Laplace-Laplace analysis of the fractional Poisson process

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Abstract

We generate the fractional Poisson process by subordinating the standard Poisson process to the inverse stable subordinator. Our analysis is based on application of the Laplace transform with respect to both arguments of the evolving probability densities. First we give an outline of basic renewal theory, then of the essentials of the classical Poisson process and its fractional generalization via replacement of the exponential waiting time density by one of Mittag-Leffler type. Turning our attention to the probability of the counting number of the fractional Poisson process assuming a given value we find in the transform domain a formula analogous to the Cox-Weiss formula in the theory of continuous time random walk. This formula contains for the jump densities (all increments being positive, in fact equal to 1) the Laplace transform instead of the customary Fourier transform. By manipulating this formula we arrive, after inversion of the transforms, to a subordination integral involving the inverse stable subordinator. Stochastic interpretation of this integral leads to the result that the fractional Poisson process can be obtained from the classical Poisson process via time change to the inverse stable subordinator.

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1 Introduction

The purpose of this paper is to revisit the fractional Poisson process and present its basic theory by treating it as a renewal process with probability of waiting time exceeding duration $t$ being given by a Mittag-Leffler type function as $E_{\beta}(-t^\beta)$ with $0 < \beta \leq 1$. Thereby we treat the general renewal process formally as a continuous time random walk (CTRW) (stressing this concept) with the counting number playing the role of position in space. We then use the known techniques of analyzing the general renewal process and its specialization to the Mittag-Leffler waiting time probability distribution, working, however, in the transform domain, with the Laplace transform not only with respect to time but also with respect to space. This is our motivation for calling our method "Laplace-Laplace analysis".

The structure of our paper is as follows. Section 2 is devoted to notations and terminology, in particular to the essential properties of the special functions needed. In Section 3 we discuss the elements of renewal theory and the CTRW concept, then in Section 4 the Poisson process and its fractional generalization. In Section 5 we consider the aspects of subordination, thereby also touching the method of parametric subordination for which we cite our papers [10, 11, 13]. Finally, Section 6 is devoted to conclusions.

There are in the literature many papers on the fractional Poisson process where the authors have outlined a number of aspects and definitions of this process, see e.g. Repin and Saichev [33], Jumarie [14], Wang et al. [38, 39], Laskin [19, 20], Mainardi et al. [22], Uchaikin et al. [37], Beghin and Orsingher [1], Cahoy et al. [2], Meerschaert et al. [29], Politi et al. [34], Kochubei [18].

To our knowledge the aspect of subordination of the Poisson process to the inverse stable subordinator seems to be dealt with only recently by Beghin and Orsingher [1]...
and Meerschaert, Nane and Vellaisamy \cite{29}. Beghin and Orsingher call the process so generated by subordination the ”first form of the fractional Poisson process” (in fact they consider other kinds of generalization, too) whereas the authors of \cite{29} call it the ”fractal Poisson process” and show that it is a renewal process with the same waiting time distribution as the fractional Poisson process.

Our approach, essentially based on the theory of Laplace transforms, is alternative to the approaches of these authors and turns out to be more direct, so we expect that the present analysis will be appreciated by applied scientists not so well acquainted with the more modern terminology and theory of stochastic processes.

2 Preliminaries

For the reader’s convenience here we present a brief introduction to the basic notions necessary for our analysis of the fractional Poisson process, including the essential elements of integral transforms, fractional calculus and special functions. The notations in these preliminaries follow our earlier papers concerning related topics, see \cite{6, 7, 8, 9, 10, 11, 13, 22, 23, 24}. For more details on general aspects the interested reader may consult the treatises by Podlubny \cite{35}, Kilbas and Saigo \cite{16}, Kilbas, Srivastava and Trujillo \cite{17}, Mathai and Haubold \cite{26}, Mathai, Saxena and Haubold \cite{27}, Mainardi \cite{21}, Diethelm \cite{4}.

**Fourier and Laplace transforms**

We generously apply the transforms of Fourier and Laplace to admissible functions or generalized functions defined on $\mathbb{R}$ or $\mathbb{R}^+$, respectively. In the following we use the symbol $\hat{\cdot}$ for the juxtaposition of a function with its Fourier or Laplace transform. A look at the superscript $\hat{\cdot}$ for the Fourier transform, $\tilde{\cdot}$ for the Laplace transform reveals their relevant juxtaposition. We use $x$ argument (associated to $\kappa$) for functions Fourier transformed, and $x$ or $t$ argument (associated to $\kappa$ or $s$, respectively) for functions Laplace transformed.

\begin{align*}
  f(x) \hat{=} \hat{f}(\kappa) := \int_{-\infty}^{+\infty} e^{ix\kappa} f(x) \, dx, & \quad \text{Fourier transform.} \\
  f(x) \hat{=} \tilde{f}(\kappa) := \int_{0}^{+\infty} e^{-\kappa x} f(x) \, dx, & \quad \text{Laplace transform.} \\
  f(t) \hat{=} \tilde{f}(s) := \int_{0}^{+\infty} e^{-st} f(t) \, dx, & \quad \text{Laplace transform.}
\end{align*}

**Convolutions**

\begin{align*}
  (u \ast v)(x) := \int_{-\infty}^{+\infty} u(x - x') v(x') \, dx', & \quad \text{Fourier convolution.}
\end{align*}
\[(u \ast v)(t) := \int_0^t u(t - t') v(t') \, dt', \quad \text{Laplace convolution.}\]

The meaning of the connective \(\ast\) will be clear from the context. For convolution powers we have:

\[u^{*0}(x) = \delta(x), \quad u^{*1}(x) = u(x), \quad u^{*(n+1)}(x) = (u^n * u)(x),\]

\[u^{*0}(t) = \delta(t), \quad u^{*1}(t) = u(t), \quad u^{*(n+1)}(t) = (u^n * u)(t),\]

where \(\delta\) denotes the Dirac generalized function.

**Fractional integral**

The Riemann-Liouville fractional integral of order \(\alpha > 0\), for a sufficiently well-behaved function \(f(t) \ (t \geq 0)\), is defined as generalization of the \(n\)-fold repeated integral, namely

\[J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) \, d\tau, \quad \alpha > 0.\]

The \textit{semi-group property} is well-known

\[J_t^\alpha J_t^\beta = J_t^{\alpha + \beta} = J_t^\beta J_t^\alpha, \quad \alpha, \beta \geq 0.\]

We have the following Laplace transform pair

\[J_t^\alpha f(t) \overset{\mathcal{L}}{=} \frac{\tilde{f}(s)}{s^\alpha}, \quad \alpha \geq 0,\]

which is the straightforward generalization of the corresponding formula for the \(n\)-fold repeated integral.

**Fractional derivatives**

The Riemann-Liouville fractional derivative of order \(\alpha > 0\), \(D_t^\alpha\), is defined as the \textit{left inverse operator} of the corresponding fractional integral \(J_t^\alpha\). Limiting ourselves to fractional derivatives of order \(\alpha \in (0, 1)\) we have, for a sufficiently well-behaved function \(f(t) \ (t \geq 0)\),

\[D_t^\alpha f(t) := D_t^1 J_t^{1-\alpha} f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t - \tau)\alpha} \, d\tau, \quad 0 < \alpha < 1,\]

while the corresponding \textit{Caputo} derivative is

\[\overset{\ast}{D_t^\alpha} f(t) := J_t^{1-\alpha} D_t^1 f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f^{(1)}(\tau)}{(t - \tau)\alpha} \, d\tau,\]

\[= D_t^\alpha f(t) - f(0^+) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} = D_t^\alpha \left[ f(t) - f(0^+) \right].\]
Both derivatives yield the ordinary first derivative as $\alpha \to 1^-$ but for $\alpha \to 0^+$ we have
\[ D_t^0 f(t) = f(t), \quad {}^*D_t^0 f(t) = f(t) - f(0^+). \]
We point out the major utility of the Caputo fractional derivative in treating initial-value problems with Laplace transform. We have
\[ \mathcal{L}[{}^*D_t^\alpha f(t); s] = s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0^+), \quad 0 < \alpha \leq 1. \]
In contrast the Laplace transform of the Riemann-Liouville fractional derivative needs the limit at zero of a fractional integral of the function $f(t)$.

**Mittag-Leffler and Wright functions**

**Mittag-Leffler functions**

\[ E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \]

The Mittag-Leffler function is entire of order $1/\alpha$. The cases $\alpha = 1, 2$ are trivial:
\[ \begin{cases} 
E_1(\pm z) = \exp(\pm z), \\
E_2(\pm z^2) = \cosh(z), \quad E_2(-z^2) = \cos(z).
\end{cases} \]

In the following we will consider, with $0 < \beta \leq 1$ and $t \geq 0$, the Laplace transform pairs
\[ \begin{cases} 
\Psi(t) = E_\beta(-t^\beta) \div \Psi(s) = \frac{s^{\beta-1}}{1 + s^\beta}, \\
\phi(t) = -\frac{d}{dt} E_\beta(-t^\beta) \div \phi(s) = \frac{1}{1 + s^\beta}.
\end{cases} \]

It is worth noting the algebraic decay of $\Psi(t)$ and $\phi(t)$ as $t \to \infty$:
\[ \Psi(t) \sim \frac{\sin(\beta \pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}, \quad \phi(t) \sim \frac{\sin(\beta \pi)}{\pi} \frac{\Gamma(\beta + 1)}{t^{\beta+1}}, \quad t \to +\infty. \]

Furthermore $\Psi(t) = E_\beta(-t^\beta)$ is the solution of the fractional relaxation equation
\[ {}^*D_t^\beta \Psi(t) = -\Psi(t), \quad t \geq 0, \quad \Psi(0) = 1. \]

We refer to [6, 7, 9] for the relevance of Mittag-Leffler functions in theory of continuous time random walk and space-time fractional diffusion and in asymptotics of power laws.

**Wright functions**

\[ W_{\lambda, \mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}. \]
We distinguish the Wright functions of the first kind \( (\lambda \geq 0) \) and of the second kind \((-1 < \lambda < 0)\). The Wright function is entire of order \(1/(1 + \lambda)\) hence of exponential type only if \(\lambda \geq 0\). The case \(\lambda = 0\) is trivial since \(W_{0,\mu}(z) = e^z/\Gamma(\mu)\).

\[ M_{\nu}(z) := W_{-\nu,1-\nu}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma[-\nu n + (1 - \nu)]}, \]

where \(0 < \nu < 1\). Special cases are

\[ M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right), \quad M_{1/3}(z) = \frac{3^{2/3}}{3} \text{Ai}\left(\frac{z}{3^{1/3}}\right). \]

Here \(\text{Ai}\) denotes the Airy function.

Mittag-Leffler functions as Laplace transforms of \(M\)-Wright functions

\[ M_{\nu}(t) \equiv E_{\nu}(-s), \quad 0 < \nu < 1. \]

Stretched Exponentials as Laplace transforms of \(M\)-Wright functions

\[ \frac{\nu}{\nu+1} M_{\nu}(1/t^\nu) \equiv e^{-s^\nu}, \quad 0 < \nu < 1. \]

\[ \frac{1}{t^\nu} M_{\nu}(1/t^\nu) \equiv \frac{e^{-s^\nu}}{s^{1-\nu}}, \quad 0 < \nu < 1. \]

Note that \(\exp(-s^\nu)\) is the Laplace transform of the extremal (unilateral) stable density \(L_{\nu}^{-\nu}(t)\), which vanishes for \(t < 0\), whereas \(\exp(-s^\nu)/s^{1-\nu}\) is related to the Laplace transform of the Green function of the time-fractional diffusion-wave equation.

The asymptotic representation of the \(M\)-Wright function

Choosing as a variable \(t/\nu\) rather than \(t\), the computation of the asymptotic representation as \(t \to \infty\) by the saddle-point approximation yields:

\[ M_{\nu}(t/\nu) \sim a(\nu) t^{(\nu-1/2)/(1-\nu)} \exp\left[-b(\nu) t^{1/(1-\nu)}\right], \]

where

\[ a(\nu) = \frac{1}{\sqrt{2\pi (1-\nu)}} > 0, \quad b(\nu) = \frac{1-\nu}{\nu} > 0. \]

The stochastic relevance of the \(M\)-Wright function

For the relevance of the \(M\)-Wright functions in fractional diffusion and related stochastic processes we refer to formulas shown in the following, proved in
Let us also point to our pair of complementary papers \[10\] and \[11\] on subordination in fractional diffusion processes and our contribution \[12\] in which we analyze some renewal processes related and contrasting to the fractional Poisson process.

From the stochastic point of view, the $M$-Wright function emerges as a natural generalization of the Gaussian density as Green function for time-fractional diffusion processes and of the pure delta-peak drift density $\delta(x-t)$ as Green function for time-fractional drift processes. In fact

- for the \textbf{time-fractional diffusion equation} with $0 < \beta \leq 1$:

$$*D_{t}^{\beta} u = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, \quad t \geq 0,$$

we have, assuming $u(x, 0) = \delta(x)$,

$$u(x, t) \equiv G_{\beta}(x, t) = \frac{1}{2t^{\beta/2}} M_{\beta/2} \left( \frac{|x|}{t^{\beta/2}} \right);$$

- for the \textbf{time-fractional drift equation} with $0 < \beta \leq 1$:

$$*D_{t}^{\beta} u(x, t) = -\frac{\partial}{\partial x} u(x, t), \quad -\infty < x < +\infty, \quad t \geq 0,$$

we have, assuming $u(x, 0) = \delta(x)$,

$$u(x, t) \equiv G_{\beta}^*(x, t) = \begin{cases} t^{-\beta} M_{\beta} \left( \frac{x}{t^{\beta}} \right), & x > 0, \\ 0, & x < 0. \end{cases}$$

In the former case the $M$-Wright function can be extended for $1 < \beta \leq 2$ to represent the Green function of the corresponding time-fractional diffusion-wave equation, intermediate between the diffusion and wave equations. In the last case the $M$-Wright function plays the role of the \textit{inverse stable subordinator}, as it will be explained later on, see Eq. (37) with $x$ replaced by $t_*$.

\section{Elements of Renewal Theory and CTRW}

\textbf{Definition} of renewal process: By a \textit{renewal process} we mean an infinite sequence $0 = t_0 < t_1 < t_2 < \cdots$ of events separated by i.i.d. (independent and identically distributed) random waiting times $T_j = t_j - t_{j-1}$, whose probability density $\phi(t)$ is given as a function or generalized function in the sense of Gel’fand and Shilov \[5\] (interpretable as a measure) with support on the positive real axis $t \geq 0$, non-negative: $\phi(t) \geq 0$, and normalized: $\int_{0}^{\infty} \phi(t) \, dt = 1$, but not having a delta peak at the origin $t = 0$. Note that the instant $t_0 = 0$ is not counted as an event.
We are interested in the counting number process $x = N(t)$ and the sojourn density $p(x,t)$ for the counting number

$$N(t) := \max \{n|t_n \leq t\} , \ n = 0, 1, 2, \cdots$$ \hspace{1cm} (1)

having the value $x$, furthermore in the expectation

$$m(t) := \langle N(t) \rangle = \int_0^\infty x p(x,t) \, dx$$ \hspace{1cm} (2)

which is the mean number of positive events in the interval $[0,t]$ and is called the renewal function, see e.g. [36]. Then, we ask for the probabilities

$$p_n(t) := P[N(t) = n] , \ n = 0, 1, 2, \cdots$$ \hspace{1cm} (3)

**Definition** of continuous time random walk (CTRW): A continuous time random walk is an infinite sequence of i.i.d. spatial positions $0 = x_0, x_1, x_2, \cdots$, separated by random jumps $X_j = x_j - x_{j-1}$, whose probability density function $w(x)$ is given as a non-negative function or generalized function (interpretable as a measure) with support on the real axis $-\infty < x < +\infty$ and normalized: $\int_0^\infty w(x) \, dx = 1$, this random walk being subordinated to a renewal process so that we have a random process $x = x(t)$ on the real axis with the property $x(t) = x_n$ for $t_n \leq t < t_{n+1}$, $n = 0, 1, 2, \cdots$.

We are interested in the sojourn probability density $u(x,t)$ of a particle wandering according to the random process $x = x(t)$ being in point $x$ at instant $t$.

Let us define the following cumulative probabilities related to the probability density function $\phi(t)$

$$\Phi(t) = \int_0^{t^+} \phi(t') \, dt' , \quad \Psi(t) = \int_{t^+}^\infty \phi(t') \, dt' = 1 - \Phi(t) .$$ \hspace{1cm} (4)

For definiteness, we take $\Phi(t)$ as right-continuous, $\Psi(t)$ as left-continuous. When the non-negative random variable represents the lifetime of technical systems, it is common to call $\Phi(t) := P(T \leq t)$ the failure probability and $\Psi(t) := P(T > t)$ the survival probability, because $\Phi(t)$ and $\Psi(t)$ are the respective probabilities that the system does or does not fail in $(0,t]$. These terms, however, are commonly adopted for any renewal process.

Now, recalling from Section 2 the definition of convolutions, we have for the solution $u(x,t)$ the Cox-Weiss series, see [3, 40],

$$u(x,t) = \left( \Psi * \sum_{n=0}^\infty \phi^n w^n \right)(x,t)$$ \hspace{1cm} (5)
which intuitively says: Before and at instant $t$ there have occurred no jumps or exactly 1 jump or exactly 2 jumps or ... and if the last jump has occurred at instant $t'$ the particle is resting there for a duration $t - t'$.

In the Fourier-Laplace domain we have

$$
\tilde{\Psi}(s) = \frac{1 - \tilde{\phi}(s)}{s},
$$

and

$$
\tilde{u}(\kappa, s) = \frac{1 - \tilde{\phi}(s)}{s} \sum_{n=0}^{\infty} \left( \tilde{\phi}(s) \tilde{w}(\kappa) \right)^n = \frac{1 - \tilde{\phi}(s)}{s} \frac{1}{1 - \tilde{\phi}(s) \tilde{w}(\kappa)}.
$$

This is the famous Montroll-Weiss solution formula for a CTRW, see [30, 40].

In the special situation of the jump density having support only on the positive semi-axis $x \geq 0$ it is convenient to replace in this formula the Fourier transform by the Laplace transform to obtain the Laplace-Laplace solution

$$
\tilde{u}(\kappa, s) = \frac{1 - \tilde{\phi}(s)}{s} \sum_{n=0}^{\infty} \left( \tilde{\phi}(s) \tilde{w}(\kappa) \right)^n = \frac{1 - \tilde{\phi}(s)}{s} \frac{1}{1 - \tilde{\phi}(s) \tilde{w}(\kappa)}.
$$

Henceforth we assume to have this special situation of jumps only in positive direction so that we will work with this Laplace-Laplace formula.

An essential trick of what follows is that we treat renewal processes as continuous time random walks with waiting time density $\phi(t)$ and special jump density $w(x) = \delta(x - 1)$ corresponding to the fact that the counting number $N(t)$ increases by 1 at each positive instant $t_n$ so that $x(t) = n$ for $t_n \leq t \leq t_{n+1}$. We then have $\tilde{w}(\kappa) = \exp(-\kappa)$ and get for the counting number process $N(t)$ the sojourn density in the transform domain

$$
\tilde{p}(\kappa, s) = \frac{1 - \tilde{\phi}(s)}{s} \sum_{n=0}^{\infty} \left( \tilde{\phi}(s) \right)^n e^{-nk} = \frac{1 - \tilde{\phi}(s)}{s} \frac{1}{1 - \tilde{\phi}(s) e^{-\kappa}}.
$$

From this formula we can find formulas for the renewal function $m(t)$ and the probabilities $p_n(t) = P\{N(t) = n\}$. Because $N(t)$ assumes as values only the
non-negative integers, the sojourn density \( p(x, t) \) vanishes if \( x \) is not equal to one of these, but has a delta peak of height \( p_n(t) \) for \( x = n \) (\( n = 0, 1, 2, 3, \cdots \)). Hence

\[
p(x, t) = \sum_{n=0}^{\infty} p_n(t) \delta(x - n).
\]  

Rewriting Eq. (9), by inverting with respect to \( \kappa \), as

\[
\sum_{n=0}^{\infty} (\Psi \ast \phi^n)(t) \delta(x - n),
\]  

we identify

\[
p_n(t) = (\Psi \ast \phi^n)(t).
\]  

According to the theory of Laplace transform we conclude from Eqs. (2) and (10)

\[
m(t) = -\frac{\partial}{\partial \kappa} \tilde{p}(\kappa, t) |_{\kappa=0} = \left( \sum_{n=0}^{\infty} p_n(t) n e^{-n \kappa} \right) |_{\kappa=0} = \sum_{n=0}^{\infty} n p_n(t).
\]

a result naturally expected, and

\[
\tilde{m}(s) = \sum_{n=0}^{\infty} n \tilde{p}_n(s) = \tilde{\Psi}(s) \sum_{n=0}^{\infty} n \left( \tilde{\phi}(s) \right)^n = \frac{\tilde{\phi}(s)}{s \left( 1 - \tilde{\phi}(s) \right)},
\]

thereby using the identity

\[
\sum_{n=0}^{\infty} n z^n = \frac{z}{(1 - z)^2}, \quad |z| < 1.
\]

Thus we have found in the Laplace domain the reciprocal pair of relationships

\[
\tilde{m}(s) = \frac{\tilde{\phi}(s)}{s(1 - \tilde{\phi}(s))}, \quad \tilde{\phi}(s) = \frac{s \tilde{m}(s)}{1 + s \tilde{m}(s)},
\]

saying that the waiting time density and the renewal function mutually determine each other uniquely. The first formula of Eq. (15) can also be obtained as the value
The first of these equations usually is called the renewal equation.

4 The Poisson process and its fractional generalization

The most popular renewal process is the Poisson process. It is characterized by its mean waiting time $1/\lambda$ (equivalently by its intensity $\lambda$), which is a given positive number, and by its survival probability $\Psi(t) = \exp(-\lambda t)$ for $t \geq 0$, which corresponds to the waiting time density $\phi(t) = \lambda \exp(-\lambda t)$. With $\lambda = 1$ we have what we call the standard Poisson process.

We generalize the standard Poisson process by replacing the exponential function by a function of Mittag-Leffler type. With $t \geq 0$ and a parameter $\beta \in (0, 1]$ we take

$$\Psi(t) = \Psi_\beta(t) = E_\beta(-t^\beta),$$
$$\phi(t) = \phi_\beta(t) = -\frac{d}{dt} E_{\beta}(-t^\beta) = \beta t^{\beta-1} E^\prime_{\beta}(-t^\beta).$$

We call the process so defined the fractional Poisson process. To analyze it we go into the Laplace domain where we have

$$\tilde{\Psi}(s) = \frac{s^{\beta-1}}{1 + s^\beta}, \quad \tilde{\phi}(s) = \frac{1}{1 + s^\beta}. \quad (18)$$

If there is no danger of misunderstanding we will not decorate $\Psi$ and $\phi$ with the index $\beta$. The special choice $\beta = 1$ gives us the standard Poisson process with $\Psi_1(t) = \phi_1(t) = \exp(-t)$.

Whereas the Poisson process has, as is well known, finite mean waiting time, the fractional Poisson process ($0 < \beta < 1$) does not have this property. In fact,

$$\langle T \rangle = \int_0^\infty t \phi(t) \, dt = \beta \left. \frac{s^{\beta-1}}{(1 + s^\beta)^2} \right|_{s=0} = \begin{cases} 1, & \beta = 1, \\ \infty, & 0 < \beta < 1. \end{cases} \quad (19)$$

Let us calculate the renewal function $m(t)$. Inserting $\tilde{\phi}(s) = 1/(1 + s^\beta)$ into Eq. (9) and taking $w(x) = \delta(x - 1)$ as in Section 3, we find for the sojourn density of the
counting function $N(t)$ the expressions

\[ \tilde{p}_\beta(\kappa, s) = \frac{s^{\beta-1}}{1 + s^\beta - e^{-\kappa}} = \frac{s^{\beta-1}}{1 + s^\beta} \sum_{n=0}^{\infty} \frac{e^{-n\kappa}}{(1 + s^\beta)^n}, \]  

(20)

and

\[ \tilde{p}_\beta(\kappa, t) = E_\beta \left(- (1 - e^{-\kappa}) t^\beta\right), \]  

(21)

and then

\[ m(t) = -\frac{\partial}{\partial \kappa} \tilde{p}_\beta(\kappa, t)|_{\kappa=0} = e^{-\kappa} t^\beta E'_\beta \left(- (1 - e^{-\kappa}) t^\beta\right)|_{\kappa=0}. \]  

(22)

Using $E'_\beta(0) = 1/\Gamma(1+\beta)$ now yields

\[ m(t) = \begin{cases} t, & \beta = 1, \\ \frac{t^\beta}{\Gamma(1+\beta)}, & 0 < \beta < 1. \end{cases} \]  

(23)

This result can also be obtained by plugging $\tilde{\phi}(s) = 1/(1+s^\beta)$ into the first equation in (15) which yields $\tilde{m}(s) = 1/s^{\beta+1}$ and then by Laplace inversion Eq. (23). Using general Taylor expansion

\[ E_\beta(z) = \sum_{n=0}^{\infty} \frac{E^{(n)}_\beta}{n!} (z - b)^n, \]  

(24)

in Eq. (21) with $b = -t^\beta$ we get

\[ \tilde{p}_\beta(\kappa, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{(n)}_\beta (-t^\beta) e^{-n\kappa}, \]  

\[ p_\beta(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{(n)}_\beta (-t^\beta) \delta(x - n), \]  

(25)

and, by comparison with Eq. (10) of Section 3, the probabilities

\[ p_n(t) = P\{N(t) = n\} = \frac{t^n}{n!} E^{(n)}_\beta (-t^\beta). \]  

(26)

Observing from Eq. (20)

\[ \tilde{p}_\beta(\kappa, s) = \frac{s^{\beta-1}}{1 + s^\beta - e^{-\kappa}} = \frac{s^{\beta-1}}{1 + s^\beta} \sum_{n=0}^{\infty} \frac{e^{-n\kappa}}{(1 + s^\beta)^n}, \]  

(27)
and inverting with respect to $\kappa$,

$$
\tilde{p}_\beta(x, s) = \frac{s^{\beta-1}}{1 + s^\beta} \sum_{n=0}^{\infty} \frac{\delta(x - n)}{(1 + s^\beta)^n},
$$

(28)

we finally identify

$$
\tilde{p}_n(s) = \frac{s^{\beta-1}}{(1 + s^\beta)^{n+1}} \sum_{n=0}^{\infty} \frac{t^n}{n!} E_\beta^{(n)}(-t^\beta) = p_n(t).
$$

(29)

En passant we have proved an often cited special case of an inversion formula by Podlubny [35]. For an alternative representation of $p_n(t)$ as an integral see Eq. (39) in Section 5.

For the Poisson process with intensity $\lambda > 0$ we have a well-known infinite system of ordinary differential equations (for $t \geq 0$), see e.g. Khintchine [15],

$$
p_0(t) = e^{-\lambda t}, \quad \frac{d}{dt} p_n(t) = \lambda (p_{n-1}(t) - p_{n}(t)),
$$

(30)

with initial conditions $p_n(0) = 0, n = 1, 2, \ldots$ which sometimes even is used to define the Poisson process. We have an analogous system of fractional differential equations for the fractional Poisson process. In fact, from Eq. (29) we have

$$
(1 + s^\beta) \tilde{p}_n(s) = \frac{s^{\beta-1}}{(1 + s^\beta)^n} = \tilde{p}_{n-1}(s).
$$

(31)

Hence

$$
s^{\beta} \tilde{p}_n(s) = \tilde{p}_{n-1}(s) - \tilde{p}_n(s),
$$

(32)

so in the time domain

$$
p_0(t) = E_\beta(-t^\beta), \quad {_\ast}D_t^\beta p_n(t) = p_{n-1}(t) - p_{n}(t),
$$

(33)

with initial conditions $p_n(0) = \delta_{n0}, n = 0, 1, 2, \ldots$, where $_\ast D_t^\beta$ denotes the time-fractional derivative of Caputo type of order $\beta$. It is also possible to introduce and define the fractional Poisson process by this difference-differential system.

Let us note that by solving the system (33), Beghin and Orsingher in [1] introduce what they call the “first form of the fractional Poisson process”, and in [29] Meerschaert, Nane and Vellaisamy show that this process is a renewal process with Mittag-Leffler waiting time density as in (17), hence is identical with the “fractional Poisson process”. 

13
5 Subordination

In order to introduce the subordination framework in a given stochastic process \( x = x(t) \) we must introduce in addition to the natural time \( t \geq 0 \) another time line \( t_\ast \geq 0 \), that we call \textit{operational time}. These two basic time lines are inter-related by two increasing processes, \( t = t(t_\ast) \) and \( t_\ast = t_\ast(t) \), inverse to each other.

We circumvent the analytical delicacies involved in inverting increasing (but not necessarily strictly monotonically increasing) functions with (possibly even in a finite interval infinitely many) jumps and intervals of constancy by considering for such functions jumps graphically represented by vertical segments and intervals of constancy (horizontal segments), in the corresponding Cartesian systems of coordinates. By inversion (reflection on the diagonal line of the first quadrant) jumps (vertical segments) become intervals of constancy (horizontal segments), and vice versa. For specialists in stochastic processes let us just say that each of the two processes (or "trajectories") can (if wanted) be represented by an equivalent process with càdlàg structure.

Let us now denote by \( x = x(t) \) the described \( \beta \)-fractional Poisson process \( x = N_\beta(t) \) (happening in \( t \geq 0 \), running along \( x \geq 0 \)) and by \( y = y(t_\ast) \) the standard Poisson process \( y = N_1(t_\ast) \) (happening in \( t_\ast \geq 0 \), running along \( y \geq 0 \)). Let us denote by \( t = t(t_\ast) \) the totally positive-skewed \( \beta \)-stable process (happening in operational time \( t_\ast \), running along natural time \( t \)). Its inverse process \( t_\ast = t_\ast(t) \) (happening in natural time \( t \), running along operational time \( t_\ast \)) is the inverse stable subordinator of order \( \beta \). These two processes \( t = t(t_\ast) \) and \( t_\ast = t_\ast(t) \) are described in our recent papers \[10, 11\], where they are referred to as the \textit{leading process} and \textit{directing process}, respectively.

Thus, from now on, we will consider the Poisson process on the time line \( t_\ast \), that we call \textit{operational time}, and we denote it by \( y = y(t_\ast) \). Its density in \( y \), evolving in operational time \( t_\ast \), is \( p_1(y, t_\ast) \). We will now state a theorem on time change and derive its underlying subordination integral.

**Theorem on Subordination and Time Change**

\[
x(t) = y(t_\ast(t)). \quad (34)
\]

The \textbf{proof} will be carried via the Laplace-Laplace method.

By Eq.\[20\] with \( \beta = 1 \) we have for the standard Poisson process (happening in operational time \( t_\ast \geq 0 \) and running on \( y \geq 0 \))

\[
\tilde{p}_1(\kappa, s_\ast) = \frac{1}{s_\ast + 1 - \exp(-\kappa)}.
\]

Then the first Laplace inversion \( s_\ast \to t_\ast \) yields by Eq.\[21\], or directly,

\[
\tilde{p}_1(\kappa, t_\ast) = \exp(-t_\ast(1 - e^{-\kappa})).
\]
and the second Laplace inversion \( \kappa \to y \) yields by Eq. (25), or directly,

\[
p_1(y, t^*) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \exp(-t^*) \delta(y - n). \tag{35}
\]

Using

\[
\int_0^\infty \exp(-at^*) \, dt^* = \frac{1}{a} \quad \text{for} \quad a = s^\beta + 1 - e^{-\kappa},
\]

we write by Eq. (20) in the form

\[
\tilde{p}_\beta(\kappa, s) = s^{\beta-1} \int_0^\infty \exp[-t^*(s^\beta + 1 - e^{-\kappa})] \, dt^*
\]

\[
= \int_0^\infty \exp(-t^*(1 - e^{-\kappa})) \{ s^{\beta-1} \exp(-t^*s^\beta) \} \, dt^*.
\]

Then, by Laplace-Laplace inversion, we obtain the subordination integral

\[
p_\beta(x, t) = \int_0^\infty p_1(x, t^*) q_\beta(t^*, t) \, dt^*, \tag{36}
\]

with the density \( q_\beta(t^*, t) \) of the inverse stable subordinator (density in operational time \( t^* \) evolving in natural time \( t \)) that we have used in [13] and later in [10, 11]. In particular, from our most recent paper [11], see Eqs. (75) and (81) in it, we have

\[
q_\beta(t^*, t) = J_t^{1-\beta} r_\beta(t, t^*) = t^{-\beta} M_\beta(t^*/t^\beta) \div s^{\beta-1} \exp(-t^*s^\beta). \tag{37}
\]

Here \( J_t^{1-\beta} \) denotes the Riemann-Liouville time-fractional integral of order \( 1 - \beta \), \( r_\beta(t, t^*) \) is the probability density (in \( t \), evolving in operational time \( t^* \)) of the extremely positively skewed stable density of index \( \beta \), \( L^{-\beta}_\beta \), namely

\[
r_\beta(t, t^*) = t^{-1/\beta} L^{-\beta}_\beta(t/t^\beta) \div e^{-t^*s^\beta}, \tag{38}
\]

and \( M_\beta \) is the so-called \( M \)-Wright function of order \( \beta \) defined in Section 2. Eqs. (37), (38) appear in various disguises in publications of other authors for which we have not done a systematic search, see e.g. [28].

**Interpretation:** Equation (36) says: The fractional Poisson process \( x(t) \) can be obtained from the standard Poisson process by time change via the inverse stable subordinator \( t^*(t) \) according to Eq. (34), namely \( x(t) = y(t^*(t)) \).

**Corollary 1:** For the fractional Poisson probabilities we have, alternatively to (26), the integral representation (the subordination integral)

\[
p_n(t) = \frac{1}{n!} \int_0^\infty t^\beta \exp(-t^*) q_\beta(t^*, t) \, dt^* = \frac{1}{n!} \int_0^\infty t^\beta \exp(-t^*) M_\beta(t^*t^{-\beta}) \, dt^*. \tag{39}
\]
**Proof:** Set $x = n$ in (35), use (26) with $E_1(-t_*) = \exp(-t_*)$, (35) and (36).

As an exercise the reader may verify (39) directly from (26) by use of the fact that $E_\beta(-s)$ is the Laplace transform of $M_\beta(t)$ (see equation (F.30) in [20]).

**Corollary 2:** With the positive-oriented extremal $\beta$-stable process $t = t(t_*)$ we have the parametric representation

$$\begin{cases} t = t(t_*), \\ x = y(t_*), \end{cases} \quad \text{for} \quad x = x(t). \quad (40)$$

**Remark** According to Meerschaert, Nane and Vellaisamy [29]

$$z_\beta(t) = z_1(y(t_*)), \quad (41)$$

where $z_\beta(t)$ is a CTRW with $\beta$-Mittag-Leffler waiting times and an arbitrary jump density, and $z_1$ is a CTRW with the exponential waiting times of the standard Poisson process having the same arbitrary jump distribution. The influence of the index $\beta$ there is transferred to the process $t_*(t)$.

IN WORDS: Define a random walk as the sum of $N$ spatial random variables, $N$ being the counting number of a renewal process. Then, the subordination of such random walk to the fractional Poisson process is equivalent to the replacement of the fractional Poisson process by the standard Poisson process evolving on the inverse stable subordinator of order $\beta$.

6 **Conclusions**

By time change via the inverse stable subordinator the standard Poisson process is transformed to the fractional Poisson process. Treating the counting number of a renewal process as a "spatial" variable and not shying away from extensive use of delta-functions and their transforms we can use the formalism of the the common theory of Continuous Time Random Walk. In this theory usually in space the Fourier transform is used, but as in our situation all jumps are of size 1 our walk proceeds only in positive direction, hence not only in time but also in space we can work with the transform of Laplace instead of that of Fourier.

The analysis carried out in this essay is based on the Laplace transform of the waiting time density as well as of the jump-width density. So, in the Montroll-Weiss solution formula written in Laplace-Laplace instead of Laplace-Fourier form, we apply the usual trick of representing the reciprocal of the denominator as an improper integral (thereby introducing an operational time variable) and can (by inverting the transforms) separate variables of time and space. Finally, we arrive at
an integral representation that allows interpretation as a time change realized (as in
the theory of time-fractional diffusion) by the ”inverse stable subordinator”.

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