Fatigue-Life Prediction of Mechanical Element by Using the Weibull Distribution

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Abstract: Applying Goodman, Gerber, Soderberg and Elliptical failure theories does not make it possible to determine the span of failure times (cycles to failure-$N_i$) of a mechanical element, and so in this paper a fatigue-life/Weibull method to predict the span of the $N_i$ values is formulated. The input’s method are: (1) the equivalent stress ($\sigma_{eq}$) value given by the used failure theory; (2) the expected $N_{eq}$ value determined by the Basquin equation; and (3) the Weibull shape $\beta$ and scale $\eta$ parameters that are fitted directly from the applied principal stress $\sigma_1$ and $\sigma_2$ values. The efficiency of the proposed method is based on the following facts: (1) the $\beta$ and $\eta$ parameters completely reproduce the applied $\sigma_1$ and $\sigma_2$ values. (2) The method allows us to determine the reliability index $R(t)$, that corresponds to any applied $\sigma_1$ value or observed $N_i$ value. (3) The method can be applied to any mechanical element’s analysis where the corresponding $\sigma_1$ and $\sigma_2$, $\sigma_{eq}$ and $N_{eq}$ values are known. In the performed application, the $\sigma_1$ and $\sigma_2$ values were determined by finite element analysis (FEA) and from the static stress analysis. Results of both approaches are compared. The steps to determine the expected $N_i$ values by using the Weibull distribution are given.

Keywords: static and fatigue reliability; mechanical design; Weibull distribution; finite element analysis; principal stresses

1. Introduction

Fatigue is a random phenomenon [1,2] that causes a mechanical component to fail [3] at a stress level lower than the material strength limit ($S_e$) [4]. Although fatigue is random and the mechanical component is subject to variable amplitude and cyclic load [$\sigma_1$, $\sigma_2$], based on single equivalent stress $\sigma_{eq}$ value, its failure analysis can be performed by applying a failure theory such as the Goodman, Gerber, Soderberg, and ASME (American Society of Mechanical Engineers) elliptical theories [5] (pp. 185,186,188) [6] (pp. 313–316). Depending on the specific research, different statistical models based on (stress, strain, crack growth, etc.) have been proposed to determine the cycles to failure [7–9]. Therefore, because rather than using the random behavior the analysis uses only a stress value, its application does not enable either the reliability of the component or its expected cycles to failure ($N_i$) values to be found. However, because the expected $N_i$ values can be determined from the related stress-cycles $S$–$N$ curve of the used material [10], and since the ultimate strength ($S_{ul}$) and $S_e$ values represent the strength of the material used, then based on the theory given in [11], and on the $S_{ul}$ and $S_e$ values, the Weibull shape $\beta$ and scale $\eta$ parameters can be determined. Therefore, the novelty of the proposed method lies in that it uses the Weibull parameters to determine both the component’s reliability and the expected $N_i$ values. Because the reliability of the component depends on both the applied stress and the inherent strength [12,13] in the product to overcome it, then the proposed method to determine the random behavior of $N$ can be based on the Weibull distribution used to model the stress, and on the Weibull distribution used to model the $N_i$ values.
Based on the theory given in [11], the Weibull stress $\beta_s$ and $\eta_s$ parameters are both fitted directly from the applied maximum and minimum principal stress $\sigma_1$ and $\sigma_2$ values, established here by static and finite element analyses (FEA). Thus, based on the fact that the reliability of the component is completely determined for either the applied stress or for the corresponding cycles to failure, the Weibull cycle to failure ($\beta_t = \beta_s, \eta_t$) parameters are found based on both the addressed ($\beta_s, \eta_s$) parameters and on the cycles to failure $N$ value, where $N$ is obtained from the S-N curve by inserting the $\sigma_{eq}$ that corresponds to the applied failure theory in Basquin’s equation. This makes the proposed method highly efficient in determining the random behavior of $N$ and the corresponding reliability indices. This efficiency is based on the fact that the Weibull stress and the Weibull cycle distributions both converge to the same reliability index [14].

To show numerically how the proposed method works, in the practical analysis the used principal stresses $\sigma_1$ and $\sigma_2$ values were determined by FEA and static analyses [15]. Because the proposed method depends only on the principal stress values, but not on the complex analysis used to determine them, then in the application a simple flat spring component made of steel AISI 4340 OQT 1300 subjected to tensile cycling stress was used. Since from the FEA simulation, the addressed principal stresses values are $\sigma_1 = 491.75$ MPa and $\sigma_2 = 184.8$ MPa, then by applying the proposed method, the corresponding Weibull stress parameters were $(\beta_s = 2.248519, \eta_s = 301.455469$ MPa). However, because $\beta_s$ and $\eta_s$ are expressed in stress units, then in order to determine the related Weibull cycle distribution, the corresponding expected cycle to failure ($N$) value of the flat spring was determined by using Basquin’s equation [see Equation (32)], where the $\sigma_{eq}$ value was estimated by using the ASME elliptical failure theory criterion [see Equation (29)]. Therefore, based on the stress $(\beta_s, \eta_s)$ parameters and on the addressed $(N)$ value, the resulting cycle Weibull parameters were $\beta_s = 2.248519, \eta_1 = 1,243,427,849$ cycles. As can be seen, although the reliability of each one of the expected cycles to failure can be determined by either the stress or the cycle Weibull families, it is important to notice that the random behavior of $N$ can only be determined by the Weibull cycle family. Finally, it is important to mention that based on the Weibull stress $(\beta_s, \eta_s)$ parameters, the steps to determine the minimal strength $S_y$ value that a component must have to feature at least the desired $R(t)$ index, and vice versa, are also given.

The structure of the paper is as follows. Section 2 offers the generalities of the fatigue analysis; and in Section 3, the generalities of the Weibull analysis is presented. The proposed method to determine the expected cycle to failure is formulated in Section 4. Section 5 contains the numerical application and a comparison between the FEA results and those given by the static analysis method. Finally, the conclusions are given in Section 5.

2. Generalities of Fatigue Analysis

Since the data from the accumulated fatigue damage analysis allow us to determine how the mechanical elements are weakened over time, then the main points considered in fatigue analysis are: (i) load, (ii) component geometry, (iii) material properties, (iv) stress-life analysis and (v) fatigue life analysis. However, fatigue analysis is performed by considering that the fatigue life depends on the average value and on the amplitude of the fluctuating loads [16]. However, since several mechanical components are subjected to fluctuating load conditions, then fatigue analysis is required in order to estimate their lifetime [17,18]. On the other hand, fatigue analysis requires an estimation of the cycles to failure, which is achieved when a set of $n$ samples are tested to failure. Thus, when the fracture occurs, the corresponding cycles are recorded as the corresponding cycle to failure. In high-cycle fatigue situations, material performance is commonly characterized by an S–N curve [19], also known as Wöhler’s curve. The S–N is a graph of the magnitude of the alternating stress ($\sigma_{eq}$) against the logarithmic of cycles to failure ($N$) [3]. Therefore, the principal stresses $\sigma_1$ and $\sigma_2$ values are used to determine the stress amplitude $\sigma_{eq}$ and mean stress $\sigma_m$ values, which are used in fatigue analysis. Then the $\sigma_{eq}$ and $\sigma_m$ values are both used in a failure theory criterion to determine the corresponding equivalent stress $\sigma_{eq}$ value used in Basquin’s equation and, thus, determine the $N$ value for the expected...
cycles to failure [20]. Because cycles to failure occur due to weakness of the strength, they can be represented by the weakest link theory represented by the Weibull distribution, which offers the smallest extreme value distribution [14]. The most useful properties of the Weibull distribution to represent fatigue data are its stability property (referred to as stability to changes of the location parameter), its stability by using the minimum operations, and its stability under scale transformation (see [14] chapter 2 for details).

Since the Weibull stress \((\beta_s, \eta_s)\) parameters used in the proposed method are both estimated directly from the applied principal stresses \(\sigma_1\) and \(\sigma_2\) values [21,22], let us present how they are determined from the static stress analysis.

2.1. Static Stress Analysis

The objective of this section is to show how to determine the principal stresses \(\sigma_1\) and \(\sigma_2\) values that we used in the proposed method (1) to determine the corresponding Weibull stress \(\beta_s\) and \(\eta_s\) parameters and (2) to perform the fatigue analysis. Although the \(\sigma_1\) and \(\sigma_2\) values are both determined depending on the axial, tension, compression, bending, torsion and shear stresses values, because the case analyzed herein deals only with bending, then \(\sigma_1\) and \(\sigma_2\) are both determined by the maximum and minimum generated moment. The procedure to determine them is explained below:

(i) By using the maximum \(M\) value determine the maximum bending stress \(\sigma_1\) value as:

\[
\sigma = \frac{Mc}{I}
\]  

(ii) Determine the maximum and minimum deflection of the element \((Y)\) as:

\[
Y = \frac{FL^3}{3EI}
\]

where, \(F\) is the maximum and minimum applied force, \(E\) is the material’s elasticity modulus, and \(L\) is the component’s length.

(iii) By solving Equation (3) for \(F\), by using the maximum and minimum \(Y\) values, determine the corresponding maximum and minimum applied forces as:

\[
F = \frac{3EIY}{L^3}
\]

(iv) Determine the inertia area moment \(I\). It is calculated as in Equation (4), where \(b\) is the element’s width, and \(t\) is the element’s thickness.

\[
I = \frac{bt^3}{12}
\]

(v) By using the maximum and minimum \(F\) value, determine the maximum and minimum bending moment \((M)\) as:

\[
M = F \times L
\]

Finally, by using the minimum \(M\) value in Equation (1), determine the corresponding minimum bending stress \(\sigma_2\) value. The proposed method is now described by using the \(\sigma_1\) and \(\sigma_2\) values.

3. Proposed Method

This section explains the proposed method to determine the cycle to failure of a mechanical component. Because the proposed method is based on both the principal applied stresses and the corresponding cycle to failure values, the section is structured to present first the steps first to determine the Weibull stress family, and second to determine the corresponding Weibull cycle to failure family. However, since the Weibull distribution is used, its generalities are stated below.
3.1. Generalities of the Weibull Analysis

The aim of this section is to present the generalities of the Weibull distribution based on which we can directly determine the $\beta$ and $\eta$ parameters of the Weibull stress $(\beta_s, \eta_s)$ and the Weibull cycles $(\beta_t = \beta_s, \eta_t)$ families. The Weibull probability density function (pdf) and its reliability function $R(t)$ [23] are given as:

$$f(t) = \frac{\beta}{\eta} \left(\frac{t}{\eta}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\}$$ \hspace{1cm} (6)

and

$$R(t) = \exp\left\{-\left(\frac{t}{\eta}\right)^{\beta}\right\}$$ \hspace{1cm} (7)

From the addressed $\sigma_1$ and $\sigma_2$ values, the $(\beta_s, \eta_s)$ parameters are directly estimated [24]. $\beta_s$ is determined as:

$$\beta_s = \frac{-4\mu_y}{0.99176 \cdot \ln\left(\frac{\sigma_1}{\sigma_2}\right)}$$ \hspace{1cm} (8)

where the constant 0.99176 is specified for the analyzed case, and the method to determine it for any case, is given in Section 4.1 in [24]. Also, $\mu_y$ is the mean of the used $Y$ vector, determined based on the median rank approach. Notice that if $n = 21$ elements are tested, then $\mu_y = -0.545624$; and $\eta_s$ is determined as:

$$\eta_s = \exp(\mu_x)$$ \hspace{1cm} (9)

where from [24] $\mu_x$ in Equation (9) is the log-mean of the observed failure-time data which based on the addressed $\sigma_1$ and $\sigma_2$ values is determined as:

$$\mu_x = \ln(\sigma_1\sigma_2)^{\frac{1}{2}}$$ \hspace{1cm} (10)

To summarize, note that by using the $\sigma_1$ and $\sigma_2$ values in Equations (8)–(10), the Weibull stress parameters are both completely determined, and that because $\mu_y = -0.545624$ is constant, then the efficiency of $\beta_s$ and $\eta_s$ only depends on the efficiency on which the $\sigma_1$ and $\sigma_2$ values were determined. Based on the addressed stress $(\beta_s, \eta_s)$ parameters, let us present the method to determine the random behavior of the addressed $\sigma_1$ and $\sigma_2$ values, based on which the random behavior of the expected cycles to failure is formulated.

3.2. Weibull Stress Family Estimation and Its Random Behavior Analysis

Before determining the random behavior of the addressed $\sigma_1$ and $\sigma_2$ values, it is necessary to mention that they should be validated by using them with the material yield strength $S_y$ value, and the desired safety factor (SF) in the yielding maximum shear stress theory (MSS) and the distortion energy theory (DE) criteria [25] as follows:

$$\text{MSS theory} = (\sigma_1) < \frac{S_y}{SF}$$ \hspace{1cm} (11)

and,

$$\text{DE theory} = \left(\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2\right)^{0.5} < \frac{S_y}{SF}$$ \hspace{1cm} (12)

If Equations (11) and (12) do not hold, then a material with higher $S_y$ for which Equations (11) and (12) holds, must be selected.

**Note 1.** Note that from Equations (11) and (12), we only know whether the design is safe or not, but at this point it is not possible to determine either the designed reliability or the corresponding expected failure times.

Based on the above, the expected behavior of $\sigma_1$ and $\sigma_2$ is determined as follows:
(i) From [24], determine the sample size \( n \) value to be used in the analysis.

\[
 n = \frac{-1}{\ln(R(t))}
\]  

(13)

where \( R(t) \) is the desired reliability of the analysis, here \( R(t) \) could be seen as the equivalent of a confidential interval level used in the quality field. Note that the reason why \( R(t) = 0.9535 \) is used because for this value, \( n \) is an integer \((n = 21)\), but any desired \( R(t) \) index can be used.

(ii) Using the \( n \) value from Equation (13) in the median rank approach function [26] given in Equation (14), determine the corresponding cumulated failure percentile \( F(t_i) \) elements as:

\[
 F(t_i) = \frac{i - 0.3}{n + 0.4}
\]  

(14)

(iii) by using the \( F(t_i) \) elements in the linear form of the reliability function defined in Equation (7), determine the corresponding \( Y_i \) elements as shown in Equation (15). Then, from the \( Y_i \) elements, determine its corresponding mean value, obtained by using Equation (16).

\[
 Y_i = \ln(-\ln(1 - ((i - 0.3)/(n + 0.4))))
\]  

(15)

and,

\[
 \mu_y = \frac{\sum_{i=1}^{n} Y_i}{n}
\]  

(16)

(iii) by inserting the \( \mu_y \) value and the principal stress \( \sigma_1 \) and \( \sigma_2 \) values into Equations (8) and (9), determine the corresponding Weibull stress parameters \((\beta_s, \eta_s)\). These estimated \((\beta_s, \eta_s)\) parameters represent the Weibull stress family used to model the random behavior of the estimated principal stresses \( \sigma_1 \) and \( \sigma_2 \) values that represent the minimum required strength the used material must present in order for it to have the designed reliability. Thus, \( \sigma_1 \) and \( \sigma_2 \) are the applied stresses from which the \( \sigma_{eq} \) value was determined; \( \sigma_{eq} \) is the equivalent constant applied stress value, that by its cyclical application demands the material presents at least the minimum strength of \( \sigma_2 \) value. Based on the \( \beta_s \) and \( \eta_s \) parameters, the minimum strength \( \sigma_{2i} \) values are determined by using the \( t_{0i} \) value that corresponds for each one of the \( Y_i \) elements as:

\[
 t_{0i} = \exp\{Y_i/\beta_s\}
\]  

(17)

Therefore, the minimum strength of \( \sigma_{2i} \) value that corresponds to each one of the \( Y_i \) elements is given as:

\[
 \sigma_{2i} = \eta_s \cdot t_{0i}
\]  

(18)

Additionally, it is important to mention that because from [24], the Weibull \( \beta_s \) and \( \eta_s \) parameters also let determine the minimum Weibull strength parameter that the used material should present if the applied constant stress is the \( \eta_s \) value, then it is determined as:

\[
 \sigma_{1i} = \eta_s/t_{0i}
\]  

(19)

Thus, in Equation (18) \( \sigma_{2i} \) is the minimum stress the used material must present if the applied stress is constant at the value given by the \( \sigma_{eq} \) value. In Equation (19), \( \sigma_{1i} \) is the minimal Weibull scale parameter that that used material must present if the applied stress is constant at the value given by the \( \eta_s \) value (for details see Equation (61) in [24]), consequently by using the \( \sigma_{1i} \) and \( \sigma_{2i} \) values in Equation (25), the corresponding mean value is determined.

Also, by using the \( \sigma_1 \) value, the \( t_{01} \) element that corresponds to the \( \sigma_1 \) and \( \sigma_2 \) values, is calculated.

\[
 t_{01} = \eta_s/\sigma_1
\]  

(20)
Consequently, by using the $t_{0\text{max}}$ and the $\beta_s$ values, the corresponding $Y_1$ value is established.

\[ Y_1 = \ln(t_{01}) \times \beta_s \] (21)

Finally, the reliability index corresponding to the $Y_1$ value is obtained.

\[ R(t) = \exp[-\exp(Y_1)] \] (22)

**Note 2.** It is important to mention that the $R(t)$ index obtained in Equation (22) corresponds to an element which has a strength equivalent to the $\sigma_1$ value. Thus, if $S_y \neq \sigma_1$, the reliability of the designed element is found by replacing the $S_y$ value with $\sigma_1$ in Equation (20).

However, note that while the reliability of the designed element can be obtained from Equation (22), the corresponding expected cycles to failure remain unknown. In this paper the cycle to failure are determined as follows:

### 3.3. Weibull Cycle Family Estimation and Its Random Behavior

To determine the Weibull cycle family and the corresponding random cycles to failure, we need first to determine the cycle ($N$) value which corresponds to the equivalent stress $\sigma_{eq}$, by using Basquin's formula. The following is the explanation of how the $N$ value is found.

#### 3.3.1. Estimation of the $N$ Value that Corresponds to the Equivalent Stress Value

The Weibull analysis of the component is performed based on the dynamic loads. That is why finding the corresponding cycles to failure occurs as follows. Depending on the component’s material, the yield stress $S_y$ and ultimate stress $S_{ut}$ values are obtained. Then the corresponding endurance limit $S_e$ value is calculated as:

\[ S_e' = 0.5(S_{ut}) \] (23)

Then based on the correction factor values, the corrected endurance limit is determined as:

\[ S_e = K_a K_b K_c K_d K_e K_f S_e' \] (24)

where, $K_a$ = surface condition modification factor, $K_b$ = size modification factor, $K_c$ = load modification factor, $K_d$ = temperature modification factor, $K_e$ = reliability factor and $K_f$ = miscellaneous-effects modification factor. After that, using Equations (25) and (26) and the values for principal stresses $\sigma_1$ and $\sigma_2$ from Section 2.1, the corresponding mid-range stress $\sigma_m$ and the alternating stress $\sigma_a$ values are set.

\[ \sigma_m = \frac{(\sigma_1 + \sigma_2)}{2} \] (25)

and,

\[ \sigma_a = \frac{(\sigma_1 - \sigma_2)}{2} \] (26)

Then, by using the $S_y$, $S_e$, $\sigma_a$ and $\sigma_m$ values in the used failure theory criterion, determine whether the designed element is safe or not. (Although as shown in Figure 1, there are several criteria, the one used here was the ASME Elliptic criterion). Thus, to determine the fatigue safety factor $n_f$, in the case of the ASME elliptical theory criterion, the $n_f$ function is:

\[ n_f = \sqrt{\left(\frac{\sigma_a}{S_e}\right)^2 + \left(\frac{\sigma_m}{S_y}\right)^2} \] (27)
If \( n_f \) is higher than one \( (n_f > 1) \), the designed element is considered to have an infinite life \( (N = \infty) \). Otherwise, it is considered to have a finite life. However, because the aspect being worked with is the probabilistic behavior of the stress, what proceeds is to determine the corresponding reversed equivalent stress \( \sigma_{eq} \) value [27]. This is obtained from the ASME elliptical failure theory criterion represented by:

\[
\left( \frac{\sigma_0}{S_e} \right)^2 + \left( \frac{\sigma_{ut}}{S_y} \right)^2 = 1
\]

(28)

The \( \sigma_{eq} \) is placed instead of the \( S_e \) element into Equation (28); thus, its value is set as:

\[
\sigma_{eq} = \frac{\sigma_0}{\sqrt{1 - \left( \frac{\sigma_{ut}}{S_y} \right)^2}}
\]

(29)

Since the calculation of the fatigue constant \( a \) and \( b \) parameters of Basquin’s equation is required, such a calculation is undertaken by using the fatigue strength factor \( f \) that is taken from [6], the ultimate stress \( S_{ut} \), and the endurance limit \( S_e \) values.

\[
a = \frac{(f S_{ut})^2}{S_e}
\]

(30)

and,

\[
b = -\frac{1}{3} \log \left( \frac{f S_{ut}}{S_e} \right)
\]

(31)

Then, by inserting the \( a, b \) and \( \sigma_{eq} \) estimated values into Equation (32), the expected average cycles to failure \( N \) value is obtained:

\[
N = \left( \frac{\sigma_{eq}}{a} \right)^{\frac{3}{2}}
\]

(32)

From Equation (32), the \( N \) value that corresponds to the equivalent stress is set. That value will be used to determine the corresponding Weibull cycle parameters.

3.3.2. Determination of the Weibull Cycle Parameters

By using the cycle to failure \( N \) value and the Weibull element \( t_{0eq} \) value that corresponds to the equivalent stress value in Equation (20), the corresponding Weibull scale cycle to failure \( \eta_t \) parameter is set.

\[
\eta_t = \frac{N}{t_{0eq}}
\]

(33)

At this point, by using the Weibull cycle \( \eta_t \) parameter, the corresponding expected cycle to failure values that correspond to each one of the \( Y_i \) elements from Equation (15) are determined.

\[
N_i = \eta_t \cdot t_{0i}
\]

(34)

Thus, by using the Weibull stress \( \beta_s \) value, the Weibull cycle to failure family is \( W (\beta_s, \eta_t) \).
Note 3. The $\beta_s$ value is used on the assumption that the failure mode remains constant.
Finally, the $N_i$ value that corresponds to the used $S_y$ value is determined based on the $t_{0by}$ element given as:

$$t_{0by} = \eta_s / S_y$$

(35)

Then by using the $t_{0by}$ value in Equation (34), the $N_i$ value that corresponds to the $S_y$ value is determined. The numerical application is as follows.

4. Mechanical Application with Weibull/Finite Element Analysis (FEA)

The numerical application is performed by using a flat spring undergoing cyclic variable stress that causes tensile deflection. The load is considered as a cantilever and is the direct cause of failure affecting a product’s reliability. By way of initial conditions, the following data are established: the minimal operational deflection of the flat spring is 3.0 mm, and the maximal operational deflection of the spring is 8.0 mm. It is also necessary to perform the spring stress analysis at point A. At this point, the convex side of the base of the spring experiences the varying tensile stresses [5]. The relevant data to determine the main applied stresses are, length $L = 65$ mm, thickness $t = 0.80$ mm, width $b = 6$ mm, and the used material is a steel AISI 4340 OQT 1300 with elasticity modulus $E = 207$ GPa. The spring is joint to the base by using the double-sided longitudinal welded joint that corresponds to the category B of joints in AWS D1.1 [28]. The structure geometry (see Figures 2 and 3) is subjected to variable cyclic load, within the elastic range, of frequency and magnitude sufficient to initiate cracking and progressive failure due to fatigue.

![Figure 2. Free-Body diagram of flat spring.](image)

![Figure 3. Cross section of spring.](image)

The relation between stress range $F_{SR}$ and the allowable cycle number $N$ under variable amplitude load condition [29] is:

$$F_{SR} = \left( \frac{C_f \times 329}{N} \right)^{0.333}$$

(36)

$$F_{SR} = \sigma_{\text{max}} - \sigma_{\text{min}}$$

(37)

in which:

- $F_{SR} =$ Allowable stress range,
- $C_f =$ Fatigue constant load,
- $N =$ Number of cycles of stress range.

With the relevant data provided, the $\sigma_1$ and $\sigma_2$ principal stress can be determined to perform the Weibull FEA stress data analysis as follows.
4.1. Weibull FEA Stress Data Analysis

The proposed Weibull analysis is performed by using the $\sigma_1$ and $\sigma_2$ principal stress values that are acting on the flat spring device; thus, in this study they are determined by performing a FEA simulation. The analysis is performed by presenting the FEA stress data procedure [30] first, in Section 4.1, and then the corresponding cycles to failure (Ni) analysis in Section 4.2. The procedure will begin by following the steps to the static stress outlined in Section 2. Therefore, from Equation (4) the estimated inertia moment is $I = 0.256 \text{ mm}^4$. Then, by using the minimum and maximum generated deflection $Y_2 = 3 \text{ mm}$ and $Y_1 = 8 \text{ mm}$ values and the elasticity module of $E = 207 \text{ GPa}$ in Equation (3), the deflection forces are $F_1 = 4.63 \text{ N}$ and $F_2 = 1.74 \text{ N}$. Those forces are introduced in the FEA simulation to determine the corresponding principal stress values used to determine the Weibull parameters. Once this is done, the principal stresses values obtained are, $\sigma_1 = 491.75 \text{ MPa}$ and $\sigma_2 = 184.8 \text{ MPa}$.

The FEA analysis of the flat spring is shown in Figure 4. The used material is a steel AISI 4340 OQT 1300, with a Young’s modulus of 207 GPa and a Poisson’s ratio of 0.29.

**Figure 4.** The finite element model of the flat spring.

The application was performed using the ANSYS software. The FEA model of the flat spring device with the dimensional features and the loading condition given in Section 4 was dispersed by using an adaptive sizing mesh of 148 mm, that uses the first-order element, with a total of 1343 nodes and 160 elements. The corresponding stress results are shown in Table 1.

**Table 1.** Stresses and deformation results from the finite element analysis (FEA).

| Maximum Principal Stress (MPa) | Minimum Principal Stress (MPa) | Deformation (mm) Force $F_1 = 4.63 \text{ N}$ | Deformation (mm) Force $F_2 = 1.74 \text{ N}$ |
|-------------------------------|-------------------------------|-----------------------------------------------|-----------------------------------------------|
| $F_1 = 4.63 \text{ N}$       | $F_2 = 1.74 \text{ N}$       | 491.75                                        | 184.8                                         |
|                               |                               | 8.02                                          | 2.99                                          |

Based on Figure 4, and to reflect the real service loads obtained by Equation (3), the stress calculations are performed based on the maximum and the minimum applied forces of 4.63 N and 1.74 N, respectively. Since to determine the principal stress values, based on which we determine the random behavior of $N$, we need to know both how the stresses are distributed and how much the element is deflected, then Figure 5 shows the stress distribution; and Figure 6 shows the corresponding deflection.

As can be seen in Figure 5, the stress concentration occurs in the red area (point A); that is why the analysis was performed there.

In Figure 6 it can be observed that the maximum deflection occurs in the blue area where the force is being applied. Then, the red area shown in Figure 5 and the blue area shown in Figure 6 are both analyzed. Finally, the estimated $\sigma_1$ and $\sigma_2$ stress values of the FEA simulation given in Table 1 is compared to the static stress analysis. Both results are given in Table 2, where it can be seen that no significant difference exists.
behavior of the corresponding cycles to failure. The designed element is considered safe, nothing can be said about its reliability nor about the expected distortion of energy theory.

### 4.1.1. Validation of the Applied Principal Stresses $\sigma_1$ and $\sigma_2$ Values

With these principal stress values from the FEA application and the material yield strength (AISI 4340 QQT 1300) value $S_y = 827$ MPa, the yielding criteria for ductile materials is ascertained through the AmesWeb tool, with a design factor of $SF = 1.65$. The maximum shear stress theory (MSS) and the distortion of energy theory (DE) output criteria are both shown in Table 3.

#### Table 3. Static Analysis Results using the theories of failure for ductile materials.

| Parameter          | Condition to Be Met for Safe Design | Status         |
|--------------------|-------------------------------------|----------------|
| Maximum Shear Stress| $(\sigma_1) < S_y/SF$                | $491.8 < 501.8$| Ok             |
| Distortion Energy  | $(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)^{0.5} < S_y/SF$ | $430.2 < 501.2$| Ok             |

As can be seen from Table 3, because the status of both failure theories is acceptable, the design element can be deemed safe. Unfortunately, it can also be pointed out at this point that although the designed element is considered safe, nothing can be said about its reliability nor about the expected behavior of the corresponding cycles to failure.

The next section will present the Weibull stress family.
4.1.2. Weibull Stress Family Determination

The corresponding Weibull stress parameters ($\beta_s$, $\eta_s$) are established as follows.

First, by selecting a reliability of $R(t) = 0.9535$ to perform the analysis, from Equation (13), $n = \frac{-1}{\ln(0.9535)} = 21$. Then by using $n = 21$ in Equation (15), the $Y_i$ elements are generated, with a mean of $\mu_Y = -0.545624$ and a standard deviation of $\sigma_Y = 1.175117$. Finally, from Equation (8), and by using the $\mu_Y$, $\sigma_1$ and $\sigma_2$ FEA values, the Weibull $\beta_s$ value is:

$$\beta_s = \frac{-4(-0.545624)}{0.99176*\ln(\frac{491.75}{184.8})} = 2.248519$$

Likewise, since from Equation (10) the logarithm average value is $\mu_s = \ln(491.75*184.8)^\frac{1}{2} = 5.708622$, then from Equation (9) the Weibull $\eta_s$ parameter is $\eta_s = \exp[5.708622] = 301.455469$ MPa. Thus, the Weibull stress family used to obtain the corresponding Weibull cycle distribution (Section 4.2) is $W(2.248519, 301.455469)$ MPa. Using the Weibull stress family results, the strength random behavior can be determined. Because these parameters only depend on the applied $\sigma_1$ and $\sigma_2$ values, then by performing the Weibull analysis their random behavior can be determined as follows.

4.1.3. Stress Random Behavior

Because the determination of the cycles to failure is based on the random behavior of the $\sigma_2$ stress value, in this section the random behavior of the stresses $\sigma_1$ and $\sigma_2$ values are both reproduced, is determined as $t_{o1} = 301.455469/491.75 = 0.617339$, and it is included in Table 4 as well.

Then, by using the $\beta_s$ value in Equation (21), the $Y_1$ value that corresponds to the $t_{o1}$ value is determined as $Y_1 = \ln(0.617339) = -1.084545$. By using the $Y_1$ value into the Equation (22), the reliability index that corresponds to the $t_{o1}$ element is $R(t) = \exp[-\exp[-1.084545]] = 0.713156$.

**Note 4.** Observe that $R(t) = 0.713156$ is not the reliability of the design component; it only represents the reliability of the $t_{o\max}$ element in the Weibull analysis. The $R(t)$ of the element is that determined by using the $Y_1$ value that corresponds to the $S_Y$ value in Equation (22). It is located between rows 2 and 3 in Table 4.

The procedure to find the reliability of the design element by using the $S_Y$ value as $\sigma_1$ is as follows:

From Equation (20), the $t_{0S_Y}$ element that corresponds to the $S_Y$ value is $t_{0S_Y} = 301.455469/827 = 0.364517$ and from Equation (21) the corresponding $Y_{S_Y}$ value is $Y_{S_Y} = \ln(0.364517) = 2.269166$. Thus, from Equation (22), the reliability index for the $Y_{S_Y}$ value is $R(t) = \exp[-\exp[-2.269166]] = 0.901768$. From the above it can be concluded that the reliability of the design element is $R(t) = 0.901768$.

Bold data highlight: (1) the $R(t)$ index that corresponds to the used $S_Y = \sigma_1 = 827$ MPa value; (2) the reproduced principal stresses values and (3) the stress Weibull eta parameter.

Based on the fact that in the Weibull analysis the reliability of the element is given by either the applied stress or the corresponding life [14], then the $t_{oi}$ elements are used to determine the corresponding random behavior of the expected cycles to failure. The analysis is as follows.
Therefore, since 

\[ \eta = b \]

and the fatigue constant

\[ S \]

From Equations (25) and (26), the mean and alternating stresses are

\[ \sigma = 4.2 \text{ Weibull Cycle to Failure (N) Analysis} \]

Now, by using the values of

\[ (0.72761583 \times 1.382091 \times 0.126168 \times 416.639 \times 208.3243 \times 312.4817 \times 104.1573 \times 112.4970518) \]

\[ (0.56250196 \times 1.284238 \times 0.172897 \times 387.1405 \times 224.1978 \times 305.6691 \times 81.47139 \times 87.68037713) \]

\[ (0.6504921 \times 0.748789 \times 0.593458 \times 225.7266 \times 384.5185 \times 305.1225 \times 79.3957 \times 85.42606) \]

\[ (0.9793812 \times 0.646989 \times 0.5045088 \times 0.799016 \times 0.546729 \times 240.8679 \times 360.3471 \times 300.6075 \times 57.39861 \times 84.1259846) \]

\[ (0.3665129 \times 0.84959 \times 0.5 \times 256.1134 \times 338.8969 \times 297.5051 \times 41.39172 \times 44.3618298) \]

\[ (0.2341223 \times 0.901115 \times 0.453271 \times 271.6459 \times 319.519 \times 295.5825 \times 23.9365 \times 25.6294661) \]

\[ (0.1052851 \times 0.954255 \times 0.406542 \times 287.6654 \times 301.7256 \times 294.6955 \times 7.030101 \times 7.52403293) \]

\[ (0.0219284 \times 1.0908 \times 0.359813 \times 304.4098 \times 285.129 \times 294.7694 \times 9.640397 \times 10.31807533) \]

\[ (0.14952577 \times 1.068761 \times 0.31804 \times 322.1837 \times 269.3992 \times 295.7915 \times 26.3926 \times 28.2618481) \]

\[ (0.279845 \times 1.135254 \times 0.266355 \times 341.4085 \times 254.2293 \times 297.8189 \times 43.58962 \times 64.7245514) \]

\[ (0.4159621 \times 1.203251 \times 0.219626 \times 362.71745 \times 239.2957 \times 301.0051 \times 61.70939 \times 66.25374946) \]

\[ (0.56250196 \times 1.268238 \times 0.172897 \times 387.1405 \times 224.1978 \times 305.6691 \times 81.47139 \times 87.68037713) \]

\[ (0.7276583 \times 1.382091 \times 0.126168 \times 416.639 \times 208.3243 \times 312.4817 \times 104.1573 \times 112.4970518) \]

\[ (0.92931067 \times 1.511797 \times 0.079439 \times 455.7394 \times 190.451 \times 323.0952 \times 132.6442 \times 144.096238) \]

\[ (1.22965981 \times 1.728746 \times 0.03271 \times 520.8685 \times 166.6371 \times 343.7528 \times 177.1157 \times 194.7355493) \]

### 4.2. Weibull Cycle to Failure (N) Analysis

From the selected material (AISI 4340 OQT 1300) \( S_{ul} = 965 \text{ MPa} \). By using it in Equation (23), \( S' = 0.5(965 \text{ MPa}) = 482.5 \text{ MPa} \). From Equation (24) the endurance stress limit is \( S_e = 354.6 \text{ MPa} \). From Equations (25) and (26), the mean and alternating stresses are \( \sigma_m = \frac{(491.75 + 184.8)}{2} = 338.28 \text{ MPa} \) and \( \sigma_a = \frac{(491.75 - 184.8)}{2} = 153.48 \text{ MPa} \). Now, by using the values of \( S_f = 827 \text{ MPa}, S_e = 354.6 \text{ MPa}, \sigma_a = 153.5 \text{ MPa} \) and \( \sigma_m = 338 \text{ MPa} \) in Equation (27), the ASME Elliptical failure theory criterion fatigue factor is \( n_f = \left( \frac{1}{\left( \frac{S'}{S} \right)^2 + \left( \frac{S_a}{S} \right)^2} \right) \text{ cycle} \). Therefore, since \( n_f > 1 \), the design element is considered as safe. Next, the endurance limit \( S_e = 354.6 \text{ MPa} \) and the \( f \) (fatigue strength factor) \( f = 0.8 \) values are used in Equations (30) and (31), and the fatigue constant \( a \) and \( b \) values are obtained as \( a = \frac{(0.8^2 + 0.6^2)}{354.6} = 1680.72194 \) and \( b = -1.125136. \) Then, the corresponding reversed equivalent stress is determined by Equation (29), \( \sigma_{eq} = \left( \frac{153.48}{\sqrt{1 - \left( \frac{338.28}{153.48} \right)^2}} \right) = 168.2 \text{ MPa} \). Finally, by using the values \( a = 1680.72194, \) \( b = -0.1125136 \) and \( \sigma_{eq} = 168.2 \text{ MPa} \) in Basquin’s equation, Equation (32), the number of cycles to failure is \( N = \left( \frac{168.2}{1680.72194} \right)^{1/111.5} = 767,615,910 \text{ Cycles} \). Once the value of \( N \) is obtained, based on that the corresponding \( \eta_f \) parameter is determined as follows.
4.2.1. Weibull Cycle Parameters

By using the cycles value \( N = 767,615,910 \) and \( t_{\text{eq}} = 0.617339 \) in Equation (33), the Weibull cycle \( \eta_t \) parameter is \( \eta_t = \frac{767,615,910}{0.617339} = 1,243,427,849 \) cycles. Therefore, the Weibull cycle to failure family used to determine the random behavior of \( N \) is \( W(2.248519, 1,243,427,849 \text{ Cycles}) \). The analysis is as follows.

4.2.2. Cycle Random Behavior

In this section the random behavior of the cycles (\( N \)) values is determined. By using the Weibull cycle \( \eta_t \) parameter and the \( t_{\text{oi}} \) elements in Equation (34), the corresponding expected cycle to failure values for each one of the \( Y_i \) elements are determined (see the seventh column in Table 5). Additionally, by using \( t_{\text{sy}} = 0.364517 \) in Equation (34) the \( N_{\text{sy}} \) value that corresponds to the \( S_y \) value is \( N_{\text{sy}} = 453,250,454 \).

The practical interpretation of data in Table 5 is that data given in the \( \sigma^2_i \) column represent the minimum expected material strength that will be required when the constant applied stress is \( \sigma_{\text{eq}} = 168.188640 \) (remember that this \( \sigma_{\text{eq}} \) is constant for all the rows in Table 5). For instance, in the first row of Table 5 by applying the \( \sigma_{\text{eq}} \) stress, the required material strength is 66.35155. Thus, from Equation (7) the expected reliability will be 0.96729 which corresponds to \( N_i = 273,683,422.9 \).

On the other hand, although the analysis was performed base on the \( \sigma_1 \) and \( \sigma_2 \) FEA results, because the input of the proposed method is only the \( \sigma_1 \) and \( \sigma_2 \) values, and due to they can also be determined by the conventional static analytical method, then let us now present how the \( \sigma_1 \) and \( \sigma_2 \) values determined by the analytical method are used in the proposed method.

4.2.3. Weibull Stress Analytical Static Family

In order to use the analytical static stress method, the mechanical element presented in Section 4 is applied, with resulting values of \( \sigma_1 = 491.75 \text{ MPa} \) and \( \sigma_2 = 184.8 \text{ MPa} \). And by applying the propose method as a complement of the FEA results we have that the expected cycle to failure of the flat spring is \( N = 453,250,454 \) cycles with a reliability index of \( R(t) = 0.901768 \).

On the other hand, although the analysis was performed base on the \( \sigma_1 \) and \( \sigma_2 \) FEA results, because the input of the proposed method is only the \( \sigma_1 \) and \( \sigma_2 \) values, and due to they can also be determined by the conventional static analytical method, then let us now present how the \( \sigma_1 \) and \( \sigma_2 \) values determined by the analytical method are used in the proposed method.

4.2.3. Weibull Stress Analytical Static Family

In order to use the analytical static stress method, the mechanical element presented in Section 4 is applied, with resulting values of \( \sigma_1 = 470 \text{ MPa} \) and \( \sigma_2 = 176 \text{ MPa} \). Based on those principal stresses values, the Weibull stress analytical family is \( W(2.240388, 287.610848 \text{ MPa}) \). Table 6 shows a summary of the comparison between the static method and the FEA simulation results.

From \( \sigma_1 \) and \( \sigma_2 \) values in Table 6, we conclude that both methods, the FEA and static methods, are equivalent. Additionally, note that for this particular case the fatigue life of the welded joint can be used to estimate the fatigue life of the flat spring mechanism [29]. This can be relevant due to the AWS D1.1 provides joint \( S–N \) curves whose allowable cycle number \( N \) is less than \( 10^7 \). The joint \( S–N \) curve for the welded joint category B is shown in Figure 7.
Table 5. Statistics of Weibull cycle to failure analysis.

| # | Equation (13) | Y | Equation (15) | t | Equation (17) | R(t) | Equation (22) | σ | Equation (18) | σ | Equation (19) | N (Cycles) | Equation (33) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −3.4034833 | 0.220104 | 0.96729 | 66.35155 | 1308.124 | 273,683,422.9 |
| 2 | −2.970195 | 0.26688 | 0.95 | 80.45246 | 1078.849 | 331,846,104.7 |
| 3 | −2.2693664 | 0.364517 | 0.901768 | 109.8856 | 501,103,112.7 |
| 4 | −1.6616495 | 0.477993 | 0.920561 | 99.53286 | 872.034 | 410,547,979.5 |
| 5 | −1.3943983 | 0.57869 | 0.873832 | 123.6689 | 701.842 | 510,103,112.7 |
| 6 | −1.1720537 | 0.698303 | 0.827103 | 143.9731 | 602.8627 | 410,547,979.5 |
| 7 | −0.9793812 | 0.84959 | 0.780374 | 162.1435 | 535.3039 | 535,250,454.6 |
| 8 | −0.8074473 | 0.954255 | 0.733832 | 195.0109 | 445.0831 | 668,801,050.2 |
| 9 | −0.6504921 | 1.068761 | 0.686916 | 210.5073 | 412.3183 | 767,615,910 |
| 10 | −0.5045088 | 1.132534 | 0.640187 | 240.8679 | 384.5185 | 868,289,756.6 |
| 11 | −0.3665129 | 1.203211 | 0.593458 | 319.519 | 338.8969 | 993,519,280.8 |
| 12 | −0.2341223 | 1.268789 | 0.546729 | 341.4085 | 338.8969 | 1,056,403,357 |
| 13 | −0.1052851 | 1.338715 | 0.496729 | 372.9654 | 338.8969 | 1,120,470,957 |
| 14 | −0.3678792 | 1.405285 | 0.446729 | 404.5185 | 338.8969 | 1,186,547,445 |
| 15 | −0.3698792 | 1.47459 | 0.406542 | 437.1405 | 338.8969 | 1,253,427,849 |
| 16 | −0.3698792 | 1.543211 | 0.360542 | 470.8679 | 338.8969 | 1,320,451,296 |

Table 6. Summary of the comparison between the analytical static method and FEA simulation results.

| Application | Maximum Principal Stress (MPa), Force $F_1 = 4.63$ N | Minimum Principal Stress (MPa), Force $F_2 = 1.74$ N | Weibull β Parameter | Cycles to Failure (N) | Reliability R(t) |
|---|---|---|---|---|---|
| Static | 470 | 176 | 2.240388 | 691,910,584 | 0.91 |
| FEA | 491.75 | 184.8 | 2.248519 | 453,250,454 | 0.90 |

The relation between the stress range $F_{SR}$ and the allowable cycle number $N$ is given by Equation (36). Therefore, by applying Equation (36) in Equation (37) with $C_f = 120 \times 10^8$, the allowable cycles $N$ are calculated. Table 7 shows a summary of the comparison between the static method and the FEA simulation fatigue life welded joint results.

As can be observed, the general conclusion is that the proposed method can be applied by using either the FEA or the static analytical stress approach. The efficiency of the proposed method depends on the accuracy on which the principal stresses are estimated. In this way the propose method can be applied in any mechanical analysis where $\sigma_1$ and $\sigma_2$ are known.
5. Conclusions

1. Because the input’s method are the $\sigma_1$ and $\sigma_2$ values, then the proposed method can be applied in any mechanical analysis where $\sigma_1$ and $\sigma_2$ are known.

2. The efficiency of the proposed method is that the Weibull $\beta$ (see Equations (8)) and $\eta$ (see Equation (9)) only depends on the $\sigma_1$ and $\sigma_2$ values. Consequently, by performing the Weibull analysis we can always reproduce them (see row between rows 5 and 6 Table 4).

3. In the proposed method the mechanical element reliability can be determined by either the stress $\sigma_{21}$ values (see column $\sigma_{21}$ in Table 5) or by the corresponding cycle to failure values.

4. The random behavior of both $\sigma_2$ and $N_i$ values is determined based on the $\sigma_{eq}$ and on the its corresponding $N_{eq}$ value, here determined by Basquin’s equation, but if they are determined in any other form the proposed method will be also efficient at determining the random behavior around them.

5. In the given application, the addressed $\sigma_1$ and $\sigma_2$ stress values, determined by the FEA and static stress analysis, were not significant, implying that the Weibull analysis given in Tables 4 and 5 can be used in both cases.

6. As demonstrated in [14], the standard fatigue methodologies based on the stress-strain analysis and on the mechanical fracture both converge to the same solution; it might then be possible to extend the proposed method to be used in the strain and mechanical fracture analysis by related the equivalent stress $\sigma_{eq}$ to the total deformation given by the elastic (Basquin’s equation) and plastic (Coffin-Manson equation) areas, although further research should be undertaken.

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