Nonlinear realizations of $W_3$ symmetry

E.Ivanov, S.Krivonos and A.Pichugin

Laboratory of Theoretical Physics, 
Joint Institute for Nuclear Research, Dubna, Head Post Office 
P.O. Box 79, 101 000 Moscow, Russia

Abstract

We deduce the $sl_3$ Toda realization of classical $W_3$ symmetry on two scalar fields in a geometric way, proceeding from a nonlinear realization of some associate higher-spin symmetry $W_3^\infty$. The Toda equations are recognized as the constraints singling out a two-dimensional fully geodesic subspace in the initial coset space of $W_3^\infty$. The proposed geometric approach can be extended to other nonlinear algebras and integrable systems.

Submitted to Phys. Lett. B

Dubna 1991

1 Dniepropetrovsk State University, Ukraine
1. Introduction. Recently, classical versions of Zamolodchikov’s $W_N$ algebras [1] received much attention (see, e.g., [2] - [7]). They were found to be the symmetry algebras of some 2D field-theory models, such as the systems of free bosonic fields [3, 5] and Toda systems [6], and certain steps towards understanding their geometric origin have been done [7]. In view of growing interest to $W$-gravities, $W$-strings and related theories it is extremely important to fully reveal, from various points of view, the geometries underlying these $W$ symmetries.

In the present letter we suggest a new geometric set-up for classical $W_N$ symmetries which relies upon the nonlinear realization approach [8, 9] and is a generalization of the treatment of $W_2$ (Virasoro) symmetry in ref. [10]. We address here the simplest nontrivial case $N = 3$, however our construction seems equally applicable to other algebras and superalgebras of this kind. The basic trick allowing us to apply the standard nonlinear realization scheme to the algebras of the type $W_N$ consists in replacing them by some infinite-dimensional linear $W_\infty$ type algebras ($W_\infty^N$ in what follows) which arise if one treats as independent all the composite higher-spin generators appearing in the commutators of the basic $W_N$ generators. We deduce the $sl_3$ Toda realization of $W_3$ [5] in a purely geometric way as a particular coset realization of $W_\infty^3$ and show that the $sl_3$ Toda equations are also intimately related to the intrinsic geometry of $W_\infty^3$: they single out a two-dimensional fully geodesic surface in the coset space of $W_\infty^3$.

2. From $W_3$ to $W_\infty^3$. The most general classical $W_3$ algebra is defined by the following relations [2]

\[
\begin{align*}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[L_n, J_m] &= (2n-m)J_{n+m} \\
[J_n, J_m] &= -24(n-m)\Lambda_{n+m} - \frac{n-m}{2} \left[ (n+m)^2 - \frac{5}{2}nm - 4 \right] L_{n+m} - \\
&- \frac{c}{48}(n^2 - 4)(n^2 - 1)n\delta_{n+m,0},
\end{align*}
\]

where $L_n$, $J_n$ are the spin 2 (Virasoro) and spin 3 generators and

\[
\Lambda_n = \frac{1}{c} \sum_m L_{n-m}L_m.
\]

The nonlinear realization techniques we intend to apply to this $W_3$ symmetry have been worked out in [8] for symmetries based on Lie algebras, i.e. linear algebras. In order to generalize these techniques to the $W_N$ type algebras we propose to treat the spin 4 generators $\Lambda_n$ in [1] and all other higher-spin composite generators present in the enveloping algebra of $L_n, J_m$ as independent ones. In other words, one replaces [1] by some linear infinite-dimensional higher-spin algebra $W_\infty^3$

\[
W_\infty^3 = \left\{ L_n, J_n, \Lambda_n, \ldots, J_n^S, \ldots \right\}, \quad S = 5, 6, \ldots
\]

in which the commutation relations between generators of the lowest spins (2 and 3) are given by [1] and all the remaining relations involving the higher-spin generators $\{\Lambda_n, \ldots, J_n^S, \ldots\}$ are successively evaluated proceeding from these basic relations and the quadratic relation (2). In principle, any commutator can be specified in this way and the generic form of higher-order
commutators can be indicated. For our purpose it is of no need to know the detailed structure of these commutators. It will be crucial that in (1) is nonvanishing and composite generators form a closed set (the last statement follows from a direct inspection of all possible products of the basic spin 2 and spin 3 currents).

Let us list some important subalgebras of (3). Of major relevance for our purpose is the following centerless infinite-dimensional subalgebra which is the genuine generalization of the truncated Virasoro algebra \{L_{-1}, L_0, L_1, \ldots \}

\[
\begin{array}{cccccc}
L_{-1} & L_0 & L_1 & L_2 & \ldots \\
J_{-2} & J_{-1} & J_0 & J_1 & J_2 & \ldots \\
\Lambda_{-3} & \Lambda_{-2} & \Lambda_{-1} & \Lambda_0 & \Lambda_1 & \Lambda_2 & \ldots \\
\end{array}
\]

We call it \(\tilde{W}_3^\infty\). To see that (4) is a subalgebra, one represents (1) and (3) as the algebra of generalized variations generated by an element \(\sum_S \int dz a^S(z) J^S(z)\), \(J^S(z) = \sum_n z^{-n-S} J_n^S, S = 2, 3, \ldots\) and notices that (4) is singled out by restriction to the parameters-functions \(a^S(z)\) regular at \(z = 0\). Then closure of (4) follows from the fact that the variety of such functions is closed under differentiation and multiplication. To avoid a possible confusing, let us point out that the higher-spin generators in (4), when treated as composite, still involve the products of generators \(L_n, J_m\) with all negative and positive conformal dimensions.

From the fact that (3) contains also a subalgebra consisting of the generators which are obtained from those in (4) by the reflection \(n \rightarrow -n\) one immediately finds out the existence of the wedge type \[\square\] subalgebra \(W_\Lambda\) in \(\tilde{W}_3^\infty\)

\[
W_\Lambda = \left\{ \begin{array}{cccccc}
L_{-1} & L_0 & L_1 \\
J_{-2} & J_{-1} & J_0 & J_1 & J_2 \\
\Lambda_{-3} & \Lambda_{-2} & \Lambda_{-1} & \Lambda_0 & \Lambda_1 & \Lambda_2 & \Lambda_3 \\
\end{array} \right\}
\]

(dots mean higher-spin generators with proper indices). One can check (using (1), (2) and the property that the composite generators form a subalgebra) that all generators in (5) with spin \(\geq 4\) constitute an infinite-dimensional ideal and the factor-algebra of (5) over this ideal is \(sl(3, R)\)

\[
W_\Lambda/\{\Lambda_{-3}, \ldots, \Lambda_3, \ldots, J_1^S \ldots J_{S-1}^S, \ldots\} \sim sl(3, R) \quad .
\]

3. Nonlinear realizations of \(W_3^\infty\). Having replaced \(W_3\) by a linear algebra \(W_3^\infty\), we are ready to construct a nonlinear realization of the \(W_3^\infty\) symmetry, following the generic prescriptions of \[\square\] \[\square\]. By reasonings of simplicity and for the correspondence with the consideration in ref. \[\square\] we limit ourselves to the truncated algebra \(\tilde{W}_3^\infty\) (4). Our ultimate aim will be to show that the \(sl_3\) Toda realization of \(W_3\) on two 2D scalar fields \[\square\] and the Toda equations of motion themselves result from a particular coset realization of \(\tilde{W}_3^\infty\) after imposing certain covariant constraints on the relevant coset parameters.

In order to have manifest 2D Lorentz symmetry, we start from the product of two commuting copies of the \(\tilde{W}_3^\infty\) symmetry groups (we denote it \(G\) with the algebra

\[
G = \tilde{W}_3^\infty_{3^+} \times \tilde{W}_3^\infty_{3^-} \quad .
\]

\footnote{It would be interesting to compare \(W_3^\infty\) (and \(W_\infty^\infty\)) with the \(W_\infty\) algebras of ref. \[\square\] \[\square\]. It is likely that they belong to different classes. For example, \(W_3^\infty\) involves several independent types of the spin 6 generators, while in \(W_\infty\) each spin occurs once.}
As a next step we need to choose an appropriate coset of $G$, which is actually reduced to
specifying the stability subgroup $H$. We have checked that no finite-dimensional cosets exist in
$G$. So, in the present case one deals with infinite-dimensional coset manifolds.

Fortunately, it is not so difficult as it could seem. First, by reasonings of the reducibility
to the Virasoro case [10] the coset should include the generators \{$L_\pm^\pm, L_0^+ + L_0^-, L_1^\pm, ... L_n^\pm$\},
with the 2D Minkowski space coordinates $x^\pm$ and the Liouville field $u(x)$ being parameters
 corresponding to the first three generators. Secondly, having as a goal to eventually come to the
$sl_3$ Toda theory, we need to reserve a place for the second Toda field $\phi(x)$ as a coset parameter.
The only appropriate generators are $J_0^\pm$, so we are led to include a linear combination of them
into the set of the coset generators. Finally, it would be desirable to place all higher-spins
generators into the stability subgroup and in what follows not to care about them.

These three requirements are satisfied in a minimal way with the two-parameter family of
stability subgroups generated by

\[
\mathcal{H} = \{ J_2^\pm - \alpha \beta J_2^\pm, \ J_1^\pm + \frac{\sqrt{3}}{2} L_1^\pm - \alpha(J_1^+ + \frac{\sqrt{3}}{2} L_1^-), \ J_0^+ - J_0^-, \ L_0^+ - L_0^-, \ J_{-1}^- - \frac{\sqrt{3}}{2} L_{-1}^- - \beta(J_1^+ - \frac{\sqrt{3}}{2} L_1^-); \Lambda_{3\pm}, \Lambda_{2\pm}, ...; J_{\pm3}^{S\pm}, ... \}, \ S \geq 5. \tag{8}
\]

The combinations of the generators $L$ and $J$ in (8) turn out to form the Borel subalgebra of
the diagonal $sl(3, R)$ in the sum of two commuting factor-algebras $sl(3, R)$ (6) contained in $\tilde{W}_3$ and $\tilde{W}_3\infty$. The parameters $\alpha, \beta$ reflect a freedom in extracting this diagonal $sl(3, R)$.

Now, an element of the coset space $G/H$ can be parametrized as follows

\[
g \equiv G/H = e^{x^\pm L_\pm^\pm} e^{\psi_1^\pm J_1^\pm} e^{\xi_1^\pm L_1^\pm} e^{\psi_2^\pm J_2^\pm} ... e^{u(L_0^+ + L_0^-)} e^{\phi(J_0^+ + J_0^-)} . \tag{9}
\]

Here, $u(x), \phi(x), \psi_1^\pm(x), \xi_1^\pm(x), ...$ constitute an infinite tower of the coset parameters-fields.
The group $G$ acts on the coset (9) from the left

\[
g_0(\lambda)g(x, u, \phi, ...) = g(x', u', \phi', ...) \cdot h \ , \tag{10}
\]

where $g_0(\lambda)$ is an arbitrary element of $G$, and $h$ belongs to the subgroup $H$. The arrangement
of the group factors in (9) is convenient in that the transformation laws of 2D coordinates $x^\pm$
and the fields $u(x), \phi(x)$ under conformal transformations ($g_0 = \exp \sum_{n=1}^{+\infty} \lambda_n L_n$) are of the standard form

\[
\delta_\lambda x^\pm = \lambda^\pm(x^\pm) = \sum_{n=-1}^{+\infty} \lambda_n^\pm(x^\pm)^{n+1} \\
\delta_\lambda u(x) = u'(x') - u(x) = \frac{1}{2}(\partial_+ \lambda^+ + \partial_- \lambda^-) \\
\delta_\lambda \phi(x) = \phi'(x') - \phi(x) = 0 \ . \tag{11}
\]

The $J_n$ transformations of the coordinates and parameters-fields can be also deduced from
the general formula (10). For $x^\pm$ and $u(x), \phi(x)$ we get the following transformations (we write

\footnote{We treat all the coset parameters other than $x^\pm$ as fields given on $x^\pm$; in other words, we actually deal
with a two-dimensional hypersurface in the coset space. This does not influence the transformation properties
of coset parameters and is typical for nonlinear realizations of space-time symmetries [6].}
down here only the transformations generated by the “+” branch of the group $G$)

$$
\begin{align*}
\delta_a x &= -\frac{\sqrt{3}}{2} a'(x) + 2\sqrt{3} (\xi_1 + \frac{\sqrt{3}}{2} \psi_1) a(x) \\
\delta_u x &= -\frac{\sqrt{3}}{2} (\xi_1 + \frac{\sqrt{3}}{2} \psi_1) a'(x) + (6\psi_2 + 2\sqrt{3}\xi_1^2) a(x) \\
\delta_a \phi(x) &= \frac{1}{4} a''(x) - \frac{3}{2} (\xi_1 + \frac{\sqrt{3}}{2} \psi_1) a'(x) - 3 \left[ \xi_2 - (\xi_1 + \frac{\sqrt{3}}{2} \psi_1)^2 \right] a(x) ,
\end{align*}
$$

(12)

where the function $a(x)$ collects constant parameters of the group element $g_0$

$$
g_0 = \exp \sum_{n=-\infty}^{+\infty} a_n J_n , \quad a(x) = \sum_{n=-\infty}^{+\infty} a_n x^{n+2} .
$$

and, for brevity, we omitted the index “+” of $x$ and higher-spin coset fields. Note that for non-zero $\alpha , \beta$ in (8) these transformations turn out to have a nontrivial action on the coset fields belonging to the “–” light-cone branch of $G$. For instance, the fields $\xi_1^-, \psi_1^-$ transform as

$$
\delta \psi^-_1 = \frac{\sqrt{3}}{2} \delta \xi^-_1 = \frac{1}{2} \alpha (a' - 4a\xi^+_1) e^{-2(u+\sqrt{3}\phi)}
$$

(13)

(for $J$-transformations from the “–” branch the situation is reversed, with $\alpha$ replaced by $\beta$).

The main peculiarity of transformations (12), (13) and their crucial difference from the conformal ones is that $J_n$ have no realizations on the coordinates $x^\pm$ alone – these generators necessarily mix the coordinates with the coset fields $\psi_1, \psi_2, \xi_1, \xi_2$. Moreover, in fact we deal here with an infinite-dimensional nonlinear representation of $W_3^\infty$ because the fields $\psi_1, \xi_1$ are transformed through higher-spin fields $\psi_2, \xi_2, \psi_3, \xi_3$ and so on. We stress that at this step the active form of transformations of the coset fields contains in each term no more than one derivative on fields, the latter arising due to the field-dependent shift of $x$ in (12). The same is true, of course, for higher-spin transformations which are generated via Lie brackets of (12), (13).

The fundamental geometric objects of nonlinear realizations, covariant Cartan’s forms, are introduced by the standard relation [8]

$$
g^{-1}dg = \sum_{n=-\infty}^{+\infty} \omega_n^\pm L_n^\pm + \sum_{n=-2}^{+\infty} \Omega_n^\pm J_n^\pm + \ldots ,
$$

(14)

where dots stand for the forms entering with the higher-spin generators. For our further purposes it will be important that the infinite set of forms associated with the coset generators is closed under the left shifts (10). Let us explicitly give a few first forms

$$
\begin{align*}
\omega^-_1 &= e^{-u} ch(\sqrt{3}\phi) dx^+ , \quad \omega^+_0 = du - 2\xi^+_1 dx^+ \\
\Omega^\pm_{-2} &= 0 , \quad \Omega^\pm_{-1} = -\frac{2}{\sqrt{3}} e^{-u} ch(\sqrt{3}\phi) dx^+ , \quad \Omega^\pm_0 = d\phi - 3\psi^+_1 dx^+ .
\end{align*}
$$

(15)

We see that the fields $\psi^\pm_1, \xi^\pm_1$ enter into some coset space Cartan forms linearly and homogeneously. A straightforward inspection shows that this is a general phenomenon: for each coset field except for $u(x), \phi(x)$ one can indicate such a Cartan form. Their set is

$$
\omega^+_0 + \omega^-_0 , \quad \Omega^+_0 + \Omega^-_0 , \quad \omega^+_n + \omega^-_n , \quad \Omega^+_n + \Omega^-_n \quad \text{for all } n \geq 1 ,
$$

(16)
where the symbols \( |_\pm \) label the projections of a given form onto \( dx^\pm \), respectively.

4. \( \mathfrak{sl}_3 \) Toda from \( W^\infty_3 \). So far our coset fields carry no any dynamics, their origin is purely geometric: they define the embedding of 2D Minkowski space \( \{ x^\pm \} \) as a hypersurface into an infinite-dimensional coset space of the group \( G = \tilde{W}^\infty_3 \times \tilde{W}^\infty_3 \). The latter generates motions of this two-dimensional hypersurface in the coset space. Now we wish to show that after imposing an infinite number of \( G \)-covariant constraints on the coset fields this hypersurface is completely specified by the two fields, \( u(x) \) and \( \phi(x) \), which are subject to the \( \mathfrak{sl}_3 \) Toda equations and for which \( G \)-transformations take the standard form of the \( W^\infty_3 \times W^\infty_3 \) ones \([4]\).

What we are going to effect is the covariant reduction procedure worked out in \([10]\) and applied there for constructing a coset realization of \( W_2 \) symmetry on a single (Liouville) field \( u(x) \). In our case it goes as follows. Given Cartan forms \([13]\) defined from the beginning on the whole algebra \( \mathcal{G} = \tilde{W}^\infty_3 \times \tilde{W}^\infty_3 \), one constrains them as

\[
g^{-1}dg = \sum_{n=-1}^{+\infty} \omega^+_n L^+_n + \sum_{n=-2}^{+\infty} \Omega^+_{n} J^+_n + \ldots = g^{-1}_{\text{red}}dg_{\text{red}} \in \tilde{\mathcal{G}} ,
\]

where \( \tilde{\mathcal{G}} \) is some subalgebra containing the stability algebra \( \mathcal{H} \) defined in eq. \([8]\). In the \( W_2 \) case \([10]\) \( \tilde{\mathcal{G}} \) was chosen to be \( \mathfrak{sl}(2,R) \) and this immediately led to the Liouville equation for \( u(x) \). In the case at hand it is natural to choose

\[
\tilde{\mathcal{G}} = \{ R^\pm = J^\pm_1 + \sqrt{3} 2 L^\pm_1 - \alpha(J^\pm_1 + \sqrt{3} 2 L^\pm_1) \}, \quad S^\pm = J^\pm_0 - \sqrt{3} 2 L^\pm_1 - \beta(J^\pm_1 + \sqrt{3} 2 L^\pm_1) \\
B^\pm = J^\pm_2 - \alpha \beta J^\pm_2, \quad U = L^+_0 - L^-_0, \quad T = J^+_0 - J^-_0 , \quad \text{All higher-spin generators} \}
\]

This algebra is an extension of \( \mathcal{H} \) by two generators, \( R^- \) and \( S^+ \). It is easy to see (remembering the reasonings of Sec.2) that all the higher-spin generators in \([18]\) form an ideal and that the factor-algebra of \([18]\) over this ideal is the diagonal \( \mathfrak{sl}(3,R) \) in the sum of the two commuting factor-algebras \( \mathfrak{sl}(3,R) \) \([6]\) (coming from \( \tilde{W}^\infty_3 \times \tilde{W}^\infty_3 \)). So this choice is indeed a natural generalization of the option of ref. \([10]\).

The covariant reduction condition \([17]\) amounts to an infinite set of the constraints

\[
\omega^+_1 = -\frac{1}{2}(\alpha + \beta) \omega^+_1 - \sqrt{3} 4 (\alpha - \beta) \Omega^+_1 , \quad \Omega^+_1 = -\frac{1}{2}(\alpha + \beta) \Omega^+_1 - \frac{1}{\sqrt{3}} (\alpha - \beta) \omega^+_1 \\
\omega^+_2 = \alpha \beta \Omega^+_2, \quad \omega^+_0 + \omega^-_0 = 0 , \quad \Omega^+_0 + \Omega^-_0 = 0 \\
\omega^+_n = 0 , \quad \Omega^+_n = 0 , \quad \text{for all } n \geq 2
\]

Eq. \([17]\) and its detailed form \([15]\) are covariant under left shifts \([10]\), which can be explicitly checked with making use of the relations \([1]\). This guarantees that the original group structure is not destroyed.

One observes that each of eqs. \([19]\) gives rise to two equations, for the projections of the \( \omega, \Omega \) onto \( dx^+ \) and \( dx^- \). An important subset of these equations is formed by the equations implying the projections \([16]\) to vanish. They are purely algebraic and serve to express higher-spin coset fields in terms of \( u(x), \phi(x) \) and derivatives of the latter (“inverse Higgs effect”, ref. \([4]\)). All such coset fields can be eliminated in this manner, e.g.,

\[
\xi^\pm_1 = \partial_{\pm} u(x) \quad , \quad \xi^\pm_2 = \frac{1}{3} [ \partial^2_{\pm} u(x) + (\partial_{\pm} u(x))^2 + (\partial_{\pm} \phi(x))^2 ] \\
\psi^\pm_1 = \frac{2}{3} \partial_{\pm} \phi(x) \quad , \quad \psi^\pm_2 = \frac{1}{6} \partial^2_{\pm} \phi(x) + \partial_{\pm} u(x) \partial_{\pm} \phi(x) \quad \text{etc.}
\]

\[\]

5
The remaining equations prove to be dynamical: they constrain \( u(x), \phi(x) \) to obey the \( sl_3 \) Toda equations:

\[
\begin{align*}
\partial_+ \partial_- u &= -\frac{1}{2} \alpha e^{-2(u+\sqrt{3} \phi)} - \frac{1}{2} \beta e^{-2(u-\sqrt{3} \phi)} \\
\partial_+ \partial_- \phi &= -\sqrt{3} \alpha e^{-2(u+\sqrt{3} \phi)} + \sqrt{3} \beta e^{-2(u-\sqrt{3} \phi)} .
\end{align*}
\] (21)

Now we substitute the expressions (21) into the geometric \( \tilde{W}_3^\infty \) transformation laws (12) and find the resulting spin 3 transformations of the fields \( u(x), \phi(x) \)

\[
\begin{align*}
\delta u(x) \equiv u'(x) - u(x) &= -\frac{1}{2} a'(x) \partial \phi + a(x) \left( \partial^2 \phi + 4 \partial u \partial \phi \right) \\
\delta \phi(x) \equiv \phi'(x) - \phi(x) &= \frac{1}{4} a''(x) - \frac{3}{2} a'(x) \partial u + a(x) \left( 2(\partial u)^2 - 2(\partial \phi)^2 - \partial^2 u \right) .
\end{align*}
\] (22)

Together with the spin 2 transformations (11) they constitute the standard \( sl_3 \) Toda realization of classical \( W_3^\infty \) symmetry (1), (2) (to be more precise, its regular part with the parameters-functions regular at \( x = 0 \)). The appropriate currents

\[
\begin{align*}
\gamma^{-2} T^{(-2)} &= -\frac{1}{2}(\partial u)^2 - \frac{1}{2}(\partial \phi)^2 - \frac{1}{2} \partial^2 u \\
\gamma^{-2} J^{(-3)} &= \frac{1}{4} \partial^3 \phi + \frac{3}{2} \partial^2 \phi \partial u + \frac{1}{2} (\partial u)^2 \partial \phi + 2(\partial u)^2 \partial \phi - \frac{2}{3} (\partial \phi)^3
\end{align*}
\] (23)

generate, via the Poisson brackets, the whole algebra (1), (2) with \( c = 3 \gamma^2 \).

Thus we have shown that the Toda realization of nonlinear \( W_3^\infty \) symmetry can be reproduced proceeding from a pure geometric coset realization of some associate linear higher-spin symmetry \( \tilde{W}_3^\infty \). In other words, \( W_3 \) can be viewed as a particular field realization of this infinite-dimensional linear algebra.

In contrast to the \( W_2 \) (Liouville) case (11), the algebraic and dynamical constraints in (19) are in general mixed under \( G \) transformations, so the exact \( W_{3+} \times W_{3-} \) structure comes out only on shell, when eqs. (21) are fulfilled. The \( sl_3 \) Toda action is still invariant under the transformations (22) with the parameters-functions of both light-cone chiralities, and the related Lie bracket structure generates two independent algebras \( W_{3+}^\infty \). However, these algebras do not commute with each other; the commutativity is restored only on shell. It is interesting to seek for another form of the \( sl_3 \) Toda action which would possess the \( W_{3+}^\infty \times W_{3-}^\infty \) symmetry off shell and originally involve an infinite number of auxiliary fields in parallel with \( u(x) \) and \( \phi(x) \). The whole set of constraints (19) could be then expected to follow from this action as the field equations.

It is worth noting that the zero-curvature representation for eqs. (21) on the \( sl(3,R) \) algebra (14) automatically arises in our approach. Indeed, after imposing the constraints (17), (13), we are left with the form valued in the algebra \( \tilde{G} \) (18) and involving only the fields \( u(x), \phi(x) \):

\[
\Omega_{\text{red}} = \Omega_{\text{sl}(3,R)} + \ldots
\] (24)

\footnote{We are at liberty to put \( \alpha \) and/or \( \beta \) equal to zero, which corresponds to some contractions of \( \tilde{G} \) (18). This gives rise to alternative reductions, with \( u + \sqrt{3} \phi \) and/or \( u - \sqrt{3} \phi \) subject to free equations.}
Here
\[
\Omega_{sl(3,R)} = \frac{1}{\sqrt{3}} e^{-u-\sqrt{3}\phi} dx^\pm R^\pm - \frac{1}{\sqrt{3}} e^{-u+\sqrt{3}\phi} dx^\pm S^\pm + 
\]
\[+(\partial_- u dx^- - \partial_+ u dx^+)U + (\partial_- \phi dx^- - \partial_+ \phi dx^+)T \quad (25)\]

and dots stand for the terms with higher-spin generators. As we have already mentioned, the higher-spin generators constitute an ideal in \( \tilde{G} \), so the Maurer-Cartan equation for the form \( \Omega_{sl(3,R)} \) closes without any reference to the higher-spin components. Thus, the Maurer-Cartan equation for \( \Omega_{red} \) immediately results in the zero-curvature condition for \( \Omega_{sl(3,R)} \):
\[
d^{\text{ext}} \Omega_{sl(3,R)} = \Omega_{sl(3,R)} \wedge \Omega_{sl(3,R)} \quad . \quad (26)\]

It is a simple exercise to verify that eq.\((26)\) is equivalent to the \( sl_3 \) Toda equations \((21)\).

Finally, let us comment on the geometric meaning of the covariant reduction procedure. As was explained in \([10, 15]\), the essence of this procedure consists in reducing a given group space to its some lower-dimensional fully geodesic subspace, in a way covariant with respect to the original nonlinear realization. The dynamical equations (Liouville equation \([10]\) and others \([15]\)) come out as the constraints accomplishing this reduction (in parallel with the inverse Higgs algebraic constraints). In the present case the relevant fully geodesic subspace is the two-dimensional coset space \( SL(3,R) / H \) (modulo higher-spin generators) which is thus the genuine analog of the pseudosphere \( SL(2,R) / SO(1,1) \) figuring in the \( W_2 \) example of ref. \([10]\).

5. Conclusion. In this paper we have explicitly demonstrated that \( W_3 \) symmetry of the \( sl_3 \) Toda system and field equations of the latter have a remarkable geometric origin: they stem from a coset realization of infinite-dimensional symmetry associated with the linear algebra \( W_3^\infty \). Doubtless, other Toda systems (based both on the algebras \( sl(N,R) \) and the algebras from other Cartan’s series, e.g. \( so(N) \)) and their \( W_N \) symmetries admit an analogous interpretation in terms of the appropriate \( W_N^\infty \) algebras. The proposed approach could provide a new insight into the classical and quantum structure of the Toda-type theories and, hopefully, lead to understanding all nonlinear algebras (and, perhaps, quantum algebras), as well as the integrable hierarchies associated with them, from a common geometric point of view. In the nearest perspective we are planning to apply our construction to other nonlinear algebras and superalgebras, e.g., Knizhnik-Bershadsky superalgebras \([14]\). On this path we expect to obtain new integrable systems and to shed more light on the geometric structure of the known ones, such as the KdV, MKdV and KP hierarchies.

Acknowledgements. It is a pleasure for us to thank S.Bellucci, A.Isaev, J.Lukierski, V.Ogievetsky, G.Sotkov and M.Vasiliev for interest in the work and discussions. E.I. & S.K. are grateful to the Theory Division of LNF-INFN in Frascati for hospitality extended to them during the course of this work.

References

[1] A.B.Zamolodchikov, Teor. Mat. Fiz., 65 (1985) 347.

[2] P.Mathieu, Phys. Lett., 208B (1988) 101.
[3] C.M.Hull, Phys. Lett., 240B (1990) 110.

[4] E.Bergshoeff, C.N.Pope, L.J.Romans, E.Sezgin, X.Shen and K.S.Stelle
Phys.Lett., 243B (1990) 350.

[5] A.Bilal and J.-L.Gervais, Phys. Lett., 206B (1988) 412; Nucl. Phys., B314 (1989) 646;
B318 (1989) 579;
S.Bellucci and E.Ivanov, Liouville realization of $W_\infty$- algebras, Preprint LNF-90/046(PT),
Frascati, 1990 (to appear in Mod. Phys. Lett. A).

[6] K.Schoutens, A.Sevrin and P. van Nieuwenhuizen, Nucl.Phys., B349 (1991) 791;
C.M.Hull, Nucl. Phys., B353 (1991) 707.

[7] A.Bilal, Phys. Lett., 249B (1990) 56;
G.Sotkov and M.Stanishkov, Nucl. Phys., B356 (1991) 439;
E.Bergshoeff, A.Bilal and K.S.Stelle, W-symmetries: Gauging and Geometry, Preprint
CERN-TH.5924/90 1990.

[8] S.Coleman, J.Wess and B.Zumino, Phys. Rev., 177B (1969) 2239;
C.Callan, S.Coleman, J.Wess and B.Zumino, Phys. Rev., 177B (1969) 2247;
D.V.Volkov, Sov. J. Part. Nucl., 4 (1973) 3;
V.I.Ogievetsky, in Proceeding of X-th Winter School of Theoretical Physics in Karpach, 1
p.117, Wroclaw 1974.

[9] E.Ivanov and V.Ogievetsky, Teor. Mat. Fiz., 25 (1975) 164.

[10] E.Ivanov and S.Krivonos, Teor. Mat.Fiz., 58 (1984)200; Lett. Math. Phys., 8 (1984) 39.

[11] C.N.Pope, L.J.Romans and X.Shen, Phys. Lett., 236B (1990) 173; 242B (1990) 401;
Nucl. Phys., B339 (1990) 191;
E. Bergshoeff, B. de Wit and M. Vasiliev, Nucl. Phys., B366 (1991) 315.

[12] K.Stelle, $w_\infty$-Geometry and $w_\infty$-Gravity, Preprint JINR, E2-91-90, Dubna 1991.

[13] E. Bergshoeff, M.P. Blencowe and K.S. Stelle, Commun. Math. Phys., 128 (1990) 213;
M. Bordemann, J. Hoppe and P. Schaller, Phys. Lett., B232 (1989) 199;
M.A. Vasiliev, Int. J. Mod. Phys., A6 (1991) 1115.

[14] A.Leznov and M.Saveliev, Comm. Math. Phys., 74 (1980) 111.

[15] E.A.Ivanov, S.O.Krivonos and V.M.Leviant, Nucl. Phys., B304 (1988) 601; J.Phys. A:
Math. Gen., 22 (1989) 345.

[16] V.G.Knizhnik, Teor. Mat. Fiz., 66 (1986) 102; M.Bershadsky, Phys. Lett., 174B (1986)
285.