A New Class of Cosmologically ‘Viable’ $f(R)$ Models

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January 16, 2018

Abstract

Instead of assuming a form of gravity and demand cosmology fit with $\Lambda CDM$, a potentially ‘viable’ $f(R)$ gravity model is derived assuming an alternative model of cosmology. Taking the ‘designer’ approach to $f(R)$, a new class of solutions are derived starting with linear coasting cosmology in which scale factor linearly increases with time during matter domination. The derived forms of $f(R)$ are presented as result.

1 Introduction

Alternative gravity models have been proposed numerous times to resolve the problems of dark matter, dark energy and to address other issues of Cosmology such as inflation. $f(R)$ theories have been of much interest as toy-models in exploring alternative gravity cosmologies. Some, through $f(R)$ models, even made attempts to explain other effects of dark matter such as flat rotation curves of galaxies without needing any exotic matter.

In context of cosmology, $f(R)$ models are generally studied as possible alternatives to either dark energy or dark matter. There are some $f(R)$ models that mimic $\Lambda CDM$ behavior under certain limits. Their viability is mostly judged on a model’s ability to reproduce scale factor evolution as predicted by $\Lambda CDM$ model. However, there seems to be no one definitive $f(R)$ model that possibly satisfies all the required criteria to be a strong contender to $\Lambda CDM$ model.

The aim of this paper is to explore the possibility of new viable $f(R)$ models assuming a universe with scale factor linearly evolving with time (at least during matter domination). This paper starts with brief background introduction to $f(R)$ gravity. After presenting generic field equations of $f(R)$ gravity we write modified FRW equations for any $f(R)$ model to study the cosmology of an isotropic and homogeneous universe. The idea of ‘designer’ approach to $f(R)$ gravity and linear coasting model are introduced in the subsequent sections. Using the designer approach, functional forms of $f$ for different geometries of a linearly coasting Universe are derived in the final section. Discussion of the results is followed by conclusions and plan for future work.

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2 \( f(R) \) Cosmology

Just as one can derive the field equations of general relativity from Einstein-Hilbert action, one can assume any functional form \( f \) in terms of Ricci scalar \( R \) and it could as well have been an alternative model of gravity. Although adding quadratic terms etc. to the Lagrangian was done just after few years of introduction of GR, the motivation to modify gravity has become more compelling in the recent times in light of latest cosmological observations.

In the simplest form of \( f(R) \), one can simply extremize the action with respect to the metric tensor alone. This is called the metric formalism of \( f(R) \) theory. We write the action as

\[
S = \frac{1}{2\chi} \int d^4x \sqrt{-g} f(R) + S_{\text{matter}} \tag{1}
\]

We extremize this action w.r.t \( \delta g_{\alpha\beta} \) to write \( \delta S = 0 \). After applying relevant boundary conditions, we get the generic field equations of the form

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f'(R) = \chi T_{\mu\nu} \tag{2}
\]

Here \( \chi = 8\pi G \) where ‘G’ is universal gravitational constant. \( g \) is the determinant of the metric and \( R \) is the Ricci scalar. Working in natural units, we, have \( c = \hbar = 1 \).

The trace of equation (2) gives us

\[
3\Box f'(R) + Rf'(R) - 2f(R) = \chi T \tag{3}
\]

Here \( T \) is the trace of the matter stress-energy tensor \( T_{\mu\nu} \).

Starting with homogeneous and isotropic Universe, we take the FLRW metric element where \( a(t) \) is the scale factor of the Universe.

\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \tag{4}
\]

for a generic \( f(R) \) model, we get modified Friedmann equations as

\[
H^2 + \frac{\kappa}{a^2} = \frac{1}{3f'(R)} \left[ \chi \rho + \frac{Rf'(R) - f(R)}{2} - 3Hf''(R)\dot{R} \right] \tag{5}
\]

and

\[
2\dot{H} + 3H^2 + \frac{\kappa}{a^2} = -\frac{1}{f'(R)} \left[ \chi P + 2H\dot{R}f''(R) + \frac{f(R) - Rf'(R)}{2} + \dot{R}f''(R) + \dot{R}^2 f'''(R) \right] \tag{6}
\]

The effective density and pressure profiles due to \( f(R) \) can be used to write the FRW equations in their standard cosmological form.

\[
H^2 + \frac{\kappa}{a^2} = \frac{1}{3f'(R)} (\chi \rho + \rho_{\text{eff}}) \tag{7}
\]

\[
2\dot{H} + 3H^2 + \frac{\kappa}{a^2} = -\frac{1}{f'(R)} (\chi P + P_{\text{eff}}) \tag{8}
\]
Here $\rho_{\text{eff}}$ and $P_{\text{eff}}$ are additional contributions to density and pressure profiles because of $f(R)$

$$\rho_{\text{eff}} = \frac{Rf'(R) - f(R)}{2} - 3Hf''(R)\dot{R}$$

(9)

$$P_{\text{eff}} = 2H\dot{R}f''(R) + \frac{f(R) - Rf'(R)}{2} + \ddot{R}f''(R) + \dot{R}^2 f'''(R)$$

(10)

As one can quickly note from (5) and (7), one cannot write the total of energy densities in fraction of critical density as ‘1’ as we do in standard cosmology.\textsuperscript{25} Although, some dynamical system of equations can be written\textsuperscript{21} they are not readily solvable.

$$w_{\text{eff}} = \frac{P_{\text{eff}}}{\rho_{\text{eff}}} = \frac{\dot{R}^2 f''' + 2H\ddot{R}f'' + \dddot{R}f'' + \frac{1}{2}(f - Rf')}{\frac{R^2f''}{2} - 3Hf''}$$

(11)

The effective pressure and density terms because of $f(R)$ are likened to contributions of unknown parameters like dark energy in the standard model equations. To judge how well a model fits with the observations, generally some limit to $\Lambda$CDM model is sought. Many attempts presented in reviews\textsuperscript{3,10,21} take a similar approach to liken effective $w$ term ($w_{\text{eff}}$) to dark energy in various versions of $f(R)$ eliminating them based on the fitness with dynamics of $\Lambda$CDM model.

### 3 The ‘Designer’ Approach to $f(R)$

In the standard approach, some form of $f(R)$ is assumed and a fit with $\Lambda$CDM is expected/produced as a consequence. Such theories, try to achieve GR limit by reproducing standard cosmology with $f(R)$ as the dark energy or early inflation replacement. In general, any model of $f(R)$ looking to explain late-time limit by reproducing standard cosmology with $f(R)$ as the dark energy or early inflation replacement. In general, any model of $f(R)$ looking to explain late-time acceleration is expected to give rise to a cosmology which also preserves the evolution sequence of the standard model viz. early inflation, radiation domination era (during which Big Bang Nucleosynthesis occurs), a matter dominated era and the present accelerated epoch.

However, one could, in principle start with the scale factor $a(t)$ and look to find the suitable functional forms of $f(R)$. One can prescribe the desired form of the scale factor $a(t)$ and integrate a differential equation for $f(R)$ that produces the desired scale factor. These are the so-called ‘Designer’ $f(R)$ gravity models. This approach of reconstruction of $f(R)$ from expansion is pioneered and explored further by Nojiri and Odintsov\textsuperscript{17,15,16}. For $\Lambda$CDM, we have to resort to numerical techniques to look for forms of $f$.\textsuperscript{20} These models, often need fine-tuning of constants.\textsuperscript{18} Dunsby \textit{et al.},\textsuperscript{7} after studying various reconstructions of $f(R)$ gravity for FRW expansion history, have concluded that only simple function of Ricci scalar $R$ that admits an exact $\Lambda$CDM expansion history is standard General Relativity with cosmological constant and additional degrees of freedom added to matter term. Moreover, the prescribed evolution of the scale factor $a(t)$ does not uniquely determine the form of $f(R)$ but could possibly give rise to a class of $f(R)$ models that need to be further explored and constrained from observational data.\textsuperscript{21}
4 Linear Coasting Cosmology

When we look beyond the standard model fit to cosmology, we could possibly find some interesting forms of $f(R)$ using the designer approach. One such model, which has a decent observational fit to cosmology is a universe with linearly coasting scale factor. Linear coasting cosmology is one of the many attempts that tried to end the ‘Dark’ age of fundamental cosmology with a coherent theoretical construct. Even recent statistical analysis of the Supernovae Ia data\textsuperscript{14} support such simple yet effective model of the Universe. Hence, linear coasting model fits Supernovae data as good as the ΛCDM model without commonly expected late-time acceleration. This model of the Universe was studied extensively by Daksh \textit{et al.} in Refs. 4–6,9.

Apart from solving the cosmological constant problem and presenting a simmering big bang nucleosynthesis, this model offers solutions to the age problem and horizon problem.\textsuperscript{9} As conventionally expected, Milne model (i.e. an empty open Universe) is not the only one with linearly coasting scale factor. One could achieve linear coasting by modified gravity also as suggested by Daksh \textit{et al.}\textsuperscript{6} as one of the possible motivations to their work, although primarily $a(t) \propto t$ is taken as an ansatz. This paper intends to derive some gravity models that could support the linear coasting model. Recently, there is lot of interest in a model on similar lines as linear coasting model called $R_h = ct$ model which also has a linearly evolving scale factor with $\ddot{a} = 0$ with $w = -1/3$. For more details please see Melia \textit{et al.} in Refs. 11,12,24. While comparisons of the results with this model might be inevitable, comments and comparisons will be subject matter of another paper.

5 $f(R)$ of a Linearly Coasting Universe

For a linearly coasting scale factor we have

$$a(t) \propto t \implies a(t) = t/t_0$$

(12)

This gives us

$$H(t) = \frac{1}{t} \text{ & } H_0 = \frac{1}{t_0}$$

(13)

For FLRW metric Ricci scalar is

$$R = 6(2H^2 + \dot{H} + \kappa/a^2)$$

(14)

5.1 For $\kappa = 0$:  

Assuming a flat Universe, we have

$$R = 6(2H^2 + \dot{H}) = \frac{6}{t^2} = 6H^2 \implies H = \sqrt{R/6}$$

(15)

Using this expression we can convert the modified Friedmann equations (5) and (6) as functions of $R$.

This leads us to

$$R^2 f'' - f/2 + \chi \rho = 0$$

(16)

$$2R^3 f''' + R^2 f'' - R f' + \frac{3}{2} f + 3 \chi P = 0$$

(17)
We do not get any new information from the trace equation as it is same as (16).

Since we are primarily interested in late-time observational data (such as supernovae data), we can assume the case of matter dominated era where \( P = 0 \). Simplifying these two equations, we get a 2\textsuperscript{nd} order differential equation

\[ f'' - \frac{1}{2R^2} f + \alpha R^{-1/2} = 0 \]  \hspace{1cm} (18)

Here we use the fact that (also using (15))

\[ \rho = \frac{\rho_0 a^3}{a^3} = \frac{\rho_0 \ell_0^3}{\ell^3} \]

\[ \chi \rho = \alpha R^{3/2} \]

where \( \alpha = \frac{\chi \rho_0 \ell_0^3}{6\sqrt{6}} \) or

\[ \alpha = \frac{\chi \rho_0}{6\sqrt{6}H_0^3} = \frac{4\pi G \rho_0}{3\sqrt{6}H_0^3} \]

Solving (18), we get the general form of \( f(R) \) as

\[ f(R) = -4\alpha R^{3/2} + C_1 R^{(\sqrt{3}+1)/2} + C_2 R^{(-\sqrt{3}+1)/2} \]  \hspace{1cm} (19)

Despite its contrived appearance, this is potentially a viable form of \( f(R) \) with constants \( C_1, C_2 \) along with \( \rho_0 \)(and hence \( \alpha \)) that need to be constrained using observational data.

Fearing divergences at \( R \to 0 \) we can set \( C_2 = 0 \) resulting in

\[ f(R) = C_1 R^{(\sqrt{3}+1)/2} - 4\alpha R^{3/2} \]  \hspace{1cm} (20)

### 5.2 For \( \kappa > 0 \):

In case of the closed universe, we have \( \kappa/a^2 = 1/t^2 \) for linear coasting. This gives us

\[ R = 6(2H^2 + \dot{H} + \kappa/a^2) = 12/t^2 = 12H^2 \implies H = \sqrt{R/12} \]  \hspace{1cm} (21)

Rest of the equations follow this definition of Ricci scalar giving rise to

\[ R^2 f'' + Rf' - f + 2\chi \rho = 0 \]  \hspace{1cm} (22)

\[ R^3 f''' + \frac{R^2}{2} f'' - \frac{3}{2} R f' + \frac{3}{2} f + 3 \chi P = 0 \]  \hspace{1cm} (23)

Solving these two equations, we have a simpler 1\textsuperscript{st} order differential equation unlike (18) as

\[ f' - \frac{1}{R} f + \frac{\beta}{2} R^2 = 0 \]  \hspace{1cm} (24)

Here

\[ \chi \rho = \beta R^{3/2} \]

where \( \beta = \frac{\chi \rho_0 \ell_0^3}{24\sqrt{3}} \) or
\[ \beta = \frac{\chi \rho_0}{24 \sqrt{3} H_0^3} = \frac{\pi G \rho_0}{3 \sqrt{3} H_0^3} \]

Solving (24), we get solution for \( f(R) \) as

\[ f(R) = C_1 R - 2\beta R^{3/2} \] (25)

### 5.3 For \( \kappa < 0 \):  
Unfortunately in the open Universe case, \( \kappa/a^2 = -1/t^2 \) gives

\[ R = 6(2H^2 + \dot{H} + \kappa/a^2) = 0 \] (26)

Both L.H.S and R.H.S of modified Friedmann equations (5) (6) will be 0 as all are written as functions of \( R \). One can, in principle, assume a power-law solution of scale factor \( a(t) \propto t^m \) in which \( m \approx 1 \). Then taking

\[ H(t) = \frac{\dot{a}}{a} = \frac{mt^{m-1}}{t^m} = \frac{m}{t} \]

one can look for possible forms of \( f(R) \). In fact, such an analysis can be done for the flat and closed cases also. Unfortunately, this leads us to a set of transcendental equations that are not readily solvable.

As for the solutions of \( f(R) \) found, we only consider cosmological viability with linear coasting as a possible alternative model of cosmology. In general, any \( f(R) \) model is considered ‘viable’ if it fulfills the following criteria as given in Ref.8

1. produces correct cosmological dynamics  
2. is stable without ghosts in theory  
3. has well-posed Cauchy problem  
4. gives correct weak-field limit (Newtonian & post-Newtonian)  
5. has cosmological perturbation theory compatible with CMB and large-scale observations

The Cauchy problem for general metric \( f(R) \) is ascertained to be well-posed.2 The other viability conditions such as stability (needing \( f''(R) \geq 0 \) etc.) and weak-field limit are studied for models that are close to GR limit.8 This makes solution given in (25) unstable if \( C_1 = 1 \). But stability of these solutions in (20) and (25) needs to be studied as they are not readily in \( R + c \phi(R) \) form.8 While some of the results may be familiar forms of \( f(R) \) having fractional powers of \( R^\frac{1}{3} \), they still need to be explored in the context of linear coasting model. Also, one needs to analyze the weak-field limit and stability of the linear perturbation solutions19 of the derived \( f(R) \) forms to consider them ‘viable’.

With well-defined alternative \( f(R) \) models of gravity in place, one should be able to carry out calculations to further investigate the structure formation and CMB spectrum results to find if these models are strong contenders for an alternative model of cosmology. One could, in principle, substitute these \( f(R) \) forms in (5) and (6) to
numerically obtain evolution of scale factor $a(t)$ during radiation domination given proper initial conditions. As late time/matter dominated solutions of $f(R)$ are not same as the radiation dominated cases, one needs to see how the radiation domination era evolves and if it produces relevant predictions in terms of nucleosynthesis and other early universe expected behavior. However, if linear coasting goes back to early enough epochs, simmering nucleosynthesis can still take place.$^9$

6 Conclusions & Future Work

Assuming a linearly coasting scale factor, we derived a potentially new ‘viable’ forms of $f(R)$. While these forms may not look suitable in terms of $\Lambda CDM$ or conventional $f(R)$ theory, they need to be re-evaluated in the light of linear coasting cosmology. As there are fewer options on constraining $f(R)$ models other than Cosmology, one can look at linear growth rate of structures and gravitational weak lensing$^{23}$ as possible observations to constrain these new classes of models. There is also need for scrutinizing stability of such solutions. CMB and structure formation theories for these models need to be studied. One can also evaluate the efficacy of these new forms in the weak field limit from the solar system tests and gravitational wave observations. These areas are to be explored in the subsequent work(s).

Acknowledgments

I thank my supervisors Prof. Daksh Lohiya & Prof. Amitabha Mukherjee for giving me freedom to pursue my own ideas along with necessary feedback. I thank CSIR (sanction no. 09/045(1324)/2014-EMR-I) for providing financial support for my research work. Special thanks are due to faculty involved in DST, India grant project SR/S2/HEP-12/2006 for encouraging my work with (partial) financial support when I had none.

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