Novel FIR Inversion with Only FIRs

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Abstract

The inversion of an FIR data sampling is usually stated to be possible with the use of a potentially unstable IIR, and in particular circumstances. It is possible to accomplish the same inversion with the doubling of an FIR sampling and with only FIRs for the sampling and the inversion. This note presents the configuration, which apparently is not in the literature, for perfect signal reconstruction.
The sampling of information with FIR filters is very common \[1\]. The signal reconstruction of a data stream sampled with these filters is typically not analyzed in a simple manner, and the invocation of a potentially unstable IIR filter bank is utilized. It is commonly stated that the inversion of a signal with only FIRs is not possible. This is not the case, and in this note a signal reconstruction is presented that performs this task without loss of information.

A digital data stream is denoted by \( X(n) \), and a k-tap filtering is performed with the transform,

\[
Y(n) = \sum_{i=1}^{k} b_i X(n - i) . \tag{1}
\]

The coefficients \( b_k \) are real for a real data stream, but the duplication of the process can be performed with complex coefficients \( c_k \). In the latter scenario, the real data stream \( X(n) \) is used and the real part of \( Y(n) \) is taken, i.e. the FIR filtering is identical to taking \( \text{Re} \ Y(n) \) with the real parts of the complex coefficients \( \text{Re} \ c_k = b_k \).

The duplication of the FIR filtering is not required in general. However, with this process of using complex coefficients \( c_k \), an perfect reconstruction of the filtered signal \( Y(n) \) can be achieved with using an additional complex FIR, with suitably chosen taps. The use of an IIR is not required.

The double process of the two FIR filterings results in the signal,

\[
Z(n) = \sum_{i=1}^{k} \sum_{j=1}^{k} d_k c_k X(n - i - j) . \tag{2}
\]

Choosing the complex coefficients \( d_k \) appropriately results in \( Z(n) = X(n) \).

Consider the example of a 4-tap filter. The reconstruction appears with the following formulae,

\[
\begin{align*}
c_0d_0 &= \rho + i\alpha_1 \\
c_1d_0 + c_0d_1 &= i\alpha_2 \\
c_1d_1 + c_0d_2 + c_2d_0 &= i\alpha_3 \\
c_2d_1 + c_1d_2 + c_3d_0 + c_0d_3 &= i\alpha_4 \\
c_2d_2 + c_3d_1 + c_1d_3 &= i\alpha_5 \\
c_3d_2 + c_2d_3 &= i\alpha_6
\end{align*}
\]
\[c_3d_3 = i\alpha_7.\]  

(3)

In general these equations are not invertible for real inputs \(c_i\) and \(d_i\), and with \(\alpha_j = 0\). Taking these taps to be complex, and with \(\alpha_i\) arbitrary, allows for a general solution. Then the signal \(\Re e Z(n) = X(n)\).

The case of a DFT is known, pertaining to \(|c_i| = 1\), that is, with coefficients on the unit circle. More general filtering requires these coefficients to be anywhere in the complex plane.

The requirements to invert a real signal are then \(\Re e c_i = b_i\) and the solution to the general system in (3). The coefficients \(\alpha_i\) can be anything, and \(\rho\) is a parameter than may rescale the output. The system in (3) has seven complex equations, in which only six real components are non-trivial as the \(\alpha_i\) are a priori free parameters (they can be chosen to solve the system of equations). There are only four inputs, the \(b_i\). There are twelve real unknowns \(\Im m c_i\) and \(d_i\).

The conditions in (3) form a matrix equation,

\[
\begin{pmatrix}
  c_0 & 0 & 0 & 0 \\
  c_1 & c_0 & 0 & 0 \\
  c_2 & c_0 & c_0 & 0 \\
  c_3 & c_2 & c_1 & c_0 \\
  0 & c_3 & c_2 & c_1 \\
  0 & 0 & c_3 & c_2 \\
  0 & 0 & 0 & c_3 \\
\end{pmatrix}
\begin{pmatrix}
  d_0 \\
  d_1 \\
  d_2 \\
  d_3 \\
\end{pmatrix}
= i
\begin{pmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3 \\
  \alpha_4 \\
  \alpha_5 \\
  \alpha_6 \\
  \alpha_7 \\
\end{pmatrix}.
\]

(4)

The inversion of the algebraic system generates the complex \(d_i\) parameters and the imaginary components of \(c_i\) (the \(b_i\)). For example, Matlab can used to invert the non-square matrix, followed by the multiplication with a vector of \(\alpha_i\) entries and solve for the \(d_i\). In general the \(\alpha_i\) are non-vanishing; a set of zero entries gives zero values for the \(d_i\) taps. The 7 equations modeling the real components of (4) can be written in a 8x8 form with the real and imaginary components of \(d = d_e + id_i\). This system is complete (7 equations in eight variables) and can be used to solve for the four \(d_i\) in terms of the complex components of \(c_i\). The solutions to \(d_i\) are then used to determine the \(\alpha_i\) parameters.

This procedure is straightforward to implement for a general tap filter, even for large numbers. The perfect inversion of the FIR filtered signal is accomplished without the use of a potentially unstable IIR on some data, and with perfect reconstruction.
The simple configuration requires the doubling of the input signal into real and imaginary parts, with another pair of filters to reconstruct. The configuration is also more cost effective than using IIRs.

References

[1] N.J. Fliege, *Multirate Digital Signal Processing*, 4th ed., John Wiley and Sons, 1994.