Higher Genus Symplectic Invariants and Sigma Model Coupled
With Gravity

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1 Introduction

This paper is a continuation of our previous paper [RT]. In [RT], among other things, we build up the mathematical foundation of quantum cohomology ring on semi-positive symplectic manifolds. We also defined higher genus symplectic invariants without gravity (topological sigma model) in terms of inhomogeneous holomorphic maps from a fixed Riemann surface, and proved the composition law they satisfy. Topological gravity, proposed by Witten, concerns the intersection theory of the moduli space of marked Riemann surfaces. Based on the physical intuition, Witten suggested a relation between those intersection numbers and the KdV hierarchy. This relation was clarified by Kontsevich (cf. [Ko]). However, the mathematical, as well as physical, phenomenon will become much more interesting if the topological sigma model is coupled with the topological gravity. In fact, in [W2] Witten proposed an approach to the topological sigma model coupled with gravity, and made a very important conjecture on the basic feature of this new model. The purpose of this paper is to establish a mathematical foundation for the theory of topological sigma model coupled with topological gravity over any semi-positive symplectic manifolds. This new theory also provides many more new geometric examples of the topological field theory coupled with gravity. For each semi-positive symplectic manifold $V$, we can associate a topological sigma model with gravity, or simply a topological field theory coupled with gravity. This theory begins with the GW-invariants

$$\Psi^V_{(A,g,k)} : H_* (\mathcal{M}_{g,k}, \mathbb{Q}) \times H_*(V, \mathbb{Z})^k \to \mathbb{Q},$$
for any $A \in H_2(V, \mathbb{Z})$ and $2g + k \geq 3$. Here $\overline{M}_{g,k}$ is the Deligne-Mumford compactification of the moduli space of genus $g$ Riemann surfaces with $k$ marked points. The GW-invariants are multilinear and supersymmetric on $H_*(V, \mathbb{Z})^k$.

At first, we will rigorously define the GW-invariant $\Psi^V$ on semi-positive symplectic manifolds (cf. section 2).

From the analytic point of view, it is the most convenient to use the inhomogeneous holomorphic maps from Riemann surfaces in $\overline{M}_{g,k}$, though other equivalent formulations may be possible, such as using stable maps and establishing a more sophisticated intersection theory. An inhomogeneous holomorphic map is a solution of an inhomogeneous Cauchy-Riemann equation (cf. Section 2).

Putting aside technical details for the time being, we can intuitively define the GW-invariants (cf. Section 2 for details) as follows: let $V$ be any symplectic manifold and $A \in H_2(V, \mathbb{Z})$. For any homology classes $[K] \in H_2(\overline{M}_{g,k}, \mathbb{Q})$ and $\alpha_i \in H_*(V, \mathbb{Z})$, represented by cycles $K, A_i$, respectively, we define $\Psi^V_{(A,g,k)}([K]; \alpha_1, \cdots, \alpha_k)$ to be the number of tuples $(\Sigma; x_1, \cdots, x_k; f)$ with appropriate sign, satisfying: $\Sigma \in K$, $f : \Sigma \mapsto V$ solves an inhomogeneous Cauchy-Riemann equation, and $f(x_i) \in A_i$, whenever

$$\sum \text{cod}(A_i) + \text{cod}(K) = 2c_1(V)(A) + 2(3 - n)(g - 1) + 2k; \quad (1.1)$$

We simply put $\Psi^V_{(A,g,k)}([K]; \alpha_1, \cdots, \alpha_k)$ to be zero if (1.1) is not satisfied.

This approach towards defining new invariants has been used before in many cases (cf. [Do], [Gr], [R], [R3], [RT], [W1]). For symplectic 4-manifolds, using unperturbed holomorphic maps, the first author already defined the invariant $\Psi$ in the very important case that $k = 0$ and $[K] = \overline{M}_{g,0}$. However, in each case, there are specified difficulties to be overcome. Using the techniques we developed in [RT], we will first prove

**Theorem A (Theorem 2.14.)** If $V$ is a semi-positive symplectic manifold, the GW-invariant $\Psi^V_{(A,g,k)}$ can be well defined for any $g, k \geq 0$ with $2g + k \geq 3$. Moreover, this $\Psi^V$ depends only the symplectic structure of $V$.

A symplectic manifold $V$ is semi-positive if it is compact and there is no $J$-holomorphic map $f : S^2 \mapsto V$ such that $3 - n \leq \int_{S^2} f^*c_1(V) < 0$, where $J$ is any given compatible almost complex structure on $V$. In particular, any algebraic manifold of dimension $\leq 3$ is semi-positive in this sense, also any algebraic manifold $V$ with $c_1(V) \geq 0$ is semi-positive.

One new consequence of our theorem, which was not obvious at all to physicists based on mathematically unjustified path integrals, is that the invariant $\Psi^V$ is a symplectic invariant. The
path integral starts from a Lagrangian. The Lagrangian for sigma model or sigma model coupled with gravity is valid for any almost complex manifolds (symplectic or not). There was a speculation that its correlation functions will be the invariants of homotopy class of almost complex structures. This is in fact false. Our invariants are symplectic invariants rather than the invariants of almost complex structures. In particular, they can distinguish different symplectic manifolds with the same homotopy class of almost complex structures (see section 5 or [R], [R1]).

One of fundamental properties of a topological field theory is the axiom on the decomposition of correlation functions. In our case, the GW-invariants serve as the correlation functions. Therefore, in order to make them more useful, or at least to construct a correct model for the topological field theory, we need to verify that our invariants satisfy the composition law.

The composition law governs how the GW-invariants change during the degeneration of stable curves. Its classical cousin in enumerative algebraic geometry is the degeneration method, which was only derived in very special cases. The classical degeneration method never became a general theory as neat as the composition law describes. One reason might be that the classical counting of holomorphic curves, particularly of higher genus, does not obey the composition laws predicted by physicists, even for the projective plane $P^2$. Namely the way of counting was not good. In [RT], we found the correct counting in terms of inhomogeneous holomorphic maps and established the composition law at least for the mixed invariants, corresponding to the $\sigma$-models without gravity. Based on the same techniques developed in [RT], we are also able to prove the composition law for all GW-invariants.

Assume $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_i + k_i \geq 3$. Fix a decomposition $S = S_1 \cup S_2$ of $\{1, \cdots, k\}$ with $|S_i| = k_i$. Then there is a canonical embedding $\theta_S : \overline{M}_{g_1, k_1+1} \times \overline{M}_{g_2, k_2+1} \rightarrow \overline{M}_{g,k}$, which assigns to marked curves $(\Sigma_i; x^i_1, \cdots, x^i_{k_i+1})$ ($i = 1, 2$), their union $\Sigma_1 \cup \Sigma_2$ with $x^i_{k_1+1}$ identified to $x^2_{k_2+1}$ and remaining points renumbered by $\{1, \cdots, k\}$ according to $S$.

There is another natural map $\mu : \overline{M}_{g-1, k+2} \rightarrow \overline{M}_{g,k}$ by gluing together the last two marked points.

Choose a homogeneous basis $\{\beta_b\}_{1 \leq b \leq L}$ of $H_*(V, \mathbb{Z})$ modulo torsion. Let $(\eta_{ab})$ be its intersection matrix. Note that $\eta_{ab} = \beta_a \cdot \beta_b = 0$ if the dimensions of $\beta_a$ and $\beta_b$ are not complementary to each other. Put $(\eta^{ab})$ to be the inverse of $(\eta_{ab})$. Now we can state the composition law, which consists of two formulas.

**Theorem B. (Theorem 2.10)** Let $[K_i] \in H_*(\overline{M}_{g_i, k_i+1}, \mathbb{Q})$ ($i = 1, 2$) and $[K_0] \in H_*(\overline{M}_{g-1, k+2}, \mathbb{Q})$. 

For any $\alpha_1, \cdots, \alpha_k$ in $H_*(V, \mathbb{Z})$. Then we have

\begin{equation}
\Psi^V_{(A,g,k)}(\theta_{S*}[K_1 \times K_2]; \{\alpha_i\}) = \sum_{A=A_1+A_2} \sum_{a,b} \Psi^V_{(A_1,g_1,k_1+1)}([K_1]; \{\alpha_i\}_{i \leq k_1}, \beta_a) \eta^{ab} \Psi^V_{(A_2,g_2,k_2+1)}([K_2]; \beta_b, \{\alpha_j\}_{j > k_1})
\end{equation}

\begin{equation}
\Psi^V_{(A,g,k)}(\mu_*[K_0]; \alpha_1, \cdots, \alpha_k) = \sum_{a,b} \Psi^V_{(A,g-1,k+2)}([K_0]; \alpha_1, \cdots, \alpha_k, \beta_a, \beta_b) \eta^{ab}
\end{equation}

There is a natural map $\pi: \overline{M}_{g,k} \to \overline{M}_{g,k-1}$ as follows: For $(\Sigma, x_1, \cdots, x_k) \in \overline{M}_{g,k}$, if $x_k$ is not in any rational component of $\Sigma$ which contains only three special points, then we define

$$\pi(\Sigma, x_1, \cdots, x_k) = (\Sigma, x_1, \cdots, x_{k-1}),$$

where a distinguished point of $\Sigma$ is either a singular point or a marked point. If $x_k$ is in one of such rational components, we contract this component and obtain a stable curve $(\Sigma', x_1, \cdots, x_{k-1})$ in $\overline{M}_{g,k-1}$, and define $\pi(\Sigma, x_1, \cdots, x_k) = (\Sigma', x_1, \cdots, x_{k-1})$.

Clearly, $\pi$ is continuous. One should be aware that there are two exceptional cases $(g, k) = (0, 3), (1, 1)$ where $\pi$ is not well defined. Associated with $\pi$, we have two $k$-reduction formulas for $\Psi^V_{(A,g,k)}$.

**Proposition C (Theorem 2.15).** Suppose that $(g, k) \neq (0, 3), (1, 1)$.

(1) For any $\alpha_1, \cdots, \alpha_{k-1}$ in $H_*(V, \mathbb{Z})$, we have

\begin{equation}
\Psi^V_{(A,g,k)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]) = \Psi^V_{(A,g,k-1)}([\pi(K)]; \alpha_1, \cdots, \alpha_{k-1})
\end{equation}

(2) Let $\alpha_k$ be in $H_{2n-2}(V, \mathbb{Z})$, then

\begin{equation}
\Psi^V_{(A,g,k)}([\pi^{-1}(K)]; \alpha_1, \cdots, \alpha_{k-1}, \alpha_k) = \alpha^*_k(A) \Psi^V_{(A,g,k-1)}([K]; \alpha_1, \cdots, \alpha_{k-1})
\end{equation}

where $\alpha^*_k$ is the Poincare dual of $\alpha_k$.

In order to formulate the generalized Witten conjecture in terms of our invariants, we need to introduce special cycles in $\overline{M}_{g,k}$. Let $\pi: \overline{U}_{g,k} \to \overline{M}_{g,k}$ be the universal family of stable curves of genus $g$ and $k$ marked points. Each marked point gives rise to a section $\sigma_i$ $(1 \leq i \leq k)$ of this fibration. Following Witten, we let $\mathcal{L}_i$ be the pull-back of the relative cotangent sheave of $\pi: \overline{U}_{g,k} \to \overline{M}_{g,k}$ by $\sigma_i$. Then we put $W_{d_1, \cdots, d_k}$ to be the Poincare dual of the cohomology class $c_1(\mathcal{L}_1)^{d_1} \cup c_1(\mathcal{L}_2)^{d_2} \cdots \cup c_1(\mathcal{L}_k)^{d_k}$. We call these $W_{d_1, \cdots, d_k}$ Witten cycles.
For convenience, as Witten did, we use

\[ < \tau_{d_1, \alpha_1}, \tau_{d_2, \alpha_2}, \ldots, \tau_{d_k, \alpha_k} >_{g,k} \]

to denote the GW-invariants \( \Psi_{(A,g,k)}([W_{d_1}, \ldots, d_k]; \alpha_1, \cdots, \alpha_k) \). Following Witten, we introduce potential functions

\[
F_g = \sum_A \sum \prod \frac{(t^a_r)_{n_{r,\alpha}}}{n_{r,\alpha}!} \cdot \prod \tau_{n_{r,\alpha}} > q^A, \quad g = 0, 1, 2, \ldots.
\]

where \( q^A \) is an element of Novikov ring \([\text{MS}], [\text{RT}] \) (section 8). We further define

\[
F^V = \sum_{g \geq 0} F_g.
\]

One of fundamental problems on \( F^V \), even to physicists, is to find the complete set of equations \( F^V \) satisfies. In Section 6, imitating the arguments of Witten in \([W2] \), we will prove (cf. Lemma 6.1, 6.2)

**Theorem C.** \( F^V \) satisfies the generalized string equation

\[
(1.7) \quad \frac{\partial F^V}{\partial t^a_0} = \frac{1}{2} \eta^a_{b0} t^b_0 t^a_0 + \sum_{i=0}^{\infty} \sum_a t^a_{i+1} \frac{\partial F^V}{\partial t^a_i}.
\]

\( F_g \) satisfies the dilation equation

\[
(1.8) \quad \frac{\partial F_g}{\partial t^1_1} = (2g - 2 + \sum_{i=1}^{\infty} \sum_a t^a_i \frac{\partial}{\partial t^a_1}) F_g + \frac{\chi(V)}{24} \delta_{g,1},
\]

where \( \chi(V) \) is the Euler characteristic of \( V \).

In general, Witten suggested

\[
U = \frac{\partial^2 F^V}{\partial t^{0,1}_0 \partial t^{0,\sigma}_0}, \quad U' = \frac{\partial^3 F^V}{\partial t^{0,1}_0 \partial t^{0,\sigma}_0}, \quad \cdots, \quad U^{(l)} = \frac{\partial^{l+2} F^V}{\partial t^{0,1}_0 \partial t^{0,\sigma}_0}, \quad \text{for} \quad l \geq 0.
\]

We will regard \( U^{(l)} \) to be of degree \( l \). By a differential function of degree \( k \) we mean a function \( G(U, U', U'', \cdots) \) of degree \( k \) in that sense. In particular, any function of form \( G(U) \) is of degree zero, and \( (U')^2 \) has degree two.

**Generalized Witten Conjecture:** For every \( g \geq 0 \), there are differential functions

\[
G_{m,a,n,\beta}(U_a, U'_a, U''_a, \cdots) \]

of degree \( 2g \) such that

\[
\frac{\partial^2 F_g}{\partial \tau_{m,a} \partial \tau_{n,\beta}} = G_{m,a,n,\beta}(U_a, U'_a, U''_a, \cdots)
\]
This conjecture was affirmed in case $V = pt$ by Kontsevich $[{Ko}]$. When $g = 0$, it is a consequence of the associativity equation proved in $[{R1}]$. But the general case is still open.

We call $\Psi^V_{(A,g,k)}(\overline{M}_{g,k}; \cdots)$ primitive GW-invariants of genus $g$. Those invariants correspond to the enumerative invariants of counting genus $g$ holomorphic curves passing through generic $k$ cycles in enumerative algebraic geometry.

**Corollary E (Proposition 6.5).** For genus $\leq 1$, the Witten invariants $<>$ can be reduced to primitive GW-invariants.

In general, we conjecture that all the Witten invariants can be derived from primitive GW-invariants.

Our invariant can be also applied to studying topology of symplectic manifolds. As an example, we will verify the Stabilizing conjecture of the first author in the case of simply connected elliptic surfaces. The conjecture claims: Suppose that $X$, $Y$ are two simply connected homeomorphic symplectic 4-manifolds. Then $X$, $Y$ are diffeomorphic if and only if $X \times S^2$, $Y \times S^2$ are deformation equivalent as symplectic manifolds. He also verified this conjecture for certain complex surfaces (cf. $[{R1}]$). By calculating our invariants for the product of simply connected elliptic surfaces with $S^2$, we will prove that

**Theorem F: (Theorem 5.1)** The stabilizing conjecture holds for simply connected elliptic surfaces.

It has been an interesting question in symplectic topology to find how many different deformation classes of symplectic structures with the same tamed almost complex structures (up to a homotopy) could exist on a fixed smooth manifold. In $[{R1}]$, for any positive integer $n$, the first author constructed examples admitting at least $n$-many different deformation classes. Using our calculation of GW-invariant, we can produce examples with infinitely many deformation classes of symplectic structures.

**Proposition H: (Proposition 5.4)** Let $X$ be the blow-up of a simply connected elliptic surface at one point. Then, the smooth 6-manifold $X \times S^2$ admits infinitely many deformation classes of symplectic structures with the same tamed almost complex structure up to a homotopy.

This paper is organized as follows. We will define the invariants and state the basic properties (including composition law) of our invariants in section 2. The section 3 is a technical section where
we will prove the various results about the compactification and transversality. All the results stated in section 2 will be proved in section 4. We will discuss the applications to the stabilizing conjecture and the Witten conjecture in section 5, 6.

Some of the results in this paper have been lectured by us in last few years. Also, the main results of this paper were announced in the paper [T] of the second author published in the proceeding of the first "Current developments in Mathematics", Boston, May, 1995. All the basic techniques were developed in [RT].

The first author wish to thank S. K. Donaldson who suggested the example of Section 5 to him.

2 Higher genus symplectic invariants and composition law

In this section, we construct the higher genus symplectic invariants. Its physical counterparts are the correlation functions of topological sigma model coupled with gravity. Some important cases of these invariants were first studied for symplectic 4-manifolds in [R3] and also discussed in [RT]. The construction here is similar to that of [R1].

First of all, let’s introduce the inhomogeneous Cauchy-Riemann equation, which plays a central role in [R1]. Compared to that of [R1], we would like to define the inhomogeneous term varying continuously as we vary the complex structures of the Riemann surfaces. This makes the construction more complicated. Let \((V, \omega)\) be a symplectic manifold and \(J\) be a tamed almost complex structure. Let \(\mathcal{M}_{g,k}\) be the moduli space of genus \(g\) Riemann surfaces with \(k\)-marked points and \(\overline{\mathcal{M}}_{g,k}\) be the Deligne-Mumford compactification. Suppose that

\[ \pi : \overline{U}_{g,k} \rightarrow \overline{\mathcal{M}}_{g,k} \]

is the universal curve. Both \(\overline{U}_{g,k}\) and \(\overline{\mathcal{M}}_{g,k}\) are projective varieties. Unfortunately, it is well-known that \(\overline{U}_{g,k}\) is not a universal family. Namely, if \(\Sigma \in \overline{\mathcal{M}}_{g,k}\) has a nontrivial automorphism, then \(\pi^{-1}(\Sigma) = \Sigma/Aut(\Sigma)\) instead of \(\Sigma\). We can not directly define the inhomogeneous term over \(\overline{U}_{g,k}\). However, this problem can be overcome by constructing some finite covers of \(\overline{\mathcal{M}}_{g,k}\).

**Definition 2.1:** A finite connected cover \(p_{\mu} : \overline{\mathcal{M}}_{g,k}^{\mu} \rightarrow \overline{\mathcal{M}}_{g,k}\) is a good cover if \(\overline{\mathcal{M}}_{g,k}^{\mu}\) is a normal projective variety with quotient singularity such that there is universal family

\[ \pi_{\mathcal{M}} : \overline{U}_{g,k}^{\mu} \rightarrow \overline{\mathcal{M}}_{g,k}^{\mu}, \]

i.e., for each \(b \in \overline{\mathcal{M}}_{g,k}^{\mu}\), \(\pi_{\mathcal{M}}^{-1}(b)\) is a stable Riemann surface isomorphic to \(p_{\mu}(b)\). Furthermore, we
have following commutative diagram

\[
p_{\mu} : \mathcal{U}_{g,k}^{\mu} \rightarrow \mathcal{U}_{g,k}
\]
\[
\downarrow \pi_{\mathcal{M}} \quad \downarrow \pi
\]
\[
p_{\mu} : \mathcal{M}_{g,k}^{\mu} \rightarrow \mathcal{M}_{g,k}
\]

It is clear that $\mathcal{U}_{g,k}^{\mu}$ is projective and unique.

To simplify the notation, we will not distinguish $b$ with $p_{\mu}(b)$ without any confusion. As we mentioned, the problem for $\mathcal{M}_{g,k}$ is that some elements have nontrivial automorphism groups. One can resolve this problem by taking finite cover locally. Hence, locally one always has a universal family. Mumford proved that such local covers can be glued together to form a global finite cover. Moreover, it can be explicit constructed via level $m$-structure. We will not give any detail of level $m$-structure. The idea is to fix a basis of $H^1$ with the coefficient in $\mathbb{Z}_m$. Then, there is no automorphism of stable Riemann surface preserving the fixed basis. We refer the reader to [Mu] for the detail.

Let $p_{\mu} : \mathcal{M}_{g,k}^{\mu} \rightarrow \mathcal{M}_{g,k}$ be a good cover. Suppose that

\[
\phi_{\mu} : \mathcal{U}_{g,k}^{\mu} \rightarrow \mathbb{P}^N
\]
is a projective embedding. There are two relative tangent bundles over $\mathbb{P}^N \times V$ with respect to $h_i$ ($i = 1, 2$), where $h_i$ is the projection from $\mathbb{P}^N \times V$ to its $i$-th factor. A section $\nu$ of $\text{Hom}(h_1^*T\mathbb{P}^N, h_2^*TV)$ is said to be anti-$J$-linear if for any tangent vector $v$ in $T\mathbb{P}^N$,

\[
(2.1) \quad \nu(j_{\mathbb{P}^N}(v)) = -J(\nu(v))
\]

where $j_{\mathbb{P}^N}$ is the complex structure on $\mathbb{P}^N$. Usually, we call such a $\nu$ an inhomogeneous term. We will often drop $h_i$ from the notation without any confusion.

**Definition 2.2.** Let $\nu$ be an inhomogeneous term. A $(J, \nu)$-perturbed holomorphic map, or simply a $(J, \nu)$-map, is a smooth map $f : \Sigma \rightarrow V$ satisfying the inhomogeneous Cauchy-Riemann equation

\[
(2.2) \quad (\bar{\partial}_J f)(x) = \nu(\phi_{\mu}(x), f(x)),
\]

where $\bar{\partial}_J$ denotes the differential operator $d + J \cdot d \cdot j_{\Sigma}$.

Let $\mathcal{M}_{g,k,\kappa}$ be the subset of $\mathcal{M}_{g,k}$ with automorphism group $\kappa$. We will use $I$ to denote the trivial automorphism group. Since $\mathcal{M}_{g,k,I}$ is smooth, without the loss of generality, we can assume that
\( \mathcal{M}_{g,k,I}^\mu \) is smooth, where \( \mathcal{M}_{g,k,\kappa}^\mu = \mathcal{P}^{-1}_\mu(\mathcal{M}_{g,k,\kappa}) \). Let \( U_{g,k,I}^\mu \) be the preimage of \( \mathcal{M}_{g,k,I}^\mu \). We denote by \( \mathcal{M}_A(\mu, g, k, J, \nu)_I \) the moduli space of \((J, \nu)\)-perturbed holomorphic maps from \((\Sigma, x_1, \ldots, x_k) \in \mathcal{M}_{g,k,I}^\mu \) into \( V \). There are some important topological properties as follows.

Let \( \pi : U_A(\mu, g, k, J, \nu)_I \rightarrow \mathcal{M}_A(\mu, g, k, J, \nu)_I \) be the universal family of curves, i.e.,

\[
\pi^{-1}(f, \Sigma, \{x_i\}) = \Sigma.
\]

We can define the evaluation map

\[
e_A(g, k) : U_A(\mu, g, k, J, \nu)_I \rightarrow V
\]

by

\[\text{(2.3)}\]

\[
e_A(g, k)(f, \Sigma, \{x_i\}, y) = f(y).
\]

Each marked point \( x_i \) defines a section

\[
\sigma_i : \mathcal{M}_A(\mu, g, k, J, \nu)_I \rightarrow U_A(\mu, g, k, J, \nu)_I
\]

by

\[\text{(2.4)}\]

\[
\sigma_i((f, \Sigma, \{x_i\})) = x_i.
\]

The composition

\[
e_i = e_A(g, k) \circ \sigma_i : \mathcal{M}_A(\mu, g, k, J, \nu)_I \rightarrow V.
\]

Let

\[
\Xi_{g,k}^A = \prod_{i=1}^k e_A(g, k) \circ \sigma_i : \mathcal{M}_A(g, k, J, \nu)_I \rightarrow V^k.
\]

Evidently, we have a map \( \Upsilon_A : \mathcal{M}_A(\mu, g, k, J, \nu)_I \rightarrow \mathcal{M}_{g,k,I}^\mu \) by assigning each \((J, \nu)\)-map to its domain. Together, we get a smooth map

\[
\Upsilon_A \times \Xi_{g,k}^A : \mathcal{M}_A(\mu, g, k, J, \nu)_I \rightarrow \mathcal{M}_{g,k,I}^\mu \times V^k.
\]

In general, \( \mathcal{M}_A(\mu, g, k, J, \nu)_I \) is not compact. However, there is a natural compactification \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)_I \), which we call GU-compactification (cf. Section 3). For our purpose, we also need to consider certain quotient \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)_I \) of \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)_I \).

**Proposition 2.3.** Suppose that \((V, \omega)\) is a semi-positive symplectic manifold. Then, there is a Baire set of second category-\( \mathcal{H} \) among all the smooth pairs \((J, \nu)\) such that for any \((J, \nu) \in \mathcal{H}\)
(1) \( \mathcal{M}_A(\mu, g, k, J, \nu)_I \) is a smooth, oriented manifold of real dimension
\[2c_1(V(A)) + 2(3 - n)(g - 1) + 2k;\]

(2) \( \Upsilon_A \) and \( \Xi_A^{g,k} \) extends to continuous maps, still denoted by the same symbols, from \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)_I \) to \( \overline{\mathcal{M}}_{g,k}^\mu \) and \( V^k \), respectively;

(3) The boundary \( \Upsilon_A \times \Xi_A^{g,k} \left( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)_I \setminus \mathcal{M}_A(\mu, g, k, J, \nu)_I \right) \) is of real codimension at least two.

We call that \( (J, \nu) \) is generic if Proposition 2.3 is satisfied.

The proof of this proposition is the main topic of Section 3.

One can construct natural cohomology classes over \( \mathcal{M}_A(\mu, g, k, J, \nu)_I \) by pulling back of the cohomology classes of \( \overline{\mathcal{M}}_{g,k}^\mu \times V^k \) through \( \Upsilon_A \times \Xi_A^{g,k} \). Then, our invariants can be defined as the paring of the cup products of those natural cohomology classes against the fundamental class of \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)_I \). The existence of such a fundamental class is a much more difficult problem, which will be discussed in [RT1]. Here, we choose to avoid this problem by considering the intersection theory as we did in [RT].

Let \( \{ \alpha_i \}_{1 \leq i \leq k} \) be integral homology classes of \( V \). Each \( \alpha_i \) can be represented by a so called pseudo-submanifold \( (P, f) \). A pseudo-submanifold is a pair \( (P, f) \), where \( P \) is a finite simplicial complex of dimension \( d_i = \text{deg}(\alpha_i) \) such that \( P^{\text{top}} = P - P_{d_i - 2} \) (\( P_{d_i - 2} \) \( - \) (\( d_i - 2 \) skeleton)) is a smooth manifold and \( f: P \rightarrow V \) is piecewise linear with respect to a triangulation of \( V \) and smooth over \( P^{\text{top}} \) in the usual sense. Any two such pseudo-submanifolds representing the same homology class are the boundary of a pseudo-submanifold cobordism in the usual sense. We refer to the Section 4 for details. We choose pseudo-submanifolds \( (Y_i, F_i) \) to represent \( \alpha_i \). Let
\[ Y = \prod_{i=1}^{k} Y_i, F = \prod_{i=1}^{k} F_i. \]
We define \( Y^{\text{top}} = \prod_{i=1}^{k} Y_i^{\text{top}} \). Clearly, \( (Y, F) \) represents \( \prod_{i=1}^{k} \alpha_i \in H_*(V^k, \mathbb{Z}) \).

In general, not every integral homology class of \( \overline{\mathcal{M}}_{g,k}^\mu \) can be represented by a pseudo-submanifold. However, some of its multiples does. Therefore, the homology classes represented by pseudo-submanifolds generate the rational homology \( H_*(\overline{\mathcal{M}}_{g,k}^\mu, \mathbb{Q}) \). We say that a pseudo-manifold \( (G, K) \) is in the general position if \( K(G^{\text{top}}) \subset \mathcal{M}_{g,k,I}^\mu \) has codimension at least two in \( K(G^{\text{top}}) \). Let \( (G, K) \) be such a pseudo-submanifold in \( \mathcal{M}_{g,k,I}^\mu \). Suppose
\[ \sum_{i=1}^{k} (2n - d_i) + (6g - 6 + 2k - \text{deg}(G)) = 2c_1(V(A)) + 2(3 - n)(g - 1) + 2k. \]
Then, we can choose a small perturbation of $F_i, K$ such that $K \times F$ is transverse to $\Upsilon_A \times \Xi_{g,k}^A$ as the PL-maps with respect to some triangulation of $V$ and as the smooth maps over $Y^{top} \times G^{top}$.

By the dimension counting, we can show that

$$\text{Im}(K \times F) \cap \text{Im}(\Upsilon_A \times \Xi_{g,k}^A(\overline{\mathcal{M}}_{A}(\mu, g, k, J, \nu)_I - \mathcal{M}_A(\mu, g, k, J, \nu)_I)) = \emptyset;$$

$$K \times F(Y \times G - Y^{top} \times G^{top}) \cap \Upsilon_A \times \Xi_{g,k}^A = \emptyset.$$

Let $\Delta \subset (\mathcal{M}_{g,k,I}^\mu \times V^k) \times (\mathcal{M}^\mu_{g,k,I} \times V^k)$ be the diagonal. Then,

$$(\Upsilon_A \times \Xi_{g,k}^A \times K \times F)^{-1}(\Delta) \subset \mathcal{M}_A(\mu, g, k, J, \nu)_I \times Y^{top} \times G^{top}$$

is a zero-dimensional smooth submanifold. Following from (2.6), it is compact and hence finite.

Suppose that

$$(\Upsilon_A \times \Xi_{g,k}^A \times K \times F)^{-1}(\Delta) = \{(f_1, s_1), \ldots, (f_m, s_m)\}$$

where $f_i \in \mathcal{M}_A(\mu, g, k, J, \nu)_I$ and $s_i$ represents other factors. For each $(f_i, s_i)$, we define a number $\epsilon(f_i, s_i) = \pm 1$ as follows: We define $\epsilon(f_i, s_i) = +1$ if the orientation, induced by the Jacobian of $\Upsilon_A \times \Xi_{g,k}^A \times K \times F$ and the orientation of $\mathcal{M}_A(\mu, g, k, J, \nu)_I \times Y^{top} \times G^{top}$ at $(f_i, s_i)$, together with the orientation of $\Delta$ matches the orientation of $(\mathcal{M}^\mu_{g,k,I} \times V^k) \times (\mathcal{M}^\mu_{g,k,I} \times V^k)$. Otherwise, we define $\epsilon(f_i, s_i) = -1$. Now, we define

$$\Psi^V_{(A,g,k,\mu)}(K; \alpha_1, \ldots, \alpha_k) = \sum_{i=1}^m \epsilon(f_i, s_i).$$

To Justify our notation, we will show

**Proposition 2.4.**

(1) $\Psi^V_{(A,g,k,\mu)}(K; \alpha_1, \ldots, \alpha_k)$ is independent of $J, \nu$, the pseudo-submanifold representatives $(Y_i, F_i)$.

(2) $\Psi^V_{(A,g,k,\mu)}(K; \alpha_1, \ldots, \alpha_k)$ is independent of semi-positive deformations of $\omega$.

(3) If $(G', K')$ is another pseudo-submanifold which is in the general position and represents the same homology class as that of $(G, K)$,

$$\Psi^V_{(A,g,k,\mu)}(K; \alpha_1, \ldots, \alpha_k) = \Psi^V_{(A,g,k,\mu)}(K'; \alpha_1, \ldots, \alpha_k).$$

Therefore, we can write $\Psi^V_{(A,g,k,\mu)}([K]; \alpha_1, \ldots, \alpha_k)$ for $\Psi^V_{(A,g,k,\mu)}(K; \alpha_1, \ldots, \alpha_k)$, where $[K]$ denotes the homology class represented by the cycle $K$. 

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We postpone the proof of Proposition 2.4 to Section 4.

**Proposition 2.5.** $\Psi^V_{(A,g,k,\mu)}(K; \alpha_1, \cdots, \alpha_k)$ is independent of the embedding $\phi_\mu$. Hence, $\Psi_{(A,g,k,\mu)}$ is a symplectic invariant.

**Proof:** Suppose that 

$$\tilde{\phi}_\mu : \overline{U}_\mu \rightarrow P^{N'}$$

is a different projective embedding. Then,

$$\phi_\mu \times \tilde{\phi}_\mu : \overline{U}_\mu \rightarrow P^N \times P^{N'}.$$ 

One can consider the inhomogeneous term $\tilde{\nu} \in \overline{\text{Hom}}_J(T(P^N \times P^{N'}), TV)$ where $\overline{\text{Hom}}_J$ means anti-complex linear homomorphism. Moreover, one can use $(J, \tilde{\nu})$ to define invariant in the same fashion and prove that the invariant is independent of $(J, \tilde{\nu})$ using the same proof of Proposition 2.4. Let $\nu, \tilde{\nu}$ be the inhomogeneous term defined through the embedding $\phi_\mu$ and $\tilde{\phi}_\mu$. Notes that $T(P^N \times P^N) = T(P^N) \times T(P^{N'})$.

Therefore, we can view both $\nu$ and $\tilde{\nu}$ as the sections of $\overline{\text{Hom}}_J(T(P^N \times P^{N'}), TV)$, where we view that $\nu$ maps the factor $T(P^{N'})$ to zero and $\tilde{\nu}$ maps the first factor to zero. Let’s denote them by $\nu$ and $\tilde{\nu}$. We observe that if $\nu(\tilde{\nu})$ is generic, so is $\tilde{\nu}(\tilde{\nu})$. It follows from the definition that we have the same invariant $\Psi$ using $(J, \nu)$ or $(J, \tilde{\nu})$. In the same way, we have the same invariant using $(J, \nu)$ or $(J, \tilde{\nu})$. As we mentioned, by repeating the proof of Proposition 2.4, we can show that the invariant defined by $(J, \tilde{\nu})$ is the same as the invariant defined by $(J, \tilde{\nu})$. Then, we finish the proof.

**Remark 2.6:** We don’t have to restrict ourself to projective embedding. In fact, we can embed $\overline{U}_{g,k}$ into any smooth complex manifold and define the inhomogeneous term in the same fashion. The proof of Proposition 2.5 shows that the resulting invariant is independent of such an embedding.

For the convenience, we define $\Psi^V_{(A,g,k,\mu)}(K, \alpha_1, \cdots, \alpha_k) = 0$ if (2.5) is not satisfied.

Let’s collect some properties of $\Psi^V_{(A,g,k,\mu)}$. The following proposition essentially follows from the definition. We will omit its proof.

**Proposition 2.7.**

1. $\Psi^V_{(A,g,k,\mu)} = 0$, if either $\omega(A) < 0$ or $c_1(V)(A) + (3 - n)(g - 1) < 0$, in particular, $\Psi_{(0,g,k,\mu)} = 0$ for any $g \geq 2$ and $n \geq 4$. 

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(2) $\Psi^V_{(A,g,k,\mu)}$ is multilinear and supersymmetry on $H_*(V,\mathbb{Z})^k$ with respect to the $\mathbb{Z}_2$-grading by even and odd degrees.

We have established the symplectic invariant

$$
\Psi^V_{(A,g,k,\mu)} : H_* (\overline{M}_{g,k}^\mu, \mathbb{Q}) \times H_*(V,\mathbb{Z})^k \to \mathbb{Q},
$$

for any $A \in H_2(V,\mathbb{Z})$ and $2g + k \geq 3$.

Now we discuss other more interesting properties of $\Psi^V_{(A,g,k,\mu)}$ associated with the structures of $\overline{M}_{g,k}$.

As we mentioned in the introduction, there is a natural map

$$
(2.9) \quad \pi : \overline{M}_{g,k} \to \overline{M}_{g,k-1}
$$

by forgetting the last marked point and contracting the unstable rational component. One should be aware that there are two exceptional cases $(g,k) = (0,3), (1,1)$ where $\pi$ is not well defined.

Suppose $\overline{M}_{g,k}^\mu \to \overline{M}_{g,k}$ is a good cover constructing through level-$m$ structure. Then, one can observe that $\overline{M}_{g,k+1}^\mu = \pi^* \overline{M}_{g,k}^\mu$ is a good cover of $\overline{M}_{g,k+1}$. Let

$$
\pi^\mu : \overline{M}_{g,k+1}^\mu \to \overline{M}_{g,k}^\mu.
$$

Then, $\pi^\mu$ induces a map on the universal families (still denoted by $\pi^\mu$.)

$$
\begin{array}{ccc}
\overline{U}_{g,k+1}^\mu & \xrightarrow{\pi^\mu} & \overline{U}_{g,k}^\mu \\
\downarrow \pi_M & & \downarrow \pi_M \\
\overline{M}_{g,k+1}^\mu & \xrightarrow{\pi^\mu} & \overline{M}_{g,k}^\mu
\end{array}
$$

Let $b \in \overline{M}_{g,k+1}^\mu$ and $\Sigma_b$ be the underline stable Riemann surface. Clearly,

$$
\pi^\mu : \Sigma_b \to \Sigma_{\pi^\mu(b)}
$$

is precisely $\pi$ defined in (2.9).

Associated with $\pi$, we have two $k$-reduction formulas for $\Psi^V_{(A,g,k,\mu)}$.

**Proposition 2.8.** Suppose that $(g,k) \neq (0,3), (1,1)$. Furthermore, suppose that $\overline{M}_{g,k+1}^\mu$ and $\overline{M}_{g,k}^\mu$ are defined as above.

(1) For any $\alpha_1, \cdots, \alpha_{k-1}$ in $H_*(V,\mathbb{Z})$, we have

$$
(2.10) \quad \Psi^V_{(A,g,k,\mu)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]) = \Psi^V_{(A,g,k-1,\mu)}((\pi^\mu)_*[K]; \alpha_1, \cdots, \alpha_{k-1})
$$

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(2) Let \( \alpha_k \) be in \( H_{2n-2}(V, \mathbb{Z}) \), then

\[
\Psi^V_{(A,g,k,\mu)}([K]; \alpha_1, \cdots, \alpha_{k-1}, \alpha_k) = \alpha_k^*(A)\Psi^V_{(A,g,k-1,\mu)}([([\pi_\mu]^{-1}(K))]; \alpha_1, \cdots, \alpha_{k-1})
\]

where \( \alpha_k^* \) is the Poincaré dual of \( \alpha_k \).

**Proof:** The proof is similar to that of Proposition 2.5. Let

\[
\phi^k_\mu : \overline{U}^\mu_{g,k} \rightarrow \mathbb{P}^N; \quad \phi^{k-1}_\mu : \overline{U}^{\mu}_{g,k-1} \rightarrow \mathbb{P}^{N'}
\]

be projective embedding and \( \nu_k, \nu_{k-1} \) be inhomogeneous terms over \( \mathbb{P}^N \) or \( \mathbb{P}^{N'} \) respectively. Consider embedding

\[
\phi^k_\mu \times (\phi^{k-1}_\mu \circ \pi_\mu) : \overline{U}^\mu_{g,k} \rightarrow \mathbb{P}^N \times \mathbb{P}^{N'}.
\]

As in Remark 2.6, we can define the invariant \( \bar{\Psi} \) using \( \phi^k_\mu \times (\phi^{k-1}_\mu \circ \pi_\mu) \) and the inhomogeneous terms over \( \mathbb{P}^N \times \mathbb{P}^{N'} \) in the same fashion. One can show that such an invariant is independent of inhomogeneous term. Furthermore, we can view \( \nu_k, \nu_{k-1} \) (denoted by \( \bar{\nu}_k, \bar{\nu}_{k-1} \)) as inhomogeneous terms over \( \mathbb{P}^N \times \mathbb{P}^{N'} \) as in the proof of Proposition 2.5. Clearly, using \( (J, \bar{\nu}_k) \), we have

\[
\Psi^V_{(A,g,k,\mu)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]) = \Psi^V_{(A,g,k,\mu)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]).
\]

Using \( (J, \bar{\nu}_{k-1}) \), we claim that

\[
\Psi^V_{(A,g,k,\mu)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]) = \Psi^V_{(A,g,k-1,\mu)}(\pi_* [K]; \alpha_1, \cdots, \alpha_{k-1}).
\]

Suppose that \( (J, \nu_{k-1}) \) is generic. We claim that \( (J, \bar{\nu}_{k-1}) \) satisfies the Proposition 2.2.

The Proposition 2.3 (1) and (2) are obvious. A remark is required for (3). In the proof of (3) in the next section, the idea is to stratify \( \overline{M}_A^r(\mu, g, k, J, \nu) \) and show that each strata is of real codimension at least 2. In the proof, we use the fact that \( \nu \) is generic over each component of stable Riemann surface called the principal components. Note that \( \bar{\nu}_{k-1} \) is zero over the rational components \( \pi_* \) contracts. So it is not generic. However, we can simply treat these rational components as bubble components (see Definition 3.6), and construct another quotient space

\[
\overline{M}_A^r \rightarrow \overline{M}_A^r
\]

in the same way as we construct \( \overline{M}_A^r \). Then, the proof of Proposition 2.3 shows that

\[
\overline{M}_A^r(\mu, g, k, J, \nu) \rightarrow \overline{M}_A(\mu, g, k, J, \nu).
\]
is of real codimension at least 2.

Once \((J, \tilde{\nu}_{k-1})\) satisfies the Proposition 2.3, we can choose \((Y_i, F_i), (G, K)\) to satisfy (2.6), (2.7).

Then,

\[
\tilde{\Psi}_{(A, g, k, \mu)}(K; \alpha_1, \cdots, \alpha_k) = \sum_{i=1}^{m} \epsilon(f_i, s_i),
\]

where \((f_i, s_i)\) is given in (2.7). Suppose that \(s_i = (\Sigma, x_1, \cdots, x_k, y_1, \cdots, y_{k-1}, y_k)\). Then,

\[
(\Sigma, x_1, \cdots, x_k) \in \mathcal{M}_{g, k, I}
\]

and

\[
\pi(\Sigma, x_1, \cdots, x_k) = (\Sigma, x_1, \cdots, x_{k-1}).
\]

Clearly, \(f \in \mathcal{M}_{A}(\mu, g, k, J, \tilde{\nu}_{k-1})\) can also be viewed as an element of \(\mathcal{M}_{A}(\mu, g, k-1, J, \nu)\). If \(Y_k = V, F_k = Id\), let

\[
\tilde{s}_i = (\Sigma, x_1, \cdots, x_{k-1}, y_1, \cdots, y_{k-1}).
\]

Clearly

\[
(2.16) \quad (\mathcal{Y}_A \times \Xi_{g, k-1} \times \pi_\mu(K) \times \prod_{i=1}^{k-1} F_i)^{-1}(\Delta) = \{(f_1, \tilde{s}_1), \cdots, (f_m, \tilde{s}_m)\}.
\]

Furthermore, it is easy to check that

\[
\epsilon(f_i, s_i) = \epsilon(f_i, \tilde{s}_i).
\]

Therefore,

\[
\Psi^V_{(A, g, k, \mu)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]) = \Psi^V_{(A, g, k-1, \mu)}((\pi_\mu)_*[K]; \alpha_1, \cdots, \alpha_{k-1}).
\]

Here, we use the fact that \((G, \pi_\mu \circ K)\) represents the homology class \((\pi_\mu)_*[K]\).

The proof of (2) is similar.

Next, we discuss the composition law. Roughly speaking, the composition law governs the change of \(\Psi\) under the surgery of Riemann surfaces. Compared to \(k\)-reduction formula, one can view the composition law as \(g\)-reduction formula, i.e., the reduction of genus. As we mentioned in the introduction, its classical cousin in enumerative algebraic geometry is the degeneration formula, which was only derived individually in very special cases. One technical reason is that it is very difficult to have a good deformation theory in algebraic geometry. But our \(\Psi\) is just a symplectic invariant. The counterpart of deformation theory in symplectic category can be realized by a gluing theorem. It has been established by the authors in \[RT\].
Recall that in the definition of $\Psi_{(A,g,k,\mu)}([K]; \alpha_1, \cdots, \alpha_k)$, we require that $K$ does not lie in the boundary of $\overline{M}_{g,k}^\mu$. We first remove this technical assumption.

**Proposition 2.9.** $\Psi_{(A,g,k,\mu)}(K; \alpha_1, \cdots, \alpha_k)$ can be defined when $K$ is in the boundary of $\overline{M}_{g,k}^\mu$. Namely, $K \subset \text{Im} \theta_S$ or $K \subset \text{Im} \bar{\mu}$, where $\theta_S, \bar{\mu}$ are defined below. Furthermore, $\Psi_{(A,g,k,\mu)}(K; \alpha_1, \cdots, \alpha_k)$ depends only on the homology class represented by $(G,K)$. Then, we extend $\Psi_{(A,g,k)}([K]; \cdots)$ for any $[K] \in H_*(\overline{M}_{g,k}^\mu, Q)$ by the linearity.

The proof follows basically from the gluing theorem in [16], Theorem 6.1. The details will appear in Section 4.

Assume $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_1 + k_1 \geq 3$. Fix a decomposition $S = S_1 \cup S_2$ of $\{1, \cdots, k\}$ with $|S_i| = k_i$. Recall that $\theta_S : \overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} \to \overline{M}_{g,k}$, which assigns to marked curves $(\Sigma_i; x_i^1, \cdots, x_i^{k_1+1})$ ($i = 1,2$), their union $\Sigma_1 \cup \Sigma_2$ with $x_{k_1+1}$ identified to $x_1^2$ and remaining points renumbered by $\{1, \cdots, k\}$ according to $S$. Suppose that $\overline{M}_{g_1,k_1+1}^\mu, \overline{M}_{g_2,k_2+1}^\mu, \overline{M}_{g,k}^\mu$ are good covers over $\overline{M}_{g_1,k_1+1}, \overline{M}_{g_2,k_2+1}, \overline{M}_{g,k}$ such that

$$\theta_S^* \overline{M}_{g,k}^\mu = \overline{M}_{g_1,k_1+1}^\mu \times \overline{M}_{g_2,k_2+1}^\mu. \tag{2.17}$$

Such good covers can be constructed using the level-$n$ structure of $\overline{M}_{g,k}^\mu$. We have another natural map defined in the introduction $\mu : \overline{M}_{g-1,k+2} \to \overline{M}_{g,k}$ by gluing together the last two marked points. Let

$$\overline{M}_{g-1,k+2}^{\mu} = \mu^* \overline{M}_{g,k}^\mu, \quad \bar{\mu} : \overline{M}_{g-1,k+2}^{\mu} \to \overline{M}_{g,k}^\mu. \tag{2.18}$$

Choose a homogeneous basis $\{\beta_b\}_{1 \leq b \leq \ell}$ of $H_*(V, \mathbb{Z})$ modulo torsion. Let $(\eta_{ab})$ be its intersection matrix. Note that $\eta_{ab} = \beta_a \cdot \beta_b = 0$ if the dimensions of $\beta_a$ and $\beta_b$ are not complementary to each other. Put $(\eta^{ab})$ to be the inverse of $(\eta_{ab})$. Now we can state the composition law, which consists of two formulas.

**Theorem 2.10.** Let $[K_i] \in H_*(\overline{M}_{g_1,k_1+1}^\mu, Q)$ ($i = 1,2$) and $[K_0] \in H_*(\overline{M}_{g-1,k+2}^{\mu}, Q)$. Suppose that $\overline{M}_{g_1,k_1+1}^\mu, \overline{M}_{g_2,k_2+1}^\mu, \overline{M}_{g,k}^\mu, \overline{M}_{g-1,k+2}^{\mu}$ are defined as (2.17), (2.18). For any $\alpha_1, \cdots, \alpha_k$ in $H_*(V, \mathbb{Z})$. Then we have

$$\Psi_{(A,g,k,\mu)}(\theta_S^*[K_1 \times K_2]; \{\alpha_i\}) = \sum_{A=A_1+A_2} \sum_{i=1}^k \Psi_{(A_1,g_1,k_1+1,\mu)}([K_1]; \{\alpha_i\}_{i \leq k}, \beta_a) \eta^{ab} \Psi_{(A_2,g_2,k_2+1,\mu)}([K_2]; \beta_b, \{\alpha_j\}_{j > k}) \tag{2.19}$$

and

$$\Psi_{(A,g,k,\mu)}(\bar{\mu}_*[K_0]; \alpha_1, \cdots, \alpha_k) = \sum_{a,b} \Psi_{(A,g-1,k+2,\mu)}([K_0]; \alpha_1, \cdots, \alpha_k, \beta_a, \beta_b) \eta^{ab}. \tag{2.20}$$

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The proof of composition law essentially follows from Proposition 2.9 by some topological arguments. We postpone it to Section 4.

So far, we are working on the covers \( \overline{\mathcal{M}}_{g,k}^\mu \). To define invariants over \( \overline{\mathcal{M}}_{g,k} \), we introduce following classical notion in algebraic topology to relate homology of \( \overline{\mathcal{M}}_{g,k}^\mu \) to homology of \( \overline{\mathcal{M}}_{g,k} \).

**Definition 2.11.** Suppose that \( f : M \to N \) be a continuous map such that both \( M \) and \( N \) have Poincare duality. Then we define transfer map

\[
 f_! : H_*(M) \to H_*(N)
\]

by \( f_!(\alpha) = PD_M \circ f^* \circ PD_N^{-1}(\alpha) \), where \( PD_M, PD_N \) are Poincare duality maps. Furthermore, transfer maps compose functorially.

For our case, we use \( \mathbb{Q} \) as coefficients since Poincare duality only holds over rational coefficient.

If \( M \) is a finite cover of \( N \), it is easy to observe that

\[
 f_* f_!(\alpha) = \lambda \alpha,
\]

where \( \lambda \) is the order of covers.

Suppose that \( \lambda_{g,k}^\mu \) is the order of cover for \( p_\mu : \overline{\mathcal{M}}_{g,k}^\mu \to \overline{\mathcal{M}}_{g,k} \).

**Definition 2.12.** For any \( [K] \in H_*(\overline{\mathcal{M}}_{g,k}, \mathbb{Q}) \) and \( \{\alpha_i\} \in H_*(V, \mathbb{Z}) \), define

\[
 (2.21) \quad \Psi_{(A,g,k)}([K]; \{\alpha_i\}) = \frac{1}{\lambda_{g,k}^\mu} \Psi_{(A,g,k,\mu)}(\,(p_\mu)_!(\,[K]\,); \{\alpha_i\}) .
\]

**Lemma 2.13.** \( \Psi_{(A,g,k)}([K]; \{\alpha_i\}) \) is independent of \( \overline{\mathcal{M}}_{g,k}^\mu \).

**Proof:** Consider the fiber product

\[
 \overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,k}^\mu \times_{\overline{\mathcal{M}}_{g,k}} \overline{\mathcal{M}}_{g,k}' .
\]

Let \( p^1 : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_{g,k}^\mu; p^2 : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_{g,k}' \) be projections. Then, we can pull back the universal family \( \overline{\mathcal{U}}_{g,k}^\mu, \overline{\mathcal{U}}_{g,k}' \) by \( p^1, p^2 \). Both are obviously the universal family over \( \overline{\mathcal{M}} \). By the uniqueness of universal family, they must be the same. In other words, let \( \overline{\mathcal{U}} \) be the universal family over \( \overline{\mathcal{M}} \).

\[
 (2.22) \quad \overline{\mathcal{U}} = (p^1)_* \overline{\mathcal{U}}_{g,k}^\mu = (p^2)_* \overline{\mathcal{U}}_{g,k}' .
\]
Let
\[ \phi_\mu : U_{g,k} \rightarrow P^N; \phi_{\mu'} : U_{g,k} \rightarrow P^{N'} \]
be projective embedding. We have a natural embedding
\[ \phi = \phi_\mu \times \phi_{\mu'} : U \rightarrow P^N \times P^{N'} \]
By Remark 2.6, we can use this embedding to define inhomogeneous term and define the analogue of Ψ. Furthermore, we can show that such invariant is independent of the choice of a generic choice of inhomogeneous term. Let’s denote this invariant as \( \tilde{\Psi}_{(A,g,k)} \). We claim that
\[ \tilde{\Psi}_{(A,g,k)}((p^1 p_\mu)!([K]), \{\alpha_i\}) = \lambda_{g,k}^{\mu'} \Psi_{(A,g,k,\mu')}((p_\mu)!([K]), \{\alpha_i\}) \]
and
\[ \tilde{\Psi}_{(A,g,k)}((p^2 p_{\mu'})!([K]), \{\alpha_i\}) = \lambda_{g,k}^{\mu} \Psi_{(A,g,k,\mu)}((p_{\mu'})!([K]), \{\alpha_i\}) . \]
For any inhomogeneous term \( \nu \) over \( P^N \), we can again view it as an inhomogeneous term over \( P^N \times P^{N'} \) and denote it by \( \tilde{\nu} \). If \((J, \nu)\) is generic, so is \((J, \tilde{\nu})\). Choose \((H, L)\) to represent \((p^1 p_\mu)!([K])\) and together with \( Y \) satisfying (2.6), (2.7). Then, \((H, p^1 \circ L)\) represents \( p^1_s((p^1 p_\mu)!([K])) = \lambda_{g,k}^{\mu} (p_\mu)!([K]) \) since the order of cover \( p^1 : \overline{M} \rightarrow \overline{M}_{g,k} \) is \( \lambda_{g,k}^{\mu} \). It is straightforward to check that
\[ (\overline{Y}_A \times \overline{\Xi}^A_{g,k} \times F \times L)^{-1}(\Delta) = (\overline{Y}_A \times \overline{\Xi}^A_{g,k} \times F \times p^1 \circ L)^{-1}(\Delta). \]
Furthermore, the orientation matches. Then,
\[ \tilde{\Psi}_{(A,g,k)}((p^1 p_\mu)!([K]), \{\alpha_i\}) = \Psi_{(A,g,k,\mu)}(p_s((p^1 p_\mu)!([K]), \{\alpha_i\}) \]
\[ = \lambda_{g,k}^{\mu'} \Psi_{(A,g,k,\mu')}((p_{\mu'})!([K]), \{\alpha_i\}) . \]
It is the same argument to show that
\[ \tilde{\Psi}_{(A,g,k)}((p^2 p_{\mu'})!([K]), \{\alpha_i\}) = \lambda_{g,k}^{\mu} \Psi_{(A,g,k,\mu)}((p_{\mu'})!([K]), \{\alpha_i\}) . \]
On the other hand, \( p^1 p_\mu = p^2 p_{\mu'} \). Therefore,
\[ \frac{1}{\lambda_{g,k}^{\mu}} \Psi_{(A,g,k,\mu)}((p_\mu)!([K]), \{\alpha_i\}) = \frac{1}{\lambda_{g,k}^{\mu'}} \Psi_{(A,g,k,\mu')}((p_{\mu'})!([K]), \{\alpha_i\}) . \]
This finishes the proof.

**Proposition 2.14.**
(1) $\Psi^V_{(A,g,k)}(K, \alpha_1, \ldots, \alpha_k)$ is a symplectic invariant.

(2) $\Psi^V_{(A,g,k)}(K, \alpha_1, \ldots, \alpha_k)$ is independent of semi-positive deformations of $\omega$.

(3) $\Psi^V_{(A,g,k)} = 0$, if either $\omega(A) \geq 0$ or $C_1(V)(A) + (3 - n)(g - 1) < 0$, in particular, $\Psi_{(0,g,k)} = 0$ for any $g \geq 2$ and $n \geq 4$.

(4) $\Psi^V_{(A,g,k)} = 0$ is multilinear and supersymmetry on $H_* (V, \mathbb{Z})^k$ with respect to the $\mathbb{Z}_2$-grading by even and odd degrees.

The proof follows from the Proposition 2.4, 2.7 and the Definition 2.12.

**Proposition 2.15.** It suppose that $(g, k) \neq (0, 3), (1, 1)$.

1. For any $\alpha_1, \ldots, \alpha_{k-1}$ in $H_* (V, \mathbb{Z})$, we have

\[
\Psi^V_{(A,g,k)}(K; \alpha_1, \ldots, \alpha_{k-1}, [V]) = \Psi^V_{(A,g,k-1)}([\pi(K)]; \alpha_1, \ldots, \alpha_{k-1})
\]

2. Let $\alpha_k$ be in $H_{2n-2}(V, \mathbb{Z})$, then

\[
\Psi^V_{(A,g,k)}([\pi^{-1}(K)]; \alpha_1, \ldots, \alpha_{k-1}, \alpha_k) = \alpha_k^*(A)\Psi^V_{(A,g,k-1)}(K; \alpha_1, \ldots, \alpha_{k-1})
\]

where $\alpha_k^*$ is the Poincare dual of $\alpha_k$.

**Proof:** Notes that $\overline{\mathcal{M}}^\mu_{g,k+1} = \pi^* \overline{\mathcal{M}}^\mu_{g,k}$ Geometrically, $(p_\mu)_!(\pi^{-1}(K))$ is represented by $(p_\mu)^{-1}(K)$. By the construction of $\mathcal{M}^\mu_{g,k+1}$,

\[
\pi_\mu(p_\mu)^{-1}(K) = (p_\mu)^{-1}(\pi_\mu(K)).
\]

Hence,

\[
(p_\mu)_!(\pi^{-1}(K)) = (p_\mu)_!(\pi_\mu)_!(K).
\]

Then, (1) follows from Proposition 2.8 and (2.32).

For (2), $[\pi^{-1}((p_\mu)_!(\pi^{-1}(K)))] = \pi([p_\mu]_!([K]))$. By the natality of transfer map,

\[
(p_\mu)_! \pi^{-1}([\pi^{-1}(K))] = (p_\mu)_! \pi ([\pi_\mu^{-1}(K)]) = (\pi_\mu)_!(\pi^{-1}(K)) = (p_\mu)_!(\pi_\mu)_!(K).
\]

Then, (2) follows from the Proposition 2.8(2) and (2.33).

**Theorem 2.16 (Composition Law).** Let $[K_i] \in H_* (\overline{\mathcal{M}}_{g_i,k_i+1}; \mathbb{Q})$ ($i = 1, 2$) and $[K_0] \in H_* (\overline{\mathcal{M}}_{g-1,k+2}; \mathbb{Q})$.

For any $\alpha_1, \ldots, \alpha_k$ in $H_* (V, \mathbb{Z})$. Then we have

\[
\Psi^V_{(A,g,k)}(\theta_{S^*}[K_1 \times K_2]; \{\alpha_i\}) = \sum_{\substack{A = A_1 + A_2 \geq 6}} \sum_{a,b} \Psi^V_{(A_1,g_1,k_1+1)}([K_1]; \{\alpha_i\}_{i \leq k}, \beta_a) \eta^{ab} \Psi^V_{(A_2,g_2,k_2+1)}([K_2]; \beta_b, \{\alpha_j\}_{j > k})
\]
(2.35) \[ \Psi^V_{(A,g,k)}(\mu_*[K_0]; \alpha_1, \cdots, \alpha_k) = \sum_{a,b} \Psi^V_{(A,g-1,k+2)}([K_0]; \alpha_1, \cdots, \alpha_k, \beta_a, \beta_b)\eta^{ab} \]

**Proof:** Let

\[ p'_\mu : \mathcal{M}^\mu_{g,k} \to \mathcal{M}^\mu_{g,k}, \quad p_{g-1,k+2}' : \mathcal{M}^\mu_{g-1,k+2} \to \mathcal{M}_{g-1,k+2}. \]

By the Proposition 2.10,

\[ \Psi^V_{(A,g,k)}(\theta_\ast_*[K_1 \times K_2]; \{\alpha_i\}) = \frac{1}{\chi_{g,k}^*} \Psi^V_{(A,g,k,\mu)}(\theta_\ast_*([K_1]) \times \theta_\ast_*([K_2]); \{\alpha_i\}) \]

\[ = \sum_{A=A_1+A_2} \sum_{a,b} \frac{1}{\chi_{g_1,k_1+1}^* \chi_{g_2,k_2+1}^*} \Psi^V_{(A_1,g_1,k_1+1,\mu)}([K_1]; \{\alpha_i\}_{i \leq k}, \beta_a) \]

\[ \eta^{ab} \Psi^V_{(A_2,g_2,k_2+1,\mu)}([K_2]; \beta_b, \{\alpha_j\}_{j > k}) \]

It is the same argument for (2).

**Remark 2.17:** The mixed invariant \( \Phi_{(A,\omega,g)} \) in [RT] can be identified with certain \( \Psi \) by choosing appropriate cycles \([K]\). More precisely, for any \( k, l \geq 0 \) with \( 2g + k \geq 3 \), put \( K_{k,l} \) to be the closure of the cycle \( K_{k,l}^0 \) in \( \mathcal{M}_{g,k,l} \), where \( K_{k,l}^0 \) is the set of all \((\Sigma, x_1, \cdots, x_{k+l})\) in \( \mathcal{M}_{g,k,l} \) with \((\Sigma, x_1, \cdots, x_k)\) being a fixed point in \( \mathcal{M}_{g,k} \). Then

\[ \Psi^V_{(A,g,k+l)}(K_{k,l}; \alpha_1, \cdots, \alpha_{k+l}) = \Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k | \alpha_{k+1}, \cdots, \alpha_{k+l}) \]

It follows from the Proposition 2.5 that \( \Phi_{(A,\omega,g)}(\alpha_1, \cdots | \cdots, \alpha_{k+l}) \) is 0 if \( \dim(\alpha_{k+l}) > 2n - 2 \) (cf. [RT]).

Recall that \( \Psi \) is only defined for so call “generic \((J,\nu) \in \mathcal{H}\)”. Following from [RT], we can relax this genericity condition as follows:

**Definition 2.18.** We call \((J,\nu) \) to be \( A \)-good if the following two conditions are satisfied.

(1) The set \( \{ f \in \mathcal{M}_A(g,k,J,\nu), | \text{Coker}(Jdf \oplus D_f) \neq 0 \} \) is of real codimension 2, where \( D_f \) is the linearization of the inhomogeneous Cauchy-Riemann equation for \((J,\nu)\)-maps at \( f \).

(2) \( \mathcal{M}_A(g,k,J,\nu) \subset \mathcal{M}_A(g,k,J,\nu) \) is of real codimension 2.

One can see from the construction that \( \Psi \) is well defined if \((J,\nu) \) is \( A \)-good.
Remark 2.19 (Relation of $\Psi$ to enumerative function): One of main applications of the composition law of the genus-0 GW-invariant $\Psi^V_{(A,0,k)}$ is to compute the enumerative invariants of rational curves in complex homogeneous spaces. In [RT], Lemma 10.1, the authors proved the equivalence of $\Psi^V_{(A,0,k)}$ with the enumerative function of rational curves on complex Grassmannian manifolds. More precisely, we showed that for Grassmannian manifolds, the integrable complex structure $J_0$ with zero perturbation term satisfies the condition of Proposition 2.3, i.e., $(J_0,0) \in \mathcal{H}$. Therefore, one doesn’t need to deform the integrable complex structure or add any perturbation terms. The integrable complex structure is already $A$-good for calculating the invariant $\Psi^V_{(A,0,k)}$. Then, by the definition, $\Psi^V_{(A,0,k)}$ is an enumerative function.

The same has been proved by Jun Li and the second author in [LT] for any complex homogeneous manifolds (also see [CF] for an alternative proof for Flag manifolds).

In contrast to the genus 0 case, $\Psi^V_{(A,g,k)}(g \geq 1)$ is not the same as the enumerative function even for the projective plane $\mathbb{P}^2$. For example, a simple computation through the composition law will yield the mixed invariant

$$\Psi^{\mathbb{P}^2}_{([L],1,3)}([K_3,0];[L],[L],[L]) = 3, \Psi^{\mathbb{P}^2}_{([L],1,9)}([K_{1,8};[L],[pt],\cdots,[pt]) = 27,$$

where $L$ is a line in $\mathbb{P}^2$ and $pt$ is a point. The first number is supposed to represent the number of degree 1 elliptic curves with fixed $j$-invariant mapping three marked points to three distinct lines. But it is well-known that there is no degree 1 elliptic curves at all in $\mathbb{P}^2$. The second number is supposed to represent the number of degree three elliptic curves with fixed $j$-invariant passing through 8 points. It is known in classical algebraic geometry that such a number should be 12. What happen was that for the integrable complex structure $J_0$ on $\mathbb{P}^2$, the boundary of the Gromov- Uhlenbeck compactification $\overline{\mathcal{M}}_A(1,k,J_0,0) - \mathcal{M}_A(1,k,J_0,0)$ has a component whose dimension is larger than the dimension of $\mathcal{M}_A(1,k,J_0,0)$ itself. Such a component consists of the maps from the union of an elliptic curve and a rational curve to $\mathbb{P}^2$, which map the elliptic curve to a point. The effect of considering the inhomogeneous Cauchy Riemann equation $\bar{\partial}_J f = \nu$ instead of the homogeneous Cauchy Riemann equation is to perturb away all those maps. In the process, we creat finitely many $(J,\nu)$-maps, which provides the correct account of the contribution of the component described above. Only adding those contributions, our invariants will satisfy the composition law, while the classical enumerative invariants do not. In fact, the composition law of the mixed invariant we proved in [RT] computes all the mixed invariants of any genus for $\mathbb{P}^N$ and many other Fano manifolds. It is still a major problem how to use it to compute the enumerative
invariants. Recall that to define the mixed invariants, we fix the complex structure on the Riemann surfaces. If we allow the complex structure of Riemann surfaces to vary, it is not clear how the composition law will even help to compute the enumerative invariants.

3 Compactification and Transversality

In this section, we discuss the structures of the moduli space $\mathcal{M}_A(\mu, g, k, J, \nu)$ and the certain quotient of its Gromov-Uhlenbeck compactifications $\overline{\mathcal{M}}_A^r(\mu, g, k, J, \nu)$. The problems we will discuss here are smoothness, the orientation of $\mathcal{M}_A(\mu, g, k, J, \nu)$ and the stratification of $\overline{\mathcal{M}}_A^r(\mu, g, k, J, \nu)$. By the nature of those questions, this section is rather technical. For the readers who only wish to get a sense of big picture, one can skip over this section. On the other hand, if reader wish to have a good understanding about the results in this paper, the properties we discuss in this section are crucial or the proof of all the results in Section 2, which will be provided in the Section 4. This section is roughly divided as two parts. In the first part, we prove the smoothness of $\mathcal{M}_A(\mu, g, k, J, \nu)$ and construct its canonical orientation. The idea of their proof is quite standard. For the smoothness, the basic tool is the Sard-Smale Transversality Theorem. We refer the readers to [M1], [R3], [M2] in the case of the Cauchy-Riemann equation. In both [R3] and [M2], the argument relies on a cumbersome norm on the space of tamed almost complex structures defined by Floer. Because the case (for the inhomogeneous Cauchy-Riemann equation) here didn’t appear in the literature, we will include an outlined proof. For the orientation, the genus 0 case was also due to McDuff [M1]. But the treatment we follow is that of the first author [R] (see also 4.12 of [RT]), which was in the spirit of Donaldson’s treatment of the orientation problem in the gauge theory. The second part will be devoted to prove Proposition 2.3, which is similar to Section 3, 4, [RT].

The idea of applying the Sard-Smale Transversality Theorem to the moduli problem was due to Freed-Uhlenbeck [FU]. Recall that the Sards-Smale Theorem says that if $X, Y$ are Banach manifolds and $\mathcal{F} : X \rightarrow Y$ is a Fredholm map of index $k$, then the set $Y_{reg}$ of regular values of $\mathcal{F}$ is of the Baire second category, provided that $\mathcal{F}$ is sufficiently differentiable. Recall that $y \in Y$ is called a regular value, if the derivative $D\mathcal{F}(x) : T_x X \rightarrow T_y Y$ is surjective at any $x$ with $\mathcal{F}(x) = y$. It then follows from the Implicit Function Theorem that $\mathcal{F}^{-1}(y)$ is $k$-dimensional manifold for every $y \in Y_{reg}$.

One obvious problem is that $\mathcal{M}^\mu_{g, k}$ may not be smooth. But it can be stratified by smooth
manifolds. Notes that $\mathcal{M}_{g,k}$ has a stratification parameterized by the automorphism group of Riemann surfaces. Namely, one can write

$$\mathcal{M}_{g,k} = \sum_{\alpha \in I} T_{g,k}^\alpha,$$

where each smooth strata $T_{g,k}^\alpha$ consists of the stable Riemann surfaces of a fixed automorphism group $\kappa$. Without the loss of generality, we can assume that

$$\mathcal{M}_{\mu,k}^\mu = \sum_{\alpha \in I} T_{\mu,k}^{\mu,\kappa},$$

where $T_{g,k}^{\mu,\kappa} = p_\mu^{-1}(T_{g,k}^\kappa)$ is smooth.

Using the same arguments as in the smooth case, we will establish the transversality theory for each stratum $T_{\mu,k}^{\mu,\kappa}$. It is rather straightforward. The precise structure of $T_{g,k}^{\mu,\kappa}$ is not needed. One only has to know that each $T_{g,k}^{\mu,\kappa}$ is smooth.

Let $\mathcal{M}_A(\mu, g, k, J, \nu)_\kappa$ consist of $(J, \nu)$-maps $f$ such that the domain of $f$ has automorphism group $\kappa$. We shall prove that

**Theorem 3.1.** There is a set $\mathcal{H}_{reg}$ of Baire second category among all the smooth pairs $(J, \nu)$ such that for any $(J, \nu)$, $\mathcal{M}_A(\mu, g, k, J, \nu)_\kappa$ is a smooth manifold of dimension $2c_1(V)(A) - 2n(g - 1) + \dim T_{g,k}^{\mu,\kappa}$.

Fix a smooth topological surface $\Sigma_g$ of genus $g$. Our basic topological object is

$$\text{Map}_A(\Sigma_g, V) = \{ f : \Sigma_g \to V \text{ such that } f \text{ is smooth and } f_*[\Sigma_g] = A \}.$$ 

To apply the Sard-Smale theorem, we need to put some Sobolev norm on $\text{Map}_A(\Sigma_g, V)$, so that it has a structure of Banach manifold.

To specify a Sobolev norm, we choose a smooth family of metrics on $\Sigma_g$, parameterized by the elements of $T_{g,k}^{\mu,\kappa}$. For example, one can choose a projective embedding

$$\phi_\mu : T_{g,k}^{\mu,\kappa} \to \mathbb{P}^N$$

in section 2 and consider the restriction of Fubini-Study metric on $\phi_\mu p_\mu^{-1}(j)$ for each $j \in T_{g,k}^{\mu,\kappa}$. Then for each $j \in T_{g,k}^{\mu,\kappa}$, $j$ defines a Sobolev $L^p_{m,j}$-norm on $\text{Map}_A(\Sigma_g, V)$. Its completion under this norm is a smooth Banach manifold if $pm > 2$. We shall also use $j$ to denote the complex structure of the underlying Riemann surface. When $j$ varies in $T_{g,k}^{\mu,\kappa}$,

$$\chi^{p,m}_{(A,\kappa, g,k)} = \bigcup_{j \in T_{g,k}^{\mu,\kappa}} L^p_{m,j}(\text{Map}_A(\Sigma_g, V)) \times \{ j \}$$
is a smooth Banach manifold as well since $\mathcal{T}^{m,k}$ is smooth. Obviously, there is a map $\chi^{m,p}_{(A,\kappa,g,k)} \to \mathcal{T}^{m,k}_{g,k}$. Let $\mathcal{H}^l$ be the completion of $\mathcal{H}$ the space of all smooth pairs $(J, \nu)$ under $C^l$-topology. Then, $\mathcal{H}^l$ is a smooth Banach manifold. Consider the universal moduli space

$$\mathcal{M}_A^l(\kappa, g, k) = \{(f, j, J, \nu) \in \chi^{m,p}_{(A,\kappa,g,k)} \times \mathcal{H}^l; \bar{\partial}_j f(x) = \nu(\phi(x), f(x))\}.$$ 

When $p > 2, 1 \leq m \leq l$, by the elliptic regularity, $\mathcal{M}_A^l(\kappa, g, k)$ is independent of $m, p$.

**Proposition 3.2.** For every $A \in H_2(V, \mathbb{Z})$ and $g \geq 0, l \geq 1$, the universal moduli space $\mathcal{M}_A^l(\kappa, g, k)$ is a smooth Banach manifold.

**Proof:** There is an infinite dimensional vector bundle

$$\mathcal{E}^{m-1,p}_{(f,j,J,\nu)} \to \chi^{m,p}_{(A,\kappa,g,k)} \times \mathcal{H}^l,$$

where the fiber $\mathcal{E}^{m-1,p}_{(f,j,J,\nu)} = W^{m-1,p}(\Lambda^0 T^* \Sigma_g \otimes J f^* TV)$. The perturbed holomorphic equation defines a section of this bundle by

$$\mathcal{F} : \chi^{m,p}_{(A,\kappa,g,k)} \times \mathcal{H}^l \to \mathcal{E}^{m-1,p}_{(f,j,J,\nu)}, \quad F(f, j, J, \nu)(x) = \bar{\partial}_j f(x) - \nu(\phi(x), f(x)).$$

Notice that the definition of $\bar{\partial}_j$ depends on the complex structure $j$ on $\Sigma_g$. Then, it is enough to show that $\mathcal{F}$ is transverse to the zero section. Suppose $\Sigma_j = (\Sigma, x_1, x_2, \ldots, x_k)$. Let $T \Sigma_j = T \Sigma \otimes \bigoplus_{i=1}^k O(-x_i)$. Then, $T_j \mathcal{T}^{m,k}_{g,k} = H^{0,1}_{(\kappa,j)}(T \Sigma_j)$-space of $\kappa$ invariant $(0,1)$-forms. Notice that one can also identity

$$H^{0,1}_{(\kappa,j)}(T \Sigma_j) = (H^{0,1}_{(\kappa,j)}(T \Sigma)) \oplus \bigoplus_{i=1}^k T_{x_i} \Sigma^\kappa,$$

where $(.)^\kappa$ means the $\kappa$-invariant subspace.

Let $\mathcal{F}(f, j, J, \nu) = 0$. We have

$$T_{(f,j)} \chi^{m,p}_{(A,\kappa,g,k)} = W^{m,p}(\Lambda^0 f^* TV) \oplus H^{0,1}_{(\kappa,j)}(T \Sigma_j);$$

$$T_{(j,\nu)} \mathcal{H}^l = C^l(End(TV, J)) \oplus C^l(\overline{\text{Hom}}_J(T \mathbb{P}^N, TV)),$$

where $End(TV, J) = \{ Y : TV \to TV; YJ + JY = 0 \}$. Furthermore, $\overline{\text{Hom}}_J(T \mathbb{P}^N, TV)$ is the space of anti-complex linear homomorphism with respect to the complex structure. There is a natural identification

$$\overline{\text{Hom}}_J(T \mathbb{P}^N, TV)|_{\Gamma_f} = \Omega^{0,1}_J(f^* TV),$$

where $\Gamma_f \subset \Sigma_g \times V \subset \mathbb{P}^N \times V$ is the graph of $f$. 

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Now, we shall show the surjectivity of the differential

\[(3.9)\]
\[DF(f, j, J, \nu) : W^{m,p}(\Lambda^0 f^*TV) \oplus H^{0,1}_{(\kappa,j)}(T\Sigma_j) \oplus C^l(End(TV, J)) \oplus C^l(\text{Hom}_J(T\mathbf{P}^N, TV)) \rightarrow W^{m-1,p}(\Lambda^{0,1}_J T^*\Sigma \otimes_J f^*TV).\]

An easy computation yields

\[(3.10)\]
\[DF(f, j, J, \nu)(\xi, s, Y, X) = D_f \xi + J \circ df \circ s + f^*Y \circ df \circ j - X|_{\Gamma_f},\]

where

\[(3.11)\]
\[D_f : W^{m,p}(\Lambda^0 f^*TV) \rightarrow W^{m-1,p}(\Lambda^{0,1}_{J,f}(f^*TV))\]

is the linearization of the Cauchy-Riemann equation at \(f\).

By the elliptic regularity theory, a \((J, \nu)\)-map \(f\) is in \(W^{l+1,q}\) for any \(q > 0\) if \((J, \nu)\) is in \(C^l\). It follows that \(D_f\) in (3.11) is well-defined. Moreover, it follows that the cokernel of \(D_f\) is contained in \(C^l(\Lambda^{0,1}_{J,f}(f^*TV))\). Since \(D_f\) is elliptic, its cokernel is of finite dimension. However, the map

\[(3.12)\]
\[X|_{\Gamma_f} : C^l(\text{Hom}_J(T\mathbf{P}^N, TV) \rightarrow C^l(\Lambda^{0,1}_{J,f}(f^*TV))\]

is surjective (here we also use the fact that \(f\) is in the space \(C^l\)). Therefore, by (3.10), \(DF(f, j, J, \nu)\) is surjective.

By the Implicit Function Theorem, we conclude that the universal moduli space \(M^l_A(\kappa, g)\) is a smooth Banach manifold.

**Proof of Theorem 3.1:** Based on Proposition 3.2, the proof is just a standard application of the Sard-Smale Theorem. For the reader’s convenience, we outline the arguments here.

Consider the projection

\[(3.13)\]
\[\pi : M^l_A(\kappa, g, k) \rightarrow \mathcal{H}^l\]

as a map between the Banach manifolds. The tangent space \(T_{(f,j,J,\nu)}M^l_A(\kappa, g, k)\) consists of \((\xi, s, Y, X)\) such that

\[(3.14)\]
\[D_f \xi + J \circ df \circ s + f^*Y \circ df \circ j - X|_{\Gamma_f} = 0.\]

The derivative

\[d\pi : T_{(f,j,J,\nu)}M^l_A(\mu, g) \rightarrow T_{(J,\nu)}\mathcal{H}^l\]
is just the projection to \((Y, X)\) factors. One can show that \(d\pi\) is a Fredholm operator whose kernel is isomorphic to the kernel of \(Df \oplus J \circ df\) and has the same index as that of \(Df \oplus J \circ df\), where

\[(3.15) \quad J \circ df : H^0_{(\nu,j)}(T\Sigma_j) \to W^{m-1,p}(\Lambda^0_{J,j}(f^*TV)).\]

Hence, the operator \(d\pi\) is onto precisely when \(Df \oplus J \circ df\) is onto for any \((J, \nu)\)-map \((f, j)\) in \(\mathcal{M}_A(\mu, g, k, J, \nu)\), where \((3.15)\)

\[\mathcal{H}^l_{\text{reg}} = \{(J, \nu) \in \mathcal{H}^l ; Df \oplus J \circ df \text{ is onto for all } (f, j) \in \mathcal{M}_A(\mu, g, k, J, \nu)\}\]

is precisely the set of the regular values of \(\pi\). By the Sard-Smale Theorem, this set is of the second category. Thus we have proved that \(\mathcal{H}^l_{\text{reg}}\) is dense in \(\mathcal{H}^l\) with respect \(C^l\)-topology. Then one can easily deduce that \(\mathcal{H}^l_{\text{reg}}\) is of the second category in \(\mathcal{H}\) with respect to \(C^\infty\) topology.

Let \(T^{\mu,k}_{g,k} = \bigcup_{K=1}^{\infty} N_K\), where \(N_K\) is compact and \(N_K \subset N_{K+1}\). Consider the set \(\mathcal{H}_{\text{reg},K} \subset \mathcal{H}\) of all smooth \((J, \nu)\) with the property that the operator \(Df \oplus J \circ df\) is onto for any \((f, j)\) satisfying:

\[||df||_{L^\infty} < K \quad \text{and} \quad j \in N_K.\]

Clearly,

\[(3.17) \quad \mathcal{H}_{\text{reg}} = \bigcap_{K>0} \mathcal{H}_{\text{reg},K}.\]

Similarly, we can define \(\mathcal{H}^l_{\text{reg},K}\).

We claim that \(\mathcal{H}_{\text{reg},K}\) is open and dense in \(\mathcal{H}\) with respect to the \(C^\infty\) - topology.

The openness is clear. It remains to prove that \(\mathcal{H}_{\text{reg},K}\) is dense in \(\mathcal{H}\) with respect to \(C^\infty\)-topology. Note that \(\mathcal{H}_{\text{reg},K} = \mathcal{H}^l_{\text{reg},K} \cap \mathcal{H}\). Then \(\mathcal{H}^l_{\text{reg},K}\) is open in \(\mathcal{H}^l\) with respect to \(C^l\)-topology. Since \(\mathcal{H}^l_{\text{reg}} \subset \mathcal{H}^l_{\text{reg},K}\) and \(\mathcal{H}^l_{\text{reg}}\) is dense in \(\mathcal{H}^l\) with respect to \(C^l\)-topology, so is \(\mathcal{H}^l_{\text{reg},K}\). It follows that \(\mathcal{H}_{\text{reg},K}\) is dense in \(\mathcal{H}\) with respect to \(C^\infty\)-topology. Notice that \(\mathcal{H}_{\text{reg}}\) is an intersection of countable open dense subsets, so it is of second category.

The dimension formula follows from the Riemann-Roch Theorem.

**Theorem 3.3.** For any \((J, \nu), (J', \nu') \in \mathcal{H}_{\text{reg}}\), there is a second category set of paths \(\mathcal{H}((J, \nu), (J', \nu'))\) connecting \((J, \nu), (J', \nu')\) among the set of all such smooth paths such that for any path \((J_t, \nu_t) \in \mathcal{H}((J, \nu), (J', \nu'))\)

\[(3.18) \quad \mathcal{M}_A(\mu, g, k, (J_t, \nu_t)) = \bigcup_{t \in [0,1]} \mathcal{M}_A(\mu, g, k, J_t, \nu_t) \times \{t\}\]

is a smooth cobordism.

The proof is identical to that of 3.1. We omit it.
**Remark 3.4:** In the case of homogeneous Cauchy-Riemann equation \((\nu = 0)\), one can use the Teichmüller space \(T_{g,k}\) (which is smooth) in the place of \(T^{\mu,\kappa}_{g,k}\). Moreover, there is no need to consider finite covers. Let \(T^*_A(g,k,J,0)\) be the set of \((f,\lambda) \in \text{Map}_A(\Sigma,g) \times T_{g,k}\) such that \(f\) is a \((J,0)\) map for the complex structure induced by \(\lambda\) but not a multiple cover of another \((J,0)\) map. The same argument implies that there is \(H_{\text{reg}}\) of second category among all the tamed almost complex structure such that for any \(J \in H_{\text{reg}}\), \(T^*_A(g,k,J,0)\) is smooth. The mapping class group \(G_g\) acts freely on \(T^*_A(g,k,J,0)\). Hence,\[(3.19)\]

\[T^*_A(g,k,J,0)/G_g \subset \mathcal{M}_A(g,k,J,0)\]

is smooth. In our case, the inhomogeneous Cauchy-Riemann equation is not preserved under the action of mapping class group. Therefore, we have to consider the smoothness for each strata of \(\mathcal{M}_g\). On the other hand, our inhomogeneous equation can handle the multiple covered map, which can not be handled by the homogeneous equation except dimension 4 \([R3]\).

Next, we construct the canonical orientation of \(\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}\) and \(\mathcal{M}_A(\mu, g, k, (J_t, \nu_t))_{\kappa}\). The construction is identical to that in \([R]\) (3.3.1) and \([RT]\) (4.12).

**Theorem 3.5.** There is a canonical orientation over \(\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}\) and \(\mathcal{M}_A(\mu, g, k, (J_t, \nu_t))_{\kappa}\).

**Proof:** Recall that the linearization of \(F\) at \((f,j) \in \mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}\) is

\[D_f \oplus J \cdot df : \Omega^0(f^*TV) \times H^{0,1}_{\kappa,j}(T\Sigma_j) \to \Omega^{0,1}(f^*TV).\]

The tangent space \(T_{f,j}\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa} = \text{Ker}(D_f \oplus J \cdot df)\). Its determinant is

\[\text{det}(T\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}) = \text{det}(D_f \oplus J \cdot df),\]

which is defined over \(\text{Map}_A(\Sigma,V) \times T^{\mu,\kappa}_{g,k}\). As usual, an orientation of \(\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}\) is just a nowhere vanishing section of \(\text{det}(T\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa})\) up to multiplication by positive functions. We shall omit “up to multiplication by positive functions” if there is no confusion. Therefore, to construct a canonical orientation of \(\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}\), it is enough to construct a canonical section of \(\text{det}(D_f \oplus J \cdot df)\) over the whole \(\text{Map}_A(\Sigma,V) \times T_{\kappa,g}\). We define\[(3.20)\]

\[D^J_f = \frac{1}{2}(D_f - J \cdot D_f \cdot J)\]

Clearly, it is \(J\)-linear. Moreover, we have\[(3.21)\]

\[D_f = D^J_f + Z_f,\]

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where $Z_f$ is the zero order term. Let

$$D_{f,t} = D_f^t + tZ_f.$$  \hspace{1cm} (3.22)

Then, $det(D_{f,t} \oplus J \cdot df)$ is isomorphic to $det(D_{f,0} \oplus J \cdot df)$. Hence, $det(D_f \oplus J \cdot df)$ is isomorphic to $det(D_f^t \oplus J \cdot df)$.

On the other hand, both $ker(D_f^t \oplus J \cdot df)$ and $coker(D_f^t \oplus J \cdot df)$ are complex vector spaces. Therefore, there is a canonical section of the determinant line bundle $det(D_f \oplus id)$ corresponding this complex structure.

Similarly, one can construct a canonical orientation on $M_A(\mu, g, k, (J, \nu))_\kappa$.

As the oriented manifolds, we have

$$M_A(\mu, g, k, J_0, \nu_0)_\kappa \times \{0\} \cup M_A(\mu, g, k, J_1, \nu_1)_\kappa \times \{1\} = M_A(\mu, g, k, J_0, \nu_0)_\kappa \cup - M_A(\mu, g, k, J_1, \nu_1)_\kappa,$$

where “-” means the opposite orientation. Let

$$M_A(\mu, g, k, J, \nu) = \bigcup_{\kappa} M_A(\mu, g, k, J, \nu)_\kappa.$$  

In the second half of this section, we focus on the compactification of $M_A(\mu, g, k, J, \nu)$.

**Definition 3.6 ([PW], [Ye], [Ko]).** Let $(\Sigma, \{x_i\})$ be a stable Riemann surface. A stable map (associated with $(\Sigma, \{x_i\})$) is an equivalence class of continuous maps $f$ from $\Sigma'$ to $V$ which are smooth at smooth points of $\Sigma'$, where the domain $\Sigma'$ is obtained by joining chains of $\mathbf{P}^1$'s at some double points of $\Sigma$ to separate the two components, and then attaching some trees of $\mathbf{P}^1$'s. We call components of $\Sigma$ principal components and others bubble components. Furthermore,

1. If we attach a tree of $\mathbf{P}^1$ at a marked point $x_i$, then $x_i$ will be replaced by a point different from intersection points on some component of the tree. Otherwise, the marked points do not change.

2. The singularities of $\Sigma'$ are normal crossing and there are at most two components intersecting at one point.

3. If the restriction of $f$ on a bubble component is constant, then it has at least three special points (intersection points or marked points). We call this component a ghost bubble [PW].

4. For each principal component, the restriction of $f$ is a $(J, \nu)$-map.
(5) The restriction of $f$ to each bubble is $J$-holomorphic.

Two such maps are equivalent if one is the composition of the other with an automorphism of bubble components fixing the special points.

Evidently, the equivalence relation is trivial unless some bubble component has one or two special points. The restriction of $f$ to each component carries a homology class. We shall use $[f]$ to denote the summation of all those homology classes.

Remark 3.7: The terminology stable maps was first used by Kontsevich and Manin in [KM]. It had appeared before in Parker-Wolfson-Ye’s proof of Gromov-Uhlenbeck compactness theorem under the name cusp curves [PW], [Ye]. Later, it was introduced to algebraic geometry by Kontsevich and Manin and known more commonly as stable maps. Here, we follow their terminology.

Then Theorem 3.1 of [RT] (see also [PW] and [Ye]) can be restated as follows:

**Theorem 3.8.** Let $f_m \in \mathcal{M}_A(\mu, g, k, J, \nu)$. Suppose that the domain $(\Sigma_m, \{x^m_i\})$ of $f$ converges to a stable Riemann surface $(\Sigma, \{x_i\})$ in the sense of Deligne-Mumford. Then, there is a subsequence $\{f_{m_t}\}$ “weakly converging” to a stable map $f$ (associated with $(\Sigma, \{x_i\})$) such that $[f] = A$. Here, by the weak convergence, we mean that the image of $f_{m_t}$ converges to the image of $f$ in the Hausdorff topology.

Strictly speaking, Proposition 3.1 of [RT] only proves the version of Theorem 3.7 without marked points. But one can easily keep track of marked points in the proof and deduce Theorem 3.7 as we stated.

We denote the space of stable maps with fundamental class $A$ by $\overline{\mathcal{M}}_A(\mu, g, k, J, \nu)$. Clearly,

\[(3.23) \quad \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \supset \mathcal{M}_A(\mu, g, k, J, \nu).\]

One can easily deduce from Theorem 3.7 the following:

**Corollary 3.9.** $\overline{\mathcal{M}}_A(\mu, g, k, J, \nu)$ is compact in the Hausdorff topology. Moreover, the evaluation map $e_i$ extends to a continuous map from $\overline{\mathcal{M}}_A(\mu, g, k, J, \nu)$.

We shall call $\overline{\mathcal{M}}_A(\mu, g, k, J, \nu)$ GU-compactification of $\mathcal{M}_A(\mu, g, k, J, \nu)$, since Gromov and Uhlenbeck first studied the compactness problem for harmonic maps and pseudo-holomorphic curves.

**Definition 3.10.** We call $f$ a reduced GU-map if $f$ satisfies (1), (4), (5) of Definition 3.6. Furthermore, it satisfies:
(2') The singularities of Σ' are of normal crossing, but it could have three or more components intersecting at one point;

(3') There are no ghost bubbles;

(6) There are no bubble components which are multiple covering maps;

(7) There are no subtrees of the bubbles whose components have the same image.

For any stable map, we can define a GU-map (with possibly different fundamental class) as follows: (i) we collapse the ghost bubbles; (ii) we replace each multiple covered bubble component by its image; (iii) we collapse each subtree of the bubbles whose components have the same image.

Clearly, this reduction may destroy the property (2) of the Definition 3.6, but still preserve the property (2'). Also in this reduction, the fundamental class may change.

Define \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \) to be the quotient of \( \mathcal{M}_A(\mu, g, k, J, \nu) \) by the above reduction. Furthermore, we define the topology on \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \) as the quotient topology.

By the definition, \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \) is a union of GU-maps with possibly different fundamental classes. We will prove the following structure theorem.

**Theorem 3.11.** Let \((V, \omega)\) be a semi-positive symplectic manifold. There is a set \( \mathcal{H}_{reg} \) of Baire second category among all the smooth pairs \((J, \nu)\) such that for any \((J, \nu) \in \mathcal{H}_{reg}, \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) - \mathcal{M}_A(\mu, g, k, J, \nu) \) consists of finitely many strata, such that each stratum is a smooth manifold of real codimension at least 2.

**Proof of Proposition 2.3.** (1) follows from Theorem 3.1. (2) is obvious (cf. Corollary 3.9).

By the construction of \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \), both \( \Upsilon \) and \( \Xi^A_{g,k} \) descend to \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \). Then (3) follows from Theorem 3.11.

**Remark 3.12:** One may ask whether or not \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \) carries a fundamental class. This is indeed the case if \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) \) admits a real analytic structure. These will be established in [RT1] by more delicate analysis. Then one can directly use the GU-compactification to prove Proposition 2.3.

In the rest of this section, we outline the proof of Theorem 3.11. The proof is identical to Section 4 of [RT]. We refer the readers to [RT] for certain details.

First we shall decompose \( \overline{\mathcal{M}}_A(\mu, g, k, J, \nu) - \mathcal{M}_A(\mu, g, k, J, \nu) \) into strata. A stratum is the set of GU-maps (possibly with total homology class different from \( A \)) satisfying: (1) their domains
with marked points are of the same homeomorphic type; (2) Each connected component carries a fixed homology class. Furthermore, for technical reasons, we need to specify those bubble components, which have the same image even though they may not be adjacent to each other, and their intersection points having the same image. Therefore, the strata of $\overline{\mathcal{M}}_{A}(\mu, g, k, J, \nu)$ are indexed by data: (i) homeomorphism type of the domain of GU-maps with marked points; (ii) a homology class associated to each component; (iii) a specification of components with the same image and their intersection points with the same image. We denote by $D$ a set of those three data. Let $\mathcal{D}_{g,k}$ be the collection of such $D$’s. Note that when we drop the multiplicity from a multiple covering map, we change the homology class. However it is still $A$-admissible in the following sense:

**Definition 3.13.** Let $D$ be given as above. We define $[D]$ to be the sum of homology classes of components in (ii). Let $P_1, \cdots, P_o$ be principal components and $B_1, \cdots, B_p$ be bubble components of $D$. We say that $D$ is called $A$-admissible if there are positive integers $b_1, \cdots, b_k$ such that

$$A = \sum_1^o [P_i] + \sum_1^p b_j [B_j]$$

(3.24)

where $[P_i], [B_j]$ are the homology classes of $P_i, B_j$. We say that $D$ is $(J, \nu)$-effective if every principal component can be represented by a $(J, \nu)$-map and every bubble component can be represented by a $J$-holomorphic map.

We will always denote by $\Sigma_i$ the domain of the $(J, \nu)$-map representing $P_i$. Let $\mathcal{D}_{g,k}^{J,\nu} \subset \mathcal{D}_{g,k}$ be the set of $A$-admissible, $(J, \nu)$-effective $D$.

**Lemma 3.14.** The set $\mathcal{D}_{g,k}^{J,\nu}$ is finite.

This is the analogue of Lemmas 4.5 of [RT] and a simple corollary of the Gromov-Uhlenbeck compactness theorem (cf. Theorem 3.8). The presence of marked points doesn’t affect the proof at all. We omit it.

One can consider $\mathcal{D}_{g,k}^{J,\nu}$ as the set of indices of strata. For each $D \in \mathcal{D}_{g,k}^{J,\nu}$, let $\mathcal{M}_D(\mu, g, k, J, \nu)$ be the space of GU-maps such that the homeomorphism type of its domain with marked points, homology class of each component, and components and their intersection points which have the same image are specified by $D$. The following lemma can be deduced from the definition.

**Lemma 3.15.**

$$\overline{\mathcal{M}}_{A}(\mu, g, k, J, \nu) = \bigcup_{D \in \mathcal{D}_{g,k}^{J,\nu}} \mathcal{M}_D(\mu, g, k, J, \nu).$$

(3.25)
By the definition, each $D$ is associated with a stable Riemann surface $\Sigma_{D,r}$, which can be obtained by contracting all the bubble components. Recall that for each principal component we have to fix the automorphism group preserving the special points to make the transversality arguments work (Theorem 3.1). Here we use $\bar{\kappa}$ to denote the $o$-tuple $(\kappa_1, \cdots, \kappa_o)$, where $\kappa_i$ is the automorphism group of the principal component $P_i$ fixing the special points (marked points and intersection points) of $\Sigma_{D,r}$. Then,

$$(3.26) \quad \mathcal{M}_D(\mu, g, k, J, \nu) = \bigcup_{\bar{\kappa}} \mathcal{M}_D(\mu, g, k, J, \nu)_{\bar{\kappa}},$$

where $\mathcal{M}_D(\mu, g, k, J, \nu)_{\bar{\kappa}}$ consists of all maps in $\mathcal{M}_D(\mu, g, k, J, \nu)$ whose $i$th-principal component has the automorphism group $\kappa_i$. Next we prove the smoothness of $\mathcal{M}_D(\mu, g, k, J, \nu)_{\bar{\kappa}}$. First we make another reduction by identifying the domains of those bubble components which have the same image, and change the homology class accordingly. Furthermore, we identify the corresponding intersection points with the same image. Suppose that the resulting new domain and homology class of each component are specified by $\bar{D}$. This process may destroy the tree structure and create some cycles in the domain. The total homology class may also change. However, it remains to be $A$-admissible.

Given such $D$ and $\bar{D}$, we can identify $\mathcal{M}_D(\mu, g, k, J, \nu)_{\bar{\kappa}}$ with the space of $(J, \nu)$-maps whose domain, homology class of each component are specified by $\bar{D}$ and the automorphism group of its principal components are specified by $\bar{\kappa}$. Let us denote this space by $\mathcal{M}_{\bar{D}}(\mu, g, k, J, \nu)_{\bar{\kappa}}$. Then,

$$(3.27) \quad \mathcal{M}_{\bar{D}}(\mu, g, k, J, \nu)_{\bar{\kappa}} = \mathcal{M}_D(\mu, g, k, J, \nu)_{\bar{\kappa}}.$$

For each $f$ in $\mathcal{M}_{\bar{D}}(\mu, g, k, J, \nu)_{\bar{\kappa}}$, the bubble components have different images. As before, let $P_1, \cdots, P_o$ be the principal components and let $B_1, \cdots, B_p$ be the bubble components. Let $\Sigma_{\bar{D}}$ be the domain of maps in the stratum $\mathcal{M}_{\bar{D}}(\mu, g, k, J, \nu)_{\bar{\kappa}}$. This is a union of $\Sigma_i$ (genus $g_i$) and some $S^2s$ intersecting each other according to the intersection pattern given by $\bar{D}$. Let $h_i$ be the number of intersection points on the component $P_i$. Note that we count a self-intersection point twice. Here, the intersection points between the components are the points in their domain, not in their image. Similarly, let $h_j^i$ be the number of intersection points on the bubble component $B_j$. Let $k_i(k_j^i)$ be the number of marked points on $P_i$-component (bubble component $B_j$), which are different from intersection points. Notice that $k_j^i = 0$ or $1$. Moreover,

$$(3.28) \quad \sum k_i + \sum k_j^i \leq 2k.$$
It may not be equal to $2k$ because the collapsing of the ghost bubbles containing a marked point forces the marked point to lie on the intersection. Let $\mathcal{M}_{\{B_j\}}^*(S^2, J, 0) \subset \mathcal{M}_{\{B_j\}}(S^2, J, 0)$ be the space of non-multiple covering maps and

$$\mathcal{M}_{\{B_j\}}^*(S^2, h^j + k^j, J, 0) = \mathcal{M}_{\{B_j\}}^*(S^2, J, 0) \times S^{2h^j+k^j}$$

where $S^{2h^j+k^j}$ is the set of distinct $h^j + k^j$-tuple points of $S^2$. Consider

\begin{equation}
\mathcal{M}_D(\mu, g, k, J, \nu)_k = \{ f : \Sigma_D \to V \mid f_{P_i} \in \mathcal{M}_{\{P_i\}}(\mu, g_i, h_i + k_i, J, \nu_i)_{\kappa_i}, \text{Im}(f_{B_j}) \neq \text{Im}(f_{B_{j'}}) \text{ if } j \neq j' \}
\end{equation}

For each bubble component, there is a parameterization group $G = PSL_2$. Therefore, $G^{p_D}$ acts on $\mathcal{M}_D(\mu, g, J, \nu)_k$, where $p_D$ is the number of bubble components. Then $\mathcal{M}_D(\mu, g, J, \nu)_k = \mathcal{M}_D(\mu, g, k, J, \nu)_k / G^{p_D}$. Clearly,

\begin{equation}
\mathcal{M}_D(\mu, g, k, J, \nu)_k \subset \prod \mathcal{M}_{\{P_i\}}(\mu, g_i, h_i + k_i, J, \nu_i)_{\kappa_i} \times \prod \mathcal{M}_{\{B_j\}}^*(S^2, h^j + k^j, J, 0),
\end{equation}

whose components intersecting each other according to the intersection pattern given by $\tilde{D}$. Consider the evaluation map

\begin{equation}
e_D : \prod \mathcal{M}_{\{P_i\}}(\mu, g, h_i + k_i, J, \nu)_{\kappa_i} \times \prod \mathcal{M}_{\{B_j\}}^*(S^2, h^j + k^j, J, 0) \to \prod V^{h_i} \times \prod V^{h^j} = V^{h_D},
\end{equation}

where $h_D = \sum h_i + \sum h^j$, and $e_D$ is defined as follows: We first define

\begin{align}
e_{P_i} : \mathcal{M}_{\{P_i\}}(\mu, g, h_i + k_i, J, \nu)_{\kappa_i} &\to V^{h_i} \\
e_{P_i}(f, x_1, \cdots, x_{h_i}, x_{h_i+1} \cdots x_{h_i+k_i}) &\equiv (f(x_1), \cdots, f(x_{h_i}))
\end{align}

For each $B_j$, we define $e_{B_j} : \mathcal{M}_{\{B_j\}}^*(S^2, h^j + k^j, J, 0) \to V^{h^j}$ by

\begin{equation}
e_{B_j}(f, y_1, \cdots, y_{h^j}, y_{h^j+1} \cdots y_{h^j+k^j}) = (f(y_1), \cdots, f(y_{h^j})).
\end{equation}

Then we define $e_D = \prod e_{P_i} \times \prod e_{B_j}$. Recall that if $M, N$ are submanifolds of $X$, $M \cap N$ can be interpreted as $M \times N \cap \Delta$, where $\Delta$ is the diagonal of $X \times X$. This means that we can realize any intersection pattern by constructing a “diagonal” in the product. Let us construct a submanifold $\Delta_D \subset V^h$ which plays the role of the diagonal. Let $z_1, \cdots, z_{\tilde{D}}$ be all the intersection points. For each $z_s$, let $I_s = \{ P_{i_1}, \cdots, P_{i_q}, B_{j_1}, \cdots, B_{j_r} \}$ be the set of components which intersect at $z_s$. Now we will construct a product $V_s$ of $V$ such that its diagonal describes the intersection at $z_s$. This is done as follows: we allocate one or two
factors from each of $V^{h_1}, \cdots, V^{h_n}$, according to whether or not $z_s$ is a self-intersection point of the corresponding principal component. We allocate one factor from each of $V^{h_1}, \cdots, V^{h_j}$. Here $V^{h_i}$ or $V^{h_j}$ are the image of $e_{P_i}$ or $e_{B_j}$. Then, we take the product of those factors and denote it by $V_s$. Let $\Delta_s$ be the diagonal of $V_s$. Then the product $\Delta_D = \Delta_1 \times \cdots \times \Delta_{t_D} \subset V^{h_D}$ is the diagonal to realize the intersection pattern between the components of $\bar{D}$. Then $e_D^{-1}(\Delta_D) \supset \tilde{M}_D(\mu, g, k, J, \nu)$. But they may not be equal because we require that bubble components have different image. But $\tilde{M}_D(\mu, g, k, J, \nu)$ is an open subset. Moreover, the group $G^{p_D}$ acts on $e_D^{-1}(\Delta_D)$.

**Theorem 3.16.** There is a set $\mathcal{H}_{reg}$ of Baire second category among all the smooth pairs $(J, \nu)$ such that for any $(J, \nu) \in \mathcal{H}_{reg}$, $\mathcal{M}_D(\mu, g, k, J, \nu)_\kappa$ is a smooth manifold of dimension

$$\sum (2c_1(V)(P_i) + 2n(1 - g_i)) + \sum \dim T_{g_i,0} + \sum (2c_1(V)(B_j) + 2n - 6) + 2h_D + \sum k_i + \sum k^2 - 2n(h_D - t_D),$$

where $g_i$ is the genus of $\Sigma_i$ and $t_D$ is the number of intersection points of $\bar{D}$.

Moreover, for any $(J, \nu)$ and $(J', \nu')$ of $\mathcal{H}_{reg}$, there is a second category set of paths $\mathcal{H}((J, \nu), (J', \nu'))$ connecting $(J, \nu)$ and $(J', \nu')$ among all the smooth paths such that for any path $(J_t, \nu_t) \in \mathcal{H}((J, \nu), (J', \nu'))$,

$$\bigcup_{t \in [0,1]} \mathcal{M}_D(\mu, g, k, J, \nu)_\kappa$$

is a smooth cobordism of one dimension higher.

By the construction of $\bar{D}$, it is evident that $t_D \leq t_D$ and $h_D \leq h_D$. But, $h_D - t_D = h_D - t_D$. Therefore,

**Corollary 3.17.** Under the conditions of Theorem 3.16, the dimension of $\mathcal{M}_D(\mu, g, k, J, \nu)_\kappa$ is less than or equal to

$$\sum (2c_1(V)(P_i) + 2n(1 - g_i)) + \sum \dim T_{g_i,0} + 2k + \sum (2c_1(V)(B_j) + 2n - 6) + 2h_D - 2n(h_D - t_D).$$

**Proof of Theorem 3.16:** The idea of the proof is similar to that in the proof of Lemma 4.8-4.11 in [M]. Also many arguments are the same as those in the proof of Theorem 4.7 in [R]. But we will avoid the Floer’s norm on the space of almost complex structures as we did before.

First of all, $\tilde{M}_D(\mu, g, k, J, \nu)_\kappa$ is an open subset of $e_D^{-1}(\Delta_D)$. Hence for the purpose of proving smoothness, we can assume that

$$\tilde{M}_D(\mu, g, k, J, \nu)_\kappa = e_D^{-1}(\Delta_D)$$
to simplify the argument.

Suppose \( p > 2, m \geq 1 \). Following (3.3),

\[
\chi_{\kappa, D}^{p,m} = \prod_{i=1}^{\infty} \bigcup_{j \in T_{g_i, h_i + k_i}} L_{m,j}^p(Map_{P_i}(\Sigma_i, V)) \times \{j\} \times \prod_{s=1}^{\infty} L_k^p(Map_{B_s}(\mathbf{P}^1, V)) \times (S^2)^{h^j + k^j}
\]

is a smooth Banach manifold. Then, we define

\[
\mathcal{M}^l(\bar{D}, \bar{\kappa}) = \{(f, j, J, \nu) \in \chi_{\kappa, D}^{p,m} \times \mathcal{H}^l; \bar{\partial}_J f(x) = \nu(\phi(x), f(x))\},
\]

where \( m \leq l \) and the equation

\[
\bar{\partial}_J f(x) = \nu(\phi(x), f(x))
\]

means the \((J, \nu)\)-holomorphic equation for each \( P_i \) component and \( J \)-holomorphic equation for each \([B_s] \) component. It is not hard to observe that

\[
\mathcal{M}^l(\bar{D}, \bar{\kappa}) = \bigcup_{(J, \nu) \in \mathcal{H}^l} \left( \prod_{i=1}^{\infty} \mathcal{M}^l_{P_i}(\mu, g_i, h_i + k_i, J, \nu)_{\kappa_i} \times \prod_{s=1}^{\infty} (\mathcal{M}^*_{[B_s]}(S^2, h^j + k^j, J, 0).\right)
\]

**Lemma 3.18.** \( \mathcal{M}^l(\bar{D}, \bar{\kappa}) \) is smooth Banach manifold.

**Proof of Lemma 3.18:** \( \mathcal{M}^l(\bar{D}, \bar{\kappa}) \) is just the analogue of \( \Theta(\Delta_J) \) in Lemma 4.9 of [RT]. The proof of Lemma 3.18 is identical to that of Lemma 4.9 in [RT]. We omit it.

Note that the evaluation map extends

\[
e_D : \mathcal{M}^l(\bar{D}, \bar{\kappa}) \rightarrow V^{h_D}.
\]

We define

\[
\tilde{\mathcal{M}}^l_D(\mu, g, \bar{\kappa}) = e_D^{-1}(\Delta_D)
\]

**Lemma 3.19.** \( \tilde{\mathcal{M}}^l_D(\mu, g, \bar{\kappa}) \) is a smooth Banach manifold.

Its proof is identical to the proof of Theorem 4.7 of [RT]. We omit it.

Then, the rest of proof of Theorem 3.16 is similar to that of Theorem 3.1. We sketch the argument.

Consider the projection

\[
\pi : \tilde{\mathcal{M}}^l_D(\mu, g, \bar{\kappa}) \rightarrow \mathcal{H}^l.
\]
Then,

\[(3.40) \quad \tilde{\mathcal{M}}_D^l(\mu, g, k, J, \nu) = (\pi)^{-1}((J, \nu)).\]

Then \(\pi\) is a Fredholm map between two Banach manifolds. It follows from the Sard-Smale Transversality Theorem that the set

\[\mathcal{H}_l^{\text{reg}} = \{(J, \nu) \in \mathcal{H}_l \mid d\pi \text{ is onto for all } f \in \mathcal{M}_D^l(\mu, g, k, J, \nu)\}\]

is of Baire second category. Let

\[\mathcal{H}_\text{reg} = \bigcap_l \mathcal{H}_l^{\text{reg}}.\]

Then,

\[(3.41) \quad \tilde{\mathcal{M}}_D(\mu, g, k, J, \nu) = \bigcap_l \mathcal{M}_D^l(\mu, g, k, J, \nu).\]

As in the proof of Theorem 3.1, we can deduce that \(\mathcal{H}_\text{reg}\) is of Baire second category. We leave it to the readers as an exercise. Then, for any \((J, \nu) \in \mathcal{H}^{\text{reg}}\),

\[\tilde{\mathcal{M}}_D(\mu, g, k, J, \nu) = \pi^{-1}((J, \nu))\]

is a smooth manifold. Since \(G_{D}^{p}\) acts freely on \(\tilde{\mathcal{M}}_D(\mu, g, k, J, \nu)\),

\[\mathcal{M}_D(\mu, g, k, J, \nu) = \tilde{\mathcal{M}}_D(\mu, g, k, J, \nu)/G_{D}^{p}\]

is smooth.

An routine counting argument yields the dimension formula.

The proof of the second part of Theorem 3.16 is identical.

Recall that if we contract all the bubble components of \(D\), we obtain the stable Riemann surface \(\Sigma_{D,r}\) in the sense of Deligne-Mumford.

An interesting special case of Theorem 3.18 is

**Corollary 3.20.** If \(\Sigma_D\) has no bubble components at all, i.e., \(\Sigma_D = \Sigma_{D,r}\), \(\mathcal{M}_{\Sigma_D}(\mu, g, k, J, \nu)\) is smooth for a generic \((J, \nu)\). Moreover, for a generic \((J_t, \nu_t)\), \(\bigcup_t \mathcal{M}_D(\mu, g, k, J_t, \nu_t)\) is a smooth cobordism. Here, the word “generic” means that it is in a set of Baire second category.

Next, we compute the codimension of \(\mathcal{M}_D(\mu, g, k, J, \nu)\). First


**Proposition 3.21.** Suppose that \((V, \omega)\) is a semi-positive symplectic manifold. Let \(\mathcal{M}_{\Sigma_D, r} \subset \overline{\mathcal{M}}_{g, k}\) be the set of stable Riemann surfaces such that their homeomorphism types are specified by \(\Sigma_D, r\). Then,

\[
(3.42) \quad \dim \mathcal{M}_D(\mu, g, k, J, \nu)_k \leq 2c_1(V)(A) + 2n(1 - g) + \dim \mathcal{M}_{\Sigma_D, r} - 2p_D,
\]

where \(p_D\) is the number of bubble components of \(D\) (not \(\overline{D}\)).

**Proof:** By Corollary 3.17, the dimension of \(\mathcal{M}_D(\mu, g, k, J, \nu)_k\) is less than or equal to

\[
\sum_i (2c_1(V)([P_i]) + 2n(1 - g_i)) + 2k + \sum_i \dim \mathfrak{T}^{\mu, \kappa_i}_{g_i, 0} + \sum_j (2c_1(V)(B_j) + (2n - 6)) + 2h_D - 2n(h_D - t_D)
\]

\[
= 2c_1(V)(|D|) + 2n \sum_i (1 - g_i) + 2k + \sum_i \dim \mathfrak{T}^{\mu, \kappa_i}_{g_i, 0} + (2n - 6)p_D + 2h_D - 2n(h_D - t_D)
\]

For a generic \(J\),

\[
(3.43) \quad 2c_1(V)(B_j) + 2n - 6 = \dim \mathcal{M}^\ast_{[B_j]}(S^2, J, 0)/PSL_2 \geq 0.
\]

If some bubble component \(B_j\) happens to be the image of two or more bubble components of \(D\), by adding \(2c_1(V)(B_j) + 2n - 6\) to the dimension formula,

\[
\dim \mathcal{M}_D(\mu, g, k, J, \nu) \leq 2c_1(V)(|D|) + 2n \sum_i (1 - g_i) + 2k + \sum_i \dim \mathfrak{T}^{\mu, \kappa_i}_{g_i, 0} + (2n - 6)p_D + 2h_D - 2n(h_D - t_D).
\]

Since \((V, \omega)\) is semi-positive, \(c_1(V)(B_j) \geq 0\) for a generic \(J\). Since \(D\) is \(A\)-admissible, \(c_1(V)(|D|) \leq c_1(V)(A)\). Let

\[
(3.44) \quad \lambda_D = (2n - 6)p_D + 2h_D - 2n(h_D - t_D).
\]

Then, Proposition 4.13 of \([RT]\) implies that

\[
(3.45) \quad \lambda_D \leq \lambda_{\Sigma_D} - 2p_D.
\]

Note that in \([RT]\), we use \(k_D\) to denote the number of bubble component instead of \(p_D\) we used in this paper (here \(k\) was used to denote the number of marked points). Therefore,

\[
\dim \mathcal{M}_D(\mu, g, k, J, \nu)_k \leq \sum_i 2c_1(V)(A) + 2n \sum_i (1 - g_i) + 2k + \sum_i \dim \mathfrak{T}^{\mu, \kappa_i}_{g_i, 0} + 2h_{\Sigma_D} - 2n(h_{\Sigma_D} - t_{\Sigma_D}) - 2p_D.
\]

Since \(\Sigma_{D, r}\) is homeomorphic to a stable curve, \(t_{\Sigma_D}\) is the number of double points and \(h_{\Sigma_D} = 2t_{\Sigma_D}\).

An easy inductive argument (Proposition of 4.13, \([RT]\)) shows that

\[
(3.46) \quad o - t_{\Sigma_D} - \sum_i g_i = 1 - g.
\]
It is easy to observe that

\begin{equation}
\text{dim} \mathcal{M}_{\Sigma, r} = 2k + \sum_i \text{dim} \mathcal{T}_{\mu, \kappa_i} + 2h_{\Sigma_r} \leq \text{dim} \mathcal{M}_{\Sigma_r}
\end{equation}

and equal iff all the \( \kappa_i \) are trivial.

**Proof of Theorem 3.11 (Structure Theorem):** It follows from Lemma 3.14, 3.15, Theorem 3.16, Proposition 3.21.

### 4 Proof of Proposition 2.4, 2.9 and 2.10

In this section, we first establish the existence of the GW-invariants \( \Psi^V \) (Proposition 2.4) and its independence from various parameters. Hence, \( \Psi^V \) is a symplectic invariant. Then, we will prove Proposition 2.9 and 2.10. Technically speaking, this section is the analogue of section 5 and 7 of [RT]. We shall repeatly use the word “generic” to mean something belonging to a set of Baire second category.

First of all, we extend the definition of pseudo-submanifolds to the singular space with quotient singularities such as \( \overline{\mathcal{M}}_{g, k}^\mu \). Furthermore, we also need to consider the transversality theory of such pseudo-submanifolds. It is well-known that over the rational coefficient \( \mathbb{Q} \), the usual theory for the smooth manifolds extends to the singular space with quotient singularities, where the Poincare duality holds over \( \mathbb{Q} \).

**Definition 4.1:** An \( n \)-dimensional finite simplicial complex \( P \) is called an abstract pseudo-manifold if \( P^{\text{top}} = P - P_{n-2} \) (\( n-2 \) skeleton) is an open smooth oriented \( n \)-dimensional manifold. \( P \) is called an abstract pseudo-manifold with boundary if \( P^{\text{top}} \) is a \( n \)-dimensional oriented smooth manifold with boundary \( \partial P^{\text{top}} \). Let \( \partial P = \overline{\partial P^{\text{top}}} \). Then \( \partial P \cap P_{n-2} \) is a subcomplex of dimension less than or equal to \( n-3 \). Let \( V \) be a stratified PL-manifold such that each stratum is even dimensional and its triangulation is compatible with the stratification. A pseudo-submanifold is a pair \( (P, f) \), where \( P \) is an abstract pseudo-manifold and \( f : P \to V \) is a piece-wise linear map (PL) with respect to some triangulation of \( V \). Furthermore, we require that \( f \) maps \( P^{\text{top}} \) into one stratum and smooth. A pseudo-submanifold cobordism between pseudo-submanifolds \( (P, f), (Q, h) \) is a pair \( (K, H) \) such that \( K \) is an abstract pseudo-manifold with boundary with \( \partial K = P \cup -Q \) and \( H \) is PL with respect to some triangulation of \( V \) and smooth over \( K^{\text{top}} \) in the sense that \( H \) maps the \( K^{\text{top}} \) smoothly in one stratum or maps the interior \( K^\circ \) of \( K^{\text{top}} \) to one stratum and \( \partial K^{\text{top}} \) to the lower strata. Moreover, \( H|_{P\cup-Q} = f \cup -h \), where \( - \) means the opposite orientation.
Furthermore, we have the following lemma on transversality.

**Lemma 4.2.** Let $V$ be a stratified PL-manifold with fundamental class $[V]$ such that the Poincaré duality holds over the rational coefficient. Then, for each homology class $\alpha$, there exists $p$ and a pseudo-submanifold representative $(P, f)$ of a homology class $p(\alpha)$. Furthermore, if $h_i : X_i \to V$ be smooth maps from smooth manifolds $X_i$ to smooth strata of $V$, then there is a small perturbation $\tilde{f} : P \to V$ such that $\tilde{f}$ is transverse to each $h_i$, i.e., $\tilde{f}$ is transverse to $h_i$ as a PL map and transverse over $P^{\text{top}}$ as a smooth map.

**Proof:** This lemma is the consequence of standard transversality results in PL topology [Mc](Theorem 5.2). One remark is that if $V$ is a smooth manifold, this lemma holds for any $\alpha$. Otherwise, the lemma holds for those class of the form $[V] \cap \beta^*$ for a cohomology class $\beta^*$ (Theorem 5.2 of [Mc]). Then, the lemma follows from the assumption that the Poincaré Duality holds over the rational coefficient $\mathbb{Q}$.

Recall that in (2.4), we have defined the evaluation map

$$e_i : \mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa} \to V.$$ (4.1)

e_i extends obviously to each $\mathcal{M}_D(\mu, g, k, J, \nu)_{\kappa}$ (Lemma 3.15). We shall still denote it by $e_i$. Furthermore,

$$\Upsilon : \mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa} \to \mathcal{M}_D^{\mu}_{g,k,\kappa},$$ (4.2)

which takes the domain of maps in $\mathcal{M}_A(\mu, g, k, J, \nu)_{\kappa}$ (section 2) extends over

$$\Upsilon : \mathcal{M}_D(\mu, g, k, J, \nu)_{\kappa} \to \mathcal{M}_D^{\mu}_{\Sigma D, r, \kappa}$$
as well. There is an obvious version of the maps $e_i$ and $\Upsilon$ for the corbordisms $\bigcup_t \mathcal{M}_A(\mu, g, k, J_t, \nu_t)_{\kappa} \times \{t\}$ and $\bigcup_t \mathcal{M}_D(\mu, g, k, J_t, \nu_t)_{\kappa} \times \{t\}$. We denote them by $e_i^{(t)}$ and $\Upsilon^{(t)}$. Then, we define

$$(\Xi_{g,k}^{A})^{(t)} = \prod i e_i^{(t)}.$$ (4.3)

**Definition 4.3.** Let $(P_i, f_i)$ be a pseudo-submanifold of $V$. We say that $(P_i, f_i)$ is transverse to $e_i$ (hence $\Xi_{g,k}^{A}$) if $(P_i, f_i)$ is transverse to $e_i$ as the maps from $\mathcal{M}_d(\mu, g, k, J, \nu)_{\kappa}$ and their extensions over $\mathcal{M}_D(\mu, g, k, J, \nu)_{\kappa}$ for each $D \in \mathcal{D}_{g,k}^{J, \nu}$ in the sense of Lemma 4.2. We say that $(P_i, f_i)$ is transverse to $e_i^{(t)}$ if it is transverse to $e_i^{(t)}$ (hence $(\Xi_{g,k}^{A})^{(t)}$) as the maps from $\bigcup_t \mathcal{M}_A(\mu, g, k, J_t, \nu_t)_{\kappa} \times \{t\}$ and their extensions over $\bigcup_t \mathcal{M}_D(\mu, g, k, J_t, \nu_t)_{\kappa} \times \{t\}$ for each $D \in \mathcal{D}_{g,k}^{J, \nu}$ in the sense of Lemma.
4.2. Similarly, we say that a pseudo-submanifold \((G, K)\) of \(\overline{\mathcal{M}}_{g,k}\) is transverse to \(\Upsilon\) (or \(\Upsilon(t)\)) if it transverse to them as the maps from \(\mathcal{M}_A(\mu, g, k, J, \nu)_K\) and its extensions over \(\mathcal{M}_D(\mu, g, k, J, \nu)_\bar{k}\) (or from \(\bigcup_t \mathcal{M}_A(\mu, g, k, J_t, \nu_t)_K\times\{t\}\) and its extensions over \(\bigcup_t \mathcal{M}_D(\mu, g, k, J_t, \nu_t)_\bar{k}\times\{t\}\) respectively.

Let us recall the construction of section 2. Let \(\alpha_i\) be integral homology classes of \(V\). We choose pseudo-submanifolds \((Y_i, F_i)\) to represent \(\alpha_i\). Let \(Y = \prod_{i=1}^{k} Y_i, F = \prod_{i=1}^{k} F_i\).

Then, \(Y^{top} = \prod_{i=1}^{k} Y_i^{top}\). Clearly, \((Y, F)\) represents \(\prod_{i=1}^{k} \alpha_i \in H_*(V^k, \mathbb{Z})\).

Note that \(\overline{\mathcal{M}}_{g,k}\) may not be smooth, but the Poincare Duality holds over rational coefficients. Hence, we can assume that each homology class can be represented by a pseudo-submanifold and the corresponding transversality holds as long as we work over \(H_*(\overline{\mathcal{M}}_{g,k}, \mathbb{Q})\). Let \((G, K)\) be a pseudo-submanifold in \(\overline{\mathcal{M}}_{g,k}\) and first we assume that \((G, K)\) is in general position i.e., \(K(G^{top}) \subset \mathcal{M}_{g,k,I}\), where \(I\) represents trivial automorphism group.

**Lemma 4.4.** By choosing small perturbations if necessary, we have that \(K \times F\) is transverse to \(\Upsilon \times \Xi_{g,k}^A\) as the PL-maps with respect to some triangulation of \(V\) and as the smooth maps over \(Y^{top} \times G^{top}\).

**Proof:** This follows obviously from Lemma 4.2.

**Corollary 4.5.** Suppose that

\[
\sum_{i=1}^{k} (2n - d_i) + (6g - 6 + 2k - deg(\mu, g)) = 2c_1(V)(A) + 2(3-n)(\mu, g - 1) + 2k.
\]

and \(F_i, K\) satisfy the statement of Lemma 4.4. Then,

\[
K \times F \cap \Upsilon \times \Xi_{g,k}^A(\overline{\mathcal{M}}_A^e(\mu, g, k, J, \nu)_I - \mathcal{M}_A(\mu, g, k, J, \nu)_I) = \emptyset;
\]

\[
K \times F(Y \times G - Y^{top} \times G^{top}) \cap \Upsilon \times \Xi_{g,k}^A = \emptyset.
\]

**Proof:** By Lemma 4.4, \(K \times F\) is transverse to \(\Upsilon \times \Xi_{g,k}^A\). To prove

\[
K \times F \cap \Upsilon \times \Xi_{g,k}^A(\overline{\mathcal{M}}_A^e(\mu, g, k, J, \nu)_I - \mathcal{M}_A(\mu, g, k, J, \nu)_I) = \emptyset,
\]

by (3.25), (3.26),

\[
\overline{\mathcal{M}}_A^e(\mu, g, k, J, \nu) = \bigcup_{D \in \mathcal{D}_{g,k}} \bigcup_{\bar{k}} \mathcal{M}_D(\mu, g, k, J, \nu)_\bar{k}.
\]
By the Proposition 3.20, except the main component $\mathcal{M}_A(\mu, g, k, J, \nu)_I$, all other components $\mathcal{M}_D(\mu, g, k, J, \nu)_\bar{\kappa}$ are smooth manifolds of dimension

\[(4.6) \quad \leq 2c_1(V)(A) + 2(3 - n)(g - 1) + 2k - 2\]

Since $K \times F$ is transverse to $\mathcal{M}_D(\mu, g, k, J, \nu)_\bar{\kappa}$, then

\[(4.7) \quad \dim(K \times F \cap \mathcal{Y} \times \Xi^{A}_{g,k}(\mathcal{M}_D(\mu, g, k, J, \nu)_\bar{\kappa})) \leq -2.\]

Hence, it is empty. Therefore,

\[(4.8) \quad K \times F \cap \mathcal{Y} \times \Xi^{A}_{g,k}(\mathcal{M}_D(\mu, g, k, J, \nu)_\bar{\kappa}) = \emptyset.\]

\[(4.9) \quad K \times F(Y \times G - Y^{top} \times G^{top}) \cap \mathcal{Y} \times \Xi^{A}_{g,k} = \emptyset.\]

follows from a similar dimension counting argument. We leave it to the readers.

Now we adopt the notations from section 2. By (2.7),

\[(4.10) \quad (\mathcal{Y} \times \Xi^{A}_{g,k} \times K \times F)^{-1}(\Delta) \subset \mathcal{M}_A(\mu, g, k, J, \nu)_I \times Y^{top} \times G^{top}\]

is a zero-dimensional smooth submanifold.

**Lemma 4.6.** $(\mathcal{Y} \times \Xi^{A}_{g,k} \times K \times F)^{-1}(\Delta)$ is compact and hence finite.

**Proof:** Suppose that there is a sequence of distinct elements

\[(f_s, X_s, x_s) \in (\mathcal{Y} \times \Xi^{A}_{g,k} \times K \times F)^{-1}(\Delta).\]

By taking a subsequence, we can assume that

\[f_s \to f \in \overline{\mathcal{M}}_A(\mu, g, k, J, \nu)\]

and

\[(X_s, x_s) \to (X, x) \in Y \times G.\]

However, $(\mathcal{Y} \times \Xi^{A}_{g,k} \times K \times F)^{-1}(\Delta)$ is smooth. Thus, either

\[(4.11) \quad (f, X, x) \in K \times F \cap \mathcal{Y} \times \Xi^{A}_{g,k}(\overline{\mathcal{M}}_A(\mu, g, k, J, \nu) - \mathcal{M}_A(\mu, g, k, J, \nu)_I) = \emptyset\]

or

\[(4.12) \quad (f, X, x) \in K \times F(Y \times G - Y^{top} \times G^{top}) \cap \mathcal{Y} \times \Xi^{A}_{g,k} = \emptyset.\]
In both cases, we have a contradiction.

Once the Lemma 4.6 is proved, as in section 2, we can define

$$\Psi_{V(A,g,k,\mu)}(K,\alpha_1,\cdots,\alpha_k)$$

as the algebraic sum of $$(\Upsilon \times \Xi_A^{g,k} \times K \times F)^{-1}(\Delta)$$. To emphasis the dependence on $$(J,\nu)$$ at this moment, we define

$$Z(A,\mu,g,k,J,\nu,F,K) = (\Upsilon \times \Xi_A^{g,k} \times K \times F)^{-1}(\Delta)$$

Sometimes (for example (4.29)), we also use $$Y,G$$ in the place of $$F,K$$ in $$Z(\cdots)$$, if there is no confusion. To abuse the notation, we denote its algebraic sum by $$|Z(A,\mu,g,k,J,\nu,F,K)|$$.

Next we have to prove Proposition 2.4. The proof will be divided into a series of Lemmas:

**Lemma 4.7.** $$|Z(A,\mu,g,k,J,\nu,F,K)|$$ is independent of the representative $$(Y,F)$$ and $$(G,K)$$, whenever $$(G,K)$$ is in general position.

**Proof:** Suppose that $$(Y',F'),(G',K')$$ are other representatives such that $$(G',K')$$ is in general position. There are corbordisms $$(Q,H)$$ and $$(L,P)$$ such that

$$\partial(Q) = Y \cup -Y', H|_{\partial(Q)} = F \cup -F'; \partial(L) = G \cup -G', P|_{\partial(L)} = K \cup -K'$$

Let’s first work on $$(Q,H)$$. By choosing a small perturbation of $$H$$ relative to $$\partial(Q)$$ if necessary, we can assume that $$H \times K$$ is transverse to $$\Upsilon \times \Xi_A^{g,k}$$. Then, by counting dimensions as one did in the proof of Lemma 4.4, one can show that

$$H \times K \cap \Upsilon \times \Xi_A^{g,k}(\overline{\mathcal{M}}_A(\mu,g,k,J,\nu) - \overline{\mathcal{M}}_A(\mu,g,k,J,\nu)_{\mathfrak{I}}) = \emptyset;$$

$$H \times K(Q \times K - Q^{\text{top}} \times K^{\text{top}}) \cap \Upsilon \times \Xi_A^{g,k} = \emptyset.$$}

Then, it follows from the same argument as in the proof of Corollary 4.5 that

$$Z(A,\mu,g,k,J,\nu,H,K) = (\Upsilon \times \Xi_A^{g,k} \times H \times P)^{-1}(\Delta) \subset \overline{\mathcal{M}}_A(\mu,g,k,J,\nu)_{\mathfrak{I}} \times Q^{\text{top}} \times K^{\text{top}}$$

is a compact, smooth, oriented 1-manifold with boundary

$$\partial(Z(A,\mu,g,k,J,\nu,H,K)) = Z(A,\mu,g,k,J,\nu,F,K) \cup -Z(A,\mu,g,k,J,\nu,F',K).$$

Hence,

$$|Z(A,\mu,g,k,J,\nu,F,K)| = |Z(A,\mu,g,k,J,\nu,F',K)|.$$
Next, we fix a \((Y, F)\) and consider \((L, P)\). By choosing a small perturbation of \(P\) relative to \(\partial(L)\), we can assume that \((L, P)\) is in general position and \(F \times P\) is transverse to \(Y \times \Xi_{g,k}^A\). Repeating the previous argument, we can show that

\[(4.19) \quad |Z(A, \mu, g, k, J, \nu, F, K)| = |Z(A, \mu, g, k, J, \nu, F, K')|.

**Lemma 4.8.** \(|Z(A, \mu, g, k, J, \nu, F, K)|\) is independent of generic \((J, \nu)\).

**Proof:** Let \((J', \nu')\) be another generic pair. Choose a generic path \((J_t, \nu_t)\) connecting \((J, \nu)\) to \((J', \nu')\), such that

\[
M_{\bar{D}}(\mu, g, k, (J_t), (\nu_t)) = \bigcup_t M_{\bar{D}}(\mu, g, k, J_t, \nu_t) \times \{t\}
\]

is a smooth, oriented cobordism between \(M_{\bar{D}}(\mu, g, k, J, \nu)\) and \(M_{\bar{D}}(\mu, g, k, J', \nu')\). By Lemma 4.2 and choosing a small perturbation if necessary, we can assume that \(F \times K\) is transverse to \(Y(t) \times (\Xi_{g,k}^A)^{(t)}\), where the choices of \(F, K\) do not affect our result by the Lemma 4.7. Then, a dimension accounting argument shows that

\[
K \times F \cap Y(t) \times (\Xi_{g,k}^A)^{(t)}(\overline{M}(\mu, g, k, (J_t), (\nu_t))) - M_{A}(\mu, g, k, (J_t), (\nu_t))_I = \emptyset,
\]

\[(4.20) \quad K \times F(Y \times G - Y^{top} \times G^{top}) \cap Y(t) \times (\Xi_{g,k}^A)^{(t)} = \emptyset.
\]

Then,

\[
Z(A, \mu, g, k, (J_t), (\nu_t), F, K) = (Y(t) \times (\Xi_{g,k}^A)^{(t)} \times K \times F)^{-1}(\Delta)
\]

\[
\subseteq M_{A}(\mu, g, k, (J_t), (\nu_t))_I \times Y^{top} \times G^{top}
\]

is a compact, smooth, oriented 1-manifold with boundary. Moreover,

\[
\partial(Z(A, \mu, g, k, (J_t), (\nu_t), F, K)) = Z(A, \mu, g, k, J, \nu, F, K) \cup -Z(A, \mu, g, k, J', \nu', F, K).
\]

Hence,

\[(4.22) \quad |Z(A, \mu, g, k, J', \nu', F, K)| = |Z(A, \mu, g, k, J, \nu, F, K)|.
\]

**Lemma 4.9.** \(|Z(A, \mu, g, k, J, \nu, F, K)|\) is independent of semi-positive symplectic deformations.

**Proof:** Let \(\omega_t\) be a family of semi-positive symplectic deformation of \(\omega_0 = \omega\). Then, we can choose a generic path \((J_t, \nu_t)\) such that \(J_t\) is \(\omega_t\)-tamed. Then, this lemma follows from the same arguments as the proof of Lemma 4.7.
Next, we prove Proposition 2.9, which gives the analytic foundation for the composition law (Theorem 2.10). Then, the composition law can be easily derived from Proposition 2.9 by using an observation of Witten.

First, we extend the definition of $\Psi$ to the case that $(G, K)$ is in the boundary, say $K(G) \subset \overline{M}^{\mu}_{g,k}$, where $\overline{M}^{\mu}_{g,k}$ is one stratum of $\overline{M}^{\mu}_{g,k}$. If $(G, K)$ has more than two components lying in different strata, the corresponding GW-invariant is just the sum of the GW-invariants of each component. Therefore, without loss of generality, we can assume that $(G, K)$ is in the one stratum of $\overline{M}^{\mu}_{g,k}$. Note that $\overline{M}^{\mu}_{g,k}$ is also a PL-manifold with local quotient singularities where the Poincare Duality holds over the rational coefficient. Therefore, we can replace $\overline{M}^{\mu}_{g,k}$ by $\overline{M}^{\mu}_{g,k}$ in our construction of $\Psi$. In this case, the proper moduli space is $\overline{M}^{\mu}_{g,k}(\mu, g, k, J, \nu)$. Note that we still have the restriction of $\Xi^{A}_{g,k}$, $\Upsilon$ to $\overline{M}^{\mu}_{g,k}(\mu, g, k, J, \nu)$. We shall denote it by $\Xi^{A}_{g,k}$ and $\Upsilon_{\Sigma}$. Then, we can choose a generic $J, \nu$ and $F$ and $K$ in general position (inside $\overline{M}^{\mu}_{g,k}$), i.e., $K(G^{top}) \subset M^{\mu}_{g,k,f}$, where $\bar{I}$ means the trivial automorphism group for each component. Repeating the construction of Lemma 4.4, 4.5, we can define

$$\Psi^{V}_{(A,g,k,\mu)}(K, \alpha_{1}, \cdots, \alpha_{k})$$

as the algebraic sum of

$$Z(A, \overline{M}^{\mu}_{g,k}, J, \nu, F, K) = (\Upsilon_{\Sigma} \times \Xi^{A}_{g,k} \times K \times F)^{-1}(\Delta) \subset M_{\Sigma}(\mu, g, k, J, \nu)_{\bar{I}} \times Y^{top} \times G^{top}$$

Before proving Proposition 2.9, we need the following family version of the gluing theorem in [RT].

Let $U \subset \bar{U} \subset M^{\mu}_{g,k}$ be an oriented submanifold (not necessarily closed) and

$$\phi : U \times [0, \epsilon) \to \overline{M}^{\mu}_{g,k}$$

be a diffeomorphism such that $\phi(U \times \{0\}) = U$ and if $t \neq 0$, $\phi(U \times \{t\}) \subset M^{\mu}_{g,k,f}$. Let

$$U_{t} = \phi(U \times \{t\}).$$

Define

$$M_{A}(U_{0}, J, \nu) = (Y)^{-1}(U_{0}) \cap M_{A,\Sigma}(\mu, g, k, J, \nu)_{\bar{I}}.$$

For each $t \neq 0$, we define

$$M_{A}(U_{t}, J, \nu) = (Y)^{-1}(U_{t}) \cap M_{A}(\mu, g, k, J, \nu)_{\bar{I}}.$$
Fix a generic \((J, \nu)\) and \(\phi\) such that for each \(t\), \(\mathcal{M}_A(U_t, J, \nu)\) is a smooth manifold of dimension

\[(4.26) \quad 2c_1(V)(A) + 2n(1 - g) + \dim U_t.\]

Then, the following is a slight modification of Theorem 6.1, [RT]. We leave the details to the readers.

**Gluing Theorem:** Let \(f_0\) be any map in \(\mathcal{M}_A(U_0, J, \nu)\). Then there is a continuous family of injective maps \(T_t\) from \(W\) into \(\mathcal{M}_A(U_t, J, \nu)\), where \(t\) is small and \(W\) is a neighborhood of \(f_0\) in \(\mathcal{M}_A(U_0, J, \nu)\), such that (1) for any \(f\) in \(W\), as \(t\) goes to zero, \(T_t(f)\) converges to \(f\) in \(C^0\)-topology on \(\Sigma_0\) and in \(C^3\)-topology outside the singular set of \(\Sigma_0\); (2) there are \(\epsilon, \delta > 0\) satisfying: if \(f'\) is in \(\mathcal{M}_A(U_t, J, \nu)\) and

\[d_V(f'(x), f_0(y)) \leq \epsilon, \quad \text{whenever} \ x \in \Sigma_t \subset U_t,\]

where \(d_V\) is the distance functions of a \(J\)-invariant metric \(h_V\) on \(V\) and then \(f'\) is in \(T_t(W)\). Moreover, for each \(t\), \(T_t\) is an orientation-preserving smooth map from \(W\) into \(\mathcal{M}_A(U_t, J, \nu)\).

**Proof of Proposition 2.9:** If \((G', K')\) is another pseudo-submanifold representing the same homology class, we can assume that there is a pseudo-manifold cobordism \((L, P)\) between \((G, K)\) and \((G', K')\).

Without loss of the generality, we may assume that \((G', K')\) is in general position. By choosing a small perturbation relative to the boundary if necessary, we can further assume that

\[(4.27) \quad P(L^o) \subset \mathcal{M}^{\mu}_g, k, I.\]

Again, by counting dimensions, we can show that

\[F \times P \cap \Upsilon \times A^{\mu, g, k, J, \nu}(\mathcal{M}_A(\mu, g, k, J, \nu) - \mathcal{M}_A(\mu, g, k, J, \nu) \cup \mathcal{M}_\Sigma(\mu, g, k, J, \nu) \cup \mathcal{M}_\Sigma(\mu, g, k, J, \nu) \cap \mathcal{M}_\Sigma(\mu, g, k, J, \nu)) = \emptyset,\]

\[(4.28) \quad F \times P(Y \times L - Y^{top} \times L^{top}) = \emptyset.\]

Then

\[(4.29) \quad Z(A, \mu, g, k, J, \nu, F, L) = (\Upsilon \times A^{\mu, g, k} \times F \times P)^{-1}(\Delta) \subset \mathcal{M}_A(\mu, g, k, J, \nu) \cup \mathcal{M}_\Sigma(\mu, g, k, J, \nu) \times Y^{top} \times L^{top}.\]

Since \(L^{top}\) is a manifold with boundary, we can choose an open subset

\[(4.30) \quad \overline{L} \subset L \subset L^{top}\]

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such that

\[(4.31)\quad Z(A, \mu, g, k, J, \nu, F, L) \subset M_A(\mu, g, k, J, \nu) \cup M_\Sigma(\mu, g, k, J, \nu) \times Y^{top} \times \tilde{L},\]

\[\partial \tilde{L} = U \cup U',\]

where \(U \subset G^{top}, U' \subset (G')^{top}\). Furthermore, there is a uniform constant \(\epsilon\) such that there are collars

\[(4.32)\quad U \times [0, \epsilon) \subset \tilde{L}; \quad U' \times [0, \epsilon) \subset \tilde{L}.\]

Thus,

\[(4.33)\quad Z(A, \mu, g, k, J, \nu, F, \tilde{L}) = Z(A, \mu, g, k, J, \nu, F, L).\]

Let \(U_t = U \times \{t\}\) and \(U'_s = U' \times \{s\}\). It follows from an ordinary cobordism argument, which is identical to that of Lemma 4.6, that \(Z(A, g, k, J, \nu, \tilde{L} - U \times [0, \epsilon) \cup U' \times [0, \epsilon))\) is a smooth, compact, oriented 1-manifold with boundary

\[(4.34)\quad \partial(Z(A, g, k, J, \nu, \tilde{L} - U \times [0, \epsilon) \cup U' \times [0, \epsilon))) = Z(A, \mu, g, k, J, \nu, F, U'_\epsilon)) \cup -Z(A, \mu, g, k, J, \nu, F, U'_\epsilon).\]

Therefore,

\[(4.35)\quad |Z(A, \mu, g, k, J, \nu, F, U_t)| = |Z(A, \mu, g, k, J, \nu, F, U'_s)|.\]

Obviously, we can use any \(U_t, U'_s\) in the place of \(U_\epsilon, U'_\epsilon\). Then, we show

\[(4.36)\quad |Z(A, \mu, g, k, J, \nu, F, U_t)| = |Z(A, \mu, g, k, J, \nu, F, U'_s)|.\]

Next, we study the behavior of \(Z(A, \mu, g, k, J, \nu, U_t)\) as \(t \to 0\). By the Gromov-Uhlenbeck compactness theorem, and by taking a subsequence if necessary, we may assume that any sequence \(f_t \in Z(A, \mu, g, k, J, \nu, U_t)\) converges to a

\[(4.37)\quad f \in \overline{M}_\Sigma(\mu, g, k, J, \nu) \times Y \times K.\]

By (4.28),

\[(4.38)\quad f \in M_\Sigma(\mu, g, k, J, \nu) \times Y^{top} \times K^{top}.\]

Hence, \(f \in Z(A, \mu, g, k, J, \nu, U_0)\) by (4.31). On the other hand, by the gluing theorem, for \(t\) small, there is a bijective map

\[(4.39)\quad T_t : Z(A, \mu, g, k, J, \nu, U_0) \to Z(A, \mu, g, k, J, \nu, U_t)\]
preserving the orientation. Therefore,

\[(4.40) \quad |Z(A, \mu, g, k, J, \nu, U_0)| = |Z(A, \mu, g, k, J, \nu, U_t)|.\]

Since \((G', K')\) is in general position, we can allow \(s = 0\) in the \((4.36)\). Therefore,

\[(4.41) \quad |Z(A, g, k, J, \nu, U_0)| = |Z(A, \mu, g, k, J, \nu, U_0')|.\]

Thus, we finish the proof of Proposition 2.9.

**Proof of Theorem 2.10.** Let’s consider the first case where we have the embedding

\[\theta_S : \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1}\]

by identifying the \(k_1 + 1\)-th marked point of the first component with the 1-st marked point of the second component. Suppose that the marked points in the first component are \(\{x_1, \ldots, x_{k_1}, y_1\}\) and the marked points in the second component are \(\{y_2, x_{k_1+1}, \ldots, x_{k_1+k_2}\}\). The map \(\theta_S\) identifies \(y_1\) with \(y_2\) and gives rise to a stable curve of genus \(g_1 + g_2\) with marked points \(\{x_1, \ldots, x_{k_1+k_2}\}\). By (2.17), for their finite covers,

\[\theta_S^*\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1}.\]

Let \([K_i] \in H_*(\overline{\mathcal{M}}^\mu_{g, k_i+1}, \mathbb{Q})\) represented by the pseudo-submanifold \(K_i\). By Proposition 2.9,

\[\Psi^V_{A, g, k, \mu}([\theta_S(K_1 \times K_2)]; \{\alpha_i\})\]

doesn’t depend on the particular representative of \([\theta_S(K_1 \times K_2)]\). In particular, it could have a representative, which is in general position. This is very important in the later applications. Here, we choose a representative \(\theta_S(K_1 \times K_2)\). Let

\[(4.42) \quad \overline{\mathcal{M}}^\mu_{\Sigma} = \theta_S(\overline{\mathcal{M}}^\mu_{g_1, k_1+1} \times \overline{\mathcal{M}}^\mu_{g_2, k_2+1})\]

Then,

\[(4.43) \quad Z(A, \mu, g, k, J, \nu, F, \theta_S(K_1 \times K_2)) = Z(A, \theta_S(\overline{\mathcal{M}}^\mu_{g_1, k_1+1} \times \overline{\mathcal{M}}^\mu_{g_2, k_2+1}), J, \nu, F, \theta_S(K_1 \times K_2)).\]

Recall (4.23) that

\[(4.44) \quad Z(A, \overline{\mathcal{M}}^\mu_{\Sigma}, J, \nu, F, \theta_S(K_1 \times K_2))
\]

\[= (\overline{\mathcal{Y}}_{\Sigma} \times \Xi_{\Sigma}^A \times F \times \theta_S(K_1 \times K_2))^{-1}(\Delta) \subset \mathcal{M}_\Sigma(\mu, g, k, J, \nu) \times Y^\text{top} \times \theta_S(K_1 \times K_2)^\text{top}.\]
But any element of $\mathcal{M}_\Sigma(\mu, g, k, J, \nu)_I$ is a pair

\[
(f_{1}, f_{2}) \in \mathcal{M}_{A_{1}}^{\mu}(\mu, g_{1}, k_{1} + 1, J, \nu)_I \times \mathcal{M}_{A_{2}}^{\mu}(\mu, g_{2}, k_{2} + 1, J, \nu)_I; \quad f_{1}(y_{1}) = f_{2}(y_{2})
\]

with $A_{1} + A_{2} = A$. Let $e_{y_{1}}^{A_{1}}$ be the evaluation map of $y_{1}$’s and $e_{y_{2}}^{A_{2}}$ be the evaluation map of $y_{2}$’s. Then,

\[
\mathcal{M}_\Sigma(\mu, g, k, J, \nu)_I = \bigcup_{A_{1} + A_{2} = A} (e_{y_{1}}^{A_{1}} \times e_{y_{2}}^{A_{2}})^{-1}(\delta),
\]

where $\delta$ is the ordinary diagonal in $V \times V$. Using this decomposition and switching the order of the components appropriately, we have

\[
\bigcup_{A_{1} + A_{2} = A} \Upsilon_{A_{1}} \times \Upsilon_{A_{2}} \times \Xi_{g_{1}, k_{1}} \times \Xi_{g_{2}, k_{2}} \times K_{1} \times K_{2} \times F \times e_{y_{1}}^{A_{1}} \times e_{y_{2}}^{A_{2}},
\]

which maps the space

\[
\bigcup_{A_{1} + A_{2} = A} \mathcal{M}_{A_{1}}(\mu, g_{1}, k_{1} + 1, J, \nu)_I \times \mathcal{M}_{A_{2}}(\mu, g_{2}, k_{2} + 1, J, \nu)_I \times K_{1}^{top} \times K_{2}^{top} \times Y^{top}
\]

into

\[
\mathcal{M}_{\Sigma_{1}, I}^{\mu} \times \mathcal{M}_{\Sigma_{2}, I}^{\mu} \times V^{k_{1} + k_{2}} \times \mathcal{M}_{\Sigma_{1}, I}^{\mu} \times \mathcal{M}_{\Sigma_{2}, I}^{\mu} \times V^{k_{1} + k_{2}} \times V \times V.
\]

We will denote this map by $\Upsilon_{\Sigma} \times \Xi_{\Sigma_{2}}^{A} \times F \times \theta_{S}(K_{1} \times K_{2})$. Let

\[
\Delta_{A_{1}, A_{2}} \subset (\mathcal{M}_{\Sigma_{1}, I}^{\mu} \times \mathcal{M}_{\Sigma_{2}, I}^{\mu} \times V^{k_{1} + k_{2}}) \times (\mathcal{M}_{\Sigma_{1}, I}^{\mu} \times \mathcal{M}_{\Sigma_{2}, I}^{\mu} \times V^{k_{1} + k_{2}})
\]

be the diagonal. Define

\[
Z(A_{1}, A_{2}, g_{1}, g_{2}, k_{1}, k_{2}, \delta, J, \nu) = (\Upsilon_{A_{1}} \times \Upsilon_{A_{2}} \times \Xi_{g_{1}, k_{1}} \times \Xi_{g_{2}, k_{2}} \times K_{1} \times K_{2} \times F \times e_{y_{1}}^{A_{1}} \times e_{y_{2}}^{A_{2}})^{-1}(\Delta_{A_{1}, A_{2}} \times \delta).
\]

Then,

\[
Z(A, \Upsilon_{\Sigma_{2}}, J, \nu, F, \theta_{S}(K_{1} \times K_{2})) = \bigcup_{A_{1} + A_{2} = A} Z(A_{1}, A_{2}, g_{1}, g_{2}, k_{1}, k_{2}, \delta, J, \nu)
\]

\[
= \bigcup_{A_{1} + A_{2} = A} (\Upsilon_{A_{1}} \times \Upsilon_{A_{2}} \times \Xi_{g_{1}, k_{1}} \times \Xi_{g_{2}, k_{2}} \times K_{1} \times K_{2} \times F \times e_{k_{1} + 1}^{A_{1}} \times e_{k_{2} + 1}^{A_{2}})^{-1}(\Delta_{A_{1}, A_{2}} \times \delta).
\]

A corbordism argument as that of Lemma 4.6 shows that $|Z(A_{1}, A_{2}, g_{1}, g_{2}, k_{1}, k_{2}, \delta, J, \nu)|$ only depends on the homology class of $[\delta] \in H_{\ast}(V \times V, \mathbf{Z})$. Choose a homogeneous basis $\{\beta_{b}\}_{1 \leq b \leq L}$ of $H_{\ast}(V, \mathbf{Z})$ modulo torsion. Let $(\eta_{ab})$ be its intersection matrix. Note that $\eta_{ab} = \beta_{a} \beta_{b} = 0$ if the
dimension of $\beta_a$ and $\beta_b$ are not complementary to each other. Let $(\eta^{ab})$ be the inverse of $(\eta_{ab})$.

Then,

\[(4.51) \quad [\delta] = \sum_{a,b} \eta^{ab} \beta_a \otimes \beta_b.\]

Choose a pseudo-submanifold representing $\beta_a$ (still denoted it by $\beta_a$). Then, one observes that

\[(4.52) \quad |Z(A_1, A_2, g_1, g_2, k_1, k_2, \delta, J, \nu)| = \sum_{a,b} \eta^{ab} |Z(A_1, \mu, g_1, k_1, \beta_a, J, \nu, Y_i, K_1)| \cdot |Z(A_2, \mu, g_2, k_2, \beta_2, J, \nu, Y_j, K_2)|.\]

Together with (4.50), it yields the first formula of Theorem 2.10

\[(4.53) \quad \Psi^V_{(A,g,k,\mu)}(\theta_{S*}[K_1 \times K_2]; \{\alpha_i\}) = \sum_{A=A_1+A_2} \sum_{a,b} \Psi^V_{A_1,g_1,k_1+1,\mu}([K_1]; \{\alpha_i\}_{i \leq k_1}, \beta_a) \eta^{ab} \Psi^V_{A_2,g_2,k_2+1,\mu}([K_2]; \beta_b, \{\alpha_j\}_{j > k_1}).\]

The second formula can be derived in the similar fashion. Here, we have an embedding

\[(4.54) \quad \mu : \overline{M}_{g-1, k+2} \to \overline{M}_{g,k}\]

by gluing last two marked points $x_{k+1}, x_{k+2}$. By (2.18), $\mu$ induces a map on the corresponding finite covers

\[\bar{\mu} : \overline{M}_{g-1, k+2}^{m_\mu} \to \overline{M}_{g,k}^{m_\mu}.\]

We choose a representative $\bar{\mu}(K)$. Let

\[(4.55) \quad \overline{M}_\Sigma = \bar{\mu}(\overline{M}_{g,k+2}^{m_\mu}).\]

Notes that

\[\overline{M}_\Sigma(\mu, g, k, I, J, \nu) = (e_{k+1} \times e_{k+2})^{-1}(\delta).\]

It implies that

\[(4.56) \quad Z(A, \mu, g, k, J, \nu, F, \mu(K)) = Z(A, \overline{M}_\Sigma, J, \nu, F, \mu(K)) = Z(A, m_\mu, g - 1, k + 2, \delta, J, \nu, F, K).\]

By (4.52),

\[(4.57) \quad |Z(A, \mu, g, k, J, \nu, F, \mu(K))| = \sum_{a,b} \eta^{ab} |Z(A, m_\mu, g - 1, k + 2, J, \nu, Y \times \beta_a \times \beta_b, K)|.\]

It yields the second formula

\[(4.58) \quad \Psi^V_{(A,g,k,\mu)}(\theta_{S*}[K_0]; \alpha_1, \cdots, \alpha_k) = \sum_{a,b} \Psi^V_{A,g-1,k+2,m_\mu}([K_0]; \alpha_1, \cdots, \alpha_k, \beta_a, \beta_b) \eta^{ab}.\]

We finish the proof of Theorem 2.10.
5 Stabilizing Conjecture

One of the initial motivations for studying the GW-invariants is to use it to distinguish nondeformation equivalent symplectic manifolds. For example, the first author had successfully calculated the genus-0 GW-invariant in [R1] to produce a large number of diffeomorphic, non-symplectic-deformation equivalent symplectic manifolds, whose existence were unknown in symplectic topology before. During the course of work in [R1], the first author observed a correspondence between the differentiable category of symplectic 4-manifolds \(X\) and the symplectic deformation category of its stabilized manifold \(X \times \mathbb{P}^1\). It can be summarized as the following stabilizing conjecture:

**Stabilizing Conjecture [R1]:** Suppose that \(X, Y\) are two simply connected homeomorphic symplectic 4-manifolds. \(X, Y\) are diffeomorphic iff the stabilized manifolds \(X \times \mathbb{P}^1, Y \times \mathbb{P}^1\) with the product symplectic structures are deformation equivalent.

It has been demonstrated by the Donaldson theory that the differentiable structure of smooth 4-manifolds is a delicate problem. However, by the results of Freedman, the two simply connected 4-manifolds \(X, Y\) are homeomorphic if and only if \(X \times \mathbb{P}^1, Y \times \mathbb{P}^1\) are diffeomorphic. Therefore, the delicate problem about the differentiable structures of smooth 4-manifolds will disappear after the stabilizing process. On the other hand, the stabilizing conjecture can be viewed as an analogy of Freedman’s theorem between the smooth and the symplectic category. The first pair of examples supporting the conjecture were constructed in [R1], where \(X\) is the blow-up of \(\mathbb{P}^2\) at 8-points and \(Y\) is a Barlow surface. Furthermore, the first author also verified the conjecture for the cases: (1). \(X\) is rational, \(Y\) is irrational; (2). \(X, Y\) are irrational but have different number of \((-1)\) curves. Since then, a lot of more evidences supporting the stabilizing Conjecture have been discovered. Note that the first Chern class is an obvious symplectic deformation invariant. The stabilizing conjecture implies that the first Chern class of a simply connected symplectic 4-manifold is a differentiable invariant, which was only proved recently by Taubes [T1]. Recently, the first author was informed by Donaldson that his results on the existence of symplectic submanifolds imply that if \(X \times \mathbb{P}^1, Y \times \mathbb{P}^1\) are symplectic deformation equivalent, then some branched covers of \(X, Y\) are diffeomorphic. In this section, we will compute some higher genus GW-invariants to prove the stabilizing conjecture for the case of simply connected elliptic surfaces \(E^n_{p,q}\). The examples \(E^n_{p,q} \times \mathbb{P}^1\) were suggested to the first author by Donaldson.

Let’s recall the construction of \(E^n_{p,q}\). Let \(E^1\) be the blow-up of \(\mathbb{P}^2\) at generic 9 points, and let \(E^n\) be the fiber connected sum of \(n\) copies of \(E^1\). Then \(E^n_{p,q}\) can be obtained from \(E^n\) by
logarithmic transformations alone two smooth fibers with multiplicity $p$ and $q$. Note that $E^n_{p,q}$ is simply connected if and only if $p, q$ are coprime. Moreover, the Euler number $\chi(E^n_{p,q}) = 12n$, and hence $n$ is a topological number.

**Theorem 5.1.** $E^n_{p,q} \times \mathbb{P}^1, E^n_{p',q'} \times \mathbb{P}^1$ with product symplectic structures are symplectic deformation equivalent if and only if $(p, q) = (p', q')$.

Combining with known results about the smooth classification of $E^n_{p,q}$ by [Ba], [FM], [MO], [MM], we can prove

**Corollary 5.2.** The stabilizing conjecture holds for $E^n_{p,q}$.

Let $F_p, F_q$ be two multiple fibers and $F$ be a general fiber. Let $A_p = [F_p], A_q = [F_q]$. It is known that $A_p = [F]_p, A_q = [F]_q$. The primitive class $A = [F]/pq$. Another piece of topological information is that the canonical class $K$ is Poincare dual to

\[(n - 2)F + (p - 1)F_p + (q - 1)F_q = ((n - 2)pq + (p - 1)p + (q - 1)q)A.\]

Then, the Theorem 5.2 follows from the following calculation

**Proposition 5.3.**

\[
\Psi_{E^n_{p,q} \times \mathbb{P}^1}^{(mA,1,1)}(\overline{\mathcal{M}}_{1,1}; \alpha) = \begin{cases} 
2q(A \cdot \alpha); & m = q(mA = A_p), \\
2p(A \cdot \alpha); & m = p(mA = A_q), \\
0; & m \neq p, q and m < pq,
\end{cases}
\]

where $\alpha$ is a 4-dimensional homology class. In particular,

\[
\Psi_{E^n_{p,q} \times \mathbb{P}^1}^{(mA,1,1)}(\overline{\mathcal{M}}_{1,1}; \cdot) \neq 0 for \ m = p, q.
\]

**Proof:** By the deformation theory of elliptic surfaces, we can choose a complex structure $J_0$ on $E^n_{p,q}$ such that all singular fibers are nodal elliptic curves. Furthermore, we can assume that the complex structures of multiple fibers are generic, i.e., whose $j$-invariants are not 0 or 1728. We shall choose $\nu = 0$. Therefore, there is no need to consider finite covers. We shall drop $\mu$ from the notation. Let $j_0$ be the standard complex structure on $\mathbb{P}^1$.

Let’s describe $\overline{\mathcal{M}}_{mA}(1, 1, J_0 \times j_0, 0)$ for $m < pq$. For any $f \in \overline{\mathcal{M}}_{mA}(1, 1, J_0 \times j_0, 0)$, the image $im(f)$ is a connected effective holomorphic curve. Namely, $im(f) = \sum a_i C_i$ where $a_i > 0$ and $C_i$
are irreducible components. Note that

\[ mA = \sum_i a_i[C_i]. \]  

(5.3)

For the product complex structure \( J_0 \times j_0 \), \( C_i = (C_i^1, C_i^2) \) where \( C_i^1 \subset E_{pq}^n, C_i^2 \subset \mathbb{P}^1 \). \( C_i^2 \) can be realized as a holomorphic map from either an elliptic curve or a rational curve to \( \mathbb{P}^1 \). Hence, \([C_i^2] = p_i[\mathbb{P}^1]\) for \( p_i \geq 0 \). By (5.3), \( p_i = 0 \) and \( C_i^2 = \{x\} \) for some \( x \in \mathbb{P}^1 \). Since \( \text{im}(f) \) is connected. We can write

\[ \text{im}(f) = (\sum a_i C_i^1) \times \{x\}, \]

where \( \sum a_i C_i^1 \) is a connected effective curve. By our assumption on singular fibers, each \( C_i^1 \) is either a multi-section or a fiber. A multi-section has positive intersection with a general fiber. A fiber has zero intersection with general fiber. Since \( mA \cdot [F] = 0 \), this implies that each \( C_i^1 \) is a fiber, i.e., a general fiber, a singular fiber or a multiple fiber. Since \( m < pq \) and a singular fiber has the same homology class of a general fiber, \( C_i^1 \) can only be a multiple fiber. Since \( \text{im}(f) \) is connected, therefore, \( \text{im}(f) \) is either \( F_p \times \{x\} \) or \( F_q \times \{x\} \). In particular,

\[ \mathcal{M}_{mA}(1,1,J_0,0) = \emptyset \text{ for } m \neq p,q \text{ and } m < pq. \]

(5.5)

Obviously, \( (J_0,0) \) is \( mA \)-good for such \( m \)'s. Hence

\[ \Psi_{(mA,1,1)}^{E_{pq}^n \times \mathbb{P}^1}(\mathcal{M}_{1,1}; \alpha) = 0 \text{ for } m \neq p,q \text{ and } m < pq. \]

(5.6)

Fix a marked point \( y_0 \),

\[ \mathcal{M}_{Ap}(1,1,J_0 \times j_0,0) = \{ f : F_p \to E_{pq}^n \times \mathbb{P}^1 \mid \text{im}(f) = F_p \times \{x\} \}/\mathbb{Z}_2 \]

(5.7)

\[ = \text{Aut}(F_p)/\mathbb{Z}_2 \times \mathbb{P}^1 = F_p \times \mathbb{P}^1, \]

because a general element of \( \mathcal{M}_{1,1} \) has automorphism group \( \mathbb{Z}_2 \). Recall the definition of \( Ap \)-goodness (Definition 2.18). Since

\[ \mathcal{M}_{Ap}(1,1,J_0 \times j_0) = \mathcal{M}_{Ap}(1,1,J_0 \times j_0,0). \]

(5.8)

Definition 2.18, (2) is automatically satisfied. Unfortunately, Definition 2.18, (1) is not satisfied. This can also be viewed from the fact that the virtual dimension

\[ 2c_1(V)(mA) + (3 - n)(g - 1) + 2 = 2, \]
but we have a moduli space of real dimension 4. For each \( f \in \mathcal{M}_{A_p}(1, 1, J_0 \times j_0, 0) \), the normal bundle

\[
N_f(E_{p,q}^n \times \mathbb{P}^1) = N_{F_p}(E_{p,q}^n) \otimes T_x \mathbb{P}^1,
\]

where \( \text{im}(f) = F_p \times \{x\} \). It is known that \( N_{F_p}(E_{p,q}^n) \) is a torsion element of order \( p \). Hence,

\[
H^1(N_f) = T_x \mathbb{P}^1.
\]

Furthermore, since \( f \) is an embedding, we have a short exact exact sequence

\[
0 \to TF_p \to T_f E_{p,q}^n \times \mathbb{P}^1 \to N_f \to 0.
\]

It induces a long exact sequence of cohomologies

\[
H^1(F_p) \to H^1(T_f E_{p,q}^n \times \mathbb{P}^1) \to H^1(N_f) \to 0.
\]

Hence, the obstruction space (3.11), (3.17)

\[
Coker(D_f \oplus J \cdot df) = H^1(N_f) = T_x \mathbb{P}^1.
\]

The obstruction bundle

\[
COK = \pi_2^* T \mathbb{P}^1,
\]

i.e., the pull-back of tangent bundle of \( \mathbb{P}^1 \).

Now we need to use following result which is an analogue of Proposition 5.7 in [R2] for genus zero invariants. The proofs are identical. We shall adapt the notation of Lemma 4.6.

**Proposition 5.4.** Suppose that \((J_0, \nu_0)\) is not \( A \)-good, but satisfies the following hypotheses:

\[
1. \quad K \times F \cap \Gamma_A \times \Xi^A_{g,k}(\overline{M}_A(g, k, J_0, \nu_0) - \mathcal{M}(g, k, J_0, \nu_0)) = \emptyset
\]

and hence \( Z(A, g, k, J, \nu, F, K) \) (cf. (4.13)) is compact.

\[
2. \quad \dim Coker(D_f \oplus J \cdot df) \text{ is constant for all } f \in Z(A, g, k, J, \nu, F, K) \text{ and } Z(A, g, k, J, \nu, F, K) \text{ is smooth manifold with the dimension } \dim Coker(D_f \oplus J \cdot df). \text{ For any generic } (J, \nu) \text{ sufficiently close to } (J_0, \nu_0), \text{ } Z(A, g, k, J, \nu, F, K) \text{ is oriented cobordant to the zero set of a transverse section of the obstruction bundle } COK. \text{ Hence it is dual to the Euler class of the obstruction bundle } COK.
\]
Now we continue the proof of Proposition 5.3. Note that $H_1(E^n_{p,q} \times \mathbb{P}^1) = H_2(E^n_{p,q}) \otimes H_2(\mathbb{P}^1)$. Without the loss of generality, suppose that $\alpha = \beta \otimes [\mathbb{P}^1]$. Choose a smooth surface $Y \subset E^n_{p,q}$ such that $Y$ represents $\beta$ and intersects $F_p$ transversely. Then,

\begin{equation}
Z(A, 1, 1, J_0, 0, Y, \mathcal{M}_{1,1}) = \bigcup_{y \in F_p \cap Y} \{y\} \times \mathbb{P}^1. \tag{5.16}
\end{equation}

Then, by Proposition 5.4,

\begin{equation}
\Psi_{(A_{p},1,1)}^{E^n_{p,q} \times \mathbb{P}^1}(\mathcal{M}_{1,1}, \alpha) = (A_p \cdot \alpha) e(T\mathbb{P}^1) = 2q(A_0 \cdot \alpha). \tag{5.17}
\end{equation}

The same proof yields that

\begin{equation}
\Psi_{(A_q,1,1)}^{E^n_{p,q} \times \mathbb{P}^1}(\mathcal{M}_{1,1}, \alpha) = 2p(A_0 \cdot \alpha). \tag{5.18}
\end{equation}

We finish the proof of Proposition 5.3.

**Proof of Theorem 5.1:** First of all, if $(p, q) = (p', q')$, $E^n_{p,q}$, $E^n_{p',q'}$ were known to be complex deformation equivalent as Kähler surfaces regardless where we perform the logarithmic transform. It follows that $E^n_{p,q} \times \mathbb{P}^1$ and $E^n_{p',q'} \times \mathbb{P}^1$ with product symplectic structures are deformation equivalent.

Conversely, suppose that $E^n_{p,q}$, $E^n_{p',q'}$ are symplectic deformation equivalent. Then, there is a diffeomorphism

\begin{equation}
F : E^n_{p,q} \times \mathbb{P}^1 \to E^n_{p',q'} \times \mathbb{P}^1 \tag{5.19}
\end{equation}

such that

\begin{equation}
\Psi_{(F \ast (A),1,1)}^{E^n_{p',q'} \times \mathbb{P}^1}(\mathcal{M}_{1,1}, F \ast (\alpha)) = \Psi_{(A,1,1)}^{E^n_{p,q} \times \mathbb{P}^1}(\mathcal{M}_{1,1}, \alpha), \tag{5.20}
\end{equation}

and

\begin{equation}
F^* c_i(E^n_{p',q'} \times \mathbb{P}^1) = c_i(E^n_{p,q} \times \mathbb{P}^1); F^* p_1(E^n_{p',q'} \times \mathbb{P}^1) = p_1(E^n_{p,q} \times \mathbb{P}^1). \tag{5.21}
\end{equation}

Let $e_0 \in H^2(\mathbb{P}^1, \mathbb{Z})$ be the positive generator. First, we claim

\begin{equation}
F^* (e_0) = e_0. \tag{5.22}
\end{equation}

Suppose that $F^*(e_0) = ne_0 + \beta$ for $\beta \in H^2(E^n_{p,q}, \mathbb{Z})$. Note that the first Pontrjagin class $p_1(E^n_{p,q} \times \mathbb{P}^1) = p_1(E^n_{p,q}) \neq 0$ and $p_1(E^n_{p',q'} \times \mathbb{P}^1) = p_1(E^n_{p',q'}) \neq 0$. Let $\gamma(E^n_{p,q}) \in H^4(E^n_{p,q}, \mathbb{Z})$ be such that
\(\gamma(E^n_{p,q})[E^n_{p,q}] = 1\). Define \(\gamma(E^n_{p',q'})\) in the same way. Then \(p_1(E^n_{p,q})\) is a nonzero multiple of \(\gamma(E^n_{p,q})\) and \(p_1(E^n_{p',q'})\) is a non-zero multiple of \(\gamma(E^n_{p',q'})\). Thus, \(F^*\gamma(E^n_{p',q'}) = \gamma(E^n_{p,q})\). Then,

\[
\begin{align*}
1 &= (\gamma(E^n_{p',q'}) \cup e_0)[E^n_{p',q'} \times P^1] = F^*(\gamma(E^n_{p',q'}) \cup e_0)[E^n_{p,q} \times P^1] \\
&= \gamma(E^n_{p,q}) \cup (ne_0 + \beta)[E^n_{p,q} \times P^1] = n.
\end{align*}
\]

Hence \(n = 1\). Furthermore, \(F^*(e_0^2) = 0\). Then \((e_0 + \beta)^2 = 2e_0\beta + \beta^2 = 0\). Therefore, \(2e_0\beta = 0\) and \(\beta^2 = 0\), consequently, \(\beta = 0\).

\[
(5.24) \quad c_1(E^n_{p,q} \times P^1) = c_1(E^n_{p,q}) + 2e_0.
\]

By (5.21), (5.22),

\[
(5.25) \quad F^*(c_1(E^n_{p',q'})) = c_1(E^n_{p,q}).
\]

However, \(F\) sends primitive classes to primitive classes, so \(F^*(A^*) = A^*\), where \(A^*\) is the Poincare dual of \(A\). Hence, \(F_*(A) = A\) and

\[
(5.26) \quad \Psi_{(E^n_{p',q'} \times P^1)}(\overline{\mathcal{M}_{1,1}}, F_*(\alpha)) = \Psi_{(E^n_{p,q} \times P^1)}(\overline{\mathcal{M}_{1,1}}, F_*(\alpha)).
\]

Suppose that \(q < p\) and \(q' < p'\). Then, \(A_p(= qA)\) and \(A_q(= pA)\) are the first and the second class of \(\{nA\}\) such that \(\Psi\) is nonzero and so are \(A_{p'}\) and \(A_{q'}\). Hence

\[
(5.27) \quad F_*(A_p) = A_{p'}, F_*(A_q) = A_{q'}.
\]

This implies that

\[
p = p', q = q'.
\]

We finish the proof of Theorem 5.1.

Even though \(E^n_{p,q} \times S^2\) are diffeomorphic to each other, they may have different first Chern classes. This problem can be resolved by blowing up \(E^n_{p,q}\) at one point. Namely, if \(E^n_{p,q} \# \overline{\mathbb{P}^2}\) is a blow-up of \(E^n_{p,q}\) at one point, \(E^n_{p,q} \# \overline{\mathbb{P}^2}\) are diffeomorphic to each other and have the same first Chern class up to a diffeomorphism. By a theorem of Wall, they have the same almost complex structure up to a homotopy. Furthermore, we can choose the blow-up loci away from multiple fibers. All the calculations in Theorem 5.3 and Theorem 5.1 can be carried through without change. Then, we show that

**Proposition 5.5.** Let \(X\) be the blow-up of a simple connected elliptic surface. Then, the smooth 6-manifold \(X \times S^2\) admits infinitely many deformation classes of symplectic structures with the same tamed almost complex structure up to a homotopy.
6 The Generalized Witten Conjecture

In this section, we formulate a conjecture on the structure of our invariants. This conjecture was originated by Witten in [W2], but he used path integrals, which are not well accepted by mathematicians. Our only contribution here is putting his arguments on a rigorous mathematical footing. During the course of discussions, we also use the results obtained in this paper to derive several other equations of the generating function rigorously. Those equations were known to physicists [W2], [Ho] in the physical context. Most arguments in this section are due to Witten.

As before, we denote by $\mathcal{U}_{g,k}$ the universal curve over $M_{g,k}$. Then each marked point $x_i$ gives rise to a section, still denoted by $x_i$, of the fibration $\mathcal{U}_{g,k} \to M_{g,k}$. If $K_{U|M}$ denotes the cotangent bundle to fibers of this fibration, we put $L_{(i)} = x_i^*(K_{U|M})$. Following Witten, we put

$$(6.1) \langle \tau_{d_1,\alpha_1} \tau_{d_2,\alpha_2} \cdots \tau_{d_k,\alpha_k} \rangle_g(q) = \sum_{A \in H_2(V,\mathbb{Z})} \Psi^V_{(A,g,k)}([K_{d_1},\cdots,d_k];\{\alpha_i\}) q^A$$

where $\alpha_i \in H_s(V,\mathbb{Q})$ and $[K_{d_1},\cdots,d_k]$ is the Poincaré dual of $c_1(L_{(1)})^{d_1} \cup \cdots \cup c_1(L_{(k)})^{d_k}$ and $q$ is an element of Novikov ring. Symbolically, $\tau_{d,\alpha}$’s denote “quantum field theory operators”. For simplicity, we only consider homology classes of even degree. Choose a basis $\{\beta_a\}_{1 \leq a \leq N}$ of $H_{s,\text{even}}(V,\mathbb{Z})$ modulo torsions. We introduce formal variables $t^a_r$, where $r = 0, 1, 2, \cdots$ and $1 \leq a \leq N$. Witten’s generating function (cf. [W2]) is now simply defined to be

$$(6.2) F^V(t^a_r;q) = \langle e^{\sum_r t^a_r \tau_{r,\beta_a}} \rangle(q) = \sum \prod_{r,a} \frac{(t^a_r)^{n_{r,a}}}{n_{r,a}!} \langle \prod_{r,a} t^a_r \tau_{r,\beta_a} \rangle(q)$$

where $n_{r,a}$ are arbitrary collections of nonnegative integers, almost all zero, labeled by $r, a$. The summation in (6.2) is over all values of the genus $g$ and all homotopy classes $A$ of $(J, \nu)$-maps. Sometimes, we write $F^V_g$ to be the part of $F^V$ involving only GW-invariants of genus $g$. It is clear that $F^V$ is a generalization of the prepotential $\Phi^V = F^V_0$ of genus 0 invariants (cf. [RT], section 9). Indeed this generalized function contains more information on the underlying manifold, for instance, using Taubes’ theorem [T2], one observes (cf. [T]) that for a minimal algebraic surface $V$ of general type,

$$(6.3) F^V(t^a_r;q) = F^V(t^a_r;0) + q^{K_V} e^{m_{0,0}} + \cdots,$$

while $\Phi^V$ depends only the intersection form of $V$.

One of Witten’s goals is to find out the equations which $F^V$ satisfies. The case that $V$ is a point corresponds to the topological gravity, where $F^V$ is governed by a complete set of solution-
the KdV hierarchy, conjectured by Witten ([W2]) and clarified by Kontsevich ([Ko]). In general, it is not clear what is (or there exists at all) the complete set of equations which \( F^V \) satisfies, though there are partial results for \( V = \mathbb{C}P^1 \) (see [EY]). However, in [W2], Witten made a conjecture on \( F^V \), which we will describe in this section.

First of all, we have obtained several important recursion formulas in section 2 about \( \Psi \). In general, we can always rewrite a recursion formula as a differential equation of the generating function. Assume that \( \beta_1 = [V] \). Following Witten’s arguments in [W2], one can deduce from (2.15) that \( F^V \) satisfies the generalized string equation:

\[
\frac{\partial F^V}{\partial t^0_0} = \frac{1}{2} \eta_{ab} t^a_0 t^b_0 + \sum_{i=0}^{\infty} \sum_{a} t^a_{i+1} \frac{\partial F^V}{\partial t^0_i}.
\]

For the reader’s convenience, we reproduce the arguments here.

**Lemma 6.1.** Suppose that \((V, \omega)\) is a semi-positive symplectic manifold. Then, the generating function \( F^V \) satisfies the generalized string equation

\[
\frac{\partial F^V}{\partial t^0_0} = \frac{1}{2} \eta_{ab} t^a_0 t^b_0 + \sum_{i=0}^{\infty} \sum_{a} t^a_{i+1} \frac{\partial F^V}{\partial t^0_i}.
\]

**Proof:** By (2.15),

\[
\Psi_{(A,g,k+1)}^V([K_{d_0,d_1,\ldots,d_k}];[V],\alpha_1,\ldots,\alpha_k) = \Psi_{(A,g,k)}^V([\pi(K_{d_0,d_1,\ldots,d_k})];\alpha_1,\ldots,\alpha_k),
\]

where for convenience, we choose to forget the first marked point instead of the last marked point as in Proposition 2.15. We choose \( d_0 = 0 \), i.e., \( c_1(L_{(1)})^{d_0} = 1 \). Next, let us find \([\pi(k_{d_0,d_1,\ldots,d_k})]\) for \((g,k) \neq (0,2),(1,0)\).

Let’s use \( L'_{(j)} \) to denote the corresponding line bundle over \( \overline{M}_{g,k} \). Then, it is natural to compare \( L_{(j)} \) with \( \pi^*L'_{(j)} \). It was known in algebraic geometry that

\[
L_{(j)} = \pi^*L'_{(j)} + D_j,
\]

where \( D_i \) is the divisor consisting of the stable curves where \( x_0, x_i \) are in a rational component with only three special points. Hence,

\[
c_1(L_{(j)})^m = c_1(\pi^*L'_{(j)})^m + D_j \sum_{i=1}^{m-1} c_1(L_{(j)})^i c_1(\pi^*L'_{(j)})^{m-i-1}.
\]

Furthermore,

\[
c_1(L_{(j)}) \cap [D_j] = 0; [D_i] \cap [D_j] = 0 \text{ for } i \neq j.
\]
Therefore,

\[(6.9) \quad c_1(\mathcal{L}(j))^m = c_1(\pi^*\mathcal{L}'(j))^m + D_j c_1(\pi^*\mathcal{L}'(j))^{m-1}\]

and

\[(6.10) \quad c_1(\mathcal{L}(1))^{d_1} \cup \cdots \cup c_1(\mathcal{L}(d_k))^{d_k} = (\pi^*\mathcal{L}'(1))^{d_1} \cup \cdots \cup (\pi^*\mathcal{L}'(1))^{d_k} + \sum_{j=1}^{k} ([D_j] \cap \bigcup_{i=1}^{n} c_1(\pi^*\mathcal{L}(i))^{d_i-\delta_{ij}}).\]

Note that \([\pi(D_j)] = \overline{\mathcal{M}}_{g,k}\). Then,

\[(6.11) \quad \Psi^V_{(A,g,k)}([\pi(K_{d_0,d_1,\ldots,d_k})]; \alpha_1, \cdots, \alpha_k) = \Psi^V_{(A,g,k)}([K_{d_1,\ldots,d_k}]; \alpha_1, \cdots, \alpha_k) + \sum_{j=1}^{k} \Psi^V_{(A,g,k)}([K_{d_1,\ldots,d_j-1,\ldots,d_k}]; \alpha_1, \cdots, \alpha_k).\]

For the dimension reason, the first term is zero. Therefore, if \((g,k) \neq (0,2), (1,0)\), we have

\[(6.12) \quad < \tau_{0,1} \prod_{i=1}^{k} \tau_{d_i,\alpha_i} > = \sum_{j=1}^{k} < \prod_{i=1}^{k} \tau_{d_i-\delta_{i,j},\alpha_i} >,\]

where we simply define \(\tau_{r,\alpha} = 0\) for \(r < 0\).

There are two exceptional cases for the previous arguments, namely, \(g = 0, k = 2\) and \(g = 1, k = 0\). In those special cases, forgetting one marked point will result in an unstable curve. For the special case \(g = 0, k = 2\), since \(\overline{\mathcal{M}}_{0,3} = pt\), the only non-zero term is

\[\Psi^V_{(0,0,3)}([\overline{\mathcal{M}}_{0,3}]; [V], \beta_a, \beta_b) = \eta_{a,b}.\]

Moreover, one can show that

\[(6.13) \quad \Psi^V_{(0,0,3)}([\overline{\mathcal{M}}_{0,3}]; [V], \beta_a, \beta_b) = \eta_{a,b}.\]

In the case that \(g = 1, k = 0\), for the dimension reason, we have that

\[(6.14) \quad \Psi^V_{(1,1,1)}([\overline{\mathcal{M}}_{1,1}]; [V]) = 0.\]

Therefore, we have an equation

\[(6.15) \quad < \tau_{0,1} \prod_{i=1}^{k} \tau_{d_i,\alpha_i} > = \sum_{j=1}^{k} < \prod_{i=1}^{k} \tau_{d_i-\delta_{i,j},\alpha_i} > + \delta_{k,2} \delta_{d_1,0} \delta_{d_2,0} \eta_{a_1,a_2},\]

The corresponding equation for the generating function is the generalized string equation

\[\frac{\partial F}{\partial t_0} = \frac{1}{2} \eta_{ab} t_0^a t_0^b + \sum_{a=0}^{\infty} \sum_{i=1} t_{i+1}^a \frac{\partial F}{\partial t_i^a}\]
We can choose $d_0 = 1$ and obtain another equation for $F^V$.

**Lemma 6.2.** $F_g$ satisfies dilation equation

\[
\frac{\partial F_g}{\partial t_1} = (2g - 2 + \sum_{i=1}^{\infty} \sum_{a} t_i^a \frac{\partial}{\partial t_i^a}) F_g + \frac{\chi(V)}{24} \delta_{g,1},
\]

where $\chi(V)$ is the Euler characteristic of $V$.

**Proof:** We choose $d_0 = 1$. Repeating the analysis we just have, we get

\[
c_1(\mathcal{L}_0) \bigcup_{j=1}^{k} c_1(\mathcal{L}_{(i)})^{d_i} = c_1(\mathcal{L}_0) \bigcup_{j=1}^{k} c_1(\mathcal{L}'_{(i)})^{d_i}.
\]

On the other hand, one has a natural identification

\[
\alpha : \overline{\mathcal{M}}_{g,k+1} \cong \overline{\mathcal{T}}_{g,k}.
\]

Furthermore,

\[
\mathcal{L}_0 = \alpha^*(\mathcal{K}_{\mathcal{M}[\mathcal{M}]}) \otimes \bigcup_{j=1}^{n} \mathcal{O}(D_j).
\]

Note that $\pi[\mathcal{K}_{\mathcal{M}[\mathcal{M}]}] = 2g - 2$. Therefore, modulo the exceptional case we have

\[
< \tau_{1,1} \prod_{i=1}^{k} \tau_{d_i, \alpha_i} >_g = (2g - 2 - n) < \prod_{i=1}^{k} \tau_{d_i, \alpha_i} >_g .
\]

Since $\overline{\mathcal{M}}_{0,3} = pt$ and $c_1(\mathcal{L}_0)$ is a nontrivial class, the contribution from the exceptional case $g = 0, k = 2$ is zero. The exceptional case $g = 1, k = 0$ corresponds to

\[
\Psi_{(0,1,1)}([K_1]; [V]).
\]

For the dimension reason, $A$ has to be zero. Moreover,

\[
\dim \overline{\mathcal{M}}_{1,1} = 1; [K_1] = \frac{1}{24} \{pt\}.
\]

Now we fix a generic element $(\Sigma_1, x) \in \overline{\mathcal{M}}_{1,1}$ and let $J_0$ be the any almost complex structure. Then

\[
K \times F \cap \mathcal{Y}_0 \times \Xi_{1,1}^0 = \{ f : (\Sigma, x) \to V | Im(f) = pt \} = V.
\]

Furthermore, $Coker(Df \oplus J \cdot df)$ can be canonically identified with $T_{Im(f)}V$. Therefore, $(J_0, 0)$ satisfies the requirement of Proposition 5.4. Hence, by Proposition 5.4,

\[
\Psi_{(0,1,1)}([pt], [V]) = c(TV) = \chi(V).
\]
Therefore, the contribution from the exceptional case is

\[(6.25) \quad \frac{1}{24} \chi(V)\]

and

\[(6.26) \quad <\tau_{1,1} \prod_{i=1}^{k} \tau_{d_i,\alpha_i} >_g = (2g - 2 - k) <\prod_{i=1}^{k} \tau_{d_i,\alpha_i} >_g + \frac{1}{24} \chi(V) \delta_{g,1} \delta_{k,0}.\]

In terms of the generating function, this is equivalent to the following differential equation

\[(6.27) \quad \frac{\partial F_g}{\partial t_1} = (2g - 2 + \sum_{i=1}^{\infty} \sum_{a} t_{a}^{n} \frac{\partial}{\partial t_{a}^{n}}) F_{g} + \frac{\chi(V)}{24} \delta_{g,1},\]

which coincides with the formula derived by Hori [Ho] using a different definition.

In the dilation equation, we have an unpleasant term $2g - 2$ to prevent us to write it as equation of $F^V$. As pointed out by Witten, there is a dimension constraint

\[(6.28) \quad c_1(V)(A) + (3 - n)(g - 1) + k = \sum_{i} (d_i + \text{cod}(\beta_i)),\]

can be rewritten as an equation

\[(6.29) \quad (\sum_{i=1}^{\infty} \sum_{a} (i - 1 + q_a)t_{a}^{n} \frac{\partial}{\partial t_{a}^{n}} - c_1(A) - (3 - n)(g - 1)) F_{g} = 0,\]

where $2q_{a} = \text{cod}(\beta_{a})$. Combining the above equations, one can deduce

**Corollary 6.3.** When $c_1 = 0$, $F^V$ satisfies dilation equation

\[(6.29) \quad \frac{\partial F^V}{\partial t_1} = \sum_{i=1}^{\infty} \sum_{a} (\frac{2}{3 - n}(i - 1 + q_a) + 1)t_{a}^{n} \frac{\partial F^V}{\partial t_{a}^{n}} + \frac{\chi(V)}{24}.\]

Similarly, we can also use (2.15) to derive the equations (for $d_0 = 0, 1$) of the generating function.

Following Witten, one can introduce

\[(6.30) \quad U = \frac{\partial^2 F^V}{\partial t_{0,1} \partial t_{0,\sigma}}, \quad U' = \frac{\partial^3 F^V}{\partial t_{0,1}^2 \partial t_{0,\sigma}}, \quad \ldots, \quad U_{(l)} = \frac{\partial^{l+2} F^V}{\partial t_{0,1}^{l+1} \partial t_{0,\sigma}}, \quad \text{for} \quad l \geq 0\]

We will regard $U_{(l)}$ to be of degree $l$. By a differential function of degree $k$ we mean a function $G(U, U', U'', \cdots)$ of degree in that sense. In particular, any function of the form $G(U)$ is of degree zero, and $(U')^2$ has degree two.

**Witten Conjecture.** For every $g \geq 0$, there are differential functions $G_{m,a,n,b}(U_\sigma, U'_\sigma, \cdots)$ of degree $2g$ such that

\[(6.31) \quad \frac{\partial^2 F_g}{\partial t_{m,a} \partial t_{n,b}} = G_{m,a,n,b}(U_\sigma, U'_\sigma, \cdots)\]
up to and including terms of genus g.

It was pointed out by Witten (cf. [W2]) that the composition law implies

\[ \frac{\partial^3 F_0}{\partial t_{d_1,a_1} \partial t_{d_2,a_2} \partial t_{d_3,a_3}} = \sum_{a,b} \frac{\partial^2 F_0}{\partial t_{d_1-1,a_1} \partial t_{0,a}} \eta^{ab} \frac{\partial^3 F_0}{\partial t_{0,b} \partial t_{d_2,a_2} \partial t_{d_3,a_3}} \]

and consequently, the conjecture for \( g = 0 \).

Recall that in the genus 0 case, WDVV equation is a direct consequence of the composition law. In the higher genus case, it is unclear if the composition law is helpful at all to derive the equation and solve Witten’s conjecture. Here, we state a closely related conjecture.

**Conjecture 6.4.** \( \langle \tau_{d_1,a_1} \tau_{d_2,a_2} \cdots \tau_{d_k,a_k} \rangle_g \) can be reduced to enumerative invariants \( \Psi^V_{(A,g,k)}(\overline{M}_{g,k}; \cdots) \).

**Proposition 6.5.** Conjecture 6.4 holds for \( g \leq 1 \)

A special case that \( g = 1 \) and \( V = \text{CP}^1 \) was checked in [W2].

**Proof:** First we assume that any \( c_1(L_{(i)}) \) is Poincare dual to a divisor in \( \overline{M}_{g,k} \setminus M_{g,k} \) for \( g \leq 1 \). Then any cycle \( [K_{d_1,\ldots,d_k}] \) can be represented by a cycle in the boundary \( \overline{M}_{g,k} \setminus M_{g,k} \) so long as \( d_1 + \cdots + d_k > 0 \). It follows from the composition law that \( \langle \tau_{d_1,a_1} \tau_{d_2,a_2} \cdots \tau_{d_k,a_k} \rangle_g \) can be computed in terms of \( \langle \tau_{d_1,a_1} \tau_{d_2,a_2} \cdots \tau_{d_l,a_l} \rangle_{g'} \) with either \( l < k \) or \( g' < g \). Then the proposition follows from a standard induction.

Next we check our assumption stated at the beginning. Given any two points \( x_1, x_2 \) in \( S^2 \), one can construct a canonical meromorphic section

\[ s_{x_1,x_2} = \frac{(x_1 - x_2)dz}{(z - x_1)(z - x_2)}. \]

This section has simple poles at \( x_1, x_2 \). Moreover, for any \( \sigma \in SL(2, \mathbb{C}) \), \( \sigma^* s_{\sigma(x_1), \sigma(x_2)} = s_{x_1,x_2} \).

The moduli space \( M_{0,k} \) (\( k \geq 3 \)) is the quotient of \( (S^2)^k \setminus \Delta_k \) by \( SL(2, \mathbb{C}) \), where \( SL(2, \mathbb{C}) \) acts on \( (S^2)^k \) diagonally, and \( \Delta \) denotes the set of \( (x_1, \ldots, x_k) \) with \( x_i = x_j \) for some \( i, j \). Notice that the universal family \( U_{0,k} \) is biholomorphic to \( S^2 \times M_{0,k} \). Then by the invariance of \( s_{x_1,x_2} \) under \( SL(2, \mathbb{C}) \), one can define a section section \( s \) of the relative cotangent bundle over \( U_{0,k} \), such that \( s(z; x_1, \ldots, x_k) = s_{x_1,x_2}(z) \). For any \( i \geq 3 \), this \( s \) restricts to an nonvanishing section \( s_i \) of \( L_{(i)} \) over \( M_{0,k} \), i.e., \( s_i(x_1, \ldots, x_k) = s(x_i) \). Clearly, each \( s_i \) extends to be a meromorphic section on \( M_{0,k} \).

Therefore, \( c_1(L_{(i)}) \) must be Poincare dual to a divisor in \( \overline{M}_{0,k} \setminus M_{0,k} \) for \( i \geq 3 \). Similarly, one can also show this for \( i \leq 3 \).

Now let \( g = 1 \). Note that each torus \( T \) is a branched covering of \( S^2 \) of degree. There are four branched points, say \( x_1, x_2, 3, x_4 \). Conversely, given any \( x_1, x_2, 3, x_4 \), one can construct a branched
covering $T$ with those as branched points. The resulting torus $T$ is uniquely determined by the orbit of $(x_1, \cdots, x_4)$ in $(S^2)^4$ by $SL(2, \mathbb{C})$. Let $\pi : T \rightarrow S^2$ be the branched covering map. Then $\pi^*(s_{x_1, x_2} s_{x_3, x_4})$ defines a nonvanishing section $s_T$ of $K_T^2$ over $T$. Using the invariance of $s_T$ under $SL(2, \mathbb{C})$, we can easily construct a nonvanishing section of the relative canonical bundle over $\mathcal{U}_{1,1}$, which can be extended meromorphically to $\overline{\mathcal{U}}_{1,1}$. It follows that any $c_1(L_{ij})$ is Poincaré dual to a divisor in $\overline{\mathcal{M}}_{1,k} \setminus \mathcal{M}_{1,k}$ for any $k \geq 1$.

**References**

[Ba] S. Bauer, Diffeomorphism types of elliptic surfaces with $p_g = 1$, (Warwick Preprint, 1992).

[CF] I. Ciocan-Fontanine, Quantum cohomology of flag varieties, Preprint.

[Do] S. Donaldson, Polynomial invariants for smooth four manifolds, Topology, 29 (1990), 257-315.

[EY] T. Eguchi and S-K. Yang, The topological $\mathbb{P}^1$ model and the large-N matrix integral, Modern Phy Letters A, Vol. 9, No. 31 (1994) 2893-2902.

[FM] R. Friedman and J. Morgan, On the diffeomorphism types of certain algebraic surfaces I, II, J. Diff. Geom., 27 (1988).

[FU] D. Freed and K. Uhlenbeck, Instantons and four-manifolds, Springer-Berlag.

[Gr] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. math., 82 (1985), 307-347.

[Ho] H. Hori, Constraints for topological strings in $D \geq 1$, preprint.

[KM] M. Kontsevich and Y. Manin, GW classes, Quantum cohomology and enumerative geometry, preprint.

[Ko] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix airy function, Comm. Math. Phys., 147 (1992).

[LT] J. Li and G. Tian, Quantum cohomology of homogeneous manifolds, to appear in J. Alg. Geom..

[Mc] C. Mccory, Cone complex and PL-transversality, Trans. Amer. Math. Soc. 207(1975), 269-291.
[M1] D. McDuff, Examples of symplectic structures, Invent. math., 89(1987), 13-36.

[M2] D. McDuff, Notes on ruled symplectic 4-manifolds, preprint.

[MS] D. McDuff and D. Salamon, J-holomorphic curves and quantum cohomology, University Lect. Series, vol. 6, AMS.

[MO] J. Morgan and K O'Grady, The smooth classification of fake K3's and similar surfaces, preprint.

[MM] J. Morgan and T. Mrowka, On the diffeomorphism classification of regular elliptic surfaces, (preprint)

[Mu] D. Mumford, Towards an enumerative geometry of the moduli space of curves, *Arithmetic and Geometry* edited by M. Artin, J. Tate.

[PW] T. Parker and J. Wolfson, A compactness theorem for Gromov's moduli space, J. Geom. Analysis, 3 (1993), 63-98.

[R] Y. Ruan, Topological Sigma model and Donaldson type invariants in Gromov theory, to appear in Duke Jour.

[R1] Y. Ruan, Symplectic topology on algebraic 3-folds, Jour. Diff. Geom., 39 (1994), 215-227.

[R2] Y. Ruan, Symplectic topology and extremal rays, Geo, Fun, Anal, Vol 3, no 4 (1993), 395-430.

[R3] Y. Ruan, Symplectic topology and complex surfaces, *Geometry and analysis on complex manifold* edited by T. Mabuchi, J. Noguchi, T. Ochiai, World Scientific.

[RT] Y. Ruan and G. Tian, A mathematical theory of quantum cohomology, To appear in J. Diff. Geom., 1995; announcement, Math. Res. Let., vol 1, no 1 (1994), 269-278.

[RT1] Y. Ruan and G. Tian, Bott-type symplectic Floer cohomology and its multiplication structures, preprint, 1995.

[T1] C. Taubes, The Seiberg-Witten invariants and symplectic forms, Math. Res. Letters 1(1995) 809-822.

[T2] C. Taubes, The Seiberg-Witten and the Gromov invariants, preprint, 1995.

[T] G. Tian, The quantum cohomology and its associativity, preprint, March, 1995.
[W1] E. Witten, Topological sigma models, Comm. Math. Phys., 118 (1988).

[W2] E. Witten, Two dimensional gravity and intersection theory on moduli space, Surveys in Diff. Geom., 1 (1991), 243-310.

[Ye] R. Ye, Gromov’s compactness theorem for pseudo-holomorphic curves, Trans. Amer. math. Soc., 1994.