The domain wall partition function
for the Izergin–Korepin nineteen-vertex
model at a root of unity

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Abstract. We study the domain wall partition function \( Z_N \) for the \( U_q(A_2^{(2)}) \)
(Izergin–Korepin) integrable nineteen-vertex model on a square lattice of size
\( N \). \( Z_N \) is a symmetric function of two sets of parameters: horizontal \( \zeta_1, ..., \zeta_N \)
and vertical \( z_1, ..., z_N \) rapidities. For generic values of the parameter \( q \) we derive
the recurrence relation for the domain wall partition function relating \( Z_{N+1} \)
to \( P_N Z_N \), where \( P_N \) is the proportionality factor in the recurrence, which is
a polynomial symmetric in two sets of variables \( \zeta_1, ..., \zeta_N \) and \( z_1, ..., z_N \). After
setting \( q = e^{i\pi/3} \) the recurrence relation simplifies and we solve it in terms of a
Jacobi–Trudi-like determinant of polynomials generated by \( P_N \).

Keywords: integrable spin chains (vertex models), quantum integrability
(Bethe Ansatz), solvable lattice models
1. Introduction

Here we study here a particular object of the nineteen-vertex model of the $U_q(A_2^{(2)})$ quantum group, also called the Izergin–Korepin model. We are interested in the partition function of the model on a square lattice in a $N \times N$ square region with domain wall boundary conditions.

Our work is motivated by the domain wall partition function (DWPF) for the six-vertex model $Z_{6v}$, constructed using the $R$-matrix of the $U_q(A_1^{(1)})$ quantum group. This partition function satisfies a set of recurrence relations found by Korepin [14]. These recurrence relations were solved by Izergin [9]. The solution is written in a form of a determinant which is called the Izergin–Korepin (IK) determinant. In statistical physics the six-vertex model represents a model for two-dimensional ice, which shows interesting critical phenomena (see [1]). The partition function $Z_{6v}$ plays a very important role in the field of integrable models. It is a crucial object in the theory of correlation functions for integrable spin chains [15] such as the XXZ spin-1/2 chain (see also [13]). In combinatorics it allowed one to count the alternating sign matrices [16, 17]. Computing domain wall partition functions for other vertex models is a very complicated problem. One of the main results generalizing the six-vertex domain wall partition function (DWPF) is due to [4], where the $U_q(A_1^{(1)})$ higher spin generalization of the DWPF is obtained in a determinant form.

Inspired by these and other results we address the question of computing the domain wall partition function for the $U_q(A_2^{(2)})$ nineteen-vertex model. This model is an integrable model associated to the Dodd–Bullough–Mikhailov equation, also known as the Jiber–Mikhailov–Shabat model [5, 20, 26]. Izergin and Korepin computed the classical and quantum $R$-matrices for this model [10]. The quantum $R$-matrix defines...
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**Figure 1.** The six vertices of the six-vertex model. The letters $a$, $b$ and $c$ are the weights of the corresponding vertices.

The Izergin–Korepin vertex model. The IK$^1$ $R$-matrix has nineteen non zero entries, which correspond to the nineteen possible vertex configurations (see figure 7). We use this $R$-matrix to build $N$ by $N$ lattice configurations which have domain wall boundary conditions, figure 2. The sum of all such configurations we call the domain wall partition function $Z_N$. In order to compute $Z_N$ we use the ideas from the six-vertex model. First, we establish the recurrence relation for the partition function and then try to find its unique solution. In the case when the deformation parameter $q$ is generic we cannot find a compact expression for $Z_N$. However, when $q = e^{i\pi/3}$ we are able to find a determinant expression.

In section 2 we shortly discuss the DWPF for the six-vertex model. In section 3 we move to the IK model. In section 4 we derive the recurrence relation using the vanishing properties of the weights of the $R$-matrix. The solution to this recurrence relation at $q = e^{i\pi/3}$ is presented in section 5. The proof follows in section 6 and with a conclusion we finish.

### 2. Six-vertex model with domain wall boundary

For the computation of the IK determinant for the six-vertex model check the papers [9, 14]. Here we present a short discussion for convenience.

The problem is to sum up configurations which are built by choosing for each vertex of a square $N \times N$ lattice one of the six vertices from figure 1. A configuration thus constructed will have on each edge one of the two states: a left arrow or a right arrow if the edge is horizontal and an up arrow or a down arrow for a vertical edge. We then impose the domain wall boundary conditions which are depicted in figure 2. Each vertex on this lattice has a position $(i, j)$ where horizontal position $i$ is counted rightwards, and the vertical position $j$ is counted downwards starting from the top left corner. The weight of the vertex at position $(i, j)$ is denoted by $w_{i,j}$ and takes one of the three values $a_{i,j}$, $b_{i,j}$ or $c_{i,j}$. The weight of a configuration $\varepsilon$ on a square domain of size $N \times N$ will be the product of all weights of its vertices

$$
\prod_{1 \leq i, j \leq L} w^{(\varepsilon)}_{i,j}.
$$

The IK partition function is the sum over all configurations (states) $\varepsilon$:

$^1$ The abbreviation IK can be a bit misleading here. Both the model and the object that we want to compute contain the IK part in their short names.
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\[ Z_{\delta v} = \sum_{\epsilon \in \text{states}} \prod_{i,j \leq N} w_{i,j}^{(\epsilon)}. \]  

(1)

The weights \( a, b \) and \( c \) are encoded in the \( R \)-matrix. The \( R \)-matrix acts on two vector spaces labeled by \( i \) and \( j \), which carry spectral parameters \( x_i \) and \( x_j \), thus we write \( R_{i,j}(x_i, x_j) \). We write the \( R \)-matrix in the spin basis: \( v_+ = (1, 0) \) and \( v_- = (0, 1) \), where \( v_+ \) corresponds to an up arrow if the edge is vertical and a right arrow if the edge is horizontal, similarly \( v_- \) corresponds to a down arrow if the edge is vertical and a left arrow if the edge is horizontal.

\[ R_{i,j}(x_i, x_j) = \begin{pmatrix} a(x_i, x_j) & 0 & 0 & 0 \\ 0 & b(x_i, x_j) & c(x_i, x_j) & 0 \\ 0 & c(x_i, x_j) & b(x_i, x_j) & 0 \\ 0 & 0 & 0 & a(x_i, x_j) \end{pmatrix}. \]  

(2)

In fact the integrable \( R \)-matrix depends on the ratio of the spectral parameters: \( R_{i,j}(x_j/x_i) = R_{i,j}(x_i, x_j) \). Using the matrix units \( e_{a,b} \) as a basis for the matrices acting in \( \mathbb{C}^2 \) we can write (the indices in the summations in this section take values – and +):

\[ R(x_2/x_1) = \sum_{a,b,c,d} r_{a,b}^{c,d}(x_2/x_1) e_{a,c} \otimes e_{b,d}, \]  

(3)

where the components of the \( R \)-matrix are denoted by \( r_{a,b}^{c,d} \); furthermore we will use their graphical representation figure 3. We will also need the \( \tilde{R} \)-matrix: \( \tilde{R} = PR \), where \( P \) is the permutation matrix:

Figure 2. The domain wall boundary conditions. The parameters \( \zeta_1, \ldots, \zeta_4 \) are associated to the horizontal lines, while the parameters \( z_1, \ldots, z_4 \) are associated to the vertical lines.
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\[ r_{a,b}^{c,d}(x_2/x_1) = x_2 \]

Figure 3. Components \( r_{a,b}^{c,d}(x_2/x_1) \). The blank arrows here define the orientations of the vector spaces. These arrows should not be confused with the arrows in figure 1 which are used to denote the configurations of edges.

\[ r_{a,b}^{c,d}(x_2/x_1) = x_2 \]

Figure 4. Components \( r_{a,b}^{c,d}(x_2/x_1) \).

\[ P = \sum_{a,b} e_{a,b} \otimes e_{b,a}, \quad (4) \]

so we have:

\[ \tilde{R}(x_2/x_1) = \sum_{a,b,c,d} r_{a,b}^{c,d}(x_2/x_1)e_{a,c} \otimes e_{b,d}. \quad (5) \]

Graphically, the components of \( \tilde{R} \) are presented in figure 4. The integrable \( R \)-matrix satisfies the Yang–Baxter equation. Using the schematic notation of the \( \tilde{R} \)-matrix this equation can be drawn as in figure 5. The Yang–Baxter equation corresponding to figure 5 is then written as

\[ \tilde{R}_{i+1}(y, x) \tilde{R}_i(z, x) \tilde{R}_{i+1}(z, y) = \tilde{R}_i(z, y) \tilde{R}_{i+1}(z, x) \tilde{R}_i(y, x), \quad (6) \]

where \( \tilde{R} \)-matrices here are: \( \tilde{R}_i = \tilde{R} \otimes Id \) and \( \tilde{R}_{i+1} = Id \otimes \tilde{R} \). This equation restricts the possible weights of the vertices. The solution reads:

\[ a(x_i, x_j) = \frac{q^2 x_i^2 - x_j^2}{(q^2 - 1)x_i x_j}, \quad b(x_i, x_j) = \frac{q(x_i^2 - x_j^2)}{(q^2 - 1)x_i x_j}, \quad c(x_i, x_j) = 1. \quad (7) \]

In the domain as on figure 2 there are \( N \) horizontal spaces carrying \( N \) parameters \( \zeta_1, \ldots, \zeta_N \) and \( N \) vertical spaces carrying \( N \) parameters \( z_1, \ldots, z_N \). The latter parameters are called inhomogeneities and the model therefore is called the inhomogeneous six-vertex model. From the form of the weights equation (7) we see that the partition function \( Z_{6v} \) is a polynomial in \( \zeta \)'s and \( \zeta \)'s divided by a common denominator that we neglect.

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\[ Z_6 v \]

in what follows. In fact, \( Z_6 v \) is symmetric separately in \( z \)'s and in \( \zeta \)'s. It can be seen by applying the \( R_{i,j} \) matrix to figure 2 and using repeatedly the Yang–Baxter equation.

If the \( R \)-matrix is applied at a position \( i \) from below or above of the domain figure 2 this action will switch two rapidities \( z_i \) and \( z_{i+1} \), if it is applied from the sides \( \zeta_i \) and \( \zeta_{i+1} \) will be switched (see figure 6).

Now we present the computation of the domain wall partition function \( Z_6 v \) for the six-vertex model. Before proceeding let us normalize \( Z_6 v \) as follows:

\[ Z_N(\zeta_1, \ldots, \zeta_N|z_1, \ldots, z_N) = Z_N(\zeta_1, \ldots, \zeta_N|z_1, \ldots, z_N) \prod_{i=1}^m (q^2 - 1)^{m-1} \zeta_i^{m-1} z_i^{m-1}, \]

where we indicated explicitly the system size \( N \) and put \( 6 v \) in the superscript. \( Z_N^{6 v} \) has two recurrence relations that correspond to setting \( \zeta_j = z_i \) and \( \zeta_j = q^{-1} z_i \). For their derivation one can consult [9, 14] or see the explanation of similar recurrences in the case of the nineteen-vertex model in section 4. The recurrence relations are:

\[ Z_N^{6 v}(\zeta_1, \ldots, \zeta_j = z_i, \ldots, \zeta_N|z_1, \ldots, z_N) = f_{i,j} Z_N^{6 v}[\zeta_j, z_i], \]
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\[ Z_N^{6v}(\zeta_1, \ldots, \zeta_j = q^{-1}z_i \ldots, \zeta_N | z_1, \ldots, z_N) = g_{i,j}^N Z_{N-1}^{6v}[\zeta_j, z_1], \]

(9)

where the square brackets indicate which variables are absent from the initial list of variables on the left hand side. The corresponding factors in the two recurrences are

\[ f_{i,j}^N = \prod_{1 \leq k \leq i \leq N} (q^2 z_i^2 - z_k^2) \prod_{1 \leq k < j \leq N} (q^2 \zeta_k^2 - z_i^2), \]

(10)

\[ g_{i,j}^N = \prod_{1 \leq k \leq i \leq N} (z_i^2 - q^2 z_k^2) \prod_{1 \leq k < j \leq N} (\zeta_k^2 - z_i^2). \]

(11)

\[ Z_N^{6v} \text{ is expressed in terms of } Z_{N-1}^{6v} \text{’s as:} \]

\[ Z_N^{6v}(\zeta_1, \ldots, \zeta_N | z_1, \ldots, z_N) = \sum_{k=1}^{N} Z_{N-1}^{6v}[\zeta_N, z_k] \prod_{i=1, i \neq k}^{N} \frac{(\zeta_N^2 - z_i^2)}{(\zeta_k^2 - z_i^2)} f_{i,N}, \]

and

(12)

\[ Z_N^{6v}(\zeta_1, \ldots, \zeta_N | z_1, \ldots, z_N) = \sum_{k=1}^{N} Z_{N-1}^{6v}[\zeta_N, z_k] \prod_{i=1, i \neq k}^{N} \frac{(q^2 \zeta_N^2 - z_i^2)}{(z_k^2 - z_i^2)} g_{i,N}. \]

(13)

The recurrence equation (8) was derived by Korepin and solved by Izergin and the solution is written as the following determinant:

\[ Z_N^{6v} = N' \det_{1 \leq i, j \leq N} \left( \frac{1}{(\zeta_i^2 - z_j^2)(q^2 \zeta_i^2 - z_j^2)} \right), \]

\[ N' = \frac{\prod_{1 \leq i, j \leq N}(\zeta_i^2 - z_j^2)(q^2 \zeta_i^2 - z_j^2)}{\prod_{1 \leq i, j \leq N}(\zeta_i^2 - \zeta_j^2)(z_j^2 - z_i^2)}. \]

(14)

\[ Z_N^{6v} \text{ is a homogenous polynomial in } z_1^2, \ldots, z_N^2, \zeta_1^2, \ldots, \zeta_N^2 \text{ of degree } N(N-1) \text{ and it satisfies} \]

the required recurrence relations together with the initial condition \( Z_1^{6v} = 1. \)

3. Nineteen-vertex model with domain wall boundary

Consider an inhomogeneous nineteen-vertex model on a lattice. States of the model are defined through assigning one of the nineteen configurations to each vertex of the lattice. Each edge of the lattice can be in three states, denoted by arrows or an empty edge, in such a way that the total number of arrows pointing towards a vertex has to be equal to the total number of arrows pointing outwards. This restriction defines the nineteen possible configurations at each vertex, figure 7.

The weights of the nineteen vertices are encoded in the \( R \)-matrix.
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\[
\begin{pmatrix}
    x_1(\zeta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & x_2(\zeta) & 0 & x_5(\zeta) & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x_3(\zeta) & 0 & x_6(\zeta) & 0 & x_7(\zeta) & 0 & 0 \\
    0 & y_5(\zeta) & 0 & x_2(\zeta) & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & y_6(\zeta) & 0 & x_4(\zeta) & 0 & x_6(\zeta) & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & x_2(\zeta) & 0 & x_5(\zeta) & 0 \\
    0 & 0 & y_7(\zeta) & 0 & y_6(\zeta) & 0 & x_3(\zeta) & 0 & 0 \\
    0 & 0 & 0 & 0 & y_5(\zeta) & 0 & x_2(\zeta) & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1(\zeta)
\end{pmatrix}. \tag{15}
\]

For the model under consideration they are defined as follows:

\[
\begin{align*}
    x_1(\zeta) &= (\zeta q^2 - 1)(\zeta q^3 + 1), \\
    x_2(\zeta) &= q(\zeta - 1)(\zeta q^3 + 1), \\
    x_3(\zeta) &= q^2(\zeta - 1)(\zeta q + 1), \\
    x_4(\zeta) &= \zeta^2 q^4 + \zeta(q - 1)(q^4 - q^2 + 1) - q, \\
    x_5(\zeta) &= \sqrt{\zeta}(q^2 - 1)(\zeta q^3 + 1), \\
    x_6(\zeta) &= -\sqrt{q\zeta}(\zeta - 1)(q^2 - 1), \\
    x_7(\zeta) &= \zeta(q^2 - 1)(\zeta q^3 + (\zeta - 1)q + 1), \\
    y_5(\zeta) &= \sqrt{\zeta}(q^2 - 1)(\zeta q^3 + 1), \\
    y_6(\zeta) &= q^2\sqrt{q\zeta}(\zeta - 1)(q^2 - 1), \\
    y_7(\zeta) &= (q^2 - 1)(\zeta q^3 - (\zeta - 1)q^2 + 1). \tag{16}
\end{align*}
\]

Figure 7. The nineteen vertices and their weights.
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The graphical notation is completely analogous to the case of the six-vertex model, figure 3. Below we will work with the $R$-matrix that depends on two parameters $R(z, \zeta) = z^2 R(\zeta/z)$, where $\zeta$ is the horizontal spectral parameter and $z$ is vertical. This $R$-matrix defines the integrable vertex model associated to the quantum group $U_q(A_2^{(2)})$ (see the derivation in [11]). This model was discovered by Izergin and Korepin and thus called the Izergin–Korepin nineteen-vertex model. The corresponding $\tilde{R}$-matrix satisfies the Yang–Baxter equation which is the same as before (6).

We are interested in counting configurations of the following object. Consider the square lattice of size $N$ filled in with the above nineteen vertices in such a way that horizontal boundary arrows are pointing in to the lattice, while the vertical ones pointing outside the lattice. These boundary conditions are called, as before, the domain wall boundary figure 2. The partition function of this object is the sum of all possible configurations with weights defined in equation (16).

\begin{equation}
Z_N = \sum_{\varepsilon \text{states}} \prod_{1 \leq i, j \leq N} w_{i,j}^{(\varepsilon)},
\end{equation}

where $w_{i,j}^{(\varepsilon)}$ is the weight of the vertex sitting at a position $i,j$ of a configuration $\varepsilon$. This partition function is a symmetric polynomial in both horizontal $\zeta_i$ and vertical $z_i$ rapidities. The fact that it is a polynomial comes from the observation that each vertex that has a $\sqrt{\zeta}$ appears necessarily with another vertex that has a $\sqrt{\zeta}$. These weights are: $x_5$, $x_6$, $y_5$ and $y_6$ and they correspond to the vertices which have a ‘turning’ of an empty line. Clearly, the number of such turnings must be even in any DWPF configuration. The fact that $Z_N$ is symmetric can be proved as in the case of the six-vertex model by attaching the $R$-matrix to two horizontal external lines in figure 2 or two vertical external lines and repetitive application of the Yang–Baxter equation. Hence the partition function $Z_N(\zeta_1, ..., \zeta_N, z_1, ..., z_N)$ is a symmetric polynomial in $z_i$’s and $\zeta_i$’s with coefficients being polynomials in $q$ with integer coefficients.

4. Recurrence relation

Before proceeding let us redefine $Z_N$ since it contains a number of trivial factors which we do not want to carry around. The variable $\zeta_i$ enters the partition function through the weights corresponding to the vertices located on the $i$th horizontal line (counting from the top). The top degree in $\zeta_i$ of the weights $x_5$, $x_6$, $y_5$ and $y_6$ is equal to $3/2$ while for all other weights the top degree is equals 2. Hence the top degree of $Z_N$ in $\zeta_i$ must be equal to $2N$ since there is always a configuration (e.g. six-vertex configuration) which does not contain any of the vertices $x_5$, $x_6$, $y_5$ and $y_6$ in the $i$th row. However, each configuration contains the trivial proportionality factor $\zeta_i$ which means that the top degree of the nontrivial part of the partition function is $2N - 1$. Indeed, due to the boundary conditions each horizontal line must contain only an even number of the weights $x_5$, $x_6$, $y_5$ and $y_6$ including zero number of these weights. Each of these weights corresponding to the $i$th line is proportional to $\zeta_i^{1/2}$, hence the weight of the configuration is divisible by $\zeta_i$. The configurations on the $i$th horizontal line containing
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\[ Z_N(\zeta_1, \zeta_N; z_1, \ldots, z_N) = (q^2 - 1)^N \prod_{i=1}^{N} \zeta_i \tilde{Z}_N(\zeta_1, \ldots, \zeta_N; z_1, \ldots, z_N), \]

where the multiplication by \((q^2 - 1)^N\) accounts for another unimportant common factor. To define uniquely this partition function as a polynomial in \(\zeta\) we need to know its values in \(2N\) points which are provided by \(2N\) recurrence relations derived below.

The partition function \(\tilde{Z}_N\) satisfies two recurrence relations in size with the initial condition \(\tilde{Z}_0 = 1^2\). They lead to:

\[ \tilde{Z}_N(\zeta_1, \ldots, \zeta_N|z_1, \ldots, z_N) = \sum_{i=1}^{N} \kappa_i(\zeta_1, \ldots, \zeta_N|z_1, \ldots, z_N) \tilde{Z}_{N-1}(\zeta_1, \ldots, \zeta_{N-1}|z_1, \ldots, z_{i-1}, z_i), \tag{18} \]

with some appropriate functions \(\kappa_i\).

By inspecting the vanishing properties of the weights of the \(R\)-matrix we notice that there are two recurrence relations in size. When we set \(\zeta_j = z_i\) in \(\tilde{Z}_N\) we get:

\[ \tilde{Z}_N(\zeta_1, \ldots, \zeta_j = z_i, \ldots, \zeta_N|z_1, \ldots, z_N) = F_{i,j}^N \tilde{Z}_{N-1}(\zeta_j, z_i). \tag{19} \]

This recurrence has a graphical interpretation shown in figure 8. Indeed, if we look at the north east corner (position \((1, N)\) on the lattice) of the domain, the boundary condition allows only for three vertices. These are vertices with the weights \(x_3, x_7\)

\[ \text{Figure 8. The recurrence relation under substitution } \zeta_i = z_i \text{ for a } 4 \times 4 \text{ lattice.} \]

Since the first row is frozen and the last column is frozen we obtain simple factors of \(x_1\)-weights, while the remaining configuration has the domain wall boundary conditions and corresponds to \(Z_3\).
and $x_6$. After setting $\zeta_1 = z_N$, $x_3$ and $x_6$ vanish, so we are left with the vertex $x_7$. This vertex has a down arrow on its vertical lower edge and a right arrow on its left edge, hence due to the boundary condition at position $(2, N)$ we are forced to put the vertex corresponding to the weight $x_1$ and at position $(1, N - 1)$ the other vertex with the weight $x_1$. In fact, all remaining vertices in the $N$th column are frozen, as well as all the remaining vertices of the first row. These vertices contribute with the products of $x_1$-weights:

$$
\prod_{1 \leq i \leq N-1} x_1(\zeta_i/z_i) \prod_{2 \leq i \leq N} x_1(\zeta_i/z_N)|_{\zeta_i = z_N}.
$$

A different recurrence appears when we set $\zeta_j$ to $-q^{-3}z_i$ in $\tilde{Z}_N$:

$$
\tilde{Z}_N(\zeta_1, \ldots, \zeta_j = -q^{-3}z_i, \ldots, \zeta_N, z_1, \ldots, z_N) = G_{i,j}^N \tilde{Z}_{N-1}[\zeta_j, z_i].
$$

The graphical explanation of this recurrence is similar to the previous recurrence. One must consider the top left corner of our domain and observe that only one vertex does not vanish under the substitution $\zeta_j = -q^{-3}z_i$. The first row and the first column freeze, while the rest returns the domain wall boundary condition for the domain of size $N - 1 \times N - 1$.

The $F$ and the $G$ are given by:

$$
F_{i,j}^N = (q^3 + 1)z_i \prod_{1 \leq k \leq i \leq N} (q^2z_i - z_k)(q^3z_i + z_k) \prod_{1 \leq k = j \leq N} (q^2\zeta_k - z_i)(q^3\zeta_k + z_i),
$$

$$
G_{i,j}^N = -q^{-N-1}(q^3 + 1)z_i \prod_{1 \leq k = i \leq N} (z_i - q^2z_k)(z_i + q^3z_k) \prod_{1 \leq k = j \leq N} (\zeta_k - z_i)(q\zeta_k + z_i).
$$

If we know $\tilde{Z}_{N-1}$ these two recurrence relations allow us to determine $\tilde{Z}_N$. We can consider $\tilde{Z}_N$ as a polynomial in $\zeta_N$ of degree $2N - 1$ with $2N$ coefficients. Since we know the values of $\tilde{Z}_N$ at $N$ points $\zeta_N = z_i$ (equation (19) with $j = N$) and at another $N$ points $\zeta_N = -q^{-3}z_i$ (equation (21) with $j = N$), therefore we can determine all the coefficients of $\tilde{Z}_N$ in its expansion in $\zeta_N$. Using the Lagrange polynomial we can write $\tilde{Z}_N$ as a sum of $\tilde{Z}_{N-1}$'s as follows:

$$
\tilde{Z}_N(\zeta_1, \ldots, \zeta_N| z_1, \ldots, z_N) = \sum_{k=1}^{N} \tilde{Z}_{N-1}[\zeta_N, z_k] \frac{\prod_{i=1, i \neq k}^{N}(\zeta_N - z_i)(\zeta_N + q^{-3}z_i)}{\prod_{i=1}^{N}(z_k - z_i)} \times \left( \frac{F_{k,N}^N}{(\zeta_N - z_k)\prod_{i=1}^{N}(z_k + q^{-3}z_i)} - \frac{q^{3(N-1)}G_{k,N}^N}{(\zeta_N + q^{-3}z_k)\prod_{i=1}^{N}(q^{-3}z_k + z_i)} \right).
$$

This is of course a polynomial because the denominators are canceled by the common prefactor and by the $F$ and $G$ respectively including the factors of $q^3 - 1$ in the definitions equations (22) and (23). Using this we write

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\[ \tilde{Z}_N(\zeta_1, \ldots, \zeta_1 z_1, \ldots, z_N) = q^{3(N-1)} \sum_{k=1}^{N} \tilde{Z}_{N-1}[\zeta_N, z_k] \prod_{i=1, i \neq k}^{N} \frac{(\zeta_N - z_i)(\zeta_N + q^{-3} z_i)}{(z_k - z_i)} \times \left( (q^3 \zeta_N + z_k) \prod_{i=k}^{N} (q^2 z_k - z_i) \prod_{1 \leq i \leq N-1} (-z_k + q^2 \zeta_i)(z_k + q^3 \zeta_i) \right. \]

\[ + q^{2(N-1)}(\zeta_N - z_k) \prod_{i=k}^{N} (z_k - q^2 z_i) \prod_{1 \leq i \leq N-1} (-z_k + \zeta_i)(z_k + q \zeta_i) \right). \] (25)

Possibly there is a way to write \( \tilde{Z}_N \) for generic \( q \) as a single determinant, for now this remains an open question. In the next section we show how to solve the recurrence relation for \( \tilde{Z}_N \) when \( q = e^{i \pi / 3} \).

5. Solution for a cubic root of unity

In this section we will assume \( q = e^{i \pi / 3} \). The recurrence relation equation (25) simplifies in this case. Upon setting \( q = e^{i \pi / 3} \) (which implies \( q^3 = -1 \) and \( q + q^{-1} = 1 \)) the weights of the \( R \)-matrix (16) have a common factor of \( \zeta - 1 \) hence \( Z_N \) factors out the product:

\[ \prod_{1 \leq i, j \leq N} (z_i - \zeta_j). \] (26)

We define \( \tilde{Z}_N \) which has a more suitable normalization

\[ Z_N(\zeta_1, \ldots, \zeta_N; z_1, \ldots, z_N) = -q^2 \prod_{1 \leq i, j \leq N} (z_i - \zeta_j) \tilde{Z}_N(\zeta_1, \ldots, \zeta_N; z_1, \ldots, z_N). \]

The initial condition becomes \( \tilde{Z}_1 = 1 \) and out of the two recurrence points only one remains. Setting \( q = e^{i \pi / 3} \) in equation (25) we find

\[ \tilde{Z}_N(\zeta_1, \ldots, \zeta_1 z_1, \ldots, z_N) = (-1)^{N-1} \sum_{k=1}^{N} \tilde{Z}_{N-1}[\zeta_N, z_k] \prod_{i=1, i \neq k}^{N} \frac{(\zeta_N - z_i)^2}{(z_k - z_i)} \times \prod_{i=1}^{N} (z_k - \zeta_i) \times \left( \prod_{i=k}^{N} (q^2 z_k - z_i) \prod_{1 \leq i \leq N-1} (-z_k + q^2 \zeta_i) + (-1)^N q^{2N-1} \prod_{i=k}^{N} (z_k - q^2 z_i) \prod_{1 \leq i \leq N-1} (z_k + q \zeta_i) \right). \]

Rewriting this in terms of \( \tilde{Z}_N \), after cancellations we get

\[ \tilde{Z}_N(\zeta_1, \ldots, \zeta_1 z_1, \ldots, z_N) = \sum_{k=1}^{N} \tilde{Z}_{N-1}[\zeta_N, z_k] \prod_{i=1, i \neq k}^{N} \frac{(\zeta_N - z_i)}{(z_k - z_i)} \times \left( \prod_{i=k}^{N} (q^2 z_k - z_i) \prod_{1 \leq i \leq N-1} (-z_k + q^2 \zeta_i) + (-1)^N q^{2N-1} \prod_{i=k}^{N} (z_k - q^2 z_i) \prod_{1 \leq i \leq N-1} (z_k + q \zeta_i) \right). \]

Set here, say \( \zeta_N = z_N \), and put \( z_N = x \). In the right hand side only one term survives in the summation over \( k \).
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\[ Z_N(\zeta_1, \ldots, \zeta_{N-1}, x|z_1, \ldots, z_{N-1}, x) = \tilde{Z}_{N-1}(\zeta_1, \ldots, \zeta_{N-1}|z_1, \ldots, z_{N-1}) \]

\[
\begin{align*}
&= \tilde{Z}_{N-1}(\zeta_1, \ldots, \zeta_{N-1}|z_1, \ldots, z_{N-1})(-1)^N q^{2N-1} \\
&= \tilde{Z}_{N-1}(\zeta_1, \ldots, \zeta_{N-1}|z_1, \ldots, z_{N-1})(-1)^N q^{2N} \\
&\quad \cdot \left( \prod_{1 \leq i < N} (x + z_i/q)(x + \zeta_i) + \prod_{1 \leq i < N} (z_i + x/q)(x + \zeta_i) \right)
\end{align*}
\]

where we used the fact that \( q = e^{i\pi/3} \). By the symmetry we can interchange \( \zeta_N \) with \( \zeta_j \) therefore we find

\[ Z_N(\zeta_1, \ldots, \zeta_j = z_i, \ldots, \zeta_N|z_1, \ldots, z_N) = P_{i,j} \tilde{Z}_{N-1}[\zeta_j, z_i]. \]  

Let us focus on the polynomial \( P_{i,j} \) and for convenience we specify \( i = N, j = N \) and set \( z_N = x \). The polynomial \( P_{N, N} = P(x|\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1}) \) is a symmetric polynomial in \( \zeta_1, \ldots, \zeta_{N-1} \) and separately in \( z_1, \ldots, z_{N-1} \)

\[ P(x|\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1}) = (-1)^N q^{2N} \left( \prod_{1 \leq i < N} (\zeta_i + qx) \prod_{i=1}^{N-1} (z_i + x/q) + \frac{1}{q} \prod_{i=1}^{N-1} (\zeta_i + x/q) \prod_{i=1}^{N-1} (z_i + qx) \right). \]  

Note, up to the overall factor of \( q^{2N} \), \( P \) is invariant under \( q \to 1/q \), which means it has to be a function of \( q^2 + q^{-1} \). When \( q^3 = -1 \) the combinations \( q^2 + q^{-1} \) are integers, hence \( P \) becomes a polynomial with purely integer coefficients. The same is therefore also true for \( \tilde{Z}_N \) itself. Let us consider now \( P \) as the generating function for some symmetric polynomials:

\[ P_N(x) = P(x|\zeta_1, \ldots, \zeta_N, z_1, \ldots, z_N) = (-1)^N q^{2N} \sum_{i=0}^{2N} x^i \Delta_{2N-i,N}(\zeta_1, \ldots, \zeta_N, z_1, \ldots, z_N). \]

We included here the factor of \( q^N \) in order to make \( \Delta_{i,N} \) \( q \)-independent. The polynomials \( \Delta_{i,N} \) are polynomials of \( 2N \) variables with the total degree \( i \). If \( i < 0 \) or \( i > 2N \) we set it equal to 0, and also \( \Delta_{0,N} = 1 \). Here is the example for \( N = 2 \):

\[
\begin{align*}
\Delta_{1,2} &= 2\zeta_1 + 2\zeta_2 - z_1 - z_2, \\
\Delta_{2,2} &= \zeta_1\zeta_2 + \zeta_1z_2 + \zeta_2z_1 + \zeta_2z_1 - 2z_1z_2, \\
\Delta_{3,2} &= -\zeta_1z_2 + 2\zeta_2\zeta_1z_1 + 2\zeta_2\z_1z_2 - \zeta_2z_1z_2, \\
\Delta_{4,2} &= \zeta_1\zeta_2z_1z_2.
\end{align*}
\]

These symmetric functions have a few nice properties which we will discuss in the next section. The solution of the recurrence relation (27) is the main result of our paper, it reads:

\[ \text{doi:10.1088/1742-5468/2016/03/033112} \]

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\[ Z_N(\zeta_1, \ldots, \zeta_N, z_1, \ldots, z_N) = \det_{1 \leq i, j \leq N-1} \Delta_{3j-i,N}(\zeta_1, \ldots, \zeta_N, z_1, \ldots, z_N), \quad (30) \]

or equivalently

\[ Z_N = \det_{1 \leq i, j \leq N-1} \sum_{i=0}^{N} (q^{-3(j-i)+2l+1} + q^{3j-i-2l-1}) E_i(z_1, \ldots, z_N) E_{3j-i-l}(\zeta_1, \ldots, \zeta_N), \]

where \( E_i(x_1, \ldots, x_N) \) are the elementary symmetric polynomials defined by \( E_i(x_1, \ldots, x_N) = 0 \) if \( i < 0 \) or \( i > N \) and otherwise

\[ E_i(x_1, \ldots, x_N) = \sum_{1 \leq n_1 < \ldots < n_i \leq N} x_{n_1}x_{n_2} \ldots x_{n_i}. \quad (31) \]

The proof of this result follows next.

6. Proof

Let us list few properties of \( \Delta_{i,N} \). First of all, looking at the definition of these polynomials (29) we can immediately express them through the elementary symmetric polynomials \( E_i \). The elementary symmetric polynomials can be defined via their generating function

\[ Q(x; z_1, \ldots, z_n) = \prod_{i=1}^{N} (x - z_i), \quad (32) \]

\[ Q(x) = \sum_{n=0}^{N} (-1)^{N-n} x^n E_{N-n}(z_1, \ldots, z_N). \quad (33) \]

We can rewrite (28) using \( Q(x) \) and \( Q(x) \)

\[ P(x; \zeta_1, \ldots, \zeta_N, z_1, \ldots, z_N) = (-q^2)^N (qQ(x)Q(x/q) + q^{-1}Q(x/q)Q(xq)). \]

Substituting equation (33), expanding in \( x \), collecting the coefficients and comparing them to (29) we get

\[ \Delta_{2N-1,N} = \sum_{1 \leq n_1, n_2 \leq N} (q^{1-n_1+n_2} + q^{1+n_1-n_2}) E_{N-n_1}(\zeta_1, \ldots, \zeta_N) E_{N-n_2}(z_1, \ldots, z_N). \]

The summation can be simplified and we obtain

\[ \Delta_{i,N} = \sum_{l=0}^{N} (q^{-s+2l+1} + q^{s-2l-1}) E_l(z_1, \ldots, z_N) E_{s-l}(\zeta_1, \ldots, \zeta_N). \quad (34) \]

Note that equation (34) is valid for generic values of \( q \). When \( q = 1 \), \( \Delta_i \) becomes the elementary symmetric polynomials of the union of \( z \)’s and \( \zeta \)’s times a factor of two. So, it can be considered as a type of \( q \)-deformation of the elementary symmetric polynomials.
Next, we set \( \zeta_N = z_N \) (without loss of generality). From the definition of \( P_N \) we see that it produces back \( P_{N-1} \):

\[
P(x|\zeta_1, \ldots, \zeta_N, z_1, \ldots, z_N)\big|_{N=z_N} = -(z_N q + x)(z_N + qx)P(x|\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1})
\]
\[
= -q(z_N^2 + xz_N + x^2)P(x|\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1}),
\]

where in the second line we took into account that \( q + q^{-1} = 1 \). Looking at equation (35) we can relate the set of \( \Delta_{i,N} \)'s in which \( \zeta_N = z_N \) to the set of \( \Delta_{i,N-1} \)'s:

\[
\Delta_{i,N}(\zeta_1, \ldots, \zeta_N) = \Delta_{i,N-1}(\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1})
\]
\[
+ z_N \Delta_{i-1,N-1}(\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1}) + z_N^2 \Delta_{i-2,N-1}(\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1}).
\]

Using this equation and a certain row–column manipulation in the matrix \( \Delta_{3j-i,N} \) we are going to show that the determinant (30) satisfies the recurrence (27).

Set \( \zeta_N = z_N \) and substitute equation (36) in every entry of the matrix in equation (30). Starting from the first row subtract from each row \( i \) row \( i + 1 \) multiplied by \( x \). Next, subtract from each column \( j \) column \( j + 1 \) multiplied by \( z_N \) starting from the \( j = (N-2) \)th column. In the resulting matrix all elements of the first column become zero except from the bottom element. The bottom element in the first column takes the form of equation (29), while the rest of the matrix is equal to \( \Delta_{3j-i,N} \) of size \( N - 1 \), and the last row is unimportant upon taking the determinant. The row–column manipulation above corresponds to the following series of equations. Upon application of the recurrence relation each entry becomes:

\[
\Delta_{3j-i,N-1} + z_N \Delta_{3j-i-1,N-1} + z_N^2 \Delta_{3j-i-2,N-1}.
\]

After the first row manipulation the last row remains as before:

\[
\Delta_{3j-N+1,N-1} + z_N \Delta_{3j-N,N-1} + z_N^2 \Delta_{3j-N-1,N-1},
\]

the rest of the matrix becomes:

\[
\Delta_{3j-i,N-1} - z_N^3 \Delta_{3j-i-3,N-1}.
\]

We notice that in the last column the first of these two terms vanish \( \Delta_{3(N-1)-i,N-1} \) for all \( i < N - 1 \), while in the first column the second term vanishes. Next, we use the last column to eliminate the unwanted terms in other entries of the matrix (except from the last row). After this, the first column except for its last element will vanish, while the last element will be:

\[
\sum_{j=1}^{N-1} z_N^{3(N-1-j)}(\Delta_{3j-N+1,N-1} + z_N \Delta_{3j-N,N-1} + z_N^2 \Delta_{3j-N-1,N-1})
\]
\[
= q^{2(N-1)}P(z_N|\zeta_1, \ldots, \zeta_{N-1}, z_1, \ldots, z_{N-1}).
\]

This completes the proof. We can alternatively view this row–column manipulation as acting on the left and on the right of equation (37) with certain matrices with unit determinant. Let us call the expression in equation (37) \( \tilde{\Delta}_{3j-i,N-1} \), and define two matrices:

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\[
A = \begin{pmatrix}
1 & -z & 0 & \ldots & 0 \\
0 & 1 & -z & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -z \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]
and
\[
B = \begin{pmatrix}
1 & z^3 & z^6 & \ldots & z^{3(N-1)} \\
0 & 1 & z^3 & \ldots & z^{3(N-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & z^3 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

We have:
\[
\det_{1 \leq k, l \leq N-1} A_{i,j} \Delta_{i-j, N-1} B_{k,l} = q^{2(N-1)} \prod_{i \leq k, l \leq N-2} \Delta_{i-j, N-1}.
\]

7. Conclusion

As we mentioned in the introduction, our study is motivated by the six-vertex model. Hence, it is natural to look at other related objects which were computed for the six-vertex model. Since the nineteen-vertex model seems to have a more complicated structure, one probably should not expect to obtain nice answers as in the six-vertex case. As we have observed, however, when \( q \) is a root of unity the nineteen-vertex model becomes ‘computable’.

Here we considered the domain wall boundary conditions for the nineteen-vertex model of Izergin and Korepin. An interesting extension of our computation would be to consider other boundary conditions, i.e. to use reflection matrices on one or two sides of the \( N \times N \) domain. In the case of the six-vertex model the corresponding partition functions are known to be determinants or Pfaffians (see [24] and also [17]). One would need to find first the recurrence relation for the partition function and then after setting \( q = e^{i\pi/3} \) it should be possible to obtain a determinantal expression. We note here that similar determinants appear in the study of the related loop model exactly when \( q = e^{i\pi/3} \). The loop model related to the IK model is called the dilute Temperley–Lieb (dTL\((n))\) loop model [21, 22]. This model has a parameter: the weight \( n \) of a loop. When \( q = e^{i\pi/3} \) this corresponds to \( n = 1 \) and the corresponding loop model is related to interesting statistical models like critical percolation, for example. In [8] it was shown that the partition function of the dTL\((1)\) model satisfies a similar recurrence as in equations (27) and (28), and has a solution similar to equation (30). Recurrence relations of the form equation (27) also appear in the study of the three-state Potts model [12].

In the context of the algebraic Bethe Ansatz the domain wall partition function for the six-vertex model represents the highest spin eigenvector of the corresponding transfer matrix with periodic boundary conditions. The parameters \( \zeta_i \) become the Bethe roots. This object is essential in the study of correlation functions of the corresponding
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model. One may similarly look at the highest spin eigenvector of the transfer matrix for the IK model given by the corresponding algebraic Bethe Ansatz (see [23] and also [19]). However, for the nineteen-vertex model the eigenvectors of the transfer matrix are much more complicated than in the case of the six-vertex model. For example, to compute the highest spin eigenvector we need to consider the nineteen-vertex model with many different boundary conditions on rectangular domains. The expression for this eigenvector for $N = 4$ pictorially is shown in figure 9. In general, the expression for this eigenvector looks very complicated. For a root of minus one $q^3 = -1$ we know few terms here, i.e. those corresponding to the domain wall boundaries. The other terms should not be expected to have a nice closed form since they, in general, are not symmetric in $\zeta$’s nor in $z$’s. The eigenvector as a whole is symmetric in $\zeta$’s and $z$’s. One then needs to find a recurrence relation for it and then, if lucky, it will be possible to find its closed form solution at $q = e^{i\pi/3}$. The knowledge of this will be helpful in understanding the other eigenvectors. In particular, we could look at the zero spin eigenvectors at $q = e^{i\pi/3}$ (i.e. when lower boundary has an equal number of up and down arrows). One such eigenvector was computed in the loop basis [7] by means of the quantum Knizhnik–Zamolodchikov equations.

Let us look carefully at the determinant expression in (30)

$$Z_N = \det \sum_{1 \leq i,j \leq N-1} \sum_{l=0}^{N} (q^{(3j-i)+2l+1} + q^{3j-i-2l-1})E_l(z_1, \ldots, z_N)E_{3j-i-l}(\zeta_1, \ldots, \zeta_N).$$
This is very close to the Lascoux determinant \[18\] for the six-vertex domain wall partition function. In particular, it can be rewritten as a determinant of a product of two rectangular matrices which can be computed applying the Cauchy–Binet formula. In this way it was shown in \[6\] that the six-vertex IK determinant is a Kadomtsev–Petviashvili (KP) tau-function. Therefore, it would be natural to ask if the determinant \((30)\) is a tau-function.

Finally, regarding the generic \(q\) expression for \(Z_N\) partition function one could try to look for its expansion in terms of symmetric polynomials. For example, it is known that \(Z_{6v}\) expands naturally in the Hall–Littlewood polynomials \([2, 3, 25]\).

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References

[1] Baxter R J 1982 Exactly Solved Models in Statistical Mechanics (London: Academic Press/Harcourt Brace Jovanovich)
[2] Betea D and Wheeler M 2014 Refined Cauchy and Littlewood identities, plane partitions and symmetry classes of alternating sign matrices arXiv:1402.0229
[3] Betea D, Wheeler M and Zinn-Justin P 2014 Refined Cauchy/Littlewood identities and six-vertex model partition functions: II. Proofs and new conjectures arXiv:1405.7035
[4] Caradoc A, Foda O and Kitanine N 2006 Higher spin vertex models with domain wall boundary conditions J. Stat. Mech. 2006 P03012
[5] Dodd R K and Bullough R K 1977 Polynomial conserved densities for the sine-Gordon equations Proc. R. Soc. A 352 481–503
[6] Foda O, Wheeler M and Zuparic M 2009 Domain wall partition functions and KP J. Stat. Mech. 2009 P03017
[7] Garbali A and Nienhuis B 2014 The dilute Temperley-Lieb \(O(n=1)\) loop model on a semi infinite strip: the ground state arXiv:1411.7020
[8] Garbali A and Nienhuis B 2014 The dilute Temperley-Lieb \(O(n=1)\) loop model on a semi infinite strip: the sum rule arXiv:1411.7160
[9] Izergin A G 1987 Partition function of the six-vertex model in a finite volume Dokl. Akad. Nauk SSSR 297 331–3
[10] Izergin A G and Korepin V E 1981 The inverse scattering method approach to the quantum Shabat-Mikhailov model Commun. Math. Phys. 79 303–16
[11] Khoroshkin S M and Tolstoy V N 1992 The uniqueness theorem for the universal \(R\)-matrix Lett. Math. Phys. 24 231–44
[12] Kirillov A N and Smirnov F A 1988 Local fields in scaling field theory associated with 3-state potts model (in Russian). Preprint ITF-88-75P Inst. Theor. Fiz. AN USSR
[13] Kitanine N, Maillet J M and Terras V 1999 Form factors of the XXZ Heisenberg spin-\(\frac{1}{2}\) finite chain Nucl. Phys. B 554 647–78
[14] Korepin V A 1982 Calculation of norms of Bethe wave functions Commun. Math. Phys. 86 391–418
[15] Korepin V E, Bogoliubov N M and Izergin A G 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[16] Kuperberg G 1996 Another proof of the alternating-sign matrix conjecture Int. Math. Res. Not. 1996 139–50
[17] Kuperberg G 2002 Symmetry classes of alternating-sign matrices under one roof Ann. Math. 156 835–66
[18] Lascoux A 1999 Square ice enumeration Seminaire Lotharingien de Combinatoire 42
[19] Lima-Santos A 1999 Bethe ansätze for nineteen vertex models arXiv:hep-th/9807219
[20] Mikhailov A V 1979 Pis’ma Zh. Eksp. Teor. Fiz 30 443
The domain wall partition function for the Izergin–Korepin nineteen-vertex model at a root of unity

[21] Nienhuis B 1990 Critical and multicritical O(\(n\)) models Physica A 163 152–7
[22] Nienhuis B 1990 Critical spin-1 vertex models and O(\(n\)) models Int. J. Mod. Phys. B 4 929–42
[23] Tarasov V O 1988 Algebraic Bethe ansatz for the Izergin-Korepin \(R\)-matrix Theor. Math. Phys. 76 793–803
[24] Tsuchiya O 1998 Determinant formula for the six vertex model with reflecting end J. Math. Phys. 39 5946–51
[25] Warnaar S O 2009 Bisymmetric functions, Macdonald polynomials and \(sl_3\) basic hypergeometric series Compos. Math. 144 2789–804
[26] Zhiber A V and Shabat A B 1979 Klein-Gordon equations with a nontrivial group Dokl. Akad. Nauk SSSR 247 1103 (in Russian)