Magic mass ratios of complete energy-momentum transfer in one-dimensional elastic three-body collisions

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Abstract

We consider a scattering of two identical blocks of mass $M$ in one dimension by exchanging momenta through elastic collisions with a ball with mass $m = \alpha M$. Initially, a ball and a block are at rest with the distance of $L$ and the other block is incident on the ball. For $\alpha < 1$, the three objects make multiple collisions. The analytic expressions for the final velocities of the three-body problem are derived by making use of the conservation of energy and momentum. The rates for energy and momentum transfer are computed as functions of $\alpha$. We find all possible values for $\alpha$ at which the initial energy and momentum of the incident block are completely transferred to the scattered block.
I. INTRODUCTION

In a head-on elastic collision of two identical objects, the momenta of the two participants are exchanged in any reference frame. In a fixed-target frame, the target acquires the energy and momentum of the incident object that makes a complete stop. Such a complete energy-momentum transfer is hardly observed in many-body collisions. An interesting example is the Newton’s cradle which consists of a series of identical pendula that are aligned along a straight line on a horizontal level. However, this can actually be understood as a series of head-on collisions of two identical objects where small gaps between each pair of adjacent pendula are assumed.

Although this small-gap model explains the observations, this model fails if some of adjacent pendula are in contact with each other so that the process cannot be decomposed into a series of two-body collisions. The reason for this failure is that the conservation laws of energy and momentum are not enough to determine the final velocities. Additional conditions such as force laws between objects are required to have a unique solution of the problem. Various studies have been carried out to find an appropriate force laws governing the actual motions of the pendula. A phenomenological model is so called the contact Hertz force of the form $F = -kx^\frac{3}{2}$. Further applications of the contact Hertz force to various collision problems can be found in Refs. In a recent work, we have studied the bouncing of a block against a rigid wall through one-dimensional multiple elastic collisions with a ball sandwiched by them. Like the small-gap model, we assumed that each collision is instantaneous and the complete trajectories of the block and the ball were uniquely determined analytically. By taking the continuum limit of the exact analytic solution of the block trajectory, we have shown that the effective force carried by the ball is proportional to $1/x^3$, where $x$ is the distance between the block and the wall. This is consistent with previous results based on the differential equation that can be derived by taking the continuum limit of the energy-momentum conservation.

In this paper, we generalize the model system of Ref. to a simple three-body system: We consider the scattering of two identical blocks of mass $M$ through multiple elastic collisions with a ball of mass $m = \alpha M$ sandwiched by the blocks. As shown in Fig. 1, the ball $C$ and the target block $B$ are initially placed at $x = 0$ and $L$, respectively, and the block $A$ is incident to the target with the initial velocity $V$. While we were interested in the trajectories
of the objects and the effective force in Ref.\textsuperscript{14}, we focus on the energy-momentum transfer from the incident block $A$ to the scattered block $B$. If $\alpha < 1$, then the system experiences multiple collisions that can be understood as a series of two-body collisions and, therefore, the velocities of all participants are uniquely determined. As a result, we find all possible values for the magic mass ratios $\alpha = \alpha_k$ at which the energy and momentum of the incident block are completely transferred to the scattered block.

The chain collisions of multiple pendula in series have been studied previously by Hart \textit{et al.}\textsuperscript{2} and by Kerwin.\textsuperscript{3} However, the pendula studied in these references are arranged in mass order so that the complete energy-momentum transfer from the incident pendulum to the target pendulum at the other end is achieved only if all of the pendula are of equal mass like the Newton’s cradle. Redner considered a similar system consisting of two identical cannonballs approaching an initially stationary ping-pong ball.\textsuperscript{17} In that reference, the author mainly focused on deriving a simple relation between that elastic collision and a corresponding billiard system, which helps to obtain the total number of collisions of the system easily. Thus, to our best knowledge, the derivation of all possible magic mass ratios $\alpha_k$ at which complete energy-momentum transfer is realized in one-dimensional three-body elastic collisions is new.

This paper is organized as follows. In Sec. II, we describe the model system and provide the definitions of kinematic variables that we use throughout this paper. In Sec. III, we construct recurrence relations for the velocity sequences of participants and solve them to determine the velocity of each object after each collision. We also compute the total number of collisions for each participant and obtain critical values for the mass ratio $\alpha$ at which the number of collisions of each object changes. In Sec. IV, we provide the verification that at the magic mass ratio $\alpha = \alpha_k$ the energy and momentum of the incident block are completely transferred to the scattered block. We also show that at the defective mass ratio $\alpha = \beta_k$, the energy and momentum transfer rates reach their local minima. Finally, we conclude in Sec. V.

II. THE MODEL AND DEFINITIONS

In this section, we describe the model system and define kinematic variables that we use throughout this paper.
As shown in Fig. 1, the model system consists of two identical blocks $A$ and $B$ with mass $M$ and a ball $C$ with mass $m = \alpha M$, where $\alpha < 1$, aligned on the $x$ axis. Initially, $C$ and $B$ are placed at rest at $x = 0$ and $L$, respectively. At time $t = 0$, block $A$ with velocity $+V$ hits the ball on the left. Then $A$ and $B$ exchange momenta through multiple collisions with $C$. We assume all of the collisions are elastic and ignore friction.

We denote by $P_n$ ($Q_n$) the $n$th collision point between $A$ ($B$) and $C$. The velocities of the three participants are defined as follows: We denote $a_n$ ($b_n$) and $c_n$ ($c'_n$) by the velocities of block $A$ ($B$) and the ball, respectively, right after the collision at $P_n$ ($Q_n$). During the ball’s motion from $P_n$ to $Q_n$, the velocity of $C$ is fixed as $c_n$. Between $Q_n$ and $P_{n+1}$, the velocity of $C$ is $c'_n$.

It is convenient to define the column-vector sequence $V_n$ as

$$V_n \equiv (a_n, b_n, \sqrt{\alpha} c'_n)^T,$$

where the scaling factor $\sqrt{\alpha}$ of the third component of $V_n$ is introduced to simplify the following analyses. The initial condition is given by

$$V_0 = (V, 0, 0)^T.$$  \hfill (2)

### III. ANALYTIC COMPUTATION

In this section, we find the general terms of the velocity sequences $a_n$, $b_n$, $c_n$, and $c'_n$ that are defined in Sec. II, by solving the recurrence relations that are constructed from the conservation of energy and momentum. By making use of these analytic solutions, we calculate the total number of collisions of each participant, and obtain the critical values of $\alpha$ at which the number of collisions of each object changes. At the end of this section, we find the expressions of the terminal velocities of the participants.

#### A. Recurrence Relation

The collision at $P_n$ transforms the velocities $a_{n-1}$ and $c'_{n-1}$ into $a_n$ and $c_n$, respectively. The collision at $Q_n$ transforms the velocities $b_{n-1}$ and $c_n$ into $b_n$ and $c'_n$, respectively. The
conservation of linear momentum and kinetic energy in the collisions at \( P_n \) and \( Q_n \) requires

\[
a_{n-1} + \alpha c_{n-1}' = a_n + \alpha c_n, \quad (3a)
\]

\[
a_{n-1}^2 + [\sqrt{\alpha c_{n-1}'}]^2 = a_n^2 + [\sqrt{\alpha c_n}]^2, \quad (3b)
\]

\[
b_{n-1} + \alpha c_n = b_n + \alpha c_n', \quad (3c)
\]

\[
b_{n-1}^2 + [\sqrt{\alpha c_n}]^2 = b_n^2 + [\sqrt{\alpha c_n'}]^2. \quad (3d)
\]

These constraints derive the following recurrence relations:

\[
\begin{pmatrix}
a_n \\
\sqrt{\alpha c_n}
\end{pmatrix} = \Gamma \begin{pmatrix}
a_{n-1} \\
\sqrt{\alpha c_{n-1}'}
\end{pmatrix}, \quad (4a)
\]

\[
\begin{pmatrix}
b_n \\
\sqrt{\alpha c_n'}
\end{pmatrix} = \Gamma \begin{pmatrix}
b_{n-1} \\
\sqrt{\alpha c_n}
\end{pmatrix}, \quad (4b)
\]

where the \( 2 \times 2 \) matrix \( \Gamma \), which is independent of \( n \), is defined by

\[
\Gamma = \frac{1}{1 + \alpha} \begin{pmatrix} 1 - \alpha & 2\sqrt{\alpha} \\ 2\sqrt{\alpha} & -(1 - \alpha) \end{pmatrix}. \tag{5}
\]

According to Eqs. (3b) and (3d), we find that the transformation matrix \( \Gamma \) does not change the norms of vectors. Therefore, the matrix \( \Gamma \) can be parametrized by the product of a rotation matrix \( \lambda(\theta) \) and a reflection matrix \( \mathbb{P} \):

\[
\Gamma = \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix} = \lambda(\theta)\mathbb{P}, \tag{6}
\]

where \( \lambda(\theta) \) and \( \mathbb{P} \) are defined by

\[
\lambda(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}, \quad (7a)
\]

\[
\mathbb{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7b)
\]

Because \( \det[\lambda(\theta)] = 1 \) and \( \det[\mathbb{P}] = -1 \), \( \det[\Gamma] = -1 \). Here, the parameter \( \theta \) is fixed by the mass ratio \( \alpha \equiv m/M \):

\[
\theta = 2 \arctan \sqrt{\alpha}. \tag{8}
\]
Because we restrict ourselves to the case $0 < \alpha < 1$, the range of $\theta$ is $0 < \theta < \frac{\pi}{2}$. For a small $\alpha$, $\theta \approx 2\sqrt{\alpha}$. In the first two rows of Table I, we list the values for trigonometric functions at $\theta$ and $\frac{1}{2}\theta$, that are particularly useful in our analysis.

By making use of Eqs. (4) and (6), we can find the recurrence relation for the three-dimensional-Euclidean-vector sequence $V_n$ of the form

$$V_n = \Lambda V_{n-1}.$$  \hspace{1cm} (9)

Then we can determine the general term for $V_n$ in terms of the initial value $V_0$ in Eq. (2) as

$$V_n = \Lambda^n V_0.$$  \hspace{1cm} (10)

Next we find the matrix representation of the matrix $\Lambda$ in Eqs. (9) and (10). We first verify that the matrix $\Lambda$ is a $3 \times 3$ matrix for a pure rotation of the three-dimensional Euclidean vector. We combine the two relations in Eq. (4) to express $\Lambda$ as a product

$$\Lambda = \Gamma_B \Gamma_A,$$  \hspace{1cm} (11)

where $\Gamma_A$ and $\Gamma_B$ are the matrices that transform the velocities at $P_n$ and $Q_n$, respectively:

$$\Gamma_A = \lambda_2(-\theta) P_3 = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & -\cos \theta \end{pmatrix},$$ \hspace{1cm} (12a)

$$\Gamma_B = \lambda_1(\theta) P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & \sin \theta & -\cos \theta \end{pmatrix},$$ \hspace{1cm} (12b)

where $\theta$ is defined in Eq. (8) and $\lambda_i(\theta)$ is the rotation matrix about the axis $i$ by an angle $\theta$ and $P_3$ is the matrix that reflects the third component. Then the explicit form of the matrix $\Lambda$ is

$$\Lambda = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ \sin^2 \theta & \cos \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta & \cos^2 \theta \end{pmatrix} = \begin{pmatrix} \frac{1-\alpha}{1+\alpha} & 0 & \frac{2\sqrt{\alpha}}{1+\alpha} \\ \frac{4\alpha}{(1+\alpha)^2} & \frac{1-\alpha}{1+\alpha} & -\frac{2(1-\alpha)\sqrt{\alpha}}{(1+\alpha)^2} \\ \frac{2(1-\alpha)\sqrt{\alpha}}{(1+\alpha)^2} & \frac{1-\alpha}{1+\alpha} & \frac{(1-\alpha)^2}{(1+\alpha)^2} \end{pmatrix}.$$  \hspace{1cm} (13)
Because \( \det[\lambda_i(\theta)] = 1 \) and \( \det[P_3] = -1 \), \( \det[\Gamma_A] = \det[\Gamma_B] = -1 \) and \( \det[\Lambda] = 1 \). In addition, \( \Lambda \) is orthogonal: \( \Lambda^{-1} = \Lambda^T \). Therefore, \( \Lambda \) is a pure rotation matrix.

**B. Computation of \( \Lambda^n \)**

In this section, we derive the analytic expression for the matrix \( \Lambda^n \) that is necessary in computing \( V_n \) in Eq. (10). Because \( \Lambda \) is a matrix for a pure rotation in the three-dimensional Euclidean space, there exists a Cartesian coordinate system in which the rotation is about \( \hat{e}_3 \) by an angle \( \psi \), where \( \hat{e}_i \) is the unit vector along the \( i \)th axis in the new coordinate system.

\[
\Lambda_R = \lambda_3(\psi) = \begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (14)

In the new coordinate system, \( \Lambda_R \) can be expressed as a single rotation about \( \hat{e}_3 \) by an angle \( n\psi \):

\[
\Lambda_R^n = \lambda_3(n\psi).
\] (15)

Then the matrix \( \Lambda \) can be expressed as

\[
\Lambda = R\lambda_3(\psi)R^T,
\]

\[
\Lambda^n = R\lambda_3(n\psi)R^T.
\] (16a) (16b)

where \( R = (\hat{e}_1, \hat{e}_2, \hat{e}_3) \), which is orthogonal: \( R^{-1} = R^T \).

By generalizing the method in Ref.\(^{14}\) for two dimensions into three dimensions, we can determine the transformation matrix \( R \). Because \( \hat{e}_3 \) is invariant under rotation \( \Lambda \), we have

\[
\Lambda\hat{e}_3 = \hat{e}_3.
\] (17)

The solution for this constraint equation is \( \hat{e}_3 = \frac{1}{\sqrt{2+\alpha}}(1, 1, \sqrt{\alpha})^T \). Choosing the remaining two bases for the Cartesian coordinate system as \( \hat{e}_1 = \sqrt{1 + \frac{1}{2}\alpha} \hat{z} \times \hat{e}_3 \) and \( \hat{e}_2 = \hat{e}_3 \times \hat{e}_1 \), we determine the triad of the new coordinate system. Here, \( \hat{z} \) is the unit vector along the third axis of the original Cartesian coordinates system in which the matrix representation of \( \Lambda \) is given in Eq. (13). As a result, we find that

\[
R = (\hat{e}_1, \hat{e}_2, \hat{e}_3) = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{\sqrt{\alpha}}{2(2+\alpha)} & \frac{1}{\sqrt{2+\alpha}} \\
\frac{1}{\sqrt{2}} & -\frac{\sqrt{\alpha}}{2(2+\alpha)} & \frac{1}{\sqrt{2+\alpha}} \\
0 & \frac{2}{\sqrt{2+\alpha}} & \frac{\sqrt{\alpha}}{2+\alpha}
\end{pmatrix}.
\] (18)
where the parameter $\psi$ is defined by

$$
\psi \equiv 2 \arctan \sqrt{\alpha (2 + \alpha)}.
$$

(19)

For $0 < \alpha < 1$, the range of $\psi$ is $0 < \psi < \frac{2}{3}\pi$. For a small $\alpha$, $\psi \approx 2\sqrt{2\alpha}$. In the last three rows of Table I, we list the values for trigonometric functions at $\psi$, $\frac{1}{2}\psi$, and $\frac{1}{4}\psi$, that are particularly useful in the following analysis.

C. General terms of velocity sequences

Substituting the initial condition $V_0$ in Eq. (2) and $\Lambda^n$, that we obtained in Sec. III B, into Eq. (10), we can determine $V_n$. The value of $c_n$ can be computed by substituting $a_{n-1}$, $a_n$, and $c'_{n-1}$ into Eq. (3a). The general terms for the velocity sequences are obtained as

$$
a_n = V_{CM} \left[ 1 + \cos \left( \frac{n\psi}{2} \right) \cos \frac{\psi}{2} \right],
$$

(20a)

$$
b_n = V_{CM} \left[ 1 - \frac{\cos(n + \frac{1}{2})\psi}{\cos \frac{\psi}{2}} \right],
$$

(20b)

$$
c_n = V_{CM} \left[ 1 + \frac{\sin(n - \frac{1}{4})\psi}{\sin \frac{\psi}{4}} \right],
$$

(20c)

$$
c'_n = V_{CM} \left[ 1 - \frac{\sin(n + \frac{1}{4})\psi}{\sin \frac{\psi}{4}} \right],
$$

(20d)

where $V_{CM}$ is the velocity of the center of mass:

$$
V_{CM} = \frac{V}{2 + \alpha} = \frac{V \cos \frac{\psi}{2}}{1 + \cos \frac{\psi}{2}}.
$$

(21)

Note that $(m + 2M)V_{CM} = MV$. In the derivation of the expressions in Eq. (20), we have used the values for trigonometric functions at $\psi$, $\frac{1}{2}\psi$, and $\frac{1}{4}\psi$ that are listed in the last three rows of Table I.

D. Special values for trigonometric functions

In order to determine the velocities in Eq. (20), we have to evaluate the trigonometric functions at special values such as $n\theta$, $(n + \frac{1}{2})\theta$, $n\psi$, $(n + \frac{1}{2})\psi$, and $(n + \frac{1}{4})\psi$. In this section, we summarize a way to evaluate them in terms of $\alpha$.
1. At angles $nx$

For $x = \theta$ or $\psi$, we can compute the values for $\cos nx$ and $\sin nx$ as

$$
\cos nx = \mathbb{R}e[e^{inx}] = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{(2k)!(n-2k)!} \sin^{2k} x \cos^{n-2k} x,
$$

(22a)

$$
\sin nx = \mathbb{I}m[e^{inx}] = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k n!}{(2k+1)!(n-2k-1)!} \sin^{2k+1} x \cos^{n-2k-1} x,
$$

(22b)

where the floor function $\lfloor x \rfloor$ is the integer $n$ such that $n \leq x < n + 1$. Special values for $\cos x$ and $\sin x$ for $\theta$ and $\psi$ are given in Table I in terms of $\alpha$.

2. At angles $(n + \frac{1}{4})x$ and $(n + \frac{1}{2})x$

For $x = \theta$ or $\psi$, we can compute the values for $\cos(n + r)x$ and $\sin(n + r)x$ for $r = \frac{1}{2}$ or $\frac{1}{4}$ as

$$
\cos(n + r)x = \cos nx \cos r x - \sin nx \sin r x,
$$

(23a)

$$
\sin(n + r)x = \sin nx \cos r x + \cos nx \sin r x,
$$

(23b)

where the values for $\cos \frac{1}{2}\theta$, $\sin \frac{1}{2}\theta$, $\cos \frac{1}{2}\psi$, $\sin \frac{1}{2}\psi$, $\cos \frac{1}{4}\psi$, and $\sin \frac{1}{4}\psi$ are given in Table I in terms of $\alpha$.

E. Total number of collisions

In this section, we compute the total number of collisions $N_i$ between the ball and the block $i$ for $i = A$ or $B$. In the following analysis, it is convenient to introduce the definitions of the ceiling ($\lceil x \rceil$), floor ($\lfloor x \rfloor$), and sawtooth ($\{ x \}$) functions. For any real number $x \in \mathbb{R}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ are defined by

$$
\lceil x \rceil = \{ n \in \mathbb{Z} | n - 1 < x \leq n, x \in \mathbb{R} \},
$$

(24a)

$$
\lfloor x \rfloor = \{ n \in \mathbb{Z} | n \leq x < n + 1, x \in \mathbb{R} \},
$$

(24b)

where $\mathbb{Z}$ is the set of integers. The sawtooth function is defined by

$$
\{ x \} = x - \lfloor x \rfloor,
$$

(25)
where $0 \leq \{x\} < 1$.

The collision at $P_n$ is allowed only if the velocity $a_{n-1}$ of $A$ is greater than that of the ball, $c'_{n-1}$. After the collision at $P_n$, the velocities of $A$ and $C$ become $a_n$ and $c_n$, respectively. If the ball $C$ experiences another collision with $B$, then it returns to $A$ with the velocity $c'_n$. If $n$ is the last collision number between $A$ and $C$, then $A$’s velocity right after $P_n$ must be smaller than or equal to that of $C$ right after $Q_n$:

$$a_n \leq c'_n. \quad (26)$$

The smallest integer that satisfies this inequality is $n = N_A$. Substituting the values for $a_n$ and $c'_n$ in Eq. (20) into Eq. (26) and making use of Table I, we find that

$$\sin(n - \frac{1}{2})\psi > 0 \quad \text{and} \quad \sin(n + \frac{1}{2})\psi \leq 0. \quad (27)$$

Thus, $N_A$ is the minimum value of $n$ satisfying $\sin(n + \frac{1}{2})\psi \leq 0$. The solution to Eq. (27) is

$$N_A - 1 < \frac{\pi}{\psi} - \frac{1}{2} \leq N_A: \quad N_A = \left\lceil \frac{\pi}{\psi} - \frac{1}{2} \right\rceil, \quad (28)$$

where the ceiling function $\lceil x \rceil$ is defined in Eq. (24a).

In a similar manner, we can find the constraint to $N_B$. The collision at $Q_n$ is allowed only if the velocity $c_n$ of the ball is greater than that of the block $B$, $b_{n-1}$. After the collision with $B$ at $Q_n$, $C$ makes another collision with $A$ at $P_{n+1}$ resulting in the velocity $c_{n+1}$. Therefore, if $n$ is the largest collision number $N_B$ between $C$ and $B$, then we have

$$c_{n+1} \leq b_n, \quad (29)$$

which leads to

$$\sin n\psi > 0 \quad \text{and} \quad \sin(n + 1)\psi \leq 0. \quad (30)$$

Then the solution for $N_B$ is determined as

$$N_B = \left\lceil \frac{\pi}{\psi} - 1 \right\rceil. \quad (31)$$

According to Eqs. (28) and (31),

$$N_A = \begin{cases} N_B = \lceil \pi/\psi \rceil - 1 & \text{for } 0 < \{\pi/\psi\} \leq \frac{1}{2}, \\ N_B + 1 = \lceil \pi/\psi \rceil, & \text{otherwise}. \end{cases} \quad (32)$$
where the sawtooth function \( \{x\} \) is defined in Eq. (25). If \( \alpha \) is small, we can make an approximation of Eq. (19) as \( \psi \approx 2\sqrt{2\alpha} \). In that case, we have
\[
N_A \approx N_B \approx \frac{\pi}{2\sqrt{2\alpha}} \text{ for a small } \alpha,
\]
which is consistent with a previous result given in Eq. (6) of Ref. [17], where the author counted the number of collisions on both sides of the ball.

**F. Critical values for mass ratio**

At the largest value of the mass ratio \( \alpha = 1 \), \( \psi = \frac{2}{3}\pi \). Therefore, \( N_A = N_B = \lceil 1/2 \rceil = 1 \) when \( \alpha = 1 \). \( N_A \) increases by unity at critical values as \( \alpha \) decreases. We define \( \alpha_k \) at which \( N_A = k \) that is \( \pi/\psi_k = k + \frac{1}{2} \), where \( \psi_k = \psi|_{\alpha=\alpha_k} \). For an \( \alpha \) that is infinitesimally smaller than \( \alpha_k \), \( \lceil (\pi/\psi) - \frac{1}{2} \rceil = k + 1 \). Therefore, \( \alpha_k \) is the minimum value of \( \alpha \) to have \( N_A = k \). \( N_B \) also increases by unity at critical values as \( \alpha \) decreases. We define \( \beta_k \) at which \( N_B = k \) that is \( \pi/\psi'_k = k + 1 \), where \( \psi'_k = \psi|_{\alpha=\beta_k} \). For an \( \alpha \) that is infinitesimally smaller than \( \beta_k \), \( N_B = \lceil (\pi/\psi) - 1 \rceil = k + 1 \). Therefore, \( \beta_k \) is the minimum value of \( \alpha \) to have \( N_B = k \). We call \( \alpha_k \) and \( \beta_k \) the \( k \)th critical values for \( N_A \) and \( N_B \), respectively. In summary,
\[
N_A = k \text{ for } \alpha_k \leq \alpha < \alpha_{k-1} \text{ for } k \geq 1,
\]
\[
N_B = k \text{ for } \beta_k \leq \alpha < \beta_{k-1} \text{ for } k \geq 1,
\]
where we set \( \alpha_0 = \infty \) and \( \beta_0 = \infty \). It is trivial to verify that the critical mass ratios are ordered as \( \cdots < \beta_2 < \alpha_2 < \beta_1 < \alpha_1 < \infty \).

The analytic solutions for \( \alpha_k \) and \( \beta_k \) can be obtained from Eqs. (28), (31) and (34) as
\[
\alpha_k = -1 + \sec \frac{\pi}{2k + 1},
\]
\[
\beta_k = -1 + \sec \frac{\pi}{2(k + 1)},
\]
for \( k \geq 1 \). For a few cases, exact values are obtained as \( \alpha_1 = 1, \beta_1 = \sqrt{2} - 1, \alpha_2 = \sqrt{5} - 2, \) and \( \beta_2 = (2 - \sqrt{3})/\sqrt{3} \).

In Table II, we list the ten largest critical values \( \alpha_k \) and \( \beta_k \) each for the mass ratio \( \alpha \). In Fig. 2, we show \( N_A \) and \( N_B \) as functions of \( \alpha \). The blue and red lines represent \( N_A \) and \( N_B \) as functions of \( \alpha \), respectively. Note that \( N_A \) and \( N_B \) changes by unity at each \( \alpha_k \) and \( \beta_k \), respectively.

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G. Terminal velocities

In this section, we determine the terminal velocities of the three objects after multiple collisions. We denote by \( a_t \), \( b_t \), and \( c_t \) the terminal velocities of \( A \), \( B \), and \( C \), respectively.

The terminal velocities \( a_t \) and \( b_t \) of the two blocks can be obtained by substituting \( n = N_A \) and \( N_B \) in Eqs. (28) and (31) into the general terms \( a_n \) and \( b_n \) in Eq. (20), respectively. The value for \( \psi \) can be found in Eq. (19). The results are

\[
\begin{align*}
a_t &= \frac{V}{2 + \alpha}[1 + (1 + \alpha) \cos N_A \psi], \quad (36a) \\
b_t &= \frac{V}{2 + \alpha}[1 - (1 + \alpha) \cos(N_B + \frac{1}{2}) \psi], \quad (36b)
\end{align*}
\]

where we have used \( \cos \frac{\psi}{2} = (1 + \alpha)^{-1} \). By making use of conservation of linear momentum, we can determine the terminal velocity \( c_t \) of \( C \) as

\[
c_t = \frac{V - a_t - b_t}{\alpha},
\]

which is the same as \( c_{N_B} \) for \( 0 < \left\{ \pi/\psi \right\} \leq \frac{1}{2} \) or, otherwise, \( c_{N_A} \).

In Fig. 3, we plot the terminal velocities \( a_t \), \( b_t \), and \( c_t \) as functions of \( \alpha \). It is remarkable that at some values of \( \alpha \), \( a_t = c_t = 0 \) and \( b_t = V \), i.e., complete transmission of the initial energy and momentum into \( B \) is achieved. We call these values the magic mass ratios, at which the initial momentum of \( A \) is completely transferred to \( B \). We denote by the defective mass ratios the values of \( \alpha \) at which the transmission rate becomes a local minimum. We shall verify in Sec. IV that \( \alpha_k \) in Eq. (35a) is a magic mass ratio and \( \beta_k \) in Eq. (35b) is a defective mass ratio.

IV. ENERGY-MOMENTUM TRANSFER

In this section, we verify that \( \alpha_k \)'s defined in Eq. (35a) are the values for the magic mass ratio at which complete momentum transfer to the block \( B \) is achieved. We also verify that \( \alpha = \beta_k \)'s defined in Eq. (35b) are the same as the defective mass ratios at which the energy-momentum transfer rates are local minima.
A. Magic mass ratio

In this section, we verify that $\alpha_k$ in Eq. (35a) is a magic mass ratio at which $a_t = c_t = 0$ and $b_t = V$.

Verification of the largest magic mass ratio $\alpha_1 = 1$ is trivial. In general, the energy-momentum conservation requires that the condition for the magic mass ratio $a_t = c_t = 0$ and $b_t = V$ can be reduced into a single condition, $b_t = V$. Applying this requirement to Eq. (36b), we find that

$$\cos(N_B + \frac{1}{2})\psi = -1. \quad (38)$$

Because the contact force on the block $B$ due to the collision with $C$ is always along the positive $x$ axis, the acceleration of $B$ is not negative. Therefore, $b_t$ in Eq. (36b) is not decreasing. This condition requires $(N_B + \frac{1}{2})\psi = \pi$, which leads to

$$N_B = \frac{\pi}{\psi} - \frac{1}{2}. \quad (39)$$

If we substitute the value for $\psi$ into Eq. (19), then the constraint is equivalent to a quadratic equation, whose unique solution is given by Eq. (35a). The exact values for the two largest values for the magic mass ratios are $\alpha_1 = 1$ and $\alpha_2 = \sqrt{5} - 2$. Ten largest numerical values for $\alpha_k$ are listed in the second column of Table II. This completes the proof of the statement that for any magic mass ratio there is an $\alpha$ that is equal to $\alpha_k$.

B. Defective mass ratio

In this section, we verify that $\beta_k$’s defined in Eq. (35b) are the values for the defective mass ratio at which both $b_t$ and $\rho_B$ reach the local minima with respect to $\alpha$. Here, $\rho_B$ is the fraction of the final state energy of $B$ relative to the total energy of the system. Likewise, $\rho_A$ and $\rho_C$ are those of $A$ and $C$, respectively.

As shown in Fig. 3, the terminal velocity $b_t$ has local minima. At each local minimum, $b_t = c_t$ since at such condition, the ball takes the local maximum terminal kinetic energy. Next, we verify the statement that $b_t = c_t$ at $\alpha = \beta_k$ for any $k \geq 1$. If we require $c_t = b_t$ to conservation of energy and momentum, then we find that

$$\begin{align*}
(1 + \alpha)b_t + a_t & = V, \quad (40a) \\
(1 + \alpha)b_t^2 + a_t^2 & = V^2. \quad (40b)
\end{align*}$$

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The coupled quadratic equations have a trivial solution \( b_t = 0 \) and \( a_t = V \) that is equivalent to the initial condition. Because the three objects do not make any penetrations, we discard this trivial solution. Then we have a unique set of solutions:

\[
\begin{align*}
a_t &= -\frac{\alpha V}{2 + \alpha}, \\
b_t &= \frac{2V}{2 + \alpha}.
\end{align*}
\]

(41a)  

(41b)

By comparing Eq. (36b) and Eq. (41b), we find that

\[
\cos(N_B + \frac{1}{2})\psi = -\frac{1}{1 + \alpha}.
\]

(42)

From the similar argument below Eq. (38), Eq. (42) gives

\[
\left(N_B + \frac{1}{2}\right)\psi = \pi - \frac{1}{2}\psi,
\]

(43)

that is,

\[
N_B = \frac{\pi}{\psi} - 1,
\]

(44)

which is equivalent to Eq. (35b) with the value for \( \psi \) in Eq. (19). The exact values for the two largest values for the defective mass ratios are \( \beta_1 = \sqrt{2} - 1 \) and \( \beta_2 = (2 - \sqrt{3})/\sqrt{3} \). Ten largest numerical values for \( \beta_k \) are given in the third column of Table II. This completes the proof of the statement that for any defective mass ratio there is an \( \alpha \) that is equal to \( \beta_k \).

Let us compute the rates for the momentum and energy transmission to \( B \) at a defective point \( \alpha = \beta_k \) analytically. The rate \( \gamma_k \) for momentum transfer to \( B \) at \( \alpha = \beta_k \) is

\[
\gamma_k = \frac{b_t}{V} = \frac{2}{2 + \beta_k},
\]

(45)

where ten largest numerical values for \( \gamma_k \) is listed in the last column of Table II. The fractions of the final state energies relative to the initial kinetic energy at \( \alpha = \beta_k \), \( \rho_i|_{\alpha=\beta_k} \) for \( i = A, B, \) and \( C \) become

\[
\begin{align*}
\rho_A|_{\alpha=\beta_k} &= \frac{a_t^2|_{\alpha=\beta_k}}{V^2} = \frac{\beta_k^2}{(2 + \beta_k)^2}, \\
\rho_B|_{\alpha=\beta_k} &= \frac{b_t^2|_{\alpha=\beta_k}}{V^2} = \frac{4}{(2 + \beta_k)^2}, \\
\rho_C|_{\alpha=\beta_k} &= \frac{\alpha c_t^2|_{\alpha=\beta_k}}{V^2} = \frac{4\beta_k}{(2 + \beta_k)^2}.
\end{align*}
\]

(46a)  

(46b)  

(46c)

Here, \( \rho_B|_{\alpha=\beta_k} \) is the local minimum of \( \rho_B \) placed at \( \alpha = \beta_k \).
In Fig. 4, we show the energy fraction \( \rho_i = E_i/E_0 \) of the final state \( i \) for \( i = A, B, \) and \( C \) as a function of \( \alpha \). Here, \( E_i \) and \( E_0 \) are the final kinetic energy of \( i \) and the initial kinetic energy of the incident block \( A \), respectively. If \( \alpha \) equals to a magic mass ratio \( \alpha_k \), then \( \rho_B = 1 \) and \( \rho_A = \rho_C = 0 \) and, therefore, the initial kinetic energy of \( A \) is completely transferred to \( B \). If \( \alpha \) equals to a defective mass ratio \( \beta_k \), then \( \rho_B = \gamma_k^2 \). In the limit \( \alpha \to 0^+ \), the final kinetic energy of \( A \) and \( C \) become vanishingly small and the corresponding transmission rates approach 1. Note that the energy loss is mainly due to the non-zero final kinetic energy of \( C \). At a defective mass ratio \( \beta_k \), the terminal energy fraction of the ball \( \rho_C \) is larger than that of the block \( A, \rho_A \), by a factor of \( 4/\beta_k \) which is much larger than 1.

V. CONCLUSION

We have considered the one-dimensional scattering of two identical blocks of mass \( M \) by exchanging momenta through elastic collisions with a ball with mass \( m = \alpha M \). Because the ball and the target block are initially not in contact with each other, the system experiences multiple two-body collisions that are well ordered and, therefore, the motion of the system is uniquely determined by the energy-momentum conservation. We have demonstrated that the velocities of the three objects in each cycle of multiple collisions construct a sequence of three-dimensional vector transforming like an Euclidean vector under the rotation about a fixed axis by a fixed angle that are completely determined by the mass ratio \( \alpha \). Recursive use of the relation derives analytic expressions for the velocities of the three objects after each collision. Based on these results, we have computed the terminal velocity and the total number of collisions of each participant in terms of \( \alpha \). It is remarkable that we have found all possible values for the magic mass ratio \( \alpha = \alpha_k \), at which the energy and momentum of the incident block are completely transferred to the target block.

Finally, it is worth mentioning that by making use of these magic mass ratios, we can devise a system which is a generalized version of the Newton’s cradle. As shown in Fig. 5 we can put \( N \) identical blocks of mass \( M \) in series along an axis and place a ball in each spacing. And let the left most block incident to the remainder of the system at rest. If each ball’s mass ratio to the block mass \( M \) is one of \( \alpha_k \) and every spacing is wide enough, then the energy and momentum of the incident block are completely transferred to the target blocks one by one. There will be many other useful applications of the magic mass ratio.
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1 Let us denote by \( u_i \) and \( v_i \) the initial and final velocities of the \( i \)th object, respectively. Then the conservation of energy and momentum requires \( u_1 + u_2 = v_1 + v_2 \) and \( u_1^2 + u_2^2 = v_1^2 + v_2^2 \). The coupled equations have a unique set of solutions: \( v_1 = u_2 \) and \( v_2 = u_1 \). If they do not make a collision, then \( v_1 = u_1 \) and \( v_2 = u_2 \).

2 J. B. Hart and R. B. Herrmann, “Energy transfer in one-dimensional collisions of many objects,” Am. J. Phys. 36, 46–49 (1968).

3 J. D. Kerwin, “Velocity, momentum, and energy transmissions in chain collisions,” Am. J. Phys. 40, 1152–1157 (1972).

4 J. D. Gavenda and J. R. Edgington, “Newton’s cradle and scientific explanation,” Phys. Teach. 35, 411–417 (1997).

5 S. Chapman, “Misconception concerning the dynamics of the impact ball apparatus,” Am. J. Phys. 28, 705–711 (1960).

6 L. Flansburg and K. Hudnut, “Dynamic solutions for linear elastic collisions,” Am. J. Phys. 47, 911–914 (1979).

7 F. Herrmann and P. Schmälzle, “Simple explanation of a well-known collision experiment,” Am. J. Phys. 49, 761–764 (1981).

8 F. Herrmann and M. Seitz, “How does the ball-chain work?,” Am. J. Phys. 50, 977–981 (1982).

9 B. Leroy “Collision between two balls accompanied by deformation: A qualitative approach to Hertz’s theory,” Am. J. Phys. 53, 346–349 (1985).

10 P. Patrició, “The Hertz contact in chain elastic collisions,” Am. J. Phys. 72, 1488–1491 (2004).

11 R. Hessel, A. C. Perinotto, R. A. M. Alfaro, and A. A. Freschi, “Force-versus-time curves during collisions between two identical steel balls,” Am. J. Phys. 74, 176–179 (2006).

12 R. Cross, “Vertical bounce of two vertically aligned balls,” Am. J. Phys. 75, 1009–1016 (2007).
13 R. Cross, “Difference between bouncing balls, springs, and rods,” Am. J. Phys. 76, 908–915 (2008).

14 J.-H. Ee and J. Lee, “A unique pure mechanical system revealing dipole repulsion,” Am. J. Phys. 80, 1078–1084 (2012).

15 Y. G. Sinai, “Dynamics of a heavy particle surrounded by a finite number of light particles,” Theor. Math. Phys. 121, 1351–1357 (1991).

16 S. De, “Derivation of an inverse square force from a gedanken construction,” Physics and Technology Quest, 2, 35–37 (1997).

17 S. Redner, “A billiard-theoretic approach to elementary one-dimensional elastic collisions,” Am. J. Phys. 72, 1492–1498 (2004).

18 Note that $\alpha = 0$ represents the case that the ball is absent.
### Tables

| $x \setminus f(x)$ | $\cos x$ | $\sin x$ | $\tan x$ |
|---------------------|----------|----------|----------|
| $\frac{1}{2} \theta$ | $\frac{1}{\sqrt{1 + \alpha}}$ | $\sqrt{\frac{\alpha}{1 + \alpha}}$ | $\sqrt{\alpha}$ |
| $\theta$            | $\frac{1 - \alpha}{1 + \alpha}$ | $\frac{2\sqrt{\alpha}}{1 + \alpha}$ | $\frac{2\sqrt{\alpha}}{1 - \alpha}$ |
| $\frac{1}{4} \psi$  | $\sqrt{\frac{2 + \alpha}{2(1 + \alpha)}}$ | $\sqrt{\frac{\alpha}{2(1 + \alpha)}}$ | $\sqrt{\frac{\alpha}{2 + \alpha}}$ |
| $\frac{1}{2} \psi$  | $\frac{1}{1 + \alpha}$ | $\frac{\sqrt{\alpha(2 + \alpha)}}{1 + \alpha}$ | $\sqrt{\alpha(2 + \alpha)}$ |
| $\psi$              | $\frac{1 - 2\alpha - \alpha^2}{(1 + \alpha)^2}$ | $\frac{2\sqrt{\alpha(2 + \alpha)}}{(1 + \alpha)^2}$ | $\frac{2\sqrt{\alpha(2 + \alpha)}}{1 - 2\alpha - \alpha^2}$ |

**TABLE I:** The values of $\cos x$, $\sin x$, and $\tan x$ at $x = \frac{1}{2} \theta$, $\theta$, $\frac{1}{4} \psi$, $\frac{1}{2} \psi$, and $\psi$ as functions of $\alpha$. 

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TABLE II: Ten largest values for the magic mass ratio $\alpha_k$ and the defective mass ratio $\beta_k$ that are defined in Eq. (35). For every $\alpha = \alpha_k$, the energy and momentum of the incident block $A$ are completely transferred to $B$. For every $\alpha = \beta_k$, the transmission rates have local minima. $\gamma_k$ is the rate of the momentum transfer to $B$ at $\alpha = \beta_k$, defined in Eq. (45).

| $k$ | $\alpha_k$ | $\beta_k$ | $\gamma_k$ |
|-----|------------|-----------|------------|
| 1   | 1          | 0.414214  | 0.828427   |
| 2   | 0.236068   | 0.154701  | 0.928203   |
| 3   | 0.109916   | 0.082392  | 0.960434   |
| 4   | 0.064178   | 0.051462  | 0.974914   |
| 5   | 0.042217   | 0.035276  | 0.982668   |
| 6   | 0.029927   | 0.025717  | 0.987305   |
| 7   | 0.022341   | 0.019591  | 0.990299   |
| 8   | 0.017321   | 0.015427  | 0.992346   |
| 9   | 0.013827   | 0.012465  | 0.993806   |
| 10  | 0.011295   | 0.010283  | 0.994885   |

FIG. 1: The initial condition of the model system. Block $A$ with the initial velocity $+V$ is approaching the ball $C$ and the block $B$ that are placed at rest at $x = 0$ and $L$, respectively. $M$ is the mass of each block and $m = \alpha M$ is the mass of the ball with $\alpha < 1$. 
FIG. 2: $N_A$ and $N_B$ as functions of $\alpha$. $\alpha_k$ and $\beta_k$ are the minimum values of $\alpha$ to have $N_A = k$ and $N_B = k$, respectively, for $k \geq 1$. For $\alpha$ such that $\alpha \geq \alpha_1 = 1$, $N_A = 1$. For $\alpha \geq \beta_1 = \sqrt{2} - 1$, $N_B = 1$. 
FIG. 3: The terminal velocities $a_t$, $b_t$, and $c_t$ of $A$, $B$, and $C$, respectively, in units of the initial velocity $V$ of $A$ as functions of the mass ratio $\alpha$. At every $\alpha = \alpha_k$, $b_t = V$ and $a_t = c_t = 0$ for $k \geq 1$. At every $\alpha = \beta_k$, $b_t = c_t$ and $b_t$ becomes a local minimum for $k \geq 1$. The two largest values for $\alpha_k$ and $\beta_k$ are $\alpha_1 = 1$, $\beta_1 = \sqrt{2} - 1$, $\alpha_2 = \sqrt{5} - 2$, and $\beta_2 = (2 - \sqrt{3})/\sqrt{3}$. More values for $\alpha_k$ and $\beta_k$ can be found in Table II.
FIG. 4: The fraction $\rho_i = E_i/E_0$ as a function of $\alpha$ for the final state $i$ for $i = A$, $B$, and $C$. Here, $E_i$ and $E_0$ are the final kinetic energy of $i$ and the initial kinetic energy of the incident block $A$, respectively. If $\alpha = \alpha_k$, then $\rho_B = 1$ and $\rho_A = \rho_C = 0$ and, therefore, the initial kinetic energy of $A$ is completely transferred to $B$. If $\alpha = \beta_k$, then $\rho_B = \gamma_k^2$, where $\gamma_k$ is listed in the last column of Table II for $1 \leq k \leq 10$.

FIG. 5: The initial condition of a generalized version of the Newton's cradle. $N - 1$ identical blocks of mass $M$ ($A_2, A_3, \cdots, A_N$) are placed at rest in equal spacing along the $x$ axis and another identical block $A_1$ is incident from the left. In the $i$th spacing a ball with mass $m_i$ is initially at rest. If $m_i/M$ equals to a magic mass ratio $\alpha_{k_i}$ for each $i = 1, 2, \cdots, N - 1$, then the initial energy and momentum of $A_1$ is completely transferred to $A_N$ in the final stage.