 0, 1$ suppose that $(M, g)$ has 1-order weak bounded geometry for $p$-forms with $p \geq 1$ and 0-order weak bounded geometry for functions. For $k \geq 2$ suppose that $M$ has $k$-order weak bounded geometry for $p$-forms with $p \geq 1$ and $k-1$-order weak bounded geometry for functions.

Then the canonical solution $u := e^{t\Delta} \omega$ of the heat equation is such that, for any $k \geq 0$ and with $\eta = \eta(n, \epsilon, i, k)$ given by the Corollary [77] of Hebey and Herzlich:

$$\forall t \in (\delta, 1), \|\nabla^k u(\cdot, t)\|_{L^r(M)} \leq c(n, r, \eta)(\frac{\delta}{t - \delta^{3/2}})\|\omega\|_{L^r(M)}.$$ 

And for $k \geq 0$ and any $p$-form $\omega$:

$$\forall t \geq 1, \|\nabla^k u(\cdot, t)\|_{L^r(M)} \leq c(n, r, \eta)\|\omega\|_{L^r(M)}.$$ 

For functions we get a better estimate for any $k \geq 1$:

$$\forall t \geq 1, \|\nabla^k u(\cdot, t)\|_{L^r(M)} \leq c(n, r, \eta)t^{-1/2}\|\omega\|_{L^r(M)}.$$ 

In order to compare with existing result, we deduce from it:

**Corollary 1.6.** Let M be a Riemannian manifold. Let $r \in [1, \infty]$ and $\omega \in L^r(M) \cap L^2(M)$. Suppose that M has 1-order weak bounded geometry for $p$-forms with $p \geq 1$ and 0-order weak bounded geometry for functions.

Then we get, for the canonical solution $u := e^{t\Delta} \omega$ of the heat equation:

$$\forall t \geq 1, \|u(\cdot, t)\|_{L^r(M), w} \leq c(n, r)\|\omega\|_{L^r(M)}.$$ 

For $t \geq 1$ and acting on $p$-forms with $p \geq 1$:

$$\forall t \geq 1, \|\nabla u\|_{L^r(M)} \leq c(n, r)\|\omega\|_{L^r(M)},$$

and acting on functions:

$$\forall t \geq 1, \|\nabla u\|_{L^r(M)} \leq c(n, r)t^{-1/2}\|\omega\|_{L^r(M)}.$$ 

Exponential decay of distribution kernels of resolvents was proved in the setting of bounded geometry of any order by Kordyukov [10].

Recall the volume doubling property for the manifold $M$:

there exists constants $C, D > 0$, such that

$$v(x, \lambda r) m < C \lambda^D v(x, r), \forall x \in M, \forall r > 0, \forall \lambda \geq 1, \quad (D)$$

where $v(x, r) = \mu(B(x, r))$ denotes the volume of the ball $B(x, r)$ of center $x$ and radius $r$.

Let now $p(t, x, y)$ be the heat kernel on functions (the heat kernel of the Laplace-Beltrami operator $\Delta$) Recall the Gaussian upper bound for the manifold $M$:

$$p(t, x, y) \leq \frac{C}{v(x, \sqrt{t})} \exp(-c\frac{\rho^2(x, y)}{t}) \forall t > 0. \quad (G)$$

Then J. Magniez and E-M. Ouhabaz [11] proved:

**Theorem 1.7.** Suppose that the manifold M has the volume doubling property (D), the Gaussian upper bound (G) and $R_k^+ \subseteq K$ Then

(i) the semi group $(e^{-t\Delta})$ acts on $L^p(\Lambda^k T^*M)$ for all $p \in [1, \infty]$ and

$$\|e^{-t\Delta}\|_{p-p} \leq C_p(\log t)^{\frac{k}{2} - \frac{1}{p}} \frac{\rho^2}{t}, \quad t > e.$$
(ii) For all \( t \geq 1 \) and \( p \geq 2 \)
\[
\|\nabla e^{-t\Delta}\|_{L^p} \leq C_p t^{-\frac{1}{2}}.
\]

(iii) There exists \( C > 0 \) such that for all \( t > 0 \) and \( x, y \in M \)
\[
|\overrightarrow{p'_k(t, x, y)}| \leq C \frac{(1+t+\frac{\rho^2(x,y)}{4})^{\frac{1}{p}}}{v(x,\sqrt{t})^\frac{1}{p}v(y,\sqrt{t})^\frac{1}{p}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right).
\]

(iv) There exists \( C, c > 0 \) such that for all \( t \geq 1 \) and \( x, y \in M \)
\[
|\overrightarrow{p'_k(t, x, y)}| \leq C \min(1, \frac{\rho^2(x,y)}{t}) \exp\left(-c\frac{\rho^2(x,y)}{t}\right).
\]

With the notation \( |\overrightarrow{p'_k(t, x, y)}| \) for the norm from \( \Lambda^k T^*_y M \) to \( \Lambda^k T^*_y M \) of the linear map \( \overrightarrow{p'_k(t, x, y)} \) between these two spaces.

Comparing to the result of J. Magneze and E-M. Ouhabaz, they get Lebesgue estimates on \( k \)-forms:

\[
\|e^{-t\Delta_k}\|_{L^p-L^p} \leq C_p (t \log t)^{\frac{1}{2}-\frac{1}{p}} \frac{\rho^2}{v(x,\sqrt{t})^\frac{1}{p}v(y,\sqrt{t})^\frac{1}{p}}, \quad t > e, \ p \in [1, \infty].
\]

And gradient estimates on functions:

\[
\|\nabla e^{-t\Delta_k}\|_{L^p-L^p} \leq C_p t^{-\frac{1}{2}}, \quad t \geq 1, \ p \geq 2.
\]

Here we need that \( M \) has order weak bounded geometry to get gradient estimates on \( p \)-forms and we need that \( M \) has 0 order weak bounded geometry to get gradient estimates on functions. Under these geometric hypotheses, our estimates are better. The methods we use are also completely different.

## 2 Admissible balls.

**Lemma 2.1.** The \( \epsilon \)-admissible radius \( R_\epsilon(x) \) is continuous.

**Proof.**
Let \( x, y \in M \). We set \( R'(x) = \sup \{ R > 0 : B(x, R) \in A(\epsilon) \} \). Suppose that \( R'(x) > d_g(x, y) \), where \( d_g(x, y) \) is the Riemannian distance between \( x \) and \( y \). Consider the ball \( B(y, \rho) \) of center \( y \) and radius \( \rho := R'(x) - d_g(x, y) \). This ball is contained in \( B(x, R'(x)) \) hence, by definition of \( R'(x) \), we have that all the points in \( B(y, \rho) \) verify the conditions 1) and 2) so, by definition of \( R'(y) \), we have that
\[
R'(y) \geq R'(x) - d_g(x, y).
\]

If \( R'(x) \leq d_g(x, y) \) this is also true because \( R'(y) > 0 \). Exchanging \( x \) and \( y \) we get that \( |R'(y) - R'(x)| \leq d_g(x, y) \).

Hence \( R'(x) \) is 1-lipschitzian so it is continuous. So the \( \epsilon \)-admissible radius \( R_\epsilon(x) \) is also continuous. \( \blacksquare \)

**Remark 2.2.** Because our admissible ball \( B(x, R_\epsilon(x)) \) is geodesic, we have that the injectivity radius \( r_{\text{inj}}(x) \) always verifies \( r_{\text{inj}}(x) \geq R_\epsilon(x) \).

**Lemma 2.3.** (Slow variation of the admissible radius) Let \( M \) be a Riemannian manifold. With \( R(x) = R_\epsilon(x) \), the \( \epsilon \)-admissible radius at \( x \in M \), \( \forall y \in B(x, R(x)) \) we have \( R(x)/2 \leq R(y) \leq 2R(x) \).

**Proof.**
Let \( x, y \in M \) and \( d(x, y) \) the Riemannian distance on \( (M, g) \). Let \( y \in B(x, R(x)) \) then \( d(x, y) \leq R(x) \) and suppose first that \( R(x) \geq R(y) \).
Then, because \( R(x) = R'(x)/2 \), we get \( y \in B(x, R'(x)/2) \) hence we have \( B(y, R'(x)/2) \subset B(x, R'(x)). \) But by the definition of \( R'(x) \), the ball \( B(x, R'(x)) \) is admissible and this implies that the ball \( B(y, R'(x)/2) \) is also admissible for exactly the same constants and the same chart; this implies that \( R'(y) \geq R'(x)/2 \) hence \( R(y) \geq R(x)/2 \), so \( R(x) \geq R(y) \geq R(x)/2 \).

If \( R(x) \leq R(y) \) then
\[
d(x, y) \leq R(x) \Rightarrow d(x, y) \leq R(y) \Rightarrow x \in B(y, R'(y)/2) \Rightarrow B(x, R'(y)/2) \subset B(y, R'(y)).
\]
Hence the same way as above we get \( R(y) \geq R(x) \geq R(y)/2 \Rightarrow R(y) \leq 2R(x) \). So in any case we proved that
\[
\forall y \in B(x, R(x)) \text{ we have } R(x)/2 \leq R(y) \leq 2R(x). \quad \blacksquare
\]

**Lemma 2.4.** The \( \epsilon \)-admissible balls \( B(x, R_\epsilon(x)) \) trivialise the bundle \( \Lambda^p \) of \( p \)-forms.

**Proof.**
Because if \( B(x, R) \) is a \( \epsilon \)-admissible ball, we have by Remark 2.2 that \( R \leq r_{inj}(x) \). Then, one can choose a local frame field for \( \Lambda^p \) on \( B(x, R) \) by radial parallel translation, as done in [15, Section 13, p. 86-87], see also [12, p. 4, eq. (1.3)]. This means that the \( \epsilon \)-admissible balls also trivialise the bundle \( \Lambda^p \). \( \blacksquare \)

### 3 Local estimates.

In order to have the local result, we choose a \((1, \epsilon)\)-admissible ball \( B(x, R) \) and the associated chart \( \varphi : B \rightarrow \mathbb{R}^n \) such that \( \varphi(x) = 0 \). We shall need to compare the laplacian \( \Delta \) in \( \mathbb{R}^n \) and the image \( \Delta_\varphi \) by \( \varphi \) of the laplacian in the Riemannian manifold \( M \). For instance for functions we have:
\[
\Delta_\varphi f = \frac{1}{\sqrt{\det(g_{ij})}} \partial_i (g^{ij} \sqrt{\det(g_{ij})}) \partial_j f.
\]
An easy computation gives:
\[
(\Delta_\varphi - \Delta)f = (g^{ij} - \delta^{ij}) \partial_i^2 f + a^{ij}(g) \partial_i g^{ij} \partial_j f,
\]
where the coefficients \( a^{ij}(g) \) are smooth functions of the metric \( g \).

Using the (1) in the definition [14] of the admissible ball, we get
\[
|\partial_i g^{ij} \partial_j f| \leq \epsilon |\partial_i^2 f|
\]
and using the (2) we get
\[
|a^{ij}(g) \partial_i g^{ij} \partial_j f| \leq C \epsilon R^{-1} |\partial f|,
\]
where the constant \( C \) depends only on the metric \( g \). So we get
\[
|\Delta_\varphi - \Delta|f| \leq \epsilon |\partial_i^2 f| + C \epsilon R^{-1} |\partial f|.
\]

To treat the case of \( p \)-forms, we shall use the Bochner-Weitzenböck formula, but in its explicit form, in order to get the dependency in the derivatives of the metric tensor.

Precisely for a \( p \) form \( \alpha, \) \( p \geq 1 \), the equation (6) p. 109 in [4] gives in \( M \):
\[
(\Delta \alpha)_{k_1...k_p} = -\nabla^i \nabla_i \alpha_{k_1...k_p} + \sum_{\nu=1}^{p} (-1)^{\nu}(\nabla_{k_\nu} \nabla^i - \nabla^i \nabla_{k_\nu})\alpha_{i_{k_1...\hat{k_\nu}...k_p}}.
\]
As is well known, the covariant derivatives are linear in the Christoffel symbols, hence in the first derivatives of the metric \( g \). Because we apply twice covariant derivation, second order derivatives of the metric tensor appear linearly in the sum, so this time we need the ball \( B \) to be \((2, \epsilon)\)-admissible and via the chart \( \varphi \) we get, the same way as for functions, for the image \( f \) in \( \mathbb{R}^n \) of the \( p \)-form \( \alpha \) in \( M \):
\[ |(\Delta \varphi - \Delta)f| \leq \epsilon |\partial^2 f| + C\epsilon R^{-2} |\partial f|. \]

So we proved:

**Lemma 3.1.** Let \( x \) be a point in a Riemannian manifold \( M \). Let \( \alpha \) be a \( p \)-form in \( L^r(B) \), with \( B := B(x,R) \) a \((1,\epsilon)\)-admissible ball in \( M \) if \( p = 0 \) and a \((2,\epsilon)\)-admissible ball in \( M \) if \( p \geq 1 \). Let \( \varphi \) be a chart on \( B \) and set \( \Delta_{\varphi} \) the image by \( \varphi \) of the laplacian in \((M,g)\) and \( f \) the image of \( \alpha \). We have that \( \Delta_{\varphi} - \Delta \) is a second order differential operator of the form:

\[ (\Delta_{\varphi} - \Delta)f = \sum_{i,j} a_{ij} \partial^j f + \sum_i b_i \partial_i f. \]

Moreover we get, for \( p = 0 \) in \( \varphi(B) \):

\[ \sum_{i,j} |a_{ij}| \leq \epsilon, \quad \sum_i |b_i| \leq C\epsilon R^{-1}, \]

and for \( p \geq 1 \):

\[ \sum_{i,j} |a_{ij}| \leq \epsilon, \quad \sum_i |b_i| \leq C\epsilon R^{-2}. \]

Now we shall use the Duhamel’s formula as in [13, Proposition 3.15]. But, instead to use it to build a parametrix, we use it to compare the heat kernel in \( \mathbb{R}^n \) and the heat kernel in the manifold \( M \).

**Proposition 3.2.** (Duhamel’s formula) Provided \( e^{-t(X+Y)} \) exists, we have

\[ e^{-t(X+Y)} = e^{-tX} - \int_0^t e^{-(t-s)(X+Y)}Ye^{-sX}ds. \]

We apply it to \( X + Y := \Delta_{\varphi}, \ X := \Delta \) hence \( Y := \Delta_{\varphi} - \Delta \) where \( \Delta_{\varphi} \) is the image by \( \varphi \) of the laplacian on \( M \), \( \Delta \) is the laplacian on \( \mathbb{R}^n \).

Given operators \( A(t), B(t) \) on our space, we set

\[ A * B := \int_0^t A(t-s)B(s)ds. \]

By [13, formula (3.17)] we get

\[ e^{-t\Delta_{\varphi}} = e^{-t\Delta} + e^{-t\Delta} * \sum_{j=1}^{\infty} (-1)^j(Ye^{-t\Delta})^j, \quad (3.1) \]

because we shall choose \( \epsilon \) small enough to make the series converging, as we shall see later on.

Now on for \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \) we set \( \partial^n f := \frac{\partial^{\gamma} f}{\partial^{\gamma_1} x_1 \cdots \partial^{\gamma_n} x_n} \) and \( |\gamma| := \gamma_1 + \cdots + \gamma_n \).

**Proposition 3.3.** Let \( r \in [1, \infty]. \) Let \( x \in M \), a Riemannian manifold. With \( B := B(x,R) \) a \((m,\epsilon)\)-admissible ball in \( M \), and any \( \delta \in (0,1) \), there is a \( \epsilon(\delta) > 0 \) such that for any \( \epsilon \leq \epsilon(\delta) \), if \( \omega \) is a \( p \)-form in \( L^r(B) \), then the \( p \)-form \( u_\varphi := e^{-t\Delta_{\varphi} \omega_\varphi} \) verifies, in \( \mathbb{R}^n \), with \( \forall \gamma \in \mathbb{N}^n, \ l := |\gamma| / 2 \)

\[ \forall t \in (\delta,1), \ |\partial^n u_\varphi|_{L^r(B_\varphi)} \leq c(n,r) \frac{\delta}{t - \delta} R^{-m} \|\omega_\varphi\|_{L^r(B_\varphi)}; \]

And

\[ \forall t \geq 1, \ |\partial^n u_\varphi|_{L^r(B_\varphi)} \leq c(n,r,\delta) t^{-l} R^{-m} \|\omega_\varphi\|_{L^r(B_\varphi)} \]

with \( m = 1 \) if \( p = 0 \) and \( m = 2 \) if \( p \geq 1 \). And also \( B_\varphi = \varphi(B) \), \( \omega_\varphi = \varphi^* \omega \) etc...

The long proof of this proposition is postponed to Appendix 1.
3.1 Sobolev comparison estimates.

Lemma 3.4. Let \( B(x, R) \in \mathcal{A}_m(\epsilon) \). We have for the Levi-Civita connection on \( M \):
\[
\forall y \in B(x, R), \, \forall k \leq m \in \mathbb{N}, \quad \left| \partial^{k-1} \Gamma_i^j(y) \right| \leq C(n, k)\epsilon R^{-k}.
\]

Proof. Let \( \Gamma^i_{kj} \) be the Christoffel coefficients of the Levi-Civita connection on the tangent bundle \( TM \). We have
\[
\Gamma^i_{kj} = \frac{1}{2} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right).
\]

On \( B(x, R) \in \mathcal{A}_m(\epsilon) \), we have \( (1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij} \) as bilinear forms. Hence
\[
\forall y \in B(x, R), \quad \left| \Gamma^i_{kj}(y) \right| \leq \frac{3}{2} (1-\epsilon)^{-1} \sum_{|\beta|=1} \sup_{i,j=1,...,n} \left| \partial^\beta g_{ij}(y) \right|
\]
in a coordinates chart on \( B(x, R) \). We also have, by definition 1.1
\[
\sum_{1\leq|\beta|\leq m} R^{i|\beta|} \sup_{i,j=1,...,n, y\in B_\epsilon(R)} \left| \partial^\beta g_{ij}(y) \right| \leq \epsilon.
\]

Hence
\[
\forall y \in B(x, R), \quad \left| \Gamma^i_{kj}(y) \right| \leq \frac{3}{2} (1-\epsilon)^{-1}\epsilon R^{-1}.
\]

Taking the first derivatives on (3.2) gives:
\[
\partial \Gamma^i_{kj} = \frac{1}{2} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) + \frac{1}{2} g^{il} \left( \frac{\partial^2 g_{kl}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{lj}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{jk}}{\partial x^l \partial x^l} \right)
\]

So
\[
\forall y \in B(x, R), \quad \left| \partial \Gamma^i_{kj}(y) \right| \leq C(1-\epsilon)^{-1} \sum_{|\beta|=1} \sup_{i,j=1,...,n} \left| \partial^\beta g_{ij}(y) \right|^2 + \sum_{|\beta|\leq 2} \sup_{i,j=1,...,n} \left| \partial^\beta g_{ij}(y) \right|.
\]

This gives, using (3.3):
\[
\forall y \in B(x, R), \quad \left| \partial \Gamma^i_{kj}(y) \right| \leq C(1-\epsilon)^{-1}(\epsilon^2 R^{-2} + \epsilon R^{-2})
\]
and, because \( \epsilon < 1 \),
\[
\forall y \in B(x, R), \quad \left| \partial \Gamma^i_{kj}(y) \right| \leq C(1-\epsilon)^{-1}\epsilon R^{-2},
\]
the constant \( C \) being independent of \( x, R \) and \( \epsilon \). Taking \( \epsilon \leq 1/2 \), we get
\[
\forall y \in B(x, R), \quad \left| \partial \Gamma^i_{kj}(y) \right| \leq C \epsilon R^{-2},
\]
again the constant \( C \) being independent of \( x, R \) and \( \epsilon \).

Derivating \( k \) times the formula (3.2), with \( k \leq m \), gives:
\[
\forall y \in B(x, R), \quad \left| \partial^{k-1} \Gamma^i_{kj}(y) \right| \leq C(n, k)\epsilon R^{-k}.
\]

The proof is complete.

Lemma 3.5. Let \( B(x, R) \) be a \((k,\epsilon)\)-admissible ball in \( M \) and \( \varphi : B(x, R) \to \mathbb{R}^n \) be the admissible chart relative to \( B(x, R) \). Set \( u_\varphi := \varphi^* u \), then, for any \( k \in \mathbb{N} \):
\[
\left\| \nabla^k u \right\|_{L^r(B(x,R))} \leq \left\| \partial^k u_\varphi \right\|_{L^r(B_\epsilon(0,1+\epsilon)R))} + \epsilon \sum_{j=0}^{k-1} (C_k R^{-j-1}) \left\| \partial^j u_\varphi \right\|_{L^r(B_\epsilon(0,1+\epsilon)R))}.
\]

and, with \( B_\epsilon(0,t) \) the euclidean ball in \( \mathbb{R}^n \) centered at 0 and of radius \( t \),
\[
\left\| u_\varphi \right\|_{W^{k,v}(B_\epsilon(0,1-\epsilon)R))} \leq c R^{-k} \left\| u \right\|_{W^{k,v}(B(x,R))}.
\]
We also have, for \( k = 0 \) and \( B(x, R) \) being \((0, \epsilon)\)-admissible:
\[
\forall u \in L^\infty_p(B(x, R)), \quad \|u\|_{L^\infty(B(x, R))} \leq (1 + C\epsilon)\|u_\varphi\|_{L^\infty(\varphi(B(x, R)))},
\]
and
\[
\|u_\varphi\|_{L^\infty(B_r(B_e(0,(1-\epsilon)R)))} \leq (1 + C\epsilon)\|u\|_{L^\infty(B(x, R))}.
\]
The constants \( c, \ C \) being independent of \( B \).

In the case of a function \( u \) on \( M \), we have better results. Let \( B(x, R) \) be a \((k - 1, \epsilon)\)-admissible ball in \( M \) and \( \varphi : B(x, R) \to \mathbb{R}^n \) be the admissible chart relative to \( B(x, R) \). Set \( u_\varphi := u \circ \varphi^{-1} \), then for \( k \geq 1 \):
\[
\|\nabla^k u\|_{L^\infty(B)} \leq \left\| \partial^k u_\varphi \right\|_{L^\infty(B_r(B_e(0,(1-\epsilon)R)))} + c\epsilon(R\epsilon^{-1}\|\partial u\varphi\|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))}) + \cdots + R^{(k-1)}\|\partial^{k-1} u_\varphi\|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))},
\]
and
\[
\|u_\varphi\|_{W^{k,r}(B_r(B_e(0,(1-\epsilon)R)))} \leq cR^{(1-k)}\|u\|_{W^{k,r}(B(x, R))}.
\]

Proof.
We have to compare the norms of \( u, \nabla u, \cdots, \nabla^m u \), with the corresponding ones for \( u_\varphi := \varphi^* u \) in \( \mathbb{R}^n \).

**Case of a function**
Let us start with the case of a function \( u \) on \( M \). In this case we have: \((\nabla u)_j := \partial_j u \) in local coordinates, so \( |\nabla u(y)| = |\partial u_\varphi(z)| \).
The components of \( \nabla^2 u \) are given by \((\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma^k_{ij} \partial_k u \) where the Christoffel symbols \( \Gamma^k_{ij} \) are those of the Levi-Civita connection. Now we have for \( B(x, R) \) a \((1, \epsilon)\)-admissible ball: \( |\Gamma^k_{ij}| \leq C\epsilon/R \).
So we get, with \(|\partial u_\varphi(z)| := \sum_j |\partial_j u_\varphi| \) and \( |\partial^2 u_\varphi(z)| := \sum_{j,k} |\partial^2_{jk} u_\varphi| \),
\[
|\nabla^2 u(y)| \leq \left| \partial^2 u_\varphi(z) \right| + c\epsilon R \left| \partial u_\varphi(z) \right|.
\]
Hence, taking the \( L^\infty \) norm, we get
\[
\|\nabla^2 u\|_{L^\infty(B(x, R))} \leq \left\| \partial^2 u_\varphi \right\|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))} + C\epsilon \|\partial u\varphi\|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))}.
\]
For controlling \( \nabla^k u \) we need only to have \( B(x, R) \) to be \((k - 1, \epsilon)\)-admissible and we get the same way:
\[
\forall y \in B(x, R), \quad |\nabla^k u(y)| \leq \left| \partial^k u_\varphi(z) \right| + \epsilon(C_1 R^{-1}) \left| \partial u_\varphi(z) \right| + \cdots + C_{k-1} R^{(1-k)} \left| \partial^{k-1} u_\varphi(z) \right|
\]
So, taking the \( L^\infty \) norm, for any \( k \in \mathbb{N} \) with \( B := B(x, R) \) to be \((k - 1, \epsilon)\)-admissible:
\[
\|\nabla^k u\|_{L^\infty(B)} \leq \left\| \partial^k u_\varphi \right\|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))} + \epsilon(C R^{-1}) \left| \partial u_\varphi \right|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))} + \cdots + R^{-1} \left| \partial^{k-1} u_\varphi \right|_{L^\infty(B_r(B_e(0,(1+\epsilon)R)))}.
\]
We shall need also the easy reverse estimate:
\[
\|u_\varphi\|_{L^\infty(B_r(B_e(0,(1-\epsilon)R)))} \leq \|u\|_{L^\infty(B)}.
\]

**Case of a p-form, \( p \geq 1 \).**
By Lemma 2.3 the \((m, \epsilon)\)-admissible ball \( B(x, R) \) trivialises the bundle \( \Lambda^p \) of p-forms on \( M \), hence the image of a p-form in \( \mathbb{R}^n \) is just a vector of functions. Precisely \( u_\varphi := \varphi^* u \in \varphi(B(x, R)) \times \mathbb{R}^N \).

We have, because \((1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij} \) in \( B(x, R) \):
\[
B_e(0,(1-\epsilon)R) \subset \varphi(B(x, R)) \subset B_e(0,(1+\epsilon)R).
\]
Let \( u \) be a p-form in \( M \). We have that \( \nabla u \) depends on the first order derivatives of the metric tensor \( g \). Precisely, using formula [1], §26, p. 106 set \( J := (i_1, \ldots, i_p) \in \mathbb{N}^p \)
\[
u_\varphi := \sum_{J \in \mathbb{N}^p} a_J dx^J = \varphi^* u \text{ in the chart } (B, \varphi) \text{ then we have that its covariant derivative } \nabla u \text{ has for components:}
\]
\[ \nabla_i \alpha_j = \frac{\partial \alpha_j}{\partial x^i} - \sum_{\nu=1}^{p} \alpha_{i_1...i_{\nu-1}k_{\nu+1}...i_p} \Gamma^k_{i_\nu i_i}. \quad (3.4) \]

with summation made with respect to the repeated index \( k \).

By Lemma 3.4 we get, with the fact that \( B(x, R) \) is \((1, \varepsilon)\)-admissible,
\[ \forall y \in B(x, R), \forall k \leq m \in \mathbb{N}, \ |\Gamma_{ij}(y)| \leq C(n)\varepsilon R^{-1}, \]
with \( C \) being independent of \( B \).

Hence
\[ \forall y \in B(x, R), \ |u(y)| = |u_\varphi(z)|, \ |\nabla u(y)| \leq |\partial u| + |\Phi|, \]
where \( \Phi \) is given by formula (3.4) and depends linearly on the coefficients of \( u \) and linearly on the first order derivatives of the metric tensor \( g \) via the Christoffel symbols \( \Gamma^k_{ij} \).

So
\[ |\nabla u(y)| \leq |\partial u_\varphi(z)| + C\varepsilon R^{-1} |u_\varphi(z)|. \quad (3.5) \]

Taking the \( L^r \) norm of this, we get
\[ \| \nabla u(y) \|_{L^r(B(x, R))} \leq \| \partial u_\varphi \|_{L^r(B(0, (1+\varepsilon)R))} + C\varepsilon R^{-1} \| u_\varphi \|_{L^r(B(0, (1+\varepsilon)R))}. \]
The same way for \( \nabla^k u \) with \( 1 < k \leq m \), by iterating formula (3.4) and still with Lemma 3.4 we have:
\[ \forall y \in B(x, R), \ |\nabla^k u(y)| \leq |\partial^k v(z)| + \varepsilon(C_0 R^{-1} |u_\varphi(z)| + C_1 R^{-2} |\partial u_\varphi(z)| + \cdots + C_{k-1} R^{-k} |\partial^{k-1} u_\varphi(z)|). \]

We deduce
\[ \| \nabla^k u \|_{L^r(B(x, R))} \leq \| \partial^k u_\varphi \|_{L^r(B(0, (1+\varepsilon)R))} + \varepsilon \sum_{j=0}^{k-1} (C_j R^{-j-1}) \| \partial^j u_\varphi \|_{L^r(B(0, (1+\varepsilon)R))}. \]

So, with a new constant \( c \) independent of \( B \):
\[ \| \nabla^k u \|_{L^r(B(x, R))} \leq \| \partial^k u_\varphi \|_{L^r(B(0, (1+\varepsilon)R))} + c\varepsilon R^{-k} \sum_{j=0}^{k-1} \| \partial^j u_\varphi \|_{L^r(B(0, (1+\varepsilon)R))}, \quad (3.6) \]
because \( R \leq 1 \).

And, using the fact that \( \| u \|_{W^{m,r}(B(x, R))} \approx \sum_{k=0}^{m} \| \nabla^k u \|_{L^r(B(x, R))} \), we also get:
\[ \| u \|_{W^{m,r}(B(x, R))} \leq c R^{-m} \| u_\varphi \|_{W^{m,r}(B(0, (1+\varepsilon)R))}. \]
The same way we get the reverse estimates
\[ \| u_\varphi \|_{W^{m,r}(B(0, (1-\varepsilon)R))} \leq c R^{-m} \| u \|_{W^{m,r}(B(x, R))}. \]
The case \( m = 0 \) is given by the equation (3.6).
All the constants here are independent of \( B \).
The proof of the lemma is complete. ■
3.2 The main local estimates.

**Theorem 3.6.** Let $r \in [1, \infty]$ and $k \in \mathbb{N}$. Let $B := B(x, R)$ be a $(\beta, \epsilon)$-admissible ball in the Riemannian manifold $M$ with $\beta := \max(k-1, 1)$ if $p = 0$ and $\beta := \max(k, 2)$ if $p \geq 1$. Then, with $\omega \in L^2_p(B) \cap L^\infty_p(B)$ and $u(x, t) = e^{-t \Delta} \omega$, the canonical solution of the heat equation, we get:

$$
\forall t \in (\delta, 1), \left\| \nabla^k u(\cdot, t) \right\|_{L^r(B(x, R))} \leq c(n, r, \delta) \frac{t^{-\beta}}{t^{\beta/2}} R^{-k-m} \| \omega \|_{L^r(B)},
$$

and

$$
\forall t \geq 1, \left\| u(\cdot, t) \right\|_{L^r(B)} \leq c(n, r) R^{-m} \| \omega \|_{L^r(B)}.
$$

This gives for $p$-forms with $p \geq 1$, $k \geq 1$,

$$
\forall t \geq 1, \left\| \nabla^k u(\cdot, t) \right\|_{L^r(B)} \leq c(n, r) R^{-k-m} \| \omega \|_{L^r(B)}.
$$

And for functions, i.e. $p = 0$, $k \geq 1$,

$$
\forall t \geq 1, \left\| \nabla^k u(\cdot, t) \right\|_{L^r(B)} \leq c(n, r) t^{-1/2} R^{-k-m} \| \omega \|_{L^r(B)}.
$$

Again we have that $m = 1$ on functions and $m = 2$ on $p$-forms with $p \geq 1$.

Proof. 

Let $\omega \in L^2(B) \cap L^r(B)$ and $u = e^{-t \Delta} \omega$ the canonical solution of the heat equation.

The ball $B$ being admissible, there is a diffeomorphism $\varphi : B \to \mathbb{R}^n$ such that $\Lambda^p$ trivialises on $B$.

So the local representation of the $p$-form $u$ is a vector of functions.

We shall apply Proposition 3.3 with $m = 1$ if $p = 0$ and $m = 2$ if $p \geq 1$, and with $\forall \gamma \in \mathbb{N}^n$, $l := |\gamma|/2:

$$
\forall t \in (\delta, 1), \left\| \partial^\gamma u(\cdot, t) \right\|_{L^r(B_{\varphi})} \leq c(n, r, \delta) \frac{t^{-1}}{t^{1+l}} R^{-m} \| \omega \|_{L^r(B_{\varphi})},
$$

$$
\forall t \geq 1, \forall \gamma \in \mathbb{N}^n, \left\| \partial^\gamma u(\cdot, t) \right\|_{L^r(B_{\varphi})} \leq c(n, r, \delta) t^{-1} R^{-m} \| \omega \|_{L^r(B_{\varphi})}.
$$

(3.7)

Where $B_{\varphi}, u_{\varphi}, \omega_{\varphi}$ are the images by $\varphi$ of $B, u, \omega$ and the image of $\Lambda^p$ is the trivial bundle $\varphi(B) \times \mathbb{R}^N$ in $\mathbb{R}^n$. The constants being independent of $B$.

First, because of the condition $(1 - \epsilon) \delta_{ij} \leq g_{ij} \leq (1 + \epsilon) \delta_{ij}$ in the definition of the $\epsilon$-admissible ball, we have that $R_{\varphi} \simeq R$. Recall that $u = e^{-t \Delta} \omega$.

Now we use the Sobolev comparison estimates given by Lemma 3.3 and, to apply it, we need to have $B \in \mathcal{A}_k(\epsilon)$ and this is the reason to define $\beta := \max(k-1, 1)$ if $p = 0$ and $\beta := \max(k, 2)$ if $p \geq 1$. And we get:

$$
\| u(\cdot, t) \|_{L^r(B)} \leq C \| u_{\varphi}(\cdot, t) \|_{L^r(B_{\varphi})},
$$

(3.8)

and, for any $k \in \mathbb{N},$

$$
\| \nabla^k u(\cdot, t) \|_{L^r(B(x, R))} \leq \| \partial^k u_{\varphi}(\cdot, t) \|_{L^r(B_{\varphi})} + \epsilon \sum_{j=0}^{k-1} (C_j R^{-j-1}) \| \partial^j u_{\varphi}(\cdot, t) \|_{L^r(B_{\varphi})},
$$

hence, because $R \leq 1,$

$$
\| \nabla^k u(\cdot, t) \|_{L^r(B(x, R))} \leq \| \partial^k u_{\varphi}(\cdot, t) \|_{L^r(B_{\varphi})} + \epsilon C R^{-k} \| u_{\varphi}(\cdot, t) \|_{W^{k-1, r}(B_{\varphi})}.
$$

(3.9)
Case of functions

In the case of functions, we need only to have \( B \in A_\beta(\epsilon) \) with \( \beta := \max(k - 1, 1) \) and we have no term in \( u_\varphi \) in the right hand side, so we get:

\[
\| \nabla^k u(\cdot, t) \|_{L^r(B(x,R))} \leq \| \partial^k u_\varphi(\cdot, t) \|_{L^r(B_\varphi)} + \epsilon \sum_{j=1}^{k-1} (C_j R^{-j-1}) \| \partial^j u_\varphi(\cdot, t) \|_{L^r(B_\varphi)},
\]

hence:

\[
\| \nabla^k u(\cdot, t) \|_{L^r(B(x,R))} \leq \| \partial^k u_\varphi(\cdot, t) \|_{L^r(B_\varphi)} + \epsilon C R^{-k} \sum_{j=1}^{k-1} \| \partial^j u_\varphi(\cdot, t) \|_{L^r(B_\varphi)},
\]

(3.10)

The constants being independent of \( B \).

Now, still by Lemma 3.3,

\[
\| \omega_\varphi \|_{L^r(B_\varphi)} \leq C \| \omega \|_{L^r(B)}.
\]

Hence replacing in (3.8) we get, with new constants (with \( m = 1 \) because we deal with functions):

\[
\forall t \in (\delta, 1), \quad \| u(\cdot, t) \|_{L^r(B)} \leq c(n, r) \| u_\varphi(\cdot, t) \|_{L^r(B_\varphi)} \leq c(n, r) \frac{\delta}{t - \delta} R^{-m} \| \omega_\varphi \|_{L^r(B_\varphi)} \leq c(n, r) \frac{\delta}{t - \delta} R^{-m} \| \omega \|_{L^r(B)}.
\]

So

\[
\forall t \in (\delta, 1), \quad \| u(\cdot, t) \|_{L^r(B)} \leq c(n, r) \frac{\delta}{t - \delta} R^{-m} \| \omega \|_{L^r(B)}.
\]

The same way:

\[
\forall t \geq 1, \quad \| u(\cdot, t) \|_{L^r(B)} \leq c(n, r) R^{-m} \| \omega \|_{L^r(B)}.
\]

For the gradient estimate on functions, we get as above, with \( \nabla \) the covariant derivative on \( M \) and \( t \in (\delta, 1) \):

\[
\| \nabla u(\cdot, t) \|_{L^r(B(x,R))} \leq \| \partial u_\varphi(\cdot, t) \|_{L^r(B_\varphi)} \leq c(n, r) \frac{\delta}{t - \delta} R^{-m} \| \omega_\varphi \|_{L^r(B_\varphi)}.
\]

Hence

\[
\forall t \in (\delta, 1), \quad \| \nabla u(\cdot, t) \|_{L^r(B)} \leq c(n, r) \frac{\delta}{t - \delta} R^{-m} \| \omega \|_{L^r(B)}.
\]

For functions we have no term in \( u_\varphi \) by Lemma 3.4,

\[
\| \nabla u(\cdot, t) \|_{L^r(B(x,R))} \leq \| \partial u_\varphi(\cdot, t) \|_{L^r(B_\varphi)},
\]

hence by 3.7

\[
\forall t \geq 1, \quad \| \nabla u(\cdot, t) \|_{L^r(B)} \leq c(n, r) R^{-m} \| \omega \|_{L^r(B)}.
\]

And more generally, for \( k \geq 2 \), by the same way, with this time \( B(x, R) \) being a \((k-1, \epsilon)\)-admissible ball for functions:

\[
\forall t \in (\delta, 1), \quad \| \nabla^k u(\cdot, t) \|_{L^r(B(x,R))} \leq c(n, r) \frac{\delta}{t - \delta^{1+k/2}} R^{-m} \| \omega_\varphi \|_{L^r(B_\varphi)} + \epsilon c(n, r) C R^{k-1} \sum_{j=1}^{k-1} \frac{\delta}{t - \delta^{1+j/2}} R^{-m} \| \omega_\varphi \|_{L^r(B_\varphi)},
\]

\[
\text{with } l = \| \gamma \| / 2 \text{ and where we used}
\]

\[
\forall t \in (\delta, 1), \quad \| \partial^\gamma u(\cdot, t) \|_{L^r(B_\varphi)} \leq c(n, r) \frac{\delta}{t - \delta^{1+l/2}} R^{-m} \| \omega_\varphi \|_{L^r(B_\varphi)}.
\]

Hence
For $t \geq 1$ we have by (3.10)

$$\forall t \geq 1, \|\nabla^k u(\cdot, t)\|_{L^r(B(x, R))} \leq c(n, r) \frac{\delta}{t - \delta/2} R^{-k-m} \|\omega\|_{L^r(B)}.$$  

Hence

$$\forall t \geq 1, \|\nabla u(\cdot, t)\|_{L^r(B)} \leq c(n, r, \delta) t^{-1/2} R^{-m} \|\omega\|_{L^r(B)}.$$  

Case of forms

This time $B(x, R)$ is a $(k, \epsilon)$-admissible ball (with $m = 2$ because we deal with $p$-forms with $p \geq 1$).

For forms because by (3.7)

$$\forall t \geq 1, \forall \gamma \in \mathbb{N}, \ 1 \leq |\gamma| / 2 : \|\partial^\gamma u(\cdot, t)\|_{L^r(B_{\gamma})} \leq c(n, r, \delta) t^{-1} R^{-m} \|\omega\|_{L^r(B_{\gamma})}.$$  

Hence

$$\forall t \geq 1, \|\nabla u(\cdot, t)\|_{L^r(B)} \leq c(n, r, \delta) t^{-1/2} R^{-m} \|\omega\|_{L^r(B)} + \epsilon C R^{-1} \|\omega\|_{L^r(B)},$$

so

$$\forall t \geq 1, \|\nabla u(\cdot, t)\|_{L^r(B)} \leq c(n, r, \delta) R^{-m} \|\omega\|_{L^r(B)}.$$  

Hence

$$\forall t \geq 1, \|\nabla u(\cdot, t)\|_{L^r(B)} \leq c(n, r, \delta) R^{-m} \|\omega\|_{L^r(B)} (t^{-1/2} + \epsilon C R^{-1}).$$

And by use of Lemma 3.6 and because for any $t \geq 1$, $t^{-1/2} \leq 1$ and $\hat{R} < 1$,

$$\forall t \geq 1, \|\nabla u(\cdot, t)\|_{L^r(B)} \leq c(n, r, \delta) R^{-1-m} \|\omega\|_{L^r(B)}.$$  

So for $p$-forms with $p \geq 1$,

$$\|\nabla^k u(\cdot, t)\|_{L^r(B(x, R))} \leq \|\partial^k u(\cdot, t)\|_{L^r(B_{\gamma})} + \epsilon C R^{-k} \|u(\cdot, t)\|_{W^{k, r}(B_{\gamma})}. \tag{3.11}$$

Hence

$$\forall t \geq 1, \|\nabla^k u(\cdot, t)\|_{L^r(B)} \leq c(n, r) R^{-k-m} \|\omega\|_{L^r(B)}.$$  

The proof is complete. ■

4 Vitali covering.

Lemma 4.1. Let $\mathcal{F}$ be a collection of balls $\{B(x, r(x))\}$ in a metric space, with $\forall B(x, r(x)) \in \mathcal{F}, \ 0 < r(x) \leq R$. There exists a disjoint subcollection $\mathcal{G}$ of $\mathcal{F}$ with the following properties:

- every ball $B$ in $\mathcal{F}$ intersects a ball $C$ in $\mathcal{G}$ and $B \subset 5C$.

This is a well known lemma, see for instance [3], section 1.5.1.

Fix $\epsilon > 0$ and let $\forall x \in M, \ r(x) := R_\epsilon(x)/120$, where $R_\epsilon(x)$ is the $(m, \epsilon)$-admissible radius at $x$, we built a Vitali covering with the collection $\mathcal{F} := \{B(x, r(x))\}_{x \in M}$. The previous lemma gives a disjoint subcollection $\mathcal{G}$ such that every ball $B$ in $\mathcal{F}$ intersects a ball $C$ in $\mathcal{G}$ and we have $B \subset 5C$. 

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We set \( G' := \{ x_j \in M : B(x_j, r(x_j)) \in G \} \) and \( C_\epsilon := \{ B(x, 5r(x)), \ x \in G' \} \): we shall call \( C(\epsilon) \) the \( m, \epsilon \) admissible covering of \((M, g)\).

We shall fix \( m \geq 0 \) and we omit it in order to ease the notation.

Then we have the Proposition 7.3 in [1]:

**Proposition 4.2.** Let \( M \) be a Riemannian manifold, then the overlap of the \( \epsilon \) admissible covering \( C(\epsilon) \) is less than \( T = \frac{1 + \epsilon^{n/2}}{(1 - \epsilon^{n/2})(120)^n} \), i.e.

\[
\forall x \in M, \ x \in B(y, 5r(y)) \ \text{where} \ B(y, r(y)) \in G \ \text{for at most} \ T \ \text{such balls.}
\]

So we have

\[
\forall f \in L^1(M), \ \sum_{j \in \mathbb{N}} \int_{B_j} |f(x)| \, dv_g(x) \leq T \| f \|_{L^1(M)}.
\]

5 The threshold.

**Theorem 5.1.** Let \( M \) be a Riemannian manifold. Let, for \( t \geq 0, \ \omega \in L^2(M) \). Then we have a solution \( u \) of the heat equation \( \partial_t u - \Delta u = 0, \ u(x, 0) = \omega(x) \), such that \( \forall t \geq 0, \ u(x, t) \in L^2(M) \) with the estimate:

\[
\forall t \geq 0, \ \| u(\cdot, t) \|_{L^2(M)} \leq \| \omega \|_{L^2(M)}.
\]

Proof.

It is well known that the Hodge laplacian is essentially positive on \( p \)-forms in \( L^2(M) \), so \( (e^{-t\Delta})_{t \geq 0} \) is a contraction semi-group on \( L^2(M) \). \( \blacksquare \)

6 Global results.

We want to globalise Theorem 3.6 by use of our Vitali covering.

First set

\[
D(\epsilon) := \{ x \in M : B(x, R_\epsilon(x)) \in C(\epsilon) \}.
\]

**Lemma 6.1.** Let \( f \) be a \( p \)-form in \( M \) and \( \tau \in [1, \infty) \). Set \( w(x) := R_\epsilon(x)^\gamma \) for a \( \gamma \in \mathbb{R} \) and \( B(x) := B(x, R_\epsilon(x)/10) \), where \( R_\epsilon(x) \) is the \( \epsilon \)-admissible radius. For \( l \geq 0 \) we have that:

\[
\forall \tau \geq 1, \ \| \nabla^l f \|_{L^\tau(M, w)} \simeq \sum_{x \in D(\epsilon)} R_\epsilon(x)^\gamma \| \nabla^l f \|_{L^\tau(B(x))}.
\]

Proof.

Let \( x \in D(\epsilon) \), this implies that \( B(x) := B(x, R_\epsilon(x)/10) \in C(\epsilon) \).

- First we start with \( l = 0 \). We shall deal with the function \( |f| \).

For any \( y \in B(x) \) we set \( R(y) := R_\epsilon(y) \). We have, because \( C(\epsilon) \) is a covering of \( M \):

\[
\| f \|_{L^\tau(M, w)} := \int_M |f(x)|^\tau w(x) dv(x) \leq \sum_{x \in D(\epsilon)} \int_{B(x)} |f(y)|^\tau R(y)^\gamma dv(y).
\]

We have, by Lemma 2.3 \( \forall y \in B, \ R(y) \leq 2R(x) \), then

\[
\sum_{x \in D(\epsilon)} \int_{B(x)} |f(y)|^\tau R(y)^\gamma dv(y) \leq \]
\[ \leq \sum_{x \in \mathcal{D}(\varepsilon)} 2^\gamma R(x)^\gamma \int_{B(x)} |f(y)|^\tau \, dv(y) \leq 2^\gamma \sum_{x \in \mathcal{D}(\varepsilon)} R(x)^\gamma \|f\|_{L^r(B(x))}^r. \]

Hence
\[ \|f\|_{L^r(M, w)}^r \leq 2^\gamma \sum_{x \in \mathcal{D}(\varepsilon)} R(x)^\gamma \|f\|_{L^r(B)}^r. \]

To get the converse inequality we still use Lemma 2.3 \( \forall y \in B, \ R(x) \leq 2R(y) \) so we get:
\[ \sum_{x \in \mathcal{D}(\varepsilon)} R(x)^\gamma \int_{B(x)} |f(y)|^\tau \, dv(y) \leq 2^\gamma \sum_{x \in \mathcal{D}(\varepsilon)} \int_{B(x)} R(y)^\gamma |f(y)|^\tau \, dv(y). \]

Now we use the fact that the overlap of \( \mathcal{C}(\varepsilon) \) is bounded by \( T \),
\[ \sum_{x \in \mathcal{D}(\varepsilon)} \int_{B(x)} R(y)^\gamma |f(y)|^\tau \, dv(y) \leq 2^\gamma T \int_{M} R(y)^\gamma |f(y)|^\tau \, dv(y) = 2^\gamma T \|f\|_{L^r(M, w)}^r. \]

So
\[ \sum_{x \in \mathcal{D}(\varepsilon)} R^\gamma \|f\|_{L^r(B)}^r \leq 2^\gamma T \|f\|_{L^r(M, w)}^r. \]

Now let \( l \geq 1 \).

We apply the case \( l = 0 \) to the covariant derivatives of \( f \).
\[ \forall \tau \geq 1, \ \|\nabla^l f\|_{L^r(M, w)}^r \simeq \sum_{x \in \mathcal{D}(\varepsilon)} R(x)^\gamma \|\nabla^l f\|_{L^r(B(x))}^r. \]

The proof is complete. \( \blacksquare \)

Let \( \omega \in L^2(M) \cap L^r(M) \) and let \( u := e^{-t\Delta} \omega \) be the canonical solution of the heat equation given by Theorem 5.1, i.e.
\[ \forall t \geq 0, \ \|u(\cdot, t)\|_{L^2(M)} \leq \|\omega\|_{L^2(M)}. \]

By Lemma 6.1 we get, replacing \( f \) by \( u \) and \( \tau \) by \( r \), with \( w(x) := R_\epsilon(x)^{\gamma} \) and using the covering \( \mathcal{C}(\varepsilon) \):
\[ \forall \tau \geq 1, \ \|\nabla^l u(\cdot, t)\|_{L^r(M, w)}^r \simeq \sum_{x \in \mathcal{D}(\varepsilon)} R_\epsilon(x)^\gamma \|\nabla^l u(\cdot, t)\|_{L^r(B(x))}^r. \] (6.12)

But Theorem 3.6 tells us with \( B := B(x, R) \in \mathcal{A}_k(\varepsilon) \) for \( p \)-form with \( p \geq 1 \) and \( B := B(x, R) \in \mathcal{A}_{k-1}(\varepsilon) \) if \( p = 0 \) and with \( m = 1 \) on functions and \( m = 2 \) on \( p \)-forms with \( p \geq 1 \):
\[ \forall t \in (\delta, 1), \ \|\nabla^k u(\cdot, t)\|_{L^r(B(x), R)} \leq c(n, r) \frac{\delta}{t - \delta^{3/2}} R^{-k-m} \|\omega\|_{L^r(B)}, \]

so, with \( l = k \) in (6.12) we get:
\[ \forall \tau \geq 1, \ \|\nabla^k u(\cdot, t)\|_{L^r(M, w)}^r \leq \sum_{x \in \mathcal{D}(\varepsilon)} R_\epsilon(x)^\gamma \|\nabla^k u(\cdot, t)\|_{L^r(B(x))}^r \leq \sum_{x \in \mathcal{D}(\varepsilon)} R_\epsilon(x)^\gamma R_\epsilon(x)^{-kr-m} \|\omega\|_{L^r(B(x))}^r \leq c(n, r)^\tau \frac{\delta}{t - \delta^{3/2}} \|\omega\|_{L^r(M, w')}. \]

Here we have set \( u'(x) := R_\epsilon(x)^{-kr-m} \) and \( w(x) := R_\epsilon(x)^{\gamma} \). Hence:
\[ \forall t \in (\delta, 1), \forall \tau \geq 1, \ \|\nabla^k u(\cdot, t)\|_{L^r(M, w)} \leq c(n, r) \frac{\delta}{t - \delta^{3/2}} \|\omega\|_{L^r(M, w')}. \]

For \( t \geq 1 \), we get by Theorem 3.6 for \( p \geq 1 \), hence \( m = 2 \):
\[ \forall \tau \geq 1, \ \|\nabla^k u(\cdot, t)\|_{L^r(B)} \leq c(n, r) R^{-k-2} \|\omega\|_{L^r(B)}, \]
Exactly the same way as above, we get:
\[ \forall t \geq 1, \forall r \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)\| \omega \|_{L^r(M, w')} . \]

Here we also set \( w'(x) := R_\gamma(x) \gamma^{-kr - 2\varepsilon} \) and \( w(x) := R_\varepsilon(x) \gamma \).

For \( p = 0 \), i.e. for functions, hence \( m = 1 \) with \( w'(x) := R_\gamma(x) \gamma^{-kr - r} \) and \( w(x) := R_\gamma(x) \gamma \), we get the better result:
\[ \forall t \geq 1, \forall r \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)t^{-1/2}\| \omega \|_{L^r(M, w')} . \]

Always with \( m = 1 \) if \( p = 0 \) and \( m = 2 \) if \( p \geq 1 \), we have: if \( k \leq 1 \) we have \( R_\varepsilon(x) := R_{m,\varepsilon}(x) \), i.e. this is the admissible radius for the \((m, \varepsilon)\)-admissible balls. If \( k \geq 2 \), then \( R_\varepsilon(x) \) is the admissible radius for the \((k, \varepsilon)\)-admissible balls for \( p \geq 1 \) and for the \((k - 1, \varepsilon)\)-admissible balls for \( p = 0 \).

Now we choose, for instance, \( \gamma = kr + rm \) and we get, with \( w(x) := R_\varepsilon(x)^{kr + rm} \):
\[ \forall t \in (\delta, 1), \forall r \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)(\delta t)\| \omega \|_{L^r(M)} . \]

And, for \( p \geq 1 \):
\[ \forall t \geq 1, \forall r \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)\| \omega \|_{L^r(M)} . \]

And for functions:
\[ \forall t \geq 1, \forall r \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)t^{-1/2}\| \omega \|_{L^r(M)} . \]

For \( r = \infty \), the passage from the local estimates to the global ones are obvious, so we proved:

**Theorem 6.2.** Let \( M \) be a Riemannian manifold. Let \( r \in [1, \infty] \) and \( \omega \in L^r(M) \cap L^2(M) \). We have, with \( u := e^{\Delta \omega} \) the canonical solution of the heat equation, for any \( k \geq 0 \):
\[ \forall t \in (\delta, 1), \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)(\delta t)\| \omega \|_{L^r(M)} . \]

Now for \( p \)-forms with \( p \geq 1 \):
\[ \forall t \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)\| \omega \|_{L^r(M)} . \]

and for functions:
\[ \forall t \geq 1, \| \nabla^k u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)t^{-1/2}\| \omega \|_{L^r(M)} . \]

where \( w(x) := R_\varepsilon(x)^{kr + rm} \).

If \( k \leq 1 \) we have \( R_\varepsilon(x) := R_{m,\varepsilon}(x) \) is the admissible radius for the \((m, \varepsilon)\)-admissible balls, with \( m = 1 \) if \( p = 0 \) and \( m = 2 \) if \( p \geq 1 \). If \( k \geq 2 \), then \( R_\varepsilon(x) \) is the admissible radius for the \((k, \varepsilon)\)-admissible balls for \( p \geq 1 \) and for the \((k - 1, \varepsilon)\)-admissible balls for \( p = 0 \).

In particular, making \( k = 0 \) and \( k = 1 \) and with the same conditions as above on the admissible balls:

**Corollary 6.3.** We have with \( r \in [1, \infty] \) and \( w(x) := R_\varepsilon(x)^{rm} \), with \( m = 1 \) if \( p = 0 \) and \( m = 2 \) if \( p \geq 1 \):
\[ \forall t \in (\delta, 1), \| u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)(\delta t)\| \omega \|_{L^r(M)} \]

and
\[ \forall t \geq 1, \| u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)\| \omega \|_{L^r(M)} . \]

For the gradient estimate, with \( w(x) := R_\varepsilon(x)^{kr + rm} \) this time:
\[ \forall t \in (\delta, 1), \| \nabla u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)(\delta t)\| \omega \|_{L^r(M)} . \]

Now for \( p \)-forms with \( p \geq 1 \):
\[ \forall t \geq 1, \| \nabla u(\cdot, t) \|_{L^r(M, w)} \leq c(n, r)\| \omega \|_{L^r(M)} . \]
and for functions:
\[
\forall t \geq 1, \quad \|\nabla u(\cdot, t)\|_{L^r(M, \omega)} \leq c(n, r)t^{-1/2}\|\omega\|_{L^r(M)}.
\]

7 Classical estimates.

We shall give some examples where we have classical estimates using that for any \(x \in M\), we have \(R_c(x) \geq \eta\), via [9] Corollary, p. 7] (see also Theorem 1.3 in the book by Hebey [8]):

**Corollary 7.1.** Let \(M\) be a Riemannian manifold. Let \(k \geq 1\); if we have the injectivity radius \(r_{\text{inj}}(x) \geq i > 0\) and \(\forall j \leq k - 1\), \(|\nabla^j R_{c(M,g)}(x)| \leq c\) for all \(x \in M\), then there exists a constant \(\eta > 0\), depending only on \(n, \epsilon, i, k\) and \(c\), such that: \(\forall x \in M, \quad R_c(x) \geq \eta\).

For \(k = 0\), if we have the injectivity radius \(r_{\text{inj}}(x) \geq i > 0\) and \(R_{c(M,g)}(x) \geq \lambda g_x\) for some \(\lambda \in \mathbb{R}\) and for all \(x \in M\), then there exists a constant \(\eta > 0\), depending only on \(n, \epsilon, i, \) and \(\lambda\), such that: \(\forall x \in M, \quad R_0(x) \geq \eta\).

**Proof.**
The Theorem of Hebey and Herzlich gives that, under these hypotheses, for any \(\alpha \in (0,1)\) there exists a constant \(\eta > 0\), depending only on \(n, \epsilon, i, k, \alpha\) and \(c\), such that:
\[
\forall x \in M, \quad r_H(1 + \epsilon, k, \alpha)(x) \geq \eta.
\]
So taking our definition with a harmonic coordinates patch, we have that:
\[
R_k(x) \geq r_H(1 + \epsilon, k, \alpha)(x).
\]
So, a fortiori, this is true when we take the sup for \(R_k(x)\) on any smooth coordinates patch, not necessarily harmonic coordinates one. \(\blacksquare\)

Then we get our "classical estimates":

**Theorem 7.2.** Let \(M\) be a Riemannian manifold. Let \(r \in [1, \infty]\) and \(\omega \in L^r(M) \cap L^2(M)\). For \(k = 0, 1\) suppose that \((M, g)\) has 1-order weak bounded geometry for \(p\)-forms with \(p \geq 1\) and 0-order weak bounded geometry for functions. For \(k \geq 2\) suppose that \(M\) has \(k\)-order weak bounded geometry for \(p\)-forms with \(p \geq 1\) and \(k - 1\)-order weak bounded geometry for functions.

Then the canonical solution \(u := e^{\Delta_{\omega}}\) of the heat equation is such that, for any \(k \geq 0\) and with \(\eta = \eta(n, \epsilon, i, k)\) given by the Corollary [7.1] of Hebey and Herzlich:

\[
\forall t \in (\delta, 1), \quad \|\nabla^k u(\cdot, t)\|_{L^r(M)} \leq c(n, r, \eta)(\frac{\delta}{t - \delta^{3/2}})\|\omega\|_{L^r(M)}.
\]

And for \(p\)-forms with \(p \geq 1\):

\[
\forall t \geq 1, \quad \|\nabla^k u(\cdot, t)\|_{L^r(M)} \leq c(n, r, \eta)\|\omega\|_{L^r(M)},
\]

and for functions:

\[
\forall t \geq 1, \quad \|\nabla^k u(\cdot, t)\|_{L^r(M)} \leq c(n, r, \eta)t^{-1/2}\|\omega\|_{L^r(M)},
\]

**Proof.**
We apply Theorem [6.2] together with Corollary [7.1] to have that there exists \(\eta > 0\) such that for any \(x \in M\), we get \(\eta \leq R_c(x) \leq 1\). Hence:

\[
\|\nabla^k u(\cdot, t)\|^r_{L^r(M, \omega)} := \int_M |\nabla^k u(x, t)|^r R_c(x)^{r+m} dx \geq \eta^{r(1+m)}\|\nabla^k u(\cdot, t)\|^r_{L^r(M)}.
\]

So Theorem [6.2] ends the proof with \(m\) as in Theorem [6.2] and with the constant \(c(n, r, \eta) := c(n, r)\eta^{-(1+m)}\).

Hence we can forget the weight. \(\blacksquare\)
8 Appendix 1

Proposition 8.1. Let \( r \in [1, \infty] \). Let \( x \in M \), a Riemannian manifold. With \( B := B(x, R) \) a \((m, \epsilon)\)-admissible ball in \( M \), and any \( \delta \in (0, 1) \), there is a \( \epsilon(\delta) > 0 \) such that for any \( \epsilon \leq \epsilon(\delta) \), if \( \omega \) is a \( p \)-form in \( L^r(B) \), then the \( \omega \)-form \( u_\omega := e^{-t\Delta} \omega \) verifies, in \( \mathbb{R}^n \), with \( \forall \gamma \in \mathbb{N}^n, l := |\gamma|/2 \nabla_t \in (\delta, 1), \| \partial^\gamma u_\omega \|_{L^r(B_\delta)} \leq c(n, r) \frac{\delta}{t - \delta t + l} R^{-m}_\omega \| \omega \|_{L^r(B_\delta)} \) And

\[
\forall t \geq 1, \| \partial^\gamma u_\omega \|_{L^r(B_\delta)} \leq c(n, r, \delta)t^{-l} R^{-m}_\omega \| \omega \|_{L^r(B_\delta)}
\]

with \( m = 1 \) if \( p = 0 \) and \( m = 2 \) if \( p \geq 1 \). And also \( B_\omega = \varphi(B) \), \( \omega_\varphi = \varphi^* \omega \) etc...

Proof.

First we work with the first convolution in (3.4).

Because we stay in \( \mathbb{R}^n \) and for easing the notation, we forget the subscript \( \varphi \), so we write \( u \) for \( u_\varphi \), \( \omega \) for \( \omega_\varphi \) etc.

Set \( Y := \Delta_\varphi - \Delta \), by Lemma (3.1) we have:

\[
Yf = \sum_{i,j} a_{ij} \partial^2_{ij} f + \sum_i b_i \partial_i f,
\]

and, for \( p = 0 \), in \( \varphi(B) \) with \( B \in \mathcal{A}_1(\epsilon) \):

(A) \( \sum_{i,j} |a_{ij}| \leq \epsilon, \sum_i |b_i| \leq C\epsilon R^{-1} \),

and for \( p \geq 1 \) with \( B \in \mathcal{A}_2(\epsilon) \):

(B) \( \sum_{i,j} |a_{ij}| \leq \epsilon, \sum_i |b_i| \leq C\epsilon R^{-2} \).

Because the (Hodge) laplacian in \( \mathbb{R}^n \) acts on \( p \)-forms componentwise, we fix \( t \) and we have:

\[
(Y e^{-t\Delta} \omega)(y, t) = Y \int \Phi(y - z, t) \omega(z) dz
\]

with \( \Phi \) the heat kernel in \( \mathbb{R}^n \).

Set \( Y_{kl} := \frac{\partial^2}{\partial y_k \partial y_l} \) and:

\[
\psi(y, t) := Y_{kl} \int (\Phi(y - z, t)) \omega(z) dz.
\]

So, again because \( \Phi \) is the heat kernel in \( \mathbb{R}^n \),

\[
\partial_x^\gamma (e^{-t\Delta} \Phi(y_k e^{-t\Delta} \omega)) = \partial_x^\gamma (e^{-t\Delta} \Phi(y_k e^{-t\Delta} \omega))(x, t) = \int \partial_x^\gamma \Phi(x - y, t) \psi(y, t) dy.
\]

Now recall that \( \psi(y, t) := \int \partial^2_{y_k y_l} (\Phi(y - z, t)) \omega(z) dz \) then, extending \( \omega \) by 0 outside \( \varphi(B) \), by Corollary (9.3) in the Appendix 2 and the inequalities (A) and (B) above:

\[
\| \psi(\cdot, t) \|_{L^r(B)} \leq \| \psi(\cdot, t) \|_{L^r(\mathbb{R}^n)} \leq \epsilon c(n, r) t^{-l} \| \omega \|_{L^r(\mathbb{R}^n)} = \epsilon c(n, r) t^{-l} \| \omega \|_{L^r(B)}.
\] (8.13)

With

\[
\theta(x, t) := \partial_x^\gamma (e^{-t\Delta} \psi)(x, t) = \int \partial_x^\gamma \Phi(x - y, t) \psi(y, t) dy,
\]

by Proposition (9.2) with \( u := \int f(y) \Phi(x - y, t) dy \) and setting \( l := |\gamma|/2 \),

\[
\| \partial^\gamma u(\cdot, t) \|_{L^r(\mathbb{R}^n)} \leq \epsilon c(n, r) t^{-l} \| f \|_{L^r(\mathbb{R}^n)},
\]

so
\[ \| \theta(\cdot, t) \|_{L^r(B)} \leq c(c(n, r)t^{-l})\| \psi(\cdot, t) \|_{L^r(B)}. \]

But by \(8.13\):
\[ \| \psi(\cdot, t) \|_{L^r(B)} \leq c(n, r)t^{-1} \| \omega \|_{L^r(\mathbb{R}^n)}. \]

So
\[ \forall t > 0, \| \theta(\cdot, t) \|_{L^r(B)} \leq c(n, r)t^{-1-l} \| \omega \|_{L^r(B)}. \]

For \( Y_k := \partial_k \) a first order derivative, we get the same way:
\[ \forall t > 0, \| \partial_x^j(e^{-t\Delta} \cdot (Y_k e^{-t\Delta} \omega)) \|_{L^r(B)} \leq c(n, r)t^{-(1+l)} \| \omega \|_{L^r(B)}. \]

Now using the complete form of \( Y \), we get, for \( t < 1 \), because \( t^{-1/2} < t^{-1} \),
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta} \omega))(\cdot, t) \|_{L^r(B)} \leq \left\| \sum_k \partial^j(e^{-t\Delta} \cdot (a_k \partial^2_k e^{-t\Delta} \omega))(\cdot, t) \right\|_{L^r(B)} + \]
\[ + \left\| \sum_k \partial^j(e^{-t\Delta} \cdot (b_k \partial_k e^{-t\Delta} \omega))(\cdot, t) \right\|_{L^r(B)} \leq c(1 + R^{-m})c(n, r)t^{-1-l} \| \omega \|_{L^r(B)}. \]

And for \( t \geq 1 \),
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta} \omega))(\cdot, t) \|_{L^r(B)} \leq c(1 + R^{-m})c(n, r)t^{-(1+l)} \| \omega \|_{L^r(B)}, \]

still with \( m = 1 \) in the case of functions, \( p = 0 \), and \( m = 2 \) in the case of \( p \)-forms, \( p \geq 1 \).

So we have, because \( R \leq 1 \):
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta} \omega))(\cdot, t) \|_{L^r(B)} \leq c(n, r)t^{-\beta} R^{-m} \| \omega \|_{L^r(B)} \]
with \( \beta = 1 + |\gamma|/2 \) for \( 0 < t < 1 \) and \( \beta = (1 + |\gamma|)/2 \) for \( t \geq 1 \).

We have to treat \( e^{-t\Delta} \cdot (Ye^{-t\Delta})^j , j \geq 2 \).
Fix any \( \delta \in (0, 1) \) and choose \( \epsilon \leq \epsilon(\delta, l) \) such that \( cc(n, r) \leq \delta^{1+l} \).

Now on we shall always suppose that the \( \epsilon \) appearing in our \((m, \epsilon)\)-admissible ball is less than \( \epsilon(\delta, l) \).

Then we have \( t \in (\delta, 1) \Rightarrow c(n, r)t^{-1-l} < 1 \) with \( 2l = |\gamma| \) and we have for \( j = 2 \):
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^2 \omega)(\cdot, t) \|_{L^r(B)} = \| \partial^j(e^{-t\Delta} \cdot ((Ye^{-t\Delta})^j)(\cdot, t)) \|_{L^r(B)} \]  

Set \( \mu := (Ye^{-t\Delta})\omega(\cdot, t) \) then
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^2 \omega)(\cdot, t) \|_{L^r(B)} = \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^j \mu)(\cdot, t) \|_{L^r(B)} \leq c(n, r)t^{-\beta} R^{-m} \| \mu \|_{L^r(B)}. \]

But, by equation \(8.13\):
\[ \| \mu \|_{L^r(B)} = \| (Ye^{-t\Delta})\omega(\cdot, t) \|_{L^r(\mathbb{R}^n)} \leq c(n, r)t^{-1} \| \omega \|_{L^r(B)} \]
so
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^2 \omega)(\cdot, t) \|_{L^r(B)} \leq \epsilon^2(c(n, r)^2t^{-1-\beta} R^{-m} \| \omega \|_{L^r(B)}. \]

Again
\[ \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^3 \omega)(\cdot, t) = \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^2 ) \mu(\cdot, t) \]
and
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^3 \omega)(\cdot, t) \|_{L^r(B)} \leq \epsilon^3(c(n, r)^3t^{-1-\beta} R^{-m} \| \mu \|_{L^r(B)} \leq c(n, r) \leq \epsilon^3(c(n, r)^3t^{-1-\beta} R^{-m} \| \omega \|_{L^r(B)}. \]

So by induction
\[ \| \partial^j(e^{-t\Delta} \cdot (Ye^{-t\Delta})^j \omega)(\cdot, t) \|_{L^r(B)} \leq \epsilon^j(c(n, r)^j t^{-j+1-\beta} R^{-m} \| \omega \|_{L^r(B)}. \]

We have, for \( c(n, r)t^{-1} < 1 \),
\[ \sum_{j=1}^{\infty} e^j c(n, r) t^{-j} = \frac{2\epsilon c(n, r)}{t - \epsilon c(n, r)}. \quad (C) \]

Hence the series
\[ \left\| \partial_y (e^{-t\Delta} \ast \sum_{j=1}^{\infty} (-1)^j (Ye^{-t\Delta})^j) \omega(\cdot, t) \right\|_{L^r(B)} \leq \left( \sum_{j=1}^{\infty} e^j c(n, r) t^{-j-1} \right) R^{-m} \| \omega \|_{L^r(B)} \]
converges for \( \delta < t < 1 \) because \( \epsilon c(n, r) \leq \delta^{-1+\epsilon} < 1 \) and we get:
\[ \sum_{j=1}^{\infty} e^j c(n, r) t^{-j} = \frac{2\epsilon c(n, r)}{t - \epsilon c(n, r)} \leq \frac{2\delta^{1+l}}{t - \delta^{1+l}}. \]

Hence, because \( \beta = 1 + l \) if \( t < 1 \), we get, with \( t^{1-\beta} = t^{-l} \):
\[ \forall t \in (\delta, 1), \left\| \partial_y (e^{-t\Delta} \ast \sum_{j=1}^{\infty} (-1)^j (Ye^{-t\Delta})^j) \omega(\cdot, t) \right\|_{L^r(B)} \leq 2t^{-l} R^{-m} \frac{\delta^{1+l}}{t - \delta^{1+l}} \| \omega \|_{L^r(B)}. \]

If \( t \geq 1 \) we have using \( (C) \)
\[ \left( \sum_{j=1}^{\infty} e^j c(n, r) t^{-j-1} \right) R^{-m} \| \omega \|_{L^r(B)} \leq 2t^{-l} R^{-m} \frac{\epsilon c(n, r)}{t - \epsilon c(n, r)} \leq 2t^{-l} R^{-m} \frac{\delta^{1+l}}{t - \delta^{1+l}} \| \omega \|_{L^r(B)}. \]

Hence again
\[ \forall t \geq 1, \left\| \partial_y (e^{-t\Delta} \ast \sum_{j=1}^{\infty} (-1)^j (Ye^{-t\Delta})^j) \omega(\cdot, t) \right\|_{L^r(B)} \leq 2t^{-l} R^{-m} \frac{\delta^{1+l}}{t - \delta^{1+l}} \| \omega \|_{L^r(B)}. \]

The formula \( (3.1) \) gives:
\[ e^{-t\Delta} \varphi = e^{-t\Delta} + e^{-t\Delta} \ast \sum_{j=1}^{\infty} (-1)^j (Ye^{-t\Delta})^j, \]

it remains to add the first term in the right hand side:
\[ \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(B)} \leq \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(B)} + \left\| \partial_y (e^{-t\Delta} \ast \sum_{j=1}^{\infty} (-1)^j (Ye^{-t\Delta})^j) \omega \right\|_{L^r(B)}. \]

Using Corollary \([9.3]\)
\[ \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(R^n)} \leq c(n, r) t^{-l} \| \omega \|_{L^r(R^n)}, \]
so adding we get
\[ \forall t \in (\delta, 1), \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(B)} \leq c(n, r) t^{-l} R^{-m} \| \omega \|_{L^r(B)} + \frac{2\delta^{1+l}}{t - \delta^{1+l}} \| \omega \|_{L^r(B)}, \]
hence
\[ \forall t \in (\delta, 1), \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(B)} \leq t^{-l} R^{-m} \| \omega \|_{L^r(B)} (c(n, r) + \frac{2\delta^{1+l}}{t - \delta^{1+l}}) \]
so with another constant \( c(n, r) \)
\[ \forall t \in (\delta, 1), \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(B)} \leq \left( c(n, r) \delta^{-l} \frac{\delta^{1+l}}{t - \delta^{1+l}} R^{-m} \| \omega \|_{L^r(B)} = \right. \]

\[ =\left. c(n, r) \frac{\delta}{t - \delta^{1+l}} R^{-m} \| \omega \|_{L^r(B)} \right). \]

And, the same way:
\[ \forall t \geq 1, \left\| \partial_y e^{-t\Delta_x} \omega \right\|_{L^r(B)} \leq t^{-l} R^{-m} \| \omega \|_{L^r(B)} (c(n, r) + \frac{\delta^{1+l}}{t - \delta^{1+l}}) \]
so again, still with \( l := |\gamma|/2 \) and with another constant \( c(n, r) \)
∀t ≥ 1, \|\partial_t e^{-t\Delta} \omega\|_{L^r(B)} \leq c(n, r) \frac{\delta^{1+t}}{1 - \delta^{1+t} t^{-1} R^{-m}} \|\omega\|_{L^r(B)}.

Recall that we dropped the index \varphi on u, R, \omega etc...
The proof of the Proposition is complete. ■

9 Appendix 2. The heat kernel in \(\mathbb{R}^n\).

All results here are very well known and they are here essentially to fix the notation. See for instance [5]. We have the heat operator \(Du := \partial_t u - \Delta u\) and the heat kernel in \(\mathbb{R}^n\):

\[
\Phi(x, t) := \begin{cases} 
\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, \; t > 0 \\
0 & x \in \mathbb{R}^n, \; t \leq 0 
\end{cases}
\]

and an easy computation gives:

\[
\partial_j \Phi(x, t) = -\frac{x_j}{2t(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}
\]

\[
\partial^2_j \Phi(x, t) = (-\frac{1}{2t} + \frac{x_j^2}{4t^2}) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}
\]

\[
\partial^2_{jk} \Phi(x, t) = \frac{x_j x_k}{4t^2(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}
\]

\[
\|\Phi(\cdot, t)\|_{L^r(\mathbb{R}^n)} = c_0(n, r) \frac{1}{(t)^{\frac{n}{2}(1-\frac{1}{r})}}.
\]

(9.14)

and

\[
\|\nabla \Phi(\cdot, t)\|_{L^r(\mathbb{R}^n)} \leq c_1(n, r) \frac{1}{(t)^{\frac{n}{2} + \frac{1}{2}(1-\frac{1}{r})}}.
\]

(9.15)

and more generally:

\[
\forall k \in \mathbb{N}, \quad \|\nabla^k \Phi(\cdot, t)\|_{L^r(\mathbb{R}^n)} \leq c_k(n, r) \frac{1}{(t)^{\frac{n}{2} + k\frac{k}{2}(1-\frac{1}{r})}}.
\]

(9.16)

These inequalities can be written for \(k \in \mathbb{N}\):

\[
\|\Phi(\cdot, t)\|_{W^k,r(\mathbb{R}^n)} = c_k(n, r) \frac{1}{(t)^{\frac{n}{2} + k\frac{k}{2}(1-\frac{1}{r})}}.
\]

We get the following lemma:

Lemma 9.1. We have for \(1 \leq r \leq s \leq \infty\):

\[
\left\| \int_{\mathbb{R}^n} f(y) \Phi(x - y, t) \, dy \right\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{t^{\frac{n}{2} + \frac{1}{2}(1 - \frac{1}{r})}} \|f\|_{L^r(\mathbb{R}^n)},
\]

where \(c(n, r, s)\) is a constant depending only on \(n, r\) and \(s\).

And

\[
\left\| \int_{\mathbb{R}^n} f(y) \nabla \Phi(x - y, t) \, dy \right\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{t^{\frac{n}{2} + \frac{1}{2}(1 - \frac{1}{r})}} \|f\|_{L^r(\mathbb{R}^n)}.
\]

And more generally:
\[ \left\| \int_{\mathbb{R}^n} f(y) \frac{\partial^k}{\partial x_1^{k_1} \cdots x_n^{k_n}} \Phi(x - y, t)dy \right\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{|k|}{2} + \frac{u - 1}{2}} f\|_{L^r(\mathbb{R}^n)}}. \]

Proof.

The convolution gives:

\[ \left\| \int_{\mathbb{R}^n} f(y) \Phi(x - y, t)dy \right\|_{L^s(\mathbb{R}^n)} \leq \|f\|_{L^r(M)} \|\Phi(\cdot, t)\|_{L^u(\mathbb{R}^n)} \]

with \( \frac{1}{s} = \frac{1}{r} + \frac{1}{u} - 1 \). Using (9.14), we get

\[ \left\| \int_{\mathbb{R}^n} f(y) \Phi(x - y, t)dy \right\|_{L^s(\mathbb{R}^n)} \leq c_0(n, r, u) \frac{1}{(t)^{\frac{u(1 - \frac{1}{s})}{2}}} \|f\|_{L^r(\mathbb{R}^n)} \]

hence,

\[ \left\| \int_{\mathbb{R}^n} f(y) \Phi(x - y, t)dy \right\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{r}{2} + \frac{u - 1}{2}}} \|f\|_{L^r(\mathbb{R}^n)}. \]

For the second part we proceed the same way with (9.15) in place of (9.14), to get:

\[ \left\| \int_{\mathbb{R}^n} f(y) \nabla \Phi(x - y, t)dy \right\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{1}{2} + \frac{r}{2} - \frac{1}{2}}} \|f\|_{L^r(\mathbb{R}^n)}. \]

The third part is the same, with (9.16) instead of (9.15), which proves the lemma. ■

**Proposition 9.2.** Let \( u(x, t) := \int_{\mathbb{R}^n} f(y) \Phi(x - y, t)dy \) then, for \( 1 \leq r \leq s \leq \infty \), we have, with \( k = (k_1, \ldots, k_n) \) and \( |k| = k_1 + \ldots + k_n \):

(i) \( \|u(\cdot, t)\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{1}{2} + \frac{u - 1}{2}}} \|f\|_{L^r(\mathbb{R}^n)}. \)

And

(ii) \( \|\nabla u(\cdot, t)\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{r}{2} + \frac{u - 1}{2}}} \|f\|_{L^r(\mathbb{R}^n)}. \)

And more generally:

(iii) \( \left\| \frac{\partial^{k_1}}{\partial x_1^{k_1} \cdots x_n^{k_n}} u(\cdot, t) \right\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{|k|}{2} + \frac{u - 1}{2}}} \|f\|_{L^r(\mathbb{R}^n)}. \)

This can also be written:

\( \|\partial^k(e^A)\|_{L^r(\mathbb{R}^n)-L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{|k|}{2} + \frac{u - 1}{2}}}. \)

Proof.

From

\[ u(x, t) := \int_{\mathbb{R}^n} f(y) \Phi(x - y, t)dy \]

we get

\[ \|u(\cdot, t)\|_{L^s(\mathbb{R}^n)} \leq \left\| \int_{\mathbb{R}^n} f(y) \Phi(\cdot - y, t)dy \right\|_{L^s(\mathbb{R}^n)}, \]

hence, by lemma 9.1

\[ \|u(\cdot, t)\|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t)^{\frac{1}{2} + \frac{u - 1}{2}}} \|f\|_{L^r(\mathbb{R}^n)}, \]

and the (i).

The same we get,
\[ \nabla u(x,t) := \int_{\mathbb{R}^n} f(y) \nabla_x \Phi(x-y,t) dy \]

hence
\[ \| \nabla u(\cdot,t) \|_{L^s(\mathbb{R}^n)} \leq \left\| \int_{\mathbb{R}^n} f(y) \nabla \Phi(x-y,t) dy \right\|_{L^s(\mathbb{R}^n)} \]

and, by Lemma 9.1, we get the (ii).
\[ \| \nabla u(\cdot,t) \|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t^{\frac{n}{2}} + \frac{r}{2} - \frac{s}{2})} \| f \|_{L^r(\mathbb{R}^n)}. \]

The same we get, by Lemma 9.1,
\[ \| \partial^k u(\cdot,t) \|_{L^s(\mathbb{R}^n)} \leq c(n, r, s) \frac{1}{(t^{\frac{|k|}{2}} + \frac{r}{2} - \frac{s}{2})} \| f \|_{L^r(\mathbb{R}^n)}. \]

This ends the proof of the proposition. ■

**Corollary 9.3.** Let \( r \in [1, \infty] \). We have, if \( u \) is the solution of the heat equation:
\[ \partial_t u - \Delta u = 0, \quad u(x,0) = f(x), \]
given by \( u(x,t) := e^{\Delta t} f = \int_{\mathbb{R}^n} f(y) \Phi(x-y,t) dy \) with \( \partial^k f := \frac{\partial^{\left| k \right|}}{\partial^{k_1} x_1 \ldots \partial^{k_n} x_n} f : \)

\[ \forall k = (k_1, \ldots, k_n) \in \mathbb{N}^n, \quad \| \partial^k u(\cdot,t) \|_{L^r(\mathbb{R}^n)} \leq c(n, r) \frac{1}{t^{\frac{|k|}{2}}} \| f \|_{L^r(\mathbb{R}^n)}, \]

or, equivalently:
\[ \forall k = (k_1, \ldots, k_n) \in \mathbb{N}^n, \quad \| \partial^k e^{t\Delta} \|_{L^r(\mathbb{R}^n) - L^r(\mathbb{R}^n)} \leq c(n, r) \frac{1}{t^{\frac{|k|}{2}}}, \]

Proof.

We apply Proposition 9.2 with \( r = s \). ■

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