DYNAMICAL SYMMETRY BREAKING AND EFFECTIVE LAGRANGIANS IN $U(n)$ FOUR-FERMION MODELS ($n = 2, 3$)

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Abstract

The formation of scalar condensates and dynamical symmetry breaking in the $U(n)$ four-fermion models (for $n = 2, 3$) with two coupling constants has been studied by the functional integration method. The bosonization procedures of the models under consideration are performed in one loop approximation. The propagators of fermions, collective Bose-fields (bound states of fermions), as well as the mass formulas for the fermions and bosons, are found. It is shown that a self-consistent consideration of four-fermion models, in the framework of dimension regularization, provides explicit mass relations for fermions (see equation (22)) for the case $n \geq 3$. The effective Lagrangian of interacting scalar bosons is also derived for the case $n = 2$.

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1 Introduction

Current research aims to derive the effective quark-meson Lagrangians from the fundamental quantum chromodynamics (QCD) Lagrangian, because QCD is the true theory of strong interactions of quarks and gluons. When the perturbation theory is valid due to the smallness of coupling constant $\alpha_s$, the predictions of QCD at high energy levels are confirmed by experimental data. However, reformulation of QCD at low energy levels (the coupling constant $\alpha_s$ is not small) in terms of hadronic fields as bounded quarks, meets serious difficulties as the nonperturbative effects play a very important role. One of the mathematical difficulties is the impossibility of integrating the generating functional for Green’s functions over gluonic fields, as the corresponding integral is not a Gaussian path integral. QCD, at low energy, can be described by local effective chiral Lagrangians (ECL) \[1\], \[2\], \[3\]. However, ECL contain many free parameters.

Under low energy quarks have approximately a contact four-fermion interaction \[4\], \[5\]. Models with four-fermion interactions are similar to the model of superconductivity. Such models take into account both quarks and mesons \[6\]. The problem of renormalization and dynamic mass formation in a four-fermion model with scalar-scalar, pseudoscalar-pseudoscalar and vector-vector interactions was investigated in \[7\]. A CP-odd, chirally non-invariant, four-fermion model with three coupling constants was studied in \[8\].

In the theory of instanton vacuum \[9\], \[10\], the chiral condensate appears to lead to breaking of the symmetry and an effective four-fermion interaction thus emerges \[11\] (see also \[12\]). It was noted by authors \[13\] that a phenomenon of chiral symmetry breaking (CSB) occurs in four-fermion models due to the self-interaction of fermion fields. The vacuum is reconstructed in models considered, and $\gamma_5$ - symmetry is broken. As a result, the condensate is formed and the condition for a vacuum expectation $\langle \bar{\psi}\psi \rangle \neq 0$ is valid, and fermions acquire masses $m = -g_0 \langle \bar{\psi}\psi \rangle$. It should be noted that low masses of pions can be explained by CSB \[14\].

The nonperturbative effects of CSB \[15\] and a confinement of quarks play a very important role in strong interactions. Thus, four-fermion models describe CSB perfectly, but not the confinement of quarks. I mention the QCD string approach \[16\] (see also \[17\]), which takes into consideration the nonperturbative effect of quark confinement. To include the confinement of
quarks, one may introduce the additional nonlocal quark interactions.

Here we consider the four-fermion models with the internal groups $U(n)$ ($n = 2, 3$) which can be identified with the flavor group of quarks but without including the colour group of symmetry. It should be noted that the symmetry group of strong interactions is the chiral $SU(n) \otimes SU(n)$ group. We investigate formal four-fermion models with the $U(n)$ group (which are chiral non-invariant models) and two coupling constants. The consideration of such models allows us to concentrate on the phenomena of dynamical symmetry breaking (DSB) which leads to the mass formation. At the same time, it is not difficult to expand the models under consideration thus introducing the additional colour symmetry (terms recovering chiral symmetry etc.) that do not influence the DSB, vacuum condensates or the mass formation.

The paper is organized as follows: In Sec. 2, the dynamical mass formation of fermions and the $U(n)$ ($n = 2, 3$) symmetry breaking in scalar-scalar four-fermion model with two coupling constants are considered. The bosonization procedure of the model under consideration is performed. Masses and propagators of collective bosonic fields (bound states of fermions) are derived in Sec. 3. In Sec. 4, the effective Lagrangian of interacting bosonic fields is evaluated for the case of the $U(2)$ group. Sec. 5 discusses results.

## 2 Dynamical mass formation and symmetry breaking

Consider a model with the internal symmetry group $U(n)$ and two coupling constants [18], [19]:

\[ L(x) = -\overline{\psi}(x)(\gamma_\mu \partial_\mu + m)\psi(x) + \frac{F}{2} \left[ \overline{\psi}(x)\psi(x) \right]^2 + \frac{G}{2} \left[ \overline{\psi}(x)T^a\psi(x) \right]^2, \]

where $T^a$ ($a = 1, \ldots, n^2 - 1$) are the generators of the group $SU(n)$, $\partial_\mu = (\partial/\partial x_i, -i\partial/\partial x_0)$ ($x_0$ is the time), $m$ is the bare mass of fermions, $\gamma_\mu$ are the Dirac matrices, $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ is the multiplet of fermions. For the $SU(2)$ group we use doublet of fermions $\psi$ and the generators $T^a \equiv \tau^a$ ($\tau^a$ are the Pauli matrices, $a = 1, 2, 3$), and for the $SU(3)$ group – triplet of fermions $\psi$ and the generators $T^a \equiv \lambda^a$ ($\lambda^a$ are the Gell-Mann matrices $a = 1, 2, \ldots, 8$). We took into consideration only scalar-scalar interactions which
lead to DSB. It is convenient to investigate DSB and the mass formation with the help of a functional integration method [20].

The generating functional for Green’s functions

\[
Z[\eta, \eta] = N_0 \int D\psi D\psi \exp \left\{ i \int d^4x \left[ \mathcal{L}(x) + \bar{\psi}(x) \eta(x) + \eta(x) \psi(x) \right] \right\}, \tag{2}
\]

where \( \eta, \eta \) are external sources, \( D\psi = \prod_x d\psi(x) \), using the replacement

\[
N_0 = N \int D\Phi_0 D\Phi_a \exp \left\{ -i \int d^4x \left[ M^2 \frac{\mu^2}{2} \Phi_a(x) + g \bar{\psi}(x) T^a \psi(x) \right] \right\}
\]

\[
+ \frac{M^2}{2} \left( \Phi_0(x) - \frac{f}{M^2} \bar{\psi}(x) \psi(x) \right) \right\}
\]

can be represented as

\[
Z[\eta, \eta] = N \int D\psi D\psi D\Phi_0 D\Phi_a \exp \left\{ i \int d^4x \left[ \bar{\psi}(x) \left[ \gamma_\mu \partial_\mu + m - f \Phi_0(x) \right. \right.
\]

\[
- \left. g \Phi_a(x) T^a \right] \psi(x) - \frac{\mu^2}{2} \phi_a^2(x) - \frac{M^2}{2} \phi_0^2(x) + \bar{\psi}(x) \eta(x) + \eta(x) \psi(x) \}
\]

\[
(3)
\]

Constants \( F = f^2/M^2, \ G = g^2/\mu^2 \) are introduced here, where \( f, g \) are dimensionless constants and the constants \( M, \mu \) have the mass dimensionality. So, dimensional constants \( M, \mu \) play the role of bare masses of bosonic fields \( \Phi_0(x), \Phi_a(x) \) and dimensionless constants \( f, g \) are bare coupling constants. The integral in Eq. (3) is Gaussian in Fermi fields, and after integrating over the \( \bar{\psi}, \psi \), we obtain

\[
Z[\eta, \eta] = N \int D\Phi_0 D\Phi_a \exp \left\{ i S[\Phi] + i \int d^4x d^4y \bar{\eta}(x) S_f(x,y) \eta(y) \right\}, \tag{4}
\]

\[
S[\Phi] = -\frac{1}{2} \int d^4x \left[ \mathcal{M}^2 \Phi_0^2(x) + \mu^2 \Phi_a^2(x) \right] + \frac{1}{2} \int d^4x \bar{\eta}(x) S_f(x,y) \eta(y) \right\}, \tag{5}
\]

where \( S[\Phi] \) is the effective action for bosonic collective fields \( \Phi_0(x), \Phi_a(x) \), and Green’s functions \( \hat{G}, S_f(x,y) \) for free fermions and for fermions in the external collective fields obey the equations

\[
(\gamma_\mu \partial_\mu + m) \hat{G}(x, y) = -\delta(x - y), \tag{6}
\]
\[
[\gamma_\mu \partial_\mu + m - f\Phi_0(x) - g\Phi_a(x)T^a] S_f(x, y) = \delta(x - y). \tag{7}
\]

In order to formulate the perturbation theory \cite{20}, we should find a solution of Eq. (7) for the vacuum averages of the fields \(\Phi_0, \Phi_a\) independent of coordinates (the case of the mean field approximation). In the momentum space Eq. (7) takes the form

\[
(i\hat{p} - A)S_f(p) = 1, \tag{8}
\]

where \(\hat{p} = p_\mu \gamma_\mu, p_\mu = (p, ip_0), A = -m + f\Phi_0 + g\Phi_a T^a\). According to Hamilton-Cayley theorem, the matrix \(A\) satisfies its characteristic equation:

\[
A^2 - b_1 A + \det A = 0 \quad \text{for} \quad U(2), \tag{9}
\]

\[
A^3 - b_1 A^2 + b_2 A - \det A = 0 \quad \text{for} \quad U(3), \tag{10}
\]

where

\[
b_1 = \text{tr} A, \quad b_2 = \frac{1}{2} \left[ (\text{tr} A)^2 - \text{tr} (A^2) \right].
\]

Let us search for a solution to Eq. (8) in the form

\[
S_f(p) = a + b\hat{p} + c_i A_i + d_i \hat{p} A^i, \tag{11}
\]

where \(i = 1\) for the \(U(2)\) group and \(i = 1, 2\) for the \(U(3)\) group, \(\hat{p} A^i\) has the meaning of the direct product of the matrices \(\hat{p}\) and \(A^i\), \(A^i\) is the \(i\)-th power of the matrix \(A\), and a summation over \(i\) (for the \(U(3)\) group) is assumed. Substituting Eq. (11) into Eq. (8), with the help of Eqs. (9), (10) and the fact that the matrices \(I\) (unit matrix), \(\hat{p}, A^i, \hat{p} A^i\) are linearly independent, we obtain a system of equations for the unknown coefficients. Solving this system, we obtain for the \(U(2)\) group

\[
a = -\frac{b_1 \det A}{\Delta_1}, \quad b = -\frac{i}{\Delta_1} \left( p^2 - \det A + b_1^2 \right), \quad d_1 = \frac{ib_1}{\Delta_1},
\]

\[
c_1 = -\frac{1}{\Delta_1} \left( p^2 - \det A \right), \quad \Delta_1 = \left( p^2 + m_1^2 \right) \left( p^2 + m_2^2 \right), \tag{12}
\]

\[
m_1 = m - f\Phi_0 - g\sqrt{\Phi_a^2}, \quad m_2 = m - f\Phi_0 + g\sqrt{\Phi_a^2},
\]

and for the \(U(3)\) group

\[
a = \frac{i \det A}{\Delta_2} \left( p^2 - b_2 \right), \quad b = -\frac{i}{\Delta_2} \left[ (p^2 - b_2)^2 + b_1 \left( p^2 b_1 - b_1 \det A \right) \right],
\]

\[
m_1 = m - f\Phi_0 - g\sqrt{\Phi_a^2}, \quad m_2 = m - f\Phi_0 + g\sqrt{\Phi_a^2}, \quad \Delta_2 = \left( p^2 + m_1^2 \right) \left( p^2 + m_2^2 \right).
\]
\[ c_1 = -\frac{1}{\Delta_2} \left[ p^4 - (b_2 + b_1^2) p^2 - b_1 \det A \right], \quad c_2 = \frac{1}{\Delta_2} \left( b_1^2 p^2 - \det A \right), \]
\[ d_1 = -\frac{i}{\Delta_2} (\det A - b_1 b_2), \quad d_2 = \frac{i}{\Delta_2} \left( p^2 - b_2 \right), \quad (13) \]
\[ \Delta_2 = \det \left( p^2 + A^2 \right) = p^2 \left( p^2 - b_2 \right)^2 + \left( \det A - p^2 b_1 \right)^2 = \left( p^2 + m_1^2 \right) \left( p^2 + m_2^2 \right) \left( p^2 + m_3^2 \right). \]

The eigenvalues of the Hermitian matrix \((-A)\) determine the real dynamical masses of the fermions (the spectrum mass). If the bare masses of fermions \(m\) are zero \((m = 0)\), they acquire the different masses because of DSB. The expressions (11)–(13) define the fermionic Green function in a covariant form, since all the coefficients are expressed through the invariants of the \(U(n)\) \((n = 2, 3)\) group. It is convenient to choose the gauge in which the matrix \(A\) is diagonal. In this case, we can put \(\Phi_0 \neq 0, \Phi_3 \neq 0, \Phi_8 \neq 8\) (for \(U(3)\)), setting the rest of \(\Phi_a\) to zero. Green’s function (11) then takes a diagonal form
\[ S_f(p) = \begin{pmatrix} \frac{-ip + m_1}{p^2 + m_1^2} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{-ip + m_n}{p^2 + m_n^2} \end{pmatrix}, \quad (14) \]
where the masses of fermions are
\[ m_1 = m - f\Phi_0 - g\Phi_3, \quad m_2 = m - f\Phi_0 + g\Phi_3 \quad \text{for} \quad U(2), \]
\[ m_1 = m - f\Phi_0 - g\Phi_3 - \frac{g}{\sqrt{3}}\Phi_8, \quad m_2 = m - f\Phi_0 + g\Phi_3 - \frac{g}{\sqrt{3}}\Phi_8, \quad (15) \]
\[ m_3 = m - f\Phi_0 + 2\frac{g}{\sqrt{3}}\Phi_8 \quad \text{for} \quad U(3). \]

It follows from this that if the fermion bare masses \(m = 0\), the fermions still acquire nonvanishing dynamical masses. From Eqs. (15) we find the values of condensates \(\Phi_0, \Phi_3, \Phi_8\) (for the case of the \(U(3)\) group) via fermion masses:
\[ 2g\Phi_3 = m_2 - m_1, \quad 3(m - f\Phi_0) = m_1 + m_2 + m_3, \]
\[ 2\sqrt{3}g\Phi_8 = 2m_3 - m_1 - m_2. \quad (16) \]

It is seen from Eq. (16) that the bare masses of fermions \(m\) are absorbed by the vacuum field \(\Phi_0\).
3 Masses and propagators of collective bosonic fields

In order to obtain the vacuum condensates $\Phi_0$, $\Phi_3$, $\Phi_8$ from Eq. (5), we solve the equations for the fields $\Phi_A(x)$ ($A = 0, 1, \ldots, 8$):

$$\frac{\delta S[\Phi]}{\delta \Phi_0(x)} = -M^2 \Phi_0(x) + if \text{tr} S_f(x, x) = 0,$$

$$\frac{\delta S[\Phi]}{\delta \Phi_a(x)} = -\mu^2 \Phi_a(x) + ig \text{tr} [S_f(x, x) T^a] = 0. \tag{17}$$

Substituting Eq. (14) into Eq. (17), we obtain a system of equations (gap equations) for the vacuum averages (for the $U(3)$ group):

$$M^2 \Phi_0 = f (I_1 m_1 + I_2 m_2 + I_3 m_3),$$

$$\mu^2 \Phi_3 = g (I_1 m_1 - I_2 m_2),$$

$$\sqrt{3} \mu^2 \Phi_8 = g (I_1 m_1 + I_2 m_2 - 2I_3 m_3), \tag{18}$$

where

$$I_j = \frac{i}{4\pi^4} \int \frac{d^4p}{p^2 + m_j^2} \quad (j = 1, 2, 3, \quad d^4p = id^3p dp_0). \tag{19}$$

In considering the $U(2)$ group, it is necessary to take into consideration the first two equations in (18) and to put $m_3 = 0$. Due to the phase transitions massless fermions ($m = 0$) become massive. The massive states of fermions correspond to the minimum of effective potential \cite{21}, \cite{22}. The integrals in Eq. (19) are quadratically divergent and we can use dimensional regularization \cite{23} or the momentum-cutoff $\Lambda$ which specifies the region of nonlocal interactions of fermions. Note that with cutoff regularization, gap equations (18) have non-trivial, non-analytic solutions \cite{13} if $F\Lambda^2 > 2\pi^2$, $G\Lambda^2 > 2\pi^2$. The integrals with cutoff ($\Lambda$) regularization play the role of form-factors. The parameter $\Lambda$ defines the region of non-locality of quark-antiquark forces. When $\Lambda \rightarrow \infty$, four-fermion interactions become local interactions.

It is known that dimensional regularization is most suited for preserving the symmetry properties of the model. With the help of the dimensional
regularization quadratic and logarithmic divergent integrals are given by [24]:

\[
\int \frac{d^{4-2\varepsilon}p}{p^2 + m^2} = i\pi^{2-\varepsilon} \Gamma(\varepsilon - 1) (m^2)^{1-\varepsilon},
\]

\[
\int \frac{d^{4-2\varepsilon}p}{(p^2 + m^2)^2} = i\pi^{2-\varepsilon} \Gamma(\varepsilon) (m^2)^{-\varepsilon}, \quad \Gamma(\varepsilon) = (\varepsilon - 1) \Gamma(\varepsilon - 1),
\]

where \(\Gamma(x)\) is the gamma-function and \(\varepsilon\) is the parameter of the dimensional regularization (we use the notation \(dp_4 = idp_0\)). In this case, we come to the constraint (see Eq. (19))

\[
I_i = \left(\frac{m_i}{m_j}\right)^2 I_j \quad (i, j = 1, 2, 3).
\]

Using Eq. (21), we find from Eq. (18) the relation \(\Phi_3/\sqrt{3}\Phi_8 = (m_1^3 - m_2^3)/m_1^3 + m_2^3 - 2m_3^3\). Comparing this relation with the equality \(\Phi_3/(\sqrt{3}\Phi_8) = (m_1 - m_2)/(m_1 + m_2 - 2m_3)\), which follows from Eq. (16), we arrive at the mass formula (for the case of the \(U(3)\) group only):

\[
(m_1 - m_2) (m_1 - m_3) (m_2 - m_3) (m_1 + m_2 + m_3) = 0.
\]

It follows from Eqs. (22), (15) that there are solutions as follows: 1) \(m_1 = m_2\) that is \(\Phi_3 = 0\); 2) \(m_1 = m_3\) that requires \(\sqrt{3}\Phi_8 = -\Phi_3\); 3) \(m_2 = m_3\) or \(\sqrt{3}\Phi_8 = \Phi_3\); 4) \(m_1 + m_2 + m_3 = 0\) that is equivalent to \(m = f\Phi_0\). So, self-consistent consideration of gap equations (18) and mass formulas (15) requires one of four conditions. The last possibility implies the negative mass of the third fermion \(m_3 = -m_1 - m_2\) if \(m_1 > 0, m_2 > 0\). If we use the momentum-cutoff \(\Lambda\), Eqs. (21), (22) are not valid.

In order to formulate the perturbation theory, we expand the fields \(\Phi_A(x), A = (0, a)\), in Eq. (4) in the neighborhood of the vacuum averages, these later being the solutions of Eqs. (18):

\[
\Phi_0(x) = \Phi_0 + \Phi'_0(x), \quad \Phi_3(x) = \Phi_3 + \Phi'_3(x), \quad \Phi_8(x) = \Phi_8 + \Phi'_8(x).
\]

After expanding the logarithm in Eq. 5 in the power of the fields \(\Phi'_A\), the effective action (5) can be represented as

\[
S[\Phi'] = -\frac{1}{2} \int d^4x d^4y \Phi'_A(x) \Delta^{-1}_{AB}(x,y) \Phi'_B(y) + \sum_{n=3}^{\infty} \frac{i}{n} \mathrm{tr} \left[ S_f \left( f \Phi'_0 + g \Phi'_a T^a \right)^n \right],
\]
\[
\Delta_{AB}^{-1}(p) = -i g_A g_B \text{tr} \left[ \int \frac{d^4k}{(2\pi)^4} S_f(k) T_A S_f(k-p) T_B \right] + \delta_{AB} M_A^2,
\]

where \( g_A = (f, g) \), \( M_A = (M, \mu) \), \( T_A = (1, T^a) \).

Generators of the \( SU(2) \) group are the Pauli matrices \( \tau^a \) \((a = 1, 2, 3)\), and for the \( SU(3) \) group – Gell-Mann matrices \( \lambda^a \) \((a = 1, 2, \ldots, 8)\). Calculating the nonvanishing elements of the inverse propagators of the auxiliary fields \( \Phi'(x) \) in the momentum space, with the accuracy of \( \mathcal{O}(g^2) \), \( \mathcal{O}(f^2) \), \( \mathcal{O}(fg) \), one finds for the \( U(2) \) group:

\[
\begin{align*}
\Delta_{11}^{-1}(p) &= \Delta_{22}^{-1}(p) = \mu^2 + g^2 (I_1 + I_2) + \left[ p^2 + (m_1 + m_2)^2 \right] Z_3^{-1}, \\
\Delta_{33}^{-1}(p) &= \mu^2 + g^2 (I_1 + I_2) + \left[ p^2 + 2 \left( m_1^2 + m_2^2 \right) \right] Z_3^{-1}, \\
\Delta_{00}^{-1}(p) &= M^2 + f^2 (I_1 + I_2) + \left[ p^2 + 2 \left( m_1^2 + m_2^2 \right) \right] \frac{f^2}{g^2} Z_3^{-1}, \\
\Delta_{03}^{-1}(p) &= fg (I_1 - I_2) + \frac{2f}{g} (m_1^2 - m_2^2) Z_3^{-1},
\end{align*}
\]

and for the \( U(3) \) group:

\[
\begin{align*}
\Delta_{11}^{-1}(p) &= \Delta_{22}^{-1}(p) = \mu^2 + g^2 (I_1 + I_2) + \left[ p^2 + (m_1 + m_2)^2 \right] Z_3^{-1}, \\
\Delta_{33}^{-1}(p) &= \mu^2 + g^2 (I_1 + I_2) + \left[ p^2 + 2 \left( m_1^2 + m_2^2 \right) \right] Z_3^{-1}, \\
\Delta_{44}^{-1}(p) &= \Delta_{55}^{-1}(p) = \mu^2 + g^2 (I_1 + I_3) + \left[ p^2 + (m_1 + m_3)^2 \right] Z_3^{-1}, \\
\Delta_{66}^{-1}(p) &= \Delta_{77}^{-1}(p) = \mu^2 + g^2 (I_2 + I_3) + \left[ p^2 + (m_2 + m_3)^2 \right] Z_3^{-1}, \\
\Delta_{60}^{-1}(p) &= M^2 + f^2 (I_1 + I_2 + I_3) + \left[ p^2 + \frac{4}{3} \left( m_1^2 + m_2^2 + m_3^2 \right) \right] 3f^2 \frac{2g^2}{Z_3^{-1}}, \\
\Delta_{88}^{-1}(p) &= \mu^2 + \frac{g^2}{3} (I_1 + I_2 + 4I_3) + \left[ p^2 + \frac{2}{3} \left( m_1^2 + m_2^2 \right) + \frac{8}{3} m_3^2 \right] Z_3^{-1}, \\
\Delta_{03}^{-1}(p) &= fg (I_1 - I_2) + \frac{2f}{g} (m_1^2 - m_2^2) Z_3^{-1}, \\
\Delta_{68}^{-1}(p) &= \frac{fg}{\sqrt{3}} (I_1 + I_2 - 2I_3) + \frac{2f}{\sqrt{3}g} (m_1^2 + m_2^2 - 2m_3^2) Z_3^{-1},
\end{align*}
\]
\[ \Delta_{3S}^{-1}(p) = \frac{g^2}{\sqrt{3}} (I_1 - I_2) + \frac{2}{\sqrt{3}} (m_1^2 - m_2^2) Z_3^{-1}, \]

where the constant of renormalization is given by

\[ Z_3^{-1} = -\frac{ig^2}{4\pi^4} \int \frac{d^4q}{(q^2 + m_1^2)^2}. \]  

(26)

Let us introduce the renormalized fields

\[ \Phi_a(x) = Z_{3}^{-1/2} \Phi'_a(x), \quad \Phi_0(x) = \sqrt{\frac{\pi}{2g}} Z_{3}^{-1/2} \Phi'_0(x) \]

and coupling constants \( g'^2 = Z_3 g^2, \ f'^2 = Z_3 f^2 \). Using the dimensional regularization, Eqs. (20), and the relation \( \lim_{\varepsilon \to 0} \varepsilon \Gamma(\varepsilon - 1) = -1 \), we arrive at the constraint (see also [25]):

\[ Z_{3}^{-1} = \frac{g^2}{m_1^2} I_1 - \frac{g^2}{4\pi^2}. \]  

(27)

Up to \( O(g^2), \ O(f^2) \), we find the renormalized free (quadratic in the collective fields) effective action

\[ S_{\text{free}}[\Phi] = -\frac{1}{2} \int d^4x \left[ (\partial_\mu \Phi_A(x))^2 + m_{AB}^2 \Phi_A(x) \Phi_B(x) \right], \]  

(28)

where \( A = (0,a) \) and the elements of the mass matrices for the \( U(2) \) group are given by

\[ m_{00}^2 = 3 \left( m_1^2 + m_2^2 \right) + \frac{2 \left( m_1^3 + m_2^3 \right)}{2m - m_1 - m_2}, \quad m_{11}^2 = m_{22}^2 = 0, \]

\[ m_{03}^2 = 3 \left( m_1^2 - m_2^2 \right), \quad m_{33}^2 = (m_1 - m_2)^2. \]  

(29)

We took into consideration that according to the gap equations (18) for the \( U(2) \) group (the first two equations with \( m_3 = 0, \ m_1 \neq m_2 \)), and Eqs. (15), the mass parameters \( M^2, \ \mu \) are given by

\[ M^2 = \frac{2f^2 I_1 \left( m_1^3 + m_2^3 \right)}{m_1^2 (2m - m_1 - m_2)}, \quad \mu^2 = -\frac{2g^2 I_1 \left( m_1^3 + m_1 m_2 + m_2^3 \right)}{m_1^2}. \]

At the particular case \( m = 0 \), one arrives from Eq. (29) to the relation \( m_{00} = m_1 + m_2 \). If masses of fermions equal, \( m_1 = m_2 \), the mass matrix
$m_{AB}$ is diagonal, and the result is the mass of scalar bosons $m_{00} = 2m_1$ and $m_{11} = m_{22} = m_{33} = 0$. The mass $m_{00}$ is identified in the quark models with the $\sigma$ meson mass [4], [5], and fields $\Phi_a$ become the Goldstone massless bosons because the symmetry $SU(2)$ is not broken. The same relation holds in the four-fermion models without internal group of symmetry [6], [7], [13].

To calculate the mass matrix $m_{AB}$ for the $U(3)$ group, it is necessary to specify the solutions of the mass equation (22). If we apply Lagrangian (1) to the real quarks interactions, one can identify the triplet of fermions $\psi = (\psi_1, \psi_2, \psi_3)$ with the triplet of light quarks ($u, d, s$). In this case we may take into consideration the real constituent quark masses (the bare masses of quarks are the same, $m$) [5] $m_1 = m_u \approx m_2 = m_d \approx 230$ MeV, $m_3 = m_s \approx 460$ MeV. Then one comes to the relation $m_3 \approx m_1 + m_2$. It should be stressed that the sign of the mass in the Dirac equation for a fermion can be changed $m_0 \rightarrow -m_0$ without loss of generality. Therefore, we arrive at our case when $m_3 = -m_1 - m_2$ ($m = f \Phi_0$). Then using the self-consistent equations (18) and Eqs. (15), one finds

$$M^2 = \frac{3f_1^2 I_1 (m_1^3 + m_2^3 + m_3^3)}{m_1^2 (3m - m_1 - m_2 - m_3)}, \quad \mu^2 = -\frac{2g_1^2 I_1 (m_1^2 + m_1 m_2 + m_3^2)}{m_1^2}.$$  

With the aid of these equations, we find from Eqs. (25) the elements of the mass matrix:

$$m_{00}^2 = 2 \left( m_1^2 + m_2^2 + m_3^2 \right) + \frac{2 (m_1^3 + m_2^3 + m_3^3)}{3m - m_1 - m_2 - m_3} = 4 \left( m_1^2 + m_2^2 + m_1 m_2 \right) - \frac{2m_1 m_2 (m_1 + m_2)}{m},$$

$$m_{11}^2 = m_{22}^2 = m_{44}^2 = m_{55}^2 = m_{66}^2 = m_{77}^2 = 0,$$

$$m_{88}^2 = 4m_3^2 - (m_1 + m_2)^2 = 3 (m_1 + m_2)^2, \quad m_{03}^2 = \sqrt{6} \left( m_1^2 - m_2^2 \right),$$

$$m_{08}^2 = \sqrt{2} \left( m_1^2 + m_2^2 - 2m_3^2 \right) = -\sqrt{2} \left( m_1^2 + 4m_1 m_2 + m_3^2 \right),$$

$$m_{38}^2 = \sqrt{3} \left( m_1^2 - m_2^2 \right), \quad m_{33}^2 = (m_1 - m_2)^2.$$  

We imply here that $m \neq 0$. If the bare mass of fermions $m = 0$, then in accordance with the gap equations (18), $\Phi_0 \neq 0$, and one arrives at the inequality (see Eqs. (16)) $m_3 \neq -m_1 - m_2$.  

11
To obtain the mass spectrum of the bosonic fields $\Phi_A(x)$, one must diagonalize the mass matrix $m_{AB}$. It follows from Eqs. (29), (30) that masses of the fields $\Phi_1(x)$, $\Phi_1(x)$ vanish, which is in agreement with the Goldstone theorem [26] concerning spontaneous (or dynamical) symmetry breaking. The other fields acquire nonzero masses. Consider the case of the $U(2)$ group. To diagonalize the matrix with the elements (29), we make the $SO(2)$-transformations

$$\Phi'_0(x) = \Phi_0(x) \cos \alpha - \Phi_3(x) \sin \alpha, \quad \Phi'_3(x) = \Phi_0(x) \sin \alpha + \Phi_3(x) \cos \alpha,$$

where

$$\tan 2\alpha = \frac{2m_{00}^2}{m_{33}^2 - m_{00}^2}.$$ 

Thus the mass matrix is diagonalized, and one comes to the following masses of bosonic fields $\Phi'_0(x)$, $\Phi'_3(x)$ (the fields $\Phi_1(x)$, $\Phi_2(x)$ remain massless):

$$m'^2_{00} = m_{00}^2 \cos^2 \alpha + m_{33}^2 \sin^2 \alpha - m_{03}^2 \sin 2\alpha,$$

$$m'^2_{33} = m_{00}^2 \sin^2 \alpha + m_{33}^2 \cos^2 \alpha + m_{03}^2 \sin 2\alpha.$$ 

It should be noted that the transformations of the collective fields (31) are generated by the corresponding fermion fields $\psi(x)$. To diagonalize the mass matrix (30) for the group $U(3)$, it is necessary to make the transformation of the fields $\Phi_0(x)$, $\Phi_3(x)$, $\Phi_8(x)$.

### 4 Effective Lagrangian of Bosonic Fields

Now we use the expression for the effective action [26]:

$$S_{eff} = -\frac{1}{2} \int d^4x d^4y \Phi_A(x) \Delta_{AB}^{-1}(x,y) \Phi_B(y)$$

$$+ \frac{1}{3!} \int d^4x d^4y d^4z \Phi_A(x) \Phi_B(y) \Phi_C(z) \Gamma_{ABC}(x,y,z)$$

$$+ \frac{1}{4!} \int d^4x d^4y d^4z d^4t \Phi_A(x) \Phi_B(y) \Phi_C(z) \Phi_D(t) \Gamma_{ABCD}(x,y,z,t).$$

Let us evaluate the terms in the sum (23), implying that parameters of expansion $g^2/4\pi^2 < 1$, $f^2/4\pi^2 < 1$. We count the components in (23) with
$n = 3$ and $n = 4$. The fermion loops at $n > 4$ give small convergent expressions at $\varepsilon \to 0$ ($\Lambda \to \infty$). Vertex functions entering Eq. (33) are defined as

$$
\Gamma_{ABC}(x, y, z) = \frac{\delta^3 S[\Phi]}{\delta \Phi_A(x) \delta \Phi_B(y) \delta \Phi_C(z)},
$$

(34)

$$
\Gamma_{ABCD}(x, y, z, t) = \frac{\delta^4 S[\Phi]}{\delta \Phi_A(x) \delta \Phi_B(y) \delta \Phi_C(z) \delta \Phi_D(t)}.
$$

In the momentum space, three- and four-point functions are given by

$$
\Gamma_{ABC}(k_1, k_2) = g_{AB}g_{CG} \text{tr} \left\{ \int \frac{d^4p}{(2\pi)^4} \left[ S_f(p + k_1 - k_2)T_{A}S_f(p)T_{B}S_f(p + k_1) + S_f(p - k_1)T_{B}S_f(p)T_{A}S_f(p + k_2 - k_1) \right] T_C \right\},
$$

(35)

$$
\Gamma_{ABCD}(k_1, k_2, k_3) = g_{AB}g_{CG} \text{tr} \left\{ \int \frac{d^4p}{(2\pi)^4} \left[ S_f(p)T_{D}S_f(p + k_2) \times \left[ T_{C}S_f(p + k_2 - k_3)T_{D}S_f(p - k_1)T_{A} + T_{C}S_f(p + k_2 - k_3)T_{A}S_f(p + k_1 + k_2 - k_3)T_{B} + T_{B}S_f(p - k_1 + k_3)T_{C}S_f(p + k_1 + k_2 - k_3)T_{B} + T_{A}S_f(p + k_1 + k_2)T_{B}S_f(p + k_3)T_C + T_{B}S_f(p - k_1 + k_3)T_{A}S_f(p + k_3)T_C \right] \right\} \right\}
$$

(36)

Inserting the Green function (14) (for $n = 2$) into integrals (35), (36), after calculations and renormalization with the help of Eq. (26), we arrive, for the case of $U(2)$ group ($T_a = \tau_a$), at the effective Lagrangian (corresponding to the action (33)) of interacting bosonic fields:

$$
\mathcal{L}_{int}(x) = -3g (m_1 + m_2) \Phi_0(x)\Phi_a^2(x) - g (m_1 + m_2) \Phi_0^3(x) - 3g (m_1 - m_2) \Phi_3(x)\Phi_0^2(x) - g (m_1 - m_2) \Phi_3(x)\Phi_a^2(x) - \frac{g^2}{4} \text{tr}\Phi^4(x),
$$

(37)

where $\Phi(x) = \Phi_0(x) + \tau^a \Phi_a(x)$. If the vacuum field $\Phi_3 = 0$, then according to Eq. (29), the equality $m_1 = m_2$ is valid and the symmetry of the group $U(2)$ is recovered. Therefore, all fields $\Phi_a(x)$ become massless, but the field $\Phi_0(x)$ is still massive. In the same manner, one can calculate the effective Lagrangian for the case of the $U(3)$ symmetry group of fermion fields.
5 Discussion

We have just considered the mass formation and DSB in the $U(n)$ four-fermion models (for $n = 2, 3$) with two coupling constants on the basis of the functional integration method and the bosonization procedure. In one loop approximation the propagators of fermions and collective Bose-fields (bound states of fermions) have been evaluated. It is interesting that, in the framework of the dimension regularization, a self-consistent consideration of gap equations and condensates (vacuum averages of the collective fields) leads to the mass formula (22) for the case of the $U(3)$ group. This new feature of four-fermion models allows us to calculate the quark masses. However, it requires the consideration of a model which possesses the additional colour symmetry, chiral symmetry, and containing boson fields with quantum numbers of real mesons (see [4], [5]). It is noted that similar original relationships of the type (22) hold in the case of the $U(n)$ ($n > 3$) group (see [27] for the case of $U(5)$ group). In our scheme the quadratic and logarithmic diverging integrals, $I_j$ and $Z_3$ are connected by the relation (27), but in the four-fermion models with the cutoff regularization, they are considered independent. Using the dimension regularization, we found the masses of collective boson fields $\Phi_A$ (29), (30) that are in agreement with the Goldstone theorem. The original effective Lagrangian of interacting scalar bosons (37) has also been derived for the case $n = 2$. So, a self-consistent consideration of four-fermion models provides the mass relations for fermions and their bound states (collective fields $\Phi_A(x)$).

It is emphasized that in the approach considered, the parameter of the dimension regularization, $\varepsilon$, possesses physical meaning because it enters the gap equations (18) which define mass formulas. All integrals in such a scheme are finite leading to the “finite renormalization”. The four-fermion models can be considered as an approximation to the description of the real quark interactions. They lead to DSB but do not provide the confinement of quarks. One may modify the model by introducing the nonlocal interactions which approximate the linear potential between quarks.

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