A Note on Doubly Nonlinear Parabolic Systems with Unilateral Constraint
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Abstract. We prove the existence and uniqueness of the solution to the doubly nonlinear parabolic systems with mixed boundary conditions. Due to the unilateral constraint the problem comes as a variational inequality. We apply the penalty method and Gronwall’s technique to prove the existence and uniqueness of the variational solution.

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1. Introduction
Let Ω be a bounded domain in $\mathbb{R}^N$, $N = 1, 2$ or 3, with a smooth boundary $\partial\Omega$ for $N = 2$ or $N = 3$. Let $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ be open disjoint subsets of $\Gamma = \partial\Omega$ (not necessarily connected) such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\text{meas}_{N-1}(\Gamma_i) > 0$ for $i = 1, 2, 3$. For a positive $T$ we denote $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$. $T$ is supposed to be fixed throughout the paper. We study the following system $(j = 1, \ldots, m)$

$$
\begin{align*}
\partial_t B^j(u) - \nabla \cdot (K^{ji}(u)\nabla u^i + e^j(u)) &= F^j(x, t, u) \quad \text{in } Q_T, \quad (1.1) \\
u(x, 0) &= u_0(x) \quad \text{in } \Omega, \quad (1.2) \\
u &= 0 \quad \text{on } \Gamma_1 \times (0, T), \quad (1.3) \\
(K^{ji}(u)\nabla u^i + e^j(u)) \cdot n &= g^j(x, t, u) \quad \text{on } \Gamma_2 \times (0, T), \quad (1.4) \\
u^j \leq 0 & \quad \text{on } \Gamma_3 \times (0, T). \quad (1.5)
\end{align*}
$$

In (1.1)–(1.5), $n$ denotes the outer unit normal to $\partial\Omega$, $u = (u^1, \ldots, u^m)$ represents the unknown fields of state variables, the vector $u_0 = (u_0^1, \ldots, u_0^m)$ describes the initial condition. By $B, K^j$ $(j = 1, \ldots, m)$, $e^j, F, g$, we denote
the vectors $B = (B^1, \ldots, B^m)$, $K^j = (K^{j1}, \ldots, K^{jm})$, $e^j = (e^{j1}, \ldots, e^{jN})$, $F = (F^1, \ldots, F^m)$, $g = (g^1, \ldots, g^m)$, which are smooth functions of primary unknowns $u$. Hence, the problem is strongly nonlinear.

Systems of equations like (1.1)–(1.5) arise in a variety of physical situations. For example, they describe the evolution of the dual water flow through porous media (cf. [10]) and, for instance, heat and moisture transfer in porous structures (see [16]).

A considerable effort has been invested into qualitative properties of scalar problems with $m = 1$ (cf. [3, 4, 5, 11, 18]). However, much less attention has been given to the qualitative properties of systems for doubly nonlinear equations of type (1.1). The global existence of weak solutions to (1.1)–(1.2) in bounded domains subject to mixed Dirichlet-Neumann boundary conditions has been shown by Alt & Luckhaus in [2] assuming the function $B^j$ to be monotone and $g \equiv 0$. This result has been extended in various different directions [9, 12, 13, 14]. For instance, Filo & Kačur [9] proved the local existence of the weak solution for the system with nonlinear Neumann boundary conditions and under more general growth conditions on nonlinearities in $u$. The uniqueness of the solution has been proven in [2] under the additional assumption $\partial_t B^j(u) \in L^1$ and assuming the elliptic term in the form $(K^{ji}(x) \nabla u^i + e^j(u))$. In [6], El Ouardi & El Hachimi proved the existence of the regular attractor for Dirichlet problem to nonlinear parabolic systems with Laplacian in the elliptic part of the problem. In [7], the same authors proved the existence of solutions for doubly nonlinear systems including the $p$-Laplacian as the principal part of the operator considering the Dirichlet boundary conditions on the whole part of the domain.

In the present paper we prove the existence and uniqueness of the variational solution to the doubly nonlinear parabolic system (1.1)–(1.4) including the unilateral constraint (1.5). We adapt ideas presented by Filo & Kačur [9] to extend their results to variational inequalities. This paper is organized as follows. In Sections 2.1, 2.2 and 2.3 we introduce basic notations, specify structure conditions and assumptions on data in the problem and recall some important auxiliary results needed below. In Section 3.1 we formulate our problem as the variational inequality and reformulate the solved problem in the operator form in appropriate function spaces. The main results, the existence and uniqueness of the variational solution, are proved in Sections 3.2 and 3.3 via the penalty method and Gronwall’s technique.

2. Preliminaries

2.1. Notations

Vectors, vector functions and operators acting on vector functions are denoted by boldface letters. Unless specified otherwise, we use Einstein summation convention for indices running from 1 to $m$. Throughout the paper, we will always use positive constants $C$, $c$, $c_1$, $c_2$, ..., which are not specified and which may differ from line to line.
For an arbitrary $p \in [1, +\infty]$, $L^p(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\| \cdot \|_{L^p(\Omega)}$, and $W^{k,p}(\Omega)$, $k \geq 0$ ($k$ need not to be an integer, see [15]), denotes the usual Sobolev space with the norm $\| \cdot \|_{W^{k,p}(\Omega)}$.

Let

$$E := \{ u \in C^\infty(\Omega)^m; \text{ supp } u \cap \Gamma_1 = \emptyset \}$$

and $\mathbb{V}$ be a closure of $E$ in the norm of $W^{1,2}(\Omega)^m$. By $\langle \cdot, \cdot \rangle$ we denote the duality between $\mathbb{V}$ and $\mathbb{V}^*$.

### 2.2. Structure and data properties

Next we introduce our assumptions on the functions in (1.1)-(1.5):

(A1) there is a strictly convex $C^1$-function $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$, $\Phi(0) = 0$, $\nabla \Phi(0) = 0$, such that

$$B(z) = \nabla \Phi(z) \quad \forall z \in \mathbb{R}^m. \quad (2.1)$$

The Legendre transform $\Psi (z) := \int_0^1 (B(z) - B(sz)) \cdot z \, ds$ satisfies

$$\Psi(z) \geq c_1 |z|^\nu - c_2 \quad (\nu > 0) \quad \forall z \in \mathbb{R}^m; \quad (2.2)$$

(A2) $K^{ji} : \mathbb{R}^m \rightarrow \mathbb{R}$ and $e^j : \mathbb{R}^m \rightarrow \mathbb{R}^N$ are continuous and $(i,j = 1, \ldots, m$ and $k = 1, \ldots, N)$

$$|K^{ji}(z)| + |e^j_k(z)| \leq c \quad \forall z \in \mathbb{R}^m. \quad (2.3)$$

$(K^{ji})$ is a positive-definite matrix satisfying

$$K^{ji}(z)\xi_i \xi^j \geq c|\xi|^2 \quad \forall \xi, z \in \mathbb{R}^m; \quad (2.4)$$

(A3) the functions $F : Q_T \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $g : \Gamma_2 \times (0, T) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ are continuous and

$$\begin{align*}
|F(x, t, z)| &\leq c(|z|^\alpha + 1), \quad \forall z \in \mathbb{R}^m, \quad [x, t] \in Q_T, \\
|g(x, t, z)| &\leq c(|z|^\alpha + 1), \quad \forall z \in \mathbb{R}^m, \quad [x, t] \in \Gamma_2 \times (0, T); \quad (2.5)
\end{align*}$$

(A4) assume $p \leq \nu$ and that either one of the following conditions is satisfied:

(i) $0 < \alpha \leq \min \{\nu, 1\}$

(ii) $1 < \alpha < (N + \alpha + 1)/N$ and

$$\alpha < \begin{cases} 
(\nu + 1)/2 & \text{for } N = 1, \\
(3\nu + 1)/(3 + \nu) & \text{for } N = 2, \\
\nu + 2 - \sqrt{\nu^2 - \nu + 3} & \text{for } N = 3;
\end{cases} \quad (2.6)$$

(A5) for initial data we assume $u_0 \in W^{1,2}(\Omega)^m$ and $u_0 \cdot B(u_0) \in L^1(\Omega)$.

### 2.3. Auxiliary results

Due to the trace theorem [14] the trace mapping $\mathcal{T} : W^{1,2}(\Omega) \rightarrow L^q(\partial \Omega)$, $q \geq 1$ for $N = 1, 2$ and $1 \leq q \leq 4$ for $N = 3$, is continuous, i.e. there exists a constant $c_{tr}$ such that

$$\|v\|_{L^q(\partial \Omega)} \leq c_{tr} \|v\|_{W^{1,2}(\Omega)} \text{ for all } v \in W^{1,2}(\Omega). \quad (2.5)$$

Let (A4) be satisfied. Then (see [9], Corollary 2)

$$\int_{\partial \Omega} |v|^\alpha + 1 \, d\Gamma \leq \eta\|v\|_{W^{1,2}(\Omega)}^\alpha + C(\eta) \int_\Omega |v|^\nu + 1 \, dx \text{ for all } v \in W^{1,2}(\Omega). \quad (2.6)$$
The following assertion is proved in [9] Lemma 2 and 3: let \( \{w_k\}_{k=1}^\infty \subset L^2(0,T;V) \cap L^\infty(Q_T)^m \) and
\[
\|w_k\|_{L^2(0,T;V)} + \|w_k\|_{L^\infty(Q_T)} < C, \; k = 1,2,\ldots
\]
Moreover, let \( w_k \to w \) a.e. on \( QT \). Then
\[
\begin{aligned}
\{ w_k \to w \; \text{ in } L^{q+1}(Q_T)^m, & \; 0 \leq q < p^*, \\
w_k \to w \; \text{ in } L^{s+1}(S_T)^m, & \; 0 < s < (N + \min\{s,\nu\} + 1)/N.
\end{aligned}
\]

\(2.7\)

3. The variational solution, existence and uniqueness

3.1. Variational solution

Let us define the closed and convex set
\[
\mathcal{K} := \{ v \in V; \; v^j \leq 0 \; \text{a.e. on } \Gamma_3, \; j = 1,\ldots,m \}.
\]

**Definition 3.1.** A vector function \( u \in L^2(0,T;\mathcal{K}) \cap L^\infty(0,T;L^{\nu+1}(\Omega)^m) \) with \( \partial_t B(u) \in L^2(0,T;V^*) \), \( B(u) \in L^1(Q_T)^m \), is a variational solution to the system (1.1)–(1.5) iff
\[
\begin{aligned}
(i) & \quad \int_0^T \langle \partial_t B(u), \varphi - u \rangle \, dt + \int_{Q_T} (K^{ji}(u) \nabla u^i + e^i(u)) \cdot \nabla(\varphi^j - u^j) \, dQ_T \\
& \quad \geq \int_0^T \int_{\Gamma_2} g(x,t,u) \cdot (\varphi - u) \, dS_T + \int_{Q_T} F(x,t,u) \cdot (\varphi - u) \, dQ_T
\end{aligned}
\]
holds for all \( \varphi \in L^2(0,T;\mathcal{K}) \cap L^\infty(Q_T)^m \) and \( u(0) = u_0 \) in \( \Omega \);
\[
(ii) \quad \int_0^T \langle \partial_t B(u), v \rangle \, dt = -\int_{Q_T} (B(u) - B(u_0)) \cdot \partial_t v \, dQ_T
\]
for all \( v \in L^2(0,T;V) \cap L^\infty(Q_T)^m \) with \( \partial_t v \in L^\infty(Q_T)^m \), \( v(T) = 0 \).

**Definition 3.2.** Define an operator \( \mathcal{F} \),
\[
\mathcal{F} : \{ \psi; \; \psi \in L^2(0,T;V), \partial_t B(\psi) \in L^2(0,T;V^*) \} \to L^2(0,T;V^*),
\]
given by the equation
\[
\int_{Q_T} \langle \mathcal{F}(\psi), v \rangle \, dt = \int_0^T \langle \partial_t B(\psi), v \rangle \, dt + \int_{Q_T} (K^{ji}(\psi) \nabla \psi^i + e^i(\psi)) \cdot \nabla v^j \, dQ_T - \int_0^T \int_{\Gamma_2} g(x,t,\psi) \cdot v \, dS_T - \int_{Q_T} F(x,t,\psi) \cdot v \, dQ_T
\]
for all \( v \in L^2(0,T;V) \cap L^\infty(Q_T)^m \).

**Remark 3.3.** If \( u \) is the variational solution to the system (1.1)–(1.5) then the inequality (3.2) can be replaced by
\[
\int_{Q_T} \langle \mathcal{F}(u), \varphi - u \rangle \, dt \geq 0
\]
(3.4)
for all \( \varphi \in L^2(0, T; \mathcal{K}) \cap L^\infty(Q_T)^m \).

### 3.2. The existence of the solution

**Theorem 3.4.** Let the assumptions (A1)–(A5) be satisfied. Then there exists the variational solution to (1.1)–(1.5).

**Definition 3.5.** Let \( S \neq \emptyset \) be a closed and convex subset of a reflexive Banach space \( Y \). An operator \( \mathcal{P} : Y \to Y^* \) is called a penalty operator associated with \( S \subset Y \) if

\[
\mathcal{P}(\zeta) = 0_{Y^*} \iff \zeta \in S.
\]

Theorem 3.4 is a consequence of the following

**Theorem 3.6.** Let \( K \) be the closed and convex subset of the space \( V \) defined by (3.1) and \( T \) be the operator given by the equation (3.3). Let the assumptions (A1)–(A5) be satisfied. Then

1. the operator \( \beta : V \to V^* \) given by the equation

\[
\langle \beta(u_\varepsilon), v \rangle = \int_{\Gamma^3} \psi^+ \cdot v \, d\Gamma \quad \text{for all } v \in V, \ (\psi^+)^j = \max \{ \psi^j(x), 0 \}, \quad (3.5)
\]

represents a penalty operator associated with \( K \).

2. For all \( \varepsilon > 0 \) there exists \( u_\varepsilon \in L^2(0, T; V) \) with \( \partial_t B(u_\varepsilon) \in L^2(0, T; V^*) \) (the variational solution of the penalized problem \( (P_\varepsilon) \)) such that

\[
\int_0^T \langle T(u_\varepsilon), \varphi \rangle \, dt + \frac{1}{\varepsilon} \int_0^T \langle \beta(u_\varepsilon), \varphi \rangle \, dt = 0 \quad (3.6)
\]

for all \( \varphi \in L^2(0, T; V) \cap L^\infty(Q_T)^m \) and \( u_\varepsilon(0) = u^0 \) in \( \Omega \).

3. Let \( \varepsilon_n \to 0^+ \) as \( n \to \infty \). The sequence \( u_{\varepsilon_n} \) of solutions to Problems \( (P_{\varepsilon_n}) \) converges weakly in \( L^2(0, T; V) \) toward the variational solution \( u \) of (1.1)–(1.5).

**Proof.**

Part (1) Due to (2.5) the penalty operator \( \beta \) is well defined and the equivalence \( \beta(u) = 0 \) iff \( u \in K \) is straightforward.

Part (2) The assertion follows from [9, Theorem 1 and Remark 1].

Part (3) Test (3.6) by \( \varphi = u_\varepsilon \chi_{(0,t)} \) (here \( \chi_{(0,t)} \) denotes the characteristic function of \( (0,t) \)) to get

\[
\int_0^T \langle T(u_\varepsilon), u_\varepsilon \chi_{(0,t)} \rangle \, ds + \frac{1}{\varepsilon} \int_0^T \langle \beta(u_\varepsilon), u_\varepsilon \chi_{(0,t)} \rangle \, ds = 0 \quad (3.7)
\]

and consequently

\[
\int_0^t \langle \partial_s B(u_\varepsilon), u_\varepsilon \rangle \, ds + \int_{Q_t} (K^{ji}(u_\varepsilon) \nabla u^{i}_\varepsilon + e^i(u_\varepsilon)) \cdot \nabla u^{j}_\varepsilon \, dQ_t + \frac{1}{\varepsilon} \int_0^t \langle \beta(u_\varepsilon), u_\varepsilon \rangle \, ds
\]

\[
= \int_0^t \int_{\Gamma_2} g(x, s, u_\varepsilon) \cdot u_\varepsilon \, dS_t + \int_{Q_t} F(x, s, u_\varepsilon) \cdot u_\varepsilon \, dQ_t. \quad (3.8)
\]
Integrating by parts in the parabolic term, (3.8) yields
\[
\int_{\Omega} \Psi(u_\varepsilon(t)) \, dx + \int_{Q_t}(K^{ji}(u_\varepsilon) \nabla u^j_\varepsilon + e^j(u_\varepsilon)) \cdot \nabla u^i_\varepsilon \, dQ_t + \frac{1}{\varepsilon} \int_0^t \langle \beta(u_\varepsilon), u_\varepsilon \rangle \, ds \\
= \int_{\Omega} \Psi(u(0)) \, dx + \int_0^t \int_{\Gamma^2} g(x, s, u_\varepsilon) \cdot u_\varepsilon \, dS_t + \int_{Q_t} F(x, s, u_\varepsilon) \cdot u_\varepsilon \, dQ_t.
\] (3.9)

Now, taking into account (A1) together with (A3), one obtains
\[
\int_{\Omega} \Psi(u_\varepsilon(t)) \, dx + \int_{Q_t}(K^{ji}(u_\varepsilon) \nabla u^j_\varepsilon + e^j(u_\varepsilon)) \cdot \nabla u^i_\varepsilon \, dQ_t + \frac{1}{\varepsilon} \int_0^t \langle \beta(u_\varepsilon), u_\varepsilon \rangle \, ds \\
\leq c_1 \int_{\Omega} \Psi(u(0)) \, dx + c_2 \int_0^t \int_{\Gamma^2} |u_\varepsilon|^\alpha \, dS_t + c_3 \int_{Q_t} |u_\varepsilon|^{p+1} \, dQ_t. \quad (3.10)
\]

Further, (3.10), interpolation inequality (2.6) and (A1)–(A4) yield
\[
\int_{\Omega} \Psi(u_\varepsilon(t)) \, dx + \int_0^t c_1 \|u_\varepsilon(s)\|^2_V \, ds + \frac{1}{\varepsilon} \int_0^t \langle \beta(u_\varepsilon), u_\varepsilon \rangle \, ds \\
\leq c_2 \int_{\Omega} \Psi(u(0)) \, dx + c_3 \int_{Q_t} \Psi(u_\varepsilon(s)) \, dQ_t + c_4. \quad (3.11)
\]

Applying the Gronwall’s inequality to (3.11) we arrive at
\[
\int_{\Omega} \Psi(u_\varepsilon(t)) \, dx \leq \left( c_1 \int_{\Omega} \Psi(u(0)) \, dx + c_2 \right) (1 + c_3 t \exp(c_3 t)) \quad (3.12)
\]
for a.e. \(0 \leq t \leq T\). Further, (3.11)–(3.12) imply
\[
\int_0^t |\langle \beta(u_\varepsilon), u_\varepsilon \rangle| \, ds \leq \varepsilon c, \quad (3.13)
\]
where \(c\) is independent of \(\varepsilon\). Hence, as \(\varepsilon \to 0\) we have
\[
\int_0^T \langle \beta(u_\varepsilon), u_\varepsilon \rangle \, ds \to 0.
\]

Analogously,
\[
\int_0^T \langle \beta(u_\varepsilon), v \rangle \, ds \to 0
\]
for every \(v \in L^2(0, T; V)\) and \(\beta(u_\varepsilon) \to 0\). Further (3.11) and (3.12) imply
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \Psi(u_\varepsilon(t)) \, dx + \int_0^T \|u_\varepsilon(t)\|^2_V \, dt \leq c, \quad (3.14)
\]
which yields (by (A1))
\[
\sup_{0 \leq t \leq T} \int_{\Omega} |u_\varepsilon(t)|^{p+1} \, dx + \int_0^T \|u_\varepsilon(t)\|^2_V \, dt \leq c. \quad (3.15)
\]
Since any bounded set in a reflexive Banach space is weakly sequentially compact, we can find a subsequence \( \{ \mathbf{u}_{\varepsilon_n} \} \) such that \( \mathbf{u}_{\varepsilon_n} \to \mathbf{u} \in L^2(0, T; \mathcal{V}) \). Let \( \mathbf{v} \in L^2(0, T; \mathcal{V}) \) be arbitrary fixed. Then

\[
\int_0^T \langle \beta(\mathbf{v}) - \beta(\mathbf{u}_{\varepsilon_n}), \mathbf{v} - \mathbf{u}_{\varepsilon_n} \rangle \, dt \geq 0 \tag{3.16}
\]
yields

\[
\int_0^T \langle \beta(\mathbf{v}), \mathbf{v} - \mathbf{u} \rangle \, dt \geq 0. \tag{3.17}
\]

Choose \( \mathbf{v} = \mathbf{u} + a\mathbf{z} \) (\( a > 0 \), \( \mathbf{z} \in L^2(0, T; \mathcal{V}) \) arbitrary), hence

\[
\int_0^T \langle \beta(\mathbf{u} + a\mathbf{z}), \mathbf{z} \rangle \, dt \geq 0 \tag{3.18}
\]
and letting \( a \to 0^+ \) we have

\[
\int_0^T \langle \beta(\mathbf{u}), \mathbf{z} \rangle \, dt \geq 0 \tag{3.19}
\]
for every \( \mathbf{z} \in L^2(0, T; \mathcal{V}) \). Hence \( \beta(\mathbf{u}) = 0 \), that is, \( \mathbf{u} \in L^2(0, T; \mathcal{K}) \). For arbitrary fixed \( \mathbf{v} \in L^2(0, T; \mathcal{K}) \), we deduce using the equation (3.6)

\[
\int_0^T \langle \mathcal{T}(\mathbf{u}_{\varepsilon_n}), \mathbf{v} - \mathbf{u}_{\varepsilon_n} \rangle \, dt = \frac{1}{\varepsilon_n} \int_0^T \langle \beta(\mathbf{v}) - \beta(\mathbf{u}_{\varepsilon_n}), \mathbf{v} - \mathbf{u}_{\varepsilon_n} \rangle \, dt \geq 0. \tag{3.20}
\]

In the rest of this section we prove that as \( \mathbf{u}_{\varepsilon_n} \to \mathbf{u} \in L^2(0, T; \mathcal{K}) \) then \( \mathbf{v} \) reads

\[
\int_0^T \langle \mathcal{T}(\mathbf{u}), \varphi - \mathbf{u} \rangle \, dt \geq 0 \quad \text{for every } \varphi \in L^2(0, T; \mathcal{K}).
\]

In order to do that we prove the following

**Lemma 3.7.** The sequence \( \mathbf{u}_{\varepsilon_n} \) satisfies

\[
\begin{align*}
\partial_t \mathcal{B}(\mathbf{u}_{\varepsilon_n}) \cdot \mathbf{v} &\to \partial_t \mathcal{B}(\mathbf{u}) \cdot \mathbf{v} \quad \text{in } L^1(Q_T), \\
\mathcal{F}(\mathbf{x}, t, \mathbf{u}_{\varepsilon_n}) \cdot \mathbf{v} &\to \mathcal{F}(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{v} \quad \text{in } L^1(Q_T), \\
g(\mathbf{x}, t, \mathbf{u}_{\varepsilon_n}) \cdot \mathbf{v} &\to g(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{v} \quad \text{in } L^1((0, T) \times \Gamma_2)
\end{align*}
\tag{3.21}
\]

for every \( \mathbf{v} \in L^2(0, T; \mathcal{V}) \cap L^\infty(Q_T)^m \) and

\[
\begin{align*}
\mathcal{K}^{ji}(\mathbf{u}_{\varepsilon_n}) \nabla u^i_{\varepsilon_n} &\to \mathcal{K}^{ji}(\mathbf{u}) \nabla u^i \quad \text{in } L^2(Q_T)^N, \\
\mathcal{E}^{ji}(\mathbf{u}_{\varepsilon_n}) &\to \mathcal{E}^{ji}(\mathbf{u}) \quad \text{in } L^2(Q_T)^N.
\end{align*}
\tag{3.22}
\]

**Proof.** Due to (A1)–(A4), (3.3) and (3.6) we have

\[
\sup \| \mathbf{v} \|_{L^2(0, T; \mathcal{V})} \leq 1 \left( \int_{Q_T} \partial_t \mathcal{B}(\mathbf{u}_{\varepsilon_n}) \cdot \mathbf{v} \, dQ_T \right) \leq c. \tag{3.23}
\]

Hence, the sequence \( \{ \partial_t \mathcal{B}(\mathbf{u}_{\varepsilon_n}) \} \) is uniformly bounded in \( L^2(0, T; \mathcal{V}^*) \) and, consequently, there exists a subsequence and \( \chi \) such that

\[
\partial_t \mathcal{B}(\mathbf{u}_{\varepsilon_n}) \to \chi \quad \text{in } L^2(0, T; \mathcal{V}^*). \tag{3.24}
\]
The identity
\[ \int_{Q_T} \partial_t B(u_{\varepsilon_n}) \cdot v \, dQ_T = - \int_{Q_T} (B(u_{\varepsilon_n}) - B(u^0)) \cdot \frac{dv}{dt} \, dQ_T \] (3.25)
holds for every \( v \in L^2(0, T; \mathbb{V}) \cap L^\infty(Q_T)^m \), \( dv/dt \in L^\infty(Q_T)^m \). Using the compactness argument one can show in the same way as in [2, Lemma 1.9] the convergence
\[ B(u_{\varepsilon_n}) \to B(u) \text{ in } L^1(Q_T). \] (3.26)
Taking the limit in (3.25) and using (3.24) and (3.26) we get
\[ \int_{Q_T} \chi \cdot v \, dQ_T = - \int_{Q_T} (B(u) - B(u^0)) \cdot \frac{dv}{dt} \, dQ_T \] (3.27)
for every \( v \in L^2(0, T; \mathbb{V}) \cap L^\infty(Q_T)^m \), \( dv/dt \in L^\infty(Q_T)^2 \). Now (3.27) yields
\[ \int_{Q_T} \chi \cdot v \, dQ_T = \int_{Q_T} \partial_t B(u) \cdot v \, dQ_T \] (3.28)
for every \( v \in L^2(0, T; \mathbb{V}) \cap L^\infty(Q_T)^m \) and therefore \( \chi = \partial_t B(u) \).

Since \( B^j \) is strictly monotone and from (3.20) it follows that [13, Proposition 3.35]
\[ u_{\varepsilon_n} \to u \text{ a.e. in } Q_T. \] (3.29)
Hence we have
\[ \begin{cases} F(x, t, u_{\varepsilon_n}) \to F(x, t, u) \text{ a.e. in } Q_T, \\ g(x, t, u_{\varepsilon_n}) \to g(x, t, u) \text{ a.e. in } (0, T) \times \Gamma_2. \end{cases} \] (3.30)
Now (2.4) and (3.30) imply that for every \( v \in L^2(0, T; \mathbb{V}) \cap L^\infty(Q_T)^m \) one obtains
\[ \begin{cases} F(x, t, u_{\varepsilon_n}) \cdot v \to F(x, t, u) \cdot v \text{ in } L^1(Q_T), \\ g(x, t, u_{\varepsilon_n}) \cdot v \to g(x, t, u) \cdot v \text{ in } L^1((0, T) \times \Gamma_2). \end{cases} \]
Further, (2.7) yields the convergence
\[ \begin{cases} F(x, t, u_{\varepsilon_n}) \cdot u_{\varepsilon_n} \to F(x, t, u) \cdot u \text{ in } L^1(Q_T), \\ g(x, t, u_{\varepsilon_n}) \cdot u_{\varepsilon_n} \to g(x, t, u) \cdot u \text{ in } L^1((0, T) \times \Gamma_2). \end{cases} \]
Now (2.3) and (3.29) give the convergence \( e^j(u_{\varepsilon_n}) \to e^j(u) \) in \( L^2(Q_T)^m \).
Using (2.3) and (3.14) we arrive at
\[ \|K^{ji}(u^1_{\varepsilon}) \nabla u^i_{\varepsilon}\|^2_{L^2(Q_T)^N} \leq C. \] (3.31)
Hence there exists \( \varphi^j \in L^2(Q_T)^N \) such that
\[ K^{ji}(u_{\varepsilon_n}) \nabla u^i_{\varepsilon} \to \varphi^j \text{ in } L^2(Q_T)^N. \] (3.32)
To prove \( \varphi^j = K^{ji}(u) \nabla u^i \) we follow the trick of Minty-Browder in reflexive spaces. Obviously, for every \( w \in L^2(0, T; \mathbb{V}) \) we have
\[ \int_{Q_T} (K^{ji}(u_{\varepsilon_n}) \nabla u^i_{\varepsilon} - K^{ji}(u_{\varepsilon_n}) \nabla w^i) \cdot (\nabla u^j_{\varepsilon} - \nabla w^j) \, dQ_T \geq 0. \] (3.33)
Letting $\varepsilon_n \to 0$ one obtains
\[
\int_{Q_T} (\varphi^i - K^{ji}(u)\nabla w^i) \cdot (\nabla u^j - \nabla w^j) \, dQ_T \geq 0. \tag{3.34}
\]
Fix any $v \in L^2(0,T;\mathbb{V})$ and set $w = u - \tau v$ ($\tau > 0$) to obtain (letting $\tau \to 0$)
\[
\int_{Q_T} (\varphi^i - K^{ji}(u)\nabla u^i) \cdot \nabla v^j \, dQ_T \geq 0. \tag{3.35}
\]
Replacing $v$ by $-v$ we deduce that equality holds above. Hence we get
\[
\varphi^i = K^{ji}(u)\nabla u^i \quad \text{a.e. in } Q_T. \tag{3.36}
\]
Now (3.32) and (3.36) yield $K^{ji}(u_{\varepsilon_n})\nabla u^i_{\varepsilon_n} \rightharpoonup K^{ji}(u)\nabla u^i$ in $L^2(Q_T)^N$. The proof of Lemma 3.7 is complete. □

By Lemma 3.7 we have
\[
\int_0^T \langle \mathcal{F}(u_{\varepsilon_n}), v \rangle \, dt \to \int_0^T \langle \mathcal{F}(u), v \rangle \, dt
\]
for every $v \in L^2(0,T;\mathbb{V})$. Thus using the inequality (3.20) it follows
\[
0 \leq \lim_{n \to \infty} \int_0^T \langle \mathcal{F}(u_{\varepsilon_n}), v - u_{\varepsilon_n} \rangle \, dt
\]
\[
= \lim_{n \to \infty} \left\{ \int_0^T \langle \mathcal{F}(u_{\varepsilon_n}), u - u_{\varepsilon_n} \rangle \, dt + \int_0^T \langle \mathcal{F}(u_{\varepsilon_n}), v - u \rangle \, dt \right\}.
\]
Since $u_{\varepsilon_n} \to u$ a.e. in $Q_T$ and $\mathcal{F}(u_{\varepsilon_n}) \rightharpoonup \mathcal{F}(u)$ we arrive at
\[
\int_0^T \langle \mathcal{F}(u), \varphi - u \rangle \, dt \geq 0 \quad \text{for every } \varphi \in L^2(0,T;\mathcal{K}).
\]
This completes the proof of Theorem 3.6. □

### 3.3. The uniqueness of the solution

In this section we prove the uniqueness of the solution. In order to do that, we assume the structure condition
\[
K^{ji}(z) = 0 \quad \text{for } j \neq i \quad \text{(i.e. } K^{ji} \text{ is a diagonal matrix)}. \tag{3.37}
\]
It is convenient to denote $K^j = K^{jj}$. In addition to (A2) and (3.37) we suppose
\[
K^j(z) = K^j(z^j) \quad \text{and} \quad c_1 \leq K^j(\xi) \leq c_2 \quad \forall \xi \in \mathbb{R}, \quad j = 1, \ldots, m. \tag{3.38}
\]

**Theorem 3.8.** Let (A1)–(A5) be satisfied and (3.37)–(3.38) hold. Moreover, assume that there exists the constant $C_L > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^m$ we have ($j = 1, \ldots, m$)
\[
\begin{cases}
|K^j(\xi_1) - K^j(\xi_2)| &\leq C_L |\xi_1 - \xi_2|, \\
|e^j(z_1) - e^j(z_2)| &\leq C_L |z_1 - z_2|, \\
|F^j(x, t, z_1) - F^j(x, t, z_2)| &\leq C_L |z_1 - z_2| \quad \forall [x, t] \in Q_T, \\
|g^j(x, t, z_1) - g^j(x, t, z_2)| &\leq C_L |z_1 - z_2| \quad \forall [x, t] \in \Gamma_2 \times (0, T).
\end{cases} \tag{3.39}
\]
Then the variational solution to \((1.1) - (1.5)\) is unique.

**Proof.** Using Kirchhoff transformation \(\mathcal{K}\) one transfers the nonlinearities in the elliptic part to the parabolic term. Introduce the new unknown variable \(h = \mathcal{K}(u)\), \(\mathcal{K} : \mathbb{R}^m \rightarrow \mathbb{R}^m\),

\[
h^j(t, x) := \int_0^{u^j(t, x)} K^j(\xi) d\xi, \quad j = 1, \ldots, m. \tag{3.40}
\]

Due to (3.38) \(\mathcal{K}\) is continuous and increasing, and one-to-one with \(\mathcal{K}^{-1}\) Lipschitz-continuous. Let \(h = \mathcal{K}(u)\) and \(\tilde{h} = \mathcal{K}(\tilde{u})\), \(\mathcal{K}\) is defined by (3.40), where \(u\) and \(\tilde{u}\) are two variational solutions to \((1.1) - (1.5)\) on \(Q_T\). Set

\[
R^j := (B^j \circ \mathcal{K}^{-1})(h) - (B^j \circ \mathcal{K}^{-1})(\tilde{h}), \quad j = 1, \ldots, m, \tag{3.41}
\]

and denote \(R = (R^1, \ldots, R^m)\). Note that \(R \in L^2(0, T; \mathbb{V}^*)\). By the Lax-Milgram theorem there is a function \(w \in L^2(0, T; \mathbb{V})\) such that

\[
\int_0^T \langle R, \phi \rangle dt = \int_{Q_T} \nabla w^j \cdot \nabla \phi^j \, dQ_T \tag{3.42}
\]

for every \(\phi \in L^2(0, T; \mathbb{V})\). Now we follow the idea presented by Alt & Luckhaus in [2]. We have

\[
\frac{1}{\tau} \int_0^\tau \langle R, w \rangle ds + \frac{2}{\tau} \int_0^{t+\tau} \langle R(s) - R(s - \tau), w(s) \rangle ds = \frac{1}{\tau} \int_0^t \langle R(s + \tau) - R(s), w(s + \tau) \rangle ds - \frac{1}{\tau} \int_0^t \langle R(s), w(s + \tau) - w(s) \rangle ds + \frac{1}{\tau} \int_t^{t+\tau} \langle R, w \rangle ds. \tag{3.43}
\]

In view of (3.42) we obtain

\[
\frac{1}{\tau} \int_0^t \langle R(s + \tau) - R(s), w(s + \tau) \rangle ds - \frac{1}{\tau} \int_0^t \langle R(s), w(s + \tau) - w(s) \rangle ds = \frac{1}{\tau} \int_{Q_t} \left( \nabla w^j(s + \tau) - \nabla w^j(s) \right) \cdot \nabla w^j(s + \tau) \, dQ_t
\]

\[
- \frac{1}{\tau} \int_{Q_t} \nabla w^j(s) \cdot \left( \nabla w^j(s + \tau) - \nabla w^j(s) \right) \, dQ_t \tag{3.44}
\]

and

\[
\frac{1}{\tau} \int_t^{t+\tau} \langle R, w \rangle ds = \frac{1}{\tau} \int_t^{t+\tau} \int_{\Omega} \nabla w^j \cdot \nabla w^j \, dx \, ds. \tag{3.45}
\]
Now the equations (3.43)–(3.45), taken together, yield

\[
\frac{1}{\tau} \int_0^\tau \langle R, w \rangle \, ds + \frac{2}{\tau} \int_\tau^{t+\tau} \langle R(s) - R(s - \tau), w(s) \rangle \, ds \\
= \frac{1}{\tau} \int_{Q_t} \left( \nabla w^j(s + \tau) - \nabla w^j(s) \right) \cdot \left( \nabla w^j(s + \tau) - \nabla w^j(s) \right) \, dQ_t \\
+ \frac{1}{\tau} \int_t^{t+\tau} \int_{\Omega} \nabla w^j \cdot \nabla w^j \, dx \, ds. \tag{3.46}
\]

Hence, as \( \tau \to 0 \), we get

\[
\int_0^t \langle \partial_s R, w \rangle \, ds = \frac{1}{2} \int_\Omega \nabla w^j(t) \cdot \nabla w^j(t) \, dx. \tag{3.47}
\]

Moreover, we have

\[
\int_{Q_t} R \cdot (h - \tilde{h}) \, dQ_t = \int_{Q_t} \nabla w^j \cdot \nabla (h^j - \tilde{h}^j) \, dQ_t. \tag{3.48}
\]

Applying the Kirchhoff transformation to (3.2) and taking \( \varphi = h \pm w \) one obtains

\[
\int_0^t \langle \partial_s (B^j \circ K^{-1})(h), w^j \rangle \, ds + \int_{Q_t} \nabla h^j \cdot \nabla w^j \, dQ_t + \int_{Q_t} \hat{e}^j(h) \cdot \nabla w^j \, dQ_t \\
= \int_{Q_t} \tilde{F}(x, s, h) \cdot w \, dQ_t + \int_t^0 \int_{\Gamma_2} \hat{g}(x, s, h) \cdot w \, dS_t. \tag{3.49}
\]

Here we denote

\[
\begin{align*}
\hat{e}^j(h) &= e^j(K^{-1}(h)), \\
\hat{F}^j(x, s, h) &= F^j(x, s, K^{-1}(h)), \\
\hat{g}^j(x, s, h) &= g^j(x, s, K^{-1}(h)).
\end{align*} \tag{3.50}
\]

Writing (3.49) for \( h \) and \( \tilde{h} \) and taking the difference of both equations we get for \( t \in (0, T) \)

\[
\int_0^t \langle \partial_s R, w \rangle \, ds + \int_{Q_t} \nabla (h^j - \tilde{h}^j) \cdot \nabla w^j \, dQ_t \\
= - \int_{Q_t} \left( \hat{e}^j(h) - \hat{e}^j(\tilde{h}) \right) \cdot \nabla w^j \, dQ_t \\
+ \int_{Q_t} \left( \hat{F}(x, s, h) - \hat{F}(x, s, \tilde{h}) \right) \cdot w \, dQ_t \\
+ \int_{\Gamma_2} \left( \hat{g}(x, s, h) - \hat{g}(x, s, \tilde{h}) \right) \cdot w \, dS_t. \tag{3.51}
\]
Estimating each integral on the right-hand side and using (3.39) together with the Young inequality one obtains, consequently,
\[
\int_{Q_t} \left( \hat{e}^j(h) - \hat{e}^j(\tilde{h}) \right) \cdot \nabla w^j \, dq_t 
\leq c_1 \delta \int_0^t ||h - \tilde{h}||^2_{L^2(\Omega)^m} \, ds + c_2 C(\delta) \int_0^t ||w||^2_{W^{1,2}(\Omega)^m} \, ds
\]  
(3.52)
and in the similar way
\[
\int_{Q_t} \left( \hat{F}(x, s, h) - \hat{F}(x, s, \tilde{h}) \right) \cdot w \, dq_t 
\leq c_1 \delta \int_0^t ||h - \tilde{h}||^2_{L^2(\Omega)^m} \, ds + c_2 C(\delta) \int_0^t ||w||^2_{W^{1,2}(\Omega)^m} \, ds.
\]  
(3.53)

Further
\[
\int_0^t \int_{\Gamma_2} \left( \hat{g}(x, s, h) - \hat{g}(x, s, \tilde{h}) \right) \cdot w \, ds \leq c_L \int_0^t ||h - \tilde{h}||_{L^2(\Gamma_2)^m} \, ||w||_{L^2(\Gamma_2)^m} \, ds 
\leq c_1 \delta \int_0^t ||h - \tilde{h}||^2_{W^{1,2}(\Omega)^m} \, ds + c_2 C(\delta) \int_0^t ||w||^2_{W^{1,2}(\Omega)^m} \, ds. 
\]  
(3.54)

Hence, we can rewrite (3.51) using the above estimates together with equations (3.47) and (3.48) to obtain
\[
\frac{1}{2} \int_\Omega |\nabla w(t)|^2 \, dx + \int_{Q_t} R \cdot (h - \tilde{h}) \, dq_t 
\leq c_1 \delta \int_0^t ||h - \tilde{h}||^2_{W^{1,2}(\Omega)^m} \, ds + c_2 C(\delta) \int_0^t ||w||^2_{W^{1,2}(\Omega)^m} \, ds.
\]  
(3.55)

Note that if \( \tilde{h} \neq h \) then the second term on the left in (3.55) is positive. Hence, provided we select \( \delta \) sufficiently small, we obtain the integral inequality
\[
||w(t)||^2_V \leq c \int_0^t ||w(s)||^2_V \, ds,
\]
for a.e. \( 0 \leq t \leq T \), from which we obtain, using the technique of Gronwall’s lemma, \( w = 0 \). Hence \( \tilde{u} = u \) and the uniqueness follows (recall that the Kirchhoff transformation \( h = \mathcal{K}(u) \) is a Lipschitz continuous one-to-one mapping). \( \square \)

Remark 3.9. All results in our paper remain valid if one assumes the non-homogeneous Dirichlet boundary condition \( u = u^D \) on \( \Gamma_1 \times (0, T) \), where \( u^D \in L^2(0, T; \mathcal{V}) \cap L^\infty(Q_T) \).

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