REPLACING A GRAPH CLASPER BY TREE CLASPERs

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Abstract. We prove that two links related by a surgery along a connected, strict graph clasper of degree $n$ are $C_n$-equivalent, i.e., related by a sequence of surgeries along strict tree claspers of degree $n$.

1. Introduction

Goussarov [6, 7] and the author [8] independently introduced topological calculus of surgery along claspers. One of the main achievements in these theories is the following characterization of the topological information carried by Goussarov-Vassiliev finite type invariants [10, 3, 2, 9, 1]: Two knots in $S^3$ have the same values for any Goussarov-Vassiliev invariant of degree $< n$ if and only if they are related by a sequence of $C_n$-moves [7, 8]. Here a $C_n$-move is defined as surgery along a certain type of tree clasper, which is a framed unitrivalent tree with each univalent vertex attached to the knot.

In [8, §8.2], the author also introduced the notion of graph claspers for links, which is a generalization of the notion of tree claspers, where the tree part is replaced by a unitrivalent graph. There we explained the idea that graph claspers may be regarded as topological realizations of unirtrivalent graphs (also called Feynman diagrams, Jacobi diagrams, etc.) used by Bar-Natan [1] to describe the structure of the graded quotients of the Goussarov-Vassiliev filtration. Recall that, in the diagram level, any connected, unitrivalent graph diagram on a 1-manifold is equivalent under the $STU$ relations to a linear combination of tree diagrams. The purpose of this short note is to prove a topological version of the above-mentioned fact: surgery along a strict graph clasper $G$ for a link can be replaced by a sequence of surgeries along strict tree claspers of the same degree as $G$.

2. Definitions

We freely use the definitions, notations and conventions in [8].

In the following, $M$ denotes a compact, connected, oriented 3-manifold.

A tangle $\gamma$ in $M$ is a “link” in the sense of [8, §1.1], i.e., a proper embedding $f: \alpha \rightarrow M$ of a compact, oriented 1-manifold $\alpha$ into $M$. As usual, we systematically confuse $\gamma$ and the image $\gamma(\alpha) \subset M$. A link in the usual sense is a tangle consisting only of circle components.

Two tangles $\gamma$ and $\gamma'$ in $M$ are equivalent, denoted by $\gamma \cong \gamma'$, if $\gamma$ and $\gamma'$ are ambient isotopic fixing the endpoints.

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Figure 1.

For the definitions of claspers, tree claspers and graph claspers, see [8, §1, §2, §8.2]. Note that a tree clasper is a special kind of connected graph clasper. A graph clasper $G$ is called strict if $G$ has no leaves. I.e., a strict graph clasper is a clasper consisting only of disk-leaves, nodes, and edges.

An important property of a strict graph clasper $G$ for a tangle $\gamma$ in $M$ is that $G$ is tame (see [8, §2.3]), and consequently surgery along $G$ does not change the 3-manifold up to canonical homeomorphism. (The proof of this fact is similar to [8, Proposition 3.3].) Thus we may regard the result $\gamma^G$ from $\gamma$ of surgery along $G$ as a tangle in $M$.

A disk-leaf $A$ in a clasper for a tangle $\gamma$ is simple if $A$ intersects $\gamma$ by only one point. A strict graph clasper is simple if all the disk-leaves are simple.

For $k \geq 1$, a $C_n$-move is a local move on a tangle defined as surgery along a strict tree clasper of degree $n \geq 1$, which is not necessarily simple. The $C_n$-equivalence on tangles is generated by $C_n$-moves and equivalence.

3. Statement and proof of the result

The purpose of this note is to prove the following.

**Theorem 1** (Stated in a different form in [8, §8.2, p.68, l.4]). Let $\gamma$ be a tangle in a compact, connected, oriented 3-manifold $M$, and let $G$ be a strict graph clasper for $\gamma$ in $M$ of degree $n \geq 1$, which is not necessarily simple. Then $\gamma$ and $\gamma^G$ are $C_n$-equivalent. (Consequently, by [8, Theorem 3.17], there are finitely many disjoint simple tree claspers $T_1, \ldots, T_p$ for $\gamma$ of degree $n$ such that $\gamma^G \cong \gamma^T_{1 \cup \cdots \cup T_p}$.)

**Proof.** We may safely assume that $G$ is connected.

The proof is by induction on the number $e(G)$ of edges in $G$. If $e(G) = 1$, then $G$ is already a strict tree clasper, and hence the assertion follows.

Let $e(G) > 1$. If we have the assertion for the case when $G$ is simple, then we have the general case by replacing a single strand by a parallel family of strands. Hence we may assume that $G$ is simple. (This assumption is just for simplifying explanations and figures.) Choose any disk-leaf $L$ of $G$. Since $e(G) > 1$, $L$ is joined by an edge to a node. Let $G'$ denote the the strict graph clasper obtained from $G$ by move 9 of [8, Proposition 2.7] and isotopy as depicted in Figure 1. We have

$\gamma^G \cong \gamma^{G'}$. There are two cases.

*Case 1.* $G'$ is connected. Since $e(G') = e(G) - 1$ and $\deg G' = \deg G$, the assertion follows from the induction hypothesis.

*Case 2.* $G'$ consists of two components $G_1$ and $G_2$. We have

\begin{align}
\gamma^{G_1 \cup G_2} &\cong \gamma^G, \\
\deg G_1 + \deg G_2 &\cong \deg G = n.
\end{align}
For $i = 1, 2$, let $N_i$ be a small regular neighborhood of $G_i$, such that $N_1 \cap N_2$ is empty. Let $\gamma_i = \gamma \cap N_i$, which is a tangle in $N_i$. Let $c \subset \gamma$ be the component which intersects $L$. Let $L_i$ denote the new disk-leaf in $G_i$, which intersects $c$. By the induction hypothesis and [8, Theorem 3.17], for $i = 1, 2$, there is a clasper $F_i$ consisting of finitely many disjoint, simple strict tree claspers of degree $\deg G_i$ for $\gamma_i$ in $N_i$, such that

$$\gamma_i^{F_i} \cong (\gamma_i)^{G_i}.$$ 

(3.3)

For $i = 1, 2$, $c \cap N_i$ consists of two components $c_i, c'_i$, where these components for $i = 1, 2$ placed in $c$ in the order $c_1, c_2, c'_1, c'_2$. For each of these arcs $c_1, c_2, c'_1, c'_2$, there are finitely many intersecting disk-leaves and finitely many winding edges, as depicted in the left-hand side of Figure 2. Slide the disk-leaves and edges of $F_2$ around $c_2$ along $c$ to traverse those of $F_1$ around $c'_1$. The result is depicted in the right-hand side of Figure 2. By [8, Propositions 4.4 and 4.6], this sliding does not change the $C_n$-equivalence class of result of surgery on $\gamma$. Let $F'_2 \subset N_2$ denote the clasper obtained from $F_2$ by the above sliding moves. It follows from the construction of $G_1$ and $G_2$ that $\gamma_{F_1 \cup F'_2} \cong \gamma_{G_1' \cup G_2'}$, where $G_1'$ and $G_2'$ are depicted in Figure 3. By an obvious graph-clasper version of [8, Proposition 3.4], we have $\gamma_{G_1' \cup G_2'} \cong \gamma$. Hence we have

$$\gamma_{F_1 \cup F'_2} \cong \gamma_{F_1 \cup F'_2} \cong \gamma_{G_1' \cup G_2'} \cong \gamma.$$ 

The assertion follows from this, (3.1), and (3.3). □

Remark. More systematic study of graph claspers as announced in [8, §8.2, §8.3] will appear elsewhere.

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