Existence and uniqueness of the globally conservative solutions for a weakly dissipative Camassa-Holm equation in time weighted $H^1(\mathbb{R})$ space

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Abstract

In this paper, we prove that the existence and uniqueness of globally weak solutions to the Cauchy problem for the weakly dissipative Camassa-Holm equation in time weighted $H^1$ space. First, we derive an equivalent semi-linear system by introducing some new variables, and present the globally conservative solutions of this equation in time weighted $H^1$ space. Second, we show that the peakon solutions are conservative weak solutions in $H^1$. Finally, given a conservative solution, we introduce a set of auxiliary variables tailored to this particular solution, and prove that these variables satisfy a particular semilinear system having unique solutions. In turn, we get the uniqueness of the conservative solution in the original variables.

Keywords: A weakly dissipative Camassa-Holm equation; Globally weak solutions; Peakon solutions; Uniqueness

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Contents

1 Introduction 2
2 Preliminary 3
3 Global solutions in Lagrange coordinates 4
   3.1 An equivalent system 4
   3.2 Global weak solutions of the equivalent system 9
4 Solutions to the original equation 11

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5 Peakon solutions

6 Uniqueness of solutions for the original equation

6.1 Uniqueness of characteristic ................................................................. 15
6.2 Proof of uniqueness .............................................................................. 23

References ........................................................................................................ 24

1 Introduction

Recently, Freire studied the weakly dissipative Camassa-Holm equation [18]
\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x \partial t} + 3u\frac{\partial u}{\partial x} + \lambda (u - u_{\partial x}) = 2uu_x + uu_{\partial x x} + \alpha u + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{\partial x x}, \quad x \in \mathbb{R}, \ t > 0, \\
&u(0, x) = u_0,
\end{aligned}
\]  
(1.1)

where $\alpha, \beta, \gamma, \Gamma$ are any real numbers, and $\lambda > 0$. The above equation (1.1) can be rewritten as
\[
\begin{aligned}
&\frac{\partial u}{\partial t} + (u + \Gamma)u_x + \lambda u = Q, \quad x \in \mathbb{R}, \ t > 0, \\
u(0, x) = u_0,
\end{aligned}
\]  
(1.2)

with $Q = \Lambda^{-2} \partial_x \left(h(u) - u^2 - \frac{1}{2} u_x^2 \right)$ and $h(u) = (\alpha + \Gamma)u + \frac{\beta}{3} u^3 + \frac{\gamma}{4} u^4$, $\Lambda^{-2} = (1 - \partial_{xx})^{-1}$. The local well-posedness of its Cauchy problem in Sobolev spaces $H^s$ with $s > \frac{3}{2}$ was studied in [12]. Meng and Yin [23] proved the local well-posedness and global strong solutions under the condition that small initial data to (1.1) in critical Besov spaces $B_{p,r}^s$ with (i) $s > 1 + \frac{1}{p}$; (ii) $s = 1 + \frac{1}{p}, \ r = 1, \ p \in [1, \infty)$. The integrability and the existence of global strong solutions were studied in Sobolev spaces [12]. In particular, the equation (1.1) has the property with $\|u\|_{H^1} = e^{-\lambda t} \|u_0\|_{H^1}$.

As $\lambda = \alpha = \beta = \gamma = \Gamma = 0$, it reduces to the Camassa-Holm (CH) equation [10][11]
\[
\begin{aligned}
&\frac{\partial u}{\partial t} - u_{\partial x x} = 3uu_x - 2u_x u_{\partial x} - uu_{\partial x x},
\end{aligned}
\]  
(1.3)

which is completely integrable, and has bi-Hamiltonian structure [4][10]. The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces were presented in [6][13][14][19][21][25][27]. The ill-posedness for the CH equation has been studied in [7][15][16]. Its existence and uniqueness of global weak solutions with initial data $u_0 \in H^1(\mathbb{R})$ were proved in [3][9][17][26]. Moreover, the CH equation has globally conservative, dissipative solutions and algebro-geometric solutions [1][2][24].

In this paper, we will prove the existence and uniqueness of the globally conservative solutions to (1.1) in time weighted $H^1$ space. Letting $k = e^{\lambda t}u$, we conclude that $\|k\|_{H^1} = \|u_0\|_{H^1}$. Hence, the existence and uniqueness of the globally conservative solutions of (1.2) in time weighted $H^1$ space can be transformed into the existence and uniqueness of the globally conservative solutions to the following equation
\[
\begin{aligned}
&k_t + (e^{-\lambda t}k + \Gamma)k_x = -\Lambda^{-2} \partial_x \left(-H(k) + e^{-\lambda t}k^2 + \frac{e^{-\lambda t}}{2}k_x^2 \right), \\
k(0, x) = \tilde{k} = u_0,
\end{aligned}
\]  
(1.4)

where $H(k) = (\alpha + \Gamma)k + \frac{\beta}{3} e^{-2\lambda t} k^3 + \frac{\gamma}{4} e^{-3\lambda t} k^4$. Noticing that (1.1) and (1.4) are equivalent. Consequently, in this paper, we mainly study the global conservative weak solutions of (1.4) with initial data $\tilde{k} \in$
$H^1(\mathbb{R})$, and prove that the equation \([1.4]\) has a unique solution, globally in time. However, in the process of proving the globally conservative solutions, in order to get the estimate $\|\Lambda^{-2}\partial_x \cdot \left((\alpha + \Gamma)k\right)\|_{L^\infty}$ (see Theorem 3.6), we use variable transformations to handle the term, which the idea comes from the previous works \[22\].

The paper is organized as follows. In Section 2, we give some definitions and estimates, which will be used in the sequel. Sections 3, 4 are devoted to construct a solution to an equivalent semi-linear system by introducing a set of new variables, this yields a conservative solution to the equation \([1.4]\). In Section 5, we prove that the peakon solution of \([1.4]\) is conserved in $H^1$. In Section 6, by constructing an ordinary differential system, we prove that the conservative solutions of \([1.4]\) is unique.

## 2 Preliminary

In this section, we first recall some definitions of globally conservative weak solutions for \([2.1]\) and give some results. We study the following equation

$$
\begin{align*}
\left\{
\begin{array}{l}
k_t + (e^{-\lambda t}k + \Gamma)k_x = -P_x, \quad t > 0, \; x \in \mathbb{R}, \\
k(0, x) = \bar{k} = \bar{u},
\end{array}
\right.
\end{align*}
$$

with $P$ is defined as a convolution:

$$
P \equiv \frac{1}{2} e^{-|x|} * \left[-\left((\alpha + \Gamma)k + \frac{\beta e^{-2\lambda t}}{3}k^3 + \frac{\gamma e^{-3\lambda t}}{4}k^4\right) + e^{-\lambda t}k^2 + e^{-\lambda t} \frac{k^2}{2} \right].
$$

For simplify the presentation, we introduce the following notation

$$
H(k) = -\left((\alpha + \Gamma)k + \frac{\beta e^{-2\lambda t}}{3}k^3 + \frac{\gamma e^{-3\lambda t}}{4}k^4\right), \quad H_2(k) = e^{-\lambda t}k^2 + e^{-\lambda t} \frac{k^2}{2}.
$$

For smooth solutions, differentiating \([2.1]\) with respect to $x$, we have

$$
\begin{align*}
k_{tx} + (e^{-\lambda t}k + \Gamma)k_x &= e^{-\lambda t}k^2 - H(k) - \frac{e^{-\lambda t}}{2}k_x^2 - P(t, x).
\end{align*}
$$

It follows from \([2.1], \; 2.3\) that

$$
\begin{align*}
(k^2)_t + \left(\frac{2e^{-\lambda t}}{3}k^3 + \Gamma k^2 + 2kP\right)_x &= 2k_x P, \\
(k_x^2)_t + \left(e^{-\lambda t}kk_x^2 + \Gamma k_x^2 + (\alpha + \Gamma)k^2 + \frac{e^{-2\lambda t}}{6}k^4 + \frac{e^{-3\lambda t}}{10}k^5 - \frac{2e^{-\lambda t}}{3}k^3\right)_x &= -2k_x P.
\end{align*}
$$

Hence

$$
E(t) = \left(\int_{\mathbb{R}} (k^2 + k_x^2)(t, x) dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} (\bar{k}^2 + \bar{k}_x^2)(x) dx\right)^{\frac{1}{2}} = E_0.
$$

Setting $w = k_x^2$, the equation \([2.4]\) yields

$$
w_t + ((e^{-\lambda t}k + \Gamma)w)_x = 2k_x (e^{-\lambda t}k^2 - H(k) - P).
$$

For $\bar{k} \in H^1(\mathbb{R})$, Young’s inequality entails that

$$
\|P\|_{L^\infty}, \; \|P_x\|_{L^\infty} \leq C \left(\|e^{-|x|}\|_{L^1} \|H(k)\|_{L^\infty} + \|e^{-|x|}\|_{L^\infty} \|H_2(k)\|_{L^1}\right).
$$
Let us briefly recall the definition of conservative weak solutions for convenience.

**Definition 2.1.** Let \( \bar{k} \in H^1(\mathbb{R}) \). Then \( k(t, x) \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R})) \) is a conservative weak solution to the Cauchy problem (2.1) when \( k(t, x) \) satisfies the following equation

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left( k \psi_t + \frac{e^{-\lambda t} k^2}{2} + \Gamma k \psi_x + P_x \psi \right)(t, x) dx dt + \int_{\mathbb{R}} \bar{k}(x) \psi(0, x) dx = 0 \tag{2.7}
\]

for any \( \psi \in C_c^\infty(\mathbb{R}^+, D) \). Moreover, the quantities \( \|k\|_{H^1} \) are conserved in time.

**Definition 2.2.** Let \( \bar{k} \in H^1(\mathbb{R}) \). If \( k(t, x) \) is a conservative weak solution for the Cauchy problem (2.1), such that the following properties hold:

1. The function \( k \) provides a solution for the Cauchy problem (2.1) in the sense of Definition 2.1.
2. If \( w = k_x^2 \) provides a distributional solution to the balance law (2.6),

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left( k^2 \phi_t + \left( e^{-\lambda t} k + \Gamma \right) w \phi_x + 2k_x (e^{-\lambda t} k^2 - H(k) - P) \phi \right) dx dt + \int_{\mathbb{R}} k_x^2(x) \phi(0, x) dx dt = 0, \tag{2.8}
\]

for any test function \( \phi \in C_c^1(\mathbb{R}^2) \).

The main theorem of this paper is as follows.

**Theorem 2.3.** For any initial data \( \bar{k} \in H^1(\mathbb{R}) \), the Cauchy problem (2.1) has a unique global conservative solution in the sense of Definition 2.2.

**Notations.** In Section 6, in order not to be ambiguous, we assume that \( k(t, -\infty) = 0 \), it follows that \( \int_{-\infty}^y k_x dx = k(t, y(t)) \).

## 3 Global solutions in Lagrange coordinates

This section is devoted to getting a system equivalent to (2.1) by introducing a coordinate transformation into Lagrange coordinates, and to proving the existence of globally conservative solutions.

### 3.1 An equivalent system

Given \( \bar{k} \in H^1(\mathbb{R}) \) be the initial data and a new variable \( \xi \in \mathbb{R} \). Define the nondecreasing map \( \xi \mapsto \bar{y}(t, \xi) \) via the following equation

\[
\int_0^{\bar{y}(\xi)} \bar{k}_x^2 dx + \bar{y} = \xi. \tag{3.1}
\]

Let \( k = k(t, x) \in H^1(\mathbb{R}) \) be the solution of equation (2.1) and the characteristic \( y(t, \xi) : t \mapsto y(t, \cdot) \) as the solutions of

\[
\begin{align*}
y_y(t, \xi) &= e^{-\lambda t} k(t, y(t)) + \Gamma, \\
y(0, \xi) &= \bar{y}.
\end{align*} \tag{3.2}
\]
Our new variables are
\[ K(t, \xi) = k(t, y(t, \xi)), \quad V(t, \xi) = \frac{k_x \circ y}{1 + k_x^2 \circ y}, \quad W(t, \xi) = \frac{k_x \circ y}{1 + k_x^2 \circ y}, \quad Q(t, \xi) = (1 + k_x^2 \circ y) \cdot y_\xi, \]
\[ (3.3) \]

From (3.2)-(3.3), we deduce that
\[ P(t, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|y(t, \xi) - x|} \left( -H(K)Q(1 - V) + e^{-\lambda t}K^2Q(1 - V) + \frac{e^{-\lambda t}}{2}QV \right)(\eta)d\eta, \]
\[ (3.4) \]
and \( G(t, \xi) \triangleq P_y(t, y) \),
\[ G(t, \xi) = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(y(t, \xi) - x)e^{-|y(t, \xi) - x|} \left( -H(K)Q(1 - V) + e^{-\lambda t}K^2Q(1 - V) + \frac{e^{-\lambda t}}{2}QV \right)(\eta)d\eta. \]
\[ (3.5) \]

Noting that the fact the \( y(t, \cdot) \) is an increasing function and letting \( x = y(t, \eta) \), we infer that
\[ P(t, \xi) = \frac{1}{2} \int_{\mathbb{R}} \text{sgn}(\xi - \eta)e^{-|\int_{\xi}^{\eta}Q(1 - V)(s)ds|} \left( -H(K)Q(1 - V) + e^{-\lambda t}K^2Q(1 - V) + \frac{e^{-\lambda t}}{2}QV \right)(\eta)d\eta, \]
\[ (3.6) \]
\[ G(t, \xi) = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(\xi - \eta)e^{-|\int_{\xi}^{\eta}Q(1 - V)(s)ds|} \left( -H(K)Q(1 - V) + e^{-\lambda t}K^2Q(1 - V) + \frac{e^{-\lambda t}}{2}QV \right)(\eta)d\eta, \]
\[ (3.7) \]
where the index \( t \) is omitted. Now, giving another variable \( Z(t, \xi) \) defined as \( Z(t, \xi) = y(t, \xi) - \xi - \Gamma t \), we obtain
\[ \begin{cases} \dot{Z}_t(t, \xi) = e^{-\lambda t}K(t, \xi), \\ Z(0, \xi) = \bar{y}(\xi). \end{cases} \]
\[ (3.8) \]
Hence, the derivatives of \( G \) and \( P \) are given by
\[ G_{\xi}(t, \xi) = -e^{-\lambda t}K^2Q(1 - V) - \frac{e^{-\lambda t}}{2}QV + H(K)Q(1 - V) + P(1 + Z_\xi), \]
\[ (3.9) \]
\[ P_{\xi}(t, \xi) = G(1 + Z_\xi). \]
\[ (3.10) \]
Combining (3.2)-(3.3) and (3.6)-(3.7), we obtain
\[ \begin{cases} y_t = e^{-\lambda t}K + \Gamma, \\ \dot{K}_t = -G, \\ \dot{V}_t = 2W \left( e^{-\lambda t}K^2(1 - V) - H(K)(1 - V) - \frac{e^{-\lambda t}}{2}V - P(1 - V) \right), \\ \dot{W}_t = (1 - 2V) \left( e^{-\lambda t}K^2(1 - V) - H(K)(1 - V) - \frac{e^{-\lambda t}}{2}V - P(1 - V) \right), \\ \dot{Q}_t = 2WQ \left( \frac{e^{-\lambda t}}{2} + e^{-\lambda t}K^2 - H(K) - P \right). \end{cases} \]
\[ (3.11) \]
Differentiating (3.11) yields

\[
\begin{align*}
    y_{\xi t} &= e^{-\lambda t}K_{\xi t}, \\
    K_{\xi t} &= e^{-\lambda t}K^2Q(1-V) + \frac{e^{-\lambda t}}{2}QV - H(K)Q(1-V) - P(1 + Z\xi), \\
    V_t &= 2W \left( e^{-\lambda t}K^2(1-V) - H(K)(1-V) - \frac{e^{-\lambda t}}{2}V - P(1-V) \right), \\
    W_t &= (1 - 2V) \left( e^{-\lambda t}K^2(1-V) - H(K)(1-V) - \frac{e^{-\lambda t}}{2}V - P(1-V) \right), \\
    Q_t &= 2WQ \left( \frac{e^{-\lambda t}}{2} + e^{-\lambda t}K^2 - H(K) - P \right).
\end{align*}
\]

(3.12)

3.2 Global weak solutions of the equivalent system

This subsection is devoted to the proof of global solution of an equivalent semi-linear system (3.11). Let \( \bar{k} \in H^1(\mathbb{R}) \). By (3.8), the system (3.11) is equivalent to

\[
\begin{align*}
    Z_t &= e^{-\lambda t}K, \\
    K_t &= -G, \\
    V_t &= 2W \left( e^{-\lambda t}K^2(1-V) - H(K)(1-V) - \frac{e^{-\lambda t}}{2}V - P(1-V) \right), \\
    W_t &= (1 - 2V) \left( e^{-\lambda t}K^2(1-V) - H(K)(1-V) - \frac{e^{-\lambda t}}{2}V - P(1-V) \right), \\
    Q_t &= 2WQ \left( \frac{e^{-\lambda t}}{2} + e^{-\lambda t}K^2 - H(K) - P \right).
\end{align*}
\]

(3.13)

Moreover, we have

\[
\begin{align*}
    Z_{\xi t} &= e^{-\lambda t}K_{\xi}, \\
    K_{\xi t} &= e^{-\lambda t}K^2Q(1-V) + \frac{e^{-\lambda t}}{2}QV - H(K)Q(1-V) - P(1 + Z\xi), \\
    V_t &= 2W \left( e^{-\lambda t}K^2(1-V) - H(K)(1-V) - \frac{e^{-\lambda t}}{2}V - P(1-V) \right), \\
    W_t &= (1 - 2V) \left( e^{-\lambda t}K^2(1-V) - H(K)(1-V) - \frac{e^{-\lambda t}}{2}V - P(1-V) \right), \\
    Q_t &= 2WQ \left( \frac{e^{-\lambda t}}{2} + e^{-\lambda t}K^2 - H(K) - P \right).
\end{align*}
\]

(3.14)

Hence, we get the following initial data \((\bar{y}, \bar{K}, \bar{V}, \bar{W}, \bar{Q})\)

\[
\begin{align*}
    \int_0^\infty \bar{k}_x^2 dx + \bar{y}(\xi) &= \xi, \\
    \bar{K}(\xi) &= \bar{k} \circ \bar{y}(\xi), \\
    \bar{V}(\xi) &= \frac{\bar{k}_x^2 \circ \bar{y}(\xi)}{1 + \bar{k}_x^2 \circ \bar{y}(\xi)}, \\
    \bar{W}(\xi) &= \frac{\bar{k}_x \circ \bar{y}(\xi)}{1 + \bar{k}_x^2 \circ \bar{y}(\xi)}, \\
    \bar{Q}(\xi) &= (1 + \bar{k}_x^2 \circ \bar{y})\bar{y}_\xi(\xi) = 1.
\end{align*}
\]

(3.15)
We will prove that the system (3.13) is a well-posed system of ordinary differential equations in the Banach space $\Omega$

$$\Omega = H^1 \cap W^{1,\infty} \times H^1 \cap W^{1,\infty} \times L^2 \cap L^\infty \times L^2 \cap L^\infty \times L^\infty.$$ 

For any $X = (Z, K, V, W, Q) \in \Omega$, the norm on $\Omega$ is given by

$$\|X\|_\Omega = \|Z\|_{H^1 \cap W^{1,\infty}} \times \|K\|_{H^1 \cap W^{1,\infty}} \times \|V\|_{L^2 \cap L^\infty} \times \|W\|_{L^2 \cap L^\infty} \times \|Q\|_{L^\infty}.$$ 

Indeed, for $\bar{X} \in H^1(\mathbb{R})$, we can see that $\bar{X} = (\bar{Z}, \bar{K}, \bar{V}, \bar{W}, \bar{Q}) \in \Omega$, we thus get $K \in W^{1,\infty}$. In the process of proving existence, it is not necessary for $K \in W^{1,\infty}$, which is used to prove uniqueness.

Before providing our main results in this paper, we first give the following lemmas.

**Lemma 3.1.** Let $X = (Z, K, V, W, Q) \in \Omega$, we define the maps $P$ and $G$ as $P(X) := P \circ y$ where $P$ and $G$ are given by (3.4)-(3.5). Then, $P$ and $G$ are Lipschitz maps on bounded sets from $\Omega$ to $H^1 \cap W^{1,\infty}$. Moreover, (3.9)-(3.10) hold.

**Proof.** Let $\Omega_M$ is a bounded subsets of $\Omega$, which is defined as

$$\Omega_M = \{X = (Z, K, V, W, Q) \in \Omega | \|X\|_\Omega \leq M\}.$$ 

**Step 1:** $P$ and $G$ are maps from $\Omega_M$ to $H^1 \cap W^{1,\infty}$. Combining (3.4)-(3.5) with Young's inequality, we have

$$\|P(X)\|_{L^1} \leq C\|\|e^{-|x|}\|_{L^1}\left((\|H(K)\|_{L^2 \cap L^\infty}\|Q(1-V)\|_{L^\infty} + e^{-\lambda t}\|K^2\|_{L^2}\|Q(1-V)\|_{L^\infty} \right) \leq CM,$

$$\|G(X)\|_{L^1} \leq C\|\|e^{-|x|}\|_{L^1}\left((\|H(K)\|_{L^2} \|Q(1-V)\|_{L^\infty} + e^{-\lambda t}\|K^2\|_{L^\infty} \|Q(1-V)\|_{L^\infty} \right) \leq CM.$$ 

Similarly, we obtain

$$\|P_x(X)\|_{L^2 \cap L^\infty} \leq C \|G_x(X)\|_{L^2 \cap L^\infty} \leq CM.$$

**Step 2:** $P$ and $G$ are Lipschitz maps from $\Omega_M$ to $H^1 \cap W^{1,\infty}$. For $X = (Z, K, W, Q, V)$ and $\tilde{X} = (\bar{Z}, \bar{K}, \bar{V}, \bar{W}, \bar{Q})$ be two elements in $\Omega_M$. According to (3.7), we deduce that

$$\|P(X) - P(\tilde{X})\|_{H^1 \cap W^{1,\infty}} \leq \|G(X) - G(\tilde{X})\|_{H^1 \cap W^{1,\infty}} \leq C\|X - \tilde{X}\|_\Omega.$$ 

Combining Step 1 and Step 2, we finish the proof of Lemma 3.1.

**Lemma 3.2.** Let $X = (Z, K, V, W, Q) \in \Omega$ be a solution the system (3.13). Then, for almost everywhere $t \in \mathbb{R}$, we have

$$W^2 + V^2 = V, \quad (3.16)$$

$$\sum_x = Q(1-V), \quad (3.17)$$

$$K_x = WQ. \quad (3.18)$$

Moreover, we have

$$\frac{d}{dt} \bar{E}(t) = \frac{d}{dt} \int_{\mathbb{R}} (K^2Q(1-V) + QV)(t, \xi) d\xi = 0. \quad (3.19)$$
Thus, the conservative law (2.5) in the new variables remains constant in time. 

\[ (W^2 + V^2)_t = V_t, \]

from which it follows

\[
(W^2 + V^2)_t = 2WW_t + 2VV_t \\
= 2W(2W - 1)\left(e^{-\lambda t}K^2(1 - V) - H(K)(1 - V) - \frac{e^{-\lambda t}}{2}V - P(1 - V)\right) \\
+ 4VW\left(e^{-\lambda t}K^2(1 - V) - H(K)(1 - V) - \frac{e^{-\lambda t}}{2}V - P(1 - V)\right) = V_t. \quad (3.20)
\]

By the same token, we deduce that

\[
y_{\xi_t} = (Q(1 - V))_t, \quad (3.21)
\]

\[
K_{\xi_t} = (WQ)_t. \quad (3.22)
\]

As \( t = 0, \) (3.20)-(3.22) remains hold. Hence, we arrive at (3.16)-(3.18). Moreover, it follows from (3.16)-(3.18) that

\[
0 \leq V \leq 1, \quad |W| \leq \frac{1}{2}. \quad (3.23)
\]

Thereby, \( V(t, \xi) \) and \( W(t, \xi) \) are uniformly bounded in \( L^\infty([0, T]; \mathbb{R}). \)

We now turn to prove (3.19). Using the fact that \( \int_\mathbb{R} (QV)_t(t, \xi)d\xi = 2 \int_\mathbb{R} WQP(t, \xi)d\xi = -2 \int_\mathbb{R} 2KQ(1 - V)G(t, \xi)d\xi \quad \text{and} \quad (3.13), \) we deduce that

\[
\frac{d}{dt} \tilde{E}(t) = \int_\mathbb{R} \left(2K_t KQ(1 - V) + K^2 Q_t - K^2 (QV)_t + (QV)_t\right)(t, \xi)d\xi \\
= \int_\mathbb{R} \left(GQ(1 - V) + 2KWQP - 2KWQP\right)(t, \xi)d\xi \\
= \int_\mathbb{R} G \cdot y_\xi(t, \xi)d\xi = \int_\mathbb{R} F_\xi(t, \xi)d\xi = 0.
\]

This means

\[
\tilde{E}(t) = \int K^2 Q(1 - V) + QV)(t, \xi)d\xi = \tilde{E}(0) = \tilde{E}_0 = E_0^2. \quad (3.24)
\]

Thus, the conservative law (2.5) in the new variables remains constant in time. \( \square \)

The following lemma and corollary which we have learned from [22] are essential.

**Lemma 3.3.** [22] Assume that \( g(x) \) is differentiable on a.e. \( [a, b], f(x) \in L^1[c, d], \) and \( g([a, b]) \subset [c, d]. \)

Then we have that \( F(g(t)) \) is absolutely continuous on \( [a, b] \) if and only if \( f(g(t))g'(t) \in L^1[c, d] \) and \( \int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt \) with \( F(x) = \int_c^x f(t)dt. \)

**Corollary 3.4.** [22] Assume that \( g(x) \) is absolutely continuous on \( [a, b], f(x) \in L^1[c, d], \) and \( g([a, b]) \subset [c, d]. \) If \( g(x) \) is monotonous or \( f(x) \in L^\infty[c, d]. \) Then we have \( \int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt. \)

We now prove the short time existence of solutions to (3.13) as follows.

**Theorem 3.5.** Given \( \bar{k} \in H^1(\mathbb{R}). \) Then there exists a time \( T > 0 \) such that the system (3.13)-(3.15) has a unique solution \( X = (Z(t), K(t), V(t), W(t), Q(t)) \in L^\infty([0, T]; \Omega). \)
Proof. For $k \in H^1(\mathbb{R})$, one can get $X = (Z, K, V, W, Q) \in \Omega$. Let $\Omega_M$ be a bounded subset of $\Omega$, defined as

$$\Omega_M = \{ X = (Z, K, V, W, Q) \in \Omega | \|X\|_{\Omega} \leq M \}.$$ 

We need to check that the right-hand side of the system (3.13) is Lipschitz continuous from $\Omega_M$ to $\Omega$. Now, we proceed as in the proof of Lemma 3.1. Therefore, the right-hand side of the system (3.13) is Lipschitz on $\Omega_M$. By the standard theory of ordinary differential equations, we conclude that there exists a unique solution $X = (Z(t), K(t), V(t), W(t), Q(t))$ be the short time solution of the system $3.13$ in $L^\infty([0, T]; \Omega)$.

Next, we turn to the proof of existence of global solutions of the Cauchy problem (3.13)-(3.15).

**Theorem 3.6.** Let $k \in H^1(\mathbb{R})$. Then the local solution $X = (Z(t), K(t), V(t), W(t), Q(t))$ of (3.13) is a unique globally conservative solution in $L^\infty(\mathbb{R}^+; \Omega)$.

**Proof.** In order to prove the existence the global solutions, we shall demonstrate the local solution $X = (Z(t), K(t), V(t), W(t), Q(t))$ is uniformly bounded in $\Omega$ on any bounded time interval $[0, T]$ with any $T > 0$. Lemma 3.2 guarantees that

$$\sup_{\xi \in \mathbb{R}} |K^2(\xi)| \leq 2 \int_{\mathbb{R}} |KK| \xi |d\xi| \leq 2 \int_{\mathbb{R}} |K^2Q(1 - V)|d\xi| \xi |(\int_{\mathbb{R}} |QV|d\xi|) \xi | \leq C\tilde{E}(0). \quad (3.25)$$

Combining (3.23) and (3.25), we infer that $K, V, W$ are uniformly bounded in $L^\infty$. However, we cannot obtain $\int_{\mathbb{R}} e^{-|Q(t)|_Z^1 KQ(1 - V)|\eta|}d\eta$ is bounded in $L^\infty$. Therefore, we use variable transformations and contradiction argument to handle the problem. According to the Cauchy problem (3.13)-(3.15), we get $\tilde{y}(\xi) \in L^\infty(\mathbb{R}^+; \Omega)$ is strictly monotonous and

$$|\tilde{y}(\xi_2) - \tilde{y}(\xi_1)| = \left| \int_{\tilde{y}(\xi_1)}^{\tilde{y}(\xi_2)} 1 dx \right| \leq \int_{\tilde{y}(\xi_1)}^{\tilde{y}(\xi_2)} (1 + \tilde{k}_y^2)dx \leq |\xi_2 - \xi_1|,$$

from which implies $\tilde{y}(\xi)$ is local Lipschitz continuous function. Lemma 3.3 entails that $K(t, \xi)$ is Lipschitz continuous as it maps $\Omega_M$ to $H^1 \cap W^{1, \infty}$. From (3.13)-(3.15), there exists a $0 \leq T < \infty$ such that $\tilde{y}(t, \xi) \in H^1_{\text{loc}}$ for $t \in [0, T)$, which means $\tilde{y}(t, \xi)$ is a local absolutely continuous function for $t \in [0, T)$. Making use of Corollary 3.4 for $t \in [0, T)$ and $[a, b] \subset \mathbb{R}$, we arrive at

$$\left\| \frac{1}{2} \int_a^b e^{-\int_a^s Q(1 - V)(\eta) d\eta} |KQ(1 - V)(\eta)| d\eta \right\|_{L^\infty} \leq \frac{1}{2} \|K\|_{L^\infty} \|\int_a^b e^{-|y(\xi) - y(\eta)|^2} d\eta \|_{L^\infty} \leq \frac{1}{2} \|K\|_{L^\infty} \int_{-\infty}^{+\infty} e^{-|y|_V^2} dy. \quad (3.26)$$

As $a \to -\infty, b \to +\infty$, the left side of (3.26) is monotonic. Applying the monotonic convergence theorem, we see that there exists a limit on the left side of (3.26). Therefore, we obtain

$$\left\| \int_{\mathbb{R}} e^{-\int_a^s Q(1 - V)(\eta) d\eta} |KQ(1 - V)(\eta)| d\eta \right\|_{L^\infty} \leq \frac{1}{2} \|K\|_{L^\infty} \int_{-\infty}^{+\infty} e^{-|s|^2} ds \leq C\tilde{E}^2_0.$$ 

Combining the above estimate and (3.23)-(3.25), we have

$$\|G\|_{L^\infty} \leq C \left( \tilde{E}(0)^2 + \|e^{-|x|^2}\|_{L^\infty} \left( \int K^2Q(1 - V) + QV d\eta \right) \right)$$

9
Remark 3.8.

Moreover, we have

\[ y \in L^\infty \cdot (\|K\|_{L^\infty} + \|K\|^2_{L^\infty}) \int K^2Q(1-V) d\eta \]

\[ \leq C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right). \]  \hspace{1cm} (3.27)

Likewise, we get

\[ \|(P, P_x)\|_{L^\infty \cap L^2}, \|(G, G_x)\|_{L^\infty \cap L^2} \leq C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right). \]  \hspace{1cm} (3.28)

Hence, one can get from (3.13) that

\[ |Q_1| \leq C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right), \]

which implies

\[ \exp\{-C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right) t\} \leq Q(t) \leq \exp\{C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right) t\}. \]

From (3.17), we have

\[ \|y_\xi\|_{L^\infty} \leq Q(t) \leq \exp\{C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right) t\}. \]  \hspace{1cm} (3.29)

It follows that

\[ \tilde{y}(\xi) - C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right) t \leq y(t, \xi) \leq \tilde{y}(\xi) + C\left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right) t. \]  \hspace{1cm} (3.30)

This means \( y(t, \xi) \) is bounded in \( L^\infty_{loc} \) for all \( t \in [0, T] \), then we have \( y(t, \xi) \in H^1_{loc} \) for \( t \in [0, T] \). According to contradiction argument, we can prove that \( T \) in the above results cannot have a upper bound, we conclude that the above the results are valid for \( t \in \mathbb{R} \).

Taking advantage of (3.27)-(3.30) and the system (3.13), we get the following estimates

\[ \frac{d}{dt} \|K\|^2_{L^2} \leq C\|K\|_{L^2} \left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right), \]

\[ \frac{d}{dt} \|V\|^2_{L^2} \leq C(\|V\|^2_{L^2} + \|V\|_{L^2}) \left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right), \]

and

\[ \frac{d}{dt} \|W\|^2_{L^2} \leq C\|W\|_{L^2} \left( \tilde{E}_0^2 + \tilde{E}_0 + \tilde{E}_0^2 + \tilde{E}_0^2 \right). \]

In addition, using (3.16)-(3.17), we have

\[ \|K_{\xi}\|_{L^2} \leq C\|W\|_{L^2} \|Q\|_{L^\infty}, \quad \|K_{\xi}\|_{L^\infty} \leq C\|W\|_{L^\infty} \|Q\|_{L^\infty}. \]

Therefore, we finish the proof of Theorem 3.6 \( \Box \)

Theorem 3.7. \([17]\) Let \( X = (Z, K, V, W, Q) \) be the corresponding solution of the system (3.13) with the initial data \( X = (\tilde{Z}, \tilde{K}, \tilde{V}, \tilde{W}, \tilde{Q}) \in L^\infty([0, T]; \Omega) \) given by Theorem 3.3. Then, \( (Z, K, V, W, Q) \) is a solution of the system (3.14), and

\[ (Z, K, V, W, Q) \in (L^2 \cap L^\infty)^4 \times L^\infty. \]

Moreover, we have \( y_\xi \geq 0 \) and \( \text{meas}(\mathcal{N}) = 0 \) with

\[ \mathcal{N} = \{(t, \xi) \in [0, T] \times \mathbb{R} \mid y_\xi = 0 \}. \]

Remark 3.8. Let \( \bar{k} \in H^1(\mathbb{R}) \), the system (3.11) also has a globally unique solution in \( L^\infty(\mathbb{R}^+; \Omega). \)
4 Solutions to the original equation

This section is devoted to proving that the globally conservative weak solution to the original equation.

**Theorem 4.1.** Let \( \bar{k} \in H^1(\mathbb{R}) \). Then, the Cauchy problem (2.1) has a globally conservative solution in the sense of Definition 2.1.

**Proof.** From Remark 3.8, we know that the system (3.11) has a unique globally conservative weak solution. Then, the mapping \( t \mapsto y(t,\xi) \) provides a solution to the Cauchy problem

\[
\begin{cases}
y_t(t,\xi) = e^{-\lambda t}K(t,\xi) + \Gamma, \\
y(0,\xi) = \bar{y}(\xi).
\end{cases}
\]

(4.1)

Set

\[
k(t,x) = K(t,\xi), \text{ if } x = y(t,\xi).
\]

(4.2)

We need to check that (4.2) is well-defined. (3.1) and (3.30) entail that

\[
\lim_{\xi \to \pm\infty} y(t,\xi) = \pm\infty.
\]

Thanks to (3.17), we see that \( y_\xi \geq 0 \) for all \( t \geq 0 \) and a.e. \( \xi \in \mathbb{R} \). Moreover, the map \( \xi \mapsto y(t,\xi) \) is nondecreasing. Assume that \( \xi_1 < \xi_2 \) but \( y(t,\xi_1) = y(t,\xi_2) \), it follows that

\[
0 = \int_{\xi_1}^{\xi_2} y_\xi(t,\eta)d\eta = \int_{\xi_1}^{\xi_2} Q(1-V)(t,\eta)d\eta.
\]

If \( Q \neq 0 \), we deduce that \( V = 1 \) and \( W = 0 \) in \( [\xi_1,\xi_2] \). Therefore, we have

\[
K(t,\xi_1) - K(t,\xi_2) = \int_{\xi_1}^{\xi_2} K_\xi(\eta)d\eta = \int_{\xi_1}^{\xi_2} W Q(\eta)d\eta.
\]

If \( Q = 0 \), the above equality also make sense. Then, for all \( t \geq 0 \) and \( x \in \mathbb{R} \), the map \( (t,x) \mapsto k(t,x) \) is well-defined. According to the definition (4.2), we give

\[
k_x(t,y(t,\xi)) = \frac{W}{1-V}, \text{ if } x = y(t,\xi), \ y_\xi \neq 0.
\]

(4.3)

Applying (3.24), (4.3) and changing the variables, we see that

\[
E(t) = \int_\mathbb{R} (k^2 + k_x^2)(x)dx = \int_{\mathbb{R} \setminus \{y_\xi \neq 0\}} (k^2 + k_x^2)(t,y(t,\xi))y_\xi d\xi
\]

\[
= \int_{\mathbb{R} \setminus \{y_\xi \neq 0\}} (K^2Q(1-V) + QV)(\xi)d\xi = \int_\mathbb{R} (K^2Q(1-V) + QV)(\xi)d\xi
\]

\[= \tilde{E}(t) = \tilde{E}_0 = \int_\mathbb{R} (\tilde{k}^2 + \tilde{k}_x^2)(x)dx,
\]

which implies \( k \) is uniformly bounded in \( H^1(\mathbb{R}) \). On the other hand, \( k \) satisfies (2.1). Indeed, for every \( \phi \in C_c^\infty(\mathbb{R}^+,\mathcal{D}) \), we have

\[
\int_{\mathbb{R}^+} \int_\mathbb{R} (-k\phi_t + (e^{-\lambda t}k + \Gamma)(t,x)k_x\phi(t,x)dxdt
\]
\[ \int_{\mathbb{R}^+} \int_{\mathbb{R}} (-k \phi_t + (e^{-\lambda t} k + \Gamma)k_x \phi)(t, y(t, \xi)) y_\xi d\xi dt \]
\[ = \int_{\mathbb{R}^+} \int_{\mathbb{R}} -K \phi_t(t, y(t, \xi)) y_\xi + (e^{-\lambda t} K + \Gamma)K_x \phi(t, y(t, \xi)) d\xi dt \]
\[ = \int_{\mathbb{R}^+} \int_{\mathbb{R}} -K(\phi(t, y(t, \xi)) y_\xi) + e^{-\lambda t}(K^2 \phi(t, y(t, \xi))) \xi + \Gamma(K \phi(t, y(t, \xi))) \xi d\xi dt \]
\[ = \int_{\mathbb{R}^+} \int_{\mathbb{R}} -U(\psi(t, y(t, \xi)) y_\xi) d\xi dt \]
\[ = \int_{\mathbb{R}^+} \int_{\mathbb{R}} K \phi(t, y(t, \xi)) y_\xi d\xi dt + \int_{\mathbb{R}} \tilde{K}(\xi) \phi(0, \xi) \tilde{y}(\xi) d\xi \]
\[ = \int_{\mathbb{R}^+} \int_{\mathbb{R}} -P_2(t, y(t, \xi)) y_\xi d\xi dt + \int_{\mathbb{R}} \tilde{K}(\xi) \phi(0, \xi) \tilde{y}(\xi) d\xi \]
\[ = \int_{\mathbb{R}^+} \int_{\mathbb{R}} -P_2(t, x) dxdt + \int_{\mathbb{R}} \tilde{k}(x) \phi(0, x) dx, \]

with
\[ (\phi(t, y(t, \xi)) y_\xi)_t = \phi_t(t, y(t, \xi)) \cdot y_\xi + \phi_x((t, y(t, \xi))(K + \Gamma)(t, \xi) \cdot y_\xi + \phi(t, y(t, \xi))(K + \Gamma)_\xi, \]

and
\[ P_2(t, y(t, \xi)) = \frac{1}{2} \left( \int_{y(t, \xi)}^{+\infty} - \int_{-\infty}^{y(t, \xi)} e^{-|y(t, \xi)-x|} (-H(k) + e^{-\lambda t} k^2 + \frac{e^{-\lambda t}}{2} k_x^2)(t, x) dx \right) \]
\[ = \frac{1}{2} \left. \left| \int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} e^{-|f_2^t Q(1-V)(t, s) ds|} (-H(K) Q(1 - V) + e^{-\lambda t} K^2 Q(1 - V) + \frac{e^{-\lambda t}}{2} Q V) d\xi' \right| \right| = G(t, \xi). \]

Likewise, we get
\[ \int_{\mathbb{R}^+} \int_{\mathbb{R}} [K_x^2 \phi_t + (e^{-\lambda t} k + \Gamma)w \phi_x + 2k_x(e^{-\lambda t} k^2 - H(k) - P)\phi]dxdt + \int_{\mathbb{R}} \tilde{k}_x^2(x) \phi(0, x) dxdt = 0. \quad (4.4) \]
Hence, we conclude that \( k(t, x) \) is a globally conservative solution to \( (1.1) \) in the sense of Definition 2.2 \( \Box \)

5 Peakon solutions

In this section, we give a conservative solution in time-weighted \( H^1 \) space with the peakon solutions of \( (1.1) \) in the following form
\[ u(t, x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|}, \quad (5.1) \]
where \( p_i(t), q_i(t), i = 1, \ldots, n \) are smooth functions with respect to \( t \).

**Theorem 5.1.** Let \( q_1(t) < q_2(t) < \ldots < q_n(t) \). Then \( (5.1) \) are weak solutions of O.D.E. in the following
\[ \begin{cases} 2\dot{p}_i - 2p_i(a_i - b_i) + 2\lambda p_i = 0, \\ -2\dot{p}_i q_i + 2p_i(a_i + b_i + p_i) + 2\Gamma p_i = 0, \end{cases} \]
where \( a_i = \sum_{j<i} p_j e^{q_j - q_i}, b_i = \sum_{j<i} p_j e^{q_j - q_i} \).
Proof. For any $i \in \{0, \ldots, n+1\}$, let

$$u_i(x, t) = \sum_{j=1}^{i} p_j(t) e^{q_j(t)} x + \sum_{j=i+1}^{n} p_j(t) e^{x-q_j(t)},$$

where $u_i(t, x) \in C^\infty$ in the space variable. Then (5.1) can be rewritten as

$$u(t, x) = \sum_{i=0}^{n} u_i(t, x) \chi_i(x),$$

which $\chi_i$ represents the characteristic function in interval $[q_i, q_{i+1})$, $i = 1, \ldots, n$ and $q_0 = -\infty$, $q_{n+1} = \infty$. Owing that $\chi_i$ has disjoint supports, we have

$$(u + \Gamma) u_x = \sum_{i=0}^{n} (u_i + \Gamma) u_{i,x} \chi_i.$$ 

Therefore

$$\left((u + \Gamma) u_x\right)_x = \sum_{i=0}^{n} \left((u_i + \Gamma) u_{i,x}\right)_x \chi_i + \sum_{i=1}^{n} \left((u_i + \Gamma) u_{i,x}\right)(q_i) \delta_{q_i} - \sum_{i=0}^{n-1} \left((u_i + \Gamma) u_{i,x}\right)(q_{i+1}) \delta_{q_{i+1}},$$

which $[v]_{q_i} = v(q_i^+) - v(q_i^-)$. Noting that $u$ is continuous, one can get $u = u_i = u_{i,xx}$ on every interval $(q_i, q_{i+1})$ and $[u^2]_{q_i} = 0$. Differentiating (5.2), we obtain

$$\left((u + \Gamma) u_x\right)_{xx} = \sum_{i=0}^{n} \left((u_i + \Gamma) u_{i,x}\right)_{xx} \chi_i + \sum_{i=1}^{n} \left((u_i + \Gamma) u_{i,x}\right)_{x} \delta_{q_i} + \sum_{i=1}^{n} \left([u_i + \Gamma u]_{q_i}\right) \delta_{q_i},$$

which $[u^2]_{q_i} = v(q_i^+) - v(q_i^-)$. Noting that $u$ is continuous, one can get $u = u_i = u_{i,xx}$ on every interval $(q_i, q_{i+1})$ and $[u^2]_{q_i} = 0$. Differentiating (5.2), we obtain

$$\left((u + \Gamma) u_x\right)_{xxx} = \sum_{i=0}^{n} \left((u_i + \Gamma) u_{i,x}\right)_{xxx} \chi_i + \sum_{i=1}^{n} \left((u_i + \Gamma) u_{i,x}\right)_{xx} \delta_{q_i} + \sum_{i=1}^{n} \left([u_i + \Gamma u]_{q_i}\right) \delta_{q_i},$$

Likewise, we have

$$\left(\frac{1}{2} u_x^2 - \alpha u - \frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4\right)_x = \sum_{i=0}^{n} \left(\frac{1}{2} u_i^2 - \alpha u - \frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4\right)_x \chi_i$$

$$+ \sum_{i=1}^{n} \left(\frac{1}{2} u_i^2 - \alpha u - \frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4\right) \delta_{q_i}.$$ 

$$u_t - u_{xxt} = \sum_{i=0}^{n} \left(u_{i,t} - u_{i,xxt}\right) \chi_i - \sum_{i=1}^{n} \left([u_{x,x}]_{q_i}\delta_{q_i} + [u_{i}]_{q_i}\delta_{q_i}\right).$$ 

It follows from (5.2) to (5.5) that

$$\sum_{i=0}^{n} \left(u_{i,t} - u_{i,xxt} + 3u_i u_{x,t} - \left((u_i + \Gamma) u_{i,x}\right)_{xx} \right) + \frac{1}{2} u_i^2 - \alpha u - \frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4 \chi_i$$

$$+ \sum_{i=1}^{n} \left(-u_{i,t} - u_{i,x}^2 + u_{i} + \Gamma u\right)_{q_i} + \frac{1}{2} u_i^2 - \alpha u - \frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4 \delta_{q_i}$$
\[ + \sum_{i=1}^{n} \left\{-\left[u_{t}\right]_{q_i} - [(u + \Gamma)u_x]_{q_i} - \lambda[u]_{q_i} \right\} \delta_{q_i}' = 0. \]

For \(\chi_i, \delta_{q_i}, \delta_{q_{ij}}, i, j, k = 1, \ldots, n\), we have
\[
\begin{cases}
- [u_i, t]_{q_i} - [u_{x}^2]_{q_i} + \Gamma[u]_{q_i} + \frac{1}{2}[u_{x}^2]_{q_i} - \alpha[u]_{q_i} - \frac{\beta}{3}[u^3]_{q_i} - \frac{\gamma}{4}[u^4]_{q_i} - \lambda[u_x]_{q_i} = 0, \\
- [u_i]_{q_i} - [(u + \Gamma)u_x]_{q_i} - \lambda[u]_{q_i} = 0.
\end{cases}
\]

According to the definition of \(a_i, b_i, i = 1, \ldots, n\), we have
\[
\begin{align*}
[u_t]_{q_i} &= 2p_i q_i, \\
[u_x]_{q_i} &= -2\dot{p}_i, \\
[uu_x]_{q_i} &= -2p_i (a_i + p_i + b_i), \\
[u_x]_{q_i} &= -2p_i, \\
[u_{x}^2]_{q_i} &= 4p_i (a_i - b_i), \\
[u]_{q_i} &= [u^2]_{q_i} = [u_{xx}]_{q_i} = 0.
\end{align*}
\]

Substituting (5.7), (5.8) into (5.6), we end up with
\[
\begin{cases}
2\ddot{p} - 2p_i (a_i - b_i) + 2\lambda p_i = 0, \\
-2p_i \dot{q}_i + 2p_i (a_i + p_i + b_i) + 2\Gamma p_i = 0.
\end{cases}
\]

Now, we consider that \(n = 1\) and \(a_1 = b_1 = 0\). Then we have
\[
\begin{cases}
2\ddot{p} + 2\lambda p = 0, \\
-2p \dot{q} + 2p^2 + 2\Gamma p = 0,
\end{cases}
\]

which implies that
\[
\begin{align*}
p &= p(0)e^{-\lambda t}, \\
q &= \frac{1}{\lambda}p(0)(1 - e^{-\lambda t}) + \Gamma t + q(0), \\
q &= p(0)t + \Gamma t + q(0),
\end{align*}
\]

It follows that
\[
\begin{align*}
u(t, x) &= p(0)e^{-\lambda t}e^{-|x - q(t)|}, \\
k(t, x) &= e^{\lambda t}u(t, x) = p(0)e^{-|x - q(t)|} \Rightarrow \|k\|_{H^1} = \|\bar{k}\|_{H^1}.
\end{align*}
\]

Figure 1 shows the evolution behavior of single peak solitary solutions with dissipative coefficient \(\lambda\) with \(p(0) = \frac{1}{2}, q(0) = 1, \Gamma = -2\),

(a) \(\lambda = 1\)  
(b) \(\lambda = \frac{1}{4}\)  
(c) \(\lambda = \frac{1}{10}\)
Figure 1: (a) $\lambda = 1$; (b) $\lambda = \frac{1}{3}$; (c) $\lambda = \frac{1}{10}$; (d) $\lambda = \frac{1}{15}$; (e) $\lambda = \frac{1}{30}$; (f) $\lambda = 0$.

Note that $(1 - \partial_{xx})u = \delta_0 \in M \hookrightarrow B_{1,\infty}^0 \hookrightarrow B_{1,\infty}^{-\frac{1}{2}}$, where $M$ is bounded measures spaces. Thanks to $B_{2,\infty}^{\frac{3}{2}} \hookrightarrow H^1(\mathbb{R})$, we get $u \in H^1(\mathbb{R})$. Hence, $k(t, x)$ is a conservative solution of equation (1.1) in $H^1$, in other words, $u(t, x)$ is a conservative solution in time weighted $H^1$ space. Moreover, we infer that

$$\lim_{\lambda \to 0} q(t) = q(0).$$

6 Uniqueness of solutions for the original equation

In this section, we consider the uniqueness solutions for (2.1). Our main result is as follows.

**Theorem 6.1.** Let $k(t, x) \in H^1$ be a conservative weak solutions to the Cauchy problem (2.1) in the sense of Definition 2.2. Then $k(t, x)$ is unique.

6.1 Uniqueness of characteristic

This subsection is devoted to the study of the uniqueness of the characteristic to (2.1). Let $k = k(t, x)$ be a conservative solution of (2.1) and satisfy (2.6). Let $y(t, \xi)$ still denote the characteristic

$$\begin{cases}
    y(t, \xi) = e^{-\lambda t}k(t, y(t)) + \Gamma, \\
    y(0, \xi) = \bar{y}.
\end{cases}$$

Introduce new coordinates $(t, \beta)$ relative to the original coordinates $(t, x)$ by the following transformation

$$y(t, \beta) + \int_{-\infty}^{y(t, \beta)} k_x^2(t, z)dz = \beta. \quad (6.2)$$

At time $t$ where the measure $\mu(t)$ is not absolutely continuous with respect to the Lebesgue measure, For any time $t$ and $\beta \in \mathbb{R}$, define $y(t, \beta)$ to be the unique $y$ such that

$$y(t, \beta) + \mu(t)\{(-\infty, y)\} \leq \beta \leq y(t, \beta) + \mu(t)\{(-\infty, y)\}. \quad (6.3)$$

Combining (3.26) and (6.1), we get

$$\frac{d}{dt} \int_{-\infty}^{y(t)} k_x^2 dx = \int_{-\infty}^{y(t)} 2k_x (e^{-\lambda t}k^2 - H(k) - P)dx. \quad (6.4)$$
Now we give the following lemma to prove the Lipschitz continuity of $x$ and $k$ as functions of the variables $t, \beta$.

**Lemma 6.2.** Let $k(t, x)$ be a conservative solution of (6.3). Then, for all $t \geq 0$, the following maps

$$
\beta \mapsto y(t, \beta), \\
\beta \mapsto k(t, y(t, \beta)),
$$
defined by (6.3), are Lipschitz continuous with constant depending only on $\beta$. Moreover, the map $t \mapsto y(t, \beta)$ is also Lipschitz continuous with a constant depending only on $\|\bar{k}\|_{H^1}$.

**Proof.**

**Step 1.** For any fixed time $t \geq 0$, the map

$$x \mapsto \beta(t, x) := x + \int_{-\infty}^{x} k^2(t, y) dy
$$
is right continuous and strictly increasing. This means the inverse $\beta \mapsto x(t, \beta)$ is well-defined, continuous, nondecreasing. Given $\beta_1 < \beta_2$, we have

$$y(t, \beta_2) - y(t, \beta_1) = (\beta_2 - \beta_1) - \int_{y(t, \beta_1)}^{y(t, \beta_2)} k^2_z(t, z) dz \leq \beta_2 - \beta_1.
$$

One can conclude that $\beta \mapsto y(t, \beta)$ is Lipschitz continuous.

**Step 2.** Let $\beta_1 < \beta_2$. Then, it follows that

$$|k(t, \beta_2) - k(t, \beta_1)| \leq \int_{y(t, \beta_1)}^{y(t, \beta_2)} |k_z(t, z)| dz \leq \int_{y(t, \beta_1)}^{y(t, \beta_2)} \frac{1}{2} (1 + k^2_x) dz
$$
$$\leq \frac{1}{2} \left[y(t, \beta_2) - y(t, \beta_1) + \int_{y(t, \beta_1)}^{y(t, \beta_2)} k^2_z(t, z) dz\right] \leq \beta_2 - \beta_1.
$$

Therefore, we arrive at the map $\beta \mapsto k(t, y(t, \beta))$ is Lipschitz continuous.

**Step 3.** According to (2.5), we have

$$\|2e^{-\lambda t} k^2 - P - H(k)\|_{L^1} \leq \| - H(k) + e^{-\lambda t} k^2 - P\|_{L^2} \|k_x\|_{L^2} \leq C_{E_0}.
$$

If $t > \tau$, we obtain

$$
\mu_{\tau}\{(-\infty, y - C_{E_0}(t - \tau))\} \leq \mu_{\tau}\{(-\infty, y)\} + \int_{\tau}^{t} \|2e^{-\lambda t} k^2 - P - H(k)\|_{L^1} dt
$$
$$\leq \mu_{\tau}\{(-\infty, y)\} + C_{E_0}(t - \tau).
$$

Defining $y^-(t) := y - C_{E_0}(t - \tau)$, we get

$$
y^-(t) + \mu_{\tau}\{(-\infty, y^-)\} \leq y - C_{E_0}(t - \tau) + \mu_{\tau}\{(-\infty, y)\} + C_0(t - \tau)
$$
$$\leq y + \mu_{\tau}\{(-\infty, y)\} \leq \beta,
$$

which implies $y(t, \beta) \geq y^-(t)$.

Likewise, defining $y^+(t) := y + C_{E_0}(t - \tau)$, it follows that

$$
y^+(t) + \mu_{\tau}\{(-\infty, y^+)\} \geq y + C_{E_0}(t - \tau) + \mu_{\tau}\{(-\infty, y)\} + C_{E_0}(t - \tau)
$$
$$\geq y + \mu_{\tau}\{(-\infty, y)\} \geq \beta.
$$

Hence, we deduce that $y(t, \beta) \leq y^+(t) := y + C_{E_0}(t - \tau)$. 

\qed
Lemma 6.3. Let $k(t, x) \in H^1(\mathbb{R})$ be a conservative solution of the Cauchy problem (2.1). Then, for any $\bar{y}(\xi) \in \mathbb{R}$, there exists a unique Lipschitz continuous map $t \mapsto y(t, \beta) := y(t, \beta(t, \xi))$ which satisfies both (6.1) and (6.4), where $y(t, \beta)$ is the solution of (6.1). Moreover, for any $0 \leq \tau \leq t$ one has

$$k(t, y(t)) - k(\tau, y(\tau)) = - \int_{\tau}^{t} P_x(s, y(s))ds.$$  \hspace{1cm} (6.5)

**Proof.** **Step 1.** Assume that $y(t)$ is the characteristic beginning at $\bar{y}(\xi) = \xi \in \mathbb{R}$, which is defined as $t \to y(t) = y(t, \beta(t))$, the map $\beta(\cdot)$ is to be determined. Let $y(t)$ be the solution of (6.1) and (6.4). For $t \notin \mathcal{N}$, we have

$$\beta(t, \xi) = y(t) + \int_{0}^{y(t)} k_x^2(z)dz$$

$$= \int_{0}^{t} \frac{d}{ds}(y(s)) + \int_{0}^{y(s)} k_x^2(s, z)dz + \bar{y}(\xi) + \int_{-\infty}^{\bar{y}(\xi)} k_x^2(z)dz$$

$$= \bar{y} + \int_{0}^{\bar{y}(\xi)} k_x^2(z)dz + \int_{0}^{t} \Gamma + \int_{-\infty}^{\bar{y}(\xi)} \left(e^{-\lambda t}k_x + 2k_x(e^{-\lambda t}k^2 - P - H(k))\right)(s, z)dz. \hspace{1cm} (6.6)$$

Set

$$F(t, \beta(t, \xi)) = \Gamma + \int_{-\infty}^{y(t, \beta(t, \xi))} \left(e^{-\lambda t}k_x + 2k_x(e^{-\lambda t}k^2 - P - H(k))\right)dy = \xi, \hspace{1cm} (6.7)$$

and

$$\bar{\beta}(t, \xi) = \bar{y}(\xi) + \int_{0}^{\bar{y}} k_x^2dy = \xi \hspace{1cm} (6.8)$$

According to (6.6)-(6.8), it follows that

$$\beta(t, \xi) = \xi + \int_{0}^{t} F(s, \beta(s, \xi))ds. \hspace{1cm} (6.9)$$

**Step 2.** Since $\|k\|_{H^1} = \|\bar{k}\|_{H^1}$, $y_\beta = \frac{1}{1+k_x^2}$ and the map $\beta \mapsto k(t, y(t, \beta))$ is Lipschitz continuous, we get

$$F_\beta = \{e^{-\lambda t}k_x + 2k_x(e^{-\lambda t}k^2 - P - H(k))\}y_\beta = \frac{e^{-\lambda t}k_x + 2k_x(e^{-\lambda t}k^2 - P - H(k))}{1+k_x^2}.$$

It follows that

$$\|F_\beta\|_{L^\infty} \leq C_{E_0}.$$ 

Moreover

$$|F(s, \beta_2) - F(s, \beta_1)| \leq C_{E_0} |\beta_2 - \beta_1|. \hspace{1cm} (6.10)$$

Then, the map $\beta \mapsto F(t, \beta(t))$ is Lipschitz continuous. Moreover, the map $\xi \mapsto \beta(t, \xi)$ is strictly monotonic and Lipschitz continuous. From (6.10), we get

$$|\beta(t, \xi_2) - \beta(t, \xi_1)| \leq |\xi_2 - \xi_1| + \int_{0}^{t} |F(s, \beta(s, \xi_2)) - F(s, \beta(s, \xi_2))|ds$$

$$\leq |\xi_2 - \xi_1| + C_{E_0} \int_{0}^{t} |\beta(t, \xi_2) - \beta(t, \xi_1)|.$$
The Gronwall inequality ensures that

\[ |\beta(t, \xi_2) - \beta(t, \xi_1)| \leq |\xi_2 - \xi_1|e^{C_{E_0}t}. \]

Consequently, for any \( \xi_2 > \xi_1 \), we get

\[
\beta(t, \xi_2) - \beta(t, \xi_1) = \xi_2 - \xi_1 + \int_0^t F(s, \beta(s, \xi_2)) - F(s, \beta(s, \xi_2))ds \geq (\xi_2 - \xi_1)(1 - C_{E_0}t),
\]

which means that monotonicity makes sense as \( t \) sufficiently small and the solution \( \beta(\cdot) \) of the integral equation (6.9) depends Lipschitz continuously on the initial data. Without loss of generality, assume that \( t \) is enough small, otherwise we can use the continuous method. In addition, the map \( \xi \mapsto F(t, y(t, \beta(t, \xi))) \) is also Lipschitz continuous.

**Step 3.** Owing to \( F \) is uniformly Lipschitz continuous, we can check that existence and uniqueness of solution of (6.9) by the fixed point theorem. We introduce the Banach space of all continuous function \( \beta : \mathbb{R}^+ \to \mathbb{R} \) with weighted norm

\[
\| \beta \| := \sup_{t \geq 0} e^{-2Ct}|\beta(t)|.
\]

For this space, we see that the Picard map

\[
(\mathcal{P}\beta)(t) = \bar{\beta} + \int_0^t F(s, \beta(s))ds\]

is a strict contraction. If \( \| \beta_2 - \beta_1 \| = h > 0 \), we obtain

\[
|\beta_2(s) - \beta_1(s)| \leq he^{2Cs}.
\]

Then, we deduce that

\[
|(\mathcal{P}\beta_2)(t) - (\mathcal{P}\beta_1)(t)| \leq \int_0^t |F(s, \beta_2) - F(s, \beta_1)|ds \leq C \int_0^t |\beta_2 - \beta_1|ds \leq |\int_0^t C|e^{2Cs}ds| \leq \frac{h}{2}e^{2Ct}.
\]

This means \( \| (\mathcal{P}\beta_2)(t) - (\mathcal{P}\beta_1)(t) \| \leq \frac{h}{2} \). The contraction mapping principle guarantees that (6.9) has a unique solution. Thanks to the arbitrary of the \( T \), we infer that the integral equation (6.9) has a unique solution on \( \mathbb{R}^+ \).

**Step 4.** Combining (6.11) and the integral equation (6.9), we infer that the uniqueness of \( x(t, \beta) \) depends on the uniqueness of \( \beta(t, \xi) \). From the previous analysis, the map \( t \mapsto y(t) = y(t, \beta(t, \xi)) \) provides the unique solution to (6.9). Owing to \( \beta(t) \) and \( x(t, \beta(t, \xi)) \) are Lipschitz continuous, then \( \beta(t, \xi) \) and \( x(t) \) are differentiable a.e. Next, we need to prove that (6.6) satisfies (6.1). Indeed, at any \( \tau > 0 \), we have

1. \( y(\tau) \) is differentiable at \( t = \tau \);
2. the measure \( \mu(t) \) is absolutely continuous.

We argue by contradiction, assume that (1) does not hold, we get \( y'(\tau) \neq e^{-\lambda \tau}k(\tau, y(\tau)) + \Gamma \). Then, there exists some \( \epsilon_0 > 0 \), such that

\[
y'(\tau) = e^{-\lambda \tau}k(\tau, y(\tau)) + \Gamma + 2\epsilon_0. \tag{6.11}
\]

Therefore, for \( \delta > 0 \) sufficiently small, it follows that

\[
y'(t) := y(\tau) + (t - \tau)[e^{-\lambda \tau}k(t, y(t)) + \Gamma + \epsilon_0] < y(t), \text{ for } t \in [\tau, \tau + \delta]. \tag{6.12}
\]
The approximation argument guarantees that \( \phi \in H^1(\mathbb{R}) \) with compact support. For any \( \epsilon > 0 \) enough small, we give the following functions

\[
\varrho^\epsilon(s, y) = \begin{cases} 
0, & y \leq -\epsilon^{-1}, \\
y + \epsilon^{-1}, & -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\
1 - \epsilon^{-1}(y - y(s)), & y^+(s) \leq y \leq y^+(s) + \epsilon, \\
0, & y \geq y^+(s) + \epsilon,
\end{cases}
\]

(6.13)

\[
\chi^\epsilon(s) = \begin{cases} 
0, & s \leq \tau - \epsilon^{-1}, \\
\epsilon^{-1}(s - \tau + \epsilon), & \tau - \epsilon \leq s \leq \tau, \\
1 - \epsilon^{-1}(s - t), & t \leq s \leq t + \epsilon, \\
0, & s \geq t + \epsilon.
\end{cases}
\]

(6.14)

Define

\[
\phi^\epsilon(s, y) = \min\{\varrho^\epsilon(s, y), \chi^\epsilon(s)\}.
\]

Let \( \phi^\epsilon \) be the test function in (2.8). Therefore, one has

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}} k_x^2 \phi_t^\epsilon + (e^{-\lambda t} k + \Gamma)k_x^2 \phi_x^\epsilon + 2(e^{-\lambda t} k^2 - P - H(k)) \phi^\epsilon dx dt = 0,
\]

which means

\[
\int_{t-\epsilon}^{t+\epsilon} \int_{s-\epsilon^{-1}}^{s+\epsilon} k_x^2 \phi_t^\epsilon + ((e^{-\lambda t} k^2 - P - H(k))) \phi^\epsilon dy dt = 0.
\]

(6.15)

If \( t \) is sufficiently close to \( \tau \), we have

\[
\lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{y(s)-\epsilon}^{y(s)+\epsilon} k_x^2 [\phi_t^\epsilon + (e^{-\lambda t} k + \Gamma) \phi_x^\epsilon] (s, x) dy ds \geq 0.
\]

Using the fact that \( e^{-\lambda s} k(s, y(s)) < e^{-\lambda \tau} k(\tau, y(\tau)) + \epsilon_0 \) and \( \phi^\epsilon_x \leq 0 \). For any \( s \in [\tau + \epsilon, t - \epsilon] \), we infer that

\[
0 = \phi_t^\epsilon + [e^{-\lambda \tau} k^2(\tau, y(\tau)) + \Gamma] \phi_x^\epsilon \leq \phi_t^\epsilon + (e^{-\lambda s} k(s, y(s))) + \Gamma) \phi_x^\epsilon.
\]

(6.16)

Noticing that the family of measures \( \mu_t \) depends continuously on \( t \) in the topology of weak convergence, taking the limit of (6.15) as \( \epsilon \to 0 \), we obtain

\[
0 = \int_{-\infty}^{y(\tau)} k_x^2(\tau, y) dy - \int_{-\infty}^{y(t)} k_x^2(t, y) dy + \int_{\tau}^{t} \int_{-\infty}^{y(s)} 2k_x((e^{-\lambda t} k^2 - P - H(k))) \phi^\epsilon(s, y) dy ds
\]

\[
+ \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{y(s)-\epsilon}^{y(s)+\epsilon} k_x^2[\phi_t^\epsilon + (e^{-\lambda t} k + \Gamma) \phi_x^\epsilon] (s, y) dy ds
\]

\[
\geq \int_{-\infty}^{y(\tau)} k_x^2(\tau, y) dy - \int_{-\infty}^{y(t)} k_x^2(t, y) dy + \int_{\tau}^{t} \int_{-\infty}^{y(s)} 2k_x((e^{-\lambda t} k^2 - P - H(k))) \phi^\epsilon(s, y) dy ds
\]

\[
+ \int_{\tau}^{t} \int_{y(s)}^{y(\tau)} 2k_x(e^{-\lambda t} k^2 - P - H(k)) \phi^\epsilon(s, y) dy ds.
\]

(6.17)
That is
\[ \mu_t\{(\infty, y^+(t))\} \geq \mu_\tau\{(-\infty, y(\tau))\} + \int_\tau^t \int_{-\infty}^{y(s)} 2k_x(e^{-\lambda t}k^2 - P - H(k))\phi^s(s, y)dyds + o_1(t - \tau). \]

From (6.12) and the map \( t \mapsto y(t) \) is Lipschitz continuity, we have
\[
|o_1(t - \tau)| \leq 2|e^{-\lambda t}k^2 - P - H(k)||L_\infty^\tau \int_{y(s)} t |k_x(s, y)|dyds
\leq 2|e^{-\lambda t}k^2 - P - H(k)||L_\infty^\tau \int_{y(s)} t (y(s) - y^+(s))^2 ds
\leq C(t - \tau)^{\frac{3}{2}},
\]
where \( o_1(t - \tau) \) satisfies \( o_1(t - \tau) \to 0 \) as \( t \to \tau \) and \( C \) depend on \( E_0 \).

Combining (6.9), (6.12) and (6.17)-(6.18), for \( t \) being close enough to \( \tau \), we have
\[
\beta(t) = \beta(\tau) + (t - \tau)\left( e^{-\lambda t}k(\tau, y(\tau)) + \Gamma + \int_{-\infty}^{y(\tau)} 2k_x[e^{-\lambda t}k^2 - P - H(k)](s, y)dyds \right) + o_2(t - \tau)
\]
\[
= y(t) + t \mu_\tau\{(-\infty, y(t))\}
\geq y(\tau) + (t - \tau)[e^{-\lambda t}k(\tau, y(\tau)) + \Gamma + \epsilon_0] + \mu_\tau\{(-\infty, y(\tau))\}
+ \int_\tau^t \int_{-\infty}^{y(s)} 2k_x[e^{-\lambda t}k^2 - P - H(k)]dyds + o_1(t - \tau).
\]

with \( o_2(t - \tau) \) satisfies \( o_2(t - \tau) \to 0 \) as \( t \to \tau \). From (6.19), we have
\[
(t - \tau)\left( \int_{-\infty}^{y(\tau)} 2k_x[e^{-\lambda t}k^2 - P - H(k)](s, y)dy \right) + o_2(t - \tau)
\geq (t - \tau)\epsilon_0 + \int_\tau^t \int_{-\infty}^{y(s)} 2k_x[e^{-\lambda t}k^2 - P - H(k)](s, y)dyds + o_1(t - \tau).
\]

Dividing both sides by \( t - \tau \) and letting \( t \to \tau \), one has \( \epsilon_0 < 0 \), which contradicts with \( \epsilon_0 > 0 \). In addition, for the case \( \epsilon_0 < 0 \), we follow the similar strategy as \( \epsilon_0 > 0 \). Hence, we conclude that \( y(t) \) is differentiable at \( t = \tau \).

**Step 5.** It follows from Definition 2.1 that
\[
\int_{R^+} \int_{R} k\phi_t + \frac{(e^{-\lambda k}k + \Gamma)}{2}k_x\phi_x + P_x\phi_x dxdt + \int_{R} \bar{k}(x)\phi(0, x)dx = 0,
\]
for any test function \( \phi \in C_c^\infty \). The approximation argument guarantees that the equation (6.20) remains hold, for any test function \( \psi \) which is Lipschitz continuous with compact support. Note that the map \( y \mapsto k(t, y) \) is absolutely continuous and integrate by parts with respect to \( x \). Therefore, for any \( \varphi \in C_c^\infty \), taking \( \phi = \varphi_x \), we have
\[
\int_{R^+} \int_{R} k_x\varphi_t + (e^{-\lambda k}k + \Gamma)k_x\varphi_x - P_x\varphi_x dxdt + \int_{R} \bar{k}(x)\varphi(0, x)dx = 0.
\]

For any \( \epsilon \geq 0 \) sufficiently small, we give the following function
\[
\varphi^\epsilon(s, y) = \begin{cases} 0, & y \leq -\epsilon^{-1}, \\ y + \epsilon^{-1}, & -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\ 1, & 1 - \epsilon^{-1} \leq y \leq y(s), \\ 1 - \epsilon^{-1}(y - y(s)), & y(s) \leq y \leq y(s) + \epsilon, \\ 0, & y \geq y(s) + \epsilon, \end{cases}
\]
\[
\int_{R^+} \int_{R} k_x\varphi_t + (e^{-\lambda k}k + \Gamma)k_x\varphi_x - P_x\varphi_x dxdt + \int_{R} \bar{k}(x)\varphi(0, x)dx = 0.
\]
and

\[ \psi^\varepsilon(s, y) = \min \{ \varphi^\varepsilon(s, y), \chi^\varepsilon(s) \}, \]

where \( \chi^\varepsilon \) is defined in (6.14). Let \( \varphi = \psi^\varepsilon \) and \( \varepsilon \to 0 \). Then, it follows from the continuity property of function \( P_x \) that

\[
\int_{-\infty}^{y(t)} k_x(t, y)dy = \int_{-\infty}^{y(\tau)} k_x(\tau, y)dy - \int_{\tau}^{t} P_x(s, y(s))ds \\
+ \lim_{\varepsilon \to 0} \int_{\tau - \varepsilon}^{t+\varepsilon} \int_{y(s)}^{y(s)+\varepsilon} k_x[\psi^\varepsilon_t + (e^{-\lambda t}k + \Gamma)\psi^\varepsilon_x]dyds.
\]

Hence, it is shown that

\[
\lim_{\varepsilon \to 0} \int_{\tau - \varepsilon}^{t+\varepsilon} \int_{y(s)}^{y(s)+\varepsilon} k_x[\psi^\varepsilon_t + (e^{-\lambda t}k + \Gamma)\psi^\varepsilon_x]dyds \\
= \lim_{\varepsilon \to 0} \left( \int_{\tau - \varepsilon}^{t} + \int_{t}^{t+\varepsilon} \right) \int_{y(s)}^{y(s)+\varepsilon} k_x[\psi^\varepsilon_t + (e^{-\lambda t}k + \Gamma)\psi^\varepsilon_x]dyds = 0. \tag{6.22}
\]

First, we claim that

\[
\lim_{\varepsilon \to 0} \int_{\tau}^{t} \int_{y(s)}^{y(s)+\varepsilon} k_x[\psi^\varepsilon_t + (e^{-\lambda t}k + \Gamma)\psi^\varepsilon_x]dyds = 0.
\]

Taking advantage of Cauchy’s inequality and \( k_x \in L^2 \), one has

\[
| \int_{\tau - \varepsilon}^{t} \int_{y(s)}^{y(s)+\varepsilon} k_x[\psi^\varepsilon_t + (e^{-\lambda t}k + \Gamma)\psi^\varepsilon_x]dyds | \\
\leq \int_{\tau - \varepsilon}^{t} \left( \int_{y(s)}^{y(s)+\varepsilon} |k_x|^2dy \right)^{\frac{1}{2}} \left( \int_{y(s)}^{y(s)+\varepsilon} [\psi^\varepsilon_t + (e^{-\lambda t}k + \Gamma)\psi^\varepsilon_x]^2dy \right)^{\frac{1}{2}} ds.
\]

Define

\[
\pi_\varepsilon(s) = \left( \sup_{x \in \mathbb{R}} \int_{y(s)}^{y(s)+\varepsilon} k_x^2(s, y)dy \right)^{\frac{1}{2}}.
\]

Note that the function \( \pi_\varepsilon(s) \) is uniformly bounded for \( \varepsilon \) and \( \lim_{\varepsilon \to 0} \pi_\varepsilon(s) = 0 \) almost every time \( t \). Hence, it follows from the dominated convergence theorem that

\[
\lim_{\varepsilon \to 0} \int_{\tau}^{t} \left( \sup_{x \in \mathbb{R}} \int_{y(s)}^{y(s)+\varepsilon} k_x^2(s, y)dy \right)^{\frac{1}{2}} ds \leq \lim_{\varepsilon \to 0} \int_{\tau}^{t} \pi_\varepsilon(s)ds = 0. \tag{6.23}
\]

In addition, for all \( s \in [\tau, t] \), one can get from the definition of \( \psi_\varepsilon \) that

\[
\psi^\varepsilon_x(s, y) = -\varepsilon^{-1}, \quad \psi^\varepsilon_t(s, y) + (e^{-\lambda s}k(s, y(s)) + \Gamma)\psi^\varepsilon_x(s, y) = 0, \tag{6.24}
\]

with \( y(s) < y < y(s) + \varepsilon \). Then, it follows from (6.24) that

\[
\int_{y(s)}^{y(s)+\varepsilon} (\psi^\varepsilon_t(s, y) + (e^{-\lambda s}k(s, x(s)) + \Gamma)\psi^\varepsilon_x(s, y))^2dy \\
= \varepsilon^{-2} \int_{y(s)}^{y(s)+\varepsilon} (k(s, y) - k(s, y(s)))^2dy
\]
Choosing \( \epsilon \) satisfies the distributional identity
\[
\epsilon^{-1} \left( \max_{y(s)<y<y(s)+\epsilon} |k(s, y) - k(s, y(s))| \right)^2 \\
\leq \epsilon^{-1} \left( \int_{y(s)}^{y(s)+\epsilon} |k(s, y) - k(s, y(s))|dy \right)^2 \\
\leq \epsilon^{-\frac{1}{2}} (\epsilon^{\frac{1}{2}} \|k_x(s)\|_{H^1})^2 \leq C \|k(s)\|_{H^1}.
\]

Using (6.23) and (6.25), we obtain
\[
\lim_{\epsilon \to 0} \int_t^t \int_{y(s)}^{y(s)+\epsilon} k_x[\psi^x_t + (e^{-\lambda s}k + \Gamma)\psi^x](s, y)dyds = 0.
\]  

The Cauchy-Schwartz inequality and (6.24) entail that
\[
\lim_{\epsilon \to 0} \left( \int_t^t + \int_{t+\epsilon}^{t+\epsilon} \right) \int_{y(s)}^{y(s)+\epsilon} k_x[\psi^x_t + (e^{-\lambda s}k + \Gamma)\psi^x](s, y)dyds \\
\leq \lim_{\epsilon \to 0} \left( \int_t^t + \int_{t+\epsilon}^{t+\epsilon} \right) \left( \int_{y(s)}^{y(s)+\epsilon} |k_x|^2dy \right)^{\frac{1}{2}} \left( \int_{y(s)}^{y(s)+\epsilon} |\psi^x_t + (e^{-\lambda s}k + \Gamma)\psi^x|^2dy \right)^{\frac{1}{2}} ds \\
\leq \lim_{\epsilon \to 0} 2\epsilon \|k(s)\|_{H^1} \left( \int_{y(s)}^{y(s)+\epsilon} 2\epsilon^{-2} \|k\|_{L^\infty}^2dy \right)^{\frac{1}{2}} \\
\leq \lim_{\epsilon \to 0} C\epsilon^{\frac{1}{2}} = 0.
\]  

Combining (6.26)-(6.27), we arrive at (6.22).

Step 6. Using the uniqueness of \( \beta(t, \xi) \), we can deduce that the uniqueness of \( y(t, \xi) \).

The following lemma is to prove the Lipschitz continuity of \( k \) with respect to \( t \) under the Lagrange coordinates.

**Lemma 6.4.** Let \( k = k(t, x) \) be a conservative solution to (2.1). Then the map \( (t, \beta) \mapsto k(t, y(t, \beta)) \) is Lipschitz continuous with a constant depending only on the norm \( \|k\|_{H^1} \).

**Proof.** Combining (6.7) and (6.9), we get
\[
|k(t, y(t, \beta)) - k(t, \beta)| \leq |k(t, y(t, \beta)) - k(t, y(t, \beta(t)))| + |k(t, y(t, \beta(t))) - k(t, y(t, \beta(t))))| \\
\leq \frac{1}{2} |\beta(t) - \beta| + (t - \tau)\|P_{x}\|_{L^\infty} \\
\leq (t - \tau)(\frac{1}{2} \|F\|_{L^\infty} + \|P_{x}\|_{L^\infty}),
\]
which implies the map \( (t, \beta) \mapsto k(t, x(t, \beta)) \) is Lipschitz continuous.

**Lemma 6.5.** Let \( k \in H^1(\mathbb{R}) \) and define the convolution \( P \) being as in (2.2). Then \( P_{x} \) is absolutely continuous and satisfies
\[
P_{xx} = P - \left( -H(k) + e^{-\lambda t}k^2 + \frac{e^{-\lambda t}}{2}k_x^2 \right).
\]  

**Proof.** The function \( \psi(s) = e^{-|s|} \) satisfies the distributional identity
\[
D_x^2\psi = \psi - \delta_0.
\]

Thanks to \( \delta_0 \) denotes a unit Dirac mass at the origins. Thus, for all function \( f \in L^1(\mathbb{R}) \), the convolution satisfies
\[
D_x^2(\psi * f) = \psi * f - f.
\]
Choosing \( f = -H(k) + e^{-\lambda t}k^2 + \frac{e^{-\lambda t}}{2}k_x^2 \), we obtain the desired result.
6.2 Proof of uniqueness

We need to seek a good characteristic, and employ how the gradient \( k_x \) of a conservative solution varies along the good characteristic, and complete the proof of uniqueness.

**Proof.**

**Step 1.** Lemmas 6.3-6.4 ensure that the map \((t, \xi) \mapsto (y, k(t, \xi))\) and \(\xi \mapsto F(t, \xi)\) are Lipschitz continuous. Thanks to Rademacher’s theorem, the partial derivatives \(y_t, y_\xi, k_t, k_\xi\) and \(F_\xi\) exist almost everywhere. Moreover, \(y(t, \xi)\) is the unique solution to (5.1), and the following holds.

**(GC)** For a.e. \(\xi\) and a.e. \(t \geq 0\), the point \((t, \beta(t, \bar{\beta}))\) is a Lebesgue point for the partial derivatives \(y_t, y_\xi, k_t, k_\xi\) and \(F_\xi\). Moreover, \(y_\xi(t, \xi) > 0\) for a.e. \(t \geq 0\).

If (GC) holds, then \(t \to y(t, \xi)\) is a good characteristic.

**Step 2.** We now construct an ODE to describe that the quantities \(k_\xi\) and \(x_\xi\) vary along a good characteristic. Supposing that \(t, \tau \notin \mathcal{N}\), and \(y(t, \xi)\) is a good characteristic, we then have

\[
y(t, \beta(t, \xi)) = \bar{y}(\xi) + \int_0^t (e^{-\lambda s}k(s, \beta(s, \xi)) + \Gamma) ds.
\]

Differentiating the above equation with respect to \(\xi\), we deduce that

\[
y_\xi = \bar{y}_\xi(\xi) + \int_0^t k_\xi(s, \xi)d\xi.
\]  

Likewise, we have

\[
k_\xi = \bar{k}_x(\bar{y}(\xi)) \bar{y}_\xi - \int_0^t G_\xi(s, \xi)d\xi.
\]  

From (6.29) - (6.30), we end up with

\[
\begin{aligned}
y_\xi &= e^{-\lambda t}k_\xi, \\
k_\xi &= -G_\xi.
\end{aligned}
\]  

**Step 3.** We now return to the original coordinates \((t, x)\) and derive an evolution equation for the partial derivative \(k_x\) along a “good” characteristic curve. For a fixed point \((t, x)\) with \(t \notin \mathcal{N}\). Suppose that \(\bar{x}\) is a Lebesgue point for the map \(x \to k_x(t, x)\), and \(\xi\) satisfies \(x = y(t, \xi)\), and suppose that \(t \to y(t, \xi)\) is a good characteristic, which implies (GC) holds. From (6.1) we have

\[
y_\beta(t, \beta) = \frac{1}{1 + k_x^2(t, y)} > 0, \quad \beta(t, \xi) > 0,
\]

which implies that \(y_\xi(t, \xi) > 0\).

Hence, the partial derivative \(k_x\) can be calculated as shown below

\[
k_x(t, y(t, \beta(t, \bar{\beta}))) = \frac{k_\xi(t, y(t, \beta(t, \xi)))}{y_\xi(t, \beta(t; \xi))}.
\]

Applying (6.31) to describe the evolution of \(k_\xi\) and \(y_\xi\), we infer that the map \(t \to k_x(t, y(t, \beta(t, \xi)))\) is absolutely continuous. It follows that

\[
\frac{d}{dt}k_x(t, y(t, \beta(t; \tau, \xi))) = \frac{d(k_x)}{dt} = \frac{y_\xi F_\xi - e^{-\lambda t}k_\xi^2}{y_\xi^2}.
\]

Hence, we conclude that as long as \(y_\beta \neq 0\), the map \(t \to k_x\) is absolutely continuous.
Step 4. Let
\[
K(t, \xi) = k(t, y(t, \xi)), \quad V(t, \xi) = \frac{k_2 \circ y}{1 + k_2 \circ y},
\]
\[
W(t, \xi) = \frac{k_2 \circ y}{1 + k_2 \circ y}, \quad Q(t, \xi) = (1 + k_2 \circ y) \cdot y_\xi.
\]
From which it follows that
\[
\begin{align*}
  y_t &= e^{-\lambda t} K + \Gamma, \\
  K_t &= -G, \\
  V_t &= 2W \left( e^{-\lambda t} K^2 (1 - V) - H(K)(1 - V) - \frac{e^{-\lambda t}}{2} V - P(1 - V) \right), \\
  W_t &= (1 - 2V) \left( e^{-\lambda t} K^2 (1 - V) - H(K)(1 - V) - \frac{e^{-\lambda t}}{2} V - P(1 - V) \right), \\
  Q_t &= 2WQ \left( \frac{e^{-\lambda t}}{2} + e^{-\lambda t} K^2 - H(K) - P \right). \\
\end{align*}
\] (6.33)

For any \( \xi \in \mathbb{R} \), we deduce that the following initial conditions
\[
\begin{align*}
  \int_0^y \tilde{K}_2 dx + \tilde{y}(\xi) &= \xi, \\
  \tilde{K}(\xi) &= \tilde{k} \circ \tilde{y}(\xi), \\
  \tilde{V}(\xi) &= \frac{\tilde{k}_2 \circ \tilde{y}(\xi)}{1 + k_2 \circ \tilde{y}(\xi)}, \\
  \tilde{W}(\xi) &= \frac{\tilde{k}_x \circ \tilde{y}(\xi)}{1 + k_2 \circ \tilde{y}(\xi)}, \\
  \tilde{Q}(\xi) &= (1 + k_2 \circ \tilde{y}) \tilde{y}_\xi(\xi) = 1. \\
\end{align*}
\] (6.34)

Making use of all coefficients is Lipschitz continuous and the previous steps again, the system (6.33)-(6.34) has a unique globally solution.

Step 5. Let \( k \) and \( \tilde{k} \) be two conservative weak solution of (2.1) with the same initial data \( \tilde{k} \in H^1(\mathbb{R}) \). For \( a.e. \ t \geq 0 \), the corresponding Lipschitz continuous maps \( \xi \mapsto y(t, \xi), \xi \mapsto \tilde{y}(t, \beta) \) are strictly increasing. Hence they have continuous inverses, say \( x \mapsto y^{-1}(t, x), x \mapsto \tilde{y}^{-1}(t, x) \). Thus, we deduce that
\[
y(t, \xi) = \tilde{y}(t, \xi), \quad k(t, y(t, \xi)) = \tilde{k}(t, y(t, \xi)).
\]
Moreover, for \( a.e. \ t \geq 0 \), we have
\[
k(t, x) = k(t, y(t, \xi)) = \tilde{k}(t, \tilde{y}(t, \xi)) = \tilde{k}(t, x).
\]
Then, we finish the proof of Theorem 6.3.

**Proof of Theorem 2.3**. Theorem 4.1 and Theorem 6.1 ensure that the equation (2.1) has a unique globally conservative solution.

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