Subspace modelling for structured noise suppression

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August 7, 2009

Abstract
The problem of structured noise suppression is addressed by i)modelling the subspaces
hosting the components of the signal conveying the information and ii)applying a non-
extensive nonlinear technique for effecting the right separation. Although the approach is
applicable to all situations satisfying the hypothesis of the proposed framework, this work
is motivated by a particular scenario, namely, the cancellation of low frequency noise in
broadband seismic signals.

1 Introduction
The problem of structured noise suppression concerns the elimination of signal components
produced by phenomena interfering with the observations of interest. This problem can be
addressed by linear techniques provided that the subspaces hosting the signal components are
complementary and well separated [1–4]. More precisely, if a signal represented by the ket
|f⟩ is produced as the superposition of two components |f1⟩ ∈ S1 and |f2⟩ ∈ S2, provided

∗This work was initiated during Z. Xu stay at the Mathematics Division of Aston University, UK
that \( S_1 \cap S_2 = \{0\} \), the components of the superposition \(|f\rangle = |f_1\rangle + |f_2\rangle\) can be separated by an oblique projection. Even when this condition is theoretically fulfilled, if the subspaces \( S_1 \) and \( S_2 \) are not well separated, the concomitant linear problem for extracting one of the signal components may be ill posed, which causes the failure to correctly split the signal by a linear operation. Hence, nonlinear techniques for determining a subspace \( \mathcal{V} \subset S_1 \), such that \(|f_1\rangle \in \mathcal{V}\), and the projection onto \( \mathcal{V} \) along \( S_2 \) is well posed, have been considered [3–5]. In those publications the theoretically complementary subspaces \( S_1 \) and \( S_2 \) are assumed to be known. Nevertheless, the condition \( S_1 \cap S_2 = \{0\} \) is strong and the possibility of meeting it depends on the ability to generate the right model for the subspaces. Unfortunately, the modelling of the complementary subspaces by pure physical considerations is not always possible and one needs to rely on more general mathematical modelling.

Although the technique for subspace modelling we introduce here is applicable to different situations, the work is motivated by a particular problem relevant to the processing of seismic signals. In the nearshore these signals may be affected by a population of low-frequency waves called infragravity waves [6]. This type of noise may be also unavoidable in bottom broadband seismic observations [7]. The interested reader is referred to [8] for explanations on how infragravity waves are generated. We restrict our consideration to the problem of reducing that type of structured low frequency noise from broadband seismic signals.

Our purpose is twofold. We aim at i) mathematically modelling the subspaces to represent the signal components ii) provide a sparse enough representation of the signals so as to make sure that the correct splitting can be realized.

Under the hypothesis that one of the signals components lies in the subspace of low frequency signals, we determine the subspace of the other component in an adaptive manner. We assume that such a component belongs to an unknown spline space and determine the knots characterizing the space by taking into account the curvature points of the signal in hand. In that sense, the space is ‘adapted’ to the particular signal being analyzed. In line with [4] we tackle the problem of finding the representation of this component through the minimization of the \( q \)-norm like quantity, which is closely related to the non-extensive entropy introduced as ingredient of a thermodynamic framework in the seminal paper by Tsallis [9] and ever since broadly applied in physics [9–16] and other disciplines [17].

The paper is organized as follows: In Section 2 we address the problem of subspaces modelling and discuss the non-extensive nonlinear technique yielding the right signal splitting. A numerical simulation concerning the filtering of low frequency noise from a seismic signal is presented in Section 3. The conclusions are drawn in Section 4.

2 Adaptive subspace modelling for structured noise filtering

As already mentioned, we are concerned with the problem of separating from a signal those components which are not relevant to the phenomenon of interest. For simplicity we consider that a signal \(|f\rangle\) is the superposition of only two components and the goal is to find a suitable model for the subspaces hosting such component. Since our work is motivated by the specific problem of filtering low frequency noise from a seismic signal, we further assume that the subspace representing that type of structured noise is spanned by a few Fourier functions. In accordance with previous works we denote such a subspace as \( \mathcal{W}^\perp \) and consider it to be fixed. The composed signal is the superposition \(|f\rangle = |f_V\rangle + |f_W^\perp\rangle\) with \(|f_V\rangle \in \mathcal{V}\). The goal is to
model the subspace \( V \) fulfilling the theoretical condition \( V \cap W^\perp = \{0\} \), regardless of the fact that the two subspaces may be too close to each other for the signal separation to be obtained via a linear approach. We allow for this difficulty by introducing the additional hypothesis that \( |f_V\rangle \) is well represented in a subspace of \( V \), which is tantamount to assuming that \( |f_V\rangle \) has a ‘sparse enough’ representation in \( V \). We further assume that \( |f_V\rangle \) is well approximated in a dedicated spline space to be constructed as described in the next section.

2.1 Finding the appropriate spline space

Let us start by stating the few definitions on spline spaces which are needed for setting up our mathematical framework. For a complete treatment of splines we refer to the fundamental books [18–20].

**Definition 1.** Given a finite closed interval \([c, d]\) we define a partition of \([c, d]\) as the finite set of points

\[
\Delta := \{x_i\}_{i=0}^{N+1}, \quad N \in \mathbb{N}, \text{ such that } c = x_0 < x_1 < \cdots < x_N < x_{N+1} = d.
\]

We further define \( N \) subintervals \( I_i, i = 0, \ldots, N \) as:

\[
I_i = [x_i, x_{i+1}), \quad i = 0, \ldots, N - 1 \text{ and } I_N = [x_N, x_{N+1}].
\]

**Definition 2.** Let \( \Pi_m \) be the space of polynomials of degree smaller or equal to \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( m \) be a positive integer and define

\[
S_m(\Delta) = \{f \in C^{m-2}[c, d] : f|_{I_i} \in \Pi_{m-1}, i = 0, \ldots, N\},
\]

where \( f|_{I_i} \) indicates the restriction of the function \( f \) on the interval \( I_i \).

An extended partition with single inner knots associated with \( S_m(\Delta) \) is a set \( \tilde{\Delta} = \{y_i\}_{i=1}^{2m+N+1} \) such that

\[
y_{m+i} = x_i, \quad i = 1, \ldots, N, \quad x_1 < \cdots < x_N
\]

and the first and last \( m \) points \( y_1 \leq \cdots \leq y_m \leq c, \quad d \leq y_{m+N+1} \leq \cdots \leq y_{2m+N} \) can be arbitrarily chosen. With each fixed extended partition \( \tilde{\Delta} \) there is associated a unique B-spline basis for \( S_m(\Delta) \), that we denote as \( \{B_{m,j}\}_{j=1}^{m+N} \). The B-spline \( B_{m,j} \) can be defined by the recursive formulae [18]:

\[
B_{1,j}(x) = \begin{cases} 
1, & t_j \leq x < t_{j+1}, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
B_{m,j}(x) = \frac{x - y_j}{y_{j+m-1} - y_j}B_{m-1,j}(x) + \frac{y_{j+m} - x}{y_{j+m} - y_{j+1}}B_{m-1,j+1}(x).
\]

For each order, \( m \), the corresponding spline space is determined by the number and position of the knots. In Fig 1 we show B-spline basis for two different cubic spline spaces \( (m = 4) \) having the same number of knots but located at different positions.

In order to find the appropriate partition \( \Delta \) giving rise to the appropriate spline space \( S_m(\Delta) \) to represent a given signal \( f \), we first determine the critical points of the signal’s curvature, i.e., we find the set \( T \) defined as

\[
T := \{t : \left(\frac{f''}{(1-f'^2)^{3/2}}\right)'(t) = 0\}.
\]
The entries in $T$ are chosen as the initial knots of $\Delta$. Extra knots are obtained by subdivisions between consecutive knots in $T$ so as to generate a partition $\Delta$ with the desired number of knots. An algorithm for implementing this procedure on a signal given as a discrete piece of data is outlined in [21]. We would like to be able to use this procedure on the signal we need to represent, namely the component $|f_V\rangle$, but, of course we do not have access to this signal; our goal is to find it! Thus, in line with [3, 4] we proceed as explained below.

Since in our framework $W^\perp$ is fixed and known, we can construct the orthogonal projector onto $W = (W^\perp)^\perp$ that we denote $\hat{P}_W$. Assuming now that $V = S_m(\Delta)$, for some order $m$ and some partition $\Delta$, we can use B-splines to span the space so that for $|f_V\rangle \in S_m(\Delta)$ we have

$$f_V(x) = \langle x | f_V \rangle = \sum_{i=1}^{M} c_i \langle x | B_{i,m} \rangle = \sum_{i=1}^{M} c_i B_{i,m}(x). \quad (4)$$

Hence, by applying the projector $\hat{P}_W$ on both sides of (4) we further have

$$|f_W\rangle = \sum_{i=1}^{M} c_i |u_i\rangle, \quad \text{where} \quad |u_i\rangle = \hat{P}_W |B_{i,m}\rangle, \quad \text{and} \quad |f_W\rangle = \hat{P}_W |f_V\rangle. \quad (5)$$

Denoting by $\hat{I}_S$ the identity operator in $S = V + W^\perp$, the projector $\hat{P}_W$ is obtainable from the relation $\hat{P}_W = \hat{I}_S - \hat{P}_{W^\perp}$. Therefore the component $|f_W\rangle$ is available and can be used to determine the knots of the spline space to represent it.

Let us suppose then that through the curvature function (3) we obtain a suitable partition for the space to represent $|f_W\rangle$. Now, in order to obtain the component $|f_V\rangle$ from $|f\rangle$ in the most usual case involving subspaces $V$ and $W^\perp$ close to each other, we need to find the representation of $|f_W\rangle$ in a subspace of $W = \text{span}\{|u_i\rangle\}_{i=1}^{M}$. The approach for achieving such an aim is discussed in the next section.

2.2 Determination of the signal representation through a nonlinear non-extensive approach

At this point we can assume that we know the spanning set for $W$ so that henceforth the problem is reduced to finding the representation (5) with sparse coefficients. For this we could
apply the approach proposed in [4], which entails to use the normal equations

\[ \langle u_n|f_W \rangle = \sum_{i=1}^{M} c_i \langle u_n|u_i \rangle, \quad n = 1 \ldots, M. \]  

(6)
as constraints for the minimization of the \( q \)-norm like quantity \( \sum_{i=1}^{M} |c_i|^q \), \( 0 < q \leq 1 \) but incorporating the equations in a stepwise manner. However, in the case motivating this work the number of necessary constraints is large enough to make the whole process slow. Hence rather than using the approach proposed in [4] we take an alternative route and apply a regularized version of the FOCUSS algorithm, which implies to minimize the functional

\[ L = \sum_{i=1}^{M} |c_i|^q + \lambda \|f_W\| - \sum_{i=1}^{M} c_i |u_i\|^2, \]

where \( \lambda \) is a regularization parameter. The algorithm for implementing the approach is based on re-weighted least squares and is given in [22]. It comprises the following simple steps

1) For each fixed \( q \) set a value for \( \lambda \) and a value for the initial vector \( |c^0\rangle = \sum_{i=1}^{M} c_i^0 |i\rangle \).

2) At each iteration, say iteration \( k \), define the operators

\[ \hat{B} = \sum_{i=1}^{M} |B_{i,m}\rangle \langle i|, \quad \hat{\Pi}_k = \sum_{i=1}^{M} \langle i|x_i^{k-1}|^{1-q/2} \langle i|, \quad \hat{B}_k = \hat{B} \hat{\Pi}_k, \]

where the representation \( \langle x|B_{i,m} \rangle i = 1 \ldots, M \) of the kets \( |B_{i,m}\rangle i = 1 \ldots, M \) are B-spline functions \( \langle x|B_{i,m} \rangle i = 1 \ldots, M \) of order \( m \).

3) Compute \( |c^k\rangle \) as

\[ |c^k\rangle = \hat{\Pi}_k (\hat{B}^* \hat{B}_k + \lambda \hat{I})^{-1} |f_W\rangle, \]

where \( \hat{B}^* \) indicates the adjoint of \( \hat{B}_k \) and \( \hat{I} \) the identity operator.

4) Given a small \( \epsilon \), while \( \| |c^k\rangle - |c^{k-1}\rangle \| > \epsilon \), repeat 2) and 3)

For the derivation of the method and discussion on convergence issues see [22].

When the numerical convergence has been reached, say at iteration \( K \), set \( c_i = c_i^K, i = 1, \ldots, M \) and compute the required component \( |f_V\rangle \) as

\[ |f_V\rangle = \sum_{i=1}^{M} c_i |B_{i,m}\rangle. \]

3 Application to filtering of structured low frequency noise from a seismic signal

We apply here the proposed approach to filtering low frequency noise from a seismic signal. As already mentioned, a common interference with broadband seismic signals is produced by long waves, generated by known or unknown sources, called infragravity waves [6]. This interference is referred to as low frequency noise as it falls in a frequency range of up to 0.05 Hz. Thus, the
model for the subspace of that type of structured noise, on a signal given by $L = 403$ samples, is

$$W^\perp = \text{span}\{e^{i2\pi n(i-1)}}, i = 1, \ldots, L\}^{21}_{n=-21}. \tag{7}$$

The particular realization of the noise we have simulated is plotted in the top left graph of Fig 2 (signal $f^n_i, i = 1, \ldots, L$). However, it is appropriate to recall that the success of the approach does not depend on the actual form of the noise (as long as it belongs to the subspace given in (7)) because the approach guarantees the suppression of the whole subspace $W^\perp$.

The seismic signal $f^s_i, i = 1, \ldots, L$ shown in the right graph of Fig 2 is a piece of a test signal distributed by the seismic industry. The left bottom graph is the superposition of the signals in the top graphs, i.e. $f_i = f^n_i + f^s_i, i = 1, \ldots, L$. We first subtract from $f$ the component in $W^\perp$ to obtain $f_W = f - P_{W^\perp} f$ and use this signal to find the points in the set (3). Then we subdivide uniformly those points to obtain 341 nonuniform knots defining the signal space. Using these knots we construct the nonuniform B-spline basis for the space (MATLAB codes for the implementation of both steps are available from [23]).

The methodology discussed in the previous section requires to fix the values for $q$ and $\lambda$. To allow for good resolution the regularization parameter $\lambda$ is given a small value $\lambda = 10^{-8}$. As for the parameter $q$ we have let it vary in the interval $(0, 1]$ with step 0.001 and the best value resulted to be $q = 0.123$. The right bottom graph of Fig 2 depicts the filtered signal arising by applying the regularized FOCUSS method for $\lambda = 10^{-8}$ and $q = 0.123$.

The left graph of Fig 3 plots, $|f^q - f^s|$, the absolute value of the difference between the approximation $f^q$ (for $q = 0.123$) and the true signal $f^s$. For the sake of comparison, in the right graph we have plotted $|f^f - f^s|$, where $f^f$ is the approximation arising by filtering with Fast
Fourier Transform (FFT). For this approximation we simply take the FFT of \( f \), eliminate the frequencies components in (7), and apply the inverse transform to obtain \( f^I \). The comparison shows the superiority of the proposed approach with respect to standard FFT filtering.

In order to analyze the dependence of the solution on the parameter \( q \) we calculated the error's norm \( \epsilon(q) = \| f^q - f^s \| \). The plot of this error, against \( q \), is depicted in Fig 4. While there is clearly an optimum value of \( q \), the one we have used in this example (and a second best value \( q = 0.312 \)) it should be mentioned that for all the values of \( q \) in \((0, 1]\) the \( f^q \) approximation is superior than that obtained by filtering with FFT.

As a final remark it may be worth stressing that, except for \( q = 1 \), the \( q \)-norm like quantity is not convex. Nevertheless, as opposed to the non-extensive \( q \)-entropy [9, 10] the \( q \)-norm like quantity is not extensive for \( q = 1 \).

4 Conclusions

The problem of structured noise suppression has been considered by modelling the subspaces of the signal components and applying a nonextensive nonlinear technique for separating them. The work was motivated by the problem of filtering infragravity waves from broadband seismic
signals. For this, the noise subspace was modelled using low frequency Fourier functions and that of the other component by a dedicated spline space (adapted to the signal in hand). A simulation involving a piece of seismic signal distributed by the seismic industry and noise up to 0.05Hz has produced encouraging results in comparison with those arising by standard Fourier Transform filtering.

Acknowledgements

Support from the Engineering and Physical Sciences Research Council (EPSRC), UK, grant EP/D06263/1, is acknowledged.
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