Comment on “Unified view of quantum correlations and quantum coherence”

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We show that contrary to what it is claimed in Phys. Rev. A 94, 022329 (2016), in general the local projective measurement that induces maximal coherence loss is not the projection onto the local basis that defines the coherence of the system, at least for all quantum-incoherent states.

For a single-partite system $\rho = \sum_i \rho_i |i\rangle\langle i|$ with a reference (coherence-defining) basis $|i\rangle$, it is obvious that the measurement that maximally eliminates the coherence in the system is the projective measurement $\Pi_B(\rho) = \sum_j |j\rangle\langle j|$. Motivated by this intuition, in the Appendix of Ref. [4], the authors attempted to prove that the local projective measurement that induces maximal coherence loss is the projection onto the local basis that defines the coherence of the system, even for bipartite or multipartite quantum states, which is formally stated in the following.

**Proposition.** For any bipartite state $\rho_{AB} = \sum_{i,j,k,l} \rho_{ijkl} |i\rangle\langle i| |k\rangle\langle k|$, where the coherence is measured with respect to the local reference bases $|i\rangle_A$ and $|j\rangle_B$, the projective measurement on subsystem $B$ that induces maximal coherence loss is $\Pi_B(\rho_{AB}) = \sum_j (|i\rangle_A \otimes |j\rangle_B) \rho_{AB} (|i\rangle_A \otimes |j\rangle_B)$.

To review the (flawed) proof, here we adopt the same notations introduced in [4]. By using the spectral decomposition of $\rho = \sum_n \rho_n |\psi^n\rangle_{AB} \langle \psi^n|$, the authors assumed the following matrix representation of $\rho_{AB}$ with respect to the local reference bases $|i\rangle_A$ and $|j\rangle_B$:

$$\rho_{AB} = \sum_n \sum_{i,j,k,l} \rho_n |\psi^n_{ij}\rangle |i\rangle_A |k\rangle_B.$$  

(1)

Note that the coherence of the system is measured with respect to these bases. Consider some complete basis $\{|\lambda_m\rangle\}$ on $B$, and corresponding projective measurement $\Pi_B(\rho_{AB}) = \sum_m (|\lambda_m\rangle \otimes |\lambda_m\rangle) \rho_{AB} (|\lambda_m\rangle \otimes |\lambda_m\rangle)$. By computing the matrix elements, the authors in fact proved that the coherence of the post-measurement state $\Pi_B(\rho_{AB})$ is lower bounded by the coherence of the reduced state of subsystem $A$, namely

$$\sum_{i,j,k,l} \left| \sum_{i,j,k,l} \rho_{ijkl} |\psi^n_{ij}\rangle |i\rangle_A |k\rangle_B \right|^2 \geq \sum_{i,j,k,l} \left| \sum_{i,j,k,l} \rho_{ijkl} |\psi^n_{ij}\rangle |i\rangle_A |k\rangle_B \right|^2,$$

(2)

where $\rho_A = \sum_{i,j} \rho_{ij} |\psi^n_{ij}\rangle |i\rangle_A |j\rangle$. For comparison, when $|\lambda_m\rangle = |j\rangle$ the absolute sum of the elements of $\Pi_B(\rho_{AB})$ should be

$$\sum_{i,j,k,l} \left| \sum_{i,j} \rho_{ijkl} \langle\psi^n_{ij}| \right|^2 \leq \sum_{i,j,k,l} \left| \sum_{i,j} \rho_{ijkl} \langle\psi^n_{ij}| \right|^2.$$

(3)

Obviously, the right hand side of Eq. (3) is generally larger than (that is, inequivalent to) that of Eq. (2) and thus the claim of the authors is not correct.

For a simple counterexample for the Proposition, we can evaluate the coherence of the following state

$$\rho_{AB} = \frac{1}{2} |+\rangle_A \langle +| \otimes |0\rangle_B \langle 0| + \frac{1}{2} |\rangle_A \langle -| \otimes |1\rangle_B \langle 1|,$$

(4)

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and the computational basis is assumed to be the coherence-defining basis. Now if we perform a local projective measurement in the computational basis of subsystem $B$, the coherence of the bipartite system is invariant and no coherence loss occurs since the whole state keeps unchanged. However, if we perform a local projective measurement in the dual basis $|\lambda\rangle_B$, the coherence of the bipartite system is completely eliminated since in this case $\rho_B = \frac{1}{2} I_A \otimes I_B$ and the identity operator $I_{AB}$ is incoherent in any basis.

Indeed, this example can be extended to arbitrary quantum-incoherent states [3], which are of the following form

$$\chi_{AB} = \sum_i p_i |\lambda_i\rangle \otimes |\tilde{i}\rangle_B,$$

(5)

where $|\tilde{i}\rangle_B$ is the incoherent basis for subsystem $B$. Inspired by the above example, we perform a local projective measurement in a mutually unbiased basis $|\lambda_i\rangle_B$ with respect to $|\tilde{i}\rangle_B$ [3, 9], which means $|\langle \lambda_i | \tilde{j}\rangle_B|^2 = \frac{1}{d_B}$ for all $i$ and $j$ ($d_B$ denotes the dimension of subsystem $B$). In this circumstance, the post-measurement state is given as

$$\Gamma_B^{[\lambda_i]} (\chi_{AB}) = \sum_i p_i |\lambda_i\rangle \otimes \sum_j \frac{1}{d_B} |\lambda_i B \langle \lambda_j|) = \rho_A \otimes \frac{1}{d_B} I_B.$$  

(6)

For any valid coherence measure $C($•$)$ defined in the
framework of $[5]$, it is easy to see that

$$C(\Pi_B^{(i)}(\chi_{AB})) = C(\chi_{AB}) = \sum_i p_i C(\rho_i^A) \geq \sum_i p_i \rho_i^A = C(\rho_A) = C(\Pi_B^{(i)}(\chi_{AB})), \quad (7)$$

where we have used the convexity of $C(\bullet)$ $[5]$. The above inequality indicates that the maximal coherence loss is alternatively induced by a projective measurement in an arbitrary mutually unbiased basis (see also Eq. (2)), which is obviously opposed to the Proposition.

Note that, in Eq. (7), we have used the following lemma.

**Lemma 1.** For any valid coherence measure, we have $C(\rho \otimes \sigma_{\text{inc}}) = C(\rho)$, where $\sigma_{\text{inc}}$ is an incoherent state.

**Proof.** Here we present two alternative proofs. First, we can adopt the methodology in Ref. $[4]$, namely, the dismissal quantum operation (partial trace) and the appending quantum operation (with an incoherent state) are all incoherent operations. Therefore, by using the monotonicity of the coherence measures under incoherent operations $[3]$, we have the inequality

$$C(\rho) \geq C(\rho \otimes \sigma_{\text{inc}}) \geq C(\rho), \quad (8)$$

which implies $C(\rho \otimes \sigma_{\text{inc}}) = C(\rho)$. Alternatively, we can also employ the framework proposed in $[1]$, which is equivalent to that of $[6]$. Yu et al. proved that a valid coherence measure should satisfy the following condition: $C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$ for block-diagonal states $\rho$ in the incoherent basis. Thus, since $\sigma_{\text{inc}} = \sum_i p_i |i\rangle\langle i|$, we have

$$C(\rho \otimes \sigma_{\text{inc}}) = C(\oplus_i p_i \rho) = \sum_i p_i C(\rho) = C(\rho), \quad (9)$$

which completes the proof. \[\square\]

In conclusion, we have proved that the Proposition raised by the authors of Ref. $[1]$ is not valid, at least for all quantum-incoherent states, where the projective measurements in mutually unbiased bases play a significant role. Furthermore, we believe this problem is highly nontrivial and probably state-dependent. A thorough solution to this problem is still left as an open question.

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