Symmetrized GA-convexity and Related Some Integral Inequalities

İmdat İşcan

Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28200, Giresun, Turkey.

Abstract. In this paper, we introduce a new concept called the symmetrized GA-convex function and give Hermite-Hadamard’s inequalities for symmetrized GA-convex functions. Furthermore, we establish Hermite-Hadamard type inequalities for the product of a GA-convex function with a symmetrized GA-convex function and also for two symmetrized GA-convex functions.

1. Introduction

Let real function $f$ be defined on some nonempty interval $I$ of real line $\mathbb{R}$. The function $f$ is said to be convex on $I$ if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The following definitions are well known in the literature.

Definition 1.1 ([9, 10]). A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Since the condition (2) can be written as

$$(f \circ \exp)(t \ln x + (1-t) \ln y) \leq t(f \circ \exp)(\ln x) + (1-t)(f \circ \exp)(\ln y),$$

then we observe that $f : I \subseteq (0, \infty) \to \mathbb{R}$ is GA-convex on $I$ if and only if $f \circ \exp$ is convex on $\ln I := \{\ln z, z \in I\}$.

In [8], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows:
\textbf{Theorem 1.2.} Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a GA-convex function and \( a, b \in I \) with \( a < b \). Let \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and geometrically symmetric with respect to \( \sqrt{ab} \) (i.e. \( g(x) = g \left( \frac{ax}{b} \right) \) for all \( x \in [a, b] \)). Then

\[
 f \left( \sqrt{ab} \right) \int_a^b \frac{g(x)}{x} \, dx \leq \int_a^b \frac{g(x)f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} \, dx.
\]

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

\textbf{Definition 1.3.} Let \( f \in L[a, b] \). The left-sided and right-sided Hadamard fractional integrals \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) of order \( \alpha > 0 \) with \( b > a \geq 0 \) are defined by

\[
 J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b
\]

and

\[
 J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad a < x < b
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt \) (see [7]).

Hermite-Hadamard’s inequalities can be represented for GA-convex functions in fractional integral forms as follows.

\textbf{Theorem 1.4 ([5]).} Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a function such that \( f \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is a GA-convex function on \( [a, b] \), then the following inequalities for fractional integrals hold:

\[
 f \left( \sqrt{ab} \right) \leq \frac{\Gamma(\alpha + 1)}{2 \left( \ln 2 \right)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2} \quad (3)
\]

with \( \alpha > 0 \).

In [6], the authors presented Hermite–Hadamard-Fejer inequalities for GA-convex functions in fractional integral forms as follows:

\textbf{Theorem 1.5.} Let \( f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a GA-convex function on \( [a, b] \) with \( a < b \) and \( f \in L[a, b] \). If \( g : [a, b] \rightarrow \mathbb{R} \) is nonnegative, integrable and geometrically symmetric with respect to \( \sqrt{ab} \), then the following inequalities for fractional integrals hold:

\[
 f \left( \sqrt{ab} \right) \left[ J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a) \right] \leq \left[ J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a) \right] \leq \frac{f(a) + f(b)}{2} \left[ J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a) \right] \quad (4)
\]

with \( \alpha > 0 \).

For a function \( f : [a, b] \rightarrow \mathbb{R} \) we consider the symmetrical transform of \( f \) on the interval, denoted by \( \tilde{f}_{[a,b]} \) or simply \( \tilde{f} \), when the interval \([a, b]\) is implicit, which
is defined by
\[
\tilde{f}(x) := \frac{1}{2} \left[ f(x) + f(a + b - x) \right], \; x \in [a, b].
\]

The anti symmetrical transform of \( f \) on the interval \([a, b]\) is denoted by \( \tilde{f}_{[a, b]} \) or simply \( \tilde{f} \) as defined by
\[
\tilde{f}(x) := \frac{1}{2} \left[ f(x) - f(a + b - x) \right], \; x \in [a, b].
\]

It is obvious that for any function \( f \) we have \( \tilde{f} + \tilde{f} = f \).

If \( f \) is convex on \([a, b]\), then \( \tilde{f} \) is also convex on \([a, b]\). But, when \( \tilde{f} \) is onvex on \([a, b]\), \( f \) may not be convex on \([a, b]\) \([1]\).

In \([1]\), Dragomir introduced symmetrized convexity concept as follow:

**Definition 1.6.** A function \( f : [a, b] \to \mathbb{R} \) is said to be symmetrized convex (concave) on \([a, b]\) if symmetrical transform \( \tilde{f} \) is convex (concave) on \([a, b]\).

Dragomir extended the Hermite-Hadamard inequality to the class of symmetrized convex functions as follow:

**Theorem 1.7** \([1]\). Assume that \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\), then we have the Hermite-Hadamard inequalities
\[
f\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

**Theorem 1.8** \([1]\). Assume that \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\). Then for any \( x \in [a, b] \) we have the bounds
\[
f\left( \frac{a + b}{2} \right) \leq \tilde{f}(x) = \frac{1}{2} \left[ f(x) + f(a + b - x) \right] \leq \frac{f(a) + f(b)}{2}.
\]

**Corollary 1.9** \([1]\). If \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\) and \( g : [a, b] \to [0, \infty) \) is integrable on \([a, b]\), then
\[
f\left( \frac{a + b}{2} \right) \int_a^b g(x)dx \leq \int_a^b g(x) \tilde{f}(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx.
\]

Moreover, if \( g \) is symmetric with respect to \( \frac{a + b}{2} \) on \([a, b]\), i.e. \( g(x) = g(a + b - x) \) for almost every \( x \in [a, b]\), then
\[
f\left( \frac{a + b}{2} \right) \int_a^b g(x)dx \leq \int_a^b g(x) f(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx.
\]

**Theorem 1.10** \([1]\). Assume that \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\). Then for any \( x, y \in [a, b] \) with \( x \neq y \) we have the Hermite-Hadamard inequalities
\[
\frac{1}{2} \left[ f\left( \frac{x + y}{2} \right) + f\left( a + b - \frac{x + y}{2} \right) \right] \leq \frac{1}{2(y - x)} \left[ \int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt \right] \leq \frac{1}{4} \left[ f(x) + f(y) + f(a + b - x) + f(a + b - y) \right].
\]
2. Symmetrized GA-Convexity

For a function \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) we consider the geometric symmetrical transform of \( f \) on the interval, denoted by \( G_{f[a,b]} \) or simply \( G_f \), when the interval \([a, b]\) is implicit, which is defined by

\[
G_f(x) := \frac{1}{2} \left[ f(x) + f\left(\frac{ab}{x}\right) \right], \quad x \in [a, b].
\]

The anti-geometric symmetrical transform of \( f \) on the interval \([a, b]\) is denoted by \( \hat{G}_{f[a,b]} \) or simply \( \hat{G}_f \) as defined by

\[
\hat{G}_f(x) := \frac{1}{2} \left[ f(x) - f\left(\frac{ab}{x}\right) \right], \quad x \in [a, b].
\]

It is obvious that for any function \( f \) we have \( \hat{G}_f + \hat{G}_f = f \).

If \( f \) is GA-convex on \([a, b]\), then \( \hat{G}_f \) is also GA-convex on \([a, b]\). Indeed, for any \( x, y \in [a, b] \) and \( t \in [0, 1] \) we have

\[
G_f(x^ty^{1-t}) = \frac{1}{2} \left[ f(x^ty^{1-t}) + f\left(\frac{ab}{x^ty^{1-t}}\right) \right] \\
\leq \frac{1}{2} \left[ f(x) + f\left(\frac{ab}{x}\right) \right] + (1 - t) \frac{1}{2} \left[ f(y) + f\left(\frac{ab}{y}\right) \right] \\
= tG_f(x) + (1 - t)G_f(y).
\]

**Remark 2.1.** Consider the real numbers \( a < 0 < b \) with \((a + b)/2 > 0\) and define the function \( f_{\cdot} : [e^a, e^b] \to \mathbb{R} \), \( f_{\cdot}(x) = \ln^3 x \). Since the function \((f \circ \exp)(x) = x^3\) is not convex on \([a, b]\), \( f_{\cdot} \) is not GA-convex on \([e^a, e^b]\). On the other hand, since the function \((\hat{G}_{f_{\cdot}} \circ \exp)(x) = \frac{1}{2} \left[ x^3 + (a + b - x)^3 \right]\) is convex on \([a, b]\), \( \hat{G}_{f_{\cdot}}(x) = \frac{1}{2} \left[ \ln^3 x + (a + b - \ln x)^3 \right] \) is GA-convex on \([e^a, e^b]\).

**Definition 2.2.** A function \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is said to be symmetrized GA-convex (GA-concave) on \([a, b]\) if geometric symmetrical transform \( \hat{G}_f \) is GA-convex (GA-concave) on \([a, b]\).

**Example 2.3.** Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \) and \( a \geq 2 \). Then the function \( f : [a, b] \to \mathbb{R}, \ f(x) = \left( \ln \frac{x}{2} \right)^{a-1} \) is GA-convex on \([a, b]\). Indeed, for any \( u, v \in [a, b] \) and \( t \in [0, 1] \) by convexity of the function \( g(\zeta) = \zeta^{a-1}, \zeta \geq 0 \), we...
have

\[ f(u^tv^{1-t}) = \left( \ln \frac{u^tv^{1-t}}{a} \right)^{a-1} \]

\[ = \left( t \ln \frac{u}{a} + (1-t) \ln \frac{v}{a} \right)^{a-1} \]

\[ \leq t \left( \ln \frac{u}{a} \right)^{a-1} + (1-t) \left( \ln \frac{v}{a} \right)^{a-1} \]

\[ = t f(u) + (1-t)f(v). \]

Thus \( \tilde{f} \) is also GA-convex on \([a, b]\). Therefore \( f \) is symmetrized GA-convex function.

**Example 2.4.** Let \( \alpha \geq 2 \). Then the function \( f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \), \( f(x) = \left( \ln \frac{x}{a} \right)^{a-1} \), is GA-convex on \([a, b]\). Therefore \( f \) is symmetrized GA-convex function.

**Example 2.5.** Let \( \alpha \geq 2 \). Then the function \( f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \), \( f(x) = \left( \ln \frac{x}{a} \right)^{a-1} + \left( \ln \frac{x}{b} \right)^{a-1} \), is symmetrized GA-convex function.

Now if \( GAC[a, b] \) is the class of GA-convex functions defined on \( I \) and \( SGAC[a, b] \) is the class of symmetrized GA-convex functions on \([a, b]\) then

\[ GAC[a, b] \not\subset SGAC[a, b]. \]

Also, if \([c, d] \subset [a, b]\) and \( f \in SGAC[a, b] \), then this does not imply in general that \( f \in SGAC[c, d] \).

**Proposition 2.6.** Let \( f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a function. \( f \) is symmetrized GA-convex on the interval \([a, b]\) if and only if \( f \circ \exp \) is symmetrized convex on the interval \([\ln a, \ln b]\).

**Proof.** Let \( f \) be a symmetrized GA-convex function on the interval \([a, b]\). If we take arbitrary \( x, y \in [\ln a, \ln b] \), then there exist \( u, v \in [a, b] \) such that \( x = \ln u \) and \( y = \ln v \)

\[ (f \circ \exp)(tx + (1-t)y) \]

\[ = \frac{1}{2} [(f \circ \exp)(tx + (1-t)y) + (f \circ \exp)(\ln a + \ln b - tx - (1-t)y)] \]

\[ = \frac{1}{2} [(f \circ \exp)(tx + (1-t)y) + (f \circ \exp)(t [\ln(ab) - x] + (1-t) [\ln(ab) - y])] \]

\[ = \frac{1}{2} \left( f(u^tv^{1-t}) + f \left( \ln \frac{u^tv^{1-t}}{ab} \right) \right) \]

\[ = \tilde{f}(u^tv^{1-t}). \]

Since \( f \) is a symmetrized GA-convex function on the interval \([a, b]\), we have

\[ \tilde{f}(u^tv^{1-t}) \leq t \tilde{f}(u) + (1-t)\tilde{f}(v) \]

\[ = \frac{1}{2} \left( f(u) + f \left( \frac{ab}{u} \right) \right) + (1-t)\frac{1}{2} \left( f(v) + f \left( \frac{ab}{v} \right) \right) \]

\[ = tf(u) + (1-t)f(v) = tf(\tilde{e}^t) + (1-t)f(\tilde{e}^t) \]

\[ = \tilde{f}(u) + (1-t)(f \circ \exp)(y) \]

By (8) and (9), we obtain that \( f \circ \exp \) is symmetrized convex on the interval \([\ln a, \ln b]\).

Conversely, if \( f \circ \exp \) is symmetrized convex on the interval \([\ln a, \ln b]\) then it is easily seen that \( f \) is symmetrized GA-convex on the interval \([a, b]\) by a similar procedure. The details are omitted. \( \square \)
Theorem 2.7. If $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a, b]$, then we have the Hermite-Hadamard inequalities

$$f \left( \sqrt[4]{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2}. \quad (10)$$

Proof. Since $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a, b]$, then by writing the Hermite-Hadamard inequality for the function $\tilde{G}_f$, we have

$$\tilde{G}_f \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{\tilde{G}_f(x)}{x} \, dx \leq \frac{\tilde{G}_f(a) + \tilde{G}_f(b)}{2}, \quad (11)$$

where, it is easily seen that

$$\tilde{G}_f \left( \sqrt{ab} \right) = f \left( \sqrt{ab} \right), \quad \frac{\tilde{G}_f(a) + \tilde{G}_f(b)}{2} = \frac{f(a) + f(b)}{2},$$

and

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{\tilde{G}_f(x)}{x} \, dx = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx.$$

Then by (11) we get required inequalities. \qed

Remark 2.8. By helping Theorem 1.7 and Proposition 2.6, the proof of Theorem 2.7 can also be given as follows:

Since $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a, b]$, $f \circ \exp$ is symmetrized convex on the interval $[\ln a, \ln b]$. So, by Theorem 1.7 we have

$$(f \circ \exp) \left( \frac{\ln a + \ln b}{2} \right) \leq \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} (f \circ \exp)(x) \, dx \leq \frac{(f \circ \exp)(\ln a) + (f \circ \exp)(\ln b)}{2},$$

i.e.

$$f \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 2.9. If $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized GA-convex on the interval $[a, b]$. Then for any $x \in [a, b]$ we have the bounds

$$f \left( \sqrt[4]{ab} \right) \leq \tilde{G}_f(x) \leq \frac{f(a) + f(b)}{2}. \quad (12)$$

Proof. Since $\tilde{G}_f$ is GA-convex on $[a, b]$ then for any $x \in [a, b]$ we have

$$f \left( \sqrt[4]{ab} \right) = \tilde{G}_f \left( \sqrt{ab} \right) \leq \frac{\tilde{G}_f(x) + \tilde{G}_f(\frac{x}{2})}{2} = \tilde{G}_f(x).$$

This gives us the first inequality in (12).

Also, for any $x \in [a, b]$ there exist a number $t_0 \in [0, 1]$ such that $x = a^{t_0}b^{1-t_0}$. By the GA-convexity of $\tilde{G}_f$ we have

$$\tilde{G}_f(x) \leq t_0 \tilde{G}_f(a) + (1 - t_0) \tilde{G}_f(b)$$

$$\tilde{G}_f(a) = \frac{f(a) + f(b)}{2}$$

which gives the second inequality in (12). \qed
Remark 2.10. By helping Theorem 1.8 and Proposition 2.6, the proof of Theorem 2.9 can also be given as follows:

Since \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized GA-convex on the interval \([a, b]\), \( f \circ \exp \) is symmetrized convex on the interval \([\ln a, \ln b]\). So, by Theorem 1.8 we have

\[
(f \circ \exp)\left(\frac{\ln a + \ln b}{2}\right) \leq (f \circ \exp)(\ln x) \leq \frac{(f \circ \exp)(\ln a) + (f \circ \exp)(\ln b)}{2},
\]

i.e.

\[
f\left(\sqrt[ab]{x}\right) \leq \mathcal{C}_f(x) \leq \frac{f(a) + f(b)}{2}
\]

for any \( x \in [a, b] \).

Remark 2.11. If \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized GA-convex on the interval \([a, b]\), then we have the bounds

\[
\inf_{x \in [a, b]} G_f(x) = f\left(\sqrt[ab]{a}\right)
\]

and

\[
\sup_{x \in [a, b]} \mathcal{C}_f(x) = \frac{f(a) + f(b)}{2}.
\]

Corollary 2.12. If \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized GA-convex on the interval \([a, b]\) and \( g : [a, b] \to [0, \infty) \) is integrable on \([a, b]\), then

\[
f\left(\sqrt[ab]{x}\right) \int_a^b \frac{g(x)\mathcal{C}_f(x)}{x} \, dx \leq \int_a^b \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} \, dx.
\]

Moreover, if \( g \) is geometrically symmetric with respect to \( \sqrt[ab]{x}\) on \([a, b]\), i.e. \( g(x) = g(ab/x) \) for all every \( x \in [a, b] \), then

\[
f\left(\sqrt[ab]{x}\right) \int_a^b \frac{g(x)f(x)}{x} \, dx \leq \int_a^b \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} \, dx.
\]

Proof. The inequality (13) follows by (12) multiplying by \( g(x)/x \geq 0 \) and integrating over \( x \) on \([a, b]\).

By changing the variable, we have

\[
\int_a^b \frac{g(x)f\left(\frac{a}{x}\right)}{x} \, dx = \int_a^b \frac{g\left(\frac{a}{x}\right)f(x)}{x} \, dx.
\]

Since \( g \) is geometrically symmetric with respect to \( \sqrt[ab]{x}\), then

\[
\int_a^b \frac{g\left(\frac{a}{x}\right)f(x)}{x} \, dx = \int_a^b \frac{g(x)f(x)}{x} \, dx.
\]

Thus

\[
\int_a^b \frac{g(x)\mathcal{C}_f(x)}{x} \, dx = \frac{1}{2} \left[ \int_a^b \frac{g(x)f(x)}{x} \, dx + \int_a^b \frac{g\left(\frac{a}{x}\right)f(x)}{x} \, dx \right]
\]

\[
= \int_a^b \frac{g(x)f(x)}{x} \, dx
\]

and by (13) we get (14). \( \square \)
Remark 2.13. The inequality (14) is known as weighted generalization of Hermite-Hadamard inequality for GA-convex functions. It has been shown now that this inequality remains valid for the larger class of symmetrized GA-convex functions \( f \) on the interval \([a,b]\).

Remark 2.14. We note that by helping Corollary 1.9 and Proposition 2.6, the proof of Corollary 2.12 can also be given. The details is omitted.

Remark 2.15. Let \( a, b, a \in \mathbb{R} \) with \( 0 < a < b \) and \( \alpha \geq 2 \). Then the function \( f : [a,b] \to \mathbb{R}, f(x) = (\ln \frac{x}{a})^{\alpha-1} \) is symmetrized GA-convex on \([a,b]\), then

i.) If we consider the function

\[
\int_a^b \frac{g(x)}{x} \, dx \leq \frac{\Gamma(\alpha+1)}{2(\ln \frac{b}{a})^{\alpha-1}} \left[ \int_a^b g(b) + \int_a^b g(a) \right] \leq \frac{1}{2} \int_a^b \frac{g(x)}{x} \, dx
\]

for any \( g : [a,b] \to [0, \infty) \) is integrable on \([a,b]\).

ii.) If we consider the function

\[
g(x) = \left( \ln \frac{x}{a} \right)^{\alpha-1} + \left( \ln \frac{b}{x} \right)^{\alpha-1}
\]

which is geometrically symmetric with respect to \( \sqrt{ab} \) in the inequality (14), then we have the following inequalities

\[
f(\sqrt{ab}) \leq \frac{\Gamma(\alpha+1)}{2(\ln \frac{b}{a})^{\alpha-1}} \left[ \int_a^b f(b) + \int_a^b f(a) \right] \leq \frac{f(a) + f(b)}{2}
\]

which are the same of inequalities in (3).

iii.) Let \( w \) be geometrically symmetric with respect to \( \sqrt{ab} \). If we consider the function

\[
g(x) = \left[ \left( \ln \frac{x}{a} \right)^{\alpha-1} + \left( \ln \frac{b}{x} \right)^{\alpha-1} \right] w(x)
\]

which is geometrically symmetric with respect to \( \sqrt{ab} \) in the inequality (14), then we have the following inequalities

\[
f(\sqrt{ab}) \left[ \int_a^b w(b) + \int_a^b w(a) \right] \leq \left[ \int_a^b f w(b) + \int_a^b f w(a) \right] \leq \frac{f(a) + f(b)}{2} \left[ \int_a^b w(b) + \int_a^b w(a) \right]
\]

which are the same of inequalities in (4).

Theorem 2.16. Assume that \( f : [a,b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized GA-convex on the interval \([a,b]\). Then for any \( x, y \in [a,b] \) with \( x \neq y \) we have the Hermite-Hadamard inequalities

\[
\frac{1}{2} f(\sqrt{xy}) + f\left( \frac{ab}{\sqrt{xy}} \right) \leq \frac{1}{2 \ln (y/x)} \left[ \int_x^y f(t) \frac{dt}{t} + \int_{ab/y}^{ab/x} f(t) \frac{dt}{t} \right] \leq \frac{1}{4} \left[ f(x) + f(y) + f\left( \frac{ab}{x} \right) + f\left( \frac{ab}{y} \right) \right].
\]
Proof. Since \( \tilde{\mathcal{G}}_{f[x,y]} \) is GA-convex on \([a, b]\), then \( \tilde{\mathcal{G}}_{f[x,y]} \) is also GA-convex on any subinterval \([x, y]\) (or \([y, x]\)) where \( x, y \in [a, b] \).

By Hermite-Hadamard inequalities for GA-convex functions we have

\[
\tilde{\mathcal{G}}_{f[x,y]} \left( \sqrt{xy} \right) \leq \frac{1}{\ln y - \ln x} \int_x^y \frac{\tilde{\mathcal{G}}_{f[x,y]}(t)}{t} dt \leq \frac{\tilde{\mathcal{G}}_{f[x,y]}(x) + \tilde{\mathcal{G}}_{f[x,y]}(y)}{2}
\]  

(16)

for any \( x, y \in [a, b] \) with \( x \neq y \).

By definition of \( \tilde{\mathcal{G}}_f \), we have

\[
\tilde{\mathcal{G}}_{f[x,y]} \left( \sqrt{xy} \right) = \frac{1}{2} \left[ f(\sqrt{xy}) + f \left( \frac{ab}{\sqrt{xy}} \right) \right],
\]

\[
\int_x^y \frac{\tilde{\mathcal{G}}_{f[x,y]}(t)}{t} dt = \frac{1}{2} \int_x^y \frac{1}{t} \left[ f(t) + f \left( \frac{ab}{t} \right) \right] dt
\]

\[
= \frac{1}{2} \int_x^y f(t) dt + \frac{1}{2} \int_x^y f \left( \frac{ab}{t} \right) dt
\]

\[
= \frac{1}{2} \int_x^y f(t) dt + \frac{1}{2} \int_{ab/y}^{ab/x} f(t) \frac{dt}{t}
\]

and

\[
\frac{\tilde{\mathcal{G}}_{f[x,y]}(x) + \tilde{\mathcal{G}}_{f[x,y]}(y)}{2} = \frac{1}{4} \left[ f(x) + f(y) + f \left( \frac{ab}{x} \right) + f \left( \frac{ab}{y} \right) \right].
\]

Thus by (16) we obtain the desired result (15).

\( \square \)

**Remark 2.17.** We note that by helping Theorem 1.10 and Proposition 2.6, the proof of Theorem 2.16 can also be given. The details is omitted.

**Remark 2.18.** If we take \( x = a \) and \( y = b \) in (15), then we get (10). If, for a given \( x \in [a, b] \), we take \( y = ab/x \), then from (15) we get

\[
f \left( \sqrt{ab} \right) \leq \frac{1}{2 \ln \frac{\sqrt{ab}}{x}} \int_x^{ab/x} f(t) \frac{dt}{t} \leq \frac{1}{2} \left[ f(x) + f \left( \frac{ab}{x} \right) \right],
\]

(17)

where \( x \neq \sqrt{ab} \), provided that \( f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \) is symmetrized GA-convex on the interval \([a, b]\).

Multiplying the inequalities (17) by \( 1/x \), then integrating the resulting inequality over \( x \) we get the following refinement of the first part of (10)

\[
f \left( \sqrt{ab} \right) \leq \frac{1}{2 \ln b/a} \int_a^b \left[ \frac{1}{x \ln \left( \sqrt{ab}/x \right)} \int_x^{ab/x} f(t) \frac{dt}{t} \right] dx \leq \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} dx,
\]

provided that \( f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \) is symmetrized GA-convex on the interval \([a, b]\).

When the function is GA-convex, we have the following inequalities as well:
Remark 2.19. If \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is GA-convex, then from (15) we have the inequalities

\[
\begin{align*}
 f(\sqrt{ab}) & \leq \frac{1}{2} \left[ f(\sqrt{x}) + f\left(\frac{ab}{\sqrt{x}}\right)\right] \\
 & \leq \frac{1}{2 \ln (y/x)} \left[ \int_x^b \frac{f(t)}{t} dt + \int_{ab/y}^{ab/x} \frac{f(t)}{t} dt \right] \\
 & \leq \frac{1}{4} \left[ f(x) + f(y) + f\left(\frac{ab}{x}\right) + f\left(\frac{ab}{y}\right)\right],
\end{align*}
\]

for any \( x, y \in [a, b] \) with \( x \neq y \).

If we multiply the inequalities (18) by \( 1/(xy) \) and integrate (18) over \( (x, y) \) on the square \( [a, b]^2 \) and divide by \((\ln b/a)^2\), then we get the following refinement of the first Hermite-Hadamard inequality for convex functions

\[
\begin{align*}
 f(\sqrt{ab}) & \leq \frac{1}{2 (\ln b/a)^2} \frac{1}{x} \int_a^b \frac{f(\sqrt{xy})}{xy} dxy + \frac{1}{2 (\ln b/a)^2} \frac{1}{y} \int_a^b \frac{f(\sqrt{xy})}{xy} dxy \\
 & \leq \frac{1}{(\ln b/a)^2} \int_a^b \frac{1}{x} f(x) \ dx
\end{align*}
\]

3. The Case of One GA-convex and the other Symmetrized GA-convex functions

In this section, we analyze the case in which one function is GA-convex (concave) in the classical sense and the other is symmetrized GA-convex (concave) on an interval \([a, b]\).

Theorem 3.1. Assume that \( g : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is GA-convex (concave) and \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized GA-convex (concave) and integrable on the interval \([a, b]\). Then we have

\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x) g(x)}{x} dx \geq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} dx + \frac{g(a) + g(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} dx - \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2},
\]

and

\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x) g(x)}{x} dx \leq \frac{g(a) + g(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} dx + f(\sqrt{ab}) \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} dx - f(\sqrt{ab}) \frac{g(a) + g(b)}{2}.
\]

Proof. Assume that \( g : [a, b] \to \mathbb{R} \) is GA-convex and \( f : [a, b] \to \mathbb{R} \) is symmetrized GA-convex on \([a, b]\), then for any \( t \in [0, 1] \)

\[
g(a^{t b^{1-t}}) \leq tg(a) + (1-t)g(b)
\]

and by (12) we have

\[
f(\sqrt{ab}) \leq \tilde{G}_f(a^{t b^{1-t}}) \leq \frac{f(a) + f(b)}{2}.
\]
By (21) and (22), we get

\[
0 \leq \left[ t g(a) + (1 - t) g(b) - g(a' b^{1-t}) \right] \left[ f(a) + f(b) \right] - \frac{f(a) + f(b)}{2} - \dot{C}_f (a' b^{1-t})
\]

\[
= \left[ t g(a) + (1 - t) g(b) \right] \left[ f(a) + f(b) \right] - \frac{g(a' b^{1-t}) f(a) + f(b)}{2}
\]

\[
- \left[ t g(a) + (1 - t) g(b) \right] \dot{C}_f (a' b^{1-t}) + \ddot{C}_f (a' b^{1-t}) g(a' b^{1-t}).
\]

That is equivalent to

\[
\left[ t g(a) + (1 - t) g(b) \right] \left[ f(a) + f(b) \right] - \frac{g(a' b^{1-t}) f(a) + f(b)}{2} + \left[ t g(a) + (1 - t) g(b) \right] \dot{C}_f (a' b^{1-t}).
\]

Integrating over \( t \) on \([0, 1]\), we get

\[
\frac{f(a) + f(b)}{2} \int_0^1 \left[ t g(a) + (1 - t) g(b) \right] dt + \int_0^1 \dot{C}_f (a' b^{1-t}) g(a' b^{1-t}) dt \geq \frac{f(a) + f(b)}{2} \int_0^1 g(a' b^{1-t}) dt + \int_0^1 \left[ t g(a) + (1 - t) g(b) \right] \dot{C}_f (a' b^{1-t}) dt.
\] (23)

Observe that

\[
\int_0^1 \left[ t g(a) + (1 - t) g(b) \right] dt = \frac{g(a) + g(b)}{2},
\]

\[
\int_0^1 g(a' b^{1-t}) dt = \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx,
\]

\[
\int_0^1 \dot{C}_f (a' b^{1-t}) g(a' b^{1-t}) dt = \frac{1}{\ln b - \ln a} \int_a^b \frac{\ddot{C}_f (x) g(x)}{x} dx,
\]

and

\[
\int_0^1 \left[ t g(a) + (1 - t) g(b) \right] \dot{C}_f (a' b^{1-t}) dt = g(a) \int_0^1 t \dot{C}_f (a' b^{1-t}) dt + g(b) \int_0^1 (1 - t) \dot{C}_f (a' b^{1-t}) dt.
\] (24)

Since \( \ddot{C}_f \) is geometrically symmetric with respect to \( \sqrt{a b} \), we have

\[
\int_0^1 (1 - t) \dot{C}_f (a' b^{1-t}) dt = \int_0^1 t \dot{C}_f (a' b^{1-t}) dt,
\]

thus by (24), we get

\[
\int_0^1 \left[ t g(a) + (1 - t) g(b) \right] \dot{C}_f (a' b^{1-t}) dt = \left[ g(a) + g(b) \right] \int_0^1 t \dot{C}_f (a' b^{1-t}) dt.
\]
Further,
\[ \int_0^1 \frac{\bar{f}}{x} dt = \frac{1}{2} \left[ \int_0^1 tf(a' b^{1-t}) dt + \int_0^1 tf(a b^{1-t}) dt \right] \]
\[ = \frac{1}{2} \left[ \int_0^1 tf(a' b^{1-t}) dt + \int_0^1 (1-t)f(a b^{1-t}) dt \right] \]
\[ = \frac{1}{2} \int_0^1 f(a' b^{1-t}) dt = \frac{1}{2} \left[ \frac{1}{\ln b - \ln a} \int_a^b f(x) dx \right]. \]

By the inequality (23), we get
\[ \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2} + \frac{1}{\ln b - \ln a} \int_a^b \bar{f}_g \frac{g(x)}{x} dx \]
\[ \geq \frac{f(a) + f(b)}{2} \frac{1}{\ln b - \ln a} \int_a^b g(x) dx + \frac{g(a) + g(b)}{2} \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \]
and the inequality (19) is proved.

By (21) and (22), we also have
\[ 0 \leq [tg(a) + (1-t)g(b) - g(a' b^{1-t})] \bar{f}_g \left( a' b^{1-t} \right) - f (\sqrt{ab}) \]
\[ = [tg(a) + (1-t)g(b)] \bar{f}_g \left( a' b^{1-t} \right) + g(a' b^{1-t}) f (\sqrt{ab}) \]
\[ - [tg(a) + (1-t)g(b)] f (\sqrt{ab}) - g(a' b^{1-t}) \bar{f}_g \left( a' b^{1-t} \right) \]
for any \( t \in [0, 1] \), which is equivalent to
\[ [tg(a) + (1-t)g(b)] f (\sqrt{ab}) + g(a' b^{1-t}) \bar{f}_g \left( a' b^{1-t} \right) \]
\[ \leq [tg(a) + (1-t)g(b)] \bar{f}_g \left( a' b^{1-t} \right) + g(a' b^{1-t}) f (\sqrt{ab}) \]
for any \( t \in [0, 1] \).

Taking the integral over \( t \in [0, 1] \), we get
\[ f (\sqrt{ab}) \int_0^1 [tg(a) + (1-t)g(b)] dt + \int_0^1 g(a' b^{1-t}) \bar{f}_g \left( a' b^{1-t} \right) dt \]
\[ = f (\sqrt{ab}) \frac{g(a) + g(b)}{2} + \frac{1}{\ln b - \ln a} \int_a^b \bar{f}_g \frac{g(x)}{x} dx \]
\[ \leq \int_0^1 [tg(a) + (1-t)g(b)] \bar{f}_g \left( a' b^{1-t} \right) dt + f (\sqrt{ab}) \int_0^1 g(a' b^{1-t}) dt \]
\[ = \frac{g(a) + g(b)}{2} \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx + f (\sqrt{ab}) \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx, \]
and the inequality (20) is proved. \( \square \)

4. Inequalities for Product of Symmetrized GA-convex functions

Theorem 4.1. Assume that both \( f, g : [a, b] \subset (0, \infty) \to \mathbb{R} \) are symmetrized GA-convex and integrable on the interval \([a, b]\). Then we have
\[ \frac{1}{\ln b/a} \int_a^b \frac{\bar{f}_g \left( x \right)}{x} dx \geq \{S_{mid} (f, g; a, b), S_{tr(a, b)} (f, g; a, b)\}, \tag{25} \]
\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x) \tilde{G}_g(x)}{x} \, dx \leq \{S_{\text{mix}}(f, g; a, b), S_{\text{mix}}(g, f; a, b)\} 
\]

(26)

where

\[
S_{\text{mix}}(f, g; a, b) = g \left( \sqrt{ab} \right) \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} \, dx + f \left( \sqrt{ab} \right) \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx - f \left( \sqrt{ab} \right) g \left( \sqrt{ab} \right),
\]

\[
S_{\text{tra}}(f, g; a, b) = \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx + \frac{g(a) + g(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} \, dx - \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2},
\]

and

\[
S_{\text{mix}}(f, g; a, b) = \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx + g \left( \sqrt{ab} \right) \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} \, dx - \frac{f(a) + f(b)}{2} g \left( \sqrt{ab} \right).
\]

Proof. By (12) we have

\[
\left( \tilde{C}_f(x) - f \left( \sqrt{ab} \right) \right) \left( \tilde{C}_g(x) - g \left( \sqrt{ab} \right) \right) \geq 0
\]

for any \( x \in [a, b] \).

This is equivalent to

\[
G_f(x)G_g(x) + f \left( \sqrt{ab} \right) g \left( \sqrt{ab} \right) \geq \tilde{C}_f(x) \left( G_f(x) + G_g(x) \right) \left( f \left( \sqrt{ab} \right) \right)
\]

(27)

for any \( x \in [a, b] \).

If we multiply the inequality (27) by 1/x and integrate over \( x \) on \([a, b]\) and divide by \( \ln b/a \), then we get

\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x) \tilde{G}_g(x)}{x} \, dx \geq f \left( \sqrt{ab} \right) g \left( \sqrt{ab} \right)
\]

and since

\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x)}{x} \, dx = \frac{1}{2} \left[ \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} \, dx + \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx \right]
\]

the inequality

\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x) \tilde{G}_g(x)}{x} \, dx \geq S_{\text{mix}}(f, g; a, b)
\]

is easily obtained.

Also, by (12) we have

\[
\left( \frac{f(a) + f(b)}{2} - \tilde{C}_f(x) \right) \left( \frac{g(a) + g(b)}{2} - \tilde{C}_g(x) \right) \geq 0
\]

for any \( x \in [a, b] \), which by the same procedure produces the inequality

\[
\frac{1}{\ln b/a} \int_a^b \frac{\tilde{G}_f(x) \tilde{G}_g(x)}{x} \, dx \geq S_{\text{tra}}(f, g; a, b).
\]
Corollary 4.3. Assume that both $f$ and $g$ are symmetrized convex, then by (12) we have

$$\left( f(a) + f(b) \right) \left( C_f(x) - g \left( \frac{\sqrt{ab}}{x} \right) \right) \geq 0$$

(28)

for any $x \in [a, b]$, which is equivalent to

$$C_f(x) C_g(x) \leq \frac{f(a) + f(b)}{2} C_f(x) + C_f(x) g \left( \frac{\sqrt{ab}}{x} \right) - \frac{f(a) + f(b)}{2} g \left( \frac{\sqrt{ab}}{x} \right)$$

for any $x \in [a, b]$. If we multiply the last inequality by $1/x$ and integrate over $x$ on $[a, b]$ and divide by $\ln b/a$, then we get the desired result

$$\frac{1}{\ln b/a} \int_a^b \frac{C_f(x) C_g(x)}{x} dx \leq S_{mix}(f, g; a, b).$$

The inequality

$$\frac{1}{\ln b/a} \int_a^b \frac{C_f(x) C_g(x)}{x} dx \leq S_{mix}(g, f; a, b)$$

follows from (28) by replacing $f$ with $g$. □

**Remark 4.2.** Observe that

$$\int_a^b \frac{C_f(x) C_g(x)}{x} dx = \frac{1}{4} \int_a^b \left[ f(x) + f \left( \frac{\sqrt{ab}}{x} \right) \right] \left[ g(x) + g \left( \frac{\sqrt{ab}}{x} \right) \right] dx$$

and

$$\int_a^b \frac{C_f(x) C_g(x)}{x} dx = \int_a^b \frac{f(x) C_g(x)}{x} dx.$$
Theorem 4.4. Assume that both \( f, g : [a, b] \subseteq (0, \infty) \rightarrow [0, \infty) \) are symmetrized GA-convex and integrable on the interval \([a, b]\). Then we have

\[
f(\sqrt[n]{ab})g(\sqrt[n]{ab}) \leq f(\sqrt[n]{ab}) \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} \, dx
\]

\[
\leq \frac{1}{\ln b/a} \int_a^b \frac{\mathcal{C}_f(x)g(x)}{x} \, dx
\]

\[
\leq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx
\]

\[
\leq \frac{f(a) + f(b)}{2} + \frac{g(a) + g(b)}{2},
\]

and

\[
f(\sqrt[n]{ab})g(\sqrt[n]{ab}) \leq g(\sqrt[n]{ab}) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx
\]

\[
\leq \frac{1}{\ln b/a} \int_a^b \frac{\mathcal{C}_g(x)g(x)}{x} \, dx
\]

\[
\leq \frac{g(a) + g(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{f(x)}{x} \, dx
\]

\[
\leq \frac{f(a) + f(b)}{2} + \frac{g(a) + g(b)}{2},
\]

Proof. If \( f, g : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R} \) are symmetrized GA-convex, then by (12) we have

\[
0 \leq f(\sqrt[n]{ab}) \leq \mathcal{C}_f(x) \leq \frac{f(a) + f(b)}{2}
\]

(31)

and

\[
0 \leq g(\sqrt[n]{ab}) \leq \mathcal{C}_g(x) \leq \frac{g(a) + g(b)}{2}
\]

(32)

for any \( x \in [a, b] \).

If we multiply (31) by \( \mathcal{C}_g(x) \), then we get

\[
0 \leq f(\sqrt[n]{ab}) \mathcal{C}_g(x) \leq \mathcal{C}_f(x) \mathcal{C}_g(x) \leq \frac{f(a) + f(b)}{2} \mathcal{C}_g(x)
\]

for any \( x \in [a, b] \). If we multiply the last inequality by \( 1/x \) and integrate over \( x \) on \([a, b]\) and divide by \( \ln b/a \), then we get

\[
0 \leq f(\sqrt[n]{ab}) \int_a^b \frac{\mathcal{C}_g(x)}{x} \, dx \leq \frac{1}{\ln b/a} \int_a^b \frac{\mathcal{C}_f(x) \mathcal{C}_g(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{\mathcal{C}_g(x)}{x} \, dx,
\]

namely

\[
0 \leq f(\sqrt[n]{ab}) \int_a^b \frac{\mathcal{C}_g(x)}{x} \, dx \leq \frac{1}{\ln b/a} \int_a^b \frac{\mathcal{C}_f(x) g(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2} \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx
\]

(33)

Since, by (10) we have

\[
g(\sqrt[n]{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} \, dx \leq \frac{g(a) + g(b)}{2},
\]
then
\[
f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \leq f\left(\sqrt{ab}\right) \frac{1}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} \, dx,
\]
and
\[
f(a) + f(b) \frac{1}{\ln b/a} \int_a^b \frac{g(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2}.
\]

By utilising (33)-(35) we get (29).

The inequality (30) follows in a similar way by making use of (32) and a similar procedure. The details are omitted.

References

[1] S. S. Dragomir, Symmetrized convexity and Hermite-Hadamard type inequalities, Journal of Mathematical Inequalities, 10 (4) (2016), 901–918.
[2] S. S. Dragomir, Hermite-Hadamard type inequalities for product of convex and symmetrized convex functions, RGMIA Research Report Collection, 20 (2017), Article 07, 11 pp.
[3] S. S. Dragomir, Hermite-Hadamard type inequalities for product of symmetrized convex functions, RGMIA Research Report Collection, 20 (2017), Article 08, 11 pp.
[4] S. S. Dragomir, Some inequalities of Hermite-Hadamard type for symmetrized convex functions and Riemann-Liouville fractional integrals. RGMIA Res. Rep. Coll., 20 (2017), Article 20, 15 pp.
[5] I. Işcan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, J. Inequal. Appl., 2013(491) (2013), 15 pages.
[6] I. Işcan, M. Kunt, Hermite-Hadamard-Fejer type inequalities for GA-convex functions via fractional integrals, Turkish J. Inequal., 2 (1) (2018), 1-20.
[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam (2006).
[8] M. A. Latif, S. S. Dragomir and E. Momaniat, Some Fejer type integral inequalities for geometrically-arithmetically-convex functions with applications, RGMIA Research Report Collection, 18 (2015), Article 15, 18pp.
[9] C. P. Niculescu, Convexity according to the geometric mean. Math. Inequal. Appl.3 (2), 155-167 (2000). Available online at http://dx.doi.org/10.7153/mia-03-19.
[10] C. P. Niculescu, Convexity according to means, Math. Inequal. Appl. 6 (4), 571-579 (2003). Available online at http://dx.doi.org/10.7153/mia-06-53.