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Takayasu cofibrations revisited

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1 Introduction

Given a natural number \( n \), let \( \tilde{\rho}_n \) be the reduced real regular representation of the elementary abelian 2-group \( V_n := (\mathbb{Z}/2)^n \). Let \( BV_n^{k\tilde{\rho}_n} \), \( k \in \mathbb{N} \), denote the Thom space over the classifying space \( BV_n \) associated to the direct sum of \( k \) copies of the representation \( \tilde{\rho}_n \). Following S. Takayasu [14], let \( M(n)_k \) denote the stable summand of \( BV_n^{k\tilde{\rho}_n} \) which corresponds to the Steinberg module of the general linear group \( GL_n(\mathbb{F}_2) \) [12].

Takayasu constructed in [14] a cofibration of the following form:

\[ \Sigma^k M(n-1)_{2k+1} \to M(n)_k \to M(n)_{k+1}. \]

This generalised the splitting of Mitchell and Priddy \( M(n) \simeq L(n) \vee L(n-1) \), where \( M(n) = M(n)_0 \) and \( L(n) = M(n)_{-1} \) [12]. Takayasu also considered the spectra \( M(n)_k \), \( k \geq 0 \), the above cofibrations are still valid for these spectra. Here and below, all spectra are implicitly completed at the prime two.

Note that the spectra \( M(n)_k \), \( k \geq 0 \), are used in the description of layers of the Goodwillie tower of the identity functor evaluated at spheres [2, 1], and the above cofibrations can also be deduced by combining Goodwillie calculus with the James fibration, as described by M. Behrens in [3, Chapter 2].

The purpose of this note is to give another proof for the existence of the above cofibrations for the cases \( k \in \mathbb{N} \). This will be carried out by employing techniques in the category of unstable modules over the Steenrod algebra [13]. Especially, the action of Lannes’ T-functor on the Steinberg unstable modules (see [14]) will play a crucial role in studying the vanishing of some extension groups of modules over the Steenrod algebra.

2 Algebraic short exact sequences

In this section, we recall the linear structure of the mod 2 cohomology of \( M(n, k) \) and the short exact sequences relating these \( \mathcal{S} \)-modules. Recall that the general linear group \( GL_n := GL_n(\mathbb{F}_2) \) acts on \( H^*V_n \cong \mathbb{F}_2[x_1, \ldots, x_n] \) by the rule:

\[ (gF)(x_1, \ldots, x_n) := F(\sum_{i=1}^n g_{i,1}x_i, \ldots, \sum_{i=1}^n g_{i,n}x_i), \]
where \( g = (g_{i,j}) \in GL_n \) and \( F(x_1, \cdots, x_n) \in \mathbb{F}_2[x_1, \ldots, x_n] \). This action commutes with the action of the Steenrod action on \( \mathbb{F}_2[x_1, \ldots, x_n] \).

By definition, the Thom class of the vector bundle associated to the reduced regular representation \( \tilde{\rho}_n \) is given by the top Dickson invariant:

\[
\omega_n = \omega_n(x_1, \ldots, x_n) := \prod_{0 \neq x \in \mathbb{F}_2[x_1, \ldots, x_n]} x.
\]

Recall also that the Steinberg idempotent \( e_n \) of \( \mathbb{F}_2[GL_n] \) is given by

\[
e_n := \sum_{b \in B, \sigma \in \Sigma_n} b \sigma,
\]

where \( B_n \) is the subgroup of upper triangular matrices in \( GL_n \) and \( \Sigma_n \) the subgroup of permutation matrices.

Let \( M_{n,k} \) denote the mod 2 cohomology of the spectrum \( M(n,k) \). By Thom isomorphism, we have an isomorphism of \( \mathcal{A} \)-modules:

\[
M_{n,k} \cong \text{Im}[\omega^k_n H^*BV_n \xrightarrow{\alpha} \omega^k_n H^*BV_n].
\]

We note that \( M_{n,k} \) is invariant under the action of the group \( B_n \).

**Proposition 2.1** ([5]). A basis for the graded vector space \( M_{n,k} \) is given by

\[
\{ e_n(\omega_{i_1}^{i_1-2i_2} \cdots \omega_{n-1}^{i_{n-1}-2i_n} \omega_{n}^{i_n}) \mid i_j > 2i_{j+1} \text{ for } 1 \leq j \leq n-1 \text{ and } i_n \geq k \}.
\]

**Theorem 2.2** (cf. [14]). Let \( \alpha : M_{n,k+1} \rightarrow M_{n,k} \) be the natural inclusion and let \( \beta : M_{n,k} \rightarrow \Sigma^k M_{n-1,2k+1} \) be the map given by

\[
\beta(\omega_{i_1, \ldots, i_n}) = \begin{cases} 0, & i_n > k, \\ \Sigma^k \omega_{i_1, \ldots, i_{n-1}}, & i_n = k. \end{cases}
\]

Then

\[
0 \rightarrow M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \rightarrow 0
\]

is a short exact sequence of \( \mathcal{A} \)-modules.

The exactness of the sequence can be proved by using the following:

**Lemma 2.3** ([6, Proposition 1.2]). We have

\[
\omega_{i_1, \ldots, i_n} = \omega_{i_1, \ldots, i_{n-1}} x_n^{i_n} + \text{terms } \omega_{j_1, \ldots, j_{n-1}} x_n^{j} \text{ with } j > j_n.
\]

Note also that a minimal generating set for the \( \mathcal{A} \)-module \( M_{n,k} \) was constructed in [5], generating the work of Inoue [8].
3 Existence of the cofibrations

A spectrum $X$ is said to be of finite type if its mod 2 cohomology, $H^*X$, is finite-dimensional in each degree. Recall that given a sequence $X \to Y \to Z$ of spectra of finite type, if the composite $X \to Z$ is homotopically trivial and the induced sequence $0 \to H^*Z \to H^*Y \to H^*X \to 0$ is a short exact sequence of $\mathcal{A}$-modules, then $X \to Y \to Z$ is a cofibration.

We wish to realise the algebraic short sequence

$$0 \to M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \to 0,$$

by a cofibration of spectra

$$\Sigma^k M(n-1)_{2k+1} \to M(n)_k \to M(n)_{k+1}.$$

The inclusion of $k\bar{\rho}_n$ into $(k+1)\bar{\rho}_n$ induces a natural map of spectra

$$i : M(n)_k \to M(n)_{k+1}.$$

It is clear that this map realises the inclusion of $\mathcal{A}$-modules $\alpha : M_{n,k+1} \to M_{n,k}$.

We wish now to realise the $\mathcal{A}$-linear map $\beta : M_{n,k} \to \Sigma^k M_{n-1,2k+1}$ by a map of spectra

$$j : \Sigma^k M(n-1)_{2k+1} \to M(n)_k$$

such that the composite $i \circ j$ is homotopically trivial. The existence of such a map is an immediate consequence of the following result.

**Theorem 3.1.** For all $k \geq 0$, we have

1. The natural map $[\Sigma^k M(n-1)_{2k+1}, M(n)_k] \to \text{Hom}_\mathcal{A}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$ is onto.

2. The group $[\Sigma^k M(n-1)_{2k+1}, M(n)_{k+1}]$ is trivial.

The theorem is proved by using the Adams spectral sequence

$$\text{Ext}_\mathcal{A}^s(H^*Y, \Sigma^t H^*X) \Rightarrow [\Sigma^t X, Y].$$

For the first part, it suffices to prove that

$$\text{Ext}_\mathcal{A}^s(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \quad \text{for} \quad s \geq 0 \text{ and } t < 0, \quad (1)$$

so that the non-trivial elements in $\text{Hom}_\mathcal{A}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$ are permanent cycles. For the second part, it suffices to prove that

$$\text{Ext}_\mathcal{A}^s(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \quad \text{for} \quad s \geq 0. \quad (2)$$

Here and below, $\mathcal{A}$-linear maps are of degree zero, and so $\text{Ext}_\mathcal{A}^s(M, \Sigma^t N)$ is the same as the group denoted by $\text{Ext}_\mathcal{A}^{s,t}(M, N)$ in the traditional notation.

The vanishing of the above extension groups will be proved in the next section.
4 On the vanishing of $\text{Ext}^s_{A}(M_{n,k}, \Sigma^{i+s}M_{m,j})$

In this section, we establish a sufficient condition for the vanishing of the extension groups $\text{Ext}^s_{A}(M_{n,k}, \Sigma^{i+s}M_{m,j})$. Note that we always consider the modules $M_{n,k}$ with $k \geq 0$.

Below we consider separately two cases for the vanishing of the groups $\text{Ext}^s_{A}(M_{n,k}, \Sigma^{i+s}M_{m,j})$: Proposition 4.1 gives a condition for the case $j = 0$ and Proposition 4.2 gives a condition for the case $j > 0$.

**Proposition 4.1.** Suppose $n > m \geq 0$ and $-\infty < i < |M_{n-m,k}|$. Then

$$\text{Ext}^s_{A}(M_{n,k}, \Sigma^{i+s}M_{m}) = 0, \quad s \geq 0.$$ 

Here $|M|$ denotes the connectivity of $M$, i.e. the minimal degree in which $M$ is non-trivial.

To consider the case $j > 0$, put $\varphi(j) = 2j - 1$ and

$$F(i, j, q) = i + j + \varphi(j) + \varphi^2(j) + \cdots + \varphi^{q-1}(j),$$

where $\varphi^t$ is the $t$-fold composition of $\varphi$. Explicitly,

$$F(i, j, q) = i + (j - 1)(2^q - 1) + q.$$ 

Note that $F(i + j, 2j - 1, q) = F(i, j, q + 1)$ and $F(i, j', q) \leq F(i, j, q)$ if $j' \leq j$.

**Proposition 4.2.** Suppose $n > m \geq 0$, $j > 0$ and $F(i, j, q) < |M_{n-m+q,k}|$ for $0 \leq q \leq m$. Then

$$\text{Ext}^s_{A}(M_{n,k}, \Sigma^{i+s}M_{m,j}) = 0, \quad s \geq 0.$$ 

Recall that Lannes’ T-functor is left adjoint to the tensoring with $H := H^*B\mathbb{Z}/2$ in the category $\mathcal{M}$ of unstable modules over the Steenrod algebra $\mathcal{M}$. We need the following result, observed by Harris and Shank [7], to prove Proposition 4.1.

**Proposition 4.3** (Carlisle-Kuhn [4, 6.1] combined with Harris-Shank [7, 4.19]). There is an isomorphism of unstable modules

$$T(L_n) \cong L_n \oplus (H \otimes L_{n-1}).$$

Here $L_n = M_{n,1}$.

**Corollary 4.4.** For $n \geq m$, we have $|T^m(M_{n,k})| = |M_{n-m,k}|$.

**Proof.** By iterating the action of $T$ on $L_n$, we see that there is an isomorphism of unstable modules

$$T^m(L_n) \cong \bigoplus_{i=0}^{m} [H^* \otimes L_{n-1}]^\otimes_{a_i},$$

where $a_i$ is the $i$-th coefficient of $T^m(L_n)$. This completes the proof.
where \(a_i\) are certain positive integers depending only on \(m\). By using the exactitude of \(T^m\) and the short exact sequences

\[
0 \to M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \to 0,
\]

it is easy to prove by induction that there is an isomorphism of graded vector spaces

\[
T^m(M_{n,k}) \cong \bigoplus_{i=0}^{m} [H^{\otimes i} \otimes M_{n-i,k}]^{|a_i|}.
\]

The corollary follows.

\[\square\]

**Proof of Proposition 4.1.** Fix \(i, s\) and take a positive integer \(q\) big enough such that \(i + s + q\) is positive. We have

\[
\operatorname{Ext}_A^s(M_{n,k}, \Sigma^{i+s} M_m) = \operatorname{Ext}_A^s(\Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m).
\]

Using the Grothendieck spectral sequence, we need to prove that

\[
\operatorname{Ext}_{\mathcal{U}}^{s-j}(\mathcal{D}j \Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m) = 0, \quad 0 \leq j \leq s.
\]

Here \(\mathcal{D}_j\) is the \(j\)th-derived functor of the destabilisation functor

\[
\mathcal{D} : \mathcal{A}\text{-mod} \to \mathcal{U}
\]

from the category of \(\mathcal{A}\)-modules to the category of unstable \(\mathcal{A}\)-modules [11].

As \(M_m\) is \(\mathcal{U}\)-injective, it is easily seen that \(\Sigma^q M_m\) has an \(\mathcal{U}\)-injective resolution \(I^*\) where \(I^t\) is a direct sum of \(M_m \otimes J(a)\) with \(a \leq \ell - t\), where \(J(a)\) is the Brown-Gitler module [10]. So we need to prove that, for \(a \leq (i+s+q)-(s-j) = i+j+q\), we have

\[
\operatorname{Hom}_{\mathcal{U}}(\mathcal{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a)) = 0.
\]

By Lannes-Zarati [11], we have

\[
\mathcal{D}_j \Sigma^q M_{n,k} = \Sigma R_j \Sigma^{j-1+q} M_{n,k} \subset \Sigma^{j+q} H^{\otimes j} \otimes M_{n,k},
\]

where \(R_j\) is the Singer functor. It follows that \(\operatorname{Hom}_{\mathcal{U}}(\mathcal{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a))\) is a quotient of

\[
\operatorname{Hom}_{\mathcal{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, M_m \otimes J(a))
\]

which is in turn a subgroup of

\[
\operatorname{Hom}_{\mathcal{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, H^{\otimes m} \otimes J(a)) = ((T^m(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}))^a)^*.
\]

This group is trivial because, by Corollary [11] we have

\[
|T^m(\Sigma^{j+q} H_i \otimes M_{n,k})| = |\Sigma^{j+q} M_{n-m,k}| = |M_{n-m,k}| + j + q > i + j + q \geq a.
\]

The proposition follows. \(\square\)
Proof of Proposition 4.2. We prove the proposition by induction on \( m \geq 0 \). By noting that \( M_{n,j} = \mathbb{Z}/2 \), the case \( m = 0 \) is a special case of Proposition 4.1.

Suppose \( m > 0 \). For simplicity, put \( E^s(\Sigma^i M_{m,j}) = \text{Ext}_{\mathcal{A}}^s(M_{n,k}, \Sigma^{i+s} M_{m,j}) \). The short exact sequence of \( \mathcal{A} \)-modules \( M_{m,j} \to M_{m,j-1} \to \Sigma^{-1} M_{m-1,2j-1} \) induces a long exact sequence in cohomology

\[
\cdots \to E^{s-1}(\Sigma^{i+j} M_{m-1,2j-1}) \to E^{s}(\Sigma^{i} M_{m,j}) \to E^{s}(\Sigma^{i} M_{m,j-1}) \to \cdots
\]

So from the cofiltration of \( M_{m,j} \)

\[
\begin{array}{c}
\Sigma^{i} M_{m,j-1} \\
\downarrow \\
\Sigma^{i+j-1} M_{m-1,2j-1} \\
\downarrow \\
\Sigma^{i+1} M_{m-1,3} \\
\downarrow \\
\Sigma^{i} M_{m-1,1}
\end{array}
\]

we see that, in order to prove \( E^s(\Sigma^i M_{m,j}) = 0 \), it suffices to prove that the groups \( E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1}) \), \( 1 \leq j' \leq j \), and \( E^s(\Sigma^i M_{m}) \), are trivial.

By Proposition 4.1, \( E^s(\Sigma^i M_{m}) \) is trivial since \( i = F(i,j,0) < |M_{n,k}| \). For \( 1 \leq j' \leq j \) and \( 0 \leq q \leq m - 1 \), we have

\[
F(i+j',2j'-1,q) = F(i,j'+q+1) < F(i,j,q+1) < |M_{n-m+1+q,k}|
\]

By inductive hypothesis for \( m - 1 \), we have \( E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1}) = 0 \). The proposition is proved.

We are now ready to prove Theorem 3.1. Recall that the connectivity of \( M_{n,k} \) is given by

\[
|M_{n,k}| = 1 + 3 + \cdots + (2^{n-1} - 1) + (2^n - 1)k.
\]

Proof of Theorem 3.1 (1). Using the Adams spectral sequence, it suffices to prove that

\[
\text{Ext}^s_{\mathcal{A}}(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \quad \text{for } s \geq 0 \text{ and } t - s < 0.
\]

For \( q \geq 0 \), we have

\[
F(k+t-s,2k+1,q) = k + t - s + 2k(2^q - 1) + q < (2^{q+1} - 1)k + q \leq |M_{q+1,k}|.
\]

The vanishing of the extension groups follows from Proposition 4.2.

Proof of Theorem 3.1 (2). Using the Adams spectral sequence, it suffices to prove that

\[
\text{Ext}^s_{\mathcal{A}}(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \quad \text{for } s \geq 0.
\]

For \( q \geq 0 \), we have

\[
F(k,2k+1,q) = k + 2k(2^q - 1) + t = (2^{q+1} - 1)k + q < |M_{q+1,k+1}|.
\]

The vanishing of the extension groups follows from Proposition 4.2.
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