Density of sets of natural numbers and the Lévy group

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Abstract

Let \(\mathbb{N}\) denote the set of positive integers. The asymptotic density of the set \(A \subseteq \mathbb{N}\) is \(d(A) = \lim_{n \to \infty} |A \cap [1,n]|/n\), if this limit exists. Let \(\mathcal{AD}\) denote the set of all sets of positive integers that have asymptotic density, and let \(\mathcal{S}_\mathbb{N}\) denote the set of all permutations of the positive integers \(\mathbb{N}\). The group \(\mathcal{L}^\sharp\) consists of all permutations \(f \in \mathcal{S}_\mathbb{N}\) such that \(A \in \mathcal{AD}\) if and only if \(f(A) \in \mathcal{AD}\), and the group \(\mathcal{L}^*\) consists of all permutations \(f \in \mathcal{L}^\sharp\) such that \(d(f(A)) = d(A)\) for all \(A \in \mathcal{AD}\). Let \(f: \mathbb{N} \to \mathbb{N}\) be a one-to-one function such that \(d(f(\mathbb{N})) = 1\) and, if \(A \in \mathcal{AD}\), then \(f(A) \in \mathcal{AD}\). It is proved that \(f\) must also preserve density, that is, \(d(f(A)) = d(A)\) for all \(A \in \mathcal{AD}\). Thus, the groups \(\mathcal{L}^\sharp\) and \(\mathcal{L}^*\) coincide.

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1. Asymptotic density and permutations

Let $A$ be a set of positive integers, and let

$$A(n) = \sum_{a \in A} 1 \text{ for } 1 \leq a \leq n$$

denote the counting function of the set $A$. The lower asymptotic density of $A$ is

$$d_L(A) = \lim \inf_{n \to \infty} \frac{A(n)}{n}.$$

The upper asymptotic density of $A$ is

$$d_U(A) = \lim \sup_{n \to \infty} \frac{A(n)}{n}.$$

The set $A$ has asymptotic density $d(A)$ if the limit

$$d(A) = \lim_{n \to \infty} \frac{A(n)}{n}$$

exists. The set $A$ has an asymptotic density if and only if $d_L(A) = d_U(A)$. We denote by $\mathcal{AD}$ the set of all sets of positive integers that have asymptotic density, that is,

$$\mathcal{AD} = \{ A \subseteq \mathbb{N} : d_L(A) = d_U(A) \}.$$

Let $S_N$ denote the group of all permutations of the positive integers $\mathbb{N}$. For any set $A \subseteq \mathbb{N}$ and permutation $g \in S_N$, we let

$$g(A) = \{ g(a) : a \in A \}.$$

Let $L^*$ be the set of all permutations that preserve density, that is, $L^*$ consists of all permutations $g \in S_N$ such that

(i) $A \in \mathcal{AD}$ if and only if $g(A) \in \mathcal{AD}$, and
(ii) $d(A) = d(g(A))$ for all $A \in \mathcal{AD}$.

The set $L^*$ is a subgroup of the infinite permutation group $S_N$, and originated in work of Paul Lévy [3] in functional analysis. This group and other related groups of permutations that preserve asymptotic density have been investigated by Obata [4,5] and Blümlinger and Obata [1].

The Lévy group $L^*$ is contained in the group $L^2$ that consists of all permutations $g \in S_N$ such that $A \in \mathcal{AD}$ if and only if $g(A) \in \mathcal{AD}$, but that do not necessarily preserve the asymptotic density of every set $A \in \mathcal{AD}$. The object of this note is to prove that $L^* = L^2$. Indeed, we prove the stronger result that if $f : \mathbb{N} \to \mathbb{N}$ is any one-to-one function, not necessarily a permutation, such that $A \in \mathcal{AD}$ implies that $f(A) \in \mathcal{AD}$, then also $d(f(A)) = \lambda d(A)$ for all $A \in \mathcal{AD}$, where $\lambda = d(f(\mathbb{N}))$. In particular, if $f$ is a permutation, then $d(f(\mathbb{N})) = d(\mathbb{N}) = 1$ and $d(f(A)) = d(A)$ for all $A \in \mathcal{AD}$. 
2. Permutations preserving density

We begin with the following “intertwining lemma.”

**Lemma 1.** Let \(A\) and \(B\) be sets of integers such that \(d(A) = d(B) = \gamma > 0\). Let \(\{\varepsilon_k\}_{k=1}^{\infty}\) be a decreasing sequence of numbers such that \(0 < \varepsilon_k < 1\) for all \(k \geq 1\) and \(\lim_{k \to \infty} \varepsilon_k = 0\). Let \(\{M_k\}_{k=1}^{\infty}\) be a sequence of positive integers such that

\[
\left| \frac{A(n)}{n} - \gamma \right| < \varepsilon_k \quad \text{and} \quad \left| \frac{B(n)}{n} - \gamma \right| < \varepsilon_k
\]

for all \(n \geq M_k\). If \(\{N_k\}_{k=1}^{\infty}\) is any sequence of integers satisfying

\[
M_{k-1} \leq N_{k-1} \leq \varepsilon_{k-1} N_k
\]

for all \(k \geq 2\) and if

\[
C = \bigcup_{k=1}^{\infty} (A \cap [N_{2k-1} + 1, N_{2k+1}]) \cup \bigcup_{k=1}^{\infty} (B \cap [N_{2k} + 1, N_{2k+1}])
\]

then

\[
d(C) = \gamma.
\]

**Proof.** If \(N_k \leq m < n\), then

\[
(\gamma - \varepsilon_k)m < A(m) < (\gamma + \varepsilon_k)m,
\]

\[
(\gamma - \varepsilon_k)n < A(n) < (\gamma + \varepsilon_k)n
\]

and so

\[
\gamma(n - m) - 2\varepsilon_k n < A(n) - A(m) < \gamma(n - m) + 2\varepsilon_k n.
\]

Similarly,

\[
\gamma(n - m) - 2\varepsilon_k n < B(n) - B(m) < \gamma(n - m) + 2\varepsilon_k n.
\]

Let \(k \geq 2\) and \(N_k < n \leq N_{k+1}\). If \(k\) is odd, then

\[
C \cap [1, n] = (A \cap [N_{k+1} + 1, n]) \cup (B \cap [N_{k-1} + 1, N_k]) \cup (C \cap [1, N_{k-1}])
\]

and so

\[
C(n) = A(n) - A(N_k) + B(N_k) - B(N_{k-1}) + C(N_{k-1}).
\]

If \(k\) is even, then

\[
C \cap [1, n] = (B \cap [N_{k+1} + 1, n]) \cup (A \cap [N_{k-1} + 1, N_k]) \cup (C \cap [1, N_{k-1}])
\]
and

\[ C(n) = B(n) - B(N_k) + A(N_k) - A(N_{k-1}) + C(N_{k-1}). \]

In both cases, since \( N_k - 1 \leq \varepsilon_{k-1}N_k \), it follows that

\[ C(n) < \gamma(n - N_k) + 2\varepsilon_k n + \gamma(N_k - N_{k-1}) + 2\varepsilon_{k-1}N_k + N_{k-1} \]
\[ < \gamma n + 5\varepsilon_{k-1}n \]

and

\[ C(n) > \gamma(n - N_k) - 2\varepsilon_k n + \gamma(N_k - N_{k-1}) - 2\varepsilon_{k-1}N_k \]
\[ > \gamma n - \gamma N_{k-1} - 4\varepsilon_{k-1}n \]
\[ > \gamma n - 5\varepsilon_{k-1}n. \]

Therefore,

\[ \left| \frac{C(n)}{n} - \gamma \right| < 5\varepsilon_{k-1} \]

for all \( n > N_k \), and so \( d(C) = \gamma \). □

**Theorem 1.** Let \( f : \mathbb{N} \to \mathbb{N} \) be a one-to-one function such that if \( A \in \mathcal{AD} \), then \( f(A) \in \mathcal{AD} \), that is, if the set \( A \) of positive integers has asymptotic density, then the set \( f(A) \) also has asymptotic density. Let \( \lambda = d(f(\mathbb{N})) \). If \( \lambda = 0 \), then \( d(f(A)) = 0 \) for all \( A \subseteq \mathbb{N} \). If \( \lambda > 0 \), then there is a unique increasing function \( \hat{f} : [0, 1] \to [0, 1] \) such that \( \hat{f}(0) = 0 \), \( \hat{f}(1) = 1 \), and

\[ d\left( f(A) \right) = \lambda \hat{f}(d(A)) \]

for all \( A \in \mathcal{AD} \).

**Proof.** We shall prove that, for every set \( A \in \mathcal{AD} \), the asymptotic density of \( f(A) \) depends only on the asymptotic density of \( A \). Equivalently, we shall prove that if \( A, B \in \mathcal{AD} \) and \( d(A) = d(B) \), then \( d(f(A)) = d(f(B)) \).

For \( \gamma \in [0, 1] \), let \( A \) and \( B \) be sets in \( \mathcal{AD} \) such that \( d(A) = d(B) = \gamma \). Suppose that

\[ 0 \leq d\left( f(A) \right) = \alpha < \beta = d\left( f(B) \right) \leq 1. \]

Let \( \{\varepsilon_k\}_{k=1}^\infty \) be a decreasing sequence of numbers such that \( 0 < \varepsilon_k < 1 \) for all \( k \geq 1 \) and \( \lim_{k \to \infty} \varepsilon_k = 0 \). For every \( k \geq 1 \) there is a positive integer \( M_k \) such that

\[ \left| \frac{A(n)}{n} - \gamma \right| < \varepsilon_k, \quad \left| \frac{B(n)}{n} - \gamma \right| < \varepsilon_k, \]
\[ \left| \frac{f(A)(n)}{n} - \alpha \right| < \varepsilon_k, \quad \left| \frac{f(B)(n)}{n} - \beta \right| < \varepsilon_k. \]
for all \( n \geq M_k \). By Lemma 1, if \( \{N_k\}_{k=1}^{\infty} \) is any sequence of integers satisfying

\[
M_{k-1} \leq N_{k-1} \leq \varepsilon_{k-1} N_k
\]  

(1)

for all \( k \geq 2 \) and if

\[
C = \bigcup_{k=1}^{\infty} \left( A \cap [N_{2k-1} + 1, N_{2k}] \right) \cup \bigcup_{k=1}^{\infty} \left( B \cap [N_{2k} + 1, N_{2k+1}] \right)
\]  

(2)

then \( d(C) = \gamma \).

We shall construct a sequence \( \{N_k\}_{k=1}^{\infty} \) satisfying (1) such that the associated set \( C \) satisfies \( d(C) = \gamma \), but \( d_L(f(C)) \leq \alpha \) and \( d_U(f(C)) \geq \beta \). This implies that the set \( f(C) \) does not have asymptotic density, which is impossible since the function \( f \) maps \( AD \) into \( AD \).

The sequence \( \{N_k\}_{k=1}^{\infty} \) and a related sequence \( \{L_k\}_{k=1}^{\infty} \) will be constructed inductively. We remark that since the function \( f \) is one-to-one, it follows that for every positive integer \( L \), there is an integer \( N' \) such that \( f(n) \leq L \) only if \( n \leq N' \). This implies that for every \( N \geq N' \) we have

\[
f(C) \cap [1, L] = f(C \cap [1, N]) \cap [1, L].
\]

Let \( N_1 = L_1 = M_1 \). Let \( k \geq 2 \) and suppose that we have constructed sequences \( N_1 < \cdots < N_{k-1} \) and \( L_1 < \cdots < L_{k-1} \). Choose an integer

\[
L_k > \max(L_{k-1}, M_k)
\]

such that \( \varepsilon_{k-1} L_k > N_{k-1} \). By the remark, there exists an integer \( N_k > L_k \) such that \( f(n) \leq L_k \) only if \( n \leq N_k \). Then

\[
\varepsilon_{k-1} N_k > \varepsilon_{k-1} L_k > N_{k-1}.
\]

We use the sequence \( \{N_k\}_{k=1}^{\infty} \) to construct the set \( C \) according to formula (2).

For \( k \geq 1 \) we have

\[
f(C) \cap [1, L_{2k}] = f(C \cap [1, N_{2k}]) \cap [1, L_{2k}]
\]

\[
= \left( \left( f(C \cap [1, N_{2k-1}]) \cap [1, L_{2k}] \right) \cup \left( f(A \cap [N_{2k-1} + 1, N_{2k}]) \cap [1, L_{2k}] \right) \right)
\]

\[
\subseteq f([1, N_{2k-1}]) \cup (f(A) \cap [1, L_{2k}])
\]

and so

\[
f(C)(L_{2k}) \leq f(A)(L_{2k}) + N_{2k-1}.
\]

It follows that

\[
\frac{f(C)(L_{2k})}{L_{2k}} \leq \frac{f(A)(L_{2k}) + N_{2k-1}}{L_{2k}} < \alpha + 2 \varepsilon_{2k-1}.
\]
Therefore,

\[ d_L(f(C)) = \liminf_{n \to \infty} \frac{f(C)(n)}{n} \leq \liminf_{k \to \infty} \frac{f(C)(L_{2k})}{L_{2k}} \leq \alpha. \]

Similarly,

\[ f(C) \cap [1, L_{2k+1}] \supseteq f(B \cap [N_{2k} + 1, N_{2k+1}]) \cap [1, L_{2k+1}] \]
\[ = (f(B \cap [1, N_{2k+1}]) \cap [1, L_{2k+1}]) \setminus (f(B \cap [1, N_{2k}]) \cap [1, L_{2k+1}]) \]
\[ \supseteq (f(B) \cap [1, L_{2k+1}]) \setminus f([1, N_{2k}]) \]

and so

\[ f(C)(L_{2k+1}) \geq f(B)(L_{2k+1}) - N_{2k}. \]

It follows that

\[ \frac{f(C)(L_{2k+1})}{L_{2k+1}} \geq \frac{f(B)(L_{2k+1}) - N_{2k}}{L_{2k+1}} > \beta - \epsilon \]

and so

\[ d_U(f(C)) = \limsup_{n \to \infty} \frac{f(C)(n)}{n} \geq \limsup_{k \to \infty} \frac{f(C)(L_{2k+1})}{L_{2k+1}} \geq \beta. \]

The inequality

\[ d_L(f(C)) \leq \alpha < \beta \leq d_U(f(C)) \]

contradicts the fact that \( f(C) \) has asymptotic density, and so \( d(f(A)) = d(f(B)) \).

If \( \lambda = d(f(N)) = 0 \), then \( d(f(A)) = 0 \) for every set \( A \subseteq N \). Suppose that \( \lambda > 0 \). Define the function \( \hat{f} \) by

\[ \hat{f}(\alpha) = \frac{d(f(A))}{\lambda} \]

where \( A \subseteq N \) and \( d(A) = \alpha \). This is well-defined, since \( d(f(A)) = d(f(A')) \) if \( d(A) = d(A') \).

Let \( 0 \leq \alpha \leq \beta \leq 1 \). There exist sets \( A \subseteq B \subseteq N \) such that \( d(A) = \alpha \) and \( d(B) = \beta \). Since \( f(A) \subseteq f(B) \subseteq f(N) \), it follows that

\[ 0 \leq d(f(A)) \leq d(f(B)) \leq d(f(N)) = \lambda \]

and so

\[ 0 \leq \hat{f}(\alpha) \leq \hat{f}(\beta) \leq 1. \]

Thus, \( \hat{f} : [0, 1] \to [0, 1] \) is an increasing function with \( \hat{f}(0) = d(f(\emptyset)) = 0 \) and \( \hat{f}(1) = d(f(N))/\lambda = 1 \). This completes the proof. \( \square \)
Theorem 2. Let \( f : \mathbb{N} \to \mathbb{N} \) be a one-to-one function such that if the set \( A \) of positive integers has asymptotic density, then the set \( f(A) \) also has asymptotic density. Let \( \lambda = d(f(\mathbb{N})) \). Then

\[
d(f(A)) = \lambda d(A)
\]

for all \( A \in \mathcal{AD} \).

**Proof.** If \( \lambda = 0 \), then \( d(f(A)) = 0 \) for all \( A \in \mathcal{AD} \) and the theorem is true.

Suppose that \( \lambda > 0 \). By Theorem 1, there is an increasing function \( \hat{f} : [0, 1] \to [0, 1] \) such that \( d(f(A)) = \lambda \hat{f}(d(A)) \) for all \( A \in \mathcal{AD} \). We shall prove that \( \hat{f}(\alpha) = \alpha \) for all positive rational numbers \( \alpha \in [0, 1] \).

Let \( \alpha = r/s \), where \( 1 \leq r \leq s \). For \( i = 1, \ldots, s \), let \( A_i = \{ a \in \mathbb{N} : a \equiv i \pmod{s} \} \). Let \( A = \bigcup_{i=1}^s A_i \). Then \( d(A_i) = 1/s \) for \( i = 1, \ldots, s \) and \( d(A) = r/s \). Since the function \( f \) is one-to-one, the set \( f(A) \) is the disjoint union of the \( r \) sets \( f(A_1), \ldots, f(A_r) \). Similarly, \( f(\mathbb{N}) \) is the disjoint union of the \( s \) sets \( f(A_1), \ldots, f(A_s) \). Since \( A, A_1, \ldots, A_s \in \mathcal{AD} \), it follows that \( f(A), f(A_1), \ldots, f(A_s) \in \mathcal{AD} \), and

\[
\lambda = d(f(\mathbb{N})) = \sum_{i=1}^s d(f(A_i)) = \lambda s \hat{f}(1/s).
\]

Then

\[
\hat{f}(1/s) = \frac{1}{s}
\]

and

\[
\hat{f}(\alpha) = \frac{d(f(A))}{\lambda} = \frac{1}{\lambda} \sum_{i=1}^r d(f(A_i)) = \sum_{i=1}^r \hat{f}(d(A_i)) = r \hat{f}(1/s) = \frac{r}{s} = \alpha.
\]

This completes the proof. \( \square \)

**Remark.** The Lévy group \( \mathcal{L}^2 \) consists of all permutations \( f \in S_\mathbb{N} \) such that \( A \in \mathcal{AD} \) if and only if \( f(A) \in \mathcal{AD} \). We can also consider the semigroup \( S^2 \) consisting of all permutations \( f \in S_\mathbb{N} \) such that \( A \in \mathcal{AD} \) implies \( f(A) \in \mathcal{AD} \). Coquet [2] proved that the group \( \mathcal{L}^2 \) is a proper subsemigroup of \( S^2 \).

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