Finite axisymmetric charged dust disks in conformatstatic spacetimes

Guillermo A. González
Escuela de Física, Universidad Industrial de Santander, A. A. 678, Bucaramanga, Colombia and Departamento de Física Teórica, Universidad del País Vasco, 48080 Bilbao, Spain

Antonio C. Gutiérrez-Piñeres and Paolo A. Ospina
Escuela de Física, Universidad Industrial de Santander, A. A. 678, Bucaramanga, Colombia

An infinite family of axisymmetric charged dust disks of finite extension is presented. The disks are obtained by solving the vacuum Einstein-Maxwell equations for conformatstatic spacetimes, which are characterized by only one metric function. In order to obtain the solutions, it is assumed that the metric function and the electric potential are functionally related, and that the metric function is functionally dependent of another auxiliary function, which is taken as a solution of Laplace equation. The solutions for the auxiliary function are then taken as given by the infinite family of generalized Kalnajs disks recently obtained by González and Reina [44], which is expressed in terms of the oblate spheroidal coordinates and represents a well behaved family of finite axisymmetric flat galaxy models. The so obtained relativistic thin disks have then a charge density that is equal, except maybe by a sign, to their mass density, in such a way that the electric and gravitational forces are in exact balance. The energy density of the disks is everywhere positive and well behaved, vanishing at the edge. Accordingly, as the disks are made of dust, their energy-momentum tensor agrees with all the energy conditions.

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I. INTRODUCTION

The study of axially symmetric solutions of Einstein and Einstein-Maxwell field equations corresponding to disklike configurations of matter, apart from its purely mathematical interest, has a clear astrophysical relevance. Indeed, thin disks can be used to model accretion disks, galaxies in thermodynamical equilibrium and the superposition of a black hole and a galaxy. Disk sources for stationary axially symmetric spacetimes with magnetic fields are also of astrophysical importance mainly in the study of neutron stars, white dwarfs and galaxy formation. Now, although normally it is considered that disks with electric fields do not have clear astrophysical importance, there exists the possibility that some galaxies be positively charged [1], so that the study of charged disks may be of interest not only in the context of exact solutions. Accordingly, through the years, many attempts had been made to find exact solutions, static and stationary, to the Einstein and Einstein-Maxwell equations that have as its source a relativistic thin disk.

Exact solutions having as sources relativistic static thin disks were first studied by Bonnor and Sackfield [2] and Morgan and Morgan [3, 4]. Since then several classes of exact solutions corresponding to static [5, 6, 7, 8, 9, 10, 11, 12, 13] and stationary [14, 15, 16, 17] thin disks have been obtained by different authors and the superposition of a static or stationary thin disk with a black hole has been considered in [18, 19, 20, 21, 22, 23, 24, 25, 26]. Relativistic disks embedded in an expanding FRW universe have been studied in [27], perfect fluid disks with halos in [28] and the stability of thin disks models has been investigated using a first order perturbation of the energy-momentum tensor in [29]. On the other hand, thin disks have been discussed as sources for Kerr-Newman fields [30, 31], magnetostatic axisymmetric metrics [32] and conformatstatic and conformatastationary metrics [33, 34], while models of electrovacuum static counterrotating dust disks were presented in [35]. Charged perfect fluid disks were also studied in [36], and charged perfect fluid disks as sources of static and Taub-NUT-type spacetimes in [37, 38].

Now, between the above mentioned works, particularly interesting are those that consider dust disks in conformatstatic spacetimes [39, 40]. In this case, the charge density of the disks is equal to their mass density and so the electric and gravitational forces are in exact balance. This kind of equilibrium configuration has been called by some authors ‘electrically counterpoised dust’ (ECD) and has been studied with some detail, both in classical and relativistic theories [39, 40, 41, 42, 43]. Now, as the matter content of the source is dust, the corresponding energy-momentum tensor very probably will agree with all the energy conditions, a fact that is particularly relevant for models of relativistic thin disks. Indeed, as can be see in some of the above mentioned works, there are many models of relativistic thin disk that do not agree with these conditions, and also many models that only fulfill these conditions partially.

The conformatstatic thin disks presented at references [33, 34] were obtained by means of the well known ‘displace, cut and reflect’ method in order to introduce a dis-

*Email address: guillego@uis.edu.co
†Email address: gutierrezp@yahoo.com
‡Email address: paolo6506@gmail.com
continuity at the first derivative of one otherwise smooth solution. The result is a solution with a singularity of the delta function type in all the $z = 0$ hypersurface and so can be interpreted as an infinite thin disk. On the other hand, solutions that can be interpreted as thin disks of finite extension can be obtained if a proper coordinate system is introduced. A coordinate system that it adapts naturally to a finite source and it presents the required discontinuous behavior is given by the oblate spheroidal coordinates. Some examples of finite thin disks solutions expressed in these coordinates can be found at references [2, 3, 5, 8].

In this paper we present an infinite family of axially symmetric charged dust disks of finite extension. In order to obtain the thin disk solutions, we will solve the Einstein-Maxwell equations for conformastatic spacetimes assuming that the metric function and the electric potential are functionally related and that the metric function is functionally dependent of another auxiliary function, which is taken as a solution of Laplace equation. Then we take the solutions for the auxiliary potential as given by the infinite family of generalized Kalnajs disks recently obtained by González and Reina [14], which is expressed in terms of the oblate spheroidal coordinates and represents a well behaved family of finite axisymmetric flat galaxy models.

Accordingly, the paper is organized as follows. First, in Sec. II we present the vacuum Einstein-Maxwell equations system and their solution for conformastatic spacetimes. We introduce the assumed functional dependences in order to explicitly integrate the equations system in such a way that the metric function and the electric potential can be expressed in terms of solutions of Laplace equation. The oblate spheroidal coordinates are introduced and the general solution of the Laplace equation expressed in these coordinates is presented.

Next, in Sec. III we obtain the surface energy-momentum tensor and the surface current density by using the distributional approach for the case of conformastatic spacetimes. The relation between the energy density and the charge density is explicitly derived and the energy density is written in terms of the mass density of the Newtonian thin disks, in such a way that is explicitly shown that the relativistic thin disks will be in agreement with all the energy conditions.

Then, in Sec. IV we restrict the general model by considering a particular family of well behaved charged dust disks. We present the particular form of the solutions of Laplace equation that describe finite Newtonian thin disk with a well behaved surface mass density, and we use this solution to obtain the corresponding relativistic charged dust disks. The behavior of the energy density is then graphically analyzed and their main properties are described. Finally, in Sec. V we present a brief discussion of our main results.

II. EINSTEIN-MAXWELL EQUATIONS AND FINITE THIN DISK SOLUTIONS

The vacuum Einstein-Maxwell equations, in geometrized units such that $c = G = \rho_0 = \epsilon_0 = 1$, can be written as

\[ G_{ab} = 8\pi T_{ab}, \]  
\[ F^{ab} \cdot b = 0, \]  
with the electromagnetic energy-momentum tensor given by

\[ T_{ab} = \frac{1}{4\pi} \left[ F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right], \]  
where

\[ F_{ab} = A_{b,a} - A_{a,b} \]  
is the electromagnetic field tensor and $A_a$ is the electromagnetic four potential.

Now then, for a conformastatic spacetime the line element can be written in cylindrical coordinates $x^i = (t, \varphi, r, z)$ as [49]

\[ ds^2 = -e^{2\lambda} dt^2 + e^{-2\lambda}(r^2 d\varphi^2 + dr^2 + dz^2), \]  
where the metric function $\lambda$ do not depends on $t$. So, if we take the electromagnetic potential as

\[ A_a = (-\phi, 0, 0, 0), \]  
where it is assumed that the electric potential $\phi$ also is independent of $t$, the vacuum Einstein-Maxwell equations reduce to

\[ \nabla^2 \lambda = e^{-2\lambda} \nabla \phi \cdot \nabla \phi, \]  
\[ \nabla^2 \phi = 2 \nabla \lambda \cdot \nabla \phi, \]  
\[ \lambda_i \lambda_j = e^{-2\lambda} \phi_i \phi_j, \]  
where $i, j = 1, 2, 3$ and $\nabla$ is the usual differential operator in cylindrical coordinates.

In order to solve the above equations system, first it is assumed that the electric potential $\phi$ is functionally dependent of the metric function $\lambda$, $\phi = \phi(\lambda)$, so that equation (7) implies that

\[ [\phi'(\lambda)]^2 = e^{2\lambda}, \]  
whose solution is given by

\[ \phi = \pm e^{\lambda} + k_1, \]  
with $k_1$ an arbitrary integration constant. With this solution, the equations system (7)-(8) reduces to only one non-linear partial differential equation,

\[ \nabla^2 \lambda = \nabla \lambda \cdot \nabla \lambda, \]
for the metric function \( \lambda \). Then, we assume an additional functional dependence for the metric function \( \lambda \), which is expressed as \( \lambda = \lambda(U) \), where \( U \) is an auxiliary function that is taken as a solution of Laplace equation. Thus then, equation (12) it reduces to

\[
\lambda''(U) = \left[ \lambda'(U) \right]^2,
\]

whence the solution is given by

\[
e^\lambda = \frac{k_3}{U + k_2},
\]

where \( k_2 \) and \( k_3 \) are arbitrary integration constants.

Now, in order to have an appropriated behavior at infinity, we will impose some boundary conditions on the metric function \( \lambda \). Accordingly, the general solution \( \eta \) implies that a polynomial in even powers of \( \lambda \) is taken as a solution of Laplace equation. So, in a next section, we will present a particular choice of these constants corresponding to an infinite family of finite thin disks with a well behaved surface mass density, the family of Kalnaj generalized disks recently presented by González and Reina [44].

### III. ENERGY-MOMENTUM TENSOR AND CURRENT DENSITY

As was pointed in the precedent section, the solutions of the Einstein-Maxwell equations corresponding to a finite disklike source are even functions of the \( z \) coordinate. So then, they are everywhere continuous functions but with their first \( z \)-derivatives discontinuous at the disk surface. Accordingly, in order to obtain the energy-momentum tensor and the current density of the source, we will express the jump across the disk of the first \( z \)-derivatives of the metric tensor as

\[
b_{ab} = [g_{ab,z}]_{z=0^+},
\]

and the jump across the disk of the electromagnetic field tensor as

\[
[F_{za}] = [A_{a,z}{}]_{z=0^+},
\]

where the reflection symmetry of the functions with respect to \( z = 0 \) has been used.

Then, by using the distributional approach [48, 49], the Einstein-Maxwell equations yield an energy-momentum tensor as

\[
T^{ab} = T^{ab}_+ \theta(z) + T^{ab} [1 - \theta(z)] + Q^{ab} \delta(z),
\]

and a current density as

\[
J^a = I^a \delta(z),
\]

where \( \theta(z) \) and \( \delta(z) \) are, respectively, the Heaviside and Dirac distributions with support on \( z = 0 \). Here \( T^{ab}_+ \) are the electromagnetic energy-momentum tensors as defined by [43] for the \( z \geq 0 \) and \( z \leq 0 \) regions, respectively, whereas that

\[
16\pi Q^{ab}_b = b^{a\xi} \delta^\xi_b - b^{a\zeta} \delta^\zeta_b + g^{a\xi} b^\xi_b - g^{a\zeta} b^\zeta_b + 2b^\xi_c (g^{\xi\zeta} \delta^\zeta_b - g^{\xi\xi} \delta^\zeta_b)
\]

gives the part of the energy-momentum tensor corresponding to the disk source, and

\[
4\pi I^a = [F^{a\zeta}] \delta(z)
\]

is the contribution of the disk source to the current density. Now, the “true” surface energy-momentum tensor.
of the disk, $S_{ab}$, and the “true” surface current density, $j^a$, can be obtained through the relationships

$$ S_{ab} = \int Q_{ab} \delta(z) \, ds_n = e^{-\lambda} Q_{ab}, \quad (25) $$

$$ j^a = \int I^a \delta(z) \, ds_n = e^{-\lambda} I^a, \quad (26) $$

where $ds_n = \sqrt{g_{zz}} \, dz$ is the physical measurement of length in the direction normal to the disk.

For the metric (5), the only non-zero component of $Q^a_0$ is

$$ Q^0_0 = -\frac{e^{2\lambda} \lambda z}{2\pi}, \quad (27) $$

whereas that the only non-zero component of $I^a$ is

$$ I^0 = -\frac{\phi z}{2\pi}. \quad (28) $$

Thus, the only non-zero component of surface energy-momentum tensor $S^0_0$ is given by

$$ S^0_0 = -\frac{e^{2\lambda} \lambda z}{2\pi}, \quad (29) $$

and the only non-zero component of the surface current density $j^0$ is

$$ j^0 = -\frac{e^{-\lambda} \phi z}{2\pi}, \quad (30) $$

where all the quantities are evaluated at $z = 0^\pm$.

The surface energy-momentum tensor of the disk and the surface current density of the disk then can be written as

$$ S^{ab} = \epsilon V^a V^b, \quad (31) $$

$$ j^a = \sigma V^a, \quad (32) $$

where

$$ V^a = e^{-\lambda}(1, 0, 0, 0) \quad (33) $$

is the velocity vector of the matter distribution. So then, the energy density and the charge density of the distribution of matter are given by

$$ \epsilon = \frac{e^{2\lambda} \lambda z}{2\pi}, \quad (34) $$

$$ \sigma = -\frac{\phi z}{2\pi}, \quad (35) $$

respectively. Now, by using equation (11), the expression (36) can be written as

$$ \sigma = \mp \epsilon \quad (36) $$

so that the charge density of the disks is equal, except maybe by a sign, to their mass density. Accordingly, the electric and gravitational forces are in exact balance, as in the configurations of ECD that were mentioned at the introduction.

Now then, in the context of classical general relativity, it is assumed that the energy-momentum tensor must fulfill certain requirements, which are embodied in the weak, strong and dominant energy conditions [51]. Indeed, for the case of a dust source, all these conditions it reduce to the condition that the energy density be greater or equal to zero, $\epsilon \geq 0$. On the other hand, from equation (14), we have that the energy density can be written as

$$ \epsilon = -\frac{k \Sigma}{(U + k)^2}, \quad (37) $$

where

$$ \Sigma = \frac{U z}{2\pi} \quad (38) $$

is the Newtonian mass density of a disklike source which gravitational potential is given by $U$. Accordingly, if the Newtonian mass density $\Sigma$ is everywhere no negative, the corresponding relativistic energy density $\epsilon$ will be no negative everywhere only if we take $k < 0$. So then, in order that the energy-momentum tensor of the disks will agree with all the energy conditions, the $C_{2n}$ constants in (18) must be properly chosen in such a way that $\Sigma \geq 0$. Furthermore, if we have a Newtonian potential $U$ that it is negative everywhere, as is expected for a compact Newtonian source, then the energy density $\epsilon$ will be non-singular everywhere at the disk.

IV. A PARTICULAR FAMILY OF DISKS

Now we shall restrict the previous general model by considering a particular family of disks with a well-behaved surface energy density. The members of the family are expressed in terms of particular solutions $U_n$ of the Laplace equation obtained by choosing properly the constants of the general solution (18). The obtained particular solutions $U_n$ represent the Newtonian gravitational potential of finite thin disks with mass density given by

$$ \Sigma_m(r) = \frac{(2n + 1) M}{2\pi a^2} \left[ 1 - \frac{r^2}{a^2} \right]^{m - \frac{1}{2}}, \quad (39) $$

where $M$ and $a$ are, respectively, the total mass and the radius of the disk and we must take $m \geq 1$. For each value of $m$, the constants $C_{2n}$ are defined through the relation (44)

$$ C_{2n} = \frac{K_{2n}}{(2n + 1) q_{2n + 1}(0)}, \quad (40) $$
TABLE I: The $C_{2n}$ constants for $m = 0, \ldots, 6$. \\

| $m$ | $C_0$ | $C_2$ | $C_4$ | $C_6$ | $C_8$ | $C_{10}$ | $C_{12}$ |
|-----|-------|-------|-------|-------|-------|----------|----------|
| 1   | $M/a$ | $M/a$ |       |       |       |          |          |
| 2   | $M/a$ | $10M/a$ | $4M/a$ |       |       |          |          |
| 3   | $M/a$ | $5M/3a$ | $9M/11a$ | $5M/33a$ |       |          |          |
| 4   | $M/a$ | $20M/11a$ | $162M/143a$ | $4M/11a$ | $7M/123a$ |          |          |
| 5   | $M/a$ | $25M/13a$ | $18M/13a$ | $10M/17a$ | $35M/247a$ | $63M/4199a$ |          |
| 6   | $M/a$ | $2M/a$ | $27M/17a$ | $260M/323a$ | $5M/19a$ | $378M/7429a$ | $33M/7429a$ |          |

where

$$K_{2n} = \frac{M}{2a} \left[ \frac{\pi^{1/2} (4n+1) (2m+1)!}{2^{2m} (m-n)! \Gamma(m+n+\frac{1}{2})} \right]$$

for $n \leq m$, and $C_{2n} = 0$ for $n > m$. In Table I we list the values of the nonzero $C_{2n}$ for the first six members of the family. So, by using these constants in the general solution we can easily see that the gravitational potential $U_m$ will be negative everywhere, as was imposed at the previous section.

With the above values for the $C_{2n}$, we can easily compute the corresponding energy density of the disks by using equation (41). Now, in order to graphically illustrate the behavior of the different particular models, first we introduce dimensionless quantities through the relations

$$\tilde{U}_m(\tilde{r}) = \frac{aU_m(\tilde{r})}{M},$$

$$\tilde{\Sigma}_m(\tilde{r}) = \frac{\pi a^2 \Sigma_m(\tilde{r})}{M},$$

$$\tilde{\epsilon}_m(\tilde{r}) = \frac{\tilde{k}\tilde{\Sigma}_m(\tilde{r})}{[\tilde{U}_m(\tilde{r}) + \tilde{k}]^2}$$

with $\tilde{k} = (ka)/M$.

Then, by using the above expressions and the values of the $C_{2n}$ constants given at Table I we obtain the following expressions

$$\tilde{\epsilon}_1 = -\frac{3\tilde{k}\sqrt{1 - \tilde{r}^2}}{2[k + \frac{3\pi}{6}(\tilde{r}^2 - 2)]^2},$$

$$\tilde{\epsilon}_2 = -\frac{5\tilde{k}(1 - \tilde{r}^2)^{3/2}}{2[k - \frac{135\pi}{216}(3\tilde{r}^4 - 8\tilde{r}^2 + 8)]^2},$$

$$\tilde{\epsilon}_3 = -\frac{7\tilde{k}(1 - \tilde{r}^2)^{5/2}}{2[k + \frac{315\pi}{3276}(5\tilde{r}^6 - 18\tilde{r}^4 + 24\tilde{r}^2 - 16)]^2},$$

$$\tilde{\epsilon}_4 = -\frac{9\tilde{k}(1 - \tilde{r}^2)^{7/2}}{2[k - \frac{315\pi}{3276}(35\tilde{r}^8 - 160\tilde{r}^6 + 288\tilde{r}^4 - 256\tilde{r}^2 + 128)]^2},$$

$$\tilde{\epsilon}_5 = -\frac{11\tilde{k}(1 - \tilde{r}^2)^{9/2}}{2[k + \frac{609\pi}{53012}(63\tilde{r}^{10} - 350\tilde{r}^8 + 800\tilde{r}^6 - 960\tilde{r}^4 + 640\tilde{r}^2 - 256)]^2},$$

$$\tilde{\epsilon}_6 = -\frac{13\tilde{k}(1 - \tilde{r}^2)^{11/2}}{2[k - \frac{3003\pi}{2097152}(231\tilde{r}^{12} - 1512\tilde{r}^{10} + 4200\tilde{r}^8 - 6400\tilde{r}^6 + 5760\tilde{r}^4 - 3072\tilde{r}^2 + 1024)]^2}.$$
are different behaviors depending of the values of $m$ and $\tilde{k}$. So, for the first two models, for $m = 1$ and $m = 2$, we find that for small values of $|\tilde{k}|$ the energy density presents a maximum near the edge of the disk, whereas that for higher values of $|\tilde{k}|$ the maximum occurs at the center of the disk. On the other hand, for $m > 3$, we find that for all the values of $|\tilde{k}|$ the maximum of the energy density occurs at the center of the disk.

We can also see that as the value of $m$ increases, the energy density is more concentrated at the center of the disks. Also, in the central part of the disks, as $m$ increases the value of the energy density also increases in such a way that, for a given value of $\tilde{k}$, the energy density is greater for a greater value of $m$. On the other hand, at
V. CONCLUDING REMARKS

We presented an infinite family of axisymmetric charged dust disks of finite extension with well behaved surface energy and charge densities. The disk were obtained by solving the vacuum Einstein-Maxwell equations system for a conformastatic spacetime. In order to obtain the solutions, a functional dependence was assumed between the electric potential and the metric function and between this one and an auxiliary function. The solutions were then expressed in terms of a solution of Laplace equation corresponding to a family of Newtonian thin disks of finite radius, the generalized Kalnajs disks \cite{44}, which describes a well behaved family of flat galaxy models.

The relativistic thin disks here presented have a charge density that is equal, up to a sign, to their energy density, and so they are examples of the commonly named ‘electrically counterpoised dust’ equilibrium configuration. The energy density of the disks is everywhere positive and well behaved, vanishing at the edge. Also, as the value of \( m \) increases, the energy density is more concentrated at the center of the disks, having a maximum at \( r = 0 \) for all the values of \( |\tilde{k}| \). However, for the first two models, for \( m = 1 \) and \( m = 2 \), for small values of \(|\tilde{k}|\) the energy density presents a maximum near the edge of the disk, whereas that for higher values of \(|\tilde{k}|\) the maximum occurs at the center of the disk.

Furthermore, as the energy density of the disks is everywhere positive and the disks are made of dust, all the models are in a complete agreement with all the energy conditions, a fact of particular relevance in the study of relativistic thin disks models. Indeed, as was mentioned at the introduction, many of the relativistic thin disks models that had been studied in the literature do not fully agrees with these conditions.

Now then, as we can see from the equations at Sec. II, the procedure here presented can be applied not only for axisymmetric conformastatic spacetimes but can also be used to obtain non-axisymmetric solutions of the vacuum Einstein-Maxwell equations. Accordingly, the thin disks models here presented can be generalized by considering for the auxiliary function \( U \) solutions of Laplace equations without the imposed axial symmetry. So, we are now working in this direction and the results will be presented in a next paper. Analogously, the generalization to conformastationary spacetimes with magnetic fields is in consideration.

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