Abstract. We attach a Dixmier algebra $B$ to the closure $\overline{O}$ of any nilpotent orbit of $G$ where $G$ is $GL(n, \mathbb{C})$, $O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. This algebra $B$ is a noncommutative analog of the coordinate ring $R$ of $\overline{O}$, in the sense that $B$ has a $G$-invariant algebra filtration and $\text{gr } B = R$.

We obtain $B$ by making a noncommutative analog of the Kraft-Procesi construction which modeled $\overline{O}$ as the algebraic symplectic reduction of a finite-dimensional symplectic vector space $L$. Indeed $B$ is a subquotient of the Weyl algebra for $L$.

$B$ identifies with the quotient of $U(g)$ by a two-sided ideal $J$, where $g = \text{Lie}(G)$. Then $\text{gr } J$ is the ideal $\mathfrak{m}(\overline{O})$ in $S(g)$ of functions vanishing on $\overline{O}$. In every case where $O$ is connected, $J$ is a completely prime primitive ideal.

1. Introduction

By means of symplectic reduction in the setting of complex algebraic varieties, Kraft and Procesi (\cite{Kraft},\cite{Procesi}) constructed a model of the closure of any nilpotent coadjoint orbit $O$ of $G$ when $G$ is one of the classical groups $GL(n, \mathbb{C})$, $O(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$. The symplectic aspect is not actually mentioned, but the construction is clearly symplectic.

In this paper we give a noncommutative analog, or quantization, of the Kraft-Procesi construction. The result is that we attach a Dixmier algebra $B$ to each orbit closure $\overline{O}$. Our algebra $B$ has a $G$-invariant algebra filtration and we show that $\text{gr } B$ is isomorphic, as a graded Poisson algebra, to the coordinate ring $R$ of $\overline{O}$.

In fact, $B$ identifies, as a filtered algebra, with the quotient $U(g)/J$ of the universal enveloping algebra $U(g)$ of $g = \text{Lie}(G)$ by some two-sided ideal $J$. Then $\text{gr } J$ is the ideal $\mathfrak{m}(\overline{O})$ in $S(g)$ defining $\overline{O}$. We find that $J$ is stable under the principal anti-automorphism of $U(g)$, and also under the anti-linear automorphism of $U(g)$ defined by a Cartan involution of $g$.

The Kraft-Procesi construction attaches to $O$ a complex symplectic vector space $L$ together with a Hamiltonian action of $G \times S$ on $L$, where $S$ is an auxiliary complex reductive Lie group. The actions of $G$ and $S$ lie inside the symplectic group $Sp(L, \mathbb{C})$. Kraft and Procesi show that $\overline{O}$ is scheme-theoretically the algebraic symplectic reduction of $L$ by $S$. In this way, they obtain $R$ as a subquotient of the algebra $P$ of polynomial functions on $L$. More precisely, $R$ is realized as $P^{\text{inv}}/I^{\text{inv}}$ where $I$ is an ideal in $P$ and the superscript $\text{inv}$ denotes taking $S$-invariants.

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It is nice from the viewpoint of representation theory to regard $\mathcal{R}$ as a subquotient of the algebra $\mathcal{P}^{even}$ of even polynomials. (We can do this since $\mathcal{P}^{inv}$ lies in $\mathcal{P}^{even}$. ) For $\mathcal{P}^{even}$ is the coordinate ring of the closure of the minimal nilpotent orbit $\mathcal{O}$ of $Sp(L, \mathbb{C})$.

To make a noncommutative analog of the Kraft-Procesi construction, we start from the fact that there is a unique Dixmier algebra attached to $\mathcal{Y}$, namely the quotient of $\mathcal{U}(sp(L, \mathbb{C}))$ by its Joseph ideal $\mathcal{J}$. We can model $\mathcal{U}(sp(L, \mathbb{C}))/\mathcal{J}$ as the even part $\mathcal{W}^{even}$ of the Weyl algebra $\mathcal{W}$ for $L$ and then $\text{gr } \mathcal{W}^{even} = \mathcal{P}^{even}$. There is an obvious quantization of the Hamiltonian $S$-action on $L$, namely the natural $(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c)$-module structure on $\mathcal{W}$. Here the subscript $c$ denotes taking a compact real form.

We define $\mathcal{B}$ to be the coinvariants for the $(\mathfrak{s} \oplus \mathfrak{s}, S_c)$-action on $\mathcal{W}^{even}$. A priori, $\mathcal{B}$ is a $(\mathfrak{g} \oplus \mathfrak{g}, G_c)$-module with a $G_c$-invariant filtration, but $\mathcal{B}$ is not an algebra. However, we easily identify $\mathcal{B}$ as the quotient by a two-sided ideal of $\mathcal{W}^{inv}$ (where the superscript again indicates taking $S$-invariants, or equivalently, $S_c$-invariants). In this way, $\mathcal{B}$ becomes a filtered algebra and a subquotient of $\mathcal{W}^{even}$.

Our main result (Theorem 6.3) is to compute the associated graded algebra $\text{gr } \mathcal{B}$. It is easy to see that $gr \mathcal{B}$ is some quotient of $\mathcal{P}^{inv}/\mathcal{I}^{inv}$, but in fact we prove $\text{gr } \mathcal{B} = \mathcal{P}^{inv}/\mathcal{I}^{inv}$. To do this, we recognize $\mathcal{B}$ as the degree zero part of the relative Lie algebra homology $H(s \oplus s, S_c; \mathcal{W}^{even})$. We consider the standard complex which computes this homology, introduce a filtration and then apply the spectral sequence for a filtered complex. We compute the $E_1$ term of the spectral sequence by using the fact proven by Kraft and Procesi that $\mathcal{I}$ is a complete intersection ideal. Then we easily show $E_1 = E_2$.

We establish some properties of $\mathcal{B}$ and the corresponding ideal $J$. If $\mathcal{O}$ is connected then $J$ is a completely prime primitive ideal (Corollary 6.5). In every case, $\mathcal{B}$ admits a unique $\mathfrak{g}^\ast$-invariant Hermitian inner product $(\cdot | \cdot)$ such that $(1|1) = 1$, where $\mathfrak{g}^\ast$ is a real form of $\mathfrak{g} \oplus \mathfrak{g}$ with $\mathfrak{g}^\ast \simeq \mathfrak{g}$ (see Proposition 3.1). This prompts the question as to whether $J$ is “good” in the sense that $J$ is maximal and $\mathcal{U}(\mathfrak{g})/J$ is unitarizable. The latter property means (since $\mathcal{B} \simeq \mathcal{U}(\mathfrak{g})/J$) that $(\cdot | \cdot)$ is positive-definite.

Attaching “good” ideals to $\mathcal{O}$ is an important problem in representation theory and the orbit method. Quite a bit of work has been done on this (see e.g., some of the references and authors cited below) but the problem for nilpotent orbits of a complex semisimple Lie group remains unsolved.

If $G = GL(n, \mathbb{C})$, then our $J$ is good (see Remark 6.6(i) and [8]). But $G = GL(n, \mathbb{C})$ is really a very special case for us as the geometry of $\mathcal{O}$ is incredibly nice, including but not limited to the fact that $\overline{\mathcal{O}}$ is always normal. For $G = O(n, \mathbb{C})$ or $G = Sp(2n, \mathbb{C})$, it is not the case that $J$ is always good. Certainly if $\overline{\mathcal{O}}$ is not normal, we should not expect $J$ to be good.

Our point of view (to be justified in [9]) is that $\mathcal{B}$ is the “canonical” quantization of the Kraft-Procesi construction, and so the failure of $J$ to be good is really a statement about $\overline{\mathcal{O}}$. The next step is then to investigate whether we can make a modification to our quantization process in order to obtain some good ideals in $\mathcal{U}(\mathfrak{g})$ attached to $\mathcal{O}$ (and even its covers).

This paper is the first in a series. In the subsequent papers we make explicit the important role of Howe duality in our project. Indeed, $\mathcal{W}^{even}$ is the Harish-Chandra module.
of the (even) oscillator representation of $Sp(L, \mathbb{C})$, and the pair $(G, S)$ constitutes a sequence of Howe dual pairs (see Remark 4.1). In taking coinvariants, we are implementing a sequence of Howe duality “operations”. Each “operation” is like implementing a Howe duality correspondence, except that we do not pass to the the irreducible quotient. In working on this project (which started in earnest in the summer of 2001 – and is part of a program we began in 1994), we have been reading the Howe duality literature. We have been influenced by especially the papers [18], [29], [1] and [25].

Our first construction of the Dixmier algebra $B$ actually came out of the ideas of Howe duality and quantization by constraints. This is given in [9] and lies more in the realm of harmonic analysis than algebra. Our starting point there is the fact ([21]) that $L$ is hyperkähler and the Kraft-Procesi construction is the algebraic analog of the hyperkähler reduction of $L$ by $S_c$.

The notion of Dixmier algebra for nilpotent orbits (including their closures and their coverings) was first developed in work of McGovern, Joseph and Vogan. See e.g. [27], [31], and [32]. The motivation for these authors and for most Dixmier algebra theorists is the search for completely prime primitive ideals. This motivation is very important for us too; we also find additional motivations coming from star products and from geometric quantization.

The results in this paper should be compared with the work in [1], [3], [4], [5], [6], [7], [10], [14], [13], [16], [21], [25], [26], [27], [28], [29] and [33]. Some of this comparison work will be done in [8] and [9].

Part of this work was carried out while I was visiting the IML and the CPT of the Université de la Méditerranée in the summer of 2001, and I thank my colleagues there for their hospitality. I especially thank Christian Duval and Valentin Ovsienko for some very valuable discussions.

It is a real pleasure to dedicate this article to Sasha Kirillov whose discoveries have opened up so many new vistas, starting of course with the Orbit Method. I warmly thank him for his friendship and his interest in my own work.

2. Dixmier Algebra for the Closure of a Complex Nilpotent Orbit

Let $G$ be a reductive complex algebraic group. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{g}^*$ be the dual space. Then $G$ acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by, respectively, the adjoint action and the coadjoint action. The symmetric algebra $S(\mathfrak{g}) = \bigoplus_{p=0}^{\infty} S^p(\mathfrak{g})$ is the algebra of polynomial functions on $\mathfrak{g}^*$. The $G$-invariants form the graded subalgebra $S(\mathfrak{g})^G = \bigoplus_{p=0}^{\infty} S^p(\mathfrak{g})^G$. We can fix some nondegenerate $G$-invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$.

The nullcone in $\mathfrak{g}^*$ is the set of $\lambda \in \mathfrak{g}^*$ which satisfy the following equivalent properties:

(i) the closure of the coadjoint orbit of $\lambda$ contains zero
(ii) the coadjoint orbit of $\lambda$ is stable under dilations of the vector space $\mathfrak{g}^*$
(iii) every nonconstant homogeneous $G$-invariant in $S(\mathfrak{g})$ vanishes on $\lambda$
(iv) $\lambda = (x, \cdot)$ where $x$ is a nilpotent in $\mathfrak{g}$

The nullcone is $G$-stable and breaks into finitely many orbits of $G$, which are then called the nilpotent coadjoint orbits, or simply the nilpotent orbits, of $G$. 

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Let $O$ be a nilpotent orbit of $G$. The closure $\overline{O}$ is a complex algebraic subvariety of $g^*$; but $\overline{O}$ may be reducible if $G$ is disconnected. The coordinate ring $\mathbb{C}[\overline{O}]$ of $\overline{O}$ is the quotient algebra

$$R = S(g)/\mathcal{I}(\overline{O})$$

(2.1)

where $\mathcal{I}(\overline{O})$ is the ideal of functions which vanish on $\overline{O}$. Then $\mathcal{I}(\overline{O})$ is a graded ideal and $R = \oplus_{p=0}^{\infty} R^p$ is a graded algebra where $R^p = S^p(g)/\mathcal{I}(\overline{O})^p$. Each space $R^p$ is a finite dimensional completely reducible representation of $G$. Kostant’s description of $S(g)$ as a module over $S(g)^G$ implies that all $G$-multiplicities in $R$ are finite.

$R$ inherits from $S(g)$ the structure of a graded Poisson algebra where $\{R^p, R^q\} \subseteq R^{p+q-1}$. This Poisson bracket $\{\cdot, \cdot\}$ on $R$ is $G$-invariant and corresponds to the holomorphic Kirillov-Kostant-Souriau symplectic form on $O$.

In this situation, we define Dixmier algebras in the following way. We fix a Cartan involution $\zeta$ of $g$. Then $\zeta$ corresponds to a compact real form $G_c$ of $G$ with Lie algebra $g_c$. Let $N = \{0, 1, 2, \ldots\}$.

**Definition 2.1.** A Dixmier algebra for $\overline{O}$ is a quadruple $(D, \xi, \tau, \vartheta)$ where

- $D$ is a filtered algebra with an increasing algebra filtration $D = \cup_{p \in \mathbb{N}} D_p$ such that $\text{gr} D$ is commutative.
- $\xi : g \to D_1$, $x \mapsto \xi^x$, is a homomorphism of Lie algebras and $\xi$ induces an isomorphism of graded Poisson algebras from $S(g)/\mathcal{I}(\overline{O})$ onto $\text{gr} D$.
- $\tau$ is a filtered algebra anti-involution of $D$ such that $\tau(\xi^x) = -\xi^x$.
- $\vartheta$ is an anti-linear filtered algebra involution such that $\vartheta(\xi^x) = \xi^{(\vartheta)}$.

Here are some explanations about the definition. First, commutativity of $\text{gr} D$ implies that $\text{gr} D$ has a natural structure of graded Poisson algebra; here the commutator in $D$ induces the Poisson bracket on $\text{gr} D$. Second, $\xi$ extends naturally to a filtered algebra homomorphism

$$\tilde{\xi} : U(g) \to D$$

(2.2)

Let $J$ be the kernel of $\tilde{\xi}$. Then $\text{gr} J$ is a Poisson ideal of $S(g)$, and $\text{gr} \tilde{\xi}$ induces a 1-to-1 homomorphism $\zeta : S(g)/\text{gr} J \to \text{gr} D$ of graded Poisson algebras. We require that $\zeta$ is surjective and

$$\text{gr} J = \mathcal{I}(\overline{O})$$

(2.3)

Notice that $\zeta$ is surjective if and only if $\tilde{\xi}$ is surjective in each filtration degree; then $\tilde{\xi}$ induces a filtered algebra isomorphism

$$U(g)/J \sim \to D$$

(2.4)

Third, $\tau$ satisfies $\tau(cA) = c \tau(A)$, $\tau(A + B) = \tau(A) + \tau(B)$, and $\tau(AB) = \tau(B)\tau(A)$ where $A, B \in D$ and $c \in \mathbb{C}$. Fourth, $\vartheta$ satisfies $\vartheta(cA) = \tau \vartheta(A)$, $\vartheta(A + B) = \vartheta(A) + \vartheta(B)$, and $\vartheta(AB) = \vartheta(A)\vartheta(B)$. Clearly $\tau \vartheta = \vartheta \tau$.

Notice that $D$ and $\xi$ (and $\zeta$) uniquely determine $\tau$ and $\vartheta$, if the latter exist. Indeed, the endomorphisms $x \mapsto -x$ and $x \mapsto \varsigma(x)$ of $g$ extend uniquely to $\tau_g$ and $\vartheta_g$, where $\tau_g$ is an algebra anti-involution of $U(g)$ and $\vartheta_g$ is an antilinear algebra involution of $U(g)$. Then, via (2.1), $\tau_g$ and $\vartheta_g$ induce $\tau$ and $\vartheta$. 

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We have an obvious notion of isomorphism of Dixmier algebras for \(\mathcal{O}\): \((\mathcal{D}, \xi, \tau, \vartheta)\) is isomorphic to \((\mathcal{D}', \xi', \tau', \vartheta')\) if there is a filtered algebra isomorphism \(\eta : \mathcal{D} \to \mathcal{D}'\) such that \(\xi' = \eta \circ \xi\), \(\eta \circ \tau = \tau' \circ \eta\), and \(\eta \circ \vartheta = \vartheta' \circ \eta\). We can easily classify Dixmier algebras.

**Observation 2.2.** Suppose \((\mathcal{D}, \xi, \tau, \vartheta)\) is a Dixmier algebra for \(\mathcal{O}\). In addition to (2.3), \(J\) satisfies

\[
\tau_g(J) = J \quad \text{and} \quad \vartheta_g(J) = J
\]

In this way, we get a bijection between (isomorphism classes of) Dixmier algebras for \(\mathcal{O}\) and two-sided ideals \(J\) of \(\mathcal{U}(\mathfrak{g})\) satisfying (2.3) and (2.5).

**Proof.** Clearly (2.3) and (2.5) imply that \((\mathcal{U}(\mathfrak{g})/J, \iota, \tau'_g, \vartheta'_g)\) is a Dixmier algebra for \(\mathcal{O}\), where \(\iota\) is the obvious map and \(\tau'_g\) and \(\vartheta'_g\) are induced by \(\tau_g\) and \(\vartheta_g\). Conversely, if \((\mathcal{D}, \xi, \tau, \vartheta)\) is given, then \((\mathcal{U}(\mathfrak{g})/J, \iota, \tau'_g, \vartheta'_g)\) is isomorphic to it via (2.4). \(\square\)

## 3. Properties of Dixmier Algebras

The hopes in constructing a Dixmier algebra are (i) \(J\) will be a completely prime primitive ideal of \(\mathcal{U}(\mathfrak{g})\), or even better, a completely prime maximal ideal, and (ii) \(\mathcal{D}\) will be unitarizable. See \([12, \S 3.1]\) for the definitions of the terms in (i).

To understand (ii), we observe that Definition 2.1 makes \(\mathcal{D}\) a \((\mathfrak{g} \oplus \mathfrak{g}, \mathcal{O})\)-module. Indeed, the natural \((\mathfrak{g} \oplus \mathfrak{g}, \mathcal{O})\)-module structure on \(\mathcal{U}(\mathfrak{g})/J\) transfers over to \(\mathcal{D}\) via (2.4). Then \(\mathfrak{g} \oplus \mathfrak{g}\) acts on \(\mathcal{D}\) through the representation

\[
\Pi : \mathfrak{g} \oplus \mathfrak{g} \to \text{End} \mathcal{D}, \quad (x, y) \mapsto \Pi^{x,y}
\]

where \(\Pi^{x,y}(A) = \xi^x A - A \xi^y\). The action of \(\mathcal{O}\) corresponds to the subalgebra \(\{(x, x) : x \in \mathfrak{g}\}\).

Next consider the subalgebra \(\mathfrak{g}^* = \{(x, \zeta(x)) : x \in \mathfrak{g}\}\) of \(\mathfrak{g} \oplus \mathfrak{g}\). We say \(\mathcal{D}\) is unitarizable if \(\mathcal{D}\) admits a \(\mathfrak{g}^*\)-invariant positive definite Hermitian inner product. In this event, by a theorem of Harish-Chandra, the operators \(\Pi^{x,\zeta(x)}\) correspond to a unitary representation of \(\mathcal{G}\) on the Hilbert space completion of \(\mathcal{D}\). This unitary representation is then a quantization of \(\mathcal{O}\) in the sense of geometric quantization, if we view \(\mathcal{O}\) as a real symplectic manifold. (If \(\mathbb{C}[\mathcal{O}] \neq \mathbb{C}[\mathcal{O}]\), then this might be just a piece of a quantization of \(\mathcal{O}\).)

Notice that the following three properties are equivalent: (i) \(J\) is maximal, (ii) \(\mathcal{D}\) is a simple ring, and (iii) the representation \(\Pi\) is irreducible.

Our formalism gives some partial results pertaining to hopes (i) and (ii). Notice that \(\mathcal{D}_0 = \mathbb{C}\) by (2.4).

**Proposition 3.1.** Suppose \((\mathcal{D}, \xi, \tau, \vartheta)\) is a Dixmier algebra for \(\mathcal{O}\) and \(J = \ker \tilde{\xi}\). Then

1. \(J\) has an infinitesimal character.
2. If \(\mathcal{O}\) is irreducible then \(J\) is a completely prime primitive ideal in \(\mathcal{U}(\mathfrak{g})\).
3. There is a a unique \(\mathcal{G}_c\)-invariant projection \(\mathcal{T} : \mathcal{D} \to \mathbb{C}\). This map \(\mathcal{T}\) is a trace, i.e., \(\mathcal{T}(AB) = \mathcal{T}(BA)\).
4. \(\mathcal{D}\) admits a unique \(\mathfrak{g}^*\)-invariant Hermitian inner product \((\cdot | \cdot)\) such that \((1|1) = 1\), and it is given by

\[
(A|B) = \mathcal{T}(AB\vartheta).
\]

where \(B\vartheta = \vartheta(B)\).
Proof. (i) This means (for any proper two-sided ideal $J$) that $J$ contains a maximal ideal of the center of $\mathcal{U}(\mathfrak{g})$. This happens if and only $\text{gr} J$ contains $S^+(\mathfrak{g})^G = \oplus_{p \geq 0} S^p(\mathfrak{g})^G$. But $\text{gr} J = \mathcal{J}(\mathcal{O})$ and $\mathcal{J}(\mathcal{O}) \supset S^+(\mathfrak{g})^G$ since $\mathcal{O}$ lies in the nullcone. (ii) If $\mathcal{O}$ is irreducible then $\mathcal{J}(\mathcal{O})$ is a prime ideal in $S(\mathfrak{g})$ and so $J$ is a completely prime ideal in $\mathcal{U}(\mathfrak{g})$. This together with (i) implies, by a result of Dixmier, that $J$ is primitive. (iii) Since $\mathcal{J}(\mathcal{O}) \supset S^+(\mathfrak{g})^G = S^+(\mathfrak{g})^{G_e}$, we have $\mathcal{R}^{G_e} = \mathbb{C}$ and so $\mathcal{D}^{G_e} = \mathbb{C}$. Thus we get a unique $G_e$-invariant projection map $T$. Now $G_e$-invariance implies $T([\xi^x, A]) = 0$ where $x \in \mathfrak{g}$. We can write this as $T(\xi^x A) = T(A \xi^x)$. Iteration gives $T(\xi_{x_1} \cdots \xi_{x_k} A) = T(A \xi_{x_1} \cdots \xi_{x_k})$. This proves $T(BA) = T(A B)$ since the $\xi^x$ generate $\mathcal{D}$. (This is the same proof as in [2, Proposition 8.4].) (iv) Suppose $(\cdot | \cdot)$ is an inner product with the desired properties. Then $(A | 1) = T(A)$. Now $\mathfrak{g}^\mathcal{J}$-invariance means that the operators $\Pi_{\xi^x(x)}$ are skew-hermitian, or equivalently, $(\xi^x A | B) = (A | B \xi^x(x))$. So for $B = \xi_{x_1} \cdots \xi_{x_k}$, we have $(A | B) = (\xi^c(x_1) \cdots \xi^c(x_k) A | 1) = (B^\vartheta A | 1) = T(B^\vartheta A)$. The result is now clear. \qed

Corollary 3.2. $\mathcal{D}$ is unitarizable if and only if the pairing defined by (3.2) is positive definite. If $\mathcal{D}$ is unitarizable, then $J$ is maximal.

Proof. Both statements follow from the uniqueness in Proposition 3.1 (iv). \qed

Example 3.3. Suppose $\mathcal{O}$ is the minimal nilpotent orbit in $\mathfrak{g}$ where $\mathfrak{g}$ is simple and $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C})$. This is a case where $\mathbb{C}[\mathcal{O}] = \mathbb{C}[\mathcal{O}]$. Then there is a unique Dixmier algebra for $\mathcal{O}$. This follows by Observation 2.2 since there is exactly one choice for $J$ satisfying (2.3) and (2.5). Moreover, (i) $J$ is a completely prime maximal ideal of $\mathcal{U}(\mathfrak{g})$, and (ii) $\mathcal{U}(\mathfrak{g})/J$ is unitarizable if $\mathfrak{g}$ is classical.

Indeed, there is a unique two-sided ideal $J$ satisfying (2.3) and $\tau_0(J) = J$; see [2, proof of Proposition 3.1]. Since the ideal $\mathcal{J}(\mathcal{O})$ is preserved by the antilinear algebra involution of $S(\mathfrak{g})$ defined by $\varsigma$, it follows by the uniqueness of $J$ that $\vartheta_0(J) = J$. Since $\mathcal{O}$ is irreducible, Proposition 3.1 (ii) implies that $J$ is completely prime. If $\mathfrak{g} \neq \mathfrak{sl}(n, \mathbb{C})$, then $J$ is the Joseph ideal and this is maximal by [19, Theorem 7.4]. If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ ($n \geq 3$), then $J$ is maximal by [30]. Finally, unitarizability is known; see [2, Theorem 9.1] for a uniform construction of these unitary representations on spaces of holomorphic functions on $\mathcal{O}$.

4. The Kraft-Procesi Construction

In this section we recall how Kraft and Procesi in [22] and [23] constructed the closures of complex classical nilpotent orbits. We add to their construction the framework of algebraic symplectic reduction.

Let $V$ be a complex vector space with $\mathfrak{b}$ a bilinear form on $V$. Let $G$ be the symmetry group of $\mathfrak{b}$. We consider only the following three cases.

(i) $\mathfrak{b}$ is identically zero. Then $G = GL(V, \mathbb{C})$.

(ii) $\mathfrak{b}$ is nondegenerate and symmetric. Then $G = O(V, \mathbb{C})$.

(iii) $\mathfrak{b}$ is nondegenerate and symplectic. Then $G = Sp(V, \mathbb{C})$.

Choose a nilpotent orbit $\mathcal{O}$ of $G$ and an element $\lambda$ in $\mathcal{O}$. Then $\lambda$ corresponds (via the trace functional on $\text{End} V$) to $x \in \mathfrak{g}$; so $x$ is in particular a nilpotent endomorphism of $V$. 
We note that $\mathcal{O}$ is connected, and so $\overline{\mathcal{O}}$ is irreducible, except in the following situation. If $G = O(V, \mathbb{C})$ where $\dim V$ is even and also the Jordan block size partition of $\mathcal{O}$ is very even (i.e., all parts are even and occur with even multiplicities), then $\mathcal{O}$ has two connected components and $\overline{\mathcal{O}}$ has two irreducible components. See [23], [11, Chapter 5].

Let $V_d$ be the image of $x^d$. Then

$$V = V_0 \supset V_1 \supset \cdots \supset V_r \supset V_{r+1} = 0$$

(4.1)

where $r$ is the largest number such that $x^r \neq 0$. We define a complex vector space $L$ by

$$L = L(V_0, V_1) \oplus L(V_1, V_2) \oplus \cdots \oplus L(V_{r-1}, V_r)$$

(4.2)

where $L(V_{d-1}, V_d)$ is obtained in the following way. If $G = GL(V, \mathbb{C})$ then

$$L(V_{d-1}, V_d) = \text{Hom}(V_d, V_{d-1}) \oplus \text{Hom}(V_{d-1}, V_d)$$

(4.3)

If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then

$$L(V_{d-1}, V_d) = \text{Hom}(V_d, V_{d-1})$$

(4.4)

Next we construct a complex Lie group $S$ of the form

$$S = S_1 \times S_2 \times \cdots \times S_r$$

(4.5)

where $S_d$ is obtained in the following way. To begin with, we put $b_0 = b$ and $S_0 = G$. If $G = GL(V, \mathbb{C})$ then for each $d$ we put $b_d = 0$ and $S_d = GL(V_d, \mathbb{C})$. If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then $V_d$ admits an intrinsic nondegenerate complex bilinear form $b_d$ and we define $S_d$ to be the symmetry group of $b_d$. In more detail, $b_d$ is the bilinear form on $V_d$ defined by $b_d(x^d(u), x^d(v)) = b(u, x^d(v))$. It turns out that $b_d$ is nondegenerate. If $b_{d-1}$ is orthogonal then $b_d$ is symplectic and we put $S_d = Sp(V_d, \mathbb{C})$. If $b_{d-1}$ is symplectic then $b_d$ is orthogonal and we put $S_d = O(V_d, \mathbb{C})$.

Next we construct commuting actions of $G$ and $S$ on $L$. If $G = GL(V, \mathbb{C})$, we make $G$ and $S$ act by

$$(g, s_1, \ldots, s_r) \cdot (A_1, B_1, A_2, B_2, \ldots, A_r, B_r) = (gA_1s_1^{-1}, s_1B_1g^{-1}, s_1A_2s_2^{-1}, s_2B_2s_1^{-1}, \ldots, s_{r-1}A_rs_r^{-1}, s_rB_rs_{r-1}^{-1})$$

(4.6)

where $A_d \in \text{Hom}(V_d, V_{d-1})$ and $B_d \in \text{Hom}(V_{d-1}, V_d)$. If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, we make $G$ and $S$ act by

$$(g, s_1, \ldots, s_r) \cdot (C_1, C_2, \ldots, C_r) = (gC_1s_1^{-1}, s_1C_2s_2^{-1}, \ldots, s_{r-1}C_rs_r^{-1})$$

(4.7)

where $C_d \in \text{Hom}(V_d, V_{d-1})$.

$L$ has a (complex) symplectic form $\Omega$ given by $\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_r$ where $\Omega_d$ is the symplectic form on $L(V_{d-1}, V_d)$ defined in the following way. If $G = GL(V, \mathbb{C})$, then $\Omega_d(A + B, A' + B') = \text{tr}(AB') - \text{tr}(BA')$. If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then $\Omega_d(C, C') = \text{tr}(C^*C)$ where $C^* \in \text{Hom}(V_{d-1}, V_d)$ is the adjoint of $C$ defined by $b_{d-1}(u, C^*(v)) = b_d(C(u), v)$.

Now $G$ and $S$ act faithfully and symplectically on $L$. Thereby $G$ and $S$ identify with commuting subgroups of the symplectic group $Sp(L, \mathbb{C})$. The action of $Sp(L, \mathbb{C})$ is Hamiltonian with canonical moment map

$$L \to sp(L, \mathbb{C})^*$$

(4.8)
Hence our actions of $G$ and $S$ are Hamiltonian with induced moment maps (obtained by projection)

$$\gamma : L \to \mathfrak{g}^* \quad \text{and} \quad \sigma : L \to \mathfrak{s}^*$$

(4.9)

Then $\gamma$ is $G$-equivariant and $S$-invariant, and $\sigma$ is $S$-equivariant and $G$-invariant.

Here are the explicit formulas for $\gamma$ and $\sigma$. We may write these as $\mathfrak{g}$-valued and $\mathfrak{s}$-valued maps, with the convention that the trace functional on $\text{End} \, L$ induces isomorphisms $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ and $\mathfrak{s} \xrightarrow{\sim} \mathfrak{s}^*$. If $G = GL(V, \mathbb{C})$, then $\gamma$ and $\sigma$ are given by

$$(-A_1B_1) \quad \text{and} \quad (B_1A_1 - A_2B_2, \ldots, B_{r-1}A_{r-1} - A_rB_r, B_rA_r)$$

(4.10)

If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then $\gamma$ and $\sigma$ are given by

$$(-C_1C_1^*) \quad \text{and} \quad (C_1^*C_1 - C_2C_2^*, \ldots, C_{r-1}^*C_{r-1} - C_rC_r^*, C_r^*C_r)$$

(4.11)

**Remark 4.1.** For each $d = 1, \ldots, r$, $S_d$ and $S_d$ act on $L(V_{d-1}, V_d)$ as a Howe dual pair.

Let $\mathcal{P}$ be the algebra of polynomial functions on $L$. Then $\mathcal{P} = S(L^*)$ is a graded Poisson algebra with respect to the Poisson bracket $\{\cdot, \cdot\}$ defined by $\Omega$. Our grading is $\mathcal{P} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \mathcal{P}^j$ where $\mathcal{P}^j = S^j(L^*)$ and then $\{\mathcal{P}^j, \mathcal{P}^k\} \subseteq \mathcal{P}^{j+k-1}$. (This choice of halving the natural degrees is convenient as the aim is to obtain $R$ as a subquotient of $\mathcal{P}$.) The momentum functions $\gamma_y (y \in \mathfrak{g})$ and $\sigma_x (x \in \mathfrak{s})$ are the component functions of $\gamma$ and $\sigma$; i.e., $\gamma_y(m) = \langle \gamma(m), y \rangle$ and $\sigma_x(m) = \langle \sigma(m), x \rangle$ where $m \in L$. The $\gamma_y$ and $\sigma_x$ lie in $\mathcal{P}^1$ and satisfy the bracket relations $\{\gamma_y, \gamma_{y'}\} = \gamma_{[y, y']}, \{\sigma_x, \sigma_{x'}\} = \sigma_{[x, x']} \text{ and } \{\gamma_y, \sigma_x\} = 0$.

Let $\mathcal{I}$ be the ideal in $\mathcal{P}$ generated by the momentum functions $\sigma_x$ where $x \in \mathfrak{s}$. Then $\mathcal{I} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \mathcal{I}^j$ is a graded ideal stable under both $G$ and $S$. Hence the quotient algebra $\mathcal{P}/\mathcal{I}$ is a graded algebra on which $G$ and $S$ act by graded algebra automorphisms. The grading is

$$\mathcal{P}/\mathcal{I} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} (\mathcal{P}/\mathcal{I})^j$$

where $(\mathcal{P}/\mathcal{I})^j = \mathcal{P}^j/\mathcal{I}^j$. Kraft and Procesi proved that $\mathcal{I}$ is the full ideal of functions vanishing on $\sigma^{-1}(0)$. Thus $\mathcal{P}/\mathcal{I}$ is the coordinate ring $\mathbb{C}[\sigma^{-1}(0)]$ of the zero locus of $\sigma$.

The **algebraic symplectic reduction** $L^{\text{red}}$ of $L$ by $S$ is the Mumford quotient of $\sigma^{-1}(0)$ by $S$. Thus $L^{\text{red}}$ is the affine complex algebraic variety with coordinate ring

$$\mathbb{C}[L^{\text{red}}] = (\mathcal{P}/\mathcal{I})^{\text{inv}} = \mathcal{P}^{\text{inv}}/\mathcal{I}^{\text{inv}}$$

(4.12)

where the superscript $\text{inv}$ denotes taking $S$-invariants. Moreover $\mathcal{I}^{\text{inv}}$ is a Poisson ideal in $\mathcal{P}^{\text{inv}}$. So $\mathbb{C}[L^{\text{red}}]$ inherits the structure of a graded Poisson algebra.

Notice that $S$ contains the center $\mathbb{Z}_2 = \{1, -1\}$ of $Sp(L, \mathbb{C})$ and the action of $\mathbb{Z}_2$ induces the decomposition $\mathcal{P} = \mathcal{P}^{\text{even}} \oplus \mathcal{P}^{\text{odd}}$ where the even part is the graded Poisson algebra

$$\mathcal{P}^{\text{even}} = \bigoplus_{d \in \mathbb{N}} \mathcal{P}^d$$

(4.13)

So $\mathcal{P}^{\text{inv}}$ lies in $\mathcal{P}^{\text{even}}$, and thus $\mathcal{P}^{\text{inv}}$ and $\mathbb{C}[L^{\text{red}}]$ are $\mathbb{N}$-graded.

The symplectic version of the Kraft-Procesi result is

**Theorem 4.2.** [22 Theorem 3.3], [23 Theorem 5.3] The algebra homomorphism $\gamma^* : S(\mathfrak{g}) \to \mathcal{P}$ defined by $y \mapsto \gamma_y (y \in \mathfrak{g})$ induces a $G$-equivariant isomorphism of $\mathbb{N}$-graded Poisson algebras from $\mathcal{R}$ onto $\mathcal{P}^{\text{inv}}/\mathcal{I}^{\text{inv}}$. 


The cited results of Kraft and Procesi are given in geometric language, and the reader who wants to read all the proofs in [22] and [23] will need some knowledge in algebraic geometry. The statements in [22] Theorem 3.3 and [23] Theorem 5.3 are easy to translate into algebra though, since we are dealing with affine varieties. Kraft and Procesi show that \( \gamma \) maps \( \sigma^{-1}(0) \) onto \( \mathcal{O} \), and moreover this surjection \( \gamma' : \sigma^{-1}(0) \to \mathcal{O} \) is a quotient map for the action of \( S \). In this setting of a reductive group acting on an affine variety, “quotient map” has a very strong meaning coming from Mumford’s geometric invariant theory, as explained in [22, §1.4] and [23, §0.11]. Precisely, \( \gamma' \) being a quotient map means that the corresponding map \( \mathcal{R} \to \mathcal{P}/\mathcal{I} \) on coordinate rings is injective and has image equal to \((\mathcal{P}/\mathcal{I})^{\text{inv}}\).

In symplectic language then, Kraft and Procesi proved that the moment map \( \gamma \) induces a \( G \)-isomorphism of affine complex algebraic varieties from \( L_{\text{red}} \) onto \( \mathcal{O} \). This isomorphism is also equivariant with respect to the natural \( \mathbb{C}^* \)-actions on \( L_{\text{red}} \) and \( \mathcal{O} \). Finally, since \( \gamma \) is a moment map it follows that \( \gamma^* \) preserves the Poisson brackets. Thus we get Theorem 4.2.

5. Weyl algebra \( \mathcal{W} \) for \( L \)

The Kraft-Procesi construction realized \( \mathcal{R} \) as a subquotient, namely \( \mathcal{P}^{\text{inv}}/\mathcal{I}^{\text{inv}} \), of \( \mathcal{P}^{\text{even}} \). Our aim is to make a noncommutative analog of their construction.

The image of the moment map \( \mathcal{L} \) is the closure \( \mathcal{Y} \) of the minimal nilpotent orbit \( \mathcal{Y} \) of \( Sp(L, \mathbb{C}) \), and \( \mathcal{P}^{\text{even}} = \mathbb{C}[\mathcal{Y}] \) as graded Poisson algebras. We know by Example 3.3 that \( \mathcal{Y} \) has a unique Dixmier algebra \( (\mathcal{D}, \xi, \tau_{\mathcal{D}}, \psi_{\mathcal{D}}) \), and then \( \mathcal{D} \) is the quotient of \( \mathcal{U}(sp(L, \mathbb{C})) \) by its Joseph ideal. In this section we will give a more concrete model for this Dixmier algebra. Then in 6 we will perform the noncommutative analog of reduction.

Let \( \mathcal{W} \) be the Weyl algebra for \( L^* \). This means that \( \mathcal{W} \) is the quotient of the tensor algebra of \( L^* \) by the two-sided ideal generated by the elements \( a \otimes b - b \otimes a - \{a, b\} \) where \( a \) and \( b \) lie in \( L^* \). Let \( a \mapsto \hat{a} \) be the natural map \( L^* \to \mathcal{W} \). We can identify \( sp(L, \mathbb{C}) \) with \( S^2 L^* \) and then we have the Lie algebra embedding

\[
\xi : sp(L, \mathbb{C}) \to \mathcal{W}, \quad \xi^{ab} = \hat{a} \hat{b} + \hat{b} \hat{a}
\]

(5.1)

There is an increasing algebra filtration \( \mathcal{W} = \bigcup_{j \in \mathbb{N}} \mathcal{W}_j \) where \( \mathcal{W}_j \) is the image of the space of tensors of degree at most \( 2j \). We have \( [\mathcal{W}_j, \mathcal{W}_k] \subset \mathcal{W}_{j+k-1} \). Thus the associated graded algebra \( \text{gr} \mathcal{W} = \bigoplus_{j \in \mathbb{N}} \mathcal{W}_j/\mathcal{W}_{j-1} \) is commutative and the commutator in \( \mathcal{W} \) induces a Poisson bracket (of degree \(-1\)) on \( \text{gr} \mathcal{W} \). In this way \( \text{gr} \mathcal{W} \) becomes a graded Poisson algebra. Then \( \text{gr} \mathcal{W} \) identifies naturally with \( \mathcal{P} \).

The symplectic group \( Sp(L, \mathbb{C}) \) acts naturally on \( \mathcal{W} \) by algebra automorphisms. This action respects \( \xi \), the filtration on \( \mathcal{W} \), the Poisson bracket on \( \text{gr} \mathcal{W} \), etc. The corresponding action of \( sp(L, \mathbb{C}) \) on \( \mathcal{W} \) is given by the operators \([\xi^{ab}, \cdot]\).

We next choose a Cartan involution \( \zeta \) of \( sp(L, \mathbb{C}) \). To do this, we go back into the Kraft-Procesi construction. Recall that each space \( V_d \) in (4.1) carried a bilinear form \( \mathbf{b}_d \) \((d = 0, \ldots, r)\). We can choose a positive definite hermitian form \( \mathbf{h}_d \) on \( V_d \) which is compatible with \( \mathbf{b}_d \) in the sense that the intersection of \( S_d \) with the unitary group of \( \mathbf{h}_d \)}
is a maximal compact subgroup $K_d$ of $S_d$. (This is an equivalent version of the setup in [21].) These $h_j$ determine naturally a positive definite hermitian form $h$ on $L$. Now we define $\varsigma(T) = -T^\dag$ for $T \in \mathfrak{sp}(L, \mathbb{C})$, where $T^\dag$ is the adjoint of $T$ with respect to $h$.

Corresponding to $\varsigma$ is a compact real form $Sp(L)$ of $Sp(L, \mathbb{C})$ with Lie algebra $\mathfrak{sp}(L)$. For later use (see [10], we notice that $G \cap Sp(L) = K_0$ and $S \cap Sp(L) = K_1 \times \cdots \times K_r$ are compact real forms of $G$ and $S$, which we will denote by $G_c$ and $S_c$.

Now $\mathcal{W}$ is a $(\mathfrak{sp}(L, \mathbb{C}) \oplus \mathfrak{sp}(L, \mathbb{C}), Sp(L))$-module, where the representation

$$\mathfrak{sp}(L, \mathbb{C}) \oplus \mathfrak{sp}(L, \mathbb{C}) \rightarrow \text{End} \mathcal{W}$$

is given by $(x, y) \cdot A = \xi^x A - A \xi^y$ and the action of $Sp(L)$ corresponds to the subalgebra $\{(x, x) : x \in \mathfrak{sp}(L)\}$. The action of the center $\mathbb{Z}_2$ of $Sp(L)$ produces the decomposition

$$\mathcal{W} = \mathcal{W}^{even} \oplus \mathcal{W}^{odd}$$

where $\mathcal{W}^{even}$ is space of invariants for $\mathbb{Z}_2$. The induced filtration on $\mathcal{W}^{even}$ satisfies $\mathcal{W}_{p+\frac{1}{2}}^{even} = \mathcal{W}_p^{even}$ if $p \in \mathbb{N}$. So we might as well just consider the algebra filtration

$$\mathcal{W}^{even} = \bigcup_{d \in \mathbb{N}} \mathcal{W}_d^{even}$$

Now we can make $\mathcal{W}^{even}$ into a Dixmier algebra.

**Proposition 5.1.** The Dixmier algebra for the closure $\overline{\mathcal{O}}$ of the minimal nilpotent orbit of $Sp(L, \mathbb{C})$ is the quadruple $(\mathcal{W}^{even}, \xi, \tau, \vartheta)$, for some unique choices of $\tau$ and $\vartheta$.

**Proof.** The map $\xi$ induces a filtered algebra isomorphism $\pi : \mathcal{U}(\mathfrak{sp}(L, \mathbb{C}))/\mathcal{J} \rightarrow \mathcal{W}^{even}$ where $\mathcal{J}$ is the Joseph ideal (see [2], §5). Clearly $\text{gr} \mathcal{W}^{even} = \mathcal{P}^{even}$ and so $\pi$ induces a graded isomorphism $S(\mathfrak{sp}(L, \mathbb{C}))/\mathcal{J}(\overline{\mathcal{O}}) \sim \rightarrow \mathcal{P}^{even}$. Now everything follows by Example 3.3. \[\square\]

6. Dixmier Algebra for $\overline{\mathcal{O}}$

$\mathcal{W}^{even}$ is, by means of (5.2), both a $(g \oplus g, G_c)$-module and an $(s \oplus s, S_c)$-module, and these two actions commute.

**Definition 6.1.** $\mathcal{B}$ is the $(g \oplus g, G_c)$-module obtained by taking the coinvariants of $\mathcal{W}^{even}$ in the category of $(s \oplus s, S_c)$-modules.

This means that $\mathcal{B}$ is the quotient $\mathcal{W}^{even}/\mathcal{M}$ where $\mathcal{M}$ is the subspace spanned by all $\xi^x A - A \xi^y$ and $A - s \cdot A$ where $x, y \in s$, $s \in S_c$ and $A \in \mathcal{W}^{even}$ (see [20, Chapter II]). Then $\mathcal{B}$ inherits from $\mathcal{W}^{even}$ an increasing $G_c$-stable vector space filtration $\mathcal{B} = \bigcup_{d \in \mathbb{N}} \mathcal{B}_d$.

Let $\mathcal{W}^{inv}$ be the algebra of invariants for $S_c$. Then $\mathcal{W}^{inv}$ lies in $\mathcal{W}^{even}$ (since $S_c$ contains $\mathbb{Z}_2$) and so $\mathcal{W}^{inv}$ inherits from $\mathcal{W}^{even}$ an algebra filtration $\mathcal{W}^{inv} = \bigcup_{d \in \mathbb{N}} \mathcal{W}_d^{inv}$.

**Lemma 6.2.** The natural map

$$\phi : \mathcal{W}^{inv} \rightarrow \mathcal{B}$$

is surjective in each filtration degree and its kernel is a two-sided ideal. In this way, $\mathcal{B}$ becomes a filtered algebra. The corresponding map $\text{gr} \phi : \mathcal{P}^{inv} \rightarrow \text{gr} \mathcal{B}$ is a surjective homomorphism of graded Poisson algebras.

**Proof.** We prove this in §7. \[\square\]
Our main result is

**Theorem 6.3.** We have \( \text{gr}\mathcal{B} = \mathcal{P}^{\text{inv}}/\mathcal{I}^{\text{inv}} \). So \( \text{gr}\mathcal{B} \simeq \mathcal{R} \) as graded Poisson algebras.

**Proof.** The proof occupies §8. \( \square \)

**Corollary 6.4.** The quadruple \( (\mathcal{B}, \xi, \tau, \vartheta) \) is a Dixmier algebra for \( \mathcal{O} \), where \( \xi, \tau \) and \( \vartheta \) are the maps induced by \( \xi, \tau, \) and \( \vartheta \).

**Proof.** We prove this in §9. \( \square \)

Let \( J \) be the kernel of the algebra homomorphism \( \tilde{\xi} : \mathcal{U}(\mathfrak{g}) \to \mathcal{B} \) defined by \( \xi \).

**Proposition 6.1.** Suppose we exclude the cases where \( \mathcal{O} \) is disconnected (so where \( G = O(2n, \mathbb{C}) \) and the Jordan block size partition of \( \mathcal{O} \) is very even). Then \( J \) is a completely prime primitive ideal of \( \mathcal{U}(\mathfrak{g}) \) with \( \text{gr}\ J = \mathcal{I}(\mathcal{O}) \) \( \tau_{\mathfrak{g}}(J) = J \), and \( \vartheta_{\mathfrak{g}}(J) = J \).

The methods we have used thus far give no information about the excluded cases.

**Remark 6.6.** (i) Suppose \( G = GL(n, \mathbb{C}) \). Then the space \( L \) has a \( G \times S \)-invariant polarization, and using this we can describe \( \mathcal{B} \) and \( \xi \) in the following way. Let \( X \) be the flag manifold of \( G \) of flags of the type in \([16]\). Let \( \mathcal{D}^{\frac{1}{2}}(X) \) be the algebra of twisted differential operators for the (locally defined) square root of the canonical bundle on \( X \) as in \([7]\). We can show \((8)\) that \( \mathcal{B} \) identifies with \( \mathcal{D}^{\frac{1}{2}}(X) \) in such a way that \( \xi \) corresponds to the canonical mapping of \( \mathfrak{g} \) into \( \mathcal{D}^{\frac{1}{2}}(X) \). Then by \([7\) Corollary 8.5], \( J \) is a maximal ideal in \( \mathcal{U}(\mathfrak{g}) \).

(ii) Suppose \( G = O(n, \mathbb{C}) \) or \( G = Sp(2n, \mathbb{C}) \). If \( \mathcal{O} \) is the minimal nilpotent orbit, then \( \mathcal{B} \) is the quotient of \( \mathcal{U}(\mathfrak{g}) \) by its Joseph ideal. This follows by the result in Example 8.3.

7. **Proof of Lemma 6.2**

The action of \( S_c \) on \( \mathcal{W} \) is completely reducible and locally finite, and \( S \) and \( S_c \) have the same invariants and the same irreducible subspaces. So we can form the decomposition

\[
\mathcal{W}^{\text{even}} = \mathcal{W}^{\text{inv}} \oplus \mathcal{X}
\]

where \( \mathcal{X} \) is the sum of all non-trivial \( S_c \)-isotypic components. Then \( \mathcal{X} \) is the span of the elements \( A - s \cdot A \) where \( A \in \mathcal{W}^{\text{even}} \) and \( s \in S_c \). Then \( \mathcal{M} = \mathcal{M}^{\text{inv}} \oplus \mathcal{X} \). Hence the natural map \( \phi \) is surjective and its kernel is \( \mathcal{M}^{\text{inv}} \). I.e., we have vector space isomorphisms

\[
\mathcal{W}^{\text{inv}}/\mathcal{M}^{\text{inv}} \sim \mathcal{W}^{\text{even}}/\mathcal{M} \sim \mathcal{B}
\]

(7.2)

The decomposition \((7.1)\) is compatible with the filtration on \( \mathcal{W}^{\text{even}} \), since the filtration is \( S_c \)-invariant. Consequently \( \phi \) is surjective in each filtration degree. So \( \mathcal{W}^{\text{even}} \) and \( \mathcal{W}^{\text{inv}} \) induce the same filtration on \( \mathcal{B} \).

Next we show that \( \mathcal{M}^{\text{inv}} \) is a two-sided ideal in \( \mathcal{W}^{\text{inv}} \). To begin with, \( \mathcal{M}^{\text{inv}} \) lies inside the subspace \( \mathcal{M}' \) of \( \mathcal{M} \) spanned by all \( \xi^x A \) and \( \Lambda \xi^x \) where \( x \in s \) and \( A \in \mathcal{W}^{\text{even}} \). This follows using \((7.1)\). So it suffices to show that \( D\mathcal{M}' \) and \( \mathcal{M}'D \) lie in \( \mathcal{M}' \) if \( D \in \mathcal{W}^{\text{inv}} \). Obviously \( DA \xi^x \) lies in \( \mathcal{M}' \). Invariance of \( D \) gives \( \xi^x D - D \xi^x = 0 \) and so \( D \xi^x A = \xi^x DA \) lies in \( \mathcal{M}' \). Thus \( D\mathcal{M}' \subseteq \mathcal{M}' \); similarly \( \mathcal{M}'D \subseteq \mathcal{M}' \).
The associated graded algebra $\text{gr } B$ is the quotient $\text{gr } W^{\text{inv}} / \text{gr } M^{\text{inv}}$. We find $\text{gr } W^{\text{inv}} = \mathcal{P}^{\text{inv}}$. Now the final assertion is clear.

**Remark 7.1.** If we replace $W^{\text{even}}$ by $\mathcal{W}$ in Definition 6.1, then we get the same thing. I.e., if $B$ is the module of coinvariants of $W$, then $B$ identifies naturally with $\mathcal{B}$. Indeed $\mathcal{B} = W/\hat{M}$ where $\hat{M}$ is the subspace spanned by all $\xi^x A - A \xi^y$ and $A - s \cdot A$ where now $A \in \mathcal{W}$. But then $\hat{M} = W^{\text{odd}} \oplus M$ and so the natural map $\tilde{\phi} : W^{\text{inv}} \to \mathcal{B}$ is surjective with the same kernel $M^{\text{inv}}$. Also the filtration $\mathcal{B} = \cup_{j \in \mathbb{N}} \mathcal{B}_j$ induced by $\mathcal{W}$ reduces to the one induced by $W^{\text{inv}}$ in the sense that $\tilde{\phi}(W^{\text{inv}}) = \mathcal{B}_d = \mathcal{B}_{d+1}$.

### 8. Proof of Theorem 6.3

We will compute $\text{gr } B$ by using a homology spectral sequence. We will consider the relative Lie algebra homology $H(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c; \mathcal{W}^{\text{even}})$. By definition (see [20], Chapter II, §6-7), $H_j(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c; \mathcal{W}^{\text{even}})$ is the $j$th derived functor, in the category of $(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c)$-modules, of the coinvariants. So

$$B = H_0(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c; \mathcal{W}^{\text{even}})$$

The idea is that we will introduce a filtration of the complex that computes the homology in such a way that the induced filtration on $H_0(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c; \mathcal{W}^{\text{even}})$ is the one we have already defined on $B$. Then we will use the usual spectral sequence of a filtered complex to compute $\text{gr } H_0(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c; \mathcal{W}^{\text{even}})$. The computation will rely on the geometric result of Kraft and Procesi that (in the notation of [11]) $\sigma^{-1}(0)$ is a complete intersection.

To begin with, we have $\mathfrak{s} \oplus \mathfrak{s} = \mathfrak{t} \oplus \mathfrak{p}$ where $\mathfrak{t} = \{(x, x) : x \in \mathfrak{s}\}$ and $\mathfrak{p} = \{(x, -x) : x \in \mathfrak{s}\}$. Then $\mathfrak{t}$ is the complexified Lie algebra of $\mathfrak{s}_c$. The standard complex ([20], page 163) for computing $H(\mathfrak{s} \oplus \mathfrak{s}, \mathfrak{s}_c; \mathcal{W}^{\text{even}})$ is

$$0 \leftarrow \Lambda^0 \mathfrak{p} \otimes \mathfrak{s}_c \mathcal{W}^{\text{even}} \leftarrow \Lambda^1 \mathfrak{p} \otimes \mathfrak{s}_c \mathcal{W}^{\text{even}} \leftarrow \cdots \leftarrow \Lambda^m \mathfrak{p} \otimes \mathfrak{s}_c \mathcal{W}^{\text{even}} \leftarrow 0$$

(8.2)

Here $m = \dim \mathfrak{s}$ and $\otimes \mathfrak{s}_c$ denotes the $\mathfrak{s}_c$-coinvariants of the tensor product. We call this complex $A$ where $^tA = \Lambda^t \mathfrak{p} \otimes \mathfrak{s}_c \mathcal{W}^{\text{even}}$.

The differential $\partial$ in (8.2) is given by

$$\partial(Y_1 \wedge \cdots \wedge Y_t \otimes D) = \sum_{l=1}^t (-1)^l Y_1 \wedge \cdots \wedge \hat{Y}_l \wedge \cdots \wedge Y_t \otimes \Pi Y_l(D)$$

(8.3)

where the $Y_i$ lie in $\mathfrak{p}$, $D \in \mathcal{W}^{\text{even}}$, and $\Pi$ is the representation (5.2). (Notice the terms involving $[Y_i, Y_j]$ are not present because $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t}$.) To make the complex more transparent, we identify $\mathfrak{p}$ with $\mathfrak{s}$ so that $(x, -x)$ corresponds to $x$. Then (8.2) becomes

$$\partial(x_1 \wedge \cdots \wedge x_t \otimes D) = \sum_{l=1}^t (-1)^l x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_t \otimes (\xi^x D + D \xi^x)$$

(8.4)

Next we define an increasing filtration of $A$ by the spaces

$$^tA^d = \Lambda^t \mathfrak{p} \otimes \mathfrak{s}_c \mathcal{W}^{\text{even}}_{d-t}$$

(8.5)

where we set $\mathcal{W}^{\text{even}}_j = 0$ if $j < 0$. Then $^tA^d \subseteq ^{t-1}A^{d+1}$. This follows since the $\xi^x$ lie in $\mathcal{W}^{\text{even}}_1$ and so $\mathfrak{p} \cdot \mathcal{W}^{\text{even}}_d \subseteq \mathcal{W}^{\text{even}}_{d+1}$. So we have in hand a filtration of the complex (8.1). We
put $A^{d,q} = d+q A^d$; then $d$ is the filtration degree and $q$ is the complementary degree. The induced filtration on the homology is $H(s \oplus s, S_c; \mathcal{W}^{even}) = \bigcup_{d \in \mathbb{N}} F^d$ where $F^d = \bigoplus_{q \in \mathbb{Z}} F^{d,q}$ and $F^{d,q}$ is the $d$th filtration piece of $H_{d+q}(s \oplus s, S_c; \mathcal{W}^{even})$. The associated graded space gr $H(s \oplus s, S_c; \mathcal{W}^{even})$ is the direct sum of the spaces

$$\text{gr}^d H_{d+q}(s \oplus s, S_c; \mathcal{W}^{even}) = F^{d,q}/F^{d-1,q+1}$$

(8.6)

Notice that $0A = \mathcal{W}^{even}$ and the filtration on $0A$ defined by (8.5) is the same one as in (5.4). So gr $H_0(s \oplus s, S_c; \mathcal{W}^{even}) = \mathfrak{B}$. Our goal is to prove

$$\text{gr}^d H_0(s \oplus s, S_c; \mathcal{W}^{even}) = (\mathcal{P}^{inv}/\mathcal{T}^{inv})^d$$

(8.7)

Now we consider the spectral sequence $E_0, E_1, \ldots$ associated to our filtered complex. (See e.g., [20, Appendix D] or [15, Chapter I, §4] for the construction of this spectral sequence in the general setting.) The $E_0$ term is given by $E_0^{d,q} = A^{d,q}/A^{d-1,q+1}$ and so

$$E_0^{d,q} = \wedge^{d+q} p \otimes_{S_c} \mathcal{W}^{even}_{-q}/\mathcal{W}^{even}_{-q-1}$$

(8.8)

The identification gr $\mathcal{W}^{even} = \mathcal{P}^{even}$ gives

$$E_0^{d,q} = \wedge^{d+q} p \otimes_{S_c} \mathcal{P}^{q}$$

(8.9)

(Thus the $E_0$ term occupies the octant of the $d, q$ plane where $q \leq 0$ and $d + q \geq 0$.) So $E_0^d$ is the complex

$$0 \leftarrow \wedge^0 p \otimes_{S_c} \mathcal{P}^d \leftarrow \wedge^1 p \otimes_{S_c} \mathcal{P}^{d-1} \leftarrow \wedge^2 p \otimes_{S_c} \mathcal{P}^{d-2} \leftarrow \ldots \leftarrow \wedge^m p \otimes_{S_c} \mathcal{P}^{d-m} \leftarrow 0$$

(8.10)

The boundary $\partial_0$ is induced by $\partial$. We can easily compute $\partial_0$ since the natural projection maps $\psi_d : \mathcal{W}_d \to \mathcal{P}^d$ are given by $\psi_d(a_1 \cdots \hat{a}_{2d}) = a_1 \cdots a_{2d}$ where $a_i \in L^*$ (cf. §5). So for $D \in \mathcal{W}_d$ we have $\psi_{d+1}(\xi^x D) = \psi_{d+1}(D \xi^x) = \sigma_x \psi_d(D)$. Thus (8.4) gives

$$\partial_0(x_1 \wedge \cdots \wedge x_t \otimes f) = \sum_l (-1)^l x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_t \otimes (2\sigma_{x_l} f)$$

(8.11)

The total complex $E_0$ is

$$0 \leftarrow \wedge^0 p \otimes_{S_c} \mathcal{P}^{even} \leftarrow \wedge^1 p \otimes_{S_c} \mathcal{P}^{even} \leftarrow \wedge^2 p \otimes_{S_c} \mathcal{P}^{even} \leftarrow \ldots \leftarrow \wedge^m p \otimes_{S_c} \mathcal{P}^{even} \leftarrow 0$$

(8.12)

The homology $H(E_0)$, together with a differential $\partial_1$, is the $E_1$ term of the spectral sequence. More precisely, $E_1^{d,q} = H_{d+q}(E_0^d)$.

To compute $E_1$, we observe that $H(E_0)$ is the $S_c$-coinvariants of the homology of the complex

$$0 \leftarrow \wedge^0 p \otimes \mathcal{P}^{even} \leftarrow \wedge^1 p \otimes \mathcal{P}^{even} \leftarrow \wedge^2 p \otimes \mathcal{P}^{even} \leftarrow \ldots \leftarrow \wedge^m p \otimes \mathcal{P}^{even} \leftarrow 0$$

(8.13)

Indeed, $E_0$ is the $S_c$-coinvariants of (8.13), and taking coinvariants commutes with taking homology. The latter follows because each space $\wedge^l p \otimes \mathcal{P}^{even}$ is a locally finite $S_c$-representation, and for any such representation $\mathcal{V}$, the natural map $\mathcal{V}^{S_c} \to \mathcal{V}_c$ from invariants to coinvariants is an isomorphism.

To compute the homology of (8.13), we recognize (8.13) as the Koszul complex $K$ of the sequence $\sigma_{y_1}, \ldots, \sigma_{y_m}$ in $\mathcal{P}^{even}$ where $y_1, \ldots, y_m$ is any basis of $s$. Recall from [14] that $\mathcal{I}$ is the ideal in $\mathcal{P}$ generated by the $\sigma_{y_i}$. Kraft and Procesi proved in [22, Theorem 3.3] and [23, Theorem 5.3] that the subscheme $\sigma^{-1}(0)$ of $L$ is a reduced complete intersection, i.e., $\sigma_{y_1}, \ldots, \sigma_{y_m}$ is a regular sequence in $\mathcal{P}$. Let us consider the Koszul complex $\tilde{K}$ of this
sequence in $\mathcal{P}$. By a well known result of commutative algebra (see [17 III, Proposition 7.10A]) the homology of $\widetilde{K}$ is concentrated in degree zero and $H_0(\widetilde{K}) = \mathcal{P}/\mathcal{I}$ as graded algebras. But $K$ is simply obtained from $\widetilde{K}$ by taking $\mathbb{Z}_2$-invariants. Hence the homology of $K$ is concentrated in degree zero and $H_0(K) = \mathcal{P}_{even}/\mathcal{I}_{even}$ as graded algebras. Then the module $H_0(K)_{S_c}$ of coinvariants identifies with $\mathcal{P}_{inv}/\mathcal{I}_{inv}$.

Thus $E_1$ is the complex

$$0 \leftarrow \mathcal{P}^{inv}/\mathcal{I}^{inv} \xleftarrow{\partial_1} 0 \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_1} 0 \leftarrow 0$$

where the differentials $\partial^{d,q}_1 : E^{d,q}_1 \rightarrow E^{d-1,q}_1$ are obviously zero and

$$E^{d,d}_1 = (\mathcal{P}^{inv}/\mathcal{I}^{inv})^d \quad \text{while} \quad E^{d,q}_1 = 0 \text{ if } q \neq -d$$

(8.15)

Now we can compute the rest of the spectral sequence. We know $E_{r+1}$ is the homology of $E_r$ with respect to a differential $\partial_r$; i.e. $E^{r,q}_{r+1} = \ker \partial^{r,q}_r / \text{im } \partial^{r,q+r}_{r+1}$ where $\partial^{r,q}_r$ maps $E^{r,q}_r$ to $E^{r-r,q+r-1}_r$. For $r \geq 1$, we find that $E^{d,q}_1 = E^{d,q}_r$ and the differentials $\partial^{r,q}_r$ are all zero.

The $E_\infty$ term of the spectral sequence satisfies

$$E^{d,q}_\infty = \text{gr}^d H_{d+q}(S \oplus S_c; \mathcal{W}^{even})$$

(8.16)

Our final step is to show our spectral sequence converges in that

$$E^{d,q}_1 = E^{d,q}_\infty$$

(8.17)

This will finish off the proof of Theorem 6.3 because then (8.15) gives the desired result (8.17).

The convergence (8.17) follows formally from the two properties: (i) $A^d = 0$ if $d < 0$ where $A^d = \oplus_{q \in \mathbb{Z}} A^{d,q}$ and (ii) $A^{d,q}$ has finite dimension. Indeed, following the notation in [15 I,§4.2], we have

$$E^{d,q}_r = Z^{d,q}_r/(B^{d,q}_r + Z^{d-1,q+1}_r)$$

$$E^{d,q}_\infty = Z^{d,q}_\infty/(B^{d,q}_\infty + Z^{d-1,q+1}_\infty)$$

(8.18)

where $Z^{d,q}_r = \{ z \in A^{d,q} : \partial z \in A^{d-r,q} \}$, $Z^{d,q}_\infty = \bigcap \ker \partial_r$, $B^{d,q}_r = A^{d,q} \cap \partial A^{d+r}$ and $B^{d,q}_\infty = A^{d,q} \cap \partial A$. Suppose we fix $d$ and $q$. Then (i) gives $Z^{d,q}_r = Z^{d,q}_\infty$ and $Z^{d-1,q+1}_r = Z^{d-1,q+1}_\infty$ if $r > d$, and (ii) gives $B^{d,q}_r = B^{d,q}_\infty$ if $r$ is large enough. Therefore $E^{d,q}_r = E^{d,q}_\infty$ for $r$ large enough. But we found $E^{d,q}_1 = E^{d,q}_r \ (r \geq 1)$ and so (8.17) follows.

We remark that it also follows that $\text{gr } H_j(S \oplus S_c; \mathcal{W}^{even}) = 0$ for $j > 0$. Thus we have proven

**Proposition 8.1.** $H_j(S \oplus S_c; \mathcal{W}^{even}) = 0$ if $j > 0$.

9. **Proof of Corollary 6.4**

$S_c$ and $G_c$ are commuting subgroups of $Sp(L)$. It follows that $\xi$ maps $\mathfrak{g}$ into $\mathcal{W}^{inv}$ and so $\xi$ induces $\xi_B$ where $\xi_B$ is the composition $\mathfrak{g} \xrightarrow{\xi} \mathcal{W}^{inv} \rightarrow \mathcal{B}$. Next $\tau$ is $Sp(L)$-invariant and so in particular is $S_c$-invariant. Then $\tau$ preserves $\mathcal{W}^{inv}$. We see that $\tau$ preserves $\mathcal{M}$, and so $\tau$ preserves $\mathcal{M}^{inv}$. Hence $\tau$ induces a filtered algebra anti-involution $\tau_B$ of $\mathcal{B} = \mathcal{W}^{inv}/\mathcal{M}^{inv}$. Finally, $\vartheta$ is $Sp(L)$-invariant and so in particular is $S_c$-invariant. Then
\( \vartheta \) preserves \( \mathcal{W}^{inv} \). We see that \( \vartheta \) preserves \( \mathcal{M} \), and so \( \vartheta \) induces an antilinear filtered algebra involution \( \vartheta_B \) of \( \mathcal{B} \).

Thus we have in place our Dixmier algebra data for \( \mathcal{O} \). It is clear because of Proposition 5.1 that the axioms are satisfied.

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