Two Theorems on Hunt’s Hypothesis (H) for Markov Processes

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Abstract

We investigate the invariance of Hunt’s hypothesis (H) for Markov processes under two classes of transformations, which are change of measure and subordination. Our first theorem shows that for two standard processes $(X_t)$ and $(Y_t)$, if $(X_t)$ satisfies (H) and $(Y_t)$ is locally absolutely continuous with respect to $(X_t)$, then $(Y_t)$ satisfies (H). Our second theorem shows that a standard process $(X_t)$ satisfies (H) if and only if $(X_{\tau_t})$ satisfies (H) for some (and hence any) subordinator $(\tau_t)$ which is independent of $(X_t)$ and has a positive drift coefficient. Applications of the two theorems are given.

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1 Introduction and main results

Let $E$ be a locally compact space with a countable base (LCCB) and $X = (X_t, P^x)$ be a standard Markov process on $E$ as described in Blumenthal and Getoor [2]. Denote by $\mathcal{B}$ and $\mathcal{B}^n$ the family of all Borel measurable subsets and nearly Borel measurable subsets of $E$, respectively. For $D \subset E$, we define the first hitting time of $D$ by

$$T_D = \inf\{t > 0 : X_t \in D\}.$$ 

A set $D \subset E$ is called thin if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P^x(T_C = 0) = 0$ for any $x \in E$. $D$ is called semipolar if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P^x(T_C < \infty) = 0$ for any $x \in E$. Let $m$ be a measure on $(E, \mathcal{B})$. $D$ is called $m$-essentially polar if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P^m(T_C < \infty) = 0$. Hereafter $P^m(\cdot) := \int_E P^x(\cdot)m(dx)$.

Hunt’s hypothesis (H) says that “every semipolar set of $X$ is polar”. This hypothesis plays a crucial role in the potential theory of (dual) Markov processes. For example, it is known that if $X$ is in duality with another standard process $\hat{X}$ on $E$ with respect to a $\sigma$-finite reference measure $m$, then (H) is equivalent to many potential principles for Markov processes, e.g., the bounded maximum principle (cf. [2]); (H) holds if and only if the fine and cofine topologies differ by polar sets ([3, Proposition 4.1] and [12, Theorem 2.2]); (H) holds if and only if every natural additive functional of $X$ is in fact a continuous additive functional ([2, Chapter VI]).

In spite of its importance, (H) has been verified only in special situations. Some fifty years ago, R.K. Getoor conjectured that essentially all Lévy processes satisfy (H). This conjecture stills remains open and is a major unsolved problem in the potential theory for Lévy processes. The reader is referred to Kesten [23], Port and Stone [31], Blumenthal and Getoor [3], Bretagnolle [5],
Forst [10], Kanda [21], Rao [32], Kanda [22], Glover and Rao [13], and Rao [33] for the results that were obtained before 1990. The reader is also referred to Hu and Sun [14], Hu et al. [17], Hu and Sun [15], and Hu and Sun [16] for the recent results on Getoor’s conjecture.

In this paper, we study Hunt’s hypothesis (H) for general Markov processes from the point of view of transformations. We will present two new theorems on (H), which imply that various classes of Markov processes satisfy (H).

We fix an isolated point ∆ which is not in \( E \) and write

\[ E_\Delta = E \cup \{ \Delta \} \]

Consider the following objects:

(i) \( \Omega \) is a set and \( \omega_\Delta \) is a distinguished point of \( \Omega \).

(ii) For \( 0 \leq t \leq \infty \), \( Z_t: \Omega \to E_\Delta \) is a map such that if \( Z_t(\omega) = \Delta \) then \( Z_s(\omega) = \Delta \) for all \( s \geq t \), \( Z_\infty(\omega) = \Delta \) for all \( \omega \in \Omega \), and \( Z_0(\omega_\Delta) = \Delta \).

(iii) For \( 0 \leq t \leq \infty \), \( \theta_t: \Omega \to \Omega \) is a map such that \( Z_s \circ \theta_t = Z_{s+t} \) for all \( s, t \in [0, \infty] \), and \( \theta_\infty(\omega) = \omega_\Delta \) for all \( \omega \in \Omega \).

We define in \( \Omega \) the \( \sigma \)-algebras

\[ F_0 = \sigma(Z_t: t \in [0, \infty]) \] and \[ F_0^t = \sigma(Z_s: s \leq t) \] for \( 0 \leq t < \infty \).

Denote

\[ \zeta(\omega) = \inf\{ \omega : Z_t(\omega) = \Delta \}, \quad \omega \in \Omega. \]

Let \( m \) be a measure on \((E, B)\). We define

\[ (H_m): \text{every semipolar set is } m\text{-essentially polar}. \]

Note that if a standard process \( X \) has resolvent densities with respect to \( m \), then \( X \) satisfies (H) if and only if \( X \) satisfies \((H_m)\) (cf. [2, Propositions II.2.8 and II.3.2]).

Now we can state the first main result of this paper.

**Theorem 1.1** Let \( X = (\Omega, M^X, \mathcal{M}_t^X, Z_t, \theta_t, P^x) \) and \( Y = (\Omega, M^Y, \mathcal{M}_t^Y, Z_t, \theta_t, Q^x) \) be two standard processes on \( E \) such that \( \mathcal{M}_t^X \cap \mathcal{M}_t^Y \supset F_0^t \) and \( \mathcal{M}_t^X \cap \mathcal{M}_t^Y \supset F_0^t \) for \( 0 \leq t < \infty \).

(i) Suppose that \( X \) satisfies (H) and for any \( x \in E \) and \( t > 0 \), \( Q^x|_{F_0^t} \) is absolutely continuous with respect to \( P^x|_{F_0^t} \) on \( \{ t < \zeta \} \). Then \( Y \) satisfies (H).

(ii) Suppose that \( X \) satisfies \((H_m)\) for some measure \( m \) on \((E, B)\) and for any \( x \in E \) and \( t > 0 \), \( Q^x|_{F_0^t} \) is absolutely continuous with respect to \( P^x|_{F_0^t} \) on \( \{ t < \zeta \} \). Then \( Y \) satisfies \((H_m)\).

A subordinator \( \tau = (\tau_t) \) is a 1-dimensional increasing Lévy process with \( \tau_0 = 0 \). Let \( X = (X_t) \) be a standard process on \( E \) and \( \tau \) be a subordinator which is independent of \( X \). The standard process \((X_\tau)\) is called the subordinated process of \((X_t)\). The idea of subordination originated from Bochner (cf. [4]). Our second theorem is motivated by the following remarkable result of Glover and Rao.

**Theorem 1.2** (Glover and Rao [13]) Let \((X_t)\) be a standard process on \( E \) and \((\tau_t)\) be a subordinator which is independent of \( X \) and satisfies (H). Then \((X_\tau)\) satisfies (H).
It is known that if a subordinator \((\tau_t)\) satisfies (H), then it must be a pure jump subordinator, i.e., its drift coefficient equals 0 ([14, Proposition 1.6]). Up to now, it is still unknown if any pure jump subordinator satisfies (H). We present the following new theorem on the equivalence between (H) for \(X\) and (H) for its time changed process.

**Theorem 1.3** Let \((X_t)\) be a standard process on \(E\) and \(m\) be a measure on \((E,\mathcal{B})\). Then,

(i) \((X_t)\) satisfies (H) if and only if \((X_{\tau_t})\) satisfies (H) for some (and hence any) subordinator \((\tau_t)\) which is independent of \((X_t)\) and has a positive drift coefficient.

(ii) \((X_t)\) satisfies \((H_m)\) if and only if \((X_{\tau_t})\) satisfies \((H_m)\) for some (and hence any) subordinator \((\tau_t)\) which is independent of \((X_t)\) and has a positive drift coefficient.

The rest of this paper is organized as follows. In Sections 2 and 3, we give the proofs of Theorems 1.1 and 1.3 respectively. Applications of Theorems 1.1 and 1.3 will be given in Sections 4 and 5 respectively.

### 2 Proof of Theorem 1.1

#### 2.1 Preliminary lemmas

Before proving Theorem 1.1, we give two lemmas. The first one is a well-known result, but we present it here for the reader’s convenience.

**Lemma 2.1** Let \(X = (X_t, P^x)\) be a standard process on \(E\) and \(m\) be a measure on \((E,\mathcal{B})\).

(i) If any thin set \(A \in \mathcal{B}\) is polar, then \(X\) satisfies (H).

(ii) If any thin set \(A \in \mathcal{B}\) is \(m\)-essentially polar, then \(X\) satisfies \((H_m)\).

**Proof.** The proof of (ii) is similar to that of (i). So we only prove (i) below.

Let \(A\) be a semipolar set. We will show that \(A\) is a polar set. By the definitions of semipolar set and polar set, we may assume without loss of generality that \(A\) is a thin nearly Borel measurable set. Then, we have \(P^x(T_A = 0) = 0\) for any \(x \in E\).

We fix an \(x \in E\). By the definition of nearly Borel measurable set, there exist two Borel sets \(A_1\) and \(A_2\) such that \(A_1 \subset A \subset A_2\) and

\[
P^x(\{\omega : X_t(\omega) \in A_2 - A_1 \text{ for some } t \in [0, \infty)\}) = 0,
\]

which implies that

\[
T_{A_1} = T_A = T_{A_2}, \quad P^x\text{-a.s.} \tag{2.1}
\]
Then, \( A_1 \in \mathcal{B} \) is a thin set. By the assumption of the lemma, we know that \( A_1 \) is a polar set. Therefore, \( P^x(T_{A_1} < \infty) = 0 \), which together with (2.1) implies that

\[
P^x(T_A < \infty) = 0.
\]

Since \( x \in E \) is arbitrary, this implies that \( A \) is a polar set. \( \square \)

Let \( X = (\Omega, \mathcal{M}^X, \mathcal{M}^T, Z_t, \theta_t, P^x) \) and \( Y = (\Omega, \mathcal{M}^Y, \mathcal{M}^T, Z_t, \theta_t, Q^x) \) be two standard processes on \( E \) such that \( \mathcal{M}^X \cap \mathcal{M}^Y \supset \mathcal{F}^0 \) and \( \mathcal{M}^T \cap \mathcal{M}^T \supset \mathcal{F}^0_t \) for \( 0 \leq t < \infty \). Suppose that for any \( x \in E \) and \( t > 0 \), \( Q^x|_{\mathcal{F}_t^0} \) is absolutely continuous with respect to \( P^x|_{\mathcal{F}_t^0} \) on \( \{t < \zeta\} \).

Let \( A \) be a subset of \( E \). For \( w \in \Omega \), we define

\[
D_A(\omega) := \inf \{t \geq 0 : Z_t(\omega) \in A\}, \quad T_A(\omega) := \inf \{t > 0 : Z_t(\omega) \in A\}.
\]

Then,

\[
\left\{ \frac{1}{l} + D_A(\theta_t^l w) \right\} \downarrow T_A(w) \text{ as } l \uparrow \infty. \tag{2.2}
\]

By (2.2), we know that for \( t > 0 \),

\[
\{T_A < t\} = \bigcup_{l=1}^{\infty} \left\{ \frac{1}{l} + D_A \circ \theta_t^l < t \right\}, \tag{2.3}
\]

and

\[
\{T_A \geq t\} = \bigcap_{l=1}^{\infty} \left\{ \frac{1}{l} + D_A \circ \theta_t^l \geq t \right\}. \tag{2.4}
\]

**Lemma 2.2** Let \( A \in \mathcal{B} \). Then, for any \( x \in E \) and \( t > 0 \),

(i) if \( P^x(T_A < t, t < \zeta) = 0 \), then \( Q^x(T_A < t, t < \zeta) = 0 \);

(ii) if \( P^x(T_A > t, t < \zeta) = 0 \), then \( Q^x(T_A > t, t < \zeta) = 0 \).

We would like to point out that the proof of Lemma 2.2 is far from trivial. Define \( \mathcal{F}^X \) to be the completion of \( \mathcal{F}^0 \) with respect to \( \{P^\mu : \mu \text{ is a finite measure on } E_\Delta\} \), and define \( \mathcal{F}_t^X \) to be the completion of \( \mathcal{F}_t^0 \) in \( \mathcal{F}^X \) with respect to \( \{P^\mu : \mu \text{ is a finite measure on } E_\Delta\} \) for \( t \in [0, \infty) \). Let \( A \in \mathcal{B} \). By [2, Theorem I.10.7], we know that \( T_A \) is a stopping time relative to \( \{\mathcal{F}_t^X\} \). Since \( \mathcal{F}_t^X \) contains all null sets in \( \mathcal{F}^X \), in general, the assumption that \( Q^x|_{\mathcal{F}_t^0} \) is absolutely continuous with respect to \( P^x|_{\mathcal{F}_t^0} \) on \( \{t < \zeta\} \) does not imply that \( Q^x|_{\mathcal{F}_t^X} \) is absolutely continuous with respect to \( P^x|_{\mathcal{F}_t^X} \) on \( \{t < \zeta\} \).

**Proof of Lemma 2.2** Step 1. If \( A \) is an open set, then \( T_A \) is an \( \{\mathcal{F}^0_{t+}\} \) stopping time, where \( \mathcal{F}^0_{t+} = \bigcap_{s> t} \mathcal{F}^0_s \). Note that

\[
\{T_A < t\} \cap \{t < \zeta\} \in \mathcal{F}^0_t \cap \{t < \zeta\},
\]
and
\[ \{ T_A > t \} \cap \{ t < \zeta \} = \bigcup_{l=1}^{\infty} \left( \{ T_A > t \} \cap \left\{ \frac{1}{l} + \frac{1}{l} < \zeta \right\} \right) \in \bigcup_{l=1}^{\infty} \left( \mathcal{F}_{t+\frac{1}{l}}^0 \cap \left\{ t + \frac{1}{l} < \zeta \right\} \right). \]

Thus, (i) and (ii) hold by the assumption that \( Q^x|_{\mathcal{F}_t^0} \) is absolutely continuous with respect to \( P^x|_{\mathcal{F}_t^0} \) on \( \{ t < \zeta \} \) for any \( t > 0 \).

**Step 2.** Suppose that \( A \) is a compact set. Then, there exists a sequence \( \{ A_n \} \) of open sets satisfying \( A_n \supseteq \overline{A}_{n+1} \) and \( \bigcap_{n \geq 1} A_n = A \). For \( s > 0 \), \( \{ s + D_{A_n} \circ \theta_s \} \) is an increasing sequence of \( \{ \mathcal{F}^0_{t+} \} \) stopping times. Define
\[ D_s = \lim_{n \to \infty} (s + D_{A_n} \circ \theta_s). \] (2.5)

Then, \( D_s \) is an \( \{ \mathcal{F}^0_{t+} \} \) stopping time.

Obviously, \( D_s \leq s + D_A \circ \theta_s \). By the right continuity of \( X \), we know that \( X(s + D_{A_n} \circ \theta_s) \in \overline{A}_n \).

Then, we obtain by the quasi-left continuity of \( X \) that for any \( x \in E \),
\[ P^x(Z(D_s) \in A, D_s < \zeta) = P^x \left( \lim_{n \to \infty} Z(s + D_{A_n} \circ \theta_s) \in \bigcap_{n \geq 1} \overline{A}_n = A, D_s < \zeta \right). \]

It follows that \( P^x(s + D_A \circ \theta_s \leq D_s, D_s < \zeta) = P_x(D_s < \zeta) \). Hence
\[ P^x(D_s \neq s + D_A \circ \theta_s) = 0. \] (2.6)

Similarly, we can show that for any \( x \in E \),
\[ Q^x(D_s \neq s + D_A \circ \theta_s) = 0. \] (2.7)

If \( P^x(T_A < t, t < \zeta) = 0 \), then we obtain by (2.3) that for any \( l \in \mathbb{N} \),
\[ P^x \left( \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} < t, t < \zeta \right) = 0. \] (2.8)

For \( l \in \mathbb{N} \), we have
\[ \left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} < t \right\} = \left( \left\{ D_{\frac{1}{l}} < t \right\} \cap \left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} = D_{\frac{1}{l}} \right\} \right) \]
\[ \cup \left( \left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} < t \right\} \cap \left\{ 1 \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \neq D_{\frac{1}{l}} \right\} \right). \] (2.9)

Then, we obtain by (2.6), (2.8) and (2.9) that
\[ P^x(D_{\frac{1}{l}} < t, t < \zeta) = 0. \] (2.10)

Note that \( \{ D_{\frac{1}{l}} < t \} \in \mathcal{F}^0_t \). By (2.10) and the assumption that \( Q^x|_{\mathcal{F}_t^0} \) is absolutely continuous with respect to \( P^x|_{\mathcal{F}_t^0} \) on \( \{ t < \zeta \} \), we get
\[ Q^x(D_{\frac{1}{l}} < t, t < \zeta) = 0. \] (2.11)
Then, we obtain by (2.7), (2.9) and (2.11) that for any \( l \in \mathbb{N} \),
\[
Q^x \left( \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} < t, \ t < \zeta \right) = 0,
\]
which together with (2.3) implies that
\[
Q^x (T_A < t, \ t < \zeta) = 0.
\]
Therefore, (i) holds.

Now we show that (ii) holds. Since \( \{T_A > t, \ t < \zeta\} = \bigcup_{n=1}^{\infty} \{T_A \geq t + \frac{1}{n}, t + \frac{1}{n} < \zeta\} \), it is sufficient to show that for any \( t > 0 \), \( P^x(T_A \geq t, \ t < \zeta) = 0 \) implies that \( Q^x(T_A \geq t, \ t < \zeta) = 0 \).

Suppose that \( P^x(T_A \geq t, \ t < \zeta) = 0 \). Then, we obtain by (2.4) that
\[
P^x \left( \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \geq t, \ t < \zeta \right) \downarrow 0 \hspace{1em} \text{as} \ l \uparrow \infty.
\] (2.12)

For \( l \in \mathbb{N} \), we have
\[
\left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \geq t \right\} = \left( \left\{ D_{\frac{1}{l}} \geq t \right\} \cap \left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} = D_{\frac{1}{l}} \right\} \right) \quad \text{\( \cup \) } \left( \left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \geq t \right\} \cap \left\{ \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \neq D_{\frac{1}{l}} \right\} \right).
\] (2.13)

By (2.6), (2.7) and (2.13), we get
\[
P^x \left( \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \geq t, \ t < \zeta \right) = P^x \left( D_{\frac{1}{l}} \geq t, \ t < \zeta \right),
\] (2.14)
and
\[
Q^x \left( \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \geq t, \ t < \zeta \right) = Q^x \left( D_{\frac{1}{l}} \geq t, \ t < \zeta \right).
\] (2.15)

Note that \( \{D_{\frac{1}{l}} \geq t\} \in \mathcal{F}_t^0 \). Then, we obtain by (2.12), (2.14), (2.15) and the assumption \( Q^x|_{\mathcal{F}_t^0} \) is absolutely continuous with respect to \( P^x|_{\mathcal{F}_t^0} \) on \( \{t < \zeta\} \) that
\[
Q^x \left( \frac{1}{l} + D_A \circ \theta_{\frac{1}{l}} \geq t, \ t < \zeta \right) \downarrow 0 \hspace{1em} \text{as} \ l \uparrow \infty,
\]
which together with (2.4) implies that \( Q^x(T_A \geq t, \ t < \zeta) = 0 \). Therefore, (ii) holds.

**Step 3.** Suppose that \( A \in \mathcal{B} \) and \( x \in E \). Let \( t > 0 \) and \( s \in (0, t) \). For \( B \subset E \), define
\[
\wedge_{s,t}(B) = \{ \omega : Z_u(\omega) \in B \text{ for some } u \in [s,t] \}.
\]
Then,
\[
\{ s + D_B \circ \theta_s < t \} \subset \wedge_{s,t}(B) \subset \{ s + D_B \circ \theta_s \leq t \},
\] (2.16)
and
\[ \{ s + D_B \circ \theta_s \geq t \} \supset (\wedge_{s,t}(B))^c \supset \{ s + D_B \circ \theta_s > t \}. \] (2.17)

Denote by \( \mathcal{O} \) the family of all open subsets of \( E \). If \( B \in \mathcal{O} \), then \( \wedge_{s,t}(B) = \bigcup_{r \in \mathbb{Q} \cap [s,t]} \{ X_r \in B \} \cup \{ X_s \in B \} \cup \{ X_t \in B \} \in \mathcal{F}^0 \). Define the set function \( I \) on \( \mathcal{O} \) by
\[ I(B) = (P^x + Q^x)(\wedge_{s,t}(B)), \quad B \in \mathcal{O}. \]

Following [11, Lemma A.2.6], we can prove the following proposition.

**Proposition 2.3** The set function \( I \) on \( \mathcal{O} \) satisfies the following conditions:

1. \((I.1)\) \( B_1, B_2 \in \mathcal{O}, B_1 \subset B_2 \Rightarrow I(B_1) \leq I(B_2) \).
2. \((I.2)\) \( I(B_1 \cup B_2) + I(B_1 \cap B_2) \leq I(B_1) + I(B_2), \quad B_1, B_2 \in \mathcal{O} \).
3. \((I.3)\) \( B_\infty \in \mathcal{O}, B_n \uparrow B \Rightarrow B \in \mathcal{O}, I(B) = \lim_{n \to \infty} I(B_n) \).

Define
\[ I^*(B) = \inf_{G \in \mathcal{O} : B \subset G} I(G), \quad B \subset E. \]

By proposition [2.3] and [11] Theorem A.1.2, we know that \( I^* \) is a Choquet capacity. By (2.5), (2.6) and (2.7), we find that
\[ I^*(B) = (P^x + Q^x)(\wedge_{s,t}(B)) \text{ if } B \text{ is a compact subset of } E. \]

Then, we obtain by the Choquet theorem (cf. [11] Theorem A.1.1) that there exist a decreasing sequence \( \{ A_n \} \) of open sets and an increasing sequence \( \{ B_n \} \) of compact sets such that
\[ B_n \subset A_n \subset A_n^{c} \text{ and } \lim_{n \to \infty} I(A_n) = \lim_{n \to \infty} I^*(B_n). \]

Consequently, we have
\[ \bigcup_{n=1}^{\infty} \wedge_{s,t}(B_n) \subset \wedge_{s,t}(A) \subset \bigcap_{n=1}^{\infty} \wedge_{s,t}(A_n) \]
and
\[ (P^x + Q^x) \left( \bigcap_{n=1}^{\infty} \wedge_{s,t}(A_n) - \bigcup_{n=1}^{\infty} \wedge_{s,t}(B_n) \right) = 0, \]
which implies that
\[ P^x(\wedge_{s,t}(A), t < \zeta) = P^x \left( \bigcup_{n=1}^{\infty} \wedge_{s,t}(A_n), t < \zeta \right), \]
\[ Q^x(\wedge_{s,t}(A), t < \zeta) = Q^x \left( \bigcap_{n=1}^{\infty} \wedge_{s,t}(A_n), t < \zeta \right), \] (2.18)

and
\[ P^x((\wedge_{s,t}(A))^c, t < \zeta) = P^x \left( \bigcup_{n=1}^{\infty} \wedge_{s,t}(A_n)^c, t < \zeta \right), \]
\[ Q^x((\wedge_{s,t}(A))^c, t < \zeta) = Q^x \left( \bigcap_{n=1}^{\infty} \wedge_{s,t}(A_n)^c, t < \zeta \right). \] (2.19)
If $P^x(T_A < t, t < \zeta) = 0$, then $P^x(s + D_A \circ \theta_s \leq t - \frac{1}{l}, t < \zeta) = 0$ for $l \in \mathbb{N}$. By (2.16), we get $P^x(\wedge_{s,t-1}(A), t < \zeta) = 0$ for $l \in \mathbb{N}$ satisfying $\frac{1}{l} < t - s$. Note that $\bigcap_{n=1}^{\infty} \wedge_{s,t-1}(A_n) \in \mathcal{F}_t^\emptyset$. By (2.18) and the assumption $Q^x|_{\mathcal{F}_t^\emptyset}$ is absolutely continuous with respect to $P^x|_{\mathcal{F}_t^\emptyset}$ on $\{t < \zeta\}$, we have that $Q^x(\wedge_{s,t-1}(A), t < \zeta) = 0$. Then, we obtain by (2.16) that $Q^x(s + D_A \circ \theta_s < t, t < \zeta) = 0$. Letting $l \to \infty$, we get $Q^x(s + D_A \circ \theta_s < t, t < \zeta) = 0$. Since $s \in (0, t)$ is arbitrary, (i) holds by (2.3).

We now prove that (ii) holds. It is sufficient to show that for any $t > 0$, $P^x(T_A \geq t, t < \zeta) = 0$ implies that $Q^x(T_A > t, t < \zeta) = 0$ since it holds that

$$
\bigcup_{n=1}^{\infty} \left\{T_A \geq t + \frac{1}{n}, t + \frac{1}{n} < \zeta\right\} = \{T_A > t, t < \zeta\} = \bigcup_{n=1}^{\infty} \left\{T_A > t + \frac{1}{n}, t + \frac{1}{n} < \zeta\right\}.
$$

Suppose that $P^x(T_A \geq t, t < \zeta) = 0$. Then, we obtain by (2.4) that

$$
P^x \left(\frac{1}{l} + D_A \circ \theta_1 \geq t, t < \zeta\right) \downarrow 0 \text{ as } l \uparrow \infty,
$$

which together with (2.17) implies that

$$
P^x \left((\wedge_{1,t}(A))^c, t < \zeta\right) \downarrow 0 \text{ as } l \uparrow \infty.
$$

(2.20)

Note that $\left(\bigcap_{n=1}^{\infty} \wedge_{1,t}(A_n)\right)^c \in \mathcal{F}_t^\emptyset$ for $l > \frac{1}{t}$. Then, we obtain by (2.17), (2.19), (2.20) and the assumption $Q^x|_{\mathcal{F}_t^\emptyset}$ is absolutely continuous with respect to $P^x|_{\mathcal{F}_t^\emptyset}$ on $\{t < \zeta\}$ that

$$
\lim_{l \to \infty} Q^x \left(\frac{1}{l} + D_A \circ \theta_1 > t, t < \zeta\right) \leq \lim_{l \to \infty} Q^x \left((\wedge_{1,t}(A))^c, t < \zeta\right) = 0.
$$

(2.21)

Therefore, we obtain by (2.4) and (2.21) that

$$
Q^x(T_A \geq t + \varepsilon, t < \zeta) = 0, \forall \varepsilon > 0,
$$

which implies that $Q^x(T_A > t, t < \zeta) = 0$.

\begin{flushright}
\Box
\end{flushright}

2.2 Proof of Theorem 1.1

(i) By Lemma 2.1, it is sufficient to show that any thin Borel measurable set for $Y$ is also polar for $Y$. Let $A \in \mathcal{B}$ be a thin set for $Y$. Define

$$
S = \{\omega : T_A(\omega) = 0\}
$$

and

$$
R = \{\omega : T_A(\omega) < \infty\}.
$$

(2.22)

Then, $Q^x(S) = 0$ for any $x \in E$. 
By [2, Theorem I.10.7], we know that $T_A$ is a stopping time relative to $\{\mathcal{F}^X_t\}$. Then, we obtain by the Blumenthal’s 0-1 law ([2, Theorem I.5.17]) that $P^x(S) = 0$ or 1 for any $x \in E$.

If $P^x(S) = 1$, then $P^x(S^c) = 0$. Note that $S^c = \bigcup_{n=1}^{\infty} \{T_A > \frac{1}{n}\}$.

Then, $P^x(T_A > \frac{1}{n}) = 0$ for $n \in \mathbb{N}$. By Lemma 2.2, we get $Q^x(T_A > \frac{1}{n}, \frac{1}{n} < \zeta) = 0$ for $n \in \mathbb{N}$. Since $Q^x(X_0 = x) = 1$, we obtain by the right continuity of $X$ that $Q^x(\zeta > 0) = 1$. Thus, we have

$$Q^x(S^c) = Q^x(T_A > 0) = \lim_{n \to \infty} Q^x \left( T_A > \frac{1}{n}, \frac{1}{n} < \zeta \right) = 0,$$

which implies that $Q^x(S) = 1$. This contradicts with $Q^x(S) = 0$. Therefore, $P^x(S) = 0$ for any $x \in E$, which implies that $A$ is a thin set for $X$.

By the assumption $X$ satisfies (H), we know that $P^x(R) = 0$ for any $x \in E$. For $t > 0$, define

$$R_t = \{\omega : T_A(\omega) < t\}. \quad (2.23)$$

Then, $P^x(R_t) = 0$. By Lemma 2.2, we obtain that $Q^x(R_t, t < \zeta) = 0$ for $t > 0$. Hence

$$Q^x(R) \leq \sum_{t \in Q \cap (0, \infty)} Q^x(R_t, t < \zeta) = 0.$$

Since $x \in E$ is arbitrary, $A$ is a polar set for $Y$.

(ii) By Lemma 2.1, it is sufficient to show that any thin Borel measurable set for $Y$ is also $m$-essentially polar for $Y$. Let $A \in \mathcal{B}$ be a thin set for $Y$. Similar to (i), we obtain by Lemma 2.2 that $A$ is a thin set for $X$.

Define $R$ and $R_t$, $t > 0$, as in (2.22) and (2.23), respectively. By the assumption that $X$ satisfies $(H_m)$, we get $P^m(R) = 0$. Then, $P^m(R_t) = 0$ for $t > 0$. By Lemma 2.2, we obtain that $Q^m(R_t, t < \zeta) = 0$ for $t > 0$. Hence

$$Q^m(R) \leq \sum_{t \in Q \cap (0, \infty)} Q^m(R_t, t < \zeta) = 0.$$

\hfill \Box

3 Proof of Theorem 1.3

The proof of (ii) is similar to that of (i). So we only prove (i) below.

Let $X = (\Omega, X_t, P^x)$ be a standard process on $E$ and $\tau = (\Theta, \tau_t, Q^0)$ be a subordinator with drift coefficient $d > 0$. We define $Y_t = X_{\tau_t}$ for $t \geq 0$. Then, $Y = (\Omega \times \Theta, Y_t, P^x \times Q^0)$ is a standard process on $E$. 

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“⇒”: Suppose $X$ satisfies (H). We will prove that $Y$ also satisfies (H).

(i) Suppose that $A \in \mathcal{B}$ is a polar set for $X$. Define

$$\Omega_A = \{\omega \in \Omega : T_A < \infty\}.$$

Then, for any $x \in E$,

$$P^x(\Omega_A) = 0. \quad (3.1)$$

Since $d > 0$, we have that

$$\{ (\omega, w) \in \Omega \times \Theta : \exists t > 0 \text{ s.t. } Y_t(\omega, w) \in A \}$$

$$= \{ (\omega, w) \in \Omega \times \Theta : \exists t > 0 \text{ s.t. } X_{\tau w}(\omega) \in A \}$$

$$\subset \Omega_A \times \Theta. \quad (3.2)$$

By (3.1), (3.2) and the independence of $X$ and $\tau$, we find that for any $x \in E$,

$$P^x \times Q^0(\{(\omega, w) \in \Omega \times \Theta : \exists t > 0 \text{ s.t. } Y_t(\omega, w) \in A \}) = 0.$$

Hence $A$ is also a polar set for $Y$.

(ii) For $y > 0$, define $\varsigma(y) = \inf\{t \geq 0 : \tau_t > y\}$. By Bertoin [1, Theorem III.5], we have that

$$Q^0(\tau_{\varsigma(y)} = y) = du(y), \quad \forall y > 0, \quad (3.3)$$

$u$ is continuous and positive on $(0, \infty)$, and

$$u(0+) = 1/d. \quad (3.4)$$

Let $A \in \mathcal{B}$. Suppose that $\omega \in \Omega$ satisfying $T_A(\omega) = 0$. By (3.3) and (3.4), we get

$$Q^0(\{ w : \exists s > 0 \text{ s.t. } \tau_s(w) \in \{ 0 < t \leq \epsilon : X_t(\omega) \in A \} \} = 1, \quad \forall \epsilon > 0.$$

Further, since $\tau_s(w) \geq ds$, we get

$$Q^0(\{ w : \exists 0 < s \leq \epsilon \text{ s.t. } \tau_s(w) \in \{ t : X_t(\omega) \in A \} \} = 1, \quad \forall \epsilon > 0. \quad (3.5)$$

Suppose that $A \in \mathcal{B}$ is not a polar set for $X$. Since $X$ satisfies (H), $A$ is not a thin set for $X$. Hence $A$ has at least one regular point, i.e., there exists some $x \in E$ such that

$$P^x(T_A = 0) = 1. \quad (3.6)$$

By (3.5), (3.6) and the independence of $X$ and $\tau$, we get

$$P^x \times Q^0(\{ (\omega, w) : T_A(\omega, w) = 0 \}) = 1.$$

Then, $x$ is a regular point of $A$ with respect to $Y$ and hence $A$ is not a thin set for $Y$. 

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We now show that $Y$ satisfies (H). By Lemma 2.1, we need only show that any thin set for $Y$ is also polar for $Y$. Suppose that $A \in B$ is a thin set for $Y$. If $A$ is not polar for $Y$, then we obtain by (i) that $A$ is also not polar for $X$. Further, we obtain by (ii) that $A$ is not a thin set for $Y$. We have arrived at a contradiction.

“$\Leftarrow$”: Suppose that $Y$ satisfies (H). We will prove that $X$ also satisfies (H).

If $X$ does not satisfy (H), then we obtain by Lemma 2.1 that there exists a $B \in B$ such that $B$ is a thin set for $X$ but not a polar set for $X$. Since $d > 0$, we obtain by the independence of $X$ and $\tau$ that $B$ is a thin set for $Y$. By the assumption that $Y$ satisfies (H), we conclude that $B$ is a polar set for $Y$.

Suppose that $\omega \in \Omega$ satisfying $T_B(\omega) < \infty$. By (3.3) and the fact that $u$ is positive on $(0, \infty)$, we get

$$Q^0(\{w : \exists s > 0 \text{ s.t. } \tau_s(w) \in \{t > 0 : X_t(\omega) \in B\}) > 0.$$  \hspace{1cm} (3.7)

Since $B$ is not a polar set for $X$, there exists $x \in E$ such that $P^x(T_B(\omega) < \infty) > 0$. Then, we obtain by (3.7) and the independence of $X$ and $\tau$ that

$$P^x \times Q^0(\{(\omega, w) : T_B(\omega, w) < \infty\}) > 0,$$

which implies that $B$ is not a polar set for $Y$. We have arrived at a contradiction. \hfill \square

4 Invariance of (H) under absolutely continuous measure change

In this section, we apply Theorem 1.1 to study Hunt’s hypothesis (H) for standard processes. By virtue of absolutely continuous measure change, we will give new examples of subprocesses, Lévy processes, and jump-diffusion processes satisfying (H).

There exists a vast literature on the absolute continuity of Markov processes. Skorohod [36, 37], Kunita and Watanabe [20], Newman [28, 29] and Jacod and Shiryaev [19] characterize the absolute continuity for Lévy processes. Itô and Watanabe [18], Kunita [24, 25] and Palmowski and Rolski [30] discuss the absolute continuity for general Markov processes. Dawson [7], Liptser and Shiryaev [27] and Kabanov, Liptser and Shiryaev [20] study absolute continuity of solutions to stochastic differential equations. We refer the reader to Cheridito, Filipović and Yor [6] for a nice summary of the references.

4.1 Subprocesses (killing transformation)

In this subsection, we will show that a standard process satisfies (H) implies that any of its standard subprocess satisfies (H). First, let us recall some definitions from Blumenthal and Getoor [2]. Consider the following objects:
By Lemma 4.2, we know that $X$ respectively. Moreover, $C$ (H). Therefore, $C$ is absolutely continuous with respect to $C$ processes.

Suppose that $X$ satisfies (H).

Theorem 4.3 Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ with state space $(E, \mathcal{B})$ is said to be of function space type provided $\Omega = W, \mathcal{M} \supset \mathcal{C}^0, \mathcal{M}_t \supset \mathcal{C}^0_t, X_t = C_t$, and $\theta_t = \varphi_t$.

(iii) Let $\varphi_t : W \to W$ be defined by $\varphi_t w(s) = w(t + s)$.

Denote by $b\mathcal{B}$ and $b\mathcal{B}^*$ the sets of all bounded measurable and bounded universally measurable functions on $(E, \mathcal{B})$, respectively.

Definition 4.1 ([2]) (i) A Markov process $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ with state space $(E, \mathcal{B})$ is said to be of function space type provided $\Omega = W, \mathcal{M} \supset \mathcal{C}^0, \mathcal{M}_t \supset \mathcal{C}^0_t, X_t = C_t$, and $\theta_t = \varphi_t$.

(ii) Two Markov processes with the same state space $(E, \mathcal{B})$ are equivalent if they have the same transition function.

(iii) Let $X$ and $Y$ be two Markov processes with state space $(E, \mathcal{B})$. Denote by $(P_t)$ and $(Q_t)$ the transition semigroups of $X$ and $Y$, respectively. $Y$ is called a subprocess of $X$ if $Q_t f(x) \leq P_t f(x)$ for all $x \in E$, $t \geq 0$ and $f \geq 0$ in $b\mathcal{B}^*$.

Lemma 4.2 Any standard process $X$ with state space $(E, \mathcal{B})$ is equivalent to a standard process $C$ of function space type with state space $(E, \mathcal{B})$. Moreover, $X$ satisfies (H) if and only if $C$ satisfies (H).

Proof. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathcal{B})$. Using the notation developed above Definition 4.1, we define a map $\pi : \Omega \to W$ by $(\pi \omega)(t) = X_t(\omega)$. Then, $C_t \circ \pi = X_t$. We define measures $\tilde{P}^x$ on $(W, \mathcal{C}^0)$ by $\tilde{P}^x = P^x \pi^{-1}$. By [2] Theorem I.4.3, we know that $C = (W, \mathcal{C}^0, \mathcal{C}^0_t, \varphi_t, \tilde{P}^x)$ is a Markov process equivalent to $X$. Define $C$ to be the completion of $\mathcal{C}^0$ with respect to $\{\tilde{P}^x : x \in \mathcal{B}\}$ for $t \in [0, \infty)$, define $C_t$ to be the completion of $\mathcal{C}^0_t$ in $\mathcal{C}$ with respect to $\{\tilde{P}^x : x \in \mathcal{B}\}$. By the assumption $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ is a standard process, we can check that $C = (W, C, \mathcal{C}_t, \varphi_t, \tilde{P}^x)$ is also a standard process. Further, we can show that $X$ satisfies (H) if and only if $C$ satisfies (H) by virtue of Lemma 2.1.

Theorem 4.3 Let $X$ be a standard process with state space $(E, \mathcal{B})$. If $X$ satisfies (H), then any standard subprocess of $X$ satisfies (H).

Proof. Suppose that $X = (X_t, P^x)$ satisfies (H) and $Y = (Y_t, Q^x)$ is a standard subprocess of $X$. By Lemma 4.2, we know that $X = (X_t, P^x)$ and $Y = (Y_t, Q^x)$ are equivalent to some standard processes $C^X = (C_t, \tilde{P}^x)$ and $C^Y = (C_t, \tilde{Q}^x)$ of function space type with state space $(E, \mathcal{B})$, respectively. Moreover, $C^X$ satisfies (H). Since $\tilde{Q}^x(A) \leq \tilde{P}^x(A)$ for any $A \in \mathcal{C}^0$ and $x \in E, C^Y$ is absolutely continuous with respect to $C^X$. Then, we obtain by Theorem 4.1 that $C^Y$ satisfies (H). Therefore, $Y$ satisfies (H) by Lemma 4.2.
Remark 4.4 Let $X = (X_t, P^x)$ be a standard process with state space $(E, \mathcal{B})$. Suppose that $X$ satisfies (H). Let $M = (M_t, 0 \leq t < \infty)$ be a right continuous multiplicative functional (MF) of $X$ satisfying $M_0 = 1$, $0 \leq M_t \leq 1$ and $M_t \in \mathcal{F}_t^0$ for all $t$. By [2 Corollary III.3.16], there is a standard subprocess $\hat{X} = (\hat{X}_t, \hat{P}^x)$ with state space $(E, \mathcal{E})$ such that $\hat{E}^x[f(\hat{X}_t)] = E^x[f(X_t)M_t]$ for any $f \in \mathcal{B}^*$. By Theorem 4.3 $\hat{X}$ satisfies (H). A concrete example of the MF $M = (M_t, 0 \leq t < \infty)$ is given by

$$M_t = \exp\left( -\int_0^t g(X_s)ds \right), \quad t \geq 0,$$

where $g \in \mathcal{B}$ and $g \geq 0$.

### 4.2 Lévy processes (density transformation)

Let $d \geq 1$ and $D = D([0, \infty), \mathbb{R}^d)$ be the space of mappings $\xi$ from $[0, \infty)$ into $\mathbb{R}^d$ which are right-continuous and have left limits. Denote $x_t(\xi) = \xi(t)$. Define $\mathcal{F}_D = \sigma(x_t, t \in [0, \infty))$ and $\mathcal{F}^0_t = \sigma(x_s, s \in [0, t])$ for $t \in [0, \infty)$. Any Lévy process on $\mathbb{R}^d$ induces a Hunt process on $(D, \mathcal{F}_D)$ (cf. [34, Section 40]). By Lemma 1.2 when we consider (H) for a Lévy process, we may assume without loss of generality that it is of the form $(x_t, P^x)$, where $P^x$ is a probability measure defined on $(D, \mathcal{F}_D)$ for $x \in \mathbb{R}^d$. Denote by $m_d$ the Lebesgue measure on $\mathbb{R}^d$.

In the sequel, we use $\Phi$ or $(a, Q, \mu)$ to denote the Lévy-Khintchine exponent of a Lévy process $(X_t, P^x)$, which means that

$$E^0[\exp\{i\langle z, X_t \rangle\}] = \exp\{-t\Phi(z)\}, \quad z \in \mathbb{R}^d, \quad t \geq 0,$$

where $E^0$ denotes the expectation with respect to $P^0$, and

$$\Phi(z) = i\langle a, z \rangle + \frac{1}{2}\langle z, Qz \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{\{|x|<1|\}} \right) \mu(dx).$$

Hereafter $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the standard Euclidean inner product and norm on $\mathbb{R}^d$, respectively.

**Theorem 4.5** ([34, Chapter IV, Theorem 4.39 c]) Let $(x_t, P^x)$ and $(x_t, P'^x)$ be two Lévy processes on $\mathbb{R}^d$ with Lévy-Khintchine exponents $(a, Q, \mu)$ and $(a', Q', \mu')$, respectively. Then the following two conditions are equivalent.

1. $P^0|_{\mathcal{F}_t^0} \ll P'^0|_{\mathcal{F}_t^0}$ for every $t \in (0, \infty)$.
2. $Q = Q'$, $\mu' \ll \mu$ with the function $K(x)$ defined by $\frac{d\mu'}{d\mu} = K(x)$ satisfying

$$\int_{\mathbb{R}^d} \left(1 - \sqrt{K(x)}\right)^2 \mu(dx) < \infty, \quad (4.1)$$

and

$$a' - a + \int_{\{|x|<1|\}} x(\mu' - \mu)(dx) \in \mathcal{R}(Q), \quad (4.2)$$

where $\mathcal{R}(Q) := \{Qx : x \in \mathbb{R}^d\}$. 

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Remark 4.6 (i) Note that finiteness of the integral appearing in (4.2) follows from (4.1) (cf. [27, Remark 33.3]).

(ii) If we let \( h(x) = 1_{\{|x|<1\}} \) in [17, Chapter IV, Theorem 4.39 c], then \( b \) and \( b' \) of [17, Chapter IV, Theorem 4.39 c] satisfy \( b = -a \) and \( b' = -a' \).

Combining Theorems [11 and 4.5] we obtain the following theorem.

**Theorem 4.7** Let \( X \) and \( X' \) be two Lévy processes on \( \mathbb{R}^d \) with Lévy-Khintchine exponents \((a, Q, \mu)\) and \((a', Q', \mu')\), respectively. Suppose that Condition (2) in Theorem 4.5 holds. Then,

(i) \( X \) satisfies (H) implies that \( X' \) satisfies (H).

(ii) \( X \) satisfies \((H_{m_d})\) implies that \( X' \) satisfies \((H_{m_d})\).

It is well-known that any symmetric Lévy process \( X \) satisfies \((H_{m_d})\) (cf. [11, Theorem 4.1.3] and [35]). As a direct consequence of Theorem 4.7, we have the following result on (H).

**Corollary 4.8** Let \( Q \) be a symmetric nonnegative-definite \( d \times d \) matrix and \( \mu \) be a measure on \( \mathbb{R}^d \) satisfying \( \mu(\{0\}) = 0 \), \( \int_{\mathbb{R}^d} (1 + |x|^2) \mu(dx) < \infty \), and \( \mu(A) = \mu(-A) \) for any \( A \in \mathcal{B}(\mathbb{R}^d) \). Suppose that \( K \) is a nonnegative measurable function on \( \mathbb{R}^d \) satisfying

\[
\int_{\mathbb{R}^d} \left( 1 - \sqrt{K(x)} \right)^2 \mu(dx) < \infty,
\]

and \( a \in \mathbb{R}^d \) satisfying

\[
a + \int_{\{|x|<1\}} x(K(x) - 1) \mu(dx) \in \mathcal{R}(Q).
\]

Let \( Y \) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a, Q, K(x))d\mu\). Then \( Y \) satisfies \((H_{m_d})\). If in addition \( Y \) has resolvent densities with respect to \( m_d \), then \( Y \) satisfies (H).

Now we give a useful lemma on the absolute continuity of measure change for Lévy processes.

**Lemma 4.9** Let \((x_t, P^x)\) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a, Q, \mu)\) and \( \mu_1 \) be a measure on \( \mathbb{R}^d \setminus \{0\} \).

(i) Suppose \( \mu_1 \leq \mu \) with the function \( k(x) \) defined by \( k(x) = \frac{d\mu_1}{d\mu} \) satisfying \( \int_{\{|x|<1\}} k^2(x) \mu(dx) < \infty \). Denote \( \mu' := \mu - \mu_1 \) and let \((x_t, P'^x)\) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a', Q, \mu')\), where \( a' := a + \int_{\{|x|<1\}} x \mu_1(dx) \). Then \( P'^x|_{\mathcal{F}_t^o} \ll P^x|_{\mathcal{F}_t^o} \) for \( x \in \mathbb{R}^d \) and \( t > 0 \).

(ii) Suppose \( \mu_1 \ll \mu \) with the function \( k(x) \) defined by \( k(x) = \frac{d\mu_1}{d\mu} \) satisfying \( \int_{\{|x|<1\}} k^2(x) \mu(dx) < \infty \) and \( \int_{\{|x|\geq 1\}} k(x) \mu(dx) < \infty \). Denote \( \mu'' := \mu + \mu_1 \) and let \((x_t, P''^x)\) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a'', Q, \mu'')\), where \( a'' := a - \int_{\{|x|<1\}} x \mu_1(dx) \). Then \( P''^x|_{\mathcal{F}_t^o} \ll P^x|_{\mathcal{F}_t^o} \) for \( x \in \mathbb{R}^d \) and \( t > 0 \).
Proof. Note that
\[ \int_{\{ |x| < 1 \}} |x| \mu_1(dx) \leq \left( \int_{\{ |x| < 1 \}} |x|^2 \mu(dx) \right)^{1/2} \left( \int_{\{ |x| < 1 \}} k^2(x) \mu(dx) \right)^{1/2} < \infty. \]
Obviously, condition (4.2) holds. By Theorem 4.5, it is sufficient to show that condition (4.1) holds.

(i) Denote \( K(x) = \frac{du'}{du} \). Then, \( K(x) = 1 - k(x) \). By the assumption that \( \mu_1 \leq \mu \), we get \( 0 \leq k(x) \leq 1 \). It follows that \( \sqrt{1 - k(x)} \geq 1 - k(x) \), which implies that \((1 - \sqrt{1 - k(x)})^2 \leq k^2(x)\). Therefore, \( \int_{\{ |x| < 1 \}} k^2(x) \mu(dx) < \infty \) implies that \( \int_{\mathbb{R}^d} \left( 1 - \sqrt{K(x)} \right)^2 \mu(dx) < \infty \).

(ii) Denote \( K(x) = \frac{d\mu'}{d\mu} \). Then, \( K(x) = 1 + k(x) \), where \( k(x) \geq 0 \). It follows that \( \sqrt{1 + k(x)} \leq 1 + k(x) \), which implies that \((\sqrt{1 + k(x)} - 1)^2 \leq k^2(x)\). Therefore, \( \int_{\{ |x| < 1 \}} k^2(x) \mu(dx) < \infty \) implies that \( \int_{\mathbb{R}^d} \left( 1 - \sqrt{K(x)} \right)^2 \mu(dx) < \infty \). \qed

**Corollary 4.10** Let \((x_t, P^x)\) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a, Q, \mu)\). Suppose \( \mu_1 \) is a finite measure on \( \mathbb{R}^d \setminus \{0\} \) such that \( \mu_1 \leq \mu \). Denote \( \mu' := \mu - \mu_1 \) and let \((x_t, P'^x)\) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a', Q, \mu')\), where \( a' := a + \int_{\{ |x| < 1 \}} x \mu_1(dx) \). Then \( P'^x|_{\mathcal{F}_t^x} \ll P^x|_{\mathcal{F}_t^x} \) for \( x \in \mathbb{R}^d \) and \( t > 0 \).

**Proof.** Denote \( k(x) = \frac{d\mu}{d\mu_1} \). Then, \( 0 \leq k(x) \leq 1 \), which together with the fact that \( \mu_1 \) is a finite measure implies that
\[ \int_{\{ |x| < 1 \}} k^2(x) \mu(dx) \leq \int_{\{ |x| < 1 \}} k(x) \mu(dx) = \int_{\{ |x| < 1 \}} \mu_1(dx) < \infty. \]
Thus, the proof is complete by Lemma 4.9(i). \qed

Combining [16] Theorem 3.1 and Proposition 3.3, Theorem 4.1 and Corollary 4.10, we obtain the following result, which implies that big jumps have no effect on the validity of (H) for any Lévy process.

**Proposition 4.11** Let \( X \) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a, Q, \mu)\). Suppose that \( \mu_1 \) is a finite measure on \( \mathbb{R}^d \setminus \{0\} \) such that \( \mu_1 \leq \mu \). Denote \( \mu' := \mu - \mu_1 \) and let \( X' \) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a', Q, \mu')\), where \( a' := a + \int_{\{ |x| < 1 \}} x \mu_1(dx) \). Then,

(i) \( X \) and \( X' \) have same semipolar sets.
(ii) \( X \) and \( X' \) have same \( m_d \)-essentially polar sets.
(iii) \( X \) satisfies (H) if and only if \( X' \) satisfies (H).
(iv) \( X \) satisfies \((H_{m_d})\) if and only if \( X' \) satisfies \((H_{m_d})\).
Finally, we give a new class of purely discontinuous Lévy processes satisfying (H).

**Proposition 4.12** Suppose that \( \rho_1 \) is a nonnegative measurable function on \( \mathbb{R}^d \) satisfying \( \rho_1(x) = \rho_1(-x) \) for \( x \in \mathbb{R}^d \) and
\[
\int_{\mathbb{R}^d} (1 \wedge |x|^2) \rho_1(x) \, dx < \infty,
\]
and \( \rho_2 \) is a nonnegative measurable function on \( \mathbb{R}^d \) satisfying
\[
\int_{\{ |x| < 1 \}} \rho_2(x) \rho_1(x) \, dx < \infty \quad \text{and} \quad \int_{\{ |x| \geq 1 \}} \rho_2(x) \rho_1(x) \, dx < \infty.
\]
Denote \( a = -\int_{\{ |x| < 1 \}} x \rho_2(x) \rho_1(x) \, dx \) and \( \mu = (1 + \rho_2(x)) \rho_1(x) \, dx \). Let \( Y \) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((a, 0, \mu)\). Then \( Y \) satisfies (H).

**Proof.** Let \( X \) be a Lévy process on \( \mathbb{R}^d \) with Lévy-Khintchine exponent \((0, 0, \rho_1(x) \, dx)\). If \( \rho_1(x) \, dx \) is a finite measure, then \( X \) is a compound Poisson process and hence satisfies (H). We now assume that \( \rho_1(x) \, dx \) is an infinite measure. Then, \( X \) is a symmetric Lévy process which has transition densities by [34, Theorem 27.7]. So \( X \) satisfies (H). Therefore, the proof is complete by Theorem [11] and Lemma 4.9(ii).

### 4.3 Jump-diffusion processes (supermartingale transformation)

Throughout this subsection, we make the following assumptions.

(A.1) \( a = (a_{ij}) \) is a bounded continuous mapping on \( \mathbb{R}^d \) with values in the set of positive definite symmetric \( d \times d \) matrices such that \( \frac{\partial a_{ij}}{\partial x_i} \) is a bounded measurable function on \( \mathbb{R}^d \) for \( 1 \leq i, j \leq d \).

(A.2) \( b \) is a bounded measurable mapping on \( \mathbb{R}^d \) with values in \( \mathbb{R}^d \).

(A.3) \( \gamma \) is a nonnegative symmetric measurable function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diagonal} \) such that
\[
\int_{\Gamma} \frac{|y|^2}{1 + |y|^2} \gamma(x, x + y) \, dy
\]
is bounded continuous for all \( \Gamma \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \).

(A.4) \( k \) is a nonnegative measurable function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diagonal} \) such that
\[
\int_{\mathbb{R}^d} k^2(x, y) \gamma(x, y) \, dy \text{ is bounded on } \mathbb{R}^d,
\]
and
\[
\int_{\Gamma} \frac{|y|^2}{1 + |y|^2} k(x, x + y) \gamma(x, x + y) \, dy
\]
is bounded continuous for all \( \Gamma \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \).
We use $C_c^\infty(\mathbb{R}^d)$ to denote the space of smooth functions on $\mathbb{R}^d$ with compact support. Define
\[
\vartheta(x, y) = [1 + k(x, y)]\gamma(x, y).
\] (4.4)

For $f \in C_c^\infty(\mathbb{R}^d)$, define
\[
\mathcal{A}^{a,b,\vartheta}f(x) := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i}
+ \int_{\mathbb{R}^d} \left( f(x + y) - f(x) - \frac{\langle y, \nabla f(x) \rangle}{1 + |y|^2} \right) \vartheta(x, x+y)dy.
\]

By (A.1)–(A.4) and Stroock [38, Theorem 4.3 and Remark, page 232], we know that the martingale problem for $\mathcal{A}^{a,b,\vartheta}$ is well-posed and the corresponding solution is a strong Feller process. Denote by $(x_t, P^x)$ the Markov process associated with $\mathcal{A}^{a,b,\vartheta}$ on $(\mathcal{D}, \mathcal{F}_D)$, where $(\mathcal{D}, \mathcal{F}_D)$ is the Skorohod space as defined in §4.2.

**Theorem 4.13** Assume that (A.1)–(A.4) hold. Then $(x_t, P^x)$ satisfies (H).

**Proof.** For $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define
\[
\mathcal{A}^{a,\vartheta} f(x) := \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial f(x)}{\partial x_j} \right) + \int_{\mathbb{R}^d} \left( f(x + y) - f(x) - \frac{\langle y, \nabla f(x) \rangle}{1 + |y|^2} \right) \gamma(x, x+y)dy.
\]
Define
\[
\varsigma_i(x) = \sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j}, \quad 1 \leq i \leq d,
\]
and set $\varsigma = (\varsigma_1, \ldots, \varsigma_d)$. Then,
\[
\mathcal{A}^{a,\vartheta} f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d \varsigma_i(x) \frac{\partial f(x)}{\partial x_i}
+ \int_{\mathbb{R}^d} \left( f(x + y) - f(x) - \frac{\langle y, \nabla f(x) \rangle}{1 + |y|^2} \right) \gamma(x, x+y)dy.
\]

Hence we obtain by (A.1)–(A.3) and Stroock [38, Theorem 4.3 and Remark, page 232] that the martingale problem for $\mathcal{A}^{a,\vartheta}$ is well-posed and the corresponding solution is a strong Feller process. Denote by $(x_t, P^{*x})$ the Markov process associated with $\mathcal{A}^{a,\vartheta}$ on $(\mathcal{D}, \mathcal{F}_D)$.

We fix a bounded continuous function $\chi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\chi(y) = y$ on a neighborhood of 0. For $x \in \mathbb{R}^d$, define
\[
\beta(x) = \varsigma(x) + \int_{\mathbb{R}^d} \left( \chi(y) - \frac{y}{1 + |y|^2} \right) \gamma(x, x+y)dy,
\] (4.5)
and
\[
\tilde{\beta}(x) = b(x) + \int_{\mathbb{R}^d} \left( \chi(y) - \frac{y}{1 + |y|^2} \right) \vartheta(x, x+y)dy.
\] (4.6)
Then, for \( f \in C_0^\infty(\mathbb{R}^d) \), we have that

\[
\mathcal{A}^{a,\gamma} f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d \beta_i(x) \frac{\partial f(x)}{\partial x_i} + \int_{\mathbb{R}^d} (f(x + y) - f(x) - \langle \nabla f(x), \chi(y) \rangle) \gamma(x, x + y) dy,
\]

(4.7)

and

\[
\mathcal{A}^{a,b,\vartheta} f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d \tilde{\beta}_i(x) \frac{\partial f(x)}{\partial x_i} + \int_{\mathbb{R}^d} (f(x + y) - f(x) - \langle \nabla f(x), \chi(y) \rangle) \vartheta(x, x + y) dy.
\]

(4.8)

Define

\[
\phi(x) = a^{-1}(x) \left( b(x) - \varsigma(x) - \int_{\mathbb{R}^d} \frac{y}{1 + |y|^2} k(x, x + y) \gamma(x, x + y) dy \right), \quad x \in \mathbb{R}^d.
\]

(4.9)

Then, \( \phi \) is locally bounded on \( \mathbb{R}^d \). By (4.4), (4.5), (4.6) and (4.9), we get

\[
\tilde{\beta}(x) = \beta(x) - \varsigma(x) + b(x) + \int_{\mathbb{R}^d} \left( \chi(y) - \frac{y}{1 + |y|^2} \right) (\partial(x, x + y) - \gamma(x, x + y)) dy
\]

\[
= \beta(x) + \left[ a(x) \phi(x) + \int_{\mathbb{R}^d} \frac{y}{1 + |y|^2} k(x, x + y) \gamma(x, x + y) dy \right]
\]

\[
+ \int_{\mathbb{R}^d} \left( \chi(y) - \frac{y}{1 + |y|^2} \right) k(x, x + y) \gamma(x, x + y) dy
\]

\[
= \beta(x) + a(x) \phi(x) + \int_{\mathbb{R}^d} \chi(y) k(x, x + y) \gamma(x, x + y) dy.
\]

(4.10)

Note that (4.3) implies that

\[
\int_{\mathbb{R}^d} [(k(x, y) + 1) \log(k(x, y) + 1) - k(x, y)] \gamma(x, x + y) dy \quad \text{is bounded on } \mathbb{R}^d.
\]

(4.11)

Then, we obtain by [6] Theorem 2.4 and Remark 2.5], (4.4), (4.7), (4.8), (4.10) and (4.11) that \( (x_t, P^x) \) is induced by a supermartingale transformation of \( (x_t, P^{sx}) \). Hence \( P^x \mid F_t \) is absolutely continuous with respect to \( P^{sx} \mid F_t \) for any \( x \in \mathbb{R}^d \) and \( t > 0 \).

We now use the theory of Dirichlet forms to show that \( (x_t, P^{sx}) \) satisfies (H). The reader is referred to Fukushima, Oshima and Takeda [11] for notation and terminology used below. First, note that \( \mathcal{A}^{a,\gamma} f \in L^2(\mathbb{R}^d, dx) \) for \( f \in C_0^\infty(\mathbb{R}^d) \). In fact, suppose \( \text{supp}[f] \subset \{ x \in \mathbb{R}^d : |x| \leq N \} \) for some \( N \in \mathbb{N} \). Then, we obtain by (A.3) that

\[
\int_{\{|x| > 2N\}} \left( f(x + y) - f(x) - \frac{\langle y, \nabla f(x) \rangle}{1 + |y|^2} \right) \gamma(x, x + y) dy \quad dx
\]

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\[= \int_{\{|x|>2N\}} \left( \int_{\{|y|\geq 1\}} f(x+y)\gamma(x, x+y)dy \right)^2 dx \]

\[\leq \left( \sup_{x \in \mathbb{R}^d} \int_{\{|y|\geq 1\}} \gamma(x, x+y)dy \right) \int_{\{|x|>2N\}} \int_{\{|y|\geq 1\}} f^2(x+y)\gamma(x, x+y)dy dx \]

\[\leq \left( \sup_{x \in \mathbb{R}^d} \int_{\{|y|\geq 1\}} \gamma(x, x+y)dy \right) \int_{\mathbb{R}^d} \int_{\{|y-x|\geq 1\}} f^2(y)\gamma(x, y)dy dx \]

\[\leq \left( \sup_{x \in \mathbb{R}^d} \int_{\{|y|\geq 1\}} \gamma(x, x+y)dy \right) \int_{\mathbb{R}^d} f^2(y)dy < \infty. \]

We consider the symmetric bilinear form on \(L^2(\mathbb{R}^d; dx)\):

\[\mathcal{E}(f, g) = -\int_{\mathbb{R}^d} \mathcal{A}^{a,\gamma} f(x)g(x)dx \]

\[= \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))dy dx, \; f, g \in C^\infty_c(\mathbb{R}^d). \]

\((\mathcal{E}, C^\infty_c(\mathbb{R}^d))\) is closable on \(L^2(\mathbb{R}^d; dx)\) and its closure \((\mathcal{E}, D(\mathcal{E}))\) is a regular symmetric Dirichlet form.

Let \((X_t^x, P^{x,x})\) be a Hunt process associated with \((\mathcal{E}, D(\mathcal{E}))\). For \(f \in C^\infty_c(\mathbb{R}^d)\), define

\[g = (1 - \mathcal{A}^{a,\gamma})f. \]

Then, \(g\) is a bounded measurable function on \(\mathbb{R}^d\) and \(g \in L^2(\mathbb{R}^d; dx)\). Denote by \(R_t^x\) the 1-resolvent of \(X_t^x\) and define

\[\mathcal{M}_t^f = R_t^x g(X_t^x) - R_0^x g(X_0^x) - \int_0^t (R_s^x g - g)(X_s^x)ds, \; t \geq 0. \]

It is known that \(\{\mathcal{M}_t^f\}\) is a martingale under \(P^{x,x}\) for \(x \in \mathbb{R}^d\) (cf. \[\square\] Chapter 4, Proposition 1.7]). Denote by \(G_1\) the 1-resolvent of \(\mathcal{E}\) and define

\[M_t^f = f(X_t^x) - f(X_0^x) - \int_0^t \mathcal{A}^{a,\gamma} f(X_s^x)ds, \; t \geq 0. \]

Since \(f = G_1 g\) \(dx\)-a.e., we get \(f = R_t^x g\) q.e.. Hence \(\{M_t^f\}\) is a martingale under \(P^{x,x}\) for q.e. \(x \in \mathbb{R}^d\).

Let \(\Psi\) be a countable subset of \(C^\infty_c(\mathbb{R}^d)\) such that for any \(f \in C^\infty_c(\mathbb{R}^d)\) there exist \(\{f_n\} \subset \Psi\) satisfying \(\|f_n - f\|_\infty, \|\partial_i f_n - \partial_i f\|_\infty, \|\partial_i \partial_j f_n - \partial_i \partial_j f\|_\infty \to 0\) as \(n \to \infty\) for any \(i, j \in \{1, 2, \ldots, d\}\).

Then, there is an \(\mathcal{E}\)-exceptional set of \(\mathbb{R}^d\), denoted by \(F\), such that \(\{M_t^f\}\) is a martingale under \(P^{x,x}\) for any \(x \in F^c\). We obtain by taking limits that \(\{M_t^f\}\) is a martingale under \(P^{x,x}\) for any \(f \in C^\infty_c(\mathbb{R}^d)\) and q.e. \(x \in \mathbb{R}^d\). Thus, by the uniqueness of solutions to the martingale problem for \(\mathcal{A}^{a,\gamma}\), we conclude that \((x_t, P^{x,x})\) is a Hunt process associated with the symmetric Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\). Since \((x_t, P^{x,x})\) is a strong Feller process, \((x_t, P^{x,x})\) must satisfy (H) (cf. \[\square\] Theorems 4.1.2 and 4.1.3]). Therefore, we obtain by Theorem \[\square\] that \((x_t, P^{x})\) satisfies (H). \(\square\)
5 Invariance of (H) under subordination and remark

As a direct application of Theorem 1.3, we obtain the following result.

**Proposition 5.1** Suppose that $X$ is a Lévy process on $\mathbb{R}^d$ with Lévy-Khintchine exponent $\varphi$. Let $c > 0$ be a constant and $\nu$ be a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x) \nu(dx) < \infty$. Then $X$ satisfies (H) if and only if the Lévy process on $\mathbb{R}^d$ with Lévy-Khintchine exponent

$$\Phi(z) := c \varphi(z) + \int_0^\infty (1 - e^{-\varphi(z)x}) \nu(dx), \quad z \in \mathbb{R}^d$$

(5.1)
satisfies (H). In particular, for $0 < \alpha < 1$, $X$ satisfies (H) if and only if the Lévy process on $\mathbb{R}^d$ with Lévy-Khintchine exponent

$$\Phi(z) = c \varphi(z) + (\varphi(z))^\alpha, \quad z \in \mathbb{R}^d$$

satisfies (H).

**Proof.** Let $\tau$ be a subordinator with drift coefficient $c$ and Lévy measure $\nu$, which is independent of $X$. Define $Y_t := X_{\tau t}$ for $t \geq 0$. Then, $Y = (Y_t)$ has the Lévy-Khintchine exponent $\Phi$ defined by (5.1). Therefore, $X$ satisfies (H) if and only if $Y$ satisfies (H) by Theorem 1.3. The second assertion is proved by letting $\tau_t = ct + \beta t$, where $\beta$ is a stable subordinator of index $\alpha$ which is independent of $X$.

Note that the uniform motion $X_t = t$ on $\mathbb{R}$ does not satisfy (H). The sufficient part of Theorem 1.3 can be regarded as a generalization of the following proposition.

**Proposition 5.2** (cf. [14, Proposition 1.6]) Let $X$ be a subordinator. If $X$ satisfies (H), then its drift coefficient equals 0.

Proposition 5.2 can be extended to the $d$-dimensional case as follows.

**Proposition 5.3** Let $X$ be a Lévy process on $\mathbb{R}^d$ with Lévy-Khintchine exponent $(a, 0, \mu)$ satisfying $\int_{\mathbb{R}^d} (1 \wedge |x|) \mu(dx) < \infty$. If $X$ satisfies (H), then its drift coefficient equals 0.

**Proof.** The 1-dimensional case follows by [23, 5]. Now we consider the case that $d > 1$. Denote by $a'$ the drift coefficient of $X$, i.e., $a' = -(a + \int_{\{|x|<1\}} x \mu(dx))$. Then, we have

$$E^0[e^{i\langle z,X_1 \rangle}] = \exp \left[ - \left( i\langle -a', z \rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle z,x \rangle}) \mu(dx) \right) \right], \quad z \in \mathbb{R}^d.$$  

(5.2)

Suppose that $X$ satisfies (H) and $a' \neq 0$. Define $S = \{ta'|t \in \mathbb{R}\}$ and let $Y = (Y_t)$ be the projection process of $X$ on $S$. Then, we obtain by [16, Lemma 3.4] that $Y$ is a one-dimensional Lévy process satisfying (H).
Denote by $P$ the projection operator from $\mathbb{R}^d$ to $S$. Then, we obtain by (5.2) that for $z \in \mathbb{R}^d$,
\[
E_0[e^{i\langle z,Y_1 \rangle}] = E_0[e^{i\langle z,P X_1 \rangle}]
= E_0[e^{i\langle P z,X_1 \rangle}]
= \exp \left\{ -\left( i\langle -a', P z \rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle P z,x \rangle}) \mu(dx) \right) \right\}
= \exp \left\{ -\left( i\langle -P a', z \rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle z,x \rangle}) \mu(dx) \right) \right\}
= \exp \left\{ -\left( i\langle -a', z \rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle z,y \rangle}) \mu_P(dy) \right) \right\},
\]
where $\mu_P$ is the image measure of $\mu$ under the map $P$. Hence the drift coefficient of $Y$ is $a'$. Since the proposition is true for the 1-dimensional case, $a' = 0$. We have arrived at a contradiction.

\begin{remark}
Let $X = (X_t)$ be a Lévy process on $\mathbb{R}^d$ with Lévy-Khintchine exponent $\Phi$ or $(a, Q, \mu)$. $X$ is called a pure jump Lévy process if $Q = 0$, $\int_{\mathbb{R}^d} (1 \wedge |x|) \mu(dx) < \infty$, and the drift coefficient $a' = -\left( a + \int_{\{ |x| < 1 \}} x \mu(dx) \right) = 0$. In this case, $\Phi$ can be expressed by $\Phi(z) = \int_{\mathbb{R}^d} (1 - e^{i\langle z,x \rangle}) \mu(dx)$ and $X$ can be expressed by $X_t = \sum_{0 < s \leq t} \Delta X_s$, where $\Delta X_s = X_s - X_{s-}$.

Based on Proposition 5.3, it is natural to ask the following question.

\begin{question}
Does any pure jump Lévy process satisfy (H)?
\end{question}

If the answer to the above question is affirmative for subordinators, i.e., any pure jump subordinator satisfies (H), then Theorems 1.2 and 1.3 imply the following claim:

Let $(X_t)$ be a standard process on an LCCB state space and $(\tau_t)$ be a subordinator which is independent of $(X_t)$. Then $(X_{\tau_t})$ satisfies (H) if and only if either $(X_t)$ or $(\tau_t)$ satisfies (H).

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