On an Elliptic Free Boundary Problem and Subsonic Jet Flows for a Given Surrounding Pressure

Chunpeng Wang ∗ (email: wangcp@jlu.edu.cn)
School of Mathematics, Jilin University, Changchun 130012, China

Zhouping Xin† (email: zpxin@ims.cuhk.edu.hk)
The Institute of Mathematical Sciences and Department of Mathematics,
The Chinese University of Hong Kong, Shatin, NT, Hong Kong

Abstract

This paper concerns compressible subsonic jet flows for a given surrounding pressure from a two-dimensional finitely long convergent nozzle with straight solid wall, which are governed by a free boundary problem for a quasilinear elliptic equation. For a given surrounding pressure and a given incoming mass flux, we seek a subsonic jet flow with the given incoming mass flux such that the flow velocity at the inlet is along the normal direction, the flow satisfies the slip condition at the wall, and the pressure of the flow at the free boundary coincides with the given surrounding pressure. In general, the free boundary contains two parts: one is the particle path connected with the wall and the other is a level set of the velocity potential. We identify a suitable space of flows in terms of the minimal speed and the maximal velocity potential difference for the well-posedness of the problem. It is shown that there is an optimal interval such that there exists a unique subsonic jet flow in the space iff the length of the nozzle belongs to this interval. Furthermore, the optimal regularity and other properties of the flows are shown.

Keywords: Free boundary, Jet flow, Mixed boundary conditions.

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1 Introduction

In this paper we study the compressible subsonic jet flows for a given surrounding pressure. Such problems arise naturally in physical experiments and engineering designs ([7]), and have received much attention for a long time. Many examples and numerical results can be found in the monographs [7, 8, 10, 20]. In general, compressible subsonic jet flows are governed by elliptic free boundary problems. The first rigorous mathematical theory was established until 1980’s. H. W. Alt, L. A. Caffarelli and A. Friedman developed a variational approach to solve free boundary problem for elliptic equations in [1, 2, 3], which can be applied to the subsonic jet flow problems. In these works, the free boundary and the solution are obtained together by solving a minimum problem with free boundary. Furthermore, the solution on the free boundary must take the extreme value so that the variational approach works. Recently, L. L. Du et al [12, 14] used this variational approach to study impinging subsonic jets and collision of two subsonic flows. There are also many works on irrotational and rotational subsonic flows past profiles or in nozzles, which are formulated

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as fixed boundary problems, and we refer to [5, 6, 11, 13, 15, 16, 18, 28, 34, 35, 36] and the references therein. Continuous subsonic-sonic flows in convergent nozzles was studied in [29, 30, 31, 32, 33], where the flows are governed by free boundary problems of a degenerate elliptic equation, and the sonic curve is a free boundary where the flow velocity is along the normal direction.

It was shown in [1, 2, 3] that for a nozzle satisfying some assumptions, if the mass flux of the flow is prescribed, then there exists a unique subsonic jet flow which is infinitely long and whose pressure on the free boundary is a constant. Here, the free boundary is the particle path connected with the wall of the nozzle. Furthermore, the pressure of the subsonic jet flow on the free boundary is a constant which is determined by the flow to be found. That is to say, the surrounding pressure cannot be given in advance in these problems. In many physical problems, the surrounding pressure should be a known constant. So a natural question is how to formulate a subsonic jet flow whose pressure on the free boundary coincides with the given surrounding pressure. In this paper, we study subsonic jet flows for a given surrounding pressure from a finitely long convergent nozzle with straight solid wall. At the inlet, the incoming mass flux is a given constant and the flow angle is prescribed, which is different from [1, 2, 3] where the boundary condition at the inlet is to prescribe the stream function. If the pressure of the flow in the nozzle is greater than the surrounding pressure, it is expected that there is an accelerating subsonic jet flow whose pressure on the free boundary coincides with the given surrounding pressure. In general, one part of the free boundary should be the particle path connected with the wall of the nozzle as in [1, 2, 3]. Because the surrounding pressure is given in advance, the subsonic jet flow may be located in a bounded domain. The outlet of the subsonic jet flow is another part of the free boundary, and one should prescribe another boundary condition on the outlet except for the coincidence between the pressure of the flow and the given surrounding pressure. As in [33], this boundary condition is prescribed that the flow velocity at the outlet is along the normal direction.

Two-dimensional steady compressible fluids satisfy the Euler system:

\[
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = 0, \quad \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(P + \rho u^2) = 0,
\]

where \((u, v)\), \(P\) and \(\rho\) represent the velocity, pressure and density of the flow, respectively. The flow is assumed to be isentropic so that \(P = P(\rho)\) is a smooth function. In particular, for a polytropic gas with adiabatic exponent \(\gamma > 1\), \(P(\rho) = \rho^\gamma / \gamma\) is the normalized pressure. Assume further that the flow is irrotational. Then the density \(\rho\) is expressed in terms of the speed \(q\) according to the Bernoulli law (1.1)

\[
\rho(q^2) = \left(1 - \frac{\gamma - 1}{2}q^2\right)^{1/(\gamma - 1)}, \quad 0 < q < \sqrt{2/(\gamma - 1)}.
\]

The sound speed \(c\) is defined as \(c^2 = P'(\rho)\). At the sonic state, it is \(c_s = \sqrt{2/(\gamma + 1)}\), which is critical in the sense that the flow is subsonic \((q < c)\) when \(q < c_s\), sonic \((q = c)\) when \(q = c_s\), and supersonic \((q > c)\) when \(q > c_s\). It is well-known that the above Euler system can be transformed into the full potential equation (1.2)

\[
\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0,
\]

where \(\varphi\) is a velocity potential with \(\nabla \varphi = (u, v)\), and \(\rho\) is given by [1, 11].

Assume that the nozzle is located symmetrically with respect to the \(x\)-axis with the vertex being \((0, 0)\) and the angle at the vertex being \(2\theta \in (0, \pi)\). In this paper we only consider the upper part of the subsonic jet flow due to the symmetry. The inlet of the nozzle \(\Gamma_{in}\) is the arc centered at \((0, 0)\) with radius \(R_0 > 0\). The upper wall of the nozzle \(\Gamma_w\) ends at \((-R \cos \theta, R \sin \theta)\) for \(R \in (0, R_0)\). For given constants \(P_e > 0\) and \(m > 0\), we seek a subsonic jet flow in \(\Omega\), which
is bounded by $\Gamma_{in}$, $\Gamma_w$, the $x$-axis, the particle path connected with the upper wall $\Gamma_{ws}$ and the 
outlet of the flow $\Gamma_{out}$, such that the incoming mass flux at $\Gamma_{in}$ is $m$, the flow velocity at $\Gamma_{in}$ and 
$\Gamma_{out}$ is along the normal direction, the flow satisfies the slip condition at $\Gamma_w \cup \Gamma_{ws}$ and the $x$-axis, 
and the pressure of the flow at $\Gamma_{ws} \cup \Gamma_{out}$ is $P_e$, where $\Gamma_{ws}$ and $\Gamma_{out}$ are free. Such a subsonic jet 
flow problem is formulated as the following free boundary problem

\begin{align}
\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) &= 0, \quad (x, y) \in \Omega, \quad (1.3) \\
- \int_{\Gamma_{in}} \rho(|\nabla \varphi(x, y)|^2) \nabla \varphi(x, y) \cdot \nu(x, y) dl &= m, \quad (x, y) \in \Omega, \quad (1.4) \\
\varphi(x, y) &= 0, \quad (x, y) \in \Gamma_{in}, \quad (1.5) \\
\nabla \varphi(x, y) \cdot \nu(x, y) &= 0, \quad (x, y) \in \Gamma_w \cup \Gamma_{ws} \cup (x\text{-axis} \cap \partial \Omega), \quad (1.6) \\
P(\rho(|\nabla \varphi(x, y)|^2)) &= P_e, \quad (x, y) \in \Gamma_{ws} \cup \Gamma_{out}, \quad (1.7) \\
\varphi(x, y) &= \xi, \quad (x, y) \in \Gamma_{out}, \quad (1.8)
\end{align}

where $\nu$ is the unit outer normal to $\partial \Omega$, $\xi$ is a free constant, and $\Gamma_{ws} \cup \Gamma_{out}$ is a free boundary. 
As to the Dirichlet boundary conditions $\text{(1.5)}$ and $\text{(1.8)}$, it means that the flow velocity at $\Gamma_{in}$ and 
$\Gamma_{out}$ is along the normal direction. Hence both $\Gamma_{in}$ and $\Gamma_{out}$ are level sets of the velocity potential, 
and the value at $\Gamma_{in}$ is normalized to be zero without loss of generality.

For the subsonic jet flow problem $\text{(1.3)}$–$\text{(1.8)}$, its pressure coincides with the given surrounding 
pressure at the free boundary. In general, this free boundary contains two parts: one is the 
particle path connected with the wall and the other is a level set of the velocity potential. Both 
the velocity potential and the stream function do not take the extreme value on the whole free 
boundary. Hence the variational approach by H. W. Alt, L. A. Caffarelli and A. Friedman cannot 
be applied to this free boundary problem. As far as we know, there are no studies on such problems 
although elliptic free boundary problems have been studied extensively (see, e.g., $[9, 19, 27]$ and 
the references therein). Indeed, it is hard to solve the problem $\text{(1.3)}$–$\text{(1.8)}$ in the physical plane 
since the characteristics of the two parts of the free boundary are completely different. It is noted 
that $\Gamma_{ws}$ and $\Gamma_{out}$ are two segments in the potential-stream coordinates. So, we study the problem 
$\text{(1.3)}$–$\text{(1.8)}$ in the potential plane since the shape but not the precise location of the free boundary 
is known in advance in the potential plane. However, for the subsonic jet flow problem in the 
potential plane, the flow satisfies a nonlinear Robin boundary condition at the inlet which causes 
a crucial difficulty for its well-posedness as in $[33]$. We need to choose a suitable space of solutions 
to ensure its well-posedness. Besides this difficulty, there are three new ones completely different 
from $[33]$. First of all, the subsonic jet flow problem in this paper is not a perturbed problem and 
there are no background solutions, while $[33]$ concerns the structural stability of a symmetric flow. 
The other two new difficulties are that the free boundary contains two different parts, and mixed 
Dirichlet-Neumann boundary conditions are prescribed on a segment. For these difficulties, we 
need some new estimates and techniques completely different from $[33]$, such as the optimal Hölder 
estimates for subsonic jet flows, the continuous dependence of subsonic jet flows with respect to 
the free boundary, the precise properties of the free boundary.

To solve the subsonic jet flow problem in the potential plane, we first study the fixed boundary 
problem, for which mixed Dirichlet-Neumann boundary conditions are prescribed on a segment. 
We use a duality argument to show the uniqueness of the solution to the fixed boundary problem. 
For the dual problem, mixed Dirichlet-Neumann boundary conditions are prescribed on a segment, 
and it may be ill-posed due to the boundary condition at the inlet. We prove the dual problem 
is well-posed under some restrictions on the upper bound for the velocity potential difference and 
the lower bound for the speed. As to the existence of solutions to the fixed boundary problem, we 
 prescribe a Neumann boundary condition instead of the nonlinear Robin one at the inlet and then 
use a fixed point argument to get solutions. To do so, we also need some restrictions on the upper 
part.
bound for the velocity potential difference and the lower bound for the speed. Since mixed Dirichlet-Neumann boundary conditions are prescribed on a segment, the regularity of solutions is weak at the joint point of the Dirichlet and Neumann data. Although there are many studies on mixed boundary value problems for elliptic equations (see, e.g., [4, 17, 22, 23, 24, 25] and the references therein), the optimal regularity of solutions to this problem is unknown yet. In this paper, we use a series of elaborate estimates to get the optimal Hölder continuity of solutions. Summing up, by restricting a suitable upper bound for the velocity potential difference, we show the well-posedness and the optimal regularity of solutions to the fixed boundary problem in a suitable space of solutions where there is a lower bound for the speed. With the well-posedness of the fixed boundary problem, together with the continuous dependence of its solutions, we can solve the subsonic jet flow problem in the potential plane. We identify a suitable space of flows in terms of the minimal speed and the maximal velocity potential difference, by which we get a complete classification of the nozzles whether there are subsonic jet flows or not. By studying the precise properties of the subsonic jet flows, one can transform them into the physical plane. The main result of this paper is that: For a given surrounding pressure and a given incoming mass flux, we solve a subsonic jet flow problem in a suitable space, which is identified in §5. The paper is arranged as follows. In §2, we state the main results of the paper and formulate the subsonic jet flow problem into a free boundary problem in the potential plane. It is proved in §3 that the fixed boundary problem is well-posed. Subsequently, the free boundary problem is solved in §5.

2 Main results and formulation in the potential plane

In this section, we first state the main results of the paper (well-posedness, nonexistence and properties of solutions). Then we formulate the subsonic jet flow problem in the potential plane and introduce the spaces of solutions.

2.1 Main results

**Definition 2.1** For \( P_c > 0 \), \( m > 0 \) and \( 0 < R < R_0 \), \((\varphi, \Omega)\) is said to be a solution to the free boundary problem \((1.3) - (1.8)\), if \( \Gamma_{\text{ws}}, \Gamma_{\text{out}} \subset C^1 \) such that \( \Gamma_w \cap \Gamma_{\text{ws}} = (-R \cos \vartheta, R \sin \vartheta) \) and \( \Gamma_{\text{out}} \) connects the \( x \)-axis and \( \Gamma_{\text{ws}} \), and \( \varphi \in C^2(\Omega \setminus \{\Gamma_w \cap \Gamma_{\text{ws}}, \Gamma_{\text{ws}} \cap \Gamma_{\text{out}}\}) \cap H^2(\Omega) \cap C^1(\Omega) \) with \( 0 < \inf_{\Omega} |\nabla \varphi| \leq \sup_{\Omega} |\nabla \varphi| < c_s \) such that \((1.3) - (1.8)\) hold, where \( \Omega \) is the domain bounded by \( \Gamma_{\text{in}}, \Gamma_w \), the \( x \)-axis, \( \Gamma_{\text{ws}} \) and \( \Gamma_{\text{out}} \).

The main results of the paper are the following theorems.
Theorem 2.1  For $P(\rho(c_e^2)) < P_e < 1/\gamma$ and $R_0 \partial c_e \rho(c_e^2) < m < R_0 \partial c_e \rho(c_e^2)$, there exists a constant $R_* \in (\hat{R}, R_0]$ depending only on $R_0$, $\vartheta$, $P_e$, $m$ and $\gamma$, such that the problem (1.3) – (1.8) admits a unique solution $(\varphi_{[R]}, \Omega_{[R]})$ with $\inf_{\Omega_{[R]}} |\nabla \varphi_{[R]}| \geq c_l$ and $\sup_{\Omega_{[R]}} \varphi_{[R]} \leq R_0 c_l$ if $R \in [\hat{R}, R_*] \cap [\hat{R}, R_0)$, while there is not such a solution if $R \in [0, \hat{R}) \cup (R_*, R_0)$, where $c_e \in (0, c_e)$ such that $P(\rho(c_e^2)) = P_e$, $\hat{R} = m/(\partial c_e \rho(c_e^2))$, and $c_l \in (0, c_e)$ is the root to

$$\rho(c_e^2) \int_{c_l}^{c_e} \frac{\rho(q^2) + 2q^2 \rho'(q^2)}{q^2} dq = 1.$$

Remark 2.1  The bounds of the incoming mass flux and restrictions of the solution in Theorem 2.1 are needed in this paper (see the argument in § 2.3 and § 2.4 below). Furthermore, Theorem 2.1 gives a complete classification of the nozzles whether there are subsonic flows in this space or not.

Theorem 2.2  Assume that $P(\rho(c_e^2)) < P_e < 1/\gamma$ and $R_0 \partial c_e \rho(c_e^2) < m < R_0 \partial c_e \rho(c_e^2)$, $(\varphi_{[R]}, \Omega_{[R]})$ with $\inf_{\Omega_{[R]}} |\nabla \varphi_{[R]}| \geq c_l$ and $\sup_{\Omega_{[R]}} \varphi_{[R]} \leq R_0 c_l$ is the solution to the problem (1.3) – (1.8) for $R \in [\hat{R}, R_*] \cap [\hat{R}, R_0)$.

(i) $\Gamma_{ws}$ and $\Gamma_{out}$ can be regarded as the graphs of $y = W_{[R]}(x) (-R \cos \vartheta \leq x \leq x_{[R]})$ and $x = J_{[R]}(y) (0 \leq y \leq y_{[R]})$, respectively, where $W_{[R]} \in C^\infty(-R \cos \vartheta, x_{[R]}) \cap C^{1,1}((-R \cos \vartheta, x_{[R]}) \cap C^{1,\alpha}([-R \cos \vartheta, x_{[R]}])$ for each exponent $\alpha \in (0, 1/2]$, $J_{[R]} \in C^\infty(0, y_{[R]}) \cap C^{1,1}(0, y_{[R]})$, and

$$-\tan \vartheta < W'_{[R]}(x) < 0, \quad W''_{[R]}(x) > 0, \quad x \in (-R \cos \vartheta, x_{[R]}),$$

$$0 < J'_{[R]}(y) < \tan \vartheta, \quad J''_{[R]}(y) > 0, \quad y \in (0, y_{[R]}).$$

Moreover, $\Gamma_{ws} = \emptyset$ if and only if $R = \hat{R}$.

(ii) $\varphi_{[R]} \in C^\infty(\overline{\Omega_{[R]}} \setminus \{(-R \cos \vartheta, R \sin \vartheta), (x_{[R]}, y_{[R]})\}) \cap C^{1,1}(\overline{\Omega_{[R]}} \setminus \{(-R \cos \vartheta, R \sin \vartheta)\}) \cap C^{1,\alpha}(\overline{\Omega_{[R]}})$ for each exponent $\alpha \in (0, 1/2)$, and

$$c_l \leq |\nabla \varphi_{[R]}(x, y)| \leq c_e, \quad -\frac{\partial \varphi_{[R]}}{\partial x}(x, y) \tan \vartheta < -\frac{\partial \varphi_{[R]}}{\partial y}(x, y) \tan \vartheta < 0, \quad (x, y) \in \Omega_{[R]}.$$

(iii) If $R_* < R_0$ additionally, then $\sup_{\Omega_{[R_*]}} \varphi_{[R_*]} = R_0 c_l$.

(iv) For each $R_1 \in [\hat{R}, R_*] \cap [\hat{R}, R_0)$,

$$\lim_{R \rightarrow R_1} \frac{\Omega_{[R]} = \Omega_{[R_1]}},$$

$$\lim_{R \rightarrow R_1} \nabla \varphi_{[R]}(x, y) = \nabla \varphi_{[R_1]}(x, y) \text{ uniformly for } (x, y) \in \Omega_{[R_1]}.$$

(v) For $R_1, R_2 \in [R_*, \hat{R}] \cap [\hat{R}, R_0)$ with $R_1 < R_2$, $\sup_{\Omega_{[R_1]}} \varphi_{[R_1]} < \sup_{\Omega_{[R_2]}} \varphi_{[R_2]}$.

Remark 2.2  $\alpha \in (0, 1/2)$ in Theorem 2.2 is almost optimal.
2.2 Formulation in the potential plane

Define a velocity potential $\varphi$ and a stream function $\psi$, respectively, by

$$
\frac{\partial \varphi}{\partial x} = u = q\cos \theta, \quad \frac{\partial \varphi}{\partial y} = v = q\sin \theta, \quad \frac{\partial \psi}{\partial x} = -\rho v = -\rho q\sin \theta, \quad \frac{\partial \psi}{\partial y} = \rho u = \rho q\cos \theta,
$$

where $\theta$ is the flow angle. The full potential equation (1.2) can be reduced to the following Chaplygin equations (2.1):

$$
\frac{\partial \theta}{\partial \psi} + \frac{\rho(q^2) + 2q^2\rho'(q^2)}{q\rho^2(q^2)} \frac{\partial q}{\partial \varphi} = 0, \quad \frac{1}{q} \frac{\partial q}{\partial \psi} - \frac{1}{\rho(q^2)} \frac{\partial \theta}{\partial \varphi} = 0
$$

in the potential-stream coordinates $(\varphi, \psi)$. And the coordinate transformations between the two coordinate systems are valid at least in the absence of stagnation points. Eliminating $\theta$ from (2.1) yields the following second-order quasilinear equation

$$
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad 0 < q < \sqrt{2/(\gamma - 1)}.
$$

Here, $B(\cdot)$ is strictly increasing in $(0, \sqrt{2/(\gamma - 1)})$, while $A(\cdot)$ is strictly increasing in $(0, c_\ast]$ and strictly decreasing in $[c_\ast, \sqrt{2/(\gamma - 1)})$. We use $A^{-1}(\cdot)$ to denote the inverse function of $A(\cdot)|_{(0,c_\ast)}$ in this paper.

Assume that $(-R_0, 0)$ in the physical plane is transformed into the origin in the potential plane without loss of generality. Then $\Omega$ is transformed into $(0, \xi) \times (0, m)$. Rewrite $\Gamma_{in}$ as

$$
x(s) = -R_0 \cos \frac{s}{R_0}, \quad y(s) = R_0 \sin \frac{s}{R_0}, \quad s \in [0, R_0 \vartheta],
$$

where $s$ is the arc length of $\Gamma_{in}$. Denote the coordinate transformation from $\{0\} \times [0, m]$ to $\Gamma_{in}$ by $S_{in}(\psi)$. Then $S_{in}(0) = 0$, $S_{in}(m) = R_0 \vartheta$, and $S^\prime_{in}(\psi) = 1/(q(0, \psi)\rho(q^2(0, \psi)))$ for $\psi \in [0, m]$. Hence

$$
\int_0^m \frac{1}{q(0, \psi)\rho(q^2(0, \psi))} d\psi = R_0 \vartheta.
$$

It follows from the first equation in (2.1) that

$$
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = \frac{\partial \theta}{\partial \psi}(0, \psi) = \frac{1}{R_0} S^\prime_{in}(\psi) = \frac{1}{R_0 q(0, \psi)\rho(q^2(0, \psi))}, \quad \psi \in (0, m).
$$

Assume that the velocity potential at $\Gamma_{ws} \cap \Gamma_{ws}$ is $\zeta$. It follows from (1.6) and the second equation in (2.1) that $\frac{\partial B(q)}{\partial \psi}(\cdot, 0)|_{(\xi, \xi)} = \frac{\partial B(q)}{\partial \psi}(\cdot, m)|_{(\xi, \xi)} = 0$. Furthermore, (1.7) yields $q(\cdot, m)|_{(\xi, \xi)} = q(\xi, \cdot)|_{(0, m)} = c_\ast$, where $c_\ast \in (0, c_\ast)$ such that $P(\rho(c_\ast^2)) = P_c$. Therefore, the subsonic jet flow problem (1.3)–(1.8) is formulated in the potential plane as the following free boundary problem

$$
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m), \quad (2.3)
$$

$$
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = \frac{1}{R_0 q(0, \psi)\rho(q^2(0, \psi))}, \quad \psi \in (0, m), \quad (2.4)
$$
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi), \quad (2.5)
\]
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta), \quad (2.6)
\]
\[
q(\varphi, m) = c_e, \quad \varphi \in (\zeta, \xi), \quad (2.7)
\]
\[
q(\xi, \psi) = c_e, \quad \psi \in (0, m), \quad (2.8)
\]
\[
\int_0^m \frac{1}{q(0, \psi) \rho(q^2(0, \psi))} d\psi = R_0 \vartheta, \quad (2.9)
\]

where \(0 < c_e < c_\ast, m > 0\) and \(\zeta > 0\) are given constants, while \((q, \xi)\) is the solution.

**Definition 2.2** For \(0 < c_e < c_\ast, m > 0\) and \(\zeta > 0\), \((q, \xi)\) is said to be a solution to the free boundary problem \((2.3)–(2.9)\), if \(q \in C^2([0, \xi] \times [0, m] \setminus \{(\xi, m), (\xi, m)\})\) with \(0 < \inf_{(0, \xi) \times (0, m)} q \leq \sup_{(0, \xi) \times (0, m)} q < c_\ast\) and \(\xi \geq \zeta\) such that \((2.3)–(2.9)\) hold.

Solutions, subsolutions and supersolutions to the fixed boundary problem \((2.3)–(2.8)\) can be defined similarly. Although this fixed boundary problem is uniformly elliptic, the well-posedness, regularity and continuous dependence of solutions are unknown yet.

### 2.3 Bounds of the incoming mass flux

If \(\xi = \zeta\), then the problem \((2.3)–(2.9)\) is symmetric, and it can be simplified into

\[
\frac{d^2}{d\varphi^2} A(\hat{q}(\varphi)) = 0, \quad \varphi \in (0, \zeta), \quad (2.10)
\]

\[
\frac{d}{d\varphi} A(\hat{q}(0)) = \frac{1}{R_0 \hat{q}(0) \rho(\hat{q}^2(0))}, \quad \hat{q}(\zeta) = c_e, \quad (2.11)
\]

\[
\frac{m}{\hat{q}(0) \rho(\hat{q}^2(0))} = R_0 \vartheta, \quad (2.12)
\]

where \(\zeta > 0\) is free. The free boundary problem \((2.10)–(2.12)\) admits a solution if and only if \(0 < m < R_0 \vartheta c_e \rho(c_e^2)\).

**Lemma 2.1** For \(0 < c_e < c_\ast\) and \(0 < m < R_0 \vartheta c_e \rho(c_e^2)\), the free boundary problem \((2.10)–(2.12)\) admits a unique solution \(\hat{q}\) with \(\zeta = \hat{c} = m(A(c_e) - A(c_m))/\vartheta\), where \(c_m \in (0, c_e)\) is the unique root to \(R_0 \vartheta c_m \rho(c_m^2) = m\). More precisely, the solution is

\[
\hat{q}(\varphi) = A^{-1}(A(c_e) - \vartheta(\hat{c} - \varphi)/m), \quad \varphi \in [0, \hat{c}].
\]

In physical plane, it is

\[
\hat{\varphi}(x, y) = \int_R^{R_0} \hat{h} \left( \frac{m}{\vartheta r} \right) dr, \quad (x, y) \in \hat{\Omega},
\]

\[
\hat{\Omega} = \{ (x, y) \in \mathbb{R}^2 : \hat{R} < \sqrt{x^2 + y^2} < R_0, x < 0, 0 < y < -x \tan \vartheta \},
\]

where \(\hat{h}\) is the inverse function of \(q \rho(q^2)\) in \(q \in (0, c_\ast)\), and \(\hat{R} = m/(\vartheta c_e \rho(c_e^2))\).

It is noted that \(\hat{q}\) solves the fixed boundary problem \((2.3)–(2.8)\) with \(\xi = \zeta = \hat{c}\). Assume that \(q_1 \) and \(q_2\) are two solutions to this problem. Set \(Q = A(q_1) - A(q_2)\). Then \(Q\) solves

\[
\frac{\partial^2 Q}{\partial \varphi^2} + \frac{\partial^2 (b(\varphi, \psi) Q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \hat{c}) \times (0, m),
\]
\[
\frac{\partial Q}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0}Q(0, \psi), \quad Q(\xi, \psi) = 0, \quad \psi \in (0, m), \\
\frac{\partial Q}{\partial \psi}(\varphi, 0) = 0, \quad \frac{\partial Q}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \hat{\xi}),
\]

where

\[
b(\varphi, \psi) = \int_0^1 \frac{B'}{A'}(A^{-1}(tA(q_1(\varphi, \psi)) + (1 - t)A(q_2(\varphi, \psi))))dt, \quad (\varphi, \psi) \in (0, \hat{\xi}) \times (0, m), \\
h(\psi) = \int_0^1 \frac{1}{A^{-1}(tA(q_1(\psi)) + (1 - t)A(q_2(\psi)))}dt, \quad \psi \in (0, m).
\]

Its dual problem is

\[
\frac{\partial^2 U}{\partial \varphi^2} + b(\varphi, \psi)\frac{\partial^2 U}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \hat{\xi}) \times (0, m), \quad (2.13)
\]

\[
\frac{\partial U}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0}U(0, \psi), \quad U(\hat{\xi}, \psi) = 0, \quad \psi \in (0, m), \quad (2.14)
\]

\[
\frac{\partial U}{\partial \psi}(\varphi, 0) = 0, \quad \frac{\partial U}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \hat{\xi}). \quad (2.15)
\]

If \(q_1\) and \(q_2\) are small perturbations of \(\hat{q}\), then \(h\) is a small perturbation of \(1/c_m\). Therefore, the eigenvalue problem for (2.13)—(2.15) is a small perturbation of

\[
\frac{\partial^2 U}{\partial \varphi^2} + b(\varphi, \psi)\frac{\partial^2 U}{\partial \psi^2} + \lambda U = 0, \quad (\varphi, \psi) \in (0, \hat{\xi}) \times (0, m), \quad (2.16)
\]

\[
\frac{\partial U}{\partial \varphi}(0, \psi) = -\frac{1}{R_0 c_m}U(0, \psi), \quad U(\hat{\xi}, \psi) = 0, \quad \psi \in (0, m), \quad (2.17)
\]

\[
\frac{\partial U}{\partial \psi}(\varphi, 0) = 0, \quad \frac{\partial U}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \hat{\xi}), \quad (2.18)
\]

where \(\lambda\) is a constant. If \(\hat{\xi} \geq R_0 c_m\), it is clear that the problem (2.16)—(2.18) admits a nonpositive eigenvalue. If \(\hat{\xi} < R_0 c_m\), one can prove that there is not a nontrivial solution to the problem (2.16)—(2.18) for each \(\lambda \leq 0\) (the proof can be found in Proposition 3.3). So, to prove that \(\hat{q}\) is the unique solution to the fixed boundary problem (2.3)—(2.8) with \(\xi = \zeta = \hat{\xi}\) by a duality argument, it is reasonable to restrict \(\hat{\xi} < R_0 c_m\), which is equivalent to

\[
R_0 \partial c_1 \rho(c_1^2) < m < R_0 \partial c_e \rho(c_e^2), \quad (2.19)
\]

where \(c_1\) is the constant given in Theorem 2.7.

Remark 2.3 If \(\hat{\xi} \geq R_0 c_m\), it is unknown whether the solution to the fixed boundary problem (2.3)—(2.8) with \(\xi = \zeta = \hat{\xi}\) is unique or not. As mentioned above, \(\hat{\xi} < R_0 c_m\) is a reasonable restriction when one proves the uniqueness by a duality argument.

Remark 2.4 \(c_1 \in (0, c_e)\) given in Theorem 2.7 satisfies \(\rho(c_1^2)(A(c_e) - A(c_1)) = 1\). Note that \(\rho(q^2)(A(c_e) - A(q))\) is strictly decreasing in \(q \in (0, c_e)\) and \(\lim_{q \to 0^+} \rho(q^2)(A(c_e) - A(q)) = +\infty\). Hence \(c_1\) is well defined.
2.4 Spaces of solutions to the fixed and free boundary problems

In order to solve the free boundary problem \((2.3) - (2.8)\), we first show the well-posedness of the fixed boundary problem \((2.3) - (2.8)\) and then determine the free boundary by \((2.9)\).

The uniqueness of the solution to the fixed boundary problem \((2.3) - (2.8)\) will be proved by a duality argument in the paper. As shown in \(\S\ 2.3\), \(m\) should satisfy \((2.4)\). For such \(m\), the solution \(\hat{q}\) to the free boundary problem \((2.10) - (2.12)\) satisfies \(\inf_{(0, \zeta)} \hat{q} > c_l\). Hence we choose

\[
\mathcal{J} = \left\{ q \in C([0, \xi] \times [0, m]) : c_l \leq \inf_{(0, \xi) \times (0, m)} q \leq \sup_{(0, \xi) \times (0, m)} q < c_e \right\}
\]

as a space of solutions to the problem \((2.3) - (2.8)\). A similar duality argument as in \(\S\ 2.3\) shows that a sufficient condition for the uniqueness of the solution to the problem \((2.3) - (2.8)\) in \(\mathcal{J}\) is

\[
0 < \zeta < R_0 c_l, \quad \zeta \leq \xi \leq R_0 c_l.
\]  

(2.20)

The existence of solutions in \(\mathcal{J}\) to the problem \((2.3) - (2.8)\) will be proved by a fixed point argument as follows: For a given \(g \in C^{1, \alpha}([0, m])\) satisfying \(c_l \leq \inf_{(0, m)} g \leq \sup_{(0, m)} g < c_e\), we solve

\[
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

(2.21)

\[
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = \frac{1}{R_0 g(\psi) \rho(g^2(\psi))}, \quad q(\xi, \psi) = c_e, \quad \psi \in (0, m),
\]

(2.22)

\[
\frac{\partial B(q)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi),
\]

(2.23)

\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta),
\]

(2.24)

\[
q(\varphi, m) = c_e, \quad \varphi \in (0, \zeta),
\]

(2.25)

and then define a mapping by \(\mathcal{J}(g) = q(0, \cdot)|_{[0, m]}\). The problem \((2.3) - (2.8)\) admits a solution if \(\mathcal{J}\) has a fixed point. It is noted that

\[
\mathcal{J}(\varphi, \psi) = c_e, \quad q(\varphi, \psi) = A^{-1}\left(A(c_e) - \frac{\xi - \varphi}{R_0 c_l \rho(c_e^2)}\right), \quad (\varphi, \psi) \in [0, \xi] \times [0, m]
\]

are super and sub solutions to the problem \((2.21) - (2.25)\), respectively. If \(\zeta\) and \(\xi\) satisfy \((2.20)\), the comparison principle yields

\[
c_l \leq A^{-1}\left(A(c_e) - \frac{\xi}{R_0 c_l \rho(c_e^2)}\right) \leq (\mathcal{J}(g))(\psi) \leq c_e, \quad \psi \in [0, m].
\]

By this estimate and other ones, one can prove that \(\mathcal{J}\) admits a fixed point.

Summing up, we will show the well-posedness of the fixed boundary problem \((2.3) - (2.8)\) in \(\mathcal{J}\) for \(\zeta\) and \(\xi\) satisfying \((2.20)\). It can be checked that \((2.20)\) holds for the symmetric flow in Lemma 2.1 if and only if \(m\) satisfies \((2.19)\). Hence, we will solve the free boundary problem \((2.3) - (2.9)\) in the space

\[
\mathcal{J} = \left\{ (q, \xi) : \xi \in (0, R_0 c_l), q \in C([0, \xi] \times [0, m]), c_l \leq \inf_{(0, \xi) \times (0, m)} q \leq \sup_{(0, \xi) \times (0, m)} q < c_e \right\}
\]

for \(R_0 \partial c_l \rho(c_e^2) < m < R_0 \partial c_e \rho(c_e^2)\) and \(0 < \zeta < R_0 c_l\).
3 Well-posedness of the fixed boundary problem

In this section, we prove the well-posedness of the fixed boundary problem \[2.3\]–\[2.8\] in \(\mathcal{J}\), where \(0 < c_e < c_s\), \(m > 0\), and \(\zeta\) and \(\xi\) satisfy \[2.20\].

3.1 Linear elliptic problem with mixed Dirichlet-Neumann boundary conditions

Assume that \(m > 0\), \(0 < \zeta \leq \xi\), \(b \in C^1((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m))\) satisfying

\[
b_1 \leq b(\varphi, \psi) \leq b_2, \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

where \(b_1 \leq b_2\) are positive constants. Consider the following problem

\[
\begin{align*}
\frac{\partial^2 U}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 U}{\partial \psi^2} &= \omega(\varphi, \psi), & (\varphi, \psi) &\in (0, \xi) \times (0, m), \\
\frac{\partial U}{\partial \varphi}(0, \psi) &= h(\psi), & U(\xi, \psi) &= 0, \\
\frac{\partial U}{\partial \psi}(\varphi, 0) &= 0, & \varphi &\in (0, \xi), \\
\frac{\partial U}{\partial \psi}(\varphi, m) &= 0, & \varphi &\in (\zeta, \xi), \\
U(\varphi, m) &= 0, & \varphi &\in (\zeta, \xi),
\end{align*}
\]

where \(\omega \in C^1([0, \xi] \times [0, m])\) and \(h \in L^\infty(0, m)\).

Definition 3.1 A function \(U \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m])\) is said to be a subsolution (supersolution, solution) to the problem \[3.2\]–\[3.6\], if

\[
\int_0^{\xi} \int_0^m \left( \frac{\partial U}{\partial \varphi} \frac{\partial \eta}{\partial \varphi} + \frac{\partial U}{\partial \psi} \frac{\partial (bn)}{\partial \psi} + \omega \eta \right) d\varphi d\psi + \int_0^\xi h(\psi) \eta(0, \psi) d\psi \leq (\geq, =) 0
\]

for any nonnegative \(\eta \in H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m])\) with \(\eta = 0\) on \((\zeta, \xi) \times \{m\} \cup \{\xi\} \times (0, m)\), and \(U(\cdot, m) \big|_{(\zeta, \xi)} \leq (\geq, =) 0\) and \(U(\xi, \cdot) \big|_{(0, m)} \leq (\geq, =) 0\) hold.

Lieberman has proved the well-posedness of general linear mixed boundary value problems for elliptic equations in weighted Hölder spaces in \[23\]–\[24\]. For the problem \[3.2\]–\[3.6\], we can show its optimal Hölder continuity by suitable sub and sup solutions in the following proposition. Moreover, the solution is still Hölder continuous if \(b \in C([0, \xi] \times [0, m])\) is relaxed to that the oscillation of \(b\) near \((\zeta, m)\) is suitably small (see Proposition 3.2 below).

Proposition 3.1 Assume that \(m > 0\), \(0 < \zeta \leq \xi\), \(b \in C^1((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m])\) satisfying \[3.1\], \(\omega \in C^1([0, \xi] \times [0, m])\) and \(h \in L^\infty(0, m)\).

(i) If \(U, \overline{U} \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m])\) are sub and super solutions to the problem \[3.2\]–\[3.6\], respectively, then \(U \leq \overline{U}\) in \((0, \xi) \times (0, m)\).

(ii) The problem \[3.2\]–\[3.6\] admits a unique solution \(U \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C^\alpha([0, \xi] \times [0, m])\) for each exponent \(\alpha \in (0, 1/2)\). Furthermore,

\[
\|U\|_{C^\alpha([0, \xi] \times [0, m])} \leq M,
\]

where \(M > 0\) is a constant depending only on \(m\), \(\zeta\), \(\xi\), \(\alpha\), \(b\), \(\|\omega\|_{L^\infty((0, \xi) \times (0, m))}\) and \(h\|_{L^\infty(0, m)}\).
Proof. It is clear that the comparison principle holds for the problem (3.2)–(3.6). As to the existence, it is assumed that $\xi > \zeta$. The proof for the case $\xi = \zeta$ is simpler. For each positive integer $n$, choose $h_n \in C^2([0, m])$ such that $\|h_n - h\|_{L^2(0, m)} \leq 1/n$ and $\|h_n\|_{L^\infty(0, m)} \leq \|h\|_{L^\infty(0, m)}$, and set

$$f_n(\varphi) = \begin{cases} \frac{m}{n}, & \text{if } 0 \leq \varphi \leq \zeta, \\ m - (\frac{\delta^4}{n^4} - (\varphi - \zeta - \delta/n)^4)^{1/4}, & \text{if } \zeta < \varphi < \zeta + \delta/n, \\ m - \delta/n, & \text{if } \zeta + \delta/n \leq \varphi \leq \xi - \delta/n, \\ m - 2\delta/n + (\frac{\delta^4}{n^4} - (\varphi - \xi + \delta/n)^4)^{1/4}, & \text{if } \xi - \delta/n < \varphi \leq \xi, \end{cases}$$

where $\delta = 1/3 \min\{m, \xi - \zeta\}$. Consider the following problem

$$\begin{align*}
\frac{\partial^2 U_n}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 U_n}{\partial \psi^2} &= \omega(\varphi, \psi), & (\varphi, \psi) &\in G_n, \\
\frac{\partial U_n}{\partial \varphi}(0, \psi) &= h_n(\psi), & \psi &\in (0, m), \\
\frac{\partial U_n}{\partial \psi}(\varphi, 0) &= 0, & \varphi &\in (0, \zeta), \\
\frac{\partial U_n}{\partial \psi}(\varphi, m) &= 0, & \varphi &\in (0, \zeta), \\
U_n(\varphi, f_n(\varphi)) &= 0, & \varphi &\in (\zeta, \xi), \\
U_n(\xi, \psi) &= 0, & \psi &\in (0, m - 2\delta/n),
\end{align*}$$

where $G_n = \{(\varphi, \psi) \in \mathbb{R}^2 : 0 < \varphi < \xi, 0 < \psi < f_n(\varphi)\}$. The classical theory on elliptic equations ([21] §6.7) yields that the problem (3.5)–(3.13) admits a unique solution $U_n \in C^2(\overline{G_n})$ such that

$$\|U_n\|_{L^\infty(G_n)} + \|U_n\|_{H^1(G_n)} \leq M_1,$$

where $M_1 > 0$ is a constant depending only on $m$, $\zeta$, $\xi$, $b$, $\|\omega\|_{L^\infty((0, \xi) \times (0, m))}$ and $\|h\|_{L^\infty(0, m)}$.

We now prove that $U_n$ is uniformly Hölder continuous. For a given exponent $\alpha \in (0, 1/2)$, set

$$\tilde{U}_n(\varphi, \psi) = -r^\alpha(\varphi, \psi) \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in [0, \xi] \times [0, m],$$

where

$$r(\varphi, \psi) = \sqrt{\lambda^2(\varphi - \zeta_n)^2 + (\psi - m)^2}, \quad (\varphi, \psi) \in [0, \xi] \times [0, m],$$

$$\beta(\varphi, \psi) = \begin{cases} \frac{1}{\mu} \arctan\frac{\psi - m}{\lambda(\varphi - \zeta_n)} - \frac{\mu - 2}{2\mu} \pi, & \text{if } (\varphi, \psi) \in (\zeta_n, \xi] \times [0, m], \\ -\frac{\mu - 1}{2\mu} \pi, & \text{if } \varphi = \zeta_n, \psi \in [0, m], \\ \frac{1}{\mu} \arctan\frac{\psi - m}{\lambda(\varphi - \zeta_n)} - \frac{1}{2} \pi, & \text{if } (\varphi, \psi) \in [0, \zeta_n) \times [0, m], \end{cases}$$

$\zeta_n = \zeta + \delta/n$, $\lambda = \sqrt{b(\zeta, m)}$ and $\mu = 1 + 1/(2\alpha)$. Then $\tilde{U}_n \in C^2(\overline{G_n})$ and for $(\varphi, \psi) \in \overline{G_n}$,

$$\begin{align*}
\frac{\partial \tilde{U}_n}{\partial \varphi} &= -\alpha \lambda^2(\varphi - \zeta_n)^{\alpha - 2} \sin \beta + \frac{\lambda}{\mu}(\psi - m)^{\alpha - 2} \cos \beta, \\
\frac{\partial \tilde{U}_n}{\partial \psi} &= -\alpha(\psi - m)^{\alpha - 2} \sin \beta - \frac{\lambda}{\mu}(\varphi - \zeta_n)^{\alpha - 2} \cos \beta, \\
\frac{\partial^2 \tilde{U}_n}{\partial \varphi^2} &= -\alpha(\alpha - 2)\lambda^4(\varphi - \zeta_n)^2 \sin \beta - \alpha \lambda^2 \sin \beta + \frac{\alpha \lambda^3}{\mu}(\varphi - \zeta_n)(\psi - m)^{\alpha - 4} \cos \beta.
\end{align*}$$
\[
\begin{aligned}
&\frac{\partial^2 U_n}{\partial \psi^2} = -\frac{\alpha(\alpha - 2)(\psi - m)^2 r^{\alpha - 4} \sin \beta}{\mu} + \frac{\lambda^2}{\mu^2}(\psi - m)^2 r^{\alpha - 4} \cos \beta, \\
&\quad - \frac{(\alpha - 2)\lambda}{\mu}(\psi - m)(\psi - \zeta_n) r^{\alpha - 4} \cos \beta.
\end{aligned}
\]

Therefore

\[
\frac{\partial^2 U_n}{\partial \varphi^2}(\varphi, \psi) + b(\varphi, \psi)\frac{\partial^2 U_n}{\partial \psi^2}(\varphi, \psi)
= \left(\alpha(2 - \alpha)b\left(\frac{\lambda^4}{b}(\psi - \zeta_n)^2 + (\psi - m)^2\right) + \frac{\lambda^2}{\mu^2}(b(\psi - \zeta_n)^2 + (\psi - m)^2) - \alpha(\lambda^2 + b)r^{\alpha - 4}\right)\sin \beta
+ \frac{2(\alpha - 1)\lambda}{\mu}(\lambda^2 - b)(\psi - \zeta_n)(\psi - m)r^{\alpha - 4}\cos \beta, \\
(\varphi, \psi) \in G_n.
\] (3.15)

Since \(\alpha \in (0, 1/2)\) and \(b \in C([0, \xi] \times [0, m])\) satisfying (3.11), it follows from the choice of \(\mu\) and \(\lambda\) that there exist two positive constants \(\tau_1\) and \(\tau_2\), which depend only on \(\alpha\) and \(b\), such that

\[
\alpha(2 - \alpha)b\left(\frac{\lambda^4}{b}(\psi - \zeta_n)^2 + (\psi - m)^2\right) + \frac{\lambda^2}{\mu^2}(b(\psi - \zeta_n)^2 + (\psi - m)^2) - \alpha(\lambda^2 + b)r^{\alpha - 4}\sin \beta
\geq (\alpha(\lambda^2 + b) + \tau_2)r^{\alpha - 4}, \\
(\varphi, \psi) \in (0, \xi) \times (0, m), (\varphi - \zeta_n)^2 + (\psi - m)^2 < \tau_1.
\] (3.16)

The definition of \(\beta\) yields \(\sin \beta \leq -\sin(1/2 - 1/\mu)\pi < 0\) in \((0, \xi) \times (0, m)\), which, together with (3.15) and (3.16), leads to

\[
\frac{\partial^2 U_n}{\partial \varphi^2}(\varphi, \psi) + b(\varphi, \psi)\frac{\partial^2 U_n}{\partial \psi^2}(\varphi, \psi)
\leq -\tau_2 r^{\alpha - 2}\sin(1/2 - 1/\mu)\pi + \frac{2(1 - \alpha)\lambda}{\mu}(\lambda^2 - b)(\psi - \zeta_n)(\psi - m) r^{\alpha - 4}
\leq -\tau_2 r^{\alpha - 2}\sin(1/2 - 1/\mu)\pi + \frac{1 - \alpha}{\mu}\lambda^2 - b)r^{\alpha - 2}, \\
(\varphi, \psi) \in G_n, (\varphi - \zeta_n)^2 + (\psi - m)^2 < \tau_1.
\]

From \(b \in C([0, \xi] \times [0, m])\) and the choice of \(\lambda\), there exists two constants \(\tau_3 \in (0, \tau_1)\) and \(\tau_4 > 0\), which depend only on \(\alpha\) and \(b\), such that

\[
\frac{\partial^2 U_n}{\partial \varphi^2}(\varphi, \psi) + b(\varphi, \psi)\frac{\partial^2 U_n}{\partial \psi^2}(\varphi, \psi) \leq -\tau_4 r^{\alpha - 2}, \\
(\varphi, \psi) \in G_n, (\varphi - \zeta_n)^2 + (\psi - m)^2 < \tau_3.
\] (3.17)

Using the comparison principle, together with (3.17) and (3.14), one gets that

\[
|U_n(\varphi, \psi)| \leq M_2 \tilde{U}_n(\varphi, \psi), \\
(\varphi, \psi) \in G_n, (\varphi - \zeta_n)^2 + (\psi - m)^2 < \tau,
\] (3.18)

where \(M_2 > 0\) and \(\tau > 0\) are suitable constants depending only on \(m, \zeta, \xi, \alpha, b, \|\omega\|_{L^\infty(0, \xi) \times (0, m)}\) and \(\|h\|_{L^\infty(0, m)}\). From (3.14) and (3.18), one can prove that the problem (3.2)–(3.6) admits a solution \(U \in H^1(0, \xi) \times (0, m)\) satisfying

\[
|U(\varphi, \psi)| \leq M_3((\varphi - \zeta_n)^2 + (\psi - m)^2)^{\alpha/2}, \\
(\varphi, \psi) \in (0, \xi) \times (0, m),
\] (3.19)

where \(M_3 > 0\) is a constant depending only on \(m, \zeta, \xi, \alpha, b, \|\omega\|_{L^\infty(0, \xi) \times (0, m)}\) and \(\|h\|_{L^\infty(0, m)}\). From (3.19), the Schauder theory and the H"older estimates for elliptic equations ([21]§6.7&8.10), one can get that \(U \in C^2((0, \xi) \times (0, m)) \cap C^\alpha([0, \xi] \times [0, m])\) satisfying (3.7).
Remark 3.1 If $b$ is a positive constant, then $\tilde{U}_n$ with $a = 1/2$ in the proof of Proposition 3.1 solves the homogeneous equation of (3.2). Hence the Hölder continuity in Proposition 3.1 is almost optimal.

Proposition 3.2 Assume that $m > 0$, $0 < \zeta \leq \xi$, $b \in C^1([0, \xi] \times (0, m)) \cap H^1([0, \xi] \times (0, m))$ satisfying (3.1). There exist an exponent $\alpha \in (0, 1/2)$ and a constant $\sigma > 0$, depending only on $b_1$ and $b_2$, such that if the oscillation of $b$ near $(\zeta, m)$ is not greater than $\sigma$, then for $\omega \in C^1([0, \xi] \times [0, m])$ and $h \in L^\infty(0, m)$, the problem (3.2)–(3.6) admits a unique solution $U \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C^3([0, \xi] \times [0, m])$.

Proof. The proof is similar to Proposition 3.1 and one needs only to construct a suitable supersolution to the problem (3.8)–(3.13). Set

$$\tilde{U}_n(\varphi, \psi) = -r^\alpha(\varphi, \psi) \sin(\varphi, \psi), \quad (\varphi, \psi) \in [0, \xi] \times [0, m],$$

where $\alpha \in (0, 1)$ is a constant to be determined,

$$r(\varphi, \psi) = \sqrt{\lambda^2(\varphi - \zeta_n)^2 + (\psi - m)^2}, \quad (\varphi, \psi) \in [0, \xi] \times [0, m],$$

$$\beta(\varphi, \psi) = \begin{cases} \frac{1}{\cos \pi} \frac{\psi - m}{\lambda(\varphi - \zeta_n)} - \frac{1}{6} \pi, & \text{if } (\varphi, \psi) \in (\zeta_n, \xi] \times [0, m], \\ -\frac{1}{3} \pi, & \text{if } \varphi = \zeta_n, \psi \in [0, m], \\ \frac{1}{\cos \pi} \frac{\psi - m}{\lambda(\varphi - \zeta_n)} - \frac{1}{2} \pi, & \text{if } (\varphi, \psi) \in [0, \zeta_n) \times [0, m], \end{cases}$$

$$\zeta_n = \zeta + \delta/n$$ and $\lambda = \sqrt{b(\zeta, m)}$. Direct calculations give that for $(\varphi, \psi) \in \overline{G}_n$,

\begin{align*}
\frac{\partial^2 \tilde{U}_n}{\partial \varphi^2}(\varphi, \psi) + b(\varphi, \psi) \frac{\partial^2 \tilde{U}_n}{\partial \psi^2}(\varphi, \psi) \\
= \left(\alpha(2 - \alpha)(\lambda^4(\varphi - \zeta_n)^2 + b(\psi - m)^2) + \frac{\lambda^2}{9}(b(\varphi - \zeta_n)^2 + (\psi - m)^2) \\
- \alpha(\lambda^2 + br^2)\right) r^{\alpha - 4} \sin \beta + \frac{2(\alpha - 1)\lambda}{3}(\lambda^2 - b)(\varphi - \zeta_n)(\psi - m)r^{\alpha - 4} \cos \beta \\
\leq - \left(\alpha(2 - \alpha)(b_2(\varphi - \zeta_n)^2 + b_1(\psi - m)^2) + \frac{b_1}{9}(b_2(\varphi - \zeta_n)^2 + (\psi - m)^2) \\
- 2\alpha b_2(b_2(\varphi - \zeta_n)^2 + (\psi - m)^2)\right) r^{\alpha - 4} \sin \frac{\pi}{6} \\
+ \frac{1}{3} |b(\zeta, m) - b(\varphi, \psi)| b_2(\varphi - \zeta_n)^2 + (\psi - m)^2) r^{\alpha - 4} \\
\leq - \left(\frac{\alpha(2 - \alpha)b_2^2}{2b_2} + \frac{b_1^2}{18b_2} - \alpha b_2 - \frac{|b(\zeta, m) - b(\varphi, \psi)|}{3} \right) b_2(\varphi - \zeta_n)^2 + (\psi - m)^2) r^{\alpha - 4}.\end{align*}

Choose $\alpha = b_2^2/(36b_2^2)$ and $\sigma = 3\alpha(2 - \alpha)b_2^2/(2b_2)$. If there exists a positive constant $\tau$ such that

$$|b(\zeta, m) - b(\varphi, \psi)| \leq \sigma, \quad (\varphi, \psi) \in (0, \xi) \times (0, m), \quad (\varphi - \zeta)^2 + (\psi - m)^2 < \tau,$$

then

$$\frac{\partial^2 \tilde{U}_n}{\partial \varphi^2}(\varphi, \psi) + b(\varphi, \psi) \frac{\partial^2 \tilde{U}_n}{\partial \psi^2}(\varphi, \psi) \leq -\frac{b_2^2}{36b_2} r^{\alpha - 2}, \quad (\varphi, \psi) \in \overline{G}_n, \quad (\varphi - \zeta)^2 + (\psi - m)^2 < \tau.$$

Subsequently, one can prove the proposition similarly to Proposition 3.1. □
3.2 Comparison principle

**Proposition 3.3 (Comparison principle)** Assume that \(0 < c_c < c_a, m > 0, \) and \(\zeta \) and \(\xi \) satisfy (2.20). Let \(q_1, q_2 \in C^2([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\}) \cap H^1((0, \xi) \times (0, m)) \cap \mathcal{F} \) be sub and super solutions to the problem (2.3)–(2.8), respectively. Then \(q_1 \leq q_2 \) in \((0, \xi) \times (0, m)\).

*Proof.* The proof is based on a duality argument. Set

\[
b(\varphi, \psi) = \int_0^1 \frac{B'}{A'} \left( A^{-1}(tA(\varphi, \psi)) + (1 - t)A(q_2(\varphi, \psi)) \right) dt, \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

\[
h(\psi) = \int_0^1 \frac{1}{A^{-1}(tA(0, \psi)) + (1 - t)A(q_2(0, \psi))} dt, \quad \psi \in (0, m).
\]

Then, \(b \in C^2([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\}) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m]) \) satisfying \(b_1 \leq b \leq b_2 \) in \((0, \xi) \times (0, m)\) with some positive constants \(b_1 \leq b_2\), and \(h \in C^2([0, m]) \) satisfying \(1/c_\ast \leq h \leq 1/c_1\) in \((0, m)\).

For each nonpositive \(\omega \in C^1([0, \xi] \times [0, m])\), consider the problem

\[
\frac{\partial^2 U}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 U}{\partial \psi^2} = \omega(\varphi, \psi), \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

(3.20)

\[
\frac{\partial U}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0} U(0, \psi), \quad U(\xi, \psi) = 0, \quad \psi \in (0, m),
\]

(3.21)

\[
\frac{\partial U}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi),
\]

(3.22)

\[
\frac{\partial U}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta),
\]

(3.23)

\[
U(\varphi, m) = 0, \quad \varphi \in (\zeta, \xi),
\]

(3.24)

We prove the well-posedness of the problem (3.20)–(3.24) by the contraction mapping principle. Set \(\mathcal{C} = \{ \mathcal{U} \in C([0, m]) : \mathcal{U} \geq 0 \text{ in } (0, m) \}\). For each \(\mathcal{U} \in \mathcal{C}\), it follows from Proposition 3.1 that the problem

\[
\frac{\partial^2 \tilde{U}}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 \tilde{U}}{\partial \psi^2} = \omega(\varphi, \psi), \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

(3.25)

\[
\frac{\partial \tilde{U}}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0} \mathcal{U}(\psi), \quad \tilde{U}(\xi, \psi) = 0, \quad \psi \in (0, m),
\]

(3.26)

\[
\frac{\partial \tilde{U}}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi),
\]

(3.27)

\[
\frac{\partial \tilde{U}}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta),
\]

(3.28)

\[
\tilde{U}(\varphi, m) = 0, \quad \varphi \in (\zeta, \xi),
\]

(3.29)

admits a unique solution \(0 \leq \tilde{U} \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m])\). Therefore, we can define a mapping \(J\) from \(\mathcal{C}\) to itself by \(J(\mathcal{U}) = U(0, \cdot)\big|_{[0,m]}\). For \(\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}\), one has

\[
J(\mathcal{U}_1) - J(\mathcal{U}_2) = \tilde{U}(0, \cdot)\big|_{[0,m]} \text{ with } \tilde{U} \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m])\)

solving

\[
\frac{\partial^2 \tilde{U}}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 \tilde{U}}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

(3.30)

\[
\frac{\partial \tilde{U}}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0} (\mathcal{U}_1(\psi) - \mathcal{U}_2(\psi)), \quad \tilde{U}(\xi, \psi) = 0, \quad \psi \in (0, m),
\]

(3.31)
\[ \frac{\partial U}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi), \quad (3.32) \]
\[ \frac{\partial U}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \xi), \quad (3.33) \]
\[ U(\varphi, m) = 0, \quad \varphi \in (\zeta, \xi). \quad (3.34) \]

It follows from Proposition 3.1 that the problem
\[
\frac{\partial^2 V}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 V}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m), \quad (3.35)
\]
\[ \frac{\partial V}{\partial \varphi}(0, \psi) = -\frac{1}{R_0c_l}, \quad V(\xi, \psi) = 0, \quad \psi \in (0, m), \quad (3.36) \]
\[ \frac{\partial V}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi), \quad (3.37) \]
\[ \frac{\partial V}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta), \quad (3.38) \]
\[ V(\varphi, m) = 0, \quad \varphi \in (\zeta, \xi). \quad (3.39) \]

admits a unique solution \( 0 \leq V \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m]). \) It is noted that
\[ \tilde{U}_\pm = \pm \|\mathcal{U}_1 - \mathcal{U}_2\|_{L^\infty(0, m)} V(\varphi, \psi), \quad (\varphi, \psi) \in [0, \xi] \times [0, m] \]
are super and sub solutions to the problem (3.30)–(3.34), respectively. Proposition 3.1 shows that \( J \) is a contraction mapping if \( V \) satisfies
\[ \|V\|_{L^\infty((0, \xi) \times (0, m))} < 1. \quad (3.40) \]

Set
\[ \nabla(\varphi, \psi) = \frac{1}{R_0c_l}(\xi - \varphi), \quad (\varphi, \psi) \in [0, \xi] \times [0, m]. \]
Then \( \nabla \) is a supersolution to the problem (3.35)–(3.39), and Proposition 3.1 leads to
\[ 0 \leq V(\varphi, \psi) \leq \nabla(\varphi, \psi) = \frac{1}{R_0c_l}(\xi - \varphi), \quad (\varphi, \psi) \in [0, \xi] \times [0, m]. \quad (3.41) \]
If \( \xi < R_0c_l \), then (3.41) yields (3.40). Turn to the other case that \( 0 < \zeta < \xi = R_0c_l \). It is noted that both \( V \) and \( \nabla \) solve (3.35). The Hopf Lemma yields that \( V(0, \psi) < \nabla(0, \psi) = 1 \) for \( \psi \in [0, m] \), which and (3.41) imply (3.40). Summing up, if \( \zeta \) and \( \xi \) satisfy (2.20), then \( J \) is a contraction mapping. Therefore, \( J \) admits a unique fixed point, and there exists a unique solution \( U \in C^2((0, \xi) \times (0, m)) \cap H^1((0, \xi) \times (0, m)) \cap C([0, \xi] \times [0, m]) \) to the problem (3.21)–(3.24). Furthermore, it follows from the classical theory on elliptic equations ([21] Theorems 6.24 and 6.30) that \( U \in C^2([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\}) \).

For a positive integer \( n \), let \( 0 \leq \eta_n \in C^2(\mathbb{R}^2) \) such that
\[
\eta_n(\varphi, \psi) = \begin{cases} 
0, & \text{if } (\varphi - \zeta)^2 + (\psi - m)^2 < \frac{1}{n^2}, \\
1, & \text{if } (\varphi - \zeta)^2 + (\psi - m)^2 > \frac{4}{n^2},
\end{cases}
\]
\[ |
abla \eta_n(\varphi, \psi)| \leq 4n, \quad \left| \frac{\partial^2 \eta_n}{\partial \varphi^2}(\varphi, \psi) \right| + \left| \frac{\partial^2 \eta_n}{\partial \psi^2}(\varphi, \psi) \right| \leq 8n^2, \quad (\varphi, \psi) \in \mathbb{R}^2. \]
It follows from the definition of sub and super solutions that
\[
\int_0^\xi \int_0^m (A(q_1) - A(q_2))\eta_m \left( \frac{\partial^2 U}{\partial \varphi^2} + b \frac{\partial^2 U}{\partial \psi^2} \right) d\varphi d\psi \\
+ \int_0^\xi \int_0^m (A(q_1) - A(q_2))U \left( \frac{\partial^2 \eta_m}{\partial \varphi^2} + b \frac{\partial^2 \eta_m}{\partial \psi^2} \right) d\varphi d\psi \\
+ 2 \int_0^\xi \int_0^m (A(q_1) - A(q_2)) \left( \frac{\partial \eta_m}{\partial \varphi} \frac{\partial U}{\partial \varphi} + b \frac{\partial \eta_m}{\partial \psi} \frac{\partial U}{\partial \psi} \right) d\varphi d\psi \\
\geq - \int_0^m (A(q_1)(0, \psi) - A(q_2)(0, \psi)) \frac{\partial \eta_1}{\partial \varphi}(0, \psi)U(0, \psi)d\psi \\
- \int_0^\xi (B(q_1)(\varphi, 0) - B(q_2)(\varphi, 0)) \frac{\partial \eta_1}{\partial \psi}(\varphi, 0)U(\varphi, 0)d\varphi \\
+ \int_0^\xi (B(q_1)(\varphi, m) - B(q_2)(\varphi, m)) \frac{\partial \eta_1}{\partial \psi}(\varphi, m)U(\varphi, m)d\varphi.
\]

Letting \( n \to \infty \) leads to
\[
\int_0^\xi \int_0^m (A(q_1)(\varphi, \psi) - A(q_2)(\varphi, \psi))\omega(\varphi, \psi)d\varphi d\psi \geq 0,
\]
which completes the proof due to the arbitrariness of \( \omega \).

Below we prove the following result for Equation (2.2) similar to the Hopf Lemma.

**Lemma 3.1** Assume that \( G \subset \mathbb{R}^2 \) is a circle. Let \( q_1, q_2 \in C^2(\overline{G}) \) with \( 0 < \inf_G q_k \leq \sup_G q_k < c_\ast \) \((k = 1, 2)\) be sub and super solutions to
\[
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in G,
\]
respectively. If \( q_1 = q_2 \) at a point \( P_0 \in \partial G \) and \( q_1 < q_2 \) on \( \overline{G} \setminus \{P_0\} \), then \( \frac{\partial q_1}{\partial \nu} > \frac{\partial q_2}{\partial \nu} \) at \( P_0 \), where \( \nu \) is the outer normal to \( \partial G \).

**Proof.** Assume that \( G \) is a circle centered at the origin with radius \( r \). Denote \( G^* = \{(\varphi, \psi) \in G : \varphi^2 + \psi^2 > r^2/4 \} \), and set \( Q = A(q_1) - A(q_2) \) on \( \overline{G^*} \). Then \( Q \in C^2(\overline{G^*}) \) is a subsolution to the following linear equation
\[
\frac{\partial^2 Q}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2} (b(\varphi, \psi)Q) = 0, \quad (\varphi, \psi) \in G^*,
\]
where
\[
b(\varphi, \psi) = \int_0^1 \frac{B'}{A'} \left( A^{-1}(tA(q_1(\varphi, \psi)) + (1 - t)A(q_2(\varphi, \psi))) \right) dt, \quad (\varphi, \psi) \in G^*.
\]
It holds that \( b \in C^2(\overline{G^*}) \) and \( b_1 \leq b \leq b_2 \) in \( G^* \) with some positive constants \( b_1 \leq b_2 \). Similar to the proof of Proposition 3.3, one can show that the comparison principle holds for the problem of Equation (3.33) with Dirichlet boundary condition. Set
\[
\tilde{Q}(\varphi, \psi) = e^{\beta \varphi^2} - e^{-\beta(\varphi^2 + \psi^2)}, \quad (\varphi, \psi) \in \overline{G^*},
\]
where $\beta$ is a positive constant to be determined. Direct calculations show that for $(\varphi, \psi) \in G^*$,
\[
\frac{\partial^2 \tilde{Q}}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2}(b(\varphi, \psi)\tilde{Q}) = -4\beta^2(\varphi^2 + b(\varphi, \psi)^2)e^{-\beta(\varphi^2 + \psi^2)} + 2\beta(1 + b(\varphi, \psi))e^{-\beta(\varphi^2 + \psi^2)}
\]
\[
+ 4\beta \frac{\partial b}{\partial \psi}(\varphi, \psi)e^{-\beta(\varphi^2 + \psi^2)} + \frac{\partial^2 b}{\partial \psi^2}(\varphi, \psi)(e^{-\beta\varphi^2} - e^{-\beta(\varphi^2 + \psi^2)})
\]
\[
\leq \left(-\min\{1, b_1\}r^2\beta^2 + 2(1 + b_2 + 2\left\|\frac{\partial b}{\partial \psi}\right\|_{L^\infty(G^*)}r)\beta + \left\|\frac{\partial^2 b}{\partial \psi^2}\right\|_{L^\infty(G^*)}e^{-\beta(\varphi^2 + \psi^2)}\right).
\]
Therefore, there exists a suitable constant $\beta > 0$ such that
\[
\frac{\partial^2 \tilde{Q}}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2}(b(\varphi, \psi)\tilde{Q}) \leq 0, \quad (\varphi, \psi) \in G^*.
\]
Since $q_1 < q_2$ on $\overline{G \setminus \{P_0\}}$, the comparison principle leads to
\[
Q(\varphi, \psi) \leq \tau \tilde{Q}(\varphi, \psi), \quad (\varphi, \psi) \in G^*, \quad \tau = \frac{1}{e^{-\beta r^2/4} - e^{-\beta r^2}} \inf_{G^*} (A(q_2) - A(q_1)) > 0,
\]
which, together with $q_1 = q_2$ at $P_0$, yields that $\frac{\partial q_1}{\partial \nu} > \frac{\partial q_2}{\partial \nu}$ at $P_0$. \qed

### 3.3 Existence of solutions

**Proposition 3.4** Assume that $0 < c_\epsilon < c_*$, $m > 0$, and $\zeta$ and $\xi$ satisfy (2.20). There is a solution $q \in C^\infty([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\}) \cap H^1((0, \xi) \times (0, m)) \cap C^{0, 1}([0, \xi] \times [0, m] \setminus \{(\zeta, m)\})$ to the problem (2.3)–(2.8) such that
\[
c_1 < q(\varphi, \psi) < c_\epsilon, \quad (\varphi, \psi) \in (0, \xi) \times (0, m) \cup (0, \zeta) \times \{m\}, \quad (3.44)
\]
\[
\frac{\partial q}{\partial \varphi}(\varphi, \psi) > 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m), \quad (3.45)
\]
\[
\frac{\partial q}{\partial \psi}(\varphi, \psi) > 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m) \cup (\zeta, \xi) \times \{m\}. \quad (3.46)
\]

**Proof.** It is assumed that $\xi > \zeta$, and the proof for the case $\xi = \zeta$ is simpler. For each positive integer $n$, consider the following problem
\[
\frac{\partial^2 A(q_n)}{\partial \varphi^2} + \frac{\partial^2 B(q_n)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in G_n, \quad (3.47)
\]
\[
\frac{\partial A(q_n)}{\partial \varphi}(0, \psi) = \frac{1}{R_0 q_n(0, \psi) \rho(\rho^2(0, \psi))}, \quad \psi \in (0, m), \quad (3.48)
\]
\[
\frac{\partial B(q_n)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi), \quad (3.49)
\]
\[
\frac{\partial B(q_n)}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta), \quad (3.50)
\]
\[
q_n(\varphi, f_n(\varphi)) = c_\epsilon, \quad \varphi \in (\zeta, \xi), \quad (3.51)
\]
\[
q_n(\xi, \psi) = c_\epsilon, \quad \psi \in (0, m - 2\delta/n), \quad (3.52)
\]
where $f_n$, $G_n$ and $\delta$ are defined in the proof of Proposition 3.1. Set
\[
\mathcal{C} = \left\{ g \in C^{1, \alpha}([0, m]) : c_l \leq g \leq c_\epsilon \text{ and } g' \geq 0 \text{ in } (0, m), g'(0) = g'(m) = 0, \right\}
\]
where \( \alpha \in (0, 1) \) and \( M_1, M_2 > 0 \) are constants to be determined. For each \( g \in \mathcal{C} \), consider the problem
\[
\begin{align*}
\frac{\partial^2 A(\mathcal{Q})}{\partial \varphi^2} + \frac{\partial^2 B(\mathcal{Q})}{\partial \psi^2} & = 0, \quad (\varphi, \psi) \in G_n, \quad (3.53) \\
\frac{\partial A(\mathcal{Q})}{\partial \varphi}(0, \psi) & = \frac{1}{R_0g(\psi)\rho(g^2(\psi))}, \quad \psi \in (0, m), \quad (3.54) \\
\frac{\partial B(\mathcal{Q})}{\partial \psi}(\varphi, 0) & = 0, \quad \varphi \in (0, \xi), \quad (3.55) \\
\frac{\partial B(\mathcal{Q})}{\partial \psi}(\varphi, m) & = 0, \quad \varphi \in (0, \zeta), \quad (3.56) \\
\mathcal{Q}(\varphi, f_0(\varphi)) & = c_e, \quad \varphi \in (\zeta, \xi), \quad (3.57) \\
\mathcal{Q}(\xi, \psi) & = c_e, \quad \psi \in (0, m - 2\delta/n). \quad (3.58)
\end{align*}
\]

It is noted that
\[
\mathcal{Q}(\varphi, \psi) = c_e, \quad \mathcal{Q}(\varphi, \psi) = A^{-1}\left(A(c_e) - \frac{\xi - \varphi}{R_0c_1\rho(c_1^2)}\right), \quad (\varphi, \psi) \in \overline{G}_n
\]

are super and sub solutions to the problem [3.53]–[3.58], respectively. By a standard fixed point argument and the theory on elliptic equations (see, e.g., [21, §15.5 and 26]), one can show that the problem [3.53]–[3.58] admits a unique solution \( \mathcal{Q} \in C^\infty(G_n) \cap C^2(\overline{G}_n) \) satisfying
\[
c_1 \leq A^{-1}\left(A(c_e) - \frac{\xi - \varphi}{R_0c_1\rho(c_1^2)}\right) \leq \mathcal{Q}(\varphi, \psi) \leq c_e, \quad (\varphi, \psi) \in \overline{G}_n. \quad (3.59)
\]

Thanks to the Harnack inequality, there exist two constants \( \tilde{\alpha} \in (0, 1) \) and \( \tilde{M}_1 > 0 \) depending only on \( m, \zeta, R_0, c_e \) and \( \gamma \), such that
\[
\|\mathcal{Q}\|_{C^\tilde{\alpha}([0, \zeta/2] \times [0, m])} \leq \tilde{M}_1. \quad (3.60)
\]

The Schauder theory gives
\[
\frac{\partial \mathcal{Q}}{\partial \psi}(0, 0) = \frac{\partial \mathcal{Q}}{\partial \psi}(0, m) = 0, \quad \|\mathcal{Q}\|_{C^{1, \alpha}([0, \zeta/2] \times [0, m])} \leq \tilde{M}_2, \quad \|\mathcal{Q}\|_{C^{2, \alpha}([0, \zeta/2] \times [0, m])} \leq \tilde{M}_3, \quad (3.61)
\]

where \( \tilde{M}_2, \tilde{M}_3 > 0 \) are constants depending only on \( m, \zeta, R_0, c_e, \gamma \) and \( M_1 \), while \( \tilde{M}_3 \) also on \( M_2 \). Multiplying [3.53] by \( \mathcal{Q} - c_e \) and then integrating over \( G_n \) by parts, one gets from [3.54]–[3.59] that
\[
\|\mathcal{Q}\|_{H^1(G_n)} \leq \tilde{M}_4, \quad (3.62)
\]

where \( \tilde{M}_4 \) is a positive constant depending only on \( m, \xi, R_0, c_e \) and \( \gamma \).

Denote \( E(s) = A(B^{-1}(s)) \) for \( s < 0 \). Then \( E'(s) > 0, E''(s) < 0 \) and \( E'''(s) < 0 \) for \( s < 0 \). Set
\[
w = \frac{\partial B(\mathcal{Q})}{\partial \varphi} \quad \text{and} \quad z = \frac{\partial B(\mathcal{Q})}{\partial \psi} \quad \text{on} \quad \overline{G}_n.
\]

Then \( w, z \in C^\infty(G_n) \cap C^1(\overline{G}_n) \) solve
\[
\begin{align*}
p_1(\varphi, \psi)\frac{\partial^2 w}{\partial \varphi^2} & + \frac{\partial^2 w}{\partial \psi^2} + p_2(\varphi, \psi)\frac{\partial w}{\partial \varphi} + p_3(\varphi, \psi)w = 0, \quad (\varphi, \psi) \in G_n,
\end{align*}
\]
\[ p_1(\varphi, \psi) \frac{\partial^2 z}{\partial \varphi^2} + \frac{\partial^2 z}{\partial \psi^2} + p_4(\varphi, \psi) \frac{\partial z}{\partial \varphi} + p_5(\varphi, \psi) \frac{\partial z}{\partial \psi} + p_6(\varphi, \psi) z = 0, \quad (\varphi, \psi) \in G_n, \]

\[ w(0, 0) = \frac{B'(2(0, 0))}{R_0 g(\psi) (g^2(\psi)) A'(2(0, 0))} > 0, \quad \psi \in (0, m), \]

\[ \frac{\partial z}{\partial \varphi}(0, \psi) + p_7(\psi) z(0, \psi) = - \frac{\rho(g^2(\psi)) g'(\psi)}{R_0 g^2(\psi)} \leq 0, \quad \psi \in (0, m), \]

\[ \frac{\partial w}{\partial \varphi}(\varphi, 0) = 0, \quad z(\varphi, 0) = 0, \quad (\varphi, \psi) \in (0, \xi), \]

\[ \frac{\partial w}{\partial \psi}(\varphi, m) = 0, \quad z(\varphi, m) = 0, \quad (\varphi, \psi) \in (0, \zeta), \]

\[ w(\varphi, f_n(\varphi)) \geq 0, \quad z(\varphi, f_n(\varphi)) \geq 0, \quad (\varphi, \psi) \in (\zeta, \xi), \]

\[ w(\xi, \psi) \geq 0, \quad z(\xi, \psi) = 0, \quad (\varphi, \psi) \in (0, m - 2\delta/n), \]

where \( p_k \in L^\infty(G_n) \) \( (1 \leq k \leq 6) \) are defined by

\[ p_1 = E'(B(\mathcal{Q})) > 0, \quad p_2 = 3E''(B(\mathcal{Q})) \frac{\partial B(\mathcal{Q})}{\partial \varphi}, \quad p_3 = E'''(B(\mathcal{Q})) \left( \frac{\partial B(\mathcal{Q})}{\partial \varphi} \right)^2 \leq 0, \]

\[ p_4 = 2 \frac{E''(B(\mathcal{Q}))}{E'(B(\mathcal{Q}))} \frac{\partial A(\mathcal{Q})}{\partial \varphi}, \quad p_5 = - \frac{E''(B(\mathcal{Q}))}{E'(B(\mathcal{Q}))} \frac{\partial B(\mathcal{Q})}{\partial \psi}, \]

\[ p_6 = \left( \frac{E'''(B(\mathcal{Q}))}{(E'(B(\mathcal{Q})))^2} - \frac{(E''(B(\mathcal{Q})))^2}{(E'(B(\mathcal{Q})))^3} \right) (\frac{\partial A(\mathcal{Q})}{\partial \varphi})^2 \leq 0, \]

and

\[ p_7(\psi) = \frac{E''(B(\mathcal{Q}), \psi))}{R_0 g(\psi) (g^2(\psi)) E'(B(\mathcal{Q}, \psi)))^2} < 0, \quad (\varphi, \psi) \in (0, m). \]

The comparison principle yields \( w \geq 0 \) and \( z \geq 0 \) on \( \overline{G_n} \). Hence

\[ \frac{\partial \mathcal{Q}}{\partial \varphi}(\varphi, \psi) \geq 0, \quad \frac{\partial \mathcal{Q}}{\partial \psi}(\varphi, \psi) \geq 0, \quad (\varphi, \psi) \in \overline{G_n}. \] (3.63)

Now take \( \alpha = \tilde{\alpha}, \) \( M_1 = \tilde{M}_1 \) and \( M_2 = \tilde{M}_2. \) It follows from (3.59)-(3.61) and (3.63) that one can define a mapping \( J \) from \( G \) to itself by \( J(g) = \mathcal{Q}(0, \cdot) \mid_{[0, m]} \). It follows from (3.61) that \( J \) is compact. One can prove the continuity of \( J \) by using its compactness and the uniqueness result for the problem (3.53)-(3.55) (see, e.g., [33] Proposition 4.7). The Schauder fixed point theorem yields that \( J \) admits a fixed point. Hence there exists a solution \( q_n \in C^\infty(G_n) \cap C^2(\overline{G_n}) \) to the problem (3.47)-(3.52). Furthermore, it follows from (3.62), (3.59) and (3.63) that \( ||q_n||_{H^1(G_n)} \leq M_4 \), and

\[ c_t \leq q_n(\varphi, \psi) \leq c_e, \quad \frac{\partial q_n}{\partial \varphi}(\varphi, \psi) \geq 0, \quad \frac{\partial q_n}{\partial \psi}(\psi, \psi) \geq 0, \quad (\varphi, \psi) \in \overline{G_n}. \] (3.64)

Then one can get by a standard limit process that the problem (2.3)-(2.8) admits a solution \( q \in H^1((0, \xi) \times (0, m)) \) satisfying

\[ c_t \leq q(\varphi, \psi) \leq c_e, \quad \frac{\partial q}{\partial \varphi}(\varphi, \psi) \geq 0, \quad \frac{\partial q}{\partial \psi}(\psi, \psi) \geq 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m). \] (3.65)

The Schauder theory shows that \( q \in C^\infty([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\}) \cap C^{0,1}([0, \xi] \times [0, m] \setminus \{(\zeta, m)\}) \). Lemma 3.1 (2.3)-(2.8) and (3.65) imply (3.44) and

\[ \frac{\partial q}{\partial \varphi}(0, \cdot) \mid_{(0, m)} > 0, \quad \frac{\partial q}{\partial \varphi}(\xi, \cdot) \mid_{(0, m)} > 0, \quad \frac{\partial q}{\partial \psi}(\cdot, m) \mid_{(\zeta, \xi)} > 0. \]
It suffices to verify (3.45) and (3.46) for \((\varphi, \psi) \in (0, \xi) \times (0, m)\). Set \(\tilde{w} = \frac{\partial B(q)}{\partial \varphi}\) and \(\tilde{z} = \frac{\partial B(q)}{\partial \psi}\) in \((0, \xi) \times (0, m)\). Then \(\tilde{w}, \tilde{z} \in C^\infty([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\})\) solve

\[
\tilde{p}_1(\varphi, \psi) \frac{\partial^2 \tilde{w}}{\partial \varphi^2} + \tilde{p}_2(\varphi, \psi) \frac{\partial \tilde{w}}{\partial \psi} + \tilde{p}_3(\varphi, \psi) \tilde{w} = 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

\[
\tilde{p}_1(\varphi, \psi) \frac{\partial^2 \tilde{z}}{\partial \varphi^2} + \tilde{p}_2(\varphi, \psi) \frac{\partial \tilde{z}}{\partial \psi} + \tilde{p}_3(\varphi, \psi) \tilde{z} = 0, \quad (\varphi, \psi) \in (0, \xi) \times (0, m),
\]

where \(\tilde{p}_k \in C^\infty((0, \xi) \times (0, m))\) \((1 \leq k \leq 6)\) are defined similarly as \(p_k\). The strong maximum principle and (3.65) yield \(\tilde{w} > 0\) and \(\tilde{z} > 0\) in \((0, \xi) \times (0, m)\).

The following proposition shows the optimal Hölder continuity of solutions to the problem (2.3) - (2.8) obtained in Proposition 3.3.

**Proposition 3.5** Assume that \(0 \leq c_e < c_*, m > 0\), and \(\zeta\) and \(\xi\) satisfy (2.20). Let \(q\) be the solution to the problem (2.3) - (2.8) obtained in Proposition 3.3. Then \(q \in C^\alpha([0, \xi] \times [0, m])\) for each exponent \(\alpha \in (0, 1/2)\), and there exists a constant \(M > 0\) depending only on \(m, \zeta, \xi, R_0, c_e, \gamma\) and \(\alpha\) such that \(\|q\|_{C^\alpha([0, \xi] \times [0, m])} \leq M\).

**Proof.** For \(\tau \in (0, \delta/\tau)\), set \(\Omega_\tau = \{(\varphi, \psi) \in \mathbb{R}^2 : (\varphi - \zeta_n)^2 + (\psi - \tau)^2 < \tau^2, \psi < m\}\). Similar to the proof of Proposition 3.3, one can prove that the comparison principle holds for the following problem

\[
\frac{\partial^2 (\tilde{Q})}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(\tilde{Q}))}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in \Omega_\tau \cap G_n, \quad (3.66)
\]

\[
\frac{\partial \tilde{Q}}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (\zeta_n - \tau, \zeta), \quad (3.67)
\]

\[
\tilde{Q}(\varphi, f_n(\varphi)) = A(\bar{c}), \quad \varphi \in (\zeta, \zeta_n + \tau), \quad (3.68)
\]

\[
\tilde{Q}(\varphi, \psi) = g(\varphi, \psi), \quad (\varphi, \psi) \in \partial \Omega_\tau \cap G_n, \quad (3.69)
\]

where \(n > \delta/\tau, \bar{c} \in [c_1, c_e]\) is a constant, and \(g \in C(\overline{G}_n)\). Note that \(A(q_n)\) is a supersolution to the problem (3.66) - (3.69) if \(A(q_n) \geq g\) on \(\partial \Omega_\tau \cap G_n\), where \(q_n \in C^\infty(G_n) \cap C^2(\overline{G}_n)\) is the solution to the problem (3.47) - (3.52).

We construct subsolutions to the quasilinear problem (3.66) - (3.69) similar to the ones in the proof of Proposition 3.2. It is noted that \((B'/A')^\alpha > 0\) in \((0, c_*)\). For \(\tilde{Q} \in C^2(\overline{\Omega}_\tau \cap \overline{G}_n)\) satisfying \(A(c_1/2) \leq \tilde{Q} \leq A(c_e)\) in \(\Omega_\tau \cap G_n\), one has

\[
\frac{\partial^2 (\tilde{Q})}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(\tilde{Q}))}{\partial \psi^2} = \frac{\partial^2 (\tilde{Q})}{\partial \varphi^2} + \frac{B'(A^{-1}(\tilde{Q}))}{A'} \frac{\partial^2 \tilde{Q}}{\partial \psi^2} + \frac{(B'/A')^\alpha(A^{-1}(\tilde{Q}))}{A'(A^{-1}(\tilde{Q}))} \left(\frac{\partial \tilde{Q}}{\partial \psi}\right)^2
\]

\[
\geq \frac{\partial^2 (\tilde{Q})}{\partial \varphi^2} + \frac{B'(A^{-1}(\tilde{Q}))}{A'} \frac{\partial^2 \tilde{Q}}{\partial \psi^2}, \quad (\varphi, \psi) \in \Omega_\tau \cap G_n, \quad (3.70)
\]

\[
\frac{B'(c_1/2)}{A'(c_1/2)} \leq \frac{B'(c_e)}{A'(c_e)} \leq \frac{B'(c_e)}{A'(c_1/2)} \leq \frac{B'(c_1/2)}{A'(c_1/2)}, \quad (\varphi, \psi) \in \Omega_\tau \cap G_n. \quad (3.71)
\]

Due to (3.71), as shown in the proof of Proposition 3.2, there exist an exponent \(\alpha_0 \in (0, 1/2)\) and a constant \(\sigma > 0\), which depend only on \(c_e\) and \(\gamma\), such that for

\[
\tilde{Q}(\varphi, \psi) = A(\bar{c}) + \tilde{M} \tau^{\alpha_0} \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_\tau \cap \overline{G}_n \quad (3.72)
\]
satisfying
\[ A(c_t/2) \leq \hat{Q} \leq A(c_e) \text{ in } \Omega_t \cap G_n \quad \text{and} \quad \operatorname{osc}_{\Omega_t \cap G_n} \frac{B'}{A'}(A^{-1}(\hat{Q})) \leq \sigma, \tag{3.73} \]
it holds that
\[ \frac{\partial^2 (\hat{Q})}{\partial \varphi^2} + \frac{B'}{A'}(A^{-1}(\hat{Q})) \frac{\partial^2 \hat{Q}}{\partial \psi^2} \geq 0, \quad (\varphi, \psi) \in \Omega_t \cap G_n, \tag{3.74} \]
where \( \hat{M} > 0 \) is a constant,
\[ r(\varphi, \psi) = \sqrt{B'(\hat{c})(\varphi - \zeta_n)^2/A'(\hat{c}) + (\psi - m)^2}, \quad (\varphi, \psi) \in [0, \xi] \times [0, m], \]
\[ \beta(\varphi, \psi) = \begin{cases} \frac{1}{3} \arctan \frac{\sqrt{A'(\hat{c})}(\psi - m)}{\sqrt{B'(\hat{c})} - (\varphi - \zeta_n)} - \frac{1}{6} \pi, & \text{if } (\varphi, \psi) \in (\zeta_n, \xi] \times [0, m], \\ - \frac{1}{3} \pi, & \text{if } \varphi = \zeta_n, \psi \in [0, m], \\ \frac{1}{3} \arctan \frac{\sqrt{A'(\hat{c})}(\psi - m)}{\sqrt{B'(\hat{c})} - (\varphi - \zeta_n)} - \frac{1}{2} \pi, & \text{if } (\varphi, \psi) \in [0, \zeta_n] \times [0, m]. \end{cases} \]
It follows from (3.70) and (3.74) that
\[ \frac{\partial^2 (\hat{Q})}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(\hat{Q}))}{\partial \psi^2} \geq 0, \quad (\varphi, \psi) \in \Omega_t \cap G_n. \]
Since \( (B'/A')' > 0 \) and \( A'' < 0 \) in \((0, c_e)\), one has
\[ L \leq \frac{r(\varphi, \psi)}{\sqrt{(\varphi - \zeta_n)^2 + (\psi - m)^2}} \leq \tau, \quad (\varphi, \psi) \in (\varphi, \psi) \in (0, \xi) \times (0, m), \]
\[ \operatorname{osc}_{\Omega_t \cap G_n} \frac{B'}{A'}(A^{-1}(\hat{Q})) \leq \mu_0 \operatorname{osc}_{\Omega_t \cap G_n} \hat{Q}, \]
where
\[ L = \min \left\{ \frac{B'(c_t/2)}{A'(c_t/2)}, 1 \right\}, \quad \tau = \max \left\{ \frac{B'(c_e)}{A'(c_e)}, 1 \right\}, \quad \mu_0 = \frac{1}{A'(c_e)} \max \left\{ \frac{B'}{A'} \right\} \]
It is clear that \( \sin \beta \leq -1/2 \) on \([0, \xi] \times [0, m]\). Therefore, if \( \tau \leq \hat{M}^{-1/\alpha_0} \tau_0 \), then (3.73) holds, where \( \tau_0 = \min \left\{ \delta, (c_t/2)^{1/\alpha_0}/\tau, (\sigma/\mu_0)^{1/\alpha_0}/\tau \right\} \). Summing up, for \( \hat{c} \in [c_t, c_e], \hat{M} > 0 \) and \( \tau \leq \hat{M}^{-1/\alpha_0} \tau_0 \), \( \hat{Q} \) given by (3.72) is a subsolution to the problem (3.66)–(3.69) if \( \hat{Q} \leq g \) on \( \partial \Omega_t \cap G_n \).

Below, we get a lower barrier function of \( g_n \) at \((\zeta, m)\) by using a sequence of subsolutions of the form (3.72) to the problem (3.66)–(3.69). For \( n > \delta/\tau_0 \), set
\[ \hat{Q}_0(\varphi, \psi) = A(c_0) + r^{\alpha_0}(\varphi, \psi) \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in \Omega_n \cap G_n, \]
where \( c_0 = \min \left\{ c_e, A^{-1}(A(c_t) + (L\tau_0)^{\alpha_0}/2) \right\} \). Then, the above discussion shows that \( \hat{Q}_0 \) is a subsolution to
\[ \frac{\partial^2 (\hat{Q})}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(\hat{Q}))}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in \Omega_n \cap G_n, \tag{3.75} \]
\[ \frac{\partial \hat{Q}}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (\zeta_n - \tau_0, \zeta), \tag{3.76} \]
\[ \hat{Q}(\varphi, f_n(\varphi)) = A(c_0), \quad \varphi \in (\zeta, \zeta_n + \tau_0), \tag{3.77} \]
\( \hat{Q}(\varphi, \psi) = A(c), \quad (\varphi, \psi) \in \partial \Omega_{\tau_0} \cap G_n. \)  

(3.78)

It follows from (3.47), (3.50), (3.51) and (3.64) that \( A(q_n) \) is a supersolution to the problem (3.75)–(3.78). Therefore,

\[
A(q_n)(\varphi, \psi) \geq \hat{Q}_0(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_0} \cap \overline{G}_n.
\]

(3.79)

Take \( \tau_1 = 2^{-1/\alpha_0} \tau_0 \). For \( n > \delta/\tau_1 \), set

\[ \hat{Q}_1(\varphi, \psi) = A(c_1) + 2\tau^{\alpha_0}(\varphi, \psi) \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_1} \cap \overline{G}_n, \]

where \( c_1 = \min \{ c_0, A^{-1}(A(c_0) + (\tau_1^{\alpha_0}/2) \} \). Then, \( \hat{Q}_1 \) is a subsolution to

\[
\frac{\partial^2 \hat{Q}}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(\hat{Q}))}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in \Omega_{\tau_1} \cap G_n,
\]

\[
\frac{\partial \hat{Q}}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (\zeta_n - \tau_1, \zeta),
\]

\[
\hat{Q}(\varphi, f_n(\varphi)) = A(c_1), \quad \varphi \in (\zeta, \zeta_n + \tau_1),
\]

\[
\hat{Q}(\varphi, \psi) = \hat{Q}_0(\varphi, \psi), \quad (\varphi, \psi) \in \partial \Omega_{\tau_1} \cap G_n,
\]

while \( A(q_n) \) is a supersolution to this problem due to (3.47), (3.50), (3.51) and (3.79). Therefore,

\[
A(q_n)(\varphi, \psi) \geq \hat{Q}_1(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_1} \cap \overline{G}_n.
\]

(3.80)

Take \( \tau_2 = 2^{-2/\alpha_0} \tau_0 \). For \( n > \delta/\tau_2 \), set

\[ \hat{Q}_2(\varphi, \psi) = A(c_2) + 2^2\tau^{\alpha_0}(\varphi, \psi) \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_2} \cap \overline{G}_n, \]

where \( c_2 = \min \{ c_0, A^{-1}(A(c_0) + (\tau_2^{\alpha_0}/2) \} \). Then, \( \hat{Q}_2 \) is a subsolution to

\[
\frac{\partial^2 \hat{Q}}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(\hat{Q}))}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in \Omega_{\tau_2} \cap G_n,
\]

\[
\frac{\partial \hat{Q}}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (\zeta_n - \tau_2, \zeta),
\]

\[
\hat{Q}(\varphi, f_n(\varphi)) = A(c_2), \quad \varphi \in (\zeta, \zeta_n + \tau_2),
\]

\[
\hat{Q}(\varphi, \psi) = \hat{Q}_1(\varphi, \psi), \quad (\varphi, \psi) \in \partial \Omega_{\tau_2} \cap G_n,
\]

while \( A(q_n) \) is a supersolution to this problem due to (3.47), (3.50), (3.51) and (3.80). Therefore,

\[
A(q_n)(\varphi, \psi) \geq \hat{Q}_2(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_2} \cap \overline{G}_n.
\]

(3.81)

Repeating the above process, one gets that for each positive integer \( k \) and each \( n > \delta/\tau_k \),

\[
A(q_n)(\varphi, \psi) \geq A(c_k) + 2^k\tau^{\alpha_0}(\varphi, \psi) \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_k} \cap \overline{G}_n, \quad \tau_k = 2^{-k/\alpha_0} \tau_0,
\]

where \( c_k = \min \{ c_0, A^{-1}(A(c_{k-1}) + 2^k(\tau_k^{\alpha_0}) \} \}. \) Note that \( 2^{k-2}(\tau_k^{\alpha_0}) = (\tau_0^{\alpha_0})/4 \) for \( k = 1, 2, \cdots \). Therefore, there exists a positive integer \( k_0 \) depending only on \( c_0 \) and \( \gamma \) such that \( c_{k_0} = c_0 \). Hence for each \( n > \delta/\tau_{k_0} \),

\[
A(q_n)(\varphi, \psi) \geq A(c_0) + 2^{k_0}\tau^{\alpha_0}(\varphi, \psi) \sin \beta(\varphi, \psi), \quad (\varphi, \psi) \in \overline{\Omega}_{\tau_{k_0}} \cap \overline{G}_n, \quad \tau_{k_0} = 2^{-k_0/\alpha_0} \tau_0.
\]
According to (3.64) and (3.81), one can prove that \( q \in C^\alpha([0, \xi] \times [0, m]) \) and
\[
\|q\|_{C^\alpha([0, \xi] \times [0, m])} \leq M_0, \tag{3.82}
\]
where \( q \) is the solution to the problem (2.3)–(2.8) obtained in Proposition 3.4 and \( M_0 > 0 \) is a constant depending only on \( m, \xi, R_0, c_e \) and \( \gamma \). Set \( Q = A(q) \) on \([0, \xi] \times [0, m]\). As the proof of (3.70), \( Q \) is a supersolution to the linear equation
\[
\frac{\partial^2 Q}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 Q}{\partial \psi^2} = 0, \quad b(\varphi, \psi) = \frac{B'}{A'}(\varphi, \psi), \quad (\varphi, \psi) \in (0, \xi) \times (0, m).
\]
For each exponent \( \alpha \in (0, 1/2) \), it follows from (3.41), (3.64), (3.82) and Proposition 3.1 that
\[
A(c_e) - M((\varphi - \zeta)^2 + (\psi - m)^2)^{\alpha/2} \leq Q(\varphi, \psi) \leq A(c_e), \quad (\varphi, \psi) \in (0, \xi) \times (0, m), \tag{3.83}
\]
where \( M > 0 \) is a constant depending only on \( m, \xi, R_0, c_e, \gamma \) and \( \alpha \). The Hölder estimates for elliptic equations (218 § 8.10) and (3.83) complete the proof of the proposition. \( \square \)

**Remark 3.2** The regularity in Proposition 3.7 is almost optimal.

### 4 Free boundary problem

In this section, we solve the free boundary problem (2.3)–(2.8) in \( \mathcal{I} \) for \( 0 < c_e < c_*, R_0 \delta c_1 \rho(c_1^2) < m < R_0 \delta c_1 \rho(c_1^2) \) and \( 0 < \zeta < R_0 c_1 \). To do so, we need the continuous dependence of solutions to the problem (2.3)–(2.8) together with its well-posedness in \( \S \).

#### 4.1 Continuous dependence of solutions

**Proposition 4.1** Assume that \( 0 < c_e < c_* \) and \( m > 0 \). For given \( \zeta_0 \) and \( \xi_0 \) satisfying \( 0 < \zeta_0 < R_0 c_1 \) and \( \zeta_0 \leq \xi_0 \leq R_0 c_1 \), it holds that
\[
\lim_{\substack{\zeta \to \zeta_0, \xi \to \xi_0 \\ 0 < \zeta \leq R_0 c_1, \xi \leq R_0 c_1}} q[\zeta, \xi](\varphi, \psi) = q[\zeta_0, \xi_0](\varphi, \psi) \quad \text{uniformly for } (\varphi, \psi) \in [0, \xi_0] \times [0, m],
\]
where \( q[\zeta, \xi] \in C^2([0, \xi] \times [0, m] \setminus \{(\zeta, m), (\xi, m)\}) \cap H^1((0, \xi) \times (0, m)) \cap \mathcal{I} \) is the solution to the fixed boundary problem (2.3)–(2.8).

**Proof.** It is assumed that \( \xi_0 > \zeta_0 \), and the proof for the case \( \xi_0 = \zeta_0 \) is similar. For convenience, we use \( M_i (1 \leq i \leq 6) \) to denote generic constants depending only on \( m, \zeta_0, \xi_0, R_0, c_e \) and \( \gamma \). Furthermore, a parenthesis after a generic constant means that this constant depends also on the variables in the parentheses.

First we prove the continuous dependence on \( \xi \). Fix \( (\zeta_0 + \xi_0)/2 \leq \xi_1 < \xi_2 \leq R_0 c_1 \). Denote \( q_k = q[\zeta_0, \xi_k] \) for \( k = 1, 2 \). It follows from Propositions 3.3 and 3.4 that
\[
c_1 \leq q_2(\varphi, \psi) \leq q_1(\varphi, \psi) \leq c_e, \quad (\varphi, \psi) \in [0, \xi_1] \times [0, m], \tag{4.1}
\]
\[
q_2(\xi_1, \psi) \geq c_e - M_1(\xi_2 - \xi_1) = q_1(\xi_1, \psi) - M_1(\xi_2 - \xi_1), \quad \psi \in [0, m]. \tag{4.2}
\]
Set \( Q = A(q_1) - A(q_2) \) on \([0, \xi_1] \times [0, m]\). Then \( 0 \leq Q \in C^2([0, \xi_1] \times [0, m] \setminus \{(\xi_0, m), (\xi_1, m)\}) \cap H^1((0, \xi_1) \times (0, m)) \cap C([0, \xi_1] \times [0, m]) \) solves
\[
\frac{\partial^2 Q}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2}(b(\varphi, \psi)Q) = 0, \quad (\varphi, \psi) \in (0, \xi_1) \times (0, m),
\]
\[ \frac{\partial Q}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0}Q(0, \psi), \quad Q(\xi_1, \psi) = A(c_\varepsilon) - A(q_2)(\xi_1, \psi), \quad \psi \in (0, m), \]
\[ \frac{\partial Q}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi_1), \]
\[ \frac{\partial Q}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta_0), \]
\[ Q(\varphi, m) = 0, \quad \varphi \in (\zeta_0, \xi_1), \]

where
\[ b(\varphi, \psi) = \int_0^1 \frac{B'}{A'}(A^{-1}(tA(q_1(\varphi, \psi)) + (1 - t)A(q_2(\varphi, \psi)))) dt, \quad (\varphi, \psi) \in (0, \xi_1) \times (0, m), \]
\[ h(\psi) = \int_0^1 \frac{1}{A^{-1}(tA(q_1(0, \psi)) + (1 - t)A(q_2(0, \psi)))} dt, \quad \psi \in (0, m). \]

Consider its dual problem
\[ \frac{\partial^2 U}{\partial \varphi^2} + b(\varphi, \psi) \frac{\partial^2 U}{\partial \psi^2} = Q(\varphi, \psi), \quad (\varphi, \psi) \in (0, \xi_1) \times (0, m), \quad (4.3) \]
\[ \frac{\partial U}{\partial \psi}(0, \psi) = -\frac{h(\psi)}{R_0}U(0, \psi), \quad U(\xi_1, \psi) = 0, \quad \psi \in (0, m), \quad (4.4) \]
\[ \frac{\partial U}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi_1), \quad (4.5) \]
\[ \frac{\partial U}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta_0), \quad (4.6) \]
\[ U(\varphi, m) = 0, \quad \varphi \in (\zeta_0, \xi_1). \quad (4.7) \]

It follows from the proof of Proposition 8.33 that the problem \((4.3) - (4.7)\) admits a unique nonpositive solution \(U \in C^2([0, \xi_1] \times [0, m] \setminus \{(\zeta_0, m), (\xi_1, m)\}) \cap H^1((0, \xi_1) \times (0, m)) \cap C([0, \xi_1] \times [0, m])\). The classical theory on elliptic equations ([21, Theorem 8.33]) yields that
\[ 0 \leq \frac{\partial U}{\partial \varphi}(\xi_1, \psi) \leq M_2, \quad \psi \in (0, m). \quad (4.8) \]

Multiplying the equation of \(Q\) by \(U\) and then integrating over \((0, \xi_1) \times (0, m)\) by parts, one gets from \((4.1), (4.2)\) and \((4.3)\) that
\[ \int_0^{\xi_1} \int_0^m Q^2(\varphi, \psi) d\varphi d\psi = \int_0^m Q(\xi_1, \psi) \frac{\partial U}{\partial \varphi}(\xi_1, \psi) d\varphi d\psi \leq M_3(\xi_2 - \xi_1). \quad (4.9) \]

For each \(\eta \in C^2([0, \xi_1] \times [0, m])\) vanishing near \(\{0\} \times [0, m] \cup \{(\zeta_0, m), (\xi_1, m)\} \times [0, m]\), it follows from \((4.9)\) and the classical theory on elliptic equations ([21, Theorem 8.12]) that \(\|\eta Q\|_{H^2((0, \xi_1) \times (0, m))} \leq M_1(\eta)(\xi_2 - \xi_1)^{1/2}\), which leads to
\[ \|\eta Q\|_{L^\infty((0, \xi_1) \times (0, m))} \leq M_2(\eta)(\xi_2 - \xi_1)^{1/2}. \quad (4.10) \]

Set
\[ \Lambda = \left( \varepsilon \left\| \frac{\partial b}{\partial \psi^2} \right\|_{L^\infty((0, \zeta_0/2) \times (0, m))} \right)^{1/2} + \frac{\varepsilon}{R_0 c_\varepsilon}, \quad \tilde{\zeta} = \min \left\{ \frac{\zeta_0}{2}, \frac{1}{\Lambda} \right\}. \]

Then \(0 \leq Q \in C^\infty([0, \tilde{\zeta}] \times [0, m])\) solves
\[ \frac{\partial^2 \tilde{Q}}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2}(b(\varphi, \psi)\tilde{Q}) = 0, \quad (\varphi, \psi) \in (0, \tilde{\zeta}) \times (0, m), \quad (4.11) \]
\[
\frac{\partial \tilde{Q}}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0} \tilde{Q}(0, \psi), \quad \tilde{Q}(\zeta, \psi) = Q(\tilde{\zeta}, \psi), \quad \psi \in (0, m), \tag{4.12}
\]

\[
\frac{\partial \tilde{Q}}{\partial \varphi}(0, \psi) = 0, \quad \frac{\partial \tilde{Q}}{\partial \varphi}(\varphi, m) = 0, \quad \varphi \in (0, \tilde{\zeta}). \tag{4.13}
\]

Similar to the proof of Proposition 3.1 one can show that the comparison principle holds for the problem (4.11)–(4.13). Set

\[
\tilde{Q}(\varphi, \psi) = \|Q(\tilde{\zeta}, \cdot)\|_{L^\infty(0, m)}(e^{\Lambda \tilde{\zeta}} - e^{\Lambda \varphi}), \quad (\varphi, \psi) \in [0, \tilde{\zeta}] \times [0, m].
\]

Then

\[
\frac{\partial^2 \tilde{Q}}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2}{\partial \psi^2}(b(\varphi, \psi)Q(\varphi, \psi)) = \|Q(\tilde{\zeta}, \cdot)\|_{L^\infty(0, m)}\left(-\Lambda^2 e^{\Lambda \varphi} + \frac{\partial^2 b}{\partial \psi^2}(\varphi, \psi)(e^{\Lambda \tilde{\zeta}} - e^{\Lambda \varphi})\right) \leq 0, \quad (\varphi, \psi) \in (0, \tilde{\zeta}) \times (0, m),
\]

\[
\frac{\partial \tilde{Q}}{\partial \varphi}(0, \psi) + \frac{h(\psi)}{R_0} \tilde{Q}(0, \psi) = \|Q(\tilde{\zeta}, \cdot)\|_{L^\infty(0, m)}\left(-\Lambda e^{\Lambda \varphi} + \frac{h(\psi)}{R_0}(e^{\Lambda \tilde{\zeta}} - e^{\Lambda \varphi})\right) \leq 0, \quad \psi \in (0, m).
\]

Hence \(\tilde{Q}\) is a supersolution to the problem (4.11)–(4.13). Thus

\[
Q(\varphi, \psi) \leq \tilde{Q}(\varphi, \psi) \leq e\|Q(\tilde{\zeta}, \cdot)\|_{L^\infty(0, m)}, \quad (\varphi, \psi) \in (0, \tilde{\zeta}) \times (0, m),
\]

which, together with (4.11), (4.10) and Proposition 3.5 leads to that

\[
\lim_{\xi_2 - \xi_1 \to 0} \|q_1 - q_2\|_{L^\infty((0, \xi_1) \times (0, m))} = 0. \tag{4.14}
\]

Turn to the continuous dependence on \(\zeta\). Fix \(\zeta_0/2 \leq \xi_1 \leq \zeta_2 \leq (\zeta_0 + \xi_0)/2\). Denote \(\tilde{q}_k = q[\zeta_k, \xi_0]\) for \(k = 1, 2\) and \(\tilde{Q} = A(\tilde{q}_1) - A(\tilde{q}_2)\). For each exponent \(\alpha \in (0, 1/2), \tilde{Q} \in C^2([0, \xi_0] \times [0, m] \setminus \{(\zeta_1, m), (\zeta_2, m), (\xi_0, m)\}) \cap H^1((0, \xi_0) \times (0, m)) \cap C^\alpha([0, \xi_0] \times [0, m])\) solves

\[
\frac{\partial^2 \tilde{Q}}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2}(\tilde{b}(\varphi, \psi)\tilde{Q}) = 0, \quad (\varphi, \psi) \in (0, \xi_0) \times (0, m),
\]

\[
\frac{\partial \tilde{Q}}{\partial \varphi}(0, \psi) = -\frac{\hat{h}(\psi)}{R_0} \tilde{Q}(0, \psi), \quad \tilde{Q}(\xi_0, \psi) = 0, \quad \psi \in (0, m),
\]

\[
\frac{\partial \tilde{Q}}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi_0),
\]

\[
\frac{\partial \tilde{Q}}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta_1),
\]

\[
\tilde{Q}(\varphi, m) = A(\tilde{q}_1)(\varphi, m) - A(\tilde{q}_2)(\varphi, m), \quad \varphi \in (\zeta_1, \xi_0),
\]

where

\[
\tilde{b}(\varphi, \psi) = \int_0^1 \frac{B'}{A'}(A^{-1}(tA(\tilde{q}_1(\varphi, \psi)) + (1 - t)A(\tilde{q}_2(\varphi, \psi)))) dt, \quad (\varphi, \psi) \in (0, \xi_0) \times (0, m),
\]

\[
\hat{h}(\psi) = \int_0^1 \frac{1}{A^{-1}(tA(\tilde{q}_1(0, \psi)) + (1 - t)A(\tilde{q}_2(0, \psi)))} dt, \quad \psi \in (0, m).
\]

Consider its dual problem

\[
\frac{\partial^2 \tilde{U}}{\partial \varphi^2} + \tilde{b}(\varphi, \psi) \frac{\partial^2 \tilde{U}}{\partial \psi^2} = \tilde{Q}(\varphi, \psi), \quad (\varphi, \psi) \in (0, \xi_0) \times (0, m),
\]
\[
\frac{\partial \tilde{U}}{\partial \varphi}(0, \psi) = -\frac{h(\psi)}{R_0} \tilde{U}(0, \psi), \quad \tilde{U}(\xi_0, \psi) = 0, \quad \psi \in (0, m),
\]

\[
\frac{\partial U}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi_0),
\]

\[
\frac{\partial U}{\partial \psi}(\varphi, m) = 0, \quad \varphi \in (0, \zeta_1),
\]

\[
\tilde{U}(\varphi, m) = 0, \quad \varphi \in (\zeta_1, \xi_0),
\]

which admits a unique solution \(\tilde{U} \in C^2([0, \xi_0] \times [0, m]) \cap H^1((0, \xi_0) \times (0, m)) \cap C^\alpha([0, \xi_0] \times [0, m])\) from the proof of Proposition 3.3. For \((\varphi, \psi) \in (0, \xi_0) \times (0, m)\) satisfying \((\varphi - (\zeta_1 + \zeta_2)/2)^2 + (\psi - m)^2 \leq (\zeta_2 - \zeta_1)^2\), it follows from the Hölder estimates for elliptic equations \((21)\) 8.11 that

\[
|\nabla \hat{Q}(\varphi, \psi)| \leq M_4(\alpha)((\varphi - \zeta_2)^2 + (\psi - m)^2)^{-(1-\alpha)/2},
\]

\[
|\nabla \hat{U}(\varphi, \psi)| \leq M_5(\alpha)((\varphi - \zeta_1)^2 + (\psi - m)^2)^{-(1-\alpha)/2}.
\]

Choose \(0 \leq \omega \in C^1([0, \xi_0] \times [0, m])\) such that for \((\varphi, \psi) \in (0, \xi_0) \times (0, m)\),

\[
\omega(\varphi, \psi) = \begin{cases} 
1, & \text{if } (\varphi - (\zeta_1 + \zeta_2)/2)^2 + (\psi - m)^2 \geq 4(\zeta_2 - \zeta_1)^2, \\
0, & \text{if } (\varphi - (\zeta_1 + \zeta_2)/2)^2 + (\psi - m)^2 \leq (\zeta_2 - \zeta_1)^2,
\end{cases}
\]

\[
|\nabla \omega(\varphi, \psi)| \leq \frac{4}{\zeta_2 - \zeta_1}, \quad \left| \frac{\partial^2 \omega}{\partial \varphi^2}(\varphi, \psi) \right| + \left| \frac{\partial^2 \omega}{\partial \psi^2}(\varphi, \psi) \right| \leq \frac{8}{(\zeta_2 - \zeta_1)^2}.
\]

Multiplying the equation of \(\hat{Q}\) by \(\omega \hat{U}\) and then integrating over \((0, \xi_0) \times (0, m)\) by parts, one gets that

\[
\int_0^{\xi_0} \int_0^m \omega(\varphi, \psi) \hat{Q}^2(\varphi, \psi) d\varphi d\psi \leq M_6(\alpha)(\zeta_2 - \zeta_1)^2a.
\]

A similar argument as (4.14) leads to \(\lim_{\zeta_2 - \zeta_1 \to 0} \|\hat{q}_1 - \hat{q}_2\|_{L^\infty((0, \xi_0) \times (0, m))} = 0\). \(\Box\)

### 4.2 Solvability and properties of solutions

Using Propositions 3.3 3.4 3.5 4.1 and Lemma 3.1 one can prove the following three lemmas.

**Lemma 4.1** Assume that \(0 < c_e < c_a\) and \(R_0 \partial \xi_1 \rho(c_e^2) < m < R_0 \partial \xi_1 \rho(c_a^2)\).

(i) For \(0 < \zeta < R_0 c_1\), there is at most one solution to the problem (2.3)–(2.9) in \(\mathcal{S}\).

(ii) For \(\hat{\zeta} < \zeta < R_0 c_1\), there is not a solution to the problem (2.3)–(2.9) in \(\mathcal{S}\).

**Proof.** For \(0 < \zeta < R_0 c_1\), assume that \((q_1, \xi_1), (q_2, \xi_2) \in \mathcal{S}\) are two solutions to the problem (2.3)–(2.9). We first prove \(\xi_1 = \xi_2\) by contradiction. If not, it is assumed that \(\xi_1 < \xi_2\) without loss of generality. Then, Propositions 3.3 3.4 3.5 lead to \(q_2(\xi_1, \psi) < c_e = q_1(\xi_1, \psi)\) for \(\psi \in (0, m)\). It follows from this estimate, Proposition 3.3 and Lemma 3.1 that \(q_2(0, \psi) < q_1(0, \psi)\) for \(\psi \in [0, m]\), which contradicts (2.9). Hence \(\xi_1 = \xi_2\). And Proposition 3.3 shows \(q_1 = q_2\).

For \(\hat{\zeta} < \zeta < R_0 c_1\), assume that \((q, \xi) \in \mathcal{S}\) is a solution to the problem (2.3)–(2.9). Then \(\xi \geq \zeta > \hat{\zeta}\). Propositions 3.3 3.4 3.5 and Lemma 2.1 lead to \(q(\hat{\zeta}, \psi) < c_e = \hat{q}(\hat{\zeta})\) for \(\psi \in (0, m)\), which, together with Proposition 3.3 and Lemma 3.1 shows that \(q(0, \psi) < \hat{q}(0)\) for \(\psi \in [0, m]\). Hence

\[
\int_0^m \frac{1}{q(0, \psi)\rho(q^2(0, \psi))} d\psi > \frac{m}{\hat{q}(0)\rho(\hat{q}^2(0))} = R_0 \vartheta,
\]

which contradicts (2.9). \(\Box\)
Lemma 4.2 Assume that for $\zeta_0 \in (0, \hat{\zeta})$, the problem (2.3)–(2.9) with $\zeta = \zeta_0$ admits a solution $(q_0, \xi_0) \in \mathcal{F}$ with $\xi_0 < R_0 c_1$. Then, there exists $\tau \in (0, \zeta_0)$, such that for each $\zeta \in (\zeta_0 - \tau, \zeta_0)$, the problem (2.3)–(2.9) admits a solution $(q, \xi) \in \mathcal{F}$.

Proof. For $\zeta = \zeta_0$, denote $q_1, q_2 \in \mathcal{F}$ to be the solutions to the problem (2.3)–(2.8) with $\xi = R_0 c_1$ and $\xi = (\zeta_0 + R_0 c_1)/2$, respectively. Then, Propositions 3.3, 3.4, 3.5, and Lemma 3.1 yield $c_1 \leq q_1(0, \psi) < q_2(0, \psi) < q_0(0, \psi)$ for $\psi \in [0, m]$. Due to Propositions 3.3, 3.4, 3.5, and 1.1, there exists $\tau \in (0, \zeta_0)$, such that for $\zeta \in (\zeta_0 - \tau, \zeta_0)$, the solution $q_+^{\zeta} \in \mathcal{F}$ to the problem (2.3)–(2.8) with $\xi = (\zeta_0 + R_0 c_1)/2$ satisfies $c_1 \leq q_+^{\zeta}(0, \psi) < q_0(0, \psi)$ for $\psi \in [0, m]$. Hence

$$\int_0^m \frac{1}{q_+^{\zeta}(0, \psi)\rho(q_+^{\zeta}(0, \psi))} d\psi > \int_0^m \frac{1}{q_0(0, \psi)\rho(q_0(0, \psi))} d\psi = R_0 \vartheta. \quad (4.15)$$

For $\zeta \in (\zeta_0 - \tau, \zeta_0)$, Propositions 3.3, 3.4, 3.5, and Lemma 3.1 show that the solution $q_-^{\zeta} \in \mathcal{F}$ to the problem (2.3)–(2.8) with $\xi = \zeta_0$ satisfies $q_-^{\zeta}(0, \psi) > q_0(0, \psi)$ for $\psi \in [0, m]$, which yields

$$\int_0^m \frac{1}{q_-^{\zeta}(0, \psi)\rho(q_-^{\zeta}(0, \psi))} d\psi < \int_0^m \frac{1}{q_0(0, \psi)\rho(q_0(0, \psi))} d\psi = R_0 \vartheta. \quad (4.16)$$

For $\zeta \in (\zeta_0 - \tau, \zeta_0)$, it follows from (4.15), (4.16), Propositions 3.3, 3.4, 3.5, and 1.1, that there exists $\xi \in (\zeta_0, R_0 c_1)$ such that the problem (2.3)–(2.9) admits a solution $(q, \xi) \in \mathcal{F}$. □

Lemma 4.3 Assume that for $\zeta_0 \in (0, \hat{\zeta})$, the problem (2.3)–(2.9) with $\zeta = \zeta_0$ admits a solution $(q_0, \xi_0) \in \mathcal{F}$. Then, for each $\zeta \in (\zeta_0, \hat{\zeta})$, the problem (2.3)–(2.9) admits a solution $(q, \xi) \in \mathcal{F}$.

Proof. Give $\zeta \in (\zeta_0, \hat{\zeta})$. Denote $q_1, q_2 \in \mathcal{F}$ to be the solutions to the problem (2.3)–(2.8) with $\xi = \zeta_0$ and $\xi = \hat{\zeta}$, respectively. Then, Propositions 3.3, 3.4, 3.5, and Lemmas 2.1, 3.1 yield $c_1 \leq q_1(0, \psi) < q_0(0, \psi)$ and $q_2(0, \psi) > q(0) > c_1$ for $\psi \in [0, m]$. Hence

$$\int_0^m \frac{1}{q_1(0, \psi)\rho(q_1(0, \psi))} d\psi > \int_0^m \frac{1}{q_0(0, \psi)\rho(q_0(0, \psi))} d\psi = R_0 \vartheta,$$

$$\int_0^m \frac{1}{q_2(0, \psi)\rho(q_2(0, \psi))} d\psi < \int_0^m \frac{m}{q(0)\rho(q(0))} = R_0 \vartheta.$$

Therefore, Propositions 3.3, 3.5, and 1.1 show that there exists $\xi \in (\zeta_0, \xi_0)$ such that the problem (2.3)–(2.9) admits a solution $(q, \xi) \in \mathcal{F}$. □

We are ready to prove the following existence and nonexistence results for the free boundary problem (2.3)–(2.9) in $\mathcal{F}$.

Theorem 4.1 For $0 < c_e < c_\ast$ and $R_0 \vartheta c_\ast \rho(c_\ast^2) < m < R_0 \vartheta c_e \rho(c_e^2)$, there exists a constant $\zeta_\ast \in (0, \hat{\zeta})$, depending only on $R_0$, $\vartheta$, $c_e$, and $\gamma$, such that the problem (2.3)–(2.9) admits a unique solution $(q[\zeta], \xi[\zeta]) \in C^2((0, \xi] \times [0, m] \setminus \{(m, m), (\xi, m)\}) \cap H^1((0, \xi] \times (0, m)) \cap \mathcal{F}$ if $\zeta \in [\zeta_\ast, \hat{\zeta}] \cap (0, \hat{\zeta})$, while there is not such a solution if $\zeta \in (0, \zeta_\ast) \cup (\hat{\zeta}, R_0 c_1)$, where $c_1$ and $\hat{\zeta}$ are given in Theorem 2.1 and Lemma 2.1, respectively. If $\zeta_\ast > 0$ additionally, then $\zeta[\zeta_\ast] = R_0 c_1$.

Proof. Set $\zeta_\ast = \inf \mathcal{E}$, where

$$\mathcal{E} = \{\zeta \in (0, R_0 c_1) : \text{the problem (2.3)–(2.9) admits a unique solution } (q[\zeta], \xi[\zeta]) \in \mathcal{F}\}.$$ 

It follows from Lemmas 4.1, 4.2, 4.3 and 2.1 that

$$\zeta_\ast \in [0, \hat{\zeta}), \quad (\zeta_\ast, \hat{\zeta}) \subset \mathcal{E}, \quad (0, \zeta_\ast) \cap \mathcal{E} = \emptyset, \quad (\hat{\zeta}, R_0 c_1) \cap \mathcal{E} = \emptyset.$$

If $\zeta_\ast > 0$ additionally, Proposition 4.1 and Lemma 4.2 show that $\zeta \in \mathcal{E}$ and $\xi[\zeta_\ast] = R_0 c_1$. □

Solutions to the problem (2.3)–(2.9) have the following properties.
Theorem 4.2 Assume that $0 < c_ε < c_τ$ and $R_0 ρ_ε (c_ε^2) < m < R_0 ρ_ε (c_τ^2)$. For $ξ ∈ [ξ_*, ξ] ∩ (0, ˆξ)$, let $(q[ξ], θ[ξ]) ∈ D(ε)$ be the unique solution to the problem (2.3)–(2.9), where $ξ_*$, $c_1$ and $ˆξ$ are given in Theorem 4.1, Theorem 2.1 and Lemma 2.1, respectively.

(i) $q[ξ], θ[ξ] ∈ C^∞([0, ξ] × [0, m] \backslash \{(ξ, m), (ξ, m), (ξ, m)\}) ∩ H^1((0, ξ) × (0, m)) ∩ C^{0,1}([0, ξ] × [0, m] \backslash \{(ξ, m)\}) ∩ C^α([0, ξ] × [0, m])$ for each exponent $α ∈ (0, 1/2)$, and

$$c_1 < q[ξ](φ, ψ) < c_ε, \quad (φ, ψ) ∈ (0, ξ) × (0, m) \cup (0, ξ) × \{m\},$$

$$−θ < θ[ξ](φ, ψ) < 0, \quad (φ, ψ) ∈ [0, ξ] × (0, m) \cup (ξ, ξ) × \{m\},$$

$$\frac{∂q[ξ]}{∂φ}(φ, ψ) > 0, \quad \frac{∂θ[ξ]}{∂ψ}(φ, ψ) < 0, \quad (φ, ψ) ∈ [0, ξ] × (0, m),$$

$$\frac{∂q[ξ]}{∂ψ}(φ, ψ) > 0, \quad \frac{∂θ[ξ]}{∂φ}(φ, ψ) > 0, \quad (φ, ψ) ∈ (0, ξ) × (0, m) \cup (ξ, ξ) × \{m\},$$

where $θ[ξ]$ is the flow angle of $q[ξ]$.

(ii) $(q[ξ], θ[ξ]) = (q, ˆθ)$ with $q$ given in Lemma 2.1.

(iii) If $ξ_∗ > 0$ additionally, then $ξ[ξ_*] = R_0 c_1$.

(iv) For each $ξ_0 ∈ [ξ_*, ξ] ∩ (0, ˆξ)$,

$$\lim_{ξ→ξ_0} ξ[ξ] = ξ[ξ_0],$$

$$\lim_{ξ→ξ_0} (q[ξ], θ[ξ])(φ, ψ) = (q[ξ_0], θ[ξ_0])(φ, ψ) \text{ uniformly for } (φ, ψ) ∈ [0, ξ[ξ_0]] × [0, m].$$

(v) For $ξ_1, ξ_2 ∈ [ξ_*, ξ] ∩ (0, ˆξ)$ with $ξ_1 < ξ_2$, it holds

$$ξ[ξ_1] > ξ[ξ_2], \quad L[ξ_1] > L[ξ_2],$$

$$q[ξ_1](φ, m) > q[ξ_2](φ, m), \quad φ ∈ [0, ξ_1),$$

$$q[ξ_1](φ, 0) < q[ξ_2](φ, 0), \quad φ ∈ [0, ξ[ξ_2]],$$

where $L[ξ]$ is the length of the upper wall of the nozzle in the physical plane for the flow $(q[ξ], ξ[ξ])$.

Proof. Theorem 4.1, Propositions 3.3, 3.4, 3.5 and 4.1, Lemmas 2.1 and 3.1 yield (i)–(iv) directly, and it suffices to verify (v). Give $ξ_1, ξ_2 ∈ [ξ_*, ξ] ∩ (0, ˆξ)$ with $ξ_1 < ξ_2$. For convenience, denote $q_k = q[ξ_k]$ and $ξ_k = ξ[ξ_k]$ for $k = 1, 2$. If $ξ_1 ≤ ξ_2$, then (i) leads to $q_2(ξ_1, ψ) < c_ε = q_1(ξ_1, ψ)$ for $ψ ∈ (0, m)$, which contradicts (2.9). Hence $ξ_1 > ξ_2$, which yields the first estimate in (4.17). It follows from $ξ_1 > ξ_2$ and (i) that

$$q_1(ξ_2, ψ) < c_ε = q_2(ξ_2, ψ), \quad ψ ∈ (0, m).$$

Set $Γ_+ = \{ξ_1, ξ_2\} × \{m\}$, $Γ_- = \{ξ_2\} × [0, m]$, and $G_+ = \{(φ, ψ) ∈ [0, ξ_2] × [0, m] : ±q_1(φ, ψ) > ±q_2(φ, ψ)\}$. Then, (i) and (4.20) show that $G_+ \subseteq G_+$. We claim that $G_+ \subseteq G_+$. If not, let $G^*_+ \subseteq G_+$ be a maximal subdomain of $G_+$ such that $Γ_+ \cap G^*_+ = \emptyset$. Note that $q_1$ and $q_2$ satisfy the same boundary conditions on $∂((0, ξ_2) × (0, m)) \backslash (Γ_+ \cup Γ_-)$, $q_1 = q_2$ on $∂G^*_+ \cap (0, ξ_2) × (0, m)$, and $G^*_+ \subseteq G_+$ satisfy the interior cone condition. Then, the comparison principle, which can be proved similar to Proposition 3.3, leads to $±q_1 ≤ ±q_2$ in $G^*_+$, which contradicts $G^*_+ ⊆ G_+$. Hence

$$G_+ \text{ and } G^- \text{ are connected and relatively open sets in } [0, ξ_2] × [0, m].$$
Next we show that
\[ [0, \xi_2] \times \{0\} \cap \mathcal{G}_+ = \emptyset \quad (4.22) \]
by contradiction. If not, from \((4.21)\) and \(\Gamma_+ \subset G_+\), there exists a \(C^1\) simple curve \(\mathcal{L}_+^* \subset (0, \xi_2) \times [0, m] \cup \{(0,0), (\xi_2,0)\}\) from \(0, \xi_2 \times \{0\}\) to \(\Gamma_+\) such that \(q_1 \geq q_2\) on \(\mathcal{L}_+^*\) and \(\mathcal{G}_+\) satisfies the interior cone condition, where \(\mathcal{G}_+ = \{ (\varphi, \psi) \in (0, \xi_2) \times (0, m) : (\varphi, \psi) \text{ is located at the left side of } \mathcal{L}_+^* \}\).

Using the comparison principle, which can be proved in a similar way as Proposition 3.3, one gets that \(q_1 \geq q_2\) on \(\mathcal{G}_+\). Then, Lemma 3.1 leads to \(q_1(0, \psi) > q_2(0, \psi)\) for \(\psi \in (0, m)\), which contradicts \((2.9)\). Similarly, one can prove that
\[ [0, \xi_1] \times \{m\} \cap \mathcal{G}_- = \emptyset \quad (4.23) \]
If not, from \((4.21)\) and \(\Gamma_- \subset G_-\), there exists a \(C^1\) simple curve \(\mathcal{L}_-^* \subset (0, \xi_2) \times (0, m) \cup [0, \xi_1] \times \{m\}\) from \(\Gamma_-\) to \([0, \xi_1] \times \{m\}\) such that \(q_1 \leq q_2\) on \(\mathcal{L}_-^*\) and \(\mathcal{G}_-\) satisfies the interior cone condition, where \(\mathcal{G}_- = \{ (\varphi, \psi) \in (0, \xi_2) \times (0, m) : (\varphi, \psi) \text{ is located at the left side of } \mathcal{L}_-^* \}\).

It follows from \((4.21)-(4.23)\) that there exists a suitable constant \(\tau \in (0, m)\) such that
\[ q_1 \leq q_2 \text{ in } [0, \xi_2] \times [0, \tau] \quad \text{and} \quad q_1 \geq q_2 \text{ in } [0, \xi_1] \times [m - \tau, m], \]
which, together with Lemma 3.1 leads to \((4.18)\) and \((4.19)\). Note that
\[ L[\zeta] = \int_0^\zeta \frac{1}{q[\zeta](\varphi, m)} d\varphi. \]
So, the second estimate in \((4.17)\) follows from the first estimate in \((4.17)\) and \((4.18)\). \(\square\)

Transforming Theorems 4.1 and 4.2 into the physical plane, one can get Theorem 2.1 and 2.2.

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