NON-DENSE ORBITS OF SYSTEMS WITH APPROXIMATE PRODUCT PROPERTY

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Abstract. We show that for any topological dynamical system with approximate product property, the set of points whose forward orbits do not accumulate to any point in a large set carries full topological pressure.

1. Introduction
Let \((X, d)\) be a compact metric space and \(f : X \to X\) be a continuous map. For \(x \in X\), denote the forward orbit of \(x\) by
\[
O_f(x) := \{f^n(x) : n \in \mathbb{N}\}.
\]
For a subset \(Z \subset X\), denote the set of points whose forward orbits do not accumulate to any point in \(Z\) by
\[
N(f, Z) := \{x \in X : \overline{O_f(x)} \cap Z = \emptyset\}.
\]
The points in \(N(f, Z)\) have non-dense (forward) orbits. Study of such sets of non-dense orbits has motivation in homogeneous dynamics, where it is connected to Diophantine approximation. The Hausdorff dimensions of such sets are intensively investigated, which sometimes led to interesting results in number theory and other fields. For example, see [10, 11, 12, 22, 21, 6, 23, 2, 19, 1, 3]. Similar results are also established for more general hyperbolic or partially hyperbolic systems [33, 9, 14, 20, 32, 35, 36, 37]. Non-dense orbits are also closely related to irregular behaviors. [15, 16] contain an elaborated classification of the sets exhibiting various statistical behaviors as well as a multifractal analysis on them for hyperbolic systems.

In this article we illustrate a new approach, which studies the topological entropy and topological pressure carried by \(N(f, Z)\) from approximate product property, a very weak variation of Bowen’s specification property [5]. We show that there is a mechanism that produces plenty of disjoint compact \(f\)-invariant sets which consist of various non-dense orbits. Approximate product property was introduced by Pfister and Sullivan [28], which is almost the weakest specification-like property [24, 30, 31]. While Bowen’s original specification property requires strong hyperbolicity, approximate product property is compatible with certain non-hyperbolic behaviors. We perceive that systems with approximate product properties (APP systems for

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short) have delicate structures in many senses and the author has obtained some interesting results [30, 31].

Let $\phi : X \to \mathbb{R}$ be a continuous potential function. For any subset $Y \subset X$, denote by $P(Y, f, \phi)$ and $h(Y, f) = P(Y, f, 0)$ the topological pressure and the topological entropy on $Y$. Denote by $P(f, \phi) := P(X, f, \phi)$ and $h(f) := h(X, f)$ the topological pressure and topological entropy of the system. We state the main result of the article as follows.

**Theorem 1.1.** Let $(X, f)$ be an APP system with positive topological entropy and $\phi : X \to \mathbb{R}$ be a continuous potential function. Suppose that $Z = \bigcup_{i=1}^{n} Z_{i}$ is a finite union of subsets of $X$ such that for each $i = 1, \ldots, n$, one of the following holds:

1. $Z_{i}$ is any single forward orbit.
2. For any given $\mu_{i} \in \mathcal{M}(X, f)$, $Z_{i}$ consists of all points whose empirical measures accumulate to $\mu_{i}$. In particular, $Z_{i}$ may contain all generic points for $\mu_{i}$.
3. $Z_{i}$ is any compact $f$-invariant subset of $X$ such that $\mathcal{M}(Z_{i}, f) \neq \mathcal{M}(X, f)$.
4. $Z_{i}$ consists of the points with weak $K_{i}$-behavior, where $K_{i}$ is any compact subset of a proper weak face (see Definition 4.1 and 4.6).

Then $P(\mathcal{N}(f, Z), f, \phi) = P(f, \phi)$. In particular, $h(\mathcal{N}(f, Z), f) = h(f)$.

The case of $Z$ in Theorem 1.1 is just a noteworthy instance but not all of them. Our key result is Theorem 3.5. In this article we adopt Pesin-Pitskel’s definition of topological pressures on non-compact sets. See [26] or [27] for details. Results in this article remain valid if another definition (e.g. by $(n, \varepsilon)$-separated sets [34]) of $P(Y, f, \phi)$ is adopted as long as it coincides with Pesin-Pitskel’s definition whenever $Y$ is compact and $f$-invariant, i.e. $f(Y) \subset Y$. They also remain valid if $\phi$ is replaced by an asymptotically additive potential $\Phi$ introduced in [17], as long as the integral of the potential is continuous. Moreover, by [31], an APP system with zero topological entropy must be uniquely ergodic. In this case every point is generic for the unique ergodic measure hence $\mathcal{N}(f, Z)$ may be empty. Finally, we notice that [38] contains a similar result for the case that the system has specification property and $Z$ consists of just a single non-transitive point.

APP systems form a broad class that includes most familiar systems. The following provides an incomplete list of them, to which our results apply:

1. Transitive sofic shifts;
2. $\beta$-shifts;
3. Ergodic toral automorphisms;
4. Transitive graph maps;
5. A homogeneous system $(G/\Gamma, g)$, where $G$ is connected semisimple Lie group without compact factors, $\Gamma$ is an irreducible cocompact lattice of $G$ and $g \in G$ is non-quasimultipotent [18];
6. Every $C^{0}$-generic map $f$ on a compact Riemannian manifold restricted to every chain-recurrent class $C$ for $f$ [4];
7. Certain partially hyperbolic diffeomorphisms, e.g. transitive time-1 maps of Anosov flows;
8. A product of an APP system and a system with tempered specification property, e.g. the product of an irrational rotation and an ergodic toral automorphism;
9. Factors and conjugates of above systems.
Note that for symbolic systems Theorem 1.1 and Theorem 3.5 directly yield the corresponding results on the Hausdorff dimension of $N(f, Z)$, which generalize [14, Theorem 1]. Moreover, in the above list there are certain homogeneous systems. Our results for these systems are in some sense related to the conjectures of Margulis [25].

Theorem 1.1 is a direct corollary of Theorem 3.5 and Proposition 4.10. Note that among the cases of $Z_i$ in Theorem 1.1, Case (2) covers Case (1) and by Lemma 4.9, Case (4) covers Case (3). We shall introduce our notations in Section 2. Then we focus on APP systems in Section 3. Finally we explain the other notions involved in Theorem 1.1 and complete the proof of the theorem in the last section.

2. Notations

Let $(X, f)$ be a topological dynamical system. Denote by $\mathcal{M}(X)$ the space of probability measures on $X$, by $\mathcal{M}(X, f)$ the subspace of all invariant probability measures for $(X, f)$ and by $\mathcal{M}_e(X, f)$ the subset consisting of the ergodic ones. As $X$ is compact, both $\mathcal{M}(X)$ and $\mathcal{M}(X, f)$ are compact metrizable spaces in the weak-$\ast$ topology [34, Theorem 6.5 and Theorem 6.10].

Denote by $D$ a metric on $\mathcal{M}(X)$ that induces the weak-$\ast$ topology on $\mathcal{M}(X)$. Denote $K(X, f) := \{K : K$ is compact subset of $\mathcal{M}(X, f)\}$. Then $K(X, f)$ is a compact metric space with the Hausdorff metric $D_H(K_1, K_2) := \max\{\max_{\mu \in K_1} \min_{\nu \in K_2} D(\mu, \nu), \max_{\nu \in K_2} \min_{\mu \in K_1} D(\mu, \nu)\}$.

Denote by $\mathbb{Z}^+$ the set of all positive integers. For $x \in X$ and $n \in \mathbb{Z}^+$, we define the empirical measure $E(x, n)$ such that

$$\int \phi dE(x, n) := \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) \text{ for every } \phi \in C(X).$$

Denote $\Omega(x) := \{\mu \in \mathcal{M}(X) : \mu \text{ is a weak-$\ast$ accumulation point of } \{E(x, n)\}_{n=1}^\infty\}$. Then every $\mu \in \Omega(x)$ is an invariant measure and $\Omega(x)$ is closed. Hence $\Omega(x) \subseteq K(X, f)$ for every $x \in X$.

Denote by $h_\mu(f)$ the metric entropy of $(X, f)$ with respect to $\mu \in \mathcal{M}(X, f)$ and by $P_\mu(f, \phi) := h_\mu(f) + \int \phi d\mu$ the pressure of $\mu$. When $Y$ is a compact $f$-invariant set, the topological entropy and topological pressure can be calculated with $(n, \varepsilon)$-separated subsets of $Y$ and we denote by $h(Y, f, \varepsilon)$ and $P(Y, f, \phi, \varepsilon)$ their values at the scale $\varepsilon$. It holds that

$$P(Y, f, \phi) = \sup\{P(Y, f, \phi, \varepsilon) : \varepsilon > 0\} = \sup\{P_\mu(f, \phi) : \mu \in \mathcal{M}(Y, f)\}.$$

Readers are referred to the books [27] and [34] for more details on measures, entropy and pressure.

3. Approximate Product Property

**Definition 3.1.** The system $(X, f)$ is said to have approximate product property, or called an APP system, if for every $\varepsilon, \delta_1, \delta_2 > 0$, there is $N > 0$ such that for
every \( n \geq N \) and every sequence \( \{x_k\}_{k=1}^{\infty} \) in \( X \), there exist an increasing sequence \( \{s_k\}_{k=1}^{\infty} \) of integers and \( z \in X \) such that

\[
s_1 = 0 \text{ and } n \leq s_{k+1} - s_k < n(1 + \delta_1) \text{ for each } k \in \mathbb{Z}^+,
\]

and

\[
|\{0 \leq j \leq n - 1 : d(f^{s_k+j}(z), f^j(x_k)) > \varepsilon\}| < \delta_2n \text{ for each } k \in \mathbb{Z}^+.
\]

Approximate product property is almost the weakest specification-like property. It is weaker than almost specification property (also called \( g \)-almost product property), tempered specification property (also called almost weak specification property or weak specification property), gluing orbit property, etc. More detailed discussions on specification-like properties can be found in [13], [24], [30] and [31].

The following is an essential fact for APP systems, which is an improved version of [28, Proposition 2.3 and Theorem 2.1].

**Proposition 3.2** ([30, Proposition 5.1]). Let \((X, f)\) be an APP system. Then for any \( \mu \in \mathcal{M}(X, f) \), any \( h \in (0, h_\mu(f)) \) and any \( \eta, \varepsilon, \delta_0 > 0 \), there are \( \delta \in (0, \delta_0) \) and a compact \( f \)-invariant subset \( \Lambda = \Lambda(\mu, h, \eta, \varepsilon, \delta) \) such that

1. There is \( N \in \mathbb{Z}^+ \) such that \( D(E(x, n), \mu) < \eta \) for every \( x \in \Lambda \) and every \( n > N \).
2. \( h < h(\Lambda, f, \delta) < h + \varepsilon \). In particular, \( h(\Lambda, f) > h \).

**Corollary 3.3.** Let \((X, f)\) be an APP system and \( \phi \) be a continuous potential. Then for any \( \mu \in \mathcal{M}(X, f) \), any \( \alpha \in \left(\int \phi d\mu, P_\mu(f, \phi)\right) \) and any \( \eta, \varepsilon, \delta_0 > 0 \), there are \( \delta \in (0, \delta_0) \) and a compact \( f \)-invariant subset \( \Lambda = \Lambda_\phi(\mu, \alpha, \eta, \varepsilon, \delta) \) such that

1. There is \( N \in \mathbb{Z}^+ \) such that \( D(E(x, n), \mu) < \eta \) for every \( x \in \Lambda \) and every \( n > N \).
2. \( \alpha < P(\Lambda, f, \phi, \delta) < \alpha + \varepsilon \). In particular, \( P(\Lambda, f, \phi) > \alpha \).

**Proof.** By continuity of \( \phi \), there is \( \eta' \in (0, \eta) \) such that

\[
|\int \phi d\nu - \int \phi d\mu| < \frac{\varepsilon}{3} \text{ whenever } D(\nu, \mu) < \eta'.
\]  \hspace{1cm} (1)

We may assume that \( \alpha + \varepsilon \leq P_\mu(f, \phi) \). Then

\[
\alpha - \int \phi d\mu + \frac{\varepsilon}{3} < h_\mu(f).
\]

Let \( \delta \in (0, \delta_0) \) and

\[
\Lambda := \Lambda \left(\mu, \alpha - \int \phi d\mu + \frac{\varepsilon}{3}, \eta', \frac{\varepsilon}{3}, \delta\right)
\]
be as obtained from Proposition 3.2. Then Condition (1) in Corollary 3.3 is satisfied as \( \eta' < \eta \). Moreover, by (1), we have

\[
P(\Lambda, f, \phi, \delta) \geq h(\Lambda, f, \delta) + \inf \left\{ \int \phi d\nu : \nu \in \mathcal{M}(\Lambda, f) \right\}
\]

\[
> \left(\alpha - \int \phi d\mu + \frac{\varepsilon}{3}\right) + \left(\int \phi d\mu - \frac{\varepsilon}{3}\right)
\]

\[= \alpha
\]
and
\[
P(\Lambda, f, \phi, \delta) \leq h(\Lambda, f, \delta) + \sup \left\{ \int \phi d\nu : \nu \in \mathcal{M}(\Lambda, f) \right\}
\]
\[
< \left( \alpha - \int \phi d\mu + \frac{\varepsilon}{3} \right) + \frac{\varepsilon}{3} + \left( \int \phi d\mu + \frac{\varepsilon}{3} \right)
\]
\[
= \alpha + \varepsilon.
\]
□

Remark 3.4. Note that the first conditions in Proposition 3.2 and Corollary 3.3 imply that for every \( x \in \Lambda \), we have
\[
D(\mu, \nu) \leq \eta \text{ for every } \nu \in \Omega(x),
\]
and hence \( D_H(\{\mu\}, \Omega(x)) \leq \eta \).

Let \( Z \) be a subset of \( X \). Denote
\[
\Omega(Z) := \{ \Omega(x) : x \in Z \} \subset K(X, f).
\]
and
\[
P_{f, \phi}^L(Z) := \sup \left\{ P_{\mu}(f) : \{\mu\} \notin \overline{\Omega(Z)}, \mu \in \mathcal{M}(X, f) \right\},
\]
where the closure of \( \Omega(Z) \) is taken with respect to the \( D_H \) metric on \( K(X, f) \). In particular, we put \( P_{f, \phi}^L(Z) := 0 \) if \( \Omega(Z) = \mathcal{M}(X, f) \).

Theorem 3.5. Let \( (X, f) \) be an APP system with positive topological entropy and \( \phi \) be a continuous potential. Then for any subset \( Z \) of \( X \), we have
\[
P(N(f, Z), f, \phi) \geq P_{f, \phi}^L(Z).
\]

Proof. If \( \overline{\Omega(Z)} = \mathcal{M}(X, f) \) then \( P_{f, \phi}^L(Z) = 0 \). The result is trivial.

Otherwise, for every \( \alpha < P_{f, \phi}^L(Z) \), there is \( \mu \in \mathcal{M}(X, f) \) such that
\[
\{\mu\} \notin \overline{\Omega(Z)} \text{ and } P_{\mu}(f) > \alpha.
\]
As \( \overline{\Omega(Z)} \) is compact, there is \( \eta > 0 \) such that
\[
\eta < \min \left\{ D_H(\{\mu\}, K) : K \in \overline{\Omega(Z)} \right\}.
\]  
(2)

Take any \( \varepsilon > 0 \). By Corollary 3.3 and Remark 3.4, there is a compact \( f \)-invariant set \( \Lambda \) such that
\[
D_H(\{\mu\}, \Omega(x)) \leq \eta \text{ for every } x \in \Lambda
\]  
(3)
and
\[
P(\Lambda, f, \phi) > P_{\mu}(f, \phi) - \varepsilon.
\]

We claim that \( \Lambda \cap Z = \emptyset \). Suppose that \( y \in \Lambda \cap Z \). By (3), \( y \in \Lambda \) implies that
\[
D_H(\{\mu\}, \Omega(y)) \leq \eta.
\]
But by (2), \( y \in Z \) implies that
\[
D_H(\{\mu\}, \Omega(y)) = \min \left\{ D_H(\{\mu\}, K) : K \in \overline{\Omega(Z)} \right\} > \eta.
\]
This is a contradiction.

As \( \Lambda \) is compact and \( f \)-invariant, we have \( \overline{O_f(x)} \subset \Lambda \) for every \( x \in \Lambda \). This implies that \( \Lambda \subset N(f, Z) \). Then
\[
P(N(f, Z), f, \phi) \geq P(\Lambda, f, \phi) > P_{\mu}(f, \phi) - \varepsilon > \alpha - \varepsilon.
\]
As $\alpha < P_{f,\phi}^\perp(Z)$ and $\varepsilon > 0$ are arbitrarily taken, we have
\[
P(\mathcal{N}(f, Z), f, \phi) \geq P_{f,\phi}^\perp(Z).
\]
\[\square\]

Let $\mu \in \mathcal{M}(X, f)$ and $\eta > 0$. Denote
\[
B(\mu, \eta) := \{\nu : D(\nu, \mu) < \eta\}.
\]
If $\Omega(x) \notin B(\mu, \eta)$ for every $x \in Z$, then $\{\mu\} \notin \overline{\Omega(Z)}$ and $P_{f,\phi}^\perp(Z) \geq P_\mu(f, \phi)$. The following is a direct corollary of Theorem 3.5.

**Corollary 3.6.** Let $(X, f)$ be an APP system with positive topological entropy and $\phi$ be a continuous potential. Let $\mu \in \mathcal{M}(X, f)$. Suppose that there is $\eta > 0$ such that $\Omega(x) \notin B(\mu, \eta)$ for every $x \in Z$. Then
\[
P(\mathcal{N}(f, Z), f, \phi) \geq P_\mu(f, \phi).
\]
In particular, if $\mu$ is an equilibrium state of $(X, f, \phi)$, then
\[
P(\mathcal{N}(f, Z), f, \phi) = P(f, \phi).
\]

We remark that it is possible that $P_{f,\phi}^\perp(Z) = 0$ when $Z$ is countable. For example, suppose that the system $(X, f)$ has periodic tempered gluing orbit property (e.g. a quasi-hyperbolic toral automorphism). Then the ergodic measures supported on periodic orbits are dense in $\mathcal{M}(X, f)$, hence $\overline{\Omega(Z)} = \mathcal{M}(X, f)$ if $Z$ is the countable set consisting of all periodic points. This case is beyond the limitation of our approach.

4. Weak Faces

**Definition 4.1** (cf. [8]). A convex subset $L$ of $\mathcal{M}(X, f)$ is called a **weak face** if for any $\mu \in L$, $\mu = \lambda \nu_1 + (1 - \lambda)\nu_2$ for $\lambda \in (0, 1)$ and $\nu_1, \nu_2 \in \mathcal{M}(X, f)$ implies that $\nu_1, \nu_2 \in L$. We say that $L$ is **proper** if $L \neq \mathcal{M}(X, f)$.

**Remark 4.2.** Existence of nonempty proper weak face requires that $(X, f)$ is not uniquely ergodic.

**Lemma 4.3.** Let $\mathcal{L} = \{L_\theta\}_{\theta \in I}$ be any family of weak faces. Then both $\bigcup_{\theta \in I} L_\theta$ and $\bigcap_{\theta \in I} L_\theta$ are weak faces.

**Proof.** Let $\mu \in \bigcup_{\theta \in I} L_\theta$, $\mu = \lambda \nu_1 + (1 - \lambda)\nu_2$ for $\lambda \in (0, 1)$ and $\nu_1, \nu_2 \in \mathcal{M}(X, f)$. Then there is $\theta_0$ such that $\mu \in L_{\theta_0}$. As $L_{\theta_0}$ is a weak face, we must have
\[
\nu_1, \nu_2 \in L_{\theta_0} \subset \bigcup_{\theta \in I} L_\theta.
\]
So $\bigcup_{\theta \in I} L_\theta$ is a weak face.

Let $\mu \in \bigcap_{\theta \in I} L_\theta$, $\mu = \lambda \nu_1 + (1 - \lambda)\nu_2$ for $\lambda \in (0, 1)$ and $\nu_1, \nu_2 \in \mathcal{M}(X, f)$. For each $\theta \in I$, we have $\mu \in L_\theta$ and $L_\theta$ is a weak face, hence $\nu_1, \nu_2 \in L_\theta$. This implies that $\nu_1, \nu_2 \in \bigcap_{\theta \in I} L_\theta$. So $\bigcap_{\theta \in I} L_\theta$ is a weak face. \[\square\]

**Lemma 4.4.** If $L = \bigcup_{i=1}^\infty L_i$ and each $L_i$ is a proper weak face, then $L$ is proper.
Proof. Fix any \( \mu_0 \in \mathcal{M}(X, f) \). For each \( i \), take \( \mu_i \in \mathcal{M}(X, f) \backslash L_i \). Let

\[
\mu_n := (1 - 2^{-n})\mu_0 + \sum_{i=1}^{n} 2^{-i} \mu_i \in \mathcal{M}(X, f).
\]

Then \( \{\mu_n\}_{n=1}^\infty \) converges to some \( \mu \in \mathcal{M}(X, f) \). For each \( i \) and each \( n > i \), we can write

\[
\mu_n = (1 - 2^{-i})\nu_{i, n} + 2^{-i} \mu_i,
\]

where

\[
\nu_{i, n} := \frac{1}{1 - 2^{-i}} \left( (1 - 2^{-n})\mu_0 + \sum_{j \in \{1, \ldots, n\} \backslash \{i\}} 2^{-j} \mu_j \right) \in \mathcal{M}(X, f).
\]

Then \( \{\nu_{i, n}\}_{n=1}^\infty \) converges to some \( \nu_i \in \mathcal{M}(X, f) \) and

\[
\mu = (1 - 2^{-i})\nu_i + 2^{-i} \mu_i.
\]

This implies that \( \mu \notin L_i \) as \( \mu_i \notin L_i \) for each \( i \). So \( \mu \notin L = \bigcup_{i=1}^\infty L_i \), hence \( L \neq \mathcal{M}(X, f) \). \( \square \)

**Corollary 4.5.** Let \( K := \bigcup_{i=1}^n K_i \) such that each \( K_i \) is a compact subset of a proper weak face. Then \( K \) is also a compact subset of a proper weak face.

**Definition 4.6.** Let \( x \in X \) and \( K \) be a subset of \( \mathcal{M}(X, f) \). We say that \( x \) has weak \( K \)-behavior if \( \Omega(x) \cap K \neq \emptyset \). We denote by \( \mathcal{H}(K) \) the set consisting of all points with weak \( K \)-behavior.

Following [8], we say that \( x \) has \( K \)-behavior if \( \Omega(x) \subset K \), and \( x \) is a point without \( K \)-behavior if \( \Omega(x) \subset \mathcal{M}(X, f) \backslash K \) (we are aware that this notion is a bit misleading). By definition, \( x \) is a point without \( K \)-behavior if and only if \( x \notin \mathcal{H}(K) \).

In particular, we have \( \mathcal{N}(f, \mathcal{H}(K)) \subset \mathcal{H}(K)^c \), i.e. every \( x \in \mathcal{N}(f, \mathcal{H}(K)) \) is a point without \( K \)-behavior.

**Definition 4.7.** A subset \( C \) of \( X \) is called the measure center of the system \( (X, f) \) if \( C \) is the smallest closed subset such that \( \mu(C) = 1 \) for any \( \mu \in \mathcal{M}(X, f) \).

**Lemma 4.8.** For any system \( (X, f) \) and any compact \( f \)-invariant subset \( Y \), the following are equivalent:

1. \( \mathcal{M}(Y, f) \neq \mathcal{M}(X, f) \).
2. \( Y \) does not include the measure center of \( (X, f) \).
3. There is \( \mu \in \mathcal{M}(X, f) \) such that \( \mu(Y) < 1 \).

**Lemma 4.9.** If \( Y \) is a compact \( f \)-invariant subset such that \( \mathcal{M}(Y, f) \neq \mathcal{M}(X, f) \), then \( Y \subset \mathcal{H}(\mathcal{M}(Y, f)) \) and \( \mathcal{M}(Y, f) \) is a compact proper weak face.

**Proof.** As \( Y \) is compact and \( f \)-invariant, \( (Y, f) \) is a subsystem. Hence \( \mathcal{M}(Y, f) \) is a compact subset of \( \mathcal{M}(X, f) \) and it is a weak face. For every \( x \in Y \) we have \( \Omega(x) \subset \mathcal{M}(Y, f) \). This implies that \( Y \subset \mathcal{H}(\mathcal{M}(Y, f)) \). As \( Y \) does not include the measure center of \( (X, f) \), we have \( \mathcal{M}(Y, f) \neq \mathcal{M}(X, f) \). So \( \mathcal{M}(Y, f) \) is a compact proper weak face. \( \square \)

We say that \( x \) is a generic point for \( \mu \in \mathcal{M}(X, f) \) if \( \Omega(x) = \{\mu\} \). Note that \( \mu \) is not necessarily an ergodic measure to have generic points. The singleton \( \{\mu\} \) may not be included in a proper weak face. In Case (2) of Theorem 1.1, \( Z_i \) consists of all points whose empirical measures accumulate to \( \mu_i \) if and only if \( Z_i = \mathcal{H}(\{\mu_i\}) \).
But this is not covered by Case (4) in the theorem. In the following proposition we consider the two cases separately.

**Proposition 4.10.** Suppose that \((X, f)\) is not uniquely ergodic. Let \(U := \bigcup_{i=1}^{m} U_i\) such that for each \(i\) we have \(U_i = \mathcal{H}(\{\mu_i\})\) for an invariant measure \(\mu_i \in \mathcal{M}(X, f)\). Let \(V := \bigcup_{j=1}^{n} \mathcal{H}(K_j)\) such that each \(K_j\) is a compact subset of a proper weak face \(L_j\). Let \(Z := U \cup V\). Then \(P_{f, \phi}^+(Z) = P(f, \phi)\).

**Proof.** Take any \(\alpha < P(f, \phi)\). There is an invariant measure \(\mu\) such that \(P_{\mu}(f, \phi) > \alpha\). Let \(L := \bigcup_{j=1}^{n} L_j\). As \((X, f)\) is not uniquely ergodic, by Lemma 4.4, we can find \(\mu_0 \in \mathcal{M}(X, f) \setminus L\) such that \(\mu_0 \neq \mu\). Then there is \(\theta \in (0, 1)\) such that for \(\mu' := \theta \mu + (1 - \theta)\mu_0\) we have

\[ P_{\mu'}(f, \phi) > \alpha \text{ and } \mu' \notin \{\mu_i : i = 1, \ldots, m\}. \]

For each \(j\), as \(L_j\) is a weak face and either \(\mu \notin L_j\) or \(\mu_0 \notin L_j\) holds, we must have \(\mu' \notin L_j\). Hence \(\mu' \notin L\).

Let

\[ K := \{\mu_i : i = 1, \ldots, m\} \cup \left( \bigcup_{j=1}^{n} K_j \right). \]

Then \(K\) is compact and \(\mu' \notin K\). There is \(\eta > 0\) such that \(D(\mu', \nu) > \eta\) for every \(\nu \in K\).

For every \(x \in U\), we have \(\mu_i \in \Omega(x)\) for some \(i\). Then

\[ D_H(\{\mu_i\}, \Omega(x)) \geq D(\mu', \mu_i) > \eta. \]

For every \(x \in V\), we have \(\Omega(x) \cap K_j \neq \emptyset\) for some \(j\). Then

\[ D_H(\{\mu_j\}, \Omega(x)) \geq \min\{D(\mu', \nu) : \nu \in K_j\} > \eta. \]

So

\[ D_H(\{\mu_j\}, \Omega(x)) > \eta \text{ for every } x \in Z. \]

This implies that \(\{\mu_j\} \notin \Omega(Z)\). Then

\[ P_{f, \phi}^+(Z) \geq P_{\mu'}(f, \phi) > \alpha. \]

As \(\alpha\) is arbitrary, we have \(P_{f, \phi}^+(Z) = P(f, \phi)\). \(\square\)

By [31], APP systems with positive topological entropy are not uniquely ergodic. So Proposition 4.10 holds for such systems and verifies Theorem 1.1 based on Theorem 3.5.

The following example provides a motivation of our consideration of compact subsets of weak faces. It also indicates that our approach may have more applications.

**Example 4.11.** Let \(M\) be a compact Riemannian manifold, \(f : M \to M\) be a \(C^1\) diffeomorphism with a dominated splitting \(TM = E \oplus F\) and \((M, f)\) is an APP system. Assume that the Lyapunov exponents are non-positive along \(E\) and non-negative along \(F\). Then by [7] and [8], the set \(PL(f)\) consisting of all physical-like measures is a compact subset of a weak face (the set consisting of all invariant measures satisfying Pesin entropy formula). In this setting we are able to generalize the results in [8]. A simple example of such a system that is not covered by [8] is the product of an irrational rotation and a quasi-hyperbolic toral automorphism. Note that when \(Z = \mathcal{H}(PL(f))\) is the set of all points with weak \(PL(f)\)-behavior,
the points in \( \mathcal{N}(f, Z) \) are not just without physical-like behavior but also have the forward orbits that do not accumulate to any point in \( Z \).

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