On nondegenerate M-stationary points for
mathematical programs with sparsity constraint

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Abstract

We study mathematical programs with sparsity constraint (MPSC) from a topological point
of view. Special focus will be on M-stationary points from [Burdakov et al. 2016]. We introduce
nondegenerate M-stationary points and define their M-index. We show that all M-stationary points
are generically nondegenerate. In particular, the sparsity constraint is active at all local minimizers
of a generic MPSC. Some relations to other stationarity concepts, such as S-stationarity, basic
feasibility, and CW-minimality, are discussed in detail. By doing so, the issues of instability and
degeneracy of points due to different stationarity concepts are highlighted. The concept of M-
stationarity allows to adequately describe the global structure of MPSC along the lines of Morse
theory. In particular, we derive a Morse relation for MPSC, which relates the numbers of local
minimizers and M-stationary points of M-index equal to one. The appearance of such saddle points
cannot be thus neglected from the perspective of global optimization. As novelty for MPSC, a
saddle point may lead to more than two different local minimizers. This is in strong contrast
with other nonsmooth optimization problems studied before, where a saddle point leads to at most
two of them. We conclude that the relatively involved structure of saddle points is the source of
well-known difficulty if solving MPSC to global optimality.

Keywords: sparsity constraint, M-stationarity, M-index, nondegeneracy, genericity,
Morse theory, saddle points

1 Introduction

We consider the following mathematical program with sparsity constraint:

\[
\text{MPSC}: \min_{x \in \mathbb{R}^n} f(x) \quad \text{s. t.} \quad \|x\|_0 \leq s,
\]

where the so-called \( \ell_0 \) ”norm” counts non-zero entries of \( x \):

\[
\|x\|_0 = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|,
\]

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the objective function \( f \in C^2(\mathbb{R}^n, \mathbb{R}) \) is twice continuously differentiable, and \( s \in \{0, 1, \ldots, n - 1\} \) is an integer. The difficulty of solving MPSC comes from the combinatorial nature of the sparsity constraint \( \|x\|_0 \leq s \). The requirement of sparsity is however motivated by various applications, such as compressed sensing, model selection, image processing etc. We refer e.g. to \cite{Donoho2006, Tibshirani1996, Shechtman2011} for further details on the relevant applications.

In the seminal paper \cite{Beck2013} necessary optimality conditions for MPSC have been stated. Namely, the notions of basic feasibility (BF-vector), \( L \)-stationarity and CW-minimality have been introduced and studied there. Note that the formulation of \( L \)-stationarity mimics the techniques from convex optimization by using the orthogonal projection on the MPSC feasible set. The notion of CW-minimum incorporates the coordinate-wise optimality along the axes. Based on both stationarity concepts, algorithms that find points satisfying these conditions have been developed. Those are the iterative hard thresholding method, as well as the greedy and partial sparse-simplex methods. In a series of subsequent papers \cite{Beck2016, Beck2018} elaborated the algorithmic approach for MPSC which is based on \( L \)-stationarity and CW-minimality.

Another line of research started with \cite{Burdakov2016}, where additionally smooth equality and inequality constraints have been incorporated into MPSC. For that, the authors coin the new term of mathematical programs with cardinality constraints. The key idea in \cite{Burdakov2016} is to provide a mixed-integer formulation whose standard relaxation still has the same solutions as MPSC. For the relaxation the notion of S-stationary points is proposed. S-stationarity corresponds to the standard Karush-Kuhn-Tucker condition for the relaxed program. The techniques applied follow mainly those for mathematical programs with cardinality constraints. In particular, an appropriate regularization method for solving MPSC is suggested. The latter is proved to converge towards so-called M-stationary points. M-stationarity corresponds to the standard Karush-Kuhn-Tucker condition of the tightened program, where zero entries of an MPSC feasible point remain locally vanishing. Further research in this direction is presented in a series of subsequent papers \cite{Cervinka2016, Bucher2018}, \cite{Cervinka2016}, \cite{Bucher2018}.

The goal of this paper is the study of MPSC from a topological point of view. The topological approach to optimization has been pioneered by \cite{Jongen1977, Jongen2000} for nonlinear programming problems, and successfully developed for mathematical programs with complementarity constraints, mathematical problems with vanishing constraints, general semi-infinite programming, bilevel optimization, semi-definite programming, disjunctive programming etc., see e.g. \cite{Shikhman2012} and references therein. The main idea of the topological approach is to identify stationary points which roughly speaking induce the global structure of the underlying optimization problem. They have not only to include minimizers, but also all kinds of saddle points – just in analogy to the unconstrained case. It turns out that for MPSC the concept of M-stationarity from \cite{Burdakov2016} is the adequate stationarity concept at least from the topological perspective. We outline our main findings and results:

1. We introduce nondegenerate M-stationary points along with their associated M-indices. The latter subsume as usual the quadratic part – the number of negative eigenvalues of the objective’s Hessian restricted to non-vanishing variables. As novelty, the sparsity constraint provides an addition to the M-index, namely, the difference between the bound and the current number of non-zero variables at a nondegenerate M-stationary point. We prove that all
M-stationary points are generically nondegenerate. In particular, it follows that all local minimizers of MPSC are nondegenerate with vanishing M-index, hence, the sparsity constraint is active. Note that M-stationary points with non-vanishing M-index correspond to saddle points. The local structure of MPSC around a nondegenerate M-stationary point is fully described just by its M-index, at least up to a differentiable change of coordinates.

2. We thoroughly discuss the relation of M-stationarity to S-stationarity, basic feasibility, and CW-minimality for MPSC. It turns out that nondegenerate M-stationary points may cause degeneracies of S-stationary points viewed as Karush-Kuhn-Tucker-points for the relaxed problem. Moreover, even under the cardinality constrained second-order sufficient optimality condition from [Bucher and Schwartz, 2018] assumed to hold at an S-stationary point, the corresponding M-stationary point does not need to be a nondegenerate local minimizer for MPSC.

As for CW-minima, we show that they are not stable with respect to data perturbations in MPSC. After an arbitrarily small $C^2$-perturbation of $f$ a locally unique CW-minimum may bifurcate into multiple CW-minima. More importantly, this bifurcation unavoidably causes the emergence of M-stationary points, being different from the CW-minima. Despite of this instability phenomenon, if a BF-vector and, hence, CW-minimum, happens to be nondegenerate as an M-stationary point, then the sparsity constraint is necessarily active.

3. We use the concept of M-stationarity in order to describe the global structure of MPSC. To this aim the study of topological properties of its lower level sets is undertaken. As in the standard Morse theory, see e.g. Milnor (1963), Goresky and MacPherson (1988), we focus on the topological changes of the lower level sets as their levels vary. Appropriate versions of deformation and cell-attachment theorems are shown to hold for MPSC. As a consequence, we derive a Morse relation for MPSC, which relates the numbers of local minimizers and M-stationary points of M-index equal to one. The appearance of such saddle points cannot be thus neglected from the perspective of global optimization. As novelty for MPSC, a saddle point may lead to more than two different local minimizers. This is in strong contrast with other nonsmooth optimization problems studied before, see e.g. Shikhman (2012), where a saddle point leads to at most two of them. We conclude that the relatively involved structure of saddle points is the source of well-known difficulty if solving MPSC to global optimality.

The paper is organized as follows. In Section 2 we discuss the notion of M-stationarity for MPSC. Section 3 is devoted to the relation of M-stationarity to other stationarity concepts from the literature. In Section 4 the global structure of MPSC is described within the scope of Morse theory.

Our notation is standard. The cardinality of a finite set $S$ is denoted by $|S|$. The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$ with the coordinate vectors $e_i$, $i = 1, \ldots, n$. For $J \subset \{1, \ldots, n\}$ we denote by $\text{conv}(e_j, j \in J)$ the convex combination of the coordinate vectors $e_j, j \in J$. Given a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $Df$ denotes its gradient as a row vector, and $D^2 f$ stands for its Hessian.
2 M-stationarity

For $0 \leq k \leq n$ we use the notation

$$\mathbb{R}^{n,k} = \{ x \in \mathbb{R}^n \mid \| x \|_0 \leq k \}.$$ 

Using the latter, the feasible set of MPSC can be written as

$$\mathbb{R}^{n,s} = \{ x \in \mathbb{R}^n \mid \| x \|_0 \leq s \}.$$ 

For a feasible point $x \in \mathbb{R}^{n,s}$ we define the following complementary index sets:

$$I_0(x) = \{ i \in \{1, \ldots, n\} \mid x_i = 0 \}, \quad I_1(x) = \{ i \in \{1, \ldots, n\} \mid x_i \neq 0 \}.$$ 

Without loss of generality, we assume throughout the whole paper that at the particular point of interest $\bar{x} \in \mathbb{R}^{n,s}$ with $\| \bar{x} \|_0 = k$ it holds:

$$I_0(\bar{x}) = \{1, \ldots, n-k\}, \quad I_1(\bar{x}) = \{n-k+1, \ldots, n\}.$$ 

Using this convention, the following local description of MPSC feasible set can be deduced. Let $\bar{x} \in \mathbb{R}^{n,s}$ be a feasible point for MPSC with $\| \bar{x} \|_0 = k$. Then, there exist neighborhoods $U_{\bar{x}}$ and $V_0$ of $\bar{x}$ and 0, respectively, such that under the linear coordinate transformation $\Phi(x) = x - \bar{x}$ we have:

$$\Phi(\mathbb{R}^{n,s} \cap U_{\bar{x}}) = (\mathbb{R}^{n-k,s-k} \times \mathbb{R}^k) \cap V_0, \quad \Phi(\bar{x}) = 0. \quad (1)$$ 

**Definition 1 (M-stationarity, Burdakov et al. (2016))** A feasible point $\bar{x} \in \mathbb{R}^{n,s}$ is called M-stationary for MPSC if

$$\frac{\partial f}{\partial x_i}(\bar{x}) = 0 \text{ for all } i \in I_1(\bar{x}).$$ 

Obviously, a local minimizer of MPSC is an M-stationary point.

**Definition 2 (Nondegenerate M-stationarity)** An M-stationary point $\bar{x} \in \mathbb{R}^{n,s}$ with $\| \bar{x} \|_0 = k$ is called nondegenerate if the following conditions hold:

- **ND1:** if $k < s$ then $\frac{\partial f}{\partial x_i}(\bar{x}) \neq 0$ for all $i \in I_0(\bar{x})$,

- **ND2:** the matrix $\left( \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) \right)_{i,j \in I_1(\bar{x})}$ is nonsingular.

Otherwise, we call $\bar{x}$ degenerate.

**Definition 3 (M-Index)** Let $\bar{x} \in \mathbb{R}^{n,s}$ be a nondegenerate M-stationary point with $\| \bar{x} \|_0 = k$. The number of negative eigenvalues of the matrix $\left( \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) \right)_{i,j \in I_1(\bar{x})}$ is called its quadratic index ($QI$). The number $s - k + QI$ is called the M-index of $\bar{x}$.
Theorem 1 (Morse-Lemma for MPSC) Suppose that \( \bar{x} \) is a nondegenerate M-stationary point for MPSC with \( \|\bar{x}\|_0 = k \) and quadratic index \( \text{QI} \). Then, there exist neighborhoods \( U_{\bar{x}} \) and \( V_0 \) of \( \bar{x} \) and 0, respectively, and a local \( C^1 \)-coordinate system \( \Psi : U_{\bar{x}} \to V_0 \) of \( \mathbb{R}^n \) around \( \bar{x} \) such that:

\[
f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{n-k} y_i + \sum_{j=n-k+1}^{n} \pm y_j^2,
\]

where \( y \in \mathbb{R}^{n-k,s-k} \times \mathbb{R}^k \). Moreover, there are exactly \( \text{QI} \) negative squares in \( \mathbb{R}_{\mathbb{R}}^n \).

**Proof:**

Without loss of generality, we may assume \( f(\bar{x}) = 0 \). By using \( \Phi \) from (1), we put \( \tilde{f} := f \circ \Phi^{-1} \) on the set \( (\mathbb{R}^{n-k,s-k} \times \mathbb{R}^k) \cap V_0 \). At the origin we have:

(i) if \( k < s \) then \( \frac{\partial \tilde{f}}{\partial y_i} \neq 0 \) for all \( i = 1, \ldots, n-k \),

(ii) \( \frac{\partial \tilde{f}}{\partial y_i} = 0 \) for all \( i = n-k+1, \ldots, n \),

(iii) the matrix \( \left( \frac{\partial^2 \tilde{f}}{\partial y_i \partial y_j} \right)_{i,j=n-k+1,...,n} \) is nonsingular.

We denote \( \tilde{f} \) by \( f \) again. Under the following coordinate transformations the set \( \mathbb{R}^{n-k,s-k} \times \mathbb{R}^k \) will be equivariantly transformed in itself. We put \( y = (Y_{n-k},Y^k) \), where \( Y_{n-k} = (y_1, \ldots, y_{n-k}) \) and \( Y^k = (y_{n-k+1}, \ldots, y_n) \). It holds:

\[
f(Y_{n-k}, Y^k) = \int_0^1 \frac{d}{dt} f(t Y_{n-k}, Y^k) \, dt + f(0,Y^k) = \sum_{i=1}^{n-k} y_i d_i(y) + f(0,Y^k),
\]

where \( d_i \in C^1, i = 1, \ldots, k \).

Due to (ii)-(iii), we may apply the standard Morse lemma on the \( C^2 \)-function \( f(0,Y^k) \) without affecting the coordinates \( Y_{n-k} \), see e.g.

in Jongen et al. (2000). The corresponding coordinate transformation is of class \( C^1 \). Denoting the transformed functions again by \( f \) and \( d_i \), we obtain

\[
f(y) = \sum_{i=1}^{n-k} y_i d_i(y) + \sum_{j=n-k+1}^{n} \pm y_j^2.
\]

In case \( k = s \), we need to consider \( f \) locally around the origin on the set

\[
\mathbb{R}^{n-k,s-k} \times \mathbb{R}^k = \mathbb{R}^{n-k,0} \times \mathbb{R}^k = \{0\}^{n-k} \times \mathbb{R}^k.
\]

Hence, \( y_i = 0 \) for \( i = 1, \ldots, n-k \), and we immediately obtain the representation (2).

In case \( k < s \), (i) provides that \( d_i(0) = \frac{\partial f}{\partial y_i}(0) \neq 0, i = 1, \ldots, n-k \). Hence, we may take

\[y_i d_i(y), i = 1, \ldots, n-k, \quad y_j, j = n-k+1, \ldots, n\]

as new local \( C^1 \)-coordinates by a straightforward application of the inverse function theorem. Denoting the transformed function again by \( f \), we obtain (2). Here, the coordinate transformation \( \Psi \) is understood as the composition of all previous ones. \( \square \)
Proposition 1 (Nondegenerate minimizers) Let \( \bar{x} \) be a nondegenerate M-stationary point for MPSC. Then, \( \bar{x} \) is a local minimizer for MPSC if and only if its M-index vanishes.

Proof:
Let \( \bar{x} \) be a nondegenerate M-stationary point for MPSC. The application of Morse Lemma from Theorem 1 says that there exist neighborhoods \( U_{\bar{x}} \) and \( V_0 \) of \( \bar{x} \) and 0, respectively, and a local \( C^1 \)-coordinate system \( \Psi : U_{\bar{x}} \to V_0 \) of \( \mathbb{R}^n \) around \( \bar{x} \) such that:

\[
f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{n-k} y_i + \sum_{j=n-k+1}^{n} \pm y_j^2,
\]

where \( y \in \mathbb{R}^{n-k,s-k} \times \mathbb{R}^k \). Therefore, \( \bar{x} \) is a local minimizer for MPSC if and only if 0 is a local minimizer of \( f \circ \Psi^{-1} \) on the set \( (\mathbb{R}^{n-k,s-k} \times \mathbb{R}^k) \cap V_0 \). If the M-index of \( \bar{x} \) vanishes, we have \( k = s \) and \( QI = 0 \), and (3) reads as

\[
f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{j=n-s+1}^{n} y_j^2,
\]

where \( y \in \{0\}^{n-s} \times \mathbb{R}^s \). Thus, 0 is a local minimizer for (4). Vice versa, if 0 is a local minimizer for (3), then obviously \( k = s \) and \( QI = 0 \), hence, the M-index of \( \bar{x} \) vanishes.

Let \( C^2(\mathbb{R}^n, \mathbb{R}) \) be endowed with the strong (or Whitney) \( C^2 \)-topology, denoted by \( C^2_s \) (see e.g. Hirsch (1976)). The \( C^2_s \)-topology is generated by allowing perturbations of the functions, their gradients and Hessians, which are controlled by means of continuous positive functions. We say that a set is \( C^2_s \)-generic if it contains a countable intersection of \( C^2_s \)-open and -dense subsets. Since \( C^2(\mathbb{R}^n, \mathbb{R}) \) endowed with the \( C^2_s \)-topology is a Baire space, generic sets are in particular dense.

Theorem 2 (Genericity for MPSC) Let \( \mathcal{F} \subset C^2(\mathbb{R}^n, \mathbb{R}) \) denote the subset of objective functions in MPSC for which each M-stationary point is nondegenerate. Then, \( \mathcal{F} \) is \( C^2_s \)-open and -dense.

Proof:
Let us fix a number of non-zero entries \( k \in \{0, \ldots, s\} \), an index set of \( k \) non-zero entries \( D \subset \{1, \ldots, n\} \), i.e. \( |D| = k \), an index subset of zero entries \( E \subset \{1, \ldots, n\} \setminus D \), and a rank \( r \in \{0, \ldots, k\} \). For this choice we consider the set \( \Gamma_{k,D,E,r} \) of \( x \) such that the following conditions are satisfied:

\begin{enumerate}
  \item[(m1)] \( x_i \neq 0 \) for all \( i \in D \), and \( x_i = 0 \) for all \( i \in \{1, \ldots, n\} \setminus D \),
  \item[(m2)] \( \frac{\partial f}{\partial x_i}(x) = 0 \) for all \( i \in D \),
  \item[(m3)] if \( k < s \) then \( \frac{\partial f}{\partial x_i}(x) = 0 \) for all \( i \in E \),
  \item[(m4)] the matrix \( \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{i,j \in D} \) has rank \( r \).
\end{enumerate}
Note that (m1) refers to feasibility, (m2) to M-stationarity, and (m3)-(m4) describe possible violations of ND1-ND2, respectively.

Now, it suffices to show that all $\Gamma_{k,D,E,r}$ are generically empty whenever $E$ is nonempty or the rank $r$ is less than $k$. By setting $I_1(x) = D$ and $I_0(x) = \{1, \ldots, n\} \setminus D$, this would mean, respectively, that at least one of the derivatives $\frac{\partial f}{\partial x_i}(x)$ vanishes for $i \in E \subset I_0(x)$ in ND1 if $k < s$, or the matrix $\left( \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{i,j \in I_1(x)}$ is singular in ND2. In fact, the available degrees of freedom of the variables involved in each $\Gamma_{k,D,E,r}$ are $n$. The loss of freedom caused by (m1) is $n - k$, and the loss of freedom caused by (m2) is $k$. Hence, the total loss of freedom is $n$. We conclude that a further nondegeneracy would exceed the total available degrees of freedom $n$. By virtue of the jet transversality theorem from Jongen et al. (2000), generically the sets $\Gamma_{k,D,E,r}$ must be empty.

For the openness result, we argue in a standard way. Locally, M-stationarity can be written via stable equations. Then, the implicit function theorem for Banach spaces can be applied to follow M-stationary points with respect to (local) $C^2$-perturbations of defining functions. Finally, a standard globalization procedure exploiting the specific properties of the strong $C^2_s$-topology can be used to construct a (global) $C^2_s$-neighborhood of problem data for which the nondegeneracy property is stable.

\[\blacksquare\]

**Theorem 3 (Genericity for minimizers)** Generically, all minimizer of MPSC are nondegenerate with the vanishing M-index.

**Proof:**

Note that every local minimizer of MPSC has to be M-stationary. Nondegenerate M-stationary points are generic by Theorem 2. Hence, generically, local minimizers are nondegenerate. Due to Proposition 4 they have vanishing M-index.

\[\blacksquare\]

By recalling Definition 3 of M-index, we deduce the following important Corollary 1 on the structure of minimizers for MPSC.

**Corollary 1 (Sparsity constraint at minimizers)** At each generic local minimizer $\bar{x} \in \mathbb{R}^{n,s}$ of MPSC the sparsity constraint is active, i.e. $\|\bar{x}\|_0 = s$.

### 3 Relation to other stationarity concepts

We relate M-stationarity to other well-known stationarity concepts for MPSC from the literature. First, we focus on S-stationarity introduced in Burdakov et al. (2016). Then, the notions of basic feasibility and CW-minimality from Beck and Eldar (2013) will be discussed.
3.1 S-stationarity

In Burdakov et al. (2016) the following observation has been made: \( \bar{x} \) solves MPSC if and only if there exists \( \bar{y} \) such that \((\bar{x}, \bar{y})\) solves the following mixed-integer program:

\[
\min_{x,y} f(x) \quad \text{s.t.} \quad \sum_{i=1}^{n} y_i \geq n - s, \quad y_i \in \{0, 1\}, \quad x_i y_i = 0, \quad i = 1, \ldots, n.
\]

(5)

Using the standard relaxation of the binary constraints \( y_i \in \{0, 1\} \), the authors arrive at the following continuous optimization problem:

\[
\min_{x,y} f(x) \quad \text{s.t.} \quad \sum_{i=1}^{n} y_i \geq n - s, \quad y_i \in [0, 1], \quad x_i y_i = 0, \quad i = 1, \ldots, n.
\]

(6)

As pointed out in Burdakov et al. (2016), MPSC and the optimization problem (6) are closely related: \( \bar{x} \) solves MPSC if and only if there exists a vector \( \bar{y} \) such that \((\bar{x}, \bar{y})\) solves (6). Additionally, the concept of S-stationarity is proposed for (6). For its formulation the following index sets are needed:

\[
I_{\pm 0}(\bar{x}, \bar{y}) = \{i \in \{1, \ldots, n\} \mid \bar{x}_i \neq 0, \bar{y}_i = 0\},
\]

\[
I_{00}(\bar{x}, \bar{y}) = \{i \in \{1, \ldots, n\} \mid \bar{x}_i = 0, \bar{y}_i = 0\}.
\]

Definition 4 (S-stationarity, Burdakov et al. (2016)) A feasible point \((\bar{x}, \bar{y})\) of (6) is called S-stationary if there exist real multipliers \(\gamma_1, \ldots, \gamma_n\), such that

\[
\nabla f(\bar{x}) + \sum_{i=1}^{n} \gamma_i e_i = 0, \quad \gamma_i = 0 \text{ for all } i \in I_{\pm 0}(\bar{x}, \bar{y}),
\]

(7)

and, additionally, it holds:

\[
\gamma_i = 0 \text{ for all } i \in I_{00}(\bar{x}, \bar{y}).
\]

In order to relate M- and S-stationarity, we introduce the canonical choice of the auxiliary variables \( \bar{y} \) for a feasible point \( \bar{x} \) of MPSC:

\[
\bar{y}_i = \begin{cases} 0, & \text{if } i \in I_1(\bar{x}), \\ 1, & \text{if } i \in I_0(\bar{x}). \end{cases}
\]

(8)

The auxiliary variables \( \bar{y} \) can be seen as counters of the zero elements of \( \bar{x} \). Note that \((\bar{x}, \bar{y})\) becomes feasible for (6).

Proposition 2 (M- and S-stationarity) If \((\bar{x}, \bar{y})\) is S-stationary for (6) then \( \bar{x} \) is M-stationary for MPSC. Vice versa, for any M-stationary point \( \bar{x} \) the canonical choice (8) of auxiliary variables \( \bar{y} \) provides an S-stationary point \((\bar{x}, \bar{y})\) for (6).

Proof:
Let \((\bar{x}, \bar{y})\) be S-stationary for (6). After a moment of reflection we see that \(I_{\pm 0}(\bar{x}, \bar{y}) = I_1(\bar{x})\) is the support of \( \bar{x} \), and (7) reads as the M-stationarity of \( \bar{x} \):

\[
\nabla_i f(\bar{x}) = 0 \text{ for all } i \in I_1(\bar{x}).
\]
Vice versa, let \( \bar{x} \) be an M-stationary point for MPSC with the canonical choice (8) of \( \bar{y} \). Then, 

\[(\bar{x}, \bar{y}) \text{ is feasible for (6), since }\]

\[
\sum_{i=1}^{n} \bar{y}_i = |I_0(\bar{x})| = n - |I_1(\bar{x})| \geq n - s.
\]

The last inequality is due to \( \|\bar{x}\|_0 \leq s \) or, equivalently, \( |I_1(\bar{x})| \leq s \). Moreover, by the choice of \( \bar{y} \) we have \( I_{\pm0}(\bar{x}, \bar{y}) = I_1(\bar{x}) \) and \( I_{00}(\bar{x}, \bar{y}) = \emptyset \). Thus, due to the M-stationarity of \( \bar{x} \), (7) is fulfilled, and \( (\bar{x}, \bar{y}) \) is S-stationary.

\[ \square \]

The importance of S-stationary points is due to the following Proposition 3.

**Proposition 3 (S-stationarity and KKT-points, Burdakov et al. (2016))** A feasible point \( (\bar{x}, \bar{y}) \) satisfies the Karush-Kuhn-Tucker condition if and only if it is S-stationary for (6).

Despite this appealing relation, nondegenerate M-stationary points of MPSC may cause degeneracies of the corresponding S-stationary points. This means that they become degenerate Karush-Kuhn-Tucker-points for (6), i.e. the linear independent constraint qualification, strict complementarity, or second-order regularity is violated. The appearance of these degeneracies is mainly due to the fact that the objective function in (6) does not depend on \( y \)-variables. We illustrate this phenomenon by means of the following Example 1.

**Example 1 (S-stationarity and degeneracies)** We consider the following MPSC with \( n = 2 \) and \( s = 1 \):

\[
\min_{x_1, x_2} (x_1 - 1)^2 + (x_2 - 1)^2 \quad s.t. \quad \| (x_1, x_2) \|_0 \leq 1.
\]

It is easy to see that the feasible point \( \bar{x} = (0, 0) \) is M-stationary with \( \| \bar{x} \|_0 = k = 0 \). Moreover, it is nondegenerate with quadratic index \( QI = 0 \). For its M-index we have

\[
s - k + QI = 1 - 0 + 0 = 1,
\]

meaning that \( \bar{x} \) is a saddle point which connects two minimizers \((1, 0)\) and \((0, 1)\). Further, by the canonical choice (5) of auxiliary \( y \)-variables, we obtain the corresponding S-stationary point \( (\bar{x}, \bar{y}) = (0, 0, 1, 1) \). Due to Proposition 3, \( (\bar{x}, \bar{y}) \) is also a Karush-Kuhn-Tucker-point for the optimization problem (6):

\[
\min_{x, y} (x_1 - 1)^2 + (x_2 - 1)^2 \quad s.t. \quad y_1 + y_2 \geq 1, \quad y_1, y_2 \in [0, 1], \quad x_1 y_1 = 0, \quad x_2 y_2 = 0.
\]

The gradients of the active constraints at \( (\bar{x}, \bar{y}) \) are linearly independent:

\[
\begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
\bar{y}_1 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
\bar{y}_2 \\
1
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
Hence, the linear independent constraint qualification holds at \((\bar{x}, \bar{y})\). Let us determine the unique Lagrange multipliers from the Karush-Kuhn-Tucker condition:

\[
\begin{pmatrix}
2(\bar{x}_1 - 1) \\
2(\bar{x}_2 - 1) \\
0 \\
0
\end{pmatrix} = \mu_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_1, \mu_2 \leq 0.
\]

We get \(\mu_1 = \mu_2 = 0\) and \(\lambda_1 = \lambda_2 = -2\). Hence, the strict complementarity is violated at \((\bar{x}, \bar{y})\). Finally, the tangential space on the feasible set vanishes at \((\bar{x}, \bar{y})\). Hence, the second derivative of the corresponding Lagrange function restricted to the tangential space is trivially nonsingular. This means that the second-order regularity is fulfilled at \((\bar{x}, \bar{y})\). Overall, we claim that \((\bar{x}, \bar{y})\) is a degenerate Karush-Kuhn-Tucker-point for \((\bar{g})\) due to the lack of strict complementarity. It remains to note that the degeneracy of S-stationary points \((\bar{x}, \bar{y})\) prevails if other choices of auxiliary \(y\)-variables are made.

\(\square\)

An attempt to define a tailored notion of nondegeneracy for S-stationary points of \((\bar{g})\) has been recently undertaken in Bucher and Schwartz (2018). Let us briefly recall their main idea. For that, the so-called CC-linearization cone \(L^{CC}(\bar{x}, \bar{y})\) at a feasible point \((\bar{x}, \bar{y})\) of \((\bar{g})\) is used, cf. Cervinka et al. (2016). Namely,

\[
(d_x, d_y) \in L^{CC}(\bar{x}, \bar{y}) \subset \mathbb{R}^n \times \mathbb{R}^n
\]

satisfies by definition the following conditions:

\[
\begin{cases}
\sum_{i=1}^{n} (d_y)_i \geq 0 & \text{if } \sum_{i=1}^{n} \bar{y}_i = n - s, \\
(d_y)_i = 0 & \text{for all } i \in I_{\leq 0}(\bar{x}, \bar{y}), \\
(d_y)_i \geq 0 & \text{for all } i \in I_{00}(\bar{x}, \bar{y}), \\
(d_y)_i \leq 0 & \text{for all } i \in I_{01}(\bar{x}, \bar{y}), \\
(d_x)_i = 0 & \text{for all } i \in I_{01}(\bar{x}, \bar{y}) \cup I_{0+}(\bar{x}, \bar{y}), \\
(d_x)_i (d_y)_i = 0 & \text{for all } i \in I_{00}(\bar{x}, \bar{y}).
\end{cases}
\]

(9)

Here, the new index sets are

\[
I_{01}(\bar{x}, \bar{y}) = \{ i \in \{1, \ldots, n\} \mid \bar{x}_i = 0, \bar{y}_i = 1 \},
\]

\[
I_{0+}(\bar{x}, \bar{y}) = \{ i \in \{1, \ldots, n\} \mid \bar{x}_i = 0, \bar{y}_i \in (0, 1) \}.
\]

**Definition 5 (CC-SOSC, Bucher and Schwartz (2018))** Let \((\bar{x}, \bar{y})\) be an S-stationary point for \((\bar{g})\). If for all directions \((d_x, d_y) \in L^{CC}(\bar{x}, \bar{y})\) with \(d_x \neq 0\), we have

\[
d_x^T \cdot D^2 f(\bar{x}) \cdot d_x > 0,
\]

then the cardinality constrained second-order sufficient optimality condition (CC-SOSC) is said to
hold at \((\bar{x}, \bar{y})\).

The role of CC-SOSC can be seen from the following Proposition [1].

**Proposition 4 (Sufficient optimality condition, Bucher and Schwartz (2018))** Let \((\bar{x}, \bar{y})\) be an S-stationary point for (6) satisfying CC-SOSC. Then, \((\bar{x}, \bar{y})\) is a strict local minimizer of (6) with respect to \(x\), i.e.

\[ f(\bar{x}) < f(x) \]

for all feasible points \((x, y)\) of (6) taken sufficiently close to \((\bar{x}, \bar{y})\), and fulfilling \(x \neq \bar{x}\).

We relate the concepts of nondegeneracy for M-stationary points and of CC-SOSC for S-stationary points.

**Proposition 5 (Nondegeneracy and CC-SOSC)** Let \(\bar{x}\) be an M-stationary point for MPSC with \(\|\bar{x}\|_0 = s\). Assume that CC-SOSC holds at the S-stationary point \((\bar{x}, \bar{y})\) for (6) with the canonical choice (8) of auxiliary variables \(\bar{y}\). Then, \(\bar{x}\) is a nondegenerate local minimizer for MPSC.

**Proof:**

Due to the canonical choice (8) of auxiliary variables \(\bar{y}\), the index sets from the definition of the CC-linearization cone \(LCC(\bar{x}, \bar{y})\) are

\[ I_{\pm 0}(\bar{x}, \bar{y}) = I_1(\bar{x}), \quad I_{00}(\bar{x}, \bar{y}) = I_{0+}(\bar{x}, \bar{y}) = \emptyset, \quad I_{01}(\bar{x}, \bar{y}) = I_0(\bar{x}). \]

Due to \(\|\bar{x}\|_0 = s\), we additionally have \(\sum_{i=1}^{n} \bar{y}_i = n - s\). Recalling (9), \((d_x, d_y) \in LCC(\bar{x}, \bar{y})\) if and only if

\[
\begin{align*}
\sum_{i=1}^{n} (d_y)_i &\geq 0, \\
(d_y)_i &= 0 \text{ for all } i \in I_1(\bar{x}), \\
(d_y)_i &\leq 0 \text{ for all } i \in I_0(\bar{x}), \\
(d_x)_i &= 0 \text{ for all } i \in I_0(\bar{x}).
\end{align*}
\]

Hence, it holds:

\[ LCC(\bar{x}, \bar{y}) = \{(d_x, 0) \mid (d_x)_i = 0 \text{ for all } i \in I_0(\bar{x})\}, \]

so that CC-SOSC says that the matrix \(\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x})\) \(i,j \in I_1(\bar{x})\) is positive definite. By Definition [2] the M-stationary point \(\bar{x}\) is then nondegenerate with the vanishing quadratic index, i.e. \(QI = 0\). Thus, using again that \(\|\bar{x}\|_0 = s\), its M-index becomes \(s - s + QI = 0\). Finally, Proposition [1] provides the assertion. \(\square\)

If the sparsity constraint is not active for an M-stationary point \(\bar{x}\) of MPSC, i.e. \(\|\bar{x}\|_0 < s\), the implication in Proposition [5] does not hold in general anymore. Namely, \(\bar{x}\) does not need to be a local minimizer for MPSC, even if CC-SOSC holds at the corresponding S-stationary point \((\bar{x}, \bar{y})\) with the canonical choice (8) of auxiliary variables \(\bar{y}\). This is illustrated by means of the following Example [2].
Example 2 (Sparsity constraint and CC-SOSC) We consider the following MPSC with $n = 2$ and $s = 1$:

$$
\min_{x_1, x_2} x_1 + x_2 \quad \text{s.t.} \quad \| (x_1, x_2) \|_0 \leq 1.
$$

It is easy to see that the feasible point $\bar{x} = (0, 0)$ is M-stationary. Note that the sparsity constraint is not active for $\bar{x}$, since $k = \| \bar{x} \|_0 = 0 < 1 = s$. By the canonical choice of auxiliary $y$-variables, we obtain the corresponding S-stationary point $(\bar{x}, \bar{y}) = (0, 0, 1, 1)$. Analogously to the proof of Proposition 4 and by recalling (8), $(d_x, d_y) \in L^{CC} (\bar{x}, \bar{y})$ if and only if

\[
\begin{cases}
(d_y)_i = 0 \text{ for all } i \in I_1 (\bar{x}), \\
(d_y)_i \leq 0 \text{ for all } i \in I_0 (\bar{x}), \\
(d_x)_i = 0 \text{ for all } i \in I_0 (\bar{x}).
\end{cases}
\]

Note that here $I_1 (\bar{x}) = \emptyset$ and $I_0 (\bar{x}) = \{1, 2\}$. Hence, the CC-linearization cone is

$$
L^{CC} (\bar{x}, \bar{y}) = \{(0, d_y) \mid (d_y)_1, (d_y)_2 \leq 0\}.
$$

Overall, CC-SOSC trivially holds at $(\bar{x}, \bar{y})$, and as follows from Proposition 4, it is a strict local minimizer of (7) with respect to $x$. Nevertheless, $\bar{x}$ is not a local minimizer. Actually, it is a nondegenerate M-stationary point with the quadratic index $QI = 0$. For its M-index we have

$$
s - k + QI = 1 - 0 + 0 = 1.
$$

We conclude that $\bar{x}$ is rather a saddle point for MPSC.  

\[\blacksquare\]

3.2 Basic feasibility and CW-minimality

We proceed by discussing stationarity concepts from Beck and Eldar (2013). Inspired by linear programming terminology, they first introduce the notion of a basic feasible vector for MPSC.

Definition 6 (Basic feasibility, Beck and Eldar (2013)) A vector $\bar{x} \in \mathbb{R}^{n,s}$ with $\| \bar{x} \|_0 = k$ is called basic feasible (BF) for MPSC if the following conditions are fulfilled:

BF1: in case $k < s$, it holds:

$$
\frac{\partial f}{\partial x_i} (\bar{x}) = 0 \text{ for all } i = 1, \ldots, n,
$$

BF2: in case $k = s$, it holds:

$$
\frac{\partial f}{\partial x_i} (\bar{x}) = 0 \text{ for all } i \in I_1 (\bar{x}).
$$

Attention has been also paid to the notion of coordinate-wise minimum for MPSC.

Definition 7 (CW-minimality, Beck and Eldar (2013)) A vector $\bar{x} \in \mathbb{R}^{n,s}$ with $\| \bar{x} \|_0 = k$ is called coordinate-wise (CW) minimum for MPSC if the following conditions are fulfilled:

\[\blacksquare\]
**CW1:** in case \( k < s \), it holds:

\[
f(\bar{x}) = \min_{t \in \mathbb{R}} f(\bar{x} + te_i) \quad \text{for all } i = 1, \ldots, n,
\]

**CW2:** in case \( k = s \), it holds:

\[
f(\bar{x}) \leq \min_{t \in \mathbb{R}} f(\bar{x} - \bar{x}_i e_i + te_j) \quad \text{for all } i \in I_1(\bar{x}) \text{ and } j = 1, \ldots, n.
\]

Basic feasibility and CW-minimality can be viewed as necessary optimality condition for MPSC.

**Proposition 6 (BF-vector and CW-minimum, Beck and Eldar (2013))** Every global minimizer for MPSC is a CW-minimum, and every CW-minimum for MPSC is a BF-vector.

It is claimed in Beck and Eldar (2013) that the basic feasibility condition is quite weak, namely, there are many BF-points that are not optimal for MPSC. The notion of CW-minimum provides a much stricter necessary optimality condition. Based on the latter, a greedy sparse-simplex method for the numerical treatment of MPSC is proposed by Beck and Eldar (2013). Let us now examine the relation between M-stationarity, basic feasibility, and CW-minimality.

**Proposition 7 (M-stationarity, BF-vector, and CW-minimum)** Every BF-vector for MPSC is an M-stationary point, in particular, so is every CW-minimum.

**Proof:**

Let \( \bar{x} \) be a BF-vector for MPSC with \( \|\bar{x}\|_0 = k \). If \( k < s \), then BF1 implies M-stationarity of \( \bar{x} \). If \( k = s \), then BF2 coincides with the latter property. Since every CW-minimum for MPSC is a BF-vector according to Proposition 6, the assertion follows.

Proposition 7 says that M-stationarity is an even weaker condition than basic feasibility and CW-minimality. Why should we care about M-stationarity then? Is it not enough to rather focus on the stricter necessary optimality condition of CW-minimality as in Beck and Eldar (2013)? It turns out that CW-minima need not to be stable with respect to data perturbations. Namely, after an arbitrarily small \( C^2 \)-perturbation of \( f \) a locally unique CW-minimum may bifurcate into multiple CW-minima. More importantly, this bifurcation unavoidably causes the emergence of M-stationary points, being different from CW-minima. Next Example 3 illustrates this instability phenomenon.

**Example 3 (CW-minimum and instability)** We consider the following MPSC with \( n = 2 \) and \( s = 1 \):

\[
\min_{x_1, x_2} x_1^2 + x_2^2 \quad \text{s. t. } \| (x_1, x_2) \|_0 \leq 1.
\]

(10)

Obviously, \( \bar{x} = (0, 0) \) is the unique minimizer of (10). Due to Proposition 6 it is also a CW-minimum, as well as a BF-vector. Further, let us perturb (10) by using an arbitrarily small \( \varepsilon > 0 \) as follows:

\[
\min_{x_1, x_2} (x_1 - \varepsilon)^2 + (x_2 - \varepsilon)^2 \quad \text{s. t. } \| (x_1, x_2) \|_0 \leq 1.
\]

(11)
It is easy to see that the perturbed problem (11) has now two solutions $\bar{x}^1 = (\varepsilon, 0)$ and $\bar{x}^2 = (0, \varepsilon)$. Both are CW-minima, and, hence, BF-points. Here, we observe a bifurcation of the CW-minimum $\bar{x}$ of the original problem (10) into two CW-minima $\bar{x}^1$ and $\bar{x}^2$ of the perturbed problem (11). Let us explain this bifurcation in terms of M-stationarity. The bifurcation is caused by the degeneracy of $\bar{x}$ viewed as an M-stationary point of the original problem (10). Note that ND1 is violated at the M-stationary point $\bar{x}$ of the original problem (10). More interestingly, although $\bar{x}$ is neither a CW-minimum nor a BF-vector of (11) anymore, it becomes a new M-stationary point for the perturbed problem. In fact, due to $\|\bar{x}\|_0 = k = 0$ and the validity of ND1, $\bar{x}$ is a nondegenerate M-stationary point of (11) with the quadratic index $QI = 0$. For its M-index we have

$$s - k + QI = 1 - 0 + 0 = 1,$$

meaning that $\bar{x}$ is a saddle point which connects two nondegenerate minimizers $\bar{x}^1$ and $\bar{x}^2$ of (11). Overall, we conclude that the degenerate CW-minimum $\bar{x}$ of the original problem (10) is not stable. Moreover, it bifurcates into two nondegenerate CW-minima $\bar{x}^1$ and $\bar{x}^2$, as well as leads to one nondegenerate saddle point $\bar{x}$ of the perturbed problem (11). $\square$

Example 3 suggests to consider nondegenerate BF-vectors or nondegenerate CW-minima for MPSC, in order to guarantee their stability with respect to sufficiently small data perturbations. Then, however, the sparsity constraint turns out to be active. This means that BF1 in Definition 6 and CW1 in Definition 7 become redundant.

**Proposition 8 (BF-vector, CW-minimum and nondegeneracy)** Let $\bar{x}$ be a BF-vector for MPSC with $\|\bar{x}\|_0 = k$. If it is nondegenerate as an M-stationary point for MPSC, then $k = s$. The same applies for CW-minima.

**Proof:**
Assume that $k < s$, then ND1 contradicts BF1, whenever $I_0(\bar{x}) \neq \emptyset$. Otherwise, we have $k = n$, and, hence, $n < s$, a contradiction. It remains to note that every CW-minimum for MPSC is a BF-vector due to Proposition 6. $\square$

## 4 Global results

Let us study the topological properties of lower level sets

$$M^a = \{ x \in \mathbb{R}^{n,s} \mid f(x) \leq a \},$$

where $a \in \mathbb{R}$ is varying. For that, we define intermediate sets for $a < b$:

$$M^b_a = \{ x \in \mathbb{R}^{n,s} \mid a \leq f(x) \leq b \}.$$

For the topological concepts used below we refer to Spanier (1966).
Let us start with Assumption 1 which is usual within the scope of Morse theory, cf. [Goresky and MacPherson (1988)]. It prevents from considering asymptotic effects at infinity.

**Assumption 1** The restriction of the objective function $f_{|\mathbb{R}^{n,s}}$ on the MPSC feasible set is proper, i.e. $f^{-1}(K) \cap \mathbb{R}^{n,s}$ is compact for any compact set $K \subset \mathbb{R}$.

**Theorem 4 (Deformation for MPSC)** Let Assumption 1 be fulfilled and $M^b_a$ contain no M-stationary points for MPSC. Then, $M^a$ is homeomorphic to $M^b$.

**Proof:**
We apply Proposition 3.2 from Part I in [Goresky and MacPherson (1988)]. The latter provides the deformation for general Whitney stratified sets with respect to critical points of proper maps. Note that the MPSC feasible set admits a Whitney stratification:

$$\mathbb{R}^{n,s} = \bigcup_{I \subset \{1, \ldots, n\}} \bigcup_{J \subset I, |J| \leq s} Z_{I,J},$$

where

$$Z_{I,J} = \{x \in \mathbb{R}^n \mid x_I^c = 0, x_J > 0, x_{I \setminus J} < 0\}.$$

The notion of criticality used in [Goresky and MacPherson (1988)] can be stated for MPSC as follows. A point $\bar{x} \in \mathbb{R}^{n,s}$ is called critical for $f_{|\mathbb{R}^{n,s}}$ if it holds:

$$Df(\bar{x})_{|T_{\bar{x}}Z} = 0,$$

where $Z$ is the stratum of $\mathbb{R}^{n,s}$ which contains $\bar{x}$, and $T_{\bar{x}}Z$ is the tangent space of $Z$ at $\bar{x}$. By identifying $I = I_1(\bar{x})$ and, hence, $I^c = I_0(\bar{x})$, we see that the concepts of criticality and M-stationarity coincide. This concludes the assertion. \[\square\]

Let us now turn our attention to the topological changes of lower level sets when passing an M-stationary level. Traditionally, they are described by means of the so-called cell-attachment. We first consider a special case of cell-attachment. For that, let $N^\epsilon$ denote the lower level set of a special linear function on $\mathbb{R}^{p,q}$, i.e.

$$N^\epsilon = \left\{x \in \mathbb{R}^{p,q} \mid \sum_{i=1}^{p} x_i \leq \epsilon \right\},$$

where $\epsilon \in \mathbb{R}$, and the integers $q < p$ are nonnegative.

**Lemma 1 (Normal Morse data)** For any $\epsilon > 0$ the set $N^\epsilon$ is homotopy-equivalent to $N^{-\epsilon}$ with \((p-1)_{\frac{q}{p}}\) cells of dimension $q$ attached. The latter cells are the $q$-dimensional simplices from the collection

$$\{ \text{conv}(e_j, j \in J) \mid J \subset \{1, \ldots, p\}, 1 \in J, |J| = q + 1 \}.$$

**Proof:**
Let $N_\epsilon$ denote the upper level set of a special linear function on $\mathbb{R}^{p,q}$, i.e.

$$N_\epsilon = \left\{ x \in \mathbb{R}^{p,q} \left| \sum_{i=1}^{p} x_i \geq \epsilon \right. \right\}.$$

In terms of upper level sets Lemma 1 can be obviously reformulated as follows: For any $\epsilon > 0$ the set $N_{-\epsilon}$ is homotopy-equivalent to $N_\epsilon$ with $(p-1)$ cells of dimension $q$ attached. Let us show the latter assertion.

First, we note that the sets $N_0$ and $N_{-\epsilon}$ are contractible. The contraction is performed via the mapping

$$(x, t) \mapsto (1 - t) \cdot x, \quad t \in [0,1].$$

For the lower level set $N^\epsilon$ we have the representation

$$N^\epsilon = \bigcup_{J \subset \{1, \ldots, p\}, |J| = q} N^\epsilon,J,$$

where

$$N^\epsilon,J = \left\{ x \in \mathbb{R}^{p,q} \left| x_{J^c} = 0, \sum_{i \in J} x_i \geq \epsilon \right. \right\}.$$

Note that $N^\epsilon,J$ is homotopy-equivalent to the set $N^J$, where

$$N^J = \left\{ x \in \mathbb{R}^{p,q} \left| x_{J^c} = 0, \sum_{i \in J} x_i = 1 \right. \right\}$$

is the $(|J| - 1)$-dimensional simplex $\mathrm{conv}(e_j, j \in J)$ of $\mathbb{R}^p$. In fact, the map

$$(x, t) \mapsto t \cdot \frac{x}{p \sum_{i=1}^{p} x_i} + (1 - t) \cdot x, \quad t \in [0,1]$$

can be used for all $N^J$. Altogether, $N_\epsilon$ is homotopy-equivalent to

$$\bigcup_{J \subset \{1, \ldots, p\}, |J| = q} \mathrm{conv}(e_j, j \in J). \quad (12)$$

Note that the set in $(12)$ is the $(q - 1)$-skeleton of the $(p - 1)$-dimensional simplex of $\mathbb{R}^p$. The $(q - 1)$-skeleton of the $(p - 1)$-dimensional simplex is the union of its simplices up to dimension $q - 1$, see e.g. Goerss and Jardine (2009).

Within the $(q - 1)$-skeleton $(12)$, we close all $q$-dimensional holes by attaching $q$-dimensional cells from the collection of simplices

$$\left\{ \mathrm{conv}(e_j, j \in J) \mid J \subset \{1, \ldots, p\}, |J| = q + 1 \right\}.$$
The attachment should result in a contractible set, as it is actually $N_0$. We note that the union of the subdivision
\[ \{ \text{conv}(e_j, j \in J) \mid J \subseteq \{1, \ldots, p\}, 1 \in J, |J| = q + 1 \} \]
(13)
is also contractible, namely, to $e_1$. To see this, we may use the map
\[ (x, t) \mapsto t \cdot e_1 + (1 - t) \cdot x, \quad t \in [0, 1]. \]
Furthermore, none of the relative interiors of the simplices in (13) can be deleted. In fact, deleting
\[ \text{gives rise to the boundary of a } q \text{-dimensional simplex and the latter is not contractible.} \]
On the other hand, for any $J^* \subset \{1, \ldots, p\} \setminus \{1\}$ with $|J^*| = q + 1$
the union
\[ \text{conv}(e_j, j \in J^*) \cup \bigcup_{J^{**} \subset J^*, |J^{**}| = q} \text{conv}(e_j, j \in J^{**} \cup \{1\}) \]
(14)
forms the boundary of the $(q + 1)$-dimensional simplex $\text{conv}(e_j, j \in J^* \cup \{1\})$. Hence, the set in (14) is not contractible. Altogether, precisely the $q$-dimensional cells in (13) can be attached to the
$(q - 1)$-skeleton (12) in order to obtain a contractible set. Its number obviously equals $\binom{p-1}{q}$. This completes the proof.

\[ \text{Theorem 5 (Cell-Attachment for MPSC)} \]
Let Assumption \[ \text{be fulfilled and } M^b_a \text{ contain exactly one M-stationary point } \bar{x} \text{ for MPSC with } \| \bar{x} \|_0 = k \text{ and the M-index equal to } s - k + QI. \]
If $a < f(\bar{x}) < b$, then $M^b$ is homotopy-equivalent to $M^a$ with \( \binom{n-k-1}{s-k} \) cells of dimension $s - k + QI$ attached, namely:
\[ \bigcup_{J \subset \{1, \ldots, n-k\}, 1 \in J, |J| = s - k + 1} \text{conv}(e_j, j \in J) \times [0, 1]^{QI}. \]
Proof:
Theorem \[ \text{allows deformations up to an arbitrarily small neighborhood of the M-stationary point } \bar{x}. \]
In such a neighborhood, we may assume without loss of generality that $\bar{x} = 0$ and $f$ has the following form as from Theorem \[ \text{Goresky and MacPherson (1988)}
\]
\[ f(x) = f(\bar{x}) + \sum_{i=1}^{n-k} x_i + \sum_{j=n-k+1}^{n} \pm x_j^2, \]
(15)
where $x \in \mathbb{R}^{n-k,s-k} \times \mathbb{R}^k$, and the number of negative squares in (15) equals $QI$. In terms of \[ \text{Goresky and MacPherson (1988)} \]
the set $\mathbb{R}^{n-k,s-k} \times \mathbb{R}^k$ can be interpreted as the product of the tangential part $\mathbb{R}^k$ and the normal part $\mathbb{R}^{n-k,s-k}$. The cell-attachment along the tangential part is standard. Analogously to the unconstrained case, one $QI$-dimensional cell has to be attached on $\mathbb{R}^k$. The cell-attachment along the normal part is more involved. Due to Lemma \[ \text{we need to attach } \binom{n-k-1}{s-k} \text{ cells on } \mathbb{R}^{n-k,s-k}, \text{ each of dimension } s - k. \]
Finally, we apply Theorem 3.7 from Part I in \[ \text{Goresky and MacPherson (1988)}, \]
which says that the local Morse data is the product of
tangential and normal Morse data. Hence, the dimensions of the attached cells add together. Here, we have then to attach \( \binom{n-k-1}{s-k} \) cells on \( \mathbb{R}^{n-k,s-k} \times \mathbb{R}^k \), each of dimension \( s-k + QI \).

Let us present a global interpretation of our results for MPSC. For that, we need to state another assumption. Following Assumption 2 is standard in the context of MPSC, cf. Beck and Eldar (2013), and gives a necessary condition for its solvability.

**Assumption 2** The restriction of the objective function \( f_{|\mathbb{R}^n} \) on the MPSC feasible set is lower bounded.

Now, we consider M-stationary points \( \bar{x} \) for MPSC with \( \|\bar{x}\|_0 = k \) and the M-index equal to one, thus, fulfilling \( s-k+QI=1 \). These so-called saddle points can be of two types:

(I) with active sparsity constraint and quadratic index equal to one, i.e.

\[ k = s, \quad QI = 1, \]

(II) with exactly \( s-1 \) non-zero entries and vanishing quadratic index, i.e.

\[ k = s-1, \quad QI = 0. \]

**Theorem 6 (Morse relation for MPSC)** Let Assumptions 1 and 2 be fulfilled, and all M-stationary points of MPSC be nondegenerate with pairwise different functional values of the objective function. Additionally, we assume that there exists a connected lower level set which contains all M-stationary points. Then, it holds:

\[
 r_I + (n-s)r_{II} \geq r - 1,
\]

where \( r \) is the number of local minimizers of MPSC, \( r_I \) and \( r_{II} \) are the numbers of M-stationary points with M-index equal to one, which correspond to the types (I) and (II), respectively.

**Proof:**

Let \( q_a \) denote the number of connected components of the lower level set \( M^a \). We focus on how \( q_a \) changes as \( a \in \mathbb{R} \) increases. Due to Theorem 4, \( q_a \) can change only if passing through a value corresponding to an M-stationary point \( \bar{x} \), i.e. \( a = f(\bar{x}) \). In fact, Theorem 4 allows homeomorphic deformations of lower level sets up to an arbitrarily small neighborhood of the M-stationary point \( \bar{x} \). Then, we have to estimate the difference between \( q_a \) and \( q_{a-\varepsilon} \), where \( \varepsilon > 0 \) is arbitrarily, but sufficiently small, and \( a = f(\bar{x}) \). This is done by a local argument. For that, let the M-index of \( \bar{x} \) be \( s-k+QI \) with \( \|\bar{x}\|_0 = k \). We use Theorem 5 which says that \( M^a \) is homotopy-equivalent to \( M^{a-\varepsilon} \) with a cell-attachment of

\[
 \bigcup_{J \subset \{1, \ldots, n-k\}, 1 \in J, |J| = s-k+1} \text{conv}(e_j, j \in J) \times [0,1)^{QI}.
\]

Let us distinguish the following cases:
1) \( \bar{x} \) is a local minimizer with vanishing M-index, i.e. \( k = s \) and \( Q_I = 0 \). Then, by (17) we attach to \( M^{a-\varepsilon} \) the cell \( \text{conv}(e_1) \) of dimension zero. Consequently, a new connected component is created, and it holds:

\[
q_a = q_{a-\varepsilon} + 1.
\]

2) \( \bar{x} \) is of type (I) with M-index equal to one, i.e. \( k = s \) and \( Q_I = 1 \). Then, by (17) we attach to \( M^{a-\varepsilon} \) the cell \( \text{conv}(e_1) \times [0, 1] \) of dimension one. Consequently, at most one connected component disappears, and it holds:

\[
q_{a-\varepsilon} - 1 \leq q_a \leq q_{a-\varepsilon}.
\]

This case is well known from nonlinear programming, see e.g. Jongen et al. (2000).

3) \( \bar{x} \) is of type (II) with M-index equal to one, i.e. \( k = s - 1 \) and \( Q_I = 0 \). Then, by (17) we attach to \( M^{a-\varepsilon} \) as many as \( n - s + 1 \) cells of dimension one, namely:

\[
\bigcup_{j = 2, \ldots, n - s + 1} \text{conv}(e_1, e_j).
\]

Consequently, at most \( n - s \) connected components disappear, and it holds:

\[
q_{a-\varepsilon} - (n - s) \leq q_a \leq q_{a-\varepsilon}.
\]

For illustration we refer to Figure 1. Case 3) is new and characteristic for MPSC.

4) \( \bar{x} \) is M-stationary with M-index greater than one, i.e. \( s - k + Q_I > 1 \). The boundary of the cell-attachment in (17) is

\[
\bigcup_{J \subset \{1, \ldots, n - k\}} (\partial \text{conv}(e_j, j \in J) \times [0, 1]^{Q_I}) \cup (\text{conv}(e_j, j \in J) \times \{0, 1\}^{Q_I}).
\]

The latter set is connected if \( s - k + Q_I > 1 \). Consequently, the number of connected components of \( M^a \) remains unchanged, and it holds:

\[
q_a = q_{a-\varepsilon}.
\]

Now, we proceed with the *global argument*. Assumption 2 implies that there exists \( c \in \mathbb{R} \) such that \( M^c \) is empty, thus, \( q_c = 0 \). Additionally, there exists \( d \in \mathbb{R} \) such that \( M^d \) is connected and contains all M-stationary points, thus, \( q_d = 1 \). Due to Assumption 1, \( M^c_d \) is compact, moreover, it contains all M-stationary points. Since nondegenerate M-stationary points are in particular isolated, we conclude that there must be finitely many of them. Let us now increase the level \( a \) from \( c \) to \( d \) and describe how the number \( q_a \) of connected components of the lower level sets \( M^a \) changes. It follows from the local argument that \( r \) new connected components are created, where \( r \) is the number of local minimizers for MPSC. Let \( q_I \) and \( q_{II} \) denote the actual number of disappearing connected components if passing the levels corresponding to M-stationary points of types (I) and (II), respectively. The local argument provides that at most \( r_I \) and \((n - s)r_{II}\)
connected components might disappear while doing so, i.e.

\[ q_I \leq r_I, \quad q_{II} \leq (n-s)r_{II}. \]

Altogether, we have:

\[ r - r_I - (n-s)r_{II} \leq r - q_I - q_{II} = q_d - q_c. \]

By recalling that \( q_d = 1 \) and \( q_c = 0 \), we get Morse relation (16).

\[ \square \]

We illustrate Theorem 6 by discussing the same MPSC as in Example 1.

**Example 4 (Saddle point)** We consider the following MPSC with \( n = 2 \) and \( s = 1 \):

\[
\min_{x_1, x_2} (x_1 - 1)^2 + (x_2 - 1)^2 \quad \text{s.t.} \quad \|(x_1, x_2)\|_0 \leq 1.
\]

As we have seen in Example 1, both M-stationary points \((1, 0)\) and \((0, 1)\) are nondegenerate minimizers. Thus, we have \( r = 2 \). Morse relation (16) from Theorem 2 provides:

\[ r_I + r_{II} \geq 1. \]

Hence, there should exist an additional M-stationary point with M-index one. In fact, \((0, 0)\) is this nondegenerate M-stationary point of type (II), cf. Example 1. Note that, due to \( r_I = 0 \) and \( r_{II} = 1 \), Morse relation (16) holds with equality here.

\[ \square \]

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