Complete regular dessins of odd prime power order

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Abstract

A dessin is a 2-cell embedding of a connected 2-coloured bipartite graph into an orientable closed surface. A dessin is regular if its group of colour- and orientation-preserving automorphisms acts regularly on the edges. In this paper we employ group-theoretic method to determine and enumerate the isomorphism classes of regular dessins with the complete bipartite underlying graphs of odd prime power order.

Keywords graph embedding, dessin d’enfant, metacyclic group.

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1 Introduction

A dessin $D$ is an embedding $i : X \hookrightarrow C$ of a 2-coloured connected bipartite graph $X$ into an orientable closed surface $C$ such that each component of $C \setminus i(X)$ is homeomorphic to an open disc. An automorphism of a dessin $D$ is a permutation of the edges which preserves the graph structure and vertex-colouring, and extends to an orientation-preserving self-homeomorphism of the supporting surface. The set of automorphisms forms the automorphism group of $D$ under composition, and it acts semi-regularly on the edges. If this action is transitive, and hence regular, then the dessin is called regular as well.

For a given dessin $D$, following Lando and Zvonkin [25] the reciprocal dessin $D^*$ of $D$ is the dessin obtained from $D$ by interchanging the vertex colours of $D$. It follows that $D^*$ has the same underlying graph, the same automorphism group and the same supporting surface as $D$. Thus dessins with a given bipartite graph $X$ appear in reciprocal pairs $(D, D^*)$. In particular, a regular dessin $D$ is symmetric if $D$ is isomorphic to $D^*$.
It follows that symmetric dessins possess an external symmetry transposing the vertex colours and therefore may be viewed as (arc)regular bipartite maps.

It is well known that compact Riemann surfaces and complex projective algebraic curves are equivalent categories. By the Riemann Existence Theorem [25, Theorem 1.8.14] every dessin on an orientable closed surface \( C \) determines a complex structure on \( C \) which makes it into a Riemann surface defined over an algebraic number field. By Belyi’s Theorem [1] the inverse is also true: if a curve \( C \) can be defined over some algebraic number field, then its complex structure can be obtained from a dessin on \( C \). The absolute Galois group \( \mathbb{G} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts naturally on the curves defined over \( \overline{\mathbb{Q}} \), the field of all algebraic numbers. Grothendieck observed that this action induces a faithful action of \( \mathbb{G} \) on the associated dessins [14]. González-Diez and Jaikin-Zapirain have recently shown that this action remains faithful even when restricted to regular dessins [12]. Thus one can study \( \mathbb{G} \) through its action on such simple and symmetrical combinatorial objects as regular dessins.

Regular dessins with complete bipartite underlying graphs will be called complete regular dessins. Because the algebraic curves associated with complete regular dessins may be viewed as generalization of Fermat curves, it is important to study complete regular dessins and the associated curves [21].

The classification of complete regular dessins is an open problem in this field, posed by Jones in [18]. In this direction, complete symmetric dessins have been classified in a series of papers [6, 7, 17, 19, 20, 23, 24, 26]. As for the general case, it has been shown that there exists a one-to-one correspondence between the following three categories: the isomorphism classes of complete regular dessins, equivalence classes of exact bicyclic group factorisations and certain pairs of skew-morphisms of the cyclic groups [11].

The correspondence between complete regular dessins and exact bicyclic group factorisations establishes a group-theoretic method to classify and enumerate the complete regular dessins. Following this method, it was shown in [8] that the graph \( K_{m,n} \) underlies a unique edge-transitive embedding if and only if \( \gcd(m, \phi(n)) = \gcd(n, \phi(m)) = 1 \) where \( \phi \) is the Euler’s totient function.

The symmetric and nonsymmetric complete regular dessins with underlying graphs \( K_{p^d,p^e} \) have been classified in [20] and [4]. Recently, the edge-transitive circular embeddings of \( K_{p^d,p^e} \) have been determined in [9], where a circular embedding is a map whose boundary walk of each face is a simple cycle without repeated edges. In this paper we consider the classification and enumeration of complete regular dessins of odd prime power order without any additional restriction. Our main result is the following

**Theorem 1.** Let \( p \) be an odd prime, and let \( \nu(d,e) \) denote the number of isomorphism classes of reciprocal pairs of regular dessins with complete bipartite underlying graph \( K_{p^d,p^e} \), \( 0 \leq d \leq e \). Then

(i) if \( d = 0 \) then \( \nu(d,e) = 1 \),

(ii) if \( 0 < d = e \) then \( \nu(d,e) = \frac{1}{2} p^{e-1} (1 + p^{e-1}) \),

(iii) if \( 0 < d < e \) then \( \nu(d,e) = p^{2d-1} \).

Moreover, in each case the dessins have type \( (p^d, p^e) \) and genus \( \frac{1}{2} (p^d - 1)(p^e - 2) \).

**Remark 1.** Dessins from (i) are exceptional, in the sense that their automorphism groups are cyclic, whereas dessins from (ii) and (iii) have noncyclic metacyclic automorphism groups; see Theorem [12]. Moreover, up to isomorphism there are \( p^{2(e-1)} \) complete regular dessins from (ii), and among them \( p^{e-1} \) are symmetric. Thus by the results obtained
in \[9\] we may conclude that all edge-transitive embeddings of \(K_{p^i, p^j}\) are circular embeddings. Finally, the most interesting case is (iii) where we find that the number \(\nu(d, e)\) is independent of the larger exponent parameter \(e\).

## 2 Preliminaries

In this section we collect some group-theoretic notations and results be to be used later, and sketch the algebraic theory on regular dessins with complete bipartite underlying graphs.

Let \(G\) be a finite \(p\)-group of exponent \(\exp(G) = p^e\). For \(0 \leq i \leq e\) we define

\[
\mathcal{U}_i(G) = \langle g^{p^i} \mid g \in G \rangle \quad \text{and} \quad \Omega_i(G) = \langle g \in G \mid g^{p^i} = 1 \rangle.
\]

The lower power series and the upper power series of \(G\) are defined to be

\[
G = \mathcal{U}_0(G) \geq \mathcal{U}_1(G) \geq \mathcal{U}_2(G) \geq \cdots \geq \mathcal{U}_e(G) = 1.
\]

and

\[
1 = \Omega_0(G) \leq \Omega_1(G) \leq \Omega_2(G) \leq \cdots \leq \Omega_e(G) = G.
\]

For a positive integer \(i\), a finite \(p\)-group \(G\) is called \(p^i\)-abelian if for any \(x, y \in G\),

\[
(xy)^{p^i} = x^{p^i}y^{p^i}.
\]

It is clear that \(G\) is \(p^i\)-abelian if and only if the mapping \(\pi_i : G \to G, a \mapsto a^{p^i}\) is a group homomorphism, and that for positive integers \(i < j\), if \(G\) is \(p^i\)-abelian then it is also \(p^j\)-abelian. Moreover, a finite \(p\)-group \(G\) is called regular if for any \(x, y \in G\) there exists \(z \in \mathcal{U}_1(\langle x, y \rangle)\) such that

\[
(xy)^p = x^py^pz
\]

where \(\langle x, y \rangle\) denotes the derived subgroup of the group generated by \(x\) and \(y\). The following result relates \(p^i\)-abelianness with regularity:

**Lemma 2.** \[27\] Lemma 5.1.7 \(\) Let \(G\) be a finite \(p\)-group with \(\mathcal{U}_1(G') = 1\) where \(i\) is a positive integer. If \(G\) is regular, then it is \(p^i\)-abelian.

Regular \(p\)-groups possess the following important property.

**Lemma 3.** \[16\] Theorem 10.5, III \(\) Let \(G\) be a finite \(p\)-group. If \(G\) is regular, then for any integer \(i \geq 1\) and any elements \(x, y \in G\), \((xy)^{-1})^{p^i} = 1\) if and only if \(x^{p^i} = y^{p^i}\).

A finite group \(G\) will be called \((m, n)\)-bicyclic if \(G\) can be factorised as a product \(G = \langle a \rangle \langle b \rangle\) of two cyclic subgroups \(\langle a \rangle\) and \(\langle b \rangle\) of orders \(m\) and \(n\). In particular, if \(\langle a \rangle \cap \langle b \rangle = 1\), then the factorisation \(G = \langle a \rangle \langle b \rangle\) will be termed exact, and in these circumstances, both the generating pair \((a, b)\) and the triple \((G, a, b)\) will be called exact \((m, n)\)-bicyclic. Moreover, if \(G\) has an automorphism transposing \(a\) and \(b\), then the pair \((a, b)\) and the triple \((G, a, b)\) will be called exact \(n\)-isobicyclic. Two exact \((m, n)\)-bicyclic triples \((G_i, a_i, b_i)\) \((i = 1, 2)\) will be called equivalent if the assignment \(a_1 \mapsto a_2\) and \(b_1 \mapsto b_2\) extends to a group isomorphism \(G_1 \to G_2\).

Given an exact \((m, n)\)-bicyclic triple \((G, a, b)\), one may construct an \((m, n)\)-complete regular dessin as follows: First define a bipartite graph \(X\) by taking the edges to be the elements of \(G\), and the black and white vertices to be the cosets of \(g\langle a \rangle\) and \(g\langle b \rangle\) of \(\langle a \rangle\).
and \( \langle b \rangle \) in \( G \), with incidence given by containment. Since \( G = \langle a \rangle \langle b \rangle \), the cosets can be written as
\[
U = \{ b^i \langle a \rangle \mid i = 0, 1, \ldots, n - 1 \} \quad \text{and} \quad V = \{ a^j \langle b \rangle \mid j = 0, 1, \ldots, m - 1 \}.
\]
It is clear that \( X \) is the complete bipartite graph \( K_{m,n} \). Moreover, the right multiplication of \( a \) and \( b \) defines a cyclic order
\[
(b^i a, b^j a^2, \ldots, b^k a^{m-1}) \quad \text{and} \quad (a^i b, a^j b^2, \ldots, a^k b^{n-1})
\]
of the edges around the vertices \( b^i \langle a \rangle \) and \( a^j \langle b \rangle \) of \( X \), and these local orientations determine an embedding of \( K_{m,n} \) into an oriented surface. The left multiplication of \( G \) induces a regular action of \( G \) as a group of dessin automorphisms on the edges. Conversely, every \((m,n)\)-complete regular dessin arises in this way. In particular, the \((m,n)\)-complete regular dessin is symmetric if and only if the corresponding triple \((G,a,b)\) is exact \(n\)-isobicyclic. To summarize we have

Proposition 4. \cite{11} The automorphism group of an \((m,n)\)-complete regular dessin is an exact \((m,n)\)-bicyclic group, and for any exact \((m,n)\)-bicyclic group \( G \), the isomorphism classes of \((m,n)\)-complete regular dessins with automorphism group isomorphic to \( G \) are in one-to-one correspondence with the equivalence classes of exact \((m,n)\)-bicyclic generating pairs of \( G \). In particular, the symmetric \((n,n)\)-complete regular dessins correspond to exact \(n\)-isobicyclic pairs of \( G \).

Proposition 5. Let \( G \) be an exact \((m,n)\)-bicyclic group, and let \( \mathcal{P}(G) \) and \( \mathcal{B}(G) \) denote the sets of exact \((m,n)\)-bicyclic pairs and exact \(n\)-isobicyclic pairs of \( G \) if \( m = n \), respectively. Define
\[
\nu(G) = \frac{|\mathcal{P}(G)|}{|\text{Aut}(G)|} \quad \text{and} \quad \nu_0(G) = \frac{|\mathcal{B}(G)|}{|\text{Aut}(G)|}.
\]
Then
\begin{enumerate}[(i)]
\item if \( m = n \), then up to isomorphism \( K_{n,n} \) underlies \( \nu(G) \) regular dessins and \( \nu_0(G) \) symmetric regular dessins;
\item if \( m \neq n \), then up to isomorphism \( K_{m,n} \) underlies \( \nu(G) \) reciprocal pairs of regular dessins.
\end{enumerate}

Proof. We note that if \( K_{m,n} \) underlies an \((m,n)\)-complete regular dessin \( D \), it also underlies an \((n,m)\)-complete regular dessin \( D^* \), the reciprocal dessin of \( D \) obtained by swapping the vertex colours of \( D \). Accordingly, every exact \((m,n)\)-bicyclic pair \((a,b)\) of \( G \) corresponds to an exact \((n,m)\)-bicyclic pair \((b,a)\) of \( G \). In particular, \( D \) is symmetric if and only if \( D \cong D^* \), or equivalently, the pair \((a,b)\) is exact \(n\)-isobicyclic.

Since the action of \( \text{Aut}(G) \) on the set \( \mathcal{P}(G) \) is semi-regular, by Proposition \cite{4} the number \( \nu(G) \) defined above is the number of isomorphism classes of \((m,n)\)-complete regular dessins with automorphism group isomorphic to \( G \). If \( m = n \), then \( \nu(G) \) is precisely the number of isomorphism classes of regular dessins with underlying graph \( K_{n,n} \) and automorphism group isomorphic to \( G \), of which the number of symmetric ones is \( \nu_0(G) \). On the other hand, if \( m \neq n \) then every \((m,n)\)-complete regular is not isomorphic to any \((n,m)\)-complete regular dessin. Since complete regular dessins occur in reciprocal pairs, \( \nu(G) \) is equal to the number of reciprocal pairs of regular dessins with underlying graphs \( K_{m,n} \) and with automorphism group isomorphic to \( G \), as claimed. 

\( \square \)
Example 1. For the abelian group defined by the presentation
\[ G = \langle a, b \mid a^m = b^n = [a, b] = 1 \rangle, \]
the triple \((G, a, b)\) determines a reciprocal pair of regular dessins with underlying graph \(K_{m,n}\), and every exact \((m, n)\)-bicyclic triple of \(G\) is equivalent to \((G, a, b)\). Therefore, the graph \(K_{m,n}\) underlies at least one reciprocal pair of regular dessins. In [11, Corollary 10] it is shown that \(K_{m,n}\) underlies a unique reciprocal pair of regular dessins if and only if \(\gcd(m, \phi(n)) = \gcd(n, \phi(m)) = 1\); see also [3]. In case of \(m = 1\) the group \(G\) is cyclic, and the underlying graph of the corresponding unique reciprocal pair of complete regular dessins is the \(n\)-star \(K_{1,n}\).

3 Exact bicyclic \(p\)-groups

In this section for odd primes \(p\) we classify the automorphism groups of regular dessins with complete bipartite underlying graphs \(K_{p^d,p^e}\).

The following result is fundamental.

Lemma 6. [16, Theorem 11.5, III] Let \(p > 2\) be a prime. If \(G\) is a finite bicyclic \(p\)-group, then \(G\) is metacyclic.

The following Hölder’s Theorem on metacyclic group is well known.

Lemma 7. [28, Theorem 20] A metacyclic group \(G\) with cyclic normal subgroup \(N\) of order \(n\) and with cyclic factor group \(F\) of order \(m\) has two generators \(x, y\) with the defining relations
\[ G = \langle x, y \mid x^n = 1, y^m = x^t, y^{-1}xy = x^r \rangle \] (1)
and with the numerical conditions
\[ r^m \equiv 1 \pmod{n} \quad \text{and} \quad t(r - 1) \equiv 0 \pmod{n}. \] (2)

Conversely, for any positive integers \(m, n, r\) and \(t\) satisfying the numerical conditions in (2), a metacyclic group of order \(mn\) with the previous given properties is defined by the three relations in (1).

Note that if \(p > 2\) then every finite \(p\)-group with cyclic derived subgroup is regular [16, Theorem 10.2, III]; thus by Lemma 6 every finite bicyclic \(p\)-group is a regular \(p\)-group.

Lemma 8. Let \(p > 2\) be a prime. If \(G\) is metacyclic, then \(G\) admits an exact bicyclic factorisation.

Proof. It is evident that every split metacyclic \(p\)-group admits an exact bicyclic factorisation, so it suffices to consider non-split metacyclic \(p\)-group \(G\). Without loss of generality we may assume that \(G\) has a presentation
\[ G = \langle a, b \mid a^{p^d} = 1, a^{p^m} = b^p, a^b = a^r \rangle, \]
where \(1 \leq m < l\). By Proposition 3 If \(m \geq n\) then \(ba^{-p^m-n} = 1\) and \(G = \langle a \rangle \langle ba^{-p^m-n} \rangle\), whereas if \(m < n\) then \(a^{-p^m-n}b = 1\) and \(G = \langle b \rangle \langle ab^{-p^m-n} \rangle\). By the product formula in either case the factorisation must be exact.

The following technical results will be useful.
Lemma 9. Let $G$ be a bicyclic $p$-group with a bicyclic factorisation $G = \langle \alpha \rangle \langle \beta \rangle$ where $|\alpha| \geq |\beta|$. Then $\exp(G) = |\alpha|$. In particular if the factorisation is exact, then for every exact bicyclic factorisation $G = \langle \alpha' \rangle \langle \beta' \rangle$ with $|\alpha'| \geq |\beta'|$ we have $|\alpha'| = |\alpha|$ and $|\beta'| = |\beta|$.

Lemma 10. [15] Let $G = AB$ be a finite abelian group, then $G$ is cyclic if and only if $\gcd(|A|/|A \cap B|, |B|/|A \cap B|) = 1$.

Lemma 11. [27] Let $p$ be an odd prime, and $n$ a positive integer. Then the multiplicative group $U(p^n)$ of $\mathbb{Z}_{p^n}$ is cyclic of order $\phi(p^n) = p^{n-1}(p-1)$, and its unique subgroup of order $p^i$ ($0 \leq i \leq n-1$) consists of elements of the form $kp^{n-1} + 1$ where $k \in \mathbb{Z}_{p^n}$.

The following theorem classifies the automorphism groups of $(p^d, p^e)$-complete regular dessins for odd primes $p$.

Theorem 12. Let $p > 2$ be a prime, and let $d$ and $e$ be positive integers, $d \leq e$. Then the automorphism group $G$ of a complete regular dessin with underlying graph $K_{p^d, p^e}$ is isomorphic to one of the following groups:

(i) $M_1(d, e, f) = \langle a, b \mid a^{p^d} = b^{p^d} = 1, a^b = a^{1+p^f} \rangle$ where either $1 \leq d \leq f \leq e \leq d + f$, or $1 \leq f < d < e \leq d + f$.

(ii) $M_2(d, e, f) = \langle a, b \mid a^{p^d} = b^{p^d} = 1, a^b = a^{1+p^f} \rangle$ where $1 \leq f < d < e$.

(iii) $M_3(d, e, h, f) = \langle a, b \mid a^{p^h} = 1, b^{p^{d+f-h}} = a^{p^d}, a^b = a^{1+p^f} \rangle$ where $h - d \leq f < d < h < e$.

Moreover, the groups are pairwise non-isomorphic.

Proof. By Proposition 1 the group $G$ admits an exact $(p^d, p^e)$-factorisation $G = AB$ where $A \cong C_{p^d}$, $B \cong C_{p^e}$ and $A \cap B = 1$. If $G$ is abelian then $G \cong M_1(d, e, e)$. In what follows we assume that $G$ is non-abelian. By Lemma 3 $G$ is metacyclic and so regular [16] Theorem 10.2, III]. Thus $G$ has a cyclic normal subgroup $N$ of order $p^h$ such that the quotient group $G/N = \overline{AB}$ is cyclic where $\overline{A} = AN/N$ and $\overline{B} = BN/N$. By Lemma 9 $\exp(G) = p^e$, so $h \leq e$. Since $\overline{G}$ is a $p$-group, by Lemma 10 either $G = AN$ or $G = BN$. We distinguish two cases:

Case (1). $G = AN$.

Since $p^e \geq p^h = |N| \geq |N : N \cap A| = |AN : A| = |G : A| = |B| = p^e$, we get $h = e$ and $|N \cap A| = 1$, so $G = N \rtimes A \cong C_{p^e} \rtimes C_{p^d}$. By Hölder’s Theorem $G$ has a presentation $G = \langle a, b_1 \mid a^{p^d} = b_1^{p^d} = 1, a^{b_1} = a^r \rangle$, where $r^{p^d} \equiv 1 \pmod{p^e}$ and $r \neq 1 \pmod{p^d}$. By Lemma 11 we may assume that $r = 1 + kp^{d}$ where $k \in \mathbb{Z}_{p^{d-1}}$, $1 \leq f < e \leq d + f$. Let $k'$ be the modular inverse of $k$ in $\mathbb{Z}_{p^{d-1}}$, then $a^{b_1^{k'}} = a^{1+kp^{d}} = a^{1+p^{f}}$. Replacing $b_1^{k'}$ with $b$ we obtain the groups in (i) where $1 \leq f < e \leq d + f$.

Case (2). $G = BN$.

If $N \cap B = 1$ then as before it is easy to show that $|N| = p^d$ and $G = N \rtimes B \cong C_{p^d} \rtimes C_{p^e}$, so $G \cong M_2(d, e, f)$. In what follows we assume that $N \cap B > 1$, then $p^{e+d} = |G| = |BN| = |N||B|/|N \cap B| = p^{h+e}/|N \cap B|$,
so \(|N \cap B| = p^{h-d}\) where \(1 \leq d < h \leq e\). Then \(G\) has a presentation

\[
G = \langle a_1, b_1 | a_1^{p^h} = b_1^{p^e} = 1, b_1^{p^{d+e-h}} = a_1^{p^d}, a_1^{b_1} = a_1^{1+kp^f} \rangle,
\]

where \(s\) and \(k\) are positive integers coprime to \(p\), \(1 \leq f < h\). Replacing \(a_1^{kp^f}\) with \(a_1^{kp^f}\) with \(b\) where \(k'\) is the modular inverse of \(k\) in \(\mathbb{Z}_{p^{h-f}}\) the presentation of \(G\) is transformed to the form

\[
G = \langle a, b | a^{p^h} = b^{p^e} = 1, b^{p^{d+e-h}} = a^{p^d}, a^b = a^{1+p^f} \rangle.
\]

Since \(a^{p^d} = (a^{p^d})^b = (a^{b})^{p^d} = a^{p^{d+(1+p^f)}},\) we have \(p^f \equiv 0 \pmod{p^{h-d}}\), and so \(h \leq d + f\). In what follows we distinguish three subcases:

If \(h = e\), then \(b^{p^d} = a^{p^d}\), so by Proposition \(\Box (a^{-1}b)^{p^d} = 1\). Thus \(G = \langle a, b \rangle = \langle a, a^{-1}b \rangle \cong C_{p^d} \rtimes C_{p^e},\) which is a group in (i).

If \(h < e\) and \(d \leq f\), then \(G' = \langle [a, b] \rangle = \langle a^{p^f} \rangle \leq \langle a^{p^d} \rangle = \langle b^{p^{d+e-h}} \rangle \leq \langle b \rangle\), so \(\langle b \rangle \leq G'.\) By Proposition \(\Box\) we have \((b^{-p^{d-e}} a)^{p^d} = 1\), so \(G = \langle a, b \rangle = \langle b, b^{-p^{d-e}} a \rangle \cong C_{p^d} \rtimes C_{p^{e-f}},\) again a group in (i).

We are left with the subcase where \(h < e\) and \(f < d\), and we obtain the groups \(M_3(d, e, h, f)\) in (iii).

Finally, to determine the isomorphism relations between the groups in (i)–(iii), we summarize the invariant types of \(G'\) and \(G/G'\) in Table 1.

| Group          | \(G'\)       | \(G/G'\)         | Condition                                      |
|----------------|--------------|------------------|------------------------------------------------|
| \(M_1(d, e, f)\) | \(C_{p^{d-f}}\) | \(C_{p^f} \times C_{p^d}\) | \(1 \leq d \leq f \leq e \leq d + f\) or \(1 \leq f < d < e \leq d + f\) |
| \(M_2(d, e, f)\) | \(C_{p^{d-f}}\) | \(C_{p^f} \times C_{p^d}\) | \(1 \leq f < d < e\) |
| \(M_3(d, e, h, f)\) | \(C_{p^{d-f}}\) | \(C_{p^f} \times C_{p^{e-h}}\) | \(h - d \leq f < d < h < e\) |

Note that each of the groups has exponent \(p^d\) and order \(p^{d+e}\). It is easily seen from the table that the groups from distinct families, or from the same family but with distinct parameters, are not isomorphic. \(\Box\)

4 Exact bicyclic pairs and automorphisms

In this section for the groups \(G\) determined in Theorem 12 we determine the set \(\mathcal{P}(G)\) of exact \((p^d, p^e)\)-bicyclic pairs \((\alpha, \beta)\) and the set \(\text{Aut}(G)\) of automorphisms of \(G\).

The following technical result will be useful.

**Lemma 13.** For the groups given in Theorem 12, the following statements hold true:

(i) The group \(G = M_1(d, e, f)\) is \(p^{d-f}\)-abelian and \(Z(G) = \langle a^{p^{d-f}} \rangle \langle b^{p^{d-f}} \rangle\).

(ii) The group \(G = M_2(d, e, f)\) is \(p^d-f\)-abelian and \(Z(G) = \langle a^{p^d-f} \rangle \langle b^{p^d-f} \rangle\).

(iii) The group \(G = M_3(d, e, h, f)\) is \(p^{h-f}\)-abelian and \(Z(G) = \langle a^{p^{h-f}} \rangle \langle b^{p^{h-f}} \rangle\).

**Proof.** To prove (i), we note that \(G' = \langle a^{p^f} \rangle \cong C_{p^{e-f}},\) so \(\bar{G}(G') = 1\). Hence by Lemma 2 the group \(G\) is \(p^{d-f}\)-abelian. It can be easily verified that \(Z(G) = \langle g^{p^{d-f}} \rangle \langle h^{p^{d-f}} \rangle\).

The proof for other cases is similar and is left to the reader. \(\Box\)

**Lemma 14.** Let \(\alpha = b^e a^l\) and \(\beta = b^e a^j\) where \(a, b\) are the generators of the group \(G\) given in Theorem 12, then
Corollary 15. (i) for $G = \mathcal{M}_1(d, e, f)$, if $(\alpha, \beta)$ is an exact $(p^d, p^e)$-bicyclic pair of $G$, then either $d = e$ and $p \nmid (il - jk)$, or $d < e$, $p^{e-d} \mid j$ and $p \nmid il$;

(ii) for $G = \mathcal{M}_2(d, e, f)$, if $(\alpha, \beta)$ is an exact $(p^d, p^e)$-bicyclic pair of $G$, then $p^{e-d} \mid i$ and $p \nmid jk$;

(iii) for $G = \mathcal{M}_3(d, e, h, f)$, if $(\alpha, \beta)$ is an exact $(p^d, p^e)$-bicyclic pair of $G$, then $p^{h-d} \mid (j + i/p^{e-h})$ and $p \nmid jk$.

Conversely, in each case if the numerical conditions are satisfied, then $(\alpha, \beta)$ is an exact $(p^d, p^e)$-bicyclic pair of $G$.

Proof. By the product formula $G = \langle \alpha \rangle \langle \beta \rangle$ is an exact $(p^d, p^e)$-bicyclic factorisation of $G$ if and only if $|\alpha| = p^d$, $|\beta| = p^e$ and $\langle \alpha \rangle \cap \langle \beta \rangle = 1$. Note that if $(\alpha, \beta)$ is an exact $(p^d, p^e)$-bicyclic pair of $G$, then by Burnside’s Basis Theorem $il - jk \not\equiv 0 \pmod{p}$.

For $G = \mathcal{M}_1(d, e, f)$, if $d = e$ then the result is obvious; see also (11). Now consider the case $d < e$. First assume that $(\alpha, \beta)$ is an exact $(p^d, p^e)$-bicyclic pair of $G$, then by Lemma 13(i) $G$ is $p^{e-f}$-abelian. Since $e - f \leq d \leq e$, by Proposition 2 we have

$$1 = \alpha^{p^d} = (b^i a^j)^{p^d} = b^{ip^d} a^{jp^d} = a^{ip^d},$$

so $j \equiv 0 \pmod{p^{e-d}}$, and hence $il \not\equiv 0 \pmod{p}$. Conversely, assume the congruences, then it can be easily verified that $|\alpha| = p^d$ and $|\beta| = p^e$. It suffices to prove that $\langle \alpha \rangle \cap \langle \beta \rangle = 1$. This is equivalent to that $\langle \alpha^{p^d-1} \rangle \cap \langle \beta^{p^e-1} \rangle = 1$. Since $e - f \leq e - 1$, by Lemma 13(i) we have $\beta^{p^e-1} = (b^i a^j)^{p^e-1} = a^{ip^{e-1}} \in \langle a \rangle$. Since $\alpha^{p^d-1} = (b^i a^j)^{p^d-1} = b^{ip^{d-1}} a^{jp^{e-1}}$, for some integer $s$, we have $\alpha^{p^d-1} \not\equiv \langle a \rangle$, so $\langle \alpha^{p^d-1} \rangle \cap \langle \beta^{p^e-1} \rangle = 1$.

The proof for the group $G = \mathcal{M}_2(d, e, f)$ is similar and omitted.

For $G = \mathcal{M}_3(d, e, h, f)$, if $G = \langle \alpha \rangle \langle \beta \rangle$ is an exact $(p^d, p^e)$-bicyclic factorisation of $G$, then $il - jk \not\equiv 0 \pmod{p}$. By Lemma 13(iii) $G$ is $p^{h-f}$-abelian. Since $h - f < d$, we have

$$1 = \alpha^{p^d} = (b^i a^j)^{p^d} = b^{ip^d} a^{jp^d} = a^{ip^d},$$

and so $j \equiv 0 \pmod{p^{e-d}}$, $e - f \leq d \leq e$, we obtain $p^{e-h} \parallel i$ and $j + i/p^{e-h} \equiv 0 \pmod{p^{h-d}}$. Since $il - jk \not\equiv 0 \pmod{p}$, $jk \not\equiv 0 \pmod{p}$. Conversely, assume the numerical conditions, then it is easy to verify that $|\alpha| = p^d$ and $|\beta| = p^e$. It suffices to prove $\langle \alpha \rangle \cap \langle \beta \rangle = 1$, or equivalently $\langle \alpha^{p^d-1} \rangle \cap \langle \beta^{p^e-1} \rangle = 1$. Since $G$ is $p^{h-f}$-abelian and $h - f \leq d < h < e$, we have

$$\beta^{p^e-1} = (b^i a^j)^{p^{d-1}} = b^{jp^{d-1}} a^{ip^{d-1}} = b^{jp^{e-h}} a^{ip^{h-d-1}} = a^{ip^{h-1}} \in \langle a \rangle.$$

Note that $p^{e-h} \parallel i$. Set $i = i' p^{e-c}$ where $p \nmid i'$. Then

$$\alpha^{p^d-1} = (b^i a^j)^{p^{d-1}} = b^{ip^{d-1}} \langle a \rangle = b^{ip^{d+c-e-1}} \langle a \rangle.$$

Since $\langle a \rangle \cap \langle b \rangle = \langle b^{p^{d+c-e}} \rangle$, we have $\alpha^{p^d-1} \not\equiv \langle a \rangle$. Thus $\langle \alpha^{p^d-1} \rangle \cap \langle \beta^{p^e-1} \rangle = 1$, as required.  

Corollary 15. (i) The number of exact $(p^d, p^e)$-bicyclic pairs of the group $\mathcal{M}_1(d, e, f)$ is equal to $p^{4e-3}(p^2 - 1)(p - 1)$ if $d = e$, and $p^{3d+e-2}(p - 1)^2$ if $d < e$.

(ii) The number of exact $(p^d, p^e)$-bicyclic pairs of the group $\mathcal{M}_2(d, e, f)$ is equal to $p^{3d+e-2}(p - 1)^2$.

(iii) The number of exact $(p^d, p^e)$-bicyclic pairs of the group $\mathcal{M}_3(d, e, h, f)$ is equal to $p^{3d+e-2}(p - 1)^2$. 

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Proof. For \( G = M_1(d, e, f) \), by Lemma \([11]\) if \( d = e \) then \((b^i a^j, b^k a^l)\) is a \((p^e, p^e)\)-bicyclic pair of \( G \) if and only if the matrix \( \begin{pmatrix} i & j \\ k & l \end{pmatrix} \) is invertible in the ring \( \mathbb{Z}_{p^e} \), so there are \( |GL(2, p^e)| = p^{4e-3}(p^2-1)(p-1) \) such pairs. On the other hand, if \( d < e \) then \((b^i a^j, b^k a^l)\) is a \((p^d, p^e)\)-bicyclic pair of \( G \) if and only if \( p^e-d \mid j \) and \( p \nmid il \), in which case the number of choices for each \( i, j, k \) and \( l \) is \( \phi(p^d), p^d, p^d \) and \( \phi(p^e) \), respectively. Multiplying these gives the desired number in (i). The proof for other cases is similar. \( \square \)

Lemma 16. Let \( p \) be an odd prime and \( n \) a positive integer, then \( p^{n-k+2} \) divides \( \binom{p^n}{k} \) for all \( 3 \leq k \leq n+2 \).

Proof. First it can be easily proved by induction on \( k \) that \( k \leq p^{k-2} \) for all integers \( k \geq 3 \). Now write \( k = p^e k_1 \) where \( e \geq 0 \) and \( p \nmid k_1 \), and denote the factor \( p^e \) of \( k \) by \((k)_p\). Since

\[
\binom{p^n}{k} = \frac{p^n(p^n-1)(p^n-2) \cdots [p^n-(k-1)]}{k!\cdots(k-1)!} = \frac{p^n}{(k)_p} \prod_{j=1}^{k-1} \frac{p^n-j}{j},
\]

and each of the numbers \( \frac{p^n-j}{j} (1 \leq j \leq k-1) \) is coprime to \( p \), we have \( \left( \binom{p^n}{k} \right)_p = \frac{p^n}{(k)_p} \). Since \( (k)_p \leq k \leq p^{k-2} \), we get \( p^{n-k+2} \mid \binom{p^n}{k} \), as claimed. \( \square \)

In what follows we determine the automorphisms of the groups in Theorem \([12]\).

Lemma 17. Let \( G = M_1(d, e, f) \), then the assignment \( a \mapsto b^r a^s, b \mapsto b^f a^u \) extends to an automorphism of \( G \) if and only if one of the following cases occur:

(i) \( f = e = d \) and \( ru - st \not\equiv 0 \pmod{p} \).

(ii) \( f < e = d \), \( r \equiv 0 \pmod{p^{e-f}} \), \( t \equiv 1 \pmod{p^{e-f}} \) and \( s \not\equiv 0 \pmod{p} \).

(iii) \( d < f = e \), \( u \equiv 0 \pmod{p^{e-d}} \) and \( st \not\equiv 0 \pmod{p} \).

(iv) \( d \leq f < e \), \( u \equiv 0 \pmod{p^{e-d}} \), \( t \equiv 1 \pmod{p^{e-f}} \) and \( s \not\equiv 0 \pmod{p} \).

(v) \( f < d < e \), \( r \equiv 0 \pmod{p^{d-f}} \), \( u \equiv 0 \pmod{p^{e-d}} \), \( t \equiv 1 \pmod{p^{e-f}} \) and \( s \not\equiv 0 \pmod{p} \).

Proof. Set \( a_1 = b^r a^s \) and \( b_1 = b^f a^u \). Then the assignment \( a \mapsto a_1, b \mapsto b_1 \) extends to an automorphism of \( G \) if and only if \( a_1^{p^e} = b_1^{p^d} = 1, a_1^{b_1} = a_1^{1+p^f} \) and \( G = \langle a_1, b_1 \rangle \).

First assume that \( a \mapsto a_1, b \mapsto b_1 \) extends to an automorphism of \( G \). Set \( q := 1 + p^f \), then \( a_1^{b_1} = a_1^q \). Since

\[
a_1^{b_1} = (b^r a^s)^{b^f a^u} = (b^f)^{a^u(^r a^s)^{b^f}} = b^f a^u(b^{r(q-1)} + sq^e)
\]

and

\[
a_1^q = (b^r a^s)^q = b^{r-q} a^{s\sigma},
\]

where \( \sigma = \sum_{i=1}^{q-1} q^{r(i-1)} \), we get \( b^r(q-1) = a^{u(q-1)} + s(q^e - \sigma) \). But \( \langle a \rangle \cap \langle b \rangle = 1 \), thus

\[
r(q-1) \equiv 0 \pmod{p^d}, \quad u(q^e-1) \equiv s(q^e - \sigma) \pmod{p^e}.
\]

Note that \( G = \langle a_1, b_1 \rangle \), so by Burnside’s Basis Theorem \( ru - st \not\equiv 0 \pmod{p} \). To simplify the numerical conditions we distinguish two cases:
Case (I). $d = e$. We distinguish two subcases:
   If $f = d = e$ then $q = 1$ and the congruences (3) and (4) are redundant.
   If $f < d = e$ then by (3) we have $r \equiv 0 \pmod{p^{f-l}}$, so $st \not\equiv 0 \pmod{p}$ and by Lemma (11) we get $q^r - 1 = (1 + p^l)^r - 1 \equiv 0 \pmod{p^r}$. Setting $z = q^r - 1$, then
   $$\sigma = \frac{q^{r^q} - 1}{q^r - 1} = \frac{1}{z}((1 + z)^q - 1) = \left(\frac{q}{1}\right) + \sum_{i=2}^{q} \left(\frac{q}{1}\right) z^{i-1}. \quad (5)$$
   Thus $\sigma \equiv q \pmod{p^r}$ and hence (4) is reduced to $s q (q^r - 1) \equiv 0 \pmod{p^r}$, which implies that $t \equiv 1 \pmod{p^{f-l}}$ by Lemma (11).

Case (II). $d < e$. Since $|b_1| = p^d$ we have $u \equiv 0 \pmod{p^{f-d}}$ and $st \not\equiv 0 \pmod{p}$. In what follows we distinguish three subcases:
   If $d < f < e$ then $q = 1$ and so (3) and (4) are redundant.
   If $d \leq f < e$ then $e + f - d \geq e$, so $u(q^r - 1) \equiv 0 \pmod{p^r}$. As before write $z = q^r - 1$ and expand $\sigma$ as $[5]$. Note that $z \equiv 0 \pmod{p^f}$. Since $2f \geq d + f \geq e$, we have $\binom{q}{i} z^{i-1} \equiv 0 \pmod{p^r}$ for all $i \geq 2$, and hence $\sigma \equiv q \pmod{p^r}$. Therefore (4) is reduced to $s q (q^r - 1) \equiv 0 \pmod{p^r}$, which implies that $t \equiv 1 \pmod{p^{f-l}}$.
   Finally, if $f < d < e$, then by (3) $r \equiv 0 \pmod{p^{f-l}}$, so by Lemma (11) we get $q^r - 1 \equiv 0 \pmod{p^d}$, and hence $u(q^r - 1) \equiv 0 \pmod{p^f}$. As before write $z = q^r - 1$ and expand $\sigma$ as [5]. Then $\binom{q}{i} z^{i-1} \equiv 0 \pmod{p^r}$ for all $i \geq 2$, so $\sigma \equiv q \pmod{p^r}$. Therefore (4) is reduced to $s q (q^r - 1) \equiv 0 \pmod{p^r}$, which implies that $t \equiv 1 \pmod{p^{f-l}}$.

   Conversely, in each case if the numerical conditions are fulfilled, then it is straightforward to verify that the above assignment extends to an automorphism of $G$, as required.

Corollary 18. Let $M_1(d, e, f)$ be the group given by Theorem (12). Then

$$|\text{Aut}(M_1(d, e, f))| = \begin{cases} 
  p^{4e-3}(p^2 - 1)(p - 1), & f = e = d, \\
  p^{2e+f-1}(p-1), & f < e = d, \\
  p^{3d+e-2}(p-1)^2, & d < f = e, \\
  p^{3d+f-1}(p-1), & d \leq f < e, \\
  p^{2(d+f)-1}(p-1), & f < d < e.
\end{cases}$$

Proof. If $f = e = d$, then by Lemma (17(i)) the size of $\text{Aut}(M_1(e, e, e))$ is equal to the number of invertible matrices $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ with entries in $\mathbb{Z}_{p^e}$, which is $p^{4e-3}(p^2 - 1)(p - 1)$.

In what follows the size of $\text{Aut}(M_1(d, e, f))$ for the remaining cases is determined by multiplying the numbers of choices for the parameters $r, s, t$ and $u$. By Lemma (17(ii)-(v)), we have the following:
   If $f < e = d$, then the number of choices for each $r, s, t$ and $u$ is $p^l$, $\phi(p^e)$, $p^f$ and $p^e$, respectively.
   If $d < f = e$, then the number of choices for each $r, s, t$ and $u$ is $p^d$, $\phi(p^e)$, $\phi(p^d)$ and $p^d$, respectively.
   If $d \leq f < e$, then the number of choices for each $r, s, t$ and $u$ is $p^d$, $\phi(p^e)$, $p^{d+f-e}$ and $p^d$, respectively.
   If $f < d < e$, then the number of choices for each $r, s, t$ and $u$ is $p^l$, $\phi(p^e)$, $p^{d+f-e}$ and $p^d$, respectively.

Lemma 19. Let $M_2(d, e, f)$ and $M_3(d, e, h, f)$ be the groups given by Theorem (12), then
(i) the assignment \( a \mapsto b^t a^s, b \mapsto b^t a^u \) extends to an automorphism of \( M_2(d, e, f) \) if and only if \( r \equiv 0 \pmod{p^{e-f}} \), \( s \not\equiv 0 \pmod{p} \) and \( t \equiv 1 \pmod{p^{d-f}} \).

(ii) the assignment \( a \mapsto b^t a^s, b \mapsto b^t a^u \) extends to an automorphism of \( M_3(d, e, h, f) \) if and only if \( r \equiv 0 \pmod{p^{d+e-h-f}}, s \equiv 1 + up^{e-h} \pmod{p^{h-d}} \), \( t \equiv 1 \pmod{p^{d-f}} \) and \( r \equiv (t-1)p^{e-h} \pmod{p^{f-e}} \).

**Proof.** Denote \( a_1 = b^t a^s \) and \( b_1 = b^t a^u \), and write \( q = 1 + p^f \).

(i) First assume that the assignment \( a \mapsto a_1, b \mapsto b_1 \) extends to an automorphism of \( G = M_2(d, e, f) \), then in terms of \( a_1 \) and \( b_1 \) the group \( G \) has the presentation

\[
G = \langle a_1, b_1 \mid a_1^{p^f} = b_1^{p^e} = 1, a_1^{b_1} = a_1^q \rangle.
\]

By Lemma 13(ii) the group \( G \) is \( p^{d-f} \)-abelian, so \( 1 = a_1^{p^f} = b_1^{p^e} = a_1^{p^d} = a_1^{pq^d} \), and hence \( r \equiv 0 \pmod{p^{d-f}} \). By Burnside’s Basis Theorem, \( ru-st \not\equiv 0 \pmod{p} \), so \( st \not\equiv 0 \pmod{p} \).

Moreover,

\[
a_1^{b_1} = (b^t a^s)^{b^t a^u} = (b^t)^a_s (a^s)^{b^t} = b^t [b^t, a^u] (a^s)^{b^t} \in b_1 \langle a \rangle
\]

and

\[
a_1^q = (b^t a^s)^q \in b_1^q \langle a \rangle,
\]

so from the relation \( a_1^{b_1} = a_1^q \) and the fact \( \langle a \rangle \cap \langle b \rangle = 1 \) we deduce that \( b^t a^u = b^s \). Thus \( r(q-1) \equiv 0 \pmod{p^e} \), and hence \( r \equiv 0 \pmod{p^{d-f}} \). By Lemma 13(ii), \( b^r \in Z(G) \), so \( b^r a^{sq^d} = b_1^{sq^d} = a_1^q = b_1^{pq^d} a^{pq^d} \), and hence \( b_1^{q^d-1} \equiv a_1^{pq^d}. \)

Conversely, if the numerical conditions are fulfilled, then it is straightforward to verify that the assignment extends to an automorphism of \( G \).

(ii) First assume that the assignment \( a \mapsto a_1, b \mapsto b_1 \) extends to an automorphism of \( G = M_3(d, e, h, f) \), then

\[
G = \langle a_1, b_1 \mid a_1^{p^h} = b_1^{p^{d+e-h}} = a_1^{p^d}, a_1^{b_1} = a_1^q \rangle,
\]

where \( q := 1 + p^f \). By Lemma 13(iii) \( G \) is \( p^{h-f} \)-abelian. Since \( h-f \leq d < h \), we have

\[
1 = a_1^{p^h} = (b^t a^s)^{p^h} = b^{rp^h} a^{sp^h} = b^{p^h},
\]

so \( r \equiv 0 \pmod{p^{e-h}} \). Since \( ru-st \not\equiv 0 \pmod{p} \), we obtain \( st \not\equiv 0 \pmod{p} \).

Moreover, we have

\[
a_1^{b_1} = (b^t a^s)^{b^t a^u} = (b^t)^a_s (a^s)^{b^t} = b^t a^{u(1-q)+sq^d},
\]

where \( \sigma = \sum_{i=1}^{q} q^{r(i-1)} \), so from the relation \( b_1^{d+e-h} = a_1^{p^d} \) we deduce that

\[
b_1^{q^d-1} = a^{u(1-q)+s(q^d-\sigma)}.
\]

Since \( \langle a \rangle \cap \langle b \rangle = (b^{d+e-h}) = \langle a^{p^d} \rangle \), we get

\[
r(q-1) \equiv 0 \pmod{p^{d+e-h}},
\]

\[
u(q^d-1) \equiv s(q^d-\sigma) \pmod{p^d}.
\]
Upon substitution (6) is transformed to
\[ a^{u(1-q^t)+s(q^t-\sigma)} = b^{r(q-1)} = (b^{p^{d+e}})^{r p^{e+h-e-d}} = a^{r p^{f+h-e}}, \]
which implies that
\[ u(1 - q^r) + s(q^t - \sigma) \equiv r p^{f+h-e} \pmod{p^h}. \] (9)

By (7) we get \( r \equiv 0 \pmod{p^{d+e-h-f}} \). Setting \( z = q^r - 1 \), then
\[ z = (1 + p^f)^{r - 1} = \left(\frac{r}{1}\right) p^f + \sum_{i=2}^{r} \left(\frac{r}{i}\right) p^f. \]

For all \( i \geq 2 \) by Lemma 16 we have \( p^{d+e-h-f-i+2+if} \mid \left(\begin{array}{c} r \\ i \end{array}\right) p^f \); since \( d+e-h-f-i+2+if \geq d-f-(i-2)+if = d+f+(i-2)(f-2) \geq d+f \geq h \), we get \( \left(\begin{array}{c} r \\ i \end{array}\right) p^f \equiv 0 \pmod{p^h} \), and so \( z = q^r - 1 \equiv r p^f \pmod{p^h} \). It follows that
\[ \sigma = \frac{q^r - 1}{q^r - 1} = \frac{1}{z}(1 + z)^q - 1 = \left(\frac{q}{1}\right) + \left(\frac{q}{2}\right) z + \sum_{i=3}^{q} \left(\frac{q}{i}\right) z^{-1} = q \pmod{p^h} \]
and so (8) and (9) are reduced to
\[ sq(q^{-1} - 1) \equiv 0 \pmod{p^d}, \]
\[ sq(q^{-1} - 1) \equiv r p^{f+h-e}(1 + up^{e-h}) \pmod{p^h}. \] (10) (11)

By Lemma 11 we deduce from (10) that \( t \equiv 1 \pmod{p^{d-f}} \). Hence
\[ q^{t-1} - 1 = (1 + p^f)^{t-1} - 1 = \left(\frac{t - 1}{1}\right) p^f + \sum_{i=2}^{t-1} \left(\frac{t - 1}{i}\right) p^f = (t - 1)p^f \pmod{p^h}, \]
and consequently, (11) is reduced to
\[ sq(t - 1)p^f \equiv r p^{f+h-e}(1 + up^{e-h}) \pmod{p^h}. \] (12)

We proceed to consider the relation \( b_1^{p^{d+e}} = a_1^{p^d} \). By Lemma 13(iii) \( G \) is \( p^{h-f} \)-abelian. Since \( h - f \leq d \leq d + e - h \), we have \( b_1^{p^{d+e-h}} = (b^a u)^{p^{d+e}} = b_1^{p^{d+e-h} a u p^{d+e-h}} \) and \( a_1^{p^d} = (b^a)^{p^d} = b^{p^d} a^{p^d} \), and so
\[ a^{(s - up^{e-h})p^d} = b_1^{p^{d+e-h-rp^d}} = (b^{p^{d+e-h}})^{t-rp^d} = c^{a(t-rp^d)}. \]

Thus
\[ (s - up^{e-h})p^d \equiv (t - rp^d) \pmod{p^h}. \] (13)

Write \( r = r_1 p^{d+e-h-f} \) and \( t = 1 + t_1 p^{d-f} \). Recall \( q = 1 + p^f \) and \( h - d \leq f \). Then (12) and (13) are reduced to
\[ st_1 \equiv r_1 (1 + up^{e-h}) \pmod{p^{h-d}}, \]
\[ s \equiv 1 + up^{e-h} + (t_1 - r_1) p^{d-f} \pmod{p^{h-d}}. \] (14) (15)

Substituting \( 1 + up^{e-h} + (t_1 - r_1) p^{d-f} \) for \( s \) in (14) we obtain \( (t_1 - r_1)(1 + up^{e-h} + p^{d-f}) \equiv 0 \pmod{p^{h-d}} \), thus \( r_1 \equiv t_1 \pmod{p^{h-d}} \) (or equivalently, \( r \equiv (t - 1)p^{e-h} \pmod{p^{e-f}} \)). Therefore (15) is reduced to \( s \equiv 1 + up^{e-h} \pmod{p^{h-d}} \).

Conversely, if the numerical conditions are fulfilled, then it is straightforward to verify that the above assignment extends to an automorphism of \( G \), as required. \( \square \)
Corollary 20. Let $M_2(d,e,f)$ and $M_3(d,e,h,f)$ be the groups given by Theorem 12. Then

(i) $|\text{Aut}(M_2(d,e,f))| = p^{d+e+2f-1}(p-1)$.

(ii) $|\text{Aut}(M_3(d,e,h,f))| = p^{2d+e+2f-h}$.

Proof. (i) By Lemma 19(i), the number of choices for each $r, s, t$ and $u$ is $p^t, p^{d-1}(p-1),$ $p^{e+f-d}$ and $p^d$, respectively, and multiplying these gives the desired number.

(ii) Note that $r, t \in \mathbb{Z}_{p^d+e-h}$ and $s, u \in \mathbb{Z}_{p^h}$. By Lemma 19(ii), we may write $r = r_1p^{d+e-h-f}$ and $t = 1 + t_1p^{d-f}$ where $r_1 \in \mathbb{Z}_{p^d}$ and $t_1 \in \mathbb{Z}_{p^d+e-h}$, so the congruence $r \equiv (t-1)p^{e-f}$ (mod $p^{e-f}$) is reduced to $r_1 \equiv t_1$ (mod $p^{h-d}$). Thus for each $r_1 \in \mathbb{Z}_{p^d}$ the number of choices for $t_1 \in \mathbb{Z}_{p^d+e-h}$ such that $r_1 \equiv t_1$ (mod $p^{h-d}$) is equal to $p^{d+e+f-2h}$, and for each $u \in \mathbb{Z}_{p^h}$ the number of choices for $s \in \mathbb{Z}_{p^d}$ such that $s \equiv 1 + up^{e-h}$ (mod $p^{h-d}$) is equal to $p^d$. Consequently, the desired number is the product $p^t p^{d+e+f-2h} p^h p^d = p^{2d+e+2f-h}$. □

5 Enumeration

In this section we calculate the number of isomorphism classes of reciprocal pairs of $(p^d, p^e)$-complete regular dessins.

The following result deals with the particular case $d = e$ where symmetric dessins may appear.

Lemma 21. For each $e \geq 1$, up to isomorphism there are $p^{2(e-1)}$ regular dessins with underlying graphs $K_{p^e, p^e}$, of which the number of symmetric ones is $p^{e-1}$.

Proof. By Theorem 12 the automorphism group of a $(p^e, p^e)$-complete regular dessin is isomorphic to $G := M_1(e,e,f)$ for some integer $f$, where $1 \leq f \leq e$. By Corollary 15 the number of exact $(p^e, p^e)$-bicyclic triples of $G$ is $p^{4e-3}(p^2-1)(p-1)$, and by Corollary 18 $|G| = p^{4e-3}(p^2-1)(p-1)$ if $f = e$, and $|\text{Aut}(G)| = p^{2(e+f)-1}(p-1)$ if $f < e$. Thus by Proposition 5 for each fixed $f$, up to isomorphism the number of $(p^e, p^e)$-complete regular dessins with automorphism group isomorphic to $G$ is 1 if $f = e$, and $p^{2e-2f-2}(p^2-1)$ if $1 \leq f < e$. Summing up we obtain the total number of $(p^e, p^e)$-complete regular dessins (up to isomorphism):

$$1 + \sum_{f=1}^{e-1} p^{2e-2f-2}(p^2-1) = p^{2(e-1)}.$$ 

By 20 Theorem 1] exactly $p^{e-1}$ of these are symmetric, as claimed. □

Combining Proposition 5 with Corollary 15, 18 and 20 we immediately obtain the following three results.

Lemma 22. Let $d < e$, then for each fixed $f$, up to isomorphism the number $\nu_1(d,e,f)$ of reciprocal pairs of $(p^d, p^e)$-complete regular dessins with automorphism group isomorphic to $M_1(d,e,f)$ is

$$\nu_1(d,e,f) = \begin{cases} 1, & \text{if } 1 \leq d < f = e, \\
p^{e-f-1}(p-1), & \text{if } 1 \leq d \leq f < e \leq d + f, \\
p^{d+e-2f-1}(p-1), & \text{if } 1 \leq f < d < e \leq d + f. \end{cases}$$

Lemma 23. Let $d < e$, then for each fixed $f$, $1 \leq f < d < e$, up to isomorphism the number $\nu_2(d,e,f)$ of reciprocal pairs of $(p^d, p^e)$-complete regular dessins with automorphism group isomorphic to $M_2(d,e,f)$ is $p^{2d-2f-1}(p-1)$. 

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Lemma 24. Let \( d < e \), then for fixed \( h \) and \( f \), \( h - d \leq f < d < h < e \), up to isomorphism the number \( \nu_3(d, e, h, f) \) of reciprocal pairs of \((p^d, p^e)\)-complete regular dessins with automorphism group isomorphic to \( M_3(d, e, h, f) \) is \( p^{d+h-2(f+1)}(p-1)^2 \).

Now we are ready to prove the main result of the paper.

Proof of Theorem 1: Let \( G \) denote the automorphism group of a \((p^d, p^e)\)-complete regular dessin. If \( d = 0 \) then \( G \) is a cyclic \( p \)-group of order \( p^e \), so by Example 1 \( \nu(d, e) = 1 \).

If \( 1 \leq d = e \), then by Lemma 21 the number of nonsymmetric dessins is \( p^{2(e-1) - p^{e-1}} \), so

\[
\nu(d, e) = p^{e-1} + \frac{1}{2}(p^{2(e-1)} - p^{e-1}) = \frac{1}{2}p^{e-1}(1 + p^{e-1}).
\]

In what follows we assume that \( 1 \leq d < e \). We distinguish four cases:

Case (1). \( 1 = d < e \). By Theorem 12, \( G \cong M_1(1, e, f) \) where the numerical conditions are reduced to \( 1 \leq f \leq e - 1 \). Thus either \( f = e - 1 \) or \( f = e \). By Lemma 22, \( \nu(d, e) = \nu_1(d, e, e - 1) + \nu_1(d, e, e) = (p - 1) + 1 = p \).

Case (2). \( 1 < d = e - 1 \). By Theorem 12 either \( G \cong M_1(e - 1, e, f) \) where \( 1 \leq f \leq e \), or \( G \cong M_2(e - 1, e, f) \) where \( 1 \leq f < e - 1 \). By Lemma 22 and 23 we get

\[
\nu(e - 1, e) = \sum_{1 \leq f < e - 1} \nu_1(e - 1, e, f) + \sum_{1 \leq f < e} \nu_2(e - 1, e, f)
\]

\[
= 1 + (p - 1) + \sum_{1 \leq f < e - 1} p^{2(e-1)-2f}(p - 1) + \sum_{1 \leq f < e - 1} p^{2(e-1)-2f-1}(p - 1)
\]

\[
= p + (p - 1)(p^{2e-2} + p^{2e-3}) \sum_{1 \leq f < e - 1} p^{-2f}
\]

\[
= p^{2e-3}.
\]

Case (3). \( 1 < d < e - 1 \) and \( e < 2d \). Combing the hypothesis with the numerical conditions in Theorem 12 we see that the following subcases may happen: (3.1) \( G \cong M_1(d, e, f) \) where \( e - d \leq f \leq e \); (3.2) \( G \cong M_2(d, e, f) \) where \( 1 \leq f < d \); (3.3) \( G \cong M_3(d, e, h, f) \) where \( d < h < e \) and \( h - d \leq f < d \). Thus, by Lemma 22, 23 and 24 we have

\[
\nu(d, e) = \sum_{e - d \leq f \leq e} \nu_1(d, e, f) + \sum_{1 \leq f < d} \nu_2(d, e, f) + \sum_{d < h < e - d \leq f < d} \nu_3(d, e, h, f)
\]

\[
= 1 + \sum_{d \leq f < e} p^{e-f-1}(p - 1) + \sum_{e - d \leq f < d} p^{d+e-2f-1}(p - 1) + \sum_{1 \leq f < d} p^{2d-2f-1}(p - 1)
\]

\[
+ \sum_{d < h < d} p^{d+h-2(f+1)}(p - 1)^2
\]

\[
= p^{e-d} + \frac{1}{p+1}(p^{3d-e+1} - p^{e-d+1}) + \frac{1}{p+1}(p^{2d-1} - p) + \frac{1}{p+1}(p^{2d} - p^{3d-e+1} - p^{e-d} + p)
\]

\[
= p^{2d-1}.
\]

Case (4). \( 1 < d < e - 1 \) and \( e \geq 2d \). By Theorem 12 the following subcases may happen: (4.1) \( G \cong M_1(d, e, f) \) where \( e - d \leq f \leq e \); (4.2) \( G \cong M_2(d, e, f) \) where \( 1 \leq f < d \); (4.3) \( G \cong M_3(d, e, f) \) where \( d < h < 2d \) and \( h - d \leq f < d \). Thus, by Lemma 22, 23 and 24 we have

\[
\nu(d, e) = \sum_{e - d \leq f \leq e} \nu_1(d, e, f) + \sum_{1 \leq f < d} \nu_2(d, e, f) + \sum_{d < h < 2d - d \leq f < d} \nu_3(d, e, f)
\]

\[
= p^d + \frac{1}{p+1}[(p^{2d-1} - p) + (p^{2d} - p^{1+d} - p^d + p)] = p^{2d-1}.
\]
Finally, for each \((p^d, p^e)\)-complete regular dessin \(D = (G, \alpha, \beta)\) where \((\alpha, \beta)\) is an exact \((p^d, p^e)\)-bicyclic pair of \(G\) given by Lemma 14 to determine its type \(|\alpha|, |\beta|, |\alpha\beta|\) it suffices to evaluate \(|\alpha\beta|\). It is easy to check that in all cases \(|\alpha\beta| = p^e\). The genus of \(D\) follows from Euler-Poincaré formula.

Remark 2. Let \((\varphi, \varphi^*)\) be a pair of skew-morphisms \(\varphi\) and \(\varphi^*\) of the cyclic groups \(\mathbb{Z}_n\) and \(\mathbb{Z}_m\), and \(\pi\) and \(\pi^*\) the associated power functions, respectively. The skew-morphism pair \((\varphi, \varphi^*)\) will be called reciprocal if they satisfy the following conditions:

(i) the orders of \(\varphi\) and \(\varphi^*\) divide \(m\) and \(n\), respectively,

(ii) \(\pi(x) = -\varphi^* - x(-1)\) and \(\pi^*(y) = -\varphi^{-y}(-1)\) are power functions for \(\varphi\) and \(\varphi^*\), respectively.

In [11, Theorem 6] the authors establish a one-to-one correspondence between the isomorphism classes of reciprocal pairs of \((m, n)\)-regular dessins and reciprocal pairs \((\varphi, \varphi^*)\) of skew-morphisms \(\varphi\) and \(\varphi^*\) of the cyclic groups \(\mathbb{Z}_n\) and \(\mathbb{Z}_m\). Therefore, Theorem 1 also gives rise to the number of reciprocal pairs of skew-morphisms \(\varphi\) and \(\varphi^*\) of the cyclic groups \(\mathbb{Z}_{p^d}\) and \(\mathbb{Z}_{p^e}\) for odd prime \(p\).

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