Elastic Trapped Modes in Solid Acoustic Resonators of Various Shapes

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Abstract. Resonators are one of the main building blocks of many acoustic, photonic, and microwave devices such as metasurfaces, sensing devices, antennas, and many more. One of the main properties of any resonator, which also determines the properties of the structure, based on the resonator, is the quality (Q) factor. Q-factor of the resonator is limited due to material and radiative losses. In this paper, we propose the existence of modes of solid resonators, immersed in a nonviscous fluid, which are non-radiative, and therefore, their Q-factor is limited only by material losses.

1. Introduction
Usually, eigenmodes of the open systems such as resonators are coupled to the modes of the freespace because their energies are lying in the same region of the spectrum, which results in a finite radiative Q-factor. However, even for open systems, one can find such modes, that do not exchange energy with the radiation continuum and therefore remain purely localized inside the system. Such modes are often called in quantum mechanics and photonics as Bound States in The Continuum [1], [2] or, in acoustics and fluidics as Trapped Modes [3], [4], [5].

Trapped modes are localized solutions of the Helmholtz equation in a bounded region, whose eigenvalues are embedded into the continuous spectrum. The first analytical proof of the existence of trapped modes was made by Ursell in 1951 [6] for an oscillating fluid, bounded by fixed surfaces and by a free surface of infinite extent. Later, trapped modes were found in infinite acoustic waveguides with an obstacle [7], [3], [8]. However, as it will be shown later, such modes can be found also in open systems of a finite size e.g. resonators.

It is well-known fact, that while solid objects support both shear and pressure waves, in nonviscous fluids there are only pressure waves with longitudinal polarization. The existence of trapped modes in finite-sized objects is due to the fact, that shear waves in solids, which are purely transverse, do not couple to pressure waves in fluids when the polarization vector of shear waves is tangential to the surface of the resonator at every point of the surface. In this case, the surface of the resonator will not shrink or expand, thus creating no vibrations in the fluid [9].

2. Formulation
Consider acoustic resonator, made of isotropic solid material with density $\rho$, speed of pressure waves $c_p$ and speed of shear waves $c_s$. Oscillations of displacement field $u$ is governed by following equation [10]

$$\rho \ddot{u} = \text{div} \sigma,$$

\[ \text{(1)} \]
where $\hat{\sigma}$ is the Cauchy stress tensor, which is connected with displacement field via the Hooke’s law

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2 \mu \varepsilon_{ij},$$  \hspace{1cm} (2)

where $\lambda = \rho (c_p^2 - 2 c_s^2)$ and $\mu = c_s^2 \rho$ are the Lame parameters and $\varepsilon_{kk} = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]$ is the infinitesimal strain tensor. It is well known fact, that Cauchy stress tensor can be decomposed into spherical and deviatoric parts [11]

$$\hat{\sigma} = \frac{1}{3} \text{Tr} \hat{\sigma} \mathbf{1} + \hat{\sigma}' ,$$  \hspace{1cm} (3)

where spherical part is connected with a change of the volume of the material. Therefore, if one can find such displacement field for which

$$\text{Tr} \hat{\sigma} \sim \text{div} \mathbf{u} = 0, \quad u_n |_{\Sigma} = 0 ,$$  \hspace{1cm} (4)

where $\Sigma$ is the surface of the resonator and $u_n$ is the component of the displacement field, normal to $\Sigma$, then one can expect, that such mode will not produce any pressure waves in the fluid outside of the resonator, and therefore will not be coupled to the radiative continuum and will have infinitely high radiative quality factor.

The so-called $\mathbf{M}$ vectorial harmonics have such properties (4) in specific geometries of the resonator, and are defined as follows [12]

$$\mathbf{M} = \text{rot} \left( \mathbf{c} \psi \right),$$  \hspace{1cm} (5)

where $\psi$ is the solution of the Helmholtz equation and $\mathbf{c}$ is the direction vector, which is chosen so, that $\mathbf{M}$ is a solution to the vectorial Helmholtz equation. For example, spherical vectorial harmonic

$$\mathbf{M}^{\text{sph}}_{\ell m}(\mathbf{r}, k) = \left[ \mathbf{e}_\theta \frac{im}{\sin \theta} j_\ell(kr) P^m_\ell(\cos \theta) - \mathbf{e}_\phi j_\ell(kr) \frac{\partial P^m_\ell(\cos \theta)}{\partial \theta} \right] e^{im\phi},$$  \hspace{1cm} (6)

where $j_\ell(kr)$ is the spherical Bessel function of the $\ell$th order, $P^m_\ell(\cos \theta)$ is the associated Legendre polynomial, $k = \omega / c_s$ is the wavevector and $\omega$ is the frequency of oscillations, will possess properties (4) in every resonator with infinite rotational symmetry around at least one axis ($D_\infty$ symmetry group). Indeed, one can show, by solving corresponding eigenmode problem for spherical resonator with radius $R$, that modes with $\mathbf{u} \sim \mathbf{M}^{\text{sph}}_{\ell m}$, with eigenfrequency equation

$$(1 - \ell) j_\ell(\alpha) + \alpha j_{\ell+1}(\alpha) = 0,$$  \hspace{1cm} (7)

where $\alpha = k R$, and $j_\ell(\alpha)$ is spherical Bessel function of first kind, are purely decoupled from modes in the freespace and therefore perfectly trapped inside the resonator.

Moreover, since boundary conditions for such modes are

$$\sigma_{nn} = 0, \quad \sigma_{n\tau} = 0, \quad u_n = 0,$$  \hspace{1cm} (8)

where indices $n$ and $\tau$ represent normal and tangential components respectively, eigenmode problem can be solved analytically even in resonators with sharp corners. For example, by solving eigenvalue problem in cylinder with radius $R$ and height $h$, one can obtain following equations for odd and even modes

$$\begin{cases} J_2(k_p R) = 0, \\ \cos(k_z h / 2) = 0 \end{cases} \text{odd mode}$$

$$\begin{cases} J_2(k_p R) = 0, \\ \sin(k_z h / 2) = 0 \end{cases} \text{even mode}$$  \hspace{1cm} (9)
which leads to the eigenfrequency equation

$$\omega^2 = c_s^2 \left[ \left( \frac{\alpha_n}{R} \right)^2 + \left( \frac{\pi q}{h} \right)^2 \right],$$

(10)

where $\alpha_n$ is the n'th root of the Bessel function of the second order $J_2(\alpha_n) = 0$, and $q = 0, 1, 2, ...$. In this case, displacement field of trapped modes will be proportional to the cylindrical vectorial harmonics with $m = 0$

$$M_{\text{cyl}}^m \mathbf{e} \left( \mathbf{r}, \mathbf{k} \right) = \left[ \hat{e}_\rho \frac{m}{\rho} Z_m(k_\rho \rho) - \hat{e}_\varphi \frac{\partial Z_m(k_\rho \rho)}{\partial \rho} \right] \cos (k_z z) \frac{\cos (k_z z) e^{im\varphi}},$$

(11)

where indices $o$ and $e$ represent odd and even harmonics with respect to $z$ coordinate. Modes with odd $q = 1, 3, 5, ...$ corresponds to odd harmonics, and modes with even $q = 2, 4, 6, ...$ corresponds to even one.

3. Numerical results

To prove existence of trapped modes in such resonators, numerical simulation was used. All numerical simulations were done in COMSOL Multiphysics® [13] in 2D geometry with axial symmetry using eigenfrequency solver. Material parameters of the resonator are the following: density $\rho = 10 \rho_0$, speed of pressure waves $c_p = 3 c_0$ and speed of shear waves $c_s = 2 c_0$, where $\rho_0 = 1.225 \text{ kg/m}^3$ and $c_0 = 343 \text{ m/s}$ are the density and speed of sound of the surrounding fluid (air). Several different geometries of the resonator were investigated: sphere, hemisphere, cylinder, cylindrical shell and torus.

3.1. Sphere and hemisphere

On the figure 1 (a) one can see the Q-factor of the eigenmodes of the spherical resonator with radius $R = 5 \text{ cm}$. Some of the eigenmodes have infinitely high Q-factor. One can show, that frequencies of such modes are in perfect match with ones, that can be obtained from the equation (7). Moreover, all trapped modes have $2\ell + 1 -$ fold degeneracy, which can be also seen from the equation (7) because it does not depend on the azimuthal number $m$.

Interestingly, the fundamental mode of such a resonator is not the dipole but the quadrupole. This happens because the time-averaged angular momentum of the sphere should be equal to zero so that the resonator does not rotate as a whole. Since the displacement field of the trapped modes is proportional to $M_{\text{ sph}}^m$ vectorial spherical harmonics, in the case, when azimuthal number $m = 0$, $u_\varphi$ is the only nonzero component. Therefore, from the figure 1 (b), where $u_\varphi$ component of the dipole, quadrupole, and octupole modes is presented it became obvious that the time-averaged angular momentum of the dipole mode is zero only in the case when the inner layer of the sphere is rotating opposite to external layer, which gives condition, that displacement field should have at least two extremes in the radial direction and therefore, $R\omega/c_s$ should be bigger than the second root of the $j_1(kR)$ spherical Bessel function. Meanwhile, for quadrupole mode and all other modes with $\ell > 1$, the condition on the time-averaged angular momentum is fulfilled automatically because, for these modes, rotation of the upper hemisphere is compensated by the lower hemisphere.

On the figure 1 (c), Q-factor of the hemispherical resonator with radius $R = 5 \text{ cm}$ is presented. One can see the existence of the modes with infinitely high Q-factor, which frequencies coincide with some of the frequencies of trapped modes from the figure 1 (a). The origin of this phenomenon can be understood from the figure 1 (d), where $u_\varphi$ component of the field of trapped modes is presented. One can see, that the field profile is identical to one for the spherical resonator. However, for the spherical resonator, there are no modes with even $\ell$. This happens because for even $\ell$ existence of the lower hemisphere is an important condition for
zeroing the time-averaged angular momentum. Another difference from the sphere is the fact that for the hemisphere trapped modes are not degenerate. Only modes with the displacement field proportional to $\mathbf{M}_{\text{sph}}^{\ell_0}$ vectorial spherical harmonics are trapped. The reason for this lies in the lower symmetry of the resonator compared to the spherical one. Another consequence of the above fact is that the displacement field has only $u_{\phi}$ component not equal to zero. This property holds not only for the trapped modes in the spherical and hemispherical resonator but for trapped modes in an arbitrary-shaped resonator with infinite rotational symmetry.

3.2. Cylinder
On the figure 2 (a) one can see Q-factor of the modes of the cylindrical resonator with radius $R = 2.5$ cm and height $h = 6$ cm. As in the case of the hemisphere, only modes which can be expanded in terms of $\mathbf{M}_{\text{sph}}^{\ell_0}$ vectorial spherical harmonics are trapped. On the field profiles presented on the figure 2 (b) one can see existence of the nodal lines along radial and $z$ directions, which are due to the conditions on $k_\varrho$ and $k_z$ components of the wavevector (equation 9). Important to mention that there are no modes with $k_\varrho = 0$ ($\alpha_n = 0$), since in these cases $\mathbf{M}_{\text{cyl}}^{0} = 0$.

3.3. Cylindrical shell and torus
Like resonators of a simple form, modes can exist in more complex-shaped resonators like cylindrical shell and torus, Q-factors of whose modes are represented on the figures 2 (c) and 2 (e). The outer and inner radii of the shell and torus are equal $a = 2.5$ cm and $b = 0.1$ cm. The height of the shell is similar to the one for the cylindrical resonator. Interestingly, some modes of these resonators look similar to the ones in the cylindrical resonator. Example of such
Figure 2. Cylindrical, cylindrical shell, and torus resonators. (a), (c), (e) - quality factor of the modes of the cylindrical, cylindrical shell, and torus resonators respectively. (b), (d), (f) - $u_{\phi}$ component of the displacement field of some of the eigenmodes of the cylindrical, cylindrical shell, and torus resonator respectively. Eigenfrequencies of the presented modes are written below the field profiles. The dotted red line represent the symmetry axis of the resonator.

modes are presented on the figures 2 (b), (d), mode with $\omega = 35.9$ kHz, and (f), mode with $\omega = 52.7$ kHz. One can see that if the width and height of the shell resonator are equal to the width and height of the cylindrical one, the eigenfrequencies of these modes are equal.

4. Conclusions
In this proceeding, we have numerically proved the existence of trapped modes in solid acoustic resonators merged in a nonviscous fluid. Such states are possible due to the polarization mismatch of the shear waves in the resonator with the pressure waves in the surrounding media. Polarization mismatch is provided by the tangential nature of the $M_{0}^{th}$ vectorial spherical harmonics to the surfaces with axial symmetry, therefore, axial symmetry of the resonator is one of the main conditions of existence of such modes. We believe that our findings can be applied to many fields of acoustic engineering including high-Q acoustic metasurfaces, acoustic sensing devices, and acoustic nanoantennas.

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