Maximum principal ratio of the signless Laplacian of graphs

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Abstract

Let $G$ be a connected graph and $Q(G)$ be the signless Laplacian of $G$. The principal ratio $\gamma(G)$ of $Q(G)$ is the ratio of the maximum and minimum entries of the Perron vector of $Q(G)$. In this paper, we consider the maximum principal ratio $\gamma(G)$ among all connected graphs of order $n$, and show that for sufficiently large $n$ the extremal graph is a kite graph obtained by identifying an end vertex of a path to any vertex of a complete graph.

Keywords: Principal ratio; Kite graph; Signless Laplacian.

AMS Classification: 05C50; 15A18.

1. Introduction

In this paper, we consider only simple, undirected graphs, i.e., undirected graphs without multiple edges or loops. The signless Laplacian of a graph $G$ is defined as $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of $G$, and $A(G)$ is the adjacency matrix of $G$. The largest eigenvalue of $Q(G)$, denoted by $q_1(G)$, is referred to as the $Q$-spectral radius of $G$. For a connected graph $G$, the Perron–Frobenius theorem implies that $Q(G)$ has a unique positive unit eigenvector $x$ corresponding to $q_1(G)$, which is called the principal $Q$-eigenvector of $G$. Let $x_{\min}$ and $x_{\max}$ be the smallest and the largest entries of $x$, respectively. We define the principal ratio $\gamma(G)$ of $Q(G)$ as

$$\gamma(G) := \frac{x_{\max}}{x_{\min}}.$$ 

Evidently, $\gamma(G) \geq 1$ with equality if and only if $G$ is regular. Therefore, it can be considered as a measure of graph irregularity. In this paper, we consider the principal ratio of graphs and determine the unique extremal graph maximizing $\gamma(G)$ among all connected graphs on $n$ vertices for sufficiently large $n$.

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The principal ratio of the adjacency matrices of graphs has been well studied. In 1958, Schneider [6] presented an upper bound on eigenvectors of irreducible nonnegative matrices; for graphs it can be described as \( \gamma(G) \leq (\lambda_1(G))^{n-1} \), where \( \lambda_1(G) \) is the largest eigenvalue of the adjacency matrix of \( G \). Nikiforov [4] improved this result for estimating the gap of spectral radius between \( G \) and its proper subgraph. Subsequently, Cioabă and Gregory [1] slightly improved the previous results for \( \lambda_1(G) > 2 \). In addition, they also proved some lower bounds on \( \gamma(G) \), which improved previous results of Ostrowski [5] and Zhang [8].

Recall that the kite or lollipop graph, denoted \( P_r \cdot K_s \), is obtained by identifying an end vertex of the path \( P_r \) to any vertex of the complete graph \( K_s \). In 2007, Cioabă and Gregory [1] initially posed the conjecture that among all connected graphs of order \( n \), the kite graph attains the maximum ratio of the largest and smallest Perron vector entries of \( A(G) \). In 2018, Tait and Tobin [7] confirmed the conjecture for sufficiently large \( n \). Recently, Liu and He [3] improved the condition as \( n \geq 5000 \). Using the method posed by Tait and Tobin [7] we can prove the main result of this paper as follows.

**Theorem 1.1.** For sufficiently large \( n \), the connected graph \( G \) on \( n \) vertices with maximum principal ratio of \( Q(G) \) is a kite graph.

## 2. Preliminaries

For graph notation and concepts undefined here, we refer the reader to [2]. Given a connected graph \( G \) on \( n \) vertices and a vertex \( v \) in \( G \), we write \( N_G(v) \) for the set of neighbors of \( v \) and \( d_G(v) \) the degree of \( v \) in \( G \). For convenience, we drop the subscript when it is understood. Recall that a pendant path is a path with one end vertex of degree one and all the internal vertices of degree two.

Let \( G \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). If \( q_1(G) > 4 \), we denote

\[
\sigma(G) := \frac{q_1(G) - 2 + \sqrt{q_1(G)^2 - 4q_1(G)}}{2}.
\]

Let \( x \) be the principal \( Q \)-eigenvector of \( G \). Hereafter, we write \( x_i \) for the entry of \( x \) corresponding to the vertex \( v_i \in V(G) \).

The following lemma gives an upper bound for \( \gamma(G) \) in terms of \( \sigma(G) \).

**Lemma 2.1.** Let \( G \) be a connected graph, and \( x \) be the principal \( Q \)-eigenvector of \( G \). Suppose that \( v_1, v_2, \ldots, v_k \) is a shortest path between \( v_1 \) and \( v_k \), where \( v_1 \) attains the minimum component of \( x \) and \( v_k \) attains the maximum. If \( q_1(G) > 4 \) and \( x_k = 1 \), then for \( 1 \leq j \leq k \),

\[
\gamma(G) \leq \frac{\sigma(G)^j - \sigma(G)^{-j} + \sigma(G)^j - 1 - \sigma(G)^{-j} - 1}{\sigma(G) - \sigma(G)^{-1}} \cdot \frac{1}{x_j},
\]

with equality if \( v_1, v_2, \ldots, v_j \) form a pendant path.

**Proof.** Set \( q := q_1(G) \) for short. By the eigenvalue equations of \( Q(G) \) we have

\[
(q - 1)x_1 \geq x_2
\]
\[(q - 2)x_2 \geq x_1 + x_3 \]
\[(q - 2)x_3 \geq x_2 + x_4 \]
\[\vdots \]
\[(q - 2)x_{j-1} \geq x_{j-2} + x_j. \]

Based on the first two inequalities, we have
\[x_1 \geq \frac{x_2}{q - 1}, \quad x_2 \geq \frac{q - 1}{(q - 2)(q - 1) - 1}x_3. \]

Now we assume that \[x_i \geq \frac{U_{i-1}-x_{i+1}}{U_i} \]
with \(U_j \) positive for all \(j \leq i \). Since \((q - 2)x_{i+1} \geq x_i + x_{i+2} \), we get
\[x_{i+1} \geq \frac{U_i}{(q - 2)U_i - U_{i-1}}x_{i+2}, \]
where \((q - 2)U_i - U_{i-1} > 0 \) as \((q - 2)x_{i+1} > x_i \geq x_{i+1}U_{i-1}/U_i \). So we let
\[U_{i+1} = (q - 2)U_i - U_{i-1} \]
with \(U_0 = 1, U_1 = q - 1 \). Solving this recurrence and using the initial conditions, we obtain
\[U_i = \frac{\sigma(G)^{i+1} - \sigma(G)^{-(i+1)} + \sigma(G)^i - \sigma(G)^{-i}}{\sigma(G) - \sigma(G)^{-1}}. \tag{2.2} \]

As a consequence,
\[x_1 \geq \frac{U_0}{U_1} \cdot x_2 \geq \prod_{i=1}^{2} \frac{U_{i-1}}{U_i} \cdot x_3 \geq \cdots \geq \prod_{i=1}^{j-1} \frac{U_{i-1}}{U_i} \cdot x_j = \frac{x_j}{U_{j-1}}. \]

Hence, for \(1 \leq j \leq k \),
\[\gamma(G) = \frac{x_k}{x_1} = \frac{1}{x_1} \leq \frac{U_{j-1}}{x_j}. \]

Finally, if \(v_1, v_2, \ldots, v_j \) form a pendant path, then we have all equalities throughout, as desired. \(\square\)

**Lemma 2.1** will be used frequently in the sequel. For the sake of convenience, we denote
\[U_i(G) := \frac{\sigma(G)^{i+1} - \sigma(G)^{-(i+1)} + \sigma(G)^i - \sigma(G)^{-i}}{\sigma(G) - \sigma(G)^{-1}} \]
for a connected graph \(G \).

**Remark 2.1.** Consider the kite graph \(P_k \cdot K_{n-k+1} \). It is straightforward to check that the smallest entry of the principal \(Q\)-eigenvector is the vertex of degree 1 and the largest is the vertex of degree \(n - k + 1 \). By **Lemma 2.1**, we have
\[\gamma(P_k \cdot K_{n-k+1}) = U_{k-1}(P_k \cdot K_{n-k+1}). \]
Lemma 2.2. Let \( j \geq 2 \) and \( q := q_1(G) > 4 \). Then
\[
(q - 1)
\left(q - 2 - \frac{1}{q-3}\right)^{j-2} \leq U_{j-1}(G) \leq (q - 1)
\left(q - 2 - \frac{1}{q}\right)^{j-2}.
\tag{2.3}
\]

Proof. Set \( U_i := U_i(G) \) for short. Recall that \( U_1 = q - 1 \) and \( U_{i+1} = (q - 2)U_i - U_{i-1} \). If \( j = 2 \), the assertion holds trivially, so we assume that \( j \geq 3 \).

Observe that
\[
\frac{U_{j-1}}{q - 1} = \frac{U_{j-1}}{U_1} = \prod_{i=2}^{j-1} \frac{U_i}{U_{i-1}},
\]
it turns to prove that for \( 2 \leq i \leq j - 1 \),
\[
q - 2 - \frac{1}{q - 3} \leq \frac{U_i}{U_{i-1}} \leq q - 2 - \frac{1}{q}.
\]
We first prove the right-hand side of (2.3) by induction on \( i \). For \( i = 2 \), recall that \( U_0 = 1 \) and \( U_1 = q - 1 \) we have
\[
\frac{U_2}{U_1} = \frac{(q - 2)U_1 - U_0}{U_1} = q - 2 - \frac{1}{q - 1} < q - 2 - \frac{1}{q}.
\]
By induction, we assume that \( U_{j-2}/U_{j-3} < q - 2 - 1/q \). Then
\[
\frac{U_{j-1}}{U_{j-2}} = \frac{(q - 2)U_{j-2} - U_{j-3}}{U_{j-2}} < q - 2 - \frac{1}{q - 2 - 1/q} < q - 2 - \frac{1}{q},
\]
as desired. Likewise, the inequality of the left-hand side of (2.3) can be proved by analogous arguments as above. \( \square \)

3. Proof of Theorem 1.1

Throughout the remainder of this paper, we always assume \( G \) is a graph maximizing \( \gamma(G) \) among all connected graphs on \( n \) vertices, where \( n \) is large enough. Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( x \) be the principal \( Q \)-eigenvector of \( G \) with maximum entry is 1. Without loss of generality, assume that \( v_1 \) attains the minimum entry of \( x \), while \( v_k \) attains the maximum, that is, \( x_k = 1 \). Suppose that \( v_1, v_2, \ldots, v_k \) is a shortest path of length \( (k-1) \) between \( v_1 \) and \( v_k \). Hereafter, we use \( C \) to denote the set \( V(G) \setminus \{v_1, \ldots, v_k\} \), and write \( S := C \cap N(v_{k-1}) \). For convenience, we always set \( q := q(G) \), \( \sigma := \sigma(G) \) and \( U_j := U_j(G) \).

In this section, the proof of our main result is presented. To this end, we divide this section into three subsections. First, we show that the vertices \( v_1, v_2, \ldots, v_{k-2} \) form a pendant path and that \( v_k \) is connected to all of the vertices that are not on this path (Subsection 3.1). Next, we show that \( v_{k-2} \) has degree exactly two (Subsection 3.2). Based on previous results, we then show that \( v_{k-1} \) also has degree exactly two (Subsection 3.3). Finally, we prove that connecting any non-edge in \( V(G) \setminus \{v_1, \ldots, v_k\} \) will increase the principal ratio, and hence, the extremal graph is exactly a kite graph.

Let us remark that \( q > 4 \). Indeed, if \( q \leq 4 \), then \( G \) must be one of the path \( P_n \), the cycle \( C_n \) and the star \( K_{1,3} \) whose principal ratio is less than that of \( P_{n-2} \cdot K_3 \).
3.1. Some auxiliary results

**Lemma 3.1.** The following statements hold.

1. \( d(v_k) = n - k + 1 \).
2. \( 2(n-k) < q < 2(n-k+1) \).
3. \( v_1, v_2, \ldots, v_{k-2} \) form a pendant path in \( G \).

**Proof.** Let \( H = P_k \cdot K_{n-k+1} \). By maximality and **Lemma 2.1**, we see

\[
U_{k-1}(G) \geq \gamma(G) \geq \gamma(H) = U_{k-1}(H).
\]

Notice that the function

\[
f(x) := \frac{x^k - x^{-k} + x^{k-1} - x^{-(k-1)}}{x - x^{-1}}\]

is increasing whenever \( x \geq 1 \). Hence, \( \sigma(G) \geq \sigma(H) \). It follows that \( q \geq q_1(H) > 2(n-k) \).

According to eigenvalue equation for \( v_k \) and \( x_k = 1 \), we find that

\[
2(n-k) < q = qx_k = d(v_k)x_k + \sum_{v \in N(v_k)} x_v < 2d(v_k), \tag{3.1}
\]

which implies \( d(v_k) \geq n - k + 1 \). On the other hand, \( v_k \) has no neighbors in \( \{v_1, \ldots, v_{k-2}\} \), as otherwise there would be a shorter path between \( v_1 \) and \( v_k \). Thus, \( d(v_k) \leq n - k + 1 \). So we obtain \( d(v_k) = n - k + 1 \). The inequality of the right-hand side of item (2) follows from (3.1) and \( d(v_k) = n - k + 1 \).

Finally, we prove the item (3). Since \( d(v_k) = n - k + 1 \), we have \( N(v_k) = C \cup \{v_{k-1}\} \). It follows that \( v_1, \ldots, v_{k-3} \) have no neighbors off the path, otherwise there would be a shorter path between \( v_1 \) and \( v_k \). Hence, \( v_1, v_2, \ldots, v_{k-2} \) form a pendant path in \( G \). \( \Box \)

To prove our main result, we need to make an estimation on \( k \), as stated in the following lemma.

**Lemma 3.2.** \( n - k = (1 + o(1)) \frac{n}{\log n} \).

**Proof.** Let \( H = P_j \cdot K_{n-j+1} \), where \( j = \lfloor n - n/\log n \rfloor \). From **Lemma 2.1** and **Lemma 2.2**, we find that

\[
\gamma(H) = U_{j-1}(H) > (q_1(H) - 3)^{j-1},
\]

and

\[
\gamma(G) \leq U_{k-1} < (q-1)^{k-1}.
\]

Since \( q_1(H) > 2(n-j) \) and \( q < 2(n-k+1) \), by the maximality of \( \gamma(G) \), we have

\[
(2(n-k) + 1)^{k-1} > (2(n-j) - 3)^{j-1}. \tag{3.2}
\]

Solving (3.2), we obtain \( n - k = (1 + o(1)) \frac{n}{\log n} \). \( \Box \)
Lemma 3.3. \( q/2 - 2 < \|x\|^2 < q/2 + 3. \)

Proof. We first prove that \( \|x\|^2 > q/2 - 2. \) Indeed,

\[
\|x\|^2 > x_k^2 + \sum_{v \in N(v_k)} x_v^2 \geq 1 + \left( \frac{\sum_{v \in N(v_k)} x_v}{n-k+1} \right)^2 = 1 + \left( \frac{q - (n - k + 1)}{n-k+1} \right)^2.
\]

Since \( q > 2(n - k), \) we see \( n - k + 1 < q/2 + 1. \) It follows that

\[
\|x\|^2 > 1 + \left( \frac{q/2 - 1}{q/2 + 1} \right)^2 > \frac{q}{2} - 2.
\]

Next, we shall prove the right-hand side. Set \( x_0 = 0. \) Then for \( 1 \leq i \leq k - 3, \) we have

\[
(q - 2)x_i = x_{i-1} + x_{i+1} < x_i + x_{i+1},
\]

which implies that

\[
x_i < \frac{x_{i+1}}{q - 3} < \cdots < \frac{x_{k-2}}{(q - 3)^{k-2-i}} < \frac{1}{(q - 3)^{k-2-i}}.
\]

It follows from the above inequalities and \( q < 2(n - k + 1) \) that

\[
\|x\|^2 < x_k + x_{k-2} + \sum_{v \in N(v_k)} x_v + \sum_{i=1}^{k-3} x_i < \frac{q}{2} + 3,
\]

as desired. \( \square \)

Lemma 3.4. For every subset \( U \) of \( N(v_k), \) we have

\[
|U| - 2 < \sum_{v \in U} x_v \leq |U|. \tag{3.3}
\]

Proof. The upper bound is clear from \( x_v \leq 1 \) for \( v \in V(G). \) The lower bound follows from the inequalities

\[
\sum_{v \in N(v_k) \setminus U} x_v \leq |N(v_k)| - |U|,
\]

and

\[
\sum_{v \in N(v_k) \setminus U} x_v = q - |N(v_k)| > |N(v_k)| - 2.
\]

The last inequality is due to the fact \( q > 2(n - k). \) This completes the proof of the lemma. \( \square \)
3.2. The vertex degree of \(v_{k-2}\) is two

**Lemma 3.5.** The vertex \(v_{k-2}\) has degree exactly 2 in \(G\).

*Proof.* Assume by contradiction that \(d(v_{k-2}) \geq 3\). Set \(T := N(v_{k-2}) \cap N(v_k)\) for short. Then \(d(v_{k-2}) = |T| + 1\) and \(|T| \geq 2\). Our proof hinges on the following claims.

**Claim 3.1.** \(\sum_{v \in T} x_v < 1\).

*Proof.* By the eigenvalue equation for \(v_{k-2}\), we get
\[
(q - (|T| + 1)) x_{k-2} = x_{k-3} + \sum_{v \in T} x_v.
\]
Noting that \(x_{k-3} = x_{k-2} U_{k-4}/U_{k-3}\) and \(U_{k-2} = (q - 2) U_{k-3} - U_{k-4}\), we have
\[
\left(\frac{U_{k-2}}{U_{k-3}} - (|T| - 1)\right) x_{k-2} = \sum_{v \in T} x_v.
\]
In light of Lemma 2.1 and Lemma 3.1, we have \(\gamma(G) = U_{k-3}/x_{k-2}\). Therefore,
\[
\gamma(G) = \frac{U_{k-3}}{x_{k-2}} = \left(U_{k-2} - (|T| - 1) U_{k-3}\right) \left(\sum_{v \in T} x_v\right)^{-1}.
\] (3.4)
If \(\sum_{v \in T} x_v \geq 1\), it follows from (3.4) that \(\gamma(G) < U_{k-2}\). On the other hand, let \(H = P_{k-1} \cdot K_{n-k+2}\). By the maximality and Lemma 2.1, we get
\[
\gamma(G) \geq \gamma(H) = U_{k-2}(H),
\]
which, together with \(\gamma(G) < U_{k-2}\), implies that \(q > q(H) > 2(n - k + 1)\), a contradiction yielding \(\sum_{v \in T} x_v < 1\).

**Claim 3.2.** \(d(v_{k-2}) \leq 3\).

*Proof.* If \(d(v_{k-2}) \geq 4\), then \(|T| \geq 3\). From Lemma 3.4, we have \(\sum_{v \in T} x_v > |T| - 2 \geq 1\), a contradiction to Claim 3.1.

Combining with our assumption for contradiction that \(d(v_{k-2}) \geq 3\), we derive that \(d(v_{k-2}) = 3\). Let \(u\) be the unique vertex in \(C\) adjacent to \(v_{k-2}\). We may assume that \(x_{k-1} \leq x_u\), otherwise we choose another path \(v_1, \ldots, v_{k-2}, u, v_k\).

Now we continue to prove this lemma by considering the vertex degree of \(v_{k-1}\). For simplicity, write \(d := d(v_{k-1})\). Recall that \(S = N(v_{k-1}) \cap C\). Then \(d = |S| + 2\).

**Case 1.** \(d(v_{k-1}) \geq 10\). Let \(G_1^- = G - \{v_{k-2}u\}\). Then \(v_1, v_2, \ldots, v_{k-1}\) form a pendant path in \(G_1^-\). Set \(q_1^- := q_1(G_1^-)\). By Rayleigh principle and Lemma 3.3,
\[
q_1^- - q \geq \frac{x(Q(G_1^-) - Q(G))}{\|x\|^2} \geq -\frac{4}{\|x\|^2} \geq -\frac{8}{q - 4}.
\] (3.5)
Let \( x^- \) be the principal \( Q \)-eigenvector of \( G_1^- \) with maximum entry 1. To compare \( \gamma(G_1^-) \) with \( \gamma(G) \), we use Lemma 2.1 and Lemma 2.2 to bound them. Then we have

\[
\gamma(G_1^-) = \frac{U_{k-2}(G_1^-)}{x_{k-1}} > \left( q_1^- - 2 - \frac{1}{q_1^- - 3} \right)^{k-3} \frac{q_1^- - 1}{x_{k-1}},
\]

and

\[
\gamma(G) = \frac{U_{k-3}}{x_{k-2}} < \left( q - 2 - \frac{1}{q} \right)^{k-4} \frac{q - 1}{x_{k-2}}. \tag{3.6}
\]

Combining the above two inequalities, we deduce that

\[
\frac{\gamma(G_1^-)}{\gamma(G)} > \frac{q_1^- - 1}{q - 1} \cdot \left( \frac{q_1^- - 2 - (q_1^- - 3)^{-1}}{q - 2 - 1/q} \right)^{k-4} \frac{(q_1^- - 3) \cdot x_{k-2}}{x_{k-1}}. \tag{3.7}
\]

To proceed further, we first consider the ratio \( x_{k-2}/x_{k-1} \). Since \( (q - 3)x_{k-2} = x_{k-3} + x_{k-1} + x_u > 2x_{k-1} \), we see

\[
\frac{x_{k-2}}{x_{k-1}} > \frac{2}{q - 3}.
\]

Furthermore, by Lemma 3.4,

\[
(q - d)x_{k-1} > x_k + \sum_{v \in S} x_v > |S| - 1 = d - 3.
\]

Then \( x_{k-1} > (d - 3)/(q - d) \). In addition, \( x_{k-1} < d/(q_1^- - d) \). In view of (3.5) and \( d \geq 10 \), we deduce that

\[
\frac{x_{k-1}}{x_{k-2}} > \left( 1 - \frac{3}{d} \right) \left( \frac{q_1^- - d}{q - d} \right) > \frac{3}{5}.
\]

As a consequence,

\[
\frac{x_{k-2}}{x_{k-1}} = \frac{x_{k-2}}{x_{k-1}} \cdot \frac{x_{k-1}}{x_{k-1}} > \frac{6}{5(q - 3)}.
\]

Combining with (3.7) and (3.5), we obtain

\[
\frac{\gamma(G_1^-)}{\gamma(G)} > \frac{11}{10} \left( \frac{q_1^- - 2 - (q_1^- - 3)^{-1}}{q - 2 - 1/q} \right)^{k-4} \frac{q_1^- - 1}{x_{k-1}} \cdot \left( \frac{q_1^- - 1}{x_{k-1}} \right) \cdot \left( \frac{q_1^- - 1}{x_{k-1}} \right)
\]

\[
> \frac{11}{10} \left( 1 - \frac{10}{q_1^-} \right)^{k-4} \frac{q_1^- - 1}{x_{k-1}} \cdot \left( \frac{q_1^- - 1}{x_{k-1}} \right) \cdot \left( \frac{q_1^- - 1}{x_{k-1}} \right)
\]

\[
> 1.
\]

The third inequality follows from Bernoulli’s inequality. Hence, \( \gamma(G_1^-) > \gamma(G) \), a contradiction.

**Case 2.** \( d(v_{k-1}) \leq 9 \). Let \( G_2^- \) be the graph obtain from \( G \) by deleting edges \( \{v_{k-1}v : v \in S\} \) and \( \{uv_{k-2}\} \). Denote \( q_2^- := q_1(G_2^-) \). We have

\[
\gamma(G_2^-) = U_{k-1}(G_2^-) > (q_2^- - 1) \left( q_2^- - 2 - \frac{1}{q_2^- - 3} \right)^{k-2}.
\]

It follows from (3.6) that
\[
\frac{\gamma(G_2^\ast)}{\gamma(G)} > \frac{q_2 - 1}{q - 1} \cdot \left(\frac{q_2 - 2 - (q_2 - 3)^{-1}}{q - 2 - 1/q}\right)^{k-4} \cdot (q_2 - 3)^2 \cdot x_{k-2}.
\]
Since \((q - 3)x_{k-2} > 2x_{k-1}\) and \((q - 2)x_{k-1} \geq (q - d(v_{k-1}))x_{k-1} > 1\), then
\[
x_{k-2} > \frac{2}{(q - 2)(q - 3)}.
\]
Using similar arguments as above, we have \(\gamma(G_2^\ast) > \gamma(G)\), a contradiction. \(\square\)

3.3. The vertex degree of \(v_{k-1}\) is two

The next lemma gives a more precise upper bound for \(q_1(G)\).

**Lemma 3.6.** \(q < 2(n - k) + 3/2\).

**Proof.** Assume for contradiction that \(q \geq 2(n - k) + 3/2\). We have the following claims.

**Claim 3.3.** \(x_v \geq 1/2\) for each \(v \in N(v_k)\).

**Proof.** If there is a vertex \(w \in N(v_k)\) such that \(x_w < 1/2\), then
\[
n - k + \frac{1}{2} \leq (q - (n - k + 1))x_k = \sum_{u \in N(v_k)} x_u < \frac{1}{2} + (n - k),
\]
a contradiction completing the proof of the claim. \(\square\)

**Claim 3.4.** There is a vertex \(z \in C\) such that \(v_{k-1}z \notin E(G)\).

**Proof.** Assume for contradiction that \(v_{k-1}v \in E(G)\) for each \(v \in C\). From the eigenvalue equations for \(v_{k-1}\) and \(v_k\), we get \((q - n + k - 1)(x_{k-1} - x_k) = x_k + x_{k-2} > 0\), and therefore \(x_{k-1} > x_k = 1\), which leads to a contradiction. This completes the proof of the claim. \(\square\)

Now, let \(G^+ = G + \{v_{k-1}z\}\), and \(x^+\) be the principal \(Q\)-eigenvector of \(G^+\) with maximum entry 1. Our goal is to show \(\gamma(G^+) > \gamma(G)\), and therefore deduce a contradiction.

Set \(q^+ := q_1(G^+)\) for short. By the Rayleigh principle and **Claim 3.3** we obtain
\[
q^+ - q \geq \frac{x^T Q(G^+) x - x^T Q(G) x}{\|x\|^2} = \frac{(x_{k-1} + x_z)^2}{\|x\|^2} > \frac{1}{\|x\|^2},
\]
which, together with **Lemma 3.3**, gives
\[
q^+ > q + \frac{2}{q + 6}.
\]
(3.8)

The next claim gives an upper bound for \(x^+_{k-1} - x_{k-1}\).

**Claim 3.5.** \(x^+_{k-1} - x_{k-1} < 4/(n - k)\).
Proof. By the eigenvalue equations for \( q \) and \( q^+ \) with respect to \( v_{k-1} \), we have

\[
(q - |S| - 2)x_{k-1} = x_{k-2} + x_k + \sum_{v \in S} x_v,
\]

\[
(q^+ - |S| - 3)x_{k-1}^+ = x_{k-2}^+ + x_k^+ + z^+ + \sum_{v \in S} x_v^+.
\]

It follows from \( q^+ > q \) that

\[
(q - |S| - 2)(x_{k-1}^+ - x_{k-1}) < (x_{k-2}^+ - x_{k-2}) + (x_k^+ + x_{k-1}^+ + z^+ - 1) + \sum_{v \in S} (x_v^+ - x_v)
\]

\[
< 3 + \sum_{v \in S} (x_v^+ - x_v).
\]

It remains to bound \( \sum_{v \in S} (x_v^+ - x_v) \). To this end, note that

\[
\sum_{v \in N(v_k) \setminus S} x_v \leq d(v_k) - |S| \quad \text{and} \quad \sum_{v \in N(v_k)} x_v = q - (n - k + 1) \geq d(v_k) - \frac{1}{2}.
\] (3.9)

We immediately obtain

\[
|S| - \frac{1}{2} \leq \sum_{v \in S} x_v \leq |S|.
\]

Obviously, \( \sum_{v \in S} x_v^+ \leq |S| \). Therefore,

\[
\sum_{v \in S} (x_v^+ - x_v) = \sum_{v \in S} x_v^+ - \sum_{v \in S} x_v \leq \frac{1}{2}.
\]

Putting the above inequalities together, we arrive at

\[
x_{k-1}^+ - x_{k-1} < \frac{7}{2(q - |S| - 2)} < \frac{4}{n - k}.
\]

The last inequality follows from the facts \(|S| \leq n - k\) and \( q > 2(n - k) \). This completes the proof of the claim.

To compare \( \gamma(G^+) \) with \( \gamma(G) \), we use Lemma 2.1 and Lemma 2.2 to bound them. On the one hand,

\[
\gamma(G^+) = \frac{U_{k-2}}{x_{k-1}^+} > \left(q^+ - 2 - \frac{1}{q^+ - 3}\right)^{k-3} \cdot \frac{q^+ - 1}{x_k^+}.
\]

On the other hand, by Lemma 3.5 we have

\[
\gamma(G) = \frac{U_{k-2}}{x_{k-1}} < \left(q - 2 - \frac{1}{q}\right)^{k-3} \cdot \frac{q - 1}{x_k}.
\]

Combining (3.8) we deduce that

\[
\frac{\gamma(G^+)}{\gamma(G)} > \left(\frac{q^+ - 2 - (q^+ - 3)^{-1}}{q - 2 - 1/q}\right)^{k-3} \cdot \frac{x_{k-1}}{x_k^+}.
\]
\[
> \left(1 + \frac{3}{2q^2}\right)^{k-3} \cdot \frac{x_{k-1}}{x_{k-1}^+}.
\]

It follows from Bernoulli's inequality that
\[
\frac{\gamma(G^+)}{\gamma(G)} > \left(1 + \frac{3(k - 3)}{2q^2}\right) \cdot \frac{x_{k-1}}{x_{k-1}^+} > \left(1 + \frac{k}{q^2}\right) \cdot \frac{x_{k-1}}{x_{k-1}^+}.
\]

(3.10)

To finish the proof, we consider the following two cases.

**Case 1.** The maximum eigenvector entry of \(x^+\) is still attained by vertex \(v_k\).

Using the same arguments as Claim 3.3, we get \(x_{k-1}^+ > 1/2\). Together with Claim 3.5 gives
\[
\frac{x_{k-1}}{x_{k-1}^+} > 1 - \frac{4}{(n-k)x_{k-1}^+} > 1 - \frac{8}{n-k}.
\]

In light of (3.10) and Lemma 3.2 we find
\[
\frac{\gamma(G^+)}{\gamma(G)} > \left(1 + \frac{k}{q^2}\right)\left(1 - \frac{8}{n-k}\right) > 1.
\]

**Case 2.** The maximum eigenvector entry of \(x^+\) is no longer attained by vertex \(v_k\).

For any vertex \(v \in C\), by the eigenvalue equations for \(v_k\) and \(v\), we get
\[
(q^+ - (n-k+1))x_k^+ = x_{k-1}^+ + \sum_{u \in C} x_u^+,
\]
\[
(q^+ - d_{G^+}(v))x_v^+ \leq x_{k-1}^+ + x_k^+ + \sum_{u \in C \setminus \{v\}} x_u^+,
\]
which imply that \(x_k^+ \geq x_v^+\). Hence the maximum entry of \(x^+\) must be attained by \(v_{k-1}\).

Therefore, \(\gamma(G^+) = U_{k-2}^+\). Applying Claim 3.5 to \(x_{k-1}^+ = 1\), we see \(x_{k-1} > 1 - 4/(n-k)\). By (3.10) again, we have \(\gamma(G^+) > \gamma(G)\).

Summing the above two cases, we see \(\gamma(G^+) > \gamma(G)\), which is a contradiction to the maximality of \(\gamma(G)\).

Based on Lemma 3.6, we can give a precise estimation for \(x_{k-1}\).

**Lemma 3.7.** \(x_{k-1} < n^{-1/6}\).

**Proof.** Let \(H = P_{k-1} \cdot K_{n-k+2}\). In view of Lemma 2.1 and Lemma 2.2 we conclude that
\[
\gamma(H) = U_{k-2}(H) \geq \left(q_1(H) - 2 - \frac{1}{q_1(H)-3}\right)^{k-3} \cdot (q_1(H) - 1)
\]
\[
> \left(2(n-k) - \frac{1}{2(n-k) - 1}\right)^{k-3} (2n - 2k + 1).
\]

On the other hand, from Lemma 3.6 we have
\[
\gamma(G) = \frac{U_{k-2}}{x_{k-1}} < \left(2(n-k) - \frac{1}{2} - \frac{1}{2(n-k) + 3/2}\right)^{k-3} \cdot \frac{2(n-k) + 1/2}{x_{k-1}}.
\]
Since $\gamma(G) \geq \gamma(H)$ we deduce that
\[
x_{k-1} < \left( \frac{(2n-k) - 1/2 - (2n-2k+3/2)^{-1}}{2(n-k) - (2n-2k-1)^{-1}} \right)^{k-3}
< \left( 1 - \frac{1}{5(n-k)} \right)^{k-3}
< e^{-\frac{k-3}{5(n-k)}}
< n^{-1/6}.
\]

The last inequality uses the fact that $(k-3)/(n-k) > 5 \cdot (\log n)/6$ by Lemma 3.2.

**Lemma 3.8.** The degree of $v_{k-1}$ is 2 in $G$.

**Proof.** It suffices to show that $|S| = 0$. Assume for contradiction that $|S| \geq 1$. Let $G^-$ be the graph obtained from $G$ by removing these $|S|$ edges, i.e., $G^- = G - \{v_{k-1}v : v \in S\}$. Our goal is to show that $\gamma(G^-) > \gamma(G)$, and therefore get a contradiction.

We first show that there is at most one vertex in $C$ such that its component in $x$ no more than $(1 - x_{k-1})/2$. Indeed, if there are $v, w \in C$ such that $x_v, x_w \leq (1 - x_{k-1})/2$, then
\[
n - k - 1 < (q - (n-k+1))x_k
= x_{k-1} + \sum_{u \in C} x_u
\leq x_{k-1} + x_v + x_w + (n-k-2)
\leq n - k - 1,
\]
which leads to a contradiction.

We consider the following two cases.

**Case 1.** $|S| \geq 2$. By eigenvalue equation we see
\[
(q - |S| - 2)x_{k-1} > 1 + \sum_{u \in S} x_u > 1 + \frac{(|S| - 1)(1 - x_{k-1})}{2} > \frac{|S|(1 - x_{k-1})}{2}.
\]  
Solving this inequality gives
\[
|S| < \frac{2(q - 2)x_{k-1}}{1 + x_{k-1}}.
\]

Combining with Lemma 3.3 and Lemma 3.7, we deduce that
\[
q - q^- \leq \frac{|S|(1 + x_{k-1})^2}{||x||^2} < \frac{2|S|(1 + x_{k-1})^2}{q - 4} < 5x_{k-1} < \frac{5}{n^{1/6}}.
\]  
By (3.11) again, we have
\[
(q - |S| - 2)x_{k-1} > 1 + \frac{(|S| - 1)(1 - x_{k-1})}{2},
\]

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which implies that
\[ x_{k-1} > \frac{|S| + 1}{2q - |S| - 5} \geq \frac{3}{2q - 7}. \]  
(3.13)
The last inequality is due to our assumption \(|S| \geq 2\).

Now we are ready to compare \(\gamma(G^-)\) with \(\gamma(G)\). By Lemma 2.2 and (3.13) we see
\[ \gamma(G) = \frac{U_{k-2}}{x_{k-1}} < \left(q - 2 - \frac{1}{q}\right)^{k-3} \cdot \frac{q - 1}{x_{k-1}} < \left(q - 2 - \frac{1}{q}\right)^{k-3} \cdot \frac{(q - 1)(2q - 7)}{3}. \]
On the other hand,
\[ \gamma(G^-) = U_{k-1}(G^-) \geq \left(q^{-} - 2 - \frac{1}{q^{-} - 3}\right)^{k-2} (q^{-} - 1). \]
Hence, the above two inequalities, together with (3.12), imply that
\[ \frac{\gamma(G^-)}{\gamma(G)} > \frac{6}{5} \cdot \left(\frac{q^{-} - 2 - (q^{-} - 3)^{-1}}{q - 2 - q^{-1}}\right)^{k-3} > \frac{6}{5} \cdot \left(1 - \frac{6}{n^{1/6}q}\right)^{k-3} > 1, \]
a contradiction.

**Case 2.** \(|S| = 1\). Let \(w\) be the unique vertex in \(C\) that adjacent to \(v_{k-1}\). If \(x_w > (1 - x_{k-1})/2\), we can deduce a similar contradiction by placing (3.13) with
\[ x_{k-1} > \frac{1 + x_w}{q - 3} > \frac{6}{5q} \]
in the proof of Case 1.

Hence, we assume \(x_w \leq (1 - x_{k-1})/2\), and therefore \(x_v > (1 - x_{k-1})/2\) for each \(v \in C \setminus \{w\}\). Evidently, there is a vertex \(z \in C \setminus \{w\}\) such that \(z\) is not adjacent to \(w\). Let \(\tilde{G}\) be the graph obtained by deleting edge \(v_{k-1}w\) and adding \(wz\). Since \(x_z > x_w\), we obtain \(q_1(\tilde{G}) > q\). Set \(\tilde{\alpha} := q_1(\tilde{G})\) for short. We have
\[ \gamma(\tilde{G}) = U_{k-1}(\tilde{G}) \geq \left(\tilde{\alpha} - 2 - \frac{1}{\tilde{\alpha} - 3}\right)^{k-2} (\tilde{\alpha} - 1) > \left(q - 2 - \frac{1}{q - 3}\right)^{k-2} (q - 1). \]
It follows from \((q - 3)x_{k-1} \geq (q - d(v_{k-1}))x_{k-1} > 1\) that
\[ \frac{\gamma(\tilde{G})}{\gamma(G)} > \left(1 - \frac{4k}{q^3}\right) \cdot (q - 2 - o(1)) \cdot x_{k-1} \]
\[ > \left(1 - \frac{4k}{q^3}\right) \cdot \frac{q - 2 - o(1)}{q - 3} \]
\[ > 1, \]
a contradiction completing the proof of Case 2, and hence finishing the proof of Lemma 3.8.
Now we are ready to prove our main theorem.

Proof of Theorem 1.1. According to Lemma 3.1, Lemma 3.5 and Lemma 3.8, we derive that \(v_1, v_2, \ldots, v_k\) form a pendant path. Hence, it remains to show that the induced subgraph \(G[C]\) is a clique. Otherwise, let \(H = P_k \cdot K_{n-k+1}\). Then \(G\) is a proper subgraph of \(H\), then \(q_1(G) < q_1(H)\). On the other hand, from Lemma 2.1 and the maximality of \(\gamma(G)\), we have

\[
U_{k-1}(G) = \gamma(G) \geq \gamma(H) = U_{k-1}(H),
\]

which yields that \(q_1(G) \geq q_1(H)\), a contradiction completing the proof of this theorem.

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