Improved Performances in Subsonic Flows of an SPH Scheme with Gradients Estimated using an Integral Approach

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ABSTRACT

In this paper we present results from a series of hydrodynamical tests aimed at validating the performance of a smoothed particle hydrodynamics (SPH) formulation in which gradients are derived from an integral approach. We specifically investigate the code behavior with subsonic flows, where it is well known that zeroth-order inconsistencies present in standard SPH make it particularly problematic to correctly model the fluid dynamics.

In particular we consider the Gresho-Chan vortex problem, the growth of Kelvin-Helmholtz instabilities, the statistics of driven subsonic turbulence and the cold Keplerian disc problem. We compare simulation results for the different tests with those obtained, for the same initial conditions, using standard SPH. We also compare the results with the corresponding ones obtained previously with other numerical methods, such as codes based on a moving-mesh scheme or Godunov-type Lagrangian meshless methods.

We quantify code performances by introducing error norms and spectral properties of the particle distribution, in a way similar to what was done in other works. We find that the new SPH formulation exhibits strongly reduced gradient errors and outperforms standard SPH in all of the tests considered. In fact, in terms of accuracy we find good agreement between the simulation results of the new scheme and those produced using other recently proposed numerical schemes. These findings suggest that the proposed method can be successfully applied for many astrophysical problems in which the presence of subsonic flows previously limited the use of SPH, with the new scheme now being competitive in these regimes with other numerical methods.

Subject headings: methods: numerical – hydrodynamics

1. Introduction

Application of computational fluid dynamics to many astrophysical problems has grown steadily over the years with advances in computational power and it has now become a standard tool for studying the non-linear evolution of baryonic structures in the Universe.

There are two methods commonly used in numerical astrophysics for solving the Euler equation. The first method makes use of a spatial grid, either fixed [Stone & Norman, 1992; Norman & Bryan, 1999; Stone et al., 2008] or adaptive [Fryxell et al., 2000; Teyssier, 2002]. The second method is a Lagrangian mesh-free numerical scheme, known as smoothed particle hydrodynamics (SPH: Lucy, 1977; Gingold & Monaghan, 1977) in which particles are used to model fluid properties. Both methods have advantages and weaknesses which are specific to the numerical approach on which each method is based.

Because of its Lagrangian nature, SPH possesses very good conservation properties, moreover the method is free of advection errors present in mesh codes and is naturally adaptative because particle trajectories trace the mass. This latter feature is particularly useful in many astrophys-
ical problems involving collapse of the structure under study.

The method, however, is not free from significant drawbacks and more specifically it has been found that in several hydrodynamical test cases there are significant differences between the results obtained using the two methods, with SPH failing to properly model the correct behavior (O’Shea et al. 2005; Agertz et al. 2007; Wadsley et al. 2008; Tasker et al. 2008; Mitchell et al. 2009; Read et al. 2010; Valcke et al. 2010; Junk et al. 2010).

More specifically, Agertz et al. (2007) found that SPH fails to resolve the formation of Kelvin-Helmholtz (KH) instabilities at fluid interfaces. This is strongly related to the fluid mixing properties of SPH as well as to the lack of a core entropy in non-radiative simulations of galaxy clusters, in contrast with what is found using mesh-based codes (Wadsley et al. 2008; Mitchell et al. 2009).

It is now widely recognized that the origin of these errors is due to two distinct problems which are present in SPH. The first problem originates from the inconsistencies of standard SPH when dealing with steep density gradients at contact discontinuities, the so-called local mixing instability (Read et al. 2010), thereby suppressing the growth of KH instabilities at the fluid interfaces. The second problem is inherent in the discrete nature of SPH, in which a finite set of particles is used to model the fluid. The discretization implies the presence of a zeroth-order error in the momentum equation due to sampling effects (Read et al. 2010), the so-called $E_0$ error. This error can be reduced if one increases the number of neighbors $N_n$ present within the kernel, but for the standard cubic spline kernel there is a threshold value for $N_n$ beyond which a clumping instability develops thus degrading the convergence rate.

In view of the benefits of the SPH method previously outlined, there have been many attempts to eliminate or reduce these difficulties. Several solutions have been proposed, concerning the problem posed by the standard formulation at contact discontinuities.

One solution is to modify the equations, so that it is the pressure rather than the density which is smoothed (Ritchie & Thomas 2001; Read et al. 2010; Hopkins 2013; Saitoh & Makino 2013; Hu et al. 2014). On the other hand, Price (2008) proposed to include in the SPH energy equation a term of artificial conductivity (AC) with the aim of smoothing the thermal energy across fluid interfaces and thus removing the associated entropy gap. This approach is similar to that used by Wadsley et al. (2008), who mimicked the effect of subgrid turbulence by adding a heat diffusion term to the equations. The method, however, requires some care in the implementation of the conduction switches to avoid the risk of getting too much diffusion.

By performing a suite of hydrodynamical tests (Valdarnini 2012, hereafter V12) it has been found that the method yields consistent results when contrasted with those obtained using mesh-based codes. In particular (V12; Biffi & Valdarnini 2015), the level of core entropies produced in simulations of non-radiative galaxy clusters are now comparable with those of grid codes. Recently Beck et al. (2016), have proposed a modification to the standard SPH code Gadget-II (Springel 2005) which incorporate the new AC term into the hydrodynamic equations.

Finally, other variants of SPH are based on Riemann solvers (Godunov-SPH: Inutsuka 2002; Cha et al. 2010; Murante et al. 2011), Voronoi tessellation techniques (Heß & Springel 2010) or on the use of high order dissipation switches (Read & Hayfield 2012).

The zeroth-order inconsistency is due to the inability of the SPH method to properly reproduce a constant function because of finite resolution (Dilts 1999; Liu et al. 2003), thus leading to poor gradient estimates (Read et al. 2010; McNally et al. 2012) and in turn affecting the momentum equations. Keeping these errors under control becomes problematic when dealing with subsonic flows, as in the case of subsonic turbulence (Bauer & Springel 2012) or with Rayleigh-Taylor instabilities (Abell 2011; Garcia-Senz et al. 2012; V12).

A possible solution is to drastically increase the number of neighbors used in the simulation. In this case, the clumping instability can be avoided either by modifying the shape of the cubic spline kernel (Read et al. 2010), or by adopting (Dehnen & Aly 2012) the Wendland kernels (Wendland 1995). These kernels are character-
ized by the specific property of not being subject to clumping instabilities in the large $N_n$ limit.

Another possibility is to consider other discretizations of the momentum equation (Morris 1996; Abell 2011), but this comes at the cost of losing energy and momentum conservation (Price 2012), thus making the scheme of little use in practice.

Finally, to overcome the difficulties of SPH mentioned above, new numerical schemes have been proposed (Springel 2010; Duffell & MacFadyen 2011; Hopkins 2015; Schaal et al. 2015; Pakmor et al. 2016) which aim to retain the advantages of using both SPH and mesh-based codes. These new schemes are quite numerical complex and, in some cases, their space discretization does not seem to be optimal as required by forthcoming parallel computing systems consisting of several million cores (Schaal et al. 2015).

A satisfactory solution to the problem of zeroth-order inconsistency in SPH has been presented by García-Senz et al. (2012), who showed how the accuracy in gradient estimates can be greatly improved by calculating first order derivatives by means of the evaluation of integrals and the use of matrix inversions. The resulting tensor scheme has been tested in a variety of hydrodynamical test cases (García-Senz et al. 2012; Rosswog 2015), showing significant improvements as compared with the standard formulation. A crucial feature of the method is that it retains the Lagrangian nature of SPH, unlike previous attempts aimed at improving gradient accuracy.

Motivated by these findings we here further investigate the performance of the new scheme, paying particular attention to its behavior in the regime of subsonic flows, where it has been found that standard SPH presents its major difficulties.

The goal of this paper is to demonstrate that, for the hydrodynamical tests considered here, the new SPH formulation gives results with accuracy comparable to that of mesh-based codes. Thus, the new code can be profitably used for many astrophysical problems without the shortcomings of standard SPH. The main advantage of the new scheme is that it keeps its fully Lagrangian nature, while retaining a relative simplicity in its implementation as compared with new numerical schemes recently proposed.

The paper is organized as follows. In Section 2 we present the hydrodynamical method and the implementation of the integral-based approach. Some basic properties of the most widely used SPH kernels are briefly reviewed in Section 3. The results of the hydrodynamical tests are given in Section 4 where we consider the Gresho-Chan vortex problem, the development of KH instabilities, the statistic of driven subsonic turbulence and finally the Keplerian disc problem. Our main results and conclusions are summarized in Section 5.

2. Hydrodynamic method

This section reviews the basic features of SPH, for a comprehensive review see Rosswog (2003) and Price (2012).

2.1. Basic equations

In SPH, the fluid is described within the solution domain by a set of $N$ particles with mass $m_i$, velocity $\vec{v}_i$, density $\rho_i$, and specific entropy $A_i$ (we use the convention of having Latin indices denoting particles and Greek indices denoting the spatial dimensions). Here, we integrate the entropy per particle (Springel & Hernquist 2002) in place of the thermal energy per unit mass $u_i$ (Hernquist & Katz 1989; Wadsley et al. 2004). The entropy $A_i$ is related to the particle pressure $P_i$ by $P_i = A_i \rho_i^2 = (\gamma - 1) \rho_i u_i$, where $\gamma = 5/3$ for a mono-atomic gas. The density estimate at the particle position $\vec{r}_i$ is given by

$$\rho_i = \sum_j m_j W(|\vec{r}_{ij}|, h_i),$$

where $W(|\vec{r}_i - \vec{r}_j|, h_i)$ is the interpolating kernel which is zero for $|\vec{r}_i - \vec{r}_j| \geq \zeta h_i$ (Price 2012). Since the kernel has compact support, the sum in Equation (1) is over a finite number of particles. The smoothing length $h_i$ is implicitly defined by

$$h_i = \eta (m_i/\rho_i)^{1/D},$$

so that in two and three dimensions, respectively, $N^{2D}_{nn} = \pi (\eta)^2$ and $N^{3D}_{nn} = 4\pi (\eta)^3 / 3$ are the mean number of neighboring particles of particle $i$ within a radius $\zeta h_i$. In principle, for a given parameter $\eta$, the solution of Equation (2) allows for non-integer values of $N_n$ (Price 2012). Here we
solve the equation for the \( h_i \) by requiring an integer value for \( N_{nn} \), to which we will generically refer in the following as the neighbor number.

Following [Price (2012)] the Euler equations are derived using a Lagrangian formulation

\[
\frac{d\vec{v}_i}{dt} = -\sum_j m_j \left[ \frac{P_i}{\Omega_j \rho_j^2} \nabla_i W_{ij}(h_i) + \frac{P_j}{\Omega_j \rho_j^2} \nabla_j W_{ij}(h_j) \right],
\]

where \( \Omega_i \) is defined as

\[
\Omega_i = \left[ 1 - \frac{\partial h_i}{\partial \rho_i} \sum_k m_k \frac{\partial W_{ik}(h_i)}{\partial h_i} \right].
\]

2.2. Artificial viscosity

The momentum equation \( \text{(3)} \) must be generalized to include an artificial viscosity (AV) term which in SPH represents the effects of shocks. This is introduced in order to prevent particle streaming and convert kinetic energy into heat at shocks; the new term reads

\[
\left( \frac{d\vec{v}_i}{dt} \right)_{AV} = -\sum j m_j \Pi_{ij} \vec{\nabla}_i \vec{W}_{ij},
\]

where \( \vec{W}_{ij} = \frac{1}{2}(W(r_{ij}, h_i) + W(r_{ij}, h_j)) \) is the symmetrized kernel and \( \Pi_{ij} \) is the AV tensor.

The latter is written following the formulation of [Monaghan (1997)], based on an analogy with the Riemann problem:

\[
\Pi_{ij} = -\frac{\alpha_{ij}}{2} \vec{v}_{ij}^{AV} \mu_{ij} f_i,
\]

where \( \rho_{ij} \) is the average density, \( \mu_{ij} = \vec{v}_{ij} \cdot \vec{r}_{ij} / |r_{ij}| \) if \( \vec{r}_{ij} \cdot \vec{r}_{ij} < 0 \) but zero otherwise and \( \vec{v}_{ij} = \vec{v}_i - \vec{v}_j \).

The signal velocity \( \vec{v}_{ij}^{AV} \) is introduced as

\[
\vec{v}_{ij}^{AV} = c_i + c_j - 3 \mu_{ij},
\]

with \( c_i \) being the sound velocity. The amount of AV is regulated by the parameter \( \alpha_i \), and \( f_i \) is a viscosity limiter which is introduced so as to suppress AV when strong shear flows are present. This is written as [Balsara (1995)]

\[
f_i = \frac{|\nabla \cdot \vec{v}_i|}{|\nabla \cdot \vec{v}_i| + |\nabla \times \vec{v}_i|},
\]

where \( |\nabla \cdot \vec{v}_i| \) and \( |\nabla \times \vec{v}_i| \) are estimated according to the SPH formalism.

The early SPH formulation [Monaghan (2005)] assumed a constant viscosity parameter \( \alpha_i \) of order unity for all the particles, thus making the scheme excessively viscous away from shocks. In the literature, the SPH scheme with this viscosity parametrization is often referred to as standard SPH, whereas here we use this term to indicate the SPH formulation which uses the AV switch which now we will describe.

To reduce the amount of AV away from shocks [Morris & Monaghan (1997)] proposed letting the \( \alpha_i \)'s vary with time according to some source term \( S_i \). The time-evolution of \( \alpha_i \) is given by

\[
\frac{d\alpha_i}{dt} = -\frac{\alpha_i - \alpha_{\text{min}}}{\tau_i} + S_i,
\]

where \( \alpha_{\text{min}} \) is a floor value and

\[
\tau_i = \frac{h_i}{c_i l_d}
\]

is a decay time scale which is controlled by the dimensionless decay parameter \( l_d \). The source term \( S_i \) is constructed so that it increases whenever \( \nabla \cdot \vec{v}_i < 0 \) [Morris & Monaghan (1997)]; here we adopt a slightly modified form [Valdarnini (2011)] which reads

\[
S_i = f_i S_0 \max \left( (\nabla \cdot \vec{v}_i)^2, 0 \right) (\alpha_{\text{max}} - \alpha_i)
\]

\[
\equiv S_i (\alpha_{\text{max}} - \alpha_i).
\]

where \( \alpha_{\text{max}} \) sets an upper limit and \( S_0 \) is unity for \( \gamma = 5/3 \). In the following, unless otherwise specified, we adopt a time-dependent AV scheme with parameters \( \{\alpha_{\text{min}}, \alpha_{\text{max}}, l_d\} = \{0.1, 1.5, 0.2\} \). This set of parameters will be denoted as AV2 (See Table 1 of Valdarnini [2011], to which we refer for more details).

To suppress AV more efficiently away from shocks, the time dependent AV scheme has been further improved by [Cullen & Dehnen (2010)]. They introduced the time derivative of \( \nabla \cdot \vec{v}_i \) to detect in advance when a flow is converging, as well as higher order gradient estimators and a more sophisticated functional form for the viscosity limiter in shear flows. The [Cullen & Dehnen (2010)] scheme will be used in some test cases and we refer to the authors’ paper for a detailed description of its implementation in SPH.
2.3. The artificial conductivity scheme

In the entropy formulation of SPH, the rate of entropy generation is given by (Springel & Hernquist 2002)

\[
\frac{dA_i}{dt} = \frac{\gamma - 1}{\rho_i} \{ Q_{AV} + Q_{AC} \},
\]

where the terms in brackets denote different sources (in the hydrodynamic test cases presented here, radiative losses are not considered).

The term \( Q_{AV} \) refers to the numerical viscosity:

\[
Q_{AV} = \left( \frac{du_i}{dt} \right)_{AV} = \frac{1}{2} \sum_j m_j \Pi_{ij} \vec{v}_{ij} \cdot \nabla_i \vec{W}_{ij}.
\]

The AC term \( Q_{AC} \) for the dissipation of energy takes the form

\[
\left( \frac{du_i}{dt} \right)_{AC} = \sum_j m_j v_{ij}^{AC} \frac{\alpha_i^C (u_i - u_j)}{\rho_i} \vec{v}_{ij} \cdot \nabla_i \vec{W}_{ij},
\]

where \( v_{ij}^{AC} \) is the AC signal velocity, \( \vec{v}_{ij} \equiv \vec{r}_{ij}/r_{ij} \), and \( \alpha_i^C \) is the AC parameter which is of order unity. The above Equation represents the SPH analogue of a diffusion equation of the form (Price 2008)

\[
\left( \frac{du_i}{dt} \right)_{AC} \approx D_i^{AC} \nabla^2 u_i,
\]

where \( D_i^{AC} \) is a numerical heat-diffusion coefficient given by

\[
D_i^{AC} \approx \frac{1}{2} \alpha_i^C v_{ij}^{AC} r_{ij}.
\]

A crucial issue concerns reducing the AC in the absence of contact discontinuities. In analogy with the AV scheme, one can define an AC switch with a source term given by

\[
S_i^C = f_C h_i \frac{\nabla^2 u_i}{\sqrt{u_i + \varepsilon}} (\alpha_i^C - \alpha_{i}^{C_{\max}}),
\]

where the Laplacian of the thermal energy is calculated as done by Brookshaw (1985)

\[
\nabla^2 u_i = 2 \sum_j m_j \left( \frac{u_i - u_j}{\rho_j} \vec{v}_{ij} \cdot \nabla_i \vec{W}_{ij}/r_{ij} \right),
\]

and for the signal velocity we use (V12)

\[
v_{ij}^{AC} = |(\vec{v}_i - \vec{v}_j) \cdot \vec{r}_{ij}|/r_{ij}.
\]

For the other parameters we set \( f_C = 1, \alpha_{i}^{C_{\min}} = 0, \alpha_{i}^{C_{\max}} = 1.5, \) and \( \varepsilon = 10^{-4} u_i \).

The time evolution of the AC parameter \( \alpha_i^C \) is similar to that of the AV

\[
\frac{d\alpha_i^C}{dt} = -\frac{\alpha_i^C - \alpha_{i}^{C_{\min}}}{\tau_i^C} + S_i^C,
\]

where \( \tau_i^C = h_i/0.2 c_i \) sets the decaying time scale away from jumps in thermal energy.

The AC term has been introduced with the purpose of smoothing the thermal energy at contact discontinuities (Price 2008), and when using the signal velocity \( \vec{v}_{ij} \) can be interpreted as a sub-grid model mimicking the effects of diffusion due to turbulence (Wadsley et al. 2008). Finally, it must be stressed that for the hydrodynamic test problems considered here, with the exception of the KH tests, thermal energy gradients are null or very small. For these tests the impact of AC on simulation results can then be considered negligible.

2.4. The Integral Approximation scheme

In SPH, the errors associated with finite sampling cannot be simply eliminated by a more accurate interpolation scheme. A gradient estimator which is exact at linear order can be constructed by using matrix inversion (Price 2012), but this comes at the cost of losing the conservation properties of SPH.

To avoid these difficulties García-Senz et al. (2012) proposed a novel approach in which SPH first-order derivatives are estimated through the use of integrals. This makes the method much less noisy than in the standard formulation, with accuracy in estimated gradients being greatly improved (García-Senz et al. 2012, Rosswog 2015).

Moreover, a significant benefit of the method is that it retains the Lagrangian nature of SPH, thereby ensuring exact conservation of linear and angular momentum.

After the paper of García-Senz et al. (2012), the performance of the scheme was investigated in detail by Rosswog (2015); here we briefly describe the essential features of the method.
Let us define the integral

\[ I(\vec{r}) = \int_V \left[ f(\vec{r}') - f(\vec{r}) \right] (\vec{r}' - \vec{r}) W(|\vec{r}' - \vec{r}|, h) d^3r', \]  
(20)

where \( W \) is a spherically symmetric and normalized kernel. By Taylor expanding \( f(\vec{r}') \) to first order around \( \vec{r} \)

\[ I_\alpha \simeq \nabla_\beta f \int \Delta_\alpha \Delta_\beta W d^3r', \]  
(21)

where we have introduced the notation \( \Delta_\alpha = (\vec{r}' - \vec{r})_\alpha \). The gradient of the function \( f \) is then given by

\[ \nabla_\alpha f = [\tau]_{\alpha\beta} I_\beta , \]  
(22)

where \( \mathcal{T} = \{ \tau \}_{\alpha\beta} \) and

\[ \tau_{\alpha\beta} = \tau_{\beta\alpha} = \int \Delta_\alpha \Delta_\beta W d^3r'. \]  
(23)

In SPH integrals are replaced by summations over particles, so that for the matrix \( \mathcal{T} \) of particle \( i \) one has

\[ \tau_{\alpha\beta}(i) = \sum_k \frac{m_k}{\rho_k} \Delta_\alpha \Delta_\beta W(r_{ik}, h_i) . \]  
(24)

In evaluating the discrete equivalent of the integral (20), a key step is to assume that the condition

\[ \sum_k \frac{m_k}{\rho_k} (\vec{r}_k - \vec{r}_i) W_{ik} \simeq 0 \]  
(25)

is fulfilled with a certain degree of accuracy. In such a case the integral (20) then becomes

\[ I_\beta(i) = \sum_k \frac{m_k}{\rho_k} f_{ik} \Delta_\beta W(r_{ik}, h_i) . \]  
(26)

Because of the approximation (25), for linear functions gradient estimates are no longer exact. However, it can easily be seen (García-Senz et al. 2012) that the gradient approximation (22), obtained using Equations (24) and (26), is now antisymmetric in the pair \( ij \). Thus, the condition (25) in the new scheme is crucial for ensuring exact conservation properties.

How well the approximation (25) is valid depends on the particle distribution within the kernel radius. The validity of the new scheme has been carefully tested (García-Senz et al. 2012; Rosswog 2015) for several hydrodynamical problems, demonstrating significant improvements in the accuracy of the results with respect to standard SPH.

To summarize, the integral approximation implies the replacement of \( \left[ \nabla_\beta W_{ik} \right]_\alpha \) in the SPH equations according to the following prescriptions:

\[ [\nabla_\beta W_{ik}(h_i)]_\alpha \to \sum_\beta C_{\alpha\beta}(i) \Delta_\beta W(r_{ik}, h_i) , \]  
(27)

and

\[ [\nabla_\beta W_{ik}(h_k)]_\alpha \to \sum_\beta C_{\alpha\beta}(k) \Delta_\beta W(r_{ik}, h_k) . \]  
(28)

where \( C = \mathcal{T}^{-1} \). In the following, the SPH formulation in which gradients are estimated according to the numerical scheme described here, will be referred to as the integral approximation (IA).

3. KERNELS

In this section we briefly review some properties of the most commonly used kernels in SPH. All of these kernels are characterized by the property of having compact support and of being continuous up to some degree.

A class of kernels which has often been considered in SPH is that of the B-spline functions (Price 2012), which are generated via the 1D Fourier transform:

\[ M_n(x, h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{\sin(kh/2)}{kh/2} \right]^n \cos(kx) dk . \]  
(29)

The degree of smoothness increases with \( n \) and the kernel approaches the Gaussian in the limit \( n \to \infty \). The function \( M_n \) is a polynomial of degree \( n - 1 \) and its derivative is continuously differentiable \( n - 2 \) times. By requiring in SPH the continuity of the first and second derivative, the first useful kernel is then \( M_4 \) (cubic spline):
Fig. 1.— Average SPH density calculated for a glass-like configuration of 128$^3$ particles in a periodic box of unit length. The theoretical expected value is $\rho = 1$ and the quantity plotted is $(\rho - 1)/10^{-3}$. Different symbols refer to different kernels; for the sake of clarity error bars are not shown. The symbol $W_4 c$ refers the Wendland kernel $W_4$, but with the self-correction term of Dehnen & Aly (2012, Eq. 18) included.

The number of neighboring particles for which the instability develops depends on the kernel degree and in 3D lies in the range between $N_n \simeq 50$ for $M_4$ up to $N_n \simeq 200$ when the $M_6$ kernel is used (Dehnen & Aly 2012).

It has been suggested that particle clumping can be avoided by modifying the kernel shape in order to have a non-zero gradient at the origin. Examples of this family of kernels are the core-triangle (CRT; Read et al. 2010) and the linear quartic (LIQ; Valcke et al. 2010). However, such adjustments reflect negatively on the density estimation ability of the kernels. By introducing a non-zero central derivative, the kernel profile becomes steeper and this in turn implies an overestimate of density when compared with the corresponding B-spline (V12, Rosswog 2015).

These attempts to fix the pairing instability by introducing ad hoc modifications in the kernel shape have recently been superseded by a new class of kernels. It has been shown (Dehnen & Aly 2012) that a necessary condition for avoiding pairing instability is that of having kernels with non-negative Fourier transforms. A class of kernels with satisfies this property and has compact support are the Wendland (1995) functions. An example in 3D of these functions is the Wendland $C^4$:

$$w(q) = \frac{495}{32\pi} \left( 1 - q \right)^6 \left( 1 + 6q + \frac{35}{3} q^2 \right), \quad (31)$$

where $w(q) = 0$ if $q > 1$. Hereafter we will refer to this kernel as $W_4$. Other classes of Wendland kernels are $C^2$ ($W_2$) and $C^6$ ($W_6$). We refer to Table 1 of Dehnen & Aly (2012) for the functional forms and normalization of these kernels. Finally, García-Senz et al. (2014) proposed using the sinc functions as another class of kernels which can be used to avoid pairing instability.

The accuracy of density estimation in SPH for different kernel families has been assessed by many authors (Dehnen & Aly 2012; V12; Rosswog 2015; Zhu et al. 2015). Here we measure the mean SPH density of $N = 128^3$ particles using a glass-like particle distribution inside a cube of sidelength unity and total mass one. Figure 1 shows the mean SPH density of the particles as a function of the neighbor number $N_n$ for different B-splines and Wendland kernels. The value of $N_n$ ranges in

$$w(q) = \frac{\sigma}{h^D} \left\{ \begin{array}{ll} \frac{1}{4}(2-q)^{3} - (1-q)^3 & 0 \leq q < 1, \\
\frac{1}{4}(2-q)^{3} & 1 \leq q < 2, \\
0 & q \geq 2, 
\end{array} \right. \quad (30)$$

where the kernel is non zero for $0 \leq q \leq \zeta = 2$ and $\zeta$ is the truncation radius in units of $h$. The normalization constant takes the values $\sigma = 10/7\pi, 1/\pi$ for $D = 2$ and $D = 3$, respectively.

The B-splines next in order which have been considered are $M_5$ ($\zeta = 2.5$) and $M_6$ ($\zeta = 3$); for a more detailed description of these kernels we refer the reader to Price (2012).

The stability properties of the $M_n$ kernel family have been investigated by a number of authors (Morris 1996; Barve et al. 2004; Read et al. 2010; Dehnen & Aly 2012, V12). A crucial result which emerges from these analyses is that all of the B-splines suffer from pairing instability.
powers of two between \( N_n = 32 \) and \( N_n = 512 \), with three distinct values of \( N_n \) being considered for each kernel according to its order (see Figure 1). To avoid overcrowding in the plot, standard deviation \( \sigma \)'s are not shown, but the general tendency is of \( \sigma \) decreasing as \( N_n \) increases with \( \sigma \approx 10^{-3} \) for the largest value of \( N_n \).

A number of conclusions can be drawn by examining, for different kernels, the accuracy behavior depicted in Figure 1. The best performances are given by \( M_5 \) and \( M_6 \), which for any given number of neighbors \( N_n \) outperform all of the other kernels. The cubic spline (\( M_4 \)) and the Wendland \( C2 (W2) \) exhibit the worst performances, regardless of the value of \( N_n \).

The Wendland kernels yield acceptable density estimates only when large values of \( N_n \) are used \((N_n \gtrsim 250)\), with \( M_5 \) and \( M_6 \) having better performances at any given \( N_n \) and stable estimates already for \( N_n = 128 \). The differences between the two families reduce progressively as \( N_n \) is increased, with the results becoming comparable only when \( N_n = 512 \).

This behavior reflects the difference in shape between the two families, with the Wendland kernels being more centrally peaked than the B-splines and systematically overestimating the density. This in turn is a direct consequence of the way in which the Wendland kernels have been constructed in order to avoid pairing instability and of their spectral properties.

It must be stressed that in making comparisons between the error behavior of kernels of different families, it is only meaningful to compare kernels with the same polynomial order \([\text{Aguilar et al.} \ 2011]\). In 3D, the Wendland equivalent of the \( M_6 \) kernel is therefore \( W2 \).

This means that in relative terms, the performances of the Wendland kernels are not very good unless one is willing to use a very large number of neighbors in the SPH simulations made with them. In this respect, it is now common practice to use the Wendland kernels \( W4 \) (or even \( W6 \)) setting \( N_n \approx 200 \). However, even small errors in the densities can have a significant impact on estimates of other hydrodynamic variables (see the results for the Sod shock tube in V12). Thus, in the case of using Wendland kernels, a conservative lower limit on the neighbor number to be used in 3D SPH runs should be \( N_n \approx 400 \).

To improve the performances of Wendland kernels \([\text{Dehnen \& Aly} \ 2012]\), proposed to subtract from the SPH density estimate \( c_i \) a fraction of the particle \( i \) self-contribution. The correction term depends on the kernel order and neighbor number, with an impact which decreases as one of the two increases. For the Wendland kernel \( W4 \) we show in Figure 1 density estimated using the self-correction term (\( W4c \)), which now brings the relative density error down to \( \sim 10^{-4} \).

However, for the hydrodynamic tests presented here the numerical set-ups consist of particle positions arranged in lattice or glass-like configurations with densities of order unity. This suggests that one can use the results of Figure 1 to assess density errors, which are already very small \((\sim 10^{-3})\) without the use of such a correction term. Therefore for the considered runs we expect a negligible impact of the self-correction term on SPH densities, and in what follows it will not be considered.

These results hold for a glass-like configuration, but for a realistic SPH distribution of particles it is difficult to assess the error behavior. \([\text{Zhu et al.} \ 2013]\) put the expected convergence rate that found for a random distribution set \((\sigma \propto N_n^{-0.5})\) and the one measured for a highly ordered distribution \((\sigma \propto N_n^{-1})\), such as a glass-like configuration. These findings strengthen the previous conclusions, suggesting that when using Wendland kernels in SPH, the number of neighbors should be kept as high as possible.

4. Hydrodynamic tests

In the following, we analyze results from some test problems aimed at assessing code performance of the new IA scheme. As already outlined in the Introduction, the problems considered here have been chosen with the specific aim of investigating code behavior when subsonic flows are present in the hydrodynamic tests.

This is motivated by the serious shortcomings which affect standard SPH in these regimes. We first discuss the Gresho-Chan test, which in this respect presents severe challenges to the SPH scheme, and then the others.
Fig. 2.— Azimuthal velocity profile of the Gresho-Chan vortex test for $M = 0.34$ at $t = 1$, with 1D resolution $N = 128$. Each panel is for a different kernel and, in each of them, lines of different color are for different neighbor number. For a given kernel in the standard run, the lowest neighbor number is used.

4.1. The Gresho-Chan vortex problem

The Gresho-Chan (Gresho & Chan 1990) vortex consists of a fluid of uniform density in differential rotation, with centrifugal forces balancing pressure gradients. The system is stationary and any change in the azimuthal velocity profile that arises during the integration is then due to numerical artifacts.

Because of sampling effects, errors in force accuracy lead to noise in the velocity field, thus generating numerical viscosity and hence particle disorder due to spurious transport of angular momentum. It is then particularly problematic for standard SPH to successfully model this problem leaving unaltered the velocity profile during the simulation, and in the literature (Springel 2010; Dehnen & Aly 2012; Kawata et al. 2014; Hu et al. 2014; Hopkins 2015; Rosswog 2015; Zhu et al. 2015) it has been widely used to validate code performances.

We next describe our initial conditions set-up. We take a gas with uniform density $\rho = 1$ within the periodic domain $0 \leq x, y < 1$, with zero radial velocity, azimuthal velocity profile

$$v_\phi(r) = \begin{cases} 
5r & (0 \leq r \leq 0.2) \\
2 - 5r & (0.2 < r \leq 0.4) \\
0 & (r > 0.4) 
\end{cases}$$

(32)

and pressure profile

$$P(r) = P_0 + \begin{cases} 
12.5r^2 & (0 \leq r \leq 0.2) \\
12.5r^2 - 20r + 4 + 4\ln(5r) & (0.2 < r \leq 0.4) \\
2(2\ln 2 - 1) & (r > 0.4)
\end{cases}$$

(33)

where $r = \sqrt{x^2 + y^2}, P_0 = (\gamma M^2)^{-1}, \gamma = 5/3$ and $M$ is the Mach number. Here we have adopted the generalized expression for the background pressure of Miczek et al. (2015), so that we can consider subsonic shear flows with low Mach numbers (Hu et al. 2014). The standard Gresho case is recovered for $M = \sqrt{3/25} \approx 0.34$, giving $P_0 = 5$.

Particle positions are initialized using an $N \times N \times 16$ lattice of particles (Zhu et al. 2015), $N$ being the effective 1D resolution. For the box thickness, we set $L_z = 16/N$ and we always consider $N > 32$. Here we use a hexagonal-close-packed (HCP) configuration for the particle coordinates. The particle velocities and pressure are set according to Equations (32) and (33). All of the simulations are run up to a final time $t_f \approx 3M$, using a fixed timestep $\Delta t = t_f/(800 \cdot 64)$, so as to ensure the same integration accuracy in runs with different Mach numbers.

We quantify the convergence rate for different runs by using the $L1$ error norm for the velocity

$$L1(v_\phi) = \frac{1}{N_b} \sum_i^{N_b} |\Phi_\phi(i) - v_\phi(r_i)|,$$

(34)
Fig. 3.— Convergence rate of the $L_1$ velocity error for the Gresho-Chan vortex test. The $L_1$ error is shown versus the 1D particle number for $M = 0.34$ at $t = 1$. Dashed lines are for the standard formulation and solid lines refer to the IA scheme. The symbols indicate different combinations of kernel and neighbor number.

where the summations is over $N_b$ bins, we set a binsize of $\Delta = 0.01$ in the range $0 \leq R \leq 0.5$ ([Hu et al. 2014]), $\tau_\phi(i)$ is the average azimuthal velocity of the particles which lie in the $i$th bin interval and $v_\phi(r_i)$ is the analytic solution at the bin radial coordinate.

Figure 2 shows the azimuthal velocity profiles for $M = 0.34$ at $t = 1$; the one dimensional resolution is $N = 128$ and each panel refers to a different kernel. Within each panel, lines with different color codes are for different neighbor numbers, the standard run always refers to the lowest neighbor number indicated in the panel. In this case, for the B-splines, values of $N_n$ are considered which are below the pairing instability threshold.

It is clear from all of the histograms that the IA scheme outperforms the standard one for all of the kernels, with the latter scheme being much more noisy. There is a tendency for the standard scheme to improve as higher order kernels are considered, but the error in the velocity profile is always significant. These results are in agreement with previous findings ([Dehnen & Aly 2012; Read & Hayfield 2012; Hu et al. 2014; Rosswog 2015]) and clearly demonstrate how, for the vortex test, inaccuracies in gradient estimates, i.e. the $E_0$ error, are the leading error sources. These errors are significantly reduced when using the IA scheme, showing how good that method is.

To quantify the performance of the IA approach, for the same test case we show in Figure 3 at $t = 1$ the velocity error $L_1$ as a function of the 1D resolution $N$. This ranges between $N = 50$ up to a maximum value of $N = 500$. For any given value of $N$ we considered different combinations of kernel and neighbor number $N_n$; these are reported in the Figure. For the same combination of resolution, kernel shape and neighbor number, we performed a simulation according to the IA formulation and a corresponding one using standard SPH.

In the case of the B-splines we employed the highest neighbor number that it is possible to use without having the pairing instability. For the Gresho-Chan vortex test the convergence rate has already been estimated for a variety of different SPH implementations, so that Figure 3 can be compared with the corresponding rates already obtained by various authors ([Dehnen & Aly 2012; Read & Hayfield 2012; Hu et al. 2014; Rosswog 2015; Zhu et al. 2015]).

From Figure 3 it can be seen that there is a resolution dependency of the $L_1$ error on $N$. In the standard case the convergence behavior is in line with those found previously, see for example Figure 10 of [Dehnen & Aly 2012] or Figure 5 of [Zhu et al. 2015]. The dependency of $M5$ on resolution parallels that of the W2 kernel and the same holds for the kernels M6/W4, this is very similar to what is seen in Figure 10 of [Dehnen & Aly 2012] (we use the same neighbor number). However for the $L1$ norm we obtained smaller errors here. For example in the M6/W4 case we found that for $N = 400$ $L1 \approx 7 \cdot 10^{-3}$, whereas for the same test run their Figure 10 shows $L1 \approx 0.015$. The same is true for the W6 runs, for which here $L1(N = 400) \approx 3 \cdot 10^{-3}$ is about a factor three smaller than in their corresponding run.

When passing from the standard scheme to the IA scheme, there is a significant reduction in the
amplitude of the $L_1$ norms. The decrease in $L_1$ is by a factor of between $\sim 5$ and $\sim 10$, with some dependency on resolution and adopted kernels. For the $M5$ ($N_n = 60$) kernel the ratio between the norms ranges from $L_1(IA)/L_1(std) \simeq 1/8$ at $N = 50$ down to $\sim 1/30$ when $N_n = 500$.

We now analyze the convergence rate of $L_1$ in the IA formulation. This depends on the adopted kernel, and for $M5$ we found $L_1 \propto N^{-1.2}$. This is close to what was given by Springel (2010) when using the moving mesh code Arepo ($L_1 \propto N^{-1.3}$).

To estimate the rate for the other kernels we adopt a conservative view and only include in the fit those points with $N \geq 80$. We then obtain $L_1 \propto N^{-1}$, which is better than that reported by Hu et al. (2014) $L_1 \propto N^{-0.7}$ for their pressure-entropy SPH formulation. The rate is also in agreement with that shown by Rosswog (2013) (Figure 10, case F3: $L_1 \propto N^{-1}$), who implemented an IA scheme using a W6 kernel. Note, however, that the value of $L_1$ at $N = 300$ found there is a factor $\sim 3$ higher than that found here ($L_1 \sim 10^{-3}$).

Higher convergence rates have been obtained by Read & Hayfield (2012) $L_1 \propto N^{-1.4}$, who employed a modified version of SPH with high-order dissipation switches, and by Zhu et al. (2013). The latter investigated the convergence behaviour of standard SPH showing that consistency in numerical convergence is achieved when the conditions $N \rightarrow \infty$ $h \rightarrow 0$ $N_n \rightarrow \infty$ are satisfied. In their varying $N_n$ case ($N_n = 120 \cdot (N/32)^{1.2}$), for the Gresho-Chan vortex test the authors report $L_1 \propto N^{-1.2}$, a much faster rate than that obtained by keeping $N_n$ constant. The value of $L_1$ in the $N = 500$ case is of the same order ($L_1 \sim 3 \cdot 10^{-3}$) as that obtained here for the W6 standard run with the same resolution.

Finally, with the exception of the lowest order kernels ($M5$ and $W2$), a comparison of the $L_1$ norms with those produced using the moving-mesh code Arepo shows that the IA formulation gives results which are comparable or better than those obtained with the mesh code (Springel 2010, Figure 29), with $L_1$ ranging here from $L_1 \simeq 8 \cdot 10^{-3}$ ($N = 80$) down to $L_1 \simeq 10^{-3}$ ($N = 500$).

To further investigate the performance of the IA scheme, we ran a suite of vortex tests with progressively lower Mach numbers. These tests are particularly challenging since, at constant velocity, the lower is the Mach number, the higher is the sound speed. This in turn implies an increase in the viscous force. Errors in the momentum equation then become progressively more important.

To aid comparison with the previous works, as in Hu et al. (2014) we considered the following Mach numbers $M = 0.02, 0.05, 0.1$ and $M = 0.34$. We ran the simulations using $N = 128$ as 1D resolution. This is a factor of two lower than that used by Hu et al. (2014) in their tests, however the results are not significantly affected by this choice. For the $M6$ kernel ($N_n = 180$) we show the azimuthal velocity profiles of the four test cases in Figure 4 so that the Figure can be compared with the corresponding histograms of Figure 2 of Hu et al. 2014.

The profiles of the standard runs largely reproduce those of Hu et al. (2014); however, a strik-
Fig. 5.— Plots at $t = 3$ of the azimuthal particle velocities for two Gresho-Chan vortex tests, both performed using a one-dimensional resolution of $N = 128$ and the $M_6$ kernel with $N_n = 180$ neighbors. Bottom panel is for $M = 0.34$ and top panel for $M = 0.06$ with $P_0 = 50$. The two plots can be compared with the corresponding ones in Figure 4 & 5 of Hopkins (2015). For the sake of clarity we show velocities of a subset of randomly selected particles. Open triangles are for the time-dependent AV scheme with settings $AV_2$ (see Section 2.2), open squares refer to the AV switch of Cullen & Dehnen (2010).

Moreover, these findings are in agreement with previous results (Read & Hayfield 2012, Hu et al. 2014) and demonstrate that in SPH simulations of the Gresho-Chan test, errors in force accuracy dominate over viscous effects.

For the considered tests we have shown until now the mean binned velocities. In order to assess the amount of noise present in the various runs, it is useful to plot directly the azimuthal particle velocities. To this end we ran two tests with 1D resolution $N = 128$, Mach numbers $M = 0.34$ and $M = 0.06$, respectively. In the latter case we set a background pressure of $P_0 = 50$. For each run performed using the time dependent AV scheme with settings $AV_2$ (see Section 2.2), we also consider a parent simulation in which has been implemented the AV switch of Cullen & Dehnen (2010).

The results are shown in Figure 5 at $t = 3$, where for the two test cases we plot the azimuthal velocities for a subset of all particles. The velocity distributions can be compared directly with those of the corresponding runs in Figure 4 & 5 of Hopkins (2015). An important feature which emerges from the plots of Figure 5 is that both AV methods show velocity distributions which are evenly scattered around the analytic solution, with the AV switch of Cullen & Dehnen (2010) exhibiting a much smaller amount of noise. It is worth noting how the IA method, even for very low Mach numbers, can accurately follow the analytic solution also at the peak vortex velocity. This behaviour is much better than that seen for the same test in the top panel of Figure 5 of Hopkins (2015).

Finally, it must be pointed out that in these simulations the amount of thermal diffusion due to the AC term is negligible and we do not include such a term in the SPH equations. This occurs because of the high sound speeds in the low Mach number regime, so that the time evolution of the $\alpha_C$ parameter is driven by the decaying rate $1/\tau_{ij}^C$, which dominates over the source term $S_{ij}^C$. To better quantify this issue we use Equation (14) to estimate at time $t$ the change in thermal energy due to the AC term: $\Delta u_{AC} \approx t(D_{ij}^{AC} \nabla^2 u_{ij})$.

The Laplacian of $u$ has a maximum at $r = 0.2$, where the azimuthal velocity reaches its peak value. The diffusion coefficient is then given by $D_{ij}^{AC} \approx \alpha_{ij}^C 5r_{ij}\tau_{ij}/2 \approx 5\alpha_{ij}^C h_i^2/2$, so that
Fig. 6.— Density maps at $t = \tau_{KH}$ for the 2D KH instability tests described in Section 4.2. The tests have been performed with a density contrast of $\chi = 2$ between the two contact layers. From left to right the different panels are for different Mach numbers: $M = 0.05$, $M = 0.1$ and $M = 0.35$. Each test case was run separately with both the standard (bottom) and the IA (top) scheme. All of the maps have been extracted from simulations performed using the M5 kernel.

\[ \Delta u_{AC} \simeq \frac{5}{2} \alpha_i^C h_i^2 \nabla^2 u_i t = \frac{5}{2} (S_i^C \tau_i^C) h_i^2 \nabla^2 u_i t , \]

where we have approximated $\alpha_i^C$ with the equilibrium solution $\alpha_i^C(eq) \simeq S_i^C \tau_i^C \simeq h_i^2 \nabla^2 u_i/(0.2 c_i \sqrt{u_i})$.

For $M = 0.34$ at $r = 0.2$ $u \sim 15/2$, $\nabla^2 u \sim 300/4$, $c_i \sim 3$ and $h_i$ can be easily computed because $\rho = 1$ so that

\[ \Delta u_{AC} \simeq 8.6 \cdot 10^3 h_i^4 t \]

For $N \gtrsim 50$ and $t \lesssim 3$ this term is always much smaller than $u$, regardless of the chosen kernel.

4.2. The Kelvin-Helmholtz instability

The KH instability has been investigated by many authors since it is a classic test in which SPH fails to properly model the development of the instability (Price 2008; Read et al. 2010; Valcke et al. 2010; Junk et al. 2010; Heß & Springel 2010; Cha et al. 2010; Murante et al. 2011; Valdarnini 2012; McNally et al. 2012; Kawata et al. 2014; Hu et al. 2014; Hopkins 2015).

The test consists of two fluid layers of different densities sliding past each other with opposite shearing velocities, and a small velocity perturbation is imposed in the direction perpendicular to the contact surface. A fluid instability develops, which is initially small and then becomes progres-
sively larger until non-linearity is reached with the appearance of KH rolls. For a sinusoidal perturbation of wavelength $\lambda$, a linear time scale can be defined as

$$\tau_{KH} = \frac{\lambda (\rho_1 + \rho_2)}{(\rho_1 \rho_2)^{1/2} v},$$

(37)

where $\rho_1$ and $\rho_2$ are the two fluid densities with a density ratio $\chi = \rho_1/\rho_2$ and $v = v_1 - v_2$ is the relative shear velocity.

To perform the test, the following conditions

$$\rho, T, v_z = \left\{ \begin{array}{ll}
\rho_1, T_1, v_1 & |y - 0.5| \leq 0.25 \\
\rho_2, T_2, v_2 & |y - 0.5| > 0.25
\end{array} \right.$$

(38)

are applied for a fluid with adiabatic index $\gamma = 5/3$ in a two-dimensional periodic domain with cartesian coordinates $x \in \{0, 1\}$, $y \in \{0, 1\}$. We set here $\rho_2 = 1$, and $\chi = 2$ for the density contrast.

The two layers are in pressure equilibrium with $P_1 = P_2 = 5/2$, so that the sound velocities in the two layers are $c_2 = \sqrt{\gamma P_2/\rho_2} = 2.04$ and $c_1 = c_2/\sqrt{\lambda} = 1.44$, respectively. The Mach number of the high-density layer is $M \approx v_1/c_1 \approx 0.7v_1$ and the KH time scale is $\tau_{KH} \approx 0.177/v_1$. We ran KH simulations with three different Mach numbers: $M = 0.05$, 0.1 and $M = 0.35$. For the latter value the initial condition set-up was similar to that of Hopkins (2013, Section 4.4.1).

The KH instability is triggered by adding in the proximity of the layer boundaries a small single-mode velocity perturbation along the $y-$direction

$$v_y = \delta v_y^{(0)} \sin(2\pi x/\lambda),$$

(39)

where $\delta v_y^{(0)} = 2 \cdot 10^{-2} v_1$, $\lambda = 1/6$ and $v_y = 0$ if $|y - \sigma| > 0.025$, where $\sigma$ takes the values 0.25 and 0.75, respectively. Note that for the amplitude of the initial velocity perturbation $\delta v_y^{(0)}$ we set here, unlike in previous runs (V12), a relative constant amplitude with respect to the streaming velocity. This was done in order to consistently compare, between runs with different Mach numbers, the impact of zeroth-order errors on $v_y \propto M$.

We performed the initial condition set-up by arranging $N^2 = 512^2$ equal mass particles inside the simulation box, setting the particle coordinates according to an HCP configuration. The lattice spacing was smoothly adjusted at the fluid interfaces so as to avoid density discontinuities; the details of the whole procedure are given in V12. For each test case the IA simulations were then performed using the $M$, $W_2$ and the $W_4$ kernels with neighbor number $N_n = 50$, $N_n = 72$ and $N_n = 162$, respectively. For the $M$, runs we also considered standard simulations. Finally, all of the simulations were performed with the AC term of Section 2.3 switched on.

For the specified range of Mach numbers, we first show in Figure 6 density plots of the KH simulations at $t = \tau_{KH}$. We show maps extracted from the $M$ runs, and contrast the IA scheme against the standard SPH scheme. For $M = 0.35$, both of the methods are able to produce KH rolls. At lower Mach numbers ($M = 0.1$) the standard method completely fails the KH test, whereas the IA scheme shows a degraded capability to resolve KH rolls. At $M = 0.05$ the rolls are absent and the differences between the two schemes are no longer present.

The relative performances of the two methods can be quantitatively assessed by measuring the $E_0$ error (Read et al. 2010; Valcke et al. 2010, V12) for the various runs. Here we first show the growth rate of the KH instability Junk et al. 2010, Heß & Springel 2010, V12), which allows one to recognize in a more visual way the differences between the KH results produced by the two schemes. The growth rate is measured by Fourier transforming, at different times, the $\lambda = 1/6$ growing mode of the $v_y$ velocity perturbation Junk et al. 2011 (V12).

The growth rates are shown in Figure 7, and their relative differences confirm the visual impressions derived from the maps of Figure 6. For $M = 0.35$ there are no significant differences between the two methods and both are able to follow the growth of the KH instability (Figure 6 right panel), the rates of the different runs being in accord with the analytic expectation. The results can also be compared with the corresponding rates in Figure 18 of Hopkins (2013), taking care about the different time scales due to the different number of modes used to seed the perturbation. Note that, unlike in Hopkins (2013), the standard version here correctly follows the development of the KH instability. We interpret this difference as being due to the small neighbor number ($\sim 32$ in 3D) adopted in his standard (PSPH) run.

As lower Mach numbers are considered, from
Fig. 7.— Growth rate of the $\lambda = 1/6$ velocity amplitude as measured by making a Fourier transform of $v_y$. The KH tests are for a density contrast $\chi = 2$ and three different values of Mach number have been considered ($M = 0.05, 0.1, 0.35$). For each KH test case we ran IA simulations using the $M_5$, $W_2$ and the $W_4$ kernels. Additionally, for the $M_5$ runs, we also performed a corresponding standard simulation. The solid line is the linear theory growth rate expectation $\propto e^{t/\tau_{KH}}$, normalized to the numerical amplitude at $t = 0$.

The panels of Figure 7 one can see a growing difficulty of the IA scheme in following the KH instability, regardless of the kernel employed in the simulation. This happens because by reducing the Mach number the shear velocity is also reduced and in turn, owing to the chosen settings, the initial velocity amplitude is reduced as well. At a fixed resolution the impact of gradient errors, and the subsequent particle disorder, on the growth of the KH instability is then higher as the Mach number decreases.

The $E_0$ error of particle $i$ is defined by Read et al. (2010)

$$E_0^i = \sum_j m_j \left( \frac{\rho_i}{\rho_j} + \frac{\rho_j}{\rho_i} \right) h_i \nabla_i \tilde{W}_{ij}, \quad (40)$$

and we show in Figure 8 the mean binned distribution of the particle errors versus $y$. The plots refer to the runs of Figure 6. A key feature is the magnitude of the errors, which in proximity of the interfaces for the IA runs are smaller by factor $\sim 5$ than the standard ones. This is in line with what expected and in accord with what seen in Figure 7 with the growth rates of the IA runs exhibiting a better behavior at low $M$. However one can see from Figure 7 that for $M = 0.05$ the KH instability is not correctly reproduced even in the IA scheme. In such a case gradient errors can be reduced by increasing the simulation resolution.

We do not undertake here a resolution study aimed at assessing the convergence rate to the KH solution in the very low ($M < 0.1$) subsonic regime. We use instead a simple argument to provide a rough estimate of the minimum number of particles $N^2$ which would be necessary to simulate the $M = 0.05$ KH test case.

An $L_1$ error norm for the KH problem has been introduced by Robertson et al. (2010) and, in analogy with their Equation 11, we conjecture here for $L_1$ a generic dependence of the form

$$L_1 \propto N^{-2\alpha} (1 + t)^{\gamma(N)}, \quad (41)$$

on the particle number $N_p = N^2$, and the simulation time $t$. We have dropped the dependency on the bulk flow velocity, present in their Equation 11, and for the power-law dependencies we generically assume the exponents $\alpha$ and $\gamma(N)$.

The simulations of Robertson et al. (2010) were performed using the Eulerian mesh code ART (Kravtsov et al. 1997); note however that their initial condition setup corresponds here to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{Growth rate of the $\lambda = 1/6$ velocity amplitude as measured by making a Fourier transform of $v_y$. The KH tests are for a density contrast $\chi = 2$ and three different values of Mach number have been considered ($M = 0.05, 0.1, 0.35$). For each KH test case we ran IA simulations using the $M_5$, $W_2$ and the $W_4$ kernels. Additionally, for the $M_5$ runs, we also performed a corresponding standard simulation. The solid line is the linear theory growth rate expectation $\propto e^{t/\tau_{KH}}$, normalized to the numerical amplitude at $t = 0$.}
\end{figure}
that seen using moving mesh schemes (Springel 2010). We therefore assume here $\alpha = 1$ as a reasonable slope on resolution convergence, thus putting a conservative lower limit on $N_2$.

In Robertson et al. (2010), the time evolution of the error norm has a weak dependency on numerical resolution: $\gamma(N) \approx 2(64/N)^{0.5}$. This slope clearly depends on the adopted numerical scheme and we simplify this dependency by assuming $\gamma_i = 1$. The impact of this assumption on estimating $N_2$ is however relatively unimportant, the ratio between the two time factors being in any case of order unity and closer to one as $\gamma_i < 1$. In fact, we further simplify the ratio (43) by just removing the time factors.

Finally, an accuracy criterion for the simulations is set by putting an upper limit on the error norm $L_1 \lesssim err(L_1)$, with $err(L_1)$ being a given threshold. We now assume for $err(L_1)$ a generic dependence on the initial velocity amplitude $\delta v_y(0)$ of the form $err(L_1) \propto (\delta v_y(0))^{\beta}$, with $\beta \geq 1$. This lower limit on $\beta$ is justified by the requirement that lower values of $\delta v_y(0)$ must correspond to lower values of $err(L_1)$. Then, for the ratio (43), we have

$$L_2/L_1 = (512/N_2)^2 \lesssim \left( \frac{\delta v_y(0)(2)}{\delta v_y(0)(1)} \right)^\beta = (M_2/M_1)^\beta = (1/7)^\beta.$$

Setting $\beta = 1$ we thus obtain for $N_2$ the lower limit $N_2 \lesssim 1500$. Note however that to achieve numerical convergence in a consistent way in SPH the number of neighbors $N_n$ must also increase when $N \to \infty$ and $h \to 0$ (Zhu et al. 2015). For their Gresho-Chan vortex test Zhu et al. (2015) adopt, when $N_n$ is allowed to vary, $N_n \propto N_1^{1.2}$. In such a case, by referring to the $M = 0.35 \ W$ run with $N_n = 162$ neighbors, we conclude that $N_n \approx 7 \cdot 162 = 1134$ neighbors and $N_p \approx 2 \cdot 10^6$ particles are the least necessary in order to simulate the $M = 0.05$ KH test case using the $W$ Wendland kernel.

Finally, it is worth noting that the difficulties of SPH to follow the formation of KH instabilities depend not only on velocity noise, but also on the local mixing instability (LMI: Agertz et al. 2007; Price 2008; Read et al. 2010; Valcke et al. 2010). This LMI occurs because, in the presence
of a density step, the entropy conservation of SPH causes a pressure blip at the boundary. These pressure discontinuities in turn lead to the presence of shock waves which then inhibit the growth of KH instabilities (Valcke et al. 2010).

Different approaches have been taken to eliminate or reduce the LMI: by introducing initial conditions with a smoothing of the density step (Valcke et al. 2010), and/or adding an AC term to give smooth entropies (Price 2008), or by reformulating the SPH density estimate (Ritchie & Thomas 2001).

Based on a suite of numerical tests, Valcke et al. (2010) argued that for low Mach numbers \( M \leq 0.2 \) the growth of KH instabilities is still suppressed by the LMI. Although the magnitude of the shocks induced by LMI has been greatly reduced by the \( \text{AV} \) scheme of Section 2.2, the growth of KH instabilities (Valcke et al. 2010) found that for low \( M \) the time scales \( \tau_{KH} \) are much higher than those set by the numerical shocks.

It is not trivial to remove from the simulations these residual shocks. For instance, they can be eliminated by applying a relaxing scheme to the initial conditions, but the growing KH instabilities are then strongly suppressed by the induced particle disorder (Valcke et al. 2010). A study on these effects is beyond the scope of this paper.

As a final point, it must be stressed that the results of the KH runs presented here have been obtained by using the AV scheme of Section 2.2 with settings \( \text{AV}_2 \). By replacing this scheme with the AV switch of Cullen & Dehnen (2010) we expect a significant reduction in the amount of AV present in the simulations (see results of the previous and next Section). This in turn will result in a more inviscid behavior and a better capacity of the code to follow the development of the KH instabilities.

4.3. Subsonic turbulence

Studies of driven isothermal subsonic turbulence (Bauer & Springel 2012) have shown substantial differences in the properties of the velocity power spectra extracted from mesh-based simulations, when compared with those produced from the same test runs using the standard formulation of SPH.

Although the use in standard SPH of a time-dependent AV scheme alleviates the problem (Price 2012b), the discrepancies are still present and their origin has been identified as being due to large errors in the SPH gradient estimates (Bauer & Springel 2012). These errors in turn imply the presence of subsonic velocity noise which is higher as lower Mach numbers are considered. As a result, SPH simulations exhibit spectra with a much smaller inertial range (i.e. Kolgomorov-like) than the ones measured using mesh codes.

A faithful numerical modeling of subsonic turbulence is particularly relevant in various astrophysical contexts (star formation, intrachannel medium, intergalactic medium), and it is therefore important to investigate the capability of the IA scheme to properly simulate this test problem.

To this end we set an HCP lattice of \( N^3 \) particles with initially zero velocities inside a periodic box of sidelength \( L = 1 \) and density \( \rho = 1 \). The gas was isothermal with \( \gamma = 1 \) and \( c_s = 1 \). Turbulence in the gas was driven by adding to the momentum Equation 3 of the particles an external stochastic driving force \( \vec{a}_{stir} \). This was constructed in \( k \)-space according to a procedure already used by previous authors (Price & Federrath 2010, Bauer & Springel 2012, Price 2012b, Hopkins 2013, 2015, Zhu et al. 2015).

The power spectrum of \( a(\vec{k})_{stir} \) varies as \( P(k) \propto k^{-5/3} \) and the Fourier modes are nonzero in the range between \( k_{min} = 2\pi/L \) and \( k_{max} = 2k_{min} \). The phases of the stirring field are drawn from an Ornstein-Uhlenbeck (O) process for which the random sequence at the step \( n \) is given by Eswaran & Pope (1988, Bartosch 2001)

\[
\frac{\text{d}z_n}{\text{d}t} = -\frac{1}{\tau} z_n + \sigma \sqrt{1 - f^2} \epsilon_n, \tag{44}
\]

where \( f = \exp (-\text{d}t/\tau) \) is a decaying factor, \( \epsilon_n \) is a Gaussian random variable with unit variance and \( \sigma \) is the variance of the UO process. The constructed sequence then has \( <x_n> = 0 \) and \( <x_{n+1}x_n> = \sigma^2 f \).

In order to obtain a pure solenoidal driving, we apply an Helmholtz decomposition in \( k \)-space:

\[
a_i(k, t) = b_i(k, t) - k_i(\vec{b} \cdot \vec{k})/k^2, \tag{45}
\]

where the vector \( \vec{b}(\vec{k}) \) is a complex vector-
Fig. 9.— Time-averaged velocity power spectra of driven subsonic ($\mathcal{M} \sim 0.3$) isothermal turbulence. The spectra are compensated by $k^{5/3}$ so that the horizontal dotted line indicates the Kolgomorov scaling. We ran simulations using the same driving routine with $N^3 = 64^3$, $128^3$ and $N = 256^3$. Dashed (black) lines are for the standard SPH runs, short-dash (blue) line is the $N^3 = 128^3$ IA run with AV settings $AV_2$ (Section 2.2), solid (red) lines are the IA runs performed using the the AV switch of [Cullen & Dehnen 2010].

As in other works [Bauer & Springel 2012, Hopkins 2015] we measure the spectral properties of the turbulent velocity field to assess the performances of the two codes. The velocity power spectrum is defined as

$$E(k) = 2\pi k^2 P(k),$$

where $k \equiv |\vec{k}|$, and $P(k)$ is the ensemble average velocity power spectrum. This is given by

$$<\vec{u}^\dagger(\vec{k}')\vec{u}(\vec{k})> = \delta_D(\vec{k}' - \vec{k}) P(k),$$

where $\vec{u}(\vec{k})$ is the Fourier transform of the velocity field $\vec{u}(\vec{x})$:

$$\vec{u}(\vec{k}) = \frac{1}{(2\pi)^3} \int \vec{u}(\vec{x}) e^{-i2\pi\vec{k} \cdot \vec{x}} d^3x .$$

In the case of incompressible turbulence, the energy spectrum follows the Kolgomorov scaling $E(k) \propto k^{-5/3}$. To measure the energy spectrum we first set inside the simulation box a cube with $N_g = (2N)^3$ grid points. From the particle velocities $\vec{u}(\vec{x}_i)$ we then estimate the grid velocity field $\vec{u}(\vec{x}_g)$ at the grid points $\vec{x}_g$, using a triangular-shaped cloud function (TSC) interpolation scheme.

We then compute the discrete Fourier transforms of $\vec{u}(\vec{x}_g)$ and the discrete power spectrum $P^d(k) = <|\hat{u}^d(\vec{k})|^2>$ is evaluated by binning the quantity $|\hat{u}^d(\vec{k})|^2$ in spherical shells of radius $k$ and averaging in the bins. The energy density of Equation (15) is then given, aside from a normalization factor, by $E(k) = 2\pi k^2 P^d(k)$. Finally, for a given simulation, the spectrum $E(k)$ is estimated by doing a time-average between $t = 10$ and $t = 25$, $N = 256^3$. For a given resolution we used in both of the schemes the same initial condition set-up and stirring force field. In all of the simulations we ran up to $t = 50$ and adopted the $M5$ kernel with $N_n = 50$ neighbors. We perform standard and IA runs by using the time-dependent AV scheme of Section 2.2 with settings $AV_2$. Additionally, we also run a set of IA simulations by using the AV method of [Cullen & Dehnen 2010]. In the following, we will refer to these IA runs with the term IA-CD, whilst we will use the term IA-AV when referring to the IA runs with AV settings $AV_2$. 

The particle accelerations are calculated at each timestep by updating the stochastic field according to the described procedure, the summation in $k$ space being performed by summing directly at the particle positions. For the driving parameters we use the values of [Bauer & Springel 2012, Hopkins 2015] we measure the spectral properties of the turbulent velocity field to assess the performances of the two codes. The velocity power spectrum is defined as

$$E(k) = 2\pi k^2 P(k),$$

where $k \equiv |\vec{k}|$, and $P(k)$ is the ensemble average velocity power spectrum. This is given by

$$<\vec{u}^\dagger(\vec{k}')\vec{u}(\vec{k})> = \delta_D(\vec{k}' - \vec{k}) P(k),$$

where $\vec{u}(\vec{k})$ is the Fourier transform of the velocity field $\vec{u}(\vec{x})$:

$$\vec{u}(\vec{k}) = \frac{1}{(2\pi)^3} \int \vec{u}(\vec{x}) e^{-i2\pi\vec{k} \cdot \vec{x}} d^3x .$$

In the case of incompressible turbulence, the energy spectrum follows the Kolgomorov scaling $E(k) \propto k^{-5/3}$. To measure the energy spectrum we first set inside the simulation box a cube with $N_g = (2N)^3$ grid points. From the particle velocities $\vec{u}(\vec{x}_i)$ we then estimate the grid velocity field $\vec{u}(\vec{x}_g)$ at the grid points $\vec{x}_g$, using a triangular-shaped cloud function (TSC) interpolation scheme.

We then compute the discrete Fourier transforms of $\vec{u}(\vec{x}_g)$ and the discrete power spectrum $P^d(k) = <|\hat{u}^d(\vec{k})|^2>$ is evaluated by binning the quantity $|\hat{u}^d(\vec{k})|^2$ in spherical shells of radius $k$ and averaging in the bins. The energy density of Equation (15) is then given, aside from a normalization factor, by $E(k) = 2\pi k^2 P^d(k)$. Finally, for a given simulation, the spectrum $E(k)$ is estimated by doing a time-average between $t = 10$ and $t = 25$, $N = 256^3$. For a given resolution we used in both of the schemes the same initial condition set-up and stirring force field. In all of the simulations we ran up to $t = 50$ and adopted the $M5$ kernel with $N_n = 50$ neighbors. We perform standard and IA runs by using the time-dependent AV scheme of Section 2.2 with settings $AV_2$. Additionally, we also run a set of IA simulations by using the AV method of [Cullen & Dehnen 2010]. In the following, we will refer to these IA runs with the term IA-CD, whilst we will use the term IA-AV when referring to the IA runs with AV settings $AV_2$. 

...
with the spectrum being sampled each $\Delta t = 0.08$ time interval.

We show in Figure 9 the spectra $E(k)$ as measured from our simulations. The spectral behavior of the standard runs is in broad agreement with previous findings [Price 2012b; Bauer & Springel 2012; Hopkins 2013, 2015; Zhu et al. 2015]. The spectra are characterized by a very narrow inertial range at low wavenumbers, with a significant decline at higher $k$. The spectra reach a minimum at a wavenumber $k_{\text{turn}}$, which increases as higher resolutions are considered, followed by a steep increase in the power at smaller scales $k \gtrsim k_{\text{turn}}$.

The precise value of $k_{\text{turn}} \sim 40 - 200$ depends on $N$, but it is still much smaller than the noise scale $\sim 2h_{\text{max}}$ set by the kernel. For incompressible turbulence one can easily approximate $h_{\text{max}}$ with the value of $\bar{h}$ given by the average density, thus obtaining $2\pi/2\bar{h} \sim \pi\zeta(N/L)(4\pi/3N_n)^{1/3} \sim 220(N/64) > > k_{\text{turn}}(N)$.

According to Price (2012b) the very limited capability of standard SPH to develop a Kolgomorov-like spectrum is due to the excess of numerical viscosity present in the scheme, which can be reduced by adopting a time-dependent AV switch. In contrast, Bauer & Springel (2012, Figure 6) showed that the rise in power at small scales is mainly a result of the subsonic velocity noise due to kernel gradient errors present in standard SPH. For the same set of initial conditions and forcing sequence, their spectra extracted from runs performed using the moving-mesh code Arepo exhibit an inertial range which extends over more than a decade in $k$. Similar results were later obtained by Hopkins (2015, Figure 27), by using for the same test problem a completely different code (Gizmo).

A similar behavior is found here for the spectra of the IA-CD runs depicted in Figure 9, which show a dramatic improvement over the corresponding standard SPH runs. The spectra exhibit now a much larger inertial range, which increases with resolution and for the $N = 256$ simulation it extends down to $k \sim 200$, close to the minimum scale $k_{\text{max}} \sim 2\pi N/5$ estimated by Hopkins (2015). The spectra are similar to the correspond-

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1 We recall that with the term standard SPH we refer here to the usual SPH scheme of Section 2.1, but incorporating the time-dependent AV switch described in Section 2.2.
strong dependence of velocity power spectra on the AV scheme as being due to the effectiveness of the IA method in removing gradient errors. This in turn implies that AV, which was previously subdominant (Bauer & Springel 2012), is now the main source of noise. The level of noise seen in the IA-AV spectra is then absent in the spectra of the IA-CD runs, because of the limited amount of AV which is generated by the employed AV switch.

Finally, in Figure 10 we show 2D maps of the density $\rho$, velocity $\vec{v}$ and $|\vec{\nabla} \times \vec{v}|^2$ extracted at $t = 50$ from simulations with resolution $N^3 = 128^3$ for both IA and standard SPH runs. A visual comparison between the maps of the two runs clearly indicates the presence in the IA simulation of well resolved small-scale features which are absent in the corresponding standard SPH map. These features of the IA maps appear qualitatively very similar to those obtained, in their tests on subsonic turbulence, by Bauer & Springel (2012, Figure 4) and by Hopkins (2012, Figure 26).

These findings confirm that in SPH simulations, kernel gradient errors play a key role in the modeling of subsonic turbulence, and demonstrate how the IA scheme can be profitably used to overcome these difficulties, with results which compare well with those obtained with other numerical schemes recently proposed.

4.4. Keplerian disc

The cold Keplerian disc problem has been investigated by many authors (Itohka & Inutsuka 2002; Cartwright et al. 2009; Cullen & Dehnen 2010; Hu et al. 2014; Hopkins 2015; Schaal et al. 2015). More specifically, we set-up the disc in a three dimensional periodic domain with boundaries defined by $(-2.5, -2.5, 0) \leq x, y, z < (2.5, 2.5, 0.25) \equiv \{L_x/2, L_y/2, L_z\}$. Within this domain the disc density satisfies

$$\rho(r) = \begin{cases} 1 & 0.5 \leq r \leq 2 \\ 0 & 2 < r \end{cases}$$

where $r$ is the cylindrical radius $r = \sqrt{x^2 + y^2}$.

Unlike in previous settings (Hopkins 2015; Schaal et al. 2015) here the disc edges are unsmoothed, moreover we put in the $x - y$ plane a small empty zone around the disc to avoid border effects so as to mimic vacuum boundary conditions. The gas initially has a very small constant pressure, $P = 10^{-6}$, and index $\gamma = 5/3$.

For the central point mass we set $GM = 1$ and the gas is subject to a static gravitational acceleration $\vec{a} = -\vec{\nabla} \Phi$, where $\Phi = -(r^2 + \epsilon^2)^{-1/2}$ is the softened potential. We introduce a softening parameter, $\epsilon = 0.25$, to avoid diverging accelerations for those particles that during the simulations escape the initial disc configuration and approach the origin at $r = 0$. The initial particle rotational velocity is then $V_\phi = r(r^2 + \epsilon^2)^{-3/4}$ and for the Keplerian orbital period $T = 2\pi r^{3/2}$ we choose as reference value that at $r = 1$. Hereafter we express time in units of $T = 2\pi$.

We implement the initial condition set-up according to the following procedure. We first construct a uniform glass-like distribution of $256^2 \times 16$ particles inside a parallelepiped of side lengths $4 \times 4 \times 0.25$. This is done by creating $16 \times 16$ replicas of a root unit cube of $16^3$ glass-like par-
Fig. 11.— For some of the SPH runs we here show at various times (in units of $T = 2\pi$), 2D density maps of the Keplerian disc. All of the SPH simulations incorporate the IA scheme, the code velocity divergence and vorticity being calculated in accordance with the scheme. Left: initial configuration for a SPH simulation which uses the AV setting $\{\alpha_{\text{min}}, \alpha_{\text{max}}, l_d\} = \{0.01, 1.5, 1.0\}$ (see Section 2.2). Middle: the same simulation but at $t = 14$. Right: here we show at $t = 20$ the disc density for an SPH run in which the AV scheme being used is that of Cullen & Dehnen (2010). To better discriminate disc structure in this run, the size of the computational domain has been reduced to $L_x = L_y = 4.5$.

.. table::

   | $t$ | $\alpha_{\text{min}}$ | $\alpha_{\text{max}}$ | $l_d$ |
   |------|-----------------------|-----------------------|------|
   | 0.00 | 0.01                  | 1.5                   | 1.0  |
   | 14.00|                       |                       |      |
   | 20.00|                       |                       |      |

For some of the simulations performed, we show in Figure 11 density maps of the simulated discs at various times. The maps are 2D slices calculated on an $800 \times 800$ grid located at $z = L_z/2$ in the simulation domain.

We do not show here results from disc simulations performed with the standard SPH implementation; these are in line with previous findings and the disc is found to be subject to disruption after few orbits ($t \sim 2 - 3$). Introducing the IA scheme significantly improves the code capability for evolving the disc, which now can be followed up to $t \sim 15$ before it begins to degrade (Figure 11, left and middle panels).

This SPH simulation (IA-AV_5) was performed by setting $\{\alpha_{\text{min}}, \alpha_{\text{max}}, l_d\} = \{0.01, 1.5, 1.0\}$ for the AV parameters, this choice being indicated with the notation AV_5 in previous calibration tests (Valdarnini 2011). With respect to the set of AV parameters adopted in the other tests performed here (AV_2), the setting AV_5 is characterized by a very low floor value ($\alpha_{\text{min}} = 0.01$) and the shortest possible decay timescale ($l_d = 1.0$). This choice of AV parameters in SPH runs, improves the disc stability though not in a significant way, with disc break-up occurring at $t \sim 12$ when the setting AV_2 is used.

Moreover, it must be stressed that in the IA runs the velocity divergence and vorticity are calculated from a velocity gradient matrix, in accord with the adopted IA scheme. The results presented here are however still valid if a standard SPH estimator is used in the calculation of the ve-
locity gradients, with disc stability being affected only marginally. This shows that errors in hydro-
dynamic forces are dominant in determining disc
stability, with respect to low-order errors affecting
the shear viscosity limiter $^{(8)}$.

A significant improvement in disc stability is
obtained by replacing the time-dependent AV
scheme of Section 2.2 with the improved method
proposed by Cullen & Dehnen (2010), which is
still based on the Morris & Monaghan (1997)
scheme but has a better shock indicator and a
more accurate AV limiter.

An IA-SPH simulation (IA-CD) performed by
incorporating the new AV scheme shows that the
small amount of AV, which still affected disc evo-
lution in the IA-AV$_5$ run, is now removed and the
disc structure is now stable up to 20 orbits. The
disc density map of the IA-CD run is shown at
t = 20 in the right panel of Figure 11, and can
be compared with the corresponding density maps
shown in Figure 6 of Hopkins (2013).

A comparison between the two suites of sim-
ulations is possible because we adopt here the
same initial condition set-up. In previous papers
(Hu et al. 2014; Beck et al. 2016) the Gaussian
ring version of the problem has been used to test
new versions of SPH, but we expect our conclu-
sions to remain unaffected by our choice of initial
conditions.

To summarize, the results presented here for
the Keplerian disc problem demonstrate that for
an SPH code both errors in gradient accuracy and
the level of AV contribute to disc stability, with
the former having a much greater impact.

5. Conclusions

In this paper we have investigated the perfor-
mance of an improved version of the standard SPH
formulation, in which an integral approach is used
to strongly reduce zeroth-order errors in gradient
estimates.

The IA method has been proposed and tested
in a variety of hydrodynamical test problems
(García-Senz et al. 2012; Rosswog 2015), but its
most significant applications are in the simulation
of subsonic flows. In the low Mach number regime
the difficulties of standard SPH have been found
to be particularly severe (Bauer & Springel 2012;
Dehnen & Aly 2012; García-Senz et al. 2012;
McNally et al. 2012; Valdarnini 2012; Hopkins
2013) and gradient accuracy is a key prerequisite
for accurate modeling the fluid dynamics.

Given the advantages of a numerical hydrody-
namical scheme based on a Lagrangian formul-
ation (for instance its natural resolution adaptativ-
ity) it is therefore crucial to assess the capability
of the proposed IA-SPH scheme to handle sub-
sonic flows. Moreover, the IA method retains the
fully conservative nature of the Lagrangian SPH
scheme, unlike previous attempts aimed at remov-
ing zeroth-order gradient errors present in SPH.

To evaluate code performance we have analyzed
results from a suite of simulations of hydrodynam-
tical test problems, performed using both the IA
and standard SPH formulations. We also con-
trast the accuracy of the results with that pro-
duced by new numerical schemes (Springel 2010;
Bauer & Springel 2012; Hopkins 2013), against
which standard SPH has been found clearly inad-
quate. Our main conclusions are as follows.

For the Gresho-Chan vortex problem it is well
known (Read & Hayfield 2012; Dehnen & Aly
2012; Hu et al. 2014; Hopkins 2015) that stan-
dard SPH is heavily affected by the $E_0$ error and
the code performances are very poor. On the
contrary, the IA formulation leads to much bet-
ter behavior, with the results of Section 4.1 being
in line with those obtained by other numerical
schemes (Springel 2010; Hopkins 2013).

The resolution study displayed in Figure 3
shows for the $L_1$ velocity error a gain in accuracy
by a factor $\sim 10$ over standard SPH. Moreover,
the validity of the approach is confirmed even in the
regime of very cold flows. This is demonstrated
by the velocity profiles of Figure 4, in which the
code is shown to be able to reproduce the ana-
lytic solution for the azimuthal velocity down to
$M = 0.02$.

The results of Section 4.2 on KH tests also indi-
cate how zeroth-order errors present in SPH affect
the growth of KH instabilities and the effective-
ness of the IA scheme in reducing these errors.
Nonetheless, in the very low subsonic regimes, the
IA method shows, at a fixed resolution, a pro-
gressively reduced capability to follow the devel-
opment of the instability. This is not surpris-
ing since, for the chosen settings, by reducing the
Mach number the perturbation amplitude is also
reduced and it becomes progressively more challenging, keeping the resolution fixed, to simulate the KH instability when $M \lesssim 0.1$.

In this respect, the heuristic arguments used in Section 4.2 to derive the necessary resolution give a lower limit for $N$ which in any case should be taken with caution, with the required value probably being much higher. In fact, to the author’s knowledge, KH simulations with very low Mach numbers ($M \lesssim 0.1$) have not previously been undertaken in the literature and it would be interesting to compare the findings of Section 4.2 with the behavior of a mesh-based code in these regimes.

The good performances of the IA formulation are confirmed by the results of Section 4.3 on simulations of driven subsonic turbulence, for which the failure of standard SPH to properly model this problem has been debated by various authors [Bauer & Springel 2012, Price 2012, Hopkins 2013, 2015, Zhu et al. 2015]. Simulations performed employing the new scheme produce velocity spectra in better agreement with the Kolmogorov law and exhibit an inertial range which now covers nearly a decade (Figure 9). Here again we see how the results, for the same initial setting and resolution, do not differ significantly from those produced by other codes [Bauer & Springel 2012, Hopkins 2013].

In the Keplerian disc problem, suppression of numerical viscosity is a critical factor for achieving stable evolution. However the results of Section 4.4 also show errors in hydrodynamical forces having a significant impact on disc stability. This suggests that the instabilities leading to disc disruption are sourced by a combination of these two factors. To successfully simulate this test problem, an SPH code must then be necessarily based on both the IA scheme and the improved AV switch of Cullen & Dehnen (2010).

The use of an IA scheme within an SPH framework also raises the issue of revisiting the choice of the kernel in SPH simulations. As discussed in Section 3, the introduction of Wendland kernels stems from the necessity of avoiding pairing instability. But this problem arose from the need to reduce zeroth-order errors present in standard SPH, which are absent or very small in the IA scheme. Therefore, if one adopts the IA-SPH framework, one can resort to the use of the $M_5$ or $M_6$ splines in place of the Wendland kernels. This choice is motivated, for the same number of neighbors, by the better accuracy of the B-splines in estimating densities, when contrasted against the Wendland kernels (see Section 4).

To summarize, the results of our tests demonstrate that by incorporating the IA method in standard SPH, the zeroth-order errors in the momentum equations are drastically reduced, with significant improvements in the performance of the new code.

These results are particularly significative given the importance of subsonic flows in many astrophysical problems. For example, in galaxy clusters subsonic turbulence adds a contribution to the intracluster medium pressure, thus biasing cluster mass estimates and in turn affecting the use of clusters as cosmological probes [Brüggen & Vazza 2013, and references cited therein].

We thus conclude that the new IA-SPH scheme, being based on a Lagrangian formulation, can be profitably used in those simulations of subsonic astrophysical flows in which the shortcomings of standard SPH prevented full exploitation of its resolution adaptativity and conservation properties.

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