Propagation Through Trapped Sets and Semiclassical Resolvent Estimates

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Let $P = -\hbar^2 \Delta + V(x)$, $V \in C_0^\infty(\mathbb{R}^n)$. We are interested in semiclassical resolvent estimates of the form

$$
\|\chi(P - E - i0)^{-1} \chi\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \frac{a(h)}{h}, \quad h \in (0, h_0],
$$

(1)

for $E > 0$, $\chi \in C^\infty(\mathbb{R}^n)$ with $|\chi(x)| \leq \langle x \rangle^{-s}$, $s > 1/2$. We ask: how is the function $a(h)$ for which (1) holds affected by the relationship between the support of $\chi$ and the trapped set at energy $E$, defined by

$$
K_E = \{ \alpha \in T^*\mathbb{R}^n : \exists C > 0, \forall t > 0, |\exp(tH_p)\alpha| \leq C \}.
$$

Here $p = |\xi|^2 + V(x)$ and $H_p = 2\xi \cdot \nabla x - \nabla V \cdot \nabla \xi$.

We have (1) with $\chi(x) = \langle x \rangle^{-s}$ and $a(h) = C$ for all $E$ in a neighborhood of $E_0 > 0$ if and only if $K_{E_0} = \emptyset$ ([6, 7]). For general $V$ and $\chi$, the optimal bound is $a(h) = \exp(C/h)$, but Burq [1] and Cardoso-Vodev [2] prove that for any given $V$, if $\chi$ vanishes on a sufficiently large compact set, for any $E > 0$ there exists $C$ such that (1) holds with $a(h) = C$. In our main theorem we improve the condition on $\chi$ and obtain a shorter proof at the expense of an a priori assumption.

**Theorem 1 ([3]).** Fix $E > 0$. Suppose that (1) holds for $\chi(x) = \langle x \rangle^{-s}$ with $s > 1/2$ and with $a(h) = h^{-N}$ for some $N \in \mathbb{N}$. Then if we take instead $\chi$ such that $K_E \cap T^* \text{supp } \chi = \emptyset$, we have (1) with $a(h) = C$. 

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In fact our result holds for more general operators, and the cutoff \( \chi \) can be replaced by a cutoff in phase space whose microsupport is disjoint from \( K_E \). In certain situations it is even possible to take a cutoff whose support overlaps \( K_E \); see [3] for more details and references.

The a priori assumption that (1) holds for \( \chi(x) = \langle x \rangle^{-s} \) with \( a(h) = h^{-N} \) is not present in [1, 2] and is not always satisfied, but there are many examples of hyperbolic trapping where it holds: see e.g. [5, 8].

To indicate the comparative simplicity of our method, we prove a special case of the Theorem, under the additional assumption that \( \text{supp} \ V \subseteq \{ |x| < R_0 \} \) and \( \text{supp} \ \chi \subseteq \{ R_0 < |x| < R_0 + 1 \} \). In other words, suppose \( (P - \lambda)u = f \), with \( \text{Re} \ \lambda = E \), and \( \text{supp} f \subseteq \{ R_0 < |x| < R_0 + 1 \} \), \( \| f \| \leq 1 \). We must prove that \( \| \chi u \| \leq C h^{-1} \), uniformly as \( \text{Im} \ \lambda \to 0^+ \). Here and below all norms are \( L^2 \) norms.

Let \( S \) denote functions in \( C^\infty(T^*\mathbb{R}^n) \) which are bounded together with all derivatives, and for \( a \in S \) define

\[
\text{Op}(a)u(x) = (2\pi h)^{-n} \int \exp(i(x-y) \cdot \xi/h) a(x, \xi) u(y) dyd\xi.
\]

Because \( P - \lambda \) has a semiclassical elliptic inverse away from \( p^{-1}(E) \) (see for example [4, Chap. 4]), we have \( \| \text{Op}(a)u \| \leq C \) whenever \( \text{supp} a \cap p^{-1}(E) = \emptyset \). Consequently it is enough to show that \( \| \text{Op}(a)u \| \leq C h^{-1} \) for some \( a \in S \) with \( a \) nowhere vanishing on \( T^* \supp \chi \cap p^{-1}(E) \). We will prove this inductively: we will show that if there is \( a_1 \) with this nowhere vanishing property such that \( \| \text{Op}(a_1)u \| \leq C h^k \), then there is \( a_2 \) with the same nowhere vanishing property such that \( \| \text{Op}(a_2)u \| \leq C h^{k+1/2} \), provided \( k \leq -3/2 \). The base case follows from the a priori assumption that \( \| u \| \leq h^{-N-1} \), so it suffices to prove the inductive step.

Take \( \varphi = \varphi(|x|) \geq 0 \) a smooth function such that \( \varphi = 1 \) when \( |x| \leq R_0 \), \( \varphi = 0 \) when \( |x| \geq R_0 + 1 \), \( \varphi' = -\psi^2 \) with \( \psi \) smooth. We require further that \( T^* \supp \psi \) be contained in the set where \( a_1 \) is nonvanishing, and in the end we will take \( a_2 = \psi \).

We will now use a positive commutator argument with \( \varphi \) as the commutant:

\[
i \langle [P, \varphi]u, u \rangle = i \langle u, \varphi f \rangle - i \langle \varphi f, u \rangle - 2 \text{Im} \lambda \| u \|^2 \geq -C \| \psi u \| \| f \|,
\]

where we used first \( (P - \lambda)u = f \) and then \( \text{Im} \lambda \geq 0 \) and \( \text{supp} f \subseteq \{ \psi \neq 0 \} \). The semiclassical principal symbol of \( i [P, \varphi] \) is

\[
h H_{\rho} \varphi = 2h \rho \varphi' = -2h \rho \psi^2,
\]

where \( \rho \) is the dual variable to \( |x| \) in \( T^*\mathbb{R}^n \).

We now define an open cover and partition of unity of \( T^* \supp \chi \) according to the regions where this commutator does and does not have a favorable sign (the favorable sign is \( H_{\rho} \varphi < 0 \), because of the direction of the inequality in (2)). Take \( c > 0 \) small enough that for \( \rho < 2c \), \( |x| > R_0 \), \( t < 0 \) we have \( x + 2pt \notin \text{supp} V \). Let \( K \) be a neighborhood of \( p^{-1}(E) \cap T^* \supp \chi \) with compact closure in \( T^* \{ R_0 < |x| < R_0 + 1 \} \), and let \( O \) be a neighborhood of \( K \) with compact closure.
in $T^*\{R_0 < |x| < R_0 + 1\}$, and let
\[ U_+ = \{ \alpha \in \mathcal{O}: \rho > c \}, \quad U_- = \{ \alpha \in \mathcal{O}: \rho < 2c \} \cup (T^*\mathbb{R}^n \setminus K). \]

Take $\phi_\pm \in C_0^\infty(O)$ with $\phi_+^2 + \phi_-^2 = 1$ on $T^*\text{supp } \chi$ and with supp $\phi_\pm \subset U_\pm$. Then
\[ H_\rho \varphi = -b^2 - 2\rho \psi^2 \phi_-^2, \quad \text{where } b = \sqrt{2\rho \psi \phi_+}, \]
and if $B = \text{Op}(b)$ and $\Phi_- = \text{Op}(\phi_-)$
\[ i[P, \varphi] = -\hbar B^* B + \hbar \Phi_- R_1 \Phi_- + \hbar^2 R_2 + O(\hbar^\infty), \]
where $R_{1,2} = \text{Op}(r_{1,2})$ for $r_{1,2} \in S$ with supp $r_{1,2} \subset \text{supp } \psi$. Combining with (2), and using $L^2$ boundedness of $R_1$, we obtain
\[ \hbar \| Bu \|^2 \leq C \hbar \| \Phi_- u \|^2 + \hbar^2 \langle R_2 u, u \rangle + C \| \psi u \| \| f \| + O(\hbar^\infty). \]

Since $\langle R_2 u, u \rangle \leq C \hbar^{2k}$ by inductive hypothesis, we have
\[
\| Bu \|^2 \leq C(\| \Phi_- u \|^2 + \hbar^{2k+1} + \hbar^{-1} \| \psi u \| \| f \|)
\leq C(\| \Phi_- u \|^2 + \hbar^{2k+1} + \delta^{-1} \hbar^{-2} + \delta \| \psi u \|^2),
\]
where we used $\| f \| \leq 1$, and where $\delta > 0$ will be specified presently. Since at least one of $B$ and $\Phi_- \in T^*$ is elliptic at each point in the interior of $T^*\text{supp } \psi$, we have
\[
\| \psi u \|^2 \leq C(\| \Phi_- u \|^2 + \| Bu \|^2), \quad (3)
\]
from which we conclude that, if $\delta$ is sufficiently small,
\[
\| Bu \|^2 \leq C_\delta(\| \Phi_- u \|^2 + \hbar^{-2} + \hbar^{2k+1}). \quad (4)
\]

Because $c$ was chosen small enough that all backward bicharacteristics through supp $\phi_-$ stay in $T^*\{|x| > R_0\}$, where $P = -\hbar^2 \Delta$, we have
\[ \| \Phi_- u \| \leq C \hbar^{-1}, \]
by standard nontrapping estimates (see, for example, [3, Sect. 6]). This, combined with (3) and (4), gives
\[ \| \psi u \|^2 \leq C_\delta(\hbar^{-2} + \hbar^{2k+1}), \]
after which taking $a_2 = \psi$ completes the proof of the inductive step.

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