GLOBAL WELL-POSEDNESS AND DECAY OF SOLUTIONS TO THE CAUCHY PROBLEM OF CONVECTIVE CAHN-HILLIARD EQUATION

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Abstract. In this paper, we consider the global well-posedness and time-decay rates of solution to the Cauchy problem for 3D convective Cahn-Hilliard equation with double-well potential via a refined pure energy method. In particular, the optimal decay rates of the higher-order spatial derivatives of the solution are obtained, the $H^{-s} (0 < s \leq \frac{1}{2})$ negative Sobolev norms is shown to be preserved along time evolution and enhance the decay rates.

1. Introduction

The convective Cahn-Hilliard equation [17, 32, 10]
(1) \[ \partial_t u + \Delta^2 u = \Delta \varphi(u) + \vec{\beta} \cdot \nabla \psi(u), \]
arises naturally as a continuous model for the formation of facets and corners in crystal growth [45, 22]. In equation (1), $u(x, t)$ denotes the slope of the interface [22], the convective term $\vec{\beta} \cdot \nabla \psi(u)$ stems from the effect of kinetic that provides an independent flux of the order parameter, similar to the effect of an external field in spinodal decomposition of a driven system [22], $\varphi(u)$ stands for the derivative of a configuration potential $\Phi(u) = \int_0^u \varphi(s) ds$, respectively. Usually, we take $\varphi(u)$ as the derivation of a double-well potential \[ \varphi(s) = \Phi'(s) = s(s^2 - 1), \quad \Phi(s) = \frac{1}{4}(s^2 - 1)^2, \]
or a singular potential (see [18, 20]) \[ \varphi_{\log}(s) = -\kappa_0 s + \kappa_1 \ln \frac{1 + s}{1 - s}, \quad 0 < \kappa_1 < \kappa_0. \]

For small driving force $\vec{\beta} \to 0$, equation (1) is reduced to the well-known classical Cahn-Hilliard equation [6, 36, 46, 19, 24, 38].

A large amount of literature has been produced about the convective Cahn-Hilliard equation in a bounded domain, subject to suitable boundary conditions. For example, Zaks et al. [47] investigated the bifurcations of stations periodic solutions of a convective Cahn-Hilliard equation; Eden and Kalantarov [15, 16] established some results on the existence of a compact attractor for the convective Cahn-Hilliard equation with periodic boundary conditions in one space dimension and three space dimension; Della Porta and
Grasselli [11] considered the initial-boundary value problem of convective nonlocal Cahn-Hilliard equation as dynamical systems and showed that they have bounded absorbing sets and global attractors; Zhao and Liu [48, 49] investigated the existence of optimal solutions and optimality condition for the initial-boundary value problem of convective Cahn-Hilliard equation; Rocca and Sprekels [37] studied a distributed control problem for a 3D convective nonlocal Cahn-Hilliard-type system involving a degenerate mobility and a singular potential. In [29, 30], Liu et. al. considered properties of solutions for the initial-boundary value problem of the convective Cahn-Hilliard equation with nonconstant mobility and degenerate mobility. There are also some papers related to the numerical solutions for the convective Cahn-Hilliard equation, please refer to [51, 1] and the reference therein.

Remark 1.1. The Cahn-Hilliard equation

\begin{equation}
\partial_t u = \Delta [-\gamma \Delta u + \varphi(u)],
\end{equation}

was used to describe phase transition problems in binary metallic alloys [36], the representation of the tumor growth process [34], color image inpainting [7] and other phenomenons. The convective Cahn-Hilliard equation can be seen as a modification of equation (2). There are various other modifications of it has been proposed in order to capture the dynamical picture of the phase transition phenomena better. To name only a few, viscous Cahn-Hilliard equation [12, 9], Cahn-Hilliard-Cook equation [2], Cahn-Hilliard-Gurtin equation [33, 26], Cahn-Hilliard-Hele-Shaw systems [43, 21], Cahn-Hilliard-Brinkman equation [3, 14] and so on.

A Cauchy problem in mathematics asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain. The Cauchy problem of the convective Cahn-Hilliard in \( \mathbb{R}^N \) \( (N \in \mathbb{Z}^+) \) has the following form:

\begin{equation}
\begin{aligned}
\partial_t u + \Delta^2 u &= \Delta \varphi(u) + \gamma \nabla \cdot \psi(u), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\
u(x, 0) &= u_0(x),
\end{aligned}
\end{equation}

where \( \gamma > 0 \) is a positive constant. It is worth pointing out that the study of problem (3) is also interesting. In [50], assuming that the initial data \( u_0(x) \) satisfies \( u_0(x) \in L^{\frac{N(u+1)}{3}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and \( \|u_0(x)\|_{L^\frac{N(u+1)}{3}} \) is sufficiently small, and the nonlinear functions \( \varphi(u) = O(1)|u|^p \) and \( \psi(u) = O(1)|u|^l \) as \( u \to 0 \), where \( p = \frac{2l+1}{2} \), the author proved that there exists a unique global smooth solution \( u \in L^\infty(0, \infty; L^{\frac{N(u+1)}{3}}(\mathbb{R}^N)) \) for problem (3). Moreover, Liu and Liu [28] studied the Cauchy problem of the degenerate convective Cahn-Hilliard equation

\begin{equation}
\begin{aligned}
\partial_t u + \Delta_{x'}^2 u &= \Delta_{x'} \varphi(u) - \bar{r} \cdot \nabla \psi(u), \quad x' \in \mathbb{R}^{N-1} \times (0, \infty), \\
u(x, 0) &= u_0(x),
\end{aligned}
\end{equation}

where \( \Delta_{x'} = \sum_{i=2}^n \frac{\partial^2}{\partial x_i^2} \) denotes the \( x' \) direction Laplacian operator with respect to \( x' = (x_2, x_3, \cdots, x_n) \), \( \varphi(u) = O(|u|^p) \) and \( \psi(u) = O(|u|^l) \) with the same growth property
and \( \theta \geq 1 \) is an integer. By using the long-short wave method and the frequency decomposition method, the authors proved the existence of the unique global classical solution with small initial data and discussed the decay estimates.

**Remark 1.2.** There are also some papers studied the global well-posedness of solutions for Cauchy problem of the Cahn-Hilliard equation (see e.g., Bricmont, Kupiainen and Taskinen [4], Caffarelli and Muler [5], Liu, Wang and Zhao [31], Cholewa and Rodriguez-Bernal [8], Duan and Zhao [13] and the reference cited therein).

It is worth pointing out that the assumptions imposed on the nonlinear functions \( \phi(u) \) and \( \psi(u) \) in [50, 28, 1, 5, 31, 8, 13] are too strict. One of the most nature assumption on the nonlinear function \( \phi(u) \) is \( \phi(u) = u^3 - u \), which is a double-well potential (the other is logarithmic potential). Moreover, we assume that \( \psi(u) = \frac{1}{2}u^2 \), which can be found in [32, 15, 16, 27] and the reference therein. Thus a natural question is how to prove that the Cauchy problem (3) with \( \phi(u) = u^3 - u \) and \( \psi(u) = \frac{1}{2}u^2 \) admits a unique global smooth solution \( u(x, t) \) and how to get the optimal temporal decay estimates? The main purpose of our present paper is devoted to the above problems. That is, we will consider the global existence and the time decay rate of solutions for the following Cauchy problem in \( \mathbb{R}^3 \):

\[
\begin{align*}
\partial_t u + \Delta^2 u &= \Delta(u^3 - u) + u \cdot \nabla u, \quad x \in \mathbb{R}^3, \ t > 0, \\
\phi(u) &= u_0, \quad x \in \mathbb{R}^3.
\end{align*}
\]

**Remark 1.3.** Since we consider the Cauchy problem of Cahn-Hilliard equation, the Laplacian operator \( (-\Delta)^\delta \) \( (\delta \in \mathbb{R}) \) can be defined through the Fourier transform, namely

\[
(-\Delta)^\delta f(x) = \Lambda^{2\delta} f(x) = \int_{\mathbb{R}^3} |x|^{2\delta} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,
\]

where \( \hat{f} \) is the Fourier transform of \( f \). Moreover, \( \nabla^l \) with an integral \( l \geq 0 \) stands for the usual spatial derivatives of order \( l \). If \( l < 0 \) or \( l \) is not a positive integer, \( \nabla^l \) stands for \( \Lambda^l \) defined by (6). We also use \( H^s(\mathbb{R}^3) \) \( (s \in \mathbb{R}) \) to denote the homogeneous Sobolev spaces on \( \mathbb{R}^3 \) with the norm \( \| \cdot \|_{H^s} \) defined by \( \| f \|_{H^s} := \| \Lambda^s f \|_{L^2} \), and we use \( H^s(\mathbb{R}^3) \) and \( L^p(\mathbb{R}^3) \) \( (1 \leq p \leq \infty) \) to describe the usual Sobolev spaces with the norm \( \| \cdot \|_{H^s} \) and the usual \( L^p \) space with the norm \( \| \cdot \|_{L^p} \).

One of the main purpose of this paper is to prove some global well-posedness results for the Cauchy problem (5) in \( \mathbb{R}^3 \). Moreover, we also derive some various algebraic decay rates of the global solutions. For \( N \geq 2 \), define the energy functional by

\[
\mathcal{E}_N(t) = \sum_{l=0}^{N} \| \nabla^l u \|_{L^2}^2,
\]

and the corresponding dissipation rate with minimum derivative counts by

\[
\mathcal{D}_N(t) = \sum_{l=0}^{N} (\| \nabla^{l+1} u \|_{L^2}^2 + \| \nabla^{l+2} u \|_{L^2}^2).
\]

Our main result on the global well-posedness of solutions of Cauchy problem (5) is stated in the following theorem.
Theorem 1.4. Let \( N \geq 2 \), suppose that the initial data \( u_0 \in H^N(\mathbb{R}^3) \), and there exists a constant \( \delta_0 > 0 \) such that if
\[
\mathcal{E}_2(0) \leq \delta_0,
\]
then there exists a unique global solution \( u(x,t) \) satisfying that for all \( t \geq 0 \),
\[
\sup_{0 \leq t \leq \infty} \mathcal{E}_2(t) + \int_0^\infty D_2(s)ds \leq C\mathcal{E}_2(0).
\]
Moreover, if \( N \geq 3 \), then for all \( t > 0 \), the following inequality holds:
\[
\sup_{0 \leq t \leq \infty} \mathcal{E}_N(t) + \int_0^\infty D_N(s)ds \leq C\mathcal{E}_N(0).
\]

Remark 1.5. For the global well-posedness of the solution, we only assume that the \( H^2 \)-norm of initial data is small, while the higher-order Sobolev norms can be arbitrarily large.

The temporal decay rate of solutions is also an interesting topic in the study of dissipative equations. One of the main tools to study the temporal decay rate is Fourier splitting method, which was introduced by Schonbek in \([39,40]\). Laterly, this method was well extended to investigate the decay for the solutions of PDE from mathematical physics. In \([31]\), by using Fourier splitting method, Liu, Wang and Zhao studied the temporal decay rate of the solution, and its derivatives for the Cauchy problem of Cahn-Hilliard equation with \( \varphi(u) = O(|u|^p) \) for some \( p > 0 \). In this paper, we improve Liu, Wang and Zhao’s results, assume that \( \varphi(u) \) is a double-well potential, study the decay rate of global solutions for problem \([5]\). More precisely, we establish the following result:

Theorem 1.6. Suppose that all assumptions in Theorem 1.4 hold. Let \( u(x,t) \) be the solution to the problem \([5]\) constructed in Theorem 1.4. Moreover, assume that \( u_0 \in \dot{H}^{-s} \) for some \( s \in [0,\frac{1}{2}] \), then
\[
\|u(t)\|_{\dot{H}^{-s}} \leq C,
\]
and
\[
\|\nabla^k u(t)\|_{H^{N-k}} \leq C(1 + t)^{-\frac{k+s}{4}}, \quad \text{for } k = 0,1,\cdots,N-2.
\]

Note that the Hardy-Littlewood-Sobolev theorem implies that for \( p \in [\frac{3}{2},2] \), \( L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3) \) with \( s = 3(\frac{1}{p} - \frac{1}{2}) \in [0,\frac{1}{2}] \). Then, on the basis of Theorem 1.6, we easily obtain the following corollary of the optimal decay estimates.

Corollary 1.7. Under the assumptions of Theorem 1.6, if we replace the \( \dot{H}^{-s}(\mathbb{R}^3) \) assumption by \( u_0 \in L^p(\mathbb{R}^3) \) \((\frac{3}{2} \leq p \leq 2)\), then the following decay estimate holds:
\[
\|\nabla^k u(t)\|_{H^{N-k}} \leq C(1 + t)^{-\sigma_k}, \quad \text{for } k = 0,1,\cdots,N-2,
\]
where
\[
\sigma_k = \frac{3}{4} \left( \frac{1}{p} - \frac{1}{2} \right) + \frac{k}{4}.
\]
The main difficulties to consider the Cauchy problem of 3D convective Cahn-Hilliard equation is how to deal with the convective term \( u \cdot \nabla u \) and the linear term of the double-well potential. Since the principle part of the convective Cahn-Hilliard equation is a fourth-order linear term and the convective term \( u \cdot \nabla u = \frac{1}{2} \nabla \cdot u^2 \) is only a first-order nonlinear term. We can’t control a first-order nonlinear term through fourth-order linear term in \( \mathbb{R}^3 \). In order to overcome this difficulty, we borrow a second-order term from the double-well potential, rewrite (5) as

\[
\partial_t u + \Delta^2 u - \Delta u = \Delta (u^3 - 2u) + u \cdot \nabla u. 
\]

Hence, one can control the convective term by the last term of the left hand side of equation (13). On the other hand, the linear term of the double-well potential also bring trouble to us. When we study the Cahn-Hilliard equations in bounded domain, it is easy to control the linear term by using Sobolev’s embedding theorem and the a priori estimates. However, in the whole space \( \mathbb{R}^3 \), since the compactness properties of Sobolev’s embedding theorem is losing, it is difficulty to deal with this linear term. By using the pure energy method \cite{25} of using a family of scaled energy estimates with minimum derivative counts and interpolations among them, we overcome the difficulty caused by the linear term of the double-well potential, obtained the suitable a priori estimates in the negative Sobolev space \( \dot{H}^{-s} \) \((0 \leq s \leq \frac{1}{2})\), prove the well-posedness and optimal decay rate of the Cauchy problem of convective Cahn-Hilliard equation in the whole spaces.

The rest of this paper is organized as follows. First of all, in Section 2, we give some useful results and lemmas which will be used in our proofs. Then, in Section 3, we establish the refined energy estimates for the solution \( u(x,t) \) of Cauchy problem (5). Section 4 is devoted to prove the local well-posedness of solutions. In Section 5, we derive the evolution of the negative Sobolev norms of the solution. The proofs of Theorem 1.4 and 1.6 are postponed in Section 6.

2. Preliminaries

In this section, we introduce some helpful results in \( \mathbb{R}^3 \).

The following Gagliardo-Nirenberg inequality was proved in \cite{35}.

**Lemma 2.1** \cite{35}. Let \( 0 \leq m, \alpha \leq l \), then we have

\[
\| \nabla^\alpha f \|_{L^p} \lesssim \| \nabla^m f \|_{L^q}^{1-\theta} \| \nabla^l f \|_{L^r}^\theta,
\]

where \( \theta \in [0,1] \) and \( \alpha \) satisfies

\[
\frac{\alpha}{3} - \frac{1}{p} = \left( \frac{m}{3} - \frac{1}{q} \right) (1-\theta) + \left( \frac{l}{3} - \frac{1}{r} \right) \theta.
\]

Here, when \( p = \infty \), we require that \( 0 < \theta < 1 \).

We also introduce the Hardy-Littlewood-Sobolev theorem, which implies the following \( L^p \) type inequality.

**Lemma 2.2** \cite{41,23}. Let \( 0 \leq s < \frac{3}{2} \), \( 1 < p \leq 2 \) and \( \frac{1}{2} + \frac{s}{3} = \frac{1}{p} \), then

\[
\| f \|_{\dot{H}^{-s}} \lesssim \| f \|_{L^p}.
\]
The following special Sobolev interpolation lemma will be used in the proof of Theorem 1.6.

**Lemma 2.3** ([42][44][41]). Let $s, k \geq 0$ and $l \geq 0$, then

$$
\| \nabla^l f \|_{L^2} \leq \| \nabla^{l+k} f \|_{L^2}^{1-\theta} \| f \|_{H^{-s}}^\theta, \quad \text{with } \theta = \frac{2}{l+k+s}.
$$

3. **Energy estimates**

The Cauchy problem (15) is equivalent to the following form:

$$
\begin{aligned}
\partial_t u + \Delta^2 u - \Delta u &= \Delta(u^3 - 2u) + u \cdot \nabla u, \quad x \in \mathbb{R}^3, \; t > 0, \\
u(\cdot, 0) &= u_0(\cdot), \quad x \in \mathbb{R}^3.
\end{aligned}
$$

In this section, on the basis of the assumptions of Theorem 1.4, we establish the energy estimates of the solution to the Cauchy problem (18).

**Lemma 3.1.** Assume $T > 0$ and $0 < \delta \ll 1$. Let

$$
\sup_{0 \leq t \leq T} \| u(t) \|_{H^2} \leq \delta,
$$

and all assumptions in Theorem 1.4 hold. Then, for any $t \in [0, T]$ and integer $k \geq 0$, we have

$$
\frac{d}{dt} \sum_{l=k}^{k+2} \| \nabla^l u \|_{L^2}^2 + \sum_{l=k}^{k+2} \| \nabla^{l+2} u \|_{L^2}^2 + \sum_{l=k}^{k+2} \| \nabla^{l+1} u \|_{L^2}^2 
\leq C_l \sum_{l=k}^{k+2} (\| u \|_{H^2} + \| u \|_{L^2}) (\| \nabla^{l+1} u \|_{L^2} + \| \nabla^{l+2} u \|_{L^2}).
$$

**Proof.** For any integer $k \geq 0$, applying $\nabla^l$ ($l = k, k+1, k+2$) to (18), multiplying the resulting identities by $\nabla^l u$, integrating over $\mathbb{R}^3$ by parts, we find that

$$
\frac{1}{2} \frac{d}{dt} \| \nabla^l u \|_{L^2}^2 + \| \nabla^{l+2} u \|_{L^2}^2 + \| \nabla^{l+1} u \|_{L^2}^2
= \int_{\mathbb{R}^3} \nabla^l (u^3 - 2u) \cdot \nabla^{l+2} u dx + \int_{\mathbb{R}^3} \nabla^l (u \cdot \nabla u) \cdot \nabla^l u dx.
$$

Note that

$$
\int_{\mathbb{R}^3} \nabla^l (u^3 - 2u) \cdot \nabla^{l+2} u dx
= \sum_{0 \leq s \leq l} C_s^l \int_{\mathbb{R}^3} \nabla^s [u(u - \sqrt{2})] \cdot \nabla^{l-s} (u + \sqrt{2}) \cdot \nabla^{l+2} u dx
$$

$$
= \sum_{0 \leq s \leq l} \sum_{0 \leq m \leq s} C_i^s C_m^l \int_{\mathbb{R}^3} \nabla^m u \cdot \nabla^{s-m} (u - \sqrt{2}) \cdot \nabla^{l-s} (u + \sqrt{2}) \cdot \nabla^{l+2} u dx
\leq \sum_{0 \leq s \leq l} \sum_{0 \leq m \leq s} C_i^s C_m^l \| \nabla^m u \|_{L^6} \| \nabla^{s-m} (u - \sqrt{2}) \|_{L^6} \| \nabla^{l-s} (u + \sqrt{2}) \|_{L^6} \| \nabla^{l+2} u \|_{L^2}
\leq \sum_{0 \leq s \leq l} \sum_{0 \leq m \leq s} C_i^s C_m^l \| \nabla^m \nabla u \|_{L^2} \| \nabla^{s-m} \nabla u \|_{L^2} \| \nabla^{l-s} \nabla u \|_{L^2} \| \nabla^{s+2} u \|_{L^2}.
$$
If \( 0 \leq s \leq \left[ \frac{l}{3} \right] \), using Lemma 2.1 it yields that

\[
\| \nabla^m \nabla u \|_{L^2} \| \nabla^{s-m} \nabla u \|_{L^2} \| \nabla^{l-s} \nabla u \|_{L^2} \tag{23}
\]

\[
\lesssim \| \nabla^\alpha u \|_{L^2}^{l+2-m} \| \nabla^{l+2} u \|_{L^2}^{1- \frac{l+2-m}{l+2}} \| u \|_{L^2}^{\frac{m+s-1}{l+2}} \| \nabla^{l+2} u \|_{L^2}^{1- \frac{m+s-1}{l+2}} \| u \|_{L^2} \| \nabla^{k+2} u \|_{L^2}^{1- \frac{1}{l+2}}
\]

\[
\lesssim \| u \|_{H^2} \| \nabla^{l+2} u \|_{L^2},
\]

where \( \alpha \) satisfies

\[
m + 1 - \frac{1}{2} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \times \frac{l + 2 - m}{l + 2} + \left( \frac{l + 2}{3} - \frac{1}{2} \right) \times \left( 1 - \frac{l + 2 - m}{l + 2} \right).
\]

Since \( 0 \leq s \leq \left[ \frac{l}{3} \right] \), we easily obtain \( \alpha = \frac{l+2}{l+2-m} \in [1, 2) \). Moreover, if \( \left[ \frac{l}{3} \right] + 1 \leq s \leq l \), using Lemma 2.1 we deduce that

\[
\| \nabla^m \nabla u \|_{L^2} \| \nabla^{s-m} \nabla u \|_{L^2} \| \nabla^{l-s} \nabla u \|_{L^2} \tag{24}
\]

\[
\lesssim \| u \|_{L^2}^{l+2-m} \| \nabla^{l+2} u \|_{L^2}^{1- \frac{l+2-m}{l+2}} \| u \|_{L^2}^{\frac{m+s-1}{l+2}} \| \nabla^{l+2} u \|_{L^2}^{1- \frac{m+s-1}{l+2}} \| u \|_{L^2} \| \nabla^{k+2} u \|_{L^2}^{1- \frac{1}{l+2}}
\]

\[
\lesssim \| u \|_{H^2} \| \nabla^{l+2} u \|_{L^2},
\]

where \( \alpha \) is defined by

\[
\frac{l + 1 - s}{3} - \frac{1}{2} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \times \frac{s + 2}{l + 2} + \left( \frac{l + 2}{3} - \frac{1}{2} \right) \times \left( 1 - \frac{s + 2}{l + 2} \right).
\]

Since \( \left[ \frac{l}{3} \right] + 1 \leq s \leq l \), we have \( \alpha = \frac{l+2}{s+2} \in [1, 2) \). Combining (23)-(24) together gives

\[
\int_{\mathbb{R}^3} \nabla^l (u^3 - 2u) \cdot \nabla^{l+2} u \, dx \lesssim \| u \|_{H^2}^2 \| \nabla^{l+2} u \|_{L^2}^2. \tag{25}
\]

On the other hand, we have

\[
\int_{\mathbb{R}^3} \nabla^l (u \cdot \nabla u) \cdot \nabla^l u \, dx = \sum_{0 \leq s \leq l} C_i^s \int_{\mathbb{R}^3} \nabla^s u \cdot \nabla \nabla^{l-s} u \cdot \nabla^l u \, dx
\]

\[
\lesssim \sum_{0 \leq s \leq l} \| \nabla^s u \cdot \nabla \nabla^{l-s} u \|_{L^6} \| \nabla^l u \|_{L^6}
\]

\[
\lesssim \sum_{0 \leq s \leq l} \| \nabla^s u \cdot \nabla \nabla^{l-s} u \|_{L^6} \| \nabla^{l+1} u \|_{L^2}.
\]

If \( 0 \leq s \leq \left[ \frac{l}{3} \right] \), from Lemma 2.1 we infer that

\[
\| \nabla^s u \cdot \nabla \nabla^{l-s} u \|_{L^6} \lesssim \| \nabla^s u \|_{L^3} \| \nabla^{l-s+1} u \|_{L^2}
\]

\[
\lesssim \| \nabla^s u \|_{L^2}^{\frac{1}{3}} \| \nabla^{l+1} u \|_{L^2}^{\frac{1}{3}} \| u \|_{L^2}^{\frac{1}{3}} \| \nabla^{l+1} u \|_{L^2}^{\frac{1}{3}} \| u \|_{L^2} \| \nabla^{l+1} u \|_{L^2}^{\frac{1}{3}}
\]

\[
\lesssim \| u \|_{H^2} \| \nabla^{l+1} u \|_{L^2}^2,
\]

where \( \alpha \) satisfies

\[
s - \frac{1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{s}{l+1} \right) + \left( \frac{l + 1}{3} - \frac{1}{2} \right) \frac{s}{l+1}.
\]
Since \(0 \leq s \leq \left[\frac{l}{2}\right]\), we easily obtain \(\alpha = \frac{l+1}{2s} \in \left[\frac{1}{2}, 1\right]\). If \(s \geq \left[\frac{l}{2}\right]+1\), by Lemma 2.1, again, we derive that
\[
\|\nabla^s u \cdot \nabla^{l-s} u\|_{L^2} \lesssim \|\nabla^s u\|_{L^2} \|\nabla^{l-s+1} u\|_{L^2}^{\alpha} \lesssim \|u\|_{L^2}^{1-\frac{s}{l+1}} \|\nabla^{l+1} u\|_{L^2}^\alpha \lesssim \|u\|_{H^2} \|\nabla^{l+1} u\|_{L^2}^2,
\]
where \(\alpha\) satisfies
\[
l - s + 1 - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right) \frac{s}{l+1} + \left(\frac{l+1}{3} - \frac{1}{2}\right) \left(1 - \frac{s}{l+1}\right).
\]
Since \(s \geq \left[\frac{l}{2}\right]+1\), we obtain \(\alpha = \frac{l+1}{2s} \in \left[\frac{1}{2}, 1\right]\). In light of (27) and (29), we deduce from (30) that
\[
\int_{\mathbb{R}^3} \nabla^l (u \cdot \nabla u) \cdot \nabla^l u dx \lesssim \|u\|_{H^2} \|\nabla^{l+1} u\|_{L^2}^2.
\]
Plugging (25) and (31) into (21), we conclude that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^l u\|_{L^2}^2 + \|\nabla^{l+2} u\|_{L^2}^2 + \|\nabla^{l+1} u\|_{L^2}^2 \leq C(\|u\|_{H^2}^2 + \|u\|_{H^2}) (\|\nabla^{l+2} u\|_{L^2}^2 + \|\nabla^{l+1} u\|_{L^2}^2),
\]
then we complete the proof. \(\square\)

4. LOCAL WELL-POSEDNESS

The main purpose of this section is to prove the local existence of solution \(u(x, t)\) in \(H^2\)-norm. To this end, we construct the solution \((u^j)_{j \geq 0}\) by solving iteratively the Cauchy problems:
\[
\begin{cases}
\partial_t u^{j+1} + \Delta^2 u^{j+1} - \Delta u^{j+1} = \Delta [(u^j)^2 u^{j+1} - 2u^{j+1}] + u^j \cdot \nabla u^{j+1}, \\
u^{j+1}|_{t=0} = u_0(x), \quad x \in \mathbb{R}^3,
\end{cases}
\]
for \(j \geq 0\), where \(u_0 \equiv 0\) holds. In the following, we shall show that \((u^j)_{j \geq 0}\) is a Cauchy sequence in Banach space \(C([0, T_1]; H^2)\) with \(T_1 > 0\) suitable small. Then, by take limit and continuous argument, one propose to prove that \(u(x, t)\) is a global solution to Cauchy problem (18).

**Lemma 4.1.** Suppose that all assumptions in Theorem 1.4 hold. There are constants \(\varepsilon > 0, T_1 > 0\) and \(M_1 > 0\) such that if \(\|u_0\|_{H^2} \leq \varepsilon\), then for each \(j \geq 0\), \(u^j \in C([0, T_1]; H^2)\) is well defined and
\[
\sup_{0 \leq t \leq T_1} \|u^j(t)\|_{H^2} \leq M_1, \quad j \geq 0.
\]
Moreover, \((u^j)_{j \geq 0}\) is a Cauchy sequence in Banach space \(C([0, T_1]; H^2)\), the corresponding limit function denoted by \(u(t)\) belongs to \(C([0, T_1]; H^2)\) with
\[
\sup_{0 \leq t \leq T_1} \|u(t)\|_{H^2} \leq M_1,
\]
and \(u(t)\) is a solution over \([0, T_1]\) to problem (18). Finally, for the Cauchy problem (18), there exists at most one solution in \(C([0, T_1]; H^2)\) satisfying (32).
PROOF. In the following, the inequality (34) will be proved by induction. First of all, since \( u^0 = 0 \), by the assumption at initial step, we easily obtain \( j = 0 \) is the trivial case. Suppose that (35) holds for \( j \geq 0 \) with \( M_1 > 0 \) small enough to be determined later. We also need to prove it for \( j + 1 \). For this goal, some energy estimates on \( u^{j+1} \) are needed. By (33), we can see that for \( k = 2 \) and \( 0 \leq l \leq k \),

\[
\frac{1}{2} \frac{d}{dt} \| \nabla^l u^{j+1} \|_{L^2}^2 + \| \nabla^{l+1} u^{j+1} \|_{L^2}^2 \\
= \int_{\mathbb{R}^3} \nabla^l [(u^j + \sqrt{2})(u^j - \sqrt{2})u^{j+1}] \cdot \nabla^{l+1} u^{j+1} dx + \int_{\mathbb{R}^3} \nabla^l (u^j \cdot \nabla u^{j+1}) \cdot \nabla^l u^{j+1} dx.
\]

(36)

Since (34) is trivial for the case \( l = 0 \), thus, we may only consider \( 1 \leq l \leq 2 \). For the last term of the right hand side of (36), by Hölder’s inequality, we derive that

\[
\int_{\mathbb{R}^3} \nabla^l (u^j \cdot \nabla u^{j+1}) \cdot \nabla^l u^{j+1} dx \\
\leq \|
\sum_{s=0}^l \nabla^s u^j \cdot \nabla^{l-s} \nabla u^{j+1} \|_{L^2} \| \nabla^l u^{j+1} \|_{L^6}
\]

\[
(37)
\]

\[
\lesssim \|
\sum_{s=0}^l \| \nabla^s u^j \|_{L^3} \| \nabla^{l-s} \nabla u^{j+1} \|_{L^2} \| \nabla^{l-1} u^{j+1} \|_{L^2}
\]

\[
\lesssim \|
\sum_{s=0}^l \| \nabla^s u^j \|_{L^2} \| \nabla^{l-s} \nabla u^{j+1} \|_{L^2} \| \nabla^{l-1} u^{j+1} \|_{L^2}
\]

\[
\lesssim \| u^j \|_{H^2} \| \nabla u^{j+1} \|_{H^2}^2,
\]

where \( \theta = 1 - \frac{s}{7} \). On the other hand, if \( l = 1 \), the first term of the right hand side of (36) can be estimated as

\[
\int_{\mathbb{R}^3} \nabla^1 [(u^j + \sqrt{2})(u^j - \sqrt{2})u^{j+1}] \cdot \nabla^3 u^{j+1} dx \\
\lesssim \| \nabla^3 u^{j+1} \|_{L^2} \|
\sum_{s=0}^l \sum_{m=0}^l \nabla^m (u^j + \sqrt{2}) \cdot \nabla^{l-s} (u^j - \sqrt{2}) \cdot \nabla^{l-s} u^{j+1} \|_{L^2}
\]

\[
(38)
\]

\[
\lesssim \| \nabla^3 u^{j+1} \|_{L^2} \| \nabla^3 (u^j + \sqrt{2}) \|_{L^6} \| \nabla^{s-m} (u^j - \sqrt{2}) \|_{L^6} \| \nabla^{l-s} u^{j+1} \|_{L^6}
\]

\[
\lesssim \| \nabla^3 u^{j+1} \|_{L^2} \| \nabla^{l-1} u^{j+1} \|_{L^2} \| \nabla^m u^j \|_{L^2} \| \nabla^{s-m} \nabla u^j \|_{L^2}
\]

\[
\lesssim \| \nabla^3 u^{j+1} \|_{L^2} \| \nabla^{l-s} u^{j+1} \|_{L^2} \| \nabla^j u^j \|_{L^2} \| \nabla^j u^j \|_{L^2} \| \nabla^j u^j \|_{L^2}
\]

\[
\lesssim \| u^j \|_{H^2}^2 \| \nabla u^{j+1} \|_{H^2}^2 + \| \nabla^2 u^{j+1} \|_{H^2}^2,
\]
where $\theta_1 = \frac{2 - m}{2}$ and $\theta_2 = \frac{2 + m - s}{2}$. Moreover, if $l = 2$, by using Sobolev's embedding theorem, we have

$$
\int_{\mathbb{R}^3} \nabla^2 [(u^j + \sqrt{2})(u^j - \sqrt{2})u^{j+1}] \cdot \nabla^4 u^{j+1} \, dx
\leq \|
abla^4 u^{j+1}\|_{L^2} \|
abla^2 [(u^j + \sqrt{2})(u^j - \sqrt{2})u^{j+1}]\|_{L^2}
\leq \|
abla^4 u^{j+1}\|_{L^2} \left( \|
abla^2 (u^j + \sqrt{2})\|_{L^2} \|
abla^4 u^{j+1}\|_{L^2} + \|
abla (u^j + \sqrt{2})\|_{L^3} \|
abla (u^j - \sqrt{2})\|_{L^6} \|
abla^4 u^{j+1}\|_{L^2} \right)
\leq \|
abla^4 u^{j+1}\|_{L^2} \left( \|
abla^2 u^j\|_{L^2} \|
abla u^j\|_{L^2}^2 + \|
abla^2 u^j\|_{L^2} \|
abla u^j\|_{L^2}^2 \|
abla^4 u^{j+1}\|_{L^2} \right)
\leq \|
abla^4 u^{j+1}\|_{L^2} \left( \|
abla^2 u^j\|_{L^2}^2 + \|
abla^2 u^j\|_{L^2}^2 \|
abla^4 u^{j+1}\|_{L^2} \right)
\|u^j\|_{H^2}^2 (\|
abla u^j\|_{H^2}^2 + \|
abla^2 u^j\|_{H^2}^2).
$$

Combining (36) and (39) together gives

$$
\frac{1}{2} \frac{d}{dt} \|u^{j+1}\|_{H^2}^2 + \|\nabla^2 u^{j+1}\|_{H^2}^2 + \|\nabla u^{j+1}\|_{H^2}^2 
\leq C (\|u^j\|_{H^2}^2 + \|u^j\|_{H^2}^2) (\|\nabla u^{j+1}\|_{H^2}^2 + \|\nabla^2 u^{j+1}\|_{H^2}^2).
$$

By taking time integration, it yields that

$$
\|u^{j+1}\|_{H^2}^2 + \int_0^t (\|\nabla^2 u^{j+1}(s)\|_{H^2}^2 + \|\nabla u^{j+1}(s)\|_{H^2}^2) \, ds
\leq C \|u_0\|_{H^2}^2 + C \int_0^t (\|u^j(s)\|_{H^2}^2 + \|u^j(s)\|_{H^2}^2) (\|\nabla u^{j+1}(s)\|_{H^2}^2 + \|\nabla^2 u^{j+1}(s)\|_{H^2}^2) \, ds,
$$

which from the inductive assumption implies

$$
\|u^{j+1}\|_{H^2}^2 + \int_0^t (\|\nabla^2 u^{j+1}(s)\|_{H^2}^2 + \|\nabla u^{j+1}(s)\|_{H^2}^2) \, ds
\leq C \varepsilon^2 + C (M_1 + M_2^2) \int_0^t (\|\nabla^2 u^{j+1}(s)\|_{H^2}^2 + \|\nabla u^{j+1}(s)\|_{H^2}^2) \, ds,
$$
for any $0 \leq t \leq T_1$. Take suitable small $\varepsilon > 0$, $T_1 > 0$ and $M_1 > 0$ such that

$$
(41) \quad \|u^{j+1}\|_{H^2}^2 + \int_0^t \left( \|\nabla^2 u^{j+1}(s)\|_{H^2}^2 + \|\nabla u^{j+1}(s)\|_{H^2}^2 \right) ds \leq M_1^2,
$$

for any $t \in [0, T_1]$. Hence, (34) is true for $j + 1$ if so for $j$, which means (34) is proved for all $j \geq 0$.

Next, on the basis of (40), we easily obtain

$$
\left\| \frac{d}{dt} \int_s^t \|u^{j+1}(\tau)\|_{H^2}^2 \, d\tau \right\|
\leq C \int_s^t \left( \|u^j(\tau)\|_{H^2} + \|u^j\|_{H^2}(\|\nabla u^{j+1}(\tau)\|_{H^2}^2 + \|\nabla^2 u^{j+1}(\tau)\|_{H^2}^2) \right) d\tau
\leq C(M_1 + M_1^2) \int_s^t \left( \|\nabla u^{j+1}(\tau)\|_{H^2}^2 + \|\nabla^2 u^{j+1}(\tau)\|_{H^2}^2 \right) d\tau,
$$

for any $t \in [0, T_1]$. Therefore, due to the time integral in the last inequality is finite due to (41), and hence $\|u^{j+1}(t)\|_{H^2}^2$ is continuous in $t$ for each $j \geq 1$. In the following, we also need to consider the convergence of the sequence $(u^j)_{j \geq 0}$. Taking the difference of (33) for $j$ and $j - 1$, we obtain

$$
\partial_t (u^{j+1} - u^j) + \Delta^2 (u^{j+1} - u^j) - \Delta (u^{j+1} - u^j)
= \Delta [(u^j)^2(u^{j+1} - u^j) + [(u^j)^2 - (u^{j-1})^2]u^j] + u^j \cdot \nabla (u^{j+1} - u^j) + (u^j - u^{j-1}) \cdot \nabla u^j.
$$

Appealing to the same energy estimate as before, we easily obtain

$$
\frac{1}{2} \frac{d}{dt} \left[ \|u^{j+1} - u^j\|_{H^2}^2 + \|\nabla^2 (u^{j+1} - u^j)\|_{H^2}^2 + \|\nabla (u^{j+1} - u^j)\|_{H^2}^2 \right]
\leq C(\|u^j - u^{j-1}\|_{H^2}^2 + \|u^j - u^{j-1}\|_{H^1}^2)(\|\nabla^2 u^j\|_{H^2}^2 + \|\nabla u^j\|_{H^2}^2)
+ C(\|u^j\|_{H^2}^2 + \|u^j\|_{H^2}^2)(\|\nabla^2 (u^{j+1} - u^j)\|_{H^2}^2 + \|\nabla (u^{j+1} - u^j)\|_{H^2}^2),
$$

which is equivalent to

$$
\frac{1}{2} \frac{d}{dt} \left( \|u^{j+1} - u^j\|_{H^2}^2 + 1 \right) + \|\nabla^2 (u^{j+1} - u^j)\|_{H^2}^2 + \|\nabla (u^{j+1} - u^j)\|_{H^2}^2
\leq C(\|u^j - u^{j-1}\|_{H^2}^2 + 1)(\|\nabla^2 u^j\|_{H^2}^2 + \|\nabla u^j\|_{H^2}^2)
+ C(\|u^j\|_{H^2}^2 + \|u^j\|_{H^2}^2)(\|\nabla^2 (u^{j+1} - u^j)\|_{H^2}^2 + \|\nabla (u^{j+1} - u^j)\|_{H^2}^2),
$$

On the basis of (41), by taking time integration, it holds that

$$
\|u^{j+1} - u^j\|_{H^2}^2 + 1 + \int_0^t \left( \|\nabla^2 (u^{j+1} - u^j)(\tau)\|_{H^2}^2 + \|\nabla (u^{j+1} - u^j)(\tau)\|_{H^2}^2 \right) d\tau
\leq CM_1^2 \sup_{0 \leq \tau \leq T_1} (\|u^j(\tau) - u^{j-1}(\tau)\|^2 + 1)
+ C(M_1 + M_1^2) \int_0^t \left( \|\nabla^2 (u^{j+1} - u^j)(\tau)\|_{H^2}^2 + \|\nabla (u^{j+1} - u^j)(\tau)\|_{H^2}^2 \right) d\tau.
$$
By the smallness of $M_1$, there exists a constant $\lambda \in (0, 1)$ such that

$$\sup_{0 \leq t \leq T_1} \|u^{j+1}(t) - u^j(t)\|_{H^2}^2 \leq \lambda \sup_{0 \leq t \leq T_1} \|u^j(t) - u^{j-1}(t)\|_{H^2}^2,$$

for any $j \geq 1$. Hence, $(u^j)_{j \geq 0}$ is a Cauchy sequence in the Banach space $C([0, T_1]; H^2)$, the limit function

$$u(t) = u_0 + \lim_{n \to \infty} \sum_{j=0}^n (u^{j+1} - u^j)$$

exists in $C([0, T_1]; H^2)$, satisfies

$$\sup_{0 \leq t \leq T_1} \|u(t)\|_{H^2}^2 \leq \sup_{0 \leq t \leq T_1} \liminf_{j \to \infty} \|u(t)\|_{H^2} \leq M_1.$$

Thus, (35) is proved. Finally, Let $u(t)$ and $\tilde{u}(t)$ are two solutions in $C([0, T_1]; H^2)$ satisfying (35). By using the same process as in (45) to prove the convergence of $(u^j)_{j \geq 0}$, we obtain

$$\sup_{0 \leq t \leq T_1} \|u(t) - \tilde{u}(t)\|_{H^2}^2 \leq \lambda \sup_{0 \leq t \leq T_1} \|u(t) - \tilde{u}(t)\|_{H^2}^2,$$

for $\lambda \in (0, 1)$, which implies that $u(t) = \tilde{u}(t)$. Then, we complete the proof of uniqueness and thus the proof of Lemma 4.1 is complete.

\[\square\]

5. Negative Sobolev estimates

In this section, we derive the evolution of the negative Sobolev norms of the solution to the Cauchy problem (5). In order to estimate the convective term and the double-well potential, we shall restrict ourselves to that $s \in [0, \frac{1}{2}]$.

For the homogeneous Sobolev space, the following lemma holds:

**Lemma 5.1.** Suppose that all the assumptions in Lemma 3.1 are in force. For $s \in [0, \frac{1}{2}]$, we have

$$\frac{d}{dt} \|u(t)\|_{H^{-s}}^2 + \|\nabla^2 u(t)\|_{H^{-s}}^2 + \|\nabla u(t)\|_{H^{-s}}^2 \lesssim \|\nabla u\|_{H^1}^2 \|u(t)\|_{H^{-s}},$$

where the parameter $\delta$ is the same as (19).

**Proof.** Applying $\Lambda^{-s}$ to (18), multiplying the resulting identities by $\Lambda^{-s} u$, and then integrating over $\mathbb{R}^3$ by parts, we deduce that

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{-s} u\|_{L^2}^2 + \|\Lambda^{-s} \nabla^2 u\|_{L^2}^2 + \|\Lambda^{-s} \nabla u\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \Lambda^{-s} (u \cdot \nabla u) \cdot \Lambda^{-s} u dx + \int_{\mathbb{R}^3} \Lambda^{-s} \Delta (u^3 - 2u) \cdot \Lambda^{-s} u dx.$$
For the first term of the right hand side of (47), we have
\[
\int_{\mathbb{R}^3} \Lambda^{-s}(u \cdot \nabla u) \cdot \Lambda^{-s} u dx \leq \|\Lambda^{-s}(u \cdot \nabla u)\|_{L^2} \|\Lambda^{-s} u\|_{L^2} \\
\lesssim \|u \cdot \nabla u\|_{L^{\frac{2}{1+s}}} \|\Lambda^{-s} u\|_{L^2} \\
\lesssim \|u\|_{L^2} \|\nabla u\|_{L^2} \|\Lambda^{-s} u\|_{L^2} \\
\lesssim \|\nabla u\|_{L^2}^{\frac{s}{2+s}} \|\nabla^2 u\|_{L^2}^{\frac{1-s}{2}} \|\nabla u\|_{L^2} \|\Lambda^{-s} u\|_{L^2} \\
\lesssim \|\Lambda^{-s} u\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).
\]

For the second term of the right hand side of (47), we have
\[
\int_{\mathbb{R}^3} \Lambda^{-s} \Delta (u^3 - 2u) \cdot \Lambda^{-s} u dx \\
= \int_{\mathbb{R}^3} \Lambda^{-s} \Delta [u(u + \sqrt{2})(u - \sqrt{2})] \cdot \Lambda^{-s} u dx \\
\lesssim \|\Lambda^{-s} u\|_{L^2} \|\Lambda^{-s} \Delta [u(u + \sqrt{2})(u - \sqrt{2})]\|_{L^2} \\
\lesssim \|\Lambda^{-s} u\|_{L^2} \left[ \|\Lambda^{-s}(u(u + \sqrt{2})\Lambda^2(u - \sqrt{2}))\|_{L^2} + \|\Lambda^{-s}(u(u - \sqrt{2})\Lambda^2(u + \sqrt{2}))\|_{L^2} \\
+ \|\Lambda^{-s}((u - \sqrt{2})(u + \sqrt{2})\Lambda^2 u)\|_{L^2} + \|\Lambda^{-s}(\nabla u \cdot \nabla (u - \sqrt{2})(u + \sqrt{2}))\|_{L^2} \\
+ \|\Lambda^{-s}(\nabla u \cdot \nabla (u + \sqrt{2}) \cdot (u - \sqrt{2}))\|_{L^2} + \|\Lambda^{-s}(\nabla (u + \sqrt{2}) \cdot \nabla (u - \sqrt{2}) \cdot u)\|_{L^2}\right] \\
\lesssim \|\Lambda^{-s} u\|_{L^2} \left[ \|u(u + \sqrt{2})\Lambda^2(u - \sqrt{2})\|_{L^\frac{1}{1+s}^\frac{1}{1+s}} + \|u(u - \sqrt{2})\Lambda^2(u + \sqrt{2})\|_{L^\frac{1}{1+s}^\frac{1}{1+s}} \\
+ \|(u - \sqrt{2})(u + \sqrt{2})\Lambda^2 u\|_{L^\frac{1}{1+s}^\frac{1}{1+s}} + \|\nabla u\|_{L^\frac{1}{1+s}^\frac{1}{1+s}} \|\nabla (u - \sqrt{2})\|_{L^\frac{1}{1+s}^\frac{1}{1+s}} \|\nabla (u + \sqrt{2})\|_{L^\frac{1}{1+s}^\frac{1}{1+s}} \right] \\
\lesssim \|\Lambda^{-s} u\|_{L^2} (\|u\|_{L^{\infty}} + \|u\|_{L^\frac{2}{1+s}}) \|\nabla^2 (u - \sqrt{2})\|_{L^2} + \|u\|_{L^{\infty}} \|u - \sqrt{2}\|_{L^\frac{3}{2}} \|\nabla^2 (u + \sqrt{2})\|_{L^2} \\
+ \|u - \sqrt{2}\|_{L^{\infty}} \|u + \sqrt{2}\|_{L^\frac{3}{2}} \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla (u + \sqrt{2})\|_{L^3} \|\nabla (u - \sqrt{2})\|_{L^3} \\
\lesssim \|\nabla u\|_{H^1} \|\Lambda^{-s} u\|_{L^2},
\]
where we have used
\[
\|v\|_{L^\infty} \lesssim \|\nabla v\|_{L^2} \|\Delta v\|_{L^2},
\]
\[
\|v\|_{L^3} \lesssim \|v\|_{L^2} \|\nabla v\|_{L^2},
\]
and
\[
\|v\|_{L^\frac{3}{2}} \lesssim \|\nabla v\|_{L^2} \|\Delta v\|_{L^2} \|\Lambda^{-s} u\|_{L^2}.
\]
Plugging the estimates (48) and (49) into (47), we deduce (46). Hence, the proof is complete. \(\square\)
6. Proofs of Theorems 1.4 and 1.6

Firstly, on the basis of the assumption that \( \|u_0\|_{H^2} \) is sufficiently small, we propose to prove the existence and uniqueness of global solution to Cauchy problem (5).

Proof of Theorem 1.4. There are two steps for us to prove Theorem 1.4.

Step 1. Global small \( E_2 \) solution.

It follows from the assumption (19), taking \( k = 0 \) in (20), we have for any \( t \in [0, T] \),

\[
\frac{d}{dt} \left( \sum_{l=0}^{2} \| \nabla^l u \|_{L^2}^2 + \sum_{l=0}^{2} \| \nabla^{l+2} u \|_{L^2}^2 + \sum_{l=0}^{2} \| \nabla^{l+1} u \|_{L^2}^2 \right) \leq C_2(\sqrt{E_2(t)} + E_2(t))D_2(t) \leq C_2(\delta + 1)D_2(t).
\]

By (50), we can choose a sufficiently small \( \delta \), such that

\[
E_2(t) + \int_{0}^{t} D_2(\tau)d\tau \leq \tilde{C}E_2(0), \quad \forall t \in [0, T].
\]

Suppose that \( \varepsilon_0 \) is a positive constant, which satisfies

\[
\varepsilon_0 = \min\{\delta + \delta^2, \varepsilon\},
\]

where \( \delta > 0 \) and \( \varepsilon > 0 \) are given in Lemmas 3.1 and 4.1 respectively. We also choose initial data \( u_0 \) and small constant \( \delta_0 \), such that

\[
\sqrt{E_2(0)} \leq \sqrt{\delta_0} := \frac{\varepsilon_0}{2\sqrt{1 + C_2}}.
\]

Next, define the lifespan of solutions of problem (18) by

\[
T := \sup \left\{ t : \sup_{0 \leq \tau \leq t} \sqrt{E_2(s)} \leq \varepsilon_0 \right\}.
\]

Note that

\[
\sqrt{E_2(0)} \leq \frac{\varepsilon_0}{2\sqrt{1 + C_2}} \leq \frac{\varepsilon_0}{2} < \frac{\varepsilon_0}{2} \leq \varepsilon,
\]

hence \( T > 0 \) holds true from the local existence result Lemma 4.1 and the continuation argument. If the time \( T \) is finite, from the definition of \( T \), we have

\[
\sup_{0 \leq \tau \leq T} \sqrt{E_2(\tau)} = \varepsilon_0.
\]

However, on the basis of the uniform a priori estimate (51), the following inequalities hold:

\[
\sup_{0 \leq \tau \leq T} \sqrt{E_2(\tau)} \leq \sqrt{\tilde{C}_2} \sqrt{E_2(0)} \leq \frac{\sqrt{\tilde{C}_2} \varepsilon_0}{2\sqrt{1 + C_2}} \leq \frac{\varepsilon_0}{2},
\]

which is a contradiction. Therefore, \( T = \infty \), and the local solution \( u(t) \) obtained in Lemma 4.1 can be extent to infinite time. Thus, there exists a unique solution \( u(t) \in C([0, \infty]; H^2) \) for the Cauchy problem (18), and the inequality (8) holds.
Recall (20), for $N \geq 3$, we have

\[ \frac{d}{dt} \sum_{l=0}^{N} \| \nabla^l u \|_{L^2}^2 + \sum_{l=0}^{N} \| \nabla^{l+2} u \|_{L^2}^2 + \sum_{l=0}^{N} \| \nabla^{l+1} u \|_{L^2}^2 \leq (\sqrt{\mathcal{E}_2(t)} + \mathcal{E}_2(t)) \mathcal{D}_N(t). \]

Since

\[ \varepsilon_0 = \min \{ \delta + \delta^2, \varepsilon \}. \]

By using the smallness of $\varepsilon_0$ and (52), we deduce that

\[ \mathcal{E}_N(t) + \int_0^t \mathcal{D}_N(t) \leq C \mathcal{E}_N(0), \quad \forall t \in [0, \infty], \]

this complete the proof of Theorem 1.4.

The second purpose of this paper is to establish the temporary decay rate of unique global solutions for Cauchy problem (5). On the basis of the conclusions of Theorem 1.4 and Lemma 5.1, we proceed to prove this result.

**Proof of Theorem 1.6.** Define

\[ \mathcal{E}_{-s}(t) := \| \Lambda^{-s} u(t) \|_{L^2}^2. \]

Then, integrating in time (46) of Lemma 5.1, by the bound (7), we obtain that for $s \in [0, \frac{1}{2}]$,

\[ \mathcal{E}_{-s}(t) \leq \mathcal{E}_{-s}(0) + C \int_0^t \| \nabla u \|_{H^1}^2 \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \]

\[ \leq C_0 \left( 1 + \sup_{0 \leq \tau \leq t} \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \right), \]

which implies (10) for $s \in [0, \frac{1}{2}]$, that is

\[ \| \Lambda^{-s} u(t) \|_{L^2}^2 \leq C_0. \]

Moreover, if $k = 1, 2, \cdots, N - 2$, we may use Lemma 2.3 to have

\[ \| \nabla^{k+2} f \|_{L^2}^2 \geq C \| \Lambda^{-1} f \|_{L^2}^2 \| \nabla^{k+1} f \|_{L^2}^{1 + \frac{2}{k+1}}. \]

Then, by this fact and (54), we get

\[ \| \nabla^{k+2} u \|_{L^2}^2 \geq C_0 \left( \| \nabla^{k} u \|_{L^2}^2 \right)^{1 + \frac{2}{k+1}}. \]

On the other hand, we may define a family of energy functions and the corresponding dissipation rates with minimum derivatives counts as

\[ \mathcal{E}^{k+2}_k := \sum_{l=k}^{k+2} \| \nabla^l u(t) \|_{L^2}^2, \]

\[ \mathcal{D}^{k+2}_k := \sum_{l=k}^{k+2} \frac{d}{dt} \| \nabla^l u(t) \|_{L^2}^2. \]
and

\[ D_k^{k+2} := \sum_{l=k}^{k+2} (\|\nabla^l \nabla u\|_{L^2}^2 + \|\nabla^{l+2} u\|_{L^2}^2). \]

Taking into account Lemma 3.1 and Theorem 1.4 we have that for \( k = 0, 1, \cdots, N - 2 \)
that

\[ \frac{d}{dt} \mathcal{E}_k^{k+2} + D_k^{k+2} \leq 0. \]

Note that

\[ D_k^{k+2} \geq \sum_{l=k}^{k+2} \|\nabla^{l+2} u\|_{L^2}^2. \]

Combining (55) and (59) together gives

\[ D_k^{k+2} \gtrsim (\mathcal{E}_k^{k+2})^{1 + \frac{2}{k+2}}. \]

From (58) and (60), we conclude that

\[ \frac{d}{dt} \mathcal{E}_k^{k+2} + (\mathcal{E}_k^{k+2})^{1 + \frac{2}{k+2}} \leq 0, \]

with \( k = 0, 1, \cdots, N - 2 \). Solving (61) directly gives

\[ \mathcal{E}_k^{k+2} \leq C_0 (1 + t)^{-\frac{k+2}{2}}, \quad \text{for} \quad k = 1, 2, \cdots, N - 2. \]

This completes the proof of Theorem 1.6.

\[ \square \]

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REFERENCES

[1] Appadu, A. R.; Djoko, J. K.; Gidey, H. H.; Lubuma, J. M. S.; Analysis of multilevel finite volume approximation of 2D convective Cahn-Hilliard equation. Jpn. J. Ind. Appl. Math. 34 (2017), no. 1, 253-304.
[2] Blomker, D.; Maier-Paape, S.; Wanner, T.; Spinodal decomposition for the Cahn-Hilliard-Cook equation. Comm. Math. Phys. 223 (2001), 553-582.
[3] Bosia, S.; Conti, M.; Grasselli, M.; On the Cahn-Hilliard-Brinkman system. Commun. Math. Sci. 13(2015), 1541-1567.
[4] Bricmont, J.; Kupiainen, A.; Taskinen, J.; Stability of Cahn-Hilliard fronts. Comm. Pure Appl. Math. 52 (1999), 839-871.
[5] Caffarelli, L. A.; Muler, N. E.; An \( L^\infty \) bound for solutions of the Cahn-Hilliard equation. Arch. Ration. Mech. Anal. 133 (1995) 129-144.
[6] Cahn, J. W.; Hilliard, J. E.; Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys. 28 (1958), 258-267.
[7] Cherfils, Laurence; Fakih, Hussein; Miranville, Alain; A Cahn-Hilliard system with a fidelity term for color image inpainting. J. Math. Imaging Vision, 54 (2016), no. 1, 117-131.
[8] Cholewa, J. W.; Rodriguez-Bernal, A.; On the Cahn-Hilliard equation in $H^1(\mathbb{R}^N)$. J. Differential Equations 253 (2012), 3678-3726.

[9] Colli, Pierluigi; Gilardi, Gianni; Spreksels, Jürgen; Optimal velocity control of a viscous Cahn-Hilliard system with convection and dynamic boundary conditions. SIAM J. Control Optim. 56 (2018), no. 3, 1665-1691.

[10] Colli, Pierluigi; Gilardi, Gianni; Spreksels, Jürgen; Optimal velocity control of a convective Cahn-Hilliard system with double obstacles and dynamic boundary conditions: a ‘deep quench’ approach. J. Convex Anal. 26 (2019), no. 2, 485-514.

[11] Della Porta, Francesco; Grasselli, Maurizio; Convective nonlocal Cahn-Hilliard equations with reaction terms. Discrete Contin. Dyn. Syst. Ser. B 20 (2015), no. 5, 1529-1553.

[12] Dlotko, T; Kania, M. B.; Sun, C.; Analysis of the viscous Cahn-Hilliard equation in $R^N$. J. Differential Equations 252 (2012), 2771-2791.

[13] Duan, N.; Zhao, X.; Global well-posedness and large time behavior to fractional Cahn-Hilliard equation in $\mathbb{R}^N$. Forum Math. 31 (2019), 803-814.

[14] Ebenbeck, Matthias; Knopf, Patrik; Optimal medication for tumors modeled by a Cahn-Hilliard-Brinkman equation. Calc. Var. Partial Differential Equations 58 (2019), no. 4, Art. 131, 31 pp.

[15] Eden, A.; Kalantarov, V. K.; The convective Cahn-Hilliard equation. Appl. Math. Lett. 20 (2007), no. 4, 455-461.

[16] Eden, Alp; Kalantarov, Varga K.; 3D convective Cahn-Hilliard equation. Commun. Pure Appl. Anal. 6 (2007), no. 4, 1075-1086.

[17] Eden, A.; Kalantarov, V. K.; Zelik, S. V.; Global solvability and blow up for the convective Cahn-Hilliard equations with concave potentials. J. Math. Phys. 54 (2013), no. 4, 041502, 12 pp.

[18] Frigeri, S.; Gal, C. G.; Grasselli, M.; Spreksels, J.; Two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with variable viscosity, degenerate mobility and singular potential. Nonlinearity 32 (2019), no. 2, 678-727.

[19] Gal, C. G.; Grasselli, M.; Longtime behavior of nonlocal Cahn-Hilliard equations, Discrete Contin. Dyn. Syst. Ser. A 34 (2014), 145-179.

[20] Gal, Ciprian G.; Giorgini, Andrea; Grasselli, Maurizio; The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property. J. Differential Equations 263 (2017), no. 9, 5253-5297.

[21] Giorgini, Andrea; Grasselli, Maurizio; Wu, Hao; The Cahn-Hilliard-Hele-Shaw system with singular potential. Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), no. 4, 1079-1118.

[22] Golovin, A. A.; Davis, S. H.; Nepomnyashchy, A. A.; A convective Cahn-Hilliard model for the formation of facets and corners in crystal growth. Phys. D 122 (1998), no. 1-4, 202-230.

[23] Grafakos, L.; Classical and Modern Fourier Analysis. Pearson Education, Inc., Prentice-Hall, 2004.

[24] Grasselli, M; Schimperna G.; Zelik, S; On the 2D Cahn-Hilliard equation with inertial term. Comm. Partial Differential Equations 34 (2009), 137-170.

[25] Guo, Yan; Wang, Yanjin; Decay of dissipative equations and negative Sobolev spaces. Commu. Partial Differential Equations 37 (2012), 2165-2208.

[26] Injrou, S.; Pierre, M.; Error estimates for a finite element discretization of the Cahn-Hilliard-Gurtin equations. Adv. Differential Equations 15 (2010), 1161-1192.

[27] Korzec, M. D.; Rybka, P.; On a higher order convective Cahn-Hilliard-type equation. SIAM J. Appl. Math. 72 (2012), no. 4, 1343-1360.

[28] Liu, Aibo; Liu, Changchun; The Cauchy problem for the degenerate convective Cahn-Hilliard equation. Rocky Mountain J. Math. 48 (2018), no. 8, 2595-2623.

[29] Liu, Changchun; On the convective Cahn-Hilliard equation with degenerate mobility. J. Math. Anal. Appl. 344 (2008), no. 1, 124-144.

[30] Liu, Changchun; Yin, Jingxue; Convective-diffusive Cahn-Hilliard equation with concentration dependent mobility. Northeast. Math. J. 19 (2003), no. 1, 86-94.
[31] Liu, Shuangqian; Wang, Fei; Zhao, Huijiang Global existence and asymptotics of solutions of the Cahn-Hilliard equation. J. Differential Equations 238 (2007), no. 2, 426-469.
[32] Mchedlov-Petrosyan, P. O.; The convective viscous Cahn-Hilliard equation: exact solutions. European J. Appl. Math. 27 (2016), no. 1, 42-65.
[33] Miranville, Alain; Consistent models of Cahn-Hilliard-Gurtin equations with Neumann boundary conditions. Phys. D 158 (2001), no. 1-4, 233-257.
[34] Miranville, Alain; Rocca, Elisabetta; Schimperna, Giulio; On the long time behavior of a tumor growth model. J. Differential Equations, 267 (2019), no. 4, 2616-2642.
[35] Nirenberg, L.; On elliptic partial differential equations, Annali della Scuola Normale Superiore di Pisa 13 (1959), 115-162.
[36] Novick-Cohen, A.; Segel, L. A.; Nonlinear aspects of the Cahn-Hilliard equation, Phys. D 10 (1984), 277-298.
[37] Rocca, E.; Sprekels, J.; Optimal distributed control of a nonlocal convective Cahn-Hilliard equation by the velocity in three dimensions. SIAM J. Control Optim. 53 (2015), no. 3, 1654-1680.
[38] Schimperna, G.; Global attractors for Cahn-Hilliard equations with nonconstant mobility. Nonlinearity 20, (2007), 2365-2387.
[39] Schonbek, M. E.; $L^2$ decay for weak solutions of the Navier-Stokes equations. Arch. Ration. Mech. Anal. 88(2) (1985), 209-222.
[40] Schonbek, M. E.; Large time behaviour of solutions to the Navier-Stokes equations. Comm. Partial Differential Equations 11(7) (1986), 733-763.
[41] Stein, E. M.; Singular integrals and Differentiability Properties of Functions, Princeton University Press: Princeton, NJ 1970.
[42] Tan, Zhong; Wu, Wenpei; Zhou, Jianfeng; Global existence and decay estimate of solutions to magneto-micropolar fluid equations. J. Differential Equations, 266 (2019), no. 7, 4137-4169.
[43] Wang, Xiaoming; Zhang, Zhifei; Well-posedness of the Hele-Shaw-Cahn-Hilliard system. Ann. Inst. H. Poincaré Anal. Non Linéaire 30(2013), 367-384.
[44] Wang, Yanjin; Decay of the Navier-Stokes-Poisson equations. J. Differential Equations 253 (2012) 273-297.
[45] Watson, Stephen J.; Otto, Felix; Rubinstein, Boris Y.; Davis, Stephen H.; Coarsening dynamics of the convective Cahn-Hilliard equation. Phys. D 178 (2003), no. 3-4, 127-148.
[46] Yin, Jingxue; On the existence of nonnegative continuous solutions of the Cahn-Hilliard equation, J. Differential Equations 97 (1992), 310-327.
[47] Zaks, Michael A.; Podolny, Alla; Nepomnyashchy, Alexander A.; Golovin, Alexander A.; Periodic stationary patterns governed by a convective Cahn-Hilliard equation. SIAM J. Appl. Math. 66 (2005), no. 2, 700-720.
[48] Zhao, Xiaopeng; Liu, Changchun; Optimal control of the convective Cahn-Hilliard equation. Appl. Anal. 92 (2013), no. 5, 1028-1045.
[49] Zhao, Xiaopeng; Liu, Changchun; Optimal control for the convective Cahn-Hilliard equation in 2D case. Appl. Math. Optim. 70 (2014), no. 1, 61-82.
[50] Zhao, Xiaopeng; Global well-posedness of solutions to the Cauchy problem of convective Cahn-Hilliard equation. Ann. Mat. Pura Appl. (4) 197 (2018), no. 5, 1333-1348.
[51] Zhao, Xiaopeng; Fourier spectral approximation to global attractor for 2D convective Cahn-Hilliard equation. Bull. Malays. Math. Sci. Soc., 41 (2018), no. 2, 1119-1138.

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