On the Generalized Hamming Weights of \((r, \delta)\)-Locally Repairable Codes

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This work was supported in part by the National Natural Science Foundation of China under Grant 61801049, and in part by the Beijing Natural Science Foundation under Grant 4184093.

**ABSTRACT** Locally repairable codes (LRCs) have attracted a lot of interests recently due to their important applications in distributed storage systems. An \((n, k, r, \delta)\)-LRC \((\delta \geq 2)\) is an \([n, k, d]\) linear code such that each of the \(n\) code symbols satisfies \((r, \delta)\)-locality and is said to be optimal if it has minimum distance \(d = n - k - \left(\lceil \frac{k}{r} \rceil - 1\right) \delta - 1 + 1\). The generalized Hamming weights (GHWs) are fundamental parameters of linear codes. Prakash et al. firstly applied GHWs to study linear codes with locality properties. In this article, we study the GHWs of \((n, k, r, \delta)\)-LRCs \((\delta \geq 2)\). Firstly, for a general \((n, k, r, \delta)\)-LRC, an upper bound on the \(i\)-th \((1 \leq i \leq k)\) GHW is presented. Then, for an optimal \((n, k, r, \delta)\)-LRC and its dual code, a lower bound on the \([i(\delta - 1)]\)-th GHW, for \(1 \leq i \leq \lceil \frac{k}{r} \rceil - 1\), of the dual code is given. Specially, when \(r \mid k\), we determine the \([i(\delta - 1)]\)-th GHW, for \(1 \leq i \leq \lceil \frac{k}{r} \rceil - 1\), of the dual code of an optimal \((n, k, r, \delta)\)-LRC. For the case of \(\delta = 2\), we obtain a lower bound on the \(i\)-th GHW for all \(1 \leq i \leq k\) of an optimal \((n, k, r, 2)\)-LRC. Moreover, it is shown that the weight hierarchy of an optimal \((n, k, r, 2)\)-LRC with \(r \mid k\) can be completely determined.

**INDEX TERMS** Locally repairable codes, erasure codes, generalized Hamming weights, locality.

**I. INTRODUCTION** Modern large distributed storage systems usually store redundant data to ensure data reliability in case of storage node failures. The redundancy scheme of 3-replication is widely used in distributed storage systems, which stores three replicas of data in different storage nodes. Due to the large volume of data, the 3-replication scheme will introduce large storage overhead. Hence, redundancy schemes based on erasure codes have become more attractive because of their higher storage efficiency compared to replication. Maximum distance separable (MDS) codes, e.g., Reed-Solomon codes, are widely used traditional erasure codes due to their good ability of fault tolerance. For storage systems using Reed-Solomon codes, the data is firstly partitioned into \(k\) packets, then the \([n, k]\) Reed-Solomon code encodes \(k\) packets into \(n\) packets and stores them across \(n\) storage nodes. The MDS property ensure the data reliability in case of any \(n - k\) failures. However, when some storage node fails, the storage systems need to repair the failed storage node. The repair process requires reading packets from \(k\) other surviving nodes, which introduces large amount of repair bandwidth. Locally repairable codes (LRCs) [2] are a class of improved erasure codes which can repair failed storage nodes efficiently and have attracted a lot of interests recently. Several real distributed storage systems, e.g., Microsoft Azure Storage [3] and Hadoop Distributed File System [4], have deployed LRCs as their redundancy schemes. Another important class of storage codes which can also achieve efficient repairing of failed storage nodes are regenerating codes [5]. Some works have proposed codes to combine the repair benefits of locally repairable codes and regenerating codes. Codes with local regeneration and erasure correction were proposed in [6]. Repair duality with locally repairable and locally regenerating codes were studied in [7].

In this article, we focus on locally repairable codes. In a \(q\)-ary \([n, k, d]\) linear code, a code symbol is said to have \(r\)-locality if it can be repaired by accessing at most \(r\) other code symbols. A \(q\)-ary \([n, k, r]\)-LRC is a \(q\)-ary \([n, k, d]\) linear code with \(r\)-locality for all code symbols. When \(r \ll k\), the storage system using LRCs as redundancy scheme only needs to read data from a small number of other available nodes...
to repair a failed storage node, which indicates low repair bandwidth. Gopalan et al. proved that the minimum distance of a $q$-ary $(n, k, r)$-LRC satisfies the following well-known Singleton-type bound [2],

$$
d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2. \tag{1}
$$

When $r = k$, the above bound reduces to the classical Singleton bound $d \leq n - k + 1$. An $(n, k, r)$-LRC is said to be optimal if its minimum distance attains the bound (1). The elegant Reed-Solomon-like optimal LRCs [8] require the field size to be just $q \geq n$. Classifications of optimal binary and ternary $(n, k, r)$-LRCs meeting the bound (1) were given in [9], [10]. Optimal cyclic $(n, k, r)$-LRCs with code length up to $q - 1$ were given in [11]. Luo et al. proposed a refined bound of $(n, k, r)$-LRCs with $d = 3, 4$ and unlimited length [12]. Wang and Zhang proposed a refined bound of $(n, k, r)$-LRCs based on integer programming methods [13].

In order to ensure local recovery of a failed node in case of more than one node failures, Prakash et al. firstly proposed the concept of $(r, \delta)$-locality [14]. The $i$-th $(1 \leq i \leq n)$ code symbol $c_i$ of a linear code $C$ is said to have $(r, \delta)$-locality if there exists a subset $\Theta_i \subset [n]$ such that

- $i \in \Theta_i$;
- $|\Theta_i| \leq r + \delta - 1$;
- the minimum distance of the punctured subcode $C|_{\Theta_i}$ obtained by puncturing these code symbols in $\{c_j, j \in [n] \setminus \Theta_i\}$, is at least $\delta$.

From the view of parity-check matrices, suppose that the punctured local subcode $C|_{\Theta_i}$ has dimension $k_i \leq r$, then there exist $|\Theta_i| - k_i$ linearly independent codewords from the dual code $C^\perp$ which form a local parity-check matrix, or local-PCM for short, $H^I$ of the local subcode $C|_{\Theta_i}$ such that the following three conditions are satisfied.

- $i \in supp(H^I)$;
- $|supp(H^I)| \leq r + \delta - 1$;
- any $\delta - 1$ columns of these $|supp(H^I)|$ columns in $H^I$ are linearly independent.

The set $supp(H^I)$ denotes the union of the supports of all the codewords in $H^I$ and $|supp(H^I)|$ is the cardinality of the set $supp(H^I)$. The code symbol $c_i$ (or the coordinate $i$) is said to be covered by the local-PCM $H^I$. The local-PCM $H^I$ contains at least $\delta - 1$ linearly independent codewords. An $(n, k, r, \delta)$-LRC $(\delta \geq 2)$ is an $(n, k, d)$ linear code with $(r, \delta)$-locality for all code symbols, i.e., each code symbol is covered by a local-PCM satisfying the above three conditions. The minimum distance of an $(n, k, r, \delta)$-LRC satisfies the following Singleton-type bound [14],

$$
d \leq n - k - \left\lceil \frac{k}{r} \right\rceil - 1(\delta - 1) + 1. \tag{2}
$$

When $\delta = 2$, the above bound reduces to the bound (1). Constructions of optimal $(n, k, r, \delta)$-LRCs attaining the bound (2) have also attracted a lot of interests. Optimal $q$-ary cyclic $(n, k, r, \delta)$-LRCs with code length $q + 1$ were proposed in [15]. Classifications of optimal binary and ternary $(n, k, r, \delta)$-LRCs with $\delta > 2$ were given in [16] and [17], respectively. Constructions of optimal $(n, k, r, \delta)$-LRCs over a small alphabet were proposed in [18]. Other optimal constructions can be found in [14], [19]–[22]. Note that there is another type of generalization of $r$-locality called $(r, t)$-locality in which each code symbol has $t$ disjoint groups of other code symbols to repair it, each group of size at most $r$. Bounds and constructions of such LRCs with multiple disjoint repair groups can be found in [23]–[28].

The generalized Hamming weights (GHWs) [29], [30] are fundamental parameters of linear codes, which were first used by Wei to characterize the performance of linear codes in a wire-tap channel of type II [29]. Let $D$ be a subcode of an $[n, k, d]$ linear code $C$. Denote the dimension of $D$ as $dim(D)$.

The support of the subcode $D$ is defined to be

$$supp(D) = \{i : \exists (c_1, c_2, \ldots, c_n) \in D, c_i \neq 0\}.$$

For $1 \leq i \leq k$, the $i$-th GHW of $C$ is defined to be

$$d_i = \min\{|supp(D)| : D \subseteq C \text{ and } dim(D) = i\}.$$

When $i = 1$, the first GHW $d_1$ is exactly the minimum distance of $C$. The $i$-th GHW, for $1 \leq i \leq k$, of $C$ satisfies the following generalized Singleton bound,

$$d_i \leq n - k + i. \tag{3}$$

When $i = 1$, the bound (3) gives the classical Singleton bound. The weight hierarchy of $C$ is the set of GHWs $\{d_i \mid 1 \leq i \leq k\}$. Moreover, it holds that $1 \leq d_1 \leq d_2 \leq \cdots \leq d_k = n$. The set of gap numbers of $C$ is $\{g_i, 1 \leq i \leq n - k\} = [n] \setminus \{d_i \mid 1 \leq i \leq k\}$, which is the complement of its weight hierarchy. Gap numbers were firstly introduced by Prakash et al. to derive the bounds on the minimum distance of $(r, \delta)$-LRCs [14] and are useful tools to study GHWs of linear codes. The weight hierarchy and gap numbers of the dual code $C^\perp$ are $\{d_i^\perp \mid 1 \leq i \leq n - k\}$ and $\{g_i^\perp \mid 1 \leq i \leq k\}$. The $i$-th GHW of $C$ satisfies [14]

$$d_i = (n + 1) - g_{k-i+1}^\perp, \quad \text{for } 1 \leq i \leq k. \tag{4}$$

Many works have conducted research to determine or estimate the GHWs of different linear codes, such as Hamming codes [29], Reed-Muller codes [29], [31], BCH codes and their dual codes [32]–[34], etc. Generally speaking, it is difficult to determine the GHWs of a linear code, especially for the complete weight hierarchy, which is known for only a few cases of linear codes. Prakash et al. firstly introduced GHWs to study linear codes with locality properties [14] and used GHWs to study codes with a sequential repair property for two erasures [35]. These two papers are pioneer work which introduced GHWs to study LRCs. Ballico and Marcolla [36] studied the GHWs of LRCs on algebraic curves. Lalitha and Lokam [37] studied the GHWs of maximally recoverable codes [38]. Some techniques in [14] can also be used to determine the weight hierarchy of maximally recoverable codes.

It is well known that for an $[n, k, d]$ MDS code attaining the classical Singleton bound, its weight hierarchy can be
uniquely determined. Moreover, the $i$-th GHW of an MDS code attains the classical generalized Singleton bound $d \leq n - k + i$ for all $1 \leq i \leq k$. The Singleton-type bound (1) and (2) are generalizations of the classical Singleton bound by taking the locality property into account. Optimal $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$ attaining the Singleton-type bound (2) can be regarded as generalizations of MDS codes with locality constraints. It is an interesting problem to investigate the properties of GHWs of $q$-ary $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$.

Note that the GHWs of $q$-ary $(n, k, r, \delta)$-LRCs were studied in [1], which are a special class of $(n, k, r, \delta)$-LRCs with $\delta = 2$. In this article, we generalize the results of [1] to study the GHWs of this more general class of $q$-ary $(n, k, r, \delta)$-LRCs with $\delta \geq 2$ and their dual codes. The upper bounds and lower bounds on GHWs of $(n, k, r)$-LRCs in [1] can be obtained by the results in this article when $\delta = 2$. The results of this article are summarized as follows.

- Firstly, for $q$-ary general $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$, an upper bound on the $i$-th $(1 \leq i \leq k)$ GHW is presented, which can be seen as a generalization of the classical generalized Singleton bound by taking the locality constraints into account. Particularly, when $i = 1$, the proposed upper bound on GHWs gives the Singleton-type upper bound (2) on the minimum distance of $(r, \delta)$-LRCs. When $r = k$, the proposed upper bound reduces to the classical generalized Singleton bound (3).

- Then, we focus on the GHWs of $q$-ary optimal $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$ and their dual codes. A lower bound on the $[\lfloor (n - 1) \rfloor]$-th GHW, for $1 \leq i \leq \lceil \frac{k}{r} \rceil - 1$, of the dual code of a $q$-ary optimal $(n, k, r, \delta)$-LRC is given. When the dimension $k$ is divisible by $r$, we determine the $[\lfloor (\delta - 1) \rfloor]$-th GHWs, for $1 \leq i \leq \lceil \frac{k}{r} \rceil - 1$, of the dual code of an optimal $(n, k, r, \delta)$-LRC. For the case of $\delta = 2$, we present a lower bound on the $i$-th GHW, for all $1 \leq i \leq k$, of a $q$-ary optimal $(n, k, r, 2)$-LRC. Moreover, it is shown that the weight hierarchy of a $q$-ary optimal $(n, k, r, 2)$-LRC with $r \mid k$ can be completely determined.

The rest of this article is organized as follows. Section II presents an upper bound on the GHWs of $q$-ary general $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$. In Section III, we focus on the GHWs of optimal $(n, k, r, \delta)$-LRCs and their dual codes. Section IV concludes the paper.

II. THE GHWs OF GENERAL LRCs

In this section, we consider the GHWs of $q$-ary general $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$. An upper bound on the $i$-th GHW, for $1 \leq i \leq k$, of $q$-ary general $(n, k, r, \delta)$-LRCs is presented, which can include the Singleton-type bound (2) and the generalized Singleton bound (3) as special cases.

**Lemma 1:** Let $C$ be a $q$-ary $(n, k, r, \delta)$-LRC $(\delta \geq 2)$ and $C^\perp$ be its dual code. Then, the $i$-th GHW, for $1 \leq i \leq \lfloor (\frac{k}{r}) - 1 \rfloor (\delta - 1)$, of the dual code $C^\perp$ satisfies
\[
d^1_i \leq \left\lceil \frac{i}{\delta - 1} \right\rceil r + i, \quad \text{for } 1 \leq i \leq \lceil \frac{k}{r} \rceil - 1 (\delta - 1).
\]

**Proof:** Since all the $n$ code symbols of $C$ satisfy $(r, \delta)$-locality, each of the $n$ coordinates is contained in a punctured local subcode with length at most $r + \delta - 1$ and minimum distance at least $\delta$. From the view of parity-check matrices, each of the $n$ coordinates is covered by a local-PCM which has at least $\delta - 1$ linearly independent codewords from the dual code $C^\perp$ and the support of such local-PCM has size at most $r + \delta - 1$. In the following, for each $1 \leq i \leq (\lfloor k/r \rfloor - 1)(\delta - 1)$, we select $i$ linearly independent codewords from some $\lceil \frac{i}{\delta - 1} \rceil + 1$ local-PCMs such that these $i$ linearly independent codewords form a group of basis of an $i$-dimensional subcode $H^\perp_i$ of the dual code $C^\perp$. Write
\[
i = \left\lceil \frac{i - 1}{\delta - 1} \right\rceil (\delta - 1) + \phi,
\]
where $1 \leq \phi \leq \delta - 1$. The $i$ selected linearly independent codewords will contain $\lceil \frac{i - 1}{\delta - 1} \rceil (\delta - 1)$ codewords from some $\lfloor \frac{i - 1}{\delta - 1} \rfloor$ local-PCMs and additional $\phi$ codewords from one local-PCM. The details of the selection procedures are as follows.

- **Step 1:** Firstly, for the first coordinate, it is covered by a local-PCM, denoted by $H^1$, which contains at least $\delta - 1$ linearly independent codewords from the dual code $C^\perp$. Select $\delta - 1$ linearly independent codewords from this local-PCM $H^1$, by the property of $(r, \delta)$-locality, these $\delta - 1$ selected codewords from $H^1$ cover at most $r + \delta - 1$ coordinates.

- **Step 2:** Then, choose a coordinate outside the union of supports of the previous selected codewords such that this coordinate satisfies the condition that ‘the coordinate is covered by a local-PCM $H^*$ in which at least $\delta - 1$ codewords are linearly independent with all the previous selected codewords’. Select $\delta - 1$ linearly independent codewords from this local-PCM $H^*$ so that all the selected codewords are linearly independent.

- **Step 3:** Repeat the above procedure in Step 2 iteratively to select $\lfloor \frac{i - 1}{\delta - 1} \rfloor (\delta - 1)$ codewords from some $\lfloor \frac{i - 1}{\delta - 1} \rfloor$ local-PCMs in total, such that all these $\lfloor \frac{i - 1}{\delta - 1} \rfloor (\delta - 1)$ selected codewords are linearly independent and the support of each group of $\delta - 1$ codewords from one local-PCM cover at most $r + \delta - 1$ coordinates.

- **Step 4:** Choose a coordinate outside the union of supports of the previous selected codewords which satisfies the condition in Step 2. Denote $H^*$ as the $\delta - 1$ codewords in the local-PCM of this coordinate which are linearly independent with all the previous selected codewords. We transform the right $\delta - 1$ columns of $H^*$ into an identity matrix by linear combinations of these $\delta - 1$ rows. Then select the first $\phi$ rows.

Note that by [16, Lemma 1], when the number of the selected local-PCMs is less than $\lfloor k/r \rfloor - 1$, a coordinate satisfying the condition in Step 2 always exist. For the complexity
of the selection procedure, since $1 \leq i \leq \left(\left\lceil \frac{r}{\delta} \right\rceil -1\right)(\delta -1)$, the above selection procedure only needs to select codewords from $\left\lceil \frac{r}{\delta} \right\rceil \leq \left\lfloor \frac{r}{\delta} \right\rfloor -1$ local-PCMs. Finally, a total of $\left\lfloor \frac{r}{\delta} \right\rfloor (\delta -1)+\phi=i$ linearly independent codewords are selected from the dual code $C^\perp$. The following figure illustrates the result of the above procedure briefly.

By the above selection procedure, all these $i$ selected codewords are linearly independent. Hence, they form a group of basis $H_i$ of an $i$-dimensional subspace $D_i^\perp$ of $C^\perp$. Moreover, the first $\left\lceil \frac{r}{\delta} \right\rceil (\delta -1)$ codewords of $H_i$ from some $\left\lceil \frac{r}{\delta} \right\rceil$ local-PCMs cover at most $\left\lceil \frac{r}{\delta} \right\rceil(r+\delta-1)$ coordinates, and the last $\phi$ codewords cover at most $r+\phi$ coordinates. Then, the support of this $i$-dimensional subspace $D_i^\perp$ satisfies

$$|\text{supp}(D_i^\perp)| \leq \left\lceil \frac{r}{\delta} \right\rceil (r+\delta-1) + (r+\phi) = \left\lceil \frac{r}{\delta} \right\rceil r + r + i = \left\lceil \frac{r}{\delta} \right\rceil r + i.$$

Therefore, for $1 \leq i \leq \left(\left\lceil \frac{r}{\delta} \right\rceil -1\right)(\delta -1)$, the $i$-th GHW of the dual code $C^\perp$ satisfies

$$d_i^x \leq |\text{supp}(D_i^\perp)| \leq \left\lceil \frac{r}{\delta} \right\rceil r + i.$$

this completes the proof. \hfill \Box

**Lemma 2:** Let $C$ be a $q$-ary $(n, k, r, \delta)$-LRC $(\delta \geq 2)$ and $C^\perp$ be its dual code. When $1 \leq i \leq \left(\left\lceil \frac{r}{\delta} \right\rceil -1\right)(\delta -1)$, the GHWs of $C^\perp$ satisfy

$$d_i^x \leq d_i^x + r + t, \text{ for } 1 \leq t \leq \delta -1. \quad (7)$$

**Proof:** Suppose that $D_i^\perp$ is an $i$-dimensional $(1 \leq i \leq \left(\left\lceil \frac{r}{\delta} \right\rceil -1\right)(\delta -1))$ subspace of the dual code $C^\perp$ whose support satisfies $|\text{supp}(D_i^\perp)| = d_i^x$. Let $H_i$ be a group of basis of the subspace $D_i^\perp$, which consists of $i$ linearly independent codewords from $C^\perp$ and

$$|\text{supp}(H_i)| = |\text{supp}(D_i^\perp)| = d_i^x. \quad (8)$$

Choose a coordinate $\chi$ from $[n] \setminus \text{supp}(H_i^\perp)$ such that this coordinate $\chi$ is covered by a local-PCM $H^{\perp\chi}$ satisfying that at least $\delta -1$ codewords of $H^{\perp\chi}$ is linearly independent with all the codewords in $H_i$. Note that combining the conclusions in Lemma 1 and [16, Lemma 1], such a coordinate always exists when $1 \leq i \leq \left(\left\lceil \frac{r}{\delta} \right\rceil -1\right)(\delta -1)$. Let $H_{i-1}^{\perp\chi}$ denote some $\delta -1$ rows of the local-PCM $H^{\perp\chi}$ which are linearly independent with all the $i$ rows in $H_i$. Then, by linear combinations of the rows in $H_{i-1}^{\perp\chi}$, it can be transformed into

$$H_{i-1}^{\perp\chi} = [H_{i-1}^{\perp\chi}, I_{(\delta-1)}],$$

where the right $\delta -1$ columns of $H_{i-1}^{\perp\chi}$ is an identity matrix. By the property of $(r, \delta)$-locality, $|\text{supp}(H_{i-1}^{\perp\chi})| \leq r + \delta -1$, thus the left part $H_{i-1}^{\perp\chi}$ of $H_{i-1}^{\perp\chi}$ has support size $|\text{supp}(H_{i-1}^{\perp\chi})| \leq r$. Hence, the first $t$ rows, for $1 \leq t \leq \delta -1$, of $H_{i-1}^{\perp\chi}$ cover at most $r+t$ coordinates. Denote the first $t$ rows of $H_{i-1}^{\perp\chi}$ as $H_i^{\perp\chi}$. Then $|\text{supp}(H_i^{\perp\chi})| \leq r+t$. Let

$$H_{i+t} = \begin{bmatrix} H_i \\ H_i^{\perp\chi} \end{bmatrix}.$$

Since all the $t$ rows of $H_i^{\perp\chi}$ are linearly independent with all the $i$ rows in $H_i$, it follows that $H_{i+t}$ has full rank. Therefore, the $i+t$ codewords in $H_{i+t}$ form a group of basis of a subspace $D_{i+t}^\perp$ of the dual code $C^\perp$ which has dimension $i+t$. Since $|\text{supp}(H_i^{\perp\chi})| \leq r+t$, we have

$$|\text{supp}(D_{i+t}^\perp)| = |\text{supp}(H_{i+t})| \leq d_i^x + (r+t). \quad (9)$$

Then, we obtain that the $(i+t)$-th GHW, for $1 \leq t \leq \delta -1$, of $C^\perp$ satisfies

$$d_{i+t}^x \leq |\text{supp}(D_{i+t}^\perp)| \leq d_i^x + r + t.$$

Hence, the conclusion holds. \hfill \Box

**Lemma 3:** Let $C$ be a $q$-ary $(n, k, r, \delta)$-LRC $(\delta \geq 2)$ and $C^\perp$ be its dual code. Then the $i$-th $(1 \leq i \leq \delta -1)$ gap number of the dual code $C^\perp$ satisfies

$$g_i^x \geq \left(\left\lceil \frac{i}{\delta} \right\rceil -1\right)(\delta -1) + i. \quad (10)$$

**Proof:** We prove the result of this lemma by induction. Firstly, when $i = 1$, the first gap number of $C^\perp$ satisfies $g_1^x = 1$. Otherwise, if $g_1^x \geq 2$, then $d_1^x = 1$. In this case, there exist at least one codeword in the dual code $C^\perp$ which has weight one. This indicates that there exist at least one specific coordinate of $C$ at which the corresponding code symbol of all codewords of $C$ always equals zero.

Next, assume that for $1 \leq i \leq k-1$, the $i$-th gap number of $C^\perp$ satisfies $g_i^x \geq \left(\left\lceil \frac{i}{\delta} \right\rceil -1\right)(\delta -1) + i$. Then consider the $(i+1)$-th gap number $g_{i+1}^x$ of $C^\perp$.

1. If $r \nmid i$, then $\left[\frac{i+1}{r}\right] = \left[\frac{i}{r}\right]$. By $g_i^x \geq g_i^x + 1$, we have

$$g_{i+1}^x \geq \left(\left[\frac{i}{r}\right] -1\right)(\delta -1) + i +1 = \left(\left[\frac{i+1}{r}\right] -1\right)(\delta -1) + (i +1).$$

2. If $r \mid i$, then $\left[\frac{i+1}{r}\right] = \left[\frac{i}{r}\right] + 1$. Let $i = sr$ for some integer $s \leq \left[\frac{i}{r}\right] = \left[\frac{i}{\delta} \right] -1$. Then, $i = sr +1$.

By Lemma 1, the $(s(\delta -1))$-th GHW of the dual code $C^\perp$ satisfies

$$d_{s(\delta -1)}^x \leq \left(\frac{s(\delta -1)}{\delta -1}\right) \cdot r + s(\delta -1) = s(r + \delta -1).$$
Hence, the set \{1, 2, \cdots, s(r + \delta - 1)\} contains at least the GHWs of $C^\perp$ in the following set
\[\{d^\perp_1, d^\perp_2, \cdots, d^\perp_{s(r + \delta - 1)}\}.\]

On the other hand, this indicates that the set \{1, 2, \cdots, s(r + \delta - 1)\} contains at most these gap numbers of $C^\perp$ in the following set
\[\{g^\perp_1, g^\perp_2, \cdots, g^\perp_{s(r + \delta - 1)}\}.\]

Therefore, the \((sr + 1)\)-th gap number of the dual code $C^\perp$ satisfies
\[g^\perp_{sr+1} \geq s(r + \delta - 1) + 1,\]
which is
\[\left(\frac{k-i+1}{r}\right) - 1 \delta - 1) + (i + 1).\]

Combining all the above discussions, the lemma holds. □

**Theorem 1:** Let $C$ be a $(n, k, r, \delta)$-LRC ($\delta \geq 2$), then the $i$-th GHW, for $1 \leq i \leq k$, of $C$ satisfies
\[d_i \leq n-k - \left(\left[\frac{k-i+1}{r}\right] - 1\right)(\delta - 1) + i.\]  

**Proof:** By (4), we know that the $i$-th GHW, for $1 \leq i \leq k$, of $C$ satisfies
\[d_i = n+1 - g^\perp_{k-i+1}.\]  

By Lemma 3, it follows that
\[g^\perp_{k-i+1} \geq \left(\left[\frac{k-i+1}{r}\right] - 1\right)(\delta - 1) + (k-i+1).\]

Combining (12) and (13), we have
\[d_i \leq n+1 - \left(\left[\frac{k-i+1}{r}\right] - 1\right)(\delta - 1) + (k-i+1)\]
\[= n-k - \left(\left[\frac{k-i+1}{r}\right] - 1\right)(\delta - 1) + i.\]

This completes the proof. □

**Remark 1:** The upper bound (11) on the GHWs of $(n, k, r, \delta)$-LRCs can be regarded as a generalization of classical generalized Singleton bound (3) by taking the $(r, \delta)$-locality properties of code symbols into account.

- When $i = 1$, the upper bound (11) gives the Singleton-type bound (2) on the minimum distance $(r, \delta)$-LRCs.
- When $r = k$, i.e., the locality properties of code symbols are not considered, the upper bound (11) reduces to the classical generalized Singleton bound (3).

By Theorem 1, when $\delta = 2$, the bound (11) gives the following upper bound of $i$-th GHW, for $1 \leq i \leq k$, of $q$-ary $(n, k, r)$-LRCs.

**Corollary 1:** Let $C$ be a $(n, k, r)$-LRC, then the $i$-th GHW, for $1 \leq i \leq k$, of $C$ satisfies
\[d_i \leq n-k - \left[\frac{k-i+1}{r}\right] + i + 1.\]

**Remark 2:** Note that there is another different kind of locality, called $(r, e)$-cooperative locality [39], which can also enable local recovery of failed storage nodes in case of at most $e$ node failures. When $e = 1$, the concept of $(r, e)$-cooperative locality reduces to $r$-locality. Abdel-Ghaffar and Weber [40] presented an upper bound on the GHWs of $q$-ary $(n, k, d)$ linear codes with $(r, e)$-cooperative locality [40, Theorem 3]. When $\delta = 2$ and $e = 1$, both our bound (11) in Theorem 1 and the bound in [40, Theorem 3] reduce to the bound on the GHWs of LRCs with $r$-locality in Corollary 1.

**III. THE GHWs OF OPTIMAL LRCs**

In this section, we consider the GHWs of $q$-ary optimal $(n, k, r, \delta)$-LRCs $(\delta \geq 2)$ and their dual codes. Firstly, a lower bound on the $[i(\delta - 1)]$-th $1 \leq i \leq \left[\frac{k}{r}\right] - 1$ GHW of the dual code of a $q$-ary optimal $(n, k, r, \delta)$-LRC is given. Specially, when $r | k$, we exactly determine the $[i(\delta - 1)]$-th GHW, for $1 \leq i \leq \left[\frac{k}{r}\right] - 1$, of the dual code. Then, for the case of $\delta = 2$, a lower bound on the $i$-th $(1 \leq i \leq k)$ GHW of a $q$-ary optimal $(n, k, r, 2)$-LRC is presented. Furthermore, it is shown that the weight hierarchy of a $q$-ary optimal $(n, k, r, 2)$-LRC with $r | k$ can be completely determined.

**Theorem 2:** Let $C$ be a $q$-ary optimal $(n, k, r, \delta)$-LRC $(\delta \geq 2)$ attaining the Singleton-type bound (2) and $C^\perp$ be its dual code. When $1 \leq i \leq \left[\frac{k}{r}\right] - 1$, the $[i(\delta - 1)]$-th GHW of the dual code $C^\perp$ satisfies
\[d^\perp_i \geq i(r + \delta - 1) - \left[\frac{k}{r}\right] r + k.\]

**Proof:** Since $C$ attains the bound (2), the 1st GHW of $C$ is the minimum distance $d_1 = n-k - \left[\frac{k}{r}\right] - 1(\delta - 1) + 1$. Hence,
\[g_{n-k-\left[\frac{k}{r}\right] - 1(\delta - 1)} = n-k - \left(\left[\frac{k}{r}\right] - 1\right)(\delta - 1).\]

By (4), it follows that the $[i(\delta - 1)]$-th $1 \leq i \leq \left[\frac{k}{r}\right] - 1\)$-th GHW is
\[d^\perp_i = (n+1) - g_{n-k-\left[\frac{k}{r}\right] - 1(\delta - 1)},\]
\[= (n+1) - \left[\left[\frac{k}{r}\right] - 1\right](\delta - 1) - 1.\]

By Lemma 2, for $1 \leq i \leq \left[\frac{k}{r}\right] - 1(\delta - 1)$, the $i$-th GHW of the dual code $C^\perp$ satisfies
\[d^\perp_i \geq d^\perp_{i+1} - r + 1.\]

Combining (16) and (17), we have
\[d^\perp_{i+\left[\frac{k}{r}\right] - 1(\delta - 1) + 1} \geq d^\perp_{i+\left[\frac{k}{r}\right] - 1(\delta - 1) + 1} - (r+1)\]
\[= \left(\left[\frac{k}{r}\right] - 1\right)(\delta - 1) - 1.\]

When $1 \leq i \leq \left[\frac{k}{r}\right] - 2(\delta - 1)$, it follows from Lemma 2 that the $i$-th GHW of the dual code $C^\perp$ satisfies
\[d^\perp_i \geq d^\perp_{i+(\delta - 1)} - (r+\delta - 1).\]
Combining (18) with (19), for \(1 \leq i \leq \lceil \frac{k}{r} \rceil - 2\), the \([i(\delta - 1)]\)-th GHW of the dual code \(C^\perp\) satisfies
\[
d_{i(\delta - 1)-1}^\perp \geq d_{i(\lceil\frac{k}{r}\rceil-1)(\delta - 1)-1}^\perp - \left(\left\lceil \frac{k}{r} \right\rceil - 1 - i \right)(r + \delta - 1) = i(r + \delta - 1) - \left\lceil \frac{k}{r} \right\rceil r + k.
\]
Combining the above discussions, the conclusion holds. □

In the following, we show that for a \(q\)-ary optimal \((n, k, r, \delta)\)-LRC \((\delta \geq 2)\) attaining the Singleton-type bound (2) with dimension \(k\) divisible by \(r\), the \([i(\delta - 1)]\)-th GHW, for \(1 \leq i \leq \frac{k}{r}, 1\), of its dual code can be exactly determined.

**Theorem 3:** Let \(C\) be a \(q\)-ary optimal \((n, k, r, \delta)\)-LRC \((\delta \geq 2)\) attaining the Singleton-type bound (2). Let \(C^\perp\) be its dual code. If the dimension \(k\) of \(C\) is divisible by \(r\), then the \([i(\delta - 1)]\)-th GHW, for \(1 \leq i \leq \frac{k}{r}, 1\), of the dual code \(C^\perp\) satisfies
\[
d_{i(\delta - 1)-1}^\perp = i(r + \delta - 1).
\]

**Proof:** By Lemma 1, for \(1 \leq i \leq \frac{k}{r} - 1\), the \([i(\delta - 1)]\)-th GHW of the dual code \(C^\perp\) satisfies
\[
d_{i(\delta - 1)-1}^\perp \leq \left\lceil \frac{i}{\delta - 1} \right\rceil r + i = i(r + \delta - 1).
\]
By Theorem 2, for a \(q\)-ary optimal \((n, k, r, \delta)\)-LRC with \(r \mid k\), when \(1 \leq i \leq \frac{k}{r} - 1\), the \([i(\delta - 1)]\)-th GHW of the dual code \(C^\perp\) satisfies
\[
d_{i(\delta - 1)-1}^\perp \geq i(r + \delta - 1) - \left\lceil \frac{k}{r} \right\rceil r + k = i(r + \delta - 1).
\]
Combining (21) and (22), the conclusion holds. □

**Example 1:** For an optimal cyclic \((n = 36, k = 20, r = 4, \delta = 3)\)-LRC [15] \(C\) over \(\mathbb{F}_{3^7}\) with \(d = 9\) and \(r \mid k\), by Theorem 3, the the \((2i)\)-th GHW, for \(1 \leq i \leq 4\), of the dual code \(C^\perp\) can be determined as follows,
\[
d_2^\perp = 6, \quad d_4^\perp = 12, \quad d_6^\perp = 18, \quad d_8^\perp = 24.
\]
Next, we consider the GHWs of optimal \((n, k, r, \delta)\)-LRC with \(\delta = 2\), i.e., optimal \((n, k, r)\)-LRCs attaining the Singleton-type bound (1).

**Lemma 4:** Let \(C\) be a \(q\)-ary optimal \((n, k, r, 2)\)-LRC \(C\) attaining the Singleton-type bound (2) and \(C^\perp\) be its dual code. Then the \(i\)-th \((1 \leq i \leq k)\) gap number of the dual code \(C^\perp\) satisfies
\[
g_i^\perp \leq \left\lceil \frac{i + \left\lceil \frac{k}{r} \right\rceil r - k}{r} \right\rceil + i - 1.
\]

**Proof:** We prove this lemma by induction. The 1st GHW of \(C\) is the minimum distance \(d_1 = n - k - \lceil \frac{k}{r} \rceil + 2\). Firstly, for \(i = k\), by (4), the \(k\)-th gap number of \(C^\perp\) is
\[
g_k^\perp = (n + 1) - d_1 = \left\lceil \frac{k}{r} \right\rceil + k - 1,
\]
which satisfies (23). Next, assume that for \(2 \leq i \leq k\), the \(i\)-th gap number of the dual code \(C^\perp\) satisfies (23). Then consider the \((i - 1)\)-th gap number \(g_{i-1}^\perp\) of \(C^\perp\).

1) If \(r \mid (k - i + 1)\), in other words, \(r \mid [i - 1 + (\lceil \frac{k}{r} \rceil r - k)]\). Then it holds that
\[
\frac{(i-1)+(\lceil \frac{k}{r} \rceil r-k)}{r} = \frac{i+(\lceil \frac{k}{r} \rceil r-k)}{r}.
\]
By \(g_{i-1}^\perp \leq g_i^\perp - 1\), it follows that
\[
g_{i-1}^\perp \leq \left\lceil \frac{i + \left\lceil \frac{k}{r} \right\rceil r - k}{r} \right\rceil + (i - 1) - 1 = \left\lceil \frac{(i - 1) + \left\lceil \frac{k}{r} \right\rceil r - k}{r} \right\rceil + (i - 1) - 1.
\]
2) If \(r \mid (k - i + 1)\), which indicates \(r \mid [i - 1 + (\lceil \frac{k}{r} \rceil r - k)]\). Then it follows that
\[
\frac{[i+(\lceil \frac{k}{r} \rceil r-k)]}{r} = \frac{[i+(\lceil \frac{k}{r} \rceil r-k)]}{r} - 1.
\]
Let \((i - 1) + (\lceil \frac{k}{r} \rceil r - k) = sr\) for some integer \(s \leq \lceil \frac{k}{r} \rceil - 1\). Then \(i+(\lceil \frac{k}{r} \rceil r-k) = sr + 1\).
By Theorem 2, when \(\delta = 2\), for \(1 \leq s \leq \lceil \frac{k}{r} \rceil - 1\), the \(s\)-th GHW of \(C^\perp\) satisfies
\[
g_s^\perp \geq s(r+1) - \left\lceil \frac{k}{r} \right\rceil r + k.
\]
Therefore, we can obtain that the set \(\{1, 2, \ldots, s(r+1) - \left\lceil \frac{k}{r} \right\rceil r - k\}\) contains at most the GHWs of \(C^\perp\) in the following set
\[
\{g_1^\perp, \ldots, g_{s(r+1)-\left\lceil \frac{k}{r} \right\rceil r-k}^\perp\}.
\]
On the other hand, this implies that the set \(\{1, 2, \ldots, s(r+1) - \left\lceil \frac{k}{r} \right\rceil r - k\}\) contains at least these gap numbers of \(C^\perp\) as follows
\[
g_s^\perp \leq s + sr - \left\lceil \frac{k}{r} \right\rceil r - k - 1 = \left\lceil \frac{(i - 1) + \left\lceil \frac{k}{r} \right\rceil r - k}{r} \right\rceil + (i - 1) - 1.
\]
Combining the above two cases, the lemma holds. □

Next, we present a lower bound on the \(i\)-th GHW, for all \(1 \leq i \leq k\), of \(q\)-ary optimal \((n, k, r, 2)\)-LRCs attaining the Singleton-type bound (2).

**Theorem 4:** Let \(C\) be a \(q\)-ary optimal \((n, k, r, 2)\)-LRC attaining the Singleton-type bound (2), then the \(i\)-th GHW, for \(1 \leq i \leq k\), of \(C\) satisfies
\[
d_i \geq n - k - \left\lceil \frac{\lceil \frac{k}{r} \rceil r - i + 1}{r} \right\rceil + i + 1.
\]

**Proof:** For \(1 \leq i \leq k\), by (4), it follows that the \(i\)-th GHW of \(C\) satisfies
\[
d_i = n + 1 - g_{k-i+1}^\perp.
\]
By Lemma 4, it follows that
\[
g_{k-i+1}^\perp \leq \left( \frac{\lfloor \frac{k}{r} \rfloor r - k - i}{\frac{k}{r}} \right) + k - i.
\] Combining (25) and (26), we have
\[
d_i \geq n + 1 - \left( \frac{\lfloor \frac{k}{r} \rfloor r - i + 1}{\frac{k}{r}} \right) + k - i.
\]
\[
d_i \geq n - k - \left( \frac{k - i + 1}{\frac{k}{r}} \right) + i + 1.
\] Hence, the conclusion holds. \(\square\)

Generally, the weight hierarchy of \(q\)-ary optimal \((n, k, r, 2)\)-LRCs attaining the Singleton-type bound (2) can not be uniquely determined. In the following, we show that for a \(q\)-ary optimal \((n, k, r, 2)\)-LRC with dimension \(k\) divisible by \(r\), its weight hierarchy can be uniquely determined.

**Theorem 5:** Let \(C\) be a \(q\)-ary optimal \((n, k, r, 2)\)-LRC attaining the Singleton-type bound (2) with dimension \(k\) divisible by \(r\). Then the \(i\)-th GHW, for \(1 \leq i \leq k\), satisfies
\[
d_i = n - k - \left( \frac{k - i + 1}{\frac{k}{r}} \right) + i + 1.
\] \(\square\)

**Proof:** By the upper bound (14) in Theorem 1, we know when \(\delta = 2\), the \(i\)-th (\(1 \leq i \leq k\)) GHW of \(C\) satisfies
\[
d_i \leq n - k - \left( \frac{k - i + 1}{\frac{k}{r}} \right) + i + 1.
\] Meanwhile, by the lower bound (24) of GHWs of \(q\)-ary optimal \((n, k, r, 2)\)-LRC in Theorem 4, together with the condition that \(r \mid k\), we know the \(i\)-th (\(1 \leq i \leq k\)) GHW of \(C\) satisfies
\[
d_i \geq n - k - \left( \frac{k - i + 1}{\frac{k}{r}} \right) + i + 1.
\] Combining (28) and (29), when \(r \mid k\), the \(i\)-th GHW, for \(1 \leq i \leq k\), of \(C\) is
\[
d_i = n - k - \left( \frac{k - i + 1}{\frac{k}{r}} \right) + i + 1.
\] this completes the proof. \(\square\)

**Example 2:** For an optimal cyclic \((n = 16, k = 8, r = 4)\)-LRC \([11] C\) over \(\mathbb{F}_{17}\) with \(d = 8\) and \(r \mid k\), by Theorem 5, the weight hierarchy of \(C\) can be completely determined as follows,
\[
d_1 = 8, \quad d_2 = 9, \quad d_3 = 10, \quad d_4 = 11, \quad d_5 = 13,
\]
\[
d_6 = 14, \quad d_7 = 15, \quad d_8 = 16.
\] It can be easily verified that \(i\)-th GHW of this optimal cyclic \((16, 8, 4)\)-LRC attains the generalized Singleton-like bound (14) for all \(1 \leq i \leq 8\).

**IV. CONCLUSION**

In this article, we study the generalized Hamming weights of \(q\)-ary \((n, k, r, \delta)\)-LRCs \((\delta \geq 2)\) and their dual codes. Firstly, for \(q\)-ary general \((n, k, r, \delta)\)-LRCs, an upper bound on the \(i\)-th GHW, for \(1 \leq i \leq k\), is presented. When \(i = 1\), the proposed upper bound gives the Singleton-type bound (2). When \(r = k\), it reduces to the classical generalized Singleton bound (3). Then, for a \(q\)-ary optimal \((n, k, r, \delta)\)-LRC attaining the Singleton-type bound (2), a lower bound on the \([i(\delta - 1)]\)-th GHW, for \(1 \leq i \leq \lfloor k/r \rfloor - 1\), of the dual code is given. When the dimension \(k\) is divisible by \(r\), we show that the \([i(\delta - 1)]\)-th GHW, for \(1 \leq i \leq \lfloor k/r \rfloor - 1\), of the dual code of a \(q\)-ary optimal \((n, k, r, \delta)\)-LRC can be exactly determined. For the case of \(\delta = 2\), a lower bound on the \(i\)-th \((1 \leq i \leq k)\) GHW of a \(q\)-ary optimal \((n, k, r, 2)\)-LRC is obtained. Moreover, it is shown that the weight hierarchy of a \(q\)-ary optimal \((n, k, r, 2)\)-LRC with \(r \mid k\) can be completely determined.

**ACKNOWLEDGMENT**

This article was presented in part at the 2017 IEEE Information Theory Workshop.

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