Stable distributions and pseudo-processes related to fractional Airy functions

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**Abstract**

In this article, we study pseudo-processes related to odd-order heat-type equations composed with Lévy stable subordinators. The aim of the article is twofold. We first show that the pseudo-density of the subordinated pseudo-process can be represented as an expectation of damped oscillations with generalized gamma distributed parameters. This stochastic representation also arises as the solution to a fractional diffusion equation, involving a higher-order Riesz-Feller operator, which generalizes the odd-order heat-type equation. We then prove that, if the stable subordinator has a suitable exponent, the time-changed pseudo-process is identical in distribution to a genuine Lévy stable process. This result permits us to obtain a power series representation for the probability density function of an arbitrary asymmetric stable process of exponent \(\nu > 1\) and skewness parameter \(\beta\), with \(0 < |\beta| < 1\). The methods we use in order to carry out our analysis are based on the study of a fractional Airy function which emerges in the investigation of the higher-order Riesz-Feller operator.

While the analysis of higher-order heat-type equations was first tackled in some special cases by Bernstein [1] and Burwell [2], a probabilistic insight on the topic was established some decades later by Krylov [3]. In his article, the author constructed a signed measure on the space \(\Omega\) of real-valued functions \(x = x(t), \ t > 0\), called sample paths, by defining the cylinder sets

\[
I_n = \{x : a_k \leq x(t_k) \leq b_k, \ k = 1, ..., n\}, \quad 0 \leq t_0 < ... < t_{n-1}.
\]

A finitely additive signed measure \(\mathbb{P}\) on \(\Omega\) is constructed by the rule

\[
\mathbb{P}(I_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{k=1}^{n} u(x_k - x_{k-1}, t_k - t_{k-1}) \ dx_1 \cdots dx_n \tag{1}
\]

where \(x_0 = 0, \ t_0 = 0\), \(u(x, t)\) is the fundamental solution to the higher-order heat equation of order \(m > 2\)

\[
\frac{\partial u}{\partial t} = c_m \frac{\partial^m u}{\partial x^m}, \quad x \in \mathbb{R}, \ t > 0
\]

and \(c_m\) is a suitable coefficient. As pointed out by Miyamoto [4], the signed measure \(\mathbb{P}\) is well defined on the algebra \(\mathcal{F}\) containing all cylinder sets and it is finitely additive on \(\mathcal{F}\). Since \(\mathbb{P}\) is signed, Kolmogorov’s extension theorem cannot be applied to extend \(\mathbb{P}\) to the
\(\sigma\)-algebra generated by \(\mathcal{F}\). Moreover, Krylov [3] proved that \(\mathbb{P}\) has infinite total variation. Therefore, no claims are made on the possibility of extending \(\mathbb{P}\) to \(\sigma(\mathcal{F})\). In Krylov's paper, only the even-order case \(m = 2n\) was considered. Hochberg [5] also developed an \(\text{Itô}\) type stochastic calculus for the even-order pseudo-processes. Similar ideas for the construction of pseudo-processes were proposed by Ladokhin [6], Daletsky and Fomin [7], and Miyamoto [4] who investigated equations in which two space derivatives appear. Debbi [8, 9] proposed a space-fractional extension of the even-order heat-type equation involving the Riesz operator. The pseudo-processes related to odd-order heat-type equations were first considered by Orsingher [10] for the third-order heat equation and were then extended to all possible orders by Lachal [11]. In these articles, the main concern was to evaluate the distribution of some functionals of pseudo-processes like the sojourn time or the maximum. Nakajima and Sato [12] have obtained the joint distribution of the hitting time and hitting place for the third-order pseudo-process. By studying the distribution of \(x(\tau_a)\) with
\[
\tau_a = \inf\{t \geq 0 : x(t) > a\}
\]
Lachal [13] and Nishioka [14, 15] showed that the measure \(\mathbb{P}\) is concentrated on sample paths \(x(t)\) of the pseudo-process which display a sort of continuity or moderate discontinuity. Alternative approaches have been proposed in the literature for the probabilistic construction of pseudo-processes. Bonaccorsi and Mazzucchi [16] and Lachal [17] obtained the solutions to higher-order heat-type equations in terms of the scaling limit of suitable random walks. Orsingher and Toaldo [18] constructed the pseudo-processes as the limit of compound Poisson processes with steps represented by pseudo-random variables with signed measures. The starting point of the present research is the paper by Orsingher and D’Ovidio [19] in which, for \(n \in \mathbb{N}\), the fundamental solution to the odd-order heat-type equation
\[
\frac{\partial u}{\partial t}(x, t) = (-1)^n \frac{\partial^{2n+1} u}{\partial x^{2n+1}}(x, t), \quad x \in \mathbb{R}, \quad t > 0
\]
is expressed as
\[
u_{2n+1}(x, t) = \frac{1}{\pi x} \mathbb{E}\left[ e^{-b_n x G_{2n+1}(1/t)} \sin(a_n x G_{2n+1}(1/t)) \right]
\]
where \(G_\gamma(\tau), \gamma, \tau > 0\), is a random variable with generalized gamma distribution having probability density function
\[
g_\gamma(y; \tau) = \gamma \frac{y^{\gamma-1}}{\tau} \exp\left(-\frac{y}{\gamma}\right), \quad y > 0
\]
and
\[
a_n = \cos\left(\frac{\pi}{2(2n + 1)}\right), \quad b_n = \sin\left(\frac{\pi}{2(2n + 1)}\right).
\]
It is known (see Orsingher [10]) that, in the case \(n = 1\), the solution to Equation (2) admits the following representation in terms of the Airy function of the first kind:
\[
u_3(x, t) = \frac{1}{\sqrt{3t}} \text{Ai}\left(\frac{x}{\sqrt{3t}}\right).
\]
For general values of \(n \in \mathbb{N}\), we start our analysis by exploring the connection between the probabilistic representation (3) and the higher-order Airy functions, which are defined by the improper Riemann integral
\[
\text{Ai}_{2n+1}(x) = \frac{1}{\pi} \int_{0}^{+\infty} \cos\left(sx + \frac{s^{2n+1}}{2n + 1}\right) ds, \quad x \in \mathbb{R}
\]
and were profoundly investigated by Askari and Ansari [20]. We then analyze the pseudo-process $X_{2n+1}(t)$ time-changed with an independent stable subordinator $S_\theta(t)$ of exponent $\theta$, $0 < \theta < 1$. In particular, denoting by $h_\theta(x,t)$ the probability density function at time $t$ of a stable subordinator having characteristic function

$$E\left[e^{i\gamma S_\theta(t)}\right] = e^{-t|\gamma|^\theta e^{-\frac{i\theta\gamma}{2} sgn(\gamma)}}, \quad \gamma \in \mathbb{R}$$

we are interested in finding the pseudo-density

$$p_{2n+1}(x,t;\theta) = \int_0^{+\infty} u_{2n+1}(x,s) h_\theta(s,t) ds, \quad x \in \mathbb{R}, \ t > 0. \quad (6)$$

By means of the construction emerging from Equation (1), the pseudo-density (6) is related to a pseudo-process which we denote as $Y_{2n+1,\theta}(t)$. The pseudo-process $Y_{2n+1,\theta}(t)$ can be represented, from a probabilistic point of view, as

$$Y_{2n+1,\theta}(t) = X_{2n+1}(S_\theta(t)).$$

We show that, as a consequence of the subordination, the probabilistic representation (3) is transformed into

$$p_{2n+1}(x,t;\theta) = \frac{1}{\pi x^\alpha} E\left[e^{-b_n x G_{(2n+1)\theta}(1/t)} \sin (a_n x G_{(2n+1)\theta}(1/t))\right]$$

where $G_\tau(\tau)$, $a_n$ and $b_n$ are defined as in formula (3). In the final section of the article, we examine a fractional extension of Equation (2) which reads, for $\alpha > 1$ and $\theta \in (0, 1]$,

$$\begin{cases}
\frac{\partial u}{\partial t}(x,t) = \chi D_\theta^\alpha u(x,t), & x \in \mathbb{R}, \ t > 0, \\
u(x,0) = \delta(x)
\end{cases} \quad (7)$$

where the space operator $\chi D_\theta^\alpha$ is a higher-order Riesz-Feller derivative of which we give an explicit representation. The analysis of Equation (7) is carried out by defining, for $\alpha > 1$, a fractional generalization of the Airy function

$$Ai_\alpha(x) = \frac{1}{\pi} \int_0^{+\infty} \cos \left( x\gamma + \frac{\gamma^\alpha}{\alpha} \right) d\gamma, \quad x \in \mathbb{R}$$

for which we also give the following power series representation:

$$Ai_\alpha(x) = \frac{1}{\pi \alpha^{\frac{\alpha-1}{2}}} \sum_{k=0}^{\infty} \Gamma \left( \frac{k+1}{\alpha} \right) \sin \left( \pi \frac{(k+1)(\alpha+1)}{2\alpha} \right) \frac{\alpha^{\frac{k}{\alpha}} x^k}{k!}.$$

For $\theta = 1$ and $\alpha > 1$, the solution to the Cauchy problem (7) is given by

$$u_\alpha(x,t) = \frac{1}{(\alpha t)^{\frac{1}{\alpha}}} Ai_\alpha \left( \frac{x}{(\alpha t)^{\frac{1}{\alpha}}} \right), \quad x \in \mathbb{R}, \ t > 0. \quad (8)$$

The extension to the case $0 < \theta < 1$ can be achieved by time-changing the pseudo-process $X_\alpha(t)$, having pseudo-distribution (8), with a stable subordinator $S_\theta(t)$ of exponent $0 < \theta < 1$. Therefore, we study the pseudo-density

$$p_\alpha(x,t;\theta) = \int_0^{+\infty} u_\alpha(x,s) h_\theta(s,t) ds, \quad x \in \mathbb{R}, \ t > 0 \quad (9)$$

which can be interpreted as the distribution of the subordinated pseudo-process

$$Y_{\alpha,\theta}(t) = X_\alpha(S_\theta(t)).$$
Throughout our analysis, we will examine the Fourier transform of the pseudo-density (9) and, in order to emphasize the probabilistic interpretation of our results, we will express the Fourier transform as an expected value with respect to a signed density. Thus, we adopt the notation

$$
E \left[ e^{ij\gamma} Y_{\alpha, \theta}(t) \right] = \int_{-\infty}^{+\infty} e^{ij\gamma} p_{\alpha}(x, t; \theta) dx. \quad (10)
$$

We prove that, for $0 < \alpha \theta \leq 2$, the Fourier transform (10) coincides with the characteristic function of a stable process

$$
E \left[ e^{ij\gamma} Y_{\alpha, \theta}(t) \right] = e^{-t|\gamma|^{\alpha\theta} \cos \left( \frac{\pi \theta}{2} \right) \left( 1 + i \tan \frac{\pi \theta}{2} \text{sgn} \gamma \right)}
$$

and the pseudo-density $p_{\alpha}(x, t; \theta)$ becomes a genuine probability density function. This implies that, by giving a power series representation for the distribution of $Y_{\alpha, \theta}(t)$, we are able to obtain the exact distribution of an arbitrary asymmetric stable process of exponent $1 < \nu < 2$ and skewness parameter $\beta$, $0 < |\beta| < 1$. Our result is consistent with that reported in the book by Zolotarev [21] and shows that the probability density function of stable processes can be represented as the pseudo-density of pseudo-processes time-changed with stable subordinators.

1. Odd-order pseudo-processes and Airy functions

For $n \in \mathbb{N}$, explicit representations for the solution to the odd-order heat-type equation

$$
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = (-1)^n \frac{\partial^{2n+1} u}{\partial x^{2n+1}}(x, t), & x \in \mathbb{R}, \ t > 0 \\
u(x, 0) = \delta(x)
\end{cases} \quad (11)
$$

have been proposed in different forms in the literature. The connection between the solution to equation (11) and the higher-order Airy function (5) was highlighted by Askari and Ansari [20]. The authors studied the higher-order Airy function as a solution to the ordinary differential equation

$$
y^{(2n)}(x) + (-1)^n xy(x) = 0. \quad (12)
$$

Similarly to the classical Airy equation, Equation (12) can be solved by applying the generalized Laplace transform method. By splitting the complex plane into $2(2n + 1)$ sectors of amplitude $\frac{\pi}{2(2n+1)}$, $2n$ linearly independent solutions are obtained among which the solution (5) arises. In the same article, the authors pointed out that, by solving Equation (11) with a Fourier transform approach, the solution reads

$$
u_{2n+1}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iyx-ity^{2n+1}} dy = \frac{1}{\pi} \int_{0}^{+\infty} \cos \left( \gamma x + t\gamma^{2n+1} \right) d\gamma = \frac{1}{((2n + 1)t)^{\frac{1}{2n+1}}} \text{Ai}_{2n+1} \left( \frac{x}{((2n + 1)t)^{\frac{1}{2n+1}}} \right). \quad (13)$$
Figure 1. The functions $f_1$ and $f_2$ are integrated, respectively, along the contours (a) and (b). Both contours enclose circular sectors with radius $R$. The arc of the contour (a) has angle ranging from $0$ to $\frac{\pi}{2(2n+1)}$, while the arc of the contour (b) has angle ranging from $-\frac{\pi}{2(2n+1)}$ to $0$. The integrals along the arcs converge to $0$ as $R \to +\infty$.

The representation (13) is crucial for our purposes since most of our probabilistic considerations emerge from the study of generalized Airy functions. For $n = 1$, the classical Airy function admits the well-known power series representation

$$Ai_3(z) = \frac{2}{3^\frac{7}{6}} \sum_{k=0}^{+\infty} \frac{\sin \left( \frac{2\pi}{3} (k + 1) \right)}{\Gamma \left( \frac{k+1}{3} \right) \left( \frac{z}{3^\frac{1}{3}} \right)^k}.$$  \hspace{1cm} (14)

The power series expansion (14) can be extended to the higher-order Airy function (5) as shown by Ansari and Askari [22]. In the following theorem, we provide a different proof which generalizes the one proposed by Watson [23], p. 189, for the classical case $n = 1$.

**Theorem 1.1.** The higher-order Airy function (5) admits the following series representation for $x \in \mathbb{R}$:

$$Ai_{2n+1}(x) = \frac{1}{\pi^{2n+1} \cdot 2^{2n+1}} \sum_{k=0}^{+\infty} \sin \left( \frac{\pi (k+1)(n+1)}{2n+1} \right) \frac{\Gamma \left( \frac{k+1}{2n+1} \right) (2n+1)^{\frac{k}{2n+1}} x^k}{k!}.$$  \hspace{1cm} (15)

Moreover, the power series (15) is absolutely convergent.

**Proof.** We start by observing that

$$Ai_{2n+1}(x) = \frac{1}{2\pi} \left( \int_{0}^{+\infty} \exp \left\{ it \left( t \right)^{2n+1} \right\} dt + \int_{0}^{+\infty} \exp \left\{ -it \left( t \right)^{2n+1} \right\} dt \right).$$  \hspace{1cm} (16)

By setting

$$f_1(t) = \exp \left\{ it \left( t \right)^{2n+1} \right\}, \quad f_2(t) = \exp \left\{ -it \left( t \right)^{2n+1} \right\}$$

the expression (16) can be reformulated by integrating $f_1$ and $f_2$ along two suitable contours encircling two different circular sectors with radius $R$. The contours are described in detail in Figure 1. By the Cauchy integral theorem, the functions $f_1$ and $f_2$ have null integrals along the considered contours. Moreover, the integrals along the arcs of the contours converge to
Thus, by summing the integrals along the two contours described in Figure 1 and taking the limit for \( R \to +\infty \), we have that

\[
\left| \int_{\Gamma_1} f_1(z) \, dz \right| = \left| iR \int_0^{\pi/(2n+1)} e^{i\theta} f_1(Re^{i\theta}) \, d\theta \right| \leq R \int_0^{\pi/(2n+1)} |f_1(Re^{i\theta})| \, d\theta
\]

\[
= R \int_0^{\pi/(2n+1)} e^{-xR} \sin \theta \leq \frac{R^{2n+1}}{2n+1} \sin((2n+1)\theta) \, d\theta \leq R \int_0^{\pi/(2n+1)} e^{-2\pi R(xR + R^{2n+1})} \, d\theta
\]

\[
= \frac{\pi}{2 \left( R^{2n+1} + x \right)} \left( 1 - e^{-\frac{R^{2n+1} + x}{2n+1}} \right)
\]  

(17)

where we have used the inequality \( \sin x \geq \frac{2x}{\pi} \) for \( x \in [0, \frac{\pi}{2}] \). Of course, the right-hand side of formula (17) tends to 0 as \( R \to +\infty \). A similar procedure permits to show that the integral of \( f_2 \) along the contour \( \Gamma_2 = \{ z \in \mathbb{C} : z = Re^{i\theta}, -\frac{\pi}{2(2n+1)} \leq \theta \leq 0 \} \) tends to 0 as \( R \to +\infty \).

Thus, by summing the integrals along the two contours described in Figure 1 and taking the limit for \( R \to +\infty \), we can write

\[
\text{Ai}_{2n+1}(x) = \frac{1}{2\pi} e^{i\pi \frac{x}{2n+1}} \int_0^{\pi/(2n+1)} \exp \left\{ x e^{i\pi \frac{n+1}{2n+1}} - \frac{x^{2n+1}}{2n+1} \right\} \, dr
\]

\[
- \frac{1}{2\pi} e^{i\pi \frac{4n+2}{2n+1}} \int_0^{\pi/(2n+1)} \exp \left\{ x e^{i\pi \frac{n+1}{2n+1}} - \frac{x^{2n+1}}{2n+1} \right\} \, dr
\]

\[
+ \frac{1}{2\pi} e^{-i\pi \frac{4n+2}{2n+1}} \int_0^{\pi/(2n+1)} \exp \left\{ x e^{-i\pi \frac{n+1}{2n+1}} - \frac{x^{2n+1}}{2n+1} \right\} \, dr
\]

(18)

The exponential functions \( f(r) = \exp \left\{ x e^{\pm i\pi \frac{n+1}{2n+1}} \right\} \) in formula (18) can be replaced by their Taylor expansions and the order of integration and summation can then be interchanged in force of the dominated convergence theorem. In order to apply the dominated convergence theorem, observe that \( f(r) = \lim_{m \to +\infty} f_m(r) \) where \( f_m(r) \) represents the sum of the first \( m \) terms of the Taylor expansion of \( f(r) \). Since \( |f_m(r)| \leq e^{\pi |x|} \) for all \( m \) and \( e^{\pi |x|} \) is integrable on \((0, +\infty)\) with respect to the measure \( \mu(dr) = e^{-\frac{r^{2n+1}}{2n+1}} \, dr \), the dominated convergence theorem can be applied. Thus, we can write

\[
\text{Ai}_{2n+1}(x) = \frac{1}{2\pi} e^{i\pi \frac{1}{2n+1}} \sum_{k=0}^{\infty} \frac{e^{i\pi \frac{k+1}{2n+1} \pi}}{k!} \int_0^{\pi/(2n+1)} r^k \exp \left\{ -\frac{r^{2n+1}}{2n+1} \right\} \, dr
\]

\[
+ \frac{1}{2\pi} e^{-i\pi \frac{1}{2n+1}} \sum_{k=0}^{\infty} \frac{e^{-i\pi \frac{k+1}{2n+1} \pi}}{k!} \int_0^{\pi/(2n+1)} r^k \exp \left\{ -\frac{r^{2n+1}}{2n+1} \right\} \, dr
\]

\[
= \frac{1}{\pi (2n+1)^{2n+1}} \sum_{k=0}^{\infty} \frac{e^{i\pi \frac{2k(n+1)+1}{4n+2}}}{2} + e^{-i\pi \frac{2k(n+1)+1}{4n+2}} \Gamma \left( k + 1 \right) \frac{(2n+1) \frac{k+1}{2n+1} \pi}{k!}
\]
where in the last step we have used the formula $\cos \theta = \sin (\theta + \frac{\pi}{2})$. The absolute convergence of power series (15) can be proved by using the Stirling approximation formula for the gamma function.

We observe that formula (15) is consistent with the result obtained by Orsingher and D’Ovidio [19] who proved the following representation for the pseudo-density:

$$u_{2n+1}(x, t) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{1}{k!} \sin \left( \frac{n\pi k}{2n+1} \right) \Gamma \left( 1 + \frac{k}{2n+1} \right) \left( -\frac{x}{t^{\frac{1}{2n+1}}} \right)^k. \quad (19)$$

The expression (19) is immediately obtained by combining formulas (13) and (15). Moreover, by using the triplication formula for the gamma function

$$\Gamma(3z) = \frac{1}{2\pi} 3^{3z-\frac{1}{2}} \Gamma(z) \Gamma \left( z + \frac{1}{3} \right) \Gamma \left( z + \frac{2}{3} \right)$$

with $z = k + 1$, formula (15) reduces to (14) for $n = 1$. Orsingher and D’Ovidio [19] discussed the behavior of the function $u_{2n+1}(x, t)$ and they observed that, while for $n = 1$ the pseudo-density is non-negative for $x > 0$, the non-negativity on the positive semi-axis is lost for $n > 1$ due to the oscillating behaviour of the function. They also pointed out that the asymmetry of $u_{2n+1}(x, t)$ seems to reduce as $n$ increases. This assertion can be supported by observing that

$$\lim_{n \to +\infty} u_{2n+1}(x, t) = -\frac{1}{\pi x} \sum_{k=1}^{+\infty} \frac{1}{k!} \sin \left( \frac{k\pi}{2} \right) \frac{(-x)^k}{k!}$$

$$= -\frac{1}{2\pi i x} \sum_{k=0}^{+\infty} \left( e^{ik\frac{\pi}{2}} - e^{-ik\frac{\pi}{2}} \right) \frac{(-x)^k}{k!}$$

$$= -\frac{1}{2\pi i x} \left( e^{-ix} - e^{ix} \right) = \frac{\sin x}{\pi x}.$$

Observe that, in the calculation performed above, the limit can be moved inside the infinite sum by applying Tannery’s theorem. This can be verified by considering, for fixed $x \in \mathbb{R}$ and $t > 0$, the sequence

$$a_k(n) = -\frac{1}{\pi x k!} \sin \left( \frac{n\pi k}{2n+1} \right) \Gamma \left( 1 + \frac{k}{2n+1} \right) \left( -\frac{x}{t^{\frac{1}{2n+1}}} \right)^k, \quad k, n \in \mathbb{N}$$

and observing that $u_{2n+1}(x, t) = \sum_{k=1}^{\infty} a_k(n)$ by formula (19). Moreover, we have that $|a_k(n)| \leq M_k$ where

$$M_k = \frac{1}{\pi |x| k!} \Gamma \left( 1 + \frac{k}{2} \right) \left| \frac{x}{\min(t, 1)} \right|^k, \quad k \in \mathbb{N}.$$
Since \( \sum_{k=1}^{\infty} M_k < \infty \) for any value of \( x \in \mathbb{R} \) and \( t > 0 \), the assumptions of Tannery’s theorem are satisfied. Therefore we can write
\[
\lim_{n \to +\infty} u_{2n+1}(x, t) = \sum_{k=1}^{\infty} \lim_{n \to +\infty} a_k(n).
\]

2. Pseudo-processes time-changed with stable subordinators

In this section, we study the pseudo-process \( X_{2n+1}(t) \), governed by the higher-order Equation (11), time-changed with and independent stable subordinator. In particular, we consider stable subordinators \( S_\theta(t) \), with exponent \( \theta \in (0, 1] \), having characteristic function
\[
\mathbb{E}\left[ e^{i\gamma S_\theta(t)} \right] = e^{-t|\gamma|^\theta e^{-\frac{2\pi i \text{sgn} \gamma}{\theta}}} , \quad \gamma \in \mathbb{R}.
\]

In the following theorem, an explicit representation for the pseudo-density (6) of the subordinated pseudo-process
\[
Y_{2n+1, \theta}(t) = X_{2n+1}(S_\theta(t))
\]
is obtained. As a preliminary result, we need the Mellin transform of the Wright function (see Prudnikov et al. [24], p. 355)
\[
\int_0^{+\infty} W_{a,b}(-x) x^{\eta-1} dx = \frac{\Gamma(\eta)}{\Gamma(b-a\eta)}, \quad \eta > 0
\]
where, for \( a > -1, \ b \in \mathbb{C}, \)
\[
W_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(ak+b)}, \quad z \in \mathbb{C}.
\]

**Theorem 2.1.** For \((2n+1)\theta > 1\), the pseudo-density (6) of the pseudo-process \( Y_{2n+1, \theta}(t) \) admits the representation
\[
p_{2n+1}(x, t; \theta) = \frac{1}{\pi x} \mathbb{E}\left[ e^{-b_n x} G_{(2n+1)\theta}(1/t) \sin \left( a_n x \ G_{(2n+1)\theta}(1/t) \right) \right]
\]
where \( G_\gamma(\tau) \) is a random variable with generalized gamma distribution having probability density given by (4) and
\[
a_n = \cos \frac{\pi}{2(2n+1)}, \quad b_n = \sin \frac{\pi}{2(2n+1)}.
\]

**Proof.** We use the Wright function representation of the probability density function of the stable subordinator
\[
h_\theta(x, t) = \frac{\theta t}{x^{\theta+1}} e^{-\frac{t}{x^\theta}} W_{-\theta, 1-\theta} \left( -\frac{t}{x^\theta} \right)
= \frac{\theta}{x} \sum_{k=0}^{\infty} \frac{(-1)^k t^{k+1}}{x^\theta(k+1)k! \Gamma(-\theta(k+1)+1)}.
\]
Thus we have
\[ p_{2n+1}(x, t; \theta) = \int_0^{+\infty} u_{2n+1}(x, s) h_\theta(s, t) \, ds \]
\[ = -\frac{\theta t}{\pi x} \sum_{k=1}^{\infty} \sin \left( \frac{n\pi k}{2n+1} \right) \Gamma \left( 1 + \frac{k}{2n+1} \right) \left( -\frac{x}{\theta} \right)^k \int_0^{+\infty} s^{-\frac{k}{2n+1}} \, W_{-\theta, 1-\theta} \left( -\frac{t}{s^{\frac{1}{\theta}}} \right) \, ds \]

Observe that, in the last step, we have used the power series representation \((19)\) for \(u_{2n+1}(x, s)\) and we have interchanged the order of summation and integration by the dominated convergence theorem. We note that \(u_{2n+1}(x, s) = \lim_{m \to +\infty} f_m(x, s)\) where \(f_m(x, s)\) represents the sum of the first \(m\) terms of the power series \((19)\), that is \(f_m(x, s) = -\frac{1}{\pi x} \sum_{k=1}^{m} \frac{1}{k!} \sin \left( \frac{n\pi k}{2n+1} \right) \Gamma \left( 1 + \frac{k}{2n+1} \right) \left( -\frac{x}{t^{\frac{1}{2n+1}}} \right)^k \). Of course \(|f_m(x, s)| \leq g(x, s)\) for all \(m\), where \(g(x, s) = \frac{1}{\pi|x|} \sum_{k=1}^{\infty} \frac{1}{k!} \Gamma \left( 1 + \frac{k}{2n+1} \right) \left( \frac{|x|}{t^{\frac{1}{2n+1}}} \right)^k \). By applying the monotone convergence theorem and integrating termwise, it can be shown that \(f_0^{+\infty} g(x, s) h_\theta(s, t) \, ds < +\infty\) for \((2n+1)\theta > 1\). Thus, the dominated convergence theorem can be applied and we can write

\[ p_{2n+1}(x, t; \theta) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{1}{k!} \sin \left( \frac{n\pi k}{2n+1} \right) \Gamma \left( 1 + \frac{k}{2n+1} \right) \left( -\frac{x}{t^{\frac{1}{2n+1}}} \right)^k \]
\[ = -\frac{(2n+1)t}{\pi x} \int_0^{+\infty} 2^n e^{-ts^{\frac{1}{\theta}}} \sum_{k=1}^{\infty} \frac{1}{k!} \sin \left( \frac{n\pi k}{2n+1} \right) \left( -xs^{\frac{1}{\theta}} \right)^k ds. \]

By using the relationship
\[ e^{x\cos \phi} \sin (x \sin \phi) = \sum_{k=0}^{\infty} \sin(\phi k) \frac{x^k}{k!} \] \hfill (23)
we finally obtain

\[ p_{2n+1}(x, t; \theta) = \frac{(2n+1)t}{\pi x} \int_0^{+\infty} e^{-x \left( \cos \frac{n\pi \theta}{2n+1} \right) \sin \left( \frac{n\pi}{2n+1} \right) \sin \left( \frac{n\pi}{2n+1} \right)} s^{2n} e^{-ts^{\frac{1}{\theta}}} ds. \]
\[ = \frac{(2n+1)t}{\pi x} \int_0^{+\infty} e^{-x \left( \cos \frac{n\pi \theta}{2n+1} \right) \sin \left( \frac{n\pi}{2n+1} \right) \sin \left( \frac{n\pi}{2n+1} \right)} s^{2n} e^{-ts^{\frac{1}{\theta}}} ds. \]

The change of variables \(y = s^{\frac{1}{\theta}}\) completes the proof.

Formula \((21)\) provides a probabilistic representation for the pseudo-density of the pseudo-process \(Y_{2n+1, \theta}(t)\) in terms of an expected value of damped oscillations with generalized gamma distributed parameters. \textbf{Theorem 2.1} is an extension of the probabilistic representation \((3)\) for the odd-order Airy function obtained by Orsingher and D’Ovidio [19]. Moreover, from the proof of the theorem, it emerges that

\[ p_{2n+1}(x, t; \theta) = -\frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{1}{k!} \sin \left( \frac{n\pi k}{2n+1} \right) \Gamma \left( 1 + \frac{k}{(2n+1)\theta} \right) \left( -\frac{x}{t^{\frac{1}{2n+1}}} \right)^k. \] \hfill (24)

The condition \((2n+1)\theta > 1\) imposed in \textbf{Theorem 2.1} ensures that the power series \((24)\) has infinite radius of convergence. This can be proved by using the Stirling approximation formula.
for the gamma function. In the next section, we extend formula (24) to non-integer values of $n$
by introducing a suitable fractional generalization of the pseudo-process $X_{2n+1}(t)$. Moreover,
we show that the pseudo-density (24) and its fractional extension are genuine non-negative
probability density functions if the parameters are chosen in a suitable way.

3. Fractional Airy functions and stable processes

In this section, we generalize the results obtained so far by studying a family of fractional-
order pseudo-processes. In particular, for $\alpha > 1$, $\theta \in (0, 1]$, we are interested in the solution
to the fractional partial differential equation

$$\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \mathcal{D}_0^\alpha u(x, t), & x \in \mathbb{R}, \ t > 0 \\
u(x, 0) = \delta(x)
\end{cases}$$

(25)

where $\mathcal{D}_0^\alpha$ represents the Riesz-Feller fractional derivative (originally defined by Feller [25])
for which we modify the usual restrictions $0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2 - \alpha\}$. As pointed
out by Mainardi et al. [26], these restrictions are usually imposed in order to ensure the
probabilistic interpretability of the Riesz-Feller operator. For the restricted parameters, the
solution to Equation (25) is indeed the probability density function of an asymmetric stable
process. However, by eliminating the upper bound on $\alpha$, we are able to obtain an explicit series
representation for the density function of asymmetric stable processes of exponent $\nu > 1$ and
skewness parameter $\beta$, with $0 < |\beta| < 1$. By using the notation

$${\mathcal{F}}\{f(x)\}(\gamma) = \int_{-\infty}^{+\infty} e^{ix\gamma} f(x) dx$$

the Riesz-Feller fractional operator can be defined implicitly by means of its Fourier transform

$${\mathcal{F}}\{\mathcal{D}_0^\alpha f(x)\}(\gamma) = -|\gamma|^{\alpha} e^{\frac{i\pi\theta}{2}} \text{sgn}(\gamma) {\mathcal{F}}\{f(x)\}(\gamma).$$

(26)

Under suitable regularity conditions on $f$, the Riesz-Feller derivative of order $\alpha$ admits, for
all values of $\theta$, the explicit integral representation

$$\mathcal{D}_0^\alpha f(x) = \frac{\Gamma(\alpha - m + 1)}{\pi} \frac{d^m}{dx^m} \left[ \sin \frac{\pi(\alpha + \theta)}{2} \int_0^{+\infty} \frac{f(x + z)}{\sqrt{2^\alpha - m + 1}} dz \\
+ (-1)^m \sin \frac{\pi(\alpha - \theta)}{2} \int_0^{+\infty} \frac{f(x - z)}{\sqrt{2^\alpha - m + 1}} dz \right]$$

(27)

with $m - 1 < \alpha < m$, $m \in \mathbb{N}$. The integral representation (27) can be proved by checking that
its Fourier transform coincides with the expression (26) (see Orsingher and Toaldo [18]). In order
to solve equations of the form (25), we use the definition (26) of Riesz-Feller derivative.
Therefore, we implicitly assume the existence of the Fourier transform of the solution to (25).
We start by considering the particular case $\alpha > 1$, $\theta = 1$:

$$\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \mathcal{D}_1^\alpha u(x, t), & x \in \mathbb{R}, \ t > 0 \\
u(x, 0) = \delta(x)
\end{cases}$$

(28)

Denoting by $u_\alpha(x, t)$ the solution to Equation (28), its Fourier transform with respect to $x$ is

$${\mathcal{F}}\{u_\alpha(x, t)\}(\gamma, t) = e^{-it \text{sgn}(\gamma)|\gamma|^\alpha}$$

(29)
from which we obtain
\[ u_\alpha(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\gamma x - it \text{sgn}(\gamma) |\gamma|^\alpha} d\gamma = \frac{1}{\pi} \int_{0}^{+\infty} \cos \left( \gamma x + t|\gamma|^\alpha \right) d\gamma. \tag{30} \]

In analogy with formula (13), by defining the generalized Airy function of order \( \alpha > 1 \)
\[ \text{Ai}_\alpha(x) = \frac{1}{\pi} \int_{0}^{+\infty} \cos \left( \frac{xs + s^\alpha}{\alpha} \right) ds, \quad x \in \mathbb{R} \tag{31} \]
we express the pseudo-density (30) in the form
\[ u_\alpha(x, t) = \frac{1}{(\alpha t)^{\frac{1}{\alpha}}} \text{Ai}_\alpha \left( \frac{x}{(\alpha t)^{\frac{1}{\alpha}}} \right). \tag{32} \]

We emphasize that, while the literature on pseudo-processes makes a clear distinction between even-order and odd-order pseudo-processes, this distinction loses its meaning when fractional-order pseudo-processes are introduced. A more precise classification of pseudo-processes related to heat-type equations is between symmetric and asymmetric pseudo-processes, these latter being strictly related to Airy functions. The first problem we tackle is the convergence of the integral defining the generalized Airy function (31).

**Theorem 3.1.** For \( \alpha > 1 \), the improper integral defining the generalized Airy function (31) is convergent for any real number \( x \).

**Proof.** For any \( x \in \mathbb{R} \) there exists \( s_0 \) such that the function
\[ u(s) = xs + \frac{s^\alpha}{\alpha} \]
is increasing for \( s \geq s_0 \). Denote by \( s(u) \) the inverse of \( u(s) \) for \( s > s_0 \). For \( c > s_0 \) we have that
\[ \int_{s_0}^{c} \cos(u(s))ds = \int_{u(s_0)}^{u(c)} \frac{\cos(u)}{x + s(u)^{\alpha-1}}du. \tag{33} \]

By setting \( f(u) = \cos(u) \) and \( g(u) = \frac{1}{x + s(u)^{\alpha-1}} \), we can write formula (33) as
\[ \int_{s_0}^{c} \cos(u(s))ds = \int_{u(s_0)}^{u(c)} f(u)g(u)du. \tag{34} \]

Since \( \lim_{u \to +\infty} s(u) = +\infty \) and \( \alpha > 1 \), for any \( x \in \mathbb{R} \) we have that \( \lim_{u \to +\infty} g(u) = 0 \). Moreover, for any \( r > s_0 \), the inequality \( \int_{s_0}^{r} f(u)du \leq 2 \) holds. Thus, by taking the limit for \( c \to +\infty \) of the integrals in formula (33), the Abel-Dirichlet test (see Zorich [27], p. 404) implies the convergence of the integral
\[ \int_{s_0}^{+\infty} \cos(u(s))ds. \]

The convergence of the integral in formula (31) follows immediately. \( \square \)

In the following theorem, we obtain a power series representation for the generalized Airy function.
Figure 2. The functions $f_1$ and $f_2$ are integrated, respectively, along the contours (a) and (b). Both contours enclose sectors of a circular annulus with inner radius $\varepsilon$ and outer radius $R$. The contour (a) has arcs with angle ranging from 0 to $\frac{\pi}{2\alpha}$, while the arcs of the contour (b) have angle ranging from $-\frac{\pi}{2\alpha}$ to 0. The integrals along the arcs converge to 0 as $\varepsilon \to 0$ and $R \to +\infty$.

**Theorem 3.2.** The generalized Airy function of order $\alpha > 1$

$$Ai_\alpha(x) = \frac{1}{\pi} \int_0^{+\infty} \cos \left( xs + \frac{s^\alpha}{\alpha} \right) ds, \quad x \in \mathbb{R}$$

admits the power series representation

$$Ai_\alpha(x) = \frac{1}{\pi\alpha} \sum_{k=0}^{\infty} \Gamma \left( \frac{k+1}{\alpha} \right) \sin \left( \frac{\pi}{2 \alpha} \left( k+1 \right) \left( \frac{\alpha}{2} + 1 \right) \right) \frac{x^k}{k!}. \quad (35)$$

Moreover, the power series (35) is absolutely convergent.

**Proof.** We omit the details of the proof since they resemble those of the proof of Theorem 1.1 where the odd integer $2n + 1$ has to be replaced by $\alpha > 1$. The only technical modification which must be made to the original proof is the choice of the integration contours. This is necessary because the functions

$$f_1(t) = \exp \left\{ itx + i\frac{t^\alpha}{\alpha} \right\}, \quad f_2(t) = \exp \left\{ -itx - i\frac{t^\alpha}{\alpha} \right\}$$

are not analytic in 0 for non-integer values of $\alpha$. Therefore, the point 0 must be excluded from the contours by introducing small arcs of radius $\varepsilon$ as in Figure 2. The integrals along these arcs vanish as $\varepsilon \to 0$.

By using formula (32) and applying Theorem 3.2, we can now express the pseudo-density of the fractional order pseudo-process in the form

$$u_\alpha(x, t) = -\frac{1}{\pi \lambda} \sum_{k=1}^{\infty} \frac{\Gamma \left( 1 + \frac{k}{\alpha} \right)}{k!} \sin \left( \frac{\pi}{2 \alpha} \left( \frac{\alpha}{2} - 1 \right) \right) \left( -\frac{x}{\frac{\alpha}{2}} \right)^k.$$

Our last step is to time-change the fractional order pseudo-process by means of and independent stable subordinator. Similarly to the odd-order case, we show that, for the pseudo-density (9) of the pseudo-process
\[ Y_{\alpha, \theta}(t) = X_{\alpha}(S_{\theta}(t)) \]

the following result holds.

**Theorem 3.3.** For \( \alpha \theta > 1 \), the pseudo-density (9) of the pseudo-process \( Y_{\alpha, \theta}(t) \) admits the representation

\[ p_{\alpha}(x, t; \theta) = \frac{1}{\pi x} \mathbb{E} \left[ e^{-b_{\alpha} x G_{\alpha \theta}(1/t)} \sin \left( a_{\alpha} x G_{\alpha \theta}(1/t) \right) \right] \]  

where \( G_{\gamma}(\tau) \) is a random variable with generalized gamma distribution having probability density given by (4) and

\[ a_{\alpha} = \cos \left( \frac{\pi}{2\alpha} \right), \quad b_{\alpha} = \sin \left( \frac{\pi}{2\alpha} \right). \]

**Proof.** The proof is identical to that of Theorem 2.1 where the odd integer \( 2n + 1 \) must be replaced by \( \alpha > 1 \).

We conclude our analysis by showing that Theorem 3.3 permits us to obtain a power series representation for the exact distribution of asymmetric stable processes with exponent \( 1 < \nu < 2 \), \( \nu \) being related to \( \alpha \) and \( \theta \), and skewness parameter \( \beta \), \( 0 < |\beta| < 1 \). Consider the pseudo-process \( X_{\alpha}(t) \) and an independent stable subordinator \( S_{\theta}(t) \) with characteristic function (20). The characteristic function of the subordinated pseudo-process

\[ Y_{\alpha, \theta}(t) = X_{\alpha}(S_{\theta}(t)) \]

is given by

\[ \mathbb{E} \left[ e^{i\gamma Y_{\alpha, \theta}(t)} \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{i\gamma X_{\alpha}(S_{\theta}(t))} \bigg| S_{\theta}(t) \right] \right] \]

\[ = \mathbb{E} \left[ e^{-i\sigma \nu |\gamma|^\nu \sin \gamma} \right] \]

\[ = e^{-t |\gamma|^\nu \cos \frac{\pi \nu}{2} \left( \sin \frac{\pi \nu}{2} \tan \frac{\pi \nu}{2} \right) \sin \gamma} \]

where we have used formulas (29) and (20). Observe that, while we have used a probabilistic notation for the steps in formula (37), from a formal point of view the expectation has to be computed in integral form by using the explicit integral representations (9) and (10). The result (37) therefore holds if and only if the order of integration can be changed, which is not trivial to prove. However, as we will briefly discuss below, it can be proved that the inverse Fourier transform of (37) coincides with the pseudo-density of \( Y_{\alpha, \theta}(t) \). For any stable process \( S_{\nu}(\sigma, \beta, \mu; t) \) of exponent \( \nu \in (0, 2) \), \( \nu \neq 1 \), dispersion parameter \( \sigma > 0 \), skewness parameter \( \beta \in [-1, 1] \), and drift parameter \( \mu \in \mathbb{R} \), the characteristic function of \( S_{\nu}(\sigma, \beta, \mu; t) \) is given by

\[ \mathbb{E} \left[ e^{i\gamma S_{\nu}(\sigma, \beta, \mu; t)} \right] = e^{-\sigma^\nu |\gamma|^\nu t (1-i\beta \sin \gamma \tan \frac{\pi \nu}{2})+i\mu t y}. \]  

By comparing formulas (37) and (38), we obtain that

\[ Y_{\alpha, \theta}(t) \overset{i.d.}{=} S_{\nu}(\sigma, \beta, \mu; t) \]
with
\[ v = \alpha \theta, \quad \beta = -\frac{\tan \frac{\pi \theta}{2}}{\tan \frac{\pi \alpha}{2}}, \quad \sigma = \left( \cos \frac{\pi \theta}{2} \right)^{\frac{1}{2}}, \quad \mu = 0. \] (40)

In order for the relationship (39) to hold, we must have that \( v \in (0, 2) \) and \( \beta \in [-1, 1] \). This poses no problem since, given \( v \in (0, 2) \) and \( \beta \in (0, 1) \), it is always possible to choose \( \theta \in (0, 1) \) and \( \alpha > 1 \) such that the formulas in (40) are satisfied. As we will see, the extension to negative values of \( \beta \) is straightforward. This permits us to represent the probability density function of stable processes as the pseudo-density of pseudo-processes time-changed with stable subordinators. For stable processes with exponent \( \nu \), \( 1 < \nu < 2 \) and skewness parameter \( \beta \in (0, 1) \), we obtain the following formula from the proof of Theorem 3.3:

\[ p_{\alpha}(x, t; \theta) = -\frac{1}{\pi \nu \cos \frac{\pi \theta}{2}} \frac{1}{k!} \sin \left( k \pi \frac{\nu - 1}{2 \alpha} \right) \Gamma \left( 1 + \frac{k}{\theta \alpha} \right) \left( -\frac{x}{t^{\frac{1}{\nu}}} \right)^k. \] (41)

Our representation (41) of the probability density function of an asymmetric stable process is consistent with that reported by Zolotarev [21] (see theorem 2.4.2). We note that, in order to verify that the power series (41) is the inverse Fourier transform of (37), it is sufficient to proceed as in Theorem 3.2 and compute the inverse Fourier transform by integrating along the contours described in Figure 2. For \( v = \alpha \theta > 1 \), \( \theta \in (0, 1) \), the pseudo-distribution (41) represents the fundamental solution to the fractional partial differential equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= x D_{\theta}^\nu u(x, t), \quad x \in \mathbb{R}, \ t > 0 \\
u(x, 0) &= \delta(x).
\end{align*}
\] (42)

This can be easily proved by showing that the Fourier transform of the solution to Equation (42) coincides with the characteristic function (37). However, as formulas (40) show the subordinated pseudo-process \( Y_{\alpha, \theta}(t) \) is identical in distribution to a genuine stochastic process only for suitable choices of \( \alpha \) and \( \theta \). The Cauchy process can be obtained as a particular case by setting \( \theta = \frac{1}{\alpha} \), which yields

\[
E \left[ e^{i \gamma Y_{\alpha, \theta}(t)} \right] = e^{-i \gamma t \sin \frac{\pi}{2 \alpha} - t |\gamma| \cos \frac{\pi}{2 \alpha}}.
\] (43)

Formula (43) represents the characteristic function of a Cauchy process with probability density function

\[ p_{\alpha}(x, t; \theta) = \frac{1}{\pi} \left( t \cos \frac{\pi \theta}{2 \alpha} \right) \frac{1}{t^2 + 2xt \sin \frac{\pi \theta}{2 \alpha} + x^2}. \] (44)

The maximum of the Cauchy density (44) is located on the negative half-axis as shown in Figure 3.

We observe that the skewness parameter in (40)

\[ \beta = -\frac{\tan \frac{\pi \theta}{2}}{\tan \frac{\pi \alpha}{2}} \]

is positive for \( 0 < \theta < 1 \) and \( 1 < \alpha \theta < 2 \). Formula (41) thus describes the probability density function of a stable process with positive skewness parameter. By using the well-known property of stable processes with exponent \( \nu \neq 1 \)

\[ -S_{\nu}(\sigma, \beta, \mu; t) \overset{\text{i.d.}}{=} S_{\nu}(\sigma, -\beta, -\mu; t), \quad t > 0 \]
Figure 3. The Cauchy distribution obtained as a subordinated pseudo-process has its modal value on the negative half-axis.

the series representation (41) can be immediately extended to stable processes with negative skewness parameter.

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